ON SOME PROBLEMS OF ELEMENTARY NUMBER THEORY

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Abstract. In this note, the following three results are proved.

1. Let \( \lfloor x \rfloor \) be the largest integer not exceeding \( x \). Then there exist a sequence \( S = \{s_1, s_2, \ldots\} \) of positive integers with \( s_n < s_{n+1} \leq 2s_n \) and a positive integer \( m \) such that
   \[
   \{2^i : 0 \leq i \leq m \} \not\subseteq \{\lfloor \alpha s_n \rfloor : n = 1, 2, 3, \ldots\}
   \]
   for any \( 0 < \alpha < 1 \).

2. Let \( r, \ell \geq 2 \) be two integers. Then there exists some integer \( k \geq 1 \) such that \( \ell^m + k \) is not \( r \)-full for all positive integers \( m \), where an integer \( n \) is called \( r \)-full if \( p^r \mid n \) implies \( p \mid n \) for any prime \( p \).

3. All but at most \( O(n/\log n) \) of the \( n \) numbers
   \[
   2^n - i \quad (1 \leq i \leq n)
   \]
   are composite, where the implied constant is absolute.

The first two results answer two problems posed by Ma–Chen and Chen–Ruzsa, respectively. The last one is only a partial result towards another problem of Chen. He asked whether there are infinitely many integers \( n \) such that all of
   \[
   2^n - i \quad (1 \leq i \leq n)
   \]
   are composite. The author could not solve it at present. What these three results in common is the accessibility of their elementary and clean proofs.

1. THE PROBLEM INVOLVING COMPLETE SEQUENCES

Let \( \mathbb{N} \) be the set of nonnegative integers. Suppose that \( A \) is a sequence of nonnegative integers whose terms are not necessarily different. Let \( P(A) \) be the set of nonnegative integers which can be represented as the sum of finitely distinct terms of \( A \). The sequence \( A \) is said to be complete if all sufficiently large integers belong to \( P(A) \). Conventionally, we declare that \( 0 \in P(A) \). Burr [3] asked that which subsets \( B \) of \( \mathbb{N} \) are equal to \( P(A) \) for a given sequence \( A \). This problem remains widely open but some partial results had already been obtained, see for example [1, 2, 5, 12, 13, 15, 16]. For \( \alpha > 0 \), let
   \[
   S_\alpha = \{\lfloor \alpha s_1 \rfloor, \lfloor \alpha s_2 \rfloor, \ldots\}.
   \]
   Typically, Hegyvári [14] proved the following interesting theorem.

Hegyvári’s Theorem. Let \( S = \{s_1 < s_2 < \cdots\} \) be a sequence of positive integers such that
   \[
   \lim_{n \to \infty} (s_{n+1} - s_n) = \infty
   \]

2010 Mathematics Subject Classification. Primary 11A41.
Key words and phrases. Complete sequences; Square–full numbers; Composite numbers; The large sieve.
and $s_{n+1} < \gamma s_n$ for all sufficiently large integers $n$, where $1 < \gamma < 2$. Suppose that $S_\alpha$ is complete for some $\alpha > 0$, then there exists a positive number $\delta$ such that $S_\beta$ are complete for all $\beta \in [\alpha, \alpha + \delta]$.

The unnecessary condition $\lim_{n \to \infty} (s_{n+1} - s_n) = \infty$ was then removed by Chen and Fang [6]. In a subsequent article, Ma and Chen [18] proved that the prerequisite is always valid in Hegvári’s Theorem. Along another line, Motivated by Ma–Chen’s Theorem. Let

$$S = \{s_1 < s_2 < \cdots \}$$

be a sequence of positive integers such that $s_{n+1} \leq 2s_n$ for all sufficiently large integers $n$. Suppose that $P(S_\alpha) \neq \mathbb{N}$ for any $0 < \alpha < 1$. Then for any positive integer $w$, there exist a real number $0 < \alpha < 1$ and a positive integer $n$ such that

$$\lfloor \alpha n + 1 \rfloor = 2^i \ (i = 0, 1, \ldots, w).$$

Ma and Chen posed the following problem.

