The Tutte polynomial of a graph, or equivalently the $q$-state Potts model partition function, is a two-variable polynomial graph invariant of considerable importance in both combinatorics and statistical physics. The computation of this invariant for a graph is NP-hard in general. In this paper, based on their self-similar structures, we recursively describe the Tutte polynomials of an infinite family of scale-free lattices. Furthermore, we give some exact analytical expressions of the Tutte polynomial for several special points at $(X, Y)$-plane.

**Keywords:** Potts model; Tutte polynomial; Spanning tree; Asymptotic growth constant; Self-similar.

1 Introduction

The Tutte polynomial $T(G; x, y)$ of a graph $G$, due to W. T. Tutte \[1\], is a two-variable polynomial and is deeply connected with many areas of both physics and mathematics. It is defined as

$$T(G; x, y) = \sum_{H \subseteq G} (x - 1)^{r(G) - r(H)} (y - 1)^{n(H)},$$

where the sum runs over all the spanning subgraphs $H$ of $G$, $r(G) = |V(G)| - k(G)$, $n(G) = |E(G)| - |V(G)| + k(G)$ and $k(G)$ is the number of components of $G$. For a thorough survey on the Tutte polynomial, we refer the reader to \[2, 3, 4\]. The importance of the Tutte polynomial comes from the rich information it contains about the underlying graph. It contains several other polynomial invariants, such as the chromatic polynomial, the flow polynomial, the Jones polynomial and the all terminal reliability polynomial as partial evaluations, and various numerical invariants such as the number of spanning trees as complete evaluations. Furthermore, the Tutte polynomial has been widely studied in the field of statistical physics where it appears as the partition function of the zero-field
Potts model [5, 6]. In fact, let \( G \) be a graph with \( n \) vertices and \( k(G) \) components and let \( Z_{\text{zero}} \) denote the partition function of the zero-field Potts model, then \( Z_{\text{zero}}(G; q, v) \) and \( T(G; x, y) \) satisfy the following relation \([7, 8]\):

\[
Z_{\text{zero}}(G; q, v) = q^{k(G)}v^{n-k(G)}T(G; (q + v)/v, v + 1).
\]

In both fields of combinatorics and statistical physics, Tutte polynomials of many graphs (or lattices) have been computed by different methods \([9, 10, 11, 12, 13, 14, 15, 16, 17]\). Recently, on the basis of the subgraph expansion definition of the Tutte polynomial, a very useful method for computing the Tutte polynomial, called the subgraph-decomposition method, was developed by Donno et al. \([18]\). This technique is highly suited for computing the Tutte polynomial of self-similar graphs, and some applications of it can be found in \([19, 20, 21]\).

The lattice (network) under consideration was introduced by Kaufman et al. \([22, 23]\) and was further studied by Zhang et al. \([24, 25]\) from the viewpoint of complex networks. The aim of this paper is to compute the Tutte polynomial of this deterministic scale-free network. For this purpose, we first partition the set of spanning subgraphs of this network into two disjoint subsets. In this way, we can express the Tutte polynomial by two summands. Then, we study the contributions of all kinds of possible combinations of this two subsets. Finally, base on the self-similar structure of the network, we obtain a recursive formula for each summand. In particular, as special cases of the general Tutte polynomial, we get:

- the recursive formula for computing the Tutte polynomial \( T(G; x, y) \);
- the number \( \tau(G) \) of spanning trees and the asymptotic growth constant \( \lim_{n \to \infty} \frac{\ln\tau(G_n)}{\ln(v(G_n))} \);
- the dimension of the bicycle space;
- the number of acyclic root-connected orientations; the number of indegree sequences of strongly connected orientations.

2 Preliminaries

In this section, we briefly discuss some necessary background that will be used for our calculations. We use standard graph terminology and the words “network” and “graph”
indistinctly. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The vertices $a$ and $b$ are the end-points of an edge $\{a, b\}$. An orientation of graph $G$ is the digraph defined by the choice of a direction for every edge of $E(G)$. A directed cycle of a digraph is a set of edges forming a cycle of the graph such that they are all directed accordingly with a direction for the cycle. A digraph is acyclic if it has no directed cycle, and strongly connected if for every pair of vertices there is a directed cycle containing them. A sink for a digraph is a vertex with no outgoing edge.

