An Elementary Proof of the Twin Prime Conjecture

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Abstract It is well known that every prime number \( p \geq 5 \) has the form \( 6k - 1 \) or \( 6k + 1 \). We will call \( k \) the generator of \( p \). Twin primes are distinguished due to a common generator for each pair. Therefore it makes sense to search for the Twin Primes on the level of their generators. This paper presents a new approach to prove the Twin Prime Conjecture by a sieve method to extract all Twin Primes on the level of the Twin Prime Generators.

We define the \( p_n \omega \)-numbers \( x \) as numbers for which holds that \( 61x - k \) and \( 61x + k \) are coprime to the prime \( p_n \).

By dint of the average distance \( \bar{d}(p_n) \) between the \( p_n \omega \)-numbers we can prove the Twin Prime Conjecture indirectly.

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Notations

We will use the following notations:
\( \mathbb{N} \) the set of the positive integers,
\( \mathbb{P} \) the set of the primes, \( \mathbb{P}^* \) primes \( \geq 5 \),
\( \mathbb{P}_- = \{ p \in \mathbb{P}^* \mid p = -1 \mod 6 \} \)
and \( \mathbb{E}_n = \{ n \in \mathbb{N} \mid 6n - 1 \in \mathbb{P}_- \} \)
\( \mathbb{P}_+ = \{ p \in \mathbb{P}^* \mid p = +1 \mod 6 \} \)
and \( \mathbb{E}_n = \{ n \in \mathbb{N} \mid 6n + 1 \in \mathbb{P}_+ \} \)
and \( \mathbb{E} = \mathbb{E}_- \cap \mathbb{E}_+ \).

1. Introduction

The question on the infinity of the twin primes keeps busy many mathematicians for a long time. 1919 V. Brun [3] had proved that the series of the inverted twin primes converges while he had tried to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4,5,6]). 2014 Y. Zhang [7] obtained a great attention with his proof that there are infinitely many consecutive primes with a distance of 70,000,000 at most. With the project "PolyMath8" this bound could be lessened down to 246 respectively to 12 assuming the validity of the Elliott-Halberstam Conjecture [8].

We present in this paper another approach as in the most works on this topic. We transfer the looking for twin primes to the level of their generators because each twin prime has a common generator.

2. Twin Prime Generators

It is well known that every prime number \( p \geq 5 \) has the form \( 6k - 1 \) or \( 6k + 1 \). We will call \( k \) the generator of \( p \). Twin primes are distinguished due to a common generator for each pair. Therefore it makes sense to search for the twin primes on the level of their generators.

Let be
\[
\kappa(p) = \begin{cases} 
\frac{p + 1}{6} & \text{for } p \in \mathbb{P}_- \\
\frac{p - 1}{6} & \text{for } p \in \mathbb{P}_+ 
\end{cases} 
\] (2.1)
an over \( \mathbb{P}^* \) defined function, the generator of the pair \((6x(p) - 1, 6x(p) + 1)\).

A number \( x \) is a member of \( \mathbb{E} \) if \( 6x - 1 \) as well as \( 6x + 1 \) are primes. This is true if the following statement holds.

Theorem 1. A number \( x \) is a member of \( \mathbb{E} \) if and only if there is no \( p \in \mathbb{P}^* \) with \( p < 6x - 1 \) where one of the following congruences holds:
\[
x = -\kappa(p) \mod p 
\] (2.2)
\[
x = +\kappa(p) \mod p 
\] (2.3)

Proof.
A. \( p \in \mathbb{P}_- \), therefore is \( p = 6\kappa(p) - 1 \):
If (2.2) is true then there is an \( n \in \mathbb{N} \) with
From (2.2) and (2.3) it is easy to see that the exclusion of a number \( x \) \( \in \mathbb{E} \) if and only if
\[
\pi(x) \leq \pi(\hat{p}(x)) - 1.
\]

For (2.3) the proof will be done with \( \pi(x) \leq \pi(\hat{p}(x)) \) if and only if
\[
x = \kappa(p_n) \text{ (mod } p_n \text{)}.
\]

With these it's shown that \( x \notin \mathbb{E} \) if the congruences (2.2) or (2.3) are valid. They cannot be true both because they exclude each other.

