ON A CLASS OF SDE WITH JUMPS AND ITS PROPERTIES

ARI ARAPOSTATHIS, ANUP BISWAS, AND LUIS CAFFARELLI

Abstract. We study stochastic differential equations with jumps with no diffusion part. We provide some basic stochastic characterizations of solutions of the corresponding non-local partial differential equations and prove the Harnack inequality for a class of these operators. We also establish key connections between the recurrence properties of these jump processes and the non-local partial differential operator.

1. Introduction

Stochastic differential equations (SDE) with jumps have received wide attention in stochastic analysis as well as in the theory of differential equations. Unlike continuous diffusion processes, SDEs with jumps have long range interactions and therefore the generator of such processes are non-local in nature. These processes arise in various applications, for instance, in mathematical finance and control [18,30] and image processing [21]. There have been various studies on such processes from a stochastic analysis viewpoint concentrating on existence, uniqueness, stability properties of the solution [1,8,10,17,25,27], as well as from a differential equation viewpoint focusing on the existence and regularity of the viscosity solution corresponding to the generator [3,5,6,13]. One of our objectives in this paper is to establish a rigorous connection between the integro-differential operator and SDEs with jumps.

Let us consider a Markov process $X$ in $\mathbb{R}^d$ with generator $\mathcal{A}$. Let $D$ be a smooth domain in $\mathbb{R}^d$. Define the exit time as $\tau = \inf\{t \geq 0 : X_t \notin D\}$. One can formally write that $\tau$ satisfies the following equation:

$$\mathcal{A}u = -1 \text{ in } D, \quad u = 0 \text{ in } D^c. \tag{1.1}$$

Now the question is when can we actually identify the solution of (1.1) as $E_x[\tau]$. When $\mathcal{A} = \Delta + b$, i.e., $X$ is a drifted Brownian motion, one can use the regularity of the solution and Itô’s formula to establish that $u(x) = E_x[\tau]$. But if we take $\mathcal{A} = \Delta^{\alpha/2} + b$, $\alpha \in (0,2)$, then it is not known if there is a $C^{1,1}$ solution (or classical solution) to (1.1) and therefore we can not apply the Itô’s formula to characterize the expected exit times. One of the main goals of this article is to characterize $u(x) = E_x[\tau]$ as a viscosity solution to (1.1). One of the hurdles in doing so is to prove that $E_x[\tau] = 0$ whenever $x \in \partial D$. When $X$ is a drifted Brownian motion this fact can easily be proved using the fact that Brownian motion has infinitely many zeros in every finite interval. But similar crossing properties are not known for $\alpha$-stable processes. We also have to restrict ourselves in the regime $\alpha \in (1,2)$ so that the jump process can dominate the drift and leads to the proof $E_x[\tau] = 0$ whenever $x \in \partial D$. The proof technique uses the exit

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time estimate of the $\alpha$-stable processes from a cone which is available for $d \geq 2$. Hence our proof works for the dimension $d \geq 2$ leaving $d = 1$ open.

A function $h$ is said to be harmonic with respect to $X$ in $D$ if $h(X_{t\wedge \tau})$ is a martingale. One of the important properties of non-negative harmonic functions for nondegenerate continuous diffusions is the Harnack’s inequality that plays crucial role in various regularity and stability estimates. The work in [11] proves the Harnack inequality for pure jump processes which is further generalized in [9] for non-symmetric kernel that can have different order. A parabolic Harnack inequality is obtained in [7] for symmetric jump processes associated with the Dirichlet form with symmetric kernel. In [31] sufficient conditions on Markov processes to satisfy the Harnack inequality are identified. Let us also mention the work in [4, 20, 32] where a Harnack inequality is established for jump processes with a non-degenerate diffusion part. Recently [23] proves the Harnack type estimate for harmonic functions that are not necessarily non-negative in all of $\mathbb{R}^d$. In this article we prove the Harnack inequality for the jump processes having a drift but no diffusion part. The idea of the proof is to establish the sufficient conditions in [31]. Later we use our Harnack estimate to obtain certain stability results for the process. Let us also mention that the estimates obtained in Section 3 and Section 4 can also be used to get a Hölder estimate for the harmonic functions following a similar method as in [10]. However we don’t do this in this paper.

In Section 5 we discuss the ergodic properties of the process such as positive recurrence, invariant measure, etc. We provide a sufficient condition for positive recurrence and the existence of an invariant measure. This is done via imposing a Lyapunov stability condition on the generator. The existence of an invariant measure is proved by following Has’minskii’s method. We establish the existence of a unique invariant measure for a fairly large class of processes. We are able to show that one can stabilize the system by just using a non-symmetric kernel (Theorem 8 below) and no drift. Let us mention that in [34] the author provides sufficient conditions for stability for a class of jump diffusion and this is by constructing suitable Lyapunov type functions. But the class of jumps considered in [34] satisfies stronger conditions than those considered here and lies in the complement of the jump kernels that we consider. Stability of 1-dimensional processes are discussed in [33] under the assumption of Lebesgue-irreducibility. We also characterize the mean hitting time as a viscosity solution of suitable integro-differential equation.

To summarize our main contribution in this paper, we have

- Provided a rigorous justification between stochastic representation and viscosity solution,
- Shown the Harnack inequality in the absence of diffusion term,
- Provided sufficient condition for the positive recurrence and proved the existence of invariant measure,
- Characterized the mean hitting time through viscosity framework.

**Notation.** The $d$-dimensional Euclidean space is denoted by $\mathbb{R}^d$. For vectors $a, b \in \mathbb{R}^d$, we denote the scalar product by $a \cdot b$. Define $\mathbb{R}_+ = [0, \infty)$. Given two real numbers $a, b$ we denote the maximum (minimum) of $a$ and $b$ by $a \vee b$ ($a \wedge b$). Define $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$. $\lfloor a \rfloor$ denotes the least integer less or equal to $a$. For $x \in \mathbb{R}^d$ and $r \geq 0$, we denote by $B_r(x)$ the open ball of radius $r$ around $x$, and $B_r$ without an argument denotes the ball of radius $r$ around
the origin. Given a metric space $S$, we denote by $B(S)$ and $B_b(S)$ the Borel $\sigma$-algebra of $S$ and respectively, the set of bounded Borel measurable functions on $S$. For an integer $n \geq 0$, $C^n_b(S)$ ($C^\infty_b(S)$) refers to the class of all real-valued functions on $S$ whose partial derivatives up to order $n$ (of any order) exist and are continuous and bounded. For any $\gamma \geq 0$, $C^\gamma(S)$ denotes the subset of $C^{[\gamma]}(S)$ which contains functions with $[\gamma]$-th partial derivatives Hölder continuous of order $\gamma - [\gamma]$. We use $| \cdot |$ to denote the Euclidean norm on $\mathbb{R}^d$. For any function $f : S \to \mathbb{R}^d$ we define $|f|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$. For any set $A \subset \mathbb{R}^d$, we use $1_A$ to denote the indicator function of $A$. The set of Borel probability measures on $S$ is denoted by $\mathcal{P}(S)$, $\| \cdot \|_{TV}$ denotes the total variation norm on $\mathcal{P}(S)$ and $\delta_x$ the Dirac mass at $x$.

The organization of the paper is as follows. In the next section we introduce the model and assumptions. Section 3 establishes the connection to the viscosity solution. In Section 4 we show the Harnack inequality. Next we establish the positive recurrent property of the process. Finally, in the Appendix we prove some auxiliary results that are used in the proofs.

2. Preliminaries

Let $\alpha \in (1, 2)$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be two given measurable functions where $\pi$ is non-negative. We define the non-local operator $I$ as follows:

$$
I f(x) := b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}}\right) \pi(x, z) \, dz.
$$

We always assume that

$$
\int_{\mathbb{R}^d} \left( |z|^2 \wedge 1\right) \pi(x, z) \, dz < \infty, \quad \forall \ x \in \mathbb{R}^d.
$$

Note that (2.1) is well-defined for any $f \in C^2_b(\mathbb{R}^d)$. Let $\Omega = D([0, \infty), \mathbb{R}^d)$, be the space of all right continuous functions: $[0, \infty) \to \mathbb{R}^d$, having finite left limits (càdlàg). Define $X_t = \omega(t)$ for $\omega \in \Omega$ and let $\{\mathcal{F}_t\}$ be the right-continuous filtration generated by the process $\{X_s\}$. In this article we always assume that given any initial distribution $\nu_0$ there exists a strong Markov process $(X, \mathbb{P}_{\nu_0})$ that satisfies the martingale problem corresponding to $I$, i.e., $\mathbb{P}_{\nu_0}(X_0 \in A) = \nu_0(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and for any $f \in C^2_b(\mathbb{R}^d)$,

$$
f(X_t) - f(X_0) - \int_0^t I f(X_s) \, ds,
$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. We denote the law of the process by $\mathbb{P}_x$ when $\nu_0 = \delta_x$. Sufficient conditions on $b$ and $\pi$ to ensure the existence of such processes are available in the literature. Unfortunately the available sufficient conditions do not cover a wide class of processes. We refer the reader to [8] for the available results in this direction, as well as [2,16,17,25,27]. When $b \equiv 0$, well-posedness of the martingale problem is obtained under some regularity assumption on $\pi$ in [11].

Let us mention again that our goal here is not to study existence and therefore we do not assume any regularity conditions on the coefficients unless otherwise stated. Before we proceed to state our assumptions and results we state the Lévy-system formula, the proof of which is a straightforward adaptation of the proof for a purely non-local operator and can be found in [11] Proposition 2.3 and Remark 2.4].
Proposition 1. If $A$ and $B$ are disjoint Borel sets in $B(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d$,
\[
\sum_{s \leq t} 1\{X_s \in A, X_s \in B\} - \int_0^t \int_B 1\{X_s \in A\} \pi(X_s, z - X_s) \, dz \, ds
\]
is a $\mathbb{P}_x$-martingale.

For any set $A \subset \mathbb{R}^d$, we define
\[
\tau(A) := \inf \{s \geq 0 : X_s / \in A\}.
\]
Therefore $\tau(A)$ denotes the exit time of $X$ from $A$.

3. Connection to the non-local PDE

The idea of the this section is to give a rigorous mathematical justification of the connection between the stochastic differential equation with jumps and the viscosity solutions to certain non-local differential equations. In this section we consider a stochastic differential equation driven by a symmetric $\alpha$-stable process. More precisely, we consider the dynamics $\{X_s\}$ where
\[
dX_t = b(X_t) \, dt + dL_t,
\]
where $L_t$ is a symmetric $\alpha$-stable process with generator given by
\[
(\Delta^{\alpha/2} f)(x) = \int_{\mathbb{R}^d} (f(x + z) - f(x)) \frac{1}{|z|^{d+\alpha}} \, dz,
\]
and $f \in C^2_b(\mathbb{R}^d)$. It is easy to see that the solution of (3.1) is also a solution to the martingale problem for $I$ for $\pi(x, z) = \frac{1}{|z|^{d+\alpha}}$.

Condition 1. There exists a positive constant $L$ such that for all $x, y \in \mathbb{R}^d$,
\[
|b(x) - b(y)| \leq M|x - y|,
\]
\[
\sup_{x \in \mathbb{R}^d} |b(x)| \leq M.
\]

Under Condition 1 and $X_0 = x \in \mathbb{R}^d$, equation (3.1) has a unique adapted strong c\'{a}dl\'{a}g solution which is a Feller process [2]. Therefore the process $X$ evolves in a continuous manner in between the jumps and the jumps are determined by $L_t$. Let us also mention the following lemma which is pretty straightforward to prove.