![Figure 1: The lattices $G_0, G_1, G_2$ and $G_3$.](image)

The network, as shown in Figure 1, can be constructed as follows.

For $n = 0$, $G_0$ is the complete graph $K_2$. For $n \geq 0$, $G_{n+1}$ can be constructed from four copies of $G_n$ by merging four groups of vertices and adding a new edge. Specifically, let $X_n$ and $Y_n$, hereafter called special vertices of $G_n$, be the leftmost and the rightmost vertex of $G_n$. $X_n$ and $X_n$ are combined into the special vertex $X_{n+1}$ of $G_{n+1}$, $Y_n$ and $Y_n$ are combined into the special vertex $Y_{n+1}$ of $G_{n+1}$, and a new edge $e_n$ is added between two vertices combined by $Y_n$ and $X_n$. The construction of $G_{n+1}$ is illustrated in Figure 2.

![Figure 2: The construction of $G_{n+1}$](image)
3 Recursive formulas

In this section, we give the computational formulas of Tutte polynomial of the lattice $G_n$. It is easy to obtain that the order and size of the lattice $G_n$ are, respectively,

$$|V(G_n)| = \frac{2 \times 4^n + 4}{3}, \quad |E(G_n)| = \frac{4^{n+1} - 1}{3}.$$ 

And the average degree after $n$ iterations is $\langle k \rangle_n = \frac{2|E(G_n)|}{|V(G_n)|}$, which approaches 4 in the infinite $n$ limit.

To investigate the Tutte polynomial $T(G_n; x, y)$. First of all, we partition the set of the spanning subgraph of $G_n$ into two disjoint subsets:

- $G_{1,n}$ denotes the set of spanning subgraphs of $G_n$, where two special vertices $X_n$ and $Y_n$ of $G_n$ belong to the same component;
- $G_{2,n}$ denotes the set of spanning subgraphs of $G_n$, where two special vertices $X_n$ and $Y_n$ of $G_n$ do not belong to the same component.

![Figure 3: The spanning subgraph of $G_n$: Type I (left), Type II (right).](image)

Observe that, for each $n \geq 0$, we have the partition $G_{1,n} \cup G_{2,n}$ of the set of spanning subgraphs of $G_n$. Next, let $T_n(x, y)$ denote the Tutte polynomial $T(G_n; x, y)$ of $G_n$. For every $n \geq 1$, $T_{1,n}(x, y)$ and $T_{2,n}(x, y)$ are the following polynomials:

- $T_{1,n}(x, y) = \sum_{H \in G_{1,n}} (x - 1)^{r(G_n) - r(H)}(y - 1)^{n(H)}$;
- $T_{2,n}(x, y) = \sum_{H \in G_{2,n}} (x - 1)^{r(G_n) - r(H)}(y - 1)^{n(H)}$.

We have

$$T_n(x, y) = T_{1,n}(x, y) + T_{2,n}(x, y).$$

In order to obtain $T_n(x, y)$, we need to find recursive formulas on $T_{1,n}(x, y)$ and $T_{2,n}(x, y)$. For this purpose, we analyze the relation between spanning subgraphs of $G_{n+1}$ and spanning subgraph of $G_n$. Observe from Figure 2 that $G_{n+1}$ is constructed from four copies of $G_n$ by merging some special vertices and adding a new edge $e_n$. Thus, a spanning subgraphs of $G_{n+1}$ is combined by the spanning subgraphs of the four copies...
$G^n_i(i = 1, 2, 3, 4)$ of $G_n$ with $S$, where $S$ may be $\{e_n\}$ or $\emptyset$. Indeed, a spanning subgraph $H$ of $G_{n+1}$ is uniquely determined by the restriction of $H$ to the four copies $G^n_i$ (denoted by $H_i(i = 1, 2, 3, 4)$, respectively) and $S$, and vice versa. Therefore, the Tutte polynomial of $G_{n+1}$ can be written as

