ASYMPTOTIC ORTHOGONALIZATION
OF SUBALGEBRAS IN II$_1$ FACTORS

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Abstract. Let $M$ be a II$_1$ factor with a von Neumann subalgebra $Q \subset M$ that has infinite index under any projection in $Q' \cap M$ (e.g., $Q$ abelian; or $Q$ an irreducible subfactor with infinite Jones index). We prove that given any separable subalgebra $B$ of the ultrapower II$_1$ factor $M^\omega$, for a non-principal ultrafilter $\omega$ on $\mathbb{N}$, there exists a unitary element $u \in M^\omega$ such that $uBu^*$ is orthogonal to $Q^\omega$.

1. Introduction

We continue in this paper the study of approximate independence properties for subalgebras in II$_1$ factors from [P7,8]. This time, we investigate the possibility of placing two subalgebras $B, Q$ of a II$_1$ factor $M$ in an orthogonal position via asymptotic unitary conjugacy of one of them, but uniformly with respect to the other. In other words, we want to find unitary elements $(u_n)_n \subset M$ such that, when viewing $u = (u_n)_n$ as an element in the ultrapower II$_1$ factor $M^\omega$ for some non-principal ultrafilter $\omega$ on $\mathbb{N}$ ([W]), the algebras $uBu^*$ and $Q^\omega$ are orthogonal, in the sense of [P1].

Recall in this respect that two $*$-subalgebras $N_1, N_2$ in a II$_1$ factor $N$ are called orthogonal, if they are $\tau$-independent, i.e., $\tau(x_1x_2) = \tau(x_1)\tau(x_2)$, $\forall x_1 \in N_1, x_2 \in N_2$, $\tau$ denoting the (unique) trace state on the ambient II$_1$ factor.

While this cannot of course be done if $Q$ is equal to $M$ and $B \neq C$, or $Q$ merely “virtually equal” to $M$ with $\dim(B)$ large enough, we will prove that once $Q$ has “infinite index” in $M$ and $B$ is separable, then such asymptotic orthogonalisation can be obtained. For instance, if $Q = A$ is abelian and $M$ itself is separable, then there exists a unitary element $u \in M^\omega$ such that $uMu^* \perp A^\omega$.

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The "infinite index" condition for a subalgebra $Q$ in a II$_1$ factor $M$ that we'll require is that for any non-zero projection $p \in Q' \cap M$, the [PP]-index of the inclusion $Qp \subset pM$ is infinite. It is equivalent to the condition that the centralizer of $M$ in the Jones basic construction algebra $\langle M, Q \rangle$ for $Q \subset M$ contains no non-zero finite projections. This amounts to $M \not \prec_M Q$ in the sense of "intertwining by bimodules" terminology (2.4 in [P6]).

We will in fact investigate this asymptotic orthogonalization problem in the more general case when the unitaries $(u_n)_n$ are subject to constraints, being required to lie in some other given von Neumann subalgebra $P \subset M$. Thus, we will prove that if $P \subset M$ is any irreducible subfactor such that $P \not \prec_M Q$, then one can indeed find $u \in U(P^\omega)$ such that $uBu^* \perp Q^\omega$. An example of such a situation is when $P$ is an irreducible hyperfinite subfactor of $M$ and $Q$ is abelian.

More generally, we prove the following:

1.1. Theorem. Let $M_n$ be a sequence of finite factors, with $\dim M_n \to \infty$. For each $n$, let $Q_n \subset M_n$ be a von Neumann subalgebra and $P_n \subset M_n$ be an irreducible subfactor. Let $\omega$ be a non-principal ultrafilter on $\mathbb{N}$. Denote by $M$ the ultraproduct $\Pi_\omega M_n$ with $Q := \Pi_\omega Q_n$, $P := \Pi_\omega P_n$ viewed as von Neumann subalgebras in $M$. Assume $P \not \prec_M Q$. Then, given any separable von Neumann subalgebra $B \subset M$, there exists a unitary element $u \in P$ such that $uBu^*$ is orthogonal to $Q^\omega$.

For the above condition $P \not \prec_M Q$ to be satisfied, it is sufficient that $P_n \not \prec_{M_n} Q_n$, $\forall n$. Another sufficient condition is that $\Pi_\omega (Q_n' \cap M_n)$ be diffuse. Or that $Q_n \subset M_n$ are all finite dimensional factors, $P_n = M_n$ and $\lim_n (\dim M_n / \dim Q_n) = \infty$. The conditions in the statement are also satisfied when $P_n, Q_n \subset M_n$ are irreducible II$_1$ subfactors of finite Jones index with $\lim_n [M_n : Q_n] / [M_n : P_n] = \infty$.

Theorem 1.1 shows that once a countably generated II$_1$ factor $M$ can be embedded in the ultrapower $R^\omega$ of the hyperfinite II$_1$ factor $R$, then one can actually embed it so that to be orthogonal to the ultraproduct of an arbitrary sequence of MASAs of $R$, or to $\Pi_\omega Q_n$ for a given sequence of irreducible subfactors $Q_n \subset R$ with $\lim_n [R : Q_n] = \infty$. This fact may be of interest in relation to Connes Approximate Embedding conjecture.

As mentioned before, when applied to the case all $M_n$ are equal to the same II$_1$ factor $M$ and all $Q_n \subset M$ are equal, with $P_n = M$, the above theorem gives:

1.2. Corollary. Let $M$ be a II$_1$ factor, $Q \subset M$ a von Neumann subalgebra such that $M \not \prec_M Q$ and $B \subset M$ a separable von Neumann subalgebra. Then there exists a unitary element $u \in M^\omega$ such that $uBu^* \perp Q^\omega$.

