Equivalent models for gauged WZW theory†

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ABSTRACT

We modify the $SL(2,\mathbb{R})/U(1)$ WZW theory, which was shown to describe strings in a 2D black hole, to be invariant under chiral $U(1)$ gauge symmetry by introducing a Steukelberg field. We impose several interesting gauge conditions for the chiral $U(1)$ symmetry. In a particular gauge the theory is found to be reduced to the Liouville theory coupled to the $c = 1$ matter perturbed by the so-called black hole mass operator. Also we discuss the physical states in the models briefly.

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1. **Introduction**

String theories have been investigated deeply as candidates for the quantum gravity theory. Recently strings in two dimensional (2D) target space have been attracting much attention and been studied in various ways. Especially $SL(2,\mathbb{R})/U(1)$ WZW theory, which was shown to describe strings on a 2D black hole background by Witten,\(^{[1]}\) seems to give us a nice toy model to examine quantum effects to dynamics of space-time such as space-time singularity, black hole evaporation and so on.

In the model the 2D black hole space-time is sliced out from three dimensional parameter space of $SL(2,\mathbb{R})$ group. The sliced manifold depends on not only the choice of the variables but also on the gauge conditions. Actually it has been pointed out that the $SL(2,\mathbb{R})/U(1)$ model is related with the Liouville theory with $c = 1$ matter, which describes strings in a flat 2D space-time, in the ref. \([1]\). Thus the $SL(2,\mathbb{R})/U(1)$ WZW theory may be expected to contain a family of string theories on various background geometries.

The relations between the $SL(2,\mathbb{R})/U(1)$ theory and the $c = 1$ Liouville theory have been discussed by many other people.\(^{[2][4][5][6][7]}\) In the ref. \([5]\) the WZW theory was found to contain the so-called new discrete states in the physical spectrum. On the other hand a isomorphism in the algebraic structures, such as $W_\infty$ algebra, the ground ring and BRST cohomology, between the two theories have been found recently \([6,7]\). Thus this problem has not been completely cleared up yet.

In this note we also study the relations between the two models by starting with the action of the gauged WZW theory. Originally in the $SL(2,\mathbb{R})/U(1)$ WZW theory, the axial $U(1)$ symmetry is explicitly broken reflecting the axial anomaly. We first restore the chiral $U(1)$ gauge symmetry by introducing a Steukelberg field. Then it will be found that a particular gauge fixing reduces the WZW theory to the $c = 1$ Liouville theory perturbed by the so-called black hole mass operator. Also another interesting gauge and the physical spectrum will be briefly examined.
2. Chiral $U(1)$ gauged WZW theory

The classical action of the $SL(2,\mathbb{R})/U(1)$ WZW theory is given by $^1$

$$S_0 = \frac{k}{8\pi} \int_{\Sigma} Tr[g^{-1}\partial_+ gg^{-1}\partial_- g] + \frac{k}{12\pi} \int_B (g^{-1}dg)^3$$

$$- \frac{k}{4\pi} \int_{\Sigma} Tr[A_- g^{-1}\partial_+ g + A_+ \partial_- gg^{-1} - A_+ gA_- g^{-1} - 2A_+ A_-],$$

where $g$ is a group element of $SL(2,\mathbb{R})$ and the gauge fields $A_\pm$ belong to the $U(1)$ subalgebra generated by $\sigma_3$. $^1$ Here we parametrize the $SL(2,\mathbb{R})$ group element by using the Gauss decomposition

$$g = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix} \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}$$

with $\rho, \psi, \chi \in \mathbb{R}$. In this parametrization the action (1) may be rewritten into $^3$

$$S_0 = \frac{k}{4\pi} \int d^2x [(\partial_+ \rho - A_+)(\partial_- \rho - A_-) + e^{2\rho}(\partial_+ + A_+)(\partial_- + A_-)\chi].$$

This action is invariant under a global $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry as well as a local (vector) $U(1)$ symmetry, whose transformations are given by

$$\delta \rho = \epsilon,$$

$$\delta \psi = -\epsilon \psi,$$

$$\delta \chi = -\epsilon \chi,$$

$$\delta A_\pm = \partial_\pm \epsilon.$$ $^4$

If we impose the unitary gauge $\rho = 0$ and integrate out the gauge fields $A_\pm$, then

$^1$ In this note we work on a Minkowski worldsheet and a Lorentz-signature space-time. The Euclidean version will be obtained by an appropriate continuation.
the classical action turns out to be

\[ S = \frac{-k}{4\pi} \int d^2x \frac{\partial_+ u \partial_- v}{1 - uv} \]  

(5)

with \( \chi = u, \psi = -v \). This is the string action on the Lorentzian black hole described by the Kruskal coordinates \( u, v \), which is given in the ref. [1].