If on the other hand \( x \notin \mathbb{E} \), then is \( 6x - 1 \) or \( 6x + 1 \) no prime. Let be \( 6x - 1 \equiv 0 \) (mod \( p \)) and \( p \in \mathbb{P}_+ \). Then we have
\[
6x - 1 = p \text{ (mod } p \text{)} = (6\kappa(p) - 1) \text{ (mod } p \text{)}
\]
\[
6x = 6\kappa(p) \text{ (mod } p \text{)}
\]
\[
x = \kappa(p) \text{ (mod } p \text{)}.
\]

For \( p \in \mathbb{P}_+ \), we have
\[
6x - 1 = -p \text{ (mod } p \text{)} = -(6\kappa(p) + 1) \text{ (mod } p \text{)}
\]
\[
6x = -6\kappa(p) \text{ (mod } p \text{)}
\]
\[
x = -\kappa(p) \text{ (mod } p \text{)}.
\]

The other both cases we can handle in the same way. Therefore either (2.2) or (2.3) is valid if \( x \notin \mathbb{E} \).

If we consider that the least proper divisor of a number \( 6x - 1 \) or \( 6x + 1 \) is less or equal to \( \sqrt{6x + 1} \) than \( p \) in the congruences (2.2) and (2.3) can be further limited by
\[
\hat{p}(x) = \min(p \in \mathbb{P}_+ | p \leq \sqrt{6x + 1}).
\]

Henceforth we will use the letter \( p \) for a general prime number and \( p_n \) if we describe an element of a sequence of primes. With \( p_n \) as the \( n \)-th prime number\(^1\) and \( \pi(x) \) as the number of primes \( \leq x \) we have with
\[
8 = \kappa(p_n) \text{ (mod } p_n \text{)} \quad \text{or} \quad 12 = \kappa(p_n) \text{ (mod } p_n \text{)}
\]
for \( 3 \leq n \leq \pi(\hat{p}(x)) \) a proofable system of criteria to exclude a number \( x \geq 4 \) as not being a member of \( \mathbb{E} \).

\(^1\) It is \( p_1 = 2 \).

### 3. The Twin Sieve

The congruences in (2.4) can be combined in the following way:
\[
x^2 = \kappa(p_n)^2 \text{ (mod } p_n \text{)} \quad \text{for } 3 \leq n \leq \pi(\hat{p}(x))
\]
(3.1)
because if \( x = \pm \kappa(p_n) \text{ (mod } p_n \text{)} \) then there is a number \( t \) with \( x = \pm \kappa(p_n) + tp_n \). Squared this produces
\[
x^2 = \kappa(p_n)^2 + p_n(t^2 - 2\kappa(p_n)) \quad \text{and we get}
\]
\[
x^2 = \kappa(p_n)^2 \text{ (mod } p_n \text{)}.
\]

This results in a system of sieves with sieve functions \( \psi(x, p_n) \) for which hold for \( 3 \leq n \leq \pi(\hat{p}(x)) \)
\[
x^2 - \kappa(p_n)^2 = \psi(x, p_n) \text{ (mod } p_n \text{)} \quad \text{respectively}
\]
\[
\psi(x, p_n) = (x^2 - \kappa(p_n)^2) \text{ (mod } p_n \text{)}.
\]

Obviously \( \psi(x, p) \) is a periodical function in \( x \) with a period length of \( p \). We'll call the sieve represented by \( \psi(x, p_n) \) as \( S_n \). For the system of the sieves \( S_3 \times \ldots \times S_n \) we'll build the aggregate sieve functions
\[
\Psi(x, p_n) = \prod_{i=3}^{n} \psi(x, p_i)
\]
(3.2)
and
\[
\hat{\Psi}(x) = \Psi(x, \hat{p}(x)).
\]

Because the value set of \( \psi(x, p) \) consists of positive integers between 0 and \( p - 1 \), \( \Psi(x, p) \) and \( \hat{\Psi}(x) \) have rational values between 0 and 1.