The orthogonality between the subalgebras $uBu^*$ and $Q^\omega$ in the above statement is the same as 1-independence in the sense of [P7, 8]. It is thus natural to push this
further, trying to find the unitary \( u \in M^\omega \) so that \( uBu^* \) becomes free independent to \( Q^\omega \). While this question remains open in this generality, let us mention here the following partial answer, which we’ll derive from Theorem 1.1 and results in [P7]:

**1.3. Corollary.** Let \( M \) be a \( \text{II}_1 \) factor and \( A \subset M \) a singular MASA. Given any separable von Neumann subalgebra \( B \subset M^\omega \), there exists a unitary element \( u \in M^\omega \) such that \( uBu^* \) is free with respect to \( A^\omega \), i.e., \( uBu^* \lor A^\omega \simeq B \lor A^\omega \).

Questions about “rotating” via unitary conjugacy a subalgebra \( B \) in a \( \text{II}_1 \) factor \( M \) so that to become (approximately) orthogonal to another subalgebra \( Q \subset M \) have been first considered in [P1]. The case when the algebra \( B \) is 2-dimensional (so “smallest possible”) has been studied in [P3], notably in the case \( Q \) is a subfactor of finite Jones index. We will comment on this and other related problems in the last section of this paper.

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2. Proof of the main results

For notations and terminology used hereafter, we send the reader to ([P7,8]; also [AP] for basics in \( \text{II}_1 \) factors theory).

To prove Theorem 1.1, we first show that for any \( F \subset M \ominus \mathbb{C} \) finite and any \( \varepsilon > 0 \) there exists a unitary element \( v \in P \) such that the expectation onto \( Q \) of any element in \( vFv^* \) is \( \varepsilon \)-close to 0 in the Hilbert norm given by the trace. Such unitary \( v \) will be constructed by patching together “incremental pieces” of it, along the lines of the technique developed in [P5,7,8]. The theorem then follows by a standard “diagonalisation along \( \omega \)” procedure of this local result.

**2.1. Lemma.** Let \( M \) be a \( \text{II}_1 \) factor, \( Q \subset M \) a von Neumann subalgebra and \( P \subset M \) an irreducible subfactor such that \( P \not\subset M \). Given any finite set \( F = F^* \subset (M \ominus \mathbb{C}1)_1 \) and any \( \varepsilon_0 > 0 \), there exists a unitary element \( v_0 \in P \) such that \( \|E_Q(v_0xv_0^*)\|_2^2 \leq \varepsilon_0 \), for all \( x \in F \).

**Proof.** Let \( \omega \) be a non-principal ultrafilter on \( \mathbb{N} \) and denote \( \mathcal{M} = (M^\omega, Q^\omega) \) be the semi finite von Neumann algebra associated with Jones basic construction for \( Q^\omega \subset M^\omega \). Thus, \( \mathcal{M} = JQ^\omega J = spM^\omega eM^\omega \subset B(L^2M^\omega) \), where \( e = e_Q^\omega \in B(L^2M^\omega) \) denotes the orthogonal projection of \( L^2M^\omega \) onto \( L^2Q^\omega \) and \( J \) is the canonical canonical conjugation on \( L^2M^\omega \).
Recall that the projection $e$ satisfies the condition $eye = E_{Q^\omega}(y)$ for any $y \in M^\omega$. Recall also that the semi-finite von Neumann algebra $M$ is endowed with a canonical normal faithful semi-finite trace $Tr$, satisfying the condition $Tr(axy) = \tau(xy)$, for all $x, y \in M^\omega$.

Fix $\varepsilon > 0$ such that $\varepsilon < \varepsilon_0$. Denote by $\mathcal{W}$ the set of partial isometries $v \in P^\omega$ with the property that $vv^* = v^*v$ and which satisfy the conditions:

\begin{equation}
\|E_{\mathcal{W}^\omega}(vxv^*)\|_2^2 \leq \varepsilon \tau(v^*v), \quad \tau(vv^*x) = 0,
\end{equation}

for all $x \in F$. We endow $\mathcal{W}$ with the order $\leq$ in which $v_1 \leq v_2$ if $v_1 = v_2v^*_1v_1$.

$(\mathcal{W}, \leq)$ is then clearly inductively ordered and we let $v \in \mathcal{W}$ be a maximal element. Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. Let $w$ be a partial isometry in $pP^\omega p$ with $w^*w = ww^*$ and denote $u = v + w$. Then $u$ is a partial isometry in $P^\omega$ with $u^*u = uu^*$. We will show that one can make an appropriate choice $w \neq 0$ such that $u = v + w$ lies in $\mathcal{W}$.

We construct $w$ by first choosing its support $q = ww^* = w^*w$, then choosing the “phase $w$” above $q$.

Note first that by writing $eux^*u^*euxu^*$ as $e(v + w)x^*(v + w)*e(v + w)x(v + w)^*$ and developing into the sum of 16 terms, we get

\begin{equation}
\|E_{Q}(xu^*)\|_2^2 = Tr(eux^*u^*euxu^*)
\end{equation}

\[= Tr(eux^*u^*euxu^*) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,
\]

where $\Sigma_i$ denotes the sum of terms having $i$ appearances of elements from $\{w, w^*\}, 1 \leq i \leq 4$. Thus, there are four terms in $\Sigma_1$, six in $\Sigma_2$, four in $\Sigma_3$, and one in $\Sigma_4$.

