Abstract

We introduce $C^*$-pseudo-multiplicative unitaries and (concrete) Hopf $C^*$-bimodules, which are $C^*$-algebraic variants of the pseudo-multiplicative unitaries on Hilbert spaces and the Hopf-von Neumann-bimodules studied by Enock, Lesieur, and Vallin [5, 6, 4, 10, 19, 20]. Moreover, we associate to every regular $C^*$-pseudo-multiplicative unitary two Hopf-$C^*$-bimodules and discuss examples related to locally compact groupoids.

1 Introduction

Multiplicative unitaries, introduced by Baaj and Skandalis [1], play a central rôle in operator-algebraic approaches to quantum groups [7, 8, 11]. Most importantly, one can associate to every locally compact quantum group a manageable multiplicative unitary, and to every manageable multiplicative unitary two Hopf $C^*$-algebras or Hopf von Neumann-algebras called the “legs” of the unitary. One of these legs coincides with the initial quantum group, and the other one is its generalized Pontrjagin dual.

In this article, we introduce $C^*$-pseudo-multiplicative unitaries and Hopf $C^*$-bimodules, and associate to every regular $C^*$-pseudo-multiplicative unitary two Hopf $C^*$-bimodules, generalizing the construction of Baaj and Skandalis [1]. We think that these two concepts can form the starting point for the development of a theory of locally compact quantum groupoids.

In the setting of von Neumann algebras, a satisfactory theory of locally compact quantum groupoids has already been developed by Lesieur [10], building on the concepts of a pseudo-multiplicative unitary and of a Hopf von Neumann bimodule introduced by Vallin [19, 20]. Our $C^*$-pseudo-multiplicative unitaries and Hopf $C^*$-bimodules turn out to be closely related to their von Neumann algebraic counterparts.

Let us mention that another approach to the problems pursued in this article was developed in the PhD thesis of the author [18]. The approach presented here allows us to drop a rather restrictive condition (decomposability) needed in [18], and to work in the framework of $C^*$-algebras instead of the somewhat exotic $C^*$-families. A comparison between the two approaches is in preparation.

This work was supported by the SFB 478 “Geometrische Strukturen in der Mathematik” [1] and partially pursued during a stay at the “Special Programme on Operator Algebras” at the Fields Institute in Toronto, Canada.

1funded by the Deutsche Forschungsgemeinschaft (DFG)
Organization  This article is organized as follows:

First, we fix notation and terminology, and summarize some background on (Hilbert) $C^*$-modules and on proper KMS-weights.

In Section 2, we introduce some convenient notation and terminology related to a $C^*$-algebraic analogue of Connes’ von Neumann-algebraic relative tensor product of Hilbert spaces. Our simple $C^*$-relative tensor product is based on the internal tensor product of $C^*$-modules and enjoys all properties that one should expect like symmetry, associativity, and functoriality.

In Section 3, we introduce a spatial fiber product of $C^*$-algebras that is based on the $C^*$-relative tensor product of Hilbert spaces. The definition is provisional and lacks several desirable properties like associativity. On the other hand, the construction is functorial, compatible with the fiber product of von Neumann algebras in a natural sense, and well-suited for the definition of Hopf $C^*$-bimodules which is given at the end of this section.

In Section 4, we define $C^*$-pseudo-multiplicative unitaries, using the $C^*$-relative tensor product of Hilbert spaces introduced in Section 2. We show that each such unitary is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [22] and restricts to a pseudo-multiplicative unitary on $C^*$-modules in the sense of Timmermann [24]. To each such unitary, we associate two algebras and two normal $*$-homomorphisms which, under favorable circumstances, form Hopf $C^*$-bimodules. In particular, we adapt the regularity condition known for (pseudo-)multiplicative unitaries [1, 4] to $C^*$-pseudo-multiplicative unitaries and show that if this condition is satisfied, then the algebras and $*$-homo morphisms mentioned above do form Hopf $C^*$-bimodules.

Section 6 is devoted to locally compact groupoids. We discuss the $C^*$-pseudo-multiplicative unitary associated to a locally compact groupoid, show that it is regular, and determine the associated Hopf $C^*$-bimodules.

Preliminaries  Given a subset $Y$ of a normed space $X$, we denote by $[Y] \subset X$ the closed linear span of $Y$.

Given a Hilbert space $H$ and a subset $X \subseteq \mathcal{L}(H)$, we denote by $X'$ the commutant of $X$. Given Hilbert spaces $H$, $K$, a $C^*$-subalgebra $A \subseteq \mathcal{L}(H)$, and a $*$-homomorphism $\pi: A \rightarrow \mathcal{L}(K)$, we put

$$\mathcal{L}^\pi(H, K) := \{T \in \mathcal{L}(H, K) \mid Ta = \pi(a)T \text{ for all } a \in A\};$$

thus, for example, $A' = \mathcal{L}^{id_A}(H)$.

We shall make extensive use of (right) $C^*$-modules, also known as Hilbert $C^*$-modules or Hilbert modules. A standard reference is [9].

All sesquilinear maps like inner products of Hilbert spaces or $C^*$-modules are assumed to be conjugate-linear in the first component and linear in the second one.

Let $A$ and $B$ be $C^*$-algebras. Given $C^*$-modules $E$ and $F$ over $B$, we denote the space of all adjointable operators $E \rightarrow F$ by $\mathcal{L}_B(E, F)$, and the subspace of all compact operators by $K_B(E, F)$.

