Mass Formula of a Five-dimensional Almost-BPS Supergravity Soliton with a Magnetic “Bolt”

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Abstract
We derive the Smarr formula for a five-dimensional spacetime which has a magnetic “bolt” in its center and is asymptotically $\mathbb{R}^{1,3} \times S^1$. Supersymmetry – and so the BPS-bound – is broken by the holonomy. We show how each topological feature of a space-like hypersurface enters the mass formula and which ones in particular give rise to the violation of the BPS-bound.
1 Introduction

Finding a proper quantum description of black holes is of major interest to solve profound problems surrounding the classical picture, like singularities and the “information paradox”.

One possible way is the in 2002 by Samir Mathur proposed “fuzzball” program [1] within the framework of string theory whose supergravity limit is smooth, horizonless and asymptotically flat solutions representing time-independent solitons.

In recent works it has been shown that horizonless solitonic solutions of supergravity can indeed be constructed purely by means of nontrivial topology; the Smarr formula has been derived in each case by means of the Komar integral formalism over cohomology [2, 3, 4, 5]. One important result is the role of Chern-Simons terms to only support the topological nature of the integral.

In this work we recall the five-dimensional results of [2] but impose a different choice of boundary conditions; in particular, we assume a spacetime which has a magnetic “bolt” in its center and is asymptotically $\mathbb{R}^{1,3} \times S^1$ – the fourth spatial dimension is periodic, and three-dimensional space is rotating along it with nonzero angular momentum even at infinity. To construct it, we define in the fashion of [6] a four-dimensional Ricci-flat base space which carries Euclidean Schwarzschild metric and magnetic flux from a “floating brane” ansatz [7] for the Maxwell fields.
The supersymmetry conditions require that the curvature tensor be either self-dual or anti-self-dual and tell how this duality has to be correlated with the one of the Maxwell-fields. Since the rotation group in four-dimensional space decomposes like $SO(4) = SU(2)_{\text{self-dual}} \times SU(2)_{\text{anti-self-dual}}$, only one half of the Killing-spinors would “feel” space’s holonomy and the other half flat space. In simple examples, this half-flatness, and the preservation or breaking of supersymmetry can be easily arranged by just changing a sign in the duality of the fields [8, 9, 10].

However, the curvature tensor for the Schwarzschild bolt is neither self-dual nor anti-self-dual, so our system is manifestly non-BPS; but, because the Schwarzschild geometry is still Ricci-flat, the (almost-)BPS-equations are still satisfied [6, 7], and hence provides a ground for more general solutions. It is also for the above stated circumstances that one speaks in the present context of “almost-BPS”-solutions.

The idea here is to derive a Smarr formula based on the Komar-integral formalism in the sense of [2] for the present boundary conditions and for non-BPS solutions and see in how far the arising mass components – especially the BPS-bound breaking terms – relate explicitly to the structure of the given spacetime, particularly space’s boundary.

Finally, expressions for the five-dimensional masses will be compared to those from [6].

2 Preliminaries

2.1 The five-dimensional Supergravity action and equations of motion

The bosonic action in five dimensions [11] is:

$$ S = \int \left( \ast_5 R - Q_{IJ} dX^I \wedge \ast_5 dX^J - Q_{IJ} F^I \wedge \ast_5 F^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right), $$

where $C_{IJK} = |\epsilon_{IJK}|$, $X^I \ (I = 1, 2, 3)$ are scalar fields and the $A^I$ are Maxwell fields.

We may consider this to arise from the on $T^6$ reduced eleven-dimensional theory, where the scalars come from the metric coefficients of the compactified dimensions,

$$ ds^2_{11} = ds^2_5 + \left( \frac{Z_2 Z_4}{Z_1^3} \right)^{\frac{1}{3}} \left( dx_5^2 + dx_6^2 + \left( \frac{Z_1 Z_3}{Z_2^3} \right)^{\frac{1}{3}} (dx_7^2 + dx_8^2) + \left( \frac{Z_1 Z_2}{Z_3^3} \right)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2) \right), $$

with the reparametrization,

$$ X^1 = \left( \frac{Z_2 Z_3}{Z_1^3} \right)^{\frac{1}{3}}, \quad X^2 = \left( \frac{Z_1 Z_3}{Z_2^3} \right)^{\frac{1}{3}}, \quad X^3 = \left( \frac{Z_1 Z_2}{Z_3^3} \right)^{\frac{1}{3}}, $$

to fulfill the constraint $X^1 X^2 X^3 = 1$. 
Moreover, there is a metric for the kinetic terms,
\[ Q_{IJ} = \frac{1}{2} \text{diag} \left( \left( \frac{1}{X^1} \right)^2, \left( \frac{1}{X^2} \right)^2, \left( \frac{1}{X^3} \right)^2 \right), \tag{4} \]
and the duals of the field strengths, \( F^I = dA^I \), are then given by
\[ G_I = Q_{IJ} \left( \star_5 F^J \right). \tag{5} \]

