NEW LAPLACE TRANSFORMS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTIONS $2F_2$ AND $3F_3$

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Motivated by the new Laplace transforms for the Kummer’s confluent hypergeometric functions $1F_1$ obtained recently by Kim et al. [Math & Comput. Modelling, 55 (2012), pp. 1068–1071], the authors aim is to establish so far unknown Laplace transforms of rather general case of generalized hypergeometric functions $2F_2(x)$ and $3F_3(x)$ by employing extensions of classical summation theorems for the series $2F_1$ and $3F_2$ obtained recently by Kim et al. [Int. J. Math. Math. Sci., 309503, 26 pages, 2010]. Certain known results obtained earlier by Kim et al. follow cases of our main findings.

Keywords: Laplace transform; Watson’s summation theorem; Dixon’s summation theorem; Whipple’s summation theorem; extension summation theorem.

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1. INTRODUCTION AND RESULTS REQUIRED

We begin with the definition of generalized hypergeometric function with $p$ numerator parameters and $q$ denominator parameters ($p$ and $q$ being nonnegative integers) by means of the following series [12, 13]:

$$pF_q\left[a_1, \cdots, a_p \atop b_1, \cdots, b_q ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!}$$  \hspace{1cm} (1.1)

whenever this series converges and elsewhere by analytic continuation. Also $(\cdot)_n$ denotes the well known Pochhammer symbol (or the shifted factorial) defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \end{cases}$$  \hspace{1cm} (1.2)

for any complex number $a$. Using the fundamental property of gamma function $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \hspace{1cm} (n \in \mathbb{N} \cup \{0\}),$$  \hspace{1cm} (1.3)

where $\Gamma$ is the well known Gamma function.

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The series $pF_q$ defined by (1.1) is convergent for all values of $z$ whenever $p \leq q$. Further, if $p = q + 1$, then the series $(1.1)$ converges when $|z| < 1$. Also it is absolutely convergent on the unite circle $|z| = 1$ if $\Re(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j) > 0$ and it is convergent on the unit circle $|z| = 1$ except at $z = 1$ provided $-1 < \Re(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j) \leq 0$. For more details about this function, we refer [12 13].

On the other hand, we define the Laplace transform of a function $f(t)$ of a real variable $t$ as the integral $g(s)$ over a range of the complex parameter $s$ by the integral

$$g(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt,$$

provided this integral exists in the Lebesgue sense. For more details about the Laplace transforms, we refer [1 2].

Now, keeping in mind the following well known and useful result

$$\int_0^\infty e^{-st} t^{a-1} dt = \Gamma(a)s^{-a},$$

provided $\Re(s) > 0$ and $\Re(a) > 0$. If we employ (1.1) with $p \leq q$, then it is a simple exercise to arrive at the following Laplace transform of a generalized hypergeometric function $pF_q$ as:

$$\int_0^\infty e^{-st} t^{v-1} pF_q \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; wt \right] dt = \Gamma(v)s^{-v} pF_q \left[ \begin{array}{c} v, a_1, \ldots, a_p \\ b_1, \ldots, b_q ; \frac{w}{s} \end{array} \right],$$

provided (i) if $p < q$, $\Re(v) > 0$, $\Re(s) > 0$ and $w$ is arbitrary or (ii) if $p = q > 0$, $\Re(v) > 0$ and $\Re(s) > \Re(w)$, especially (iii) if $p = q > 0$, $s = w$, $\Re(v) > 0$, $\Re(s) > 0$ and $\Re(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j - v) > 0$.

Further, it is not out of place to mention here that interchanging the order of summation and integration (in the proof of (1.6)) is easily seen to be justified due to the uniform convergence of the series defined by (1.1).

