Fermionic Casimir effect in an external magnetic field

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Abstract

The influence of an external constant uniform magnetic field on the Casimir energy density of a Dirac field under antiperiodic (and periodic) boundary condition is computed by applying Schwinger’s proper time method. The result thus obtained shows that in principle, under suitable conditions, the magnetic field can enhance the fermionic Casimir energy density.

Introduction

The Casimir effect can be generally defined as the effect of non-trivial space topology on the vacuum fluctuations of relativistic quantum fields. The corresponding change in the vacuum fluctuations appears as a shift in the vacuum energy density and a resulting vacuum pressure. In the case of the electromagnetic Casimir effect, there are three experiments involving metallic surfaces. The results, particularly the two more recent ones, are in accord with theoretical predictions.

The Casimir effect has been computed for fields other than the electromagnetic and boundary conditions different from the one implemented by conducting surfaces. The fermionic Casimir effect is of particular importance due, e.g., to the fundamental role played by the electron in QED and the quarks in QCD; it was first computed by Johnson for applications in the MIT bag model, in which the Casimir energy density is an important ingredient. For a massless Dirac particle Johnson’s result predicts an energy density 7/4 times the energy density of the electromagnetic Casimir effect. In the case of the Casimir effect of an electrically charged quantum field it is natural and important to ask how an external electromagnetic

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field may influence the fluctuations and, consequently, the effect. Indeed, we should expect on physical grounds the existence of such an influence and it is necessary to calculate its features and magnitude in order to clarify its role and to obtain a deeper understanding of the Casimir effect. Even if the conclusion turns out to be that the magnitude of the effect is negligible in realistic situations, nonetheless it can be argued that similarly to other vacuum effects, such as the Scharnhorst effect [7], it is a matter of first principles and as such it must be investigated. Here we propose to investigate the influence of an external field by considering a Dirac quantum field under antiperiodic (and periodic) boundary conditions [8]. These choices of geometry and quantum and external fields avoid technicalities in the formalism and will permit us to focus our attention on the fundamental issue, which is the physical effect of the external field on the fermionic vacuum energy density. Once this influence is understood the path is open to consider more complicated geometries and external fields as well as other quantum vacua.

**Fermionic Casimir energy density for antiperiodic boundary conditions**

In order to accomplish the investigation which we have described above we will employ the method proposed by Schwinger which stems originally from his source theory [9]. Since the method has been clearly explained by Schwinger [9] and already applied to several situations [10, 11] we will recall briefly its main features. The vacuum energy is given by

\[ E_0 = -\frac{W^{(1)}}{T}, \]  

where \( W^{(1)} \) is the one-loop effective action and \( T \) is the duration of the measurement. In the case of a fermionic field Schwinger’s proper time formula for the effective action is given by

\[ W^{(1)} = \frac{i}{2} \int_{s_o}^\infty \frac{ds}{s} Tr e^{-isH}, \]

where \( s_o \) is a cutoff in the proper time \( s \), \( Tr \) means the total trace and \( H \) is the proper time Hamiltonian, which for charged fermions in an external electromagnetic field reads

\[ H = (P - eA)^2 - (e/2)\sigma_{\mu\nu}F^{\mu\nu} + m^2, \]

where \( P \) has components \( P_\mu = -i\partial_\mu \), \( e \) is the charge of the Dirac particle, \( A \) is the electromagnetic potential, \( F \) is the electromagnetic field, which here appears contracted with the combination of gamma matrices \( \sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2 \), and \( m \) is the mass of the Dirac particle. We will consider first the influence of a uniform external magnetic field in the case of antiperiodic boundary condition on the \( OZ \)-direction. The imposition of antiperiodic boundary conditions gives for the \( z \)-component of \( P \) the eigenvalues \( \pi n/a \), where \( n \) runs in the set of odd integers. The other space components of \( p \) are constrained into the Landau levels created by the magnetic field \( B \); we call \( B \) the component of \( B \) along the \( OZ \)-axis and for convenience the axis positive direction is chosen in such a way that \( eB \) is positive. The trace in (2) is then given by:

\[ Tr e^{-isH} = e^{-ism^2} \sum_{\alpha = \pm 1} 2 \sum_{n = -\infty}^{\infty} e^{-is[\pi(2n + 1)/a]} \sum_{n' = 0}^{\infty} e^{BA} e^{-iseB(2n' + 1 - \alpha)} \int \frac{dt d\omega}{2\pi} e^{i\omega^2}, \]
where the first sum takes care of the four components of the Dirac spinor, the second sum is over the eigenvalues stemming from the antiperiodic boundary condition, the third sum is over the Landau levels with the corresponding multiplicity factor due to degeneracy, and the integral range is given by the measurement time $T$ and by the continuum of eigenvalues $\omega$ of the operator $P^0$. Following Schwinger we apply Poisson sum formula \cite{12} to the second sum and sum straightforwardly over the Landau levels to recast the trace into the following form:

$$Tr e^{-isH} = \frac{T aA e^{-ism^2}}{4\pi^2 i} \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{i(an)^2/4s} \right] [1 + iseB L(iseB)] ,$$

where $L(\xi) = \coth \xi - \xi^{-1}$ is the Langevin function. Taking (3) into (2) we obtain for the effective action:

$$W^{(1)} = L^{(1)}(B) TaA - \rho_{\text{total}}^{\text{AP}}(a, B) aAT ,$$

where on the r.h.s. the first term gives the (unrenormalized) Heisenberg-Euler effective Lagrangian \cite{13}:

$$L^{(1)}(B) = \frac{1}{8\pi^2} \int_{s_0}^{\infty} ds s^3 e^{-ism^2} iseB \coth(iseB) ,$$

and the second term gives the (still cutoff-dependent) Casimir energy density

$$\rho_{\text{total}}^{\text{AP}}(a, B) = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} (-1)^n \int_{s_0}^{\infty} ds s^3 e^{-ism^2+i(an)^2/4s} [1 + iseB L(iseB)] ,$$

which is the quantity we are interested in. The Heisenberg-Euler Lagrangian makes no contribution to the Casimir energy density, since it exhibits no dependence on the separation $a$. Therefore it does not take part of the expression for the Casimir energy density, which is set to zero when $a \to \infty$. We now make the change of the integration variable to $\sigma = a^2/4s$. The limit in which the cutoff $s_0$ goes to zero can be taken and in the resulting expression the part of the Casimir energy density which exists in the absence of the external magnetic field can be expressed in terms of the modified Bessel function $K_2$ (formula 3.471.9 in \cite{14}). The resulting expression for the total vacuum energy density is:

$$\rho_{\text{total}}^{\text{AP}}(a, B) = 2 \frac{(am)^2}{\pi^2 a^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(amn)$$

$$+ \frac{eB}{4\pi^2 a^2} \sum_{n=1}^{\infty} (-1)^n \int_{0}^{\infty} d\sigma e^{-n^2\sigma / 4 - (am)^2 / \sigma} L(eBa^2 / \sigma) .$$

When there is no external magnetic field $B$ the Casimir energy density is given by the first term on the r.h.s. of equation (4)

$$\rho_{\text{AP}}(a, 0) = 2 \frac{(am)^2}{\pi^2 a^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(amn) .$$

Here we are interested in the second term on the r.h.s. of equation (4), which measures the influence of the external magnetic field in the Casimir energy density. The contribution of the magnetic field is given by a quadrature, which is strictly positive, decreases monotonically as $n$ increases and goes to zero in the limit $n \to \infty$. Consequently, we have by Leibnitz criterion a convergent alternating series in (4) and we may conclude that the external magnetic field
increases the fermionic Casimir energy density. This is the main result of this work, which elucidates part of the interplay between two of the most fundamental phenomena in relativistic quantum field theory, namely: the Casimir effect and the fermionic vacuum properties described by the Euler-Heisenberg effective Lagrangian. Remark that the same fermionic vacuum acted by an external field under antiperiodic boundary condition can be studied from a different point of view, in which we look for changes in the vacuum constitutive relations due to confinement. This point of view requires a different analysis and leads to quite different physical phenomena\[15].

**Strong magnetic field regime for antiperiodic boundary conditions**

Consider the strong field regime in which changes in the charged vacuum may be easier to occur \[16\]. The integral in equation (9) is dominated by the exponential function, whose maximum is \( \exp(-amn) \) and occurs at \( \sigma = 2am/n \). Therefore, in the strong field regime we are justified in substituting the Langevin function by \( 1 - \xi - 1 \), which in the cases \( am \ll 1 \) and \( am \gg 1 \) is characterised, respectively, by \( |B| \gg |\phi_o|/a^2 \) and \( |B| \gg (|\phi_o|/a^2)(a/\lambda_c) \), where \( \phi_o \) is the fundamental flux \( 1/e \) and \( \lambda_c \) is the Compton wavelength \( 1/m \). For antiperiodic boundary conditions in the strong field regime, in both cases, the second term in (9) can also be expressed in terms of a modified Bessel function (formula 3.471,9 in \[14\]), and the Casimir energy density can be written as:

\[
\rho_{AP}(a, B) = \frac{eBm}{\pi^2 a^2} (am)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} K_1(amn) .
\]  (11)

