THE ENERGY-MOMENTUM TENSOR AS A SECOND FUNDAMENTAL FORM

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Abstract. We show that it is natural to consider the energy-momentum tensor associated with a spinor field as the second fundamental form of an isometric immersion. In particular we give a generalization of the warped product construction over a Riemannian manifold leading to this interpretation. Special sections of the spinor bundle, generalizing the notion of Killing spinor, are studied. First applications of such a construction are then given.

1. Introduction

In this paper, we show that there exists an analogy between the role of the energy-momentum tensor associated with an eigenspinor field of the Dirac operator and the role of the second fundamental form of a hypersurface.

In [9], O. Hijazi proved that on a compact Riemannian spin manifold, any eigenvalue \( \lambda \) of the Dirac operator \( D \), to which is attached an eigenspinor \( \psi \), satisfies
\[
\lambda^2 \geq \inf_M \left( \frac{1}{4} S + |T^\psi|^2 \right),
\]
where \( S \) is the scalar curvature of the manifold, and \( T^\psi \) is the energy-momentum tensor associated with \( \psi \). This lower bound gives a non-trivial information on the spectrum of \( D \) without requiring \( S \) to be positive (compare with Friedrich’s inequality [5]). Even though the r.h.s. of (\( \star \)) depends on the eigenspinor \( \psi \), we shall show that, in the case of a hypersurface, the limiting case can be geometrically interpreted.

Therefore, we begin by recalling basic facts regarding spin geometry of hypersurfaces, such as the identification of the restriction of the spin bundle of an ambient space with the spin bundle of a hypersurface, and the spinorial Gauß formula.

We then prove that the extrinsic lower bound for the eigenvalue of the Dirac operator on a compact hypersurface bounding a compact domain given in [10] is related to the intrinsic estimate (\( \star \)). Equality cases are characterized by the existence of special sections of the spinor bundle, called \( T \)-Killing spinors, which generalize the notion of Killing spinors (these particular sections have been studied by E.C. Kim and Th. Friedrich in [7]).

Recall that complete simply connected Riemannian spin manifolds carrying real Killing spinors are characterized in [2]. For this, C. Bär proved that the usual cone constructed

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over such a manifold has to be Ricci flat. The geometry of the cone being studied by S. Gallot in [8], it suffices then to apply the already known characterization of irreducible, simply connected Riemannian spin manifolds carrying parallel spinors [11, 17].

In the last three sections, we show that the above argument can be generalized to Riemannian spin manifolds carrying $T$-Killing spinors in the following way:

We start by constructing a generalized warped product over a Riemannian manifold $(M, g)$, deforming the initial metric in the direction of the energy-momentum tensor $T^\psi$ associated with a $T$-Killing spinor field $\psi$. Therefore, we can consider $(M, g)$ as a hypersurface of the constructed manifold $\mathcal{M}$ whose second fundamental form is given by $T^\psi$. In the case where this 2-symmetric tensor is parallel, we then show how this construction extends the one of the usual warped product. We finally prove that in the case of a manifold carrying a $T$-Killing spinor field whose energy-momentum tensor is a projector, the manifold $\mathcal{M}$ has to be Ricci flat (Theorem 5.1).

2. The restriction principle and $T$-Killing spinors

2.1. Restricting spinors to a hypersurface. Let $(N^{n+1}, g)$ be an oriented Riemannian spin manifold of dimension $n + 1$, with a fixed spin structure. Denote by $\Sigma N$ the spinor bundle associated with this spin structure. If $M^n$ is an oriented hypersurface isometrically immersed into $N$, denote by $\nu$ its unit, globally defined, normal vector field. Then $M$ is endowed with a spin structure, canonically induced by the one on $N$. Denote by $\Sigma M$ the corresponding spinor bundle. Recall that the spinor bundle $\Sigma N$ splits into

$$\Sigma N = \Sigma^+ N \oplus \Sigma^- N$$

where $\Sigma^\pm N$ is the $\pm 1$-eigenspace for the action of the complex volume form $\omega_{n+1} = i^{(n+1)/2} \omega$. The following proposition is essential for what follows (see for example [3, 4, 12, 16]):

**Proposition 2.1.** There exists an identification of $\Sigma N|_M$ (resp. $\Sigma^+ N|_M$) if $n$ is even (resp. odd) with $\Sigma M$, which after restriction to $M$, sends every spinor field $\psi \in \Gamma(\Sigma N)$ to the spinor field denoted by $\psi^* \in \Gamma(\Sigma M)$. Moreover, if $\cdot_N$ (resp. $\cdot_M$) stands for Clifford multiplication on $\Sigma N$ (resp. $\Sigma M$), then one has

$$(X \cdot_N \nu \cdot_N \psi)^* = X \cdot\psi^*, \quad (1)$$

for any vector field $X$ tangent to $M$.

