Packing Three Cubes in D-Dimensional Space

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Abstract: Denote \( V_n(d) \) the least number that every system of \( n \) cubes with total volume 1 in \( d \)-dimensional (Euclidean) space can be packed parallelly into some rectangular parallelepiped of volume \( V_n(d) \). New results \( V_3(5) \approx 1.802803792 \), \( V_3(7) \approx 2.05909680 \), \( V_3(9) \approx 2.21897778 \), \( V_3(10) \approx 2.27220126 \), \( V_3(11) \approx 2.31533581 \), \( V_3(12) \approx 2.35315527 \), \( V_3(13) \approx 2.38661963 \) can be found in the paper.

Keywords: packing of cubes; extreme

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1. Introduction

In 1966, L. Moser [1] raised the following problem: Determine the smallest number \( A \) so that any system of squares of total area 1 can be packed parallelly into some rectangle of area \( A \). This problem can also be found in [2–6]. The problem has been extended to higher dimensions and has been studied for a specific number of squares (cubes). To distinguish so that any system of squares of total area 1 can be packed parallelly into some rectangular parallelepiped of volume \( V_n(d) \). \( V(d) \) denotes the maximum of all \( V_n(d) \), \( n = 1, 2, 3, \ldots \).

Some results are known in 2-dimensional space. Kleitman and Krieger [7] proved that every finite system of squares with total area 1, can be packed into the rectangle with sides of lengths \( \frac{2}{\sqrt{3}} \) and \( \sqrt{2} \), so \( V(2) \leq \frac{4}{\sqrt{6}} \approx 1.632993162 \). After twenty years Novotný [8] showed that \( V_3(2) = 1.2277589 \) and \( V(2) \geq \frac{2 + \sqrt{3}}{\sqrt[3]{2}} > 1.244 \). The exact results are known also for \( n = 4, 5, 6, 7, 8 \), Novotný [9] proved \( V_4(2) = V_5(2) = \frac{2 + \sqrt{3}}{\sqrt[3]{2}} \) and in Novotný [10] proved \( V_6(2) = V_7(2) = V_8(2) = \frac{2 + \sqrt{3}}{\sqrt[3]{2}} \). The estimate of \( V(2) \) was improved by Novotný [11] \( V(2) < 1.53 \). Later, this result was improved by Hurgady [12] \( V(2) \leq \frac{2607}{2000} < 1.4 \).

In 3-dimensional space, the estimate of \( V(3) \) was gradually improved. Meir and Moser [13] proved \( V(3) \leq 4 \) and later Novotný [14] proved \( V(3) \leq 2.26 \). The exact results are known for \( n = 2, 3, 4, 5 \): Novotný [15] \( V_2(3) = \frac{2}{3} \), \( V_3(3) = 1.44009951 \), Novotný [16] \( V_4(3) = 1.5196303266 \), and in Novotný [14] proved \( V_5(3) = V_4(3) \).

The results for \( n = 3 \) and \( d = 4, 6, 8 \) are known too: \( V_3(4) = 1.63369662 \), Bálint and Adamko in [17]; \( V_3(6) = 1.94449161 \), Bálint and Adamko in [18]; \( V_3(8) = 2.14930699 \), Sedliačková in [19].

Adamko and Bálint proved \( \lim_{d \to \infty} V_n(d) = n \) for \( n = 5, 6, 7, \ldots \) in [20]. Theorem holds also for \( n = 2, 3, 4 \).
2. Main Results

The main part of this section is the proof of $V_3(5) \doteq 1.802803792$. We use the same method as [17,18]. At the end of the section, we offer (without proof) the values of $V_3(d)$ for $d \in \{7,9,10,11,12,13\}$.

