Dynamical analysis of anisotropic scalar-field cosmologies for a wide range of potentials

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Abstract

We perform a detailed dynamical analysis of anisotropic scalar-field cosmologies, and in particular of the most significant Kantowski–Sachs, locally rotationally symmetric (LRS) Bianchi I and LRS Bianchi III cases. We follow the new and powerful method of \(f\)-devisers, which allows us to perform the whole analysis for a wide range of potentials. Thus, one can just substitute the specific potential form in the final results and obtain the corresponding behavior, without the need of new calculations. We find a very rich behavior, and amongst others the universe can result in isotropized solutions with observables in agreement with observations, such as de Sitter, quintessence-like, or stiff-dark energy solutions. In particular, all expanding, accelerating, stable attractors are isotropic. Additionally, we prove that as long as matter obeys the null energy condition, bounce behavior is impossible. Finally, applying the general results to the well-studied exponential and power-law potentials, we find that some of the general stable solutions disappear. This feature may be an indication that
such simple potentials might restrict the dynamics in scalar-field cosmology,
opening the way to the introduction of more complicated ones.

Keywords: anisotropic cosmology, dark energy, Bianchi, Kantowski–Sachs,
scalar-field cosmology, dynamical analysis, cosmological bounce
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(Some figures may appear in colour only in the online journal)

1. Introduction

According to combined observations, the current observable universe is homogeneous and
isotropic at a great accuracy [1–3]. Although inflation is the most successful candidate for
explaining such a behavior, strictly speaking the problem is not fully solved, since in the
literature one usually starts from the beginning with a homogeneous and isotropic Friedmann–
Robertson–Walker (FRW) metric, examining the evolution of fluctuations, instead of starting
with an arbitrary metric and show that inflation is indeed realized and that the universe
evolves towards homogeneity and isotropy. However, this complete analysis is hard even for
numerical elaboration [4], and thus in order to extract analytical information one should impose
one assumption more, namely to consider anisotropic but homogeneous cosmologies. This
class of geometries [5] exhibits very interesting cosmological features, both in inflationary
and post-inflationary epochs [6], and in these lines isotropization is a crucial issue, namely
whether the universe can result in isotropic solutions without the need of fine-tuning the model
parameters. Finally, the class of anisotropic geometries has recently gained a lot of interest,
under the light of the recently announced Planck Probe results [7], in which some small
anisotropic ‘anomalies’ seem to appear.

Amongst the various families of homogeneous but anisotropic geometries the most
well-studied are the Bianchi type [8] (see [9] and references therein) and the Kantowski–
Sachs metrics [10–13]. Furthermore, amongst the Bianchi subclass the simplest but still very
interesting geometries are the Bianchi I [12–25] and Bianchi III [12, 26–28]. Thus, in these
geometries one can analytically examine the rich behavior, and incorporate additionally the
matter content of the universe [12–53].

On the other hand, there are now strong evidences that the observable universe is
accelerating [54, 55], which led scientists to follow two directions in order to explain it. The
first is to modify the gravitational sector itself (see [56, 57] for reviews and references therein).
The second is to introduce the concept of dark energy (see [58] and references therein), which
could be the simple cosmological constant, a quintessence scalar field [59–62], a phantom
field [63–66], or the combination of both these fields in a unified scenario named quintom
[67–74]. In all these scalar-field-based scenarios a crucial quantity is the field-potential, since
it can radically affect the cosmological behavior.

In the present work we are interested in performing a dynamical analysis of anisotropic
scalar-field cosmology. This phase-space and stability examination allows us to bypass the
nonlinearities and complications of the cosmological equations, which prevent complete
analytical treatments, obtaining a qualitative description of the global dynamics of these
scenarios, which is independent of the initial conditions and the specific evolution of
the universe. Moreover, in these asymptotic solutions we calculate various observable
quantities, such as the dark-energy and total equation-of-state parameters, the deceleration
parameter, the various density parameters, and of course the isotropization measure. However,
in order to remain general, we extend beyond the usual procedure [12, 75–91], and we follow the method of the $f$-devisers introduced in [92], which allows us to perform the whole analysis without the need of an a priori specification of the potential. Thus, after such a general analysis one can just substitute the specific potential form in the final results, instead of having to repeat the whole dynamical elaboration from the start. This general investigation appears for the first time in the literature and it eliminates almost any calculation need from future relevant investigations.

The paper is organized as follows: in section 2 we briefly present the general features of scalar-field cosmology in anisotropic geometry, introducing the kinematic variables and the various observables. In section 3 we perform a complete dynamical analysis for a wide range of potentials, and we discuss the isotropization and bounce issues. In section 4 we apply the obtained results to the specific case of an exponential potential with a cosmological constant, while in section 5 to the case of a power-law potential. Finally, in section 6 we summarize the obtained results.

2. Anisotropic cosmology

In this section we present the basic features of scalar-field cosmology in anisotropic locally rotationally symmetric (LRS) Bianchi I, LRS Bianchi III and Kantowski–Sachs geometries. The corresponding action takes the usual form, namely [59–62]

$$S_{\text{metric}} = \int \! d^4x \sqrt{-g} \left[ \frac{R}{2} + \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) + \mathcal{L}^{(m)} \right],$$  

(1)

where $g^{\mu\nu}$ is the anisotropic metric described in detail below (we use the metric signature $(-1, 1, 1, 1)$), $R$ is the Ricci scalar, Greek indices run from 0 to 3, and we impose the standard convenient units in which $c = 8\pi G = 1$. In the above action we have added a canonical minimally-coupled scalar field $\phi$, with $V(\phi)$ its potential, in which as usual one attributes the dark-energy sector. Additionally, $\mathcal{L}^{(m)}$ accounts for the matter content of the universe, which is assumed to be an orthogonal perfect fluid with energy density $\rho_m$ and pressure $p_m$, and thus its equation-of-state parameter is $w_m = p_m/\rho_m$, while its barotropic index reads $\gamma \equiv w_m + 1$. Finally, for simplicity we do not consider an explicit cosmological constant, which if necessary can arise from a constant term in the potential $V(\phi)$.

Let us make a comment here concerning the relation of anisotropic geometry with the universe content. In order to have full consistency, the consideration of anisotropic geometry would need the consideration of an anisotropic matter, and a vector instead of a scalar field or the vector coupled to the scalar field (similarly to models of anisotropic inflation [93–100] where a scalar inflaton is coupled to a vector, or similarly to anisotropic Einstein-aether scenarios [102], that is a vector-tensor model of gravitational Lorentz violation [103, 104]). However, since the post-inflationary anisotropies are small, as a first approach one can neglect the anisotropies of the various fluids at the post-inflationary universe, and focus only on the geometrical anisotropies [17, 18, 20, 21, 25, 38, 40–44, 49, 53, 89]. Therefore, in this work we consider a scalar field, and the matter to be homogeneous and isotropic.

Let us now present the cosmological application of the above general scalar-field action in a background of anisotropic geometry. Without loss of generality we focus on the most interesting and well-studied cases, introducing an anisotropic metric of the form [41]:

$$ds^2 = -N(t)^2 dt^2 + [e_1^1(t)]^{-2} dr^2 + [e_2^2(t)]^{-2} [d\theta^2 + S(\theta)^2 d\phi^2],$$  

(2)
where $1/e_1(t)$ and $1/e_2(t)$ are the expansion scale factors. This metric can describe three geometric classes, given by three corresponding forms of $S(\theta)$, namely

$$S(\theta) = \begin{cases} \sin \theta : & \text{Kantowski–Sachs}, \\ e^\theta : & \text{LRS Bianchi I}, \\ \sinh \theta : & \text{LRS Bianchi III}, \end{cases}$$  

(3)

known respectively as Kantowski–Sachs, LRS Bianchi I and LRS Bianchi III geometries. It proves convenient to express the above three cases in a unified way following [109], which will later allow to perform the dynamical analysis in a unified way too, using a dominant normalization variable. This approach was introduced in [110], where the LRS Bianchi type IX models were reduced to a regularized first-order system of differential equations for some bounded gravitational variables. In particular, the LRS Bianchi type I model naturally appears as a boundary subset in LRS Bianchi type III, which appears as an invariant boundary of the LRS Bianchi type VIII models, which can be viewed as an invariant boundary of the LRS Bianchi type IX models [109, 110] (see [111, 112] for a similar global dynamical analysis for LRS Bianchi type VIII models, which can be viewed as an invariant boundary of the LRS Bianchi type II, III, and IX models [109, 110] (see [111, 112] for a similar global dynamical analysis for spatially self-similar, spherically symmetric, perfect-fluid models, and [113, 114] under the addition of a scalar field).

Following [109] we redefine the metric (2) as:

$$ds^2 = -N(t)^2 dr^2 + [e_1(t)]^2 d\varphi^2 + [e_2(t)]^2 [d\theta^2 + k^{-1} \sin(\sqrt{k} \vartheta)^2 d\varphi^2].$$  

(4)

Henceforth, we can deal with the aforementioned three metrics as an 1-parameter family of metrics, where the choices $k = +1$ and $k = -1$ correspond respectively to Kantowski–Sachs and LRS Bianchi III, and $k \to 0$ corresponds to Bianchi I.

We next consider relativistic fluid dynamics (see for instance [105]) in such a geometry. For any given fluid 4-velocity vector field $u^\mu$, the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects into the instantaneous rest-space of a comoving observer (letters from the second half of the Greek alphabet denote covariant spacetime indices). As usual we decompose the covariant derivative $\nabla_\mu u_\nu$ into its irreducible parts

$$\nabla_\mu u_\nu = -u_\mu u_\nu + \sigma_{\mu\nu} + \frac{1}{2} \Theta h_{\mu\nu} - \omega_{\mu\nu},$$

where $\hat{u}_\mu$ is the acceleration vector defined as $\hat{u}_\mu = u^\nu \nabla_\nu u_\mu$ (the dot denotes a derivative with respect to $t$ for the choice of the 4-velocity vector field $u = \delta_t$), $\sigma_{\mu\nu}$ is the symmetric and trace-free shear tensor ($\sigma_{\mu\nu} = \sigma_{\nu\mu}$, $\sigma_{\mu\nu} u^\nu = 0$, $\sigma_{\nu\mu} u^\nu = 0$), and $\omega_{\mu\nu}$ is the antisymmetric vorticity tensor ($\omega_{\nu\mu} = \omega_{\mu\nu}$, $\omega_{\nu\mu} u^\nu = 0$). Moreover, in the above expression we have also introduced the volume expansion scalar $\Theta = \nabla_\mu u^\mu$, which as usual defines a representative length scale $\ell$ along the flow lines [106]:

$$\Theta \equiv \frac{3 \ell}{\ell} = \frac{3 \dot{\ell}}{\ell},$$  

(5)

describing the volume expansion or contraction. That is, $\ell$ is a form of average ‘scale factor’, while $\Theta$ is a form of average Hubble parameter. In particular, it is standard to define the Hubble parameter as $H = \Theta / 3$ [106], and thus it becomes clear that in FRW geometries $\ell$ coincides with the usual scale factor. Furthermore, one can show that these kinematic fields are related through [105, 106]

$$\sigma_{\mu\nu} := \hat{u}_\nu u^\mu + \nabla_\nu (u^\mu) - \frac{1}{2} \Theta h_{\mu\nu}$$  

(6)

$$\omega_{\mu\nu} := -u_{[\nu} \hat{u}_{\mu]} - \nabla_{[\mu} u_{\nu]},$$  

(7)

It proves more convenient to work into the synchronous temporal gauge, where we can set $N$ to any positive function of $t$, with the simplest choice being $N = 1$. Therefore, one obtains the following constraints on kinematical variables:

$$\sigma^\mu_\nu = \text{diag}(0, -2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\mu\nu} = 0,$$  

(8)
where we have defined
\[ \sigma_+ \equiv \frac{1}{3} \frac{d}{dt} \left[ \ln \frac{e_1^1}{e_2^2} \right]. \] (9)

Thus, the Hubble parameter can be written in terms of \( e_1^1 \) and \( e_2^2 \) as
\[ H = -\frac{1}{3} \frac{d}{dt} [\ln e_1^1 (e_2^2)^2], \] (10)
while one can additionally define the Gauss curvature of the 3-spheres \([106]\) as
\[ K = (e_2^2)^2. \] (11)

Finally, in order to relate the above analysis to observations, one defines various density parameters, namely \([75]\): the curvature one
\[ \Omega_k \equiv -\frac{kK}{3H^2}, \] (12)
the matter one
\[ \Omega_m \equiv \frac{\rho_m}{3H^2}, \] (13)
the shear one
\[ \Omega_\sigma \equiv \left( \frac{\sigma_+}{H} \right)^2, \] (14)
and the dark-energy one, which is just the scalar-field one,
\[ \Omega_{DE} \equiv \Omega_\phi \equiv \frac{\frac{1}{2} \dot{\phi}^2 + V(\phi)}{3H^2}, \] (15)
where we have assumed that both matter and scalar field are homogeneous, that is they depend only on time.