Let us first take care of the terms $Tr(X)$ with $X$ containing a pattern of the form $...eux^*...e...$, or $...eux^*w^*...$, for a given $x \in F$. There are seven such terms: the one in $\Sigma_4$, all four in $\Sigma_3$, and two in $\Sigma_2$. We denote by $\Sigma'$ the sum of the absolute values of these terms. Note that for each such $X$, we have $Tr(X) = Tr(ww^*ey)$ for some $y \in (Q)_1$ and thus, by applying the Cauchy-Schwartz inequality and taking into account the definition of $Tr$, we get the estimate

\begin{equation}
|Tr(X)| = |Tr(...eux^*e...)| = |Tr(ww^*ey)|
\end{equation}

\[\leq (Tr(eux^*w^*ww^*e))^1/2(Tr(qey^*yeq))^1/2 \leq \|qxq\|_2\|q\|_2,
\]

where the last inequality is due to the fact that $Tr(qey^*yeq) \leq Tr(qeq) = \tau(q)$ and $Tr(eux^*w^*ww^*e) = Tr(eux^*qwx^*e) = \tau(wx^*qwx) = \tau(qx^*qxq)$.

By (Corollary 2.2.(i) in [P4]), the irreducible subfactor $pP^\omega p$ of the II$_1$ factor $pM^\omega p$ contains a diffuse abelian subalgebra that’s 2-independent to $F$ with respect
to the trace state \( \tau(\cdot)/\tau(p) \) on \( pM^\omega p \). This implies that there exists a projection \( q \in \mathcal{P}(pM^\omega p) \) of trace \( \tau(q) = \varepsilon^2 \tau(p)^2/64 \) such that \( \tau(qx) = 0 \) and \( \|qxq\|_2^2/\tau(p) = \tau(q)/\tau(p))^2 \). All this, the elements \( \{\tau(q)\}_{q=1}^\infty \) of the above \( 8 \) terms of the form \( \tau(\cdot) \) are 2 majorized by 1.

It follows that \( \|qxq\|_2 \leq \varepsilon\tau(q)^{1/2}/8 \), \( \forall x \in F \). Hence, for this choice of \( q \), the right hand side term in (3) will be majorized by \( \varepsilon\tau(q)/8 \). By summing up over the seven terms in \( \Sigma' \), we get \( \Sigma' \leq 7\varepsilon\tau(q)/8 \).

We now estimate the sum \( \Sigma'' \) of \( |Tr(X)| \) with \( X \) running over the remaining four terms in \( \Sigma_2 \), and the sum \( \Sigma''' \) of four terms \( Tr(X) \) with \( X \) having only one occurrence of \( w, w^* \). We’ll show that one can choose the “phase \( w \)” above the (fixed by now) projection \( q \) in a way that makes \( \Sigma'' + \Sigma''' \) be majorized by \( \varepsilon\tau(q)/8 \).

At this point, it is convenient to enumerate the elements in \( F = \{x_1, ..., x_n\} \). For each \( i = 1, 2, ..., n \) we have

\[
\Sigma'' = |Tr(ewx_i^*v^*evx_iw^*)| + |Tr(evx_i^*w^*ewx_i^*)| \\
+|Tr(ewx_i^*v^*ewx_i^*)| + |Tr(evx_i^*w^*evx_iw^*)| \\
= |Tr(w^*ewY_{1,i})| + |Tr(w^*ewY_{2,i})| \\
+|Tr(wY_{3,i}wY_{4,i})| + |Tr(w^*Y_{5,i}w^*Y_{6,i})|
\]

where each one of the terms \( Y_{j,i} \) depends on \( x_i \in F \) and belongs to the set \( S_0 := q((M^\omega)_1 e(M^\omega)_1)q \subset qL^2(M, Tr)q \).

Note that, as \( i = 1, 2, ..., n \), the number of possible indices \( (j, i) \) in (4) is 4n. There are \( 2n \) terms of the form \( |Tr(w^*ewY)| \), \( n \) terms of the form \( |Tr(wXwY)| \) and \( n \) terms of the form \( |Tr(w^*Xw^*Y)| \), which by using the fact that \( |Tr(w^*Xw^*Y)| = |Tr(wX^*wY^*)| \) we can view as \( n \) additional terms of the form \( |Tr(wXwY)| \). In all this, the elements \( X, Y \) belong to \( S_0 \subset qL^2(M, Tr)q \), and are thus bounded in operator norm by 1 and are supported (from left and right) by projections of trace \( Tr \) majorized by 1.

Similarly, as \( i \) runs over \( \{1, 2, ..., n\} \), the four terms in \( \Sigma''' \) give rise to \( 4n \) terms of the form \( |Tr(wX)| \) with \( X \in qL^2M^\omega q \).

We want to prove that for any \( \delta > 0 \) there exists \( w \in \mathcal{U}(qM^\omega q) \) such that each one of the above \( 8n \) terms is less than \( \delta \).

To take care of the terms in \( \Sigma''' \), note that by results in ([P4] or [P8]) for any finite subset \( E \subset qM^\omega q \), there exists a finite dimensional subfactor \( P_0 \subset qP^\omega q \) such that \( \|E_{P_0 \cap qP^\omega q(x) - \tau(x)/\tau(q)}1\|_2 \leq \delta/2 \), \( \forall x \in E \). By applying this to the elements in \( \Sigma''' \), which are of the form \( \tau(wx) \) with \( x \) running over a finite set \( E \subset qM^\omega q \), and using the Cauchy-Schwartz inequality, one obtains that for each unitary element \( w \in N := P_0' \cap qP^\omega q \) of trace satisfying \( |\tau(w)| \leq \delta\tau(q)/20 \), we have \( |\tau(wx)| < \delta\tau(q) \).
\( \forall x \in E. \) Taking \( \delta \) sufficiently small, one obtains that for any \( 1 \leq i \leq n \) one has \( \Sigma'' \leq \varepsilon \tau(q)/32, \) for any unitary element \( w \in N \) of trace 0.

