A Note on q-Deformed Two-Dimensional Yang-Mills and Open Topological Strings

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Abstract

In this note we make a test of the open topological string version of the OSV conjecture, proposed in hep-th/0504054, in the toric Calabi-Yau manifold \( X = O(-3) \to \mathbb{P}^2 \) with background D4-branes wrapped on Lagrangian submanifolds. The D-brane partition function reduces to an expectation value of some inserted operators of a q-deformed Yang-Mills theory living on a chain of \( \mathbb{P}^1 \)'s in the base \( \mathbb{P}^2 \) of \( X \). At large \( N \) this partition function can be written as a sum over squares of chiral blocks, which are related to the open topological string amplitudes in the local \( \mathbb{P}^2 \) geometry with branes at both the outer and inner edges of the toric diagram. This is in agreement with the conjecture.
1 Introduction

Microstate counting of four-dimensional BPS black holes, arising from compactifications of type II superstring theory on Calabi-Yau 3-folds, has received significant progress in recent years. Ooguri, Strominger and Vafa [1] proposed a remarkable relation between this counting problem with the topological string theory. They argued that the mixed ensemble partition function $Z^{BH}$ of BPS black holes is related to the topological string amplitude $\psi^{\text{top}}$ in the same 3-fold as

$$Z^{BH} = |\psi^{\text{top}}|^2,$$

(1.1)

to all orders of the 't Hooft $1/N$ expansion. Since the left hand side has definite sense even for finite $N$, the OSV relation (1.1) can also be viewed as a non-perturbative completion of the corresponding topological string theory.

This conjecture has been tested in [2, 3] by realizing BPS black holes through D-branes wrapped on cycles in a special class of Calabi-Yau manifolds which is a vector bundle of rank 2 on a Riemann surface $\Sigma_g$ with genus $g$. In this case the black hole partition function can be effectively obtained from a q-deformed two-dimensional Yang-Mills defined on $\Sigma_g$. Due to the noncompactness of these Calabi-Yau spaces the original relation (1.1) is modified by an additional summation over an integer, which is interpreted as measuring the Ramond-Ramond fluxes through $\Sigma_g$, and the appearance of ghost branes, which encode the extra closed string moduli (of the physical type II theory) at infinity. In [4] the OSV relation is extended to open topological strings, which capture the information of BPS states involving some “background” D-branes wrapped on certain Lagrangian submanifolds of the Calabi-Yau 3-folds. In [5] the OSV is tested in more general toric geometries, see also [6, 7, 8, 9, 10]. Another direction of development is the study of large $N$ phase transition of the q-deformed 2-d Yang-Mills on the sphere, see e.g. [11, 12, 13, 14, 15]. Remarkably, the authors of [16] give a physical argument why the OSV relation is true from the point of view of M-theory and the AdS$_3$/CFT$_2$ correspondence. Some related works can be found in [17, 18, 19, 20].

In this short note we will make a test of the open topological string version of the OSV conjecture, proposed in [4], in the toric Calabi-Yau manifold

$$X = O(-3) \to \mathbb{P}^2$$

(1.2)

with background D4-branes wrapped on proper Lagrangian submanifolds. The organization of this note is as follows: In section 2 we review this open topological string version of OSV. In section 3 we explain the brane configurations and construct the corresponding partition function. We wrap three stacks of $N$ D4-branes on three toric divisors...
respectively, and also three stacks of $M (M < N)$ “background” D4-branes on three Lagrangian submanifolds. Besides degrees of freedom on each brane, there are additional bifundamental matter fields localized on the intersections of these branes. To engineering the D0 and D2 charges we need to turn on some proper Chern-Simons couplings. The partition function of this brane system can be effectively reduce to a q-deformed two-dimensional Yang-Mills theory living on a necklace of $P^1$’s in the base $P^2$ of $X$. The matter arising from to the intersections of two divisors reduces to a point operator, from the two-dimensional viewpoint, inserted at the point where two $P^1$’s meet together. While the matter due to the intersection of the divisors and the Lagrangians corresponds to a so-called holonomy freezing operator on each of three $P^1$’s. The partition function we needed can be constructed by gluing some basic building-block amplitudes properly. In section 4 we study the large $N$ factorization of the partition function and related it to the open topological string amplitude with “background” D-branes sitting on the inner edges of the toric diagram and ghost branes on the outer edges. This result is completely in agreement with the conjecture proposed in [4].

