Thermodynamic Equations of State for Dirac and Majorana Fermions

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Abstract

The thermodynamic equations of state for Majorana and Dirac fermions are quite different even in the limit of zero mass. The corresponding equations are derived from general principles, and then applied to neutrinos. The nature of the approach to equilibrium is explored, along with the subtleties of low or zero mass neutrinos interacting via purely left-handed currents. Possible applications to cosmology, supernovae, and supersymmetry are discussed.

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1 Introduction

Fermions are usually described with 4-component complex spinors which satisfy the Dirac equation. The four complex components describe two spin states for a particle and two spin states for the antiparticle. These are the usual “Dirac” fermions, such as electrons.

In the event that the fermions have no charges or electric or magnetic dipole moments, the possibility exists that the particle and antiparticle states are identical, just as the photon or neutral pion states and their antiparticle states are identical. This constraint applied to a Dirac fermion yields a 2-complex component (or 4-real component) spinor representing a particle which is identical to its antiparticle: a “Majorana” fermion.

Perhaps surprisingly, it remains in many workers minds an open question as to whether or not neutrinos are their own antiparticles. The argument has been that in the Standard Model with only left-handed couplings, a putative right-handed neutrino would not (beyond gravity) interact. It is then conceivable that the two observed neutrino states, one called a left-handed neutrino and the other called a right-handed antineutrino are actually two helicity states of the same Majorana particle. The antineutrino of the Dirac theory here appears as the right neutrino of the Majorana theory and observed neutrinos might then be Majorana.

In the absence of right-handed couplings, and with massless Dirac neutrinos with no amplitudes connecting left and right-handed components, the right handed neutrinos would appear to be not detectable. For neutrinos of small masses $m$, amplitudes involving the right-handed components are often suppressed by factors of order $m/M_W$, where $M_W$ is the W-boson mass, characteristic of the weak scale. This leads to the often-quoted “practical Majorana-Dirac confusion theorems” [1], essentially stating that if one does experiments on a left-handed neutrino, it gets harder and harder to tell if it’s Majorana or Dirac as its mass tends to zero.

This note addresses a different situation, where we consider systems of
more than one neutrino. Our purpose here is to exhibit thermodynamic
equations of state for an ideal gas of Dirac neutrinos and for an ideal gas of
Majorana neutrinos. Thermodynamic equations of state have some impor-
tance for astrophysical and cosmological models. Before considering ther-
modynamic systems with large numbers of particles let us first consider the
(anti-symmetry) properties of wave functions for systems of more than one
neutrino, and apply them to the simplest multineutrino process: production
of pairs \[3\].

2 Anti-symmetry for Dirac and Majorana Fermions

The central point is that all Dirac wave functions of momenta \( p_i \) and spins
\( s_i \) for particles and antiparticles,

\[
\Psi_{\text{Dirac}} = \Psi_{\text{Dirac}}(p_1, s_1, p_2, s_2, \ldots; \bar{p}_1, \bar{s}_1, \bar{p}_2, \bar{s}_2, \ldots),
\]

are anti-symmetric with respect to particle exchange \( (p_i, s_i) \leftrightarrow (p_j, s_j), \)
\( (i \neq j) \), and with respect to antiparticle exchange \( (\bar{p}_i, \bar{s}_i) \leftrightarrow (\bar{p}_j, \bar{s}_j), \)
\( (i \neq j) \). However, there is no particular enforced exchange symmetry between a par-
ticle exchanged with an antiparticle since these two objects have different
fermion number.

For the Majorana case, there is no distinction between a particle and an
antiparticle, and no conserved fermion number.

