Universal First-passage Properties of Discrete-time Random Walks and Lévy Flights on a Line: Statistics of the Global Maximum and Records

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In these lecture notes I will discuss the universal first-passage properties of a simple correlated discrete-time sequence \( \{x_0 = 0, x_1, x_2, \ldots, x_n\} \) up to \( n \) steps where \( x_i \) represents the position at step \( i \) of a random walker hopping on a continuous line by drawing independently, at each time step, a random jump length from an arbitrary symmetric and continuous distribution (it includes, e.g., the Lévy flights). I will focus on the statistics of two extreme observables associated with the sequence: (i) its global maximum and the time step at which the maximum occurs and (ii) the number of records in the sequence and their ages. I will demonstrate how the universal statistics of these observables emerge as a consequence of Pollaczek-Spitzer formula and the associated Sparre Andersen theorem.

I. INTRODUCTION

Since the remarkable founding paper by Einstein in 1905 \([1]\), followed closely by two seminal papers respectively by Smoluchowski \([2]\) and Langevin \([3]\), random walks and the associated continuous-time Brownian motion have remained as fundamental cornerstones of statistical physics with an amazingly impressive number of applications \([4–9]\) that range from traditional ‘natural’ sciences such as physics, chemistry, biology, mathematics and astronomy all the way to ‘man-made’ subjects such as computer science and finance. Even though many aspects of this classical subject are extremely well understood and form textbook materials, it is fascinating that new questions with non-trivial answers, arising from new applications, continue to spring unexpected surprises.

In these lectures I will discuss some of these recent applications. The general area of random walks is vast with an enormous literature. My goal for these lectures is rather modest. I will just focus on a rather simple and restricted model: a discrete-time random hopper on a continuous line. Starting from the origin \( x_0 = 0 \), the position of the particle at step \( n \) evolves via the Markov rule, \( x_n = x_{n-1} + \xi_n \), where \( \xi_n \)'s denote the random jumps at different time steps. These jumps are independent and identically distributed (i.i.d.) random variables, each drawn from the same distribution \( \phi(\xi) \) which is symmetric and continuous. If the walk evolves up to step \( n \), one generates a sequence or a discrete-time series: \( \{x_0 = 0, x_1, x_2, \ldots, x_n\} \). Clearly the members of this sequence are correlated random variables. Such a sequence is perhaps the simplest possible correlated sequence that appears rather naturally in many different contexts. A classic example of such a walk can be found in bacterial chemotaxis, where a bacteria, in search of food, jumps from one position to another at discrete time steps \([12]\). In the context of queueing theory \( x_n \) represents the length of a single queue at time \( n \) \([10]\). In the context of the evolution of stock prices \( x_n \) represents the logarithm of the price of a stock at time \( n \) \([11]\). It can also represent the \( x \) coordinates of the beads of a Rouse polymer chain in thermal equilibrium in \( d \)-dimensions (when the jump distribution is Gaussian) \([13]\) (see also \([14]\)). When the jump distribution has a power law tail with a divergent second moment, \( \phi(\xi) \sim |\xi|^{-1-\mu} \) (with \( 0 < \mu \leq 2 \)) for large \(|\xi|\), this sequence represents a Lévy flight which also has enormous number of applications \([3,15,19]\).

Here I will focus on two extreme observables associated with such a correlated sequence: (i) the global maximum \( M_n = \max\{0, x_1, x_2, \ldots, x_n\} \) of the sequence and the associated time step \( m \) at which this maximum is realized in a given sample (ii) the number and ages of records of this sequence where a record is set to happen at step \( i \) if \( x_i \) is bigger than all the previous values: \( x_i > x_k \) for all \( 0 \leq k < i \). Age of a record is simply the number of steps up to which this record survives, i.e., till it gets surpassed by the next record breaking event.

Now, if the number of steps \( n \) of the sequence is large and if the second moment of the jump length distribution \( \sigma^2 = \int_\infty^\infty \xi^2 \phi(\xi) \, d\xi \) is finite, one would expect, correctly, to recover the continuous-time limit results of the Brownian motion as a consequence of the central limit theorem, at least for the global maximum (records are not very well defined in the continuous-time limit). However, it turns out, as I will show here in some detail, that many properties associated with extreme events such as the global maximum or the number of records are completely universal for all \( n \), i.e., they do not depend on the jump length distribution \( \phi(\xi) \) at all whatever be the value of \( n \), as long as \( \phi(\xi) \) is symmetric and continuous. Note, in particular, that this universality does not even require a finite \( \sigma^2 \), e.g., it holds even for long range Lévy flights.

In fact, this universality has nothing to do with the central limit theorem. Instead, it will turn out to be a consequence of the Sparre Andersen theorem \([20]\) concerning the first-passage properties of such a random walk sequence \([3,21]\). This is a rather deep combinatorial theorem and the final result looks deceptively simple though its derivation is far from simple. Here I will provide a derivation of this result using another result on the generating
function of the maximum of such a sequence, known as the Pollaczek-Spitzer formula \textsuperscript{22, 23}. Somehow these results are not so well known among physicists. So, I’ll discuss these results in some detail and use them to derive some universal and some nonuniversal properties associated with the statistics of the maximum and the records of this random walk sequence.

In the latter half of my lectures in the school, I also discussed the statistical properties of the functionals of Brownian motion via the Feynman-Kac formula and in particular, various interesting applications of the so called first-passage Brownian functionals, where one considers the Brownian motion till its first-passage time. They turn out to have various applications: in queueing theory, in finance, in simple models of particle moving in a disordered random potential and in astrophysics where one is interested in the distribution of the life time of a comet in the solar system. However, I will not include this interesting topic in these lecture notes, as I have already discussed it in another article \textsuperscript{24}. The interested readers may consult this article and also another review on Brownian functionals with interesting applications in the localisation theory \textsuperscript{25}.

This article is organised as follows. In Section II, I define the model precisely and review some basic preliminaries to remind the readers the central limit theorem and the Lévy stable laws. In Section III, I discuss the first-passage properties associated with the random walk sequence and discuss the Pollaczek-Spitzer formula and how this formula leads to the Sparre Andersen theorem. Section IV is devoted to the statistics of the global maximum and the universal statistics of the time of its occurrence where we use the Sparre Andersen theorem. In Section V, we discuss the statistics of the number of records and their ages and show how universal properties emerge again as a consequence of the Sparre Andersen theorem. Finally, I conclude in Section VI with a summary and some open problems.

II. RANDOM WALKS, BROWNIAN MOTION, LÉVY FLIGHTS: SOME PRELIMINARIES

A. Definitions

Let us start with a simple discrete-time random walker moving on a continuous line. The position \( x_n \) of the walker after \( n \) steps evolves for \( n \geq 1 \) via,

\[
x_n = x_{n-1} + \xi_n
\]

starting at \( x_0 = 0 \), where the step lengths \( \xi_n \)'s are i.i.d. random variables with zero mean and each drawn from a normalized (to unity) distribution \( \phi(\xi) \) which is symmetric, \( \phi(\xi) = \phi(-\xi) \) (see Fig. 1).

Few examples of the jump length distribution \( \phi(\xi) \) are:

(i) \( \phi(\xi) = \frac{1}{\xi} e^{-|\xi|} \) (Exponential)

(ii) \( \phi(\xi) = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\xi^2/2\sigma_0^2} \) (Gaussian)

(iii) \( \phi(\xi) = \frac{1}{2} \left[ \theta(\xi+1) - \theta(\xi-1) \right] \) (Uniform)

(iv) \( \phi(\xi) \sim |\xi|^{-1-\mu} \) for large \( |\xi| \) with \( 0 \leq \mu \leq 2 \) such that \( \sigma^2 = \int |\xi|^2 \phi(\xi) \, d\xi \) does not exist (Lévy flights)

(v) \( \phi(\xi) = \frac{1}{\pi} \delta(\xi+1) + \frac{1}{\pi} \delta(\xi-1) \) (Lattice random walk where the lattice spacing is unity).

In first 4 of these examples, the cumulative jump distribution \( \Psi(x) = \int_{-\infty}^{x} \phi(\xi) \, d\xi \) is a continuous function. In the last example (v), where the walker is restricted to move on a one dimensional lattice with unit lattice spacing, the cumulative jump distribution \( \Psi(\xi) \) is a non-continuous function. We will see later that this continuity property of \( \Psi(\xi) \) will play an important role. Note further that in examples (i)-(iii) and (v), the variance of the step length, \( \sigma^2 = \int_{-\infty}^{\infty} \xi^2 \phi(\xi) \, d\xi \) is finite. We will see that in such cases the central limit theorem holds. In the Lévy case (iv), the central limit theorem breaks down.

The evolution equation (1) is Markovian since the position \( x_n \) at step \( n \) depends only on the position at just the previous time step \( x_{n-1} \) (and not on the full history before the \((n-1)\)-th step) and on the current noise, i.e., the noise \( \xi_n \) at step \( n \). This Markovian property makes life simple as we will see later. As a simple example of a non-Markovian evolution consider the rule

\[
x_n = 2x_{n-1} - x_{n-2} + \xi_n
\]

where \( \xi_n \)'s are again i.i.d random variables. This is just the discrete-time version of the continuous-time random acceleration problem: \( \frac{d^2 x}{dt^2} = \xi(t) \) where \( \xi(t) \) is a Gaussian white noise with zero mean \( \langle \xi(t) \rangle = 0 \) and delta
FIG. 1: A trajectory of a random walker starting at the initial position $x_0$ and evolving with the number of steps $n$.

correlator $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. It turns out that the first-passage properties of even this simple non-Markovian evolution is highly nontrivial [26–28]. We will not consider non-Markov evolution rules further in these lectures and focus only on the Markov evolution (1). For the first-passage properties of non-Markovian stochastic processes see [28] and references therein.

