Rigidity-controlled crossover: from spinodal to critical failure

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Failure in disordered solids takes place when elasticity (reversibility) breaks down \cite{1}. The implied abrupt mechanical degradation can be associated with brittle rupture \cite{2}, large plastic avalanche \cite{3}, or result from other nucleation type event \cite{4}. In strain controlled experiments, failure may be accompanied by a dramatic stress drop, and the challenge is to predict and control such undesirable events.

The mechanism of failure in random elastic systems is nontrivial because of the intricate interplay between threshold-type nonlinearity, quenched disorder and long-range interactions. While the strength of disorder, the threshold-type nonlinearity, quenched disorder and long-range interactions are known to affect the failure mechanism \cite{5-8}, here we focus on the role of system’s rigidity, which has recently emerged as another relevant factor in failure-related phenomena \cite{9,11}.

Failure in disordered solids is characterized by scale-free statistics of large events. The associated intermittency has been linked to the existence of either spinodal \cite{3,12-15} or critical points \cite{7,16-18}. At large disorder and infinite system size, failure is known to be linked to percolation \cite{19-24}; however, the physical nature of failure at finite disorder remains a subject of debate \cite{3,7,15,24}.

In this Letter, we use an analytically tractable mean-field model to show that both spinodal and critical scaling behaviors can coexist near the threshold of the brittle-to-ductile transition \cite{25,31}. Ductile response is understood here in the sense of stable development of small avalanches representing micro-failure events \cite{32,33}. Brittle response necessarily involves large events representing system size instabilities \cite{34,35}.

Our starting point is the fiber bundle model (FBM) with global stress redistribution \cite{36}. This model was used to explain a variety of physical phenomena from failure of textiles \cite{37}, and acoustic emission in loaded composites \cite{38} to earthquake dynamics \cite{39}. It is usually studied in the stress control setting, where failure is brittle and scaling is spinodal \cite{2,40}. To address failure under strain control and to be able to tune the system to criticality, we drive the system differently, using an external harmonic spring \cite{41,44}.

In our analysis, brittle failure emerges as a supercritical, while ductile failure as a subcritical phenomenon. The critical behavior can be associated with the brittle-to-ductile transition and we show that due to superuniversality of the mean-field models \cite{12}, the equilibrium and out-of-equilibrium exponents are the same.

The main focus of this Letter, however, is the role of the system’s rigidity \cite{33} as the regulator of the brittle-to-ductile transition. It is known that rigid, crystal-like solids subjected to stresses fail catastrophically \cite{44}. Instead loose, marginally jammed solids fail gradually \cite{9,11}. In view of the minimal nature of our model, we could construct analytically the rigidity-disorder phase diagram delineating the domain of ductile behavior at low rigidity and high disorder from the domain of brittle behavior at low disorder and high rigidity.

One of our crucial findings is that in the brittle-to-ductile crossover region, which bridges robust spinodal criticality with tuned classical criticality, the transitional exponents are non-universal, depending sensitively on system size, disorder, and rigidity. We also show that when rigidity can be conditioned by the system size, failure becomes brittle in the thermodynamic limit, and scaling survives only as a finite size effect, cf. \cite{7,9}.

Equilibrium (static) avalanches, corresponding to jumps between different globally minimizing configurations, have been previously linked to Burgers shocks \cite{15,40}. Here we extend this analogy showing that if rigidity is interpreted as "time", and strain as "space", the brittle-to-ductile transition and the associated critical behavior can be viewed as a "finite time" Burgers turbulence \cite{47}. Given that our model is essentially a mean-field version of the Burridge-Knopoff model \cite{39}, the developed analogy reinforces a conceptual link between earthquakes (fracture) and turbulence \cite{48}.

Consider a discrete system with dimensionless energy:

\[ \mathcal{H} = \frac{1}{N} \sum_{i=1}^{N} \left[ u_i(x_i) + \frac{\lambda}{2} (X - x_i)^2 \right] + \frac{\Lambda}{2} (\epsilon - X)^2, \]  

(1)
where $u_i(x)$ is a Lennard-Jones type potential of a breakable element, $X$ is a Weiss-type mean field accounting for the interaction among breakable elements, and $\varepsilon$ is the controlling parameter representing the harmonic interaction of the field $X$ with the loading device, see Fig. 1(a). For determinacy, we assume that the potential $u_i(x)$ is piece-wise quadratic: $u_i(x) = (x^2/2)\Theta(l_i - x) + ((l_i^2/2)\Theta(x - l_i)$, where $\Theta$ is the Heaviside function; for $x \leq l_i$, the element is intact, while for $x > l_i$, it is broken. Here, $l_i$ are random numbers drawn from the probability distribution $f(l)$. In our numerical illustrations, we use Weibull’s distribution with density $f(x) = \rho x^{\rho - 1} \exp(-x^{\rho})$; broad disorder corresponds to small $\rho$. In our generalization of the FBM, we introduced two new parameters: the internal stiffness $\lambda$, and the external stiffness $\Lambda = \kappa/N$, where $\kappa \sim N$ is the effective elasticity of the elastic environment.

