GROSS-PITAEVSKII-POISSON EQUATIONS FOR DIPOLAR BOSE-EINSTEIN CONDENSATE WITH ANISOTROPIC CONFINEMENT

WEIZHU BAO†, NAOUFEL BEN ABDALLAH‡, AND YONGYONG CAI§

Abstract. Ground states and dynamical properties of dipolar Bose-Einstein condensate are analyzed based on the Gross-Pitaevskii-Poisson system (GPPS) and its dimension reduction models under anisotropic confining potential. We begin with the three-dimensional (3D) Gross-Pitaevskii-Poisson system and review its quasi-2D approximate equations when the trap is strongly confined in z-direction and quasi-1D approximate equations when the trap is strongly confined in x-, y-directions. In fact, in the quasi-2D equations, a fractional Poisson equation with the operator \((-\Delta)^{1/2}\) is involved which brings significant difficulties into the analysis. Existence and uniqueness as well as nonexistence of the ground state under different parameter regimes are established for the quasi-2D and quasi-1D equations. Well-posedness of the Cauchy problem for both equations and finite time blowup in 2D are analyzed. Finally, we rigorously prove the convergence and linear convergence rate between the solutions of the 3D GPPS and its quasi-2D and quasi-1D approximate equations in weak interaction regime.

Key words. Gross-Pitaevskii-Poisson system, dipolar Bose-Einstein condensate, ground state, dimension reduction

AMS subject classifications. 35Q55, 35A01, 81Q99

1. Introduction. Quantum degenerate gases have received considerable interests both theoretically and experimentally, since the first observation of Bose-Einstein condensate (BEC) with dilute bosonic gas in 1995. The properties of these ultracold dilute quantum gases are determined by the short-range, isotropic contact interactions between the particles, which have been studied extensively. For those particles with large permanent magnetic or electric dipole moment, dipole-dipole interactions are non-negligible, and the dipolar interactions are long-range and anisotropic, different from contact interactions. Due to these remarkable properties of dipolar interactions, there have been great interests to study dipolar BEC in the last decade. In 2005, the first dipolar BEC with \(^{52}\)Cr atoms was successfully realized in experiments at the Stuttgart University \[15\]. Very recently in 2011, a dipolar BEC with \(^{164}\)Dy atoms, whose dipole-dipole interaction is much stronger than that of \(^{52}\)Cr, has been performed in experiments at the Stanford University \[21\]. These success of experiments have renewed interests in theoretically studying dipolar BECs.

In this paper, we will consider the zero temperature mean-field model of dipolar BEC, the three-dimensional (3D) Gross-Pitaevskii equation (GPE) with dipolar interaction in dimensionless form \[2\] \[12\] \[24\] \[31\] \[32\]

\[i\partial_t \psi(r, t) = \left[-\frac{1}{2} \nabla^2 + V(r) + \beta |\psi|^2 + \lambda \left(U_{\text{dip}} * |\psi|^2\right)\right] \psi, \quad r = (x, z) \in \mathbb{R}^3, \quad t > 0, \tag{1.1}\]

where \(t\) is time, \(x = (x, y) \in \mathbb{R}^2\) and \(r = (x, z) = (x, y, z) \in \mathbb{R}^3\) are the Cartesian coordinates, \(\psi = \psi(r, t)\) is the dimensionless complex-valued wave function, \(V(r)\) is a given real-valued trapping potential in the experiments, \(\beta\) and \(\lambda\) are dimensionless constants representing the contact interaction and dipolar interaction, respectively, and \(U_{\text{dip}}(r)\) is given as

\[U_{\text{dip}}(r) = \frac{3}{4\pi} \frac{1 - 3 (r \cdot n)^2 / |r|^2}{|r|^3} = \frac{3}{4\pi} \frac{1 - 3 \cos^2(\theta)}{|r|^3}, \quad r \in \mathbb{R}^3, \tag{1.2}\]

with the dipolar axis \(n = (n_1, n_2, n_3) \in \mathbb{R}^3\) satisfying \(|n| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1\). Although the kernel \(U_{\text{dip}}\) is highly singular near the origin, the convolution is well-defined for \(p \in L^p(\mathbb{R}^3)\) with \(U_{\text{dip}} * p \in L^p(\mathbb{R}^3)\) \((p \in (1, \infty))\) \[12\]. In the context of BEC, the initial data is usually normalized such that \(\|\psi(\cdot, 0)\|_{L^2} = 1\).

†Department of Mathematics and Center for Computational Science and Engineering, National University of Singapore, Singapore 119076 (bao@math.nus.edu.sg).
‡IMT, UMR CNRS 5219, Université Paul Sabatier, 31062 Toulouse Cedex, France.
§Department of Mathematics, National University of Singapore, Singapore 119076 (caiyongyong@nus.edu.sg).
Denote the differential operators $\partial_n = n \cdot \nabla$ and $\partial_{nn} = \partial_n \partial_n$, and notice the identity \cite{2}

$$U_{\text{dip}}(r) = \frac{3}{4\pi |r|^3} \left(1 - \frac{3(r \cdot n)^2}{|r|^2}\right) = -\delta(r) - 3\partial_{nn} \left(\frac{1}{4\pi |r|}\right), \quad r \in \mathbb{R}^3,$$  \hspace{1cm} (1.3)

with $\delta$ being the Dirac distribution, we can re-formulate the GPE (1.1) as the following Gross-Pitaevskii-Poisson system (GPPS) \cite{2,11}

$$i \partial_t \psi(r, t) = \left[-\frac{1}{2} \nabla^2 + V(r) + (\beta - \lambda) |\psi|^2 - 3\lambda \partial_{nn} \phi \right] \psi, \quad r \in \mathbb{R}^3, \quad t > 0,$$  \hspace{1cm} (1.4)

$$\nabla^2 \phi(r, t) = -|\psi(r, t)|^2, \quad r \in \mathbb{R}^3, \quad \lim_{|r| \to \infty} \phi(r, t) = 0, \quad t \geq 0,$$  \hspace{1cm} (1.5)

The above GPPS in 3D conserves the mass, or the normalization condition,

$$\|\psi(\cdot, t)\|_2^2 = \int_{\mathbb{R}^3} |\psi(r, t)|^2 \, dr = \int_{\mathbb{R}^3} |\psi(r, 0)|^2 \, dr = 1, \quad t \geq 0,$$  \hspace{1cm} (1.6)

and energy per particle

$$E_{3D}(\psi) = \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla \psi|^2 + V(r) |\psi|^2 + \frac{\beta - \lambda}{2} |\psi|^4 + \frac{3\lambda}{2} |\partial_n \nabla \phi|^2 \right] \, dr, \quad \varphi = \frac{1}{4\pi |r|} \ast |\psi|^2.$$  \hspace{1cm} (1.7)

It was proven \cite{2} that when $\beta \geq 0$ and $-\frac{\beta}{2} \leq \lambda \leq \beta$, there exists a unique positive ground state $\Phi_g$, which is defined as the minimizer of the energy functional, i.e. $E_{3D}(\Phi_g) = \min_{\|\phi\|_2 = 1} E_{3D}(\Phi)$ and the Cauchy problem of the GPPS (1.4)-(1.5) is globally well-posed; otherwise there exists no ground state and the Cauchy problem is locally well-posed and finite time blow-up may happen under certain conditions \cite{2}.

In many physical experiments of dipolar BECs, the condensates are confined with strong harmonic trap in one or two axes directions, resulting in a pancake- or cigar-shaped dipolar BEC, respectively. Mathematically speaking, this corresponds to the anisotropic potentials $V(r)$ of the form:

Case I (pancake-shaped), potential is strongly confined in the vertical $z$ direction with

$$V(r) = V_2(x) + \frac{x^2}{2\varepsilon^2}, \quad r = (x, z) \in \mathbb{R}^3,$$  \hspace{1cm} (1.8)

Case II (cigar-shaped), potential is strongly confined in the horizontal $x = (x, y) \in \mathbb{R}^2$ plane with

$$V(r) = V_1(z) + \frac{x^2 + y^2}{2\varepsilon^2}, \quad r = (x, z) \in \mathbb{R}^3,$$  \hspace{1cm} (1.9)

where $0 < \varepsilon \ll 1$ is a small parameter describing the strength of confinement. In such cases, the above GPPS in 3D can be formally reduced to 2D and 1D, respectively \cite{11}.

In Case I, when $\varepsilon \to 0^+$, evolution of the solution $\psi(r, t)$ of GPPS (1.4)-(1.5) in $z$-direction would essentially occur in the ground state mode of $L_z := -\frac{1}{2} \partial_{zz} + \frac{1}{2\varepsilon^2} \Delta$, which is spanned by $w_\varepsilon(z) = \varepsilon^{-1/2} e^{-\frac{1}{4} - \frac{z^2}{2\varepsilon^2}} [11, 13]$. By taking the ansatz

$$\psi(x, z, t) = e^{-it/2\varepsilon^2} \phi(x, t) w_\varepsilon(z), \quad (x, z) \in \mathbb{R}^3, \quad t \geq 0,$$  \hspace{1cm} (1.10)

the 3D GPPS (1.4)-(1.5) will be formally reduced to a quasi-2D equation \cite{11}:

$$i \partial_t \phi = \left[-\frac{1}{2} \Delta + V_2 + \frac{\beta - \lambda + 3\lambda n_3^2}{\sqrt{2\pi \varepsilon}} |\phi|^2 - \frac{3\lambda}{2} (\partial_{n_\perp} n_\perp - n_\perp^2 \Delta) \phi^{2D} \right] \phi, \quad x \in \mathbb{R}^2, \quad t > 0,$$  \hspace{1cm} (1.11)

where $x = (x, y)$, $n_\perp = (n_1, n_2)$, $\partial_{n_\perp} = n_\perp \cdot \nabla$, $\partial_{n_\perp} n_\perp = \partial_{n_\perp} (\partial_{n_\perp} n_\perp)$, $\Delta = \partial_{xx} + \partial_{yy}$ and

$$\phi^{2D}(x, t) = U_{\varepsilon}^{2D} \ast |\phi|^2, \quad U_{\varepsilon}^{2D}(x) = \frac{1}{2\sqrt{2\pi \varepsilon^3/2}} \int_{\mathbb{R}} \frac{e^{-s^2/2}}{\sqrt{x^2 + y^2 + \varepsilon s^2}} \, ds, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$  \hspace{1cm} (1.12)
In addition, as $\varepsilon \to 0^+$, $\varphi^{2D}$ can be approximated by $\varphi_{2D}^{\infty}$ as:

$$\varphi_{2D}^{\infty}(x, t) = U_{\text{dip}}^{2D} * |\phi|^2, \quad \text{with} \quad U_{\text{dip}}^{2D}(x) = \frac{1}{2\pi \sqrt{x^2 + y^2}}, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (1.13)$$

which can be re-written as a fractional Poisson equation:

$$(-\Delta)^{1/2} \varphi_{2D}^{\infty}(x, t) = |\phi(x, t)|^2, \quad x \in \mathbb{R}^2, \quad \lim_{|x| \to \infty} \varphi_{2D}^{\infty}(x, t) = 0, \quad t \geq 0. \quad (1.14)$$

Thus an alternative quasi-2D equation II can be obtained as [11]:

$$i\partial_t \phi = \left[ -\frac{1}{2}\Delta + V_2 + \frac{\beta - \lambda + 3\lambda n_2^2}{\sqrt{2\pi \varepsilon}} |\phi|^2 - \frac{3\lambda}{2}(\partial_{n_z} n_z - n_3^2 \Delta)(-\Delta)^{-1/2}(|\phi|^2) \right] \phi. \quad (1.15)$$

Similarly, in Case II, evolution of the solution $\psi(x, z, t)$ of GPPS (1.4)-(1.5) in $x = (x, y)$-directions would essentially occur in the ground state mode of $L_x := -\frac{1}{2}(\partial_{xx} + \partial_{yy}) + \frac{x^2 + y^2}{2\varepsilon}$, which is spanned by $w_\varepsilon(x) = \varepsilon^{-1/2} e^{-\frac{x^2}{2\varepsilon}}$ [11]. Again, by taking the ansatz

$$\psi(x, z, t) = e^{-it/\varepsilon^2} \varphi(z, t) w_\varepsilon(x), \quad (x, z) \in \mathbb{R}^3, \quad t \geq 0, \quad (1.16)$$

the 3D GPPS (1.4)-(1.5) will be formally reduced to a quasi-1D equation [11]:

$$i\partial_z \phi = \left[ -\frac{1}{2}\partial_{zz} + V_1 + \frac{\lambda (1 - 3n_3^2)}{2\pi \varepsilon^2} |\phi|^2 - \frac{3\lambda (3n_3^2 - 1)}{8\sqrt{2\pi \varepsilon}} \partial_{zz} \varphi^{1D} \right] \phi, \quad z \in \mathbb{R}, \quad t \geq 0, \quad (1.17)$$

where

$$\varphi^{1D}(z, t) = U_1^{1D} * |\phi|^2, \quad U_1^{1D}(z) = \frac{\sqrt{2\pi \varepsilon^2}}{\sqrt{2\pi \varepsilon}} \int_{|z|}^{\infty} e^{-s^2/2\varepsilon^2} ds, \quad z \in \mathbb{R}, \quad t \geq 0. \quad (1.18)$$

The above effective lower dimensional models in 2D and 1D are very useful in the study of dipolar BEC since they are much easier and cheaper to be simulated in practical computation. In fact, for the GPE without the dipolar term, i.e. $\lambda = 0$, there have been extensive studies on this subject. For formal analysis and numerical simulation, the convergence rate of such dimension reduction was investigated numerically in [3, 5] and a nonlinear Schrödinger equation with polynomial nonlinearity in reduced dimensions was proposed in [23]. For rigorous analysis, convergence of the dimension reduction under anisotropic confinement has been proven in the weak interaction regime [9, 8], i.e. $\beta = O(\varepsilon)$ in 2D and $\beta = O(\varepsilon^2)$ in 1D. However, with the dipolar term, i.e. $\lambda \neq 0$, there were few works towards the mathematical analysis for this dimension reduction except some preliminary results in [12] where different scalings and formalism were adopted. In fact, our quasi-2D models (1.11) and (1.15) and quas-1D model (1.17) are much easier to be used in mathematical analysis and practical numerical computation.

The main aim of this paper is to establish existence and uniqueness of the ground states and well-posedness of the Cauchy problems associated to the quasi-2D equations I and II and quasi-1D equation, and to analyze the convergence and convergence rate of the dimension reduction from 3D to 2D and 1D. In order to do so, without loss of generality, we assume the potential $V_d(\eta) \geq 0$ for $\eta \in \mathbb{R}^d$ ($d = 1, 2, 3$). It is natural to consider the energy space in $d$-dimensions ($d = 1, 2, 3$) defined as

$$X_d = \left\{ u \in H^1(\mathbb{R}^d) \mid \|u\|_{X_d}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^d} V_d(\eta) |u(\eta)|^2 d\eta < \infty \right\},$$

and the unit sphere of $X_d$ defined as

$$S_d = X_d \bigcap \{ u \in L^2(\mathbb{R}^d) \mid \|u\|_{L^2(\mathbb{R}^d)} = 1 \}.$$
This paper is organized as follows. In Sections 2, 3 and 4, we study quasi-2D equation I (1.11), II (1.13) and quasi-1D equation (1.17), respectively. In each section, we first establish existence and uniqueness as well as nonexistence of the ground state under different parameter regimes, and then study the well-posedness of the corresponding Cauchy problem. In Section 5, we rigorously prove the validity of dimension reduction from 3D GPPS (1.4)-(1.5) to 2D and 1D in the weak interaction regimes. Our approach is based on a-priori estimates from the energy and mass conservation together with the Strichartz estimates.

