GRÖBNER BASES AND BETTI NUMBERS OF MONOIDAL COMPLEXES

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To Mel Hochster on his 65th birthday

ABSTRACT. In this note we consider monoidal complexes and their associated algebras, called toric face rings. These rings generalize Stanley-Reisner rings and affine monoid algebras. We compute initial ideals of the presentation ideal of a toric face ring, and determine its graded Betti numbers. Our results generalize celebrated theorems of Hochster in combinatorial commutative algebra.

1. INTRODUCTION

Combinatorial commutative algebra is a branch of combinatorics, discrete geometry and commutative algebra. On the one hand problems from combinatorics or discrete geometry are studied using techniques from commutative algebra. On the other hand questions in combinatorics motivated various results in commutative algebra. Since the fundamental papers of Stanley (see [13] for the results) and Hochster [8, 9] combinatorial commutative algebra is a growing and very active field of research. See also Bruns-Herzog [7], Villarreal [16], Miller-Sturmfels [11] and Sturmfels [15] for classical and recent results and new developments in this area of mathematics.

Stanley-Reisner rings and affine monoid algebras are two of the classes of rings considered in combinatorial commutative algebra. In this paper we consider toric face rings associated to monoidal complexes. They generalize Stanley-Reisner rings by allowing a more general incidence structure than simplicial complexes, and more general rings associated with their faces, namely affine monoid algebras instead of polynomial rings.

In cooperation with M. Brun and B. Ichim the authors have studied the local cohomology of toric face rings in previous work [1], [3], [10], and one of the main results is a general version of Hochster’s formula for the local cohomology of a Stanley-Reisner ring (see [7] or [13]), even beyond toric face rings.

In this paper we want to generalize Hochster’s formulas for the graded Betti numbers of a Stanley-Reisner ring [9] and affine monoid rings [11] Theorem 9.2] to toric face rings. Such a generalization is indeed possible for monoidal complexes, that, roughly speaking, can be embedded into a space \( \mathbb{Q}^d \). As counterexamples show, full generality does not seem possible. One of the problems encountered is to construct a suitable grading. This forces us to consider grading monoids that are not necessarily cancellative.

Another topic treated are initial ideals (of the defining ideals) of toric face rings with respect to monomial (pre)orders defined by weights. Indeed, toric face rings come up naturally in the study of initial ideals of affine monoid algebras. In this regard we generalize results of Sturmfels [15]. We will pay special attention to the question when the initial ideal is radical, monomial or both. This gives an opportunity to indicate a “simplicial”
proof of Hochster’s famous theorem on the Cohen-Macaulay property of affine normal monoid domains [8]. For unexplained terminology we refer the reader to [6] and [7].

2. MONOIDAL COMPLEXES AND TORIC FACE RINGS

A cone is a subset of a space \( \mathbb{R}^d \) of type \( \mathbb{R}_+x_1 + \cdots + \mathbb{R}_+x_n \) with \( x_1, \ldots, x_n \in \mathbb{R}^d \). The dimension of a cone \( C \) is the vector space dimension of \( \mathbb{R}C \). A face of \( C \) is a subset of type \( C \cap H \) where \( H \) is a support hyperplane of \( C \), i.e. a hyperplane \( H \) for which \( C \) is contained in one of the two closed halfspaces \( H^+, H^- \) determined by \( H \). A rational cone is generated by elements \( x \in \mathbb{Q}^d \). A pointed cone has \( \{0\} \) as a face.

A fan in \( \mathbb{R}^d \) is a finite collection \( \mathcal{F} \) of cones in \( \mathbb{R}^d \) satisfying the following conditions:

(i) all the faces of each cone \( C \in \mathcal{F} \) belong to \( \mathcal{F} \), too;
(ii) the intersection \( C \cap D \) of \( C, D \in \mathcal{F} \) is a face of \( C \) and of \( D \).

We want to investigate more general configurations of cones, giving up the condition that all cones are contained in a single space, but retaining the incidence structure. A conical complex consists of

(i) a finite set \( \Sigma \) of sets,
(ii) a cone \( C_c \subseteq \mathbb{R}^{\delta_c} \), \( \delta_c = \dim \mathbb{R}C_c \), for each \( c \in \Sigma \),
(iii) and a bijection \( \pi_c : C_c \rightarrow c \) for each \( c \in \Sigma \) such that the following conditions are satisfied:
(a) for each face \( C' \) of \( C_c \), \( c \in \Sigma \), there exists \( c' \in \Sigma \) with \( \pi_c(C') = c' \);
(b) for all \( c, d \in \Sigma \) there exist faces \( C' \) of \( C_c \) and \( D' \) of \( D_d \) such that \( c \cup d = \pi_c(C') \cap \pi_d(D') \) and the restriction of \( \pi_d^{-1} \circ \pi_c \) to \( C' \) is an isomorphism of the cones \( C' \) and \( D' \).

Here an isomorphism of cones \( C, D \) is a bijective map \( \varphi : C \rightarrow D \) that extends to an isomorphism of the vector spaces \( \mathbb{R}C \) and \( \mathbb{R}D \). Simplifying the notation, we write \( \Sigma \) also for the conical complex. A fan \( \mathcal{F} \) is a conical complex in a natural way: fans are nothing but embedded conical complexes.

As introduced in the definition, \( \delta_c \) will always denote the dimension of \( C_c \) so that \( \mathbb{R}C_c \) can be identified with \( \mathbb{R}^{\delta_c} \). The elements \( c \in \Sigma \) are called the faces of \( \Sigma \). Similarly, one defines rays and facets of \( \Sigma \) as 1-dimensional and maximal faces of \( \Sigma \). The dimension of \( \Sigma \) is the maximal dimension of a facet of \( \Sigma \). We denote by \( |\Sigma| = \bigcup_{c \in \Sigma} c \) the support of \( \Sigma \). Identifying \( C_c \) with \( c \), we may consider \( C_c \) as a subset of \( |\Sigma| \). Then we can treat \( |\Sigma| \) almost like an (embedded) fan. The main difference is that it makes no sense to speak of concepts like convexity globally. However, locally in the cones \( C_c \) we may consider convex subsets. The complex \( \Sigma \) is rational and pointed, respectively, if all cones \( C_c \), \( c \in \Sigma \), are rational and pointed respectively. We call \( \Sigma \) simplicial, if all cones \( C_c \), \( c \in \Sigma \), are simplicial, i.e. they are generated by linearly independent vectors.

In order to define interesting algebraic objects associated to a conical complex one needs a corresponding discrete structure. A monoidal complex \( \mathcal{M} \) supported by a conical complex \( \Sigma \) is a set of monoids \( (M_c)_{c \in \Sigma} \) such that

(i) for each \( c \in \Sigma \) the monoid \( M_c \) is an affine (i.e. finitely generated) monoid contained in \( \mathbb{Z}^{\delta_c} \);
(ii) \( M_c \subseteq C_c \) and \( \mathbb{R}_+M_c = C_c \) for every \( c \in \Sigma \);
(iii) for all \( c, d \in \Sigma \) the map \( \pi_d^{-1} \circ \pi_c \) restricts to a monoid isomorphism between 
\[ M_c \cap \pi_d^{-1}(c \cap d) \] 
and 
\[ M_d \cap \pi_d^{-1}(c \cap d) . \]
In other words, we have chosen an affine monoid \( M_c \) for every \( c \in \Sigma \) which generates \( C_c \) 
and whose intersection with a face \( C_d \) of \( C_c \) is just \( M_d \). The monoidal complex naturally 
associated to a single affine monoid \( M \) is simply denoted by \( M \); it is supported on the 
conical complex formed by the faces of the cone \( \mathbb{R}_+ M \).

The simplest examples of conical complexes are those associated with rational fans \( \mathcal{F} \). 
For each cone \( C \in \mathcal{F} \) we choose \( M_C = C \cap \mathbb{Z}^d \). These monoids are finitely generated by 
Gordan’s lemma. Moreover, they are normal: recall that an affine monoid \( M \) is normal if 
\( M = \text{gp}(M) \cap \mathbb{R}_+ M \).

**Remark 2.1.** Let \( \text{gp}(M) \) denote the group of differences of a monoid \( M \). The groups 
\( \text{gp}(M_C) \) of the monoids in a monoidal complex associated with a fan again a monoidal 
complex in a natural way since \( \text{gp}(M_D) = \text{gp}(M_C) \cap \mathbb{R} D \) if \( D \) is a face of \( C \).

In general the compatibility condition between the passage to faces and the formation 
of groups of differences need not be satisfied. Nevertheless, the rational structures defined 
by the monoids \( M_c \), namely the rational subspaces \( \mathbb{Q} \text{gp}(M_c) \) of \( \mathbb{R}^d \) are compatible with 
the passage to faces. This follows from condition (ii): both monoids \( \text{gp}(M_c) \cap \mathbb{R} C_d \) and 
\( \text{gp}(M_d) \) are contained in \( \mathbb{Z}^d \) and have the same rank \( \delta_d \).