Furthermore, we can straightforwardly write the deceleration parameter as
\[ q = -1 - \frac{H}{H^2}, \] (16)
the dark-energy equation-of-state parameter \( w_{DE} \) as
\[ w_{DE} \equiv w_\phi = \frac{\frac{1}{2} \dot{\phi}^2 - 2V(\phi)}{\frac{1}{2} \dot{\phi}^2 + 2V(\phi)}, \] (17)
while it proves convenient to define as usual the ‘total’ equation-of-state parameter
\[ w_{tot} \equiv -1 - \frac{2H}{3H^2} = 2q - \frac{1}{3}. \] (18)

We mention here that in anisotropic scalar-field cosmology there is a discussion on the precise definition of the dark-energy sector, namely whether it should include the shear term along the scalar-field one. Such a question is related to the observational capabilities, and in particular on whether one can measure the shear and the scalar-field energy densities separately. Since \( \Omega_\sigma \) can indeed be estimated separately by measuring the luminosity distance in different directions (if statistically there is a significant anisotropy), one can define the average deviation from \( \Omega_m \) as \( \Omega_{DE} \equiv \Omega_\phi \), while the anisotropic component will be just \( \Omega_\sigma \) \([107]\). Therefore, in this work we define the dark-energy sector to be the scalar-field alone as usual, and in the various tables we provide \( \Omega_{DE} \), \( \Omega_\sigma \) and \( w_{DE} \) separately.

In summary, we have now all the required information to proceed to a detailed investigation of scalar-field anisotropic cosmology, namely the action (1), the kinematic variables (9), (10) and (11), and the observables (12)–(18). In the following subsection we extract the cosmological equations for the three geometries of (3), namely the Kantowski–Sachs, the LRS Bianchi I and the LRS Bianchi III ones in a unified way \([109]\).
2.1. Unified approach for the field equations

Taking the variation of the action (1) for the 1-parameter family of metrics (4) leads to

\[ 3H^2 + kK = 3\sigma_+^2 + \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi) \]  
(19)

\[ -3(\sigma_+ + H)^2 - 2\dot{\sigma}_+ - 2\dot{H} - kK = (\gamma - 1)\rho_m + \frac{1}{2}\dot{\phi}^2 - V(\phi) \]  
(20)

\[ -3\sigma_+^2 + 3\sigma_+ H - 3H^2 + \dot{\sigma}_+ - 2\dot{H} = (\gamma - 1)\rho_m + \frac{1}{2}\dot{\phi}^2 - V(\phi), \]  
(21)

where we have used the Ricci-scalar relation

\[ R = \frac{1}{12}H^2 + \frac{1}{6}\sigma_+^2 + \frac{1}{6}\dot{\sigma}_+ + \frac{2}{3}kK. \]

In the above equations the choice \( k = 1 \) corresponds to Kantowski–Sachs metric [12, 30–40], \( k = -1 \) corresponds to LRS Bianchi III [12, 26–28] and \( k = 0 \) corresponds to LRS Bianchi I [12, 14–25].

Additionally, the evolution of the Gauss curvature is given by

\[ \dot{K} = -2(\sigma_+ + H)K, \]  
(22)

while the evolution equation for \( e_1^{1} \) writes as [108]

\[ \dot{e}_1^{1} = -(H - 2\sigma_+)e_1^{1}. \]  
(23)

Combining (20) and (21) we obtain the shear evolution equation

\[ \ddot{\sigma}_+ = -3H\sigma_+ - \frac{kK}{3}. \]  
(24)

Furthermore, relations (19), (20), (21) and (24) give the Raychaudhuri equation

\[ \dot{H} = -H^2 - 2\sigma_+^2 - \frac{1}{\gamma}(3\gamma - 2)\rho_m - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}V(\phi) \]  
(25)

where we have used the matter equation of state \( p_m = (\gamma - 1)\rho_m \).

Finally, variation of the action (1) with respect to the matter energy density and the scalar field, gives respectively the corresponding evolution equations, namely

\[ \dot{\rho}_m = -3H\gamma\rho_m \]  
(26)

and

\[ \ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi}, \]  
(27)

where the last one can equivalently be written as \( \dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0 \), since \( \rho_\phi = \dot{\phi}^2/2 + V(\phi) \) and \( w_\phi \) is given by (17).

In summary, the relevant cosmological equations for scalar-field cosmology for the 1-parameter family of metrics (4) are: the Gauss constraint or Hamiltonian constraint in the Hamiltonian formulation (which reduces to the Friedmann equation in the isotropic case) (19), the evolution equation for 2-curvature \( K \) (22), the evolution equation for the anisotropies (24), the Raychaudhuri equation (25), and the matter and scalar-field evolution equations (26) and (27) (note that the evolution equation for \( e_1^{1} \) (23) decouples from the rest, allowing for a dimensional reduction of the dynamical system). Lastly, the various density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters are given by (12)–(18).
Table 1. The function $f(s)$ for the most common quintessence potentials [92].

| Potential | References | $f(s)$ |
|-----------|------------|--------|
| $V(\phi) = V_0 e^{-\lambda \phi} + \Lambda$ | [13, 117, 118] | $-s(s - \lambda)$ |
| $V(\phi) = V_0 [e^{\alpha \phi} + e^{\beta \phi}]$ | [119–121] | $-(s + \alpha)(s + \beta)$ |
| $V(\phi) = V_0 [\cosh(\xi \phi) - 1]$ | [59, 60, 83–85, 117, 122–126] | $-\frac{1}{2}(s^2 - \xi^2)$ |
| $V(\phi) = V_0 \sinh^{-\alpha}(\beta \phi)$ | [59, 60, 84, 85, 117, 124, 127] | $\frac{s^2}{\alpha} - \alpha \beta^2$ |

3. Local dynamical analysis for a wide range of potentials

In the previous section we presented scalar-field cosmology in three anisotropic geometries, namely the Kantowski–Sachs, the LRS Bianchi I, and the LRS Bianchi III ones using a unified approach. In this section we perform the full phase-space analysis of these scenarios, however in order to remain general we do not specify the scalar potential form, keeping it arbitrary.

As it is known, in order to perform the dynamical analysis of a cosmological model we have to introduce suitable auxiliary variables in order to transform the cosmological equations into an autonomous dynamical system [12, 75–78], that is in a form $X' = f(X)$, where $X$ is the column vector of the auxiliary variables, $f(X)$ is the column vector of the autonomous equations, and primes denote derivatives with respect to $\ln a$. The critical points $X_c$ are extracted by the requirement $X' = 0$, and in order to determine their stability properties we expand around $X_c$ as $X = X_c + U$, with $U$ the column vector constituted by the perturbations of the auxiliary variables. Therefore, up to first order we obtain $U' = \Xi \cdot U$, where the coefficients of the perturbation equations are contained inside the matrix $\Xi$, and then the type and stability of each critical point is determined by the eigenvalues of $\Xi$ (stable (unstable) point for eigenvalues with negative (positive) real parts, or saddle point for eigenvalues with real parts of different sign).

In order to follow the above procedure and in order to handle the involved differentiations, it is necessary to determine a specific potential form $V(\phi)$ of the scalar field $\phi$, the parameters of which will determine the aforementioned eigenvalues [12, 75–91]. The disadvantage of this procedure is that for each different potential one must repeat all the calculations from the beginning. Therefore, it would be very helpful to develop an extended method that could handle the potential differentiations in a unified way, without the need of any a priori specification. This is exactly the method of $f$-devisers introduced in [92] and applied in [122] for scalar-field FRW cosmologies in the presence of a generalized Chaplygin Gas.

In this extended dynamical analysis method, one introduces two new dynamical variables $s$ and $f$ as

$$s \equiv -\frac{V_\phi(\phi)}{V(\phi)},$$

$$f \equiv \frac{V_{\phi\phi}(\phi)}{V(\phi)} - \frac{V_\phi(\phi)^2}{V(\phi)^2},$$

where the subscript ‘$\phi$’ denotes differentiation with respect to $\phi$. In principle $f$ can be expressed as an explicit function of $s$, that is $f = f(s)$. Therefore, following the above procedure, we can transform our cosmological system into a closed dynamical system for a set of normalized, auxiliary, variables and $s$. Such a procedure is possible for a wide range of potentials, especially for the usual ansatzes of the cosmological literature, where it results to very simple forms for $f(s)$, as can be seen in table 1 (note that for the single exponential potential the $s$-variable is not required, since it is a constant and thus $f$ is automatically zero). However, we mention that the...
method cannot be applied to arbitrary potentials, but only to those that allow for a solution of $f$ in terms of $s$. For instance, in some specific forms such is the logarithmic $V(\phi) \propto \phi^p \ln(\phi)$ [128] and the generalized exponential one $V(\phi) \propto \phi^p e^{-\phi^m}$ [129], used in the inflationary context, $f$ cannot be expressed as a single-valued function of $s$. In these cases one should apply asymptotic techniques in order to extract the dominant branch at large $\phi$-values as in [128, 129]. However, in general, for a wide range of potentials the introduction of the variables $f$ and $s$ adds an extra direction in the phase-space, whose neighboring points correspond to ‘neighboring’ potentials. Therefore, after the general analysis has been completed, the substitution of the specific $f(s)$ for the desired potential gives immediately the specific results, through a form of intersection of the extended phase-space.

On the other hand, when $f(s)$ is given we can straightforwardly reconstruct the corresponding potential form starting with

\begin{equation}
\frac{ds}{d\phi} = -f(s) \tag{30}
\end{equation}

\begin{equation}
\frac{dV}{d\phi} = -sV \tag{31}
\end{equation}

which lead to

\begin{equation}
\phi(s) = \phi_0 - \int_{s_0}^{s} \frac{1}{f(K)} dK \tag{32}
\end{equation}

\begin{equation}
V(s) = e^{\int_{s_0}^{s} \frac{K}{f(K)} dK} V_0 \tag{33}
\end{equation}

with the integration constants satisfying $\phi(s_0) = \phi_0, V(s_0) = \tilde{V}_0$. Note that relations (32) and (33) are always valid, giving the potential in an implicit form. However, for the usual cosmological cases of table 1 we can additionally eliminate $s$ between (32) and (33), and write the potential explicitly as $V = V(\phi)$.

As a specific example let us take the inverse hyperbolic-sine potential. In this case, substitution of the corresponding $f(s)$ of table 1 in (32) and (33) gives $\phi(s) = [\beta \phi_0 + \coth^{-1}(\alpha \phi/s) - \coth^{-1}(\alpha \phi/s^0)]/\beta$ and $V(s) = \tilde{V}_0(s^2 - \alpha^2 \beta^2 s^{2/2}(s^2 - \alpha^2 \beta^2 s^{2/2})^{-\alpha/2}$. Thus, using the parameter values $\tilde{V}_0 = V_0 \sinh^{-\alpha}(\beta \phi_0)$ and $s_0 = -\alpha \beta \coth(\beta \phi_0)$ we obtain $V(\phi) = V_0 \sinh^{-\alpha}[\beta (\phi - 2\phi_0)]$ and the potential $V = V_0 \sinh^{-\alpha}(\beta \phi)$ is recovered at $\phi_0 = 0$.

Finally, note that the $f$-devisers method allows also to reconstruct a scalar-field potential from a model with stable fixed points. In particular, choosing a function $f(s)$ with the requested properties (existence of minimum, intervals of monotony, differentiability) to have late-time stable attractors, one uses (32) and (33) to explicitly obtain $V(\phi)$. This is similar to the superpotential construction method [130], which allows for the construction of stable kink-type solutions in scalar-field cosmological models, starting from the dynamics, and specifically for the Lyapunov stability.

3.1. Kantowski–Sachs scalar-field cosmology

For Kantowski–Sachs scalar-field cosmology the cosmological equations are (19), (22), (24), (25)–(27) for the choice $k = +1$. In order to analyze possible recollapsing models, such as the Kantowski–Sachs ones, it is useful to use a dominant normalization variable. This idea was introduced in [110], where the LRS Bianchi type IX perfect-fluid models were reduced to a regularized first-order system of differential equations for some bounded gravitational
variables. Thus, in order to transform this cosmological system in its autonomous form we introduce the auxiliary variables:

\[Q = \frac{H}{D}, \quad \Sigma = \frac{\sigma^+}{D}, \quad x = \frac{\dot{\phi}}{\sqrt{6D}}, \quad y = \frac{\sqrt{V}}{\sqrt{3D}}, \quad z = \frac{\rho_m}{3D^2}, \quad \mathcal{K} = \frac{kK}{3D^2}, \quad (34)\]

where \(D = \sqrt{H^2 + \frac{1}{3}K}\) is the dominant normalization variable. Furthermore, we introduce the variable \(\tau\) through \(d\tau = D \, dt\), and from now on primes will denote derivatives with respect to \(\tau\) (in the case of flat geometry \(\tau\) becomes just \(\ln a\)). We stress here that apart from the above variables, we will use the two variables \(s\) and \(f\) defined in (28), (29), in order to handle the a wide range of potential \(V(\phi)\).

In terms of these auxiliary variables the first Friedmann equation (19) becomes \(x^2 + y^2 + z + \Sigma^2 = 1\), the \(D\)-definition becomes \(Q^2 + \mathcal{K} = 1\), while from (34) we see that \(y \geq 0, z \geq 0\) and \(\mathcal{K} \geq 0\). Therefore, we conclude that the auxiliary variables obey \(-1 \leq Q \leq 1, -1 \leq \Sigma \leq 1, -1 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1, 0 \leq \mathcal{K} \leq 1\), while \(s\) is unconstrained. By construction the normalized variables are bounded in the above intervals, but \(s\) can take values over the real line. Thus, a complete investigation should include an analysis of the behavior at infinity, using the Poincaré projection method, as in [41, 90], however for simplicity we do not proceed to such a detailed analysis in the present work.

Now, concerning the physical meaning of the auxiliary variables, we deduce that the sign of \(Q\) (which is the sign of \(H\)) determines whether the universe is expanding or contracting, \(\mathcal{K}\) which is related with \(\Omega_k\) (defined in (12) by \(\Omega_k = -\frac{\mathcal{K}}{2}\)) determines the curvature \((\mathcal{K} = 0\) corresponds to flat universe), while \(\Sigma\) determines the anisotropy level \((\Sigma = 0\) corresponds to isotropization).

