An alternative non-Markovianity measure by divisibility of dynamical map

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Identifying non-Markovianity with non-divisibility, we propose a measure for non-Markovianity of quantum process. Three examples are presented to illustrate the non-Markovianity, measure for non-Markovianity is calculated and discussed. Comparison with other measures of non-Markovianity is made. Our non-Markovianity measure has the merit that no optimization procedure is required and it is finite for any quantum process, which greatly enhances the practical relevance of the proposed measure.

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I. INTRODUCTION

A quantum process is said to be Markovian if the future states of the process depends only on the state of present time. In contrast, dependence on past states should then be a characteristic feature of non-Markovian processes. With the development of technology to manipulate quantum system, the quantum non-Markovian process has attracted increasing attention in recent years\textsuperscript{[1,2]}. On one hand the inevitable interaction of a quantum system with its environment leads to dissipation of energy and loss of quantum coherence, on the other hand the quantum system may temporarily regain some of the previously lost energy and/or information due to non-Markovian effects in the dynamics. This motivates the study on the non-Markovianity and a measure for the degree of non-Markovianity is indeed needed.

Several approaches are proposed to quantify non-Markovianity, including the measure based on the increase of trace distance\textsuperscript{[3]}, the measure by quantifying the increase of entanglement shared between the system and an isolated ancilla and the divisibility of the dynamical map\textsuperscript{[4]}, the measure based on the decay rate of master equation itself\textsuperscript{[5,6]}, and the measure through the Fisher information flow\textsuperscript{[7,8]}. Although several approaches to quantifying the non-Markovianity are proposed, the definition of non-Markovianity still remains elusive and, in some sense, controversial\textsuperscript{[9]}

It has been proven that all divisible dynamical maps are Markovian, this divisibility property holds for a larger class of quantum processes than those described by the Lindblad master equation, for example, the time-local master equation with positive decay rates. This indicates that the divisibility may be a good starting point to quantify non-Markovianity. In this paper, we will propose a measure for non-Markovianity based on the divisibility of dynamical maps, three dynamical maps are presented and the corresponding non-Markovian measures are calculated. These results suggest that the measure can capture the feature of non-Markovian dynamics, and provide an easy way to calculate the non-Markovianity.

II. NON-DIVISIBILITY AND NON-MARKOVIANITY

In quantum Markovian process, the future state of the quantum system depends only on the state of present time. However, writing this statement in a precise mathematical representation is not an easy task. Instead, we use the following description for quantum Markovian process. A quantum evolution is Markovian if it is an element of any one-parameter continuous completely positive semigroup\textsuperscript{[10]}. The quantum evolution governed by the master equation

\begin{equation}
\frac{d\rho}{dt} = \mathcal{L}\rho,
\end{equation}

is an example, where $\mathcal{L}$ is a time-independent generator of the well-known Lindblad form,

\begin{equation}
\mathcal{L}\rho = -i[H, \rho] + \sum_\alpha \gamma_\alpha (V_\alpha \rho V_\alpha^\dagger - \frac{1}{2} V_\alpha^\dagger V_\alpha \rho - \frac{1}{2} \rho V_\alpha V_\alpha^\dagger) \tag{2}
\end{equation}

with $\gamma_\alpha \geq 0$. This generator leads to completely positive trace-preserving maps $\Lambda(t) = e^{t\mathcal{L}}$ and it satisfies the composition law,

\begin{equation}
\Lambda(t_1 + t_2) = \Lambda(t_2)\Lambda(t_1). \tag{3}
\end{equation}

