Hypersurface of a Finsler space subjected to an $h$-exponential change of metric

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Abstract

Recently we have obtained the Cartan connection for the Finsler space whose metric is given by an exponential change with an $h$-vector. In this paper, we discuss certain geometric properties of a Finslerian hyperspace subjected to an $h$-exponential change of metric.

Keywords: Finsler space, hypersurface, exponential change, $h$-vector.

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1 Introduction

In 2006, YU Yao-yong and YOU Ying [12] studied a Finsler space with metric function given by exponential change of Riemannian metric. In 2012, H. S. Shukla et. al. [10] considered a Finsler space $\mathcal{F}^n = (M^n, L)$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$\mathcal{L} = L e^{\beta/L},$$

where $\beta = b_i(x)y^i$ is 1-form on manifold $M^n$.

H. Izumi [7] introduced the concept of an $h$-vector $b_i(x, y)$ which is $v$-covariant constant with respect to the Cartan connection and satisfies $L C^h_{ij} b_h = \rho h_{ij}$, where $\rho$ is a non-zero scalar function and $C^h_{jk}$ are components of Cartan tensor. Thus if $b_i$ is an

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From the above definition, we have

\[(1.2) \quad L \dot{\partial}_j b_i = \rho_{ij},\]

which shows that \(b_i\) is a function of directional argument also. H. Izumi \[7\] proved that the scalar \(\rho\) is independent of directional argument. Gupta and Pandey \[6\] proved that if the \(h\)-vector \(b_i\) is gradient then the scalar \(\rho\) is constant.

B. N. Prasad \[9\] obtained the Cartan connection of Finsler space whose metric is given by \(h\)-Rander’s change of a Finsler metric. Gupta and Pandey \[9\] obtained the Cartan connection of Finsler space whose metric is given by \(h\)-Kropina change of Finsler metric. Present authors \[2\] studied the Cartan connection of Finsler space whose metric is given by \(h\)-exponential change of Finsler metric.

The theory of hypersurfaces in a Finsler space has been introduced by by E. Cartan \[1\]. A. Rapcsàk \[11\] introduced three kinds of hyperplanes and M. Matsumoto \[8\] has classified the hyersurfaces and developed a systematic theory of Finslerian hypersurfaces. Gupta and Pandey \[4, 5\] discussed the hypersurface of a Finsler space whose metric is given by certain transformation with an \(h\)-vector.

In the present paper, we discuss the geometric properties of hypersurface of a Finsler space \(\ast F^n = (M^n, \ast L)\), whose metric function \(\ast L\) is given by an \(h\)-exponential change of a Finsler metric function \(i.e.

\[(1.3) \quad \ast L = L e^{\beta},\]

where \(\beta = b_i(x, y)y^i\) and \(b_i\) is an \(h\)-vector.

\section{Preliminaries}

Let \(F^n = (M^n, L)\) be an \(n\)-dimensional Finsler space equipped with the Fundamental function \(L(x, y)\). The metric tensor, angular metric tensor and Cartan tensor are defined by \(g_{ij} = \frac{1}{2} \partial_i \partial_j L^2\), \(h_{ij} = g_{ij} - l_i l_j\) and \(C_{ijk} = \frac{1}{2} \partial_i g_{jk}\) respectively, where \(\partial_k = \frac{\partial}{\partial y^k}\). The
Cartan connection is given by \( C = (F^i, N^i, C^i) \). The \( h \)- and \( v \)-covariant derivatives \( X_{ij} \) and \( X_i|_j \) of a covariant vector field \( X_i \) are defined by

\[
X_{ij} = \partial_j X_i - N_j^r \dot{\partial}_r X_i - X_r F^r_{ij},
\]

and

\[
X_i|_j = \dot{\partial}_j X_i - X_r C^r_{ij},
\]

where \( \dot{\partial}_k = \frac{\partial}{\partial x^k} \).

