Shadow of Schwarzschild-Tangherlini black holes

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Abstract

We study the shadow cast by the HD Schwarzschild-Tangherlini black hole, and analytically calculate the influence of extra dimensions on the shadow of a black hole. A black hole casts a shadow as an optical appearance because of its strong gravitational field which is known to be a dark zone covered by a circle for a Schwarzschild black hole. We demonstrate that the null geodesic equation can be integrated by Hamilton-Jacobi approach, which enable us to investigate the shadow cast by the HD Schwarzschild-Tangherlini black holes. Interestingly, it turns out that, for fixed values of the mass parameter, the shadow in HD spacetimes are smaller when compared with 4D Schwarzschild black hole. Further, the shadows of HD Schwarzschild-Tangherlini black holes are concentric circles with radius of the circle decreases with increase in D. We visualize the photon regions and the shadows in various dimensions for different values of the parameters, and the energy emission rates are also investigated. Our results, in the limit D = 4, reduced exactly to vis-à-vis Schwarzschild black hole case.

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I. INTRODUCTION

There is a great interest to investigate nature of the black holes, i.e., mass and spin of the black hole, which can be possibly determined by observation of black hole shadow [1–4]. Now, it is a general belief that a black hole, if it is in front of a luminous background, will cast a shadow. To observe the shadow of the black hole at the center of the Milky way, one will be looking for a ring of light around a region of darkness, which is called the black holes shadow. That light is produced by matter that is circling at the very edge of the event horizon, and its shape and size are determined by the black holes mass and spin. A shadow is the optical appearance cast by a black hole, and its existence was first studied by Bardeen [5]. The shadow of a nonrotating black hole is circular [6] while a distorted circle for rotating black holes and this is due to the presence of the spin parameter [5, 7, 8]. It was Synge [9] who studied shadow of Schwarzschild black holes and thereafter Luminet [10] discussed the optical properties of static, spherically symmetric Schwarzschild black holes and constructed a simulated photograph of the shadow. Later, it received a significant attention and has become a quite active research field (for a review, See [11]). More discussion on shadow of Schwarzschild black hole [12], other spherically symmetric black holes [13] have been intensively studied and have been extended to rotating black holes [14–25].

Over the past decade there has been an increasing interest in the study of black holes, and related objects, in $\text{HD}$, motivated to a large extent by developments in string theory. Recently, it has been proposed that our universe may have emerged from a black hole in a higher-dimensional (HD) Universe. Black holes are very interesting gravitational as well as geometrical objects to study in $4D$ which may also exist in HD spacetimes. Although, at present, the work on HD can probably be most fairly described as extended theoretical speculation and it has no direct observational and experimental support, in contrast to $4D$ general relativity. However, this theoretical work has led to the possibility of proving the existence of extra dimensions which is best demonstrated by Reall and Emparan to show that there is a 'black ring' solution in $5D$ [26]. If such a 'black ring' could be produced in a particle accelerator such as the Large Hadron Collider, this would provide the evidence that HD exist. There are other reasons for this interest in HD black holes [27–31] in particular, e.g., the statistical calculation of black hole entropy using string theory was first done for certain $5D$ black holes [32] and also the possibility of producing tiny HD black
holes at LHC in certain brane-world scenarios \[33\]. This study of shadow is extended to HD spacetime by several researchers, e.g., Papnoi et. al. \[34\] have studied the shadow cast by 5D rotating Myers-Perry black holes, for pure Gauss-Bonnet gravity rotating black holes by \[35\], also in Kaluza-Klein gravity \[19\]. However, the results of these work can’t go over to the Schwarzschild black hole.

Hence, it is pertinent to investigate the apparent shape of HD Schwarzschild-Tangherlini black holes to visualize the shape of the shadow and compare the results with images for 4D Schwarzschild black hole. An apparent Shape of black hole is determined via boundary of the shadow which can be studied by the null geodesic equations. Clearly, the extra spacetime dimension shall change the equations of motion which may lead to the modification of black hole shadow.

