The unique ergodicity of equicontinuous laminations

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Abstract. We prove that a transversely equicontinuous minimal lamination on a locally compact metric space $Z$ has a transversely invariant Radon measure. Moreover if the space $Z$ is compact, then the transversely invariant Radon measure is shown to be unique up to a scaling.

1. Introduction

Let $Z$ be a locally compact metric space, $\mathcal{L}$ a $p$-dimensional lamination on $Z$. We assume throughout that $\mathcal{L}$ is minimal. Let $h : \mathbb{R}^p \times X \to Z$ be a lamination chart, i.e., a homeomorphism onto an open subset $h(\mathbb{R}^p \times X)$ such that the plaque $h(\mathbb{R}^p \times \{x\})$ lies on a leaf of $\mathcal{L}$ for any $x \in X$. We identify $X$ with the image $h(\{0\} \times X)$ and call it a cross section of $\mathcal{L}$. With the metric induced from $Z$, $X$ is also locally compact. Notice that any leaf of $\mathcal{L}$ intersects $X$.

Given a leafwise curve joining two points $x$ and $y$ on $X$, a holonomy map along $c$ is defined as usual to be a local homeomorphism $\gamma$ from an open neighbourhood $\text{Dom}(\gamma)$ of $x$ onto an open neighbourhood $\text{Range}(\gamma)$ of $y$. We say that $\mathcal{L}$ is transversely equicontinuous w.r.t. a cross section $X$ if the family of all the corresponding holonomy maps is equicontinuous.

Theorem 1.1. Let $\mathcal{L}$ be a minimal lamination on a locally compact metric space $Z$, transversely equicontinuous w.r.t. a cross section $X$. Then there is a Radon measure on $X$ which is left invariant by any holonomy map. If further $Z$ is compact, then the invariant measure is unique up to a scaling.

The existence of invariant measure was already shown by R. Sacksteder in [S] for a pseudogroup acting on a compact metric space. But the compactness condition for a cross section is too strong to obtain a corresponding result for laminations or foliations (even on compact spaces or manifolds). In section 2, we include a slightly general theorem applicable to laminations; the proof closely follows an argument in Lemme 4.4 in [DKN], which is meant for codimension one foliations.

In section 3 we show the uniqueness for a compact lamination. The argument here which is adapted for pseudo*groups as defined in section 2 is rather messy, but the original idea is quite simple, which the reader can find in section 4.

1991 Mathematics Subject Classification. Primary 53C12, secondary 37C85.

Key words and phrases. lamination, foliation, transversely invariant measure, unique ergodicity.

The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540096.
In section 4, we deal with an equicontinuous group action on a compact metric space, together with a random walk on a group. We show that the corresponding harmonic probability measure on the space is unique.

The uniqueness of harmonic measures for tangentially sufficiently smooth foliations and laminations \((\mathcal{C},\mathcal{G})\) remains an open question.

### 2. The existence

Let \(Y\) be a Hausdorff space. By a local homeomorphism, we mean a homeomorphism \(\gamma\) from an open subset \(\text{Dom}(\gamma)\) of \(Y\) onto an open subset \(\text{Range}(\gamma)\).

A set \(\Gamma\) of local homeomorphisms of \(Y\) is called a pseudo\(^*\)group, if it satisfies the following conditions.

1. If \(\gamma \in \Gamma\) and \(U\) is an open subset of \(\text{Dom}(\gamma)\), then the restriction \(\gamma|_U\) is in \(\Gamma\).
2. The identity \(\text{id}_X\) belongs to \(\Gamma\).
3. If \(\gamma, \gamma' \in \Gamma\) and \(\text{Dom}(\gamma') = \text{Range}(\gamma)\), then the composite \(\gamma' \circ \gamma\) is in \(\Gamma\).
4. If \(\gamma \in \Gamma\), then \(\gamma^{-1} \in \Gamma\).

This differs from the usual definition of pseudogroups in that it does not assume the axiom for taking the union. Thus for example the set of all the holonomy maps w. r. t. a cross section given in section 1 forms a pseudo\(^*\)group, while the pseudogroup they generate might be bigger. There are two reasons for introducing the concepts of pseudo\(^*\)groups: one is that in Theorem 1.1, assuming the equicontinuity for the pseudogroup generated by the holonomy maps may be stronger than what we have tacitly in mind: the other is that some part of the argument in section 3 cannot be put into the framework of the usual pseudogroups.

