Instability criterion for oblique modes in stratified circular Couette flow

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Abstract

An analytical approach is carried out that provides an inviscid stability criterion for the strato-rotational instability (in short SRI) occurring in a Taylor-Couette system. The control parameters of the problem are the rotation ratio $\mu$ and the radius ratio $\eta$. The study is motivated by recent experimental [1] and numerical [2, 3] results reporting the existence of unstable modes beyond the Rayleigh line for centrifugal instability ($\mu = \eta^2$). The modified Rayleigh criterion for stably stratified flows provides the instability condition, $\mu < 1$, while in experiments unstable modes were never found beyond the line $\mu = \eta$. Taking into account finite gap effects, we consider non axisymmetric perturbations with azimuthal wavenumber $l$ in the limit $lFr << 1$, where $Fr$ is the Froude number. We derive a necessary condition for instability: $\mu < \mu^*$ where $\mu^*$, a function of $\eta$, takes the asymptotic values, $\mu^* \rightarrow 1$ in the narrow gap limit, and $\mu^* \rightarrow 2\eta^2(1 + \eta)$ in the wide gap limit, in agreement with recent numerical findings. A stronger condition, $\mu < \eta$, is found when $\eta > 0.38$, in agreement with experimental results obtained for $\eta = 0.8$. Whatever the gap size, instability is predicted for values of $\mu$ larger than the critical one, $\mu_c = \eta^2$, corresponding to centrifugal instability.
1 Introduction

Despite its long-lasting study since the work of Taylor [4], the stability of cylindrical Couette flow remains a vivid research area attracting many investigators. A survey of the literature on the topic can be found in [5]. The flow occurs in the annular gap between two concentric cylinders of radii $R_1 < R_2$ rotating independently at the angular velocities $\Omega_1$ and $\Omega_2$, respectively. The control parameters are the radius ratio $\eta = R_1/R_2$ and the rotation ratio $\mu = \Omega_2/\Omega_1$. Inside the gap, the angular velocity profile of the laminar steady flow is $\Omega(r)$. Its linear stability with respect to axisymmetric perturbations is governed, in the inviscid limit, by the Rayleigh criterion [6]: $d(r^2\Omega)^2/dr > 0$. A generalized Rayleigh criterion for non-axisymmetric centrifugal instabilities has been derived for a free axisymmetric vortex using a large axial wavenumber WKB approximation [7]. The instability takes the form of a spatially oscillating mode localized between two turning points where it matches with the exponentially decaying solutions outside. The generalization to take into account a background rotation and a stable stratification did not reveal fundamental changes in the Rayleigh criterion. This contrasts with what was found for bounded flows [8, 9, 10]. In these studies, it was shown that a stable stratification drastically changes the stability criterion, showing a strong analogy with the modified Rayleigh criterion obtained when a magnetic field parallel to the cylinders’ axis is present. In the hydromagnetic case [11, 12] the condition for stability with respect to axisymmetric perturbations is: $d(\Omega^2)/dr > 0$. The presence of a stable stratification has a similar effect on the Rayleigh criterion though its validity is in that case restricted to non-axisymmetric perturbations [8, 9, 10].

In the astrophysical context, the magneto-rotational instability (MRI) was recognized as a potential source of turbulence in accretion disks [13], especially Keplerian disks, with angular velocity $\Omega(r) \sim r^{-3/2}$, unable to sustain centrifugal instability according to the standard Rayleigh criterion. The MRI mechanism is robust, it occurs in bounded as well as unbounded flows and for compressible or incompressible fluid. The capacity of the strato-rotational instability (SRI) to trigger turbulence in Keplerian disks raises the open question of what are the appropriate radial boundary conditions at the edges of the disk. The existence of SRI has been demonstrated in the inviscid limit for perturbations that satisfy no normal flow conditions on the channel walls. In the viscous case, both the no-slip [2] and stress-free [9] conditions give rise to instability. Mixed boundary conditions in which no normal flow
is imposed on one side of the channel and zero pressure on the other side were not able to sustain growing modes [10]. The effect of a vertically varying stratification was also examined [10] showing that SRI persists in that case.

Beside possible applications in geophysics and astrophysics, SRI was studied in the laboratory. Until very recently the conclusion of experimental studies was that a stable vertical stratification stabilizes the flow [14, 15, 16]. The experimental evidence of the strato-rotational instability (SRI) in a Taylor-Couette system was definitely assessed in [1]. Moreover, the values of the control parameters for which instability occurs were found in good agreement with numerical predictions [2] that yields the condition : \( \mu < \eta \), for instability.

The aim of the present contribution is to derive an instability criterion by an entirely analytical analysis that improves previous studies achieved in the small gap limit [8, 9, 10]. When curvature effects are neglected the problem under consideration reduces to the stability of a stratified plane Couette flow rotating at constant angular velocity \( \Omega \). In cartesian coordinates \((x, y, z)\) the flow velocity is \( V = (0, Sx, 0) \) where \( S \) is the constant shear. The condition for instability in the stratified case is \( S/2\Omega < 0 \), while the standard Rayleigh-Pedley criterion gives : \( S/2\Omega < -1 \), when there is no stratification. The substitution \( S \rightarrow r\Omega' \), is often used to deduce the modified Rayleigh criterion for stratified flows with curved streamlines : \( d(\Omega^2)/dr < 0 \). When applied to the circular Couette flow, the instability criterion simply yields : \( \mu < 1 \). However, in experiments [1] achieved for a finite value of the gap, unstable modes were never found beyond the line \( \mu = \eta \), in agreement with previous numerical results [2]. Quite recently, the situation has changed since numerical calculations [3] have shown that the stability line has a more complicated dependence on \( \eta \). In the narrow gap limit, the stability limit was found beyond the line \( \mu = \eta \), while in the wide gap limit it was found in between the lines \( \mu = \eta^2 \) and \( \mu = \eta \). The narrowing of the instability range, when the gap size increases, is possibly due to curvature effects that were not taken into account with sufficient accuracy in the transposition \( S \rightarrow r\Omega' \). In the present contribution a more appropriate treatment of curvature effects is carried out leading to an instability criterion that involves the two parameters \( \mu \) and \( \eta \). Our stability results will be compared to the experimental ones [1] obtained for a value of the radius ratio \( \eta = 0.8 \), and to the numerical ones [2, 3] for three values of the gap in the range \( 0.3 \leq \eta \leq 0.78 \).
2 Stratified circular Couette flow