A number \( x \) will be “sieved” by \( S_n \) if and only if \( \psi(x, p_n) = 0 \). With (3.3) in this case also is \( \Psi(x, p_n) = 0 \). In contrast to the sieve of ERATOSTHENES in our sieve the exclusion of a number \( x \) will be not controlled by \( x \text{ Mod } p = 0 \), but by \( (x^2 - \kappa(p_n)^2) \text{ Mod } p = 0 \).

Let be
\[
O_n = \min(x \in \mathbb{N} | \hat{p}(x) = p_n).
\]

For \( x \geq O_n \) “works” the sieve \( S_n \), i.e. \( O_n \) is the origin of the sieve \( S_n \). Every sieve has up from \( O_n \) in every \( \psi \)-period just \( p_n - 2 \) positions with \( \psi(x, p_n) \neq 0 \) and two positions with \( \psi(x, p_n) = 0 \), once if (2.2) and on the other hand if (2.3) is valid. We speak about \( a \)- and \( b \)-bars of the sieve \( S_n \). From (2.2) and (2.3) it is easy to see that the distance between an \( a \)- and a \( b \)-bar is \( 2\kappa(p_n) \).

It is \( p_n \leq \hat{p}(x) \leq \sqrt{6x + 1} \) and therefore \( p_n^2 \leq 6x + 1 \).

Then
\[
O_n = \frac{p_n^2 - 1}{6}
\]
(3.5)
is the least number which meets this relation. It is easy to prove that for every prime \( p \) holds that \( p^2 - 1 \) is an integer divisible by 6.

**Theorem 2.** Every sieve \( S_n \) with \( n \geq 3 \) starts at its origin \( O_n \) with a sieve bar and we have \( \psi(O_n, p_n) = 0 \).

**Proof.** We substitute \( p_n \) by \( 6\kappa(p_n) \pm 1 \). With this and (3.5) holds

\[
O_n = \frac{(6\kappa(p_n) \pm 1)^2 - 1}{6} = \frac{6\kappa(p_n)(6\kappa(p_n) \pm 2)}{6} = \kappa(p_n)(6\kappa(p_n) \pm 1) \pm \kappa(p_n) = \kappa(p_n)^2 \pm \kappa(p_n) = \pm \kappa(p_n) \pmod{p_n} \rightarrow \psi(O_n, p_n) = 0.
\]

We see that \( S_n \) starts for \( p_n \in \mathbb{P}_- \) with an \( a \)-bar (2.2) and in the other case with a \( b \)-bar (2.3).

For every \( x \geq O_n \), the local position in the sieve \( S_n \) relative to the phase start can be determined by the position function \( \tau(x, p_n) \):

\[
x + \kappa(p_n) = \tau(x, p_n) \pmod{p_n} \quad \text{respectively}
\tau(x, p_n) = (x + \kappa(p_n)) \pmod{p_n}.
\]

**Between the sieve function** \( \psi(x, p) \) and the position function \( \tau(x, p) \) there is the following relationship:

\[
\psi(x, p) = \tau(x, p) \cdot (x - \kappa(p)) \pmod{p} = \tau(x, p) \cdot (\tau(x, p) - 2\kappa(p)) \pmod{p}.
\]

Obviously \( \psi(x, p) = 0 \) if and only if \( \tau(x, p) = 0 \) (\( a \)-bar) or \( \tau(x, p) = 2\kappa(p) \) (\( b \)-bar).

4. The Permeability of the Sieves \( S_3 \times \ldots \times S_n \)

For every \( x \) in the interval

\[
A_n := [O_n, O_n + 1]
\]

\( \hat{p}(x) \) persists constant on the value \( p_n \). The length of this interval will be denoted as \( d_n \). It is depending on the distance between successive primes. Since they can only be even, we have with \( a = 2, 4, 6, \ldots \)

\[
d_n = \frac{(p_n + a)^2 - 1}{6} - \frac{p_n^2 - 1}{6} = \frac{2ap_n + a^2}{6}.
\]

On the other hand it results because of \( p_{n+1} < 2p_n \),

\[
d_n = \frac{p_{n+1}^2 - 1}{6} - \frac{p_n^2 - 1}{6} = \frac{p_{n+1}^2 - p_n^2}{6} = \frac{(p_n + a)^2 - 1 - p_n^2}{6} = \frac{2ap_n + a^2}{6}.
\]

\[
\frac{p_{n+1}^2 - p_n^2}{6} = \frac{3p_n^2 - p_n^2}{6} = \frac{p_n^2}{2}.
\]