The Einstein equations \[2\] are
\[ R_{\mu\nu} = Q_{IJ} \left( F^I_{\mu\rho} F^J_{\nu\rho} - \frac{1}{6} g_{\mu\nu} F^I_{\rho\sigma} F^{J\rho\sigma} + \partial_{\mu} X^I \partial_{\nu} X^J \right), \tag{6} \]
and the Maxwell equations,
\[ \nabla_\rho \left( Q_{IJ} F^J_{\rho\mu} \right) = J^C_{I\mu} \tag{7} \]
with the five-dimensional Chern-Simons 1-form current \[1\],
\[ J^C_{I\mu} = \frac{1}{16} C_{IJK} \bar{\epsilon}_{\mu\rho\sigma\kappa\lambda} F^{J\rho\sigma} F^{K\kappa\lambda}. \tag{8} \]

Moreover, this can be expressed in terms of the dual field strengths,
\[ dG_I = \frac{1}{4} C_{IJK} F^J \wedge F^K. \tag{9} \]

Eq. (6) can be rewritten such that the RHS is free of any trace terms,
\[ R_{\mu\nu} = Q_{IJ} \left( \frac{2}{3} F^I_{\mu\rho} F^J_{\nu\rho} + \partial_{\mu} X^I \partial_{\nu} X^J \right) + \frac{1}{6} Q^{IJ} G_{I\mu\rho\sigma} G^{J\rho\sigma}_{\nu\sigma}, \tag{10} \]
especially since this form is much more helpful for the derivation of the Komar mass formula.

### 2.2 Invariances and the Komar integral

We assume the metric to have a time-like Killing vector, \( K^\mu \), and can hence write the five-dimensional mass formula in terms of a Komar integral in five dimensions,
\[ M = \frac{3}{32 \pi G_5} \int_{X^3} \star_5 dK, \tag{11} \]
where \( X^3 \) is the 3-boundary of the five-dimensional spacetime. Smoothness of spatial sections, \( \Sigma_4 \), allows in virtue of properties of the Killing vector to rewrite this formula as an integral over such

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\(^1\)The Levi-Civita tensor for curved spacetime is related to the Levi-Civita symbol of Minkowski spacetime like \( \bar{\epsilon}^{\mu_1 \cdots \mu_5} = (-g)^{-\frac{1}{2}} \epsilon^{\mu_1 \cdots \mu_5} \Leftrightarrow \bar{\epsilon}_{\mu_1 \cdots \mu_5} = (-g)^{\frac{1}{2}} \epsilon_{\mu_1 \cdots \mu_5} \) with the convention \( \epsilon_{01234} = 1 \).
by $X^3$ bound space-like hypersurfaces:

$$M = \frac{3}{32\pi G_5} \int_{X^3} \star_5 dK = \frac{3}{16\pi G_5} \int_{\Sigma_4} K^{\mu} R_{\mu\nu} d\Sigma^\nu. \tag{12}$$

Assuming, furthermore, that the matter fields have the symmetries of the metric, means them to be invariant under the Lie-derivative along the Killing vector, $K$,

$$\mathcal{L}_K F = 0 = \mathcal{L}_K G, \tag{13}$$

where $\mathcal{L}_K$ is the corresponding Lie derivative, we get, in the same fashion as in [2], the equations,

$$0 = d (i_K F) \iff i_K F^I = d\lambda^I \text{ and } i_K G_I = d\Lambda_I - \frac{1}{2} C_{IJK} \lambda^J F^K + H_I^{(2)}, \tag{14}$$

where $\lambda^I$ are magnetostatic potentials of the $G_I$ and electrostatic potentials of the $F^I$, respectively; $\Lambda_I$ are globally defined 1-forms and $H_I^{(2)} \in H^2 (\mathcal{M}_5)$ closed but not exact 2-forms.

With (14) the Einstein equations (10) become

$$K^{\mu} R_{\mu\nu} = \frac{1}{3} \nabla_\rho (2Q_{IJ} \lambda^I F^J_\rho + Q^I_\rho J_\sigma G_{J\sigma}^{\rho}) + \frac{1}{6} Q^{IJ} H_I^{(2)\rho\sigma} G_{J\nu\rho\sigma}. \tag{15}$$

From this follows the Komar mass integral [12, 13, 14, 15, 2] over the spatial hypersurface, $\Sigma_4$, including the boundary terms over $X^3$:

$$M = -\frac{1}{16\pi G_5} \left[ \int_{\Sigma_4} H_I^{(2)} \wedge F^I - \int_{X^3} (2\lambda^I G_I - \Lambda_I \wedge F^I) \right]. \tag{16}$$

Since in [2] the spacetime was assumed to be asymptotic to $\mathbb{R}^{1,4}$, the boundary integral was taken over $X^3 = S^3$. Here the spacetime will be asymptotic to $\mathbb{R}^{1,3} \times S^1$ and so we have $X^3 = S^2 \times S^1$.

