Holographic Entanglement Entropy with Momentum Relaxation

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ABSTRACT

We studied the holographic entanglement entropy for a strip and sharp wedge entangling regions in momentum relaxation systems. In the case of strips, we found analytic and numerical results for the entanglement entropy and showed the effect on the minimal surface by the electric field. We also studied the entanglement entropy of wedges and confirmed that there is a linear change in the electric field. This change is proportional to the thermoelectric conductivity, $\bar{\alpha}$, that can be measured.

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1 Introduction

Entanglement entropy plays a prominent role in various areas of physics such as quantum field theory, gravity theory and condensed matter physics. However, it is very difficult to evaluate this amount because of non-local property. Fortunately, an effective way to calculate entanglement entropy has recently been developed by Ryu and Takayanagi [1]. Since this method is based on AdS/CFT correspondence [2] which relates gravitational systems to quantum field theories, it may shed a light on understanding quantum gravity [3, 4]. Apart from this, the method is useful to study various interesting cases. For examples, it has been used to distinguish different phases e.g., [5, 6, 7] and to study thermalization under quantum quenches e.g., [8, 9, 10, 11]. Further studies with the method show us new universal properties of generic CFT e.g., [12, 13, 14, 15, 16, 17]. Even for a particular non-conformal field theory, it turns out that Ryu and Takayanagi’s proposal is still useful by considering a top-down study on the field theory and the corresponding supergravity solution [18, 19, 20, 21, 22, 23, 24].

Entanglement entropy is a very important quantity but, in most cases, it is difficult to calculate. It is also unclear whether this amount can be measured easily[1]. Therefore, it is hard to tell if Ryu and Takayanagi’s method is valid in various cases. In this study, we try to find examples where entanglement entropy changes can be represented by other easily measurable physical quantities. Like thermal entropy, the entanglement entropy responses to other external sources. One of such sources which can be produced easily in a laboratory is the electric field. Thus we will investigate how the entanglement entropy changes under the electric field using holographic methods.

Accordingly, to demonstrate a system in the electric field holographically, we need to consider a gravity dual describing the system. The Ryu-Takayanagi’s formula and the gravity dual allow us to study the response of entanglement entropy to the electric field. For the sake of simplicity, we introduce a time-independent electric field, i.e, DC electric field. To realize such a situation, we take the momentum relaxation into account. This is because infinite currents can occur without the momentum relaxation. The gravity duals have been developed to study holographic DC conductivities in AdS/CMT e.g., [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. Among these backgrounds, we adopted a background geometry of an axion model, where a scalar field breaks translation invariance explicitly [28].

In the absence of the electric field, holographic entanglement entropy with small momentum relaxation was studied in [47]. In our work, we performed analytic study on a limit of the entanglement entropy with the strip entangling region. In addition to this, we provide numerical results without taking any limit. This calculation shows that entanglement entropy becomes larger with more charge density and stronger momentum relaxation.

Furthermore, we studied how the strip type entanglement entropy changes by turning on the constant electric field. And we found that the minimal surface anchored to the strip is tilted by

\footnote{Recently quantum purity, Rényi entanglement entropy and mutual information were measured in experiments [25].}
the electric field and the deformation is proportional to the thermoelectric conductivity $\bar{\alpha}$ in the conductivity tensor. This is a quite interesting result because the thermoelectric conductivity is a clearly measurable quantity. However, the entanglement entropy changes from the deformation do not appear in the linear order of the electric field. To see the effect, we need to consider the quadratic order gravity dual but it is beyond our present work.

Instead we consider the wedge type entangling region whose symmetric axis is not orthogonal to the electric field\(^2\). Finally, we took a sharp wedge limit and obtained the entanglement entropy changes at the linear level of the electric field. And we confirmed that the response to the electric field is proportional to the anticipated thermoelectric conductivity $\bar{\alpha}$ as we expected. This result is also remarkable because the thermoelectric conductivity or the Nernst signal could reflect the existence of the quantum critical point even in the normal phase of the superconductor\(^4\). Thus it is suggested that the entanglement entropy may be associated with the quantum critical point.

This paper is organized as follows: In section 2 we introduce the bulk solution dual to a momentum relaxation system with the electric field and summarize how the conductivities can be related to the bulk fields. In section 3, we study on the holographic entanglement entropy for the strip entangling region. In section 4, we consider the holographic entanglement entropy for the wedge region. In section 5, we conclude our work.

## 2 A Gravity Dual of Momentum Relaxation with Electric Field

In this section we review a gravity dual to a momentum relaxation system in the electric field. The geometry has been discussed to study finite DC conductivity \(e.g.\), [28, 40]. The momentum relaxation in physical systems is essential to obtain finite DC conductivity. Let us start with a simple model with momentum relaxation:

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R + 6 - \frac{1}{4} F^2 - \frac{1}{2} \sum_{I=1}^{2} (\partial \chi_I)^2 \right).
\] (1)

This system admits a black brane solution as follows:

\[
ds^2 = -U(r)dt^2 + r^2 (dx^2 + dy^2) + \frac{dr^2}{U(r)},
\] (2)

\[A = q \left( \frac{1}{r_h} - \frac{1}{r} \right) dt, \quad \chi_I = (\beta x, \beta y),
\] (3)

with \(U(r) = \left( r^2 - \frac{\beta^2}{r} - \frac{M}{r} + \frac{q^2}{4\pi^2} \right)^{\frac{1}{2}}\). The massless scalar field, $\chi_I$, is associated with momentum relaxation, breaking the translational symmetry and inducing finite DC conductivity. To see this, let

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\(^2\)The holographic entanglement entropy for the wedge entangling region includes the corner contribution that represents a universal characteristic of CFT [14, 15].

\(^3\)We set the AdS radius to 1 and $F = dA$. 

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us turn on the external electric field and other related fields along the $x$-direction

$$\delta A_x = -E_x t + a_x(r), \quad \delta g_{tx} = r^2 h_{tx}(r), \quad \delta g_{rx} = r^2 h_{rx}(r), \quad \delta \chi_x = \psi_x(r), \quad \delta \chi_x = \psi_x(r),$$

where $E_x$ corresponds to the external field and other fluctuations are dual to the electric current and the heat current in the boundary system.

