Black $p$-branes and their Vertical Dimensional Reduction

H. Lü†, C.N. Pope†

Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843

K.-W. Xu

International Center for Theoretical Physics, Trieste, Italy
and
Institute of Modern Physics, Nanchang University, Nanchang, China

ABSTRACT

We construct multi-center solutions for charged, dilatonic, non-extremal black holes in $D = 4$. When an infinite array of such non-extremal black holes are aligned periodically along an axis, the configuration becomes independent of this coordinate, which can therefore be used for Kaluza-Klein compactification. This generalises the vertical dimensional reduction procedure to include non-extremal black holes. We then extend the construction to multi-center non-extremal $(D - 4)$-branes in $D$ dimensions, and discuss their vertical dimensional reduction.

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1 Introduction

The supergravity theories that arise as the low-energy limits of string theory or M-theory admit a multitude of $p$-brane solutions. In general, these solutions are characterised by their mass per unit $p$ volume, and the charge or charges carried by the field strengths that support the solutions. Solutions can be extremal, in the case where the charges and the mass per unit $p$-volume saturate a Bogomol’nyi bound, or non-extremal if the mass per unit $p$-volume exceeds the bound. There are two basic types of solution, namely elementary $p$-branes, supported by field strengths of rank $n = p + 2$, and solitonic $p$-branes, supported by field strengths of degree $n = D - p - 2$, where $D$ is the spacetime dimension. Typically, we are interested in considering solutions in a “fundamental” maximal theory such as $D = 11$ supergravity, which is the low-energy limit of M-theory, and its various toroidal dimensional reductions. A classification of extremal supersymmetric $p$-branes in M-theory compactified on a torus can be found in [1].

The various $p$-brane solutions in $D \leq 11$ can be represented as points on a “brane scan” whose vertical and horizontal axes are the spacetime dimension $D$ and the spatial dimension $p$ of the $p$-brane world-volume. The same process of Kaluza-Klein dimensional reduction that is used in order to construct the lower-dimensional toroidally-compactified supergravities can also be used to perform dimensional reductions of the $p$-brane solutions themselves: Since the Kaluza-Klein procedure corresponds to performing a consistent truncation of the higher-dimensional theory, it is necessarily the case that the lower-dimensional solutions will also be solutions of the higher-dimensional theory. There are two types of dimensional reduction that can be carried out on the $p$-brane solutions. The more straightforward one involves a simultaneous reduction of the spacetime dimension $D$ and the spatial $p$-volume, from $(D, p)$ to $(D - 1, p - 1)$; this is known as “diagonal dimensional reduction” [2, 3]. It is achieved by choosing one of the spatial world-volume coordinates as the compactification coordinate. The second type of dimensional reduction corresponds to a vertical descent on the brane scan, from $(D, p)$ to $(D - 1, p)$, implying that one of the directions in the space transverse to the $p$-brane world-volume is chosen as the compactification coordinate. This requires that one first construct an appropriate configuration of $p$-branes in $D$ dimensions that has the necessary $U(1)$ isometry along the chosen direction. It is not a priori obvious that this should be possible, in general. However, it is straightforward to construct such configurations in the case of extremal $p$-branes, since these satisfy a no-force condition which means that two or more $p$-branes can sit in neutral equilibrium, and thus multi $p$-brane solutions exist [4]. By taking a limit corresponding to an infinite continuum of
p-branes arrayed along a line, the required $U(1)$-invariant configuration can be constructed

In this paper, we shall investigate the dimensional-reduction procedures for non-extremal $p$-branes. In fact, the process of diagonal reduction is the same as in the extremal case, since the non-extremal $p$-branes also have translational invariance in the spatial world-volume directions. The more interesting problem is to see whether one can also describe an analogue of vertical dimensional reduction for non-extremal $p$-branes. There certainly exists an algorithm for constructing a non-extremal $p$-brane at the point $(D - 1, p)$ from one at $(D, p)$ on the brane scan. It has been shown that there is a universal prescription for “blackening” any extremal $p$-brane, to give an associated non-extremal one [9]. Thus an algorithm, albeit inelegant, for performing the vertical reduction is to start with the general non-extremal $p$-brane in $D$ dimensions, and then take its extremal limit, from which an extremal $p$-brane in $D - 1$ dimensions can be obtained by the standard vertical-reduction procedure described above. Finally, one can then invoke the blackening prescription to construct the non-extremal $p$-brane in $D - 1$ dimensions. However, unlike the usual vertical dimensional reduction for extremal $p$-branes, this procedure does not give any physical interpretation of the $(D - 1)$-dimensional $p$-brane as a superposition of $D$-dimensional solutions.

At first sight, one might think that there is no possibility of superposing non-extremal $p$-branes, owing to the fact that they do not satisfy a no-force condition. Indeed, it is clearly true that one cannot find well-behaved static solutions describing a finite number of black $p$-branes located at different points in the transverse space. However, we do not require such general kinds of multi-$p$-brane solutions for the purposes of constructing a configuration with a $U(1)$ invariance in the transverse space. Rather, we require only that there should exist static solutions corresponding to an infinite number of $p$-branes, periodically arrayed along a line. In such an array, the fact that there is a non-vanishing force between any pair of $p$-branes is immaterial, since the net force on each $p$-brane will still be zero. The configuration is in equilibrium, although of course it is highly unstable. For example, one can have an infinite static periodic array of $D = 4$ Schwarzschild black holes aligned along an axis. In fact, the instability problem is overcome in the Kaluza-Klein reduction, since the extra coordinate $z$ is compactified on a circle. Thus there is a stable configuration in which the $p$-branes are separated by precisely the circumference of the compactified dimension. Viewed from distances for which the coordinates orthogonal to $z$ are large compared with this circumference, the fields will be effectively independent of $z$, and hence $z$ can be used as the compactification coordinate for the Kaluza-Klein reduction, giving rise to a non-
extremal $p$-brane in $D - 1$ dimensions.

