Reflection positivity and complex analysis of the Yang-Mills theory from a viewpoint of gluon confinement

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In order to understand the confining decoupling solution of the Yang-Mills theory in the Landau gauge, we consider the massive Yang-Mills model which is defined by just adding a gluon mass term to the Yang-Mills theory with the Lorentz-covariant gauge fixing term and the associated Faddeev-Popov ghost term. First of all, we show that massive Yang-Mills model is obtained as a gauge-fixed version of the gauge-invariantly extended theory which is identified with the gauge-scale model with a single fixed-modulus scalar field in the fundamental representation of the gauge group. This equivalence is obtained through the gauge-independent description of the Brout-Englert-Higgs mechanism proposed recently by one of the authors. Then, we reconfirm that the Euclidean gluon and ghost propagators in the Landau gauge obtained by numerical simulations on the lattice are reproduced with good accuracy from the massive Yang-Mills model by taking into account one-loop quantum corrections. Moreover, we demonstrate in a numerical way that the Schwinger function calculated from the gluon propagator in the Euclidean region exhibits violation of the reflection positivity at the physical point of the parameters. In addition, we perform the analytic continuation of the gluon propagator from the Euclidean region to the complex momentum plane towards the Minkowski region. We give an analytical proof that the reflection positivity is violated for any choice of the parameters in the massive Yang-Mills model, due to the existence of a pair of complex conjugate poles and the negativity of the spectral function for the gluon propagator to one-loop order. The complex structure of the propagator enables us to explain why the gluon propagator in the Euclidean region is well described by the Gribov-Stingl form. We try to understand these results in light of the Fradkin-Shenker continuity between Confinement-like and Higgs-like regions in a single confinement phase in the complementary gauge-scale model.

PACS numbers: 12.38.Aw, 21.65.Qr

I. INTRODUCTION

It is still a challenging problem in particle physics to explain quark and gluon confinement in the framework of quantum gauge field theories [1]. The very first question to this problem is to clarify what criterion should be adopted to understand confinement. For quark confinement, there is a well-established gauge-invariant criterion given by Wilson [2], namely, the area law falloff of the Wilson loop average leading to the linear static quark potential with a non-vanishing string tension. For gluon confinement, on the other hand, there is no known gauge-invariant criterion to the best of the author's knowledge. This is also the case for more general hypothesis of color confinement including quark and gluon confinement as special cases. Once the gauge is fixed, however, there are some proposals. For instance, the Kugo-Ojima criterion for color confinement is given for the Lorentz covariant Landau gauge [3]. Indeed, it is rather difficult to prove the color confinement criterion even in a specific gauge, although there appeared an announcement for a proof of the Kugo-Ojima criterion for color confinement in the covariant Landau gauge [4]. Even if color confinement is successfully proved in a specific gauge, this does not automatically guarantee color confinement in the other gauges. Therefore the physical picture for confinement could change gauge by gauge.

The information on confinement is expected to be encoded in the gluon and ghost propagators which are obtained by fixing the gauge. Recent investigations have confirmed that in the Lorentz covariant Landau gauge the decoupling solution [5, 6] is the confining solution of the Yang-Mills theory in the three- and four-dimensional spacetime, while the scaling solution is realized in the two-dimensional spacetime. Therefore, it is quite important to understand the decoupling solution in the Lorentz covariant Landau gauge. Of course, there are so many approaches towards this goal. In this paper, we focus on the approach [7–9] which has been developed in recent several years and has succeeded to reproduce some features of the decoupling solution with good accuracy. We call this approach the mass-deformed Yang-Mills theory with the gauge fixing term or the massive Yang-Mills model in the covariant gauge in short.

However, the reason why this approach is so successful is not fully understood yet in our opinion. In the original works [7], the massive Yang-Mills model in the Landau gauge was identified with a special parameter limit of the Curci-Ferrari model [10]. However, the Curci-Ferrari model is not invariant under the usual Becchi-Rouet-Stora-Tyutin (BRST) transformation, but invariant just under the modified BRST transformation which does not respect the usual nilpotency.
In this paper we show based on the previous works \[11, 12\] that the mass-deformed Yang-Mills theory with the covariant gauge fixing term has the \textit{gauge-invariant extension} which is given by a gauge-scalar model with a single fixed-modulus scalar field in the fundamental representation of the gauge group, provided that a constraint called the \textit{reduction condition} is satisfied. We call such a model the \textit{complementary gauge-scalar model}. This equivalence is achieved based on the gauge-independent description \[11, 12\] of the Brout-Englert-Higgs (BEH) mechanism \[13–15\] which does not rely on the spontaneous breaking of gauge symmetry \[16, 17\]. This description enables one to give a \textit{gauge-invariant mass term of the gluon field} in the Yang-Mills theory which can be identified with the gauge-invariant kinetic term of the scalar field in the complementary gauge-scalar model.

In this paper, we first confirm that the massive Yang-Mills model with one-loop quantum corrections being included in the Euclidean region reproduces with good accuracy the \textit{gluon and ghost propagators} of the decoupling solution of the Yang-Mills theory in the Landau gauge obtained by numerical simulations on the lattice. In fact, the resulting gluon and ghost propagators in the massive Yang-Mills model can be well fitted to those on the lattice by adjusting the parameters, namely, the coupling constant $g$ and the gluon mass parameter $M$.

For gluon confinement, the violation of \textit{reflection positivity} is regarded as a necessary condition for confinement. In fact, it is known that the gluon propagator in the Yang-Mills theory exhibits the violation of reflection positivity. This fact was directly shown by the numerical simulations on the lattice, e.g., in the covariant Landau gauge \[18, 19\]. In this paper, by using the relevant gluon propagator in the massive Yang-Mills model, we calculate the \textit{Schwinger function} in a numerical way to demonstrate that the reflection positivity is violated at the physical point of parameters reproducing the Yang-Mills theory.

In order to understand these facts and consider the meaning of gluon confinement, we perform the analytic continuation of the gluon and ghost propagators in the Euclidean region to those in the Minkowski region on the complex momentum squared plane. The consideration of the complex structure of the propagator enables us to give an analytical proof that the reflection positivity is violated for any choice of the parameters without restricting to the physical point of the Yang-Mills theory in the massive Yang-Mills model with one-loop quantum corrections being included. For this proof, it is enough to show that the Schwinger function necessarily becomes negative in some region, which is achieved by calculating separately the contributions to the gluon Schwinger function from the pole part and the continuous (branch cut) part of the gluon propagator based on the generalized spectral representation in the massive Yang-Mills model to one-loop order. It turns out that the violation of reflection positivity is an immediate consequence of the facts that the gluon propagator has a pair of complex conjugate poles and that the spectral function of the gluon propagator has negative value on the whole range, see \[20\]. See e.g., \[21\] for the construction of the spectral function from the Euclidean data of numerical simulations on the lattice.

The complex structure of the propagator enables us to explain why the gluon propagator in the Euclidean region is well described by the \textit{Gribov-Stingl form} \[22\], as demonstrated in the numerical simulations on the lattice \[23\]. Indeed, the pole part of the gluon propagator due to a pair of \textit{complex conjugate poles} exactly reproduces the Gribov-Stingl form which is fitted to the numerical simulations to very good accuracy, after subtracting the small contribution coming from the continuous part represented by the spectral function obtained from the discontinuity across the \textit{branch cut} on the positive real axis on the complex momentum plane.

The above result suggests that gluon confinement is not restricted to the confinement phase of the ordinary Yang-Mills theory, and can be extended into more general situations, namely, anywhere represented by the massive Yang-Mills model, which includes the Higgs phase in the complementary gauge-scalar model. In the lattice gauge theory, it is known that the confinement phase in the pure Yang-Mills theory is analytically continued to the Higgs phase in the relevant gauge-scalar model, which is called the \textit{Fradkin-Shenker continuity} \[24\] as a special realization of the Osterwalder-Seiler theorem \[25\]. There are no local order parameters which can distinguish the confinement and Higgs phases. There is no thermodynamic phase transition between confinement and Higgs phases \[26\], in sharp contrast to the adjoint scalar case \[27\] where there is a clear phase transition between the two phases. Therefore, Confinement and Higgs phases are just subregions of a single Confinement-Higgs phase \[28, 30\]. Therefore, permanent violation of positivity can be understood in light of the Fradkin-Shenker continuity between Confinement-like and Higgs-like regions in a single confinement phase in the gauge-scalar model.

This paper is organized as follows. In sec. II, we introduce the massive Yang-Mills model in the covariant gauge. In sec. III, we show that the massive Yang-Mills model with quantum corrections to one-loop order well reproduces the gluon and ghost propagators of the decoupling solution. In sec. IV, we show that the gluon propagator exhibits violation of reflection positivity through the calculation of the Schwinger function. In sect. V, we perform the analytic continuation of the propagator to the complex momentum to examine the complex structure. In the final section we draw the conclusion and discuss the future problems to be tackled. In Appendix A, we give a recursive construction of the transverse and gauge-invariant gluon field to show the gauge-invariant extension of the massive Yang-Mills model. In Appendix B, we give another way for solving the reduction condition. In Appendix C, we give the expression for the vacuum polarization tensor for gluons and the self-energy function for ghosts for an arbitrary gauge fixing.
II. GAUGE-IN Variant EXTENSION OF THE MASS-DEFORMED YANG-MILLS THEORY IN THE COVARIANT LANDAU GAUGE

A. Mass deformation of the Yang-Mills theory in the covariant Landau gauge

We introduce the mass-deformed Yang-Mills theory in the covariant gauge which is defined just by adding the naive mass term $L_m$ to the ordinary massless Yang-Mills theory in the (manifestly Lorentz) covariant gauge fixing. The total Lagrangian density $\mathcal{L}_\text{YM}^{\text{tot}}$ of the massive Yang-Mills model consists of the Yang-Mills Lagrangian $\mathcal{L}_\text{YM}$, the gauge-fixing (GF) term $\mathcal{L}_\text{GF}$, the associated Faddeev-Popov (FP) ghost term $\mathcal{L}_\text{FP}$, and the mass term $\mathcal{L}_m$,

$$\mathcal{L}_\text{YM}^{\text{tot}} = \mathcal{L}_\text{YM} + \mathcal{L}_\text{GF} + \mathcal{L}_\text{FP} + \mathcal{L}_m,$$

$$\mathcal{L}_\text{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a + \frac{\alpha}{2} N^A A^A,$$

$$\mathcal{L}_\text{GF} = N^A \partial^\mu A^A \left[ A^A \right] + \frac{\alpha}{2} N^A N^A,$$

$$\mathcal{L}_\text{FP} = i \tilde{C}^A \partial^\mu D^A_{\mu} \left[ \mathcal{A} \right]^{AB} C^B,$$

$$\mathcal{L}_m = \frac{1}{2} M^2 A^A_i A^A_i,$$

where $A^A_i$ denotes the Yang-Mills field, $N^A$ the Nakanishi-Lautrup field and $\mathcal{C}^A_i, \mathcal{C}^A$ ($A = 1, \ldots, \dim G$) the Faddeev-Popov ghost and antighost fields, which take their values in the Lie algebra $\mathfrak{g}$ of a gauge group $G$ with the structure constants $f_{ABC}$ ($A, B, C = 1, \ldots, \dim G$).

We call this theory the massive Yang-Mills model in the covariant gauge for short.

The expectation value of the operator $O[A]$ of $A^A_i$ is given according to the path integral quantization using the BRST-invariant total action $S_{\text{YM}}^{\text{tot}}[A, \mathcal{C}, \overline{\mathcal{C}}, N]$ and the BRST-invariant integration measure $D A^A D \mathcal{C}^A D \overline{\mathcal{C}} D N$

$$\langle O[A] \rangle_{\text{YM}} := \frac{\int D A^A D \mathcal{C}^A D \overline{\mathcal{C}} D N e^{i S_{\text{YM}}^{\text{tot}}[A, \mathcal{C}, \overline{\mathcal{C}}, N]} O[A]}{\int D A^A D \mathcal{C}^A D \overline{\mathcal{C}} D N e^{i S_{\text{YM}}^{\text{tot}}[A, \mathcal{C}, \overline{\mathcal{C}}, N]}}.$$

In the Landau gauge $\alpha = 0$, especially, the average is cast into a simpler form by integrating the Nakanishi-Lautrup field $N^A$ and subsequently the ghost and antighost field $\mathcal{C}^A_i, \mathcal{C}^A$ as

$$\langle O[A] \rangle_{\text{YM}} = \frac{\int D A^A D \mathcal{C}^A D \overline{\mathcal{C}} D N e^{i S_{\text{YM}}^{\text{tot}}[A, \mathcal{C}, \overline{\mathcal{C}}, N] + i S_{\text{FP}}[A, \mathcal{C}, \overline{\mathcal{C}}] + i S_m[A]} O[A]}{\int D A^A D \mathcal{C}^A D \overline{\mathcal{C}} D N e^{i S_{\text{YM}}^{\text{tot}}[A, \mathcal{C}, \overline{\mathcal{C}}, N] + i S_{\text{FP}}[A, \mathcal{C}, \overline{\mathcal{C}}] + i S_m[A]}}.$$

The massive Yang-Mills model just defined is a special case of a massive extension of the massless Yang-Mills theory in the most general renormalizable gauge having both BRST and anti-BRST symmetries given by $\Delta_{\text{FP}}[\mathcal{A}] := \det(\partial^\mu D^A_{\mu} [\mathcal{A}]^{AB}).$

The massive Yang-Mills model consists of the Yang-Mills Lagrangian $L$, the gauge-fixing (GF) term $L_{\text{GF}}$, the associated Faddeev-Popov (FP) ghost term $L_{\text{FP}}$, and the mass term $L_m$,

$$L_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a + \frac{\alpha}{2} N^A N^A,$$

$$L_{\text{GF}} = \frac{\alpha}{2} N^A N^A + \frac{\beta}{2} N^A N^A,$$

$$L_{\text{FP}} = i \overline{\mathcal{C}}^A \partial^\mu D^A_{\mu} \left[ \mathcal{A} \right]^{AB} \mathcal{C}^B,$$

where $\alpha$ and $\beta$ are parameters which correspond to the gauge-fixing parameters in the $M \to 0$ limit, $D^A_{\mu} \left[ \mathcal{A} \right]^{AB} := \partial_\mu \mathcal{A}^A \left( \mathcal{A} \right) + g f_{ABC} \mathcal{A}^B_\mu \mathcal{C}^C$, and $\mathcal{N} := -N + g \overline{\mathcal{C}} \times \mathcal{C}$. The $\alpha = 0$ case is the Curci-Ferrari model with the coupling constant $g$, the mass parameter $M$ and the parameter $\beta$. In the Abelian limit with vanishing structure constants $f_{ABC} = 0$, the FP ghosts decouple and the Curci-Ferrari model reduces to the Nakanishi model. For $M \neq 0$, the physics depends on the parameter $\beta$. This result should be compared with the $M = 0$ case, in which $\beta$ is a gauge fixing parameter and hence the physics should not depend on $\beta$. In the $M = 0$ case, indeed, any choice of $\beta$ gives the same physics. However, this is not the case for $M \neq 0$. See e.g., [32] for more details. The massive Yang-Mills model is regarded as a $\beta = 0$ case of the Curci-Ferrari model. This point of view taken in the preceding works is good from the viewpoint of renormalizability, since the Curci-Ferrari model is known to be renormalizable. However, the Curci-Ferrari model lacks the physical unitarity at least in the perturbation theory. This issue will be avoided in the gauge-invariant extension which retains the nilpotent BRST symmetry, as discussed later.

We show that the massive Yang-Mills model in a covariant gauge has the gauge-invariant extension which is given by the gauge-scalar model with a single radially-fixed (or fixed modulus) scalar field in the fundamental representation of a gauge group if the theory is subject to an appropriate constraint which we call the reduction condition. We call such a gauge-scalar model the complementary gauge-scalar model. In other words, the complementary-gauge scalar model with a single radially fixed scalar field in the fundamental representation reduces to the mass-deformed Yang-Mills theory in a fixed gauge if an appropriate reduction condition is imposed.

with the Faddeev-Popov determinant,

$$\Delta_{\text{FP}}[\mathcal{A}] := \det(\partial^\mu D^A_{\mu} [\mathcal{A}]^{AB}).$$
For $G = SU(2)$, the complementary gauge-scalar model is given by
\[ \mathcal{L}_{RF} = \mathcal{L}_{YM} + \mathcal{L}_{kin}, \]
\[ \mathcal{L}_{YM} = -\frac{1}{2} \text{tr}[(\mathcal{F}_{\mu\nu})^2], \]
\[ \mathcal{L}_{kin} := (D_{\mu}[\alpha']\Phi)^\dagger \cdot (D^{\mu}[\alpha']\Phi), \]
with a single fundamental scalar field $\Phi$ subject to the radially fixed condition,
\[ f(\Phi(x)) := (\Phi(x)^\dagger \cdot \Phi(x) - \frac{1}{2} v^2) = 0, \]
where $v$ is a positive constant and $\Phi(x)$ is the $SU(2)$ doublet formed from two complex scalar fields $\phi_1(x), \phi_2(x)$,
\[ \phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \phi_1(x), \phi_2(x) \in \mathbb{C}, \]
where $D_{\mu}[\alpha']$ is the covariant derivative in the fundamental representation $D_{\mu}[\alpha'] := \partial_{\mu} - ig\alpha_{\mu}^\prime$.