If a dynamical map can be written in this decomposition with both $\Lambda(t_2)$ and $\Lambda(t_1)$ being completely positive, the
dynamical map is called divisible. This composition law can be extended to a general case, where the generator in Eq. (2) is time-dependent, namely,

$$\frac{d\rho}{dt} = \mathcal{L}(t)\rho \quad (4)$$

with

$$\mathcal{L}(t)\rho = -i[H(t),\rho] + \sum_\alpha \gamma_\alpha(t)(V_\alpha(t)\rho V_\alpha^\dagger(t) - \frac{1}{2}V_\alpha^\dagger(t)V_\alpha(t)\rho - \frac{1}{2}V_\alpha(t)V_\alpha^\dagger(t)\rho V_\alpha(t)) \quad (5)$$

where \(\gamma_\alpha(t) > 0\). This is known as time-dependent markovian [12]. The solution to Eq. (4) can be written in terms of the two-parameter family of dynamical maps \(\Lambda(t_2,t_1)\) \((t_2 \geq t_1 \geq 0)\). The composition law corresponding to Eq. (5) becomes

$$\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0), \quad (6)$$

and the map \(\Lambda(t_2,t_1)\) can be written as \(\Lambda(t_2,t_1) = Te^{\int_{t_1}^{t_2} \mathcal{L}(t')dt'}\), \(T\) is the chronological operator. The composition law (divisibility of the map) implies that the dynamical map \(\Lambda(t_2,t_1)\)

$$\Lambda(t_2,t_1) = Te^{\int_{t_1}^{t_2} \mathcal{L}(t')dt'} \quad (t_2 \geq t_1 \geq 0), \quad (7)$$

transforming a state at \(t_1\) into a state at \(t_2\) (for systems governed by time independent master equation [11], \(\Lambda(t_2,t_1) = \Lambda(t_2-t_1,0) = \Lambda(t)\)) must be trace-preserving and completely positive, regardless of which dynamics it describes, e.g., it is from the time-dependent master equation or the time-independent master equation. Note that the starting time \(t_1\) is not zero.

A measure for non-Markovianity should quantify the deviation of a dynamical map from Markovian evolution. Noticing that when a dynamics is non-Markovian, the dynamical map \(\Lambda(t_2,t_1)\) may not be completely positive, we may use the non-divisibility to quantify the non-Markovianity. In fact, this is the underlying reason that the trace distance can increase [8], and the system gains entanglement with an isolated ancilla [8].

It is worth stressing that there is no contradiction between the requirement on non-completely positivity and that on physics. Consider a quantum evolution in a time interval \((0,t_2)\), we always have \(\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0)\) due to the time continuity. For \(\Lambda(t_2,0)\) to be a dynamical map, it is required that \(\Lambda(t_2,0)\) must be completely positive, however, \(\Lambda(t_2,t_1)\) may not be completely positive. Therefore these two-parameter maps in non-Markovian dynamics do not generate a quantum dynamical semi-group. Then one may wonder: does there exist a \(\Lambda(t_2,t_1)\) that is not completely positive but \(\Lambda(t_2,0)\) does? The answer is yes. First, a wide range of non-Markovian process can be described by time-local master equations via time-convolutionless projection operator [13,18]. Second, it has been shown that any quantum dynamics described by memory kernel master equation may be written in a time-local form [19]. Note that the decay rates in these time-local master equation are different from that in Eq. (5), they can be negative. With this time-local master equation, the dynamical map with non-zero starting time in Eq. (5) may violate the complete positivity due to the negative decay rates. This implies that the non-complete positivity of the map \(\Lambda(t_2,t_1)\) is an essential feature of non-Markovian process. The time-dependent decay rate may be negatively infinite at some points of time, where the revival of population or regaining of quantum coherence happen [7,17]. We call these points of time singular points \(t_s\). When \(t_1 = t_s\) or \(t_2 = t_s\), \(\Lambda(t_2,t_1)\) may not exist. However, we can use \(\Lambda(t_2,t_1)\) in the limit that \(t_1 \to t_s\) instead of \(\Lambda(t_2,t_s)\). It is convenient to discuss \(\Lambda(t_2,t_1)\) with a specific time-local master equation, but this is not necessary.

