A weak type estimate for regular fractional sparse operators

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Abstract

In this note the weak type estimates for fractional integrals are studied. More precisely, we adapt the arguments of Domingo-Salazar, Lacey, and Rey to obtain improvements for the end point weak type estimates for regular fractional sparse operators.

1 Introduction

Let \( M \) denote the Hardy–Littlewood maximal function. The Muckenhoupt–Wheeden Conjecture stems from vector-valued estimates for the maximal function. In 1971, C. Fefferman and E. Stein [4] showed that

\[
\| Mf \|_{L^{1,\infty}(w)} \lesssim \int_{\mathbb{R}^n} |f(x)Mw(x)| \, dx,
\]

or equivalently

\[
L^1(Mw) \xrightarrow{M} L^{1,\infty}(\omega),
\]

for arbitrary weight \( w \). Throughout this note a weight \( w \) is a nonnegative locally integrable function. The above relation gave rise to the following natural question, formulated by Muckenhoupt and Wheeden in 70’s: Suppose that \( T \) is a Calderón–Zygmund singular integral operator, can we have the same mapping property if we replace \( M \) in (1.1) by \( T \), that is

\[
L^1(Mw) \xrightarrow{T} L^{1,\infty}(w).
\]

This problem, known as the Muckenhoupt–Wheeden Conjecture, remained puzzled for quite some time, and was believed to be true. In addition, during the development of this open problem, C. Pérez [7] verified that

\[
L^1(M_{L(logL)}w) \xrightarrow{T} L^{1,\infty}(w).
\]

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In particular, if $M^2$ denotes the second iteration of $M$, then

$$L^1(M^2 \omega) \xrightarrow{T} L^{1, \infty}(\omega).$$

We remark here that $M_{L(log L)^s}$ is a smaller object than $M^2$ pointwise. Although there are so many remarkable progresses on this problem, no one can achieve the origin estimate. The breakthrough that surprised people is that the Muckenhoupt–Wheeden Conjecture was finally shown to be false in a series of works of M. Reguera and C. Thiele [8], [9]. The question we are concerned in this note is based on the point of view of (1.3) for fractional integral operators $I_\alpha$. It was C. Pérez [2] who proved that

$$L^1(M_\alpha(M_{L(log L)^s} w)) \xrightarrow{I_\alpha} L^{1, \infty}(w), \forall \delta \in (0, 1),$$

(1.4)

for general weight $w$. In the manner of logarithm scale estimate, Domingo-Salazar, Lacey, and Rey [3] obtained a better estimate. They imposed an assumption to the Young function $\phi$ so that for Calderón–Zygmund operator $T$, any weight $w$ on $\mathbb{R}^d$, it holds that

$$\sup_{\lambda > 0} \lambda w (T^* f > \lambda) \lesssim c_{\phi} \int_{\mathbb{R}^d} |f(x)| M_{\phi(L)} w(x) dx.$$  

The proofs in their work can be extended to other dyadic operators which bring us to have the following observation.

To begin with, let $\varphi : [0, \infty) \to [0, \infty)$ be a Young function, that is, a convex, increasing function such that $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. From these properties, on can deduce that its inverse $\varphi^{-1}$ exists on $(0, \infty)$. Moreover, given a Young function, we can define its complementary function $\psi$ by

$$\psi(s) = \sup_{t > 0} \{st - \varphi(t)\}.$$  

We will assume that $\lim_{t \to \infty} \varphi(t)/t = \infty$ to ensure that $\psi$ is finite valued. Under these conditions, $\psi$ is also a Young function and it is associated with the dual space of $\varphi(L)$.

We now state our main result.

**Theorem 1.1.** Let $\phi$ be any Young function satisfying

$$c_{\phi} := \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^k)} < \infty,$$

where $\psi$ is the complementary function of $\phi$. Then, for any $N$-regular sparse family of cubes $S$ and any weight $w$ on $\mathbb{R}^d$

$$\|A_{\alpha, \nu} f\|_{L^{1, \infty}(w)} \lesssim c_{\phi} \|f\|_{L^1(M_\alpha(M_{\phi}w))}, \forall \nu \geq 1.$$  

(1.5)

It is worth to mention that the original open problem regarding (1.4) which was formulated in the work of C. Pérez [2] is that whether the following is true:

$$L^1(M_\alpha(M w)) \xrightarrow{I_\alpha} L^{1, \infty}(w), \forall \delta \in (0, 1).$$  

