Convergence analysis of a characteristics-based hybrid method for multicomponent transport in porous media

Prabir Daripa* and Sourav Dutta†
Department of Mathematics, Texas A&M University
College Station, TX 77843-3368

May 26, 2018

Abstract

In this paper, convergence of a characteristics-based hybrid method recently introduced in Daripa & Dutta (J. Comput. Phys., 335:249-282, 2017) has been proved. This method which combines a discontinuous finite element method and a modified method of characteristics (MMOC) has been successfully applied to solve a coupled, nonlinear system of elliptic and transport equations that arise in multicomponent two-phase porous media flows. The novelty in this paper is the convergence analysis of the MMOC procedure for a nonlinear system of transport equations. For this purpose, an analogous single-component system of transport equations has been considered and possible extension to multicomponent systems has been discussed. Error estimates have been obtained and these estimates have also been validated by realistic numerical simulations of flows arising in enhanced oil recovery processes.

Keywords — multicomponent two-phase flow, finite difference method, method of characteristics, convergence analysis, error estimate, numerical simulations

AMS Subject Classifications — 65M12, 65M25, 65M06, 76S05

1 Introduction

Recently, in Daripa and Dutta [1] we developed a hybrid numerical method for solving a coupled system of elliptic and transport equations that arise in modeling multicomponent multiphase porous media flow in the context of chemical Enhanced Oil Recovery (EOR) by Surfactant-Polymer-flooding (SP-flooding). The model there involves two immiscible fluids (water and oil) with two components (polymer and surfactant) present in one of the fluids (aqueous phase). The hybrid method is derived from a non-traditional discontinuous finite element method and a time implicit finite difference method based on the Modified Method Of Characteristics (MMOC). Numerical results obtained with this method for a variety of initial data in rectilinear and radial geometries are in excellent agreement qualitatively with physics based expectation and converge under mesh refinement. In some cases where exact solutions are available, numerical results are in excellent agreement with the exact ones as well.

*Author for correspondence (email:daripa@math.tamu.edu, ORCID ID: orcid.org/0000-0002-8771-0149)
†email:sdutta@math.tamu.edu, ORCID ID: orcid.org/0000-0002-7051-175X
In this paper, we present a convergence analysis of the numerical method. For the analysis, we consider a reduced system of equations in one spatial dimension involving only one component (polymer). This reduced system models chemical Enhanced Oil Recovery (EOR) by polymer flooding in one-dimension as opposed to polymer flooding in two-dimensions \[2, 3\]. Previous works \[4, 5\] on the convergence analysis of the MMOC-based methods have been without components and the present work builds on that work by adding a component which complicates the analysis in a significant way. This is due to the coupling of the coefficients involved in the transport equations, as will be abundantly clear from the analysis presented here. As discussed in the paper, the convergence analysis presented here can be extended to the original problem involving two components and in two-dimensions. For the purpose of validating the error estimates obtained from the analysis, we carry out numerical simulations of polymer flooding and compute $L^2$ and $L^\infty$ error norms for the numerical solutions.

To this end, it is worth citing some methods, without being exhaustive by any means, on numerically solving similar set of partial differential equations arising in porous media flows. These methods can be broadly categorized into two classes: purely Eulerian and Eulerian-Lagrangian. Some of the methods which fall under Eulerian class are locally mass conservative finite volume methods \[6, 7\] and finite element based methods such as control volume, discontinuous Galerkin \[8, 9\] and mixed finite element \[10\] which have high order accuracy and have been applied for numerical simulations of porous media flows. Some of the methods which can be grouped under Eulerian-Lagrangian class are front-tracking methods \[2, 3\] and MMOC based methods \[4, 5\]. There are many variants of these methods such as modified method of characteristics with adjusted advection \[11\], the Eulerian-Lagrangian localized adjoint method \[12\], and the characteristic mixed finite element method \[13\], to name a few. Error estimates and convergence analysis of most of these methods have been carried out (see \[4, 13, 14\]). But none of these analyses include systems with components. The present analysis is on a system with components.

The analysis involves estimation of errors introduced by the finite difference discretization of the derivatives, by the linear interpolation to compute solutions at points where the characteristic curves intersect the computational grid and by the linearization of the coefficients. These coefficient functions in the two transport equations depend on both the wetting phase saturation and the component concentration. This coupling creates an additional challenge for the analysis of the multicomponent system. The finite difference discretization errors are estimated using multi-variable Taylor Series. The errors due to the linear interpolation are estimated using the Peano kernel theorem \[4\] and the errors due to quasi-linear approximation of the nonlinear coefficients are estimated using various inequalities including the Cauchy-Schwarz and generalized arithmetic mean-geometric mean (AM-GM) inequalities. The transport equations are tested against the error variables and some of the resulting inner products are replaced by the estimates mentioned above. The transport equations are then rewritten in forms that allow us to estimate the discrete $L^2$ errors in the aqueous phase saturation and the component concentration. Taking into account the time discretization, a discrete Gronwall type inequality is finally used to obtain the desired estimates.

The rest of the paper is laid out as follows. In section 2.1, the governing equations for incompressible, multicomponent, immiscible two-phase flow of fluids through porous media. In section 2.2, we present the numerical method: the computational grids, the non-traditional discontinuous finite element method, the MMOC based finite difference scheme for the transport equations and the computational algorithm. In section 3, we present the convergence analysis of the method and the error estimates obtained. We present the numerical results and compare them with the theoretical
error estimates in section 4. Finally section 5 contains concluding remarks.

2 Background

2.1 Model

In [1], a system of equations governing two-phase, two-component (SP-flooding) flow through porous media has been presented. With the possibility of some potential overlap, here we present the model for a single component flow (polymer flooding) system which will be later used for the analysis of the numerical method.

Let \( \Omega \subset \mathbb{R}^2 \) represent a porous medium with boundary \( \partial \Omega \). The incompressible and immiscible flow of the wetting phase (water or an aqueous solution of polymer and/or surfactant) and the non-wetting phase (oil) is described by a combination of the multiphase extension of Darcy’s law (see [15]) for each phase and transport equations for each component. Let \( s_j \) denote the saturation (volume fraction), \( v_j \) denote the velocity, \( p_j \) denote the phase pressure and \( q_j \) denote the volumetric injection/production rate of phase \( j \) where \( j = o \) and \( j = a \) denote the non-wetting and the wetting phases respectively. We recall from Daripa & Dutta [1] the transport equations

\[
\phi \frac{\partial s_j}{\partial t} + \nabla \cdot v_j = q_j, \quad (x, t) \in \Omega \times (0,T], \quad j = a, o,
\]

and the equation for conservation of mass of any component dissolved in the aqueous phase

\[
\phi \frac{\partial (c s_a)}{\partial t} + \nabla \cdot (c v_a) = c^i q_a \delta (x - x^i) + c q_a \delta (x - x^p), \quad (x, t) \in \Omega \times (0,T],
\]

where \( c \) is the concentration (volume fraction in the aqueous phase) of the dissolved component and \( c^i \) is the concentration of the component in the injected fluid. Also, \( x^i \) is the location of the injection well, \( x^p \) is the location of the production well and \( \delta \) is the Dirac-delta function. Using conservation of momentum of each phase, the phase velocity \( v_j \) is given by the Darcy-Muskat law

\[
v_j = -K(x) \lambda_j \nabla p_j, \quad x \in \Omega, \quad j = a, o.
\]

Here \( \phi \) is the porosity (taken to be constant in the numerical experiments in this study), \( K(x) \) is the absolute permeability tensor of the porous medium, \( \lambda = \lambda_a(s, c) + \lambda_o(s, c) \) is the total mobility and \( \lambda_j = k_{rj}/\mu_j \) is the phase mobility where \( k_{rj} \) is the relative permeability and \( \mu_j \) is the viscosity of phase \( j \). In addition to the above, the capillary pressure \( p_c \) is defined by

\[
p_c = p_o - p_a.
\]

Since the porous medium is initially saturated with the two phases, we have

\[
\sum_{j=o,a} s_j = 1.
\]

The combination of the above equations produces a system of strongly coupled nonlinear equations which can be potentially degenerate. In order to avoid this difficulty, we reformulate the problem by
using a fictitious pressure \( p \), to be called the global pressure below, for incompressible, immiscible two-phase flows with a single component (see [1]) defined by

\[
p = \frac{1}{2} (p_o + p_a) + \frac{1}{2} \int_{s_c}^s \left( \hat{\lambda}_o(\zeta, c) - \hat{\lambda}_a(\zeta, c) \right) \frac{dp_c}{d\zeta}(\zeta)d\zeta
- \frac{1}{2} \int \left( \int_{s_c}^s \frac{\partial}{\partial c} \left( \hat{\lambda}_o(\zeta, c) - \hat{\lambda}_a(\zeta, c) \right) \frac{dp_c}{d\zeta}(\zeta)d\zeta \right) \left( \frac{\partial c}{\partial x} \partial x + \frac{\partial c}{\partial y} \partial y \right),
\]

where \( \hat{\lambda}_j = \lambda_j/\lambda \) for \( j = a, o \) and \( s_c \) is the value of the aqueous phase saturation for which \( p_c(s_c) = 0 \). The global pressure is well defined for all values of \( s_a \) in \([s_{ra}, 1 - s_{ra}]\) where \( s_{ra} \) (resp. \( s_{ro} \)) is the residual saturation of the wetting phase (resp. non-wetting phase). If we write \( s_a = s \), an equivalent formulation of the problem is obtained in terms of the primary variables \((p, s, c)\)

\[- \nabla \cdot \left( K(x) \lambda \nabla p \right) = q_a + q_o, \quad x \in \Omega, \ t \in [0, T], \quad (7a)\]

\[\frac{\partial s}{\partial t} + \frac{\partial f}{\partial s} \cdot \nabla s + \nabla \cdot \left( D \nabla s \right) = G_s - \frac{\partial f}{\partial c} \cdot \nabla c, \quad x \in \Omega, \ t \in [0, T], \quad (7b)\]

\[\frac{\partial c}{\partial t} + \left( \frac{f}{s} \nabla + \frac{D}{s} \nabla s \right) \cdot \nabla c = G_c, \quad x \in \Omega, \ t \in [0, T], \quad (7c)\]

where \( D(s, c) = K(x) \lambda_o(s) f(s, c) \frac{dp_c(s)}{ds} \) and \( q_a + q_o \) is an appropriate source term for the pressure equation which denotes net volume of fluid containing the non-wetting phase \( (q_o) \) and the wetting phase \( (q_a) \), injected per unit volume per unit time. For numerical purposes, this is modeled by a finite number of point sources and sinks located at isolated points \( x^i \) and \( x^p \) respectively so that \( q_a + q_o = \sum_{k=i,p} q_k \delta(x - x^k) \). Also, \( G_s = (1 - f) \xi \xi \delta(x - x^i) \) and \( G_c = (c^1 - c) \frac{\partial}{\partial c}, \frac{c}{c} \xi \xi \delta(x - x^i) \) are the source terms for the transport equation where \( \xi \xi \xi \xi \) is the volumetric flux at the injection/production points. We also make following practical and physical assumptions, namely

\[q_a = Q \quad \text{and} \quad q_o = 0 \quad \text{at} \quad x = x^i, \quad (8a)\]

\[q_a = -\frac{\lambda_a}{\lambda} Q \quad \text{and} \quad q_o = -\frac{\lambda_o}{\lambda} Q \quad \text{at} \quad x = x^p, \quad (8b)\]

which mean that oil is never injected and the fluid mixture obtained at the production well is proportional to the resident fluid at the point. The following initial and boundary conditions are prescribed.

\[\forall x \in \Omega : \quad \xi(s(x), 0) = s_0(x) \quad \& \quad c(x, 0) = c_0(x), \quad (9a)\]

\[\forall (x, t) \in \partial \Omega \times (0, T) : \quad \nabla s \cdot \hat{n} = 0, \quad \nabla c \cdot \hat{n} = 0 \quad \& \quad v_j \cdot \hat{n} = 0 \quad (j = a, o), \quad (9b)\]

where \( \hat{n} \) denotes the outward unit normal to \( \partial \Omega \). Several models of relative permeability, \( k_{rj} \) and capillary pressure are available in the literature (see [16, 17]). For the numerical simulations presented in this study, we use the following modification to the van Genuchten model made by Parker et al. ([18]),

\[k_{ra}(s) = s^{1/2} \left( 1 - (1 - s^{1/m})^m \right)^2, \quad (10a)\]

\[k_{ro}(s) = (1 - s)^{1/2} \left( 1 - s^{1/m} \right)^{2m}, \quad (10b)\]

\[p_c(s) = \frac{1}{\alpha_0} \left( s^{1/m} - 1 \right)^{1-m}, \quad (10c)\]
where \( s_e = (s - \sigma_{ra})/(1 - \sigma_{ra}) \) is the effective saturation. The values of the parameters \( m \) and \( \alpha_0 \) in the above model are known to depend on the interfacial tension, \( \sigma_0 \) between the non-wetting and the wetting phases. In our study below we take \( m = 2/3 \) and \( \alpha_0 = 0.125 \) (see [19]). Alternatively Corey-type imbibition relations can also be used (see [16]). These models assume that \( p_c \) and \( k_{ij} (j = a, o) \) are nonlinear functions of only the wetting phase saturation, \( s \), which is a valid assumption for polymer flooding.