In section 2, we obtain the equations of motion for axially symmetric $p$-branes in an arbitrary dimension $D$. We then construct multi-center non-extremal $D = 4$ black hole solutions in section 3, and show how they may be used for vertical dimensional reduction of non-extremal black holes. In section 4, we generalise the construction to non-extremal $(D - 4)$-branes in arbitrary dimension $D$.

2 Equations of motion for axially symmetric $p$-branes

We are interested in describing multi-center non-extremal $p$-branes in which the centers lie along a single axis in the transverse space. Metrics with the required axial symmetry can be parameterised in the following way:

$$ds^2 = -e^{2U}dt^2 + e^{2A}dx^i dx^i + e^{2V}(dz^2 + dr^2) + e^{2B}r^2d\Omega^2,$$

where $(t, x^i)$, $i = 1, \ldots, p$, are the coordinates of the $p$-brane world-volume. The remaining coordinates of the $D$ dimensional spacetime, i.e. those in the transverse space, are $r, z$ and the coordinates on a $\tilde{d}$-dimensional unit sphere, whose metric is $d\Omega^2$, with $\tilde{d} = D - p - 3$. The functions $U, A, V$ and $B$ depend on the coordinates $r$ and $z$ only. We find that the Ricci tensor for the metric (1) is given by

$$R_{00} = e^{2U-2V}(U'' + \dot{U} + U'^2 + \ddot{U} + p(U'A' + \dot{U}A) + \tilde{d}(U'B' + \dot{U}B) + \frac{\tilde{d}}{r}U'),$$

$$R_{rr} = -\left(U'' - U'V' + U'^2 + \dot{U}V + \ddot{V} + V'' + p(A'' - V'A' + A'^2 + \dot{V}A) + \tilde{d}(B'' - V'B' - \frac{1}{r}V' + \dot{V}B + B'^2 + \frac{2}{r}B')\right),$$

$$R_{zz} = -\left(\ddot{U} - \dot{U}V + \dddot{U} + U'V' + \dot{V} + V'' + p(\dddot{A} - \dot{V}A + \dot{A}^2 + V'A') + \tilde{d}(\dddot{B} - \dot{V}B + V'B' + B^2 + \frac{1}{r}V')\right),$$

$$R_{rz} = \left(-\dddot{U} + \dot{V}U' - \dddot{U} + \dot{U}V' + \dddot{V} + p(-\dddot{A} + \dot{V}A' - \dot{A}A' + V'A) + \tilde{d}(-\dot{B}' + \dot{V}B' + \frac{1}{r}V - \frac{1}{r}B - \dot{B}B' + V'B)\right),$$

$$R_{ab} = -e^{2B-2V}(B'' + \dot{B} + U'B' + \ddot{B} + \frac{1}{r}U' + p(A'B' + \dot{A}B' + \frac{1}{r}A') + \tilde{d}(B'^2 + B^2 + \frac{2}{r}B'))r^2\tilde{g}_{ab} + (\tilde{d} - 1)(1 - e^{2B-2V})\tilde{g}_{ab},$$

$$R_{ij} = -e^{2A-2V}(A'' + \dddot{A} + U'A' + \dddot{U} + p(A'^2 + \dot{A}^2) + \tilde{d}(A'B' + \dot{A}B' + \frac{1}{r}A'))\delta_{ij},$$

where the primes and dots denote derivatives with respect to $r$ and $z$ respectively, $\tilde{g}_{ab}$ is the metric on the unit $\tilde{d}$-sphere, and the components are referred to a coordinate frame.
Let us consider axially-symmetric solutions to the theory described by the Lagrangian
\[ e^{-1}L = R - \frac{1}{2} ( \partial \phi )^2 - \frac{1}{2n!} e^{-a\phi} F_n^2 , \] (3)
where \( F_n \) is an \( n \)-rank field strength. The constant \( a \) in the dilaton prefactor can be parameterised as
\[ a^2 = \Delta - \frac{2(n-1)(D-n-1)}{D-2} , \] (4)
where the constant \( \Delta \) is preserved under dimensional reduction. (For supersymmetric \( p \)-branes in M-theory compactified on a torus, the values of \( \Delta \) are \( 4/N \) where \( N \) is an integer \( 1 \leq N \leq N_c \), and \( N_c \) depends on \( D \) and \( p \). Non-supersymmetric \((D-3)\)-branes with \( \Delta = 24/(N(N+1)(N+2)) \) involving \( N \) 1-form field strengths were constructed in \([1]\). Their equations of motion reduce to the \( SL(N+1,R) \) Toda equations. Further non-supersymmetric \( p \)-branes with other values of \( \Delta \) constructed in \([1]\) however cannot be embedded into M-theory owing to the complications of the Chern-Simons modifications to the field strengths.) We shall concentrate on the case where \( F_n \) carries an electric charge, and thus the solutions will describe elementary \( p \)-branes with \( p = n-2 \). (The generalisation to solitonic \( p \)-branes that carry magnetic charges is straightforward.) The potential for \( F_n \) takes the form
\[ A_{0i_1\cdots i_p} = \gamma e_{i_1\cdots i_p} , \]
where \( \gamma \) is a function of \( r \) and \( z \). Thus the equations of motion will be
\[ \square \phi = -\frac{1}{2}a S^2 , \quad R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + S_{MN} , \quad \partial_M ( \sqrt{-g} e^{-a\phi} F^{M_1\cdots M_n} ) = 0 , \] (5)
where
\[ S_{00} = \frac{d}{2(D-2)} S^2 e^{2V} ( \gamma^2 + \gamma'^2 ) , \quad S_{rr} = \frac{1}{2(D-2)} S^2 e^{2V} ( d\gamma^2 - \bar{d}\gamma'^2 ) , \]
\[ S_{zz} = \frac{1}{2(D-2)} S^2 e^{2V} ( -d\gamma^2 + d\gamma'^2 ) , \quad S_{rz} = -\frac{1}{2} S^2 e^{2V} \gamma' \gamma' , \]
\[ S_{ab} = \frac{d}{2(D-2)} S^2 ( \gamma^2 + \gamma'^2 ) e^{2B} r^2 g_{ab} , \quad S_{ij} = -\frac{d}{2(D-2)} S^2 e^{2A} ( \gamma^2 + \gamma'^2 ) \delta_{ij} , \]
\[ S^2 = e^{-2pA-2V-a\phi-2U} \text{ and } d = p+1 . \]