Note that the monoids \( M_c \) form a direct system of sets with respect to the embeddings 
\( \pi_d^{-1} \circ \pi_c : M_c \to M_d \) where \( c, d \in \Sigma \) and \( c \subseteq d \). We set 
\[ \mathcal{M} = \lim_{\rightarrow} M_c . \]
In general, there exists no global monoid structure on \( \mathcal{M} \), but it carries a partial monoid 
structure since we can consider each monoid \( M_c \) as a subset of \( \mathcal{M} \) in the natural way. 
Whenever there exists \( c \in \Sigma \) such that \( a, b \in M_c \) then \( a + b \) is their sum in \( M_c \), and as an 
element of \( \mathcal{M} \) the sum is independent of the choice of \( c \).

Next we choose a field \( K \) and define the toric face ring \( K[\mathcal{M}] \) of \( \mathcal{M} \) (over \( K \)) as follows. 
As a \( K \)-vector space let 
\[ K[\mathcal{M}] = \bigoplus_{a \in \mathcal{M}} K t^a . \]
We set 
\[ t^a \cdot t^b = \begin{cases} t^{a + b} & \text{if } a, b \in M_c \text{ for some } c \in \Sigma, \\ 0 & \text{otherwise}. \end{cases} \]
Multiplication in \( K[\mathcal{M}] \) is defined as the \( K \)-bilinear extension of this product. It turns 
\( K[\mathcal{M}] \) into a \( K \)-algebra. In the following, the elements of \( \mathcal{M} \) are called monomials.

There exist at least two other natural descriptions of toric face rings of a monoidal 
complex. The first is a realization as an inverse limit of the affine monoid rings \( K[M_c] \), 
\( c \in \Sigma \). For \( c \in \Sigma \) and a face \( d \) of \( c \) there exists a natural projection map 
\( K[M_c] \rightarrow K[M_d] \) which sends monomials \( t^a \) to zero if \( a \notin M_d \), the face projection map. With respect to 
these maps we may consider the inverse limit \( \lim_{\leftarrow} K[M_c] \): 

**Proposition 2.2.** Let \( \mathcal{M} \) be a monoidal complex supported on a conical complex \( \Sigma \). Then 
\[ K[\mathcal{M}] \cong \lim_{\leftarrow} K[M_c] . \]
For the proof of the proposition we introduce some more notation. Let \( c \in \Sigma \) and let \( p_c \) be the ideal of \( K[\mathcal{M}] \) which is generated by all monomials \( t^a \) with \( a \notin M_c \). Then there is a natural isomorphism of \( K \)-algebras \( K[M_c] \cong K[\mathcal{M}] / p_c \). In particular, \( p_c \) is a prime ideal. Moreover, if \( d \subset c, c, d \in \Sigma \), then the natural epimorphism \( K[\mathcal{M}] / p_c \to K[\mathcal{M}] / p_d \) coincides with the map induced from the projection map \( K[M_c] \to K[M_d] \), and we identify these maps in the following.

**Proof of Proposition 2.2.** Observe that each of the ideals \( p_c \) has a \( K \)-basis consisting of monomials of \( K[\mathcal{M}] \). Therefore the following equations are satisfied for \( c, d, e \in \Sigma \):

(i) \( p_c + p_d = p_{c \cap d} \),
(ii) \( p_c \cap (p_d + p_e) = p_c \cap p_d + p_c \cap p_e \),
(iii) \( p_c \cap p_d \cap p_e = p_c \cap p_d + p_c \cap p_e \) for all \( i, j, k \).

Now it follows easily that \( \lim_{c \in \Gamma} K[\mathcal{M}] / p_c \) is isomorphic to \( K[\mathcal{M}] / \bigcap_{c \in \Gamma} p_c \) (for example, see Example 3.3 in [1]). But \( \bigcap_{c \in \Gamma} p_c = 0 \), and so

\[
\lim_{c \in \Gamma} K[M_c] \cong \lim_{c \in \Gamma} K[\mathcal{M}] / p_c \cong K[\mathcal{M}] / \bigcap_{c \in \Gamma} p_c \cong K[\mathcal{M}].
\]

Second, we want to describe a toric face ring as a quotient of a polynomial ring. It is not difficult to compute the defining ideal of such a presentation. In view of Theorem 3.4 below we have to consider elements of \( K[\mathcal{M}] \) that are either monomials \( t^a, a \in \mathcal{M} \), or 0. For a uniform notation we augment \( |\mathcal{M}| \) by an element \(-\infty\) and set \( t^{-\infty} = 0 \).

**Proposition 2.3.** Let \( \mathcal{M} \) be a monoidal complex supported on a conical complex \( \Sigma \), and let \( (a_e)_{e \in E} \) be a family of elements of \( |\mathcal{M}| \cup \{-\infty\} \) generating \( K[\mathcal{M}] \) as an \( K \)-algebra. (Equivalently, \( \{a_e : e \in E\} \cap M_c \) generates \( M_c \) for each \( c \in \Sigma \).) Then the kernel \( I_{\mathcal{M}} \) of the surjection

\[
\phi : K[X_e : e \in E] \to K[\mathcal{M}], \quad \phi(X_e) = t^{a_e},
\]

is generated by

(i) all monomials \( \prod_{h \in H} X_h \) where \( H \) is a subset of \( E \) for which \( \{a_h : h \in H\} \) is not contained in any monoid \( M_c \), \( c \in \Gamma \), and
(ii) all binomials \( \prod_{g \in G} X^i_g - \prod_{h \in H} X^j_h \) where \( G, H \subset E \), all \( a_g, a_h \) are contained in a monoid \( M_c \) for some \( c \in \Sigma \), and \( \sum_{g \in G} i_g a_g = \sum_{h \in H} j_h a_h \).

Moreover, the monomials in (i) are all monomials contained in \( I_{\mathcal{M}} \). A binomial is contained in \( I_{\mathcal{M}} \) if either both its monomials are contained in the family of monomials given in (i), or it is in the list of the binomials in (ii).

**Proof.** It is clear that \( I_{\mathcal{M}} \) contains the ideal \( J \) generated by all the monomials and binomials listed in (i) and (ii).

For the converse, let \( f \) be a polynomial such that \( \phi(f) = 0 \). Then we can assume that all monomials of \( f \) map to elements of \( |\mathcal{M}| \) since all other monomials belong to \( I_{\mathcal{M}} \). Now let \( c \in \Sigma \), and define \( f_c \) to be the polynomial that arises as the sum of those terms of \( f \) whose monomials are mapped to elements of \( M_c \subset |\mathcal{M}| \). Then \( \phi(f_c) = 0 \) as well. It is well-known and easy to show that \( f_c \) then belongs to the ideal in \( K[X_e : e \in E] \) generated by all those binomials in (ii) for which \( a_g, a_h \in M_c \). (Equivalently, the binomials in (ii) for \( M_c \) generate the presentation ideal of \( K[M_c] \) over a polynomial ring in the variables \( X_e \).)
where \( e \in E \) and \( a_e \in M_c \). Therefore we may replace \( f \) by \( f - f_c \), and finish the proof by induction on the number of terms of \( f \).

It is clear that a monomial belongs to \( I_M \) if and only if it is contained in the family of monomials given in (i). If a binomial is an element of \( I_M \), then either both monomials belong to this ideal, or none of the monomials. In the latter case it must be one of the binomials of the family of binomials given in (ii), since no other binomials belong to the kernel of the map \( K[X_e : e \in E] \to K[M] \). This follows directly from the construction of the ring \( K[M] \).

\[ \square \]

**Example 2.4.**

(i) Let \( \mathcal{F} \) be a rational fan in \( \mathbb{R}^d \), and let \( \mathcal{M} \) be the conical complex associated with it. Then the algebra \( K[M] \) is the toric face ring introduced by Stanley [12].

(ii) Let \( \Delta \) be an abstract simplicial complex on the vertex set \( [n] = \{1, \ldots, n\} \). Then \( \Delta \) has a geometric realization by considering the simplices \( \text{conv}(e_{i_1}, \ldots, e_{i_m}) \) such that \( \{i_1, \ldots, i_m\} \) belongs to \( \Delta \) (here \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \)). The cones over the faces of the geometric realization form a fan \( \mathcal{F} \), and its toric face ring \( R \) given by (i) is nothing but the Stanley-Reisner ring of \( \Delta \). In fact, according to Proposition 2.3, the kernel of the natural epimorphism \( K[X_1, \ldots, X_n] \to R \) is generated by those monomials \( X_{j_1} \cdots X_{j_r} \) such that \( \{j_1, \ldots, j_r\} \notin \Delta \).

Algebras associated with monoidal complexes therefore generalize Stanley-Reisner rings by allowing arbitrary conical complexes as their combinatorial skeleton and, consequently, monoid algebras as their ring-theoretic flesh.

(iii) The polyhedral algebras of [5] are another special case of the algebras associated with monoidal complexes. For them the cones are generated by lattice polytopes and the monoids are the polytopal monoids considered on [5].

**Remark 2.5.** In [2] toric face rings were defined by their presentation ideals given in Proposition 2.3. Thus Proposition 2.2 is equivalent to [2, Theorem 4.7]. In [2] the generators of the affine monoids \( M_c \) for \( c \in \Sigma \) were fixed right in the beginning while in this paper we fix only the monoids and are free to choose generators whenever we like. The new approach leads directly to the natural description of the toric face rings. Using arguments as above (e.g. the prime ideals as in [2,2]) one obtains alternative and slightly more compact proofs than those in [2].

