Well-posedness of a fully nonlinear evolution inclusion of second order

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Abstract

The well-posedness of the abstract Cauchy problem for the doubly nonlinear evolution inclusion equation of second order

\begin{align*}
&u''(t) + \partial\Psi(u'(t)) + B(t, u(t)) \ni f(t), \quad t \in (0, T), \quad T > 0, \\
&u(0) = u_0, \quad u'(0) = v_0
\end{align*}

in a real separable Hilbert space $\mathcal{H}$, where $u_0 \in H, v_0 \in D(\partial\Psi) \cap D(\Psi), f \in L^2(0, T; \mathcal{H})$. The functional $\Psi : \mathcal{H} \to (-\infty, +\infty]$ is supposed to be proper, lower semicontinuous, and convex and the nonlinear operator $B : [0, T] \times \mathcal{H} \to \mathcal{H}$ is supposed to satisfy a (local) Lipschitz condition. Existence and uniqueness of strong solutions $u \in H^2(0, T^*; \mathcal{H})$ as well as the continuous dependence of solutions from the data re shown by employing the theory of nonlinear semigroups and the Banach fixed-point theorem. If $B$ satisfies a local Lipschitz condition, then the existence of strong local solutions are obtained.

Keywords evolution inclusion of second order · Well-posedness · Orlicz space · Fixed point argument · Nonlinear semigroups · Subdifferential operator

Mathematics Subject Classification 34G25 · 46N10 · 47H20 · 47J35

1 Introduction

Throughout the article, let $(\mathcal{H}, | \cdot |, (\cdot, \cdot))$ be a real separable Hilbert space equipped with the inner product $(\cdot, \cdot)$ and the induced norm $| \cdot | := (\cdot, \cdot)^{1/2}$ that we identify with its topological dual space $\mathcal{H}^*$ through the Riesz representation theorem, see, e.g., BRÉZIS [Bré11, Theorem 4.11, p. 97]. Then, we investigate the abstract Cauchy problem

\begin{equation}
\begin{cases}
    u''(t) + \partial\Psi(u'(t)) + B(t, u(t)) \ni f(t) & \text{for a.e. } t \in (0, T), \quad T > 0, \\
    u(0) = u_0, \quad u'(0) = v_0
\end{cases}
\end{equation}

where $\partial\Psi : \mathcal{H} \ni \mathcal{H}$ is the Fréchet subdifferential of the functional $\Psi : \mathcal{H} \to (-\infty, +\infty], B : [0, T] \times \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, and $f : [0, T] \to \mathcal{H}$ is an external source.

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We recall that for a proper, lower semi continuous and convex functional \( f : \mathcal{H} \to (-\infty, +\infty] \), the subdifferential of \( f \) in \( u \in D(f) \) is given by

\[
\partial f(u) = \{ \xi \in \mathcal{H} : f(u) - f(v) \leq \langle \xi, u - v \rangle \}.
\]

1.1 Literature review

Multivalued evolution equations (or evolution inclusions) of second order of the form

\[
u''(t) + A(t)u'(t) + B(t)u(t) \ni f(t), \quad t \in (0, T),
\]

have been studied for several cases and under certain conditions by several authors. They are mainly two cases to be distinguished: In the first case, the principal part of \( A \) is linear and in the second case, the principal part of the operator \( B \) is linear. Within the first case, B. Emmrich & Thalhammer showed in [Em,Th] that for a proper, lower semi continuous and convex functional \( \Psi \rightarrow \mathcal{H} \), the subdifferential of \( \Psi \) in \( u \in D(\Psi) \) is given by

\[
\partial \Psi(u) = \{ \xi \in \mathcal{H} : \Psi(u) - \Psi(v) \leq \langle \xi, u - v \rangle \}.
\]

The existence and uniqueness of strong solutions in the case where \( B \) is a linear operator has also been discussed by several authors. In B. Bac22, the author investigated the Cauchy problem for the doubly nonlinear equation

\[
u''(t) + \partial \Psi_{u(t)}(u'(t)) + \partial \mathcal{E}_t(u(t)) + B(t, u(t), u'(t)) = f(t), \quad t \in (0, T),
\]

where \( \partial \Psi_{u(t)}(u'(t)) \) is multi-valued and nonlinear in \( u' \) as well as \( u \), and the principal part of \( \partial \mathcal{E}_t(u(t)) \) is a linear, bounded, positive, and symmetric operator. The operator \( B \) can be considered a strongly continuous perturbation of \( \partial \Psi_{u(t)}(u'(t)) \) and \( \partial \mathcal{E}_t(u(t)) \). The existence and uniqueness of strong solutions in the case where \( A \) is a linear, bounded, symmetric, and positive and \( B \) is a maximal monotone operator has been shown by Barbu [Bar10].