Then, the linearized equations of motion are given as follows:

$${h''_{tx}(r) + \frac{4h'_{tx}(r)}{r} - \frac{\beta^2 h_{tx}(r)}{r^2 U(r)} + \frac{q a'_x(r)}{r^4} = 0, \quad \psi_x(r) - \beta h_{rx}(r) - \frac{qE_x}{\beta r^2 U(r)} = 0,}$$

$${a''_x(r) + \frac{U'(r)a'_x(r)}{U(r)} + \frac{q h'_{tx}(r)}{U(r)} = 0,}$$

$${\psi''_x(r) + \left( U'(r) \frac{a'_x(r)}{U(r)} + \frac{2}{r} \right) \psi'_x(r) - \beta h'_{tx}(r) - \beta \left( \frac{U'(r)}{U(r)} + \frac{2}{r} \right) h_{rx}(r) = 0,}$$

where Eq. (5) comes from the Einstein equation, and Eq. (6) and (7) are the equations of motion of the matter fields. Note that all the equations above are not independent because Eq. (7) can be obtained by rearranging the other equations. As a consequence, only three equations in (5) and (6) determine the profile of the fluctuations for given boundary conditions. Another thing we need to know is that $h_{rx}$ is not a dynamical field and it plays a role of a Lagrange multiplier. In fact, this is a shift vector in an ADM decomposition along the holographic radial direction. We will discuss the gauge fixing later.

The electric current and the heat current using gauge/gravit dualit are given by

$${J^x = \lim_{r \to \infty} J(r), \quad Q^x = T^{tx} - \mu J^x = \lim_{r \to \infty} Q(r)},$$

where

$$J(x) \equiv \sqrt{-g} F^{xt} = U(r) a'_x(r) + q h_{tx}(r),$$

$$Q(r) \equiv U^2(\delta g_{tx})' - A_t(r) J(r).$$

In addition, it can be easily checked that $J(r)$ and $Q(r)$ remain as constants along the holographic radial direction $r$ by using the equations of motion (5) and (7). Therefore, the boundary currents can be computed at the horizon $r = r_h$

$${J^x = \lim_{r \to r_h} J(r), \quad Q^x = \lim_{r \to r_h} Q(r),}$$

Near the horizon the fields behave as

$$a_x \sim \frac{E_x \log(r - r_h)}{4\pi T} + a_x^{(0)} + O(r - r_h), \quad h_{tx} \sim h_{tx}^{(0)} + O(r - r_h), \quad h_{rx} \sim \frac{E_{rx}}{r^2 U(r)} + h_{rx}^{(0)} + O(r - r_h), \quad \psi_x \sim \psi_x^{(0)} + O(r - r_h),$$

$$U(r) \sim 4\pi T(r - r_h) + \cdots.$$
where the Hawking temperature $T$ is given by $T = \frac{U'(r_h)}{4\pi}$ and the logarithmic term appears due to the ingoing boundary condition. By solving the equations of motion near the horizon, one can obtain

$$H_{r,x} = r_h^2 h_{t_x}^{(0)} , \quad h_{t_x}^{(0)} = \frac{-q E_x}{\beta^2 r_h^2} .$$

Using these results, the linear response theory yields the following electric conductivity ($\sigma = J^x/E_x$) and thermoelectric conductivity ($\bar{\alpha} = Q^x/E_x$):

$$\sigma = 1 - \frac{q h_{t_x}^{(0)}}{E_x} = 1 + \frac{\mu^2}{\beta^2} , \quad \bar{\alpha} = -\frac{4\pi r_h^2 h_{t_x}^{(0)}}{E_x} = \frac{4\pi q}{\beta^2} .$$

So the constant $H_{r,x}$ is given by a combination of physical parameters as $H_{r,x} = -\frac{1}{4\pi} \bar{\alpha} E_x$.

The following sections will consider minimum surfaces in this geometry. It is convenient to use a formal expansion parameter $\lambda$, which will be taken as 1 after the calculation. This trick allows us to write the metric in the following form:

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2(dx^2 + dy^2) + 2\lambda \left( \delta_{tx}(r)dt dx + \delta_{rx}(r)dr dx \right) + O(\lambda^2) .$$

In addition to the conductivities, other physical quantities such as temperature, energy density, charge density, and entropy density can be read from the geometry as follows:

$$T = \frac{U'(r_h)}{4\pi} = \frac{3r_h}{4\pi} \left( 1 - \frac{\beta^2}{6r_h^2} - \frac{q^2}{12r_h^4} \right) ,$$

$$\epsilon = 2M = \frac{4r_h^4}{2} - \frac{q^2}{2r_h^4} , \quad \rho = q , \quad s = 4\pi r_h^2 .$$

These quantities characterize the dual system to the unperturbed geometry with $\lambda = 0$.

### 3 Holographic Entanglement Entropy in Electric Field: Strip Entangling Region

#### 3.1 Basic Setup

In this subsection, we use the Ryu-Takayanagi formula to obtain entanglement entropy for the strip region. The strip also extends along a direction perpendicular to the electric field. To describe the minimal surface anchored to the strip, we may consider a two dimensional surface with the following map in the target geometry (14):

$$t = 0 , \quad r = r(\sigma_1) , \quad x = \sigma_1 , \quad y = \sigma_2 .$$
where $y$ coordinate indicates the infinitely extending direction. Then, one can find the action for the minimal surface as follows:

$$A = \int_{-L/2}^{L/2} dy \int_{-1/2}^{1/2} dx \sqrt{\frac{r^2 r''}{U(r)}} + r^4 + 2\lambda r^2 \delta g_{r x} r' ,$$  \hspace{1cm} (18)$$

where $L$ is the length of the strip along $y$-direction. This is our basic action for the holographic entanglement entropy. Or, if we define another convenient coordinates $z \equiv \frac{1}{r}$ and $\sigma \equiv x/l$, we can write down the action in terms of surfaces, $z$ as follows:

$$A = \frac{L}{l} \int_{-1/2}^{1/2} d\sigma \sqrt{\frac{z^2}{z f(z)}} + \frac{1}{z^4} - 2\lambda \frac{z'}{z^4} \delta g_{r x}(z) ,$$  \hspace{1cm} (19)$$

where $f(z) = \frac{1}{z^2} - \frac{\tilde{\beta}^2}{z} - \tilde{M} z + \frac{\tilde{q}^2}{4} z^2$ with $\tilde{\beta} = \beta l$, $\tilde{M} = M l^3$ and $\tilde{q} = q/l^2$. $r(x)$ can always be recovered by $r(x) = \frac{1}{lz(x/l)}$.

