Decisional states
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Abstract

This article introduces the decisional states of system, and provides a practical algorithm for computing them. The decisional states are defined as the internal states of a system that lead to the same decision, based on a user-provided utility or pay-off function. The utility function encodes some a priori knowledge external to the system, it quantifies how bad it is to make mistakes. The intrinsic underlying structure of the system is modeled by an ϵ-machine and its causal states. The decisional states are the emerging patterns corresponding to the utility function. In a complex systems perspective, these patterns thus form a partition of the lower-level system states that is defined according to the higher-level user’s knowledge.

The transitions between these decisional states correspond to events that lead to a change of decision. An algorithm is provided so as to estimate the states and their transitions from data. Application examples are given for hidden model reconstruction, cellular automata filtering, and edge detection in images.

Keywords: ϵ-machines; decisional states; utility.

1 Motivation

We are monitoring a system, and we are given a utility/cost function for comparing predictions made about this system to what happens really. For example, we are monitoring the weather. We have a pay-off function $U(y, z)$ related to setting an equipment outdoor, with $y$ the weather we predict to take our decision and $z$ what really happens. We benefit from the equipment in the case it is outside when the weather is good, so $U(y = \text{sunny}, z = \text{sunny}) = 1$, while we gain nothing when it is inside and it is raining : $U(y = \text{rain}, z = \text{rain}) = 0$. We miss an opportunity when we keep the equipment indoors when it could have been useful, so $U(y = \text{rain}, z = \text{sunny}) = -1$. The equipment gets damaged under the rain, so $U(y = \text{sunny}, z = \text{rain}) = -2$. We would like events telling us when to set up the equipment or not based solely on the current system state $x$. These events are determined by maximising the expected utility of our predictions $y$ based on $x$.

This simple scenario is easy to transpose to more elaborated contexts. This article presents the theoretical background for this problem, as well as a concrete algorithm for computing the above events from data and a utility function only. The main contributions of this article are:

- The way information is encoded in the utility function, which represents a new and clear way to represent the user knowledge independently of the system’s intrinsic dynamics.
- The decisional states concept, allowing a modeller to represent system states with equivalent decisions for the user based on the preceding utility function.
- The practical algorithm for computing the system states from data, which is also applicable for the reconstruction of ϵ-machines [5].

Section 2 relates the context in which we’ll pose the above problem informally. Section 3 introduces formally how the knowledge brought by a utility function can be used in order to compute the internal states of the process leading to the same decisions. Section 4 gives mathematical examples of the theory introduced in Section 3. Section 5 details how to effectively compute decisional and other utility-related process states from data. Section 6 gives application examples, including inferring hidden states from symbolic time series, detecting patterns in cellular automata and edges in images. A general conclusion is then given, followed by an appendix providing more details about predictions in a physical context and a second appendix highlighting the differences and commonalities with previous work approaches.

Free/libre source code is available and a link to the reference implementation is given at the end of the document.

2 Background information

The idea of using the expected utility in order to determine which decision to take is not new\(^1\) and the first contribution of this paper listed in introduction lies in the way to define the utility function. Usually the utility is defined as a real-valued function associated to each outcome and quantifies the user’s interest in that outcome. Probability theory is then used to estimate the expected utility one may get when taking different actions. The

\(^1\)The Wikipedia entry [http://en.wikipedia.org/wiki/Expected_utility](http://en.wikipedia.org/wiki/Expected_utility) traces the history of the concept back to Bernoulli’s work in the 18th Century.
action with maximal expected utility is then usually retained, although risk-aversion effects are sometimes taken into account. There is an abundant literature on probability and decision theory \[2, 3, 23\] which detail these ideas. Markov Decision Processes \[10, 4\] are a successful framework for modelling a system using this kind of utility functions.

The present article introduces utility functions not on the outcomes themselves, but rather on the effect of acting according to predictions. This is a more natural way of thinking in many contexts, as illustrated by the weather forecasting scenario in introduction: the user tries to predict the future and acts accordingly. The utility is determined by the consequences of what really happens compared to the predictions, hence it is a function of two variables. The utility quantifies the consequences of making mistakes or being right about what happens next.

The basis and theoretical fundation for the present context is thus rooted in predictive models. The \(\epsilon\)-machine is precisely such a model, where the system states are clustered in equivalence classes for making predictions \[17, 8\]. The \(\epsilon\)-machine is also a Markov model, in the sense that transitions between states do not depend on which other state was previously visited. \(\epsilon\)-machines are in addition the minimal Markov automaton leading to optimal predictions while keeping deterministic transitions. See also \[16\] and \[13\] for other classes of Markov models with different properties. For readers familiar to Hidden Markov Models (HMM) and Markov Decision Processes Appendix B highlights the differences between a classical HMM model and the \(\epsilon\)-machine.

When dealing with utility functions based on the effect of predictions the \(\epsilon\)-machine naturally becomes the underlying model that is inferred from data. In other words the utility function determines a structure corresponding to the user knowledge on top of the \(\epsilon\)-machine, while the \(\epsilon\)-machine itself represents the system’s internal relations independently of the user. This clear separation of internal structure vs. external knowledge is a neat secondary effect of defining the utility in terms of the effect of user predictions instead of attributing a utility directly to each outcome.

The third contribution of this paper listed in introduction is to present a family of algorithms (with a practical implementation) for reconstructing \(\epsilon\)-machines and their extension to the new framework introduced in this document (See Section 5). This new family of algorithm offers more flexibility in its data representation and choice of parameters than the previous one \[19\], while providing a computational performance that makes it suitable for a large class of practical applications (See the examples in Section 6). It is possible to call only the \(\epsilon\)-machine reconstruction part (See Section 6.1) of the algorithm, and thus apply it to other frameworks than the one presently considered.

This section aimed at giving the reader an intuitive idea of what the framework proposed in this document is about. The next section describes the framework formally.

### 3 Decisional states framework

#### 3.1 General problem targeted by the proposed framework

Let \(X\) be a measurable space comprising configurations \(x\) of the system under investigation. Let \(Z\) be a measurable space of all entities that we wish to predict from the current system state. For example:

- In a symbolic series context, \(X\) is the set of all past strings up to the current point \(x^− \in A^*\) (Appendix B), and \(Z\) is the set of all future strings after the current point \(x^+ \in A^*\) in Appendix B). The concrete example in Section 6.1 highlights this case.

- More generally for temporal systems with state space, an \(x \in X\) should include all causal influence from the past that might possibly affect the present (i.e., a past light cone). Similarly, \(z \in Z\) is the set of future light cones. Appendix A details this approach, and the concrete example in Section 6.2 highlights this case.

- In the case of a non-temporal system, \(X\) is defined as the relevant space of parameters that have an influence on the system state at the point under investigation, and similarly for \(Z\) being the space of parameters influenced by \(X\). A common example in physics is the Markov Random Field representation of lattice systems \[4\]. In an image context \(X\) is the neighbourhood of a given pixel up to a range that we assume determines the statistical distribution of that pixel, and \(Z\) is the value of the pixel \[4\]. The concrete example in Section 6.3 highlights this case.

For each configuration \(x \in X\), we’d like to associate a prediction \(y_z \in Z\) amongst all possible outcomes. The actual outcome \(z \in Z\) can differ from \(y_z\): we have a range of possible \(z \in Z\), and they occur with a probability distribution \(p(Z|x)\).

Let us now consider that the loss incurred by having acted according to prediction \(y\) when \(z\) is the future that actually happens is quantified by \(L(y, z)\), independently of the particular \(x\) for which \(y\) was chosen instead of \(z\) (so, \(L\) is a real-valued function defined on \(Z^2\)). We could equivalently define a utility function with \(U(y, z) = -L(y, z)\). Minimising the loss is equivalent to maximising the utility, both concepts will be used interchangeably when needed. An important difference between the present context and MDP \[10, 23\] is thus that utility functions have two arguments: the utility quantifies our knowledge of how bad it is to make mistakes. Actions are based on predictions on what we think will happen, and are thus mapped to subsets of possible futures (ex: “going out for a hike” is
mapped to “it won’t rain in the next hours”). Actions are implicit in the utility function: The utility function quantifies the effect of having taken an action based on a prediction $y$, while $z$ actually happens. It is not a globally-defined quantity at the current system state.

