A GENERAL INFORMATION THEORETICAL PROOF FOR
THE SECOND LAW OF THERMODYNAMICS

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Abstract

We show that the conservation and the non-additivity of the information, together with the additivity of the entropy make the entropy increase in an isolated system. The collapse of the entangled quantum state offers an example of the information non-additivity. Nevertheless, the later is also true in other fields, in which the interaction information is important. Examples are classical statistical mechanics, social statistics and financial processes. The second law of thermodynamics is thus proven in its most general form. It is exactly true, not only in quantum and classical physics but also in other processes, in which the information is conservative and non-additive.

Keywords: Information conservation, Non-additivity of information, Entropy increase

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To understand the foundation of the second law of thermodynamics is a long standing problem in physics. Text books tell us[1], the state of a macroscopic system with larger entropy is more probable. However, whether the system always goes from a less probable state to a more probable state is still an open question. The H-theorem of Boltzmann is a classical proof for definite approaching to equilibrium. It is based on a model of colliding classical particle system for the macroscopic matter, therefore is not general enough, even from the view point of the classical statistical physics. In 1948, Shannon[2, 3] discovered a powerful theory of information. It is applicable in analysis of all statistical processes, including statistical physics. Here we show that by use of its fundamental ideas, one can simply prove the second law of thermodynamics in its most general form. In quantum statistical mechanics, it is based on the state entanglement in time development and the state collapse in measurement, therefore is quite general.

The time development of the density operator $\rho(t)$ for an isolated system is governed by the von Neumann equation. Its solution is

$$\rho(t) = U(t, t_0)\rho(t_0)U(t_0, t) ,$$

(1)
in which \( U(t, t_0) \) is the time displacement operator of the state from time \( t_0 \) to time \( t \). Defining

\[
\mathcal{I}(t) = \text{Tr}[\rho(t) \ln \rho(t)]
\]

at time \( t \), we see from (1)

\[
\mathcal{I}(t) = \text{Tr}[U(t, t_0)\rho(t_0) \ln \rho(t_0)U(t_0, t)U(t, t_0)] = \mathcal{I}(t_0),
\]

because of \( U(t_0, t)U(t, t_0) = 1 \). It is the information conservation in quantum mechanics[4].

To measure the entropy of a system, one has to divide the system into macroscopically infinitesimal parts. The entropy of the \( i \)th part is defined to be \( S_i = -k_B \text{Tr} (\rho_i \ln \rho_i) \), in which \( \rho_i \) is the reduced density operator of the part \( i \). The entropy of the whole system is defined to be the sum

\[
S = \sum_i S_i = -k_B \sum_i \text{Tr} (\rho_i \ln \rho_i)
\]

of the entropies of these parts, as an extensive thermodynamical variable should be. When one measures the entropy of the system at time \( t_0 \), he has destroyed the entanglement of the states of various parts of the system. The state and the density operator of the system are therefore factorized. Under this condition, the entropy of the system is

\[
S(t_0) = -k_B \sum_i \text{Tr} [\rho_i(t_0) \ln \rho_i(t_0)]
\]

\[
= -k_B \text{Tr} [\rho(t_0) \ln \rho(t_0)] = -k_B \mathcal{I}(t_0).
\]

For an isolated system, the information conservation (3) works. The information of the system at \( t > t_0 \) is therefore

\[
\mathcal{I}(t) = S(t_0)/k_B.
\]

During the period from time \( t_0 \) to \( t \), the interaction between different parts of the system makes their states be entangled again. It means the states of different parts are correlated. If one measures the entropy of the system at time \( t \), he has to measure the entropies of every part of the system, and therefore destroy this entanglement once more. This is the state collapse, and causes the loss of correlation information. Since the parts of the system are not isolated, their information is not conserved. It makes the entropy

\[
S(t) = -k_B \sum_i \text{Tr} [\rho_i(t) \ln \rho_i(t)]
\]

at time \( t \) does not equal \( S(t_0) \) in general. By intuition we see, the sum of information of all parts of the system should not be more than the information of the system, since the correlation information of various parts is not included in the sum. It is

\[
\sum_i \text{Tr}[\rho_i(t) \ln \rho_i(t)] \leq \mathcal{I}(t).
\]
If this is true, we obtain
\[ S(t) \geq S(t_0) \] (9)
from (6)-(8) for an isolated system.

To prove the statement (8), let us remind you some mathematical inequalities. We also collect the proofs of these inequalities here, to make our description be self-contained, although their original forms may be found in text books\[4, 5\]. By the way, in the following we understand that 0 ln 0 \(\equiv\) \(\lim_{\xi \to 0} (\xi \ln \xi) = 0\).

**Lemma 1.** For any non-negative number \(x\) we have
\[ x \ln x \geq x - 1 \, , \] (10)
the equality holds when and only when \(x = 1\).