$$T_{n+1}(x, y) = \sum_{H_i \subseteq G^n_i \cup \{\cup_i \cup S = H\}} (x - 1)^{r(G_{n+1}) - r(H)}(y - 1)^{n(H)},$$

where the sum runs over all spanning subgraphs $H_i$ of $G^n_i (i = 1, 2, 3, 4)$ and $S$. Now, we need to know how $r(H)$ and $n(H)$ depend on $r(H_i)$ and $n(H_i)$, for $i = 1, 2, 3, 4$. Note that $|V(G_{n+1})| = 4|V(G_n)| - 4$, $|E(H)| = \sum_{i=1}^{4} |E(H_i)|$ if $S = \{e_n\}$ and $|E(H)| = \sum_{i=1}^{4} |E(H_i)|$ if $S = \emptyset$. So, there are two cases to be considered.

**Case 1.** $S = \{e_n\}$.

In this case, we consider the spanning subgraph $H$ of $G_{n+1}$, which contains the new adding edge $\{e_n\}$, and $|E(H)| = \sum_{i=1}^{4} |E(H_i)| + 1$.

**Subcase 1.** If $k(H) = \sum_{i=1}^{4} k(H_i) - 3$, then

$$r(H) = |V(H)| - k(H) = (4|V(G_n)| - 4) - \left(\sum_{i=1}^{4} k(H_i) - 3\right) = \sum_{i=1}^{4} r(H_i) - 1.$$

Moreover, we have

$$n(H) = |E(H)| - r(H) = \left(\sum_{i=1}^{4} |E(H_i)| + 1\right) - \left(\sum_{i=1}^{4} r(H_i) - 1\right) = \sum_{i=1}^{4} n(H_i) + 2$$

d\text{and}\n
$$r(G_{n+1}) - r(H) = (|V(G_{n+1})| - 1) - \left(\sum_{i=1}^{4} r(H_i) - 1\right) = \sum_{i=1}^{4} (r(G_n) - r(H_i)).$$

Thus,

$$(x - 1)^{r(G_{n+1}) - r(H)}(y - 1)^{n(H)} = (y - 1)^2 \prod_{i=1}^{4} (x - 1)^{r(G_n) - r(H_i)}(y - 1)^{n(H_i)}.$$
and
\[ r(G_{n+1}) - r(H) = (|V(G_{n+1})| - 1) - \left( \sum_{i=1}^{4} r(H_i) \right) = \sum_{i=1}^{4} (r(G_n) - r(H_i)) - 1. \]

Hence,
\[ (x - 1)^{r(G_{n+1})-r(H)} (y - 1)^{n(H)} = \frac{y - 1}{x - 1} \prod_{i=1}^{4} (x - 1)^{r(G_n) - r(H_i)} (y - 1)^{n(H_i)}. \]

**Subcase 3.** If \( k(H) = \sum_{i=1}^{4} k(H_i) - 5 \), then we can obtained, similarly, that
\[ (x - 1)^{r(G_{n+1})-r(H)} (y - 1)^{n(H)} = \frac{1}{(x - 1)^2} \prod_{i=1}^{4} (x - 1)^{r(G_n) - r(H_i)} (y - 1)^{n(H_i)}. \]

**Case 2.** \( S = \emptyset \).

In this case, we consider the spanning subgraph \( H \) of \( G_{n+1} \), which does not contain the new adding edge \( e_n \), and \( |E(H)| = \sum_{i=1}^{4} |E(H_i)| \).