In particular, when $p = q = 1$, for Kummer’s confluent hypergeometric function $1F_1$ (also referred to as the confluent hypergeometric function of the first kind), we see that its Laplace transform is

$$\int_0^\infty e^{-st} t^{v-1} 1F_1 \left[ \begin{array}{c} a \\ c \end{array} ; wt \right] dt = \Gamma(b)s^{-b} 2F_1 \left[ \begin{array}{c} a, b \\ c ; \frac{w}{s} \end{array} \right],$$

provided $\Re(b) > 0$ and (i) $\Re(s) > \max\{\Re(w), 0\}$ or (ii) $s = w$, $\Re(s) > 0$ and $\Re(c - a - b) > 0$.

Also, when $p = q = 2$ and $p = q = 3$ for generalized hypergeometric functions $2F_2$ and $3F_3$, we see their Laplace transforms, respectively, are given by

$$\int_0^\infty e^{-st} t^{v-1} 2F_2 \left[ \begin{array}{c} a_1, a_2 \\ b_1, b_2 \end{array} ; wt \right] dt = \Gamma(v)s^{-v} 3F_2 \left[ \begin{array}{c} v, a_1, a_2 \\ b_1, b_2 ; \frac{w}{s} \end{array} \right],$$

provided $\Re(v) > 0$ and (i) $\Re(s) > \max\{\Re(w), 0\}$ or (ii) $s = w$, $\Re(s) > 0$ and $\Re(b_1 + b_2 - a_1 - a_2 - v) > 0$, and

$$\int_0^\infty e^{-st} t^{v-1} 3F_3 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} ; wt \right] dt = \Gamma(v)s^{-v} 4F_3 \left[ \begin{array}{c} v, a_1, a_2, a_3 \\ b_1, b_2, b_3 ; \frac{w}{s} \end{array} \right].$$

provided $\Re(v) > 0$ and (i) $\Re(s) > \max\{\Re(w), 0\}$ or (ii) $s = w$, $\Re(s) > 0$ and $\Re(b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - v) > 0$.

By employing classical summation theorems such as those of Gauss second, Kummer and Bailey for the series $1F_1$; Watson, Dixon and Whipple for the series $2F_2$ and their generalizations [7 8 9]. Recently Kim et al. [5 6] have obtained a large number of Laplace transforms for the confluent hypergeometric function $1F_1$ and generalized hypergeometric function $2F_2$. Here, in our present investigation, will
mention a few of them, which are:

\[
\int_0^\infty e^{-st}t^{b-1} \, _1F_1 \left[ \frac{a}{2} ; \frac{1}{2}ts \right] dt = s^{-b} \frac{\Gamma(\frac{1}{2}) \Gamma(b) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)},
\]

(1.10)

provided \( \Re(b) > 0 \) and \( \Re(s) > 0 \).

\[
\int_0^\infty e^{-st}t^{-a} \, _1F_1 \left[ \frac{a}{2} ; \frac{1}{2}ts \right] dt = s^{a-1} \frac{\Gamma(1-a) \Gamma\left(\frac{1}{2}c + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}c \right) \Gamma\left(\frac{1}{2}c - \frac{1}{2}a + \frac{1}{2}\right)},
\]

(1.11)

provided \( \Re(1-a) > 0 \) and \( \Re(s) > 0 \).

\[
\int_0^\infty e^{-st}t^{b-1} \, _1F_1 \left[ \frac{a}{1 + a - b} ; -ts \right] dt = s^{-b-2a} \frac{\Gamma(\frac{1}{2}) \Gamma(b) \Gamma(1 + a - b)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}b \right) \Gamma(1 + \frac{1}{2}a - b)},
\]

(1.12)

provided \( \Re(b) > 0 \) and \( \Re(s) > 0 \).

\[
\int_0^\infty e^{-st}t^{c-1} \, _2F_2 \left[ \frac{a, b}{\frac{1}{2}(a + b + 1), 2c} ; st \right] dt = s^{-c} \frac{\Gamma(\frac{1}{2}) \Gamma(c) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)},
\]

(1.13)

provided \( \Re(c) > 0 \), \( \Re(s) > 0 \) and \( \Re(2c - a - b) > -1 \).