Throughout the paper, we adopt the standard notation of Sobolev space and use \( \|f\|_p^p := \int_{\mathbb{R}^d} |f(\eta)|^p \, d\eta \) for \( p \in (0, \infty) \) when there is no confusion about the space \( \mathbb{R}^d \), denote \( C \) as a generic constant which is independent of \( \varepsilon \), let \( X^* \) as the dual space of \( X \), and adopt the Fourier transform of a function \( f(\eta) \in L^1(\mathbb{R}^d) \) as

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(\eta) e^{-i\xi \cdot \eta} \, d\eta, \quad \xi \in \mathbb{R}^d. \tag{1.19}
\]

### 2. Results for the quasi-2D equation I. In this section, we prove existence and uniqueness as well as nonexistence of the ground state under different parameter regimes, and then

#### 2.1. Existence and uniqueness of ground state. Associated to the quasi-2D equation I (1.11)-(1.12), the energy is

\[
E_{2D}(\Phi) = \int_{\mathbb{R}^2} \left[ \frac{1}{2} \nabla |\Phi|^2 + V_2(x)|\Phi|^2 + \frac{\beta - \lambda + 3n_3^2 \lambda}{2\sqrt{2\pi} \varepsilon} |\Phi|^4 - \frac{3\lambda}{4} |\Phi|^2 \varphi_{2D}^2 \right] \, dx, \quad \Phi \in X_2, \tag{2.2}
\]

where

\[
\varphi_{2D} = (\partial_{n_x} n_1 - n_3^2 \Delta) \varphi_{2D}, \quad \varphi_{2D} = U_{2D}^2 * |\Phi|^2. \tag{2.3}
\]

The ground state \( \Phi_g \in S_2 \) of (1.11) is the minimizer of the nonconvex minimization problem:

\[
\text{Find } \Phi_g \in S_2, \quad \text{such that } E_{2D}(\Phi_g) = \min_{\Phi \in S_2} E_{2D}(\Phi). \tag{2.4}
\]

For the ground state, we have the following results:

**Theorem 2.1.** (Existence and uniqueness of ground state) Assume \( 0 \leq V_2(x) \in L^\infty_\text{loc}(\mathbb{R}^2) \) and \( \lim_{|x| \to \infty} V_2(x) = 0, \) then we have

(i) There exists a ground state \( \Phi_g \in S_2 \) of the system (1.11)-(1.12) if one of the following conditions holds

(A1) \( \lambda \geq 0 \) and \( \beta - \lambda > -\sqrt{2\pi} C_b \varepsilon; \)

(A2) \( \lambda < 0 \) and \( \beta + \frac{1}{2}(1 + 3|2n_3^2 - 1|) \lambda > -\sqrt{2\pi} C_b \varepsilon. \)

(ii) The positive ground state \( |\Phi_g| \) is unique under one of the following conditions:

(A’1) \( \lambda \geq 0 \) and \( \beta - \lambda \geq 0; \)

(A’2) \( \lambda < 0 \) and \( \beta + \frac{1}{2}(1 + 3|2n_3^2 - 1|) \lambda \geq 0. \)

Moreover, any ground state is of the form \( \Phi_g = e^{i\theta_0} |\Phi_g| \) for some constant \( \theta_0 \in \mathbb{R}. \)

(iii) If \( \beta + \frac{1}{2}(1 - 3n_3^2) < -\sqrt{2\pi} C_b \varepsilon, \) there exists no ground state of the equation (1.11).

In order to prove this theorem, we first study the property of the nonlocal term.

**Lemma 2.2.** (Kernel \( U_{2D}^2 \) in (1.12)) For any real function \( f(x) \) in the Schwartz space \( \mathcal{S}(\mathbb{R}^2) \), we have

\[
U_{2D}^2 \ast f(\xi) = \hat{f}(\xi) \frac{\hat{\imath}(\xi)}{\pi} \int_{\mathbb{R}} e^{-\xi^2 s^2/2} \, ds, \quad f \in \mathcal{S}(\mathbb{R}^2). \tag{2.5}
\]
Moreover, define the operator
\[ T_{\alpha\alpha'}(f) = \partial_{\alpha\alpha'}(U_{x'}^2 * f), \quad \alpha, \alpha' = x, y, \]
then we have
\[ \|T_{\alpha\alpha'}f\|_2 \leq \frac{\sqrt{2}}{\sqrt{\pi \varepsilon}} \|f\|_2, \quad \|T_{\alpha\alpha'}f\|_2 \leq \|\nabla f\|_2, \quad \tag{2.6} \]
hence \( T_{\alpha\alpha'} \) can be extended to a bounded linear operator from \( L^2(\mathbb{R}^2) \) to \( L^2(\mathbb{R}^2) \).

**Proof.** From (1.12), we have
\[ |U_{\varepsilon}^{2D}(x)| = \left| \frac{1}{2\sqrt{2\pi}^{3/2}} \int_{\mathbb{R}^2} \frac{e^{-s^2/2}}{\sqrt{|x|^2 + \varepsilon^2 s^2}} ds \right| \leq \frac{1}{2\pi|\varepsilon|}, \quad 0 \neq x \in \mathbb{R}^2. \quad \tag{2.7} \]
This immediately implies that \( U_{\varepsilon}^{2D} * g \) is well-defined for any \( g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \) since the right hand side in the above inequality is the singular kernel of Riesz potential. Re-write \( U_{\varepsilon}^{2D}(x) \) as [11]
\[ U_{\varepsilon}^{2D}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{w_\varepsilon^2(z)w_\varepsilon^2(z')}{\sqrt{|x|^2 + (z-z')^2}} dzdz', \quad x \in \mathbb{R}^2, \]
using the Plancherel formula, we get
\[ \hat{U}_{\varepsilon}^{2D}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\hat{w}_\varepsilon^2(\xi_\lambda)\hat{w}_\varepsilon^2(\xi_\lambda)}{|\xi|^2 + s^2} d\xi = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2}}{|\xi|^2 + s^2} ds, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad \tag{2.8} \]
which immediately implies (2.5). Here \( \hat{c} \) denotes the complex conjugate of \( c \). Concerning \( T_{\alpha\alpha'} \), we only need to prove the results for \( T_{xx} \) since others are similar. Applying the Fourier transform, we have
\[ \left| \hat{T}_{xx}f(\xi) \right| = \left| \frac{\hat{f}(\xi)}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2} |\xi|^2}{|\xi|^2 + s^2} ds \right| \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2} |\xi|^2}{|\xi|^2 + s^2} ds \leq \frac{\sqrt{2}}{\sqrt{\pi \varepsilon}} \|\hat{f}(\xi)\|, \quad \xi \in \mathbb{R}^2. \quad \tag{2.9} \]
Thus we can get the first inequality in (2.6) and know that \( T_{xx} : L^2 \rightarrow L^2 \) is bounded. Moreover, from
\[ \left| \hat{T}_{xx}f(\xi) \right| = \left| \frac{\hat{f}(\xi)}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2} |\xi|^2}{|\xi|^2 + s^2} ds \right| \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2} |\xi|^2}{|\xi|^2 + s^2} ds \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e^{-\varepsilon^2 s^2/2} |\xi|^2}{|\xi|^2 + s^2} ds \leq |\xi| \|\hat{f}(\xi)\|, \tag{2.10} \]
we obtain the second inequality in (2.6) and know that \( T_{xx} : H^1 \rightarrow L^2 \) is bounded too.

**Remark 2.1.** In fact, \( T_{\alpha\alpha'} \) is bounded from \( L^p \rightarrow L^p \), i.e., there exists \( C_p > 0 \) independent of \( \varepsilon \), such that
\[ \|T_{\alpha\alpha'}(f)\|_p \leq \frac{C_p}{\varepsilon} \|f\|_p, \quad p \in (1, \infty). \quad \tag{2.11} \]
This can be obtained by Minkowski inequality and \( L^p \) estimates for Poisson equation.

**Lemma 2.3.** For the energy \( E_{2D}(\cdot) \) in (2.3), we have
(i) For any \( \Phi \in S_2 \), denote \( \rho(\Phi) = |\Phi(x)|^2 \), then we have
\[ E_{2D}(\Phi) \geq E_{2D}(|\Phi|) = E_{2D}(\sqrt{\rho}), \quad \forall \Phi \in S_2, \quad \tag{2.12} \]
so the ground state \( \Phi_0 \) of (2.2) is of the form \( e^{i\theta_0} |\Phi_0| \) for some constant \( \theta_0 \in \mathbb{R} \).
(ii) Under the condition (A1) or (A2) in Theorem 2.1, \( E_{2D}(\sqrt{\rho}) \) is bounded below.
(iii) Under the condition (A1') or (A2') in Theorem 2.1, \( E_{2D}(\sqrt{\rho}) \) is strictly convex.
Proof. (i) For any $\Phi \in S_2$, then $|\Phi| \in S_2$, and a simple calculation shows
\[
E_{2D}(\Phi) - E_{2D}(|\Phi|) = \frac{1}{2} \| \nabla \Phi \|^2 - \frac{1}{2} \| \nabla |\Phi| \|^2 \geq 0, \quad \Phi \in S_2,
\] (2.13)
where the equality holds iff [20]
\[
|\nabla \Phi(x)| = |\nabla |\Phi(x)||, \quad \text{a.e. } x \in \mathbb{R}^2,
\] (2.14)
which is equivalent to
\[
\Phi(x) = e^{i\theta_0}|\Phi(x)|, \quad \text{for some } \theta_0 \in \mathbb{R}.
\] (2.15)
Then the conclusion follows.

(ii) For $\sqrt{\rho} = \Phi \in S_2$, we split the energy $E_{2D}$ into two parts, i.e.
\[
E_{2D}(\Phi) = E_1(\Phi) + E_2(\Phi) = E_1(\sqrt{\rho}) + E_2(\sqrt{\rho}),
\] (2.16)
where
\[
E_1(\sqrt{\rho}) = \int_{\mathbb{R}^2} \left[ \frac{1}{2} \| \nabla \sqrt{\rho} \|^2 + V_2(x)\rho \right] dx,
\] (2.17)
\[
E_2(\sqrt{\rho}) = \int_{\mathbb{R}^2} \left[ \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi \varepsilon} \rho} |\rho|^2 - \frac{3\lambda}{4} \rho \varphi^{2D} \right] dx,
\] (2.18)
with
\[
\varphi^{2D} = (\partial_{n_a n_a} - n_3^2 \Delta) U^2D * \rho.
\] (2.19)
Applying the Plancherel formula and Lemma [2,2] there holds
\[
\int_{\mathbb{R}^2} \varphi^{2D}(x)\rho(x) dx = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \varphi^{2D}(\xi)\tilde{\rho}(\xi) d\xi
\]
\[
= \frac{1}{4\pi^3} \int_{\mathbb{R}^3} \frac{(n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2}{|\xi|^2 + s^2} e^{-\varepsilon^2 s^2/2} d\xi d\xi d\xi.
\] (2.20)
Recalling the Cauchy inequality and $n_1^2 + n_2^2 + n_3^2 = 1$, we have
\[
-n_3^2|\xi|^2 \leq (n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2 \leq (1 - 2n_3^2)|\xi|^2, \quad \xi \in \mathbb{R}^2.
\] (2.21)
Denoting $C_0 = \max\{|n_3|^2, |1 - 2n_3^2|\}$, we can derive that
\[
\left| \int_{\mathbb{R}^2} \varphi^{2D}(x)\rho(x) dx \right| \leq \frac{C_0}{4\pi^3} \int_{\mathbb{R}^3} e^{-\varepsilon^2 s^2/2} |\rho|^2 d\xi d\xi d\xi = \frac{\sqrt{2}C_0}{\sqrt{\pi \varepsilon} \varepsilon} \| \rho \|^2.
\] (2.22)
Hence, $E_2(\sqrt{\rho})$ can be bounded below by $\| \rho \|^2$. In fact, under the condition (A1), i.e. $\lambda \geq 0$ and $\beta - \lambda > -\sqrt{2\pi} C_b \varepsilon$, we have
\[
E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi \varepsilon} \varepsilon} \| \rho \|^2 - \frac{3\sqrt{2}n_3^2\lambda}{4\sqrt{\pi \varepsilon} \varepsilon} \| \rho \|^2 > -\frac{C_b}{2} \| \rho \|^2.
\] (2.23)
Similarly, under the condition (A2), if $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$, then
\[
E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi \varepsilon} \varepsilon} \| \rho \|^2 > -\frac{C_b}{2} \| \rho \|^2.
\] (2.24)
Recalling the choice of the best constant $C_b$, under either condition (A1) or (A2), the energy

$$E_{2D}(\sqrt{\rho}) = E_1(\sqrt{\rho}) + E_2(\sqrt{\rho}) > \frac{1}{2} \|\nabla \sqrt{\rho}\|_2^2 - \frac{C_b}{2} \|\rho\|_0^2 \geq 0.$$  

(iii) Again, we split the energy as (2.10). It is well known that $E_1(\sqrt{\rho})$ is strictly convex in $\rho$ [20]. It remains to show that $E_2(\sqrt{\rho})$ is convex in $\rho$. For any real function $u \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, let

$$H(u) = \int_{\mathbb{R}^2} \left( \frac{\beta - \lambda + 3n^2\lambda}{2\sqrt{2\pi}} \right) |u|^2 - \frac{3\lambda}{4} u \left( \partial_{\nu_\perp} n_{\perp} - n_{\perp}^2 \Delta_{\perp} \right) (U_{\varepsilon}^{2D} + u) \, dx.$$  

Then $E_2(\sqrt{\rho}) = H(\rho)$. It suffices to show that $H(\rho)$ is convex in $\rho$. For this purpose, let $\sqrt{\rho_1} = \Phi_1 \in S_2$ and $\sqrt{\rho_2} = \Phi_2 \in S_2$, for any $\theta \in (0,1]$, consider $\rho_\theta = \theta \rho_1 + (1 - \theta) \rho_2$ and $\sqrt{\rho_\theta} \in S_2$, then we compute directly and get

$$\theta H(\rho_1) + (1 - \theta) H(\rho_2) - H(\rho_\theta) = \theta(1 - \theta) H(\rho_1 - \rho_2).$$  

Similar to (2.20), looking at the Fourier domain, we can obtain the lower bounds for $H(\rho_1 - \rho_2)$ under the condition (A1') or (A2'), while replacing $C_b$ with 0 in the above proof of (ii), i.e.,

$$H(\rho_1 - \rho_2) \geq 0.$$  

This shows that $H(\rho)$, i.e. $E_2(\sqrt{\rho})$, is convex in $\rho$. Thus $E_{2D}(\sqrt{\rho})$ is strictly convex in $\rho$. \Box

**Proof of Theorem 2.1** (i) We first prove the existence results. Lemma 2.3 ensures that there exists a minimizing sequence of nonnegative function $\{\Phi^n\}_{n=0}^\infty \subset S_2$, such that $\lim_{n \to \infty} E_{2D}(\Phi^n) = \inf_{\Phi \in S_2} E(\Phi)$. Then, under condition (A1) or (A2), there exists a constant $C$ such that

$$\|\nabla \Phi^n\|_2 + \|\Phi^n\|_4 + \int_{\mathbb{R}^2} V_2(x)|\Phi^n(x)|^2 \, dx \leq C, \quad n \geq 0.$$  

Therefore $\Phi^n$ belongs to a weakly compact set in $L^4(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, and $L^{2}_{V_2}(\mathbb{R}^2)$ with a weighted $L^2$-norm given by $\|\Phi\|_{L^{2}_{V_2}} = \left( \int_{\mathbb{R}^2} |\Phi(x)|^2 V_2(x) \, dx \right)^{1/2}$. Thus, there exists a $\Phi^\infty \in W := H^1(\mathbb{R}^2) \cap L^{2}_{V_2}(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$ and a subsequence of $\{\Phi^n\}_{n=0}^\infty$ (which we denote as the original sequence for simplicity), such that

$$\Phi^n \to \Phi^\infty, \quad \text{in } W, \quad \nabla \Phi^n \to \nabla \Phi^\infty, \quad \text{in } L^2.$$  