Finally, let us take care of the terms in \( \Sigma'' \). To do this, recall that we are under the assumption \( P \not\subseteq M \), which in turn implies \( P_{\omega} \not\subseteq M_{\omega} Q_{\omega} \). Thus, \( (P_{\omega})' \cap M \) contains no finite non-zero projections of \( M = \langle M_{\omega}, Q_{\omega} \rangle \) and so \( N' \cap qMq \) contains no finite non-zero projections of \( M \) either.

To estimate \( \Sigma'' \), we first show that for any \( \delta > 0 \) and any two \( m \)-tuples of elements \( (Z_1, ..., Z_m), (Z'_1, ..., Z'_m) \) in \( S_0 \cap M_+ \), there exists a unitary element \( w \in N \) such that

\[
(5) \quad \Sigma_i Tr(w^*Z_iwZ'_i) \leq \delta.
\]

To see this, let \( \mathcal{H} \) denote the Hilbert space \( L^2(qMcq, Tr)^{\otimes m} \) and note that we have a unitary representation \( \mathcal{U}(N) \ni w \mapsto \pi(w) \in \mathcal{U}(\mathcal{H}) \), which on an \( m \)-tuple \( X = (X_i)_{i=1}^m \in \mathcal{H} \) acts by \( \pi(w)(X) = (w^*X_iw)_i \).

Now note that this representation has no (non-zero) fixed point. Indeed, for if \( X \in \mathcal{H} \) satisfies \( \pi(w)(X) = X, \forall w \in \mathcal{U}(N) \), then on each component \( X_i \in L^2(qMcq, Tr) \) of \( X \) we would have \( w^*X_iw = X_i, \forall w \). Thus \( X_iw = wX_i \) and since the unitaries of \( N \) span linearly the algebra \( N \), this would imply \( X_i \in N' \cap L^2(qMcq, Tr) \). Hence, \( X_i^*X_i \in N' \cap L^1(qMcq, Tr) \) and therefore all spectral projections of \( X_i^*X_i \) corresponding to intervals \([t, \infty)\) with \( t > 0 \) would be projections of finite trace in \( N' \cap qMcq \), forcing them all to be equal to 0. Thus, \( X_i = 0 \) for all \( i \).

With this in mind, denote by \( K_Z \subset \mathcal{H} \) the weak closure of the convex hull of the set \( \{ \pi(w)(Z) \mid w \in \mathcal{U}(N) \} \), where \( Z = (Z_1, ..., Z_m) \) is viewed as an element in \( \mathcal{H} \). Since \( K_Z \) is bounded and weakly closed, it is weakly compact, so it has a unique element \( Z^0 \in K_Z \) of minimal norm \( || Z^0 ||_{2, Tr} \). Since \( K_Z \) is invariant to \( \pi(w) \) and \( || \pi(w)(Z^0) ||_{2, Tr} = || Z^0 ||_{2, Tr} \), it follows that \( \pi(w)(Z^0) = Z^0 \). But we have shown that \( \pi \) has no non-zero fixed points, and so \( 0 = Z^0 \in K_Z \).

Let us deduce from this that if \( Z = (Z_i)_i, Z' = (Z'_i)_i \) are the two \( m \)-tuples of positive elements in \( S_0 \), then we can find \( w \in \mathcal{U}(N) \) such that \( (5) \) holds true. If there would exist \( \delta_0 > 0 \) such that \( \Sigma_i Tr(\pi(w)(Z_i)Z'_i) \geq \delta_0, \forall w \in \mathcal{U}(N) \), then by taking convex combinations and weak closure, one would get \( 0 \geq \langle Z^0, Z' \rangle \geq \delta_0 \), a contradiction.

This finishes the proof of \( (5) \). Note that by taking for one of the \( i \) the elements \( Y_i, Y'_i \) to be equal to \( e \), one can get \( w \in \mathcal{U}(N) \) to also satisfy \( || \pi(w) ||^2 \leq \delta \).

We will now use this fact to prove that, given any \( m \)-tuples \( (X_i)_i, (Y_i)_i, (X'_i)_i, (Y'_i)_i \in S_0^m \) (not necessarily having positive operators as entries), there exists \( w \in \mathcal{U}(N) \) such that \( |Tr(w^*X_iwX'_i)| \leq \delta, |Tr(w^*Y_iwY'_i)| \leq \delta \), for all \( i \). Indeed, because if we denote by \( e_i \) the left support of \( X_i \) and \( f_i \) the left support of \( Y_i \), then by the
Cauchy-Schwartz inequality we simultaneously have for all \( i \) the estimates

\[
|\text{Tr}(w^*X_iwX'_i)|^2 \leq \text{Tr}(w^*X_i^*X_iwX'_iX'_i^*)\text{Tr}(e_i) \leq \text{Tr}(w^*X_i^*X_iwX'_iX'_i^*)
\]

and respectively

\[
|\text{Tr}(wY_iwY'_i)|^2 \leq \text{Tr}(w^*Y_i^*Y_iwY'_iY'_i^*)\text{Tr}(f_i) \leq \text{Tr}(w^*Y_i^*Y_iwY'_iY'_i^*).
\]

Since all \( X_i^*X_i, X_i'^*X_i'^*, Y_i^*Y_iwY_i'^*Y'_i'^* \) are positive elements in \( S_0 \), we can now apply (5) to deduce that there exist \( w \in \mathcal{U}(N) \) of arbitrarily small trace such that all the \( 4n \) elements appearing in \( \Sigma'' \) for \( i = 1, 2, ..., n \) are arbitrarily small, making \( \Sigma'' \leq \varepsilon \tau(q)/32, \forall i. \)

Altogether, we then get for \( u = v + w \) the estimate

\[
\|E_Q((xu^*)^2)\|_2 = |E_Q((vxv^*)^2)\|_2 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]

\[
\leq \varepsilon \tau(vv^*) + \Sigma' + \Sigma'' + \Sigma'''
\]

\[
\leq \varepsilon \tau(vv^*) + 15\varepsilon \tau(ww^*)/16 + 2\varepsilon \tau(ww^*)/32 = \varepsilon \tau(uu^*),
\]

for any \( x \in F. \) Thus, \( u \in \mathcal{W} \) and \( u \geq v, u \neq v, \) contradicting the maximality of \( v. \)