Figure 1. (a) Schematic representation of the system; (b) brittle response at $\Lambda = 0.4$; (c) ductile response at $\Lambda = 5$. Solid black lines: equilibrium path, thin black line: out-of-equilibrium paths; grey lines: metastable states. Parameters: $N = 100$, $\lambda = 1$, $\rho = 4$.

In Fig. 1(b,c), we illustrate the typical behavior of the local and global minima of (1) by showing the relation between the applied strain $\varepsilon$ and the conjugate stress $\sigma = \Lambda(\varepsilon - X)$, see also [54]. Our Fig. 1(b) shows the brittle behavior, which includes a system size transition from the partially broken to the fully broken state. In contrast, our Fig. 1(c) illustrates the ductile behavior, characterized by the gradual accumulation of damage. The equilibrium (global minimum) deformation paths are shown in Fig. 1(b,c) by thick black lines. We assume that failure is reversible, and show by thin black lines the out-of-equilibrium (marginally stable) paths that are different for loading and unloading.

The boundary separating brittle and ductile regimes depends on the strength of the disorder (our parameter $\rho$) and on the dimensionless parameter

$$\nu = \frac{\lambda}{\Lambda(1 + \lambda)},$$

which we interpret as a measure of the structural rigidity of the system [39, 51, 52]. When $\nu$ is small, meaning that either $\Lambda$ is large or $\lambda$ is small, individual breakable elements interact weakly and the limit $\lambda \to 0$ can be associated with the (jamming) threshold beyond which the rigidity is lost [51]. Instead, when $\nu \to \infty$ the system can be viewed as overconstrained [9, 10].

The ensemble averaged brittleness/ductility threshold can be found by solving the system of equations $-2f(x_c) = f'(x_c)x_c$ and $1 - F(x_c) = f(x_c)x_c - 1/\nu$, where $F(x) = \int_0^x f(x')dx'$ is the cumulative distribution of thresholds [30]. In particular, for Weibull-distributed thresholds, the line separating brittle from ductile behavior is given by the equation $\nu_* = \exp(1/\rho + 1)/\rho$, see Fig. 2(a).

In the limit $N \to \infty$ the avalanche distribution in the model (1) can be computed analytically [39, 50]

$$p(s) = \frac{s^{\nu-1}}{\nu !} \int_0^\infty \left(1 - g(x)\right) f(x) e^{-h(x)s}dx,$$

where $g(x) = f(x)x/(1 - F(x) + \nu^{-1})$, and $h(x) = g(x) - \ln g(x)$. In the large-event-size asymptotics the universal pre-integral multiplier $s^{\nu-1}/s! \sim s^{-3/2}$ represents the classical mean-field contribution, reflecting the built-in statistics of Brownian return times [39, 54]. In the limit $s \to \infty$ the integrated distribution can be obtained by the saddle-point approximation around the global minimum, $x_0$, of the function $h(x)$. It is a root of either $g(x_0) = 1$ or $g'(x_0) = 0$, and the emergence of such two cases is a general feature of mean-field models [54].

Consider first the out-of-equilibrium path (dynamic avalanches). Then, if $g'(x_0) = 0$ while $g(x_0) \neq 1$, we obtain $p(s) \sim s^{-2} e^{-s(h(x_0) - 1)}$. This is a sub-critical dis-
distribution describing the ductile (POP) regime [57], dominated by uncorrelated random events. If \( g(x_0) = 1 \) but \( g'(x_0) \neq 0 \), the point \( x_0 \) is spinodal, and the distribution is super-critical, characterizing the brittle (SNAP) regime [57]. Neglecting the system-size events, we can write the corresponding local distribution in the form

\[
p(s, x) \sim s^{-3/2}(x - x_0)e^{-s^{3/2}(x-x_0)^2}.
\]

The avalanche size diverges near \( x_0 \), and the integrated distribution takes the classical form \( p(s) \sim s^{-5/2} \) [50]. Finally, if \( g(x_0) = 1 \) and \( g'(x_0) = 0 \), the local distribution reads

\[
p(s, x) \sim s^{-3/2}(x - x_0)^2e^{-s^{3/2}(x-x_0)^2}.
\]

The characteristic avalanche size again diverges near \( x_0 \) and the integrated distribution takes the form \( p(s) \sim s^{-3/2} \). This is the critical (crackling) regime [57] associated with brittle-to-ductile transition; the exponent 9/4 has appeared previously in the context of composite FBM involving breakable and unbreakable springs [58]. Other values of the exponents also appeared in the more complex FBM based models describing richer physics [59].