2.2 Numerical scheme

The system of coupled transport equations given by eqs. (7b) and (7c) is solved using a combination of the MMOC and an implicit time finite difference scheme. For the computational grid, we partition the domain \( \Omega \) into rectangular cells. Given positive integers \( I, J \in \mathbb{Z}^+ \), set \( \Delta x = (x_{\text{max}} - x_{\text{min}})/I = 1/I \) and \( \Delta y = (y_{\text{max}} - y_{\text{min}})/J = 1/J \). We define a uniform Cartesian grid \((x_i, y_j) = (i \Delta x, j \Delta y)\) for \( i = 0, ..., I \) and \( j = 0, ..., J \). Each \((x_i, y_j)\) is called a grid point. For the case \( i = 0, I \) or \( j = 0, J \), a grid point is called a boundary point, otherwise it is called an interior point. In general, the grid size is defined as \( h = \max(\Delta x, \Delta y) > 0 \). However, in this paper we use an uniform spatial grid: \( \Delta x = \Delta y = h = 1/N \).

The elliptic flow equation (7a) for global pressure is solved using a discontinuous finite element method on a non-body-fitted grid which is constructed in the following way. We introduce uniform triangulations inside the grid generated for the transport equations (7b) and (7c). This means every rectangular region \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\) is cut into two pieces of right triangular regions: one is bounded by \( x = x_i, y = y_j \) and \( y = \frac{y_{j+1} - y_j}{x_{i+1} - x_i} (x - x_i) + y_j \), the other is bounded by \( x = x_{i+1}, y = y_{j+1} \) and \( y = \frac{y_{j+1} - y_j}{x_{i+1} - x_i} (x - x_{i+1}) + y_j \). Collecting all those triangular regions, also called elements, we obtain a uniform triangulation, \( L^h = \{\kappa|\kappa\text{ is a triangular element}\} \). We may also choose the hypotenuse to be \( y = \frac{y_{j+1} - y_j}{x_{i+1} - x_i} (x - x_i) + y_j \), and get another uniform triangulation from the same Cartesian grid. There is no conceptual difference on these two triangulations for our method.

2.2.1 Pressure equation

The elliptic equation describing the evolution of global pressure is given by

\[
\begin{align}
-\nabla \cdot (K(x) \lambda \nabla p) &= \tilde{q}, \quad x \in \Omega \setminus \Sigma, \\
(K(x) \lambda \nabla p) \cdot \hat{n} &= 0, \quad x \in \partial \Omega, 
\end{align}
\]  

where \( \tilde{q} = q_a + q_o \) and \( \Sigma \) denotes the union of the interfaces that separate \( \Omega \) into several subdomains. However, for simplicity of exposition we assume here that we have only two separated subdomains, \( \Omega^+ \) and \( \Omega^- \) separated by an interface \( \Sigma \) (see Figure 1) which, as we will see later, is also the initial configuration of the quarter five-spot domain for all of our numerical simulations.

The following kinematic condition holds at the interface \( \Sigma \).

\[
[K(x) \lambda \nabla p \cdot \hat{n}]_\Sigma = 0, 
\]  

where \( \hat{n} \) is the outward unit normal which points from \( \Omega^- \) to \( \Omega^+ \) and \([\cdot]\) denotes a jump. We assume the boundary \( \partial \Omega \) and the interface \( \Sigma \) to be Lipschitz continuous. Hence a unit normal vector, \( \hat{n} \) can be defined a.e. on \( \Sigma \). This problem is solved using a non-traditional finite element formulation.
Figure 1: Initial configuration of the solution domain with the (blue) arc representing the initial position of the discontinuity (see [20]) which is second order accurate in the $L^\infty$ norm for matrix coefficient elliptic equations with discontinuities across the interfaces. The weak formulation of eqs. (11a) and (11b) in the usual Sobolev spaces $H^1(\Omega)$ with $\psi \in H^1(\Omega)$ is given by

$$
\int_{\Omega^+} K\lambda \nabla p \nabla \psi + \int_{\Omega^-} K\lambda \nabla p \nabla \psi - \int_{\partial\Omega} K\lambda \psi \nabla p \cdot \hat{n} = \int_{\Omega} \tilde{q} \psi.
$$

(13)

The elements, $\kappa$ of triangulation, $L^h$, are classified into regular cells and interface cells. We call $\kappa$ a regular cell if its vertices are in the same subdomain and an interface cell when its vertices belong to different subdomains. For an interface cell, $\kappa = \kappa^+ \cup \kappa^-$ where $\kappa^+$ and $\kappa^-$ are separated by a line segment $\Sigma^h_k$, obtained by joining the two points where the interface $\Sigma$ intersects the sides or the vertices of that interface cell. A set of grid functions, $H_{1,h} = \{\omega^h \mid \omega^h = \omega_{i,j} ; 0 \leq i \leq I, 0 \leq j \leq J\}$ are defined on the grid points of the mesh $L^h$. An extension operator $U^h : H_{1,h} \to H^1(\kappa)$ is constructed as follows. For any $\phi^h \in H_{1,h}$, $U^h(\phi^h)$ is a piecewise linear function and matches $\phi^h$ on the grid points. In a regular cell, it is a linear function that interpolates the values of $\phi^h$ at the grid points. In an interface cell, it consists of two pieces of linear functions, one each defined on $\kappa^+$ and $\kappa^-$. The location of the discontinuity of the extended function $U^h(\phi^h)$ in an interface cell is on the line segment $\Sigma^h_k$. Hence an interface jump condition on the pressure, $p$, if there is one, can be imposed on the two end points of this line segment at $\{\partial\kappa\} \cap \{\Sigma^h_k\}$ while the interface jump condition, eq. (12), is imposed at the middle point of $\Sigma^h_k$. For the construction of such extension operators for discontinuous coefficient elliptic equations, see [20, 21]. The extension operators for the pressure equation, given by eqs. (11a) and (11b), have been explicitly constructed in [1]. Using the extension functions as discussed above (also see [1]), the discrete version of the weak
formulation eq. (13) can be reformulated to finding a discrete function \( \phi^h \in H^{1,h} \) such that

\[
\sum_{K \in L^h} \left( \int_{K^+} K \lambda \nabla U^h(\phi^h) \nabla U^h(\psi^h) + \int_{K^-} K \lambda \nabla U^h(\phi^h) \nabla U^h(\psi^h) \right) - \sum_{K \in L^h} \int_{\partial K} K \lambda U^h(\psi^h) \nabla U^h(\phi^h) \cdot \hat{n} = \sum_{K \in L^h} \left( \int_{K^+} \tilde{q} U^h(\psi^h) + \int_{K^-} \tilde{q} U^h(\psi^h) \right), \quad \forall \psi^h \in H^{1,h}.
\]

(14)

It can be shown that if \( K(x) \) is positive definite, then the matrix obtained for the linear system of the discretized weak form, eq. (14), is also positive definite and is therefore invertible.

### 2.2.2 Transport equations

The transport equations (7b) and (7c) are solved using a combination of a finite difference method with the Modified Method Of Characteristics (MMOC). At first we rewrite eqs. (7b) and (7c) as

\[
\phi \frac{\partial s}{\partial t} + \frac{\partial f}{\partial s} \mathbf{v} \cdot \nabla s + \nabla \cdot (D \nabla s) = g_s - \frac{\partial f}{\partial c} \mathbf{v} \cdot \nabla c, \quad \text{(15a)}
\]

\[
\phi \frac{\partial c}{\partial t} + \left( \frac{f}{s} \mathbf{v} + \frac{D}{s} \nabla s \right) \cdot \nabla c + cg = gc, \quad \text{(15b)}
\]

where \( D(s,c) = K(x) \lambda_o(s) f(s,c) \frac{dp_o(s)}{ds}, g_s = (1-f)Q \delta(x-x^i), g = \frac{Q_s}{s} \delta(x-x^i) \) and \( g_c = \frac{sQ}{s} \delta(x-x^i) \).

In eq. (15a) we replace the advection term \( \phi \frac{\partial s}{\partial t} + \frac{\partial f}{\partial s} \mathbf{v} \cdot \nabla s \) by a derivative along its characteristic direction in the following way

\[
\frac{\partial}{\partial \tau_s} = \frac{1}{\psi_s} \left( \phi \frac{\partial}{\partial t} + \frac{\partial f}{\partial s} \mathbf{v} \cdot \nabla \right),
\]

(16)

where \( \tau_s \) is used to parametrize the characteristics. Here \( \psi_s \) is a suitable normalization that simplifies the numerical discretization of the characteristic derivative and is defined by

\[
\psi_s = \left[ \phi^2 + \left( \frac{\partial f}{\partial s} \right)^2 |\mathbf{v}|^2 \right]^{1/2}.
\]

(17)

Then eq. (15a) is equivalently written in the form

\[
\psi_s \frac{\partial s}{\partial \tau_s} + \nabla \cdot (D \nabla s) = g_s - \frac{\partial f}{\partial c} \mathbf{v} \cdot \nabla c.
\]

(18)

For computation, we use the spatial grid described in the beginning of § 2.2 and the time interval [0, T] is uniformly divided into \( L \) subintervals of length \( \Delta t \) such that \( t^n = n \Delta t \) and \( T = L \Delta t \). We denote the grid values of the variables by \( w_{ij}^n = w(x_{ij}, t^n) \) where \( x_{ij} = x(ih, jh) \). Consider that the solution is known at some time \( t^n \) and the solution at a subsequent time \( t^{n+1} \) needs to be computed. Then starting from any point \( (x_{ij}, t^{n+1}) \) we trace backward along the characteristics to a point \( (\bar{x}_{ij}, t^n) \) where the solution is already known. As shown in Figure 2, the points \( p_1 = (x_{ij}, t^{n+1}) \) and
Figure 2: Discrete approximation of the characteristic curve from \( x_{ij} \) to \( x_{ij} \) in 1D

\( p_2 = (x_{ij}, t^n) \) lie on the same characteristic curve. From the equation of the characteristic curves given by

\[
\frac{dx}{dr_s} = \frac{1}{\phi} \frac{\partial f}{\partial s} v,
\]

we use numerical discretization to obtain an approximate value of \( \bar{x}_{ij} \) in the following way

\[
\bar{x}_{ij} = x_{ij} - \frac{\partial f}{\partial s}(s^n_{ij}, c^n_{ij}) v^n_{ij} \Delta t/\phi.
\]

Using the above equation, the derivative in the characteristic direction, defined by eq. (16), is approximated by

\[
\psi_s \frac{\partial s}{\partial \tau_s} \approx \psi_s \frac{s(x_{ij}, t^{n+1}) - s(\bar{x}_{ij}, t^n)}{[|x_{ij} - \bar{x}_{ij}|^2 + (\Delta t)^2]^{1/2}} = \phi \frac{s^n_{ij} - \bar{s}^n_{ij}}{\Delta t} \quad \text{(see § 3)}.
\]

This leads to the following implicit-time finite difference formulation for eq. (18)

\[
\phi \frac{s^n_{ij} - \bar{s}^n_{ij}}{\Delta t} + \nabla_h(\bar{D}\nabla_h s)^{n+1}_{ij} = (g_s)_{ij} - \left( \frac{\partial f}{\partial c} \right)_{ij}^n \left( v^n_{ij} \cdot \nabla_h c^n_{ij} \right), \quad (19)
\]

where

\[
\bar{s}^n_{ij} = s(\bar{x}_{ij}, t^n) \quad \& \quad \bar{D}^n_{ij} = D(\bar{s}^n_{ij}, c^n_{ij}),
\]

\[
\nabla_h(\bar{D}\nabla_h s)^{n+1}_{ij} = \bar{D}_{i+1/2,j} \frac{s^{n+1}_{i+1,j} - s^n_{i,j}}{\Delta x^2} - \bar{D}_{i-1/2,j} \frac{s^{n+1}_{i-1,j} - s^n_{i,j}}{\Delta x^2} + \bar{D}_{i,j+1/2} \frac{s^{n+1}_{i,j+1} - s^n_{i,j}}{\Delta y^2} - \bar{D}_{i,j-1/2} \frac{s^{n+1}_{i,j} - s^n_{i,j-1}}{\Delta y^2},
\]

\[
\bar{D}_{i+1/2,j} = \frac{D(\bar{s}^n_{i+1,j}, c^n_{i+1,j}) + D(\bar{s}^n_{i,j}, c^n_{i,j})}{2},
\]

\[
\bar{D}_{i,j+1/2} = \frac{D(\bar{s}^n_{i,j+1}, c^n_{i,j+1}) + D(\bar{s}^n_{i,j}, c^n_{i,j})}{2}.
\]
Following the same procedure as before we define the following equations, analogous to eqs. (16) and (17), for the concentration equation eq. (15b).