A hypersurface \( M^{n-1} \) of the underlying smooth manifold \( M^n \) may be parametrically represented by the equation \( x^i = x^i(u^\alpha) \), where \( u^\alpha \) are Gaussian coordinates on \( M^{n-1} \) (Latin indices run from 1 to \( n \) while Greek indices run from 1 to \( n-1 \)). Here, we shall assume that the matrix consisting of the projection factors \( B^i_\alpha = \partial x^i / \partial u^\alpha \) is of rank \( n-1 \). If the supporting element \( y^i \) at a point \( u = (u^\alpha) \) of \( M^{n-1} \) is assumed to be tangent to \( M^{n-1} \), we may then write \( y^i = B^i_\alpha(u) v^\alpha \) so that \( v = (v^\alpha) \) is thought of as the supporting element of \( M^{n-1} \) at a point \( u^\alpha \). Since the function \( L(u, v) = L(x(u), y(u, v)) \) gives arise a Finsler function on \( M^{n-1} \), we get an \((n-1)\)-dimensional Finsler space \( F^{n-1} = (M^{n-1}, L(u, v)) \).

At each point \( u^\alpha \) of \( F^{n-1} \), the unit normal vector \( N^i(u, v) \) is defined as

\[
g_{ij} B^i_\alpha N^j = 0, \quad g_{ij} N^i N^j = 1,
\]

The inverse projection factors \( B^i_\alpha(u, v) \) of \( B^i_\alpha \) are defined as

\[
B^i_\alpha = g^{\alpha\beta} g_{ij} B^j_\beta,
\]

where \( g^{\alpha\beta} \) is the inverse of metric tensor \( g_{\alpha\beta} \) of \( F^{n-1} \).

from (2.3) and (2.4), it follows that

\[
B^i_\alpha B^\beta_\delta = \delta^\beta_\delta, \quad B^i_\alpha N_i = 0, \quad N_i B^i_\alpha = 0, \quad N_i N^i = 1,
\]

and further

\[
B^i_\alpha B^\beta_\delta + N^i N_j = \delta^i_\beta.
\]

For the induced Cartan connection \( ICT = (F^\alpha_{\beta\gamma}, G^\alpha_{\beta}, C^\alpha_{\beta\gamma}) \) on \( F^{n-1} \), the second fundamental \( h \)-tensor \( H_{\alpha\beta} \) and the normal curvature vector \( H_\alpha \) are given by

\[
H_{\alpha\beta} = N_i (B^i_\alpha B^\beta_\delta + F^i_{\beta\gamma} B^\gamma_\alpha B^\delta_\beta) + M_\alpha H_\beta,
\]

\[
H_\alpha = N_i B^i_\alpha + F^i_{\beta\gamma} B^\gamma_\alpha B^\delta_\beta + M_\alpha H_\beta.
\]
and
\begin{equation}
(2.8) \quad H_\alpha = N_i (B^i_{0\alpha} + G^i_j B^j_\alpha),
\end{equation}
where \( M_\alpha = C_{ijk} B^i_\alpha N^j N^k \), \( B^i_\alpha = \frac{\partial^2 x^i}{\partial u^\alpha \partial v^\beta} \) and \( B^i_{0\alpha} = B^i_{\beta\alpha} v^\beta \).

The equations (2.7) and (2.8) yield
\begin{equation}
(2.9) \quad H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha, \quad H_{\alpha 0} = H_{\alpha\beta} v^\beta = H_\alpha + M_\alpha H_0.
\end{equation}

The second fundamental \( \nu \)-tensor \( M_{\alpha\beta} \) is defined as
\begin{equation}
(2.10) \quad M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k.
\end{equation}

The relative \( h \)- and \( \nu \)-covariant derivatives of \( B^i_\alpha \) and \( N^i \) are given by
\begin{equation}
(2.11) \quad B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \quad B^i_{\alpha|\beta} = M_{\alpha\beta} N^i,
N^i_{|\beta} = -H_{\alpha\beta} B^a_\beta g^{ij}, \quad N^i_{|\beta} = -M_{\alpha\beta} B^a_\beta g^{ij}.
\end{equation}

Let \( X_i(x, y) \) be a vector field on \( F^n \). Then the relative \( h \)- and \( \nu \)-covariant derivatives of \( X_i \) are given by
\begin{equation}
(2.12) \quad X_i|_{\beta} = X_{ij} B^j_\beta + X_i|_j N^j H_\beta, \quad X_i|_{\beta} = X_i|_j B^j_\beta.
\end{equation}

A. Rapcsák [11] introduced three kinds of hyperplanes. M. Matsumoto [8] obtained their characteristic conditions, which are given in the following lemmas:

**Lemma 2.1.** A hypersurface \( F^{n-1} \) is a hyperplane of first kind if and only if \( H_\alpha = 0 \) or equivalently \( H_0 = 0 \).