### 2.3 Spacetime with a “running bolt” and conserved charges

The five-dimensional metric, called the “running bolt” [6] is a time fibration over Euclidian Schwarzschild:

$$ds_5^2 = -Z^{-2} (dt + k) + Z ds_4^2$$

$$= -Z^{-2} (dt + k) + Z \left[ (1 - \frac{2m}{r}) d\tau^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \tag{17}$$

2The here used convention $\text{div } X = -\delta X \Rightarrow \delta dZ_I = -\nabla^2 Z_I$, where $\delta$ is the to $d$ adjoint exterior derivative, means for here: $\nabla^2 K_\mu = R_{\mu\nu} K^\nu$, so the opposite sign as in [2].
The Maxwell fields are set up by the “floating brane” ansatz [7],

\[ A^I = -\varepsilon Z_I^{-1} (dt + k) + B^I, \]  

(18)

where \( \varepsilon \) is set by the (anti-)self-duality of the fields. The magnetic field strengths are

\[ \Theta^{(I)} = dB^{(I)}. \]  

(19)

The three forms, \( Z_I, \Theta^{(I)} \) and \( k \), are determined through the equations [16, 17, 8, 6]:

\[ \Theta^{(I)} = \varepsilon \ast_4 \Theta^{(I)}, \]

(20)

\[ \hat{\nabla}^2 Z_I = \frac{1}{2} \varepsilon C_{IJK} \ast_4 [\Theta^{(J)} \wedge \Theta^{(K)}], \]

(21)

\[ dk + \varepsilon \ast_4 dk = \varepsilon Z_I \Theta^{(I)}. \]

(22)

Note, that (20) − (22) are purely represented on the base manifold.

Following the choice of solution for the field strength made in [6],

\[ \Theta^{(I)} = q_I \left( \frac{1}{r^2} d\tau \wedge dr + \varepsilon d\Omega_2 \right), \]

(23)

we have also

\[ Z_I = 1 - \frac{1}{2m} \frac{1}{r^2} C_{IJK} q_J q_K \]

(24)

\[ k = \mu (r) d\tau = \varepsilon \left( \frac{1}{r} - \frac{1}{2m} \right) \left[ \Sigma_{I=1}^3 q_I - \frac{3}{4m^2} q_1 q_2 q_3 \left( \frac{1}{r} + \frac{1}{2m} \right) \right] d\tau, \]

(25)

where the \( q_I \) are M5-charges associated with the magnetic field strength component.

It is important to note that harmonic terms \( \propto d\tau \) have been chosen such that \( k \) vanishes on the bolt, which is essential to remove closed timelike curves. With this choice, the asymptotic limit of the angular momentum does not vanish but has a finite value:

\[ \mu \xrightarrow{r \rightarrow \infty} \gamma = -\frac{\varepsilon}{2m} \left( \Sigma_{I=1}^3 q_I - \frac{3}{4m^2} q_1 q_2 q_3 \right). \]

(26)

It is this finite limit which led to the name “running bolt”.

Transforming (26) leads to a formula for the magnetic charges:

\[ \Sigma_{I=1}^3 q_I = -2\varepsilon m \gamma + \frac{3}{4m^2} q_1 q_2 q_3. \]

(27)

Equations (9), (5) and (4), lead to the conserved charge-densities:

\[ 0 = d \left( 2Q_{IJ} \ast_5 dA^J - \frac{1}{2} C_{IJK} A^J \wedge dA^K \right). \]

(28)
Using (18), (19) and (23) – (25), one can define and compute the conserved electric M2-charges [6],

\[ Q^I = \int_{S_1 \times S_\infty} (2G^I - \frac{1}{2}C_{IJK} A^J \wedge dA^K) \]

\[ = -(8\pi m)(4\pi) \frac{1}{2} C_{IJK} \left[ \frac{\varepsilon}{m} q_J q_K + \frac{\gamma}{2} (q_J + q_K) \right]. \]

3 Komar mass in five-dimensional almost-BPS supergravity

3.1 Setting up the mass formula

The generic timelike Killing vector may be equipped with an extra \( \tau \)-component associated with the angular momentum, so near infinity we can write:

\[ K = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \tau}, \]

where \( \alpha \) is a constant.

In the following we derive expressions for the fields and fluxes from the RHS of (16) to understand their contributions to the mass formula.

From (18), (19) and (23) follows the Maxwell-field strength, \( F^I = dA^I \), which decomposes into an exact and a harmonic part,

\[ F^I = dA^I + \varepsilon q_I d\Omega_2, \]

where

\[ A^I = -\varepsilon Z^{-1} (dt + \mu d\tau) + q_I \left( \frac{1}{r} - \frac{1}{2m} \right) d\tau. \]

Note that we chose a gauge such that the \( A^{(I)} \) vanish at the bolt and are thus globally smooth.