\[
\frac{\partial}{\partial \tau_c} = \frac{1}{\psi_c} \left( \phi \frac{\partial}{\partial t} + f_s v \cdot \nabla + \frac{D_s}{s} \nabla s \cdot \nabla \right),
\]

\[
\psi_c = \left[ \phi^2 + \left( \frac{f_s}{s} \right)^2 |v|^2 + \left( \frac{D_s}{s} \right)^2 |\nabla s|^2 \right]^{1/2}.
\]

The advection term \( \phi \frac{\partial c}{\partial t} + f_s v \cdot \nabla c + \frac{D_s}{s} \nabla s \cdot \nabla c \) is replaced by the derivative in the characteristic direction \( \tau_c \) given by \( \psi_c \frac{\partial c}{\partial \tau_c} \) where

\[
\psi_c \frac{\partial c}{\partial \tau_c} \approx \psi_c \frac{c(x_{ij}, t^{n+1}) - c(\bar{x}_{ij}^{c}, t^n)}{|x_{ij} - \bar{x}_{ij}|^2 + (\Delta t)^2} = \phi \frac{c_{ij}^{n+1} - c_{ij}^n}{\Delta t}.
\]

Here, \( \bar{c}_{ij}^n = c(\bar{x}_{ij}^{c}, t^n) \) is computed using an approximate value of \( \bar{x}_{ij}^{c} \) given by

\[
\bar{x}_{ij}^{c} = x_{ij} - \left( \left( \frac{f}{s} \right) (s_{ij}^n, c_{ij}^n) v + \left( \frac{D}{s} \right) (s_{ij}^n, c_{ij}^n) \nabla s \right) \Delta t \phi,
\]

where, as before, \( (\bar{x}_{ij}^{c}, t^n) \) and \( (x_{ij}, t^{n+1}) \) lie on the same characteristic curve. Here the superscript ‘c’ is used to denote the characteristic curves associated with the polymer transport equation Eq. (15b). Thus we arrive at the following implicit-time finite difference formulation for Eq. (15b)

\[
\phi \frac{c_{ij}^{n+1} - c_{ij}^n}{\Delta t} + (g)_{ij}^n c_{ij}^{n+1} = (g_c)_{ij}^n.
\]

Hence Eq. (19) and Eq. (20) form the finite difference approximation of the transport equations, Eq. (15a) and Eq. (15b) respectively.

The pseudocode (see Algorithm 1) for the method is given below. Here, \( s_0^{\sigma_0} \) is the initial resident wetting phase saturation or the amount of water (wetting phase) present in the reservoir before the flood simulation starts.

### 3 Convergence study and error analysis

Let \( s_i^n = s(x_i, t^n) \) be the grid values of the actual solution of the saturation equation (15a) and \( w_i^n = w(x_i, t^n) \) be the grid values of the numerical solution of that equation where \( x_i = ih \) and \( t^n = n\Delta t \). Similarly, let \( p_i^n \) and \( r_i^n \) be the grid values of the actual and the numerical solutions respectively of eq. (11a). Also, let \( c_i^n \) and \( m_i^n \) be the grid values of the actual and the numerical solutions respectively of eq. (15b). Finally, let \( v_i^n \) and \( z_i^n \) be the grid values of the actual and the numerical solutions respectively of the total velocity given by \( v = -K\lambda \nabla p \). The errors in the numerical approximation are defined as follows.

\[
\zeta_i^n = s_i^n - w_i^n, \quad \pi_i^n = p_i^n - r_i^n, \quad \& \quad \theta_i^n = c_i^n - m_i^n.
\]
Algorithm 1 Polymer flooding simulation

1: procedure
   Set up Cartesian grid and FE Mesh and a permeability field

2:   \( i, j \leftarrow 1, \ldots, N; \ h \leftarrow \frac{1}{N} \) \hspace{1cm} \( \triangleright (N \times N \text{ is the grid size}) \)
3:   \( \Sigma \leftarrow \text{Initial interface} \) \hspace{1cm} \( \triangleright \Sigma = \partial \Omega^+ \cup \partial \Omega^- \)
4:   \( K(x) \leftarrow \text{choose type of heterogeneity} \)

Set model parameters

5:   \( \mu_o, \mu_w, s_{ro}, s_{ra}, Q, c_0, s_{0} \) \( \leftarrow \) values from Table 1

Initialization

6:   \( (s, c) \leftarrow \begin{cases} (1 - s_{ro}, c_0) & x \in \Omega^+ \\ (s_{0}, 0) & x \in \Omega^- \end{cases} \)
7:   \( t \leftarrow 0 \)
8:   \( \Delta t \leftarrow \text{value} \) \hspace{1cm} \( \triangleright \Delta t \text{ chosen for desired accuracy} \)

Computation loop

9:   while \( (s(x_{N,N}, t) \leq 1 - s_{0} \) \( \text{ and } t < T_{stop} \) \( ) \) do
10:      Compute \( \{\mu_a, \lambda_a, \lambda_o, \lambda, p_c\} \) using \( (s^n, c^n, v^{n-1}) \)
11:      Solve the global pressure equation for \( p^n, v^n \)
12:      Recompute \( \{\mu_a, \lambda_a, \lambda_o, \lambda, p_c\} \) using \( (s^n, c^n, v^n) \)
13:      Solve the transport equations for \( s^{n+1} \) and \( c^{n+1} \)
14:      \( t \leftarrow t + \Delta t \)
15:      close;
We define the following discrete norms for any \( u \in W^{l,p}(\Omega) \), \( v \in L^2(\Omega) \) and \( w \in L^\infty(\Omega) \) where \( \Omega = [0,1]^2 \).

\[
\|u\|_{l,p} = \left( \sum_{k=0}^{l} \left( \sum_i h \left| \frac{d^k u_i}{dx^k} \right|^p \right) \right)^{1/p}, \quad |u|_{l,p} = \left( \sum_i h \left| \frac{d^k u_i}{dx^k} \right|^p \right)^{1/p} \\
\|v\| = \left( \sum_i h|v_i|^2 \right)^{1/2}, \quad w_\infty = \max_i |w_i|
\]

In particular, \( \| . \| \) and \( \langle . \rangle \) denote the discrete \( L^2 \) norm and the associated inner product respectively. For analysis, we consider a reduced system of equations (see eqs. (15a) and (15b)) in one spatial dimension as given by

\[
\phi \frac{\partial s}{\partial t} + b \frac{\partial s}{\partial x} + \frac{\partial}{\partial x} \left( D \frac{\partial s}{\partial x} \right) = F, \quad s(x,0) = s_0(x); \quad x \in \Omega \setminus \partial \Omega \tag{21a}
\]

\[
\phi \frac{\partial c}{\partial t} + a \frac{\partial c}{\partial x} + Gc = H, \quad c(x,0) = c_0(x); \quad x \in \Omega \setminus \partial \Omega \tag{21b}
\]

where \( b(s,c) = \frac{\partial f}{\partial s} v, a(s,c) = (\frac{\partial f}{\partial c} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x}), F(s,c) = g_s - \frac{\partial f}{\partial c} \frac{\partial c}{\partial x}, G(s) = g \) and \( H(s) = g_c \). Then the characteristic finite difference approximation of Eq. (21a) and Eq. (21b) are given by

\[
\phi_i \frac{w_i^n - w_i^{n-1}}{\Delta t} + \delta_x (D \delta_x w_i^n) = F_i^n \quad w_i^0 = s_0(x_i), \tag{22}
\]

\[
\phi_i \frac{m_i^n - \bar{m}_i^{n-1}}{\Delta t} + G_i m_i^n = H_i^n \quad m_i^0 = c_0(x_i), \tag{23}
\]

where \( w_i^{n-1} = w(x_i, t_{n-1}^i), \quad \bar{x}_i^n = x_i - b(w_i^{n-1}, m_i^{n-1}) \Delta t/\phi_i, \tag{24}\)

\( \text{and} \quad \bar{m}_i^{n-1} = m(x_i, t_{n-1}^i), \quad \bar{x}_i^n = x_i - a(w_i^{n}, m_i^{n-1}) \Delta t/\phi_i. \tag{25}\)

From Eq. (21a) and Eq. (22) we have the following

\[
\psi_s \frac{\partial s}{\partial \tau} + \frac{\partial}{\partial x} \left( D \frac{\partial s}{\partial x} \right) = F(s^n, c^n), \quad s(x,0) = s_0(x), \tag{26a}
\]

\[
\phi_i \frac{w_i^n - w_i^{n-1}}{\Delta t} + \delta_x (D \delta_x w_i^n) = F(w_i^n, m_i^n) \quad w_i^0 = s_0(x_i). \tag{26b}
\]

In the following analysis \( M, \bar{M}, \tilde{M}, M_k(k \in \mathbb{Z}^+) \) and \( C \) are generic constants independent of the time step and space discretizations \( \Delta t \) and \( h \) respectively. We will also assume the following bounds on the porosity, \( \phi_0 \leq \phi(x) \leq \phi^* \). For the rest of the analysis of the water saturation equation (26b), with slight abuse of notation, we will write \( \bar{x}_i^n \) and \( \bar{m}_i^n \) to mean \( \bar{x}_i^{s,n} \) and \( \bar{m}_i^{s,n} \) respectively. In the next two lemmas we estimate the errors introduced by approximating the derivative in the characteristic direction and the second order derivative term in eq. (26a) with their finite difference discretizations given in eq. (26b).

Lemma 3.1 The error in approximating the characteristic derivative in eq. (26a) is given by

\[
\psi_i \left( \frac{\partial s}{\partial \tau} \right)_i^n \bar{m}_i^{n} - \phi_i \frac{s_i^n - s_i^{n-1}}{\Delta t} = O \left( \frac{\partial^2 s^*}{\partial \tau^2} \right) \Delta s.
\]
where \( s_{i}^{n-1} = s(x_i, t_{i}^{n-1}) \) with \( x_i = x_i = b(s_i^n, c_i^n) \Delta t / \phi_i \).

**Proof.** Let \( p_1 = (x, t^n) \) be a point on the grid (see Figure 2) and the characteristic that passes through this point intersects the previous time level at \( p_2 = (\bar{x}, t^{n-1}) \) where \( \bar{x} = x - b(s, c)/\phi(x) \Delta t \) and let \( \Delta \tau = [(x - \bar{x})^2 + (t^n - t^{n-1})^2]^{1/2} \). Hence \( \Delta \tau = \psi_s / \phi \Delta t \). Using the Taylor series expansion along the characteristic direction, we write

\[
s(p_1 - \Delta \tau) = s(p_1) - \Delta \tau \frac{\partial s}{\partial \tau} + \frac{\Delta \tau^2}{2} \frac{\partial^2 s}{\partial \tau^2},
\]

where \( \frac{\partial^2 s}{\partial \tau^2} \) is some evaluation of the second derivative along the characteristic segment between \( p_2 \) and \( p_1 \). In the convection dominated case, this second derivative is relatively much smaller than \( \frac{\partial^2 x}{\partial \tau^2} \) or \( \frac{\partial^2 z}{\partial \tau^2} \) [4]. This is rewritten as

\[
\Delta \tau \frac{\partial s}{\partial \tau} = s^n - s^{n-1} + \frac{\Delta \tau^2}{2} \frac{\partial^2 s}{\partial \tau^2}.
\]

Using \( \frac{\psi_s}{\Delta \tau} = \frac{\phi}{\Delta t} \) we obtain

\[
\frac{\psi_s}{\Delta \tau} \frac{\partial s^n}{\partial \tau} - \phi \frac{s^n - s^{n-1}}{\Delta t} = \frac{\psi_s}{\Delta \tau} \frac{\Delta \tau^2}{2} \frac{\partial^2 s}{\partial \tau^2} = \frac{\psi_s}{\Delta \tau} \frac{\Delta \tau^2}{2} \frac{\partial^2 s}{\partial \tau^2} = \frac{\psi_s}{\Delta \tau} \frac{\partial^2 s}{\partial \tau^2}.
\]

This leads to the final result:

\[
\psi_s,i \left( \frac{\partial s}{\partial \tau} \right)_i^n - \phi_i \frac{s_i^n - s_i^{n-1}}{\Delta t} = O \left( \left| \frac{\partial^2 s}{\partial \tau^2} \right| \Delta \tau \right).
\]