**Lemma 2.2.** A hypersurface \( F^{n-1} \) is a hyperplane of second kind if and only if \( H_{\alpha\beta} = 0 \).

**Lemma 2.3.** A hypersurface \( F^{n-1} \) is a hyperplane of third kind if and only if \( H_{\alpha\beta} = 0 = M_{\alpha\beta} \).

### 3 The Finsler space \( *F^n=(M^n, *L) \)

Let us denote \( b_i y^i \) by \( \beta \), then indicatory property of \( h_{ij} \) yield \( \partial_i \beta = b_i \). The quantities corresponding to \( *F^n \) is denoted by asterisk over that quantity. We shall use following notations \( L_i = \partial_i L = l_i, \quad L_{ij} = \partial_i \partial_j L, \quad L_{ijk} = \partial_i \partial_j \partial_k L \). From (1.3), we get
\begin{equation}
(3.1) \quad *L_{ij} = e^\tau (1 + \rho - \tau) L_{ij} + \frac{e^\tau}{L} m_i m_j,
\end{equation}
\[ *L_{ijk} = e^\tau (1 + \rho - \tau) L_{ijk} + (\rho - \tau) \frac{e^\tau}{L} [m_i L_{jk} + m_j L_{ik} + m_k L_{ij}] \]
\[ - \frac{e^\tau}{L} [m_j m_k l_i + m_i m_k l_j + m_i m_j l_k - m_j m_k], \]

where \( \tau = \frac{\var}{T} \), \( m_i = b_i - \tau l_i \). The normalised supporting element and the metric tensor of \( ^*F^m \) are obtained as \[2\]
\[ *l_i = e^\tau (m_i + l_i), \]
\[ *g_{ij} = \nu e^{2\tau} g_{ij} + e^{2\tau} (2\tau^2 - \tau - \rho) l_il_j + e^{2\tau} (1 - 2\tau)(b_il_j + b_jl_i) + 2e^{2\tau}b_ib_j. \]

Differentiating the angular metric tensor \( h_{ij} \) with respect to \( y^k \), we get
\[ \dot{h}_{ij} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} + l_j h_{ik}), \]
which gives
\[ L_{ijk} = \frac{2}{L} C_{ijk} - \frac{1}{L^2} (h_{ij} l_k + h_{ik} l_j + h_{kj} l_i). \]

Using this, the equation (3.2) may be re-written as
\[ \nu e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau} (2\nu - 1)(m_i h_{kj} + m_j h_{ki} + m_k h_{ij}), \]

where \( \nu = 1 + \rho - \tau \).

The inverse metric tensor of \( ^*F^m \) is derived as follows\[2\]:
\[ *g^{ij} = \frac{e^{-2\tau}}{\nu} \left[ g^{ij} - \frac{1}{m^2 + \nu} b^i b^j + \frac{\tau - \nu}{m^2 + \nu} (b^i l^j + b^j l^i) - \left\{ \frac{\tau - \nu}{m^2 + \nu} (m^2 + \tau) - \rho \right\} l_il_j \right], \]

where \( b \) is magnitude of the vector \( b^i = g^{ij} b_j \).

The relation between cartan connection coefficients of \( ^*F^m \) and \( F^m \) is given by
\[ \nu e^{2\tau} = F^i_{jk} + D^i_{jk}. \]