**Subcase 1.** If \( k(H) = \sum_{i=1}^{4} k(H_i) - 3 \), then
\[ r(H) = |V(H)| - k(H) = (4|V(G_n)| - 4) - (\sum_{i=1}^{4} k(H_i) - 3) = \sum_{i=1}^{4} r(H_i) - 1. \]
Moreover, we have
\[ n(H) = |E(H)| - r(H) = (\sum_{i=1}^{4} |E(H_i)|) - (\sum_{i=1}^{4} r(H_i) - 1) = \sum_{i=1}^{4} n(H_i) + 1 \]
and
\[ r(G_{n+1}) - r(H) = (|V(G_{n+1})| - 1) - \left( \sum_{i=1}^{4} r(H_i) - 1 \right) = \sum_{i=1}^{4} (r(G_n) - r(H_i)) \]
Thus,
\[ (x - 1)^{r(G_{n+1})-r(H)} (y - 1)^{n(H)} = (y - 1) \prod_{i=1}^{4} (x - 1)^{r(G_n) - r(H_i)} (y - 1)^{n(H_i)}. \]

**Subcase 2.** If \( k(H) = \sum_{i=1}^{4} k(H_i) - 4 \), then
\[ r(H) = |V(H)| - k(H) = (4|V(G_n)| - 4) - (\sum_{i=1}^{4} k(H_i) - 4) = \sum_{i=1}^{4} r(H_i). \]
Moreover, we have
\[ n(H) = |E(H)| - r(H) = \left( \sum_{i=1}^{4} |E(H_i)| \right) - \left( \sum_{i=1}^{4} r(H_i) \right) = \sum_{i=1}^{4} n(H_i) \]
and
\[ r(G_{n+1}) - r(H) = (|V(G_{n+1})| - 1) - \left( \sum_{i=1}^{4} r(H_i) \right) = \sum_{i=1}^{4} (r(G_i) - r(H_i)) - 1 \]
Thus we have
\[ (x - 1)^{r(G_{n+1})-r(H)}(y - 1)^{n(H)} = \frac{1}{x-1} \prod_{i=1}^{4} (x - 1)^{r(G_i) - r(H_i)}(y - 1)^{n(H_i)}. \]

For the convenience of discussion, we use solid lines to join two special vertices when the corresponding spanning subgraph of \( G_n \) belongs to \( G_{1,n} \); Otherwise, we use dotted lines instead of solid lines. We distinguish two types of the spanning subgraph as shown in Figure 3.

**Theorem 1.** The Tutte polynomial \( T_{n+1}(x, y) \) of \( G_{n+1} \) is given by
\[ T_{n+1}(x, y) = T_{1,n+1}(x, y) + T_{2,n+1}(x, y), \]
where the polynomials \( T_{1,n+1}(x, y) \) and \( T_{2,n+1}(x, y) \) satisfy the following recursive relations:
\[ T_{1,n+1}(x, y) = y(y - 1)T_{1,n}^4 + \frac{4y}{x-1}T_{1,n}^3T_{2,n} + \frac{2x + 2}{(x - 1)^2}T_{1,n}T_{2,n}^2, \quad (1) \]
\[ T_{2,n+1}(x, y) = \frac{2y + 2}{x - 1}T_{1,n}^2T_{2,n}^2 + \frac{4x}{(x - 1)^2}T_{1,n}T_{2,n}^3 + \frac{x}{(x - 1)^2}T_{2,n}^4, \quad (2) \]
with initial conditions \( T_{1,0}(x, y) = 1, \ T_{2,0}(x, y) = x - 1. \)

