Scattering problem for Klein-Gordon equation with cubic convolution nonlinearity *

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Abstract
The scattering problem for the Klein-Gordon equation with cubic convolution nonlinearity is considered. Based on the Strichartz estimates for the inhomogeneous Klein-Gordon equation, we prove the existence of the scattering operator, which improves the known results in some sense.

Keywords: Asymptotic of solution; Klein-Gordon equation; scattering operator

Subject class: 35P25, 81Q05, 35B05

1 Introduction
This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form
\[
\begin{cases}
\partial_t^2 u - \Delta u + u = F_\gamma(u) & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x)
\end{cases}
\tag{1.1}
\]
where \( u \) is a real-valued or a complex-valued unknown function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\). The nonlinearity is a cubic convolution term \( F_\gamma(u) = -(V_\gamma(x) * |u|^2)u \) with \( |V_\gamma(x)| \leq C|x|^{-\gamma} \). Here, \( 0 < \gamma < n \) and * denotes the convolution in the space variables. The term \( F_\gamma(u) \) is an approximative expression of the nonlocal interaction of specific elementary particles. The equation (1.1) was studied by Menzala and Strauss in [1].

In order to define the scattering operator for (1.1), we first give some Banach spaces. The usual Lebesgue space is given by \( L^p = \{ \phi \in S' : \|\phi\|_{L^p} < +\infty \} \), where the norm \( \|\phi\|_{L^p} = \left(\int_{\mathbb{R}^n} |\phi(x)|^p dx \right)^{1/p} \) if \( 1 \leq p < +\infty \) and \( \|\phi\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |\phi(x)| \) if \( p = +\infty \). The weighted Sobolev space \( H_0^{\beta,k} \) is defined by

\[
H_0^{\beta,k} = \{ \phi \in S' : \|\phi\|_{H_0^{\beta,k}} = \|\langle x \rangle^k \langle i\nabla \rangle^\beta \phi\|_{L^p} < +\infty \},
\]
with \( \langle x \rangle = \sqrt{1 + x^2} \) and \( \langle i\nabla \rangle = \sqrt{1 - \Delta} \). We also write \( H^{\beta,k} = H_0^{\beta,k} \) and \( H^\beta = H_2^{\beta,0} \) if it does not cause a confusion. A Hilbert space \( X^{\beta,k} \) is denoted by \( H^{\beta,k} \bigoplus H^{\beta-1,k} \). Let \( X_\rho^{\beta,k} \) be a ball of a radius \( \rho > 0 \) with a center in the origin in the space \( X^{\beta,k} \). The scattering operator of (1.1) is defined as the mapping \( S : X_\rho^{\beta,k} \ni (f_-, g_-) \to (f_+, g_+) \in X^{\beta,0} \) if the following condition holds:

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For $(f_-, g_-) \in X_{\rho,k}^\beta$, there uniquely exists a time-global solution $u \in C(R; H^\beta)$ of (1.1), and data $(f_+, g_+) \in X_{\rho,k}^\beta$ such that $u(t)$ approaches $u_{\pm}(t)$ in $H^\beta$ as $t \to \pm \infty$, where $u_{\pm}(t)$ are solutions of linear Klein-Gordon equations whose initial data are $(f_{\pm}, g_{\pm})$, respectively.

We say that $(S, X_{\beta,k}^\rho)$ is well-defined if we can define the scattering operator $S : X_{\rho,k}^\beta \to X_{\rho,k}^\beta$ for some $\rho > 0$. In [2], Mochizuki prove that if $n \geq 3$, $\beta \geq 1$, $\gamma < n$ and $2 \leq \gamma \leq 2\beta + 2$, then $(S, X_{\beta,k}^\beta)$ is well-defined. Hidano [3] see that if $n \geq 2$, $\beta \geq 1$, $4/3 < \gamma < 2$ and $k > 1/3$, then $(S, X_{\beta,k}^\beta)$ is well-defined. By using the Strichartz estimate for pre-admissible pair and the complex interpolation method for the weighted Sobolev space, Hidano [4] shows that $(S, X_{\beta,k}^\beta)$ is well-defined if $n \geq 2$, $\beta \geq 1$, $4/3 < \gamma < 2$ and $k > (2 - \gamma)/2$.

Our aim of this article is to show that $(S, X_{\beta,1}^\beta)$ is well-defined if $n \geq 2$.

More precisely, we prove the following theorem.