Another important formula is the well-known spinorial Gauss formula: if $\nabla^N$ and $\nabla$ stand for the covariant derivatives on $\Gamma(\Sigma N)$ and $\Gamma(\Sigma M)$ respectively, then, for all $X \in TM$ and $\psi \in \Gamma(\Sigma N)$

$$(\nabla^N_X \psi)^* = \nabla_X \psi^* + \frac{1}{2} h(X) \cdot \psi^*, \quad (2)$$

where $h$ is the second fundamental form of the immersion $M \hookrightarrow N$ viewed as a symmetric endomorphism of the tangent bundle of $M$. 
Recall that the ambient spinor bundle $\Sigma N$ can be endowed with a Hermitian inner product $(\ ,)_N$ for which Clifford multiplication by any vector tangent to $N$ is skew-symmetric. This product induces another Hermitian inner product on $\Sigma M$, denoted by $(\ ,)$ making the identification of Proposition 2.1 an isometry. Now, relation (II) shows that Clifford multiplication by any vector tangent to $M$ is skew-symmetric with respect to $(\ ,)$.

2.2. On Hijazi’s inequality involving the energy-momentum tensor. We now discuss the role of the energy-momentum tensor associated with a special section of the spinor bundle when it is involved in lower bounds for the first eigenvalue of the Dirac operator. Recall the following estimate

**Theorem 2.2** (Hijazi, [9]). On a compact Riemannian spin manifold $(M^n, g)$, any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor $\psi$, satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} S + |T^\psi|^2 \right),$$

where $S$ is the scalar curvature of $(M^n, g)$, and $T^\psi$ the field of symmetric endomorphisms defined on the complement of the set of zeros of $\psi$ by

$$T^\psi(X, Y) = \frac{1}{2} \text{Re}(X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2}).$$

By the Cauchy-Schwarz inequality, one can easily check that this lower bound improves the one obtained by Th. Friedrich [5]

$$\frac{n}{4(n-1)} \inf_M S,$$

and in particular, it may give a non-trivial lower bound in the case where $\inf_M S$ is non-positive whereas Friedrich’s inequality requires $S$ to be positive.

If equality holds in (3), then $\psi$ has constant length,

$$\text{tr}(T^\psi)^2 = \frac{1}{4} S + |T^\psi|^2 = \text{constant}$$

and

$$\text{div}(T^\psi) = 0.$$ (5)

Interpreting the energy-momentum tensor associated with a spinor field as a second fundamental form (see also [7]) leads to the following proposition and remarks:

**Proposition 2.3.** Let $M^n \hookrightarrow (N^{n+1}, g)$ be any compact oriented hypersurface isometrically immersed in an oriented Riemannian spin manifold $(N^{n+1}, g)$, with constant mean curvature $H$ and second fundamental form $h$. Assume $(N^{n+1}, g)$ admits a parallel spinor field, then $M$ satisfies the equality case in (3).

Moreover, the energy-momentum tensor associated with the spinor field $\psi$ in Theorem 2.2 satisfies

$$2T^\psi = h.$$
Proof. Let $\Phi$ be any parallel spinor field on $N$. Then Proposition 2.1 and Gauß formula (2) yield
\[ \nabla_X \psi + \frac{1}{2} h(X) \cdot \psi = 0, \tag{6} \]
with $\psi := \Phi^*$. Let $(e_1, \ldots, e_n)$ be a positively oriented local orthonormal basis of $\Gamma(TM)$. Then for $j = 1, \ldots, n$ we have
\[ \nabla_{e_j} \psi = - \sum_{k=1}^n \frac{1}{2} h_{jk} e_k \cdot \psi. \]
Taking Clifford multiplication by $e_i$ and the scalar product with $\psi$, we get
\[ \Re(e_i \cdot \nabla_{e_j} \psi, \psi) = - \sum_{k=1}^2 \frac{1}{2} h_{jk} \Re(e_i \cdot e_k \cdot \psi, \psi). \]
Since $\Re(e_i \cdot e_k \cdot \psi, \psi) = -\delta_{ik} |\psi|^2$, it follows, by the symmetry of $h$
\[ \Re(e_i \cdot \nabla_{e_j} \varphi + e_j \cdot \nabla_{e_i} \psi, \psi) = h_{ij} |\psi|^2. \]
Therefore, $2T^\psi = h$. Moreover, $N$ has to be Ricci-flat, and the Gauß Equation yields
\[ \text{tr}(2T^\psi)^2 = n^2 H^2 = S + |2T^\psi|^2 = \text{constant}. \]
Tracing Equation (6) completes the proof. \qed

Remark 2.4. Since the Gauß equation implies
\[ \frac{n^2}{4} H^2 = \frac{1}{4} S + |T^\psi|^2 = \text{constant}, \]
we point out that we are exactly in the equality case of the extrinsic estimate given in [10] (Theorem 6), as expected.

Remark 2.5. Conversely, we can not conclude that if $(M, g)$, dim $M \geq 3$, satisfies the equality case in (3), there exists an isometric immersion of $M$ as a hypersurface with constant mean curvature into a Riemmanian spin manifold with parallel spinors. However, this is true if dim $M = 2$ (see [6]).