The confining condition $\lim_{|x| \to \infty} V_2(x) = \infty$ will give that $\|\Phi^\infty\|_2 = 1$ [19] 2 [1]. Hence $\Phi^\infty \in S_2$ and $\Phi^n \to \Phi^\infty$ in $L^2(\mathbb{R}^2)$ due to the $L^2$-norm convergence and weak convergence of $\{\Phi^n\}_{n=0}^\infty$. By the lower semi-continuity of the $H^1$- and $L^{2}_{V_2}$-norm, for $E_1$ in (2.17), we know

$$E_1(\Phi^\infty) \leq \lim_{n \to \infty} \inf_{n \to \infty} E_1(\Phi^n).$$  

By the Sobolev inequality, there exists $C(p) > 0$ depending on $p \geq 2$, such that $\|\Phi^n\|_p \leq C(p)(\|\nabla \Phi^n\|_2 + \|\Phi^n\|_2) \leq C(p)(1 + C)$, uniformly for $n \geq 0$. Applying the Hölder’s inequality, we have

$$\|\Phi^n\|^2 - (\Phi^\infty)^2 \|_2^2 \leq C_1(\|\Phi^n\|_6^3 + \|\Phi^n\|_6^3)\|\Phi^n - \Phi^\infty\|_2,$$  

and if $\lambda < 0$ and $n_\perp = \frac{1}{2}$, then

$$E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n^2\lambda}{2\sqrt{2\pi}} \|\rho\|_2^2 + \frac{3\sqrt{2}(1 - 2n_\perp^2)\lambda}{4\sqrt{\pi}} \|\rho\|_2^2 > - \frac{C_b}{2} \|\rho\|_2^2.$$  

(2.25)
which shows \( \rho^n = (\Phi^n)^2 \to \rho^\infty = (\Phi^\infty)^2 \) in \( L^2(\mathbb{R}^2) \). Using the Fourier transform of \( U^{2D}_\varepsilon \) in Lemma 2.2 and (2.22), it is easy to derive the convergence for \( E_2 \) in (2.18), i.e.,

\[
E_2(\Phi^\infty) = \lim_{n \to \infty} E_2(\Phi^n).
\]

Hence

\[
E_{2D}(\Phi^\infty) = E_1(\Phi^\infty) + E_2(\Phi^\infty) \leq \lim \inf_{n \to \infty} E_{2D}(\Phi^n).
\] (2.35)

Now, we see that \( \Phi^\infty \) is indeed a minimizer. For the uniqueness part, it is straightforward by the strict convexity of \( E_{2D}(\sqrt{\rho}) \) shown in Lemma 2.3.

(ii) Since the nonlinear term in the equation behaves as a cubic nonlinearity, it is natural to consider the following. Let \( \Phi \in S_2 \) be a real function that attains the best constant \( C_b \) [29], then \( \Phi(x) \) is radially symmetric. Choose \( \Phi_\delta(x) = \delta^{-1} \Phi(\delta^{-1} x) \) with \( \delta > 0 \), then \( \Phi_\delta \in S_2 \). Denote \( \varphi_\delta = (\partial_{n_1} n_\perp - n_3^2 \Delta_{\perp} ) (U_{\varepsilon}^{2D} * \Phi_\delta^2) \), by the same computation as in Lemma 2.3, there holds

\[
\int_{\mathbb{R}^2} \varphi_\delta |\Phi_\delta|^2 \, dx = -\frac{1}{4\pi} \varepsilon \int_{\mathbb{R}^2} \frac{(n_1 \xi_1 + n_2 \xi_2)^2 - n_3^2 \xi_3^2}{|\xi|^2 + s^2} e^{-s^2/2} \left| \frac{\Phi^2(\delta \xi)}{2} \right|^2 \, dsd\xi = -\frac{1}{4\pi \varepsilon \delta^2} \int_{\mathbb{R}^2} \frac{(n_1 \xi_1 + n_2 \xi_2)^2 - n_3^2 \xi_3^2}{|\xi|^2 + \delta^2 s^2} e^{-s^2/2} \left| \frac{\Phi^2(\xi)}{2} \right|^2 \, dsd\xi.
\]

Using the fact that \( \Phi(x) \) is radially symmetric, \( \frac{\Phi^2(\xi)}{2} \) is also radially symmetric, then we obtain

\[
\int_{\mathbb{R}^2} \varphi_\delta |\Phi_\delta|^2 \, dx = -\frac{(n_1^2 + n_2^2 - 2n_3^2) + o(1)}{\sqrt{2\pi \varepsilon \delta^2}} \Phi^4, \quad \text{as } \delta \to 0^+. \quad (2.36)
\]

Hence, as \( \delta \to 0^+ \), we get

\[
E_{2D}(\Phi_\delta) = \frac{1}{2\delta^2} \left( \| \nabla \Phi \|^2_2 + \beta + \frac{1}{4\pi} \lambda (1 - 3n_3^2) + o(1) \right) \| \Phi \|^4_4 + \int_{\mathbb{R}^2} V_2(\delta x) |\Phi_\delta|^2(x) \, dx.
\]

Recalling that \( \| \nabla \Phi \|^2_2 = C_b \| \Phi \|^4_4 \), we know \( \lim_{\delta \to 0^+} E_{2D}(\Phi_\delta) = -\infty \) if \( \beta + \frac{1}{4\pi} \lambda (1 - 3n_3^2) < -\sqrt{2\pi} C_b \varepsilon \), i.e. there is no ground state in this case. \( \square \)

2.2. Well-posedness for the Cauchy problem. Here, we study the well-posedness of the Cauchy problem corresponding to the quasi-2D equation (1.11)-(1.12). Using the Fourier transform of the kernel \( U_{\varepsilon}^{2D} \) in Lemma 2.2, it is straightforward to see that the nonlinear term introduced by \( U_{\varepsilon}^{2D} \) behaves like cubic term. Thus, those methods for classic cubic nonlinear Schrödinger equation would apply [18, 20, 27]. In particular, we have the following theorem concerning the Cauchy problem of (1.11)-(1.12).

**Theorem 2.4.** (Well-posedness of Cauchy problem) Suppose the real-valued trap potential satisfies \( V_2(x) \geq 0 \) for \( x \in \mathbb{R}^2 \) and

\[
V_2(x) \in C^\infty(\mathbb{R}^2) \text{ and } D^k V_2(x) \in L^\infty(\mathbb{R}^2), \quad \text{for all } k \in \mathbb{N}_0^3 \text{ with } |k| \geq 2, \quad (2.37)
\]

then we have

(i) For any initial data \( \phi(x, t = 0) = \phi_0(x) \in X_2 \), there exists a \( T_{\max} \in (0, +\infty] \) such that the problem (1.11)-(1.12) has a unique maximal solution \( \phi \in C([0, T_{\max}), X_2) \). It is maximal in the sense that if \( T_{\max} < \infty \), then \( \| \phi(\cdot, t) \|_{X_2} \to \infty \) when \( t \to T_{\max} \).

(ii) As long as the solution \( \phi(x, t) \) remains in the energy space \( X_2 \), the \( L^2 \)-norm \( \| \phi(\cdot, t) \|_2 \) and energy \( E_{2D}(\phi(\cdot, t)) \) in (2.2) are conserved for \( t \in [0, T_{\max}) \).

(iii) Under either condition (A1) or (A2) in Theorem 2.1 with constant \( C_b \) being replaced by \( C_b/\| \phi_0 \|^2_2 \), the solution of (1.11)-(1.12) is global in time, i.e., \( T_{\max} = \infty \).
Proof. The proof is standard. We shall use the known results for semi-linear Schrödinger equation [13]. For \( \phi \in X_2 \), denote \( \rho = |\phi|^2 \) and consider the following

\[
G(\phi, \bar{\phi}) := G(\rho) = \frac{1}{2} \int_{\mathbb{R}^2} |\phi|^2 \left( \partial_{t_n} n_{-} - n_2^2 \Delta \right) (U_{2D}^2 * |\phi|^2) \, dx,
\]

\[
g(\phi) = \frac{\delta G(\phi, \bar{\phi})}{\delta \phi} = \phi \left( \partial_{t_n} n_{-} - n_2^2 \Delta \right) (U_{2D}^2 * |\phi|^2), \quad \phi \in X_2.
\]

Then the equations (1.11)-(1.12) read

\[
i \partial_t \phi = \left[ -\frac{1}{2} \Delta + V_2(x) \right] \phi + \beta_0 |\phi|^2 \phi - 3\lambda g(\phi), \quad x \in \mathbb{R}^2, \quad t > 0,
\]

(2.38)

where \( \beta_0 = \frac{\beta - \lambda + 3\alpha^2 \lambda}{\sqrt{2}\pi} \). Using the \( L^p \) boundedness of \( T_{jk} \) (cf. Lemma 2.2 and Remark 2.1) and the Sobolev inequality, for \( \|u\|_{X_2} + \|v\|_{X_2} \leq M \), it is easy to prove the following

\[
\|g(u) - g(v)\|_{4/3} \leq C(M)\|u - v\|_4.
\]

(2.39)

In view of the standard Theorems 9.2.1, 4.12.1 and 5.7.1 in [13] and [27] for the well-posedness of the nonlinear Schrödinger equation, we can obtain the results (i) and (ii) immediately. The global existence (iii) comes from the uniform bound for \( \|\phi(\cdot, t)\|_{X_2} \) which can be derived from energy and \( L^2 \)-norm conservation.

When the initial data is small, there also exists global solutions [13, 12]. Otherwise, blow-up may happen in finite time, and we have the following results.

**THEOREM 2.5.** (Finite time blow-up) For any initial data \( \phi(x, t = 0) = \phi_0(x) \in X_2 \) with

\[
\int_{\mathbb{R}^2} |x|^2 |\phi_0(x)|^2 \, dx < \infty,
\]

if conditions (A1) and (A2) with constant \( C_b \) being replaced by \( C_b/\|\phi_0\|_2^2 \) are not satisfied and assume \( V_2(x) \) satisfies \( 2V_2(x) + x \cdot \nabla V_2(x) \geq 0 \), and let \( \phi := \phi(x, t) \) be the solution of the problem (1.11), there exists finite time blow-up, i.e., \( T_{\text{max}} < \infty \), if \( \lambda = 0 \), or \( \lambda > 0 \) and \( n_3^2 \geq \frac{1}{7} \), and one of the following holds:

(i) \( E_{2D}(\phi_0) < 0 \);

(ii) \( E_{2D}(\phi_0) = 0 \) and \( \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi}_0(x) \cdot (x \cdot \nabla \phi_0(x)) \, dx \right) < 0 \);

(iii) \( E_{2D}(\phi_0) > 0 \) and \( \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi}_0(x) \cdot (x \cdot \nabla \phi_0(x)) \, dx \right) < -\sqrt{2E_{2D}(\phi_0)}\|x\phi_0\|_2 \);

where \( \text{Im}(f) \) denotes the imaginary part of \( f \).

**Proof.** Define the variance

\[
\sigma_V(t) := \sigma_V(\phi(\cdot, t)) = \int_{\mathbb{R}^2} |x|^2 |\phi(x, t)|^2 \, dx = \sigma_x(t) + \sigma_y(t), \quad t \geq 0,
\]

(2.40)

where

\[
\sigma_\alpha(t) := \sigma_\alpha(\phi(\cdot, t)) = \int_{\mathbb{R}^2} \alpha^2 |\phi(x, t)|^2 \, dx, \quad \alpha = x, y.
\]

(2.41)

For \( \alpha = x, y \), differentiating (2.41) with respect to \( t \), integrating by parts, we get

\[
\frac{d}{dt} \sigma_\alpha(t) = -i \int_{\mathbb{R}^2} \left[ \bar{\alpha}\phi(x, t) \partial_\alpha \phi(x, t) - \alpha \phi(x, t) \partial_\alpha \bar{\phi}(x, t) \right] \, dx, \quad t \geq 0.
\]

(2.42)

Similarly, we have

\[
\frac{d^2}{dt^2} \sigma_\alpha(t) = \int_{\mathbb{R}^2} \left[ 2|\partial_\alpha \phi|^2 + \beta_0 |\phi|^4 + 3\lambda |\phi|^2 \alpha \partial_\alpha (\partial_{t_n} n_{-} - n_2^2 \Delta) \phi - 2\alpha |\phi|^2 \partial_\alpha V_2(x) \right] \, dx
\]

(2.43)
where $\beta_0 = \frac{\beta - \lambda + 3n_3^2}{2\pi}\sqrt{2}\varphi = U_{\varepsilon}^{2D} \ast |\varphi|^2$. Writing $\rho = |\varphi|^2$, $\dot{\varphi} = (\partial_{n_\perp n_{\perp}} - n_3^2\Delta)\varphi$, $n_\xi(\xi) = (n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2$ and noticing that $\rho$ is a real function, by the Plancherel formula, we have
\[
\int_{\mathbb{R}^2} |\varphi|^2 (x \cdot \nabla \dot{\varphi}) \, dx = -\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \dot{\rho}(\xi) \nabla \cdot \left( \xi \ddot{\varphi} \right) \, d\xi = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \dot{\rho}(\xi) \nabla \cdot \left( \xi n_\xi U_{\varepsilon}^{2D} \rho \right) \, d\xi
\]
\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \delta \nabla (n_\xi U_{\varepsilon}^{2D} + n_\xi U_{\varepsilon}^{2D} \xi \cdot \nabla \rho) \, d\xi
\]
\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left( |\dot{\rho}|^2 \nabla (n_\xi U_{\varepsilon}^{2D}) + n_\xi U_{\varepsilon}^{2D} \xi \cdot \frac{1}{2} \nabla |\rho|^2 \right) \, d\xi
\]
\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (n_\xi U_{\varepsilon}^{2D} + \frac{1}{2} \xi \cdot \nabla (n_\xi U_{\varepsilon}^{2D})) |\dot{\rho}|^2 \, d\xi
\]
\[
= - \int_{\mathbb{R}^2} |\varphi|^2 \ddot{\varphi} \, dx + \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{n_\xi s^2 e^{-s^2/2}|\rho|^2}{(|\xi|^2 + s^2)^2} \, dsd\xi.
\]

Denote
\[
I(t) := I(\phi(\cdot, t)) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{n_\xi s^2 e^{-s^2/2}|\rho|^2}{(|\xi|^2 + s^2)^2} \, dsd\xi, \quad t \geq 0,
\]

using $n_\xi \in [-n_3^2|\xi|^2, (1 - 2n_3^2)|\xi|^2]$, we obtain
\[
-\sqrt{2}n_3^2 \frac{V(t)}{\sqrt{\varepsilon}} \leq I(t) \leq \sqrt{2}(1 - 2n_3^2) \frac{V(t)}{\sqrt{\varepsilon}}, \quad t \geq 0.
\]

If $\lambda = 0$, or $\lambda > 0$ and $n_3 \geq \frac{1}{2}$, noticing $\lambda I(t) \leq 0$ in these cases, summing the $V(t)$ for $\alpha = x, y$, and using the energy conservation, we have
\[
\frac{d^2}{dt^2} E_2D(\phi(\cdot, t)) = 2 \int_{\mathbb{R}^2} \left[ \nabla \varphi^2 + \beta_0 |\varphi|^4 + \frac{3}{2} \lambda |\varphi|^2 (x \cdot \nabla \varphi) - |\varphi|^2 x \cdot \nabla V_2(x) \right] \, dx
\]
\[
= 4E_2D(\phi(\cdot, t)) + 3\lambda I(t) - 2 \int_{\mathbb{R}^2} |\varphi|^2 (2V_2(x) + x \cdot \nabla V_2(x)) \, dx
\]
\[
\leq 4E_2D(\phi(\cdot, t)) \equiv 4E_2D(\phi_0), \quad t \geq 0.
\]

Thus,
\[
\sigma_V(t) \leq 2E_2D(\phi_0)t^2 + \sigma'_V(0)t + \sigma_V(0), \quad t \geq 0,
\]
and the conclusion follows in the same manner as those in $[27, 13]$ for the standard nonlinear Schrödinger equation.