This gauge-scalar model is invariant under the gauge transformation,
\[ \alpha_{\mu}(x) \rightarrow \alpha^\prime_{\mu}(x) := U(x)\alpha_{\mu}(x)U(x)^\dagger + ig^{-1}U(x)\partial_{\mu}U(x)^\dagger, \]
\[ \phi(x) \rightarrow \phi^\prime(x) := U(x)\phi(x), \quad U(x) \in G. \]

It is more convenient to convert the scalar field into the gauge group element. For this purpose, we introduce the matrix-valued scalar field $\Theta$ by adding another $SU(2)$ doublet $\Phi := e\Phi^*$ as
\[ \Theta(x) := \begin{pmatrix} \Phi(x) \\ \Phi^*(x) \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Then the complementary $SU(2)$ gauge-scalar model with a single radially-fixed scalar field in the fundamental representation is defined by
\[ \mathcal{L}_{RF} = \mathcal{L}_{YM} + \mathcal{L}_{kin} + \mathcal{L}_{c}, \]
\[ \mathcal{L}_{YM}[\alpha'] = \frac{1}{2} \text{tr}(\mathcal{F}_{\mu\nu}[\alpha'](x)\mathcal{F}^{\mu\nu}[\alpha'](x)), \]
\[ \mathcal{L}_{kin}[\alpha', \Theta] = \frac{1}{2} \text{tr}((D_{\mu}[\alpha']\Theta(x))^\dagger D^{\mu}[\alpha']\Theta(x)), \]
\[ \mathcal{L}_{c}[\alpha', \Theta] = u(x) f(\Theta), \quad f(\Theta) := \text{tr}(\Theta^\dagger \Theta - \frac{1}{2} v^2 1)/\text{tr}(1), \]
where $u$ is the Lagrange multiplier field to incorporate the holonomic constraint (7) written in the matrix form $f(\Theta) = 0$. The radially-fixed gauge-scalar model with the Lagrangian density (11) is invariant under the gauge transformation,
\[ \alpha_{\mu}(x) \rightarrow \alpha^\prime_{\mu}(x) := U(x)\alpha_{\mu}(x)U(x)^\dagger + ig^{-1}U(x)\partial_{\mu}U(x)^\dagger, \]
\[ \Theta(x) \rightarrow \Theta^\prime(x) := U(x)\Theta(x), \quad U(x) \in G. \]

Then we introduce the normalized matrix-valued scalar field $\hat{\Theta}$ by
\[ \hat{\Theta}(x) = \Theta(x)/\left(\frac{v}{\sqrt{2}}\right), \quad v > 0. \]

The above constraint (7) implies that the normalized scalar field $\hat{\Theta}$ obeys the conditions: $\hat{\Theta}(x)\hat{\Theta}(x)^\dagger = \Theta(x)\Theta(x)^\dagger = 1$, and $\text{det} \Theta(x) = 1$. Therefore, $\Theta$ is an element of $SU(2)$,
\[ \hat{\Theta}(x) \in G = SU(2). \]

This is an important property to provide a gauge-independent BEH mechanism.

The massive vector boson field $\mathcal{W}_{\mu} \in \mathcal{G} = su(2)$ is defined in terms of the original gauge field $\alpha_{\mu} \in \mathcal{G} = su(2)$ and the normalized scalar field $\hat{\Theta} \in G = SU(2)$ as shown in a previous paper (12),
\[ \mathcal{W}_{\mu}(x) := ig^{-1}(D_{\mu}[\alpha']\hat{\Theta}(x)\hat{\Theta}(x)^\dagger = -ig^{-1}\hat{\Theta}(x)(D_{\mu}[\alpha']\hat{\Theta}(x)^\dagger = \frac{1}{2} ig^{-1}[(D_{\mu}[\alpha']\hat{\Theta}(x)\hat{\Theta}(x)^\dagger - \hat{\Theta}(x)(D_{\mu}[\alpha']\hat{\Theta}(x)^\dagger)]]. \]

According to the gauge-independent BEH mechanism (11) (12), the kinetic term of the scalar field $\Theta$ is identical to the mass term of $\mathcal{W}_{\mu}$,
\[ \mathcal{L}_{kin}[\alpha', \hat{\Theta}] = \frac{1}{2} \text{tr}((D_{\mu}[\alpha']\hat{\Theta}(x))^\dagger D^{\mu}[\alpha']\hat{\Theta}(x)) = M^2 \text{tr}(\mathcal{W}_{\mu}(x)\mathcal{W}^{\mu}(x)), \quad M = g\frac{v}{2}. \]

The massive vector field $\mathcal{W}_{\mu}$ is rewritten using $\hat{\Theta}(x)\hat{\Theta}(x)^\dagger = 1$
\[ \mathcal{W}_{\mu}(x) = \mathcal{W}_{\mu}(x) - ig^{-1}\hat{\Theta}(x)\partial_{\mu}\hat{\Theta}(x)^\dagger. \]

Then it is shown that the massive vector boson field $\mathcal{W}_{\mu}$ has the expression, \(^1\)
\[ \mathcal{W}_{\mu}(x) = \hat{\Theta}(x)\mathcal{W}_{\mu}^{\hat{\Theta}}(x)\hat{\Theta}(x)^\dagger, \]
where $\mathcal{W}_{\mu}^{\hat{\Theta}}$ denotes the gauge transform of $\mathcal{W}_{\mu}$ by $\hat{\Theta} \in G$. Notice that $\mathcal{W}_{\mu}$ transforms according to the adjoint representation under the gauge transformation,
\[ \mathcal{W}_{\mu}(x) \rightarrow \mathcal{W}_{\mu}^{U}(x) = U(x)\mathcal{W}_{\mu}(x)U(x)^\dagger, \]
whereas $\mathcal{W}_{\mu}^{\hat{\Theta}}$ is gauge invariant,
\[ \mathcal{W}_{\mu}^{\hat{\Theta}}(x) \rightarrow (\mathcal{W}_{\mu}^{\hat{\Theta}})^U(x) = \mathcal{W}_{\mu}^{\hat{\Theta}}(x). \]

\(^1\) In (12), $\mathcal{W}_{\mu}^{\hat{\Theta}}(x)$ was written as $\hat{\mathcal{W}}_{\mu}(x)$. 

Therefore, the mass term can be written in terms of the gauge-invariant field $A^\dagger_\mu$ as

$$L_{\text{kin}}[\mathcal{A}, \hat{\Theta}^\dagger] = M^2 \text{tr}(A^\dagger_\mu(x) A^\mu_\mu(x)), \quad M = \frac{g}{2}.$$

(21)

This theory is supposed to obey the reduction condition for the massive vector field mode $\mathcal{W}_\mu(x)$. The stationary form of the reduction condition is given by

$$\chi(x) := \mathcal{D}^\mu_\mu[\mathcal{A}] \mathcal{W}_\mu(x) = 0,$$

(22)

where $\mathcal{D}^\mu_\mu[\mathcal{A}]$ is the covariant derivative in the adjoint representation $\mathcal{D}^\mu_\mu[\mathcal{A}] := \partial_\mu - ig[\mathcal{A}, \cdot]$. The stationary reduction condition is cast into

$$\chi(x) := \mathcal{D}^\mu_\mu[\mathcal{A}] \mathcal{W}_\mu(x),$$

where $\mathcal{D}^\mu_\mu[\mathcal{A}]$ is the covariant derivative in the adjoint representation $\mathcal{D}^\mu_\mu[\mathcal{A}] := \partial_\mu - ig[\mathcal{A}, \cdot]$. The stationary reduction condition is given by

$$\chi(x) := \mathcal{D}^\mu_\mu[\mathcal{A}] \mathcal{W}_\mu(x) = 0,$$

(22)

This implies that imposing the reduction condition $\chi(x) := \mathcal{D}^\mu_\mu[\mathcal{A}] \mathcal{W}_\mu(x) = 0$ is equivalent to imposing the Landau gauge condition $\partial^\mu A^\mu_\mu(x) = 0$ (or transverse condition) for the gauge-invariant field $A^\dagger_\mu$. Therefore, we can use the (gauge-transformed) reduction condition $\tilde{\chi}$ written as

$$\tilde{\chi}^\dagger(\hat{x}) := \hat{\Theta}(\hat{x})^\dagger \chi(\hat{x}) \hat{\Theta}(\hat{x}) = \partial^\mu A^\mu_\mu(x) = 0,$$

(24)

and the associated Faddeev-Popov determinant $\Delta_{\text{FP}}$ reads

$$\Delta_{\text{FP}}[\mathcal{A}, \hat{\Theta}^\dagger] := \det \left[ \frac{\delta^{\mu\mu} \hat{\Theta}^\dagger_\mu}{\delta \hat{\Theta}^\dagger} \right].$$

Notice that the reduction condition $\chi$ and the associated FP determinant $\Delta_{\text{FP}}$ are written in terms of $A^\dagger_\mu$ alone, $\tilde{\chi} = \chi[A^\dagger_\mu]$ and $\Delta_{\text{red}}[A^\dagger_\mu] = \Delta_{\text{red}}[\tilde{\chi}]$, and hence they are gauge invariant.

We show that the massive Yang-Mills (mYM) model in the Landau gauge can be converted to the complementary gauge-scalar (CGS) model, namely, radially-fixed gauge-scalar model subject to the reduction condition. In fact, the vacuum expectation value of a gauge-invariant operator $\mathcal{O}[\mathcal{A}]$ of $A^A$ reads

$$\langle \mathcal{O}[\mathcal{A}] \rangle_{\text{mYM}} := \frac{\int D\hat{\Theta} \int D\mathcal{W}[\mathcal{A}] \delta(\partial^\mu A^\mu_\mu) e^{iS_{\text{YM}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}]} \mathcal{O}[\mathcal{A}]}{\int D\hat{\Theta} \int D\mathcal{W}[\mathcal{A}] \delta(\partial^\mu A^\mu_\mu) e^{iS_{\text{YM}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}] + iS_{\text{kin}}[\mathcal{A}]}},$$

(26)

where the normalized matrix-scalar field $\hat{\Theta}$ is introduced and the integration over the gauge volume $\int D\hat{\Theta}^\dagger$ is inserted in the second equality, the integration variable $\mathcal{A}$ is renamed to $A^\dagger_\mu$ in the third equality, the gauge invariance of the Yang-Mills action $S_{\text{YM}}[A^\dagger_\mu] = S_{\text{YM}}[\mathcal{A}]$, the integration measure $D\mathcal{A}^\dagger = D\mathcal{A}$ and the operator $\mathcal{O}[A^\dagger_\mu] = \mathcal{O}[\mathcal{A}]$ is used in the fourth equality, and the FP determinant $\Delta_{\text{FP}}[\mathcal{A}]$ for the Landau gauge $\partial^\mu A^\mu_\mu = 0$ in the massive Yang-Mills model is identified with the FP determinant $\Delta_{\text{FP}}$ for the reduction condition $\chi$ in the fifth equality. In the last step, the delta function $\delta(\hat{\Theta}^\dagger, h[\mathcal{A}])$ on the group $G$ satisfying $\int D\hat{\Theta}^\dagger \delta(\hat{\Theta}^\dagger, h[\mathcal{A}]) = 1$ is used to rewrite

$$\delta(\hat{\Theta}^\dagger, h[\mathcal{A}]) = \delta(\hat{\Theta}^\dagger, h[\mathcal{A}]) = \Delta_{\text{FP}}[\mathcal{A}] \delta(\partial^\mu A^\mu_\mu),$$

(27)

which is valid when the following equation for a given $\mathcal{A}$ has a unique solution of $h = h[\mathcal{A}] \in G$,

$$\partial^\mu A^\mu_\mu(x) = 0.$$  

(28)
This uniqueness of the solution corresponds to assuming that there are no Gribov copies if $\partial_\mu \mathcal{A}_\mu^{h[\mathcal{A}]}(x) = 0$ is regarded as the gauge-fixing condition. Notice that we have taken into account the radially-fixed constraint (7) for replacing the scalar field $\Theta$ by the normalized matrix-valued (or group-valued) scalar field $\hat{\Theta}$ in the last step.

B. Solving the reduction condition

In the complementary gauge-scalar model, the scalar field $\Phi$ and the gauge field $\mathcal{A}$ are not independent field variables, because we intend to obtain the massive pure Yang-Mills theory which does not contain the scalar field $\Phi$. Therefore, the scalar field $\Phi$ is to be eliminated in favor of the gauge field $\mathcal{A}$. This is in principle achieved by solving the reduction condition as an off-shell equation, which is different from solving the field equation for the scalar field $\Phi$ as adopted in the preceding studies [34–39]. Consequentially, the resulting massive Yang-Mills model in the covariant gauge-fixing term and the associated Faddeev-Popov ghost term becomes power-counting renormalizable in the perturbative framework, as demonstrated to one-loop order in the next section.

Moreover, the entire theory is invariant under the usual Becchi-Rouet-Stora-Tyutin (BRST) transformation $\delta_{\text{BRST}}$. The nilpotency $\delta_{\text{BRST}}^2 = 0$ of the usual BRST transformations ensures the unitarity of the theory in the physical subspace of the total state vector space determined by zero BRST charge according to Kugo and Ojima [3]. This situation should be compared with the Curci-Ferrari model [10] which is not invariant under the ordinary BRST transformation, but instead can be made invariant under the modified BRST transformation $\delta'_{\text{BRST}}$. Nevertheless, this fact does not guarantee the unitarity of the Curci-Ferrari model due to the lack of usual nilpotency of the modified BRST transformation satisfying $\delta'_{\text{BRST}}^2 = 0$, see e.g., [32].

We proceed to eliminate the scalar field $\Phi$ or $\Theta$ by solving the reduction condition to obtain the massive Yang-Mills model from the complementary gauge-scalar model

$$\langle \mathcal{O}[\mathcal{A}] \rangle_{\text{CGS}} = \langle \mathcal{O}[\mathcal{A}] \rangle_{\text{YM}}. \quad (29)$$

Notice that introducing the reduction condition does not break the original gauge symmetry. The general form of the transverse and gauge-invariant Yang-Mills gauge field $\mathcal{A}_\mu^{h[\mathcal{A}]}$ satisfying (24) can be obtained explicitly by order by order expansion in powers of the gauge field $\mathcal{A}$ up to the Gribov copies. Indeed, $\mathcal{A}_\mu^{h[\mathcal{A}]}$ satisfying the transverse condition,

$$\partial_\mu \mathcal{A}_\mu^{h[\mathcal{A}]} = 0, \quad (30)$$

is obtained as a power series in $\mathcal{A}$,

$$\mathcal{A}_\mu^{h[\mathcal{A}]} = \mathcal{A}_\mu^{T} - i g \frac{\partial_\mu}{\partial^2} \left[ \mathcal{A}_\nu, \partial_\nu \frac{\partial \cdot \mathcal{A}}{\partial^2} \right]$$

$$- i \frac{2}{g^2} \left[ \partial \cdot \mathcal{A}, \frac{1}{\partial^2} \partial \cdot \mathcal{A} \right] + ig \left[ \mathcal{A}_\mu, \frac{1}{\partial^2} \partial \cdot \mathcal{A} \right]$$

$$+ i \frac{1}{g^2} \left[ 1 \frac{\partial^2}{\partial^2} \partial \cdot \mathcal{A}, \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial^2} \right] + \mathcal{O}(\mathcal{A}^3), \quad (31)$$

where we have defined the transverse field $\mathcal{A}_\mu^{T}$ in the lowest order term linear in $\mathcal{A}$ as

$$\mathcal{A}_\mu^{T} := \mathcal{A}_\mu - \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial^2}. \quad (32)$$

Then we find that the transverse field $\mathcal{A}_\mu^{h[\mathcal{A}]}$ is rewritten into

$$\mathcal{A}_\mu^{h[\mathcal{A}]} = \left( \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Psi_\nu, \quad (33)$$

$$\Psi_\nu = \mathcal{A}_\nu - i g \left[ \frac{1}{\partial^2} \partial \cdot \mathcal{A}, \mathcal{A}_\nu \right]$$

$$+ i \frac{2}{g^2} \left[ \frac{1}{\partial^2} \partial \cdot \mathcal{A}, \partial_\nu \frac{\partial \cdot \mathcal{A}}{\partial^2} \right] + \mathcal{O}(\mathcal{A}^3). \quad (34)$$

Under an infinitesimal gauge transformation $\delta_\Lambda$, defined by $\delta_\Lambda \mathcal{A}_\mu = \mathcal{D}_\mu \left[ \mathcal{A}, \Lambda \right] := \partial_\mu \Lambda - ig \left[ \mathcal{A}_\mu, \Lambda \right]$, $\Psi_\nu$ transforms as

$$\delta_\Lambda \Psi_\nu = \partial_\nu \left( \Lambda - i \frac{g}{2} \left[ \partial \cdot \mathcal{A}, \Lambda \right] \right) + \mathcal{O}(g^2). \quad (35)$$

Therefore, $\mathcal{A}_\mu^{h}$ given by (33) is left invariant by infinitesimal gauge transformations order by order of the expansion,

$$\delta_\Lambda \mathcal{A}_\mu^{h[\mathcal{A}]}(x) = 0. \quad (36)$$

In Appendix A, we give a recursive construction of the transverse field $\mathcal{A}_\mu^{h[\mathcal{A}]}$ and the proof of gauge invariance of the resulting $\mathcal{A}_\mu^{h[\mathcal{A}]}$.