**Proof.** The initial conditions are easily verified. The strategy of the proof is to study all possible configurations of the spanning subgraph \( H_i \) in \( G_n^i \ (i = 1, 2, 3, 4) \), and analyze which kind of contributions they give to \( T_{1,n}(x, y) \) and \( T_{2,n}(x, y) \). As shown in Table 1, a configuration produces a basic term of form \( T_{1,n}^iT_{2,n}^j(i + j = 4) \), and by the previous analysis, each basic term has to be multiplied by a factor \((y - 1)^2, \frac{y - 1}{x - 1}, \frac{1}{(y - 1)x} \) or \( y - 1, \frac{1}{x - 1} \) according to Case 1 and Case 2. More precisely, (i) the factor is \((y - 1)^2\) if \( S = \{ e_n \} \) and \( k(H) = \sum_{i=1}^{4} k(H_i) - 3; \) (ii) the factor is \( \frac{y - 1}{x - 1} \) if \( S = \{ e_n \} \) and \( k(H) = \sum_{i=1}^{4} k(H_i) - 4; \) (iii) the factor is \( \frac{1}{(y - 1)x} \) if \( S = \{ e_n \} \) and \( k(H) = \sum_{i=1}^{4} k(H_i) - 5; \) (iv) the factor is \((y - 1)\) if \( S = \emptyset \) and \( k(H) = \sum_{i=1}^{4} k(H_i) - 3; \) (v) the factor is \( \frac{1}{x - 1} \) if \( S = \emptyset \) and \( k(H) = \sum_{i=1}^{4} k(H_i) - 4. \)
According to Eq. (2), it is easy to prove by induction that $x - 1$ divides $T_{2,n}(x, y)$ in $\mathbb{Z}[x, y]$. Thus, we can rewrite $T_{2,n}(x, y)$ as $(x - 1)N_n(x, y)$ in $\mathbb{Z}[x, y]$, and Theorem 1 can be reduced to the following:

**Theorem 2.** The Tutte polynomial $T_{n+1}(x, y)$ of $G_{n+1}$ is given by

$$T_{n+1}(x, y) = T_{1,n+1}(x, y) + (x - 1)N_{n+1}(x, y),$$
where the polynomial $T_{1,n+1}(x, y), N_{n+1}(x, y)$ satisfy the following recursive relations:

$$T_{1,n+1}(x, y) = y(y - 1)T_{1,n}^4 + 4yT_{1,n}^3N_n + (2x + 2)T_{1,n}^2N_n^2,$$

$$N_{n+1}(x, y) = (2y + 2)T_{1,n}^2N_n^2 + 4xT_{1,n}N_n^3 + x(x - 1)N_n^4$$

with initial conditions $T_{1,0}(x, y) = 1, N_0(x, y) = 1$.

It is well-known that the evaluation of the Tutte polynomial for a particular point at $(X, Y)$-plane is related to some combinatorial information and algebraic properties of the graph considered.

(1) $T(G; 1, 0)$ is the number of acyclic root-connected trees of $G$;

(2) $T(G; 1, 1)$ is the number of spanning trees;

(3) $T(G; 0, 1)$ is the number of indegree sequences of strongly connected orientations of $G$;

(4) $T(G; -1, -1) = (-1)^{|E(G)|}(-2)^{|\text{dim}(B)|}$, where $B$ is the bicyclic space of $G$.

**Theorem 3.** The number of acyclic root-connected orientations of $G_n$ is given by $T_n(1, 0) = \prod_{i=0}^{n} (i + 1)^{2 \times 4^{n-i}}$; The number of indegree sequences of strongly connected orientations of $G_n$ is given by $T_n(0, 1) = \frac{n}{2} \prod_{i=0}^{n} (i + 1)^{2 \times 4^{n-i}}$.

**Proof.** By taking $x = 1$ and $y = 0$ in Theorem 2 we have $T_n(1, 0) = T_{1,n}(1, 0)$, and

$$T_{1,n}(1, 0) = 4T_{1,n-1}(1, 0)N_n^2(1, 0),$$

$$N_n(1, 0) = 2T_{1,n-1}(1, 0)N_{n-1}^2(1, 0) + 4T_{1,n-1}(1, 0)N_{n-1}^3(1, 0)$$

A useful relation yields from Eqs. (3) and (4)

$$\frac{N_n(1, 0)}{T_{1,n}(1, 0)} = \frac{1}{2} + \frac{N_{n-1}(1, 0)}{T_{1,n-1}(1, 0)}.$$  (5)

It implies that

$$\frac{N_n(1, 0)}{T_{1,n}(1, 0)} = \frac{n}{2} + \frac{N_0(1, 0)}{T_{1,0}(1, 0)},$$

and

$$N_n(1, 0) = \frac{n + 2}{2}T_{1,n}(1, 0)$$

since $T_0(1, 0) = 1, N_0(1, 0) = 1$. 


