On the Degrees of Freedom of $K$-User SISO Interference and $X$ Channels with Delayed CSIT

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Abstract

The $K$-user single-input single-output (SISO) AWGN interference channel and $2 \times K$ SISO AWGN X channel in i.i.d. fading environment are considered where the transmitters have the delayed channel state information (CSI) through noiseless feedback links. It is known that both channels have no more than one degree of freedom (DoF) without CSI at transmitters. Transmission schemes based on a new multiphase interference alignment approach are proposed for these channels that achieve DoF values greater than one with delayed CSI at transmitters (delayed CSIT), except for the two-user interference channel. Moreover, the achieved DoFs are strictly increasing in $K$, and as $K \to \infty$, approach limiting values of $\frac{4}{6 \ln 2 - 1} \approx 1.2663$ and $\frac{1}{\ln 2} \approx 1.4427$ for interference and X channels, respectively. The achieved DoFs are greater than the best previously reported DoFs for these channels with delayed CSI at transmitters.

I. INTRODUCTION

The impact of feedback on the performance of a communication system has been extensively investigated from different perspectives. Although a negative result was first established by Shannon [2] indicating that feedback does not increase the capacity of a memoryless point-to-point channel, there are various results affirming the significant effect of feedback on other performance criteria such as complexity and error probability of this channel.

In multi-terminal networks, however, it has been shown that feedback can enlarge the capacity region of a multi-user channel. The capacity region enlargements for multiple access and broadcast channels with access to noiseless output feedback are reported for AWGN cases in [3], [4] and for discrete cases
The capacity region enlargements in the AWGN multiple access and broadcast channels are “additive” which are bounded with increase of SNR. Suh et al. in [7] characterized the capacity region of a two-user AWGN interference channel (to within two bits) under the assumption that each transmitter has access to a noiseless feedback link from its respective receiver. They have shown that such feedback links can provide “multiplicative” gain in the capacity region of AWGN interference channels, i.e., the gap between the feedback and non-feedback capacity regions can be arbitrarily enlarged as SNR increases.

An important performance measure for a power constrained communication channel is its degrees of freedom (DoF) which determines the behavior of the sum-capacity in high SNR regime. While it is always assumed that the receiver(s) know the channel coefficients perfectly and instantaneously, the CSI knowledge of transmitter(s) (CSIT) is usually subject to some limitations. At one extreme, it is assumed that the transmitter(s) know the CSI instantaneously and perfectly (full CSIT assumption). Under this condition, the capacity region (and hence, the DoF region) of the multiple-input multiple-output (MIMO) broadcast channel was characterized in [8]. The DoF of the $K$-user SISO interference channel was shown to be $\frac{K}{2}$ with full CSIT [9]. It was shown in [10] that the $S \times K$ SISO X channel can achieve $\frac{SK}{S+K-1}$ DoF with full CSIT. At the other extreme, the transmitter(s) have no knowledge about CSI (no CSIT assumption). In this case, the $K$-user MISO broadcast channel was studied in [11]. Other works include [12], [13] which investigate the DoF region of two-user MIMO broadcast and interference channels without CSIT. Later in [14], the DoF regions of $K$-user MIMO broadcast, interference and X channels with no CSIT were characterized. In specific, by developing some upper and lower bounds, they showed that the MISO broadcast, SISO interference and SISO X channel can achieve no more than one DoF.

The access of transmitter(s) to the full instantaneous CSI is essentially an ideal assumption. While this assumption can be realized when the channel is subject to slow fading, it is not practically viable when channel variations are fast. In this situation, it is curious whether any type of feedback can improve the DoF performance of the channel. Recently, Maddah-Ali and Tse in [15], [16] considered the MISO broadcast channel in fast fading environment where the transmitter obtains the past CSI through noiseless feedback (delayed CSIT assumption). Surprisingly, they showed that even in a fast fading setup, the delayed version of CSIT significantly improves the channel DoF. In particular, it was shown that the $K$-user MISO broadcast channel with $M \geq K$ antennas at transmitter has $\frac{K}{1+\frac{1}{2}+\cdots+\frac{1}{K}}$ DoF under delayed CSIT assumption. Subsequently, Abdoli et al. in [17] investigated the DoF of the three-user MIMO broadcast channel under delayed CSIT assumption and the DoF region of the two-user MIMO broadcast channel with delayed CSIT was obtained in [18]. The main idea in broadcast channels lies behind the following observations: Since the transmitter has access to both past CSI and past transmitted data symbols, it perfectly knows the whole past interference at each receiver. Also, an interference term at a receiver can be
a useful piece of information for some other receivers about their data symbols. Therefore, retransmission of such interference terms not only aligns the interference at some receivers, but also provides other receivers with a desired piece of information about their data symbols.

In contrast to the broadcast channel, in networks with distributed transmitters such as interference and X channels, there is a fundamental constraint in using the knowledge of past CSI at transmitters: Each transmitter has only access to its own data symbols. Indeed, a transmitter cannot obtain the whole past interference at a receiver when the interference is due to more than one interferer. This restriction turns out to be a performance limiting factor in terms of DoF of the system for networks with more than two users. The DoF of the two-user MIMO interference channel with delayed CSIT is addressed in [19], [20]. In [21], Maleki et al. showed that the $2 \times 2$ SISO X and 3-user SISO interference channel can achieve $\frac{8}{7}$ and $\frac{5}{7}$ DoF, respectively, under delayed CSIT assumption. Soon thereafter, Ghasemi et al. in [22] achieved greater DoF of $\frac{6}{5}$ for the $2 \times 2$ X channel and $\frac{3}{2}$ for the $3 \times 3$ X channel. They also proposed an immediate generalization of their transmission scheme to the $K \times K$ case, $K \geq 2$, and achieved $\frac{4}{3} - \frac{2}{3(3K-1)}$ DoF with delayed CSIT.

In this paper, we first investigate the 3-user SISO interference channel with delayed CSIT and show that $\frac{36}{31}$ DoF is achievable in this channel. This is greater than the previously reported $\frac{9}{8}$ DoF in [21]. Then, we consider the $K$-user SISO interference channel for $K > 3$ with delayed CSIT, and propose a transmission scheme that achieves DoF values which are strictly increasing in $K$ and approach the limiting value of $\frac{4}{6 \ln 2 - 1} \approx 1.2663$ as $K \to \infty$. Thereafter, we consider the $2 \times 3$ SISO X channel and show that this channel can achieve $\frac{7}{2}$ DoF under delayed CSIT assumption. By generalizing our transmission scheme to the $2 \times K$ SISO X channel with delayed CSIT, we achieve DoF values which are strictly increasing in $K$ and approach the limiting value of $\frac{1}{\ln 2} \approx 1.4427$ as $K \to \infty$. For $K \geq 3$, our achievable DoFs for the $2 \times K$ X channel are strictly greater that the achievable DoFs reported in [22] for the $K \times K$ X channel with delayed CSIT.

The schemes proposed in this paper for the $K$-user interference and $2 \times K$ X channels operate in $K$ main phases: In phase 1, the transmitters send fresh data symbols together with some redundancy over time. The redundancy is such that “part” of the interference can be removed at each receiver by the end of this phase. Then, each transmitter exploits its knowledge of past CSI and its own transmitted data symbols to obtain the interference terms it caused at the non-intended receivers (if not already removed). Each of these interference terms, if being retransmitted, can align the past interference at a receiver while providing a useful linear combination for another receiver. Hence, they can be considered as common messages of order 2 and are fed to the system in phase 2 together with some redundancy over time. The transmitted redundancy again helps some receivers remove part of the interference. The transmitters again
using the past CSI and their own transmitted order-2 messages, will obtain their non-removed interference terms at non-intended receivers. This yields generation of common messages for subsets of cardinality 3 of receivers. These order-3 messages, in turn, will be transmitted in phase 3, towards generation of order-4 messages. This procedure goes on phase by phase up to phase $K$ where order-$K$ messages will be delivered to all receivers without generating higher order messages.

The rest of the paper is organized as follows: the system model is described in Section II. Section III presents the main results of the paper. In Sections IV and V our transmission schemes for SISO interference and X channels with delayed CSIT are introduced, respectively, and their achievable DoFs are obtained. Finally, Section VI concludes the paper.

II. System Model

For any integer $K \geq 2$, the discrete-time $K$-user SISO AWGN interference channel (IC) with private messages is defined by a set of $K$ transmitter-receiver pairs $(\text{TX}_i, \text{RX}_i), 1 \leq i \leq K$, where $\text{TX}_i$ wishes to transmit message $W[i] \in \mathcal{W}[i]$ to $\text{RX}_i$. Moreover, the input-output relationship of this channel in time slot $t$, $t = 1, 2, \cdots$, is specified by

$$y_j(t) = \sum_{i=1}^{K} h_{ji}(t)x_i(t) + z_j(t), \quad 1 \leq j \leq K,$$

where $x_i(t) \in \mathbb{C}$ is the complex channel input symbol transmitted by $\text{TX}_i$, $y_j(t) \in \mathbb{C}$ and $z_j(t) \sim \mathcal{CN}(0, 1)$ are the complex received symbol and additive complex Gaussian noise at the input of $\text{RX}_j$, respectively, and $h_{ji}(t) \in \mathbb{C}$ is the complex channel coefficient from $\text{TX}_i$ to $\text{RX}_j$. The channel input is subject to the average power constraint

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \mathbb{E}[|x_i(t)|^2] \leq P,$$

where $\tau$ is the block length. The channel coefficients are assumed to be i.i.d. across the transmitters and receivers, and the noise is assumed to be i.i.d. across receivers as well as time.

We further assume that the channel is subject to fast fading, that is, the channel coefficients are i.i.d. across time. Define the $K \times K$ channel matrix $H(t) \triangleq [h_{ji}(t)]_{1 \leq i,j \leq K}$. We make the following assumptions on the knowledge of transmitters and receivers about the CSI:

- At the beginning of time slot $t$, $t \geq 2$, all transmitters and receivers perfectly know $\{H(t')\}_{t'=1}^{t-1}$ (via feedback links).
- At the beginning of time slot $t$, $t \geq 1$, each receiver $\text{RX}_j$, $1 \leq j \leq K$, perfectly knows all the channel coefficients terminating at $\text{RX}_j$ (i.e., $\{h_{ji}(t)\}_{i=1}^{K}$).

We investigate this channel under the above set of two assumptions which is referred to as delayed CSIT.
assumption. Let $\mathbf{R} \triangleq (R_1, R_2, \ldots, R_K) \in (\mathbb{R}^+)^K$ be a $K$-tuple of rates corresponding to the transmitter-receiver pairs $(TX_1, RX_1)$, $(TX_2, RX_2)$, \ldots, $(TX_K, RX_K)$. A $(2^\tau \mathbf{R}, \tau)$ code of block length $\tau$ and rate $\mathbf{R}$ with delayed CSIT consists of $K$ sets of encoding functions $\{\varphi_{t,\tau}^{[i]}\}_{1 \leq t \leq \tau}, 1 \leq i \leq K$,

$$\varphi_{t,\tau}^{[i]} : \mathcal{W}^{[i]} \times \mathbb{C}^{K \times (t-1)} \rightarrow \mathbb{C},$$

$$x_i(t) = \varphi_{t,\tau}^{[i]}(W^{[i]}, \{H(t')\}_{t'=1}^{t-1}), \quad 1 \leq t \leq \tau,$$

satisfying the power constraint of Eq. (2), together with $K$ decoding functions $\psi_{\tau}^{[i]}, 1 \leq i \leq K$,

$$\psi_{\tau}^{[i]} : \mathbb{C}^\tau \rightarrow \mathcal{W}^{[i]},$$

$$\hat{W}_{\tau}^{[i]} = \psi_{\tau}^{[i]}(\{y_i(t)\}_{t=1}^{\tau}).$$

All encoding and decoding functions are revealed to all transmitters and receivers before the transmission begins. Defining the average probability of error at $RX_i$ for this code as

$$P_{e,\tau}^{[i]} \triangleq \mathbb{P}\left(\hat{W}_{\tau}^{[i]} \neq W^{[i]}\right) = \frac{1}{2^{\tau R_i}} \sum_{w=1}^{2^{\tau R_i}} \mathbb{P}\left(\hat{W}_{\tau}^{[i]} \neq W^{[i]}|W^{[i]} = w\right), \quad 1 \leq i \leq K,$$

the average probability of error of this code is given by

$$P_{e,\tau} \triangleq \max_{1 \leq i \leq K} P_{e,\tau}^{[i]}.$$

A rate tuple $\mathbf{R}$ is said to be achievable if there exists a sequence $\{(2^\tau \mathbf{R}, \tau)\}_{\tau=1}^{\infty}$ of codes, for which $\lim_{\tau \rightarrow \infty} P_{e,\tau} = 0$. The closure of all achievable rate tuples $\mathbf{R}$ is called the capacity region of this channel with delayed CSIT and power constraint $P$ and is denoted by $\mathcal{C}(P)$.

We consider this channel with $P \rightarrow \infty$, and define an achievable sum degrees of freedom (simply called “achievable degrees of freedom”, or “achievable DoF”) as

$$\text{DoF}^{IC}_1(K) \triangleq \lim_{P \rightarrow \infty} \sum_{i=1}^{K} \frac{R_i}{\log_2 P}.$$  

(7)

The supremum of all achievable DoFs is called the DoF of this channel with delayed CSIT, and is denoted by $\text{DoF}^{IC}_1(K)$. More formally,

$$\text{DoF}^{IC}_1(K) \triangleq \lim_{P \rightarrow \infty} \sup_{\mathbf{R} \in \mathcal{C}(P)} \sum_{i=1}^{K} \frac{R_i}{\log_2 P}.$$  

(8)

We indeed consider a more general transmission setup as follows: Fix an integer $m, 1 \leq m \leq K$. Define $\mathcal{S}_m$ as a subset of cardinality $m$ of $\{1, 2, \cdots, K\}$. Obviously, $\mathcal{S}_K = \{1, 2, \cdots, K\}$. For every subset $\mathcal{S}_m \subset \{1, 2, \cdots, K\}$, and for every $i \in \mathcal{S}_m$, $TX_i$ wishes to transmit a common message $W^{[i|\mathcal{S}_m]} \in \mathcal{W}^{[i|\mathcal{S}_m]}$ of rate $R^{[i|\mathcal{S}_m]}$ to all receivers $RX_j, j \in \mathcal{S}_m$. We call $W^{[i|\mathcal{S}_m]}$ an order-$m$ message. The case $m = 1$
represents the interference channel with private messages as described earlier. The codes, probabilities of error, achievable rates, capacity region, and degrees of freedom are similarly defined as before, now for a $K\binom{K}{m-1}$-tuple of rates. For any $1 \leq m \leq K$, the DoF (resp. achievable DoF) of transmission of order-$m$ messages over the SISO IC with delayed CSIT is denoted by $\operatorname{DoF}^\text{IC}_m(K)$ (resp. $\operatorname{DoF}^\text{IC}_m(K)$).

For any integers $S, K \geq 2$, the discrete-time $S \times K$ SISO AWGN X channel is defined by a set of $S$ transmitters $\text{TX}_i$, $1 \leq i \leq S$, and $K$ receivers $\text{RX}_j$, $1 \leq j \leq K$, with the following input-output relationship:

$$y_j(t) = \sum_{i=1}^{S} h_{ji}(t)x_i(t) + z_j(t), \quad 1 \leq j \leq K,$$

where the parameters are defined as in Eq. (1). The delayed CSIT assumption is defined exactly as in the IC case. Fix an integer $m$, $1 \leq m \leq K$. For every subset $S_m \subset \{1, 2, \cdots, K\}$, and for every $i \in \{1, 2, \cdots, S\}$, $\text{TX}_i$ wishes to transmit a common message $W^{[i|S_m]} \in \mathcal{W}^{[i|S_m]}$ of rate $R^{[i|S_m]}$ to all receivers $\text{RX}_j$, $j \in S_m$. The case $m = 1$ corresponds to the X channel with private messages.

The achievable rates, capacity region, and degrees of freedom are similarly defined, now for a $S\binom{K}{m}$-tuple of rates. The DoF (resp. achievable DoF) of this channel under delayed CSIT assumption is denoted by $\operatorname{DoF}^X_m(S, K)$ (resp. $\operatorname{DoF}^X_m(S, K)$) for $1 \leq m \leq K$.