The paper is organized as follows: in Sect. II, we review the Schwarzschild-Tangherlini black hole solutions and present the associated thermodynamical quantities. In Sect. III, we have presented the particle motion around the Schwarzschild Tangherlini black hole by using the Hamilton-Jacobi approach necessary to discuss the shadow. The observables are introduced Sect. IV to plot the apparent shapes of the black hole shadows and finally in Sect. V, we have concluded with the results.

II. THE SCHWARZSCHILD-TANGHERLINI SPACETIME

We present the basic framework for general relativity in HD, and introduce the Schwarzschild Tangherlini solutions that generalize the 4D Schwarzschild solution to the HD. The action in HD space-time

\[ I = \int d^Dx \sqrt{-g} R. \]  \hspace{1cm} (1)

This is a straightforward generalization of Einstein-Hilbert action to HD and the only aspect that deserves some attention is the implicit definition of Newton’s constant \( G_D \) in HD, without loss of generality we use units such that \( G_D = c = 1 \). Using variational principle, one obtains the Einstein equation in HD as

\[ R_{ab} - \frac{1}{2} R g_{ab} = 0 \quad \text{or} \quad R_{ab} = 0, \]  \hspace{1cm} (2)

where \( R_{ab} \), \( R \) and \( g_{ab} \) are respectively Ricci tensor, Ricci scalar and metric tensor. Tangherlini \[36\] has found the asymptotically flat, static and spherically symmetric vacuum solution
of (2) as a generalization of the Schwarzschild black hole

\[ ds^2 = -\left(1 - \frac{\mu}{r^{D-3}}\right)dt^2 + \frac{1}{\left(1 - \frac{\mu}{r^{D-3}}\right)}dr^2 + r^2d\Omega_{D-2}^2, \]  

where

\[ d\Omega_{D-2}^2 = d\theta_1^2 + \sin^2 \theta_1^2 d\theta_2^2 + \cdots + \prod_{i=1}^{D-3} \sin^2 \theta_i^2 d\theta_{D-2}^2, \]

is a metric on \((D-2)\)-dimensional unit sphere. The parameter \(\mu\) is related to black hole mass \(M\) via

\[ \mu = \frac{16\pi M}{(D-2)\Omega_{D-2}}, \]

where \(\Omega_{D-2}\) is the volume of the \((D-2)\) dimensional sphere given by

\[ \Omega_{D-2} = \frac{2\pi^\frac{D-1}{2}}{\Gamma\left(\frac{D-1}{2}\right)}. \]

This suggests that the Schwarzschild solution generalizes to HD in the form given by the metric (3). First thing to be noticed is that this simplifies to the normal Schwarzschild metric when \(D = 4\). Secondly, the HD version is very similar to the 4D one and the notable difference is when we go to HD then the fall off term \(1/r\) replaced with the \(1/r^{D-3}\). As shown by Tangherlini [36], this turns out to give the correct solution and that the metric (3) is indeed Ricci flat, and this solution is called the Schwarzschild-Tangherlini solution.

As can be seen from equation (3) we have no black-hole solution in \(D = 3\). If the mass parameter \(\mu < 0\) then we get a naked singularity, which is not physical. If \(\mu > 0\), the black hole horizon radius \(r_h\) which is obtained by solving \(g_{tt}(r_h) = 0\) as

\[ r_h = \left[\frac{16\pi M}{(D-2)\Omega_{D-2}}\right]^{\frac{1}{D-3}}. \]

The event horizon is the nonrotating Killing horizon and its spatial cross-section is around \((D-2)\)-sphere. The mass of the black hole in terms of the horizon radius \(r_h\) gives

\[ M = \frac{(D-2)\Omega_{D-2}}{16\pi} r_h^{D-3}. \]

The area of the event horizon for the metric (3), is

\[ A = \Omega_{D-2} r_h^{D-2}. \]

The black hole entropy is expected to obey area law [37]. Explicitly the entropy \(S\) in terms of \(M\), can be written as

\[ S = \frac{\Omega_{D-2}}{4} \left[\frac{16\pi M}{(D-2)\Omega_{D-2}}\right]^{\frac{D-2}{2}}. \]
The black hole obeys the first law of thermodynamics $dM = TdS$, which can be used to obtain Hawking Temperature as

$$T_{BH} = \frac{D - 3}{4\pi r_h}.$$  \hspace{1cm} (11)

Clearly, all above result reduces to the 4D Schwarzschild black holes when $D = 4$. Although a black hole is invisible, it can cast a shadow when it is in front of a bright object. The aim of this work is to discuss the effect of extra dimension in a black hole shadow in the background of Schwarzschild-Tangherlini.