The computed critical exponents coincide with the ones known for the mean-field RFIM [1] [60], because the energy [1] can be mapped on the soft-spin RFIM. To this end we need to minimize out the variable \( X \), which gives \( \mathcal{H} = -(1/N^2)\sum_{i,j} J x_i x_j - (1/N) \sum_{i} [H x_i - v_i(x_i)] \), where \( v_i(x) = u_i(x) = x^i + x^i + \lambda \varepsilon /2(\lambda + 0) \), see also [60]. Note that the Lenard-Jones type potential \( u_i(x) \) was transformed along the way into the double-well potential \( v_i(x) \). Other mean-field formulations leading to the same spinodal and critical exponents that are relevant for amorphous plasticity are discussed in [18] [24]; the same two main regimes have been also identified for some sandpile automata [59]. Interestingly, a numerical analysis of a non-mean-field model of a structural phase transition reveals the possibility of a similar coexistence of two scaling behaviors [4].

![Figure 3](image-url)  
Figure 3. Finite size crossover associated with brittle to ductile transition. Each curve gives the value of the scaling exponent averaged over 250 realizations. Parameters \( \lambda = 1 \) and \( \nu = 0.873 \) (critical value at \( \rho = 4 \)). The exponents and the uncertainty were computed using the maximum likelihood method [61].

In finite size systems, the crossover from the robust spinodal scaling in the brittle regime (exponent 5/2) to the non-robust critical scaling (exponent 9/4) takes place in an extended transition zone, where the system exhibits non-universal exponents, see Fig. 3. The ubiquity of such transitional phenomena may explain the large scatter in reported scaling behavior of disordered solids [62] [64].

The mean-field model [1] can be used to demonstrate directly the super-universality of the critical regime [12] [65] [68]. For instance, one can show that the exponent 9/4 is valid for both out-of-equilibrium and equilibrium paths [50]. Instead, the spinodal criticality, which exists in the out-of-equilibrium model, disappears in the equilibrium model because the SNAP event takes place before the spinodal point is reached. Integrating the avalanches should be then performed only up to some Maxwellian \( x_* < x_0 \), and since in this case the function \( h(x) \) attains its minimum at the boundary, we obtain \( p(s) \propto s^{-5/2}e^{-s^{3/2}(x-x_0)} \). While this distribution has the same exponent 5/2 as in the case of the out-of-equilibrium path, the scaling is now obscured by the exponential cut off.

![Figure 4](image-url)  
Figure 4. (a) Time (rigidity) evolution of the randomly distributed initial Burgers data \( \sigma_0(\varepsilon) \) (red) at \( \rho = 2 \), and \( \lambda = 1 \); black line corresponds to \( \nu = \exp(1/\rho + 1)/\rho \). (b) Shock merging with critical complexity appearing at \( \nu = \nu_* \). Thick black line shows the shock in the averaged system which emerges in the limit \( N \to \infty \).

We now turn to an intriguing analogy between the equilibrium version of the model [1] and Burgers turbulence [47]. If we minimize out the variables \( x_i \) in [1] and consider the thermodynamic limit \( N \to \infty \) [50], the equilibrium problem reduces to finding \( \mathcal{H}(\varepsilon, \nu) \sim \min_{X \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon}(\varepsilon - X)^2 + q^\infty(X) \right\} \), where

\[
q^\infty(z) = \left[ 1 - F(\sqrt{\lambda/(\lambda + 1)}z) \right]^2 + \lambda/(\lambda + 1) \int_0^z f(\sqrt{\lambda/(\lambda + 1)}z')^2/2dz'.
\]

We can now use the Hopf-Lax formula [69] to turn this variational problem into a Cauchy problem for a Hamilton-Jacobi equation \( \partial_t \mathcal{H} + 1/2(\partial_{\varepsilon} \mathcal{H})^2 = 0 \), where the rigidity \( \nu \) plays the role of time. This equation must be supplemented by the initial condition \( \mathcal{H}(\varepsilon, 0) = q^\infty(\varepsilon) \). Then the tension \( \sigma = \partial_{\varepsilon} \mathcal{H} \) satisfies the inviscid Burgers equation

\[
\partial_t \sigma + \sigma \partial_\varepsilon \sigma = 0 \quad (4)
\]
with initial condition $\sigma_0 = \partial_\varepsilon q^\infty(\varepsilon)$. Interestingly, the viscous Burgers equation for $\sigma$ and the corresponding KPZ equation \cite{KPZ} for $H$ emerge as a finite size effect in the model 1 with finite temperature.