We can recover a global quantity by computing the expected utility, integrated over all possible futures that may happen. The expected utility, when it exists, is:

$$E[U] = \int_{x \in X} \int_{z \in Z} U(y, z) p(x, z) \, dx \, dz$$

or in a discrete scenario:

$$E[U] = \sum_{x \in X} \sum_{z \in Z} U(y, z) p(x, z)$$

And for all $x \in X$ with non-null probability\footnote{Technically we should introduce here a set $X' = X \setminus \{x : p(x) = 0\}$ of all $x \in X$ with non-null probabilities. In practice we are dealing with observed system configurations with non-null probabilities, and will act as if $X' = X$.}

$$E[U] = \sum_{x \in X} p(x) \sum_{z \in Z} U(y, z) p(z \mid x)$$

The goal is usually to find a function $y_x$ that maximises the expected utility: this would correspond to making the best predictions on average (and implicitly acting accordingly). By analogy with causal states [8] we now cluster together configurations $x$ according to their statistical properties and look at conditions for which these clustering lead to maximal expected utility $E[U]$.

The user can then decide on an action based on the set of predictions leading to maximal expected utility, and/or based on the value of the utility itself.

3.2 Equivalence relations

$E[U]$ is maximal when each term $T(x, y) = p(x) \sum_{z \in Z} U(y, z) p(z \mid x)$ is maximal (see Eq. \ref{eq:maximal}). Since $p(x)$ is constant for a given $T(x, y)$, and assuming we can choose the $y$ for each $x$ independently, maximising $T$ is equivalent to maximising each $\sum_{z \in Z} U(y, z) p(z \mid x)$. Let us note $U(y \mid x) = E_{z \in Z}[U(y, z) \mid x] = \sum_{z \in Z} U(y, z) p(z \mid x)$, the expected utility of choosing the prediction $y$ for a given $x$.

Another assumption is implicit in this argumentation: that making a decision does not modify the system. The weather forecasting example in the introduction falls in this category. However, sometimes taking a decision modifies the system. For example, when monitoring a patient’s health in order to decide whether to administrate a drug or not. In that case we have to rely on approximations (usually an additional assumption that the change is effective only at a different time scale than that of the observations) so we can still aggregate them on a recent past sliding window. Alternatively, other frameworks like Interactive Learning [21] might be better suited for these situations.

Let us now recall the causal states construction [17]. Appendix A explains with more details how the notion might be derived in a physical context:

**Causal state equivalence relation:** $x_1 \equiv x_2$ if, and only if, the conditional distributions $p(Z \mid x_1) = p(Z \mid x_2)$ are the same. The equivalence classes $\sigma(x) = \{w : p(Z \mid w) = p(Z \mid x)\}$ are called the *causal states*. See Appendix A for a discussion on this term.

By analogy with the causal states construction, let us now define the following equivalence relations:

**Utility equivalence relation:** $x_1 \equiv u x_2$ if, and only if, $\max_{y \in Z} U(y \mid x_1) = \max_{y \in Z} U(y \mid x_2)$. That is, the maximal expected utility is the same at points $x_1$ and $x_2$, even if the sets of optimal predictions $Y(x_1) = \arg\max_{y \in Z} U(y \mid x_1)$ and $Y(x_2)$ that induce this utility might differ for $x_1$ and $x_2$.

**Prediction equivalence relation:** $x_1 \equiv p x_2$ if, and only if, $\arg\max_{y \in Z} U(y \mid x_1) = \arg\max_{y \in Z} U(y \mid x_2)$. That is, the sets of optimal predictions $Y(x_1) = Y(x_2)$ are the same, even if the utility induced by these predictions might differ for $x_1$ and $x_2$.

Let us call *iso-utility states* $v \in \Psi$ and *iso-prediction states* $\psi \in \Psi$, the partitions of $X$ corresponding to these equivalence relations: $v(x) = \{w : w \equiv x\}$ and $\psi(x) = \{w : w \equiv x\}$.

Let us call decisional states $\omega \in \Omega$ the intersection of both: $\omega(x) = \{w : w \equiv x \text{ and } w \equiv x\}$. When both the expected utility and the optimal predictions are the same, we assume the decisions that are taken on the system are the same, hence the name. In other words, we suppose the utility function encodes all that a user needs to take a decision.

These equivalence relations partition the configuration space $X$ into clusters, with the corresponding properties common to all points in the cluster. It should be noted that $E[U]$ as defined on the whole space does not consider which specific decisional state the process is in. Knowing which is the current cluster for any given point $x$ allows us to refine the expected utility to a local $E_v[U]$ (with $x \in v$ the iso-utility state) and which decision to take to reach this utility (by refining again to the decisional state). Section 3.4 details how to derive notions of complexity from these local expected values.
3.4 Transition graphs

In the discrete case the causal states form a deterministic automaton, the \( \epsilon \)-machine \([8]\). Since the causal states sub-partition the other states as mentioned in the previous section, the iso-utility and iso-prediction states (and their intersections) also form automata: these are coarser-grained versions of the \( \epsilon \)-machine. Figure 2 shows decisional states gathering causal states of an underlying \( \epsilon \)-machine, with iso-utility and iso-prediction states on top of the decisional states.

Let \( x_1 \) and \( x_2 \) be in the same causal state \( \sigma \). Then by definition of the causal states \( p(Z|x_1) = p(Z|x_2) \). In that case, the expected utility of any prediction \( y \in Z \) is the same for \( x_1 \) and \( x_2 \): \( \mathbb{E}(y|x_1) = \sum_{z \in Z} U(y,z)p(z|x_1) = \sum_{z \in Z} U(y,z)p(z|x_2) = \mathbb{E}(y|x_2) \). Therefore the optimal predictions and induced utilities are the same: \( x_1 \equiv x_2 \) and \( x_1 \equiv x_2 \), and so is the combination of both.

Thus the causal states sub-partition both the iso-utility, iso-prediction and decisional states.

The converse is not true: we can have two distinct causal states \( \sigma_1 \) and \( \sigma_2 \) with the same maximum value of \( \mathbb{E}(y|x) = \sum_{z \in Z} U(y,z)p(z|x) \) at the same \( y \) points, but with different \( p(z|x) \) for at least one \( z \in Z \).

Figure 1 shows the relations between the different states defined on the process.

3.3 Relation between the causal, iso-utility, iso-prediction and decisional states

Let \( A \) be the next observed symbol, in a discrete scenario with current system decisional state (so automaton, the \( \epsilon \)-machine the transitions are defined using the symbols of \( A \), \( \epsilon \)-machines are further refined by labelling the transitions with the involved symbol in \( A \): \( \epsilon_w \rightarrow \gamma \). However in the present case this is not necessary:

- Iso-utility state transitions are events that change the expected utility, irrespectively of the implied symbols. Several causal states might belong to the same iso-utility state, as depicted in Fig. 1.
- Iso-prediction state transitions are events that change the possible optimal prediction choice, with the same comment.
- Decisional state transitions change at least one of the above.

Moreover, as shown in Figure 2 the transitions between decisional states might cover several transitions between different causal states that sub-partition the decisional ones, in addition to different symbols. While for the \( \epsilon \)-machine the transitions are defined using the symbols of the discrete alphabet, in the present case, there can be at most one transition from one state to another, irrespectively of the implied symbols.

In computational mechanics \([8]\) the mutual information \( C = I(x;\sigma) \) between a configuration \( x \) and the causal state \( \sigma \) for \( x \) is referred to as the statistical complexity. In the present context we might define by analogy a decisional complexity \( D \) as the amount of information necessary to retain about the configuration of a process in order to be able to make an optimal decision, given a utility function. Once that utility function is fixed we can compute the decisional states and define \( D = I(x;\omega) \), where \( x \) is a configuration of the system and \( \omega \) the decisional state for \( x \).
In the discrete case, by definition $D = I(x; \omega) = H(\omega) - H(\omega|x)$ with $H$ proper entropies (differential entropies in the continuous case). But then $H(\omega|x) = 0$ by construction of the $\omega$, which in the discrete case leads to well-defined transition graphs. Thus $D = H(\omega)$ in this case, the amount of information necessary to encode the decisional states. Since $-p(\omega) \log p(\omega) = -(\sum_{\sigma \in \omega} p(\sigma)) \log (\sum_{\sigma \in \omega} p(\sigma)) \leq -\sum_{\sigma \in \omega} p(\sigma) \log p(\sigma)$ because $-\log$ is monotonically decreasing, then $D \leq C$ in the discrete case.

The same construction also works for defining similar quantities:

- $P = I(x; \psi)$ is the difficulty to get the best predictions, tentatively called here the optimal prediction complexity.
- $V = I(x; v)$ is the difficulty to estimate the expected utility of $x$.