Theorem 1.1 Let the function $V_\gamma(x)$ satisfy

$$|V_\gamma(x)| \leq C|x|^{-\gamma}, \quad |\nabla V_\gamma(x)| \leq C|x|^{-(1+\gamma)}.$$ 

Assume that $n \geq 2$, $\gamma$ and $\beta$ satisfy (1.3). Then there exists a positive number $\delta_0 > 0$ satisfying the following properties:

(1). For $(f, g) \in X_{\beta,1}^\beta$ with $\|(f, g)\|_{X_{\beta,1}^\beta} \leq \delta_0$, there uniquely exist final states $(f_{\pm}, g_{\pm}) \in X_{\rho,k}^\beta$ and a solution $u(t) \in C(R; H^\beta)$ of (1.1) such that $u(t)$ approaches $u_{\pm}(t)$ in $X_{\rho,k}^\beta$ as $t \to \pm \infty$, where $u_{\pm}(t)$ are solutions of the linear Klein-Gordon equation with initial data $(f_{\pm}, g_{\pm})$, respectively. Moreover, as $\pm t$ large enough we have

$$\|(u(t), \partial_t u(t)) - (u_{\pm}(t), \partial_t u_{\pm}(t))\|_{X_{\rho,k}^\beta} \leq C(t)^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$.

(2). For $(f_-, g_-) \in X_{\beta,1}^\beta$ with $\|(f_-, g_-)\|_{X_{\beta,1}^\beta} \leq \delta_0$, there uniquely exists a final state $(f_+, g_+) \in X_{\rho,k}^\beta$ and a solution $u(t) \in C(R; H^\beta)$ of (1.1) such that $u(t)$ approaches $u_{\pm}(t)$ in $X_{\rho,k}^\beta$ as $t \to \pm \infty$, where $u_{\pm}(t)$ are solutions of the linear Klein-Gordon equation with initial data $(f_{\pm}, g_{\pm})$, respectively. Moreover, as $\pm t$ large enough we have

$$\|(u(t), \partial_t u(t)) - (u_{\pm}(t), \partial_t u_{\pm}(t))\|_{X_{\rho,k}^\beta} \leq C(t)^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$.

In this article we denote by $J_\varepsilon = (i\nabla)x + i\varepsilon t \nabla$, $L_\varepsilon = i\partial_t - \varepsilon \langle t \nabla \rangle$ and $P = t \nabla + x \partial_t$ with $\varepsilon \in \{+, -\}$. For a given Banach space with norm $\| \cdot \|$ and a vector $v = (v^+, v^-)$, denote by

$$\|v\| = \|v^+\| + \|v^-\|, \quad \|Pv\| = \|Pv^+\| + \|Pv^-\|,$$

$$\|Jv\| = \|J_+ v^+\| + \|J_- v^-\|, \quad \|Lv\| = \|L_+ v^+\| + \|L_- v^-\|.$$ 

We also denote by the space-time norm

$$\|\phi\|_{L_t^p(I, L_x^q)} = \|\|\phi(t)\|_{L_x^q(R^n)}\|_{L_t^p(I)},$$

where $I$ is a bounded or unbounded time interval, and denote different positive constants by the same letter C.

The rest of the article is organized as follows. In Section 2 we give some preliminary calculations. Then Section 3 is devoted to the proof of Theorem 1.1.
2 Preliminaries

In this section, we prove some lemmas for our main results. Let \( w^\varepsilon = i\partial_t (i\nabla)^{-1} u - \varepsilon u \) with \( \varepsilon \in \{-, +\} \). Then the Klein-Gordon equation (1.1) can be be rewritten as a system of equations

\[
\begin{align*}
L_\varepsilon w^\varepsilon &= (i\nabla)^{-1} F_\gamma(u) \\
w^\varepsilon|_{t=0} &= w^\varepsilon_0
\end{align*}
\]

(2.1)

where \( L_\varepsilon = i\partial_t - \varepsilon (i\nabla) \), \( w^\varepsilon_0 = i (i\nabla)^{-1} 1 + \varepsilon f \). By the fact that

\[
\left. \begin{array}{l}
u = \frac{1}{2}(w^+ - w^-), \\
\partial_t \nu = -\frac{i}{2} (i\nabla)(w^+ - w^-),
\end{array} \right\}
\]

we can rewrite the term \( F_\gamma(u) \) as

\[
F_\gamma(u) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}} C_{\varepsilon_1\varepsilon_2\varepsilon_3} (V_\gamma \ast w^{\varepsilon_1} w^{\varepsilon_2}) w^{\varepsilon_3}
\]

with some constants \( C_{\varepsilon_1\varepsilon_2\varepsilon_3} \). Denote \( U_\varepsilon(t) \varphi = e^{-\varepsilon (i\nabla)t} \varphi \) and for given \( T \in \mathbb{R} \),

\[
\Psi_\varepsilon[g] = \int_T^t U_\varepsilon(t-\tau) (i\nabla)^{-1} g(\tau) d\tau,
\]