3. Results for the quasi-2D equation II. In this section, we investigate the existence, uniqueness as well as nonexistence of ground state of the quasi-2D equation II $[14, 15]$ and the well-posedness of the corresponding Cauchy problem.

3.1. Existence and uniqueness of ground state. Associated to the quasi-2D equation II $[14, 15]$, the energy is
\[
E_2D(\Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ |\nabla \Phi|^2 + V_2(x)|\Phi|^2 + \frac{\beta - \lambda + 3n_3^2}{2\sqrt{2\pi}\varepsilon} |\Phi|^4 - \frac{3\lambda}{4} |\Phi|^2 \varphi \right] \, dx, \quad \Phi \in X_2.
\]

where
\[
\varphi(x) = (\partial_{n_\perp n_{\perp}} - n_3^2\Delta)((-\Delta)^{-1/2}|\Phi|^2).
\]
Moreover, the ground state $\Phi_g \in S_2$ of the equation (1.15) is defined as the minimizer of the nonconvex minimization problem:

$$\text{Find } \Phi_g \in S_2, \text{ such that } \tilde{E}_{2D}(\Phi_g) = \min_{\Phi \in S_2} \tilde{E}_{2D}(\Phi). \quad (3.3)$$

For the above ground state, we have the following results.

**Theorem 3.1.** (Existence and uniqueness of ground state) Assume $0 \leq V_2(x) \in L^\infty_{loc}(\mathbb{R}^2)$ and $\lim_{|x| \to \infty} V_2(x) = \infty$, then we have

1. There exists a ground state $\Phi_g \in S_2$ of the equation (1.15) if one of the following conditions holds
   - (B1) $\lambda = 0$ and $\beta > -\sqrt{2\pi}C_b \varepsilon$;
   - (B2) $\lambda > 0$, $n_3 = 0$ and $\beta - \lambda > -\sqrt{2\pi}C_b \varepsilon$;
   - (B3) $\lambda < 0$, $n_3^2 \geq \frac{1}{2}$ and $\beta - (1 - 3n_3^2)\lambda > -\sqrt{2\pi}C_b \varepsilon$.

2. The positive ground state $|\Phi_g|$ is unique under one of the following conditions
   - (B1') $\lambda = 0$ and $\beta \geq 0$;
   - (B2') $\lambda > 0$, $n_3 = 0$ and $\beta \geq \lambda$;
   - (B3') $\lambda < 0$, $n_3^2 \geq \frac{1}{2}$ and $\beta - (1 - 3n_3^2)\lambda \geq 0$.

Moreover, any ground state $\Phi_g = e^{i\theta_0}|\Phi_g|$ for some constant $\theta_0 \in \mathbb{R}$.

3. There exists no ground state of the equation (1.15) if one of the following conditions holds
   - (B1'') $\lambda > 0$ and $n_3 \neq 0$;
   - (B2'') $\lambda < 0$ and $n_3^2 < \frac{1}{2}$;
   - (B3'') $\lambda = 0$ and $\beta < -\sqrt{2\pi}C_b \varepsilon$.

Again, in order to prove this theorem, we first analyze the nonlocal part in the equation (1.15). In fact, following the standard proof in [25], we can get

**Lemma 3.2.** (Property of fractional Poisson equation (1.13)) Assume $f(x)$ is a real valued function good enough, for the fractional Poisson equation

$$(-\Delta)^{-1/2}\varphi(x) = f(x), \quad x \in \mathbb{R}^2, \quad \lim_{|x| \to \infty} \varphi(x) = 0,$$

we have

$$\varphi(x) = \int_{\mathbb{R}^2} \frac{f(x')}{2\pi|x - x'|} dx' = \left(\frac{1}{2\pi|x|}\right) * f, \quad x \in \mathbb{R}^2,$$

and the Hardy-Littlewood-Sobolev inequality implies

$$\|\varphi\|_{p^*} \leq C_p\|f\|_p, \quad p^* = \frac{2p}{2 - p}, \quad p \in (1, 2). \quad (3.4)$$

Moreover, the first order derivatives of $\varphi$ are the Riesz transforms of $f$ and satisfy

$$\|\partial_\alpha \varphi\|_q \leq C_q\|f\|_q, \quad q \in (1, \infty), \quad \alpha = x, y, \quad (3.5)$$

and the second order derivatives satisfy

$$\|\partial_{\alpha\alpha'} \varphi\|_q \leq \|\partial_\alpha \left((-\Delta)^{-1/2}\partial_\alpha' f\right)\|_q \leq C_q\|\partial_{\alpha'} f\|_q, \quad q \in (1, \infty), \quad \alpha, \alpha' = x, y. \quad (3.6)$$

**Remark 3.1.** Similar results hold for $T_{\alpha\alpha'}$ defined in Lemma 2.2, i.e.

$$\|T_{\alpha\alpha'} f\|_p \leq C_p\|\nabla f\|_p, \quad \text{for } p \in (1, \infty). \quad (3.7)$$
Since the fractional Poisson operator \((-\Delta)^{-1/2}\) is taken as an approximation of \(U_{\varepsilon}^{2D}\), we consider the convergence regarding with the derivatives.

**Lemma 3.3.** For any real-valued function \(f \in L^p(\mathbb{R}^2)\), let

\[
T_\alpha^\varepsilon(f) = \partial_\alpha(U_{\varepsilon}^{2D} * f), \quad R_\alpha(f) = \partial_\alpha(-\Delta)^{-1/2}f, \quad \alpha = x, y,
\]

then \(T_\alpha^\varepsilon\) is bounded from \(L^p\) to \(L^p\) for \(1 < p < \infty\) with the bounds independent of \(\varepsilon\). Specially, for any fixed \(f \in L^p(\mathbb{R}^2)\) with \(p \in (1, \infty)\), we have

\[
\lim_{\varepsilon \to 0^+} \|T_\alpha^\varepsilon(f) - R_\alpha(f)\|_p = 0, \quad p \in (1, \infty).
\]

**Proof.** We can write \(R_\alpha\) and \(T_\alpha^\varepsilon\) as

\[
R_\alpha(f) = K_\alpha * f, \quad T_\alpha^\varepsilon(f) = K_\alpha^\varepsilon * f,
\]

where \(R_\alpha\) is the Riesz transform and

\[
K_\alpha(x) = \frac{\alpha}{2\pi |x|^3}, \quad K_\alpha^\varepsilon(x) = \frac{1}{2\sqrt{2\pi}^{3/2}} \int_{\mathbb{R}^2} \frac{e^{-|x-s|^2/\varepsilon}}{|x-s|^3} ds, \quad x \in \mathbb{R}^2, \quad \alpha = x, y.
\]

It is easy to check that \(K_\alpha^\varepsilon\) satisfies

\[
|K_\alpha^\varepsilon(x)| \leq B|x|^{-2}, \quad |\nabla K_\alpha^\varepsilon(x)| \leq B|x|^{-3}, \quad |x| > 0,
\]

\[
\int_{R_1 < |x| < R_2} K_\alpha^\varepsilon(x) dx = 0, \quad 0 < R_1 < R_2 < \infty,
\]

for some \(\varepsilon\)-independent constant \(B\). Then the standard theorem on singular integrals \([25]\) implies that \(T_\alpha^\varepsilon\) is well defined for \(L^p\) functions and is bounded from \(L^p\) to \(L^p\) with \(\varepsilon\)-independent bound.

Thus, we only need to prove the convergence in \(L^2\), other cases can be derived by an approximation argument and interpolation. For the \(L^2\) convergence, looking at the Fourier domain, we find that

\[
\|T_\alpha^\varepsilon(f) - R_\alpha(f)\|^2 \leq \frac{1}{4\pi^2} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left[ \frac{\alpha}{|\xi|} - \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{e^{-|s|^2/\varepsilon}}{|\xi|^2 + s^2} ds \right]^2 d\xi.
\]

Notice that for fixed \(0 \neq \xi \in \mathbb{R}^2\), the dominated convergence theorem suggests that

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \frac{(1 - e^{-|s|^2/\varepsilon})|\xi|}{|\xi|^2 + s^2} ds = 0,
\]

hence, the conclusion in \(L^2\) case is obvious by using the dominated convergence theorem again. Using approximation and noticing that \(L^2 \cap L^q\) is dense in \(L^p\) for \(q \in (1, \infty)\), noticing the uniform bound on \(T_\alpha^\varepsilon : L^p \to L^p\) for \(p \in (1, \infty)\), we can complete the proof. \(\square\)

**Lemma 3.4.** For the energy \(\tilde{E}_{2D}(\cdot)\) in \((\ref{e1.1})\), the following properties hold

(i) For any \(\Phi \in S_2\), denote \(\rho(x) = |\Phi(x)|^2\), then we have

\[
\tilde{E}_{2D}(\Phi) \geq \tilde{E}_{2D}(|\Phi|) = \tilde{E}_{2D}(\sqrt{\rho}), \quad \forall \Phi \in S_2,
\]

so the ground state \(\Phi_0\) of \((\ref{e1.1})\) is of the form \(e^{i\theta_0}|\Phi|\) for some constant \(\theta_0 \in \mathbb{R}\).

(ii) If condition \((B1)\) or \((B2)\) or \((B3)\) in Theorem \((\ref{t3.1})\) holds, then \(\tilde{E}_{2D}\) is bounded below.

(iii) If condition \((B1')\) or \((B2')\) or \((B3')\) in Theorem \((\ref{t3.1})\) holds, then \(\tilde{E}_{2D}(\sqrt{\rho})\) is strictly convex.
Proof. (i) It is similar to the case of Lemma 2.3.
(ii) Similar as Lemma 2.3 for $\Phi \in S_2$, denote $\rho = |\Phi|^2$, we only need to consider the lower bound of the following functional
\[
\tilde{H}(\rho) = -\lambda \int_{\mathbb{R}^2} \rho \left( \partial_{n_z n_z} - n_3^2 \Delta \right) \left[ |(-\Delta)^{-1/2} \rho| \right] d\mathbf{x}.
\]
Using the Plancherel formula and Cauchy inequality, for $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$, we have
\[
\tilde{H}(\rho) = \frac{\lambda}{4\pi^2} \int_{\mathbb{R}^2} \frac{(n_1 \xi_1 + n_2 \xi_2)^2 - n_3^2 \xi_3^2}{|\xi|} |\hat{\rho}(\xi)|^2 d\xi \geq \frac{\lambda}{4\pi^2} \int_{\mathbb{R}^2} (1 - 2n_3^2) |\xi| |\hat{\rho}(\xi)|^2 d\xi \geq 0. \quad (3.15)
\]
For $\lambda > 0$ and $n_3 = 0$, it is easy to see $\tilde{H}(\rho) \geq 0$. Hence, assertion (ii) is proven.
(iii) Similar as Lemma 2.3, it is sufficient to prove the convexity of $\tilde{H}(\rho)$ in $\rho$. For $\sqrt{\rho_1} \in S_2$, $\sqrt{\rho_2} \in S_2$ and any $\theta \in [0,1]$, denote $\rho_\theta = \theta \rho_1 + (1 - \theta) \rho_2$, we have
\[
\theta \tilde{H}(\rho_1) + (1 - \theta) \tilde{H}(\rho_2) - \tilde{H}(\rho_\theta) = \theta(1 - \theta) \tilde{H}(\rho_1 - \rho_2), \quad (3.16)
\]
where the RHS is nonnegative under the given condition, i.e., $\tilde{H}(\rho)$ is convex. \[\square\]

Proof of Theorem 3.1. (i) We only need to consider the existence since the uniqueness is a consequence of the convexity of $E_{2D}(\sqrt{\rho})$ in Lemma 3.3. For existence, we may apply the same arguments in Theorem 2.1 where instead, for sequence $\rho^n = (\Phi^n)^2$, we have to show
\[
\liminf_{n \to \infty} \tilde{H}(\rho^n) \geq \tilde{H}(\rho^\infty), \quad \text{with} \quad \rho^\infty = |\Phi^\infty|^2. \quad (3.17)
\]
Denote
\[
\varphi^n = (\partial_{n_z n_z} - n_3^2 \Delta) \left[ |(-\Delta)^{-1/2} \rho^n| \right], \quad n = 0, 1, \ldots, \text{ or } n = \infty.
\]
Using $\Phi^n \to \Phi^\infty$ in $L^2(\mathbb{R}^2)$ and $\Phi^n \to \Phi^\infty$ in $H^1(\mathbb{R}^2)$, then $\rho^n \to \rho^\infty$ in $L^p(\mathbb{R}^2)$ $p > 1$, and Lemma 3.2 shows that $\varphi^n \to \varphi^\infty$ in $W^{-1,p'}(\mathbb{R}^2)$ which is the dual space of $W^{1,p}$ with $p' = p/(p-1)$. Thus (3.14) is true and the existence of ground state follows.
(ii) To prove the nonexistence results, we try to find the case where $E_{2D}$ doesn’t have lower bound. For any $\Phi \in S_2$, denote $\rho(\mathbf{x}) = |\Phi(\mathbf{x})|^2$ and let $\theta \in \mathbb{R}$ such that $(\cos \theta, \sin \theta) = \frac{1}{\sqrt{n_1^2 + n_2^2}}(n_1, n_2)$ when $n_1^2 + n_2^2 \neq 0$ and $\theta = 0$ if $n_1 = n_2 = 0$. For any $\varepsilon_1, \varepsilon_2 > 0$, consider the following function
\[
\Phi_{\varepsilon_1, \varepsilon_2}(x, y) = \varepsilon_1^{-1/2} \varepsilon_2^{-1/2} \Phi(\varepsilon_1^{-1}(x \cos \theta + y \sin \theta), \varepsilon_2^{-1}(-x \sin \theta + y \cos \theta)), \quad (3.18)
\]
denoting $\rho_{\varepsilon_1, \varepsilon_2} = |\Phi_{\varepsilon_1, \varepsilon_2}|^2$, then
\[
\mathbf{\hat{r}}_{\varepsilon_1, \varepsilon_2}(\xi_1, \xi_2) = \rho(\varepsilon_1(\xi_1 \cos \theta + \xi_2 \sin \theta), \varepsilon_2(-\xi_1 \sin \theta + \xi_2 \cos \theta)). \quad (3.19)
\]
By the Plancherel formula and changing variables, we get
\[
\tilde{H}(\rho_{\varepsilon_1, \varepsilon_2}) = \frac{\lambda}{4\pi^2} \int_{\mathbb{R}^2} \frac{(n_1 \xi_1 + n_2 \xi_2)^2 - n_3^2 \xi_3^2}{|\xi|} |\hat{\rho}_{\varepsilon_1, \varepsilon_2}|^2 d\xi = \frac{\lambda}{4\pi^2} \int_{\mathbb{R}^2} \frac{(n_1^2 + n_2^2) \eta_1^2 - n_3^2 |\eta|^2}{|\eta|} |\hat{\rho}|^2(\varepsilon_1 \eta_1, \varepsilon_2 \eta_2) d\eta = \frac{\lambda}{4\varepsilon_1^2 \varepsilon_2^2 \pi^2} \int_{\mathbb{R}^2} \frac{(1 - n_3^2) \eta_1^2 - n_3^2 (\eta_1^2 + \varepsilon_1^2 \eta_2^2)}{\sqrt{\eta_1^2 + \varepsilon_1^2 \eta_2^2}} |\hat{\rho}|^2(\eta_1, \eta_2) d\eta.
\]
Let \( \kappa = \frac{\varepsilon_1}{\varepsilon_1} \), then the dominated convergence theorem implies
\[
\bar{H}(\rho_{\varepsilon_1, \varepsilon_2}) = \begin{cases} 
\frac{1-2n_3+o(1)}{4\varepsilon_1^2} \lambda \int_{\mathbb{R}^2} |\eta_1| |\tilde{\rho}(\eta_1, \eta_2)|^2 \, d\eta, & \kappa \to 0^+, \\
\frac{-n_3+o(1)}{4\varepsilon_1^2} \lambda \int_{\mathbb{R}^2} |\eta_1| |\tilde{\rho}(\eta_1, \eta_2)|^2 \, d\eta, & \kappa \to +\infty.
\end{cases}
\] (3.20)