Now we get from (14a):

\[ \lambda^I = (1 + \alpha \mu) \varepsilon Z^{-1} - \alpha \frac{\mu}{r} - \beta^I, \]

where \( \beta^I \) are constants.

With (18) and (5) we have

\[ G_I = \frac{1}{2} \left[ -\varepsilon r^2 \left( 1 - \frac{2m}{r} \right) Z_I + \varepsilon r^2 Z_I Z^{-3} \mu \alpha \right] \]

\[ + \frac{1}{2} Z_I Z^{-3} \left( \varepsilon r^2 \mu + q_I Z_I \right) dt \wedge d\Omega_2 + \frac{\varepsilon q_I Z_I^2 Z^{-3}}{2r^2} dt \wedge d\tau \wedge dr, \]
and find from this together with (34),

\[
i_K G_I + \frac{1}{2} C_{IJK} \lambda^J F^K = -\frac{\alpha \epsilon}{4m} C_{IJK} q_J q_K d\Omega_2 - \frac{1}{2} C_{IJK} \beta^J F^K \\
- \frac{1}{2} d \left[ Z_I Z^{-3} \left( 1 + \alpha \mu + \alpha r \mu' + \frac{\alpha \epsilon}{r} q_I Z_I \right) dt \right] \\
- \frac{1}{2} d \left\{ \mu Z_I Z^{-3} \left( 1 + \alpha \mu + \alpha r \mu' + \frac{\alpha \epsilon}{r} q_I Z_I \right) + \frac{\alpha C_{IJK} q_J q_K}{2r} \right\} d\tau,
\]

which is a manifestly closed expression.

Using (14b) we can directly read that the total derivative terms of (36) flow into \( d\Lambda_I \), together with all exact pieces of (32), such that we have:

\[
\Lambda_I = -\frac{1}{2} \left\{ Z_I Z^{-3} \left( 1 + \alpha \mu + \alpha r \mu' + \frac{\alpha \epsilon}{r} q_I Z_I \right) (dt + \mu d\tau) + \frac{\alpha C_{IJK} q_J q_K}{2r} d\tau \\
+C_{IJK} \beta^J \left[ -\epsilon Z_I^{-1} (dt + \mu d\tau) + \frac{1}{2} q_J d\tau \right] \right\} + \tilde{\Lambda}_I,
\]

where \( \tilde{\Lambda}_I \) is a constant closed 1-form which must be fixed such that \( \Lambda_I \) is smooth at the bolt:

\[
\tilde{\Lambda}_I = \frac{1}{4m} C_{IJK} q_J \left( \frac{\alpha}{4m} q_K + \beta^K \right) d\tau + \text{const.} \cdot dt.
\]

The remaining harmonic term in (36), along with the nontrivial piece in (32), sum up to the 2-form harmonic:

\[
H^{(2)}_I = -\frac{\epsilon}{2} C_{IJK} q_J \left( \frac{\alpha}{2m} q_K + \beta^K \right) d\Omega_2.
\]

The harmonic term is (up to a coefficient) completely given by the bolt’s volume, like the harmonic part of the field strengths. Hence, with (32) and (39) we see directly that the bulk integral in eq. (16) becomes

\[
\int_{\Sigma_4} H^{(2)}_I \wedge F^I = \int_{\Sigma_4} H^{(2)}_I \wedge dA^I = \int_{\Sigma_4} d \left( H^{(2)}_I \wedge A^I \right) = \int_{S^1 \times S^2_\infty} H^{(2)}_I \wedge A^I \\
= \frac{1}{2} C_{IJK} q_J \left( \frac{\alpha}{2m} q_K + \beta^K \right) \int_{S^1 \times S^2_\infty} \left( \epsilon Z_I^{-1} \gamma + \frac{1}{2m} q_I \right) d\tau \wedge d\Omega_2.
\]

Put in other words, the homology of our base space has no self-intersection, as opposed to the Gibbons-Hawking base which is elaborated in detail in [2]. Because the only non-trivial topology lies in the volume of the \( S^2 \) and so has canceled out in the bulk term, the mass in the present case is a pure boundary integral.

### 3.2 The asymptotic mass formula

In this section we will evaluate (16). As noted above, it is a pure boundary integral, so it is convenient to first derive the asymptotic expressions for the potentials and fields to see what is left of each term at infinity.
From (32) − (35) and (37) − (38) follows:

\[ \lambda^I \rightarrow (1 + \alpha \gamma) \varepsilon - \beta^I, \]  
\[ \Lambda_I \rightarrow -\frac{1}{2} [(1 + \alpha \gamma) \gamma - \frac{\alpha_m}{2m} C_{IJK} q_J q_K - \Sigma_{I=1}^3 C_{IJK} (\varepsilon \gamma + \frac{1}{2m} q_I) \beta^K] \, d\tau + ..., \]  
\[ F^I \rightarrow \varepsilon q_I d\Omega_2 + .... \]  
\[ G_I \rightarrow \frac{1}{2} \left( 2 \varepsilon m^2 - \frac{\varepsilon}{2m} \gamma_1 q_2 q_3 - \frac{\varepsilon}{2m} C_{IJK} q_J q_K + q_I \gamma \right) \, d\tau \wedge d\Omega_2 + .... \]  

where the remaining terms don’t contribute to the boundary’s volume form, \( d\text{vol} (S^1 \times S^2) \propto d\tau \wedge d\Omega_2 \).