Let \(X\) be a locally compact metric space and \(\Gamma\) a pseudo\(^*\)group of local homeomorphisms of \(X\). We assume that the action is minimal, i.e., the \(\Gamma\)-orbit of any point is dense in \(X\), and that the action is equicontinuous, i.e., for any \(\epsilon > 0\), there is \(\delta(\epsilon) > 0\) such that if \(\gamma \in \Gamma\), \(x, x' \in \text{Dom}(\gamma)\) and \(d(x, x') < \delta(\epsilon)\), then we have \(d(\gamma x, \gamma x') \leq \epsilon\).

Denote by \(C_c(X)\) the space of real valued continuous functions \(\zeta\) whose support \(\text{supp}\zeta\) is compact. A Radon measure \(\mu\) on \(X\) is called \(\Gamma\)-invariant if whenever \(\zeta \in C_c(X)\) and \(\gamma \in \Gamma\) satisfy \(\text{supp}\zeta \subset \text{Dom}(\gamma)\), we have \(\mu(\zeta \circ \gamma^{-1}) = \mu(\zeta)\). In fact if \(\mu\) is \(\Gamma\)-invariant, we get a bit more, e.g. for any bounded continuous function \(\zeta : X \to \mathbb{R}\) which vanishes outside \(\text{Dom}(\gamma)\), we have \(\mu(\zeta \circ \gamma^{-1}) = \mu(\zeta)\), as the dominated convergence theorem shows. In this case the both hand sides might be \(\infty\). This will be used in section 3.

Let \(X_0\) be a relatively compact open subset of \(X\), and denote by \(\Gamma_0\) the restriction of \(\Gamma\) to \(X_0\) i.e.,

\[
\Gamma_0 = \{ \gamma \in \Gamma \mid \text{Dom}(\gamma) \cup \text{Range}(\gamma) \subset X_0 \}.
\]

The purpose of this section is to show the following theorem.

**Theorem 2.1.** There exists a finite \(\Gamma_0\)-invariant Radon measure \(\mu\) on \(X_0\).

The minimality assumption shows then the existence of \(\Gamma\)-invariant measure on \(X\) and the proof of the existence part of Theorem [1.1] will be complete.

Let us define

\[
C_c(X)_{\geq 0} = \{ \zeta \in C_c(X) \mid \zeta \geq 0 \} \quad \text{and} \quad C_c(X)_{> 0} = \{ \zeta \in C_c(X)_{\geq 0} \mid \zeta(x) > 0, \text{ } \exists x \in X \}.
\]
For any $\psi \in C_c(X)$ and $\gamma \in \Gamma$, extend the function $\psi \circ \gamma^{-1}$ to the whole $X$ so as to vanish outside $\text{Range}(\gamma)$ and still denote it by $\psi \circ \gamma^{-1}$. It may no longer be continuous. For any $\zeta \in C_c(X)_{\geq 0}$ and $\psi \in C_c(X)_{>0}$, define $(\zeta : \psi)$ by

$$(\zeta : \psi) = \inf \left\{ \sum_{i=1}^{n} c_i \mid \zeta \leq \sum_{i=1}^{n} c_i \psi \circ \gamma_i^{-1}, \ c_i > 0, \ \gamma_i \in \Gamma, \ n \in \mathbb{N} \right\}.$$ 

Notice that the minimality of $\Gamma$ implies that $(\zeta : \psi) < \infty$ and $(\zeta : \psi) = 0$ if and only if $\zeta = 0$.

Fix once and for all a function $\chi \in C_c(X)_{>0}$ such that $\chi = 1$ on $X_0$, and define a map $L_\psi : C_c(X)_{\geq 0} \to \mathbb{R}$ by

$$L_\psi(\zeta) = (\zeta : \psi)/\chi(\psi).$$

It is routine to show the following properties of $L_\psi$.

\begin{align*}
(2.1) \quad L_\psi(c\zeta) &= cL_\psi(\zeta), \ \forall c \geq 0, \\
(2.2) \quad L_\psi(\zeta_1 + \zeta_2) &\leq L_\psi(\zeta_1) + L_\psi(\zeta_2), \\
(2.3) \quad \zeta_1 \leq \zeta_2 \Rightarrow L_\psi(\zeta_1) \leq L_\psi(\zeta_2), \\
(2.4) \quad \text{supp}\zeta \subset \text{Dom}(\gamma) \Rightarrow L_\psi(\zeta \circ \gamma^{-1}) = L_\psi(\zeta), \\
(2.5) \quad L_\psi(\zeta) \geq 1/(\chi : \zeta).
\end{align*}

