ON MAGNETOELASTIC SOLITONS IN FERROMAGNET

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Abstract

We study the solitonic excitations in the compressible ferromagnetic Heisenberg chain (in the continuum limit).
INTRODUCTION

Solitons in magnetically ordered crystals have been widely investigated from both theoretical and experimental points of view[1-16]. In particular, the existence of coupled magnetoelastic solitons in the Heisenberg compressible spin chain has been extensively demonstrated[17-23]. In [23] were presented the new class integrable and nonintegrable spin systems. In this letter we consider the some of these nonlinear models of magnets - the some of the Myrzakulov equations(ME), which describe the nonlinear dynamics of compressible magnets.

A. THE 0-CLASS OF THE SPIN-PHONON SYSTEMS

The Myrzakulov equations with the potentials have the form[23]:

the $M_{00}^{10}$ - equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$  \hspace{1cm} (1)

the $M_{00}^{20}$ - equation:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$  \hspace{1cm} (2)

the $M_{00}^{30}$ - equation:

$$2iS_t = \{(\mu S_x^2 - u + m)[S, S_x]\}_x + h[S, \sigma_3]$$  \hspace{1cm} (3)

the $M_{00}^{40}$ - equation:

$$2iS_t = n[S, S_{xxxx}] + 2\{(\mu S_x^2 - u + m)[S, S_x]\}_x + h[S, \sigma_3]$$  \hspace{1cm} (4)

the $M_{00}^{50}$ - equation:

$$2iS_t = [S, S_{xx}] + auS_x + bS_x$$  \hspace{1cm} (5)

where $v_0, \mu, \lambda, n, m, a, b, \alpha, \beta, \rho, h$ are constants, $u$ is a scalar function(potential), subscripts denote partial differentiations, $[,] (\{,\})$ is commutator (anticommutator),

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad r^2 = \pm 1, \quad S^2 = I.$$  

The solutions of these ME for the potential

$$u = Usech^2k(x - x_0)$$  \hspace{1cm} (6a)

and for the boundary condition

$$S |_{x=\pm \infty} = \sigma_3, \quad u |_{x=\pm \infty} = 0$$  \hspace{1cm} (7)

are given by

$$S^+ = AW shz \cdot sech^2z, \quad S^- = 1 - 2sech^2z$$  \hspace{1cm} (6b)
and the following formulas, respectively \( r^2 = 1, A^2 = 4, W = \exp(i(\omega t + \phi)), \phi = \text{const}, z = k(x - x_0) \):

\[
M_{00}^{30} : w = mk^2 - h, k^2 = U/4\mu, \lambda = \frac{1}{4} \tag{8a}
\]

\[
M_{00}^{40} : w = nk^4 + 2mk - h, k^2 = U/(4\mu - 5n) \tag{8b}
\]

**B. THE 1-CLASS OF THE SPIN-PHONON SYSTEMS**

Here we present the following ME[23]:

the \( M_{00}^{11} \) - equation:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{9a}
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S_3)_{xx} \tag{9b}
\]

the \( M_{00}^{12} \) - equation:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{10a}
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S_3)_{xx} \tag{10b}
\]

the \( M_{00}^{13} \) - equation:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{11a}
\]

\[
u_t + u_x + \lambda(S_3)_x = 0 \tag{11b}
\]

the \( M_{00}^{14} \) - equation:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3] \tag{12a}
\]

\[
u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(S_3)_x = 0 \tag{12b}
\]

The some properties of these Myrzakulov equations were considered in refs.[25-28].