For fixed \( \kappa > 0 \) and letting \( \varepsilon_1 \to 0^+ \), we have
\[
\int_{\mathbb{R}^2} V_2(x) |\Phi_{\varepsilon_1, \varepsilon_2}|^2 \, dx = O(1)
\]
and
\[
\|\nabla \Phi_{\varepsilon_1, \varepsilon_2}\|_2^2 = \frac{1}{\varepsilon_1^2} \|\partial_x \Phi\|_2^2 + \frac{1}{\varepsilon_2^2} \|\partial_y \Phi\|_2^2,
\]
\[
\|\Phi_{\varepsilon_1, \varepsilon_2}\|_4^4 = \frac{1}{\varepsilon_1 \varepsilon_2} \|\Phi\|_4^4.
\] (3.21)

Thus under the condition (B1’), i.e. \( n_3 \neq 0 \) and \( \lambda > 0 \), choosing \( \kappa \) large enough, we get
\[
\tilde{E}_{2D}(\Phi_{\varepsilon_1, \varepsilon_2}) = C_1 \frac{1}{\varepsilon_1^2} + C_2 \frac{C_3}{\kappa^2 \varepsilon_1^2} + C_4 \frac{1-2n_3+o(1)}{\varepsilon_1^2} + O(1),
\] (3.22)
where \( C_k (k = 1, 2, 3, 4) \) are constants independent of \( \kappa, \varepsilon_1 \) and \( C_4 > 0 \). Since \( n_3 \neq 0 \), the last term is negative for \( \kappa \) large, sending \( \varepsilon_1 \to 0^+ \), one immediately finds that
\[
\lim_{\varepsilon_1 \to 0^+, \varepsilon_2 = \varepsilon_1} \tilde{E}_{2D}(\Phi_{\varepsilon_1, \varepsilon_2}) = -\infty,
\]
which justifies the nonexistence. Under the condition (B2’), i.e. \( n_3^2 < \frac{3}{2} \) and \( \lambda < 0 \), by choosing \( \kappa \) small enough in (3.20), sending \( \varepsilon_1 \to 0^+ \), we will have the same results. Case (B3’’) will reduce to Theorem 2.1. \( \square \)

3.2. Existence results for the Cauchy problem. Let us consider the Cauchy problem of equation (1.15), noticing the nonlinearity \( \phi (\partial_{x \cdot n_1} - n_3^2 \Delta) ((-\Delta)^{-1/2} |\phi|^2) \) is actually a derivative nonlinearity, it would bring significant difficulty in analyzing the dynamic behavior. The common approach to solve the Schrödinger equation is trying to solve the corresponding integral equation by fixed point theorem. However, the loss of order 1 derivative due to the nonlocal term will cause trouble. This can be overcome by the smoothing effect of inhomogeneous problem \( iu_t + \Delta u = g(x, t) \), which provides a gain of order 1 derivative [10, 17]. When \( V_2(x) = 0 \), i.e. without external trapping potential which corresponds to the free expansion of a dipolar BEC after turning off the confinement, the above approach can be extended straightforward. However, when \( V_2(x) \neq 0 \), i.e. with an external trapping potential, especially a confinement trapping potential with \( \lim_{|x| \to \infty} V_2(x) = \infty \), the approach in [10, 17] has some difficulties. By configuring that \( (\partial_{x \cdot n_1} - n_3^2 \Delta)((-\Delta)^{-1/2} |\phi|^2) \) is almost a first order derivative, we are able to establish the well-posedness of (1.15) with a general external potential \( V_2(x) \) by a different approach.

The Cauchy problem of the Schrödinger equation with derivative nonlinearity has been investigated extensively in the literatures [16, 18]. Here, we present an existence result in the energy space with the special structure of our nonlinearity, which will show that the approximation (1.15) of (1.11) is reasonable in suitable sense. We are interested in the case of \( \lambda \neq 0 \).

**Theorem 3.5.** (Existence for Cauchy problem) Suppose the real potential \( V_2(x) \) satisfies (3.37) and \( \lim_{|x| \to \infty} V_2(x) = \infty \), and initial value \( \phi_0(x) \in X_2 \), either condition (B2) or (B3) in Theorem 2.1 holds with constant \( C_0 \) being replaced by \( C_0 / \|\phi_0\|_2^2 \), then there exists a solution \( \phi \in L^2((0, \infty); X_2) \cap W^{1, \infty}([0, \infty); X_2^*) \) for the Cauchy problem of (1.15). Moreover, there holds for \( L^2 \) norm and energy \( \tilde{E}_{2D} \) conservation, i.e.
\[
\|\phi(\cdot, t)\|_{L^2(\mathbb{R}^2)} = \|\phi_0\|_{L^2(\mathbb{R}^2)}, \quad \tilde{E}_{2D}(\phi(\cdot, t)) \leq \tilde{E}_{2D}(\phi_0), \quad \forall t \geq 0.
\] (3.23)

**Proof.** We first consider the Cauchy problem for the following equation,
\[
i \partial_t \phi^0(x, t) = H_x^0 \phi^0 + g_1(\phi^0) + g_2(\phi^0), \quad x \in \mathbb{R}^2, \quad t > 0,
\] (3.24)
with the initial data \( \phi^0(x, 0) = \phi_0(x) \), \( \beta_0 = \frac{\beta - \lambda + 3n_3^2}{\sqrt{2n_3}} \), \( \varphi^0 = U_{\delta}^{2D} * |\phi^0|^2 \) where \( U_{\delta}^{2D} \) is given in (1.12) as
\[
U_{\delta}^{2D}(x) = \frac{1}{2\sqrt{2} \pi \sqrt{sd}} \int_{\mathbb{R}^2} e^{-s^2/2} \, ds \quad (\delta > 0),
\]
and
\[
H_x^0 = \frac{1}{2} \Delta + V_2(x), \quad g_1(\phi^0) = \beta_0 |\phi^0|^2 \phi^0, \quad g_2(\phi^0) = -\frac{3\lambda}{2} \phi^0 (\partial_{x \cdot n_1} - n_3^2 \Delta) \phi^0.
\] (3.25)

The proof involves the standard energy and contraction arguments, and we refer to [10, 17] for the details.
Thus, from (3.32) and (3.34), there exists a sequence by a similar proof in Theorem 2.1, the weak convergence of \( \phi \) to \( \phi_n \). We denote the pairing of \( X_2 \) and its dual \( X_2^* \) by \( \langle \cdot, \cdot \rangle_{X_2, X_2^*} \) as

\[
\langle f_1, f_2 \rangle_{X_2, X_2^*} = \text{Re} \int_{\mathbb{R}^2} f_1(x) \overline{f_2(x)} \, dx,
\]

where \( \text{Re}(f) \) denotes the real part of \( f \). Using the results in Theorem [3.24] and [13], we see that there exists a unique maximal solution \( \phi^\delta \in C([-T^\delta_{\text{min}}, T^\delta_{\text{max}}], X_2) \cap C^1([-T^\delta_{\text{min}}, T^\delta_{\text{max}}], X_2^*) \). Here maximal means that if either \( t \uparrow T^\delta_{\text{max}} \) or \( t \downarrow -T^\delta_{\text{min}} \), \( \|\phi^\delta(t)\|_{X_2} \to \infty \). We want to show that as \( \delta \to 0^+ \), \( \phi^\delta \) will converge to a solution of equation (1.11).

First, we show that \( T^\delta_{\text{min}} = +\infty \) and \( T^\delta_{\text{max}} = +\infty \). The energy conservation for (3.24) is

\[
E_\delta(t) := \frac{1}{2} \|\nabla \phi^\delta\|^2 + \frac{1}{2} \|\phi^\delta\|_4^4 + \int_{\mathbb{R}^2} V_2(x)|\phi^\delta|^2 \, dx + E^\delta_{\text{dip}}(t) \equiv E_\delta(0), \quad t \geq 0,
\]

where

\[
E^\delta_{\text{dip}}(t) = -\frac{3\lambda}{4} \int_{\mathbb{R}^2} |\phi^\delta|^2 |\partial_{n_1 n_2} - n_3^2 \Delta| \phi^\delta \, dx, \quad t \geq 0.
\]

Similar computation as in Lemma 3.4 confirms that \( E^\delta_{\text{dip}} \geq 0 \). Hence energy conservation will imply that \( \|\phi^\delta(t)\|_{X_2} < \infty \) for all \( t \), i.e. \( T^\delta_{\text{max}} = T^\delta_{\text{min}} = +\infty \).

We notice that

\[
X_2 \hookrightarrow H^1 \hookrightarrow L^2 \hookrightarrow H^{-1} \hookrightarrow X_2^*,
\]

where \( H^{-1} \) is viewed as the dual of \( H^1 \). Consider a bounded time interval \( I = [-T, T] \), it follows from energy conservation that there exists a constant \( C_1(\phi_0) > 0 \) such that

\[
\|\phi^\delta\|_{C([-T, T]; X_2)} \leq C_1(\phi_0).
\]

Moreover, Lemma 2.2 and Remark 3.1 would imply

\[
\|\phi^\delta |\partial_{n_1 n_2} - n_3^2 \Delta| \phi^\delta |\|_q \leq C \|\phi^\delta\|_{q'} \|\nabla |\phi^\delta|\|^2 \|_p \leq C \|\phi^\delta\|_{q'} \|\phi^\delta\|_{2p/(2-p)} \|\nabla |\phi^\delta|\|_2
\]

for \( q, p \in (1, 2), \frac{1}{q} + \frac{1}{p} = \frac{1}{q} \). Then we have

\[
\|\phi^\delta\|_{C^1([-T, T]; X_2^*)} \leq C_2(\phi_0).
\]

Thus, from (3.22) and (3.24), there exists a sequence \( \delta_n \to 0^+ \) \((n = 1, 2, \ldots)\) and a function \( \phi \in L^\infty([-T, T]; X_2) \cap W^{1,\infty}([-T, T]; X_2^*) \) [13], such that

\[
\phi^{\delta_n}(t) \to \phi(t) \quad \text{in} \ X_2, \text{ for all } t \in [-T, T].
\]

For each \( t \in [-T, T] \), due to the mass conservation of the equation (3.24), we know \( \|\phi^{\delta_n}(t)\|_2 = \|\phi_0\|_2 \), by a similar proof in Theorem 2.1, the weak convergence of \( \phi^{\delta_n}(t) \) in \( X_2 \) would imply that \( \phi^{\delta_n}(t) \) converges strongly in \( L^2 \), which is a consequence of the fact that \( V_2(x) \) is a confining potential. So,

\[
\phi^{\delta_n} \to \phi, \quad \text{in} \ C([-T, T]; L^2(\mathbb{R}^2)).
\]
In view of (3.35), (3.36) and the Gagliardo-Nirenberg’s inequality, we obtain
\[ \phi^{\delta_n} \to \phi, \quad \text{in } C([-T, T]; L^p(\mathbb{R}^2)), \quad \text{for all } p \in [2, \infty). \tag{3.37} \]

We now try to say that \( \phi \) actually solves equation (1.13). For any function \( \psi(x) \in X_2 \) and \( f(t) \in C^\infty_c([-T, T)), \) from equation (3.24), we have
\[ \int_{-T}^{T} \left[ \langle i\phi^{\delta_n}, \psi \rangle_{X_2, X_2^*} f'(t) + \langle H^V_x \phi^{\delta_n} + g_1(\phi^{\delta_n}) + g_2(\phi^{\delta_n}), \psi \rangle_{X_2, X_2^*} f(t) \right] dt = 0. \tag{3.38} \]

Recalling \(|g_1(u) - g_1(v)| \leq C(|u|^2 + |v|^2)|u - v|, \), (3.37) implies that \( g_1(\phi^{\delta_n}(t)) \to g_1(\phi(t)), \quad \text{in } L^p(\mathbb{R}^2) \) for some \( p \in [1, \infty), \)
\[ \langle g_1(\phi^{\delta_n}(t)), \psi(t) \rangle_{X_2, X_2^*} \to \langle g_1(\phi(t)), \psi(t) \rangle_{X_2, X_2^*}. \tag{3.39} \]

For \( g_2(\phi^{\delta_n}) \), consider \( \varphi^{\delta_n}(x, t), \) noticing \( \partial_{\alpha} \varphi^{\delta_n} = T^{\delta_n}_\alpha(\varphi^{\delta_n}) \) (\( \alpha = x, y \)) (defined in Lemma 3.3), we have proven in Lemma 3.3 that \( T^{\delta_n}_\alpha \) is uniformly bounded from \( L^p \) to \( L^p \) and
\[ T^{\delta_n}_\alpha(\langle \varphi(t) \rangle^2) \to R_\alpha(\langle \varphi(t) \rangle^2) = \partial_\alpha(-\Delta)^{-1/2}(\langle \varphi(t) \rangle^2) \quad \text{in } L^p(\mathbb{R}^2), \quad p \in (1, \infty), \quad \delta_n \to 0^+. \tag{3.41} \]

Rewriting
\[ T^{\delta_n}_\alpha(\langle \varphi(t) \rangle^2) = T^{\delta_n}_\alpha(\langle \varphi(t) \rangle^2 - |\varphi(t)|^2) + T^{\delta_n}_\alpha(\langle \varphi(t) \rangle^2), \tag{3.42} \]
recalling the fact (3.37), we immediately have
\[ T^{\delta_n}_\alpha(\langle \varphi(t) \rangle^2) \to R_\alpha(\langle \varphi(t) \rangle^2) \quad \text{in } L^p(\mathbb{R}^2), \quad \text{for some } p \in (1, \infty), \tag{3.43} \]
which is actually
\[ \partial_\alpha \varphi^{\delta_n}(t) \to \partial_\alpha \left( (-\Delta)^{-1/2}|\varphi(t)|^2 \right), \quad \text{in } L^p(\mathbb{R}^2), \quad \text{for some } p \in (1, \infty). \tag{3.44} \]

Hence, integration by parts, for \( \alpha' = x, y, \)
\[ \langle \delta_n(t) \partial_{\alpha} \varphi^{\delta_n}(t), \psi(t) \rangle_{X_2, X_2^*} = \text{Re} \int_{\mathbb{R}^2} \bar{\varphi}^{\delta_n}(t) \partial_{\alpha} \varphi^{\delta_n}(t) \psi(t) \, dx \]
\[ = -\text{Re} \int_{\mathbb{R}^2} \partial_{\alpha} \bar{\varphi}^{\delta_n}(t) \varphi^{\delta_n}(t) \psi(t) + \phi^{\delta_n}(t) \partial_{\alpha} \psi(t) \, dx, \]
passing to the limit as \( n \to \infty, \)
\[ \lim_{n \to \infty} \langle \delta_n(t) \partial_{\alpha} \varphi^{\delta_n}(t), \psi(t) \rangle_{X_2, X_2^*} = -\text{Re} \int_{\mathbb{R}^2} R_\alpha(\langle \varphi(t) \rangle^2) \partial_{\alpha} \varphi(t) \psi(t) + \phi(t) \partial_{\alpha} \psi(t) \, dx \]
\[ = \langle \varphi(t) \partial_{\alpha} \varphi(t) \rangle_{X_2, X_2^*}, \]
in view of (3.44) and (3.35), we obtain
\[ \lim_{n \to \infty} \langle g_2(\varphi^{\delta_n}(t)), \psi(t) \rangle_{X_2, X_2^*} = \langle \tilde{g}_2(\varphi(t)), \psi(t) \rangle_{X_2, X_2^*}. \tag{3.45} \]

Combining the above results and (3.40) together, sending \( n \to \infty, \) dominated convergence theorem will yield
\[ \int_{-T}^{T} \left[ \langle i\phi, \psi \rangle_{X_2, X_2^*} f'(t) + \langle H^V_x \phi + g_1(\phi) + \tilde{g}_2(\phi), \psi \rangle_{X_2, X_2^*} f(t) \right] dt = 0, \]
which proves that

\[ i\partial_t \phi = H^V \phi + g_1(\phi) + \tilde{g}_2(\phi), \quad \text{in } X_2^*, \quad \text{a.a. } t \in [-T, T], \quad (3.46) \]

with \( \phi(t = 0) = \phi_0 \), and \( \phi \in L^\infty([-T, T]; X_2) \cap W^{1, \infty}([-T, T]; X_2^*) \). Moreover, by lower semi-continuity of \( X_2 \) norm, (3.39) and (3.45), the energy \( \tilde{E}_{2D} \) (3.1) satisfies

\[ \tilde{E}_{2D}(\phi(t)) \leq \tilde{E}_{2D}(\phi_0). \quad (3.47) \]

It is easy to see that we can choose \( T = \infty \).