**Theorem 1.** $V_3(5) \doteq 1.802803792$.

Outline of the proof

1. We show that there are only two important packing configurations. Their volumes are $W_1 = x^4(x + y + z)$ and $W_2 = x^3(x + y)(y + z)$, see Figure 1. Firstly, we need to find $\min \{W_1, W_2\}$ for each $\{x, y, z\}$. The maximum from $\min \{W_1, W_2\}$ is the final result, we denote it $\max \min \{W_1, W_2\}$;
2. Cubes with sides $x \doteq 0.946629932$, $y \doteq 0.690148624$, and $z \doteq 0.608279275$ have $W_1 = W_2 = 1.802803792$. We prove that this volume is sufficient for packing any three cubes with a total volume of 1 in dimension 5;
3. We obtain an estimation of the side size of the largest cube: $0.9445 \leq x \leq 0.9939$;
4. Using $1 = x^5 + y^5 + z^5$ and $1 > x \geq y \geq z > 0$, we obtain constraints $x^5 + y^5 \leq 1$ and $x^5 + 2y^5 \geq 1$;
5. $z = (1 - x^5 - y^5)^{1/5}$. Therefore, it is sufficient to work only with $x$ and $y$. $M$ is a region of $\{x, y\}$ bounded by constraints from steps 3 and 4. We obtain a curve $C$ from $W_1 = W_2$, see Figure 2. Curve $C$ divides the region $M$ into continuous regions $C_1$ and $C_2$, see Figure 3;
6. We clarify:
   (a) $W_1(X) < W_2(X)$ holds for $X \in C_1$. Therefore, $\max \min \{W_1, W_2\} = \max W_1(X)$, $X \in C_1$;
   (b) $W_1(X) > W_2(X)$ holds for $X \in C_2$. Therefore, $\max \min \{W_1, W_2\} = \max W_2(X)$, $X \in C_2$;
7. We show that the asked maximum is on curve $C$;
   (a) We use critical points for region $C_1$;
   (b) We were unable to use critical points on the whole $C_2$, so we gradually numerically exclude subregions. We start with comparison of maximum of subregions and 1.8 (packing with $V_3(5) > 1.8$ exists).

**Proof.** Consider three cubes with edge lengths $x, y, z$ in the 5-dimensional Euclidean space, where $1 > x \geq y \geq z > 0$ and the total volume $x^5 + y^5 + z^5 = 1$.

We are looking for the smallest volume of a parallelepiped containing all three cubes. Therefore, from several ways of packing, we can ignore the packing that leads in any circumstances to a larger volume.

Let $X, Y, Z$ denote the cubes (sorted from the largest). We attach cubes $X$ and $Y$ to each other, for example, in the direction of the fifth dimension. Parallelepiped containing cubes $X$ and $Y$ has volume $x^4(x + y)$.

If we place the cube $Z$ to the cube $X$ in the direction of the fifth dimension, we receive volume $x^4(x + y + z)$. We obtain volume $x^3(x + y)(x + z)$ for other four directions.

If we place the cube $Z$ to the cube $Y$ in the direction of the fifth dimension, we receive again $x^4(x + y + z)$. We obtain (after appropriate shifting of the cube $Y$) volume $x^3(x + y)(y + z)$ for other four directions.

Because $x^3(x + y)(y + z) \leq x^3(x + y)(x + z)$, we can ignore packings that lead to the volume $x^3(x + y)(x + z)$.

If we start with cubes $X$ and $Z$, or $Z$ and $Y$, the same results are obtained.

Therefore, it is sufficient to consider only two cases of packing three cubes, see Figure 1a,b. In the first case, the volume $W_1 = x^4(x + y + z)$ is sufficient for packing, in the second case, the volume $W_2 = x^3(x + y)(y + z)$ is sufficient.
The function \( W = C \), \( 2, 2046 \)

\[
\frac{y}{x} \leq \frac{y}{x} = 0.596398035.
\]

Let us consider only the case that \( y + z > x \). From \( y^5 + y^5 = 1 - x^5 \), we find
\[
y \leq \sqrt[5]{1 - x^5}, \text{ and, therefore, } y + z \leq 2y \leq 2\sqrt[5]{1 - x^5}. \text{ Then, } x < y + z \leq 2\sqrt[5]{1 - x^5} \text{ and, therefore, } x^5 < 2\sqrt[5]{1 - x^5}.
\]

We attain the upper bound for \( x, x \leq \frac{2}{\sqrt[5]{3}} \). \( x \geq y \geq z \) and \( x^5 + y^5 + z^5 = 1 \), therefore, \( x^5 \geq \frac{1}{2} \) and \( x \geq \frac{1}{\sqrt[5]{3}} \). This implies that we can consider only \( x \in \left[ \frac{1}{\sqrt[5]{3}}, \frac{2}{\sqrt[5]{3}} \right] \), i.e., \( 0.8027 \leq x \leq 0.9939 \).

