Representations
of The Coordinate Ring of $GL_q(n)$

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Abstract

It is shown that the finite dimensional irreducible representations of the quantum matrix algebra $M_q(n)$ (the coordinate ring of $GL_q(n)$) exist only when $q$ is a root of unity ($q^n = 1$). The dimensions of these representations can only be one of the following values: $\frac{N^k}{2^k}$ where $N = \frac{n(n-1)}{2}$ and $k \in \{0, 1, 2, ...N\}$ For each $k$ the topology of the space of states is $(S^1)^{(N-k)} \times [0, 1]^{(k)}$ (i.e. an $N$ dimensional torus for $k = 0$ and an $N$ dimensional cube for $k = N$).
1. Introduction

As far as $q$ is not a root of unity, there is a well-known parallelism [1-5] between the representation theory of the universal enveloping algebras in the deformed and the undeformed which results from the parallelism between the root space decompositions in these cases. However for the dual objects (i.e. quantum matrix algebras or the coordinate ring of the algebra of functions on the group) such a parallelism does not exist due to the obvious fact that the classical limit of these objects is a free abelian algebra.

In this letter which is an extension of a previous one [6] we pose and answer the following two questions:

1) Is there a natural definition of $q$-analogue of root systems for the quantum matrix algebra $M_q(n)$ (the coordinate ring of $GL_q(n)$)?

2) What is the character of the representations of $M_q(n)$?

In answering the above questions we follow the following line of reasoning.

We present in section (2), the most essential properties of $M_q(3)$ which allowed us to prove in ref. [6] the special ($n = 3$) case of the results stated in the abstract of this letter. In section (3) we show that these properties are also true for $M_q(n)$ and define a canonical root system for $M_q(n)$. The commutation relations in this root system are very simple and allows us to characterize various representations of this algebra. The theorems presented in section (3), are stated without proof. Their proof constitute the material of section (4).

2. A resume of Representations of $GL_q(3)$

Consider a matrix $T \in GL_q(3)$

$$T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

For the commutation relations between the elements of this matrix see [6]. The elements of this matrix generate the quantum algebra $M_q(3)$. The strategy introduced in [6], for constructing the representations of $M_q(3)$, was based on the following observations.

a- The commuting elements $c$, $e$ and $g$ can be simultaneously diagonalized. They play the role of Cartan subalgebra generators. We denote the Cartan subalgebra by $\Sigma^0$
The positive (respectively negative) root systems are generated by the elements $f, h$ and $\Delta = ek - qfh$ (respectively, $b,d$ and $\Delta' = ae - qbd$). The nice properties of this choice of roots are that:

**b-1** They have multiplicative commutation relations with the elements of $\Sigma^0$. By multiplicative relation between two elements $x$ and $y$, we mean a relation of the form $xy = q^\alpha yx$, where $\alpha$ is an integer.

**Remark**: In the rest of this paper a multiplicative relation between $x$ and $y$ is indicated as $xy \approx yx$.

**b-2** All the positive (resp. negative) roots commute among themselves.

**b-3** Each negative root has multiplicative relation with all the positive roots except one, which we call the positive root corresponding to that negative root. ($f, h$ and $\Delta$ correspond respectively to $b, d$ and $\Delta'$), with commutation relations:

\[
bf - fb = (q - q^{-1})ce \\
dh - hd = (q - q^{-1})eg \\
q^{-1}\Delta'\Delta - q\Delta\Delta' = (q^{-1} - q)\det_q(T)e
\]

We denote the set of positive and negative roots by $\Sigma^+$ and $\Sigma^-$ respectively.

**c-** When $q$ is a root of unity, $q^n = 1$, the $p$-th power of all the elements in $\Sigma \equiv \Sigma^0 \cup \Sigma^+ \cup \Sigma^-$ are central.

**d-** From the representations of $\Sigma$ one can reconstruct the representations of $M_q(n)$ (i.e. one can reconstruct the representations of $a$ and $k$).

The basic strategy is then very simple. One considers a cube of states

\[
W = \{|l, m, n > = f^l h^m \Delta^n | 0 > : 0 \leq l, m, n \leq p - 1\}
\]

where $|0 >$ is a common eigenvector of $c, e$ and $g$ and then shows that $W$ is invariant under the action of $\Sigma$. (see [6] for details)

**Property b1**) shows that the states of $W$ are all eigenstates of $c, e$ and $g$.

**Property b-2**) shows that each positive root acts a raising operator in one direction of the cube independent of the other positive roots. (i.e. $f$ in the $l$ direction, $h$ in the $m$ direction, etc).