This shows that \( v \) must be a unitary element. Thus, if we represent \( v \in P^\omega \) as a sequence of unitary elements \( (v_n)_n \) in \( P \), then we have

\[
\lim_{n \to \omega} \|E_Q(v_nx_n^*)\|_2 = \|E_Q(vx^*)\|_2 \leq \varepsilon < \varepsilon_0,
\]

for all \( x \in F. \) Thus, if we let \( v_0 = v_n \) for some large enough \( n \), then \( v_0 \) satisfies \( \|E_Q(v_0x_0^*)\|_2 \leq \varepsilon_0 \), for all \( x \in F. \)

\( \square \)

**Proof of Theorem 1.1.** Let \( \{b_j\}_1 \subset B \) be a sequence of elements that’s dense in \( (B)_1 \) in the Hilbert norm \( \| \|_2. \) By applying the above lemma to the factor \( M = \Pi_\omega M_n, \) with its von Neumann subalgebra \( Q = \Pi_\omega Q_n \) and its irreducible subfactor \( P = \Pi_\omega P_n, \) the finite set \( F = \{b_1, ..., b_m\} \) and \( \varepsilon = 2^{-m-1}, \) one gets a unitary element \( u_m \in P \) such that \( \|E_Q(u_m^*b_ju_m^*)\|_2 \leq 2^{-m-1} \) for all \( 1 \leq j \leq m. \)

Let us now take representations \( b_j = (b_{j,n})_n \in B \subset M \) and \( u_m = (u_{m,n})_n, \) with \( u_{m,n} \in \mathcal{U}(P_n) \) and \( b_{j,n} \in (M_n)_1. \) Thus, we have

\[
\lim_{n \to \omega} \|E_{Q_n}(u_{m,n}b_{j,n}u_{m,n}^*)\|_2^2 \leq 2^{-m-1}.
\]

Denote by \( V_m \) the set of all \( n \in \mathbb{N} \) such that \( \|E_{Q_n}(u_{m,n}b_{j,n}u_{m,n}^*)\|_2 < 2^{-m}, \) for all \( 1 \leq j \leq m. \) Note that \( V_m \) corresponds to an open closed neighborhood of \( \omega \) in \( \Omega, \)
under the identification \( \ell^\infty \mathbb{N} = C(\Omega) \). Let now \( W_m, \ m \geq 0 \), be defined recursively as follows: \( W_0 = \mathbb{N} \) and \( W_{m+1} = W_n \cap V_{n+1} \cap \{ n \in \mathbb{N} \mid n > \min W_n \} \). Note that, with the same identification as before, \( W_m \) is a strictly decreasing sequence of neighborhoods of \( \omega \) in \( \Omega \).

Define \( u = (u_m)_m \) by letting \( u_n = u_{m,n} \) for \( n \in W_{m-1} \setminus W_m \). It is then easy to see that \( u \) is a unitary element in \( \mathcal{P} \) and that \( \lim_{\omega} \| E_{Q_n}(u_{n,b_j,n}u_n^*) \|_2 = 0 \) for any \( j \). In other words, \( E_Q(ub_ju^*) = 0, \ \forall j \). By the density of the set \( \{ b_j \}_j \) in \( (B)_1 \), it follows that \( uBu^* \perp Q \).

**Proof of Corollary 1.2.** This is just the case where all \( M_n \) are equal to the same \( \text{II}_1 \) factor \( M \), all \( P_n \subset M \) are equal to \( M \) and \( Q_n = Q, \ \forall n \), of Theorem 1.1.

**Proof of Corollary 1.3.** By Theorem 1.1 (or Corolary 1.2), there exists \( u_0 \in \mathcal{U}(M^\omega) \) such that \( u_0Bu_0^* \perp A^\omega \). But then one can apply (Theorem 4.1 or Theorem 5.2.1 in [P7]) to get a unitary element \( u_1 \in \mathcal{U}(A^\omega) \) such that \( u_1(u_0Bu_0^*)u_1^* \) is free independent to \( A^\omega \), showing that \( u = u_1u_0 \) satisfies the required condition.

### 3. Further Comments

#### 3.1. Initial motivation

The orthogonalization relation between subalgebras \( B, Q \) of a \( \text{II}_1 \) factor \( M \), as well as questions about conjugating a subalgebra \( B \) by a unitary element \( u \in M \) such that \( uBu^* \) becomes orthogonal to \( Q \), have been first considered in [P1]. They were used as a tool for calculating normalizers, and more generally the intertwining space between subalgebras, in the spirit of what has later become the intertwining by bimodules techniques [P6]. For instance, it was shown in [P1] that if a unitary \( u \in M \) has the property that \( uA_0u^* \perp Q \) for some diffuse abelian subalgebra \( A_0 \subset Q \), then \( u \) is perpendicular to the normalizer of \( Q \) in \( M \). This allowed to prove several indecomposability properties (e.g., absence of Cartan subalgebras) for ultraproduct \( \text{II}_1 \) factors and for free group factors with uncountable number of generators.

Related to asymptotic orthogonalization, it has been shown in (Lemma 2.5 in [P1] and Corollary 2.4 in [P9]) that in order for a unitary \( u \) to be orthogonal to the intertwining space \( \mathcal{I}(P,Q) \) between subalgebras \( P, Q \subset M \), it is necessary and sufficient that there exists a diffuse subalgebra \( B \subset P^\omega \) such that \( uBu^* \perp P^\omega \).