In cylindrical coordinates \((r, \varphi, z)\) the velocity field in the basic state is \(V = (0, r\Omega(r), 0)\) with the angular velocity given by [2]

\[
\Omega(r) = \Omega_1 \left( \frac{A}{r^2} + B \right) = \Omega_1 \tilde{\Omega}(r) \quad \text{where} \quad A = R_1^2 \frac{1 - \mu}{1 - \eta^2}, \quad \text{and} \quad B = \frac{\mu - \eta^2}{1 - \eta^2}
\] (1)

The fluid is assumed incompressible with a stable density stratification along the vertical cylinders’ axis \((\partial \rho / \partial z < 0)\).

2.1 Linearized equations for perturbations

The governing linearized equations for the perturbed velocity \(u = (\tilde{u}, \tilde{v}, \tilde{w})\), the pressure and entropy perturbations, respectively \(\tilde{p}\) and \(\tilde{h}\) are given in [8]. In the study of stratified plane Couette flow [9, 10] it was particularly convenient to reduce the full set of equations to a set of coupled equations for the radial velocity component \(\tilde{u}\) and the pressure \(\tilde{p}\). Following the same procedure, the perturbed quantities are sought in the form \((\tilde{u}, \tilde{p}) = \exp (imz + i\varphi + \omega t)\) where \(m\) and \(l\) are respectively the axial and azimuthal wavenumbers, these notations are consistent with [8] but not with [2] where \(m\) has a different meaning. The radial dependencies \(u(r)\) and \(p(r)\) satisfy

\[
\sigma D_s u + \frac{l}{r} Z u = -i \frac{G}{K} p
\] (2)

\[
(\sigma^2 - 2\Omega Z)u - 2i \frac{l}{r} \Omega p = -i \sigma Dp,
\] (3)

where \(D \equiv \partial / \partial r\) and \(D_s = D + 1/r\). Here, \(\sigma = \omega - l\Omega\), and \(Z = 2\Omega + r\Omega'\). The quantities \(K\) and \(G\) are given below

\[
K = \sigma^2 - N^2, \quad \text{and} \quad G = \sigma^2 m^2 + \frac{l^2}{r^2} K
\] (4)

where \(N\) is the Brunt-Väisälä frequency. For circular Couette flow with angular velocity [11], the value of \(Z = 2\Omega_1 B\), is a constant. Thus, the Rayleigh discriminant, \(\Phi = 2\Omega Z\), is proportional to \(\Omega\) and it can be written:

\[
\Phi = 4\Omega_1^2 B \tilde{\Omega} \equiv \Omega_1^2 \tilde{\Phi}
\] (5)

The system (2)-(3) can be reduced to a second order differential equation for the radial component \(u\), as in [7 8]. However, such a formulation leads to tractable results only in a
few cases, for instance in the large axial wavenumber limit ($m >> 1$) or the small gap limit ($\eta \to 1$). The present stability analysis will closely follow the procedure done in [10] for the plane Couette flow. The main difference is the consideration of curvature effects. However, the asymptotic analysis [10] based on the small azimuthal wavenumber assumption cannot be transposed directly in the circular case where $l$ takes integer values: $l = 1, 2, \cdots$. In the circular case, we shall assume the following scalings

$$\omega = l\Omega_0 \bar{\omega}, \quad \Omega = \Omega_0 \bar{\Omega}, \quad \Rightarrow \quad \sigma = l\Omega_0 \bar{\sigma}, \quad K = -N^2 (1 - l^2 Fr^2 \bar{\sigma}^2)$$

where $\Omega_0 = (\Omega_1 + \Omega_2)/2$, is the mean angular velocity and $Fr = \Omega_0 / N$, the Froude number. The asymptotic value $K \to -N^2$ obtained in the limit $l << 1$ [10] is recovered here in the limit $lFr << 1$, that can be reached either when $l << 1$ and $Fr \approx O(1)$ or $l \approx O(1)$ and $Fr << 1$. Another important assumption concerns the term $H = \sigma^2 - 2\Omega Z$ in Eq. (3) that was approximated in [10] by $H \to -\Phi$ where $\Phi$ the Rayleigh discriminant given in Eq. (5) is now written $\Phi = \Omega_0^2 \bar{\Phi}$ leading to $H = \Omega_0^2 (l^2 \bar{\sigma}^2 - \bar{\Phi})$. Noticing that $\bar{\Phi}$ can takes values as large as $4\Omega_1^2 / \Omega_0^2$, the assumption $l << 1$ will be replaced by the weaker condition: $l < l_{max}$ with $l_{max} = 2\Omega_1 / \Omega_0$. To illustrate our purpose, this gives the following constraints: $l < 2.2$ for $\mu = 0.8$ and $l < 3$ for $\mu = 0.3$. Thus, the lower is $\mu$, the higher is the allowed value of $l$. In the following, we shall consider the limit $lFr << 1$ with $l \leq l_{max}$. This leads to simplifications in Eqs. (2)-(3) that are listed below

$$K \to -N^2, \quad H \to -\Phi, \quad \frac{G}{K} = -l^2 \left(m^2 \bar{\sigma}^2 - \frac{1}{r^2} \right) = -l^2 g(r)$$

where $\bar{m} = mFr$.

It must be stressed that the generalization of the Rayleigh criterion [7] performed in the limit $m >> 1$, is not equivalent to the approach described here. The scaling used in [7] for the frequency, suitable in describing inertial-gravity waves, could explain why SRI was not found in this study.