**Lemma 2.1** Let \( 2 \leq q < \frac{2n}{n-2}, \frac{2}{q} = \frac{n}{2}(1 - \frac{2}{n}) \). Then for any time interval \( I \) and for any given \( T \in I \) the following estimates are true:

\[
\|\Psi_\varepsilon[g]\|_{L^r(I,L^q)} \leq \|g\|_{L_t^r(I,H^{\mu-1})},
\]

\[
\|\Psi_\varepsilon[g]\|_{L^\infty(I,L^2)} \leq \|g\|_{L_t^{r'}(I,H^{\mu-1})},
\]

and

\[
\|U_\varepsilon(t) \varphi\|_{L^r(I,L^q)} \leq \|\varphi\|_{H^\mu},
\]

where \( r' = \frac{r}{r-1}, q' = \frac{q}{q-1} \) and \( \mu = \frac{1}{2}(1 + \frac{n}{2})(1 - \frac{2}{n}) \).

The proof of Lemma 2.1 is based on the duality argument along with the \( L^p - L^q \) time decay estimates. The similar result be found in [5].

**Lemma 2.2** Assume \( 2 \leq p < \frac{2n}{n-2} \) for \( n \geq 3 \) \( (2 \leq p < +\infty \) for \( n = 2 ) \), denote by \( \alpha = (1 + \frac{n}{2})(1 - \frac{2}{p}) \). The estimate is valid

\[
\|\phi\|_{L^p} \leq C(t)^{-\frac{n}{2}(1 - \frac{2}{p})} \|\phi\|_{H^{\alpha}} + \|J_\varepsilon \phi\|_{H^{\alpha-1}},
\]

for all \( t \in \mathbb{R} \), provided that the right-hand side is finite.

This lemma comes from Lemma 2.1 in [5] and the fact that \( \|\phi\|_{L^p} \leq C\|\phi\|_{H^{\alpha}} \) when \( p \geq 2 \).

**Lemma 2.3** Assume \( |V_\gamma(x)| \leq |x|^{-\gamma} \) with \( 0 < \gamma < n, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\} \).

(1). For \( 1 < r < +\infty, 1 < p_1, p_2 < +\infty \) and \( p_3 > r \) satisfying \( 1 + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \), we have

\[
\| (V_\gamma \ast w^{\varepsilon_1} w^{\varepsilon_2}) w^{\varepsilon_3} \|_{L^r} \leq \| w^{\varepsilon_1} \|_{L^{p_1}} \| w^{\varepsilon_2} \|_{L^{p_2}} \| w^{\varepsilon_3} \|_{L^{p_3}}.
\]
3 Proof of Theorem 1.1

For $1 < \gamma < \min\{\frac{2(n+1)}{n+2}, \frac{3n-2}{n+2}\}$, we choose
\[
\frac{(n+2)(\gamma+1)}{4n} + \frac{1}{2} < \beta < \frac{(n+2)(\gamma+1)}{2n}, q = \left(\frac{2\beta}{n+2} + \frac{1}{2} - \frac{\gamma+1}{n}\right)^{-1},
\]
They satisfy
\[
1 \leq \beta \leq 2, 2 < q < \frac{2n}{n+2(1-\gamma)}, 1 < \gamma < \frac{3n\beta}{n+2}.
\]
Let $\mu = \frac{1}{2}(1 + \frac{n}{2})(1 - \frac{q}{2})$, we also have
\[
\mu + \beta - 2 \leq 0, \mu \leq \beta - 1, \text{ and } 0 < \mu \leq \frac{1}{2}.
\]
Let $r, p$ and $s$ be chosen as
\[
\frac{2}{r} = \frac{n}{2} (1 - \frac{2}{q}), \frac{2}{p} + \frac{\gamma}{n} = 2 - \frac{2}{q}, \frac{2}{s} = 1 - \frac{2}{r}.
\]
The proof of Theorem 1. Introduce the function space

\[ X = \{ v = (v^+, v^-) \in C(R; (L^2(R^n))^2); \quad \| v \|_X < +\infty \} \]

with the norm

\[
\| v \|_X = \| v \|_{L^\infty_t (R; H^\beta)} + \| v \|_{L^1_t (R; H^\beta - \mu)} + \| \partial_t v \|_{L^\infty_t (R; H^\beta - 1)} + \| \partial_x v \|_{L^1_t (R; L^3)} + ||P v||_{L^\infty_t (R; H^\beta - 1)} + ||P v||_{L^1_t (R; L^3)} + ||J v||_{L^\infty_t (R; H^\beta - 1)}.
\]