We have already used the fact that the zero ideal of \( K[M] \) is the intersection of the prime ideals \( p_c \). This implies that \( K[M] \) is reduced.

Let \( \Sigma \) be a conical complex. A conical complex \( \Gamma \) is a subdivision of \( \Sigma \) if \( |\Gamma| = |\Sigma| \) and each face \( c \in \Sigma \) is the union of faces \( d \in \Gamma \). The subdivision is called a triangulation if \( \Gamma \) is simplicial. We call a subdivision \( \Gamma \) rational, if all cones \( C_d, d \in \Gamma \) are rational.

Suppose that \( \Gamma \) is a subdivision of \( \Sigma \), let \( \mathcal{M} \) be a monoidal complex supported by \( \Gamma \), and \( c \) a face of \( \Sigma \). In the situation of Proposition 2.3 for the toric face ring \( K[M] \) we let \( S_c \) be the polynomial subring of \( S = K[X_e : e \in E] \) generated by those \( X_e \) for which \( a_e \in C_c \). Furthermore let \( \mathcal{M}_c \) be the monoidal subcomplex of \( \mathcal{M} \) consisting of all faces \( D_d \) of \( \Gamma \), \( d \subset c \), and their associated monoids.
Since $\mathcal{M}_c$ is a monoidal subcomplex, one has a natural epimorphism $K[\mathcal{M}] \to K[\mathcal{M}_c]$, generalizing the face projection. It is given by $t^a \mapsto t^a$ whenever $a \in C_c$, and $t^a \mapsto 0$ otherwise.

But we have also an embedding $K[\mathcal{M}_c] \to K[\mathcal{M}]$, since points of $|\mathcal{M}_c|$ that are contained in a face of $\Gamma$, are also contained in a face of $\mathcal{M}_c$.

In order to encode the incidence structure of $\Sigma$ we let $A_\Sigma$ denote the ideal in $S$ generated by the squarefree monomials $\prod_{h \in H} X^h$ for which $\{a_h : h \in H\}$ is not contained in a face of $\Sigma$.

**Proposition 2.6.** With the notation introduced in Proposition 2.3 we have:

(i) The embedding $K[\mathcal{M}_c] \to K[\mathcal{M}]$ is a section of the projection $K[\mathcal{M}] \to K[\mathcal{M}_c]$, and thus makes $K[\mathcal{M}_c]$ a retract of $K[\mathcal{M}]$.  
(ii) Let $c_1, \ldots, c_n$ be the facets of $\Sigma$, and set $\mathcal{M}_i = \mathcal{M}_{c_i}$. Then

$$I_{\mathcal{M}} = A_\Sigma + SI_{\mathcal{M}_1} + \cdots + SI_{\mathcal{M}_n}.$$  
Moreover, for each face $c \in \Sigma$ we have $I_{\mathcal{M}_c} = S_c \cap I_{\mathcal{M}}$.

**Proof.** Part (i) is evident, and the representation of $I_{\mathcal{M}}$ in part (ii) follows immediately from Proposition 2.3: none of the binomial relations is lost on the right hand side, and it contains also all the monomial relations because these are either contained in one of the $I_{\mathcal{M}_i}$ or in $A_\Sigma$. The equation $I_{\mathcal{M}_c} = S_c \cap I_{\mathcal{M}}$ restates part (i), lifted to the presentations of the algebras. \qed

In particular we can apply Proposition 2.6 in the case $\Gamma = \Sigma$.

### 3. Toric Face Rings and Initial Ideals

Next we want to compute initial ideals of the presentation ideals of monoidal complexes considered in Proposition 2.3. Recall that a **weight vector** for a polynomial ring $S = K[X_1, \ldots, X_n]$ is an element $w \in \mathbb{N}^n$ where $\mathbb{N}$ denotes the set of non-negative integers. Given this vector we assign $X_i$ the weight $w_i$. It is easy to see that this is equivalent to endow $S$ with a positive $\mathbb{Z}$-grading under which the monomials are homogeneous. Thus the whole terminology of graded rings (with the prefix $w$) can be applied. In particular, we can speak of the $w$-degree of a monomial; it is defined by

$$\deg_w X^a = \sum_{i=1}^n a_i w_i = a \cdot w.$$  

A weight vector $w$ determines a **weight (pre-)order** if one sets

$$X^a \leq_w X^b \iff a \cdot w \leq b \cdot w.$$  

The only axiom of a monomial order (as considered below) not satisfied is antisymmetry: for $n > 1$ there always exist distinct monomials $X^a$ and $X^b$ such that simultaneously $X^a \leq_w X^b$ and $X^b \leq_w X^a$.

The **$w$-initial component** $\text{in}_w(f)$ of a polynomial $f$ is simply its $w$-homogeneous component of highest degree. Let $V \subseteq S$ be a subspace. Then the **$w$-initial subspace** $\text{in}_w(V)$ is the subspace generated by the polynomials $\text{in}_w(f)$, $f \in V$. Observe that for an ideal $I \subseteq S$ the $w$-initial subspace $\text{in}_w(I)$ is again an ideal of $S$. Now well-known results for monomial
orders (see below) hold also for weight orders. E.g. for subspaces $V_1 \subseteq V_2 \subseteq S$ we have $\text{in}_w(V_1) = \text{in}_w(V_2)$ if and only if $V_1 = V_2$.

A monomial order $<$ on $S$ is a total order of the monomials of $S$ such that $1 < X^a$ for all monomials $X^a$, and $X^a < X^b$ implies $X^{a+c} < X^{b+c}$ for all monomials $X^a$, $X^b$, $X^c$. Now we can speak similarly of initial terms $\text{in}_<(f)$ and initial subspaces $\text{in}_<(V)$ with respect to $<$. Recall that a Gröbner basis of $I$ is a set of elements of $I$ whose initial monomials generate $\text{in}_<(I)$. Such a set always exists and then also generates $I$.

It is an important fact that a monomial order can always be approximated by a weight order if only finitely many monomials are concerned: for an ideal $I$ of $S$ there exists a weight vector $w \in \mathbb{N}^n$ such that $\text{in}_<(I) = \text{in}_w(I)$. Conversely, given a weight vector $w \in \mathbb{N}^n$ and a monomial order $< $ we can refine the weight order $<_w$ to a monomial order $< $ by setting $X^a < X^b$ if either $a \cdot w < b \cdot w$, or $a \cdot w = b \cdot w$ and $X^a < X^b$. Observe also that the $w$-initial terms of a Gröbner basis of $I$ with respect to $< $ generate $\text{in}_w(I)$. For more details and general results on weight orders and monomial orders we refer to [4] or [15].

The ideal given in Proposition 2.3 has a special structure. It is generated by monomials and binomials. This property persists in the passage to initial ideals.

**Lemma 3.1.** Let $I \subset K[X_1, \ldots, X_n]$ be an ideal generated by monomials and binomials and $w \in \mathbb{N}^n$ a weight vector. Then $\text{in}_w(I)$ is generated by the monomials and the initial components of the binomials in $I$.

**Proof.** We refine the weight order to a monomial order $<$. Using the Buchberger algorithm to compute a Gröbner basis for $I$, one enlarges the given set of generators of $I$ consisting of monomials and binomials only by more monomials and binomials. The corresponding initial components with respect to the weight order $<_w$ then generate $\text{in}_w(I)$. \hfill \Box

It is a useful consequence of Lemma 3.1 that the decomposition of the ideal $I_{\mathcal{M}}$ in Proposition 2.6 is passed onto their initial ideals. We need it only for the trivial subdivision of $\Sigma$ by itself, but it can easily be generalized to the setting of Proposition 2.6. (Also see [3] Theorem 5.9 for a related result.)

**Proposition 3.2.** Consider the presentation of $K[\mathcal{M}]$ as a residue class ring of $S = K[X_e : e \in E]$ as in Proposition 2.3, a weight vector $w$ on $S$ and the induced weight vectors for the subalgebras $S_c = K[X_e : a_e \in M_c]$, $c \in \Sigma$. Then

\[ \text{in}_w(I_{e}) = A_{\mathcal{M}} + S \cdot \text{in}_w(I_{M_1}) + \cdots + S \cdot \text{in}_w(I_{M_n}) \]

where again $c_1, \ldots, c_n$ are the facets of $\Sigma$, and $M_i = M_{C_i}$. Moreover, $\text{in}_w(I_{M_c}) = S_c \cap \text{in}_w(I_{\mathcal{M}})$ for all $c \in \Sigma$.

**Proof.** It is clear that the right hand side is contained in $\text{in}_w(I_{e})$. For the converse inclusion it is enough to consider the system of generators of $I_{\mathcal{M}}$ described in Proposition 3.1 and there is nothing to say about the monomials in $I_{\mathcal{M}}$. Let $f$ be the initial component of a binomial $g$ in $I_{\mathcal{M}}$. According to Proposition 2.3 there are two cases: (1) $g$ belongs to $A_{\mathcal{M}}$, then so does $f$. (2) $g \in I_{M_i}$ for some $i$; then $f \in \text{in}_w(I_{M_i})$, and we are done with the decomposition of $\text{in}_w(I_{e})$.