3 \hspace{1em} \textbf{\( D = 4 \) black holes and their dimensional reduction}

3.1 \hspace{1em} \textbf{Single-center black holes}

Let us first consider black hole solutions in \( D = 4 \), whose charge is carried by a 2-form field strength. By appropriate choice of coordinates, and by making use of the field equations,
we may set $B = -U$. Defining also $V = K - U$, we find that equations of motion (5) for the metric
\[ ds^2 = -e^{2U} \, dt^2 + e^{2K - 2U} \left( dr^2 + dz^2 \right) + e^{-2U} \, r^2 \, d\theta^2 \] (7)
can be reduced to
\[
\begin{align*}
\nabla^2 U &= \frac{1}{4} e^{-\phi - 2U} (\dot{\gamma}^2 + \gamma'^2), \quad \nabla^2 K - \frac{2}{r} K' &= \frac{1}{2} e^{-\phi - 2U} \gamma'^2 - 2U'^2 - \frac{1}{2} \phi'^2, \\
\nabla^2 K &= \frac{1}{2} e^{-\phi - 2U} \dot{\gamma}^2 - 2\dot{U}^2 - \frac{1}{2} \dot{\phi}^2, \quad \frac{1}{r} \dot{K} = -\frac{1}{2} e^{-\phi - 2U} \dot{\gamma} \dot{\gamma}' + 2\dot{U} U' + \frac{1}{2} \dot{\phi} \dot{\phi}', \\
\nabla^2 \phi &= \frac{1}{2} e^{-\phi - 2U} (\dot{\gamma}^2 + \gamma'^2), \quad \nabla^2 \gamma = (a \dot{\phi}' + 2U') \gamma' + (a \dot{\phi} + 2U) \dot{\gamma}
\end{align*}
\] (8)
where $\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{d^2}{dz^2}$ is the Laplacian for axially-symmetric functions in cylindrical polar coordinates.

We shall now discuss three cases, with increasing generality, beginning with the pure Einstein equation, with $\phi = 0$ and $\gamma = 0$. The equations (8) then reduce to
\[ \nabla^2 U = 0, \quad K' = r (U'^2 - \dot{U}^2), \quad \dot{K} = 2r U' \dot{U}, \] (9)
thus giving a Ricci-flat axially-symmetric metric for any harmonic function $U$. The solution for $K$ then follows by quadratures. A single Schwarzschild black hole is given by taking $U$ to be the Newtonian potential for a rod of mass $M$ and length $2M$ \cite{12}, i.e.
\[ U = \frac{1}{2} \log \left( \frac{\sigma + \tilde{\sigma} - 2M}{\sigma + \tilde{\sigma} + 2M} \right), \] (10)
where $\sigma = \sqrt{r^2 + (z - M)^2}$ and $\tilde{\sigma} = \sqrt{r^2 + (z + M)^2}$. The solution for $K$ is
\[ K = \frac{1}{2} \log \left( \frac{\sigma + \tilde{\sigma} - 2M)(\sigma + \tilde{\sigma} + 2M)}{4\sigma \tilde{\sigma}} \right). \] (11)
Now we shall show that this is related to the standard Schwarzschild solution in isotropic coordinates, i.e.
\[
\begin{align*}
\quad ds^2 &= -(1 - \frac{M}{2R})^2 \, dt^2 + \left( \frac{M}{2R} \right)^4 \left( d\rho^2 + d\eta^2 + \rho^2 d\phi^2 \right), \\
\quad \text{where } R &\equiv \sqrt{\rho^2 + \eta^2}. \quad \text{To do this, we note that (12) is of the general form (1), but with } B \neq -U. \quad \text{As discussed in \cite{13}, setting } B = -U \text{ depends firstly upon having a field equation for which the } R_{00} \text{ and } R_{\theta\theta} \text{ components of the Ricci tensor are proportional, and secondly upon performing a holomorphic coordinate transformation from } \eta \equiv r + iz \text{ to new variables } \xi \equiv \rho + iy. \quad \text{Comparing the coefficients of } d\phi^2 \text{ in (1) and (12), we see that we must have } \Re(\eta) = \Re(\xi)(1 - m^2/(4\xi \dot{\xi})), \quad \text{and hence we deduce that the required holomorphic transformation is given by} \\
\quad \eta &= \xi - \frac{m^2}{4\xi}. \] (13)
It is now straightforward to verify that this indeed transforms the metric (10), with \( U \) and \( K \) given by (11) and (12), into the standard isotropic Schwarzschild form (12).

