On orbifold elliptic genus

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0 Introduction.

Elliptic genus was derived as the partition function in quantum field theory \cite{26}. Mathematically it is the beautiful combination of topology of manifolds, index theory and modular forms (cf. \cite{15}, \cite{10}). Elliptic genus for smooth manifolds has been well-studied. Recently, Borisov and Libgober ([3], [4]) proposed some definitions of elliptic genus for certain singular spaces, especially for complex orbifolds which is a global quotient $M/G$, here the finite group $G$ acts holomorphically on complex variety $M$. Similar definitions have been introduced by string theorists in the 80s, in the study of orbifold string theory. One of their guiding principles is the modular invariance. More recently orbifold string theory has attracted the attention of geometers and topologists. For example Chen and Ruan (cf. \cite{7}, \cite{22}) have defined orbifold cohomology and orbifold quantum cohomology groups.

One of most important properties for elliptic genus is its rigidity property under compact Lie group action. For smooth manifolds, the rigidity and its generalizations have been well studied. Since orbifold elliptic genus is the partition function of orbifold string theory, it is natural to expect the rigidity property for orbifold elliptic genus. Although the global quotients from a very important class of orbifolds, many interesting orbifolds are not global quotients. For example, most of the Calabi-Yau hypersurfaces of weighted projective spaces are not global quotients. In this paper we define elliptic genus for general orbifolds which generalizes the definition of Borisov and Libgober, and prove their rigidity property. We actually introduce the more general elliptic genus involving twisted bundles and proved its rigidity. The idea of considering the weights in the definition of orbifold elliptic genus comes from \cite{4} and our proof of the K-theory version of Witten rigidity theorems \cite[§4]{21}. The proof of the rigidity is again a combination of modular invariance and the index theory. Only more complicated combinatorics are involved in the definition and the proof.

This paper is organized as follows: In Sections 1 and 2 we review the equivariant index theorem on orbifolds. We define orbifold elliptic genus and prove its rigidity for almost complex orbifolds in Section 3. Finally in Section 4 we introduce orbifold elliptic genus for spin orbifolds, and will study its rigidity property on a later occasion.

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1 Equivariant index theorem for spin orbifolds

In this section and the next section we recall the notations on orbifolds, and explain the equivariant index theorem for orbifolds (cf. [8, Chap. 14], [25]).

We first recall the definition of orbifolds, which are called V-manifolds in [11], [23]. We consider the pair \((G, V)\), here \(V\) is a connected smooth manifold, \(G\) is a finite group acting smoothly and effectively on \(V\). A morphism \(\Phi : (G, V) \rightarrow (G', V')\) is a family of open embedding \(\varphi : V \rightarrow V'\) satisfying:

i) For each \(\varphi \in \Phi\), there is an injective group homomorphism \(\lambda_{\varphi} : G \rightarrow G'\) such that \(\varphi\) is \(\lambda_{\varphi}\)-equivariant.

ii) For \(g \in G', \varphi \in \Phi\), we define \(g\varphi : V \rightarrow V'\) by \((g\varphi)(x) = g\varphi(x)\) for \(x \in V\). If \((g\varphi)(V) \cap \varphi(V) \neq \emptyset\), then \(g \in \lambda_{\varphi}(G)\).

iii) For \(\varphi \in \Phi\), we have \(\Phi = \{g\varphi, g \in G'\}\).

The morphism \(\Phi\) induces a unique open embedding \(i_{\Phi} : V/G \rightarrow V'/G'\) of orbit spaces.

**Definition 1.1** An orbifold \((X, \mathcal{U})\) is a paracompact Hausdorff space \(X\) together with a covering \(\mathcal{U}\) of \(X\) consisting of connected open subsets such that

i) For \(U \in \mathcal{U}\), \(\mathcal{V}(U) = ((G_U, \tilde{U}) \xrightarrow{\tilde{\varphi}} U)\) is a ramified covering \(\tilde{U} \rightarrow U\) giving an identification \(U \simeq \tilde{U}/G_U\).

ii) For \(U, V \in \mathcal{U}, U \subset V\), there is a morphism \(\varphi_{VU} : (G_U, \tilde{U}) \rightarrow (G_V, \tilde{V})\) that covers the inclusion \(U \subset V\).

iii) For \(U, V, W \in \mathcal{U}, U \subset V \subset W\), we have \(\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}\).

In the above definition, we can replace \((G, V)\) by a category of manifolds with an additional structure such as orientation, Riemannian metric or complex structure. We understand that the morphisms (and the groups) preserve the specified structure. So we can define oriented, Riemannian or complex orbifolds.

**Remark:** Let \(G\) be a compact Lie group and \(M\) a smooth manifold with a smooth \(G\)-action. We assume that the action of \(G\) is effective and infinitesimally free. Then the quotient space \(M/G\) is an orbifold. Reciprocally, any orbifold \(X\) can be presented this way. For example, let \(O(X)\) be the total space of the associated tangential orthonormal frame bundle. We know that \(O(X)\) is a smooth manifold and the action of the orthogonal group \(O(n)\) (\(n = \dim X\)) is infinitesimally free on \(O(X)\). The \(X\) is identified canonically with the orbifold \(O(X)/O(n)\).

Let \(X\) be an oriented orbifold, with singular set \(\Sigma X\). For \(x \in X\), there exists a small neighbourhood \((G_x, \tilde{U}_x) \xrightarrow{\tilde{\varphi}} U_x\) such that \(\tilde{x} = \tau_x^{-1}(x) \in \tilde{U}_x\) is a fixed point of \(G_x\). Such \(G_x\) is unique up to isomorphisms for each \(x \in X\) [23, p468]. Let \((1), (h_x^1), \cdots (h_x^{\rho_x})\) be all the conjugacy classes in \(G_x\). Let \(Z_{G_x}(h_x^j)\) be the centralizer of \(h_x^j\) in \(G_x\). One also denotes by \(\tilde{U}_x^{h_x^j}\) the fixed points of \(h_x^j\) in \(\tilde{U}_x\). There is a natural bijection

\[
(y, (h_y^j))|y \in U_x, j = 1, \cdots \rho_y\} \\
\simeq \Pi_{j=1}^{\rho_x} \tilde{U}_x^{h_x^j} / Z_{G_x}(h_x^j).
\]
So we can define globally [14, p77],

\[(1.2) \quad \Sigma X = \{(x, (h^j_x))| x \in X, G_x \neq 1, j = 1, \ldots, \rho_x\}.
\]

Then \(\Sigma X\) has a natural orbifold structure defined by

\[(1.3) \quad \{ (ZG_x(h^j_x)/K^j_x, \tilde{U}^j_x) \rightarrow \tilde{U}^j_x / ZG_x(h^j_x) \}_{(x, U, j)}.
\]

Here \(K^j_x\) is the kernel of the representation \(ZG_x(h^j_x) \rightarrow \text{Diffeo}(\tilde{U}^j_x)\). The number \(m = |K^j_x|\) is called the multiplicity of \(\Sigma X\) in \(X\) at \((x, h^j_x)\). Since the multiplicity is locally constant on \(\Sigma X\), we may assign the multiplicity \(m_i\) to each connected component \(X_i\) of \(\Sigma X\). In a sense \(\Sigma X\) is a resolution of singularities of \(X\).

**Definition 1.2** A mapping \(\pi\) from an orbifold \(X\) to an orbifold \(X'\) is called smooth if for \(x \in X, y = \pi(x)\), there exist orbifold charts \((G_x, \tilde{U}_x), (G'_y, \tilde{U}'_y)\) together with a smooth mapping \(\phi : \tilde{U}_x \rightarrow \tilde{U}'_y\) and a homomorphism \(\rho : G_x \rightarrow G'_y\) such that \(\phi\) is \(\rho\)-equivariant and \(\tau'_y \circ \phi = \pi \circ \tau_x\).