2 Open topological string version of OSV

In [4] the original OSV relation is extended to open topological strings. Besides the D4-D2-D0 branes wrapped various cycles to engineer black hole charges, they also introduce additional “background” D4-branes which wrap Lagrangian 3-cycles and fill a 1+1 dimensional subspace of the Minkowski spacetime. The background branes induce a new gauge theory in this 1+1 dimensions. Now the brane system contain more BPS microstates. The electric charges with respect to the 1+1 dimensional gauge field are open D2-branes ending on the Lagrangian 3-cycles, while the magnetic charges are domain walls in the 1+1 dimensional theory. Having taken these facts into account the authors of [4] propose that, just as the original OSV, the open topological string amplitude also captures the chiral sector of the brane partition function at large $N$.

In their paper this proposal is checked on a particular class of Calabi-Yau 3-folds

$$X = O(-p) \oplus O(p - 2) \to P^1.$$  \hspace{1cm} (2.1)

On the divisor $\mathcal{D} = O(-p) \to P^1$ there are $N$ D4-branes wrapped. The D2 and D0 charges are introduced by turning on proper Chern-Simons couplings. The Lagrangian 3-cycle $L$, where $M (M < N)$ background D4-branes wrapped, is chosen such that $L$ meets $\mathcal{D}$ along a circle $\gamma$. There are additional bifundamental matter fields localized along $\gamma$. It is argued that, from the point of view of the D4-branes on $\mathcal{D}$, the only effect of these
matter fields is that the first $M$ eigenvalues of the holonomy along $\gamma$ are fixed. The theory on $D$ can be reduced to a q-deformed Yan-Mills on the base $\mathbb{P}^1$. Suppose $\gamma$ lies entirely on $\mathbb{P}^1$ the effect of background branes is the insertion of the operator $\delta_M \left( e^{i\mathbf{H} \gamma}, A, e^{i\phi} \right)$, where $A$ is the two-dimensional gauge connection on $\mathbb{P}^1$. However there is an ambiguity due to the partition of $p$ pinched points on $\mathbb{P}^1$ to two sides of $\gamma$, i.e. a choice of two integers $p_1$ and $p_2$ such that $p_1 + p_2 = p$. They find that this fact corresponds to the framing ambiguity in the open topological string description. Eventually they gain the large $N$ factorization of the brane partition function and identify the chiral blocks with the corresponding open topological string amplitudes as

$$Z_{YM}(N, g_s, \theta, \phi) = \sum_{l \in \mathbb{Z}} \int \int dH \phi'_1 dH \phi'_2 \times$$

$$\psi^a_{\text{top}}(g_s, t + lp_1 g_s, u + lp_1 g_s, u') \overline{\psi}^a_{\text{top}}(g_s, t - lp_1 g_s, u - lp_1 g_s, u'). \quad (2.2)$$

The parameters are related as

$$t = \frac{1}{2} g_s N (p - 1) - ip \theta, \quad (2.3)$$

$$u = \frac{1}{2} g_s N (p_1 - 1) - i(p_1 \theta - \phi), \quad (2.4)$$

$$u'_1 = \frac{1}{2} g_s N + i \phi'_1, \quad (2.5)$$

$$u'_2 = \frac{1}{2} g_s N + i \phi'_2. \quad (2.6)$$

The summation over the integer $l$ is interpreted as the RR fluxes through $\mathbb{P}^1$. The appearance of ghost and anti-ghost branes is due to the physical closed string moduli at infinity. At the level of topological strings either open or closed string viewpoint is available. In the physical type II theory, however, only the closed one remains.