Now let us consider the pure Einstein-Maxwell case, where \( \gamma \) is non-zero, but \( a = 0 \) and hence \( \phi = 0 \). We find that the equations of motion (8) can be solved by making the ansatz
\[
e^{-U} = e^{-\tilde{U}} - c^2 e^{2\tilde{U}}, \quad \gamma = 2c e^{2\tilde{U}} \left( 1 - c^2 e^{2\tilde{U}} \right)^{-1},
\]
where \( c \) is an arbitrary constant and \( \tilde{U} \) satisfies \( \nabla^2 \tilde{U} = 0 \). Substituting into (8), we find that all the equations are then satisfied if
\[
K' = r \left( \tilde{U}'^2 - \dot{\tilde{U}}^2 \right), \quad \dot{K} = 2r \tilde{U}' \dot{\tilde{U}}.
\]
(Our solutions in this case are in agreement with [14], after correcting some coefficients and exponents.) The solution for a single Reissner-Nordstrøm black hole is given by taking the harmonic function \( \tilde{U} \) to be the Newtonian potential for a rod of mass \( \frac{1}{2}k \) and length \( k \), implying that \( \tilde{U} \) and \( K \) are given by
\[
\tilde{U} = \frac{1}{2} \log \frac{\sigma + \tilde{\sigma} - k}{\sigma + \tilde{\sigma} + k}, \quad K = \frac{1}{2} \log \frac{(\sigma + \tilde{\sigma} - k)(\sigma + \tilde{\sigma} + k)}{4\sigma \tilde{\sigma}},
\]
where \( \sigma = \sqrt{r^2 + (z - k/2)^2} \) and \( \tilde{\sigma} = \sqrt{r^2 + (z + k/2)^2} \). The metric can be re-expressed in terms of the standard isotropic coordinates \( (\hat{t}, \rho, y, \theta) \) by performing the transformations
\[
\eta = \frac{1}{1 - c^2} (\xi - \frac{k^2}{16\xi}), \quad t = (1 - c^2) \hat{t},
\]
where \( \xi = \rho + iy \) and \( \hat{k} = (1 - c^2)k \), giving
\[
ds^2 = - \left( 1 + \frac{\hat{k}R}{(R + \frac{1}{4}k)^2} \sinh^2 \mu \right)^{-2} \left( R - \frac{1}{4}k \right)^2 dt^2 \\
+ \left( 1 + \frac{\hat{k}R}{(R + \frac{1}{4}k)^2} \sinh^2 \mu \right)^2 \left( 1 + \frac{\hat{k}}{4R} \right)^4 (d\rho^2 + dy^2 + \rho^2 d\theta^2),
\]
where \( c = \tanh \mu \), and again \( R \equiv \sqrt{\rho^2 + y^2} \). (It is necessary to rescale the time coordinate, as in (17), because the function \( e^{-U} \) given in (14) tends to \((1 - c^2)\) rather than 1 at infinity.) Equation (18) is the standard Reissner-Nordstrøm metric in isotropic coordinates, with mass \( M \) and charge \( Q \) given in terms of the parameters \( \hat{k} \) and \( \mu \) by
\[
M = \frac{1}{2} \hat{k} \sinh^2 \mu + \frac{1}{2} \hat{k}, \quad Q = \frac{1}{4} \hat{k} \sinh 2\mu.
\]
The extremal limit is obtained by taking \( \hat{k} \to 0 \) at the same time as sending \( \mu \to \infty \), while keeping \( Q \) finite, implying that \( Q = \frac{1}{2} M \). This corresponds to setting \( c \to 1 \) in (14).
description in the form (8) becomes degenerate in this limit, since the length and mass of the Newtonian rod become zero. However, the rescalings (17) also become singular, and the net result is that the metric (18) remains well-behaved in the extremal limit. The previous pure Einstein case is recovered if $\mu$ is instead sent to zero, implying that $Q = 0$ and $c = 0$.

Finally, let us consider the case of Einstein-Maxwell-Dilaton black holes. We find that the equations of motion (8) can be solved by making the ansätze
\[
\phi = 2a(U - \tilde{U}) , \quad e^{-\Delta U} = (e^{-\tilde{U}} - c^2 e^{\tilde{U}}) e^{-a^2 \tilde{U}} , \\
\gamma = 2c e^{2\tilde{U}} (1 - c^2 e^{2\tilde{U}})^{-1} ,
\]
where, as in the pure Einstein-Maxwell case, $c$ is an arbitrary constant and $\tilde{U}$ satisfies $\nabla^2 \tilde{U} = 0$. Substituting the ansätze into (8), we find that all the equations are satisfied provided that the function $K$ satisfies (15). The solution for a single dilatonic black hole for generic coupling $a$ is given by again taking the harmonic function $\tilde{U}$ to be the Newtonian potential for a rod of mass $\frac{1}{2} k$ and length $k$. After performing the coordinate transformations
\[
\eta = (1 - c^2)^{-\frac{1}{2}} \left( \xi - \frac{k^2}{16} \right) , \quad t = (1 - c^2)^{\frac{1}{2}} \tilde{t} ,
\]
where $\hat{k} = (1 - c^2)^{1/2} k$, and writing $c = \tanh \mu$, we find that the metric takes the standard isotropic form
\[
ds^2 = -\left( 1 + \frac{\hat{k} R}{(R + \frac{1}{4} k)^2} \sinh^2 \mu \right)^2 \left( \frac{R - \frac{1}{4} \hat{k}}{R + \frac{1}{4} k} \right)^2 d\hat{t}^2 \\
+ \left( 1 + \frac{\hat{k} R}{(R + \frac{1}{4} k)^2} \sinh^2 \mu \right)^2 \left( 1 + \frac{\hat{k}}{4 R} \right)^4 (d\rho^2 + d\eta^2 + \rho^2 d\theta^2) .
\]
The mass $M$ and charge $Q$ are given by
\[
M = \frac{\hat{k}}{\Delta} \sinh^2 \mu + \frac{1}{2} \hat{k} , \quad Q = \frac{\hat{k}}{4 \sqrt{\Delta}} \sinh 2\mu .
\]
Again, the extremal limit is obtained by taking $\hat{k} \to 0, \ mu \to \infty$, while keeping $Q$ finite, implying that $Q = \sqrt{\Delta} M/2$.

### 3.2 Vertical dimensional reduction of black holes

The vertical dimensional reduction of a $p$-brane solution requires that the Kaluza-Klein compactification coordinate should lie in the space transverse to the world-volume of the extended object. In order to carry out the reduction, it is necessary that the higher-dimensional solution be independent of the chosen compactification coordinate. In the case
of extremal $p$-branes, this can be achieved by exploiting the fact that there is a zero-force condition between such objects, allowing arbitrary multi-center solutions to be constructed. Mathematically, this can be done because the equations of motion reduce to a Laplace equation in the transverse space, whose harmonic-function solutions can be superposed. Thus one can choose a configuration with an infinite line of $p$-branes along an axis, which implies in the continuum limit that the solution is independent of the coordinate along the axis.