**Definition 1.3** An orbifold vector bundle \(\xi\) over an orbifold \((X, \mathcal{U})\) is defined as follows: \(\xi\) is an orbifold and for \(U \in \mathcal{U}\), \((G^U_x, \tilde{\xi}_U : \tilde{\xi}_U \rightarrow \tilde{U})\) is a \(G^U_x\)-equivariant vector bundle and \((G^U_x, \tilde{\xi}_U)\) (resp. \((G^U_x/K_U, \tilde{U}), K_U = \text{Ker}(G^U_x \rightarrow \text{Diffeo}(\tilde{U}))\) is the orbifold structure of \(\xi\) (resp. \(X\)). In general, \(G^U_x\) does not act effectively on \(\tilde{U}\), i.e. \(K_U \neq \{1\}\). If \(G^U_x\) acts effectively on \(\tilde{U}\) for \(U \in \mathcal{U}\), we say \(\xi\) is a proper orbifold vector bundle.

**Remark:** Let \(G\) be a compact Lie group acting effectively and infinitesimally freely on \(M\). Then each \(G\)-equivariant bundle \(E \rightarrow M\) defines a proper orbifold vector bundle \(E/G \rightarrow M/G\), and vice versa.

In the following, we will always denote by \((G_x, \tilde{U}_x)\) \((x \in X)\) the orbifold chart as above. For \(h \in G_x\), we have the following \(h\)-equivariant decomposition of \(T\tilde{U}_x \otimes_\mathbb{R} \mathbb{C}\) as a real vector bundle on \(\tilde{U}_x^h\),

\[(1.4) \quad T\tilde{U}_x \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{\lambda \in \mathbb{Q}[0,1]} N_{\lambda(h)} \bigoplus T\tilde{U}_x \otimes_\mathbb{R} \mathbb{C}.
\]

Here \(N_{\lambda(h)}\) is the complex vector bundle over \(\tilde{U}_x^h\) with \(h\) acting by \(e^{2\pi i \lambda}\) on it. The complex conjugation provides a \(\mathbb{C}\) anti-linear isomorphism between \(N_{\lambda(h)}\) and \(\overline{N_{(1-\lambda)(h)}}\). If the order of \(h\) is even, this produces a real structure on \(N_{\frac{1}{2}(h)}\), so this bundle is the complexification of a real vector bundle \(N_{\frac{1}{2}(h)}^R\) on \(\tilde{U}_x^h\). Thus, \(T\tilde{U}_x\) is isomorphic, as a real vector bundle, to

\[(1.5) \quad T\tilde{U}_x \simeq \bigoplus_{\lambda \in \mathbb{Q}[0, \frac{1}{2}]} N_{\lambda(h)} \oplus N_{\frac{1}{2}(h)}^R \bigoplus T\tilde{U}_x^h.
\]
Note that $N_{\lambda(h)}$ (resp. $N_{\lambda(h)}^R$) extends to complex (resp. real) vector bundle on $\Sigma X$. We will still denote them by $N_{\lambda(h)}, N_{\lambda(h)}^R$.

Assume that a compact Lie group $H$ acts differentially on $X$. For $\gamma \in H$, let $X^\gamma = \{ x \in X, \gamma x = x \}$. In the index theorem, we will use the following orbifold as fixed point set of $\gamma$ which is a resolution of singularities of $X^\gamma$ [3, p180]. For $x \in X^\gamma$, then on local chart $(G_x, \tilde{U}_x)$, $\gamma \tilde{U}$ acts on $\tilde{U}_x$ as a linear map. The compatibility condition for $\gamma \tilde{U}$ means that there exists an automorphism $\alpha$ of $G_x$ such that for each $g \in G_x$, $\gamma \tilde{U} \circ g \circ \gamma \tilde{U}^{-1} = \alpha(g)$. For $h \in G_x$, let $(h, \gamma) = \{ gh \alpha(g)^{-1}; g \in G_x \}$ be the $\gamma$ conjugacy class in $G_x$. Let

\[(1.6) \quad \tilde{U}_x^{h, \gamma} = \{(y, h_1) \in \tilde{U}_x \times G_x | (h_1 \circ \gamma \tilde{U})(y) = y, h_1 \in (h, \gamma) \}.\]

Let $\tilde{U}_x^{h, \gamma}$ be the fixed point set of $h \circ \gamma \tilde{U}$ in $\tilde{U}_x$, then $\tilde{U}_x^{h, \gamma}$ is connected, and $x \in \tilde{U}_x^{h, \gamma}$.

For $g \in G_x$, $g$ acts on $\tilde{U}_x^{h, \gamma}$ by the transformation \[(y, h) \to (g(y), g \circ h \circ \alpha(g)^{-1}).\]
Indeed, if $(h \circ \gamma \tilde{U})(y) = y$, as $\alpha(g)^{-1} \circ \gamma \tilde{U} = \gamma \tilde{U} \circ g^{-1} \circ \gamma \tilde{U}^{-1} = \gamma \tilde{U} \circ g^{-1}$, we know

\[(1.7) \quad (gh \circ \alpha(g)^{-1}) \gamma \tilde{U} \circ g(y) = gh \circ \gamma \tilde{U}(y) = g(y).\]

Let $Z_{h, G_x}^\gamma = \{ g \in G_x, gh \circ \gamma(g)^{-1} = h \}$, $K_{h, G_x}^\gamma = \text{Ker}\{Z_{h, G_x}^\gamma \rightarrow \text{Diffeo}(\tilde{U}_x^{h, \gamma})\}$. Then

\[(1.8) \quad (Z_{h, G_x}^\gamma / K_{h, G_x}^\gamma, \tilde{U}_x^{h, \gamma}) \to \tilde{U}_x^{h, \gamma} / Z_{h, G_x}^\gamma = \tilde{U}_x^{(h), \gamma} / G_x\]

defines an orbifold. We denote it by $\tilde{X}^\gamma$. Clearly, $m(\tilde{X}^\gamma) = |K_{h, G_x}^\gamma|$ is local constant on $\tilde{X}^\gamma$.

**Definition 1.4** The oriented orbifold $X$ is spin if there exists 2-sheeted covering of $SO(X)$ ($SO(X)$ is the oriented orthonormal frame bundle of $TX$), such that for $U \in U$, there exists a principal $\text{Spin}(n)$ bundle $\text{Spin}(U)$ on $\tilde{U}$, such that $\text{Spin}(X)|_U \rightarrow SO(X)|_U$ is induced by $\text{Spin}(U) \rightarrow SO(U)$, and $\text{Spin}(U)$ also verifies the corresponding compatible condition.

Then $\text{spin}(X)$ is clearly a smooth manifold.

Assume that orbifold $X$ is spin. Let $h^TX$ be a metric on $TX$ and $S(TX) = S^+(TX) \oplus S^-(TX)$ the corresponding orbifold spinor bundle on $X$. Let $c(\cdot)$ be the Clifford action of $TX$ on $S(TX)$. Let $\nabla^{S(TX)}$ be the connection on $S(TX)$ induced by the Levi-Civita connection $\nabla^{TX}$ on $TX$. Let $W$ be a complex orbifold vector bundle on $X$. Let $\nabla^W$ be a connection on $W$. Then $\nabla^{S(TX)} \otimes W = \nabla^{S(TX)} \otimes 1 + 1 \otimes \nabla^W$ is a connection on $X$. Let $\Gamma(S^+(TX) \otimes W)$ be the set of $C^\infty$ sections of $S^+(TX) \otimes W$ on $X$. Let $DX \otimes W$ be the Dirac operator on $\Gamma(S^+(TX) \otimes W)$ to $\Gamma(S^-(TX) \otimes W)$, defined by

\[(1.9) \quad DX \otimes W = c(e_i) \nabla^{S^+(TX)} \otimes W.\]

Here $\{e_i\}$ is an orthonormal basis of $TX$. 