In the setting of single-valued operators, this has been studied in the seminal work [LiS65] of Lions & Strauss, where the authors have shown the well-posedness of the Cauchy problem for the doubly nonlinear evolution equation

\[
u''(t) + A(t, u(t), u'(t)) + Bu(t) = f(t), \quad t \in (0, T),
\]

where \( B \) is a linear, self-adjoint, and unbounded operator and \( A \) is a nonlinear operator being nonlinear in \( u' \) and linear in \( u' \) and satisfying a monotonicity type of condition. Emmrich & Thalhammer showed in [EST15] the existence of solutions where for each \( t \in [0, T] \), \( A(t) : \mathcal{V}_a \to \mathcal{V}_a^* \) is a hemicontinuous operator that satisfy a suitable growth condition such that \( A + \kappa I \) is monotone and coercive, and the operator \( B(t) = B_0 + C(t) \):
$V_B \to V_B^*$ is the sum of a linear, bounded, symmetric, and strongly positive operator and a strongly continuous perturbation $C(t)$ with the same assumptions on $V_A$ and $V_B$ as above. In all the previous cases where the principal part of $B$ is linear, the operators $A$ and $B$ are, in general, defined on different spaces.

Doubly nonlinear evolution inclusions where the leading parts of $A$ and $B$ are both nonlinear and that are defined on different spaces, are unfortunately not treatable in our framework. However, in some concrete problems, this has been shown by exploiting the special structure of the operators. For example, Puhst [Puh15] showed the existence of weak solutions under the assumption that the operators $A$ and $B$ are nonlocal operators. Friedman & Nečas [FrN88] showed the existence of weak solutions under the assumptions that the operators are potential operators that are twice differentiable such that the Hessian matrices are uniformly positive definite and bounded. Bulíček, Málek & Rajagopal [BMR12] and Bulíček, Kaplický & Steinhauser [BKS13] showed the existence of weak solutions under the assumptions that the operators satisfy strong monotonicity, Lipschitz, and growth conditions, which has been shown to be classical solutions under stronger regularity conditions on the operators.

However, to the author’s best knowledge, there are no abstract results for fully nonlinear evolution inclusions.

For further results on nonlinear evolution equations, we refer to Leray [Ler53], Dionne [Dio63], Emmrich & Thalhammer [EmT10, EmT11], Emmrich, Šiška [EmŠ11] including stochastic perturbations, Emmrich, Šiška & Thalhammer [EST15] for a numerical analysis, Emmrich, Šiška & Wróblewska-Kamińska [EvWK16] and Ruf [Ruf17] for results on Orlicz spaces, and the monographs Lions [Lio69], Lions & Magenes [LiM68, Chapitre 3.8], Barbu [Bar76, Chapter V], Wloka [Wlo82, Chapter V], Zeidler [Zei90, Chapter 33], Roubíček [Rou13, Chapter 11] and the references therein. The list of literature presented in this section is of course not intended to be exhaustive.

2 Assumptions and main results

In this section, we establish the existence and uniqueness of solutions as well as the continuous dependence of the solution from the data specified below. Before we state the main result, we collect all the assumptions concerning the functional $\Psi$, the nonlinear operator $B$ as well as the external force $f$.

Assumption A

Let $\Psi : \mathcal{H} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex functional such that $\partial \Psi(0) \neq \emptyset$.

Assumption B

\begin{enumerate}
\item Let $B : [0, T] \times \mathcal{H} \to \mathcal{H}$ satisfy the following local LIPSCHITZ condition: For all $R > 0$, there exists a function $\alpha_R \in L^2(0, T; \mathbb{R}_0^+)$ such that

$$|B(t, u) - B(t, v)| \leq \alpha_R(t)||u - v|| \quad \text{for all } u, v \in B(0, R),$$

(2.1)

and almost all $t \in [0, T]$, where $B(0, R)$ denotes the closed ball in $\mathcal{H}$ with radius $R > 0$ and center $0 \in \mathcal{H}$. Furthermore, there exists a function $g \in L^2(0, T; \mathbb{R}_0^+)$,
such that
\[ |B(t,0)| \leq g(t) \quad \text{for a.a. } t \in [0,T]. \]

\( ii) \) For all strongly measurable \( v : [0,T] \to \mathcal{H} \), the map \( t \mapsto B(t,v(t)) \) is strongly measurable as a mapping from \([0,T]\) to \( \mathcal{H} \).

**Remark 2.1** From Assumption A, it follows in particular that the subdifferential \( \partial \Psi \) is a maximal monotone operator in the sense of Brézis, see [Bré73].

**Remark 2.2** From the Lipschitz continuity and the square-integrability of \( \alpha \), we infer that the map \( t \mapsto B(t,u) \) for all \( u \in \mathcal{H} \).

Having collected all assumptions, we are in the position to state the main result.

