Conservation laws for the nonlinear Schrödinger equation in Miwa variables

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Abstract

A compact expression for the generating function of the constants of motion for the nonlinear Schrödinger equation is derived using the functional representation of the AKNS hierarchy.

1. Introduction

In this paper we would like to discuss once more the conservation laws for the nonlinear Schrödinger equation (NLSE),

\begin{align}
    i\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + 2q^2r &= 0 \\
    -i\frac{\partial r}{\partial t} + \frac{\partial^2 r}{\partial x^2} + 2qr^2 &= 0.
\end{align}

The existence of an infinite number of conserved quantities is a characteristic feature of integrable partial differential equations (PDEs), and this question in the context of the NLSE was discussed in the very first works devoted to this model, where the authors established the integrability of the NLSE and elaborated the corresponding inverse scattering transform (IST). The IST is based on a representation of the equation in question as a compatibility condition for an overdetermined linear system (the so-called zero-curvature representation) which in the case of the NLSE can be written as

\begin{align}
    \frac{\partial}{\partial x} \Psi(x, t; \lambda) &= U(x, t; \lambda) \Psi(x, t; \lambda) \\
    \frac{\partial}{\partial t} \Psi(x, t; \lambda) &= V(x, t; \lambda) \Psi(x, t; \lambda).
\end{align}

Here \( \Psi \) is 2-column, the matrix \( U \) is given by

\[
    U = i \begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix}
\]
and \( V \) is some \( 2 \times 2 \) matrix which is a second-order polynomial in \( \lambda \) (in what follows we will not need its explicit form).

The \( U-V \) representation (3), (4) of the NLSE is enough to obtain answers to a wide range of questions related to this equation. In particular, one can derive from (3) and (4) an infinite number of integrals of motion of the NLSE. This can be done as follows (here we will only outline some key points; more elaborated description of the IST can be found in various textbooks on this topic, e.g. [1–3]). Introducing the so-called Jost functions \( \Psi_\pm \) of the scattering problem (3) (i.e. solutions of (3) satisfying different boundary conditions) and the scattering matrix \( T(x, t; \lambda) \), \( \Psi_+ = \Psi_- T \), one can obtain from the zero-curvature representation that the diagonal elements of the \( 2 \times 2 \) matrix \( T \) do not depend on time. Hence they (or their logarithms) can be used as generating functions of constants of motion (the latter are the coefficients of the power series in \( \lambda(\lambda^{-1}) \) of the former).

Another approach stems from the viewpoint when one considers an integrable equation (the NLSE in our case) as a member of an integrable hierarchy (the AKNS in our case). The \( U-V \) representation of the equations of the AKNS hierarchy can be written as

\[
\frac{\partial}{\partial x} \Psi = U \Psi \quad (6)
\]
\[
\frac{\partial}{\partial t_n} \Psi = V_n \Psi \quad (7)
\]

where \( U \) is given again by (5) and \( V_n \) are different matrices (\( V_n \) is some \( n \)th-order polynomial in \( \lambda \)). All equations of the hierarchy are compatible and one can solve them simultaneously, i.e. one can think of \( q \) and \( r \) as functions of an infinite set of variables \( q = q(t_1, t_2, t_3, \ldots) \), \( r = r(t_1, t_2, t_3, \ldots) \) with the evolution with respect to \( t_n \) being given by the \( k \)th NLSE (\( k \)th member of the AKNS hierarchy). In some situations such a standpoint leads to more transparent results, and the main aim of this paper is to apply it to the question of the description of the constants of motion of the NLSE (read constants of motion of the AKNS hierarchy).