The mass term of $\mathcal{W}_\mu$ is equal to that of $\mathcal{A}_\mu^{\Phi^1}$,

$$\mathcal{L}_{\text{kin}} = M^2 \text{tr} \left( \mathcal{W}_\mu(x) \mathcal{W}_\mu(x) \right) - M^2 \text{tr} \left( \mathcal{A}_\mu^{\Phi^1}(x) \mathcal{A}_\mu^{\Phi^1}(x) \right). \quad (37)$$

Therefore, the “mass term” of gauge-invariant field $\mathcal{A}_\mu^{h}$ is used to rewrite the kinetic term of the scalar field:

$$S_{\text{kin}}^\wedge_{[\mathcal{A}]} = \int d^D x \mathcal{M}^2 \text{tr} \left( \mathcal{A}_\mu^{h[\mathcal{A}]} \mathcal{A}_\mu^{h[\mathcal{A}]} \right) + \mathcal{O}(\mathcal{A}^3), \quad (37)$$

$\mathcal{L}_{\text{kin}} = M^2 \text{tr} \left( \mathcal{W}_\mu(x) \mathcal{W}_\mu(x) \right) - M^2 \text{tr} \left( \mathcal{A}_\mu^{\Phi^1}(x) \mathcal{A}_\mu^{\Phi^1}(x) \right). \quad (37)$

---

2 See e.g., [10] [11] for reviews of the Stückelberg field. Notice that the reduction condition is an off-shell condition. Therefore, solving the reduction condition is different from solving the field equation for the Stückelberg field as done in the preceding works [32]. This means that the solution of the reduction condition does not necessarily satisfy the field equation, while the solution of the field equation of the complementary gauge-scalar model automatically satisfies the reduction condition [32].
In this way, we have eliminated the scalar field by solving the reduction condition.

Only when we adopt the covariant Landau gauge $\partial \cdot \mathcal{A} = 0$ as the gauge-fixing condition, the infinite number of nonlocal terms disappear so that $S_{\text{kin}}^*$ reduces to the naive mass term of $\mathcal{A}$,

$$S_m[\mathcal{A}] = \int d^Dx \, M^2 \text{tr}(\mathcal{A}_\mu(x) \mathcal{A}_\mu(x)).$$ (38)

In the Landau gauge, thus, the complementary gauge-scalar model with the reduction condition reduces to the massive Yang-Mills model with the naive mass term.

The explicit expression of the massive vector field $\mathcal{W}_\mu$ in terms of $\mathcal{A}_\mu$ is given in Appendix B. Notice that $\mathcal{W}_\mu$ agrees with $\mathcal{A}_\mu^\dagger = \mathcal{A}_\mu$ in the Landau gauge $\partial \cdot \mathcal{A} = 0$.

### III. MASSIVE YANG-MILLS THEORY AND DECOUPLING SOLUTIONS

In order to reproduce the decoupling solution of the Yang-Mills theory in the covariant Landau gauge, we calculate one-loop quantum corrections to the gluon and ghost propagators in the massive Yang-Mills model. The Nakanishi-Lautrup field $\mathcal{N}^A$ can be eliminated so that the gauge-fixing term reduces to

$$\mathcal{L}_{GF} \to -\frac{1}{2} \alpha^{-1}(\partial^\mu \mathcal{A}^A)^2.$$ (39)

The results in the Landau gauge is obtained by taking the limit $\alpha \to 0$ in the final step of the calculations. Only in the Landau gauge $\alpha = 0$ the massive Yang-Mills model with a naive mass term $\mathcal{L}_m$ has the gauge-invariant extension. In order to obtain the gauge-independent results in the other gauges with $\alpha \neq 0$, we need to include an infinite number of non-local terms in addition to the naive mass term $\mathcal{L}_m$ for gluons, as shown in the previous section.

#### A. Feynman rules for the massive Yang-Mills model

The Feynman rules for the massive Yang-Mills model are given as follows. The diagrammatic representations of the Feynman rules are given in Fig. 1

**P1** gluon propagator $\langle \mathcal{A} \mathcal{A} \rangle$

$$\hat{D}^{AB}_{\mu \nu} (k) := -\frac{i \delta^{AB}}{k^2 - M^2} \left[ g_{\mu \nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 - \alpha M^2} \right].$$ (40a)

$$= \delta^{AB} \left[ -\frac{1}{k^2 - M^2} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{M^2} \right) - \frac{k_\mu k_\nu}{M^2 (k^2 - \alpha M^2)} \right].$$ (40b)

$$= \delta^{AB} \left[ -\frac{1}{k^2 - M^2} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) - \frac{\alpha}{k^2 - \alpha M^2} \frac{k_\mu k_\nu}{k^2} \right].$$ (40c)

**P2** ghost propagator $\langle \mathcal{C} \bar{C} \rangle$

$$\Delta^{AB}_{gh} (k) := -\frac{i \delta^{AB}}{k^2 + i \epsilon}.$$ (41)

**V1** three-gluon vertex function $\langle \mathcal{A} \mathcal{A} \mathcal{A} \rangle$

$$\Gamma^{ABC}_{\mu \nu \lambda} (p, q, r) = g f^{ABC} [(q - r)_\mu g_{\nu \lambda} + (r - p)_\nu g_{\mu \lambda} + (p - q)_\lambda g_{\mu \nu}],$$ (42)

**V2** gluon-ghost-antighost vertex function $\langle \mathcal{A} \mathcal{C} \bar{C} \rangle$

$$\Gamma^{ABC}_{\mu} (p, q, r) := i g f^{ABC} r_\mu,$$ (43)

**V3** four-gluon vertex function $\langle \mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A} \rangle$

$$\Gamma^{ABCD}_{\mu \nu \lambda \rho} (p, q, r, k) = -ig^2 [f^{ABE} f^{ECF} (g_{\mu \lambda} g_{\nu \rho} - g_{\rho \nu} g_{\mu \lambda}) + f^{ADE} f^{EBC} (g_{\mu \rho} g_{\nu \lambda} - g_{\rho \lambda} g_{\nu \mu}) + f^{ACE} f^{EBD} (g_{\mu \nu} g_{\rho \lambda} - g_{\nu \rho} g_{\mu \lambda})].$$ (44)

Here the momentum conservation is omitted and the momentum flow at each vertex is regarded as incoming, while the momentum of antighost as outgoing. Notice that the Feynman rules are the same as those of the ordinary Yang-Mills theory in the Lorenz gauge except for the gluon propagator which was replaced by the massive propagator (40).

The gluon propagator has the same form as that in the renormalizable $R_\xi$ gauge where unitarity is not manifest. For any finite values of $\alpha$, the gluon propagator has good high-energy behavior, namely, the asymptotic behavior $O(1/k^2)$ as $k \to \infty$, and hence the theory is renormalizable by power counting. For example, the choice $\alpha = 1$ leads to the propagator $\frac{1}{k^2 - M^2} g_{\mu \nu}$. In the limit $\alpha \to \infty$, the gluon propagator reduces to the standard form for a massive spin-one particle, as can be seen in the second form. We recover the unitary gauge in which particle content is manifest, since there are no unphysical fields, and...
hence unitarity is transparent, while renormalizability is not transparent.

For any finite values of \( \alpha \), the gluon propagator has an extra unphysical pole at \( k^2 = \alpha M^2 \) besides the physical pole (massive gauge bosons) at \( k^2 = M^2 \), as can be seen in the second form of (40). In order to preserve unitarity, the unphysical poles must be eliminated or mutually cancel in the S-matrix element involving only physical particles. In the spontaneously broken gauge theory, the would-be Nambu-Goldstone boson field has the propagator with the unphysical pole at \( k^2 = \alpha M^2 \), and this unphysical pole of the would-be Nambu-Goldstone particle cancels one of the gauge boson in order to preserve unitarity. This is not the case in our model, since there are no Nambu-Goldstone particles without spontaneous symmetry breaking. The above type of cancellation of unphysical poles can be proven to all orders in perturbation theory by using the generalized Ward-Takahashi identities which are a consequence of the gauge invariance of the theory.

In the limit \( \alpha \to 0 \), however, the gluon propagator reduces to the simple form for a massive spin-one particle with the transverse projector \( \frac{1}{\kappa^2 - M^2} \text{tr} \left( g_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{\kappa^2} \right) \), as can be seen in the third form of (40), and the contribution from the unphysical pole at \( k^2 = 0 \) disappears in this limit. Therefore, the Landau gauge is the very special gauge which guarantees renormalizability and unitarity in agreement with the gauge invariance of the theory. This result is consistent with our point of view that the massive Yang-Mills model has the gauge-invariant extension only in the Landau gauge.

B. Renormalization

We now take into account quantum corrections to the gluon and ghost propagators to one loop order. In Fig. 2 we enumerate the one-loop diagrams which contribute to the gluon and ghost propagators to one-loop order.

In the massive Yang-Mills model we introduce the renormalization factors \( Z_{\varphi}, Z_{\bar{\varphi}} = Z_{\varphi}, Z_{\bar{\varphi}}, Z_{M^2}, Z_{\alpha} \) to connect the bare unrenormalized fields (gluon \( \varphi_B \), ghost \( \bar{\varphi}_B \) and antighost \( \bar{\varphi}_B \)) and bare parameters (the coupling constant \( g_B \), the mass parameter \( M_B \) and the gauge-fixing parameter \( \alpha_B \)) to the renormalized fields \( (\varphi_R, \bar{\varphi}_R) \) and renormalized parameters \( (g_R, M_R \text{ and } \alpha_R) \) respectively [43–45].

\[
\begin{align*}
\varphi_B &= Z_{\varphi}^{1/2} \varphi_R, \quad \bar{\varphi}_B = Z_{\bar{\varphi}}^{1/2} \bar{\varphi}_R, \\
g_B &= Z_g g_R, \quad M_B^2 = Z_{M^2} M_R^2, \quad \alpha_B = Z_{\alpha}^{-1} Z_{\varphi} \alpha_R.
\end{align*}
\]

For comparison with the lattice data, we move to the Euclidean region and use \( k_E \) to denote the Euclidean momentum so that \( k^2 = -k_E^2 \), and introduce the dimensionless squared momentum

\[ s := \frac{k_E^2}{M^2}. \]

For gluons, we introduce the two-point vertex function \( \Gamma^{(2)}(k) \) as the inverse of the transverse part \( \mathcal{D}_T \) of the propagator \(^3\) and the vacuum polarization function \( \Pi_T \) as

\[
\Gamma^{(2)}(k_E) := \left[ \mathcal{D}_T(k_E) \right]^{-1} = k_E^2 + M^2 + \Pi_T(k_E^2) + k_E^2 \delta_Z + M^2 \delta_{M^2} = k_E^2 + M^2 + \Pi_T^{\text{fin}}(k_E^2),
\]

where \( \delta_Z \) and \( \delta_{M^2} \) are counterterms to cancel the divergence coming from the vacuum polarization function \( \Pi_T \) to obtain the finite renormalized one \( \Pi_T^{\text{fin}} \)

\[
\Pi_T^{\text{fin}}(k_E^2) := \Pi_T(k_E^2) + k_E^2 \delta_Z + M^2 \delta_{M^2},
\]

under the suitable renormalization conditions to be discussed shortly, and they are related to the renormalization factors as

\[
\delta_Z = Z_{\varphi} - 1, \quad \delta_{M^2} = Z_{M^2} Z_{\varphi} - 1.
\]

We denote also the dimensionless versions \( \hat{\mathcal{D}}_T(s) \) and \( \hat{\Pi}(s) \) of \( \mathcal{D}_T(k_E) \) and \( \Pi_T(k_E^2) \) with the hat respectively

\[
\Gamma^{(2)}(k_E)/M^2 := \left[ \hat{\mathcal{D}}_T(s) \right]^{-1} = s + 1 + \hat{\Pi}_T(s) + s \delta_Z + \delta_{M^2} = s + 1 + \hat{\Pi}_T^{\text{fin}}(s),
\]

with

\[
\hat{\Pi}_T^{\text{fin}}(s) := \hat{\Pi}_T(s) + s \delta_Z + \delta_{M^2}.
\]

The gluon vacuum polarization function in the covariant Landau gauge \( \alpha = 0 \) calculated using the dimensional

\(^3\) In this paper we focus on the Landau gauge. For the gluon propagator, therefore, we discuss the transverse part alone.
regularization in Euclidean space is given to one-loop order as the power-series expansion in $\epsilon := 2 - \frac{D}{2}$

$$\hat{\Pi}_T(s) = \frac{g^2 C_2(G)}{16\pi^2} \frac{1}{12} \times \frac{9}{s - 26} \left\{ \epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta \right\}$$

$$+ \frac{63}{s} - \frac{121}{3} + h(s) \right\},$$

(52)

where $C_2(G)$ is the quadratic Casimir operator of a gauge group $G$, $\gamma$ is the Euler constant, and $\eta$ is the value of $s$ at the scale $\bar{\mu}$ introduced through the dimensional regularization for dimensional reasons

$$\eta := \frac{\bar{\mu}^2}{M^2}.$$  

(53)

Here we have defined the functions of $s$,

$$h(s) := K_1(s) + K_2(s) + K_3(s),$$

$$K_1(s) := -\frac{1}{s^2} + \frac{3}{2} \ln s,$$

$$K_2(s) := \left( 1 + \frac{1}{s} \right)^3 (s^2 - 10s + 1) \ln (s + 1),$$

$$K_3(s) := \frac{1}{2} \left( 1 + \frac{4}{s} \right)^\frac{3}{2} (s^2 - 20s + 12) \ln \left( \frac{\sqrt{4+s} - \sqrt{s}}{\sqrt{4+s} + \sqrt{s}} \right),$$

(54)

Notice that there are no singular term in the finite part $\hat{\Pi}^\text{fin}_T(s)$ even at $s = 0$, because there does not exist $O(s^{-2})$ term in the bracket $[\ldots]$ of (52), since the expansion of $h(s)$ around $s = 0$ reads

$$h(s) = \frac{111}{2s} + \frac{389s}{60} + s^2 \left( -\frac{1}{2} \ln s - \frac{141}{140} \right) + \frac{269s^3}{420} + O(s^4),$$

(55)

which follows from

$$K_2(s) = \frac{1}{s^2} - \frac{15}{6} + \frac{133}{s} - \frac{187}{12} - \frac{43s^2}{60} + \frac{23s^3}{30} + O(s^4),$$

$$K_3(s) = -\frac{48}{s} + \frac{331}{15} - \frac{61s^2}{210} - \frac{53s^3}{420} + O(s^4).$$

(56)

Thus, the finite part of the gluon vacuum polarization to one-loop reads

$$\hat{\Pi}^\text{fin}_T(s) = \frac{g^2 C_2(G)}{16\pi^2} \frac{1}{12} s \left[ \frac{63}{s} - \frac{121}{3} + h(s) \right].$$

(57)

Therefore, the finite part $\hat{\Pi}^\text{fin}_T(s)$ has the $s = 0$ limit,

$$\hat{\Pi}^\text{fin}_T(s = 0) = \frac{g^2 C_2(G)}{16\pi^2} \frac{1}{12} \frac{15}{2}.$$  

(58)

C. Naive (zero-momentum) renormalization conditions

For gluons, we can take a naive renormalization condition such that

$$\Gamma^{(2)}_{af}(k_E = 0) = M^2 \iff \hat{\Pi}^\text{fin}_T(s = 0) = 0.$$  

(59)

A renormalization condition first adopted by Tissier and Wschebor is written in terms of $\Gamma^{(2)}_{af}$ or equivalently $\hat{\Pi}^\text{fin}_T$ as

$$\left\{ \begin{array}{l}
\Gamma^{(2)}_{af}(k_E = 0) = M^2 \\
\Gamma^{(2)}_{af}(k_E = \mu) = \mu^2 + M^2 \\
\hat{\Pi}^\text{fin}_T(s = 0) = 0 \\
\hat{\Pi}^\text{fin}_T(s = \nu) = 0
\end{array} \right\} \text{ (at } \mu = 1 \text{ GeV),}$$

(60)

where we have introduced the dimensionless ratio of the renormalization scale $\mu$ to the mass defined by

$$\nu := \frac{\mu^2}{M^2}.$$  

(61)

Adopting the renormalization condition [TW1], we obtain the renormalized gluon vacuum polarization function,

$$\hat{\Pi}^\text{fin}_{TW}(s) = \frac{g^2 C_2(G)}{16\pi^2} \frac{1}{12} s \left[ \frac{111}{2s} + h(s) - (s \to \nu) \right].$$

(62)

Note that constant terms in $[\ldots]$ are canceled by the subtraction: $-(s \to \nu)$.