Plugging (40) into (16) we find with (41) − (44) the purely boundary induced mass:

\[ M = \pi \frac{2}{G_5} \left[ 12 m^2 \gamma^2 (1 + \alpha \gamma) - (3 \alpha \gamma + 2) \Sigma_{I=1}^3 C_{IJK} q_J q_K - \frac{9\varepsilon}{2m} (\alpha \gamma^2 + \gamma + \alpha) q_I q_2 q_3 \right] \]

\[ = \frac{2\pi}{G_5} \left( \frac{2 + 3\alpha \gamma}{64\pi^2} \varepsilon \Sigma_{I=1}^3 Q^I - \frac{3\alpha + \gamma}{2} \varepsilon m \Sigma_{I=1}^3 q_I - 3 \alpha \gamma m^2 \right). \]  

Note that the complete \( \beta^I \)-dependence has canceled out, as it must for the mass to be gauge-invariant.

The above result could have also been achieved by directly calculating

\[ M = \frac{3}{32\pi G_5} \int_{S^1 \times S^2_\infty} \star_5 dK = \frac{3}{32\pi G_5} \int_{S^1 \times S^2_\infty} \star_5 \left[ \partial_r (g_{00} + \alpha g_{01}) \, dr \wedge dt + \partial_r (g_{01} + \alpha g_{11}) \, dr \wedge d\tau \right]. \]  

It is very helpful to use the frames of the five-dimensional metric,

\[ e^0 = Z^{-1} (dt + \mu d\tau) \quad e^1 = Z^{\frac{1}{2}} \left( 1 - \frac{2m}{r} \right)^{\frac{3}{2}} d\tau \]
\[ e^2 = Z^{\frac{1}{2}} \left( 1 - \frac{2m}{r} \right)^{-\frac{3}{4}} dr \quad e^3 \wedge e^4 = Z r^2 d\Omega_2 \]  

(47)

to easily compute the duals in the above integral.

Now we consider two natural special cases for the Killing vector – the rest frame, \( K = \frac{\partial}{\partial t} \), and the asymptotically static frame, \( K = \frac{\partial}{\partial \hat{t}} \), where the latter is based upon the at infinity co-rotating coordinate system with time-coordinate, \( \hat{t} \).

For that purpose, it is convenient to first write a more general form of the Killing vector:

\[ K = \alpha_0 \frac{\partial}{\partial t} + \alpha_1 \frac{\partial}{\partial \hat{t}}. \]  

(48)

In this form eq. (45) becomes with (27) and (30):

\[ M = \frac{2\pi}{G_5} \left( \frac{2\alpha_0 + 3\alpha_1 \gamma}{64\pi^2} \varepsilon \Sigma_{I=1}^3 Q^I - \frac{3\alpha_1 + \gamma \alpha_0}{2} \varepsilon m \Sigma_{I=1}^3 q_I - 3 \alpha_1 \gamma m^2 \right). \]  

(49)

In order to get the mass in the asymptotically static frame one has to consider the coordinates
given in [6], eqs. (3.20), and choose the Killing vector accordingly, \( K = \frac{\partial}{\partial t} = \sqrt{1 - \gamma^2} \frac{\partial}{\partial t} + \sqrt{1 - \gamma^2} \frac{\partial}{\partial \tau} \):

\[
M_{a.s.} = \frac{2\pi}{G_5 \sqrt{1 - \gamma^2}} \left[ \frac{\gamma^2 + 2}{64 \pi^2} \varepsilon \Sigma_{I=1}^3 Q_I + \frac{\gamma(\gamma^2 - 4)}{2} \varepsilon m \Sigma_{I=1}^2 q_I - 3m^2 \gamma^2 \right].
\]  

(50)

This one will become handy for the later discussion of the mass terms.

To go to the rest frame, one has to take \( K = \frac{\partial}{\partial t} \), that is, choose \( \alpha_0 = 1 \) and \( \alpha_1 = 0 \):

\[
M_0 = \frac{\pi}{16 \sqrt{5}} \varepsilon \left( \frac{1}{4 \pi^2} \Sigma_{I=1}^3 Q_I - 4m \gamma \Sigma_{I=1}^2 q_I \right).
\]  

(51)

We see already here that the BPS-bound break is due to the magnetic \( M5 \)-charges which are driving the bolt; a closer analysis follows in the next subsection.

3.3 Mass term analysis

In the following we want to illuminate the explicit origin of the extra mass term which violates the BPS-bound. For this purpose, we will substitute the total Maxwell-charge into the mass formula and consider the remaining terms in detail.