### III. MEASURE FOR NON-MARKOVIANITY

To construct a measure for non-Markovianity, we resort to the Choi-Jamiolkowski isomorphism [20,21], it asserts that a linear map \(\Lambda : M_d \to M_d\) is isomorphic to the Choi matrix,

$$C_\Lambda = \sum_{i,j=1}^d |i\rangle \langle j| \otimes \Lambda(|i\rangle \langle j|), \quad (8)$$

where \(|i\rangle\) are orthogonal bases. An familiar form of the Choi matrix is

$$\rho_\Lambda = (\Lambda \otimes I)|\phi\rangle \langle \phi|, \quad (9)$$

where \(|\phi\rangle\) is the maximally entangled state \(|\phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle \otimes |i\rangle\), \(I\) is an identity map acting on the ancilla, and \(\rho_\Lambda\) is the normalized \(C_\Lambda\). It turns out that \(\Lambda\) is completely positive if and only if \(\rho_\Lambda\) is positive semidefinite. In other words, the sufficient and necessary condition of non-complete positivity is the negativity of \(\rho_\Lambda\). Hence, the sum of negative eigenvalues of \(\rho_\Lambda\) can be taken as a measure for the non-complete positivity of the dynamical map. However, the summation may sometimes be an infinite value in some models due to singular decay rates, this suggests to use a normalized quantity

$$Ncp = \arctan(-\sum \lambda_k) \quad (10)$$

as a measure for non-complete positivity of the map \(\Lambda\), where \(\lambda_k\) is the \(k-th\) negative eigenvalue of \(\rho_\Lambda\). Clearly if \(\rho_\Lambda \geq 0\), \(Ncp = 0\). Then we have \(0 \leq Ncp \leq \frac{\pi}{2}\). Based on the complete positivity property of the map \(\Lambda(t_1,t_2)\), a measure of non-Markovianity has been proposed [8]. This measure is different from ours in that we use the an averaged negativity of the map \(\Lambda(t_1,t_2)\) to quantify the non-Markovianity. Moreover, insightful examples are presented to shed light on the non-Markovianity measure.

To calculate Eq. (10) with a given time-local master equation, the exact form of \(\Lambda(t_1,t_2)\) is not necessary.
What we need is to extend the time-local master equation from one system to two systems, taking an isolated ancilla attached to the system. The Hilbert space is extended from $H_2$ to $H_d \otimes H_d$ accordingly. All operators, say $\hat{O}$, are replaced by $\hat{O} \otimes I$. By this extension, we get a new master equation, which describes the system and the ancilla. The system evolves in the same manner as before, while the ancilla is isolated from both the system and environment. The master equation can be solved starting from $|\phi\rangle$ at time $t_1$ and the state at time $t_2$ ($t_2 > t_1$) can be obtained.

We aim at finding a measure $NM$ for non-Markovianity which captures the feature of non-complete positivity of all possible $\Lambda(t_2,t_1)$. Note that $Ncp$ is a function of $t_1$ and $t_2$. Let $S$ count the number of $Ncp$ in all time interval with $Ncp > 0$, i.e.,

$$S = \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \ c(t_2,t_1), \quad c = \begin{cases} 1, & \text{if } Ncp > 0 \\ 0, & \text{if } Ncp = 0 \end{cases}.$$  \hfill (11)

If $S = 0$, i.e., all $\Lambda(t_2,t_1)$ (for any $t_1$ and $t_2$, as long as $t_2 > t_1$) are completely positive, the non-Markovianity $NM$ should be defined to be zero. If $S > 0$, we define

$$NM = \lim_{T \to +\infty} \frac{\int_0^T dt_1 \int_{t_1}^T dt_2 \ Ncp(t_2,t_1)}{\int_0^T dt_1 \int_{t_1}^T dt_2 \ c(t_2,t_1)}$$  \hfill (12)

as a measure of non-Markovianity. This can be understood as an averaged non-complete positivity of all the non-completely positive maps in interval $t = (0, +\infty)$. Therefore, from $0 \leq Ncp \leq \frac{2}{\pi}$, we have $0 \leq NM \leq \frac{2}{\pi}$.

