SHRINKING TARGET PROBLEM FOR RANDOM ITERATED FUNCTION SYSTEMS

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ABSTRACT. We describe the shrinking target problem for random iterated function systems which semi-conjugate to a random subshifts of finite type. We get the Hausdorff dimension of the set based on shrinking target problems with given targets. The main idea is an extension of ubiquity theorem which plays an important role to get the lower bound of the dimension. Our method can be used to deal with the sets with respect to an more general targets and the sets based on the quantitative Poincaré recurrence properties.

1. INTRODUCTION

This paper investigate the shrinking target problem for random iterated function systems. On a base probability space \((\Omega, \mathcal{F}, P)\) with a measure preserving ergodic transformation \(\sigma\) on it, we can define a family of random attractors generated by random iterated function systems. We will consider the Hausdorff dimension of the points whose orbit can be well approximated.

Shrinking target problem is considering such a set:
\[
\{x \in X : T^n x \in A_n, \text{ for infinitely many } n \in \mathbb{N}\},
\]
where \(\{A_n\}_{n \in \mathbb{N}}\) is a given (decreasing in some sense) sequence of subsets of the giving compact space \(X\) and \(T : X \to X\) is a continuous (normally expanding) map. It was proposed in [13, 14] where they consider an expanding rational map on Riemann sphere acting on its Julia sets. There are two field to study such a set: in the sense of measure and dimension. See the results of measures in [5, 7, 3, 20, 11, 9, 17] and the results of dimension in [25, 28, 27, 23, 6, 18, 24, 22] for instances.

Shrinking target problem for nonautonomous dynamical systems corresponding to Cantor series expansions has been considered in [10] (in the sense of dimension) and [26] (in the sense of measure). There is a strong relationship between nonautonomous dynamical systems and random dynamical systems. In this paper, we study the dimension result of such problem for random iterated function systems, which can be seen as an extension of [10]. We derive the formula for Hausdorff dimension of the considering set after some reasonable assumptions.

The outline of the paper is as follows: Section 2 develops background about random iterated function systems and presents our main results, namely theorem 2.1 and corollary 1 and 2. Section 3 provides the basic properties that will be used in the proof of our results. Section 4 will give the upper bound while section 5 will give the lower bound by building a theorem of an extension of ubiquity theorem which can go back to [8, 1, 4, 2] to deal

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with limsup set. Section 6 will return to explain the corollaries which can be seen as an extension of theorems 2.1.

## 2. Setting and Main Result

### 2.1. Random Subshift and the Pressure

Denote by $\Sigma$ the symbolic space $(\mathbb{Z}^+)^\mathbb{N}$, and endow it with the standard ultrametric distance: for any $u = u_0u_1 \cdots$ and $v = v_0v_1 \cdots$ in $\Sigma$, $d(u, v) = e^{-\inf\{n \in \mathbb{N} : u_n \neq v_n\}}$, with the convention $\inf(\emptyset) = +\infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\sigma$ a $\mathbb{P}$-preserving ergodic map. The product space $\Omega \times \Sigma$ is endowed with the $\sigma$-field $\mathcal{F} \otimes \mathcal{B}(\Sigma)$, where $\mathcal{B}(\Sigma)$ stands for the Borel $\sigma$-field of $\Sigma$.

Let $l$ be a $\mathbb{Z}^+$ valued random variable such that

$$\int \log(l) \, d\mathbb{P} < \infty \text{ and } \mathbb{P}\{\{\omega \in \Omega : l(\omega) \geq 2\}\} > 0.$$

Let $A = \{A(\omega) = (A_{r,s}(\omega)) \in \Omega \}$ be a random transition matrix such that $A(\omega)$ is a $l(\omega) \times l(\sigma^k(\omega))$-matrix with entries 0 or 1. We suppose that the map $\omega \mapsto A_{r,s}(\omega)$ is measurable for all $(r, s) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and each $A(\omega)$ has at least one non-zero entry in each row and each column. Let

$$\Sigma = \{\omega = v_0v_1 \cdots : 1 \leq v_k \leq l(\sigma^k(\omega)) \text{ and } A_{v_k,v_{k+1}}(\sigma^k(\omega)) = 1 \text{ for all } k \in \mathbb{N}\},$$

and $F_{\omega} : \Sigma \to \Sigma_{\sigma\omega}$ be the left shift $(F_{\omega})i = i+1$ for any $v = v_0v_1 \cdots \in \Sigma_{\omega}$. Define $\Sigma_{\Omega} = \{(\omega, \omega) : \omega \in \Omega, \omega \in \Sigma_{\omega}\}$. The space $\Sigma_{\Omega}$ is endowed with the $\sigma$-field obtained as the trace of $\mathcal{F} \otimes \mathcal{B}(\Sigma)$, Define the map $F : \Sigma_{\Omega} \to \Sigma_{\Omega}$ as $F((\omega, \omega)) = (\sigma\omega, F_{\omega}\omega)$. The corresponding family $F = \{F_{\omega} : \omega \in \Omega\}$ is called a random subshift. We assume that this random subshift is topologically mixing, i.e. there exists a $\mathbb{Z}^+$-valued random variable $M$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-almost every (a.e.) $\omega$, $A(\omega)A(\sigma\omega) \cdots A(\sigma^{M(\omega)-1}(\omega)$ is positive (i.e. each entry of the matrix is positive).

For each $n \in \mathbb{N}$, define $\Sigma_{\omega,n}$ as the set of words $v = v_0v_1 \cdots v_{n-1}$ of length $n$, i.e. such that $1 \leq v_k \leq l(\sigma^k(\omega))$ for all $0 \leq k \leq n-1$ and $A_{v_k,v_{k+1}}(\sigma^k(\omega)) = 1$ for all $0 \leq k \leq n-2$. Define $\Sigma_{\omega,n} = \cup_{n \in \mathbb{N}} \Sigma_{\omega,n}$. For $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, we write $\|v\|$ for the length $n$ of $v$, and we define the cylinder $[v]_\omega$ as $[v]_\omega := \{w \in \Sigma_{\omega} : w_i = v_i \text{ for } i = 0, \ldots, n-1\}$ and $v^* = v_0v_1 \cdots v_{n-2} \in \Sigma_{\omega,n-1}$.

For any $s \in \Sigma_{\omega,1}$, $p \geq M(\omega)$ and $s' \in \Sigma_{\sigma^p+1,\omega,1}$, there is at least one word $v(s,s') \in \Sigma_{\omega,p+1}$ such that $sv(s,s')s' \in \Sigma_{\omega,p+1}$. We fix such a $v(s,s')$ and denote the word $sv(s,s')s'$ by $s \ast s'$. Similarly, for any $w = w_0w_1 \cdots w_{n-1} \in \Sigma_{\omega,n}$ and $w' = w'_0w'_1 \cdots w'_{m-1} \in \Sigma_{\sigma^{n-p},\omega,m}$ with $n, p, m \in \mathbb{N}$ and $p \geq M(\sigma^{n-1}(\omega))$, we fix $v(w_{n-1},w'_0) \in \Sigma_{\omega,n+p}$ (a word depending on $w_{n-1}$ and $w'_0$ only) so that $w \ast w' := w_0w_1 \cdots w_{n-1}v(w_{n-1},w'_0)w'_0w'_1 \cdots w'_{m-1} \in \Sigma_{\omega,n+m+p-1}$.

We say that a measurable function $\Upsilon : \Sigma_{\Omega} \to \mathbb{R}$ is in $L^1(\Omega, C(\Sigma))$ if

$$C_{\Upsilon} := \int_{\Omega} \|\Upsilon(\omega)\|_{\infty} \, d\mathbb{P}(\omega) < \infty,$$

where $\|\Upsilon(\omega)\|_{\infty} := \sup_{\omega \in \Sigma} |\Upsilon(\omega, \omega)|$.

(2) for any $n \in \mathbb{N}$, there exists a random variable $\text{var}_n(\Upsilon, \cdot)$ such that

$$\sup \{|\Upsilon(\omega, \omega) - \Upsilon(\omega, w)| : v_i = w_i, \forall i < n\} \leq \text{var}_n(\Upsilon, \omega).$$

Furthermore, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\text{var}_n(\Upsilon, \omega) \to 0$ as $n \to \infty$. 


Now, if $\Upsilon \in L^1(\Omega, C(\Sigma))$, due to Kindsman’s subadditive ergodic theorem,

$$P(\Upsilon) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{v \in \Sigma_{\omega,n}} \sup \exp (S_n \Upsilon(\omega, v))$$

exists for $\mathbb{P}$-a.e. $\omega$ and does not depend on $\omega$, where $S_n \Upsilon(\omega, v) = \sum_{i=0}^{n-1} \Upsilon(F^i(\omega, v))$. This limit is called topological pressure of $\Upsilon$.

### 2.2. A model of random iterated function systems

We present the model of random iterated function systems. Fixed any nonempty compact set $U \subset \mathbb{R}^d$ such that $U = \text{Int}(\overline{U})$.

We assume that for any $s \in \mathbb{N}$, for any $\omega \in \Omega$, there exists a homeomorphism map $g^s_\omega$ from $U_{\omega,s} = U$ to $g^s_\omega(U) \subset U$. Denote $g^s_\omega = g^s_{\omega,n} = g^s_{\sigma_{\omega,n-1},\omega} \circ \cdots \circ g^s_{\sigma_{\omega,1},\omega}$ for $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$.

We can define the following sets:

$$U^v_\omega = g^v_\omega(U), \quad \forall v \in \Sigma_{\omega,n},$$

$$X_\omega = \bigcap_{n \geq 1} \bigcup_{v \in \Sigma_{\omega,n}} U^v_\omega \quad \text{and} \quad X^v_\omega = X_\omega \cap U^v_\omega,$$

$$X_\Omega = \{ (\omega, x) : \omega \in \Omega, x \in X_\omega \},$$

Denote $T^s_\omega = (g^s_\omega)^{-1} : U_{\omega,s} \to U_{\omega,s} = U$ as its inverse function and $T^v_\omega = T^{v_{n-1}}_\omega \circ T^{v_{n-2}}_\omega \circ \cdots \circ T^{v_0}_\omega$ for $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, we need to point out that $g^v_\omega$ is a map from $U$ to $U^v_\omega$ and $T^v_\omega : U^v_\omega \to U$ is the inverse map of $g^v_\omega$.

It is easy to check that $g^{v_0v_1 \cdots v_{n-1}} \omega(x_{\omega,s}) = X^v_\omega$ for any $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n+1}$ with $n \in \mathbb{N}$.

We say that a function $\tilde{\psi} : \mathcal{U}\Omega = \{(\omega, s, x) : \omega \in \Omega, 1 \leq s \leq l(\omega), x \in U \} \to \mathbb{R}$ is in $L^1(\Omega, C(U))$ if

1. for any $1 \leq s \leq l(\omega), x \in U$, the map $\omega \mapsto \tilde{\psi}(\omega, s, x)$ is measurable,
2. $\int_{\Omega} \| \tilde{\psi}(\omega) \|_\infty d\mathbb{P}(\omega) < \infty$, where $\| \tilde{\psi}(\omega) \|_\infty := \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U} |\tilde{\psi}(\omega, s, x)|$,
3. for any $\varepsilon > 0$, there exists a random variable $\text{var}(\tilde{\psi}, \cdot, \varepsilon)$ such that

$$\sup_{1 \leq s \leq l(\omega)} \sup_{x, y \in U, |x - y| \leq \varepsilon} |\tilde{\psi}(\omega, s, x) - \tilde{\psi}(\omega, s, y)| \leq \text{var}(\tilde{\psi}, \omega, \varepsilon).$$

Furthermore, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\text{var}(\tilde{\psi}, \omega, \varepsilon) \to 0$ as $\varepsilon \to 0$.