The congruences from (3.6)

\[
x + \kappa(p_i) = \tau(x, p_i) \pmod{p_i}, \quad 3 \leq i \leq n.
\]

meet the requirements of the Chinese Remainder Theorem (see [1], p. 89). Therefore it is modulo \( 5 \cdot 7 \ldots \cdot p_n \) uniquely resolvable. With

\[
p_n \#_5 := \prod_{i=3}^{n} p_i = 5 \cdot 7 \ldots \cdot p_n
\]

it’s \( \pmod{p_n \#_5} \) uniquely resolvable. Therefore the sieves \( S_3 \times \ldots \times S_n \) have the period length \( p_n \#_5 \) and for the aggregate sieve function holds:

\[
\psi(x + a \cdot p_n \#_5, p_n) = \psi(x, p_n) \quad a \in \mathbb{N}.
\]

**Definition 4.1.** A positive integer \( x \) will be called an “\( \omega_p \)-number” if both \( 6x - 1 \) and \( 6x + 1 \) are coprime to \( p \). In this case is \( \Psi(x, p) > 0 \).

Let be

\[
\mathcal{P}_n := [O_n, O_n + p_n \#_5 - 1]
\]

the interval of the period of the sieves \( S_3 \times \ldots \times S_n \). We’ll denote it henceforth as period section. Evidently is \( \mathcal{A}_n \subset \mathcal{P}_n \) for all \( n \geq 3 \).

The values of the function \( \tau(x, p) \) are the numbers \( 0, 1, \ldots, p - 1 \). Two of them result in the excluding of \( x \) and \( p - 2 \) don’t. Therefore by working of the sieves \( S_3 \times \ldots \times S_n \) we have

\[
\varphi(p_n) = \prod_{i=3}^{n}(p_i - 2)
\]

\( \omega_{p_n} \)-numbers in \( \mathcal{P}_n \). If a lot of them are in \( \mathcal{A}_n \), they are members of \( \mathbb{E} \) because the sieves \( S_3 \times \ldots \times S_n \) here are working only. The relation between (4.6) and the period length of (4.4) results in

\[
\eta(p_n) = \varphi(p_n) \cdot \prod_{i=3}^{n} \frac{p_i - 2}{p_i},
\]

as a measure of the mean “permeability” of working of the sieves \( S_3 \times \ldots \times S_n \) or as the density of the \( \omega_{p_n} \)-numbers in \( \mathcal{P}_n \). Obviously \( \eta(p) \) is a strong monotonously decreasing function. Its inversion

\[
\overline{\eta}(p) = \frac{1}{\eta(p)}
\]

\( \# \) It is \( p_n \#_5 = \frac{p_n \#}{6} \), with the primorial \( p_n \# \)

\( ^5 \) Then is \( \text{gcd}(36x^2 - 1, p \#_5) = 1 \).
discriminates the average distance between the $\omega_p$-numbers in their period section.

**Theorem 3.** The density of the $\omega_p$-numbers in their period section is lower bounded by

$$\eta(p) > \frac{2}{3} p \text{ for } p \in \mathbb{P}^*.$$ 

**Proof.** Let be $Q_p = \{q \in \mathbb{P}^* | q \leq p\}$ and $U_p = \{u \equiv 1(\text{mod } 2) | 5 \leq u \leq p\}$.

Because all primes $> 2$ are odd numbers it holds $Q_p \subset U_p$ for $p > 7^8$. All factors of $\eta(p)$ are less than 1. It results

$$\eta(p) > \prod_{u \in U} \frac{u}{u} = \frac{3 \cdot 5 \cdot 7 \cdot 9}{7} \cdots \frac{p-4}{p} \frac{p-2}{p} = \frac{3}{p}.$$ 

By inverting this relationship, we obtain for the average distance

$$\overline{s}(p) < \frac{p}{3}.$$ 

(4.9)

Under consideration of (4.6) we obtain furthermore

$$2 \overline{s}(p_n) < \frac{2p_n}{3} < \frac{2}{3}(p_n + 1) \leq d_n \Rightarrow \overline{s}(p_n) < \frac{d_n}{2}.$$ 

(4.10)

This means that the average distance between $\omega_{p_n}$-numbers remains ever less than the half of the length of $A_n$, the interval where $\omega_{p_n}$-numbers are twin prime generators.