**Theorem 2.3** Let \( \Psi : \mathcal{H} \to (-\infty, +\infty] \) and \( B : [0,T] \times \mathcal{H} \to \mathcal{H} \) be given and satisfy Assumption A and Assumption B. Then, for every initial values \( u_0 \in \mathcal{H}, v_0 \in D(\partial \Psi) \cap D(\Psi) \) and every external source term \( f \in L^2(0,T; \mathcal{H}) \), there exists a unique local in time strong solution to (1), i.e., there exits \( \tilde{T} > 0 \) and functions \( u \in H^2(0,\tilde{T}; \mathcal{H}) \) and a \( \eta : [0,T] \to \mathcal{H} \) strongly measurable such that

\[
\begin{align*}
    u''(t) + \eta(t) + B(t, u(t)) &= f(t) \quad \text{for a.e. } t \in (0, \tilde{T}), \\
    \eta(t) &\in \partial \Psi(u'(t)) \quad \text{for a.e. } t \in (0, \tilde{T}),
\end{align*}
\]

and the initial conditions \( u(0) = u_0 \) and \( u'(0) = v_0 \) are fulfilled. If \( B \) satisfies the global Lipschitz condition: There exists a function \( \alpha \in L^2(0, T) \) such that

\[ |B(t, v) - B(t, w)| \leq \alpha(t)|v - w| \quad \text{for all } v, w \in \mathcal{H}, \]

and almost every \( t \in (0,T) \), then there exists a unique global solution \( u \in H^2(0,T; \mathcal{H}) \) to (1). Furthermore, the solution depends continuously on the data, i.e., let \( u_1 \) and \( u_2 \) be the solution to (1) associated with the data \((f, u_1^0, v_1^0)\) and \((g, u_2^0, v_2^0)\) from \( L^2(0,T; \mathcal{H}) \times \mathcal{H} \times D(\partial \Psi) \cap D(\Psi) \), respectively. Then, there exists a constant \( M > 0 \) such that

\[ ||u_1 - u_2||_{C([0,T]; \mathcal{H})} \leq M \int_0^T \alpha(t) dt \left( |u_0^1 - u_0^2|^2 + |v_0^1 - v_0^2|^2 + ||f - g||^2_{L^2(0,T; \mathcal{H})} \right). \]

**Proof.** The main idea consists in rewriting the evolution equation as two coupled first order evolutions equations

\[
\begin{align*}
    v'(t) + \partial \Psi(v(t)) + B(t, u(t)) &\ni f(t) \quad \text{for a.e. } t \in (0,T), \\
    u'(t) &= v(t) \quad \text{for all } t \in (0,T), \\
    u(0) &= u_0, \quad v(0) = v_0
\end{align*}
\]

and considering for fixed \( u \in C([0,T]; \mathcal{H}) \) the auxiliary problem

\[
\begin{cases}
    \tilde{v}'(t) + \partial \Psi(\tilde{v}(t)) \ni \tilde{f}(t) \quad \text{for a.e. } t \in (0,T), \\
    \tilde{v}(0) = v_0,
\end{cases}
\]

where \( \tilde{f}(t) = f(t) - B(t,u(t)), t \in [0,T] \). We notice that since \( u \) is continuous and Assumption B holds, \( \tilde{f} \in L^2(0,T; \mathcal{H}) \) is ensured.

Then, based on the theory of nonlinear semigroups, existence and uniqueness of strong
solutions \( \tilde{v} \in H^1(0,T;\mathcal{H}) \) for the auxiliary problem (2.7) such that the differential inclusion in (2.7) is satisfied almost everywhere in \((0,T)\) and \(\Psi(u) \in W^{1,1}(0,T)\) for every initial value \(v_0 \in \overline{D(\partial \Psi)} \cap D(\Psi)\) is well known, see for instance BRÉZIS [Bré73, Proposition 3.12, p. 106] or BARBU [Bar10, Theorem 4.11, p. 158].

Denoting with \( J : C([0,T];\mathcal{H}) \rightarrow H^1([0,T];\mathcal{H}) \) the solution operator which maps the function \(u \mapsto J(u)\) to the unique solution of (2.7), we obtain for \(u_1, u_2 \in B_{C([0,T];\mathcal{H})}(u_0, R)\) for fixed \(R > 0\), the inequality

\[
|J(u_1)(t) - J(u_2)(t)|^2 \leq 2 \int_0^t |B(r, u_1(r)) - B(r, u_2(r))| |J(u_1)(r) - J(u_2)(r)| dr
\]

\[
\leq \int_0^t |J(u_1)(r) - J(u_2)(r)|^2 dr + \int_0^t \alpha_R(r) |u_1(r) - u_2(r)|^2 dr
\]

for all \(0 \leq t \leq T\). Then, with GRONWALL’s lemma, we obtain

\[
|J(u_1)(t) - J(u_2)(t)|^2 \leq e^t \int_0^t \alpha_R(r) |u_1(r) - u_2(r)|^2 dr \quad \text{for all } t \in [0,T].
\] (2.8)