### III. MOTION OF A TEST PARTICLE

When a black hole is in front of the light source, the light reaches the observer when deflect due to gravitational field of the black hole. However, it turns out that some of the photons may fall into the black hole, which result a dark zone called the shadow, and the apparent shape of the black hole is the boundary of the shadow. We present the necessary calculations for obtaining the shape of Schwarzschild-Tangherlini black hole shadow, which demands the study of the motion of test particle. We employ the Lagrangian and Hamilton-Jacobi equation to obtain equations of motion, which requires a study of geodesics equation of a particle near Schwarzschild-Tangherlini black hole. We begin with the Lagrangian which reads

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu},$$  \hspace{1cm} (12)

where an over dot is derivative with respect to affine parameter $\tau$ and $g_{\mu\nu}$ is the metric tensor. The canonically conjugate momentum for the Schwarzschild-Tangherlini black holes metric (3) can be calculated as

$$P_t = \left(1 - \frac{\mu}{r_{D-3}}\right) \dot{t} = E,$$  \hspace{1cm} (13)

$$P_r = \left(1 - \frac{\mu}{r_{D-3}}\right)^{-1} \dot{r},$$  \hspace{1cm} (14)

$$P_{\theta_i} = r^2 \left(\prod_{n=1}^{D-3} \sin^2 \theta_n \dot{\theta}_{D-3}, \quad i = 1, ..., D - 3 \right),$$  \hspace{1cm} (15)

$$P_\phi = r^2 \left(\prod_{i=1}^{D-2} \sin^2 \theta_i \dot{\theta}_{D-2} = L, \right)$$  \hspace{1cm} (16)

where $P_\phi = P_{\theta_{D-2}}$, $E$ and $L$ are respectively energy and angular momentum of the test particle. For $D = 4$, we have $i = 1$ and the quantities $P_{\theta_i}$ and $P_\phi$ reduces to Schwarzschild
case, can be reads

\[ P_{\theta_1} = r^2 \dot{\theta}_1, \]
\[ P_{\phi} = r^2 \sin^2 \theta_1 \dot{\theta}_2. \]

We use Hamilton-Jacobi method to analyze photon orbits around the black hole and use the formulation of geodesic equations by Carter approach for Schwarzschild black hole \[39\], which we extended here to HD. The Hamilton-Jacobi method in the HD reads

\[
\frac{\partial S}{\partial \tau} = \mathcal{H} = -\frac{1}{2} g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\] (17)

where \( S \) is the Jacobi action. On using (3) in Eq. (17), we obtain

\[
-2 \frac{\partial S}{\partial \tau} = -\frac{1}{1 - \frac{\mu}{r^{D-3}}} \left( \frac{\partial S_r}{\partial t} \right)^2 + \left( 1 - \frac{\mu}{r^{D-3}} \right) \left( \frac{\partial S_r}{\partial r} \right)^2 + \sum_{i=1}^{D-3} \frac{1}{r^2 \prod_{n=1}^{i-1} \sin^2 \theta_n} \left( \frac{\partial S_{\theta_i}}{\partial \theta_i} \right)^2 \]
\[
+ \frac{1}{r^2 \prod_{i=1}^{D-3} \sin^2 \theta_i} \left( \frac{\partial S_{\phi}}{\partial \phi} \right)^2.
\] (18)