**5. The Sieve Process on Average**

The intervals $A_n, n \geq 3$ defined by (4.1) cover the positive integers $\geq 4$ gapless and densely. It is

$$\mathbb{N} = \{1, 2, 3\} \cup \bigcup_{n=3}^{\infty} A_n \text{ and } \bigcap_{n=3}^{\infty} A_n = \emptyset.$$ 

They are the beginnings of the period sections $P_n$ of the $\omega_{p_n}$-numbers. Hereafter let’s say A-sections to the intervals $A_n$. Every $\omega_{p_n}$-number which lies in an A-section is a twin prime generator (see above). In contrast to the A-sections the period sections $P_n$ overlap each other very densely. So the period section $P_3$ reaches over 1739 A-sections up to the beginning of the period section $P_{748}$ and the next $P_{10}$ over 7863 A-sections up to the beginning of $P_{7873}$.

### Theorem 4.

Each origin $O_n$ cannot be located at the beginning $O_m + a \cdot p_m \#_5 \mid a \in \mathbb{N}$ of any period of the sieves $S_3 \times \cdots \times S_m$ for $m < n$. Therefore it holds for $m < n$

$$O_n \neq O_m \text{ (mod } p_m \#_5).$$

**Proof.** The equation

$$\frac{p_n^2}{6} + a \cdot p_m \#_5 = \frac{p^2_n - 1}{6} \text{ and thus } p_n^2 + a \cdot p_m \#_5 = p_n^2$$

is for no primes $p_m < p_n$ solvable, because of

$$\gcd(p_m, p_n) = 1.$$ 

Vice versa holds that every period section $P_{n+1}$ starts always inside of the previous period section $P_n$ nearby to its origin because (see (4.3) also)

$$O_{n+1} = O_n + d_n \text{ and } d_n < \frac{p_n^2}{2} \leq \frac{p_n \#_5}{2}.$$ 

Let be

$$P_n^k := \left[O_n + k \cdot p_n \#_5, O_n + (k + 1) \cdot p_n \#_5 - 1\right]$$

and $A_n^+ := O_n + p_{n+1} \#_5, O_{n+1} - 1 + p_{n+1} \#_5$.

With these we can show the recursive structure of the period sections

$$P_{n+1} = P_n \setminus A_n \cup \bigcup_{k=1}^{p_{n+1}-1} P_n^k \cup A_n^+.$$ 

(5.1)

We can clearly see that the period section $P_n$ overlaps $P_{n+1}$ up to the end of $P_n \setminus A_n$ and $P_{n+1}$ has much space for A-sections $A_t$ with $t \geq n + 1$.

The one consequence of this dense overlapping of the period sections is that a plurality of the $\omega_{p_n}$-gaps from the period section $P_n$ persist constant as also $\omega_{p_n}$-gaps for $n > m$ but in a shifted position relative to their origin $O_n$ (see Theorem 4 and (5.1)).

On the other hand this dense overlapping guarantees that extreme anomalies of the distribution of the $\omega_{p_n}$-numbers cannot occur.

For the quantity of the $\omega_{p_n}$-numbers in $P_n$ is corresponding with (4.6)

$$\varphi(p_n) = \varphi(p_{n-1}) \cdot (p_n - 2).$$

In $P_n$ the $\varphi(p_n)$ $\omega_{p_n}$-numbers are spread over $p_n \#_5$ positions. According to (5.1) we have

$$p_{n+1} \cdot \varphi(p_n) \omega_{p_n} \text{ numbers in } P_{n+1}.$$ 

In comparison

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*For $p \leq 7$ is $Q_p = U_p$.

* We can even prove that $\overline{s}(p_n) < d_n$ for $p_n > 200(n > 45)$. 