For the later purpose let us rewrite the action in first-order formalism with respect to \( \chi \) and \( \psi \) by introducing canonical momenta \( \pi_\chi \) and \( \pi_\psi \)

\[
S_0 = \frac{1}{4\pi} \int d^2x \left[ k(\partial_+ \rho - A_+)(\partial_- \rho - A_-) + \pi_\psi(\partial_- + A_-)\psi + \pi_\chi(\partial_+ + A_+)\chi - \frac{1}{k} \pi_\psi \pi_\chi e^{-2\rho} \right].
\]  

(6)

Now we may recover the broken axial \( U(1) \) gauge symmetry by introducing a Steukelberg field \( \lambda \). The chiral \( U(1) \) invariant action will be given by

\[
S = S_0 - \frac{k}{4\pi} \int d^2x(\partial_+ \lambda - A_+)(\partial_- \lambda - A_-).
\]  

(7)

This action is indeed invariant under the following chiral gauge transformations;

\[
\delta \rho = \frac{1}{2} \theta_L + \frac{1}{2} \theta_R, \\
\delta \lambda = \frac{1}{2} \theta_L + \frac{1}{2} \theta_R, \\
\delta \psi = -\theta_L \psi, \\
\delta \chi = -\theta_R \chi, \\
\delta A_+ = \partial_+ \theta_L, \\
\delta A_- = \partial_- \theta_R.
\]  

(8)

So far we have been discussing the classical action of the \( SL(2, \mathbb{R})/U(1) \) WZW theory. In the quantum theory we need to take into account of quantum corrections
to the classical action. If we integrate out $\pi_\psi$, $\pi_\chi$, $\psi$ and $\chi$ in the action (7) then we will obtain the determinant factor, $\det[e^{-2\rho}(\partial_- - A_-)e^{2\rho}(\partial_+ + A_+)]^{-1}$. This determinant may be easily evaluated by using well-known technique. The $\rho$ dependent part of the determinant is found to be

$$\exp[-\frac{1}{2\pi} \int d^2x (\partial_+ \rho \partial_- \rho - A_+ \partial_- \rho - A_- \partial_+ \rho - \frac{1}{2} \sqrt{-\hat{g}} \hat{R}(\rho))]$$

(9)

where $\hat{g}$ denotes the reference metric in the conformal gauge. On the other hand if we quantize the action (7) naively in the operator formalism, then we will encounter "anomalies" in the operator algebras of the $SL(2,\mathbb{R})$ currents and the energy-momentum tensor. However when we add the quantum correction term given by (9), then the currents will be found to satisfy proper quantum algebras. Details in such analysis will be reported elsewhere.

Anyhow the effective action in the conformal gauge would be given by

$$S = \frac{1}{4\pi} \int d^2x \left[ (k-2)\partial_+ \rho \partial_- \rho + \sqrt{-\hat{g}} \hat{R} \rho - k \partial_+ \lambda \partial_- \lambda \\
+ \pi_\psi \partial_- \psi + \pi_\chi \partial_+ \chi - \frac{1}{k} \pi_\psi \pi_\chi e^{-2\rho} + b \partial_- c + \bar{b} \partial_+ \bar{c} \\
+ A_- \{ \pi_\psi \psi - (k-2)\partial_+ \rho + k \partial_+ \lambda \} \\
+ A_+ \{ \pi_\chi \chi - (k-2)\partial_- \rho + k \partial_- \lambda \} \right]$$

(10)

where the ghosts $(c, b)$ are introduced to fix the diffeomorphism and $\hbar$ is set to 1 simply. Moreover if we rescale the variables as $\rho = \frac{1}{\sqrt{2(k-2)}} \phi$, $\lambda = \frac{1}{\sqrt{2k}} X$, $\psi = a \gamma$, $\pi_\psi = \frac{1}{a} \beta$, $\chi = b \bar{\gamma}$, $\pi_\chi = \frac{1}{b} \bar{\beta}$, this action (10) turns out to be

$$S = \frac{1}{4\pi} \int d^2x \left[ \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{1}{\sqrt{2(k-2)}} \sqrt{-\hat{g}} \hat{R} \phi - \frac{1}{2} \partial_+ X \partial_- X \\
+ \beta \partial_- \gamma + \bar{\beta} \partial_+ \bar{\gamma} + \mu \beta \bar{\beta} e^{-\sqrt{2/k} \phi} + b \partial_- c + \bar{b} \partial_+ \bar{c} \\
+ A_- J_+ + A_+ J_- \right]$$