Let $H$ be a compact Lie group. If $\gamma \in H$ acts on $X$ and lifts to Spin$(X)$ and $W$. Then $\nabla^{S(TX)}$ is $\gamma$ invariant and we can always find $\gamma$ invariant connection $\nabla^W$ on $W$. Note that $D^X \otimes W$ is a $\gamma$ invariant elliptic operator on $X$. For $x \in X$, let $K^W_x = \text{Ker}(G^W_x \rightarrow G_x)$. On $\tilde{U}(h)$, let $N$ be the normal bundle of $\tilde{U}h\cdot \gamma_0$ in $\tilde{U}_x$. Let $W^0$ be the subbundle of $W$ on $\tilde{U}_x$ which is $K^W_x$-invariant. Then $W^0$ extends to a proper orbifold vector bundle on $X$. We have the following decompositions:

$$(1.10)\quad N = \oplus_{0<\theta<\pi} N_\theta \oplus N_\pi,$$

$$W^0 = \oplus_{0\leq \theta<2\pi} W_\theta,$$

where $N_\theta, W_\theta$ (resp. $N_\pi$) are complex (resp. real) vector bundle on which $h \circ \gamma_0$ acts as multiplication by $e^{i\theta}$. Then $\nabla^{TX}$ induces connection $\nabla^N_\theta$ on $N_\theta$, and $\nabla^{TX} = \oplus \nabla^N_\theta \oplus \nabla^{TX^\gamma}$. Let $R^W, R^{W^0}, R^N, R^{TX^\gamma}$ be the curvatures of $\nabla^W, \nabla^{W^0}, \nabla^N, \nabla^{TX^\gamma}$ ($\nabla^{W^0}$ is the connection on $W^0$ induced by $\nabla^W$).

**Definition 1.5** For $h \in G_x, g = h \circ \gamma_0$, $0 < \theta \leq \pi$, we write

$$(1.11)\quad \text{ch}_g(W, \nabla^W) = \frac{1}{|K_x|} \sum_{h_1 \in G_x, \tau(h_1) = h} \text{Tr} \left[ (h_1 \circ \gamma_0) \exp\left( \frac{R^W_{U}(\theta)}{2\pi} \right) \right] = \text{Tr} \left[ g \exp\left( \frac{R^{W^0}_{U}(\theta)}{2\pi} \right) \right],$$

$$\hat{A}(T\tilde{U}^g, \nabla^{T\tilde{U}}) = \text{det}^{1/2} \left( \frac{iR^{T\tilde{U}^g}}{\sinh \left( \frac{i}{2\pi} R^{T\tilde{U}^g} \right)} \right),$$

$$\hat{A}_\theta(N_\theta, \nabla^N_\theta) = \frac{i^{\frac{1}{2} \dim N_\theta}}{\text{det}^{1/2} \left( 1 - g \exp\left( \frac{i}{2\pi} R^N_\theta \right) \right)},$$

$$\hat{A}_g(N, \nabla^N) = \prod_{0<\theta_{\leq \pi}} \hat{A}_\theta(N_\theta, \nabla^N_\theta).$$

If we denote by $\{x_j, -x_j\}$ ($j = 1, \cdots, l$) the Chern roots of $N_\theta, T\tilde{U}^g$ (Here we consider $N_\theta$ as a real vector bundle) such that $\Pi x_j$ defines the orientation of $N_\theta$ and $T\tilde{U}^g$, then

$$(1.12)\quad \hat{A}(T\tilde{U}^g, \nabla^{T\tilde{U}}) = \Pi_j \frac{x_j}{\sinh \left( \frac{x_j}{2} \right)},$$

$$\hat{A}_\theta(N_\theta, \nabla^N_\theta) = 2^{-l} \prod_{j=1}^l \frac{1}{\sinh \left( \frac{1}{2} (x_j + i\theta) \right)} = \prod_{j=1}^l \frac{e^{1/2(x_j + i\theta)}}{e^{x_j + i\theta} - 1}.$$

Recall that for $\gamma \in H$, the Lefschetz number $\text{Ind}_\gamma(D^X \otimes W)$, which is the index of $D^X \otimes W$ if $\gamma = 1$, is defined by

$$(1.13)\quad \text{Ind}_\gamma(D^X \otimes W) = \text{Tr} \gamma |_{\text{Ker}D^X \otimes W} - \text{Tr} \gamma |_{\text{Coker}D^X \otimes W}.$$  

By using heat kernel, as in [8] Th. 14.1, we get

**Theorem 1.1** For $\gamma \in H$, we have the following equality:

$$(1.14)\quad \text{Ind}_\gamma(D^X \otimes W) = \sum_{F \in X^\gamma} \frac{1}{m(F)} \int_F \alpha_F,$$

where $\alpha_F$ is the characteristic class

$$\hat{A}(T\tilde{U}h \cdot \gamma_0, \nabla^{T\tilde{U}h \cdot \gamma_0}) \prod_{0<\theta_{\leq \pi}} \hat{A}_\theta(N_\theta, \nabla^N_\theta) \text{ch}_{h \cdot \gamma}(W, \nabla^W)$$

on $\tilde{U}_x^{h \cdot \gamma_0}$.  

5
Let $S^1$ act differentiably on $X$. Let $X^{S^1} = \{ x \in X, \gamma(x) = x, \text{for all} \gamma \in S^1 \}$. Let $V$ be the canonical basis of $\text{Lie}(S^1) = \mathbb{R}$. For $x \in X$, let $V_x$ be the smooth vector field on $(G_x, \tilde{U}_x)$ corresponding to $V$. Then $V_x$ is $G_x$-invariant [8, p181]. We still denote by $V_x$ the corresponding smooth vector field on $X$. We have $X^{S^1} = \{ x \in X, V_x(x) = 0 \}$. 

For $x \in X$, let $(1, \cdots) \cdot (h^j_x, \cdots)$ be the conjugacy classes of $G_x$. Let $\tilde{X}^{S^1} = \{(x, (h^j_x)) | x \in X^{S^1}, h^j_x \in G_x \}$. Then $\tilde{X}^{S^1}$ has a natural orbifold structure defined by

\begin{equation}
(1.15) \{ Z_{G_x}(h^j_x)/K^j_{x,V}, \tilde{U}^h_x = \tilde{U}^h_x \cap \{ y \in \tilde{U}_x | V_x(y) = 0 \} \} \to (\tilde{U}^h_x/Z_{G_x}(h^j_x), (h^j_x)),
\end{equation}

where $K^j_{x,V}$ is the kernel of $Z_{G_x}(h^j_x) \to \text{Diffeo}(\tilde{U}^h_x)$. We have the following decomposition of smooth vector bundles on $\tilde{U}^h_x$:

\begin{align}
N_{\lambda(h)} &= \bigoplus_j N_{\lambda,j}, \\
N_{\mathcal{R},\Pi}^{2j} &= \bigoplus_{j>0} N_{\lambda,j} \oplus N_{\mathcal{R},0}^{2j}, \\
T \tilde{U}^h &= \bigoplus_{j>0} N_{\lambda,j} \oplus T \tilde{U}^h_{\Pi}, \\
W^0 &= \bigoplus_{\lambda,j} W^0_{\lambda,j}.
\end{align}

Note that $N_{\lambda,j}, N_{\mathcal{R},\Pi}^{2j}, N_{0,j}$ and $W^0_{\lambda,j}$ extend to complex vector bundles on $\tilde{X}^{S^1}$, and $\gamma = e^{2\pi it} \in S^1$ acts on them as multiplication by $e^{2\pi ij}$. Also, $N_{\mathcal{R},\Pi}^{2j,0}$ and $T \tilde{U}^h_{\Pi}$ extend to real vector bundles on $\tilde{X}^{S^1}$, and $S^1$ acts on them as identity. In fact, $T \tilde{U}^h = T \tilde{U}^h_{\Pi} \oplus_{\nu \neq 0} N_{0,j,\Pi}$, where $N_{0,j,\Pi}$ denotes the undelang real bundle of the complex vector bundle $N_{0,j,\Pi}$ on which $g \in S^1$ acts by multiplying by $g^\nu$. Since we can choose either $N_{0,j,\Pi}$ as the complex vector bundle for $N_{0,j,\Pi}$, in what follows, we always assume $N_{0,j,\Pi}$ are zero if $j < 0$.