III. MAIN RESULTS AND DISCUSSION

A. Main Results

The main results of this paper are formulated in the following two theorems:

Theorem 1: In the $K$-user SISO interference channel with delayed CSIT and $K \geq 2$, $\operatorname{DoF}^\text{IC}_1(K)$ degrees of freedom is achievable almost surely, where $\operatorname{DoF}^\text{IC}_1(K)$ is obtained by

$$\operatorname{DoF}^\text{IC}_1(K) = \left[1 - \frac{K - 2}{K(K - 1)^2} \cdot \frac{K - 2}{K - 1} A_2(K)\right]^{-1},$$

and $A_2(K)$ is given by

$$A_2(K) \triangleq -\frac{(K - 2)(K - 3)}{4(4(K - 2)^2 - 1)} + \sum_{\ell_1 = 0}^{K - 3} \frac{(K - \ell_1 - 1)(3\ell_1^2 + \ell_1 - 1)}{2(K - \ell_1)(4\ell_1^2 - 1)} \prod_{\ell_2 = \ell_1 + 1}^{K - 2} \frac{\ell_2}{2\ell_2 + 1}. \tag{11}$$

Moreover, for $2 \leq m \leq K$, $\operatorname{DoF}^\text{IC}_m(K)$ degrees of freedom is achievable in transmission of order-$m$ messages, where

$$\operatorname{DoF}^\text{IC}_m(K) = \left[1 + \frac{(K - m)(K - m - 1)}{2m[4(K - m)^2 - 1]} \cdot \sum_{\ell_1 = 0}^{K - m - 1} \frac{(K - m - \ell_1 + 1)(3\ell_1^2 + \ell_1 - 1)}{2(K - \ell_1)(4\ell_1^2 - 1)} \prod_{j = \ell_1 + 1}^{K - m} \frac{\ell_2}{2\ell_2 + 1}\right]^{-1}. \tag{12}$$
TABLE I

| K   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| Our achievable DoF for the K-user IC | 36/31 | 45/38 | 1400/1171 |

**Proof:** See Section IV.

**Theorem 2:** In the $2 \times K$ SISO X channel with delayed CSIT and $K \geq 2$, $\text{DoF}^X_1(2, K)$ degrees of freedom is achievable almost surely, where

$$ \text{DoF}^X_1(2, K) = \left[ 1 - \sum_{\ell_1=0}^{K-2} \frac{(K - 1 - \ell_1)(\ell_1 + 1)}{(K - \ell_1)(2\ell_1 + 1)} \prod_{\ell_2=\ell_1+1}^{K-1} \frac{\ell_2}{2\ell_2 + 1} \right]^{-1}. \quad (13) $$

More generally, for $2 \leq m \leq K$, $\text{DoF}^X_m(2, K)$ degrees of freedom is achievable in transmission of order-$m$ messages, where

$$ \text{DoF}^X_m(2, K) = \left[ 1 - \sum_{\ell_1=0}^{K-m-1} \frac{(K - m - \ell_1)(\ell_1 + 1)}{(K - \ell_1)(2\ell_1 + 1)} \prod_{\ell_2=\ell_1+1}^{K-m} \frac{\ell_2}{2\ell_2 + 1} \right]^{-1}. \quad (14) $$

**Proof:** See Section V.

**Remark 1:** Using scaled versions of the schemes proposed in Sections IV and V, $N\text{DoF}^I_1(K)$ and $N\text{DoF}^X_1(2, K)$ are achievable in the $K$-user MIMO IC and $2 \times K$ MIMO X channel, respectively, with $N$ antennas available at each node and with delayed CSIT, where $\text{DoF}^I_1(K)$ and $\text{DoF}^X_1(2, K)$ are given by Eqs. (10) and (13), respectively.

**B. Discussion**

Our achievable DoFs for the $K$-user SISO IC and $2 \times K$ SISO X channel with private messages and delayed CSIT are plotted in Figs. 1 and 2 for $2 \leq K \leq 75$, respectively. For the sake of comparison, the achievable DoF reported in [22] for the $K \times K$ SISO X channel with delayed CSIT is also plotted in Fig. 2. As it is seen in the figure, for $K \geq 3$, our achievable DoF for the $2 \times K$ X channel, i.e., $\text{DoF}^X_1(2, K)$ presented in Theorem 2, is strictly greater than $\frac{4}{3} - \frac{2}{3(3K-1)}$ which is achieved in [22] for the $K \times K$ X channel. It can be also easily shown that our achievable DoFs are strictly increasing in $K$, and it is proved in Appendix E that, as $K \rightarrow \infty$, the achievable DoFs approach limiting values of $\frac{4}{6\ln 2 - 1} \approx 1.2663$ and $\frac{1}{\ln 2} \approx 1.4427$ for the IC and X channel, respectively. Tables I and II list our achievable DoFs for the $K$-user IC and $2 \times K$ X channel with delayed CSIT and $2 \leq K \leq 5$. For $K = 3$, we achieve $36/31$ DoF which is greater than the previously reported value of $9/8$ DoF in [21].
Fig. 1. Our achievable DoF for the $K$-user SISO interference channel with delayed CSIT and $3 \leq K \leq 75$.

Fig. 2. Our achievable DoF for the SISO X channel with delayed CSIT and $2 \leq K \leq 75$. 
TABLE II
ACHIEVABLE DOFS FOR THE SISO X CHANNEL WITH DELAYED CSIT

| K   | 2    | 3    | 4    | 5    |
|-----|------|------|------|------|
| Our achievable DoF for the $2 \times K$ X channel | $\frac{6}{5}$ | $\frac{9}{7}$ | $\frac{105}{79}$ | $\frac{1575}{1163}$ |
| Achievable DoF in [22] for the $K \times K$ X channel | $\frac{6}{5}$ | $\frac{5}{4}$ | $\frac{14}{11}$ | $\frac{9}{7}$ |

IV. PROOF OF THEOREM 1

In this section, we prove that $\text{DoF}_{IC}^{m}(K)$, $1 \leq m \leq K$, stated in Theorem 1 can be achieved in the $K$-user SISO IC with delayed CSIT. To this end, we first elaborate on our achievable scheme for the case of $K = 3$. We then propose our transmission scheme for the general $K$-user setting.

Before proceeding with the transmission schemes, let us define some notations which will be widely used throughout the paper. We use $u[i|S_m:S_n]$ to denote a symbol which is:

- available at TX$_i$,
- available at RX$_j$, for every $j \in S_n$,
- intended to be decoded at RX$_k$, for every $k \in S_m$.

We refer to $u[i|S_m:S_n]$ as an $(S_m:S_n)$-symbol available at TX$_i$. The order of symbol $u[i|S_m:S_n]$ is defined as the ordered pair $(m, n)$ containing the cardinalities of $S_m$ and $S_n$, respectively. For instance, $u[2|1,5;3]$ is a $(1,5;3)$-symbol of order $(2,1)$ which is available at TX$_2$ and RX$_3$, and is intended to be decoded at both RX$_1$ and RX$_5$, where the set braces “{” and “}” have been omitted to avoid cumbersome notations. For ease of notation, a symbol $u[i|S_m:S_n]$ with $S_n = \{\}$ is denoted by $u[i|S_m]$ and is called an $S_m$-symbol of order $m$.

A. The 3-user SISO Interference Channel

Consider the 3-user SISO IC with delayed CSI available at the transmitters as depicted in Fig. 3. In order to achieve $\text{DoF}_{IC}^{m}(3) = \frac{36}{31}$, suggested by Eq. (10), transmission is accomplished in three distinct phases. The fresh data symbols are fed to the channel in the first phase. In the remaining phases, extra linear equations are delivered to the receivers in such a way that the interference is properly aligned at each receiver. At the end of transmission scheme, the receivers are left with the desired number of equations in terms of their respective data symbols.

It is important to point out that we will use several random coefficients during our transmission scheme to construct and transmit different channel input symbols. These coefficients are randomly generated before the beginning of communication and revealed to all transmitters and receivers. The transmission phases are described in detail as follows:
Fig. 3. The 3-user SISO interference channel with delayed CSIT.

- **Phase 1 (3-user IC):**

This phase takes 5 time slots, during which each transmitter feeds 4 fresh data symbols to the channel. Since in the interference channel there exists no \( u^{[i,j]} \) for \( j \neq i \), we simply use \( u^{[i]} \) instead of \( u^{[i,j]} \). Let \( u^{[i]} \triangleq [u^{[i]}_1, u^{[i]}_2, u^{[i]}_3, u^{[i]}_4]^T \) denote the vector containing the data symbols of TX\( _i \), \( 1 \leq i \leq 3 \).

In each time slot, every transmitter transmits a random linear combination of its 4 data symbols. Let \( c^{[i]}(t) \triangleq [c^{[i]}_1(t), c^{[i]}_2(t), c^{[i]}_3(t), c^{[i]}_4(t)]^T \) denote the vector containing the random coefficients of the linear combination transmitted by TX\( _i \), \( 1 \leq i \leq 3 \), over time slot \( t \), \( 1 \leq t \leq 5 \), i.e., \( x_i(t) = (c^{[i]}(t))^T u^{[i]} \). Ignoring the noise terms at receivers, the received signal at RX\( _j \), \( 1 \leq j \leq 3 \), in time slot \( t \), \( 1 \leq t \leq 5 \), is equal to

\[
y_j(t) = h_{j1}(t)x_1(t) + h_{j2}(t)x_2(t) + h_{j3}(t)x_3(t) \\
= h_{j1}(t) (c^{[1]}(t))^T u^{[1]} + h_{j2}(t) (c^{[2]}(t))^T u^{[2]} + h_{j3}(t) (c^{[3]}(t))^T u^{[3]}.
\]  

(15)

Therefore, by the end of phase 1, RX\( _j \) obtains the following system of linear equations in terms of all transmitted data symbols:

\[
y_j = D_{j1}C^{[1]} u^{[1]} + D_{j2}C^{[2]} u^{[2]} + D_{j3}C^{[3]} u^{[3]}, \quad 1 \leq j \leq 3,
\]  

(16)

where \( y_j \) is the \( 5 \times 1 \) vector of received symbols at RX\( _j \) during 5 time slots, \( D_{ji} \) is the \( 5 \times 5 \) diagonal matrix containing \( h_{ji}(t) \), \( 1 \leq t \leq 5 \), on its main diagonal, and \( C^{[i]} \) is the \( 5 \times 4 \) matrix containing the random coefficients employed by TX\( _i \) during these 5 time slots,

\[
C^{[i]} \triangleq [c^{[i]}(1)|c^{[i]}(2)|c^{[i]}(3)|c^{[i]}(4)|c^{[i]}(5)]^T, \quad 1 \leq i \leq 3.
\]  

(17)
Therefore, RX\textsubscript{1} equations are linearly independent almost surely, and therefore, RX\textsubscript{1} \((\mathbb{C}[i])\) is a diagonal matrix with i.i.d. elements on its main diagonal, and thereby, it is also full rank almost surely, i.e., \(\text{rank}(\mathbb{C}[i]) = 5\). Since \(\mathbb{C}[i]\) and \(\mathbb{D}[ji]\) are independent of each other, their multiplication is also full rank almost surely. This means \(\text{rank}(\mathbb{Q}[ji]) = 4\), where \(\mathbb{Q}[ji] \triangleq \mathbb{D}[ji] \mathbb{C}[i]\), \(1 \leq i, j \leq 3\). Since \(\mathbb{Q}[ji]\) is a full rank \(5 \times 4\) matrix, its left null space is one dimensional almost surely. As a result, for each \((i, j)\), \(1 \leq i, j \leq 3\), there exists a nonzero \(5 \times 1\) vector \(\boldsymbol{\omega}[ji] = [\omega_{ji1}, \omega_{ji2}, \omega_{ji3}, \omega_{ji4}, \omega_{ji5}]^T\) such that

\[
\mathbb{Q}[ji]^T \boldsymbol{\omega}[ji] = \mathbf{0}_{4 \times 1}, \quad 1 \leq i, j \leq 3. \tag{18}
\]

Note that by the end of phase 1, all transmitters and receivers have access to \(\mathbb{Q}[ji], 1 \leq i, j \leq 3\), and thus, can calculate \(\boldsymbol{\omega}[ji], 1 \leq i, j \leq 3\). Using Eqs. (16) and (18), RX\textsubscript{1} can obtain

\[
\begin{align*}
\mathbf{y}_1^T \omega_{13} &= (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{13} + (\mathbf{u}[2]^T \mathbb{Q}[12]^T \omega_{13} + (\mathbf{u}[3]^T \mathbb{Q}[13]^T \omega_{13} = (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{13} + (\mathbf{u}[2]^T \mathbb{Q}[12]^T \omega_{13}, \tag{19}
\mathbf{y}_1^T \omega_{12} &= (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{12} + (\mathbf{u}[2]^T \mathbb{Q}[12]^T \omega_{12} + (\mathbf{u}[3]^T \mathbb{Q}[13]^T \omega_{12} = (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{12} + (\mathbf{u}[3]^T \mathbb{Q}[13]^T \omega_{12}. \tag{20}
\end{align*}
\]

Similarly, RX\textsubscript{2} can obtain

\[
\begin{align*}
\mathbf{y}_2^T \omega_{21} &= (\mathbf{u}[2]^T \mathbb{Q}[22]^T \omega_{21} + (\mathbf{u}[3]^T \mathbb{Q}[23]^T \omega_{21}, \tag{21}
\mathbf{y}_2^T \omega_{23} &= (\mathbf{u}[2]^T \mathbb{Q}[22]^T \omega_{23} + (\mathbf{u}[1]^T \mathbb{Q}[21]^T \omega_{23}, \tag{22}
\end{align*}
\]

and RX\textsubscript{3} can obtain

\[
\begin{align*}
\mathbf{y}_3^T \omega_{31} &= (\mathbf{u}[3]^T \mathbb{Q}[33]^T \omega_{31} + (\mathbf{u}[2]^T \mathbb{Q}[32]^T \omega_{31}, \tag{23}
\mathbf{y}_3^T \omega_{32} &= (\mathbf{u}[3]^T \mathbb{Q}[33]^T \omega_{32} + (\mathbf{u}[1]^T \mathbb{Q}[31]^T \omega_{32}. \tag{24}
\end{align*}
\]

If we somehow deliver \((\mathbf{u}[1]^T \mathbb{Q}[21]^T \omega_{23}, (\mathbf{u}[2]^T \mathbb{Q}[12]^T \omega_{13}, (\mathbf{u}[1]^T \mathbb{Q}[31]^T \omega_{32}, and (\mathbf{u}[3]^T \mathbb{Q}[13]^T \omega_{12}\) to RX\textsubscript{1}, then it can obtain enough equations to resolve its four desired data symbols as follows:

- \((\mathbf{u}[1]^T \mathbb{Q}[21]^T \omega_{23}\) and \((\mathbf{u}[1]^T \mathbb{Q}[31]^T \omega_{32}\) are two desired equations in terms of \(4 \times 1\) data vector \(\mathbf{u}[1]\).
- \((\mathbf{u}[2]^T \mathbb{Q}[12]^T \omega_{13}\) can be subtracted from \(\mathbf{y}_1^T \omega_{13}\) to yield \((\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{13}, which is a desired equation in terms of \(\mathbf{u}[1]\).
- \((\mathbf{u}[3]^T \mathbb{Q}[13]^T \omega_{12}\) can be subtracted from \(\mathbf{y}_1^T \omega_{12}\) to yield \((\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{12}, which is a desired equation in terms of \(\mathbf{u}[1]\).

Therefore, RX\textsubscript{1} will have a system of four linear equations in terms of \(4 \times 1\) data vector \(\mathbf{u}[1]\), namely, \((\mathbf{u}[1]^T \mathbb{Q}[21]^T \omega_{23}, (\mathbf{u}[1]^T \mathbb{Q}[31]^T \omega_{32}, (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{13}, and (\mathbf{u}[1]^T \mathbb{Q}[11]^T \omega_{12}. As we prove in Appendix A, these equations are linearly independent almost surely, and therefore, RX\textsubscript{1} can solve them to obtain \(\mathbf{u}[1]\).
a similar argument, having \((u[1]^i)^{\top}Q_{21}^{T}\omega_{23}, \ (u[2]^i)^{\top}Q_{12}^{T}\omega_{13}, \ (u[3]^i)^{\top}Q_{32}^{T}\omega_{31}\), and \((u[3]^i)^{\top}Q_{23}^{T}\omega_{21}\), RX2 can obtain four linearly independent equations in terms of \(u[2]^i\), and so, it can solve them to obtain \(u[2]^i\). Also, after providing RX3 with \((u[1]^i)^{\top}Q_{31}^{T}\omega_{32}, \ (u[3]^i)^{\top}Q_{13}^{T}\omega_{12}, \ (u[2]^i)^{\top}Q_{32}^{T}\omega_{31}\), and \((u[3]^i)^{\top}Q_{23}^{T}\omega_{21}\), it can obtain enough equations to solve for \(u[3]^i\).

