Nominal Logic and Abstract Syntax

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1 Introduction

A great deal of research in programming languages, type theory, and security is based on proving properties such as strong normalization, type soundness or noninterference by induction or co-induction on the structure of typing derivations, operational semantics rules, or other syntactic constructs. Such proofs are essentially combinatorial in nature, usually involving \(O(n^p)\) cases, where \(n\) is the number of syntactic constructs, typing rules, operational transitions, etc., and \(p\) is small. Usually, only a small number of cases are “interesting”, and published proofs often give only a few illustrative cases. This provides little assurance that a proof is correct. It is widely felt that machine assistance for constructing such proofs is desirable [60, 3]. Providing such assistance is severely complicated by the problem of dealing with names and binding in abstract syntax.

Logicians since Frege have grappled with the problem of dealing with the syntax of logical expressions. In mathematical logic textbooks it is not unusual to see a formal definition of the concrete syntax of the object-language as a particular set of strings over an alphabet including not only variables, function and predicate symbols, and logical connectives, but also punctuation such as parentheses, brackets, and commas. Then various technical lemmas such as the fact that a string may represent at most one formula, that parentheses match, etc. are proved, along with structural

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induction principles. These results are necessary to show that the use of strings and punctuation to represent formulas satisfies our intuitions about the “real” structure of formulas.

However, we now have both a highly developed theory and advanced programming techniques for taking care of these syntactic details automatically; many high-level programming languages (such as Prolog and ML) provide advanced features for parsing strings into abstract syntax trees and computing with the results. Nowadays it is more common (and by far more agreeable) to specify a logical or mathematical language using abstract syntax, that is, as a set of abstract syntax trees defined by an inductive construction. This has the positive effect of isolating the low-level technical details of parsing from the high-level hierarchical representations of terms which are most convenient for reasoning. Moreover, the theory of abstract syntax trees, term languages, etc., is now well-understood, so that a large number of definitions and results are standard and can be reused for any term language (and are usually taken for granted).

While this approach works well for languages with variables, constants, and operator symbols, it does not work so well as soon as variable binding enters the picture. This is because variable binding and substitution interact in complex ways. Subtle errors can arise if care is not taken with definitions; this problem has plagued both famous logicians\(^2\) and well-known programming languages.\(^3\)

It is standard practice in mathematical logic to assume that there is some infinite set \(V\) of variable symbols, for example integers or strings, and to treat binding term constructors as ordinary function symbols taking variables as arguments or parameters. For example, the \(\forall\) symbol in a universally quantified formula \(\forall x.\varphi\) may be viewed, from a syntactic point of view, as a binary function symbol \(\text{forall} : V \times \text{Prop} \to \text{Prop}\), or as a family of unary formula constructors \((\forall x : \text{Prop} \to \text{Prop} \mid x \in V)\) taking a formula as an argument. If one of these approaches is employed, then a number of basic syntactic definitions and results (such as the “alphabetic variance” or \(\alpha\)-equivalence relation, capture-avoiding substitution function and related lemmas) again need to be proved in order to establish that this approach to implementing binding matches our intuitive understanding.

Once these low-level details of binding have been presented and proved correct, mathematical rigor is usually reserved for high-level issues, and low-level syntactic and binding issues are left implicit. For example, the Barendregt Variable Convention is often taken for granted in mathematical exposition. After defining capture-avoiding substitution, renaming, and \(\alpha\)-equivalence and proving their properties in detail, Barendregt states:

2.1.13. VARIABLE CONVENTION. If \(M_1, \ldots, M_n\) occur in a certain mathematical context (e.g., definition, proof), then in these terms all bound variables are chosen to be different from free variables.\(^5\)

At best, paper definitions or proofs signal the use of such a convention with a “without loss of generality”; at worst, the convention (and the argument that its use is sound) is implicit. While clear enough for human readers, however, such conventions still leave a considerable gap between mathematical exposition and correct formalizations or computer implementations of programming languages and logics involving names and binding \(^4\). \(^5\)

In this column, I will first survey the state of the art for solving these problems, and then present a new approach called nominal abstract syntax that has gained popularity since its introduction six years ago by Gabbay and Pitts. I will then discuss applications and future directions for this work.

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\(^2\)including Hilbert and Ackermann \(^12\) among others, according to Stoy \(^19\)

\(^3\)for example, LISP’s well-known “dynamic variable scoping” bug \(^6\)
2 Approaches to dealing with names and binding

2.1 First-order abstract syntax

In programming, abstract syntax with binding is usually implemented using ordinary abstract syntax techniques, and then defining capture-avoiding renaming/substitution and $\alpha$-equivalence explicitly; that is, by making all the implicit syntactic manipulations explicit. This first-order abstract syntax approach requires explicit management of fresh name generation (e.g. using a side-effecting gensym function) as well as writing a lot of repetitive “boilerplate” code. Despite these drawbacks, it is by far the most popular technique for real compilers, interpreters, theorem provers, and other symbolic programs. However, reasoning about languages defined in this way is considered by most experts in this area to be impractical for all but the simplest examples, because of the large number of intermediate reasoning steps, renaming and substitution lemmas, etc. that must be verified.

2.2 Name-free approaches

Another popular technique for managing abstract syntax with binding is to use a name-free notation for functions. Name-free approaches have a long history, beginning with Schönfinkel’s development of combinatory logic in the 1920s [71], and have had considerable influence on both theory and practice of logic and programming. Schönfinkel [71] and later Curry and Feys [19, 20] developed combinatory logic, a logic of applicative expressions defined using rewriting rules. In combinatory logic, a $\lambda$-term such as $\lambda x.\lambda y.\lambda z.\lambda w.\lambda u.(x y)(z w)$ can be expressed as the combinator expression $S(S(KS)(S(KK)I))(KI)$; here $S$, $K$, and $I$ are basic functional expressions with the same meaning as $\lambda xyz.(xz)(yz)$, $\lambda xy.x$, and $\lambda x.x$, respectively. N. G. de Bruijn [21] proposed two encodings (often called de Bruijn indices and de Bruijn levels) for the $\lambda$-calculus which neatly circumvent the difficulties arising from $\alpha$-equivalence by representing variables as integer references (or pointers) to their binding sites. For example, the de Bruijn index version of both $\lambda x.\lambda y.\lambda z.\lambda w.\lambda u.(x y)(z w)$ is $\lambda\lambda 21$. Thus, $\alpha$-equivalence collapses to syntactic equality. Stoy diagrams [79] are a graphical representation of $\lambda$-terms often used to explain variable binding; an example is shown in Figure 1.

