Dirac Operator in Matrix Geometry

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Dedicated to the Memory of Vladimir Fock and Dmitri Ivanenko

We review the construction of the Dirac operator and its properties in Riemannian geometry and show how the asymptotic expansion of the trace of the heat kernel determines the spectral invariants of the Dirac operator and its index. We also point out that the Einstein-Hilbert functional can be obtained as a linear combination of the first two spectral invariants of the Dirac operator. Next, we report on our previous attempts to generalize the notion of the Dirac operator to the case of Matrix Geometry, when instead of a Riemannian metric there is a matrix valued self-adjoint symmetric two-tensor that plays a role of a "non-commutative" metric. We construct invariant first-order and second-order self-adjoint elliptic partial differential operators, that can be called "non-commutative" Dirac operator and non-commutative Laplace operator. We construct the corresponding heat kernel for the non-commutative Laplace type operator and compute its first two spectral invariants. A linear combination of these two spectral invariants gives a functional that can be considered as a non-commutative generalization of the Einstein-Hilbert action.
1 Introduction

Dirac operator was discovered by Dirac in 1928 as a “square root” of the D’Alambert operator in a flat Minkowskian space in an attempt to develop a relativistic theory of the electron. In 1929, almost immediately after Dirac’s paper, Fock and Ivanenko [1, 2] showed how to generalize the Dirac’s equation for the case of General Relativity. Fock completed the geometrization of the theory of spinors on Riemannian manifolds in [3, 4, 5]. This development was purely local. Only much later, at the end of forties in a global setting it was understood that there are topological obstructions to the existence of the spinor structure and spinor fields and Dirac operators cannot be introduced on every Riemannian manifold (see, for example, [6, 7, 8]).

The construction of a square root of the Laplacian naturally leads to the study of complex representations of the Clifford algebra. The spinors are then introduced as the elements of the corresponding vector space. It turns out that there are no non-trivial representations of the orthogonal group in the vector space of spinors compatible with Clifford multiplication. Therefore, spinors on a Riemannian manifold cannot be introduced as sections of a vector bundle associated with the frame bundle of the manifold. Instead of the special orthogonal group one considers its double covering group, so called spin group. Contrary to the orthogonal group the spin group has a representation in the vector space of spinors compatible with Clifford multiplication. Now, if the frame bundle of the manifold allows a reduction to the spin group, then the manifold is said to admit a spin structure and one can define the spinor bundle, which is a vector bundle associated with this reduction via the representation of the spin group. The spinors are the sections of the spinor bundle.

The spinor bundle inherits a connection from the canonical Levi-Civita connection, which enables one to define the Dirac operator. The Dirac operator on a spinor bundle is now called the Dirac operator in the narrow sense, while the general (also called twisted) Dirac operator (or Dirac type operator) is any self-adjoint first-order operator whose square is equal to the D’Alambert operator (or the Laplace operator).

The Dirac operator in Riemannian geometry is defined with the help of a Riemannian metric and the spin connection. Following the ideas of our papers [9, 10, 11] we are going to generalize this formalism to the case of Matrix Geometry, when instead of a single Riemannian metric there is a matrix-valued symmetric 2-tensor, which we call a “non-commutative met-
Matrix Geometry is motivated by the relativistic interpretation of gauge theories and is intimately related to Finsler geometry (rather a collection of Finsler geometries) (see [9, 10, 11]). In the present paper we will not discuss the origin of Matrix Geometry, but simply assume the existence of such a structure. We will not be concerned about the global issues as well. We will simply assume that there are no topological obstructions to all the structures introduced below. Our “non-commutative” Dirac operator is a first-order elliptic partial differential operator such that its square is a second-order self-adjoint elliptic operator with positive definite leading symbol (not necessarily of Laplace type).

The outline of the paper is as follows. First, we review the construction of the Dirac operator in Riemannian geometry. In section 2.1 we introduce the Clifford algebra and describe its properties. In section 2.2 we introduce the spin group and show that it is a double cover of the orthogonal group. In section 2.3 the spin representation of the spin group is introduced and the spinors are defined. In section 2.4 the derivation of the spin connection and its curvature is described. In section 2.5 we define the Dirac operator and its index. In section 2.6 we introduce the heat kernel. It is explained how the asymptotic expansion of the trace of the heat kernel generates the spectral invariants of the Dirac operator, in particular, its index.

In section 3.1 we introduce non-commutative (or matrix) generalization of the Riemannian metric and the Dirac matrices as a deformation of the commutative limit. In section 3.2 we introduce non-commutative versions of the Hodge star operator acting on space of matrix valued $p$-forms. In section 3.3 the relation of the matrix geometry to Finsler geometry is explored. In section 3.4 we promote the vector spaces introduced above to vector bundles and define the corresponding Hilbert spaces. Since we do not have a Riemannian volume element, we work here not with tensors but rather with densities of various weight. In section 3.5 we develop an exterior calculus for matrix-valued densities and in section 3.6 we introduce a “non-commutative connection”. Sections 3.7 and 3.8 discuss the construction of the non-commutative versions of the Dirac operator and the Laplacian. In sections 3.9 and 3.10 we discuss the spectral asymptotics of the operators introduced before and compute the first two spectral invariants.

In section 4 we construct a non-commutative deformation of the Einstein-Hilbert functional.
2 Dirac Operators in Riemannian Geometry

2.1 Clifford Algebra

Let $M$ be an $n$-dimensional manifold, $x$ be a point in $M$ and $V = T_x M$ be the tangent space at the point $x$ (which is isomorphic to $\mathbb{R}^n$) equipped with a positive-definite scalar product $\langle \cdot , \cdot \rangle$. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$ so that for even dimension $n = 2m$ and for odd dimension $n = 2m + 1$. Let also $N = 2^m$.

The real Clifford algebra $\text{Cliff}(n)$ is the universal associative algebra with unit generated multiplicatively by the range $\gamma(V)$ of a linear map $\gamma : V \to \text{Cliff}(n)$ (1)

satisfying

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2\langle u,v \rangle I,$$

where $I$ is the unit in the algebra. In particular,

$$\gamma(u)\gamma(u) = |u|^2 I,$$

where $|u|^2 = \langle u,u \rangle$, and, therefore, for any unit vector $u$, $|u| = 1$, the element $\gamma(u)$ is invertible and

$$[\gamma(u)]^{-1} = \gamma(u).$$

Let $\otimes V$ be the tensor algebra

$$\otimes V = \bigoplus_{k=0}^{\infty} \otimes^k V,$$

where $\otimes^0 V = \mathbb{R}$, and $\mathcal{I}$ be the ideal generated by the set

$$\{ u \otimes v + v \otimes u - 2\langle u,v \rangle | u,v \in V \}.$$

Then the Clifford algebra can be identified with the quotient

$$\text{Cliff}(n) = \otimes V / \mathcal{I}. $$

The corresponding complex Clifford algebra is obtained by tensoring the real algebra with the complex numbers

$$\text{Cliff}_C(n) = \text{Cliff}(n) \otimes_{\mathbb{R}} \mathbb{C}.$$
The tensor algebra \( \otimes V \) has a natural \( \mathbb{N} \)-grading, which after reduction mod 2 leads to a natural \( \mathbb{Z}_2 \)-grading of the Clifford algebra

\[
\text{Cliff}(n) = \text{Cliff}_+(n) \oplus \text{Cliff}_-(n),
\]

(8)

where \( \text{Cliff}_+(n) \) and \( \text{Cliff}_-(n) \) are the even and the odd parts respectively consisting of the sums of products of even and odd number of elements from \( \gamma(V) \) with

\[
\text{Cliff}_k(n) \text{Cliff}_j(n) \subset \text{Cliff}_{jk}(n),
\]

(9)

where \( k, j = \pm 1 \). Therefore, the even part \( \text{Cliff}_+(n) \) is a subalgebra of the Clifford algebra \( \text{Cliff}(n) \).

Let \( F_0(n) = \mathbb{R} \) and for \( 1 \leq k \leq n \)

\[
F_k(n) = \text{Span}\{\gamma(u_1) \cdots \gamma(u_j) \mid 1 \leq j \leq k, \ u_i \in V\}
\]

(10)

be the subspace of \( \text{Cliff}(n) \) consisting of the sums of the products of at most \( k \) elements from \( \gamma(V) \). Then the Clifford algebra has a natural increasing filtration

\[
F_0(n) \subset F_1(n) \subset \cdots \subset F_n(n) = \text{Cliff}(n)
\]

(11)

such that

\[
F_j(n)F_k(n) \subset F_{k+j}(n),
\]

(12)

where, by definition \( F_j(n) = F_n(n) \) if \( j > n \).

Further, let

\[
C_0(n) = \mathbb{R}, \quad C_k(n) = F_k(n)/F_{k-1}(n), \quad 1 \leq k \leq n.
\]

(13)

Then

\[
F_{2k}(n) = \bigoplus_{j=0}^{k} C_{2k-2j}(n)
\]

(14)

and

\[
F_{2k+1}(n) = \bigoplus_{j=0}^{k} C_{2k-2j+1}(n).
\]

(15)

The space \( C_k(n) \) is isomorphic to \( \wedge^k V \) and the Clifford algebra is a graded algebra, which, as a vector space, has the form

\[
\text{Cliff}(n) = \bigoplus_{k=0}^{n} C_k(n),
\]

(16)
with
\[ C_j(n)C_k(n) = \bigoplus_{0 \leq 2l \leq k+j} C_{k+j-2l}(n), \quad (17) \]
and is naturally isomorphic to the exterior algebra \( \wedge V \)
\[ \wedge V = \bigoplus_{k=0}^{n-1} \wedge^k V, \quad (18) \]
where \( \wedge^0 V = \mathbb{R}. \)

Therefore the map \( \gamma : V \to \text{Cliff}(n) \) can be extended to an isomorphism
\[ \gamma : \wedge V \to \text{Cliff}(n) \quad (19) \]
of the exterior algebra and the Clifford algebra such that
\[ \gamma(1) = 1. \quad (20) \]

We also have
\[ \text{Cliff}_+(n) = \bigoplus_{0 \leq 2k \leq n} C_{2k}(n), \quad \text{Cliff}_-(n) = \bigoplus_{0 \leq 2k+1 \leq n} C_{2k+1}(n). \quad (21) \]

There are natural projections
\[ \text{Pr}_k : \text{Cliff}(n) \to C_k(n). \quad (22) \]
The projection onto the unit element
\[ \text{Pr}_0 : \text{Cliff}(n) \to C_0(n) = \mathbb{R} \quad (23) \]
defines a natural linear functional on the Clifford algebra, which satisfies a very important property
\[ \text{Pr}_0(AB) = \text{Pr}_0(BA), \quad (24) \]
and a normalization condition
\[ \text{Pr}_0(1) = 1. \quad (25) \]

One can conclude from this that
\[ \text{Pr}_0 C_k(n) = 0 \quad \text{for} \ k \neq 0, \quad (26) \]
and
\[ \text{Pr}_0 F_{2k+1}(n) = 0. \] (27)

There is a natural involution
\[ \alpha : \text{Cliff}(n) \to \text{Cliff}(n), \] (28)
such that
\[ \alpha^2 = \text{Id} \quad \text{and} \quad \alpha(AB) = \alpha(A)\alpha(B) \] (29)
defined by
\[ \alpha[\gamma(u_1) \cdots \gamma(u_k)] = (-1)^k\gamma(u_1) \cdots \gamma(u_k). \] (30)
Then
\[ \alpha(A) = (-1)^k A \quad \text{for } A \in C_k(n). \] (31)
and
\[ \alpha(A) = \varepsilon(A) A \] (32)
where
\[ \varepsilon(A) = \pm 1 \quad \text{for } A \in \text{Cliff}_\pm(n) \] (33)
is the parity of the element $A$.

There is a natural transpose on the tensor algebra $\otimes V$ defined by
\[ u_1 \otimes \cdots \otimes u_k \mapsto u_k \otimes \cdots \otimes u_1. \] (34)
Since the ideal $I$ is preserved under this action, there is a natural linear anti-involution of the Clifford algebra (reversing map, or transposition)
\[ \tau : \text{Cliff}(n) \to \text{Cliff}(n) \] (35)
such that
\[ \tau^2 = \text{Id}, \quad \text{and} \quad \tau(AB) = \tau(B)\tau(A) \] (36)
defined by
\[ \tau[\gamma(u_1) \cdots \gamma(u_k)] = \gamma(u_k) \cdots \gamma(u_1). \] (37)

The composition of the above maps defines another anti-involution (conjugation)
\[ * = \tau \circ \alpha : \text{Cliff}(n) \to \text{Cliff}(n) \] (38)
such that
\[ *^2 = \text{Id}, \quad \text{and} \quad (AB)^* = B^*A^* \] (39)
by
\[
[\gamma(u_1) \cdots \gamma(u_k)]^* = (-1)^k \gamma(u_k) \cdots \gamma(u_1) .
\] (40)

Note that
\[
A^* = \varepsilon(A) \tau(A)
\] (41)
so that for the even Clifford subalgebra the anti-involutions \( \tau \) and \( * \) coincide.