By (1.10), for each $a \in \mathbb{C}$, the eigenspace of $h \circ \gamma_{\tilde{U}}$ with eigenvalue $a$ is equal to the sum of the above elements $N_{\lambda,j}$ such that

\begin{equation}
(1.17) e^{2\pi i(\lambda + tj)} = a.
\end{equation}

Let $A \subset \mathbb{R}$ consist of $a \in \mathbb{R}$ such that there exists $x \in X^{S^1}$, more than one non-zero $N_{\lambda,j}$ on $\tilde{U}^h_{\Pi}$ are in the eigenspace of $h \circ \gamma_{\tilde{U}}$ with eigenvalue $e^{2\pi i a}$. As $X$ is compact, $A$ is a discrete set of $\mathbb{R}$.

If $\gamma = e^{2\pi it}, t \in \mathbb{R} \setminus A$, then $\tilde{X}^\gamma = X^{S^1}$ by the construction. An immediate consequence of Theorem 1.1 is the following.

**Theorem 1.2** Under the condition of Theorem 1.1, for $t \in \mathbb{R} \setminus A$, $\gamma = e^{2\pi it}$, we have

\begin{equation}
(1.18) \text{Ind}_\gamma(D^X \otimes W) = \sum_{F \in \tilde{X}^{S^1}} \frac{1}{m(F)} \int_F \alpha_F,
\end{equation}

where $\alpha_F$ is the characteristic class

\begin{align}
\tilde{A}(T \tilde{U}^h, \nabla T \tilde{U}^h) \sum_{\lambda,j} e^{2\pi i(\lambda + tj)} \text{ch}(W^0_{\lambda,j}, \nabla W^0_{\lambda,j})/\Pi \lambda_j i^{2 \dim N_{\lambda,j}} \det^{1/2}(1 - e^{2\pi i(\lambda + tj)} \exp(\frac{i}{2\pi} R N_{\lambda,j})).
\end{align}

on $\tilde{U}^h_{\Pi}$.
2 Equivariant index theorem for almost complex orbifolds

If $X$ is an almost complex orbifold, then on the orbifold chart $(G_x, \tilde{U}_x)$ for $x \in X$, we have the following $h$-equivariant decomposition of $T\tilde{U}_x$ as complex vector bundles on $\tilde{U}_x$

$$T\tilde{U}_x \simeq \bigoplus_{\lambda \in \mathbb{Q} \cap [0,1]} N_{\lambda(h)}.$$  

(2.1)

Here $N_{\lambda(h)}$ are complex vector bundles over $\tilde{U}_x$ with $h$ acting by $e^{2\pi i\lambda}$ on it, and $N_{0(h)}$ is $T\tilde{U}_x^h$. Again $N_{\lambda(h)}$ extend to complex vector bundles on $\tilde{\Sigma}X$. We will still denote it by $N_{\lambda(h)}$.

Let $F(x, h) = \sum_\lambda \lambda \dim_{\mathbb{C}} N_{\lambda(h)}$ be the fermionic shift, then $F : X \cup \tilde{\Sigma}X \to \mathbb{Q}$ is locally constant. For a connected component $X_i \subset X \cup \tilde{\Sigma}X$, we define $F(X_i)$ to be the values of $F$ on $X_i$.

Let $W$ be an orbifold complex vector bundle on $X$. Let $D^X \otimes W$ be the Spin$^c$ Dirac operator on $\Lambda(T^*(0,1)X) \otimes W$ [16, Appendix D].

Let $H$ be a compact Lie group acting on $X$. We assume that the action $H$ on $X$ lifts on $W$, and preserves the complex structures of $TX$ and $W$. Now for $\gamma \in H$, the decomposition (1.10) on $\hat{U}_x^h\gamma$ also preserves the complex structure of the normal bundle $N$. We denote by $R_N$ the curvature of $\nabla_N$ as complex vector bundle. Then

$$N = \oplus_{0<\theta<2\pi} N_\theta, \quad W^0 = \oplus_{0\leq \theta<2\pi} W_\theta.$$  

(2.2)

Here $N_\theta, W_\theta$ are complex vector bundles on which $h \circ \gamma$ acts as multiplication by $e^{i\theta}$. The following theorem is proved in [8, Th. 14.1].

**Theorem 2.1** Let

$$\text{Td}(T\tilde{U}^{h\circ\gamma\tilde{U}}, \nabla T\tilde{U}^{h\circ\gamma\tilde{U}}) = \det \left( \frac{-R_{T\tilde{U}^{h\circ\gamma\tilde{U}}}/2i\pi}{1 - \exp(-R_{T\tilde{U}^{h\circ\gamma\tilde{U}}}/2i\pi)} \right)$$

be the Chern-Weil Todd form of $T\tilde{U}^{h\circ\gamma\tilde{U}}$. Then we have

$$\text{Ind}_\gamma(D^X \otimes W) = \sum_{F \in X^\gamma} \frac{1}{m(F)} \int_F \alpha_F.$$  

(2.3)

Here on $\tilde{U}^{h\circ\gamma\tilde{U}}, \alpha_F$ is the characteristic class

$$\text{Td}(T\tilde{U}^{h\circ\gamma\tilde{U}}, \nabla T\tilde{U}^{h\circ\gamma\tilde{U}}) \text{ch}_{h\circ\gamma}(W, \nabla W) / \det(1 - (h \circ \gamma) \exp(\frac{i}{2\pi} R_N)).$$

If $H = S^1$, on $\tilde{U}_V^h$ as in (1.13), we have the following decomposition of complex vector bundles,

$$N_{\lambda(h)} = \oplus_j N_{\lambda(h),j}, \quad T\tilde{U}_V^h = \oplus_{0,j} \oplus T\tilde{U}_V^{h,j}.$$  

(2.4)
Here $N_{\lambda(h), j}, N_{h, j}$ extend to complex vector bundles on $\tilde{X}^S_1$, and $\gamma = e^{2\pi it} \in S^1$ acts on them as multiplication by $e^{2\pi ij t}$.

By Theorem 2.1, we get,

**Theorem 2.2** Under the condition of Theorem 2.1, for $t \in \mathbb{R} \setminus A$, $\gamma = e^{2\pi it}$, we have

$$\text{Ind}_\gamma(D^{X} \otimes W) = \sum_{F \in \tilde{X}^S_1} \frac{1}{m(F)} \int_F \alpha_F.$$  \hspace{1cm} (2.5)

Here on $\tilde{U}^h_{V}$, $\alpha_F$ is the characteristic class

$$\text{Td}(T\tilde{U}^h_{V}, \nabla T\tilde{U}^h_{V}) \sum_{\lambda, j} e^{2\pi i(\lambda + tj)} \chi(W^0_{\lambda, j}, \nabla W^0_{\lambda, j})/\Pi \lambda, j \det \left(1 - e^{2\pi i(\lambda + tj)} \exp \left(\frac{i}{2\pi} R_{\lambda, j}^N \right) \right).$$

Note that we can also get Theorems 1.2 and 2.2 from [25, Theorem 1].

### 3 Elliptic genus for almost complex orbifolds

In this section, we define the elliptic genus for a general almost complex orbifold and prove its rigidity property. We are using the setting of Section 2.