Let $E$ and $F$ be $C^*$-modules over $A$ and $B$, respectively, and let $\pi: A \rightarrow \mathcal{L}_B(F)$ be a $*$-homomorphism. Then one can form the internal tensor product $E \otimes_\pi F$, which is a $C^*$-module over $B$ [9 Chapter 4]. This $C^*$-module is the closed linear span of elements $\eta \otimes_A \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $(\eta \otimes_A \xi \otimes_B \eta') = (\xi \otimes_B (\eta \otimes_\pi \eta'))$ and $(\eta \otimes_B \eta) b = \eta \otimes_B (\eta b)$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. We denote the internal tensor product by “$\otimes$”; thus, for example, $E \otimes_\pi F = E \otimes F$. If the representation $\pi$ or both $\pi$ and $A$ are understood, we write “$\otimes_A$” or “$\otimes$”, respectively, instead of “$\otimes_\pi$”.

Given $E$, $F$ and $\pi$ as above, we define a flipped internal tensor product $F \otimes_\pi E$ as follows. We equip the algebraic tensor product $F \otimes E$ with the structure maps $\langle (\xi \otimes_\pi \eta) \xi' \otimes_\pi \eta' \rangle := \langle \xi \pi(\eta)\xi' \otimes_\pi \eta' \rangle$, $\langle (\xi \otimes \eta) b \rangle := \langle \eta b \rangle$, and by factoring out the null-space of the semi-norm $\zeta \mapsto \|\langle \zeta \rangle\|^1/2$ and taking completion, we obtain a $C^*$-$B$-module $F \otimes_\pi E$. This is the closed linear span of elements $\xi \otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle (\xi \otimes \eta) \xi' \otimes \eta' \rangle = \langle \xi \pi(\eta) \xi' \otimes \eta' \rangle$. 

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with representations \( \pi \) we write "\( \pi \odot \)" or simply "\( \odot \)" instead of "\( \pi \circ \)" if the representation \( \pi \) or both \( \pi \) and \( A \) are understood, respectively.

Evidently, the usual and the flipped internal tensor product are related by a unitary map \( \Sigma : F \otimes E \xrightarrow{\cong} E \otimes F \), \( \eta \otimes \xi \mapsto \xi \otimes \eta \).

We shall frequently use the following result [3 Proposition 1.34]:

**Proposition 1.1.** Let \( E_1, E_2 \) be \( C^* \)-modules over \( A \), let \( F_1, F_2 \) be \( C^* \)-modules over \( B \) with representations \( \pi_i : A \to \mathcal{L}(E_i) \) \((i = 1, 2)\), and let \( S \in \mathcal{L}_A(E_1, E_2) \), \( T \in \mathcal{L}_B(F_1, F_2) \) such that \( T\pi_i(a) = \pi_i(a)T \) for all \( a \in A \). Then there exists a unique operator \( S \otimes T \in \mathcal{L}_A(E_1 \otimes F_1, E_2 \otimes F_2) \) such that \( (S \otimes T)(\eta \otimes \xi) = S\eta \otimes T\xi \) for all \( \eta \in E_1 \) and \( \xi \in F_1 \). Moreover, \( (S \otimes T)^* = S^* \otimes T^* \).

We shall frequently consider the following kind of \( C^* \)-modules: Let \( H \) and \( K \) be Hilbert spaces. We call a subset \( \Gamma \subseteq \mathcal{L}(H, K) \) a concrete \( C^* \)-module if \( [\Gamma^* \Gamma] = \Gamma \). If \( \Gamma \) is such a concrete \( C^* \)-module, then evidently \( \Gamma^* \) is a concrete \( C^* \)-module as well, the space \( B := [\Gamma^* \Gamma] \) is a \( C^* \)-algebra, and \( \Gamma \) is a full right \( C^* \)-module over \( B \) with respect to the inner product given by \( \langle \xi | \xi' \rangle = \zeta^* \zeta' \) for all \( \zeta, \zeta' \in \Gamma \).

**Lemma 1.2.** Let \( H, K \) and \( L \) be Hilbert spaces and \( \Delta \subseteq \mathcal{L}(H, K) \) \( \Gamma \subseteq \mathcal{L}(K, L) \) be concrete \( C^* \)-modules such that \( [\Gamma^* \Gamma \mathcal{L}] \subseteq \Delta \). Then \( [\Gamma \mathcal{L}] \subseteq \mathcal{L}(H, L) \) is a concrete \( C^* \)-module, and there exists an isomorphism of right \( C^* \)-modules \( \Gamma \otimes_{[\Gamma \mathcal{L}]} \mathcal{L} \cong [\Gamma \mathcal{L}] \), \( \gamma \otimes \delta \mapsto \gamma \delta \).

We shall be primarily interested in the case where \( H = K \) and \( \Delta \subseteq \mathcal{L}(H) \) is a \( C^* \)-algebra.

## 2 The \( C^* \)-relative tensor product

The construction of a \( (C^*)^* \)-relative tensor product of Hilbert spaces is needed for the definition of \( (C^*)^* \)-pseudo-multiplicative unitaries and almost everywhere in the theory of locally compact quantum groupoids.

In the setting of von Neumann algebras, the relative tensor product is constructed as follows. Given a Hilbert space \( K \) with a (normal, faithful, nondegenerate) antirepresentation of a von Neumann algebra \( N \) equipped with a (normal, faithful, semifinite) weight \( \nu \), one defines a subspace \( D(K; \nu) \subseteq K \) of bounded elements and an \( N \)-valued inner product \( \langle \cdot | \cdot \rangle_\nu \) on \( D(K; \nu) \). Given another Hilbert space \( H \) with a normal faithful nondegenerate representation of \( N \), one defines the relative tensor product \( H \odot K \) as the completion of the algebraic tensor product \( H \otimes D(K; \nu) \) with respect to the inner product \( \langle \eta \otimes \xi | \eta' \otimes \xi' \rangle_\nu = \langle \eta | [\xi \otimes \eta']_N \rangle_\nu \).