Using Lemma 3.1 in Eq. (26a), we estimate the error introduced by numerical discretization of the characteristic derivative as

\[
\phi_i \left( \frac{s_i^n - s_i^{n-1}}{\Delta t} + \frac{\partial}{\partial x} \left( D \frac{\partial s^n}{\partial x} \right) \right)_i = F(s_i^n, c_i^n) + O \left( \left| \frac{\partial^2 s}{\partial \tau^2} \right| \Delta \tau \right). \tag{27}
\]

Now we estimate the approximation error for the second order derivative term in the left hand side of Eq. (26a). By definition,

\[
\delta_x(D \delta_x w^n)_i = \frac{1}{h} \left( \bar{D}_{i+1/2}(\delta_x w^n)_{i+1/2} - \bar{D}_{i-1/2}(\delta_x w^n)_{i-1/2} \right) = \frac{1}{h^2} \left( \bar{D}_{i+1/2}(w^n_{i+1} - w^n_i) - \bar{D}_{i-1/2}(w^n_i - w^n_{i-1}) \right), \tag{28}
\]

where

\[
\bar{D}_{i+1/2} = \frac{1}{2} \left[ D(x_i, w^{n-1}_i) + D(x_{i+1}, w^{n-1}_{i+1}) \right] \quad \text{and} \quad \bar{D}_{i-1/2} = \frac{1}{2} \left[ D(x_{i-1}, w^{n-1}_{i-1}) + D(x_i, w^{n-1}_i) \right].
\]

The numerical approximation of the second order derivative in Eq. (21a) is given by

\[
\delta_x(D \delta_x s^n)_i = \frac{1}{h^2} \left[ D_{i+1/2}(s^n_{i+1} - s^n_i) - D_{i-1/2}(s^n_i - s^n_{i-1}) \right],
\]

where

\[
D_{i+1/2} = \frac{1}{2} \left[ D(x_i, s^n_i) + D(x_{i+1}, s^n_{i+1}) \right] \quad \text{and} \quad D_{i-1/2} = \frac{1}{2} \left[ D(x_{i-1}, s^n_{i-1}) + D(x_i, s^n_i) \right].
\]
Lemma 3.2 The finite difference approximation error of the second derivative term in eq. (26a) is given by

\[
\frac{d}{dx} \left( D \frac{d}{dx} s^n \right)_i - \delta_x (D \delta_x s^n)_i = O(h \|s^n\|_{3,\infty}).
\]

Proof. From the Taylor series expansion, we know that

\[
\left( \frac{du}{dx} \right)_i - u(x + h/2) - u(x - h/2) \frac{1}{h} = O \left( h^2 \|u\|_{3,\infty} \right).
\]

This can be rewritten as

\[
\left( \frac{du}{dx} \right)_i - \delta_x(u_i) = O(h^2 \|u\|_{3,\infty}).
\]

Using this estimate for the second derivative term, we obtain

\[
\frac{d}{dx} \left( D \frac{d}{dx} s^n \right)_i - \delta_x (D \delta_x s^n)_i = O \left( h^2 \left\| D \frac{d}{dx} s^n \right\|_{3,\infty} \right).
\]

Using the definition (28), we obtain

\[
\frac{d}{dx} \left( D \frac{d}{dx} s^n \right)_i = \frac{1}{h} \left( \left( D \frac{d}{dx} s^n \right)_{i+1/2} - \left( D \frac{d}{dx} s^n \right)_{i-1/2} \right) = O \left( h^2 \left\| D \frac{d}{dx} s^n \right\|_{3,\infty} \right).
\]

Using \((D \frac{d}{dx} s^n)_{i+1/2} = D_{i+1/2} \frac{ds^n}{dx}\)_{i+1/2} = D_{i+1/2} \left( \frac{s^n_{i+1} - s^n_i}{h} \right) + O(h^2 \left\| s^n \right\|_{3,\infty}) \), we continue as

\[
\frac{d}{dx} \left( D \frac{d}{dx} s^n \right)_i = \frac{1}{h} \left[ D_{i+1/2} \frac{s^n_{i+1} - s^n_i}{h} - O(h^2 \left\| s^n \right\|_{3,\infty}) \right] + \frac{1}{h} \left[ D_{i-1/2} \frac{s^n_i - s^n_{i-1}}{h} + O(h^2 \left\| s^n \right\|_{3,\infty}) \right] = O \left( h^2 \left\| D \frac{ds^n}{dx} \right\|_{3,\infty} \right).
\]

This leads to the final estimate

\[
\frac{d}{dx} \left( D \frac{d}{dx} s^n \right)_i = \frac{1}{h} \left[ D_{i+1/2} \frac{s^n_{i+1} - s^n_i}{h} - D_{i-1/2} \frac{s^n_i - s^n_{i-1}}{h} \right] = O(h \left\| s^n \right\|_{3,\infty}) + O(h^2 \left\| s^n \right\|_{4,\infty}) = O(h \left\| s^n \right\|_{3,\infty}).
\]

Using the result of Lemma 3.2, we rewrite Eq. (27) as

\[
\phi_i \frac{s^n_i - s^{n-1}_i}{\Delta t} + \delta_x (D \delta_x s^n)_i = F(s^n_i, c^n_i) + O \left( \left\| \frac{\partial^2 s^*}{\partial \tau^2} \right\| \Delta \tau \right) + O(h \left\| s^n \right\|_{3,\infty}). \tag{29}
\]

Subtracting Eq. (26b) from Eq. (29), we obtain

\[
\phi_i \frac{s^n_i - s^{n-1}_i}{\Delta t} - \phi_i \frac{w^n_i - \bar{w}^{n-1}_i}{\Delta t} + \delta_x (D \delta_x s^n)_i - \delta_x (D \delta_x w^n)_i
\]

\[
= F(s^n_i, c^n_i) - F(w^n_i, m^n_i) + O \left( \left\| \frac{\partial^2 s^*}{\partial \tau^2} \right\| \Delta \tau, h \left\| s^n \right\|_{3,\infty} \right).
\]
Recall that we define the numerical error in saturation as \( \zeta^*_i = s^*_i - w^*_i \). Using the definition of \( \zeta^*_i \) and rearranging terms, we rewrite the above as

\[
\frac{\phi_i}{\Delta t} \left( \frac{\zeta^*_i - (s^*_i - w^*_i)}{\Delta t} - \delta_x(\tilde{\mathcal{D}}\delta_x \zeta^*_i) \right) = F(s^*_i, c^*_i) - F(w^*_i, m^*_i) \\
+ O \left( \frac{\partial^2 s^*}{\partial t^2} \right) - \delta_x((\tilde{\mathcal{D}} - \mathcal{D})\delta_x s^*_i). \tag{30}
\]

In the above, the capillary dissipation coefficients \( D \) and \( \tilde{D} \) have been replaced with \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \) respectively and the signs associated with the terms have been reversed. This is because the definition of capillary pressure, given in Eq. (4), ensures that \( \frac{dp}{ds} \leq 0 \) which implies that \( D, \tilde{D} \leq 0 \). Hence, in Eq. (30) and in the rest of the analysis of the water transport equation (26b), we will use \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \) to denote the absolute values of the capillary dissipation coefficients \( D \) and \( \tilde{D} \) respectively. We assume that these will be bounded by \( D_s \leq \mathcal{D}, \tilde{\mathcal{D}} \leq D^* \).

Consider the first term on the left hand side of Eq. (30). Let \( \zeta^*_i = \mathcal{D}^n \zeta^*_i \) be the piecewise linear interpolant of \( \zeta^*_i \) such that \( \tilde{\zeta}^{n-1} = \mathcal{D}^n \tilde{\zeta}^{n-1}(\bar{x}_i) = \mathcal{D}^n s^{n-1}(\bar{x}_i) - w^{n-1}(\bar{x}_i) = \mathcal{D}^n s^{n-1}(\bar{x}_i) - w^{n-1} - i. \)

Then,

\[
\zeta^*_i - (s^*_i - w^*_i) = (\zeta^*_i - \tilde{\zeta}^{n-1}_i) + \mathcal{D}^n s^{n-1}(\bar{x}_i) - w^{n-1} - \tilde{\zeta}^{n-1}_i + w^{n-1} \\
= (\zeta^*_i - \tilde{\zeta}^{n-1}_i) - (s^{n-1}(\bar{x}_i) - s^{n-1}(\bar{x}_i)) - ((1 - \mathcal{D}^n) s^{n-1}(\bar{x}_i)). \tag{31}
\]

Below, we find estimates for the last two terms, A and B, of the right hand side of Eq. (31), followed by the estimate of the source term \( (F(s^*_i, c^*_i) - F(w^*_i, m^*_i)) \) on the right hand side of Eq. (30). Once we have these estimates, we can substitute Eq. (31) in Eq. (30), take inner products with \( \zeta^*_i \) and use the estimates to rewrite the equation.

**A. Estimate of the term A on the right hand side of Eq. (31):** This is carried out in several steps below.

\[
s^{n-1}(\bar{x}_1) - s^{n-1}(\bar{x}_i) \\
\leq \hat{M} \| s^{n-1} \|_{1,\infty} |\bar{x}_i - \bar{x}_1| \quad (\hat{M} \text{ is a constant}) \\
= \hat{M} \| s^{n-1} \|_{1,\infty} \left| \frac{\partial f}{\partial s}(w^{n-1}_i, m^{n-1}_i) z^{n-1}_i \right| \frac{\Delta t}{\phi_i} \\
\leq \hat{M} \| s^{n-1} \|_{1,\infty} \left( \left| \frac{\partial f}{\partial s}(w^{n-1}_i, m^{n-1}_i) \right| |z^{n-1}_i - v^n_i| \right) \\
+ |v^n_i| \left| \frac{\partial f}{\partial s}(w^{n-1}_i, m^{n-1}_i) - \frac{\partial f}{\partial s}(s^*_i, c^*_i) \right| \frac{\Delta t}{\phi_i}. \tag{32}
\]

Next we estimate the terms A-1 and A-2 on the right hand side of (32).

**A-1. Estimate of the term A-1 on the right hand side of Eq. (32):**

We rewrite the term A-1 as

\[
|z^{n-1}_i - v^n_i| \leq |z^{n-1}_i - v^{n-1}_i| + |v^n_i - v^{n-1}_i| \tag{33}
\]
Recall that \( z_i^n = -K\lambda(w_i^{n-1}, m_i^{n-1})\frac{\partial}{\partial x} + v_i^n = -K\lambda(s_i^n, c_i^n)\frac{\partial}{\partial x}. \) Then the first term A-1-1 on the right hand side of the above inequality (33) is written as

\[
|z_i^{n-1} - v_i^{n-1}| = \left| K\lambda(w_i^{n-1}, m_i^{n-1})\frac{\partial}{\partial x}(p_i^{n-1} - r_i^{n-1}) + K\left(\lambda(s_i^{n-1}, c_i^{n-1}) - \lambda(w_i^{n-1}, m_i^{n-1})\right)\frac{\partial p_i^{n-1}}{\partial x} \right|
\]

\[
\leq K \left\| \lambda \left\|_{\infty} \middle\| \frac{\partial}{\partial x}(p_i^{n-1}) \right\|_{\infty} + K \left\| \lambda \left(\lambda(s_i^{n-1}, c_i^{n-1}) - \lambda(w_i^{n-1}, m_i^{n-1})\right) \middle\| \frac{\partial p_i^{n-1}}{\partial x} \right\|_{\infty}. \tag{34}
\]

The non-traditional discontinuous finite element method adopted here for solving the pressure equation gives us the following estimates [20],

\[
\left\| \frac{\partial p_i^{n-1}}{\partial x} \right\|_{\infty} = O(h). \tag{35}
\]

The numerical scheme will still converge if a different finite element formulation is used as long as it preserves or improves upon the above error estimate. Using Taylor series we write,

\[
\left| \lambda(s_i^{n-1}, c_i^{n-1}) - \lambda(w_i^{n-1}, m_i^{n-1}) \right| \leq |s_i^{n-1} - w_i^{n-1}| \left\| \frac{\partial \lambda}{\partial s} \right\|_{\infty} + |c_i^{n-1} - m_i^{n-1}| \left\| \frac{\partial \lambda}{\partial c} \right\|_{\infty}
\]

\[
\leq M \left( |\zeta_i^{n-1}| + |\theta_i^{n-1}| \right). \tag{36}
\]

Using Eq. (35) and Eq. (37) in Eq. (34), we obtain following estimate for the first term A-1-1 of the righthand side of (33).

\[
|z_i^{n-1} - v_i^{n-1}| \leq M(h + |\zeta_i^{n-1}| + |\theta_i^{n-1}|). \tag{38}
\]

To estimate the term A-1-2 of the inequality (33) we observe

\[
|v_i^n - v_i^{n-1}| \leq \Delta t \left\| \frac{\partial v}{\partial t} \right\|_{\infty}. \tag{39}
\]

Using Eq. (38) and Eq. (39) in (33), we obtain the following estimate for A-1 (see Eq. (32)).

\[
|z_i^{n-1} - v_i^n| \leq M(h + \Delta t + |\zeta_i^{n-1}| + |\theta_i^{n-1}|). \tag{40}
\]

This concludes the estimate for the term A-1 in Eq. (32).