Then, in order to show the existence solutions to the initial value problem (1), it suffices to show that the map \( \mathfrak{F} : B_{C([0,T];\mathcal{H})}(u_0, R) \rightarrow B_{C([0,T];\mathcal{H})}(u_0, R) \) with

\[
\mathfrak{F}(u)(t) := u_0 + \int_0^t J(u(s)) ds, \quad t \in [0, \tilde{T}]
\] (2.9)

possesses a fix point for a time-point \(0 < \tilde{T} \leq T\), where \(B_{C([0,T];\mathcal{H})}(u_0, R)\) denotes the closed ball in \(C([0,\tilde{T};\mathcal{H})\) of radius \(R > 0\) and center \(u_0\) which can be seen as constant function in \(C([0,\tilde{T};\mathcal{H})\). As soon as existence of a fixed point \(u\) of \(\mathfrak{F}\) is shown, it follows

\[
u(t) = \mathfrak{F}(u)(t) = u_0 + \int_0^t J(u(s)) ds = u_0 + \int_0^t v(s) ds, \quad t \in [0, \tilde{T}],
\] (2.10)

i.e., relation (2.6) holds and it follows \(u \in H^2(0,\tilde{T};\mathcal{H})\). Since the operator \(J\) maps the fixed point to the unique solution of the auxiliary problem (2.7), the differential inclusion (2.5) holds as well. Finally, taking into account that the initial conditions are also satisfied, we deduce the existence of a strong solution to (1). We notice that since the resolvent operator \(J\) maps continuous functions into (absolutely) continuous functions, the operator \(\mathfrak{F}\) itself maps continuous functions into continuous functions.

**Uniqueness:**

Before showing the existence of strong solutions, we establish uniqueness of solutions on the whole interval \([0,T]\). For this, we assume there are two solutions \(u_1, u_2 \in H^2(0,T;\mathcal{H})\) of (1) to the same initial data. Then, by making use of (2.8) and (2.10), we obtain

\[
\sup_{s \in [0,t]} |u_1(t) - u_2(t)|^2 \leq \sup_{s \in [0,t]} \int_0^s |J(u_1)(\tau) - J(u_2)(\tau)| d\tau^2
\]

\[
\leq \left( \int_0^t |J(u_1)(s) - J(u_2)(s)| ds \right)^2
\]

\[
\leq \sqrt{t} \int_0^t |J(u_1)(s) - J(u_2)(s)|^2 ds
\]

\[
\leq \sqrt{T} \int_0^t e^s \int_0^s 2\alpha_R(\tau) |u_1(\tau) - u_2(\tau)|^2 d\tau ds
\]

\[
\leq \sqrt{T} e^T \int_0^t \|u_1 - u_2\|^2_{C([0,s];\mathcal{H})} \int_0^s 2\alpha(\tau) d\tau ds,
\]
where $R := \sup_{t \in [0,T]} (|u_1(t)| + |u_2(t)|)$. Defining $a(t) := \|u_1 - u_2\|_{C([0,t];\mathcal{W})}^2$ and $\lambda(t) := \int_0^t 2\alpha_R(\tau) d\tau$, there holds $a, \lambda \in L^\infty(0,T)$ with $\lambda \geq 0$ a.e. in $(0,T)$ such that

$$a(t) \leq \int_0^t \lambda(s)a(s)ds \quad \text{for all } t \in [0,T].$$

**Gronwall's Lemma** yields immediately $a \equiv 0$ on $[0,T]$ so that $u_1 = u_2$.

**Existence:**

In order to prove existence of local solutions, we make use of the Banach fixed-point theorem which provides the existence of a (unique) solution on a possibly small time-interval, i.e., we show existence of local solutions. Then, by iterating this procedure and making sure that the time interval do not minimize in each iteration step, global solution can be constructed. Therefore, we need to check that the conditions of the Banach fixed-point theorem are fulfilled. Primarily, we show that for fixed $R > 0$ the map $\mathfrak{F} : B_{C([0,T];\mathcal{W})}(u_0, R) \to B_{C([0,T];\mathcal{W})}(u_0, R)$ is well defined for sufficiently small $\tilde{T} > 0$, i.e., it maps the closed ball in $C([0, \tilde{T}]; \mathcal{W})$ of radius $R > 0$ and center $u_0$ into itself. In order to do that, we need the following a priori estimate:

$$\frac{1}{2} \frac{d}{dt} |J(u)(t)|^2 = \frac{1}{2} \frac{d}{dt} |v(t)|^2 = (v'(t), v(t)) \leq (v'(t), v(t)) + (\xi(t) - \eta, v(t)) = (f(t) - B(t, u(t)) - \eta, v(t)) \leq (|f(t)| + |B(t, u(t))| + |\eta|)|v(t)| \leq (|f(t)| + \alpha_R(t)|u(t)| + g(t) + |\eta|)|J(u)(t)| \quad \text{for a.e. } t \in [0, T],$$