This construction is symmetric in the sense that one can equivalently define \( H \odot K \) as the completion of an algebraic tensor product \( D(K; \nu) \odot K \), where \( D(H; \nu) \subseteq H \) is a subspace equipped with an \( N^\text{opp} \)-valued inner product.

In this section, we formalize a \( C^* \)-algebraic analogue of this construction. It seems that in the setting of \( C^* \)-algebras, the analogues of the spaces \( D(K; \nu) \) and \( D(H; \nu^\text{opp}) \) can not be reconstructed from an (anti)representation of a \( C^* \)-algebra equipped with a weight alone, but need to be given explicitly in the form of \( C^* \)-factorizations (Subsection 2.1). The definition of the \( C^* \)-relative tensor product is then straightforward (Subsection 2.2) and essentially coincides with the relative tensor product in the setting of von Neumann algebras (Subsection 2.3).

### 2.1 \( C^* \)-bases and \( C^* \)-factorizations

To define a \( C^* \)-algebraic analogue of the relative tensor product \( H \odot K \) described above, we replace

- the von Neumann algebra \( N \) and the weight \( \nu \) by a \( C^* \)-base \( \epsilon \) (Definition 2.1 and Example 2.2), and
• the (anti)representations of $N$ on $H$ and $K$ by $C^*$-(anti)factorizations of $H$ and $K$ (Definition 2.3) relative to $\epsilon$.

**Definition 2.1.** A $C^*$-base is a triple $(\mathfrak{B}, \mathfrak{H}, \mathfrak{B}^1)$, shortly written $\mathfrak{B}_\mathfrak{H}$, consisting of a Hilbert space $\mathfrak{H}$ and two commuting nondegenerate $C^*$-algebras $\mathfrak{B}, \mathfrak{B}^1 \subseteq \mathcal{L}(\mathfrak{H})$.

Two $C^*$-bases $\mathfrak{B}_\mathfrak{H}$ and $\mathfrak{B}'_\mathfrak{H}$ are equivalent if $\mathfrak{C} = \text{Ad}_U(\mathfrak{B})$ and $\mathfrak{C}' = \text{Ad}_U(\mathfrak{B}'^1)$ for some unitary $U : \mathfrak{H} \to \mathfrak{H}$.

Every proper KMS-weight on a $C^*$-algebra gives rise to a $C^*$-base.

**Example 2.2.** Let $B$ be a $C^*$-algebra with a proper KMS-weight $\mu$. As usual, we put $\mathfrak{N}_\mu := \{ b \in B \mid \mu(b^*b) < \infty \}$ and denote by $(H_\mu, \Lambda_\mu, \pi_\mu)$ a GNS-construction for $\mu$, i.e., $H_\mu$ is a Hilbert space, $\Lambda_\mu : \mathfrak{N}_\mu \to H_\mu$ is a linear map with dense image and $\pi_\mu : B \to \mathcal{L}(H_\mu)$ is a representation such that for all $b, b' \in \mathfrak{N}_\mu$ and $c, c' \in B$.

Moreover, we denote by $J_\mu : H_\mu \to H_\mu$ the modular conjugation, which is a conjugate-linear isometric isomorphism. Then the $C^*$-algebras $\pi_\mu(B)$ and $J_\mu \pi_\mu(B) J_\mu$ commute and the triple $(H_\mu, \pi_\mu(B), J_\mu \pi_\mu(B) J_\mu)$ is a $C^*$-base. Moreover, there exists a unitary $\hat{J}_\mu$ such that for all $\beta, \xi \in H_\mu$, we have

$$H_{\mu^{op}} := H_\mu, \quad \Lambda_{\mu^{op}}(b^{op}) := J_\mu \Lambda_\mu(b^*), \quad \pi_{\mu^{op}}(c^{op}) := J_\mu \pi_\mu(c^*) J_\mu$$

for all $b \in \mathfrak{N}_\mu$ and $c \in B$ is a GNS-construction for $\mu^{op}$. In particular, $\pi_{\mu^{op}}(B^{op}) = J_\mu \pi_\mu(B) J_\mu$ and replacing $\mu$ by $\mu^{op}$, we find $\pi_\mu(B) = J_\mu \pi_{\mu^{op}}(B^{op}) J_\mu$.

**Definition 2.3.** A $C^*$-factorization of a Hilbert space $H$ with respect to a $C^*$-base $\mathfrak{B}_\mathfrak{H}$ is a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{H})$ satisfying $[\mathfrak{H}^\alpha, \mathfrak{H}^\alpha] = \mathfrak{B}$, $[\mathfrak{H}^\alpha, \mathfrak{B}] = \alpha$, and $[\mathfrak{H}^\alpha, \mathfrak{H}^\alpha] = H$. We denote the set of all $C^*$-factorizations of a Hilbert space $H$ with respect to a $C^*$-base $\mathfrak{B}_\mathfrak{H}$ by $C^*$-fact$(H; \mathfrak{B}_\mathfrak{H})$.