(1.6)
It has been proved in [2] that
\[ L^1(M_\alpha(M_{L\log L})^\delta w)) \xrightarrow{I_\alpha} L^{1,\infty}(w), \forall \delta \in (0, 1). \]
However,
\[ L^1(M_\alpha w) \xrightarrow{I_\alpha} L^{1,\infty}(w), \]
for general weight \( w \). Observe that
\[ M_\alpha(M_{L\log L})^\delta w) \geq M_\alpha(M_{L})^\delta w) = M_\alpha(Mw). \]
In summary, for the Calderón–Zygmund operator \( T \) and the fractional integral operator \( I_\alpha \), what we have known is that
\[ \left\{ \begin{array}{l}
L^1(M_\alpha(M_{L\log L})^\delta w)) \xrightarrow{I_\alpha} L^{1,\infty}(w), \\
L^1(M_{L\log L})^\delta w) \xrightarrow{T} L^{1,\infty}(w)
\end{array} \right. \]
and
\[ L^1(Mw) \xrightarrow{T} L^{1,\infty}(w). \]
Our main result states that we can have smaller logarithm scales for any \( N \)-regular fractional sparse operators.

2 Dyadic grid and sparse operators

Definition 2.1 (Dyadic Grid).
A dyadic grid, denoted \( \mathcal{D} \), is a collection of cubes in \( \mathbb{R}^d \) with the following properties:
- for any \( Q \in \mathcal{D} \), there is \( k \in \mathbb{Z} \) such that \( |Q| = 2^{kd} \);
- if \( Q, P \in \mathcal{D} \), then \( Q \cap P = Q, P, \) or \( \emptyset \);
- for each \( k \in \mathbb{Z} \), the family \( \mathcal{D}_k := \{ Q \in \mathcal{D} : |Q| = 2^{kd} \} \) forms a partition of \( \mathbb{R}^d \).

Notation 2.2.
- Given any subset \( \mathcal{S} \subset \mathcal{D} \), \( \text{Ch}_\mathcal{S}(Q) \subset \mathcal{S} \) denote the immediate successors/children of \( Q \in \mathcal{D} \);
- \( \mathbb{P} \subset \mathcal{D} \) denotes a dyadic system with its localized counterpart:
  \[ \mathbb{P}(Q) := \{ \text{Dyadic sub-cubes of } Q \}, \text{ where } Q \in \mathcal{D}. \]

Definition 2.3 (Fractional Sparse Operator [1]).
The fractional sparse operator is given by
\[ \mathcal{A}^{\mathcal{S}}_{\alpha, \nu} f(x) := \left( \sum_{Q \in \mathcal{S}} (f)^{\nu}_{a,Q} \cdot \chi_Q(x) \right)^{\frac{1}{\nu}}, \]
and
\[ (f)_{\alpha, Q} := |Q|^{\frac{\alpha}{d} - 1} \int_Q f(x)dx, \]
where \( \nu > 0, 0 \leq \alpha < d \) and \( \mathcal{S} \) is a sparse collection [3] of dyadic cubes in the sense that
\[ \exists \Lambda_0 \in (0, 1) \text{ s.t. } \bigcup_{P \in \mathbb{P}(Q) \cap \mathcal{S}} |P| \leq \frac{1}{\Lambda_0} |Q|. \]
It has been known that the operator $A_{\alpha,\nu}$ dominate some classes of classical operators. For instance, for $\nu = 1$ and $\nu = 2$ with $\alpha = 0$, these operators dominate large classes of Calderón–Zygmund singular integrals and Littlewood–Paley square functions, respectively. For $\nu = 1$ with $0 < \alpha < d$, $A_{\alpha,\nu}$ dominate the fractional integral operator $I_\alpha$. However in this note we are unable to prove the result for all the fractional sparse operators. We require the dyadic operators to be even more sparse which we use the following definition.

**Definition 2.4 (N-Regular Sparse).**

We say that a family of sparse cubes $S$ is $N$-regular if

$$\exists N \in \mathbb{N} \text{ s.t. } \forall Q \in S \implies \#Ch_S(Q) \leq N.$$  

## 3 Proof of Theorem 1.1

**Proof.** Suppose that $S$ is a family of sparse cubes satisfies (2.1). Recall that

$$L^1(M_\alpha w) \xrightarrow{M_\alpha} L^{1,\infty}(w).$$  

(3.1)

Note that (1.5) is equivalent to show that

$$w \left( \{ x \in \mathbb{R}^d : \Lambda_1 \leq A_{\alpha,\nu}^S f(x) < 2\Lambda_1 \} \right) \lesssim c_\phi \cdot \| f \|_{L^1(M_\alpha(M_\nu w))}.$$  