However, it has been shown that the vanishing-momentum renormalization condition ([59]: $\Gamma^{(2)}_{af}(k_E = 0) = M^2$ or $\hat{\Pi}^\text{fin}(s = 0) = 0$ yields the infrared Landau pole, namely, the coupling constant diverges at a certain momentum in the infrared region. Therefore, we use another renormalization condition explained in the next subsection.

D. Infrared safe renormalization condition

For ghost, we introduce the two-point vertex function $\Gamma^{(2)}_{gh}$ and the propagator $\Delta_{gh}$ and the self-energy function $\hat{\Pi}_{gh}$,

$$\Gamma^{(2)}_{gh}(k_E) := \left[ \Delta_{gh}(k^2_E) \right]^{-1}$$

$$= k^2_E + \Pi_{gh}(k^2_E) + k^2_E g, C,$$

$$= k^2_E + \Pi^\text{fin}_{gh}(k^2_E),$$

(63)

These expressions are obtained by taking the limit $\alpha \to 0$ of those with an arbitrary $\alpha$ given in [30] as summarized in the Appendix C.
where $\delta_C$ is a counterterm to cancel the divergence coming from the ghost self-energy function $\Pi_{gh}$ to obtain the finite one $\tilde{\Pi}_{gh}^{\mathrm{fin}}$.

$$\tilde{\Pi}_{gh}^{\mathrm{fin}}(k_E^2) := \Pi_{gh}(k_E^2) + k_E^2 \delta_C,$$

(64)

and is related to the renormalization factor as

$$\delta_C = Z_{\partial} - 1.$$  

(65)

We also define the dimensionless versions $\hat{\Delta}_{gh}(s)$ and $\hat{\Pi}_{gh}(s)$ of $\Delta_{gh}(k_E^2)$ and $\Pi_{gh}(k_E^2)$ as

$$\Gamma_{gh}^{(2)}(k_E^2)/M^2 := [\Delta_{gh}(s)]^{-1}$$

$$= s + \hat{\Pi}_{gh}(s) + s \delta_C$$

$$= s + \tilde{\Pi}_{gh}^{\mathrm{fin}}(s),$$

(66)

with

$$\tilde{\Pi}_{gh}^{\mathrm{fin}}(s) = \Pi_{gh}(s) + s \delta_C.$$  

(67)

The ghost self-energy function $\Pi_{gh}(s)$ is calculated using the dimensional regularization and the dimensionless version $\tilde{\Pi}_{gh}(s)$ is given to one-loop order by

$$\tilde{\Pi}_{gh}(s) = \frac{g^2 C_2(G)}{16 \pi^2} \left[ -3 \left\{ \epsilon^1 - \gamma + \ln(4\pi) + \ln \eta \right\} ight.$$

$$- 5 + f(s) \bigg],$$

$$f(s) := \frac{s}{s} \ln s + \frac{1 + s}{s^2} \ln(1 + s)$$

$$= \frac{5}{2} + \frac{11}{6} - \ln s \bigg] + \frac{s^2}{4} + O(s^3).$$

(68)

For ghosts, we impose the renormalization condition

$$\Gamma_{gh}^{(2)}(k_E = \mu) = \mu^2 \iff \tilde{\Pi}_{gh}^{\mathrm{fin}}(s = \nu) = 0.$$  

(69)

The renormalization condition (69) determines the counterterm $\delta_C$ as

$$\delta_C^{(1)} = -\tilde{\Pi}_{gh}(s = \nu)/\nu$$

$$= -\left\{ \frac{g^2 C_2(G)}{16 \pi^2} \right\} \left[ -3 \left\{ \epsilon^1 - \gamma + \ln(4\pi) + \ln \eta \right\} ight.$$

$$- 5 + f(\nu) \bigg] \bigg].$$

(70)

Then we obtain the renormalized ghost self-energy function under the renormalization condition (69)

$$\tilde{\Pi}_{gh}^{\mathrm{fin}}(s) = \frac{g^2 C_2(G)}{16 \pi^2} \frac{1}{4} s [f(s) - f(\nu)].$$

(71)

We now return to the gluon renormalization. To avoid the infrared Landau pole for the coupling, we replace this renormalization condition by the following one:

$$[\mathrm{TW}2] \begin{cases}
Z_{M^2} Z_{\alpha} \alpha Z_{\partial} = 1 \\
\Gamma_{\alpha\partial}^{(2)}(k_E = \mu) = \mu^2 + M^2 \iff \tilde{\Pi}_{\alpha\partial}^{\mathrm{fin}}(s = \nu) = 0 \\
& \text{at } \mu = 1 \text{ GeV}.
\end{cases}$$

(72)

There is a well-known non-renormalization for the coupling in the Taylor scheme [43] which also holds in the massive Yang-Mills model in the Landau gauge: The identity

$$Z_g Z_{\alpha}^{1/2} Z_{\partial} = \tilde{Z}_\alpha^2,$$

(73)

implies in the Landau gauge

$$Z_g Z_{\alpha}^{1/2} Z_{\partial} = 1,$$

(74)

since in the Landau gauge,

$$\tilde{Z}_\alpha = 1$$ for $\alpha = 0.$

(75)

The meaning of the first renormalization condition is explained as follows. For the massive Yang-Mills model in the Landau gauge $\alpha = 0$ as a special limit of the Curci-Ferrari model, the non-renormalization theorem holds in the sense that a combination of renormalization factors is finite to all orders in the loop expansions [44, 45]: The identity

$$Z_{M^2} Z_{\alpha} Z_{\partial} = \tilde{Z}_\alpha^2,$$

(76)

implies in the Landau gauge

$$Z_{M^2} Z_{\alpha} Z_{\partial} = 1.$$  

(77)

As $Z_{M^2} Z_{\alpha} = 1 + \delta_{M^2}$ from (49) and $Z_{\partial} = 1 + \delta_C$ from (65), the non-renormalization theorem (76) in the Landau gauge reduces to

$$\delta_{M^2} = Z_{M^2} Z_{\alpha} - 1 = Z_{\partial}^{-1} - 1 = 1 + \delta_C - 1,$$

(78)

which means in the one-loop level

$$\delta_{M^2}^{(1)} = -\delta_C^{(1)}.$$  

(79)

This is the first renormalization condition of (72).

Then $\delta_Z$ is determined from the second renormalization condition of (72): $\tilde{\Pi}_{\alpha\partial}^{\mathrm{fin}}(s = \nu) = \Pi_T(s = \nu) + s \delta_C^{(1)} = 0$ by using $\delta_C^{(1)} = 0$ as

$$\delta_C^{(1)} = \left[ \tilde{\Pi}_{\alpha\partial}(s = \nu) - \delta_C^{(1)} \right]/\nu$$

$$= -\left\{ \frac{g^2 C_2(G)}{16 \pi^2} \right\} \frac{1}{12} [26 \left\{ \epsilon^1 - \gamma + \ln(4\pi) + \ln \eta \right\}$$

$$+ \frac{48}{\nu} - \frac{121}{3} + h(\nu) + \frac{3}{\nu} f(\nu)] \bigg].$$

(80)

Then, by substituting (80) and (79), the renormalized gluon vacuum polarization function is modified into [20]

$$\tilde{\Pi}_{\alpha\partial}^{\mathrm{fin}}(s) = \frac{g^2 C_2(G)}{16 \pi^2} \frac{1}{12} s \left\{ 48 + 3 f(\nu) + h(s) - (s \to \nu) \right\}. $$

(81)
The gluon vacuum polarization at $s = 0$ has a positive value

$$\hat{\Pi}^\text{fin}_T(s = 0) = \frac{g^2 C_2(G)}{16\pi^2} \left[ 3f(\nu) - \frac{15}{2} \right] > 0,$$

(82)

where we have used $f(s)$ is a monotonically increasing function of $s$ with $f(0) = \frac{5}{2}$.

The renormalization factors are obtained as functions of $g^2$ and $\nu$

$$Z^{(1)}_{z(0)} = -\delta^{(1)}_z = \frac{C_2(G)g^2}{16\pi^2} \left[ \frac{26}{3} \left\{ \epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta \right\} ight. 
- \frac{48}{\nu} + \frac{121}{3} - h(\nu) - \frac{3}{\nu} f(\nu) \Bigg],$$

(83)

$$Z^{(1)}_\phi = -\delta^{(1)}_C = \frac{C_2(G)g^2}{16\pi^2} \left[ \frac{3}{4} \left\{ \epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta \right\} 
+ 5 - f(\nu) \right],$$

(84)

$$Z^{(1)}_g = \frac{1}{2} Z^{(1)}_{\phi} - Z^{(1)}_e = -\frac{1}{2} \delta^{(1)}_z - \delta^{(1)}_C 
= \frac{C_2(G)g^2}{16\pi^2} \left[ -22 \left\{ \epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta \right\} 
+ \frac{24}{\nu} - \frac{211}{6} + 2 h(\nu) + 3 f(\nu) + \frac{3}{2\nu} f(\nu) \right],$$

(85)

and

$$Z^{(1)}_{M^2} = - Z^{(1)}_{z(0)} - Z^{(1)}_e = -\delta^{(1)}_z - \delta^{(1)}_C 
= \frac{C_2(G)g^2}{16\pi^2} \left[ -35 \left\{ \epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta \right\} 
+ \frac{48}{\nu} - \frac{166}{3} + h(\nu) + 3 f(\nu) + \frac{3}{2\nu} f(\nu) \right].$$

(86)

We can obtain the renormalization group functions using these renormalization factors. For instance, the anomalous dimension of the field $\Phi$ is obtained from the renormalization factor $Z_\Phi = 1 + Z^{(1)}_\Phi + \cdots$ according to

$$\gamma_\Phi(g^2, M^2) := \left. \frac{\partial \ln Z_\Phi}{\partial \ln \mu} \right|_{g_B, M_B} = \left. \frac{\partial \ln [1 + Z^{(1)}_\Phi + \cdots]}{\partial \ln \mu} \right|_{g_B, M_B} 
= 2\nu \left. \frac{\partial Z^{(1)}_\Phi}{\partial \nu} \right|_{g_B, M_B} + \cdots,$$

(87)

where the replacement of the derivative with respect to $\mu^2$ by $\nu = \mu^2/M^2$ is valid to one-loop order, since $M$ is the renormalized mass which depends on the renormalization scale $\mu$. Therefore, the ghost field has the anomalous dimension to one-loop order

$$\gamma_{\Phi}(g^2, M^2) := \frac{\partial \ln Z_\Phi}{\partial \ln \mu} \bigg|_{g_B, M_B} = - \frac{C_2(G)g^2}{16\pi^2} \frac{\nu}{\nu^2} \left\{ \gamma \nu = 0 \left. \frac{\partial}{\partial \nu} \frac{\partial \ln Z_{\Phi}}{\partial \ln \mu} \right|_{g_B, M_B} \right.$$

(88)

Similarly, the anomalous dimension of the gluon field is calculated to one-loop order as

$$\gamma_{\alpha}(g^2, M^2) := \left. \frac{\partial \ln Z_{\alpha}}{\partial \ln \mu} \right|_{g_B, M_B} = - \frac{C_2(G)g^2}{16\pi^2} \left[ \frac{48}{\nu^2} - h(\nu) + \frac{3}{\nu^2} f(\nu) - \frac{3}{\nu} f'(\nu) \right].$$

(89)

The $\beta$ function for the gauge coupling constant is obtained from

$$\beta_{g^2}(g^2, M^2) := \frac{\partial g^2}{\partial \ln \mu} = g^2 \frac{\partial \ln g^2}{\partial \ln \mu} = - 2g^2 \left. \frac{\partial \ln Z_{\Phi}}{\partial \ln \mu} \right|_{g_B, M_B} = - 4g^2 \nu \frac{\partial Z^{(1)}_{\Phi}}{\partial \nu} + \cdots,$$

(90)

which is indeed calculated to one-loop as

$$\beta_{g^2}(g^2, M^2) = g^2 [\gamma_{\alpha}(g^2, M^2) + 2\gamma_{\Phi}(g^2, M^2)]$$

(91)

$$= \frac{C_2(G)g^4}{16\pi^2} \left[ \frac{16}{\nu^2} - \frac{1}{\nu^2} f(\nu) + \frac{1}{\nu^2} f'(\nu) - \frac{1}{\nu^2} f''(\nu) \right].$$

Notice that $\gamma_\Phi$ is always negative ($\gamma_\Phi = 0$ at $\nu = 0$). We find that $\gamma_{\alpha}$ is negative for $\nu > 0.28$, becomes zero at $\nu \sim 0.28 \sim 0.53^2$ and positive for $\nu < 0.28$ ($\gamma_{\alpha} = 1/3$ at $\nu = 0$). We find that the beta function $\beta_{g^2}$ is negative for $\nu > 0.07$, becomes zero at $\nu \sim 0.07 \sim 0.26^2$ and positive for $\nu < 0.07$. This implies that the coupling constant increases monotonically in decreasing the scale $\mu$ until $\mu$ reaches the value $\mu/M > 0.26$, and it turns over at $\mu/M \sim 0.26$ and decreases towards the infrared limit $\mu \rightarrow 0$.

In the ultraviolet region $\nu \gg 1$ or $\mu \gg 1$, the beta function $\beta_{g^2}$ in the massive Yang-Mills theory is negative
for $\nu \ll 1$, since (91) has the expansion for $\nu \gg 1$

$$\beta_{g^2}(g^2, M^2) = \frac{C_2(G)g^4}{16\pi^2} \left[ \frac{22}{3} + \frac{59}{2} - \frac{9}{2} \ln \left( \frac{1}{\nu} \right) \nu - O \left( \nu^{-5/2} \right) \right]. \quad (92)$$

This result is in agreement with the standard, universal beta function of the usual Yang-Mills model reflecting the ultraviolet asymptotic freedom

$$g^2(\mu) \downarrow 0 \, (\mu \nearrow \infty). \quad (93)$$

In the infrared region $\nu \ll 1$ or $\mu \ll 1$, on the other hand, the beta function $\beta_{g^2}$ of the massive Yang-Mills model becomes remarkably positive in the deep infrared regime, since (91) has the expansion for $\nu \ll 1$

$$\beta_{g^2}(g^2, M^2) = \frac{C_2(G)g^4}{16\pi^2} \left[ \frac{1}{3} + \left( \ln \nu - \frac{367}{180} \right) \nu + O \left( \nu^{-5/2} \right) \right]. \quad (94)$$

This implies that the running coupling constant $g^2(\mu)$ decreases towards the infrared region and vanishes as $\mu \to 0$

$$g^2(\mu) \downarrow 0 \, (\mu \nearrow 0). \quad (95)$$

Therefore the RG flow drives the system towards a weak coupling region as $\mu$ goes to zero. This fact justifies the use of the one-loop approximation to study the Yang-Mills theory even in infrared region.

E. Fitting to the numerical simulations

We utilize the data obtained by the numerical simulations on the lattice for the Yang-Mills theory in the covariant Landau gauge \[17\] to determine the parameters, the coupling constant $g$ and the gluon mass parameter $m$, in the massive Yang-Mills model.

First, we attempted to fit the data of numerical simulations for the gluon propagator on the lattice \[17\] to the analytical expression $\Delta$ for the gluon propagator with one-loop quantum corrections by adjusting the two parameters $g$ and $M$ in the fitting range $0 < k_E \leq 1\text{GeV}$ at $\mu = 1\text{GeV}$ for $G = SU(3)$. However, as shown in Fig. 3, the fitting does not work so well and the appropriate parameters cannot be obtained in this setting. The reason of this failure will be attributed to the fact that the renormalization conditions adopted in the lattice simulations are different from those adopted in this paper, leading to the different scale for the gluon propagator. We introduce an overall scale factor $Z$ which can scale...
the gluon propagator as a whole to absorb the difference of the renormalization conditions. In [47], indeed, such a scaling of data obtained by numerical simulations for the gluon propagator was adopted to satisfy the renormalization condition $\mathcal{Z}(k_E^2 = \mu^2) = 1/\mu^2$ at $\mu = 4$ GeV. This kind of rescaling was also adopted in [1]. By performing the fitting anew with an additional scale parameter, the fitting result is dramatically improved to give the precise values for the parameters as shown in Fig. 4 where the fitting parameter with errors are given by

$$g = 4.1 \pm 0.1 \Leftrightarrow \lambda := \frac{g^2 C_2(G)}{16\pi^2} = 0.32 \pm 0.02,$$

$$M = g = 0.454 \pm 0.004 \Leftrightarrow M^2 = 0.206 \pm 0.004,$$

$$Z = 2.65 \pm 0.02.$$  \hfill (96)

We use these parameters to plot the ghost propagator using the analytical expression by including quantum corrections to one-loop order in the massive Yang-Mills model, as shown in Fig. 5.