Combinators and de Bruijn index representations are powerful and useful ideas; the former serves as the basis for efficient functional language implementations [18], while the latter has been used as an efficient internal representation in many theorem provers (beginning with de Bruijn’s AUTOMATH) and in efficient functional programming implementation techniques such as explicit substitutions [4]. Nevertheless, combinators can increase the size of an expression exponentially, and neither encoding is human-readable so neither is well suited for high-level programming or reasoning tasks. Moreover, name-free approaches by definition do not provide any assistance for dealing with free names.
2.3 Higher-order abstract syntax

An elegant alternative is to employ higher-order abstract syntax (HOAS) [61]. In this technique, we work with an enriched meta-language that provides some form of binding (such as the typed lambda-calculus). Variables and binding term constructors at the object level are encoded as variables and higher-order constants in the metalanguage. This is best illustrated by an example. Using higher-order abstract syntax, a quantified formula like $\forall x : \mathbb{N}. P(x)$ would be encoded as $\text{forall} (\lambda x. P x)$, where $\text{forall} : (\mathbb{N} \rightarrow \text{Prop}) \rightarrow \text{Prop}$, $P : \mathbb{N} \rightarrow \text{Prop}$. This powerful idea, first used in Church’s higher-order logic [17], is used in many advanced programming languages and logical frameworks (e.g. λProlog [57] and Twelf [62] among others [7, 22]).

However, in my opinion, higher-order abstract syntax is not without drawbacks, both from the point of view of programming with and reasoning about languages involving binding. The problems can be broken down into five areas:

1. Higher-order abstract syntax is based on complex semantic and algorithmic foundations (higher-order logic [17], recursive domain equations [77], higher-order unification [46]) so requires a fair amount of ingenuity to learn, implement and analyze [27, 43, 74].

2. Properties of the metalanguage (such as weakening and substitution lemmas) are inherited by object languages, whether or not this is desirable; this necessitates modifications to handle logics and languages with unusual behavior. Examples include linear logic [10] and concurrency [80].

3. Variable names are “second class citizens”; they only represent other object-language expressions and have no existence or meaning outside of their scope. This complicates formalizing languages with generativity (for example, datatype names in ML), program logics with mutable variables such as Hoare logic [51] or dynamic logic [44] and translations such as closure conversion that rely on the ability to test for name-equality.

4. Higher-order language encodings are often so different from their informal “paper” presentations that proving “adequacy” (that is, equivalence of the encoding and the real language) is nontrivial, and elegant-looking encodings can be incorrect for subtle reasons. Hannan [39] developed and proved partial correctness of a closure conversion translation in LF, but did not prove adequacy of the encoding; careful inspection suggests that it is not adequate. Abel [2] investigated an elegant and natural-seeming but inadequate third-order and less elegant but adequate second-order HOAS encoding of the $\lambda\mu$-calculus.

5. Higher-order abstract syntax is less expressive than first-order abstract syntax: it apparently cannot deal with situations involving “open terms” mentioning an indefinite number of free variables. For example, HOAS apparently cannot model the behavior of ML-style let-bound polymorphism as usually implemented [11], though a simulation is possible [10].

To be fair, when it works, higher-order abstract syntax is highly satisfying and clearly superior to first-order abstract syntax, and research on higher-order abstract syntax has shown that many of these problems can be alleviated. In addition, any of these properties can also be seen in a more positive light:

I. Higher-order abstract syntax is based on powerful and elegant semantic and algorithmic foundations (higher-order logic, recursive domain equations, higher-order unification) involving deep ideas of computer science and logic.
II. Properties of the metalanguage (such as weakening and substitution lemmas) are inherited by object languages, thus saving a lot of work re-proving them for typical languages.

III. Programmers do not have to painstakingly re-implement efficient fresh name generation, α-equivalence, or capture-avoiding substitution operations, freeing them to focus on high-level problems.

IV. Higher-order language encodings encourage “refactoring” object-languages in a way that makes all variable binding explicit; the results are often much more elegant and uniform than “paper” versions.

V. Higher-order abstract syntax encourages “one binding at a time” definitions and proofs, and discourages complicated (and frequently unnecessary) reasoning about open terms.

I am not arguing that points (1–5) are right and (I–V) are wrong (or vice versa); both views have merit. Anyone interested in formalizing programming languages or logics should consider higher-order abstract syntax, because it enjoys several mature implementations such as Twelf and λProlog. Nevertheless, I believe it is worthwhile to investigate alternatives.

3 Nominal abstract syntax

Recently, Gabbay and Pitts [35, 36] have developed an alternative approach to abstract syntax with binding. This approach is based on the idea of taking names to be an abstract but first-class data type, and name-binding (or name-abstraction) to be an abstract data type construction involving the type of names and an arbitrary type. Thus, as in first-order abstract syntax, names denote semantic values, and are not just syntactic entities; on the other hand, like higher-order abstract syntax, access to the internal representations of names and name-binding operations is restricted so that low-level implementation issues are separated from high-level concerns. I call this approach nominal abstract syntax, because of its focus on names.

The key technical insight in nominal abstract syntax is the fact that one-to-one (or, equivalently, invertible) renamings can play a central role in explicating name-binding (and, in fact, many other uses of names). At an intuitive level, the reason is that in many situations, the only properties of names that are of interest are equality/inequality among names and freshness, that is, the property that a name does not appear “free” in a term. Non-invertible capture-avoiding renamings do preserve equality (e.g., $t = u$ implies $t[x/y] = u[x/y]$) but may not preserve inequality or freshness. For example, $x \neq y$ but $x[x/y] = x = y[x/y]$, and $x \notin \text{FV}(\lambda x.f \; x \; y)$ but $x[x/y] = x = \text{FV}(\lambda x'.f \; x' \; x) = \text{FV}((\lambda x.f \; x)[x/y])$. Invertible renamings, on the other hand, preserve all of these properties: for example, writing $(-)(x\leftrightarrow y)$ for simultaneous capture-avoiding substitution of $x$ for $y$ and $y$ for $x$, we have $x(x\leftrightarrow y) = y \neq x = y(x\leftrightarrow y)$ and $x(x\leftrightarrow y) = y \notin \text{FV}(\lambda x'.f \; x' \; x) = \text{FV}((\lambda x.f \; x \; y)(x\leftrightarrow y))$.

Although an approach based on invertible renamings may seem unnatural to modern logicians and computer scientists, Gabbay and Pitts were not the first to recognize the importance of invertible renamings. Permutations are frequently used to define the equivalence class of alphabetic variants of an object (e.g., in logic programming, two logic program clauses or unifiers which differ only by a permutation of variables are equivalent; graphs and automata are considered equivalent up to invertible renaming of state names). Also, in a formalization of the λ-calculus, McKinna and Pollack [52] identified invertible renamings as an important concept that can be used to define α-equivalence.

In fact, the basic idea of using one-to-one renamings to understand name-binding dates to Frege’s Begriffsschrift, the first systematic treatment of symbolic predicate logic. In Frege’s work,
bound names were syntactically distinguished from unbound names: the former were written using German letters \(a, b, c\), and the latter using Roman (italic) letters \(x, y, z\). For Frege, bound names were subject to renaming using the following principle:

\[\ldots\text{Replacing a German letter everywhere in its scope by some other one is, of course, permitted, so long as in places where different letters initially stood different ones also stand afterward. This has no effect on the content.}\]

On the other hand, Frege did not give a completely explicit formal treatment of substitution, and as a result much of the complexity resulting from the interaction of substitution and name-binding was hidden.

Gabbay and Pitts’ original approach was based on ideas from Fraenkel-Mostowski permutation models of set theory (FM set theory) \[28, 23, 47\], originally developed as an early attempt to prove the independence of the Axiom of Choice from ZF-set theory. In fact, Gabbay and Pitts’ work was carried out in a form of FM set theory that does not satisfy the Axiom of Choice. While this is a very interesting approach, it has led many observers to believe that nonstandard FM set theory and rejection of the Axiom of Choice is necessary for working with nominal abstract syntax, not just sufficient.