Denote by \( X_\rho \) a ball of a radius \( \rho > 0 \) with a center in the origin in the space \( X \). Let us consider the linearized version of (2.1)

\[
\begin{align*}
L_t w^\varepsilon &= (i \nabla)^{-1} F_\gamma (v) \\
w^\varepsilon |_{t=0} &= w_0^\varepsilon
\end{align*}
\]

with a given vector \( v = (v^+, v^-) \in X_\rho \), where

\[ F_\gamma (v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3} \]

with some given constants \( C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \). The integration of the linearized Cauchy problem (3.1) with respect to time yields

\[ w^\varepsilon = U_\varepsilon (t) w_0^\varepsilon + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \Psi_\varepsilon ((V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3}). \]

Taking the \( L^\infty_t (R; H^\beta) \)-norm of (3.2), applying the Hölder inequality, Lemma 2.11 and Lemma 2.3 we find

\[
\| w^\varepsilon \|_{L^\infty_t (R; H^\beta)} \leq \| U_\varepsilon (t) w_0^\varepsilon \|_{L^\infty_t (R; H^\beta)} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \| \Psi_\varepsilon ((V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3}) \|_{L^\infty_t (R; H^\beta)}
\]

\[
\leq \| w_0^\varepsilon \|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \| (V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3} \|_{L^1_t (R; H^\beta + \mu - 1)}
\]

\[
\leq \| w_0 \|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \| \| v^{\varepsilon_1} \|_{H^\beta + \mu - 1} \| v^{\varepsilon_2} \|_{L^p} \| v^{\varepsilon_3} \|_{L^p} \|_{L^1_t (R; L^p)}
\]

\[
\leq \| w_0 \|_{H^\beta} + C \rho \| v \|_{L^1_t (R; L^p)}^2
\]

(3.3)

since \( p > 2 > q', q > 2 > q', \mu \leq \frac{1}{2} \) and \( 2 - \frac{2}{q} = \frac{q}{n} + \frac{2}{p} \). Similarly, taking the \( L^1_t (R; H^\beta - \mu) \) we obtain

\[
\| w^\varepsilon \|_{L^1_t (R; H^\beta - \mu)} \leq \| U_\varepsilon (t) w_0^\varepsilon \|_{L^1_t (R; H^\beta - \mu)} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \| \Psi_\varepsilon ((V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3}) \|_{L^1_t (R; H^\beta - \mu)}
\]

\[
\leq \| w_0^\varepsilon \|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \| (V_\gamma * \bar{v}^{\varepsilon_1_2} v^{\varepsilon_3}) v^{\varepsilon_3} \|_{L^1_t (R; H^\beta + \mu - 1)}
\]

\[
\leq \| w_0 \|_{H^\beta} + C \| v \|_{L^1_t (R; H^\beta + \mu - 1)} \| v \|_{L^1_t (R; L^p)}^2
\]

(3.4)
since $\mu \leq \frac{1}{2}$, $p > 2 > q'$, $q > 2 > q'$ and $2 - \frac{2}{q} = \frac{2}{n} + \frac{2}{p}$. Applying the operator $\partial_t$ to (3.1) we deduce that $\partial_t w^\varepsilon$ satisfies the following system
\[
\begin{align*}
L_\varepsilon \partial_t w^\varepsilon &= \langle i \nabla \rangle^{-1} \partial_t F_\gamma(v) \\
\partial_t w^\varepsilon|_{t=0} &= -i\varepsilon \langle i \nabla \rangle w^\varepsilon_0 - i\langle i \nabla \rangle^{-1} F_\gamma(v)|_{t=0}
\end{align*}
\]
with
\[F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * v^{\varepsilon_1} v^{\varepsilon_2}) v^{\varepsilon_3}.
\]
Then by integrating with respect to time,
\[\partial_t w^\varepsilon = U_\varepsilon(t)(\partial_t w^\varepsilon|_{t=0}) + \Psi_\varepsilon(\partial_t F_\gamma(v)).\]
Taking the $L^\infty_t (R; H^{\beta - 1})$-norm and $L^1_t (R, L^q)$-norm, applying the Hölder inequality and Lemma 4.1 we find that, since $\beta \geq 1$, $\mu \leq \beta - 1$ and $\mu + \beta - 2 \leq 0$,
\[
\begin{align*}
\|\partial_t w^\varepsilon\|_{L^\infty_t (R; H^{\beta - 1})} + \|\partial_t w^\varepsilon\|_{L^1_t (R, L^q)} &
\leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta - 1}} + \|\partial_t F_\gamma(v)\|_{L^1_t (R, H^{\mu + \beta - 2})} \\
&\leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta - 1}} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * (\overline{\partial_t v^{\varepsilon_1} v^{\varepsilon_2}} + \overline{v^{\varepsilon_1} \partial_t v^{\varepsilon_2}})) v^{\varepsilon_3} + (V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) \partial_t v^{\varepsilon_3}\|_{L^1_t (R, L^q)} \\
&\leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta - 1}} + C \|\partial_t v\|_{L^1_t (R, L^q)} \|v\|_{L^2_t (R, L^p)}^2 \\
&\leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta - 1}} + C \rho \|v\|_{L^2_t (R, L^p)}^2
\end{align*}
\]
On the other hand, we have
\[\|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta - 1}} \leq \|w^\varepsilon_0\|_{H^{\beta}} + \|F_\gamma(v)\|_{L^\infty_t (R, H^{\beta - 2})},\]
and for $p_1 > 2$ satisfying $\frac{3}{2} = \frac{n}{p} + \frac{3}{p_1}$,
\[\|F_\gamma(v)\|_{L^\infty_t (R, H^{\beta - 2})} \leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}\|_{L^\infty_t (R, L^2)} \\
\leq C \|v\|_{L^\infty_t (R, L^{p_1})}^3 \leq C \|v\|_{L^\infty_t (R, H^{\beta})}^3 \leq C \rho^3\]
since $\beta \leq 2, \gamma \leq 3\beta$ and $\|v\|_{L^p_1} \leq C \|v\|_{H^{\beta}}$. Then
\[
\|\partial_t w^\varepsilon\|_{L^\infty_t (R, H^{\beta - 1})} + \|\partial_t w^\varepsilon\|_{L^1_t (R, L^q)} \leq C \|w_0\|_{H^{\beta}} + C \rho \|v\|_{L^2_t (R, L^p)}^2.
\]
Notice that $P = t \nabla + x \partial_t, J_\varepsilon = \langle i \nabla \rangle x + i\varepsilon \langle i \nabla \rangle$ and $L_\varepsilon = i\partial_t - \varepsilon \langle i \nabla \rangle$. We get
\[
J_\varepsilon - \varepsilon L_\varepsilon, [L_\varepsilon, P] = -i\varepsilon \langle i \nabla \rangle^{-1} \nabla L_\varepsilon,
\]
and
\[
P((V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}) = (V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}})P(v^{\varepsilon_3}) + (t \nabla V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}.
\]
Applying the operator $P$ to (5.1) yields