As a result of the reduction of the problem 1 to 4, avalanches become shock waves \cite{shocks}. In the averaged model, the ductile-to-brittle transition can be then associated with the shock formation at a finite rigidity $\nu_* = \min_{\varepsilon \in \mathbb{R}} \left\{ -1/\partial_\varepsilon \sigma_0(\varepsilon) \right\}$, see Fig. 4(b); in the $(\varepsilon, \nu)$ plane this "event" becomes a direct analog of the liquid-vapor critical point.

At finite $N$, the "evolution" equation for the stress remains the same as in the case $N \to \infty$, while the initial condition changes to $\sigma_0 = \partial_\varepsilon q(\varepsilon) = N^{-1} \sum_{i=1}^{N} \varepsilon q(\lambda_i - \sqrt{\lambda/(\lambda + 1)} \varepsilon)$, see \cite{initial_condition} for details. In Fig. 4(a) we show how the increase of rigidity transforms the ductile response, where avalanches take the form of small Burgers shocks (POP events), into the brittle response with a single Burgers shock representing a system size SNAP event. In Fig. 4(b) we track the position of individual shocks and visualize their merging sequence.

To highlight the critical nature of the system with rigidity value close to $\nu_*$, we studied the $\nu$ dependence of the number of shocks $n$. In Fig. 5 we show the standard deviation $\Delta n = \left[ K^{-1} \sum_{i=1}^{K} (n_i - K^{-1} \sum_{i=1}^{K} n_i)^2 \right]^{1/2}$, where different realizations of disorder are indexed by $i = 1, 2, ..., K$. Note the peak indicating the anomalous broadening of the distribution around the critical point $\nu = \nu_*$. The situation is fundamentally different in the conventional decaying Burgers turbulence where the initial data have zero average, which infinitely delays the emergence of scaling.

So far we were assuming that the rigidity measure $\nu$ is finite as $N \to \infty$. A broader class of elastic environments can be modeled if we assume that $\kappa \sim N^\alpha$, with $0 \leq \alpha \leq 1$. For instance, if the load is transmitted through a surface of a 3D body we have $\alpha = 2/3$ and $\nu \sim N^{1/3}$. In this setting, small systems would be necessarily ductile, while brittle behavior would dominate in the thermodynamic limit. At a given disorder, the scaling will be then seen in a window of system sizes, while the (percolation type) critical regime will emerge only at infinite size and infinite disorder \cite{percolation}.

To conclude, we used an analytically transparent model to quantify the role of system’s rigidity (global connectivity) as a control parameter for the transition from brittle to ductile failure. We showed that this transition can be associated with the crossover from spinodal to classical criticality, generating, in finite size systems, a scaling region with non-universal exponents. Such behavior is generic for a broad class of systems, encompassing fracture, plasticity, structural phase transitions, and now we established a new link to fluid turbulence.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Time (rigidity) evolution of the normalized standard deviation for the number of shocks. The statistics was obtained from $K = 1000$ realizations of the quenched disorder with $p = 3$, and $\lambda = 1$. Inset plots: (a) $\nu = 0.1$ (b) $\nu = 1$; (c) $\nu = 6$.}
\end{figure}

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[68] tem. Here we assume that the distribution of thresholds for the sub-parts has a compact support.

[69] See Supplemental Material at [URL will be inserted by publisher] for the study of the structure of the metastable states, the computation of the statistics of avalanche distribution, mapping on the RFIM and the reduction to Burgers model in the case of finite N.

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Each value of $k$ defines an equilibrium branch extending between the two limits induced by the inequalities $\bar{x}_0(k,\varepsilon) < \bar{x}_k$ and $\bar{x}_1(k,\varepsilon) > \bar{x}_k$. For the failure thresholds we can then write

$$
\varepsilon^f_k = \frac{\lambda + 1}{\lambda} \left[ 1 - \frac{k}{N} \right] \nu + 1 \bar{x}_k,
$$

(10)

where $0 \leq k < N$. Similar expressions can be obtained for the rebuilding thresholds

$$
\varepsilon^r_k = \left[ 1 - \frac{k}{N} \right] \nu + 1 \bar{x}_k,
$$

(11)

where $0 < k \leq N$. The ensuing equilibrium branches are represented by the gray lines in Fig. 1 (b, c) in the main text.