Additionally, in terms of the auxiliary variables the various density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters, defined in (12)–(18), read:

\[
\begin{align*}
\Omega_k &= 1 - \frac{1}{Q^2}, \\
\Omega_m &= 1 - \frac{(x^2 + y^2 + \Sigma^2)}{Q^2}, \\
\Omega_{DE} &= \Omega_\phi = \frac{x^2 + y^2}{Q^2}, \\
\Omega_\sigma &= \left(\frac{\Sigma}{Q}\right)^2, \\
q &= -3x^2(y - 2) + 3y(y^2 + \Sigma^2 - 1) - 6\Sigma^2 + 2, \\
u_{DE} &= -1 + \frac{2x^2}{x^2 + y^2}, \\
u_{tot} &= \frac{x^2(y - 2) + y(y^2 + \Sigma^2 - 1) - \Sigma^2 + 1}{\Sigma^2 - 1}. \quad (35)
\end{align*}
\]

In summary, using the dimensionless auxiliary variables (34), along with the two constraints, we result to the five-dimensional dynamical system:

\[Q' = -\frac{1}{2}(Q^2 - 1)[3(y - 2)x^2 + 2(Q - 3\Sigma)\Sigma + 3\Sigma(y^2 + \Sigma^2 - 1) + 2], \quad (36)\]

\[\Sigma' = -\frac{3}{2}Q(y - 2)\Sigma^3 + (1 - Q^2)(\Sigma^2 - 1) - \frac{3}{2}Q[(y - 2)x^2 + (y^2 - 1)y + 2]\Sigma, \quad (37)\]
\[ x' = -\frac{3}{2}Q(\gamma - 2)x^3 + \frac{1}{2}x[2\Sigma - Q(2Q - 3\Sigma)\Sigma + 3\gamma(y^2 + \Sigma^2 - 1) + 6] + \sqrt{\frac{3}{2}}sy^2, \quad (38) \]

\[ y' = \frac{1}{2}y[-\sqrt{6}xx + 2\Sigma + Q(-3(\gamma - 2)x^2 - 3(\gamma - 2)\Sigma^2 + 3\gamma - 2Q\Sigma)] - \frac{3}{2}Qy^3\gamma, \quad (39) \]

\[ s' = -\sqrt{6}xf(s), \quad (40) \]

defined on the phase-space
\[ \{(Q, \Sigma, x, y, s) \in \mathbb{R}^5 : -1 \leq Q \leq 1, -1 \leq \Sigma \leq 1, -1 \leq x \leq 1, 0 \leq y \leq 1, \]
\[ 0 \leq x^2 + y^2 + \Sigma^2 \leq 1. \quad (41) \]

There are several invariant sets, that is areas of the phase-space that evolve to themselves under the dynamics, for the dynamical system (36)–(40). Firstly, we have the invariant sets \( Q = \pm 1 \), corresponding to \( K = 0 \), where in particular \( Q = 1 \) corresponds to expanding universe, whereas \( Q = -1 \) corresponds to contracting one. Furthermore, there exist invariant sets corresponding to specific subclasses of the general scenario, such as the set \( x^2 + y^2 + \Sigma^2 = 1 \), that is \( z = 0 \), which corresponds to absence of standard matter, and which includes the isotropic invariant set \( x^2 + y^2 = 1, \Sigma = 0 \), or the set \( y = 0 \) which corresponds to potential absence, or the set \( x = y = 0 \) which corresponds to scalar-field absence.

The scenario of Kantowski–Sachs scalar-field cosmology (the system (36)–(40)) admits 12 isolated critical points (6 corresponding to expanding universe and 6 corresponding to contracting one) and 10 curves of critical points (5 corresponding to expanding universe and 5 corresponding to contracting one), which are displayed in table 2 along with their existence and stability conditions.\(^4\) We use the notation \( \epsilon = \pm 1 \), that is \( \epsilon = +1 \) corresponds to expanding universe, while \( \epsilon = -1 \) corresponds to contracting one. The details of the analysis and the calculation of the various eigenvalues of the \( 5 \times 5 \) perturbation matrix \( \mathbf{X} \) are presented in section A.1 of the appendix. Moreover, for each critical point we calculate the values of the various density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters, given by (35), and we summarize the results in table 3.

Once again we stress that all these results hold for a wide range of potentials, that is why there is a large variety of critical points and curves of critical points. Interestingly enough, as we are going to see in detail in the specific applications of the following sections, for some of the usual potentials of the literature the majority of these stable points disappear or are not stable anymore, and that is why they have not been obtained in the specific-potential literature \([12, 41, 76, 77]\). This feature may be an indication that some of the simple potentials of the literature, such is the exponential and the power-law one, might restrict the dynamics in scalar-field cosmology.

Amongst the above various critical points, we are interested those which correspond to an expanding universe (that is with \( \epsilon = +1 \)), which moreover are stable and thus they can be the late-time state of the universe. In particular:

- **Point \( P^*_1 \)**, which is asymptotically stable for potentials having \( f(0) > 0 \), corresponds to the de Sitter (\( w_\text{tot} = -1 \)), isotropic (\( \Omega_r = 0 \)), accelerating (\( q < 0 \)) universe, which is dark-energy dominated (\( \Omega_{\text{DE}} = 1 \)), with the dark-energy behaving like a cosmological constant (\( w_{\text{DE}} = -1 \)).

\(^4\) Note that for particular cases of \( f(0) \) and \( f'(0) \) some of these points become non-hyperbolic, and therefore in order to extract their stability properties a center manifold analysis \([131]\) is necessary. However, since such an investigation in the general case lies beyond the scope of the present work, we prefer to perform it straightforward in the specific applications of the following sections, whenever it is necessary.
Table 2. The critical points and curves of critical points of the system (36)–(40) of Kantowski– Sachs scalar-field cosmology. We use the notation $\epsilon = \pm 1$, where $\epsilon = +1$ corresponds to expanding universe and $\epsilon = -1$ to contracting one, with the stability conditions outside parentheses corresponding to $\epsilon = +1$ while those inside parentheses to $\epsilon = -1$. We use the notation $s^*$ for the real values of $s$ satisfying $f(s^*) = 0$, and $s_0$ for an wide range of real value. Furthermore, the variable $u$ varies in $[0, 2\pi]$. Finally, note that the points $P^0_1$ and $P^0_2$ are special points of the curves $C_0(0)$, and are given separately for clarity.

| Name       | $Q$ | $\Sigma$ | $x$ | $y$ | $s$ | Existence   | Stability                  |
|------------|-----|----------|-----|-----|-----|-------------|-----------------------------|
| $P^0_1$    | $e$ | $e$      | 0   | 0   | $s_c$ | Always      | Unstable (stable)           |
| $P^0_2$    | $e$ | $-e$     | 0   | 0   | $s_c$ | Always      | Unstable (stable)           |
| $P^0_3$    | $\frac{2\epsilon}{3\epsilon - 1}$ | 0   | 0   | $s_0$ | $0 \leq y \leq \frac{3}{2}$ | Saddle            |
| $P^0_4$    | $0$ | 0        | 0   | 0   | $s_c$ | Always      | Saddle                     |
| $P^0_5$    | $0$ | 0        | 0   | 1   | $0$   | Always      | Stable (unstable)           |
| $P^0_6(\epsilon)$ | $\frac{\epsilon}{\sqrt{1 - \frac{\epsilon^2}{4}}} \sqrt{\frac{1}{\epsilon^2 - 1}}$ | $s^* < 0 < y \leq \frac{3}{2}$ | $s^* > \frac{\epsilon}{\sqrt{1 - \frac{\epsilon^2}{4}}} \sqrt{\frac{1}{\epsilon^2 - 1}}$ | Saddle |
| $P^0_7(s^*)$ | $\frac{2\epsilon s^* + 2}{5\epsilon s^* + 4}$ | $\frac{2\epsilon s^* - 2}{5\epsilon s^* + 4}$ | $s^* > 0 < (s^*)^2 \leq 2$ | Saddle |
| $P^0_8(\epsilon)$ | $\frac{\epsilon}{\sqrt{1 - \frac{\epsilon^2}{4}}} \sqrt{\frac{1}{\epsilon^2 - 1}}$ | $s^* < 0 < (s^*)^2 \leq 6$ | Stable (unstable) |
| $P^0_9(s^*)$ | $\frac{\epsilon}{\sqrt{1 - \frac{\epsilon^2}{4}}} \sqrt{\frac{1}{\epsilon^2 - 1}}$ | $s^* > \frac{\epsilon}{\sqrt{1 - \frac{\epsilon^2}{4}}} \sqrt{\frac{1}{\epsilon^2 - 1}}$ | Saddle otherwise |
| $C_-(s^*)$ | $\cos u$ | $\sin u$ | 0   | $s^*$ | Always      | Stable for $0 \leq y < 2.0 < u < \pi$, $s^* > \sqrt[3]{\csc(a)}f(s^*) > 0$ or $0 \leq y < 2.0 < u < 2\pi$, $s^* < \sqrt[3]{\csc(a)}f(s^*) < 0$ | Saddle otherwise |
| $C_+(s^*)$ | $1$  | $\cos u$ | $\sin u$ | 0   | $s^*$ | Always      | Unstable for $0 \leq y < 2.0 < u < \pi$, $s^* < \sqrt[3]{\csc(a)}f(s^*) < 0$ or $0 \leq y < 2.0 < u < 2\pi$, $s^* > \sqrt[3]{\csc(a)}f(s^*) > 0$ | Saddle otherwise |

- Point $P^0_1(s^*)$ corresponds to an isotropic, dark-energy dominated universe, with the dark-energy equation-of-state parameter lying in the quintessence regime, which can be accelerating or not according to the potential parameters. In the case of exponential potential this point becomes the most important one, since it is both stable and compatible with observations [76].
- Point $P^0_1(s^*)$ corresponds to an isotropic universe with $0 < \Omega_{DE} < 1$, that is it can alleviate the coincidence problem since dark-energy and dark-matter density parameters can be of
the same order. However, it has the disadvantage that for the usual case of dust matter ($\gamma = 1$) it is not accelerating and moreover it leads to $w_{\text{DE}} = 0$, which are not favored by observations.

Finally, note that in the present case the regions of expanding ($\epsilon = +1$) and contracting ($\epsilon = -1$) universe are not disconnected, and thus theoretically one could have heteroclinic orbits from one to the other region, which is the realization of a cosmological bounce. However, as we prove in the end of this section, this is not possible as long as we have a real minimally-coupled scalar field and a matter sector satisfying the null energy condition.

### 3.2. LRS Bianchi III and Bianchi I scalar-field cosmology

Contrary to the Kantowski–Sachs case, where Hubble normalized variables are not compact and thus a separate analysis is needed, in LRS Bianchi III and Bianchi I geometries one can perform the dynamical analysis in a unified way. The corresponding cosmological equations are (19), (22), (24)–(27) for the choices $k = -1$ and $k = 0$, respectively. In order to transform these cosmological system in its autonomous form we introduce the auxiliary variables:

$$
x = \frac{\phi}{\sqrt{6H}}, \quad y = \frac{\sqrt{V}}{\sqrt{3H}}, \quad z = \frac{\rho_m}{3H^2}, \quad \Sigma = \frac{\sigma^+}{H}, \quad \Omega_k = -\frac{kK}{3H^2},
$$

(42)

using also the variable $\tau$ defined through $d\tau = |H|d\tau \equiv \epsilon Hdt$, where $\epsilon = \text{sgn}(H)$ (primes will denote derivatives with respect to $\tau$). Additionally, we use the two variables $s$ and $f$ defined in (28), (29), in order to handle the unspecified potential $V(\phi)$.

We mention here that for the Bianchi I case $\Omega_\kappa$ becomes automatically zero (since $k = 0$), and thus examining both LRS Bianchi III and LRS Bianchi I in a unified way, Bianchi I corresponds to the invariant set $\Omega_\kappa = 0$. However, note that in Bianchi III one can also have

5 An alternative way to obtain the LRS Bianchi I case is from the corresponding Kantowski–Sachs one, setting $Q = \pm 1$ and $D = |H|$, since $K$ does not appear explicitly in the LRS Bianchi I cosmological equations (note however that this does not mean that $K = 0$).
some critical points with $\Omega_k = 0$ as expected, that also belong to the Bianchi I boundary. The difference is that in the latter case the stability will be in principle different, since the system can evolve in the $\Omega_k$-direction, which is not the case in Bianchi I.

From the definitions (42) we deduce that $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $z \geq 0$ and $\Omega_k \geq 0$, while $s$ is un-constrained. Moreover, in terms of the auxiliary variables the Gauss constraint (19) becomes $x^2 + y^2 + z + \Sigma^2 + \Omega_k = 1$, which allows us to reduce the dimension of the phase-space, eliminating one dynamical variable, for instance $z$.