where we have tested the auxiliary problem (2.3) with its unique solution $v = J(u)$. This again yields by **Gronwalls Lemma**

$$\sup_{s \in [0,t]} |J(u)(t)| \leq |v_0| + \int_0^t 2(|f(s)| + \alpha(s)|u(s)| + g(s) + |\eta|)ds = C + \int_0^t 2(|f(s)| + \alpha(s)|u(s)| + g(s))ds \quad \text{for all } t \in [0,T]. \quad (2.11)$$

where we used the fact that $J(u)(0) = \tilde{v}(0) = v_0$ and defined $C := (|v_0| + T|\eta|)$. Employing (2.6), we obtain for $u \in B_{C([0,T];\mathcal{W})}(u_0, R)$

$$\|\mathfrak{F}(u) - u_0\|_{C([0,\tilde{T}];\mathcal{W})} \leq \int_0^{\tilde{T}} |J(u)(t)|dt \leq \int_0^{\tilde{T}} C + 2 \int_0^t (|f(s)| + \alpha(s)|u(s)| + g(s))ds dt \leq \tilde{T}(C + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}\|u\|_{C([0,T];\mathcal{W})} + \|g\|_{L^1(0,T)})) \leq \tilde{T}(C + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}(R + |u_0|) + \|g\|_{L^1(0,T)})) \leq R$$

with $\tilde{T} \leq T_1 := R(|v_0| + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}(R + |u_0|) + \|g\|_{L^1(0,T)}))^{-1} > 0$. Second, we show that for sufficiently small $\tilde{T} > 0$, the map $\mathfrak{F}$ is also a contraction. Let $u, v \in$
Then, doing the same calculations as in the uniqueness part, we obtain

\[
\| \mathcal{F}(u) - \mathcal{F}(v) \|_{C([0, \bar{T}] ; \mathcal{H})} = \sup_{t \in [0, \bar{T}]} | \int_0^t (J(u)(s) - J(v)(s)) ds |
\]

\[
\leq \int_0^{\bar{T}} |J(u)(s) - J(v)(s)| ds
\]

\[
\leq \int_0^{\bar{T}} \|u - v\|_{C([0, \bar{T}] ; \mathcal{H})} \int_0^s 2\alpha(\tau) d\tau ds
\]

\[
\leq \bar{T}(2\|\alpha\|_{L^1(0, \bar{T})}) \|u - v\|_{C([0, \bar{T}] ; \mathcal{H})}
\]

\[
\leq L\|u - v\|_{C([0, \bar{T}] ; \mathcal{H})},
\]

where \( L := \bar{T}2\|\alpha\|_{L^1(0, \bar{T})} < 1 \) for \( \bar{T} < T_2 := (2\|\alpha\|_{L^1(0, \bar{T})})^{-1} > 0 \). Thus, by the BANACH fixed-point theorem, there exists a unique solution \( u \in C([0, \bar{T}], \mathcal{H}) \) to (1) on the time interval \([0, \bar{T}]\) with \( 0 < \bar{T} < \min\{T_1, T_2\} \). We assumed here without loss of generality that \( \alpha \neq 0 \) in \( L^2(0, \bar{T}) \), otherwise \( B \) would be constant almost everywhere and the assertion would be trivial.

Now, there are two possibilities to show global existence of solutions in the case where \( B \) satisfies the global LIPSCHITZ condition. The first possibility is to show the boundedness of the derivative of a solutions on the whole interval, such that blow ups of not only the solution itself but also of its derivative in finite time can not occur. This would lead to an interval of existence independent of the initial values. Then, applying successively the BANACH fixed point theorem to the new initial value problem where the initial values are determined by the solution of the previous step, so that this procedure would cover the whole interval. Another possibility is to define the operator \( \mathcal{F} \) on the whole space \( C([0, \bar{T}] ; \mathcal{H}) \) equipped with a norm equivalent to the standard one and employ again the BANACH fixed point theorem, where we need the equivalent norm to ensure the contractivity of \( \mathcal{F} \). We tackle the problem with the latter option and define the operator \( \mathcal{F} : C(0, T; \mathcal{H}) \to C(0, T; \mathcal{H}) \) as in (2.5), where we equip the space \( C(0, T; \mathcal{H}) \) with the norm \( \|v\|_x := \sup_{t \in [0, T]} e^{-Lt}|v(t)| \) with \( \bar{L} = 2\|\alpha\|_{L^1(0, T)} \). Since \( \mathcal{F} \) is obviously a self map, it
remains to show that \( \mathfrak{F} \) is a contraction:
\[
\| \mathfrak{F}(u) - \mathfrak{F}(v) \|_X = \sup_{t \in [0, T]} e^{-Lt} \int_0^t |J(u)(s) - J(v)(s)| ds \leq \sup_{t \in [0, T]} e^{-Lt} \int_0^t \| J(u)(s) - J(v)(s) \| ds \\
\leq \sup_{t \in [0, T]} 2e^{-Lt} \int_0^t \int_0^s \alpha(\tau) |u(\tau) - v(\tau)| d\tau ds \\
\leq \sup_{t \in [0, T]} 2e^{-Lt} \int_0^t \sup_{\tau \in [0, a]} |u(\tau) - v(\tau)| \int_0^s \alpha(\tau) d\tau ds \\
\leq \sup_{t \in [0, T]} 2e^{-Lt} \| \alpha \|_{L^1(0, T)} \int_0^t \sup_{\tau \in [0, a]} \int_0^s \alpha(\tau) ds \| u - v \|_X \\
= \sup_{t \in [0, T]} 2e^{-Lt} \| \alpha \|_{L^1(0, T)} \frac{e^{Lt} - 1}{L} \| u - v \|_X \\
= \sup_{t \in [0, T]} (1 - e^{-Lt}) \| u - v \|_X \\
= (1 - e^{-LT}) \| u - v \|_X
\]