For a Majorana wave function

\[
\Psi_{\text{Majorana}} = \Psi_{\text{Majorana}}(p_1, s_1, p_2, s_2, \ldots)
\]

there is an anti-symmetry under the exchange \( (p_i, s_i) \leftrightarrow (p_j, s_j), \)
\( (i \neq j) \), for all neutrinos.
3 Pair Production of Neutrinos

The exchange symmetry of wave functions is above and beyond any explicit (standard or otherwise) model assumptions about interactions \[3\]. Exchange anti-symmetry has some profound implications. As an example of the consequences of anti-symmetric wave functions, contrast the following neutrino reaction for the two cases of interest,

\[ e^- + e^+ \rightarrow \nu_1 + \bar{\nu}_1 \quad (\text{Dirac}), \quad e^- + e^+ \rightarrow \nu_1 + \nu_2 \quad (\text{Majorana}). \]  

For the Dirac case, in the limit of zero mass and in the center of mass reference frame, there is a parity violating forward-backward asymmetry in the scattering angle $\theta$ of the differential cross section

\[ d \sigma(e^- e^+ \rightarrow \nu_1 \bar{\nu}_1) = |f_{\text{Dirac}}(\theta)|^2 d\Omega \neq |f_{\text{Dirac}}(\pi - \theta)|^2 d\Omega. \]  

For the Majorana case, in the limit of zero mass and in the center of mass reference frame, the opposite helicities of the two neutrinos would require that the two neutrino spins are parallel (say at angle $\theta$ the parallel spins $\langle \uparrow, \uparrow \rangle$). Quantum mechanics for identical fermions with parallel spins dictates that

\[ f_{\text{Majorana}}(\theta) = f(\theta) - f(\pi - \theta), \]  

so no Majorana forward-backward asymmetry can exist:

\[ d \sigma(e^- e^+ \rightarrow \nu_1 \nu_2) = |f_{\text{Majorana}}(\theta)|^2 d\Omega = |f_{\text{Majorana}}(\pi - \theta)|^2 d\Omega. \]

This process evades the “practical confusion theorem” by having more than one neutrino in the problem, and quantum mechanics makes a sharp distinction, independent of mass, in the production of distinct or identical particles. In the following sections we will study the thermodynamics of systems with many Dirac or Majorana neutrinos.


4 Equations of State for Gases of Fermions

In the following sections we will call the Dirac of Majorana fermions “neutrinos” without loss of generality, and with the understanding that the arguments apply to any fermions.

4.1 Majorana Neutrino Thermal Equations of State

Before calculating the equation of state for an ideal gas of Majorana neutrinos, let us first consider an ideal gas of photons. For the photon case, the following holds true: (i) The photon wave function

\[ \Psi_{\text{photon}}(K_1, S_1, K_2, S_2, \ldots) \]

is symmetric under the exchange \((K_i, S_i) \leftrightarrow (K_j, S_j), (i \neq j)\); (ii) The distinction between particle and antiparticle for the photon is null and void; (iii) Photon number is not conserved. From a statistical mechanical viewpoint, the above requires that only zero chemical potential needs to be considered for the photon. The grand canonical ensemble pressure is then

\[ P_{\text{photon}} = -k_B T \left\{ 2 \int \frac{d^3 K}{(2\pi)^3} \ln(1 - e^{-\hbar c K/k_B T}) \right\}. \quad (8) \]

Equation 8 yields the usual result,

\[ P_{\text{photon}}(T) = \left( \frac{\pi^2 \hbar c}{45} \right) \left( \frac{k_B T}{\hbar c} \right)^4, \quad (9) \]

from which all other thermal properties follow.

For Majorana fermions the following holds true: (i) The neutrino wave function in Eq.(2) is anti-symmetric; (ii) The distinction between particle and antiparticle for the Majorana neutrino is null and void; (iii) Fermion number is not conserved. From a statistical mechanical viewpoint, the above requires that we set the chemical potential to zero for the Majorana neutrino. The grand canonical ensemble pressure is then

\[ P_{\text{neu}} = \frac{\pi^2 \hbar c}{45} \left( \frac{k_B T}{\hbar c} \right)^4, \]
\[ P_{\text{Majorana}} = k_B T \left\{ 2 \int \left( \frac{d^3 p}{(2\pi \hbar)^3} \right) \ln(1 + e^{-\varepsilon/k_B T}) \right\}, \]  

(10)

where

\[ \varepsilon = \sqrt{c^2 p^2 + m^2 c^4}. \]  

(11)

