Nonmeasurability in Banach spaces

Robert Rałowski

Abstract. We show that for a $\sigma$-ideal $\mathcal{I}$ with a Steinhaus property defined on Banach space, if two non-homeomorphic Banach with the same cardinality of the Hamel basis then there is a $\mathcal{I}$ nonmeasurable subset as image by any isomorphism between of them. Our results generalize results from [2].

1. Notation and Terminology

Throughout this paper, $X$, $Y$ will denote uncountable Polish spaces and $\mathcal{B}(X)$ the Borel $\sigma$-algebra of $X$. We say that the ideal $\mathcal{I}$ on $X$ has Borel base if every element $A \in \mathcal{I}$ is contained in a Borel set in $\mathcal{I}$. (It is assumed that an ideal is always proper.) The ideal consisting of all countable subsets of $X$ will be denoted by $[X]^{\leq \omega}$ and the ideal of all meager subsets of $X$ will be denoted by $\mathbb{K}$. Let $\mu$ be a continuous probability measure on $X$. The ideal consisting of all $\mu$-null sets will be denoted by $L_\mu$. By the following well known result, $L_\mu$ can be identified with the $\sigma$-ideal of Lebesgue null sets.

Theorem 1.1 ([5], Theorem 3.4.23). If $\mu$ is a continuous probability on $\mathcal{B}(X)$, then there is a Borel isomorphism $h : X \to [0, 1]$ such that for every Borel subset $B$ of $[0, 1]$, $\lambda(B) = \mu(h^{-1}(B))$, where $\lambda$ is a Lebesgue measure.

Definition 1.1. We say that $(Z, \mathcal{I})$ is Polish ideal space if $Z$ is Polish uncountable space and $\mathcal{I}$ is a $\sigma$-ideal on $Z$ having Borel base and containing all singletons. In this case, we set

$$\mathcal{B}_+(Z) = \mathcal{B}(Z) \setminus \mathcal{I}.$$ 

A subset of $Z$ not in $\mathcal{I}$ will be called a $\mathcal{I}$-positive set; sets in $\mathcal{I}$ will also be called $\mathcal{I}$-null. Also, the $\sigma$-algebra generated by $\mathcal{B}(Z) \cup \mathcal{I}$ will be denoted by $\overline{\mathcal{B}}(Z)$, called the $\mathcal{I}$-completion of $\mathcal{B}(Z)$.

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It is easy to check that $A \in \overline{B}(Z)$ if and only if there is an $I \in \mathcal{I}$ such that $A \triangle I$ (the symmetric difference) is Borel.

**Example 1.1.** Let $\mu$ be a continuous probability measure on $X$. Then $(X, [X]^{\leq \omega}), (X, K), (X, \mathbb{L}_\mu)$ are Polish ideal spaces.

**Definition 1.2.** A Polish ideal group is 3-tuple $(G, \mathcal{I}, +)$ where $(G, \mathcal{I})$ is Polish ideal space and $(G, +)$ is an abelian topological group with respect to the Polish topology of $G$.

Now we are ready to recall the crucial property which was introduced by Steinhaus see [6].

**Definition 1.3.** Let $(X, +)$ be any topological group with topology $\tau$. We say that ideal $\mathcal{I} \subset \mathcal{P}(X)$ have Steinhaus property iff

$$\forall A_1, A_1 \in \mathcal{P}(X) \exists B_1, B_2 \in \mathcal{B}(X) \exists U \in \tau \ B_1 \subset A_1 \land B_2 \subset A_2 \land U \subset A_1 + A_2.$$  

In the same paper [6] was proven that ideal of null sets poses the Steinhaus property.

**Definition 1.4.** Let $(X, +)$ be any topological group and let $\mathcal{I} \subset \mathcal{P}(X)$ be any invariant $\sigma$-ideal with singletons then $\mathcal{I}$ has strong Steinhaus property iff

$$\forall B \in \mathcal{B}, \forall A \in \mathcal{P}(X) \setminus \mathcal{I} \text{ int}(A + B) \neq \emptyset.$$  

It is well known that ideal of meager sets in any topological group $(G, +)$ has strong Steinhaus property see [4]. Moreover the ideal of the null sets respect to Haar measure on the locally compact topological group $(G, +)$ has also strong Steinhaus property see [1].

**Definition 1.5.** Let $(X, \mathcal{I})$ be a Polish ideal space and $A \subseteq X$. We say that $A$ is $\mathcal{I}$–nonmeasurable, if $A \notin \overline{B}(X)$. Further, we say that $A$ is completely $\mathcal{I}$–nonmeasurable if

$$\forall B \in \mathcal{B}(X) \ A \cap B \neq \emptyset \land A^c \cap B \neq \emptyset.$$  

Clearly every completely $\mathcal{I}$–nonmeasurable set is $\mathcal{I}$–nonmeasurable. In the literature, completely $[X]^{\leq \omega}$–nonmeasurable sets are called Bernstein sets. Also, note that $A$ is completely $\mathbb{L}_\mu$–nonmeasurable if and only if the inner measure of $A$ is zero and the outer measure one.