2.2 Governing equations in the limit $lFr << 1$ and $l < l_{max}$

The perturbations $u$ and $p$ are scaled according to

$$u = l\bar{u}, \quad \text{and} \quad p = \Omega_0 \bar{p}$$

(8)
and substituted in (2)-(3) leading to the following set of equations:

\[ \bar{\sigma} D_x \bar{u} + \frac{\bar{Z}}{r} \bar{u} = i g \bar{p} \tag{9} \]

\[ \bar{\Phi} \bar{u} + 2i \frac{\bar{\Omega}}{r} \bar{p} = i \bar{\sigma} D \bar{p} \tag{10} \]

where \( \bar{\Omega} = \Omega/\Omega_0 \) and \( \bar{Z} = Z/\Omega_0 \). We have reported in Appendix A the equations that should be solved for \( l \geq l_{\text{max}} \) when the limit \( H \to -\Phi \) is no longer valid.

Eqs. (9)-(10) are the circular version of those obtained for a plane Couette flow \([10]\), they constitute a set of coupled equations provided \( \Phi \neq 0 \). It was shown in \([10]\) that elimination of \( \bar{u} \) leads to a simple equation for \( \bar{p} \), its two independent solutions being exponential functions. Then substituting \( \bar{p} \) in (10) gives immediately the expression for \( \bar{u} \). Finally, satisfaction of the conditions \( \bar{u} = 0 \), on the two boundaries confining the flow leads to the instability criterion.

The same procedure is applied here to the cylindrical case. After some calculations, the first step gives the equation satisfied by \( \bar{p} \)

\[ \bar{\sigma} \left[ D^2 \bar{p} + \left( \frac{1}{r} - \frac{\Phi'}{\Phi} \right) D \bar{p} - \bar{m}^2 \bar{\Phi} \bar{p} \right] + 2 \frac{\bar{\Omega}}{r} \left[ \frac{\partial}{\partial r} \log \left( \frac{\bar{\Phi}}{\bar{\Omega}} \right) \right] \bar{p} = 0 \tag{11} \]

In the general case, the solutions of (11) cannot be expressed in terms of simple analytical functions. However, for circular Couette flow with angular velocity given by (1) the Rayleigh discriminant is proportional to \( \bar{\Omega} \) and the derivative of the logarithmic term in (11) vanishes. This remarkable feature occurs for angular velocity profiles of the type \( \Omega(r) = Cr^x + D \), where \( C \) and \( D \) are constant coefficients. This allows considerable simplification of Eq. (11) that, provided \( \bar{\sigma} \neq 0 \), becomes

\[ D^2 \bar{p} + \left( \frac{1}{r} - \frac{\Omega'}{\Omega} \right) D \bar{p} - \bar{m}^2 \bar{\Phi} \bar{p} = 0. \tag{12} \]

It should be noticed that the term in the left-hand-side of Eq. (12) is reminiscent of the potential vorticity describing the content of a Rossby wave as mentioned in [10]. In the small gap limit, \( \eta \to 1 \), with \( r/R_1 = 1 + (1 - \eta)(x/\eta) \) and \( \Omega \) constant, the term in front of \( D \bar{p} \) disappears from (12) thus recovering the solution \( \bar{p} = \exp(\pm \sqrt{\bar{\Omega} x}) \) found in [10] for the stratified rotating plane Couette flow when the Rayleigh criterion for stability, \( \Phi > 0 \), is satisfied. Then, introducing \( \hat{p} = \bar{p}/\sqrt{\bar{\Omega}} \), it satisfies

\[ \mathcal{L}_0 \hat{p} + \left[ \frac{A}{\Omega^2 r^4} \left( 2 - \frac{3A}{\Omega^2 r^2} \right) - 4\bar{m}^2 B \bar{\Omega} \right] \hat{p} = 0 \tag{13} \]
with the differential operator $\mathcal{L}_0 = D_s D$ accounting for curvature effects. In Eq. (13) use has been made of the identity $\bar{m}^2 \bar{\Phi} = \hat{m}^2 \hat{\Phi}$, where $\hat{m} = m\Omega_1/N$ and $\hat{\Phi} = 4B\hat{\Omega}$. We shall assume that $\hat{m}$ is large enough so that the term in $\hat{m}^2$ dominates in Eq. (13). Our analysis differs from previous studies that linearized the angular velocity: $\Omega(r) = \Omega(r_0) + (r - r_0)\Omega' + \cdots$ around a mean radius $r_0$ as in the thin gap limit. Here, the exact expression of $\hat{\Omega}(r)$ given in (11) is substituted in (13) that becomes

$$\mathcal{L}_0 \hat{p} - 4\hat{m}^2 B \left( \frac{A}{r^2} + B \right) \hat{p} = 0.$$  

(14)

When $A$ and $B$ have the same sign, the solutions of (14) are the modified Bessel functions $I_n(\alpha r)$ and $K_n(\alpha r)$ of general order $n$ real positive with $n = 2\hat{m}\sqrt{AB}$ and $\alpha = 2\hat{m}B$. The expression for $\hat{p}$ is

$$\hat{p} = A_0 I_n(\alpha r) + B_0 K_n(\alpha r)$$  

(15)

where the unknown coefficients $A_0$ and $B_0$ will be determined by the satisfaction of the boundary conditions on $\bar{u}$. Substituting (15) in (10) and using the relation between the Bessel functions and their derivatives, one gets $\bar{u} = \hat{u}/\sqrt{\bar{\Omega}}$ with

$$\hat{u} = A_0 U_1(r) + B_0 U_2(r)$$  

(16)

Using vector notations, the two functions $U_1(r)$ and $U_2(r)$ are considered as components of the bidimensional vector $\mathbf{U} = (U_1, U_2)$. Similarly, introducing the vector $\mathbf{\Psi}_\nu = (I_\nu(\alpha r), K_\nu(\alpha r))$ for $\nu \in (n, n-1)$ gives the expression for $\mathbf{U}$

$$\mathbf{U} = 2\frac{\bar{\Omega}}{r} \mathbf{\Psi}_n + \bar{\sigma} \left( \frac{n}{r} - \frac{\Omega'}{2\bar{\Omega}} \right) \mathbf{\Psi}_n + \alpha \mathbf{\Psi}_{n-1}.$$  

(17)

In the following, we shall neglect the terms in $\Omega'/\bar{\Omega}$ that appear in factor of $\bar{\sigma}$, which are small compared to the terms proportional to $n$ or $\alpha$ which behave like $\hat{m}$.