**Problem 1.** Does there exist a sequence $S = \{s_1, s_2, \ldots \}$ of positive integers with $s_n < s_{n+1} \leq 2s_n$ and a positive integer $m$ such that for each $0 < \alpha < 1$,

$$\{2^i : 0 \leq i \leq m\} \not\subseteq S_\alpha?$$

They further remarked that $S = \{n^2 : n = 1, 2, \ldots \}$ is not the required sequence for their problem. Since Ma and Chen did not indicate any idea or detail of their claim, the author would like to offer an alternative proof of it for convenience of the readers. The aim is to prove that for any integer $m$, there exists $0 < \alpha < 1$ such that

$$\lfloor \alpha n + 1 \rfloor = 2^i \ (i = 0, 1, \ldots, m).$$

The argument goes as follows: Let $\alpha$ be the number satisfying the requirement. For any $0 \leq i \leq m$, it suffices to find some $n_i$ with $2^i = \lfloor \alpha n_i^2 \rfloor$, which is equivalent to

$$\sqrt{\alpha - 1}2^i \leq n_i < \sqrt{\alpha - 1}(2^i + 1).$$

The above inequalities would be satisfied if we have

$$\sqrt{\alpha - 1} \left( \sqrt{2^i + 1} - \sqrt{2^i} \right) > 1 \quad (0 \leq i \leq m).$$

The proof ends up with the choice of $\alpha = 4^{-1}(2^m + 1)^{-1}$.

As we mentioned in the abstract, the answer to Problem 1 is affirmative. Actually, we have the following much stronger result.

**Theorem 1.1.** Let $8/5 \leq \gamma < 2$ be a given number and $S = \lfloor \gamma^n \rfloor$. Then

$$\{2^j, 2^{j+1}\} \not\subseteq S_\alpha$$

for any $0 < \alpha < 1$, where

$$j = \left\lfloor \log \left( \frac{2\gamma}{2 - \gamma} \right) / \log 2 \right\rfloor + 1.$$
Remark. Taking $\gamma = 8/5$ in Theorem 1.1, then $S = \left\lfloor \left(\frac{8}{5}\right)^n \right\rfloor$ and $j = 4$. So we have
\[
\{2^4, 2^5\} \not\subseteq \{\lfloor \alpha \lfloor (8/5)^n \rfloor \rfloor : n = 1, 2, \ldots \}\.
\]
for any $0 < \alpha < 1$.

Proof. Suppose that $2^j \in S_\alpha$ for some $\alpha \in (0, 1)$, then there exists a positive integer $k$ such that
\[
2^j = \lfloor \alpha \lfloor \gamma^k \rfloor \rfloor.
\]
Getting removed of the floor function, we have
\[
2^j \leq \alpha \lfloor \gamma^k \rfloor < 2^j + 1
\]
and hence
\[
2^j \alpha^{-1} \leq \gamma^k < (2^j + 1)\alpha^{-1} + 1.
\]
From which we deduce that
\[
2^j \gamma \alpha^{-1} \leq \gamma^{k+1} < (2^j \gamma + \gamma)\alpha^{-1} + \gamma
\]
and
\[
2^j \gamma^2 \alpha^{-1} \leq \gamma^{k+2} < (2^j \gamma^2 + \gamma^2)\alpha^{-1} + \gamma^2.
\]
By the right–hand side inequality of equation (1.1), we have
\[
\lfloor \alpha \lfloor \gamma^{k+1} \rfloor \rfloor \leq \alpha \lfloor \gamma^{k+1} \rfloor < 2^j \gamma + \gamma + \gamma \alpha < (2^j + 2)\gamma.
\]
Whereas from the left–hand side inequality of equation (1.2), we shall get
\[
\lfloor \alpha \lfloor \gamma^{k+2} \rfloor \rfloor > \alpha \lfloor \gamma^{k+2} \rfloor - 1 > \alpha(2^j \gamma^2 \alpha^{-1} - 1) - 1 > 2^j \gamma^2 - 2.
\]
Recall that
\[
j = \left\lfloor \log \left( \frac{2\gamma}{2 - \gamma} \right) / \log 2 \right\rfloor + 1, \quad \text{i.e.,} \quad \frac{2\gamma}{2 - \gamma} < 2^j \leq \frac{4\gamma}{2 - \gamma},
\]
then a slight computations lead to
\[
\frac{2^j+1}{2^j+4} \leq \gamma < \frac{2^j+1}{2^j+2}.
\]
Thus, it can be deduced that
\[
\lfloor \alpha \lfloor \gamma^{k+1} \rfloor \rfloor < (2^j + 2)\gamma < 2^{j+1}
\]
from equations (1.3), (1.5) and
\[
\lfloor \alpha \lfloor \gamma^{k+2} \rfloor \rfloor > 2^j \gamma^2 - 2 \geq 2^j \left( \frac{2^j+1}{2^j+4} \right)^2 - 2 > 2^{j+1} \quad \text{(since } j \geq 4 \text{ for } \gamma \geq 8/5),
\]
from equations (1.4), (1.5), which certainly means $2^{j+1} \not\in S_\alpha$. □
2. The problems involving $r$–full numbers

In 1981, Erdős [11] proved that if $a_n \in \mathbb{N}$ and $\lim_{n \to \infty} (a_{n+1} - a_n) = \infty$, then $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ is irrational. It is surely that square–free numbers $n$ do not satisfy the requirement. Thus, Erdős further conjectured that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is irrational. Later, this conjecture was resolved by Chen and Ruzsa [7]. Now, let’s fix some terminologies which shall occur in the context below. If $p \nmid n$ for any prime $p$, then the integer $n$ is said to be $r$–free. An integer $n$ is called $r$–full if $p \mid n$ implies $p^r \mid n$ for any prime $p$. 2–free and 2–full are called square–free and square–full respectively conventionally. Chen and Ruzsa proved the following more general theorem.