**Property b-3**) shows that each negative root acts as a lowering operator only in the same direction of the cube which has been generated by its corresponding positive root. Therefore the study of the representations effectively reduces to the analysis of the action of a
typical pair of positive and corresponding negative root say (f and b) on a string of states (i.e. |l, m, n> for fixed m and n). This analysis is very much like the one carried out for the case of $GL_q(2)$ in [7].

Property c) allows the identification of opposite sides of the cube $W$, independent of each other.

These are the basic steps which lead to the classification of $M_q(3)$ modules. In fact what simplifies the representation theory of $GL_q(3)$ is the observation that the q-analogue of non simple roots are the q-minors of the matrix $T$. Therefore what we need is to prove that the properties (a - d) hold exactly for $M_q(n)$, this is the subject of the next section where we define the natural root system of this algebra.

3. The Root system of $M_q(n)$

The quantum matrix algebra $M_q(n)$ is generated by 1 and the elements of an $n \times n$ matrix $T$, subject to the relations [8]:

$$RT_1T_2 = T_2T_1R$$

where $R$ is the solution of the Yang-Baxter equation corresponding to $SL_q(n)$ [9]. The commutation relations derived from (1) can be neatly expressed in the following way.

For any for elements $a, b, c,$ and $d$ in the respective positions specified by rows and columns (ij), (ik), (lj) and (lk), the following relations hold:

$$ab = qba \quad cd = qdc$$

$$ac = qca \quad bd = qdb$$

$$bc = cb \quad ad - da = (q - q^{-1})bc$$

The quantum determinant of $T$ is also defined as:

$$D_q(T) = \Sigma_{i=1}^{n}(-q)^{i-1}t_{1i}\Delta_{1i}$$

where $\Delta_{1i}$ is the q-minor corresponding to $t_{1i}$ and is defined by a similar formula. The q-cofactor of the element $t_{1i}$ is defined to be $(-q)^{i-1}\Delta_{1i}$ and is denoted by $C_i$. In eq. 2 $D_q(T)$ has been expanded in terms of the elements in the first row of $T$. Another useful expansion is in term of the last column of $T$:

$$D_q(T) = \Sigma_{i=1}^{n}(-q)^{n-i}\Delta_{in}t_{in}$$

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In order to present the multiplicative relations in (2), we devise a notation which will be also convenient later on, when we will derive the relations between the q-minors. Any element of the matrix \( T \) will be shown by a \( \bullet \) and any q-minor of any size by a square. The positions of the dots or squares, represent their relative positions in the matrix \( T \), the order of the elements in a multiplicative relation is shown by an arrow, the factor which is obtained when one reverses the sense of arrow is indicated on the arrow. Thus the multiplicative relations in (2) are depicted as follows:

Let us label the elements of the matrix \( T \) as follows:

\[
T = \begin{pmatrix}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Y_1 & H_1 \\
  \cdot & \cdot & \cdot & \cdot & Y_2 & H_2 & X_1 \\
  \cdot & \cdot & \cdot & Y_3 & H_3 & X_2 \\
  \cdot & \cdot & Y_4 & H_4 & X_3 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  Y_{n-1} & H_{n-1} & X_{n-2} & \cdot & \cdot & \cdot & \cdot \\
  H_n & X_{n-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot 
\end{pmatrix}
\]

Consider the elements \( H_i, X_i \) and \( Y_i \) together with the q-minors (q-determinants of the submatrices)

\[
H_{ij} = det_q \begin{pmatrix}
  \cdot & \cdot & \cdot & H_i \\
  \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot \\
  H_j & \cdot & \cdot & \cdot 
\end{pmatrix}
\]
\[ X_{ij} = \text{det}_q \begin{pmatrix} 
\ddots & \cdots & X_i \\
\vdots & \ddots & \iddots \\
X_j & \ddots & \iddots 
\end{pmatrix} \]

\[ Y_{ij} = \text{det}_q \begin{pmatrix} 
\ddots & \cdots & Y_i \\
\vdots & \ddots & \iddots \\
Y_j & \ddots & \iddots 
\end{pmatrix} \]

(i.e. \( H_{12} = \text{det}_q \begin{pmatrix} Y_1 & H_1 \\
H_2 & X_1 \end{pmatrix} = Y_1X_1 - qH_1H_2 \)) For convenience we sometimes denote \( H_i, X_i \) and \( Y_i \) by \( H_{ii}, X_{ii} \) and \( Y_{ii} \) respectively.