Substituting (5) into (4) and using the initial condition \( T_{1,0}(1,0) = 1 \), we have

\[
T_{1,n}(1,0) = (n + 1)^2 T_{1,n-1}^4(1,0) = \prod_{i=0}^{n} (i + 1)^2 \times 4^{n-i}.
\]

Similarly, by taking \( x = 0 \) and \( y = 1 \) in Theorem 2, we have \( T_n(0,1) = T_{1,n}(0,1) - N_n(0,1) \), and

\[
\begin{align*}
T_{1,n}(0,1) &= 4T_{1,n-1}^3(0,1)N_n(0,1) + 2T_{1,n-1}^2(0,1)N_{n-1}^2(0,1), \\
N_n(0,1) &= 4T_{1,n-1}^2(0,1)N_{n-1}^2(0,1).
\end{align*}
\]

Using the same techniques, we can obtained

\[
T_{1,n}(0,1) = \frac{n + 2}{2} N_n(0,1), \quad N_n(0,1) = \prod_{i=0}^{n} (i + 1)^2 \times 4^{n-i}.
\]

And

\[
T_n(0,1) = \frac{n + 2}{2} N_n(0,1) - N_n(0,1) = \frac{n}{2} N_n(0,1) = \frac{n}{2} \prod_{i=0}^{n} (i + 1)^2 \times 4^{n-i}.
\]

From above, the number \( T_n(0,1) \) of acyclic root-connected trees and the number \( T_n(1,0) \) of indegree sequences of strongly connected orientations in \( G_n \) are satisfied \( T_n(0,1) = \frac{n}{2} T_n(1,0) \).

**Theorem 4.** For a positive integer \( n \geq 1 \), the Tutte polynomial of \( G_n \), \( T_n(x,y) \) along the line \( y = x \) is given by

\[
T_n(x,x) = x(x^2 + 5x + 2)^{\frac{n-1}{3}}.
\]

**Proof.** By taking \( y = x \) in Theorem 2 we have

\[
T_{1,n}(x,x) = x(x - 1)T_{1,n-1}^4 + 4xT_{1,n-1}^3N_{n-1} + (2x + 2)T_{1,n-1}^2N_{n-1}^2,
\]

\[
N_n(x,x) = (2x + 2)T_{1,n-1}^2N_{n-1}^2 + 4xT_{1,n-1}N_{n-1}^3 + x(x - 1)N_{n-1}^4,
\]

and \( T_{1,0}(x,x) = N_0(x,x) = 1 \). It can be obtained easily that \( T_{1,n}(x,x) = N_n(x,x) \) by induction. Substituting it into Eq.(7) and using the initial condition \( N_0(x,x) = 1 \), we have

\[
N_n(x,x) = (x^2 + 5x + 2)N_{n-1}^4 = (x^2 + 5x + 2)^{\frac{n-1}{3}}.
\]
Thus, $T_n(x, x) = T_{1,n}(x, x) + (x - 1)N_n(x, x) = xN_n(x, x) = x(x^2 + 5x + 2)\frac{4^n-1}{3}$. $\square$

Since the number of spanning trees is $\tau(G) = T(G; 1, 1)$, from Theorem 4, we can obtain immediately the following result, which was also obtained in [25] by employing the decimation technique.