The expressions for \( D^i_{00}, D^i_{0k} \) and \( D^i_{jk} \) are given by \[2\]
\[ D^i_{00} = \frac{L}{\nu e^\tau} \left[ \frac{e^\tau}{L} \beta_0 m^i + 2e^\tau F_0 \right] + \frac{l^i}{e^\tau} \left[ E_{00} - \frac{L}{e^\tau} (m^2 + \nu)^{-1} \left( \frac{e^\tau}{L} \beta_0 m^2 + 2e^\tau F_{00} \right) \right] \]
\[ - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} \left[ \frac{e^\tau}{L} \beta_0 m^2 + 2e^\tau F_{00} \right], \]
\[ D^i_{0j} = \frac{L G^i_{0j}}{\nu e^\tau} + \frac{l^j}{e^\tau} \left[ G_{0} - L (m^2 + \nu)^{-1} G_{0j} \right] - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} G_{0j}, \]
\[ D^i_{jk} = \frac{L G^i_{jk}}{\nu e^\tau} + \frac{l^j}{e^\tau} \left[ G_{jk} - L (m^2 + \nu)^{-1} G_{jk} \right] - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} G_{jk}. \]
Lemma 3.1. \[ \text{If the h-vector } b_i \text{ is gradient then the scalar } \rho \text{ is constant.} \]

4 The Hypersurface \( ^*F^{n-1} \) of the space \( ^*F^n \)

Let us consider Finslerian hypersurfaces \( F^{n-1} = (M^{n-1}, L(u,v)) \) of \( F^n \) and \( ^*F^{n-1} = (M^{n-1}, ^*L(u,v)) \) of \( ^*F^n \). Let \( N^i \) be the unit normal vector at a point of \( F^{n-1} \). The
functions $B^i_\alpha(u)$ may be considered as component of $(n-1)$ linearly independent vectors tangent to $F^{n-1}$ and they are invariant under h-exponential change of Finsler metric. The unit normal vector $^*N^i(u, v)$ of $^*F^{n-1}$ is uniquely determined by

\[(4.1) \quad ^*g_{ij}B^i_\alpha N^j = 0, \quad ^*g_{ij}^*N^i N^j = 1.\]

The inverse projection factors $^*B^\alpha_i(u, v)$ of $B^i_\alpha$ along $^*F^{n-1}$ are defined as

\[(4.2) \quad ^*B^\alpha_i = ^*g^\alpha_\beta g_{ij} B^j_\beta,\]

where $^*g^\alpha_\beta$ is the inverse of metric tensor $^*g_{\alpha\beta}$ of $^*F^{n-1}$.

From (4.2), it follows that

\[(4.3) \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad ^*N_i B^j_\alpha = 0, \quad ^*N_i N^i = 1,\]

and further

\[(4.4) \quad B^i_\alpha B^\beta_j + N^i N_j = \delta_i^j.\]

Now, Transvection of (2.3) by $v^\alpha$ gives

\[(4.5) \quad y_j N^j = 0.\]

Transvecting (3.4) by $N_i N^j$ and by using (4.5), we have

\[(4.6) \quad ^*g_{ij} N^i N^j = \nu e^{2\tau} + 2e^{2\tau} (b_i N^i)^2,\]

this implies that

\[(4.7) \quad \frac{N^j}{e^\tau \sqrt{\nu + 2(b_i N^i)^2}}\]

is unit vector.

Also, Transvection (3.4) by $B^i_\alpha N^j$ and using (4.5), gives us

\[(4.8) \quad ^*g_{ij} B^i_\alpha N^j = (b_j N^j) e^{2\tau} \left\{ (1 - 2\tau) l_i B^i_\alpha + 2b_i B^i_\alpha \right\}.\]

This shows that $N^j$ is normal if and only if R.H.S. of equation (4.8) is zero. Since $e^{2\tau} \left\{ (1 - 2\tau) l_i B^i_\alpha + 2b_i B^i_\alpha \right\}$ can not be zero, otherwise transvection of $e^{2\tau} \left\{ (1 - 2\tau) l_i B^i_\alpha + 2b_i B^i_\alpha \right\}$ by $v^\alpha$ gives $L = 0$, which is not possible. Hence $N^j$ is normal to $^*F^{n-1}$ if and
only if \( b_j N^j = 0 \).

From (4.7) and (4.8), we may state that

\[
(4.9) \quad ^*N^i = \frac{N^i}{e^\tau \sqrt{\nu}}
\]

is unit normal vector of \( ^*F^{n-1} \).