Therefore, our goal in phase 2 boils down to delivering \((u[1]^i)^{\top}Q_{21}^{T}\omega_{23}\) and \((u[2]^i)^{\top}Q_{12}^{T}\omega_{13}\) to both RX1 and RX2, delivering \((u[1]^i)^{\top}Q_{31}^{T}\omega_{32}\) and \((u[3]^i)^{\top}Q_{13}^{T}\omega_{12}\) to both RX1 and RX3, and delivering \((u[2]^i)^{\top}Q_{32}^{T}\omega_{31}\) and \((u[3]^i)^{\top}Q_{23}^{T}\omega_{21}\) to both RX2 and RX3. Therefore, the following order-2 symbols can be defined:

\[
\begin{align*}
    u[1][i,2] & \triangleq (u[1]^i)^{\top}Q_{21}^{T}\omega_{23}, \quad u[1][i,3] \triangleq (u[1]^i)^{\top}Q_{31}^{T}\omega_{32}, \\
    u[2][i,2] & \triangleq (u[2]^i)^{\top}Q_{12}^{T}\omega_{13}, \quad u[2][i,3] \triangleq (u[2]^i)^{\top}Q_{32}^{T}\omega_{31}, \\
    u[3][i,3] & \triangleq (u[3]^i)^{\top}Q_{13}^{T}\omega_{12}, \quad u[3][i,2] \triangleq (u[3]^i)^{\top}Q_{23}^{T}\omega_{21}. \tag{25-27}
\end{align*}
\]

- **Phase 2 (3-user IC):**

This phase takes 12 time slots to transmit 18 order-2 symbols generated in phase 1. Since we have generated only 6 order-2 symbols in phase 1, we simply repeat phase 1 three times to obtain 18 order-2 symbols required in phase 2. This takes \(3 \times 5 = 15\) time slots and hence, phase 2 begins at time slot \(t = 16\). Consequently, at the beginning of phase 2, for every \((i, j), 1 \leq i, j \leq 3, i \neq j\), there are three order-2 symbols \(u[1][i,j]^i, u[2][i,j]^i, \) and \(u[3][i,j]^i\) at TXi and three order-2 symbols \(u[1][i,j]^j, u[2][i,j]^j, \) and \(u[3][i,j]^j\) at TXj. The transmission in phase 2 is then carried out as follows:

In the first time slot of phase 2, TX1 transmits a random linear combination of \(u[1][1,2]^i\) and \(u[2][1,2]^i\) while TX2 transmits \(u[1][2,1]^i\). In the second time slot, TX1 transmits another random linear combination of \(u[1][1,2]^i\) and \(u[2][1,2]^i\) while TX2 repeats \(u[2][1,2]^i\). TX3 is silent during these two time slots. After these two time slots, every receiver obtains two linearly independent equations in terms of three \((1, 2)\)-symbols \(u[1][1,2]^i, u[2][1,2]^i, \) and \(u[2][1,2]^i\) almost surely. Thus, each of RX1 and RX2 in order to resolve these three order-2 symbols, needs an extra equation. Consider the equations received at RX3 during these two time slots:

\[
y[3]_3(t) = h[31](t)x[1]_3(t) + h[32](t)x[2]_3(t) = h[31](t) \left( c[1][1,2]^i(t) \right)^{\top} u[1][1,2]^i + h[32](t)u[2][1,2]^i, \quad t = 16, 17, \tag{28}
\]

where \(u[1][1,2]^i \triangleq \left[ u[1][1,2]^i, u[2][1,2]^i \right]^T\), and \(c[1][1,2]^i(t) \triangleq \left[ c[1][1,2]^i(t), c[2][1,2]^i(t) \right]^T\) is the \(2 \times 1\) vector of random coefficients employed by TX1 in time slot \(t\). Now, RX3 can form

\[
\frac{1}{h[32](16)} y[3]_3(16) - \frac{1}{h[32](17)} y[3]_3(17) = \left[ \frac{h[31](16)}{h[32](16)} (c[1][1,2]^i(16))^{\top} \frac{h[31](17)}{h[32](17)} (c[1][1,2]^i(17))^{\top} \right] u[1][1,2]^i, \tag{29}
\]

which is an equation solely in terms of the elements of \(u[1][1,2]^i\). This is the side information that RX3 has about the order-2 symbols of RX1 and RX2, and can provide the extra equation required by both
RX$_1$ and RX$_2$ to resolve their order-2 symbols. Based on our terminology, this quantity is denoted by $u^{[1,2,3]}$. The next two time slots are dedicated to the transmission of another three order-2 (1, 2)-symbols. However, this time, the roles of TX$_1$ and TX$_2$ are exchanged. Specifically, during time slots $t = 18, 19$, TX$_2$ transmits two random linear combinations of $u^{[2,1,2]}_2$ and $u^{[2,1,2]}_3$ while TX$_1$ repeats the same symbol $u^{[1,2]}_3$. The side information $u^{[2,1,2,3]}$ is similarly formed at RX$_3$ by the end of these two time slots.

Up to this point, we have sent $6$ order-2 (1, 2)-symbols in 4 time slots, and generated two pieces of side information at RX$_3$. Analogously, for each of receiver pairs $\{1, 3\}$ and $\{2, 3\}$, the above procedure can be repeated using their respective transmitters. Therefore, by spending another $2 \times 4 = 8$ time slots, we will transmit $2 \times 6 = 12$ order-2 symbols and generate the side information $u^{[2,2,3,1]}$ and $u^{[3,2,3,1]}$ at RX$_1$, and $u^{[1,1,3,2]}$ and $u^{[3,1,3,2]}$ at RX$_2$. Therefore, our goal is reduced to

(a) delivering $u^{[1,1,2,3]}$ and $u^{[2,1,2,3]}$ to both RX$_1$ and RX$_2$,

(b) delivering $u^{[1,1,3,2]}$ and $u^{[3,1,3,2]}$ to both RX$_1$ and RX$_3$,

(c) delivering $u^{[2,2,3,1]}$ and $u^{[3,2,3,1]}$ to both RX$_2$ and RX$_3$.

To this end, consider a random linear combination $\alpha_1 u^{[1,1,2,3]} + \alpha_2 u^{[1,1,3,2]}$. If we somehow deliver this quantity to all three receivers, then

- RX$_1$ obtains a linear equation in terms of its own desired symbols,
- since RX$_2$ has $u^{[1,1,3,2]}$, it can cancel $u^{[1,1,3,2]}$ to obtain $u^{[1,1,2,3]}$,
- since RX$_3$ has $u^{[1,1,2,3]}$, it can cancel $u^{[1,1,2,3]}$ to obtain $u^{[1,1,3,2]}$.

Therefore, $\alpha_1 u^{[1,1,2,3]} + \alpha_2 u^{[1,1,3,2]}$ is desired by all three receivers. By similar arguments, one can conclude that $\beta_1 u^{[2,2,1,3]} + \beta_2 u^{[2,2,3,1]}$ and $\gamma_1 u^{[3,3,1,2]} + \gamma_2 u^{[3,2,3,1]}$ are desired by all three receivers, where $\beta_1$, $\beta_2$, $\gamma_1$, and $\gamma_2$ are random coefficients. According to our terminology, we define the following order-3 symbols:

$$u^{[1,2,3]} \triangleq \alpha_1 u^{[1,1,2,3]} + \alpha_2 u^{[1,1,3,2]},$$

$$u^{[2,1,2,3]} \triangleq \beta_1 u^{[2,2,1,3]} + \beta_2 u^{[2,2,3,1]},$$

$$u^{[3,1,3,2]} \triangleq \gamma_1 u^{[3,3,1,2]} + \gamma_2 u^{[3,2,3,1]}.$$

Although delivering $u^{[1,1,2,3]}$, $u^{[2,1,2,3]}$, and $u^{[3,1,2,3]}$ to all three receivers will provide each of them with useful information about its desired symbols as discussed above, it is not still sufficient to achieve the goals (a), (b), and (c). To be more specific, recall that RX$_1$ needs to obtain both symbols $u^{[1,1,2,3]}$ and $u^{[1,1,3,2]}$. Thus, assuming $u^{[1,1,2,3]}$ has been delivered to all three receivers, RX$_1$ still needs an extra equation in terms of $u^{[1,1,2,3]}$ and $u^{[1,1,3,2]}$. To obtain this extra equation, we notice that by delivering $u^{[1,1,2,3]}$ to all three receivers, both RX$_2$ and RX$_3$ will have both symbols $u^{[1,1,2,3]}$ and $u^{[1,1,3,2]}$. Therefore, any random linear combination $\alpha'_1 u^{[1,1,2,3]} + \alpha'_2 u^{[1,1,3,2]}$ can be considered as the extra equation required at RX$_1$ which
is also available at RX2 and RX3. Therefore, we can define the following \((1; 2, 3)\)-symbol at TX1:

\[
u^{[1;2,3]}_1 \triangleq \alpha'_1 u^{[1,2,3]}_1 + \alpha'_2 u^{[1,3,2]}_1.
\] (33)

By repeating the same argument for RX2 and RX3, the following \((2; 1, 3)\)-symbol and \((3; 1, 2)\)-symbol can be defined:

\[
u^{[2;1,3]}_2 \triangleq \beta'_1 u^{[2,1,3]}_1 + \beta'_2 u^{[2,3,1]}_2,
\] (34)

\[
u^{[3;1,2]}_3 \triangleq \gamma'_1 u^{[3,1,2]}_1 + \gamma'_2 u^{[3,2,3]}_2,
\] (35)

where \(\beta'_1, \beta'_2, \gamma'_1, \text{ and } \gamma'_2\) are random coefficients. To summarize, one can achieve the goals (a), (b), and (c) if:

I. \(u^{[1,2,3]}_1, u^{[2,1,3]}_2, \text{ and } u^{[3,1,2]}_3\) are delivered to all three receivers.

II. \(u^{[1,2,3]}_1, u^{[2,1,3]}_2, \text{ and } u^{[3,1,2]}_3\) are respectively delivered to RX1, RX2, and RX3.

The goals I and II will be accomplished in the next phase.

- **Phase 3-I (3-user IC):**

  In this subphase, which takes three time slots, we fulfill the goal I as follows: Using time division in three consecutive time slots, the three symbols \(u^{[1,2,3]}_1, u^{[2,1,3]}_2, \text{ and } u^{[3,1,2]}_3\) will be delivered to all three receivers.

- **Phase 3-II (3-user IC):**

  In this subphase, the goal II is accomplished in one time slot by simultaneous transmission of symbols \(u^{[1,2,3]}_1, u^{[2,1,3]}_2, \text{ and } u^{[3,1,2]}_3\) by TX1, TX2, and TX3, respectively.

Finally, in order to compute the achieved DoF, we note that a total of \(3 \times 12 = 36\) fresh data symbols were fed to the system in phase 1. To deliver these data symbols to their intended receivers, we spent \(3 \times 5 = 15\) time slots in phase 1, \(3 \times 4 = 12\) time slots in phase 2, three time slots in subphase 3-I, and one time slot in subphase 3-II. Therefore, our achieved DoF is equal to

\[
\text{DoF}^{\text{IC}}_1(3) = \frac{36}{15 + 12 + 3 + 1} = \frac{36}{31}.
\] (36)

One finally notes that the proposed transmission scheme starting from the phase 2 was dedicated to transmission of order-2 messages to the receivers. Therefore, we have proved that \(\text{DoF}^{\text{IC}}_2(3) = \frac{18}{12+3+1} = \frac{9}{8}\) is achievable in the 3-user IC with delayed CSIT as suggested by Eq. (12). Also, \(\text{DoF}^{\text{IC}}_3(3) = 1\) was trivially achieved using time division in the phase 3-I.
B. The $K$-user SISO Interference Channel

In this section, we generalize our multiphase transmission scheme to the $K$-user SISO IC with delayed CSIT and $K > 3$. The transmission scheme is a multiphase scheme wherein the fresh data symbols are fed to the system in phase 1 towards generating order-2 symbols. The remaining phases are responsible for generating higher order symbols and finally providing each receiver with appropriate equations to resolve its own data symbols. Fig. 4 depicts a high-level block diagram for the proposed multiphase scheme.

- **Phase 1 ($K$-user IC):**

In this phase, each transmitter transmits $(K-1)^2 + 1$ random linear combinations of $(K-1)^2$ data symbols in $(K-1)^2 + 1$ time slots. Let $u^{[i]} = \begin{bmatrix} u^{[i]}_1, u^{[i]}_2, \ldots, u^{[i]}_{(K-1)^2} \end{bmatrix}^T$ be the data vector of TX$_i$, $1 \leq i \leq K$. Define

$$C^{[i]} = \begin{bmatrix} c^{[i]}(1) | c^{[i]}(2) | \cdots | c^{[i]}((K-1)^2 + 1) \end{bmatrix}^T, \quad 1 \leq i \leq K,$$

(37)

where $c^{[i]}(t)$ is the $(K-1)^2 \times 1$ vector of the random coefficients employed by TX$_i$ in time slot $t$, $1 \leq t \leq (K-1)^2 + 1$. Then, ignoring the noise, after these $(K-1)^2 + 1$ time slots, RX$_j$ receives the following vector of $(K-1)^2 + 1$ channel output symbols:

$$y_j = D_{j1}C^{[1]}u^{[1]} + D_{j2}C^{[2]}u^{[2]} + \cdots + D_{jK}C^{[K]}u^{[K]}, \quad 1 \leq j \leq K,$$

(38)

where $D_{ji}$ is the diagonal matrix of size $[(K-1)^2 + 1] \times [(K-1)^2 + 1]$ which contains the channel coefficients $h_{ji}(t)$, $1 \leq t \leq (K-1)^2 + 1$, on its main diagonal.

Define $Q_{ji} \triangleq D_{ji}C^{[i]}$, $1 \leq i, j \leq K$. Since $D_{ji}$ and $C^{[i]}$ are full rank almost surely and independent of each other, their multiplication is also full rank almost surely. Hence, $Q_{ji}$ is a full rank matrix of size $[(K-1)^2 + 1] \times (K-1)^2$ almost surely, and so, its left null space is one dimensional. Therefore, there exist nonzero vectors $\omega_{ji} = [\omega_{ji1}, \omega_{ji2}, \cdots, \omega_{ji((K-1)^2+1)}]^T$ such that

$$Q_{ji}^T\omega_{ji} = 0_{(K-1)^2 \times 1}, \quad 1 \leq i, j \leq K.$$
Thus, for any $1 \leq j \leq K$ and any $i \in S_K \setminus \{j\}$, RX$_j$ can construct

$$y_j^T \omega_{ji} = \sum_{i' \in S_K \setminus \{i\}} (u[i'])^T Q_{ji'}^T \omega_{ji} = (u[j])^T Q_{jj}^T \omega_{ji} + \sum_{i' \in S_K \setminus \{i,j\}} (u[i'])^T Q_{ji'}^T \omega_{ji}. \quad (40)$$

We note that $(u[i'])^T Q_{ji'}^T \omega_{ji}$, $i' \in S_K \setminus \{i,j\}$, is an equation solely in terms of $u[i']$, and thus, it is desired by RX$_{i'}$. It is easy to see that if we deliver all $K-2$ quantities $(u[i'])^T Q_{ji'}^T \omega_{ji}$, $i' \in S_K \setminus \{i,j\}$, to RX$_j$, then RX$_j$ can cancel their summation from Eq. (40) to obtain $(u[j])^T Q_{jj}^T \omega_{ji}$, which is a desired equation for RX$_j$. Therefore, one can define $K-2$ order-2 $(i',j)$-symbols available at TX$_{i'}$ by

$$u[i'j] \triangleq (u[i'])^T Q_{ji'}^T \omega_{ji}, \quad i' \in S_K \setminus \{i,j\}. \quad (41)$$

Since there are $K-1$ choices of $i$, $i \in S_K \setminus \{j\}$, a total of $(K-1)(K-2)$ order-2 symbols of the form $u[iij]$, $i \in S_K \setminus \{j\}$, will be constructed for a fixed $j$. These symbols, if delivered, will provide RX$_j$ with $K-1$ equations solely in terms of $u[i]$ while providing every RX$_i$, $i \in S_K \setminus \{j\}$, with $K-2$ equations in terms of $u[i]$.

