ON THE NLS APPROXIMATION FOR THE NONLINEAR KLEIN-GORDON EQUATION

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ABSTRACT. In this paper, developing a new approach based on Fourier analysis methods for dispersive PDEs, we establish a low regularity NLS approximation for the one-dimensional cubic Klein-Gordon equation. Our main result includes energy class solutions which are formally asymptotically in $L^2(\mathbb{R})$. A precise rate of convergence is also obtained assuming more regularity.

1. INTRODUCTION

The nonlinear Schrödinger equation (NLS) is a universal model describing the envelope dynamics of slowly modulating small amplitude wave packets. By multi-scaling analysis, the NLS is derived from various Hamiltonian systems, including the Korteweg-de Vries equation [19, 21], the Boussinesq equation [7], water wave problems [32, 10, 14, 24, 26, 31, 17, 30, 8, 12, 6], nonlinear optics [5] and discrete models [22, 20, 14, 11]. For extensive references on the universality of the NLS, we refer to the books by Schneider and Uecker [23] and by Sulem and Sulem [27].

The purpose of this article is to develop a new approach to justify the NLS approximation based on Fourier analysis methods for dispersive PDEs. In this way, we aim to include a larger class of solutions compared to the previously known dynamical system approach as well as exploring a possibility of extending the interval of approximation using the conservation laws.

To make our discussion concrete, we restrict ourselves to the one-dimensional nonlinear Klein-Gordon equation (NLKG), that is, the simplest expository textbook example [23, Chapter 11], given by

\[ \partial_t^2 u - \partial_x^2 u + u + u^3 = 0, \]  

where $u = u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For a small parameter $\epsilon > 0$, we seek a solution of the form

\[ u_\epsilon(t, x) = \epsilon A_\epsilon(\epsilon^2 t, \epsilon(x - c_g t))e^{i(kx - \omega t)} + c.c., \]  

where $A_\epsilon = A_\epsilon(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the amplitude function and $c.c.$ stands for the complex conjugate of the former term. Here, the dispersion relation and the group velocity are chosen respectively as

\[ \omega = \langle k \rangle \quad \text{and} \quad c_g = \omega'(k) = \frac{k}{\langle k \rangle} \]
where \( \langle \cdot \rangle = \sqrt{1 + \cdot} \) denotes the Japanese bracket, to cancel out \( O(\epsilon) \)- and \( O(\epsilon^2) \)-terms. Then, from the next \( O(\epsilon^3) \)-order terms, the cubic NLS

\[
2i\omega \hat{\psi}^{(\text{NLS})} + (1 - c_3^2)\hat{\psi}^{(\text{NLS})} + 3|\psi^{(\text{NLS})}|^2\psi^{(\text{NLS})} = 0, \tag{1.3}
\]

where \( \psi^{(\text{NLS})} = \psi^{(\text{NLS})}(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \), is derived so that the sum of two wave packets

\[
\epsilon \psi^{(\text{NLS})}(\epsilon^2 t, \epsilon(x - c_3 t)) e^{i(kx - \omega t)} + \text{c.c.}
\]

approximates the NLKG flow. For a more detailed formal derivation and a rigorous proof for sufficiently regular flows, we refer to [23, Chapter 11].

In this paper, we include a larger class of solutions, namely the energy class, for the NLS approximation. Note that solutions to the NLKG (1.1) preserve the energy

\[
\mathcal{E}(u) = \frac{1}{2} \left( \| \partial_x u \|_{L^2(\mathbb{R})}^2 + \| \partial_t u \|_{L^2(\mathbb{R})}^2 + \| u \|_{L^4(\mathbb{R})}^4 \right) + \frac{1}{4} \| u \|_{L^4(\mathbb{R})}^4.
\]

By the \( \epsilon \)-scaling in the ansatz (1.2), it is natural to introduce the rescaled Sobolev space \( H^s_{\epsilon}(\mathbb{R}) \), for \( s \in \mathbb{R} \) and \( \epsilon \in (0, 1] \), equipped with the norm

\[
\| u \|_{H^s_{\epsilon}(\mathbb{R})} = \| \epsilon D \| u \|_{L^2(\mathbb{R})},
\]

where \( D = -i\partial_x \), and define \( m(D) \) as the Fourier multiplier operator with symbol \( m(\xi) \), i.e., \( m(D) \hat{f}(\xi) = m(\xi) \hat{f}(\xi) \). In particular, we call \( H^s_{\epsilon}(\mathbb{R}) \) the energy class.

Throughout the article, we assume that initial data satisfies

\[
\sup_{\epsilon \in (0, 1]} \| \psi_{\epsilon, 0} \|_{H^1_{\epsilon}(\mathbb{R})} \leq R < \infty \tag{H1}
\]

and

\[
\lim_{\epsilon \to 0} \| 1_{|D| > \delta \epsilon^{-1/3}} \psi_{\epsilon, 0} \|_{H^1_{\epsilon}(\mathbb{R})} = 0 \quad \text{for any } \delta > 0, \tag{H2}
\]

where \( 1_{|D| > N} \) is a high frequency cut-off with symbol \( 1_{|\xi| > N} \).

Remark 1.1. (i) Some regularity must be imposed on initial data, because the NLKG (1.1) is well-posed in \( H^s(\mathbb{R}) \) only if \( s \geq \frac{1}{2} \) (see [15, Appendix D]). In (H1), the regularity \( s \) is chosen to be one, but we may say it is asymptotically zero due to the formal norm convergence \( \| \cdot \|_{H^1_{\epsilon}(\mathbb{R})} \to \| \cdot \|_{L^2(\mathbb{R})} \) as \( \epsilon \to 0 \) even though \( H^1_{\epsilon}(\mathbb{R}) = H^1(\mathbb{R}) \) as sets.

(ii) The energy class solutions are considered with a hope to find potential applications of the energy conservation law, for instance, extending the interval of approximation. Certainly, there is a room to reduce the regularity further in the assumption (H1), but such low regularities will not be pursued here to avoid additional technical complications.

(iii) (H2) ensures that initial data \( \epsilon \psi_{\epsilon, 0}(\epsilon x) e^{i k x} + \text{c.c.} \) is tightly localized at two frequencies \( \pm k \), with \( |\xi \pm k| \ll \epsilon^{-\frac{2}{3}} \), and thus the two-wave packet structure (1.2) remains for long time.

Our first main result asserts that the NLS approximation is valid under the above two assumptions.
Theorem 1.2 (NLS approximation to the NLKG). We assume (H1)-(H2) for \( \{\psi_{\epsilon,0}\}_{\epsilon \in (0,1]} \). For \( \epsilon \in (0,1] \), let \( u_{\epsilon}(t) \in C_t(\mathbb{R}; H^1(\mathbb{R})) \) be the solution to the NLKG (1.1) with initial data

\[
(u_{\epsilon}(0), \partial_t u_{\epsilon}(0)) = (\epsilon \psi_{\epsilon,0}(\epsilon)e^{ikx} + \text{c.c.}, \langle D \rangle (-ie\psi_{\epsilon,0}(\epsilon)e^{ikx} + \text{c.c.})) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}),
\]

and let \( \psi_{\epsilon}(\text{NLS})(t) \in C_t(\mathbb{R}; H^1(\mathbb{R})) \) be the solution to the NLS (1.3) with initial data \( \psi_{\epsilon,0} \in H^1(\mathbb{R}) \). Then, there exists \( T > 0 \), independent of \( \epsilon \in (0,1] \), such that

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \left\| u_{\epsilon}(t, x) - \left( \epsilon \psi_{\epsilon}(\text{NLS})(\epsilon^2 t, \epsilon(x - c_g t))e^{i(kx-\omega t)} + \text{c.c.} \right) \right\|_{C_t([-\frac{T}{16}, \frac{T}{16}]; H^1(\mathbb{R}))} = 0.
\]

Remark 1.3. (i) The convergence (1.5) is well-known for higher regularity solutions. By a dynamical system approach, it is shown provided that the NLS flow \( \psi_{\epsilon}(\text{NLS})(t) \) is in \( C_t([-T, T]; H^s(\mathbb{R})) \) for \( s \geq 5 \) [23, Theorem 11.2.6]. In [18], the regularity is reduced to \( s > 1 \). In our main theorem, the required regularity is reduced to one, but which is formally asymptotically zero (see Remark 1.1 (i)).

(ii) Higher-order corrections will not be discussed in this paper, because our main focus is on including rough solutions while higher-order corrections are typically valid for more regular solutions (see [23, Theorem 11.2.6]).

Remark 1.4 (Optimality). The assumptions (H1)-(H2) are optimal in the sense that the linear part of the reformulated equation can be approximated by the linear Schrödinger flow only under (H1)-(H2) (see Lemma 3.1 and Remark 3.2).

Remark 1.5 (Rate of convergence in (1.5)). (i) By the \( \epsilon \)-scaling, both the NLKG flow \( u_{\epsilon}(t) \) and the NLS ansatz \( \epsilon \psi_{\epsilon}(\text{NLS})(\epsilon^2 t, \epsilon(x - c_g t))e^{i(kx-\omega t)} + \text{c.c.} \) are of \( O(\sqrt{\epsilon}) \) in \( H^1(\mathbb{R}) \). The main result (1.5) justifies the NLS approximation with \( o(\sqrt{\epsilon}) \)-difference.