As we saw in the previous section, the equations of motion for an axially-symmetric non-extremal black-hole configuration can also be cast in a form where the solutions are given in terms of an arbitrary solution of Laplace’s equation. Thus again we can superpose solutions, to describe multi-black-hole configurations. We shall discuss the general case of dilatonic black holes, since the $a = 0$ black holes and the uncharged black holes are merely special cases. Specifically, a solution in which $\tilde{U}$ is taken to be the Newtonian potential for a set of rods of mass $\frac{1}{2}k_n$ and length $k_n$ aligned along the $z$ axis will describe a line of charged, dilatonic black holes:

$$\tilde{U} = \frac{1}{2} \sum_{n=1}^{N} \log \frac{\sigma_n + \tilde{\sigma}_n - k_n}{\sigma_n + \tilde{\sigma}_n + k_n}, \quad (24)$$

$$K = \frac{1}{4} \sum_{m,n=1}^{N} \log \left[ \frac{[\sigma_m \sigma_n + (z - z_m - \frac{1}{2}k_m)(z - z_n + \frac{1}{2}k_n) + r^2]}{[\sigma_m \sigma_n + (z - z_m - \frac{1}{2}k_m)(z - z_n - \frac{1}{2}k_n) + r^2]} \right]$$

$$+ \frac{1}{4} \sum_{m,n=1}^{N} \log \left[ \frac{[\tilde{\sigma}_m \sigma_n + (z - z_m + \frac{1}{2}k_m)(z - z_n - \frac{1}{2}k_n) + r^2]}{[\tilde{\sigma}_m \sigma_n + (z - z_m + \frac{1}{2}k_m)(z - z_n + \frac{1}{2}k_n) + r^2]} \right], \quad (25)$$

where $\sigma_n^2 = r^2 + (z - z_n - \frac{1}{2}k_n)^2$ and $\tilde{\sigma}_n^2 = r^2 + (z - z_n + \frac{1}{2}k_n)^2$. (Multi-center Schwarzschild solutions were obtained in [12], corresponding to (24) and (25) with $k_n = 2M_n$, and $U$ equal to $\tilde{U}$ rather than the expression given in (20).) This describes a system of $N$ non-extremal black holes, which remain in equilibrium because of the occurrence of conical singularities along the $z$ axis. These singularities correspond to the existence of (unphysical) “struts” that hold the black holes in place [13, 14]. If, however, we take all the constants $k_n$ to be equal, and take an infinite sum over equally-spaced black holes lying at points $z_n = nb$ along the entire $z$ axis, the conical singularities disappear [10]. In the limit when the separation goes to zero, the resulting solution (25) becomes independent of $z$. For small $k = k_n$, we have $U \sim -\frac{1}{2}k(r^2 + (z - nb)^2)^{-1/2} + O(k^3/r^3)$, and thus in the limit of small $b$, the sum giving $\tilde{U}$ in (24) can be replaced by an integral:

$$\tilde{U} \sim -\frac{k}{2b} \int_{L}^{L'} \frac{dz'}{\sqrt{r^2 + z'^2}}, \quad (26)$$
in the limit \( L \to \infty \). Subtracting out the divergent constant \(-k(\log 2 + \log L)/b\), this gives the \( z \)-independent result \([10]\)

\[
\tilde{U} = \frac{k}{b} \log r .
\]

(27)

Similarly, one finds that \( K \) is given by

\[
K = \frac{k^2}{b^2} \log r .
\]

(28)

One can of course directly verify that these expressions for \( \tilde{U} \) and \( K \) satisfy the equations of motion \((8)\). Since the associated metric and fields are all \( z \)-independent, we can now perform a dimensional reduction with \( z \) as the compactification coordinate, giving rise to a solution in \( D = 3 \) of the dimensionally reduced theory, which is obtained from \([3]\), with \( D = 4 \) and \( n = 2 \), by the standard Kaluza-Klein reduction procedure. A detailed discussion of this procedure may be found, for example, in \([3]\). From the formulae given there, we find that the relevant part of the \( D = 3 \) Lagrangian, namely the part involving the fields that participate in our solution, is given by

\[
e^{-1}\mathcal{L} = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{4} e^{-\varphi - a\phi} F_2^2 ,
\]

(29)

where \( \varphi \) is the Kaluza-Klein scalar coming from the dimensional reduction of the metric, \( i.e. \) \( ds_4^2 = e^\varphi ds_3^2 + e^{-\varphi} dz^2 \). A “standard” black hole solution in \( D = 3 \) would be one where only the combination of scalars \((-\varphi - a\phi)\), occurring in the exponential prefactor of the field strength \( F_2 \) that supports the solution, is non-zero. In other words, the orthogonal combination should vanish, \( i.e. \) \( a\varphi - \phi = 0 \). Since our solution in \( D = 4 \) has \( \phi = 2a(U - \tilde{U}) \), it follows that \( \varphi = 2U - 2\tilde{U} \), and hence we should have

\[
ds_4^2 = e^{2U - 2\tilde{U}} ds_3^2 + e^{2\tilde{U} - 2U} dz^2 .
\]

(30)

Comparing this with the \( D = 4 \) solution, whose metric takes the form \([9]\), we see that the \( D = 3 \) solution will have the above single-scalar structure if \( K = \tilde{U} \). From \((27)\) and \((28)\), this will be the case if the parameter \( k \) setting the scale size of the rods, and the parameter \( b \) determining the spacing between the rods, satisfy \( k = b \).

It is interesting to note that since \( k \) is the length of each rod, and \( b \) is the period of the array, the condition \( k = b \) implies that the rods are joined end to end, effectively describing a single rod of length \( L \) and mass \( \frac{1}{2}L \) in the limit \( L \to \infty \). In other words, the \( D = 4 \) multi-black-hole solution becomes a single black hole with \( k = L \to \infty \) in this case. If \( r \) is large compared with \( z \), the solution is effectively independent of \( z \), and thus one can reduce to \( D = 3 \) with \( z \) as the Kaluza-Klein compactification coordinate. (This is rather
different from the situation in the extremal limit; in that case, the lengths and masses of
the individual rods are zero, and the sum over an infinite array does not degenerate to a
single rod of infinite length.)