Since there are $K$ choices for RX$_j$, $1 \leq j \leq K$, a total of $K(K-1)(K-2)$ order-2 symbols $u[iij]$, $i \in S_K \setminus \{j\}$, are generated by the end of phase 1. After delivering all these symbols to their intended pairs of receivers, every receiver will be provided with $K-1 + (K-1)(K-2) = (K-1)^2$ linear equations in terms of its own data symbols. Namely, RX$_j$ will obtain the following $(K-1)^2$ linear equations in terms of $u[i]$:

$$\begin{align*}
(u[i])^T Q_{ji}^T \omega_{ji}, & \quad i_1 \in S_K \setminus \{j\}, \quad (42) \\
(u[i])^T Q_{i_2i}^T \omega_{i_2i_3}, & \quad i_2, i_3 \in S_K \setminus \{j\}, i_2 \neq i_3. \quad (43)
\end{align*}$$

It is proved in Appendix A that these $(K-1)^2$ equations are linearly independent almost surely, and thus, each receiver can resolve all its $(K-1)^2$ data symbols.

Finally, it takes $\frac{K(K-1)(K-2)}{\text{DoF}^I_{IC}(K)}$ time slots to deliver all the order-2 symbols generated in phase 1 to their intended pairs of receivers. Hence, one can write

$$\text{DoF}^I_{IC}(K) = \frac{(K-1)^2 K}{(K-1)^2 + 1 + \frac{K(K-1)(K-2)}{\text{DoF}^I_{IC}(K)}}. \quad (44)$$

- Phase $m$-$I$, $2 \leq m \leq K-1$ (K-user IC):

This subphase takes a total of $N^I_{m-1}$ order-$m$ symbols of the form $u[i|S_m]$, $S_m \subset S_K$, $i \in S_m$, and transmits them to the receivers in $T^I_m$ time slots. Then, a total of $N^I_{m+1}$ order-$(m+1)$ symbols of the form $u[i|S_{m+1}]$, $S_{m+1} \subset S_K$, $i \in S_{m+1}$, together with $N^I_{m+1}$ symbols of the form $u[i|S_{m+1}\setminus \{i\}]$, $S_{m+1} \subset S_K$, $i \in S_{m+1}$, are
generated such that if the generated symbols are somehow delivered to their intended receiver(s), then every subset $S_m$ of cardinality $m$ of receivers will be able to decode all the $S_m$-symbols transmitted in this subphase. The parameters $N_{m}^{\text{IC-I}}$, $T_{m}^{\text{IC}}$, $N_{m+1}^{\text{IC-I}}$, and $N_{m+1}^{\text{IC-II}}$ are given by

$$N_{m}^{\text{IC-I}} = m[2(K - m) + 1] \binom{K}{m},$$  \hspace{1cm} (45)

$$T_{m}^{\text{IC}} = m(K - m + 1) \binom{K}{m},$$  \hspace{1cm} (46)

$$N_{m+1}^{\text{IC-I}} = (m^2 - 1) \binom{K}{m+1},$$  \hspace{1cm} (47)

$$N_{m+1}^{\text{IC-II}} = (m + 1) \binom{K}{m+1}. $$  \hspace{1cm} (48)

The following is a detailed description of this subphase:

Fix $S_m \subset S_K$ and sort the elements of $S_m$ in ascending cyclic order. Fix $i_1 \in S_m$ and let $i_2 \in S_m$ be the element immediately after $i_1$ in that ordering. Consider vector $u^{[i_1|S_m]} \triangleq \left[ u_1^{[i_1|S_m]}, u_2^{[i_1|S_m]}, \ldots, u_{K-m+1}^{[i_1|S_m]} \right]^T$ of $K-m+1$ $S_m$-symbols available at TX$_{i_1}$ and vector $u^{[i_2|S_m]} \triangleq \left[ u_1^{[i_2|S_m]}, u_2^{[i_2|S_m]}, \ldots, u_{K-m}^{[i_2|S_m]} \right]^T$ of $K-m$ $S_m$-symbols available at TX$_{i_2}$. In the first $K-m+1$ time slots of this subphase, TX$_{i_1}$ and TX$_{i_2}$ transmit $K-m+1$ random linear combinations of elements of $u^{[i_1|S_m]}$ and $u^{[i_2|S_m]}$, respectively, while the rest of transmitters are silent. Let $c^{[i_1|S_m]}(t)$ (resp. $c^{[i_2|S_m]}(t)$) be the $(K - m + 1) \times 1$ vector (resp. $(K - m) \times 1$ vector) of the random coefficients employed by TX$_{i_1}$ (resp. TX$_{i_2}$) in time slot $t$, $1 \leq t \leq K - m + 1$. Then, ignoring the noise, by the end of these time slots, RX$_j$, $1 \leq j \leq K$, will have the following vector of $K-m+1$ channel output symbols:

$$y_j = D_{j_{i_1}}C^{[i_1|S_m]}u^{[i_1|S_m]} + D_{j_{i_2}}C^{[i_2|S_m]}u^{[i_2|S_m]}$$  \hspace{1cm} (49)

$$= Q_{j_{i_1}}u^{[i_1|S_m]} + Q_{j_{i_2}}u^{[i_2|S_m]},$$  \hspace{1cm} (50)

where $C^{[i_1|S_m]}$ and $C^{[i_2|S_m]}$ are defined as

$$C^{[i_1|S_m]} \triangleq \left[ c^{[i_1|S_m]}(1), c^{[i_1|S_m]}(2), \ldots, c^{[i_1|S_m]}(K-m+1) \right]^T$$  \hspace{1cm} (51)

$$C^{[i_2|S_m]} \triangleq \left[ c^{[i_2|S_m]}(1), c^{[i_2|S_m]}(2), \ldots, c^{[i_2|S_m]}(K-m+1) \right]^T$$  \hspace{1cm} (52)

$D_{j_{i_1}}$ and $D_{j_{i_2}}$ are the diagonal matrices of size $(K - m + 1) \times (K - m + 1)$ containing the channel coefficients $h_{j_{i_1}}(t)$ and $h_{j_{i_2}}(t)$, $1 \leq t \leq K - m + 1$, on their main diagonal, respectively, and $Q_{j_{i_1}}$ and $Q_{j_{i_2}}$ are defined as $Q_{j_{i_1}} \triangleq D_{j_{i_1}}C^{[i_1|S_m]}$ and $Q_{j_{i_2}} \triangleq D_{j_{i_2}}C^{[i_2|S_m]}$.

Therefore, in specific, each receiver RX$_j$, $j \in S_m$, obtains $K-m+1$ desired linearly independent equations in terms of the $2(K-m)+1$ transmitted $S_m$-symbols, and thus, needs $K-m$ extra equations to resolve all the transmitted $S_m$-symbols. It is easily verified that $Q_{j_{i_2}}$, $1 \leq j \leq K$, is a full rank matrix.
of size \((K - m + 1) \times (K - m)\) almost surely, and so, its left null space is one dimensional. Specifically, there exist nonzero vectors \(\omega_{j'_{i_2}}\) such that

\[
Q_{j'_{i_2}}^T \omega_{j'_{i_2}} = 0, \quad j' \in S_K \setminus S_m. \tag{53}
\]

Hence, each receiver \(RX_j', j' \in S_K \setminus S_m\), can construct \(y_{j'}^T \omega_{j'_{i_2}} = (u_{[i_1]|S_m])^T Q_{j'_{i_1}}^T \omega_{j'_{i_2}}\) which is a linear combination in terms of \(u_{[i_1]|S_m}\) and thus, if delivered to all receivers \(RX_j, j \in S_m\), can provide each of them with an extra equation in terms of their desired \(S_m\)-symbols. On the other hand, the above linear combination is solely in terms of \(u_{[i_1]|S_m}\) (available at \(TX_{i_1}\)) and the channel coefficients (available at \(TX_{i_1}\), due to the delayed CSIT assumption, by the end of these \(K - m + 1\) time slots). Therefore, based on our terminology, one can define

\[
u_{[i_1]|S_m; j'] \triangleq (u_{[i_1]|S_m})^T Q_{j'_{i_1}}^T \omega_{j'_{i_2}}, \quad j' \in S_K \setminus S_m. \tag{54}
\]

After delivering all these side information symbols to all receivers \(RX_j, j \in S_m\), each of them will obtain \(K - m + 1\) linear equations in terms of the \(K - m + 1\) transmitted \(S_m\)-symbols. Namely, \(RX_j, j \in S_m\), will obtain the following equations:

\[
Q_{j_{i_1}} u_{[i_1]|S_m} + Q_{j_{i_2}} u_{[i_2]|S_m}
\]

\[
(u_{[i_1]|S_m})^T Q_{j'_{i_1}}^T \omega_{j'_{i_2}}, \quad j' \in S_K \setminus S_m. \tag{55}
\]

It is shown in Appendix B that the above equations are linearly independent almost surely, which enables \(RX_j\) to solve them for \(u_{[i_1]|S_m}\) and \(u_{[i_2]|S_m}\).

We repeat the same procedure for every choice of \(i_1 \in S_m\), i.e., for each choice, we spend \(K - m + 1\) time slots to transmit \(2(K - m) + 1\) \(S_m\)-symbols and generate \(K - m\) side information symbols. This implies the transmission of a total of \(m[2(K - m) + 1]\) \(S_m\)-symbols in \(m(K - m + 1)\) time slots and generation of \(m(K - m)\) side information symbols. Since \(S_m \subset S_K\) could be any subset with cardinality \(m\), we transmit a total of \(N_{m}^{IC-1}\) order-\(m\) symbols in \(T_{m}^{IC}\) time slots and generate \(m(K - m)(K - m + 1)\) side information symbols, where \(N_{m}^{IC-1}\) and \(T_{m}^{IC}\) are given by Eqs. (45) and (46).

In order to deliver the generated side information symbols to their respective intended receivers, fix a subset \(S_{m+1} \subset S_K\) and an index \(i_1 \in S_{m+1}\). For every \(j' \in S_{m+1} \setminus \{i_1\}\), we have generated exactly one side information symbol \(u_{[i_1]|S_{m+1} \setminus \{i_1\}; j']\). Since there are \(m\) different choices for \(j', j' \in S_{m+1} \setminus \{i_1\}\), we can identify \(m\) symbols of the form \(u_{[i_1]|S_{m+1} \setminus \{i_1\}; j']\) for fixed \(S_{m+1} \subset S_K\) and \(i_1 \in S_{m+1}\). Moreover, every receiver \(RX_{j'}\), \(j' \in S_{m+1} \setminus \{i_1\}\), has exactly one of these \(m\) symbols and wishes to obtain the rest, while \(RX_{i_1}\) wishes to obtain all the \(m\) symbols. Therefore, if we deliver \(m - 1\) random linear combinations of these \(m\) symbols to all receivers in \(S_{m+1}\), then each of them (except for \(RX_{i_1}\)) will remove its known
side information and obtain $m - 1$ linearly independent equations in terms of the $m - 1$ desired symbols almost surely and hence, decode all desired symbols. Thus, we define $m - 1$ $S_{m+1}$-symbols as follows

$$u_{\ell}[i;S_{m+1}] \triangleq \sum_{j' \in S_{m+1}\setminus\{i\}} \alpha_{\ell}^{[i;S_{m+1}\setminus\{j'\};j']} u_{\ell}[i;S_{m+1}\setminus\{j'\};j'], \quad 1 \leq \ell \leq m - 1, \quad (57)$$

where $\alpha_{\ell}^{[i;S_{m+1}\setminus\{j'\};j']}, j' \in S_{m+1}\setminus\{i\}, 1 \leq \ell \leq m - 1$, is a random coefficient.

Since $RX_{i_1}$ wishes to obtain all the $m$ symbols $u_{\ell}[i;S_{m+1}\setminus\{j'\};j'], j' \in S_{m+1}\setminus\{i\}$, after delivering the above $m - 1$ linear equations to $RX_{i_1}$, it still requires one extra linearly independent equation to resolve all its desired symbols. However, recall that after delivering all the $S_{m+1}$-symbols defined in Eq. (57) to all receivers $RX_{j'}, j' \in S_{m+1}$, every receiver $RX_{j'}, j' \in S_{m+1}\setminus\{i\}$, will be able to obtain all the $m$ symbols $u_{\ell}[i;S_{m+1}\setminus\{j'\};j'], j' \in S_{m+1}\setminus\{i_1\}$. Thereafter, any linear combination of the symbols $u_{\ell}[i;S_{m+1}\setminus\{j'\};j'], j' \in S_{m+1}\setminus\{i_1\}$, will be available at every receiver $RX_{j'}, j' \in S_{m+1}\setminus\{i_1\}$. In specific, we can define a new random linear combination

$$u_{i_1}[i;S_{m+1}\setminus\{i_1\}] \triangleq \sum_{j' \in S_{m+1}\setminus\{i_1\}} \alpha_{m}^{[i;S_{m+1}\setminus\{j'\};j']} u_{i_1}[i;S_{m+1}\setminus\{j'\};j'], \quad (58)$$

as a symbol which is available at $TX_{i_1}$ and at every receiver $RX_{j'}, j' \in S_{m+1}\setminus\{i_1\}$, and is desired by $RX_{i_1}$.

Since there are $\binom{K}{m+1}$ choices of $S_{m+1} \subset S_K$, and $m + 1$ choices of $i_1 \in S_{m+1}$ for each $S_{m+1} \subset S_K$, a total of $N^{IC}_m$ order-$(m + 1)$ $S_{m+1}$-symbols and $N^{IC-II}_m$ order-$(1, m)$ $(i_1; S_{m+1}\setminus\{i_1\})$-symbols will be generated where $N^{IC-I}_m$ and $N^{IC-II}_m$ are given by Eqs. (47) and (48). If we deliver all the $S_{m+1}$-symbols and $(i_1; S_{m+1}\setminus\{i_1\})$-symbols, $S_{m+1} \subset S_K$, $i_1 \in S_{m+1}$, defined in Eqs. (57) and (58) to their intended receiver(s), then each receiver will be able to decode all its desired order-$m$ symbols transmitted in this subphase. This will be accomplished during the next phases.

- **Phase K-I (K-user IC):**

  In this subphase, in each time slot, an order-$K$ symbol of the form $u_{i_1}[i;S_K], i \in S_K$, is transmitted by $TX_i$ while the other transmitters are silent. After each time slot, ignoring the noise, each receiver receives the transmitted symbol without any interference. This implies that

  $$\text{DoF}^{IC}_K(K) = 1. \quad (59)$$

- **Phase m-II, 3 \leq m \leq K (K-user IC):**

  In this subphase, each time slot is dedicated to transmission of the order-$(1, m - 1)$ symbols $u_{i_1}[i;S_{m}\setminus\{i\}]$, ...
Let $i \in S_m$, for a fixed $S_m$, $S_m \subset S_K$. In specific, in the time slot dedicated to $S_m$, every transmitter $TX_i$, $i \in S_m$, transmits $u_i^{[i; S_m \backslash \{i\}]},$ simultaneously. Since each receiver $RX_j$, $j \in S_m$, has all symbols $u_i^{[i; S_m \backslash \{i\}]}, i \in S_m \backslash \{j\}$, it will decode its desired symbol (i.e., $u_j^{[j; S_m \backslash \{j\}]})$ after this time slot. If we denote by $\text{DoF}^{IC-II}_m(K)$ the achievable DoF of transmitting all $(i; S_m \backslash \{i\})$-symbols over the $K$-user SISO IC with delayed CSIT, then one can write

$$\text{DoF}^{IC-II}_m(K) = m, \quad 3 \leq m \leq K. \quad (60)$$

Combining Eqs. (45) to (48) and (60), we conclude that

$$\text{DoF}^{IC}_m(K) = N^{IC-I}_m T^{IC}_m + \text{DoF}^{IC-II}_m(K) + \text{DoF}^{IC-I+1}_m(K), \quad 2 \leq m \leq K - 1. \quad (61)$$

In Appendix C, it is shown that Eq. (12) is a closed form solution to the recursive Eq. (61) with the initial condition (59) and $2 \leq m \leq K$. As a result, for $m = 2$, it is shown that

$$\text{DoF}^{IC}_2(K) = \frac{1}{1 - A^2(K)}, \quad (62)$$

where $A^2(K)$ is given in Eq. (11). Equation (10) immediately follows from Eqs. (11), (44) and (62).

V. PROOF OF THEOREM 2

For $K = 2$, our transmission scheme reduces to a modified version of the scheme proposed in [22] and achieves the same DoF of $\frac{6}{5}$. Hence, we would rather start with $K = 3$ and elaborate on our transmission scheme for the $2 \times 3$ X channel with delayed CSIT. We show that it achieves $\text{DoF}^X_1(2, 3) = \frac{9}{7}$ and $\text{DoF}^X_2(2, 3) = \frac{9}{8}$, as suggested by Eqs. (13) and (14). Finally, we will proceed with the general $2 \times K$ case.