**C. THE 2-CLASS OF THE SPIN-PHONON SYSTEMS**

In this section we consider the following ME[23]

the \( M_{00}^{21} \) - equation:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{13a}
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S_3^2)_{xx} \tag{13b}
\]

the \( M_{00}^{22} \) - equation:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{14a}
\]

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3] \tag{14b}
\]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \lambda (S_3^2)_{xx} \quad (14b) \]

the \( M_{00}^{23} \) - equation:

\[ 2i S_t = [S, S_{xx}] + (u S_3 + h)[S, \sigma_3] \quad (15a) \]

\[ u_t + u_x + \lambda (S_3^2)_x = 0 \quad (15b) \]

the \( M_{00}^{24} \) - equation:

\[ 2i S_t = [S, S_{xx}] + (u S_3 + h)[S, \sigma_3] \quad (16a) \]

\[ u_t + u_x + \alpha (u^2)_x + \beta u_{xxxx} + \lambda (S_3^2)_x = 0 \quad (16b) \]

Some of these ME are studied in [25-28].

D. THE 3-CLASS OF THE SPIN-PHONON SYSTEMS

Now we consider the following ME([23]):

the \( M_{00}^{31} \) - equation:

\[ 2i S_t = \{(\mu S_3^2 - u + m)[S, S_x]\}_x \quad (17a) \]

\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda (S_3^2)_{xx} \quad (17b) \]

the \( M_{00}^{32} \) - equation:

\[ 2i S_t = \{(\mu S_3^2 - u + m)[S, S_x]\}_x \quad (18a) \]

\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \lambda (S_3^2)_{xx} \quad (18b) \]

the \( M_{00}^{33} \) - equation:

\[ 2i S_t = \{(\mu S_3^2 - u + m)[S, S_x]\}_x \quad (19a) \]

\[ u_t + u_x + \lambda (S_3^2)_x = 0 \quad (19b) \]

the \( M_{00}^{34} \) - equation:

\[ 2i S_t = \{(\mu S_3^2 - u + m)[S, S_x]\}_x \quad (20a) \]

\[ u_t + u_x + \alpha (u^2)_x + \beta u_{xxxx} + \lambda (S_3^2)_x = 0 \quad (20b) \]

The soliton solitons of these ME - the \( M_{00}^{31}, M_{00}^{32}, M_{00}^{33} \) and \( M_{00}^{34} \) equations are given by (6) and the following formulas, respectively \((r^2 = 1, A^2 = 4, W = \exp(i(\omega t + \varphi)), \omega = mk^4, \varphi = \text{const}, z = k(x - x_0 U = 4\mu k^2)):\)

\[ M_{00}^{31} : \lambda = \mu \nu_0^2 \quad (21a) \]

\[ M_{00}^{32} : \alpha = 3\beta/2\mu, \quad k^2 = -(\lambda + \nu_0^2\mu)/(4\mu \beta), \quad \lambda = -\mu \nu_0^2 - 4\mu \beta k^2 \quad (21b) \]

\[ M_{00}^{33} : \lambda = -\mu, k^2 = U/4\mu \quad (21c) \]

\[ M_{00}^{34} : \alpha = 3\beta/2\mu, k^2 = -(\lambda + \mu)/4\mu \beta \quad (21d) \]
E. THE 4-CLASS OF THE SPIN-PHONON SYSTEMS

These ME look like[23]:

the $M_{00}^{41}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\bar{S}_x^2 - u + m\}[S, S_x]$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^2_x)_{xx}$$  \hspace{1cm} (22a)  

the $M_{00}^{42}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\bar{S}_x^2 - u + m\}[S, S_x]$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_{xx}$$  \hspace{1cm} (22b)  

the $M_{00}^{43}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\bar{S}_x^2 - u + m\}[S, S_x]$$

$$u_t + u_x + \lambda(S^2_x)_x = 0$$  \hspace{1cm} (23a)  

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_{xx}$$  \hspace{1cm} (23b)  

the $M_{00}^{44}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\bar{S}_x^2 - u + m\}[S, S_x]$$

$$u_t + u_x + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_x = 0$$  \hspace{1cm} (24a)  

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_{xx}$$  \hspace{1cm} (24b)  