The Lagrangian in (19) has no explicit dependence of $\sigma$. So there is a conserved quantity $H$ given by

$$H = \frac{L}{l} \frac{1 - \lambda \delta g_{r x} z'}{z \sqrt{f(z)}} + z^2 (1 - 2\lambda z^2 \delta g_{r x}) = \frac{L}{lz^2} ,$$  \hspace{1cm} (20)$$

where $z_* \equiv z(\sigma_*)$ is the location of the tip of the minimal surface. Since we are considering regular surfaces, $z'(\sigma)$ vanishes at the tip of the surface. From this relation one can find the integrated first order equations of motion as follows:

$$z' = \pm \sqrt{\frac{f(z)}{z}} \left( \frac{z_*^4 - z^4}{z} \right) - \lambda \frac{f(z) \delta g_{r x}(z)}{z^2} \left( \frac{z_*^4 - z^4}{z^2} \right) + \mathcal{O}(\lambda^2)$$

$$= \pm \frac{\sqrt{f(z)} (z_*^4 - z^4)}{z} + \lambda E_x \frac{\tilde{\alpha} l^4 (z_*^4 - z^4)}{4\pi} + \mathcal{O}(\lambda^2) ,$$  \hspace{1cm} (21)$$

where we took a gauge $\delta g_{r x} = \frac{\tilde{h}_{r x}}{\sqrt{f(z)}}$, which is consistent with the boundary conditions. Plugging the above equations into (19), we obtained the regularized area for the minimal surface as follows:

$$\mathcal{A}_{\text{reg}}^{\text{strip}} = 2\frac{L}{l} \int_{\epsilon/l}^{z_*} dz \frac{z_*^2}{z^3 \sqrt{f(z)} (z_*^4 - z^4)} ,$$  \hspace{1cm} (22)$$

where $\epsilon$ is the length scale corresponding to the UV cutoff. Here, one can notice that the effect from the electric field is encoded in the location of the tip $z_*$. By using the Ryu-Takayanagi formula, the entanglement entropy for the strip is:

$$S_{EE}^{\text{strip}} = \frac{\mathcal{A}_{\text{reg}}^{\text{strip}}}{4 G_N} ,$$  \hspace{1cm} (23)$$

\footnote{In the calculation we took a gauge $h_{r x} = \frac{\tilde{h}_{r x}}{\sqrt{f(z)}}$. It is legitimate because $h_{r x}$ plays a role of a Langrange multiplier at the linear level and the choice satisfies the regularity condition at the horizon and near the boundary of AdS space. Furthermore, our final result in this work depends on the gauge invariant quantities, such as the thermoelectric conductivity $\tilde{\alpha}$ and geometric angle $\delta$. For this reason, the gauge fixing is believed to be adequate to consider this calculation. In addition, we speculate that the gauge fixing would be clearer if we consider the second order effect of the electric field on the geometry. This is because additional constraints must be considered for $g_{r x}$.}
where \( G_N \) is the 4-dimensional Newton constant. In order to show the numerical result, we define the following refined function which is \( \frac{4G_N l}{L} \) times the finite part of the entanglement entropy up to the minus sign:

\[
S_{\text{strip}}^R = \frac{2l}{\varepsilon} - \frac{4G_N l}{L} S_{\text{EE}}^\text{strip}.
\]  

(24)

Now, let us look at the equation of motion for the minimal surface. The equation of motion can be derived from the action (19) and the solution is expanded in terms of \( \lambda \) as follows:

\[
z(\sigma) = z_0(\sigma) + \lambda z_1(\sigma) + O(\lambda^2).
\]  

(25)

Then the equation of motion for each order is given by

\[
\begin{align*}
z_0'' - \frac{z_0' f'(z_0)}{2 f(z_0)} &+ 2 z_0 f(z_0) + \frac{z_0'^2}{z_0} = 0, \\
z_1'' + \frac{z_1 f''(z_0)}{2 f(z_0)} &+ 2 z_1 f'(z_0) + \frac{z_0' (-z_1 f''(z_0) z_0' + \delta g_{rr}(z_0) f'(z_0) z_0'^2 - 2 f'(z_0) z_1')}{2 f(z_0)} \\
&+ 2 f(z_0) (z_1 - 3 z_0 \delta g_{rr}(z_0) z_0') + \delta g_{rr}'(z_0) z_0' z_0'^3 + \frac{2 z_0' z_1' - \delta g_{rr}(z_0) z_0'^3}{z_0} - \frac{z_1 z_0'^2}{z_0^2} = 0.
\end{align*}
\]  

(26)

(27)

Here, the zeroth order solution is given by an even function from the symmetry of (26). Furthermore, suppose \( z_1(\sigma) \) is a solution of (27), then one can easily show that \( -z_1(-\sigma) \) is also a solution. Therefore the solution \( z_1 \) of (27) is an odd function. By using this fact, one can recognize that \( z_* \) is larger than \( z_0(0) \) because \( z_0'(0) \) vanishes but there is a non-vanishing \( z_1'(0) \). We found a solution, \( z_0(\sigma) + \lambda z_1(\sigma) \), numerically and plot the solution in Fig. 1.

Figure 1: The intersection of a deformed minimal surface by the electric field (Solid curve) with \( \tilde{\beta} = 1, \tilde{M} = 1 \) and \( \tilde{q} = 1 \): The dotted line denotes the minimal surface before applying the electric field.
To find the effect on entanglement entropy, let us come back to (22). In order to get the $z_*$, one can consider a shift of the tip along the $\sigma$ direction. It is denoted by $\lambda \Delta$, then the regularity condition of the tip is given by

$$z_0' (\lambda \Delta) + \lambda z_1' (\lambda \Delta) = 0 .$$

(28)

This condition gives us $\Delta = -z_1'(0)/z_0''(0)$. Therefore, it turns out that $z_*$ is

$$z_* \sim z_0 (\lambda \Delta) + \lambda z_1 (\lambda \Delta) = z_0 (0) + \frac{1}{2} \lambda^2 \Delta^2 z_0''(0) + \lambda^2 \Delta z_1'(0) + \mathcal{O}(\lambda^3)
= z_0 (0) - \frac{\lambda^2 z_1'(0)^2}{2 z_0''(0)} + \mathcal{O}(\lambda^3) .$$

(29)

From this, we found that the tip change is the order of $\lambda^2$. However, our background (14), is only valid in the linear order. Therefore, one needs a quadratic background metric to see the effect on entanglement entropy for the strip case. We leave this as a future work.

### 3.2 Numerical Result for Holographic Entanglement Entropy

Now we study the holographic entanglement entropy (22) numerically. As examined in the previous subsection, there is no linear variation of the entanglement entropy to the electric field. Therefore, we will focus only on the zeroth-order results without the electric field.