The following assumptions will be needed throughout the paper.

1. $\text{Int}(U^v_\omega) \cap \text{Int}(U^{v'}_\omega) = \emptyset$ for all $\omega \in \Omega$ and for all $1 \leq s_1, s_2 \leq l(\omega)$ with $s_1 \neq s_2$.
2. $\sup_{1 \leq s \leq l(\omega)} \sup_{x \in U} \tilde{\psi}(\omega, s, x) d\mathbb{P}(\omega) > 0$

and

$$\exp(-\text{var}(\psi, \omega, ||x - y||)) \leq \frac{||g^s_\omega(x) - g^s_\omega(y)||}{||x - y|| \exp(\psi(\omega, s, x))} \leq \text{exp}(\text{var}(\psi, \omega, ||x - y||)).$$

Under the assumptions, fixed $x_0 \in U$, there is $\mathbb{P}$-almost surely a natural projection $\pi_\omega : \Sigma_{\omega} \to X_\omega$ defined as

$$\pi_\omega(y) = \lim_{n \to \infty} g^{v_0}_{\sigma_{\omega,n}} \circ g^{v_1}_{\sigma_{\omega,n-1}} \circ \cdots \circ g^{v_{n-1}}_{\sigma_{\omega,1}}(x_0).$$

Noticing that this mapping may not be injective, but it should be surjective.
2.3. Shrinking target problem and result. Let $\phi \in \mathbb{L}^1(\Omega, \mathcal{C}(U))$ such that
\[
c_\phi := -\int_{\Omega} \sup_{1 \leq s \leq (\omega) x \in U} \phi(\omega, s, x) d\mathbb{P}(\omega) > 0.
\]
For any $\omega \in \Omega$, fix any point $z_\omega \in X_\omega$. Now we consider the points in $X_\omega$ whose orbits are well approximated by the sequence $\{z_{\sigma^n \omega}\}_{n \in \mathbb{N}}$ with rate corresponding to $\phi$, that is
\[
W(\phi, \omega) = \left\{ x \in X_\omega : x \in U_\omega^v \text{ and } \|T_\omega^v x - z_{\sigma^n | \omega}\| \leq \exp(S_v(\phi(x, x))) \right\},
\]
where for any $n \in \mathbb{N}$, for any $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, for $x \in U_\omega^v$, $S_v(\phi(x, x)) := \phi(\omega, x) + \sum_{i=1}^{n-1} \phi(\sigma \omega, T_\omega^{v_{i-1}} x)$. The main aim of the shrinking target problem is to consider the dimension and the measure of the set $W(\phi, \omega)$. In this paper we will fix our effort on the Hausdorff dimension of it.

Define
\[
\Psi(\omega, \underline{v}) = \psi(\omega, v_0, \pi(\underline{v})) \text{ and } \Phi(\omega, \underline{v}) = \phi(\omega, v_0, \pi(\underline{v})),
\]
for any $\omega \in \Omega$ and any $\underline{v} = v_0 v_1 \cdots \in \Sigma_\omega$. By construction, $\Psi$ and $\Phi$ are in $\mathbb{L}^1(\Omega, C(\Sigma))$.

**Theorem 2.1.** Under our assumption, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the Hausdorff dimension of the set $W(\phi, \omega)$ is the unique root $q_0$ of the equation $P(q(\Psi + \Phi)) = 0$, where the pressure function $P$ is defined in (1).

In theorem 2.1, for a fixed $\omega \in \Omega$, the targets are $\{z_{\sigma^n \omega}\}_{n \in \mathbb{N}}$ which are just depends on $n$. Using a similar method of the proof of the theorem, we can deal with general targets $\{z_{\sigma^n \omega}\}_{n \in \mathbb{N}}$ which are depends on $v$. It is described in the following corollary which will be explained in section 6.

**Corollary 1.** For any $\omega \in \Omega$, for any $n \in \mathbb{Z}^+$, for any $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, fix $z_{\sigma^n \omega} \in X_{\sigma^n \omega}$. We make the following assumption: there exists a $\mathbb{Z}^+$-valued r.v. $M'$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that
\[
\mathbb{P}\left( \left\{ \omega \in \Omega : k \leq M'(\sigma^{n-1} \omega) \text{ and } v' \in \Sigma_{\omega,n,k} \right\} \right) = 1
\]
(5)

Now we consider the following set
\[
W'(\phi, \omega) = \left\{ x \in X_\omega : x \in U_\omega^v \text{ and } \|T_\omega^v x - z_{\sigma^n | \omega}\| \leq \exp(S_v(\phi(x, x))) \right\},
\]
for infinitely many $v \in \Sigma_{\omega,v}$. The Hausdorff dimension of $W'(\phi, \omega)$ is equal to the unique root $q_0$ of the equation $P(q(\Psi + \Phi)) = 0$.

Here, the equation (5) means that the target can be hit. In the fullshifts situation for each point $z_{\sigma^n \omega} \in X_{\sigma^n \omega}$, it can be hit for some $x \in X_\omega^v$ for each $v \in \Sigma_{\omega,n}$ which means $T_\omega^v x = z_{\sigma^n \omega}$ since $T_\omega^v = X_{\omega}$. We make a more general condition that we do not need to wait too long to get a point that can be hit (that is $g_{\omega}^{\sigma v'}(z_{\sigma^n + k \omega}^{v'}) \in X_{\omega}^{v'}$). In fact, here the random variable $M'$ play the same role as $M$. Instead of the equation (5) in the assumption, we can also use the following condition: For $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of positive numbers decreasing to 0 such that for all $n \in \mathbb{Z}^+$, for all $v \in \Sigma_{\omega,n}$, there exist $k \in \mathbb{N}, k \leq \gamma_n$ and $v' \in \Sigma_{\omega,n,k}$ satisfying $v v' \in \Sigma_{\omega,n+k}$ and $g_{\omega}^{\sigma v'}(z_{\sigma^n + k \omega}^{v'}) \in X_{\omega}^{v'}$. 
Corollary 2. We make the following assumption: there exists a $\mathbb{Z}^+$-valued r.v. $M^n$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that
\[
P\left( \omega \in \Omega: \begin{array}{c}
\forall n \in \mathbb{Z}^+, \forall v \in \Sigma_{\omega,n}, \text{ there exist } k \in \mathbb{N}, \\
v \in M^n(\sigma^{n-1}\omega) \text{ and } v' \in \Sigma_{\sigma^n\omega,k} \text{ such that:} \\
v' \in \Sigma_{\omega,n+k} \text{ and } T^{v'}_\omega x_{\omega} = x_{\omega} \in \Sigma^{v'} \right) = 1. \tag{6}
\]

Consider the following set
\[
W^n(\phi, \omega) = \left\{ x \in X_\omega: \begin{array}{c}
x \in U_{\omega}^v \text{ and } \|T^n_x x - x\| \leq \exp(S_{\|v\|\phi(\omega, x)}) \text{ for infinitely many } v \in \Sigma_{\omega,s} \\
\text{such that:} \\
\{s, s_1, s_2, \ldots, s_l(\omega)\} \subseteq \text{int}(\Sigma_{\omega,ns}), \\
\text{there exist } n \in \mathbb{N} \text{ such that} \\
\|S_n(\omega, x)\| \geq 0,
\end{array} \right\}
\]

The Hausdorff dimension of $W^n(\phi, \omega)$ is equal to the unique root $q_0$ of the equation $P(q(\psi + \Phi)) = 0$.

$W^n(\phi, \omega)$ can be seen as the points $x$ whose orbits come back closer and closer to $x$ at a rate $\phi$ possibly depending on $x$ which is also called "quantitative Poincaré recurrence property". Equation (6) in the assumption of corollary 2 is not surprising. First, for $\mathbb{P}$-a.e. $\omega \in \Omega$, for any $v \in \Sigma_{\omega,n}$ with $n$ large enough, the fixed $x^n_\omega$ of $T^n_\omega$ (also $g^n_\omega$) exists. In fact we just need to notice (2) and (3), using Birkhoff ergodic theorem, we know that the map $g^n_\omega$ is a contraction, the contraction mapping principle ensures the existence of the fixed point. Second, we want to deal with $\|T^n_\omega x - x\| \leq \exp(S_{\|v\|\phi(\omega, x)})$ which can be seen as they are very near to the fixed point of $T^n_\omega$. Third, since we want to deal with the fixed point of $T^n_\omega: X^n_\omega \to X^n_{\sigma^{n}\omega}$, it is reasonable to assume the fixed point is in $X^n_\omega \cap X^n_{\sigma^{n}\omega}$. At last, in the deterministic situation with full coding, it is easy to see that the fixed point is inside the attractor.

Remark 1.

- In fact if we have the separation condition, that is $U^n_{\omega_{s_1}} \cap U^n_{\omega_{s_2}} = \emptyset$ for any $1 \leq s_1, s_2 \leq l(\omega)$ with $s_1 \neq s_2$, we can define $T_\omega: \cup_{1 \leq s \leq l(\omega)} U^n_{\omega_{s}} \to U$ such that $T_\omega|U^n_{\omega_{s}} = T^n_{\omega_{s}}$. In our setting we know their inners are pairwise disjoint, but they may intersect on their boundary where there is a trouble to define $T_\omega$.
- In fact, both (2) and (4) can be replaced as: there exists $n \in \mathbb{N}$ such that
\[
-\int_{\Omega} \sup_{v \in \Sigma_{\omega,n}} \sup_{x \in U^n_{\omega}} S_n(\omega, x) d\mathbb{P}(\omega) > 0,
\]
where $\tilde{\psi} \in \{\psi, \phi\}$

Let us make some comments on our setting and result:

- In many works on the shrinking target problems (for example [13, 14, 25]), there will be conformal situation which fails in our case. This takes a big trouble to control measures of balls. Since we are dealing with higher dimension, it is not easy to use the trick in [29, 30] where we give a good control of neighbor cylinders. Without the method in [28, 23], we overcome such a trouble (see the proof of lemma 5.2) by using the cylinders with radius about $r$ to cover a ball $B(x, r)$, and noticing the number of them is not so big since they are inside $B(x, 2r)$.
- As was noted, the maps $g^n_\omega$ with $1 \leq s \leq l(\omega)$ may not be contraction, but they are contraction in average which means they are contractions if we look for a long time.
- We now deal with subshift situation which have not been considered before.
- As in [25, 28, 27, 23, 6, 18, 24, 22], they are dealing the deterministic situation. In our case, we deal with the random iterated function systems with respect to
Lemma 3.1. \cite{29} to prove the following lemma. We use the same method in \cite[proposition 3]{29}, we can proof:

Bowen-Ruelle formula holds, i.e.

\[ \text{Let } u \text{ be any extension and } C \text{ be an extension and } \lambda(\omega) = \lambda(\omega) > 0 \text{ and a probability measure } \tilde{\mu}_\omega = \tilde{\mu}_\omega^\Psi \text{ on } \Sigma_\omega \text{ such that } (C^\Psi)^* \tilde{\mu}_{\sigma,\omega} = \lambda(\omega) \tilde{\mu}_\omega. \]

We call the family \( \{ \tilde{\mu}_\omega : \omega \in \Omega \} \) a random weak Gibbs measure on \( \{ \Sigma_\omega : \omega \in \Omega \} \) associated with \( \Psi \). If we want, we can use \( \{ \tilde{\mu}_\omega^\Psi : \omega \in \Omega \} \) to declare that it is with respect to \( \Psi \in L^1(\Omega, C(\Sigma)) \).