(ii) It does not seem possible to improve the \( o(\sqrt{\epsilon}) \)-difference in (1.5) for general solutions in Theorem 1.2, because our proof relies on a density argument (see Remark 3.2).

The next theorem provides a precise rate of convergence assuming more regularity.

Theorem 1.6 (NLS approximation to the NLKG; rate of convergence). In Theorem 1.2, we further assume that for some \( s > 0 \),

\[
\sup_{\epsilon \in (0,1]} \| \psi_{\epsilon,0} \|_{H^s(\mathbb{R})} < \infty.
\]

(H2')

Then, for any small \( \eta > 0 \), we have

\[
\left\| u_{\epsilon}(t, x) - \left( \epsilon \psi_{\epsilon}(\text{NLS})(\epsilon^2 t, \epsilon(x - c_g t))e^{i(kx-\omega t)} + \text{c.c.} \right) \right\|_{C_t([-\frac{T}{16}, \frac{T}{16}]; L^2(\mathbb{R}))} \lesssim \epsilon^{\min\left(\frac{s}{2}, \frac{1}{2}, \frac{3}{2} - \eta\right)}.
\]

Remark 1.7. (i) (H2') is a stronger assumption, because (H2') implies (H2).

(ii) From the linear flow approximation, one can see that the \( O(\epsilon^{\frac{s}{2} + \frac{1}{2}}) \)-rate of convergence is optimal (see Remark 3.2).
The main contribution of this paper is to introduce a new approach for the NLS approximation which we think is robust. Our approach is based on the reformulation of the problem as a system of integral equations (2.4), or the Duhamel representation (see Section 2). It turns out that this integral representation has several crucial advantages in reducing regularity. First of all, we note that in a dynamical system approach [23, Section 11.2], it involves estimating the residual
\[
\text{Res}(u) = -\partial_t^2 u + \partial_x^2 u - u^3, \quad u(t, x) = \epsilon \psi^{(NLS)}(\epsilon^2 t, \epsilon(x - c_g t)) e^{i(kx - \omega t)} + c.c.,
\]
where \(\psi^{(\text{NLS})}\) is a solution to the NLS (1.3). Thus, proving smallness of the residual in \(H^1(\mathbb{R})\) requires high Sobolev norm bounds, \(\|\psi^{(\text{NLS})}\|_{H^s(\mathbb{R})} < \infty\). However, such derivative terms do not appear in the integral equation (2.4).

Secondly, the reformulation provides more detailed information about the limit procedure. Indeed, derivation of the integral equations (2.4) leads us to notice that the amplitude \(A_\epsilon(t)\) in (1.2) includes very high frequency waves, and thus the amplitude must be separated into the core profile \(\psi_\epsilon\) and the high frequency remainder \(r_\epsilon\) (see Remark 2.1). Consequently, they must be measured separately in Sobolev norms with different scales.

Lastly, we mention that from the integral representation (2.4), dispersive effects can be captured properly. By the reformulation, we see that the linear evolution in (2.4) is given by the propagator \(S_\epsilon(t) = e^{-\frac{\epsilon t}{2}((1+\epsilon D) - \sqrt{\epsilon x^2 + \epsilon^{-1} D})}\). Then, employing well-known Fourier analysis methods as Strichartz estimates and multilinear estimates involving Fourier restriction norms, one can deduce uniform bounds for nonlinear solutions which are useful to prove the NLS approximation.

**Remark 1.8.** The NLS is derived from the NLKG in a different context, namely as a non-relativistic limit [16], but it does not have the technical issue caused by the high frequency remainder.

**Remark 1.9.** An interesting question is whether the \(O(\frac{1}{\epsilon})\)-interval of approximation can be extended. In [9], a positive answer is given for more complicated quadratic NLKG in a periodic setting but with more regular solutions. As an alternative approach, one may attempt to use the energy conservation law to extend the interval of approximation. Nevertheless, we are currently unable to do that. Indeed, our proof heavily relies on the two-wave packet structure \(\epsilon \psi_{\epsilon, 0}(\epsilon) e^{ikx} + c.c.\) at initial time, but the structure will be broken up as time goes, because the nonlinearity immediately generates different frequency modes. Unfortunately, the energy conservation law does not seem to control this procedure straightforwardly. Therefore, it will be left to our future work.

1.1. **Organization of the paper.** In Section 2, we derive the system of integral equations (2.4). In Section 3 and 4, we investigate the properties of the linear part of (2.4) and use them to prove basic well-posedness of the system. Then, in Section 5 and 6, we prove more refined estimates for the core profile (smallness of high frequencies) and the remainder (smallness). Finally, in Section 7, we complete the proof of the main results.
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2. **Reformulation of the problem**

To begin with, we present a reformulation of the cubic nonlinear Klein-Gordon equation (1.1) in a way that Fourier analysis methods can be properly employed.

2.1. **Derivation of the equation for the amplitude function.** For numerical simplicity, we fix $k = 1$ by scaling, and look for a solution of the form

$$ u(t, x) = \epsilon A(t, \epsilon(x - \frac{t}{\epsilon^2})) e^{i(x - \sqrt{2}t)} + c.c. $$

In order to find the equation for the amplitude $A(t, x)$, rescaling by $v_t(t, x) = \frac{1}{\epsilon} u_t(\frac{t}{\epsilon^2}, \frac{x}{\epsilon})$, we deduce

$$ \partial_t^2 v_t + \frac{1}{\epsilon^4} (1 - \epsilon^2 \partial_x^2) v_t - \frac{1}{\epsilon^2} v_t^3 = 0 \quad (2.1) $$

with initial data $(v_t(0, \partial_t v_t(0))) = (2 \text{Re}(e^{i\xi} \psi_{t,0}), \frac{2}{\epsilon} \partial_x \text{Im}(e^{i\xi} \psi_{t,0}))$. We note that the equation (2.1) in a strong form is given by

$$ v_t(t) = \left\{ e^{-\frac{it}{\epsilon^2} (\xi D)} e^{i\xi} \psi_{t,0} - \frac{i}{2} \int_0^t e^{-\frac{i(t-t_1)}{\epsilon^2} (\xi D)} \frac{1}{\langle \xi D \rangle} v_t(t_1)^3 dt_1 \right\} + c.c. $$

with

$$ v_t(t, x) = e^{-\frac{it}{\epsilon^2} (x - \frac{t}{\epsilon^2})} A(t, x) + c.c. $$

Next, complexifying the equation, we impose that the amplitude function $A(t)$ obeys the equation

$$ A(t) = e^{-\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} e^{-\frac{i}{\epsilon^2} ((\xi D) - \frac{\mu}{\epsilon^2})} e^{i\xi} \psi_{t,0} $$

$$ - \frac{i}{2} \int_0^t e^{-\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} e^{-\frac{i(t-t_1)}{\epsilon^2} ((\xi D) - \frac{\mu}{\epsilon^2})} \frac{1}{\langle \xi D \rangle} \left( e^{\frac{i}{\epsilon^2} (x - \frac{t_1}{\epsilon^2})} A(t_1) + c.c. \right)^3 dt_1. $$

We observe that $e^{-\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} e^{-\frac{i}{\epsilon^2} ((\xi D) - \frac{\mu}{\epsilon^2})} e^{i\xi}$ is the Fourier multiplier of symbol $e^{-itp_x(\xi)}$, where

$$ p_x(\xi):= \frac{1}{\epsilon^2} \left( \langle 1 + \epsilon \xi \rangle - \sqrt{2} - \frac{\epsilon}{\sqrt{2}} \xi \right), \quad (2.2) $$

and similarly, $e^{-\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} e^{-\frac{i(t-t_1)}{\epsilon^2} ((\xi D) - \frac{\mu}{\epsilon^2})} \frac{1}{\langle \xi D \rangle} e^{\frac{i}{\epsilon^2} (x - \frac{t_1}{\epsilon^2})}$ has symbol $\frac{e^{-i(t-t_1)p_x(\xi)}}{\langle 1 + \epsilon \xi \rangle}$. Thus, introducing the linear propagator

$$ S_t(t) = e^{-itp_x(D)} $$

the equation can be written as

$$ A(t) = S_t(t) \psi_{t,0} - \frac{i}{2} \int_0^t S_t(t-t_1) \frac{1}{\langle 1 + \epsilon D \rangle} e^{-\frac{i}{\epsilon^2} (x - \frac{t_1}{\epsilon^2})} \left( A(t_1) e^{\frac{i}{\epsilon^2} (x - \frac{t_1}{\epsilon^2})} + c.c. \right)^3 dt_1. $$

Finally, expanding the nonlinear term as

$$ e^{-\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} \left( A(t) e^{\frac{i}{\epsilon^2} (x - \frac{t}{\epsilon^2})} + c.c. \right)^3 = 3(|A|^2 A + \mathcal{R}_t(A)), $$

where

$$ \mathcal{R}_t(A) := \frac{1}{2} \int_0^t e^{-\frac{i}{\epsilon^2} (x - \frac{t-t_1}{\epsilon^2})} \left( A(t_1) e^{\frac{i}{\epsilon^2} (x - \frac{t_1}{\epsilon^2})} + c.c. \right)^3 dt_1. $$
we obtain the amplitude equation in a compact form

\[ R_\epsilon(A_\epsilon) := \frac{1}{3} e^{\frac{2i}{\epsilon}(x-\frac{y}{\sqrt{2}})} A_\epsilon^3 + e^{\frac{-2i}{\epsilon}(x-\frac{y}{\sqrt{2}})} |A_\epsilon|^2 \tilde{A}_\epsilon + \frac{1}{3} e^{\frac{-4i}{\epsilon}(x-\frac{y}{\sqrt{2}})} \tilde{A}_\epsilon^3, \]

we obtain the amplitude equation in a compact form

\[ A_\epsilon(t) = S_\epsilon(t)\psi_{\epsilon,0} - \frac{3i}{2} \int_0^t S_\epsilon(t-t_1) \frac{1}{\langle 1 + \epsilon D \rangle} \left( |A_\epsilon|^2 A_\epsilon + R_\epsilon(A_\epsilon) \right) (t_1) dt_1. \quad (2.3) \]

2.2. Decomposition of the amplitude equation. It turns out, however, that the equation (2.3) by itself is rather difficult to analyze, because on the frequency side, the solution consists of a large localized part and small very high frequency pieces.