This is not the case; Pitts \[63\] showed that the basic principles of nominal abstract syntax can be formalized as nominal logic, an extension of typed first-order equational logic that can be analyzed within ZFC just like any other logic. In this approach, there is no conflict with the Axiom of Choice or the mainstream foundations of mathematics. On the other hand, nominal logic lacks some of the Choice-like properties of first-order logic (such as unrestricted Skolemization), but this is not a foundational problem.

In the rest of this section I will provide a brief overview of nominal abstract syntax and nominal logic, followed by an example of reasoning in nominal logic. Much more detail can be found in the papers \[63, 81, 34, 15\].

### 3.1 Nominal logic

The key ingredients of nominal logic are:

- a syntactic class of names \(a, b, \ldots \in A\), partitioned into name-types \(\nu, \nu', \ldots\),
- a swapping operation \((a b) t\) that swaps two names of the same type within an value,
- a freshness relation \(a \# t\) that relates a name to a value when the name does not appear “free” in the value,
- an abstraction operation \(\langle a \rangle t\) that binds a name within an value, and admits equality up to \(\alpha\)-equivalence, and
- a self-dual new-quantifier \(\forall a: \nu. \varphi\) that quantifies over all fresh names (equivalently, some fresh name) of type \(\nu\).

In addition, nominal logic embodies two key logical principles:

- **Fresh name generation.** A name fresh for any value (or for each of finitely many values) can always be found.
- **Equivariance.** Relations are invariant up to swapping; the choice of particular names in a formula is irrelevant.
The syntax of nominal logic is as follows:

(Forms of terms) \( t \) ::= \( a \) | \( c \) | \( f(t) \) | \( x \) | \( \langle a \rangle t \)  

(Atomic formulas) \( A \) ::= \( p(t) \) | \( t \approx u \) | \( a \neq t \)  

(Formulas) \( \varphi \) ::= \( A \) | \( \varphi \lor \psi \) | \( \bot \) | \( \forall x: \sigma. \varphi \) | \( \forall a: \nu. \varphi \)

A language \( L \) consists of a set of data types \( \delta \), name types \( \nu \), constants \( c : \delta \), function symbols \( f : \delta \to \delta \), and relation symbols \( p : \nu \to o \). Well-formed terms and atomic formulas are defined as follows:

$$
\begin{align*}
\Sigma \vdash a : \nu & \quad \Sigma \vdash b : \nu & \quad \Sigma \vdash \langle a \rangle t : (\nu)\sigma \\
\Sigma \vdash c : \delta & \quad \Sigma \vdash f(t) : \delta & \quad \Sigma \vdash \langle a \rangle t : (\nu)\sigma \\
\Sigma \vdash c : \delta & \quad \Sigma \vdash t : \sigma & \quad \Sigma \vdash u : \sigma \\
\Sigma \vdash a : \nu & \quad \Sigma \vdash t : \sigma & \quad \Sigma \vdash \langle a \rangle t : o \\
\Sigma \vdash a : \nu & \quad \Sigma \vdash a \neq t & \quad \Sigma \vdash p(t) : o 
\end{align*}
$$

Here we write \( o \) for the type of propositions; however, quantification over types mentioning \( o \) is not allowed. Well-formedness for the first-order connectives are defined in the usual way; well-formedness for \( \forall a: \nu. \varphi \) is defined as for \( \forall a: \nu. \varphi \). Other formulas such as truth \( \top \), conjunction \( \varphi \land \psi \), disjunction \( \varphi \lor \psi \), logical equivalence \( \varphi \iff \psi \), and existential quantification \( \exists x: \sigma. \varphi \) are defined as usual in classical logic.

While ordinary capture-avoiding renaming/substitution needs to be defined carefully to prevent variable capture, invertible renamings can be defined by a simple structural induction:

\[
\begin{align*}
(a \ b) \cdot a &= b \\
(a \ b) \cdot b &= a \\
(a \ b) \cdot a' &= a' \quad (a \neq a' \neq b) \\
(a \ b) \cdot (\langle a \rangle b) &= \langle (a \ b) \cdot a \rangle (a \ b) \cdot b
\end{align*}
\]

Note that the names appearing in “binding” positions of abstractions are not \( \alpha \)-renamed prior to applying a swapping; instead, the swapping is applied to both the name and body. This would be incorrect for a non-invertible renaming, e.g.

\[
\lambda x.y[x/y] = \lambda x'.x \neq \lambda x.x = \lambda (x[x/y]).y[x/y]
\]

however, invertibility ensures that “variable capture” is avoided:

\[
(a \ b) \cdot (\langle a \rangle b) = (b)a = ((a \ b) \cdot a)(a \ b) \cdot b
\]

That is, invertible renamings are inherently capture-avoiding.

Next, we define what it means for a name to be independent of (or fresh for) a term. Intuitively, a name \( a \) is fresh for a term \( t \) (that is, \( a \neq t \)) if \( t \) possesses no occurrences of \( a \) unenclosed by an abstraction of \( a \). We define this using the following inference rules:

\[
\begin{align*}
\frac{a \neq b}{a \neq \# b} & \quad \frac{a \neq \# c}{a \neq \# f(i_1, \ldots, i_n)} & \quad \frac{a \neq \# t}{a \neq \# \langle b \rangle t} & \quad \frac{a \neq \# \langle a \rangle t}{a \neq \# \langle a \rangle t}
\end{align*}
\]

We sometimes refer to the set of “free” names of a term \( FN(t) = A - \{ a \mid a \neq \# t \} \) as its support.

Finally, we define an appropriate equality relation on nominal terms that identifies abstractions up to “safe” renaming.

\[
\begin{align*}
\frac{a \approx a}{a \approx \# c} & \quad \frac{t \approx u}{f(i_1, \ldots, i_n) \approx f(u_1, \ldots, u_n)} & \quad \frac{t \approx \langle a \rangle u}{a \neq \# u & \quad t \approx \langle a \rangle u}
\end{align*}
\]
For example, \( \langle a \rangle f(a, b) \approx \langle c \rangle f(c, b) \not\approx \langle b \rangle f(b, a) \); the first equation is derived as follows:

\[
\frac{\vdash a \not\approx c}{\vdash a \not\approx b} \quad \frac{\vdash a \not\approx a}{\vdash b \not\approx b} \quad \frac{\vdash \langle a \rangle f(a, b) \approx \langle c \rangle f(c, b)}{\vdash \langle a \rangle f(a, b) \not\approx \langle c \rangle f(c, b)}
\]

The second rule for abstraction may seem asymmetric because we do not check that \( b \not\approx t \). In fact, this check is redundant: If \( a \not\approx t \) and \( t \approx \langle a \rangle \cdot u \), then \( a \not\approx \langle a \rangle \cdot u \); by applying the swapping \( \langle a \rangle \rangle \cdot \langle b \rangle \) to both sides, we get \( b \not\approx u \), since \( \langle a \rangle \cdot a = b \) and \( \langle a \rangle \cdot \langle a \rangle \cdot u = u \). (It is straightforward to show that \( a \not\approx t \) implies \( \langle b \rangle \cdot a \not\approx \langle b \rangle \cdot t \) for any \( a, b, b', t \).