Both gluon propagator and ghost propagator in the decoupling solution of the Yang-Mills theory are well reproduced by the values of parameters $g$ and $M$. In what follows we call these values of the parameters the physical point for the Yang-Mills theory.

IV. REFLECTION POSITIVITY AND THE VIOLATION IN THE YANG-MILLS THEORY

In this section, we observe that the Euclidean gluon propagator in the massive Yang-Mills model exhibits violation of reflection positivity. This result suggests gluon confinement in the Yang-Mills theory.

A. Reflection positivity and the Schwinger function

The Osterwalder-Schrader axioms [48] are general properties to be satisfied for the quantum field theory formulated in the Euclidean space, which are the Euclidean version of the Wightman axioms for the relativistic quantum field theory formulated in the Minkowski spacetime. A relativistic quantum field theory described by a set of the Wightman functions satisfying the Wightman axioms can be constructed from a set of Schwinger functions (Euclidean Green’s functions) if they obey the Osterwalder-Schrader axioms. In particular, the axiom of reflection positivity is the Euclidean counterpart to the positive definiteness of the norm in the Hilbert space of the corresponding Wightman quantum field theory. If the reflection positivity is violated, a particular Euclidean correlation function cannot have the interpretation in terms of stable particle states, which is regarded as a manifestation of confinement. To demonstrate the violation of reflection positivity in the Osterwalder-Schrader axioms, one counterexample suffices.

For the special case of a single propagator, the reflection positivity reads

$$\int d^Dx \int d^Dy f^*(x, -x_D) \mathcal{D}(x - y, x_D - y_D)f(y, y_D) \geq 0,$$

$$f \in \mathcal{S}_+(\mathbb{R}^D),$$  \hfill (97)

where $\mathcal{S}_+(\mathbb{R}^D)$ denotes a complex-valued test (Schwartz) function with support in \{(x, x_D); x_D > 0\}. The reflection positivity is rewritten as

$$\int dx_D \int dy_D \int d^{D-1}p f^*(p, -x_D) f(p, y_D) \Delta(p, x_D - y_D) = \int_0^\infty dt \int_0^\infty dt' \int d^{D-1}p f^*(p, t) f(p, t') \Delta(p, -(t + t')) \geq 0,$$  \hfill (98)

where we defined $\Delta(p, x_D - y_D)$ by

$$\mathcal{D}(x - y) := \int d^{D-1}p \ e^{ip(x - y)} \Delta(p, x_D - y_D).$$  \hfill (99)

In what follows we call $\Delta(p, x_D - y_D)$ the Schwinger function. For this inequality to hold for any test function $f \in \mathcal{S}_+(\mathbb{R}^D)$, the Schwinger function $\Delta$ must satisfy the positivity

$$\Delta(p, -(t + t')) \geq 0.$$  \hfill (100)

We consider a particular Schwinger function in the D-dimensional spacetime defined by the Fourier transform of the Euclidean propagator $\mathcal{D}(p, p_E^D)$,

$$\Delta(t) := \Delta(p = 0, t) := \int d^{D-1}x \ e^{-ip \cdot x} \mathcal{D}(x, t)|_{p = 0}$$

$$= \int_{-\infty}^{+\infty} \frac{dp_E^D}{2\pi} e^{ip_E^D t} \mathcal{D}(p = 0, p_E^D).$$ \hfill (101)

If $\mathcal{D}(p, p_E^D)$ is even in $p_E^D$, namely, $\mathcal{D}(0, -p_E^D) = \mathcal{D}(0, p_E^D)$, the Schwinger function reduces to

$$\Delta(t) = 2 \int_0^{+\infty} \frac{dp_E^D}{2\pi} \cos(p_E^D t) \mathcal{D}(0, p_E^D).$$ \hfill (102)

To demonstrate the violation of reflection positivity, one counterexample suffices. Therefore, non-positivity of the Schwinger function $\Delta(t)$ at some value of $t$ leads to the violation of reflection positivity. Consequently, the reflection positivity is violated for the gluon propagator. The corresponding states cannot appear in the physical particle spectrum. This is consistent with gluon confinement.

For the free massive propagator,

$$\tilde{\mathcal{D}}(p) = \frac{1}{p^2 + m^2} \quad (m > 0),$$ \hfill (103)

we find $\Delta(t)$ is positive for any $t$:

$$\Delta(t) = \int_{-\infty}^{+\infty} \frac{dp_E^D}{2\pi} e^{ip_D^D t} \frac{1}{p_E^D^2 + m^2} = \frac{1}{2m} e^{-m|t|} > 0.$$ \hfill (104)
Therefore, there is no reflection-positivity violation for the free massive propagator, as expected. For unconfined particles, the reflection positivity should hold.

### B. Positivity violation for the decoupling solution of the Yang-Mills theory

In order to examine the violation of the reflection positivity through the behavior of the gluon Schwinger function, we first construct a set of gluon and ghost propagators in such a way that they are renormalized to satisfy the renormalization conditions \([TW2]\) and \([69]\) in the massive Yang-Mills model to reproduce the decoupling solution in the Yang-Mills theory to one-loop order. The integral in obtaining the Schwinger function as the Fourier transform of the gluon propagator is not so easy to be performed analytically, hence we resort to the numerical calculations for this definite integral.

In Fig. 6, we give the plot for the gluon propagator and the associated Schwinger function in the Landau gauge \(\alpha = 0\) for the \(SU(3)\) massive Yang-Mills model at the physical point of parameters \(g = 4.1\) and \(M/\mu = 0.454\). We observe that the Schwinger function takes negative values for \(\mu t > 6\) and hence the reflection positivity is violated. Therefore, this result suggests that the reflection positivity is violated for the decoupling solution in the Yang-Mills theory. The more detailed analysis of the reflection positivity will be given in the next section from the viewpoint of the complex structure of the gluon propagator.

### C. Positivity violation in the complementary gauge-scalar model

In what follows, we examine how the gluon propagator and the Schwinger function are modified if the parameters \(g\) and \(M\) deviate from the physical point. In this case the massive Yang-Mills model is no longer regarded as a low-energy effective theory of the original Yang-Mills theory. However, the resulting model can be regarded as the gauge-scalar model with the complementarity between Higgs and Confinement in the sense that the confinement phase in the Yang-Mills theory is analytically connected with no phase transition to the Higgs phase in the gauge-scalar model through the BEH mechanism, which is called the Fradkin-Shenker continuity.

#### 1. Smaller coupling constant

First, we take smaller values for the coupling constant \(g\) than the physical value \(g = 4.1\) and keep the mass parameter \(M\) fixed to the physical value \(M/\mu = 0.454\). In Fig. 7, the gluon propagator and the associated Schwinger functions are given for a smaller value \(g = 2.3\). For a more smaller value \(g = 1\), they are given in Fig. 8.

![Fig. 6: The gluon propagator and the Schwinger function at the physical point of the parameters \(g = 4.1\), \(M/\mu = 0.454\): (top) gluon propagator \(\mu^2 \mathcal{D}\) as a function of \(k_E/\mu\) and (bottom) the Schwinger function \(\mu \Delta\) as a function of \(\mu t\), where all quantities are made dimensionless using the rescaling of appropriate powers of \(\mu\).](image1)

![Fig. 7: The same plots as those given in Fig. 6 for a smaller coupling constant \(g = 2.3\) with a physical value \(M/\mu = 0.454\).](image2)

![Fig. 8: The same plots as those given in Fig. 6 for a more smaller coupling constant \(g = 1\) with a physical value \(M/\mu = 0.454\).](image3)
For smaller coupling constant $g$, the gluon propagator $\mathcal{D}$ seems to be monotonically decreasing in $k_E$. The Schwinger function falls off very slowly from $t = 0$ value and keeps its positivity until very large value of $t$, although it is difficult to see the difference from the graphs. Consequently, the smallest value of $t$ giving the negative value of the Schwinger function shifts to larger values of $t$, and eventually go to infinity as $g \to 0$. This result is reasonable, since, in the vanishing coupling limit $g \to 0$, the gluon propagator must reduce to the free massive propagator in the tree level. Therefore, the reflection positivity must be recovered and the Schwinger function keeps positivity everywhere in the limit $g \to 0$. As far as the results of the numerical calculations are concerned, the positivity seems to be not violated and restored for relatively smaller coupling constants.

However, this observation turns out to be wrong. In fact, we can prove analytically that the reflection positivity of the gluon Schwinger function is violated for any value of the parameters $g$ and $M$ in the massive Yang-Mills model with one-loop quantum corrections being included. The proof will be given in the next section. The Schwinger function $\Delta$ is an oscillating exponentially fall-off function of $t$ approaching zero finally as $t \to \infty$. Therefore, it is difficult to examine the violation of positivity in the large $t$ region in the numerical way due to the restriction on the precision of numerical calculations. For smaller coupling constant $g$, therefore, the Schwinger function takes a smaller but negative value for larger $t$, until the negativity disappears only in the limit $g \to 0$.

For quite small mass parameter $M/\mu^2$ or large coupling constant $g$, the gluon propagator $\mathcal{D}(k_E^2)$ becomes singular at two values of $k_E^2$ and takes negative values in between. In Fig. 11, the gluon propagator is given for the parameters $g = 4.1$ and $M/\mu = 0.08$. This result is consistent with the statement [20] that the gluon propagator has poles in the Euclidean region (namely, tachyonic poles) with multiplicity two or a pair of complex conjugate poles under some assumptions on the propagator and the spectral function. The related issue will be discussed in the next section.

Therefore, this singular behavior affects the associated Schwinger function $\Delta(t)$. This feature will be an artifact due to the limitation of one-loop calculations. Therefore, we exclude the relevant region of parameters from the following considerations.

2. Smaller mass parameter

Next, we keep the coupling constant fixed to the physical value $g = 4.1$, and take smaller gluon mass parameter $M/\mu$ than the physical value $M/\mu = 0.454$. For a smaller value $M/\mu = 0.2$, the gluon propagator and the associated Schwinger functions are given in Fig. 9. For a more smaller value $M/\mu = 0.141$, they are given in Fig. 10.

As the value of mass parameter $M/\mu$ is chosen to be smaller and smaller than the physical value for the Yang-Mills theory, the gluon propagator $\mathcal{D}(p)$ exhibits sizable non-monotonic behavior and the Schwinger function exhibits more enhanced negativity, leading to the clearer violation of reflection positivity.

For smaller mass $M$ or larger coupling constant $g$ than the physical value for the Yang-Mills theory, the gluon propagator $\mathcal{D}_t(p)$ exhibits stronger non-monotonic behavior.

3. Presence of Euclidean poles
For small $g^2$ and large $M^2/\mu^2$, namely, for large $1/g^2$ and large $v^2 \simeq (M^2/\mu^2)/g^2$, the Schwinger function exhibits small violation of positivity. This region corresponds to the Higgs-like region in the complementary gauge-scalar model. For large $g^2$ and small $M^2/\mu^2$, namely, for small $1/g^2$ and small $v^2 \simeq (M^2/\mu^2)/g^2$, the Schwinger function exhibits large violation of positivity. This region corresponds to the Confinement-like region in the complementary gauge-scalar model.

However, there is no phase transition between the positivity violation and restoration. There is just a smooth crossover separating large and small violation of positivity. The massive Yang-Mills model has only one confinement phase. This result is interpreted as the Fradkin-Shenker continuity in the complementary gauge-scalar model from the viewpoint of the gauge-invariant extension from the massive Yang-Mills model to the gauge-invariant complementary gauge-scalar model explained in section II.

V. COMPLEX STRUCTURE OF THE GLUON PROPAGATOR

In the previous section we have investigated the propagator in the Euclidean region. In this section, we study the propagator on the complex plane of the squared momentum $k^2$, which follows from the analytic continuation of the propagator from the Euclidean region to the Minkowski region. We find that the violation of the reflection positivity in the Euclidean region is understood from the existence of a pair of complex conjugate poles and the discontinuity across the branch cut yielding the negative spectral function represented by the generalized spectral representation of the gluon propagator. As a consequence of the complex structure, we give an analytical proof that the reflection positivity is always violated for any choice of the parameters $M$ and $g$ in the massive Yang-Mills model to one-loop order.

A. Spectral representation of a propagator

It is well-known that a propagator $\mathcal{D}(k^2)$ in the Minkowski region $k^2 > 0$ (for the time-like momentum $k$) has the spectral representation of the Källén–Lehmann form under assumptions of the general principles of the quantum field theory such as the spectral condition, the Poincaré invariance and the completeness of the state space $[21]$. The full propagator $\mathcal{D}(k^2)$ of the field $\phi$ is written as the weighted sum of the free propagator,

$$\mathcal{D}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \quad k^2 \geq 0,$$  \tag{105}
with the weight function \( \rho(\sigma^2) \) called the spectral function being obtained from the state sum

\[
\theta(k_0)\rho(k^2) := (2\pi)^d \sum_{n} |\langle 0 | \phi(0) | P_n \rangle|^2 \delta^D(P_n - k),
\]

where \( d \) is the space dimension, \( D \) is the spacetime dimension, the sum is over all the intermediate states with the total momentum \( P_n \), and \( \theta(k_0) \) is a step function ensuring the positivity \( k_0 \geq 0 \). The spectral function \( \rho \) has contributions from a stable single-particle state with physical mass \( m_P \) (pole mass) and intermediate many-particle states \( |p_1, \ldots, p_n \rangle \) with a continuous spectrum, such as two-particle states, three-particle states, and so on,

\[
\rho(k^2) = Z \delta(k^2 - m_P^2) + \tilde{\rho}(k^2), \quad k^2 \geq 0,
\]

\[
\tilde{\rho}(k^2) = (2\pi)^d \sum_{n=2}^{\infty} |\langle 0 | \phi(0) | p_1, \ldots, p_n \rangle|^2 \delta^D(p_1 + \ldots + p_n - k).
\]

Then the spectral representation is written as the sum of the contributions from the real pole \( k^2 = m_P^2 \) and the branch cut

\[
\mathcal{D}(k^2) = \frac{Z}{m_P^2 - k^2} + \int_0^\infty d\sigma^2 \frac{\tilde{\rho}(\sigma^2)}{\sigma^2 - k^2}, \quad k^2 \geq 0.
\]

This spectral representation can be extended to the complex momentum \( k^2 \in \mathbb{C} \). See the left panel of Fig. 14. A propagator \( \mathcal{D}(k^2) \) as a complex function of the complex variable \( z = k^2 \in \mathbb{C} \) has the spectral representation with the spectral function \( \rho \),

\[
\mathcal{D}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \quad k^2 \in \mathbb{C} - [s_{\text{min}}, \infty),
\]

\[
\rho(\sigma^2) := \frac{1}{\pi} \text{Im} \mathcal{D}(\sigma^2 + i\epsilon).
\]

This representation \( \text{(109)} \) is applied to an arbitrary \( k^2 \) in the complex plane except for the singularities located on the positive real axis \([s_{\text{min}}, \infty)\). The spectral function \( \rho \) known as the dispersion relation is obtained from the discontinuity across the branch cut, \( \mathcal{D}(z + i\epsilon) - \mathcal{D}(z - i\epsilon) = \mathcal{D}(z + i\epsilon) - \mathcal{D}(z + i\epsilon)^* = 2i \text{Im} \mathcal{D}(z + i\epsilon) \). It is explicitly checked that the two definitions of the spectral functions \( \text{(107)} \) and \( \text{(110)} \) agree with each other once the theory is specified. The representation \( \text{(110)} \) is obtained under the following assumptions \( 17, 52 \):

1. \( \mathcal{D}(z) \) is holomorphic except singularities on the positive real axis.
2. \( \mathcal{D}(z) \to 0 \) as \( |z| \to \infty \).
3. \( \mathcal{D}(z) \) is real on the negative real axis.

This is indeed the case of the quantum Yang-Mills theory, see e.g., [20].

The spectral representation has a straightforward generalization in the presence of complex simple poles, see e.g., [20, 53]. Suppose that the propagator has simple complex poles at \( z = z_\ell \) \( (\ell = 1, \ldots, n) \). See the right panel of Fig. 14. Then the propagator \( \mathcal{D}(k^2) \) has the generalized spectral representation,

\[
\mathcal{D}(k^2) = \mathcal{D}_p(k^2) + \mathcal{D}_c(k^2), \quad k^2 \in \mathbb{C} - ([s_{\text{min}}, \infty) \cup \{z_\ell\}^n),
\]

\[
\mathcal{D}_p(k^2) := \sum_{\ell=1}^n \frac{Z_\ell}{z_\ell - k^2} + \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \quad \mathcal{D}_c(k^2) := \int_{\gamma_{0}} d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2},
\]

where \( \gamma_0 \) is a small contour circulating clockwise around the pole at \( z_\ell \). Here we have separated the propagator \( \mathcal{D} \) into the contribution from the complex poles \( \mathcal{D}_p \) and that from the branch cut \( \mathcal{D}_c \). This is derived from the following assumptions \( 20 \):

1. \( \mathcal{D}(z) \) is holomorphic except singularities on the positive real axis and a finite number of simple poles.
2. \( \mathcal{D}(z) \to 0 \) as \( |z| \to \infty \).
3. \( \mathcal{D}(z) \) is real on the negative real axis.