**Lemma 2.2.** If $\eta > 0$ and $\xi, \xi' \in C_c(X)_{\geq 0}$ satisfies $\xi + \xi' = \chi$, then there is $\delta > 0$ such that if $\psi \in C_c(X)_{>0}$, $\text{diam}(\text{supp}\psi) < \delta$ and $\zeta \in C_c(X_0)_{\geq 0}$ we have

$$L_\psi(\xi) + L_\psi(\xi') \leq (1 + 2\eta)L_\psi(\zeta).$$

**Proof.** Given $\eta > 0$, there is $\epsilon > 0$ such that if $x, x' \in X_0$ and $d(x, x') \leq \epsilon$, then $|\xi(x) - \xi(x')| \leq \eta$. Also this implies $|\xi'(x) - \xi'(x')| \leq \eta$. Choose $\delta = \delta(\epsilon)$. Let $\psi$ be as in the lemma and assume

$$(\zeta : \psi) \leq \sum_{i} c_i \psi \circ \gamma_i^{-1}.$$ 

Notice that if we restrict $\gamma_i$ in $\text{Dom}(\gamma_i) \cap \text{supp}\psi \cap \gamma_i^{-1}(\text{supp}\zeta)$, still the inequality (2.6) holds. Hence if we choose $x_i$ from $\text{Range}(\gamma_i) \subset \text{supp}(\zeta) \subset X_0$, then for any $x \in \text{Range}(\gamma_i)$, we have

$$|\xi(x) - \xi(x_i)| \leq \eta \quad \text{and} \quad |\xi'(x) - \xi'(x')| \leq \eta.$$ 

Moreover the following inequality

$$\xi(x) \psi \circ \gamma_i^{-1}(x) \leq (\xi(x_i) + \eta) \psi \circ \gamma_i^{-1}(x)$$

holds for any $x \in X$, since if $x \not\in \text{Range}(\gamma_i)$ the both hand sides are 0. Then we have

$$\zeta(x) \xi(x) \leq \sum_{i} c_i \xi(x_i) \psi \circ \gamma_i^{-1}(x)$$

$$\leq \sum_{i} c_i (\xi(x_i) + \eta) \psi \circ \gamma_i^{-1}(x).$$

This shows

$$(\zeta : \psi) \leq \sum_{i} c_i (\xi(x_i) + \eta).$$
We have a similar inequality for $\xi'$. Since $x_i \in X_0$ and thus $\xi(x_i) + \xi'(x_i) = 1$, we have
\[
(\xi : \psi) + (\xi' : \psi) \leq (2\eta + 1) \sum_i c_i.
\]
The lemma follows from this.

Continuing the proof of Theorem 1.1, let us extend the operator $L_\psi : C_c(X_0)_{\geq 0} \to \mathbb{R}$ to $C_c(X_0)$ by just putting
\[
L_\psi(\zeta) = L_\psi(\zeta_+) - L_\psi(\zeta_-),
\]
where $\zeta_+ \text{ (resp. } \zeta_-)$ is the positive (resp. negative) part of $\zeta$.

Then we have:
\[
|L_\psi(\zeta)| \leq \|\zeta\|_{\infty}, \quad \forall \zeta \in C_c(X_0)_{\geq 0}.
\]
In fact if $\zeta \geq 0$, then $\zeta \leq \|\zeta\|_{\infty} \chi$, and thus $L_\psi(\zeta) \leq \|\zeta\|_{\infty}$, the general case following easily from this.

Let us identify $L_\psi$ with the following point of a compact Hausdorff space:
\[
L_\psi = \{L_\psi(\zeta)\}_{\zeta} \in \prod_{\zeta \in C_c(X_0)} [-\|\zeta\|_{\infty}, \|\zeta\|_{\infty}].
\]

Let $\{\psi_n\}$ be a sequence in $C_c(X_0)_{\geq 0}$ such that $\text{diam}(\text{supp} \psi_n) \to 0$. Choose an operator $L \in \bigcap_m \text{Cl}\{L_{\psi_n} \mid n \geq m\}$. This means that for any finite number of elements $\psi_n \in C_c(X_0)$ and any $\epsilon > 0$, there is a sequence $n_i \to \infty$ such that $|L(\psi_n) - L_{\psi_n}(\zeta_0)| < \epsilon$. Now we have the following properties of the map $L : C_c(X_0) \to \mathbb{R}$.

\begin{align*}
(2.8) \quad &L(c\zeta) = cL(\zeta), \quad \forall c \in \mathbb{R}, \\
(2.9) \quad &L(\zeta_1 + \zeta_2) \leq L(\zeta_1) + L(\zeta_2), \quad \forall \zeta_1, \zeta_2 \geq 0, \\
(2.10) \quad &\zeta_1 \leq \zeta_2 \Rightarrow L(\zeta_1) \leq L(\zeta_2), \\
(2.11) \quad &\text{supp} \zeta \subset \text{Dom}(\gamma), \quad \gamma \in \Gamma \Rightarrow L(\zeta \circ \gamma^{-1}) = L(\zeta), \\
(2.12) \quad &\zeta \in C_c(X_0)_{\geq 0} \Rightarrow L(\zeta) \geq 1/(\chi : \zeta), \\
(2.13) \quad &|L_\psi(\zeta)| \leq \|\zeta\|_{\infty}.
\end{align*}

