Integral Representations of Equally Positive
Integer-Indexed Harmonic Sums at Infinity*

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Abstract

We identify a partition-theoretic generalization of Riemann zeta function and the equally positive integer-indexed harmonic sums at infinity, to obtain the generating function and the integral representations of the latter. The special cases coincide with zeta values at positive integer arguments.

Keywords: harmonic sum, integral representation, zeta value

1 Introduction

The harmonic sum of indices \( a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\} \) is defined as (see \[1\] eq. 4, pp. 1)

\[
S_{a_1, \ldots, a_k} (N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{\text{sign} (a_1)^{n_1}}{n_1^{[a_1]}} \times \cdots \times \frac{\text{sign} (a_k)^{n_k}}{n_k^{[a_k]}},
\]

which is naturally connected to the Riemann zeta function, by noting that \( N = \infty, k = 1 \) and \( a_1 > 0 \) gives \( S_{a_1} (\infty) = \zeta (a_1) \). A variety of the study can be found in the literature. For instance, Hoffman \[4\] established the connection between harmonic sums and multiple zeta values. We especially focus on the equally positively indexed harmonic sums, given by the case \( a_1 = \cdots = a_k = a > 0 \)

\[
S_{a_k} (N) := S_{a, \ldots, a} (N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{(n_1 \cdots n_k)^a}, \quad (1.1)
\]

and also the equally positive integer-indexed harmonic sums (EPIIHS), namely \( a = m \in \mathbb{Z}_{>0} \). If \( N = \infty \), we additionally assume \( m \in \mathbb{Z}_{>1} \) for convergence.

*This work is supported by the Austrian Science Fund (FWF) grant SFB F50 (F5006-N15 and F5009-N15)
Recently, Schneider [7] studied the generalized \( q \)-Pochhammer symbol and obtained [7, pp. 3]

\[
\prod_{n \in X} \frac{1}{1 - f(n) q^n} = \sum_{\lambda \in \mathcal{P}_X} q^{\left| \lambda \right|} \prod_{\lambda_i \in \lambda} f(\lambda_i), \tag{1.2}
\]

where

- \( X \subseteq \mathbb{Z}_{>0} \) and \( f : \mathbb{Z}_{>0} \rightarrow \mathbb{C} \) such that if \( n \not\in X \) then \( f(n) = 0 \);
- \( \mathcal{P}_X \) is the set of partitions into elements of \( X \);
- \( \lambda \vdash n \) means \( \lambda \) is a partition of \( n \), the size \( |\lambda| \) is the sum of the parts of \( \lambda \), i.e., the number \( n \) being partitioned, and \( \lambda_i \in \lambda \) means \( \lambda_i \in \mathbb{Z}_{>0} \) is a part of partition \( \lambda \).

Further define \( l(\lambda) := k, n_\lambda := \lambda_1 \cdots \lambda_k \) and denote \( \mathcal{P} := \mathcal{P}_{\mathbb{Z}_{>0}} \). Noting \( \lambda_1 \geq \cdots \geq \lambda_k \geq 1 \), a partition-theoretic generalization of Riemann zeta function [7, eq. 11, pp. 4] is defined and identified as

\[
\zeta_p \left( \{ a \}^k \right) := \sum_{l(\lambda) = k} \frac{1}{n_\lambda} = \sum_{\lambda_1 \geq \cdots \geq \lambda_k \geq 1} \frac{1}{\lambda_1 \cdots \lambda_k} = S_{a_k} (\infty), \tag{1.3}
\]

which leads to the generating function and the integral representation of \( S_{m_k} (\infty) \), presented in the next section.

### 2 Main results

We first apply (1.2) to the case \( X = \{1, 2, \ldots, N\} \) and \( f(n) := \frac{t^n}{n^a} \), obtaining

\[
\prod_{n=1}^{N} \frac{1}{1 - \frac{t^n}{n^a} q^n} = \sum_{\lambda \in \mathcal{P}_X} q^{\left| \lambda \right|} \prod_{\lambda_i \in \lambda} \frac{t^{\lambda_i}}{\lambda_i^a} = \sum_{\lambda \in \mathcal{P}_X} q^{\left| \lambda \right|} \frac{l(\lambda)^a}{n_\lambda^a},
\]

which, by further letting \( q \to 1 \), yields the following generating function.

**Theorem 1.** The generating function of \( S_{a_k} (N) \) is given by

\[
\sum_{k=0}^{\infty} S_{a_k} (N) t^k = \prod_{n=1}^{N} \frac{n^a}{n^a - t^a}. \tag{2.1}
\]

**Remark 2.** The special case for \( a = 1 \) is [2] eq. 9, pp. 1272]

\[
\sum_{k=0}^{\infty} t^k S_{1_k} (N) = \frac{N!}{(1-t) \cdots (N-t)} = N \cdot B(N, 1-t), \tag{2.2}
\]

involving the beta function \( B \), defined by

\[
B(x, y) := \int_{0}^{1} z^{x-1} (1-z)^{y-1} \, dz = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \tag{2.3}
\]

where the integral representation holds for \( \text{Re}(x), \text{Re}(y) > 0 \).
Corollary 3. For \( m \in \mathbb{Z}_{>1} \), denote \( \xi_m := \exp \left( \frac{2\pi i}{m} \right) \) with \( i^2 = -1 \). Then,

\[
\sum_{k=0}^{\infty} S_m(k) t^m = m \prod_{j=0}^{m-1} \Gamma \left( 1 - \xi_j^m t \right).
\]

