Degree of entanglement in a quantum measurement process

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We suggest a quantum measurement model in an ion trap which specifies the probability distribution of two, distinct internal ground states of a trapped four-level ion. The external degrees of motion of the four-level ion constitute the meter which, in turn, is coupled to the environment by engineered reservoirs. In a previous publication, a similar measurement model was employed to test decoherence effects on quantum nonlocality in phase space on the basis of coincidence measurements of the entangled system-meter scheme. Here, we study the effects of decoherence on the entanglement of formation characterized by the concurrence. The concurrence of the system enables to find the maximum possible violation of the Bell inequality. Surprisingly, this model gives illustrative insights into the question to what extent the Bell inequality can be considered as a measure of entanglement.

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I. INTRODUCTION

Entanglement constitutes the single most characteristic property that makes quantum mechanics distinct from any classical theory [1]. It has been in the center of interest since the early days of quantum theory, mainly due to its relation to nonlocality [2]. It was the discovery of the Bell inequalities [3] that opened the possibility to test nonlocality in laboratory experiments [4, 5]. Surprisingly, entanglement has found its application in a newly emerging area of research. It has been recognized that entanglement forms a fundamental resource for quantum information processing (QIP) [6, 7]. For application’s purposes, it became essential to quantify entanglement and, accordingly, a number of useful measures of the degree of entanglement have been introduced [8]. One of them, the entanglement of formation [9], is the subject of this paper. The entanglement of formation quantifies the resources needed to create a given entangled state.

In this paper we study the dynamics of the entanglement of formation in a quantum measurement model which is a slightly modified version of our model, proposed previously [10]. We are interested in how the entanglement of formation is created and lost during the course of the measurement. Environmentally induced decoherence prevents us from observing quantum effects in the macroscopic world [11, 12, 13, 14, 15, 16, 17]. However, we have shown that it is possible to greatly reduce the decoherence rate by assuming the environment to be a squeezed reservoir [10]. This enables us to observe violations of Bell-type inequalities in phase space even when the meter is already in the macroscopic domain. The entanglement of formation and, in particular, the closely related concurrence allows us to find the full time-dependence of the maximum possible violation of Bell-type inequalities. We will show how the dynamics of the concurrence is affected by coupling to a squeezed reservoir and present an analytical study of the dynamics of the entanglement of formation of decoherent, non-orthogonal and mixed qubits exposed to an open environment.

We also address the question as to what extent violations of the Bell inequalities represent a good measure of the degree of entanglement. We find that not only the degree of mixture of bipartite qubits plays a role but also the degree of overlap between them. In particular, the meter generates a non-orthogonal and mixed qubit which can be rewritten as a superposition of orthogonal qubit states. Surprisingly, in this context, violations of the Bell inequalities can be stronger in a system which, compared to other systems, has a larger degree of mixture but the same amount of entanglement of formation. We explain this apparently contradictory behavior on the basis of the measurement model.

The outline of the paper is as follows: First, in Sec. II, we present the measurement model which consists of a four-level atom in a Paul trap coupled to engineered reservoirs. Then, in Sec. III, we derive the concurrence and the entanglement of formation and discuss the time evolution of these quantities for different squeezing parameters of the environment. We also analyze the role of the Bell inequality as a measure of entanglement and explain some counterintuitive findings on the basis of indistinguishability between qubit states. In Sec. IV we conclude with some discussion.
II. MEASUREMENT MODEL IN A PAUL TRAP

The measurement model consists of a four-level atom harmonically bound in a three-dimensional trap where it oscillates along the three principal axes with frequencies \( \nu_1 = \nu_2 = \nu_3 = \nu \). The level structure of the four-level atoms with its relevant polarization-sensitive dipole transitions is shown in Fig. 1 [18]. We determine the population distribution of the internal atomic ground-states which, in turn, form the system to be measured. The external or motional degrees of freedom establish the meter and its coupling to the environment is accomplished by engineered reservoirs [19, 20]. The trapped four-level atom is driven in a Raman configuration with two classical, \( \sigma \)-polarized laser fields of frequencies \( \omega_1 \) and \( \omega_2 = \omega_1 + \nu = \omega_0 \). The laser frequencies are off-resonant with respect to the electronic \( \sigma \)-transitions \(|1\rangle \leftrightarrow |4\rangle\) and \(|2\rangle \leftrightarrow |3\rangle\) by a detuning of \( \Delta \) \((\Delta \ll \omega_1, \omega_2)\), see Fig. 2. A similar “system-meter” interaction was investigated by Wallentowitz and Vogel [21], in order to realize the quantum-mechanical counterpart of nonlinear optical phenomena in the motional (mechanical) degrees of freedom. In contrast to our four-level system, however, theirs consists of a two-level atom where polarization dependent features do not play a role.