2.3. On $T$-Killing spinors. We now give basic properties of $T$-Killing spinors, which satisfy equality case in (3). They are generalizations of Killing spinors, and have been studied by Th. Friedrich and E.C. Kim in [7].

Let $(M^n, g)$ be a $n$-dimensional Riemannian spin manifold. A $T$-Killing spinor field $\psi \in \Gamma(\Sigma M)$ is a spinor field which satisfies
\[ \nabla_X \psi = -T^\psi(X) \cdot \psi, \quad \forall X \in TM \tag{7} \]
and
\[ \text{tr}(T^\psi) = \text{constant}, \]
where $T^\psi$ is the energy-momentum tensor associated with $\psi$, defined as in Theorem 2.2. It is easy to see that such a spinor field is an eigenspinor for the Dirac operator and since it satisfies the equality case in (3), it also satisfies Equations (4) and (5). Moreover $\psi$ has constant length.
Indeed, computing the action of the spinorial curvature tensor $\mathcal{R}$ on the spinor $\psi$, we see that necessarily, for all vector fields $X,Y \in \Gamma(TM)$,

$$\mathcal{R}(X,Y)\psi = \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi$$

$$= \left( T^\psi(Y) \cdot T^\psi(X) - T^\psi(X) \cdot T^\psi(Y) \right) \cdot \psi$$

$$+ \left( (\nabla_Y T^\psi)(X) - (\nabla_X T^\psi)(Y) \right) \cdot \psi \tag{8}$$

and

$$\text{Ric}(X) \cdot \psi = 2 \sum_{i=1}^{n} e_i \cdot \mathcal{R}(e_i, X)\psi$$

$$= 4\text{tr}(T^\psi)T^\psi(X) \cdot \psi - 4T^2_{\psi}(X) \cdot \psi - 2 \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i} T^\psi)(X) \cdot \psi \tag{9}$$

Recall the following definition

**Definition 2.6.** A symmetric 2-tensor $T \in S^2(M)$ is called a **Codazzi tensor** if it satisfies the Codazzi-Mainardi equation, i.e.

$$(\nabla_X T)(Y) = (\nabla_Y T)(X) \quad \forall X,Y \in \Gamma(TM),$$

($T$ being viewed in this formula via the metric $g$ as a symmetric endomorphism of the tangent bundle).

To every nowhere vanishing spinor field $\psi$, we can associate the real vector field $V_\psi$ defined by

$$g(V_\psi, X) = i(\psi, T^\psi(X) \cdot \psi), \quad \forall X \in \Gamma(TM).$$

Note that $T$-Killing spinors have the following property:

**Proposition 2.7.** Let $\psi$ be a $T$-Killing spinor field. If $T^\psi$ is a Codazzi tensor, then the associated vector field $V_\psi$ is a Killing vector field.

**Proof.** For all $X,Y \in TM$, we compute

$$g(\nabla_X V_\psi, Y) = X g(V_\psi, Y) - g(V_\psi, \nabla_X Y)$$

$$= i(\nabla_X \psi, T^\psi(Y) \cdot \psi) + i(\psi, \nabla_X (T^\psi(Y)) \cdot \psi)$$

$$+ i(\psi, T^\psi(Y) \cdot \nabla_X \psi) - i(\psi, T^\psi(\nabla_X Y) \cdot \psi)$$

and since Clifford multiplication by vector fields is skew-symmetric with respect to $(\ldots, \cdot)$, we have

$$g(\nabla_X V_\psi, Y) = i(T^\psi(Y) \cdot T^\psi(X) \cdot \psi, \psi) - i(T^\psi(X) \cdot T^\psi(Y) \cdot \psi, \psi) + i(\psi, \nabla_X (T^\psi(Y)) \cdot \psi)$$

which is clearly skew-symmetric if (10) holds for $T^\psi$. \hfill \Box

**Remark 2.8.** A spinor field satisfying Equation (7) and whose associated energy-momentum tensor $T^\psi$ is a Codazzi tensor is called a **Codazzi Energy-Momentum spinor**. This notion generalizes the notion of Killing spinors (see [14] for a study of these particular spinor fields). It is known that given a Riemannian manifold $(M,g)$ of dimension 3, the
existence of a non trivial Codazzi Energy-Momentum spinor field is equivalent to the
existence of an isometric immersion of the universal covering of $M$ into the Euclidean
4-dimensional space (see [13]).