If the uniqueness of the \( L^\infty([-T, T]; X_2) \cap W^{1, \infty}([-T, T]; X_2^*) \) solution to the quasi-2D equation II (1.10) is known, we can prove that the solution constructed above in Theorem 3.3 is actually \( C([-T, T]; X_2) \cap C^1([-T, T]; X_2^*) \) and conserves the energy.

Next, we discuss possible finite time blow-up for the continuous solutions of the quasi-2D equation II (1.10). To this purpose, the following assumptions are introduced:

(A) Assumption on the trap and coefficient of the cubic term, i.e. \( V_2(x) \) satisfies \( 3V_2(x) + x \cdot \nabla V_2(x) \geq 0, \frac{\beta - \lambda + 3\lambda n_3^2}{\sqrt{2\pi} \varepsilon} \geq - \frac{C_b}{\|\phi_0\|_2^2} \), with \( \phi_0 \) being the initial data of equation (1.15);

(B) Assumption on the trap and coefficient of the nonlocal term, i.e. \( V_2(x) \) satisfies \( 2V_2(x) + x \cdot \nabla V_2(x) \geq 0, \lambda = 0 \) or \( \lambda > 0 \) and \( n_3^2 \geq \frac{1}{2} \).

**Theorem 3.6.** (Finite time blow-up) For any initial data \( \phi(x, t = 0) = \phi_0(x) \in X_2 \) with \( \int_{\mathbb{R}^2} |x|^2 |\phi_0(x)|^2 \, dx < \infty \), if conditions (B1), (B2) and (B3) with constant \( C_b \) being replaced by \( C_b/\|\phi_0\|_2^2 \) are not satisfied, let \( \phi := \phi(x, t) \) be a \( C([0, T_{max}]; X_2) \) solution of the problem (1.10) with \( L^2 \) norm and energy conservation, then there exists finite time blow-up, i.e., \( T_{max} < \infty \), if one of the following condition holds:

(i) \( \tilde{E}_{2D}(\phi_0) < 0 \), and either Assumption (A) or (B) holds;

(ii) \( \tilde{E}_{2D}(\phi_0) = 0 \) and \( \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi}_0(x) (x \cdot \nabla \phi_0(x)) \, dx \right) < 0 \), and either Assumption (A) or (B) holds;

(iii) \( \tilde{E}_{2D}(\phi_0) > 0 \), and \( \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi}_0(x) (x \cdot \nabla \phi_0(x)) \, dx \right) < - \sqrt{3 \tilde{E}_{2D}(\phi_0)} \|x\phi_0\|_2 \) if Assumption (A) holds, or \( \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi}_0(x) (x \cdot \nabla \phi_0(x)) \, dx \right) < - \sqrt{2 \tilde{E}_{2D}(\phi_0)} \|x\phi_0\|_2 \) if Assumption (B) holds.

**Proof.** Calculating derivatives of the variance defined in (2.40), for \( \alpha = x, y \), we have

\[ \frac{d}{dt} \sigma_{\alpha}(t) = 2 \text{Im} \left( \int_{\mathbb{R}^2} \bar{\phi} \alpha \partial_\alpha \phi \, dx \right), \quad t \geq 0, \quad (3.48) \]

and

\[ \frac{d^2}{dt^2} \sigma_{\alpha}(t) = \int_{\mathbb{R}^2} \left[ 2|\partial_\alpha \phi|^2 + \beta_0 |\phi|^4 + 3\lambda |\phi|^2 \alpha \partial_\alpha (\partial_{\alpha n_1} n_1 - n_3^2 \Delta) \phi - 2\alpha |\phi|^2 \partial_\alpha V_2(x) \right] \, dx, \quad (3.49) \]

where \( \beta_0 = \frac{\beta - \lambda + 3\lambda n_3^2}{\sqrt{2\pi} \varepsilon} \), \( (-\Delta)^{1/2} \phi = |\phi|^2 \). Writing \( \rho = |\phi|^2, \tilde{\phi} = (\partial_{n_1} n_1 - n_3^2 \Delta) \phi \) and noticing that \( \rho \) is a real function, by the Plancherel formula, similar as Theorem 2.3, we get

\[ \int_{\mathbb{R}^2} |\phi|^2 (x \cdot \nabla \tilde{\phi}) \, dx = - \frac{3}{2} \int_{\mathbb{R}^2} |\phi|^2 \tilde{\phi} \, dx. \quad (3.50) \]

Hence, summing (3.49) for \( \alpha = x, y \), and using the energy conservation, if Assumption (A) holds, we have

\[ \frac{d^2}{dt^2} \sigma_{V}(t) = 2 \int_{\mathbb{R}^2} \left( \nabla \phi \right)^2 + \beta_0 |\phi|^4 + \frac{9}{4} \lambda |\phi|^2 (\partial_{n_1} n_1 - n_3^2 \Delta) \varphi - |\phi|^2 (3V_2(x) + x \cdot \nabla V_2(x)) \, dx \]

\[ = 6\tilde{E}_{2D}(\phi) - \int_{\mathbb{R}^2} (|\nabla \phi|^2 + \beta_0 |\phi|^4) \, dx - 2 \int_{\mathbb{R}^2} |\phi|^2 (3V_2(x) + x \cdot \nabla V_2(x)) \, dx \]

\[ \leq 6\tilde{E}_{2D}(\phi(\cdot, t)) \leq 6\tilde{E}_{2D}(\phi_0), \quad t \geq 0. \quad (3.51) \]
Thus,
\[ \sigma_V(t) \leq 3\tilde{E}_{2D}(\phi_0)t^2 + \sigma_V(0)t + \sigma_V(0), \quad t \geq 0, \]
and the conclusion follows as in Theorem 2.5. If Assumption (B) holds, the energy contribution of the nonlocal part is non-positive and we have
\[
\frac{d^2}{dt^2}\sigma_V(t) = 2 \int_{\mathbb{R}^2} \left[ |\nabla \phi|^2 + \beta_0|\phi|^4 - \frac{9}{4}\lambda|\phi|^2 \left( \partial_n \nabla \phi - n_3 \Delta \right) \right] dx
\]
\[
+ \frac{3\lambda}{2} \int_{\mathbb{R}^2} |\phi|^2 \varphi d\mathbf{x} - 2 \int_{\mathbb{R}^2} |\phi|^2 \left( 2V_2(\mathbf{x}) + \mathbf{x} \cdot \nabla V_2(\mathbf{x}) \right) dx
\]
\[
\leq 4\tilde{E}_{2D}(\phi_0), \quad t \geq 0,
\]
and the conclusion follows in a similar way as the Assumption (A) case. \[ \square \]

4. Results for quasi-1D equation. In this section, we prove the existence and uniqueness of the ground state for quasi-1D equation (1.17) and establish the well-posedness for dynamics.

4.1. Existence and uniqueness of ground state. Associated to the quasi-1D equation (1.17), the energy is
\[
E_{1D}(\Phi) = \int_\mathbb{R} \left[ \frac{1}{2}\partial_z \Phi^2 + V_1(\xi)|\Phi|^2 + \frac{1}{2}\beta_{1D}|\Phi|^4 + \frac{3\lambda(1 - 3n_3^2)}{16\sqrt{2\pi}\varepsilon}|\Phi|^2 \varphi \right] dz,
\]
where \( \beta_{1D} = \frac{\beta + \lambda(1 - 3n_3^2)}{2n_3\pi} \) and
\[
\varphi(z) = \partial_z (U_{\varepsilon}^{1D} * |\Phi|^2), \quad U_{\varepsilon}^{1D}(z) = \frac{2e^{-\varepsilon z^2}}{\sqrt{\pi}} \int_{|z|}^{\infty} e^{-s^2/\varepsilon} ds.
\]
Again, the ground state \( \Phi_g \in S_1 \) of the equation (1.17) is defined as the minimizer of the nonconvex minimization problem:
\[
\text{Find } \Phi_g \in S_1, \text{ such that } E_{1D}(\Phi_g) = \min_{\Phi \in S_1} E_{1D}(\Phi).
\]

For the above ground state, we have the following results.

**Theorem 4.1.** (Existence and uniqueness of ground state) Assume \( 0 \leq V_1(\xi) \in L_{\text{loc}}^\infty(\mathbb{R}) \) and \( \lim_{|\xi| \to \infty} V_1(\xi) = \infty \), for any parameter \( \beta, \lambda, \varepsilon > 0 \), there exists a ground state \( \Phi_g \in S_1 \) of the quasi-1D equation (1.17)-(1.18), and the positive ground state \( |\Phi_g| \) is unique under one of the following conditions:

\( C1 \) \( \lambda(1 - 3n_3^2) \geq 0 \) and \( \beta - (1 - 3n_3^2)\lambda \geq 0 \);
\( C2 \) \( \lambda(1 - 3n_3^2) < 0 \) and \( \beta + \lambda(1 - 3n_3^2) \geq 0 \).

Moreover, \( \Phi_g = e^{\theta_0}|\Phi_g| \) for some constant \( \theta_0 \in \mathbb{R} \).

To complete the proof, we first study the property of the convolution kernel \( U_{\varepsilon}^{1D} \) (1.18).

**Lemma 4.2.** (Kernel \( U_{\varepsilon}^{1D} \) (1.18)) For any \( f(\xi) \) in the Schwartz space \( S(\mathbb{R}) \), we have
\[
U_{\varepsilon}^{1D} * f(\xi) = \hat{f}(\xi)U_{\varepsilon}^{1D}(\xi) = \frac{\sqrt{2\pi}\varepsilon \hat{f}(\xi)}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2/2s} \frac{e^{-s^2/\varepsilon}}{|\xi|^2 + s} ds, \quad \xi \in \mathbb{R}.
\]

Hence
\[
\|\partial_z (U_{\varepsilon}^{1D} * f)\|_2 \leq \frac{2\sqrt{2}}{\sqrt{\pi}\varepsilon} \|f\|_2.
\]
Proof. For any \( f(z) \in S(\mathbb{R}) \), rewrite the kernel as \( \|f\|_2^2 \leq \|f\|_\infty^2 \|f\|_2^2 \). Then we have

\[
E_{1D}(\rho) = \int_{\mathbb{R}} \left[ \beta + \lambda \frac{(1 - 3n_2^2)/2}{4\pi^2} \rho^2 + \frac{3\lambda(1 - 3n_2^2)}{16\sqrt{2\pi^2}} \rho \partial_{zz}(u_{1D}(\rho)) \right] dz. \tag{4.9}
\]

Then under condition (C1) or (C2), using the Plancherel formula and Lemma 4.2 after similar computation as in Lemma 2.2, we would have

\[
E_{1D}(\sqrt{\rho}) \leq E_{1D}(\sqrt{\rho}) \quad \forall \Phi \in S_1,
\]

so the ground state \( \Phi_\rho \) of (4.4) is of the form \( e^{i\theta_0} |\Phi_\rho| \) for some constant \( \theta_0 \in \mathbb{R} \).

(iii) We come to the convexity of \( E_{1D}(\sqrt{\rho}) \). Following Lemma 2.3, we only need to consider the functional

\[
H_{1D}(\rho) = \int_{\mathbb{R}} \left[ \beta + \lambda(1 - 3n_2^2)/2 \rho^2 + \frac{3\lambda(1 - 3n_2^2)}{16\sqrt{2\pi^2}} \rho \partial_{zz}(u_{1D}(\rho)) \right] dz.
\]

5. Convergence rate of dimension reduction. In this section, we discuss the dimension reduction of 3D GPPS to lower dimensions. Inspired by the previous work of Ben Abdallah et al. \[8, 9\] for GPE without the dipolar term (i.e. \( \lambda = 0 \)) and \[6, 7\] for Schrödinger-Poisson systems, we are going to find a limiting \( \varepsilon \)-independent equation as \( \varepsilon \to 0^+ \). Thus in quasi-2D equation I (1.11), II (1.15) and quasi-1D equation (1.17), we have to consider the coefficients to be \( O(1) \). The existence of the global
solution for the full 3D system \((1.4), (1.5)\) has been proven in [12] when \(\beta \geq 0\) and \(\lambda \in [-\frac{1}{2}, \beta]\), hence we would expect the limiting equation in lower dimensions valid in a similar regime. Thus in lower dimensions, we require that in the quasi-2D case, \(\beta = O(\epsilon), \lambda = O(\epsilon)\), and in the quasi-1D case, \(\beta = O(\epsilon^2), \lambda = O(\epsilon^2)\), i.e. we are considering the weak interaction regime, then we would get an \(\epsilon\)-independent limiting equation. In this regime, we will see that GPPS will reduce to a regular GPE in lower dimensions.