Equality \( W_1 = W_2 \) holds if, and only if, \( x^2 = y^2 + yz \). In this case, \( z = \frac{x^2 - y^2}{y} \) and \( W_1 = W_2 = x^5 + \frac{x^6}{y} \). When we substitute \( z = \frac{x^2 - y^2}{y} \) into \( x^5 + y^5 + z^5 = 1 \), we find the curve \( C: x^5y^5 + y^{10} - y^5 + (x^2 - y^2)^5 = 0 \) (Figure 2).

The interval for \( x \) can be reduced. If we choose \( x \in [a, b], 0 < a < b < 1 \), then \( 1 - b^5 \leq 1 - x^5 \leq 1 - a^5 \). If \( y = z \), then \( 1 - x^5 = y^5 + z^5 = 2y^5 \) and, therefore, \( y = \sqrt[5]{\frac{1 - x^5}{2}} \).

The function \( W_1 = x^4(x + y + z) \) has the greatest value if \( y = z \), i.e., \( y = \sqrt[5]{\frac{1 - x^5}{2}} \). For \( x \in [a, b] \), we find \( W_1 \leq x^4(x + 2y) \leq b^4 \left( b + 2\sqrt[5]{\frac{1 - a^5}{2}} \right) \).

Denote \( W_1(a, b) = b^4 \left( b + 2\sqrt[5]{\frac{1 - a^5}{2}} \right) \).

The inequality \( W_1(a, b) < 1.8021 \) is valid for the intervals: \( x \in [0.8027, 0.9190], x \in [0.9190, 0.9360], x \in [0.9360, 0.9410], x \in [0.9410, 0.9420], x \in [0.9420, 0.9430], x \in [0.9430, 0.9440], x \in [0.9440, 0.9445] \), hence for the asked maximum holds \( x \geq 0.9445 \).

Therefore, we have shown that the asked \( \text{max min} \{ W_1, W_2 \} \) will be attained for \( x \in [0.9445, 0.9939] \).

From the assumption \( 0 < z \leq y \leq x < 1 \) follows that \( x^5 + y^5 \leq x^5 + y^5 + z^5 = 1 \) and also \( 1 = x^5 + y^5 + z^5 \leq x^5 + 2y^5 \).
Consider the closed region $M$ determined by inequalities $0.9445 \leq x \leq 0.9939$, $x^5 + y^5 \leq 1$, $x^5 + 2y^5 \geq 1$. The curve $C$ divides the region $M$ into two open regions $C_1$, $C_2$, (Figure 3).

![Figure 3. Regions $C_1$, $C_2$.](image)

We are looking for $\max \min \{W_1, W_2\}$, when $W_1 = x^4(y + z)$, $W_2 = x^3(y + z).$ From the condition $x^5 + y^5 + z^5 = 1$ we find

$$W_1 = W_1(x, y) = x^4(y + z) + \sqrt[5]{1 - x^5 - y^5},$$

(1)

$$W_2 = W_2(x, y) = x^3(y + z)(y + z) + \sqrt[5]{1 - x^5 - y^5}.$$  (2)

Let $C_1$ denote the closure of the set $C$. The functions $W_1$, $W_2$ are continuous on $M$ and the equality $W_1 = W_2$ holds just in the points of the curve $C$.

Take the point $A_1 = (0.945, 0.70) \in C_1$. The inequality $W_1(X) < W_2(X)$ holds in every point $X \in C_1$, because of $W_1(A_1) < W_2(A_1)$. Therefore, for the asked maximum holds $\max \min \{W_1(X), W_2(X)\} = \max \{W_1(X)\}.$

Take the point $A_2 = (0.965, 0.65) \in C_2$. The inequality $W_1(X) > W_2(X)$ holds in every point $X \in C_2$, because of $W_1(A_2) > W_2(A_2)$. Therefore, for the asked maximum holds $\max \min \{W_1(X), W_2(X)\} = \max \{W_2(X)\}.$

On the compact set $C_1$ the function (1) has its maximum in some point $B$.