### 3.5 Interpretation and notes

Decisional states are equivalent to merging those causal states which lead to the same decisions relatively to our utility function. In this case the causal states have lost their maximality property due to the fact we’re only interested in making a prediction and not in keeping the full conditional distributions. We have, in the general case, clustered together the causal states that lead to the same optimal predictions and maximal expected utility value, based on a given utility function.

Conversely this defines an equivalence relation amongst utility functions: Two utility functions $U_1$ and $U_2$ are equivalent when they induce the same clustering of causal states into the decisional ones, with the same expected values and optimal predictions. These utility functions would induce the same decisions in a system: they are functionally equivalent. Isomorphisms between utility functions leading to the same predictions but with different utility values could also be defined: these are transformations of utility functions that preserve the iso-prediction states. Similarly, transformations could be defined that only preserve the iso-utility states.

The transitions between the iso-utility states correspond to events that provoke a change in the expected utility of the system. Identifying these events might become a crucial practical application, for example for detecting when the expected utility reaches a predefined threshold.

The transitions between the iso-prediction states correspond to events that provoke a change in the optimal predictions that can be chosen. Similarly, a user might be interested in monitoring these changes, for example, to maintain the current action as long as it is appropriate (as long as it matches one of the possible predictions for the system’s evolving iso-prediction state).

The hypothesis made here is that when the cost/utility is defined in terms of a functional (high-level) value, when it has a signification in high-level terms, then the transition events also correspond to interesting high-level objects to look at. This might form the basis for an automated search for meaningful events in a given system’s evolution.

In any case, the utility function encodes external information not available in the original data. So long as one stays with causal states, only information present in the low-level data can be obtained. Much like introducing a prior in a Bayesian framework, here the utility function can be seen as encoding an a priori information not available in the original data. This has at least three consequences from an emergent point of view:

- The causal states represent the finest scale at which we can meaningfully associate a utility function and take decisions. Since the decisional states are supersets of the causal states, then any partition of the data defined with respect to a utility function cannot go below that scale, whatever the chosen utility function.
- Macro-level information needs not be computationally reducible [6] to the lower level in order to be incorporated: The utility function is defined on $Z^2$, not $X$, and it can possibly be incompressible, stated as a value table and not explicitly computed in terms of the lower-level scale. The data $x \in X$ is then clustered into sets which need not have a meaning defined at that level.
- If the hypothesis that “emergent structures are sub-machines of the $\epsilon$-machine” [17, sec. 11.2.2] is correct, then the decisional states are the emergent structures corresponding to a given utility function. Rather than looking for emergent entities directly we might then encode our knowledge in a utility function, and look at the decisional states in order to find good emergent entity candidates. If these do not suffice, we might then refine the utility function iteratively.

Finally, it should be noted the utility function is not the only source of external introduction of knowledge in the system. Additional assumptions are made either implicitly or explicitly if the system is able to generalise to unknown values. For example, the hypothesis that $p(Z|x)$ can be decomposed using kernels or Bayesian networks could be one such assumption. The accuracy of the proposed method for finding decisional states depends on how well these extra assumptions are verified, independently of the chosen utility function.

In particular, results where a bad utility value is expected might be either intrinsic to the data (what we would like to detect) or an artifact indicating the generalisation/sampling/etc. assumptions are not appropriate at this point. Running the same data with different generalisation methods implying different assumptions would be
4 Analytic examples

4.1 Example 1: when bad predictions are useless

In this subsection utility is given to a prediction only if it is correct; otherwise, the prediction is declared useless: \( U(z, z) = 1 \) and \( U(y, z \neq y) = 0 \). In a continuous scenario the delta function \( U(y, z) = \delta(y, z) \) is used instead.

From Section 3.2:

\[
U(y|x) = \sum_{z \in \mathbb{Z}} U(y, z) p(z|x)
\]

then becomes:

\[
U(y|x) = p(y|x)
\]

The set \( Y(x) \) of predictions \( y \) realising an optimal gain becomes:

\[
Y(x) = \{ y : p(y|x) = \max_{z \in \mathbb{Z}} p(z|x) \}
\]

And Eq. (1) leads to:

\[
\mathbb{E}_{\max}[U] = \sum_{x \in X} p(x) \max_{z \in \mathbb{Z}} p(z|x)
\]

But for each causal state \( \sigma \subset \omega \) in each decisional state \( \omega \) the conditional probability \( p(Z|x) = p(Z|\sigma) \) is constant for each \( x \in \sigma \). The decisional states are found by gathering causal states with the same maxima points \( y \) for \( p(Z|\omega) \). We can then write in this special case the above formula as:

\[
\mathbb{E}_{\max}[U] = \sum_{\omega} \sum_{\sigma \subset \omega} p(\sigma)p(y_\omega|\sigma)
\]

where \( y_\omega \) is taken as any maxima of \( p(Z|\sigma) \) common to all \( \sigma \subset \omega \).

Under the condition \( U(z, z) = 1 \) and \( U(y, z \neq y) = 0 \) (or \( U(y, z) = \delta(y, z) \) in the continuous case) the full conditional probability distributions \( p(Z|x) \) do not matter, what’s important is that these distributions peak at the same maxima.

4.2 Example 2: Loss defined by error squared

This section investigates the case where the loss function \( L(y, z) \) can be written as a squared difference between the actual event and the prediction: \( L(y, z) = (z - y)^2 \), provided this operation is meaningful in \( Z \).

We re-develop the treatment from [5 section 1.5.5] in our new context:

With the above loss function Eq. (1) becomes:

\[
\mathbb{E}[L] = \int_{z \in Z} \int_{x \in X} L(y, z) p(x, z) dx dz
\]

\[
\mathbb{E}[L] = \int_{z \in Z} \int_{x \in X} (z - y)^2 p(x, z) dx dz
\]

As in Section 3.2, the goal is to find an \( y_x \) function that minimises the expected loss: \( \mathbb{E}_{\min}[L] \).

The extrema of \( \mathbb{E}[L] \) are given by the functional equation \( \frac{\partial \mathbb{E}[L]}{\partial y_x} = 0 \), with:

\[
\frac{\partial \mathbb{E}[L]}{\partial y_x} = 2 \int_{z \in Z} (z - y_x) p(x, z) dz
\]

Solving \( \frac{\partial \mathbb{E}[L]}{\partial y_x} = 0 \) gives:

\[
\int_{z \in Z} z p(x, z) dz = y_x \int_{z \in Z} p(x, z) dz
\]

So except for a set of \( x \) with null probability mass:

\[
p(x) \int_{z \in Z} z p(z|x) dz = y_x p(x)
\]

\[
y_x = \mathbb{E}_{z \in Z} [z|x]
\]

For a given causal state \( \sigma \), \( p(Z|x) \) is constant for \( x \in \sigma \) so we can write \( y_\sigma = \mathbb{E}_{z \in Z} [z|\sigma] \).

The decisional states are in this example obtained by clustering together the causal states with the same expected value of \( z \) within the state.

These results are obtained because the utility function can be treated analytically; in the general case we do not have such simple formula available. The next section presents an algorithm that can infer the structure of the decisional states from observed data and numerical integration.
5 Estimating the decisional states from data

5.1 General presentation of the algorithm

There are two distinct tasks the algorithm must perform:

- Estimating the probability distributions \( p(Z|x) \) and \( p(X) \) from data. The probability distribution estimator must act on the whole space \( X \). It is responsible for providing values for unobserved data (generalisation ability). It might use all available observations: \( \hat{p}(Z|x) = F(O) \), where \( O = \{(x_i, z_i)\}_{i=1\ldots N} \) represents the data in the form of observation pairs \((x_i, z_i)\), and \( F \) a generic function.

- Clustering \( X \) into causal, iso-prediction, iso-utility and decisional states according to the user needs. This implies as a sub-task estimating the maxima for \( U(y|x) \). The first step is to build \( \hat{U}(y|x) = \int_Z \hat{p}(z|x)U(y,z)dz \) with a user-provided integrator and the utility function. Then, an optimiser might be invoked so as to compute \( Y(x) = \text{Argmax}_{y\in Z} \hat{U}(y|x) \).

