On the Correspondence
between the Variational Principles
in the Eulerian and Lagrangian Descriptions

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Abstract. The relationship between the variational principles for equations of continuum mechanics
in Eulerian and Lagrangian descriptions is considered. It is shown that, for a system of differential
equations in Eulerian variables, the corresponding Lagrangian description is related to introducing
nonlocal variables. The connection between the descriptions is obtained in terms of differential cov-
erings. The relation between the variational principles of a system of equations and its symplectic
structures is discussed. It is shown that, if a system of equations in Lagrangian variables can be de-
derived from a variational principle, then there is no corresponding variational principle in the Eulerian
variables.

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1. INTRODUCTION

In continuum mechanics, it is well known that, for a given system of equations in the Eulerian variables,
there exists a natural analogue in the Lagrangian variables, which describes the same continuum motion.
The connection between the initial system and its analogue in Lagrangian variables is given by a differential
covering [1]. From the Eulerian point of view, the Lagrangian description contains nonlocal variables which
can affect on all the main geometrical structures, including symmetries, conservation laws (see, e.g., [2]), and
symplectic and Hamiltonian structures as well.

There are different kinds of variational principles in continuum mechanics (see, e.g., [3]). In our paper,
we mean a stationary-action principle by a variational principle. The action functional corresponds to some
differential form (Lagrangian) \( L \)

\[ \mathcal{L} = \int L, \]

and its stationary points are solutions of the corresponding Euler–Lagrange equation \( E(L) = 0 \), where \( E \)
stands for the Euler operator.

The paper [4] is also devoted to the study of the connection between the variational principles in the
Eulerian and Lagrangian descriptions. In this paper, the authors use nonlocal variables, and hence, they deal
with an intermediate description, instead of a purely Eulerian one. In our paper, we obtain a relation between
the variational principles in the Eulerian description (without nonlocal variables) and the Lagrangian one.
The relation is based on the concept of symplectic structure for a system of differential equations.

2. LAGRANGIAN DESCRIPTION AS DIFFERENTIAL COVERING

Consider the mass conservation law in continuum mechanics,

\[ \rho_t + (u\rho)_x + (v\rho)_y + (w\rho)_z = 0. \]  

(1)

Here \( \mathbf{v} = u\partial_x + v\partial_y + w\partial_z \) is the velocity field, \( \rho \) is the mass density. Below we assume that \( \rho > 0 \).

Choosing suitable nonlocal variables, one can introduce a potential for the mass conservation law, which
satisfies the following relation:

\[ \rho \, dx \wedge dy \wedge dz - u\rho \, dt \wedge dy \wedge dz + v\rho \, dt \wedge dx \wedge dz - w\rho \, dt \wedge dx \wedge dy = d(\xi^1 \, d\xi^2 \wedge d\xi^3) = d\xi^1 \wedge d\xi^2 \wedge d\xi^3. \]  

(2)
Relation (2) is equivalent to the following system of equations:

$$\rho = \det \left( \frac{\partial \xi}{\partial x} \right), \quad \xi_i^i + u\xi_i^x + v\xi_i^y + w\xi_i^z = 0, \quad i = 1, 2, 3.$$  \hfill (3)

Here $\partial \xi / \partial x$ stands for the corresponding Jacobi matrix. Note that the functions $\xi_i(t, x, y, z)$ are Lagrangian variables.

**Remark 1.** Transformations that preserve the volume form in the $(\xi^1, \xi^2, \xi^3)$-space form the symmetry group for the equations in the Lagrangian variables.

Since the only consistency condition for the system (3) is the mass conservation law (1), it follows that, for any system of equations in the Eulerian variables, the potential (2) determines its differential covering. If a system of equations at hand is of the form (in the Eulerian variables)

$$F^1 = 0, \quad \ldots, \quad F^m = 0,$$  \hfill (4)

then the mass conservation law allows us to derive the covering system in the following form:

$$F^1 = 0, \quad \ldots, \quad F^m = 0, \quad \rho = \det \left( \frac{\partial \xi}{\partial x} \right), \quad \xi_i^i + u\xi_i^x + v\xi_i^y + w\xi_i^z = 0, \quad i = 1, 2, 3.$$  \hfill (5)

One can choose $x, y, z$ as new dependent variables for (5) and obtain an ordinary Lagrangian representation of system (4). Therefore, for a system of equations in Eulerian variables, the corresponding Lagrangian description can be regarded as a differential covering. This fact allows us to lift some geometrical structures from the Eulerian description to the Lagrangian one.