5.1. Reduction to 2D. We consider the weak interaction regime, i.e., \(\beta \rightarrow \epsilon \beta, \lambda \rightarrow \epsilon \lambda\). In Case I [8], for full 3D GPPS \((1.4)-(1.5)\), introduce the re-scaling \(z \rightarrow \epsilon z, \psi \rightarrow \epsilon^{-1/2}\psi\) which preserves the normalization, then

\[
i \partial_t \psi^\epsilon(x, z, t) = \left[ H_x^\epsilon + \frac{1}{\epsilon^2} H_z + (\beta - \lambda) |\psi^\epsilon|^2 - 3\epsilon \lambda \partial_n \cdot \nabla \psi^\epsilon \right] \psi^\epsilon, \quad (x, z) \in \mathbb{R}^3, \ t > 0, \tag{5.1}\]

where \(x = (x, y) \in \mathbb{R}^2\), and

\[
H_x^\epsilon = -\frac{1}{2} (\partial_{xx} + \partial_{yy}) + V_2(x, y), \quad H_z = -\frac{1}{2} \partial_{zz} + \frac{z^2}{2}, \quad \n_x = (n_1, n_2, n_3/\epsilon), \quad \partial_n = \n_x \cdot \nabla, \quad \partial_{n, n} = \partial_{n_x} (\partial_{n_x}), \quad (-\partial_{xx} - \partial_{yy} - \frac{1}{\epsilon^2} \partial_{zz}) \varphi^\epsilon = \frac{1}{\epsilon} |\psi^\epsilon|^2, \quad \lim_{|x| \to \infty} \varphi^\epsilon(x) = 0. \tag{5.2}\]

It is well-known that \(H_z\) has eigenvalues \(\mu_k = k + 1/2\) with corresponding eigenfunction \(w_k(z)\) \((k = 0, 1, \ldots)\), where \(\{w_k\}_{k=0}^\infty\) forms an orthonormal basis of \(L^2(\mathbb{R})\) [11][28], specifically, \(w_0(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/2}\). Following [8], it is convenient to consider the initial data concentrated on the ground mode of \(H_z\), i.e.,

\[
\psi^\epsilon(x, z, 0) = \phi_0(x)w_0(z), \quad \phi_0 \in X_2 \text{ and } \|\phi_0\|_{L^2(\mathbb{R}^2)} = 1. \tag{5.3}\]

In Case I [8], when \(\epsilon \to 0^+\), quasi-2D equation I \((1.11), II(1.15)\) will yield an \(\epsilon\)-independent equation in the weak interaction regime,

\[
i \partial_t \phi(x, t) = H_x^\epsilon \phi + \frac{\beta - (1 - 3n_3^2)\lambda}{\sqrt{2\pi}} |\phi|^2 \phi, \quad x = (x, y) \in \mathbb{R}^2, \tag{5.4}\]

with initial condition \(\phi(x, 0) = \phi_0(x)\). We follow the ideas in [6][8][9] to show the convergence from the 3D GPPS to the 2D approximation. First, let us state the main result.

**Theorem 5.1.** (Dimension reduction to 2D) Suppose \(V_2\) satisfies condition (2.37), \(-\frac{1}{2} \leq \lambda \leq \beta\) and \(\beta \geq 0\), let \(\psi^\epsilon \in C([0, \infty); X_3)\) and \(\phi \in C([0, \infty); X_2)\) be the unique solutions of equations (5.1)-(5.5) and (5.6), respectively, then for any \(T > 0\), there exists \(C_T > 0\) such that

\[
\left\| \psi^\epsilon(x, z, t) - e^{-i\mu t/\epsilon^2} \phi(x, t)w_0(z) \right\|_{L^2(\mathbb{R}^3)} \leq C_T \epsilon, \quad \forall t \in [0, T]. \tag{5.7}\]

Under the assumption, we have the global existence of \(\psi^\epsilon\) [12] as well as \(\phi\) [8][13]. Define the projection operator onto the ground mode \(H_z\) by

\[
\Pi \psi^\epsilon(x, z, t) = e^{-i\mu t/\epsilon^2} \phi(x, t)w_0(z), \quad (x, z) \in \mathbb{R}^3, \ t \geq 0, \tag{5.8}\]

where

\[
\phi(x, t) = e^{i\mu t/\epsilon^2} \int_{\mathbb{R}} \psi^\epsilon(x, z, t)w_0(z)dz, \quad x = (x, y) \in \mathbb{R}^2, \quad t \geq 0. \tag{5.9}\]

Since the space \((x, z) \in \mathbb{R}^3\) is anisotropic, we introduce the \(L_p^\epsilon L_q^\epsilon\) space by the norm

\[
\|f\|_{(p, q)} := \|f\|_{L_p^\epsilon L_q^\epsilon} = \|f(\cdot, z)\|_{L^q}, \quad p, q \in [1, \infty]. \tag{5.10}\]
The corresponding anisotropic Sobolev inequalities are available [5].

**Lemma 5.2.** (Uniform bound) Let \( \psi^\varepsilon \) and \( \phi \) be the solutions of \((5.7)\) and \((5.6)\), respectively, \( \phi^\varepsilon \) be defined in \((5.9)\), \( \lambda \in [-\frac{3}{4}, \beta] \) and \( \beta \geq 0 \), we have

\[
\psi^\varepsilon \in L^\infty((0, \infty), H^1(\mathbb{R}^3)), \quad \phi, \phi^\varepsilon \in L^\infty((0, T), H^1(\mathbb{R}^2)),
\]

(5.11)

with uniform bound in \( \varepsilon \). Moreover, for \( p \in [2, \infty] \),

\[
\| \psi^\varepsilon - \Pi \psi^\varepsilon \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t (\psi^\varepsilon - \Pi \psi^\varepsilon) \|^2_{L^2(\mathbb{R}^3)} \leq C \varepsilon^2, \quad \| \psi^\varepsilon - \Pi \psi^\varepsilon \|_{(p, 2)} \leq C \varepsilon,
\]

(5.12)

with \( C \) depending on \( \| \phi_0 \|_{X_2} \), uniform in time \( t \).

**Proof.** From energy conservation of equation \((5.1)\), we know that the following energy is constant,

\[
E(t) := (H_V^\varepsilon \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2} (H_x \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{\beta - \lambda}{2} \| \psi^\varepsilon \|^4_{L^4(\mathbb{R}^3)} + \frac{3 \lambda \varepsilon^2}{2} \| \partial_n \nabla \psi^\varepsilon(t) \|^2_{L^2(\mathbb{R}^3)},
\]

where \((\cdot, \cdot)\) denotes the standard \( L^2 \) inner product in \( \mathbb{R}^3 \). Using standard \( L^p \) estimates for Poisson equation \((5.3)\), we have \( \| \partial_n \nabla \phi^\varepsilon(t) \|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\varepsilon} \| \phi^\varepsilon(t) \|^2_{L^2(\mathbb{R}^3)} \), which implies

\[
\frac{\beta - \lambda}{2} \| \psi^\varepsilon \|^4_{L^4(\mathbb{R}^3)} + \frac{3 \lambda \varepsilon^2}{2} \| \partial_n \nabla \phi^\varepsilon(t) \|^2_{L^2(\mathbb{R}^3)} \geq 0, \quad \text{and} \quad E(0) = \frac{\mu_0}{\varepsilon^2} + C_0,
\]

(5.13)

where \( C_0 \) depends on \( \| \phi_0 \|_{X_2} \). Writing \( \psi^\varepsilon(t) = \sum_{k=0}^{\infty} \phi_k(x, t) \varepsilon_k(z) \), and using the \( L^2 \) conservation

\[
\sum_{k=0}^{\infty} \| \phi_k(t) \|^2_{L^2(\mathbb{R}^2)} = 1,
\]

we can deduce from the energy conservation that

\[
\frac{\mu_0}{\varepsilon^2} + C_0 \geq (H_V^\varepsilon \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2} (H_x \psi^\varepsilon(t), \psi^\varepsilon(t)) = (H_V^\varepsilon \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2} \sum_{k=0}^{\infty} \mu_k \| \phi_k(t) \|^2_{L^2(\mathbb{R}^2)}
\]

\[
= (H_V^\varepsilon \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} (\mu_k - \mu_0) \| \phi_k(t) \|^2_{L^2(\mathbb{R}^2)} + \frac{\mu_0}{\varepsilon^2}.
\]

Hence,

\[
\| \partial_x \psi^\varepsilon \|^2_{L^2(\mathbb{R}^3)} + \| \partial_y \psi^\varepsilon \|^2_{L^2(\mathbb{R}^3)} \leq (H_V^\varepsilon \psi^\varepsilon(t), \psi^\varepsilon(t)) \leq C_0,
\]

(5.14)

\[
\| \partial_z \psi^\varepsilon \|^2_{L^2(\mathbb{R}^3)} \leq (H_x \psi^\varepsilon, \psi^\varepsilon) \leq \mu_0 + C_0 \varepsilon^2,
\]

(5.15)

\[
\| \psi^\varepsilon - \Pi \psi^\varepsilon \|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\mu_1 - \mu_0} \sum_{k=1}^{\infty} (\mu_k - \mu_0) \| \phi_k(t) \|^2_{L^2(\mathbb{R}^2)} \leq 2 C_0 \varepsilon^2,
\]

(5.16)

\[
\| \partial_z (\psi^\varepsilon - \Pi \psi^\varepsilon) \|^2_{L^2(\mathbb{R}^3)} \leq \sum_{k=1}^{\infty} \frac{\mu_k}{\mu_k - \mu_0} (\mu_k - \mu_0) \| \phi_k(t) \|^2_{L^2(\mathbb{R}^2)} \leq \frac{3}{2} C_0 \varepsilon^2.
\]

(5.17)

Estimate on \( \| \psi^\varepsilon - \Pi \psi^\varepsilon \|_{(p, 2)} \) follows from Sobolev embedding. \( \Box \)

We also need the following Strichartz estimates for the unitary group \( e^{-itH_Y^\varepsilon} \), which is valid when \( V_2 \) satisfies condition \((2.37)\) [13].

**Definition 5.3.** In two dimensions, let \( q', r' \) be the conjugate index of \( q \) and \( r \) \((1 \leq q, r \leq \infty)\) respectively, i.e. \( 1 = 1/q' + 1/q = 1/r' + 1/r \), we call the pair \((q, r)\) admissible and \((q', r')\) conjugate admissible if

\[
\frac{2}{q} = 2 \left( \frac{1}{2} - \frac{1}{r} \right), \quad 2 \leq r < \infty.
\]

(5.18)
The following estimates are established in [13] [12] [26].

**Lemma 5.4. (Strichartz’s estimates)** Let \((q, r)\) be an admissible pair and \((\gamma, \rho)\) be a conjugate admissible pair, \(I \subset \mathbb{R}\) be a bounded interval satisfying \(0 \in I\), then we have

(i) There exists a constant \(C\) depending on \(I\) and \(q\) such that

\[
\left\| e^{-iHt} \varphi \right\|_{L^q(I; L^r(\mathbb{R}^2))} \leq C(I, q) \| \varphi \|_{L^2(\mathbb{R}^2)}. \tag{5.19}
\]

(ii) If \(f \in L^q(I; L^r(\mathbb{R}^2))\), there exists a constant \(C\) depending on \(I\), \(q\) and \(\rho\), such that

\[
\left\| \int_I \mathcal{S}(t-s)H^\rho f(s) \, ds \right\|_{L^q(I; L^r(\mathbb{R}^2))} \leq C(I,\rho, q) \| f \|_{L^q(I; L^r(\mathbb{R}^2))}. \tag{5.20}
\]

Now, we are able to prove the theorem.

**Proof of Theorem 5.1.** In view of Lemma 5.2, we can derive

\[
\| \psi^\varepsilon - e^{-i \frac{\mu_0}{\varepsilon^2} \phi_0 \varepsilon} \|_{L^2(\mathbb{R}^3)} \leq \| \psi^\varepsilon - \Pi \psi^\varepsilon \|_{L^2(\mathbb{R}^3)} + \| \Pi \psi^\varepsilon - e^{-i \frac{\mu_0}{\varepsilon^2} \phi_0 \varepsilon} \|_{L^2(\mathbb{R}^3)} \leq C \varepsilon + \| \phi^\varepsilon(t) - \phi(t) \|_{L^2(\mathbb{R}^3)}. \tag{5.21}
\]

Hence, we need to estimate the difference between \(\phi^\varepsilon\) and \(\phi\). By (5.1) and (5.4), we know \(\phi^\varepsilon(x, t)\) in (5.9) satisfies the following equation

\[
i \partial_t \phi^\varepsilon = H^\rho \phi^\varepsilon + (\beta - \lambda + 3n_3^2 \lambda) e^{i \mu_0 t / \varepsilon^2} \int_\mathbb{R} \left| \psi^\varepsilon \right|^2 \psi^\varepsilon w_0(z) \, dz + \varepsilon \eta^\varepsilon, \]

\[
g^\varepsilon = e^{i \mu_0 t / \varepsilon^2} \int_\mathbb{R} P_\varepsilon(\phi^\varepsilon) \psi^\varepsilon w_0(z) \, dz,
\]

where the differential operator \(P_\varepsilon\) is defined as

\[
P_\varepsilon(\phi^\varepsilon) = -3 \lambda \left( (n_1^2 - n_3^2) \partial_{xx} + (n_2^2 - n_3^2) \partial_{yy} + 2n_1 n_2 \partial_{xy} + \frac{2}{\varepsilon} (n_1 n_3 \partial_{xz} + n_2 n_3 \partial_{yz}) \right) \phi^\varepsilon. \tag{5.22}
\]

Denote \(\chi^\varepsilon(x, t) = \phi^\varepsilon - \phi\), noticing that \(\| w_0 \|_{L^4}^4 = 1 / \sqrt{2\pi}\), \(\chi^\varepsilon\) satisfies the following equation

\[
i \partial_t \chi^\varepsilon = H^\rho \chi^\varepsilon + f_1^\varepsilon + f_2^\varepsilon + \varepsilon \eta^\varepsilon, \quad \chi^\varepsilon(t = 0) = 0,
\]

\[
f_1^\varepsilon = \frac{\lambda e^{i \mu_0 t / \varepsilon^2} \left( |\phi^\varepsilon|^2 \phi^\varepsilon - |\phi|^2 \phi \right)}{\sqrt{2\pi}},
\]

\[
f_2^\varepsilon = (\beta - \lambda + 3n_3^3 \lambda) e^{i \mu_0 t / \varepsilon^2} \int_\mathbb{R} \left( |\psi^\varepsilon|^2 \psi^\varepsilon - e^{-i \mu_0 t / \varepsilon^2} |\phi|^2 w_0^2 \phi^\varepsilon \right) w_0(z) \, dz.
\]

Applying Strichartz estimates on bounded interval \([0, T]\) and recalling that \((\infty, 2)\) is an admissible pair, we can obtain

\[
\left\| \chi^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C \left\{ \left\| f_1^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} + \left\| f_2^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} + \varepsilon \left\| g^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \right\},
\]

where \((\rho^*, \rho), (\gamma^*, \gamma)\) and \((q^*, q)\) are some conjugate admissible pairs. By a similar argument in [9], we have the estimates for \(f_1^\varepsilon\) and \(f_2^\varepsilon\) which comes from the cubic nonlinearity, i.e. for appropriate \(\rho \in (1, 2)\) and \(\gamma \in (1, 2)\),

\[
\left\| f_1^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C \left\| \chi^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))}, \quad \left\| f_2^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C \varepsilon. \tag{5.23}
\]

The basic tools involved are the Hölder’s inequality, Sobolev inequalities and the estimates in Lemma 5.1 and we omit the proof of this part here for brevity. So,

\[
\left\| \chi^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C \left( \left\| \chi^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} + \varepsilon \left\| g^\varepsilon \right\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} + \varepsilon \right). \tag{5.24}
\]
Next, we shall estimate $g^\varepsilon$. Let $\varphi^\varepsilon_1$ and $\varphi^\varepsilon_2$ be the solution of the rescaled Poisson equation \eqref{eq:5.3} with $|\varphi^\varepsilon|^2$ replaced by $|\Pi\varphi^\varepsilon|^2$ and $|\varphi^\varepsilon|^2 - |\Pi\varphi^\varepsilon|^2$, respectively, then $\varphi^\varepsilon = \varphi^\varepsilon_1 + \varphi^\varepsilon_2$ and we can rewrite

$$
g^\varepsilon = J^\varepsilon_1 + J^\varepsilon_2 + J^\varepsilon_3,
$$
where

$$
J^\varepsilon_1 = \int_\mathbb{R} P_\varepsilon(\varphi^\varepsilon_1)\varphi^\varepsilon_1 w_0 \, dz,
J^\varepsilon_2 = e^{\frac{\varepsilon^2}{4\delta}} \int_\mathbb{R} P_\varepsilon(\varphi^\varepsilon_2) (\varphi^\varepsilon_2 - \Pi\varphi^\varepsilon_2) w_0 \, dz,
J^\varepsilon_3 = e^{\frac{\varepsilon^2}{4\delta}} \int_\mathbb{R} P_\varepsilon(\varphi^\varepsilon_3) \Pi\varphi^\varepsilon_3 w_0 \, dz.
$$