Lemma 1. Assume Condition 1 holds and $T > 0$. Then there exists a constant $C = C(L, T)$ such that if $x_n \to x$ as $n \to \infty$ and $X^n, X$ denotes the solution to (3.1) with initial data $X_0 = x_n, X_0 = x$, respectively, then
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{s \in [0, T]} |X^n_s - X_s|^2 \right] = 0.
\]

The generator associated to (3.1) is given by
\[
I f(x) = b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} (f(x + z) - f(x)) \frac{1}{|z|^{d+\alpha}} \, dz,
\]
where $f$ is in $C^2_b(\mathbb{R}^d)$. Now we recall the definition of a viscosity solution [5,13]. Let $D$ be a bounded domain with $C^1$ boundary.
Definition 1. A function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ upper (lower) semi-continuous on $\bar{D}$ is said to be a sub-solution (super-solution) to
\[ \mathcal{I}u = f \quad \text{in } D, \]
\[ u = g \quad \text{in } D^c, \]
where $\mathcal{I}$ is given by (3.4) if for any $x \in \bar{D}$ and a function $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi(x) = u(x)$ and $\varphi(z) > u(z)$, $(\varphi(z) < u(z))$ on $\mathbb{R}^d \setminus \{x\}$ it holds that
\[ \mathcal{I}\varphi(x) \geq f(x) \quad (\mathcal{I}\varphi(x) \leq f(x)), \quad \text{if } x \in D, \]
and if $x \in \partial D$ then,
\[ \max \{(\mathcal{I}\varphi(x) - f(x), g(x) - u(x)) \geq 0 \quad \text{(min \{\mathcal{I}\varphi(x) - f(x), g(x) - u(x)\} \leq 0)} \right), \]
A function $u$ is said to be a viscosity solution if it is both sub- and super-solution.

Let $f$ and $g$ be two bounded, continuous functions on $\mathbb{R}^d$. Given a domain $D$, we define
\[ u(x) = \mathbb{E}_x \left[ \int_0^{\tau(D)} f(X_s) \, ds + g(X_{\tau(D)}) \right] \quad \text{for } x \in \mathbb{R}^d, \quad (3.5) \]
where $\mathbb{E}_x$ denotes the expectation w.r.t. $\mathbb{P}_x$. Our main theorem in this section is the following:

Theorem 1. Let $d \geq 2$. The function $u(\cdot)$ defined by (3.5) is bounded, continuous and is the unique viscosity solution to the equation
\[ \mathcal{I}u = -f \quad \text{in } D, \]
\[ u = g \quad \text{in } D^c. \quad (3.6) \]

We start by proving several lemmas that will lead to the proof of Theorem 1. We say that $b$ is locally bounded if for any compact set $K$, $\sup_{x \in K} |b(x)| < \infty$.

Lemma 2. Let $D$ be a bounded domain. Suppose $\{X_s\}$ is a strong Markov process associated with $\mathcal{I}$ in (2.1), with $b$ locally bounded, and that the following hold: for any compact set $K$,
\[ \sup_{x \in K} \int_{\{|z|>1\}} |z| \pi(x,z) \, dz < \infty, \quad \text{and} \quad \inf_{x \in K} \int_{\mathbb{R}^d} |z|^2 \pi(x,z) \, dz = \infty. \]
Then we have $\sup_{x \in D} \mathbb{E}_x[(\tau(D))^m] < \infty$, for any positive integer $m$.

Proof. Without loss of generality we assume that $0 \in D$. Otherwise we inflate the domain to include 0. Let $\bar{d} = \text{diam}(D)$ and $M_D = \sup_{x \in D} |b(x)|$. Recall that $B_R$ denotes the ball of radius $R$ around the origin. We can choose $R > 2(\bar{d} \vee M_D)$, $R > 1$, and large enough to satisfy the following
\[ \inf_{x \in D} \int_{B_R} |z|^2 \pi(x,z) \, dz > 1 + 2\bar{d}M_D + 2\bar{d} \sup_{x \in D} \int_{\{1<|z|\leq R\}} |z| \pi(x,z) \, dz. \quad (3.7) \]
Now we take \( f \in C^2_b(\mathbb{R}^d) \) be a radially increasing function such that \( f(x) = |x|^2 \) for \( |x| \leq 2R \) and \( f(x) = 8R^2 \) for \( |x| \geq 2R + 1 \). Then for any \( x \in D \),

\[
\mathcal{I}f(x) = b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} (f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}}) \pi(x, z) \, dz
\]

\[
\geq -2\bar{d}M_D + \int_{B_R} (f(x + z) - f(x) - \nabla f(x) \cdot z) \pi(x, z) \, dz
\]

\[
+ \int_{\{1 < |z| \leq R\}} \nabla f(x) \cdot z \pi(x, z) \, dz + \int_{B_R^c} (f(x + z) - f(x)) \pi(x, z) \, dz.
\]

Now for any \( |z| \geq R \), it holds that \( |x + z| \geq \bar{d} \geq |x| \). Therefore \( f(x + z) \geq f(x) \). Hence

\[
\mathcal{I}f(x) \geq -2\bar{d}M_D + \int_{\{1 < |z| \leq R\}} \nabla f(x) \cdot z \pi(x, z) \, dz
\]

\[
+ \int_{B_R} (f(x + z) - f(x) - \nabla f(x) \cdot z) \pi(x, z) \, dz
\]

\[
\geq -2\bar{d}M_D - 2\bar{d} \int_{\{1 < |z| \leq R\}} |z| \pi(x, z) \, dz + \int_{B_R} |z|^2 \pi(x, z) \, dz
\]

\[
\geq 1.
\]

Hence we have for \( x \in D \),

\[
\mathbb{E}_x[f(X_{\tau(D) \land t})] - f(x) = \mathbb{E}_x\left[\int_0^{\tau(D) \land t} \mathcal{I}(f)(X_s) \, ds\right]
\]

\[
\geq \mathbb{E}_x[\tau(D) \land t].
\]

Letting \( t \to \infty \) we obtain \( \mathbb{E}_x[\tau(D)] \leq 8R^2 \). Since \( x \in D \) is arbitrary this shows that \( \sup_{x \in D} \mathbb{E}_x[\tau(D)] \leq 8R^2 \).

We continue by the method of induction. We have proved the result for \( m = 1 \). Assume that it is true for \( m \), i.e., \( M_m := \sup_{x \in D} \mathbb{E}_x[(\tau(D))^{m}] < \infty \). Define \( h(x) = M_m f(x) \) where \( f \) is defined above. Then from the calculations above we have for \( x \in D \),

\[
\mathbb{E}_x[h(X_{\tau(D) \land t})] - h(x) \geq \mathbb{E}_x[M_m(\tau(D) \land t)].
\]

(3.8)
Now denoting $\tau(D)$ by $\tau$ we have
\[
\mathbb{E}_x[\tau^{m+1}] = \mathbb{E}_x \left[ \int_0^\infty (m+1)(\tau-t)^m 1_{\{t<\tau\}} \, dt \right]
\]
\[
= \mathbb{E}_x \left[ \int_0^\infty (m+1)\mathbb{E}_x \left[ (\tau-t)^m 1_{\{t<\tau\}} \mid \mathcal{F}_{t\wedge \tau} \right] \, dt \right]
\]
\[
= \mathbb{E}_x \left[ \int_0^\infty (m+1)1_{\{t\wedge \tau<\tau\}} \mathbb{E}_x \left[ (\tau-t)^m 1_{\{t<\tau\}} \mid \mathcal{F}_{t\wedge \tau} \right] \, dt \right]
\]
\[
\leq \sup_{x \in D} \mathbb{E}_x[\tau^m] \mathbb{E}_x \left[ \int_0^\infty (m+1)1_{\{t\wedge \tau<\tau\}} \, dt \right]
\]
\[
\leq M_m (m+1) \mathbb{E}_x[\tau].
\]
This completes the proof from (3.8).

Following lemma is a careful modification of [32, Lemma 2.1].

**Lemma 3.** Let $D$ be a given domain. There exists a constant $\kappa_1 > 0$ such that for any $x \in D$ and $r \in (0, 1)$
\[
\mathbb{P}_x \left( \sup_{0 \leq s \leq t} |X_s - X_0| > r \right) \leq \kappa_1 t r^{-\alpha},
\] (3.9)
where $X$ satisfies (3.1).

**Proof.** Let $f \in C_b^2(\mathbb{R}^d)$ be such that $f(x) = |x|^2$ for $|x| \leq \frac{1}{2}$ and $f(x) = 1$ for $|x| \geq 1$. Therefore we have a constant $c$ such that
\[
\sup_{x \in \mathbb{R}^d} |\nabla f(x)| \leq c,
\]
and for all $x, z \in \mathbb{R}^d$, we have
\[
|f(x + z) - f(x) - \nabla f(x) \cdot z| \leq c|z|^2.
\]
Define $f_r(y) = f(\frac{y-x}{r})$ where $x$ is a point in $D$. Now for $y \in B_r(x)$, we obtain
\[
\left| \int_{|z| \leq r} (f_r(y+z) - f_r(y) - \nabla f_r(y) \cdot z 1_{\{|z| \leq 1\}}) \frac{1}{|z|^{d+\alpha}} \, dz \right|
\]
\[
\leq \left| \int_{|z| \leq r} (f_r(y+z) - f_r(y) - \nabla f_r(y) \cdot z) \frac{1}{|z|^{d+\alpha}} \, dz \right| + \left| \int_{|z| > r} (f_r(y+z) - f_r(y)) \frac{1}{|z|^{d+\alpha}} \, dz \right|
\]
\[
\leq c \frac{1}{r^2} \int_{|z| \leq r} |z|^{2-d-\alpha} \, dz + 2 \int_{|z| > r} |z|^{-d-\alpha} \, dz
\]
\[
\leq \frac{c_3}{r^\alpha},
\]
for some constant $c_3$. Since $\alpha > 1$ we have for $y \in \bar{B}_r(x)$ that
\[
|I f_r(y)| \leq \frac{c_4}{r^\alpha},
\]
where \( c_4 \) is a positive constant depending on \( c_3 \) and \( M \). Therefore using the Itô’s formula we have
\[
\frac{c_4}{r^\alpha} \mathbb{E}_x \left[ \tau(B_r(x)) \wedge t \right] \geq \mathbb{E}_x \left[ f_r(X_{\tau(B_r(x)) \wedge t}) \right].
\]
Since \( f_r = 1 \) on \( B_r^c(x) \) we have \( \mathbb{P}_x(\tau(B_r) \leq t) \leq c_4 r^{-\alpha} t \). This completes the proof. \( \square \)

Now we define the following process
\[
Y_t = x + L_t. \tag{3.10}
\]
It is easy to see that \( Y \) is symmetric \( \alpha \)-stable Lévy process starting at \( x \). Therefore it is easy to check from the martingale property that for any measurable function \( f : D(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R} \), we have
\[
\mathbb{E}_x[f(Y_t)] = \mathbb{E}_{\bar{x}}[f(a Y_{a-\cdot})]. \tag{3.11}
\]
Now we recall the following theorem from [28, Theorem 1].

**Theorem 2.** Let \( \theta \in (0, \pi) \). Let \( G \) be a closed cone in \( \mathbb{R}^d, d \geq 2 \), of angle \( \theta \) with vertex at 0. Define
\[
\eta(G) = \inf \{ t \geq 0, Y_t \not\in G \}.
\]
Then there exists \( \alpha_0(\theta) > 0 \) such that
\[
\mathbb{E}_x[(\eta(G))^p] < \infty \quad \text{for } p < \alpha_0(\theta),
\]
\[
\mathbb{E}_x[(\eta(G))^p] = \infty \quad \text{for } p > \alpha_0(\theta),
\]
for all \( x \in G \setminus \{0\} \).