With an appropriate geometry of the lasers the motion can only be affected in \( z \)-direction assuming that both laser fields are traveling in \( z \)-direction (see Fig. 3). Since the lasers are strongly detuned from the atomic transition frequencies the atom stays in its ground states during the interaction. Further, if we assume the resolved sideband limit, we are able to influence the motional quantum state of the atom in a controlled manner. In particular, the system-meter interaction Hamiltonian, \( H_{SM} \), for the given geometry and frequencies of the lasers in the vibrational rotating-wave approximation is given as [21]

\[
H_{SM} = \frac{1}{2}\hat{\sigma}_z(i\hbar\Omega\eta\hat{a} - i\hbar\Omega^*\hat{a}^\dagger) .
\]
Here,
\begin{align}
\Omega &= \frac{\Omega_1 \Omega_2}{2 \Delta} , \\
\Delta &= \omega_{23} - \omega_0 , \\
\Omega_i &= \frac{2dE_i}{h} , \\
\hat{\sigma}_z &= |2\rangle \langle 2| - |1\rangle \langle 1| + |4\rangle \langle 4| - |3\rangle \langle 3| ,
\end{align}
where \( \eta \) is the Lamb-Dicke parameter, \( d \) is the dipole moment which we assume to be the same for all possible dipole transitions in the four-level atom, and \( E_i \) are the electric field amplitudes of the applied lasers. We have assumed a small Lamb-Dicke parameter, \( \eta \ll 1 \), in the interaction Hamiltonian which allows us to neglect nonlinear terms in the motional operators \( \hat{a} \) and \( \hat{a}^\dagger \). The phase of \( \Omega = \Omega e^{i \theta} \) can be adjusted by the phase difference of the two lasers. From now on, we assume \( \phi = \pi \) which leads to the following interaction Hamiltonian
\begin{equation}
H_{\text{SM}} = \frac{i}{2} \hbar \hat{\sigma}_z (\eta \Omega \hat{a} - \eta |\Omega\rangle \langle \hat{a}|).
\end{equation}
This type of system-meter Hamiltonian is of the same structure as the interaction Hamiltonian between a four-level atom and a cavity field which we recently employed to study decoherence effects on the visibility of interference fringes \(^{(22)}\) and on nonlocality in phase space \(^{(10)}\).

The replacement of the optical meter by the mechanical one in the system-meter interaction has considerable advantages in the realization of the measurement model. In particular, the experimental progress in reservoir engineering for trapped ions \(^{(11, 20)}\) makes this measurement model a feasible testing ground of decoherence effects in engineered reservoirs. Further, the interaction Hamiltonian \(^{(1)}\) can be easily modified to include nonlinear terms in the motional operators \( \hat{a} \) and \( \hat{a}^\dagger \) by appropriate settings of the frequency and geometry of the applied lasers. The nonlinear interaction of parametric type together with specific reservoirs, such as dissipative two-phonon processes, makes it possible to generate nonclassical, macroscopic motional states in dissipative environments (Gilles and Knight \(^{(23)}\)).