Finally, in terms of the auxiliary variables the various density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters, defined in (12)–(18), read:

$$
\Omega_m = 1 - x^2 - y^2 - \Sigma^2 - \Omega_k,
\Omega_{\text{DE}} = \Omega_\phi = x^2 + y^2,
\Omega_\sigma = \Sigma^2,
q = \frac{3}{2}(2 - y)(x^2 + \Sigma^2) - \frac{3}{2}y^2 + \frac{1}{2}(3y^2 - 2)(1 - \Omega_k),
u_{\text{DE}} = -1 + \frac{2x^2}{x^2 + y^2},
u_{\text{tot}} = -1 + y + \frac{(2 - y)x^2 - y^2}{1 - \Sigma^2 - \Omega_k}.
$$

In summary, using the dimensionless auxiliary variables (42), along with the constraint equation, we result to the five-dimensional dynamical system:

$$
\Sigma' = \epsilon \left[3 - \frac{3y}{2}\right] \Sigma^3 - \frac{1}{2} \Sigma [3(y - 2)x^2 + 3y(y^2 + \Omega_k - 1) - 2(\Omega_k - 3)] + \Omega_k,
$$

$$
x' = \epsilon \left[\frac{3}{2}(2 - y)x^3 + \frac{1}{2}x[2(3\Sigma^2 + \Omega_k - 3) - 3y(\Sigma^2 + y^2 + \Omega_k - 1)] + \sqrt{\frac{3}{2}y^2}\right],
$$

$$
y' = \epsilon \left[-\frac{3y^3}{2} + \frac{1}{2}y[-3y(\Sigma^2 + \Omega_k - 1) - \sqrt{6}\Sigma x + 2(3\Sigma^2 + \Omega_k) - 3(\gamma - 2)x^2]\right],
$$

$$
\Omega'_k = \epsilon \left[(2 - 3y)\Omega_k^2 + \Omega_k[-3(y - 2)\Sigma^2 + 3y - 2\Sigma + (6 - 3y)x^2 - 3y^2 - 2]\right],
$$

$$
s' = -\epsilon \sqrt{6}xf(s),
$$

defined on the phase-space

$$
\{(\Sigma, x, y, \Omega_k, s) \in \mathbb{R}^5 : -1 \leq \Sigma \leq 1, -1 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq \Omega_k \leq 1,
0 \leq x^2 + y^2 + \Omega_k + \Sigma^2 \leq 1\}.
$$

As before, $\epsilon = +1$ corresponds to an expanding universe while $\epsilon = -1$ to a contracting one. However, we mention that in the present case, and contrary to the previous Kantowski–Sachs one, these two regions, that is $y < 0$ and $y > 0$, are not connected, since the sign of $y$ is an invariant, as can be deduced from (46). Therefore, a transition from one to the other is impossible, and thus LRS Bianchi III and LRS Bianchi I scalar-field cosmologies do not allow for a cosmological bounce. This is verified by the analysis of subsection 3.3, where we prove that a bounce is not possible as long as we have a real minimally-coupled scalar field and a matter sector satisfying the null energy condition.
Observe that for LRS Bianchi I models the perturbation matrix is $\frac{1}{\Sigma_1}$.

We use the notation $s^*$ for the real values of $s$ satisfying $f(s^*) = 0$, and $s_\epsilon$ for an arbitrary real value. Finally, the variable $u$ varies in $[0, 2\pi]$.

### Table 4. The critical points and curves of critical points of the system (44)–(48) of LRS Bianchi III and Bianchi I scalar-field cosmologies, in the expanding universe subspace ($\epsilon = +1$). We use the notation $s^*$ for the real values of $s$ satisfying $f(s^*) = 0$, and $s_\epsilon$ for an arbitrary real value. Finally, the variable $u$ varies in $[0, 2\pi]$.

| Name | $\Sigma$ | $x$ | $y$ | $\Omega_k$ | $s$ | Existence | BIII | BI |
|------|---------|-----|-----|-------------|-----|-----------|------|-----|
| $R_1^+$ | 1 | 0 | 0 | 0 | $s_\epsilon$ | Always | ✓ | ✓ |
| $R_1^-$ | $-1$ | 0 | 0 | 0 | $s_\epsilon$ | Always | ✓ | ✓ |
| $R_2^+$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $s_\epsilon$ | Always | ✓ | ✓ |
| $R_3^+$ | 0 | 0 | 0 | 0 | $s_\epsilon$ | Always | ✓ | ✓ |
| $R_4^+$ | $-1 + \frac{3\sqrt{2}}{2}$ | 0 | 0 | $\frac{1}{2}(2 - \gamma)(3\gamma - 2)$ | $s_\epsilon$ | $\frac{2}{3} \leq \gamma \leq 1$ | ✓ | ✗ |
| $R_5^+$ | 0 | 0 | 1 | 0 | $s_\epsilon$ | Always | ✓ | ✓ |
| $R_6(s^*)$ | 0 | $\frac{3\sqrt{2}}{2}$ | 0 | $\frac{3\sqrt{2}}{2}(2 - \gamma)(3\gamma - 2)$ | $s^*$ | $(s^*)^2 \geq 3\gamma, s^* \neq 0$ | ✓ | ✓ |
| $R_7(s^*)$ | $\frac{3\sqrt{2}}{2} - 1$ | $\frac{3\sqrt{2}}{2}$ | 0 | $\frac{3\sqrt{2}}{2}(2 - \gamma)(3\gamma - 2)$ | $s^*$ | $\frac{2}{3} \leq \gamma < 1, \ (s^*)^2 \geq \frac{3\gamma}{1-\gamma}$ | ✓ | ✗ |
| $R_8(s^*)$ | 0 | $\frac{3\sqrt{2}}{2}$ | $\frac{3\sqrt{2}}{2}$ | 0 | $s^*$ | $(s^*)^2 \leq 6, s^* \neq 0$ | ✓ | ✓ |
| $R_9(s^*)$ | 0 | 1 | 0 | 0 | $s^*$ | Always | ✓ | ✓ |
| $R_{10}(s^*)$ | 0 | $-1$ | 0 | 0 | $s^*$ | Always | ✓ | ✓ |
| $C(s^*)$ | $\cos u$ | $\sin u$ | 0 | 0 | $s^*$ | Always | ✓ | ✓ |

Furthermore, observe that the system (44)–(48) is form-invariant under the change

$$\{\epsilon, \Sigma, x, y, \Omega_k, s\} \rightarrow \{-\epsilon, -\Sigma, -x, -y, \Omega_k, s\}$$

and thus if a point is unstable in the expanding branch then its symmetrical partner in the contracting branch will be stable for the same conditions, and vice versa. Therefore, it is sufficient to discuss the behavior of one half of the phase-space, and the dynamics in the other half will be obtained via the transformation (50). Hence, we restrict our analysis to the positive $\epsilon = +1$ (expanding) branch.

There are several invariant sets, for the system (44)–(48) of LRS Bianchi III and I expanding ($\epsilon = +1$) scalar-field cosmology. Amongst them there exists the invariant set $\Omega_k = 0$ (corresponding to $k = 0$ since $e_2^2 = 0$), which is the boundary corresponding to LRS Bianchi I scalar-field cosmology. This invariant set contains the isotropic invariant set $\Omega_k = 0$, $\Sigma = 0$. Finally, there exist the invariant sets $y = 0$ (potential absence), and the invariant set $x = y = 0$ (scalar-field absence).

The scenario of LRS Bianchi III scalar-field cosmology (the system (44)–(48)), in the expanding universe subspace, admits six isolated critical points and seven curves of critical points, which are displayed in table 4 along with their existence conditions. Amongst them, the points satisfying $\Omega_k = 0$ exists for LRS Bianchi I too. However, as we mentioned above, the stability of the points with $\Omega_k = 0$ will be different in the two geometry classes. In particular, the stability analysis of LRS Bianchi I does not require the examination of the stability along the $\Omega_k$-axis, while in LRS Bianchi III this is necessary since the system can move along the $\Omega_k$-direction too. In table 5 we summarize the stability conditions for LRS Bianchi III and LRS Bianchi I cases. The details of the analysis and the calculation of the various eigenvalues of the $5 \times 5$ perturbation matrix $\Xi$ are presented in section A.2 of the appendix.\(^6\)

Finally, for each critical point we calculate the values of the various density parameters, the deceleration parameter, and the dark-energy and total equation-of-state parameters, given

\[^6\] Observe that for LRS Bianchi I models the perturbation matrix is $4 \times 4$, since the direction $\Omega_k$ is absent. The eigenvalues will be the same as for the $5 \times 5$ matrix, apart from the first one that corresponds to the extra $\Omega_k$-direction.
Moreover are stable and thus they can be the late-time state of the universe. In particular:

\[ \text{dynamics in scalar-field cosmology}. \]

...disappear or are not stable anymore, which may be an indication that some of the simple

potentials of the literature, such is the exponential and the power-law one, might restrict the

observables are given by the expanding ones under the transformation \( (50) \).

Lastly, we stress that all these results hold for a wide range of potentials, that is why

there is a large variety of critical points and curves of critical points. Similarly to the previous

Kantowski–Sachs case, for some of the usual potentials the majority of these stable points

disappear or are not stable anymore, which may be an indication that some of the simple

potentials of the literature, such is the exponential and the power-law one, might restrict the

dynamics in scalar-field cosmology.

The physically interesting critical points are those which correspond to expansion, which

moreover are stable and thus they can be the late-time state of the universe. In particular:

\[ \text{by (43), and we summarize the results in table 6. We mention that the above points and curves} \]

\[ \text{correspond only to the half phase-space of expanding solutions. Thus, the whole phase-space} \]

\[ \text{admits also their symmetric partners corresponding to contractions, which coordinates and} \]

\[ \text{observables are given by the expanding ones under the transformation (50).} \]

\[ \text{Lastly, we stress that all these results hold for a wide range of potentials, that is why} \]

\[ \text{there is a large variety of critical points and curves of critical points. Similarly to the previous} \]

\[ \text{Kantowski–Sachs case, for some of the usual potentials the majority of these stable points} \]

\[ \text{disappear or are not stable anymore, which may be an indication that some of the simple} \]

\[ \text{potentials of the literature, such is the exponential and the power-law one, might restrict the} \]

\[ \text{dynamics in scalar-field cosmology.} \]

\[ \text{The physically interesting critical points are those which correspond to expansion, which} \]

\[ \text{moreover are stable and thus they can be the late-time state of the universe. In particular:} \]

---

**Table 5.** Stability conditions for the critical points and curves of critical points of the system (44)–(48) of both LRS Bianchi III and LRS Bianchi I scalar-field cosmologies, in the expanding universe subspace \((\epsilon = +1)\). We use the same notations as in table 4.

| Name    | Stability of LRS BIII models | Stability restricted to LRS BI boundary |
|---------|-----------------------------|--------------------------------------|
| \( R^* \) | Unstable                    | Unstable                             |
| \( R^- \) | Unstable                    | Unstable                             |
| \( R_2 \) | Saddle                      | \( x \)                               |
| \( R_3 \) | Saddle                      | Saddle                               |
| \( R_4 \) | Saddle                      | \( x \)                               |
| \( R_5 \) | Stable for \( f(0) > 0 \) or saddle otherwise | Same as for LRS BIII |
| \( R_6(s^*) \) | Stable for \( 0 < \gamma < \frac{3}{4}, s^* < -\sqrt{3\gamma} \) and \( f'(s^*) < 0 \), or \( 0 < \gamma < \frac{3}{4}, s^* \geq \sqrt{3\gamma} \) and \( f'(s^*) > 0 \); saddle otherwise | Stable for \( \frac{3}{4} < \gamma < \frac{3}{2}, \gamma \neq \frac{2}{3} \), \( s^* < -\sqrt{3\gamma} \) and \( f'(s^*) < 0 \), or \( 0 < \gamma < \frac{3}{2}, s^* > \sqrt{3\gamma} \) and \( f'(s^*) > 0 \); saddle otherwise |
| \( R_7(s^*) \) | Stable for \( \frac{3}{4} < \gamma < 1, s^* < -\sqrt{\frac{3\gamma}{1 - \gamma}} \), \( f'(s^*) < 0 \), or \( \frac{3}{4} < \gamma < 1, s^* > \sqrt{\frac{3\gamma}{1 - \gamma}} \), \( f'(s^*) > 0 \); saddle otherwise | Stable for \( \frac{3}{4} < \gamma < 1, s^* < -\sqrt{\frac{3\gamma}{1 - \gamma}} \), \( f'(s^*) < 0 \), or \( \frac{3}{4} < \gamma < 1, s^* > \sqrt{\frac{3\gamma}{1 - \gamma}} \), \( f'(s^*) > 0 \); saddle otherwise |
| \( R_8(s^*) \) | Stable for \( 0 < \gamma < \frac{3}{4}, -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0 \), or \( 0 < \gamma < \frac{3}{4}, 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0 \); saddle otherwise | Stable for \( \frac{3}{4} < \gamma < \frac{3}{2}, \gamma \neq \frac{2}{3} \), \( s^* < -\sqrt{3\gamma} \) and \( f'(s^*) < 0 \), or \( 0 < \gamma < \frac{3}{2}, s^* > \sqrt{3\gamma} \) and \( f'(s^*) > 0 \); saddle otherwise |
| \( R_9(s^*) \) | Unstable for \( 0 < \gamma < 2, s^* < 0, f'(s^*) < 0 \); saddle otherwise | Unstable for \( 0 < \gamma < 2, s^* < 0, f'(s^*) < 0 \) |
| \( R_{10}(s^*) \) | Stable for \( \frac{3}{4} < \gamma < 1, -\sqrt{\frac{3\gamma}{1 - \gamma}} < s^* < -\sqrt{\gamma} \), \( f'(s^*) < 0 \), or \( 1 < \gamma < 2, s^* < -\sqrt{\gamma} \), \( f'(s^*) < 0 \); saddle otherwise | Stable for \( \frac{3}{4} < \gamma < 1, -\sqrt{\frac{3\gamma}{1 - \gamma}} < s^* < -\sqrt{\gamma} \), \( f'(s^*) < 0 \), or \( 1 < \gamma < 2, s^* < -\sqrt{\gamma} \), \( f'(s^*) < 0 \); saddle otherwise |
| \( C(s^*) \) | Unstable for \( 0 \leq \gamma < 2, 0 < u < \pi, s^* < \sqrt{\frac{2}{\csc'(u)}}, f'(s^*) < 0 \) or \( 0 \leq \gamma < 2, \pi < u < 2\pi, s^* > \sqrt{\frac{2}{\csc'(u)}}, f'(s^*) > 0 \); saddle otherwise | Unstable for \( 0 \leq \gamma < 2, 0 < u < \pi, s^* < \sqrt{\frac{2}{\csc'(u)}}, f'(s^*) < 0 \) or \( 0 \leq \gamma < 2, \pi < u < 2\pi, s^* > \sqrt{\frac{2}{\csc'(u)}}, f'(s^*) > 0 \); saddle otherwise |
Table 6. The critical points and curves of critical points of the system (44)–(48) of both LRS Bianchi III and Bianchi I scalar-field cosmologies, in the expanding universe subspace ($\epsilon = +1$), and the corresponding values of the basic observables, namely the curvature density parameter $\Omega_k$, the shear density parameter $\Omega_\sigma$, the deceleration parameter $q$, the dark-energy equation-of-state parameter $w_{\text{DE}}$, the dark-energy density parameter $\Omega_{\text{DE}}$, and the total equation-of-state parameter $w_{\text{tot}}$, calculated using (43). The notation is as in table 4.