Nominal logic proper consists of first-order logic extended with a first-order axiomatization of swapping, freshness, and abstraction, and with a new form of quantified formula, \( \mathcal{N} a. \varphi \). For the purposes of this column, we restrict attention to term models of nominal logic, in which function symbols are interpreted as themselves (with swapping and abstraction having the special meanings given by the above rules). The intended objects of study in nominal logic are usually terms, and focusing on term models means that we can avoid some model-theoretic subtleties (for more on this issue, see [12]).

Since there is no choice in the interpretation of the constants and function symbols, a term model \( M \) can be represented as a set of ground atomic formulas \( A \); we require that this set be closed under renaming, so that \( \langle a \rangle \cdot A \in M \) if and only if \( A \in M \). For term models, the semantics of closed nominal logic formulas can be defined as follows:

\[
\begin{align*}
M \not\models \bot & \iff A \in M \\
M \models A & \iff \vdash t \approx u \\
M \models a \not\approx u & \iff \vdash a \not\approx u \\
M \models \varphi \supset \psi & \iff M \models \varphi \text{ implies } M \models \psi \\
M \models \forall x : \sigma. \varphi(x) & \iff M \models \varphi(t) \text{ for every } t : \sigma \\
M \models \mathcal{N} a : \nu. \varphi(a) & \iff M \models \varphi(a) \text{ for some } a : \nu \not\in FN(\varphi(a))
\end{align*}
\]

As an example, we consider the following (valid) property of nominal logic:

\[
\vdash \forall a, b : \nu, x : \sigma. a \not\approx x \land b \not\approx x \supset (a \cdot b) \cdot x \approx x
\]

To verify the validity of this formula, it suffices to show by structural induction on terms \( t \) that for any concrete names \( a, b \not\approx t \), \( \langle a \rangle \cdot t \approx t \). This is straightforward for all but the abstraction cases; for abstractions, if the abstracted name is \( a \) or \( b \), then the second equality rule for abstractions must be used.

We next establish further properties of nominal logic mentioned above. In particular, it is easy to show by induction (on \( n \)-tuples of terms \( \overline{t} \)) that the freshness principle

\[
(F) \quad \forall \overline{x}. \exists a : \nu. a \not\approx \overline{x}
\]

is valid. Similarly, it is straightforward to show by induction on \( \varphi \) that the equivariance principle

\[
(EV) \quad \varphi \iff (a \cdot b) \cdot \varphi
\]

is valid. We also made the (possibly counterintuitive) claim that \( \mathcal{N} \) is self-dual; that is, \( \neg \mathcal{N} a. \varphi(a) \iff \mathcal{N} a. \neg \varphi(a) \). To prove this, suppose \( M \not\models \mathcal{N} a. \varphi(a) \). Then for every \( a \not\in FN(\varphi(a)) \), \( M \not\models \varphi(a) \). Since
\( FN(\varphi(a)) \) is finite and \( \Lambda \) is infinite, we may choose a particular \( a \not\in FN(\varphi(a)) \) such that \( M \models \varphi(a) \). Hence, \( M \models \neg \varphi(a) \), and since \( a \not\in FN(\varphi(a)) \) we can conclude that \( M \models \forall a. \neg \varphi(a) \). Using similar reasoning it is not difficult to show that

\[
\forall a. \varphi(a, \overline{x}) \iff \exists a. a \not\equiv \overline{x} \land \varphi(a, \overline{x}) \iff \forall a. a \not\equiv \overline{x} \supset \varphi(a, \overline{x})
\]

Because of this self-duality property, we can use particularly simple proof rules for the \( \forall \)-quantifier. For example, sequent-style rules have the following form:

\[
\frac{\Sigma \# a : \Gamma \Rightarrow \varphi(a) \quad \Sigma \# a : \Gamma, \varphi(a) \Rightarrow C}{\Sigma : \Gamma \Rightarrow \forall a. \varphi(a) \Rightarrow C} \quad \frac{\Sigma \# a : \Gamma \Rightarrow \varphi(a) \quad \Sigma : \Gamma, \forall a. \varphi(a) \Rightarrow C}{\Sigma : \Gamma \Rightarrow \forall a. \varphi(a) \Rightarrow C}
\]

Here, the context \( \Sigma \) contains variables (introduced as \( \forall \) or \( \exists \) parameters) and names (introduced by the \( \forall \)-rules). The context \( \Sigma \# a \) indicates that \( a \) is assumed to be distinct from all names in \( \Sigma \) and fresh for all values of variables in \( \Sigma \). Intuitively, these rules state that to either prove a \( \forall \)-quantified conclusion or make use of a \( \forall \)-quantified hypothesis, it suffices to instantiate the conclusion or hypothesis with a completely fresh name and proceed. In the example to follow, we won’t be completely formal about proofs in nominal logic; instead we will reason at the semantic level about term models. More detail about the proof theory can be found in the papers \[34, 15\].

### 3.2 A theory of the syntax of the \( \lambda \)-calculus

As an example of the expressiveness of nominal logic, we show how the abstract syntax of the \( \lambda \)-calculus \[5\] can be formalized as a theory \( \Gamma_A \) of nominal logic. We also prove some simple properties concerning capture-avoiding substitution. We consider a language including one data type \( \text{exp} \) for \( \lambda \)-terms, one name-type \( \text{var} \) for \( \lambda \)-calculus variable names, and the following function symbols:

\[
\begin{align*}
\text{var} : & \quad \text{var} \rightarrow \text{exp} \\
\text{lam} : & \quad \langle \text{var} \rangle \text{exp} \rightarrow \text{exp} \\
\text{app} : & \quad \text{exp} \times \text{exp} \rightarrow \text{exp}
\end{align*}
\]

We use \( a, b, c \) for variables of type \( \text{var} \), and \( M, N \) for variables of type \( \text{exp} \). Since we are interpreting nominal logic over syntactic models only, we assume the following axioms expressing that \( \text{var} \), \( \text{lam} \), and \( \text{app} \) are injective functions, and their ranges are disjoint:

\[
\begin{align*}
\text{var}(a) & \approx \text{var}(b) \supset a \approx b \\
\text{app}(M, N) & \approx \text{app}(M', N') \supset M \approx M' \land N \approx N' \\
\text{lam}(M) & \approx \text{lam}(M') \supset M \approx M' \\
\text{var}(a) & \not\approx \text{app}(M, N) \\
\text{var}(a) & \not\approx \text{lam}(M) \\
\text{app}(M, M') & \not\approx \text{lam}(N)
\end{align*}
\]

Let \( P(x) \) be a formula with a free parameter \( x : \text{exp} \) (and possibly other parameters). We can express a structural induction principle over expressions as follows:

\[
(\Lambda_{\text{ind}}) \quad (\forall a : \text{var. } P(\text{var}(a))) \\
\land (\forall M, N : \text{exp. } P(M) \land P(N) \supset P(\text{app}(M, N))) \\
\land (\forall a : \text{var. } \forall M : \text{exp. } P(M) \supset P(\text{lam}(\langle a \rangle M))) \\
\supset \forall x : \text{exp. } P(x)
\]

\( ACM SIGACT News \)
Theorem 1. Let $R_1, \ldots, R_n$ be fresh relation symbols and let $\Gamma$ be a set of nominal Horn clauses, that is, closed formulas of the form

$$\forall \alpha. \forall \beta. A_1(\alpha, \beta) \land \ldots \land A_n(\alpha, \beta) \supset R_i(\bar{\alpha}, \bar{\beta})$$

where $A_1, \ldots, A_n$ are either freshness, equality, or $R_i$ formulas. Then $\Gamma$ has a unique least term model $M$.

