Application of Microlocal Analysis to the Theory of Quantum Fields Interacting with a Gravitational Field

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It is explained how techniques from microlocal analysis can be used to settle some long-standing questions that arise in the study of the interaction of quantum matter fields with a classical gravitational background field.

1 Introduction

Quantum field theory (QFT) is the theory of elementary particles and their fundamental interactions. The very successful standard model which describes the electromagnetic, weak and strong interactions of the observed elementary particles does however not incorporate gravity. The quantisation of gravity ("quantum gravity") is a very difficult problem which is presently still far from its solution. An easier, but only approximate approach to the interaction of matter and gravitational fields is the "semiclassical theory" (or "QFT in curved spacetimes") where the gravitational field is described classically as a 4-dim. Lorentzian manifold \((M, g)\) and only the matter fields are quantized to operator valued Wightman fields. This theory, which we will describe in the following more closely, should have a wide range of physical applicability, from quantum effects in the early universe to the Hawking radiation of massive stars collapsing to a black hole, and should also lead to a better understanding of the local features of standard QFT (for a good introduction see e.g. \[2\]).

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2 The basic setting of QFT

We only deal with the simple example of a free (= linear) massive scalar field (Klein-Gordon field).

Let \((\mathcal{M}, g)\) be a 4-dim. globally hyperbolic (i.e. there exists a spacelike Cauchy hypersurface \(\Sigma\) such that \(\mathcal{M} = \mathbb{R} \times \Sigma\)) Lorentzian manifold \((g_{\mu\nu} \text{ having signature } + - - -)\) and consider the linear Klein-Gordon equation

\[
(\Box_g + m^2)\Phi = \left( g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2 \right) \Phi
= \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \Phi) + m^2 \Phi = 0
\tag{2.1}
\]

in some coordinate system, where \(m > 0\) is the mass of the scalar field \(\Phi: \mathcal{M} \to \mathbb{R}\), \(\nabla_\mu\) the covariant derivative on \(\mathcal{M}\) w.r.t. \(g_{\mu\nu}\) and \(\sqrt{g} := |\det(g_{\mu\nu})|^{1/2}\). By the linearity of the field equation (2.1) the field \(\Phi\) has no self-interaction (“free” field), but is coupled via the metric tensor to the gravitational field. (2.1) is a hyperbolic partial differential equation with variable coefficients. It possesses unique retarded and advanced fundamental solutions \(E_{\text{ret}}, E_{\text{av}}\).

Quantizing the classical field \(\Phi\) means to construct a Hilbert space \(\mathcal{H}\) of physical states and an operator-valued distribution \(\hat{\Phi}(f)\) \((f \in \mathcal{D}(\mathcal{M})\) a testfunction) acting on \(\mathcal{H}\) which describes the field observables localized in \(\text{supp } f \subset \mathcal{M}\) subject to the following axioms:

(i) \(\forall f \in \mathcal{D}(\mathcal{M}): \hat{\Phi}(f)\) is a linear (unbounded) closable operator on \(\mathcal{H}\) with dense domain \(D \subset \mathcal{H}\) such that \(\hat{\Phi}(f)^* \supset \hat{\Phi}(f)\) (hermiticity) and \(\hat{\Phi}(f) D \subset D\).

(ii) \(\forall \psi \in D: \Lambda^{(n)}(f_1, \ldots, f_n) := \langle \psi, \hat{\Phi}(f_1) \ldots \hat{\Phi}(f_n) \psi \rangle \in \mathcal{D}'(\mathcal{M}^n)\) (n-point distributions).

(iii) \(\forall f_1, f_2 \in \mathcal{D}(\mathcal{M}): \left[ \hat{\Phi}(f_1), \hat{\Phi}(f_2) \right] = \frac{i}{2} \langle f_1, E f_2 \rangle, E := E_{\text{ret}} - E_{\text{av}}\) (commutation relations).

(iv) \(\forall f \in \mathcal{D}(\mathcal{M}): \hat{\Phi} \left((\Box_g + m^2) f \right) = 0\) (field equations).