In the limit \( am \ll 1 \), after discarding second order terms in the expansions of the corresponding Bessel functions (formula 8.446 in \[14\]), we obtain from (10) and (11):

\[
\rho_{AP}(a, 0) = \frac{7}{180a^4} \pi^2 (am \ll 1) ,
\]  (12)

which is essentially the result obtained in \[8\] and

\[
\rho_{AP}(a, B) = -\frac{eB}{12a^2} \left( am \ll 1, |B| \gg |\phi_o|/a^2 \right) .
\]  (13)

From (12) and (13) we can see that the ratio between the Casimir energy density with the strong external magnetic field and the Casimir energy without this field is

\[
\frac{\rho_{AP}(a, B)}{\rho_{AP}(a, 0)} = \frac{15}{17\pi^2} \frac{B}{\phi_o/a^2} \left( am \ll 1, |B| \gg |\phi_o|/a^2 \right) .
\]  (14)

A rough numerical estimation can be produced. If we set \( a = 1\mu m \) we find that \( \rho_{AP}(a, B) \approx \rho_{AP}(a, 0) \times 10^{-4} B/Tesla \). We now examine the opposite limit, namely \( am \gg 1 \). In this limit we can use the asymptotic expansion of the corresponding Bessel functions (formula 8.451,6 in \[14\]) to obtain from (10) and (11):

\[
\rho_{AP}(a, 0) = -4 \left( \frac{am}{2\pi^3} \right)^{3/2} e^{-am} \left( am \gg 1 \right) ,
\]  (15)
\[
\rho_{AP}(a, B) = -\frac{eB}{a^2} \left( \frac{am}{2\pi^3} \right)^{1/2} e^{-am} \quad (am \gg 1, |B| \gg (|\phi_o|/a^2)(a/\lambda_c)) .
\]  

(16)

and from (15) and (16):
\[
\frac{\rho_{AP}(a, B)}{\rho_{AP}(a, 0)} = \frac{B}{(2\phi_o/a^2)(a/\lambda_c)} \quad (am \gg 1, |B| \gg (|\phi_o|/a^2)(a/\lambda_c)) .
\]  

(17)

We can also try a rough numerical estimation for this case. For \(a = 1\mu m\) we have \(\rho_{AP}(a, B) \approx 10^{-10}\rho_{AP}(a, 0)B/Tesla\).

**Final remarks and conclusion**

Results for the periodic case can also be obtained in the same way, here we will only give the main result. The total Casimir energy density for periodic boundary conditions is given by
\[
\rho_A(a, B) = 2 \frac{(am)^2}{\pi^4 a^4} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(amn)
\]
\[+ \frac{eB}{4\pi^2 a^2} \sum_{n=1}^{\infty} \int_0^{\infty} d\sigma e^{-n^2\sigma/(4-(am)^2)/\sigma} L(eB a^2/\sigma) .
\]  

(18)

The first term is the Casimir energy density in the absence of the external magnetic field, the second one takes into account the presence of this field. Notice that the main difference between the antiperiodic and periodic results is the absence of the factor \((-1)^n\) in the summations. As in the antiperiodic case, we can investigate the strong field regime (and also the weak field regime) and repeat the analysis we did for the antiperiodic case. These results will be published elsewhere.

Summing up, we have obtained the general expression of the fermionic Casimir energy density under the effect of an external magnetic field for antiperiodic and periodic boundary conditions. The results show that the external field increases the Casimir energy density and reveals the interplay between two agents which are known to affect vacuum fluctuations, namely: external fields and non-euclidean space topology. We have derived an expression for the vacuum energy density in the regime of strong magnetic field and in this regime we have also obtained the small and large mass limits of this energy density. Our formalism has a natural extension to more complicated gauge groups and consequently may be useful in the investigation of the QCD vacuum; this will be the subject of forthcoming work. As a final comment we observe that it may be also interesting to investigate the combined action of confinement and applied magnetic field in the analogues of Casimir effect which occurs in condensed matter physics and critical systems [17].

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