| Name | $\Omega_k$ | $\Omega_\sigma$ | $\Omega_{\text{DE}}$ | $\Omega_*$ | $q$ | $w_{\text{DE}}$ | $w_{\text{tot}}$ |
|------|------------|-----------------|----------------------|-----------|----|----------------|----------------|
| $R_1^+$ | 0 | 0 | 0 | 1 | 2 | Arbitrary | 1 |
| $R_1^-$ | 0 | 0 | 0 | 1 | 2 | Arbitrary | 1 |
| $R_2$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | Arbitrary | 0 |
| $R_3$ | 1 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | Arbitrary | $\gamma - 1$ |
| $R_4$ | $\frac{1}{2}(2 - \gamma)(3\gamma - 2)$ | 3 | $3\gamma$ | 0 | $\left(1 - \frac{3\gamma}{2}\right)^2$ | $\frac{3\gamma}{2} - 1$ | Arbitrary | $\gamma - 1$ |
| $R_5$ | 0 | 1 | 0 | $\frac{3\gamma}{2}$ | $\frac{3\gamma}{2} - 1$ | $\gamma - 1$ | $\gamma - 1$ |
| $R_6(s^*)$ | 0 | $\frac{3\gamma}{2}$ | $\frac{3\gamma}{2} - 1$ | $\frac{3\gamma}{2} - 1$ | $\frac{3\gamma}{2} - 1$ | $\frac{3\gamma}{2} - 1$ | $\frac{3\gamma}{2} - 1$ |
| $R_7(s^*)$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $R_8(s^*)$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $R_9(s^*)$ | $\frac{3(\gamma^2 - 1)}{4(\gamma^2 + 1)^2}$ | 0 | $\frac{3}{(\gamma^2 + 1)^2}$ | $\frac{3}{4(\gamma^2 + 1)^2}$ | $\frac{3}{2(\gamma^2 + 1)}$ | $\frac{1}{(\gamma^2 + 1)}$ | $\frac{1}{(\gamma^2 + 1)}$ |
| $C(s^*)$ | 0 | 0 | $\frac{3}{\gamma^2 + 1}$ | $\frac{3}{2(\gamma^2 + 1)}$ | $\frac{1}{(\gamma^2 + 1)}$ | $\frac{1}{(\gamma^2 + 1)}$ | $\frac{1}{(\gamma^2 + 1)}$ |

- Point $R_3$, which is asymptotically stable for potentials having $f(0) > 0$, corresponds to the de Sitter ($w_{\text{tot}} = -1$), isotropic ($\Omega_\sigma = 0$), accelerating ($q < 0$) universe, which is dark-energy dominated ($\Omega_{\text{DE}} = 1$), with the dark-energy behaving like a cosmological constant ($w_{\text{DE}} = -1$). This point exists for both LRS Bianchi III and Bianchi I geometries.
- Point $R_6(s^*)$ corresponds to an anisotropic universe with $0 < \Omega_{\text{DE}} < 1$, that is it can alleviate the coincidence problem since dark-energy and dark-matter density parameters can be of the same order. However, it has the disadvantage that for the usual case of dust matter ($\gamma = 1$) it is not accelerating and moreover it leads to $w_{\text{DE}} = 0$, which are not favored by observations. This point exists for both LRS Bianchi III and Bianchi I geometries, however the stability intervals in the two cases are different.
- Point $R_7(s^*)$ corresponds to an anisotropic universe, with $0 < \Omega_{\text{DE}} < 1$, $0 < \Omega_\sigma < 1$ and non-vanishing $\Omega_\sigma$. It is not accelerating and for usual dust matter it leads to $w_{\text{DE}} = 0$. Thus, this point is not favored by observations. However, it is still very interesting that this point, although stable, maintains a non-zero anisotropy. We discuss on these issues in the subsection 3.3. This point exists only for LRS Bianchi III geometry.
- Point $R_9(s^*)$ corresponds to an anisotropic, dark-energy dominated universe, with the dark-energy equation-of-state parameter lying in the quintessence regime, which can be accelerating or not according to the potential parameters. In the case of exponential potential this point becomes the most important one, since it is both stable and compatible with observations [76]. This point exists for both LRS Bianchi III and Bianchi I geometries, however the stability intervals in the two cases are different.
- Point $R_{10}(s^*)$ corresponds to an anisotropic universe, with $0 < \Omega_{\text{DE}} < 1$, $0 < \Omega_\sigma < 1$, non-vanishing $\Omega_\sigma$ and $\Omega_\sigma = 0$, with the dark-energy equation-of-state parameter lying in the quintessence regime. It is not accelerating and thus this point is not favored by observations. However, interestingly enough this point, although it can be stable, maintains
a non-zero anisotropy. We discuss on these issues in the subsection 3.3. This point exists only for LRS Bianchi III geometry.

3.3. Late-time isotropization and Bounce behavior

Having performed the dynamical analysis for the three geometrical classes, we desire to address two important issues of physical significance, namely the late-time isotropization and the bounce behavior.

3.3.1. Late-time isotropization. The criterion of late-time isotropization in an expanding universe \((H > 0)\) is the vanishing of the shear \(\sigma_+\) [36], or alternatively we can use the stronger condition [12, 41, 115]:

\[
\frac{\sigma_+}{H} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

For LRS Bianchi I and III geometries, with a scalar field with a positive and convex potential, which is expanding at a given time \(t = t_0\), that is \(H(t_0) > 0\), it is well-known that [41]:

(i) \(H(t) \geq 0, \quad \dot{H}(t) \leq 0\) for all \(t \geq t_0\),

(ii) \(\sigma_+(t), \dot{\phi}(t), (\mathcal{R}(t) \rightarrow 0\) for \(t \rightarrow \infty\),

(iii) \(H \rightarrow \sqrt{\frac{V_0}{3}}, \) where \(V_0\) is the minimum of the potential and in particular \(H \rightarrow 0\) for an exponential potential.

These results can be also straightforwardly proven in the case of an additional barotropic fluid satisfying the strong energy conditions namely \(\rho_m + p_m > 0\) and \(\rho_m + 3p_m > 0\), since at late times \(\rho_m \rightarrow 0\) and then the scalar field dominates. However, the above theorem does not give an answer to isotropization for the cases where the potential minimum is zero, or in the case of exponential potentials, since \(\sigma_+\) and \(H\) vanish simultaneously, thus the rate \(\sigma_+/H\) cannot be calculated \(a\ priori\) from the field equations. Furthermore, for more general potentials, not necessarily convex, the theorem may not be true.

Additionally, the well-known Wald theorem, namely that all initially expanding Bianchi models, except type IX, approach the de Sitter spacetime if all energy conditions are satisfied [116], cannot be applied in the present work, since the scalar field can violate the strong energy condition. However, there are extensions of the Wald theorem for sources violating the strong energy condition, known as ‘Cosmic No-Hair theorem’ (see [132–134]), where in particular a scalar field with an exponential potential is introduced in addition to a matter field satisfying the strong and dominant energy conditions, and all initially expanding Bianchi models except type IX, are proved to result to isotropic power-law or exponential inflationary solutions depending on the potential’s slope values (the shear, the 3-curvature and all components of the energy–momentum tensor of the matter field are strongly suppressed by the scalar-field potential). Moreover, the ‘Cosmic No-Hair theorem’ can be extended in non-minimal scalar field too [135, 136]. Nevertheless, the ‘Cosmic No-Hair theorem’ has been proven only for constant or exponential potential.

Finally, note that in the case of an additional coupling of the scalar to a vector field, inflation with an anisotropic stable hair is cosmologically viable for the case of Bianchi I [93–100, 137, 138], as well as for LRS Bianchi II, III and Kantowski–Sachs geometries [101]. However, these studies are also restricted to exponential or power-law potentials.

One can go beyond the above specific investigations, applying the powerful method of dynamical analysis and extracting the asymptotic behavior, as we do in the present work. In
this way one can explicitly see whether the late-time stable points are always isotropic or not.

In particular, as we showed, for expanding LRS Bianchi I models, the late-time attractors are always isotropic. Note that the fact that we may have stable attractors that are not always de Sitter (solutions dominated by the scalar field, or scaling solutions) is consistent with the non-validity of the Wald’s theorem in our case. The same results hold for the Kantowski–Sachs models which satisfy \(10R > 0\), and in particular we found that all stable expanding solutions are isotropic. However, in the case of expanding LRS Bianchi III models there exist two anisotropic stable late-time attractors, in addition to the isotropic ones that exist in Bianchi I too. In particular, we found the stable fixed points \(R_1(s^*)\) and \(R_{10}(s^*)\), which maintain a non-zero anisotropy even at asymptotically late times. However, these points, apart from being inconsistent with observations \((R_1(s^*)\) for usual dust dark matter has \(w_{DE} = 0\), while \(R_{10}(s^*)\) has \(\Omega_m = 0\), they are always non-accelerating\(^7\). Therefore, we conclude that in the examined geometries, all the expanding, accelerating, late-time attractors are always isotropic.

Summarizing, applying the phase-space analysis it is possible to prove that for LRS Bianchi I, III and Kantowski–Sachs models, the late-time accelerating attractors are isotropic, at least in the finite region of the phase-space. A complete investigation of these features should include the examination of the behavior at infinity since the variable \(s\) can diverge (we recall that the other normalized variables are bounded). This can be done using the Poincaré projection method as in [41, 90], however for simplicity we do not proceed to such a detailed analysis in the present work.

### 3.3.2. Bounce behavior

Let us now investigate another issue of great physical importance, namely whether anisotropic scalar-field cosmology, in the framework of General Relativity, allows for the bounce realization. Note that the bounce is impossible in flat FRW scalar-field cosmology, in the case where the scalar field is minimally-coupled to General Relativity, since its realization requires the violation of the null energy condition \(\rho + p \geq 0\), which cannot be obtained by a canonical scalar field [139–141].

Following the approach and conventions of [49], we first define the bounce realization. Introducing the scale factors \(X = e^{1}(t)^{-1}\) and \(Y = e^{2}(t)^{-1}\), and the corresponding expansion parameters \(H_t = X'X^{-1}\) and \(H_y = Y'Y^{-1}\), a bounce in \(X\) occurs at time \(t = t_0\) if and only if \(H_t(t_0) = 0\) and \(H_y(t_0) > 0\), while a bounce in \(Y\) occurs at time \(t = t_1\) if and only if \(H_t(t_1) = 0\) and \(H_y(t_1) > 0\). Although in principle it may be possible to have a bounce in one of the scale factors but not in the other, since this does not lead to a new expanding region in the universe, in the following we assume that the bounce occurs in both \(X\) and \(Y\) scale factor, but not necessarily at the same time [49]. Finally, we assume that matter obeys the null energy condition \(\rho_m + p_m \geq 0\).

Defining the 3-curvature scalar as \((3R) = \frac{2k}{R'} = 2kK\), where \(k = +1, 0, -1\), for Kantowski–Sachs, LRS Bianchi I and LRS Bianchi III respectively, it is easy to re-write the Raychaudhuri and the Friedmann equations, and the evolution equation for anisotropies, in a unified way as:

\[
\begin{align*}
\dot{H} &= -H^2 - 2\sigma^2_+ - \frac{1}{2}(\rho_m + 3p_m) - \frac{1}{2}V'(\phi), \\
(3R) &= \dot{\phi}^2 + 2V(\phi) - 6H^2 + 6\sigma^2_+ + 2\rho_m, \\
\dot{\sigma}_+ &= -3H\sigma_+ - \frac{1}{6}(3R).
\end{align*}
\]

Note that, as we show in table 4, in order for \(R_1(s^*)\) or \(R_{10}(s^*)\) to be stable \(s^*\) and \(f'(s^*)\) must have the same sign, which is not the case for simple potentials such are the exponential and the power-law ones, that is why this anisotropized point was not found to be stable in exponential potentials [132–134], which is just the ‘Cosmic No-Hair theorem’. One should go to more complicated potentials, like the inverse hyperbolic-sine one: \(V(\phi) = \frac{1}{2}\sigma_0 \sinh^{-1}(2\sigma_0 \phi), \alpha > 0\) [50, 60, 84, 85, 117, 124, 127], which leads to \(f(s) = s^2 - \alpha^2\), in which case \(R_1(s^*)\) is stable. That is, we do verify the ‘Cosmic No-Hair theorem’, which has been proven for exponential potentials, but we show that its extension to a wide range of potentials is not possible.
Using that $\sigma_+ = -\frac{1}{3}(H_x - H_y)$ and $H = \frac{1}{3}(H_x + 2H_y)$, as well as the matter equation of state $p_m = (\gamma - 1)\rho_m$, equations (52) lead to

$$\dot{\phi}^2 = -\gamma \rho_m - 2(H_y + H_y^2 - H_x H_y). \quad (53)$$

Let us assume that we have a bounce in the $Y$ direction at the time $t = t_1$ (that is $H_y(t_1) = 0$ and $\dot{H}_y(t_1) > 0$). In this case (53) gives

$$\dot{\phi}^2(t_1) = -\gamma \rho_m(t_1) - 2\dot{H}_y(t_1). \quad (54)$$

Thus, taking into account the matter’s null energy condition (that is $\gamma \rho_m \geq 0$) we deduce that at the bounce $\dot{\phi}^2(t_1) < 0$, that is the bounce in the $Y$ direction, and thus the total bounce, is impossible for real scalar fields.