Define $\varepsilon := \{ \Lambda_1 < A_{\alpha,\nu}^S f \leq 2\Lambda_1 \} - \{ Mf > \Lambda_1^{-1} \}$. It suffices to check that

$$w(\varepsilon) \leq \frac{1}{\Lambda_1} \int_{\varepsilon} A_{\alpha,\nu}^S f(x)w(x)dx \lesssim \int_{\mathbb{R}^d} |f(x)|M_\alpha(M_\phi(L)w)(x)dx.$$  

By getting rid of the set $\{ M_\alpha f > \Lambda_1^{-1} \}$, we can eliminate from $S$ all those cubes $Q$ such that $\langle f \rangle_{\alpha,Q} > \Lambda_1^{-1}$. For $k \in \mathbb{N}$, define the set

$$S_k := \{ Q \in S : \Lambda_1^{-k-1} < \langle f \rangle_{\alpha,Q} \leq \Lambda_1^{-k} \},$$  

and set

$$A_{\alpha,\nu}^{S_k} f(x) := \left( \sum_{Q \in S_k} \langle f \rangle_{\alpha,Q}^{\nu} \cdot \chi_Q(x) \right)^{\frac{1}{\nu}}.$$  

The critical lemma is the following:

**Lemma 3.1.**

*For each $k \in \mathbb{N}$, there is an absolute constant $C$ such that*

$$\int_{\varepsilon} A_{\alpha,\nu}^{S_k} f(x)w(x)dx \leq \frac{2^k}{\Lambda_1^k} w(\varepsilon) + C \cdot \frac{1}{\psi^{-1}(2^k)} \cdot \int_{\mathbb{R}^d} |f(x)|M_\alpha(M_\phi(L)w)(x)dx.$$  

**Proof.** In general, the family of sparse cubes may not have layer structure. However, we can give it some kind of layer structure.

- **Layer Decomposition:**
Write \( S_k \) as the union of \( S_{k,v} \), for \( v = 0, 1, \ldots \), where \( S_{k,0} \) are the maximal elements of \( S_k \), and \( S_{k,v+1} \) are the maximal elements of \( S_k \setminus \bigcup_{l=0}^v S_{k,l} \). We are free to assume that \( S_{k,v} = \emptyset \) if \( v > 4^{k+1} \).

- **Pairwise Disjoint Dense Subset:**

Define that

\[
E_Q := Q \setminus \bigcup_{P \in \text{Ch}_{S_{k,v+1}}(Q)} P, \quad \forall Q \in S_{k,v}.
\]

Note that \( E_Q \) is disjoint \( \subset S_k \). Set \( u := 2^k \). It follows from (1.5) that for each \( v \geq 0 \), and \( Q \in S_{k,v} \).

Then

\[
Q_u := \bigcup_{P \in \mathcal{P}(Q) \cap S_{k,v+u}} \mathcal{P}, \quad \forall Q \in S_{k,v} \implies |Q_u| \leq \Lambda_0^{-u} |Q|.
\]

For each \( Q \in S_{k,v} \), we decompose the set \( \varepsilon \cap Q \) into

\[
\varepsilon \cap Q = \varepsilon \cap \left( \bigcup_{Q_u} \bigcup_{l=0}^{u-1} \bigcup_{Q' \in \mathcal{P}(Q) \cap S_{k,v+l}} E_{Q'} \right),
\]

and hence

\[
\int_A S_{k,v} f(x) w(x) dx \leq \sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in S_{k,v}} \sum_{l=0}^{u-1} \sum_{Q' \in \mathcal{P}(Q) \cap S_{k,v+l}} \langle f \rangle_{\alpha,Q} \cdot w(\varepsilon \cap Q).
\]

We estimate (3.2) by split it into **Layer Part** and **Bottom Part**.

- **Layer Part:**

\[
\sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in S_{k,v}} \sum_{l=0}^{u-1} \sum_{Q' \in \mathcal{P}(Q) \cap S_{k,v+l}} \langle f \rangle_{\alpha,Q} \cdot w(\varepsilon \cap E_{Q'}) \leq \Lambda_1^{-k} u \sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in S_{k,v}} w(\varepsilon \cap E_Q)
\]

\[
\leq \frac{2^k}{\Lambda_1^k} w(\varepsilon).
\]