Let \( u = \{ u_{n,\omega} \} \) be an extension and \( \Psi \in L^1(\Omega, C(\Sigma)) \). Then for \( (n, \omega) \in \mathbb{N} \times \Omega \)

\[ Z_{u,n}(\Psi, \omega) := \sum_{v \in \Sigma_{\omega,n}} \exp \left( S_n \Psi(\omega, u_{n,\omega}(v)) \right) \]

is called \( n \)-th partition function of \( \Psi \) in \( \omega \) with respect to \( u \).

Due to the assumption \( \log(l) \in L^1(\Omega, \mathbb{P}) \), using the same method as in \cite{12, 16}, it is easy to prove the following lemma.

Lemma 3.1. \cite{29} Let \( u \) be any extension and \( \Phi \in L^1(\Omega, C(\Sigma)) \).

Then \( \lim_{n \to \infty} \frac{1}{n} \log Z_{u,n}(\Phi, \omega) = P(\Phi) \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). This limit is independent of \( u \).

Using a standard approach, it can be easily proven that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), the Bowen-Ruelle formula holds, i.e. \( \dim_H X_\omega = t_0 \) where \( t_0 \) is the unique root of the equation \( P(t(\Psi)) = 0 \).

Noting the assumptions in subsection 2.2, especially \( \psi \in L^1(\Omega, C(U)) \) and (3), using the same method in \cite[proposition 3]{29}, we can proof:
Proposition 2. For \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there are sequences \( (\epsilon(\psi, \omega, n))_{n \in \mathbb{N}} \) (also denote as \( (\epsilon(\Psi, \omega, n))_{n \in \mathbb{N}} \) and \( (\epsilon(\Upsilon, \omega, n))_{n \in \mathbb{N}} \) of positive numbers, decreasing to 0 as \( n \to +\infty \), such that for all \( n \in \mathbb{N} \), for all \( v = v_0v_1 \ldots v_n \in \Sigma_{\omega, n} \), we have:

1. For all \( z \in U^v_\omega \),

\[
|U^v_\omega| \leq \exp(S_n\psi(\omega, z) + n\epsilon(\psi, \omega, n)),
\]

furthermore, there exists a ball \( B \) with radius \( \exp(S_n\psi(\omega, z) - n\epsilon(\psi, \omega, n)) \) such that \( B \subset U^v_\omega \).

Hence for all \( v \in [v]_\omega \),

\[
|U^v_\omega| \leq \exp(S_n\psi(\omega, v) + n\epsilon(\psi, \omega, n)),
\]

and there exists a ball \( B \) with radius \( \exp(S_n\psi(\omega, v) - n\epsilon(\psi, \omega, n)) \) such that \( B \subset U^v_\omega \).

2. The measure \( \tilde{\mu}_\omega^T \) (see details in proposition 1) exists, and for all \( v \in [v]_\omega \), we have

\[
\exp(-n\epsilon(\Upsilon, \omega, n)) \leq \frac{\tilde{\mu}_\omega^T([v]_\omega)}{\exp(S_n\Upsilon(\omega, v) - nP(\Upsilon))} \leq \exp(n\epsilon(\Upsilon, \omega, n)).
\]

Remark 2.  
- By Maker’s ergodic theorem from [19], we can get that for \( \Upsilon \in \mathbb{L}^1(\Omega, C(\Sigma)) \), for \( \mathbb{P} \)-a.e. \( \omega \in \omega \) we have \( \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \text{var}_{\omega, n}(\Upsilon, \sigma^i_\omega)}{n} = 0 \). Without any difficulty, we can ask

\[
\sum_{i=0}^{n-1} \text{var}_{\omega, n-1}(\Upsilon, \sigma^i_\omega) \leq \epsilon(\Upsilon, \omega, n)
\]

in proposition 2

- Using item 3 in the definition of \( \mathbb{L}^1(\Omega, \bar{C}(U)) \), also by Maker’s ergodic theorem, we can ask that for any \( n \in \mathbb{Z}^+ \), for any \( v \in \Sigma_{\omega, n} \) and any \( x, y \in U^v_\omega \), we have

\[
|S_n\phi(\omega, x) - S_n\phi(\omega, y)| \leq n\epsilon(\Phi, \omega, n)
\]

for any \( x, y \in U^v_\omega \).

- From item 1 of proposition 2, we can easily get that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there is a sequence \( (\epsilon'(\psi, \omega, n))_{n \in \mathbb{N}} \) of positive numbers, decreasing to 0 as \( n \to +\infty \), such that for all \( n \in \mathbb{N} \), for any \( v \in \Sigma_{\omega, n} \), \( U^v_\omega \) contains a ball with radius \( |U^v_\omega|^{1+\epsilon'(\psi, \omega, n)} \). We do not distinguish between \( \epsilon'(\Psi, \omega, n) \) and \( \epsilon(\Psi, \omega, n) \) for the convenient of writing.

We now start a series of estimations that will be useful later. These estimations can be seen from [29, 30]. At first, choose \( \tilde{M} \in \mathbb{N} \) large enough such that

\[
\mathbb{P}(\{\omega \in \Omega : M(\omega) \leq \tilde{M}\}) > 7/8.
\]

Second, Birkhoff ergodic theorem and Egorov’s theorem yields there exist \( C > 0 \) and a measurable subspace \( \overline{\Omega} \subset \{\omega \in \Omega : M(\omega) \leq \tilde{M}\} \) such that \( \mathbb{P}(\overline{\Omega}) > 6/7 \) and for all \( \omega \in \overline{\Omega} \), \( n \in \mathbb{N} \) and for \( \Upsilon \in \{\Phi, \Psi\} \), one has

\[
\max \left( \frac{1}{n} S_n \|\Upsilon(\omega)\|_\infty, \frac{1}{n} S_n \|\Upsilon(\sigma^{-n+1}\omega)\|_\infty \right) \leq C.
\]

Furthermore, there exist \( c > 0 \) small enough and \( N \in \mathbb{N} \) with \( N > \tilde{M} \) large enough such that for any \( n \geq N \) one has

\[
S_n\Psi(\omega, v) \leq -cn
\]
Third, we say that a function $\Upsilon : \Sigma_\Omega \to \mathbb{R}$ is a random Hölder continuous potential if

1. $\Upsilon \in L^1(\Omega, C(\Sigma))$;
2. $\exists \kappa \in (0, 1]: \text{var}_n \Upsilon(\omega) \leq K_\kappa(\omega)e^{-\kappa n}$, with $\int \log K_\kappa(\omega)d\mathbb{P}(\omega) < \infty$.

For any $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers, let $\{\Psi_i\}_{i \in \mathbb{N}}, \{\Phi_i\}_{i \in \mathbb{N}}$ be two sequences of random Hölder potentials such that

$$\int \Omega \max\left\{\|\Psi - \Psi_i(\omega)\|_\infty, \|\Phi - \Phi_i(\omega)\|_\infty\right\} d\mathbb{P}(\omega) < \varepsilon_i^3. \quad (11)$$

For each $i \in \mathbb{N}$, using Birkhoff ergodic theorem and Egorov’s theorem, there exists a measurable set $\Omega(i) \subset \Omega$ and $N_i \in \mathbb{N}$ such that

- $\mathbb{P}(\Omega(i)) > 3/4$,
- for any $n \geq N_i$,

$$\left| S_n \|\Psi - \Psi_i(\sigma^N \omega)\|_\infty - n \int \|\Psi - \Psi_i(\omega)\|_\infty d\mathbb{P} \right| \leq n \varepsilon_i^3, \quad (12)$$

and

$$\left| S_n \|\Phi - \Phi_i(\sigma^N \omega)\|_\infty - n \int \|\Phi - \Phi_i(\omega)\|_\infty d\mathbb{P} \right| \leq n \varepsilon_i^3. \quad (13)$$

- by using proposition 2, we can ask that for each $\omega \in \Omega(i)$, the measure $\tilde{\mu}_{\sigma^N \omega}^{i} =: \tilde{\mu}_{\sigma^N \omega}(\Psi_i + \Phi_i)$ is well defined such that for any $n \in \mathbb{N}$, for any $v \in \Sigma_{\sigma^N \omega, n}$

$$\exp(n \varepsilon(q_i(\Psi_i + \Phi_i), \sigma^N \omega, n)) \leq \frac{\tilde{\mu}_\omega^i([v])}{\exp(q_is_n(\Psi_i + \Phi_i)(\sigma^N \omega, v))} \leq \exp(n \varepsilon(q_i(\Psi_i + \Phi_i), \sigma^N \omega, n)),$$

and

$$\varepsilon(q_i(\Psi_i + \Phi_i), \sigma^N \omega, n) \leq \varepsilon_i^3, \text{ for } n \geq N_i.$$

Let

$$F_{i,\beta,n}(\sigma^N \omega, \varepsilon) =: \left\{ v \in \Sigma_{\sigma^N \omega} : \forall v' \in \Sigma_{\sigma^N \omega} \text{ with } |v \wedge v'| \geq n \right\}$$

and

$$E_{i,\beta}(\sigma^N \omega, N, \varepsilon) = \bigcap_{n \geq N} F_{i,\beta,n}(\sigma^N \omega, \varepsilon) \text{ and } E_{i,\beta}(\sigma^N \omega, \varepsilon) = \bigcup_{N \geq 1} E_{i,\beta}(\sigma^N \omega, N, \varepsilon).$$

Using the same method as in [29, lemma 3.15], more easily we can get:

**Proposition 3.** For all $i \in \mathbb{N}$, for any $\varepsilon > 0$, for all $\omega \in \Omega(i)$, the set $E_{i,\alpha_i}(\sigma^N \omega, \varepsilon)$ has full $\tilde{\mu}_{\sigma^N \omega}^{i}$-measure, where $\alpha_i = T_i(q_i)$ and $P(q\Phi_i - T_i(q)\Psi_i) = 0$. Then we can choose $\mathcal{M}_i(\omega)$ large enough such that $\tilde{\mu}_{\sigma^N \omega}(E_{i,\alpha_i}(\sigma^N \omega, \mathcal{M}_i(\omega), \varepsilon)) > 1/2$. Furthermore we can also choose a set $\Omega_i \subset \Omega(i)$ with $\mathbb{P}(\Omega_i) > 1/2$ and $N_i \in \mathbb{N}$ with $\mathcal{M}_i(\omega_i) \leq N_i$ for each $\omega \in \Omega_i$, so that $\tilde{\mu}_{\sigma^N \omega}^{i}(E_{i,\alpha_i}(\sigma^N \omega, N_i, \varepsilon)) > 1/2$. 
Remark 3. For any \( v \in E_{i,\beta}(\sigma^n,\omega,N_i,\varepsilon) \), for any \( n \geq N_i \), \( \forall v' \in \Sigma_{\sigma^n,\omega} \) with \( |v \wedge v'| \geq n \) we have \( \left| S_n\Phi(N,v') - \beta \right| \leq \varepsilon \). The notation \( v \wedge v' \) means the longest common prefix of \( v \) and \( v' \), that is, for \( v = v_0 v_1 \cdots v_n \cdots \) and \( v' = v'_0 v'_1 \cdots v'_n \cdots \) with \( v_n = v'_n \) for \( 0 \leq n \leq p - 1 \) and \( v_p \neq v'_p \), define \( v \wedge v' = v_0 v_1 \cdots v_{p-1} \).