* It is easy to prove that the $\omega_p$-numbers in their period section are symmetrically distributed around $\frac{p \#_5}{2}$ and $p \#_5$. Nevertheless the distribution is non-uniform.
between them and the \( \omega_{pn+1} \) --numbers resulting from the working of the sieve \( S_{n+1} \) we see

\[
p_{n+1} \cdot \varphi(p_n) - \varphi(p_{n+1}) = p_{n+1} \cdot \varphi(p_n) - (p_{n+1} - 2) \cdot \varphi(p_n) = 2\varphi(p_n).
\]

(5.2)

We loose by the working of \( S_{n+1} \) in the period section \( \mathcal{P}_{n+1} \) just \( 2\varphi(p_n) \) potential generators of twin primes. In other words, the sieve \( S_{n+1} \) has \( 2\varphi(p_n) \) "beating bars" in \( \mathcal{P}_{n+1} \). At these positions \( x \) holds

\[
\Psi(x, p_n) > 0 \text{ and } \psi(x, p_{n+1}) = 0.
\]

(5.3)

Only the beating bars let grow the gaps by exclusion of the \( \omega_{pn} \) --numbers between two \( \omega_{pn} \) --gaps to one \( \omega_{pn+1} \) --gap. By the working of the sieve \( S_{n+1} \) we obtain the following sieve balance "on average":

The distances between the \( \omega_{pn} \) --numbers persist unchanged at \( \delta(p_n) \) on average except of those \( 2\varphi(p_n) \) \( \omega_{pn} \) --numbers which are met by the beating bars of the sieve \( S_{n+1} \). Thereby a distance \( D \) occurs between the adjacent \( \omega_{pn+1} \) --numbers on average:

\[
p_{n+1} \#_5 = \text{changed} + \text{unchanged}
\]

\[
= 2\varphi(p_n) \cdot D + (\varphi(p_{n+1}) - 2\varphi(p_n)) \cdot \delta(p_n)
\]

\[
= 2\varphi(p_n) \cdot D + \left( \varphi(p_n) \cdot (p_{n+1} - 2) \right) \frac{p_n \#_5}{\varphi(p_n)}
\]

\[
= 2\varphi(p_n) \cdot D + p_{n+1} \#_5 - 4p_n \#_5
\]

and therefore

\[
0 = 2\varphi(p_n) \cdot D - 4p_n \#_5
\]

and hence

\[
D = \frac{2p_n \#_5}{\varphi(p_n)} = 2\delta(p_n).
\]

Therefore even the gaps between the \( \omega_{pn+1} \) --numbers (\( \omega_{pn+1} \) --gaps) which result from the beating bars persist less than \( d_{n+1} \) on average because

\[
D = 2\delta(p_n) < \frac{2p_n}{3} < \frac{2p_{n+1}}{3} < \frac{2}{3} (p_{n+1} + 1) \leq d_{n+1}. \tag{5.4}
\]

6. Proof of the Twin Prime Conjecture

The proof will be done indirectly. We assume that there is only a finite number of twin primes and therefore there is only a finite number of twin prime generators. Let be \( y_o \) the greatest one. It lies in the A-section \( \mathcal{A}_{y_o} \) with \( n_o = \pi \left( \hat{p}(y_o) \right) \), the beginning of the period section \( \mathcal{P}_{n_o} \).

In the subsequent A-sections \( \mathcal{A}_t \) with \( t > n_o \) consequently there cannot be any twin prime generators and therefore no \( \omega_{pt} \) --numbers. But then we have \( \omega_{pt} \) --gaps with lengths \( > d_i \) in all (infinitely many) period sections \( \mathcal{P}_t \) for \( t > n_o \).

Because

- all period sections \( \mathcal{P}_t \) are very densely overlapped and therefore extreme anomalies of the distribution of the \( \omega_{pt} \) --numbers cannot occur,
- the average distances between the \( \omega_{pt} \) --numbers are less than \( \frac{d_i}{2} \),
- and even the \( \omega_{pt} \) --gaps which are generated by beating bars of the sieves \( S_t \) are less than \( d_i \) on average,

therefore it is not possible to have for all \( t > n_o \) only period sections \( \mathcal{P}_t \) with \( \omega_{pt} \) --gaps at their beginnings which are all greater than \( d_i \).

Therefore the proof assumption cannot be valid and thus the Twin Prime Conjecture must be true.

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