Moreover by Lemma 2.2 and 2.3, we have

**Lemma 2.3.** If $\zeta \in C_c(X_0)_{\geq 0}$ and $\xi, \xi' \in C_c(X_0)_{\geq 0}$ satisfy $\xi + \xi' = \chi$, then
\[
L(\xi\zeta) + L(\xi'\zeta) = L(\zeta).
\]

From this one can derive the linearity of $L$. First of all notice that
\[
(2.14) \quad \zeta, \zeta' \in C_c(X_0)_{\geq 0} \Rightarrow |L(\zeta) - L(\zeta')| \leq \|\zeta - \zeta'\|_{\infty}.
\]

In fact we have
\[
L(\zeta') = L(\zeta + \zeta' - \zeta) \leq L(\zeta + (\zeta' - \zeta)_+) \leq L(\zeta) + L((\zeta' - \zeta)_+) \\
\leq L(\zeta) + \|(\zeta' - \zeta)_+\|_{\infty} \leq L(\zeta) + \|\zeta' - \zeta\|_{\infty}.
\]

Continuing the proof of the linearity, notice that it suffices to show it only for those functions $\zeta_1, \zeta_2 \in C_c(X_0)_{\geq 0}$. Choose $\epsilon > 0$ small and let
\[
\xi_j = (\zeta_j + \epsilon \chi)/(\zeta_1 + \zeta_2 + 2\epsilon)
\]
for \( j = 1, 2 \). Then we have \( \xi_1 + \xi_2 = \chi \). Now

\[
\xi_1(\zeta_1 + \zeta_2) - \zeta_1 = \epsilon(\zeta_2 - \zeta_1)/(\zeta_1 + \zeta_2 + \epsilon).
\]

Therefore by (2.14), we have

\[
|L(\xi_1(\zeta_1 + \zeta_2)) - L(\zeta_1)| \leq \epsilon.
\]

On the other hand by Lemma 2.3,

\[
L(\xi_1(\zeta_1 + \zeta_2)) + L(\xi_2(\zeta_1 + \zeta_2)) = L(\zeta_1 + \zeta_2).
\]

Since \( \epsilon \) is arbitrary, we have obtained

\[
L(\zeta_1) + L(\zeta_2) = L(\zeta_1) + L(\zeta_2),
\]

as is required.

Now \( L \), being a positive operator, corresponds to a Radon measure \( \mu \). By (2.12), the measure \( \mu \) is nontrivial, and since (2.13) implies

\[
\inf \{L(\zeta) \mid \zeta \in C_c(X_0), \|\zeta\|_\infty \leq 1\} \leq 1,
\]

the measure \( \mu \) satisfies \( \mu(X_0) \leq 1 \). Finally (2.11) means the \( \Gamma_0 \)-invariance of \( \mu \).

3. The uniqueness

In this section \( \Gamma \) is again an equicontinuous and minimal pseudo*group of local homeomorphisms of a locally compact metric space \( X \). The modulus of equicontinuity is also denoted by \( \epsilon \rightarrow \delta(\epsilon) \). Denote by \( B_r(x) \) the open \( r \)-ball in \( X \) centered at \( x \in X \).

We make the following additional assumption on the pseudo*group \( \Gamma \).

Assumption 3.1. There is a relatively compact open subset \( X_0 \) of \( X \) and \( a > 0 \) such that if \( \gamma \in \Gamma \), \( x \in X_0 \), \( x \in \text{Dom}(\gamma) \subset B_a(x) \) and \( \gamma x \in X_0 \), then there is \( \hat{\gamma} \in \Gamma \) such that \( \text{Dom}(\hat{\gamma}) = B_a(x) \) and \( \hat{\gamma}|_{\text{Dom}(\gamma)} = \gamma \).

The purpose of this section is to show the following theorem.

Theorem 3.2. Let \( \Gamma \) be an equicontinuous and minimal pseudo*group on \( X \) satisfying Assumption 3.1. Then the \( \Gamma \)-invariant Radon measure on \( X \) is unique up to a scaling.

First of all let us show that the holonomy pseudo*group \( \Gamma \) on a cross section \( X \) of a minimal lamination on a compact space \( Z \), equicontinuous w. r. t. \( X \) satisfies Assumption 3.1. Choose any relatively compact open subset \( X_0 \) of \( X \).