So, in summary, the user must provide:

- A probability density estimator \( \hat{p}(Z|X) \) from data observations \( O \).
- A utility function \( U \) acting on \( Z^2 \) with \( Z \) the space of predictions.
- An integrator for computing the expected value of \( U \) with respect to the estimated density.
- A multi-modal optimiser in order to compute \( Y(x) = \text{Argmax}_{y\in Z} \hat{U}(y|x) \).
- A clustering algorithm for gathering probability distributions (for causal states), utility values (for iso-utility states), or similar sets \( Y(x) \) (for iso-prediction states). Decisional states are found by intersection of the iso-utility and iso-prediction states.

Optionally, the user may associate a symbol to each \((x_i, x_{i+1})\) consecutive configurations. This step is detailed in Section 5.2 once the main algorithm is explained and we can see why and when this step may become necessary.

Figure 3 recapitulates these points and shows the algorithm steps that will be detailed in the next sections.

The reference C++ algorithm implementation (see Appendix C) handles common cases by proposing discrete and kernel density estimation, deterministic or Monte-Carlo integration, an exhaustive optimiser especially suited to small search spaces and a simple Genetic Algorithm for larger spaces, and a single-link hierarchical clustering algorithm that finds connected components.
with a user-defined match predicate and threshold. The algorithm is parallelised, allowing the use of multi-core CPUs. The reference implementation is additionally fully templatised and generic, allowing users to plug in their favourite routines: These are the functional inputs of the algorithm in Fig. 3.

5.2 Kernel density estimation

This section describes one way to perform Step 1 in Fig. 3. A discrete probability estimator is suitable for small $X$ spaces where a sufficient amount of data was observed, so that $p(Z|X)$ can be reliably estimated by counting occurrences of all $x$ and $z$. For larger spaces or when unknown or continuous $X$ might be encountered, the system must be able to generalise. We now present the case for a Kernel Density Estimation (KDE) of the probability density $p(Z|X) = F(O)$. In general, the kernel $K(a;b)$ with $a$ and $b$ in the joint space $\{a,b\} \subset X \times Z$ is not separable: The density estimate is $\hat{p}(x,z) \propto \sum_{i=1}^{N} K(x_i,z_i;x,z)$, summing over all observation pairs $(x_i,z_i) \in O$. In the particular case of separable kernels for the configuration space $X$ and the prediction space $Z$ we have instead: $\hat{p}(x,z) \propto \sum_{i=1}^{N} K(x_i;\hat{z}(x_i)/K(z_i;z)$. Even when the kernel is separable the user may benefit from the joint kernel approach: For analysing time series it is natural to consider a moving window of $\text{dim}(X \times Z)$ values and perform the density estimation on the joint space. In another example in Section 6.3 an image is considered as the limit distribution of a Markov Random Field, and the density estimation is also performed on the joint space (with $Z$ being in that example the space of pixel values and $X$ the space of pixel neighbourhoods).

In any case, the conditional probability density is estimated by integrating out the $p(X)$ factor over $Z$: $\hat{p}(z|x) = \int_{\hat{z}(x) \in Z} \hat{p}(x,\hat{z})$. Several sampling mechanisms are provided over $Z$ for the integration, including exhaustive listing of $Z$ for small search spaces. The adequate method depends on the particular user application.

Computing the causal states (and the other states built on top of the causal states) only requires the conditional distributions, and not the joint ones. Without loss of generality it is thus possible to request that $K(a,a) = 1$ with $a$ being $x$ or $z$ or the joint data $(x,z)$ depending on the above cases. For example, the radial basis function $K(a,b) = e^{-||a-b||^2/h}$, with $h$ the kernel width. Indeed, dividing by $\hat{p}(X)$ absorbs the change of scale. Better numerical accuracy is however achieved by requesting $K(a,a) = 1$, especially in high dimensions where the multivariate Normal kernel would lead to very small $K(a,a)$.

The discrete case is recovered when choosing the delta function as a kernel. In that case, similar observations $(x_i,z_i)$ are effectively summed up for a given $x_i$ and the probability estimator is an histogram. In practice it is preferable to use a specialised discrete estimator implementation for efficiency reasons in small search spaces.

Finally, the kernel width $h$ can be chosen according to a variety of estimators from the data. In practice, it has been observed that results ultimately depend on the final task for which the algorithm is applied to. $h$ is then considered as a free parameter, which can be determined for example by cross-validation or by using a genetic algorithm. This gave the best results for classification tasks based on the decisional complexity feature (unpublished author work in progress). An hypothesis is that while the kernel width $h$ found this way does not realise an a priori form of optima (like the AMISE [9]), it realises an a posteriori ideal compromise between bias and variance in the estimated density for the particular task the algorithm is applied to. This is similar to the approach in [3] except that we have reduced the meta-parameter search to $h$ and got rid of the histogram boundaries by using a KDE.

The default implementation proposes a reasonable choice based on the average distance between nearest data points, from which the aforementioned cross-validation and search techniques can build on.

5.3 Using the probability estimates

Two operations are performed using $\hat{p}(Z|x)$:

- Comparison: We need to check whether $\hat{p}(Z|x_1)$ and $\hat{p}(Z|x_2)$ are similar for clustering or not $x_1$ and $x_2$ in the same causal state.
- Expectation: We need to estimate the expected utility of a prediction $y \in Z$ for a given $x \in X$: $\hat{U}(y|x) = \int_{y \in Z} \hat{p}(z|x)U(y,z)$.

Comparison is handled by choosing a similarity measure between probability distributions. The reference implementation proposes the $\chi^2$ statistic, the Bhattacharyya, Variational and Harmonic mean distances, and the Jensen-Shannon divergence. The Bhattacharyya distance is the default for the Kernel Density Estimation, and a $\chi^2$ test is the default for the discrete case.

Let $S \subseteq Z$ be a set of sample points used over $Z$ for comparing the probability distributions (possibly with $S = Z$ for an exhaustive approach). Expectation of the utility for a candidate $y \in Z$ is simply performed numerically over $Z$ at the chosen sample points $S \subseteq Z$: $\hat{U}(y|x) = (\sum_{s \in S} \hat{p}(s|x)U(y,s)) / (\sum_{s \in S} \hat{p}(s|x))$.

5.4 Clustering

Clustering of the causal states is performed directly by matching the probability distributions as defined in the previous subsection.
For iso-utility and iso-prediction states an additional step is necessary: Optimising $\hat{U}(y|x)$ in order to find $Y(x) = \text{Argmax}_{y \in Z} \hat{U}(y|x)$. Any multi-modal optimisation scheme can be invoked at this point. Equivalent best predictions $y \in Y(x)$ must be found, so uni-modal search schemes returning only one candidate are not adapted.

Once the prediction sets $Y(x)$ and the optimal utility values are computed it is possible to cluster them. Approximate matchers might become necessary for checking the appropriate equivalence relations for each state definition, due to numerical precision, limited data size, etc.

False positives are when $x_1$ and $x_2$ are clustered together when they are mathematically not equivalent, false negatives are when the points are in different states when they should not. These risks are minimised by providing more sample points to look at and by increasing the data size. In the limit of an infinite number of data points and samples, consistency is determined by the chosen approximate matchers (ex: the Bhattacharyya distance in the previous section) and by whether the data respects or not the mathematical assumptions needed for the theory to work (ex: conditional stationarity of the $p(Z|x)$).

Additionally we can exploit the fact that causal states sub-partition the decisional ones. If we compute the iso-utility, iso-prediction and decisional states first we might then restrict the search for similar probability distributions $p(Z|x_1)$ and $p(Z|x_2)$ to points $\{x_1, x_2\} \subset \omega$ within each decisional state $\omega$.

If we compute the causal states first (as in Fig. 3), we might use representative $p(Z|x)$ distributions for each causal state, by averaging all $p(Z|x)$ for $x \in \sigma$ in order to reduce numerical discrepancies: $p(Z|\sigma) = \text{avg}_{x \in \sigma} p(Z|x)$. As aforementioned all distributions should be the same mathematically but in practice the estimators are not perfect. This step becomes an effective way to reduce the discrepancies. The expected utility $\hat{U}(y|x)$ is then set to $\hat{U}(y|\sigma)$ for all $x \in \sigma$. This approach (causal states first) was found to give better results in practice.

The chosen clustering algorithm must also handle the unavoidable errors in the estimation of the clustered quantities from finite data: probability distributions $\hat{p}(Z|x)$ for causal states, and some tolerance for floating-point utility value comparison. Ideally, all $x$ with exactly equal such quantities should be put together by definition of the equivalence relations for each state (see Section 3.2). However due to the estimation errors some tolerance has to be given for the equality, which then introduces side-effects like the loss of the transitivity relation, etc.

