A REMARK ON THE HIGHER TORSION INVARIANTS
FOR FLAT VECTOR BUNDLES WITH FINITE HOLONOMY

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ABSTRACT. We show that the Igusa-Klein topological torsion and the Bismut-Lott analytic torsion are equivalent for any flat vector bundle whose holonomy is a finite subgroup of $\text{GL}_n(\mathbb{Q})$. Our proof uses Artin’s induction theorem in representation theory to reduce the problem to the special case of trivial flat line bundles, which is a recent result of Puchol, Zhu and the second author. The idea of using Artin’s induction theorem appeared in a paper of Ohrt on the same topic, of which our present work is an improvement.

Keywords: analytic torsion, Reidemeister-Franz torsion.

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0. INTRODUCTION

The theory of topological torsion was developed by Franz [12], Reidemeister [30], de Rham [10], Milnor [23], Whitehead [34] and many others. The analytic torsion, which is an analogue of the topological torsion, was defined by Ray and Singer [29]. Cheeger [9] and Müller [24] independently proved that the topological torsion and the analytic torsion coincide for unitarily flat vector bundles. This result is now known as the Cheeger-Müller theorem. Bismut, Zhang and Müller simultaneously considered the extension of the Cheeger-Müller theorem. Müller [25] extended the theorem to the unimodular case. Bismut and Zhang [5] extended the theorem to the general case. There are also various extensions to equivariant cases [6, 20, 21].

Wagoner [33] conjectured the existence of higher topological/analytic torsion invariants. The conjectured invariant should be an invariant for pairs $(M \to S, F)$, where $M \to S$ is a smooth fibration with compact fiber and $F$ is a flat vector bundle over $M$. Bismut and Lott [4] confirmed the analytic side of Wagoner’s conjecture by

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constructing the so-called Bismut-Lott analytic torsion. Igusa [18] confirmed the topological side of Wagoner’s conjecture by constructing the so-called Igusa-Klein topological torsion. Goette, Igusa and Williams [17, 16] used Igusa-Klein topological torsion to detect the exotic smooth structure of fiber bundles. Dwyer, Weiss and Williams [11] constructed another topological torsion. The relation among these higher torsion invariants (in the most general case) is still unknown.

Bismut and Goette [2] showed that the Bismut-Lott torsion and the Igusa-Klein torsion are equivalent if there exists a fiberwise Morse function $f : M \to \mathbb{R}$ satisfying the Morse-Smale transversality [32]. In fact, Bismut and Goette extended the Bismut-Lott torsion to the equivariant case and proved their result in the equivariant context. Goette [13, 14] extended the results in [2] to arbitrary fiberwise Morse functions. There are also related works in [3, 7]. We refer to the survey by Goette [15] for an overview on higher torsion invariants. Goette also proposed a program extending the argument in [13, 14] to functions with both non-degenerate critical points and birth-death critical points.

Igusa [19] axiomatized higher torsion invariants for trivial flat line bundles. He showed that any invariant satisfying the additivity axiom and the transfer axiom is essentially the Igusa-Klein torsion. Badzioch, Dorabiala, Klein and Williams [1] showed that the Dwyer-Weiss-Williams torsion satisfies Igusa’s axioms. Using the results in [22] and [27], Puchol, Zhu and the second author [28] showed that the Bismut-Lott torsion satisfies Igusa’s axioms. As a result, all these higher torsion invariants are equivalent for trivial flat line bundles.

As for arbitrary flat vector bundles, Ohrt [26] proposed a similar axiomatization approach for higher torsion invariants. Under the assumption that the fibrations under consideration have simple fibers, he showed that any invariant satisfying his axioms is essentially the Igusa-Klein torsion. Puchol, Zhu and the second author [28] showed that the Bismut-Lott torsion also satisfies Ohrt’s axioms.

The purpose of this paper is to explore the relation between the Igusa-Klein torsion and the Bismut-Lott torsion without restrictions on the fibrations. Instead, we need to assume that the holonomy of the flat vector bundle in question lies in $GL_n(\mathbb{Q})$. Our result is related to the transfer index conjecture proposed by Bunke and Gepner [8].