It holds $\frac{\partial W_1}{\partial y} = x^4 \left(1 - \frac{y^4}{\sqrt[5]{1 - x^5 - y^5}}\right).$ The equality $\frac{\partial W_1}{\partial y} = 0$ holds if $x^5 + 2y^5 - 1 = 0$ but the points of the curve $x^5 + 2y^5 - 1 = 0$ do not belong to the region $C_1$. For every point $X \in C_1$ holds $\frac{\partial W_1}{\partial y} < 0$. Therefore, the point $B$ must lie on the curve $C$.

For every point $X = (x, y), x \in [a, b], y \in [c, d]$ the inequality $z \leq \sqrt[5]{1 - a^5 - c^5}$ holds, and so $W_1 = x^4(y + z) \leq b^4(b + d + \sqrt[5]{1 - a^5 - c^5})$, $W_2 = x^3(y + z) \leq b^3(b + d)(d + \sqrt[5]{1 - a^5 - c^5})$.

Denote

$$W_{11}(a, b, c, d) = b^4(b + d + \sqrt[5]{1 - a^5 - c^5}),$$

$$W_{22}(a, b, c, d) = b^3(b + d)(d + \sqrt[5]{1 - a^5 - c^5}).$$

Examine the region $C_2$.

For $x \in [0.9900, 0.9939], y \in [0.43, 0.60]$ is $W_{22} < 1.8$. For $x \in [0.9850, 0.9900], y \in [0.47, 0.60]$ is $W_{22} < 1.8$. For $x \in [0.9800, 0.9850]$ and, step by step, for $y \in [0.51, 0.56], [0.56, 0.60], [0.60, 0.65]$ is always $W_{22} < 1.8$. 
For $x \in [0.975, 0.980]$ and, step by step, for $y \in [0.54, 0.60], [0.60, 0.64], [0.64, 0.7]$ is always $W_{22} < 1.8$.

For $x \in [0.970, 0.975]$ and, step by step, for $y \in [0.56, 0.61], [0.61, 0.63], [0.63, 0.65], [0.65, 0.68]$ is always $W_{22} < 1.8$.

For $x \in [0.965, 0.970]$ and, step by step, for $y \in [0.58, 0.61], [0.61, 0.63], [0.63, 0.64], [0.64, 0.65], [0.65, 0.66], [0.66, 0.67], [0.67, 0.69], [0.69, 0.75]$ is always $W_{22} < 1.8$.

For $x \in [0.960, 0.965]$ and, step by step, for $y \in [0.60, 0.62], [0.62, 0.63], [0.63, 0.65], [0.65, 0.64], [0.64, 0.644], [0.644, 0.647], [0.647, 0.65], [0.65, 0.652], [0.652, 0.654], [0.654, 0.656], [0.656, 0.658], [0.658, 0.66], [0.66, 0.662], [0.662, 0.664], [0.664, 0.666], [0.666, 0.668], [0.668, 0.67], [0.67, 0.673], [0.673, 0.677], [0.677, 0.68], [0.68, 0.685], [0.685, 0.695], [0.695, 0.72]$ is always $W_{22} < 1.8$.

For $x \in [0.955, 0.960]$ and, step by step, for $y \in [0.62, 0.63], [0.63, 0.635], [0.635, 0.638], [0.638, 0.64], [0.64, 0.641], [0.641, 0.642], [0.697, 0.698], [0.698, 0.7], [0.7, 0.703], [0.703, 0.709], [0.709, 0.724], [0.724, 0.73]$, is always $W_{22} < 1.8$.

We do not exclude the region $x \in [0.955, 0.960], y \in [0.642, 0.697]$ in this way, it is not effective.