(2.4)

Proof. From (2.1) and (2.2), we have

\[
\sum_{k=0}^{\infty} S_m(k) t^m = \prod_{n=1}^{N} \frac{n^m}{(n - \xi_0^m t) \cdots (n - \xi_{m-1}^m t)} = \prod_{j=0}^{m-1} N \cdot B(N, 1 - \xi_j^m t).
\]

Then, apply the limit (see [6, pp. 254, ex. 5]) \( \Gamma(z) = \lim_{n \to \infty} N^z B(N,z) \) to \( z_j = 1 - \xi_j^m t, \ j = 0, \ldots, m-1 \), by noting \( \xi_0^m + \cdots + \xi_{m-1}^m = 0 \), to complete the proof.

Remark 4. An alternative proof can be given by letting \( N = \infty \) in (2.1) and applying [3, Thm. 1.1, pp. 547].

Remark 5. For general \( a > 0 \), we failed to obtain a closed form of

\[
\prod_{n=1}^{\infty} n^a (n^a - t).
\]

Example 6. When \( m = 2 \), we apply (2.4) to get

\[
B(1 + t, 1 - t) = \Gamma(1 + t) \Gamma(1 - t) = \sum_{k=0}^{\infty} S_{2k}(\infty) t^{2k}.
\]

From the integral representation (2.3), we obtain (also see Remark 7)

\[
B(1 + t, 1 - t) = \int_{0}^{1} z^t (1 - z)^{-t} \, dz = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{0}^{1} \log^k \left( \frac{z}{1 - z} \right) \, dz.
\]

(2.5)

Then it follows, by comparing coefficients of \( t \),

\[
S_{2k}(\infty) = \frac{1}{(2k)!} \int_{0}^{1} \log^k \left( \frac{z}{1 - z} \right) \, dz.
\]

In particular, \( k = 1 \) yields

\[
\frac{\pi^2}{6} = \zeta(2) = S_2(\infty) = \frac{1}{2} \int_{0}^{1} \log^2 \left( \frac{z}{1 - z} \right) \, dz.
\]

Remark 7. We may interchange the integral and the sum of the series in (2.5), by restricting \( t \) to a closed compact set, e.g., \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \), satisfying \( \text{Re}(1-t), \text{Re}(1+t) > 0 \) as that in (2.3), in order to guarantee uniform convergence of the integral representation. (Similar discussion is omitted for the multiple beta function, defined next.)
Definition 8. The multiple beta function \([5, \text{Ch. 49}]\) is defined as

\[
B(\alpha_1, \ldots, \alpha_m) := \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_m)}{\Gamma(\alpha_1 + \cdots + \alpha_m)} = \int_{\Omega_m} \prod_{i=1}^{m} x_i^{\alpha_i-1} dx, \quad (2.6)
\]

where \(\Omega_m = \{(x_1, \ldots, x_m) \in \mathbb{R}_{>0}^m : x_1 + \cdots + x_{m-1} < 1, \ x_1 + \cdots + x_m = 1\}\) and the integral representation requires \(\text{Re}(\alpha_1), \ldots, \text{Re}(\alpha_m) > 0\).

Following the same idea as that in Example 6, we first have, from (2.4),

\[
B\left(1 - \xi_0^m t, \ldots, 1 - \xi_{m-1}^m t\right) = \frac{1}{(m-1)!} \sum_{k=0}^{\infty} S_{m_k}(\infty) t^{mk}.
\]

Then, apply the integral representation (2.6), expand the integrand as a power series in \(t\), and compare coefficients of \(t\), to obtain the following integral representation.

Theorem 9. For all \(m, k \in \mathbb{Z}_{>0}\) with \(m \geq 2\),

\[
S_{m_k}(\infty) = \frac{(-1)^{mk} (m-1)!}{(mk)!} \int_{\Omega_m} \log^{mk} \left(\prod_{j=0}^{m-1} x_j^{\xi_j^m} \right) dx.
\]

Corollary 10. In particular, the case \(k = 1\) implies for integer \(m \in \mathbb{Z}_{>1}\) that

\[
\zeta(m) = \frac{(-1)^m}{m} \int_{\Omega_m} \log^{m} \left(\prod_{j=0}^{m-1} x_j^{\xi_j^m} \right) dx,
\]

or alternatively

\[
\zeta(m) = \frac{(-1)^m}{m} \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \log^{m} \left(x_1^{\xi_0^m} \cdots x_{m-1}^{\xi_{m-1}^m} \right) dx_{m-1} \cdots dx_1.
\]

3 Acknowledgment

The author would like to thank Dr. Jakob Ablinger for his help on harmonic sums; Prof. Johannes Blümlein for his handwritten notes on the proof of (2.2); and especially his mentors, Prof. Peter Paule and Prof. Carsten Schneider, for their valuable suggestions.

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