### 3  Branes in $O(-3) \rightarrow \mathbb{P}^2$

The toric Calabi-Yau 3-fold $X = O(-3) \rightarrow \mathbb{P}^2$ can be described, in the language of the two dimensional $\mathcal{N} = 2$ gauged linear sigma model, by four chiral fields $X_{\mu}$ ($\mu = 0, 1, 2, 3$), with $U(1)$ charges $(-3, 1, 1, 1)$, as

$$|X_1|^2 + |X_2|^2 + |X_3|^2 - 3|X_0|^2 = t. \quad (3.1)$$

This equation is just the D-flatness condition of the GLSM, which defines a submanifold $\bar{X}$ in $\mathbb{C}^4$. The 3-fold $X$ is defined by the quotient of $\bar{X}$ mod by the action of the gauge
group $U(1)$. It is a holomorphic line bundle of degree $-3$ over the base $\mathbb{P}^2$ which is represented by $X_0 = 0$. It is a toric manifold, and its toric diagram is

![Toric Diagram]

We wrap $N$ D4-branes respectively on the divisors $\mathcal{D}_i$ which is defined, in addition to (3.1), by the equations

$$\mathcal{D}_i : \quad X_i = 0, \quad i = 1, 2, 3. \quad (3.2)$$

These submanifolds are just the total space of a line bundle of degree $-3$ over $\mathbb{P}^1$, i.e.

$$\mathcal{D}_i \sim O(-3) \to \mathbb{P}^1. \quad (3.3)$$

Any two divisors $\mathcal{D}_i$ and $\mathcal{D}_j$ intersect along a complex plane specified by $X_i = X_j = 0$. There are additional bifundamental matters localized at these intersections. The base of $\bigcup \mathcal{D}_i$ is a chain of three $\mathbb{P}^1$'s which meet pairwise at a single point $p_i$.

Since we want to test the relation [4] between the open topological string amplitude and the two dimensional Yang-Mills thoery, we also need to add “background” D4-branes wrapped on proper Lagrangian submanifolds. The Lagrangian submanifolds of some simple toric varieties have been studied in e.g. [21, 22]. We choose following three Lagrangian submanifolds $\mathcal{L}_i$ as

$$\mathcal{L}_1 : \quad |X_1|^2 - |X_0|^2 = 0, \quad |X_2|^2 - |X_0|^2 = c_1. \quad (3.4)$$

$$\mathcal{L}_2 : \quad |X_2|^2 - |X_0|^2 = 0, \quad |X_3|^2 - |X_0|^2 = c_2. \quad (3.4)$$

$$\mathcal{L}_3 : \quad |X_3|^2 - |X_0|^2 = 0, \quad |X_1|^2 - |X_0|^2 = c_3. \quad (3.4)$$

Here the constants $c_i$ satisfy $0 < c_i < t$. Now we wrap $M$ D4-branes on each of these three Lagrangians. Combine (3.2) and (3.4) it can be seen that $\mathcal{D}_i$ intersects with $\mathcal{L}_i$ along a circle $\gamma_i$ which is entirely in three base $\mathbb{P}^1$’s respectively. As before there are also bifundamental matters along $\gamma_i$. 