Iterating the Markov evolution rule (1) up to $n$ steps, it follows that the position $x_n$ of the walker after $n$ steps, starting at $x_0 = 0$, is simply a sum of $n$ i.i.d. random variables

$$x_n = \sum_{k=1}^{n} \xi_k. \quad (3)$$

In the case when $\sigma^2$ is finite, using the independence property of the step lengths $\xi_k$’s, it follows that the mean square displacement of the particle after $n$ steps, for all $n$, is simply

$$\langle x_n^2 \rangle = n \sigma^2. \quad (4)$$

**Brownian limit:** At this point, it is useful to consider the continuous-time limit where the random walk reduces to a Brownian motion. Let us define $\Delta t$ as a small time interval and set $t = n\Delta t$. Then (4) gives

$$\langle x_n^2 \rangle = \frac{\sigma^2}{\Delta t} t. \quad (5)$$

If one now takes the limit $\Delta t \to 0$, it follows that $\sigma^2 \to 0$ also in order that $\langle x^2(t) \rangle$ remains finite at finite time $t$. Thus, to have a meaningful continuous-time limit, the mean square step length $\sigma^2 \to 2D\Delta t$ as $\Delta t \to 0$ with a finite diffusion constant $D$, leading to the diffusive law of Brownian motion $\langle x^2(t) \rangle = 2Dt$ for all $t$. In this continuous-time limit, one can also rewrite the Markov evolution rule (1) as

$$\frac{\Delta x_n}{\Delta t} = \frac{\xi_n}{\Delta t} = \xi(t) \quad (6)$$

where $\xi(t)$ is random noise with zero mean that is uncorrelated at two different times, $\langle \xi(t_1)\xi(t_2) \rangle = 0$ for $t_1 \neq t_2$. At the same time instant, however, $\langle \xi^2(t) \rangle = \sigma^2/(\Delta t)^2 = 2D/\Delta t$. Thus, as $\Delta t \to 0$, $\langle \xi^2(t) \rangle$ diverges. A useful physicist’s
way of writing this correlation function of the noise is \( \langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2) \). In this limit it is called the white noise and one writes \( \xi(t) \) as a stochastic Langevin equation

\[
\frac{dx}{dt} = \xi(t)
\]

(7)

where \( \xi(t) \) is the white noise with zero mean and a correlator \( \langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2) \). Note that for all practical purposes, such as in numerical simulation, one will interpret the delta function as \( \delta(0) \equiv 1/\Delta t \).

We will see later that in the Brownian limit many properties of the walk, such as its first-passage probability, become much simpler. In contrast, for discrete time evolution, even though the process is Markov, some of these properties are quite nontrivial.

### B. Green’s Function

Let us get back to our basic discrete-time Markov evolution \( \xi \). In this subsection, let us compute a basic object namely the free (bare) Green’s function \( G(x, x_0, n) \) defined as the probability density of the position of the walker after step \( n \) at \( x \), given that it started from \( x_0 \) at step 0. Using the Markov property, one can easily write down a recursion relation for the evolution of \( G(x, x_0, n) \)

\[
G(x, x_0, n) = \int_{-\infty}^{\infty} G(x', x_0, n-1) \phi(x - x') dx'
\]

(8)

which counts the event of particle jumping from its position \( x' \) at step \( n - 1 \) to its position \( x \) at step \( n \) by an amount \( (x - x') \) drawn from the distribution \( \phi(x - x') \). This is called the forward Kolmogorov equation, since one considers the current position \( x \) of the walker as a variable. Alternatively, one can also write down a backward Kolmogorov equation where one considers the starting position of the walker \( x_0 \) as a variable

\[
G(x, x_0, n) = \int_{-\infty}^{\infty} G(x, x_0', n-1) \phi(x_0' - x_0) dx_0'.
\]

(9)

Here one considers the displacement of the particle at the first step from \( x_0 \) to \( x_0' \) and for the subsequent evolution up to \( (n-1) \) steps the starting position of the walker is at \( x_0' \). Both equations are completely equivalent to each other. We will see later, however, that for certain first-passage related quantities, the backward equation is often computationally more advantageous than the forward one.

These integral equations (8) or (9) can be easily solved using Fourier transforms. For example, for the forward equation, we define

\[
\tilde{G}(k, x_0, n) = \int_{-\infty}^{\infty} G(x, x_0, n) e^{ikx} dx
\]

(10)

and use the convolution form of (8) to get \( \tilde{G}(k, x_0, n) = \tilde{G}(k, x_0, n-1) \tilde{\phi}(k) \) where \( \tilde{\phi}(k) \) is the Fourier transform of \( \phi(x) \). Iterating \( n \) times and using the initial condition, \( G(x, x_0, 0) = \delta(x - x_0) \) and hence \( \tilde{G}(k, x_0, 0) = e^{ikx_0} \) one gets \( \tilde{G}(k, x_0, n) = [\tilde{\phi}(k)]^n e^{ikx_0} \). Inverting the Fourier transform one obtains the exact Green’s function

\[
G(x, x_0, n) = \int_{-\infty}^{\infty} [\tilde{\phi}(k)]^n e^{-ik(x-x_0)} \frac{dk}{2\pi}
\]

(11)

Let us now see what happens for large \( n \). In cases where \( \sigma^2 \) is finite, one has, for small \( k \), \( \tilde{\phi}(k) \approx 1 - \sigma^2 k^2/2 + O(k^4) \). Now, for large \( n \), the dominant contribution to the integral in (11) comes from small \( k \) region. Substituting the small \( k \) behavior, exponentiating and performing the Gaussian integral, one gets, for large \( n \), the standard Gaussian behavior

\[
G(x, x_0, n) \approx \frac{1}{\sqrt{2\pi n \sigma^2}} \exp \left[ -\frac{(x - x_0)^2}{2n\sigma^2} \right]
\]

(12)

which is essentially the statement of the central limit theorem (CLT). Note that the universal Gaussian form holds only near the central peak but not at the tails which are described by nonuniversal large deviation function that I will not discuss here [3]. On the other hand, for jump distributions with a divergent \( \sigma^2 \) (such as for Lévy flights in example (iv)), the CLT breaks down [9, 15, 16]. For Lévy flights, one can write for small \( k \), \( \tilde{\phi}(k) \approx 1 - |a k|^\mu \) where
0 \leq \mu \leq 2$ is the Lévy index and $a$ is a microscopic length. Substituting this in (11) and rescaling $n(ak)^\mu \to k^\mu$ one gets, for large $n$,

$$G(x, x_0, n) \simeq \frac{1}{a n^{1/\mu}} \Phi_\mu \left( \frac{(x - x_0)}{a n^{1/\mu}} \right)$$

(13)

where the function

$$\Phi_\mu(z) = \int_{-\infty}^{\infty} e^{-|k|^{\mu} - i k z} \frac{dk}{2\pi}$$

(14)

is called the Lévy stable function of index $\mu$. Note that this function $\Phi_\mu(z)$, for large $z$, has the same power law tail, $\Phi_\mu(z) \sim |z|^{-1-\mu}$ as the jump distribution itself. For some special values of $\mu$, one can compute this function explicitly [9]. Thus, the result in (13) is the statement of Lévy stable law [14, 16]: the sum of i.i.d Lévy variables is itself Lévy distributed (up to a rescaling by $n^{1/\mu}$), i.e., the Lévy distribution is stable under addition [13,17]. This is thus the counterpart of the CLT which is the analogous statement for the sum of i.i.d random variables with a finite $\sigma^2$: the stable law for CLT is Gaussian. Note that from (13) it follows that the typical distance traversed by the particle in step $n$ scales super-diffusively: $x \sim n^{1/\mu}$ for $0 < \mu \leq 2$.

**Brownian limit:** In the continuous-time limit, when $\sigma^2$ is finite and hence the CLT holds, the integral equations (8) or (9) reduce to partial differential equations. For example, given the Langevin evolution in (7), the forward Kolmogorov equation (8) reduces to

$$G(x, x_0, t + \Delta t) = \int_{-\infty}^{\infty} G(x - \xi(t)\Delta t, x_0, t) \phi(\xi(t)) d\xi(t)$$

(15)

Expanding the Green's function on the rhs in a Taylor series, keeping terms up to $O((\Delta t)^2)$ and using the property that $\langle \xi(t)^2 \rangle = \int_{-\infty}^{\infty} \xi^2 \phi(\xi) d\xi = 2D/\Delta t$, one gets, taking $\Delta t \to 0$, the well known diffusion equation for the Green's function

$$\frac{\partial G}{\partial t} = D \frac{\partial^2 G}{\partial x^2}$$

(16)

starting from the initial condition, $G(x, x_0, 0) = \delta(x - x_0)$. Similarly one can write down a backward diffusion equation with $x$ in (16) replaced by $x_0$. The solution of the forward (or the backward) diffusion equation can be easily found using Fourier transforms and one recovers, as expected, the Gaussian behavior

$$G(x, x_0, t) = \frac{1}{\sqrt{4\pi D t}} \exp \left[ -\frac{(x - x_0)^2}{4Dt} \right].$$

(17)

For Lévy flights, where $\sigma^2$ is infinite and the CLT breaks down, one can still formally define a continuous-time limit, and obtain the so called Lévy fractional diffusion equation (for a review and discussion, see [18]). This simply follows by rewriting the basic recursion relation (8) as

$$G(x, x_0, n) = \int_{-\infty}^{\infty} G(x - \xi_n, x_0, n - 1) \phi(\xi_n) d\xi_n.$$  

(18)

Next we write $G(x - \xi_n, x_0, n - 1) = \int_{-\infty}^{\infty} \tilde{G}(k, x_0, n - 1) e^{ik(x - \xi_n)} dk$ and substitute it in (18). This gives

$$G(x, x_0, n) = \int_{-\infty}^{\infty} dk \tilde{G}(k, x_0, n - 1) \hat{\phi}(k).$$

(19)