A single-linked hierarchical clustering is then performed by default: this is equivalent to finding connected components with respect to the given match predicate. The reasons for this choice are:

- Occam’s razor: we want to find the simplest model able to handle the data. Connected components maximise the clusters size by gathering data when a matching path is found between them. This leads to a minimal number of states in the discrete case while ensuring that data in different states do not match (consistency), hence minimal statistical $C = H(\sigma)$ or decisional $D = H(\omega)$ complexity values.

- An interpretation for the continuous case. Connected components ensure $d(a,b) > \Delta$ for $a$ and $b$ points in different clusters, $d$ a dissimilarity measure (ex: in utility values for iso-utility states, on probability distributions for causal states, etc), and $\Delta$ a threshold for the mismatch between clusters. This is equivalent to single-linked hierarchical clustering where we cut the hierarchy at level $\Delta$. In the continuous case the transition graph construction fails on a continuum of infinitely many nearby states. In that case connected components with threshold $\Delta$ ensure that the system state changed at least by that amount when transiting from one point to the next in a different component. For example when monitoring a system expected utility value, a decision might be taken only when a sudden change is detected, but not for a gradual change of the same magnitude.

5.5 Ensuring the $\epsilon$-machine determinism

Causal State Splitting Reconstruction (CSSR) is the reference algorithm for reconstructing $\epsilon$-machines on discrete strings of symbols. It works by recursively splitting the current causal state estimates as the string length is increased. The consistency on shorter string lengths is maintained while the causal states are refined to take into account more symbols. In the limit, it provably converges to the true causal states.

In the present case we do not act on strings of symbols but on $(x,z)$ mappings. Hence it is not possible to refine iteratively the current causal state estimates by enlarging the dimensions of $X$ and $Z$. Yet the “symbols” of discrete data are implicitly present in the $(x_i, x_{i+1})$ transitions when monitoring the system and the order of the data presentation matters (the index $i$ then corresponds to ordered time steps). It would be possible to recover a symbolic representation of the data set from all such transitions, and apply CSSR if so desired. Here we directly cluster the system configurations $x \in X$, not necessarily represented as strings of symbols. For example, each $x \in X$ might correspond to a past light cone (see Appendix A).

The drawback is that the proposed algorithm does not so far ensure that the resulting automaton is deterministic in terms of symbol transitions. The labelled transitions between states can be recovered by looking at the symbol suffix implied by passing from $x_i$ to $x_{i+1}$. But there is no guarantee that a given (state+symbol) combination always lead to the same state deterministically.

Example: Suppose that $x_i = aaba$ and $w_i = abba$ are in the same causal state: $p(Z|x_i)$ and $p(Z|w_i)$ match and
were clustered together with string length limited to depth 4. We observe that \( x_{i+1} = abac \) and \( w_{i+1} = bbac \), with the same suffix \( c \), yet \( p(Z| x_{i+1}) \) and \( p(Z| w_{i+1}) \) do not match anymore and therefore were not clustered in the same state. This is a violation of the \( \epsilon \)-machine determinism: from the same state and with the same symbol, the transition leads to different states. Yet this case is possible when clustering independently \( x_i, w_i, x_{i+1}, w_{i+1} \) into their own states as we do.

For iso-utility, iso-prediction and decisional states this is not a problem: As explained in Section 3.4 transitions are determined in terms of changes in utility related quantities, the string symbols are irrelevant in that case. For an \( \epsilon \)-machine reconstruction however the proposed algorithm needs to be augmented by an additional step.

The user can optionally express symbol values together with each \( x_i \rightarrow x_{i+1} \) transition. These are used as constraints for the clustering algorithm when they are available. The following procedure is implemented:

- Before clustering: If several \( x_i = x \rightarrow x_{i+1} = w \) transitions are observed, with the same configuration value \( x \in X \) and symbol \( a \in A \), then all corresponding \( w \in X \) are pre-clustered in the same state.

- Clustering is performed as described in Section 5.4

- After clustering, iterate the following steps:
  - Split step: It might be that data \( x_1 \) and \( x_2 \) for the transitions \( x_1 \rightarrow w_1 \) and \( x_2 \rightarrow w_2 \) were clustered together, while \( w_1 \) and \( w_2 \) are not clustered together (see the above example). In that case, the state containing \( x_1 \) and \( x_2 \) is split in order to restore determinism.
  - Merge step: As before clustering. Note that splitting acts on the states of mismatching configurations before a transition, while merging acts on the states of matches after a transition, so both can be applied without undoing each other.
  - Break the loop in the case of incompatible constraints and there is no convergence.

Convergence of the loop would effectively ensure determinism of the reconstructed automaton in the perfect case where all distribution estimations are exact.

Unfortunately this is not the case in practice. Indeed, clustering from finite data is necessarily imperfect. If \( x \in \sigma_2 \) is wrongly affected to causal state \( \sigma_1 \) then forcing symbol determinism might create spurious states: \( \sigma_2 \) is erroneously split until the transitions are consistent, while the source of the inconsistency is not detected. Or similarly states are merged when they should not.

We had to accept a threshold for clustering distributions together (ex: a significance level for the Chi-Square test), due to the imperfect distribution estimation. In turn, we have no choice but to accept that some \( x \in \sigma \) might be misclassified and might generate spurious transitions. The same way we ignore small discrepancies in distribution clustering, the solution is to ignore small discrepancies in the automaton determinism. Formally:

Let \( \sigma \) be a causal state, \( a \in A \) a symbol in the alphabet \( A \). The automaton is deterministic when each time a data value \( x \in \sigma \) is followed by the symbol \( a \) then \( w = xa \) falls in a unique causal state \( \varphi, \forall x \in \sigma \). When the automaton is not deterministic there is instead a distribution \( p(\varphi|\sigma, a) \) with \( \varphi \in \Phi \).

We propose here to set a threshold \( \theta = 1 - \epsilon > \frac{1}{2} \) for ignoring small discrepancies up to \( \epsilon \): When \( 3\varphi/p(\varphi|\sigma, a) > \theta \) then the unique such \( \varphi \) is taken as the automaton transition. This is completely independent from the probability of the transition itself \( p(\lambda|\sigma) \).

Concretely, the split and merge step described above are applied only on such transitions \( \varphi \), ignoring the spurious transitions.

5.6 Complexity of the algorithm

Depending on the user context, one or the other of these tasks might become the dominant algorithm cost:

- Estimating the probability distributions \( \hat{p}(Z|x) \). Using the above Kernel Density Estimation the complexity is roughly \( O(N(M + q(N))) \) with \( N \) the number of data and \( M \) the number of samples \( s \in S \subseteq Z \) at which \( \hat{p}(Z = s|x) \) is estimated. \( q() \) is the cost of performing a nearest neighbours query in the joint space (so negligible kernel values are quickly eliminated, the worse-case limit of \( q = O(N) \) is the summing of all kernel values at all data points).

- Clustering tasks. In particular comparing probability distributions is more costly than comparing points, so clustering causal states is usually expensive. Finding connected components might then use up to \( O(KN) \) calls to the dissimilarity measure, with \( K \) the final number of clusters.

- Evaluating the utility function. For the analytical examples in the Section 4 \( U(y, z) \) is simple enough so its evaluation was not the main issue. However in a different scenario the algorithm complexity might have to be defined in terms of the number of evaluations of the cost function.

- Optimising \( \hat{U}(y|x) \) in order to find \( \hat{Y}(x) = \text{Argmax}_{y \in Z} \hat{U}(y|x) \). An exhaustive search of \( Z \) is only feasible for small discrete spaces. Advanced multimodal optimisation techniques might become necessary and induce large computation times.

Finally the memory requirements for running these computations might exceed by far the current processing capabilities if care is not taken. For example it might not be
possible to store all \( \hat{p}(Z|x) \) distributions for each unique \( x \in X \) present in the data set, especially if a large number of samples \( s \in Z \) are needed (ex: the Monte-Carlo sampling error decreases as \( O(1/\sqrt{|S|}) \)).

6 Application examples

6.1 Hidden model reconstruction of the Even process

This first application example demonstrates the capability of performing an \( \epsilon \)-machine reconstruction. Since the \( \epsilon \)-machine is the minimal and optimal deterministic automaton for reproducing a process statistically, and since the decisional states transition graph is a sub-machine of the \( \epsilon \)-machine (see Section 3.5), the proposed algorithm needs to perform well on this task. Moreover recovering a hidden generative model from symbolic observations is a common task for many practical applications, as in [16] where a comparison is explicitly made with \( \epsilon \)-machines and finite memory Markov models.