The equality $\text{in}_w(I_{M_c}) = S_c \cap \text{in}_w(I_{\mathcal{M}})$ is left to the reader. It is easily derived from Proposition 2.3 and Proposition 3.1. \hfill \Box
Recall that a function \( f : X \to \mathbb{R} \) on a convex set \( X \) is called \textit{convex} if \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) for all \( x, y \in X \) and \( t \in [0,1] \). A function \( f : [\Sigma] \to \mathbb{R} \) on a conical complex \( \Sigma \) is called \textit{convex} if it is convex on all the cones \( C_c \) for \( c \in \Sigma \). For a function \( f : [\Sigma] \to \mathbb{R} \) a connected subset \( W \) of a facet \( C_c \) of \( \Sigma \) is a \textit{domain of linearity} if it is maximal with respect to the following property: \( g|_W \) can be extended to an affine function on \( \mathbb{R} C_c \).

Now a subdivision \( \Gamma \) of a conical complex \( \Sigma \) is said to be \textit{regular}, if there exists a convex function \( f : [\Sigma] \to \mathbb{R} \) whose domains of linearity are facets of \( \Gamma \). Such a function is called a \textit{support function} for the subdivision \( \Gamma \).

Let \( (a_e)_{e \in E} \) be a family of elements of \( |\mathcal{M}| \) such that \( \{a_e : e \in E\} \cap M_c \) generates \( M_c \) for each \( c \in \Sigma \). Now we choose a polynomial ring \( S = K[X_e : e \in E] \) and define the surjective homomorphism \( \varphi : K[X_e : e \in E] \to K[\mathcal{M}] \) which maps \( X_e \) to \( t^{a_e} \) as considered in Proposition 2.3. Let \( w = (w_e)_{e \in E} \) be a weight vector for \( S \).

On the one hand, the weight vector \( w \) determines initial ideals, especially the initial ideal \( \mathfrak{in}_w(I, \mathcal{M}) \). On the other hand, \( w \) determines also a conical subdivision \( \Gamma_w \) of the conical complex \( \Sigma \) as follows. First, every cone \( C_c \subseteq \mathbb{R}^\delta_c \) and the weight vector \( w \) define the cone

\[
C'_c = \mathbb{R}_+((a_e, w_e) : e \in E \text{ such that } a_e \in C_c) \subseteq \mathbb{R}^{\delta_c+1}.
\]

The projection on the first \( \delta_c \) coordinates maps \( C'_c \) onto \( C_c \). The \textit{bottom} of \( C'_c \) with respect to \( C_c \) consists of all points \( (a, h_a) \in C'_c \) such that the line segment \( [(a,0),(a,h_a)] \) intersects \( C'_c \) only in \( (a,h_a) \). In other words, \( h_a = \min\{h' : (a,h') \in C'_c\} \). Clearly \( h_a > 0 \) for all \( a \in C_c, a \neq 0 \). The bottom is a subcomplex of the boundary of \( C'_c \) (or \( C_c \) itself). Note that its projection onto \( C_c \) defines a conical subdivision of the cone \( C_c \). Second, the collection of these conical subdivisions of the cones \( C_c \) constitutes a conical subdivision of \( \Sigma \).

Now we show that this subdivision is regular as defined above. To this end we define the function \( \text{ht}_w : [\Sigma] \to \mathbb{R} \) as follows. For \( a \in [\Sigma] \) there exists a minimal face \( c \in \Sigma \) such that \( a \in C_c \). Construct \( C'_c \) as above using the weight vector \( w \). Then we define

\[
\text{ht}_w(a) = \min\{h' \in \mathbb{R} : (a,h') \in C'_c\},
\]

i.e. \( \text{ht}_w(a) \) is the unique vector in the bottom of \( C'_c \) which is projected on \( C_c \) via the projection map on the first \( \delta_c \) coordinates.

**Proposition 3.3.** Let \( \mathcal{M} \) be a monoidal complex supported on a conical complex \( \Sigma \), let \( (a_e)_{e \in E} \) be a family of elements of \( |\mathcal{M}| \) such that \( \{a_e : e \in E\} \cap M_c \) generates \( M_c \) for each \( c \in \Sigma \), and let \( w = (w_e)_{e \in E} \) be a weight vector. Then:

(i) For \( c \in \Sigma \), \( b_1, \ldots, b_m \in C_c \) and \( \alpha_i > 0 \), \( i = 1, \ldots, m \), we have

\[
\text{ht}_w\left(\sum_{i=1}^{m} \alpha_i b_i\right) \leq \sum_{i=1}^{m} \alpha_i \text{ht}_w(b_i).
\]

In particular, \( \text{ht}_w \) is a convex function on \( [\Sigma] \).

(ii) Its domains of linearity are the cones \( D_d \) for facets \( d \) of \( \Gamma_w \), i.e. equality holds in \( (i) \) if and only if there exists a facet of \( \Gamma_w \) containing \( b_1, \ldots, b_m \).

Therefore \( \Gamma_w \) is a regular subdivision of \( \Sigma \).

Part (i) uses only the definition of \( \text{ht} \) and that the cones \( C'_c \) are closed under \( \mathbb{R}_+ \)-linear combinations, and part (ii) reflects the fact that an \( \mathbb{R}_+ \)-linear combination of points in
the boundary of a cone $C$ lies in the boundary if and only if all points (with nonzero coefficients) belong to a facet of $C$. (Also see [6, Lemma 7.16].)

Since the weights $w_e$ are positive, the cone $C'_e$ are pointed, even if $C_e$ is not. Thus all faces of $\Gamma_w$ are pointed, too.

For each $D_d$ with $d \in \Gamma_w$ we let $N_{d,w}$ be the monoid generated by all $a_e \in D_d$ for which $\text{ht}_w(a_e) = w_e$. The cones $D_d$ and the monoids $N_{d,w}$ form a monoidal complex $\mathcal{M}_w = \mathcal{M}_{\Gamma_w}$ supported by the conical complex $\Gamma_w$, the monoidal complex defined by $w$. Observe that each extreme ray of a cone $D_d$ of $\Gamma_w$ is the image of an extreme ray of $C'_c$ for some $c \in \Sigma$. The latter contains a point $(a_e, w_e)$, and therefore $w_e = \text{ht}_w(a_e)$. This implies $D_d = \mathbb{R}_+ N_{d,w}$. The remaining conditions for a monoidal complex are fulfilled as well. It is important to note that the monoidal complex $\mathcal{M}_w$ is not only dependent on $\Gamma_w$ or on the pair $(\Gamma_w, E)$, but also on the chosen weight $w$.

The algebra $K[\mathcal{M}_w]$ is again a residue class ring of the polynomial ring $K[X_e : e \in E]$ under the assignment

$$X_e \mapsto \begin{cases} t^{a_e} & \text{if } a_e \in \mathcal{M}_w, \\ 0 & \text{else.}\end{cases}$$

The kernel of this epimorphism is denoted by $J_{\mathcal{M}_w}$. It is of course just the presentation ideal of the toric face ring $K[\mathcal{M}_w]$ supported by the conical complex $\Gamma_w$ (and here we must allow that indeterminates $X_e$ go to 0).

One cannot expect that $\text{in}_w(I_{\mathcal{M}}) = J_{\mathcal{M}_w}$ since $J_{\mathcal{M}_w}$ is always a radical ideal, but $\text{in}_w(I_{\mathcal{M}})$ need not be radical. However, this is the only obstruction. The next theorem generalizes a result of Sturmfels (see [14] and [15]) who proved it in the case that conical complex is induced from a single monoid and that the subdivision $\Gamma_w$ is a triangulation. It is essentially equivalent to [2, Theorem 5.11]. See Remark 2.5 for the difference of the two approaches.

**Theorem 3.4.** Let $\mathcal{M}$ be a monoidal complex supported on a conical complex $\Sigma$, let $(a_e)_{e \in E}$ be a family of elements of $\mathcal{M}|$, such that $\{a_e : e \in E\} \cap M_c$ generates $M_c$ for each $c \in \Sigma$, and let $w = (w_e)_{e \in E}$ be a weight vector. Moreover, let $\mathcal{M}_w$ be the monoidal complex defined by $w$. Then the ideal $J_{\mathcal{M}_w}$ is the radical of the initial ideal $\text{in}_w(I_{\mathcal{M}})$.