Let $\alpha$ be a $C^*$-factorization of a Hilbert space $H$ with respect to a $C^*$-base $\mathfrak{B}_\mathfrak{H}$. Then $\alpha$ is a concrete $C^*$-module and a full right $C^*$-module over $\mathfrak{B}$ with respect to the inner product $(\xi|\xi') := \xi^* \xi'$. Moreover, there exists a unitary $\alpha \otimes \mathfrak{B} \xrightarrow{\alpha} H$, $\xi \otimes \zeta \mapsto \xi \zeta$. (1)

From now on, we shall identify $\alpha \otimes \mathfrak{B}$ with $H$ as above without further notice.

By Proposition 1.1 there exists a unique representation

$$\rho_\alpha : \mathfrak{B}^1 \to \mathcal{L}(\alpha \otimes \mathfrak{B}) \cong \mathcal{L}(H)$$

such that for all $b^j \in \mathfrak{B}^1$ and $\xi, \zeta \in \mathfrak{B}$,

$$\rho_\alpha(b^j)(\xi \otimes \zeta) = \xi \otimes b^j \zeta \quad \text{or, equivalently,} \quad \rho_\alpha(b^j)\xi\zeta = \xi b^j \zeta.$$ 

Clearly, this representation is nondegenerate and faithful.

Let $K$ be a Hilbert space. Then each unitary $U : H \to K$ induces a map

$$V_u : C^*$-fact$(H; \mathfrak{B}_\mathfrak{H}) \to C^*$-fact$(K; \mathfrak{B}_\mathfrak{H})$, $\alpha \mapsto V \alpha$.

Let $\beta$ be a $C^*$-factorization of $K$ with respect to $\mathfrak{B}_\mathfrak{H}$. We put

$$\mathcal{L}(H_\alpha, K_\beta) := \{ T \in \mathcal{L}(H, K) \mid T \alpha \subseteq \beta, T^* \beta \subseteq \alpha \}.$$ 

Evidently, $\mathcal{L}(H_\alpha, K_\beta)^* = \mathcal{L}(K_\beta, H_\alpha)$. Let $T \in \mathcal{L}(H_\alpha, K_\beta)$. Then the map

$$T_\alpha : \alpha \to \beta, \quad \xi \mapsto T \xi$$

is a $C^*$-representation of $K$. Let $T_\alpha$. Then

$$T_\alpha : \alpha \to \beta, \quad \xi \mapsto T \xi.$$
is an adjointable operator of $C^*$-modules with adjoint $(T_\alpha)^* = (T^*)_\beta$. With respect to the isomorphism $\alpha \otimes \beta \cong H$, we have $T \equiv T_\alpha \otimes \text{id}_\beta$ and

$$T \rho_\alpha (b^1) \equiv (T_\alpha \otimes \text{id}_\beta)((\text{id}_\alpha \otimes b^1)T_\alpha \otimes \text{id}_\beta) \equiv \rho_\beta (b^1)T \quad (2)$$

for all $b^1 \in B^1$.

We shall consider Hilbert spaces equipped with several $C^*$-factorizations that are compatible in the following sense:

**Definition 2.4.** Let $\alpha$ be a $C^*$-factorization of a Hilbert space $H$ with respect to a $C^*$-base $\mathfrak{B}_{\mathfrak{M}_1}$, and let $\mathfrak{C}_{\mathfrak{K}_1}$ be another $C^*$-base. We call a $C^*$-factorization $\beta \in C^*$-fact($H; \mathfrak{C}_{\mathfrak{K}_1}$) compatible with $\alpha$, written $\alpha \perp \beta$, if $[\rho_\alpha (\mathfrak{B}_1) \beta] = \beta$ and $[\rho_\beta (\mathfrak{C}_1) \alpha] = \alpha$, and put

$$C^*$-fact($H_\alpha; \mathfrak{C}_{\mathfrak{K}_1}$) := $\{ \beta \in C^*$-fact($H; \mathfrak{C}_{\mathfrak{K}_1}$) | \alpha \perp \beta \}.$$

**Example 2.5.** If $\mathfrak{B} \in C^*$-fact($\mathfrak{B}_1; \mathfrak{M}_{\mathfrak{M}_1}$), $\rho_\alpha = \text{id}_{\mathfrak{B}_1}$, $\mathfrak{C}^t \in C^*$-fact($\mathfrak{C}_1; \mathfrak{M}_{\mathfrak{M}_1}$), $\rho_{\mathfrak{B}_1} = \text{id}_{\mathfrak{B}_1}$, and hence $\mathfrak{B} \perp \mathfrak{C}^t$.

**Remark 2.6.** Let $H$, $\mathfrak{B}_{\mathfrak{M}_1}$, $\mathfrak{C}_{\mathfrak{K}_1}$ and $\alpha$, $\beta$ be as in Definition 2.4. If $\alpha \perp \beta$, then $\rho_\alpha (\mathfrak{B}^t) \subseteq \mathcal{L}(H_\beta)$, $\rho_\beta (\mathfrak{C}^t) \subseteq \mathcal{L}(H_\alpha)$, and Equation (2) implies that $\rho_\alpha (\mathfrak{B}^t)$ and $\rho_\beta (\mathfrak{C}^t)$ commute.

### 2.2 Definition and basic properties

Assume that we are given the following data:

Hilbert spaces $H, K$, a $C^*$-base $\mathfrak{B}_{\mathfrak{M}_1}$, and

$C^*$-factorizations $\alpha \in C^*$-fact($H; \mathfrak{B}_{\mathfrak{M}_1}$), $\beta \in C^*$-fact($H; \mathfrak{B}_{\mathfrak{M}_1}$) \quad (3)$

Then we can form the internal tensor product

$H_\alpha \otimes_\beta K := \alpha \otimes \beta$,

and the isomorphism $\alpha$ induces isomorphisms

$$\alpha \otimes \rho_\beta K \cong H_\alpha \otimes_\beta K \cong H_\rho_\alpha \otimes_\beta, \quad \xi \otimes \eta \zeta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta \quad (4)$$

From now on, we shall use these isomorphisms without further mentioning.