The Even process is used as a benchmark in [19]. A similar experiment is conducted with the proposed algorithm for comparison.

The Even process consists of two states, and generates binary strings where blocks of an even number of 1s are separated by an arbitrary number of 0s. Despite its apparent simplicity the Even process does not correspond to any finite-length Markov chain [16], and requires the power of an \( \epsilon \)-machine to be reconstructed. Figure 4 shows the process states and transitions.

Figure 4: Definition of the Even process.

Data is generated according to the Even process as a series of symbols. The goal of the experiment is to reconstruct the underlying transition graph from these observations.

The algorithm described in Section 5 is set up with the following parameters:

- System configurations \( x \in X \) are taken as the symbols in a sliding window of size \( L \) past data values. The predictions \( z \in \{0,1\} \) are the symbol in the series following this window.
- Discrete distributions are built by monitoring \((x, z)\) pairs in the training set of size \( N \) generated associations.
- A Chi-Square test is used in order to match distributions, with 5% accuracy.
- The clustering algorithm finds the connected components (see Section 5.4). Symbol constraints are available and implemented as described in Section 5.5 with a tolerance threshold \( \theta = 0.95 \).
- We are not interested in this example in decisional states, so we do not set a utility function.

The result of one typical reconstruction, using \( N = 10^6 \) associations and a past window of 10 points, is shown in Fig. 5. The recurrent causal states of this \( \epsilon \)-machine correctly correspond to the definition of the Even process. Close inspection of the data shows that the transient state corresponds to strings formed of 10 symbols 1 in a row. Due to the limit in window size the algorithm cannot distinguish whether the last symbol 1 was emitted from recurrent state A or B. Logically, it observes that \( \frac{1}{3} \) of the time the next symbol is a 0 in the data set and \( \frac{2}{3} \) of the time it is a 1, matching the proportions of the symbols in the data set: \( p(s = 1) = p(s = 1|A)p(A) + p(s = 1|B)p(B) \) as the process is really in either the state A or the state B.

Parameters: \( 10^6 \) data points, using a past window size of 10 points (default random seed 42).

Figure 5: Reconstruction of the Even process.

The proposed algorithm classifies every single training point in a causal state, hence creates transient states if necessary to match data. The underlying process recurrent states can be found from the strongly connected component of the \( \epsilon \)-machine graph. Note that despite the Even process not being equivalent to any finite-length Markov chain, the proposed algorithm reconstructs it fairly well with a window size of 10.

In [19] an experiment is conducted to study the behaviour of the CSSR algorithm depending on the history size. The transposition of this experiment is conducted in the current framework in order to highlight the differences between both algorithm behaviours.
6.2 Cellular automaton

Another test case where causal states were applied is the detection of moving particles in cellular automata, and their interactions [18]. The introduction of a utility function in this context provides a simple yet effective way to demonstrate the concepts presented in this document. The next section considers the usage of decisional states in a larger-scale application.

Figure 6 shows the result of that experiment: how many recurrent states are found on average (over 30 independent trials) by the proposed algorithm, depending on the window size. This diagram is directly comparable to Fig. 4.

- Results for \( L = 1 \) and \( L = 2 \) produce an incorrect transition graph, there is not enough history to reliably determine the states. The two states for \( L = 1 \) in Fig. 6 are thus not the correct ones.

- CSSR may split the states with each increase in the window size, whereas the present algorithm clusters states using the whole window and symbol constraints. When there is not enough data to estimate the distributions properly CSSR over-splits the states. The current algorithm merges them thanks to the tolerance on state transition determinism as explained in Section 5.5. In either case the underlying problem remains the same: Distributions over large \( L \) cannot be correctly estimated from a small number of observations. As the number of states determines the estimated statistical complexity the user can select which algorithm to apply depending on whether over or under estimated complexities are preferred.

- When there is enough data to estimate the distributions correctly the proposed algorithm gets more precise as the window length increases.

6.2 Cellular automaton

Figure 7 shows the evolution rule and configuration of an elementary one-dimensional automaton. Each row of the regular grid contains the system state at a given time. An evolution rule dictates how the cell binary states evolve at each time step. The evolution rule has a support, 3 cells in figure 7, from which the next cell configuration is deduced. Propagation of this support in time defines “light-cones” according to the terminology of Appendix A, within the implementation constraint of a limited depth.

In this context the data set \( X \) is the space of all past light-cones (in blue on Figure 7). From the current system state we would like to predict the future of the system, so \( Z \) is the space of all future light-cones (in red on Figure 7). Even though the cellular automaton is completely deterministic, the state of cells in the future cone depend on information which is outside the past cone, so we observe a distribution of different futures for each past. See also Appendix A. With cyclic boundary conditions and a fixed evolution rule for the whole automaton, all cells have exactly the same distribution so we can aggregate the observations across all cells.

The utility function is chosen by the user according to the application needs. Here we chose to define the utility of a prediction as the number of correctly predicted cell states in the future cone. Hence the utility takes in this example integer values between 0 and the maximum \( d^2 - 1 \) where \( d \) if the future cone depth (we could also have used a proportion between 0 and 1).

Given the discrete nature and relatively small search space of the problem, the algorithm described in Section 5.4 is setup with:

- A simple discrete probability density estimator based on \((x, z)\) observation counts: \( \hat{p}(z|x) = \frac{\text{count}(x, z)}{\text{count}(x)} \).

- An exhaustive integrator weighting the utility of all possible future cones by their probability: \( \hat{U}(y|x) = \sum_{z \in Z} \hat{p}(y|z)U(y, z) \). In practice unobserved \( z \) values would induce a null contribution so the summing occurs only on observed \( z \).

- An exhaustive search optimiser, computing \( \hat{U}(y|x) \) for all possible \( y \in Z \). The maximum utility value as well as the set \( Y(x) = \text{Argmax}_{y \in Z} \hat{U}(y|x) \) of best predictions are maintained during the search.
6 APPLICATION EXAMPLES

6.3 Image filtering and edge detection

Top-left: Raw cellular automaton field.
Top-right: Statistical complexity field (difficulty to get the future distribution).
Bottom-left: Iso-Utility complexity field (difficulty to get the maximal expected utility).
Bottom-right: Iso-prediction complexity field (difficulty to get an optimal prediction).

Parameters: Past depth 6, future depth 4, 400 cells, 300 steps, 100 initial transient dropped, default random seed 42.

Figure 8: Raw cells and complexity fields of a cellular automaton.

- The connected component, single-link hierarchical clustering algorithm described in Section 5.4, with exact match predicates.

Figure 8 shows the results of the experiment for the rule “110” introduced in Figure 7. It can be compared with [18, Fig. 3]. The raw cellular automaton field is the direct application of this rule. The statistical, iso-utility and iso-prediction complexity fields are mapped to a grey scale range where white represents their respective minimum complexity value and black their respective maximum.

As expected the causal states sub-cluster the iso-predictive and iso-utility states: we observe alternative versions of the particles. The utility is based on the number of correctly predicted cells, irrespectively of their position in the future cone. Information that is irrelevant to this utility function is masked out in the iso-prediction field, whereas it was present in the statistical complexity field.

Some information was lost. But if all the user cares is encoded in the utility function, that information was noise and clarity was gained in the result. In extension to [18], we have defined a new family of automatic filters based on utility functions.

This framework may possibly be adapted for generic feature detection with a variant setup. In this example scenario we are considering edge detection.

Following the construction in [11], the data space $X$ is defined as the neighbourhoods of image pixels $z \in Z$. The prediction problem is to find the value of $z$ from the neighbourhood. Figure 9 shows how the neighbourhood is defined in this experiment: up to two pixels in each direction except corners. Larger or smaller neighbourhoods were tested: smaller regions make the prediction more difficult, larger regions lead to thicker detected edges. The adequate size of the neighbourhood depends on the application. Similarly defining $z$ as a centre block instead of a single pixel was also tried, with similar observed effects (precision and edge size). For a 8-bits grey image the data space is thus $X = [0 \ldots 255]^{20}$ and the prediction space $Z = [0 \ldots 255]$.

The data space $X$ is thus considerably larger than in the previous examples. Fortunately, unlike the cellular...
6.3 Image filtering and edge detection

automaton case where a difference in $x$ can lead to completely different predictions, usually images are not significantly altered when pixel values differ by a small amount. In the present context we exploit this nearby consistency in $X$ and $Z$ in order to apply kernel density estimators. These are more reliable than the simple count-based estimator used in the previous section, especially since $X$ has a higher dimension.