Here, we consider a additive separable solution for Jacobi action \( S \), which can be expressed as

\[
S = \frac{1}{2} m^2 \tau - Et + L\phi + S_r(r) + \sum_{i=1}^{D-3} S_{\theta_i}(\theta_i),
\] (19)

where \( S_r(r) \) and \( S_{\theta_i}(\theta_i) \) are respectively functions of \( r \) and \( \theta_i \) and \( m \) is the mass of the test particle, which is zero for the photon. On the right hand side, the second and third term related with the conservation of energy and the angular momentum respectively. The Hamilton-Jacobi Eq. (17), on using (19) can be recast as

\[
r^4 \left( 1 - \frac{\mu}{r^{D-3}} \right)^2 \left( \frac{\partial S_r}{\partial r} \right)^2 = E^2 r^4 - r^2 \left( 1 - \frac{\mu}{r^{D-3}} \right) (K + L^2),
\] (20)

\[
\sum_{i=1}^{D-3} \frac{1}{\prod_{n=1}^{i-1} \sin^2 \theta_n} \left( \frac{\partial S_{\theta_i}}{\partial \theta_i} \right)^2 = K - \prod_{i=1}^{D-3} L^2 \cot^2 \theta_i,
\] (21)

where \( K \) is the Carter constant \[39\]. By using Eqs. (13)-(15) in Eqs. (20) and (21), we get
FIG. 1: Plot showing the dependency of effective potential on radial coordinate $r$ in different dimension $D$ and angular momentum $L$. 
the complete null geodesics equation for Schwarzschild-Tangherlini black hole as

\[ \dot{t} = \frac{E}{1 - \frac{\mu}{r^{D-3}}}, \quad (22) \]

\[ \dot{\phi} = \frac{L}{r^{D-3} \prod_{i=1}^{D-1} \sin^2 \theta_i}; \quad (23) \]

\[ r^2 \ddot{r} = \pm \sqrt{\mathcal{R}}, \quad (24) \]

\[ r^2 \sum_{i=1}^{D-3} \prod_{n=1}^{i-1} \sin \theta_i \dot{\theta}_i = \pm \sqrt{\Theta_i}, \quad (25) \]

where “+” and “−” sign respectively gives a motion of photon in outgoing and ingoing radial direction and a dot denote the derivative with respect to affine parameter \( \tau \). For the null curves the expressions of \( \mathcal{R}(r) \) and \( \Theta_i(\theta_i) \) in Eqs. (24) and (25) can takes the form as

\[ \mathcal{R}(r) = E^2 r^4 - r^2 \left[ 1 - \frac{\mu}{r^{D-3}} \right] \left[ \mathcal{K} + L^2 \right], \quad (26) \]

\[ \Theta_i(\theta_i) = \mathcal{K} - \prod_{i=1}^{D-3} L^2 \cot^2 \theta_i. \quad (27) \]

The Eqs. (22)-(25) govern the motion of photon in Schwarzschild-Tangherlini space-time. The characteristics of photon near the black hole can be defined by two impact parameters, which are functions of constants of motion \( E, L, L \) and \( \mathcal{K} \). For the general orbits around the black hole the impact parameters are \( \xi = L/E, \eta = \mathcal{K}/E^2 \). It is important to discuss the effective potential for determining the boundary of the shadow of a black hole. We can find out the effective potential by rewriting the radial null geodesic equation for Schwarzschild-Tangherlini black hole which is given by

\[ \left( \frac{dr}{d\tau} \right)^2 + V_{eff}(r) = 0, \quad (28) \]

where \( V_{eff} \) is the effective potential for radial motion given by

\[ V_{eff} = \frac{1}{r^2} \left( 1 - \frac{\mu}{r^{D-3}} \right) (\mathcal{K} + L^2) - E^2. \quad (29) \]

Thus it is straight forward to show that the effective potential for Schwarzschild-Tangherlini is maximum for critical radius \( r_c \): \[ 41 \]

\[ r_c = \frac{1}{2^{D-3}} \left[ \frac{16\pi M(D-1)}{\Omega_{D-2}(D-2)} \right]^{D-3}. \quad (30) \]
TABLE I: Variation of mass parameter $\mu$ and $\alpha^2+\beta^2$ with dimension $D$.