Nominal Horn clauses are also written in a Prolog-like notation, in which $\forall$-quantified variables are replaced by constant names:

$$R_i(\bar{\alpha}, \bar{\beta}) := A_1(\alpha, \beta), \ldots, A_n(\alpha, \beta)$$

For example, Figure 2 shows a typechecking judgment and a relation defining capture-avoiding substitution. (We also use Prolog-like notation for lists and a list membership predicate $\text{mem.}$) Reading $b \# N$ as $b \not\in FV(N)$, the axioms for $\text{subst}$ correspond precisely to Barendregt’s relational definition of capture-avoiding substitution [5].

It is inconvenient to work exclusively with relations, so we introduce a recursive definition principle which justifies adding function symbols to the language. First, we observe that nominal logic has a limited Skolemization property:

Theorem 2. If $M \vDash \forall x : \sigma. \exists y : \sigma'. F(\bar{\alpha}, y)$, then we may consistently extend the language with a constant $f : \sigma \rightarrow \sigma'$ satisfying $\forall x : \sigma. F(x, f(x))$.

Suitable generalizations to multiple-argument functions also hold.

It is not difficult to show that the $\text{subst}$ relation defined in Figure 2 is total and functional in its first three arguments, so we can Skolemize it as $-\{ / - \} : \exp \times \exp \times \var \rightarrow \exp$ satisfying the following properties:

\begin{align*}
\text{Ia:}\var. \forall N : \exp. \var(a)\{N/\alpha\} &\approx N \\
\text{Ia:}\var, b : \var. \forall N : \exp. \var(b)\{N/\alpha\} &\approx \var(b) \\
\text{Ia:}\var. \forall N, M_1, M_2 : \exp. \text{app}(M_1, M_2)\{N/\alpha\} &\approx \text{app}(M_1\{N/\alpha\}, M_2\{N/\alpha\}) \\
\text{Ia:}\var, \text{Ib:}\var. \forall N, M : \exp. b \# N \supset \text{lam}(b)\{N/\alpha\} &\approx \text{lam}(b)\{N/\alpha\} 
\end{align*}
The $\beta$-reduction and $\eta$-reduction relations are also definable, using the following formulas:

\[
\begin{align*}
(\beta) \quad & \text{app}(\lambda(a)M), N \rightarrow_\beta M\{N/a\} \\
(\eta) \quad & \forall M. \text{var}(\lambda(a)\text{app}(M, a)) \rightarrow_\eta M
\end{align*}
\]

along with reflexivity, transitivity, symmetry, and congruence properties, if desired. Note that the implicit constraint $a \neq M$ arising from the quantifier ordering in $(\eta)$ corresponds to the traditional side-condition $a \notin \text{FV}(M)$ on $\eta$-reduction.

We now prove two elementary properties of capture-avoiding substitution.

**Proposition 3.** $\forall a:\text{var}, N, M:\text{exp}. \; a \neq M \supset M\{N/a\} \approx M$.

**Proof.** Proof is by the structural induction principle ($\Lambda_{\text{ind}}$) applied to $M$. If $M \approx \text{var}(b)$, then we must have $a \neq b$ since $\text{var}$ is injective. So $M\{N/a\} \approx \text{var}(b)\{N/a\} \approx \text{var}(b) \approx M$. If $M \approx \text{app}(M_1, M_2)$, then $a \neq M$ implies $a \neq M_1, M_2$, so by induction, $\text{app}(M_1, M_2)\{N/a\} \approx \text{app}(M_1, M_2)\{N/a\} \approx \text{app}(M_1, M_2)\{N/a\} \approx \text{app}(M_1, M_2)\{N/a\} \approx \text{app}(M_1, M_2)\{N/a\} \approx M$. \(\square\)

**Proposition 4.** $\forall a, b:\text{var}. \forall M, N, N':\text{exp}. \; a \neq N' \supset M\{N/a\}\{N'/b\} \approx M\{N'/b\}\{N\{N'/b\}/a\}$.

**Proof.** Let $a, b$ be fresh names. Proof is by the structural induction principle ($\Lambda_{\text{ind}}$) applied to $M$. If $M \approx \text{var}(c)$, then there are three cases, depending on whether $c \approx a$, $c \approx b$, or $a \neq c \neq b$. If $c \approx a$, then $M\{N/a\} \approx N$ and $M\{N'/b\} \approx \text{var}(a)$, so

\[
M\{N/a\}\{N'/b\} \approx N\{N'/b\} \approx \text{var}(a)\{N\{N'/b\}/a\} \approx M\{N'/b\}\{N\{N'/b\}/a\}
\]

If $c \approx b$, then

\[
M\{N/a\}\{N'/b\} \approx \text{var}(b)\{N'/b\} \approx N' \approx N'\{N\{N'/b\}/a\} \approx M\{N'/b\}\{N\{N'/b\}/a\}
\]

where $N' = N\{N\{N'/b\}/a\}$ because $a \neq N'$ (by Proposition 3). If $a \neq c \neq b$ then

\[
\text{var}(c)\{N/a\}\{N'/b\} \approx \text{var}(c) \approx \text{var}(c)\{N\{N'/b\}/a\}
\]

Next, if $M = \text{app}(M_1, M_2)$, then by induction we have

\[
\begin{align*}
M_1\{N/a\}\{N'/b\} & \approx M_1\{N'/b\}\{N\{N'/b\}/a\} \\
M_2\{N/a\}\{N'/b\} & \approx M_2\{N'/b\}\{N\{N'/b\}/a\}
\end{align*}
\]

so we can calculate that

\[
\begin{align*}
\text{app}(M_1, M_2)\{N/a\}\{N'/b\} & \approx \text{app}(M_1\{N/a\}\{N'/b\}, M_2\{N/a\}\{N'/b\}) \\
& \approx \text{app}(M_1\{N'/b\}\{N\{N'/b\}/a\}, M_2\{N'/b\}\{N\{N'/b\}/a\}) \\
& \approx \text{app}(M_1, M_2)\{N'/b\}\{N\{N'/b\}/a\}
\end{align*}
\]

Finally, for the case of $\lambda$-abstraction, suppose that $c \neq a, b, N, N'$ and $M \approx \lambda(a)M'$; the induction hypothesis is

\[
M'\{N/a\}\{N'/b\} \approx M'\{N'/b\}\{N\{N'/b\}/a\}
\]
Under these assumptions, we can conclude

\[
\text{lam}(\langle c \rangle M')\{N/a\}\{N'/b\} \approx \text{lam}(\langle c \rangle M'\{N/a\}\{N'/b\}) \\
\approx \text{lam}(\langle c \rangle M'\{N'/b\}\{N\{N'/b\}/a\}) \\
\approx \text{lam}(\langle c \rangle M'\{N'/b\}\{N\{N'/b\}/a\}) .
\]

Note that the last step relies on the fact that since \(c \# N, b, N',\) it follows that \(c \# N\{N'/b\}.\) This completes the proof.