Remark 2.1. The solution \( A_\epsilon \) to (2.3) is never localized in frequencies of order one. Indeed, even if \( \tilde{A}_\epsilon(t, \xi) \) is mostly localized near the origin at time \( t = 0 \), the nonlinear term

\[ \frac{i}{2} \int_0^t S_\epsilon(t-t_1) \frac{1}{\langle 1 + \epsilon D \rangle} \left( e^{\frac{2i}{\epsilon}(x-\frac{y}{\sqrt{2}})} \tilde{A}_\epsilon^3 + 3e^{\frac{-2i}{\epsilon}(x-\frac{y}{\sqrt{2}})} |A_\epsilon|^2 \tilde{A}_\epsilon + e^{\frac{-4i}{\epsilon}(x-\frac{y}{\sqrt{2}})} \tilde{A}_\epsilon^3 \right) (t_1) dt_1 \]

immediately generates very high frequencies \( \xi = \pm \frac{2}{\epsilon}, -\frac{4}{\epsilon} \). One may expect that these high frequency pieces vanish as \( \epsilon \to 0 \) due to fast dispersion. However, they might be measured largely in a standard Sobolev norm \( \| \cdot \|_{H^s(\mathbb{R})} \). On the other hand, if one employs the rescaled Sobolev norm \( \| \cdot \|_{H^s_\epsilon(\mathbb{R})} \) for (2.3), one becomes unable to capture how much portion of \( A_\epsilon \) is localized in frequencies.

A key idea to resolve the problem is to separate the amplitude function into the core profile and the high frequency remainder,

\[ A_\epsilon(t, x) = \psi_\epsilon(t, x) + r_\epsilon(t, x), \]

and impose that \((\psi_\epsilon(t), r_\epsilon(t))\) solves the system of equations

\[
\begin{aligned}
\psi_\epsilon(t) &= S_\epsilon(t)\psi_{\epsilon,0} - \frac{3i}{2} \int_0^t S_\epsilon(t-t_1) \frac{1}{\langle 1 + \epsilon D \rangle} \left( |\psi_\epsilon|^2 \psi_\epsilon \right) (t_1) dt_1, \\
r_\epsilon(t) &= -\frac{3i}{2} \int_0^t S_\epsilon(t-t_1) \frac{1}{\langle 1 + \epsilon D \rangle} \left( |A_\epsilon|^2(A_\epsilon) - |\psi_\epsilon|^2 \psi_\epsilon + R_\epsilon(A_\epsilon) \right) (t_1) dt_1.
\end{aligned}
\]

(2.4)

In a sequel, they will be estimated separately.

3. Linear rescaled Klein-Gordon flow

In this section, we investigate properties of the linear rescaled Klein-Gordon flow

\[ S_\epsilon(t) = e^{-itp_\epsilon(D)} = e^{-\frac{4i}{\epsilon}(1+\epsilon D)-\sqrt{2}-\frac{y}{\sqrt{2}} D} \]

focusing on its connection to the linear Schrödinger flow \( e^{it\sqrt{2} D^2} \).
3.1. Convergence of the linear flow. By the formal Taylor series expansion
\[
\langle 1 + \epsilon D \rangle = \sqrt{2} + \frac{\epsilon}{\sqrt{2}} D + \frac{\epsilon^2}{4\sqrt{2}} D^2 - \frac{\epsilon^3}{8\sqrt{2}} D^3 + \cdots,
\]
it is expected that \( S_\epsilon(t) \) converges to \( e^{it\sqrt{2} \partial_x^2} \) as \( \epsilon \to 0 \). It can be stated rigorously as follows.

**Lemma 3.1** (Convergence of the linear flow \( S_\epsilon(t) \)). Let \( \epsilon \in (0, 1] \). (i) For any sufficiently small \( \delta > 0 \), we have
\[
\| S_\epsilon(t) u_0 - e^{it\sqrt{2} \partial_x^2} u_0 \|_{L^2(\mathbb{R})} \lesssim \delta^3 \| u_0 \|_{L^2(\mathbb{R})} + \| P_{>\delta^{-1/3}} u_0 \|_{L^2(\mathbb{R})}.
\]
(ii) If \( u_0 \in H^s(\mathbb{R}) \), then
\[
\| S_\epsilon(t) u_0 - e^{it\sqrt{2} \partial_x^2} u_0 \|_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{s}{2}} \| u_0 \|_{H^s(\mathbb{R})}.
\]

Proof. By the Plancherel theorem, we have
\[
\| S_\epsilon(t) u_0 - e^{it\sqrt{2} \partial_x^2} u_0 \|_{L^2(\mathbb{R})} \sim \| (e^{-it(\hat{p}_\epsilon(\xi) - \frac{1}{4\sqrt{2}} \xi^2)} - 1) \hat{u}_0 \|_{L^2(\mathbb{R})}.
\]
Note that
\[
| e^{-it(\hat{p}_\epsilon(\xi) - \frac{1}{4\sqrt{2}} \xi^2)} - 1 | \lesssim \min \{ \epsilon |\xi|^3, 1 \}, \tag{3.1}
\]
because \( |\hat{p}_\epsilon(\xi) - \frac{1}{4\sqrt{2}} \xi^2| \lesssim \epsilon |\xi|^3 \) by Taylor’s theorem. Thus, the lemma follows. \( \square \)

**Remark 3.2.** From the proof of Lemma 3.1 (see (3.1) in particular), one can see that \( (H2') \) is the minimal requirement for the convergence \( \| S_\epsilon(t) u_0 - e^{it\sqrt{2} \partial_x^2} u_0 \|_{H^s(\mathbb{R})} \to 0 \) as well as \( (H2') \) is optimal for the \( O(\epsilon^{s/2}) \)-rate of convergence.

3.2. Uniform linear Strichartz estimates. For \( \epsilon \in (0, 1] \), we define the rescaled Sobolev norms by
\[
\| u \|_{W^{s,r}_\epsilon(\mathbb{R})} := \| (\epsilon D)^s u \|_{L^r(\mathbb{R})} \quad \text{and} \quad \| u \|_{H^{s,2}_\epsilon(\mathbb{R})} := \| u \|_{W^{s,2}_\epsilon(\mathbb{R})}.
\]
Using a smooth cut-off \( \chi \in C_0^\infty(\mathbb{R}) \) such that \( \text{supp} \chi \subset \{ \xi : \frac{1}{2} < |\xi| < 4 \} \), \( \chi \equiv 1 \) on \( 1 \leq |\xi| \leq 2 \) and \( \sum_{N \in \mathbb{Z}} \chi \left( \frac{\xi}{N} \right) = 1 \), we define the Littlewood-Paley projection \( P_N \) by
\[
\overline{P_N \hat{f}(\xi)} = \chi \left( \frac{\xi}{N} \right) \hat{f}(\xi).
\]

Our analysis heavily relies on dispersive properties of the linear Klein-Gordon flow \( S_\epsilon(t) \) in the form of Strichartz estimates. Indeed, such estimates are well-known for the standard Klein-Gordon equation, and in essence, the same holds for the rescaled one. However, in consideration of the \( \epsilon \to 0 \) limit, possible \( \epsilon \)-dependences in the inequalities must be clarified.