The definition of $H_\alpha \otimes_\beta K$ is functorial in the following sense. Let $L, M$ be Hilbert spaces, $\gamma \in C^*$-fact($L; \mathfrak{M}_{\mathfrak{M}_1}$), $\delta \in C^*$-fact($M; \mathfrak{M}_{\mathfrak{M}_1}$), and $S \in \mathcal{L}(H, L)$, $T \in \mathcal{L}(K, M)$. We define an operator

$S \otimes T \in \mathcal{L}(H_\alpha \otimes_\beta K, L_\gamma \otimes_\delta M)$

in the following cases, using Propositions 1.1 and 1.2 and the isomorphisms (4):

i) If $S \in \mathcal{L}(H_\alpha, L_\gamma)$ and $T \rho_\beta (b) = \rho_\beta (b)T$ for all $b \in \mathfrak{B}$, we put $S \otimes T \equiv S \otimes T \in \mathcal{L}(\alpha \otimes \rho_\beta K, \gamma \otimes_\delta M)$.

ii) If $T \in \mathcal{L}(K_\beta, M_\delta)$ and $S \rho_\alpha (b^1) = \rho_\alpha (b^1)S$ for all $b^1 \in \mathfrak{B}$, we put $S \otimes T \equiv S \otimes T \in \mathcal{L}(\rho_\alpha \otimes_\beta K, L_\rho_\delta \otimes M)$. 

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If \( S \in \mathcal{L}(H_\alpha, L_\gamma) \) and \( T \in \mathcal{L}(K_\beta, M_\delta) \), we get \( S \otimes T = S_\alpha \otimes \text{id}_\beta \otimes T_\beta \) in both cases. In the special case where \( S = \text{id}_H \) or \( T = \text{id}_K \), we abbreviate

\[
S_{[1]} := S \otimes \text{id}: H_\alpha \otimes \beta K \to L_\alpha \otimes \beta K, \\
T_{[2]} := \text{id} \otimes T: H_\alpha \otimes \beta K \to H_\alpha \otimes \beta M.
\]

Given C*-algebras \( C, D \) and *-homomorphisms \( \rho: C \to \rho_\alpha(\mathcal{B}^1)' \subseteq \mathcal{L}(H), \sigma: D \to \rho_\beta(\mathcal{B}^2)' \subseteq \mathcal{L}(K) \), we put

\[
\rho_{[1]}: C \to \mathcal{L}(H_\alpha \otimes \beta K), \quad c \mapsto \rho(c) \otimes \text{id}_\beta, \\
\sigma_{[2]}: D \to \mathcal{L}(H_\alpha \otimes \beta K), \quad d \mapsto \sigma(d) \otimes \text{id}_\alpha.
\]

Combining the leg notation and the ket-bra notation, we define for each \( \xi \in \alpha \) and \( \eta \in \beta \) two pairs of adjoint operators

\[
|\xi\rangle_{[1]}: K \to H_\alpha \otimes \beta K, \quad \xi \mapsto \xi \otimes \zeta, \\
|\eta\rangle_{[2]}: H \to H_\alpha \otimes \beta K, \quad \eta \mapsto \zeta \otimes \eta.
\]

We put \( |\alpha\rangle_{[1]} := \{ |\xi\rangle_{[1]} | \xi \in \alpha \} \) and similarly define \( |\alpha\rangle_{[1]}, |\beta\rangle_{[2]}, \langle \beta|_{[2]} \).

**Proposition 2.7.** Let \( H, K, \mathcal{B}, \mathcal{D}, \alpha, \beta \) be as in \( \mathbf{6} \), and let \( \varepsilon \mathcal{K}_{\mathcal{D}^1}, \mathcal{D} \mathcal{L}_{\mathcal{D}^1} \) be C*-bases. Then there exist compatibility-preserving maps

\[
\begin{align*}
C^*\text{-fact}(H_\alpha; \varepsilon \mathcal{K}_{\mathcal{D}^1}) &\to C^*\text{-fact}(H_\alpha \otimes \beta K; \varepsilon \mathcal{K}_{\mathcal{D}^1}), \\
C^*\text{-fact}(K_\beta; \mathcal{D} \mathcal{L}_{\mathcal{D}^1}) &\to C^*\text{-fact}(H_\alpha \otimes \beta K; \mathcal{D} \mathcal{L}_{\mathcal{D}^1}),
\end{align*}
\]

For all \( \gamma \in C^*\text{-fact}(H_\alpha; \varepsilon \mathcal{K}_{\mathcal{D}^1}) \) and \( \delta \in C^*\text{-fact}(K_\beta; \mathcal{D} \mathcal{L}_{\mathcal{D}^1}) \),

\[
\rho_{(\gamma \ominus \delta)} = (\rho_{\gamma})_{[1]}, \quad \rho_{(\delta \ominus \beta)} = (\rho_{\delta})_{[2]}, \quad \gamma \ominus \beta \perp \alpha \ominus \delta.
\]