The above proof should seem trivial, and this is the point: nominal abstract syntax facilitates a rigorous style of reasoning with names and binding that is close to intuition and informal practice. Moreover, it provides an equational theory for dealing with expressions involving names and binding using standard algebraic and logical techniques. This advantage is shared by name-free approaches such as combinatory logic or de Bruijn indices. However, the latter approaches rely on cleverly getting rid of explicit names. As a result, it can be awkward or impossible to reason about situations involving free names, and even when possible, such reasoning is very unlike informal reasoning. In contrast, we can reason directly with free names in nominal abstract syntax in a formal, yet intuitive way.

More advanced approaches such as higher-order abstract syntax often provide properties like the substitution lemma above “for free”, that is, as a consequence of the metatheory of the higher-order metalanguage. This means that for languages whose metatheory does not match that of the metalanguage, such properties must again be proved in detail, and higher-order abstract syntax is too high-level for this, since names and fresh name generation are no longer accessible. In contrast, nominal techniques require explicit definitions of and reasoning about substitution, but are more flexible.

4 Applications

4.1 Programming techniques

Probably the most immediately useful application of nominal abstract syntax is in providing more advanced support for symbolic programming with languages with bound names. One obvious approach is to extend an existing programming language with support for nominal abstract syntax, much as the languages λProlog, Twelf, ML\(\lambda\) \cite{55}, and Delphin \cite{73} extend logic or functional programming paradigms with support for higher-order abstract syntax.

FreshML \cite{65,75,76} is an extension of the ML programming language that provides built-in support for nominal abstract syntax. The main additions to the language are the \texttt{let x:name = fresh} construct, which chooses a fresh name and binds it to \(x\), and the abstraction type/term constructor. Abstractions are considered equal up to \(\alpha\)-renaming, and pattern matching against abstractions automatically freshens the bound name. In early versions of FreshML, a complicated type analysis was employed to ensure that name-generating functions were “pure” (side-effect-free); this analysis was found to be overly restrictive and has been dropped in recent versions resulting in a language that is more permissive but has side-effects. There are no constant names in FreshML; instead, names are always obtained via fresh name generation and manipulated via variables. On the other hand, in recent versions of FreshML, names may be attached to data such as strings and integers; also, data structures containing names may be bound, not just individual names. The current implementation is available as an extension to the Objective Caml language, and so inherits the mature compiler and libraries available for that language.
datatype lam = Var of name | Lam of <name>lam | App of lam * lam;

datatype sem = L of (unit -> sem) -> sem | N of neu
and neu = V of name | A of neu * sem;

fun reify(L f) = let fresh x:name in Lam(<x>(reify(f(fn () => N(V x)))))
and reify(N n) = reifyn n
and reifyn(V x) = Var x
and reifyn(A(n,d)) = App(reifyn n, reify d);

fun evals [] (Var x) = N(V x)
| evals((x,v)::env) (Var y) = if x = y then v()
| evals env (Lam(<x>t)) = L(fn v => evals ((x,v)::env) t)
| evals env (App(t1,t2)) = case evals env t1
| evals env (App(t1,t2)) = case evals env t1
| evals env (App(t1,t2)) = case evals env t1
| L f => f(fn () => evals env t2)
| N n = N(A(n, evals env t2));

fun eval t = evals [] t;
fun norm t = reify(eval t);

Figure 3: Normalization by evaluation in FreshML

Figure 3 shows an interesting example FreshML program (from Shinwell et al. [75]). It implements normalization by evaluation, an advanced technique for optimization that uses ML’s higher-order features to simplify lambda-calculus expressions. In ordinary ML, fresh names must be generated by a user-defined gensym function; in contrast, in FreshML, built-in binding and fresh name generation can be used. In addition, although this program uses side-effects, it is not difficult to prove that it is actually a pure function (as one would expect).

αProlog is a Prolog-like language that supports nominal abstract syntax; roughly speaking, it is to Prolog as FreshML is to ML. Unlike ordinary Prolog, αProlog is strongly typed. Some simple Horn clause programs involving λ-terms were shown in Figure 2. Types in αProlog are useful for describing the binding structure of term languages, and help catch many more errors statically. The unification algorithm used in αProlog is essentially the nominal unification algorithm developed by Urban, Pitts, and Gabbay [81]. In this algorithm (and in αProlog) names are treated as concrete constants, rather than requiring that names are only manipulated via variables. However, name constants in program clauses are not interpreted as global constants, but as \( \forall \)-quantified within the clause. Thus, two names appearing in the clause may be instantiated with any two other distinct names, but not with the same name. In particular, this means that freshness and inequality constraints between names and other data can be employed to correctly implement informal freshness side-conditions, as shown in the closure conversion example of Figure 4.
\[ c\text{conv}(\{x\mid G\}, \text{var}(x), \text{Env}, \pi_1(E)). \]
\[ c\text{conv}(\{x\mid G\}, \text{var}(y), \text{Env}, E') \quad \begin{align*} &:- c\text{conv}(G, \text{var}(y), \pi_2(\text{Env}), E') \\ &c\text{conv}(G, \text{app}(T_1, T_2), \text{Env}, E') \quad \begin{align*} &:- c\text{conv}(G, T_1, E_1), c\text{conv}(G, T_2, \text{Env}, E_2), \\ &E' = \text{let}(E_1, \langle c\rangle \text{app}(\pi_1(\text{var}(c)), \text{pair}(E_2, \pi_2(\text{var}(c))))). \end{align*} \]
\[ c\text{conv}(G, \text{lam}((x)T), \text{Env}, \text{pair}(\text{lam}(\langle y \rangle E'), E)) \quad \begin{align*} &:- x \not\in G, y \not\in G, \\ &c\text{conv}(\{x\mid G\}, T, \text{var}(y), E'). \end{align*} \]

Figure 4: Closure conversion in \(\alpha\)Prolog

\(\alpha\)Prolog is an example of a nominal logic programming language, that is, its logical foundation is nominal logic. Initially, the connection between nominal logic as originally formulated by Pitts and the operational behavior of unification and proof search in \(\alpha\)Prolog was less than clear. Nominal logic as reformulated by Cheney [16, 12] (and presented in simplified form in this paper) now provides a robust logical and semantic foundation for \(\alpha\)Prolog; however, as currently implemented, \(\alpha\)Prolog is logically incomplete (that is, there are queries whose answers cannot be found, even in principle.) The reason is that \(\alpha\)Prolog’s proof search and unification techniques address only the equational theory of nominal logic. Because of the equivariance principle, this is not enough; for example, the two atomic formulas \(p(a)\) and \(p(b)\) are logically equivalent but not provably equal as terms.