If \( k \) and \( b \) are not equal, the dimensional reduction of the \( D = 4 \) array of black holes
will of course still yield a 3-dimensional solution of the equations following from (29), but
now with the orthogonal combination \( a\phi - \phi \) of scalar fields active also. Such a solution
lies outside the class of \( p \)-brane solitons that are normally discussed; we shall examine such
solutions in more detail in the next section.

4 Reductions of higher-dimensional black \((D - 4)\)-branes

The equations of motion (3) for general black \( p \)-branes in \( D \) dimensions become rather
difficult to solve in the axially symmetric coordinates, owing to the presence final term
involving \( \tilde{d} - 1 \) in \( R_{ab} \) given in (2). This term vanishes if \( \tilde{d} = 1 \), as it did in
the case of 4-dimensional black holes discussed in section 3. The simplest generalisation of
these 4-dimensional results is therefore to consider \((D - 4)\)-branes, which have \( \tilde{d} = 1 \) also.
They will arise as solutions of the equations of motion following from (3) with \( n = D - 2 \).
The required solutions can be obtained by directly solving the equations of motion (3),
with Ricci tensor given by (2). However, in practice it is easier to obtain the solutions
by diagonal Kaluza-Klein oxidation of the \( D = 4 \) black hole solutions. The ascent to \( D \)
dimensions can be achieved by recursively applying the inverse of the one-step Kaluza-Klein
reduction procedure.

The one step reduction of the metric from \((\ell + 1)\) to \( \ell \) dimensions takes the form
\[
d_{\ell+1}^2 = e^{2\alpha_{\ell+1}\phi_{\ell+1}}d_{\ell}^2 + e^{-2(\ell-2)\alpha_{\ell+1}\phi_{\ell+1}}dx_{5-\ell}^2,
\]
where \( \alpha_{\ell}^{-2} = 2(\ell - 1)(\ell - 2) \). (We have omitted the Kaluza-Klein vector potential since
it is not involved in the solutions that we are discussing.) The kinetic term for the
field strength \( F_{\ell-1} \) in \((\ell + 1)\) dimensions, \( i.e. \) \( e^{-a_{\ell+1}\phi_{\ell+1}}F_{\ell-1}^2 \), reduces to the kinetic term
\( e^{-\alpha_{\ell+1}\phi_{\ell+1} + 2\alpha_{\ell}\phi_{\ell}}F_{\ell-2}^2 \) in \( \ell \) dimensions for the relevant field strength \( F_{\ell-2} \). We may define
\( -a_{\ell+1}\phi_{\ell+1} + 2\alpha_{\ell}\phi_{\ell} \equiv -a_{\ell}\phi_{\ell} \), where \( a_{\ell}^2 = a_{\ell+1}^2 + 4\alpha_{\ell}^2 \). In fact although the dilaton coupling
constant \( a_{\ell} \) is different in different dimensions \( \ell \), the related quantity \( \Delta \), defined in (4), is
preserved under dimensional reduction \( \tilde{3} \). The solutions that we are considering have the
feature that the combination of scalar fields orthogonal to \( \phi_{\ell} \) in \( \ell \) dimensions vanishes, \( i.e. \)
\( 2a_{\ell}\phi_{\ell+1} + a_{\ell+1}\phi_{\ell} = 0 \). This ensures that a single-scalar solution in \( D \) dimensions remains
a single-scalar solution in all the reduction steps. Thus we have the following recursive relations

\[
\frac{\phi_{\ell+1}}{a_{\ell+1}} = \frac{\phi_{\ell}}{a_{\ell}} = \cdots = \frac{\phi_4}{a_4} = 2(U - \tilde{U}) ,
\]

\[
\varphi_{\ell} = -2\alpha_{\ell} \frac{\phi_{\ell}}{a_{\ell}} = -\frac{4}{(\ell - 1)(\ell - 2)}(U - \tilde{U}) ,
\]

(32)

where \(U\) and \(\tilde{U}\) are the functions for the four-dimensional dilatonic black holes given in section 3. We find that the metric for the \((D - 4)\)-brane in \(D\) dimensions is then given by

\[
ds^2_D = e^{\frac{-2(D-4)}{D-2}(U-\tilde{U})} ds_4^2 + e^{\frac{4}{D-2}(U-\tilde{U})} (dx_1^2 + \cdots + dx_p^2)
= e^{\frac{4}{D-2}(U-\tilde{U})} (-e^{2\tilde{U}} dt^2 + dx^i dx^i) + e^{\frac{2(D-4)}{D-2}\tilde{U} - \frac{4(D-3)}{D-2} U} \left( e^{2K} (dr^2 + dz^2) + r^2 d\theta^2 \right),
\]

(33)

and the dilaton is given by \(\phi_D = 2a_D(U - \tilde{U})\). If the functions \(\tilde{U}\) and \(K\) are those for a Newtonian rod, given by (16), and the function \(U\) is given by (20), the metric describes a single black \((D - 4)\)-brane. The coordinate transformations

\[
\eta = (\xi - \frac{\hat{k}^2}{16\xi})(\cosh \mu)^{\frac{4(D-3)}{\Delta(D-2)}} , \quad \tilde{x}^\mu = \hat{x}^\mu(\cosh \mu)^{-\frac{4}{\Delta(D-2)}} ,
\]

(34)

where \(\hat{k} = (\cosh \mu)^{-4/(\Delta(D-2))}\), put the metric into the standard isotropic form for a black \((D - 4)\)-brane, where \(c = \tanh \mu\). The further transformation \(\tilde{r} = (R + \frac{1}{2}\hat{k})^2/R\) puts the metric into the form

\[
ds^2_D = \left(1 + \frac{\hat{k}^2}{\tilde{r}} \sinh^2 \mu \right)^{\frac{4}{\Delta(D-2)}} \left(-e^{2f} dt^2 + dx^i dx^i \right)
\left(1 + \frac{\hat{k}^2}{\tilde{r}} \sinh^2 \mu \right)^{\frac{4(D-3)}{\Delta(D-2)}} \left( e^{-2f} dr^2 + r^2 d\theta^2 \right),
\]

(35)

where \(e^{2f} = 1 - \frac{\hat{k}^2}{\tilde{r}}\). This is the standard form for black \((D - 4)\)-branes discussed in [1].