For any set $E$, $|E|$ will denote the cardinality of $E$.

The rest of our notations and terminology are standard. For other notation and terminology in Descriptive Set Theory we follow [5].

The main motivation of this paper is the Theorem about the nonmeasurability of the images under isomorphism over $\mathbb{Q}$ between $\mathbb{R}^n$ and $\mathbb{R}^m$ whenever $m \neq n$. More precisely in [2] the following Theorem was proved.
Theorem 1.2. Let $I$ is nontrivial invariant $\sigma$-ideal of subsets of the group $(\mathbb{R}, +)$ which has strong Steinhaus property such that $[\mathbb{R}]^\omega \subset I$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any linear isomorphism over $\mathbb{Q}$. Then

1. If $A \subset \mathbb{R}^2$ is a bounded subset of the real plane such that $A + T = \mathbb{R}^2$ for some $T \in [\mathbb{R}^2]^\omega$ then $f[A]$ is $I$-nonmeasurable subset of the real line.
2. If $A \subset \mathbb{R}^2$ is a bounded subset of the real plane with non empty interior $\text{int}(A) \neq \emptyset$. Then the set $f[A]$ is completely $I$–nonmeasurable subset of the real line.

2. Results

Here we present the main results of this paper.

Theorem 2.1. Let $X, Y$ be a Banach spaces and let us suppose that

1. $I \subset \mathcal{P}(Y)$ be an $\sigma$ - ideal with Steinhaus property,
2. $\forall n \in \omega \forall A \in I$ $n \neq 0 \to nA \in I$,
3. $f : X \to Y$ by any isomorphism between $X, Y$ which is not homeomorphism.

Then the image of the unit ball $f[K]$ is $I$ non-measurable in $Y$, where $K$ a unit ball of the space $X$ with the center equal $0 \in X$.

Theorem 2.2. Let $X, Y$ be any Banach spaces for which there exists linear isomorphism $f : X \to Y$ which is not continuous then image of the unit ball $K \subset X$ by $f$ has not Baire property.

Theorem 2.3. Let $X, Y$ be a Banach spaces and let us assume that

1. $I \subset \mathcal{P}(Y)$ be an $\kappa$ - complete additive invariant ideal with Steinhaus property,
2. let $\min\{|S| : S \in \mathcal{P}(X) \land S \text{ is dense set in } X\} < \kappa$,
3. $f : X \to Y$ by any isomorphism between $X, Y$, which is not homeomorphism.

Then the image of the unit ball $f[K]$ is $I$ non-measurable in $Y$, where $K$ a unit ball of the space $X$ with the center equal $0 \in X$.

3. Proofs o the main results

Proof. of the Theorem 2.1. Let assume that image $f[K]$ is $I$-measurable in space $Y$. First of all let observe that

$$Y = f[X] = f[\bigcup_{n=1}^{\infty} nK] = \bigcup_{n=1}^{\infty} f[nK] = \bigcup_{n=1}^{\infty} nf[K]$$

then we have that $f[K] \notin I$ is $I$ positive. Then by Steinhaus property of the ideal $I$ there exist unit ball $0 \in U$ consisting $0$ for which $U \subset f[K] - f[K] = f[K - K]$ is hold. But $f$ is linear isomorphism then $f^{-1}[U] \subset K - K$ then $f^{-1}$ is bounded linear
operator (so is continuous) then by Banach Theorem on invertible operator we have
that \( f \) is bounded then \( f \) is homeomorphism what is impossible by (3).

Here we give the following Lemma:

**Lemma 3.1.** Let \((X, \| \cdot \|)\) be any Banach space then \(\sigma\)-ideal \(K\) of the meager sets
has Steinhaus property.