**Corollary 5.** The number of spanning trees of $G_n$ is given by $T_n(1, 1) = 8\frac{4^n-1}{3} = 2^{4^n-1}$.

The asymptotic growth constant is

$$\lim_{n \to \infty} \frac{\ln \tau(G_n)}{|V(G_n)|} = \frac{3}{2} \ln 2 \approx 1.0397.$$  

Similarly, $T_n(-1, -1) = (-1)\times(-2)\frac{4^n-1}{3} = (-1)^{\frac{4^n-1}{3}}(-2)\frac{4^n-1}{3} = (-1)^{|E(G_n)|}(-2)^{\dim(B)}$
by taking $x = -1$ in Theorem 4. So, we have

**Corollary 6.** The dimension of the bicycle space of $G_n$ is $\frac{4^n-1}{3}$.

### Appendices

#### 4 Tutte polynomial of two other scale-free networks

In this section, we consider the Tutte polynomial of two scale-free networks $(2,2)$-flower and $(1,3)$-flower which have some similar characteristics: identical degree sequences, without crossing edges and always connected. Most of the topological properties of $(2,2)$-flower and $(1,3)$-flower can be determined exactly [26].

**4.1 $(2,2)$-flower**

If the newly added edge is ignored at each iterative generation, then the fractal lattice $G_n$ considered above become the $(2,2)$-flower $F_n$, see Figure 4. And the contributions to $T_{1,n}(x, y)$ and $T_{2,n}(x, y)$ are degraded into the case of $S = \emptyset$, and listed on the right of Table 1. So, we can obtain

$$T_n(x, y) = T_{1,n}(x, y) + (x - 1)N_n(x, y),$$

where the polynomials $T_{1,n}(x, y), N_n(x, y)$ satisfy the following recursive relations:

$$T_{1,n}(x, y) = (y - 1)T_{1,n-1}^4 + 4T_{1,n-1}^3N_{n-1} + 2(x - 1)T_{1,n-1}^2N_{n-1}^2,$$

$$N_n(x, y) = 4T_{1,n-1}^2N_{n-1}^2 + 4(x - 1)T_{1,n-1}N_{n-1}^3 + (x - 1)^2N_{n-1}^4.$$
If $x = y = 1$, then $T_n(1,1) = T_{1,n}(1,1)$ and $T_{1,n}(1,1) = N_n(1,1)$. Thus, $T_n(1,1) = 4T_{n-1}^4(1,1)$. Since the initial value $T_0(1,1) = 1$, we can obtain

$$\tau(F_n) = T_n(1,1) = 2^{2^n}(4^n-1)$$

and

$$\lim_{n \to \infty} \frac{\ln \tau(F_n)}{|V(F_n)|} = \ln 2 \simeq 0.6931.$$

4.2 (1,3)-flower

For the (1,3)-flower $F_n$, see Figure 5, choosing two adjacent vertices with largest degree as special vertices in each iteration, we can obtain similarly the following recursive relation:

$$T_n(x, y) = T_{1,n}(x, y) + T_{2,n}(x, y),$$

where the polynomials $T_{1,n}(x, y)$ and $T_{2,n}(x, y)$ satisfy the following recursive relations:

$$T_{1,n}(x, y) = (y - 1)T_{1,n-1}^4 + \frac{4}{x - 1}T_{1,n-1}^3T_{2,n-1} + \frac{3}{x - 1}T_{1,n-1}^2T_{2,n-1}^2 + \frac{1}{x - 1}T_{1,n-1}T_{2,n-1}^3,$$

$$T_{2,n}(x, y) = \frac{3}{x - 1}T_{1,n-1}^2T_{2,n-1}^2 + \frac{3}{x - 1}T_{1,n-1}T_{2,n-1}^3 + \frac{1}{x - 1}T_{2,n-1}^4.$$

with the initial conditions $T_{1,0}(x, y) = 1$, $T_{2,0}(x, y) = x - 1$. And

$$T_n(x, y) = T_{1,n}(x, y) + (x - 1)N_n(x, y),$$
where the polynomials $T_{1,n}(x, y)$ and $N_n(x, y)$ satisfy the following recursive relations:

\[
T_{1,n}(x, y) = (y - 1)T_{1,n-1}^4 + 4T_{1,n-1}^3N_{n-1} + 3(x - 1)T_{1,n-1}^2N_{n-1}^2 + (x - 1)^2T_{1,n-1}N_{n-1}^3,
\]

\[
N_n(x, y) = 3T_{1,n-1}^2N_{n-1} + 3(x - 1)T_{1,n-1}N_{n-1}^3 + (x - 1)^2N_{n-1}^4
\]

with the initial conditions $T_{1,0}(x, y) = 1$ and $N_0(x, y) = 1$.