**Notation.** For any \( \omega \in \Omega \), for any \( i \in \mathbb{N} \) define

\[
\theta(i, \omega, 1) = \inf \{ n \in \mathbb{N} : \sigma^n \in \Omega_i \}
\]

and for any \( p \in \mathbb{N} \) with \( p > 1 \) define

\[
\theta(i, \omega, p) = \inf \{ n \in \mathbb{N} : n > \theta(i, \omega, p - 1) \text{ and } \sigma^n \in \Omega_i \}
\]

From Birkhoff ergodic theorem, for any \( i \in \mathbb{N} \), for \( \mathbb{P}\text{-a.e. } \omega \in \Omega \)

\[
\lim_{p \to \infty} \frac{p}{\theta(i, \omega, p)} = \mathbb{P}(\Omega_i) > 1/2,
\]

then

\[
\lim_{p \to \infty} \frac{\theta(i, \omega, p) - \theta(i, \omega, p - 1)}{\theta(i, \omega, p - 1)} = 0.
\]

Since \( \mathbb{N} \) is countable, \( \mathbb{P}\text{-a.e. } \omega \in \Omega \), for any \( i \in \mathbb{N} \), equations (14) and (15) also hold. There is no trouble that we can assume that they hold for all \( \omega \in \Omega \).

Remark 4. It is important that we introduce the sequences \( \{\Phi_i\}_{i \in \mathbb{N}}, \{\Psi_i\}_{i \in \mathbb{N}} \) of random Hölder potentials to approximate \( \Psi \) and \( \Phi \). First, this can be done, see the details in [29].

Second, if we define a function \( T \) that is the root of \( P(q \Phi - T(q) \Psi) \), we just know the function \( T \) is concave. But for the function \( T_i \) which is the root of \( P(q \Phi_i - T_i(q) \Psi_i) \), it is not only differentiable but also analytic (see [21, chapter 9] and [12]). This will provide us lots of good informations. Furthermore, let \( q_i \) be the root of \( P(q(\Psi_i + \Phi_i)) = 0 \), we can easily get that \( \lim_{i \to \infty} q_i = q_0 \). Recall that \( q_0 \) is the root of \( P(q(\Psi + \Phi)) = 0 \).

4. Upper bound

It is easy to check:

\[
W(\phi, \omega) = \cap_{N \in \mathbb{N}} \cup_{n \geq N} \cup_{v \in \Sigma_{\omega,n}} \{ x \in U^n_v : ||T^n_{\omega}x - z_{\omega v^{|\omega|},\omega}|| \leq \exp(S_{|v|}(\phi(\omega,v)) \}
\]

Define \( V^n_v = \{ x \in U^n_v : ||T^n_{\omega}x - z_{\omega v^{|\omega|},\omega}|| \leq \exp(S_{|v|}(\phi(\omega,v)) \} \), using (3) and the same method in proposition 2, we can easily get there exists a sequence \( \{\epsilon_n(\omega)\} \) of positive number such that \( \epsilon_n(\omega) \) decreasing to 0 as \( n \) increasing to \( \infty \) and for any \( y \in U^n_v \),

\[
|V^n_v| \leq \exp(S_{|v|}(\psi + \Phi)(\omega, y)) + |v|\epsilon_{|v|}(\omega)
\]

so that for any \( z \in \{\omega\}

\[
|V^n_v| \leq \exp(S_{|v|}(\Psi + \Phi)(\omega, z)) + |v|\epsilon_{|v|}(\omega)
\]

**Theorem 4.1.** \( \dim_H W(\phi, \omega) \leq q_0 \).

**Proof.** For any \( q > 0 \) and \( \delta > 0 \) denote by \( \mathcal{H}_q^\delta \) the \( q \)-dimensional Hausdorff pre-measure computed using coverings by sets of diameter less than \( \delta \). For any \( N \in \mathbb{N} \), define \( \delta_N := \sup_{n \geq N} \sup_{v \in \Sigma_{\omega,n}} |V^n_v| \). Then for any \( M \geq N \)

\[
\mathcal{H}_{\delta_n}^q(W(\phi, \omega)) \leq \sum_{n \geq M} \sum_{v \in \Sigma_{\omega,n}} |V^n_v|^q.
\]
Since (16), we have
\[
\sum_{v \in \Sigma_{\omega,n}} |V_v^n|^q \leq \sum_{v \in \Sigma_{\omega,n}} \exp(qS_n(\Psi + \Phi)(\omega, v) + nq\epsilon_n(\omega)).
\]

If \( q > q_0 \), \( P(q(\Psi + \Phi)) < 0 \). From lemma 3.1, for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there exists \( N'(\omega) \in \mathbb{N} \), for \( n > N'(\omega) \) we get
\[
\sum_{v \in \Sigma_{\omega,n}} \exp(qS_n(\Psi + \Phi)(\omega, v) + nq\epsilon_n(\omega)) \leq \exp(n\frac{P(q(\Psi + \Phi))}{2}).
\]

This implies
\[
\mathcal{H}_n^q(W(\phi, \omega)) \leq \sum_{n \geq \max\{N'(\omega), N\}} \exp(n\frac{P(q(\Psi + \Phi))}{2}) < \sum_{n \geq N'(\omega)} \exp(n\frac{P(q(\Psi + \Phi))}{2}),
\]
then \( \mathcal{H}^q(W(\phi, \omega)) \leq \sum_{n \geq N'(\omega)} \exp(n\frac{P(q(\Psi + \Phi))}{2}) < \infty \). So \( \dim_H W(\phi, \omega) \leq q \). Since \( q > q_0 \) is arbitrary, we get that \( \dim_H X_\omega \leq q_0 \). □

5. LOWER BOUND

**Theorem 5.1.** For \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there exists a set \( K_\omega \subset W(\phi, \omega) \) and a probability measure \( \eta_\omega \) on it such that \( \dim_H \eta_\omega \geq q_0 \). Then we have \( \dim_H W(\phi, \omega) > q_0 \).

**Proof.** From (9) and (10), we know that for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), there exists \( K = K(\omega) \in \mathbb{N} \) such that for all \( n \geq K \), for all \( v \in \Sigma_{\Omega,n} \) we have
\[
\exp(-2Cn) \leq |U_v^n| \leq \exp(-cn/2).
\]

We now fixed a sequence \( \{\epsilon_i\} \in \mathbb{N} \) of positive numbers deceasing to 0 such that for any \( i \in \mathbb{N} \), one has:
\[
2(18 + 4(\alpha_i + \epsilon_i))\epsilon_i/c < 1,
\]
and
\[
\epsilon_i < \min\{\frac{1}{d + 2}, \frac{1}{q_i(2 + \alpha_i) + d + 1}\}.
\]

**Step 1:** Define \( Y(\omega) = S_{\theta(1,\omega,1)}||(\Psi(\omega)||_\infty + S_{\theta(1,\omega,1)}||(\Phi(\omega)||_\infty \cdot choose m = \theta(1, \omega, s) \in \mathbb{N} \) with \( s \in \mathbb{N} \) large enough such that
- \( m \geq K(\omega) \),
- \( N + N_1 \leq m\epsilon_i^{3} \),
- for all \( k \geq m \),
  \[ \max\{\epsilon(\Psi, \omega, k), \epsilon(\Phi, \omega, k)\} \leq \epsilon_1^{3} \].

Fix a word \( w \in \Sigma_{\omega,m} \), then \( \sigma^m w \in \Omega_1 \).

Now, define
\[
E(\omega, w) = \{\pi_\omega(w * v) : v \in E_{1,\alpha_1}(\sigma^m + N_1, \epsilon_1)\}
\]
and noticing \( \tilde{\mu}_{\sigma^m+N_1}(E_{1,\alpha_1}(\sigma^m + N_1, \epsilon_1)) > 1/2 \) by proposition 3, define
\[
\tilde{\zeta}_{\omega,w}(w * v) = \frac{\tilde{\mu}_{\sigma^m+N_1}(E_{1,\alpha_1}(\sigma^m + N_1, \epsilon_1))}{\tilde{\mu}_{\sigma^m+N_1}(E_{1,\alpha_1}(\sigma^m + N_1, \epsilon_1))},
\]
and
\[
\zeta_{\omega,w} = \pi_\omega \circ \tilde{\zeta}_{\omega,w} = \tilde{\zeta}_{\omega,w} \circ \pi_\omega^{-1}.
\]

We will easily obtain that \( \zeta_{\omega,w}(E(\omega, w)) = 1 \).
For $n \geq \mathcal{N}_1$, define
\[
\mathcal{F}_{w,n} = \{ B(x, 2|U^w_x|) : x \in E(\omega, w) \cap U^w_x, v \in \Sigma_{m+\mathcal{N}, n} \}.
\]
From Besicovitch covering theorem $Q(d)$ families of disjoint balls namely
\[
\mathcal{F}^1_{w,n}, \ldots, \mathcal{F}^{Q(d)}_{w,n}
\]
can be extracted from $\mathcal{F}_{w,n}$, so that
\[
E(w) \subset \bigcup_{j=1}^{Q(d)} \bigcup_{B \in \mathcal{F}_{w,n}} B.
\]
Since $\zeta_{\omega,w}(E(w)) = 1$, there exists $j$ such that
\[
\zeta_{\omega,w}\left(\bigcup_{B \in \mathcal{F}_{w,n}} B\right) \geq \frac{1}{Q(d)}.
\]
Again, we extract from $\mathcal{F}_{w,n}^j$ a finite family of pairwise disjoint balls $D(w, n) = \{B_1, \ldots, B_j\}$ such that
\[
\zeta_{\omega,w}\left(\bigcup_{B_l \in D(w,n)} B_l\right) \geq \frac{1}{2Q(d)}.
\]
For each $B_l \in D(w, n)$, there exists $y_l \in U^w_x \cap E(w)$ such that
\[
B_l = B(y_l, 2|U^w_x|) \supset U^w_x.
\]
Now we can get (see lemma 5.2 in general situation):
For any $x \in E(\omega, w)$, for $r \leq |U^w_x| \exp(-(C + 4)m\varepsilon_1^2)$, one has that
\[
\zeta_{\omega,w}(B(x, r)) \leq (3r^{-\varepsilon_1^2}) \left(\frac{r}{|U^w_x|}\right)^{q_1(1+\alpha_1-2\varepsilon_1)}.
\] (20)
Now choose $p_1 \geq \mathcal{N}_1$ large such that
\begin{itemize}
  \item $\sigma^{m+\mathcal{N}+p_1, \omega} \in \Omega_1$,
  \item for all $k \geq m + \mathcal{N} + p_1$,
    \[
    \max\{\epsilon(\Psi, \omega, k), \epsilon(\Phi, \omega, k)\} \leq \varepsilon_2^3,
    \]
  \item
    \[
    p_1 \geq \frac{(m + 2\mathcal{N})C + Y(\omega)}{c\varepsilon_1^2/2},
    \]
\end{itemize}
and
\[
p_1 \geq \frac{(q_1(1+\alpha_1-2\varepsilon_1) + d + 1) \log 2 + \log(Q(d))}{c\varepsilon_1^2/2},
\]
\begin{itemize}
  \item $\mathcal{N} + \mathcal{N}_2 \leq (m + \mathcal{N} + p_1)\varepsilon_2^3$, that is $m + \mathcal{N} + p_1 \geq \frac{\mathcal{N} + \mathcal{N}_2}{\varepsilon_2^3}$,
  \item for any $k \in \mathbb{N}$ such that $\theta(2, \omega, k) \geq m + \mathcal{N} + p_1$, we have
    \[
    \frac{\theta(2, \omega, k) - \theta(2, \omega, k - 1)}{\theta(2, \omega, k - 1)} \leq \varepsilon_2^3
    \]
\end{itemize}
Noticing that there exists \( 1 \leq s \leq l(\sigma^{m+N+p_1+N} \omega) \) such that
\[
z_{\sigma^{m+N+p_1+N+1} \omega} \in X_{\sigma^{m+N+p_1+N+1} \omega} \subset X_{\sigma^{m+N+p_1+N+1} \omega}.
\]
Since \( \sigma^{m+N+p_1-1} \omega \in \Omega_1 \), we know that there exists \( w \ast v(l) \ast s \in \Sigma_\omega, m+N+p_1+N+1 \) such that \( y = g_{\omega}^{w+v(l)}(z_{\sigma^{m+N+p_1+N+1} \omega}) \in X_{\omega}^{w+v(l)\ast s} \subset U_{\omega}^{w+v(l)\ast s} \).