**Lemma 3.3** (Uniform linear Strichartz estimates). Let \( \epsilon \in (0, 1] \). Suppose that \( q, r, \tilde{q}, \tilde{r} \geq 2 \) and
\[
\frac{2}{q} + \frac{1}{r} = \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2}.
\]
Then, we have
\[
\| S_\epsilon(t) u_0 \|_{L^2_t L^q_x(\mathbb{R})} \lesssim \| u_0 \|_{H^{s,2}_\epsilon(\mathbb{R})}.
\]
and 
\[ \left\| \int_0^t S_\epsilon(t-t_1)F(t_1)dt_1 \right\|_{L^p_x(\mathbb{R};L^r_x(\mathbb{R}))} \leq \|F\|_{L^p_x(L_t^q(\mathbb{R};W_t^{3/\epsilon,1+1/r}(\mathbb{R}))}. \]

**Proof.** Since \( p_\epsilon'(\epsilon\xi) = \frac{1}{(1+\epsilon\xi)^s} \), the van der Corput lemma deduces that 
\[ \|S_\epsilon(t)P_Nu_0\|_{L^\infty_x} = \left\| \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} e^{-ip_\epsilon(\xi)}e^{i(x-y)\xi} \chi(\xi)dy \right\} u_0(y)dy \right\|_{L^\infty_x} \leq \frac{1}{\langle \epsilon N \rangle^{3/2|t|^{1/2}}} \|u_0\|_{L^1}. \]

Thus, the standard interpolation argument (see Keel-Tao [13] for instance), together with the Littlewood-Paley inequality, yields the desired inequalities. \( \square \)

**Remark 3.4.** Strichartz estimates for the rescaled flow \( S_\epsilon(t) \) require some regularities like the standard linear Klein-Gordon flow \( e^{it(D^2)} \). On the other hand, in the \( \epsilon \to 0 \) limit, they are closely related to the Strichartz estimates for the linear Schrödinger flow \( e^{it\Delta} \), i.e.,
\[ \|e^{i\frac{t}{\sqrt{\epsilon}}}\xi^2u_0\|_{L^2_t(\mathbb{R};L^r_x(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})}, \]
\[ \left\| \int_0^t e^{i\frac{t}{\sqrt{\epsilon}}}\xi^2F(t_1)dt_1 \right\|_{L^p_x(\mathbb{R};L^r_x(\mathbb{R}))} \lesssim \|F\|_{L^p_x(L_t^q(\mathbb{R};L^r_x(\mathbb{R}))}} \]
as the rescaled Sobolev norm \( \|\cdot\|_{W^s_{r,r}(\mathbb{R})} \) formally converges to the \( L^r(\mathbb{R}) \)-norm.

Next, we introduce two types of Fourier restriction norm spaces, namely the **Bourgain** spaces [1], associated with the linear propagator \( S_\epsilon(t) \). We define \( X^{s,b} \) (resp., \( X_{\epsilon}^{s,b} \)) as the completion of Schwartz functions with respect to
\[ \|u\|_{X^{s,b}} := \|\langle \xi \rangle^{s+\rho}(\xi)\|^b u(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R} \times \mathbb{R})}, \]

(resp., \( \|u\|_{X_{\epsilon}^{s,b}} := \|\langle \epsilon \xi \rangle^{s+\rho}(\epsilon \xi)\|^b u(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R} \times \mathbb{R})} \)

where \( u(\tau,\xi) \) denotes the space-time Fourier transform of \( u(t,x) \). Then, given \( T > 0 \), we define \( X^{s,b}([-T,T]) \) (resp., \( X_{\epsilon}^{s,b}([-T,T]) \)) with the norm
\[ \|u\|_{X^{s,b}([-T,T])} := \inf \left\{ \|v\|_{X^{s,b}} : v = u \text{ on } [-T,T] \right\}, \]

(resp., \( \|u\|_{X_{\epsilon}^{s,b}([-T,T])} := \inf \left\{ \|v\|_{X_{\epsilon}^{s,b}} : v = u \text{ on } [-T,T] \right\} \)).

Later, we will frequently use the following basic mapping properties (refer to [1] [29]) and the trilinear estimate involving the Fourier restriction spaces.

**Lemma 3.5.** (i) **(Inhomogeneous estimate)** For any \( b > \frac{1}{2} \),
\[ \left\| \int_0^t S_\epsilon(t-t_1)F(t_1)dt_1 \right\|_{X^{0,-(1-b)}([-T,T])} \lesssim \|F\|_{X^{0,-(1-b)}([-T,T])}. \]

(ii) **(Transference principle)** If \( b > \frac{1}{2} \) and \( \frac{2}{q} + \frac{1}{r} = \frac{1}{4} \) and \( q, r \geq 2 \), then
\[ \|u\|_{L^q_t([-T,T];L^r_x)} \lesssim \|u\|_{X_{\epsilon}^{1/2,b}([-T,T])}. \]
Lemma 3.6 (Trilinear estimate). For \( b \in (\frac{1}{2}, 1] \), we have
\[
\|v_1 v_2 v_3\|_{X^{\alpha, -(1-b)}([-T, T])} \lesssim \|v_1 v_2 v_3\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}))} \lesssim T^{\frac{1-b}{2p}} \|v_1\|_{X^{1, b}([-T, T])} \|v_2\|_{X^{1, b}([-T, T])} \|v_3\|_{X^{0, b}([-T, T])}.
\]

Proof. Interpolating the two basic inequalities \( \|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{0, b}} \) and \( \|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{0, 0}} \), we obtain \( \|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{0, 1-b}} \) whose dual inequality is given by \( \|u\|_{X^{0, -(1-b)}} \lesssim \|u\|_{L^p_t L^q_x} \). Thus, by Hölder’s inequality and Strichartz estimates, we prove that
\[
\|v_1 v_2 v_3\|_{X^{\alpha, -(1-b)}} \lesssim \|v_1 v_2 v_3\|_{L^p_t L^q_x} \lesssim T^{\frac{1-b}{2p}} \|v_1\|_{L^p_t L^q_x} \|v_2\|_{L^p_t L^q_x} \|v_3\|_{C^1([-T, T])} \lesssim T^{\frac{1-b}{2p}} \|v_1\|_{X^{1, b}} \|v_2\|_{X^{1, b}} \|v_3\|_{X^{0, b}}.
\]

4. Well-posedness of the nonlinear Klein-Gordon equation

We present basic well-posedness results for the rescaled NLKG (2.1) and the system (2.4). Consequently, we confirm that \( \psi_\epsilon(t, x) + r_\epsilon(t, x) + \text{c.c., from (2.4)} \), is a unique solution to the NLKG (2.1) with initial data \( (2\text{Re}(\bar{e}^{ix\epsilon} \psi_{\epsilon,0}), \frac{1}{\epsilon} \langle \bar{e}^{ix\epsilon} \psi_{\epsilon,0} \rangle \text{Im}(\bar{e}^{ix\epsilon} \psi_{\epsilon,0})) \) (see Remark 4.2 below).

The well-posedness of the standard NLKG (1.1) is well-known. For the rescaled one, it is rephrased as follows.

Proposition 4.1 (Global well-posedness for the rescaled NLKG). Let \( \epsilon \in (0, 1] \). Given initial data \((v_\epsilon(0), \partial_t v_\epsilon(0)) \in H^1_t(\mathbb{R}) \times L^2(\mathbb{R})\), there exists a unique global strong solution \( v_\epsilon(t) \in C_T(\mathbb{R}; H^1_\epsilon) \) to the rescaled NLKG (2.1). Moreover, \( v_\epsilon(t) \) preserves the energy
\[
E^\epsilon[v] = \frac{\epsilon^2}{2} \|\partial_x v\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|v\|_{L^2(\mathbb{R})}^2 + \epsilon^2 \|\partial_t v\|_{L^2(\mathbb{R})}^2 + \frac{\epsilon^2}{4} \|v\|_{L^4(\mathbb{R})}^4.
\]

Sketch of Proof. The proof follows from a standard contraction mapping argument. Indeed, by the Sobolev embedding \( H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R}) \), one can show that the corresponding integral equation is locally well-posed in \( C_T([-T, T]; H^1_\epsilon) \) for sufficiently small \( T_\epsilon > 0 \) possibly depending on \( \epsilon \in (0, 1] \). Then, it can be upgraded to global well-posedness, because the conservation law prevents the solution from blowing up in finite time. \qed

Remark 4.2. As for well-posedness of the rescaled NLKG (2.1), it is not necessary to employ Strichartz estimates in the previous section. Nevertheless, if one uses \( \epsilon \)-dependent inequalities, e.g., the Sobolev inequality \( \|v\|_{L^4(\mathbb{R})} \lesssim \frac{1}{\epsilon} \|v\|_{H^1_\epsilon(\mathbb{R})} \), the size of the interval \([-T_\epsilon, T_\epsilon]\) in the contraction mapping argument would shrink to \( \{0\} \) as \( \epsilon \to 0 \).

Next, using Strichartz estimates, we establish well-posedness for the system (2.4).