4
3.1 Dynamics on branes

The low energy dynamics on the D4-branes wrapped on $D_i = O(-3) \to \mathbb{P}^1$, as argued in [2, 3], is just the topological twist of the $\mathcal{N} = 4$ gauge theory, studied in [23]. To turn on the D0 and the D2 charges we need to add the following terms

$$
\frac{1}{2g_s} \int_{D_i} \text{Tr} \, F \wedge F + \frac{\theta}{g_s} \int_{D_i} \text{Tr} \, F \wedge \mu
$$

(3.5)

to the original $\mathcal{N} = 4$ theory. Here $\mu$ is the area form on $\mathbb{P}^1$. As discussed in [2, 3] the theory localized to modes which are invariant under the $U(1)$ rotation acting on the fiber $\mathbb{C}^1$. This is a crucial observation which effectively reduces the theory to a two-dimensional gauge theory on $\mathbb{P}^1$. To see this introduce the variable $\Phi(z)$ by

$$
\Phi(z) = \int_{\mathbb{C}_1^1} \mathcal{F} = \oint_{\mathbb{S}_1^1} \mathcal{A}.
$$

(3.6)

Now what is needed is the partition function of the theory on $\mathbb{P}^1$ defined by the action

$$
S = \frac{1}{g_s} \int_{\mathbb{P}^1} \text{Tr} \, \Phi \, F + \frac{\theta}{g_s} \int_{\mathbb{P}^1} \mu \, \text{Tr} \, \Phi - \frac{1}{2g_s} \int_{\mathbb{P}^1} \mu \, \text{Tr} \, \Phi^2.
$$

(3.7)

Integrating out the field $\Phi$ it becomes the standard Yang-Mills action with a $\theta$-term. However there is a subtlety noticed in [3]: Although the action is just standard, the integration measure of $\Phi$ is not as usual. It is proper to consider the variable $e^{i\phi}$, rather than $\Phi$, as the fundamental field. Taking account of this fact the resulting quantum theory is the so-called q-deformed two-dimensional Yang-Mills theory.

Because of the intersections of the D4-branes wrapped on $D_i$ with each other and with the background D4-branes wrapped on $L_i$, there should be some additional bifundamental matter fields localized at these intersections. The effect of integrating out these matter fields, from the viewpoint of the two-dimensional theory, is the insertion of some operators on the base sphere $\mathbb{P}^1$. As argued in [5], the matter fields localized on $D_i \cap D_j$ correspond to inserting the operator

$$
\mathcal{V} = \sum_R \text{Tr}_R V^{-1}_{(i)} \text{Tr}_R V_{(i+1)} ,
$$

(3.8)

at the point $p_i$ where two base spheres meet, with

$$
V_{(i)} = e^{i\Phi_{(i)}} e^{-i f A_{(i)}} , \quad V_{(i+1)} = e^{i\Phi_{(i+1)}} .
$$

(3.9)

The integral contour is just a small loop around the meet points.
The divisor $D_i$ intersects with the Lagrangian submanifold $L_i$ along a curve $\gamma_i$. There are also bifundamental matters localized on $\gamma_i$. As discussed in [4], the effect of these matters, from the point of view of the gauge theory on $D_i$, is just the insertion of the holonomy freezing operator $\delta_M \left( e^{i\mathcal{F} A}, e^{i\phi} \right)$, which can be written in a Weyl invariant way as follows

$$
\delta_M \left( e^{i\mathcal{F} A}, e^{i\phi} \right) = D ( \oint A )^{-1} \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{\alpha=1}^{M} \delta \left( (e^{i\mathcal{F} A})_{\sigma(\alpha)}, e^{i\phi_{\alpha}} \right). \tag{3.10}
$$

The insertion of this delta-function type operator fixes the first $M$ eigenvalues of the holonomy $e^{i\mathcal{F} A}$ along $\gamma_i$, which is a $N \times N$ matrix, to be $M$ specified numbers $e^{i\phi_{\alpha}}$ with $\alpha = 1, \ldots, M$. Since intersection curve $\gamma_i$ is totally contained in the base $\mathbb{P}^1$, the holonomy freezing operator $\delta_M \left( e^{i\mathcal{F} A}, e^{i\phi} \right)$ just simply reduce to

$$
\delta_M \left( e^{i\mathcal{F} A}, e^{i\phi} \right), \tag{3.11}
$$

in the two-dimensional theory, where $A$ is the gauge connection on $\mathbb{P}^1$. As noted in [4], there is an ambiguity which corresponds to the number of the pinched points on each side of $\gamma_i$, and parametrized by a choice of $a_i$ and $a_i'$ such that $a_i + a_i' = 3$, since the degree of the divisor $D_i$ is $-3$.