Following similar arguments as in the Brownian case, in the large $n$ limit, one needs to keep only the small $k$ contribution of $\hat{\phi}(k) = 1 - |ak|^\mu$ in (19). This gives

$$G(x, x_0, n) - G(x, x_0, n - 1) \simeq -a^\mu \int_{-\infty}^{\infty} dk |k|^\mu \tilde{G}(k, x_0, n - 1).$$

(20)

Now, we need to divide both sides by the time increment $\Delta t$ and take the limit $\Delta t \to 0$. To obtain a sensible limit, one needs to take $a \to 0$ limit as well, keeping the ratio $a^\mu/\Delta t = K$ fixed. This gives a continuous-time integro-differential equation

$$\frac{\partial G}{\partial t} = -K \int_{-\infty}^{\infty} dk \tilde{G}(k, x_0, n - 1) |k|^\mu = -K \left( -\partial_x^2 \right)^{\mu/2} G$$

(21)
where the integral in the $k$ space can be formally interpreted as a fractional derivative. Note that for $\mu = 2$, one recovers the standard diffusion equation. But for $0 \leq \mu < 2$, one still needs to solve an integral equation even in the continuous-time limit. Thus for the Lévy walks, even though one can formally write down a continuous-time equation, it is not as useful as the ordinary Brownian case where one has a true differential equation in real space whose solution can be easily obtained. This continuous-time fractional diffusion equation has been studied extensively in the recent past (for a review see [18]), and many interesting results, in particular concerning first-passage properties have been derived (see for instance [24, 30]). However, in these lectures I will not use this approach and will rather stick to the discrete-time evolution.

### III. RANDOM WALKS: SURVIVAL AND FIRST-PASSAGE

Having done with these standard basic preliminaries, let us now turn to the first-passage properties of a random walk evolving in discrete time via the Markov rule [11] with arbitrary symmetric jump length distribution $\phi(\xi)$. We first define the restricted Green’s function $G^+(x, x_0, n)$ as the probability (density) that the walker, starting at $x_0 > 0$ at step 0, reaches the position $x > 0$ at step $n$ but without crossing the origin in between, i.e., it stays positive at all intermediate steps and lands at $x$ exactly at the $n$-th step

$$G^+(x, x_0, n) = \text{Prob}[x_n = x, x_{n-1} \geq 0, x_{n-2} \geq 0, \ldots, x_1 \geq 0|x_0].$$  \hspace{1cm} (22)

Using the Markov property of the evolution, one can again write down the evolution equation for the restricted Green’s function, both forward and backward as in case of free Green’s function in the previous subsection

$$G^+(x, x_0, n) = \int_0^\infty G^+(x', x_0, n-1) \phi(x-x') \, dx'; \quad \text{(forward)}$$  \hspace{1cm} (23)

$$G^+(x, x_0, n) = \int_0^\infty G^+(x, x_0', n-1) \phi(x_0' - x_0) \, dx_0'; \quad \text{(backward)}$$  \hspace{1cm} (24)

The interpretation is as before. For example, in the forward case, one considers the walker reaching $x'$ at step $(n-1)$ (staying positive always) and then making a final jump $x' \to x$ at step $n$ by drawing a random length $x - x'$ from the distribution $\phi(\xi)$. Similarly, in the backward equation, the particle at step 1 jumps from its initial position $x_0$ to a new position $x_0'$ and subsequently evolves for $(n-1)$ steps starting from this new initial position $x_0'$ while staying positive all along. One then integrates over all possible jumps at the first step, but making sure that $x_0'$ is positive.

The survival probability or the persistence is defined as the probability $Q(x_0, n)$ that the particle survives (i.e. stays positive) up to step $n$, no matter what the final position $x$ at step $n$ is. Thus

Survival Probability: $Q(x_0, n) = \text{Prob}[x_n \geq 0, x_{n-1} \geq 0, x_{n-2} \geq 0, \ldots, x_1 \geq 0|x_0] = \int_0^\infty G^+(x, x_0, n) \, dx$ \hspace{1cm} (25)

Thus, one can either solve first the forward equation [23], obtain the restricted Green’s function $G^+(x, x_0, n)$ for all $x$ and then integrate over $x$ in [23] to obtain the survival probability $Q(x_0, n)$. Alternatively, and in a much easier way, one can integrate the backward equation [24] over $x$ and write directly a backward evolution equation for the survival probability itself

$$Q(x_0, n) = \int_0^\infty Q(x_0', n-1) \phi(x_0' - x_0) \, dx_0'. \hspace{1cm} (26)$$

Thus one saves an extra integration step [24] and just needs to solve only the integral equation [24] starting from the initial condition $Q(x_0, 0) = 1$ for all $x_0 \geq 0$. This initial condition follows from the fact that the walker definitely (with probability 1) does not cross 0 in 0 step. One thus sees why the backward equation [26] is more advantageous compared to the forward equation, at least as far as the persistence properties are concerned.

Once we have obtained the survival probability $Q(x_0, n)$, the first-passage probability can be easily computed from it. The first-passage probability $F(x_0, n)$ is defined as the probability that the walker, starting initially at $x_0$, crosses the origin for the first time immediately after step $n-1$ (i.e., it is positive at step $n-1$, but becomes negative at step $n$). It then follows that

$$F(x_0, n) = Q(x_0, n-1) - Q(x_0, n) \hspace{1cm} (27)$$

as it counts the fraction of paths that survived up to step $(n-1)$, but not up to step $n$. 

So, to compute the first-passage or the survival probability, we need to solve the integral equations (23), (24) or just directly (26). Note the important differences in these equations compared to the free Green’s functions in (8) and (9): they look almost similar, but not quite. In equations (23), (24) or (26), the limit of integration on the rhs is from 0 to \( \infty \), as opposed to \(- \infty \) to \( \infty \) in the free Green’s function equations (8) and (9). This makes a huge difference! The reason is, even though (26) apparently seems to have a convolution form, the limit of integration is only over half-space \([0, \infty)\) and not the full space \([-\infty, \infty]\). If the limits were over the full space, as in the case of free Green’s functions, one can simply use the Fourier transform methods. But for the half-space problem, unfortunately one can not use simple Fourier transform technique. In fact, such half-space integral equations have been well studied in mathematics and are known as Wiener-Hopf integral equations [31]. For a general kernel \( \phi(x - x') \), they are notoriously difficult to solve! However, for the particular case where the kernel \( \phi(x - x') \) has the interpretation of a probability density function (i.e., non-negative and normalizable function), one can obtain explicit solution [23] (as discussed later).

The discussion above makes it clear the technical reason as to why computing the first-passage properties of even a simple random walker (but with arbitrary jump distribution \( \phi(\xi) \)) is nontrivial. Before we write the solution explicitly, let us see first how this problem simplifies in the continuous-time Brownian limit.

**Brownian limit:** In the Brownian limit, one can reduce the discrete time backward integral equation (26) for the survival probability into a partial differential equation. Let us consider the survival probability \( Q(x_0, t + \Delta t) \) up to time \( t + \Delta t \). Let us break the interval \([0, t + \Delta t]\) into two intervals \([0, \Delta t]\) and \([\Delta t, t + \Delta t]\). In the first small interval the particle evolves from its initial position \( x_0 \) to a new random position \( x_0 + \xi(0)\Delta t \) where \( \xi(0) \) is the initial noise in the Langevin equation (7). Subsequently the particle evolves in the interval \([\Delta t, t + \Delta t]\) starting from its new initial position \( x_0 + \xi(0)\Delta t \). Thus, the analogue of (26) is

\[
Q(x_0, t + \Delta t) = \int_0^\infty Q(x_0 + \xi(0)\Delta t, t) \phi(\xi(0)) d(\xi(0))
\]  

(28)

Expanding in a Taylor series as in the case of free Green’s function and using the properties of the white noise, one then gets the backward Fokker-Planck equation for the survival probability

\[
\frac{\partial Q}{\partial t} = D \frac{\partial^2 Q}{\partial x^2}
\]  

(29)

valid for all \( x_0 \geq 0 \) and to be solved with the boundary conditions: (a) \( Q(x_0 = 0, t) = 0 \) for all \( t \) and (b) \( Q(x_0 \to \infty, t) = 1 \) for all \( t \) and subject to the initial condition \( Q(x_0, 0) = 1 \) for all \( x_0 > 0 \). Thus in the Brownian limit, we are able to reduce the Wiener-Hopf integral equation into a partial differential equation (PDE): that’s already a big simplification!

The solution to this PDE can be obtained by various standard methods. Let me just mention here a slightly non-standard but quick method. Clearly, using the diffusive scaling \( x \sim t^{1/2} \), it follows that the function \( Q(x_0, t) \) must have a scaling form: \( Q(x_0, t) = U \left( \frac{x_0}{\sqrt{4Dt}} \right) \). Substituting this scaling form in the PDE (29) one obtains an ordinary differential equation (that’s what scaling always does: reduces a function of two variables into a function of a single scaled variable) valid for \( z \geq 0 \)

\[
\frac{d^2 U}{dz^2} + 2z \frac{dU}{dz} = 0.
\]  

(30)

The initial and the boundary conditions of the PDE translates into two boundary conditions: \( U(0) = 0 \) and \( U(z \to \infty) = 1 \). The solution can be easily obtained: \( U(z) = \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du \). Thus we get the explicit well known solution [5, 21]

\[
Q(x_0, t) = \text{erf} \left( \frac{x_0}{\sqrt{4Dt}} \right)
\]  

(31)

Note that even though we had assumed scaling (without really proving it!), the solution (31) is exact for all \( t \) as one can directly verify by substituting it in the PDE (29). One also sees that for large \( t \) and fixed \( x_0 \) the survival probability decays as a power law

\[
Q(x_0, t) \simeq \frac{x_0}{\sqrt{\pi D t}}
\]  

(32)

The first-passage probability is given by \( F(x_0, t) = -\frac{\partial Q}{\partial t} \) which is just the continuous-time limit of (27). Using (31) one then gets

\[
F(x_0, t) = \frac{x_0}{\sqrt{4\pi Dt^3}} e^{-x_0^2/4Dt}
\]  

(33)
which decays, for large $t$ and fixed $x_0$, as $t^{-3/2}$ with the famous first-passage exponent $3/2$ \cite{4,5,21}.