Proposition 4.3 (Local well-posedness for the system (2.4)). Let \( \epsilon \in (0, 1] \) and \( b \in (\frac{1}{2}, 1) \). We assume that \( \{\psi_{\epsilon,0}\}_{\epsilon \in [0, 1]} \) satisfies (H1) for some \( R > 0 \). Then, there exists \( 0 <
Then, by Strichartz estimates and the trilinear estimate (Lemma 3.6), one can show that
\[ \psi \in X^{s,b}([-T,T]) \]
with
\[ p \leq r \quad \text{if} \quad q \leq t \]
provided that
\[ \psi \in X^{s,b}([-T,T]) \]
Moreover, we have
\[ \|\psi\|_{X^{s,b}([-T,T])} \leq \|\psi,0\|_{H^s(\mathbb{R})}. \]

**Proof.** We define
\[ \Phi_1(\psi)(t) := S_t(\psi,0) - \frac{3i}{2} \int_0^t S_t(t - s) \frac{1}{(1 + sD)^{\frac{1}{2}}} |\psi|^2 \psi(s) \, ds. \]

Then, by Strichartz estimates and the trilinear estimate (Lemma 3.6), one can show that
\[ \|\Phi_1(\psi)\|_{X^{1,b}} \leq cR + cT^{\frac{1}{2b}} \|\psi\|^3_{X^{1,b}} \leq cR + cT^{\frac{1}{2b}} (2cR)^3, \]
\[ \|\Phi_1(\psi) - \Phi_1(\psi')\|_{X^{1,b}} \leq cT^{\frac{1}{2b}} (\|\psi\|^2_{X^{1,b}} + \|\psi'\|^2_{X^{1,b}}) \|\psi - \psi'\|_{X^{1,b}} \]
\[ \leq 2cT^{\frac{1}{2b}} (2cR)^2 \|\psi - \psi'\|_{X^{1,b}} \leq \frac{1}{2} \|\psi - \psi'\|_{X^{1,b}}, \]
provided that \( \|\psi\|_{X^{1,b}}, \|\psi'\|_{X^{1,b}} \leq cR. \) We take small \( T > 0 \) such that \( cT^{\frac{1}{2b}} (2cR)^2 \leq \frac{1}{4}. \)

Then, it follows that \( \Phi_1 \) is contractive on \( \{\psi \in X^{1,b} : \|\psi\|_{X^{1,b}} \leq 2cR\} \), and it has a unique fixed point \( \psi \) with \( \|\psi\|_{X^{1,b}} \leq 2cR. \)

Next, we consider
\[ \Phi_2(r)(t) := -\frac{3i}{2} \int_0^t S_t(t - s) \frac{1}{(1 + sD)^{\frac{1}{2}}} (|\psi + r|^2 (\psi + r) - |\psi|^2 \psi + \mathcal{R}_s(\psi + r)) \, ds. \]

We again use Strichartz estimates and the trilinear estimate (Lemma 3.6) to obtain
\[ \|\Phi_2(r)\|_{X^{1,b}} \leq cT^{\frac{1}{2b}} (\|\psi\|^3_{X^{1,b}} + \|r\|^3_{X^{1,b}}) \leq 2cT^{\frac{1}{2b}} (2cR)^3, \]
\[ \|\Phi_2(r) - \Phi_2(r')\|_{X^{1,b}} \leq cT^{\frac{1}{2b}} (\|\psi\|^2_{X^{1,b}} + \|r\|^2_{X^{1,b}} + \|r'\|^2_{X^{1,b}}) \|r - r'\|_{X^{1,b}} \]
\[ \leq 3cT^{\frac{1}{2b}} (2cR)^3 \|r - r'\|_{X^{1,b}} \]
if \( \|r\|_{X^{1,b}}, \|r'\|_{X^{1,b}} \leq 2cT^{\frac{1}{2b}} (2cR)^3 \leq cR. \) Therefore, \( r = \Phi_2(r) \) has a unique solution \( r \)
with \( \|r\|_{X^{1,b}} \leq 2cT^{\frac{1}{2b}} (2cR)^3. \)

It remains to show (4.2). Indeed, by Strichartz estimates, we have
\[ \|\psi\|_{X^{s,b}} \leq \|\psi,0\|_{H^s} + \|\psi\|^2 \psi \|_{X^{s,-1-b}}. \]

Slightly modifying the proof of Lemma 3.6, one can show that
\[ \|\psi\|^2 \psi \|_{X^{s,-1-b}} \leq \|\psi\|^2 \psi \|_{L^2_T L^2} \|\psi\|_{C_t H^s} \]
\[ \leq T^{\frac{1}{2b}} \|\psi\|^2_{X^{1,b}} \|\psi\|_{X^{s,b}} \leq T^{\frac{1}{2b}} R^2 \|\psi\|_{X^{s,b}}. \]
Thus, taking smaller \( T \ll \frac{1}{4b} \) if necessary, we obtain \( \|\psi\|_{X^{s,b}} \leq \|\psi,0\|_{H^s}. \)
Remark 4.4. From the construction, summing two equations in the system (2.4) and their c.c.’s, one can see that $\psi_0(t, x) + r_\epsilon(t, x) + c.c.$ is a strong solution to the rescaled NLKG (2.1) and that it is included in $C_t([-T, T]; H^1(\mathbb{R}))$ by the transference principle. Therefore, by uniqueness, the solution constructed from the system (2.4) is a unique strong solution to the rescaled NLKG with initial data $(\text{Re}(e^{it\epsilon} \psi_{\epsilon, 0}), \frac{2}{\epsilon} \langle \partial_x e^{it\epsilon} \psi_{\epsilon, 0} \rangle)$.

Remark 4.5. We have an analogous well-posedness result for the NLS

$$i\partial_t \psi_{\text{NLS}} + \frac{1}{4\sqrt{2}} \partial_x^2 \psi_{\text{NLS}} - \frac{3}{2\sqrt{2}} |\psi_{\text{NLS}}|^2 \psi_{\text{NLS}} = 0. \quad (4.4)$$

Indeed, by the standard argument (see Cazenave [2]) but with the Strichartz estimate

$$\|e^{4\sqrt{2} \partial_x^2 t} u_0\|_{C_t(\mathbb{R}; L^2(\mathbb{R})) \cap L^4(\mathbb{R}; L^8(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})},$$

one can show that there exists $T > 0$, independent of $\epsilon \in (0, 1]$, such that the following hold.

(i) If (H1) holds, the solution $\psi_{\epsilon, 0} \in C_t([-T, T]; H^1(\mathbb{R}))$ to the NLS (4.4) with initial data $\psi_{\epsilon, 0}$ exists, and

$$\|\psi_{\epsilon, 0}\|_{C_t([-T, T]; H^1(\mathbb{R})) \cap L^4([-T, T]; L^8(\mathbb{R}))} \lesssim \|\psi_{\epsilon, 0}\|_{H^1(\mathbb{R})}. \quad (4.5)$$

(ii) If we further assume (H2*) for some $s > 0$, then

$$\|\psi_{\epsilon, 0}\|_{C_t([-T, T]; H^s(\mathbb{R}))} + \|\psi_{\epsilon, 0}\|_{L^4([-T, T]; H^s(\mathbb{R}))} \lesssim \|\psi_{\epsilon, 0}\|_{H^s(\mathbb{R})}. \quad (4.6)$$

5. High frequency estimate for the core profile

We recall that the convergence of the linear flow $S_\epsilon(t)$ requires smallness of high frequencies (see Lemma 3.1 and Remark 3.2). Thus, one may expect that a similar estimate would be needed for the nonlinear problem. In this section, we prove the following proposition for the core profile.

Proposition 5.1 (High frequency estimates for the core profile). Let $\epsilon \in (0, 1]$ and $b \in (\frac{4}{3}, 1)$. We assume (H1) and (H2) for $\{\psi_{\epsilon, 0}\}_{\epsilon \in (0, 1]}$, and let $(\psi_{\epsilon}, r_{\epsilon}) \in C_t([-T, T]; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ be the solution to the system (2.4). Then, for any $\delta > 0$, we have

$$\lim_{\epsilon \to 0} \left( \|P_{>\delta^e-1/3} \psi_{\epsilon}\|_{X^{1,b}_{\epsilon}([-T, T])} + \|P_{>\delta^e-1/3} |\psi_{\epsilon}|^2 \psi_{\epsilon}\|_{L^2([-T, T]; L^2(\mathbb{R}))} \right) = 0,$$

where $P_{>N} = \sum_{M \geq N} \mathbb{P}_M$.

For the proof, one may try to estimate the $\psi_{\epsilon}$-equation in the system (2.4) putting the high frequency projection $P_{>\delta^e-1/3}$. However, one would immediately realize that the frequency cut-off does not work properly, because frequency cut-offs are blurred in products. Indeed, the difference between $P_{>\delta^e-1/3} |\psi_{\epsilon}|^2 \psi_{\epsilon}$ and $|P_{>\delta^e-1/3} \psi_{\epsilon}|^2 P_{>\delta^e-1/3} \psi_{\epsilon}$ is nonzero in general.
To resolve the technical issue, we employ the operator \( m_N(D) \) with multiplier

\[
m_N(\xi) := \begin{cases} 
\frac{|\xi|}{N} & \text{if } |\xi| \leq N, \\
1 & \text{if } |\xi| \geq N.
\end{cases}
\]

Remark 5.2. (i) \( m_N(D) \) acts like a high frequency cut-off in the sense that \( m_N(\xi) = 1 \) for high frequencies \( |\xi| \geq N \) but it is arbitrarily small in very low frequencies, i.e., \( m_N(\xi) \leq \delta \) if \( |\xi| \leq \delta N \) for any small \( \delta > 0 \).

(ii) \( m_N(\xi_1) \) and \( m_N(\xi_2) \) are comparable if \( |\xi_1| \sim |\xi_2| \), i.e., \( m_N(\xi) \sim m_N(2\xi) \) for all \( \xi \). For this reason, \( m_N(D) \) has an advantage in handling the blurring effect in a product.