From Eqs.(10) and (11) it follows that

\[ P_{\text{Majorana}} = \hbar c \left( \frac{k_B T}{\hbar c} \right)^4 F_{\text{Majorana}} \left( \frac{m c^2}{k_B T} \right), \]  

(12)

where

\[ F_{\text{Majorana}}(x) = \left( \frac{1}{\pi^2} \right) \int_x^{\infty} \left( \frac{\sqrt{y^2 - x^2}}{y} \right) \ln(1 + e^{-y}) dy. \]  

(13)

In the limit of zero mass,

\[ \lim_{m \to 0} P_{\text{Majorana}}(T) = \left( \frac{7\pi^2 \hbar c}{360} \right) \left( \frac{k_B T}{\hbar c} \right)^4, \]  

(14)

where \( F_{\text{Majorana}}(0) = (7\pi^2/360) \) has been employed. Thus, in the zero mass limit and with zero chemical potential, \( P \sim (\hbar c/\lambda^4) \) where the thermal wave length \( \lambda \sim (\hbar c/k_B T) \). The above results from dimensional analysis hold true for both zero mass photons and zero mass Majorana neutrinos.

### 4.2 Dirac Neutrino Thermal Equations of State

For the case in which fermion number \( N \) is conserved, one may choose the grand canonical partition function with a chemical potential \( \mu \) to be

\[ Z_{\text{Grand}} = Tr \left\{ e^{-(H - \mu N)/k_B T} \right\}, \]  

(15)

The grand canonical pressure (in the thermodynamic limit of infinite volume \( V \)),

\[ P(T, \mu) = \lim_{V \to \infty} \left( \frac{k_B T}{V} \right) \ln Z_{\text{Grand}}(T, \mu, V), \]  

(16)
provides a complete determination of thermodynamic equations of state. The thermodynamic law reads

\[ dP = s dT + n d\mu, \]  

(17)

where \( s \) and \( n \) denote, respectively, the entropy and fermion number per unit volume.

For an ideal gas of Dirac neutrinos, the pressure is

\[ P_{\text{Dirac}} = k_B T \left\{ 2 \int \frac{d^3 p}{(2\pi \hbar)^3} \ln \left[ \left( 1 + e^{(\mu - \epsilon)/k_B T} \right) \left( 1 + e^{-(\mu + \epsilon)/k_B T} \right) \right] \right\}. \]  

(18)

The point is that both the Dirac particle and antiparticle have a positive energy \( \epsilon \) but the fermion number is +1 for the particle and −1 for the antiparticle. Thus, the particle gets a chemical potential \( \mu \) while the antiparticle gets a chemical potential \( -\mu \).

Equations 11 and 16 imply that

\[ P_{\text{Dirac}} = \hbar c \left( \frac{k_B T}{\hbar c} \right)^4 G_{\text{Dirac}} \left( \frac{mc^2}{k_B T}, \frac{\mu}{k_B T} \right), \]  

(19)

where

\[ G_{\text{Dirac}}(x, z) = \left( \frac{1}{\pi^2} \right) \int_x^\infty \left[ y \sqrt{y^2 - x^2} \right] \ln \left[ \left( 1 + e^{(z-y)} \right) \left( 1 + e^{-(z+y)} \right) \right] dy. \]  

(20)

In the zero mass limit

\[ \lim_{m \to 0} P_{\text{Dirac}}(T, \mu) = \hbar c \left( \frac{k_B T}{\hbar c} \right)^4 G \left( \frac{\mu}{k_B T} \right), \]  

(21)

where

\[ G(z) = \left( \frac{1}{\pi^2} \right) \int_0^\infty y^2 \ln \left[ \left( 1 + e^{(z-y)} \right) \left( 1 + e^{-(z+y)} \right) \right] dy. \]  

(22)

In the Fermi-Dirac degenerate regime

\[ P_o(\mu) = \lim_{T \to 0} \lim_{m \to 0} P_{\text{Dirac}}(T, \mu), \]  

(23)
one finds for the pressure and lepton number per unit volume, $n_o = (dP_o/d\mu)$,

$$P_o(\mu) = \left(\frac{\hbar c}{12\pi^2}\right)\left(\frac{\mu}{\hbar c}\right)^4, \quad n_o = \left(\frac{1}{3\pi^2}\right)\left(\frac{\mu}{\hbar c}\right)^3. \quad (24)$$