These equations describe the nonlinear interaction of the spin and phonon subsystems[23]. For the case $\mu = 0$ the soliton solitons of these equations were obtained in [7,28]. Here we consider the case $\mu \neq 0$. In this case we present the soliton solitons of the Myrzakulov equations(22)-(25) for the boundary condition (7). The soliton solitons of the ME $M_{00}^{41}, M_{00}^{42}, M_{00}^{43}$ and $M_{00}^{44}$ equations are given by (6) and the following formulas, respectively ($\nu^2 = 1, A^2 = 4, W = exp(i(\omega t + \varphi)), \omega = k^4 + 2mk^2, U = -yk^2, y = 1 - 4\mu, \varphi = const, z = k(x - x_0)$):

$$M_{00}^{41} : \lambda = \nu_0^2/4$$  \hspace{1cm} (25a)  

$$M_{00}^{42} : \alpha = -6\beta/y, \quad k^2 = (4\lambda + \nu_0^2y)/(4y\beta)$$  \hspace{1cm} (25b)  

$$M_{00}^{43} : \lambda = y/4$$  \hspace{1cm} (26a)  

$$M_{00}^{44} : \alpha = -6\beta/y, \quad k^2 = (4\lambda - y)/(4y\beta)$$  \hspace{1cm} (26b)  

F. THE 5-CLASS OF THE SPIN-PHONON SYSTEMS

Finally we consider the following equations[23]:

the $M_{00}^{51}$ - equation:

$$2iS_t = [S, S_{xx}] + auS_x + bS_x$$  \hspace{1cm} (27a)  

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(f)_{xx}$$  \hspace{1cm} (27b)
the $M_{00}^{52}$ - equation:
\[ 2iS_t = [S, S_{xx}] + auS_x + bS_x \] (28a)
\[ \rho u_t = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxx} + \lambda(f)_{xx} \] (28b)

the $M_{00}^{53}$ - equation:
\[ 2iS_t = [S, S_{xx}] + auS_x + bS_x \] (29a)
\[ u_t + u_x + \lambda(f)_x = 0 \] (29b)

the $M_{00}^{54}$ - equation:
\[ 2iS_t = [S, S_{xx}] + auS_x + bS_x \] (30a)
\[ u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(f)_x = 0 \] (30b)

Here $f$ is a scalar function[23].

Appendix. **A LIST OF THE 2+1 DIMENSIONAL INTEGRABLE SPIN EQUATIONS**

Here we want present the some integrable (2+1)-dimensional spin systems - the Ishimori and some Myrzakulov equations.

Consider a n-dimensional space with the basic unit vectors: $\vec{e}_1 = \vec{S}, \vec{e}_2, ..., \vec{e}_n$ and $\vec{e}_1^2 = E = \pm 1$. Then the 2+1 dimensional Myrzakulov-0 equation[23] has the form
\[ \vec{S}_t = \sum_{i=2}^{n} a_i \vec{e}_i \] (31a)
\[ \vec{S}_x = \sum_{i=2}^{n} b_i \vec{e}_i \] (31b)
\[ \vec{S}_y = \sum_{i=2}^{n} c_i \vec{e}_i \] (31c)

where $a_i, b_i, c_i$ are real functions, $\vec{S} = (S_1, S_2, ..., S_n), \vec{S}^2 = E = \pm 1$. This equation admits the many interesting class integrable and nonintegrable reductions. Below we present only the some integrable reductions of the Myrzakulov-0 equation.

1) The Myrzakulov-IV(M-IV) equation
\[ \vec{S}_t + \{ \vec{S}_{xy} + V \vec{S} + E \vec{S}_x \wedge (\vec{S} \wedge \vec{S}_y) \}_x = 0 \]
\[ V_x = \frac{E}{2}(\vec{S}_x^2)_y \]