On one hand by compactness of \( Z \) there is \( L > 0 \) such that the germ of any element of the restriction \( \Gamma_0 \) to \( X_0 \) is a finite composite of the holonomy maps along leaf curves of length \( \leq L \) that join two points in \( X_0 \). On the other hand there is \( a' > 0 \) such that each leaf curve of length \( \leq L \) starting at \( x \in X_0 \) and ending at a point in \( X_0 \) admits a holonomy map defined on the ball \( B_{a'}(x) \). An easy induction shows that Assumption 3.1 is satisfied for \( a = \delta(a') \).

Let us embark upon the proof of Theorem 3.2. Choosing \( a \) even smaller, one may assume that there is a nonempty open subset \( X_1 \) of \( X_0 \) such that the \( a \)-neighbourhood \( B_a(x) \) of any point \( x \) of \( X_1 \) is contained in \( X_0 \) and that if \( \gamma \in \Gamma \) and \( x' \in X_0 \) satisfies \( \text{Dom}(\gamma) = B_a(x') \) and \( \gamma x' \in X_1 \), then the image \( \text{Range}(\gamma) = \)
\[ \gamma(B_a(x')) \] is contained in \( X_0 \). Choose \( b > 0 \) so that \( b \leq \delta(a/3) \), and assume there is \( x_0 \in X_1 \) such that \( C = \text{Cl}(B) \subset X_1 \), where \( B = B_b(x_0) \).

Let \( M \) be the space of continuous maps from \( C \) to \( X_0 \), with the supremum distance \( d_\infty \). Define
\[
\Gamma_C = \{ \gamma \mid C \subset \text{Dom}(\gamma), \gamma C \subset X_0 \}
\]
and let \( G \) be the closure of \( \Gamma_C \) in \( M \).

**Lemma 3.3.** (1) \( G \) is a locally compact metric space.
(2) Any \( g \in G \) is a homeomorphism onto a compact subset \( gC \) in \( X_0 \) and \( g \), as well as the inverse map \( g^{-1} \), is \( \delta(\epsilon) \)-continuous.

**Proof.** All that needs proof is the \( \delta(\epsilon) \)-continuity of \( g^{-1} \). Assume \( \gamma_n \in \Gamma_C \) converge to \( g \in G \) in the \( d_\infty \)-distance. If \( x, x' \in C \) satisfy \( d(x, x') > 0 \), then \( d(\gamma_n x, \gamma_n x') \geq \delta(\epsilon) \) by the equicontinuity of the inverse map \( \gamma_n^{-1} \). Thus \( d(gx, gx') \geq \delta(\epsilon) \), as is required.

Recall the notations \( B = B_b(x_0) \) and \( C = \text{Cl}(B) \).

**Lemma 3.4.** If \( g_n \to g \) in \( G \), and \( y \in gB \), then for any large \( n \) we have \( y \in g_n B \) and \( g_n^{-1}y \to g^{-1}y \).

**Proof.** Choose an arbitrary point \( x \in B \) and \( \epsilon > 0 \) such that \( \text{Cl}(B_\epsilon(x)) \subset B \). First let us show that for any \( \gamma \in \Gamma_C \),
\[
(3.1) \quad B_{\delta(\epsilon)}(\gamma x) \subset \gamma \text{Cl}(B_\epsilon(x)).
\]
In fact, by the choice of the number \( b \), we have \( \gamma(B) \subset \text{Cl}(B_{a/3}(x_0)) \). That is, \( \gamma(B) \subset B_a(\gamma x) \), and thus \( (\gamma|_B)^{-1} \) admits an extension \( \gamma^{-1} \) defined on \( B_a(\gamma x) \). Choose an arbitrary point \( y \in B_{\delta(\epsilon)}(\gamma x) \). Then by the \( \delta(\epsilon) \)-continuity of \( \gamma^{-1} \), the point \( x' = \gamma^{-1}y \) lies in \( \text{Cl}(B_\epsilon(x)) \subset B \). On the other hand \( x' = \gamma^{-1}x' = \gamma^{-1}\gamma x' \). Since \( \gamma^{-1} \) is injective, we have \( y = \gamma x' \). This finishes the proof of (3.1).