Which in view of (3.4) and (4.5), gives

\[
(4.10) \quad ^*N_i = N_i e^\tau \sqrt{\nu}.
\]

Thus, we have:

**Theorem 4.1.** Let \( ^*F^n \) be the Finsler space obtained from \( F^n \) by \( h \)-exponential change given by (1.3). Further if \( ^*F^{n-1} \) and \( F^{n-1} \) are the hypersurfaces of these spaces. Then the vector \( b_i \) is tangential to hypersurface \( F^{n-1} \) if and only if every vector normal to \( F^{n-1} \) is also normal to \( ^*F^{n-1} \). And then the normal vector is given by (4.9).

Let \( b_i \) is gradient vector, i.e. \( b_\text{ji} = b_{ij} \), then

\[
(4.11) \quad F_{ij} = 0,
\]

which in view of Lemma (3.1), gives

\[
(4.12) \quad \rho_i = 0.
\]

Now, if \( b_i \) is tangent to hyperplane \( F^{n-1} \) i.e.

\[
(4.13) \quad b_j N^j = 0.
\]

Using (4.5), (4.11) and (4.13), we have

\[
(4.14) \quad D^i_{00} N_i = 0.
\]

The normal curvature tensor \( ^*H_\alpha \) for hypersurface \( ^*F^{n-1} \) is given by

\[
^*H_\alpha = ^*N_i (B^i_{0\alpha} + ^*G^i_j B^j_\alpha),
\]

by use of (2.8) and (4.9), above equation becomes

\[
(4.15) \quad ^*H_\alpha = \sqrt{\nu} e^\tau \left( H_\alpha + N_i D^i_{0j} B^j_\alpha \right),
\]

which on transvection by \( e^\alpha \) and using (4.14), gives

\[
(4.16) \quad ^*H_0 = \sqrt{\nu} e^\tau H_0.
\]

Thus in view of Lemma (2.1), we have:
**Theorem 4.2.** Let the $h$-vector $b_i$ be a gradient and tangent to hypersurface $F^{n-1}$. Then the hypersurface $F^{n-1}$ is a hyperplane of first kind if and only if hypersurface $^*F^{n-1}$ is hyperplane of first kind.

Taking the relative $h$-covariant differentiation of (4.13) with respect to the Cartan connection of $F^{n-1}$, we get

$$b_{i|\beta}N^i + b_iN_{\beta} = 0.$$ 

Using (2.11) and (2.12), the above equation gives

$$(b_{i|j}B^j_{\beta} + b_{i|j}N_{i}^{\beta}H_{\beta})N^i - b_iH_{\alpha\beta}B^\alpha_jg^{ij} = 0.$$ 

Travecting by $v^\beta$ and using (2.9), we get

$$b_{i|0}N^i = (H_\alpha + M_\alpha H_0)B^\alpha_jb^j - b_{i|j}H_0N^iN^j.$$ 

For the hypersurface to be first kind, $H_0 = 0 = H_\alpha$. Then above equation reduces to $b_{i|0}N^i = 0$. If the vector $b_i$ is gradient, i.e. $b_{i|j} = b_{j|i}$, then we get

$$E_{i0}N^i = b_{i|0}N^i = \beta_iN^i.$$ 

The tensors $D^i_{00}$, $D^i_{0j}$, $G_{ij}$ and $G_j$ satisfies the following, which can be easily verified:

$$D^i_{00}N_i = 0, \quad D^r_{0j}L_{jr}N^j = 0$$

(4.17)

$$L_{ijr}D^r_{00} = (E_{00} - (m^2 + \nu)^{-1}\beta_0m^2)\left[\left(\frac{2\rho}{L^2\nu} - \frac{1}{L^2}\right)h_{ij} - \frac{1}{L^2\nu}(m_jl_i + m_il_j)\right],$$

$$G_{ij}N^iB^j_{\alpha} = 0, \quad D^i_{0j}N_iB^j_{\alpha} = 0, \quad G_jN^j = 0, \quad G_{ij}b^jN^j = 0,$$

$$D^i_{0j}b_iN^j = 0, \quad D^i_{0j}N^jB^k_{\alpha}h_{jk} = 0, \quad D^r_{0j}l_rN^j = 0, \quad G_r^j{l_r}N^j = 0.$$ 