We work in the rest frame and set \( \alpha = 0 \).

With the identity,

\[
F \wedge F = (dA)^2 + dA \wedge F_{\text{harmonic}} + F_{\text{harmonic}} \wedge dA + F_{\text{harmonic}}^2 = 2F_{\text{harmonic}} \wedge dA,
\]  

(52)

which follows from the fact that \( (dA)^2 \propto (dr \wedge d\tau)^2 = 0 \) and \( F_{\text{harmonic}}^2 \propto (d\Omega_2)^2 = 0 \) as a direct consequence of the non-intersecting homology, and (33) and (39) one can rewrite (40) like

\[
-\int_{\Sigma_4} H_i^{(2)} \wedge F^I = \frac{\varepsilon}{2} C_{IJK} \beta^I q_J \int_{\Sigma_4} d\Omega_2 \wedge dA^K = \frac{1}{2} C_{IJK} \beta^I \int_{\Sigma_4} F_{\text{harmonic}}^J \wedge dA^K
\]

\[
= \frac{1}{2} C_{IJK} \beta^I \int_{\Sigma_4} F^J \wedge F^K = \beta^I \int_{\Sigma_4} dG_I = \beta^I \int_{S^1 \times S^3_{\infty}} G_I.
\]  

(53)

Now, with this, (41) and (29) eq. (16) becomes

\[
M_0 = \frac{1}{16 \pi G_5} \left\{ \beta^I \int_{S^1 \times S^3_{\infty}} G_I + \int_{S^1 \times S^3_{\infty}} \left[ 2\Sigma_{I=1}^3 (\varepsilon - \beta^I) G_I - \Lambda_I \wedge F^I \right] \right\}
\]

\[
= \frac{1}{16 \pi G_5} \int_{S^1 \times S^3_{\infty}} \left[ \Sigma_{I=1}^3 (2\varepsilon - \beta^I) G_I - \Lambda_I \wedge F^I \right]
\]  

(54)

\[
= \frac{1}{16 \pi G_5} \left\{ \varepsilon \Sigma_{I=1}^3 Q_I + \int_{S^1 \times S^3_{\infty}} \left[ \frac{\varepsilon}{2} \Sigma_{I=1}^3 C_{IJK} A^J \wedge F^K - \beta^I G_I - \Lambda_I \wedge F^I \right] \right\},
\]

where the \( \beta \)-term cancels with the \( \beta \)-term inherent to \( \Lambda_I \wedge F^I \) (see (42)).
There will be no contribution by $G_I$ left, if we choose $\beta^I = 0$; if, on the other hand, we say $\beta^I = \beta = \frac{2m^2 \gamma \Sigma q_I}{4m \gamma \Sigma q_I + \Sigma I C_{IJK} q_J q_K}$, then $\Lambda_I \wedge F^I$ vanishes from the integral. Hence, the degrees of freedom, $\beta^I$, allow one to move the contributions of these two terms around, which sum up to

$$-\frac{1}{16\pi G_5} \int_{S^1 \times S^2} (\beta^I G_I + \Lambda_I \wedge F^I) = \frac{\pi m}{G_5} \varepsilon \gamma \Sigma_{I=1}^3 q_I. \quad (55)$$

Furthermore, the substitution of the total Maxwell-charge into the mass formula brought up another boundary term:

$$\frac{1}{16\pi G_5} \varepsilon \Sigma_{I=1}^3 C_{IJK} \int_{S^1 \times S^2} A^J \wedge F^K = -\frac{2\pi m}{G_5} \varepsilon \gamma \Sigma_{I=1}^3 q_I. \quad (56)$$

The contributions to the BPS-bound violation by (55) and (56) are obviously opposed to each other.

Note: One could also choose $\beta^I$ to make both the $A \wedge F$ term and the $\Lambda \wedge F$ term vanish and so to turn the extra-mass into a pure integral over $G_I$,

$$\beta^I = \beta = \varepsilon \left( \frac{2m^2 \gamma \Sigma q_I}{4m \gamma \Sigma q_I + \Sigma I C_{IJK} q_J q_K} - 1 \right), \quad (57)$$

in which case

$$M_0 = \frac{1}{16\pi G_5} \varepsilon \Sigma_{I=1}^3 q_I - \int_{S^1 \times S^2} \beta^I G_I. \quad (58)$$

In any case, one can see that the main agents are the field strengths, $F^I$ and $G_I$: They do not fall off towards infinity but have according to (43) and (44) “surviving” legs in the bolt’s volume, $F^I \rightarrow \varepsilon q_I d\Omega_2$, and the whole boundary’s volume, $G_I \rightarrow [...]. d\tau \wedge d\Omega_2$, respectively.