\[
\int_0^\infty e^{-st}t^{c-1} \, _2F_2 \left[ \frac{a, b}{1 + a - b, 1 + a - c} ; st \right] dt = s^{-c} \frac{\Gamma(c) \Gamma(1 + \frac{1}{2}a) \Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(1 + a) \Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a - c) \Gamma(1 + a - b - c)},
\]

(1.14)

provided \( \Re(c) > 0 \), \( \Re(s) > 0 \) and \( \Re(a - 2b - 2c) > -2 \).

\[
\int_0^\infty e^{-st}t^{c-1} \, _2F_2 \left[ \frac{a, b}{d, e} ; st \right] dt = s^{-c} \pi \frac{\Gamma(c) \Gamma(d) \Gamma(e)}{2^{2c-1} \Gamma\left(\frac{1}{2}a + \frac{1}{2}d\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}e\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}d\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}e\right)},
\]

(1.15)

provided \( \Re(c) > 0 \) and \( \Re(s) > 0 \) with \( a + b = 1 \) and \( d + e = 1 + 2c \).

**Remark.** The results (1.10) and (1.11) are also recorded in [13].

The aim of this research paper is to obtain certain new and useful (potentially) Laplace transforms for the generalized hypergeometric functions \(_2F_2\) and \(_3F_3\), so far not recorded in the literature, by using (1.8) and (1.9) with the help of known extensions of the classical summation theorems obtained earlier by Kim et al. [4]. For this, we shall require the following summation formulae due to Kim et al. [3].
Extension of Gauss second summation theorem

\[
\begin{align*}
\sum_{F_2} \left[ \begin{array}{c} a, b, d + 1 \\ \frac{1}{2} + (a + b + 3), d \\ \frac{1}{2} \\
\end{array} \right] \\
= \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} a + \frac{1}{2} b + \frac{3}{2}) \Gamma(\frac{1}{2} a - \frac{1}{2} b - \frac{1}{2})}{\Gamma(\frac{1}{2} a - \frac{1}{2} b + \frac{1}{2})} \left\{ \frac{1}{2}(a + b - 1) - \frac{ab}{2} \right\} + \frac{a+b+1}{\Gamma(\frac{1}{2} a) \Gamma(\frac{1}{2} b)}
\end{align*}
\]

provided \( \Re(d) > 0 \).

Extension of Bailey summation theorem

\[
\begin{align*}
\sum_{F_2} \left[ \begin{array}{c} a, 1-a, d + 1 \\ c + 1, d \\ \frac{1}{2} \\
\end{array} \right] = 2^{-c} \Gamma(\frac{1}{2}) \Gamma(c + 1)
\end{align*}
\]

\[
\times \left\{ \frac{1}{2} \Gamma(\frac{1}{2} a + \frac{1}{2} c) \Gamma(\frac{1}{2} c - \frac{1}{2} a + \frac{1}{2}) + \frac{1 - \frac{c}{2}}{\Gamma(\frac{1}{2} a + \frac{1}{2} c) \Gamma(\frac{1}{2} c - \frac{1}{2} a + 1)} \right\}
\]

provided \( \Re(d) > 0 \).

Extension of Kummer summation theorem

\[
\begin{align*}
\sum_{F_2} \left[ \begin{array}{c} a, b, d + 1 \\ 2+a-b, d \\ -1 \\
\end{array} \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(2 + a - b)}{2\pi (1-b)} \left\{ \frac{1+a-b}{\Gamma(\frac{1}{2} a) \Gamma(\frac{1}{2} c - \frac{1}{2} a + \frac{1}{2})} + \frac{1 - \frac{a}{2}}{\Gamma(\frac{1}{2} a + \frac{1}{2} c) \Gamma(1 + \frac{1}{2} a - b)} \right\}
\end{align*}
\]

provided \( \Re(d) > 0 \).