3. Generalized warped product over a Riemannian manifold

Proposition 2.3 and Remarks 2.4 and 2.8 show that it is natural to consider the energy-
momentum tensor associated with a spinor field satisfying (7) as the second fundamental
form of an appropriate immersion. In this section, we give a natural construction of a
momentum tensor associated with a spinor field satisfying (7) as the second fundamental

Recall that if $\pi : M \to N$ is a smooth map between two manifolds, then two vector
fields $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ are said to be $\pi$-related ($X \overset{\pi}{=} Y$) if $d\pi(X_p) = Y_{\pi(p)}$ at all point $p$ in $M$. Two vector fields $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ are $\pi$-related if and only
if $X(f \circ \pi) = Y(f) \circ \pi$ for all smooth function $f$ defined on $N$. Moreover, it is a classical
fact that $X_1 \overset{\pi}{=} X_2$ and $Y_1 \overset{\pi}{=} Y_2$ imply $[X_1, Y_1] \overset{\pi}{=} [X_2, Y_2]$.

Now let $(M^n, g)$ be a compact Riemannian spin manifold. We denote by $\nabla$ the Levi-
Civita connection on $M$. Let $h \in S^2(M)$ be a symmetric 2-tensor, and $f : I \to \mathbb{R}$ a
smooth function, with $f(1) = 0$ and $I = [1 - \varepsilon, 1 + \varepsilon] \subset \mathbb{R}$, such that the metric

$$ g_t = g + f(t)h $$

is well-defined on $M$ for all $t \in I$. We endow $M \times I$ with the metric

$$ g = \pi_M^*(g_t) \oplus \pi_I^*(d^2t), $$

where $\pi_M$ and $\pi_I$ are respectively the first and second canonical projections of $M \times I$ on
$M$ and $I$. We denote by $\mathcal{M} = (M \times I, g)$ the generalized Riemannian warped product
obtained by this construction.

Note that for $f(t) = t^2 - 1$ and $h = g$, this construction corresponds to the usual cone
over $M$, the metric $g$ being then defined by $g = t^2g + d^2t$ (see [8]). In the following, we
will call $\mathcal{M}$ a generalized cone if by definition $f(t) = t^2 - 1$

The lift of a vector field $X \in \Gamma(TM)$ to $\mathcal{M}$ will be called a horizontal vector field.
By definition, it is the only vector field, $\widetilde{X} \in \Gamma(T\mathcal{M})$, such that $d\pi_M(\widetilde{X}) = X$ and
d$\pi_I(\widetilde{X}) = 0$. The set of horizontal vector fields will be denoted by $\mathcal{H}(\mathcal{M})$. In the
following, $\partial_t$ will stand for the unit vector field spanning $\Gamma(TI)$, as well as its lift to
$\Gamma(T\mathcal{M})$. Then we have the following classical proposition (see [15] for example):

**Proposition 3.1.** If $\widetilde{X}, \widetilde{Y} \in \mathcal{H}(\mathcal{M})$ are horizontal vector fields, then

$$ [\widetilde{X}, \widetilde{Y}] = [\overline{X}, \overline{Y}], \quad \text{and} \quad [\widetilde{X}, \partial_t] = 0. $$

If $q : I \to \mathbb{R}$ is a smooth function, then the gradient of $\overline{q} = q \circ \pi_I : \mathcal{M} \to \mathbb{R}$ is the lift of
the gradient of $q$.

For all $X \in \Gamma(TM)$ and $t \in I$ we denote by $H_t(X)$ the vector field defined by

$$ g_t(H_t(X), Y) = h(X, Y), \quad \forall Y \in \Gamma(TM). $$
\textbf{Proposition 3.2.} If $\nabla$ stands for the Levi-Civita connection of the generalized Riemannian warped product $\mathcal{M}$, and $\nabla^t$ for the Levi-Civita connection of $(M,g_t)$, then for all $X,Y \in \Gamma(TM)$, we have the following generalized O’Neill formulas:

\begin{align}
\nabla_{\partial_t} \partial_t &= 0 , \\
\nabla_{\tilde{X}} \partial_t &= \nabla_{\partial_t} \tilde{X} = \frac{f'(t)}{2} H_t(X) , \\
\nabla_{\tilde{X}} \tilde{Y} &= \nabla^{t}_{X}Y - \frac{f'(t)}{2} h(X,Y) \partial_t .
\end{align}

\textit{Proof.} Equality (13) is trivial. Hence, note that since $g(\tilde{X}, \partial_t) = 0$, we have

$$g(\nabla_{\partial_t} \tilde{X}, \partial_t) = 0 .$$

On the other hand, $g(\partial_t, \partial_t) = 1$ implies $g(\nabla_{X} \partial_t, \partial_t) = 0$. By the Koszul formula, we get

$$2g(\nabla_{\partial_t} \tilde{X}, \tilde{Y}) = \frac{f'(t)}{2} H_t(X) ,$$

which proves (14). Again, by the Koszul formula

$$2g(\nabla_{\tilde{X}} \tilde{Y}, \partial_t) = f'(t) g_t(H_t(X), Y) = f'(t) \tilde{g}(H_t(X), \tilde{Y}) ,$$

and

$$2g(\nabla_{\tilde{X}} \tilde{Y}, \tilde{Z}) = \tilde{X} g_t(Y, Z) + \tilde{Y} g_t(Z, X) - \tilde{Z} g_t(X, Y) = 2g_t(\nabla^t_{X}Y, Z) = 2g(\nabla^t_{X}Y, \tilde{Z}) .$$