- **Bottom Part:**

\[
\forall Q \in S_{k,v} \implies \langle f |_{E_Q} \rangle_{\alpha,Q} = \langle f \rangle_{\alpha,Q} - \sum_{P \in \text{Ch}_{S_{k,v+1}}(Q)} \left( \frac{|P|}{|Q|} \right)^{1-\frac{\alpha}{d}} \langle f \rangle_{\alpha,P}
\]

\[
\geq \langle f \rangle_{\alpha,Q} - \Lambda_1^{-k} \left( \sum_{P \in \text{Ch}_{S_{k,v+1}}(Q)} \frac{|P|}{|Q|} \right)^{1-\frac{\alpha}{d}}
\]

\[
\geq \langle f \rangle_{\alpha,Q} - \Lambda_1^{-k} \Lambda_0^{1-\frac{\alpha}{d}} > \left( 1 - \Lambda_1 \Lambda_0^{1-\frac{\alpha}{d}} \right) \langle f \rangle_{\alpha,Q}.
\]
We note that "\( \gtrsim \)" holds if \( S \) is \( N \) regular.

\[
\sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in \mathcal{S}_{k,v}} \langle f \rangle_{\alpha,Q} \cdot w(\varepsilon \cap Q_u) \lesssim \sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in \mathcal{S}_{k,v}} f(y) dy \cdot \langle w(\varepsilon \cap Q_u) \rangle_{\alpha,Q}
\]

**Bottom Part**

\[
\lesssim \sum_{v=0}^{\Lambda_1^{k+2}} \sum_{Q \in \mathcal{S}_{k,v}} \frac{1}{\psi^{-1}(2^{2k})} \cdot \int_{E_Q} f(x) M_\alpha(M_{\phi(L)} w)(x) dx
\]

\[
\leq \frac{1}{\psi^{-1}(2^{2k})} \cdot \int_{\mathbb{R}^d} f(x) M_\alpha(M_{\phi(L)} w)(x) dx.
\]

Thus, the proof of this lemma is complete.

Now, let us finish the main theorem:

\[
w(\varepsilon) \leq \frac{1}{\Lambda_1} \int_\varepsilon A_{\alpha,\nu} f(x) w(x) dx = \frac{1}{\Lambda_1} \sum_{k=1}^{\infty} \int_\varepsilon A_{\alpha,\nu}^k f(x) w(x) dx
\]

\[
\leq \frac{1}{\Lambda_1} \sum_{k=1}^{\infty} \left( \frac{2^k}{\Lambda_1^k} w(\varepsilon) + \frac{C}{\psi^{-1}(2^{2k})} \cdot \int_{\mathbb{R}^d} |f(x)| \cdot M_\alpha(M_{\phi(L)} w)(x) dx \right),
\]

which implies that

\[
w(\varepsilon) \lesssim c_\phi \cdot \int_{\mathbb{R}^d} |f(x)| \cdot M_\alpha(M_{\phi(L)} w)(x) dx, \ \forall r \geq 1,
\]

provided that

\[
\Lambda_1 > 2, \ \frac{1}{\Lambda_1} \cdot \sum_{k=1}^{\infty} \frac{2^k}{\Lambda_1^k} < 1 \text{ and } \Lambda_1 \cdot \Lambda_0^{1-\frac{\alpha}{d}} < 1. \quad (3.3)
\]

Moreover, we are free to assume that \( \Lambda_0 \) is small enough such that \([3,3]\) holds for suitable \( \Lambda_1 \).

**References**

[1] D. Cruz-Uribe and K. Moen, *fractional Muckenhoupt-Wheeden theorem and its consequences*, Integral Equations Operator Theory. **76** (2013), no. 3, 421–446.

[2] M.J. Carro, C. Pérez, F. Soria and J. Soria *Maximal functions and the control of weighted inequalities for the fractional integral operator*, Indiana Univ. Math. J., **54** (2005), no. 3, 627–644.

[3] C. Domingo-Salazar, M. Lacey and G. Rey, *Borderline weak-type estimates for singular integrals and square functions*, Bulletin London Math. Soc., **48** (2015), no. 1, 63–73.

[4] C. Fefferman and E.M. Stein, *Some maximal inequalities*, Amer. J. Math., **93** (1971), no. 1, 107–115.

[5] A.K. Lerner and F. Nazarov, *Intuitive dyadic calculus: the basics*, Expo. Math., **37** (2019), no. 3, 225–265.
[6] A.K. Lerner, *A simple proof of the $A_2$ conjecture*, Int. Math. Res. Not. IMRN. **14** (2013), 3159–3170.

[7] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc., **49** (1994), no. 2, 296–308.

[8] M.C. Reguera, *On Muckenhoupt–Wheeden conjecture*, Adv. Math. **227** (2011), no. 4, 1436–1450.

[9] M.C. Reguera and C. Thiele, *The Hilbert transform does not map $L^1(Mw)$ to $L^{1,\infty}(w)$*, Math. Res. Lett. **19** (2012), no. 1, 1–7,