| $D$ | $\mu$   | $\alpha^2+\beta^2$ |
|-----|---------|------------------|
| 4   | $2M$    | $27M^2$          |
| 5   | $\frac{8M}{3\pi}$ | $\frac{32M}{3\pi}$ |
| 6   | $\frac{3M}{2\pi}$   | $\left(\frac{7M}{20\pi}\right)^{\frac{3}{2}}$ |
| 7   | $\frac{16M}{5\pi^2}$ | $\left(\frac{108M}{5\pi^2}\right)^{\frac{1}{2}}$ |
| 8   | $\frac{5M}{2\pi^2}$   | $\left(\frac{10M}{\pi^2}\right)^{\frac{1}{2}}$ |
| 9   | $\frac{48M}{7\pi^3}$ | $\left(\frac{1446M}{\pi^3}\right)^{\frac{1}{3}}$ |

For $D=4$, the HD equation of effective potential (29) reduces to Schwarzschild spacetime. We have plotted the radial dependency of effective potential in Fig. 1. As we know for Schwarzschild case $V_{\text{eff}}$ for photon has maximum at $r = 3M$, which shows a unstable circular orbit and as we go $r \to \infty$, effective potential asymptotes to a constant value. One can see from Fig. 2, as we go to HD the maximum value of effective potential increases, which emphasizes that as we go to the HD unstable circular orbits becomes smaller. The photon orbits are circular and unstable corresponding to the maximum value of effective potential. The unstable circular orbit determine the boundary of apparent shape of the black hole and can be obtained by maximizing the effective potential, which demands

\[
V_{\text{eff}} = \frac{\partial V_{\text{eff}}}{\partial r} = 0 \quad \text{or} \quad R(r) = \frac{\partial R(r)}{\partial r} = 0, \quad (31)
\]

we obtain impact parameters $\eta$ and $\xi$ related to dimension $D$ via,

\[
\eta + \xi^2 = \frac{D - 1}{D - 3} \left[ \frac{16\pi M(D - 1)}{2(D - 2)\Omega_{D-2}} \right]^{\frac{2}{D-3}}. \quad (32)
\]

The contour of the Eq. (32) can describe the apparent shape of Schwarzschild-Tangherlini black hole. For the Schwarzschild black hole the effective potential has maxima at $r_c = 3M$.

IV. SHADOW OF THE BLACK HOLE

The shadow of black hole and its photon orbit can be determined by geometrical optics. The apparent shape of a black hole is defined by the boundary of the shadow. From
FIG. 2: Shadow cast by Schwarzschild-Tanglerlini black holes at $\theta_i = \pi/2$, in different dimension $D$ and mass parameter $M$. 
Eq. (32) the size of Schwarzschild-Tangherlini black holes depends upon mass and dimension of space-time. To visualize the shadow of the black hole, it is appropriate to use the celestial coordinates $\alpha$ and $\beta$ \[40\]. For Schwarzschild-Tangherlini black hole the celestial coordinate modified as \[1\]

$$
\alpha = \lim_{r_0 \to \infty} \left( \frac{r_0 P(\phi)}{P(t)} \right),
$$

$$
\beta_i = \lim_{r_0 \to \infty} \left( \frac{r_0 P(\theta_i)}{P(t)} \right), \quad i = 1, \ldots, D - 3,
$$

where $[P(t), P(\phi), P(\theta_i)]$ are the vi-tetrad component of momentum. By using Eqs. (13)-(16) and geodesics equations of motion (22)-(25), one can obtain the expressions of celestial co-ordinates $\alpha$ and $\beta_i$, given by

$$
\alpha = \frac{-\xi}{D-3 \prod_{i=1}^{D-3} \sin \theta_i},
$$

$$
\beta_i = \pm \sqrt{\eta - \xi^2 \prod_{i=1}^{D-3} \cot^2 \theta_i}.
$$

The Eqs. (35)-(36) relates the celestial coordinate $\alpha, \beta_i$ to constant of motion $\eta$ and $\xi$. If we take equatorial plane ($\theta_i = \pi/2$), the celestial coordinate reduces to

$$
\alpha = -\xi, \quad \beta = \pm \sqrt{\eta},
$$

the Eq. (37) must follow the condition

$$
\eta + \xi^2 = \alpha^2 + \beta^2 = \frac{D - 1}{D - 3} \left[ \frac{16\pi M(D - 1)}{2(D - 2)\Omega_{D-2}} \right] \frac{\eta^2}{\xi^2}.
$$