Therefore, we conclude that in LRS Bianchi type I, III and Kantowski–Sachs geometries, with a minimally-coupled scalar field to General Relativity and a perfect fluid satisfying the null energy condition, a cosmological bounce is not permitted (unless the reality condition is violated). This result is an extension of the no-bounce theorem proved in [49], if we include matter additionally to the scalar field. We stress that this result holds only in General Relativity, since going to modified gravity constructions a bounce is possible in both isotropic and anisotropic geometries [139, 140, 142–144].

4. Application 1: exponential potential with a cosmological constant

In the previous section we performed a complete dynamical analysis of anisotropic scalar-field cosmology, using the advanced $f$-devisers method, which allowed us to extract the results without specifying the potential form. In order to give a specific application, in this section we just substitute the exponential potential with a cosmological constant in the general results, obtaining the corresponding dynamical behavior without repeating the whole analysis from the beginning.

We are interested in a potential of the form

$$V(\phi) = V_0 e^{-\lambda \phi} + \Lambda, \quad (55)$$

since exponential potentials have been extensively studied in cosmological frameworks [12, 17, 42, 43, 76, 81, 84, 113, 114, 117, 119, 145–153].

As we mentioned in table 1, the corresponding $f(s)$ function is given by

$$f(s) = -s(s - \lambda), \quad (56)$$

and thus its roots $f(s^*) = 0$ and the corresponding derivatives $f'(s^*)$ read simply

$$s^* = 0, \quad f'(s^*) = \lambda. \quad (57)$$

$$s^* = \lambda, \quad f'(s^*) = -\lambda. \quad (58)$$

In summary, all we have to do is to substitute these values in the general results of section 3. Those expressions that were independent from $s^*$, will be the same in the present specific potential case too. Thus, only the results that were depending on $s^*$, $f(s^*)$ and $f(0)$ will be now specified. Finally, note that since the potential at hand has two roots $s^*$, the points of the general analysis depending on $s^*$, will split to two. In the following subsections we examine the three anisotropic geometries separately.
4.1. Kantowski–Sachs metric

In Kantowski–Sachs geometry the results were summarized in tables 2 and 3 of subsection 3.1. Inserting the specific values (57),(58) we obtain the following.

Concerning the physically interesting critical point $P^+_5$, we see that it now becomes non-hyperbolic, since one of its eigenvalues becomes zero since $f(0) = 0$ (see section A.1 of the appendix). In order to analyze its stability we apply the center manifold analysis [131] and we introduce the coordinate transformation

$$u = \frac{\sqrt{6}}{6}s, \quad v_1 = x - \frac{\sqrt{6}}{6}s, \quad v_2 = \Sigma - 2(Q - 1), \quad v_3 = y - 1, \quad v_4 = Q - 1,$$  

(59)

finding that the center manifold is given by the graph

$$\{(u, v_1, v_2, v_3, v_4) : v_1 = \sqrt{\frac{2}{3}}\lambda u^2 + 2(\lambda^2 - 1)u^3 + O(u)^4, \quad v_2 = O(u)^4, \quad v_3 = -\frac{u^2}{2} - \sqrt{\frac{2}{3}}\lambda u^3 + O(u)^4, \quad v_4 = O(u)^4, \quad |u| < \delta\},$$  

(60)

where $\delta$ is a sufficiently small positive constant. The dynamics on the center manifold obeys the gradient-like equation $u' = -\sqrt{6}\lambda u^2 + (6 - 2\lambda^2)u^3 + O(u)^4$, and thus the center manifold of the origin is locally asymptotically of saddle type, with an 1D center manifold and 4D stable manifold. Interestingly enough, although the point $P^+_5$ is in general stable for general potentials (except in the special case $f(0) = 0$), it is not stable anymore in the specific case of exponential potential with a cosmological constant (where we have exactly $\lambda > 0$) of (58), they become saddle points (since the stability conditions of table 2 cannot be fulfilled), and thus they cannot be the late-time state of the universe. Note that in the case where the cosmological constant is zero, the points $P^-_{11}(s^*)$ and $P^+_{12}(s^*)$ become conditionally stable again, and they are the usual isotropized, quintessence-like and stiff-dark energy points of the literature [76]. However, a non-zero cosmological constant always dominates over the scalar-field terms at late times, that is $P^-_{11}(s^*)$ and $P^+_{12}(s^*)$ become saddle, and the possible expanding attractor is the cosmological-constant, de Sitter point $P^-_8$.  

Finally, as we mentioned in the general-potential analysis of Kantowski–Sachs geometry of subsection 3.1, the contracting points $P^-_7$ or $P^-_8$ become stable for particular parameter values.

Now, in order to present the aforementioned results in a more transparent way, we perform a numerical elaboration of our cosmological system. In figure 1 we depict the projection of the phase-space on the $x−y−x$ subspace and on the $γ−Σ$ plane, in the case of dust matter ($γ = 1$). For these parameter choices, and in agreement with table 2 and the discussion in the text, depending on the initial conditions, which determine the basin of attraction, the universe at late times can result either in the isotropized, dark-energy dominated, de Sitter point $P^-_5$, or in the contracting solution $P^-_8$.

As we see, through the application of our general analysis in this specific potential, we re-obtained the results of the literature [41], without the need of any calculation, which reveals the capabilities of the our method.

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8 For $\lambda > 0$ ($\lambda < 0$) the orbits with initially $u > 0$ ($u < 0$) tend asymptotically to the origin as time passes, while orbits with initially $u < 0$ ($u > 0$) depart from the origin and become unbounded.

9 Note that $P^+_5$ for $\lambda > 0$ ($\lambda < 0$) is a local attractor for $s > 0$ ($s < 0$), while for $s < 0$ ($s > 0$) the orbits can be unbounded along the $s$-direction at late times.
Figure 1. Projection of the phase-space on the: (a) $x - y - s$ subspace and (b) $y - \Sigma$ plane, in the case of Kantowski–Sachs geometry with exponential potential plus a cosmological constant, for $\lambda = 1$ in units where $8\pi G = 1$, in the case of dust matter ($\gamma = 1$). For these parameter choices, and in agreement with table 2 and the discussion in the text, depending on the initial conditions (which determine the basin of attraction) the universe at late times can result either in the isotropized, dark-energy dominated, de Sitter point $P^+_5$, or in the contracting solution $P^-_8$. Note that since in the projection of the graphs, the axis $Q$ that distinguishes expanding ($Q > 0$) from contracting ($Q < 0$) solution is absent, there is a degeneracy and all expanding points coincide with their contracting partners.
We close this subsection by stressing that, as we see, some of the stable points that exist for general potentials, are not stable or do not even exist in the specific case of exponential potential with a cosmological constant. This feature may be an indication that the exponential potentials restrict the dynamics in scalar-field cosmology.

4.2. LRS Bianchi III and Bianchi I metrics

In LRS Bianchi III and Bianchi I geometries the results for a general potential were summarized in tables 4, 5 and 6 of subsection 3.2. Inserting the specific values (57), (58) of the exponential potential with a cosmological constant we obtain the following.

The physically interesting critical point $R_5$ now becomes non-hyperbolic, since one of its eigenvalues becomes zero since $f(0) = 0$ (see section A.2 of the appendix). In order to analyze its stability we apply the center manifold analysis [131] and we introduce the coordinate transformation

$$u = \frac{\sqrt{6}}{3} s, \quad v_1 = x - \frac{\sqrt{6}}{3} s, \quad v_2 = -x^2 - (y - 1)^2 - \Omega_k,$$

$$v_3 = [1 - x^2 - y^2 - \Sigma - \Omega_k],$$

$$v_4 = y - 1 + \frac{[1 - x^2 - y^2 - \Sigma - \Omega_k]}{2},$$

finding that the center manifold is given by the graph

$$\{(u, v_1, v_2, v_3, v_4) : v_1 = \frac{\sqrt{2}}{\sqrt{3}} \lambda u^2 + 2(\lambda^2 - 1)u^3 + O(u)^4, \quad v_2 = -u^2 - \frac{2\sqrt{6}}{3} u^3 + O(u)^4,$$

$$v_3 = O(u)^4, \quad v_4 = -\frac{u^2}{2} - \frac{2\sqrt{2}}{\sqrt{3}} \lambda u^3 + O(u)^4, \quad |u| < \delta, \}$$

where $\delta$ is a sufficiently small positive constant. The dynamics on the center manifold obeys the gradient-like equation $u' = -\sqrt{6}\lambda u^2 + (6 - 2\lambda^2)u^3 + O(u)^4$, and thus the center manifold of the origin is locally asymptotically of saddle type, with an 1D center manifold and 4D stable manifold. However, since it has a high-dimensional stable manifold, for $\lambda > 0$ ($\lambda < 0$) it can still attract the majority of the orbits having $s > 0$ ($s < 0$) initially. Note that for $R_5$ the center manifold analysis and the stability conditions are the same for the LRS Bianchi III and Bianchi I cases.

The physically interesting critical points $R_6(s^*)$, $R_7(s^*)$, $R_8(s^*)$ and $R_{10}(s^*)$ do not exist for the first solution $s^* = 0$ of (57). For the second solution $s^* = \lambda$ they all become saddle points since $s^* f'(s^*) = -\lambda^2 < 0$, and thus they cannot be the late-time state of the universe (unless the cosmological constant is zero, in which case they remain stable).

As we see, through the application of our general analysis in this specific potential, we re-obtained the results of the literature [41], without the need of any calculation, which reveals the capabilities of our method.

Similarly to the Kantowski–Sachs case, we mention that some of the stable points that exist for general potentials, are not stable in the specific case of exponential potential with a cosmological constant. This feature may be an indication that the exponential potentials restrict the dynamics in scalar-field cosmology.

Now, in order to present the aforementioned results in a more transparent way, we perform a numerical elaboration of our cosmological system. In figure 2 we depict some orbits in the phase-space, in the case of dust matter ($\gamma = 1$).

Additionally, in figure 3 we depict some orbits in the phase-space, in the case of dust matter ($\gamma = 1$) for the boundary set of LRS Bianchi I geometry.
5. Application 2: power-law potential

In this section we give a second specific application, of the general-potential analysis of section 3. In particular, we are interested in the power-law potential of the form \[ V(\phi) = V_0 \phi^n. \] (63)

As far as we are aware, the potential (63) in the anisotropic context has not been investigated up to now.

In this particular case, the use of the general definitions (28), (29) gives \( s = -\frac{n}{\phi} \) and \( f = -\frac{n}{s^2} \) and thus

\[ f(s) = -\frac{s^2}{n}. \] (64)

and therefore its roots \( f(s^*) = 0 \) and the corresponding derivatives \( f'(s^*) \) read simply

\[ s^* = 0, \quad f'(s^*) = 0. \] (65)

In summary, all we have to do it to substitute these values in the general results of section 3. Those expressions that were independent from \( s^* \), will be the same in the present specific potential case too. Thus, only the results that were depending on \( s^* \), \( f(s^*) \) and \( f(0) \) will be now specified. In the following subsections we examine the three anisotropic geometries separately.
Figure 3. Orbits in the phase-space, in the case of LRS Bianchi I geometry with exponential potential plus a cosmological constant, for $\lambda = 1$ in units where $8\pi G = 1$, in the case of dust matter ($\gamma = 1$). At late times the universe results in the isotropized, dark-energy dominated, de Sitter solution $R^5$. For completeness, we also depict the dotted circles corresponding to the two curves of critical points $C(0)$ and $C(1)$, that is for $s^* = 0$ and $s^* = \lambda = 1$, respectively (see table 4).

5.1. Kantowski–Sachs metric

In Kantowski–Sachs geometry the results were summarized in tables 2 and 3 of subsection 3.1. Inserting the specific values (65) we obtain the following.

Concerning the physically interesting critical point $P^+_5$, we see that it now becomes non-hyperbolic, since one of its eigenvalues becomes zero since $f(0) = 0$ (see section A.1 of the appendix). In order to analyze its stability we apply the center manifold analysis [131] and we introduce the coordinate transformation (59), finding that the center manifold is given by the graph

\[
\{(u, v_1, v_2, v_3) : v_1 = -\frac{2u^3}{n} + \mathcal{O}(u)^4, v_2 = \mathcal{O}(u)^4, v_3 = -\frac{u^2}{2} + \mathcal{O}(u)^4, v_4 = \mathcal{O}(u)^4, |u| < \delta\},
\]

(66)

where $\delta$ is a sufficiently small positive constant. The dynamics on the center manifold obeys the gradient-like equation $u' = \frac{2}{5}u^3 + \mathcal{O}(u)^5$, and thus for $n > 0$ the center manifold of the origin is locally asymptotically of saddle type, with 1D center manifold and 4D stable manifold, while for $n < 0$ point $P^+_5$ is completely stable. Furthermore, the other two physically interesting critical points, namely $P_{11}$ and $P_{12}$, do not exist anymore, since $s^* = 0$.