2. Functional representation of the AKNS hierarchy

Our starting point is the so-called functional representation of the AKNS hierarchy, which can be written as

\[
\frac{i \zeta}{\partial_t} q = q - q^- + \zeta^2 q^* r^-
\]
\[
-\frac{i \zeta}{\partial_t} r = r - r^+ + \zeta^2 q^* r^2. \quad (8)
\]

Here \( q \) and \( r \) are functions of an infinite set of times, \( q = q(t), r = r(t) \),

\[
f(t) = f(t_1, t_2, t_3, \ldots) \quad (10)
\]

\( \partial_t = \partial/\partial t_1 \) and the designation \( f^\pm \) stands for the function with shifted arguments (Miwa shifts),

\[
f^\pm = f(t \pm i\zeta) \quad (11)
\]
\[
f(t) = f(t_1 \pm i\zeta, t_2 \pm i\zeta^2/2, t_3 \pm i\zeta^3/3, \ldots) \quad (12)
\]

These equations, which can be termed an ‘AKNS hierarchy in Miwa variables’, may be derived in different ways. First, this can be done by careful analysis of the linear problems (6) and (7). One can find the shifts \( t_k \to t_k \pm i\zeta^k/k \) in some textbooks on the IST. For example, in the book by Newell [2] one can find the representation of a solution of (6), (7) which in our terms (i.e. after interchanging \( q \) and \( r \) and rescaling the times) can be written as

\[
\Psi = \frac{1}{\tau} \left( \begin{array}{cc} \tau^+ & -\zeta \rho^- \\ \xi \sigma^+ & \tau^- \end{array} \right) \exp \left( \frac{i x}{2 \zeta} \sigma_3 \right) \quad (13)
\]
where $\sigma_3 = \text{diag}(1, -1)$ and $\rho, \sigma, \tau$ are the tau-functions of the AKNS hierarchy, defined by

$$ q = \frac{\sigma}{\tau}, \quad r = \frac{\rho}{\tau}, \quad \frac{\partial^2}{\partial x^2} \ln \tau = qr. \quad (14) $$

Substituting (13) in (3) one can get after simple manipulations equations (8) and (9). Another way is to use the (generalized) Hirota bilinear identities which are one of the most important formulae of the Kyoto school approach to integrable systems [5]. One should also mention paper [6] where a functional representation has been derived for the Davey–Stewartson system. The AKNS hierarchy is known to be an integrable reduction of the Davey–Stewartson hierarchy, and one can obtain from the formulae of [6] a similar representation for the AKNS hierarchy. Explicitly equations (8) and (9) have been written down in the paper [4].

We will not repeat here the derivation of (8) and (9) and only demonstrate that the first equations of the AKNS hierarchy (the NLSE in particular) can be easily obtained from them. Indeed, using the multidimensional Taylor series for $f^\pm = f(t \pm i[\xi])$,

$$ f^\pm = f \pm i\xi \partial_t f + \frac{\xi^2}{2}(\pm i\partial_2 f - \partial_{11} f) + \frac{\xi^3}{6}(\pm 2 i\partial_3 f - 3 \partial_{21} f \mp i \partial_{111} f) + \cdots \quad (15) $$

(here $\partial_j$ stands for $\partial/\partial_{t_j}$, $\partial_{jk}$ for $\partial^2/\partial_{t_j}\partial_{t_k}$, etc) and expanding equations (8), (9) in power series in $\xi$ one will obtain that the functions $q$ and $r$ satisfy an infinite number of PDEs. The first non-trivial equations (the $\xi^2$ terms),

$$ i\partial_2 q + \partial_{11} q + 2 q^2 r = 0 \quad (16) $$

$$ -i\partial_r + \partial_{11} r + 2qr^2 = 0 \quad (17) $$

are nothing else than the NLSE. Hereafter we will identify variables $x, t$ and $t_1, t_2$

$$ x = t_1, \quad t = t_2. \quad (18) $$

The next equations

$$ 2\partial_t q - 3i\partial_{21} q - \partial_{111} q - 6q^2\partial_r q = 0 \quad (19) $$

$$ 2\partial_t r + 3i\partial_{21} r - \partial_{111} r - 6r^2\partial_q q = 0 \quad (20) $$

can be rewritten using (16), (17) as

$$ \partial_3 q + \partial_{11} q + 6qr \partial_q q = 0 \quad (21) $$

$$ \partial_3 r + \partial_{11} r + 6qr \partial_r r = 0. \quad (22) $$

These are the third-order NLSEs. Thus one can view (8), (9) as a ‘condensed’ form of the AKNS hierarchy.