First extension of Watson summation theorem

\[
\begin{align*}
\sum_{F_3} \left[ \begin{array}{c} a, b, c \\ \frac{1}{2} (a + b + 1), 2c + 1, d \\ 1 \\
\end{array} \right] = \frac{2^{a+b-2} \Gamma(c + \frac{1}{2}) \Gamma(\frac{1}{2} a + \frac{1}{2} b + \frac{1}{2}) \Gamma(\frac{1}{2} c - \frac{1}{2} a - \frac{1}{2} b + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(a) \Gamma(b)}
\end{align*}
\]

\[
\times \left\{ \frac{\Gamma(\frac{1}{2} a) \Gamma(\frac{1}{2} b)}{\Gamma(c - \frac{1}{2} a + \frac{1}{2}) \Gamma(c - \frac{1}{2} b + \frac{1}{2})} + \frac{2c - d}{d} \frac{\Gamma(c - \frac{1}{2} a + \frac{1}{2}) \Gamma(c - \frac{1}{2} b + \frac{1}{2})}{\Gamma(c - \frac{1}{2} a + 1) \Gamma(c - \frac{1}{2} b + 1)} \right\}
\]

provided \( \Re(d) > 0 \) and \( \Re(2c - a - b) > -1 \).

Second extension of Watson summation theorem

\[
\begin{align*}
\sum_{F_3} \left[ \begin{array}{c} a, b, c \\ \frac{1}{2} (a + b + 3), 2c, d \\ 1 \\
\end{array} \right] = \frac{2^{a+b-2} \Gamma(c + \frac{1}{2}) \Gamma(\frac{1}{2} a + \frac{1}{2} b + \frac{3}{2}) \Gamma(\frac{1}{2} c - \frac{1}{2} a - \frac{1}{2} b - \frac{1}{2})}{(a - b - 1)(a - b + 1) \Gamma(\frac{1}{2}) \Gamma(a) \Gamma(b)}
\end{align*}
\]

\[
\times \left\{ \alpha \frac{\Gamma(\frac{1}{2} a) \Gamma(\frac{1}{2} b)}{\Gamma(c - \frac{1}{2} a + \frac{1}{2}) \Gamma(c - \frac{1}{2} b + \frac{1}{2})} + \beta \frac{\Gamma(\frac{1}{2} a + \frac{1}{2}) \Gamma(\frac{1}{2} b + \frac{1}{2})}{\Gamma(c - \frac{1}{2} a) \Gamma(c - \frac{1}{2} b)} \right\}
\]
provided $\Re(d) > 0$ and $\Re(2c - a - b) > -1$ with
\[
\alpha = a(2c - a) + b(2c - b) - 2c + 1 - \frac{ab}{d}(4c - a - b - 1) \quad \text{and} \quad \beta = 8\left\{\frac{1}{2d}(a + b + 1) - 1\right\}.
\]

**Extension of Dixon summation theorem**

\[
\begin{align*}
\quad 4F3 \left[ a, \quad b, \quad c, \quad d + 1 \quad : \quad 2 + a - b, 1 + a - c, \quad d \right] &= \frac{\alpha}{b - 1} \cdot \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(2 + a - b)\Gamma(1 + a - c)\Gamma\left(\frac{1}{2}a - b + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma(2 + a - b - c)\Gamma\left(\frac{1}{2}a - c + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - b + \frac{3}{2}\right)} \\
&+ \frac{\beta}{b - 1} \cdot \frac{2^{-a-1}\Gamma\left(\frac{1}{2}\right)\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma(1 + a - b - c)\Gamma(1 + \frac{1}{2}a - c)\Gamma(1 + \frac{1}{2}a - c)};
\end{align*}
\]
provided $\Re(d) > 0$ and $\Re(a - 2b - 2c) > -2$ with
\[
\alpha = 1 - \frac{1}{d}(1 + a - b) \quad \text{and} \quad \beta = \frac{1 + a - b}{1 + a - b - c}\left\{\frac{a}{d}(1 + a - b - 2c) - 2(1 + \frac{1}{2}a - b - c)\right\}.
\]