Therefore, the map \( \mathfrak{F} \) is a contraction on \( C([0, T], \mathcal{H}) \) and by the \textsc{Banach} fixed point theorem there exists a unique fixed point \( u \in C([0, T], \mathcal{H}) \) which is a solution to (1).

**Stability:**

Finally, we want to show the continuous dependence of the solution from the data. Let \( u_1 \) and \( u_2 \) be the solution to (1) associated with \((f, u_1^0, v_1^0), (g, u_0^2, v_0^2) \in L^2(0, T; \mathcal{H}) \times \mathcal{H} \times D(\mathfrak{F})\), respectively. With the same reasoning as for (2.8), we derive with \textsc{Gronwall’s} lemma
\[
|J(u_1)(t) - J(u_2)(t)|^2 \leq e^t \left( |v_1^0 - v_0^2|^2 + \| f - g \|^2_{L^2(0, T; \mathcal{H})} \right) \int_0^t \alpha(\tau) |u_1(\tau) - u_2(\tau)|^2 d\tau
\]
for all \( t \in [0, T] \). Then, continuing as in the uniqueness part, we obtain
\[
\sup_{s \in [0, t]} |u_1(t) - u_2(t)|^2 \leq |u_0^1 - u_0^2|^2 + \sup_{s \in [0, t]} \int_0^s |J(u_1)(\tau) - J(u_2)(\tau)| d\tau|^2 \\
\leq |u_0^1 - u_0^2|^2 + \left( \int_0^t |J(u_1)(s) - J(u_2)(s)| ds \right)^2 \\
\leq |u_0^1 - u_0^2|^2 + \sqrt{t} \int_0^t |J(u_1)(s) - J(u_2)(s)|^2 ds \\
\leq |u_0^1 - u_0^2|^2 + \sqrt{t} \int_0^t e^s \left( |v_0^1 - v_0^2|^2 + \| f - g \|^2_{L^2(0, T; \mathcal{H})} \right) ds \\
+ \int_0^t e^s \int_0^s 2\alpha(\tau) |u_1(\tau) - u_2(\tau)|^2 d\tau ds \\
\leq M \left( |u_0^1 - u_0^2|^2 + |v_0^1 - v_0^2|^2 + \| f - g \|^2_{L^2(0, T; \mathcal{H})} \right) \\
+ \sqrt{T} e^t \int_0^t \| u_1 - u_2 \|^2_{C([0, s]; \mathcal{H})} \int_0^s 2\alpha(\tau) d\tau ds,
\]
for a constant $M > 0$ independent of the data. Defining again $a(t) := \|u_1 - u_2\|^2_{C([0,t]; \mathcal{H})}$ and $\lambda(t) := \int_0^t 2\alpha_R(\tau)d\tau$ as well as $b = M \left( |u_0^1 - u_0^1|^2 + |v_0^1 - v_0^1|^2 + \|f - g\|^2_{L^2(0,T; \mathcal{H})} \right)$, there holds $a, \lambda \in L^\infty(0,T)$ with $\lambda \geq 0$ a.e. in $(0,T)$ such that

$$a(t) \leq b + \int_0^t \lambda(s)a(s)ds \quad \text{for all } t \in [0,T].$$

Again, with the Gronwall lemma, we obtain the desired estimate.

\[ \square \]

**Corollary 2.4** In the case, when $B$ satisfies the local Lipschitz condition, there exists a maximal solution, i.e., there exists a time interval $[0,\bar{T}) \subset [0,T]$ and a function $u$ such that for each compact subinterval $[0,S] \subset [0,\bar{T}]$ there holds $u \in H^2(0,S; \mathcal{H})$ and $u$ solves problem (1) pointwise almost every on $(0,\bar{T})$. Furthermore, for every sequence $(t_n) \subset [0,\bar{T})$ with $t_n \nearrow \bar{T}$ as $n \to \infty$, there holds $|u(t_n)| \to +\infty$ as $n \to \infty$.