For $J^\varepsilon_1$, this one reduces to the quasi-2D equation \eqref{eq:1.11}, where we have that

$$
J^\varepsilon_1 = -3\lambda (\partial_{n_1 n_1} - \nu_3^2 \Delta) \varphi^\varepsilon_{2D},
$$
with $U^2_{2D}$ given in \eqref{eq:1.12}. In view of the property of $U^2_{2D}$ in Lemma \ref{lem:2.2} and Remark \ref{rem:3.1}, recalling $\varphi^\varepsilon \in L^\infty([0, \infty); H^1(\mathbb{R}^3))$, using Hölder’s inequality and Sobolev inequality, we obtain

$$
\| J^\varepsilon_1 \|_{p} \leq \| P_\varepsilon(\varphi^\varepsilon_{2D}) \|_{p1} \| \varphi^\varepsilon \|_{p2} \leq C \| \nabla|\varphi^\varepsilon|^2 \|_{p1} \| \varphi^\varepsilon \|_{p2} \leq C,
$$
where $1 < p < p_1 < 2$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

For $J^\varepsilon_2$, applying Minkowski inequality, Hölder’s inequality and Sobolev inequality, as well as estimates for Poisson equation, noticing $\psi^\varepsilon \in L^\infty([0, \infty); H^1(\mathbb{R}^3))$ and Lemma \ref{lem:5.2}, we estimate

$$
\| J^\varepsilon_2 \|_p \leq \| P_\varepsilon(\varphi^\varepsilon_2)(\varphi^\varepsilon_2 - \Pi\varphi^\varepsilon_2) w_0 \|_{(1,p)} \leq C \| P_\varepsilon(\varphi^\varepsilon_2) \|_{L^{p_3}([0,T];L^1(\mathbb{R}^3))} \| \varphi^\varepsilon_2 - \Pi\varphi^\varepsilon_2 \|_{(\infty,2)} \leq C \| \varphi^\varepsilon \|_{L^\infty([0,T];L^1(\mathbb{R}^3))} \| \varphi^\varepsilon \|_{L^p(\mathbb{R}^3)} \leq C,
$$
where $p^* = 2p/(2-p) \leq 3$.

For $J^\varepsilon_3$, similar as $J^\varepsilon_1$, $J^\varepsilon_2$, we have

$$
\| J^\varepsilon_3 \|_p \leq \| P_\varepsilon(\varphi^\varepsilon_3) \Pi\varphi^\varepsilon_2 w_0 \|_{(1,p)} \leq C \| P_\varepsilon(\varphi^\varepsilon_3) \|_{L^{p_3}([0,T];L^1(\mathbb{R}^3))} \| \varphi^\varepsilon_2 \|_{L^{p_3}([0,T];L^2(\mathbb{R}^3))} \leq C \| \varphi^\varepsilon \|_{L^\infty([0,T];L^1(\mathbb{R}^3))} \| \varphi^\varepsilon \|_{L^{p_3}([0,T];L^2(\mathbb{R}^3))} \leq C,
$$
where $p_3 = 2p^3/(2-p_1) \leq 6$. Hence, by choosing $p = 6/5$, and $p_1 = 4/3$ would satisfy all the conditions for $J^\varepsilon_k$ ($k=1,2,3$), where we shall derive that uniformly in $t$,

$$
\| g^\varepsilon \|_{L^p(\mathbb{R}^3)} \leq \| J^\varepsilon_1 \|_{L^p(\mathbb{R}^3)} + \| J^\varepsilon_2 \|_{L^p(\mathbb{R}^3)} + \| J^\varepsilon_3 \|_{L^p(\mathbb{R}^3)} \leq C.
$$
Then choose $q = p$ in \eqref{eq:5.24}, we have

$$
\| \chi^\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq C \left[ \| \chi^\varepsilon \|_{L^p([0,T];L^2(\mathbb{R}^3))} + \varepsilon \right].
$$
Applying the results for all $t \in [0, T]$, we find

$$
\| \chi^\varepsilon(t) \|_{L^p}^2 \leq C \left[ \int_0^t \| \chi^\varepsilon(s) \|^2_{L^p} \, ds + \varepsilon \right], \quad t \in [0, T],
$$
and Gronwall’s inequality will give that $\| \chi^\varepsilon(t) \|_2 \leq C \varepsilon$ for all $t \in [0, T]$. Combining with \eqref{eq:5.24}, we can draw the desired conclusion.

**5.2. Reduction to 1D.** In this case, we again consider the weak interaction regime $\beta \to \varepsilon^2 \beta$, $\lambda \to \varepsilon \lambda$. In Case II \eqref{eq:1.9}, for the full 3D GPPS \eqref{eq:1.4}--\eqref{eq:1.5}, introducing the re-scaling $x \to \varepsilon x$, $y \to \varepsilon y$, $\psi \to \varepsilon^{-1} \psi^\varepsilon$ which preserves the normalization, then

$$
i\partial_x \psi^\varepsilon(x, z, t) = \left[ H_x^\varepsilon + \frac{1}{\varepsilon^2} H_x + (\beta - \lambda) |\psi^\varepsilon|^2 - 3\varepsilon \lambda \partial_{n_n} \varphi^\varepsilon \right] \psi^\varepsilon, \quad (x, z) \in \mathbb{R}^3,
$$
(5.31)
where $x = (x, y) \in \mathbb{R}^2$ and

$$\mathbf{H}^V = -\frac{1}{2} \partial_{zz} + V(z), \quad \mathbf{H}_x = -\frac{1}{2} (\partial_{xx} + \partial_{yy} + x^2 + y^2),$$

$$\mathbf{n}_\varepsilon = (n_1/\varepsilon, n_2/\varepsilon, n_3), \quad \partial_{\mathbf{n}_\varepsilon} = \mathbf{n}_\varepsilon \cdot \nabla, \quad \partial_{\mathbf{n}_\varepsilon} = \partial_{n_1} (\partial_{n_1}),$$

$$(-\frac{1}{\varepsilon^2} \partial_{xx} - \frac{1}{\varepsilon^2} \partial_{yy} - \partial_{zz})\varphi^\varepsilon = \frac{1}{\varepsilon^2} |\varphi^\varepsilon|^2, \quad \lim_{|x| \to \infty} \varphi^\varepsilon(x) = 0. \quad (5.34)$$

Note that the ground state mode of $\mathbf{H}_x$ would be given by $w_0(x)w_0(y)$ with eigenvalue 1, and the initial data is then assumed to be

$$\psi^\varepsilon(x, z, 0) = \phi_0(z)w_0(x)w_0(y), \quad \phi_0 \in X_1 \quad \text{and} \quad \|\phi_0\|_{L^2(\mathbb{R})} = 1. \quad (5.35)$$

In Case II (1.9) in the introduction, when $\varepsilon \to 0^+$, the quasi-1D equation (1.17) will lead to an $\varepsilon$-independent equation in the weak interaction regime,

$$i\partial_t \phi(z, t) = \mathbf{H}^V \phi + \frac{\beta + \frac{\lambda_0}{\varepsilon}(1-3\lambda_0)}{2\pi} |\phi|^2 \phi, \quad z \in \mathbb{R}, \quad t > 0, \quad (5.36)$$

with the initial condition $\phi(z, 0) = \phi_0(z)$. Following the steps in the last subsection, we can prove the following results.

**Theorem 5.5.** (Dimension reduction to 1D) Suppose the real-valued trap potential satisfies $V_1(z) \geq 0$ for $z \in \mathbb{R}$ and $V_I(z) \in C^\infty(\mathbb{R})$, $D^k V_I(z) \in L^\infty(\mathbb{R})$ for all $k \geq 2$. Assume $-\frac{\beta}{2} \leq \lambda \leq \beta$ and $\beta \geq 0$, and let $\psi^\varepsilon \in C([0, \infty); X_k)$ and $\phi \in C([0, \infty); X_1)$ be the unique solutions of the equations (5.31), (5.32) and (5.36), respectively, then for any $T > 0$, there exists $C_T > 0$ such that

$$\left\| \psi^\varepsilon(x, z, t) - e^{-i\varepsilon^2/2}(\phi(z, t))w_0(x)w_0(y) \right\|_{L^2(\mathbb{R}^3)} \leq C_T \varepsilon, \quad \forall t \in [0, T]. \quad (5.37)$$

6. Conclusion. We have analyzed the effective lower dimensional models for three dimensional Gross-Pitaevskii-Poisson system (GPPS) describing dipolar Bose-Einstein condensates (BEC) in anisotropic confinement. The quasi-2D approximate equations I (1.11) and II (1.15) are introduced in the case that the trap is strongly confined in the vertical $z$-direction, and the quasi-1D approximate equation (1.17) is presented in the case that the trap is strongly confined in the $x$, $y$-directions. Properties of ground states for all equations, such as existence and uniqueness as well as non-uniqueness results were studied. Well-posedness of the Cauchy problems for both equations and possible finite time blow-up in 2D case were discussed. Finally, we rigorously proved the linear convergence rate of the dimension reduction from 3D GPPS to its quasi-2D and quasi-1D approximations in the weak interaction regime, i.e. $\beta = \lambda = O(\varepsilon^{3-d})$, in lower $d \ (d = 1, 2)$ dimensions. In such situation, all the nonlocal terms in the effective equation (1.11), (1.15) and (1.17) vanish, resulting in a regular GPE in lower dimensions. We remark that the results in the paper hold true for a larger class of confinements rather that the harmonic ones. In fact, effective two-dimensional models have been derived and analyzed recently for a multilayer stack of dipolar BEC formed by a strong lattice potential [22].

Acknowledgment This work was supported in part by the Academic Research Fund of Ministry of Education of Singapore grant R-146-000-120-112 (W. B. and Y.C.). This work was partially done while W.B. and Y.C. were visiting the Institut de Mathématiques de Toulouse at Université Paul Sabatier in 2010. Y.C. would like to thank the support from the European Union programme “Differential Equations with Applications in Science and Engineering” MEST-CT-2005-021122 during his visit.

REFERENCES

[1] W. Bao and Y. Cai, Ground states of two-component Bose-Einstein condensates with an internal atomic Josephson junction, East Asia J. Appl. Math., 1 (2010), pp. 49-81.
[2] W. Bao, Y. Cai and H. Wang, Efficient numerical methods for computing ground states and dynamics of dipolar Bose-Einstein condensates, J. Comput. Phys., 229 (2010), pp. 7674–7892.

[3] W. Bao, Y. Ge, D. Jaksch, P. A. Markowich and R. M. Weishäupl, Convergence rate of dimension reduction in Bose-Einstein condensates, Comput. Phys. Commun., 177 (2007), pp. 832–850.

[4] W. Bao, D. Jaksch and P. A. Markowich, Numerical solution of the Gross-Pitaevskii equation for Bose-Einstein condensation, J. Comput. Phys., 187 (2003), pp. 318–342.

[5] W. Bao, P. A. Markowich, C. Schmeiser and R. M. Weishäupl, On the Gross-Pitaevskii equation with strongly anisotropic confinement: formal asymptotics and numerical experiments, Math. Models Meth. Sci., 15 (2005), pp. 767–782.

[6] N. Ben Abdallah, F. Castella, F. Delbecque-Fendt and F. Méhats, The strongly confined Schrödinger-Poisson system for the transport of electrons in a nanowire, SIAM J. Appl. Math., 69 (2009), pp. 1162–1173.

[7] N. Ben Abdallah, F. Méhats and O. Pinaud, Adiabatic approximation of the Schrödinger-Poisson system with a partial confinement, SIAM J. Math. Anal., 36 (2005), pp. 986–1013.

[8] N. Ben Abdallah, F. Méhats, C. Schmeiser and R. M. Weishäupl, The nonlinear Schrödinger equation with strong anisotropic harmonic potential, SIAM J. Math. Anal., 37 (2005), pp. 189–199.

[9] N. Ben Abdallah, F. Castella and F. Méhats, Time averaging for the strongly confined nonlinear Schrödinger equation, using almost periodicity, J. Diff. Equ., 245 (2008), pp. 154–200.

[10] J. Bourgain, Global solutions of nonlinear Schrödinger equations, Colloquium publications, Amer. Math. Soc., 1999.

[11] Y. Cai, M. Rosenkranz, Z. Lei and W. Bao, Mean-field regime of trapped dipolar Bose-Einstein condensates in one and two dimensions, Phys. Rev. A, 82 (2010), article 043623.

[12] R. Carles, P. A. Markowich and C. Sparber, On the Gross–Pitaevskii equation for trapped dipolar quantum gases, Nonlinearity, 21 (2008), pp. 2569–2590.

[13] T. Cazenave, Semilinear Schrödinger equations, Courant Lect. Notes Math., 10, Amer. Math. Soc., Providence, RI 2003.

[14] D. Funaro, Polynomial approximations of differential equations, Springer-Verlag, Berlin, 1992.

[15] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler and T. Pfau, Bose-Einstein condensation of Chromium, Phys. Rev. Lett., 94 (2005), article 160401.

[16] N. Hayashi and T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, Differ. Integral Equ., 2 (1994), pp. 453–461.

[17] C. E. Kenig, G. Ponce and L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. Henry Poincaré, 10 (1993), pp. 255–288.

[18] C. E. Kenig, G. Ponce and L. Vega, The Cauchy problem for quasi-linear Schrödinger equations, Invent. Math., 158 (2004), pp. 343–388.

[19] E. H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, Amer. Math. Soc., 2nd ed., 2001.

[20] E. H. Lieb, R. Seiringer and J. Yngvason, Bosons in a trap: a rigorous derivation of the Gross-Pitaevskii energy functional, Phys. Rev. A, 61 (2000), article 043602.

[21] M. Lu, N. Q. Burdick, S.-H. Youn, and B. L. Lev, A strongly dipolar Bose-Einstein condensate of Dysprosium, Phys. Rev. Lett., 107 (2011), article 190401.

[22] M. Rosenkranz, Y. Cai and W. Bao, Effective dipole–dipole interactions in multilayered dipolar Bose-Einstein condensates, preprint.

[23] L. Salasnich, Generalized nonpolynomial Schrödinger equations for matter waves under anisotropic transverse confinement, J. Phys. A: Math. Theor., 42 (2009), article 335205.

[24] L. Santos, G. Shlyapnikov, P. Zoller and M. Lewenstein, Bose-Einstein condensation in trapped dipolar gases, Phys. Rev. Lett., 85 (2000), 1791–1797.

[25] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, New Jersey, 1970.

[26] R. S. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations, Duke Math. J., 44 (1977), pp. 705–714.

[27] C. Sulem and P.-L. Sulem, The nonlinear Schrödinger equation, self-focusing and wave collapse, Springer-Verlag, New York, 1999.

[28] G. Szegő, Orthogonal polynomials 4th ed., Amer. Math. Soc. Colloq. Publ. 23, AMS, Providence, RI, 1975.

[29] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commum. Math. Phy., 87 (1983), pp. 567–576.

[30] S. Yi and H. Pu, Vortex Structures in dipolar condensates, Phys. Rev. A, 73 (2006), article 061602.

[31] S. Yi and L. You, Trapped atomic condensates with anisotropic interactions, Phys. Rev. A, 61 (2000), article 041604.

[32] S. Yi and L. You, Trapped condensates of atoms with dipole interactions, Phys. Rev. A, 63 (2001), article 053607.