Satisfaction of the boundary conditions $\bar{u}(R_1) = \bar{u}(R_2) = 0$ provides the algebraic system

$$\mathbf{M} \mathbf{A} = 0$$  

(18)

with the vector $\mathbf{A} = (A_0, B_0)$ and the elements of the matrix $\mathbf{M}$ given by $M_{ij} = U_j(R_i)$. Their expressions are easily deduced from

$$\mathbf{U}(R_i) = \frac{\lambda_i}{R_i} \mathbf{\Psi}_n^i + \alpha \sigma_i \mathbf{\Psi}_{n-1}^i$$  

(19)
where $\sigma_i = \bar{\omega} - \bar{\Omega}_i$ and $\lambda_i = 2\bar{\Omega}_i + n\sigma_i$ with $\bar{\Omega}_i = \bar{\Omega}(R_i)$ for $i = 1, 2$. We have also introduced $\Psi_n^i = (I_n^{(i)}, K_n^{(i)})$ where the abbreviations $I_n^{(i)} = I_n(\alpha R_i)$ and $K_n^{(i)} = K_n(\alpha R_i)$ have been used.

2.3 Dispersion relation

The vanishing of the determinant associated to the algebraic system in Eq. (18) leads to the dispersion relation

$$\frac{\lambda_1 \lambda_2}{R_1 R_2} S_n - \alpha^2 \sigma_1 \sigma_2 S_{n-1} + \alpha \sigma_2 \frac{\lambda_1}{R_1} C_{12} - \alpha \sigma_1 \frac{\lambda_2}{R_2} C_{21} = 0 \quad (20)$$

where the following quantities have been introduced :

$$S_n = I_n^{(1)} K_n^{(2)} - I_n^{(2)} K_n^{(1)} \quad \text{and} \quad C_{ij} = I_n^{(i)} K_n^{(j)} + I_n^{(j)} K_n^{(i)} \quad \text{for} \quad (i, j) \in (1, 2). \quad (21)$$

To simplify the calculations we shall use the asymptotic expansions at large arguments of the modified Bessel functions $I$ and $K$. At the lowest order, the asymptotic behaviors of expressions (21) are:

$$S_n = S_{n-1} \approx \text{sinh}(\alpha_1 - \alpha_2) \quad \text{and} \quad C_{12} = C_{21} \approx \text{cosh}(\alpha_1 - \alpha_2) \quad (22)$$

Introducing $q = \alpha(R_2 - R_1)$, the dispersion relation (20) reads

$$T_1 - T_2 \coth q = 0 \quad (23)$$

where $T_1 = \left(\frac{\lambda_1 \lambda_2}{R_1 R_2} - \alpha^2 \sigma_1 \sigma_2\right)$ and $T_2 = \alpha \left(\frac{\sigma_1}{R_1} - \frac{\sigma_2}{R_2}\right) \quad (24)$

are second order polynomials in $\bar{\omega}$ that will be expressed in terms of the rate of shear $S$ given below:

$$S = \frac{\mu - 1}{\mu + 1} < 0, \quad \Rightarrow \quad \begin{cases} \sigma_1 = \bar{\omega} - 1 + S \\ \sigma_2 = \bar{\omega} - 1 - S \end{cases} \quad (25)$$

Doing the change of variable $\hat{\omega} = \bar{\omega} - 1$, expressions (24) becomes

$$T_1 = X(\hat{\omega}^2 - S^2) + \frac{4n}{R_1 R_2} (\hat{\omega} + S^2) + \frac{4(1 - S^2)}{R_1 R_2} \quad (26)$$

$$T_2 = \frac{\alpha}{R_1} \left\{2 \left[(1 - \eta)(\hat{\omega} + S^2) - (1 + \eta)S(1 + \hat{\omega})\right] + n(1 - \eta)(\hat{\omega}^2 - S^2)\right\} \quad (27)$$

The quantity $X$ involved in expression (26) for $T_1$ can be expressed in terms of the control parameters, $\mu$ and $\eta$, as follows

$$X = \frac{n^2}{R_1 R_2} - \alpha^2 \equiv 4m^2 B \frac{\eta - \mu}{1 - \eta} \quad (28)$$
2.4 Instability criterion

It is worth while noticing that the sign of the quantity $\eta - \mu$ which appears in expression (28) for $X$ seems determinant for the stability of the system according to what has been observed both in experiments [11] and in numerical computations [2]. To check whether our calculations support this finding we shall determine the roots of the dispersion relation (23). Instability could occur if there is a pair of complex conjugate roots, one with negative imaginary part corresponding to instability growth, the other to decay. The calculation of the discriminant and the determination of its sign is facilitated by introducing the following quantities

$$t_0 = X - n(1 - \eta)Q, \quad \text{with} \quad Q = \frac{\alpha}{R_1} \coth q, \quad (29)$$

and

$$t_1 = \frac{4n}{R_1 R_2} - 2(1 - \eta)Q, \quad t_2 = 2(1 + \eta)SQ. \quad (30)$$

The dispersion relation (23) now reads

$$\hat{\omega}^2 t_0 + \hat{\omega}(t_1 + t_2) - S^2(t_0 - t_1) + t_2 + \frac{4}{R_1 R_2}(1 - S^2) = 0 \quad (31)$$

and the associated discriminant is

$$\Delta = (t_1 + t_2)^2 - 4t_0 \left[-S^2(t_0 - t_1) + t_2 + \frac{4}{R_1 R_2}(1 - S^2)\right] \quad (32)$$