(11)
where \( J_\pm \) are the chiral \( U(1) \) gauge currents defined by

\[
J_+ = \beta \gamma - \sqrt{\frac{k-2}{2}} \partial_+ \phi + \sqrt{\frac{k}{2}} \partial_+ X, \\
J_- = \bar{\beta} \bar{\gamma} - \sqrt{\frac{k-2}{2}} \partial_- \phi + \sqrt{\frac{k}{2}} \partial_- X.
\]

This action may be regarded as the chiral \( U(1) \) gauged version of the WZW action given in the ref. [2]. The parameter \( \mu (\neq 0) \), which relates to the black hole mass, is completely arbitrary, since it can be changed by the scaling. It should be noted that the action (11) is not invariant under the chiral \( U(1) \) transformations classically, however it is invariant in the quantum level.

3. Gauge fixing and \( c = 1 \) string

In this section we are going to perform gauge fixing to the chiral \( U(1) \) gauge symmetry in the quantum action (11) by using several gauge conditions. First let us impose conditions \( A_\pm = 0 \). Through the standard BRST gauge fixing procedure we may find the gauge fixed action to be

\[
S = \frac{1}{4\pi} \int d^2x \left\{ \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{1}{\sqrt{2k'}} \sqrt{-g} \hat{R} \phi - \frac{1}{2} \partial_+ X \partial_- X \\
+ \beta \partial_- \gamma + \bar{\beta} \partial_+ \bar{\gamma} + \mu \beta \bar{\beta} e^{-\sqrt{k'} \phi} \\
+ b \partial_- c + \bar{b} \partial_+ \bar{c} + \eta \partial_- \xi + \bar{\eta} \partial_+ \bar{\xi} \right\},
\]

where \((\xi, \eta)\) are the ghost fields corresponding to the chiral \( U(1) \) parameter appeared in (8) and \( k' \) denotes \( k - 2 \) hereafter. The BRST charge is also obtained by considering the chiral \( U(1) \) transformations as well as diffeomorphism, and is found to be\(^2\)

\[
Q_{BRST} = \int d\sigma \{ e T_{++}^{\text{matter}} + \xi J_+ + c b \partial_+ c + c \eta \partial_+ \xi \}.
\]

Here \( T_{++}^{\text{matter}} \) is the energy-momentum tensor in the matter sector, which are given

\(^2\) Hereafter we are going to show only the left moving part of the BRST charge, since the right moving part can be always given in a similar form.
explicitly by

\[ T_{++}^{\text{matter}} = T_{++}^{c=1} + T_{++}^{\beta, \gamma} = -\frac{1}{2}(\partial_+ \phi)^2 - \frac{1}{\sqrt{2k'}} \partial_+^2 \phi + \frac{1}{2}(\partial_+ X)^2 - \beta \partial_+ \gamma. \]  \hspace{1cm} (15)

If we discard the term \( \frac{1}{\mu} \beta \bar{\beta} e^{-\sqrt{2/k'} \phi} \) in the action (11), then this model turns out to be a free conformal field theory. In this case the BRST charge (13) is exactly same as one given in the ref. [6,7]. The BRST nilpotency can be achieved when \( k = 9/4 \). However the effects of the discarded term can be incorporated by acting the screening operator \( \beta e^{-\sqrt{2/k'} \phi} \), which plays an important role to determine the physical spectrum and to compute the correlation functions.\(^{[10]} \)

One of the gauge conditions which we would like to examine newly in this note is

\[ \gamma = \bar{\gamma} = 1. \]  \hspace{1cm} (16)

In this case the gauge fixed action is given by

\[ S = \frac{1}{4\pi} \int d^2 x \left[ \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{1}{\sqrt{2k'}} \sqrt{-g} \hat{R} \phi - \frac{1}{2} \partial_+ X \partial_- X + \mu \beta \bar{\beta} e^{-\sqrt{2/k'} \phi} \right. \\
+ A_- \{ \beta - \sqrt{\frac{k'}{2}} \partial_+ \phi + \sqrt{\frac{k}{2}} \partial_+ X \} + A_+ \{ \bar{\beta} - \sqrt{\frac{k'}{2}} \partial_- \phi + \sqrt{\frac{k}{2}} \partial_- X \} \\
\left. + b \partial_- c + \bar{b} \partial_+ \bar{c} + \eta \xi + \bar{\eta} \bar{\xi} \right]. \]  \hspace{1cm} (17)