Therefore the brane partition function has been reduced to the partition function of the q-deformed Yang-Mills defined on a chain of three $\mathbb{P}^1$’s, with the observable (3.8) inserted at each point $p_i$ where two $\mathbb{P}^1$’s meet together, and the holonomy freezing operator (3.11) along the curve $\gamma_i$ on each $\mathbb{P}^1$.

![Diagram](image.png)

### 3.2 Partition function of 2d qYM

The q-deformed two-dimensional Yang-Mills theory can be solved exactly by an operatorial approach [3]. What we need is some basic amplitudes as building blocks. The
partition function with various insertions can be gained by the gluing properly these simple amplitudes. The Hilbert space $\mathcal{H}$ of the q-deformed two-dimensional Yang-Mills can be viewed as the space of class functions on the gauge group. This is because the path integral over the fields of the theory on a Riemann surface $\Sigma$ with boundary $\partial \Sigma$ gives a function $\Psi(U)$. The group element $U$ is just the holonomy along the boundary. A convenient basis of $\mathcal{H}$ is the characters $\text{Tr}_R U$, which are also the energy eigenfunctions of the 2-d qYM. These functions satisfy the orthonormality condition

$$\langle R | Q \rangle = \delta_{RQ}. \quad (3.12)$$

To construct the brane partition function we can cut the chain of $P^1$’s into several parts: three “vertexes” and three annulus with a holonomy freezing operator inserted. In followings we will review each of these amplitudes.

The partition function on an annulus with area $a$ is \[3\]

$$\langle R | A(a) | Q \rangle = \delta_{RQ} e^{-a \left( \frac{1}{2} g_s C_2(R) - i \theta C_1(R) \right) } \equiv \delta_{RQ} A_R^{(a)}, \quad (3.13)$$

where $C_1(R)$ and $C_2(R)$ are respectively the first and the second Casimir of the representation $R$. It can be viewed as an “evolution” amplitude from state $|Q\rangle$ to state $|R\rangle$ with the Euclidean time parameter $a$. It is zero unless the initial and final states are the same one.

The “vertex” amplitude, i.e. the expectation value of the observable (3.8), has been worked out in [5]. It takes the form as

$$V_{RQ} \equiv \langle R | V | Q \rangle = \Theta^N(q) S_{RQ}(q) q^{-\frac{1}{2} C_2(R) - \frac{1}{2} C_2(Q)}, \quad q = e^{-g_s}. \quad (3.14)$$

Here $\Theta^N(q) = \sum_{m \in \mathbb{Z}} q^{m^2/2}$, and the function $S_{RQ}(q)$ is related to the S-matrix of the WZW model through

$$S_{RQ}(q) = S_{RQ}(q^{-1}) = q^{-\rho^2 - \frac{N}{24}} \sum_{\sigma \in S_N} (-1)^{\sigma} q^{\sigma(R+\rho)-(Q+\rho)}, \quad (3.15)$$

where $\rho = \frac{1}{2}(N-1, N-3, \ldots, 3-N, 1-N)$ is the half of the sum of all positive roots of $U(N)$. The dot in the exponential denotes the standard inner product of $\mathbb{R}^N$, and $R$ here denotes $(R_1, \ldots, R_N)$ — number of boxes in each row of the Young diagram.