**Proof.** For a single monoid the theorem is [6, Theorem 7.18], and we reduce the general case to it.

As remarked above, the ideal $J_{\mathcal{M}_w}$ is the presentation ideal of a toric face ring by construction. The underlying complex is $\Gamma_w$, a subdivision of $\Sigma$. We apply Proposition 2.6 to this subdivision of $\Sigma$ and the facets of $\Sigma$. The latter correspond to single monoids $M_1, \ldots, M_n$. Thus

$$J_{\mathcal{M}_w} = A_\Sigma + J_{(M_1)_w} + \cdots + J_{(M_n)_w}. \tag{2}$$

By [6, Theorem 7.18] we have $J_{(M_1)_w} = \text{Rad} \text{in}_w(I_{M_1})$, and therefore

$$J_{\mathcal{M}_w} = A_\Sigma + \text{Rad} S \cdot \text{in}_w(I_{M_1}) + \cdots + \text{Rad} S \cdot \text{in}_w(I_{M_n}).$$

The right hand side is certainly contained in $\text{Rad} \text{in}_w(I_{\mathcal{M}})$, and contains $\text{in}_w(I_{\mathcal{M}})$ by Proposition 3.2 Since $J_{\mathcal{M}_w}$ is a radical ideal, we are done. □
Because of the equation $\text{Rad} \, \text{in}_w(I_\mathcal{M}) = J_\mathcal{M}_w$, we have always the inclusion $\text{in}_w(I_\mathcal{M}) \subseteq J_\mathcal{M}_w$. It is a natural question to characterize the cases in which we have equality. It holds exactly when the monoids $N_{d,w}$ are determined by their cones:

**Corollary 3.5.** With the hypotheses of Theorem 3.4, the following statements are equivalent:

(i) $\text{in}_w(I_\mathcal{M})$ is a radical ideal;

(ii) For all facets $d \in \Gamma_w$ one has $N_{d,w} = M_c \cap D_d$ where $c \in \Sigma$ is the smallest face such that $d \subseteq c$.

**Proof.** Condition (ii) evidently depends only on the facets of $\Sigma$, but this holds for (i) likewise. The equality $J_\mathcal{M}_w = \text{in}_w(I_\mathcal{M})$ is passed to the facets, since we obtain the corresponding ideals for the facets $c_i$ by intersection with $S_{c_i}$, and in the converse direction we use equation (2) and Proposition 3.2. Therefore it is enough to consider the case of a single cone, in which the corollary is part of [6, Corollary 7.20].

Before presenting another corollary we have to characterize the cases in which $\text{in}_w(I_\mathcal{M})$ is a monomial ideal. We say that a monoidal complex is free if all its monoids are free commutative monoids. Evidently this implies that the associated conical complex is simplicial, but not conversely. The free monoidal complexes are exactly those derived from abstract simplicial complexes (compare Example 2.4 (ii)). We note the following obvious consequence of Theorem 3.4:

**Lemma 3.6.** Under the hypothesis of Theorem 3.4 the following statements are equivalent:

(i) $\text{Rad} \, \text{in}_w(I_\mathcal{M})$ is a (squarefree) monomial ideal;

(ii) $\mathcal{M}_w$ is a free monoidal complex.

In particular, if these equivalent conditions hold, then $\Gamma_w$ is a regular triangulation of $\Sigma$.

For the next result we recall the definition of unimodular cones. Let $L \subseteq \mathbb{R}^d$ be a lattice, i.e. $L$ is a subgroup of $\mathbb{R}^d$ generated by $\mathbb{R}$-linearly independent elements, and we assume that $L \subseteq \mathbb{Q}^d$. Let $C \subseteq \mathbb{R}^d$ be a rational pointed cone. Since for each extreme ray $R$ of $C$ the monoid $R \cap L$ is normal and of rank 1, there exists a unique generator $e$ of this monoid. We call these generators the extreme generators $C$ with respect to $L$. If $C$ is simplicial, then we call $C$ unimodular with respect to $L$ if the sublattice of $L$ generated by the extreme generators of $C$ with respect to $L$ generate a direct summand of $L$.

**Theorem 3.7.** With the same assumptions as in Theorem 3.4, the following statements are equivalent:

(i) The ideal $\text{in}_w(I_\mathcal{M})$ is a monomial radical ideal;

(ii) The conical complex $\Gamma_w$ is a triangulation of $\Sigma$, the extreme generators of a cone $D_d$ for $d \in \Gamma_w$ with respect to $\text{gp}(M_c)$ generate the monoid $N_{d,w}$, and $D_d$ is unimodular with respect to $\text{gp}(M_c)$.

**Proof.** It follows from 3.5 and 3.6 that $\text{in}_w(I_\mathcal{M})$ is a monomial radical ideal if and only if:

(a) $\mathcal{M}_w$ is free.

(b) For a facet $d \in \Gamma_w$ let $c \in \Sigma$ be the smallest face such that $d \subseteq c$. Then $N_{d,w} = M_c \cap D_d$. 

It remains to show the equivalence of (a) and (b) to (ii). But both sides of this equivalence depend only on the single monoids \( M_c \) and the restrictions of \( \mathcal{M}_w \) to them. In the case of a single monoid the theorem is part of [6, Corollary 7.20].

Now we can give a nice criterion for the normality of the monoids in a monoidal complex in terms of an initial ideal with respect to a weight vector.

**Theorem 3.8.** The following statements are equivalent:

(i) All monoids \( M_c \) of the monoidal complex \( \mathcal{M} \) are normal.

(ii) There exists a family of elements \((a_e)_{e \in \mathcal{E}} \) of \( |\mathcal{M}| \) such that \( \{a_e : a_e \in M_c\} \) generates \( M_c \) for each monoid \( M_c \) of \( \mathcal{M} \) and a weight vector \( w = (w_e)_{e \in \mathcal{E}} \) such that \( \text{in}_w(I_{\mathcal{M}}) \) is a monomial radical ideal.

**Proof.** (ii) implies (i) is again reduced to the case of a single monoid \( M \) by Proposition 3.2. In this case (ii) implies that \( M \) is the union of free monoids with the same group as \( M \). Then the normality of \( M \) follows immediately.

For the converse we have to construct a regular unimodular triangulation \( \Gamma \) of \( \mathbb{R}_+M \) by elements of \( M \), which we choose as a system of generators. The weight of \( X_e \) is then chosen as the value of the support function of the triangulation at \( a_e \).

The existence of such a triangulation is a standard result. E. g. see [6, Theorem 2.70] where it is stated for a single monoid \( M \). The construction goes through for monomial complexes as well (and the proof implicitly makes use of this fact). However, there is one subtle point to be taken into account: if \( M \) is normal and \( F \) is a face of the cone \( \mathbb{R}_+M \), then \( \text{gp}(M \cap F) = \text{gp}(M) \cap \mathbb{R}F \). This condition ensures that the groups \( \text{gp}(M_c) \) form again a monoidal complex, and that unimodularity of a free submonoid does not depend on the monoid \( M_c \) in which it is considered.

In the investigation of a normal monoid \( M \) one is usually not interested in an arbitrary system of generators of \( M \), but in \( \text{Hilb}(M) \). It is well-known that one can not always find a (regular) unimodular triangulation by elements of \( \text{Hilb}(M) \), and this limits the value of results like Theorem 3.8 considerably. Nevertheless, it is very powerful when the unimodularity of certain triangulations is given automatically.

Theorem 3.8 can be used to prove that monoid algebras of normal affine monoids are Cohen-Macaulay. This result is due to Hochster [8].

**Corollary 3.9.** Let \( M \) be a normal affine monoid. Then the monoid algebra \( K[M] \) is Cohen-Macaulay for every field \( K \).

**Proof.** We may assume that \( M \) is positive. In fact, \( M = U(M) \oplus M' \) where \( U(M) \) is the group of units of \( M \) and \( M' \) is a normal affine monoid which is positive. Moreover, \( K[M] \) is a Laurent polynomial extension of \( K[M'] \) and thus we may replace \( M \) by \( M' \).

It follows from Theorem 3.8 that there exists a system of generators \((a_e)_{e \in \mathcal{E}} \) of \( M \) and a weight vector \( w = (w_e)_{e \in \mathcal{E}} \) such that \( K[M] = S/I_M \) where \( S = K[X_e : e \in \mathcal{E}] \) and \( \text{in}_w(I_M) \) is a monomial radical ideal. Thus \( \text{in}_w(I_M) = I_\Delta \) for an abstract simplicial complex \( \Delta \) on the vertex set \( E \). Now standard results from Gröbner basis theory yield that \( K[M] \) is Cohen-Macaulay if the Stanley-Reisner ring \( K[\Delta] = S/I_\Delta \) is Cohen-Macaulay.

Observe that \( \Delta \) is a triangulation of a cross section of \( \mathbb{R}_+M \). Now one can use e. g. a theorem of Munkres [7, 5.4.6] which states that the Cohen-Macaulay property of \( K[\Delta] \)
only depends on the topological type of $|\Delta|$. A cross-section of a pointed cone is homeomorphic to a simplex whose Stanley-Reisner ring is certainly Cohen-Macaulay. 

4. Betti Numbers of Toric Face Rings

A consequence of Proposition 2.3 is a presentation of a toric face ring $K[\mathcal{M}]$ over a polynomial ring $S$. It is a natural question to determine the Betti numbers of $K[\mathcal{M}]$ over $S$, and the graded Betti numbers if there exists a natural grading. The first question is of course which grading is a natural one to consider. Recall that in general $|\mathcal{M}|$ has only a partial monoid structure and cannot be used directly. But even if $\Sigma$ is a fan in $\mathbb{R}^d$ and the monoids in $\mathcal{M}$ are embedded in $\mathbb{Z}^d$, then $\mathbb{Z}^d$ may not be the best choice to start with.

At first we recall a few facts from graded homological algebra. Let $H$ be an (additive) commutative monoid which is positive, i.e. $H$ has no invertible elements except 0. Usually one defines graded structures on rings and modules via groups. If $H$ is cancellative, i.e. if $a + b = a + c$ implies $b = c$ for $a, b, c \in H$, then $H$ can be naturally embedded into the abelian (Grothendieck) group $G$ of $H$. Therefore one can defines terms like $H$-graded by considering $G$-graded objects whose homogeneous components with degrees not in $H$ are zero. But we will have to consider noncancellative monoids, and thus it may be impossible to embed $H$ into a group.