Thus, in the degenerate regime,

$$P_o = \left(\frac{\hbar c}{12\pi^2}\right)(3\pi^2|n_o|)^{4/3} \quad (Dirac). \quad (25)$$

The chemical potential variations for the Dirac neutrino Eq.(21) have no counterpart in the Majorana Eq.(14). In fact, in the general thermodynamic Eq.(17) holds for Dirac neutrinos but must be restricted to $\mu_{Majorana} = 0$ for the Majorana case. Thus the equilibrium thermodynamic regime of Eq.(25) does not exist for Majorana neutrinos.

5 Weak Interactions and the Approach to Equilibrium

The arguments in the previous section are described in some mathematical detail, but it is instructive to have a physical picture of what is happening. Imagine the familiar filling up of energy levels with Dirac fermions such as electrons. Each level can take two fermions in different spin states. Now if the same procedure is attempted with the less-familiar Majorana fermions, the two different spin states can now annihilate, leaving that energy level vacant. This, physically, is the reason that there is no Fermi energy is that there is no conserved lepton number and thus no Lagrange multiplier (chemical potential) to enforce lepton conservation.

In the Standard Model, with the added hypothesis of Majorana neutrinos and under normal conditions, the interactions of neutrinos with each other are quite feeble, and the chance of a neutrino-number violating reaction is rather small. In this case, true equilibrium may take a long time to be achieved. One might then wish to consider an approximately conserved quantity like the number of left-handed neutrinos less the number of right-handed neutrinos,
assuming then that left and right handed neutrinos annihilate or are produced in pairs. An attendant Lagrange multiplier analogous to chemical potential might be applied in the ideal gas case, however, the corresponding equation of state will not correspond to local thermal equilibrium, and whether or not this is acceptable in a given situation is a purely dynamical question.

There are situations in which neutrino densities are very high, and energy densities are high enough that the weak interactions are effectively unsuppressed. The difference in thermodynamic equations of state for Dirac and Majorana neutrinos would then clearly have physical consequences.

6 Application to Astrophysics and Cosmology

While the main point of this paper is to point out the differences in the thermodynamics of Dirac and Majorana fermions, we would also wish to explore some physical systems where these ideas may be applied.

The thermodynamics of reactions in stars depends in large part on the nature of the chemical potentials. Consider a reaction written as

\[ \sum_k z_k C_k \leftrightarrow 0, \quad (26) \]

where \( C_k \) is a particle component and \( z_k \) is an integer (positive or negative). A condition for thermal equilibrium is that

\[ \sum_k z_k \mu_k = 0, \quad (equilibrium). \quad (27) \]

One simply replaces the symbolic component \( C_k \) in the reaction with the chemical potential \( \mu_k \).

For example, thermal equilibrium for \( e^+ + e^- \leftrightarrow 2\gamma \), implies \( \mu_{e^+} + \mu_{e^-} = 2\mu_{\gamma} \). Since \( \mu_{\gamma} = 0 \), electrons and positrons are in equilibrium if \( \mu_{e^+} + \mu_{e^-} = 0 \). One may now add \( e^+ + e^- \leftrightarrow \nu + \bar{\nu} \), (Dirac), to derive the equilibrium condition \( \mu_{e^+} + \mu_{e^-} = \mu_{\nu} + \mu_{\bar{\nu}}, \) (Dirac) so that \( \mu_{\nu} + \mu_{\bar{\nu}} = 0 \), (Dirac). For
the Majorana case we have (in close analogy with photons), \( e^+ + e^- \leftrightarrow 2\nu \), (Majorana). This requires \( \mu_\nu = 0 \), (Majorana).