The result in [28] is proven for open cones. The statement in Theorem 2 follows from the fact that every closed cone is contained in an open cone except for the vertex of the cone and with probability 1 the exit location from an open cone is not the vertex. The following result is also obtained in [16] using the estimates in transition density. However our proof technique is different and does not use the estimates on transition density.

**Lemma 4.** Let \( d \geq 2 \). For any \( x \in \partial D, \mathbb{P}_x(\tau(D) > 0) = 0 \).

**Proof.** Let \( x_0 \in \partial D \) be a fixed point. We consider an open cone \( G \) in the complement of \( D \) at a distance \( r \) from the boundary \( \partial D \) such that the distance between the cone and the boundary is achieved by \( x_0 \) and the vertex of the cone \( x_r \). In fact, we can find an angle \( \theta \) and axis of the cone that can be kept fixed for all \( r \) small and satisfies the above mentioned properties. It is quite clear that this can be done for some truncated cone. So first we assume that the full cone \( G \) with angle \( \theta \) and vertex \( x_r \) lies completely in \( D^c \). Let \( \eta(G^c) \) be the exit time of \( Y_t \) that starts at \( x_0 \). Since translation of coordinate does not affect the exit time, we can assume that \( x_r = 0 \). Then from Theorem 2 we have \( 0 < p < \alpha_0(\theta) \), satisfying
\[
\mathbb{E}_{x_0}[(\eta(G^c))^p] = |x_0|^\alpha p \mathbb{E}_{\frac{x_0}{|x_0|}}[(\eta(G^c))^p] < \infty,
\]
where we used the property (3.11). Now by the upper-semi continuity property we have
\[
\sup \{ \mathbb{E}_u[(\eta(G^c))^p] : u \in G^c, |u| = 1 \} < \infty.
\]
Therefore we can find a constant \( \kappa_2 > 0 \) not depending on \( r \) (for \( r \) small) such that
\[
\mathbb{E}_{x_0}[(\eta(G^c))^p] \leq \kappa_2 |r|^\alpha p. \tag{3.12}
\]
Let \( \alpha' \in (1, \alpha) \). Then for any \( \varepsilon > 0 \), we can choose \( r \) small enough so that

\[
\mathbb{P}_{x_0}(\eta(G^c) > r^{\alpha'}) \leq \kappa_2 r^p (\alpha - \alpha')^p < \varepsilon, \tag{3.13}
\]

where we used (3.12). Again using Condition [1] and (3.1), (3.10), we have

\[
\sup_{s \in [0, r^{\alpha'}]} |X_s - Y_s| \leq Mr^{\alpha'}, \tag{3.14}
\]

with probability 1. Hence on \( \{\eta(G^c) \leq r^{\alpha'}\} \) we have \( |Y_{\eta(G^c)} - X_{\eta(G^c)}| \leq Mr^{\alpha'} \), (3.14). But \( Y_{\eta(G^c)} \in G \) and \( \text{dist}(x_0, G) = r \). Since \( Mr^{\alpha'} < r \) for \( r \) small enough we have \( X_{\eta(G^c)} \in D^c \) on \( \{\eta(G^c) \leq r^{\alpha'}\} \). Therefore from (3.13) we get

\[
\mathbb{P}_{x_0}(\tau(D) > r^{\alpha'}) < \varepsilon,
\]

for all \( r \) small. This concludes the proof for the case when we can fit whole cone in \( D^c \) near \( x_0 \). For any other scenario we can modify the domain locally around \( x_0 \) and get that the exit time from new domain is 0. Now we use Lemma 3 to obtain that with high probability the paths spends \( r^{\alpha} \) amount of time in a ball of radius of order \( r \). Combining these two facts we can conclude the proof. \( \square \)

**Remark 1.** Note that the result of Lemma 4 still holds if \( X \) satisfies (3.1) in weak-sense (see also (17)) for some locally bounded measurable drift \( b(\cdot) \).

We have immediate corollary from Lemma [4].

**Corollary 1.** For any domain \( D \) with \( C^1 \) boundary, \( \mathbb{P}_x(\tau(D) = \tau(D^0)) = 1 \) for all \( x \in D^0 \).

**Lemma 5.** For \( x \in D \) we have

\[
\mathbb{P}_x(X_{\tau(D)^-} \in \partial D, X_{\tau(D)} \in D^c) = 0,
\]

\[
\mathbb{P}_x(X_{\tau(D)^-} \in \partial D, X_s \in D^0 \text{ for some } s \in [0, \tau(D)]) = 0.
\]

**Proof.** We only prove the first one as the proof for the second one follows along the same lines. From Condition [1] we obtain that \( X_t \) has density for every \( t > 0 \) [12]. Define

\[
D_R = \{z \in D^c : \text{dist}(z, D) \geq R\}.
\]

It is enough to prove that \( \mathbb{P}_x(X_{\tau(D)^-} \in \partial D, X_{\tau(D)} \in D_R) = 0 \) for every \( R > 0 \). Now for any \( t > 0 \), we have from Proposition [1]

\[
\mathbb{P}_x(X_{t \wedge \tau(D)^-} \in \partial D, X_{t \wedge \tau(D)} \in D_R) \leq \mathbb{E}_x \left[ \sum_{s \leq t} 1_{\{X_s \in \partial D, X_s \in D_R\}} \right]
\]

\[
= \mathbb{E}_x \left[ \int_0^t 1_{\{X_s \in \partial D\}} \int_{D_R} \frac{1}{|X_s - z|^{d+\alpha}} \, dz \, ds \right]
\]

\[
\leq \frac{\kappa_3}{R^\alpha} \mathbb{P}_x \left[ \int_0^t 1_{\{X_s \in \partial D\}} \, ds \right],
\]

for some constant \( \kappa_3 \). But the RHS is 0 from the fact the \( X_s \) has density. Hence we have \( \mathbb{P}_x(X_{t \wedge \tau(D)^-} \in \partial D, X_{t \wedge \tau(D)} \in D_R) = 0 \) for arbitrary \( t > 0 \). This completes the proof by letting \( t \to \infty \). \( \square \)
Proof of Theorem 1. Uniqueness follows by the comparison principle in [22, Corollary 2.9]. Since \( f \) and \( g \) are bounded, it is easy to see from Lemma 2 that \( u \) is bounded. Also in view of Lemma 4 we have \( u(x) = g(x) \) for \( x \in \partial D \). Now we prove that \( u \) is continuous in \( D \). Let \( x_n \to x \) in \( D \) as \( n \to \infty \). We need to show that \( |u(x_n) - u(x)| \to 0 \) as \( n \to \infty \). In view of Lemma 2 it is enough to show that for any \( T > 0 \) we have

\[
E_{x_n} \left[ \int_0^{\tau(D) \wedge T} f(X_s) \, ds + g(X_{\tau(D)}) \right] - E_x \left[ \int_0^{\tau(D) \wedge T} f(X_s) \, ds + g(X_{\tau(D)}) \right] \xrightarrow{n \to \infty} 0. 
\]  
(3.15)

To simplify the notation, we denote \( \tau_n \) for \( \tau(D) \) for the process that starts at \( x_n \). Similarly we define \( \tau \). From Lemma 1 we have

\[
E \left[ \sup_{s \in [0,T+1]} |X^n_s - X_s|^2 \right] \xrightarrow{n \to \infty} 0 .
\]  
(3.16)

Passing to a subsequence we can assume that

\[
\sup_{s \in [0,T+1]} |X^n_s - X_s| \to 0 ,
\]  
(3.17)

as \( n \to \infty \). Now one of the following happen:

(a) \( X \) is discontinuous at \( \tau \wedge T \),

(b) \( X \) is continuous at \( \tau \wedge T \).

Also in light of Lemma 5 and Corollary 1 we see that for any given \( \varepsilon > 0 \) the trajectory \( \{X_s, s \in [0, \tau \wedge T - \varepsilon]\} \) has a positive distance from the boundary provided \( x \in D^0 \). Therefore from (3.16) we get

\[
\liminf_{n \to \infty} \tau_n \wedge T \geq \tau \wedge T \quad \text{for} \ x \in D^0 .
\]  
(3.18)

This is trivially true for \( x \in \partial D \) due to Lemma 4. But exit times are upper semi-continuous which together with (3.17) gives

\[
\lim_{n \to \infty} \tau_n \wedge T = \tau \wedge T , \quad \forall x \in D ,
\]  
(3.19)

with probability 1. Next we prove that \( X^n_{\tau_n} \to X_\tau \) on \( \{\tau < T\} \). First we note from (3.18) that on \( \{\tau < T\} \), \( \tau^n < T \) for all \( n \) large. If \( X \) is discontinuous at \( \tau \) then using (3.16) and an argument similar to above we have \( X^n_{\tau_n} \to X_\tau \) as \( n \to \infty \) on \( \{\tau < T\} \). Again if \( X \) is continuous at \( \tau \) using (3.16) and (3.18) we have \( X^n_{\tau_n} \to X_\tau \). Now to conclude (3.15) we notice that \( \mathbb{P}_x(\tau \geq T_1) \to 0 \) as \( T_1 \to \infty \). Thus we prove (3.15) along a sub-sequential limit. Convergence of the full sequence is established by the uniqueness of the limit. This proves that \( u \) is continuous in \( \mathbb{R}^d \).

Next we show that \( u \) is a viscosity solution to (3.6). It is straightforward to check using the Markov property of \( X \) that for any \( t \geq 0 \),

\[
u(x) = E_x \left[ \int_0^{T(D) \wedge T} f(X_s) \, ds + u(X_{\tau(D) \wedge T}) \right] .
\]  
(3.19)
Let \( \varphi \in C^2_b(\mathbb{R}^d) \) be such that \( \varphi(x) = u(x) \) and \( \varphi(z) > u(z) \) for all \( z \in \mathbb{R}^d \setminus \{x\} \). Then from (3.19) and Itô’s formula we have
\[
\mathbb{E}_x \left[ \int_0^{\tau(D) \wedge t} \mathcal{I} \varphi(X_s) \, ds \right] = \mathbb{E}_x \left[ \varphi(X_{\tau(D) \wedge t}) \right] - \varphi(x)
\geq \mathbb{E}_x \left[ u(X_{\tau(D) \wedge t}) \right] - u(x)
= \mathbb{E}_x \left[ \int_0^{\tau(D) \wedge t} -f(X_s) \, ds \right].
\]
Now dividing both sides by \( t \) and letting \( t \to 0 \) we get \( \mathcal{I} \varphi(x) \geq -f(x) \) and thus \( u \) is a sub-solution. Similarly we can show that \( u \) is super-solution and so is a viscosity solution. \( \square \)

The following theorem proves the regularity of viscosity solution w.r.t. domains.

**Theorem 3.** Let \( D_n, D \) be a collection of \( C^1 \) domains and \( D_n \to D \) in Hausdorff topology. Then \( u_n \to u \) where \( u_n, u \) are the viscosity solutions to (3.16) in \( D_n \) and \( D \), respectively.

**Proof.** Let \( \tau^n, \tau \) denote the exit time from \( D_n \) and \( D \), respectively. To complete the proof we only need to establish that for any \( T > 0 \), \( \tau^n \wedge T \to \tau \wedge T \) with probability 1 and \( X_{\tau^n} \to X_\tau \) as \( n \to \infty \) on \( \{\tau < T\} \). This can be shown following the same argument as above. \( \square \)

Rest of this section we consider the jump kernel of following form.
\[
\pi = \pi_1 = \frac{k(x,z)}{|z|^{d+\alpha}}, \quad k(x,z) = 1, \quad \forall \, x \in \mathbb{R}^d \text{ and } |z| \leq 1,
\]
\[
0 < \inf_{x \in D, z \in \mathbb{R}^d} k(x,z) \leq \sup_{x \in D, z \in \mathbb{R}^d} k(x,z) < \infty, \text{ for every compact set } D.
\]

By \( \mathcal{I} \) we denote the operator \( \mathcal{I} f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x)) \pi_1(x,z) \, dz \). Note that the gradient terms disappears due to the symmetry of \( \pi_1 \) in the unit ball.