To analyze their (local) stability, we need to study the positive definiteness of the Hessian matrix for the energy $H(x,\varepsilon)$

$$
\mathcal{M} = \begin{pmatrix}
M_1 & 0 & \ldots & 0 & -\lambda \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & 0 & M_N & -\lambda \\
-\lambda & \ldots & -\lambda & -\lambda & N(\lambda + \Lambda)
\end{pmatrix},
$$

(12)

where $M_i$ is either $\lambda + 1$, for $1 \leq i < N - k$, or $\lambda$, for $N - k \leq i \leq N$. The sufficient condition for stability is that all the principal minors of $\mathcal{M}$ are positive. The first $N$ minors are just the product of diagonal terms are therefore always positive. The last principal minor, the determinant

$$
\det(\mathcal{M}) = \prod_{i=1}^{N} M_i \prod_{i=1}^{N} \left( \lambda + \Lambda - \frac{\lambda^2}{M_i} \right).
$$

(13)

is also positive implying stability of the obtained equilibrium configurations; the unstable configurations must contain at least one element in the spinodal state represented in our model by a single point.

### Equilibrium (global minimum) path

For large $N$ we can write

$$
S_k = \frac{1}{N} \sum_{i=0}^{k} \frac{\bar{x}_i^2}{2} \approx \int_{\bar{x}_1}^{\bar{x}_k} \frac{x^2}{2} f(x) dx,
$$

(14)

where we used the fact that for ordered distributions we can use the approximation $k/N \sim F(\bar{x}_k)$ [2]. We can then write the continuous approximation of the discrete energy in the form

$$
\mathcal{H}(x,\varepsilon) = \frac{\lambda \Lambda(1 - F(x))}{\lambda(1 - F(x)) + \Lambda(\lambda + 1)} \frac{\varepsilon^2}{2} + \int_{0}^{x} f(x') \frac{x'^2}{2} dx',
$$

(15)
Using the equilibrium condition $\partial H(x, \varepsilon)/\partial x = 0$, and applying it for the discrete values $\bar{x}_k$, we obtain

$$\varepsilon_k^0 = \frac{1}{\sqrt{\lambda N}} \left( 1 - \frac{k}{N} \right) \nu + 1 \bar{x}_k.$$  \hspace{1cm} (16)

Note that the three formulas (10), (11) and (16) are different only by constant multipliers.

Out-of-equilibrium (zero viscosity limit) path. Each microscopic configuration characterized by parameter $k$ exists in an extended domain of the loading parameter $\varepsilon$ between the failure strain $\varepsilon_k^0$ and the rebinding strain $\varepsilon_k^r$. At large $N$, we can use the approximation

$$\varepsilon_f(x) = \frac{1}{\varepsilon_k^0} \left[ (1 - F(x)) \nu + 1 \right] x.$$ \hspace{1cm} (17)

and $\sigma_f(x) = [1 - F(x)]x$. Similarly, along the reverse path,

$$\varepsilon_r(x) = \left[ (1 - F(x)) \nu + 1 \right] x.$$ \hspace{1cm} (18)

Figure 6. First row: averaged stress-strain relations, second row: averaged strain dependence on the internal variable $x$; blue (red) curves correspond to the loading (unloading) out-of-equilibrium paths; black curves correspond to the equilibrium (global minimum) path.

Brittle to ductile transition. It is easier to see if the system is brittle if we consider the out-of-equilibrium (marginal stability) path, even though the actual ductility threshold would be the same if we consider the global minimum path. All we need to check is the condition that the curve $\varepsilon(x)$ has a local maximum, which reads $[1 - F(x)] - f(x_c)x_c + \nu^{-1} = 0$. To locate the brittle to ductile transition we need to find the inflection point on the curve $\varepsilon(x)$ characterized by the condition $-2f(x_c) - f'(x_c)x_c = 0$.

In the case of Weibull distribution, we obtain from the first of these two conditions

$$x_c = \left[ 1 - W\left( -\frac{\exp(1/\rho)}{\rho \nu} \right) \right]^{1/\rho},$$ \hspace{1cm} (19)

where $W(x)$ is the Lambert function, defined through the equation $x = W(x)e^{W(x)}$. Then the second condition gives $\nu = e^{\varepsilon_f^0 - 1}/\rho$, which delineates the boundary between brittle and ductile regimes.

Statistics of avalanches. We first compute the avalanche distribution for the case of the out-of-equilibrium loading path; it will be clear that the same procedure can be adapted for the out-of-equilibrium unloading path and for the reversible equilibrium paths.

For an avalanche of size $s$ to take place along the out-of-equilibrium loading path and be associated with the failure of $k$th (in strength) spring, we must have $\varepsilon_{k+j} \leq \varepsilon_k$, for $j = 1, 2, \ldots, s - 1$, and $\varepsilon_{k+s} > \varepsilon_k$, which, following [3], we called the forward condition; to secure that $\varepsilon_k$ is larger than previous thresholds, we must also require that $\varepsilon_j \leq \varepsilon_k$, for all $j < k$, which we call the backwards condition. Given that the re-binding sequence for the unloading out-of-equilibrium path is $\varepsilon_k^r = \varepsilon_k^0$ and for the equilibrium path is $\varepsilon_k^0 = \sqrt{\frac{1}{\lambda + 1}} \varepsilon_k$, the avalanche condition in those two cases and the ensuing avalanche statistics will be the same as in the case of out-of-equilibrium loading path, so it is sufficient to deal with this case only.