The upper limit of the integral $T$ in the definition Eq. (12) is taken to be infinite. However, the distribution of $Ncp$ on $(t_1,t_2)$ plane is often periodic or is limited in a small area, this suggests that the integration can be taken merely in one period of time or taken in a finite area. When $Ncp$ is neither periodic nor limited in a small region, the upper limit $T$ in Eq. (12) should be large enough to get a convergent $NM$. The definition can be written into a simple form,

$$NM = E(Ncp(A_N)), \quad \hfill (13)$$

where $A_N$ represents all the non-completely positive maps and $E$ means expectation value. Therefore, $NM$ can be numerically calculated by averaging a large number of non-completely positive maps with equal weight (or randomly) in a representative area as discussed above. The representative area means one period in time or a limited region where $Ncp > 0$.

To illustrate the measure of non-Markovianity, we present three examples in the next section. We work in the interaction picture for simplicity to calculate the measure, since unitary transformation does not change the eigenvalues of $\rho_s$ as well as $Ncp$ of $\Lambda(t_2,t_1)$, hence the non-Markovianity measure under unitary transformation remains unchanged.

IV. EXAMPLES

A. damping J-C model

The first example is a two-level system coupling to a reservoir at zero temperature. The reservoir consists of infinite number of harmonic oscillators that is also referred in the literature as the spin-boson model. This model is exactly solvable [17]. The Hamiltonian for such a system reads,

$$H = H_0 + H_I.$$  \hfill (14)

with

$$H_0 = \hbar \omega_0 \sigma_+ \sigma_- + \sum_k \hbar \omega_k b_k^\dagger b_k,$$

$$H_I = \sigma_+ B + \sigma_- B^\dagger,$$  \hfill (15)

where $B = \sum_k g_k b_k$. The Rabi frequency of the two-level system and the frequency for the $k-th$ harmonic oscillator are denoted by $\omega_0$ and $\omega_k$, respectively. $b_k^\dagger$ and $b_k$ are the creation and annihilation operators of $k-th$ oscillator, which couples to the system with coupling constant $g_k$.

Assuming the system and the reservoir are initially uncorrelated, we can obtain a time-dependent master equation in the interaction picture,

$$\dot{\rho} = -i \frac{s(t)}{2} [\sigma_+ \sigma_- , \rho] + \gamma(t) (\sigma^- \rho^+ - \frac{1}{2} \sigma^+ \sigma^- \rho - \frac{1}{2} \rho \sigma^+ \sigma^-), \quad \hfill (17)$$

where $s(t) = -2 \text{Im}[\frac{c(t)}{\omega_0(t)}]$ and $\gamma(t) = -2 \text{Re}[\frac{c(t)}{\omega_0(t)}]$. $\Omega(t)$ plays the role of Lamb shift and $\gamma(t)$ is the decay rate. Both $\Omega(t)$ and $\gamma(t)$ are time-dependent. $c(t)$ is determined by $c(t) = -\int_0^t f(t - \tau) \sigma(\tau) d\tau$, where $f(t - \tau) = \int d\omega J(\omega) \exp(i(\omega_0 - \omega)(t - \tau))$ is the environmental correlation function. In the derivation of the master equation, the reservoir is assumed in its vacuum at $t = 0$.

Consider the following spectral density,

$$J(\omega) = \frac{1}{\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2},$$  \hfill (18)

where $\gamma_0$ represents the coupling constant between the system and reservoir, $\lambda$ defines the spectral width of the coupling at the resonance point $\omega_0$. For the spectral density [18], we have $s(t) = 0$, $c(t) = c_0 \exp(-\lambda t/2) [\cosh(d t/2) + \lambda \sinh(d t/2)]$, and $\gamma(t) = \frac{2 \gamma_0 \lambda \sinh(d t/2)}{d \cosh(d t/2) + \lambda \sinh(d t/2)}$ with $d = \sqrt{\lambda^2 - 2 \gamma_0 \lambda}$. Note that Eq. (17) is derived without any approximations, hence it is non-Markovian and it exactly describes the dynamics of the open system.

It is well-known that $\lambda$ characterizes the correlation time $\tau_R$ of the reservoir through $\tau_R = \lambda^{-1}$. The time scale $\tau_S$ on which the state of the system changes is given by $\tau_S = \gamma_0^{-1}$. So the degree of non-Markovianity should be relevant to the rate $\dot{R} = \tau_R/\tau_S$. Namely, when $\dot{R}$ is
A straightforward calculation yields

\[ N_{cp}(t_1, t_2) = \begin{cases} \arctan\left(\frac{c(t_2)^2}{c(t_1)^2} - 1\right), & c(t_2)^2 > 1 \\ 0, & c(t_2)^2 \leq 1, \end{cases} \]

where \( c(t) \) was defined below Eq. (17) and \( t_2 \geq t_1 \geq 0 \).