**A-2. Estimate of the term A-2 on the right hand side of Eq. (32):**

\[
\left| \frac{\partial f}{\partial s}(w_i^{n-1}, m_i^{n-1}) - \frac{\partial f}{\partial s}(s_i^n, c_i^n) \right|
\]

\[
\leq \left| (w_i^{n-1} - s_i^{n-1}) \left\| \frac{\partial^2 f}{\partial s^2} \right\|_{\infty} + m_i^{n-1} - c_i^{n-1} \left\| \frac{\partial^2 f}{\partial c \partial s} \right\|_{\infty}
\]

\[
+ |s_i^{n-1} - s_i^n| \left\| \frac{\partial f}{\partial s} \right\|_{\infty} + |c_i^{n-1} - c_i^n| \left\| \frac{\partial f}{\partial c} \right\|_{\infty} \left\| \frac{\partial c}{\partial t} \right\|_{\infty}
\]

\[
\leq M(|\zeta_i^{n-1}| + |\theta_i^{n-1}| + \Delta t). \tag{41}
\]
Using the estimates for A-1 and A-2, as given by Eq. (40) and Eq. (41) respectively, in Eq. (32) we finally obtain the estimate for the term A of Eq. (31) as

$$|s^{n-1}(\bar{x}_i) - s^{n-1}(\hat{x}_i)| \leq M\Delta t(|\zeta_i^{n-1}| + |\theta_i^{n-1}| + h + \Delta t). \quad (42)$$

**B. Estimate of the term B on the right hand side of Eq. (31):**

Using the Peano kernel Theorem in the spirit of the paper by Douglas and Russell [4], we obtain the following,

$$(1 - \mathcal{S})s^{n-1}(\hat{x}_i) = O\left(h^2\|s^{n-1}\|_{2,\infty}\right). \quad (43)$$

**C. Estimate of the source term** $(F(s_i^n, c_i^n) - F(w_i^n, m_i^n))$ in Eq. (30):

$$F(s_i^n, c_i^n) - F(w_i^n, m_i^n) \leq |s_i^n - w_i^n|\left\|\frac{\partial F}{\partial s}\right\|_{\infty} + |c_i^n - m_i^n|\left\|\frac{\partial F}{\partial c}\right\|_{\infty} \leq M(|\zeta_i^n| + |\theta_i^n|). \quad (44)$$

Equation (31) is substituted into Eq. (30) and the resulting equation is tested against $\zeta_i^n$. Using the estimates (42), (43) and (44) to replace some of the inner products, we rewrite Eq. (30) as

$$\langle \phi_i \frac{\zeta_i^n - \bar{\zeta}_i^{n-1}}{\Delta t}, \zeta_i^n \rangle - \langle \delta_x(\bar{\mathcal{D}}_x \zeta_i^n), \zeta_i^n \rangle \leq M\left(h + \Delta t + h^2/\Delta t + |\zeta_i^{n-1}| + |\theta_i^{n-1}| + |\zeta_i^n| + |\theta_i^n|\right) \langle \zeta_i^n, \zeta_i^n \rangle$$

$${\phantom{=}} + \langle \epsilon_i^n, \zeta_i^n \rangle - \langle \delta_x((\bar{\mathcal{D}} - \mathcal{D})s_i^n), \zeta_i^n \rangle \quad (45)$$

where $\epsilon_i^n = O(\|\frac{\partial^2 s}{\partial x^2}\|_{\infty}\Delta t, \|s^n\|_{3,\infty} h)$. Below, we obtain the estimates for the terms D-1, D-2, D-3 and D-4 in order to rewrite Eq. (45) in terms of only the errors $\zeta$ and $\theta$.

**D-1. Estimate of the term D-1:** The inner product is rewritten as

$$\langle \phi_i \frac{\zeta_i^n - \bar{\zeta}_i^{n-1}}{\Delta t}, \zeta_i^n \rangle = \langle \phi_i \frac{\zeta_i^n - \zeta_i^{n-1}}{\Delta t}, \zeta_i^n \rangle - \langle \phi_i \frac{\bar{\zeta}_i^{n-1} - \zeta_i^{n-1}}{\Delta t}, \zeta_i^n \rangle \quad \text{D-1-1}$$

Using the inequality $|a - b||a| \geq |a|^2 - |b|^2$ we estimate the term D-1-1 as

$$\langle \phi_i \frac{\zeta_i^n - \zeta_i^{n-1}}{\Delta t}, \zeta_i^n \rangle \geq \frac{M}{\Delta t}\left(\|\zeta^n\|^2 - \|\zeta^{n-1}\|^2\right)$$

**D-2. Estimate of the term D-2 in Eq. (45):**

Using summation by parts, we write

$$\langle \delta_x(\bar{\mathcal{D}}_x \zeta_i^n), \zeta_i^n \rangle = -\langle (\bar{\mathcal{D}}_x \zeta_i^n), (\delta_x \zeta_i^n) \rangle$$

and similarly for D-4, we have

$$\langle \delta_x((-\mathcal{D})_x s_i^n), \zeta_i^n \rangle = -\langle ((-\mathcal{D})_x s_i^n), (\delta_x \zeta_i^n) \rangle.$$
D-3. Estimate of the term D-3 in Eq. (45): Finally testing the term D-3 against \( \zeta^n_i \), we get
\[
\langle c^n_i, \zeta^n_i \rangle \leq M(h + \Delta t) \sum_i h|\zeta^n_i| \leq M(h^2 + \Delta t^2 + \| \zeta^n \|^2)
\]
Substituting the estimates for the term D-1-1 and replacing the terms D-2, D-3 and D-4 as shown before, we rewrite Eq. (45) as
\[
\frac{M}{\Delta t}(\| \zeta^n \|^2 - \| \zeta^{n-1} \|^2) + \langle (\bar{\theta} \delta_x \zeta^n)_{i, \zeta}, (\delta_x \zeta^n)_{i, \zeta} \rangle \leq \hat{M}(\| \zeta^n \|^2 + \langle \theta^n_i, \zeta^n_i \rangle + h^2 + \Delta t^2 + \frac{h^4}{\Delta t^2} + \| \zeta^{n-1} \|^2 + \| \theta^{n-1} \|^2)
\]
We now estimate the remaining inner product terms E-1, E-2, E-3 and E-4 in Eq. (46).

E-1: \( \langle (\bar{\theta} \delta_x \zeta^n)_{i}, (\delta_x \zeta^n)_{i} \rangle \geq D_x|\zeta^n|_{1,2} \),

E-2: \( \langle \theta^n_i, \zeta^n_i \rangle \leq M(\| \theta^n \|^2 + \| \zeta^n \|^2) \) (using Cauchy-Schwarz),

E-3: \( \langle ((\bar{\theta} - \theta) \delta_x s^n)_{i}, (\delta_x \zeta^n)_{i} \rangle = \sum_i \bar{h}((\bar{\theta} - \theta) \delta_x s^n)_{i, \delta_x \zeta^n} \leq \left\| \frac{\partial \bar{s}_i}{\partial x} \right\| \sum_i h|\bar{\theta} \delta_x s^n_{i} - \theta(s^n_i, c^n_i)\left| \frac{\partial \zeta^n_i}{\partial x} \right| \),

Using Taylor series we write
\[
|\bar{\theta}(\bar{w}_i^{n-1}, m_i^{n-1}) - \theta(s^n_i, c^n_i)| \leq |s^n_i - \bar{w}_i^{n-1}| \left\| \frac{\partial \theta}{\partial s} \right\|_{\infty} + |c^n_i - m_i^{n-1}| \left\| \frac{\partial \theta}{\partial c} \right\|_{\infty} \leq M(\Delta t + h\Delta t + h^2 + |\theta_i^{n-1}| + |\phi_i|h^2)\right),
\]
where we use the estimate \( |c^n_i - m_i^{n-1}| \leq M(\| \theta_i^{n-1} \| + \Delta t) \). Also,
\[
|s_i^n - w_i^{n-1}| \leq |s_i^n - s_i^{n-1}| + |s_i^{n-1} - w_i^{n-1}| \leq |s_i^n - s_i^{n-1}| + |\bar{\theta} s_i^{n-1}(\bar{x}_i) - \bar{s}_i^{n-1} - \bar{\theta}_i^{n-1}|
\]
\[
\leq M \left(\Delta t + |\bar{x}_i - x_i| s_i^{n-1} \right)_{1,\infty} + |((1 - \bar{\theta}) s_i^{n-1}(\bar{x}_i)) + \bar{\theta}_i^{n-1}| 
\]
\[
\leq M \left(\Delta t + \left\| \frac{\partial \bar{f}}{\partial s} \right\| \| s_i^{n-1} \|_{1,\infty} \right)_{\infty} + \| \bar{\theta}_i^{n-1} \|_{1,\infty} \right)
\]
\[
\leq M \left(\Delta t + h\Delta t + h^2 + |\bar{\theta}_i^{n-1}| \right) \]

Above, we have used Eq. (43) and that \( \| z_i^{n-1} \|_{\infty} \) is bounded which has been proved after Eq. (56) below. Hence we have an estimate for E-3 as
\[
\langle ((\bar{\theta} - \theta) \delta_x s^n)_{i}, (\delta_x \zeta^n)_{i} \rangle \leq M_1(s^n_{1,\infty} \sum_i h|\zeta_i^{n-1}| \left\| \frac{\partial \zeta_i^n}{\partial x} \right\|_{1,\infty} + M_2(s^n_{1,\infty} \sum_i h|\theta_i^{n-1}| \left\| \frac{\partial \zeta_i^n}{\partial x} \right\|_{1,\infty} \right)
\]
\[
+ M_3(s^n_{1,\infty} \sum_i h(\Delta t + h\Delta t + h^2) \left\| \frac{\partial \zeta_i^n}{\partial x} \right\|_{\infty} \right) \]
\[
\leq M(\| \zeta_i^{n-1} \|^2 + \| \theta_i^{n-1} \|^2 + |\zeta_i^{n-1}|_{1,2} + \Delta t^2 + h^2 \Delta t^2 + h^4).\]
**E-4. Estimate of the term E-4 in Eq. (46):** Using the fundamental theorem of calculus,

\[
\zeta_i^{n-1} - \zeta_i^{n-1} = \int_{x_i} \frac{\partial \zeta_i^{n-1}}{\partial x} \frac{\vec{x}_i - x_i}{|\vec{x}_i - x_i|} \, d\sigma
\]

Hence

\[
|\zeta_i^{n-1} - \zeta_i^{n-1}| \leq \int_{x_i} \left| \frac{\partial \zeta_i^{n-1}}{\partial x} \right| \, d\sigma \leq \left( \int_{x_i} \left| \frac{\partial \zeta_i^{n-1}}{\partial x} \right|^2 \, d\sigma \right)^{1/2} \left( \int_{x_i} \left| \frac{\partial \zeta_i^{n-1}}{\partial x} \right|^2 \, d\sigma \right)^{1/2}.
\]

Therefore,

\[
\langle \phi_i \zeta_i^{n-1} - \zeta_i^{n-1} \rangle, \zeta_i^n \rangle \leq \frac{\phi^*}{\Delta t} \left( \sum_i h_i |\zeta_i^n|^2 \right)^{1/2} \left( \sum_i h_i |\zeta_i^{n-1} - \zeta_i^{n-1}|^2 \right)^{1/2}
\]

\[
\leq M |\zeta_i^{n-1}|_{1,2} |\zeta_i^n|_{\infty} ||z_i^{n-1}||_{\infty}
\]

\[
\leq M |\zeta_i^{n-1}|_{1,2} |\zeta_i^n|_{1,2} (1 + h) (\log 1/h)^{1/2} \quad \text{[Using a result from [22]]}
\]

\[
\leq M (\log 1/h)^{1/2} (1 + h) \left( |\zeta_i^{n-1}|_{1,2}^2 + |\zeta_i^n|_{1,2}^2 \right).
\]

Above we have again used that $||z_i^{n-1}||_{\infty}$ is bounded. Using all of the above estimates for E-1, E-2, E-3, E-4 in Eq. (46) we get,

\[
M (||\zeta_i^n||^2 - ||\zeta_i^{n-1}||^2) + D_s \Delta t |\zeta_i^{n-1}|_{1,2}^2 \leq M \Delta t (h^2 + \Delta t^2 + \frac{h^4}{\Delta t^2} + h^2 \Delta t^2 + h^4) + M \Delta t \left( ||\zeta_i^n||^2 + ||\theta_i^n||^2 + ||\zeta_i^{n-1}||^2 + ||\theta_i^{n-1}||^2 \right)
\]

\[
+ M \Delta t (1 + (1 + h) (\log 1/h)^{1/2}) \left( |\zeta_i^{n-1}|_{1,2}^2 + |\zeta_i^n|_{1,2}^2 \right).
\]