The approach of considering the image as the limit distribution of a Markov Random Field [11] is applied to this example. Concretely, this amounts to estimating the probability densities in the joint space $X \times Z$ and inferring the conditional distributions by integration of $p(X)$, as described in Section 5.2.

The prediction space $Z$ can be run through exhaustively in the case considered here: only one centre grey pixel. Sub-sampling for numerical integration is thus not necessary, and we set $S = Z$.

The utility of a prediction $y$ when the true value $z$ happens is defined as $U(y, z) = -\max(0, |y - z| - \tau)$. In other words, we accept small prediction discrepancies up to $\tau$ at no cost, reflecting the fact the image is not significantly altered by small variations in pixel grey levels. Then the utility decreases (the loss increases) with each grey level difference between the predicted and the true value.

In a first experiment we compute the decisional complexities of each pixel, and plot these values on a grey scale map. The bottom row of Figure 10 corresponds to this scenario. In a second experiment a pre-processing is performed. For each region $x$ we compute the minimal and the average grey level. Either of these can be subtracted from both $x$ and $z$ without loss of genericity (we would still be able to reconstruct the original grey level for a prediction $z$ by adding back the value shift defined on $z$). In practice it was observed that subtracting the minimal value of the neighbourhood leads to better results than subtracting the average. The middle row of Figure 10 uses this pre-processing. Finally the best results are obtained when ordering the states by their complexity values and plotting the ranks of the states instead of the complexity values themselves on a grey scale. The left image on the top row shows the effect of this transform, to be compared to the left image on the middle row. The result of a Sobel filtering is presented for comparison, as well as the original image. All filter results are normalised in grey scale range.

Compared to the Sobel filtering the proposed technique has several distinctive characteristics:

- The filter is defined globally: features on the whole picture are taken into account. For example the flat zones in the original image are detected as having low

complexity. When the filter is applied locally this information is taken into account. While the Sobel filter is sensitive to noise in the local grey level gradients, the proposed filter assigns a low complexity to similar regions. The extreme example would be the cellular automaton background in Figure 8: the Sobel filter would detect many edges for each small triangle pattern, while the proposed filter assigns a low (blank) value for these patterns.

- Fine details are similarly considered statistically on the whole picture. In the bottom-left region of the picture, the filaments attached to the hat are detected as single units: each light-dark transition has its own complexity, low values are whitened out by the ranking. The Sobel filter detects two transitions, one on each side of the filament, according to grey level differences only. Note that this is not a perfect effect: some filaments (on top of the hat light background) are also detected as two transitions with the proposed filter, depending on the global statistical properties of each involved light-dark transition. Yet in general the proposed algorithm produces finer details and less artifacts.

- In a different image where each light/dark edge pattern would occur with a different frequency the proposed algorithm would produce a different result locally, while the Sobel filter would be unaffected by the rest of the picture. This might be an advantage or not, depending on the application.

- Making global statistics and clustering probability distributions comes with a computational cost. While Sobel filtering is very efficient, applying the proposed algorithm requires comparatively large computation times (in the order of one to two hours with a dual-core 3.16GHz Intel CPU).

The effect of increasing the kernel size is shown in the middle and bottom rows. A small Gaussian kernel size effectively induce a bad generalisation (over fitting in another context). This is apparent in the form of a noisy background. The effect of using a rank-based representation attenuates this noise while keeping the high-frequency edge components that have high complexity values (top-left vs middle-left images). Larger kernel sizes smooth more and more details. In the bottom row, with no pre-processing, this effect helps to obtain large regions with approximately the same complexity. When the kernel size is too high black speckles appear on the image, corresponding to spurious $(x, z)$ joint distributions that are unique to this pixel and not found in the rest of the picture.

Finally, no additional well-known image-processing technique were performed on the images, including no hysteresis thresholding, no double-edge detection, and no smoothing/multi-resolution treatment. It would be interesting to combine the complexity-based filters to other

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4 Obtained with The Gimp, http://www.gimp.org/
5 Using ImageMagick "normalize" filter, http://www.imagemagick.org
6 APPLICATION EXAMPLES

6.3 Image filtering and edge detection

Top: Left: Decisional complexity ranks for $h = 0.75$, $\tau = 5$ (preprocessed, normalised). Middle: Result of a Sobel filter (normalised). Right: The Lenna benchmark image (grey scale, unnormalised).

Middle: Left: Decisional complexity values for $h = 0.75$, $\tau = 5$ (preprocessed, normalised). Middle: Decisional complexity values for $h = 5$, $\tau = 15$ (preprocessed, normalised). Right: Decisional complexity values for $h = 7$, $\tau = 17$ (preprocessed, normalised).

Bottom: Decisional complexity values for $h = 5$, $\tau = 15$ (normalised). Middle: Decisional complexity values for $h = 7$, $\tau = 17$ (normalised). Right: Decisional complexity values for $h = 10$, $\tau = 20$ (normalised).

Figure 10: Proposed Image filter (top left) and some variants.
well-established edge detection, segmentation or noise removal algorithms. The goal of this article is to introduce the decisional states and their applications in different contexts. The proposed edge detection is only a demonstration of how the main concept might be used in practice.

7 Conclusion

We derived the decisional states notion for a system: the internal states that lead to the same decision, given a user-defined utility function. Compared to alternative approaches in the domain [10, 24], here the utility function is defined on the space of predictions: $U(y, z)$ quantifies what gain/loss is incurred when $y$ is predicted while $z$ happens. This makes the present work suited for applications like time series processing and detection of anomalous/more complex zones in a system, while less suited for reinforcement learning [22].

The framework chosen for applying the utility function is the $\epsilon$-machine [8], which is a Markovian graph model with higher genericity than finite-memory Markov chains [16]. In this context the $\epsilon$-machine corresponds to the internal structure of the system, irrespectively of any user-defined utility. The decisional states are built on top of this internal structure in a way that reflects the external knowledge (encoded in the utility function) brought in the system.

Coming with the decisional states are definitions of complexity measures on the system. It is possible to quantify precisely, in number of bits, the difficulty there is to make an optimal prediction in terms of the chosen utility. Another consequence is a way to identify events that provoke a change of decision, represented as transitions in a state diagram, assuming decisions are based on the expected utility.

A family of algorithms was introduced in order to compute the decisional states from data. This family of algorithms is adaptable to specific application needs with: The utility function, a probability density estimator, an integrator, a multi-modal optimiser, etc. This algorithm is usable both for computing the $\epsilon$-machine, and for computing the newly introduced decisional states on top of it.

The decisional states were exemplified mathematically on analytically tractable examples, and numerically on practical problems like image filtering. The technique presented in this paper is generic and applicable to a wide range of topics. A reference implementation is provided, see Appendix C. This C++ code is highly adaptable as well as optimised for a variety of common cases. It is available as free-libre software.

Appendix A: Causality, predictions and causal states in a physical context

Let us consider what a prediction means in a physical framework, where information transfer is limited in speed. Fig. 11 displays a schematic view of a system’s past and possible future.

![Figure 11: Light-cone representation of a prediction problem](image)

In this view the system present is a single point in state-space. Contrast this with dynamical systems where the present is the whole state vector, the line in Fig. 11. Here there is no instant propagation of information, and only a small portion of the state vector is accessible.

The past light-cone is the collection of all points that could possibly have an a priori causal influence on the present. The future cone is the collection of all points that might possibly be influenced by the present state. The problem is that in order to infer correctly the state of a point $F$ in the future cone we might potentially need all points in the past light cone of $F$. It would theoretically be possible to have access to the points like $P$ in the system current past, provided in practice that we indeed recorded the value of $P$. However there is by definition no way of getting the value of points like $O$ that are outside the current system past light cone. Since both points belong to the past light cone of a point $F$ in our own future, the consequence is that even for deterministic systems we get a statistical distribution of possible futures for a given observed past, depending on what information present outside the current past cone is necessary to predict the future. In other words, boundary and/or conditions in inaccessible regions may determine part of the future, which is well-known in physics.

Let us now consider grouping two past cones $x_1^-$ and $x_2^-$ together if they lead to the same distribution of futures $p(X^+|x_1^-) = p(X^+|x_2^-)$. Suppose that point $P$ in the past have a distinct value for pasts $x_1$ and $x_2$. Then there
is no way to recover what the value of \( P \) was by new observations: we cannot use future knowledge to decide between \( x_1 \) and \( x_2 \) since \( p(X^+|x_1) = p(X^+|x_2) \). For all practical matter these two pasts are then equivalent. Mathematically the associated equivalence relation \( x_1 \sim x_2 \) partitions the system past light cones in sets \( \sigma(x^-) = \{y^- : p(X^+|y^-) = p(X^+|x^-)\} \).