Instability could occur if the discriminant is negative. To determine the sign of $\Delta$ two cases will be considered according to the sign of $t_0$. Expression (29) for $t_0$ shows that it is necessarily negative when $X < 0$, which occurs for $\mu > \eta$. On the opposite side, when $X > 0$ for $\mu < \eta$, $t_0$ can take positive as well as negative values. The change of sign of $t_0$ occurs when

$$X = n(1 - \eta)Q \quad (33)$$

Replacing $X$, $n$ and $Q$ by their expressions in terms of $\mu$ and $\eta$, the above equation becomes

$$\coth q = b_0, \quad \text{with} \quad b_0 = \frac{(1 + \eta)(\eta - \mu)}{(1 - \eta)\sqrt{(1 - \mu)(\mu - \eta^2)}} \quad (34)$$

Eq. (34) can only be satisfied if $b_0 > 1$. Writing that the square of $b_0$ is larger than unity, gives the following relation between $\mu$ and $\eta$

$$2(1 + \eta^2)\mu^2 - \mu[(1 + \eta^2)^2 + 4\eta^2] + 2(1 + \eta^2)\eta^2 > 0 \quad (35)$$
The left-hand-side of (35) is a second order polynomial in \( \mu \) with coefficients depending on \( \eta \). It takes a positive value for values of \( \mu \) ranging outside the interval between the two roots \( \mu_- = 2\eta^2/(1+\eta^2) \) and \( \mu_+ = (1+\eta^2)/2 \). The root \( \mu_+ > \eta \), introduced when taking the square of \( b_0 \) in (34) is spurious and will not be considered. Thus, the meaningful condition for the satisfaction of \( b_0 > 1 \) is:

\[
\mu < \mu_- \quad \text{with} \quad \mu_- = \frac{2\eta^2}{(1+\eta^2)}
\]  

(36)

Numerical values of \( \mu_- \) are reported in Table 1 for different values of \( \eta \). The above results are summarized as follows. For \( \mu > \mu_- \) and \( b_0 < 1 \), we have \( t_0 < 0 \), whatever the value of \( q \). For \( \mu < \mu_- \), and \( b_0 > 1 \), one can write \( b_0 = \coth q_0 \) and the sign of \( t_0 \) depends on the value of \( q \). One gets

\[
t_0 > 0 \quad \text{when} \quad q > q_0 \quad \text{or} \quad t_0 < 0 \quad \text{when} \quad q < q_0.
\]  

(37)

Considering separately the two cases, \( t_0 < 0 \) and \( t_0 > 0 \), we shall derive the conditions for instability (\( \Delta < 0 \)) in the next sections.

2.4.1 \( t_0 < 0 \)

It was shown in the previous section that \( t_0 \) can be negative for any value of \( \mu \) but with the additional constraint \( \coth q > b_0 \) when \( \mu < \mu_- \). When \( t_0 < 0 \), a necessary condition for instability could be derived from the requirement that the term in factor of \( t_0 \) in Eq. (32) is negative

\[
-S^2(t_0 - t_1) + t_2 + \frac{4}{R_1 R_2} (1 - S^2) < 0
\]  

(38)

Replacing \( t_0, t_1 \) and \( t_2 \) by their expressions (29)-(30) and proceeding to some manipulations, one gets

\[
-S^2 X + SQ [2a + nS(1 - \eta)] + \frac{4}{R_1 R_2} [(1 - S^2) + nS^2] < 0
\]  

(39)

where the positive quantity \( a \) is expressed in terms of \( \mu \) and \( \eta \)

\[
a = 1 - S + \eta(1 + S) = \frac{2(1 + \mu \eta)}{1 + \mu} \leq 2
\]  

(40)

Then, expression (28) for \( X \) is substituted in the left-hand-side of Eq. (39) that is written as the sum of two terms, leading to

\[
P_1 + S^2 P_2 < 0,
\]  

(41)
with
\[ P_1 = \alpha^2 S^2 + 2\alpha S \frac{\alpha}{R_1} \coth q + \frac{4}{R_1 R_2} \]  
\[ P_2 = -\frac{(n-2)^2}{R_1 R_2} + (1-\eta)n \frac{\alpha}{R_1} \coth q \]  

It is worth noticing that in the small gap limit \((\eta \to 1)\) the term \(P_2\) is irrelevant since the terms in \(n\) do not exist and \(S^2 \ll 1\). In this limit, when \(R_1 \sim R_2\) and \(a \sim 2\), Eq. (42) reduces to: \(P_1 < 0\), where \(P_1\) can be factorized, to get
\[ \left(\alpha S + \frac{2}{R_1} \coth \frac{q}{2}\right) \left(\alpha S + \frac{2}{R_1} \tanh \frac{q}{2}\right) < 0 \]  
thus recovering the condition for instability, \(S < 0\), found for stratified rotating plane Couette flows. The instability occurs for values of \(q\) that belong to an interval bounded by the two values, \(q_-\) and \(q_+\), deduced from
\[ \tanh \frac{q_-}{2} = \gamma \frac{q_-}{2}, \quad \text{and} \quad \coth \frac{q_+}{2} = \gamma \frac{q_+}{2} \quad \text{with} \quad \gamma = \frac{\eta |S|}{(1-\eta)}. \]  
For \(\gamma > 1\), the instability region is \(q < q_+\) while for \(\gamma < 1\) the values of \(q\) are in the interval \(q_- < q < q_+\). It is found that \(\gamma = 1\) when \(\mu = 2\eta - 1\). For \(\eta < 0.5\), the value of \(\gamma\) is lower than unity, whatever the value of \(\mu\).