Our choice for the Cartan subalgebra, and the positive and negative root system is as follows. These subalgebras are generated respectively by the elements \( H_i, X_{ij} \) and \( Y_{ij} \). We denote these subalgebras by \( \Sigma^0, \Sigma^+ \) and \( \Sigma^- \) respectively. This choice is rather unique because of the following propositions.

**proposition 1**

\[ [H_{ij}, H_{kl}] = 0 \]
\[ [X_{ij}, X_{kl}] = 0 \]
\[ [Y_{ij}, Y_{kl}] = 0 \]

*note*: the elements \( H_{ij} \) do not belong to \( \Sigma^0 \). We have included their relation here for later use.

**proposition 2**

\[ H_iX_{ij} = qX_{ij}H_i \quad \forall j \]
\[ H_{j+1}X_{ij} = qX_{ij}H_{j+1} \quad \forall i \]
\[ H_kX_{ij} = X_{ij}H_k \quad k \neq i, j + 1 \]
with \((q \rightarrow q^{-1}, X_{ij} \rightarrow Y_{ij})\)

**proposition 3.**

\[
H_{ij}X_{kl} \approx X_{kl}H_{ij} \quad X \rightarrow Y
\]

\[
Y_{kl}X_{ij} \approx X_{ij}Y_{kl} \quad (k, l) \neq (i, j)
\]

\[
Y_iX_i - X_iY_i = (q - q^{-1})H_iH_{i+1}
\]

\[
q^{-1}Y_{ij}X_{ij} - qX_{ij}Y_{ij} = (q^{-1} - q)H_{i,j+1}H_{i+1,j}
\]

**proposition 4.** For \(q^p = 1\) the \(p\)-th power of all the elements of \(\Sigma\) are central.

Let \(V\) be a \(\Sigma\) module. We call this module trivial if, the action of one or more of the elements of \(\Sigma\) on it, is identically zero. We are interested in nontrivial \(\Sigma\) -modules. (the trivial one’s are representations of reductions of \(\Sigma\).

**proposition 5.** A \(\Sigma\) module \(V\) is nontrivial only if the following condition holds. all the subspaces

\[
K_{ij} \equiv \{v \in V|H_{ij}v = 0\}
\]

must be zero dimensional.

**Proof.** The proof is exactly parallel to the case of \(M_q(3)\), and is based on the multiplicativity of \(H_{ij}\)’s with all the elements of \(\Sigma\).

**proposition 6:**

i) Finite dimensional irreducible representations of \(\Sigma\) exist only when \(q\) is a root of unity.

ii) Any non-trivial \(\Sigma\) module \(V\) is also an \(M_q(n)\) module and vice versa.

**proof:** The proof of this proposition is exactly parallel to the case of \(M_q(3)\). One uses the expressions (2)(resp. 3) for the \(q\)-determinants \(Y_{ij}\)(resp. \(X_{ij}\)) (starting from \(j = i + 1\), continuing to \(j = i + 2, i + 3\ldots\)) and uses the fact that in the representation of \(\Sigma\), all the elements \(H_{ij}\) are invertible diagonal matrices. Note that invertibility of \(H_{ij}\)’s (due to proposition. 5) is crucial here, otherwise one can not define the actions of the remaining elements of \(T\) or \(V\).
At this stage one considers a hypercube of states $W$:

$$W := \{ | \mathbf{l} > = \prod_{i=1}^{N} E_i^{l_i} | 0 >, \quad E_i \in \Sigma^+, \quad 0 \leq l_i \leq p - 1 \}$$

where $| 0 >$ is a common eigenstate of $H_i, s$ and shows that $W$ is invariant under the action of $\Sigma$. Denote the negative root corresponding to $E_i$ by $F_i$ and let the values of their $p$-th power on $W$ be respectively the numbers $\eta_i^+$ and $\eta_i^-$. Depending on the values of these parameters this pair of roots either traverse a full $p$-dimensional cycle (when $\eta_i^+ \neq 0, \eta_i^- \neq 0$) or traverse a $\frac{p}{2}$ dimensional line segment with highest and lowest weight (when $\eta_i^+ = \eta_i^- = 0$) in the $i$-th direction of the cube. An intermediate case also occurs when only one of the parameters is zero in which case the representation is semicyclic in that direction. The detailed analysis is a word by word repetition of that carried out in [6] for $GL_q(3)$. This proves our statement made in the abstract of this letter.

We now proceed to the next section where the general method for deriving the the commutation relations in $\Sigma$ are presented.

5. Commutation relations in $\Sigma$

We use the graphical notation introduced in section 4. The first basic fact is presented in the following lemma.

Lemma 7.