The center of the Clifford algebra is one-dimensional in even dimension \( n = 2m \) and two-dimensional in odd dimension \( n = 2m + 1 \). More precisely,
\[
\mathcal{Z}(\text{Cliff}(2m)) = C_0(n) \] (42)
\[
\mathcal{Z}(\text{Cliff}(2m+1)) = C_0(2m+1) \oplus C_{2m+1}(2m+1) .
\] (43)

This simply means that the only elements that commute with all elements of the Clifford algebra have the form \( aI \) in even dimension \( n = 2m \) and \( aI + b\gamma(e_1) \cdots \gamma(e_{2m+1}) \) in odd dimension \( n = 2m + 1 \), where \( a, b \) are scalars and the vectors \( \{e_a\} \) are orthogonal to each other.

Let \( \{e_a\} = \{e_1, \ldots, e_n\} \), where \( a = 1, \ldots, n \), be an oriented orthonormal basis of \( V \), that is
\[
\langle e_a, e_b \rangle = \delta_{ab} ,
\] (44)
where \( \delta_{ab} \) is the Kronecker symbol. We use small Latin letters running over \( 1, \ldots, n \) to denote vectors from the vector space \( V \). We also use the standard summation convention to sum over repeated indices. Such indices will be raised and lowered by the Euclidean metric (the Kronecker symbol \( \delta_{ab} \)). Then the elements
\[
\gamma_a = \gamma(e_a)
\] (45)
of the Clifford algebra satisfy the anti-commutation relations
\[
\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}I .
\] (46)
Thus, \( \gamma_a \) are involutions that anti-commute with each other
\[
(\gamma_a)^2 = I ,
\] (47)
\[
\gamma_a \gamma_b = -\gamma_b \gamma_a , \quad \text{for } a \neq b .
\] (48)
The Clifford algebra \( \text{Cliff}(n) \) is multiplicatively generated by the elements \( \gamma_a \).
Let $S_k$ be the permutation group of integers $(1, \ldots, k)$. The signature $\text{sgn}(\sigma)$ (or the sign, or the parity) of a permutation $\sigma \in S_k$ is defined to be $+1$ if $\sigma$ is even and $-1$ if $\sigma$ is odd. The complete antisymmetrization of a tensor $T_{a_1 \cdots a_k}$ over the indices $a_1, \ldots, a_k$, is denoted by the square brackets, and is defined by

$$T_{[a_1 \cdots a_k]} = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T_{a_{\sigma(1)} \cdots a_{\sigma(k)}} ,$$

where the summation is taken over the $k!$ permutations of $(1, \ldots, k)$. Let us further define the anti-symmetrized products of $\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_k}$

$$\gamma_{a_1 \cdots a_k} = [\gamma_{a_1} \cdots \gamma_{a_k}] .$$

Of course, these elements are completely anti-symmetric in all their indices. They are non-zero only when all indices are different. In this case

$$\gamma_{a_1 \cdots a_k} = \gamma_{a_1} \cdots \gamma_{a_k} .$$

Obviously, for $1 \leq k \leq n$ we have

$$C_k(n) = \text{Span} \{ \gamma_{a_1 \cdots a_k} \mid 1 \leq a_j \leq n, \ 1 \leq j \leq k \} ,$$

and

$$\text{Cliff}(n) = \text{Span} \{ \mathbb{I}, \gamma_{a_1 \cdots a_k} \mid 1 \leq a_j \leq n, \ 1 \leq j \leq k, \ 1 \leq k \leq n \} .$$

That is each element of the Clifford algebra $\text{Cliff}(n)$ is a linear combination of the elements $\gamma_{a_1 \cdots a_k}$ with real coefficients.

The extension of the map $\gamma$ to the whole exterior algebra $\wedge V$ is defined by

$$\gamma(1) = \mathbb{I} , \quad \gamma(e_{a_1} \wedge \cdots \wedge e_{a_k}) = \gamma_{a_1 \cdots a_k} .$$

Therefore, the elements

$$\mathbb{I}, \ \gamma_{a_1}, \ \gamma_{a_1 a_2}, \ \ldots, \ \gamma_{a_1 \cdots a_n} ,$$

where $(1 \leq a_1 < a_2 < \cdots < a_n \leq n)$ form a basis in the vector space $\text{Cliff}(V)$. The number of elements in the basis is $2^n$. Therefore, the Clifford algebra $\text{Cliff}(n)$ has the dimension

$$\text{dim Cliff}(n) = 2^n .$$
The dimension of the even subalgebra $\text{Cliff}_+(n)$ is equal to one half of $\dim \text{Cliff}(n)$, that is, $2^{n-1}$.

Let us define the chirality element $\Gamma \in \text{Cliff}_C(n)$ by

$$\Gamma = i^{n(n-1)/2} \gamma_{1\ldots n} = \frac{i^{n(n-1)/2}}{n!} \varepsilon_{a_1\ldots a_n} \gamma_{a_1\ldots a_n}, \quad (57)$$

where $\varepsilon_{a_1\ldots a_n}$ is the completely antisymmetric Levi-Civita symbol normalized by $\varepsilon_{1\ldots n} = +1$. There is an ambiguity of choosing the sign of the chirality operator $\Gamma$ corresponding to the choice of the orientation of the vector space $V$. The chirality operator is an involution, that is

$$\Gamma^2 = I \quad (58)$$

that anticommutes with all $\gamma_a$ in even dimensions and commutes with all $\gamma_a$ in odd dimensions. That is,

$$\Gamma \gamma_a = -\gamma_a \Gamma, \quad \text{for even } n, \quad (59)$$

$$\Gamma \gamma_a = \gamma_a \Gamma, \quad \text{for odd } n. \quad (60)$$

Thus in odd dimension $\Gamma$ lies in the center of the Clifford algebra, and in even dimension we have

$$\Gamma \gamma_{a_1\ldots a_k} = (-1)^k \gamma_{a_1\ldots a_k} \Gamma \quad \text{for even } n. \quad (61)$$

The involutions defined above act on the basis elements as follows: for any $1 \leq k \leq n$ we have

$$\alpha(\gamma_{a_1\ldots a_k}) = (-1)^k \gamma_{a_1\ldots a_k},$$
$$\tau(\gamma_{a_1\ldots a_k}) = (-1)^{k(k-1)/2} \gamma_{a_1\ldots a_k},$$
$$\gamma_{a_1\ldots a_k}^* = (-1)^{k(k+1)/2} \gamma_{a_1\ldots a_k},$$
$$\text{Pr}_0(\gamma_{a_1\ldots a_k}) = 0. \quad (62)$$

In even dimension the chirality operator can be used to define the main involution $\alpha$

$$\alpha(\gamma_{a_1\ldots a_k}) = \Gamma \gamma_{a_1\ldots a_k} \Gamma \quad \text{for even } n. \quad (63)$$

We list below some properties of the basis elements of the Clifford algebra. All elements $\gamma_{a_1\ldots a_k}$ satisfy the normalization condition

$$\tau(\gamma_{a_1\ldots a_k}) \gamma_{a_1\ldots a_k} = I, \quad (64)$$
and, therefore, are invertible
\[
(\gamma_{a_1\cdots a_k})^{-1} = \tau(\gamma_{a_1\cdots a_k}) .
\] (65)

Moreover, the set of elements
\[
\pm \mathbb{I}, \pm \gamma_{a_1}, \pm \gamma_{a_1a_2}, \ldots, \pm \gamma_{a_1\cdots a_n}, \quad (1 \leq a_1 < a_2 < \cdots < a_n \leq n)
\] (66)
forms a finite multiplicative group.

There holds
\[
\text{Pr}_0(\tau(\gamma_{a_1\cdots a_k})\gamma^{b_1\cdots b_j}) = 0, \quad \text{for } k \neq j ,
\] (67)
\[
\text{Pr}_0(\tau(\gamma_{a_1\cdots a_k})\gamma^{b_1\cdots b_k}) = k!\delta^{b_1}_{a_1}\cdots\delta^{b_k}_{a_k} .
\] (68)

Therefore, there is a natural inner product in the Clifford algebra defined by
\[
\langle A, B \rangle = \text{Pr}_0(\tau(A)B) .
\] (69)

The basis introduced above is orthonormal in this inner product. Thus, every element \(A \in \text{Cliff}(n)\) can be presented in the form
\[
A = A_{(0)} \cdot \mathbb{I} + \sum_{k=1}^{n} \frac{1}{k!} A^{a_1\cdots a_k}_{(k)} \gamma_{a_1\cdots a_k} ,
\] (70)
where
\[
A_{(0)} = \langle \mathbb{I}, A \rangle = \text{Pr}_0 A ,
\] (71)
\[
A^{a_1\cdots a_k}_{(k)} = \langle \gamma^{a_1\cdots a_k}, A \rangle = \text{Pr}_0 (\tau(\gamma^{a_1\cdots a_k})A) .
\] (72)

The product of the basis elements of the Clifford algebra is given by
\[
\gamma_{a_1\cdots a_k} \gamma^{b_1\cdots b_j} = \sum_{p=0}^{n} (-1)^{p(2k-p-1)/2} \frac{k!j!}{p!(k-p)!(j-p)!} \delta^{b_1}_{a_1} \cdots \delta^{b_p}_{a_p} \gamma_{a_{p+1}\cdots a_k} \gamma^{b_{p+1}\cdots b_j} .
\] (73)

In particular,
\[
\gamma_{a_1a_2} \gamma^{b_1\cdots b_k} = \gamma_{a_1a_2}^{b_1\cdots b_k} - 2k\delta_{[a_1}^{b_1} \gamma_{a_2]}^{b_2\cdots b_k} - k(k-1)\delta^{[b_1}_{a_1} \delta^{b_2}_{a_2} \gamma^{b_3\cdots b_k]} ;
\] (74)
\[
\gamma_{b_1\cdots b_k} \gamma^{a_1a_2} = \gamma_{b_1\cdots b_k}^{a_1a_2} - 2k\delta_{b_1}^{[a_1} \gamma_{b_2]}^{a_2} \gamma_{b_3\cdots b_k]} - k(k-1)\delta^{[a_1}_{b_1} \delta^{a_2}_{b_2} \gamma^{a_3\cdots b_k]} .
\] (75)
which for \( k = 2 \) takes the form
\[
\gamma_{a_1 a_2} \gamma^{b_1 b_2} = \gamma_{a_1 a_2} b_1 b_2 - 4 \delta_{[a_1}^{b_1} \gamma_{a_2] b_2} - 2 \delta_{[a_1}^{[b_1} \delta_{a_2]}^{b_2]}.
\] (76)

Therefore,
\[
[\gamma_{a_1 a_2}, \gamma^{b_1 \cdots b_k}] = -4 k \delta_{[a_1}^{b_1} \gamma_{a_2] [b_2} \cdots b_k],
\] (77)

and, in particular,
\[
[\gamma_{ab}, \gamma_{cd}] = 2 (\gamma_{abcd} - \delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad} + \delta_{ad} \delta_{bc}).
\] (78)

Thus \( \gamma_{ab} \) form a representation of the Lie algebra of the orthogonal group \( SO(n) \).

On the other hand, the anti-commutator of the elements \( \gamma_{ab} \) is
\[
\gamma_{ab} \gamma_{cd} + \gamma_{cd} \gamma_{ab} = 2 (\gamma_{abcd} - \delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad}).
\] (79)

### 2.2 Spin Group

For any unit vector \( u \in V \) we have
\[
[\gamma(u)]^{-1} = \gamma(u),
\] (80)

More generally, let \( u_1, \ldots, u_k \) be a collection of unit vectors from \( V \), and let
\[
T = \gamma(u_1) \cdots \gamma(u_k).
\] (81)

Then
\[
\tau(T)T = \mathbb{I}.
\] (82)

Thus, the elements of the Clifford algebra \( \text{Cliff}(n) \) of the form \( \gamma(u_1) \cdots \gamma(u_k) \), where \( u_1, \ldots, u_k \) are unit vectors from \( V \), are invertible and form a multiplicative group
\[
\text{Pin}(n) = \{ \mathbb{I}, \gamma(u_1) \cdots \gamma(u_k) \mid u_j \in V, |u_j| = 1, k \in \mathbb{N} \} \subset \text{Cliff}(n).
\] (83)

Alternatively,
\[
\text{Pin}(n) = \{ T \in \text{Cliff}(n) \mid \tau(T)T = \mathbb{I}, TC_1(n)T^{-1} \subset C_1(n) \}.
\] (84)
The group \( \text{Pin}(n) \) naturally splits into two parts,

\[
\text{Pin}(n) = \text{Spin}(n) \cup \mathcal{P}\text{Spin}(n),
\]

(85)
an even part \( \text{Spin}(n) \), called the spin group, consisting of products of even number of elements

\[
\text{Spin}(n) = \{ \mathbb{I}, \pm \gamma(u_1) \cdots \gamma(u_{2k}) \mid u_j \in V, |u_j| = 1, k \in \mathbb{N} \} \subset \text{Cliff}_+(n)
\]

(86)
and the odd part \( \mathcal{P}\text{Spin}(n) \) consisting of products of odd number of elements \( \gamma(u_1) \cdots \gamma(u_{2k+1}) \) (which do not form a group). Here \( \mathcal{P} = \gamma(e) \) with some unit vector \( e \). It is easy to see that the group \( \text{Pin}(n) \) is generated multiplicatively by reflections in all hyperplanes. The spin group \( \text{Spin}(n) \) is the subgroup of the group \( \text{Pin}(n) \) generated by even number of reflections.