**Proof:** It may be verified by differentiation, that \(x \ln x - (x - 1)\) as a continuous function of non-negative variable \(x\) has unique minimum 0 at \(x = 1\). The lemma is therefore proven.

**Lemma 2.** For sets \([w_i]\) and \([x_i]\) of non-negative numbers with \(\sum_i x_i = 1\), we have
\[ \sum_i x_i w_i \ln \sum_{i'} x_{i'} w_{i'} \leq \sum_i x_i w_i \ln w_i \, . \] (11)

**Proof:** The average \(\bar{w} \equiv \sum_i x_i w_i\) is non-negative. For \(\bar{w} > 0\), by lemma 1 we see
\[ \sum_i x_i w_i \ln \sum_{i'} x_{i'} w_{i'} - \sum_i x_i w_i \ln w_i \\
= - \sum_i x_i \bar{w} \frac{w_i}{\bar{w}} \ln \frac{w_i}{\bar{w}} \leq - \sum_i x_i \bar{w} \left(\frac{w_i}{\bar{w}} - 1\right) = 0 \, , \] (11) is true. Since two sides of (11) are continuous functions of non-negative variables \([w_i]\) and \([x_i]\), it is also true for the limit case \(\bar{w} = 0\). The lemma is therefore proven.

**Lemma 3.** For sets \([W_i]\) and \([T_{ij}]\) of non-negative numbers with
\[ \sum_i W_i = 1 \quad \text{and} \quad \sum_i T_{ij} = \sum_j T_{ij} = 1 \, , \] (12)
we have
\[ W_j' \equiv \sum_i W_i T_{ij} \geq 0 \quad \text{for every} \ j \, , \] (13)
\[ \sum_j W_j' = 1 \, , \] (14)
and
\[ \sum_j W_j' \ln W_j' \leq \sum_i W_i \ln W_i \, . \] (15)

**Proof:** (13) and (14) are obvious. By (12) and lemma 2 we see
\[ \sum_j W_j' \ln W_j' = \sum_j \left(\sum_i W_i T_{ij}\right) \ln \left(\sum_{i'} W_{i'} T'_{i'j}\right) \\
\leq \sum_i W_i T_{ij} \ln W_i = \sum_i W_i \ln W_i \, . \]
The lemma is therefore proven.

**Lemma 4.** For positive numbers \([W_{ij}], W_i = \sum_j W_{ij} \text{ and } W'_j = \sum_i W_{ij}, \text{ with } \sum_{ij} W_{ij} = 1, \) we have
\[
\sum_i W_i = 1, \quad \sum_j W'_j = 1, \tag{16}
\]
and
\[
\sum_{ij} W_{ij} \ln W_{ij} \geq \sum_i W_i \ln W_i + \sum_j W'_j \ln W'_j. \tag{17}
\]
The equality holds when and only when \(W_{ij} = W_i W'_j\) for all \(ij\), it is that the \(W_{ij}\) may be factorized.

Proof: (16) is obvious. By lemma 1 we see
\[
\frac{W_{ij}}{W_i W'_j} \ln \frac{W_{ij}}{W_i W'_j} \geq \frac{W_{ij}}{W_i W'_j} - 1, \tag{18}
\]
the equality holds when and only when \(W_{ij} = W_i W'_j\). Multiplying two sides of (18) by the positive number \(W_i W'_j\) and summing up over \(ij\), one obtains
\[
\sum_{ij} W_{ij} \ln W_{ij} - \sum_i W_i \ln W_i - \sum_j W'_j \ln W'_j \geq 0.
\]
This is exactly (17). The lemma is therefore proven.

Suppose \([L]\) is a complete set of commutative dynamical variables of the system, with a complete orthonormal set of eigenstates \([|n\rangle]\). The \([L]\) representation of density operator \(\rho\) is a matrix with elements \(\rho_{n,n'} = \langle n|\rho|n'\rangle\). If \(\rho\) itself is included in the set \([L]\), the \([L]\) representation of \(\rho\) is called natural. In a natural representation, the density matrix is diagonal: \(\rho_{n,n'} = W_n \delta_{n,n'}\), in which \(W_n\) is the \(n\)th eigenvalue of \(\rho\), denoting the probability of finding the system being in the state \(|n\rangle\). The information (2) may be written in the form
\[
\mathcal{I} = \sum_n W_n \ln W_n, \tag{19}
\]
with a set \([W_n]\) of non-negative numbers. One may also consider the information about a specially chosen complete set of commutative dynamical variables \([L']\), with complete set of orthonormal eigenstates \([|m\rangle]\). For an ensemble of the systems with the density operator \(\rho\), the probability of finding the system in the state \(|m\rangle\) is
\[
W'_m = \sum_n \langle m|n\rangle W_n \langle n|m\rangle. \tag{20}
\]
The definition of the information about the variables \([L']\) is
\[
\mathcal{I}_{[L']} \equiv \sum_m W'_m \ln W'_m. \tag{21}
\]
Since \(|\langle n|m\rangle|^2\) are non-negative, and \(\sum_n |\langle n|m\rangle|^2 = \sum_m |\langle n|m\rangle|^2 = 1\), according to lemma 3 and equation (19) we have
\[
\mathcal{I}_{[L']} \leq \mathcal{I}. \tag{22}
\]
Now, let us divide the system into two parts $a$ and $b$. Suppose $[L_i]$, with $i = a$ or $b$, is a complete set of commutative dynamical variables of part $i$, $|n_i⟩$ is their $n_i$th eigenstate, and $[|n_i⟩]$ is a complete set of states of part $i$. Therefore $[|n_a n_b⟩] ≡ [|n_a⟩|n_b⟩]$ is a complete orthonormal set of states of the system. In the $[L_a L_b]$ representation, the density operator of the system is a matrix, with elements