\[
\begin{cases}
  L_\epsilon P w^\epsilon = i\varepsilon (i\nabla)^{-2} \nabla F_\gamma(v) - \langle i\nabla \rangle^{-1} P F_\gamma(v) - \langle i\nabla \rangle^{-3} \nabla \partial_t F_\gamma(v)
  \\
Pw^\epsilon|_{t=0} = x \partial_t w^\epsilon|_{t=0} = x (-i\varepsilon \langle i\nabla \rangle w^\epsilon_0 - i\langle i\nabla \rangle^{-1} F_\gamma(v)|_{t=0}
\end{cases}
\]

with

\[P F_\gamma(v) = \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}} C_{\epsilon_1, \epsilon_2, \epsilon_3} (V_\gamma * v^{\epsilon_1} v^{\epsilon_2}) P V^{\epsilon_3} + (t \nabla V_\gamma * v^{\epsilon_1} v^{\epsilon_2}) v^{\epsilon_3}.
\]

Integrating with respect to time, we get

\[P w^\epsilon = U_\epsilon(t) (P w^\epsilon|_{t=0}) - \Psi_\epsilon (i\varepsilon \langle i\nabla \rangle^{-1} \nabla F_\gamma(v)) + \Psi_\epsilon (P F_\gamma(v)) + \Psi_\epsilon (\langle i\nabla \rangle^{-2} \nabla \partial_t F_\gamma(v)). \tag{3.6}
\]

Taking the $L^\infty_t(R; H^{\beta-1})$-norm and the $L^{q'}_t(R, L^p)$-norm of (3.6), applying the Hölder inequality and Lemma 2.2, we find

\[
\begin{align*}
  \|P w^\epsilon\|_{L^\infty_t(R; H^{\beta-1})} + \|P w^\epsilon\|_{L^{q'}_t(R, L^p)} & \\
  \leq \|P w^\epsilon\|_{t=0} \|H^{\beta-1}\|_{L^{q'}_t(R, L^p)} + \|P F_\gamma(v)\|_{L^{q'}_t(R, L^p)} + \|\langle i\nabla \rangle^{-2} \nabla \partial_t F_\gamma(v)\|_{L^{q'}_t(R, L^p)} & \\
  \leq \|P w^\epsilon\|_{t=0} \|H^{\beta-1}\|_{L^{q'}_t(R, L^p)} + \|P F_\gamma(v)\|_{L^{q'}_t(R, L^p)} + \|\partial_t F_\gamma(v)\|_{L^{q'}_t(R, L^p)}
\end{align*}
\]