**A. Pollaczek-Spitzer formula and Sparre Andersen Theorem**

Let us now go back to the basic Wiener-Hopf integral equation \cite{20} that describes the evolution of the survival probability $Q(x_0, n)$. As mentioned before, the solution is nontrivial for a general kernel $\phi(x-x')$. However, when the cumulative distribution $\Psi(x) = \int_{-\infty}^{x} \phi(\xi) d\xi$ is a continuous function such as in examples (i)-(iv) in Section-I (but not for lattice random walk (v) where $\Psi(x)$ is a discontinuous function), an explicit solution was first found by Pollaczek \cite{22} and later independently by Spitzer \cite{23} in a slightly different context. Pollaczek was interested in finding the distribution of the ordered partial sums of a set of i.i.d. variables, whereas Spitzer was interested in finding the distribution of the maximum of the set of partial sums, which is related (see later) to the survival probability. Spitzer’s derivation was more combinatorial. The same integral equation also appeared previously in a variety of half-space transport problems in physics and astrophysics (see \cite{32} and references therein) and several other derivations of the solution of this equation, mostly algebraic in nature, are known \cite{32}. Unfortunately, all these derivations, both the combinatorial as well as the algebraic ones, are highly technical in nature and there is no easy way! Here I will avoid these technical steps and instead just state the final result and discuss its applications. Readers who are interested in the algebraic derivation may consult \cite{33} where we have listed systematically the steps that lead to the final solution.

The solution of \cite{20}, with the initial condition $Q(x_0, 0) = 1$ for all $x_0 > 0$, is in terms of a double Laplace transform of $Q(x_0, n)$

\[
\int_{0}^{\infty} \left[ \sum_{n=0}^{\infty} Q(x_0, n) s^n \right] e^{-p x_0} dx_0 = \frac{1}{p \sqrt{1 - s}} \exp \left[ - \frac{p}{\pi} \int_{0}^{\infty} \ln \left( 1 - s \tilde{\phi}(k) \right) \frac{dk}{p^2 + k^2} \right] \tag{34}
\]

where $\tilde{\phi}(k) = \int_{-\infty}^{\infty} \phi(\xi) e^{ik \xi} d\xi$ is the Fourier transform of the jump length distribution. We will refer this solution in \cite{34} as the Pollaczek-Spitzer formula.

Let us now discuss some consequences of this explicit result.

**B. Sparre Andersen Theorem**

Although the survival probability $Q(x_0, n)$ for arbitrary $x_0$ depends explicitly on the jump length distribution $\phi(\xi)$ as evident in \cite{34}, it turns out that $Q(0, n)$ (the survival probability of the particle up to $n$ steps starting at the origin) becomes, somewhat miraculously, independent of the distribution $\phi(\xi)$ as long as it is a continuous function. To see this, let us take the $p \to \infty$ limit in \cite{34}. Making a change of variable $px_0 = y$ on the lhs of \cite{34} and taking $p \to \infty$ limit, the lhs reduces, to leading order, to $\frac{1}{p} \sum_{n=0}^{\infty} Q(0, n) s^n$. On the rhs, taking $p \to \infty$ limit gives $\frac{1}{p \sqrt{1 - s}}$. Equating the leading order terms (of $O(1/p)$ for large $p$) on both sides gives the identity, for all $s$,

\[
\sum_{n=0}^{\infty} Q(0, n) s^n = \frac{1}{\sqrt{1 - s}}. \tag{35}
\]

Equating powers of $s$ one gets the Sparre Andersen theorem \cite{21}

\[
q(n) = Q(0, n) = \binom{2n}{n} 2^{-2n} \tag{36}
\]

where we have used, for convenience, a shorthand notation $q(n)$ for $Q(0, n)$. Thus, quite amazingly, the survival probability $q(n) = Q(0, n)$ (starting from the origin) is completely universal and that too for all $n$ (and not just for large $n$). No matter whether the jump length distribution is exponential, Gaussian or uniform, $q(n)$ is the same and is given by the simple formula in \cite{36}. Sparre Andersen derived this formula originally using rather involved combinatorial approach. This simple looking formula is however a bit deceptive and led several authors to try to derive it in a ‘simple’ way! Unfortunately, all attempts led to equally complicated derivation (see \cite{34} and references therein). Deriving this formula as a special case of the Pollaczek-Spitzer solution is instructive as it shows that the role of the starting point $x_0 = 0$ is important for this universality. One looses this universality the moment $x_0$ is nonzero.
Let us also note another interesting fact. In the limit of large \( n \), the survival probability \( q(n) \) in (36) decays, to leading order, as
\[
q(n) = Q(0, n) \simeq \frac{1}{\sqrt{n}}
\]  
(37)

Let us emphasise again that this result holds for arbitrary continuous jump distribution \( \phi(\xi) \) including even the Lévy flights! One may ‘naively’ remark that this \( n^{-1/2} \) asymptotic decay is equivalent to the \( t^{-1/2} \) decay of the survival probability in the Brownian limit derived in (32). However, this is not correct and is actually rather subtle as was shown in [33]. Consider first a continuous and symmetric jump distribution with a finite second moment \( \sigma^2 = \int_{-\infty}^{\infty} \xi^2 \phi(\xi) \, d\xi \). To derive the Brownian limit from the Pollaczek-Spitzer formula (34), one first considers the scaling limit, one may ‘naively’ remark that this scaling limit \( x_0 \to \infty \) and \( n \to \infty \) but keeping the ratio \( x_0/\sqrt{n} \) fixed. A careful asymptotic analysis of (31) shows that in this limit the first two leading terms for large \( n \) are given by
\[
Q(x_0, n) \simeq \text{erf} \left( \frac{x_0}{\sqrt{2\sigma^2 n}} \right) + \frac{1}{\sqrt{\pi n}} e^{-x_0^2/2\sigma^2 n}
\]  
(38)

If one now takes the \( x_0 << \sqrt{n} \) limit, one recovers the universal Sparre Andersen result in (37) from the second term on the rhs of Eq. (38). On the other hand, if one keeps the scaling ratio \( x_0/\sqrt{n} \) fixed and takes the strict \( n \to \infty \) limit, the second term in (38) becomes subleading and the first term on the rhs (which remains nonuniversal in this limit as it contains \( \sigma^2 \) explicitly) becomes the leading term that provides the Brownian result in (31) upon identifying \( \sigma^2 n = 2D t \). Thus the \( n^{-1/2} \) universal decay of the survival probability (for \( x_0 = 0 \)) is not quite related to the Brownian result \( t^{-1/2} \); they originate from two different terms in (38).

**Generalization to asymmetric jump distribution:** Actually there exists a generalized Sparre Andersen theorem [20] which holds for non-symmetric (but still continuous) jump length distribution \( \phi(\xi) \). Unlike in the symmetric case, for asymmetric jump distribution of a random walk starting at \( x_0 = 0 \), the probability that the walker is on the positive side up to \( n \) steps is different from the probability that it is on the negative side up to \( n \) steps. Thus one needs to define two different survival probabilities
\[
q_+(n) = \text{Prob}[x_n \geq 0, x_{n-1} \geq 0, \ldots, x_1 \geq 0|x_0 = 0]
\]
\[
q_-(n) = \text{Prob}[x_n \leq 0, x_{n-1} \leq 0, \ldots, x_1 \leq 0|x_0 = 0]
\]  
(39) (40)

For symmetric jump distribution \( q_+(n) = q_-(n) = q(n) \). In the asymmetric case, the generalized Sparre Andersen theorem reads
\[
\tilde{q}_+(s) = \sum_{n=0}^{\infty} q_+(n) s^n = \exp \left[ \sum_{n=1}^{\infty} \frac{p_+}{n} s^n \right]
\]
\[
\tilde{q}_-(s) = \sum_{n=0}^{\infty} q_-(n) s^n = \exp \left[ \sum_{n=1}^{\infty} \frac{p_-}{n} s^n \right]
\]  
(41) (42)

where \( p_+ = \text{Prob}(x_n \geq 0) = \int_0^{\infty} G(x, x_0, n) \, dx \) and \( p_- = \text{Prob}(x_n \leq 0) = \int_{-\infty}^{0} G(x, x_0, n) \) are just the probabilities that exactly at the \( n \)-th step the particle position is positive and negative respectively. For the symmetric (zero bias) case, \( p_+ = p_- = 1/2 \) (by symmetry) and then both equations (41) and (42) reduce to (35).