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1. Main result

Let $M \to S$ be a smooth fibration. Let $Z$ be the fiber. Let $F$ be a flat vector bundle over $M$. We assume that

- $\pi_1(S)$ is finite;
- $Z$ is closed and oriented;
- the holonomy group of $F$ is finite.
These assumptions appear in [26]. For $M \to S$ and $F$ as above, we denote by

$$\tau^{\text{BL}}(M/S, F) \in H^{\text{even} \geq 2}(S)$$

its Bismut-Lott analytic torsion class [4] (cf. [28, Definition 2.1]), and denote by

$$\tau^{\text{IK}}(M/S, F) \in H^{\text{even} \geq 2}(S)$$

its Igusa-Klein topological torsion class [18]. For $k \in \mathbb{N}$ and a class $a \in H^k(S)$, let

$$a^{[k]} \in H^k(S)$$

be its component of degree $k$. Set

$$\tau^{\text{an}}(M/S, F) = \sum_k \left\{ \frac{2^{2k}(k!)^2}{(2k+1)!} \tau^{\text{BL}}(M/S, F) \right\}^{[2k]},$$

$$\tau^{\text{top}}(M/S, F) = \sum_k \left\{ -\frac{k!}{(2\pi)^k} \tau^{\text{IK}}(M/S, F) + \frac{\zeta(-k)rk}{2} \int_Z e(TZ) \text{ch}(TZ) \right\}^{[2k]},$$

where $\int_Z : H^*(M) \to H^*(S)$ is the integration along the fiber, $e(TZ)$ (resp. $\text{ch}(TZ)$) is the Euler class of the relative tangent bundle $TZ$ (resp. the Chern character of $TZ \otimes_R \mathbb{C}$), and $\zeta$ is the Riemann zeta function. The first identity in (1.3) is the Chern normalization introduced by Bismut and Goette [2, Definition 2.37].

Let $\pi_1(M)$ be the fundamental group of $M$. Let $\tilde{M}$ be the universal cover of $M$, which is canonically equipped with a right group action of $\pi_1(M)$. For a group homomorphism $\rho : \pi_1(M) \to \text{GL}_n(\mathbb{C})$ with finite image, set

$$F_\rho = \tilde{M} \times_\rho \mathbb{C}^n,$$

which is a flat vector bundle over $M$ with finite holonomy. For convenience, we denote

$$\tau^{\text{an/top}}(M/S, \rho) = \tau^{\text{an/top}}(M/S, F_\rho).$$

For a finite Galois extension $K/\mathbb{Q}$, we denote by $\text{Gal}(K/\mathbb{Q})$ its Galois group. For $g \in \text{Gal}(K/\mathbb{Q})$ and a group homomorphism $\rho : \pi_1(M) \to \text{GL}_n(K)$, we define

$$g.\rho : \pi_1(M) \to \text{GL}_n(K)$$

$$\gamma \mapsto (g(\rho(\gamma)_{i,j}))_{1 \leq i,j \leq n},$$

where $\rho(\gamma)_{i,j} \in K$ are the entries of the matrix $\rho(\gamma) \in \text{GL}_n(K)$.

**Theorem 1.** For a smooth manifold $S$ with finite fundamental group, a smooth fibration $M \to S$ with closed oriented fiber, a finite Galois extension $K/\mathbb{Q}$ and a group homomorphism $\rho : \pi_1(M) \to \text{GL}_n(K)$ with finite image, we have

$$\sum_{g \in \text{Gal}(K/\mathbb{Q})} \tau^{\text{an}}(M/S, g.\rho) = \sum_{g \in \text{Gal}(K/\mathbb{Q})} \tau^{\text{top}}(M/S, g.\rho).$$

In particular, for a homomorphism $\rho : \pi_1(M) \to \text{GL}_n(\mathbb{Q})$ with finite image, we have

$$\tau^{\text{an}}(M/S, \rho) = \tau^{\text{top}}(M/S, \rho).$$
2. A consequence of Artin’s induction theorem

Let $G$ be a finite group. Let $R(G)$ be its representation ring with rational coefficients. In other words, as a $\mathbb{Q}$-vector space,

$$R(G) = \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Q}\rho,$$

where $\text{Irr}(G)$ is the set of isomorphism classes of irreducible (complex) representations of $G$. The ring structure of $R(G)$ is given by tensor products.