Since we again have general solutions given in terms of the harmonic function \(\tilde{U}\), we may superpose a set of Newtonian rod potentials, by taking \(\tilde{U}\) and \(K\) to have the forms (24) and (25). Equilibrium can again be achieved, without conical singularities on the \(z\) axis, by taking an infinite line of such rods, with equal masses \(\frac{1}{2}k\), lengths \(k\), and spacings \(b\). As discussed in section 3, the resulting functions \(\tilde{U}\) and \(K\) become \(z\)-independent, and are given by (27) and (28). Thus we can perform a vertical dimensional reduction of the black \((D - 4)\)-brane metric (33) in \(D\) dimensions to a solution in \((D - 1)\) dimensions. For generic values of \(k\) and \(b\), this solution will involve two scalar fields. However, as discussed in section 3, it will describe a single-scalar solution if \(k = b\). In this case, we have \(K = \tilde{U} = \log r\),
and hence we find that the \((D - 1)\)-dimensional metric \(d\tilde{s}^2_{D-1}\), obtained by taking \(z\) as the compactification coordinate, so that \(ds^2_D = e^{2\alpha \varphi} d\tilde{s}^2_{D-1} + e^{-2(1+D-2)\alpha \varphi} dz^2\), is given by

\[
\begin{align*}
\tilde{d}s^2_{D-1} &= -e^{2\tilde{U}} dt^2 + dx^i dx^i + e^{4\tilde{U}} - 4U dr^2 + r^2 e^{2\tilde{U}} - 4U d\theta^2, \\
&= -r^2 dt^2 + dx^i dx^i + (1 - c^2 r^2)^\frac{1}{2} (dr^2 + d\theta^2). \tag{36}
\end{align*}
\]

Although the condition that the length \(k\) of the rods and their spacing \(b\) be equal is desirable from the point of view that it gives rise to a single-scalar solution in the lower dimension, it is clearly undesirable in the sense that the individual single \(p\)-brane solutions are being placed so close together that their horizons are touching. This reflects itself in the fact that the sum over the single-rod potentials is just yielding the potential for one rod, of infinite length and infinite mass, and accordingly, the higher-dimensional solution just describes a single infinitely-massive \(p\)-brane. The corresponding vertically-reduced solution \(36\), which one might have expected to describe a black \(((D - 1) - 3)\)-brane in \((D - 1)\) dimensions, thus does not have an extremal limit. This can be understood from another point of view: A vertically-reduced extremal solution is in fact a line of uniformly distributed extremal \(p\)-branes in one dimension higher. In order to obtain a black \(((D - 1) - 3)\)-brane in \((D - 1)\)-dimension that has an extremal limit, we should be able to take a limit in the higher dimension in which the configuration becomes a line of extremal \(p\)-branes. Thus a more appropriate superposition of black \(p\)-branes in the higher dimension would be one where the spacing \(b\) between the rods was significantly larger than the lengths of the rods. In particular, we should be able to pass to the extremal limit, where the lengths \(k\) tend to zero, while keeping the spacing \(b\) fixed. In this case, the functions \(\tilde{U}\) and \(K\) will take the form \(27\) and \(28\) with \(k < b\). Defining \(\beta = k/b\), we then find that the lower-dimensional metric, after compactifying the \(z\) coordinate, becomes

\[
\begin{align*}
\tilde{d}s^2_{D-1} &= r^{\frac{2\beta(\beta-1)}{D-3}} \left( -r^{2\beta} dt^2 + dx^i dx^i \right) + r^{-2\beta + \frac{2\beta(\beta-1)}{D-3}} (1 - c^2 r^{2\beta})^\frac{1}{2} \left( r^{2\beta} dr^2 + r^2 d\theta^2 \right). \tag{37}
\end{align*}
\]

This can be interpreted as a black \(((D - 1) - 3)\)-brane in \((D - 1)\) dimensions. (In other words, what is normally called a \((D - 3)\)-brane in \(D\) dimensions.) The extremal limit is obtained by sending \(k = b\beta\) to zero and \(\mu\) to infinity, keeping \(b\) and the charge parameter \(Q = (k \sinh 2\mu) / (4\sqrt{\Delta})\) finite. At the same time, we must rescale the \(r\) coordinate so that \(r \rightarrow r (\cosh \mu)^{1/\Delta}\), leading to the extremal metric

\[
\begin{align*}
\tilde{d}s^2 &= -dt^2 + dx^i dx^i + \left( 1 - \frac{4\sqrt{\Delta} Q}{b} \log r \right) \frac{1}{2} (dr^2 + r^2 d\theta^2). \tag{38}
\end{align*}
\]

\(^1\)The somewhat clumsy notation is forced upon us by the lack of a generic \(D\)-independent name for a \((D - 3)\)-brane in \(D\) dimensions.
Thus the solution (37) seems to be the natural non-extremal generalisation of the extremal \((D - 1) - 3\)-brane (38). Note that the black solutions (37) involve two scalar fields, as we discussed previously, although in the extremal limit the additional scalar decouples.