For weak nuclear reactions in stars, e.g.

\[
\nu + (Z, A) \leftrightarrow (Z - 1, A) + e^-, \tag{28}
\]

we have the equilibrium condition

\[
\mu_\nu + \mu_{(Z,A)} = \mu_{(Z-1,A)} + \mu_{e^-}, \quad (Dirac). \tag{29}
\]

Viewed as a Majorana neutrino reaction, Eq.(28) yields the equilibrium condition

\[
\mu_{(Z,A)} = \mu_{(Z-1,A)} + \mu_{e^-}, \quad (Majorana). \tag{30}
\]

Thus, with a nuclear chemical potential energy difference \( \varepsilon^* = \mu_{(Z,A)} - \mu_{(Z-1,A)} \) there will be a different nuclear reaction equilibrium condition

\[
\varepsilon^* = \mu_{e^-} - \mu_\nu \quad (Dirac), \quad \varepsilon^* = \mu_{e^-} \quad (Majorana). \tag{31}
\]

Eqs.(4,6) are equally true for the equilibrium of the beta decay reaction

\[
(Z, A) \leftrightarrow (Z-1, A) + e^- + \bar{\nu}, \quad (Dirac), \quad (Z, A) \leftrightarrow (Z-1, A) + e^- + \nu, \quad (Majorana). \tag{32}
\]

Since the equilibrium distribution of nuclear particles is different in models where the neutrinos are considered as Dirac particles or where the neutrinos are considered as Majorana particles, it is evident that the dynamics of the approach to equilibrium must also depend on whether the neutrinos are considered as Dirac particles or as Majorana particles.

Of particular interest are supernovae, where most of the dynamics in the core is driven by neutrinos and in fact, the neutrinos are assumed to be degenerate [1], a situation impossible for Majorana neutrinos in local thermal equilibrium. Significant annihilation of neutrinos and antineutrinos (or perhaps left and right handed Majorana neutrinos) has been discussed by Goodman et al. [3], indicating that there are astrophysical systems in which there are significant (Standard Model) neutrino-neutrino interactions.
Estimates of effects of a neutrino number non-conserving interactions in the context of supernova explosions and equilibrium equations of states for Dirac and Majorana equations of state would be of interest.

There are possible cosmological implications as well, where the process $\nu + \bar{\nu} \rightarrow e^+ + e^-$ is assumed to bring the neutrinos to equilibrium at temperatures greater than about $1 \, MeV$. Indeed, it is possible that the neutrinos were, if Dirac, degenerate in the early universe, a situation impossible for Majorana neutrinos in equilibrium. Shapiro et al. have pointed out difficulties in right-handed massive Dirac neutrinos coming to thermal equilibrium in the early universe with only Standard Model interactions, indicating again, that the approach to equilibrium and the approximations are dynamical questions.

More exotic than neutrinos, but of great current interest, are supersymmetric theories, which hypothesize a massive Majorana partner for every observed boson. In particular, there should be neutral particles such as partners to the photon, which are fermions and which are their own antiparticles. These particles, in equilibrium, must obey the Majorana equation of state as given above. At low energies they must be quite massive to have avoided production and detection at accelerator experiments, but in the early universe before the symmetry breaking that gave them mass they were presumably massless Majorana fermions.

7 Zero Chemical Potential for Dirac Fermions with Lepton Number Violation

We note in passing that it is not only the possibility that neutrinos are Majorana that could force their chemical potential to be zero and generate a Majorana-like equation of state. In fact, the presence of any interactions violating lepton number conservation, such as those generically present in grand-unified theories and, indeed, in the Standard Model itself, are sufficient, if one is concerned about local equilibrium distributions. At some
level, given the apparent matter-antimatter asymmetry in the universe, it seems likely that some lepton (and indeed baryon) number violation took place in the early universe.

8 Conclusions

We have pointed out that the thermodynamic equations of state for Majorana and Dirac particles are quite different, even in the massless limit. This difference is essentially due to the fact that Majorana fermions are their own antiparticles and there is no conserved fermion number, no corresponding chemical potential, and no possibility of forming a degenerate fermion gas in local equilibrium for the Majorana case. We have noted that there is a subtlety involved, since the electro-weak approach to equilibrium can be quite slow if the interactions between fermions of the same helicities are sufficiently small, and we have discussed the case of Dirac and Majorana neutrinos in some detail. Possible applications have been explored.

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