Summing over $1 \leq n \leq L$ (with $L \Delta t = T$) we get,

\[
M (||\zeta_i^n||^2 - ||\zeta_i^0||^2) + D_s \Delta t \sum_{n=1}^L |\zeta_i^n|_{1,2}^2 \leq MT (h^2 + \Delta t^2 + \frac{h^4}{\Delta t^2} + h^2 \Delta t^2 + h^4) + M \Delta t \sum_{n=1}^L \left( ||\zeta_i^n||^2 + ||\theta_i^n||^2 + ||\zeta_i^{n-1}||^2 + ||\theta_i^{n-1}||^2 \right)
\]

\[
+ M \Delta t (1 + (1 + h) (\log 1/h)^{1/2}) \sum_{n=1}^L |\zeta_i^n|_{1,2}^2.
\]

Using discrete Gronwall’s inequality and noting that $\zeta_i^0 = 0$ and $\theta_i^0 = 0$ this can be rewritten as

\[
M ||\zeta_i^n||^2 + (D_s \Delta t - \rho_1) \sum_{n=1}^L |\zeta_i^n|_{1,2}^2 \leq M \Delta t \sum_{n=1}^L \left( ||\theta_i^n||^2 \right) + M \max(h^2 + \Delta t^2, h^4/\Delta t^2)
\]

where $\rho_1 = M \Delta t (1 + (1 + h) (\log 1/h)^{1/2}) \to 0$ faster than $D_s \Delta t$ as $(h, \Delta t) \to 0$. This concludes the analysis of the water transport equation (Eq. (21a)).
Next we consider the polymer transport equation (Eq. (21b)). Replacing the advective terms with a derivative along the characteristic direction, Eq. (21b) becomes

$$\psi_c \frac{\partial c}{\partial \tau} + Gc = H,$$

whose finite difference approximation is given by

$$\phi_i \frac{m_i^n - \bar{m}_i^{n-1}}{\Delta t} + G_i m_i^n = H_i^n.$$

Recall that $$\theta_i^n = c_i^n - \bar{m}_i^n$$. Using an analogue of Lemma 3.1 for the characteristic derivative of the polymer transport equation in Eq. (49) and subtracting Eq. (50) from the result we obtain

$$\phi_i \frac{\theta_i^n - (\bar{c}_i^{n-1} - \bar{m}_i^{n-1})}{\Delta t} + G_i \theta_i^n = H(s_i^n) - H(w_i^n) + O\left(\left\|\frac{\partial^2 c}{\partial \tau^2}\right\|_{\infty} \Delta \tau\right)$$

$$\leq |H(s_i^n) - H(w_i^n)| + M \Delta t$$

As before the source terms are estimated as

$$|H(s_i^n) - H(w_i^n)| \leq |s_i^n - w_i^n| \left\|\frac{\partial H}{\partial s}\right\|_{\infty} \leq M |\xi_i^n|$$

In the following, with slight abuse of notation, we suppress the superscript “c” from $$\tilde{x}_i^{c,n}$$ and $$\tilde{x}_i^{c,n}$$ to denote the points on the characteristic curves of the polymer transport equation. Continuing with the analysis, we rewrite the numerator of the first term on the left side of Eq. (51) as

$$\theta_i^n - (\bar{c}_i^{n-1} - \bar{m}_i^{n-1}) = (\theta_i^n - \bar{\theta}_i^{n-1}) - (\bar{c}_i^{n-1} - \bar{m}_i^{n-1}) - (1 - \mathcal{F}) c_i^{n-1} (\bar{x}_i)$$

The term G is estimated by the Peano kernel theorem, as was done in Eq. (43).

**F. Estimate of the term F:** This estimate is carried out in a series of steps.

$$|\bar{c}_i^{n-1} (\bar{x}_i) - c_i^{n-1} (\bar{x}_i)| \leq \|c^{n-1}\|_{1,\infty} |\bar{x}_i - \bar{x}_i|$$

$$\leq M \frac{\Delta t}{\phi_s} \left[ \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) z_i^{n-1} - \frac{f}{s} \left( s_i^n, c_i^n \right) v_i^n \right]$$

$$+ M \frac{\Delta t}{\phi_s} \left[ \frac{D}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) \frac{\partial w_i^n}{\partial x} - \frac{D}{s} \left( s_i^n, c_i^n \right) \frac{\partial s_i^n}{\partial x} \right]$$

**F-1. Estimate of the term F-1:**

$$\left| \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) z_i^{n-1} - \frac{f}{s} \left( s_i^n, c_i^n \right) v_i^n \right| \leq \left| \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) \right| |z_i^{n-1} - v_i^n|$$

$$+ \left| \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) - \frac{f}{s} \left( s_i^n, c_i^n \right) \right| |v_i^n|$$

$$\leq \left| \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) \right| |z_i^{n-1} - v_i^n|$$

$$+ \left| \frac{f}{s} \left( \bar{w}_i^n, m_i^{n-1} \right) - \frac{f}{s} \left( s_i^n, c_i^n \right) \right| |v_i^n|$$
Out of the two pieces F-1-1 and F-1-2 required to obtain an estimate of F-1, we have already estimated the term F-1-1 in Eq. (40) which we recall here: \(|z_i^n - v_i^n| \leq M(h + \Delta t + |\xi_i^{n-1}| + |\theta_i^{n-1}|).

We next estimate the term F-1-2 in Eq. (40).

**F-1.2. Estimate of the term F-1-2 in Eq. (55):**

\[
\left| \frac{f}{s}(\bar{w}_i^n, m_i^{n-1}) - \frac{f}{s}(s_i^n, c_i^n) \right| \\
\leq |\bar{w}_i^n - s_i^n| \left| \frac{\partial}{\partial s} \left( \frac{f}{s} \right) \right|_\infty + |\theta_i^{n-1}| \left| \frac{\partial c_i^n}{\partial t} \right|_\infty + \Delta t \left| \frac{\partial c_i^n}{\partial s} \right|_\infty \\
\leq M \left( |\bar{w}_i^n - s_i^n| + |\theta_i^{n-1}| + \Delta t \right)
\]

**F-1.2-a. Estimate of the term F-1.2-a:**

\[
\bar{w}_i^n - s_i^n = \mathcal{J} s^n(\bar{x}_i) - s^n(x_i) - \bar{\xi}_i^n = (\bar{x}_i - x_i) \frac{\partial s^n}{\partial x} - (1 - \mathcal{J}) s^n(\bar{x}_i) - \bar{\xi}_i^n
\]

Therefore

\[
|\bar{w}_i^n - s_i^n| \leq |\bar{x}_i - x_i| \|s^n\|_{1,\infty} + Ch^2 + |\bar{\xi}_i^n| \quad \text{(Using the Peano-kernel theorem)}
\]

\[
\leq M \Delta t \left\{ \left| \frac{f}{s} \right|_\infty \|z_i^{n-1}\|_\infty + \left| \frac{D}{s} \right|_\infty \left( \left| \frac{\partial s_i^n}{\partial x} \right|_\infty + \frac{\partial \xi_i^n}{\partial x} \right) \right\} + Ch^2 + |\bar{\xi}_i^n| \\
\leq M \Delta t \left\{ h + C + \left| \frac{\partial \xi_i^n}{\partial x} \right| \right\} + M h^2 + |\bar{\xi}_i^n|
\]

(56)

The last step of the above estimate in Eq. (56) requires a bound on \(\|z_i^{n-1}\|_\infty\) which was also used while estimating the term E-4. Before further analysis, we prove this statement here. Note that, even though we prove the result for \(\|z_i^n\|_\infty\), it is true for any other time \(t^n\) with \(n \in (0, T)\).

\[
z_i^n = -K \lambda(w_i^n, m_i^n) \frac{\partial s_i^n}{\partial x} = K \lambda(w_i^n, m_i^n) \left[ \frac{\partial s_i^n}{\partial x} - \frac{\partial p_i^n}{\partial x} \right]
\]

\[
\|z_i^n\|_\infty \leq \|K\|_\infty \|\lambda\|_\infty \left( 1 + \left| \frac{\partial p_i^n}{\partial x} \right|_\infty \right) \leq M(1 + \beta h); \quad (\beta \text{ is a constant})
\]

Using Eq. (56) we obtain an estimate for F-1-2 as

\[
\left| \frac{f}{s}(\bar{w}_i^n, m_i^{n-1}) - \frac{f}{s}(s_i^n, c_i^n) \right| \leq M \left( h^2 + \Delta t + h \Delta t + |\theta_i^{n-1}| + |\bar{\xi}_i^n| + \Delta t \left| \frac{\partial \xi_i^n}{\partial x} \right| \right)
\]

(57)

Using these estimates of F-1-1 and F-1-2 in Eq. (55) we obtain an estimate of F-1 as

\[
\left| \frac{f}{s}(\bar{w}_i^n, m_i^{n-1}) z_i^{n-1} - \frac{f}{s}(s_i^n, c_i^n) v_i^n \right| \leq M \left( h + \Delta t + h^2 + h \Delta t + |\xi_i^{n-1}| + |\theta_i^{n-1}| + |\bar{\xi}_i^n| + \Delta t \left| \frac{\partial \xi_i^n}{\partial x} \right| \right)
\]

(58)
F-2. Estimate of the term F-2 of Eq. (54):

\[
\left| \frac{D}{s} (\bar{\omega}_i^n, m_i^{n-1}) \frac{\partial \omega_i^n}{\partial x} - \frac{D}{s} (s_i^n, c_i^n) \frac{\partial s_i^n}{\partial x} \right| \\
\leq \left( |\bar{\omega}_i^n - s_i^n| \right) \left| \frac{\partial}{\partial s} \left( \frac{D}{s} \right) \right|_\infty + |m_i^{n-1} - c_i^n| \left| \frac{\partial}{\partial c} \left( \frac{D}{s} \right) \right|_\infty \left| \frac{\partial s_i^n}{\partial x} \right| + \left| \frac{D}{s} (\bar{\omega}_i^n, m_i^{n-1}) \right| \left| \frac{\partial \bar{s}_i^n}{\partial x} \right|
\]

Using the estimate for F-1-2-a given in Eq. (56) in Eq. (59), we obtain

\[
\left| \frac{D}{s} (\bar{\omega}_i^n, m_i^{n-1}) \frac{\partial \omega_i^n}{\partial x} - \frac{D}{s} (s_i^n, c_i^n) \frac{\partial s_i^n}{\partial x} \right| \leq M \left( |\bar{\omega}_i^n - s_i^n| + |\theta_i^{n-1}| + \Delta t + \left| \frac{\partial \theta_i^n}{\partial x} \right| \right) \\
\leq M \left( \Delta t \left( C + h \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right) + h^2 + |\bar{\zeta}_i^n| + |\theta_i^{n-1}| + \Delta t + \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right) \\
\leq M \left( \Delta t + h^2 + h\Delta t + |\bar{\zeta}_i^n| + |\theta_i^{n-1}| + (1 + \Delta t) \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right)
\]

(60)

Using Eq. (58) and Eq. (60) in Eq. (54) we obtain the following estimate for the term F in Eq. (53).

\[
|c^{n-1}(\bar{x}_i) - c^n(\bar{x}_i)| \leq M \Delta t \left( h + \Delta t + h^2 + h\Delta t + |\zeta_i^{n-1}| + |\theta_i^{n-1}| + \Delta t \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right) \\
+ M \Delta t \left( \Delta t + h^2 + h\Delta t + |\bar{\zeta}_i^n| + |\theta_i^{n-1}| + (1 + \Delta t) \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right) \\
\leq M \Delta t \left( h + \Delta t + h^2 + h\Delta t + |\zeta_i^{n-1}| + |\theta_i^{n-1}| + |\bar{\zeta}_i^n| + (1 + \Delta t) \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right)
\]

(61)

We test Eq. (51) against \( \theta_i^n \) and using Eq. (52) and Eq. (61), we get

\[
\langle \phi_i \left( \frac{\theta_i^n - \bar{\theta}_i^{n-1}}{\Delta t} \right), \theta_i^n \rangle \\
\leq \langle M \left( h + h^2 + \Delta t + h\Delta t + |\zeta_i^{n-1}| + |\theta_i^{n-1}| + |\bar{\zeta}_i^n| + (1 + \Delta t) \left| \frac{\partial \zeta_i^n}{\partial x} \right| \right), \theta_i^n \rangle \\
+ \langle M \left( h^2 \Delta t + \Delta t + |\zeta_i^n| \right), \theta_i^n \rangle.
\]