(3.7)

since $\beta \geq 1$ and $\mu + \beta - 2 \leq 0$ and $\mu \leq \beta - 1$. As in the proof of (3.5) we deduce

\[
\|F_\gamma(v)\|_{L^{q'}_t(R, L^p)} + \|\partial_t F_\gamma(v)\|_{L^{q'}_t(R, L^p)} \leq C \|v\|_{L^{q'}_t(R, L^p)} \|v\|_{L^{q'}_t(R, L^p)} \|v\|_{L^{q'}_t(R, L^p)} \leq C \rho \|v\|_{L^{q'}_t(R, L^p)}.
\]

(3.8)

Let $p_3 > 2$ and $s_3 > 2$ satisfy

\[
\frac{3}{2} - \frac{1}{q} = \frac{\gamma + 1}{n} + \frac{1}{p_3}, \quad 1 - \frac{1}{r} = \frac{2}{s_3}.
\]

The Hölder inequality and Lemma 2.2 imply

\[
\begin{align*}
  \|P F_\gamma(v)\|_{L^{q'}_t(R, L^p)} & \\
  \leq C \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}} \left[ \| (V_\gamma * v^{\epsilon_1} v^{\epsilon_2}) P v^{\epsilon_3} \|_{L^{q'}_t(R, L^p)} + \| (t \nabla V_\gamma * v^{\epsilon_1} v^{\epsilon_2}) v^{\epsilon_3} \|_{L^{q'}_t(R, L^p)} \right] & \\
  \leq C \| P v \|_{L^{q'}_t(R, L^p)} \|v\|_{L^{q'}_t(R, L^p)}^{2} + C \| \nabla v \|_{L^{q'}_t(R, L^p)} \|v\|_{L^{q'}_t(R, L^p)}^{2} & \\
  \leq C \rho \|v\|_{L^{q'}_t(R, L^p)} + C \rho \|v\|_{L^{q'}_t(R, L^p)} \|v\|_{L^{q'}_t(R, L^p)}^{2}
\end{align*}
\]

(3.9)

here we use the condition $\|\nabla V_\gamma\| \leq C|x|^{-(\gamma+1)}$. By Lemma 2.2 we have

\[
\|v\|_{L^{q'}_t(R, L^p)} \leq C \left( \|t\|_{H^{\alpha}}^{\frac{1}{2}} \|v\|_{H^{\alpha}} + \|J v\|_{H^{\alpha-1}} \|v\|_{L^{q'}_t(R)} \right) \leq C \left( \|v\|_{L^\infty_t(R; H^{\beta})} + \|J v\|_{L^\infty_t(R; H^{\beta-1})} \right) \leq C \rho
\]

(3.10)
since $\alpha = (1 + \frac{n}{2})(1 - \frac{2}{p}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p}) > \frac{1}{2}$. Similarly,

\[
\|t^{1/2}v\|_{L_t^p(R,L^p)} \leq C\|t^{-\frac{n}{2}(1 - \frac{2}{p}) + \frac{1}{2}}\|v\|_{H^{\alpha_2}} + \|Jv\|_{H^{\alpha_3}}\|L_t^p(R) \leq C\left(\|v\|_{L_t^p(R,H^{\beta})} + \|Jv\|_{L_t^p(R,H^{\beta-1})}\right) \leq C\rho,
\]

(3.11)
since $\alpha_3 = (1 + \frac{n}{2})(1 - \frac{2}{p}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p}) > \frac{1}{2}$, then we obtain, from (3.7)-(3.11),

\[
\|P\rho^3\|_{L_t^p(R,H^{\beta-1})} \leq \|P\rho^3\|_{L_t^p(R,H^{\beta})} \leq \|P\rho^3\|_{L_t^p(R,H^{\beta-1})} \leq C\rho^3,
\]

(3.12)

\[
\|w^\mu\|_{L_t^p(R,H^{\beta})} + \|w^\mu\|_{L_t^p(R,H^{\beta-1})} \leq \|w^\mu\|_{H^{\beta}} + \|w^\mu\|_{H^{\beta-1}} \leq \|w^\mu\|_{H^{\beta}} + C\rho^3
\]

(3.13)

\[
\|\partial_t w^\mu\|_{L_t^p(R,H^{\beta})} + \|\partial_t w^\mu\|_{L_t^p(R,H^{\beta-1})} \leq \|w^\mu\|_{H^{\beta}} + \|w^\mu\|_{H^{\beta-1}} \leq \|w^\mu\|_{H^{\beta}} + C\rho^3
\]

(3.14)

To estimate the term $\|Pw^\mu\|_{L_t^p(R,H^{\beta-1})}$, we give some estimates. It follows from the Sobolev embedding theorem that

\[
\|F_\gamma(v)\|_{L_t^p(R,L^2)} \leq C \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+,\} \} \|(V_\gamma * \bar{v}t^2v^2)\|_{L_t^p(R,L^2)} \leq \|v\|^2_{L_t^p(R,L^p)} \leq C\|v\|_{L_t^p(R,H^{\beta})}^3 \leq C\rho^3,
\]