### 4 Relation between the five- and the four-dimensional mass

The difference between the above derived masses is:

$$M_0 = \frac{1}{\sqrt{1-\gamma^2}} \left( M_{\text{a.s.}} - \frac{3}{2} \gamma Q_e \right), \quad (59)$$

where $Q_e$ is the Kaluza-Klein charge,

$$Q_e = -\frac{\pi}{G_5 \sqrt{1-\gamma^2}} \left[ \frac{3 \varepsilon}{2m} (1 + \gamma^2) q_1 q_2 q_3 + \gamma \Sigma_{I=1}^3 C_{IJK} q_J q_K - 4m^2 \gamma^3 \right]. \quad (60)$$

So, the asymptotically static mass, $M_{\text{a.s.}}$, is related to the in the rest frame moving mass, $M_0$, by the usual relativistic factor, $\sqrt{1-\gamma^2}$, and an additional shift by the Kaluza-Klein charge.
induced by the bolt

4.1 Relating the masses

The mass of the solution found in [6] was obtained after dimensional reduction to four dimensions, whereas here we have considered the intrinsic five-dimensional Komar mass. In the following we examine the relation between these in more detail.

The dimensional reduction in [6] happened in the asymptotically static frame along the $\hat{\tau}$-circle. The asymptotic mass was then read off from the $(0,0)$-coefficient of the emergent four-dimensional Einstein-metric. We denote this mass by $M_{a.s.}^{(4)}$ to distinguish it from the asymptotically static mass, $M_{a.s.}$, derived earlier.

This mass can also be obtained from a four-dimensional Komar integral:

$$M_{a.s.}^{(4)} = -\frac{1}{8\pi G_4} \int_{S^2} \star_4 dK_E,$$

where $E$ denotes the connection to the four-dimensional Einstein-metric, $K_E = g^{E}_{00}\hat{d}$, and the normalization is received in the same manner as for $M_{a.s.}$ with the different assumption of the mass density referring to a three-dimensional volume, $M_{a.s.}^{(4)} = \int_{\Sigma_3} T^E_{00}d\Sigma_3$.

It is useful to show the explicit relation between the five-dimensional metric and the four-dimensional Einstein-metric in order to deduce the relation between the masses. This can be obtained from eq. (3.21) of [6],

$$ds^2_5 = w^{-2} (d\hat{\tau} + \omega d\hat{t})^2 + w ds^2_E,$$

$$ds^2_E = -\hat{I}_4^{-\frac{1}{4}} d\hat{t}^2 + \hat{I}_4^{\frac{1}{4}} [dr^2 + (1 - \frac{2m}{r}) r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]$$

with the warp factors,

$$w = Z \left[ \frac{1 - \gamma^2}{Z^3(1 - \frac{2m}{r}) - \mu^2} \right]^\frac{1}{4}$$

and

$$\hat{I}_4^{\frac{1}{4}} = - (g^{E}_{00})^{-1} = \frac{1 - \frac{2m}{r}}{\sqrt{Z^3(1 - \frac{2m}{r}) - \mu^2}}.$$

$$\omega = w^2 g_{01} = \gamma - (1 - \frac{2m}{r})^{-2} \mu \hat{I}_4^{-1}.$$
Furthermore, the two sets of frames are connected like

five-dimensional: \[ e^0 = \sqrt{w} e_E^0 \quad e^2 = \sqrt{w} e_E^2 \quad e^3 \wedge e^4 = w e_E^3 \wedge e_E^4 \quad e^1 = w^{-1} (d \hat{\tau} + \omega d \hat{t}) \]
four-dimensional: \[ e^0_E = \hat{I}_4^{\frac{1}{2}} d \hat{t} \quad e^2_E = \hat{I}_4^{\frac{1}{2}} d r \quad e^3_E \wedge e^4_E = \hat{I}_4^{\frac{3}{2}} (1 - \frac{2m}{r}) r^2 d \Omega_2 \]

In order to derive the relation between the masses, we take a closer look at the Killing-vectors, \( K \) and \( K_E \); in the asymptotically static frame they are both equal to \( \frac{\partial}{\partial \hat{t}} \), but the associated one-forms result from lowering with the different metrics,

\[ K = g_{0\mu} d\hat{x}^\mu = g_{00} d\hat{t} + g_{01} d\hat{\tau} \quad \text{and} \quad K_E = g_{0\mu}^E d\hat{x}^\mu = g_{00}^E d\hat{t}. \quad (66) \]

Working with frames, one can easily show that

\[
\begin{align*}
\ast_5 dK &= \ast_5 (\partial_r g_{00} dr \wedge d\hat{t} + \partial_r g_{01} dr \wedge d\hat{\tau}) \\
&= w^{-1} (\partial_r g_{00} - \omega \partial_r g_{01}) \ast_5 (e^2 \wedge e^0) + \hat{I}_4^{\frac{1}{2}} w^2 \partial_r g_{01} \ast_5 (e^2 \wedge e^1) \\
&= -w^{-1} (\partial_r g_{00} - \omega \partial_r g_{01}) e^1 \wedge e^3 \wedge e^4 - \hat{I}_4^{\frac{1}{2}} \frac{1}{w} \partial_r g_{01} e^0 \wedge e^3 \wedge e^4 \\
&= -2\hat{I}_4^2 \left( w \partial \hat{I}_4 \right)^{-1} (\partial_r g_{00} - \omega \partial_r g_{01}) d\hat{\tau} \wedge \ast_4 dK_E + \text{"terms with } d\hat{t}\text{"}. \quad (67)
\end{align*}
\]