A. The $2 \times 3$ SISO X Channel

In this section, we prove that $\text{DoF}^X_1(2, 3) = \frac{9}{7}$ and $\text{DoF}^X_2(2, 3) = \frac{9}{8}$ are achievable in the $2 \times 3$ SISO X channel with delayed CSIT which is depicted in Fig. [5]. To this end, we propose a transmission scheme which has three distinct phases:

• **Phase 1 (2 × 3 X Channel):**
Fig. 5. The $2 \times 3$ SISO X channel with delayed CSIT.

This phase takes 9 time slots to transmit 15 data symbols as follows: Fix $i_1 = 1$ and $i_2 = 2$. During the first 3 time slots, 5 data symbols $u^{[i_1]} ≜ [u_1^{[i_1]}, u_2^{[i_1]}, u_3^{[i_1]}]^T$ and $u^{[i_2]} ≜ [u_1^{[i_2]}, u_2^{[i_2]}]^T$ (all intended for RX$_1$) are transmitted by TX$_{i_1}$ and TX$_{i_2}$, respectively. In specific, in each of these 3 time slots, TX$_{i_1}$ transmits a random linear combination of $u_1^{[i_1]}$, $u_2^{[i_1]}$, and $u_3^{[i_1]}$ while TX$_{i_2}$ transmits a random linear combination of $u_1^{[i_2]}$ and $u_2^{[i_2]}$. Let $c^{[i_1]}(t) ≜ [c_1^{[i_1]}(t), c_2^{[i_1]}(t), c_3^{[i_1]}(t)]^T$ and $c^{[i_2]}(t) ≜ [c_1^{[i_2]}(t), c_2^{[i_2]}(t)]^T$ denote the vectors containing the random coefficients of the linear combinations transmitted by TX$_{i_1}$ and TX$_{i_2}$, respectively, over time slot $t$, $1 \leq t \leq 3$.

After these 3 time slots, every receiver obtains 3 linearly independent equations in terms of the 5 transmitted data symbols almost surely. Thus, RX$_1$ in order to resolve these 5 desired data symbols, needs two more linearly independent equations. Now, consider the equations received at each of RX$_2$ and RX$_3$ in time slot $t$, $1 \leq t \leq 3$:

$$y_j(t) = \sum_{k=1}^{2} h_{jik}(t)x_{ik}(t)$$

$$= \sum_{k=1}^{2} h_{jik}(t) (c^{[i]}(t))^T u^{[i]}(t), \quad j = 2, 3.$$  \hspace{1cm} (63)

The system of linear equations received at RX$_j$, $j = 2, 3$, by the end of these 3 time slots can be written as

$$y_{j|1} = \sum_{k=1}^{2} D_{jik|1} C^{[i]} u^{[i]}(t), \quad j = 2, 3,$$  \hspace{1cm} (64)

where $y_{j|1}$ is the $3 \times 1$ vector of received symbols at RX$_j$ during these 3 time slots, $D_{jik|1}$ is the $3 \times 3$ diagonal matrix containing $h_{jik}(t)$, $1 \leq t \leq 3$, on its main diagonal, and $C^{[i]}$ (resp. $C^{[i]}$) is the $3 \times 3$
(resp. $3 \times 2$) matrix containing the random coefficients employed by $\text{TX}_{i_1}$ (resp. $\text{TX}_{i_2}$) during these 3 time slots, i.e.,

$$
C^{[i_1][1]}_1 \triangleq [c^{[i_1][1]}(1)|c^{[i_1][1]}(2)|c^{[i_1][1]}(3)]^T, \quad k = 1, 2.
$$

(65)

Since the elements of $C^{[i_1][1]}$ and $C^{[i_2][1]}$ are i.i.d., they are full rank almost surely, i.e., $\text{rank}(C^{[i_1][1]}) = 3$ and $\text{rank}(C^{[i_2][1]}) = 2$. One can verify that $D_{ji_1|1}$ is also full rank almost surely and is independent of $C^{[i_1][1]}$. Therefore, $Q_{ji_1|1} \triangleq D_{ji_1|1} C^{[i_1][1]}$ is full rank almost surely. Specifically, $Q_{ji_2|1}$ is a full rank $3 \times 2$ matrix, and thus, its left null space is one dimensional almost surely. Let the $3 \times 1$ vector $\omega_{ji_2|1}$ be in the left null space of $Q_{ji_2|1}$, i.e.,

$$
Q_{ji_2|1}^T \omega_{ji_2|1} = 0_{2 \times 1}, \quad j = 2, 3.
$$

(66)

After these 3 time slots, every receiver can calculate $\omega_{ji_2|1}$, $j = 2, 3$. Then, using Eqs. (64) and (66), $\text{RX}_j$, $j = 2, 3$, can obtain

$$
y_{j|1}[\omega_{ji_2|1}] = (u^{[i_1][1]})^T Q_{ji_1|1}^T \omega_{ji_2|1} + (u^{[i_2][1]})^T Q_{ji_1|1}^T \omega_{ji_2|1}
$$

$$
= (u^{[i_1][1]})^T Q_{ji_1|1}^T \omega_{ji_2|1},
$$

(67)

which is an equation solely in terms of $u^{[i_1][1]}$. Therefore, if we somehow deliver $(u^{[i_1][1]})^T Q_{ji_1|1}^T \omega_{ji_2|1}$, $j = 2, 3$, to $\text{RX}_1$, then it will have enough equations to resolve its 5 desired data symbols (it can be easily shown that these equations are linearly independent almost surely). Hence, two symbols $u^{[i_1][1:2]}$ and $u^{[i_1][1:3]}$ can be defined as

$$
u^{[i_1][1:j]} \triangleq (u^{[i_1][1]})^T Q_{ji_1|1}^T \omega_{ji_2|1}, \quad j = 2, 3.
$$

(68)

In the same way, the following 5 fresh data symbols (now, all intended for $\text{RX}_2$) are transmitted during the next 3 time slots:

$$
u^{[i_1][2]} \triangleq [u_1^{[i_1][2]}, u_2^{[i_1][2]}, u_3^{[i_1][2]}]^T,
$$

(69)

$$
u^{[i_2][2]} \triangleq [u_1^{[i_2][2]}, u_2^{[i_2][2]}]^T,
$$

(70)

and the following two side information symbols are generated:

$$
u^{[i_1][1:2]} \triangleq (u^{[i_1][2]})^T Q_{ji_1|2}^T \omega_{ji_2|2}, \quad j = 1, 3,
$$

(71)

where $Q_{ji_1|2}$ and $\omega_{ji_2|2}$ are similarly defined.
The same procedure is followed during the last 3 time slots to transmit another 5 fresh data symbols

\[ u^{[i_1|3]} \triangleq [u_1^{[i_1|3]}, u_2^{[i_1|3]}, u_3^{[i_1|3]}]^T, \]
\[ u^{[i_2|3]} \triangleq [u_1^{[i_2|3]}, u_2^{[i_2|3]}]^T, \]

which are all intended for RX_3, and generate the two side information symbols

\[ u^{[i_1|3;j]} \triangleq (u^{[i_1|3]})^T Q_{ji_{1|3}}^T \omega_{ji_{1|3}}, \quad j = 1, 2, \]

with similar definitions of \( Q_{ji_{1|3}}^T \) and \( \omega_{ji_{1|3}} \).

After these 9 time slots, if we deliver the side information symbols defined in Eqs. (68), (71) and (74) to their respective receivers, then each receiver will be able to decode all its own 5 data symbols. To this end, consider the linear combination \( u^{[i_1|1;2]} + u^{[i_1|2;1]} \). If we deliver this linear combination to both RX_1 and RX_2, then RX_1 can cancel \( u^{[i_1|2;1]} \) to obtain \( u^{[i_1|1;2]} \). Similarly, RX_2 can cancel \( u^{[i_1|1;2]} \) to obtain \( u^{[i_1|2;1]} \). Note also that both \( u^{[i_1|1;2]} \) and \( u^{[i_1|2;1]} \) are available at TX_{1i}, and so is their summation. Therefore, one can define the following order-2 symbol available at TX_{1i}:

\[ u^{[i_1|1;2]} \triangleq u^{[i_1|1;2]} + u^{[i_1|2;1]}, \]

The following order-2 symbols can be similarly defined:

\[ u^{[i_1|1;3]} \triangleq u^{[i_1|1;3]} + u^{[i_1|3;1]} \]
\[ u^{[i_1|2;3]} \triangleq u^{[i_1|2;3]} + u^{[i_1|3;2]} \]

Our goal in phase 2 is to deliver the above three order-2 symbols to their respective pairs of receivers.

- **Phase 2 (2 × 3 X Channel):**

  This phase takes 12 time slots to transmit 18 order-2 symbols generated in phase 1. Recall that in phase 1 we generated only three order-2 symbols \( u^{[i_1|1;2]}, u^{[i_1|1;3]}, \) and \( u^{[i_1|2;3]} \) which are all available at TX_{1i}, where \( i_1 = 1 \). As we will see later, the following 18 order-2 symbols are required for phase 2:

\[ u_k^{[i_1|1;2]}, u_k^{[i_1|1;3]}, u_k^{[i_1|2;3]}, \quad i = 1, 2, \quad 1 \leq k \leq 3. \]

Therefore, we repeat phase 1 three times with \((i_1, i_2) = (1, 2)\) and three times with \((i_1, i_2) = (2, 1)\) to generate the above 18 order-2 symbols required for phase 2. The transmission in phase 2 is then accomplished as follows:

The first 4 time slots of phase 2 are dedicated to transmission of 6 \((1, 2)\)-symbols \( \{u_k^{[1|1;2]}\}_{k=1}^3 \) and
\( \left\{ u_k^{[2][1,2]} \right\}_{k=1}^3 \). This is accomplished in exactly the same way as the first 4 time slots of phase 2 in Section IV-A and the side information symbols \( u^{[1][1,2][3]} \) and \( u^{[2][1,2][3]} \) will be generated at RX_3. Similar to phase 2 of Section IV-A, the next 8 time slots are dedicated to transmission of 6 \((1,3)\)-symbols and 6 \((2,3)\)-symbols. However, in contrast to Section IV-A, the \((1,3)\)-symbols and \((2,3)\)-symbols are here transmitted by TX_1 and TX_2. Hence, after these 8 time slots, the side information \( u^{[1][2,3][1]} \) and \( u^{[2][2,3][1]} \) will be generated at RX_1 and the side information \( u^{[1][1,3][2]} \) and \( u^{[2][1,3][2]} \) will be generated at RX_2.

Therefore, after these 12 time slots, our goal is reduced to

(a) delivering \( u^{[1][1,2][3]} \) and \( u^{[2][1,2][3]} \) to both RX_1 and RX_2,
(b) delivering \( u^{[1][1,3][2]} \) and \( u^{[2][1,3][2]} \) to both RX_1 and RX_3,
(c) delivering \( u^{[1][2,3][1]} \) and \( u^{[2][2,3][1]} \) to both RX_2 and RX_3.

Now, consider \( u^{[1][1,2][3]} \), \( u^{[1][1,3][2]} \), and \( u^{[1][2,3][1]} \). Note that these three symbols are available at TX_1, and so is any linear combination of them. Another observation is that each receiver has exactly one symbol out of these three symbols and requires the other two. Hence, if we deliver two random linear combinations of these three symbols to all receivers, then RX_1 can remove \( u^{[1][2,3][1]} \) from the two linear combinations to obtain two random linear combinations solely in terms of \( u^{[1][1,2][3]} \) and \( u^{[1][1,3][2]} \), and so, solve them for \( u^{[1][1,2][3]} \) and \( u^{[1][1,3][2]} \). Likewise, RX_2 (resp. RX_3) can remove \( u^{[1][1,3][2]} \) (resp. \( u^{[1][1,2][3]} \)) from the two random linear combinations and obtain two random linear equations solely in terms of its own pair of desired symbols, and resolve its desired symbols. Thus, the following two random linear combinations can be considered as order-3 symbols to be delivered to all three receivers in the next phase:

\[
\begin{align*}
u_1^{[1][1,2][3]} & \triangleq \alpha_1 u^{[1][1,2][3]} + \alpha_2 u^{[1][1,3][2]} + \alpha_3 u^{[2][2,3][1]}, \\
u_2^{[1][1,2][3]} & \triangleq \alpha'_1 u^{[1][1,2][3]} + \alpha'_2 u^{[1][1,3][2]} + \alpha'_3 u^{[2][2,3][1]}.
\end{align*}
\]

Using the same arguments for \( u^{[2][1,2][3]} \), \( u^{[2][1,3][2]} \), and \( u^{[2][2,3][1]} \), one can define the following order-3 symbols:

\[
\begin{align*}
u_1^{[2][1,2][3]} & \triangleq \beta_1 u^{[2][1,2][3]} + \beta_2 u^{[2][1,3][2]} + \beta_3 u^{[2][2,3][1]}, \\
u_2^{[2][1,2][3]} & \triangleq \beta'_1 u^{[2][1,2][3]} + \beta'_2 u^{[2][1,3][2]} + \beta'_3 u^{[2][2,3][1]},
\end{align*}
\]

where \( \beta_i \) and \( \beta'_i \), \( 1 \leq i \leq 3 \), are random coefficients.

**Phase 3 (2 × 3 X Channel):**

Using time division in 4 time slots, the 4 order-3 symbols \( u_1^{[1][1,2][3]} \), \( u_2^{[1][1,2][3]} \), \( u_1^{[2][1,2][3]} \), and \( u_2^{[2][1,2][3]} \) will be delivered to all three receivers.

At the end, in view of the fact that we have fed a total of \( 6 \times 15 = 90 \) fresh data symbols to the system in \( 6 \times 9 = 54 \) time slots in phase 1, and spent 12 time slots in phase 2 and 4 time slots in phase 3, the
achieved DoF is equal to
\[ \text{DoF}_X^{(2, 3)} = \frac{90}{54 + 12 + 4} = \frac{9}{7}. \] (83)

Also, regarding the phases 2 and 3, we have \( \text{DoF}_X^{(2, 3)} = \frac{18}{12+4} = \frac{9}{8} \), and \( \text{DoF}_X^{(3, 2)} = 1 \).

**B. The \( 2 \times K \) SISO X Channel**

Our transmission scheme for the \( 2 \times K \) SISO X channel with delayed CSIT is a multiphase scheme as depicted in Fig. 6. In particular, for every \( m, 1 \leq m \leq K - 1 \), phase \( m \) takes \( N_X^m \) order-\( m \) symbols of the form \( u_{[i]|S_m} \), \( S_m \subset S_K \), \( i \in \{1, 2\} \), and transmits them to the receivers in \( T_X^m \) time slots. Then, a total of \( N_X^{m+1} \) order-\( (m + 1) \) symbols of the form \( u_{[i]|S_{m+1}} \), \( S_{m+1} \subset S_K \), \( i \in \{1, 2\} \) are generated such that if the generated symbols are somehow delivered to their intended receivers, then every subset \( S_m \) of cardinality \( m \) of receivers will be able to decode all the \( S_m \)-symbols transmitted in phase \( m \). The parameters \( N_X^m \), \( T_X^m \), and \( N_X^{m+1} \) are given by

\[ N_X^m = 2[2(K - m) + 1] \binom{K}{m}, \] (84)
\[ T_X^m = 2(K - m + 1) \binom{K}{m}, \] (85)
\[ N_X^{m+1} = 2m \binom{K}{m+1}. \] (86)

The following is a detailed description of phase \( m \):

- **Phase \( m \), \( 1 \leq m \leq K - 1 \) (\( 2 \times K \) X Channel):

  Fix \( i_1 = 1 \) and \( i_2 = 2 \). For every \( S_m \subset S_K \), consider the following two vectors of \( S_m \)-symbols:

  \[ u_{[i_1]|S_m} \triangleq \begin{bmatrix} u_{[i_1]|S_m}^{[1]}, & u_{[i_1]|S_m}^{[2]}, & \cdots, & u_{[i_1]|S_m}^{[K-m+1]} \end{bmatrix}^T, \] (87)
  \[ u_{[i_2]|S_m} \triangleq \begin{bmatrix} u_{[i_2]|S_m}^{[1]}, & u_{[i_2]|S_m}^{[2]}, & \cdots, & u_{[i_2]|S_m}^{[K-m]} \end{bmatrix}^T, \] (88)

  and transmit them exactly as the phase \( m \)-I of Section [V-B]. More specifically, in \( K - m + 1 \) time slots, \( TX_{i_1} \) and \( TX_{i_2} \) transmit \( K - m + 1 \) random linear combinations of elements of \( u_{[i_1]|S_m} \) and \( u_{[i_2]|S_m} \), respectively. Using the same arguments as in the phase \( m \)-I of Section [V-B], \( K - m \) side information symbols of the form \( u_{[i_1]|S_{m};j'] \), \( j' \in S_K \setminus S_m \), are generated after these \( K - m + 1 \) time slots (see Eq. (54)).
If we somehow deliver all symbols \( u^{[i_1|S_m:j]} \), \( j' \in S_K \setminus S_m \), to every receiver \( RX_j, j \in S_m \), then every receiver \( RX_j, j \in S_m \), will be obtain enough linearly independent equations to decode all the \( S_m \)-symbols in \( u^{[i_1|S_m]} \) and \( u^{[i_2|S_m]} \).