Let us mention here a special case with drift, noted by Le Doussal and Wiese [70], that is explicitly solvable and that gives rise to a power law decay of the survival probability with a continuously dependent exponent. Consider the evolution,
\[
x_n = x_{n-1} + \mu + \xi_n
\]  
(43)

with \( x_0 = 0 \). Here \( \mu \) represents a drift and \( \xi_n \)'s are i.i.d noise variables each drawn from a symmetric Cauchy distribution
\[
\phi(\xi) = \frac{a}{\pi(\xi^2 + a^2)}
\]  
(44)

In this case, the variable \( y_n = x_n - \mu n \) undergoes a symmetric random walk, \( y_n = y_{n-1} + \xi_n \). Hence, the probability distribution of \( y_n \) at step \( n \), starting from \( y_0 = 0 \), can be easily computed from the free Green’s function discussed in Section I. In fact, the Cauchy distribution corresponds to the Lévy laws in [13] with index \( \mu = 1 \). Hence,
\[
G(y, 0, n) = \frac{1}{an} \Phi_1 \left( \frac{y}{an} \right) = \frac{an}{\pi(y^2 + a^2n^2)}
\]  
(45)
Thus

\[ p_n^+ = \text{Prob}(x_n \geq 0) = \text{Prob}(y_n &geq; -\mu n) = \int_{-\mu n}^{\infty} \frac{an}{\pi(y^2 + a^2n^2)} dy = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(\mu/a) \] (46)

\[ p_n^- = \text{Prob}(x_n \leq 0) = \text{Prob}(y_n \leq -\mu n) = \int_{-\infty}^{-\mu n} \frac{an}{\pi(y^2 + a^2n^2)} dy = \frac{1}{2} - \frac{1}{\pi} \tan^{-1}(\mu/a) \] (47)

Substituting these results in (41) and (42) one gets

\[ \tilde{q}_\pm(s) = \sum_{n=0}^{\infty} q_\pm(n)s^n = \frac{1}{(1-s)^{\zeta_\pm}}; \quad \zeta_\pm = \frac{1}{2} \pm \frac{1}{\pi} \tan^{-1}(\mu/a). \] (48)

Inverting the generating function one then finds that for large \( n \)

\[ q_\pm(n) \simeq \frac{1}{\Gamma(\zeta_\pm)} \frac{1}{n^{\theta_\pm(\mu)}}; \quad \theta_\pm(\mu) = 1 - \zeta_\pm = \frac{1}{2} \mp \frac{1}{\pi} \tan^{-1}(\mu/a). \] (49)

Thus the persistence exponents \( \theta_\pm(\mu) \) are nontrivial and vary continuously with the drift \( \mu \). For example, as \( \mu \to \infty \) (drift away from the origin), \( \theta_+ \to 0 \) (the particle always remains positive) and as \( \mu \to -\infty \) (drift towards the origin), \( \theta_+ \to 1 \) leading to a faster decay than the driftless (\( \mu = 0 \)) case where \( \theta_\pm(0) = 1/2 \).

### IV. FIRST APPLICATION: STATISTICS OF THE MAXIMUM OF THE WALK

The study of the statistics of the maximum of a set of i.i.d. random variables goes back a long way and the subject is called Extreme Value Statistics (EVS) \[33\]. The results are well established and have found a lot of applications in a wide variety of fields \[35\]. However, the standard EVS, developed for i.i.d. variables, does not apply when the random variables are correlated. Recently there has been growing interests in the statistics of the maximum of a set of strongly correlated variables.

More precisely, let us consider again the sequence (1) starting from \( x_0 = 0 \) and the successive noise variables \( \xi_k \)'s are as usual i.i.d variables each drawn from a symmetric and continuous \( \phi(\xi) \). Let us define the global maximum of the walk up to \( n \) steps

\[ M_n = \max(0, x_1, x_2, \ldots, x_n). \] (50)

Clearly \( M_n \) is a random variable taking different values for different realizations of the walk and we would like to compute the distribution of \( M_n \). Note that even though the noise variables \( \xi_k \)'s are uncorrelated, the position of the walker \( x_k \)'s are correlated. For example, when \( \sigma^2 = \langle \xi^2 \rangle \) is finite, it is easy to see from (11) that

\[ \langle x_m x_n \rangle = \sigma^2 \min(m, n) \] (51)

Thus, this is clearly an example where one is trying to compute the distribution of a set of correlated random variables.

The distribution of \( M_n \), as we will see now, is actually closely related to the survival probability \( Q(x_0, n) \) discussed in the previous section. To establish this connection, let us first define the cumulative distribution \( \text{Prob}(M_n \leq y) \). This is just the probability that the walk, starting at \( x_0 = 0 \) at step 0, stays below the level \( x = y \) up to step \( n \), i.e.,

\[ \text{Prob}(M_n \leq y) = \text{Prob}[x_1 \leq y, x_2 \leq y, \ldots, x_n \leq y]. \] (52)

Let us make a shift and define \( z_k = y - x_k \). Then, \( z_k \)'s evolve via the same Markov rule (11), but starting from the initial position \( z_0 = y \) (since \( x_0 = 0 \)). Thus (52) reduces to

\[ \text{Prob}(M_n \leq y) = \text{Prob}[z_1 \geq 0, z_2 \geq 0, \ldots, z_n \geq 0|z_0 = y] = Q(y, n) \] (53)

where \( Q(y, n) \) is precisely the survival probability of the walk up to \( n \) steps, starting at \( y \). The solution of \( Q(y, n) \) is given by the Pollaczek-Spitzer formula (34) for arbitrary continuous distribution \( \phi(\xi) \).
A. Expected Maximum

The exact solution for $Q(y, n)$ in (34) thus also provides an exact solution (or rather the double Laplace transform) of the probability distribution of the maximum, at least in principle. In practice however, the extraction of the moments of the maximum from this explicit Pollaczek-Spitzer formula (34) turns out to be rather nontrivial. For instance, even the first moment, i.e., the expected maximum $E[M_n]$ is hard to extract for all $n$ and arbitrary continuous noise distribution $\phi(\xi)$. This question first arose in the context of a packing problem in two dimensions where $n$ rectangles of variable sizes are packed in a semi-infinite strip of width one [37, 38]. It was shown in Ref. [38] that for the special case of the uniform jump distribution, $\phi(\xi) = 1/2$ for $-1 \leq \xi \leq 1$ and $\phi(\xi) = 0$ outside, for large $n$,

$$E[M_n] = \sqrt{\frac{2n}{3\pi}} - 0.297952 \cdots + O(n^{-1/2}). \quad (54)$$

The leading $\sqrt{n}$ behavior is easy to understand and can be derived from the corresponding behavior of a continuous-time Brownian motion after a suitable rescaling [38]. However, the leading finite-size correction term turns out to be a nontrivial constant $-c$ with $c = 0.29795219028 \cdots$ that was computed in Ref. [38] by enumerating an intricate double series obtained after a lengthy calculation by a different method. It is important to compute the leading finite size correction term very precisely as it provides a sharper estimate of the efficiency of rectangle packing algorithms studied in computer science [37, 38].

Recently, we were able to show [14], starting from the Pollaczek-Spitzer formula (34), that for arbitrary continuous and symmetric jump distribution $\phi(\xi)$ with a finite second moment $\sigma^2 = \int_{-\infty}^{\infty} \xi^2 \phi(\xi) d\xi$, the expected maximum has a similar asymptotic behavior as in the uniform case, namely,

$$E[M_n] = \sigma \sqrt{\frac{2n}{\pi}} - c + O(n^{-1/2}). \quad (55)$$

Moreover, an exact expression for the constant $c$ was found [14]

$$c = -\frac{1}{\pi} \int_{0}^{\infty} \frac{dk}{k^2} \ln \left( \frac{1 - \hat{\phi}(k)}{\sigma^2 k^2 / 2} \right), \quad (56)$$

where $\hat{\phi}(k)$ is the Fourier transform of $\phi(\xi)$. In particular, for the uniform distribution (example (iii)), one has $\hat{\phi}(k) = \sin(k)/k$ and (56) gives

$$c = -\frac{1}{\pi} \int_{0}^{\infty} \frac{dk}{k^2} \ln \left( \frac{6}{k^2} \left( 1 - \frac{\sin k}{k} \right) \right) = 0.29795219028 \cdots \quad (57)$$

The extraction of the constant correction term (56) explicitly from (34) turned out to be highly nontrivial and required a certain number of delicate mathematical manipulations [14]. Interestingly, the same constant $c$ also appears in an apparently different problem when one tries to compute the average flux to a spherical trap in 3-dimensions of particles undergoing Rayleigh flights [39]. The origin of this connection has now been understood–both problems are effectively described by exactly the same Wiener-Hopf integral equation, albeit with two different initial conditions [33]. Many other interesting nontrivial exact results for this spherical trap problem have been recently computed in [33, 40, 41].

For the jump distributions where $\sigma^2$ is infinite, as in the case of Lévy flights, a similar formula for the expected maximum can be derived [14] from the Pollaczek-Spitzer formula. For example, for $\hat{\phi}(k) = 1 - |ak|^\mu + O(k^2)$ (for small $k$ and with $1 < \mu \leq 2$), the expected maximum is given by [14]

$$\frac{E(M_n)}{a} = \frac{\mu}{\pi} \Gamma \left( 1 - \frac{1}{\mu} \right) n^{1/\mu} + \gamma + O(n^{3/\mu - 1}) \quad (58)$$

where the constant

$$\gamma = \frac{1}{\pi} \int_{0}^{\infty} \frac{dz}{z^2} \ln \left( \frac{1 - \frac{\hat{\phi}(z)}{z^\mu}}{2^\mu} \right). \quad (59)$$

For example, for $\hat{\phi}(k) = \exp[-|ak|^\mu]$ with $1 < \mu \leq 2$, one obtains [14]

$$\gamma = \frac{1}{\pi} \int_{0}^{\infty} \frac{dk}{k^2} \ln \left( \frac{1 - e^{-k^\mu}}{k^\mu} \right) = \frac{\zeta(1/\mu)}{(2\pi)^{1/\mu} \sin(\pi/2\mu)}. \quad (60)$$
Note that for $0 < \mu \leq 1$, the expected maximum is strictly infinite.

We close this subsection by just pointing out another completely different problem where the expected maximum of a random walk plays an important role. Recently we showed that the expected perimeter $\langle L_n \rangle$ of the convex hull of a 2-dimensional random walk of $n$ steps is exactly equal (up to a factor $2\pi$) to the expected maximum of the $x$-components of this 2-d random walk: $\langle L_n \rangle = 2\pi(M_n)$ where $M_n = \max(0, x_1, x_2, \ldots, x_n)$ \cite{42, 43}. This connection allowed us to obtain a number of exact results for the statistics of the convex hulls of random walks in two dimensions. We do not discuss this problem in detail here, but refer the interested readers to \cite{42, 43} for details.