First, we define the following function to facilitate numerical computation:

$$\tilde{A}_0 \equiv A_{reg} = 2L \int_{\tilde{z}}^{z_0^*} dz z_0^2 \sqrt{f(z) (z_0^* - z)} ,$$

where $z_0^* \equiv z_0 (0)$ and $\tilde{z} \equiv \epsilon/l$. For convenience, we replace $\tilde{M}$ with $z_h$ and the other parameters using the relation $f(z_h) = 0$. Then, the useful expression for $f(z)$ is given by

$$f(z) = \left( z - z_h \right) \left( \frac{z^3 q^2 z_h^3 + 2 \beta^2 z_h^3 - 4 z_h^2 - 4 z z_h - 4 z^2}{4 z^2 z_h^3} \right) .$$

(31)

By adopting a scaled coordinate $u \equiv z/z_0^*$, (30) is written in terms of variables $\xi \equiv z_0^*/z_h$, $\tilde{q} \equiv \tilde{q} z_h^2$ and $\tilde{\beta} = \beta z_h$. Then the area functional becomes

$$\tilde{A}_0 = \frac{4L}{z_0^*} \int_{\tilde{z}}^{1} du \frac{1}{u^2 \sqrt{(1 - u^4) (1 - \xi u) (4 + 4 \xi u + 4 \xi^2 u^2 - 2 \beta^2 \xi^2 u^2 - \tilde{q}^2 \xi^3 u^3)}} .$$

(32)

In addition the temperature of the system can be written in terms of the dimensionless parameters as follows:

$$\tilde{T} \equiv T\ell = \frac{1}{16 \pi z_h} \left( 12 - \tilde{q}^2 z_h^2 - 2 \tilde{\beta}^2 z_h^2 \right) = \frac{\xi}{16 \pi z_0^*} \left( 12 - q^2 - 2 \beta^2 \right) .$$

(33)

Thus the parameters in terms of the original physical quantities are given by $\xi = \frac{z_h}{z_0^*}$, $z_h = 1/(r_h \ell)$, $\tilde{\beta} = \beta / r_h$ and $\tilde{q} = q / r_h$. 

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Figure 2: The refined function $\hat{S}_{\text{strip}}^{\text{R}}$ at $\hat{\beta} = 0$ (a), $\hat{\beta} = 0.5$ (b), $\hat{q} = 0$ (c) and $\hat{q} = 1$ (d) : The lines on the surfaces denote $\hat{T} = 0$ (Solid), 0.2 (Shortdashed), 0.3 (Dotdashed), 0.4 (Longdashed) and these lines are plotted in Fig. 3.

Also, (21) and the regularity determines $z_0^*$ as follows:

$$z_0^* = \frac{1}{\int_0^1 du \frac{4u^2}{\sqrt{(1-u^4)(1-\xi u)}} \left(4 + 4\xi u + 4\xi^2 u^2 - 2\beta^2 \xi^2 u^2 - \bar{q}^2 \xi^3 u^3\right)}.$$  \hspace{1cm} (34)

Therefore, for given $\xi, \bar{q}$ and $\bar{\beta}$, the regularized area of minimal surface is determined in terms of $\hat{\epsilon}$. In general, this area has a leading term that is proportional to the inverse of $\hat{\epsilon}$. Subtracting this term and taking the suitable rescaling by the definition in (24), we can get the finite part of the minimal surface area that does not depend on $\hat{\epsilon}$. We show our numerical results in Fig. 2 and Fig. 3.

### 3.3 Analytic Result for Holographic Entanglement Entropy

Now we try to obtain analytic expression for the holographic entanglement entropy in a certain limit. Our formulas (32) and (34) are also useful to study the analytic form. In order to study the analytic expression, one may take a small $\xi$ limit on both equations. This limit then allows to integrate the above equations. We obtained the entanglement entropy in this limit as the following form :

$$S_{\text{EE}}^{\text{strip}} = \frac{L}{4G_N l} \left[ \frac{2l}{\epsilon} + \frac{\sqrt{2} \pi^2 \Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})^3} l^2 + \frac{\pi r_h^2 \gamma_1 \Gamma(\frac{1}{4})}{\sqrt{2} \Gamma(-\frac{1}{4})^2 \Gamma(\frac{1}{4})} l^3 + \frac{\pi r_h^3 (2 \gamma_1 - 4 - \gamma_2) \Gamma(\frac{1}{4})}{\sqrt{2} \Gamma(-\frac{1}{4})^3} l^4 \right]$$ 

$$+ \frac{r_h^4 (576 \pi^3 (3 \gamma_1^2 - 4 \gamma_2) \Gamma(\frac{1}{4}) - 5 \sqrt{2} \gamma_1^2 \Gamma(\frac{1}{4})^7)}{92160 \pi^2 \Gamma(\frac{3}{4})^5} l^5$$

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Figure 3: The refined function $\hat{S}_{\text{strip}}$ for (a) $\tilde{\beta} = 0$, (b) $\tilde{\beta} = 0.5$, (c) $\tilde{q} = 0$ and (d) $\tilde{q} = 1$. These curves represent the lines on the surfaces in Fig. 2 and correspond to the temperatures $\bar{T} = 0$(Solid), 0.2(Shortdashed), 0.3(Dotdashed) and 0.4(Longdashed).

\[ + \frac{8 \pi^2 r_h^5 \gamma_1 (2 \gamma_1 - 4 - \gamma_2) (\pi^3 \Gamma(-\frac{1}{4}) + 48 \Gamma(\frac{3}{4})^5 l^5}{3 \Gamma(-\frac{1}{4})^7 \Gamma(\frac{3}{4})^6} + O(l^6) \],

where $\varepsilon$ is taken as 1 after calculation as a formal expansion parameter and $r_h$ is given by $1/(l z_h)$. In addition, the $\gamma_1$ and $\gamma_2$ are defined by

\[ \gamma_1 \equiv \tilde{\beta}^2 z_h^2, \quad \gamma_2 \equiv \tilde{q}^2 z_h^2. \] (36)

The physical meaning of this limit ($\xi \ll 1$) is that the tip distance($z_*$) from the boundary is much smaller than that of the horizon of the black hole($z_h$). Roughly speaking, this limit is similar to $\frac{l}{z_h} \ll 1$, but it is not the equivalent limit in a strict sense. The analytic result (35) is compared with the previous numerical calculations in Fig. 4.