However here we may drop out the variables \( \beta, \bar{\beta}, A_\pm, \xi, \eta, \bar{\eta} \) by using the non-dynamical equations of motion, \( \beta = \sqrt{\frac{k'}{2}} \partial_+ \phi - \sqrt{\frac{k}{2}} \partial_+ X, \bar{\beta} = \sqrt{\frac{k'}{2}} \partial_- \phi - \sqrt{\frac{k}{2}} \partial_- X, \eta = \bar{\eta} = 0, \xi = \bar{\xi} = 0 \). After performing this the action turns out to be

\[ S = \frac{1}{4\pi} \int d^2 x \left[ \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{1}{\sqrt{2k'}} \sqrt{-g} \hat{R} \phi - \frac{1}{2} \partial_+ X \partial_- X \\
+ \mu \left( \sqrt{\frac{k'}{2}} \partial_+ \phi - \sqrt{\frac{k}{2}} \partial_+ X \right) \left( \sqrt{\frac{k'}{2}} \partial_- \phi - \sqrt{\frac{k}{2}} \partial_- X \right) e^{-\sqrt{2/k'} \phi} \right. \\
\left. + b \partial_- c + \bar{b} \partial_+ \bar{c} \right]. \]  \hspace{1cm} (18)

7
Also we will find the BRST charge to be

\[ Q_{BRST} = \int d\sigma \{ cT_{++} + cb\partial_+ c \}, \]  

(19)

which is identical to the BRST charge of the Liouville theory coupled to a \( c = 1 \) matter \( X \). The term like a gravitational vertex operator in the action (18) is an \((1,1)\) operator called the "black hole mass operator".\(^2\) Thus we have seen that the \( SL(2,\mathbb{R})/U(1) \) WZW theory is completely equivalent to the \( c = 1 \) string theory perturbed by the black hole mass operator. In this model also the screening operator \( \oint (\sqrt{k} \frac{\partial \phi}{2} - \sqrt{k} \frac{\partial X}{2}) \exp(-\sqrt{k} \frac{\phi}{2}) \) coming from the black hole mass operator may be important to calculate the correlation functions.

We may choose another gauge condition

\[ \beta = \bar{\beta} = 1. \] 

(20)

By repeating similar manipulation the gauge fixed action and the BRST charge are found to be\(^3\)

\[ S = \frac{1}{4\pi} \int d^2x \left[ \frac{1}{2} \partial_+ \phi \partial_- \phi + \left( \frac{1}{\sqrt{2}k'} + \sqrt{\frac{k'}{2}} \right) \sqrt{-g} \tilde{R} \phi ight. 
\left. - \frac{1}{2} \partial_+ X \partial_- X - \sqrt{\frac{k}{2}} \sqrt{-g} \tilde{R} X + b\partial_- c + \bar{b}\partial_+ \bar{c} + \mu e^{-\sqrt{k} \frac{\phi}{2}} \right] \]

(21)

and

\[ Q_{BRST} = \int d\sigma \left[ c \left( -\frac{1}{2} (\partial_+ \phi)^2 - \frac{1}{\sqrt{2k'}} \partial_+^2 \phi + \frac{1}{2} (\partial_+ X)^2 - \sqrt{\frac{k'}{2}} \partial_+^2 \phi + \sqrt{\frac{k}{2}} \partial_+^2 X \right) + cb\partial_+ c \right]. \]

(22)

This BRST charge also satisfies it’s nilpotency if \( k = 9/4 \). At a glance this model looks to differ from the \( c = 1 \) string. However if we perform the following "Lorentz" transformation:

\[ 3) \text{The Coulomb term would be deformed correspondingly to the modification of the energy-momentum tensor.}\]
transformation to the 2D target space coordinate \((\phi, X)\),

\[
\begin{align*}
\phi &= (k-1)\tilde{\phi} \pm \sqrt{k(k-2)}\tilde{X}, \\
X &= -\sqrt{k(k-2)}\tilde{\phi} \mp (k-1)\tilde{X},
\end{align*}
\]

then the BRST charge (22) as well as the gauge fixed action (21) written in terms of \(\tilde{\phi}\) and \(\tilde{X}\) are found to be identical to those of the \(c = 1\) string. At the same time the "cosmological constant" term \(\exp(-\sqrt{2/k}\phi)\) is also transformed into an \((1,1)\) operator, \(V_{1-k,-k} = \exp(\sqrt{2/k}(1-k)\tilde{\phi} + \sqrt{2/k}\tilde{X})\). Thus it is shown that the gauge \(\beta = \bar{\beta} = 1\) also gives rise to the Liouville theory with \(c = 1\) matter perturbed by the operator \(V_{1-k,-k}\).