Next let us show that for any \( g \in G \), we have
\[
(3.2) \quad B_{\delta(\epsilon)/2}(gx) \subset g \text{Cl}(B_\epsilon(x)).
\]
Again assume \( \gamma_n \in \Gamma_C \) converge to \( g \). Since \( \gamma_n x \to gx \), we have for any large \( n \) that \( B_{\delta(\epsilon)/2}(gx) \subset B_{\delta(\epsilon)}(\gamma_n x) \). Thus if \( y \in B_{\delta(\epsilon)/2}(gx) \), then by (3.1), \( y = \gamma_n x_n \) for some \( x_n \in \text{Cl}(B_\epsilon(x)) \). Passing to a subsequence, assume that \( x_n \to x' \in \text{Cl}(B_\epsilon(x)) \). Now in the following inequality
\[
d(gx, y) = d(gx', \gamma_n x_n) \leq d(gx', \gamma_n x') + d(\gamma_n x', \gamma_n x_n),
\]
both terms of the RHS can be arbitrarily small if \( n \) is sufficiently large. That is, \( y = gx' \), showing (3.2).

To finish the proof of the lemma, assume \( g_n \to g \) in \( G \) and \( y \in gB \). By (3.2), for any sufficiently small \( \epsilon > 0 \) we have \( B_{\delta(\epsilon)/2}(g_n^{-1}y) \subset g_n \text{Cl}(B_\epsilon(g^{-1}y)) \). Since \( g_n g^{-1}y \to y \), we have \( y \in g_n \text{Cl}(B_\epsilon(g^{-1}y)) \) for any large \( n \) and therefore \( g_n^{-1}y \in \text{Cl}(B_\epsilon(g^{-1}y)) \). Since \( \epsilon \) is arbitrarily small, this shows the lemma. q. e. d.
Let $\Gamma_0$ be the restriction of the pseudo*group $\Gamma$ to $X_0$. We shall construct a pseudo*group $\Gamma_2$ of local homeomorphisms of $G$. For any $\gamma \in \Gamma_0$, define

$$\text{Dom}(\gamma_2) = \{g \in G | gC \subset \text{Dom}(\gamma)\},$$

$$\text{Range}(\gamma_2) = \{g \in G | gC \subset \text{Range}(\gamma)\},$$

$$\gamma_2 g = \gamma \circ g, \forall g \in \text{Dom}(\gamma_2).$$

It may happen that for some $\gamma \in \Gamma_0$, $\text{Dom}(\gamma) = \text{Range}(\gamma) = \emptyset$. In that case $\gamma_2$ is not defined.

**Lemma 3.5.** The subsets $\text{Dom}(\gamma_2)$ and $\text{Range}(\gamma_2)$ are open in $G$, and $\gamma_2$ is $\delta(\epsilon)$-continuous w. r. t. the metric $d_\infty$.

**Proof.** The easy proof is omitted. q. e. d.

Denote by $\Gamma_2$ the pseudo*group consisting of all the elements $\gamma_2$ for $\gamma \in \Gamma_0$ and their restrictions to open subsets of the domains.

**Lemma 3.6.** The action of $\Gamma_2$ on $G$ is minimal.

**Proof.** First let us show that for $\gamma_1, \gamma_2 \in \Gamma C$, there is $\gamma_3 \in \Gamma_2$ such that $\gamma_1 \in \text{Dom}(\gamma_3)$ and that $\gamma_3(\gamma_1) = \gamma_2$. Since $\gamma_1 C \subset B_a(\gamma_1 x_0)$, there is an element $\gamma' \in \Gamma$ defined on $B_a(\gamma_1 x_0)$ which extends $\gamma_2 \circ \gamma_1^{-1}$. Let $\gamma \in \Gamma_0$ be the restriction of $\gamma'$ to $\Gamma_0$, i. e. the restriction such that $\text{Dom}(\gamma) = B_a(\gamma_1 x_0) \cap X_0 \cap \gamma_1^{-1} X_0$. Clearly $\gamma_1 C$ is contained in $\text{Dom}(\gamma)$, showing the claim.

Thus we have shown that $\Gamma_2$-orbit of $\text{id}C$ is nothing but $\Gamma C$ and hence dense in $G$. To finish the proof, we shall show that for any $g \in G$, the $\Gamma_2$-orbit of $g$ visits an arbitrarily small neighbourhood of any element $\gamma_2 \in \Gamma C$. Let $\epsilon$ be any small number such that the $2\epsilon$-neighbourhood of $\gamma_2 C$ is contained in $X_0$. Take $\gamma_1 \in \Gamma C$ such that $d_\infty(g, \gamma_1) < \delta(\epsilon)$. Choosing $\epsilon$ and hence $\delta(\epsilon)$ even smaller, one may very well assume that $gC$ is contained in $B_a(\gamma_1 x_0)$. Then the element $\gamma \in \Gamma_0$ constructed above (for $\gamma_1$ and $\gamma_2$) contains $gC$ in its domain, i. e. $g$ is contained in $\text{Dom}(\gamma_2)$, and furthermore $d_\infty(\gamma_2 g, \gamma_2) < \epsilon$.

q. e. d.