Here, however, we do not consider nonlinear effects in the system-meter interaction. The environment, as in Ref. \(^{(10)}\), is taken to be a squeezed reservoir which can be engineered according to Refs. \(^{(19, 20)}\). The master equation for the system-meter density operator in the Markov approximation is given as
\begin{equation}
\frac{\partial}{\partial t} \rho = \frac{1}{i \hbar} [H_{\text{SM}}, \rho] + \frac{\gamma}{2} \left\{ (N + 1)[2b \rho b^\dagger - b^\dagger b \rho - \rho b b^\dagger] + N[2b^\dagger \rho b - bb^\dagger \rho - \rho b b^\dagger] + M[2b^\dagger \rho b - b^\dagger b \rho - \rho b b^\dagger] + M^*[2b \rho b^\dagger - b \rho b^\dagger] \right\} ,
\end{equation}
where \( N \) is the number of photons and \( M = -|M|e^{2i \theta} \) is the squeezing parameter which characterizes the degree of phase-dependent correlations, with the squeezing phase \( \theta \), in the squeezed reservoir. The master equation can be analytically solved with a characteristic function approach, leading to the following time evolution of the system-meter density operator \(^{(10)}\)
\begin{equation}
\rho(t) = \sum_{n,m=1}^{2} \rho_{nm} e^{-i \gamma_{nm}(t)} |n\rangle \langle m| \otimes \frac{\langle \hat{\alpha}_n(t), \varepsilon | \hat{\alpha}_m(t), \varepsilon \rangle}{\langle \hat{\alpha}_m(t), \varepsilon | \hat{\alpha}_n(t), \varepsilon \rangle}.
\end{equation}
The summation is over the internal atomic ground states, \( |1\rangle \) and \( |2\rangle \), and \( \rho_{nm} \) are the initial atomic density matrix elements which we assume to be given as \( \rho_{nm} = 1/2 \) for all \( n, m \). The amplitudes of the ensuing squeezed coherent states, \( |\alpha, \varepsilon\rangle = \exp\left(\frac{1}{2} \varepsilon \hat{a}^\dagger - \frac{1}{2} \varepsilon \hat{a}\right) |\alpha\rangle \), are given by
\begin{equation}
\hat{\alpha}_n(t) = \frac{|\Omega|\eta(-1)^n}{\gamma} \left[ \cosh(r) - \sinh(r)e^{2i \theta} \right] \left( 1 - e^{-\frac{\gamma t}{2}} \right) .
\end{equation}
Here \( \varepsilon = re^{2i \theta} \) with the squeezing parameter \( r \) and the squeezing phase \( \theta \) defined in the usual way. These states form the pointer states of the meter.

The squeezed coherent states \( |\tilde{\alpha}_n(t), \varepsilon\rangle \), \( n = 1, 2 \), follow from first displacing the vacuum by the amplitude \( ((-1)^n |\Omega|\eta/\gamma)(1 - e^{-\frac{\gamma t}{2}}) \) and then squeezing the resulting coherent state. The exponent \( \Gamma_{nm}^{sq}(t) \) is responsible for the decoherence and is given as
\begin{equation}
\Gamma_{nm}^{sq}(t) = \left\{ \frac{|(-1)^n - (-1)^m|^2 |\Omega| \eta}{\gamma^2} \left[ 1 + 2 \left[ \sinh^2(r) - \cosh(r) \sinh(r) \cos(2\theta) \right] \right] \left( 1 - e^{-\frac{\gamma t}{2}} \right) \right\} .
\end{equation}
FIG. 3: Comparison of the decoherence, \( \exp[\Gamma_{\text{sq,vac}}(t)] \), in a squeezed reservoir (sq) to that in an ordinary vacuum (vac) (full line) vs. the dimensionless time, \( \gamma t/2 \). The squeezing parameters are given by \( r = 2 \) and \( \theta = 0 \) (dashed line) and \( r = 3.5 \) and \( \theta = 0 \) (dotted line). The amplitudes of the corresponding squeezed states (see text) are given by \( \pm \alpha_0 = 100 \) respectively.

In Ref. [10] we demonstrated how to reduce the decoherence rate of the measurement apparatus with the help of the squeezed reservoir by adjusting the squeezing phase to \( \theta = 0 \) and increasing the squeezing parameter \( r \). This is displayed in Fig. 3 where the amplitude of the ensuing squeezed coherent state \( |\tilde{\alpha}, \varepsilon \rangle \) is given as \( \alpha_0 = |\Omega| \eta/\gamma = 100 \). The amplitude can be engineered by suitable settings of the laser strengths and the engineered reservoir damping constant \( \gamma \) keeping in mind that the constraints \( \eta \ll 1 \) and \( \Omega_i/2\Delta \ll 1 \) (with \( i = 1, 2 \)) must be satisfied. In spite of these restrictions, it is possible to achieve large amplitudes, \( \Omega \eta/\gamma \gg 1 \). As a result of this highly reduced decoherence rate, we could predict the existence of distinctive quantum features of the meter even in a macroscopic domain.