Integrating the last equation as in (11) yields

\[
M_{a.s.} = \frac{3}{32\pi G_5} \int_{S^1 \times S^2_\infty} \ast_5 dK = \left. -\frac{3 \text{vol}(S^1)}{16\pi G_5} \frac{\partial_r g_{00}}{\partial_r \hat{I}_4} \right|_{r \to \infty} \int_{S^2_\infty} \ast_4 dK_E \\
= \left. \frac{3 \partial_r g_{00}}{4 \partial_r g_{01}} \right|_{r \to \infty} M_{a.s.}^{(4)} = \frac{3}{4} \left( 1 - \frac{2\partial_r \omega}{\partial_r \hat{I}_4} \right|_{r \to \infty} M_{a.s.}^{(4)}, \quad (68)
\]

where we have used that the gravitational constants are linked like

\[ G_4 = \frac{1}{\text{vol}(S^1)} G_5 = \frac{1}{8\pi m \sqrt{1-\gamma}} G_5. \quad (69) \]

In any case, we see that the masses are proportional.

One can show that for a special choice of \( \alpha_0 \) and \( \alpha_1 \) in (48) the masses match, but it involves unintuitive ratios of the derivatives of the metrics and will not be further elaborated at this point.

### 4.2 Smarr formula for a four-dimensional mass through “dimensional extension”

If, on the other hand, one has with a four-dimensional spacetime and would like to express the asymptotic mass by means of an integral over topology in the sense of a Smarr formula, then one would face the problem of singularities. However, one could circumvent that issue by “dimensional extension”.
extension”, that is, assuming a curled up extra-dimension – like the $S^1$ – and use the inverse form of (68) to write a bit more generally:

$$M^{(4)}_{a.s} = \zeta_E \int_{S^2_\infty} \star_4 dK_E = \frac{\zeta_E}{\text{vol}(S^1)} \int_{S^1 \times S^2_\infty} d\hat{\tau} \wedge \star_4 dK_E = -\left(1 - \left. \frac{\partial w}{\partial r} \right|_{r \to \infty}\right)^{-1} \frac{\zeta_E}{\text{vol}(S^1)} \int_{S^1 \times S^2_\infty} \star_5 dK,$$

where $\zeta_E$ is the four-dimensional normalization and $\text{vol}(S^1)$ the extra-dimension’s volume in the asymptotically static frame. (In the situation above they are $\zeta_E = -\frac{1}{8\pi G_4}$ and $\text{vol}(S^1) = 8\pi m \sqrt{1 - \gamma^2}$). The warp-factor, $w$, results from embedding the four-dimensional into the five-dimensional metric, as in (62).

Now, with the five-dimensional Komar integral in (70) one can work in the same manner as done in subsection 3.1 to set up a five-dimensional Smarr formula for the four-dimensional asymptotic mass.

5 Conclusion

We have derived the Komar mass for an almost-BPS solution of supergravity in a five-dimensional stationary spacetime where we gave space a “bolt” at the center and made it asymptotically $S^1 \times \mathbb{R}^3$.

The very goal was to determine explicitly how each mass component follows from topology and especially which field and flux terms account particularly for the extra-mass causing the violation of the BPS-bound.

At first, the whole mass formula turned out to be a pure boundary integral. This is due to the fact that the only harmonic form in the present spacetime is the volume form of the bolt, $d\Omega_2$, which squares to zero, and so the wedge-product term in the bulk integral leaves only exact pieces. In other words, the topology of the base space does not inhabit any self-intersecting homology as opposed to the Gibbons-Hawking base.

It could be shown that, how much each field and flux term contributes to the extra-mass, is variable by a degree of freedom; one could, for example, gauge such that the extra-mass becomes a boundary integral solely over the dual field strengths, $G_I$. In general, the other terms involve the field strengths, $F^I$, whose common harmonic part in the bolt’s volume is finite at infinity and so gives nonvanishing terms in the boundary integral.

Furthermore, the deviation between the masses obtained here and in [6] could be explained by the fact that different-dimensional spacetimes underlie them. The mass computed in this work ensued from the Komar integral in the present five dimensions, but the one from [6] is footed on the four-dimensional Einstein-metric of the along the $S^1$ reduced spacetime. This specific nature of the latter mass was explicitly pointed out by writing it in terms of a four-dimensional Komar-integral. Based on this, a formula was set up to relate these masses.
Also, this formula was shown to be invertible to a formalism allowing the mass of a four-dimensional spacetime to be written in terms of a five-dimensional Smarr formula and hence an integral over topology without singularities.

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