For $\rho \in R(G)$, let $\chi_\rho : G \to \mathbb{C}$ be its character. We denote

$$R_{\text{rat}}(G) = \{ \rho \in R(G) : \chi_\rho(g) \in \mathbb{Q} \text{ for any } g \in G \}.$$

For any subgroup $H \leq G$, let $1 \in R(H)$ be the one-dimensional trivial representation, let $\text{Ind}_H^G 1 \in R(G)$ be the induced representation. Clearly, we have

$$\text{Ind}_H^G 1 \in R_{\text{rat}}(G).$$

Lemma 2. The vector space $R_{\text{rat}}(G)$ is spanned by $(\text{Ind}_H^G 1)_{H \leq G}$.

The lemma above is a consequence of [31, Exercise 13.8]. We still give a proof for the sake of completeness.

Proof. Let $S(G) \subseteq R(G)$ be the vector subspace spanned by $(\text{Ind}_H^G 1)_{H \leq G}$.

Claim 1. We have

$$\text{Ind}_H^G S(H) \subseteq S(G) \text{ for any } H \leq G.$$

This is an immediate consequence of the identity $\text{Ind}_H^G \text{Ind}_J^H = \text{Ind}_J^G$ for $J \leq H \leq G$.

Claim 2. For a surjective homomorphism $f : G \to G'$, we have

$$f^* S(G') \subseteq S(G),$$

where $f^* : R(G') \to R(G)$ is defined by $f^* \rho = \rho \circ f$. This is an immediate consequence of the identity $f^* \text{Ind}_{H'}^G = \text{Ind}_{f^{-1}(H')}^G$ for any $H' \leq G'$ and $H = f^{-1}(H')$.

Claim 3. For finite groups $G$ and $G'$, we have

$$R(G \times G') = R(G) \otimes R(G'),$$

$$R_{\text{rat}}(G \times G') = R_{\text{rat}}(G) \otimes R_{\text{rat}}(G'),$$

$$S(G \times G') \supseteq S(G) \otimes S(G').$$

Now we are ready to prove the lemma by induction. If $|G| = 1$, the lemma obviously holds. Assume that

$$S(H) = R_{\text{rat}}(H) \text{ for any finite group } H \text{ with } |H| < N.$$

We consider a finite group $G$ with $|G| = N$. We need to show that $R_{\text{rat}}(G) \subseteq S(G)$. Let $K/\mathbb{Q}$ be a finite Galois extension such that all the representations of all the groups of order $\leq N$ may take values in $\text{GL}_n(K)$. There are three cases.

Case 1. The group $G$ is not cyclic.

Let $\rho \in R_{\text{rat}}(G)$. We obviously have

$$\rho = \frac{1}{|K : \mathbb{Q}|} \sum_{g \in \text{Gal}(K/\mathbb{Q})} g \cdot \rho.$$
On the other hand, by Artin’s induction theorem, there exist cyclic subgroups \(H_1, \ldots, H_m \leq G\) and \((\varphi_k \in R(H_k))_{k=1, \ldots, m}\) such that

\[
\rho = \sum_{k=1}^{m} \text{Ind}_{H_k}^{G} \varphi_k .
\]

By (2.8) and (2.9), we have

\[
\rho = \frac{1}{[K : \mathbb{Q}]} \sum_{k=1}^{m} \text{Ind}_{H_k}^{G} \left( \sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k \right).
\]

Note that \(H_k\) is cyclic while \(G\) is not, by our hypothesis (2.7), we have

\[
\sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k \in R_{\text{rat}}(H_k) = S(H_k).
\]

From (2.10), (2.11) and Claim 1, we obtain \(\rho \in S(G)\). Hence \(R_{\text{rat}}(G) \subseteq S(G)\).