**Extension of Whipple summation theorem**

\[
\begin{align*}
\quad 4F3 \left[ a, 1 - a, c, d + 1 \quad : \quad e + 1, 2c - e + 1, d \right] &= \frac{2^{-2a}\Gamma(e + 1)\Gamma(e - c)\Gamma(2c - e + 1)}{\Gamma(e - a + 1)\Gamma(e - c + 1)\Gamma(2c - a - e + 1)} \\
\times &\left\{(1 - \frac{2c - e}{d})\Gamma\left(\frac{1}{2}e - \frac{1}{2}a + \frac{1}{2}\right)\Gamma(1 + a - \frac{1}{2}e + \frac{1}{2})\right\} \\
&+ \left\{(e - 1)\Gamma\left(\frac{1}{2}e - \frac{1}{2}a + \frac{1}{2}\right)\Gamma(1 + a - \frac{1}{2}e + \frac{1}{2})\right\} \\
&\times \Gamma\left(\frac{1}{2}a + \frac{1}{2}e + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - \frac{1}{2}e\right); \quad (1.22)
\end{align*}
\]
provided $\Re(d) > 0$ and $\Re(c) > 0$.

### 2. Three general Laplace transforms of $2F_2(x)$

In this section, we shall list three general Laplace transforms of $2F_2(x)$ obtained with the help of (1.8) and the extensions of summation formulas (1.11), (1.17) and (1.18). Clearly, since (1.8) is the most general case, so it is desirable to find, as much as possible, less general case involving various particular values of the parameters $a$, $b$, $d$ and $e$. Below, in (2.1), (2.2) and (2.2), we give three new and very general Laplace transforms of $2F_2$, which are not listed in the standard table of the Laplace transforms books [11][12][13][14].

\[
\begin{align*}
\int_0^\infty e^{-st}t^{b-1}2F_2 \left[ a, \quad d + 1 \quad : \quad \frac{1}{2}(a + b + 3), \quad d \right] dt &= \frac{s^{-b}\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \left\{\frac{\frac{1}{2}(a + b - 1) - \frac{ab}{d}}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} + \frac{a + b + 1 - 2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right\}, \quad (2.1)
\end{align*}
\]
provided $\Re(b) > 0$, $\Re(d) > 0$ and $\Re(s) > 0$. 


Remark: In (2.1), (2.2) and (2.3), if we respectively take \( d = \frac{1}{2}(a + b + 1), \) \( d = c \) and \( d = 1 + a - b, \) we recover (1.10), (1.11) and (1.12) obtained earlier by Kim et al. [3].

3. Four general Laplace transforms of \( 3F_3(x) \)

As described in section 2, here we shall mention from general Laplace transforms of \( 3F_3(x) \) with the help of (1.19) and the extensions of summation formulas (1.19), (1.20), (1.21) and (1.22). Below, in (3.1), (3.2), (3.3) and (3.4), we give four new and very general Laplace transforms of \( 3F_3, \) which are not listed in the standard tables of Laplace transforms books [1, 3, 10, 11].

\[
\int_0^\infty e^{-st} t^{-a} \binom{3}{2}F_2 \left[ \begin{array}{c} a, \ b, \ d + 1 \\ c + 1, \ d \end{array} ; \frac{1}{2} ts \right] dt = \frac{s^{a-1} \Gamma\left(\frac{1}{c}\right) \Gamma(1-a) \Gamma(c+1)}{2^c} \\
\frac{\Gamma\left(\frac{d}{c} a + \frac{1}{c} b\right) \Gamma\left(\frac{d}{c} c - \frac{1}{c} b + \frac{1}{c} \right) + \Gamma\left(\frac{d}{c} a + \frac{1}{c} c + \frac{1}{c} \right) \Gamma\left(\frac{d}{c} c - \frac{1}{c} a + 1 \right)}{2^c},
\]

provided \( \Re(1-a) > 0, \Re(d) > 0 \) and \( \Re(s) > 0. \)