The Eq. (38) governs the complete orbit of photon around black hole which cast shadow and appears as circle. Now we take the contour plot of Eq. (38) which shows the shadow of Schwarzschild-Tangherlini black hole, clearly shown in Fig. [2 and 3]. When we go to the HD the effective size of shadow decreases, which physically represents a photon sphere decreases with the HD of spacetime. For $D = 4$, the Eq. (38) reduces to the most general case of a Schwarzschild black hole, reads

$$
\eta + \xi^2 = \alpha^2 + \beta^2 = 27M^2,
$$

and the contour plot of Eq. (39) represent shadow of Schwarzschild black hole which has been shown in first plot of Fig. [3].
FIG. 3: Shadow cast by Schwarzschild-Tanglerlini black holes at $\theta_i = \pi/2$ for different mass parameter $M$ and dimension $D$. 
In this section, we study the energy emission rate from Schwarzschild-Tangherlini black hole. The energy emission rate can be calculated via \[ \frac{d^2E(\omega)}{d\omega dt} = \frac{2\pi^2\sigma_{lim}}{\exp (\omega/T_{BH}) - 1}\omega^3, \] where $T_{BH}$ is Hawking temperature for Schwarzschild-Tangherlini black holes which is shown in Eq. (11) and $\sigma_{lim}$ is the limiting constant value and in HD it can be expressed as $\sigma_{lim} \approx \frac{\pi^{(D-2)}}{\Gamma \left(\frac{D}{2}\right)} R_s^{D-2}$. 

FIG. 4: Plot showing the behavior of the radius of black hole shadow $R_s$ with dimension $D$.

FIG. 5: Plot showing the variation of energy emission rate for frequency $\omega$ in different dimension $D$.

A. Energy emission rate
for Schwarzschild case \((D = 4)\) the value of limiting constant reduced to

\[
\sigma_{\text{lim}} \approx \pi R_s^2,
\]

where \(R_s\) is radius of shadow. As we have shown in Fig. 2 and 3, the shadow of black hole is circle and the radius of shadow \(R_s\), which can also be a observable as \([14]\)

\[
R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2(\alpha_t - \alpha_r)},
\]

where \((\alpha_t, \beta_t)\) and \((\alpha_r, 0)\) are the top, bottom and right positions of co-ordinates from where’s reference circle passes. The complete form of energy emission of black hole in terms of dimensions \(D\) can be reads

\[
d^2E(\omega) \frac{d\omega}{dt} = \frac{2\pi^{D+2} R_s^{D-2}}{(e^{\omega/T} - 1)\Gamma\left(\frac{D}{2}\right)}\omega^3.
\]

The variation of \(d^2E(\omega)/d\omega dt\) vs \(\omega\) can be seen from Fig. 5 for different dimension \(D\).

V. CONCLUSION

The black hole shadow in the near future may be realistic due to an observation of black hole SgrA* in the center of our galaxy. It was Synge \([9]\) and Luminet \([10]\) who first calculated the shadow of non-rotating Schwarzschild black hole, and demonstrated that circular light orbit exists on photon sphere at \(r = 3m\), which is the critical radius where the effective potential has a maximum. We have generalized this and other results to study the shadow cast by Schwarzschild-Tangherlini black holes by studying the motion of a test particle and derive the complete null geodesics equations by applying Hamilton-Jacobi equation and Carter separable method. We also derive the expression of effective potential from the radial equation of motion and found that it has spacetime dimension dependence. It turns out that apparent shape of the Schwarzschild-Tangherlini black holes is also a function of spacetime dimension and that the size of black hole shadows increases with increase in the spacetime dimensions and so is energy emission rate. We also observe the deviation of the peak of effective potential towards the central object.

The results presented here are the generalization of previous discussions, on the Schwarzschild black hole shadow, in more general setting, and the possibility of a further generalization of these results to HD rotating Kerr black hole, Myers-Perry black hole and Lovelock black holes is an interesting problem for future research.
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