Case 2. The group \(G\) is cyclic, and there exist non trivial cyclic groups \(G'\) and \(G''\) such that \(G = G' \times G''\).

By our hypothesis (2.7) and Claim 3, we have

\[
R_{\text{rat}}(G) = R_{\text{rat}}(G') \otimes R_{\text{rat}}(G'') = S(G') \otimes S(G'') \subseteq S(G).
\]

Case 3. The group \(G\) is cyclic, and \(|G| = p^r\) where \(p\) is a prime number.

Let \(a \in G\) be a generator of \(G\). For \(k = 0, \ldots, p^r - 1\), let \(\varphi_k\) be the one dimensional representation defined by \(\varphi_k(a) = \exp(2k\pi i/p^r)\). Then \((\varphi_k)_{k=0, \ldots, p^r - 1}\) is a basis of \(R(G)\). Thus \(R_{\text{rat}}(G)\) is spanned by

\[
\left( \sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k \right)_{k=0, \ldots, p^r - 1}.
\]

If \(k\) is a multiple of \(p\), then \(\varphi_k : G \to \mathbb{C}^*\) is not injective. There exists a surjective group homomorphism \(f : G \to G'\) with \(|G'| < |G|\) and \(\varphi_k' \in R(G')\) such that \(\varphi_k = f^*\varphi_k'\). We have

\[
\sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k = f^* \left( \sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k' \right) \in f^*R_{\text{rat}}(G') .
\]

Then, by our hypothesis (2.7) and Claim 2, we have

\[
\sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k \in f^*S(G') \subseteq S(G) .
\]

If \(k\) is not a multiple of \(p\), we can directly verify that

\[
\frac{1}{[K : \mathbb{Q}]} \sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\varphi_k = \frac{1}{p^r-1(p-1)} \left( \sum_{k=0}^{p^r-1} \varphi_k - \sum_{k=0}^{p^r-1} \varphi_{kp} \right)
\]

\[
= \frac{1}{p^r-1(p-1)} \left( \text{Ind}_{e}^{G_{p^r-1}} \mathbb{1} - \text{Ind}_{\langle a^{p^r-1} \rangle}^{G_{p^r-1}} \mathbb{1} \right) \in S(G) ,
\]

where \(\{e\}\) is the trivial subgroup and \(\langle a^{p^r-1} \rangle\) is the subgroup generated by \(a^{p^r-1}\). In conclusion, each element in (2.13) lies in \(S(G)\). Hence \(R_{\text{rat}}(G) \subseteq S(G)\). \(\square\)
The key ingredient in the proof of Lemma 2 is Artin’s induction theorem, which is also used in [26, §5].

3. PROOF OF THE MAIN RESULT

Now we assume that there is a surjective group homomorphism \( \mu : \pi_1(M) \to G \). Then any linear representation of \( G \) may be viewed a representation of \( \pi_1(M) \). Since

\[
\tau^{\text{an/top}}(M/S, \rho \oplus \rho') = \tau^{\text{an/top}}(M/S, \rho) + \tau^{\text{an/top}}(M/S, \rho')
\]

for \( \rho, \rho' \) linear representations of \( G \), we have a \( \mathbb{Q} \)-linear map

\[
\delta : R(G) \to H^{\text{even}\geq 2}(S)
\]

\[
\rho \mapsto \tau^{\text{an}}(M/S, \rho) - \tau^{\text{top}}(M/S, \rho) .
\]