Now we can choose the smallest \( p \) and \( v'(l) \in \Sigma_{\sigma^{m+N} \omega} \cap (2, \omega, \delta - m - N) \) such that
- \( v(l) \ast s \) is the prefix of \( v'(l) \), which is equivalent to \( U_{\omega}^{w+v'(l)} \subset U_{\omega}^{w+v(l)\ast s} \),
- define \( r_{w+v'(l)} = \exp(S_{m+N+p_1+N} \phi(\omega, y) - (m + N + p_1 + N)\varepsilon_1^3) \), then
\[
T_{\omega}^{w+v(l)}(U_{\omega}^{w+v'(l)}) \subset B(z_{\sigma^{m+N+p_1+N+1} \omega}^{r_{w+v'(l)}}).
\]

Then for any \( x \in U_{\omega}^{w+v'(l)} \subset U_{\omega}^{w+v(l)\ast s} \),
\[
|T_{\omega}^{w+v(l)}x - z_{\sigma^{m+N+p_1+N+1} \omega}| \leq \exp(S_{m+N+p_1+N} \phi(\omega, x)),
\]
and
\[
|U_{\omega}^{w+v(l)}| \geq |U_{\omega}^{w+v(l)}|^{1+\alpha_1+2\varepsilon_1},
\]
(see (32) and (33) in general situation)

Let \( G(\omega, w) = \{ U_{\omega}^{w+v'} : B_1 \in D_{\omega, p_1} \} \), \( G_{\omega, i} = G(\omega, w) \) and define a set function \( \eta_\omega^i \) as follows,
\[
\eta_\omega^i(U_{\omega}^{w+v'}) = \frac{\sum_{B \in G(\omega)} \zeta_{\omega, w}(B)}{|\Sigma_{\omega, w}|}.
\]

From (20) and (21), we will get (see (36) in general situation):
\[
\eta_\omega^i(U_{\omega}^{w+v'}) \leq |U_{\omega}^{w+v'}|^{q_i(1-4\varepsilon_1) - \varepsilon_1^3}.
\]

**Step 2:** Suppose that \( G_{\omega, i} \) is well defined and so is the set function \( \eta_\omega^i \) on it. For any \( w \) such that \( U_{\omega}^{w} \subset G_{\omega, i} \), set \( m = |w| \), then \( \sigma^m \omega \in \Omega_{i+1} \) and the following hold
- \( N_{i+1} \leq m \varepsilon_{i+1} \),
- for all \( k \geq m \),
\[
\max\{\varepsilon(\Psi, \omega, k), \varepsilon(\Phi, \omega, k)\} \leq \varepsilon_{i+1}^3.
\]

Now, we define
\[
E(\omega, w) = \{ \pi_\omega(w \ast v) \in U_{\omega}^{w} : v \in E_{i+1, \alpha_{i+1}}(\sigma^{m+N} \omega, N_{i+1}, \varepsilon_{i+1}) \}
\]
and noticing \( \tilde{\pi}_{\sigma^{m+N} \omega}^{i+1}(E_{i+1, \alpha_{i+1}}(\sigma^{m+N} \omega, N_{i+1}, \varepsilon_{i+1})) > 1/2 \) in proposition 3, we can define
\[
\zeta_{\omega, w}(w) = \frac{\tilde{\pi}_{\sigma^{m+N} \omega}^{i+1}(w \ast \sigma^{m+N} \omega \cap \Sigma_{\omega, w}^{i+1} \epsilon_{i+1})}{\tilde{\pi}_{\sigma^{m+N} \omega}^{i+1}(E_{i+1, \alpha_{i+1}}(\sigma^{m+N} \omega, N_{i+1}, \varepsilon_{i+1})),
\]
and
\[
\zeta_{\omega, w} = \pi_\omega \circ \zeta_{\omega, w} = \zeta_{\omega, w} \circ \pi_\omega^{-1}.
\]

We will easily obtain that \( \zeta_{\omega, w}(E(\omega, w)) = 1 \).

For any \( n \geq N_{i+1} \), define
\[
\mathcal{F}_{w, n} = \{ B(x, 2U_{\omega}^{w+v}) : x \in E(\omega, w) \cap U_{\omega}^{w+v}, v \in \Sigma_{\sigma^{m+N}, n} \}
\]
From Besicovitch covering theorem \( Q(\delta) \) families of disjoint balls namely
\[
\mathcal{F}_{w, n}^1, \ldots, \mathcal{F}_{w, n}^{Q(\delta)}
\]
can be extracted from $F_{w,n}$, so that

$$E(\omega, w) \subset \bigcup_{j=1}^{Q(d)} \bigcup_{B \in F_{w,n}} B.$$  

Since $\zeta_{\omega,w}(E(\omega, w)) = 1$, there exists $j$ such that

$$\zeta_{\omega,w}\left(\bigcup_{B \in F_{w,n}} B\right) \geq \frac{1}{Q(d)}.$$  

Again, we extract from $F_{w,n}^j$ a finite family of pairwise disjoint balls $D(w, n) = \{B_1, \ldots, B_r\}$ such that

$$\zeta_{\omega,w}\left(\bigcup_{B \in D(w, n)} B\right) \geq \frac{1}{2Q(d)}. \quad (23)$$  

For each $B_i \in D(w, n)$, there exists $y_i \in U_{\omega}^{w*v(l)} \cap E(\omega)$ such that

$$B_i = B(y_i, 2|U^{w*v(l)}_{\omega}|) \supset U_{\omega}^{w*v(l)}.$$  

Now we turn to prove the following lemma.

**Lemma 5.2.** For any $x \in E(\omega)$, for $r \leq |U^w_{\omega}| \exp(-(C + 4)m\varepsilon^3_{i+1})$, one has that

$$\zeta_{\omega,w}(B(x, r)) \leq (2r^{-\varepsilon^3_{i+1}})d\left(\frac{r}{|U^w_{\omega}|}\right)^{q_{i+1}+1(1+\alpha_i+1-2\varepsilon_{i+1})}.$$  

**Proof.** Define

$$\Delta_{w,r} = \{U_{\omega}^{w*v} : |v|_{\sigma^{m+N}_{\omega}} \cap E_{t, \varepsilon_{i+1}}(\sigma^{m+N}_{\omega}, N_i, \varepsilon_i) \neq \emptyset \},$$

recall the definition of $\star$ in subsection 2.1. Let $k_1 = \inf\{|v|, U_{\omega}^{w*v} \in \Delta_{w,r}\}$ and $k_2 = \sup\{|v|, U_{\omega}^{w*v} \in \Delta_{w,r}\}$. Noticing the following fact:

- $k_1 \geq N_{i+1}$, in fact since $|w*v| = m+N+n$ and $|U_{\omega}^{w*v}| < r \leq |U^w_{\omega}| \exp(-(C + 4)m\varepsilon^3_{i+1})$, we will get

$$\exp(S_{\psi+n}(F^m(\omega, \Psi)) - 2(m + N + n)e(\Psi, \omega, m + N + n)) < \exp(-(C + 4)m\varepsilon^3_{i+1}).$$

From (9) and (22), we have

$$C(n+N) + 2(m + n + N)\varepsilon^3_{i+1} \geq (C + 4)m\varepsilon^3_{i+1},$$

which yield $n + N \geq m\varepsilon^3_{i+1}$, and then $n \geq N_{i+1}$ by $N + N_{i+1} \leq m\varepsilon^3_{i+1}$.

- from (11), (12) and (13), we have that for any $v \in [w*v]_\omega$ with $U_{\omega}^{w*v} \in \Delta_{w,r},$

$$|S_{[v]}\Phi(\omega, \Psi) - S_{[v]}\Phi_{i+1}(\omega, \Psi)| \leq 2|v|\varepsilon^3_{i+1}, \quad (24)$$

$$|S_{[v]}\Psi(\omega, \Psi) - S_{[v]}\Psi_{i+1}(\omega, \Psi)| \leq 2|v|\varepsilon^3_{i+1} \quad (25)$$

and

$$\frac{|S_{[v]}\Phi_{i}(\omega, \Psi) - S_{[v]}\Psi_{i}(\omega, \Psi)|}{|S_{[v]}\Phi_{i}(\omega, \Psi) - \alpha_{i+1}|} \leq \varepsilon_{i+1} \quad (26)$$
• for any $U^w_{\omega} \in \Delta_{w,r}$, using (24),(25) and (26) and the definition of $\zeta_{\omega,w}$ and $\xi_{\omega,w}$, we can get
  \[
  \frac{\log \zeta_{\omega,w}\left((w * v)_\omega\right)}{\log |U^w_{\omega}| - \log |U^w_{\omega}|} - q_{i+1}(1 + \alpha_i + 1) \leq 2q_{i+1} \epsilon_{i+1}.
  \]  
(27)

• for any $U^w_{\omega} \in \Delta_{w,r}$ with $U^w_{\omega} \cap B(x, r) \neq \emptyset$, we will get $U^w_{\omega} \subset B(x, 2r)$. Noticing $|U^w_{\omega}| \geq r$ then $U^w_{\omega}$ contains a ball $B_{w,r}$, with radius $r^{1+\epsilon_{i+1}}$. Since $\{B_{w,r} \mid U^w_{\omega} \in \Delta_{w,r}\}$ have no inner intersection. So there are at most $(2r^{-\epsilon_{i+1}})^d$ of $U^w_{\omega} \in \Delta_{w,r}$ can intersect with $B(x, r)$.