Therefore the proof of (15) is completed. \hfill \Box

\textbf{Proposition 3.3.} For all $X,Y,Z \in \Gamma(TM)$, we have

\begin{align}
\nabla_{\tilde{X}} \nabla_{\partial_t} \partial_t &= \nabla_{\partial_t} \nabla_{\partial_t} \partial_t = 0 , \\
\nabla_{\partial_t} \nabla_{\tilde{X}} \partial_t &= \nabla_{\partial_t} \nabla_{\partial_t} \tilde{X} = \frac{f''(t)}{4} H_t(X) - \frac{(f'(t))^2}{4} H_t(H_t(X)) , \\
\nabla_{\tilde{X}} \nabla_{\partial_t} \tilde{Y} &= \nabla_{\tilde{X}} \nabla_{\tilde{Y}} \partial_t = \frac{f'(t)}{2} H_t(Y) - \frac{(f'(t))^2}{4} h(X, H_t(Y)) \partial_t , \\
\nabla_{\partial_t} \nabla_{\tilde{X}} \tilde{Y} &= \frac{f'(t)}{2} H_t(Y) - \frac{f'(t)}{2} \tilde{H}_t(\nabla^t_{X}Y) - \frac{f''(t)}{2} h(X,Y) \partial_t \partial_t , \\
\nabla_{\tilde{X}} \nabla_{\partial_t} \tilde{Z} &= \frac{f'(t)}{2} H_t(Y) - \frac{f'(t)}{2} h(X, \nabla^t_{Y}Z) \partial_t - \frac{(f'(t))^2}{4} h(Y, Z) H_t(X) .
\end{align}
where $A_t(X,Y)$ is the vector field defined on $M$ by

$$g_t(A_t(X,Y), Z) = X h(Y, Z) + Y h(Z, X) - Z h(X, Y)$$
$$- h(X, [Y, Z]) + h(Y, [Z, X]) + h(Z, [X, Y]).$$

**Proof.** Equality (16) is trivial. Since $g(∇_X ∂_t, ∂_t) = 0$, we have $g(∇_{∂_t} ∇_X ∂_t, ∂_t) = 0$. Moreover,

$$g(∇_{∂_t} ∇_X ∂_t, Y) = ∂_t g(∇_X ∂_t, Y) - g(∇_X ∂_t, ∇_{∂_t} Y)$$
$$= ∂_t \left( \frac{f'(t)}{2} g(H_t(X), Y) \right) - \left( \frac{f'(t)}{4} \right)^2 g(H_t(X), H_t(Y))$$
$$= \partial_t \left( \frac{f'(t)}{2} g_t(H_t(X), Y) \right) - \left( \frac{(f'(t))^2}{4} g_t(H_t(X), H_t(Y)) \right)$$
$$= \frac{f''(t)}{2} g_t(H_t(X), Y) - \left( \frac{(f'(t))^2}{4} g_t(H_t(H_t(X))), Y \right)$$
$$= g(\frac{f''(t)}{2} \tilde{H}_t(X) - \left( \frac{(f'(t))^2}{4} \tilde{H}_t(H_t(X)), Y \right),$$

which proves (17). On the other hand,

$$g(∇_X ∇_{∂_t} Y, Z) = \tilde{X} g(∇_{∂_t} Y, Z) - g(∇_{∂_t} Y, ∇_X Z)$$
$$= \frac{f'(t)}{2} \left( \tilde{X} g_t(H_t(Y), Z) - g_t(H_t(Y), ∇_X Z) \right)$$
$$= \frac{f'(t)}{2} g(∇_X \tilde{H}_t(Y), Z),$$

and

$$g(∇_X ∇_{∂_t} Y, ∂_t) = \tilde{X} g(∇_{∂_t} Y, ∂_t) - g(∇_{∂_t} Y, ∇_X ∂_t)$$
$$= - \left( \frac{(f'(t))^2}{4} g_t(H_t(Y), H_t(X)) \right)$$
$$= - \left( \frac{(f'(t))^2}{4} h(H_t(X), Y) \right).$$

Therefore, we proved (18). For (19), we show that

$$g(∇_{∂_t} ∇_X Y, ∂_t) = \partial_t g(∇_X Y, ∂_t) - g(∇_X Y, ∇_{∂_t} ∂_t)$$
$$= - \partial_t \left( \frac{f'(t)}{2} h(X, Y) \right)$$
$$= - \frac{f''(t)}{2} h(X, Y)$$