After some simplification, we get

\[
\phi_i (\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq \phi_i \|\theta^n\|^2 + (\phi_i - \phi_i^n) \|\theta_i^{n-1}\|^2 + M \Delta t \left( h^4 + \Delta t^2 + h^2 + h^2\Delta t^2 + \frac{h^4}{\Delta t^2} \right) \\
+ M \Delta t \left( \|\theta_i^{n-1}\|^2 + \|\zeta_i^{n-1}\|^2 + \|\theta_i^n\|^2 + \|\zeta_i^n\|^2 + (1 + \Delta t)(|\zeta_i^n|^2 + \|\theta_i^n\|^2) \right).
\]

(62)

Adding Eq. (47) and Eq. (62), summing over \( 1 \leq n \leq L \) and after some further simplification, we
get

\[ M(\|\varsigma^L\|^2 + \|\theta^L\|^2) + M\Delta t \sum_{n=1}^{L} |\zeta^n|_{1,2}^2 \]

\[ \leq M(1 + \Delta t + \Delta t^2) \sum_{n=0}^{L} \|\theta^n\|^2 + M\Delta t \sum_{n=0}^{L} |\zeta^n|_2^2 + MT \left( h^2 + \Delta t^2 + h^2 \Delta t^2 + h^4 + \frac{h^4}{\Delta t^2} \right) \]

\[ + M\Delta t \left( (1 + \Delta t) + (1 + h)(\log 1/h)^{1/2} \right) \sum_{n=0}^{L} |\zeta^n|_{1,2}^2 \]

(63)

where \( T = L\Delta t \). Let \( \rho = ((1 + \Delta t) + (1 + h)(\log 1/h)^{1/2}) \) such that \( \rho \to 0 \). Then using the discrete Gronwall’s inequality in Eq. (63), we get

\[ \|\varsigma^L\|^2 + \|\theta^L\|^2 + \Delta t \sum_{n=0}^{L} |\zeta^n|_{1,2}^2 \leq M(h^2 + \Delta t^2), \]

(64)

where \( M = M \left( \|s\|_{L^\infty(W^3,\infty)}, \|s\|_{W^1,\infty(L^\infty)}, \left\| \frac{\partial^2 s}{\partial T^2} \right\|_{L^\infty(L^\infty)}, \|c\|_{L^\infty(W^3,\infty)}, \|c\|_{W^1,\infty(L^\infty)}, \right. \]

\[ \left. \left\| \frac{\partial^2 c}{\partial T^2} \right\|_{L^\infty(L^\infty)}, \|p\|_{L^\infty(W^1,\infty)}, \|p\|_{W^1,\infty(W^1,\infty)} \right) . \]

It is also assumed that the auxiliary functions \( f(s,c), \lambda(s,c), D(s,c) \) have sufficient regularity in the discrete Sobolev norms. We anticipate an \( L^2 \) error of the order \( O(h) \) and consequently, we assume that \( \Delta t = O(h) \) as \( h \to 0 \) which implies \( \max(h^2 + \Delta t^2, h^4/\Delta t^2) = h^2 + \Delta t^2 \). In particular, it follows that

\[ \|\varsigma^L\|_{L^2} \leq Mh, \quad \|\theta^L\|_{L^2} \leq Mh. \]

(65)

Note that \( \Delta t = O(h) \) hypothesis is very reasonable since in the case of a one-dimensional parabolic equation the basic method can only be expected to have an \( O(h + \Delta t) \) estimate. Also, with a less stringent restriction like \( \Delta t = O(h^\gamma) \) for some \( \gamma < 2 \), we will have an \( L^2 \) error estimate of the order \( O(h^{1-\gamma/2}) \). The final error estimate is summarized in the following theorem.

**Theorem 3.3** Let \( s \) and \( c \) be the solutions of Eq. (21a) and Eq. (21b) respectively. Let \( w \) and \( m \) be the solutions of Eq. (26b) and Eq. (23) respectively where \( \tilde{w}_i^{n-1} \) is given by Eq. (24) and \( \tilde{m}_i^{n-1} \) is given by Eq. (25). Then, the errors \( \varsigma = s - w \) and \( \theta = c - m \) satisfy the inequalities given in Eq. (65) and the convergence result given in Eq. (64).

### 3.1 Extension to two dimensions in space

Here we discuss how to extend the analysis of Eq. (23) and Eq. (26b) to two spatial dimensions. The error estimates for the discretization of the characteristic derivatives and the capillary dissipation terms, obtained in Lemma 3.1 and Lemma 3.2 respectively, can be easily extended to two spatial dimensions without changing the order of the estimates. The various inequalities and tools used at various stages of the analysis like the Cauchy-Schwarz inequality, discrete Gronwall’s inequality
and the Taylor series have multidimensional analogues. The Peano kernel theorem can also be used in a similar manner for estimating the error introduced due to the bilinear interpolation required in the two-dimensional analysis. The $\| \frac{\partial \pi}{\partial x} \|_\infty$ estimate obtained from the finite element solution of the elliptic pressure equation is also available in two or higher spatial dimensions [20]. The $L^\infty$ estimate of a mesh function [22], that has been used to estimate the term E4, is also applicable for a two-dimensional domain. In the analysis of the one dimensional system, the spatial grid has been taken to be uniform with a fixed spatial grid size $h$. In the two dimensional system, the grid can be taken to be uniform in each spatial dimension with constant $h_x = \Delta x$ and $h_y = \Delta y$. The quasilinear treatment of the nonlinearity in the functions $f, D, \lambda$ will allow us to obtain analogous estimates of the two-dimensional inner products involving these terms without affecting the convergence results. Hence, a similar analytical calculation can be made to obtain an $O(h_x + h_y + \Delta t)$ error estimate for the two dimensional problem.

3.2 Extension to two component systems

Here we discuss the possibility of extending this analysis to the case of two-component two-phase flows like surfactant-polymer flooding. Such a system has been studied recently in Daripa & Dutta [1] which we present below.

\begin{align}
- \nabla \cdot (K(x)\lambda(s,c,\Gamma)\nabla p) &= q_a + q_o, \\
\phi \frac{\partial s}{\partial t} + \frac{\partial f_a}{\partial s} v \cdot \nabla s + \nabla \cdot \left( D \frac{\partial p_c}{\partial s} \nabla s \right) &= g_s - \frac{\partial f_a}{\partial c} v \cdot \nabla c - \frac{\partial f_a}{\partial \Gamma} v \cdot \nabla \Gamma - \nabla \cdot \left( D \frac{\partial p_c}{\partial \Gamma} \nabla \Gamma \right), \\
\phi \frac{\partial c}{\partial t} + \left( \frac{f_a}{s} v + \frac{D}{s} \frac{\partial p_c}{\partial s} \nabla s + \frac{D}{s} \frac{\partial p_c}{\partial \Gamma} \nabla \Gamma \right) \cdot \nabla c &= g_c, \\
\phi \frac{\partial \Gamma}{\partial t} + \left( \frac{f_a}{s} v + \frac{D}{s} \frac{\partial p_c}{\partial s} \nabla s + \frac{D}{s} \frac{\partial p_c}{\partial \Gamma} \nabla \Gamma \right) \cdot \nabla \Gamma &= g\Gamma,
\end{align}

where $D(s,c,\Gamma) = K(x)\lambda_o(s,\Gamma)f_o(s,c,\Gamma)$, $\Gamma$ is the surfactant concentration and the source terms $q_a, q_o, g_s, g_c, g\Gamma$ are defined similar to the one-component flow model. As seen from the above model, the transport equations for polymer and surfactant have a similar structure. Hence the surfactant transport equation can be analyzed in a similar fashion to obtain error estimates for this two-component two-phase flow system. However, such an exercise also poses certain challenges. The functions $p_c$ and $D, f, \lambda, \lambda_a, \lambda_o$ are not always dependent on all three components $s, c, \Gamma$. For instance, the capillary pressure $p_c = p_c(s,\Gamma)$ is only affected by changes in water saturation $s$ and surfactant concentration $\Gamma$, whereas the fractional flow functions $f_a = f_a(s,c,\Gamma)$ and $f_o = f_o(s,c,\Gamma)$, depend on all three. Similarly, $\lambda_a = \lambda_a(s,c,\Gamma)$ but $\lambda_o = \lambda_o(s,\Gamma)$. This means that the estimates are not always symmetric with respect to the two transport variables $c$ and $\Gamma$. Hence, an analogous error estimate for the two-component system is difficult to obtain as a direct extension of the one-component system and it needs further non-trivial analysis. However, due to the similarity in the structure of the transport equations for $c$ and $\Gamma$ and because the numerical method for the two-component system is a direct extension of the one-component system, we anticipate an equivalent $L^2$ estimate of the order $O(h + \Delta t)$ even for the error in surfactant concentration.
4 Numerical results

In [1], an exact solution for the two-dimensional, immiscible, two-phase flow problem has been constructed and used for numerical verification of the convergence and the order of accuracy of the numerical method. The $L^2$ and $L^\infty$ error norms and the respective orders of accuracy with spatial grid refinement have been presented there for the water saturation $s$, the pressure $p$ and the velocity $v$. The $L^2$ and $L^\infty$ error norms for $s$ with time step refinement at different fixed spatial grid sizes have been also presented along with the corresponding convergence rates. The $L^2$ error in saturation has been shown to be of the order $O(h)$. This is consistent with the estimate obtained from our one-dimensional analysis presented in this paper and, as discussed above, is expected to be true in two-dimensional case. In [1], the $O(h)$ error in the $L^\infty$ norm of the gradient of pressure has also been observed in the numerical results obtained with an exact solution. Additionally, there the numerical water saturation profiles at various spatial resolutions have been found to compare favorably with the same for the exact solution, thus providing support for the convergence of the numerical method.

Several numerical experiments have been carried out using this numerical method to simulate practical two-phase flow problems, both with and without components, that arise in the context of chemical enhanced oil recovery. Mainly two different types of computational domains have been employed for these simulations - a quarter of a five-spot geometry (or radial flow) that mimics oil reservoir conditions near the location of the physical sources, and a rectilinear geometry that mimics the flow conditions far from the location of the sources. In the first case, the sources (injection or production wells) have been treated as point source terms while in the second case the two opposite boundary walls are treated as wall sources. A variety of different types of heterogeneous permeability fields have also been used. These include rectangular inclusions, channelised domains, a multiscale permeability field generated using a stationary, isotropic, fractal Gaussian field and sections of the SPE10 permeability field. These simulation results have been used to qualitatively validate the numerical method by comparing with results from existing literature. Also, several comparison studies have been performed between different combinations of single or multi-component, two phase flows which demonstrate the capability of the method to capture the intricate details of the flow characteristics and produce numerical results that are consistent with expectations based on physics. The reader is directed to Daripa & Dutta [1] for further details.

In this section, we present numerical results obtained from solving the two-phase single component system of equations (polymer flooding), given by eqs. (7a)–(7c), subject to realistic initial and boundary data. These are intended to support the error calculations with respect to an exact solution that were presented in [1] and provide further quantitative evidence about the accuracy and the order of convergence of the method, even in the case of practical numerical simulations. In section 4.1, the input data is given in table 1. The numerical errors are measured in the discrete norms.

\begin{align}
    e_{s,\text{max}} &= \max_{ij} |s(x_{ij}) - w_{ij}| \equiv \|s - w\|_{L^\infty}, \\
    e_{s,2} &= \sqrt{\sum_{ij} |s(x_{ij}) - w_{ij}|^2 \Delta x \Delta y} \equiv \|s - w\|_{L^2}.
\end{align}

Here, $w_{ij}$ is the numerical solution $w$ evaluated at the grid point $(x_i, y_j) = x_{ij}$ whereas $s(x_{ij})$ is the finest grid numerical solution in section 4.1. The errors for the pressure and the velo-
ity are computed in a similar fashion. The order of accuracy is computed using the formula
\[ \log_2 \left( \frac{e_\alpha(h)}{e_\alpha(h/2)} \right) (\alpha = 2, \infty) \]

4.1 Two-dimensional polymer flood problem

We perform simulation of polymer flooding on a quarter five-spot homogeneous geometry \( \Omega = [0, 1]^2 \) with absolute permeability, \( K = 1 \) and input parameters listed in Table 1. The transport source terms in eqs. (7b) and (7c) for a quarter-five spot flow geometry are taken as

\[ g_s = \begin{cases} \frac{1 - f_a}{Q} & \text{at } x = \{x^i \equiv (0, 0) \} \\ 0 & \text{elsewhere} \end{cases} \]

\[ g_c = \begin{cases} \frac{(c^i - c)}{s} & \text{at } x = \{x^i \equiv (0, 0) \} \\ 0 & \text{elsewhere} \end{cases} \]

The source terms for the pressure eq. (7a) are taken as

\[ q_a = \begin{cases} Q & \text{at } x = \{x^i \equiv (0, 0) \} \\ -(\lambda_a/\lambda) Q & \text{at } x = \{x^p \equiv (1, 1) \} \\ 0 & \text{elsewhere} \end{cases} \]

\[ q_o = \begin{cases} 0 & \text{at } x = \{x^i \equiv (0, 0) \} \\ -(\lambda_o/\lambda) Q & \text{at } x = \{x^p \equiv (1, 1) \} \\ 0 & \text{elsewhere} \end{cases} \]