The sets \( \sigma \) are called the causal states of the system \([17]\).

In a discrete scenario a new observation leads to a transition from a state \( s_1 \) to a state \( s_2 \). The causal states and their transitions form a deterministic automaton: the \( \epsilon \)-machine \([8]\). A neat result is the abstraction of the time dependencies into the states. The transitions between states include all dependencies from the past that could have an influence on the future, hence the \( \epsilon \)-machine actually forms a Markovian automaton \([17]\).

The causal states construction is not limited to light-cones as described above. We can also cluster together data (parameter) points \( x \in X \) according to the conditional distributions \( p(Z|x) \) of points \( z \in Z \) in a space of predictions. The same equivalence relation as above can be defined, except that now care must be exercised on the interpretation: all we have defined are internal states with the same predictive power, without referring to causality. For example, sneezing and coughing are good predictors for being ill, though they are the symptoms and not the cause of the illness. However, when the space \( X \) is restricted to past time (and \( Z \) to future time) as is the case in this section, it is a reasonable assumption that a causal relation indeed provides the desired predictive ability. We refer to the equivalence classes induced by the above relation as causal states in this document, following the current usage of the term, but keep in mind that prediction and causation are different issues.

**Appendix B : \( \epsilon \)-machines presentation for HMM practitionners**

In the present approach we model the data as being produced by an \( \epsilon \)-machine, on top of which we seek the structure imposed by a user-defined utility function. The result in the discrete case (see section 3.4) is a deterministic Markovian automaton, in the sense that transitions between states do not depend on which other state was previously visited.

The framework is thus of determining hidden states and their transitions from data, and the \( \epsilon \)-machine is a class of Markov model \([16]\). In the present case however the state-to-state transitions also correspond to symbol emissions, so we are dealing with an edge-emitting Markov model \([13] \). Fig. 12 shows the difference between the usual Hidden Markov Models and the \( \epsilon \)-machine.

Let us consider an alphabet \( \mathcal{A} = \{A, B\} \). In the Hidden Markov Model framework \([15] \), the hidden states \( \mathcal{S} = \{s_1, s_2, s_3\} \) each have a different probability distribution \( q(X|s_i) \) over \( X \in \mathcal{A} \) for emitting the symbols while in state \( s_i \). Transitions between states are determined by probability distributions \( p(s_j|s_i) \) for reaching state \( s_j \) while in state \( s_i \). Symbols are produced according to \( q_d \) while in state \( s_j \), with possibly a state-to-state transition according to \( p_i \) before symbol production.

In the \( \epsilon \)-machine framework each hidden state \( s_i \in \mathcal{S} \) corresponds to a distinct conditional probability \( p(X^+|s_i) \) of future strings \( x^+ \in X^+ \subset A^+ \). Each state \( s_i \) gathers past strings \( x^- \in A^- \) with the same conditional probability of future observations. The states \( \mathcal{S} \) form a partition of all possible past strings. Symbols are produced on state transitions: when a symbol is emitted, the updated state \( x_{t+1} \) including the newly emitted symbol might fall in the same or in another state \( s_j \in \mathcal{S} \) (see Fig. 12). \( \epsilon \)-machine is the minimal deterministic automata that is able to reconstruct the statistical aspects of a process.

The differences between Markov processes with finite memory and \( \epsilon \)-machines is expressly highlighted in \([16] \) in terms of shift spaces \([12] \): Markov processes with finite memory cover finite-type shifts while \( \epsilon \)-machines can capture the structure of the more general sofic shifts. A yet more generic type of models is proposed by \([13] \), with a tighter internal representation while retaining an optimal generative power, but at the cost of losing the automaton determinism.

In Markov Decision Processes (MDP) \([10, 7] \), the previous observations are used to build a model of the world (possibly with hidden states). Actions are then chosen according to this model in order to maximise the expected utility that would result of that choice (including costs and rewards). MDP consider a feasible set of actions that can be taken at any internal state, and the effect of these actions. The utility can be expressed as a function of histories \( U(x^-) \) or it can be expressed in terms of a reward \( R(S = s) \) of being in state \( s \) and a cost function \( C(S = s, A = a) \) of taking action \( a \) while in state \( s \). In \([21] \) each hidden state of a the Hidden Markov Model is attributed a mean and variance utility of the process to reconstruct data exactly. Here we require only that the reconstructed data have the same statistical properties as the original data, and we are not dealing with exact reconstruction of specific strings of events.

6Compare this with the Kolmogorov-Chaitin complexity, which is defined in terms of a minimal program on a Turing machine that is able to reconstruct the data exactly. Here we require only that the reconstructed data have the same statistical properties as the original data, and we are not dealing with exact reconstruction of specific strings of events.
be in that state. This approach fares well when the utility is itself a global quantity and may evolve over time.

In the present framework, previous observations are also used to build a model (the $\epsilon$-machine). However, the $\epsilon$-machine captures equivalence classes of probability distributions of futures by definition. Hence, all past observations within the same state will lead to the same optimal decisions (see section 3.2): these decisions are taken based on predictions of what will happen next. The utility is a function defined by the user, and quantifies the benefits/costs incurred by comparing what we thought would happen (next observations, a future $y^+$) to what really happens (the true future $x^+$).

An assumption is that the utility function encodes all the information needed to take a decision: When both the expected utility and the predictions are the same, we assume the user takes the same decisions. This framework is well suited to the scenario given in introduction, but perhaps not so well suited to reinforcement learning [13] [22]. However the same formalism is applicable to any system in which a prediction “usefulness” can be defined, including classical loss/utility functions like the minimum sum of squares error between the prediction and the actual future.

Table 1 recapitulates the differences between MDP and Decisional States.

A framework that is closely related to reinforcement learning and that also makes use of $\epsilon$-machines has recently been proposed [21]. It shows that a balance between exploration and control emerges as a consequence of using the $\epsilon$-machine formalism without having to introduce that balance explicitly. That framework also relates learning with energy minimisation, and it makes explicit the agent actions. The present approach differs in that it introduces utility functions and considers that all futures are not equivalent for the agent. It would be interesting to try combining both approaches.

### Appendix C: Web information

The reference implementation of the algorithm presented in this article is available on the author web site: http://nicolas.brodu.numerimoire.net

The code is highly templatised and the classes might be directly included into a user project. The code is available as free-libre software (GNU LGPL v2.1 or more recent) and contributions are welcome.

The latest experimental version of the code as well as any previous version are available at the source repository at http://source.numerimoire.net/decisional_states.

The experiments presented in the main document are reproducible with the version tagged ArXiv_v2.

| Observations | MDP | Decisional States |
|--------------|-----|-------------------|
| Provided to the agent at each time step as a random variable $O^t$. The extent to which the observation includes the current system state (fully and partially observable cases) determines how well the agent can build a model of the world, and how well it can estimate the result of its actions. | Provides as pairs of past and future histories (or light-cones, see Appendix A). An observation is $o = (x^-, x^+)$. The agent has access to the current history $x^-$ at each time. Internal states are not directly accessible to the agent: the information necessary to infer the system state from $x^-$ is a characteristic of the system (See section 3.4). |
| Utility | Provided to the agent as the result of its actions. $U(x^-)$ quantifies the utility of the history $x^-$. It is usually separated in a reward $R(S = s)$ of being in state $s$ and a cost function $C(S = s, A = a)$ of taking action $a$ while in state $s$. | Encodes the knowledge of the experimentalist, the cost of making mistakes on predictions of the future. $U(y^+, x^+)$ quantifies the utility of predicting the future $y^+$ when $x^+$ really happens. |
| Actions | A feasible subset of possible actions can be taken for each state. The effects of an action $A = a$ at state $S^t = s$ are described as $P(S^{t+1} | s, a)$. Actions are chosen on the expected utility of the history that would result of this choice. | Actions are implicit and need not be defined. We rather assume that the user acts based on predictions of the future (see the example in introduction). Actions are thus mapped to sets of predictions $y^+$ of the future. When both the maximal expected utility $\max_{y^+} E_y [U(y^+, x^+) | x^-]$ and the predictions $\{y^+\}$ that reach this maximum are the same, we assume the same action will be taken. See Section 3. |

Table 1: Differences between MDP and decisional states
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