**Lemma 6.** Let \( f \) be a locally bounded function and \( D \) be any compact set. Then there is a constant \( C \) such that for any sub-domain \( D' \subset D \), and a viscosity solution \( u \) to the equation
\[
\mathcal{I} u = f \quad \text{in} \quad D',
\]
\[
u = 0 \quad \text{in} \quad (D')^c,
\]
we have \( \sup_{x \in \mathbb{R}^d} |u(x)| \leq C. \)

**Proof.** From Lemma 2 we find a function \( \xi \in C^2_b(\mathbb{R}^d) \) such that \( \xi \) is non-negative, nondecreasing and \( \mathcal{I} \xi(x) > \sup_{x \in D} |f(x)| \) for all \( x \in D \). Let \( M > 0 \) be the smallest number so that \( M - \xi \) touches \( u \) from above at least at one point. We claim that \( M \leq \sup_{x \in \mathbb{R}^d} |\xi(x)|. \) If not, then \( M - \xi(x) > 0 \) for all \( x \in (D')^c \). Therefore \( M - \xi \) touches \( u \) in the interior of \( D' \) from above. Hence by definition of viscosity solution we get \( \mathcal{I}(M - \xi(x)) \geq f(x) \) where \( x \in D' \) is a point of contact from above. But this is contradiction to the definition of \( \xi \). Hence we have \( M \leq \sup_{x \in \mathbb{R}^d} |\xi(x)|. \) Also definition of \( M \) we have
\[
\sup_{x \in D'} u(x) \leq \sup_{x \in D'} (M - \xi(x)) \leq M \leq \sup_{x \in \mathbb{R}^d} |\xi(x)|. \]
The result follows by applying the same argument on \( -u. \) \( \square \)
The following theorem holds in more general domain. However we present it here in a simpler form.

**Theorem 4.** Let $B$ be a ball and $f$ be any locally Hölder continuous function. Let $k(\cdot, z)$ in \[3.20\] be locally Hölder continuous uniformly w.r.t. $z$. Let $\tau(B)$ be the exit time from $B$ of the process $X$ with generator $\pi_1$. Then $u(x) = \mathbb{E}_x[\int_0^{\tau(B)} f(X_s)ds]$ is a viscosity solution to the equation

$$\tilde{I}u = -f \quad \text{in} \quad B,$$

$$u = 0 \quad \text{in} \quad B^c.$$ 

**Proof.** By $B_\varepsilon$ we denote the $\varepsilon$-neighborhood of $B$. In view of Lemma 6 we can have a family of viscosity solution $\{u_\varepsilon, \varepsilon \in (0, 1]\}$ such that $u_\varepsilon$ satisfies the above viscosity solution in the domain $B_\varepsilon$ and the family is uniformly bounded. For the existence of viscosity solution we refer [15]. Now we can represent the solution $u_\varepsilon$ as the solution a viscosity solution to a integro-differential equation with kernel $\frac{1}{|z|^{d+\alpha}}$ by moving the error terms on the r.h.s. with $f$.

Therefore by [29, Proposition 1.1] we get that $\{u_\varepsilon\}$ is a family of globally Hölder continuous functions with a fixed Hölder exponent. Hence $u_\varepsilon \to u$ as $\varepsilon \to 0$ along some subsequence.

Again from [26, Theorem 6.2] we see that for each $\varepsilon > 0$, $u_\varepsilon$ is in $C^{\alpha+\delta}$ for some $\delta > 0$ in the interior of $B_\varepsilon$. Hence we can apply Itô formula to obtain

$$u_\varepsilon(x) = \mathbb{E}_x[u_\varepsilon(X_{\tau(B)})] + \mathbb{E}_x[\int_0^{\tau(B)} f(X_s)ds].$$

Now let $\varepsilon \to 0$ to obtain the result. \hfill $\Box$

### 4. Harnack Inequality

In this section, we prove Harnack inequality for harmonic functions. The operator considered here is more general than the one in previous section. Define for $f \in C^2_b(\mathbb{R}^d), d \geq 1,$

$$I(f)(x) = b(x) \cdot \nabla f(x) + \int_0^{d} \left(f(x + z) - f(x) - \nabla f(x) \cdot z 1_{|z| \leq 1}\right) \frac{k(x, z)}{|z|^{d+\alpha}} dz,$$ \hspace{1cm} (4.1)

where

- $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is Borel measurable function such that $k(x, z) = k(x, -z)$ and $d_1 \leq k(x, z) \leq d_2$ \hspace{1cm} $\forall x, z \in \mathbb{R}^d,$

  - for some positive constant $d_1, d_2,$

- $b : \mathbb{R}^d \to \mathbb{R}^d$ be any Borel measurable map that is locally bounded.

A measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic in $D$ if for any $B \subset D$, it satisfies,

$$h(x) = \mathbb{E}_x[h(X_{\tau(B)})] \quad \forall x \in B,$$

where $(X, \mathbb{P}_x)$ is the strong Markov process associated to (4.1).

**Theorem 5.** Let $D$ be a bounded domain of $\mathbb{R}^d$ and $K \subset D$ be compact. Then for any bounded, non-negative function that is harmonic with respect to the operator (4.1) in $D$, we have a constant $C$, not depending on $h$, such that

$$h(x) \leq Ch(y) \quad \text{for all } x, y \in K.$$
Proof. By following the method as in Lemma 3 we obtain a constant for $t$ $\kappa_4 > 0$ such that

$$\inf_{x \in B_2(x)} \mathbb{E}_x[\tau(B_r(x))] \geq \kappa_4^{-1} r^\alpha,$$

(a) $\sup_{x \in B_r(x)} \mathbb{E}_x[\tau(B_r(x))] \leq \kappa_4 r^\alpha$.

Lemma 7. There exists constants $\kappa_1$ and $r_0 > 0$ such that for any $x \in D$ and $r \in (0, r_0)$,

(a) $\inf_{z \in B_r(x)} \mathbb{E}_z[\tau(B_r(x))] \geq \kappa_1^{-1} r^\alpha$,

(b) $\sup_{z \in B_r(x)} \mathbb{E}_z[\tau(B_r(x))] \leq \kappa_1 r^\alpha$.

Proof. By following the method as in Lemma 3 we obtain a constant $\kappa_1$ such that

$$\mathbb{P}_x(\tau(B_r(x)) \leq t) \leq \kappa_1 t r^{-\alpha},$$

(4.2)

for $t \geq 0$ and $x \in D_2 := \{y : \text{dist}(y, D) \leq 2\}$. Choose $t = \beta r^{-\alpha}$ where $2^\alpha \beta \kappa_1 < 1/2$. Then for $z \in B_{\frac{r}{2}}(x)$ and using (4.2), we get

$$\mathbb{E}_z[\tau(B_r(x))] \geq \mathbb{E}_z[\tau(B_{\frac{r}{2}}(z))] \geq \beta r^{\alpha} \mathbb{P}_z(\tau(B_{\frac{r}{2}}(z)) > \beta r^{\alpha}) \geq \frac{1}{2} \beta r^{\alpha}.$$

This proves the first part. Now we come to the second part. Let us take a radially non-decreasing function $f$ that is in $C^2_b(\mathbb{R}^d)$ and convex in $B_1(0)$ with the property that there exists a positive $c$ and

$$f(x + z) - f(x) - z \cdot f(x) \geq c|z|^2 \quad \text{for } |x| \leq 1, \quad |z| \leq 3. \quad (4.3)$$

For example we can take $f(x) = |x|^2$ in $B_4(0)$. Now for any point $x_0 \in D$ we define $g_r(x) = f\left(\frac{x - x_0}{r}\right)$. Then for $x \in B_r(x_0)$,

$$\int_{\mathbb{R}^d} \left( g_r(x + z) - g_r(x) - z \cdot \nabla g_r(x) 1_{\{|z| \leq 1\}} \right) \frac{k(x, z)}{|z|^2} \, dz$$

$$= \int_{|z| \leq 3r} \left( g_r(x + z) - g_r(x) - z \cdot \nabla g_r(x) 1_{\{|z| \leq 3r\}} \right) \frac{k(x, z)}{|z|^2} \, dz$$

$$+ \int_{|z| > 3r} \left( g_r(x + z) - g_r(x) \right) \frac{k(x, z)}{|z|^2} \, dz$$

$$\geq d_1 \frac{c}{r^2} \int_{|z| \leq 3r} |z|^{2-\alpha} \, dz$$

$$= \frac{3^{2-\alpha}}{4 - \alpha} d_1 c r^{-\alpha},$$

where in the second equality we used the fact that $k(x, z) = k(x, -z)$ and for third inequality $g(x + z) \geq g(x)$ we use for $|z| \geq 3r$. Now we choose $r_0$ small enough so that

$$\mathcal{T}(g_r)(x) \geq \kappa_5 \frac{1}{r^\alpha}$$

for $r \in (0, r_0)$ and $x \in B_r(x_0)$. Here we use the fact that $\alpha > 1$. Then using Itô’s formula we have

$$\mathbb{E}_{x_0}[\tau(B_r(x_0))] \leq \kappa_5^{-1} r^\alpha \sup_{x \in \mathbb{R}^d} |f(x)|.$$
This completes the proof.

\[ \text{Lemma 8. There exists a constant } \kappa_6 > 0 \text{ such that for any } r \in (0, 1), \ x \in D \text{ and } A \subset B_r(x) \text{ we have} \]
\[ \mathbb{P}_z(\tau(A^c) < \tau(B_{3r}(x))) \geq \kappa_6 \frac{|A|}{|B_r(x)|} \quad \forall z \in B_{2r}(x). \]

\[ \text{Proof. Denote } \tau = \tau(B_{3r}(x)). \text{ Let } \mathbb{P}_z(\tau(A^c) < \tau) < 1/4 \text{ for some } z \in B_{2r}(x). \text{ Otherwise there is nothing to prove as } \frac{|A|}{|B_r(x)|} \leq 1. \text{ Again from Lemma 3 we have } \beta > 0 \text{ such that } \mathbb{P}_z(\tau \leq t) \leq 1/4 \text{ for } t = \beta r^\alpha \text{ and } z \in B_{2r}(x). \text{ Hence using the Levy-system formula we get} \]
\[ \mathbb{P}_z(\tau(A^c) < \tau) \geq \mathbb{E}_z \left[ \sum_{s \leq \tau(A^c) \wedge t} 1_{\{X_s \neq X_s, X_s \in A\}} \right] \]
\[ = \mathbb{E}_z \left[ \int_0^{\tau(A^c) \wedge t} \int_A \frac{k(x, z - X_s)}{|z - X_s|^{d+\alpha}} \, dz \, ds \right] \]
\[ \geq \mathbb{E}_z \left[ \int_0^{\tau(A^c) \wedge t} \int_A \frac{d_1}{(6r)^{d+\alpha}} \, dz \, ds \right] \]
\[ \geq \kappa_7 r^{-\alpha} \frac{|A|}{|B_r(x)|} \mathbb{E}_z[\tau(A^c) \wedge \tau \wedge t], \]

where in the third inequality we use the fact that \(|X_s - z| \leq 6r\) for \(s < \tau, z \in A\). Now
\[ \mathbb{E}_z[\tau(A^c) \wedge \tau \wedge t] \geq t \mathbb{P}_z(\tau(A^c) \geq \tau \geq t) \]
\[ = \beta r^\alpha [1 - \mathbb{P}_z(\tau(A^c) < \tau) - \mathbb{P}_z(\tau < t)] \]
\[ \geq \frac{\beta}{2} r^\alpha. \]

Therefore combining with above we have \( \mathbb{P}_z(\tau(A^c) < \tau) \geq \frac{\beta_6 r^\alpha}{2} \frac{|A|}{|B_r(x)|}. \)

\[ \text{Lemma 9. There exists positive constants } \kappa_i, i = 9, 10, \text{ such that if } x \in D, \ r \in (0, 1), \ z \in B(x, r) \text{ and } H \text{ is a bounded non-negative function with support in } B_{2r}(x), \text{ then} \]
\[ \mathbb{E}_z[H(X_{\tau(B_r(x))})] \leq \kappa_9 (\mathbb{E}_z[\tau(B_r(x))]) \int H(u) \frac{k(x, u - x)}{|u - x|^{d+\alpha}} \, du, \]

and
\[ \mathbb{E}_z[H(X_{\tau(B_r(x))})] \geq \kappa_{10} (\mathbb{E}_z[\tau(B_r(x))]) \int H(u) \frac{k(x, u - x)}{|u - x|^{d+\alpha}} \, du. \]

The proof follows using the same argument as in \[31\] Lemma 3.5.

\[ \text{Proof of Theorem 3. From Lemmas 7, 8 and 9, we see that the conditions (A1)-(A3) in } \[31\] \text{ are satisfied. Hence the proof follows from } \[31\] \text{ Theorem 2.4.} \]
5. Positive recurrence and Invariant measure

In this section we study the long time stability property for the Markov process associated to the operator \([4, 1]\). The study is based on the assumption of the existence of a Lyapunov function.