The expectation value of the holonomy freezing operator is

$$\Delta_{RQ} \equiv \langle R | \delta_M \left( e^{i\delta A}, e^{i\phi} \right) | Q \rangle = \sum_A \text{Tr}_{R/A}(e^{-i\phi}) \text{Tr}_{Q/A}(e^{i\phi}). \quad (3.16)$$

Here $\text{Tr}_{R/A}$ is the skew trace, defined by $\text{Tr}_{R/A}(U) = \sum_B B_{AQ} B_{AR}^{-1} \text{Tr}(U)$ with $B_{AR}$ being the branching coefficients. More details can be found in [4].
The total partition function can be constructed by the gluing above basic amplitudes as followings

\[
Z^{q\text{YM}}(g, \theta, \phi_i) = Z_0 \sum_{R_i, R'_i} \mathcal{V}_{R_i R'_i} A_{R_2}^{(a_2)} \Delta_{R_2 R'_2} A_{R'_2}^{(a'_2)} \mathcal{V}_{R'_2 R_3} A_{R_3}^{(a_3)} \Delta_{R_3 R'_3} A_{R'_3}^{(a'_3)} \mathcal{V}_{R'_3 R_1} A_{R_1}^{(a_1)} \Delta_{R_1 R'_1} A_{R'_1}^{(a'_1)}
\]

\[
= Z_0 6^{3N}(q) \sum_{R_i, R'_i} S_{R'_i R_2} S_{R'_2 R_3} S_{R'_3 R_1} \Delta_{R_1 R'_1} \Delta_{R_2 R'_2} \Delta_{R_3 R'_3} \prod_{i=1}^{3} q^{\frac{a_i + a'_i}{2}} \prod_{i=1}^{3} e^{i \theta_i (a_i C_1(R_i) + a'_i C_1(R'_i))}.
\]

(3.17)

There is an ambiguity due to a choice of \(a_i\) and \(a'_i\) such that \(a_i + a'_i = 3\). The normalization factor \(Z_0\) is determined by the requirement of the Large \(N\) factorization in the next section. The first line of (3.17) can be read off form the following graph.

4 Large \(N\) factorization and open topological strings

In this section we will study the Large \(N\) factorization the the \(q\)-deformed 2-d YM into chiral and anti-chiral sectors, and relate them with the open topological string amplitudes. To do this an essential technical tool introduced in [24] is the notion of a composite representation, whose contribution survives in the large \(N\) limit. A composite representation \(R\) of \(U(N)\) can be labeled by two Young diagrams \(R_\pm\) and a integer \(l_R\) as

\[
R = R_+ \overline{R_-}[l_R].
\]

(4.1)

The meaning of the integer \(l_R\) is just the powers of the determinant representation of \(U(N)\). The Casimir operators for a composite representation \(R = R_+ \overline{R_-}[l_R]\) is

\[
C_1(R) = Nl_R + |R_+| + |R_-|, \tag{4.2}
\]

\[
C_2(R) = \kappa_{R_+} + \kappa_{R_-} + N(|R_+| + |R_-|) + Nl_R^2 + 2l_R(|R_+| + |R_-|), \tag{4.3}
\]
where \(|R|\) denotes the total number of the Young diagram corresponding to the representation \(R\), and

\[
\kappa_R = \sum_i R_i (R_i - 2i + 1),
\]

which is independent of \(N\). The factorization of the \(S_{RQ}\) has been gained in \([4,5]\)

\[
S_{RQ}(q) = M(q^{-1}) \eta^N(q^{-1}) (-q^2)^{\frac{N}{2}} q^{\frac{1}{2}N_+ + \frac{1}{2}N_-} \sum_{\nu} (-1)^{|\nu|} q^{-N(|\nu|)} C_{Q'_{1,R+} P}(q^{-1}) C_{Q'_{-R-} P}(q^{-1}).
\]

Here \(M(q)\) and \(\eta(q)\) is the McMahon and Dedekind eta functions, and \(C_{Q'_{1,R+} P}(q^{-1})\) is the topological vertex amplitude of \([25]\), \(Q^t\) denotes the transpose representation. This result can be argued by the geometric transition of the deformed conifold \(T^*S^3\) to the resolved conifold \(O(-1) \oplus O(-1) \to P^1\), both with four stacks of non-compact D-branes.