Finally, the contracting points $P^+_7$ and $P^+_8$ become non-hyperbolic too. Applying the center manifold analysis we introduce the coordinate transformation

\[
u_1 = \frac{1}{2}(2\Sigma - Q - 1), u_2 = s, v_1 = Q + 1, v_2 = x - \eta, v_3 = y,
\]

(67)
Figure 4. Orbits in the phase-space, in the case of Kantowski–Sachs geometry with power-law potential, for \( n = 2 \), in the case of dust matter (\( \gamma = 1 \)). For these parameter choices, and in agreement with table 2 and the discussion in the text, depending on the initial conditions (which determine the basin of attraction) the universe at late times can result either in the isotropized, dark-energy dominated, de Sitter point \( P^+_{5} \), or in the contracting solutions \( P^-_{7} \) or \( P^-_{8} \).

where \( \eta = -1 \) for \( P^-_{7} \) and \( \eta = +1 \) for \( P^-_{8} \), finding that the center manifold is given by the graph

\[
\{(u_1, u_2, v_1, v_2, v_3) : v_1 = O(u_1^2),
\quad v_2 = -\frac{\eta}{2}u_1^2 + O(u_1^3),
\quad v_3 = O(u_1^3),
\quad |(u_1, u_2)| < \delta\}.
\] (68)

The dynamics on the center manifold of \( P^-_{7,8} \) is dictated by \( u'_1 = O(u_1^4), u'_2 = \frac{\sqrt{2n}}{2}u_2^2 + O(u_1^2u_2^2) \). Hence, for \( \eta = -1 \), that is for \( P^-_{7} \), and for \( n > 0 \) \((n < 0)\) the orbits with initially \( u_2 > 0 \) \((u_2 < 0)\) tend asymptotically to the origin as time passes, while orbits with initially \( u_2 < 0 \) \((u_2 > 0)\) depart from the origin and become unbounded along the \( u_2 \)-axis. For \( \eta = +1 \), that is for \( P^-_{8} \), and for \( n < 0 \) \((n > 0)\) the orbits with initially \( u_2 > 0 \) \((u_2 < 0)\) tend asymptotically to the origin as time passes, while orbits with initially \( u_2 < 0 \) \((u_2 > 0)\) depart from the origin and become unbounded along the \( u_2 \)-axis. In summary \( P^-_{7} \) and \( P^-_{8} \) possess a 3D stable manifold and a 2D center manifold. Thus, although in the general case \( P^-_{7} \) and \( P^-_{8} \) can be stable (with the exception of \( f'(0) = 0 \), they are not completely stable in this special case of the power-law potential (where we have exactly \( f'(0) = 0 \)). However, they can still conditionally attract the majority of the orbits.

In order to present the above results in a more transparent way, we perform a numerical elaboration of our cosmological system. In figure 4 we depict some orbits in the phase-space, in the case of dust matter (\( \gamma = 1 \)). For these parameter choices, and in agreement with table 2 and the discussion in the text, depending on the initial conditions, which determine the basin of attraction, the universe at late times can result either in the isotropized, dark-energy dominated, de Sitter point \( P^+_{5} \), or in the contracting solutions \( P^-_{7} \) or \( P^-_{8} \).
Finally, and similarly to the previous section, we see here too that some of the stable points that exist for general potentials, do not exist in the specific case of power-law potential. This feature may be an indication that power-law potentials restrict the dynamics in scalar-field cosmology.

5.2. LRS Bianchi III and Bianchi I metrics

In LRS Bianchi III and Bianchi I geometries the results for a general potential were summarized in tables 4 and 6 of subsection 3.2. Inserting the specific values (65) of the power-law potential we obtain the following.

The physically interesting critical point $R_5$ now becomes non-hyperbolic, since one of its eigenvalues becomes zero since $f(0) = 0$ (see section A.2 of the appendix). In order to analyze its stability we apply the center manifold analysis \[131\] and we introduce the coordinate transformation (61), finding that the center manifold is given by the graph

\[
\{(u, v_1, v_2, v_3, v_4) : v_1 = -\frac{2u^3}{n} + \mathcal{O}(u)^4, v_2 = -u^2 + \mathcal{O}(u)^4, v_3 = \mathcal{O}(u)^4, v_4 = -\frac{u^2}{2} + \mathcal{O}(u)^4, |u| < \delta\},
\]

where $\delta$ is a sufficiently small positive constant. The dynamics on the center manifold obeys the gradient-like equation $\dot{u} = -\frac{2u^3}{n} + \mathcal{O}(u)^4$, and thus for $n > 0$ the center manifold of the origin is locally asymptotically of saddle type, with an 1D center manifold and 4D stable manifold, while for $n < 0$ point $R_5$ is completely stable. However, since it has a high-
dimensional stable manifold it can still attract the majority of the orbits. Note that for $R_3$ the center manifold analysis and the stability conditions are the same for the LRS Bianchi III and Bianchi I cases.

Finally, the other three physically interesting critical points, namely $R_6(s^*)$, $R_7(s^*)$, $R_8(s^*)$, and $R_{10}(s^*)$ do not exist anymore, since $s^* = 0$. Similarly to the previous Kantowski–Sachs case, we see that some of the stable points that exist for general potentials, do not exist in the specific case of power-law potential. This feature may be an indication that power-law potentials restrict the dynamics in scalar-field cosmology.

Lastly, we perform a numerical elaboration of our cosmological system and in figure 5 we depict some orbits in the phase-space, in the case of dust matter ($\gamma = 1$).

6. Conclusions

In this work we performed a detailed dynamical analysis of anisotropic scalar-field cosmologies, and in particular of the most significant Kantowski–Sachs, LRS Bianchi I and LRS Bianchi III cases. These geometries are very interesting, especially if one finds late-time isotropized solutions, since they provide an explanation for the observed universe isotropy, instead of having to assume it from the start, like in FRW cosmology. Additionally, the whole discussion may be relevant for the explanation of the anisotropic ‘anomalies’ reported in the recently announced Planck probe results [7].

The investigation of the phase-space behavior is very helpful and necessary in every cosmology, since it bypasses the nonlinearities and complications of the cosmological equations, which prevent complete analytical treatments, and it reveals the late-time behavior of the universe. However, a disadvantage of the relevant literature up to now was that the whole analysis had to be performed separately and from the start, for every different scalar-field potential [12, 75–91]. Apart from the huge calculation needs, this feature put limits in the whole approach, weakening the idea of the dynamical analysis itself, since the results were potential-dependent.

Therefore, in the present work we extended beyond the usual procedure, and we followed the powerful method of the $f$-devisers, which allows us to perform the whole analysis without the need of an a priori specification of the potential. In particular, with the introduction of the new variables $f$ and $s$, one adds an extra direction in the phase-space, whose neighboring points correspond to ‘neighboring’ potentials. Thus, after the general analysis has been completed, the substitution of the specific $f(s)$ for the desired potential gives immediately the specific results, through a form of intersection of the extended phase-space. That is, one can just substitute the specific potential form in the final results and obtain the corresponding behavior, instead of having to repeat the whole dynamical elaboration from the start. This general investigation appears for the first time in the literature and it eliminates almost any calculation need from future relevant investigations.

Following the aforementioned extended procedure, we extracted and we presented the complete and detailed phase-space behavior of Kantowski–Sachs, LRS Bianchi I and LRS Bianchi III geometries for a wide range of scalar-field potentials, calculating also the values of basic observables such is the various density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters. We found a very rich behavior: in all cases the universe can result in isotropized solutions with observables in agreement with observations, independently of the initial conditions and its specific evolution. In particular, all expanding, accelerating, stable attractors are isotropic. Amongst others, the universe can result in a dark-energy dominated, accelerating, de Sitter solution, in a dark-energy dominated
quintessence-like solution, or in a stiff-dark energy solution with dark-energy and dark-matter densities of the same order.

Additionally, we proved that in the examined geometries, namely in LRS Bianchi type I, III and Kantowski–Sachs models, with a real, minimally-coupled scalar field to General Relativity and a perfect fluid satisfying the null energy condition, a cosmological bounce is impossible.

Finally, for completeness, after the extraction of the general-potential results, and in order to show how the analysis is applied to specific potentials, we presented the application to two well-studied potentials, namely the exponential plus cosmological constant one and the power-law one. In the first case we re-obtained the results of the literature, while in the second case, the relevant analysis appears for the first time. However, the interesting point is that, as we showed, some of the stable points that exist for general potentials, are not stable or do not even exist in the two specific potentials. This feature may be an indication that the exponential and power-law potentials might restrict the dynamics in scalar-field cosmology, opening the way to the introduction of more complicated ones.

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Appendix. Eigenvalues of the perturbation matrix $\Xi$ for each critical point

A.1. Kantowski–Sachs models

The system (36)–(40) admits 12 isolated critical points (6 corresponding to expanding universe and 6 corresponding to contracting one) and 10 curves of critical points (5 corresponding to expanding universe and 5 corresponding to contracting one), presented in table 2. In this appendix we calculate the eigenvalues of the perturbation matrix $\Xi$ for each critical point and curve of critical points. We use the notation $\epsilon = \pm 1$.

For the curves of critical points $P^\epsilon_1$ the associated eigenvalues read $\{3\epsilon, 2\epsilon, 3\epsilon(2 - \gamma), 0, 0\}$, while for the curves of critical points $P^\epsilon_2$ they are $\{6\epsilon, 3\epsilon, 3\epsilon(2 - \gamma), 0, 0\}$. That is they are both non-hyperbolic, behaving as curves of unstable points (for $\epsilon = +1$) or curves of saddle points (for $\epsilon = -1$).

For the curves of critical points $P^\pm_3$ the associated eigenvalues read

$$
\begin{align*}
\{0, \pm \frac{3\gamma}{4 - 3\gamma}, \mp \frac{3(2 - \gamma)}{4 - 3\gamma}, \mp \frac{3[(2 - \gamma) + \sqrt{(2 - \gamma)(18 - 41\gamma + 24\gamma^2)}]}{2(4 - 3\gamma)},
\mp \frac{3[(2 - \gamma) - \sqrt{(2 - \gamma)(18 - 41\gamma + 24\gamma^2)}]}{2(4 - 3\gamma)}\},
\end{align*}
$$
and thus both curves $P^\pm_{s}$ are normally hyperbolic (curves of saddle points) since the eigenvector associated to the zero eigenvalue is tangent to the $s$-axis.

For the curves of critical points $P^s_0$, the associated eigenvalues are

$$\left\{ -\frac{3\epsilon}{2}(2-\gamma), -\frac{3\epsilon}{2}(2-\gamma), \frac{3\epsilon \gamma}{2}, -\epsilon(2-3\gamma) \right\},$$

hence, they are curves of saddle points.

For the critical points $P^s_1$ the associated eigenvalues are

$$\left\{ -3\epsilon, -2\epsilon, -3\gamma\epsilon \frac{1}{2}, -\frac{3\epsilon}{4} \left[ 1 - \sqrt{1 - 4f(0)} \right], -\frac{3\epsilon}{4} \left[ 1 + \sqrt{1 - 4f(0)} \right] \right\},$$

and therefore they are asymptotically stable (for $\epsilon = +1$) or unstable (for $\epsilon = -1$) provided $f(0) > 0$.

For the critical points $P^s_5$ the associated eigenvalues are

$$\left\{ -3\epsilon, -2\epsilon, -3\gamma\epsilon \frac{1}{2}, -\frac{3\epsilon}{4} \left[ 1 + \sqrt{1 - 4f(0)} \right], -\frac{3\epsilon}{4} \left[ 1 - \sqrt{1 - 4f(0)} \right] \right\},$$

and thus they are saddle points.

For the critical points $P^s_7$ the associated eigenvalues write

$$\{3\epsilon, 0, 4\epsilon, 3\epsilon(2-\gamma), \sqrt{6}f'(0)\},$$

while for $P^s_8$ they are

$$\{3\epsilon, 0, 4\epsilon, 3\epsilon(2-\gamma), -\sqrt{6}f'(0)\}.$$

In both cases, as can be seen by a simple center manifold analysis [131], the zero eigenvalue possesses an eigen-direction tangent to the $\Sigma$-axis passing through the critical point, and the eigenvector associated to the eigenvalue $\pm \sqrt{6}f'(0)$ is tangent to the $s$-axis. Thus, and assuming $\gamma \neq 2$, for $f'(0) = 0$ both $P^s_1$ and $P^s_5$ have a 2D center manifold tangent at the corresponding fixed point to the $\Sigma$-$s$ plane. Moreover, for $f'(0) < 0$, $P^s_1$ behaves as saddle point with a 4D stable manifold, while $P^s_5$ behaves as saddle point with a 4D-unstable manifold. Similarly, for $f'(0) > 0$ $P^s_1$ behaves as saddle point with a 4D unstable manifold, while $P^s_5$ behaves as saddle point with a 4D-stable manifold. Finally, note that the points $P^s_4$ and $P^s_6$ are special points of the curves of critical points $C_{\Sigma}(0)$ (see below), and are given separately for clarity.