**Condition 2.** We say that the operator \([4, 1]\) satisfies the Lyapunov stability condition if there exists a \(V \in C^2(\mathbb{R}^d)\) such that \(\inf_{x \in \mathbb{R}^d} V(x) > -\infty\) and there exists a compact set \(B\) and \(\varepsilon > 0\) satisfying

\[
\mathcal{I}(V)(x) \leq -\varepsilon, \quad \forall x \in B^c.
\]

It is trivial to check that if \(V\) satisfies the Lyapunov condition then we have a compact set \(K\) such that \(\int_{K^c} |V(z)| \frac{1}{|z|^{d+\alpha}} \, dz < \infty\).

**Proposition 2.** If there exists a \(\gamma \in (1, \alpha)\) such that \(\frac{b(x) \cdot x}{x^{\gamma}} \to -\infty\) as \(|x| \to \infty\) then operator \([4, 1]\) satisfies the Lyapunov stability condition.

**Proof.** Let us take a non-negative function \(f \in C^2(\mathbb{R}^d)\) such that \(f(x) = |x|^\gamma\) for \(|x| \geq 1\). For \(|x| \neq 0\) we compute the following:

\[
\left| \int_{\mathbb{R}^d} (|x + z|^\gamma - |x|^\gamma) \frac{k(x, z)}{|z|^{d+\alpha}} \, dz \right| = |x|^{\gamma - \alpha} \int_{\mathbb{R}^d} \left( |x| + z \right)^\gamma |z|^{d+\alpha} \, dz \left| k(x, |z|) \right|
\leq d_2 |x|^{\gamma - \alpha} \sup_{|y|=1} \int_{\mathbb{R}^d} |y + z|^\gamma - 1 \frac{1}{|z|^{d+\alpha}} \, dz.
\]

Now

\[
\sup_{|y|=1} \int_{\mathbb{R}^d} |y + z|^\gamma - 1 \frac{1}{|z|^{d+\alpha}} \, dz
= \sup_{|y|=1} \left[ \int_{\{|z| \leq \frac{1}{2} \}} |y - z|^\gamma - 1 \frac{1}{|z|^{d+\alpha}} \, dz + \int_{\{|z| > \frac{1}{2} \}} |y - z|^\gamma - 1 \frac{1}{|z|^{d+\alpha}} \, dz \right]
\leq \sup_{|y|=1} \left[ \int_{\{|z| \leq \frac{1}{2} \}} \kappa_{12} |z|^2 \frac{1}{|z|^{d+\alpha}} \, dz + \int_{\{|z| > \frac{1}{2} \}} |y - z|^\gamma - 1 \frac{1}{|z|^{d+\alpha}} \, dz \right]
\leq \kappa_{13} \sup_{|y|=1} \left[ 1 + \int_{\{|z| > \frac{1}{2} \}} |z|^\gamma \frac{1}{|z|^{d+\alpha}} \, dz \right] \leq C,
\]

for some positive constants \(\kappa_{12}, \kappa_{13}, C\). Hence combining the previous two displays we obtain

\[
\left| \int_{\mathbb{R}^d} (|x + z|^\gamma - |x|^\gamma) \frac{k(x, z)}{|z|^{d+\alpha}} \, dz \right| \leq C|x|^{\gamma - \alpha}.
\]
For $|x| \geq 2$ and $|x+h| \leq 1$ implies that $|h| \geq 1$. Therefore for $|x| \geq 2$, 
\[
\int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
= \int_{\{|x+z| \leq 1\}} (f(x+z) - |x+z|^\gamma) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz + \int_{\mathbb{R}^d} (|x+z|^\gamma - |x|^\gamma) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
\leq d_2 (\sup_{|y| \leq 1} f(y) + 1) \int_{\{|z| \geq 1\}} \frac{1}{|z|^{d+\alpha}} \, dz + \int_{\mathbb{R}^d} (|x+z|^\gamma - |x|^\gamma) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
\leq C_1 (1 + |x|^\alpha),
\]
for some constant $C_1$, where in the last line we use (5.2). Therefore from the assertion on $b$ it is clear that for any given $\varepsilon > 0$ we can find a compact set $B$ such that
\[
\mathcal{I} f(x) \leq -\varepsilon,
\]
for all $x \in B^c$. \hfill \Box

We say that a function $g : \mathbb{R}^d \to \mathbb{R}$ has at most linear growth if there exists a constant $L$ so that $|g(x)| \leq L (1 + |x|)$.

**Lemma 10.** Let $\{X_t\}$ be the Markov process associated to the generator (4.1) and the operator satisfies Condition (3). We also assume that $b$ has at most linear growth. Then for any $x \in B^c$ we have $\mathbb{E}_x [\tau(B^c)] \leq \frac{2}{\varepsilon} (\mathcal{V}(x) + \inf \mathcal{V}^-)$.

**Proof.** Define $B_R = \{|x| \leq R\}$. Now $R_0$ be a fixed large number. We choose a cut-off function $\chi$ that is 1 on $B_{R_1}$, $R_1 >> R_0$ and vanishes outside of $B_{R_1+1}$. Then $f := \chi \mathcal{V}$ is in $C^2_b(\mathbb{R}^d)$. For $x \in B_{R_0}$, we see that we can choose $R_1$ large enough so that
\[
|x+z| \geq R_1 \Rightarrow |z| \geq \frac{1}{2} R_1 \quad \text{and thus} \quad |x+z| \leq \frac{3}{2} |z|.
\]
Then for $|x| \leq R_0$,
\[
\left| \int_{\mathbb{R}^d} (f(x+z) - \mathcal{V}(x+z)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz \right| \leq 2 \int_{\{|x+z| \geq R_1\}} \left| \mathcal{V}(x+z) \frac{k(x,z)}{|z|^{d+\alpha}} \right| \, dz
\]
\[
\leq 2d_2 (\frac{\varepsilon}{2}) \frac{1}{|z|^{d+\alpha}} \int_{\{|x+z| \geq R_1\}} \, dz
\]
\[
\leq 2d_2 (\frac{\varepsilon}{2}) \frac{1}{|z|^{d+\alpha}} \int_{\{|z| \geq R_1\}} \, dz
\]
\[
\leq \frac{\varepsilon}{2},
\]
promised $R_1$ is chosen large enough. Hence for these choices of $R_1$ we have
\[
\mathcal{I} f(x) \leq -\frac{\varepsilon}{2} \quad \forall x \in B_{R_0} \setminus B.
\]
Denote $\tau_R = \tau(B^c) \wedge \tau(\bar{B}_R)$. Then applying Itô formula we have for $x \in B_{R_0} \setminus B$
\[
\mathbb{E}_x [f(X_{\tau_{R_0}}) - \mathcal{V}(x)] \leq -\frac{\varepsilon}{2} \mathbb{E}_x [\tau_{R_0}],
\]
implying that
\[ \mathbb{E}_x[\tau_{R_0}] \leq \frac{2}{\varepsilon} (\mathcal{V}(x) + [\inf \mathcal{V}^\cdot]). \] (5.3)

Now from Lemma [12] we see that \( \tau(\bar{B}_R) \to \infty \) as \( R \to \infty \) with probability 1. Hence applying Fatou’s lemma we get the result from (5.3). □

A Markov process is said be to positive recurrent if for any compact set \( G \) with positive Lebesgue measure \( \mathbb{E}_x[\tau(G^c)] < \infty \), for any \( x \in \mathbb{R}^d \).

**Theorem 6.** If the operator (4.1) satisfies Condition [2] and \( b \) has at most linear growth then the associated Markov process is positive recurrent.

**Proof.** First we note that if Lyapunov condition is satisfied w.r.t. a compact set \( B \), then it is also satisfied for any compact set containing \( B \). Hence we can assume that \( B \) is a ball centered at origin. Now we consider another ball \( B_1 \) with center at origin and containing \( B \). Now we define
\[ \tau := \inf \{ t \geq 0, X_t \notin \bar{B}_1 \}, \quad \hat{\tau} := \inf \{ t > \tau, X_t \in B \}. \]

Therefore for \( X_0 = x \in \bar{B}_1 \), \( \hat{\tau} \) denotes the return time to \( B \) after hitting \( \bar{B}_1^c \). Now we prove that
\[ \sup_{x \in \bar{B}} \mathbb{E}_x[\hat{\tau}] < \infty. \] (5.4)

From Lemma [10] we have already seen that \( \mathbb{E}_x[\tau(B^c)] \leq \frac{\varepsilon}{2} [\mathcal{V}(x) + (\inf \mathcal{V})^-] \) for \( x \in B^c \). From Lemma [2] we have \( \sup_{x \in B} \mathbb{E}_x[\tau] < \infty \). Let \( \mathbb{P}_\tau(x, \cdot) \) be the exit distribution of \( \{X_s\} \) that starts from \( x \in \bar{B} \). Hence to prove (5.4) it is enough to show that
\[ \sup_{x \in \bar{B}} \int_{B_1^c} \mathcal{V}(y) + (\inf \mathcal{V})^- \mathbb{P}_\tau(x, dy) < \infty. \]

Again \( \mathcal{V} \) being finite on compact sets it is enough if we can find a compact set \( K \) such that
\[ \sup_{x \in B} \int_{K^c} \mathcal{V}(y) + (\inf \mathcal{V})^- \mathbb{P}_\tau(x, dy) < \infty. \] (5.5)

To do this we choose \( R \) large enough so that
\[ \frac{|x - z|}{|z|} > \frac{1}{2} \quad \text{for} \quad |z| \geq R, \ x \in \bar{B}_1. \]

Define \( K = \{ z : |z| \leq R \} \). Then for any Borel set \( A \subset K^c \), we have from Proposition [11] that
\[
\mathbb{P}_x(X_{\tau \wedge t} \in A) = \mathbb{E}_x \left[ \sum_{s \leq \tau \wedge t} 1_{\{X_s \in B_1, X_s \in A\}} \right] \\
= \mathbb{E}_x \left[ \int_0^{\tau \wedge t} 1_{\{X_s \in B_1\}} \int_A \frac{k(X_s, z - X_s)}{|X_s - z|^{d+\alpha}} dz \, ds \right] \\
\leq d_2 d^{d+\alpha} \mathbb{E}_x \left[ \int_0^{\tau \wedge t} \int_A \frac{1}{|z|^{d+\alpha}} dz \, ds \right] \\
= d_2 d^{d+\alpha} \mathbb{E}_x[(\tau \wedge t) \mu(A)],
\]
Thus by strong Markov property we get
\[ P_\tau(x, A) \leq d_2 2^{d+\alpha} \sup_{x \in B} E_x[\tau] \mu(A). \]
Thus using the standard approximation argument we see that for any non-negative function \( g \) that vanishes outside of \( K^c \) we have
\[ \int_{K^c} g(y)P_\tau(x, dy) \leq \tilde{\kappa} \int_{K^c} g(y)\mu(dy), \]
for some constant \( \tilde{\kappa} \). This proves (5.5) since \( V \) is integrable on \( K^c \) w.r.t. \( \mu \) and \( \mu(K^c) < \infty \).