For $\tau \in H = \{ \tau \in \mathbb{C}; \text{Im} \tau > 0 \}$, $q = e^{2\pi i \tau}$, $t \in \mathbb{C}$, let

$$\theta(t, \tau) = c(q) q^{1/8} \sin(\pi t) \Pi_{k=1}^\infty (1 - q^k e^{2\pi it}) \Pi_{k=1}^\infty (1 - q^k e^{-2\pi it}).$$  \hspace{1cm} (3.1)

be the classical Jacobi theta function [3], where $c(q) = \Pi_{k=1}^\infty (1 - q^k)$. Set

$$\theta'(0, \tau) = \frac{\partial \theta(\cdot, \tau)}{\partial t} |_{t=0}. $$ \hspace{1cm} (3.2)

Recall the following transformation formulas for the theta-functions [3]:

$$\theta(t + 1, \tau) = -\theta(t, \tau), \quad \theta(t + \tau, \tau) = -q^{-1/2} e^{-2\pi it} \theta(t, \tau),$$

$$\theta(\frac{t}{2}, -\frac{1}{2}) = \frac{1}{1 \sqrt{\tau}} q^{\frac{\pi t^2}{4}} \theta(t, \tau), \quad \theta(t, \tau + 1) = e^{\frac{\pi i}{4}} \theta(t, \tau).$$ \hspace{1cm} (3.3)

For a complex or real vector bundle $F$ on a manifold $X$, let

$$\text{Sym}_q(F) = 1 + q F + q^2 \text{Sym}^2 F + \cdots,$$ \hspace{1cm} (3.4)

$$\Lambda_q(F) = 1 + q F + q^2 \Lambda^2 F + \cdots,$$

be the symmetric and the exterior power operations $F$, respectively.

Let $X$ be an almost complex orbifold, and $\text{dim}_\mathbb{C} X = l$. In this Section, all vector bundles are complex vector bundles. Let $W$ be a proper orbifold complex vector bundle on $X$, and $\text{dim}_\mathbb{C} W = m$. Then $W^0$ in (2.2) is $W$. Now for the vector bundle $W$, the fermionic shift $F(X_i, W) = \sum_\lambda \lambda \dim W_\lambda$ is well define on each connected component $X_i \subset X \cup \tilde{\Sigma} X$. For $x \in X$, $y = e^{2\pi iz}$, we use the orbifold chart $(G_x, \tilde{U}_x)$. For $h \in G_x$, by
(2.4), we define on $\tilde{U}_x^h$,

\begin{equation}
\Theta^z_{q,x,(h)}(TX) = \bigotimes_{\lambda \in Q \cap [0,1]} \left( \bigotimes_{k=1}^{\infty} \left( \Lambda_{-y^{-1}q^{k-1}+\lambda(h)} N^*_\lambda(h) \otimes \Lambda_{-yq^{k-\lambda(h)}} N_{\lambda(h)}(h) \right) \right)
\end{equation}

\begin{equation}
\Theta^z_{q,x,(h)}(TX|W) = \bigotimes_{\lambda \in Q \cap [0,1]} \left( \bigotimes_{k=1}^{\infty} \left( \Lambda_{-y^{-1}q^{k-1}+\lambda(h)} W^*_\lambda(h) \otimes \Lambda_{-yq^{k-\lambda(h)}} W_{\lambda(h)}(h) \right) \right)
\end{equation}

It is easy to verify that each coefficient of $q^a$ ($a \in Q$) in $\Theta^z_{q,x,(h)}(TX)$, $\Theta^z_{q,x,(h)}(TX|W)$ defines an orbifold vector bundle on $\tilde{\Sigma} X$. We will denote it by $\Theta^z_{q,x,i}(TX)$, $\Theta^z_{q,x,i}(TX|W)$ on the connected component $X_i$ of $X \cup \tilde{\Sigma} X$. It is the usual Witten element on $X$ (see [10] and [20]),

\begin{equation}
\Theta^z_q(TX) = \bigotimes_{k=1}^{\infty} \left( \Lambda_{-y^{-1}q^{k-1}} T^* X \otimes \Lambda_{-yq^{k}} T X \right) \bigotimes_{k=1}^{\infty} \left( \text{Sym}_{q^k} T^* X \otimes \text{Sym}_{q^k} T X \right).
\end{equation}

**Definition 3.1** The orbifold elliptic genus of $X$ is defined to be

\begin{equation}
F(y, q) = y^{\frac{1}{2}} \sum_{X_i \subset X \cup \tilde{\Sigma} X} y^{-F(X_i)} \text{Ind}(D^{X_i} \otimes \Theta^z_{q,X_i}(TX)).
\end{equation}

More generally, we define the orbifold elliptic genus associated to $W$ as

\begin{equation}
F(y, q, W) = y^{\frac{m}{2}} \sum_{X_i \subset X \cup \tilde{\Sigma} X} y^{-F(X_i, W)} \text{Ind}(D^{X_i} \otimes \Theta^z_{q,X_i}(TX|W)).
\end{equation}

If $X$ is a global quotient $M/G$ where the action of finite group $G$ on almost complex manifold $M$ preserves its complex structure, the equation (3.7) coincides with [4, Definition 4.1].

We next prove that the orbifold elliptic genus is rigid for $S^1$ action on $X$. 


Let $S^1$ act on $X$, preserving the complex structure on $TX$, and lifting on $W$. Naturally, we define the Lefschetz number for $\gamma \in S^1$,

$$F_\gamma(y, q, W) = y^{\frac{m}{2}} \sum_{X_i \subseteq X} y^{-F(X_i, W)} \text{Ind}_\gamma(D^X_{y, q} \otimes \Theta^g_{q, x_i}(TX|W))$$

(3.9)

Let $P$ be a compact manifold acted infinitesimally freely by a compact Lie group $G$, and $X = P/G$ the corresponding orbifold. We still denote $W$ the corresponding vector bundle on $P$ for $W$. Then $K_X = \det(T^{1,0} X)$, $K_W = \det W$ are naturally induced by complex line bundles on $P$, we still denote it by $K_X$, $K_W$. We may also consider $K_X$, $K_W$ as an orbifold line bundle on $X$.

Recall that the equivariant cohomology group $H^*_S(P, \mathbb{Z})$ of $P$ is defined by

$$H^*_S(P, \mathbb{Z}) = H^*(P \times_{S^1} ES^1, \mathbb{Z}).$$

(3.10)

where $ES^1$ is the universal $S^1$-principal bundle over the classifying space $BS^1$ of $S^1$. So $H^*_S(P, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\pi: P \times S^1 ES^1 \to BS^1$. Let $p_1(W)_{S^1}, p_1(TX)_{S^1} \in H^*_S(P, \mathbb{Z})$ be the equivariant first Pontrjagin classes of $W$ and $TX$ respectively. Also, recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[u]$$

(3.11)

with $u$ a generator of degree 2.

Recall from [7, Theorem B] that for smooth manifold $X$ one needs the conditions

$$p_1(W - TX)_{S^1} = 0, \ c_1(W - TX)_{S^1} = 0.$$  

(3.12)

for the rigidity theorem.

Note that if the connected component $X_i$ of $X \cup \Sigma \overline{X}$ is defined by $(\overline{U}_x, Z_{G_x}(h))$, for $g \in Z_{G_x}(h)$, set $U_{V, V, h} = \{b \in \overline{U}_x | h b = gb, V_X(b) = 0\}$. Then the connected component $X_{ik}$ of $X_{ik}^{S^1}$ is defined by $(U_{V, V, h}, Z_{G_x}(g))$. We have the following decomposition of complex vector bundles on $U_{V, V, h}$,

$$T \overline{U}_x = \sum_{\lambda(h), \lambda(g) \in Q^r[0, 1], \nu \in \mathbb{Z}} N_{\lambda(h), \lambda(g), \nu},$$

$$W = \sum_{\lambda(h), \lambda(g) \in Q^r[0, 1], \nu \in \mathbb{Z}} W_{\lambda(h), \lambda(g), \nu},$$

(3.13)

where $g, h \in G_x$ (resp. $\gamma = e^{2\pi i t} \in S^1$) act on $N_{\lambda(h), \lambda(g), \nu}, W_{\lambda(h), \lambda(g), \nu}$ as multiplication by $e^{2\pi i\lambda(g)}$, $e^{2\pi i\lambda(h)}$ (resp. $e^{2\pi i\nu t}$). Let $2\pi i x_{\lambda(h), \lambda(g), \nu}^j, 2\pi i w_{\lambda(h), \lambda(g), \nu}^j$ be the formal Chern roots of $N_{\lambda(h), \lambda(g), \nu}, W_{\lambda(h), \lambda(g), \nu}$ respectively. To simplify the notation, we will omit the superscript $j$.