Let \( u \in V \) and \( T \in \text{Pin}(n) \). Then there is a vector \( v \in V \) such that

\[
T\gamma(u)T^{-1} = \gamma(v).
\]

(87)
This defines an orthogonal transformation of \( V \)

\[
\tilde{\rho}(T) : u \mapsto v = \tilde{\rho}(T)u.
\]

(88)
We slightly modify this definition by including an additional factor

\[
\tilde{\rho}(T) = (\alpha \circ \rho)(T) = \varepsilon(T)\rho(T),
\]

(89)
where \( \alpha \) is the main involution and \( \varepsilon(T) \) is the parity of the element \( T \). Hence, this modification does not affect the spin group. Clifford algebra \( \text{Cliff}(n) \) carries a natural action of the orthogonal group \( O(n) \) inherited from the tensor algebra. Thus the homomorphism \( \rho \) is defined by

\[
\varepsilon(T)T\gamma(u)T^{-1} = \rho(T)\gamma(u),
\]

(90)
or

\[
T\gamma(u)T^* = \rho(T)\gamma(u).
\]

(91)
In particular,

\[
\varepsilon(T)T\gamma^aT^{-1} = \rho^a_b(T)\gamma^b.
\]

(92)
Hence there is a continuous surjective two-to-one homomorphism

\[ \rho : \text{Pin}(n) \rightarrow O(n), \]  

(93)

defined by

\[ \rho^a{}_b(T) = \varepsilon(T) \text{Pr}_0(T\gamma^a T^{-1} \gamma_b) = \text{Pr}_0(T\gamma^a T^* \gamma_b) \]  

(94)

so that

\[ O(n) = \text{Pin}(n)/\mathbb{Z}_2. \]  

(95)

Similarly,

\[ \rho : \text{Spin}(n) \rightarrow \text{SO}(n), \]  

(96)

is a continuous surjective two-to-one homomorphism and

\[ \text{SO}(n) = \text{Spin}(n)/\mathbb{Z}_2. \]  

(97)

This means that the group PIN(n) is a double covering group of the orthogonal group O(n). The group O(n) is disconnected and has two connected components: the proper subgroup \text{SO}(n) containing the proper orthogonal transformations (with determinant equal to +1), and PSO(n) consisting of orthogonal transformations with determinant equal to (−1). The elements of PSO(n) are products of a proper orthogonal transformation from \text{SO}(n) and a reflection \( P \). Thus,

\[ O(n) = \text{SO}(n) \cup \text{PSO}(n). \]  

(98)

The group \text{SO}(n) is connected but not simply connected. The spin group Spin(n) is a double covering group of the special orthogonal group \text{SO}(n). For \( n = 2 \) the group Spin(2) is connected but not simply connected, whereas for \( n \geq 3 \) the group Spin(n) is simply connected and is the universal covering group of \text{SO}(n).

The eq. (98) shows that the space \( C_2(n) \) is closed under the algebra commutator. Therefore, it forms a Lie algebra with the Lie bracket identified with the Clifford algebra commutator. This Lie algebra is the Lie algebra of the spin group Spin(n). The Lie algebra of the group Pin(n) is, of course, the same. The generators of this Lie algebra are the basis elements \( \gamma_{ab} \), which form a representation of the Lie algebra of the orthogonal group \text{SO}(n). Thus, the Lie algebra of the spin group is isomorphic to the Lie algebra of the orthogonal group \text{SO}(n).
In other words, the spin group Spin(\(n\)) is obtained by exponentiating the Lie algebra of the group SO(\(n\)) inside the Clifford algebra

\[
\text{Spin}(n) = \exp[C_2(n)].
\] (99)

Let \(\theta\) be an element of the Lie algebra of the group SO(\(n\)) represented by an antisymmetric matrix \((\theta_{ab})\). Then \(\theta_{ab}\gamma^{ab} \in C_2(n)\) is an element of the Lie algebra of the spin group Spin(\(n\)) and the double covering homomorphism \(\rho : \text{Spin}(n) \rightarrow \text{SO}(n)\) is given by

\[
\rho \left[ \exp \left( -\frac{1}{4} \theta_{ab} \gamma^{ab} \right) \right] = \exp(\theta).
\] (100)

### 2.3 Spin Representation

Recall that \(N = 2^m\). Let \(S\) be a \(N\)-dimensional complex vector space (which is, of course, isomorphic to \(\mathbb{C}^N\)), \(S^*\) be the dual space of linear functionals \(S \rightarrow \mathbb{C}\) and \(\text{End}(S)\) be the algebra of linear endomorphisms \(S \rightarrow S\) of the vector space \(S\) (which is isomorphic to the vector space \(\text{Mat}(N, \mathbb{C})\) of complex square matrices of order \(N\)). We will call the elements of the vector space \(S\) the Dirac spinors (or complex spinors). Let \(\langle \cdot, \cdot \rangle : S \times S \rightarrow \mathbb{C}\) be an inner product on \(S\). Then the elements of the dual space \(S^*\) are naturally identified with the adjoint vectors by

\[
\bar{\psi}(\varphi) = \langle \psi, \varphi \rangle,
\] (101)

the space of endomorphisms \(\text{End}(S)\) is identified with \(S \otimes S^*\), and the adjoint \(\bar{T}\) of an endomorphism \(T\) is defined with respect to this inner product, that is

\[
\langle \psi, T\varphi \rangle = \langle \bar{T}\psi, \varphi \rangle.
\] (102)

Finally, we denote by \(\text{Aut}(S)\) the group of automorphisms (invertible linear endomorphisms) of the vector space \(S\) (which is isomorphic to the general linear group \(\text{GL}(N, \mathbb{C})\) of complex non-degenerate square matrices of order \(N\)) and by \(U(S)\) the group of unitary endomorphisms (which is isomorphic to \(U(N)\)) that preserve the inner product, that is

\[
\bar{U}U = \mathbb{I}.
\] (103)

Then, in even dimension \(n = 2m\) the complex Clifford algebra \(\text{Cliff}_C(2m)\) is isomorphic to the algebra of endomorphisms \(\text{End}(S)\)

\[
\text{Cliff}_C(2m) = \text{End}(S).
\] (104)
In odd dimension $n = 2m + 1$ the Clifford algebra $\text{Cliff}_C(2m+1)$ is isomorphic to the direct sum of two copies of $\text{End}(S)$

$$\text{Cliff}_C(2m + 1) = \text{End}(S) \oplus \text{End}(S).$$

(105)

Thus in even dimension $n = 2m$ one can identify the elements of the complex Clifford algebra $\text{Cliff}_C(2m)$ with the complex square matrices of order $N$. The unit element is identified with the unit matrix and the elements $\gamma_a$ become then the Dirac matrices.

In odd dimension $n = 2m + 1$ the dimensionality of the representation space should be doubled. That is the elements of the complex Clifford algebra $\text{Cliff}_C(2m + 1)$ if odd dimension $n = 2m + 1$ are identified with the complex block matrices of order $2N$. Of course, now the unit element is the unit matrix of order $2N$. Let $\{\gamma'_a\}$, where $a = 1, \ldots, 2m$, be the Dirac matrices of order $N$ in even dimension $n = 2m$ and $\Gamma'$ be the corresponding chirality operator. Then the elements $\gamma_a$ of the complex Clifford algebra $\text{Cliff}_C(2m+1)$ if odd dimension $n = 2m + 1$ are

$$\gamma_a = \begin{pmatrix} \gamma'_a & 0 \\ 0 & \gamma'_a \end{pmatrix}, \quad \gamma_{2m+1} = \begin{pmatrix} \Gamma' & 0 \\ 0 & -\Gamma' \end{pmatrix}.$$  

(106)

The basis elements in odd dimension are

$$\gamma_{a_1 \ldots a_k} = \begin{pmatrix} \gamma'_{a_1 \ldots a_k} & 0 \\ 0 & \gamma'_{a_1 \ldots a_k} \end{pmatrix}, \quad \gamma_{a_1 \ldots a_k, (2m+1)} = \begin{pmatrix} \gamma'_{a_1 \ldots a_k} \Gamma' & 0 \\ 0 & -\gamma'_{a_1 \ldots a_k} \Gamma' \end{pmatrix},$$

where $1 \leq k \leq 2m$ and the indices $a_j$ run over $1, \ldots, 2m$. The unit matrix and the chirality operator in odd dimension, which determine the center of the Clifford algebra $\text{Cliff}_C(2m + 1)$ in odd dimension, are

$$\mathbb{I} = \begin{pmatrix} \mathbb{I}' & 0 \\ 0 & \mathbb{I}' \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \mathbb{I}' & 0 \\ 0 & -\mathbb{I}' \end{pmatrix}.$$  

(107)

Note that the projection $\text{Pr}_0$ onto the identity element is nothing but the matrix trace normazed so that $\text{Pr}_0(\mathbb{I}) = 1$.

The spin representation of the Clifford algebra $\text{Cliff}(n)$ is a representation with the representation space $S$, that is with complex square matrices of order $N$. Thus in even dimension $n = 2m$ there is only one irreducible faithful representation of the Clifford algebra $\text{Cliff}(2m)$. In odd dimension $n = 2m + 1$ the spin representation of the Clifford algebra $\text{Cliff}_C(2m + 1)$
is obtained by an additional projection onto either the first or the second component of \( \text{End}(S) \oplus \text{End}(S) \). Thus there are two non-equivalent faithful irreducible spin representations of the Clifford algebra \( \text{Cliff}_C(2m+1) \) by complex square matrices of order \( N \), one obtained by the set of matrices \( \{\gamma'_a, \Gamma'\} \) and the other by the set of matrices \( \{\gamma'_a, -\Gamma'\} \).

Since the spin group \( \text{Spin}(n) \) is embedded in the Clifford algebra \( \text{Cliff}_C(n) \), this also defines the spin representation of the spin group. The elements of the Clifford algebra act on the vector space \( S \), and, therefore, \( S \) becomes the Clifford module, that is a module over the Clifford algebra.

In even dimension the chirality operator \( \Gamma \) is a nontrivial involution, which has the eigenvalues \(+1\) and \(-1\). Since it is not in the center of the Clifford algebra, it splits the whole spinor space \( S \) into the eigenspaces \( S_+ \) and \( S_- \) corresponding to these eigenvalues. Thus the spin representation of the spin group \( \text{Spin}(2m) \) in even dimension decomposes into the eigenspaces of the chirality operator, that is

\[
S = S_+ \oplus S_- ,
\]

where the subspaces \( S_\pm \) are defined by

\[
S_\pm = \{ \psi \in S \mid \Gamma \psi = \pm \psi \} .
\]

The spinors from the spaces \( S_+ \) and \( S_- \) are called right and left (or positive and negative) Weyl spinors (or half-spinors) respectively. Of course, the dimension of the subspaces \( S_\pm \) is equal to one half of the dimension of the space \( S \)

\[
\dim S_\pm = \frac{N}{2} .
\]

Also, since \( \Gamma \) anticommutes with \( \gamma_a \), the Clifford multiplication intertwines the chiral subspaces

\[
C_1(n)S_\pm = S_\mp .
\]

In odd dimension the chirality operator is trivial, it is either \( \Gamma = +\mathbb{I} \) or \( \Gamma = -\mathbb{I} \), depending on the spin representation of the Clifford algebra. Therefore, there is only one irreducible spin representation of the spin group \( \text{Spin}(2m+1) \) in odd dimension, i.e. there are no half-spinors in odd dimension.

Finally, in the spinor space \( S \) there exists a Hermitian positive-definite inner product \( \langle \cdot , \cdot \rangle \) such that for any unit vector \( u \in V \) the element \( \gamma(u) \) is self-adjoint and unitary

\[
\bar{\gamma}(u) = \gamma(u) = [\gamma(u)]^{-1} .
\]
In this representation the chirality operator $\Gamma$ is also self-adjoint and unitary
\[ \bar{\Gamma} = \Gamma = \Gamma^{-1}. \] (114)

In even dimension, the chiral subspaces $S_+$ and $S_-$ are orthogonal in this inner product.