Therefore, for every \( S_m \subset S_K \), a total of \( 2(K - m) + 1 \) \( S_m \)-symbols are transmitted in \( K - m + 1 \) time slots, and \( K - m \) side information symbols are generated. Since there are \( \binom{K}{m} \) choices of \( S_m \subset S_K \), this implies the transmission of \( [2(K - m) + 1]\binom{K}{m} \) order-\( m \) symbols in \( (K - m + 1)\binom{K}{m} \) time slots and generation of \( (K - m)\binom{K}{m} \) side information symbols. Now, in order to deliver the generated side information symbols to their respective intended receivers, fix a subset \( S_{m+1} \subset S_K \). For every \( j' \in S_{m+1} \), we have generated exactly one side information symbol \( u^{[i_1|S_{m+1}\setminus\{j'\}:j']} \). Since there are \( m + 1 \) different choices for \( j', j' \in S_{m+1} \), we can identify \( m + 1 \) symbols of the form \( u^{[i_1|S_{m+1}\setminus\{j'\}:j']} \) for a fixed \( S_{m+1} \subset S_K \).

Moreover, every receiver \( RX_{j'}, j' \in S_{m+1} \), has exactly one of these \( m + 1 \) symbols and wishes to obtain the rest. Therefore, if we deliver \( m \) random linear combinations of these \( m + 1 \) symbols to all receivers in \( S_{m+1} \), then each of them will remove its known side information and obtain \( m \) linearly independent equations in terms of the \( m \) desired symbols almost surely and hence, decode all desired symbols. Thus, we define \( m \) \( S_{m+1} \)-symbols as follows:

\[
u^{[i_1|S_{m+1}]}_\ell \triangleq \sum_{j' \in S_{m+1}} \beta^{[i_1|S_{m+1}\setminus\{j'\}:j']}\nu^{[i_1|S_{m+1}\setminus\{j'\}:j']}_\ell, \quad 1 \leq \ell \leq m, \tag{89}\]

where \( \beta^{[i_1|S_{m+1}\setminus\{j'\}:j']} \), \( j' \in S_{m+1} \), \( 1 \leq \ell \leq m \), is a random coefficient. Since there are \( \binom{K}{m+1} \) choices of \( S_{m+1} \), \( S_{m+1} \subset S_K \), a total of \( m\binom{K}{m+1} \) order-\( (m+1) \) symbols will be generated as above.

Finally we note that, so far, we have only generated order-\((m+1)\) symbols of the form \( u^{[i_1|S_{m+1}]} \), with \( i_1 = 1 \), which are all available at \( TX_1 \). However, in order for phase \( m+1 \) to work, we need order-\((m+1)\) symbols of both forms \( u^{[1|S_{m+1}]} \) and \( u^{[2|S_{m+1}]} \). This can be seen from Eqs. (87) and (88). To resolve this issue, we simply repeat phase \( m \) with \( (i_1, i_2) = (2, 1) \). This together with the previous round of phase \( m \) implies the transmission of a total of \( N^X_m \) order-\( m \) symbols in \( T^X_m \) time slots, and generation of \( N^X_{m+1} \) order-\((m+1)\) symbols, where \( N^X_m, T^X_m \), and \( N^X_{m+1} \) are given by Eqs. (84) to (86). If we deliver all these \( S_{m+1} \)-symbols to their intended subsets of receivers, then each receiver will be able to decode all its desired order-\( m \) symbols transmitted in this phase. This will be accomplished during the next phases.

- **Phase \( K \) (2 × \( K \) X Channel):**

In this phase, in each time slot, an order-\( K \) symbol of the form \( u^{[i|S_K]} \), \( i \in \{1, 2\} \), is transmitted by \( TX_i \) while the other transmitter is silent. Therefore,

\[
\text{DoF}_K^X(2, K) = 1. \tag{90}\]
Finally, using Eqs. (84) to (86), we can express $\text{DoF}_m^X(2, K)$, the achieved DoF of transmission of order-$m$ symbols in the $2 \times K$ SISO X channel with delayed CSIT, as

$$\text{DoF}_m^X(2, K) = \frac{N_m^X}{T_m^X + \frac{N_{m+1}^X}{\text{DoF}_{m+1}^X(2,K)}}$$

$$= \frac{2[2(K - m) + 1](K)}{2(K - m + 1)(K_m) + \frac{2m(K_{m+1})}{\text{DoF}_{m+1}^X(2,K)}}$$

$$= \frac{(m + 1)[2(K - m) + 1]}{(m + 1)(K - m + 1) + \frac{m(K-m)}{\text{DoF}_{m+1}^X(2,K)}}, \quad 1 \leq m \leq K - 1. \quad (91)$$

It is proved in Appendix D that Eqs. (13) and (14) are indeed closed form expressions for $\text{DoF}_m^X(2, K)$, $1 \leq m \leq K$, satisfying the recursive Eq. (91) together with the initial condition (90).

VI. CONCLUSIONS

We proposed new multiphase interference alignment schemes and obtained new achievable results on the DoF of the $K$-user SISO interference channel and $2 \times K$ SISO X channel under delayed CSIT assumption. Our results show that the DoF of these channels with the outdated CSI at transmitters is strictly greater than that with no CSIT. The achieved DoFs were shown to be strictly increasing in $K$ and approach limiting values of $\frac{4}{6 \ln 2 - 1}$ and $\frac{1}{\ln 2}$, respectively, for the interference and X channels as $K \rightarrow \infty$. This is in contrast to the no CSIT assumption wherein it is known that both channels have only one DoF for all values of $K$. For the interference channel, we improved the best previously known result on the DoF of the 3-user case with delayed CSIT, and to the best of our knowledge, this paper presents the first DoF results for the $K$-user case with $K > 3$. For the $2 \times K$ X channel, our achievable DoF is strictly greater than the best previously reported result on that of the $K \times K$ X channel.

In the lack of tight upper bounds, no optimality argument can be made; however, it is conjectured that the DoF of both the $K$-user IC and $S \times K$ X channel with delayed CSIT is bounded above by a constant (i.e., it does not scale with number of users).

APPENDIX A

PROOF OF LINEAR INDEPENDENCE IN PHASE 1 FOR THE $K$-USER IC

In this appendix, we show that after phase 1 of the proposed transmission scheme for the $K$-user SISO IC with delayed CSIT, the $(K - 1)^2$ linear equations obtained by each receiver in terms of its data symbols are linearly independent almost surely (see Section IV-B, Eqs. (42) and (43)). To this end, consider the aforementioned equations at RX$_j$, $1 \leq j \leq K$:

$$(u^{[j]})^T Q^T_{ji} \omega_{i_1}, \quad i_1 \in S_K \backslash \{j\}, \quad (92)$$
which are equivalent to the system of linear equations \((\mathbf{u}^{[j]})^T \mathbf{Q}_{i2j}^T \omega_{i2i3}, \quad i_2, i_3 \in S_K \setminus \{j\}, i_2 \neq i_3,\) (93)

where \(\mathbf{P}^{[j]}\) is a \((K-1)^2 \times (K-1)^2\) matrix defined as

\[
\mathbf{P}^{[j]} \triangleq \begin{cases} 
\{ \mathbf{Q}_{jj}^T \omega_{j1} \}_{i_1 \in S_K \setminus \{j\}}, & \{ \mathbf{Q}_{i2j}^T \omega_{i2i3} \}_{i_2, i_3 \in S_K \setminus \{j\}, i_2 \neq i_3} \\
(\mathbf{C}^{[j]})^T \begin{cases} 
\{ \mathbf{D}_{jj} \omega_{j1} \}_{i_1 \in S_K \setminus \{j\}}, & \{ \mathbf{D}_{i2j} \omega_{i2i3} \}_{i_2, i_3 \in S_K \setminus \{j\}, i_2 \neq i_3}
\end{cases}
\end{cases}
\] (94)

\[
= (\mathbf{C}^{[j]})^T \begin{cases} 
\{ \mathbf{D}_{jj} \omega_{j1} \}_{i_1 \in S_K \setminus \{j\}}, & \{ \mathbf{D}_{i2j} \omega_{i2i3} \}_{i_2, i_3 \in S_K \setminus \{j\}, i_2 \neq i_3}
\end{cases}.
\] (95)

Let \(\mathbf{h}_{ij}\) denote the vector of length \((K-1)^2 + 1\) containing the main diagonal of \(\mathbf{D}_{ij}\) and define \(\mathbf{v}_{\ell} \triangleq [1, 1, \ldots, 1]^T\). Then, one can write

\[
\mathbf{P}^{[j]} = (\mathbf{C}^{[j]})^T (\mathbf{H}^{[j]} \circ \Omega^{[j]}),
\] (96)

where

\[
\mathbf{H}^{[j]} \triangleq \begin{cases} 
\mathbf{h}_{jj} \mathbf{v}_{K-1}^T, & \{ \mathbf{h}_{ij} \mathbf{v}_{K-2}^T \}_{i \in S_K \setminus \{j\}}
\end{cases},
\] (97)

\[
\Omega^{[j]} \triangleq \begin{cases} 
\{ \omega_{j1} \}_{i_1 \in S_K \setminus \{j\}}, & \{ \omega_{i2} \}_{i_2 \in S_K \setminus \{j\}, i_2 \neq i_3}
\end{cases} = [\omega_{j1} i_1 \in S_K \setminus \{j\}, j_1 \in S_K \setminus \{i_1\}],
\] (98)

and “\(\circ\)” denotes the element-wise product operator. Recall that \(\mathbf{Q}_{jj}^T \omega_{j1} i_1 = (\mathbf{C}^{[i_1]})^T \mathbf{D}_{jj} \omega_{j1} i_1 = 0_{(K-1)^2 \times 1}\).

Hence, the vector \(\mathbf{D}_{jj1} \omega_{j1} i_1\) lies in the left null space of \(\mathbf{C}^{[i_1]}\). However, \(\mathbf{C}^{[i_1]}\) is a random \([(K-1)^2 + 1] \times (K-1)^2\) matrix, and thus, it is full rank almost surely and its left null space is one dimensional, denoted by the nonzero unit vector \(\mathbf{n}^{[i_1]}\). It immediately follows that, for every \(j_1 \in S_K \setminus \{i_1\}\), there exists a nonzero scalar \(a_{j1i1}\) such that \(\mathbf{D}_{jj1} \omega_{j1i1} = a_{j1i1} \mathbf{n}^{[i_1]}\), or equivalently, \(\omega_{j1i1} = a_{j1i1} \mathbf{D}^{-1}_{jj1} \mathbf{n}^{[i_1]}\). Note that \(\mathbf{D}_{jj1}\) is full rank, and so, invertible almost surely. Therefore, \(\Omega^{[j]}\) can be rewritten as follows

\[
\Omega^{[j]} = [a_{j1i1} \mathbf{D}^{-1}_{jj1} \mathbf{n}^{[i_1]}]_{i_1 \in S_K \setminus \{j\}, j_1 \in S_K \setminus \{i_1\}},
\] (99)

Since \(a_{j1i1}\)'s are nonzero and each of them scales a whole column of \(\mathbf{H}^{[j]} \circ \Omega^{[j]}\), they do not affect the rank. Hence,

\[
\operatorname{rank}(\mathbf{H}^{[j]} \circ \Omega^{[j]}) = \operatorname{rank}(\mathbf{H}^{[j]} \circ [\mathbf{D}^{-1}_{jj1} \mathbf{n}^{[i_1]}]_{i_1 \in S_K \setminus \{j\}, j_1 \in S_K \setminus \{i_1\}}).
\] (100)

One also can write

\[
\mathbf{H}^{[j]} \circ [\mathbf{D}^{-1}_{jj1} \mathbf{n}^{[i_1]}]_{i_1 \in S_K \setminus \{j\}, j_1 \in S_K \setminus \{i_1\}} = \mathbf{H}^{[j]} \circ \mathbf{N}^{[j]} \circ (\mathbf{H}^{[j]})^{\circ(-1)} = \mathbf{N}^{[j]} \circ (\mathbf{H}^{[j]})^{\circ(-1)},
\] (101)
where

\[ \hat{H}^{[j]} = \left[ \hat{h}_{ji} \right]_{i, j \in S_K \setminus \{j\} \setminus \{i\}} \]  
\[ N^{[j]} = \left[ n^{[i]}_j v_{K-1}^T \right]_{i \in S_K \setminus \{j\}} \]  
\[ \Phi^{[j]} \triangleq \hat{H}^{[j]} \circ N^{[j]}, \]  

and \((\hat{H}^{[j]})^{-1}\) denotes the element-wise inverse of \(\hat{H}^{[j]}\). We note that \(\hat{H}^{[j]}\) and \(N^{[j]}\) are independent of each other, since \(N^{[j]}\) is a function of \(\{C^{[i]}\}_{i \in S_K \setminus \{j\}}\) which are independent of \(\hat{H}^{[j]}\). Also, \(\hat{H}^{[j]}\) and \(\hat{H}^{[j]}\) are independent of each other, since the channel coefficients are i.i.d. across the transmitters and receivers. Hence, \(\Phi^{[j]}\) is independent of \(\hat{H}^{[j]}\).

On the other hand, it can be easily verified that the elements of \(\hat{H}^{[j]}\), and thereby \((\hat{H}^{[j]})^{-1}\), are i.i.d.. Also, it is easy to show that all elements of \(\Phi^{[j]}\) are nonzero almost surely. Therefore, for any given \(\Phi^{[j]}\), the elements of \(\Phi^{[j]} \circ (\hat{H}^{[j]})^{-1}\) are also independent of each other, since \(\Phi^{[j]}\) is independent of \((\hat{H}^{[j]})^{-1}\). This implies that, for any given \(\Phi^{[j]}\), \(\Phi^{[j]} \circ (\hat{H}^{[j]})^{-1}\) is full rank almost surely. This means that \(\Phi^{[j]} \circ (\hat{H}^{[j]})^{-1}\) is full rank almost surely.

Finally, we note that \(C^{[j]}\) is independent of \(\hat{H}^{[j]}, N^{[j]}\), and \(\hat{H}^{[j]}\), and thereby, of \(\hat{H}^{[j]} \circ \Omega^{[j]}\). Therefore, regarding Eqs. (96), (100) and (101) and applying Lemma 1, one can conclude that \(P^{[j]}\) is full rank almost surely.

**Lemma 1:** Let \(A_{m \times n}\) and \(B_{n \times m}\) be two independent (not necessarily i.i.d.) random matrices with continuous probability distributions and let \(m \leq n\). If \(A\) and \(B\) are full rank almost surely, then \(AB\) is full rank almost surely.

**Proof:** If \(m = n\), then the lemma is obviously true. Assume \(m < n\). Let \(a_i, 1 \leq i \leq n\), and \(b_j, 1 \leq j \leq m\), be the \(i\)th and \(j\)th column of \(A\) and \(B\), respectively. Then, the \(j\)th column of \(AB\) can be written as \(\sum_{i=1}^{n} b_j a_i\). Now, assume a linear combination of the columns of \(AB\) are equal to zero, namely

\[ \sum_{j=1}^{m} \gamma_j \sum_{i=1}^{n} b_j a_i = 0_{m \times 1}. \]  

Therefore, exchanging the order of the summations, we have \(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} \gamma_j b_{ji}\right) a_i = 0_{m \times 1}\). This can be written in matrix form as follows

\[ A \left( \sum_{j=1}^{m} \gamma_j b_j \right) = 0_{m \times 1}. \]  

Thus, the vector \(\sum_{j=1}^{m} \gamma_j b_j\) either is equal to zero or lies in the null space of \(A\). In the former case, we get \(\gamma_j = 0, 1 \leq j \leq m\), since \(B\) is full rank almost surely. In the latter case, since \(A\) is full rank almost surely, its null space is \(n - m\) dimensional. Let \(N_{n \times (n-m)} \triangleq [n_1, n_2, \ldots, n_{n-m}]\) denote the basis of the
null space of $A$. Then, there should exist $\xi_\ell$, $1 \leq \ell \leq n - m$, such that

$$
\sum_{j=1}^{m} \gamma_j b_j = \sum_{\ell=1}^{n-m} \xi_\ell n_\ell.
$$

(107)

Note that $N$ is independent of $B$, since $A$ and $B$ are independent of each other. Consider the square matrix $[B|N]_{n \times n}$. Since $B$ and $N$ are full rank almost surely (with continuous distributions) and independent of each other, one can easily show that $[B|N]$ is full rank almost surely. This together with Eq. (107) yields to $\gamma_j = 0$, $1 \leq j \leq m$, and $\xi_\ell = 0$, $1 \leq \ell \leq n - m$.