There are two ways around this: solve the more general problem, or limit the programs so that the special case is enough. Both approaches have been explored. Complete proof search for general nominal logic programs requires solving equivariant unification problems [11, 13], that is, unifying atomic formulas up to both a substitution for free variables and a name-permutation. This process is \(NP\)-complete and appears nontrivial to implement. Urban and Cheney [83] studied a fragment of nominal logic programs for which proof search based on nominal unification is complete. The idea of this result is that clauses that only manipulate bound names in simple ways are automatically equivariant, so explicit unification modulo equivariance is unnecessary. Although it excludes some interesting programs (such as closure conversion), this fragment includes many interesting \(\alpha\)Prolog programs, though slight modifications are sometimes necessary.

It is important to point out that higher-order abstract syntax offers benefits for programming with names and binding that nominal abstract syntax still lacks; in particular, HOAS provides capture-avoiding substitution as an efficient, built-in operation, whereas nominal techniques typically do not. But higher-order techniques seem much more difficult to incorporate into existing languages, because of the need for higher-order unification and matching to manipulate higher-order terms. On the other hand, while capture-avoiding substitution is annoying to have to implement in FreshML or \(\alpha\)Prolog, there is no conceptual problem in doing so; instead, the problem is that there are a large number of similar “boilerplate” cases that are conceptually uninteresting but have to be written anyway.

The need to write such boilerplate code and the need to switch to a new language are two potential drawbacks for programmers considering nominal abstract syntax. Recently, some progress has been made on both fronts, by projects providing nominal abstract syntax as a library or lightweight language translation rather than as a language extension, and employing generic programming techniques to alleviate the burden of implementing name-related boilerplate (or nameplate). Cheney [14] developed a Haskell class library called FreshLib that provides much of the functionality of FreshML.
within Haskell, and also showed how to use Lämmel and Peyton Jones’ *scrap your boilerplate* approach to generic programming [48] to provide reusable, generic definitions of capture-avoiding substitution and other nameplate. Pottier [69] has developed Coml, a language tool for Objective Caml that translates type declarations decorated with binding specifications to plain Objective Caml programs that automatically deal with name-binding. Coml’s binding specifications can describe more complex binding structure than the one-name-at-a-time binding present in nominal logic; for example, Coml can express pattern matching and letrec binding forms. In addition, Coml provides visitor classes that can easily be overridden to implement capture-avoiding substitution. While these developments are encouraging, it is not clear yet whether nominal techniques can provide the same combination of convenience and efficiency as higher-order techniques.

4.2 Automated reasoning

A second major application area for nominal techniques is specifying and proving properties of formal systems, including logics, programming language calculi, concurrency calculi, and security protocols.

Initial work of this form was carried out by Gabbay. Gabbay [30] implemented FM set theory in the Isabelle theorem prover, as a variant of Isabelle’s implementation of ZF set theory (Isabelle/ZF). Unfortunately, Isabelle/ZF relies heavily on the Axiom of Choice, even in places where it is not strictly necessary. Because FM set theory is incompatible with the Axiom of Choice, it was necessary to re-develop a significant amount of set theory in Isabelle/FM. Gabbay also investigated FM-HOL, a form of higher-order logic based on FM set theory [31]. As far as I know this has not been implemented; implementing FM-HOL as a variation on Isabelle/HOL would likely involve considerable effort because Isabelle/HOL also relies extensively on the Axiom of Choice.

Gabbay’s work was foundational in the sense that it attempted to incorporate nominal techniques into the surrounding mathematical foundations. While this is attractive because it builds several desirable properties into the foundations, it has a high start-up cost and requires potential users to adapt to the new foundations. One motivation for Pitts’ development of nominal logic was to show how to work with nominal abstract syntax without leaving the mathematical mainstream. Pitts’ recent paper [64] continues this theme by showing how to relate classical and nominal approaches to $\alpha$-equivalence. Based on this insight, Norrish [58], Urban and Tasson [82] and Urban and Norrish [80] have performed non-foundational formalizations of properties of the $\lambda$-calculus in classical higher-order logic. Rather than build swapping, freshness, etc. into the logic, Urban and others have defined swapping functions and constructed nominal abstract syntax trees explicitly, and proved explicit induction or recursion principles. Although some additional subgoals need to be proved in this approach relative to a foundational approach, the start-up cost of implementing this approach appears lower, and the learning curve for users already familiar with Isabelle/HOL seems gentler. Urban and others are currently working on extending Isabelle/HOL’s datatype package so that induction and recursion principles can be derived automatically for datatypes employing nominal abstract syntax.

Nominal logic appears to be related to two other recently investigated approaches to formal reasoning about languages with names: the Theory of Contexts [45] and $\text{FO}\lambda\Delta$ [56]. Both have been used to carry out complete machine-checkable formalizations of properties of interesting languages, including the $\pi$-calculus.

The Theory of Contexts (TOC) is an axiomatic extension to the Calculus of Inductive Constructions (or CIC), the type theory of the Coq system. In TOC, names are represented by an abstract base type $V$. Equality is assumed to be decidable for names. Moreover, there is a freshness
relation \textit{notin} relating names and arbitrary values, and fresh names are always assumed to exist. Name-binding is represented using the function type $V \rightarrow X$. The Theory of Contexts appears to be similar in some respects to nominal logic; in fact, Miculan, Scagnetto, and Honsell [53] have developed a translation from nominal logic specifications to TOC specifications. This translation is sound (translates derivable formulas of nominal logic to derivable formulas of CIC/TOC) but not complete (some non-derivable formulas have derivable translations). This is because the Theory of Contexts is set in a higher-order type theory that is stronger than the first-order setting of nominal logic. Nevertheless, it seems fair to say that TOC is approximately equivalent to nominal logic in expressive power. The chief difference seems to be the handling of binding via a higher-order function encoding rather than an explicit axiomatization.

Miller and Tiu [54] have introduced $FO\lambda\Delta\nabla$, which stands for First-Order logic with $\lambda$-terms, Definitions, and the $\nabla$-quantifier. As its title suggests, this logic includes function types populated by $\lambda$-terms, but only permits quantification over “first-order” types (that is, types not mentioning $o$, the type of propositions). In addition, $FO\lambda\Delta\nabla$ includes the ability to make definitions and perform case-based reasoning on the structure of definitions, and a novel self-dual quantifier $\nabla$. Though $\nabla$ and $\nabla$ behave differently, Gabbay and Cheney [51] developed a partial (sound but incomplete) translation from $FO\nabla$ (the definition-free part of $FO\lambda\Delta\nabla$) to nominal logic, and Cheney [15] has developed an improved, sound and complete translation. Miculan and Yemane [54] have investigated the idea of using the semantics of nominal logic as the basis of a denotational semantics for $FO\lambda\nabla$.

Computational techniques such as unification, constraint solving, and rewriting are very relevant to automated deduction. As noted earlier, Urban, Pitts, and Gabbay [51] first studied unification for nominal terms. Cheney [11,16,13] noted that the unification problems their algorithm solves are only special cases of the problems that must be solved in general nominal logic programming or rewriting. The general cases of nominal unification (solving equations $t \approx u$), freshness constraint solving ($a \# t$), and equivariant unification ($\exists \pi. \pi \cdot t \approx u$) are all NP-complete. In fact, even equivariant matching (that is, equivariant unification where one side is ground) is NP-complete. Despite this, several tractable special cases of these problems are known.