The above result can also have an extremal limit. At zero temperature, the charge density is given by $\tilde{q} = \sqrt{12 - 2 z_h^2 \tilde{\beta}/z_h^2}$. So the holographic entanglement entropy at zero temperature is determined to be:

\[ S_{EE}^{\text{strip}} = \frac{L}{4 G_N l} \left[ \frac{2 l}{\varepsilon} + \frac{\sqrt{2} \pi^2 \Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})^3} + \frac{64 \pi^2 \beta^2}{3 \Gamma(-\frac{1}{4})^4} l^2 + \frac{16 \pi^2 r_h (4 r_h^2 - \beta^2)}{\Gamma(-\frac{1}{4})^4} l^3 

+ \frac{8 \pi (160 \pi^2 \beta^4 \Gamma(-\frac{3}{4})^2 + (3 \beta^4 + 8 r_h^2 \beta^2 - 48 r_h^4) \Gamma(-\frac{1}{4})^6}{5 \Gamma(-\frac{1}{4})^{10}} l^4

- \frac{8 \pi^2 r_h \beta^2 (4 r_h^2 - \beta^2) (\pi^3 \Gamma(-\frac{1}{4}) + 48 \Gamma(\frac{3}{4})^5 l^5}{\Gamma(-\frac{1}{4})^7 \Gamma(\frac{3}{4})^5 l^5} + O(l^6) \right]. \] (37)
This shows the effect of momentum relaxation on the holographic entanglement entropy when the temperature is zero.

4 Holographic Entanglement Entropy in Electric Field: Wedge Entangling Region

In this section, we consider the holographic entanglement entropy with wedge entangling regions. This type of entangling surfaces has been studied in [14, 15] to consider the corner contribution of entanglement entropy. It turns out that this corner contribution contains universal information of conformal field theory.

See Fig. 5 for a cartoon of a minimal surface anchored to a wedge region. It is useful to take polar coordinates $(\rho, \theta)$ in the $x - y$ plane as the coordinates of the minimal surface. Our main concern is how much the holographic entanglement entropy changes at the linear level under the weak electric field. To see this linear response, the direction of the electric field must be rotated by an angle $\delta$, and one of the Cartesian coordinates $x$ is given by $x = -\rho \sin(\theta - \delta)$. The need for the rotation was
inferred by lessons of the previous section.

Considering the above configuration, the background geometry can be written as follows:

\[
\begin{align*}
    ds^2 &= -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2(d\rho^2 + \rho^2d\theta^2) \\
    &\quad - 2\lambda \left( g_{tx}(r)dt + g_{rx}(r)dr \right) \left( \rho \cos(\theta - \delta)d\theta + \sin(\theta - \delta)d\rho \right) + O(\lambda^2).
\end{align*}
\]

And we may identify the coordinates of the minimal surface as the following way:

\[
    t = 0, \quad r = r(\sigma_1, \sigma_2) = 1/\tilde{z}(\sigma_1, \sigma_2), \quad \rho = \sigma_1, \quad \theta = \sigma_2
\]

Then the action for the minimal surface is given by

\[
    A = \int_{-\Omega/2}^{\Omega/2} d\theta \int_0^\infty d\rho \left( \frac{\rho}{\tilde{z}} \sqrt{1 + \frac{\tilde{z}'^2 + \rho^2\tilde{z}^2}{\rho^2\tilde{z}^2 f(\tilde{z})}} + \lambda \frac{\Xi_{rx} \left( \tilde{z}' \cos(\delta - \theta) - \rho \tilde{z} \sin(\delta - \theta) \right)}{\tilde{z}^2 f(\tilde{z}) \sqrt{1 + \frac{\tilde{z}'^2 + \rho^2\tilde{z}^2}{\rho^2\tilde{z}^2 f(\tilde{z})}}} \right).
\]

We are interested only in a variation of the action under the electric field and regard the wedge as a part of a larger minimal surface like the cartoon (b) in Fig. 5. So we consider a part of the minimal surface with the region, \( \rho < R \) and we will see how the surface changes for this finite region. By this reason, the integration for the radial coordinate \( \rho \) is replaced by the integration between 0 to \( R \).

Furthermore, we will take a convenient choice of parameters obtained by a scaling with respect to \( \Omega \):

\[
    \sigma = \theta/\Omega, \quad \tilde{z} = \Omega w, \quad \tilde{\beta} = \Omega \beta, \quad \tilde{M} = \Omega^3 M, \quad \tilde{q} = \Omega^2 q.
\]

Then, the action becomes

\[
    A = \frac{1}{\Omega} \int_{-1/2}^{1/2} d\sigma \int_0^R d\rho \left( \frac{\rho}{w^2} \sqrt{1 + \frac{w'' + \Omega^2 \rho^2 w'^2}{\rho^2 w^2 f(w)}} \right. \\
    \quad \left. + \lambda \Omega^2 \Xi_{rx} \left( w' \cos(\delta - \Omega \sigma) - \rho w \sin(\delta - \Omega \sigma) \right) \right) \\
    \quad \left( \frac{w'^2 + \Omega^2 \rho^2 w'^2}{w^2 f(w)} \right) \sqrt{1 + \frac{w'^2 + \Omega^2 \rho^2 w'^2}{w^2 f(w)}}
\]

where \( f(w) \) is given by

\[
    f(w) = \frac{1}{w^2} - \frac{\tilde{\beta}^2}{2} - \tilde{M} w + \frac{\tilde{q}^2}{4} w^2.
\]

The above action determines the minimal surface by solving the equation of motion for \( w(\rho, \sigma) \). In general, the equation is not easy to solve.

Now let us take a particular limit considering a sharp wedge. Then one can use \( \Omega \) as another small parameter. The action up to the quadratic order in \( \Omega \) is given by

\[
    A = \frac{1}{\Omega} \int_{-1/2}^{1/2} d\sigma \int_0^R d\rho \left( \sum_{n=0}^3 \mathcal{L}_{(n)}^{0th} \Omega^n + \lambda \sum_{n=2}^3 \mathcal{L}_{(n)}^{1st} \Omega^n + O(\Omega^4) \right)
\]
In this regard, where \( \alpha \) where \( \lambda \), \( \sigma \), \( \alpha \) and \( \beta \) as expanded as \( A_{reg} \sim e^\omega \) and \( A_{reg} \) is given in Appendix A. In addition, the regularized area is given by

\[
L^{0th}_{(1)} = 0, \quad L^{0th}_{(2)} = \frac{\beta w^2 (w')^2 + 2 \rho w^2}{4 \rho w^2 \sqrt{\frac{(w')^2}{\rho^2} + 1}}, \quad L^{0th}_{(3)} = \frac{M w w^2}{2 \rho \sqrt{\frac{w^2}{\rho^2} + 1}}.
\]