**APPENDIX B**

**Proof of Linear Independence in Phase $m$-I for the $K$-user IC and Phase $m$ for the $K$-user X Channel**

Consider the following system of linear equations:

$$
Q_{ji_1} u_{[i_1|S_m]} + Q_{ji_2} u_{[i_2|S_m]} = (u_{[i_1|S_m]})^T Q_{ji_1}^T \omega_{ji_2}, \quad j' \in S_K \setminus S_m.
$$

(108)

(109)

which are equivalent to the system of linear equation $(u_{[S_m]})^T G[j]$, where $G[j]$ and $u_{[S_m]}$ are defined as

$$
G[j] \triangleq \begin{bmatrix}
(Q_{ji_1})^T & \{Q_{ji_1}^T \omega_{ji_2}\}_{j' \in S_K \setminus S_m} \\
(Q_{ji_2})^T & \bigcirc
\end{bmatrix},
$$

(110)

$$
u_{[S_m]} \triangleq (u_{[i_1|S_m]})^T, (u_{[i_1|S_m]})^T.
$$

(111)

Note first that, by definition, $Q_{ji_1} = D_{ji_1} C_{[i_1|S_m]}$ and $Q_{ji_2} = D_{ji_2} C_{[i_2|S_m]}$. These matrix multiplications are nothing but scaling the columns of $C_{[i_1|S_m]}$ and $C_{[i_2|S_m]}$ by the diagonal elements of $D_{ji_1}$ and $D_{ji_2}$, respectively. Since the diagonal elements of $D_{ji_1}$ and $D_{ji_2}$ are nonzero almost surely and since scaling the columns of a matrix by nonzero factors does not affect its rank, one can write

$$
\text{rank}(Q_{ji_1}) = \text{rank}(C_{[i_1|S_m]}) = K - m + 1,
$$

(112)

$$
\text{rank}(Q_{ji_2}) = \text{rank}(C_{[i_2|S_m]}) = K - m.
$$

(113)

Also, if a linear combination of some columns is added to a (nonzero) scaled version of a column in a matrix then its rank does not change. Therefore, if we replace the $K - m + 1$’th column of $G[j]$ with a linear combination of its first $K - m + 1$ columns, its rank will not change. If we choose the coefficients of such a linear combination to be the elements of $\omega_{ji_2}$ (which are all nonzero almost surely), then since
by definition, \((Q_{ji}^T \omega_{ji}) = 0_{(K-m) \times 1}\), we get

\[
\text{rank}(G^{[j]}) = \text{rank}(\tilde{G}^{[j]}),
\]

where

\[
\tilde{G}^{[j]} \triangleq \begin{bmatrix}
(Q_{ji}^T) & \{Q_{ji}^T \omega_{ji} \}_{j' \in (S_K \setminus S_m) \cup \{j\}} \\
(Q_{ji2}^T) & 0
\end{bmatrix},
\]

and \(Q_{ji1}^T\) and \(Q_{ji2}^T\) are respectively the submatrices of \(Q_{ji1}\) and \(Q_{ji2}\) including their first \(K - m\) rows. Hence, it suffices to show \(\tilde{G}^{[j]}\) is full rank. To do so, we note that \(\tilde{Q}_{ji2}\) is a \((K-m) \times (K-m)\) matrix with \(\text{rank} (\tilde{Q}_{ji2}) = \text{rank}(Q_{ji2}) = K-m\). If we show that the matrix \(\{Q_{ji1}^T \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}\) is also a square full rank matrix of size \((K-m+1) \times (K-m+1)\), then using Lemma 2 it immediately follows that \(\tilde{G}^{[j]}\) is full rank. Now, we rewrite \(\{Q_{ji1}^T \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}\) as:

\[
[Q_{ji1}^T \omega_{ji2}]_{j' \in (S_K \setminus S_m) \cup \{j\}} = (C^{[i1 | S_m]})^T [D_{ji1} \omega_{ji2}]_{j' \in (S_K \setminus S_m) \cup \{j\}}.
\]

Since the matrices are square, we have

\[
\text{det}(\{Q_{ji1}^T \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}) = \text{det}(C^{[i1 | S_m]}) \cdot \text{det}(\{D_{ji1} \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}),
\]

and since \(C^{[i1 | S_m]}\) is full rank almost surely, \(\text{det}(C^{[i1 | S_m]}) \neq 0\). Thus, it remains to show \(\{D_{ji1} \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}\) is full rank. Using the same argument as in Appendix A one can write

\[
\omega_{ji2} = a_{ji2} D_{ji2}^{-1} n^{[i2]},
\]

where \(a_{ji2}\) is a nonzero scalar. Therefore,

\[
\text{rank}(\{D_{ji1} \omega_{ji2}\}_{j' \in (S_K \setminus S_m) \cup \{j\}}) = \text{rank}(\{a_{ji2} D_{ji1} D_{ji2}^{-1} n^{[i2]}\}_{j' \in (S_K \setminus S_m) \cup \{j\}})
\]

\[(a) \Rightarrow \text{rank}(\{D_{ji1} D_{ji2}^{-1} n^{[i2]}\}_{j' \in (S_K \setminus S_m) \cup \{j\}})
\]

\[(b) \Rightarrow \text{rank}(\begin{bmatrix} h_{ji1}(t) \\ h_{ji2}(t) \end{bmatrix}_{j' \in (S_K \setminus S_m) \cup \{j\}})
\]

\[(c) \Rightarrow K - m + 1,
\]

where (a) follows from the fact that scaling the columns of a matrix by nonzero factors \((a_{ji2}\)'s) will not change its rank; (b) follows from the fact that scaling the rows of a matrix by nonzero factors (elements of \(n^{[i2]}\)) will not change its rank; and (c) is true since \(h_{ji1}(t), h_{ji2}(t)\)'s are i.i.d. for \(1 \leq t \leq K - m + 1\) and \(j' \in (S_K \setminus S_m) \cup \{j\}\).

**Lemma 2:** Let \(A = [a_{ij}]_{m \times m}\) and \(B = [b_{ij}]_{n \times n}\) be two square matrices which are full rank almost
surely and let $C = [c_{ij}]_{m \times n}$ be an arbitrary matrix. Then the following matrix is full rank almost surely:

$$D = \begin{bmatrix} C & A \\ B & \circ \end{bmatrix}.$$  

(120)

**Proof:** Denote by $a_j$, $b_j$, and $d_j$ the $j$'th columns of $A$, $B$, and $D$, respectively. Assume that

$$\sum_{j=1}^{m+n} \alpha_j d_j = 0_{(m+n) \times 1},$$  

(121)

for some $\alpha_1, \alpha_2, \ldots, \alpha_{m+n} \in \mathbb{C}$. Then, since $d_{ij} = 0$ for $m + 1 \leq i \leq m + n$ and $n + 1 \leq j \leq m + n$, one can write $\sum_{j=1}^{n} \alpha_j b_j = 0_{n \times 1}$ and since $B$ is full rank almost surely, we have $\alpha_j = 0$, $1 \leq j \leq n$. This together with Eq. (121) yields $\sum_{j=n+1}^{m+n} \alpha_j d_j = 0_{(m+n) \times 1}$. Considering the first $m$ elements of these columns, it follows that $\sum_{j=n+1}^{m+n} \alpha_j a_{j-n} = 0_{m \times 1}$ and since $A$ is full rank almost surely, we have $\alpha_j = 0$, $n + 1 \leq j \leq m + n$.

**APPENDIX C**

**CLOSED FORM SOLUTION TO THE RECURSIVE EQUATION (61) FOR THE $K$-USER IC**

In this appendix, we derive a closed form solution to the recursive equation

$$\text{DoF}^{IC}_{K-i}(K) = \frac{(K - i) (2i + 1)}{(K - i)(i + 1) + \frac{i}{K-i+1} + \frac{(K-i-1)_{i}}{\text{DoF}^{IC}_{K-i+1}(K)}}, \quad 1 \leq i \leq K-2,$$  

(122)

$$\text{DoF}^{IC}_{K}(K) = 1.$$  

(123)

We start by rearranging Eq. (122) in the following form:

$$1 - \frac{1}{\text{DoF}^{IC}_{K-i}(K)} = \frac{i}{(K - i)(2i + 1)} \left[ (K - i - 1) \left( 1 - \frac{1}{\text{DoF}^{IC}_{K-i+1}(K)} \right) + \frac{K - i}{K - i + 1} \right], \quad 1 \leq i \leq K-2.$$  

(124)

Define $A_{K-i}(K) \triangleq 1 - \frac{1}{\text{DoF}^{IC}_{K-i}(K)}$. Then, we have

$$A_{K-i}(K) = \frac{i}{(K - i)(2i + 1)} \left[ (K - i - 1) A_{K-i+1}(K) + \frac{K - i}{K - i + 1} \right], \quad 1 \leq i \leq K-2,$$  

(125)

$$A_{K}(K) = 0.$$  

(126)

Express $A_{K-i}(K)$ as

$$A_{K-i}(K) = \sum_{\ell=0}^{i} a_{K-i}^{[K-i]} \frac{1}{K - \ell},$$  

(127)
where \( a_{K-\ell}^{[K-i]} \) is found using
\[
a_{K-\ell}^{[K-i]} = [(K-\ell)A_{K-i}(K)]_{K=\ell}, \quad 0 \leq \ell \leq i.
\]
(128)

Substituting the expansion of Eq. (127) for \( A_{K-i+1}(K) \) in Eq. (125), we get
\[
A_{K-i}(K) = \frac{i}{(K-i)(2i+1)} \left[ \sum_{\ell=0}^{i-1} \frac{(K-i-1)a_{K-i+1}^{[K-i+1]}}{K-\ell} + \frac{K-i}{K-i+1} \right].
\]
(129)

Equations (128) and (129) lead to three recursive equations as follows:
\[
a_{K-i}^{[K-i]} = \frac{(i-\ell+1)i}{(i-\ell)(2i+1)} a_{K-i}^{[K-i+1]}, \quad 0 \leq \ell \leq i-2,
\]
(130)
\[
a_{K-i}^{[K-i+1]} = \frac{i}{2i+1} \left( 2a_{K-i+1}^{[K-i+1]} + 1 \right),
\]
(131)
\[
a_{K-i}^{[K-i]} = -\frac{i}{2i+1} \sum_{\ell=0}^{i-1} a_{K-\ell}^{[K-i+1]} - \sum_{\ell=0}^{i-2} \frac{a_{K-\ell}^{[K-i]}}{i-\ell+1},
\]
(132)

where Eq. (132) follows from Eq. (130). Applying Eq. (130) \( i-\ell-1 \) times, we will have
\[
a_{K-i}^{[K-i]} = \frac{1}{2} a_{K-i}^{[K-\ell-1]} (i-\ell+1) \prod_{j=\ell+2}^{i-j} \frac{j}{2 + j}, \quad 0 \leq \ell \leq i-2.
\]
(133)

Substituting Eq. (133) in Eq. (132), we get
\[
a_{K-i}^{[K-i]} = -\frac{i}{2i+1} \sum_{\ell=0}^{i-2} a_{K-\ell}^{[K-i+1]} \prod_{j=\ell+2}^{i} \frac{j}{2 + j} - \frac{1}{2} \sum_{\ell=0}^{i-3} a_{K-\ell}^{[K-i-1]} \prod_{j=\ell+2}^{i-1} \frac{j}{2 + j} - \frac{1}{2} \sum_{\ell=0}^{i-2} a_{K-\ell}^{[K-i-1]} \prod_{j=\ell+2}^{i} \frac{j}{2 + j}
\]
\[
= \frac{i(i-1)}{2(2i+1)} a_{K-i+2}^{[K-i+2]} - \frac{i}{2(2i+1)} a_{K-i+2}^{[K-i+1]}
\]
(134)

where (a) results from reapplying Eq. (133) to \( a_{K-i+1}^{[K-i+1]} \), and (b) follows from applying Eq. (131) to \( a_{K-i+2}^{[K-i+1]} \).
Finally, using Eqs. (127) and (135) to (137), we have
\[
a_{K^{-i}}^{[K-i+1]} = \frac{i}{2i + 1} \left[ 1 - \frac{(i-1)(i-2)}{4(i-1)^2 - 1} \right], \quad 0 \leq i \leq K - 2. \tag{136}
\]

It follows from substituting Eq. (136) in Eq. (133) that
\[
a_{K^{-i}}^{[K-i+1]} = \frac{(i-1)(3i^2 + \ell - 1)}{2(4i^2 - 1)} \prod_{j=\tilde{l}+1}^i \frac{j}{2j + 1}, \quad 0 \leq \ell \leq i - 2. \tag{137}
\]

Finally, using Eqs. (127) and (135) to (137), we have
\[
A_{K^{-i}}(K) = -\frac{i(i-1)}{2(4i^2 - 1)(K-i)} + \sum_{\ell=0}^{i-1} \frac{(i-\ell+1)(3\ell^2 + \ell - 1)}{2(K-\ell)(4\ell^2 - 1)} \prod_{j=\tilde{l}+1}^i \frac{j}{2j + 1}, \quad 0 \leq i \leq K - 2. \tag{138}
\]

Since, by definition, \(D\text{Of}_{K^{-i}}^\text{IC}(K) = \frac{1}{1-A_{K^{-i}}(K)}\), we have the following closed form expression for \(D\text{Of}_{K^{-i}}^\text{IC}(K)\), \(0 \leq i \leq K - 2\):
\[
D\text{Of}_{K^{-i}}^\text{IC}(K) = \left[ 1 + \frac{i(i-1)}{2(4i^2 - 1)(K-i)} - \sum_{\ell=0}^{i-1} \frac{(i-\ell+1)(3\ell^2 + \ell - 1)}{2(K-\ell)(4\ell^2 - 1)} \prod_{j=\tilde{l}+1}^i \frac{j}{2j + 1} \right]^{-1}. \tag{139}
\]

**APPENDIX D**

**CLOSED FORM SOLUTION TO THE RECURSIVE EQUATION (91) FOR THE 2 \times K X CHANNEL**

In this appendix, we derive the closed form solution to the recursive equation
\[
D\text{Of}_{K^{-i}}^X(2, K) = \frac{(K-i+1)(2i+1)}{(K-i+1)(i+1) + \frac{(K-i+1)i}{D\text{Of}_{K^{-i+1}}^X(2, K)}}, \quad 1 \leq i \leq K - 1, \tag{140}
\]

\[D\text{Of}_{K}^X(2, K) = 1. \tag{141}\]

Rearranging Eq. (140) in the form of
\[
1 - \frac{1}{D\text{Of}_{K^{-i}}^X(2, K)} = \frac{i}{(K-i+1)(2i+1)} \left[ (K-i) \left( 1 - \frac{1}{D\text{Of}_{K^{-i+1}}^X(2, K)} \right) + 1 \right], \quad 1 \leq i \leq K - 1, \tag{142}
\]

and defining \(B_{K^{-i}}(K) \triangleq 1 - \frac{1}{D\text{Of}_{K^{-i}}^X(2, K)}\), one can write
\[
B_{K^{-i}}(K) = \frac{i}{(K-i+1)(2i+1)} \left[ (K-i)B_{K^{-i+1}}(K) + 1 \right], \quad 1 \leq i \leq K - 1, \tag{143}
\]

\[B_{K}(K) = 0. \tag{144}\]
Express $B_{K-i}(K)$ as

$$B_{K-i}(K) = \sum_{\ell=0}^{i-1} b_{K-\ell}^{[K-i]} / K - \ell,$$  \hspace{1cm} (145)

where $b_{K-\ell}^{[K-i]}$ is found using

$$b_{K-\ell}^{[K-i]} = [(K - \ell)B_{K-i}(K)]_{K=\ell}, \quad 0 \leq \ell \leq i - 1.$$  \hspace{1cm} (146)