**Remark 2.** Other conservation laws of systems of equations also allow us to introduce nonlocal variables and to obtain differential coverings.

### 3. INFINITE PROLONGATION OF A SYSTEM OF DIFFERENTIAL EQUATIONS

Consider the system of differential equations

$$F^1 = 0, \quad \ldots, \quad F^m = 0,$$  \hfill (6)

where $F^i$ are functions of independent variables $x^1, \ldots, x^n$, dependent variables $u^1, \ldots, u^m$, and their derivatives up to some finite order. Denote the multi-index of the form $\alpha, x^i$ by $\alpha$. Here and below, we assume the summation over repeated indices. Write

$$D_\alpha = D_{\alpha_1}^{\alpha_1} \circ \ldots \circ D_{\alpha_n}^{\alpha_n}, \quad u_\alpha^i = D_\alpha(u^i),$$

where $D_{x^i}$ are the operators of total derivatives. Denote the infinite prolongation of the system of equations $F = 0$ by $\mathcal{E}$,

$$\mathcal{E}: F^i = 0, \quad D_x^1(F^i) = 0, \quad D_x^n(F^i) = 0, \quad \ldots$$

Denote by $l_F$ the universal linearization operator [1] for $F$. It acts on a vector-function $\varphi = (\varphi^1, \ldots, \varphi^m)^T$ by the formula

$$(l_F(\varphi))^i = l_{F^i}(\varphi^j) = \frac{\partial F^i}{\partial u_\alpha^j} D_\alpha(\varphi^j).$$

Denote by $l_\mathcal{E}$ the restriction of the operator $l_F$ to the system $\mathcal{E}$.

Below we deal with $l$-normal [1] systems only. We say that a system of equations of the form

$$u_{b_1 x^n}^1 = \Phi^1, \quad \ldots, \quad u_{b_m x^n}^m = \Phi^m$$  \hfill (7)

is of extended Kovalevskaya form if all $b_i$ are positive integers and the right-hand side $(\Phi^1, \ldots, \Phi^m)^T$ does not depend on the variables $u_{b_i x^n}^i$ and their derivatives. The systems of equations that can be written in extended Kovalevskaya form are $l$-normal. Most systems of equations in continuum mechanics can be written in an extended Kovalevskaya form.
Example 1. In dimensionless variables, the Navier–Stokes system of equations for an incompressible fluid,

\[ \begin{align*}
  u_t + uu_x + vu_y + wu_z &= -p_x + u_{xx} + u_{yy} + u_{zz}, \\
  v_t + uv_x + vv_y + wv_z &= -p_y + v_{xx} + v_{yy} + v_{zz}, \\
  w_t + uw_x + vw_y + ww_z &= -p_z + w_{xx} + w_{yy} + w_{zz}, \\
  u_x + v_y + w_z &= 0,
\end{align*} \]

can be written in extended Kovalevskaya form for \( x^n = z \) and \( b = (2, 2, 1, 1) \). One can eliminate \( w_z \) from the fourth equation; eliminate \( u_{zz} \) from the first equation; eliminate \( v_{zz} \) from the second equation; eliminate \( p_z \) from the third equation. The Euler system of equations can also be written in extended Kovalevskaya form.

4. SYMPLECTIC STRUCTURES AND VARIATIONAL PRINCIPLES

For \( l \)-normal systems of differential equations, similarly to classical differential geometry, there are two equivalent representations of symplectic structures, namely, as equivalence classes of differential forms and as equivalence classes of operators in total derivatives (see, e.g., [1]).

1. A symplectic structure of a system of equations \( \mathcal{E} \) is a closed variational 2-form on \( \mathcal{E} \), i.e., an element of the kernel of the variational differential

\[ \delta : E^{2, n-1}_1(\mathcal{E}) \to E^{3, n-1}_1(\mathcal{E}). \]

Here \( E^{p, n-1}_1(\mathcal{E}) \) are groups of variational \( p \)-forms from the \( \mathcal{E} \)-spectral sequence.