(v) If \(\mathcal{M} = \mathbb{R}^4\) (Minkowski space) with metric \(g = \text{diag}(+1, -1, -1, -1)\) one demands Poincaré covariance of the theory, in particular that the translations \(T_a: x \mapsto x + a, a \in \mathbb{R}^4\), can be implemented in \(\mathcal{H}\) by a strongly continuous unitary group \(U(a) = \exp(i a_\mu P^\mu)\) whose generator \(P^\mu\) has spectrum in the positive forward light cone (spectrum condition). The vacuum state \(\Omega \in D \subset \mathcal{H}\) is the unique eigenstate of \(P^\mu\) to eigenvalue 0.

Whereas conditions (i)–(iv) can equally well be formulated on a curved manifold \(\mathcal{M}\) as on \(\mathbb{R}^4\) this axiom (v) depends in an essential way on the special global structure of \(\mathbb{R}^4\). Thus, the vacuum state does not exist on a generic manifold, and it has been the main problem of QFT in curved spacetimes to
find a substitute for (v) in this case. To study this question we consider only quasifree states $\psi \in D$ where (by def.) all the physical information is contained in the 2-point correlation function $\Lambda^{(2)} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$.

3 Hadamard states and their wavefront set

The background geometry reacts on the energy-momentum content of the matter fields via the semiclassical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \langle \psi | \hat{T}_{\mu\nu}(x) | \psi \rangle$$

(3.1)

where $\langle \psi | \hat{T}_{\mu\nu}(x) | \psi \rangle$ is the expectation value in some state $\psi \in \mathcal{H}$ of the energy-momentum tensor operator $\hat{T}_{\mu\nu}$ of $\hat{\Phi}$ at a point $x \in \mathcal{M}$, the l.h.s. of (3.1) is the Einstein tensor. Looking at the classical expression for $T_{\mu\nu}$

$$T_{\mu\nu}(x) = (\nabla_\mu \Phi)(\nabla_\nu \Phi) - \frac{1}{2}(\nabla_\gamma \Phi \nabla^\gamma \Phi + m^2 \Phi^2)$$

one notes that it contains terms quadratic in $\Phi(x)$ and its derivatives, but $\langle \psi | \hat{\Phi}(x)\hat{\Phi}(x) | \psi \rangle$ is in general not defined (remember that $\langle \psi | \hat{\Phi}(x)\hat{\Phi}(y) | \psi \rangle$ is a distribution). Therefore, it was an old idea to admit only those states $\psi$ as physical whose two-point functions $\Lambda^{(2)}$ have all the same singular kernel $G(x,y)$ and differ only in their smooth parts, since then one can unambiguously define

$$\langle \psi | \hat{\Phi}(x)\hat{\Phi}(x) : | \psi \rangle := \lim_{x \rightarrow y} \left[ \langle \psi | \hat{\Phi}(x)\hat{\Phi}(y) | \psi \rangle - G(x,y) \right]$$

and $\langle \psi | : \hat{T}_{\mu\nu}(x) : | \psi \rangle$ on the r.h.s. of (3.1) can be made a well defined quantity. These are the so-called Hadamard states. Without giving their precise definition in terms of their singular kernel (which can be found in [4]) we state the following important theorem which was recently proven by M. Radzikowski [5] and which characterizes these states in a local and covariant manner by the wavefront set of $\Lambda^{(2)}$:

**Theorem 3.1.** [Theorem 5.1 of [3]]

A quasifree state of the linear Klein-Gordon quantum field on a globally hyperbolic spacetime manifold $(\mathcal{M}, g)$ is an Hadamard state

$$\Leftrightarrow WF(\Lambda^{(2)}) = \left\{ (x_1, k_1; x_2, -k_2) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\}; (x_1, k_1) \sim (x_2, k_2); k_0^0 \geq 0 \right\}$$

(3.2)
where \((x_1, k_1) \sim (x_2, k_2) :\iff x_1 \text{ and } x_2 \text{ are connected by a null geodesic } \gamma, k_1, k_2 \text{ are cotangent to } \gamma \text{ at } x_1 \text{ resp. } x_2 \text{ and parallel transported to each other along } \gamma.\)

(Note that a globally hyperbolic manifold is time orientable, hence the condition \(k_0^1 \geq 0\) makes good sense.) This theorem says that only positive frequencies occur in \(WF(\Lambda^{(2)})\). This is the sought for microlocal remnant of the spectrum condition.