(iii) \( m_N(D) \) acts like the differential operator \( \frac{1}{N} \partial_x \) in low frequencies \( |\xi| \leq N \).

Proof of Proposition 5.4. We claim that

\[
\|m_N(D)\psi_\epsilon\|_{X^{1,b}_1} \lesssim \|m_N(D)\psi_\epsilon,0\|_{H^1_b}, \quad \|m_N(D)(|\psi_\epsilon|^2\psi_\epsilon)\|_{L_t^{2b}L_x^2} \lesssim \|m_N(D)\psi_\epsilon,0\|_{H^1_b}. \tag{5.1}
\]

Indeed, applying the Fourier multiplier \( m_N(D) \) to the \( \psi_\epsilon \)-equation in the system (2.4) and then estimating by Strichartz estimates, we obtain

\[
\|m_N(D)\psi_\epsilon\|_{X^{1,b}_1} \lesssim \|m_N(D)\psi_\epsilon,0\|_{H^1_b} + \|m_N(D)(|\psi_\epsilon|^2\psi_\epsilon)\|_{X^{0,-(1-b)}_2} \lesssim \|m_N(D)\psi_\epsilon,0\|_{H^1_b} + \|m_N(D)(|\psi_\epsilon|^2\psi_\epsilon)\|_{L_t^{2b}L_x^2}. \tag{5.2}
\]

For the nonlinear term, we decompose

\[
m_N(D)(|\psi_\epsilon|^2\psi_\epsilon) = m_N(D)(|P_{\leq N}\psi_\epsilon|^2P_{\leq N}\psi_\epsilon) + m_N(D)(|\psi_\epsilon|^2\psi_\epsilon - |P_{\leq N}\psi_\epsilon|^2P_{\leq N}\psi_\epsilon) =: I + II,
\]

where \( P_{\leq N} = \sum_{M \leq N} P_M \). For the former term \( I \), we note that \( |P_{\leq N}\psi_\epsilon|^2P_{\leq N}\psi_\epsilon \) is localized in \( |\xi| \leq 6N \) on the Fourier side, where \( m_N(D) \) behaves like \( \sim \frac{|\xi|}{N} \) (see Remark 5.2 (ii) and (iii)). Thus, the Leibniz rule and the trilinear estimate (Lemma 3.3) yield

\[
\|I\|_{L_t^{2b}L_x^2} \sim \frac{1}{N} \|\partial_x(P_{\leq N}\psi_\epsilon)|^2P_{\leq N}\psi_\epsilon\|_{L_t^{2b}L_x^2} \lesssim T^{\frac{1-b}{2b}}\|\psi_\epsilon\|_{X^{1,b}_1} \frac{1}{N} \|\partial_xP_{\leq N}\psi_\epsilon\|_{X^{0,b}_2} \lesssim T^{\frac{1-b}{2b}}\|m_N(D)\psi_\epsilon\|_{X^{0,b}_2}.
\]

For the latter term \( II \), using the trivial bound \( m_N(\xi) \leq 1 \) and the trilinear estimate (Lemma 3.3), we show that

\[
\|II\|_{L_t^{2b}L_x^2} \leq \|\psi_\epsilon|^2\psi_\epsilon - |P_{\leq N}\psi_\epsilon|^2P_{\leq N}\psi_\epsilon\|_{L_t^{2b}L_x^2} \lesssim T^{\frac{1-b}{2b}}\|\psi_\epsilon\|_{X^{1,b}_1} P_{> N}\psi_\epsilon\|_{X^{0,b}_2} \sim T^{\frac{1-b}{2b}}\|m_N(D)\psi_\epsilon\|_{X^{0,b}_2},
\]

where \( m_N(\xi) \sim 1 \) for \( |\xi| \geq N \) is used in the last step. Hence, taking smaller \( T > 0 \) if necessary, we prove that

\[
\|m_N(D)(|\psi_\epsilon|^2\psi_\epsilon)\|_{L_t^{2b}L_x^2} \leq \frac{1}{2}\|m_N(D)\psi_\epsilon\|_{X^{0,b}_2},
\]

from which (5.1) follows (see (5.2)).
Now, we fix $\delta > 0$ and set $N = \delta \epsilon^{-1/3}$. Then, by the claim (5.1), it suffices to show that
$$m_{\delta \epsilon^{-1/3}}(D) \phi_{\epsilon,0} \in L^2 \to 0$$
(see Remark 5.2 (i)). Indeed, for any $\delta_0 \in (0, \delta]$, we have
$$m_{\delta \epsilon^{-1/3}}(D) \phi_{\epsilon,0} \in L^2 \leq m_{\delta \epsilon^{-1/3}}(D)P_{\epsilon \delta_0 \epsilon^{-1/3}} \phi_{\epsilon,0} \in L^2 + m_{\delta \epsilon^{-1/3}}(D)P_{\epsilon \delta_0 \epsilon^{-1/3}} \phi_{\epsilon,0} \in L^2 \leq \frac{\delta_0}{\delta} \|\phi_{\epsilon,0}\|_{L^2} + \|P_{\epsilon \delta_0 \epsilon^{-1/3}} \phi_{\epsilon,0}\|_{L^2}.$$
where we used the properties of $m_N(D)$ in the last step (see Remark 5.2). Consequently, by (H1) and (H2), we obtain $m_{\delta \epsilon^{-1/3}}(D) \phi_{\epsilon,0} \in L^2 \leq \frac{\delta_0}{\delta} + o_\epsilon(1)$. However, since $\delta_0 > 0$ is arbitrarily, this completes the proof of the proposition.

\textbf{Remark 5.3.} Repeating the proof of Proposition 5.1 to the NLS (4.4), one can show that for any $\epsilon > 0$,
$$\lim_{\epsilon \to 0} \| P_{\epsilon \delta_0 \epsilon^{-1/3}} (|\psi^{(\text{NLS})}_\epsilon|^2 \phi^{(\text{NLS})}_\epsilon) \|_{L^1([-T,T];H^1(\mathbb{R}))} = 0.$$  
(5.3)
Indeed, the proof of (5.3) is easier, since one may employ the simpler Strichartz norm $\| \cdot \|_{C_t([-T,T];H^1(\mathbb{R})) \cap L^2([-T,T];L^2(\mathbb{R}))}$ rather than the Fourier restriction norm.

6. Remainder estimate

In Proposition 4.3, a preliminary bound $\| r_\epsilon \|_{X^s([-T,T])} \lesssim T^{\frac{1}{3} - \frac{s}{2}} \mathbb{R}^3$ is obtained for the remainder term. In this section, we show that the remainder vanishes as $\epsilon \to 0$.

\textbf{Proposition 6.1} (Remainder estimate). Let $\epsilon \in (0, 1]$. We assume (H1) and (H2) for $\{\psi_{\epsilon,0}\}_{\epsilon \in (0,1]}$, and let $(\psi_\epsilon, r_\epsilon) \in C_t([-T,T];H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ be the solution to the system (2.4). Then, we have
$$\lim_{\epsilon \to 0} \| r_\epsilon(t) \|_{C_t([-T,T];H^1(\mathbb{R}))} = 0.$$  
(6.1)
If we further assume (H2') for some $s > 0$, then
$$\| r_\epsilon(t) \|_{C_t([-T,T];H^1(\mathbb{R}))} \lesssim \epsilon^{\min \{s, 1 - \eta\}} \text{ for any small } \eta > 0.$$  
(6.2)

The proof of the proposition is reduced to that of the following lemma.

\textbf{Lemma 6.2} (Reduction to the core profile estimate). Let $\epsilon \in (0, 1]$, and let $\eta > 0$ be a sufficiently small number which does not depend on $\epsilon \in (0, 1]$. We assume that $\{\psi_{\epsilon,0}\}_{\epsilon \in (0,1]}$ satisfies (H1). Then, the solution $(\psi_\epsilon, r_\epsilon) \in X^{1,1-\eta}([-T,T]) \times X^{1,1-\eta}([-T,T])$ to the system (2.4) satisfies
$$\| r_\epsilon \|_{X^{1,1-\eta}([-T,T])} \lesssim \epsilon^{1-2\eta} + \| P_{\frac{1}{100\epsilon}} \psi_\epsilon \|_{X^{1,1+\eta}([-T,T])}.$$  

\textit{Proof of Proposition 6.1, assuming Lemma 6.2}. By Proposition 5.1, Lemma 6.2 yields (6.1). For (6.2), we recall from Proposition 4.3 that $\| \psi_\epsilon \|_{X^s,\frac{1}{2}+\eta} \lesssim 1$ if (H2') holds. Thus, it follows that $\| P_{\frac{1}{100\epsilon}} \psi_\epsilon \|_{X^{0,1+\eta}} \lesssim \epsilon^\eta$. \hfill \Box

For the proof of the lemma, we employ the following integral estimate.
Lemma 6.3. For $\tau, \xi \in \mathbb{R}$, we define
\[ I^{\pm}(\tau, \xi) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1_{\{\xi_1, \xi_2, |\xi - \xi_1 - \xi_2| \leq \frac{1}{10e}\}} d\xi_1 d\xi_2}{\langle \tau + \frac{1}{e^2}(\pm(1 + e\xi_1) + \langle 1 + e\xi_2 \rangle + \langle 1 + e(\xi - \xi_1 - \xi_2) \rangle) \rangle^{2(1-\eta)}}. \]