2) The Myrzakulov-I(M-I) equation looks like
\[ \vec{S}_t = (\vec{S} \wedge \vec{S}_y + u\vec{S})_x \]
\[ u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y) \]
3) The Myrzakulov-II (M-II) equation

$$\vec{S}_t = (\vec{S} \wedge \vec{S}_y + u \vec{S})_x + 2cb^2 \vec{S}_y - 4cv \vec{S}_x$$

$$u_x = -\vec{S} (\vec{S}_x \wedge \vec{S}_y),$$

$$v_x = \frac{1}{16b^2c^2} (\vec{S}_{1x})_y$$

4) The Myrzakulov-III (M-III) equation

$$\vec{S}_t = (\vec{S} \wedge \vec{S}_y + u \vec{S})_x + 2(b(c + d) \vec{S}_y - 4cv \vec{S}_x$$

$$u_x = -\vec{S} (\vec{S}_x \wedge \vec{S}_y),$$

$$v_x = \frac{1}{4(2bc + d)^2} (\vec{S}_{1x})_y$$

5) The Myrzakulov-VIII (M-VIII) equation looks like

$$iS_t = \frac{1}{2} [S_{xx}, S] + iuS_x$$

$$u_y = \frac{1}{4} tr(S[S_y, S_x])$$

where the subscripts denote partial derivatives and $S$ denotes the spin matrix ($r^2 = \pm 1$)

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix},$$

$$S^2 = I$$

6) The Ishimori equation

$$iS_t + \frac{1}{2} [S, M_{10} S] + A_{20} S_x + A_{10} S_y = 0$$

$$M_{20} u = \frac{\alpha}{4i} tr(S[S_y, S_x])$$

where $\alpha, b, a=\text{ consts}$ and

$$M_{j0} = M_j, \quad A_{j0} = A_j \quad \text{as} \quad a = b = -\frac{1}{2}.$$ 

7) The Myrzakulov-IX (M-IX) equation has the form

$$iS_t + \frac{1}{2} [S, M_1 S] + A_2 S_x + A_1 S_y = 0$$

$$M_2 u = \frac{\alpha}{4i} tr(S[S_y, S_x])$$

where $\alpha, b, a=\text{ consts}$ and

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + (a^2 - 2ab - b) \frac{\partial^2}{\partial x^2};$$
\[ M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + a(a + 1) \frac{\partial^2}{\partial x^2}, \]
\[ A_1 = i\alpha \{(2ab + a + b)u_x - (2b + 1)\alpha u_y\} \]
\[ A_2 = i\{\alpha(2ab + a + b)u_y - (2a^2b + a^2 + 2ab + b)u_x\}. \]

The M-IX eqs. admit the two integrable reductions. As \(b=0\), after the some manipulations reduces to the M-VIII equation and as \(a = b = -\frac{1}{2}\) to the Ishimori equation. In general we have the two integrable cases: the M-IXA equation as \(\alpha^2 = 1\), the M-IXB equation as \(\alpha^2 = -1\). We note that the M-IX equation is integrable and admits the following Lax representation

\[ \alpha \Phi_y = \frac{1}{2} [S + (2a + 1)I] \Phi_x \]
\[ \Phi_t = \frac{i}{2} [S + (2b + 1)I] \Phi_{xx} + \frac{i}{2} W \Phi_x \]

where

\[ W_1 = W - W_2 = (2b + 1)E + (2b - a + \frac{1}{2})SS_x + (2b + 1)FS \]
\[ W_2 = W - W_1 = FI + \frac{1}{2} S_x + ES + \alpha SS_y \]
\[ E = -\frac{i}{2\alpha} u_x, \quad F = \frac{i}{2} \left( \frac{(2a + 1)u_x}{\alpha} - 2u_y \right) \]

Hence we get the Lax representations of the M-VIII as \(b = 0\) and for the Ishimori equation as \(a = b = -\frac{1}{2}\). The M-IX equation admit the different type exact solutions (solitons, lumps, vortex-like, dromion-like and so on).

8) The Myrzakulov-XXII (M-XXII) equation has the form

\[ -iS_t = \frac{1}{2} ([S, S_y] + 2iuS)_x + \frac{i}{2} V_1 S_x - 2ia^2 S_y \]
\[ u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y) \]
\[ V_{1x} = \frac{1}{4a^2} (\vec{S}^2)_y \]

and so on.

All of these equations admit the corresponding Lax representations, which were presented in [23]. The gauge equivalent counterparts of the above presented Myrzakulov equations are found in [28].

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