Note that the poles must appear as real poles or pairs of complex conjugate poles as a consequence of the Schwarz reflection principle \( \mathcal{D}(z^*) = [\mathcal{D}(z)]^* \).

From now on, we focus on a propagator with a pair of complex conjugate simple poles. This is indeed the case for the gluon propagator of the massive Yang-Mills model as will be shown in the next subsection. For a propagator with one pair of complex conjugate simple poles at \( k^2 = v \pm iw \), the generalized spectral representation \( \text{(111)} \) reduces to

\[
\mathcal{D}(k^2) = \mathcal{D}_p(k^2) + \mathcal{D}_c(k^2),
\]

\[
\mathcal{D}_p(k^2) := \frac{Z}{(v + iw) - k^2} + \frac{Z^*}{(v - iw) - k^2},
\]

\[
\mathcal{D}_c(k^2) := \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2},
\]
B. Gluon propagator on the complex momentum plane

We first perform the analytic continuation of the propagator $\mathcal{D}$ in the Euclidean region $k^2 = -k_E^2 < 0$ to the entire complex plane $k^2 \in \mathbb{C}$. Fig. 15 is the plot of the real and imaginary parts of the complex-valued gluon propagator $\mathcal{D}(k^2)$ on the complex momentum plane $k^2 \in \mathbb{C}$, at the physical point of the parameters (96) in the massive Yang-Mills model. Note that the gluon propagator $\mathcal{D}(k^2)$ is real-valued only on the negative real axis (space-like momentum) $k^2 = -k_E^2 < 0$, since the imaginary part $\text{Im} \mathcal{D}(k^2)$ is zero only on the negative real axis (space-like momentum). The real part $\text{Re} \mathcal{D}(k^2)$ on the negative real axis $k^2 = -k_E^2 < 0$ is identical to the Euclidean propagator. We observe that the gluon propagator has a pair of complex conjugate poles and the imaginary part has discontinuities across the branch cut on the positive real axis $\mathcal{D}(k^2 + i\epsilon) \neq \mathcal{D}(k^2 - i\epsilon)$ ($k^2 > 0$, $\epsilon \downarrow 0$), while there are no discontinuities on the negative real axis $\mathcal{D}(k^2 + i\epsilon) = \mathcal{D}(k^2 - i\epsilon)$ ($k^2 < 0$, $\epsilon \downarrow 0$). Therefore, in discussing the behavior of the propagator on the positive real axis, we must specify which side is used. In what follows we use the limit $\mathcal{D}(k^2 + i\epsilon)$ ($\epsilon \downarrow 0$).

Next, we focus on the real axis $k^2 \in \mathbb{R}$ to see the behavior of the complex-valued gluon propagator $\mathcal{D}(k^2)$ as a function of a real-valued momentum $k^2 \in \mathbb{R}$. Fig. 16 is the plot of the real and imaginary parts of the complex-valued gluon propagator on the real axis $k^2 \in \mathbb{R}$ at the physical point of the parameters (96) in the massive Yang-Mills model. On the negative real axis $k^2 > 0$ (the Euclidean region), $\text{Re} \mathcal{D}(k^2)$ changes the sign such that it is positive for small $k^2$, and negative for large $k^2$, which implies the existence of (at least one) zeros of $\text{Re} \mathcal{D}(k^2)$ in the Minkowski region $k^2 > 0$. The scaled imaginary part $\text{Im} \mathcal{D}(k^2 + i\epsilon)/\pi$ is identical to the spectral function $\rho(k^2)$. Therefore, the spectral function is identically zero in the Euclidean region,

$$\rho(k^2) \equiv 0 \text{ for } k^2 = -k_E^2 < 0. \quad (115)$$

FIG. 15: The gluon propagator $\mathcal{D}(k^2)$ as a complex function of the complex squared momentum $k^2 \in \mathbb{C}$, (top) the real part $\text{Re} \mathcal{D}(k^2)$, (bottom) the imaginary part $\text{Im} \mathcal{D}(k^2)$, at the physical point of the parameters $\lambda := N g^2/(4\pi)^2 = 0.32$, $M^2/\mu^2 = 0.206$.

FIG. 16: The gluon propagator $\mathcal{D}(k^2)$ as a function of $k^2$ restricted on the real axis $k^2 \in \mathbb{R}$, (top) the real part $\text{Re} \mathcal{D}(k^2)$, (bottom) the scaled imaginary part $\text{Im} \mathcal{D}(k^2)/\pi$ which is equal to the spectral function $\rho(k^2)$, at the physical point of the parameters $\lambda := N g^2/(4\pi)^2 = 0.32$, $M^2/\mu^2 = 0.206$. 
However, it is non-trivial in the Minkowski region. It is remarkable that the spectral function is always negative,

\[ \rho(k^2) := \frac{1}{\pi} \text{Im} \mathcal{D}(k^2 + i\epsilon) < 0 \text{ for } k^2 > 0, \tag{116} \]

in the massive Yang-Mills model to one-loop order.

For a given propagator \( \mathcal{D}(k^2) \), we can decompose it into the contribution from the branch cut \( \mathcal{D}_c(k^2) \) and that from the poles \( \mathcal{D}_p(k^2) \). Fig. 17 gives this decomposition of the gluon propagator for the Euclidean momentum \( \mathcal{D}(k^2) = \mathcal{D}_p(k^2) + \mathcal{D}_c(k^2) \) for \( k^2 < 0 \).

According to the separation of the propagator, the Schwinger function is also separated into the two parts: the continuous cut part \( \Delta_c(t) \) coming from the spectral function and the pole part \( \Delta_p(t) \) coming from the pole part \( \mathcal{D}_p \) of the propagator \( \mathcal{D} \),

\[ \Delta(t) = \Delta_p(t) + \Delta_c(t), \]

\[ \Delta_p(t) := \int_{-\infty}^{+\infty} \frac{dk_E}{2\pi} e^{ik_Et} \mathcal{D}_p(k_E^2), \]

\[ \Delta_c(t) := \int_{-\infty}^{+\infty} \frac{dk_E}{2\pi} e^{ik_Et} \mathcal{D}_c(k_E^2). \tag{117} \]

Especially, the cut part \( \Delta_c(t) \) is directly written as an
The same procedure is also applied to the Schwinger function. Fig. 18 shows the respective ratio $\Delta_c(t)/\Delta(t)$ of the pole or cut part $\Delta_{c,p}(t)$ to the total Schwinger function $\Delta(t)$. Using the already known spectral function $\rho(k^2)$ calculated according to $\rho(k^2) = \text{Im} \mathcal{D}(k^2)/\pi$, the cut part $\Delta_c(t)$ of the Schwinger function is obtained by integrating $\rho(k^2)$ according to (118). Then the pole part $\Delta_p(t)$ of the Schwinger function is obtained as the difference $\Delta_p(t) = \Delta(t) - \Delta_c(t)$ from the total Schwinger function $\Delta(t)$. Note that the ratio can become divergent at a zero $t_0$ of the Schwinger function $\Delta(t_0) = 0$, which should be ignored as an artifact of this procedure.

C. A pair of complex conjugate poles and Gribov-Stingl form

If the propagator has no complex poles besides the singularities on the real positive axis, the complex pole part vanishes $\Delta_{c,p}(k^2) = 0$ and the gluon propagator obeys the usual spectral representation

$$\Delta_c(k^2) = \mathcal{D}_c(k^2) = \int_0^\infty d\sigma^2 \rho(\sigma^2) \frac{1}{\sigma^2 - k^2}.$$ (119)

Then the Schwinger function is calculated from the cut part alone

$$\Delta(t) = \Delta_c(t) = \int_0^\infty d\sigma^2 \rho(\sigma^2) \frac{1}{2\sqrt{\sigma^2}} e^{-\sqrt{\sigma^2}t}.$$ (120)

In this case, we find that the positivity of the spectral function implies the positivity of the Schwinger function

$$\rho(\sigma^2) \geq 0 \Rightarrow \sigma^2 \geq 0 \Rightarrow \Delta(t) \geq 0 \text{ for } t \geq 0,$$ (121)

which implies that non-positivity of the Schwinger function implies non-positivity of the spectral function

$$\Delta(t) < 0 \text{ for } t \geq 0 \Rightarrow \rho(\sigma^2) < 0 \text{ for } \sigma^2 \geq 0.$$ (122)

Thus, when the propagator has no singularities other than the positive real axis, the positivity of the spectral function is directly related to the positivity of the Schwinger function, or the reflection positivity. The violation of reflection positivity can be seen as the non-positivity of the spectral function. However, this is not the case for the Yang-Mills theory, as demonstrated in the massive Yang-Mills model shortly.

Suppose that the propagator has a pair of complex conjugate poles at $k^2 = v \pm iw$ ($v \in \mathbb{R}, w > 0$) with the respective residues $Z, Z^* \in \mathbb{C}$. Then the pole part of the propagator in the Euclidean region is represented as

$$\Delta_p(k^2 = -k_E^2) = \frac{Z^*}{-k_E^2 - (v + iw)-k_E^2 - (v - iw)} = \frac{k_E^2 + (v + iw) + k_E^2 + (v - iw)}{k_E^2 + 2vk_E^2 + (v^2 + w^2)}.$$ (123)

This pole part of the propagator agrees with the Gribov-Stingl form (22) with real parameters $c_0, c_1, c_2, d_0, d_1 \in \mathbb{R}$,

$$\Delta_{GS}(k_E^2) = \frac{d_0 + d_1 k_E^2}{c_0 + c_1 k_E^2 + c_2 k_E^4} = \frac{d_0}{c_0} + \frac{d_1}{c_1} + \frac{d_2}{c_2} k_E^2,$$ (124)

$c_0, c_1, c_2, d_0, d_1 \in \mathbb{R}$.

Note that all the coefficients $c_0, c_1, c_2, d_0, d_1$ are not independent. The Gribov-Stingl form actually has four independent parameters, since one of them is eliminated by the rescaling. This number of independent parameters agrees with that of the pole part of the propagator with a pair of complex conjugate poles characterized by the four parameters $v, w, \text{Re}[Z], \text{Im}[Z]$. For instance, the correspondence between two sets of parameters is given as

$$\begin{cases} \frac{c_0}{c_1} = v^2 + w^2, \\ \frac{c_0}{c_2} = 2v, \\ \frac{d_0}{c_0} = 2(\text{Re}[Z] + w \text{Im}[Z]), \\ \frac{d_0}{c_2} = 2 \text{Re}(Z), \end{cases}$$ (125)

which has the inverse relation

$$\begin{cases} v = \frac{1}{2} \frac{c_0}{c_2}, \\ w = \sqrt{\frac{c_0}{c_2} - v^2} = \sqrt{\frac{c_0}{c_2} - \left(\frac{1}{2} \frac{c_0}{c_2}\right)^2}, \\ \text{Re}[Z] = \frac{1}{2} \frac{d_0}{c_2}, \\ \text{Im}[Z] = \frac{1}{w} \left(\frac{1}{2} \frac{d_0}{c_2} - \frac{1}{2} \frac{d_1}{c_2} v\right) = \frac{\frac{1}{2} \frac{d_0}{c_2} - \frac{1}{2} \frac{d_1}{c_2} \sqrt{\frac{c_0}{c_2} - \left(\frac{1}{2} \frac{c_0}{c_2}\right)^2}}{\sqrt{\frac{c_0}{c_2} - \left(\frac{1}{2} \frac{c_0}{c_2}\right)^2}}. \end{cases}$$ (126)

For $v + iw$ to be a complex number (namely, $w$ to be a real number), the parameters of the Gribov-Stingl form must satisfy the restriction

$$\frac{c_0}{c_2} - \left(\frac{1}{2} \frac{c_0}{c_2}\right)^2 > 0 \Leftrightarrow \frac{c_2}{4c_0c_2} < 1.$$ (127)

Assuming this condition, we can obtain the closed form for the pole part of the Schwinger function

$$\Delta_p(t) = \int_0^\infty \frac{dk_E}{2\pi} e^{ik_Et} \left[\frac{Z}{k_E^2 + (v + iw)} + \frac{Z^*}{k_E^2 + (v - iw)}\right].$$ (128)
Indeed, the contribution from one of the poles is exactly obtained as
\[
\int_{-\infty}^{\infty} \frac{dk E}{2\pi} e^{ikEt} \frac{Z}{k_f^2 + (v+iw)} = i \text{Res} \left[ \frac{Z}{(k_E - \alpha)(k_E + \alpha)} \right]_{k_E = \pm \alpha} = i e^{ia|t|} \frac{Z}{2\alpha} = i \exp \left[ -t r^{1/2} e^{i\varphi} - i\varphi \right] \frac{Z}{2\alpha t^{1/2}} \times \exp \left[ -it r^{1/2} \sin \varphi - i\varphi \right],
\]
where we have defined
\[
\alpha^2 = -(v+iw) = -\sqrt{v^2 + w^2} e^{i\theta} = -r e^{i\theta} \Rightarrow \alpha = \left( v^2 + w^2 \right)^{1/4} e^{i\theta/2 + \pi/2} = \pm r^{1/2} e^{i\varphi},
\]
\[
r = \sqrt{v^2 + w^2}, \quad \theta = \arctan \frac{w}{v}, \quad \varphi = \frac{\theta}{2} = \frac{1}{2} \arctan \frac{w}{v},
\]
where \( \alpha \) must be located on the upper half plane of the complex \( k_E \) plane. Therefore, the pole part of the Schwinger function coming from a pair of complex conjugate poles is exactly obtained as [49]
\[
\Delta_p(t) = \frac{\sqrt{\text{Re}(Z)^2 + \text{Im}(Z)^2}}{(v^2 + w^2)^{1/4}} \exp \left[ -t (v^2 + w^2)^{1/4} \cos \varphi - t(v^2 + w^2)^{1/4} \sin \varphi + \varphi - \delta \right],
\]
\[
\varphi := \frac{1}{2} \arctan \frac{w}{v}, \quad \delta := \arctan \frac{\text{Im}(Z)}{\text{Re}(Z)}. \quad \text{(131)}
\]
At \( t = 0 \), \( \Delta_p(0) \) has the value,
\[
\Delta_p(0) = \frac{\sqrt{\text{Re}(Z)^2 + \text{Im}(Z)^2}}{(v^2 + w^2)^{1/4}} \cos(\varphi - \delta). \quad \text{(132)}
\]
We find that \( \Delta_p(t) \) is oscillating between positive and negative values, although the absolute value \( |\Delta_p(t)| \) becomes smaller for larger \( t > 0 \).\(^5\)

On the other hand, the cut part \( \Delta_c(t) \) of the Schwinger function is estimated using the integral representation [118]:
\[
\Delta_c(t) = \int_0^\infty d\sigma^2 \rho(\sigma^2) \frac{1}{2\sqrt{\sigma^2}} e^{-\sqrt{\sigma^2}|t|}. \quad \text{(134)}
\]

\(^5\) The pole part of the Schwinger function becomes positive only when the poles become real values \( w = 0 \) (or \( \varphi = 0 \))
\[
\Delta_p^{CI}(t) = \frac{|\text{Re}(Z)|}{\sqrt{|w|}} \exp[-t \sqrt{|w|}]. \quad \text{(133)}
\]
This representation is an exact relation between the spectral function and the cut part of the Schwinger function, which holds irrespective of the existence or non-existence of complex poles.

At least to one-loop order in the massive Yang-Mills model, the spectral function \( \rho(\sigma^2) \) takes the negative value \( \rho(\sigma^2) < 0 \) for all \( \sigma^2 > 0 \):
\[
\rho(\sigma^2) < 0 \quad \text{for} \quad \forall \sigma^2 > 0, \quad \text{(135)}
\]
as demonstrated numerically in Fig. [16] (at the physical point) in this paper and shown analytically for any value of the parameters \( g \) and \( M \) in [20]. According to (134), therefore, the cut part \( \Delta_c(t) \) of the Schwinger function takes negative value for any value of \( t \),
\[
\Delta_c(t) < 0 \quad \text{for} \quad \forall t \geq 0, \quad \text{(136)}
\]
although \( \Delta_c(t) \) takes smaller and smaller negative value for larger and larger \( t > 0 \).

Moreover, the negative spectral function (135) yields the existence of one pair of complex conjugate poles or two real poles in the Euclidean region as shown in [20]. According to (131), therefore, the pole part \( \Delta_p(t) \) of the Schwinger function due to a pair of complex conjugate poles takes negative value for a certain value of \( t \):
\[
\Delta_p(t) < 0 \quad \text{for} \quad t \geq 0. \quad \text{(137)}
\]
Thus, the Schwinger function \( \Delta(t) \) obtained as a sum of two parts, \( \Delta(t) = \Delta_p(t) + \Delta_c(t) \) has necessarily negative value at a certain value of \( t \),
\[
\Delta(t) = \Delta_c(t) + \Delta_p(t) < 0 \quad \text{for} \quad \exists t \geq 0. \quad \text{(138)}
\]
Thus we complete the analytical proof that the reflection positivity is always violated irrespective of the choice of the parameters \( g \) and \( M \) in the massive Yang-Mills model to one-loop order.