Since we are interested in the asymptotics for the avalanche distribution at large $N$, we assume that $s \ll N$. Using the ordered thresholds $\bar{x}_i$, and Eq. (10) for the sequence $\varepsilon_k$, we can then rewrite $\varepsilon_{k+j} \geq \varepsilon_k$ in the form

$$\bar{x}_{k+j} \geq \bar{x}_k \left( 1 + \frac{j}{N - k + j + N\nu^{-1}} \right).$$ \hspace{1cm} (20)

Defining $\delta_k = \bar{x}_k/\left( 1 + \frac{j}{N - k + j + N\nu^{-1}} \right)$, and using the assumption that $j \ll N - k$, we can simplify these relation further

$$\bar{x}_{k+j} \geq \bar{x}_k + j\delta_k.$$ \hspace{1cm} (21)

Note next that breaking of one spring at the elongation $\varepsilon_k$, corresponding to a threshold $\bar{x}_k = x$, raises the load on the remaining fiber by $\delta_k$. The average number of fibers that breaks as a result of this load increase is equal to the number of thresholds in the interval $(x, x + \delta_k)$, which is $Nf(x)\delta_k$. Thus, the average number of fibers breaking as a result of the failure of the $k$th fiber is

$$g(x) = \frac{f(x)x}{1 - F(x) + \nu^{-1}},$$ \hspace{1cm} (22)

where we again used the approximation $k/N \sim F(x)$ [12]. For an avalanche of size $s$, the increase in load will be approximately $s\delta_k$, which leads to $g(x)s$ broken springs. The (forward) probability that the additional $s - 1$ springs
break is then given by a Poisson distribution with the rate \( g(x)s \),
\[
\tilde{p}_f(s, x) = \frac{(g(x)s)^{s-1}}{(s-1)!} e^{-g(x)s}.
\] (23)

To complete this expression, we still need to secure the condition stating that all the \( s - 1 \) inequalities \( \bar{x}_{k+1} < x + \delta_k, \bar{x}_{k+2} < x + 2\delta_k, \ldots, \bar{x}_{k+s-1} < x + (s - 1)\delta_k \) are satisfied. To this end, we divide the interval \((x, x + s\delta_k)\) into \( s \) sub-intervals of size \( \delta_k \). For our condition to be satisfied, we must have at least one threshold value in the first interval \((x, x + \delta_k)\), at least two in the first two intervals, and at least \( s - 1 \) in the first \( s - 1 \) intervals. To ensure that \( \varepsilon_{k+s} > \varepsilon_k \), there should be no threshold values in the last interval \((x + (s - 1)\delta_k, x + s\delta_k)\). It can be shown that such combinatorial problem can be solved giving \( p[s - 1, s] \sim \frac{1}{s} \). [3]. We can then write the probability that the forward condition is satisfied in the form
\[
p_f(s, x) = \tilde{p}_f(s, x)p[s - 1, s] = \frac{(g(x)s)^{s-1}}{s!} e^{-g(x)s}.
\] (24)

We still need to satisfy the backward condition that the threshold \( \varepsilon_k \) is necessarily bigger than its predecessors. To find the corresponding (backward) probability, we consider a finite number \( n \) of such elements, \( k - 1, k - 2, \ldots, k - n \) and search for the condition that \( \bar{x}_{k-1} < x - \delta_k, \bar{x}_{k-2} < x - 2\delta_k, \ldots, \bar{x}_{k-n} < x - n\delta_k \). If there are no thresholds in \((x - \delta_k, x)\), at most one in \((x - 2\delta_k, x)\), at most two in \((x - 3\delta_k, x)\), and at most \( n - 1 \) in \((x - n\delta_k, x)\), then all our inequalities are fulfilled. This implies that the number \( m \), not exceeding \( n - 1 \), must be in the interval \((x - n\delta_k, x - \delta_k)\), while all the remaining \( k - 1 - m \) thresholds must be smaller than \( x - n\delta_k \). The corresponding probability is given again by a Poisson distribution,
\[
\tilde{p}_b(s, x) = \frac{(g(x)n)^m}{m!} e^{-g(x)n}.
\] (25)