For the critical points $P^s_9$ the associated eigenvalues write as

$$\left\{ -\frac{3\epsilon}{2}(2-\gamma)s^* + \sqrt{(2-\gamma)[4(3\gamma^2 - \sqrt{9\gamma^2 + \Delta_1}) + \Delta_2]} \right\},$$

where $\Delta_1 = [\gamma(3\gamma - 4) + 2\gamma(s^*)^2 + 2\gamma\gamma(3\gamma - 4) + 2](s^*)^2$ and $\Delta_2 = [10 - (25 - 12\gamma)\gamma](s^*)^2$. Hence, they are non-hyperbolic for $\gamma = \frac{7}{3}$, otherwise they are of saddle type.
For the critical points $P^c_0(s^c)$ the associated eigenvalues are
\[
\left\{ \frac{-3\epsilon[(s^c)^2 - 2]}{(s^c)^2 + 4}, \frac{-6\epsilon[\gamma(s^c)^2 - (s^c)^2 + \gamma]}{(s^c)^2 + 4}, \frac{6\epsilon s^c f'(s^c)}{(s^c)^2 + 4},
\frac{3\epsilon[(s^c)^2 + 2 + \sqrt{2 + (s^c)^2}][18 - 7(s^c)^2]}{2[(s^c)^2 + 4]},
\frac{-3\epsilon[(s^c)^2 - 2 - \sqrt{2 + (s^c)^2}][18 - 7(s^c)^2]}{2[(s^c)^2 + 4]} \right\},
\]
thus they are non-hyperbolic for $(s^c)^2 \in \{0, 2, 2\}$. For $0 < \gamma < 2$ and $-\sqrt{2} < s^c < \sqrt{2}$ they are saddle points.

For the critical points $P^c_1(s^c)$ the associated eigenvalues read
\[
\left\{ \frac{1}{2}\epsilon[(s^c)^2 - 6], \frac{1}{2}\epsilon[(s^c)^2 - 6], \epsilon[(s^c)^2 - 3\gamma], \epsilon[(s^c)^2 - 2], -\epsilon s^c f'(s^c) \right\}.
\]
Thus, $P^c_1(s^c)$ is non-hyperbolic for $s^2 \in \{0, 2, 6, 3\gamma\}$ or $f'(s^c) = 0$. Furthermore, $P^c_1(s^c)$ ($P^c_2(s^c)$) is asymptotically stable (unstable) for $0 < \gamma < \frac{3}{2}$, $-\sqrt{3\gamma} < s^c < 0$, $f'(s^c) < 0$ or $0 < \gamma < \frac{3}{2}$, $0 < s^c < \sqrt{3\gamma}$, $f'(s^c) > 0$ or $\frac{3}{2} < \gamma < 2$, $-\sqrt{2} < s^c < 0$, $f'(s^c) < 0$ or $\frac{3}{2} < \gamma < 2$, $0 < s^c < \sqrt{2}$, $f'(s^c) > 0$. In any other case they are saddle points.

For the critical points $P^c_2(s^c)$ the associated eigenvalues are
\[
\left\{ \epsilon(3\gamma - 2), -\frac{3\epsilon}{2}(2 - \gamma), -\frac{3\epsilon}{4} \left\{ -\gamma + \sqrt{(2 - \gamma) \left[ \frac{24\gamma^2}{(s^c)^2} - 9\gamma + 2 \right] + 2} \right\},
\frac{3\epsilon}{4} \left\{ -\gamma - \sqrt{(2 - \gamma) \left[ \frac{24\gamma^2}{(s^c)^2} - 9\gamma + 2 \right] + 2} \right\}, \frac{3\epsilon f'(s^c)}{s^c} \right\}.
\]
Therefore, $P^c_2(s^c)$ ($P^c_3(s^c)$) is asymptotically stable (unstable) if either $0 < \gamma < \frac{3}{2}$, $s^c < -\sqrt{3\gamma}$ and $f'(s^c) < 0$, or $0 < \gamma < \frac{3}{2}$, $s^c > \sqrt{3\gamma}$ and $f'(s^c) > 0$, while they are non-hyperbolic if either $\gamma = 0$, or $s^c = \frac{3}{2}$, or $s^2 = 3\gamma$, or $s^c = 0$, or $f'(s^c) = 0$. In any other case they are saddle points.

Finally, for the curves of critical points $C_c(s^c)$ the associated eigenvalues are
\[
\left\{ 0, 3\epsilon(2 - \gamma), -2(-2\epsilon + \cos u), 3\epsilon - \gamma - \sqrt{3} s^c \sin u, -\sqrt{6} f'(s^c) \sin u \right\}
\]
and thus $C_c(s^c)$ are normally hyperbolic and their stability is determined by the sign of the non-zero eigenvalues. Hence, $C_c(s^c)$ are unstable for $0 \leq \gamma < 2$, $0 < u < \pi$, $s^c < \sqrt{\gamma} \csc(u)$, $f'(s^c) < 0$ or $0 \leq \gamma < 2$, $\pi < u < 2\pi$, $s^c > \sqrt{\gamma} \csc(u)$, $f'(s^c) > 0$, otherwise they are saddle points. Similarly, $C_-(s^c)$ are stable for $0 \leq \gamma < 2$, $0 < u < \pi$, $s^c > -\sqrt{\gamma} \csc(u)$, $f'(s^c) > 0$, or $0 \leq \gamma < 2$, $\pi < u < 2\pi$, $s^c < -\sqrt{\gamma} \csc(u)$, $f'(s^c) < 0$, otherwise they are saddle points.

**A.2. Expanding LRS Bianchi III and Bianchi I models**

The system (44)–(48), in the expanding universe subspace, admits six isolated critical points and six curves of critical points, presented in table 4. We mention that these points and curves correspond only to the half phase-space of expanding solutions. Thus, the whole phase-space admits also their symmetric partners corresponding to contractions, which coordinates and observables are given by the expanding ones under the transformation (50). In this appendix we calculate the eigenvalues of the perturbation matrix $\mathbf{\Xi}$ for each critical point and curve of critical points of the expanding solutions.
For the curve of critical points $R_4^*$ the associated eigenvalues read $\{2, 3, 0, 0, 3(2 - \gamma)\}$, while for the curve of critical points $R_1^*$ they read $\{6, 3, 0, 0, 3(2 - \gamma)\}$. Thus, they are both non-hyperbolic, with 3D unstable manifold. For Bianchi I models the eigenvalues for $R_1^*$ are the same, apart from the first one since the extra $\Omega_2$-direction is not required for the analysis. Thus, the stability conditions of Bianchi I are the same as for LRS Bianchi III.

For the curve of critical points $R_2$, the associated eigenvalues are $\{-\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, 0, 3(1 - \gamma)\}$, and thus they are non-hyperbolic (the zero-eigenvalue is associated with an eigenvector tangent to the $s$-axis), behaving as saddle. This point does not belong to the LRS Bianchi I boundary set.

For the curve of critical points $R_3$ the eigenvalues are $\{3\gamma - 2, 0, \frac{3\gamma}{2}, \frac{3\gamma}{2} - 3, \frac{3\gamma}{2} - 3\}$, and thus they are non-hyperbolic (the zero-eigenvalue is associated with an eigenvector tangent to the $s$-axis), behaving as a saddle. For Bianchi I models the eigenvalues for $R_3$ are the same, apart from the first one since the extra $\Omega_2$-direction is not required for the analysis. Thus, the stability conditions of Bianchi I are the same as for LRS Bianchi III.

For the curve of critical points $R_4$ the associated eigenvalues are
\[
\left\{\frac{3(\gamma - 2)}{2}, \frac{3\gamma}{2}, -\frac{3}{4}[2 - \gamma + \sqrt{(2 - \gamma)(\gamma(24\gamma - 41) + 18)}], -\frac{3}{4}[2 - \gamma - \sqrt{(2 - \gamma)(\gamma(24\gamma - 41) + 18)}]\right\},
\]
and thus they are non-hyperbolic (the zero-eigenvalue is associated with an eigenvector tangent to the $s$-axis), behaving as saddle. This point does not belong to the LRS Bianchi I boundary set.

For the isolated critical point $R_5$ the eigenvalues write as
\[
\left\{-2, -3, -3\gamma, \frac{3}{2}[-\sqrt{9 - 12f(0) - 3}], \frac{3}{2}[-\sqrt{9 - 12f(0) - 3}]\right\},
\]
therefore it is stable for $f(0) > 0$ or saddle for $f(0) < 0$. For Bianchi I models the eigenvalues for $R_5$ are the same, apart from the first one since the extra $\Omega_2$-direction is not required for the analysis. Thus, the stability conditions of Bianchi I are the same as for LRS Bianchi III.

For the critical point $R_6(s^*)$ the eigenvalues read
\[
\left\{3\gamma - 2, \frac{3}{2}(2 - \gamma), -\frac{3}{4}\left[-\gamma + \sqrt{(2 - \gamma)\left[\frac{24\gamma^2}{s^*} - 9\gamma + 2\right]} + 2\right]\right\},
\]
that is it is asymptotically stable if either $0 < \gamma < \frac{3}{4}$, $s^* < -\sqrt{3\gamma}$ and $f'(s^*) < 0$, or $0 < \gamma < \frac{3}{4}$, $s^* > \sqrt{3\gamma}$ and $f'(s^*) > 0$, otherwise it is a saddle point. For the Bianchi I subcase the eigenvalues associated to $R_6(s^*)$ are the same apart from the first one. This leads to a broader interval of stability conditions than that of Bianchi III, for which the new bifurcation value $\gamma = \frac{3}{4}$ appears. Summarizing, for the LRS Bianchi I class, $R_6(s^*)$ is stable for $0 < \gamma < 2$, $s^* < -\sqrt{3\gamma}$ and $f'(s^*) < 0$, or $0 < \gamma < 2$, $s^* > \sqrt{3\gamma}$ and $f'(s^*) > 0$.

For the critical point $R_7(s^*)$ the eigenvalues are
\[
\left\{-\frac{3}{4}\left[2 - \gamma + \sqrt{(2 - \gamma)[\Delta_2 + 4(3\gamma^2 + \sqrt{\Delta_1})]}\right]s^*, -\frac{3}{4}\left[2 - \gamma - \sqrt{(2 - \gamma)[\Delta_2 + 4(3\gamma^2 + \sqrt{\Delta_1})]}\right]s^*\right\},
\]

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belong to the LRS Bianchi I boundary set. Class. Quantum Grav. Let γ < 0 thus it is non-hyperbolic for s < 0. For Bianchi I models the eigenvalues associated to R are ±√a R. For the critical point R(γ) the associated eigenvalues are
$$\left\{ \left(s^2 - 2, \frac{3}{2} \left[ (s^2)^2 - 6 \right], \frac{1}{2} \left[ (s^2)^2 - 6 \right], (s^2)^2 - 3 \gamma, -s^2 f'(s^*) \right) \right\},$$
thus it is non-hyperbolic for s^2 ∈ [0, 2, 6, 3γ] or f'(s^*) = 0. It is asymptotically stable for 0 < γ < 2, −√3γ < s^* < 0, f'(s^*) < 0 or 0 < γ < 2, 0 < s^* < √3γ, f'(s^*) > 0 or 2 < γ < 2, −2 < s^* < 0, f'(s^*) < 0 or 2 < γ < 2, 0 < s^* < √2, f'(s^*) > 0, otherwise it is a saddle point. For the Bianchi I subcase the eigenvalues associated to R(γ) are the same apart from the first one (corresponding to Ω-direction). Thus, we have a broader interval of stability conditions than that of Bianchi III, for which the new bifurcation values γ = ±1 and s^* = ±√2 appear. Hence, the stability conditions for R(γ) in the case of LRS Bianchi I are 0 < γ < 2, −√3γ < s^* < 0, f'(s^*) < 0, or 0 < γ < 2, 0 < s^* < √3γ, f'(s^*) > 0. In any other case it is a saddle point.

For the critical points R^+_1 (R^−_1) the eigenvalues are \{4, 0, 3 ± √2s^*, 6 − 3γ, ±√6f'(s^*)\}. Therefore, R^+_1 (R^−_1) is unstable for 0 < γ < 2, s^* < 0 (s^* > 0), f'(s^*) < 0 (f'(s^*) > 0), otherwise they are saddle points. For Bianchi I models the eigenvalues associated to R^±_1 are the same, apart from the first one, and thus the stability conditions of Bianchi I are the same as those of LRS Bianchi III.

For the critical points R^±_2 the associated eigenvalues are
$$\left\{ \left(3(s^2)^2 - 2, \frac{1}{2} \left[ (s^2)^2 - 6 \right], \frac{1}{2} \left[ (s^2)^2 - 6 \right], (s^2)^2 - 3 \gamma, -s^2 f'(s^*) \right) \right\},$$
thus they are non-hyperbolic for (s^2)^2 ∈ [0, 2, 6, 3γ] or f'(s^*) = 0. They are stable for \(\frac{2}{3} < \gamma < 1, -\frac{\sqrt{3}}{\gamma - 1} < s^* < -\sqrt{2}, f'(s^*) < 0\); or \(1 < \gamma < 2, s^* < -\sqrt{2}, f'(s^*) < 0\); or \(\frac{2}{3} < \gamma < 1, \sqrt{2} < s^* < \frac{\sqrt{3}}{\gamma - 1}, f'(s^*) > 0\); or \(1 < \gamma < 2, s^* > \sqrt{2}, f'(s^*) > 0\). This point does not belong to the LRS Bianchi I boundary set.

Finally, for the curve of critical points C(s^*) the associated eigenvalues are
$$\{4 - 2 \cos(u), 0, 6 - 3 \gamma, 3 - \sqrt{2} \sin(u)s^*, -\sqrt{6} \sin(u)s^* f'(s^*)\}.$$
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