In particular, the propagator of the Gribov type is a special case corresponding to \( c_1 = 0 \) and \( d_0 = 0 \)
\[
\tilde{G}_C(p) = \frac{d_1}{c_2} \frac{p^2_E}{c_2} + p^2_E, \quad \text{(139)}
\]
which has a pair of pure imaginary poles
\[
v = 0, \quad \pm iw, \quad w = \sqrt{c_0/c_2}, \quad \text{(140)}
\]
with the real-valued residue
\[
\text{Re}(Z) = \frac{1}{2} \frac{d_1}{c_2}, \quad \text{Im}(Z) = 0. \quad \text{(141)}
\]
The pole part of the Schwinger function for the propagator of the Gribov type (139) is given by
\[
\Delta_p^{G}(t) = \frac{d_1}{2c_2 r^{1/2}} e^{-\frac{r^{1/2}}{\sqrt{2}} t} \cos \left( \frac{r^{1/2}}{\sqrt{2}} t + \frac{\pi}{4} \right)
\]
\[
= \frac{Z}{\sqrt{|w|}} e^{-\frac{\sqrt{|w|}}{r^{1/2}} t} \cos \left( \frac{\sqrt{|w|}}{\sqrt{2}} t + \frac{\pi}{4} \right), \quad \text{(142)}
\]
where we have used \( \varphi = \frac{\pi}{2} \) and \( r^{1/2} = (c_0/c_2)^{1/4} = \sqrt{|w|} \).
The fitting parameters are determined as \[ \begin{align*}
\hat{c}_0 &= 1.7678 \pm 3 \times 10^{-5}, \\
\hat{c}_1 &= 0.73006 \pm 5 \times 10^{-5}, \\
\hat{c}_2 &= 0.32505 \pm 8 \times 10^{-5}, \\
\hat{d}_0 &= 1.4268 \pm 3 \times 10^{-5}, \\
\hat{d}_1 &= 0.2512 \pm 1 \times 10^{-4},
\end{align*} \] (144)
which is subject to the restriction (127)
\[ \frac{\hat{c}_1^2}{4\hat{c}_0\hat{c}_2} = 0.2319 \pm 1 \times 10^{-4}. \] (145)

This result is translated into the complex pole and the residue of the gluon propagator
\( \{ \hat{v} = 1.123 \pm 4 \times 10^{-4}, \hat{w} = 2.044 \pm 2 \times 10^{-4}, \quad \text{Re}[Z] = 0.3863 \pm 2 \times 10^{-4}, \quad \text{Im}[Z] = 0.8615 \pm 2 \times 10^{-4} \} \). (146)

It is ensured that this data reproduces the location of the poles given in Fig. 13. We find that the fitting errors are very small and to good accuracy the pole part of the gluon propagator is identical to the Gribov-Stingl form. This result strongly suggests that the pole part of the gluon propagator indeed stems from a pair of complex conjugate poles. For the other argument for understanding the Gribov-Stingl form, see [19].

The pole part of the Schwinger function can be calculated according to (131) once all the parameters of the Gribov-Stingl form are determined. If our analysis of the complex structure of the propagator is correct, the result should agree with the pole part of the Schwinger function given in Fig. 18. In fact, Fig. 20 shows excellent agreement between pole part of the Schwinger function obtained from the spectral function and the expression (131) with the parameters obtained through the fit of the gluon propagator to the Gribov-Stingl form. This result supports the validity of our arguments.

Thus we have shown that the gluon propagator consist of the pole part due to a pair of complex conjugate poles and the cut part due to the branch cut on the positive real axis, in agreement with the generalized spectral representation (114). This is also the case for the associated Schwinger function. In this way we can conclude that the reflection positivity is violated in the massive Yang-Mills model at the physical point.
E. Parameter dependence other than the physical point

We investigate the gluon propagator and the associated Schwinger function at the choice of the parameters other than the physical point.

1. Smaller gauge coupling

For smaller value of the coupling constant $\lambda := Ng^2/(4\pi)^2 = 0.1$ with the physical value for $M$, we obtain the gluon propagator in Fig. 21 and the Schwinger function in Fig. 22. The cut part is relatively large and has the opposite sign to the pole part to cause cancellation. The fall-off of both parts of the Schwinger function is slow for smaller value of the coupling constant. Therefore, violation of reflection positivity largely comes from the pole part.

We find that the cut parts $\Delta_c$ and $\Delta_p$ of the gluon propagator and the Schwinger function are always negative. This result indicates that the spectral function is negative irrespective of the choice of the parameters. Thus, the reflection positivity is always violated for any choice of the parameters.

VI. CONCLUSION AND DISCUSSION

We have examined the mass-deformed Yang-Mills theory or the massive Yang-Mills model in the covariant Landau gauge with two parameters, the coupling constant $g$ and the mass parameter $M$, in order to reproduce the confining decoupling solution of the pure Yang-Mills theory. By choosing appropriate values for a set of two parameters $g$ and $M$, we have shown that the massive Yang-Mills model well reproduces simultaneously the gluon and ghost propagators of the decoupling solution obtained by

FIG. 21: The same plots for the gluon propagator in the Euclidean region as those in Fig. 17 for the choice of parameters with a smaller coupling constant, $\lambda = 0.1$, $M^2/\mu^2 = 0.206$.

FIG. 22: The same plots for the gluon Schwinger function in the Euclidean region as those in Fig. 18 for the choice of parameters with a smaller coupling constant, $\lambda = 0.1$, $M^2/\mu^2 = 0.206$.

2. Smaller gluon mass

For smaller value of the gluon mass at $M^2/\mu^2 = 0.02$, we obtain the gluon propagator in Fig. 23 and the Schwinger function in Fig. 24. The cut part is relatively rather small. Therefore, violation of reflection positivity largely comes from the pole part.

We find that the cut parts $\Delta_c$ and $\Delta_c$ of the gluon propagator and the Schwinger function are always negative. This result indicates that the spectral function is negative irrespective of the choice of the parameters. Thus, the reflection positivity is always violated for any choice of the parameters.
the numerical simulations on the lattice (at least) in the low-momentum region. Such a choice of the parameters is called the physical point for the Yang-Mills theory.

Then we have shown that the reflection positivity is violated in the massive Yang-Mills model at the physical point of the parameters by observing the negativity of the Schwinger function which is obtained in a numerical way as the Fourier transform of the gluon propagator.

The violation of reflection positivity was also confirmed by examining the complex structure of the complex-valued gluon propagator obtained by performing the analytic continuation of the Euclidean propagator to the entire complex squared momentum plane. We have verified that the violation of reflection positivity in the Euclidean region detected by the Schwinger function associated with the Euclidean gluon propagator is a consequence of the complex structure of the complex-valued gluon propagator, namely, (i) the existence of a pair of complex conjugate poles in the gluon propagator and (ii) the negativity of the spectral function obtained from the discontinuity of the gluon propagator across the branch cut on the positive real axis on the complex squared momentum plane. At the physical point, the contribution from the cut part to the gluon propagator in the Euclidean region is relatively small compared with that from the pole part. Therefore, the propagator in the Euclidean region is well approximated by the contribution from a pair of complex conjugate poles in the complex region, which implies that the propagator in the Euclidean region is well described by the Gribov-Stingl form, in agreement with the lattice result [23]. The violation of reflection positivity is regarded as a necessary condition for gluon confinement. Therefore, our results of reflection positivity violation at the physical point of the massive Yang-Mills model support strongly gluon confinement in the Yang-Mills theory.

We have regarded the massive Yang-Mills model at the physical point as the low-energy effective model of the pure Yang-Mills theory. However, the massive Yang-Mills model with the parameters $g$ and $M$ other than the physical point has another meaning. We have discussed that the massive Yang-Mills model in the covariant Landau gauge has the gauge-invariant extension, which is identified with the complementary gauge-scalar model with a radially fixed fundamental scalar field which is subject to an appropriate reduction condition. In other words, the gauge-scalar model with a radially fixed fundamental scalar field subject to the reduction condition can be gauge-fixed to becomes the massive Yang-Mills model in the covariant Landau gauge. The gauge-invariant extension of a non-gauge theory is performed through the gauge-independent description of the BEH mechanism [12] without relying on the spontaneous symmetry breaking which was first proposed for the adjoint scalar field...
Therefore, the Yang-Mills theory in the confinement phase characterized by the decoupling solution is identified with the massive Yang-Mills model with the physical point of the parameters \( g \) and \( M \). This physical point of the massive Yang-Mills model represents a point in the parameter space of the complementary gauge-scalar model obtained as a gauge-invariant extension of the massive Yang-Mills model. Thus, the violation of reflection positivity in the massive Yang-Mills model for any value of the parameters \( g \) and \( M \) is consistent with the Fradkin-Shenker continuity in the sense that the massive Yang-Mills model describes both Confinement-like and Higgs-like regions in the single confinement phase of the complementary gauge-scalar model. Our result seems to be consistent with the other approaches \[55\] \[56\].

Let us make comments on the sum rule for the spectral function called the superconvergence relation \[57\] \[58\],

\[
\int_0^\infty d\sigma^2 \rho(\sigma^2) = 0. \tag{147}
\]

It is obvious that this sum rule cannot be satisfied for the negative spectral function. In \[20\], remarkably, the generalized sum rule for the spectral function has been derived in the presence of a pair of complex conjugate poles

\[
2 \text{Re} Z + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0, \tag{148}
\]

provided that the propagator has the asymptotic behavior

\[
\lim_{|k^2| \to \infty} k^2 \mathcal{D}(k^2) = 0, \tag{149}
\]

in the region far from the origin of the complex \( k^2 \) plane. In fact, it is shown that the gluon propagator fulfills this condition in Yang-Mills theories in the Landau gauge due to the asymptotic freedom and the negativity of the anomalous dimension \[57\] \[58\]. The assumption (149) is enough to obtain

\[
Z = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \ \mathcal{D}(x + i\epsilon), \quad x = \text{Re} k^2. \tag{150}
\]

The real part of (150) leads to a generalized sum rule for the spectral function (148) by taking into account the relation (112):

\[
\text{Im} \ \mathcal{D}(x + i\epsilon) = \begin{cases} 0 & (x < 0) \\ \pi \rho(x) & (x > 0) \end{cases}, \tag{151}
\]

while the imaginary part leads to another relation,

\[
\text{Im} Z = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk^2 \ \text{Re} \mathcal{D}(k^2 + i\epsilon). \tag{152}
\]

Therefore, the usual superconvergence relation (147) does not hold unless the residue of the complex pole is pure imaginary \( \text{Re} Z = 0 \). The preliminary results for the massive Yang-Mills model to one-loop order at the physical point are \[54\]

\[
\text{Re} Z = 0.386322, \quad \text{Im} Z = 0.861514,
\]

\[
\int_0^\infty d\sigma^2 \rho(\sigma^2) = -0.694533 < 0,
\]

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} dk^2 \ \text{Re} \mathcal{D}(k^2 + i\epsilon) = 1.74006 > 0, \tag{153}
\]

which leads to

\[
2 \text{Re} Z + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0.0781108,
\]

\[
2 \text{Im} Z - \frac{1}{\pi} \int_{-\infty}^{\infty} dk^2 \ \text{Re} \mathcal{D}(k^2 + i\epsilon) = -0.0170369. \tag{154}
\]

It will be interesting to examine whether the generalized sum rule holds or not, and to what extent it is satisfied beyond one-loop level when the gluon propagator has a pair of complex conjugate pole.

Moreover, it is desirable to extend the results obtained in this paper to a finite temperature to see whether or not the reflection positivity violated in the low-temperature confinement phase is recovered in the high-temperature deconfinement phase even in the pure Yang-Mills theory. Then we can ask whether or not the transition detected by the positivity violation/restoration agrees with the confinement/deconfinement transition detected by the Polyakov loop average. It is also interesting to examine how the relevant complex structure changes depending on the temperature. These issues will be discussed in future works.

**Acknowledgements**

The authors would like to thank Professor Oliveira for providing us with the data of numerical simulation for gluon and ghost propagators presented in this paper. K.-I. K. thanks Matthieu Tissier, Nicolas Wschebor, Julien Serreau, and Urko Reinosoa for the discussion on positivity violation. This work was supported by Grant-in-Aid for Scientific Research, JSPS KAKENHI Grant Number (C) No.15K05042. R. M. was supported by Grant-in-Aid for JSPS Research Fellow Grant Number 17J04780.

**Appendix A: Recursive construction of the gauge-invariant transverse field**

In this section, we derive the recurrence relation for obtaining the power-series solution of the reduction condition which is equivalent to the transverse condition,

\[
\partial_\mu \mathcal{A}_\mu^{\lfloor \alpha \rfloor} = 0. \tag{A1}
\]
First, we expand \( h[\mathcal{A}] \) into the power series in the gauge field \( \mathcal{A} \),
\[
  h[\mathcal{A}] = 1 + h^{(1)} + h^{(2)} + \cdots, \tag{A2}
\]
\[
  h^{-1}[\mathcal{A}] = 1 + (h^{-1})^{(1)} + (h^{-1})^{(2)} + \cdots. \tag{A3}
\]
where the superscript \((k)\) denotes \(k\)-th term in the power series. Note that \((h^{-1})^{(k)} = (h^{(k)})\dagger\) because \(h^{-1} = h^\dagger\).

Then the \(k\)-th term of \( \mathcal{A}_\mu^h \) is given by
\[
  (\mathcal{A}_\mu^h)^{(k)} = \sum_{l=0}^{k-1} h^{(l)}(\mathcal{A}_\mu)h^{(k-l-1)\dagger} - ig^{-1}\sum_{l=0}^{k} \partial_\mu h^{(l)}h^{(l-k)\dagger} \tag{A4}
\]
where we have defined \( h_{\mu} \), the \(k\)-th term of \( h[\mathcal{A}] \) into the power series in the gauge field \( \mathcal{A} \). By taking into account the fact that \( (h^{-1})^{(k)} = (h^{(k)})\dagger \), the \(k\)-th term of \( \mathcal{A}_\mu^h = h[\mathcal{A}]h^\dagger - ig^{-1}\partial_\mu hh^\dagger \) is given by
\[
  (\mathcal{A}_\mu^h)^{(k)} = \sum_{l=0}^{k-1} h^{(l)}(\mathcal{A}_\mu)h^{(k-l-1)\dagger} - ig^{-1}\sum_{l=0}^{k} \partial_\mu h^{(l)}h^{(l-k)\dagger} \tag{A6}
\]
where the \(k\)-th term of \( \mathcal{A}_\mu^h \) can be written as
\[
  \mathcal{A}_\mu^h = \left( \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Psi_\nu. \tag{A7}
\]
This relation is shown to hold with mathematical induction as follows. For \( k = 0 \), it follows from (A5) that
\[
  ig^{-1}h^{(1)} = \frac{\partial_\mu}{\partial^2} \mathcal{A}_\mu. \tag{A15}
\]

Next, we show that \( \mathcal{A}_\mu^h \) is gauge invariant by using (A5). Now let the gauge transformation of \( \mathcal{A}_\mu \) by \( V \)
\[
  \mathcal{A}_\mu \to V\mathcal{A}_\mu V^{-1} + ig^{-1}V\partial_\mu V^{-1}. \tag{A10}
\]
We observe that \( \mathcal{A}_\mu^h \) is indeed gauge invariant if the gauge transformation of \( h \) obeys
\[
  h \to hV^{-1}, \tag{A11}
\]
whose infinitesimal form for \( V = e^{-ig\Lambda} \) is given by
\[
  \delta h = -igh\Lambda. \tag{A12}
\]

In the following, we show that (A12) holds order by order.