**Proof.** Let \( \gamma \in C^*\text{-fact}(H_\alpha; \varepsilon \mathcal{K}_{\mathcal{D}^1}) \). Then \( \gamma \ominus \beta = [|\beta\rangle\rangle_{[2]} |\beta\rangle\rangle] \subseteq \mathcal{L}(\mathcal{K}, H_\alpha \otimes \beta K) \) is a C*-factorization of \( H_\alpha \otimes \beta K \) with respect to \( \varepsilon \mathcal{K}_{\mathcal{D}^1} \) because

\[
[|\gamma\rangle\rangle_{[2]} |\beta\rangle\rangle |\beta\rangle\rangle] = [|\gamma\rangle\rangle \rho_{\beta}(\mathcal{B}) |\beta\rangle\rangle] = [|\gamma\rangle\rangle] = \varepsilon, \\
[|\beta\rangle\rangle |\gamma\rangle\rangle |\beta\rangle\rangle] = [|\beta\rangle\rangle |\beta\rangle\rangle |\beta\rangle\rangle] = [|\beta\rangle\rangle |\beta\rangle\rangle |\beta\rangle\rangle] = [|\beta\rangle\rangle |\beta\rangle\rangle |\beta\rangle\rangle].
\]

The relation \( \rho_{(\gamma \ominus \beta)} = (\rho_{\gamma})_{[1]} \) is evident. If \( \gamma' \in C^*\text{-fact}(H_\alpha; \varepsilon \mathcal{K}_{\mathcal{D}^1}) \) and \( \gamma' \perp \gamma \), then \( \gamma' \ominus \beta \perp \gamma \ominus \beta \) because

\[
[\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle
\]

and similarly \( [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\beta\rangle\rangle |\beta\rangle\rangle]. \)

Let \( \delta \in C^*\text{-fact}(K_\beta; \mathcal{D} \mathcal{L}_{\mathcal{D}^1}) \). Similar arguments as above show that \( \alpha \ominus \delta \in C^*\text{-fact}(H_\alpha \otimes \beta K; \mathcal{D} \mathcal{L}_{\mathcal{D}^1}) \), that \( \rho_{(\alpha \ominus \delta)} = (\rho_{\delta})_{[1]} \), and that the map \( \delta' \mapsto \alpha \ominus \delta' \) preserves compatibility. Finally, \( \gamma \ominus \beta \perp \alpha \ominus \delta \) because

\[
[\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\alpha\rangle)_{[1]} |\delta\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\alpha\rangle)_{[1]} |\delta\rangle = [\rho_{(\gamma \ominus \beta)}(\mathcal{C}^1)] |\alpha\rangle)_{[1]} |\delta\rangle
\]

and similarly \( [\rho_{(\alpha \ominus \delta)}(\mathcal{D}^1)] |\beta\rangle)_{[2]} |\gamma\rangle = [\rho_{(\alpha \ominus \delta)}(\mathcal{D}^1)] |\beta\rangle)_{[2]} |\gamma\rangle \). \( \square \)

Evidently, the relative tensor product is symmetric in the following sense:
Proposition 2.8. Let $H, K, \gamma, \alpha, \beta$ be as in (3). There exists a unitary
\[
\Sigma: H_\alpha \otimes_\beta K \cong \alpha \otimes \gamma \otimes \beta \rightarrow \beta \otimes \gamma \otimes \alpha \cong K_\beta \otimes_\delta H,
\]
\[
\xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.
\]
For each $C^*$-base $\epsilon \mathcal{K}_\alpha$ and all $\gamma \in C^*$-fact($H_\alpha; \epsilon \mathcal{K}_\alpha$), $\delta \in C^*$-fact($K_\beta; \epsilon \mathcal{K}_\beta$),
\[
\Sigma_\epsilon(\gamma \circ \beta) = \beta \ast \gamma, \quad \Sigma_\epsilon(\alpha \circ \delta) = \delta \ast \alpha.
\]
The space $\gamma$ together with the $C^*$-factorizations $\mathcal{B}$ or $\mathcal{B}^1$, respectively, is a left or right
unit for the relative tensor product:

Proposition 2.9. Let $H, K, \gamma, \alpha, \beta$ be as in (3). There exist unitaries
\[
\Phi: \gamma \mathcal{B} \otimes_\beta K \rightarrow K,
\]
\[
\Psi: H_\alpha \otimes_\beta \gamma \rightarrow H,
\]
\[
b \otimes \zeta \otimes \eta \mapsto \eta b \zeta,
\]
\[
\xi \otimes \zeta \otimes \eta \mapsto \eta \xi \zeta.
\]
For each $C^*$-base $\epsilon \mathcal{K}_\alpha$ and all $\gamma \in C^*$-fact($H_\alpha; \epsilon \mathcal{K}_\alpha$), $\delta \in C^*$-fact($K_\beta; \epsilon \mathcal{K}_\beta$),
\[
\Phi_\epsilon(\gamma \circ \beta) = \beta, \quad \Phi_\epsilon(\gamma \circ \beta) = \delta, \quad \Psi_\epsilon(\alpha \circ \delta) = \alpha , \quad \Psi_\epsilon(\gamma \circ \beta) = \gamma.
\]

Proof. $\Phi$ and $\Psi$ are obtained by combining (1) and (2). The formulas for $\Phi_\epsilon$ hold because
\[
\Phi_\epsilon(\gamma \circ \beta) = [\Phi(\beta)]_{\epsilon \mathcal{K}_\alpha} = [\beta \mathcal{B}^1] = \beta \ast \gamma \circ \beta, \quad \text{and} \quad \Psi_\epsilon(\alpha \circ \delta) = [\Psi(\delta)]_{\epsilon \mathcal{K}_\beta} = [\alpha \circ \delta] = \alpha \ast \delta \circ \alpha; \text{the}
\]
formulas for $\Psi_\epsilon$ follow similarly.