So
\[
\zeta_{\omega,w}(B(x, r) \leq \zeta_{\omega,w}(\pi^{-1}(B(x, r)))
\leq \sum_{U^w_{\omega} \in \Delta_{w,r}, U^w_{\omega} \cap B(x, r) \neq \emptyset} \zeta_{\omega,w}(w * v) \leq (2r^{-\epsilon_{i+1}})^d \sup_{U^w_{\omega} \in \Delta_{w,r}} \zeta_{\omega,w}(w * v)
\leq (2r^{-\epsilon_{i+1}})^d \sup_{U^w_{\omega} \in \Delta_{w,r}} \left(\frac{|U^w_{\omega}|}{U^w_{\omega}}\right) q_{i+1}(1 + \alpha_i + 1 - 2\epsilon_{i+1})
\leq (2r^{-\epsilon_{i+1}})^d \left(\frac{r}{|U^w_{\omega}|}\right) q_{i+1}(1 + \alpha_i + 1 - 2\epsilon_{i+1}).
\]

Now choose $p_{i+1} \geq N_{i+1}$ large such that
• $\sigma^{m+N+p_{i+1}} \omega \in \Omega_{i+1}$
• for all $k \geq m + N + p_{i+1}$,
  \[
  \max\{\epsilon(\Psi, \omega, k), \epsilon(\Phi, \omega, k)\} \leq \epsilon^3_i + 2.
  \]
  
  \[
  p_{i+1} \geq \frac{(m + 2N)C + Y(\omega)}{e^3_{i+1}/2},
  \]
(28)
and
\[
   p_{i+1} \geq \frac{(q_{i+1}(1 + \alpha_i + 1 - 2\epsilon_{i+1}) + d + 1) \log 2 + \log(Q(d))}{e^3_{i+1}/2},
\]
(29)
• $N + N_{i+2} \leq (m + N + p_{i+1}) \epsilon^3_{i+2}$, that is $m + N + p_{i+1} \geq \frac{N + N_{i+2}}{\epsilon^3_{i+2}}$,
• for any $k \in \mathbb{N}$ such that $\theta(i + 2, \omega, k) \geq m + N + p_{i+1}$, we have
  \[
  \frac{\theta(i + 2, \omega, k) - \theta(i + 2, \omega, k - 1)}{\theta(i + 2, \omega, k - 1)} \leq \epsilon^3_{i+2}
  \]
(30)
Noticing that there exists $1 \leq s \leq l(\sigma^{m+N+p_{i+1}+N} \omega)$ such that
\[
\exists_{\sigma^{m+N+p_{i+1}+N} \omega} \in X^{ss}_{\sigma^{m+N+p_{i+1}+N} \omega} \subset X^{m+N+p_{i+1}+N} \omega.
\]
Since $\sigma^{m+N+p_{i+1}+N} \omega \in \Omega_{i+1}$, we know that there exists
\[
   w \ast v(l) * s \in \Sigma_{w,m+N+p_{i+1}+N+1}
\]
such that
\[
y = \eta^{w \ast v(l)}(z_{\sigma^{m+N+p_{i+1}+N} \omega}) \in X^{w \ast v(l) \ast s}_{w \ast v(l) \ast s} \subset U^{w \ast v(l) \ast s}_{\omega}.
\]
Now we can choose the smallest $p$ and $v'(l) \in \Sigma_{\sigma}^{m+N} \theta((i+2, \omega, p)-m-N}$ such that

- $v(l) * s$ is the prefix of $v'(l)$, which is equivalent to $U_{v}^{\omega} v'(l) \subset U_{v}^{\omega} v'(l) s$.
- $y \in U_{v}^{\omega} v'(l)$

2. define $r_{v}^{\omega} v'(l) = \exp(S_{m+N+p+1} \phi(\omega, y) - (m + N + p + 1 + N) \varepsilon_{i+1}^{3})$ then

$$T_{\omega}^{\omega} v'(l) (U_{v}^{\omega} v'(l)) \subset B(z_{\sigma}^{m+N+p+1+N} \phi, r_{v}^{\omega} v'(l)).$$  (31)

We claim that: for any $\omega$ we have

$$\omega \in U_{\omega}^{\omega} v'(l) \subset U_{\omega}^{\omega} v'(l) s,$$

$$T_{\omega}^{\omega} v'(l) x - z_{\sigma}^{m+N+p+1+N} \phi(\omega, x) \leq \exp(S_{m+N+p+1} \phi(\omega, x)).$$  (32)

In fact, (31) implies $|T_{\omega}^{\omega} v'(l) x - z_{\sigma}^{m+N+p+1+N} \phi(\omega, x)| \leq r_{\omega}^{\omega} v'(l)$, then noticing $x, y \in U_{\omega}^{\omega} v'(l) s$, (7) and (22), we have

$$S_{m+N+p+1} \phi(\omega, y) - (m + N + p + 1 + N) \varepsilon_{i+1}^{3} \leq S_{m+N+p+1} \phi(\omega, x),$$

from the definition of $r_{\omega}^{\omega} v'(l)$, the claim holds.

**Lemma 5.3.**

$$|U_{\omega}^{\omega} v'(l)| \geq |U_{\omega}^{\omega} v'(l)|^{1+\varepsilon_{i+1}^{2}}$$  (33)

Although the proof of the lemma is bit complex, but the main idea is very simple. First, since we have choose the smallest $p$, and $\theta(i+2, \omega, p)-\theta(i+2, \omega, p-1)$ is smaller than $\theta(i+2, \omega, p) \varepsilon_{i+1}^{3}$. Second, use (24),(25) and (26). We can skip the details of the proof at first sight.

**Proof.** From the choice of $p$ and $v'(l)$, define $v''(l) = v'(l)|_{\theta(i+2, \omega, p-1)-m-N}$. We claim that

$$|U_{\omega}^{\omega} v'(l)| \geq |U_{\omega}^{\omega} v''(l)|^{1+\varepsilon_{i+1}^{2}}$$  (34)

In fact, for any $u \in [w * v'(l)]_{\omega}$

$$|U_{\omega}^{\omega} v'(l)| \geq \exp(S_{\theta(i+2, \omega, p)}(\omega, u) - \theta(i+2, \omega, p)\epsilon(\Psi, \omega, \theta(i+2, \omega, p))).$$

$$\geq \exp(S_{\theta(i+2, \omega, p-1)}(\omega, u) - (\theta(i+2, \omega, p) - \theta(i+2, \omega, p-1))\epsilon C)$$

$$\cdot \exp((\theta(i+2, \omega, p) - (1 + \varepsilon_{i+1}^{3})\epsilon_{i+1}^{3}.$$

$$\geq \exp(S_{\theta(i+2, \omega, p-1)}(\omega, u) - C\theta(i+2, \omega, p-1)\varepsilon_{i+1}^{3}.$$

$$\cdot \exp((\theta(i+2, \omega, p) - (1 + \varepsilon_{i+1}^{3})\epsilon_{i+1}^{3}).$$

Since $\epsilon(\Psi, \omega, \theta(i+2, \omega, p) < \varepsilon_{i+1}^{3} + C\varepsilon_{i+1}^{3} < 1$

Therefore,

$$|U_{\omega}^{\omega} v''(l)| \geq |U_{\omega}^{\omega} v''(l)|^{1+\varepsilon_{i+1}^{3} \cdot (C+3)\theta(i+2, \omega, p-1)\varepsilon_{i+1}^{3}.$$

Now there will be two cases:

**Case 1.** $U_{\omega}^{\omega} v''(l) \not\subset U_{\omega}^{\omega} v''(l) s$. Since $w * v''(l)$ and $w * v$ are the prefixes of $w * v'(l)$, we have $U_{\omega}^{\omega} v''(l) \subset U_{\omega}^{\omega} v'(l) s$, using a similar method as the proof of (34), we can get

$$|U_{\omega}^{\omega} v''(l)| \geq |U_{\omega}^{\omega} v''(l)|^{1+\varepsilon_{i+1}^{3}}.$$
now (34) can imply
\[ |U_w^{w,v(x)}| \geq |U_w^{w,v(x)}|^{1+\varepsilon^2_{i+1}} \geq |U_w^{w,v(l)}|^{(1+\varepsilon^2_{i+1})^2} \geq |U_w^{w,v(l)}|^{1+\alpha_{i+1}+2\varepsilon_{i+1}}. \]

**Case 2.** \( U_w^{w,v(x)} \subset U_w^{w,v(l)} \). Since we have chosen the smallest \( p \), so from (31), there exist \( x \in U_w^{w,v(l)} \) such that
\[ |T_w^{w,v(l)}x - z_{\sigma_m+N+p_{i+1}+\varepsilon_{i+1}}| > r_w^{w,v(l)}. \]

From (3), noticing \( y = g_w^{w,v(l)}(z_{\sigma_m+N+p_{i+1}+\varepsilon_{i+1}}) \), \( x = g_w^{w,v(l)}(T_w^{w,v(l)}x) \), for any \( v \in [w * v(l)] \), and (22), we will get
\[ |U_w^{w,v(l)}| \geq ||x - y|| \geq \exp(S_{m+p_{i+1}+2\mathcal{N}}(\Psi,\Phi)(\omega,\varepsilon)) - 4(m + p_{i+1} + 2\mathcal{N})\varepsilon^3_{i+1}, \]
so that we can claim that
\[ |U_w^{w,v(l)}| \geq \exp(S_{p_{i+1}}(\Psi + \Phi)F^{m+\mathcal{N}}(\omega,\varepsilon)) - 5(m + p_{i+1} + 2\mathcal{N})\varepsilon^3_{i+1}. \]

In fact we just need to notice the following claim: for \( \Upsilon \in \{\Phi,\Psi + \Phi\} \) we have
\[ |S_{m+p_{i+1}+2\mathcal{N}}(\Upsilon(\omega,\varepsilon)) - S_{p_{i+1}}(\Upsilon(F^{m+\mathcal{N}}(\omega,\varepsilon)))| \leq p_{i+1}\varepsilon^3_{i+1}, \]
see (9) and (28).