A significant area for future work in this area is the investigation of nominal equational unification: that is, unification modulo an extension to the equational theory of nominal logic. Many structural congruences considered in concurrency calculi can be expressed by a nominal equational theory. For example, $\pi$-calculus terms are considered structurally congruent modulo axioms such as:

\[
x \# P \supset \nu u(\langle x \rangle P) \approx P \quad \nu u(\langle x \rangle \nu u(\langle y \rangle P)) \approx \nu u(\langle y \rangle \nu u(\langle x \rangle P))
\]

Fernandez, Gabbay, and Mackie [25] have investigated nominal term rewriting systems. They study conditions for establishing confluence and show how existing higher-order rewriting formalisms can be simulated using nominal rewriting. They also discuss the implications of the NP-hardness of equivariant matching and present a syntactic condition on rewriting rules that ensure nominal matching is sufficient. Fernandez and Gabbay [24] investigate an extension of nominal rewriting with a “hidden name” operation $\nu a. t$; this operation behaves like the name-restriction operation $\nu a. P$ in the $\pi$-calculus. Gabbay [37] has also proposed a novel approach to reasoning about contexts (that is, terms with “holes”) based on nominal techniques.

So far most techniques for automating reasoning with nominal abstract syntax have focused on general-purpose formal systems and proof tools such as Isabelle/HOL or Coq. In contrast, there are several lightweight, domain-specific applications for formalizing, programming with, and reasoning about various forms of higher-order abstract syntax, including $\lambda$Prolog [54], Twelf [62], Linear LF [10], and Concurrent LF [86]. These systems seem to have a much gentler learning curve than...
the general-purpose systems, yet are suitable for a wide range of applications. I am particularly interested in developing an analogous logical framework for nominal abstract syntax. The $\alpha$Prolog language can be viewed as a first step in that direction.

On the other hand, the fact that nominal abstract syntax can be constructed explicitly in general-purpose reasoning systems such as Isabelle/HOL opens up another new direction: relating denotational and operational semantics, and proving results via denotational rather than via operational techniques.

### 4.3 Semantics

Besides providing a foundation for programming and reasoning in the presence of name-binding, nominal techniques have several applications in logic and programming language semantics.

Early work by Pitts and Stark [66] and Odersky [59] studied name-generation in functional programming languages as a simplified case of general side-effects. Pitts and Stark’s nu-calculus [66] analyzed name generation as an effectful computation, analogous to reference generation in ML. Names could be introduced with a “fresh name” binder $\nu n.t$, and tested for equality. The fresh name construction was interpreted operationally by maintaining a name-store: on encountering a $\nu n.t$ term, a fresh name is bound to $n$ and added to the store. In some ways, this work can be seen as an early precursor to that of Pitts and Gabbay on FreshML [65]. Odersky developed a quite different functional theory of local names [59]. His $\lambda\nu$-calculus is syntactically essentially the same as Pitts and Stark’s nu-calculus. But instead of treating name-generation as an effect, the $\lambda\nu$-calculus deals with names in a local and functionally pure way. Odersky developed a denotational semantics for $\lambda\nu$ in terms of name-swapping and support. This development clearly foreshadows the later developments underlying nominal logic, FreshML, and $\alpha$Prolog, although, of course, without the application to binding.

Schöpp and Stark [72] have developed a form of type theory based on nominal logic and the logic of bunched implications (BI) [69]. Intuitively, the idea of this system is to identify the “resources” of BI with the sets of names supporting values in nominal logic. This theory axiomatizes a type of names and includes “fresh” dependent products and sums $\Pi^*, \Sigma^*$ (corresponding to $\forall^{\text{new}}, \exists^{\text{new}}$ in BI) in addition to ordinary dependent products and sums. Though very interesting, there are many unresolved practical problems in this setting (for example, it is not yet known whether strong normalization holds).

One area in which reasoning about name-generation is of particular interest is in concurrency calculi, in particular the $\pi$-calculus. A number of researchers have investigated applications of nominal techniques to the $\pi$-calculus and similar systems [33, 26, 78]. Ideas from nominal logic have also been incorporated into logics for reasoning about concurrency or data structures with name-hiding [8, 9].

### 5 Future directions

The equivariance principle is an integral component of nominal logic as currently formulated. Without it, the $\nu$-quantifier would no longer be self-dual, and reasoning about function and relation symbols would be significantly more complicated. On the other hand, it has several undesirable consequences. As noted earlier, because of equivariance, $\alpha$Prolog’s intuitively appealing proof search strategy based on nominal unification is incomplete; to obtain completeness, it is necessary to either place significant restrictions on the language or solve NP-complete equivariant unification problems. Furthermore, equivariance implies that no linear order on the set of names (or even any
finite nonempty subset) can be denoted by a relation symbol, since \( a < b \) and \( b < a \) cannot both hold. For this reason, I believe that finding an alternative approach to nominal logic that supports names, \( \mathcal{N} \)-quantification, and binding without relying on equivariance is an important open issue.

Model theory and database theory \cite{model} may be interesting places to look for inspiration concerning how to attack this problem. In model theory, the groups of automorphisms or order-preserving automorphisms play important roles. In database theory, \textit{generic} queries whose answers are invariant under permutations of the domain elements are often of interest. Thus, generic queries are similar to equivariant formulas. In addition, a great deal of research has concerned the impact of having a total order on the domain of individuals on expressive power. In this setting, one typically considers \textit{order-invariant} queries whose meaning is independent of the linear order.

Another important direction for future work is to reconcile nominal logic with well-known constructive or type-theoretic principles. I believe nominal abstract syntax is entirely satisfactory from the point of view of mathematical constructiveness; for example, nominal abstract syntax trees can be defined as an inductive construction similar to that used for ordinary abstract syntax trees, and by design nominal logic does not rely on the Axiom of Choice. Moreover, the proof theory and semantics of intuitionistic nominal logic has been investigated by Gabbay and Cheney \cite{gabbay2015intuitionistic}. However, nominal logic seems challenging to integrate with type-theoretic approaches to computation and reasoning, because of its use of explicit freshness constraints, its non-confluent equational theory, and explicit fresh name generation. Some of these problems have already been encountered in purely functional versions of FreshML and in Schöpp and Stark’s dependent type theory with names.

6 Conclusion

Gabbay and Pitts’ approach to formalizing abstract syntax with names and binding (i.e., \textit{nominal abstract syntax}) is an important foundational development relevant to logic and computer science. It provides a level of abstraction for reasoning about languages with binding that lies between first-order abstract syntax (which is usually too low-level) and higher-order abstract syntax (which is powerful, but too high-level for some applications). Nominal abstract syntax also seems to provide justification for the kind of reasoning people already perform, rather than requiring new proof techniques.

Nominal logic is an extension of first-order logic formalizing the principles of nominal abstract syntax. It has numerous applications, ranging from providing a foundation for a logic programming language to machine-checked or assisted proofs of language properties. In this column I have attempted to give an impression of the key ideas underlying nominal logic, how it can be used, how it is being applied, and how it might be improved.

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