4 Construction of Hadamard states

Having recognized the physical importance of Hadamard states we are immediately led to the following mathematical problem: Construct (if possible all) Hadamard states for a given spacetime \((\mathcal{M}, g)\), i.e. (all) \(\Lambda^{(2)} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})\) fulfilling the following five conditions (which are the axioms (i)–(v) from above rewritten in terms of \(\Lambda^{(2)}\)):

(i) \(\Lambda^{(2)}(f_1, f_2) = \Lambda^{(2)}(f_2, f_1) \quad \forall f_1, f_2 \in \mathcal{D}(\mathcal{M})\) (hermiticity)

(ii) \(\Lambda^{(2)}(f, f) \geq 0 \quad \forall f \in \mathcal{D}(\mathcal{M})\) (positivity of scalar product in \(\mathcal{H}\))

(iii) \(\text{Im}\Lambda^{(2)}(f_1, f_2) = \frac{1}{2}\langle f_1, Ef_2 \rangle\) (commutation relations)

(iv) \(\Lambda^{(2)}((\Box_g + m^2)f_1, f_2) = 0 = \Lambda^{(2)}(f_1, (\Box_g + m^2)f_2)\) (field equations)

(v) \(WF(\Lambda^{(2)})\) as in Eq. (3.2) (microlocal spectrum condition).

To solve this problem we first give a parametrization of (pure) quasifree states in terms of two operators \(R\) and \(I\):

\[\Lambda^{(2)}(f_1, f_2) = \frac{1}{2} \left\langle \left(\Box_g - n^\mu \nabla_\mu\right)Ef_1, I^{-1}\left(\Box_g - n^\mu \nabla_\mu\right)Ef_2 \right\rangle_{L^2(\Sigma, d^3\sigma)} \tag{4.1}\]

is the two-point function of a pure quasifree state, i.e. fulfills (i)–(iv).

A proof of this theorem and the following ones and a generalization to mixed states can be found in [3]. Next we give conditions on \(R\) and \(I\) such that \(\Lambda^{(2)}_\Sigma\) of Eq. (4.1) has the correct wavefront set (property (v) above). The idea is to project out from the fundamental solution \(E\) (whose WF is known to contain
positive and negative frequencies) only the positive frequencies. To this end we consider a foliation of $\mathcal{M}$ into hypersurfaces $\Sigma_t$, $t \in [-T,T] \subset \mathbb{R}$, in a small neighborhood of the Cauchy surface $\Sigma$: $\mathcal{M} = [-T,T] \times \Sigma$, $\Sigma_t = \{t\} \times \Sigma$, and take the operators $R$ and $I$ as depending on the parameter $t$.

**Theorem 4.2.** [Theorem 3.12 of [3]]

Let $I(t), R(t)$ be pseudodifferential operators on $\Sigma_t$, $t \in [-T,T]$ such that $I$ is elliptic and such that there exists a pseudodifferential operator $Q$ on $[-T,T] \times \Sigma$ which has the property

$$Q(R - iI - n^\mu \nabla_\mu) = \Box_g + m^2 + r$$

for some smoothing operator $r$, and which possesses a principal symbol $q(x,k)$ with the property

$$q^{-1}(0) \setminus \{0\} \subset \{(x,k) \in T^*\mathcal{M}; k^0 > 0\}.$$  

(4.3)

Then $\Lambda^{(2)}_\Sigma$, Eq. (4.4), is an Hadamard state (i.e. $\Lambda^{(2)}_\Sigma$ has the wavefront set (2.2)).