The holonomy freezing operator amplitude \(\Delta_{RQ}\) in \((3.16)\) with respect to two composite representations is \([4]\)

\[
\Delta_{RQ} = \delta_{l_{RQ}} \left( \sum_{A_+} s_{R/A_+}(e^{-i\phi}) s_{Q/A_+}(e^{i\phi}) \right) \left( \sum_{A_-} s_{R/A_-}(e^{i\phi}) s_{Q/A_-}(e^{-i\phi}) \right).
\]

Here \(s_{R/A}(x)\) is the skew Schur function, which is related to the Schur function through

\[
s_{R/A}(x) = \sum_Q N_{AQ}^R s_Q(x),
\]

where the quantity \(N_{AQ}^R\) is the Littlewood-Richardson number. The formula \((4.6)\) follows from the relation between the skew trace and the skew Schur function \([4]\)

\[
\text{Tr}_{R/A}(e^{i\phi}) = s_{R/A_+}(e^{i\phi}) s_{R/A_-}(e^{-i\phi}) \det e^{iR\phi}.
\]

Put every things needed into \((3.17)\) and introduce the parameter \(l = l_1 + l_2 + l_3\). Up to \(O(e^{-N})\) terms, the partition function takes the following factorization form

\[
Z_{YM}^{qY}(g_s, \theta, \phi) = \sum_{l \in \mathbb{Z}} \sum_{P_i} (-1)^{|P_1| + |P_2| + |P_3|} \psi^{\text{top}}_{P_1 P_2 P_3}(g_s, t + lg_s, u_i) \psi^{\text{top}}_{P_1' P_2' P_3'}(g_s, \bar{t} - lg_s, u_i).
\]

Note that the integers \(l'_i\) disappear due to \(\delta_{l_l l'_l}\) in \(\Delta_{R_i R_i'}\). Here \(\psi^{\text{top}}_{P_1 P_2 P_3}(g_s, t, u_i)\) is the open topological string amplitude with three stacks of ghost branes at each outer edge of the
toric diagram, and three stacks of background branes at each of the inner edges. The summations over representations $R_{i+}$ and $R'_{i+}$ have been absorbed in $\psi^{\text{top}}_{P_1 P_2 P_3}$, while $R_{i-}$ and $R'_{i-}$ in $\psi^{\text{top}}_{P'_1 P'_2 P'_3}$. This result is in complete agreement with the conjecture proposed in [4] by Aganagic, Neitzke and Vafa. In the followings we will give some details.

The parameters of the topological string theory is

$$t = \frac{3}{2} g_s N - 3i\theta,$$  \hspace{1cm} (4.10)

$$u_1 = \phi_1 - a_1 \theta - \frac{i}{2} a_1 g_s N - ig_s ((a_1 - 1)l_1 + l_3),$$ \hspace{1cm} (4.11)

$$u_2 = \phi_2 - a_2 \theta - \frac{i}{2} a_2 g_s N - ig_s ((a_2 - 1)l_2 + l_1),$$ \hspace{1cm} (4.12)

$$u_3 = \phi_3 - a_3 \theta - \frac{i}{2} a_3 g_s N - ig_s ((a_3 - 1)l_3 + l_2).$$ \hspace{1cm} (4.13)

As discussed in [4], $2 \text{Re} \pi u_i/g_s$ is the chemical potential corresponding to the additional electric charges due to the Lagrangian branes, while $\text{Im} u_i/\pi$ is the magnetic charges due to the domain walls in 1+1 subspace. The above relation between parameters can be determined as follows. In the second line of (3.17), we choose

- $q^{\frac{N}{2}(|R'_{i+}| + |R_{i+}|)}$ from $S_{R'_{i+} R_{i+}}$ and similar terms from $S_{R_{i+} R_{i+}}$ and $S_{R'_{i+} R_{i+}}$;

- $s_{R_{i+}/A_{i+}} (e^{-i\phi_i}) s_{R'_{i+}/A_{i+}} (e^{i\phi_i})$ from $\Delta_{R_i R'_i}$;

- $N|R_{i+}| + 2l_1|R_{i+}|$ from $C_2(R_i)$, and similar terms in $C_2(R'_i)$;

- $|R_{i+}|$ and $|R'_{i+}|$ from $C_1(R_i)$ and $C_1(R'_i)$, respectively.