When curvature effects are present, the occurrence of instability could be suppressed when the term \(P_2\) is positive and \(S^2 P_2 > |P_1|\). As it will be intricate to derive a global condition for satisfaction of Eq. (41) it will be replaced by a stronger constraint that consists to impose separately \(P_1 < 0\) and \(P_2 \leq 0\). Assuming that \((n-2)^2 \approx n^2\) in Eq. (43), then \(P_2\) will be negative for values of \(q\) satisfying the condition given below
\[ \coth q \leq \frac{b_1}{(1-\eta)} \quad \text{where} \quad b_1 = \frac{n}{\alpha R_2} = \eta \left(\frac{1-\mu}{\mu-\eta^2}\right)^{1/2} \]  
The upper bound for \(\coth q\) in (46) has to be larger than unity, this occurs for
\[ \mu \leq \mu^* \quad \text{with} \quad \mu^* = \frac{\eta^2[1 + (1-\eta)^2]}{\eta^2 + (1-\eta)^2} \]  
For values of \(\mu\) lower than \(\mu^*\), \(P_2\) will be negative for values of \(q\) such that \(q > q^*\) with \(\coth q^* = b_1/(1-\eta)\). We have checked for some representative values of \(\mu\) and \(\eta\) that values of \(q\) inside the interval \([q_-, q_+]\) are larger than \(q^*\). Therefore, the two conditions \(P_1 < 0\) and \(P_2 < 0\) can be satisfied simultaneously. The stability limit \(\mu^*\) admits asymptotic values. In
the narrow gap limit \((\eta \rightarrow 1)\) it is found that \(\mu^* \rightarrow 1\), which is the stability line for plane Couette flows, while in the wide gap limit \(\mu^* \rightarrow 2\eta^2(1 + \eta)\). Moreover, the value of the gap size for which \(\mu^* = \eta\) corresponds to \(\eta = 0.38\). Numerical values of \(\mu^*\) are reported in Table 1 for different values of \(\eta\).

Eq. \((47)\) provides a necessary condition for instability. To be a sufficient one the positive term \((t_1 + t_2)^2\) in expression \((32)\) for the discriminant should not exceed the negative term. The term \((t_1 + t_2)^2\) could even vanish if

\[
\coth q = b_1 \frac{(1 + \mu)}{(1 - \mu \eta)} \equiv \coth q_{12}
\]  

(48)

The value \(\coth q_{12}\) is consistent with the values of \(\coth q\) allowed by Eq. \((46)\) if

\[
\frac{(1 + \mu)}{(1 - \mu \eta)} \leq \frac{1}{(1 - \eta)} \implies \mu < \eta
\]

(49)

When \(\eta > 0.38\), the condition, \(\mu < \eta\), provides a stronger constraint than the condition \(\mu < \mu^*\) coming from Eq. \((47)\).

We have not found the specific value, \(\mu = \mu_s\), for which the positive and negative terms exactly cancel in the expression for the discriminant written in Eq. \((32)\). Thus, the upper bound \(\eta\) found in Eq. \((49)\) is a low estimate of the exact value \(\mu_s\) which is more likely expected in the range \(\eta \leq \mu_s \leq \mu^*\).

The above results are strictly valid for values of the rotation rate such that \(\mu_- < \mu < \mu^*\). In that case, the constraints on the value of \(q\), which are respectively \(q \in [q_-, q_+]\) and \(q > q^*\), can be satisfied simultaneously. When \(\mu < \mu_-\), an additional constraint has to be satisfied: \(q < q_0\), as shown in Eq. \((37)\). Instability could occur if there is an overlap of the two intervals \([q_-, q_+]\) and \([q^*, q_0]\) which requires that \(q_0 > q_-\). In that case instability occurs for \(q \in [q_-, q_0]\).

2.4.2 \(t_0 > 0\)

In the previous section, when investigating the case \(t_0 < 0\), the existence of instability was assessed for \(\mu_- < \mu < \eta\) and \(q \in [q_-, q_+]\). For \(\eta^2 < \mu < \mu_-\), the instability is restricted to a narrower range of values of \(q\) and its existence was not demonstrated in a systematic way. For \(\mu < \mu_-\), we shall now investigate the case \(t_0 > 0\) that could be more favorable to SRI.

When \(t_0 > 0\), the terms in the left-hand-side of Eq. \((32)\) are rearranged to read

\[
\Delta = (t_1 + t_2 - 2St_0)^2 + 4^2(1 - S)t_0 t_3 \quad \text{with} \quad t_3 = S \left(\frac{n}{R_1R_2} - Q\right) - \frac{(1 + S)}{R_1R_2}
\]

(50)
In that case, the discriminant could be negative if $t_3$ is negative. The change of sign of $t_3$ occurs when

$$\coth q = \frac{1}{R_2} \left( \frac{n}{\alpha} + \frac{(1+S)}{\alpha|S|} \right) = b_1 + b_2$$

(51)

The value of $\coth q$ in Eq. (51) is the sum of two positive contributions. The first contribution coincides with $b_1$ given in Eq. (46), it is independent of $\hat{m}$ and for $\mu \leq \mu_-$ it stands in the range : $1 \leq b_1 \leq b_0$, so that we can write $b_1 = \coth q_1$. The second contribution $b_2 \sim \alpha^{-1}$, that behaves like $\hat{m}^{-1}$ can be neglected for large values of $\hat{m}$. In that case, $t_3$ will be negative for values of $q$ such that

$$\coth q \leq b_1 \quad \text{or} \quad q > q_1,$$

(52)

which is consistent with the condition : $\coth q \leq b_0$, allowing for $t_0 > 0$. As soon as the value of $\hat{m}$ decreases, the contribution $b_2$ increases until it reaches a value such that $b_1 + b_2 = b_0$. In that case, $t_3$ will be negative for values of $q$ corresponding to $\coth q \leq b_0$ which exactly coincides with the condition ensuring $t_0$ is positive. Therefore, considering smaller values of $\hat{m}$ will never give a stronger condition and moreover it will be contradictory with the assumption leading to Eq. (13).