Since \( \Omega \) is considered small, the solution \( w \) is closed to the form of \( \rho \) times a function of \( \sigma \). Thus we assume that \( w \) can be represented by a polynomial of \( \rho \). Under this assumption, we found a solution of the equation of motion as the following form:

\[
w(\rho, \sigma) = w_0(\rho, \sigma) + \lambda w_1(\rho, \sigma)
\]

\[
= \rho h_{0,1}(\sigma) + \Omega^2 \left( \sum_{q=1}^{3} h_{2,q}(\sigma) \rho^q \right) + \Omega^3 h_{3,4}(\sigma) \rho^4
\]

\[
+ \lambda \mathbb{H}_r \left( \Omega^2 g_{2,3}(\sigma) \rho^3 \cos \delta + \Omega^3 g_{3,3}(\sigma) \rho^3 \sin \delta \right).
\]

The differential equations for the functions, \( h_{p,q} \) and \( g_{p,q} \) are given in Appendix A. In addition, the regularized action to the linear order in \( \lambda \) is

\[
A_{reg} = \frac{1}{\Omega} \int_{\sigma_{min}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma) + \lambda \beta(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} L^{0th}_{(n)} \Omega^n + \lambda \sum_{n=2}^{3} L^{1st}_{(n)} \Omega^n \right),
\]

where \( \sigma_{max} \) and \( \sigma_{min} \) denote the maximum and minimum values of \( \sigma \) satisfying \( \alpha(\sigma) + \lambda \beta(\sigma) = R \). And they can be expanded as \( \sigma_{max, min} \sim \sigma^{(0)}_{max, min} + \lambda \sigma^{(1)}_{max, min} \), respectively. The regularised action to the linear order in \( \lambda \) is

\[
A_{reg} = \frac{1}{\Omega} \left( \int_{\sigma_{min}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma) + \lambda \beta(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} L^{0th}_{(n)} \Omega^n \right) \right)_{w_0 + \lambda w_1} + \lambda \left( \sum_{n=2}^{3} \int_{\alpha(\sigma) + \lambda \beta(\sigma)}^{R} d\rho \right) \left( \sum_{n=0}^{3} L^{0th}_{(n)} \Omega^n \right)_{w_0} + \mathcal{O}(\lambda^2).
\]

Plugging (47) into the above action, we can find the on-shell action as follows:

\[
A_{reg} \sim \frac{1}{\Omega} \int_{\sigma_{min}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} L^{0th}_{(n)} \Omega^n \right)_{w_0} - \frac{\lambda \mathbb{H}_r}{\Omega} \cos \delta \int_{\sigma_{max}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \Omega^2 \left( 2g_{2,3} \left( h_{1,0}^2 + 1 \right) - h_{0,1} g_{2,3} h_{1,0}^3 \right) \frac{h_{0,1} h_{1,0}^2}{h_{0,1}^2 + 1} + \mathcal{O}(\lambda^2).
\]
\[-\frac{\lambda \mathbb{H}_{rx}}{\Omega} \sin \delta \int_{-\sigma_{\text{max}}}^{\sigma_{\text{max}}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \frac{\rho \Omega^2 (2g_{3,3} (h_{0,1}^2 + 1) - h_{0,1} g_{3,3} h_{0,1})}{h_{0,1}^3 / h_{0,1}^2 + 1} \]

\[+ \frac{\lambda \mathbb{H}_{rx}}{\Omega} \int_{-\sigma_{\text{max}}}^{\sigma_{\text{max}}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \rho \left( \frac{\Omega^2 h_{0,1} \cos \delta}{\sqrt{h_{0,1}^2 + 1}} + \frac{\Omega^3 \left( \sigma h_{0,1} - h_{0,1} \sin \delta \right)}{\sqrt{h_{0,1}^2 + 1}} \right) \]

\[+ \frac{\lambda}{\Omega} \left( \sigma_{\text{max}}^{(1)} - \sigma_{\text{min}}^{(1)} \right) \left( \int_{\alpha(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} \mathcal{L}_{(n)^2}^{(0)} \right)_{\sigma=\sigma_{\text{max}}^{(0)}} \right) \]

\[-\frac{\lambda}{\Omega} \int_{-\sigma_{\text{max}}}^{\sigma_{\text{max}}} d\sigma \beta(\sigma) \left( \sum_{n=0}^{3} \mathcal{L}_{(n)^2}^{(0)} \right)_{\rho=\alpha(\sigma)} \],

(50)

where we used \(\sigma_{\text{min}}^{(0)} = -\sigma_{\text{max}}^{(0)}\). And the first order correction for \(\sigma_{\text{min}}\) and \(\sigma_{\text{max}}\) is given by

\[\sigma_{\text{min},\text{max}}^{(1)} = -\frac{\beta(\sigma_{\text{min},\text{max}}^{(0)})}{\alpha'(\sigma_{\text{min},\text{max}}^{(0)})}.\]

(51)

This can be easily derived from \(R = \alpha(\sigma_{\text{min},\text{max}}) + \lambda \beta(\sigma_{\text{min},\text{max}})\).

We know that the zeroth order in \(\lambda\) is also very interesting, but our main interest here is the linear order variation. Thus, if we write down only the first order part after some algebra in Appendix B, the linear action in \(\lambda\) is:

\[\delta A_{\text{reg}} = -\lambda \mathbb{H}_{rx} \Omega^2 \sin \delta \int_{-\sigma_{\text{max}}}^{\sigma_{\text{max}}} d\sigma \left( I_1 + I_2 \right) + \mathcal{O}(\Omega^3),\]

(52)

where

\[I_1 = \frac{(R^2 - \alpha(\sigma)^2) \left( 2g_{3,3}(\sigma) \left( h_{0,1}(\sigma)^2 + 1 \right) - h_{0,1}(\sigma) g_{3,3}(\sigma) h_{0,1}(\sigma) \right)}{2h_{0,1}(\sigma)^3 / h_{0,1}^2 + 1},\]

(53)

\[I_2 = (R^2 - \alpha(\sigma)^2) \left( \frac{\lambda \left( \sigma h_{0,1}(\sigma) - h_{0,1}(\sigma) \right)}{2 / \sqrt{h_{0,1}(\sigma)^2 + 1}} \right).\]

(54)

Here the contribution of \(\alpha(\sigma)^2\) term to the integration is much smaller than the contribution of \(R^2\) term. See Appendix B for the explicit form of \(\alpha(\sigma)\) that is order of \(\mathcal{E}\). Thus we drop the \(\alpha(\sigma)^2\) term.