### B. Time at which the Random Walker’s Trajectory Achieves its Maximum

In the previous subsection we discussed the statistics of the maximum $M_n$ of an $n$-step walker. Another interesting question is the following: given an $n$-step walker that started at the origin at step 0, at which step $m$ does the maximum $M_n$ happen? In other words, at which time step the $n$-step walker is farthest (in the positive direction) from the origin. This time step $m$ of the occurrence of the maximum is itself a random variable. It turns out that the probability distribution of this time step $P(m|n)$ (given the total number of steps $n$ and that $x_0 = 0$) is also closely related to the survival probability $Q(0, n)$ discussed above.

Before we discuss this, let us remark that for a continuous-time Brownian motion $\xi(t)$ of total duration $t$ and starting at the origin, the analogous probability density $P(t_m|t)$ of the time $t_m$ at which the Brownian motion is maximally away from the origin in the positive direction was computed by Lévy \cite{44}

$$P(t_m|t) = \frac{1}{\pi \sqrt{t_m(t-t_m)}}, \quad 0 \leq t_m \leq t \tag{61}$$

known as the celebrated Lévy’s arcsine law. The name ‘arcsine’ is due to the fact that the cumulative distribution of $t_m$ has the arcsine form: $\text{Prob}(t_m \leq zt) = \frac{2}{\pi} \arcsin(\sqrt{z})$ for $0 \leq z \leq 1$. Thus the maximum is more likely to occur at the beginning $t_m = 0$ or at the end $t_m = t$ of the time window, a fact slightly counterintuitive given that the walk is symmetric around 0. Note that Lévy’s arcsine law also appears in the distribution of the occupation time of a Brownian motion \cite{44}. Let $t_+ = \int_0^t \theta(x(\tau)) d\tau$ be the time spent by a Brownian motion of total duration $t$ on the positive side of the origin. Then the probability density function of $t_+$ has exactly the same form as in (61)

$$P(t_+|t) = \frac{1}{\pi \sqrt{t_+(t-t_+)}}; \quad 0 \leq t_+ \leq t \tag{62}$$

This result looks rather simple, but again is nontrivial to derive. For a derivation using Feynman-Kac path integral technique, see \cite{24}.

The two random variables $t_m$ and $t_+$ represent two rather different observables even though they share the same probability distribution. The derivation in the two cases are also quite different. In mathematical terms, one would say that $t_m \equiv t_+$ where $\equiv$ means that these two random variables have the same statistical law. For the Brownian motion, one can prove this equivalence in law directly \cite{5}, without actually deriving the distribution separately in each case. In fact, this equivalence between $t_m$ and $t_+$ holds for many other Markov processes as well \cite{5}.

Coming back to the random variable $t_m$ of our interest, we note that the distribution of $t_m$ has rather different shapes if one puts various constraints on the Brownian motion. For example, in case of a Brownian bridge i.e. a Brownian motion conditioned to be at $x(0) = 0$ and $x(t) = 0$, the probability density of $t_m$ is known to be uniform \cite{5}

$$P(t_m|t) = \frac{1}{t}; \quad 0 \leq t_m \leq t \tag{63}$$

Recently, using path integral methods, this distribution $P(t_m|t)$ was computed for a variety of other constrained Brownian motions, such as Brownian excursion, Brownian meander, reflected Brownian bridge etc. \cite{45, 48}. Interestingly, $P(t_m = x(t) = L)$ is also precisely the disorder-averaged equilibrium probability density of a particle, moving in an external disordered potential in one dimension, at position $x$ in a box of size $L$ \cite{49}. Some of these results have been recently rederived by a functional renormalization group method \cite{50}. In addition, in the context of the convex hull of Brownian motion in 2-dimensions, it turns out that to compute the mean area of the convex hull of a 2-d Brownian motion, one needs to compute the distribution $P(t_m|t)$ of the corresponding one dimensional Brownian motion \cite{42, 43}. Very recently, the distribution $P(t_m|t)$ has been computed exactly \cite{51} for the random acceleration process (the continuous-time version of the non-Markov evolution rule in \cite{24}). This, to my knowledge, is perhaps the first exact result on $P(t_m|t)$ for a non-Markov process.

The analogous distribution $P(m|n)$ for the discrete-time random walk process in \cite{11} for arbitrary continuous and symmetric jump length distribution $\phi(\xi)$ can be computed exactly from the knowledge of the survival probability...
FIG. 2: A trajectory of a random walker of \( n \) steps, starting at the initial position 0, achieving its maximum \( M_n \) at an intermediate step \( m \).

\[ q(n) = Q(0, n) \]

To see this, consider Fig. 2. Let us just invert this figure and look at the trajectory from the position \( M_n \), i.e., make a change of variable: \( z_k = M_n - x_k \). Next we decompose the trajectory into two parts: the left side for time steps between 0 and \( m \) and the right side for time steps between \( m \) and \( n \). Using the Markov property, these two parts are independent of each other. In the inverted picture, for the left side, let us also invert the ‘time’, i.e., propagate backwards. One has to thus consider all \( z_k \) paths that start at \( z = 0 \) and stays positive up to \( m \) steps (which is equivalent to saying that \( x_k \)’s stays below \( M_n \)). Note that finally we have to integrate over all possible \( M_n \) which means in the inverted picture the final value of \( z_m \) is integrated over. Thus the contribution from this left part is just \( q(m) = Q(0, m) \). A similar reasoning shows that the contribution from the right part is \( q(n-m) = Q(0, n-m) \). Multiplying one gets, upon using the Sparre Andersen result (64),

\[ P(m|n) = q(m)q(n-m) = \binom{2m}{m} \binom{2(n-m)}{(n-m)} 2^{-2n}. \]

One can check easily that this distribution is normalized to unity: \( \sum_{m=0}^{n} P(m|n) = 1 \). Amazingly, thanks to the Sparre Andersen result, the distribution \( P(m|n) \) is again universal for all \( m \) and \( n \), i.e., independent of the jump length distribution as long as it is continuous. Thus, it is given by the same formula (64) for Gaussian, uniform or even for Lévy flights!

In the limit of large \( m \) and \( n \) (keeping the ratio \( m/n = x \) fixed), one gets

\[ P(m|n) \simeq \frac{1}{\pi \sqrt{m(n-m)}} \]

which, once again, may naively look like the arcsine law for the Brownian motion (61). However, note that this asymptotic result in (65) is valid even for Lévy flights. This ‘arcsine’ looking law, valid for arbitrary distribution, is not quite the same as the ‘arcsine’ law in the Brownian limit for the same reason discussed before in the context of the survival probability.
V. SECOND APPLICATION: STATISTICS OF RECORDS

In this section we will discuss another beautiful recent application of the Sparre Andersen theorem \[^{38}\] that results in the universal statistics of records in a random walk sequence (including the Lévy flights) \[^{52}\]. Statistics of records forms an integral part of diverse fields including meteorology \[^{53, 54}\], hydrology \[^{55}\], economics \[^{56}\], sports \[^{57, 59}\] and entertainment industries among others. In popular media such as television or newspapers, one always hears and reads about record breaking events. It is no wonder that Guinness Book of Records has been a world’s best-seller since 1955. Understanding the statistics of records is particularly important in the context of current issues of climatology such as global warming.

Consider any discrete time series \(\{x_0, x_1, x_2, \ldots, x_n\}\) of \(n\) entries that may represent, e.g., the daily temperatures in a city or the stock prices of a company or the budgets of Hollywood films. A record happens at step \(i\) if the \(i\)-th entry \(x_i\) is bigger than all previous entries \(x_0, x_1, \ldots, x_{i-1}\). Statistical questions that naturally arise are: (a) how many records occur up to step \(n\)? (b) How long does a record survive? (c) what is the age of the longest surviving record? Answering these questions is the main goal of the theory of records.

The mathematical theory of records has been studied for over 50 years \[^{60–63}\] and the questions posed in the previous paragraph are well understood in the case when \(x_i\)’s are i.i.d random variables. Recently, there has been a resurgence of interest in the record theory due to its multiple applications in diverse complex systems such as spin glasses \[^{64}\], adaptive processes \[^{65}\] and evolutionary models of biological populations \[^{66, 67}\] and models of growing networks \[^{69}\]. The results in the record theory of i.i.d variables have been rather useful in these different contexts. Recently, Krug has studied the record statistics when the entries have non-identical distributions but still retaining their independence \[^{68}\]. However, in most realistic situations the entries of the time series are correlated. Very little seems to be known about the statistics of records for a correlated time series. Recently, we developed a general formalism \[^{52}\] to study the statistics of records in a random walk sequence evolving via \(\sum\) with an arbitrary jump distribution \(\phi(\xi)\). We showed \[^{52}\] that for symmetric and continuous jump distributions, the statistics of records have universal properties as a consequence of the Sparre Andersen theorem discussed before. Below we discuss this formalism developed in \[^{52}\] in some details.

To proceed, let us consider a realization of the sequence \(x_i\)’s in \((\sum)\) up to \(n\) steps. The discussion below is general and holds even for asymmetric jump distribution \(\phi(\xi)\). Let \(R\) be the number of records in this realization. We use the convention that the first entry \(x_0\) is counted as a record. Evidently \(R\) is an integer. Let \(l_i\) denote the time interval between the \(i\)-th and the \((i+1)\)-th record. Thus, \(l_i\) is the age of the \(i\)-th record, i.e., it denotes the time up to which the \(i\)-th record survives. We will use the shorthand notation \(\bar{l} = \{l_1, l_2, \ldots, l_R\}\) to denote the set of \(R\) successive intervals (see Fig. 3). Note that the last record, i.e., the \(R\)-th record still stays a record at the \(n\)-th step since there is no more record breaking events after it. Hence \(l_R\) (the last one in Fig. 3) denotes the number of steps after the occurrence of the last record till the last step \(n\). The main idea is to first calculate the joint probability distribution \(P(\bar{l}, R|n)\) of the ages \(\bar{l}\) and the number \(R\) of records, given the length \(n\) of the sequence.