#### 3.2. The orthogonalization problem

Given von Neumann subalgebras \( B, Q \) in a \( \text{II}_1 \) factor \( M \), the problem of finding a unitary \( u \in M \) such that \( uBu^* \perp Q \) will be called the orthogonalization problem for \( B, Q \subset M \). For a given von Neumann subalgebra \( Q \subset M \) and a fixed finite dimensional subalgebra \( B \), with its trace
inherited from $M$, all isomorphic copies of $B$ are unitary conjugate in $M$. Thus, the orthogonalization problem becomes a question about whether there exist copies of $B$ that are perpendicular to $Q$. This provides a source of isomorphism invariants for the inclusion $Q \subset M$.

3.3. The two dimensional case. The simplest case of this problem is when $B$ is two dimensional, i.e., $B = \mathbb{C}f + \mathbb{C}(1-f)$, for some projection $f \in M$ of trace $\tau(f) = \alpha$. Thus, since all projections of same trace are unitary conjugate in $M$, the question of whether $B$ can be unitarily conjugated to an algebra orthogonal to the von Neumann subalgebra $Q \subset M$ becomes: for what $\alpha \in (0,1]$ does there exist $f \in \mathcal{P}(M)$ such that $E_Q(f) = \alpha 1$.

This problem has been systematically investigated in [P3], where the answers are formulated in terms of the invariant

$$\Lambda(Q \subset M) := \{ \alpha \in [0,1] \mid \exists f \in \mathcal{P}(M), E_Q(f) = \alpha 1 \}.$$ 

Similarly, one denotes its approximate version by

$$\Lambda_{app}(Q \subset M) := \{ \alpha \in ([0,1] \mid \forall \varepsilon > 0, \exists f \in \mathcal{P}(M), \|E_Q(f) - \alpha 1\|_2 \leq \varepsilon \}.$$ 

It obviously coincides with $\Lambda(Q^\omega \subset M^\omega)$ and satisfies the property: $\alpha \in \Lambda_{app}$ if and only if the algebra $B = \mathbb{C}f + \mathbb{C}(1-f)$, with $f$ a projection of trace $\alpha$, can be asymptotically conjugated to an algebra orthogonal to $Q$. Note also that $\Lambda_{app}(Q^\omega \subset M^\omega) = \Lambda(Q^\omega \subset M^\omega)$.

It was already noticed in [P3] that if $Q \subset M$ is an irreducible subfactor of infinite index, then (2.4 in [PP]) can be used to show that $\Lambda_{app}(Q \subset M) = \Lambda(Q^\omega \subset M^\omega) = [0,1]$. This is of course implied by Theorem 1.1, which in fact applies to all cases when $M \not\subset M$, for instance when $Q = A$ is a MASA in $M$.

The calculation of the invariant $\Lambda(Q \subset M)$ is in general quite difficult, but some partial answers could be obtained in [P3] in several particular cases. For instance, if $Q = A$ is a MASA then $\Lambda(A \subset M)$ contains all rationals in $[0,1]$ and in addition $N_M(A)''$ is of type $\text{II}_1$ then $\Lambda(A \subset M) = [0,1]$ (this ought to be the case for any MASA).

In the finite index case, the results obtained in [P3] depend on weather $[M : Q] < 4$ (thus $[M : Q] = 4 \cos^2(\pi/n + 2)$ for some $n \geq 1$, by Jones celebrated results in [J]), or $[M : Q] \geq 4$. To describe them, denote $[M : Q]^{-1} = \lambda$ and define recursively the polynomials $P_k(x)$ by $P_{-1} \equiv 1$, $P_0 \equiv 1$, $P_{k+1}(x) = P_k(x) - xP_{k-1}(x)$, $k \geq 0$.

Then, (Theorem in [P3]) shows that if $[M : Q] = \lambda^{-1} = 4 \cos^2(\pi/n + 2)$ then we have $\Lambda(Q \subset M) = \{0\} \cup \{\lambda P_{k-1}/\lambda P_k(\lambda) \mid 0 \leq k \leq n - 1\}$. While if $[M : Q] \geq 4$ and we let $0 < t \leq 1/2$ be so that $t(1 - t) = \lambda$, then $\Lambda(Q \subset M) \cap (0,t) = \{\lambda P_{k-1}(\lambda)/P_k(\lambda) \mid k \geq 0\}$. 
Thus, the situation is quite rigid when the index is under the threshold 4, with the set Λ being finite and completely understood. While above the threshold 4 the set is always infinite, being completely determined in the intervals \([0, t) \cup (1 - t, 1]\), but with the calculation of \(\Lambda(Q \subset M) \cap [t, 1 - t]\) still open in general. It is interesting to note that in both cases, (Theorem in [P3]) also provides the following uniqueness result: any two projections \(f_1, f_2 \in M\) satisfying \(E_Q(f_1) = \alpha 1 = E_Q(f_2)\), with \(\alpha = \lambda P_{k-1}(\lambda) / P_k(\lambda)\) for some \(k \geq 0\), are conjugate by a unitary element in \(Q\). Moreover, since \([M^\omega : Q^\omega] = [M : Q]\), the results in [P3] show that \(\Lambda(Q \subset M) = \Lambda_{app}(Q \subset M)\) for index < 4 and \(\Lambda(Q \subset M) \cap (0, t) = \Lambda_{app}(Q \subset M) \cap (0, t)\) for index \(\geq 4\), with any projection that’s close to expect on a scalar in these sets being close to a projection that actually expects on a scalar.