\[
\int_0^\infty e^{-st} t^{b-1} \binom{3}{2}F_2 \left[ \begin{array}{c} a, \ \frac{1}{2} \left(2 + a - b\right), \ d \end{array} ; -ts \right] dt = \frac{s^{-b} \Gamma\left(\frac{1}{c}\right) \Gamma(b) \Gamma(2 + a - b)}{2^{c(1-b)}} \left\{ \frac{1 + a - b}{\Gamma\left(\frac{1}{c} a + \frac{1}{c} b\right) \Gamma\left(\frac{1}{c} a - b + \frac{1}{c} \right)} + \frac{1 - a}{\Gamma\left(\frac{1}{c} a + \frac{1}{c} c + \frac{1}{c} \right) \Gamma\left(\frac{1}{c} c - \frac{1}{c} a + 1 \right)} \right\},
\]

provided \( \Re(b) > 0, \Re(d) > 0 \) and \( \Re(s) > 0. \)

Remark: In (2.1), (2.2) and (2.3), if we respectively take \( d = \frac{1}{2}(a + b + 1), \) \( d = c \) and \( d = 1 + a - b, \) we recover (1.10), (1.11) and (1.12) obtained earlier by Kim et al. [3].
New Laplace transforms for the generalized hypergeometric functions $2F_2$ and $3F_3$

\[
\int_0^\infty e^{-st}t^{e-1} \, 3F_3 \left[ \begin{array}{ccc} a, & b, & d + 1 \\ 2 + a - b, 1 + a - c, & d \\ \end{array} ; st \right] dt
\]

\begin{equation}
(3.3)
\end{equation}

\[
= \frac{s^{-c-2}a\Gamma(\frac{1}{a})\Gamma(c)}{b - 1} \left\{ (1 + a - b)\Gamma(1 + a - c)\Gamma(\frac{1}{2}a - b + \frac{1}{2}) \right. \\
\left. + \frac{\beta}{2} \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma((1 + a - b)\Gamma(1 + \frac{1}{2}a - b - c))} \right\},
\]

provided $\Re(c) > 0$, $\Re(s) > 0$ and $\Re(a - 2b - 2c) > -2$ with

\[
\alpha = 1 - \frac{1}{d}(1 + a - b) \quad \text{and} \quad \beta = \frac{1 + a - b}{1 + a - b - c} \left( \frac{a}{\alpha} \right)(1 + a - b - 2c) - 2(1 + \frac{1}{2}a - b - c). \]

\[
\int_0^\infty e^{-st}t^{e-1} \, 3F_3 \left[ \begin{array}{ccc} a, & 1 - a, & d + 1 \\ e + 1, 2c - e + 1, & d \\ \end{array} ; st \right] dt
\]

\begin{equation}
(3.4)
\end{equation}

\[
= \frac{s^{-c-2}a\Gamma(c)\Gamma(e + c + 1)\Gamma(2c - e + 1)}{\Gamma(e - a + 1)\Gamma(e - c + 1)\Gamma(2c - e - a + 1)} \\
\times \left\{ \left( 1 - \frac{2c - e}{d} \right) \frac{\Gamma(1/2e - 1/2a + 1/2c - 1/2a - 1/2e + 1/2)}{\Gamma(1/2a + 1/2e + 1/2)} \right. \\
\left. + \frac{e}{d - 1} \frac{\Gamma(1/2e - 1/2a + 1/2c - 1/2a - 1/2e + 1/2)}{\Gamma(1/2a + 1/2e + 1/2)} \right\},
\]

provided $\Re(c) > 0$ and $\Re(s) > 0$.

**Special cases:** In (3.1), if we take $d = 2c$ or in (3.2), if we take $d = \frac{1}{2}(a + b + 1)$, we recover (1.13). While in (3.3) and (3.4), if we take $d = 1 + a - b$ and $d = e$, we respectively recovered (1.14) and (1.15).

**Concluding remark:** we conclude this paper by remarking that the Laplace transforms for the generalized hypergeometric functions $2F_2$ and $3F_3$ established in this paper may be useful in Mathematics, Statistics, Physics and Engineering.

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