Ishikawa and Kato [7] pointed out that the BRST cohomology of the \(SL(2,\mathbb{R})/U(1)\) WZW theory is splitted into a product of BRST cohomologies of the \(c = 1\) string and of a \(U(1)/U(1)\) topological theory. Our gauge choices kill out the topological freedom explicitly.

4. The physical states

If we admit naively that the string physics described by the gauged WZW model does not depend on the choice of the gauge fixing conditions, up to zero mode, then the three models found in the previous section would be equivalent mutually. However it seems to be still necessary and interesting to find out the explicit relations between these models in order to see how strings in a black hole and strings in a flat space-time are linked to each other.

In the \(A_{\pm} = 0\) gauge, the series of the physical states including the so-called new discrete states have been found by Distler and Nelson [5]. The simplest states among them are given in the form of the \(SL(2,\mathbb{R})\) primaries dressed by the Steukelberg field \(X\) (we consider only the "holomorphic" part below) as

\[
V_{j,m} = \gamma^r \exp\left(\frac{\sqrt{2}}{k}j\phi - \frac{\sqrt{2}}{k}mX\right).
\]

The physical condition \(Q_{BRST}|_{phys} = 0\) restricts the parameters appearing in
(24) to be \( r = j - m, \ j = \pm \frac{m}{3} - \frac{1}{2} \) in the case of \( k = 9/4 \). We may obtain a series of discrete states \( \mathcal{C} \) and \( \mathcal{D} \) in the terminology in the ref. [5] by acting a operator

\[
I^- = \oint \gamma^2 e^{\sqrt{2}X}
\]

(25)
successively to the physical states given by \( V_{j,m} \).

Another series of the discrete states can be obtained from \(^2\)

\[
\tilde{V}_{j,m} = \beta^s \exp \left( \sqrt{\frac{2}{k'}} j \phi - \sqrt{\frac{2}{k}} mX \right),
\]

(26)

where \( s \) is a positive integer and \( s = -j + m \). For \( k = 9/4 \) there are two series of solutions of the physical state condition; (a) \( j = \frac{m}{3} - \frac{1}{2} \) and (b) \( j = -\frac{m}{3} - 1 \). The series (a) give a subset of the new discrete states in \( \tilde{D} \) \(^5\). On the other hand the series (b) are contained in the series \( \mathcal{C} \) or \( \mathcal{D} \).

We can show that these physical states are reduced to the tachyon and the discrete states which appear in the \( c = 1 \) string theory in the case of other gauge fixing examined in the previous section.

It is obvious that \( \gamma = 1 \) gauge reduces the states \( V_{j,m} \) in (24) and the series of the discrete states obtained by acting the operator (25) on them to the tachyonic states and the discrete states in the \( c = 1 \) string theory. On the other hand by substituing the equation of motion \( \beta = \sqrt{\frac{k'}{2}} \partial_+ \phi - \sqrt{\frac{k}{2}} \partial_+ X \), the operators \( \tilde{V}_{j,m} \) given in (26) are replaced by

\[
\tilde{V}_{j,m} = \left( \frac{\sqrt{2}}{4} \partial_+ \phi - \frac{3\sqrt{2}}{4} \partial_+ X \right)^s \exp \left( 2\sqrt{2} j \phi - \frac{2\sqrt{2}}{3} mX \right).
\]

(27)

We would like to examine only the lowest case \( s = 1 \) here. The first discrete state in the series (a) is given by \( (j, m) = (1/8, 9/8) \). Then the operator \( \tilde{V}_{1/8,9/8} \) is found
to be a total divergence,

\[
\tilde{V}_{1,\frac{9}{8}} = \left( \frac{\sqrt{2}}{4} \partial_+ \phi - \frac{3\sqrt{2}}{4} \partial_+ X \right) \exp \left( \frac{\sqrt{2}}{4} \phi - \frac{3\sqrt{2}}{4} X \right)
\]

\[= \partial_+ \left( \exp \left( \frac{\sqrt{2}}{4} \phi - \frac{3\sqrt{2}}{4} X \right) \right),
\]