To compute this joint distribution we need two quantities as inputs. First, let \(q_\cdot(l)\) denote the probability that a walk, starting initially at \(x_0\), stays below its starting position \(x_0\) up to step \(l\). Clearly \(q_\cdot(l)\) does not depend on the starting position \(x_0\) due to translational invariance and one can just set \(x_0 = 0\). Then \(q_\cdot(l)\) is precisely the survival probability defined in \[^{10}\] whose generating function \(\tilde{q}_\cdot(s)\) is given by the generalized Sparre Andersen result in \[^{12}\]. Recall that for the symmetric case \(q_\cdot(l) = q_+(l) = q(l)\) is universal and its generating function is given exactly in \[^{39}\]

\[
\tilde{q}(s) = \sum_{l=0}^{\infty} q(l) s^l = \frac{1}{\sqrt{1-s}}.
\] (66)

The second input is the first-passage probability \(f_\cdot(l)\) that the walker crosses its starting point \(x_0\) for the first time between steps \((l - 1)\) and \(l\) from below \(x_0\) (see Fig. 3). Once again, \(f_\cdot(l)\) does not depend on the starting point \(x_0\) due to translational invariance and one can set \(x_0 = 0\). Setting \(x_0 = 0\), it follows that \(f_\cdot(l) = q_\cdot(l - 1) - q_\cdot(l)\) whose generating function can be expressed in terms of that of \(q_\cdot(l)\)

\[
\tilde{f}_\cdot(s) = \sum_{l=1}^{\infty} f_\cdot(l) s^l = 1 - (1-s)\tilde{q}_\cdot(s).
\] (67)

In the symmetric case, \(f_+(l) = f_-(l) = f(l)\) with a generating function

\[
f(s) = 1 - (1-s)\tilde{q}(s) = 1 - \sqrt{1-s}
\] (68)

where we have used \[^{66}\].
FIG. 3: A realization of the random walk sequence \( \{x_0 = 0, x_1, x_2, \ldots, x_n\} \) of \( n \) steps with \( R \) records. Records are shown as big red dots. Note that a local maximum of the walk is not necessarily a record. The set \( \{l_1, l_2, \ldots, l_R\} \) denotes the time intervals between successive records.

Armed with these two ingredients \( q_{-}(l) \) and \( f_{-}(l) \), we can then write down explicitly the joint distribution of the ages \( \vec{l} \) and the number \( R \) of records

\[
P(\vec{l}, R | n) = f_{-}(l_1) f_{-}(l_2) \cdots f_{-}(l_{R-1}) q_{-}(l_R) \delta_{\sum_{i=1}^{R} l_i, n}
\]

(69)

where we have used the Markov renewal property of random walks which dictates that the successive intervals are statistically independent, except for the global sum rule that the total interval length is \( n \) (see Fig. 3) which is incorporated by the delta function. Note that since the \( R \)-th record is the last one (i.e., no more records have happened after it), the interval to its right has distribution \( q_{-}(l) \) rather than \( f_{-}(l) \). One can check that \( P(\vec{l}, R | n) \) is normalized to unity when summed over \( \vec{l} \) and \( R \).

Note that in the case of symmetric jump distribution, since \( q_{-}(l) = q(l) \) and \( f_{-}(l) = f(l) \) are universal due to the symmetric Sparre Andersen theorem, it follows that \( P(\vec{l}, R | n) \) and all marginals of it are also universal. Below we will focus on the symmetric case only.

A. Universal Distribution of the Number of Records up to step \( n \)

Let us focus here on the case of symmetric jump distribution \( \phi(\xi) = \phi(-\xi) \) where \( \phi(\xi) \) is continuous. In this case we can replace \( q_{-}(l) \) by \( q(l) \) and \( f_{-}(l) \) by \( f(l) \) in the joint distribution (69). Let us first compute the probability distribution of the number of records \( R \), \( P(R|n) = \sum_{\vec{l}} P(\vec{l}, R | n) \). To perform this sum, it is easier to consider its generating function. Multiplying (69) by \( s^n \) and summing over \( \vec{l} \), one gets

\[
\sum_{n=R-1}^{\infty} P(R|n) s^n = [\hat{f}(s)]^{R-1} \hat{q}(s) = \frac{(1 - \sqrt{1 - s})^{R-1}}{\sqrt{1 - s}}
\]

(70)

where we have used the explicit expressions for \( \hat{q}(s) \) and \( \hat{f}(s) \) from Eqs. \( 66 \) and \( 68 \).

By expanding in powers of \( s \) and computing the coefficient of \( s^N \) one gets the explicit result

\[
P(R|n) = \binom{2n-R+1}{n} 2^{-2n+R-1}
\]

(71)

which is universal for all \( R \) and \( n \). The moments of \( R \) are also naturally universal and can be computed for all \( n \). For
example, the first three moments are

\[
\langle R \rangle = (2n + 1) {2n \choose n} 2^{-2n},
\]

\[
\langle R^2 \rangle = 2n + 2 - \langle R \rangle,
\]

\[
\langle R^3 \rangle = -6n - 6 + (7 + 4n)\langle R \rangle.
\]  

(72)

In particular, for large \( n \), the mean, variance and the skewness behave as

\[
\text{Mean} : \quad \langle R \rangle \simeq \frac{2}{\sqrt{\pi}} \sqrt{n}
\]

\[
\text{Variance} : \quad \langle R^2 \rangle - \langle R \rangle^2 \simeq 2 \left( 1 - \frac{2}{\pi} \right) n
\]

\[
\text{Skewness} : \quad \frac{\langle (R - \langle R \rangle)^3 \rangle}{\langle (R - \langle R \rangle)^2 \rangle^{3/2}} \simeq \frac{4(4 - \pi)}{(2\pi - 4)^{3/2}}
\]

(73)

In \cite{52}, these results were also verified numerically for different jump length distributions (uniform, Gaussian, Cauchy) all giving the same universal answer.

The results in \cite{73} suggest that there is only a single scale for the number of records \( R \sim n^{1/2} \). This is confirmed by analysing the full distribution \( P(R|n) \) of \( R \) in \cite{71} in the limit of large \( n \). One finds that \( P(R|n) \) actually has the following scaling form for large \( n \) \cite{61}

\[
P(R|n) \simeq \frac{1}{n^{1/2}} g \left( \frac{R}{n^{1/2}} \right) ; \quad g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/4}.
\]  

(74)

Thus the distribution is broad in the sense that the mean and the standard deviation measuring the fluctuation around the mean, both scale as \( \sim n^{1/2} \). Also, the mode of this distribution, i.e., the most probable (typical) value of \( R \) is at \( R = 0 \). It is interesting to compare this result for the random walk sequence \cite{1} with that of an uncorrelated i.i.d sequence where each entry \( x_i \) is a random variable drawn from some distribution \( p(x) \). In the latter case, it is well known \cite{61} that the distribution of the number of records \( P(R|n) \) does not depend on \( p(x) \), and for large \( n \), it approaches a Gaussian,

\[
P(R|n) \simeq \frac{1}{\sqrt{2\pi \log n}} \exp \left[ - \frac{(R - \log n)^2}{2 \log n} \right]
\]  

(75)

with mean \( \langle R \rangle = \log n \) and the standard deviation \( \sqrt{\log n} \). This distribution has its peak at \( R = \log n \), in stark contrast to the random walk case where the most probable value of \( R \) is zero. In addition, even the fluctuations of \( R \) are small compared to the mean for large \( n \), again in contrast to the random walk case where the fluctuations are large \( \sim O(\sqrt{n}) \) for large \( n \). Thus the effect of correlation in the random walk sequence manifests itself in a broad scaling distribution for the number of records.

\section*{B. Universal Age Distribution of Records}

Since the mean number of records grows as \( \langle R \rangle \sim n^{1/2} \), it follows that the typical age of a record grows also as \( \langle l \rangle = \langle R \rangle / \langle R \rangle \sim n^{1/2} \) for large \( n \). However there are rare records that are not typical and their ages follow different statistics. For example, what is age distribution of the longest lasting and the shortest lasting records? These extreme statistics of ages can also be derived from the joint distribution in \cite{69} and hence they are also universal and independent of \( \phi(\xi) \).

Let us first consider the longest lasting record with age \( l_{\text{max}} = \max(l_1, l_2, \ldots, l_R) \). It is easier to compute its cumulative distribution \( Y(l|n) = \text{Prob}[l_{\text{max}} \leq l] \) given \( n \). Now, if \( l_{\text{max}} \leq l \), it follows that each of the intervals \( l_i \leq l \) for \( i = 1, 2, \ldots, R \). Thus, we need to sum up \cite{69} over all \( l_i \)'s and \( R \) such that \( l_i \leq l \) for each \( i \). As usual it is easier to carry out this summation by considering the generating function and we get

\[
\sum_n Y(l|n) s^n = \frac{\sum_{l'=1}^l q(l') s^{l'}}{1 - \sum_{l'=1}^l f(l') s^{l'}}.
\]  

(76)
One can extract, in principle, the distribution \( Y(l|n) \) from this general expression. In particular, the asymptotic large \( n \) behavior of the average \( \langle l_{\text{max}} \rangle = \sum_{n=1}^{\infty} [1 - Y(l|n)] \) can be extracted explicitly \[52\]

\[
\langle l_{\text{max}} \rangle \simeq c_1 n; \quad c_1 = 2 \int_{0}^{\infty} dy \log \left[ 1 + \frac{1}{2\sqrt{\pi}} \Gamma(-1/2, y) \right] = 0.626508 \ldots
\]  

(77)

where \( \Gamma(-1/2, y) = \int_{y}^{\infty} dx x^{-3/2} e^{-x} \) is the incomplete Gamma function. Thus, the age of the longest record \((\sim n)\) is much large than the typical age \((\sim \sqrt{n})\) for large \( n \).