The key point is that, if we deal not only with solutions of the NLSE, $q(x, t), r(x, t)$, but consider them as solutions of all equations of the hierarchy, $q = q(t_1, t_2, \ldots), r = r(t_1, t_2, \ldots)$, then we can formally solve the auxiliary linear problem. Indeed, matrix (13), which can be rewritten without invoking the tau-functions as

$$ \Psi = \begin{pmatrix} 1 & -\xi q^\ast r^- \\ \xi q^\ast & 1 \end{pmatrix} \begin{pmatrix} \exp(iu_1) & 0 \\ 0 & \exp(-iu_2) \end{pmatrix} \quad (23) $$

where

$$ u_1 = \frac{x}{2\xi} + \xi \int dx \, q^\ast r^- \quad (24) $$

$$ u_2 = \frac{x}{2\xi} + \xi \int dx \, qr^- \quad (25) $$

solves (6) with $\lambda = (2\xi)^{-1}$. Hence we can now rewrite the results which were presented in terms of solutions of (6) (i.e. in terms of the Jost or Baker–Akhiezer functions) in terms of $q$, $r$ themselves. This is also valid for the generating function of the integrals of motion.
Of course, matrix (23) is a formal solution of (6) and one must be ready to face some problems when, e.g., one tries to construct the Jost functions (i.e. to satisfy some boundary conditions). However, for our purposes this is not an obstacle. Moreover, we will not repeat the ‘classical’ algorithm, Jost functions $\Psi_\pm(\lambda)\rightarrow$ scattering matrix $T(\lambda)\rightarrow$ generating function $\ln T_{11}(\lambda)$. Knowing the answer, we will first present the final result, and then, using the functional representation (8), (9) of the AKNS hierarchy, will prove it.

3. Conservation laws

The main result of this work can be presented as follows: the function

$$J(t, \zeta) = q(t + i[\zeta])r(t)$$

(26)

is the generating function for the constants of motion.

Indeed, it follows from (16), (17) that

$$i\partial_t q^* r = \frac{\partial}{\partial x} \left( q^* \frac{\partial r}{\partial x} - \frac{\partial q^*}{\partial x} r \right) - 2q^* r(q^* r - qr).$$

(27)

Using again (16), (17), this time with shifted arguments, one can easily get

$$i\zeta \frac{\partial}{\partial x} q^* r = q^* r - qr$$

(28)

which leads to

$$\frac{\partial}{\partial t} J(\zeta) = \frac{\partial}{\partial x} F(\zeta)$$

(29)

where

$$F = i\partial q^* r - q^* \frac{\partial r}{\partial x} - \zeta(q^* r)^2$$

(30)

(recall that $x = t_1$ and $t = t_2$).

Thus we have obtained an infinite number of divergent-like conservation laws

$$\frac{\partial}{\partial t} J_m = \frac{\partial}{\partial x} F_m$$

(31)

where $J_m$ and $F_m$ are coefficients of the Taylor series for $J(\zeta)$ and $F(\zeta)$

$$J(\zeta) = \sum_{m=0}^{\infty} J_m \zeta^m$$

(32)

$$F(\zeta) = \sum_{m=0}^{\infty} F_m \zeta^m.$$  

(33)

Some of the first conserved densities are given by

$$J_0 = qr$$

(34)

$$J_1 = \frac{\partial q}{\partial x} r$$

(35)

$$J_2 = \frac{1}{2} ((\partial_2 q - \partial_{11} q) r = -\left( \frac{\partial^3 q}{\partial x^3} + q^2 r \right) r$$

(36)

$$J_3 = \left( \frac{i}{3} \partial_3 q - \frac{1}{2} \partial_2 q - \frac{i}{6} \partial_{11} q \right) r = -i \left( \frac{\partial^3 q}{\partial x^3} + 4qr \frac{\partial q}{\partial x} + q^2 \frac{\partial r}{\partial x} \right) r.$$  

(37)