We can now compute the probability that \( m \) thresholds are randomly distributed among these \( n \) intervals such that no threshold value lies in the interval \((x - \delta_k, x)\), at most one in the interval \((x - 2\delta_k, x - \delta_k)\), at most two in the interval \((x - 3\delta_k, x - 2\delta_k)\), and so on. This is again a combinatorial problem whose solution is \( p(m, n) \sim 1 - \frac{m}{n} \) [3]. The probability for the backwards condition to be fulfilled is
\[
p_b(s, x) = \tilde{p}_b(s, x)p[m, n] = e^{-g(x)n} \sum_{m=0}^{n-1} \frac{(g(x)n)^m}{m!n} (n - m).
\] (26)

Rearranging the summation in Eq. (26), we can re-write it as,
\[
p_b(s, x) = (1 - g) e^{-g(x)n} \sum_{m=0}^{n-1} \frac{(g(x)n)^m}{m!} + e^{-g(x)n} \left( \frac{(g(x)n)^n}{n!} \right).
\] (27)

In the limit \( n \to \infty \), we have \( \sum_{m=0}^{n-1} \frac{(g(x)n)^m}{m!} \to e^{g(x)n} \); and with the use of the Stirling approximation, \( n! \approx n^n e^{-n} \sqrt{2\pi n} \), we can show that the last term in Eq. (27) vanishes for \( g \leq 1 \). The (backwards) probability is then
\[
p_b(s, x) = 1 - g(x).
\] (28)

The probability of the avalanche of size \( s \) starting at the element \( k \) with the threshold value \( x_k = x \) can be now written as the product of the forward (24) and the backward (28) probabilities,
\[
p(s, x) = \frac{s^{s-1}}{s!} g(x)^{s-1} e^{-sg(x)} (1 - g(x)).
\] (29)

The final expression for the integrated avalanche distribution takes the form
\[
p(s) = \int_0^s \phi(x) e^{-g(x) + \ln g(x)} dx,
\] (30)

where \( \phi(x) = \left[ 1 - g(x) \right] \frac{f(x)}{g(x)} \) and \( x_c \) is the maximum of the averaged curve \( \varepsilon(x) \).

Asymptotic analysis. We now focus on the tail of the distribution \( p(s) \) assuming that \( N \to \infty \). We use the saddle-point approximation, which implies that the main contribution to the integral will come from the vicinity of \( x = x_0 \), where the function \( h(x) = g(x) - \ln g(x) \) reaches its global minimum. To find \( x_0 \), we need to solve the equation \( h'(x) = \frac{g'(x)}{g(x)} g(x) - 1 = 0 \). There are three possibilities,

1. \( g(x_0) \neq 1 \) and \( g'(x_0) = 0 \) (ductile regime),
2. \( g(x_0) = 1 \) and \( g'(x_0) = 0 \) (critical regime)
3. \( g(x_0) = 1 \) and \( g'(x_0) \neq 0 \) (brittle regime).

If \( g(x_0) \neq 1 \) and \( g'(x_0) = 0 \), we can write, \( h(x) \approx g(x_0) - \ln g(x_0) + \frac{g'(x_0)}{2g(x_0)} (g(x_0) - 1)(x - x_0)^2 \). Then using the saddle-point approximation in (30), and applying the Stirling approximations \( n! \approx s^n e^{-s} \sqrt{2\pi n} \), we obtain
\[
p(s) \sim \frac{s^{s-1}}{s!} e^{-sh(x_0)} \phi(x_0) \sqrt{\frac{2\pi}{s[h''(x_0)]}}
\] (31)

When simultaneously \( g(x_0) = 1 \) and \( g'(x_0) = 0 \) we have \( h''(x_0) = 0 \), and \( h'''(x_0) = 0 \); therefore the Taylor
Explicitly, and using Stirling approximations, we obtain
\[ p(s) = \frac{s^{N-1}e^{-s}}{s!} \int_0^{s_0} -f(x_0)g''(x_0)(x-x_0)^2 dx. \]

(32)

Computing the integral explicitly and using Stirling’s approximation, we obtain \( p(s) \sim s^{-N/2}. \)

In the brittle regime, we need to consider separately equilibrium and out-of-equilibrium paths.