and

$$g(∇_{∂_t} ∇_X Y, Z) = \partial_t g(∇_X Y, Z) - g(∇_X Y, ∇_{∂_t} Z)$$
$$= \partial_t g(∇_X Y, Z) - \frac{f'(t)}{2} g(∇_X Y, H_t(Z))$$
$$= \frac{f'(t)}{2} g(A_t(X, Y), Z) - \frac{f'(t)}{2} g(H_t(∇_X Y), Z).$$
On the other hand, we have
\[
g(\nabla_X \nabla_Y \tilde{Z}, \partial_t) = \tilde{X}g(\nabla_Y \tilde{Z}, \partial_t) - g(\nabla_Y \tilde{Z}, \nabla_X \partial_t) \\
= - \frac{f'(t)}{2} \left( Xh(Y, Z) + g_t(\nabla_Y Z, H_t(X)) \right) \\
= - \frac{f'(t)}{2} \left( Xh(Y, Z) + h(\nabla_Y Z, X) \right)
\]
and
\[
g(\nabla_X \nabla_Y \tilde{Z}, \tilde{V}) = \tilde{X}g(\nabla_Y \tilde{Z}, \tilde{V}) - g(\nabla_Y \tilde{Z}, \nabla_X \tilde{V}) \\
= \tilde{X}g_t(\nabla_Y \tilde{Z}, \nabla_X \tilde{V}) - g_t(\nabla_Y \tilde{Z}, \nabla_X \tilde{V}) - \frac{(f'(t))^2}{4} h(Y, Z) h(X, V) \\
= g_t(\nabla_X \nabla_Y \tilde{Z}, \tilde{V}) - \frac{(f'(t))^2}{4} h(Y, Z) g(H_t(X), \tilde{V}) \\
= g(\nabla_X \nabla_Y \tilde{Z}, \tilde{V}) - \frac{(f'(t))^2}{4} h(Y, Z) g(\nabla_X \nabla_Y \tilde{Z}, \tilde{V})
\]
Therefore the proof of \[20\] is completed. \(\square\)

**Proposition 3.4.** If \(R\) and \(R^t\) denote respectively the Riemann curvature tensor of \(M\) and of \((M, g_t)\), then we have the following relations:

\[
R(\partial_t, \partial_t) = R(\partial_t, \partial_t)\tilde{X} = 0
\]

\[
R(\tilde{X}, \partial_t) = -\frac{f''(t)}{2} H_t(X) + \frac{(f'(t))^2}{4} H_t(H_t(X))
\]

\[
R(\tilde{X}, \partial_t)\tilde{Y} = \frac{f'(t)}{2} \left( \nabla_X H_t(Y) + H_t(\nabla_X Y) - A_t(\tilde{X}, \tilde{Y}) \right) \\
+ \frac{f''(t)}{2} h(X, Y) \partial_t - \frac{(f'(t))^2}{4} h(X, H_t(Y)) \partial_t
\]

\[
R(\tilde{X}, \tilde{Y}) \partial_t = \frac{f'(t)}{2} \left[ \nabla_X (H_t(Y)) - \nabla_Y (H_t(X)) \right]
\]

\[
R(\tilde{X}, \tilde{Y}) \tilde{Z} = R^t(\nabla_X Y, Z) + \frac{f'(t)}{2} \left( (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z) \right) \partial_t \\
+ \frac{(f'(t))^2}{4} \left( h(X, Z) H_t(Y) - h(Y, Z) H_t(X) \right).
\]

**Proof.** Straightforward from the preceding proposition. \(\square\)

**Remark 3.5.** Let \(q\) be a positive smooth function on \(I\). Define \(f := q^2 - 1\) and \(h = g\). Then \(g_t = q^2(t)g\) and \(M\) is the usual warped product constructed over \(M\). We have \(f' = 2qq'\) and \(f'' = 2(qq'' + (q')^2)\), and since

\[
q^2(t)g(H_t(X), Y) = g_t(H_t(X), Y) = h(X, Y) = g(X, Y),
\]
we get

\[
H_t(X) = \frac{1}{q^2(t)} X.
\]
Therefore, it is straightforward to see that Propositions 3.2, 3.3 and 3.4 correspond precisely to O’Neill Formulas ([15] p. 206 and 210).

In the following, take \( f(t) = t^2 - 1 \). If \( X, Y, Z, W \in \Gamma(T(M \times f I)|_M) \) are vector fields tangent to \( M \) and \( \partial_t \) the unit normal vector field on \( M \), then, Proposition 3.4 yields

\[
\begin{align*}
g(R(\partial_t)\partial_t, Y) &= h(H(X), Y) - h(X, Y) \\
g(R(\partial_t, Z) &= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\
g(R(X, Y)Z, W) &= g(R(X, Y)Z, W) + h(X, Z)h(Y, W) - h(Y, Z)h(X, W) .
\end{align*}
\]

As expected, the construction of \( \mathcal{M} \) allows the interpretation of \( h \) as the second fundamental form of the hypersurface \( M \times \{1\} \subset \mathcal{M} \).

4. **T-Killing spinors with parallel Energy-Momentum tensor**

Assume now that \((M^n, g)\) admits a non trivial \( T \)-Killing spinor field \( \psi \) with parallel energy-momentum tensor \( T^\psi \). Consider the generalized cone \( \mathcal{M} \) over \( M \), with \( h = 2T^\psi \).