If we drop out the discussed higher orders in \(\epsilon\) and the integrations of some odd functions,\(^6\) the leading change of the regularized minimal surface to the external electric field is given by

\[\delta A_{\text{reg}} = \lambda R^2 \Omega^2 \mathbb{H}_{rx} \sin \delta \int_{-\sigma_{\text{max}}}^{\sigma_{\text{max}}} d\sigma \frac{-\sigma h(\sigma)^3 h'(\sigma) + h(\sigma)^4 + h(\sigma) h'(\sigma) g'(\sigma) - 2g(\sigma) \left( h'(\sigma)^2 + 1 \right)}{2h(\sigma)^3 / h'(\sigma)^2 + 1},\]

(55)

\(^6\)One can easily see that all \(h_{p,q}\)'s and \(g_{3,3}\) are even functions and \(g_{2,3}\) is an odd function from the equations in Appendix A.
Figure 5: Cartoons for a minimal surface (a) and its top view (b): We regard the wedge type minimal surface as a part of a larger structure, such as a tail of a strip type minimal surface. We have depicted that imagination in the top view (b).

where \( h(\sigma) = h_{0,1}(\sigma) \) and \( g(\sigma) = -g_{3,3}(\sigma) \). In addition \( g \) and \( h \) satisfy the following equations:

\[
h'' = -\frac{2 \left(h'^2 + 1\right)}{h},
\]

\[
g'' = -\frac{4g'h'}{h} + \frac{2g \left(1 + h'^2\right)}{h^2} - 2h^2 + 6\sigma hh' + 4h^2 h'^2.
\]

(56)\hspace{1cm} (57)

One can solve the above equations numerically and perform the numerical integration in (55). Finally, we can get the linear response of the holographic entanglement entropy to the electric field as follows.

\[
\frac{\delta S_{EE}}{\delta E_x} = R^2 \frac{\Omega^2}{4G_N^2} N \frac{\tilde{\alpha}}{4\pi} \sin \delta,
\]

(58)

where \( N \) is the integration in (55). The integrand of \( N \) is a positive function over \(-\frac{1}{2} \leq \sigma \leq \frac{1}{2}\), thus \( N \) does not vanish and it is about 0.05. Therefore, the linear response of the holographic entanglement entropy to the electric field is proportional to the thermoelectric conductivity \( \tilde{\alpha} \) which is a measurable quantity.

5 Discussion

In this work, we have studied the entanglement entropy affected by the external electric field. Though the entanglement entropy has usually been considered as an important and fundamental quantity representing a variety of quantum natures, it still remains as a big issue to measure the quantum entanglement entropy in the laboratory. One of the main goals of this work is how we can relate the quantum entanglement entropy to the measurable quantities like the transport coefficients. This would be useful to figure out the relation between the quantum phenomena and various macroscopic quantities. Moreover, the present work may provide new intuitions about how to measure the entanglement entropy in the laboratory.
More specifically, we have taken into account the entanglement entropy of the strip and wedge entangling region with turning on the electric field and momentum relaxation. If there is no momentum relaxation which breaks the translation symmetry, the DC conductivity usually has an infinite value because of the translation invariance. In order to obtain a finite conductivity in the zero frequency limit, we first considered the geometry with the non-vanishing momentum relaxation which resembles introducing impurities to the dual field theory. This momentum relaxation usually modifies the electric and thermoelectric conductivities. Theoretically, those changes of the transport coefficients can be governed by the response theory and we can easily measure such changes experimentally in the laboratory. In this situation, we can ask how the entanglement entropy is affected by the external electric field and what is the relation between the modified entanglement entropy and transport coefficients. In this paper, we showed by using the holographic method how the entanglement entropy modified by the external electric field can be connected to the transport coefficient, especially the thermoelectric conductivity.

For the strip case, we found that turning on the external electric field tilted the minimal surface as expected. In the holographic model, the area of the minimal surface is directly associated with the entanglement entropy. Thus, we may expect the change of the entanglement entropy due to the external electric field. We found that such a change of the entanglement entropy does not occur at least in the linear response theory. This is because the holographic formula of the entanglement entropy in (18) has an invariant form under the parity transformation like $x \rightarrow -x$. Despite the tilted minimal surface, the invariance under the parity transformation does not allow the change of the entanglement entropy at least at the linear order. However, the higher order corrections can affect the resulting entanglement entropy. In fact, we explicitly showed that the second order correction caused by the tilted minimal surface can change the entanglement entropy, although we did not regard an additional contribution caused by the background metric deformation. However we do not present the result in this note because we leave it as part of future works. The additional contribution is related to the response theory at the second order and we hope to report more results on this issue in future works. Anyway, the results in the strip entangling region showed the fact that the change of the entanglement entropy can be represented in terms of transport coefficients.

In order to get more clear and explicit relation at the linear order, we further considered the entanglement entropy in the wedge entangling region which is utilized to extract universal information about the corner contribution. As shown in the strip case, the entanglement entropy invariant under the parity transformation does not give any nontrivial contribution at the linear order, so that we took into account the external electric field which is rotated with an arbitrary angle. Since the external electric field with an arbitrary rotation usually breaks the parity invariance, one can expect the nontrivial contribution to the entanglement entropy even at the linear order. We showed with explicit calculations that the linear order correction to entanglement entropy really occurs in this setup. Intriguingly, we furthermore found that the change of the entanglement entropy is directly
related to the thermoelectric conductivity. This is the first example showing how the entanglement entropy can be connected to the transport coefficients. In future works, we hope to report more evidence and understanding on the underlying structure of the connection between the entanglement entropy and transport coefficients.