(28)

which means the state given by \(\tilde{V}_{1,\frac{9}{8}}\) is BRST trivial. Also the first state in the series (b), \((j,m) = (-1,0)\), is found to be the discrete state \(W_{1,0}^{(-)}\) in the \(c = 1\) string theory up to a total divergence. Because

\[
\tilde{V}_{-1,0} = \left( \frac{\sqrt{2}}{4} \partial_+ \phi - \frac{3\sqrt{2}}{4} \partial_+ X \right) \exp(-2\sqrt{2}\phi)
\]

\[= -\frac{1}{8} \partial_+ \exp(-2\sqrt{2}\phi) - \frac{3\sqrt{2}}{4} \partial_+ X \exp(-2\sqrt{2}\phi).
\]

(29)

Thus it is seen that the physical spectrum in the \(SL(2,\mathbb{R})/U(1)\) WZW theory indeed coincides with that in the \(c = 1\) string theory in this level. At higher level \((s > 1)\) we may expect a similar mechanism to work to reduce the discrete states generated by \(\tilde{V}_{j,m}\) to BRST trivial states or to the physical states of the \(c = 1\) string.

Next we shall examine the case of the \(\beta = 1\) gauge also. After performing the "Lorentz" transformation (23), the operators given in (26) turn out to be

\[
\tilde{V}_{j,m} = \exp \left( 2\sqrt{2}\tilde{j} \phi - \frac{2\sqrt{2}}{3} \tilde{m} \tilde{X} \right),
\]

(30)

where \(\tilde{j} = \frac{5}{4}j - \frac{1}{4}m\) and \(\tilde{m} = \pm(\frac{1}{4}j - \frac{5}{4}m)\). From the relations of \(j\) and \(m\) for the states \(\tilde{V}_{j,m}\) in (26), \(4\) we will find \(\tilde{j} = \pm\frac{1}{3} \tilde{m} - \frac{1}{2}\), which are the condition that the operators have conformal dimension one. Thus the states given by (30) correspond to the tachyonic states in the case of the \(c = 1\) string theory.

4) In this reduction we should suppose that the original representation is given by a continuous parameter \(s\) and positive integers \(r\).
How about the states given by $V_{j,m}$ (24) in this gauge? By using the equation of motion $\gamma = \sqrt{\frac{k'}{2}} \partial_+ \phi - \sqrt{\frac{k}{2}} \partial_+ X$, the operators $V_{j,m}$ may be represented as

$$\left( -\frac{\sqrt{2}}{4} \partial_+ \tilde{\phi} \pm \frac{3\sqrt{2}}{4} \partial_+ \tilde{X} \right)^r \exp \left( 2\sqrt{2} \tilde{j} \tilde{\phi} - \frac{2\sqrt{2}}{3} \tilde{m} \tilde{X} \right)$$

through the Lorentz transformation. In the case of $r = 1$, the physical states are given by $(\tilde{j}, \tilde{m}) = (-5/4, -9/4)$ and $(\tilde{j}, \tilde{m}) = (-1/8, -9/8)$, or equivalently $(\tilde{j}, \tilde{m}) = (-1, 0)$ and $(\tilde{j}, \tilde{m}) = (1/8, \pm 9/8)$. Then the operators $V_{j,m}(r = 1)$ are found to be reduced to

$$V_{-\frac{5}{4}, -\frac{9}{4}} = \pm \frac{3\sqrt{2}}{4} \partial_+ \tilde{X} \exp(-2\sqrt{2}\tilde{\phi}),$$

$$V_{-\frac{1}{8}, -\frac{9}{8}} = 0$$

up to total divergences. The operator appearing in the right hand side of (32) is just the $W_{1,0}^{(-)}$ written in terms of $\tilde{\phi}$ and $\tilde{X}$.

Thus it seems to be true that the gauged WZW theory is completely equivalent to the Liouville theory coupled to $c = 1$ matter as far as it is described in terms of the variables defined by the Gauss decomposition (2). However the analysis of the physical spectrum given here is far from complete. It would be necessary to find out a complete description of the BRST cohomology in terms of the free fields in order to confirm the identical structure of the physical spectrum. Also the implications from this observation to the 2D black hole physics should be investigated.

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