2. The variational 2-forms of a system \( \mathcal{E} \) can be described as operators in total derivatives \( \Delta \) that satisfy the relation

\[ \Delta^* \circ l_{\mathcal{E}} = l_{\mathcal{E}}^* \circ \Delta, \]

modulo operators of the form \( \nabla \circ l_{\mathcal{E}} \), where \( \nabla = \nabla^* \). Here the operator \( \Delta^* \) is formally adjoint to the operator \( \Delta \).

Every symplectic structure of an \( l \)-normal system of differential equations determines a mapping from its symmetries to the variational 1-forms (i.e., it determines a Noether theorem). This mapping can equivalently be described in both ways: in terms of variational forms and in terms of operators. Such operators take symmetries of \( \mathcal{E} \) to \( \text{Ker} l_{\mathcal{E}}^* \), which is isomorphic to the group of variational 1-forms \( E^{1, n-1}_1(\mathcal{E}) \) in \( l \)-normal case.

Remark 3. The definition of a symplectic structure as a closed variational 2-form allows us to lift symplectic structures in coverings.

We say that a system of equations \( \mathcal{E} \) is variational if there exists a Lagrangian \( L \) such that the following relation holds

\[ F = E(L), \]

where \( E \) is the Euler operator. We say also that a system of equations \( \mathcal{E} \) admits a variational principle if, for some operator in total derivatives \( A \), there exists a Lagrangian \( L \) such that the following relation holds:

\[ A(F) = E(L). \]

In this situation, the operator \( \Delta = A^*|_{\mathcal{E}} \) determines a symplectic structure for the corresponding system \( \mathcal{E} \). Let us say that a symplectic structure of \( \mathcal{E} \) which can be obtained in this way is related to a variational principle.

5. MAIN RESULTS

Informally speaking, the symplectic structures of a system of differential equations can be regarded as “Noether theorems” of the system. Therefore, it is quite natural to expect that, at least for an \( l \)-normal system of differential equations, its symplectic structures can be used for deriving variational principles.

Theorem 1. If an \( l \)-normal system of equations \( \mathcal{E} \) has a trivial de Rham cohomology group \( H^{n+1}(\mathcal{E}) \), then every symplectic structure of \( \mathcal{E} \) is related to a variational principle (at least locally).
Proof. Every symplectic structure can be regarded as an equivalence class of differential forms. As follows from [5], a symplectic structure \( \omega \in E_1^{2, n-1}(\mathcal{E}) \) is related to a variational principle (locally) if and only if it is generated by an exact differential form (as an equivalence class of differential forms). Every closed variational 2-form \( \omega \) either belongs to \( \delta E_1^{1, n-1}(\mathcal{E}) \) or generates a nontrivial element \([\omega]\) of the cohomology group \( E_2^{2, n-1}(\mathcal{E}) \). If \( \omega \in \delta E_1^{1, n-1}(\mathcal{E}) \), then it is generated by an exact differential form. Assume now that \([\omega]\) \( \neq 0 \). According to the two-line theorem (see [1]), the group \( E_2^{2, n-2}(\mathcal{E}) \) is trivial, and hence, \([\omega]\) is a cocycle. The group \( E_3^{2, n-1}(\mathcal{E}) \) is also trivial and, therefore, \([\omega] \in \text{Im} d_2^{0, n} \), where
\[
d_2^{0, n} : E_2^{0, n}(\mathcal{E}) \to E_2^{2, n-1}(\mathcal{E}).
\]
Then \([\omega] = \omega + \delta E_1^{1, n-1}(\mathcal{E}) \) is generated by an exact differential form. Hence, the symplectic structure \( \omega \) is still generated by an exact differential form and is related to a variational principle.

A system of equations in extended Kovalevskaya form admits a canonical way to derive a (global) variational principle from a symplectic structure [6]. Therefore, if a system of equations is written in an extended Kovalevskaya form in both Eulerian and Lagrangian descriptions, then the relation between the variational principles in these descriptions can be obtained in terms of symplectic structures. Every variational principle in the Eulerian description generates a unique symplectic structure, which can be lifted to the Lagrangian description. The last step is to derive the corresponding variational principle from this lift.

Let \( F = 0 \) be a system of differential equations in Eulerian variables, \( \mathcal{E} \) be its infinite prolongation, \( \tilde{\mathcal{E}} \) be the corresponding system in Lagrangian variables. Denote the covering from \( \mathcal{E} \) to \( \tilde{\mathcal{E}} \) by \( \tau \).