Then, for $\epsilon \in (0, 1]$ and sufficiently small $\eta > 0$, we have
\[ I^{\pm}(\tau, \xi) \leq \max \left\{ \frac{1}{\langle \tau + \frac{2\sqrt{2} + \sqrt{2} - \frac{1}{10} \rangle^{1-2\eta}}}, \frac{1}{\langle \tau + \frac{2\sqrt{2} + \sqrt{2} + \frac{1}{10} \rangle^{1-2\eta}} \right\}. \]

Proof. In a sequel, we always assume that $|\xi_1|, |\xi_2|, |\xi - \xi_1 - \xi_2| \leq \frac{1}{10e}$. First, in the integral $I^{\pm}(\tau, \xi)$, we will substitute the function in the bracket by
\[ y_2 = y_2(\xi_2) := \tau + \frac{1}{e^2} \left( \pm(1 + e\xi_1) + \langle 1 + e\xi_2 \rangle + \langle 1 + e(\xi - \xi_1 - \xi_2) \rangle \right). \]

Indeed, $y_2(\xi_2)$ is a convex function having its unique minimum at $\xi_2 = \frac{\xi - \xi_1}{2}$, because $y_2'(\frac{\xi - \xi_1}{2}) = 0$ and $y_2''(\frac{\xi - \xi_1}{2}) = \frac{1}{(1 + e\xi_2)^3} + \frac{1}{(1 + e(\xi - \xi_1 - \xi_2))^3} \sim 1$. As a consequence, by the mean value theorem, we have $|\frac{dy_2}{d\xi_2}| = \frac{y_2'(\xi_2)}{2} |\xi_2 - \frac{\xi - \xi_1}{2}| \sim \frac{\xi - \xi_1}{2}$, while by Taylor’s theorem, $y_2(\xi_2) - y_2(\frac{\xi - \xi_1}{2}) = \frac{y_2''(c)}{2} (\xi_2 - \frac{\xi - \xi_1}{2})^2 \sim (\xi_2 - \frac{\xi - \xi_1}{2})^2$. Hence, it follows that
\[ \left| \frac{dy_2}{d\xi_2} \right| \sim \sqrt{y_2(\xi_2) - y_2(\frac{\xi - \xi_1}{2})}. \]

Moreover, we have
\[ \left| y_2 - \tau - \frac{1}{e^2} (2\sqrt{2} \pm \sqrt{2}) \right| \leq \frac{1}{10e^2}. \]

Therefore, substituting $\xi_2$ by $y_2$ separately on $(-\infty, \frac{\xi - \xi_1}{2})$ and $[\frac{\xi - \xi_1}{2}, \infty)$, and switching the order of integration, we obtain
\[ I^{\pm}(\tau, \xi) \leq \int_{\tau + \frac{2\sqrt{2} + \sqrt{2} - \frac{1}{10} \rangle}^{\tau + \frac{2\sqrt{2} + \sqrt{2} + \frac{1}{10} \rangle}} \int_{|\xi_1| \leq \frac{1}{10e}} \frac{1_{\{y_2 \geq y_2(\frac{\xi - \xi_1}{2})\}}}{\langle y_2 \rangle^{2(1-\eta)}} \frac{d\xi_1}{\sqrt{y_2 - y_2(\frac{\xi - \xi_1}{2})}} dy_2. \]

Next, in the inner integral $\cdots$, we will substitute by
\[ y_1 = y_1(\xi) := y_2(\frac{\xi - \xi_1}{2}) = \tau + \frac{1}{e^2} \left( \pm(1 + e\xi_1) + 2\langle 1 + e(\frac{\xi - \xi_1}{2}) \rangle \right), \quad (6.3) \]
whose first and second derivatives are given by
\[ y_1'(\xi) = \frac{1}{e} \left( \pm \frac{1 + e\xi_1}{\langle 1 + e\xi_1 \rangle} - \frac{1 + e(\frac{\xi - \xi_1}{2})}{\langle 1 + e(\frac{\xi - \xi_1}{2}) \rangle} \right), \quad (6.4) \]
\[ y_1''(\xi) = \pm \frac{1}{\langle 1 + e\xi_1 \rangle^3} + \frac{1}{2\langle 1 + e(\frac{\xi - \xi_1}{2}) \rangle^3}. \]
For $I^+(\tau, \xi)$, we note as before that $y_1(\xi_1)$ is convex, $\frac{d}{d\xi_1}y_1(\xi_1) = 0$, $|\frac{d}{d\xi_1}| \sim |\xi_1 - \frac{\xi}{2}|$ and $y_1(\xi_1) - y_1(\frac{\xi}{2}) \sim (\xi_1 - \frac{\xi}{2})^2$. Thus, the substitution $y_1 = y_2(\frac{\xi - x_1}{2})$ yields

$$I^+(\tau, \xi) \leq \int_{r_0 - \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}}^{r_0 + \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \left\{ \int_{\tau + \frac{\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}}^{\tau + \frac{\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \frac{dy_1}{y_2^{2(1-\eta)}} \right\} dy_2.$$  

Then, applying the elementary inequality

$$\int_{\alpha}^{\beta} \frac{dy}{\sqrt{\beta - y_1 \sqrt{y_1 - \alpha}}} = \int_{0}^{\beta - \alpha} \frac{dy_1}{(\beta - \alpha) - y_1 \sqrt{y_1}} = \int_{0}^{1} \frac{dy_1}{\sqrt{1 - y_1 \sqrt{y_1}}} \sim 1 \quad (\alpha < \beta),$$

we conclude that

$$I^+(\tau, \xi) \leq \int_{r_0 - \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}}^{r_0 + \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \left\{ \frac{dy_2}{y_2^{2(1-\eta)}} \right\} \max \left\{ \frac{1}{\tau + \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \right\}. $$

For $I^-(\tau, \xi)$, we again substitute by $y_1(\xi_1) = y_2(\frac{\xi - x_1}{2})$, but we instead use that $|y(\xi_1) - \tau - \frac{\sqrt{\tau}}{c^2}| \leq \frac{1}{10e\tau}$ (see (6.3)) and $-\frac{d}{d\xi_1} \sim \frac{1}{\epsilon}$ (see (6.4)). Then, it follows that

$$I^-(\tau, \xi) \leq \int_{r_0 - \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}}^{r_0 + \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \left\{ \frac{dy_2}{y_2^{2(1-\eta)}} \right\} \max \left\{ \frac{1}{\tau + \frac{3\sqrt{\tau + \frac{1}{\epsilon^2}}}{c^2}} \right\}. $$

\[ \square \]

**Proof of Lemma 6.2** For convenience, we denote $P_{\text{low}} = P_{\frac{1}{100}}$ and $P_{\text{high}} = P_{\frac{1}{100}}$. For the $r_\epsilon$-equation in the system (2.4), after taking out the low frequency piece $R_\epsilon(P_{\text{low}} \psi_\epsilon)$ from $R_\epsilon(\psi_\epsilon)$, we apply Strichartz estimates. Then, it follows that

$$\|r_\epsilon\|_{X^{1,\frac{1}{2}+\eta}} \leq \|A_\epsilon\|^2 A_\epsilon - \|\psi_\epsilon\|^2 \psi_\epsilon + R_\epsilon(A_\epsilon) - R_\epsilon(P_{\text{low}} \psi_\epsilon) - \|R_\epsilon(P_{\text{low}} \psi_\epsilon)\|_{X^{1,-\frac{1}{2}+\eta}} + \|R_\epsilon(P_{\text{low}} \psi_\epsilon)\|_{X^{1,-\frac{1}{2}+\eta}}$$

$$= I + II.$$  

For $I$, we apply the trilinear estimate (Lemma 3.6) with the bounds $\|\psi_\epsilon\|_{X^{1,1-\eta}}, \|r_\epsilon\|_{X^{1,1-\eta}} \leq 1$ (see Proposition 4.3). Then, it follows that

$$I \leq T^{1-2\eta} \left( \|r_\epsilon\|_{X^{0,\frac{1}{2}+\eta}} + \|P_{\text{high}} \psi_\epsilon\|_{X^{0,\frac{1}{2}+\eta}} \right).$$

Hence, replacing $T > 0$ by smaller one if necessary, we obtain

$$\|r_\epsilon\|_{X^{1,\frac{1}{2}+\eta}} \leq II + \|P_{\text{high}} \psi_\epsilon\|_{X^{0,\frac{1}{2}+\eta}}.$$
It remains to estimate $II$. Indeed, it is obvious that

$$II \leq \left\| e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} (P_{low} \psi_\varepsilon)^3 \right\|_{L^0_{\psi_{\varepsilon}}} + \left\| e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} (e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} P_{low} \psi_\varepsilon)^2 (P_{low} \psi_\varepsilon) \right\|_{L^0_{\psi_{\varepsilon}}}
$$

$$+ \left\| e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} (e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} P_{low} \psi_\varepsilon)^3 \right\|_{L^0_{\psi_{\varepsilon}}}$$

$$=: II_1 + II_2 + II_3.$$ 

Thus, it suffices to show that for $j = 1, 2, 3$,

$$II_j \lesssim \epsilon^{1-2\eta} \left\| \psi_\varepsilon \right\|^3_{L^0_{\psi_{\varepsilon}}} \lesssim_R \epsilon^{1-2\eta}. \quad (6.5)$$