We have

From (2):

$$\frac{\partial W_2}{\partial x} = \sqrt{\frac{x^2}{(1-x^2-y^2)^3}} \left[ (4x + 3y)(y\sqrt{(1-x^2-y^2)^4 + 1} - x^5) - 5x^6 - 4x^5y \right]$$

and

$$\frac{\partial W_2}{\partial y} = \sqrt{\frac{x^3}{(1-x^2-y^2)^3}} \left[ (x + 2y)(\sqrt{(1-x^2-y^2)^4 + 1} - x - 2y - xy^4) \right].$$

If $\frac{\partial W_2}{\partial x} > 0$ and $\frac{\partial W_2}{\partial y} > 0$, therefore, for every point $X = (x, y), x \in [a, b], y \in [c, d]$ we have two inequalities:

$$(4x + 3y)(y\sqrt{(1-x^2-y^2)^4 + 1} - x^5) - x^5(5x + 4y) \leq$$

$$\leq (4b + 3d)(d\sqrt{(1-a^2-c^5)^4 + 1} - a^5) - a^5(5a + 4c)$$

and

$$(x + 2y)(\sqrt{(1-x^2-y^2)^4 + 1} - x^5 - 2y - xy^4) \geq$$

$$\geq (a + 2c)(\sqrt{(1-b^5-d^5)^4 + 1} - b^5 - 2d^5 - bd^4).$$

Denote

$$DW_{2x}(a, b, c, d) = (4b + 3d)(d\sqrt{(1-a^2-c^5)^4 + 1} - a^5) - a^5(5a + 4c),$$

$$DW_{2y}(a, b, c, d) = (a + 2c)(\sqrt{(1-b^5-d^5)^4 + 1} - b^5 - 2d^5 - bd^4).$$

For $x \in [0.955, 0.960]$ and $y \in [0.642, 0.670]$ is $DW_{2x}(a, b, c, d) < 0$ and, therefore, $\frac{\partial W_2}{\partial x} < 0$.

For $x \in [0.955, 0.960]$ and $y \in [0.670, 0.697]$ is also $DW_{2x}(a, b, c, d) < 0$ and, therefore, $\frac{\partial W_2}{\partial x} < 0$.

Therefore, the asked maximum cannot be achieved for $x \in [0.955, 0.960]$.

For $x \in [0.950, 0.955]$ and, step by step, for $y \in [0.63, 0.636], [0.636, 0.639], [0.639, 0.64], [0.64, 0.641], [0.717, 0.718], [0.718, 0.72], [0.72, 0.725], [0.725, 0.73], [0.73, 0.75]$ is always $W_{22} < 1.8$.

We do not exclude the region $x \in [0.950, 0.955], y \in [0.641, 0.717]$ in this way, it is not effective.

For $x \in [0.950, 0.955]$ and $y \in [0.641, 0.671]$ is $DW_{2y}(a, b, c, d) > 0$ and, therefore, $\frac{\partial W_2}{\partial y} > 0$. 
For \( x \in [0.950, 0.955] \) and \( y \in [0.671, 0.700] \) is \( \frac{\partial W_2}{\partial x} < 0 \) and, therefore,

\[
\frac{\partial W_2}{\partial x} < 0.
\]

This implies that the asked maximum cannot be achieved for \( x \in [0.950, 0.955] \).

For \( x \in [0.9475, 0.9500] \) and, step by step, for \( y \in [0.640, 0.649], [0.649, 0.653], [0.653, 0.655], [0.655, 0.656], [0.656, 0.657], [0.719, 0.720], [0.720, 0.722], [0.722, 0.726], [0.726, 0.735], [0.735, 0.750] \) is always \( W_{22} < 1.8 \).

We do not exclude the region \( x \in [0.9475, 0.9500], y \in [0.657, 0.719] \) in this way, it is not effective.

For \( x \in [0.9475, 0.9500] \) and \( y \in [0.657, 0.684] \) is \( \frac{\partial W_2}{\partial y} > 0 \).

For \( x \in [0.9475, 0.9500] \) and \( y \in [0.684, 0.719] \) is \( \frac{\partial W_2}{\partial x} < 0 \) and, therefore,

\[
\frac{\partial W_2}{\partial x} < 0.
\]

This implies that the asked maximum cannot be achieved for \( x \in [0.9475, 0.9500] \), see Figure 4.

![Figure 4](image-url)

**Figure 4.** The Region \( M \) after the final reduction.

For \( x \in [0.9445, 0.9475] \) and, step by step, for \( y \in [0.650, 0.653], [0.653, 0.655], [0.655, 0.656], [0.656, 0.657] \) is always \( W_{22} < 1.8 \).

For \( x \in [0.9445, 0.9475] \) and \( y \in [0.657, 0.690] \) is \( \frac{\partial W_2}{\partial y} > 0 \).