**Remark 2.5** We notice that we did not impose any compactness assumption neither on the sublevels of the dissipation potential $Ψ$ nor that the operator $B$ is a strongly continuous perturbation in order to show existence of solutions.

### 3 Example

In this section, we want to apply the abstract result to a concrete example. Let, $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $T > 0$. We consider the initial-boundary value problem

\[
(P) \quad \begin{cases}
    \partial_t u(x,t) - \nabla_x \cdot p(x,t) + b(t,x,u) = f(x,t) & \text{in } \Omega \times (0,T),
    \\
    p(x,t) \in \partial_x \psi(x,\nabla_x \partial_t u(x,t)) & \text{a.e. in } \Omega \times (0,T),
    \\
    u(x,0) = u_0(x) & \text{on } \Omega,
    \\
    u'(x,0) = v_0(x) & \text{on } \partial \Omega \times [0,T],
    \\
    u(x,t) = 0 & \text{on } \partial \Omega \times [0,T]
\end{cases}
\]

where $f : \Omega \times [0,T] \to \mathbb{R}, b : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\psi : \Omega \times \mathbb{R}^d \to (-\infty, +\infty]$ are measurable functions satisfying the following conditions:

(3.a) The function $\psi : \Omega \times \mathbb{R}^d \to [0, +\infty]$ is a non-negative Carathéodory function such that $\psi(x,\cdot)$ is a proper, lower semicontinuous, and convex, and $\psi(x,0) = 0$ for almost every $x \in \Omega$.

(3.b) There exists a strictly increasing, convex, and lower semicontinuous function $\Phi : \mathbb{R} \to [0, +\infty]$ with

$$\lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty$$

such that

$$\Phi(|\xi|) \leq \psi(x,\xi)$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$. 
(3.c) The function \( b : [0, T] \times \Omega \to \mathbb{R} \) is a CARATHÉODORY function satisfying the following LIPSCHITZ condition: there exist a constant \( C_b > 0 \) such that
\[
|b(t, x, u) - b(t, x, v)| \leq C_b|u - v| \quad \text{for all } u, v \in \mathbb{R}.
\]

(3.d) There holds \( f \in L^2(\Omega \times (0, T)) \).

Then, we naturally choose \( \mathcal{H} = L^2(\Omega) \). Then, the functional \( \Psi : \mathcal{H} \to \mathbb{R} \) and operator \( B : [0, T] \times \mathcal{H} \to \mathcal{H} \) are given by
\[
\Psi(v) = \int_{\Omega} \psi(x, \nabla v(x)) \, dx
\]
and
\[
(B(t, u), w) = \int_{\Omega} b(t, x, u(x))w(x) \, dx \quad \text{for all } w \in \mathcal{H}.
\]

An important class examples for the function \( \psi \) is given by the class of convex superlinear and anisotropic ORLICZ functions, e.g.,
\[
\begin{align*}
1 & \quad \psi(\xi) = |\xi| \log(1 + |\xi|) \text{ with } \partial \psi(\xi) = \text{Sgn}(\xi) \log(1 + |\xi|) + \frac{\xi}{1 + |\xi|}, \\
2 & \quad \psi(\xi) = |\xi| \exp(|\xi|) \text{ with } \partial \psi(\xi) = \text{Sgn}(\xi) \exp(|\xi|) + \xi \exp(|\xi|), \\
3 & \quad \psi(\xi) = \frac{1}{p} |\xi|^p + |\xi| \text{ for } p > 1 \text{ with } \partial \psi(\xi) = \text{Sgn}(\xi) + \xi |\xi|^{p-2}, \\
4 & \quad \psi(\xi) = \exp\left(\frac{1}{p} |\xi|^p + |\xi|\right) \text{ for } p > 1 \text{ with } \partial \psi(\xi) = \exp\left(\frac{1}{p} |\xi|^p + |\xi|\right)(\text{Sgn}(\xi) + \xi |\xi|^{p-2}), \\
5 & \quad \psi(\xi) = \exp(|\xi| \log(1 + |\xi|)) \text{ with } \partial \psi(\xi) = \exp(|\xi| \log(1 + |\xi|))(\text{Sgn}(\xi) \log(1 + |\xi|) + \frac{\xi}{1 + |\xi|}),
\end{align*}
\]
and so on, where \( \text{Sgn} : \mathbb{R}^d \to \mathbb{R}^d \) denotes the multi-valued and multi-dimensional sign function defined by
\[
\text{Sgn}(\xi) = \begin{cases} B(0, 1), & \text{if } \xi = 0, \\
\frac{\xi}{|\xi|}, & \text{otherwise}. \end{cases}
\]

For more example, see also [EmW13, CaZ09, Don74, GwS11, Zei85]. In that case, the effective domain of the functional \( \Psi \) is given by an anisotropic ORLICZ space which is, in general, neither reflexive nor separable which makes the analysis in general difficult. In the following lemma, we show that conditions of Theorem 2.3 are satisfied.