Hence we introduce $H$-graded objects directly. Let $R$ be a commutative ring and $M$ an $R$-module (where we as always assume that $R$ is commutative and not trivial). An $H$-grading of $R$ is a decomposition $R = \bigoplus_{h \in H} R_h$ of $R$ as abelian groups such that $R_h \cdot R_g \subseteq R_{h+g}$ for all $h, g \in H$. A graded ring together with an $H$-grading is called an $H$-graded ring. Now assume that $R$ is an $H$-graded ring. A grading of $M$ is a decomposition $M = \bigoplus_{h \in H} M_h$ of $M$ as abelian groups such that $R_h \cdot M_g \subseteq M_{h+g}$ for all $h, g \in H$. An $H$-graded $R$-module $M$ together with an $H$-grading is called an $H$-graded module. $M_h$ is called the $h$-homogeneous component of $M$ and an element $x \in M_h$ is said to be homogeneous of degree $\deg x = h$.

From now on we assume that $R$ is a Noetherian $H$-graded ring. The f. g. $H$-graded $R$-modules build a category. The morphisms are the homogeneous $R$-module homomorphisms $\phi : M \to M'$, i.e. $\phi(M_h) \subseteq M'_h$ for all $h \in H$. For $h \in H$ we let $M(-h)$ be the $H$-graded $R$-module with homogeneous components $M(-h)_g = \bigoplus_{h' \in H, h' + h = h} R_{g-h'} M_{h'}$ for $g \in H$. In particular, $R(-h)$ is a free $R$-module of rank 1 with generator sitting in degree $h$. Since kernels of homogeneous maps of f. g. $H$-graded $R$-modules are again f. g. $H$-graded and there exist f. g. free $H$-graded $R$-modules, every f. g. $H$-graded $R$-module has a free (hence projective) resolution

$$F_* : \cdots \to F_n \to \cdots \to F_0 \to 0$$

where $F_n$ is a finite direct sum of free modules of the form $R(-h)$ for some $h \in H$ and all maps are homogeneous and $R$-linear.

Next we want to pose a condition on $H$ and specialize the considered class of rings. We say that $H$ is cancellative with respect to 0 if $a + b = a$ implies $b = 0$ for $a, b \in H$. Let $K$ be a field. An $H$-graded $K$-algebra $R$ is a Noetherian $K$-algebra $R = \bigoplus_{h \in H} R_h$ with $R_0 = K$. Since $H$ is positive all homogeneous units of $R$ must belong to $R_0$ and $R$ has the unique $H$-graded maximal ideal $m = \bigoplus_{h \in H \setminus \{0\}} R_h$. We see that $R$ is an $H$-graded local
ring, a notion defined in the obvious way. Observe that m is also maximal in R. The ring R behaves like a local ring because of the next lemma.

**Lemma 4.1.** Assume that H is cancellative with respect to 0 and R is an H-graded K-algebra. Then Nakayama’s lemma hold, i.e. if M is a f. g. H-graded module and N \( \subseteq M \) is a f. g. H-graded submodule s.t. \( M = N + mM \), then \( M = N \). In particular, homogeneous elements \( x_1, \ldots, x_n \) are a minimal system of generators of \( M \) if and only if their residue classes are a \( K \)-vector space basis of \( M/mM \) and then we write \( n = \mu(M_m) \).

**Proof.** We may assume without loss of generality that \( N = 0 \). Now let \( x_1, \ldots, x_n \) be a minimal system of generators of homogeneous elements of \( M \). Since \( M = mM \) we have an equation

\[
x_n = \sum_{i=1}^{n} a_i x_i
\]

where \( a_i \in m \). For all homogeneous components \( a_{ij} \) of some \( a_i \) we may without loss of generality assume that \( \deg x_n = \deg a_{ij} + \deg x_i \). Fix a homogeneous components \( a_{in} \) of \( a_n \). Then \( \deg x_n = \deg a_{in} + \deg x_n \) implies \( \deg a_{in} = 0 \) because \( H \) is cancellative with respect to 0. Thus \( a_{in} \in K \cap m \), and so \( a_{in} = 0 \). Hence \( a_n = 0 \) and \( x_n \) is a linear combination of \( x_1, \ldots, x_{n-1} \), in contradiction to the minimality of the system of generators.

**Example 4.2.** Let \( F = \mathbb{N}^n \) for some \( n \geq 0 \), and set \( \deg a = \sum a_i \) for \( a \in F \). Let \( M \) a quotient of \( F \) by a homogeneous congruence, i.e. a congruence in which \( x \sim y \) implies \( \deg x = \deg y \). Then \( M \) is cancellative at 0, but in general it is not cancellative.

Now we can re-prove many well-known results from local ring or \( \mathbb{Z} \)-graded ring theory. (E. g. See [7] Section 1.5.)

For example, let \( x_1, \ldots, x_n \) be a minimal system of generators of \( M \) where \( \deg x_i = h_i \in H \), let \( \varphi : F = \bigoplus_{i=1}^{n} R(-h_i) \to M \) be the homogeneous map sending the generator \( e_i \) of \( R(-h_i) \) to \( x_i \). Then we claim that \( \text{Ker} \varphi \subseteq mF \). Indeed, otherwise it follows that the residue classes of \( x_1, \ldots, x_n \) are not a \( K \)-vector space basis of \( M/mM \) and thus by Nakayama’s lemma \( x_1, \ldots, x_n \) is not minimal system of generators. This is a contradiction. Consequently there exist minimal \( H \)-graded free resolutions, i.e. given a f. g. \( H \)-graded module \( M \), there exists an \( H \)-graded free resolution \( F_\ast \) of \( M \) such that \( \text{Ker} \partial_n \subseteq mF_n \) for all \( n \). If we write \( F_n = \bigoplus_{h \in H} R(-h) \beta_{n,h}^R(M) \), then we call the \( \beta_{n,h}^R(M) \) the \( H \)-graded Betti numbers of \( M \). Up to homogeneous isomorphism of complexes, \( F_\ast \) is uniquely determined by the requirement that \( \text{Ker} \partial_n \subseteq mF_n \) all \( n \). The numbers \( \beta_{n,h}^R(M) \) are also uniquely determined. Indeed, \( \text{Tor}_n^R(M,K) \) is an \( H \)-graded module considered as an \( R \)- or \( K = R/m \)-module and we have \( \dim_h \text{Tor}_n^R(M,K) = \beta_{n,h}^R(M) \) as is easily verified. Some more results in this direction can easily be verified: a f. g. \( H \)-graded \( R \)-module is projective if and only if it is free, \( \text{proj dim} M = \text{proj dim} M_m \) and so forth.

Next we want to apply the theory discussed so far, to the situation of toric face rings. Let \( \mathcal{M} \) be a monoidal complex supported on a pointed conical complex \( \Sigma \), and let \( \{a_e\}_{e \in E} \) be a family of elements of \( |\mathcal{M}| \) such that \( \{a_e : e \in E\} \cap M_c \) generates \( M_c \) for each \( c \in \Sigma \).

According to Proposition 2.3 the defining ideal \( I_{\mathcal{M}} \) of the toric face ring \( K[|\mathcal{M}|] \) of a monoidal complex is a sum

\[
I_{\mathcal{M}} = A_{\mathcal{M}} + B_{\mathcal{M}}
\]
where $A_{\mathcal{M}}$ is an ideal generated by squarefree monomials and $B_{\mathcal{M}}$ is a binomial ideal containing no monomials: This is a consequence of the fact that every binomial generator vanishes on the vector $(1)_{e \in E}$, but a monomial has here value 1.

Recall that a congruence relation on a commutative monoid $M$ is an equivalence relation $\sim$ such that for $a, b, c \in M$ with $a \sim b$ we have $a + c \sim b + c$. Now $M / \sim$ is again a commutative monoid in a natural way.

Consider the free monoid $\mathbb{N}^E$ with generators $f_e$ for $e \in E$. Note that $S = K[X_e : e \in E]$ is the monoid algebra of $\mathbb{N}^E$, the monomials in $S$ are denoted by $X^a = \prod_{e \in E} X_e^{a_e}$. On $\mathbb{N}^E$ we define the congruence relation $a \sim b$ for $a, b \in \mathbb{N}^E$ if and only if $X^a - X^b \in B_{\mathcal{M}}$ is a binomial. We let $H_{\mathcal{M}}$ denote the monoid $\mathbb{N}^E / \sim$. It is not to hard to see and well-known that $S/B_{\mathcal{M}}$ is exactly the monoid algebra of the monoid $H_{\mathcal{M}}$.

Lemma 4.3. $H_{\mathcal{M}}$ is a commutative positive monoid with monoid algebra $S/B_{\mathcal{M}}$.