**Proof:** Expand the determinant and note that all of the relations are of type (4-c) except the ones on the lower edge, which is of type (4-a).

We combine diagram 1 with three similar relations in the following diagram.
Lemma 8.

**Proof:** Let the minor be $n \times n$. For $n = 2$ direct calculation verifies the statement. We use induction on $n$. Consider fig.(1). Writing $\Delta_{n+1}$ as

$$\Delta_{n+1} = \Sigma d_i C_i$$

where $C_i$ is the cofactor of $d_i$ in $\Delta_{n+1}$ and passing $a$ through $d_i$, we have:

$$a\Delta_{n+1} = \Sigma (d_i a + (q - q^{-1}) bc_i) C_i$$

We now use the assumption of induction ($aC_i = qC_ia$) and the property of the determinant ($\Sigma a_i C_i = 0$) to arrive at the final result.

It’s combined with three other relations in the following diagram.

Lemma. 9

**Proof:** Write the q-minor as which symbolically means that the q-minor is the sum of the products of the elements of A and q-cofactors in B. Passing the $\bullet$ through A gives the factor 1 and passing it through B gives q.

**Corollary:** a)

$$\Delta \Delta' = q \Delta' \Delta$$

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Proof: expand the left hand minor and use lemma (9).

The • ( resp. the small minor) can be in other similar positions as in the previous two lemmas, with appropriate factors of $q$ or $q^{-1}$ ( resp. $q^l$ or $q^{-l}$ )

Lemma 10 :

**Proof:** Consider fig. (2). We use induction on $l'$. For $l' = 0$ the result is true. Assume that its true for $l'$ and write $\Delta$ as: $\Delta = \sum_{i=1}^{m} a_i C_i$ where $C_i$ is the q-cofactor of $a_i$ in $\Delta$ and use the results of the previous lemmas:i.e:

$$a_i \Delta' = \Delta' a_i \quad C_i \Delta' = q^{-l'} q^{l-1} \Delta' c_i \quad 1 \leq i < l$$

$$a_i \Delta' = q^{-l} \Delta' a_i \quad C_i \Delta' = q^{-l'} q^{l} \Delta' c_i \quad l \leq i \leq m$$

from which we obtain the result for $l' + 1$.

**Important Remark:**

In the matrix $T$, there are many more positions of q-minors which give rise to very complicated commutation relations. But in $\Sigma$ there is none (as the reader can verify) other than those between $X_{ij}$ and $Y_{ij}$, which we now compute exactly.

**Proof of the last relation in proposition 3.**

Consider fig. (3). Write $H_{i,j+1}$ as $H_{i,j+1} = a_1 C_1 + a_2 C_2 + ...$ where $C_1 = X_{ij}$ is the q-cofactor of $a_1$ in the big matrix. $H_{i,j+1}$, being the determinant of the big matrix, commutes with $a_1$. On the other hand:

$$a_1 H_{i,j+1} = a_1 (a_1 X_{ij} + \Sigma_{i \geq 2} a_i C_i)$$

We now use the fact that for $i \geq 2$, $a_i a_i = qa_i a_1$, $a_1 C_i = qC_i a_1$ (lemma 8) and pass $a_1$ through $\Sigma a_i C_i$ to find:

$$a_1 H_{i,j+1} = a_1^2 X_{ij} + q^2(H_{i,j+1} - a_1 X_{ij})a_1$$
Multiplying both sides from the left by $a^{-1}$ we obtain:

$$a_1X_{ij} - q^2X_{ij}a_1 = (1 - q^2)H_{i,j+1} \hspace{1cm} (4)$$

(Note: direct calculations which do not need invertibility of $a_1$ confirm this equation).

Now we expand $Y_{ij}$ in:

$$Y_{ij}X_{ij} = (a_1\hat{C}_1 + \sum_{k \geq 2} a_k\hat{C}_k)X_{ij}$$

where $\hat{C}_k$ is the cofactor of $a_k$ in the matrix $Y_{ij}$. Note that: $\hat{C}_1 = H_{i+1,j}$.

From lemmas (8-9) we have for $k \geq 2$:

$$(a_k\hat{C}_k)X_{ij} = q^2X_{ij}(a_k\hat{C}_k)$$

Therefore

$$Y_{ij}X_{ij} = a_1X_{ij}H_{i+1,j} + q^2X_{ij}(Y_{ij} - a_1H_{i+1,j})$$

Combining this with eq.(5) we obtain the final result: i.e.

$$q^{-1}Y_{ij}X_{ij} - qX_{ij}Y_{ij} = (q^{-1} - q)H_{i,j+1}H_{i+1,j}$$

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