$$\rho_{n_a n_b, n'_a n'_b} \equiv ⟨n_a n_b|ρ|n'_a n'_b⟩ .$$

From (20) we see the probability of finding part $a$ in the state $|n_a⟩$ and part $b$ in the state $|n_b⟩$ is

$$W_{n_a n_b} = \sum_n ⟨n_a n_b|n⟩ W_n ⟨n|n_a n_b⟩ ,$$

with normalization

$$\sum_{n_a n_b} W_{n_a n_b} = 1 .$$

The information of dynamical variables $[L_a, L_b]$ is

$$I_{L_a, L_b} = \sum_{n_a n_b} W_{n_a n_b} \ln W_{n_a n_b} ≤ I .$$

The probability of finding part $a$ in the state $|n_a⟩$ and the probability of finding the part $b$ in the state $|n_b⟩$ are

$$W_{n_a} = \sum_{n_b} W_{n_a n_b} \quad \text{and} \quad W'_{n_b} = \sum_{n_a} W_{n_a n_b} .$$

respectively. In (25-27), it is understood that the summation is over those $n_a$ and $n_b$ only, for which $W_{n_a n_b} > 0$.

The density operator $ρ_a$ of part $a$ is reduced from the density operator $ρ$ of the system. In the representation $[L_a]$, it is a matrix with elements

$$(ρ_a)_{n_a, n'_a} = \sum_{n_b} ρ_{n_a n_b, n'_a n_b} = \sum_{n_b} ⟨n_a n_b|ρ|n'_a n_b⟩ ,$$

and may be written in a compact form

$$ρ_a = \text{Tr}_b ρ .$$

The subscript $b$ denotes that the trace is a sum of matrix elements diagonal with respect to quantum numbers of part $b$. Likewise, $ρ_b = \text{Tr}_a ρ$. Suppose $ρ_i$ is included in the set $[L_i]$, the probability of finding the part $i$ in state $|n_i⟩$ is its eigenvalue $W_{n_i}$, and is expressed in (27). The information about part $i$ is

$$I_i = \text{Tr} ρ_i \ln ρ_i = \sum_{n_i} W_{n_i} \ln W_{n_i} .$$

From lemma 4 and equations (26,27) we see

$$\text{Tr} ρ_a \ln ρ_a + \text{Tr} ρ_b \ln ρ_b ≤ \text{Tr} ρ \ln ρ .$$
The equality holds when and only when the density operator of the system may be factorized into a direct product of density operators of its parts, it is when and only when its parts do not correlate with each other. We may further subdivide the parts and apply (31) to them again and again, the result is the statement (8). As we showed before, this statement proves (9), which is the

**Theorem:** The entropy of an isolated system if changes can only increase.

It is exactly the second law of thermodynamics. This law is therefore finally proven. According to the relationship between the entropy and the probability of a macroscopic state[1] referred at the beginning of this paper, it in turn shows that an isolated macroscopic system always goes from the less probable state to the more probable state.

The proof here is quite general. It looks like relying on the quantum mechanical effects of state entanglement and its collapse. However, it is still more general. It is an information theoretical proof, relies only on the conservation (3) and the non-additivity (31) of the information. The extensive character (4) (additivity) of the entropy is also important. Information conservation is a character of dynamics. It is shared by quantum dynamics and classical dynamics, as well as some possible dynamics not yet have been discovered. The non-additivity of information is purely mathematical. It may be deduced from the general relations (27) of the probabilities by use of mathematical inequalities stated before. State entanglement and its collapse is only a special way of their realization. They may be realized in classical mechanics or in some unknown mechanics as well. The second law of thermodynamics is therefore almost dynamics independent, except the requirement of information conservation. It may be still exactly true in the future, even though one day people find that the quantum mechanics is only approximate. It is also quite generally applicable, not only to thermodynamics but also to some other statistical sciences, if the information conservation is true for them. To consider its possible applications in the social and financial sciences is interesting.

From the proof we learn that the entropy of an isolated system increases only because one loses the correlation information between different parts of the system. It emphasizes the importance of the correlation information in a complete statistical science.

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