Then $N_{\lambda(h), \lambda(g), \nu}, W_{\lambda(h), \lambda(g), \nu}$ extend to orbifold vector bundles on $X_{ik}$. Now the natural generalization of (3.12) for orbifold is the following: there exists $n \in \mathbb{N}$, such that on each connected component $X_{ik}$ of $X_{ik}^{S^1}$,

$$\sum_{\lambda(h), \lambda(g), \nu, \nu, \nu, \nu} \left[ (w_{\lambda(h), \lambda(g), \nu}^j + \lambda(g) - \tau \lambda(h) + vu)^2 - (x_{\lambda(h), \lambda(g), \nu}^j + \lambda(g) - \tau \lambda(h) + vu)^2 \right] = n \pi^2 u^2 \in H^*(X_{ik}, \mathbb{Q})[\tau, t],$$

(3.14)
vector bundle on $\tilde{V}$

As the vector field as in (1.3), the normal bundle $N$ \( (3.18) \)

Ind

Recall that \( (3.16) \)

Let \( (3.17) \)

G take the orbifold chart \((3.11)\)

Theorem 3.1 Assume that \(S^1\) acts on \(P\) which induces the \(S^1\)-action on \(X\), and lifts to \(W\), and \(c_1(W) = 0 \mod N\) in \(H^*(P, \mathbb{Z})\) for some \(1 < N \in \mathbb{N}\). Also assume that there exists \(n \in \mathbb{N}\) such that equations \((3.14)\) and \((3.15)\) hold. Then for any \(N\)-th root of unity \(y = e^{2\pi i z}\) we have

i) If \(n = 0\), then \(F_\gamma(y, q, W)\) is constant on \(\gamma \in S^1\).

ii) If \(n < 0\), then \(F_\gamma(y, q, W) = 0\).

Note that, in case \(W = TX\), the condition \((3.14), (3.15)\) are automatic, and as a consequence we get the rigidity and vanishing theorems for the usual orbifold elliptic genus \(F(y, q)\). In particular we know that for a Calabi-Yau almost complex manifold \(X\), \(F(y, q)\) is rigid for any \(y\).

Proof: Using Theorem 2.2, for \(\gamma = e^{2\pi it}, t \in \mathbb{R} \setminus A\), \(y = e^{2\pi iz}, q = e^{2\pi i \tau}\), we get

\[
(3.16) \quad F_\gamma(y, q, W) = y^{\frac{m}{2}} \sum_{X_i \subset X \cup \Xi X} y^{-F(X_i, W)} \sum_{F \in \mathcal{X}_s^{S^1}} \frac{1}{m(F)} \int_F \alpha_F,
\]

Recall that \(V_X\) is the smooth vector field generated by \(S^1\)-action on \(X\). For \(x \in X\), take the orbifold chart \((G_x, \tilde{U}_x)\). If \(X_i \subset X \cup \Xi X\) is represented by \(\tilde{U}_x^h / ZG_x(h)\) on \(\tilde{U}_x\) as in \((1.3)\), the normal bundle \(N_{X_i, g, V} = N_{\partial U_{V_x}^h / U_{V_x}^h, g}\) of \(U_{V_x}^h, g\) in \(\tilde{U}_x^h\) extends to an orbifold vector bundle on \(\tilde{X}_s^{S^1}\). By Theorem 2.2, the contribution of the chart \((G_x, \tilde{U}_x)\) for \(\text{Ind}_\gamma(DX_i \otimes \Theta^2_{q, X_i}(TX|W))\) is

\[
(3.17) \quad \frac{1}{|ZG_x(h)|} \sum_{gh = h_g, g \in G_x} \int_{U_{V_x}^h} \frac{Td(TU_{V_x}^{h, g}) ch_{g, \gamma}(\Theta^2_{q, X_i, (h)}(TX|W))}{\det(1 - (g \circ \gamma)) e^{-\frac{\pi i}{h} R_{N_{X_i, g, V}}}}. \]

Let

\[
(3.18) \quad N_v = \sum_{\lambda(h), \lambda(g) \in \mathbb{Q} \cap [0, 1]} N_{\lambda(h), \lambda(g), v},
\]

\[
W_v = \sum_{\lambda(h), \lambda(g) \in \mathbb{Q} \cap [0, 1]} W_{\lambda(h), \lambda(g), v}.
\]

As the vector field \(V_X\) commutes with the action of \(G_x\), \(N_v\) and \(W_v\) extend to vector
bundles on $\tilde{X}_i^{S_i}$. So the contribution of the chart $(G_x, \tilde{U}_x)$ for $F_\gamma(y, q, W)$ is

$$\text{(3.19)}$$
$$\frac{1}{|G_x|} \sum_{g_h=g, g, h \in G_x} y^{\frac{m}{2} - F(X_i, W)} \int_{\Gamma_{H^2}} \frac{\text{Td}(TU_{V, h}^g) \Theta_{q, x, (h)}(TX|W)}{\det(1 - (g \circ \gamma) e^{-\frac{1}{2}\pi i} F)} H_{x, g}^h.$$  

$$= \frac{1}{|G_x|} \sum_{g_h=g, g, h \in G_x} \int_{U_{V, h}^g} (2\pi i x_0(0, h, 0)) y^{\frac{m}{2} - F(X_i, W)}$$

$$\Pi_{\lambda(h), v} \prod_{k=1}^{\infty} \frac{1 - e^{-2\pi i (\lambda(h), v)}}{(1 - q^{-1} e^{-2\pi i (\lambda(h), v)})} \Pi_{\lambda(h) > 0, \lambda(g), v}$$

$$\prod_{\lambda(h), v} \prod_{k=1}^{\infty} \frac{1 - e^{-2\pi i (\lambda(h), v)}}{(1 - q^{-1} e^{-2\pi i (\lambda(h), v)})} \Pi_{\lambda(h) > 0, \lambda(g), v}$$

$$= (i^{-1} c(q) q^{1/8})^{l-m} \frac{1}{|G_x|} \sum_{g_h=g, g, h \in G_x} \int_{U_{V, h}^g} (2\pi i x_0(0, h, 0))$$

$$\Pi_{\lambda(h), v} \prod_{k=1}^{\infty} \frac{1 - e^{-2\pi i (\lambda(h), v)}}{(1 - q^{-1} e^{-2\pi i (\lambda(h), v)})} \Pi_{\lambda(h) > 0, \lambda(g), v}$$

To get the last equality of (3.19), we use (3.1), (3.15).

We will consider $F_\gamma(y, q, W)$ as a function of $(t, z, \tau)$, we can extend it to a meromorphic function on $\mathbb{C} \times \mathbb{H} \times \mathbb{C}$. From now on, we denote $(i^{-1} c(q) q^{1/8})^{l-m} \theta(t, z, \tau) F_\gamma(y, q, W)$ by $F(t, z, \tau)$. We also denote by $F_\gamma(t, z, \tau) \tilde{U}_x$ the function defined by (3.19). We set

$$\text{(3.20)}$$
$$F(t, z, \tau) \tilde{U}_x = (i^{-1} c(q) q^{1/8})^{l-m} \theta(t, z, \tau) F_\gamma(t, z, \tau) \tilde{U}_x.$$

Now, the equation (3.14) implies the equalities

$$\text{(3.21)}$$
$$\sum_{\lambda(h), \lambda(g), v} w^2_{\lambda(h), \lambda(g), v} - \sum_{\lambda(h), \lambda(g), v} x^2_{\lambda(h), \lambda(g), v} = 0, \quad \sum_{\lambda(h), \lambda(g), v} w_{\lambda(h), \lambda(g), v} - \sum_{\lambda(h), \lambda(g), v} x_{\lambda(h), \lambda(g), v} = 0,$$

$$\sum_{\lambda(h), \lambda(g), v} \lambda(h) \lambda(g) (\dim W_{\lambda(h), \lambda(g)} - \dim N_{\lambda(h), \lambda(g)}) = 0,$$

$$\sum_{\lambda(h), \lambda(g), v} \lambda(g) v (\dim W_{\lambda(h), \lambda(g), v} - \dim N_{\lambda(h), \lambda(g), v}) = 0,$$

$$\sum_{\lambda(h), \lambda(g), v} \lambda(g)^2 (\dim W_{\lambda(g)} - \dim N_{\lambda(g)}) = 0, \quad \sum_{\lambda(h), \lambda(g), v} v^2 (\dim W_v - \dim N_v) = n.$$

By (3.1), for $a, b \in 2\mathbb{Z}, k \in \mathbb{N},$

$$\text{(3.22)}$$
$$\theta(x + k(t + a \tau + b), \tau) = e^{-\pi i (2kax + 2ka^2 + k^2 a^2 \tau)} \theta(x + kt, \tau).$$

As $c_1(K_X) = 0$ mod $N$ in $H^*(P, \mathbb{Z})$, by the same argument as [10, §8] or [21, Lemma 2.1, Remark 2.6], $\sum_v v \dim N_v$ mod $N$ is constant on each connected component of $X$.