(3.15)

where $p_5 = \frac{6n}{3n - 2\gamma}$, which satisfies $p_5 \leq \frac{2n}{n - 2\beta}$ because of $\gamma \leq 3\beta$. Using the relation $x = (i\nabla)^{-1}J_x - it(i\nabla)^{-1}v$ we deduce

\[
\|xF_\gamma(v)\|_{L_t^p(R,L^2)} \leq \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+,\} \} \|(V_\gamma * \bar{v}t^2v^2)(xv^3)\|_{L_t^p(R,L^2)} \leq C\|v\|^2_{L_t^p(R,L^p)} \leq C\|v\|^3_{L_t^p(R,H^{\beta})} \leq C\rho^3,
\]

(3.16)

where $p_4 = \frac{6n}{3n - 2\gamma}$, which satisfies

\[
2 < p_4 \leq \frac{2n}{n - 2\beta} \cdot \frac{n}{2}(1 - \frac{2}{p_4}) \geq \frac{1}{3}(1 + \frac{n}{2})(1 - \frac{2}{p_4}) \leq \beta
\]

because of $1 < \gamma \leq \frac{3n\beta}{n+2}$. Using the relation $[(i\nabla)^{\beta-1},x] = -(\beta - 1)(i\nabla)^{\beta-3}v$ we deduce

\[
\|Pw^\mu\|_{L_t^p(R,H^{\beta-1})} \leq \|x(i\nabla)w^\mu\|_{H^{\beta-1}} + \|x(i\nabla)^{-1}F_\gamma(v)\|_{L_t^p(R,H^{\beta-1})} \leq \|x(i\nabla)w^\mu\|_{H^{\beta-1}} + \|x(i\nabla)^{-1}F_\gamma(v)\|_{L_t^p(R,H^{\beta-1})} \leq \|x(i\nabla)w^\mu\|_{L_t^p(R,L^2)} + C\|xF_\gamma(v)\|_{L_t^p(R,L^2)} + C\|F_\gamma(v)\|_{L_t^p(R,L^2)} \leq \|x(i\nabla)w^\mu\|_{L_t^p(R,L^2)} + C\rho^3 + C\rho^3 \leq \|w^\mu\|_{H^{\beta-1}} + C\rho^3,
\]
which, combining with (3.12), yields
\begin{equation}
\|Pw^\varepsilon\|_{L_t^\infty(R,H^\beta-1)} + \|Pw^\varepsilon\|_{L_t^1(R,L^\beta)} \leq \|w_0\|_{H^\beta,1} + C\rho^3.
\end{equation}
(3.17)

Notice that
\[ [L_\varepsilon, x] = -\varepsilon (i\nabla)^{-1}\nabla, \quad [x, (i\nabla)^{-1}] = - (i\nabla)^{-3}\nabla. \]

Then we deduce that \( xw^\varepsilon \) satisfies
\begin{equation}
L_\varepsilon(xw^\varepsilon) = -\varepsilon (i\nabla)^{-1}\nabla w^\varepsilon - (i\nabla)^{-1}(xF_\gamma(v)) + (i\nabla)^{-1}\nabla F_\gamma(v).
\end{equation}

Using \( J_\varepsilon = i\varepsilon P - \varepsilon L_\varepsilon x \) and (3.13) yields
\begin{equation}
\|J_\varepsilon w^\varepsilon\|_{L_t^\infty(R,H^\beta-1)} \leq \|Pw^\varepsilon\|_{L_t^\infty(R,H^\beta-1)} + \|L_\varepsilon(xw^\varepsilon)\|_{L_t^\infty(R,H^\beta-1)},
\end{equation}
with
\begin{align*}
\|L_\varepsilon(xw^\varepsilon)\|_{L_t^\infty(R,H^\beta-1)} &\leq \|w^\varepsilon\|_{L_t^\infty(R,H^\beta)} + \| (i\nabla)^{-2} F_\gamma(v)\|_{L_t^\infty(R,H^\beta-1)} + \| (i\nabla)^{-1}(xF_\gamma(v))\|_{L_t^\infty(R,H^\beta-1)} \\
&\leq \|w^\varepsilon\|_{L_t^\infty(R,H^\beta)} + \| F_\gamma(v)\|_{L_t^\infty(R,L^2)} + \|xF_\gamma(v)\|_{L_t^\infty(R,L^2)} \leq C\|w_0\|_{H^\beta} + C\rho^3.
\end{align*}

Then we get
\begin{equation}
\|J_\varepsilon w^\varepsilon\|_{L_t^\infty(R,H^\beta-1)} \leq C\|w_0\|_{H^\beta,1} + C\rho^3.
\end{equation}
(3.18)