Now by Lemmata 3.3, 3.5 and 3.6, one can apply Theorem 2.1 to $(\Gamma_2, G)$ to find a $\Gamma_2$-invariant Radon measure $m$ on $G$. (This is the point where the concept of pseudo*group is useful. Notice that even if $\Gamma_2$ is equicontinuous, it does not necessarily imply that the pseudogroup generated by $\Gamma_2$ is equicontinuous.) One can assume $m$ is a probability measure since $G$ is in fact a precompact open subset of a bigger space. Now let $\mu$ and $\mu'$ be distinct $\Gamma_0$-invariant probability measures on $X_0$. Then their restrictions to $B$ are also distinct, by the minimality of the $\Gamma_0$-action. That is, there is a function $\zeta \in C_c(B)$ such that $\mu(\zeta) \neq \mu'(\zeta)$. One may assume further that $\zeta$ is nonnegative valued.

**Lemma 3.7.** For any $g \in G$, we have

$$\int_{X_0} \zeta(g^{-1} x) \mu(dx) = \int_{X_0} \zeta(x) \mu(dx).$$

**Proof.** For $g \in \Gamma C$, this is just the $\Gamma_0$-invariance of $\mu$. For general $g$, assume $\gamma_n \rightarrow g$ for $\gamma_n \in \Gamma C$. Then by Lemma 3.4, if $x \in gB$, then $x \in \gamma_n B$ for any large $n$ and $\gamma_n^{-1} x \rightarrow g^{-1} x$. If $x \notin gB$, then since $\gamma_n \text{supp}(\zeta) \rightarrow \text{gsupp}(\zeta)$ in the Hausdorff distance, $\zeta(\gamma_n^{-1} x) = 0$ for any large $n$, as well as $\zeta(g^{-1} x)$. In any case for
any $x \in X_0$, we have $\zeta(\gamma^{-1}x) \to \zeta(g^{-1}x)$. The lemma follows from the dominated convergence theorem. q. e. d.

Now recall the space $X_1$. It is an open subset of $X_0$ which contains $C$ such that the $a$-neighbourhood $B_a(x)$ of any point $x$ of $X_1$ is contained in $X_0$ and that if $\gamma \in \Gamma$ and $x' \in X_0$ satisfies $\text{Dom} (\gamma) = B_a(x')$ and $\gamma x' \in X_1$, then the image $\text{Range}(\gamma) = \gamma (B_a(x'))$ is contained in $X_0$.

**Lemma 3.8.** The function

$$Z(x) = \int_G \zeta(g^{-1}x)m(dg)$$

is constant on $X_1$.

**Proof.** Define a function $\zeta_x : G \to \mathbb{R}$ by $\zeta_x(g) = \zeta(g^{-1}x)$. Lemma 3.3 and an additional argument as above shows that $\zeta_x$ is a continuous function.

Choose $x, x' \in X_1$ on the same $\Gamma$-orbit. By the assumption of $X_1$, there is $\gamma \in \Gamma_0$ such that $\gamma x = x'$ and $\text{Dom} (\gamma) = B_a(x) \subset X_0$ and $\text{Range}(\gamma) \subset X_0$. Then we have

$$\{g \in G \mid \zeta_x(g) > 0\} \subset \text{Dom} (\gamma).$$

In fact if $\zeta_x(g) = \zeta(g^{-1}x) > 0$, then $x \in gB$. On the other hand, $\text{diam}(gB) \leq 2a/3$, and thus $gC \subset B_a(x) = \text{Dom}(\gamma)$, i. e. $g \in \text{Dom} (\gamma)$. By the $\Gamma_x$-invariance of the measure $m$, we have

$$Z(x) = \int_G \zeta_x(g)m(dg) = \int_G \zeta_x(\gamma^{-1}g)m(dg) = \int_G \zeta_x(g^{-1} \circ \gamma)m(dg)$$

$$= \int_G \zeta(g^{-1}x)m(dg) = \int_g \zeta_{\gamma x}(g)m(dg) = Z(x').$$

That is, the function $Z$ is constant along a $\Gamma$-orbit in $X_1$. On the other hand it is continuous, since $\zeta \circ g^{-1}$ has the same modulus of continuity. Now the minimality of $\Gamma_0$-action on $X_1$ shows the lemma. q. e. d.