Moreover, we studied nonlocal properties of the coupled system-meter scheme and demonstrated violations of Bell-type inequalities [3, 24] in the Clauser-Horne-Shimony-Holt form [25] of phase space observables [26, 27] even when the meter reached a macroscopic state [10].

III. ENTANGLEMENT OF FORMATION

Underlying nonlocality is the concept of entanglement. It is responsible for the fact that a composite system possesses properties which can not be understood by considering the parts of the system separately. In other words, there is no element of “reality” in the parts considered by their own but only a created reality which depends on what is measured in the other part. This uniquely quantum concept has proved to be a fundamental resource for quantum information processing and the quantification of entanglement is essential to assess the full performance of an information theory based on quantum mechanics [6, 7].

There are several good measures of the degree of entanglement for both pure and mixed quantum states [8]. Perhaps the most seminal one is the entanglement of formation which quantifies the resources needed to create an entangled state [9]. Entanglement of formation is a measurable quantity, at least for a pair of qubits which is the case we are dealing with here. The underlying quantity is called concurrence [9]. For pure states, concurrence is strongly connected with two-particle visibility [28, 29] which is a property that cannot exist separately in the parts of a bipartite system. The expression relating the concurrence to the density operator, \( \rho \), of a mixed state is given as

\[
C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
\]

where the \( \lambda_i \)'s are the square roots of the eigenvalues of \( \rho^* \tilde{\rho} \) in descending order. Here \( \tilde{\rho} \) results from applying the spin-flip operation to \( \rho^* \),

\[
\tilde{\rho} = (\sigma_y \otimes \sigma_y)^* \rho^* (\sigma_y \otimes \sigma_y).
\]

Here \( \sigma_y \) is the Pauli spin operator in the standard basis and \( \rho^* \) is the complex conjugate of \( \rho \). The entanglement of formation, \( E_f(\rho) \), of the state \( \rho \) is connected with the concurrence, \( C(\rho) \), via the formula

\[
E_f(\rho) = h(\frac{1 + \sqrt{1 - C^2(\rho)}}{2}),
\]

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).
\]
There is another remarkable property of the concurrence which is directly related to nonlocality. One can show that the maximum possible violation of the Bell inequality \([3, 24]\) in the Clauser-Horne-Shimony-Holt form \([25]\)
\[
\mathcal{B}(\rho) = |E(c, d) + E(c', d) + E(c, d') - E(c', d')| \leq 2,
\]
for a state \(\rho\) is given as \([21, 30, 31, 32]\)
\[
B_{\text{max}}(\rho) = 2\sqrt{M(\rho)}.
\]
Here \(c, c'\) are two dichotomous variables of the first system and \(d, d'\) are two of the second, and \(E(c, d)\) is the expectation value of the correlation of \(c\) and \(d\), and so on for the other expectation values. The quantity \(M(\rho)\) is the sum of the two larger eigenvalues of \(T_p T_p^\dagger\), where \(T_p\) is the 3x3 matrix with the \((m, n)\) element given by
\[
t_{mn} = \text{tr}(\rho \sigma_n \otimes \sigma_m),
\]
and the \(\sigma_i\)'s are the Pauli matrices.

Based on the previous considerations, a number of interesting questions arise with respect to our measurement model concerning the generation of the entanglement of formation and its dynamical properties. Among them is the control of the dynamical properties of the entanglement of formation and the possibility of observing its quantum features in the macroscopic domain of the meter in a squeezed reservoir environment. Another one is the determination of the maximum possible violation of the Bell inequality \([15]\) and the time when it is achieved in the measurement.