A combination of (3.12) with (3.13), (3.14), (3.17) and (3.18) yields
\begin{equation}
\|w\|_X \leq C\|w_0\|_{H^\beta,1} + C\rho^3.
\end{equation}
(3.19)

Therefore the map \( M : w = M(v) \) defined by the problem (3.1), transforms a ball \( X_\rho \) with a small radius \( \rho = C\|w_0\|_{H^\beta,1} \) into itself. Denote \( \tilde{w} = M(\tilde{v}) \), then in the same way as in the proof of (3.19) we have
\begin{equation}
\|M(v) - M(\tilde{v})\|_X \leq C\rho^2\|v - \tilde{v}\|_X.
\end{equation}

Thus \( M \) is a contraction mapping in \( X_\rho \) and so there exists a unique solution \( w = M(w) \) of (3.1) if the norm \( \|w_0\|_{H^\beta,1} \) is small enough.

To prove the asymptotic of the solution \( w(t,x) \), we use the equation, for \( |t| > |t'| \),
\begin{equation}
U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t') = \int_{t'}^t U_\varepsilon(-\tau)(i\nabla)^{-1}F_\gamma(w(\tau))d\tau.
\end{equation}

Taking the \( H^\beta \)-norm of this equation, using the similar proof of (3.3) and (3.4), we deduce
\begin{equation}
\|U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t')\|_{H^\beta} \leq C\rho^2(t')^{-\delta}
\end{equation}
with \( \delta = \frac{2n\beta}{n+2} - 2 > 0 \), since we have \( \|w\|_X \leq \rho \) and
\begin{equation}
\|\langle t\rangle^{-\frac{n}{2}(1-\frac{1}{\delta})}\|_{L_t^1([t',t])} \leq C\langle t'\rangle^{-\delta}.
\end{equation}

Then there uniquely exist finial states \( w^\varepsilon_\pm \in H^\beta \) satisfying, for \( \pm t \) large enough,
\begin{equation}
\|w^\varepsilon(t) - U_\varepsilon(t)w^\varepsilon_\pm\|_{H^\beta} \leq C\rho^2\langle t\rangle^{-\delta}.
\end{equation}
Set \( u(t) = \frac{1}{2}(w^+(t) - w^-(t)) \), \( f_\pm(x) = \frac{1}{2}(w^\pm_+ - w^\pm_-) \), \( g_\pm(x) = -\frac{i}{2}(i\nabla)(w^\pm_+ + w^\pm_-) \) and 
\( u_\pm(t) = \frac{1}{2}(U_+^\pm(t)w^\pm_+ - U_-^\pm(t)w^\pm_-) \). Then \( u(t) \) and \( u_\pm(t) \) satisfy Theorem 1.1(1).

Proof of Theorem 1.1(2). For given \((f_-, g_-) \in X^{\beta,1}\) and \( v = \{v^+, v^-\} \in X_\rho \), we consider the linearized version of the final state problem of (3.1)

\[
\begin{cases}
L_\varepsilon w^\varepsilon = -\langle i\nabla \rangle^{-1}F_\gamma(v) \\
\|U_\varepsilon(t)w^\varepsilon - w^\varepsilon_-(x)\|_{H^\beta} \to 0 \text{ as } t \to \infty
\end{cases}
\]

with \( w^\varepsilon_-(x) = i\langle i\nabla \rangle^{-1}g_-(x) - \varepsilon f_-(x) \in H^{\beta,1} \). The integration with respective to time yields

\[
w^\varepsilon(t) = U_\varepsilon(t)w^\varepsilon_- + \int_{-\infty}^t U_\varepsilon(t - \tau)\langle i\nabla \rangle^{-1}F_\gamma(v(\tau))d\tau.
\]

In the same way as in the proof of Theorem 1.1(1), we find that, if \( \|(f_-, g_-)\|_{X^{\beta,1}} \leq \rho \) small, there uniquely exists a global solution \( w^\varepsilon(t) \in C(R, H^\beta) \) and a final state \( w^\varepsilon_+ \in H^\beta \) such that, as \( t \to +\infty \),

\[
\|w^\varepsilon(t) - U_\varepsilon(t)w^\varepsilon_+\|_{H^\beta} \leq C(t)^{-\delta}
\]

with \( \delta = \frac{2\beta}{\beta + 2} - 2 > 0 \). Set \( u(t) = \frac{1}{2}(w^+(t) - w^-(t)) \), \( f_+(x) = \frac{1}{2}(w^+_+ - w^+_-) \), \( g_+(x) = -\frac{1}{2}(i\nabla)(w^+_+ + w^+_-) \) and \( u_+(t) = \frac{1}{2}(U_+(t)w^+_+ - U_-(t)w^+_-) \). Then \( u(t) \) and \( u_+(t) \) satisfy Theorem 1.1(2).

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