Proof. It only remains to show that $H_{\mathcal{M}}$ is positive. Let $g, h \in H_{\mathcal{M}}$ for $g, h \in \mathbb{N}^E$ such that $g + h = 0$ and assume that $\overline{g}, \overline{h} \neq 0$. It follows from the definition of $H_{\mathcal{M}}$ that $X^{g+h} - 1 \in B_{\mathcal{M}}$. But $B_{\mathcal{M}}$ is generated by binomials that vanish on the zero vector $(0)_{e \in E}$ because all monoids $M_c$ for $c \in \Sigma$ are positive. The binomial $X^{g+h} - 1$ does not vanish on $(0)$, and this yields a contradiction.

We saw that from the algebraic point of view it is very useful if $H_{\mathcal{M}}$ is cancellative with respect to 0. But it is not strong enough for a combinatorial description of the Betti numbers, as a counterexample will show. The next lemma describes a stronger cancelation property for monoidal complexes associated with fans.

Lemma 4.4. Assume that $\Sigma$ is a rational pointed fan in $\mathbb{R}^n$ such that $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$.

(i) If $\overline{i} + \overline{j} = \overline{i} + \overline{k}$ for $\overline{i}, \overline{j}, \overline{k} \in H_{\mathcal{M}}$ where $i, j, k \in \mathbb{N}^E$, then $X^j - X^k \in I_{\mathcal{M}}$.

(ii) The monoid $H_{\mathcal{M}}$ is cancellative with respect to 0.

(iii) If $X^j - X^k \in I_{\mathcal{M}}$ and $X^j, X^k \not\in I_{\mathcal{M}}$, then $\overline{i} = \overline{j}$ in $H_{\mathcal{M}}$.

Proof. (i) Note that the toric face ring $K[\mathcal{M}]$ has a natural $\mathbb{Z}^n$-grading induced by the embeddings $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$. Then also the polynomial ring $K[X_e : e \in E]$ is $\mathbb{Z}^n$-graded if we give $X_e$ the degree $a_e \in \mathbb{Z}^n$. Observe that the ideal $B_{\mathcal{M}}$ is then $\mathbb{Z}^n$-graded since the generators are homogeneous with respect to this grading. Then $K[X_e : e \in E] / B_{\mathcal{M}}$ is $\mathbb{Z}^n$-graded. Equivalently we obtain a monoid homomorphism $\varphi : H_{\mathcal{M}} \to \mathbb{Z}^n, \overline{i} \mapsto \sum_{e \in E} i_e a_e$.

Now $\overline{i} + \overline{j} = \overline{i} + \overline{k}$ implies that $\sum_{e \in E} (i_e + j_e) a_e = \sum_{e \in E} (i_e + k_e) a_e$ in $\mathbb{Z}^n$, and thus $\sum_{e \in E} j_e a_e = \sum_{e \in E} k_e a_e$. It follows that $X^j - X^k \in \text{Ker}(K[X_e : e \in E] \to K[\mathcal{M}] = I_{\mathcal{M}}$.

(ii) It follows from (i) that $H_{\mathcal{M}}$ is cancellative with respect to 0, because $X^j - 1 \not\in I_{\mathcal{M}}$.

(iii) If $X^j - X^k \in I_{\mathcal{M}}$ and $X^j, X^k \not\in I_{\mathcal{M}}$, then $X^j - X^k \in B_{\mathcal{M}}$ by the last observation of Proposition 2.3. Hence $\overline{i} = \overline{j}$ in $H_{\mathcal{M}}$. $\square$

Let $S = K[X_e : e \in E]$. Observe that all rings $S, S / B_{\mathcal{M}}$ and $K[\mathcal{M}]$ are naturally $H_{\mathcal{M}}$-graded. For $S$ we set $\text{deg} X^j = \overline{i} \in H_{\mathcal{M}}$. Since $X^a - 1 \not\in I_{\mathcal{M}}$ we have that $H_{\mathcal{M}}$ is positive, and $S$ is an $H_{\mathcal{M}}$-graded local ring. If $H_{\mathcal{M}}$ is cancellative with respect to 0, then one can apply Lemma 4.3 to f. g. $H_{\mathcal{M}}$-graded $S$-modules like $K[\mathcal{M}]$. In particular, we can speak about minimal $H_{\mathcal{M}}$-graded resolutions. The next goal is to determine the corresponding $H_{\mathcal{M}}$-graded Betti numbers.
For \( \overline{h} \in H_{\mathcal{M}} \) we define
\[
\Delta_{\overline{h}} = \{ F \subseteq E : \overline{h} = \overline{g} + \sum_{e \in F} \overline{f}_e \text{ for some } \overline{g} \in H_{\mathcal{M}} \}.
\]

We see immediately that \( \Delta_{\overline{h}} \) is a simplicial complex on the finite vertex set \( E \), and we call \( \Delta_{\overline{h}} \) the squarefree divisor complex of \( \overline{h} \). Moreover, we need a special subcomplex of \( \Delta_{\overline{h}} \) defined as follows
\[
\Delta_{\overline{h}, \mathcal{M}} = \{ F \subseteq E : \overline{h} = \overline{g} + \sum_{e \in F} \overline{f}_e \text{ for some } \overline{g} \in H_{\mathcal{M}} \text{ such that } X^g \in I_{\mathcal{M}} \}.
\]

For an arbitrary simplicial complex \( \Delta \) on some ordered vertex set \( E \) (with order \( < \)) we let \( \tilde{\mathcal{C}}(\Delta) \) denote the augmented oriented chain complex of \( \Delta \) with coefficients in \( K \), i.e. the complex
\[
\tilde{\mathcal{C}}_*(\Delta) : 0 \to \tilde{\mathcal{C}}_{\text{dim} \Delta} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \tilde{\mathcal{C}}_0 \xrightarrow{\partial} \tilde{\mathcal{C}}_{-1} \to 0
\]
where \( \tilde{\mathcal{C}}_i = \bigoplus_{F \subseteq \Delta, \dim F = i} KF \) and \( \partial(F) = \sum_{F' \subseteq \Delta, \dim F' = i-1} e(F,F')F' \). Here \( e(F,F') \) is 0 if \( F' \not\subseteq F \). Otherwise it is \((-1)^k\) if \( F = \{ e_0, \ldots, e_i \} \) for elements \( e_0 < \cdots < e_i \) in \( E \), and \( F' = \{ e_0, \ldots, e_{k-1}, e_{k+1}, \ldots, e_i \} \). Further we let \( \tilde{H}(\Delta) = \tilde{H}_i(\tilde{\mathcal{C}}(\Delta)_*) \) be the \( i \)-th reduced simplicial homology group of \( \Delta \). If \( \Delta' \) is a subcomplex of \( \Delta \) we let \( \tilde{\mathcal{C}}(\Delta, \Delta')_* = \tilde{\mathcal{C}}(\Delta)_*/\tilde{\mathcal{C}}(\Delta')_* \) denote the relative augmented oriented chain complex of \( \Delta \) and \( \Delta' \), and \( H_i(\Delta, \Delta') \) the \( i \)-th homology of this complex.

**Theorem 4.5.** Assume that \( \Sigma \) is a rational pointed fan in \( \mathbb{R}^n \) such that \( M_c \subseteq \mathbb{Z}^n \) for \( c \in \Sigma \). Let \( (a_e)_{e \in E} \) be a family of elements of \( \mathcal{M} \) such that \( \{ a_e : e \in E \} \cap M_c \) generates \( M_c \) for each \( c \in \Sigma \). Then
\[
\beta_{\text{th}}^S(K[\mathcal{M}]) = \dim_K \tilde{H}_{i-1}(\Delta_{\overline{h}}, \Delta_{\overline{h}, \mathcal{M}})
\]
for all \( \overline{h} \in H_{\mathcal{M}} \) and \( i \in \mathbb{N} \).

**Proof.** Let \( S = K[X_e : e \in E] \). We fix an arbitrary total order \( < \) on \( E \). Let \( \mathcal{K}(K[\mathcal{M}]) \) denote the Koszul complex of \( X_e, e \in E \) tensored with \( K[\mathcal{M}] \). This complex is naturally \( H_{\mathcal{M}} \)-graded and its \( H_{\mathcal{M}} \)-graded homology is exactly \( \text{Tor}(K, K[\mathcal{M}]) \). (E. g. see [7] for details.) Thus we can use this complex to determine the numbers \( \beta_{\text{th}}^S(K[\mathcal{M}]) \). We have
\[
\mathcal{K}_i(K[\mathcal{M}]) = \bigoplus_{F \subseteq E, |F| = i} K[\mathcal{M}] \left( - \sum_{e \in F} \overline{f}_e \right)
\]
and the differential \( \partial_i : \mathcal{K}_i(K[\mathcal{M}]) \to \mathcal{K}_{i-1}(K[\mathcal{M}]) \) is given on the component
\[
K[\mathcal{M}] \left( - \sum_{e \in F} \overline{f}_e \right) \to K[\mathcal{M}] \left( - \sum_{e \in F'} \overline{f}_e \right)
\]
for \( F', F \subseteq E \) as the zero map for \( F' \not\subseteq F \), or otherwise as multiplication \( e(F,F')X_{e_k} \) where
\[
e(F,F') = \begin{cases} 0 & \text{if } F' \not\subseteq F, \\ (-1)^{k-1} & \text{if } F = \{ e_1 < \cdots < e_i \}, F' = F \setminus \{ e_k \}. \end{cases}
\]