### 2.4 Spin Connection

Let $(M, g)$ be a smooth compact orientable $n$-dimensional Riemannian spin manifold without boundary and with a positive-definite Riemannian metric $g$. Let the tangent bundle $TM$ be oriented by choosing a smooth oriented basis. Since $M$ is orientable the transition functions are matrices from $SO(n)$. Let $SO(M)$ be the frame bundle, i.e. the principal fiber bundle of oriented orthonormal frames with the structure group $SO(n)$. The typical fiber of the frame bundle $SO(M)$ is $SO(n)$. The spin group $Spin(n)$ is a double cover of the group $SO(n)$ (for $n \geq 3$ it is the universal cover and, thus, simply connected). A spin structure on $M$ is a principal bundle $Spin(M)$ with the structure group $Spin(n)$ together with a double covering homomorphism $Spin(M) \to SO(M)$ which preserves the group action. The necessary and sufficient conditions for a manifold to have a spin structure are the vanishing of the first two Stiefel-Whitney classes of the manifold $M$. There can be several possible inequivalent spin structures, which are parametrized by representations of the fundamental group $\pi_1(M)$. For simply connected manifolds the spin structure is unique.

The spinor bundle $S$ is the associated vector bundle with the structure group $Spin(n)$ whose typical fiber is the spinor space $S$. Spinor fields are sections of this vector bundle. We denote by $C^\infty(S)$ the space of smooth sections of the spinor bundle. Using the Hermitian inner product $\langle \cdot, \cdot \rangle$ on the spinor space $S$ and the invariant Riemannian volume element $d\text{vol}$ on $M$ one defines the natural $L^2$-inner product $\langle \cdot, \cdot \rangle$ in $C^\infty(S)$ and the Hilbert space of square integrable sections $L^2(S)$ as the completion of $C^\infty(S)$ in this norm.

To define the Dirac operator on a Riemannian manifold $M$ we need a connection (covariant derivative) on the spinor bundle $S$

\[ \nabla^S : C^\infty(S) \to C^\infty(T^*M \otimes S), \] (115)

which we assume to be compatible with the Hermitian inner product on the spinor bundle $S$. This connection is naturally extended to bundles in the ten-
sor algebra over $S$ and $S^*$. Any Riemannian manifold has a unique symmetric connection $\nabla^{TM}$ compatible with the metric, the Levi-Civita connection. In fact, using the Levi-Civita connection together with $\nabla^S$, we naturally obtain connections on bundles in the tensor algebra over $S, S^*, TM, T^*M$; the resulting connection will be denoted just by $\nabla$. It will usually be clear which bundle’s connection is being referred to, from the nature of the section being acted upon.

All the homomorphism and involutions of the Clifford algebra are naturally extended to bundle maps, in particular,

$$\gamma : T^*M \to S.$$  \hspace{1cm} (116)

Since the principal bundle Spin$(M)$ is a double cover of the orthonormal frame bundle $SO(M)$, it inherits the Levi-Civita connection. The exact form of this correspondence is obtained from the differential of the homomorphism $\rho$:

$$\rho : \text{Spin}(M) \to SO(M).$$  \hspace{1cm} (117)

Since for any two spinors $\psi, \varphi \in C^\infty(S)$, $\langle \psi, \gamma_{a_1\ldots a_k}\varphi \rangle \in C^\infty(\wedge^k T^*M)$ transforms like a tensor (in fact, like a $k$-form), then the spin connection can be defined by requiring it to satisfy the Leibnitz rule

$$\nabla_b\langle \psi, \gamma_{a_1\ldots a_k}\varphi \rangle = \langle \nabla_b\psi, \gamma_{a_1\ldots a_k}\varphi \rangle + \langle \psi, \gamma_{a_1\ldots a_k}\nabla_b\varphi \rangle.$$  \hspace{1cm} (118)

We label the local coordinates $x^\mu$ on the manifold $M$ by Greek indices which run over $1, \ldots, n$. Let $\partial_\mu$ be a coordinate basis for the tangent space $T_xM$ at a point $x \in M$ and let $\gamma_\mu = \gamma(\partial_\mu)$. Then

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}I,$$  \hspace{1cm} (119)

where $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ is the Riemannian metric. Let

$$e_a = e_a^\mu \partial_\mu$$  \hspace{1cm} (120)

be an orthonormal basis for the tangent space $T_xM$. Let $e_a^\mu$ be the matrix inverse to $e_a^\mu$, defining the dual basis

$$\omega^a = e_a^\mu dx^\mu$$  \hspace{1cm} (121)

in the cotangent space $T_x^*M$. Then

$$g_{\mu\nu}e_a^\mu e_b^\nu = \delta_{ab}, \quad g^{\mu\nu}e_a^\mu e_b^\nu = \delta^{ab}.$$  \hspace{1cm} (122)
and the matrices $\gamma_\mu$ are related to the constant matrices $\gamma_a$ forming a representation of the Clifford algebra by

$$
\gamma_a = e_\mu^a \gamma_\mu, \quad \gamma_\mu = e^a_\mu \gamma_a.
$$

Similarly, we define

$$
\gamma_{\mu_1 \cdots \mu_k} = \gamma_{a_1 \cdots a_k} e_{\mu_1}^{a_1} \cdots e_{\mu_k}^{a_k}.
$$

Thus, in local coordinates one obtains for the spin connection

$$
\nabla_\mu \psi = \left( \partial_\mu + \frac{1}{4} \gamma_{ab} \omega^{ab}_\mu \right) \psi,
$$

where $\omega^{ab}_\mu$ is the spin connection one-form defined by

$$
\omega^{ab}_\mu = e^a_{[\mu} e^b_{\nu]} - e^b_{[\mu} e^a_{\nu]} + e_{ca} e^a_{[\nu} e^b_{\lambda]} \gamma^c_{\mu]}.
$$

This is nothing but the Fock-Ivanenko coefficients [1, 2].

We will generalize the above setup as follows. Let $G$ be a compact semi-simple Lie group and $G$ be the principal fiber bundle over the manifold $M$ with the structure group $G$. Let $W$ be the associated vector bundle with the structure group $G$ whose typical fiber is a vector space $W$. Then the vector bundle $W \otimes S$ is a twisted spinor bundle. The sections of the twisted spinor bundle are represented locally by $k$-tuples of spinors, where $k = \dim W$ is the dimension of the vector space $W$. For a twisted spinor bundle $W \otimes S$ the covariant derivative is defined by

$$
\nabla^\nu_\mu \psi = \left( \partial_\mu + \frac{1}{4} \gamma_{ab} \omega^{ab}_\mu + A_\mu \right) \psi,
$$

where $A_\mu$ is the connection 1-form on the vector bundle $W$ taking values in the Lie algebra of the gauge group $G$. In the following, we redefine the definition of the spinor bundle. We will denote the twisted spinor bundle $W \otimes S$ by $S$ and call it just the spinor bundle. The meaning of the bundle (twisted or not) is usually clear from the context. Note that the dimension of the fiber of the twisted spinor bundle is $2^m \cdot \dim W$. So, when dealing with the twisted spinor bundle we will redefine the definition of the number $N$. It will always mean the dimension of the fiber of the spinor bundle, whether twisted or not.
The curvature of the spin connection is described by the commutator of the covariant derivatives
\[ [\nabla_\mu, \nabla_\nu] \psi = \mathcal{R}_{\mu\nu} \psi, \quad (128) \]
where
\[ \mathcal{R}_{\mu\nu} = \frac{1}{4} \gamma^{\alpha\beta} R_{\alpha\beta\mu\nu} + \mathcal{F}_{\mu\nu}, \quad (129) \]
\( R_{\alpha\beta\mu\nu} \) is the Riemann curvature of the metric \( g \) and
\[ \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (130) \]

### 2.5 Dirac Operator

The Dirac operator is a first order partial differential operator acting on smooth sections of the spinor bundle
\[ D : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S) \quad (131) \]
defined by the composition of the covariant derivative with the Clifford multiplication
\[ D = i \gamma \nabla = i \gamma^\mu \nabla_\mu. \quad (132) \]
The leading symbol of the Dirac operator is
\[ \sigma_L(D; x, \xi) = -\gamma^\mu(x) \xi_\mu, \quad (133) \]
where \( \xi \in T^*_x M \) is a covector at a point \( x \in M \). Since it is self-adjoint and non-degenerate for any \( \xi \neq 0, x \in M \), the Dirac operator \( D \) is elliptic.

One can also easily check that the Dirac operator is symmetric, (or formally self-adjoint), that is, for any two smooth spinor fields \( \psi, \varphi \in \mathcal{C}^\infty(S) \)
\[ (D \psi, \varphi) = (\psi, D \varphi). \quad (134) \]

The Laplacian is a second order partial differential operator acting on smooth sections of the spinor bundle
\[ \Delta : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S) \quad (135) \]
defined by
\[ \Delta = -\nabla \nabla = g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad (136) \]
where
\[ \nabla : C^\infty(T^* M \otimes S) \to C^\infty(S) \] (137)
is the formal adjoint of the covariant derivative operator with respect to the \(L^2\) inner product on the spinor bundle \(S\).

The square of the Dirac operator is
\[ D^2 = -\Delta - \frac{1}{2} \gamma^{\mu\nu} R_{\mu\nu}. \] (138)

By using the curvature of the spin connection (129), the eq. (79) and the Bianci identity we obtain the Lichnerowicz formula
\[ D^2 = -\Delta + \frac{1}{4} R I - \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}, \] (139)
where \(R\) is the scalar curvature.

The leading symbol of the operator \(D^2\)
\[ \sigma_L(D^2; x, \xi) = g^{\mu\nu}(x) \xi_\mu \xi_\nu I \] (140)
is, of course, elliptic, self-adjoint, scalar and positive-definite.

The Dirac operator \(D\) is a formally self-adjoint elliptic operator acting on smooth sections of spinor bundle over a compact manifold without boundary. One can show that \(D\) is essentially self-adjoint, that is, its closure is self-adjoint and, hence, it has a unique self-adjoint extension to \(L^2(S)\). The same is true for its square \(D^2\). It is well known that the operator \(D\) has a discrete real spectrum \((\lambda_n)_{n=1}^\infty\), which can be ordered according to
\[ 0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_n^2 \leq \cdots. \] (141)
Moreover, each eigenspace is finite-dimensional and the eigenspinors \((\varphi_n)_{n=1}^\infty \in C^\infty(S)\) are smooth sections of the spinor bundle that form an orthonormal basis in \(L^2(S)\).

Let the dimension of the manifold \(n = 2m\) be even. Then the spinor bundle \(S\) has a \(\mathbb{Z}_2\) grading
\[ S = S_+ \oplus S_-, \] (142)
where \(S_\pm\) are the subbundles of the right (left) Weyl spinors. It is easy to see that the Dirac operator anticommutes with the chirality operator
\[ \Gamma D = -D \Gamma \] (143)
and, therefore, interchanges the parity of the spinors, that is, in fact,

\[ D : C^\infty(S_\pm) \to C^\infty(S_\mp). \]  

(144)

In other words, the Dirac operator has odd parity, and, therefore, its square \( D^2 \) is an even operator

\[ D^2 : C^\infty(S_\pm) \to C^\infty(S_\mp). \]  

(145)

Let

\[ P_\pm = \frac{1}{2}(I \pm \Gamma) \]  

(146)

be the projections onto the subbundles \( S_\pm \) and

\[ D_\pm = P_\mp DP_\pm. \]  

(147)

Then

\[ D_\pm^2 = 0, \quad D_+ = D_-. \]  

(148)

and

\[ D = D_+ + D_-, \quad D^2 = D_- D_+ + D_+ D_- . \]  

(149)

The operators

\[ D_\mp D_\pm : C^\infty(S_\pm) \to C^\infty(S_\pm) \]  

(150)

are second-order self-adjoint non-negative differential operators of even parity

\[ D_\mp D_\pm = D_\pm D_\mp . \]  

(151)

Thus, \( D^2 \) acts in the chiral subbundles of the spinor bundle \( S_\pm \). One can easily show that for all non-zero eigenvalues there is an isomorphism between the right and left eigenspaces. In particular, their dimensions, that is the multiplicities \( d_\pm \) of the right and left eigenspinors corresponding to the same non-zero eigenvalue \( \lambda^2 \), are equal. This clearly does not work for the zero eigenvalues; so there could be any number of right or left eigenspinors corresponding to zero eigenvalue.