To answer these questions it is essential to recognize that the potentially entangled system-meter state consists of nonorthogonal “qubits”. In particular, the squeezed coherent states \(|\hat{\alpha}_n(t), \varepsilon\rangle\) of the meter are not completely orthogonal during the course of the measurement process. It is, however, possible to define concurrence and, consequently, entanglement of formation for nonorthogonal bipartite systems by introducing an orthonormal basis in the two subsystems of the bipartite state \([33, 34]\). In the measurement model under consideration the orthonormal, time-dependent basis for the meter is formed by
\[
|\hat{0}(t)\rangle = |-\hat{\alpha}(t), \varepsilon\rangle,
\]
\[
|\hat{1}(t)\rangle = \frac{|\hat{\alpha}(t), \varepsilon\rangle - P(t)| - \hat{\alpha}(t), \varepsilon\rangle}{\sqrt{1 - |P(t)|^2}},
\]
\[
\hat{\alpha}(t) = \alpha_0 [\cosh(r) - \sinh(r)] \left(1 - e^{-\frac{\theta}{2}}\right),
\]
\[
P(t) = |P(t)| = |\langle -\hat{\alpha}(t), \varepsilon|\hat{\alpha}(t), \varepsilon\rangle|.
\]
Here we set the squeezing phase \(\theta = 0\) in order to maximize the effect of the squeezed reservoir on the decoherence rate (see Fig. \([3]\)). The orthonormal basis states for the system, of course, are given by the two ground states of the four-level atom, \(|1\rangle\) and \(|2\rangle\). We can now construct the spin-flip operators for the meter, \(\sigma^M_y(t)\), and for the system, \(\sigma^S_y(t)\), as
\[
\sigma^M_y(t) = i \{[\hat{1}(t)]\langle0(t)| - |\hat{0}(t)\rangle\langle\hat{1}(t)|\},
\]
\[
\sigma^S_y = i \{[2\rangle(1\rangle - |1\rangle2\rangle)\}.
\]
With these results it is straightforward to calculate the concurrence \([14]\), from Eqs. \([12]\) and \([8]\) of the system-meter state, in spite of its complicated dynamical properties. Its analytical expression takes a rather simple form,
\[
C(\rho(t)) = \exp[\Gamma_{12}(t)] \frac{\sqrt{1 - P^2(t)}}{P(t)}.
\]
Surprisingly, it is possible, at least in principle, to determine the concurrence of the system-meter state by directly measuring the (time-dependent) operator \(\sigma^M_y(t) \otimes \sigma^S_y(t)\)
\[
C(\rho(t)) = -\langle \rho(t) \sigma^M_y(t) \otimes \sigma^S_y(t) \rangle.
\]
When we recall that our measurement model contains a rather involved decoherence mechanism for the system-meter and, therefore, its state rapidly approaches a mixture, this is an amazing result that gives the concurrence (at least in this model) an operational meaning. Analogously, we can calculate the maximum violation of the Bell inequality, Eq. \((16)\), yielding
\[
B_{\text{max}}(\rho(t)) = 2\sqrt{1 + C^2(\rho(t)) + \exp[2\Gamma_{12}(t)] - P^2(t)}.
\]
FIG. 4: Time dependence of the concurrence, $C_{SV,UV}(\rho, t)$, (left figure) and entanglement of formation, $E^f_{SV,UV}(\rho, t)$, (right figure) for different squeezing parameters. The notation as well as the parameters are the same as in Fig. 3. SV denotes squeezed vacuum and UV denotes ordinary vacuum.

Based on Eq. (24), we have plotted the concurrence as well as the entanglement of formation, Eqs. (13) and (14), for different squeezing parameters in Fig. 4. With increasing squeezing parameter, the entanglement of formation approaches its maximum at a much later time than in a regular vacuum. A squeezed environment which monitors the meter is capable to maintain nonclassical properties of the system-meter, i.e. the correlations between them, over a much longer period of time. The system-meter state has already entered its macroscopic domain when the entanglement of formation reaches its maximum. This can be seen in Fig. 4 for a squeezed reservoir with a squeezing parameter of $r = 3.5$ and for the corresponding squeezed coherent states of amplitude $\alpha_0 = 100$, for example. In addition, we also see that the entanglement of formation does not reach its maximum possible value of 1, irrespective of the type of reservoir which monitors the system-meter. This, however, is not surprising for a mixed state which contains partly nonorthogonal qubits. The maximum possible value of the entanglement of formation of the system-meter is additionally reduced in a squeezed environment, since there is a competition between the positive effect of the highly reduced decoherence rate (which maintains the purity of the system-meter on a greatly enhanced time-scale) and the negative effect of the larger overlap (i.e. the intrinsic indistinguishability of the meter) between the nonorthogonal meter states. In Fig. 5 we display the time evolution of the maximum possible violation of the Bell inequality, Eq. (26), in the measurement model. Again, we can observe violations of the Bell inequality on a greatly enhanced time scale in a squeezed reservoir with increasing squeezing parameter. We have also found this result in Ref. 10 based on a phase-space equivalent of the Bell-inequality [26, 27]. However, the approach with the concurrence of the system-meter has a number of advantages. First, we are able to observe the full time-dependence of the “formation” of entanglement during the course of the measurement. Second, we can display the maximum possible violation of the Bell inequality at every time step. In general, it is hard to find this quantity on the basis of the inequality, Eq. (15), which depends on four parameters. Beyond it, the phase-space approach of Banaszek and Wódkiewicz [26, 27]
can not approach the maximum possible value of the violation of the Bell inequality because of the smoothing effect of the Wigner function.