When $t_0 > 0$, the necessary condition for SRI is of the type

$$\coth q < \coth q_{\text{max}} \quad \text{with} \quad q_0 \leq q_{\text{max}} \leq q_1,$$

(53)

meaning that instability might occur for $q > q_{\text{max}}$, the value of $q_{\text{max}}$ depending on the value of $\hat{m}$. To find a sufficient condition for SRI it will be argued as in the previous section where we have looked for the value of $\coth q$ corresponding to the vanishing of the positive terms in the discriminant. In Eq. (50) it occurs when $t_1 + t_2 = 2St_0$, that could be solved only if $t_1 + t_2 < 0$. Having determined in Eq. (48) the value of $\coth q = \coth q_{12}$ for which the sum $t_1 + t_2$ vanishes, the sum will be negative for $\coth q > \coth q_{12}$ or correspondingly $q < q_{12}$. As $\coth q_{12} > \coth q_1$, and consequently $q_{12} < q_1$, the simultaneous satisfaction of $q > q_{\text{max}}$ and $q < q_{12}$ is only possible when $q_{\text{max}}$ belongs to the interval $q_0 < q_{\text{max}} < q_{12}$ which implies to consider values of $\hat{m}$ not too large. In that case, instability will occurs for $q_{\text{max}} < q < q_{12}$. Hence, whatever the sign of $t_0$, the values of $q$ leading to instability are restricted to a limited band.

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2.4.3 Comparison with experimental and numerical results

The experimental results [1] obtained for $\eta = 0.8$ are in qualitative agreement with a first set of numerical results [2] obtained for $\eta = 0.78$. Both studies show that the strato-rotational instability occurs for $\mu < \eta$, in agreement with the condition found in Eq. (49). However, quite recently we have had knowledge of a second set of numerical results [3] that do not confirm these previous findings. The stability line found in [3] is better represented by $\mu = \mu^*$, where $\mu^*$ is given explicitly in Eq. (47). We have reported in Table 1 some values of $\mu^*$ corresponding to values of $\eta$ used in experiments and computations. In the narrow gap limit, the asymptotic value $\mu^* = 1$ is found to agree with the stability line for stratified plane Couette flows. The computations performed in [3] for three values of the gap size and different values of the Froude number ($0.5 < Fr < 2.2$) exhibit stability lines $\mu = \mu_s$ with values of $\mu_s$ above the line $\mu = \eta$ for $\eta = 0.78$ and $\eta = 0.5$, in disagreement with numerical and experimental results obtained earlier [2, 1] for respectively $\eta = 0.78$ and $\eta = 0.8$. For these values of $\eta$, the stability lines in [3] are in the range $\eta < \mu_s < \mu^*$, in reasonable agreement with the results in Eq. (47).

In the wide gap limit, the necessary instability condition (47) takes the asymptotic form $: \mu < 2\eta^2(1 + \eta)$, that fits with a good accuracy the numerical results found in [3] for $\eta = 0.3$. The behavior of the numerical neutral curves (Reynolds number versus $\mu$) is strongly dependent on the azimuthal wavenumber [2] and on the Froude number [3]. These features are not be reproduced by the present analysis based on Eqs. (9)-(10) which are independent of $l$. Although the Froude number appears in these equations through $\tilde{m} = mFr$, the instability conditions derived here are independent of $\tilde{m}$.

| $\eta$ | $\eta^2$ | $\mu_-$ | $\mu^*$ |
|--------|----------|--------|--------|
| 0.8    | 0.64     | 0.78   | 0.978  |
| 0.78   | 0.608    | 0.756  | 0.971  |
| 0.5    | 0.25     | 0.4    | 0.625  |
| 0.3    | 0.09     | 0.165  | 0.231  |

Table 1: Stability limits for centrifugal instability ($\mu_c = \eta^2$) and for SRI ($\mu = \mu^*$) as functions of $\eta$. The line $\mu = \mu_-$ plays a fundamental role to determine the sign of $t_0$. 

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3 Conclusion

We performed an inviscid stability analysis of SRI in a Taylor-Couette system characterized by a Froude number, \( Fr \), measuring the relative importance of rotation and stratification. Non-axisymmetric disturbances were considered with azimuthal wavenumber \( l \) satisfying \( lFr << 1 \) and \( l < l_{max} \).

Finite gap effects were taken into account more appropriately than in previous inviscid approaches. Although assumptions have been made, they never concerned the angular velocity profile which is kept equal to \( \hat{\Omega} = A/r^2 + B \). Thus, gap size effects manifest themselves through the quantities \( A \) and \( B \) which depend on \( \mu \) and \( \eta \), the control parameters of the system. We derived a necessary instability condition, \( \mu < \mu^* \), that fits with a good accuracy the recent numerical results of Ref. [3]. A stronger condition, \( \mu < \eta \), found for \( \eta > 0.38 \), better fits with earlier experimental and numerical results [1, 2]. For the small gap value \( \eta = 0.3 \), the numerical results [2, 3] exhibit a stability line \( \mu = \mu_s \) where \( \mu_s \) is in good agreement with the asymptotic value \( \mu^* \rightarrow 2\eta^2(1 + \eta) \) found for \( \eta \rightarrow 0 \). Unfortunately, in the wide gap limit, experimental results are still lacking for comparison.

The angular velocity profile of circular Couette flow is peculiar since it allows an analytical resolution for the pressure perturbations in terms of Bessel functions. This is an essential step in the above derivation of the stability criterion for SRI in incompressible fluid. A slight change in \( \Omega(r) \) can lead to completely different stability results. A flow with constant angular momentum (\( \Omega \sim r^{-2} \)) obtained when \( B = 0 \) in [1], was considered in a thin cylindrical shell [17]. This type of flow is generally assumed centrifugally stable, its Rayleigh discriminant being equal to zero. When \( \Phi = 0 \), Eqs. (9) and (10) are decoupled and the above analysis for SRI cannot be applied. In that case, the existence of nonaxisymmetric unstable modes was proved for unstratified flow in a compressible fluid [17] with an equation of state of the type \( p \sim \rho^7 \). It will be interesting in future work to extend the present analytical approach to compressible fluids and to other angular velocity profiles. A first step in this direction was achieved in a recent theoretical approach [18] based on shallow-water approximation for annular sections of Keplerian disks.