It is just from (9) and (28), then
\[ |S_{m+p_{i+1}+2\mathcal{N}}(\Upsilon(\omega,\varepsilon)) - S_{p_{i+1}}(\Upsilon(F^{m+\mathcal{N}}(\omega,\varepsilon)))| \]
\[ = |S_{\theta(\omega,\varepsilon)}(\Upsilon(\omega,\varepsilon)) + S_{m+\mathcal{N} - \theta(\omega,\varepsilon)}(\Upsilon(F^{\theta(\omega,\varepsilon)}(\omega,\varepsilon))) + S_{\mathcal{N}}(F^{m+p_{i+1}+\mathcal{N}}(\omega,\varepsilon))| \]
\[ \leq S_{\theta(\omega,\varepsilon)} \|\Upsilon(\omega)\|_\infty + (m + 2\mathcal{N}C) \text{ see (9)} \]
\[ \leq p_{i+1}\varepsilon^3_{i+1} \text{ see (28).} \]

Now
\[ |U_w^{w,v(l)}| \geq \exp(S_{p_{i+1}}(\Psi + \Phi)(F^{m+\mathcal{N}}(\omega,\varepsilon))) - 5(m + p_{i+1} + 2\mathcal{N})\varepsilon^3_{i+1} \]
\[ \geq \exp(-10p_{i+1}\varepsilon^3_{i+1} + (S_{p_{i+1}}(\Psi + \Phi)(F^{m+\mathcal{N}}(\omega,\varepsilon))) (\text{since } p_{i+1} \geq m + 2\mathcal{N}) \]
\[ \geq \exp(-10p_{i+1}\varepsilon^3_{i+1} + (S_{p_{i+1}}(\Psi_{i+1} + \Phi_{i+1})(F^{m+\mathcal{N}}(\omega,\varepsilon))) - 4p_{i+1}\varepsilon^3_{i+1}) \]
\[ \text{see (24) and (25)} \]
\[ \geq \exp(-14p_{i+1}\varepsilon^3_{i+1} + (1 + \alpha_{i+1} + \varepsilon_{i+1})(S_{p_{i+1}}(\Psi_{i+1})(F^{m+\mathcal{N}}(\omega,\varepsilon)))) \]
\[ \text{see (26)} \]
\[ \geq \exp(-14p_{i+1}\varepsilon^3_{i+1} + (1 + \alpha_{i+1} + \varepsilon_{i+1})(S_{p_{i+1}}(\Psi(F^{m+\mathcal{N}}(\omega,\varepsilon))) - p_{i+1}\varepsilon^3_{i+1}) \]
\[ \text{see (25)} \]
\[ \geq \exp((1 + \alpha_{i+1} + \varepsilon_{i+1})(S_{m+\mathcal{N} + p_{i+1}}(\Psi(\omega,\varepsilon)) - 2p_{i+1}\varepsilon^3_{i+1}) - 14p_{i+1}\varepsilon^3_{i+1}) \]
\[ \text{see (35), but change } 2\mathcal{N} \text{ to } \mathcal{N} \]
\[ \geq \exp((1 + \alpha_{i+1} + \varepsilon_{i+1})(S_{m+\mathcal{N} + p_{i+1}}(\Psi(\omega,\varepsilon))) - (16 + 2(\alpha_{i+1} + \varepsilon_{i+1}))p_{i+1}\varepsilon^3_{i+1}). \]

Noticing item 1 of proposition 2 and (22), we have
\[ |U_w^{w,v(l)}| \leq \exp(S_{m+\mathcal{N} + p_{i+1}}(\Psi(\omega,\varepsilon)) + (m + \mathcal{N} + p_{i+1})\varepsilon^3_{i+1}) \leq \exp(-cp_{i+1}/2). \]
Then
\[ |U^{w*\nu'}_\omega(l)| \geq |U^{w*\nu}_\omega(l)|^{(1 + \alpha_{i+1} + \varepsilon_{i+1})} \exp(-(1 + \alpha_{i+1} + \varepsilon_{i+1})(m + N + \rho_{i+1})\varepsilon_{i+1}^3) \]
\[ \cdot \exp(-(16 + 2(\alpha_{i+1} + \varepsilon_{i+1}))\rho_{i+1}\varepsilon_{i+1}^3) \]
\[ \geq |U^{w*\nu}_\omega(l)|^{(1 + \alpha_{i+1} + \varepsilon_{i+1})} \exp(-(18 + 4(\alpha_{i+1} + \varepsilon_{i+1}))\rho_{i+1}\varepsilon_{i+1}^3) \]
\[ \geq |U^{w*\nu}_\omega(l)|^{(1 + \alpha_{i+1} + \varepsilon_{i+1})+2(18+4(\alpha_{i+1}+\varepsilon_{i+1}))\varepsilon_{i+1}^3}/c \text{ see (17)} \]
\[ \geq |U^{w*\nu}_\omega(l)|^{1 + \alpha_{i+1} + \varepsilon_{i+1} + \varepsilon_{i+1}^2} \]
see (18).

That is
\[ |U^{w*\nu'}_\omega(l)| \geq |U^{w*\nu}_\omega(l)|^{1 + \alpha_{i+1} + \varepsilon_{i+1} + \varepsilon_{i+1}^2}, \]
also by (34), we have
\[ |U^{w*\nu'}_\omega(l)| \geq |U^{w*\nu}_\omega(l)|^{(1 + \varepsilon_{i+1}^2)(1 + \alpha_{i+1} + \varepsilon_{i+1} + \varepsilon_{i+1}^2)} \geq |U^{w*\nu}_\omega(l)|^{1 + \alpha_{i+1} + 2\varepsilon_{i+1}}. \]

Let \( G(\omega, w) = \{U^{w*\nu}_\omega(l) : B_l \in D(w, \rho_{i+1})\} \), \( G_{\omega, i+1} = \bigcup U^{w}_{\omega} \subset G_{\omega, i} \) \( G(\omega, w) \) and define \( \eta_{i+1}^{+1} \) as follows,
\[ \eta_{i+1}^{+1}(U^{w*\nu'}_\omega(l)) = \frac{\zeta_{\omega, w}(B_l)}{\sum_{B \in G(\omega)} \zeta_{\omega, w}(B_i)} \eta^{+1}(U^{w'}_\omega(l)). \]

Now we turn to estimate \( \eta_{i+1}^{+1}(U^{w*\nu'}_\omega(l)) \).

Lemma 5.2 and (23) tells us that
\[ \eta_{i+1}^{+1}(U^{w*\nu'}_\omega(l)) \leq 2Q(d)\eta^{+1}(U^{w'}_\omega)(4|U^{w*\nu}_\omega(l)|^{\varepsilon_{i+1}^2}d(\frac{2|U^{w*\nu}_\omega(l)|}{|U^{w'}_\omega|})\eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1}), \]
then using lemma 5.3,
\[ \eta_{i+1}^{+1}(U^{w*\nu'}_\omega(l)) \leq 2Q(d)\eta^{+1}(U^{w'}_\omega)(2(2|U^{w*\nu}_\omega(l)|)^{\varepsilon_{i+1}^2}d(\frac{2|U^{w*\nu}_\omega(l)|}{|U^{w'}_\omega|})\eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1}) \]
\[ \leq 2^{\eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1}) + d + 1}Q(d)|U^{w*\nu}_\omega(l)|^{\eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1}) - \delta_{i+1}^3|U^{w'}_\omega| - \eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1})}. \]

Noticing (28) and (29), we have
\[ \frac{\log(2^{\eta_{i+1}^{+1}(1 + \alpha_{i+1} - 2\varepsilon_{i+1}) + d + 1}Q(d))}{\log(|U^{w*\nu}_\omega(l)|)} \leq \varepsilon_{i+1}^3, \]
\[ \frac{\log(|U^{w*\nu}_\omega(l)|)}{\log(|U^{w*\nu'}_\omega(l)|)} \leq \frac{Y(\omega) + C(m + N)}{c\rho_{i+1}/2} \leq \varepsilon_{i+1}^3, \]
then
\[ \eta^{i+1}(U^{\omega_{w^v}v}(l)) \]
\[ \leq |U^{\omega_{w^v}v}(l)|q_{i+1}(1+\alpha_{i+1}-2\epsilon_{i+1})-d_{i+1}^{3}q_{i+1}(1+\alpha_{i+1}-2\epsilon_{i+1})e_{i+1}^{3}-e_{i+1}^{3} \]
\[ \leq |U^{\omega_{w^v}v}(l)|q_{i+1}(1+\alpha_{i+1}-2\epsilon_{i+1})-(q_{i+1}(1+\alpha_{i+1}-2\epsilon_{i+1})-d_{i+1})e_{i+1}^{3}+1 \]
\[ \leq |U^{\omega_{w^v}v}(l)| \frac{q_{i+1}(1+\alpha_{i+1}-2\epsilon_{i+1})-e_{i+1}+1}{1+\alpha_{i+1}+2\epsilon_{i+1}} \text{ see (33) in lemma 5.3 and (19)} \]
\[ \leq |U^{\omega_{w^v}v}(l)|q_{i+1}(1-4\epsilon_{i+1})-e_{i+1}. \]

So that for any \( U^{\omega_{w^v}v}(l) \in G_{\omega,i+1} \), we have
\[ \eta^{i+1}(U^{\omega_{w^v}v}(l)) \leq |U^{\omega_{w^v}v}(l)|q_{i+1}(1-4\epsilon_{i+1})-e_{i+1}. \]  

**step 3:** Now define \( K_\omega = \cap_{i \in \mathbb{N}} U_j \supseteq \bigcup_{i \in \mathbb{N}} U_j \subseteq G_{\omega,1} \) \( U^{\omega}_w = \cap_{i \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} U_{i,w} \). From the construction we can easily get \( K_\omega \subseteq W(\varphi, \omega) \). Noticing the definition of \( \{ \eta_i \}_{i \in \mathbb{N}} \), we can distribute a probability measure \( \eta_\omega \) on the algebra generated by \( U_{i,w} \) such that for any \( U^{\omega}_w \in G_{\omega,i} \), \( \eta_\omega(U^{\omega}_w) = \eta^i_\omega(U^{\omega}_w) \).

We have the following properties:

(i) If \( U^{\omega_{w^v}v}(l_1) \) and \( U^{\omega_{w^v}v}(l_2) \) are different elements that belong to \( G(\omega, w) \subset G_{\omega,i+1} \), \( B_{l_1} \cap B_{l_2} = \emptyset \) recall that \( B_{l_j} = B(y_{l_j}, 2|U^{\omega_{w^v}v}(l_j)|) \) and \( y_{l_j} \in U^{\omega_{w^v}v}(l_j) \) for \( j = 1, 2 \). So their distance is at least \( \max_{j \in \{1, 2\}} |U^{\omega_{w^v}v}(l_j)| \).

(ii) For any \( U^{\omega_{w^v}v}(l) \in G(\omega, w) \subset G_{\omega,i+1} \), we denote the corresponding \( y \) by \( y_l \), then \( y_l \in U^{\omega_{w^v}v}(l) \cap E(\omega, w) \subset B_l \cap E(\omega, w) \neq \emptyset \).

(iii) For any \( U^{\omega_{w^v}v}(l) \in G(\omega, w) \subset G_{\omega,i+1} \),
\[ \eta_\omega(U^{\omega_{w^v}v}(l)) \leq |U^{\omega_{w^v}v}(l)|q_{i+1}(1-4\epsilon_{i+1})-e_{i+1}. \]

(iv) Any \( U^{\omega_{w^v}v}(l) \in G(\omega, w) \subset G_{\omega,i+1} \) is contained in an element \( U^{\omega}_w \in G_{\omega,i} \) such that
\[ \eta_\omega(U^{\omega_{w^v}v}(l)) \leq 2Q(d)\eta^{i}_\omega(U^{\omega}_w)\tau_{\omega,w}(B(y_l, 2|U^{\omega_{w^v}v}|)), \]
where \( q_i \in Q_i \) is such that \( \tau_{i}(q_i) = d_i \).

Because of the separation property (i), the probability measure \( \eta_\omega \) can be extended to the \( \sigma \)-algebra generated by \( U_{i \in \mathbb{N}} G_{\omega,i} \) and it is easy to notice that \( \eta_\omega(K_\omega) = 1 \).