We compute the \( L^\infty \) and \( L^2 \) error norms of the numerical solutions for the saturation on a sequence of uniformly refined meshes \( h = 1/8, 1/16, 1/32, 1/64 \) using eq. (67), but with \( s(x_{ij}) \) representing the solution on the finest grid size \( h = 1/128 \). A similar procedure is applied to estimate the error norms and the order of accuracy for the pressure and the velocity. The numerical errors and the order of accuracy are presented in Table 2. In Table 3 we present the numerical errors and convergence rates with respect to time step size refinement \( \Delta t = 1/20, \ldots, 1/160 \) by keeping the spatial grid size fixed at three different levels \( h = 1/16, 1/32 \) and 1/64 for the quarter five-spot flooding problem. The error calculations for both the tables have been performed at the time of water breakthrough which is given by the time at which the water saturation at \( x^p \) reaches a chosen threshold value. We observe (see Table 2) the following approximate orders of accuracy in space.

\[ ||s - w||_{L^2} = O(h), \quad ||p - r||_{L^2} = O(h^2) \quad \& \quad ||v - z||_{L^2} = O(h^2) \]

---

**Table 1: Simulation input data**

| Model parameter          | Symbol | Value       |
|--------------------------|--------|-------------|
| Spatial grid size        | \( h \times k \) | variable    |
| Porosity                 | \( \phi \) | 1           |
| Initial resident water saturation | \( s^0 \) | 0.21        |
| Oil viscosity            | \( \mu_o \) | 12.6        |
| Pure water viscosity     | \( \mu_w \) | 1.26        |
| Residual aqueous phase saturation | \( s_{ra} \) | 0.1         |
| Residual oleic phase saturation | \( s_{ro} \) | 0.2         |
| Parameters of capillary pressure relation [eq. (10c)] | \( \alpha_0, m \) | 0.125, 2/3  |
| Concentration of polymer in injected fluid | \( c_0 \) | 0.1         |
| Injection rate           | \( Q \) | 200         |
| Time step size           | \( \Delta t \) | 1/50        |
The $O(h)$ error in the $L^2$ norm for saturation $s$ directly matches with the estimate obtained from our one-dimensional analysis in eq. (65). The $O(h)$ error in the gradient of the pressure $p$ (as seen in eq. (35)) is also observed in the $L^\infty$ norm $\|v - z\|_{L^\infty}$ for the velocity in Table 2. Moreover, the orders of accuracy in Table 2 for the $L^2$ and $L^\infty$ errors of all the three variables $s$, $p$ and $v$ are consistent with the orders of accuracy obtained using an exact solution (see Table 1 in [1]).

The order of accuracy in the $L^\infty$ norm of the error in saturation, as presented in the upper part of the last column of Table 2 can be seen to reduce significantly with reduction in spatial grid size. This is because the saturation and its $L^\infty$ norm are both highly sensitive to minor changes in the flow and domain parameters, especially the ones whose $L^\infty$ bounds enter the generic coefficient $M$ used in Eq. (64) and in various other intermediate estimates obtained in Section 3. Hence the $L^\infty$ error norms of saturation in Table 2 are at least an order higher than the corresponding norms for pressure and velocity, both of which are less sensitive to minor changes in the parameter space. To overcome this, much finer spatial grid size and time step size (data not shown here) need to be adopted for the numerical solution of the transport equations.

Table 3 shows the $L^2$ error in saturation and the rate of convergence with respect to time. The results confirm that approximately a first order convergence rate in time can be obtained using this method. This compares favorably with results obtained using an exact solution (see Table 2 in [1]) and also with the convergence rate expected from a first order time discretization scheme. We believe that with higher order time-stepping methods, the method will be able to preserve the accuracy and the expected second or third order convergence rates.

In Figure 3 snapshots of the water saturation profile are shown at the same time level $t = 0.5$ for four different spatial grid resolutions. The increase in the quality and sharpness of the saturation profiles with grid refinement can be easily observed while preserving the accuracy of the solution. This provides a qualitative numerical validation that the method converges with spatial grid refinement.

5 Conclusions

Recently, in Daripa & Dutta [1] a hybrid method for solving a two-phase two-component flow through porous media has been proposed and implemented in two-dimensions. The hybrid method uses a non-traditional discontinuous finite element method for solving the elliptic equation and a time implicit finite difference method based modified method of characteristics for solving the transport equations. Numerical results obtained there are in excellent agreement with the physics of flow as well as with exact solutions when available. Numerical results are also found to converge under mesh refinement. In this paper, we perform numerical analysis of the method to prove convergence by considering a reduced system, namely two-phase one component porous media flow in one-dimension. The novelty in the paper is the consideration of the transport equation for one component in the proof which significantly complicates the analysis previously performed by others [4, 5] without any component. Basic ideas of the proof can be extended to two-dimensions and to two-components which has been discussed in the paper but needless to say, the extension will be even more involved technically as one can see from the proof for the case presented in this paper.

In the analysis presented here, the convergence behavior of the MMOC based finite difference part of this hybrid numerical method has been studied. An optimal order $O(h)$ error for the pressure gradient obtained by the finite element part has been assumed and this has been numerically validated in [20]. Using this result, $L^2$ error estimates of the wetting phase saturation $s$ and the
component concentration $c$ have been computed. Numerical experiments have been performed to simulate two-phase one-component flow in a quarter five-spot geometry. The $L^2$ error norm of $s$ and the $L^\infty$ error norm of the velocity obtained numerically have been used to verify our theoretical error estimates. These $L^2$ and $L^\infty$ error norms have been also used to demonstrate the numerical

| $h$  | $\|s - w\|_{L^2}$ | Order | $\|s - w\|_{L^\infty}$ | Order |
|------|------------------|-------|------------------|-------|
| 1/8  | 4.39e-3          | –     | 3.64e-2          | –     |
| 1/16 | 1.85e-3          | 1.247 | 1.76e-2          | 1.048 |
| 1/32 | 7.84e-4          | 1.239 | 1.08e-2          | 0.704 |
| 1/64 | 3.22e-4          | 1.284 | 6.86e-3          | 0.656 |

| $h$  | $\|p - r\|_{L^2}$ | Order | $\|p - r\|_{L^\infty}$ | Order |
|------|------------------|-------|------------------|-------|
| 1/8  | 4.12e-3          | –     | 1.61e-3          | –     |
| 1/16 | 9.30e-4          | 2.147 | 4.75e-4          | 1.761 |
| 1/32 | 2.10e-4          | 2.147 | 1.35e-4          | 1.815 |
| 1/64 | 3.85e-5          | 2.448 | 3.10e-5          | 2.123 |

| $h$  | $\|v - z\|_{L^2}$ | Order | $\|v - z\|_{L^\infty}$ | Order |
|------|------------------|-------|------------------|-------|
| 1/8  | 1.18e-3          | –     | 6.94e-3          | –     |
| 1/16 | 2.94e-4          | 2.005 | 2.98e-3          | 1.220 |
| 1/32 | 7.46e-5          | 1.979 | 1.39e-3          | 1.110 |
| 1/64 | 1.96e-5          | 1.928 | 6.89e-4          | 1.002 |

Table 2: Error and order for saturation, pressure and velocity at water breakthrough of a quarter five-spot polymer flooding simulation

| $\Delta t$ | $h = 1/16$ | $h = 1/32$ | $h = 1/64$ |
|-------------|------------|------------|------------|
|             | $\|s - w\|_{L^2}$ Rate | $\|s - w\|_{L^2}$ Rate | $\|s - w\|_{L^2}$ Rate |
| 1/20        | 9.34e-3    | –          | 8.93e-3    | –          |
| 1/40        | 4.36e-3    | 1.100      | 4.69e-3    | 0.923      | 3.48e-3    | 0.901      |
| 1/80        | 2.27e-3    | 0.941      | 2.51e-3    | 0.899      | 1.94e-3    | 0.846      |
| 1/160       | 1.25e-3    | 0.867      | 1.48e-3    | 0.762      | 1.17e-3    | 0.722      |

Table 3: Error and rates for saturation with time step refinement at water breakthrough of a quarter five-spot polymer flooding simulation.
Figure 3: Comparison of saturation contours during a polymer flood in a homogeneous quarter five-spot reservoir with four different spatial resolutions. The contours are plotted at the same time level, $t = 0.5$ to compare the convergence with grid refinement.

The convergence of the method as well as the order of accuracy with spatial grid refinement and the convergence rates with respect to the time step refinement.

Acknowledgments

Numerical simulations have been performed using high performance research computing resources provided by Texas A&M University (http://hprc.tamu.edu). The research reported in this paper has been supported by the U.S. National Science Foundation grant DMS-1522782.

References

[1] P. Daripa and S. Dutta. Modeling and simulation of surfactant-polymer flooding using a new hybrid method. *J. Comput. Phys.*, 335:249–282, 2017.

[2] P. Daripa, J. Glimm, B. Lindquist, M. Maesumi, and O. McBryan. On the simulation of heterogeneous petroleum reservoirs. In *Numerical Simulation in Oil Recovery, IMA Vol. Math. Appl. 11*, pages 89–103, New York, NY, 1988. Springer.

[3] P. Daripa, J. Glimm, B. Lindquist, and O. McBryan. Polymer floods: A case study of nonlinear wave analysis and of instability control in tertiary oil recovery. *SIAM J. Appl. Math.*, 48:353–373, 1988.
[4] J. Douglas Jr and T. F. Russell. Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. *SIAM J. Numer. Anal.*, 19(5):871–885, 1982.

[5] J. Douglas Jr. Finite difference methods for two-phase incompressible flow in porous media. *SIAM J. Numer. Anal.*, 20(4):681–696, 1983.

[6] L. J. Durlofsky. A triangle based mixed finite element-finite volume technique for modeling two phase flow through porous media. *J. Comput. Phys.*, 105(2):252–266, 1993.

[7] C. Cancès, I. S. Pop, and M. Vohralík. An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow. *Math. Comput.*, 83(285):153–188, 2014.

[8] B. Rivière and M. F. Wheeler. Discontinuous Galerkin methods for flow and transport problems in porous media. *Commun. Numer. Methods Eng.*, 18(1):63–68, 2002.

[9] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab. Optimal a priori error estimates for the $hp$-version of the local discontinuous Galerkin method for convection–diffusion problems. *Math. Comput.*, 71(238):455–479, 2001.

[10] D. Nayagam, G. Schäfer, and R. Mosé. Modelling two-phase incompressible flow in porous media using mixed hybrid and discontinuous finite elements. *Comput. Geosci.*, 8(1):49–73, 2004.

[11] J. Douglas Jr, C.-S. Huang, and F. Pereira. The modified method of characteristics with adjusted advection. *Numer. Math.*, 83(3):353–369, 1999.

[12] M. A. Celia, T. F. Russell, I. Herrera, and R. E. Ewing. An Eulerian-Lagrangian localized adjoint method for the advection-diffusion equation. *Adv. Water Resour.*, 13(4):187–206, 1990.

[13] T. Arbogast and M. F. Wheeler. A characteristics-mixed finite element method for advection-dominated transport problems. *SIAM J. Numer. Anal.*, 32(2):404–424, 1995.

[14] H. Wang. An optimal-order error estimate for an ELLAM scheme for two-dimensional linear advection-diffusion equations. *SIAM J. Numer. Anal.*, 37(4):1338–1368, 2000.

[15] M. Muskat and M. W. Meres. The flow of heterogeneous fluids through porous media. *J. Appl. Phys.*, 7(921):346–363, 1936.

[16] A. T. Corey. *Mechanics of Immiscible Fluids in Porous Media*. Water Resources Publications, Littilton, Colorado, 1986.

[17] M. Th. van Genuchten. A closed form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil Sci. Soc.*, 44:892–898, 1980.

[18] J. C. Parker, R. J. Lenhard, and T. Kuppusamy. A parametric model for constitutive properties governing multiphase flow in porous media. *Water Resour. Res.*, 23(4):618–624, 1987.
[19] B. Ghanbarian-Alavijeh, A. Liaghat, G.-H. Huang, and M. Th. van Genuchten. Estimation of the van Genuchten soil water retention properties from soil textural data. *Pedosphere*, 20(4):456–465, 2010.

[20] S. Hou, W. Wang, and L. Wang. Numerical method for solving matrix coefficient elliptic equation with sharp-edged interfaces. *J. Comput. Phys.*, 229:7162–7179, 2010.

[21] Z. Li, T. Lin, and X. Wu. New Cartesian grid methods for interface problems using the finite element formulation. *Numer. Math.*, 96:61–98, 2003.

[22] J. H. Bramble. Second order finite difference analogue of the first biharmonic boundary value problem. *Numer. Math.*, 9:236–249, 1966.