Let

\[ \text{Ker} (D) = \{ \psi \in C^\infty(S) \mid D\psi = 0 \} \]  

(152)

be the kernel of the operator \( D \), that is the vector space of its zero eigenspinors. Then

\[ \text{Ker} (D_\pm) = \text{Ker} (D) \cap C^\infty(S_\pm) = \{ \psi \in C^\infty(S_\pm) \mid D\psi = 0 \} \]  

(153)
are invariant subspaces of the right and left zero eigenspinors of the Dirac operator and
\[ \text{Ker}(D) = \text{Ker}(D_+) \oplus \text{Ker}(D_-). \]  \hspace{1cm} (154)

The index of the Dirac operator is a topological invariant of the manifold \( M \) and the spinor bundle defined by
\[ \text{Ind}(D) = \dim \text{Ker}(D_+) - \dim \text{Ker}(D_-). \]  \hspace{1cm} (155)

2.6 Heat Kernel

Thus, \( D^2 \) is a self-adjoint elliptic second-order partial differential operator with a positive definite scalar leading symbol acting on sections of spinor bundle over a compact manifold without boundary. Such operators are called Laplace type operators. For \( t > 0 \) the heat semigroup
\[ \exp(-tD^2) : L^2(S) \rightarrow L^2(S) \]  \hspace{1cm} (156)
is a bounded operator (in fact, it is a smoothing operator \( L^2(S) \rightarrow C^\infty(S) \)). The integral kernel of this operator, called the heat kernel, is
\[ U(t; x, x') = \sum_{n=1}^{\infty} e^{-t\lambda_n^2} \varphi_n \otimes \bar{\varphi}_n(x'), \]  \hspace{1cm} (157)
where each eigenvalue is counted with its multiplicity. The heat kernel satisfies the heat equation
\[ (\partial_t + D^2)U(t; x, x') = 0 \]  \hspace{1cm} (158)
with the initial condition
\[ U(0^+; x, x') = \delta(x, x'), \]  \hspace{1cm} (159)
where \( \delta(x, x') \) is the Dirac distribution.

For \( t > 0 \) the heat kernel \( U(t; x, x') \) is a smooth function near the diagonal of \( M \times M \) and has a well defined diagonal value \( U(t; x, x) \). Moreover, the heat semigroup is a trace-class operator with a well defined \( L^2 \)-trace
\[ \text{Tr}_{L^2} \exp(-tD^2) = \int_M \text{dvol}(x) \text{tr}_S U(t; x, x), \]  \hspace{1cm} (160)
where $\text{tr}_S$ is the trace in the spinor space. The trace of the heat kernel is a spectral invariant of the Dirac operator since

$$\text{Tr}_{L^2} \exp(-tD^2) = \sum_{n=1}^{\infty} e^{-t\lambda_n^2}. \quad (161)$$

Similarly, let $F \in C^\infty(\text{End}(S))$ be a smooth section of the endomorphism bundle of the spinor bundle. We can define the trace

$$\text{Tr}_{L^2} [F \exp(-tD^2)] = \int_M d\text{vol}(x) \text{tr}_S [F(x)U(t; x, x)]. \quad (162)$$

Note, however, that, in general, this is not a spectral invariant.

In a particular case, when the dimension of the manifold is even and $F$ is the chirality operator, $F = \Gamma$, we obtain

$$\text{Tr}_{L^2} [\Gamma \exp(-tD^2)] = \text{Tr}_{L^2} \exp(-tD_-D_+) - \text{Tr}_{L^2} \exp(-tD_+D_-). \quad (163)$$

Since the nonzero spectra of the operators $D_-D_+$ and $D_+D_-$ are isomorphic, we obtain

$$\text{Tr}_{L^2} (\Gamma \exp(-tD^2)) = \text{Ind}(D). \quad (164)$$

That is, this trace does not depend on $t$ and is a topological invariant equal to the index of the operator $D$.

One can show that there is an asymptotic expansion of the diagonal of the heat kernel as $t \to 0$ [13] (for a review, see also [14, 15, 16, 17])

$$U(t; x, x) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} a_k(D^2; x), \quad (165)$$

and the corresponding expansion of the trace of the heat kernel

$$\text{Tr}_{L^2} [F \exp(-tD^2)] \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} A_k(F, D^2). \quad (166)$$

The coefficients $A_k(F, D^2)$, called the global heat invariants, are invariants determined by the integrals over the manifold

$$A_k(F, D^2) = \int_M d\text{vol}(x) \text{tr}_S F a_k(D^2; x), \quad (167)$$
of local heat invariants $a_k(D^2; x)$ constructed polynomially from the jets of the symbol of the Dirac operator $D$, so that they are polynomial in curvatures and their covariant derivates.

In the particular case $F = \mathbb{I}$ the heat invariants $A_k(\mathbb{I}, D^2)$ are spectral invariants of the Dirac operator, and in the case $F = \Gamma$ (in even dimension $n$) all heat invariants $A_k(\Gamma, D^2)$ vanish except for one that determines the index of the Dirac operator, that is

$$A_k(\Gamma, D^2) = 0, \quad k \neq \frac{n}{2},$$  \hspace{1cm} (168)

$$A_{n/2}(\Gamma, D^2) = (4\pi)^{n/2}\text{Ind}(D) .$$ \hspace{1cm} (169)

The first two spectral invariants are given by

$$A_0(\mathbb{I}, D^2) = N \int_M \text{dvol} \, 1 ,$$ \hspace{1cm} (170)

$$A_1(\mathbb{I}, D^2) = -\frac{1}{12} N \int_M \text{dvol} \, R .$$ \hspace{1cm} (171)

where $N = \text{dim} \, S$. 

### 3 Dirac Operators in Matrix Geometry

In this section we closely follow our papers [9, 10, 11].

#### 3.1 Non-commutative Metric and Dirac Matrices

Now, let $S$ be a $N$-dimensional complex vector space with a positive definite Hermitean inner product $\langle \, , \rangle$, $S^*$ be its dual vector space and $\text{End}(S)$ be the space of linear endomorphisms of the vector space $S$. The vector space $S$ is isomorphic to $\mathbb{C}^N$ and $\text{End}(S)$ be is isomorphic to the vector space $\text{Mat}(N, \mathbb{C})$ of complex square matrices of order $N$. The group of automorphisms $\text{Aut}(S)$ of the vector space $S$ is isomorphic to the general linear group $GL(N, \mathbb{C})$ of complex square nondegenerate matrices of order $N$ and the group of unitary endomorphisms $G(S)$ is isomorphic to $SU(N)$; the dimension of the group $G$ is $\dim G = N^2$. The group $G$ acts on vectors and covectors by left and right
action
\[ \varphi' = U \varphi, \quad \varphi \in S \tag{172} \]
\[ \bar{\varphi}' = \bar{\varphi} U^{-1}, \quad \bar{\varphi} \in S^*. \tag{173} \]

Now, let \( M \) be a smooth compact orientable \( n \)-dimensional manifold without boundary and
\[ V = T_x M \quad \text{and} \quad V^* = T^*_x M \tag{174} \]
be the tangent and contangent spaces at a point \( x \) in \( M \). We introduce the following notation for the vector spaces of vector-valued and endomorphism-valued tensors
\[ \Lambda_p = \wedge^p V^* \otimes S, \quad \Lambda^p = \wedge^p V \otimes S. \tag{175} \]
\[ E_p = \wedge^p V^* \otimes \text{End}(S), \quad E^p = \wedge^p V \otimes \text{End}(S). \tag{176} \]
Suppose we are given a map
\[ \Gamma : V^* \to \text{End}(S) \tag{177} \]
determined by a self-adjoint endomorphism-valued vector \( \Gamma \in V \otimes \text{End}(S) \) given locally by the matrix-valued vector \( \Gamma^\mu \). Let us define an endomorphism-valued tensor \( a \in V \otimes V \otimes \text{End}(S) \) by
\[ a_{\mu\nu} = \frac{1}{2} (\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu). \tag{178} \]
Then \( a^{\mu\nu} \) is self-adjoint and symmetric
\[ a^{\mu\nu} = a^{\nu\mu} \quad \overline{a^{\mu\nu}} = a^{\mu\nu}. \tag{179} \]

One of our main assumptions about the matrix \( a \) is that it defines an isomorphism between the spaces \( \Lambda_1 = V^* \otimes S \) and \( \Lambda^1 = V \otimes S \), i.e.
\[ a : \Lambda_1 \to \Lambda^1. \tag{180} \]
Let us consider the matrix
\[ H(\xi) = a^{\mu\nu} \xi_\mu \xi_\nu = [\Gamma(\xi)]^2, \tag{181} \]
with \( \xi \in T^*_x M \) being a cotangent vector and \( \Gamma(\xi) = \Gamma^\mu \xi_\mu \). Our second assumption is that this matrix is positive definite, i.e.
\[ H(\xi) > 0 \quad \text{for any } \xi \neq 0. \tag{182} \]
Thus, all eigenvalues of this matrix are real and positive for $\xi \neq 0$. We will call the matrix $a^{\mu\nu}$ the non-commutative metric and the matrices $\Gamma^\mu$ the non-commutative Dirac matrices.

We will also need a self-adjoint non-degenerate endomorphism $\rho \in \text{End}(S)$ (given locally by a matrix-valued function). In the case when $S$ is a spinor space described in section 2 there is a very simple particular solution

$$
\Gamma^\mu = \gamma^\mu, \quad a^{\mu\nu} = g^{\mu\nu}I, \quad \rho = g^{1/4}I,
$$

(183)

where $\gamma^\mu$ are Dirac matrices in a Riemannian manifold with a Riemannian metric $g^{\mu\nu}$ and

$$
g = |\det g^{\mu\nu}|^{-1}.
$$

(184)

These matrices satisfy all the above conditions. We will refer to this particular case as the commutative limit. In general, we represent these objects as a deformation of the commutative limit

$$
\Gamma^\mu = \gamma^\mu + \varkappa \alpha^\mu, \quad a^{\mu\nu} = g^{\mu\nu}I + \varkappa h^{\mu\nu}, \quad \rho = g^{1/4} \exp(\varkappa \phi),
$$

(185)

where $\varkappa$ is a deformation parameter, $\alpha^\mu$ and $\phi$ are some matrices and and

$$
h^{\mu\nu} = (\alpha^\mu \gamma^\nu + \gamma^\mu \alpha^\nu) + (\alpha^\nu \gamma^\mu + \gamma^\mu \alpha^\nu) + \varkappa (\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu).
$$

(186)

Our construction should make sense in the limit $\varkappa \to 0$ as a power series in the deformation parameter.

Since the map $a$ is an isomorphism, the inverse map

$$
b = a^{-1} : \Lambda^1 \to \Lambda^1,
$$

(187)

is well defined. In other words, for any $\psi \in \Lambda^1$ there is a unique $\varphi^\nu \in \Lambda^1$ satisfying the equation $a^{\mu\nu} \varphi^\nu = \psi^\mu$, and, therefore, there is a unique solution of the equations

$$
a^{\mu\nu} b^\nu_\alpha = \delta^\mu_\alpha, \quad b^\alpha_\nu a^{\nu\mu} = \delta^\mu_\alpha.
$$

(188)

Notice that the matrix $b^{\mu\nu}$ has the property

$$
\bar{b}^{\mu\nu} = b^{\nu\mu},
$$

(189)

but is neither symmetric $b^{\mu\nu} \neq b^{\nu\mu}$ nor self-adjoint $\bar{b}^{\mu\nu} \neq b^{\mu\nu}$.

The isomorphism $a$ naturally defines the maps

$$
A : \Lambda_p \to \Lambda^p, \quad B : \Lambda^p \to \Lambda_p,
$$

(190)
as follows

\[(A\varphi)_{\mu_1 \cdots \mu_p} = A^{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} \varphi_{\nu_1 \cdots \nu_p}, \quad (191)\]

where

\[A^{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} = \text{Alt}_{\mu_1 \cdots \mu_p} \text{Alt}_{\nu_1 \cdots \nu_p} a^{\mu_1 \nu_1 \cdots a^{\mu_p \nu_p}} \quad (192)\]

and

\[(B\varphi)_{\mu_1 \cdots \mu_p} = B_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} \varphi^{\nu_1 \cdots \nu_p}, \quad (193)\]

where

\[B_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} = \text{Alt}_{\mu_1 \cdots \mu_p} \text{Alt}_{\nu_1 \cdots \nu_p} b_{\mu_1 \nu_1 \cdots b_{\mu_p \nu_p}} \quad (194)\]

Here \(\text{Alt}_{\mu_1 \cdots \mu_p}\) denotes the complete antisymmetrization over the indices \(\mu_1, \ldots, \mu_p\).