Note that because \( \delta \mathcal{A}_\mu = \partial_\mu \Lambda - ig[\mathcal{A}_\mu, \Lambda] \), \( \delta h^{(k)} \) contains \(k\)-th term and \((k-1)\)-th term of power series of \( \delta h \). For this reason, we separate \( \delta h^{(k)} \) into two parts as
\[
  \delta h^{(k)} = \delta h^{(k)}_\perp + \delta h^{(k)}_\perp \tag{A13}
\]
where \( \delta h^{(k)}_\perp \) has the order of \( k-1 \) and \( \delta h^{(k)}_\perp \) has the order of \( k \). Therefore, the \(k\)-th order term of (A12) is written as
\[
  ig^{-1}\delta h^{(k)}_\perp + ig^{-1}\delta h^{(k+1)}_\perp = h^{(k)}\Lambda. \tag{A14}
\]

This relation is shown to hold with mathematical induction as follows. For \( k = 0 \), it follows from (A5) that
\[
  ig^{-1}h^{(1)} = \frac{\partial_\mu}{\partial^2} \mathcal{A}_\mu. \tag{A15}
\]

By taking the variation under the gauge transformation, we obtain
\[
  ig^{-1}\delta h^{(1)}_\perp = \Lambda. \tag{A16}
\]

Thus we have shown (A14) holds for \( k = 0 \). Next, suppose that (A14) holds for \( k-1 \). Then we proceed to show that it holds for \( k \). By taking variation of (A5) for \( k + 1 \) under the gauge transformation we obtain
\[ i g^{-1} \delta h_{(k+1)} = \frac{\partial}{\partial x^\mu} \sum_{l=0}^{k} \left( \delta h_{(l)} \mathcal{A}_\mu(h_{(k-l)})^\dagger + h_{(l)} \mathcal{A}_\mu(\delta h_{(k-l)})^\dagger + h_{(l)} \partial_\mu \Lambda(h_{(k-l)})^\dagger \right. \\
\left. - ig^{-1} \partial_\mu \delta h_{(l)}(h_{(k-l+1)})^\dagger \right) \\
= \frac{\partial}{\partial x^\mu} \left[ h(k) \partial_\mu \Lambda - ig^{-1} \partial_\mu h(k)(\delta h_{(l)})^\dagger \right. \\
+ \sum_{l=0}^{k-1} \left( h(l) \partial_\mu \Lambda(h_{(k-l)})^\dagger - ig^{-1} \partial_\mu h(l)(\delta h_{(k-l+1)})^\dagger \right) \\
\left. + \sum_{l=0}^{k} \left( \delta h_{(l)} \mathcal{A}_\mu(h_{(k-l)})^\dagger + h_{(l)} \mathcal{A}_\mu(\delta h_{(k-l)})^\dagger - ig^{-1} \partial_\mu \delta h_{(l)}(h_{(k-l+1)})^\dagger \right) \right] \\
= h(k) \Lambda + \frac{\partial}{\partial x^\mu} \sum_{l=0}^{k-1} \left( h(l) \partial_\mu \Lambda(h_{(k-l)})^\dagger - ig^{-1} \partial_\mu h(l)(\delta h_{(k-l+1)})^\dagger \right. \\
\left. + \delta h_{(l)} \mathcal{A}_\mu(h_{(k-l+1)})^\dagger + h_{(l)} \mathcal{A}_\mu(\delta h_{(k-l)})^\dagger - ig^{-1} \partial_\mu \delta h_{(l+1)}(h_{(k-l)})^\dagger \right). \tag{A17} \]

By taking the variation of \[ (A5) \] for \( k \) we obtain
\[ i g^{-1} \delta h_{(k)} = \frac{\partial}{\partial x^\mu} \sum_{l=0}^{k-1} \left( \delta h_{(l)} \mathcal{A}_\mu(h_{(k-l)})^\dagger + h_{(l)} \mathcal{A}_\mu(\delta h_{(k-l)})^\dagger - igh_{(l)}[\mathcal{A}_\mu, \Lambda](h_{(k-l)})^\dagger \\
- ig^{-1} \partial_\mu h_{(l)}(h_{(k-l)})^\dagger - ig^{-1} \partial_\mu h_{(l)}(\delta h_{(k-l)})^\dagger \right). \tag{A18} \]

By summing up the underlined part, double-underlined part, wavy-lined part and broken-lined part respectively, we obtain

\[
\begin{align*}
\text{underlined part} &= - ngh^0 \Lambda \mathcal{A}_\mu(h_{(k-1)})^\dagger, \tag{A19} \\
\text{double-underlined part} &= ngh^1 \mathcal{A}_\mu \Lambda(h_{(k-1)})^\dagger, \tag{A20} \\
\text{wavy-lined part} &= \partial_\mu(h^0 \Lambda)(h_{(k-1)})^\dagger, \tag{A21} \\
\text{broken-lined part} &= - \partial_\mu(h^1 \Lambda)(h_{(k-1)})^\dagger. \tag{A22}
\end{align*}
\]

where we have used the assumption of induction. The sum of \[ (A19) \] and \[ (A20) \] cancels the third term in the parentheses of \[ (A18) \]. The sum of \[ (A21) \] and \[ (A22) \] cancels the first term in the parentheses of \[ (A17) \]. Therefore \[ (A14) \] is satisfied for \( k \). Thus we have shown that \( \mathcal{A}^h \) is invariant under a gauge transformation.

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**Appendix B: Solving the reduction condition in another way**

By using the massive vector field mode \( \mathcal{W}_\mu \) \[ 17 \], the reduction condition reads
\[
\chi(x) = \chi^A(x) T_A = \partial_\mu \mathcal{W}_\mu(x) - ig[\mathcal{A}_\mu(x), \mathcal{W}_\mu(x)] \\
= \partial_\mu \mathcal{A}_\mu(x) - ig^{-1} \partial_\mu (\Theta(x) \partial_\mu \Theta(x)^\dagger) \\
- [\mathcal{A}_\mu(x), \Theta(x) \partial_\mu \Theta(x)^\dagger]. \tag{B1}
\]

For the scalar field \( \Theta(x) \), we introduce the Lie algebra \( \mathcal{G} \)-valued field \( \theta(x) \) as
\[
\Theta(x) = e^{-ig \theta(x)} \in \mathcal{G}, \quad \theta(x) := \theta^A(x) T_A \in \mathcal{G}. \tag{B2}
\]

In the following, we solve the reduction condition by expressing the scalar field \( \Theta(x) \) as a power series in the gauge field \( \mathcal{A}_\mu \). The Lie algebra form of the pure gauge
reads

\[ \hat{\Theta} \partial_\mu \hat{\Theta}^\dagger = \left(1 - ig \theta - \frac{1}{2} g^2 \partial_\theta \theta \right) \partial_\mu \left(1 + ig \theta - \frac{1}{2} g^2 \partial_\theta \theta \right) + O(\theta^3) \]

\[ = \left(1 - ig \theta - \frac{1}{2} g^2 \partial_\theta \theta \right) \left(ig \partial_\mu \theta - \frac{1}{2} g^2 \partial_\mu \partial_\theta \theta \right) + O(\theta^3) \]

\[ = -ig \partial_\mu \theta - \frac{1}{2} g^2 \partial_\mu \partial_\theta \theta + \frac{1}{2} g^2 \partial_\mu \theta + g^2 \partial_\theta \mu + O(\theta^3), \]

\[ = -ig \partial_\mu \theta + \frac{1}{2} g^2 [\theta, \partial_\mu \theta] + O(\theta^3). \] (B3)

The more general expression is given as

\[ \hat{\Theta}(x) \partial_\mu \hat{\Theta}(x)^\dagger = -\sum_{n=0}^{\infty} \frac{(-ig)^{n+1}}{(n+1)!} [Ad \theta(x)]^n \partial_\mu \theta(x), \] (B4)

By substituting this result into the reduction condition \( \chi = 0 \), we have

\[ 0 = \partial_\mu \mathcal{A}_\mu - ig^{-1} \partial_\mu (\partial_\mu \hat{\Theta}^\dagger) = [\mathcal{A}_\mu, \hat{\Theta} \partial_\mu \hat{\Theta}^\dagger] \]

\[ = \partial_\mu \mathcal{A}_\mu - ig^{-1} \partial_\mu (ig \partial_\mu \theta + \frac{1}{2} g^2 [\theta, \partial_\mu \theta]) \]

\[ - \left[ \mathcal{A}_\mu, ig \partial_\mu \theta + \frac{1}{2} g^2 [\theta, \partial_\mu \theta] \right] + O(\theta^3) \]

\[ = \partial_\mu \mathcal{A}_\mu + \partial_\mu \partial_\mu \theta - \frac{1}{2} ig [\theta, \partial_\mu \partial_\mu \theta] - ig \mathcal{A}_\mu, \partial_\mu \theta] + O(\theta^3). \] (B5)

This is recast into

\[ \partial^2 \theta = - \partial \cdot \mathcal{A} + \frac{1}{2} ig [\theta, \partial^2 \theta] + ig \mathcal{A}_\mu, \partial_\mu \theta] + O(\theta^3), \] (B6)

which yields

\[ \theta = - \frac{1}{\partial_\mu} \partial \cdot \mathcal{A} + \frac{1}{2} ig \frac{1}{\partial_\mu} [\theta, \partial^2 \theta] + ig \frac{1}{\partial_\mu} \mathcal{A}_\mu, \partial_\mu \theta] + O(\theta^3). \] (B7)

Substituting recursively for \( \theta \), we obtain a power series,

\[ \theta(x) = - \frac{1}{\partial_\mu} \partial \cdot \mathcal{A}(x) + \frac{1}{2} ig \frac{1}{\partial_\mu} \left[ \frac{1}{\partial_\mu} \partial \cdot \mathcal{A}(x), \partial \cdot \mathcal{A}(x) \right] \]

\[ - ig \frac{1}{\partial_\mu} \mathcal{A}_\mu(x), \partial_\mu \frac{1}{\partial_\mu} \partial \cdot \mathcal{A}(x) ] + O(\theta^3). \] (B8)

The massive vector field mode \( \mathcal{W}_\mu \) [17] is written as

\[ \mathcal{W}_\mu^\dagger = \mathcal{A}_\mu(x) - ig^{-1} \hat{\Theta}(x) \partial_\mu \hat{\Theta}(x)^\dagger. \]

\[ = \mathcal{A}_\mu + \partial_\mu \theta - ig^{-1} \hat{\Theta}(x) \partial_\mu \hat{\Theta}(x)^\dagger \]

\[ = \mathcal{A}_\mu + \frac{1}{2} ig \frac{1}{\partial_\mu} \left[ \partial \cdot \mathcal{A}, \partial \cdot \mathcal{A} \right] \]

\[ - \frac{1}{2} \left[ \frac{\partial \cdot \mathcal{A}}{\partial_\mu}, \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial_\mu} \right] + O(\theta^3), \] (B9)

where we have defined the transverse field \( \mathcal{A}_\mu^T \) in the lowest order term linear in \( \mathcal{A} \) as

\[ \mathcal{A}_\mu^T := \mathcal{A}_\mu - \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial_\mu}. \] (B10)

Notice that \( \mathcal{W}_\mu \) agrees with \( \mathcal{A}_\mu \) in the Landau gauge \( \partial \cdot \mathcal{A} = 0 \).

Thus, by substituting \( \mathcal{W} \) of (B9) into (16), the term \( S_{\text{kin}} \) reads

\[ S_{\text{kin}}[\mathcal{A}] = \int d^D x \ M^2 \text{tr} \{ \mathcal{W}_\mu \mathcal{W}_\mu^\dagger \}
\]

\[ = \int d^D x \ M^2 \text{tr} \left\{ \mathcal{A}_\mu^T \mathcal{A}_\mu^T \right\}
\]

\[ + ig \mathcal{A}_\mu^T \partial_\mu \frac{1}{\partial_\mu} \left[ \frac{\partial \cdot \mathcal{A}}{\partial_\mu}, \partial \cdot \mathcal{A} \right] \]

\[ - 2ig \mathcal{A}_\mu^T \partial_\mu \frac{1}{\partial_\mu} \left[ \mathcal{A}_\lambda, \partial_\lambda \frac{\partial \cdot \mathcal{A}}{\partial_\mu} \right] \]

\[ - ig \mathcal{A}_\mu^T \left[ \frac{\partial \cdot \mathcal{A}}{\partial_\mu}, \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial_\mu} \right] \} + O(\mathcal{A}^4). \] (B11)

By performing integration by parts and taking into account the transversality \( \partial_\mu \mathcal{A}_\mu^T = 0 \), the action \( S_{\text{kin}} \) takes the form,

\[ S_{\text{kin}}[\mathcal{A}] = \int d^D x \ M^2 \text{tr} \left\{ \mathcal{A}_\mu^T \mathcal{A}_\mu^T - ig \mathcal{A}_\mu^T \left[ \frac{\partial \cdot \mathcal{A}}{\partial_\mu}, \partial_\mu \frac{\partial \cdot \mathcal{A}}{\partial_\mu} \right] \right\} \]

\[ + O(\mathcal{A}^4). \] (B12)

This indeed agrees with the expression (37).

**Appendix C: One loop correction in the arbitrary gauge parameter**

We keep the gauge fixing parameter \( \alpha \) arbitrary. The gluon propagator is decomposed into the transverse and longitudinal parts,

\[ \mathcal{D}_{\mu\nu}(k) = \mathcal{D}_T(k) \perp_{\mu\nu}(k) + \mathcal{D}_L(k) \parallel_{\mu\nu}(k), \]

\[ \perp_{\mu\nu}(k) := g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \parallel_{\mu\nu}(k) := \frac{k_\mu k_\nu}{k^2}, \] (C1)
where \( \perp_{\mu\nu} \) and \( \parallel_{\mu\nu} \) are respectively the transverse and longitudinal projectors. The full gluon propagator \( \tilde{D}_{\mu\nu}(k) \) and the bare gluon propagator \( \tilde{D}_{\mu\nu}^{(0)}(k) \) are related to the vacuum polarization tensor \( \Pi_{\mu\nu}(k) \) as

\[
\tilde{D}_{\mu\nu}^{-1}(k) = \tilde{D}_{\mu\nu}^{(0)}(k) + \Pi_{\mu\nu}(k). \tag{C2}
\]

The gluon vacuum polarization tensor is decomposed as

\[
\Pi_{\mu\nu}(k) = \Pi_T(k) \perp_{\mu\nu}(k) + \Pi_L(k) \parallel_{\mu\nu}(k). \tag{C3}
\]

Therefore, we have

\[
\tilde{D}_T^{-1}(k) = k^2 + M^2 + \Pi_T(k) + k^2 \delta_Z + M^2 \delta_M^2, \tag{C4}
\]

\[
\tilde{D}_L^{-1}(k) = \frac{1}{\alpha} (k^2 + \alpha M^2) + \Pi_L(k) + M^2 \delta_M^2. \tag{C5}
\]

The gluon vacuum polarization tensor \( \Pi_{\mu\nu}^{AB}(k) = \delta^{AB} \Pi_{\mu\nu}(k) \) is calculated to one loop as (after the Wick rotation)

\[
\Pi_T(k) = \frac{g^2 C_2(G) \, k^2}{48\pi^2} \times \left[ \left( \frac{9}{s} (1 + \alpha) - (26 - 6\alpha) \right) (\epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta) - \frac{12}{3} - \alpha^2 + \frac{1}{s} (63 + (9 - 2\alpha) \alpha) - \frac{(1 - \alpha)^2}{s^2} + \ln(s) - \frac{1}{2} \left( s (7 + 3\alpha) + (26 + 3\alpha (2 + \alpha)) + \frac{1}{s} (26 \alpha^2 (3 + \alpha)) - \frac{1}{s^2} (-7 + \alpha (6 + (3 - 2\alpha) \alpha)) - \frac{(1 - \alpha)^3}{s^3} \right) \ln(\alpha) + \frac{1}{2 s^2} (s + 4)^2 \frac{s^2 - 20s + 12}{s^2 + 2s (\alpha - 5) + (1 - \alpha)^2} \sqrt{s^2 + (1 - \alpha)^2 + 2s (1 + \alpha)} \times \ln \left( \frac{2\sqrt{\alpha}}{1 + s + \alpha - \sqrt{s^2 + (1 - \alpha)^2 + 2s (1 + \alpha)}} \right) \right], \tag{C6}
\]

and

\[
\Pi_L(k) = \frac{g^2 C_2(G) \, 3 \, k^2}{48\pi^2} \times \left[ \left( \frac{3}{s} \frac{1}{(1 + \alpha)} (\epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta) + \frac{5}{s} (1 + \alpha) + \frac{1}{s^2} (1 - \alpha)^2 + \ln(s) - \frac{1}{2} \left( 1 + \frac{3}{s} (1 + \alpha) + \frac{3}{s^2} (1 - \alpha) + \frac{1}{s^3} (1 - \alpha)^3 \right) \ln(\alpha) - \frac{1}{s^3} (s^2 + (1 - \alpha)^2 + 2s (1 + \alpha))^2 \ln \left( \frac{2\sqrt{\alpha}}{1 + s + \alpha - \sqrt{s^2 + (1 - \alpha)^2 + 2s (1 + \alpha)}} \right) \right) \right]. \tag{C7}
\]

The ghost propagator \( \Delta_{gh}^{AB}(k) = \delta^{AB} \Delta_{gh}(k) \) is related to the ghost self-energy \( \Pi_{gh}^{AB}(k) = \delta^{AB} \Pi_{gh}(k) \) as

\[
\Delta_{gh}(k) = \frac{i}{k^2 + \Pi_{gh}(k) + k^2 \delta_G}. \tag{C8}
\]

The ghost self-energy \( \Pi_{gh}(k) \) is calculated to one loop as (after the Wick rotation)

\[
\Pi_{gh}(k) = -\frac{g^2 C_2(G)}{16\pi^2} \frac{1}{4k^2} \left[ (3 - \alpha) (\epsilon^{-1} - \gamma + \ln(4\pi) + \ln \eta) + \frac{5}{s} \alpha + \frac{1}{s} (1 + \alpha)^2 \ln(\alpha) - \frac{(\alpha + s)^2 (\alpha - s)}{s^2} \ln(s + \alpha) - \frac{(1 + s)^3}{s^2} \ln(s + 1) \right]. \tag{C9}
\]
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