Chen–Ruzsa’s theorem. Let $r$ be an integer with $r \geq 2$ and $\{a_n\}$ a sequence of distinct positive integers such that each $a_n$ is either $r$–free or square–full. Then $\sum_{n=1}^{\infty} a_n \ell^{-a_n}$ is irrational.

Observation the proof of their proof, Chen and Ruzsa naturally raised the following two problems.

Problem 2. Let $r, \ell \geq 2$ be two given integers. Then for any $k \geq 1$, is there a positive integer $m$ such that each of $\ell^m + i (1 < i < k)$ is $r$–full.

Problem 3. Let $r, \ell \geq 2$ be two given integers. Then for any $k \geq 1$, is there a positive integer $m$ such that $\ell^m + k$ is $r$–full.

Without doubt, Problem 2 implies Problem 3. We shall show that the answer to Problem 3 is negative, hence the same to Problem 2. That is,

**Theorem 2.2.** Let $r, \ell \geq 2$ be two given integers. Then there exists a positive integer $k$ such that $\ell^m + k$ is not $r$–full for any positive integers $k$.

**Proof.** The proof of our theorem is divided into three cases.

**Case I.** Assuming first $\ell = 2$, we consider the number $2^m + 10$. If $m = 1$, then $2^1 + 10 = 12$ is not 2–full, hence not $r$–full. If $m \geq 2$, then $2|2^m + 10$ but $2^2 \nmid 2^m + 10$, which means that $2^m + 10$ is not 2–full for $m \geq 2$, hence not $r$–full either.

**Case II.** We next assume that there exists a prime $p$ such that $p^2 \mid \ell$. It is surely that $p|\ell^m + p$ but $p^2 \nmid \ell^m + p$. Hence, $\ell^m + p$ is not $r$–full for any positive integers $m$.

**Case III.** Finally, we assume that $\ell$ is a square–free number with at least one odd prime factor $q$. By Dirichlet’s theorem in arithmetic progressions (see for example [9, Chapter 4]), there is some integer $s \geq 2$ such that $\ell s - 1$ is another prime $q_s$. Let $k = \ell(q_s - 1)$. Then

$$\ell^m + k = \ell(\ell^{m-1} + q_s - 1).$$

If $m = 1$, then $\ell + k = \ell q_s$. Thus we have $q_s|\ell + k$ but $q_s^2 \nmid \ell + k$. If $m \geq 2$, then $q|\ell^m + k$ whereas $q^2 \nmid \ell^m + k$. Therefore $\ell + k$ is not 2–full for any $m$, hence not $r$–full either. □
3. THE PROBLEM INVOLVING COMPOSITE NUMBERS

In 1849, de Polignac [22] conjectured that every odd number greater than 3 is the sum of a prime and a power of 2. However, two counterexamples 127 and 959 were found out by himself [23] soon. He then revised his conjecture by requiring the odd numbers to be sufficiently large. In 1934, Romanoff [24] proved a remarkable theorem, which states that the odd numbers which can be represented by the form \( p + 2^m \) \( (p \in \mathbb{P}, m \in \mathbb{N}) \) have a positive lower density. Later, the opposite direction was achieved by van der Corput [8], who illustrated that the odd numbers which can not be written as the form \( p + 2^k \) \( (p \in \mathbb{P}, k \in \mathbb{N}) \) still possess a positive lower density. To answer a question of Romanoff, Erdős [10] constructed a specific arithmetic progression, none of which can be written as the sum of a prime and a power of 2.