The results in [P3] also show that for subfactors \(Q \subset M\) of index \(\geq 4\) that are locally trivial, i.e., for which \(Q' \cap M = C_f \cap C(1 - f), FMf = Qf, (1 - f)M(1 - f) = Q(1 - f)\), with \(\tau(f) = t \leq 1/2\) where \(t(1 - t) = \lambda = [M : Q]^{-1}\), the invariant \(\Lambda(Q \subset M)\) contains no points in the interval \((t, 1 - t)\), being equal to the set \([0, t) \cup (\lambda P_{k-1}(\lambda) / P_k(\lambda) | k \geq 0)\) when intersected with \([0, 1/2]\). This is in particular the case when \(Q = \{e_n \mid n \geq 1\}'' \subset \{e_n \mid n \geq 0\}'' = M\) is a subfactor generated by Jones projections of trace \(\tau(e_n) = \lambda < 1/4\).

The opposite phenomenon may hold true for subfactors \(Q \subset M\) with graph \(A_{\infty}\) and \([M : Q] = \lambda^{-1} > 4\). Namely, it is quite possible that in all such cases one has \([t, 1 - t] \subset \Lambda(Q \subset M)\). As a supporting evidence, consider the standard representation \(\mathcal{Q}^{st} \subset \mathcal{E} \mathcal{M}^{st}\) of \(Q \subset M\), as described in [P10]. This is an inclusion of discrete type I von Neumann algebras (i.e., direct sums of type \(I^\infty\) factors) with inclusion graph \(A_{\infty}\) and a conditional expectation \(\mathcal{E}\) having the property that \(\mathcal{E}|_M = E_Q^M\). Also, \(\mathcal{E}\) is the unique expectation preserving the trace \(\text{Tr}\) on \(\mathcal{M}^{st}\) whose weights are proportional to the square roots of indices of irreducible subfactors in the Jones tower. Since \(Q \subset M\) is embedded with commuting squares into \(Q^{st} \subset \mathcal{E} \mathcal{M}^{st}\), one has \(\Lambda(Q \subset M) \subset \Lambda(Q^{st} \subset \mathcal{E} \mathcal{M}^{st})\), and it is an easy exercise to see that the latter contains the entire interval \([t, 1 - t]\).

3.4. The finite dimensional case. The orthogonalization problem is certainly interesting for any finite dimensional abelian algebra \(B = \Sigma_{i=1}^n \mathbb{C} f_i\), with \(f_1, \ldots, f_n\) a partition of 1 with projections in \(M\) of trace \(\tau(f_i) = \alpha_i\), beyond the case \(n = 2\). To state this problem properly, for each \(n \geq 2\) we consider the set \(F_n\) of all \(n\)-tuples \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(0 \leq \alpha_i \leq 1\) and \(\Sigma_i \alpha_i = 1\). If \(Q\) is a von Neumann subalgebra of \(M\), we denote

\[
\Lambda_{n-1}(Q \subset M) := \{(\alpha_i)_i \in F_n \mid \exists f_1, \ldots, f_n \in \mathcal{P}(M), \Sigma_i f_i = 1, E_Q(f_i) = \alpha_i, \forall i\}.
\]

Thus, \(\Lambda_1(Q \subset M) = \{(\beta, 1 - \beta) \mid \beta \in \Lambda(Q \subset M)\}\). Also, any \((\alpha_i)_i \in \Lambda_{n-1}(Q \subset M)\) produces elements in \(\Lambda_{k-1}(Q \subset M)\) with \(k \leq n - 2\) by taking \(k\)-tuples \((\beta_j)_{j=1}^k\)
corresponding to partitions of \( \{1, \ldots, n\} \) into \( k \) subsets \( S_j \) and letting \( \beta_j = \Sigma_{i \in S_j} \alpha_i \). Thus, the restrictions on \( \Lambda = \Lambda_1 \) propagate into a set of restrictions for \( \Lambda_n, n \geq 2 \). In particular, any entry of an element in \( \Lambda_n(Q \subset M) \) for \( n \geq 1 \) is at least \( [M : Q]^{-1} \).

The question here is to calculate (or at least estimate) the invariants \( \Lambda_n(Q \subset M) \) for all \( n \geq 1 \). The case when \( Q \) is an irreducible subfactor of finite index is particularly interesting. One source of \( \Lambda_1(Q \subset M) \) corresponding to partitions of \( \{1, \ldots, n\} \) is to take irreducible subfactors \( P \subset Q \) and look for partitions of \( 1 \) with \( n + 1 \) projections \( f_1, \ldots, f_n, f_{n+1} \) in \( P' \cap M \). Indeed, because then \( E^M_Q(f_i) \in P' \cap Q = \mathbb{C}1 \) must be scalars. Such \( P \subset Q \) can be taken to be an irreducible subfactor obtained by reducing inclusions from a Jones tunnel \( Q_{-m} \subset Q_{-m+1} \subset \ldots \subset Q_{-1} \subset Q \subset M \) associated with \( Q \subset M \) by a minimal projection in \( Q'_{-m} \cap Q \). Thus, an \( (n+1) \)-tuple in \( \Lambda_n \) will appear whenever one has an \( (n+1) \)-point in the principal graphs of \( Q \subset M, M \subset \langle M, Q \rangle \). For instance, any triple point in \( \Gamma_{Q \subset M} \), will produce an element in \( \Lambda_2(Q \subset M) \), whose entries are proportional to square roots of indices of the corresponding irreducible subfactors in the Jones tunnel/tower. When combined with the restrictions on entries coming from the obstructions on \( \Lambda(Q \subset M) = \Lambda_1(Q \subset M) \), this can provide restrictions on the existence of triple points (and thus of graphs of subfactors). More generally, one can apply this same reasoning to the universal graph of \( \Gamma^u_{Q \subset M} \), as defined in [P10].