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Appendix

A. Equations for $h_{p,q}$ and $g_{p,q}$

\[
\begin{align*}
\eta'_{0,1} &= -2 \left( \frac{\left(h'_{0,1}\right)^2 + 1}{h_{0,1}} \right) \\
\eta''_{2,1} &= \frac{h^3_{0,1} \left( \left(h'_{0,1}\right)^2 - 1 \right) - 4h_{0,1}h'_{0,1}h'_{2,1} + 2h_{2,1} \left( \left(h'_{0,1}\right)^2 + 1 \right)}{h^2_{0,1}} \\
\eta''_{2,2} &= \frac{2 \left(h_{2,2} \left( \left(h'_{0,1}\right)^2 + 1 \right) - 2h_{0,1}h'_{0,1}h'_{2,2} \right)}{h^2_{0,1}} \\
\eta''_{2,3} &= \frac{-\beta^2 h^3_{0,1} \left( \left(h'_{0,1}\right)^2 - 2 \right) + 8h_{0,1}h'_{0,1}h'_{2,3} - 4h_{2,3} \left( \left(h'_{0,1}\right)^2 + 1 \right)}{2h^2_{0,1}} \\
\eta''_{3,4} &= \frac{M h^4_{0,1} \left( 4 - 3 \left(h'_{0,1}\right)^2 \right) - 8h_{0,1}h'_{0,1}h''_{3,4} + 4h_{3,4} \left( \left(h'_{0,1}\right)^2 + 1 \right)}{2h^2_{0,1}} \\
g'_{2,3} &= -\frac{4g'_{2,3}h'_{0,1}}{h_{0,1}} + \frac{2g_{2,3} \left( \left(h'_{0,1}\right)^2 + 1 \right)}{h^2_{0,1}} - 6h_{0,1}h'_{0,1} \\
g'_{3,3} &= -\frac{4g'_{3,3}h'_{0,1}}{h_{0,1}} + \frac{2g_{3,3} \left( \left(h'_{0,1}\right)^2 + 1 \right)}{h^2_{0,1}} + \frac{2 \left(-3\sigma h^3_{0,1}h'_{0,1} - 2h^4_{0,1} \left(h'_{0,1}\right)^2 + h^4_{0,1} \right)}{h^2_{0,1}} \\
\alpha(\sigma) &= \frac{\mathcal{E}}{h_{0,1}} + \Omega^2 \left( -\frac{\mathcal{E}^3h_{2,3}}{h^4_{0,1}} - \frac{\mathcal{E}^2h_{2,2}}{h^3_{0,1}} - \frac{\mathcal{E}h_{2,1}}{h^2_{0,1}} \right) - \Omega^3 \frac{\mathcal{E}^4h_{3,4}}{h^5_{0,1}} \\
\beta(\sigma) &= -\frac{\mathcal{E}^3\Omega g_{2,3}^{\mathbb{H}_{\text{rx}}}}{h^4_{0,1}} \sin \delta - \frac{\mathcal{E}^3\Omega^2 g_{3,3}^{\mathbb{H}_{\text{rx}}}}{h^4_{0,1}} \cos \delta ,
\end{align*}
\]

where $\mathcal{E} = \frac{\epsilon}{n}$ is a scaled cut-off.
B. Detailed Calculation for (22)

Since \( \alpha(\sigma) \) and \( h_{0,1} \) are even functions and \( g_{2,3} \) is an odd function,

\[
- \frac{\lambda H_{rx}}{\Omega} \cos \delta \int_{-\sigma_{\text{max}}^{(0)}}^{\sigma_{\text{max}}^{(0)}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \frac{\rho \Omega^2 (2g_{2,3}(h_{0,1}^{2} + 1) - h_{0,1}g_{2,3}' h_{0,1}')}{{h_{0,1}^{3}}\sqrt{h_{0,1}^{2} + 1}} = 0 \ .
\]

(68)

By the same reason,

\[
\frac{\lambda H_{rx}}{\Omega} \sin \delta \int_{-\sigma_{\text{max}}^{(0)}}^{\sigma_{\text{max}}^{(0)}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \frac{\Omega^2 h_{0,1}' \cos \delta + \Omega^3 (\sigma h_{0,1}' - h_{0,1}) \sin \delta}{\sqrt{h_{0,1}^{2} + 1}} = \frac{\lambda H_{rx}}{\Omega} \sin \delta \int_{-\sigma_{\text{max}}^{(0)}}^{\sigma_{\text{max}}^{(0)}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \rho \left( \frac{\Omega^3 (\sigma h_{0,1}' - h_{0,1})}{\sqrt{h_{0,1}^{2} + 1}} \right) .
\]

(69)

In addition,

\[
\frac{\lambda}{\Omega} \left( \sigma_{\text{max}}^{(1)} - \sigma_{\text{min}}^{(1)} \right) \left( \int_{\alpha(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} \mathcal{L}_{(n)}^{0\text{th}} \Omega^n \right) \right)_{\sigma = \sigma_{\text{max}}^{(0)}} = \frac{\lambda}{\Omega} \left( -\beta \left( -\sigma_{\text{max}}^{0} \right) + \beta \left( \sigma_{\text{max}}^{0} \right) \right) \left( \int_{\alpha(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} \mathcal{L}_{(n)}^{0\text{th}} \Omega^n \right) \right)_{\sigma = \sigma_{\text{max}}^{(0)}}
\]

\[
= \frac{\lambda}{\Omega} \left( -\frac{2g_{2,3}\varepsilon^2}{h_{0,1}^{2}h_{0,1}'} \right)_{\sigma = \sigma_{\text{max}}^{(0)}} \mathbb{H}_{rx} \Omega^3 \left( \int_{\alpha(\sigma)}^{R} d\rho \left( \sum_{n=0}^{3} \mathcal{L}_{(n)}^{0\text{th}} \Omega^n \right) \right)_{\sigma = \sigma_{\text{max}}^{(0)}}
\]

\[
\sim \frac{\lambda}{\Omega} \left( -\frac{2g_{2,3}\varepsilon^2}{h_{0,1}^{2}h_{0,1}'} \right)_{\sigma = \sigma_{\text{max}}^{(0)}} \mathbb{H}_{rx} \Omega^3 \left( \sqrt{h_{0,1}^{2}(\sigma)^2 + 1} \left( \log(R) - \log \left( \frac{\varepsilon}{h_{0,1}(\sigma)} \right) \right) \right)_{\sigma = \sigma_{\text{max}}^{(0)}}
\]

\[
= \mathcal{O} \left( \Omega^2 \right) ,
\]

(70)

because \( h_{0,1} \left( \sigma_{\text{max}}^{(0)} \right) = R/\varepsilon + \mathcal{O} \left( \Omega \right) \). The last term in (22) is given by

\[
- \frac{\lambda}{\Omega} \int_{-\sigma_{\text{max}}^{(0)}}^{\sigma_{\text{max}}^{(0)}} d\sigma \beta(\sigma) \left( \sum_{n=0}^{3} \mathcal{L}_{(n)}^{0\text{th}} \Omega^n \right)_{\rho = \alpha(\sigma)}
\]

\[
= \left( -\frac{\lambda}{\Omega} \right) \varepsilon^2 \int_{-\sigma_{\text{max}}^{(0)}}^{\sigma_{\text{max}}^{(0)}} d\sigma \left( -\frac{\mathbb{H}_{rx} \Omega^3 g_{3,3}(\sigma) \sqrt{h_{0,1}^{2}(\sigma)^2 + 1}}{h_{0,1}(\sigma)} \right) + \mathcal{O} \left( \Omega^2 \right) ,
\]

(71)

where the integration is finite and so this term is quadratic in \( \varepsilon \). We may ignore this.
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