We will assume that these maps are isomorphisms as well. Strictly speaking, one has to prove this. This is certainly true for the weakly deformed maps (maps close to the identity). Then the inverse operator

\[A^{-1} : \Lambda^p \to \Lambda_p, \quad (195)\]

is defined by

\[(A^{-1}\varphi)_{\mu_1 \cdots \mu_p} = (A^{-1})_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} \varphi^{\nu_1 \cdots \nu_p}, \quad (196)\]

where \(A^{-1}\) is determined by the equation

\[(A^{-1})_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} A^{\nu_1 \cdots \nu_p \alpha_1 \cdots \alpha_p} = \delta^{\alpha_1}_{\mu_1} \cdots \delta^{\alpha_p}_{\mu_p}. \quad (197)\]

Notice that because of the noncommutativity, the inverse operator \(A^{-1}\) is not equal to the operator \(B\), so that \(A^{-1}B \neq \text{Id}\).

This is used further to define the natural inner product on the space of \(p\)-forms \(\Lambda_p\) via

\[\langle \psi, \varphi \rangle = \frac{1}{p!} \bar{\psi}_{\mu_1 \cdots \mu_p} A^{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} \varphi_{\nu_1 \cdots \nu_p}. \quad (198)\]

### 3.2 Non-commutative Star Operators

Of course, (on orientable manifolds) we always have the standard volume form \(\varepsilon\), which is a tensor from \(E_n\) given by the completely antisymmetric Levi-Civita symbol \(\varepsilon_{\mu_1 \cdots \mu_n}\). The contravariant Levi-Civita symbol \(\tilde{\varepsilon}\) with components

\[\varepsilon^{\mu_1 \cdots \mu_n} = \varepsilon_{\mu_1 \cdots \mu_n}, \quad (199)\]
is a tensor from $E^n$. These forms are used to define the standard isomorphisms

$$
\varepsilon : \Lambda^p \to \Lambda_{n-p}, \quad \tilde{\varepsilon} : \Lambda_p \to \Lambda^{n-p}
$$

(200)

by

$$
(\varepsilon \varphi)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \varphi^{\mu_1 \cdots \nu_p}, \quad (\tilde{\varepsilon} \varphi)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \tilde{\varepsilon}_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \varphi_{\nu_1 \cdots \nu_p}.
$$

(201)

By using the well known identity

$$
\varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \varepsilon_{\mu_1 \cdots \mu_{n-p} \lambda_1 \cdots \lambda_p} = (n-p)! \frac{\delta_{\lambda_1}}{\nu_1} \cdots \frac{\delta_{\lambda_p}}{\nu_p}
$$

(202)

we get

$$
\tilde{\varepsilon} \varepsilon = \varepsilon \tilde{\varepsilon} = (-1)^{p(n-p)} \text{Id}.
$$

(203)

By combining $\varepsilon$ and $\tilde{\varepsilon}$ with the endomorphism $\rho$ we get the forms $\varepsilon \rho^2 \in E_n$ and $\tilde{\varepsilon} \rho^{-2} \in E^n$. Notice, however, that, in general, the contravariant form $\tilde{\varepsilon} \rho^{-2}$ is not equal to that obtained by raising indices of the covariant form $\varepsilon \rho^2$, i.e. $\tilde{\varepsilon} \rho^{-2} \neq A \varepsilon \rho^2$ or

$$
\varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \varepsilon_{\mu_1 \cdots \mu_{n-p} \lambda_1 \cdots \lambda_p} = (n-p)! \frac{\delta_{\lambda_1}}{\nu_1} \cdots \frac{\delta_{\lambda_p}}{\nu_p} A_{\mu_1 \nu_1} \cdots A_{\mu_n \nu_n} \varepsilon_{\nu_1 \cdots \nu_n} \rho^2.
$$

(204)

If we require this to be the case then the matrix $\rho$ should be defined by

$$
\rho = \eta^{-1/4},
$$

(205)

where

$$
\eta = \frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} \varepsilon_{\nu_1 \cdots \nu_n} a_{\mu_1 \nu_1} \cdots a_{\mu_n \nu_n}.
$$

(206)

Since $a^{\mu \nu}$ is self-adjoint, we also find that $\eta$ and, hence, $\rho$ is self-adjoint. The problem is that in general $\eta$ is not positive definite. Notice that in the commutative limit

$$
\eta = \det g^{\mu \nu} = (\det g_{\mu \nu})^{-1},
$$

(207)

which is strictly positive.

Therefore, we can finally define two different star operators

$$
*, \tilde{*} : \Lambda_p \to \Lambda_{n-p}
$$

(208)

by

$$
* = \varepsilon \rho A \rho, \quad \tilde{*} = \rho^{-1} A^{-1} \rho^{-1} \tilde{\varepsilon}
$$

(209)
that is
\[ (\ast \phi)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \rho A^{\nu_1 \cdots \nu_p \alpha_1 \cdots \alpha_p} \rho \varphi_{\alpha_1 \cdots \alpha_p}, \] (210)
\[ (\tilde{\ast} \phi)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \rho^{-1} (A^{-1})_{\mu_1 \cdots \mu_{n-p} \beta_1 \cdots \beta_{n-p}} \rho^{-1} \varepsilon_{\beta_1 \cdots \beta_{n-p} \alpha_1 \cdots \alpha_p} \varphi_{\alpha_1 \cdots \alpha_p}, \] (211)

The star operators are self-adjoint in the sense
\[ \langle \varphi, \ast \psi \rangle = \langle \ast \varphi, \psi \rangle, \quad \langle \varphi, \tilde{\ast} \psi \rangle = \langle \tilde{\ast} \varphi, \psi \rangle, \] (212)
and satisfy the relation: for any \( p \) form
\[ \ast \tilde{\ast} = \tilde{\ast} \ast = (-1)^{p(n-p)} \text{Id}. \] (213)

### 3.3 Finsler geometry

The above construction is closely related to Finsler geometry [18]. Let \( h(\xi) \) be an eigenvalue of the matrix \( H(\xi) = a^{\mu \nu} \xi_\mu \xi_\nu \). First of all, we note that \( h(\xi) \) is a homogeneous function of \( \xi \) of degree 2, i.e. for any \( \lambda > 0 \)
\[ h(\lambda \xi) = \lambda^2 h(\xi). \] (214)

Next, for each eigenvalue \( h(\xi) \) we define the Finsler metric
\[ g^{\mu \nu}(\xi) = \frac{1}{2} \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} h(\xi). \] (215)

All these metrics are positive definite. In the case when a Finsler metric does not depend on \( \xi \) it is simply a Riemannian metric. The Finsler metrics are homogeneous functions of \( \xi \) of degree 0
\[ g^{\mu \nu}(\lambda \xi) = g^{\mu \nu}(\xi), \] (216)
so that they depend only on the direction of the covector \( \xi \) but not on its magnitude. This leads to a number of identities, in particular,
\[ h(\xi) = g^{\mu \nu}(\xi) \xi_\mu \xi_\nu \] (217)
and
\[ \frac{\partial}{\partial \xi_\mu} h(\xi) = 2 g^{\mu \nu}(\xi) \xi_\nu. \] (218)
Next, again for each eigenvalue we define the tangent vector $u \in T_x M$ by
\[ u^\mu(\xi) = g^{\mu\nu}(\xi)\xi_\nu, \] (219)
and the inverse (covariant) Finsler metric by
\[ g_{\mu\nu}(u(\xi))g^{\nu\alpha}(\xi) = \delta^\alpha_\mu, \] (220)
so that
\[ \xi_\mu = g_{\mu\nu}(u(\xi))u^\nu(\xi). \] (221)

The existence of Finsler metrics allows one to define various connections, curvatures etc (for details see [18]).

### 3.4 Vector Bundles

Now, we assume that the manifold $M$ admits the promotion of all vector spaces introduced locally above to smooth vector bundles over the manifold $M$. We use script letters to distinguish the vector bundles from the vector spaces. Moreover, we can slightly generalize the setup and introduce vector bundles of densities of weight $w$ over the manifold $M$. For each bundle we indicate the weight explicitly in the notation of the vector bundle. For example, $S[w]$ is a vector bundle of densities of weight $w$ with the typical fiber $S$. The sections $\varphi$ of the vector bundle $S[w]$ are vector-valued functions $\varphi(x)$ that transform under diffeomorphisms $x^\mu = x'^\mu(x)$ according to
\[ \varphi'(x') = J^{-w}(x)\varphi(x), \] (222)
where
\[ J(x) = \det \left[ \frac{\partial x'^\mu(x)}{\partial x^\alpha} \right]. \] (223)

We will consider mostly the bundles of densities of weight $\frac{1}{2}$, $S[\frac{1}{2}]$, and, more generally, $\Lambda_p[\frac{1}{2}]$. If $dx = dx^1 \wedge \cdots \wedge dx^n$ is the standard Lebesgue measure in a local chart on $M$, then we define the diffeomorphism-invariant $L^2$-inner product
\[ (\psi, \varphi) = \int_M dx \langle \psi(x), \varphi(x) \rangle, \] (224)
and the $L^2$ norm
\[ ||\varphi||^2 = (\varphi, \varphi) = \int_M dx \langle \varphi(x), \varphi(x) \rangle. \] (225)
The completion of $C^\infty(\Lambda_p[\frac{1}{2}])$ in this norm defines the Hilbert space $L^2(\Lambda_p[\frac{1}{2}])$.

To avoid misunderstanding we stress here the weights of the objects introduced above. The matrices $\Gamma^\mu$ and $a^{\mu\nu}$ have weight 0 and the matrix $\rho$ is assumed to be a density of weight $\frac{1}{2}$. The square of this matrix, $\rho^2$, has weight 1 and plays the role of a “non-commutative measure”.

The operators $\varepsilon$ and $\tilde{\varepsilon}$ introduced above change the weight by 1. The operator $\varepsilon$ raises the weight by 1, and the operator $\tilde{\varepsilon}$ lowers the weight by 1. More precisely, for any $w$

$$\varepsilon : \Lambda^p[w] \rightarrow \Lambda_{n-p}[w-1]$$

$$\tilde{\varepsilon} : \Lambda^p[w] \rightarrow \Lambda^{n-p}[w+1]$$

The star operators $*$ and $\tilde{*}$ do not change the weights, however,

$$*, \tilde{*} : \Lambda_p[w] \rightarrow \Lambda_p[w].$$

This is precisely the reason for the introduction of the matrix $\rho$, which is a density of weight $\frac{1}{2}$.

Our goal is to construct covariant self-adjoint first-order and second-order differential operators acting on smooth sections of the bundles $\Lambda_p[\frac{1}{2}]$ and $\Lambda^{p+1}[\frac{1}{2}]$, that are covariant under both diffeomorphisms,

$$L'\varphi'(x') = J^{-1/2}(x)L\varphi(x),$$

and the gauge transformations

$$L'\varphi' = UL\varphi.$$}

3.5 Non-commutative Exterior Calculus

Next, we define invariant differential operators on smooth sections of the bundles $\Lambda_p[0]$ and $\Lambda_p[1]$. The exterior derivative (the gradient) on tensors

$$d : C^\infty(\Lambda_p[0]) \rightarrow C^\infty(\Lambda_{p+1}[0])$$

is defined by

$$(d\varphi)_{\mu_1\ldots\mu_{p+1}} = (p + 1)\partial_{[\mu_1}\varphi_{\mu_2\ldots\mu_p]}, \quad \text{if } p = 0, \ldots, n - 1,$$

$$d\varphi = 0 \quad \text{if } p = n.$$
where the square brackets denote the complete antisymmetrization. The coderivative (the divergence) on densities of weight 1
\[ \tilde{d} : C^\infty(\Lambda^p[1]) \to C^\infty(\Lambda^{p-1}[1]) \] (234)
is defined by
\[ \tilde{d} = (-1)^{np+1} \tilde{\varepsilon} d \varepsilon. \] (235)
By using (202) one can easily find
\[ (\tilde{d}\varphi)^{\mu_1\cdots\mu_{p-1}} = \partial_\mu \varphi^{\mu\mu_1\cdots\mu_{p-1}} \] if \( p = 1, \ldots, n \),
(236)
\[ \tilde{d}\varphi = 0 \] if \( p = 0 \). (237)
One can also show that these definitions are covariant and satisfy the standard relations
\[ d^2 = \tilde{d}^2 = 0. \] (238)
Recall that the endomorphism \( \rho \) is a section of the bundle \( \text{End}(S)[\frac{1}{2}] \). Therefore, if \( \varphi \) is a section of the bundle \( \Lambda_p[\frac{1}{2}] \), the quantity \( \rho^{-1}\varphi \) is a section of the bundle \( \Lambda_p[0] \). Hence, the derivative \( d(\rho^{-1}\varphi) \) is well defined as a smooth section of the vector bundle \( \Lambda_{p+1}[0] \). By scaling back with the factor \( \rho \) we get an invariant differential operator on densities of weight \( \frac{1}{2} \)
\[ \rho d \rho^{-1} : C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda_{p+1} \left[ \frac{1}{2} \right] \right). \] (239)
Similarly, we can define the invariant operator of codifferentiation on densities of weight \( \frac{1}{2} \)
\[ \rho^{-1} \tilde{d} \rho : C^\infty \left( \Lambda^p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda^{p-1} \left[ \frac{1}{2} \right] \right). \] (240)

### 3.6 Non-commutative Connection

Now, let \( B \) be a smooth anti-self-adjoint section of the vector bundle \( E_i[0] \), defined by the matrix-valued covector \( B_\mu \), that transforms under the gauge transformations as
\[ B'_\mu = U B_\mu U^{-1} - (\partial_\mu U)U^{-1}. \] (241)
Such a section naturally defines the maps:
\[ B : \Lambda_p \left[ \frac{1}{2} \right] \to \Lambda_{p+1} \left[ \frac{1}{2} \right]. \] (242)
by
\[(B\varphi)_{\mu_1\cdots\mu_{p+1}} = (p+1)B_{\mu_1\varphi\mu_2\cdots\mu_{p+1}}\] (243)
and
\[\tilde{B} : \Lambda^p \left[ \frac{1}{2} \right] \rightarrow \Lambda^{p-1} \left[ \frac{1}{2} \right]\] (244)
by
\[(\tilde{B}\varphi)_{\mu_1\cdots\mu_{p-1}} = B_\mu\varphi^{\mu\mu_1\cdots\mu_{p-1}}.\] (245)
Notice that
\[\tilde{B} = (-1)^{np+1}\tilde{\varepsilon}B\varepsilon\] (246)
similar to (235).