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Appendix: Limit $lFr << 1$

When there is no other restriction on the value of $l$ than $lFr << 1$, the quantity $H = l^2 \sigma^2 - \Phi$ cannot be simplified. In that case, the governing equations are

\[
\bar{\sigma} D_s \bar{u} + \frac{Z}{r} \bar{u} = ig \bar{p} \tag{A.1}
\]
\[
\bar{H} \bar{u} - 2i \frac{\Omega}{r} \bar{p} = -i \bar{\sigma} D \bar{p} \tag{A.2}
\]

that depend on the azimuthal wavenumber $l$ through $\bar{H}$. Elimination of $\bar{u}$ gives the equation for $\bar{p}$

\[
\bar{\sigma} \left[D^2 \bar{p} + \left(\frac{1}{r} - \frac{H'}{H}\right) D \bar{p}\right] + \left[\bar{\sigma}(\bar{m}^2 \bar{H} - \frac{l^2}{r^2}) + 2 \frac{\Omega}{r} \frac{\partial}{\partial r} \log \left(\frac{\bar{H}}{\Omega}\right)\right] \bar{p} = 0 \tag{A.3}
\]

which has a structure analogous to Eq. (11), the main difference is that $\bar{H}$ appears instead of $\Phi$. The derivative of the logarithmic term is given by

\[
\frac{H'}{H} - \frac{\Omega'}{\Omega} = -l^2 \bar{\sigma} \bar{\Omega}' \frac{(2 \bar{\Omega} + \bar{\sigma})}{H \Omega}
\]

Provided $\bar{\sigma} \neq 0$, Eq. (A.3) becomes

\[
D^2 \bar{p} + \left(\frac{1}{r} - \frac{H'}{H}\right) D \bar{p} + \left[\bar{m}^2 \bar{H} - \frac{l^2}{r^2} - 2l^2 \frac{\bar{H}}{rH} (2 \bar{\Omega} + \bar{\sigma})\right] \bar{p} = 0 \tag{A.4}
\]

The above equation can be simplified by assuming $\bar{m} >> l$ or $m >> lFr^{-1}$. For $l = 1$ and $Fr = 0.5$ this leads to $m >> 2$. As the critical value of the axial wavenumber is not mentioned in [2, 3], it cannot be checked if the assumption is satisfied. After introducing $\bar{p} = \bar{H}^{1/2} \hat{p}$, one gets the following equation for $\hat{p}$

\[
\mathcal{L}_0 \hat{p} + \bar{m}^2 (l^2 \sigma^2 - \Phi) \hat{p} = 0 \tag{A.5}
\]

Eq. (A.5) is a generalization of Eq. (14) that takes into account the value of the azimuthal wavenumber $l$, its resolution is left for future work.

References

[1] M. Le Bars and P. Le Gal, Experimental analysis of the Strato-Rotational Instability in a cylindrical Couette flow, Phys. Rev. Lett. 99, 064502 (2007).

[2] D. A. Shalybkov and G. Rüdiger, Stability of density-stratified viscous Taylor-Couette flows, Astronom. and Astrophys. 438, 411-417 (2005).
[3] G. Rüdiger and D. A. Shalybkov, Stratorotational instability in MHD Taylor-Couette flows, accepted for publication in Astronom. and Astrophys.

[4] G. I. Taylor, Stability of viscous fluid contained between two rotating cylinders, Phil. Trans. Roy. Soc. London A 223, 289-343 (1923).

[5] R. Tagg: A guide to literature related to the Taylor-Couette problem, in Ordered and turbulent patterns in Taylor-Couette flow, edited by C. D. Andereck and F. Hayot (Plenum, New York, 1992).

[6] Lord Rayleigh, On the dynamics of revolving fluids, Proc. Roy. Soc. Cambridge, A 93, 148-154 (1916).

[7] P. Billant and F. Gallaire, Generalized criterion for non-axisymmetric centrifugal instabilities, J. Fluid Mech. 542, 365 (2005).

[8] I. Yavneh, J.C. McWilliams and M.J. Molemaker, Non-axisymmetric instability of centrifugally-stable stratified Taylor-Couette flow, J. Fluid Mech. 448, 1-21 (2001).

[9] B. Dubrulle, L. Marié, C. Normand, D. Richard, F. Hersant and J.-P. Zahn, A hydrodynamic shear instability in stratified disks, Astronom. and Astrophys. 429, 1-13 (2005).

[10] O. M. Umurhan, On the stratorotational instability in the quasi-hydrostatic semigeostrophic limit, Mon. Not. R. Astron. Soc. 365, 85-100 (2006).

[11] S. Chandrasekhar, The stability of non-dissipative Couette flow in hydromagnetics, Proc. Nat. Acad. Sci. 46, 253 (1960).

[12] E. P. Velikhov, Stability of an ideally conducting liquid flowing between cylinders rotating in a magnetic field, J. Exp. Theoret. Phys. 36, 1398-1404 (1959).

[13] S. A. Balbus and J. F. Hawley, Instability, turbulence and enhanced transport in accretion disk. Rev. Mod. Phys. 70, 1-53 (1998).

[14] E. M. Withjack and C. F. Chen, An experimental study of Couette instability of stratified fluids, J. Fluid Mech. 66, 725 (1974).

[15] B. M. Boubnov, E. B. Gledzer and E. J. Hopfinger, Stratified circular Couette flow: instability and flow regimes, J. Fluid Mech. 292, 333-358 (1995).
[16] F. Caton, B. Janiaud and E. J. Hopfinger, Stability and bifurcations in stratified Taylor-Couette flow, J. Fluid Mech. 419, 93-124 (2000).

[17] J. C. B. Papaloizou and J. E. Pringle, The dynamical stability of differentially rotating discs with constant specific angular momentum, Mon. Not. R. Astronom. Soc. 208, 721-750 (1984).

[18] O. M. Umurhan, A shallow-water theory for annular sections of Keplerian disks, to be published in Astronom. and Astrophys. [arXiv:0802.3486v5 [astro-ph]] 2008.