Now we prove that the Markov process is positive recurrent. We need to show that for any compact set \( G \) with positive Lebesgue measure, \( E_x[\tau(G^c)] < \infty \) for any \( x \in \mathbb{R}^d \). Given any such \( G \) and \( x \) we choose \( B \) and \( B_1 \) as above so that \( G \cup \{x\} \subset B \). Now we define a sequence of stopping times as follows:
\[
\tilde{\tau}_0 = 0 \\
\tilde{\tau}_{2n+1} = \inf\{t > \tilde{\tau}_{2n} : X_t \notin B_1\}, \\
\tilde{\tau}_{2n+2} = \inf\{t > \tilde{\tau}_{2n+1} : X_t \in B\}, \quad n = 0, 1, \ldots .
\]
Using strong Markov property we see that \( E_x[\tilde{\tau}_n] < \infty \) for all \( n \). Next using a similar argument as in Lemma 8 we can find a \( \delta > 0 \) such that
\[ \inf_{x \in B} P_x(\tilde{\tau}(G^c) < \tau(B_1)) > \delta. \]
Hence
\[ p := \sup_{x \in B} P_x(\tilde{\tau}(B_1) < \tau(G^c)) \leq 1 - \delta < 1. \]
Thus by strong Markov property we get
\[ P_x(\tau(G^c) > \tilde{\tau}_{2n}) \leq p P_x(\tau(G^c) > \tilde{\tau}_{2n-2}) \leq \cdots \leq p^n. \]
This implies \( P_x(\tau(G^c) < \infty) = 1 \). Thus
\[
E_x[\tau(G^c)] \leq \sum_{n=1}^{\infty} E_x[\tilde{\tau}_{2n}1_{\{\tilde{\tau}_{2n-2} < \tau(G^c) \leq \tilde{\tau}_{2n}\}}]
\]
\[ = \sum_{n=1}^{\infty} \sum_{l=1}^{n} E_x[(\tilde{\tau}_{2l} - \tilde{\tau}_{2l-2})1_{\{\tilde{\tau}_{2n-2} < \tau(G^c) \leq \tilde{\tau}_{2n}\}}]
\]
\[ = \sum_{l=1}^{\infty} E_x[(\tilde{\tau}_{2l} - \tilde{\tau}_{2l-2})1_{\{\tilde{\tau}_{2l-2} < \tau(G^c) \leq \tilde{\tau}_{2l}\}}]
\]
\[ \leq \sum_{l=1}^{\infty} pl^{-1} \sup_{x \in B} E_x[\tilde{\tau}_2]
\]
\[ = \sup_{x \in B} \frac{E_x[\tilde{\tau}_2]}{1 - p} < \infty. \]
This completes the proof.

**Theorem 7.** Let \( \{X_s\} \) be the Markov process associated to the generator \( \mathcal{L} \) and \( b \) has at most linear growth. Then \( \{X_s\} \) has an invariant measure.

**Proof.** The proof is based on Has’minkin’s construction. Consider two sets \( B \subset B_1 \) as in Theorem 5 and define \( \tau \), \( \tau \). We define a Markov process \( \hat{X} \) on \( B \) with transition kernel given by

\[
\hat{P}_x(dy) = \mathbb{P}_x(X_t \in dy).
\]

Let \( f \) be any bounded, non-negative measurable function on \( B_1 \). Define \( Q_f(x) = \mathbb{E}_x[f(X_t)] \). We show that \( Q_f \) is harmonic in \( B_1 \). To prove this it is enough to show that \( Q_f(x) = \mathbb{E}_x[f(X_{\tau})] \) for some bounded non-negative function \( f \). Define \( f(x) = \mathbb{E}_x[f(X_{\tau}(B_1))] \) for \( x \in B_1 \). Therefore using strong Markov property it is easy to check that \( Q_f(x) = \mathbb{E}_x[f(X_{\tau})] \). Therefore from Theorem 5 we have a positive constant \( C_H \), independent of \( f \), satisfying

\[
Q_f(x) \leq C_H Q_f(y) \quad \forall \ x, y \in B.
\]  

(5.6)

We note that \( Q_1 \equiv 1 \). Denote for \( A \subset \tilde{B}, \ Q_{1_A}(x) = Q(x, A) \). For any pair of probability measure \( \mu, \mu' \) on \( B \), we claim that

\[
\left\| \int_B (\mu(dx) - \mu'(dx))Q(x, \cdot) \right\|_{TV} \leq \frac{C_H - 1}{C_H} \|\mu - \mu'\|_{TV}.
\]  

(5.7)

This implies that the map \( \mu \to \int_B Q(x, \cdot)\mu(dx) \) is a contraction and hence it has a unique fixed point \( \hat{\mu}(A) = \int_B Q(x, A)\hat{\mu}(dx) \) for any Borel set \( A \subset \tilde{B} \). In fact, \( \hat{\mu} \) is the invariant measure of the Markov chain \( \hat{X} \). Now we prove the claim (5.7). Given any two probability measure \( \mu, \mu' \) on \( B \), we can find subsets \( C, D \subset \tilde{B} \) so that

\[
\left\| \int_B (\mu(dx) - \mu'(dx))Q(x, \cdot) \right\|_{TV} = 2 \int_B (\mu(dx) - \mu'(dx))Q(x, C),
\]

\[
\|\mu - \mu'\|_{TV} = 2(\mu - \mu'(D)).
\]

In fact, the restriction \( (\mu - \mu') \) to \( D \) is a non-negative measure and on \( D^c \) it is non-positive measure. Hence for any point \( x_0 \in \tilde{B} \),

\[
\left\| \int_B (\mu(dx) - \mu'(dx))Q(x, \cdot) \right\|_{TV}
\]

\[
= 2 \int_D (\mu(dx) - \mu'(dx))Q(x, C) + 2 \int_{D^c} (\mu(dx) - \mu'(dx))Q(x, C)
\]

\[
\leq 2 \int_D (\mu(dx) - \mu'(dx))Q(x, C) + 2 \frac{1}{C_H} \int_{D^c} (\mu(dx) - \mu'(dx))Q(x_0, C)
\]

\[
= 2 \int_D (\mu(dx) - \mu'(dx)) (Q(x, C) - Q(x_0, C)) + 2(1 - C_H^{-1})Q(x_0, C)(\mu - \mu'(D))
\]

\[
\leq \|\mu - \mu'\|_{TV} \left( \sup_{x \in B} Q(x, C) - Q(x_0, C) \right) + (1 - C_H^{-1})\|\mu - \mu'\|_{TV}.
\]
Now to complete the proof of (5.7) we observe that we can choose $x_0$ to satisfy $\sup_{x \in B} Q(x, C) \leq Q(x_0, C) + \varepsilon$ for any $\varepsilon > 0$ small enough.

Now we define a probability measure $\nu$ on $\mathbb{R}^d$ as follows.

$$
\int_{\mathbb{R}^d} f(x) \nu(dx) = \int_B E_x [\int_0^{\tau_x} f(X_s) ds] \tilde{\mu}(dx), \quad f \in C_b(\mathbb{R}^d).
$$

It is straightforward to check that $\nu$ is an invariant measure of $\{X_s\}$ (see for example, [3, Theorem 2.6.9]).

**Remark 2.** The above argument proves the existence of an invariant measure. If $k(\cdot, \cdot) = 1$ and the drift $b$ belongs to certain Kato class, in particular bounded, (12) then the transition probability has density and therefore $\nu$ has a density. We also refer to [19] where the existence of a transition density is proved for non-constant $k(\cdot, \cdot)$. Also it is easy to show that there exists a unique invariant measure with density.

The following result provides sufficient conditions for the existence of moments of the invariant measure.

**Proposition 3.** Let $\mathcal{V}$ be a non-negative $C^2$ function, $\mathcal{V}(x) \to \infty$ as $|x| \to \infty$, such that $\mathcal{I}\mathcal{V} \leq 0$ outside a compact set $K$ where $\mathcal{I}$ is given by (4.1). Let $\nu$ be an invariant measure of the Markov process associated to the generator $\mathcal{I}$. Then

$$
\int_{\mathbb{R}^d} |\mathcal{I}\mathcal{V}(x)| \nu(dx) \leq 2 \int_K |\mathcal{I}\mathcal{V}(x)| \nu(dx).
$$

**Proof.** Let $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth non-decreasing, concave, function such that

$$
\varphi(x) = \begin{cases} 
  x & \text{for } x \leq n, \\
  n + 1 & \text{for } x \geq n + 1.
\end{cases}
$$

We also assume that $\varphi(x) \leq |x|$ for all $x \in \mathbb{R}_+$. Then $\mathcal{V}_n(x) := \varphi(\mathcal{V}(x))$ is in $C^2_b(\mathbb{R}^d)$. Then $\mathcal{I}\mathcal{V}_n(x) \to \mathcal{I}\mathcal{V}(x)$ as $n \to \infty$. Now $\nu$ being an invariant measure we have

$$
\int_{\mathbb{R}^d} \mathcal{I}\mathcal{V}_n(x) \nu(dx) = 0. \quad (5.8)
$$

$\varphi$ being a concave function we have $\varphi_n(z) \leq \varphi_n(x) + (z - x) \cdot \varphi_n'(x)$ for all $x, z \in \mathbb{R}_+$. Hence

$$
\mathcal{I}\mathcal{V}_n(x) = \int_{\mathbb{R}^d} (\mathcal{V}_n(x + z) - \mathcal{V}_n(x)) \frac{k(x, z)}{|z|^{d+\alpha}} dz + \varphi_n'(\mathcal{V}(x)) b(x) \cdot \nabla \mathcal{V}(x)
$$

$$
\leq \int_{\mathbb{R}^d} (\mathcal{V}(x + z) - \mathcal{V}(x)) \varphi_n'(\mathcal{V}(x)) \frac{k(x, z)}{|z|^{d+\alpha}} dz + \varphi_n'(\mathcal{V}(x)) b(x) \cdot \nabla \mathcal{V}(x)
$$

$$
= \varphi_n'(\mathcal{V}(x)) \mathcal{I}\mathcal{V}(x),
$$
which is negative for $x \in K^c$. Therefore using (5.8) we get
\[
\int_{\mathbb{R}^d} |IV_n(x)| \nu(dx) = \int_K |IV_n(x)| \nu(dx) - \int_{K^c} IV_n(x) \nu(dx)
= \int_K |IV_n(x)| \nu(dx) + \int_K IV_n(x) \nu(dx)
\leq 2 \int_K |IV_n(x)| \nu(dx).
\]
Now letting $n \to \infty$ and using Fatou’s lemma on the l.h.s. and dominated convergence theorem on the r.h.s. we obtain the result. □

**Corollary 2.** Let $V$ be a non-negative $C^2$ function, $V(x) \to \infty$ as $|x| \to \infty$, satisfying $IV(x) \leq -h(x)$ for $x \in K^c$, $K$ compact and $h$ is a non-negative function. If $\nu$ is an invariant measure then $\int_{\mathbb{R}^d} h(x) \nu(dx) < \infty$.