The relative tensor product is functorial in the following sense:

Proposition 2.10. Let $H, K, \gamma, \alpha, \beta$ be as in (3), $\epsilon \mathcal{K}_\alpha$ a $C^*$-base, and $\gamma \in C^*$-fact($H_\alpha; \epsilon \mathcal{K}_\alpha$), $\delta \in C^*$-fact($K_\beta; \epsilon \mathcal{K}_\beta$),
\[
S \otimes T \in \mathcal{L}(H_\alpha \otimes_\beta K, \gamma) \text{ for all } S \in \rho_\alpha(\mathcal{B}^1) \cap \mathcal{L}(H_\alpha), T \in \mathcal{L}(K_\beta).
\]
\[
S \otimes T \in \mathcal{L}(H_\alpha \otimes_\beta K, \gamma, \delta) \text{ for all } S \in \rho_\alpha(\mathcal{B}^1) \cap \mathcal{L}(H_\alpha), T \in \rho_\beta(\mathcal{B}^1) \cap \mathcal{L}(K_\beta).
\]

Proof. We only prove the first inclusion; the second one follows similarly. For all $S \in \rho_\alpha(\mathcal{B}^1) \cap \mathcal{L}(H_\alpha)$ and $T \in \mathcal{L}(K_\beta)$, we have
\[
(S \otimes T)(\gamma \circ \beta) = [T_\beta S_\gamma] \subseteq [\beta S_\gamma] = \gamma \circ \beta,
\]
and similarly $(S \otimes T)^* \gamma \circ \beta \subseteq \gamma \circ \beta$. The relative tensor product is associative in the following sense:

Proposition 2.11. Let $H, K, L$ be Hilbert spaces, $\gamma, \alpha, \beta$ be as in (3), $\epsilon \mathcal{K}_\alpha$, $\mathcal{B}^1$, $C^*$-bases, and $\alpha \in C^*$-fact($H; \mathcal{B}^1$), $\alpha, \beta \in C^*$-fact($K; \mathcal{B}^1$), $\gamma \in C^*$-fact($K; \mathcal{K}_\alpha$), $\delta \in C^*$-fact($L; \mathcal{K}_\beta$) such that $\alpha \perp \gamma$. Then there exists an isomorphism
\[
\Theta: (H_\alpha \otimes_\beta K) \otimes_\beta L \rightarrow (\alpha \otimes L_\beta) \otimes_\beta K_\alpha \otimes_\beta L,
\]
\[
\Theta_{\ast} \text{ is given by}
\]
\[
(\epsilon \circ \beta) \otimes \delta \mapsto \epsilon \circ (\beta \circ \delta), \quad (\alpha \circ \beta) \otimes \delta \mapsto \alpha \circ (\beta \circ \delta), \quad \text{for all } \epsilon \in C^*$-fact($H_\alpha; \mathcal{B}^1$),
\]
\[
(\alpha \circ \beta) \otimes \delta \mapsto (\alpha \circ \beta) \otimes \delta, \quad \text{for all } \epsilon \in C^*$-fact($K_\beta; \mathcal{B}^1$) s.t. $\beta \perp \epsilon \perp \gamma, (\alpha \circ \beta) \otimes \delta \mapsto (\alpha \circ \beta) \otimes \delta, \quad \text{for all } \epsilon \in C^*$-fact($L_\beta; \mathcal{B}^1$).
2.3 Relation to the relative tensor product

Let \( H, K, \mathfrak{B}_\mathfrak{B} \), \( \alpha, \beta \) be as in \((7)\). If \( \mathfrak{B}_\mathfrak{B} \) arises from a proper KMS-weight \( \mu \) on a \( C^* \)-algebra \( B \) as in Example \( 2.2 \), then \( H\alpha \otimes \beta K \) can be identified with a von Neumann-algebraic relative tensor product as follows.

We use the same notation as in Example \( 2.2 \). The proper KMS-weight \( \mu \) extends to a normal semifinite faithful weight \( \bar{\mu} \) on the von Neumann algebra \( N := \mathfrak{B}'' \), and the maps \( \Lambda_\mu \) and \( \pi_\mu \) extend uniquely to maps \( \Lambda_{\bar{\mu}} \) and \( \pi_{\bar{\mu}} \) such that \((H_\mu, \Lambda_\mu, \pi_\mu)\) becomes a GNS-construction for \( \bar{\mu} \). As in the case of \( C^* \)-algebras, we obtain from this GNS-construction and the modular conjugation \( J_\mu = J_\bar{\mu} \) a GNS-construction \((H_\mu, \Lambda_{\bar{\mu}}; \pi_{\bar{\mu}})\) for the opposite weight \( \bar{\mu}^{op} \) on \( N^{op} \).

Since \( \pi_{\bar{\mu}}; \bar{\mu}^{op} \) commutes with \( \mathfrak{B} = \pi_\mu(B) \) and \( \pi_{\bar{\mu}}(N) \) commutes with \( \mathfrak{B}^{op} = \pi_{\bar{\mu}}(B^{op}) \), respectively, we can extend \( \bar{\rho}_\alpha \) and \( \bar{\rho}_\beta \) to representations

\[
\bar{\rho}_\alpha : N^{op} \longrightarrow \mathcal{L}(\alpha \otimes \bar{\delta}) \cong \mathcal{L}(H),
\]

\[
y \mapsto \text{id}_\alpha \otimes \pi_{\bar{\mu}}^{op}(y^{op}),
\]

\[
x \mapsto \text{id}_\beta \otimes \pi_\mu(x). \]

**Lemma 2.12.** The representations \( \bar{\rho}_\alpha \) and \( \bar{\rho}_\beta \) are faithful, nondegenerate, and normal.