For \( x \in [0.9445, 0.9475] \) and \( y \in [0.690, 0.700] \) is \( \frac{\partial W_2}{\partial x} < 0 \) and, therefore,

\[
\frac{\partial W_2}{\partial x} < 0.
\]

For \( x \in [0.9445, 0.9475] \) and, step by step, for \( y \in [0.720, 0.726], [0.726, 0.743], [0.743, 0.760] \) is always \( W_{11} < 1.8 \).

So function (2) on the compact set \( C_2 \) must achieve its maximum in some point of the curve \( C \). It is the same point \( B \) as above.

We ask constrained maximum of the function

\[
W(x, y) = x^5 + \frac{x^6}{y} \tag{3}
\]

on the curve \( C \)

\[
C(x, y) = x^5 y^5 + y^{10} - y^5 + (x^2 - y^2)^5 \tag{4}
\]

for \( x \in [0.9445, 0.9475] \).
System of equations \( \frac{\partial W}{\partial x} \frac{\partial C}{\partial y} - \frac{\partial W}{\partial y} \frac{\partial C}{\partial x} = 0 \) and \( C(x, y) = 0 \) has the form

\[
7x^6y^5 + 12xy^{10} - 6xy^5 + 5x^5y^6 + 10y^{11} - 5y^6 + (x^2 - y^2)^4(2x^3 - 10y^3 - 12xy^2) = 0,
\]
\[
x^5y^6 + y^{10} - y^5 + (x^2 - y^2)^5 = 0.
\]

The solution is \( x = 0.946629932, y = 0.690148624, \) and then \( z = 0.608279275. \)

If we generalize considerations from the proof, we will achieve the curve \( C: x^d y^d + y^{2d} - y^d + (x^2 - y^2)^d = 0, \) where \( d \) is dimension. The graph of the curve \( C \) depends on the parity of \( d \), see Figures 5 and 6. Considering only the values \( 1 > x \geq y > 0 \), the shape of the curve \( C \) is similar, regardless of parity, see Figure 2.

For \( d \leq 10 \) the asked maximum is achieved on the curve \( C \). For dimensions 7, 9 and 10 the results are:

\[
V_3(7) = 2.05909680 \quad \text{and} \quad x = 0.978852925, \quad y = 0.703495386, \quad z = 0.658493716,
\]
\[
V_3(9) = 2.21897778 \quad \text{and} \quad x = 0.991008397, \quad y = 0.704394561, \quad z = 0.689849087,
\]
\[
V_3(10) = 2.27220126 \quad \text{and} \quad x = 0.993961280, \quad y = 0.702901846, \quad z = 0.702641521.
\]

Let \( P \) is intersection the constraint curve \( x^d + 2y^d - 1 = 0 \) and the curve \( C \). If \( d = 11 \), then the constrained extreme on the curve \( C \) does not meet the required assumption \( y \geq z \). Therefore, the asked maximum must be on the constraint curve to the left of point \( P \) or on the curve \( C \) above \( P \), see Figure 7. The same situation occurs for \( d = 12 \) and \( d = 13 \).

\[
V_3(11) = 2.31533581 \quad \text{and} \quad x = 0.994989464, \quad y = z = 0.719809616,
\]
\[
V_3(12) = 2.35315527 \quad \text{and} \quad x = 0.995762712, \quad y = z = 0.734956999,
\]
\[
V_3(13) = 2.38661963 \quad \text{and} \quad x = 0.996369617, \quad y = z = 0.748358875.
\]
3. Conclusions

The issue of packing squares is an old problem and even though there are multiple partial results, it remains unresolved. We investigated a modified problem: packing three cubes in 5-dimensional space. We also calculated results for dimensions 7, 9, 10, 11, 12, 13.

Considering the previous results by [17–19], we can say that solution is located on the curve $C$ for dimensions 4 . . . 10. It means, that there are two (different) packings that give (the same) the largest volume.

There seems to be only a single maximal packing for dimensions greater than 10. In this packing, two smallest cubes are the same. However, the paper confirms it only for dimensions 11, 12, 13.

There is a space for several improvements in our work: Is it possible to find a $V_3(d)$ without long numerical calculations? Is it true that two different maximum packings exist only for dimensions less than 11?

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