It is quite evident from the Theorem 7 that Harnack inequality plays a crucial role in the analysis. Therefore one might wish to get stability from an asymmetric kernel and deploy the Harnack inequality from [9] to prove a similar result as in Theorem 7.

**Theorem 8.** Let $\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a non-negative measurable function satisfying the following property: there exists constants $q_i, i = 1, 2$, and $1 < \alpha' < \alpha < 2$, such that

- For all $x \in \mathbb{R}^d$, $1_{\{|z| > 1\}} \pi(x, z) \leq \frac{q_1}{|z|^{d+\alpha'}}$;
- For all $x, y, z$,
  \[
  \pi(x, z - x) \leq q_2 \pi(y, y - z), \quad \text{whenever} \quad |z - x| \wedge |z - y| \geq 1, \ |x - y| \leq 1;
  \]
- For any $R > 0$ we have $q_R > 0$ such that for $|z| \leq R$ and $x \in \mathbb{R}^d$
  \[
  \frac{q_R^{-1}}{|z|^{d+\alpha}} \leq \pi(x, z) \leq \frac{q_R}{|z|^{d+\alpha}};
  \]
- For any $R > 0$ we have $R_1, \beta \in (1, 2)$ and $q_\beta > 0$ such that for any $|x| \leq R, \ |z| \geq R_1$
  \[
  \frac{q_\beta^{-1}}{|z|^{d+\beta}} \leq \pi(x, z) \leq \frac{q_\beta}{|z|^{d+\beta}};
  \]
- There exists $V \in C^2(\mathbb{R}^d)$ that is bounded from below and
  \[
  \int_{\mathbb{R}^d} \left(V(x + z) - V(x) - \nabla V(x) \cdot z 1_{\{|z| \leq 1\}}\right) \pi(x, z) dz < -\varepsilon,
  \]
  outside a compact set for some $\varepsilon > 0$.

Then the Markov process associated with the above kernel has an invariant measure.

The first three assumption establishes Harnack property of the Markov process from [9]. Then the proof of Theorem 8 follows using a similar argument as Theorem 7. Next we give an example of a kernel $n(\cdot, \cdot)$ that satisfies the conditions in Theorem 8.
Example 1. Let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be a non-negative smooth function such that

\[
\varphi(x) = \begin{cases} 
1 & \text{for } |x| \leq \frac{1}{2}, \\
0 & \text{for } |x| \geq 1.
\end{cases}
\]

Define for \( 1 < \alpha' < \beta' < \alpha < 2 \),

\[
\gamma(x, z) := \varphi \left( 2 \frac{x + z}{1 + |x|} \right) (1 - \varphi(4x))(\alpha' - \beta'),
\]

and let

\[
\pi_1(x, z) := \frac{1}{|z|^{d + \beta' + \gamma(x, z)}}, \\
\pi(x, z) := \frac{1}{|z|^{d + \alpha} + \pi_1(x, z)}.
\]

We prove that \( \pi \) satisfies the conditions of Theorem 8. Let us also comment that there exists a unique solution to the martingale problem \([22, 23]\) corresponding to kernel \( \pi \). We only show that the second and the last conditions hold.

Note that \( \alpha' - \beta' \leq \gamma(x, z) \leq 0 \) for all \( x, z \). Take \( x, y, z \in \mathbb{R}^d \) such that \( |x - z| \wedge |y - z| \geq 1, |x - y| \leq 1 \). Then \( |z - y| \leq 1 + |z - x| \). It is easy to see that

\[
\pi_1(x, z - x) \leq \left( 1 + \frac{1}{|z - x|} \right)^{d + \beta' + \gamma(x, x - z)} \frac{1}{|z - y|^{d + \beta' + \gamma(x, x - z)}} \leq 2^{d + \beta'} \frac{1}{|z - y|^{d + \beta' + \gamma(y, z - y)}} |z - y|^{-\beta'(x, z - x) + \gamma(y, z - y)}.
\]

Hence it is enough to show that \( |z - y|^{-\gamma(x, z - x) + \gamma(y, z - y)} < \varrho \) where \( \varrho \) is independent of \( x, y \) and \( z \). We note that if \( |x| \leq 2 \Rightarrow |y| \leq 3 \) then for \( |z| \geq 4 \) we have \( \gamma(x, z - x) = 0 = \gamma(y, z - y) \). Therefore for \( |x| \leq 2 \), \( |z - y|^{-\gamma(x, z - x) + \gamma(y, z - y)} \leq 7^{\beta' - \alpha'} \). So we consider \( |x| \geq 2 \) and so \( |y| \geq 1 \). Since we only need to consider the case where \( \gamma(x, z - x) \neq \gamma(y, z - y) \) we consider \( z \in \mathbb{R}^d \) such that \( |z| \leq 2(1 + |x|) \). Then

\[
\log(|z - y|)(-\gamma(x, z - x) + \gamma(y, z - y)) \leq \log(3(1 + |x|))|\varphi|_{\infty} \frac{2|z|(|\beta' - \alpha'|)}{(1 + |x|)(1 + |y|)} \leq \log(3(1 + |x|))|\varphi|_{\infty} \frac{4(1 + |x|)(|\beta' - \alpha'|)}{(1 + |x|)|x|} \leq \varrho.
\]

Now we prove the Lyapunov property. In view of Proposition 2 it is enough to show that for \( |x| \geq 4 \), we have \( c_1, c_2 \) such that

\[
\int_{\mathbb{R}^d} (|x + z|^\theta - |x|^\theta) \pi_1(x, z) \, dz \leq c_1 - c_2 |x|^{\theta - \alpha'},
\]

for some \( \theta \in (\alpha', \beta') \). Note that the gradient term disappears as the \( \pi_1 \) is symmetric for \( |x| \geq 4, |z| \leq 1 \). To do this we first note that for \( |x| \geq 1 \), \( \pi_1(x, z) = \varphi(2 \frac{x + z}{1 + |x|}) (\alpha' - \beta') \).
Denote $A_1 = \{ z : \frac{x+|z|}{1+|x|} \leq \frac{1}{4} \}$ and $A_2 = \{ z : \frac{1}{4} < \frac{x+|z|}{1+|x|} \leq \frac{1}{2} \}$. Then

$$
\int_{\mathbb{R}^d} \left( |x+z|^\theta - |x|^\theta \right) \pi_1(x,z) \, dz = |x|^{-\beta'+\theta} \int_{\mathbb{R}^d} \left( \left| \frac{x}{|x|} + z \right|^\theta - 1 \right) \frac{|x|^{-\gamma(x,|x|)}}{|z|^{d+\beta'+\gamma(x,|x|)}} \, dz
$$

$$
= |x|^{-\beta'+\theta} \int_{A_1} \left( \left| \frac{x}{|x|} + z \right|^\theta - 1 \right) \frac{|x|^{-\alpha'+\beta'}}{|z|^{d+\alpha'}} \, dz
$$

$$
+ |x|^{-\beta'+\theta} \int_{A_2} \left( \left| \frac{x}{|x|} + z \right|^\theta - 1 \right) \frac{|x|^{-\alpha'+\beta'}}{|z|^{d+\beta'+\gamma(x,|x|)}} \, dz
$$

$$
+ |x|^{-\beta'+\theta} \int_{(A_1 \cup A_2)^c} \left( \left| \frac{x}{|x|} + z \right|^\theta - 1 \right) \frac{1}{|z|^{d+\beta'}} \, dz.
$$

Now note that $A_1 \cup A_2 \subset \{ z : \frac{|x|}{|x|} + z \leq 1 \}$ and $\{ z : \frac{|x|}{|x|} + z \leq \frac{1}{4} \} \subset A_1$. Thus

$$
\int_{\mathbb{R}^d} \left( |x+z|^\theta - |x|^\theta \right) n(x,z) \, dz \leq |x|^{\theta-\alpha'} \int_{\{|\frac{x}{|x|} + z\| \leq \frac{1}{4}\}} \left( \left| \frac{x}{|x|} + z \right|^\theta - 1 \right) \frac{1}{|z|^{d+\alpha'}} \, dz
$$

$$
+ |x|^{-\beta'+\theta} \int_{\mathbb{R}^d} \left| \frac{x}{|x|} + z \right|^\theta - 1 \frac{1}{|z|^{d+\beta'}} \, dz
$$

$$
\leq c_1 - c_2 |x|^{\theta-\alpha'}.
$$

\[ \square \]

**Proposition 4.** Let $B$ be any bounded open subset of $\mathbb{R}^d$ and $\{X_s\}$ be the Markov processes associated to (4.1) or $\pi$. Assume that for any compact set $K$ and open set $G$, $\sup_{x \in K} \mathbb{P}_x (\tau(G^c) > T) \to 0$ as $T \to 0$. Then for any invariant measure $\nu$ of $\{X_s\}$ we have $\nu(B) > 0$.

**Proof.** The idea of the proof relies on the fact that the process spends certain amount of time in each open set and it is positive recurrent. Let $\nu(B) = 0$. We assume that $B_{2r}(x_0) \subset B$ for some $r < 1$. Then by Lemma 3 (also [8, Proposition 3.1]) we have

$$
\sup_{x \in B_r(x_0)} \mathbb{P}_x (\tau(B_r(x)) \leq t) \leq \kappa t,
$$

for some constant $\kappa > 0$ and $t > 0$. Therefore we can find $t_0 > 0$ such that

$$
\inf_{x \in B_r(x_0)} \mathbb{P}_x (\tau(B_r(x)) \geq t_0) \geq \frac{1}{2}.
$$

(5.10)
Let $K$ be compact set satisfying $\nu(K) > \frac{1}{4}$. Then assertion we have $T_0 > 0$ such that
\[
0 = \nu(B) \geq \frac{1}{T_0 + t_0} \int_0^{T_0 + t_0} \nu(dx) P(t, x; B_{2r}(x_0)) \, dt
\]
\[
= \frac{1}{T_0 + t_0} \int_{\mathbb{R}^d} \nu(dx) \mathbb{E}_x \left[ \int_0^{T_0 + t_0} 1_{B_{2r}(x_0)}(X_s) \, ds \right]
\]
\[
\geq \frac{1}{T_0 + t_0} \nu(K) \inf_{x \in K} \mathbb{P}_x(\tau(B_r^c(x_0) \leq T_0) \inf_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_{2r}(x_0)) \geq t_0) \, d\nu_t
\]
\[
\geq \frac{1}{T_0 + t_0} \nu(K) \inf_{x \in K} \mathbb{P}_x(\tau(B_r^c(x_0) \leq T_0) \inf_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_r(x)) \geq t_0) \, d\nu_t
\]
\[
\geq \frac{t_0}{T_0 + t_0} \frac{\nu(K)}{4} > 0.
\]
But this is a contradiction. Hence $\nu(B) > 0$. \qed

The rest of the section is devoted to characterize the mean hitting time of the process for certain class of kernel. For the following theorem we assume that the corresponding Markov process does not have finite time explosion.

**Theorem 9.** Let $\pi_1$ be the kernel given by (3.20). We assume that $k(\cdot, z)$ is locally Hölder continuous uniformly w.r.t. $z$ and $\pi_1$ satisfies (5.9) with $\beta \leq \alpha$. Let $B$ be a ball. Furthermore assume that for any compact set $D$, $\sup_{x \in D} \mathbb{E}_x[\tau(B^c)] < \infty$. Then $u(x) := \mathbb{E}_x[\tau(B^c)]$ is a viscosity solution to
\[
\tilde{I}u = -1 \quad \text{in} \quad B^c,
\]
\[
u = 0 \quad \text{in} \quad B,
\]
where $\tilde{I}$ denotes the integro-differential operator w.r.t. the kernel $\pi_1$.