Let \( f(n) \) be the representation function of \( n \) written as the sum of a prime and a power of 2, i.e.,

\[
f(n) = \# \{(p, k) : n = p + 2^k, \; p \in \mathbb{P}, \; k \in \mathbb{N}\}.
\]

Erdős [10] showed that

\[
\limsup_{n \to \infty} \frac{f(n)}{\log \log n} > 0, \tag{3.1}
\]

which gave an affirmative answer to a question of Turán. It turns out that the magnitude of the function \( f(n) \) is quite difficult to decide. Erdős conjectured that \( f(n) = o(\log n) \). However, Erdős commented that he can even not prove that not all the integers

\[n - 2^k \; (1 \leq n < \log n / \log 2)\]

are primes for sufficiently large numbers \( n \). By checking of the prime table (up to 203775), Erdős conjectured that the largest exceptional integer is probably 105. For integers \( n \leq x \), Vaughan [25] proved that the number of the exceptional integers is \( O(x \exp(-c_1 \log x \log \log x / \log \log x)) \) for some constant \( c_1 > 0 \) by the Montgomery sieve [19] (i.e., the arithmetic large sieve). Under the assumption of the extended Riemann Hypothesis, Hooley [17] proved that the exceptional numbers are less than \( O(x^{1-\lambda+\varepsilon}) \) for any sufficiently small positive number \( \varepsilon \), where \( \lambda = \prod_p \left(1 - \frac{1}{p(p-1)}\right) \).

Narkiewicz [21] improved this to \( O(x^{1-\lambda/\log 2+\varepsilon}) \) under the same assumption.

In view of the above conjecture investigated by Erdős, Chen [4] asked the following dual problem.

**Problem 4.** Are there infinitely many integers \( n \) such that all of

\[2^n - i \; (1 \leq i \leq n)\]

are composite?
The author cannot solve this problem presently. Here we only give some considerations regarding it.

**Theorem 3.3.** Let \( n \geq 4 \) be a positive integer. Then all but at most \( O(n/\log n) \) of 
\[
2^n - i \quad (1 \leq i \leq n)
\]
are composite.

**Proof.** For any prime \( p < \sqrt{2^n - n} \), suppose that
\[
2^n \equiv a_p \pmod{p}.
\]
Let \( \mathcal{A} \) be the set of positive integers \( j \) up to \( n \) such that \( 2^n - j \) is prime. Then
\[
j \not\equiv a_p \pmod{p} \quad (\forall p < \sqrt{2^n - n}).
\]
We now employ the large sieve due to Montgomery [19, Corollary to Theorem 2] to give an upper bound of \( \mathcal{A} \).

**Lemma.** Let \( M \) and \( N \) be integers, \( N > 0 \) and let
\[
\mathcal{A} = \{ n_i : i = 1, \ldots, Z \}
\]
be a set of \( Z \) integers with \( M + 1 \leq n_1 < n_2 < \cdots < n_Z \leq M + N \). Let \( Q \geq 1 \) and for each prime \( p \leq Q \) suppose that there are at least \( \omega(p) \) (\( \geq 0 \)) residue classes modulo \( p \) that contain no elements of \( \mathcal{A} \). Then
\[
Z \leq (N^{1/2} + Q)^2 S^{-1},
\]
where
\[
S = \sum_{q \leq Q} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p - \omega(p)}.
\]
In our situation, we have \( \omega(p) = 1 \) for any prime \( p < \sqrt{2^n - n} \). Let \( Q < \sqrt{2^n - n} \) be a parameter depending on \( n \) to be decided later. By the Lemma, we have
\[
|\mathcal{A}| \leq (n^{1/2} + Q)^2 S^{-1}, \tag{3.2}
\]
where
\[
S = \sum_{q \leq Q} \mu(q)^2 \prod_{p|q} \frac{1}{p - 1}.
\]
We are in a position to provide a lower bound of \( S \). It is clear that
\[
S > \sum_{q \leq Q} \mu(q)^2 \prod_{p|q} \frac{1}{p} = \sum_{q \leq Q} \frac{\mu(q)^2}{q}. \tag{3.3}
\]
Let
\[
M(t) = \sum_{q \leq t} \mu(q)^2.
\]
Then it is well-known that (see for example [20, Theorem 2.2])

$$M(t) = \frac{6}{\pi^2} t + O(\sqrt{t}).$$

Integrating by parts, we have

$$\sum_{q \leq Q} \frac{\mu(q)^2}{q} = \frac{M(Q)}{Q} + \int_1^Q \frac{M(t)}{t^2} dt \gg \int_1^Q \frac{1}{t} dt \gg \log Q.$$  \hspace{1cm} (3.4)

Gathering equations (3.2), (3.3) and (3.4), we conclude that

$$|A| \ll \left(n^{1/2} + Q\right)^2 (\log Q)^{-1}.$$  \hspace{1cm} (3.5)

The theorem follows immediately from making $Q = n^{1/2}$.

**ACKNOWLEDGMENTS**

The author is supported by the Natural Science Foundation of Jiangsu Province of China, Grant No. BK20210784, China Postdoctoral Science Foundation, Grant No. 2022M710121, the foundations of the projects "Jiangsu Provincial Double-Innovation Doctor Program", Grant No. JSSCBS20211023 and "Golden Phoenix of the Green City–Yang Zhou" to excellent PhD, Grant No. YZLYJF2020PHD051.

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