For \( \beta_{\text{th}}^S(K[\mathcal{M}]) \) we have first to determine \( \mathcal{K}_i(K[\mathcal{M}])_{\overline{h}} \). Thus we compute
\[
K[\mathcal{M}] \left( - \sum_{e \in F} \overline{f}_e \overline{h} \right) = \bigoplus_{\overline{F} \in H_{\mathcal{M}}, \sum_{e \in F} \overline{f}_e = \overline{h}} K[\mathcal{M}]_{\overline{F}}.
\]
Such an $\overline{H}$ exists if and only if $F \in \Delta_{\mathbb{H}}$. If $K[\mathcal{M}]_{\mathbb{H}} \neq 0$, then $X^{h} \notin I_{\mathcal{M}}$. Assume that there exists another $\overline{h''}$ such that $\overline{h''} + \sum_{e \in F} \overline{f_{e}} = \overline{h}$ and $K[\mathcal{M}]_{\mathbb{H}} \neq 0$. It follows $X^{h''} \notin I_{\mathcal{M}}$. We obtain from (4.4) that $X^{h''} \cong X^{h'} \in B_{\mathcal{M}}$. Hence $\overline{h'} = \overline{h''}$ in $H_{\mathcal{M}}$, i.e. $\overline{H}$ is uniquely determined if it exists. Moreover, we have $K[\mathcal{M}]_{\mathbb{H}} = 0$ if and only if $F \in \Delta_{\mathbb{H}}$. Hence

$$\mathcal{H}_{i}(K[\mathcal{M}]) \cong \bigoplus_{F \in \Delta_{\mathbb{H}} \setminus \Delta_{\mathbb{H}} \mathcal{M}, |F| = i} KF.$$ 

Consider $F' \in \Delta_{\mathbb{H}} \setminus \Delta_{\mathbb{H}} \mathcal{M}$ such that $|F'| = i - 1$. Choose $\overline{H}$ such that $\overline{h'} + \sum_{e \in F} \overline{f_{e}} = \overline{h}$ and $\overline{h''}$ such that $\overline{h''} + \sum_{e \in F'} \overline{f_{e}} = \overline{h}$. The differential $\partial_{i}(\mathcal{H}_{i}(K[\mathcal{M}]) \to \mathcal{H}_{i-1}(K[\mathcal{M}]))$ on the component $KF \to KF'$ (which corresponds to $K[\mathcal{M}]_{\mathbb{H}} \to K[\mathcal{M}]_{\mathbb{H}}$) is given by

$$\partial_{i}(F) = \begin{cases} 0 & \text{if } F' \not\subset F, \\ \varepsilon(F, F')F' & \text{if } F = \{e_{1} < \cdots < e_{i}\}, F' = F \setminus \{e_{k}\}. \end{cases}$$

Hence we see that the complex $\mathcal{H}(\mathcal{M}[\mathcal{M}])_{\mathbb{H}}$ coincides with $\mathcal{H}_{\mathcal{M}}(\mathcal{M}[\mathcal{M}])$ and this yields

$$\beta_{ih}(\mathcal{M}[\mathcal{M}]) = \dim_{K} \mathcal{H}_{i-1}(\Delta_{\mathbb{H}} \setminus \Delta_{\mathbb{H}} \mathcal{M})$$

as desired. $\square$

One can easily generalize Theorem 4.5 in the following way: if $\mathcal{M}$ satisfies the properties of Lemma 4.4, then the proof of Theorem 4.5 works for $\mathcal{M}$. However, in general one can not expect that the compact combinatorial formula is true for all monoidal complexes without any further assumptions. Indeed, a counterexample is

**Example 4.6.** We consider the M"obius strip as a monoidal complex $\mathcal{M}$ by considering each quadrangle as a unit square and choosing the monoid over it as the corresponding monoid. Together with the compatibility conditions this determines $\mathcal{M}$ completely.

![Figure 1. Möbius strip as a monoidal complex](image)

The ideal $I_{\mathcal{M}}$ is generated by the binomials resulting from the unit squares and monomials

$$X_{y}X_{z} - X_{u}X_{w}, \quad X_{y}X_{w} - X_{y}X_{z}, \quad X_{y}X_{v} - X_{u}X_{y}, \quad X_{u}X_{v}X_{w} \text{ and } X_{u}X_{w}X_{z}.$$ 

The other monomials are redundant. E.g. $X_{y}X_{z}X_{z} = X_{y}(X_{u}X_{v} - X_{u}X_{w}) + X_{v}(X_{u}X_{w} - X_{u}X_{w}) + X_{v}X_{u}X_{w}$. Since the binomial relations are homogeneous, $H_{\mathcal{M}}$ is cancellative with respect to 0.
Let \( x_d \) stand for the residue class of \( X_d \), and choose the degree
\[
\overline{h} = x_{u\overline{x}_y} - x_{u\overline{x}_w} - x_{u\overline{x}_y} = x_{u\overline{x}_w} - x_{u\overline{x}_y} = x_{v\overline{x}_y} \in \mathcal{H}_{\mathcal{M}}.
\]
This equation shows that Lemma 4.4 does not hold for \( \mathcal{M} \).

Since \( \chi^h \in I_{\mathcal{M}} \), one has \( K[\mathcal{M}]_{\overline{h}} = 0 \). The degree \( \overline{h} \) component of the Koszul complex is
\[
\mathcal{K}(K[\mathcal{M}])_{\overline{h}}: 0 \rightarrow K^4 \rightarrow K^{12} \rightarrow K^9 \rightarrow 0 \rightarrow 0
\]
where \( K^9 \) is in homological degree 1, and the Betti numbers are
\[
\beta_{i\overline{h}}^S(K[\mathcal{M}]) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{else.} \end{cases}
\]

Now we consider the complex \( \mathcal{C}_i^S(\Delta_{\overline{h}}, \Delta_{\overline{h}}) \) which is given by
\[
0 \rightarrow K^4 \rightarrow K^{12} \rightarrow K^6 \rightarrow 0 \rightarrow 0
\]
with \( K^6 \) in homological degree 0. Hence \( \mathcal{H}_1(\Delta_{\overline{h}}, \Delta_{\overline{h}}) = K^d \) for some \( d \geq 2 \), and the formula of Theorem 4.5 does not hold in this case.

The results of this section imply, in particular, the well-known Tor formula of Hochster for Stanley-Reisner rings (see [9]). Recall that if \( \Delta \) is a simplicial complex on the vertex set \( [n] \) and we let \( S = K[X_1, \ldots, X_n] \), then \( K[\Delta] = S/I_{\Delta} \) is the Stanley-Reisner ring of \( \Delta \) where \( I_{\Delta} = (\prod_{i \in F} X_i : F \notin \Delta) \) is the Stanley-Reisner ideal of \( \Delta \). Now all considered rings have a natural \( \mathbb{Z}^n \)-grading. It is well-known that \( \beta_{ia}^S(K[\Delta]) = 0 \) if \( a \) is not a squarefree vector, i.e. a 0-1 vector. (Either one shows this by using the results of this section, or proves this directly). For a squarefree vector \( a \) with support \( W = \{ i \in [n] : a_i = 1 \} \), we write \( \beta_{iW}^S(K[\Delta]) = \beta_{ia}^S(K[\Delta]) \) for the corresponding Betti-number.

**Corollary 4.7** (Hochster). Let \( \Delta \) be a simplicial complex on the vertex set \( [n] \). Then for \( W \subseteq [n] \) one has
\[
\beta_{iW}^S(K[\Delta]) = \dim_K \mathcal{H}_{|W| - i - 1}(\Delta_W)
\]
where \( \Delta_W = \{ F \in \Delta : F \subseteq W \} \).

**Proof.** In Example 2.4 it was observed that there exists a rational pointed fan \( \Sigma \) and an embedded monoidal complex \( \mathcal{M} \) such that \( K[\mathcal{M}] = K[\Delta] \). Thus the binomial ideal \( B_{\mathcal{M}} \) is 0, and \( I_{\mathcal{M}} = I_{\Delta} \) is generated by squarefree monomials. The monoid \( H_{\mathcal{M}} \) is nothing but the free monoid \( \mathbb{N}^\Sigma \) in this case. Thus the induced grading is just the natural \( \mathbb{N}^\Sigma \)-grading on \( K[\Delta] \). It remains to observe that the complex \( \mathcal{C}_{\Sigma - 1}(\Delta_{\overline{h}}, \Delta_{\overline{h}}) \) coincide with the complex \( \mathcal{C}_{|W| - \Sigma - 1}(\Delta_W) \) which determines the homology \( \mathcal{H}_{|W| - i - 1}(\Delta_W) \). This concludes the proof.

**Remark 4.8.** Let \( \Delta \) be a simplicial complex on \( [n] \). Hochster computed also the local cohomology of the Stanley-Reisner ring as a \( \mathbb{Z}^n \)-graded \( K \)-vector space in terms of combinatorial data of the given complex (for example, see [7] or [13]). While the Tor formula is restricted to embedded monoidal complexes (or complexes which behave like these), one can prove a Hochster formula for the local cohomology in great generality. In fact, to show such a formula for toric face rings was one of the starting points of the systematic
study of toric face rings. See [10] for the case of embedded monoidal complexes and [1] for classes of rings which include toric face rings as a special case.

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