For the shortest lasting record \( l_{\text{min}} = \min(l_1, l_2, \ldots, l_R) \), it is also useful to consider the cumulative distribution \( Z(l|n) = \text{Prob}[l_{\text{min}} \geq l] \) given \( n \). This event is equivalent to having the lengths, \( l_i \geq l \) for all \( i = 1, 2, \ldots, R \). Following similar procedure as in the case of the longest lasting record, one finds the generating function

\[
\sum_{n} Z(l|n) s^n = \frac{\sum_{i=1}^{\infty} q(l_i) s^i}{1 - \sum_{l'=l} f(l) s^{l'}}
\]  

(78)

One can then extract, in a similar way, the asymptotic large \( n \) behavior of \( \langle l_{\text{min}} \rangle \sim \sqrt{n/\pi} \[52\]. Thus, the mean age of the shortest lasting record grows in a similar way as that of a typical record, i.e., as \( \sqrt{n} \), albeit with a smaller prefactor \( 1/\sqrt{\pi} = 0.56419 \ldots \) compared with \( \sqrt{\pi/4} = 0.88623 \ldots \).

### C. Two Generalizations

In the discussion above for the statistics of records, we had assumed that the jump length distribution \( \phi(\xi) \) is symmetric and continuous. However, the basic renewal equation \[59\] is valid for continuous but asymmetric jump distribution as well. The only difference is that we have to use the appropriate expressions for \( f_- (l) \) and \( q_- (l) \) from the generalized Sparre Andersen theorem. For example, the generating function for the distribution \( P(R|n) \) for the number of records up to step \( n \) is given by the asymmetric version of \[70\]

\[
\sum_{n=1}^{\infty} P(R|n) s^n = [\tilde{f}_- (s)]^{R-1} \tilde{q}_- (s) = [1 - (1-s)\tilde{q}_- (s)]^{R-1} \tilde{q}_- (s)
\]  

(79)

where \( \tilde{q}_- (s) \) is given by \[82\].

Indeed, for the special case of a random walk sequence in presence of a drift \( \mu \) and Cauchy distributed jumps as in \[13\], one can obtain explicit results \[70\] for \( P(R|n) \) in \( \text{(79)} \). Using \( \text{(18)} \) one gets the exact generating function: \( \tilde{q}_- (s) = (1-s)^{-\zeta_-} \) and substituting this in \( \text{(79)} \) gives

\[
\sum_{n} P(R|n) s^n = \frac{[1 - (1-s)^{1-\zeta_-}]}{(1-s)^{\zeta_-}}^{R-1}
\]  

(80)

from which it follows that the average number of records grows anomalously \( (R) \sim n^{1-\zeta_-} \sim n^{\zeta_+} \) for large \( n \). Using \( \zeta_+ = 1/2 + \tan^{-1}(\mu/a)/\pi \), one sees that as \( \mu \to \infty \) (positive drift away from the origin), \( \zeta_+ \to 1 \) and thus the average number of records grows linearly with the number of steps \( n \), i.e., at every step a new record happens on an average. Of course, this is expected in presence of an infinite drift since the particle moves ballistically in the positive semi-axis. On the other hand, as \( \mu \to -\infty \), \( \zeta_+ \to 0 \) indicating that the average number of records do not grow with \( n \) for large \( n \). This is also expected since the particle mostly stays on the negative side of the origin when \( \mu \to -\infty \) and thus hardly ever makes a positive record. These results were then used to understand the anomalous avalanche size distribution in a model of a particle moving in a random potential \[70\].

Another interesting generalization of these results emerged from the following observation: it turns out that the constant \( c_1 = 0.626508 \ldots \) that appears as the prefactor of the linear growth of the longest lasting record in \( \text{(77)} \) also appears in the excursion theory of Brownian motion \[71\]. Let us consider a Brownian motion over a time interval \([0,t]\) and consider the set of successive zero crossing intervals or excursions (see Fig. \[4\]). Let us denote the maximum excursion length up to time \( t \) by \( l_{\text{max}}(t) \)

\[
l_{\text{max}}(t) = \max(\tau_1, \tau_2, \ldots, \tau_N, A(t))
\]  

(81)

where \( A(t) \) denotes the length of the last interval before \( t \) (see Fig. \[4\]). Let \( Q(t) = \text{Prob}[l_{\text{max}}(t) = A(t)] \) denote the probability the last incomplete excursion is the longest one. Then it turns out \[71\] that \( Q(t) \) tends, for large \( t \), to
the same constant $Q(t) \rightarrow c_1 = 0.626508 \ldots$ as in Eq. (77). We were able to understand recently why this same constant $c_1$ appears in apparently different observables namely (i) in the length of the longest lasting record and (ii) the probability $Q(t)$ that the last excursion is the longest [72]. This understanding led us to study the statistics of $l_{\text{max}}(t)$ and that of $Q(t)$ for generic stochastic processes going beyond the simple Brownian motion [72]. The statistics of $l_{\text{max}}(t)$ turns out to have interesting universal features that allowed us to distinguish between stochastic processes that are smooth (i.e., with a finite density of zero crossings) versus the ones that are rough (where the density of zero crossings is infinite as in the case of the Brownian motion) [72].

VI. SUMMARY AND CONCLUSION

In summary, I have discussed the universal first-passage properties associated with a discrete-time random walk sequence consisting of $n$ steps, where the walker starts at the origin $x_0 = 0$ and at each step jumps by a random amount drawn independently at each step from a symmetric and continuous distribution $\phi(\xi)$. The first-passage probability is universal, i.e., independent of the jump length distribution due to the Sparre Andersen theorem. We have then used the consequence of this result on the statistics of two extreme random variables: (i) the global maximum of the walk and the step at which it occurs and (ii) the number and ages of records. We have seen that the distribution of the time of the maximum as well as the record statistics become universal as a consequence of the Sparre Andersen theorem.

The distribution of the value of the maximum, however, is non-universal and depends explicitly on $\phi(\xi)$. The random variables belonging to this sequence are correlated. For the distribution of the maximum, the standard EVS of i.i.d. random variables does not apply due to these correlations. The computation of the distribution of the maximum for this discrete-time sequence is thus nontrivial due to these correlations, even though in the corresponding continuous-time Brownian motion it is easy to compute. However, thanks to the Pollaczek-Spitzer formula, one knows, at least in principle, how to compute the generating function of this maximum distribution for arbitrary symmetric and continuous $\phi(\xi)$. The leading large $n$ behavior of the moments of the maximum can be extracted relatively easily from this explicit Pollaczek-Spitzer formula. However, extracting the subleading finite size correction term turns out to be much trickier. At least for the expected maximum, we have seen how to compute exactly the leading finite size
correction term, for the case when the jump distribution has a finite variance and also for the case of Lévy flights with index $1 < \mu \leq 2$. These results are interesting because the expected maximum of a discrete-time random walk is exactly related to the perimeter of the convex hull of a planar random walk which has important applications in the estimation of home range of animals in ecology \cite{42, 43}.

It would also be interesting to compute the expected maximum and the distribution of the time of its occurrence in presence of a drift. In the Brownian limit, in presence of a drift, the distribution of the time at which the maximum occurs has been computed using a path integral method \cite{47}, with interesting applications in finance. However, for the discrete-time case, I am not aware of any result so far and it would be interesting to compute this distribution.

There are interesting generalizations of the results presented here. For example, concerning the statistics of records, we have studied only the statistics of ‘positive’ records, i.e., when the value $x_i$ of a record that occurs at step $i$ is bigger than all previous values, given that the sequence started at $x_0 = 0$. It would be interesting to investigate the statistics of the records of the absolute values of the sequence, i.e., of $\{|0, |x_1|, |x_2|, \ldots, |x_n|\}$ which, to my knowledge, has not yet been studied \cite{72}.

As I already mentioned, the record statistics of this Markov sequence has been studied in presence of a constant drift with interesting applications in avalanche dynamics \cite{70}. In particular, we have seen one case, namely the Cauchy distribution with drift, where the average number of records grows with the sequence size $n$ anomalously with a nontrivial drift-dependent exponent \cite{70}. It would not be difficult to compute the distribution of the ages of records in this particular case. The study of the age distribution of records for arbitrary asymmetric jump distribution remains an open problem.

Another interesting generalization is to consider the Markov sequence generated by the recursion: $x_n = r x_{n-1} + \xi_n$ where $0 \leq r \leq 1$ is a parameter and $\xi_n$’s are, as before, symmetric i.i.d. noise variables. This is just a discrete-time analogue of the continuous-time Orstein-Uhlenbeck (OU) process of a particle moving in a harmonic potential. This is seen by writing, $x_n = x_{n-1} - (1-r)x_{n-1} + \xi_n$ which, in the continuous-time limit (alongwith $r \to 1$ limit), becomes the process OU process, $dx/dt = -\lambda x + \xi(t)$ where $\xi(t)$ is a zero mean Gaussian white noise. This discrete-time sequence has many applications, e.g., it appears in the context of the practical sampling of experimental data on the persistence of a stochastic process \cite{74, 77} and also in the simple system of a ball bouncing non-elastically on a noisy platform \cite{78}. In the latter context, the parameter $0 \leq r \leq 1$ represents the coefficient of restitution of the collision of the ball with the platform \cite{78} and the Brownian limit $r = 1$ corresponds to elastic collision. The first-passage properties of this sequence for generic $0 < r < 1$ turns out to be highly nontrivial even for a Gaussian noise distribution \cite{74}. Explicit exact result is known only for the exponential noise distribution \cite{78}. While, for generic $0 < r < 1$, these first-passage properties are nonuniversal, one recovers several interesting universal properties in the elastic limit $r \to 1$ \cite{78}. It would be interesting to study the statistics of the maximum and that of the records in this simple Markov sequence for arbitrary $0 < r < 1$ and arbitrary noise distribution.

In conclusion, there are still many unresolved questions associated with even simple one dimensional random walks. Depending on the new applications, new questions emerge requiring new techniques to solve them which are often nontrivial and interesting.

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