**Lemma 3.9.** The function $Z$ is constant on $X_0$.

**Proof.** It suffices to show that for any $x' \in X_0$ and $x \in X_1$ on the same $\Gamma_0$-orbit, we have $Z(x') = Z(x)$. By the assumption of $X_1$, there exists an element $\gamma \in \Gamma_0$ such that $\gamma x' = x$ and $\text{Dom}(\gamma) = B_a(x') \cap X_0$. Then just as before, one can show

$$\{g \in G \mid \zeta_{x'}(g) > 0\} \subset \text{Dom} (\gamma).$$

Again by the $\Gamma_x$-invariance of $\mu$, we have $Z(x) = Z(x')$. q. e. d.

Now let us finish the proof of Theorem 3.2. By Lemma 3.5 the function $Z$ is constant on $X_0$, depending only on $\zeta$ and $m$. We have on one hand

$$\int_{X_0} \int_G \zeta(g^{-1}x)m(dg)\mu(dx) = \int_{X_0} Z\mu(dx) = Z.$$ 

On the other hand by Fubini and by Lemma 3.7

$$Z = \int_G \int_{X_0} \zeta(g^{-1}x)\mu(dx)m(dg) = \int_G \mu(\zeta)m(dg) = \mu(\zeta).$$

Since $Z$ does not depend on the choice of $\mu$, we have $\mu(\zeta) = \mu'(\zeta)$, contrary to the assumption.
4. The uniqueness of harmonic measures for group actions

Here the notations of the previous sections are all abandoned. Let $\alpha : \Gamma \times X \to X$ be an effective (i.e. faithful) action of a countable group $\Gamma$ on a compact metric space $X$, and let $p$ be a probability measure on $\Gamma$, i.e. a function $p : \Gamma \to [0, 1]$ such that $\sum_{\gamma \in \Gamma} p(\gamma) = 1$. We assume that $\text{supp}(p) = \{ \gamma \in \Gamma \mid p(\gamma) > 0 \}$ generates $\Gamma$ as a semigroup. A probability measure $\mu$ on $X$ is called $p$-harmonic if $\mu = \alpha^* (p \times \mu)$, that is, for any continuous function $f$ on $X$, we have

$$\int_X f(x) \mu(dx) = \int_X \sum_{\gamma \in \Gamma} p(\gamma) f(\gamma x) \mu(dx).$$

This section is devoted to the proof of the following theorem.

**Theorem 4.1.** If the action $\alpha$ is equicontinuous and minimal, then the $p$-harmonic probability measure $\mu$ on $X$ is unique.

**Proof.** Let $M$ be the space of continuous maps from $X$ to $X$, endowed with the supremum metric $d_\infty$, and let $G$ be the closure of $\Gamma$ in $M$. Then as in section 3, Lemmata 3.3 and 3.4, we can show that $G$ is a compact metrizable topological group, (with the topology induced from the metric $d_\infty$).

Let $f$ be an arbitrary continuous function on $X$. Let $m$ be a Haar probability measure on $G$. Define a function $f_m : X \to \mathbb{R}$ by

$$f_m(x) = \int_G f(gx) m(dg).$$

The function $f_m$ is on one hand continuous since the functions $f \circ g$ have the same modulus of continuity, and on the other hand constant on $\Gamma$-orbits by the right invariance of $m$. Hence by the minimality of the action, $f_m$ is a constant, which we denote by $c(f, m)$.

Let $\mu$ be a $p$-harmonic probability measure on $X$, and define a function $f_\mu : G \to \mathbb{R}$ by

$$f_\mu(g) = \int_X f(gx) \mu(dx).$$

Then $f_\mu$ is a continuous function w. r. t. $d_\infty$, and by the $p$-harmonicity of $\mu$, it satisfies

$$f_\mu(g) = \sum_{\gamma \in \Gamma} p(\gamma) f_\mu(g\gamma).$$

If $f_\mu$ takes the maximal value at $g \in G$, then for any $\gamma \in \text{supp}(p)$, the value of $f_\mu$ at $g\gamma$ is also the maximal. Repeating this arguments one can show that $f_\mu$ takes the maximal value on the coset $g\Gamma$, since $\text{supp}(p)$ generates $\Gamma$ as a semigroup. Because $\Gamma$ is dense in $G$, the function $f_\mu$ must be a constant, equal to $f_\mu(e) = \mu(f)$.

Now we have

$$\int_X \int_G f(gx) m(dg) \mu(dx) = \int_X c(f, m) \mu(dx) = c(f, m).$$

On the other hand by Fubini,

$$c(f, m) = \int_G \int_X f(gx) \mu(dx) m(dg) = \int_G \mu(f) m(dg) = \mu(f).$$

Then the value $\mu(f)$, being equal to $c(f, m)$, does not depend on $\mu$, showing the uniqueness of $\mu$. q. e. d.
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