By (3.19), (3.21), (3.22), we know for $a, b \in 2\mathbb{Z},$

$$\text{(3.23)}$$
$$F(t + a \tau + b, \tau, z) = e^{-\pi i z \sum_v v \dim N_v} F(t, \tau, z).$$
For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we define its modular transformation on $\mathbb{C} \times H$ by

\begin{equation}
A(t, \tau) = \left( \begin{array}{c}
t \\
\frac{a\tau + b}{c\tau + d} \\
\end{array} \right), \tag{3.24}
\end{equation}

By (3.19), under the action $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

\begin{equation}
\begin{aligned}
\theta(A(t_1, \tau)) &= e^{\pi iz(t_1^2 - t^2) / c\tau + d} \theta(t_1, \tau) \\
\theta(A(t_2, \tau)) &= \theta(t_2, \tau), \\
\theta'(A(0, \tau)) &= (c\tau + d) e^{-\pi iz^2 / c\tau + d} \theta'(0, \tau)
\end{aligned} \tag{3.25}
\end{equation}

For $g, h \in G_x$, by looking at the degree 2 $\dim_{\mathbb{C}} U_{U_V}^{h,g}$ part, that is the $\dim_{\mathbb{C}} U_{U_V}^{h,g}$-th homogeneous terms of the polynomials in $x, w$'s, on both sides of the following equation, we get

\begin{equation}
\begin{aligned}
&\int_{U_{U_V}^{h,g}} (x_{0(h), 0(g), 0}) \Pi_{\lambda(h), \lambda(g), \nu} e^{2\pi i cz\lambda(h), \lambda(g), \nu(c\tau + d)} \\
&\theta(w_{\lambda(h), \lambda(g), \nu} (c\tau + d) + \lambda(g)(c\tau + d) - (a\tau + b)\lambda(h) + z(c\tau + d) + tv, \tau)
\end{aligned} \tag{3.26}
\end{equation}

By (3.3), (3.19), (3.21), (3.25) and (3.26), we easily derive the following identity:

\begin{equation}
\begin{aligned}
F(A(t, \tau), z)_{G_x} &= \frac{1}{|G_x|} \left( c\tau + d \right)^l e^{\pi icm^2 / (c\tau + d)} \frac{\theta'(0, \tau)^l}{\theta(z(c\tau + d), \tau)^m} \\
&\sum_{g\bar{h} = hg, \bar{h} \in G_x} \int_{U_{U_V}^{h,\bar{h}}} (2\pi i x_{0(h), 0(g), 0}) \\
&\times \left\{ \Pi_{\lambda(h), \lambda(g), \nu} e^{2\pi i cz\lambda(h), \lambda(g), \nu + (\lambda(g) - (a\tau + b)\lambda(h)) (c\tau + d) + tv) e^{-2\pi i z\lambda(h)}} \right\} \\
&\times \Pi_{\lambda(h), \lambda(g), \nu} \theta(w_{\lambda(h), \lambda(g), \nu} + \lambda(g)(c\tau + d) - (a\tau + b)\lambda(h) + z(c\tau + d) + tv, \tau)
\end{aligned} \tag{3.27}
\end{equation}
By (3.3), (3.15), (3.21), (3.22), (3.27), we have

(3.28)

\[ (c\tau + d)^{-l}e^{-\pi icnt^2/(c\tau + d)} \frac{\theta(z(c\tau + d), \tau)^m}{\theta'(0, \tau)^l} F(A(t, \tau), z) \tilde{\nu}_z \]

\[ = \frac{1}{|G_\delta|} \sum_{gh=g.h, \tau \in G_\delta} \int_{U^{h,g}_V} (2\pi i \chi(0,0,0)) \]

\[ \Pi_{\lambda(h), \lambda(g), v, j} \left\{ e^{2\pi ic\tau(d\lambda(g) - b\lambda(h) + \tau(c\lambda(g) - a\lambda(h)))} e^{2\pi ic\tau(w_{\lambda(h), \lambda(g), v} + tv)} \right\} \]

\[ \times e^{-2\pi ic\tau(c\lambda(g) - a\lambda(h) + \lambda(g\tau h^a))} ((c\tau + d) e^{-2\pi ic\tau h\lambda(h)}) \]

\[ \times \frac{\Pi_{\lambda(h), \lambda(g), v} \theta(w_{\lambda(h), \lambda(g), v} + \lambda(g\tau h^b) - \tau \lambda(g\tau h^a) + z(c\tau + d) + tv, \tau)}{\Pi_{\lambda(h), \lambda(g), v} \theta(x_{\lambda(h), \lambda(g), v} + \lambda(g\tau h^b) - \tau \lambda(g\tau h^a) + tv, \tau)} \]

\[ = \frac{1}{|G_\delta|} \sum_{gh=g.h, \tau \in G_\delta} \int_{U^{h,g}_V} (2\pi i \chi(0,0,0)) \Pi_{\lambda(h), \lambda(g), v} \left( e^{2\pi ic\tau(w_{\lambda(h), \lambda(g), v} + tv)} e^{-2\pi ic\tau(c\tau + d) \lambda(g\tau h^a)} \right) \]

\[ \times \frac{\Pi_{\lambda(h), \lambda(g), v} \theta(w_{\lambda(h), \lambda(g), v} + \lambda(g\tau h^b) - \tau \lambda(g\tau h^a) + z(c\tau + d) + tv, \tau)}{\Pi_{\lambda(h), \lambda(g), v} \theta(x_{\lambda(h), \lambda(g), v} + \lambda(g\tau h^b) - \tau \lambda(g\tau h^a) + tv, \tau)} \].

Recall that \( c_1(K_W) \equiv 0 \mod N \) where \( K_W = \det W \), this implies that the line bundle \( K_W^{cz} \) is well defined on \( P \), also as an orbifold vector bundle on \( X \). Let

(3.29)

\[ F^A(t, \tau, z) = (i^{-1}c(q)q^{1/8})^{m-l} \frac{\theta'(0, \tau)^l}{\theta(z(c\tau + d), \tau)^m} \sum_{X_i \subset X \cup \tilde{X}} g^\frac{m}{2} F(X_i, W) \text{Ind}_i(D \otimes K_0^{cz} \otimes \Theta_{q, X_i}^{(c\tau + d)z}(TX|W)). \]

Now, observe that as \( A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) \( \in SL_2(\mathbb{Z}) \), when \( g, h \) run through all pair of \( \sum_{gh=g.h, \tau \in G_\delta} \), then \( g^{-c}h^a, g^d h^b \) run through all pair of \( \sum_{gh=g.h, \tau \in G_\delta} \). Then by (3.28), (3.29)

(3.30)

\[ F(A(t, \tau), z) = (c\tau + d)^l e^{\pi icnt^2/(c\tau + d)} F^A(t, \tau, z). \]

The following lemma implies that the index theory comes in to cancel part of the poles of the functions \( F \).

**Lemma 3.1** The function \( F^A(t, \tau, z) \) is holomorphic in \( (t, \tau) \) for \( (t, \tau) \in \mathbb{R} \times H \).

The proof of the above Lemma is the same as the proof of [17, Lemma 1.3] or [13, Lemma 2.3].