**Lemma 3.1** Let the conditions (3.a)-(3.c) be fulfilled and assume that there exists \( \tilde{v} \in W^{1,\infty}(\Omega) \) such that \( \Psi(\tilde{v}) < +\infty \). Then, the functional \( \Psi : \mathcal{H} \to [0, +\infty] \) is proper, lower semicontinuous, and convex and the operator \( B : [0, T] \times \mathcal{H} \to \mathcal{H} \) is LIPSCHITZ continuous. Furthermore, let \( v \in \text{D}(\partial \Psi) \). Then, there holds \( \zeta \in \partial \Psi(v) \) if and only if \( \zeta = \nabla \cdot p \) and \( p(x) \in \partial \psi(x, \nabla v(x)) \) for almost every \( x \in \Omega \).

**Proof.** It is easily checked that \( \Psi \) is proper and convex on \( \mathcal{H} \). We show that \( \Psi \) is lower semicontinuous on \( \mathcal{H} \). For that, we equivalently show that the sublevel sets \( J_\alpha := \{ v \in \mathcal{H} : \Psi(v) \leq \alpha \} \) are closed for all \( \alpha \in \mathbb{R} \). Let \( \alpha \in \mathbb{R} \) and let \( (v_n)_{n \in \mathbb{N}} \subset J_\alpha \) such that \( v_n \rightharpoonup v \) in \( \mathcal{H} \) as \( n \to \infty \). By Condition (3.b) and the DE LA VALLÉE POUSSIN criterion,
we know that \((\nabla v_n)_{n \in \mathbb{N}}\) is weakly compact in \(L^1(\Omega)\) and therefore (up to a subsequence) \(\nabla v_n \rightharpoonup \nabla v\) in \(L^1(\Omega)\). By uniqueness of the limit, we infer the convergence of the whole sequence. Now, choose a subsequence \(n_k\) such that
\[
\lim_{k \to \infty} \int_\Omega \psi(x, \nabla v_{n_k}(x)) \, dx = \liminf_{n \to \infty} \int_\Omega \psi(x, \nabla v_n(x)) \, dx.
\]
Then, by Ekeland and Temam [EkT99, Theorem 2.1, p. 243], there holds
\[
\int_\Omega \psi(x, \nabla v(x)) \, dx \leq \liminf_{k \to \infty} \int_\Omega \psi(x, \nabla v_{n_k}(x)) \, dx 
\leq \liminf_{n \to \infty} \int_\Omega \psi(x, \nabla v_n(x)) \, dx,
\]
which shows the lower semicontinuity of \(\Psi\) on \(\mathcal{H}\).

Now, we show the Lipschitz continuity of \(B\): 
\[
(B(t, u) - B(t, v), w) = \int_\Omega (b(t, x, u(x)) - b(t, x, v(x))) \, w(x) \, dx 
\leq \left( \int_\Omega |b(t, x, u(x)) - b(t, x, v(x))|^2 \, dx \right)^{1/2} \|w\|_{L^2(\Omega)} 
\leq \left( \int_\Omega C^2_b |u(x) - v(x)|^2 \, dx \right)^{1/2} \|w\|_{L^2(\Omega)} \quad \text{for all } w \in \mathcal{H}
\]
and hence the Lipschitz continuity of \(B\). The last assertion follows from Proposition 5.1, p. 21, Proposition 5.7, p. 27, and Proposition 2.1, p. 271, in Ekeland and Temam [EkT99] which completes the proof.

We are now in the position, to state the main result.

**Theorem 3.2** Let the Conditions (3.a)-(3.d) be satisfied and assume that there exists \(\tilde{v} \in W^{1,\infty}(\Omega)\) such that \(\Psi(\tilde{v}) < +\infty\). Then, for every \(u_0 \in L^2(\Omega)\) and \(v_0 \in D(\partial \Psi) \cap D(\Psi)\), there exists a unique weak solution \(u \in H^2(0, T, \mathcal{H})\) to (P). Furthermore, the stability estimate 2.4 is fulfilled.

**Proof.** Since the assumptions are all fulfilled, this follows immediately from Theorem 2.3.

**Remark 3.3** We notice again, that we did not impose any compactness condition on \(\Psi\) but rely on the maximal monotonicity of its subdifferential. This implies that we do not identify the nonlinearity in \(B\) by compactness but a fixed point argument. Compare with [EŠWK16], where a time discretization method has been employed to show with compactness arguments the existence of weak solutions in an Orlicz space.

**Remark 3.4** Instead of the global Lipschitz condition (3.c), we can impose a local Lipschitz condition on \(B\). Consequently, we obtain the local existence result given in Theorem 2.3.

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