The sufficient conditions of this theorem allow to prove the Hadamard property for many examples of quantum states which have already been constructed in the literature on certain spacetime manifolds (in the sense of Theorem 4.1), but which have so far not been shown to be of the Hadamard type (which is a necessary condition for being physically acceptable). Among these are the ground and thermodynamic equilibrium states on static spacetimes [3, Theorems 3.18 and 3.19] and the so-called adiabatic vacuum states on Robertson-Walker spacetime models [3, Theorem 3.24]. It can also be shown that the frequently employed method of Hamiltonian diagonalization for constructing quantum states on a curved manifold does in general not lead to Hadamard states and is therefore unphysical [3, Theorem 3.27].

But even more important, Theorem 4.2 gives us a guide to a method of explicitly constructing Hadamard states on an arbitrary globally hyperbolic spacetime manifold. The idea is to construct the pseudodifferential operators $R(t)$, $I(t)$ and $Q$ by an asymptotic expansion of their symbols such that Eq. (4.2) is fulfilled. To achieve this we choose Gaussian normal coordinates in a neighborhood of $\Sigma$ such that the metric reduces to the simple form

$$g_{\mu\nu} = \begin{pmatrix} 1 & -h_{ij}(t, \bar{x}) \\ -h_{ij}(t, \bar{x}) & \end{pmatrix}$$
where $h_{ij}(t, \vec{x})$ is the Riemannian metric on $\Sigma_t$ induced by $g$, and the Klein-Gordon operator reads

$$
\Box_g + m^2 = \frac{1}{\sqrt{h}} \partial_t (\sqrt{h} \partial_t) - \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j) + m^2. \tag{4.4}
$$

Then we make the following Ansatz for a factorization of (4.4)

$$
\Box_g + m^2 = (-a - \frac{1}{\sqrt{h}} \partial_t \sqrt{h})(a - \partial_t) - r \tag{4.5}
$$

$$
a(t, x, k) \sim -i \sqrt{h} i^2 k_j + m^2 + \sum_{\nu=1}^{\infty} b^{(\nu)}(t, x, k)
$$

with $r$ a smoothing operator and $b^{(\nu)}$ symbols of order $1 - \nu$. It turns out that the $b^{(\nu)}$ can be determined successively such that Eq. (4.5) holds, and that the operators $I, R, Q$ corresponding to the symbols

$$
I(t, x, k) := -\frac{1}{2i} \left[ a(t, x, k) - a(t, x, -k) \right]
$$

$$
R(t, x, k) := \frac{1}{2} \left[ a(t, x, k) + a(t, x, -k) \right] \tag{4.6}
$$

$$
Q(t, x, k) := -a(t, x, k) - \frac{1}{\sqrt{h}} \partial_t \sqrt{h}
$$

satisfy all the requirements of Theorems (4.1) and (4.2) (for details see [3, Section 3.7]). Therefore we have the following

**Theorem 4.3.** (Theorem 3.29 of [3])

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with Cauchy surface $\Sigma$. In a neighborhood of $\Sigma$ let the pseudodifferential operators $I$ and $R$ be constructed as shown in Eq. (4.6).

Then Eq. (4.3) is the two-point distribution of a pure Hadamard state of the Klein-Gordon quantum field on $(\mathcal{M}, g)$.

## 5 Outlook

The new characterization of physical quantum states in gravitational background fields by the wavefront set opens a new era in the study of QFT on curved spacetimes. The following problems can and will be treated on globally hyperbolic manifolds using microlocal techniques:
- generalization to self-interacting scalar quantum fields (e.g. $\Phi^4$-theory) by giving a microlocal spectrum condition for all $n$-point functions [1]
- perturbation and renormalization theory (R. Brunetti & K. Fredenhagen, in preparation)
- generalization of the results of Section 4 to Dirac- and electromagnetic fields on manifolds using the polarization set
- construction of states on physically interesting spacetime models and calculation of physical effects
- construction of a Euclidean version of QFT on curved spacetimes (W. Junker, in preparation)
- treatment of (non-abelian) gauge-theories in this frame...

References

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