Now, we return to the proof of Theorem 3.1. Note that the possible polar divisors of \( F \) in \( C \times H \) are of the form \( t = \frac{k}{j}(c\tau + d) \) with \( k, c, d, j \) integers and \( (c, d) = 1 \) or \( c = 1 \) and \( d = 0 \).
We can always find integers $a, b$ such that $ad - bc = 1$, and consider the matrix
\[ A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{Z}). \]

(3.31) \[ F^A(t, \tau, z) = (-c\tau + a)\cdot t e^{\pi i c t^2 / (-c\tau + a)} F\left(A(t, \tau), z\right) \]

Now, if $t = \frac{k}{j}(c\tau + d)$ is a polar divisor of $F(t, \tau, z)$, then one polar divisor of $F^A(t, \tau, z)$ is given by
\[ \frac{t}{-c\tau + a} = \frac{k}{j} \left(\frac{c}{-c\tau + a} + d\right), \]
which exactly gives $t = k/j$. This contradicts Lemma 3.3, and completes the proof of Theorem 3.1.

\[ \square \]

4 Elliptic genus for spin orbifolds

We are following the setting of Section 1.

For $\tau \in \mathbb{H} = \{ \tau \in \mathbb{C}; \text{Im}\tau > 0 \}$, $q = e^{2\pi i \tau}$, let
\[ \theta_3(v, \tau) = c(q)\Pi_{k=1}^{\infty} (1 + q^{k-1/2}e^{2\pi i v}) \Pi_{k=1}^{\infty} (1 + q^{k-1/2}e^{-2\pi i v}). \]

(4.1) be the other three classical Jacobi theta-functions \[ 3 \], where $c(q) = \Pi_{k=1}^{\infty} (1 - q^k)$. Let $X$ be a compact orbifold, $\text{dim}_{\mathbb{R}} X = 2n$. We assume that $X$ and $\Sigma X$ are spin in the sense of Definition 1.4. For $x \in X$, taking the orbifold chart $(G_x, \tilde{U}_x)$. By (1.4), for $h \in G_x$, we define on $\tilde{U}_x^h$
\[ \Theta'_{q,x,(h)}(TX) = \otimes_{\lambda \in \mathbb{Q} \cap [0, 1]} \left( \otimes_{k=1}^{\infty} \left( \Lambda_{q^{k-1+\lambda(h)}}N_{\lambda(h)}^* \otimes \Lambda_{q^{k-\lambda(h)}}N_{\lambda(h)} \right) \right) \]
\[ \otimes_{k=1}^{\infty} \left( \text{Sym}_{q^{k-1+\lambda(h)}}N_{\lambda(h)}^* \otimes \text{Sym}_{q^{k-\lambda(h)}}N_{\lambda(h)} \right) \otimes \otimes_{k=1}^{\infty} \left( \Lambda_{q^k}(T\tilde{U}_x^h) \otimes \text{Sym}_{q^k}(T\tilde{U}_x^h) \right). \]

(4.2) It is easy to verify that each coefficient of $q^a \ (a \in \mathbb{Q})$ in $\Theta'_{q,x,(h)}(TX)$ defines an orbifold vector bundle on $X \cup \Sigma X$. We denote as $\Theta'_{q,x}(TX)$ on the connected component $X_i$ of $X \cup \Sigma X$. Especially, $\Theta'_{q,x}(TX)$ is the usual Witten elements on $X$
\[ \Theta'_q(TX) = \otimes_{k=1}^{\infty} \left( \Lambda_{q^k}(TX) \otimes \text{Sym}_{q^k}(TX) \right). \]

(4.3) We propose the following definition for the elliptic genus on spin orbifold:

Definition 4.1 The orbifold elliptic genus of $X$ is
(4.4) \[ F(q) = \sum_{X_i \subset X \cup \Sigma X} \text{Ind} \left( D^{X_i} \otimes (S^+(TX_i) \oplus S^-(TX_i)) \otimes \Theta'_{q,x_i}(TX) \right). \]
Let $S^1$ act on $X$ and preserve the spin structure of $X \cup \Sigma X$. Naturally, we define the Lefschetz number for $\gamma \in S^1$

$$F_{d_\gamma}(q) = \sum_{x_i \in X \cup \Sigma X} \text{Ind}_\gamma \left( DX_i \otimes (S^+(TX_i) \oplus S^-(TX_i)) \otimes \Theta_{q,x_i}(TX) \right)$$

On local chart $(G_x, \tilde{U}_x)$, for $h, g \in G_x$, $gh = hg$, then as in (4.5), on $\tilde{U}^h_x$, we have

$$T\tilde{U}_x = N_0 \oplus \lambda(h) \in [0, \frac{1}{2}] N_0(h) \oplus N^{\mathbb{R}}_{\frac{1}{2}(h)}.$$ 

here $h$ acts on the real vector bundles $N_0 = T\tilde{U}^h_x$, $N^{\mathbb{R}}_{\frac{1}{2}(h)}$ as multiplication by $1, e^{\pi i}$. $h$ acts on complex vector bundles $N_{\lambda(h)}$ as multiplication by $e^{2\pi i \lambda(h)}$. Now on $\tilde{U}^{h,g}_x$, the fixed point set of $g$ on $\tilde{U}^h_x$, we have the following decomposition

$$N_{\lambda(h)} = \oplus_{\lambda(g) \in [0, \frac{1}{2}]} N_{\lambda(h), 0} \oplus \lambda(g) \in [0, \frac{1}{2}] \left[ N_{\lambda(h), 1 - \lambda(g)} \right]$$

for $0 < \lambda(h) < \frac{1}{2}$.

$$N_0 = \oplus_{\lambda(g) \in [0, \frac{1}{2}]} N_0(h) = \oplus_{\lambda(g) \in [0, \frac{1}{2}]} N^\mathbb{R}_{\frac{1}{2}(h), \lambda(g)}.$$ 

Here $N_{\lambda(h), \lambda(g)}$ ($\lambda(h), \lambda(g) \in \{0, \frac{1}{2}\}$) are real vector bundles on $\tilde{U}^{h,g}_x$. And $N_{\lambda(h), \lambda(g)}$ ($\lambda(h)$ or $\lambda(g)$ not in $\{0, \frac{1}{2}\}$) are complex vector bundles on $\tilde{U}^{h,g}_x$. $h, g$ act on $N_{\lambda(h), \lambda(g)}$ as multiplication by $e^{2\pi i \lambda(h)}$, $e^{2\pi i \lambda(g)}$ respectively. Again $N_{\lambda(h), \lambda(g)}$ extends to a vector bundle on $\tilde{X}_i^{S^1}$.

For $f(x)$ a holomorphic function, we denote by $f(x_{\lambda(h), \lambda(g)})(N_{\lambda(h), \lambda(g)}) = \Pi_j f(x_{\lambda(h), \lambda(g), j})$, the symmetric polynomial which gives characteristic class of $N_{\lambda(h), \lambda(g)}$. we get that the contribution of the chart $(G_x, \tilde{U}_x)$ for $F_{d_\gamma}(q)$ is

$$F_{d_\gamma}(t, \tau) = \frac{i^{-n}}{\pi \sqrt{\det G_x}} \sum_{gh = hg, g \in G_x} \int_{\tilde{U}^{h,g}_x} \left( 2\pi i x_{0(h), 0(g)} \theta_1(x_{0(h), 0(g)}, \tau) \right) (N_{0(h), 0(g)})$$

$$\Pi_{\lambda(h) \in (0, \frac{1}{2}]} \theta_1(x_{\lambda(h), \lambda(h)} + \lambda(g) - \tau \lambda(h), \tau) (N_{\lambda(h), \lambda(g)})$$

$$\Pi_{0 < \lambda(h) < \frac{1}{2}} \theta_1(x_{\lambda(h), \lambda(h)} + \lambda(g) - \tau \lambda(h), \tau) (N_{\lambda(h), \lambda(g)})$$

We plan to return to the study of their rigidity and vanishing properties on a later occasion.

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