Back to the construction of the cone, given \( X, Y \in TM \), it is straightforward to see that,

\[
\nabla^t_X Y = \nabla_X Y + (t^2 - 1)B^t(X, Y)
\]

where the vector valued 2-symmetric tensor \( B^t \) is defined by

\[
g_t(B^t(X, Y), Z) = (\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) - (\nabla_Z h)(X, Y) , \quad \forall Z \in TM .
\]

Therefore, Propositions 3.2, 3.4 and formula (9) yield

**Proposition 4.1.** Assume that \((M^n, g)\) admits a non trivial \( T \)-Killing spinor field \( \psi \) with parallel energy-momentum tensor \( T^\psi \). Then the Levi-Civita connection and the Riemann curvature tensor of the generalized cone \( \mathcal{M} \) over \( M \), with \( h = 2T^\psi \) satisfy

\[
\begin{align*}
\nabla_{\partial_t} \partial_t &= 0 \\
\nabla_{\partial_t} \partial_t &= \nabla_{\partial_t} \tilde{X} = t \overset{\sim}{H_t(X)} \\
\nabla_{\tilde{X}} \tilde{Y} &= \nabla_{\tilde{X}} \tilde{Y} - th(X, Y)\partial_t ,
\end{align*}
\]

and

\[
\begin{align*}
R(\partial_t, \partial_t)\partial_t &= R(\partial_t, \partial_t)\tilde{X} = R(\tilde{X}, \tilde{Y})\partial_t = 0 \\
R(\tilde{X}, \partial_t)\partial_t &= -\overset{\sim}{H_t(X)} + t^2H_t(H_t(X)) \\
R(\tilde{X}, \partial_t)\tilde{Y} &= h(X, Y)\partial_t - t^2h(X, H_t(Y))\partial_t \\
R(\tilde{X}, \tilde{Y})\tilde{Z} &= R(\tilde{X}, \tilde{Y})Z + t^2\left(h(X, Z)H_t(Y) - h(Y, Z)H_t(X)\right) .
\end{align*}
\]

Moreover, the Ricci tensor of \((M^n, g)\), viewed as a field of symmetric endomorphism, satisfies for all \( X \in \Gamma(TM) \)

\[
\text{Ric}(X) = \text{tr}(H)H(X) - H^2(X)
\]

where \( H \) stands for \( H_1 \) with the above notations.
With the help of this proposition, we can give an explicit relation between the Ricci tensor of the generalized cone $\mathcal{M}$ and the Ricci tensor of $M$. For this, we have to define a canonical local oriented orthonormal basis of $T\mathcal{M}$. At a fixed $t \in I$, define the symmetric positive endomorphism $G_t$ of $T\mathcal{M}$ by

$$g_t(X, Y) = g(G_t(X), Y), \quad \forall X, Y \in TM.$$ 

Now, given a local oriented orthonormal basis $(e_1, \ldots, e_n)$ of $T\mathcal{M}$, we get the local oriented orthonormal basis $(E_1, \ldots, E_n, \partial_t)$ of $T\mathcal{M}$ by defining, at each point $(x, t) \in \mathcal{M}$, the vector $E_i(x, t)$ as the lift of the vector $e_i^t(x) := G_t^{-\frac{1}{2}}(e_i(x)) \in T_x M$ (where for all positive symmetric endomorphism $B$, $B^{\frac{1}{2}}$ stands for its positive square root).

Since $g_t = g + (t^2 - 1)h$, it is easy to see that $G_t = \text{Id} + (t^2 - 1)H$ for all $t \in I$. Hence, at a fixed point $t \in I$, $G_t$ is parallel since we assumed that $H$ is parallel.

We then have the following

**Corollary 4.2.** Denote the Ricci tensor of $\mathcal{M}$ by $\tilde{\text{Ric}}$, then for all horizontal vector fields $\tilde{X}, \tilde{Y} \in \mathcal{H}(\mathcal{M})$, we have at a fixed point $(x, t) \in \mathcal{M}$

$$\tilde{\text{Ric}}(\tilde{X}, \partial_t) = 0$$

$$\tilde{\text{Ric}}(\partial_t, \partial_t) = t^2 \sum_{i=1}^{n} h(e_i^t, H_t(e_i^t)) - \sum_{i=1}^{n} h(e_i^t, e_i^t)$$

$$\tilde{\text{Ric}}(\tilde{X}, \tilde{Y}) = \text{tr}(H)h(X, Y) - h(H(X), Y) - t^2 \sum_{i=1}^{n} h(e_i^t, e_i^t)h(X, Y) + t^2 h(H_t(X), Y).$$

**Proof.** Straightforward from Proposition 4.1. Remark that since $G_t$ is parallel, we have

$$g_t(R(e_i^t, X)e_i^t, Y) = g(R(e_i, X)e_i, Y).$$

$\square$

**Remark 4.3.** As in Remark 3.5 assuming $h = g$, then $g_t = t^2 g$ and $\mathcal{M}$ is the usual cone over $M$. We get

$$H_t(X) = \frac{1}{t^2}X \quad \text{and} \quad e_i^t = \frac{1}{t} e_i.$$

Since $2T^\psi = h$, the spinor field $\psi \in \Gamma(\Sigma\mathcal{M})$ is actually a real Killing spinor field with Killing number $\frac{1}{2}$ and we recover C. Bär’s result which says that $\mathcal{M}$ has to be Ricci flat (2).