The second fundamental $h$- tensor $^*H_{\alpha\beta}$ for hyperplane $^*F^{n-1}$ is given by

$$^*H_{\alpha\beta} - ^*M_\alpha^*H_\beta = ^*N_i(B^i_{\alpha\beta} + ^*F^i_{jk}B^j_{\alpha}B^k_{\beta}),$$

then by use of (2.7), (3.8) and (4.10), above equation gives

(4.18) $$^*H_{\alpha\beta} - ^*M_\alpha^*H_\beta = e^\tau\sqrt{\nu}\left[H_{\alpha\beta} + N_iD^i_{jk}B^j_{\alpha}B^k_{\beta}\right] - e^\tau\sqrt{\nu}M_\alpha H_\beta.$$
Contracting (3.11) by $B^i_\alpha B^k_\beta N_j$ and using $m^j N_j = 0$, $\nu^i N_j = 0$, we get

$$D^j_{ik} B^i_\alpha B^k_\beta N_j = \frac{L}{\nu e^\tau} H^j_{ik} B^i_\alpha B^k_\beta N_j = -\frac{L}{2\nu e^\tau} H_{jik} N^j B^i_\alpha B^k_\beta,$$

which in view of (3.14) and (4.17), gives

(4.19) $$D^j_{ik} B^i_\alpha B^k_\beta N_j = -\frac{L}{2\nu e^\tau} [L^r_{ij} D^r_{0k} + L_{jkr} D^r_{0i} - L_{kir} D^r_{0j}] N^j B^i_\alpha B^k_\beta.$$

Now we calculate each terms of the above equation separately.

Transvecting (3.12) by $N^j$, we have

(4.20) $$G_{ij} N^j = \mu N_i,$$

where

$$\mu = \frac{1}{2L^2} \left[ -e^\tau (E_{00} - (m^2 + \nu)^{-1} \beta_0 m^2) (2\rho - \nu) - e^\tau \nu D^r_{00} m_r + \nu (\nu - 1) e^\tau \beta_0 \right].$$

Contracting $L_{ijr}$ by $N^j B^i_\alpha B^k_\beta D^r_{0k}$ and using (1.2), (3.10) and above equation, we obtain

(4.21) $$L_{kijr} N^j B^i_\alpha B^k_\beta D^r_{0j} = \frac{2\mu}{\nu e^\tau} M_{\alpha\beta},$$

Transvecting $L_{ijr}$ by $N^j B^i_\alpha B^k_\beta$ and using (1.2), (3.10) and (3.12), we get

(4.22) $$L_{kjr} N^j B^i_\alpha B^k_\beta D^r_{0j} = \frac{2}{\nu e^\tau} \left[ \lambda M_{\alpha\beta} - \frac{e^\tau}{2L} \beta_r C^r_{ij} N^j B^i_\alpha B^k_\beta m_k \right],$$

where

$$\lambda = \frac{1}{2L^2} \left[ -e^\tau (E_{00} - (m^2 + \nu)^{-1} \beta_0 m^2) (2\rho - \nu) - e^\tau \nu (\nu - 1) D^s_{00} m_s + \nu (\nu - 1) e^\tau \beta_0 \right].$$

Similarly, transvecting $L_{kjr}$ by $N^j B^i_\alpha B^k_\beta D^r_{0i}$ and using $M_{\alpha\beta} = M_{\beta\alpha}$, we have

(4.23) $$L_{kjr} N^j B^i_\alpha B^k_\beta D^r_{0j} = \frac{2}{\nu e^\tau} \left[ \lambda M_{\alpha\beta} - \frac{e^\tau}{2L} \beta_r C^r_{ij} N^j B^i_\alpha B^k_\beta m_k \right].$$

Plugging (4.21), (4.22), (4.23) in equation (4.19), we obtain

(4.24) $$D^j_{ik} N^j B^i_\alpha B^k_\beta = \frac{L(\mu - 2\lambda)}{e^\tau \nu} M_{\alpha\beta} + \frac{e^\tau}{2L} \beta_r C^r_{ij} \left[ N^j B^i_\alpha B^k_\beta m_k + N^j B^i_\alpha B^k_\beta m_k \right].$$

Now, suppose that $h$-vector $b_i$ satisfies the condition

(4.25) $$b_r|_0 C^r_{ij} = \kappa h_{ij},$$

where $\kappa$ is a constant.
then
\begin{equation}
\beta_r C_{ij}^r = \kappa h_{ij},
\end{equation}
where \( \kappa \) is a scalar function.