Similarly, if $x = y = 1$, then $T_n(1,1) = T_{1,n}(1,1)$ and

\[
T_{1,n}(1, 1) = 4T_{1,n-1}^3(1, 1)N_{n-1}(1, 1), \quad (8)
\]

\[
N_n(1, 1) = 3T_{1,n-1}^2(1, 1)N_{n-1}^2(1, 1). \quad (9)
\]

From Eqs. (8) and (9), we have

\[
T_{1,n}(1, 1) = (\frac{4}{3})^n N_n(1, 1).
\]

Since the initial value $N_0(1, 1) = 1$, we can obtain

\[
\tau(F_n) = T_n(1, 1) = 3^{(4^n-3n-1)/9} 4^{(2^n+4^n+3n-2)/9}
\]

and

\[
\lim_{n \to \infty} \frac{\ln \tau(F_n)}{|V(F_n)|} = \frac{1}{6} (4 \ln 2 + \ln 3) \simeq 0.6452.
\]

which are coincides with the results in [27] based on the relationship between determinants of submatrices of the Laplacian matrix.

**References**

[1] Tutte W T, 1954 *Canad. J. Math.* 6 80

[2] Jaeger F, Vertigan D L and Welsh D J A, 1990 *Math. Proc. Cambridge Philos. Soc.* 108 35

[3] Welsh D J A and Merino C, 2000 *J. Math. Phys.* 41 1127

[4] Ellis-Monahan J and Merino C, 2011 Graph polynomials and their applications I: the Tutte polynomial *Structural Analysis of Complex Networks* ed D Matthias (Boston: Birkhäuser)

[5] Potts R B, 1952 *Proc. Cambridge Philos. Soc.* 48 106
[6] Wu F Y, 1992 Rev. Mod. Phys. 64 1099

[7] Fortuin C M and Kasteleyn P W, 1972 Physica 57 536

[8] Beaudin L, Ellis-Monahan J, Pangborn G and Shrock R, 2010 Discrete Math. 310 2037

[9] Shrock R, 2000 Physica A 283 388

[10] Chang S C and Shrock R, 2000 Phys. A 286 189

[11] Chang S C and Shrock R, 2001 Phys. A 296 234

[12] Chang S C and Shrock R, 2004 Adv. Appl. Math. 32 44

[13] Chang S C and Shrock R, 2008 J. Stat. Phys. 130 1011

[14] Shrock R, 2011 J. Phys. A: Math. Theor. 44 145002

[15] Shrock R and Xu Y, 2012 J. Phys. A: Math. Theor. 45 055212

[16] Salas J and Sokal A D, 2009 J. Stat. Phys. 135 279

[17] Alvarez P D, Canfora F, Reyes S A and Riquelme S, 2012 Eur. Phys. J. B 85 99

[18] Donno A and Iacono D, 2013 Adv. Geom. 13 663

[19] Liao Y H, Hou Y P and Shen X L, 2013 EPL 104 38001

[20] Liao Y H, Fang A X and Hou Y P, 2013 Phys. A 392 4584

[21] Gong H L and Jin X A, 2014 Phys. A 414 143

[22] Kaufman M and Griffiths R B, 1981 Phys. Rev. B 24 496

[23] Griffiths R B and Kaufman M, 1982 Phys. Rev. B 26 5022

[24] Zhang Z Z, Zhou S G, Zou T and Chen G S, 2008 J. Stat. Mech. P09008

[25] Zhang Z Z, Liu H X, Wu B and Zhou T, 2011 Phys. Rev. E 83 016116

[26] Zhang Z Z, Zhou S G, Zou T, Chen L C and Guan J H, 2009 Phys. Rev. E 79 031110

[27] Lin Y, Wu B, Zhang Z Z and Chen G, 2011 J. Math. Phys. 52 113303