In fact the above proposal for the non-extremal generalisation of \((D - 3)\)-branes in \(D\) dimensions receives support from a general analysis of non-extremal \(p\)-brane solutions. The usual prescription for constructing black \(p\)-branes, involving a single scalar field, as described for example in [9], breaks down in the case of \((D - 3)\)-branes in \(D\) dimensions, owing to the fact that the transverse space has dimension 2, and hence \(\tilde{d} = 0\). Specifically, one can show in general that there is a universal procedure for “blackening” the extremal single-scalar \(p\)-brane

\[
\begin{align*}
\text{ds}^2 &= e^{2A}(-e^{-2f}dt^2 + dx^i dx^i) + e^{2B}(dr^2 + r^2 d\Omega^2), \\
&= e^{2A}(-dt^2 + dx^i dx^i) + e^{2B}(e^{-2f}dr^2 + r^2 d\Omega^2),
\end{align*}
\]

(39)

where \(e^{2f} = 1 - \hat{k}r^{\tilde{d}}\), and the functions \(A\) and \(B\) take the same form as in the extremal solution, but with rescaled charges:

\[
e^{\frac{\Delta(D-2)}{2d}}A = e^{\frac{\Delta(D-2)}{2d}}B = 1 + \frac{\hat{k}}{e^{\tilde{d}}} \sinh^2 \mu.
\]

(40)

However, the case where \(\tilde{d} = 0\) must be treated separately, and we find that the black solutions then take the form

\[
\begin{align*}
\text{ds}^2 &= -(1 - \hat{k} \log r)dt^2 + dx^i dx^i + \frac{1}{r^2} \left(1 + \hat{k} \sinh^2 \mu \log r\right) \frac{4}{\Delta} \left((1 - \hat{k} \log r)^{-1} dr^2 + d\theta^2\right),
\end{align*}
\]

(41)

In the extremal limit, i.e., \(\hat{k} \to 0\) and \(\mu \to \infty\), the metric becomes

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + dx^i dx^i + (1 + QR) \frac{4}{\Delta} (dR^2 + d\theta^2),
\end{align*}
\]

(42)

where \(R = \log r\). Unlike the situation for non-zero values of \(\tilde{d}\), where the analogous limit of the black \(p\)-branes gives a normal isotropic extremal \(p\)-brane, in this \(\tilde{d} = 0\) case the extremal limit describes a line of \((D - 3)\)-branes in \(D\) dimensions, lying along the \(\theta\) direction, rather than a single \((D - 3)\)-brane. (In fact this line of \((D - 3)\)-branes can be further reduced, by compactifying the \(\theta\) coordinate, to give a domain-wall solution in one lower dimension [17, 18].) Thus it seems that there is no appropriate single-scalar non-extremal generalisation of an extremal \((D - 3)\)-brane in \(D\) dimensions, and the two-scalar solution (37) that we obtained by vertical reduction of a black \((D - 3)\)-brane in one higher dimension is the natural non-extremal generalisation.
5 Conclusions

In this paper, we raised the question as to whether one can generalise the procedure of vertical dimensional reduction to the case of non-extremal \( p \)-branes. It is of interest to do this, since, combined with the more straightforward procedure of diagonal dimensional reduction, it would provide a powerful way of relating the multitude of black \( p \)-brane solutions of toroidally-compactified M-theory, analogous to the already well-established procedures for extremal \( p \)-branes. Vertical dimensional reduction involves compactifying one of the directions transverse to the \( p \)-brane world-volume. In order to achieve the necessary translational invariance along this direction, one needs to construct multi-center \( p \)-brane solutions in the higher dimension, which allow a periodic array of single-center solutions to be superposed. This is straightforward for extremal \( p \)-branes, since the no-force condition permits the construction of arbitrary multi-center configurations that remain in neutral equilibrium. No analogous well-behaved multi-center solutions exist in general in the non-extremal case, since there will be net forces between the various \( p \)-branes. However, an infinite periodic array along a line will still be in equilibrium, albeit an unstable one. This is sufficient for the purposes of vertical dimensional reduction.

The equations of motion for general axially-symmetric \( p \)-brane configurations are rather complicated, and in this paper we concentrated on the simpler case where the transverse space is 3-dimensional. This leads to simplifications in the equations of motion, and we were able to obtain the general axially-symmetric solutions for charged dilatonic non-extremal \( (D-4) \)-branes in \( D \) dimensions. These solutions are determined by a single function \( \tilde{U} \) that satisfies a linear equation, namely the Laplace equation on a flat cylindrically-symmetric 3-space, and thus multi-center solutions can be constructed as superpositions of basic single-center solutions. The single-center \( p \)-brane solutions correspond to the case where \( \tilde{U} \) is the Newtonian potential for a rod of mass \( k \) and length \( \frac{1}{2}k \).

The rather special features that allowed us to construct general multi-center black solutions when the transverse space is 3-dimensional also have a counterpart in special features of the lower-dimensional solutions that we could obtain from them by vertical dimensional reduction. The reduced solutions are expected to describe non-extremal \( (D-3) \)-branes in \( D \) dimensions. Although a general prescription for constructing single-scalar black \( p \)-branes from extremal ones for arbitrary \( p \) and \( D \) was given in [9], we found that an exceptional case arises when \( p = D - 3 \). In this case the general analysis in [9] degenerates, and the single-scalar black solutions take the form (11), rather than the naive \( \tilde{d} \to 0 \) limit of (39) and (40) where one would simply replace \( r^{-\tilde{d}} \) by \( \log r \). The extremal limit of (11) in fact
fails to give the expected extremal \((D-3)\)-brane, but instead gives the solution (42), which describes a line of \((D-3)\)-branes. Interestingly enough, we found that the vertical reduction of the non-extremal \(p\)-branes obtained in this paper gives a class of \((D-3)\)-branes which are much more natural non-extremal generalisations of extremal \((D-3)\)-branes. In particular, their non-extremal limits do reduce to the standard extremal \((D-3)\)-branes. The price that one pays for this, however, is that the non-extremal solutions involve two scalar fields (\(i.e.\) the original dilaton of the higher dimension and also the Kaluza-Klein scalar), rather than just one linear combination of them. Thus we see that a number of special features arise in the cases we have considered. It would be interesting to see what happens in the more generic situation when \(\tilde{d} > 0\).

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