For a typical non-Markovian case \((R = 5)\), we plot \( N_{cp} \) as a function of \( t_1 \) and \( \Delta t \) in Fig. 1(a) \((\Delta t = t_2 - t_1, \text{since } t_2 \geq t_1, \text{we use } \Delta t \text{ instead of } t_2 \text{ for convenience})\). This

very small, the evolution is Markovian, when \( \tau_R \) is comparable with \( \tau_S \), the memory effect of reservoir should be taken into account, the dynamics of the open system is then non-Markovian.

Let us first analyze the non-Markovianity of the dynamics by examining \( \gamma(t) \). For \( R < \frac{1}{\lambda} \), \( \gamma(t) \) is always positive and it is a monotonically increasing function of time, all \( \Lambda(t_2, t_1) \) are completely positive, hence the dynamics is Markovian. When \( R > \frac{1}{\lambda} \), \( \gamma(t) \) is a periodic function of time, it takes negative values sometimes. In particular, \( \gamma(t) \) has discrete singular points where the upper level gains population, this is a typical feature of non-Markovianity.

Now we see if our measure can capture all these features of non-Markovianity. In order to apply our measure, we have to extend the time-local master equation to the compound system, i.e., the operators \( \sigma_\pm \) in Eq. (17) is replaced by \( \sigma_\pm \otimes I \) with \( I \) being the ancilla’s \( 2 \times 2 \) identity operator. For a given time interval \((t_1, t_2)\), a straightforward calculation yields

\[ N_{cp}(t_1, t_2) = \begin{cases} \arctan\left(\frac{1}{2}\left(c(t_2)^2 - c(t_1)^2\right)\right), & c(t_2)^2 > 1 \\ 0, & c(t_2)^2 \leq 1, \end{cases} \]

\[ \text{where } c(t) = \frac{\sigma_{\pm} - \sigma_{\|}}{2} \text{ was defined below Eq. (17) and } t_2 \geq t_1 \geq 0. \]

For a typical non-Markovian case \((R = 5)\), we plot \( N_{cp} \) as a function of \( t_1 \) and \( \Delta t \) in Fig. 1(a) \((\Delta t = t_2 - t_1, \text{since } t_2 \geq t_1, \text{we use } \Delta t \text{ instead of } t_2 \text{ for convenience})\). This

plot shows the non-zero area and its value of \( N_{cp} \) versus \( t_1 \) and \( t_1 - t_2 \). As \( N_{cp} \) is a periodic function of \( t_1 \), and the area where \( N_{cp} > 0 \) decays very fast with \( \Delta t \). \( NM \) can be given by averaging all \( N_{cp} \) in one period, it yields \( NM = 0.835 \).

As expected, when \( R < 0.5 \), \( NM = 0 \), and when \( R > 0.5 \) the non-markovianity is finite. We plot the measure of non-Markovianity with different \( R \) in Fig. 1(b). Here \( R \) is chosen from 0.5 to 10. Note that when \( R = 0.5 \), \( \gamma(t) \) does not exist due to the zero denominator. Our result for \( R = 0.5 \) is obtained at \( R = 0.5 + \varepsilon \) with \( \varepsilon \) an infinitesimal positive number. Intuitively, the larger \( R \) is, the stronger the non-Markovianity. The results in Fig. 1(b) demonstrate that this is indeed the case.

B. J-C model with detuning

The second example is similar to the first one, but here we consider the system in a cavity whose center frequency is detuned from the system Rabi frequency \( \omega_0 \). The dynamics in the interaction picture is governed by Eq. (17), but \( s(t) \) and \( \gamma(t) \) are determined by the Lorenz spectral density

\[ J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2 / 2}{(\omega_0 - \Delta - \omega)^2 + \lambda^2}, \]
where $\Delta$ denotes the detuning.