**5. The Case of a Projector**

If $M$ carries a $k$-dimensional parallel smooth distribution $\mathcal{K}$, we can define on $M$ a parallel field of symmetric endomorphisms $H$ satisfying $H^2 = H$ as the projector on $\mathcal{K}^\perp$, the orthogonal of $\mathcal{K}$. We can also consider the generalized cone $\mathcal{M}$ over $M$ constructed
with $H$. Then, if $(e_1, \ldots, e_n)$ is an oriented orthonormal basis of $TM$ such that the vectors $e_1, \ldots, e_k$ span $\mathcal{K}$, we get

$$H_t(X) = \begin{cases} 0 & \text{if } X \in \mathcal{K} \\ \frac{1}{t}X & \text{if } X \in \mathcal{K}^\perp \end{cases}$$

and

$$e_i^t = \begin{cases} e_i & \text{if } 1 \leq i \leq k \\ \frac{1}{t}e_i & \text{if } k + 1 \leq i \leq n. \end{cases}$$

Hence, Corollary 4.2 implies the following

**Theorem 5.1.** Let $(M^n, g)$ be a compact Riemannian spin manifold admitting a smooth parallel distribution $\mathcal{K}$ and let $\psi$ be a non-trivial $T$-Killing spinor whose energy-momentum tensor $T^\psi$ corresponds to the orthogonal projection on $\mathcal{K}^\perp$. Then the generalized cone $\mathcal{M}$ has to be Ricci flat.

The argument used by C. Bär in [2] seems to be still usefull in this situation. Therefore, we can formulate the following question in the case where $(M^n, g)$ is a compact Riemannian spin manifold admitting a smooth parallel distribution.

*Is it possible to find lower bounds for the first eigenvalue of the Dirac operator on $(M^n, g)$, such that the limiting cases are characterized by the existence of a non-trivial $T$-Killing spinor whose energy-momentum tensor is a projector?*

Such an estimate on a compact Riemannian spin manifold admitting a parallel 1-form is given in [1]. One can note that the problem formulated above is a particular case of the problem of studying the spectrum of the Dirac operator on a manifold admitting a parallel $k$-form.

**References**

[1] B. Alexandrov, G. Grantcharov, and S. Ivanov, *An estimate for the first eigenvalue of the Dirac operator on compact Riemannian spin manifold admitting parallel one-form*, J. Geom. Phys. **28** (1998), 263–270.

[2] C. Bär, *Real Killing Spinors and Holonomy*, Com. Math. Phys. **154** (1993), 525–576.

[3] ______, *Metrics with Harmonic Spinors*, Geom. Func. Anal. **6** (1996), 899–942.

[4] J.P. Bourguignon, O. Hijazi, J.-L. Milhorat, and A. Moroianu, *A Spinorial approach to Riemannian and Conformal Geometry*, Monograph, En préparation, 2003.

[5] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalar-Krümmung*, Math. Nach. **97** (1980), 117–146.

[6] ______, *On the spinor representation of surfaces in Euclidean 3-space*, J. Geom. Phys. **28** (1998), no. 1-2, 143–157.

[7] Th. Friedrich and E.-C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. **37** (2001), no. 1-2, 1–14.

[8] S. Gallot, *Equations différentielles caractéristiques de la sphère*, Ann. Sci. Ec. Norm. Sup. **12** (1979), 235–267.

[9] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys. **16** (1995), 27–38.

[10] O. Hijazi, X. Zhang, and S. Montiel, *Dirac operator on embedded hypersurfaces*, Math. Res. Let. **8** (2001), no. 1-2, 195–208.
[11] N. Hitchin, *Harmonic Spinors*, Advances in Math. **14** (1974), 1–55.

[12] B. Morel, *Eigenvalue Estimates for the Dirac-Schrödinger Operators*, J. Geom. Phys. **38** (2001), 1–18.

[13] ______, *Surfaces in $S^3$ and $H^3$ via Spinors*, Preprint, math.DG/0204090 (2002).

[14] ______, *Codazzi Energy-Momentum Spinors*, In preparation (2003).

[15] B. O’Neill, *Semi-Riemannian Geometry*, Acad. Press, New York, 1983.

[16] A. Trautman, *Spinors and the Dirac operator on hypersurfaces I. General Theory*, Journ. Math. Phys. **33** (1992), 4011–4019.

[17] M.Y. Wang, *Parallel spinors and parallel forms*, Ann. Glob. Anal. Geom. **7** (1989), 59–68.

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