So, using \( h_{ij} B^i_\alpha N^j = 0 \), equation (4.24) yields
\begin{equation}
D_{ik}^j B^i_\alpha B^k_\beta N_j = \frac{L(\mu - 2\lambda) M_{\alpha\beta}}{\nu e^r}.
\end{equation}

And then (4.18) becomes
\begin{equation}
\ast H_{\alpha\beta} = \ast M_{\alpha\beta} e^r e^\nu.
\end{equation}

Next, transvecting (3.6) by \( B^i_\alpha B^j_\beta N^k \) and using (2.10), we have
\begin{equation}
\ast M_{\alpha\beta} = \sqrt{e^r} e^\nu M_{\alpha\beta}.
\end{equation}

Thus from (2.28) and (2.29), we have:

**Theorem 4.3.** For the exponential change with an \( h \)-vector, let the \( h \)-vector \( b_i \) be a gradient and tangential to hypersurface \( F^{n-1} \) and satisfies condition (4.25). Then

1. \( \ast F^{n-1} \) is a hyperplane of second kind if \( F^{n-1} \) is hyperplane of second kind and \( M_{\alpha\beta} = 0 \).

2. \( \ast F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

**5 Example**

A Finsler space \( F^n \) is called \( \ast P \)-Finsler space if the \( (v)hv \)-torsion tensor \( P_{ij}^r \) satisfies
\begin{equation}
P_{ij}^r := C_{ij}^r|0 = \lambda C_{ij}^r.
\end{equation}

Taking \( h \)-covariant derivative of (1.1) and using \( L|k = 0 = h_{ijk} \) and \( \rho_i = 0 \), we get
\begin{equation}
b_{r|k} C_{ij}^r + b_r C_{ij|k} = 0.
\end{equation}

Contracting the above equation by \( y^k \) and using (5.1), we get
\begin{equation}
b_{r|0} C_{ij}^r + \lambda b_r C_{ij}^r = 0,
\end{equation}
which in view of (1.1), becomes
\begin{equation}
b_{r|0} C_{ij}^r = \kappa h_{ij}, \quad \kappa = -\frac{\lambda \rho}{L},
\end{equation}
which is required condition (4.25). Thus, we have:
Theorem 5.1. For the exponential change with an \( h \)-vector, let the \( h \)-vector \( b_i \) be a gradient and tangential to hypersurface \( F^{n-1} \) of a *P-Finsler space \( F^n \). Then

1. \( *F^{n-1} \) is a hyperplane of second kind if \( F^{n-1} \) is hyperplane of second kind and \( M_{\alpha\beta} = 0 \).

2. \( *F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

A Landsberg space is *P-Finsler space for \( \kappa = 0 \).

Thus, we have:

Corollary 5.1. For the exponential change with an \( h \)-vector, let the \( h \)-vector \( b_i \) be a gradient and tangential to hypersurface \( F^{n-1} \) of a Landsberg space \( F^n \). Then

1. \( *F^{n-1} \) is a hyperplane of second kind if \( F^{n-1} \) is hyperplane of second kind and \( M_{\alpha\beta} = 0 \).

2. \( *F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

Discussion

Gupta and Pandey [3] have proved that for Kropina change with an \( h \)-vector (let the \( h \)-vector \( b_i \) be a gradient and tangential to hypersurface \( F^{n-1} \) and satisfies condition \( \beta_i C_{ij}^r = 0 \)),

\( *F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

In present paper, authors proved that for exponential change with an \( h \)-vector (same conditions),

\( *F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

Notice that Kropina change with an \( h \)-vector is finite in nature (in the sense that number of terms) whereas exponential change with an \( h \)-vector is infinite in nature, although in both cases (finite and infinite) same result holds.

The question is that Is there any particular type of change with an \( h \)-vector (same conditions) for which \( *F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind?
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