We extend Eq. (17) to a compound system by introducing an ancilla as we did in the first example, then we calculate $N_{cp}(t_1, t_2)$ numerically. We plot $N_{cp}$ as a function of $t_1$ and $\Delta t$ in Fig. 2(a) for a typical non-Markovian case. We see that $N_{cp}(t_1, t_2)$ is finite in contrast to infinite values in the same region for the first example. On the other hand, $N_{cp}$ of all $\Lambda(t_2, t_1)$ (with different $t_1$ and $t_2$) are far less than $\frac{\pi}{2}$, indicating weaker non-Markovianity in comparison with the first example. Finally, from Eq. (12), or Eq. (13) we have $NM = 3.66 \times 10^{-4}$ in this case.

Now we discuss the dependence of non-Markovianity on the detuning. We plot $NM$ in Fig. 2(b) with different $\Delta$. We find that $NM$ appears non-zero at about $\Delta = 4$, it first increases then decreases with $\Delta$. This result is similar to that in [7], where the non-Markovianity is measured by the decreases of trace distance.

C. a two-level system coupling to a finite spin bath

In the third example, we consider a central spin-$\frac{1}{2}$ coupling to a bath of $N$ spin-$\frac{1}{2}$s. The interaction Hamiltonian is,

$$H = \sum_{k=1}^{N} A_{k} \sigma_{z} \sigma_{z}^{k}$$

where $A_{k} = A/\sqrt{N}$ represents the coupling constants. The non-Markovianity of the central spin in this model is discussed in [1]. Assume the initial state of the whole system is $\rho_{s}(0) \otimes (\frac{1}{2} I)$, i.e., all spins in the reservoir are in a maximal mixed state. The density matrix of the central spin at time $t$ takes,

$$\rho(t) = \left(\begin{array}{cc}
\rho_{11} & \rho_{12} \cos^{N}(\frac{2At}{\sqrt{N}}) \\
\rho_{21} \cos^{N}(\frac{2At}{\sqrt{N}}) & \rho_{22}
\end{array}\right).$$

(22)

In terms of dynamical map, the dynamics can be represented as, $\Lambda(t, 0)\rho = \frac{1}{2}(1 - \cos^{N}(\frac{2At}{\sqrt{N}}))\sigma_{z} \rho \sigma_{z} + \frac{1}{2}(1 + \cos^{N}(\frac{2At}{\sqrt{N}}))\rho$. This is equivalent to the following master equation,

$$\dot{\rho} = \gamma(t) \mathcal{L}(\rho),$$

(23)

where $\mathcal{L}(\rho) = \sigma_{z} \rho \sigma_{z} - \rho$, and the time-dependent $\gamma(t) = \frac{N}{2} \tan(\frac{2At}{\sqrt{N}})$. This example is discussed in several papers as a classical example to quantify non-Markovianity, and the non-Markovianity is infinite [1, 8]. By our definition of non-Markovianity, it is finite. This allows us to establish a relation between non-Markovianity $NM$ and the number of spin $N$ in the reservoir.

It is easy to calculate $\rho_{\Lambda}$ (defined in Eq. 9) by Eq. (23),

$$\Lambda(t_2, t_1) \otimes I) \psi \langle \psi = \left(\begin{array}{cccc}
0.5 & 0 & 0 & 0.5k \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.5k & 0 & 0 & 0.5
\end{array}\right)$$

(24)
that $\frac{24T \sqrt{N}}{N}$ is close to zero. The off-diagonal element of the density matrix Eq. (24) then takes $\rho_{12} e^{-\frac{2A^2}{\lambda^2}}$, indicating that the density matrix in Eq. (24) describes a typical Markovian process.

V. SUMMARY

We have presented a measure for non-markovianity based on the divisibility of dynamical map. This measure has the advantage that it is easy to calculated and no optimization is required. Three examples are illustrated, which show that the measure can nicely manifest the non-Markovianity. We also compare our measure with others in the literature and find that it almost agrees with the trace-distance based measure.

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