Define $B_n = \{ x \in \mathbb{R}^d : \text{dist}(x, B) \leq n \}$. Then the same argument as in Theorem 4 gives that $u_n := \mathbb{E}_x[\tau(B_n \cap B)]$ is a viscosity solution of
\[
\tilde{I}u_n = -1 \quad \text{in} \quad B_n \cap B,
\]
\[
u = 0 \quad \text{in} \quad (B_n \cap B)^c.
\]

**Lemma 11.** Let the conditions of Theorem 9 hold. Then for any compact sets $D, K$ such that $B_1(0) \subset K$ we have
\[
\sup_{x \in D} \int_{K^c} \mathbb{E}_z[\tau(B^c)] \pi_1(x, z) \, dz < \infty.
\]
Proof. We denote $\tau = \tau(B^c)$. Fix $n_0$ large enough so that $B \subset B_{n_0}$. Denote $\tau_{n_0} = \tau(B_{n_0} \cap B)$. We know that for any $x \in B_{n_0} \cap B$ we have

$$E_x[1_{\{\tau_{n_0} \leq \tau\}} E_{X_{\tau_{n_0}}} \{\bar{\tau}\}] \leq E_x[\bar{\tau}].$$

(5.12)

By (5.9) we see that we have $R_1 > 0, \beta \in (1, 2)$ such that

$$q_\beta^{-1} \frac{1}{|z|^{d+\beta}} \leq \pi_1(x, z - x), \quad \pi_1(x, z) \leq \frac{q_\beta}{|z|^{d+\beta}},$$

(5.13)

for some $q_\beta > 0$ and all $x \in B_{n_0}, |z| \geq R_1$. Define $K_1 = \{x : |x| \leq R_1\}$. Therefore using Proposition 1 we get for any Borel set $A \subset K_1$, that

$$P_x(X_{\tau \wedge \tau_{n_0}} \in A) = E_x \left[ \sum_{s \leq \bar{\tau} \wedge \tau_{n_0}} 1_{\{X_s \in B_{n_0} \cap B, X_s \in A\}} \right]$$

$$= E_x \left[ \int_0^{\bar{\tau} \wedge \tau_{n_0}} 1_{\{X_s \in B_{n_0} \cap B\}} \int_A \pi_1(X_s, z - X_s) \, dz \, ds \right]$$

$$\geq q_\beta^{-1} E_x \left[ \int_0^{\bar{\tau} \wedge \tau_{n_0}} \int_A \frac{1}{|z|^{d+\beta}} \, dz \, ds \right]$$

$$= q_\beta^{-2} E_x[\bar{\tau} \wedge \tau_{n_0} \wedge t] \int_A \pi_1(x, z) \, dz.$$

Now let $t \to \infty$ to obtain

$$P_x(X_{\tau \wedge \tau_{n_0}} \in A) \geq q_\beta^{-2} E_x[\bar{\tau} \wedge \tau_{n_0}] \int_A \pi_1(x, z) \, dz.$$  

(5.14)

By Lemma 3 we get $E_x[\bar{\tau} \wedge \tau_{n_0}] > 0$. Hence combining (5.12) and (5.14) we see that there exists a positive constant $d_3$ such that

$$\int_{K^c} E_x[\tau(B^c)] \pi_1(x, z) \, dz \leq d_3 E_x[1_{\{X_{\tau \wedge \tau_{n_0}} \in K_1\}} E_{X_{\tau_{n_0}}} \{\bar{\tau}\}] \leq d_3 E_x[\bar{\tau}].$$

This completes the proof together with the fact that $E_x[\bar{\tau}]$ is locally bounded. \qed

Proof of Theorem 9: Consider the sequence of solution $\{u_n\}$ defined in (5.11). First we note that $u_n(x) \leq E_x[\tau(B^c)]$ for all $x$ and thus the family $\{u_n\}$ is uniformly locally bounded. By the condition in (5.20) we can modify the equation (5.11) to an equation with fractional laplacian as mentioned in Theorem 1. Then using Lemma 11 we get uniform interior regularity [44, Theorem 26] and uniform boundary regularity (see the proof of Proposition 1.1 in [29] and also [26]) near $\partial B$. Hence the family $\{u_n\}$ is equi-continuous over any compact subset of $B^c$. Therefore we obtain the result by passing to the limit. \qed

6. Appendix

Lemma 12. Let $\{X_s\}$ be the Markov process associated to the generator (4.1) and the drift $b$ has at most linear growth. Then for any $T > 0$, $P_x(\sup_{s \in [0, T]} |X_s| < \infty) = 1$. 
Proof. We select a non-negative, radial function $f \in C^2(\mathbb{R}^d)$ with the following properties:

$$f(x) = |x|^{1/2} \text{ for } |x| \geq 1, \quad \text{and} \quad |f(x)| \leq |x|^{1/2} \text{ for } |x| > 1.$$ 

Now we consider a collection of $C^2_b(\mathbb{R}^d)$ functions $\{\chi_R\}$ such that $\chi_R$ is non-negative and non-decreasing and for $R > 1$,

$$\chi_R(x) = \chi_R(|x|) = \begin{cases} f(x) & \text{for } |x| \leq R, \\ \sqrt{R+1} & \text{for } |x| > R + 1. \end{cases}$$

In fact, we can choose such sequence with the property that $\chi_R(x) \leq f(x)$ and $|\chi'_R| \leq |\chi''_R|(x) \leq D_1 \frac{1}{\sqrt{|x|}}$ for some constant $D_1$ and $|x| \geq 1$. It is easy to see that $\chi_R \uparrow f$ as $R \to \infty$. Also for $|x| \geq 1$,

$$|x|\chi'(|x|) \leq |x|D_1 \frac{1}{\sqrt{|x|}} \leq \sqrt{2}D_1 \chi_R(x).$$

Therefore if $b$ has at most linear growth then

$$|b(x) \cdot \nabla \chi_R(x)| \leq D_2 [1 + \chi_R(x)], \quad (6.1)$$

for some constant $D_2$. Now we show that there exists $\bar{\kappa} > 0$, independent of $R$, so that

$$\sup_{|x| \leq R+1} \left| \int_{\mathbb{R}^d} (\chi_R(x+z) - \chi_R(x))^2 \frac{k(x,z)}{|z|^{d+\alpha}} \, dz \right| \leq \bar{\kappa}. \quad (6.2)$$

In what follows $M_1, M_2, \ldots$ are constants not depending on $R$. If $|x| \leq 1$, we have

$$\int_{\mathbb{R}^d} (\chi_R(x+z) - \chi_R(x))^2 \frac{k(x,z)}{|z|^{d+\alpha}} \, dz \leq d_2 D_2^2 \int_{\{|z| \leq 2\}} |z|^2 \frac{1}{|z|^{d+\alpha}} \, dz + 2d_2 \int_{\{|z| > 2\}} |z| \frac{1}{|z|^{d+\alpha}} \, dz$$

$$+ 4d_2 \int_{\{|z| > 2\}} |x| \frac{1}{|z|^{d+\alpha}} \, dz$$

$$\leq M_1.$$
Let $1 \leq |x| \leq R+1$. Then $(\chi_{R}(x+z) - \chi_{R}(x))^2 \leq (f(x+z) - \chi_{R}(x))^2$. Thus we have
\[
\int_{R^d} (\chi_{R}(x+z) - \chi_{R}(x))^2 \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
\leq d_2 D_2 \left( \int_{\{|z| \leq 2\}} \frac{1}{|z|^{d+\alpha}} \, dz + 2 \int_{\{|z| > 2\}} (f(x+z) - \sqrt{|x|}) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz \right)
\]
\[
+ 2 \int_{\{|z| > 2\}} (\sqrt{|x|} - \chi_{R}(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
\leq d_2 D_2 \left( \int_{\{|z| \leq 2\}} \frac{1}{|z|^{d+\alpha}} \, dz + 4 \int_{\{|z| > 2\}} (\sqrt{|x+z|} - \sqrt{|x|}) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz \right)
\]
\[
+ 4 \int_{\{|z| > 2\}} (\sqrt{|x+z|} - f(x+z)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
+ 2 \int_{\{|z| > 2\}} (\sqrt{|x|} - \chi_{R}(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz
\]
\[
\leq M_2,
\]
for some constant $M_2$, where in the third inequality is computed using the fact that $|\sqrt{|x|} - \chi_{R}(x)| \leq 1$, $\sqrt{|x+z|} - f(x+z)| \leq 1$, and for the second term we follow the computation in Proposition 2. This proves (6.2). By $t_R$ we denote the exists time of $\{X_s\}$ from $B_R(0)$. Let $M_R(t)$ be the martingale that we obtain applying Itô formula on $\chi_{R}(X_s)$. Then
\[
\chi_{R}(X_s \wedge t_R) = \chi_{R}(x) + \int_0^{s \wedge t_R} \mathcal{L}(\chi_{R}(X_u)) \, du + M_R(s \wedge t_R),
\]
and so applying (6.1) and (6.2) we have a constant $M_3$ such that for $s \leq T$,
\[
\chi_{R}(X_{s \wedge t_R}) \leq \sqrt{|x|} + M_3 \left( T + \int_0^{s \wedge t_R} \chi_{R}(X_u) \, du \right) + \sup_{s \leq T} |M_R(s \wedge t_R)|. \tag{6.3}
\]
Hence applying Gronwall’s inequality we have from (6.3) that
\[
\sup_{s \leq T \wedge t_R} \chi_{R}(X_s) \leq M_4 \left( 1 + \sup_{s \leq T} |M_R(s \wedge t_R)| \right), \tag{6.4}
\]
for some constant not depending on $R$. Now by Doob’s martingale theorem
\[
\mathbb{E}_x \left[ \sup_{s \leq T} |M_R(s \wedge t_R)| \right] \leq 2 \left( \mathbb{E}_x \left[ (M_R(T \wedge t_R))^2 \right] \right)^{1/2}
\]
\[
= 2 \left( \mathbb{E}_x \left[ \sum_{s \leq T \wedge t_R} (\Delta M_R(s))^2 \right] \right)^{1/2}
\]
\[
= 2 \left( \mathbb{E}_x \left[ \int_0^{T \wedge t_R} \int_{\mathbb{R}^d} (\chi_{R}(X_s + z) - \chi_{R}(X_s)) \frac{k(X_s, z)}{|z|^{d+\alpha}} \, dz \, ds \right] \right)^{1/2}
\]
\[
\leq 2\sqrt{\kappa T}.
\]
Thus from (6.4) we have a constant \( M_5 \), not depending on \( R \), such that
\[
\mathbb{E}_x \left[ \sup_{s \leq T \wedge \tau_R} \chi_R(X_s) \right] \leq M_5,
\]
and thus
\[
\limsup_{R \to \infty} \mathbb{E}_x \left[ \sup_{s \leq T \wedge \tau_R} \chi_R(X_s) \right] \leq M_5.
\] (6.5)

Suppose there exist \( \varepsilon > 0 \) such that \( \mathbb{P}_x(\sup_{s \leq T} |X_s| = \infty) > \varepsilon \). Then \( \mathbb{P}_x(\sup_{s \leq T} |X_s| \geq R) > \varepsilon \) for all \( R \), implying \( \mathbb{P}_x(\sup_{s \leq T \wedge \tau_R} |X_s| \geq R) > \varepsilon \) for all \( R \). Then from (6.5) we get
\[
\limsup_{R \to \infty} \varepsilon \sqrt{R} \leq M_5,
\]
which is a contradiction. \( \square \)

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**Department of Electrical and Computer Engineering, The University of Texas at Austin, 1 University Station, Austin, TX 78712**

*E-mail address: anupbiswas@utexas.edu*

**Department of Mathematics, The University of Texas at Austin, 1 University Station, Austin, TX 78712**

*E-mail address: caffarel@math.utexas.edu*