By noting that the skew Schur function $s_{R/A}$ is a homogeneous function with degree $|R| - |A|$, together with the fact $a_i + a'_i = 3$, we can simplify the product of these factors as follows

$$\prod_{i=1}^{3} \left( s_{R_{i+}/A_{i+}} (e^{-iu_i}) s_{R'_{i+}/A_{i+}} (e^{iu_i}) e^{-\left(\frac{3}{2} g_s N - 3i\theta_i + l g_s\right)|R'_{i+}|} \right).$$ \hspace{1cm} (4.14)

Here $u_i$’s are defined as above. If requiring three Kähler moduli $\frac{3}{2} g_s N - 3i\theta_i$ are same, we must have $\theta_1 = \theta_2 = \theta_3$ denoted by $\theta$, i.e. $t = \frac{3}{2} g_s N - 3i\theta$ as above. If we choose the anti-chiral sectors from above four kind of terms, their product is exactly

$$\prod_{i=1}^{3} \left( s_{R_{i-}/A_{i-}} (e^{iu_i}) s_{R'_{i-}/A_{i-}} (e^{-iu_i}) e^{-\left(\frac{3}{2} g_s N - 3i\theta_i \right)|R'_{i-}|} \right).$$ \hspace{1cm} (4.15)
In the paper [25] the gluing rule for the A-model topological string amplitude, with brane sitting on inner edges of the toric diagram, is

\[
\sum_{ABB'} (-1)^s e^{-t|A|} C_{PQA\otimes B} C_{P'R'A\otimes B'} s_B(e^{-iu}) s_{B'}(e^{-t+iu}). \tag{4.16}
\]

Since the topological vertex make sense only for irreducible representations, so \(C_{PQA\otimes B}\) is just the abbreviation for \(\sum_A N^R_{AB} C_{PQR}\), and \(C_{P'R'A\otimes B'}\) for \(\sum_A N^R'_{AB'} C_{P'R'R}\). The sign factor \(s\) and the frame factor \(f\) only depend on the irreducible representation \(R\) and \(R'\). Do the summation over \(B\) and \(B'\) and note the relation (4.7) between the Schur and skew Schur functions, then (4.16) becomes

\[
\sum_{ARR'} (-1)^{s(R,R')} e^{-t|R'|} C_{PQR} C_{P'R'R} s_R/e^{-iu}) s_{R'}/e^{iu} \tag{4.17}
\]

where we have used the homogeneity of the skew Schur functions to extract \(e^{-t}\) which cancels \(e^{-t|A|}\) and gives the factor \(e^{-t|R'|}\). The structure in (4.17) is exactly the same as what has appeared in the 2d qYM amplitude with holonomy freezing operator \(\Delta_{RR'}\) inserted, c.f. (4.14) and (4.15).

At last we determine the normalization factor \(Z_0\). Note that there are quadratic terms of \(l_i\) in the expressions of \(C_2(R)\), see (4.3), and \(S_{RQ}\), see (4.5). The product of these factors just gives \(q^{l/2N^2}\) with \(l = l_1 + l_2 + l_3\). This structure is the same as that of [5] although now there are holonomy freezing operators, since we have \(a_i + a'_i = 3\). Therefore the normalization factor \(Z_0\) is just that of [5], i.e.

\[
Z_0 = e^{-\frac{3}{2}g_s N^3 + \frac{g_s^2}{4\pi^2} N} \tag{4.18}
\]

This normalization factor is crucial for the large \(N\) factorization and the relation between the topological string amplitude.

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