This enables us to define the covariant exterior derivative
\[\mathcal{D} : C^\infty (\Lambda^p \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda^{p+1} \left[ \frac{1}{2} \right]),\] (247)
by
\[\mathcal{D} = \rho(d + B)\rho^{-1}\] (248)
and the covariant coderivative
\[\tilde{\mathcal{D}} : C^\infty (\Lambda^p \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda^{p-1} \left[ \frac{1}{2} \right]),\] (249)
by
\[\tilde{\mathcal{D}} = (-1)^{np+1}\tilde{\varepsilon}\mathcal{D}\varepsilon = \rho^{-1}(\tilde{d} + \tilde{B})\rho.\] (250)
These operators transform covariantly under both the diffeomorphisms and the gauge transformations.

One can easily show that the square of the operators $\mathcal{D}$ and $\tilde{\mathcal{D}}$
\[
\begin{align*}
\mathcal{D}^2 & : C^\infty (\Lambda^p \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda^{p+2} \left[ \frac{1}{2} \right]) \\
\tilde{\mathcal{D}}^2 & : C^\infty (\Lambda^{p+2} \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda^p \left[ \frac{1}{2} \right])
\end{align*}
\] (251) (252)
are zero-order differential operators. In particular, in the case $p = 0$ they define the gauge curvature $\mathcal{R}$, which is a section of the bundle $E_2[0]$, by
\[\begin{align*}
(\mathcal{D}^2\varphi)_{\mu\nu} = \rho\mathcal{R}_{\mu\nu}\rho^{-1}\varphi, & \quad (\tilde{\mathcal{D}}^2\varphi = \rho^{-1}\mathcal{R}_{\mu\nu}\rho\varphi^{\nu\mu},
\end{align*}\] (253)
where
\[\mathcal{R} = dB + [B, B] ,\] (254)
and the brackets \([,]\) denote the Lie bracket of two matrix-valued 1-forms, i.e.
\[
[A, B]_{\mu\nu} = A_\mu B_\nu - B_\nu A_\mu .
\]
(255)

The gauge curvature is anti-self-adjoint and transforms covariantly the gauge transformations
\[
\mathcal{R}'_{\mu\nu} = U \mathcal{R}_{\mu\nu} U^{-1} .
\]
(256)

3.7 Non-commutative Laplacians

Finally, by using the objects introduced above we can define second-order differential operators that are covariant under both diffeomorphisms, and the gauge transformations. In order to do that we need first-order differential operators (divergences)
\[
\text{Div} : C^\infty (\Lambda_p \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda_{p-1} \left[ \frac{1}{2} \right]) ,
\]
(257)

First of all, by using the \(L^2\) inner product on the bundle \(\Lambda_p \left[ \frac{1}{2} \right]\) we define the adjoint operator \(\bar{D}\) by
\[
(\varphi, D \psi) = (\bar{D} \varphi, \psi) .
\]
(258)

This gives
\[
\bar{D} = -A^{-1} \tilde{D} A = -(-1)^{np+1} A^{-1} \varepsilon D \varepsilon A = -A^{-1} \rho^{-1} (\tilde{d} + \tilde{B}) \rho A ,
\]
(259)

which in local coordinates reads
\[
(\bar{D} \varphi)_{\mu_1 \cdots \mu_p} = - (A^{-1})_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} \rho^{-1} (\partial_\nu + \mathcal{B}_\nu) \rho A^{\nu_1 \cdots \nu_p \alpha_1 \cdots \alpha_{p+1}} \varphi_{\alpha_1 \cdots \alpha_{p+1}} .
\]
(260)

The problem with this definition is that usually it is difficult to find the matrix \((A^{-1})_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p}\).

Then we define the second order operators
\[
\bar{D} \bar{D}, D \bar{D}, \Delta : C^\infty (\Lambda_p \left[ \frac{1}{2} \right]) \rightarrow C^\infty (\Lambda_p \left[ \frac{1}{2} \right]) ,
\]
(261)

where the “non-commutative Laplacian” is a self-adjoint operator defined by
\[
\Delta = -\bar{D} \bar{D} - D \bar{D}
\]
(262)
\[
= A^{-1} \tilde{D} A \Delta + \Delta A^{-1} \tilde{D} A
\]
\[
= A^{-1} \rho^{-1} (\tilde{d} + \tilde{B}) \rho A \rho (d + \mathcal{B}) \rho^{-1} + \rho (d + \mathcal{B}) \rho^{-1} A^{-1} \rho^{-1} (\tilde{d} + \tilde{B}) \rho A .
\]
In local coordinates this reads

\[(\Delta \varphi)_{\mu_1 \cdots \mu_p} = (263)\]

\[\begin{align*}
(p + 1)A_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p}^{-1}(\partial_\nu + B_\nu)\rho A^{\nu p_1 \cdots p_\alpha_1 \cdots \alpha_p} \rho(\partial_\alpha + B_\alpha)\rho^{-1} \\
+ \rho(\partial_{\mu_1} + C_{[\mu_1})\rho^{-1} A_{\mu_2 \cdots \mu_{p-1]} \nu_1 \cdots \nu_{p-1})\rho^{-1}(\partial_\nu + B_\nu)\rho A^{\nu p_1 \cdots p_{p-1} \alpha_1 \cdots \alpha_p} \bigg\} \varphi_{\alpha_1 \cdots \alpha_p}.
\end{align*}\]

In the special case \(p = 0\) the “non-commutative Laplacian” \(\Delta\) reads

\[\Delta = \rho^{-1}(\bar{d} + \bar{B})\rho A d + B)\rho^{-1},\] (264)

which in local coordinates has the form

\[\Delta = \rho^{-1}(\partial_\mu + B_\mu)\rho a^{\mu \nu} \rho(\partial_\nu + B_\nu)\rho^{-1}.\] (265)

The leading symbol of the operator \((-\Delta)\) for \(p = 0\)

\[\sigma_L(-\Delta; x, \xi) = a^{\mu \nu}(x)\xi_\mu \xi_\nu,\] (266)

is self-adjoint and positive definite for \(\xi \neq 0\). Therefore, the Laplacian is an elliptic operator. The same is true for any \(p\).

We could have also defined the coderivatives by

\[\bar{D}_1 = -\ast D\ast, \quad \bar{D}_2 = -BD A, \quad \bar{D}_3 = -\ast D\ast, \quad \bar{D}_4 = -\ast D\ast.\] (267)

These operators have the advantage that \(\bar{D}_1\) is polynomial in the matrix \(a^{\mu \nu}\) and \(\bar{D}_2\) is polynomial in the matrices \(a^{\mu \nu}\) and \(b_{\mu \nu}\). However, the second order operators \(D_j \bar{D}_j, D\bar{D}_j\) and \(\Delta_j = -\bar{D}_j D - D\bar{D}_j, (j = 1, 2, 3, 4)\), are not self-adjoint, in general. In the commutative limit all these definitions coincide with the standard de Rham Laplacian.

### 3.8 Non-commutative Dirac Operator

Notice first that the matrix \(\Gamma\) introduced above naturally defines a map

\[\Gamma : C^\infty(\Lambda^p \frac{1}{2}) \to C^\infty(\Lambda^{p+1} \frac{1}{2})\] (268)

by

\[(\Gamma \varphi)^{\mu_1 \cdots \mu_{p+1}} = (p + 1)\Gamma^{[\mu_1} \varphi^{\mu_2 \cdots \mu_{p+1}]}\] (269)
and the map
\[ \bar{\Gamma} : C^\infty(\Lambda_p \left[ \frac{1}{2} \right]) \to C^\infty(\Lambda_{p-1} \left[ \frac{1}{2} \right]) \] (270)

by
\[ (\bar{\Gamma}\varphi)_{\mu_1 \ldots \mu_{p-1}} = \Gamma^\mu \varphi_{\mu_1 \ldots \mu_p}. \] (271)

Therefore, we can define first-order invariant differential operator ("non-commutative Dirac operator")
\[ D : C^\infty(\Lambda_p \left[ \frac{1}{2} \right]) \to C^\infty(\Lambda_p \left[ \frac{1}{2} \right]) \] (272)

by
\[ D = i\bar{\Gamma}D = i\bar{\Gamma}\rho(d + B)\rho^{-1}, \] (273)
which in local coordinates reads
\[ (D\varphi)_{\mu_1 \ldots \mu_p} = i(p + 1)\Gamma^\mu \rho(\partial_\mu + B_\mu)\rho^{-1}\varphi_{\mu_1 \ldots \mu_p}. \] (274)

The adjoint of this operator with respect to the \( L^2 \) inner product is
\[ \bar{\bar{D}} = iA^{-1}\bar{\bar{D}}\Gamma A = iA^{-1}\rho^{-1}(\bar{\bar{d}} + \bar{\bar{B}})\rho\Gamma A, \] (275)
which in local coordinates becomes
\[ (\bar{\bar{D}}\varphi)_{\mu_1 \ldots \mu_p} = i(p + 1)A^{-1}_{\mu_1 \ldots \mu_p\nu_1 \ldots \nu_p}\rho^{-1}(\partial_\nu + B_\nu)\rho\Gamma^{[\nu}_{\alpha_1 \ldots \alpha_p}\varphi_{\alpha_1 \ldots \alpha_p}. \] (276)

In the case \( p = 0 \) these operators simplify to
\[ D = i\bar{\Gamma}D = i\Gamma^\mu \rho(\partial_\mu + B_\mu)\rho^{-1}, \] (277)
\[ \bar{\bar{D}} = i\bar{\bar{D}}\Gamma = i\rho^{-1}(\partial_\nu + B_\nu)\rho\Gamma^{\nu}. \] (278)

These operators have the same leading symbol
\[ \sigma_L(D; x, \xi) = \sigma_L(\bar{\bar{D}}; x, \xi) = -\Gamma^\mu(x)\xi_\mu, \] (279)
which is self-adjoint and non-degenerate. Therefore, the Dirac operator and its adjoint \( \bar{\bar{D}} \) are elliptic. One can show that the same is true for any \( p \).

These operators can be used then to define second order differential operators
\[ D\bar{\bar{D}} = -\bar{\bar{D}}DA^{-1}\bar{\bar{D}}\Gamma A \]
\[ = -\bar{\bar{D}}\rho(d + B)\rho^{-1}A^{-1}\rho^{-1}(\bar{\bar{d}} + \bar{\bar{B}})\rho\Gamma A, \] (280)
\[ \bar{\bar{D}}D = -\bar{\bar{A}}^{-1}\bar{\bar{D}}\Gamma A\bar{\bar{D}} \]
\[ = -\bar{\bar{A}}^{-1}\rho^{-1}(\bar{\bar{d}} + \bar{\bar{B}})\rho\Gamma A\bar{\bar{D}}\rho(d + B)\rho^{-1}. \] (281)
The operators $D\bar{D}$ and $\bar{D}D$ are self-adjoint elliptic and non-negative. They have the same non-zero spectrum. That is, if $\lambda \neq 0$ is an eigenvalue of the operator $D\bar{D}$ with the eigensection $\varphi$, then $\bar{D}\varphi$ is the eigenfunction of the operator $\bar{D}D$ with the same eigenvalue. Conversely, if $\psi$ is an eigensection of the operator $\bar{D}D$ with an eigenvalue $\lambda \neq 0$, then $D\psi$ is an eigensection of the operator $D\bar{D}$ with the same eigenvalue. Of course, if the Dirac operator is self-adjoint, i.e. $D = \bar{D}$, then $D\bar{D} = \bar{D}D$. However, if $D$ is not self-adjoint, then the zero eigenspaces of these operators can be different, and one can define an index

$$\text{Ind}(D) = \dim \text{Ker}(\bar{D}) - \dim \text{Ker}(D).$$

In the case $p = 0$ these operators have the form

$$D\bar{D} = -\bar{\Gamma}D\bar{D}\Gamma$$
$$= -\Gamma^\mu \rho(\partial_\mu + B_\mu)\rho^{-2}(\partial_\nu + B_\nu)\rho\Gamma^\nu, \quad (283)$$

$$\bar{D}D = -\bar{D}\bar{\Gamma}\bar{D}\Gamma$$
$$= -\rho^{-1}(\partial_\nu + B_\nu)\rho\Gamma^\mu\rho(\partial_\mu + B_\mu)\rho^{-1}. \quad (284)$$

These operators have the same leading symbol as the non-commutative Laplacian. Therefore, one can obtain a non-commutative version of the Lichnerowicz formula.