**Theorem 2.** If the system of equations \( \tilde{\mathcal{E}} \) is an l-normal variational system, then the corresponding symplectic structure of \( \mathcal{E} \) is not a lift of a symplectic structure of \( \mathcal{E} \).

Proof. Consider the algebra \( \tau_\ast \text{-sym} \tilde{\mathcal{E}} \) of \( \tau \)-projectable symmetries of \( \tilde{\mathcal{E}} \). Then, for any variational 2-form \( \omega \in E_1^{2, n-1}(\mathcal{E}) \), the following diagram is commutative:
\[
\begin{array}{ccc}
\tau_\ast \text{-sym} \tilde{\mathcal{E}} & \xrightarrow{\tau^\ast(\omega)} & E_1^{1, n-1}(\tilde{\mathcal{E}}) \\
\tau_\ast & \downarrow & \downarrow \tau^\ast \\
\text{sym} \mathcal{E} & \xrightarrow{\omega} & E_1^{1, n-1}(\mathcal{E})
\end{array}
\]
Thus the relation \( \tau^\ast(\omega)(\varphi) = 0 \) holds for a symmetry \( \varphi \in \tau_\ast \text{-sym} \tilde{\mathcal{E}} \) which acts in a fiber of \( \tau \), i.e., the lift of the symplectic structure of \( \mathcal{E} \) is degenerate, and the equivalence class of operators which corresponds to \( \tau^\ast(\omega) \) can not contain the identity operator.

This theorem shows that, if an l-normal system of equations in Lagrangian variables is variational, then the corresponding variational principle has no analogues in the Eulerian variables. A similar result holds for any covering from an l-normal system of differential equations such that the fiber symmetry algebra is nontrivial. In particular, it holds for coverings from l-normal systems of differential equations which are based on the introduction of potentials for conservation laws.

6. EXAMPLES

Consider two examples of variational l-normal systems in Lagrangian variables.

**Example 2.** Equations of motion of a polytropic gas (\( p = C \rho^\gamma \)) in the Lagrangian variables are Euler–Lagrange equations for the Lagrangian
\[
L = \left( \frac{x^2 + y^2 + z^2}{2} - V - U \right) dt \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3,
\]
where \( V \) is the potential energy and \( U \) is the internal energy. The corresponding system of equations can be written in extended Kovalevskaya form,
\[
x_{tt} = -\frac{\delta(V + U)}{\delta x}, \quad y_{tt} = -\frac{\delta(V + U)}{\delta y}, \quad z_{tt} = -\frac{\delta(V + U)}{\delta z}.
\]
Thus, this system of equations is l-normal, and there is no corresponding variational principle for the equations of motion of a polytropic gas in (purely) Eulerian variables.
Example 3. The Green–Naghdi equation in Lagrangian variables can be derived from the variational principle for the Lagrangian [7]

\[
L = \frac{x_t^2}{2} \left( 1 + \varepsilon \left( H''^2 + \frac{x_{mm}x''_{m}}{x''_{m}} \right) \right) + \frac{1}{6x_m^4} \left( \varepsilon x_t^2 - 3gx_m^2(2Hx_{mm} + x_m) \right) \mathrm{d}t \wedge \mathrm{d}m,
\]

where the function \( H'(x) \) describes the bottom topography, \( g \) is the gravitational acceleration, \( \varepsilon \) is a small parameter. After the \( \pi/4 \)-rotation in the \((t, m)\)-space, the corresponding equation can be written in extended Kovalevskaya form. Thus, this equation is also \( l \)-normal, and there is no corresponding variational principle for the Green–Naghdi equations in the Eulerian variables.

7. CONCLUSION

The construction of a symplectic structure for a system of differential equations enables one to connect variational principles in Eulerian and Lagrangian variables. The connection is based on the fact that one can consider Lagrangian variables as nonlocal variables in a differential covering. Moreover, the result thus obtained allows us to show that a nondegenerate variational principle in the Lagrangian variables is not related to any variational principle in the Eulerian variables. However, every variational principle in the Eulerian variables is related to some variational principle in the Lagrangian variables. Thus, from this point of view, the Lagrangian description is preferable to the Eulerian one.

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