Another interesting question for subfactors of finite index \( Q \subset M \) is whether \( \Lambda(Q \subset M) \) (more generally \( \Lambda_n(Q \subset M), n \geq 1 \)) depends solely on the standard invariant of the subfactor \( G_{Q \subset M} \). The results in [P3] show that if \( [M : Q] < 4 \), then in fact \( \Lambda(Q \subset M) = \Lambda_1(Q \subset M) \) only depends on \( [M : Q] \). This may be the case for all \( \Lambda_n(Q \subset M), \) when \( [M : Q] \leq 4 \), but it is quite unclear for index \( > 4 \), where however the irreducibility condition \( Q' \cap M = \mathbb{C} \) should be imposed. A test case is when \( \Gamma_{Q \subset M} = A_\infty \) (i.e., when \( Q \subset M \) has TLJ standard invariant) with the index running over the interval \((4, \infty)\) (cf. [P5]).

3.5. Unitaries in orthonormal basis. One case of particular interest is to decide whether for some given \( 2 \leq k \leq n \) one can have \( \alpha = (\alpha_i)_i \in \Lambda_n(Q \subset M) \) with \( \alpha_1 = \alpha_2 = \ldots = \alpha_k = \lambda = [M : Q]^{-1} \). Thus, in such a case one has \( s\lambda \in \Lambda(Q \subset M) \) for any \( s = 0, 1, 2, \ldots, k \). By (Proposition 1.7 in [PP]), this is equivalent to whether \( Q_{-1} \subset Q \) has an orthonormal basis \( \{m_i\}_i \) with the first \( k \) terms \( 1 = m_1, \ldots, m_k \) being unitary elements.

If \([M : Q] \notin \mathbb{N}\) and \( n \) denotes its integer part then, as pointed out in (1.4.2° of [PP]), the formula \( \Sigma_i m_im_i^* = [M : Q]1 \) implies \( k \leq n - 1 \).

The [P3] restrictions on \( \Lambda(Q \subset M) \) can be used to obtain further restrictions on the maximal number of unitaries that can appear in an orthonormal basis of \( M \) over \( Q \). Indeed, if the index \([M : Q] \) is less than 4 but \( \neq 3 \), then the equations \( \lambda P_{m-1}(\lambda)/P_m(\lambda) = 2\lambda \) do not have solutions for \( \lambda^{-1} = 4 \cos^2(\pi/n+2) \) and \( n \neq 2, 4 \).
If in turn $[M : Q] > 4$ and we let $[M : Q] = n + \varepsilon$, with $1 > \varepsilon > 0$, then having $n - 1$ mutually orthogonal projections $f_1, \ldots, f_{n-1} \in M$ with $E_Q(f_i) = \lambda = [M : Q]^{-1}$ would imply that $f = 1 - \sum_i f_i$ satisfies $\tau(f) = 1 - (n - 1)\lambda \in \Lambda(Q \subset M)$ and $\tau(f) \geq \lambda/(1 - \lambda)$. This in turn implies $\varepsilon \geq \lambda/(1 - \lambda)$. Thus, $\varepsilon < \lambda/(1 - \lambda)$ forces $k \leq n - 2$.

In turn, it is an open problem whether an irreducible subfactor $Q \subset M$ with integer index $n \geq 5$ always has an orthonormal basis with $n$ unitaries. Note that by a result in [P2], if a subfactor $Q$ of a $\Pi_1$ factor $M$ contains a Cartan subalgebra of $M$, then $Q \subset M$ does have an orthonormal basis of unitaries (even if its index is infinite). However, if a subfactor $Q \subset M$ has $A_\infty$-graph (e.g., when $Q \subset M$ is constructed by the universal amalgamated free product method in [P5]), then finding even one single unitary $u \in M$ with $E_Q(u) = 0$ is an open question (see also the conjecture at the end of 3.3 above).

3.6. A related dilation problem. The question about whether a scalar $\alpha \in (0, 1)$ is the expected value on $Q$ of a projection in $M$ is viewed in [P3] as a dilation problem. More generally, one can ask this same question for an arbitrary element $b \in Q$ with $0 \leq b \leq 1$: can $b$ be dilated to a projection $p \in M$, i.e., does there exist a projection $p \in M$ such that $E_Q(p) = b$?

Alternatively, one can attempt the calculation of the entire set $E_Q(\mathcal{P}(M))$, or of the set $\tilde{\Lambda}(Q \subset M)$ of all functions $g : [0, 1] \to [0, 1]$ that can be spectral distributions of elements in $E_Q(\mathcal{P}(M))$, i.e., with the property that there exists $p \in \mathcal{P}(M)$ with $g(\alpha) = \tau(e_\alpha(E_Q(p)))$, $\forall \alpha \in [0, 1]$. These sets should be calculable for subfactors of index $< 4$.

On the other hand, note that if $Q \subset M$ satisfies the infinite index condition $M \not\prec_M Q$, then Corollary 1.2 easily yields a calculation of the approximate versions of these sets, showing that $E_{Q^\omega}(\mathcal{P}(M^\omega)) = \{b \in Q \mid 0 \leq b \leq 1\}$ and thus $\tilde{\Lambda}(Q^\omega \subset M^\omega)$ is equal to the set of non-increasing functions from $[0, 1]$ to $[0, 1]$. To see this, note first that $E_{Q^\omega}(\mathcal{P}(M^\omega))$ is $\| \|_2$-closed (by the usual $\omega$-diagonalisation procedure). Then note that any $b \in Q^\omega$ with $0 \leq b \leq 1$ can be approximated uniformly by elements of finite spectrum $b' = \sum_i \alpha_i q_i$ with $q_i \in \mathcal{P}(Q^\omega)$, $0 \leq \alpha_i \leq 1$. Finally, as noticed in 3.3 above, any $\alpha_i q_i$ can be dilated to a projection $p_i \in q_i M^\omega q_i$, thus $p = \sum_i p_i$ satisfies $E_{Q^\omega}(p) = b'$.

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