These constructions can be used to develop non-commutative generalization of the standard theory of elliptic complexes, in particular, spin complex, de Rham complex, index theorems, cohomology groups, heat kernel etc. If the bundle $S$ is $\mathbb{Z}_2$-graded, then, similarly to the Riemannian case discussed in section 2, there is an index of the Dirac operator even it is self-adjoint. This is a very interesting topic that requires further study.

### 3.9 Spectral Asymptotics

Since the non-zero spectra of the operators $\bar{D}D$ and $D\bar{D}$ are isomorphic, this also means that the spectral invariants of the operators $\bar{D}D$ and $D\bar{D}$ are equal except possibly for the invariant $A_{n/2}(I, \bar{D}D)$ which determines the index in even dimension. Thus, for $n > 2$ the spectral invariants $A_0$ and $A_1$ of the operators $\bar{D}D$ and $D\bar{D}$ are the same. Therefore, we can pick any of these operators $\bar{D}D$ or $D\bar{D}$ to compute the invariants $A_0$ and $A_1$. In present
paper we will restrict ourselves to the case $p = 0$. The operators $\bar{D}D$ and $D\bar{D}$ have the same leading symbol equal to

$$
\sigma_L(\bar{D}D; x, \xi) = \sigma_L(D\bar{D}; x, \xi) = H(x, \xi) = [\Gamma^\mu(x)\xi_\mu]^2,
$$

(285)

with $\xi \in T^*_x M$ a cotangent vector. Since by our assumption this matrix is self-adjoint and positive definite, these operators are elliptic. In fact, all non-commutative Laplacians and Dirac operators introduced in the previous subsection are elliptic.

It is well known that a self-adjoint elliptic partial differential operator with positive definite leading symbol on a compact manifold without boundary has a discrete real spectrum bounded from below [13]. Since the operator $\bar{D}D$ transforms covariantly under the diffeomorphisms as well as under the gauge transformations (173) the spectrum is invariant under these transformations.

The heat semigroup $\exp(-t\bar{D}D)$ is a trace-class operator with a well defined $L^2$ trace

$$
\text{Tr}_{L^2} \exp(-t\bar{D}D) = \int_M dx \text{tr}_S U(t; x, x).
$$

(286)

Moreover for any smooth endomorphism-valued function $F \in C^\infty(\text{End}(S)[0])$ the following trace is defined

$$
\text{Tr}_{L^2}[F \exp(-t\bar{D}D)] = \int_M dx \text{tr}_S F(x)U(t; x, x).
$$

(287)

We have defined the heat kernel $U(t; x, x')$ in such a way that it transforms as a density of weight $\frac{1}{2}$ at both points $x$ and $x'$. More precisely, it is a section of the exterior tensor product bundle $S[\frac{1}{2}] \boxtimes S^*[\frac{1}{2}]$. Therefore, the heat kernel diagonal $U(t; x, x)$ transforms as a density of weight 1, i.e. it is a section of the bundle $\text{End}(S)[1]$, and the trace $\text{Tr}_{L^2}\exp(-t\bar{D}D)$ is invariant under diffeomorphisms.

As in the case of Laplace type operators there is an asymptotic expansion as $t \to 0^+$ of the heat kernel diagonal

$$
U(t; x, x) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} a_k(\bar{D}D; x),
$$

(288)

and of the heat trace as $t \to 0^+$ [13]

$$
\text{Tr}_{L^2}[F \exp(-t\bar{D}D)] \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} A_k(F, \bar{D}D),
$$

(289)
where

\[ A_k(F, \bar{D}D) = \int_M dx \, \text{tr}_S F(x) a_k(\bar{D}D; x). \tag{290} \]

are the global heat invariants.

A second-order differential operator is called Laplace type if it has a scalar leading symbol. Most of the calculations in quantum field theory and spectral geometry are restricted to the Laplace type operators for which nice theory of heat kernel asymptotics is available \[13, 14, 15, 16, 17\]. However, the operators considered in the present paper have a matrix valued principal symbol \(H(x, \xi)\) and are, therefore, not of Laplace type. The study of heat kernel asymptotics for non-Laplace type operators is quite new and the methodology is still underdeveloped. As a result even the invariant \(A_2\) is not known, in general. For some partial results see \[19, 20, 11\].

### 3.10 Heat Invariants

For so called natural non-Laplace type differential operators, which are constructed from a Riemannian metric and canonical connections on spin-tensor bundles the coefficients \(A_0\) and \(A_1\) were computed in \[20\]. For general non-Laplace type operators they were computed in \[11\]. Following these papers we will use a rather formal method that is sufficient for our purposes of computing the asymptotics of the heat trace of the second-order elliptic self-adjoint operator \(\bar{D}D\).

First, we present the heat kernel diagonal for the operator \(\bar{D}D\) in the form

\[ U(t, x, x) = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} e^{-i\xi x} \exp(-t\bar{D}D) e^{i\xi x}, \tag{291} \]

where \(\xi x = \xi_\mu x^\mu\), which can be transformed to

\[ U(t, x, x) = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \exp \left[ -t \left( H + K + \bar{D}D \right) \right] \cdot I, \tag{292} \]

where \(H\) is the leading symbol of the operator \(\bar{D}D\)

\[ H = [\Gamma(\xi)]^2, \tag{293} \]

with \(\Gamma(\xi) = \Gamma^\mu(x)\xi_\mu\), and \(K\) is a first-order self-adjoint operator defined by

\[ K = -\Gamma(\xi) D - \bar{D} \Gamma(\xi). \tag{294} \]
Here the operators in the exponent act on the unity matrix $I$ from the left.

By changing the integration variable $\xi \to t^{-1/2} \xi$ we obtain

$$\int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \exp \left( -H - \sqrt{t} K - t \bar{D}D \right) \cdot I,$$

and the problem becomes now to evaluate the first three terms of the asymptotic expansion of this integral in powers of $t^{1/2}$ as $t \to 0$.

By using the Volterra series

$$\exp(A + B) = e^A + \sum_{k=1}^{\infty} \int_0^1 d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \times$$

$$\times e^{(1-\tau_k)A} B e^{(\tau_k-\tau_{k-1})A} \cdots e^{(\tau_2-\tau_1)A} B e^{\tau_1 A}, \quad (296)$$

we get

$$\exp \left( -H - \sqrt{t} K - t \bar{D}D \right) = e^{-H} - t^{1/2} \int_0^1 d\tau_1 e^{-(1-\tau_1)H} K e^{-\tau_1 H}$$

$$+ t \left[ \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right.$$

$$\left. - \int_0^1 d\tau_1 e^{-(1-\tau_1)H} \bar{D}D e^{-\tau_1 H} \right] + O(t^2). \quad (297)$$

Now, since $K$ is linear in $\xi$ the term proportional to $t^{1/2}$ vanishes after integration over $\xi$. Thus, we obtain the first two coefficients of the asymptotic expansion of the heat kernel diagonal

$$U(t; x, x) = (4\pi t)^{-n/2} \left[ a_0(x) + ta_1(x) + O(t^2) \right] \quad (298)$$

in the form

$$a_0 = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} e^{-H}, \quad (299)$$
\[ a_1 = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \left[ \int_0^1 d\tau_2 \int_0^\tau_2 d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right. \\
- \left. \int_0^1 d\tau_1 e^{-(1-\tau_1)H} D D e^{-\tau_1 H} \right] . \] (300)

These quantities are matrix-valued densities. The coefficient \( a_0 \) is constructed from the matrix \( a \) but not its derivatives, whereas the coefficient \( a_1 \) is constructed from the matrix \( a \) and its first and second derivatives as well as from the first derivatives of the field \( B \) and the matrix \( \rho \) and its first and second derivatives. Moreover, it is polynomial in the derivatives of \( a^{\mu\nu} \), \( \rho \) and \( B_\mu \), more precisely, linear in second derivatives of \( a \) and \( \rho \) and the first derivatives of \( B \) and \( \rho \) and quadratic in first derivatives of \( a \) and \( \rho \). By tracing the local invariants and integrating over the manifold we finally get the global heat invariants

\[ A_0 = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \text{tr}_S e^{-H} . \] (301)

\[ A_1 = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \text{tr}_S \left[ \int_0^1 d\tau_2 \int_0^\tau_2 d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right. \\
- \left. \int_0^1 d\tau_1 e^{-(1-\tau_1)H} D D e^{-\tau_1 H} \right] . \] (302)

The global heat invariants are invariant under both the diffeomorphisms and the gauge transformations. Since the operator \( \bar{D}D \) is self-adjoint, the heat kernel diagonal \( U(t; x, x) \) is a self-adjoint matrix-valued density, and, therefore, the heat trace \( \text{Tr}_{L^2} \exp(-t\bar{D}D) \) is a real invariant. Therefore, the coefficients \( a_0 \) and \( a_1 \) are self-adjoint matrix densities and the invariants \( A_0 \) and \( A_1 \) are real.

4 Non-commutative Einstein-Hilbert Functional

It is an interesting fact that a linear combination of the first two spectral invariants of the Dirac operator on Riemannian manifold determines the
Einstein-Hilbert functional. Indeed, by using the eqs. (170), (171) we obtain

\[
S_{EH}(g) = -\frac{1}{16\pi G N} \left\{ 12A_1(\mathbb{I}, D^2) + 2\Lambda A_0(\mathbb{I}, D^2) \right\}
\]

\[
= \frac{1}{16\pi G} \int_M d\text{vol} \ (R - 2\Lambda) ,
\]  
(303)

where \( G \) and \( \Lambda \) are positive parameters. This functional is the action functional of the general theory of relativity which determines the Einstein equations of the gravitational field. In the general theory of relativity the Riemannian metric \( g \) (rather its pseudo-Riemannian version) is identified with the gravitational field and the parameters \( G \) and \( \Lambda \) with the Newtonian gravitational constant and the cosmological constant respectively.

In differential geometry the extremals of the Einstein-Hilbert functional are the Einstein spaces, that is Riemannian metrics \( g \) satisfying the vacuum Einstein equations with the cosmological constant

\[
R_{\mu\nu} = \Lambda g_{\mu\nu} ,
\]  
(304)

where \( R_{\mu\nu} \) is the Ricci tensor. The study of Einstein spaces is a very important area in differential geometry and mathematical physics.

In full analogy with the above one can build an invariant functional of the non-commutative metric \( a^{\mu\nu} \) (or the non-commutative Dirac matrices \( \Gamma^\mu \)), the endomorphism \( \rho \) and the endomorphism-valued covector \( B_\mu \) as a linear combination of the first two spectral invariants of the non-commutative operator \( DD \). Such a functional can be called a non-commutative deformation of the Einstein-Hilbert functional. The extremals of this functional are then “non-commutative Einstein equations”, whose solutions determine the structures that can be called “non-commutative Einstein spaces”. One can show that this functional does not depend on the derivatives of the field \( B_\mu \). Therefore, variation with respect to \( B_\mu \) gives just a constraint which expresses \( B_\mu \) in terms of derivatives of the functions \( a^{\mu\nu} \) and \( \rho \). One can also impose some additional consistency conditions to express the extra ingredients, like the matrix \( \rho \) in terms of the non-commutative metric \( a^{\mu\nu} \) (or non-commutative Dirac matrices \( \Gamma^\mu \)). For example, the requirement that the non-commutative Dirac operator should be self-adjoint, gives a constraint which can be used to fix the connection \( B \). The question of uniqueness of such consistency conditions is one of many open problems in this approach. The study of
these structures is an extremely interesting problem in differential geometry. It might also have applications in modern gravitational and high-energy physics. Such attempts are discussed in our previous papers \[9, 10, 11\].

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