The right insert of Fig. 3 displays the time dependence of the difference between the maximum possible violations of the Bell inequality, Eq. (26), and between the concurrences, Eq. (24), in two different squeezed reservoir environments with squeezing parameters \( r_1 = 2, r_2 = 3.5 \) and \( \theta_1 = \theta_2 = 0 \). From the figure one can get some insight as to what extent a violation of the Bell inequality tells us something about the nature of entanglement [34, 35]. Munro et al. [35] suggested that the more mixed a system is made the more entanglement (or concurrence) is generally required to violate the Bell inequality to the same degree. This, however, is not generally true as pointed out by Ghosh et al. [22] but they could not find a simple explanation. Based on the right part of Fig. 3 we can answer this question as follows. At \( \gamma t \approx 0.3 \) in Fig. 3 the difference of the concurrences between the two systems \( C_{SV}^{diff}(t) = C_{SV}^{max,1}(t) - C_{SV}^{max,2}(t) \) is slightly larger than zero. In contrast, the degree of mixedness of system 1, corresponding to a squeezed environment with parameters \( r_1 = 2 \) and \( \theta_1 = 0 \), is obviously larger than that of system 2 (with squeezing parameters \( r_2 = 3.5 \) and \( \theta_2 = 0 \)) as a consequence of the advanced decoherence in system 1 (see Fig. 3). Thus, this system contradicts the statement by Munro et al. [35]. We get a larger violation of the Bell inequality in a system having the same degree of entanglement but, at the same time, more mixedness than the other system. We suggest the following explanation for this apparently peculiar behavior. It is the overlap between the nonorthogonal meter states that reduces the maximum possible violation, Eq. (26). In order to violate the Bell inequality it is necessary for the components of the state of the composite system to be distinguishable. When the degree of distinguishability gets smaller the amount of the maximum possible violation of the Bell inequality will also be reduced. This explains why the difference of the maximum possible violations of the Bell inequality in Fig. 3 is slightly larger than zero in spite of the fact that system 1 is more mixed but has the same amount of entanglement as system 2. Obviously, the overlap of the meter states of system 2 at this particular time is much larger than that of system 1. This observation can give a novel direction to investigations of the problem as to what extent the Bell inequality is a measure of entanglement and connects it not only with mixedness but also with the degree of indistinguishability of nonorthogonal qubits.

IV. SUMMARY

In summary, we have investigated dynamical properties of the entanglement of formation in a measurement model. We have also demonstrated the ability to influence the time evolution of the entanglement of formation by a squeezed reservoir and found a way to maintain this nonclassical property in a macroscopic domain of the meter, in spite of it being monitored by the environment. Furthermore, this model gives some insight into dynamical properties of the entanglement of decoherent and nonorthogonal entangled qubits which is of central interest in quantum information theory [33].

Finally, we note that it is possible to implement this measurement model with a trapped “four-level” \(^{198}\)Hg\(^{+}\)-ion [18] which is exposed to engineered reservoirs [19, 20]. In addition, it seems, at least in principle, possible to directly measure the concurrence with time-dependent Pauli spin-flip operators. The present measurement model is well suited to study the entanglement in dissipative environments and helps to clarify some of the underlying physical principles. In particular, it gives new insights into the question as to what extent the Bell inequality is a measure of entanglement and explains how to get larger amounts of violation in a system with more mixedness but the same amount of entanglement as a reference system.

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