The measure \( \eta_\omega \) can be also extended to \( U \) by setting, for any \( B \in B(U) \), \( \eta_\omega(B) := \eta_\omega(B \cap K_\omega) \).

**step 4:** Now let us estimate the lower Hausdorff dimension of \( \eta_\omega \) and then get the lower bound for the Hausdorff dimension \( K_\omega \). If \( \eta_0 = 0 \), there is nothing to prove. So we assume that \( \eta_0 > 0 \).

Let us fix a ball \( B \) which is a subset of \( U \) with length smaller than that of every element in \( G_{\omega,1} \), and assume that \( B(x, r) \cap K_\omega \neq \emptyset \). Let \( U^{\omega}_w \) be the element of largest diameter in \( \bigcup_{i \geq 1} G_{\omega,i} \) such that \( B \) intersects at least two elements of \( G_{\omega,i+1} \) and is included in \( U^{\omega}_w \in G_{\omega,i} \). We remark that this implies that \( B \) does not intersect any other element of \( G_{\omega,i+1} \) and as a consequence \( \eta_\omega(B) \leq \eta_\omega(U^{\omega}_w) \).

Let us distinguish three cases:
\[ \eta_\omega(B) \leq \eta_\omega(U^w_\omega) \leq |U^w_\omega|^{q_i(1-4\epsilon_i)-\epsilon_i^2} \leq |B|^{q_i(1-4\epsilon_i)-\epsilon_i^2}. \] (39)

\[ |B| \leq \frac{1}{4} |U^w_\omega| \exp(-(C + 4)|w|\epsilon_{i+1}^3) \]. Assume \( U^w_\omega \in \ldots \), are the elements of \( G(\omega, w) \subset G_{\omega,i+1} \) which have non-empty intersection with \( B \). From (38) in property (iv), we get

\[ \eta_\omega(B) = \sum_{l=1}^k \eta_\omega(B \cap U^w_\omega(l)) \leq 2Q(d) \eta_\omega((U^w_\omega) \sum_{l=1}^k \zeta_{\omega,w}(U^w_\omega(l)). \]

From property (i) we can also deduce that \( \max\{|U^w_\omega(l)| : 1 \leq l \leq k\} \leq |B| \).

\[ \zeta_{\omega,w}(B(y, 3|B|)) \leq (2 \cdot (3|B|)^{-\epsilon_i^2+1})^d \left( \frac{3|B|}{|U^w_\omega|} \right)^{q_{i+1}(1+\alpha_{i+1}+2\epsilon_{i+1})}. \]

So that

\[ \eta_\omega(B) \leq 2Q(d) \eta_\omega((U^w_\omega)) \sum_{l=1}^k \zeta_{\omega,w}(U^w_\omega(l)) \]

\[ \leq 2Q(d) \eta_\omega((U^w_\omega)) \zeta_{\omega,w}(B(y, 3|B|)) \]

\[ \leq 2Q(d)|U^w_\omega|^{q_i(1-4\epsilon_i)-\epsilon_i^2} \left( 2 \cdot (3|B|)^{-\epsilon_i^2+1} \right)^d \left( \frac{3|B|}{|U^w_\omega|} \right)^{q_{i+1}(1+\alpha_{i+1}+2\epsilon_{i+1})} \]

\[ \leq 2Q(d)(2 \cdot (3|B|)^{-\epsilon_i^2+1})^d |U^w_\omega|^{q_i(1-4\epsilon_i)-\epsilon_i^2} \]

\[ \cdot \left( \frac{3|B|}{|U^w_\omega|} \right)^{q_i(1-4\epsilon_i)-\epsilon_i^2} \left( q_{i+1}(1+\alpha_{i+1}+2\epsilon_{i+1}) - q_i(1-4\epsilon_i) + \epsilon_i^2 \right) \]

\[ \leq 2 \cdot 2dQ(d)(3|B|)^{q_i(1-4\epsilon_i)-\epsilon_i^2-\epsilon_{i+1}^3} \left( \frac{3|B|}{|U^w_\omega|} \right)^{q_{i+1}(1+\alpha_{i+1}+2\epsilon_{i+1}) - q_i(1-4\epsilon_i)}. \]

From \( \lim_{i \to \infty} \alpha_i > 0 \) and \( \lim_{i \to \infty} q_i = q_0 > 0 \) we have \( q_{i+1}(1+\alpha_{i+1}+2\epsilon_{i+1}) - q_i(1-4\epsilon_i) \geq 0 \) for \( i \) large enough. Further more since \( 3|B|/|U^w_\omega| \leq 1 \), we have:

\[ \eta_\omega(B) \leq 2d+1Q(d)(3|B|)^{q_i(1-4\epsilon_i)-\epsilon_i^2-\epsilon_{i+1}^3}. \] (40)

\[ \frac{1}{4} |U^w_\omega| \exp(-(C + 4)|w|\epsilon_{i+1}^3) \leq |B| \leq |U^w_\omega| : \text{ we need at most } M(B) \text{ balls } (B(k))_{1 \leq k \leq M(B)} \text{ with diameter } \frac{1}{4} |U^w_\omega| \exp(-(C + 4)|w|\epsilon_{i+1}^3) \text{ that can cover } B \text{ with } M(B) \leq \Gamma(d)(4 \exp((C + 4)|w|\epsilon_{i+1}^3))^d, \text{ where } \Gamma \text{ is a function just depends on } d \text{ since we are dealing with problems in } \mathbb{R}^d. \] For these Balls we have the estimate above. Consequently,
\[ \eta_\omega(B) \leq \sum_{k=1}^{M(B)} \eta_\omega(B(k)) \]
\[ \leq \sum_{k=1}^{M(B)} 2^{d+1} Q(d)(3|B(k)|)^{q_1(1-4\varepsilon_1)} - \varepsilon_1^2 - \varepsilon_1^3 \]
\[ \leq \Gamma(d)(4 \exp((C + 4)|w|)(\varepsilon_i + 1)^3) \cdot 2^{d+1} Q(d)(3|B|)^{q_1(1-4\varepsilon_1)} - \varepsilon_1^2 - \varepsilon_1^3 \]
\[ \leq 2^{3d+1} \cdot Q(d) \cdot \Gamma(d)(3|B|)^{q_1(1-4\varepsilon_1)} - \varepsilon_1^2 - \varepsilon_1^3 \]
\[ \leq 2^{3d+1} \cdot Q(d) \cdot \Gamma(d)(3|B|)^{q_1(1-4\varepsilon_1)} - \varepsilon_1^2 - \varepsilon_1^3 |B|^{-\frac{(C+4)^3}{c^2}} \]

Finally,
\[ \eta_\omega(B) \leq 2^{3d+1} \cdot Q(d) \cdot \Gamma(d)(3|B|)^{q_1(1-4\varepsilon_1)} - \varepsilon_1^2 - \varepsilon_1^3 |B|^{-\frac{(C+4)^3}{c^2}}. \] (41)

It follows from the estimations (39),(40) and (41) that
\[ \dim_H(\eta_\omega) \geq \lim_{t \to \infty} q_t = q_0. \]

then \( \dim_H K_\omega \geq q_0, \) so is \( \dim_H W(\omega, \phi) \geq q_0. \)

6. Remarks of Corollary 1 and Corollary 2

This section is mainly to explain the result in corollary 1 and corollary 2. For the upper bound of the Hausdorff dimension, it is the same in section 4, where we use a very natural cover to get the control of the upper bound.

Now we begin to explain the lower bound.

For corollary 1. The lower bound is almost the same of section 5 except the preparation of the choice of \( v'(l) \) in step 2.

The important procedure of choosing \( p \) and \( v'(l) \) in step 2 is to search a point \( x \in X_{\omega}^{w*ve(l)} \) so that to make sure the existence of them.

Let us compare \( z_{\sigma,\omega} \in X_{\sigma,\omega}^{w} \) and (5) in assumption of corollary 1.

In step 2 of the proof of theorem 5.1, for given \( w \) and \( v(l), (8) \) implies for \( 1 \leq s \leq l(\sigma^{m+n+N+n+k}+N') \), we can joint the words \( w*ve(l) \) and \( s = w*ve(l)*s \) and \( z_{\sigma^{m+n+N+n+k}+N'} \in X_{\sigma^{m+n+N+n+k}+N'} \) ensures that we can find \( s \) with \( z_{\sigma^{m+n+N+n+k}+N'} \in X_{\sigma^{m+n+N+n+k}+N'}^{w*ve(l)}. \) So we have
\[ x = g_{w*ve(l)}^{w*ve(l)}(z_{\sigma^{m+n+N+n+k}}) \in X_{\omega}^{w*ve(l)} \subseteq X_{\omega}^{w*ve(l)}, \]
and then the existence of \( p \) and \( v'(l) \).

For corollary 1, we asked \( M \) large enough such that (8) can be changed as
\[ P(\{\omega \in \Omega : M(\omega) \leq M, M'(\omega) \leq M\}) > 7/8. \]

Then continue the process in section 3. For given \( w \) and \( v(l) \), assumption (5) implies that there exists \( v \) in \( \Sigma_{\sigma^{m+n+N+n+k}} \) with \( k \leq M \leq N \) such that \( w*ve(l)v \) in \( \Sigma_{\sigma^{m+n+N+n+k}} \) and
\[ x = g_{w*ve(l)}^{w*ve(l)}(z_{\sigma^{m+n+N+n+k}}) \in X_{\omega}^{w*ve(l)}, \]
This will also implies the existence of \( p \) and \( v'(l) \). In fact the word \( w*ve(l)v \) plays the similar role as \( w*ve(l)*s \) in the proof of theorem 5.1.
For corollary 2. Define $z^v_\omega = T^v_\omega x^v_\omega = x^v_\omega \in X_{\sigma|\omega}^{v}$ and $x^v_\omega \in X_\omega$, we get $z^v_\omega \in X_{\sigma|\omega}^{v}$ (6) in the assumption implies (5) (if we do not distinguish $M'$ and $M''$), so for given $w$ and $v(l)$, we can also find $\tilde{v}$ such that $w \ast v(l) \tilde{v} \in \Sigma_{\omega, m+N+n+k}$, and $z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \in X_{\omega}^{w \ast v(l) \tilde{v}}$ with $k \leq N$. So that we can also choose $p$ and $v'(l)$ as in step 2 in the proof of theorem 5.1.

Noticing $T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} = z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \in X_{\omega}^{w \ast v(l) \tilde{v}} \cap X_{\omega}^{w \ast v(l) \tilde{v}}$,

$$\|T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - x\| \leq \|T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \|$$

$$+ \|T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \|$$

$$+ \|z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - x\|$$

$$= \|T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \|$$

$$+ \|z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - x\|$$

Since $T_{\omega}^{w \ast v(l) \tilde{v}}$ is essentially expanding, the term $\|z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - x\|$ can be ignored with respect the term $\|T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} - T_{\omega}^{w \ast v(l) \tilde{v}} z^{w \ast v(l) \tilde{v}}_{\sigma m+N+n+k, \omega} \|$, which means for any $x \in U_{\omega}^{w \ast v(l) \ast}$, we can also imply (32). The control of the lower bound of the Hausdorff dimension for $W''(\phi, \omega)$ is almost the same as for $W'(\phi, \omega)$ if we choose proper $\{z^v_\omega : \omega \in \Omega, v \in \Sigma_{\omega, s}\}$.  

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