We consider $II_1$ first. Note that by the Plancherel theorem and the Cauchy-Schwarz inequality, the proof of (6.5) with $j = 1$ is reduced to show a uniform bound for the integral

$$\frac{1}{(\tau - \frac{\sqrt{2}}{\varepsilon^2} + p_\varepsilon (\xi + \frac{2}{\varepsilon})^{1-2\eta}} \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} \frac{\mathbb{1}_{|\xi_1|, |\xi_2|, |\xi - \xi_1 - \xi_2| \leq \frac{1}{\varepsilon}}}{} \langle \tau - \tau_1 - \tau_2 + p_\varepsilon (\xi - \xi_1 - \xi_2) \rangle^{2(1-\eta)}$$

Indeed, eliminating $\tau_1, \tau_2$-integrations by the elementary inequality

$$\int_{\mathbb{R}} \frac{d\tau}{(\tau - \alpha)^{2(1-\eta)}(\tau - \beta)^{2(1-\eta)}} \leq \frac{1}{(\alpha - \beta)^{2(1-\eta)}},$$

it is further reduced to show that

$$I_1 := \frac{1}{(\tau - \frac{\sqrt{2}}{\varepsilon^2} + p_\varepsilon (\xi + \frac{2}{\varepsilon})^{1-2\eta}} \int_{\mathbb{R} \times \mathbb{R}} \frac{\mathbb{1}_{|\xi_1|, |\xi_2|, |\xi - \xi_1 - \xi_2| \leq \frac{1}{\varepsilon}}}{} \langle \tau + p_\varepsilon (\xi_1) + p_\varepsilon (\xi_2) + p_\varepsilon (\xi - \xi_1 - \xi_2) \rangle^{2(1-\eta)} \leq \epsilon^{2(1-2\eta)}.$$

Similarly but using that the Fourier transform of $e^{\frac{2i}{\varepsilon^2} (x^2 - \varepsilon^2)} A_\xi$ is given by $\hat{A}_\xi (-\tau + \frac{\sqrt{2}}{\varepsilon^2}, -\xi - \frac{2}{\varepsilon})$ and $-\tau + \frac{\sqrt{2}}{\varepsilon^2} + p_\varepsilon (-\xi - \frac{2}{\varepsilon}) = -\tau + \frac{\sqrt{2}}{\varepsilon^2} + \frac{\xi}{\varepsilon^2} + \frac{1}{\varepsilon^2} (1 + \epsilon \xi)$, the proof of (6.5) for $j = 2, 3$ can

\[1\] It is a typical reduction. We refer to [28, Lemma 3.1] for instance.
be respectively reduced to show that

\[
\sup_{\tau \in \mathbb{R}, |\xi| \leq \frac{\epsilon}{50\epsilon}} \left\langle \tau - \frac{\xi}{\epsilon \sqrt{2}} + \frac{\sqrt{2}}{\epsilon^2} + \frac{1}{\epsilon^2} (1 - \epsilon^2) \right\rangle^{1-2\eta} \left( - \tau + \frac{\xi}{\epsilon \sqrt{2}} + \frac{3\sqrt{2}}{\epsilon^2}, \xi \right) \leq \epsilon^{2(1-2\eta)},
\]

\[
\sup_{\tau \in \mathbb{R}, |\xi| \leq \frac{\epsilon}{50\epsilon}} \left\langle \tau - \frac{\xi}{\epsilon \sqrt{2}} + \frac{\sqrt{2}}{\epsilon^2} + \frac{1}{\epsilon^2} (1 - \epsilon^2) \right\rangle^{1-2\eta} \left( - \tau + \frac{\xi}{\epsilon \sqrt{2}} + \frac{3\sqrt{2}}{\epsilon^2}, \xi \right) \leq \epsilon^{2(1-2\eta)},
\]

but these integral inequalities immediately follow from Lemma [6.3] \(\square\)

7. Proof of the main results

We are ready to prove Theorem 1.2 and Theorem 1.6. Indeed, by the remainder estimate (Proposition 6.1), it is enough to consider the core profile \(\psi \) in the system (2.4).

Subtracting the NLS

\[
\psi_{\epsilon}^{(\text{NLS})}(t) = e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \psi_{\epsilon,0} - \frac{3i}{2\sqrt{2}} \int_0^t e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \left( |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right)(t_1) dt_1
\]

from the core profile equation, the difference of the two flows is written as

\[
\psi_{\epsilon}(t) - \psi_{\epsilon}^{(\text{NLS})}(t) = (S_\epsilon(t) - e^{\frac{i(1-1)\tau^2}{4\epsilon^2}}) \psi_{\epsilon,0} \\
- \frac{3i}{2} \int_0^t \left( S_\epsilon(t - t_1) - e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \right) \frac{1}{\langle 1 + \epsilon D \rangle} (|\psi_{\epsilon}|^2 \psi_{\epsilon})(t_1) dt_1 \\
- \frac{3i}{2} \int_0^t e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \frac{1}{\langle 1 + \epsilon D \rangle} \left( |\psi_{\epsilon}|^2 \psi_{\epsilon} - |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right)(t_1) dt_1 \\
- \frac{3i}{2} \int_0^t e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \left( \frac{1}{\langle 1 + \epsilon D \rangle} - \frac{1}{\sqrt{2}} \right) \left( |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right)(t_1) dt_1.
\]

For the third term on the right hand side, we note that by the uniform bounds for nonlinear solutions (4.5) for \(\psi_{\epsilon}^{(\text{NLS})}\) and (4.1) with the transference principle for \(\psi_{\epsilon}\),

\[
\left\| |\psi_{\epsilon}|^2 \psi_{\epsilon} - |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right\|_{L^1_t L^2_x} \leq T^{1/2} \left( \left\| |\psi_{\epsilon}|^2 \right\|_{L^1_t L^2_x} + \left\| |\psi_{\epsilon}^{(\text{NLS})}|^2 \right\|_{L^1_t L^2_x} \right) \left\| \psi_{\epsilon}(t) - \psi_{\epsilon}^{(\text{NLS})}(t) \right\|_{C_t L^2_x} \\
\leq T^{1/2} \left\| \psi_{\epsilon}(t) - \psi_{\epsilon}^{(\text{NLS})}(t) \right\|_{C_t L^2_x},
\]

where \(T > 0\) is sufficiently small but independent of \(\epsilon > 0\) and the time interval \([-T, T]\) in norms is omitted for convenience. Hence, we have that for \(s = 0, 1,\)

\[
\left\| \psi_{\epsilon}(t) - \psi_{\epsilon}^{(\text{NLS})}(t) \right\|_{C_t H^s_x} \leq \left\| (S_\epsilon(t) - e^{\frac{i(1-1)\tau^2}{4\epsilon^2}}) \psi_{\epsilon,0} \right\|_{C_t H^s_x} \\
+ \int_0^T \left\| \left( S_\epsilon(t - t_1) - e^{\frac{i(1-1)\tau^2}{4\epsilon^2}} \right) (|\psi_{\epsilon}|^2 \psi_{\epsilon})(t_1) \right\|_{C_t L^2_x} dt_1 \\
+ \int_0^T \left\| \left( \frac{1}{\langle 1 + \epsilon D \rangle} - \frac{1}{\sqrt{2}} \right) \left( |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right) \right\|_{L^1_t H^s_x} dt_1.
\]

For Theorem 1.2 we take \(s = 1\) in (7.2), and recall that by the high frequency estimates ((H2), Proposition 5.3 and (5.3)),

\[
\left\| P_{> \delta^e - 1/3} \psi_{\epsilon,0} \right\|_{H^1}, \left\| P_{> \delta^e - 1/3} \left( |\psi_{\epsilon}|^2 \psi_{\epsilon} \right) \right\|_{L^1_t L^2_x}, \left\| P_{> \delta^e - 1/3} \left( |\psi_{\epsilon}^{(\text{NLS})}|^2 \psi_{\epsilon}^{(\text{NLS})} \right) \right\|_{L^1_t L^2_x} \to 0
\]
for any $\delta > 0$. Thus, by Lemma 3.1 we conclude that $\|\psi_\epsilon(t) - \psi_\epsilon^{(\text{NLS})}(t)\|_{C_tL^2_x} \to 0$.

For Theorem 1.6 we assume that (H2') holds, and take $s = 0$ in (7.2). Then, by (4.3) and (4.6), we have that $\|\psi_\epsilon,0\|_{H^s_x}, \|\psi_\epsilon^2\|_{L^1_t H^s_x}$ and $\|\psi_\epsilon^{(\text{NLS})}\|_{L^1_t H^s_x}$ are uniformly bounded. Thus, following the proof of Lemma 3.1 one can show that $\|\psi_\epsilon(t) - \psi_\epsilon^{(\text{NLS})}(t)\|_{C_tL^2_x} \leq \epsilon^\gamma$. Finally, combining with the remainder estimate (Proposition 6.1), we complete the proof.

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