Consider first the out-of-equilibrium path. We need to expand the function \( h(x) = g(x) - \ln g(x) \) up to second order to obtain \( h(x) \approx 1 + \frac{g''(x_0)}{2}(x-x_0)^2. \) We can also expand \( \phi(x) \) to obtain \( \phi(x) \approx -g'(x_0)f(x_0)(x-x_0). \) These expansions allow us to approximate the integral \([30]\) by
\[ p(s) = \frac{s^{N-1}e^{-s}}{s!} \int_0^{s_0} g'(x_0)f(x_0)(x-x_0)e^{-s\frac{g''(x_0)}{2}(x-x_0)^2} dx. \]

(33)

Along the out-of-equilibrium path, the avalanches are counted up to \( x = x_0 \); if we compute the integral explicitly, and use the Stirling approximations, we obtain \( p(s) \sim s^{-N/2}. \)

Consider now the equilibrium path. The actual equilibrium SNAP event takes place at some \( x_0 < x_0 \), given by the Maxwell construction. The counting of avalanches should be then performed only up to the point \( x_0 \), and in the integral \([30]\), we must put \( x_0 = x_0 \). The function \( h(x) \) will attain its minimum in the boundary point \( x_0 \), which is the upper limit of integration. In such case, the following asymptotic representation holds at \( N \to \infty \) \([7]\)
\[ \int_{x_{\text{inf}}}^{x_{\text{sup}}} e^{-N h(x)} dx \to \frac{e^{N h(x_0)}}{N h'(x_0)} \]

(34)

This allows to write, \( p(s) \sim s^{-N/2}e^{-s(1-h(x_0))}. \)

Mapping on RFIM. Using the condition \( \partial_x \mathcal{H} = 0 \), we obtain \( X(x, \varepsilon) = \frac{1}{\lambda + 1} \left( \lambda \varepsilon + \lambda \sum_{i=1}^N x_i \right). \) If we substitute this expression back into \( \mathcal{H} \) we obtain
\[ \mathcal{H} = -\frac{1}{N^2} \sum_{i,j} J x_i x_j - \frac{1}{N} \sum_i [H x_i - v_i(x_i)], \]

(35)

where \( J = \frac{\lambda^2}{2(\lambda + \Lambda)} \), \( H = \frac{\lambda \Lambda}{\Lambda + \Lambda} \), and
\[ v_i(x_i) = u_i(x_i) + x_i^2 + \frac{\lambda \Lambda \varepsilon}{2(\lambda + \Lambda)}. \]

Initial condition for the Burgers equation. In the case of finite \( N \), the equilibrium condition \( \partial_x \mathcal{H} = 0 \) allows us to write
\[ \mathcal{H}(X, \varepsilon) = \frac{1}{N} \sum_{i=1}^N e_i(X) + \frac{\Lambda}{2} (\varepsilon - X)^2. \]

Here, two metastable branches \( e_i = \frac{X^2}{\lambda + 1} + \frac{\lambda i}{2} \Theta(X - l_j) \) defined in each interval \( X \in [l_i, \frac{\lambda + 1}{2} l_i] \). If we choose the branch with the minimal energy, the remaining problem reduces to finding
\[ \tilde{\mathcal{H}}(\varepsilon, \nu) = \min_{X \in \mathbb{R}} \left\{ \frac{1}{2\nu} (\varepsilon - X)^2 + q(X) \right\}, \]

where \( q(X) = \frac{1}{N} \sum_{i=1}^N \frac{X^2}{2} \Theta(l_i - \sqrt{\frac{\lambda}{\lambda + 1}} X) + \frac{\lambda + 1}{2} \Theta(X - \sqrt{\frac{\lambda}{\lambda + 1}} l_i). \) The initial data for the associated Burgers equation are
\[ \sigma_0 = \partial_x q = \frac{1}{N} \sum_{i=1}^N \varepsilon X \Theta(l_i - \frac{\lambda}{\lambda + 1} \varepsilon). \]

In the limit \( N \to \infty \) we have \( \frac{1}{N} \sum_{i=1}^N \Theta(l_i - \sqrt{\frac{\lambda}{\lambda + 1}} X) \sim \int_{-\sqrt{\frac{\lambda}{\lambda + 1}} X}^{\infty} f(l) dl \) and \( \frac{1}{N} \sum_{i=1}^N \frac{\lambda i}{2} \Theta(X - \sqrt{\frac{\lambda}{\lambda + 1}} l_i) \sim \int_0^{\sqrt{\frac{\lambda}{\lambda + 1}} X} f(l) \frac{\lambda i}{2} dl. \) Then, in this limit,
\[ \tilde{\mathcal{H}}(\varepsilon, \nu) = \min_{X \in \mathbb{R}} \left\{ \frac{1}{2\nu} (\varepsilon - X)^2 + q^\infty(X) \right\}, \]

where \( q^\infty(X) = \sqrt{\frac{\lambda}{\lambda + 1}} \int_{-\sqrt{\frac{\lambda}{\lambda + 1}} X}^{\infty} f(X) (X^2/2) dX + [1 - F(\sqrt{\frac{\lambda}{\lambda + 1}} X)](X^2/2). \) The initial condition for the associated Burgers equation is \( \sigma_0(\varepsilon) = \varepsilon [1 - F(\sqrt{\frac{\lambda}{\lambda + 1}} \varepsilon)]. \)

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