Substituting the expansion of Eq. (145) for $B_{K-i+1}(K)$ in Eq. (143), we get

$$B_{K-i}(K) = \frac{i}{(K-i+1)(2i+1)} \left[ \sum_{\ell=0}^{i-2} \frac{(K-i)b_{K-\ell}^{[K-i+1]}}{K-\ell} + 1 \right].$$  \hspace{1cm} (147)

Equations (146) and (147) result in two recursive equations as follows:

$$b_{K-\ell}^{[K-i]} = \frac{(i-\ell)i}{(i-\ell-1)(2i+1)} b_{K-\ell}^{[K-i+1]}, \quad 0 \leq \ell \leq i - 2,$$  \hspace{1cm} (148)

$$b_{K-i+1}^{[K-i]} = \frac{i}{2i+1} \left[ 1 - \sum_{\ell=0}^{i-2} \frac{b_{K-\ell}^{[K-i+1]}}{i-\ell - 1} \right]$$

$$= \frac{i}{2i+1} \left[ 1 - \sum_{\ell=0}^{i-2} \frac{b_{K-\ell}^{[K-i]} - b_{K-\ell}^{[K-i-1]}}{i-\ell} \right],$$ \hspace{1cm} (149)

where Eq. (149) follows from Eq. (148). Applying Eq. (148) $i - \ell - 1$ times, we will have

$$b_{K-\ell}^{[K-i]} = b_{K-\ell}^{[K-i-\ell-1]} (i-\ell) \frac{i}{j} \frac{j}{2j+1}, \quad 0 \leq \ell \leq i - 2.$$  \hspace{1cm} (150)

Substituting Eq. (150) in Eq. (149), it follows that

$$b_{K-\ell}^{[K-i]} = \frac{i}{2i+1} - \sum_{\ell=0}^{i-2} b_{K-\ell}^{[K-i-\ell-1]} \frac{i}{j} \frac{j}{2j+1}$$

$$= \frac{i}{2i+1} - \frac{i}{2i+1} b_{K-\ell+2}^{[K-i]} - \sum_{\ell=0}^{i-3} b_{K-\ell}^{[K-i-\ell-1]} \frac{i}{j} \frac{j}{2j+1}$$

$$= \frac{i}{2i+1} - \frac{i}{2i+1} b_{K-\ell+2}^{[K-i]} - \frac{i}{2i+1} \sum_{\ell=0}^{i-3} b_{K-\ell}^{[K-i-\ell-1]} \frac{i-1}{j} \frac{j}{2j+1}$$

$$= \frac{i}{2i+1} - \frac{i}{2i+1} \left[ b_{K-\ell+2}^{[K-i]} + \sum_{\ell=0}^{i-3} b_{K-\ell}^{[K-i-\ell-1]} \frac{i-1}{j} \frac{j}{2j+1} \right]$$

$$= \frac{i}{2i+1} - \frac{i}{2i+1} \left[ b_{K-\ell+2}^{[K-i]} + \sum_{\ell=0}^{i-3} b_{K-\ell}^{[K-i-\ell-1]} \frac{i-1}{2(i-1)+1} \right]$$

$$= \frac{i}{2i+1} - \frac{i}{2i+1} \times \frac{i-1}{2(i-1)+1}$$

$$= \frac{i^2}{4i^2 - 1}, \quad 0 \leq i \leq K - 1,$$ \hspace{1cm} (152)
where (a) simply follows from an application of Eq. \[(151)\] for \(b_{K-i+2}^{[i]}\). Substituting Eq. \[(152)\] for \(b_{K-i}^{[i]}\) in Eq. \[(150)\], we obtain
\[
b_{K-i}^{[i]} = \frac{(i - \ell)(\ell + 1)}{2(\ell + 1) - 1} \prod_{j=\ell+1}^{i} \frac{j}{2j + 1}, \quad 0 \leq \ell \leq i - 2.
\] (153)
Combining Eqs. \[(145)\], \[(152)\] and \[(153)\], we can write
\[
B_{K-i}^{[i]}(K) = \sum_{\ell=0}^{i-1} \frac{(i - \ell)(\ell + 1)}{(K - \ell)(2(\ell + 1) - 1)} \prod_{j=\ell+1}^{i} \frac{j}{2j + 1}, \quad 0 \leq i \leq K - 1.
\] (154)
which together with \(\text{DoF}_{K-i}^{X}(2, K) = \frac{1}{1 - B_{K-i}^{[i]}(K)}\), yields
\[
\text{DoF}_{K-i}^{X}(2, K) = \left[1 - \sum_{\ell=0}^{i-1} \frac{(i - \ell)(\ell + 1)}{(K - \ell)(2(\ell + 1) - 1)} \prod_{j=\ell+1}^{i} \frac{j}{2j + 1}\right]^{-1}, \quad 0 \leq i \leq K - 1.
\] (155)

**APPENDIX E**

**ASYMPTOTIC BEHAVIOR OF THE ACHIEVABLE DOFS**

In this appendix, we show that
\[
\lim_{K \to \infty} \text{DoF}_{1}^{C}(K) = \frac{4}{6 \ln 2 - 1},
\] (156)
\[
\lim_{K \to \infty} \text{DoF}_{1}^{X}(2, K) = \frac{1}{\ln 2}.
\] (157)
Regarding Eqs. \[(10)\], \[(11)\] and \[(13)\], it suffices to show that
\[
\lim_{K \to \infty} \Psi(K) = \frac{21}{16} - \frac{3}{2} \ln 2.
\] (158)
\[
\lim_{K \to \infty} \Phi(K) = 1 - \ln 2.
\] (159)
where
\[
\Psi(K) \triangleq \sum_{\ell_1=0}^{K-3} \frac{(K - \ell_1 - 1)(3\ell_1^2 + \ell_1 - 1)}{2(K - \ell_1)(4\ell_1^2 - 1)} \prod_{\ell_2=\ell_1+1}^{K-2} \frac{\ell_2}{2\ell_2 + 1},
\] (160)
\[
\Phi(K) \triangleq \sum_{\ell_1=0}^{K-2} \frac{(K - \ell_1 - 1)(\ell_1 + 1)}{(K - \ell_1)(2\ell_1 + 1)} \prod_{\ell_2=\ell_1+1}^{K-1} \frac{\ell_2}{2\ell_2 + 1}.
\] (161)
To do so, for any integers \(K, p \geq 0\), define the functions \(\Gamma_p(K)\) and \(\Lambda_p(K)\) as
\[
\Gamma_p(K) \triangleq \sum_{\ell=0}^{K-p} \frac{K - \ell - 1}{(K - \ell)2^{K-\ell}},
\] (162)
\[
\Lambda_p(K) \triangleq \sum_{\ell=0}^{K-p} \frac{\ell(K - \ell - 1)}{K(K - \ell)2^{K-\ell}}.
\] (163)
Using $\sum_{n=1}^{\infty} \frac{1}{n^{2^n}} = \ln 2$, $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$, and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, it is easily verified that, for any integer $p \geq 0$,

$$\lim_{K \to \infty} \Gamma_p(K) = \lim_{K \to \infty} \Lambda_p(K) = -\ln 2 + 2^{1-p} + \sum_{n=1}^{p-1} \frac{1}{n2^n}. \quad (164)$$

In specific,

$$\lim_{K \to \infty} \Gamma_2(K) = \lim_{K \to \infty} \Lambda_2(K) = 1 - \ln 2, \quad (165)$$

$$\lim_{K \to \infty} \Gamma_3(K) = \lim_{K \to \infty} \Lambda_3(K) = \frac{7}{8} - \ln 2. \quad (166)$$

Now, using the following two lemmas together with the Squeeze Theorem, Eqs. (158) and (159) are immediate.

**Lemma 3:** The following inequalities hold for $K \geq 3$:

$$\frac{3K}{2K-3} \Lambda_3(K) < \Psi(K) < \frac{3}{2} \Gamma_3(K) + \frac{K-2}{5(K-1)2^K}. \quad (167)$$

**Proof:**

(i) **Upper bound:**

$$\Psi(K) = \sum_{\ell_1=0}^{K-3} \frac{(K - \ell_1 - 1)(3\ell_1^2 + \ell_1 - 1)}{2(K - \ell_1)(4\ell_1^2 - 1)} \prod_{\ell_2=\ell_1+1}^{K-2} \frac{\ell_2}{2\ell_2 + 1}$$

$$= \sum_{\ell_1=0}^{K-3} \frac{(K - \ell_1 - 1)(3\ell_1^2 + \ell_1 - 1)(\ell_1 + 1)}{2(K - \ell_1)(4\ell_1^2 - 1)(2\ell_1 + 3)} \prod_{\ell_2=\ell_1+2}^{K-2} \frac{\ell_2}{2\ell_2 + 1}$$

$$= \frac{K-2}{6K} \prod_{\ell_2=2}^{K-2} \frac{\ell_2}{2\ell_2 + 1} + \frac{K-2}{5(K-1)} \prod_{\ell_2=3}^{K-2} \frac{\ell_2}{2\ell_2 + 1}$$

$$+ \sum_{\ell_1=2}^{K-3} \frac{(K - \ell_1 - 1)(3\ell_1^2 + \ell_1 - 1)(\ell_1 + 1)}{2(K - \ell_1)(4\ell_1^2 - 1)(2\ell_1 + 3)} \prod_{\ell_2=\ell_1+2}^{K-2} \frac{\ell_2}{2\ell_2 + 1}$$

$$\leq (a) \frac{K-2}{5 \times 2^4(K-1)} \prod_{\ell_2=3}^{K-2} \frac{\ell_2}{2\ell_2 + 1} + \left\{ \frac{3(K-1)}{2^4K} \prod_{\ell_2=2}^{K-2} \frac{\ell_2}{2\ell_2 + 1} + \frac{3(K-2)}{2^4(K-1)} \prod_{\ell_2=3}^{K-2} \frac{\ell_2}{2\ell_2 + 1} \right\}$$

$$+ \sum_{\ell_1=2}^{K-3} \frac{3(K - \ell_1 - 1)}{2^4(K - \ell_1)} \prod_{\ell_2=\ell_1+2}^{K-2} \frac{\ell_2}{2\ell_2 + 1}$$

$$\leq (b) \frac{K-2}{5(K-1)2^K} + \frac{3}{2} \sum_{\ell_1=0}^{K-3} \frac{K - \ell_1 - 1}{(K - \ell_1)2^{K-\ell_1}}$$

$$= \frac{3}{2} \Gamma_3(K) + \frac{K-2}{5(K-1)2^K}. \quad (168)$$
where (a) follows from the fact that \( \frac{(3\ell_1^2 + \ell_1 - 1)(\ell_1 + 1)}{(4\ell_1^2 - 1) (2\ell_1 + 3)} < \frac{3}{8} \) for \( \ell_1 \geq 2 \) together with inequality \( \frac{1}{6} < \frac{3}{16} \), and (b) is valid since \( \frac{\ell_2}{2\ell_2 + 1} < \frac{1}{2} \) for \( \ell_2 \geq 2 \).

(ii) Lower bound:

\[
\Phi(K) = \sum_{\ell_1 = 0}^{K-3} \frac{(K - \ell_1 - 1)(\ell_1 + 1)}{(4\ell_1^2 - 1)} \frac{\ell_2}{2(\ell_2 - 1)} \prod_{\ell_2 = \ell_1 + 1}^{\ell_2 = \ell_1 + 2} \frac{\ell_2}{2\ell_2 + 1}
\]

\[
= \sum_{\ell_1 = 0}^{K-3} \frac{(K - \ell_1 - 1)(\ell_1 + 1)(\ell_1 + 1)(\ell_1 + 2)}{(2K - 3)(K - \ell_1)(4\ell_1^2 - 1)} \prod_{\ell_2 = \ell_1 + 2}^{\ell_2 = \ell_1 + 2} \frac{\ell_2}{2\ell_2 - 1}
\]

\[
= \frac{3K}{2K - 3} \sum_{\ell_1 = 0}^{K-3} \ell_1(K - \ell_1 - 1)K(K - \ell_1)2^{K - \ell_1}
\]

\[
= \frac{3K}{2K - 3} \Lambda_3(K),
\]

where (a) follows from the fact that \( \frac{(3\ell_1^2 + \ell_1 - 1)(\ell_1 + 1)}{4\ell_1^2 - 1} > \frac{3}{4} \) for \( \ell_1 \geq 0 \), and \( \frac{\ell_2}{2\ell_2 - 1} > \frac{1}{2} \) for \( \ell_2 \geq 2 \). \( \blacksquare \)

Lemma 4: The following inequalities hold for \( K \geq 2 \):

\[
\frac{2K}{2K - 1} \Lambda_2(K) < \Phi(K) < \Gamma_3(K) + \frac{(K - 1)^2}{(2K - 1)(2K - 3)} + \frac{K - 1}{15K2^K}.
\]

Proof:

(i) Upper bound:

\[
\Phi(K) = \sum_{\ell_1 = 0}^{K-2} \frac{(K - \ell_1 - 1)(\ell_1 + 1)}{(K - \ell_1)(2\ell_1 + 1)} \prod_{\ell_2 = \ell_1 + 1}^{\ell_2 = \ell_1 + 1} \frac{\ell_2}{2\ell_2 + 1}
\]

\[
= \frac{(K - 1)^2}{(2K - 1)(2K - 3)} + \sum_{\ell_1 = 0}^{K-3} \frac{(K - \ell_1 - 1)(\ell_1 + 1)(\ell_1 + 2)(\ell_1 + 3)(\ell_1 + 5)}{(2\ell_1 + 1)(2\ell_1 + 3)(2\ell_1 + 5)} \prod_{\ell_2 = \ell_1 + 3}^{\ell_2 = \ell_1 + 3} \frac{\ell_2}{2\ell_2 + 1}
\]

\[
\leq \frac{(K - 1)^2}{(2K - 1)(2K - 3)} + \frac{(K - 1)^2}{15K2^K} + \sum_{\ell_1 = 0}^{K-3} \frac{K - \ell_1 - 1}{2\ell_2 + 1}\prod_{\ell_2 = \ell_1 + 3}^{\ell_2 = \ell_1 + 3} \frac{\ell_2}{2\ell_2 + 1}
\]

\[
\leq \frac{(K - 1)^2}{(2K - 1)(2K - 3)} + \frac{(K - 1)^2}{15K2^K} + \sum_{\ell_1 = 0}^{K-3} \frac{K - \ell_1 - 1}{2\ell_2 + 1}\prod_{\ell_2 = \ell_1 + 3}^{\ell_2 = \ell_1 + 3} \frac{\ell_2}{2\ell_2 + 1}
\]

\[
= \Gamma_3(K) + \frac{(K - 1)^2}{(2K - 1)(2K - 3)} + \frac{K - 1}{15K2^K},
\]

where (a) follows from the fact that \( \frac{(\ell_1 + 1)^2(\ell_1 + 2)}{(2\ell_1 + 1)(2\ell_1 + 3)(2\ell_1 + 5)} < \frac{1}{8} \) for \( \ell_1 \geq 1 \), and (b) is true since \( \frac{\ell_2}{2\ell_2 + 1} < \frac{1}{2} \) for \( \ell_2 \geq 3 \).
(ii) Lower bound:

\[
\Phi(K) = \sum_{\ell_1=0}^{K-2} \frac{(K-\ell_1-1)(\ell_1+1)}{(K-\ell_1)(2\ell_1+1)} \prod_{\ell_2=\ell_1+1}^{K-1} \frac{\ell_2}{2\ell_2+1}
\]

\[
= \sum_{\ell_1=0}^{K-2} \frac{(K-\ell_1-1)(\ell_1+1)^2}{(2K-1)(K-\ell_1)(2\ell_1+1)} \prod_{\ell_2=\ell_1+2}^{K-1} \frac{\ell_2}{2\ell_2-1}
\]

\[
> \frac{2K}{2K-1} \sum_{\ell_1=0}^{K-2} \frac{\ell_1(K-\ell_1-1)}{K(K-\ell_1)2^{K-\ell_1}}
\]

\[
= \frac{2K}{2K-1} \Lambda_2(K),
\]

where (a) follows from the fact that \(\frac{(\ell_1+1)^2}{2\ell_1+1} > \frac{1}{2}\ell_1\) for \(\ell_1 \geq 0\), and \(\frac{\ell_2}{2\ell_2-1} > \frac{1}{2}\) for \(\ell_2 \geq 2\). 

\[
(172)
\]

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