**Proof.** Let \(A, B \in \mathcal{P}(X) \setminus K\) be any meager positive subsets of the Banach space
then there exists positive radius \(r > 0\) of two balls \(K_1 = K(x, \frac{r}{2}), K_2 = K(y, r) \subset X\)
with the centers \(x, y \in X\) such that \(A_1 = A \cap K_1\) and \(B_1 = B \cap K_2\) are comeager subsets in the balls \(K_1(x, r)\) and \(K_2(y, r)\) respectively. Let \(z = y - x\) and \(r_0 = \frac{r}{2}\) and
let \(K_0 = K(z, r_0)\) then we will show that
\[
\forall t \in K_0 \ (t + A) \cap B_1 \neq \emptyset
\]
First let observe that if \(t \in K_0\) and \(s \in K_1\) then
\[
\| (t + s) - y \| = \| (t - z + z) + (s - x + x) - y \| = \| (t - z) + (s - x) + z + x - y \|
\]
\[
= \| (t - z) + (s - x) \| \leq \| t - z \| + \| s - x \| < r_0 + \frac{r}{2} = r
\]
thus we have \(K_0 + K_1 \subset K_2\) and then \((K + A_1) \cap B_1\) is comeager in the open set
\((K_0 + K_1) \cap K_2\) then \((K + A_1) \cap B_1\) is nonempty set.
Thus there exists open set \(K_0\) such that
\[
\forall t \in K_0 \ (t + A) \cap B \neq \emptyset
\]
Finally we have
\[
\forall t \in K_0 \ (t + A) \cap B \neq \emptyset \iff \forall t \in K_0 \exists a \in A \exists b \in Bt + a = b \iff \forall t \in K_0 t \in B - A
\]
Thus we have \(K_0 \subset B - A\) what finishes proof of this lemma.

By the above Lemma we can give the Theorem 2.2.

**Proof.** (of the Theorem 2.2). By Lemma 3.1 the ideal of the meager sets of the space \(Y\) has Steinhaus property and let observe that for any \(\alpha \in \mathbb{R} \setminus \{0\}\) the map
\[
Y \ni y \mapsto \alpha \cdot y \in Y
\]
is the homeomorphism then second condition of the Theorem 2.1 is fullfiled then we are getting assertion.

Immediately we have

**Corollary 3.1.** Let \(X < \| \cdot \|\) be infinite dimensional Banach space then there
exists linear automorphism of \(X\) for which image of the unit ball do not has Baire property.

**Proof.** Let \(B\) is Hamel base which is subset of the unit ball of the space \(X\). Let \(B_0 = \{e_n \in X : n \in \omega\} \subset B\) be any countable subset of the our Hamel base \(B\).
Let define \( g : \mathcal{B} \to \mathcal{B} \) as follows:
\[
g(x) = \begin{cases} (n + 1) \cdot x & x = e_n \in \mathcal{B}_0 \\ x & x \in \mathcal{B} \setminus \mathcal{B}_0 \end{cases}
\]
Now we are ready to define \( f : X \to X \) let \( x \in X \) and let \( A \in [\mathcal{B}]^{<\omega} \) and assume that
\[
x = \sum_{e \in A} \alpha_e \cdot e
\]
then \( f(x) = \sum_{e \in A} \alpha_e \cdot g(e) \). It is easy to see that \( f \) is noncontinuous linear automorphism of \( X \). Then by Theorem 2.2 proof is finished. ■

Proof. of the Theorem 2.3 Let assume that image \( f[K] \) is \( \mathcal{I} \)-measurable in space \( Y \). Let \( \gamma < \kappa \) be such that \( \gamma = |S| \) and the subset \( S \subset X \) be dense in space \( X \) then we have
\[
Y = f[X] = f[\bigcup_{x \in S} \{x\} + K] = \bigcup_{x \in S} f[\{x\} + K] = \bigcup_{x \in S} \{f(x)\} + f[K]
\]
thus \( f[K] \not\in \mathcal{I} \) is \( \mathcal{I} \) positive. Then by Steinhaus property of the ideal \( \mathcal{I} \) there exist unit ball \( 0 \in U \) consisting \( 0 \) for which \( U \subset f[K] - f[K] = f[K - K] \) is hold. But \( f \) is linear isomorphism then \( f^{-1}[U] \subset K - K \) then \( f^{-1} \) is bounded linear operator (so is continuous) then by Banach Theorem on invertible operator we have that \( f \) is bounded then \( f \) is homeomorphism what is impossible by (3). ■

Immediately we are getting the following

Corollary 3.2. Let \( X, Y \) be a Banach spaces with the following properties:
\begin{itemize}
  \item[(1)] \( X \) is separable Banach space,
  \item[(2)] \( \mathcal{I} \subset \mathcal{P}(Y) \) be additive invariant \( \sigma \)-ideal with Steinhaus property,
  \item[(3)] \( f : X \to Y \) by any isomorphism between \( X, Y \), which is not homeomorphism.
\end{itemize}
Then the image of the unit ball \( f[K] \) is \( \mathcal{I} \) non-measurable in \( Y \), where \( K \) a unit ball of the space \( X \) with the centre equal \( 0 \in X \).

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Robert Ralowski, Institute of Mathematics, Wroclaw University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.
E-mail address: robert.ralowski@pwr.wroc.pl