**Lemma 3.** For any subgroup \( H \leq G \), we have \( \text{Ind}_{H}^{G} \mathbb{1} \in \ker \delta \).

**Proof.** Let \( M' = \tilde{M} \times_{\mu} (G/H) \), which is a finite covering of \( M \). By [26, Remark 5.2] and the induction formula for Bismut-Lott torsion [28, Theorem 2.4], we have

\[
\tau^{\text{an}}(M'/S, \mathbb{1}) = \tau^{\text{an}}(M/S, \text{Ind}_{H}^{G} \mathbb{1}) .
\]

By [26, Remark 5.2] and the induction formula for Igusa-Klein torsion [26, Definition 2.1, Theorem 3.1], we have

\[
\tau^{\text{top}}(M'/S, \mathbb{1}) = \tau^{\text{top}}(M/S, \text{Ind}_{H}^{G} \mathbb{1}) .
\]

On the other hand, by the higher Cheeger-Müller/Bismut-Zhang theorem for trivial flat line bundles [28, Theorem 0.1], we have

\[
\tau^{\text{an}}(M'/S, \mathbb{1}) = \tau^{\text{top}}(M'/S, \mathbb{1}) .
\]

From (3.2)-(3.5), we obtain \( \delta (\text{Ind}_{H}^{G} \mathbb{1}) = 0 \). \( \square \)

**Proof of Theorem 1.** Let \( \rho : \pi_1(M) \to \text{GL}_n(K) \) be as in Theorem 1. Denote its image by \( G \), which is a finite group by assumption. We may and we will view \( \rho \) as a representation of \( G \). We obviously have

\[
\sum_{g \in \text{Gal}(K/\mathbb{Q})} g.\rho \in R_{\text{rat}}(G) .
\]

On the other hand, by Lemma 2 and Lemma 3, we have

\[
R_{\text{rat}}(G) \subseteq \ker \delta .
\]

From (3.2), (3.6), (3.7) and the linearity of \( \delta \), we obtain (1.7). \( \square \)

4. FURTHER DISCUSSIONS

Let \( M \to S \) and \( \rho : \pi_1(M) \to \text{GL}_n(K) \) be as in Theorem 1.

The following result was proved in [26, §5].

**Lemma 4.** If \( \tau^{\text{an}}(M/S, \rho) = \tau^{\text{top}}(M/S, \rho) \) for any \( M \to S \) and any one dimensional representation \( \rho \), then \( \tau^{\text{an}}(M/S, \rho) = \tau^{\text{top}}(M/S, \rho) \) for any \( M \to S \) and any \( \rho \).

Now we give an observation, which is a direct consequence of Theorem 1 and Lemma 4.
Corollary 5. If \( \tau_{\text{an}}(M/S, \rho - g \rho) = \tau_{\text{top}}(M/S, \rho - g \rho) \) for any \( M \to S \), any one dimensional representation \( \rho \) and any \( g \in \text{Gal}(K/Q) \), then \( \tau_{\text{an}}(M/S, \rho) = \tau_{\text{top}}(M/S, \rho) \) for any \( M \to S \) and any \( \rho \).

Proof. By our assumption, if \( \rho \) is one dimensional, we have
\[
\tau_{\text{an}}(M/S, \rho) - \tau_{\text{top}}(M/S, \rho) = \tau_{\text{an}}(M/S, g \rho) - \tau_{\text{top}}(M/S, g \rho)
\]
for any \( g \in \text{Gal}(K/Q) \). By Theorem 1 and (4.1), we have
\[
\tau_{\text{an}}(M/S, \rho) - \tau_{\text{top}}(M/S, \rho) = \frac{1}{[K:Q]} \sum_{g \in \text{Gal}(K/Q)} \left( \tau_{\text{an}}(M/S, g \rho) - \tau_{\text{top}}(M/S, g \rho) \right) = 0.
\]
Hence \( \tau_{\text{an}}(M/S, \rho) = \tau_{\text{top}}(M/S, \rho) \) for any \( M \to S \) and any one dimensional \( \rho \). Now, applying Lemma 4, we get \( \tau_{\text{an}}(M/S, \rho) = \tau_{\text{top}}(M/S, \rho) \) for any \( M \to S \) and any \( \rho \). \( \square \)

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