Note that, to present $J_m$ for $m = 2, \ldots$ in a standard way, i.e. in terms of $q$, $r$ and their derivatives with respect to $x$, one has to use evolution equations of the hierarchy (21), (22).
and higher. However, it is possible not to use these equations but instead to ‘iterate’ the identity (8), which can be rewritten as

\[ q^* = q + i \zeta q_x - \zeta^2(q^*)^2 r \]

\[ = q + i \zeta q_x - \zeta^2[q_{xx}^* + (q^*)^2 r] - i \zeta^3 [(q^*)^2 r]_x \]

or to return to the traditional inverse scattering scheme: it follows from (38) that \( J(t, \zeta) \) satisfies

\[ J = qr + i \zeta r \frac{\partial J}{\partial x} r - \zeta^2 J^2 \]

which leads to the recurrence relation

\[ J_0 = qr, \quad J_1 = q_x r \]

\[ J_{m+1} = r \frac{\partial}{\partial x} J_m - \sum_{l=0}^{m-1} J_l J_{m-1-l}, \quad m \geq 1. \]

One can easily identify these equations with the standard for the inverse scattering approach equations for the generating function. Thus, the main result (26) of this paper can be interpreted as follows. Equation (41), if considered as an ordinal differential equation, is the famous Riccati equation which cannot be solved explicitly for arbitrary functions \( q \) and \( r \). However in our case \( q \) and \( r \) are not arbitrary but related by an infinite number of PDEs of the hierarchy, and it turns out that in this situation, though these restrictions do not determine \( q \) and \( r \) uniquely, equation (41) can be solved formally and this solution is given by (26).

Equations (31) can be rewritten as

\[ \frac{\partial}{\partial t} I_m = 0 \]

where \( I_m \) are integrals of the densities \( J_m \). In the case when \( q \) and \( r \) vanish (sufficiently rapidly) as \( x \to \pm \infty \) \( I_m \) are given by

\[ I_m = \int_{-\infty}^{\infty} dx J_m. \]

In the periodical case, \( q, r(x + L) = q, r(x) \),

\[ I_m = \int_{0}^{L} dx J_m \]

while in the case of non-trivial boundary conditions, say the finite-density ones, the integral on the right-hand side should be in some way regularized.

4. Conclusion

The aim of this paper is twofold. First we want to demonstrate that the Kyoto school approach can be used not only to reveal and study some mathematical structures behind integrable equations, but also to solve some ‘practical’ problems, as discussed above. Another purpose of this paper is to attract attention to the functional representation of the AKNS hierarchy. This system is one of the first studied integrable models for which the IST has been developed, and since the 1970s the IST has been the main tool to study the NLSE and related problems. As to the methods which were developed later for such equations, e.g. KP and two-dimensional Toda equations, to our knowledge their application to the AKNS hierarchy is rather limited. At the same time the formulation of a problem in terms of the functional equations seems to be
promising for, e.g., developing some perturbation schemes for various NLSE-related problems (say, spectral ones, which are based on the so-called trace formulae and extract deformation of parameters of solution from the perturbations of the constants of motion and other scattering data).

To conclude, we would like to make a few remarks on the following question. The conservation laws are a topic of particular interest for those who study the integrable systems and there are many approaches to this problem. The inverse scattering technique and Miwa shifts are not the only ones. For example, considering the AKNS hierarchy one should mention papers [7] and [8] where constants of motion were studied using the additional symmetries and noncommutative calculus. Thus, it would be interesting to derive the relationships between these rather different approaches. Concerning [7] one can notice that in this paper we used the simultaneous action of all evolutionary flows (equations) of the hierarchy, while Orlov and Schulman came to similar results using additional symmetries, which do not belong to the hierarchy. We presume that these symmetries can also be constructed using the Miwa shifts (this time with $\zeta$-dependence having no $\zeta \to 0$ limit, as in the case of the Bäcklund transformations). Probably, more a convenient standpoint to this question is one of Bergvelt and ten Kroode who considered the extended AKNS hierarchy, which contains both continuous (original AKNS) and discrete (Toda-like) equations (see [9]). As to the work of Dimakis and Muller-Hoissen [8], it is interesting to find out how the Miwa shifts manifest themselves in the bi-differential calculus and whether it is possible to model the noncommutative structures of the bicomplex formulation using the flows of the commutative AKNS hierarchy. These problems are out of the scope of this paper and we hope to get some answers during the following studies.

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