**Proof.** We only prove the assertions concerning \( \bar{\rho}_\alpha \). This representation is faithful and nondegenerate because \( \mathfrak{B} \) is nondegenerate. Let us show that \( \bar{\rho}_\alpha \) is normal. Every normal linear functional on \( \mathcal{L}(H) \) can be approximated in norm by functionals of the form \( \langle \xi| \cdot |\xi' \rangle \), where \( \xi, \xi' \in \alpha \) and \( \zeta, \zeta' \in \bar{\delta} \). Therefore, it suffices to show that for each such \( \omega \), the composition \( \omega \circ \bar{\rho}_\alpha \) is normal. But this holds because \( (\omega \circ \bar{\rho}_\alpha)(x) = \langle \xi|\rho_\alpha(x)|\xi' \rangle = \langle \xi|\xi' \rangle = \langle \xi'|\xi \rangle \) for all \( x \in N \).

The definition of the von-Neumann-algebraic relative tensor product involves the subspace

\[
D(H_\mu; \bar{\mu}^{op}) := \{ \xi \in H \mid \exists C > 0 \forall y \in \mathfrak{B}^{op} : \|\bar{\rho}_\alpha(y)\xi\| \leq C\|\Lambda_{\bar{\mu}}^{op}(y)\| \}. \]

Evidently, an element \( \zeta \in H \) belongs to \( D(H_\mu; \bar{\mu}^{op}) \) if and only if the map \( \Lambda_{\bar{\mu}}^{op}(\mathfrak{B}^{op}) \rightarrow H \) given by \( \Lambda_{\bar{\mu}}^{op}(y) \mapsto \bar{\rho}_\alpha(y)\zeta \) extends to a bounded linear map \( L(\zeta) : H_\mu \rightarrow H \).

**Lemma 2.13.** \( \alpha_0 \Lambda_{\bar{\mu}}(\mathfrak{B}_{\bar{\mu}}) \subseteq D(H_\mu; \bar{\mu}^{op}) \) and \( L(\xi_0 \Lambda_{\bar{\mu}}(x)) = \xi_0 \pi_\mu(x) \) for all \( \xi_0 \in \alpha \) and \( x \in \mathfrak{B}_{\bar{\mu}} \).

**Proof.** By Tomita-Takesaki theory, \( \pi_\mu(x) \Lambda_{\bar{\mu}}^{op}(y) = \pi_{\bar{\mu}}(y) \Lambda_{\bar{\mu}}(x) \) for all \( x \in \mathfrak{B}_{\bar{\mu}} \) and \( y \in \mathfrak{B}^{op} \). Consequently,

\[
\xi_0 \pi_\mu(x) \Lambda_{\bar{\mu}}^{op}(y) = \xi_0 \pi_{\bar{\mu}}(y) \Lambda_{\bar{\mu}}(x) = \bar{\rho}_\alpha(y)\xi_0 \Lambda_{\bar{\mu}}(x) \quad \text{for all} \quad \alpha, x \in \mathfrak{B}_{\bar{\mu}}. \]

The claims follow.

Recall that the relative tensor product \( H_\mu \hat{\otimes}_{\bar{\mu}} K \) of \( H \) and \( K \) with respect to the representations \( \bar{\rho}_\alpha \), \( \bar{\rho}_\beta \) and the weight \( \bar{\mu} \) is the Hilbert space obtained from the algebraic tensor product \( D(H_\mu; \bar{\mu}^{op}) \otimes K \) and the sesquilinear form given by

\[
\langle \zeta \odot \omega | \zeta' \odot \omega' \rangle := \langle \omega | \bar{\rho}_\beta(L(\zeta^*) L(\zeta)) \omega' \rangle
\]

for all \( \zeta, \zeta' \in D(H_\mu; \bar{\mu}^{op}) \) and \( \omega, \omega' \in K \). For all \( \zeta \in D(H_\mu; \bar{\mu}^{op}) \) and \( \omega \in K \), we denote the image of \( \zeta \odot \omega \) in \( H_\mu \hat{\otimes}_{\bar{\mu}} K \) by \( \zeta \hat{\odot} \omega \).

**Proposition 2.14.** There exists a unique unitary

\[
\Phi : H_\mu \hat{\otimes}_{\bar{\mu}} K \rightarrow H_\mu \hat{\otimes}_{\bar{\mu}} K \cong H_\alpha \otimes \beta K \cong \alpha \hat{\otimes}_{\beta} K
\]

such that

\[
\Phi(\theta \hat{\otimes} \eta \zeta) \equiv L(\theta) \zeta \odot \eta \quad \text{for all} \theta \in D(H_\mu; \bar{\mu}^{op}), \eta \in \beta, \zeta \in \bar{\delta}, \quad (5)
\]

\[
\Phi(\xi \Lambda_{\mu}(x) \hat{\otimes} \omega) \equiv \xi \pi_\mu(x) \odot \omega \quad \text{for all} \xi \in \alpha, x \in \mathfrak{B}_{\bar{\mu}}, \omega \in K. \quad (6)
\]
Proof. Uniqueness follows from the relation $[\beta\Phi] = K$ and (4). Existence of a unitary $\Phi$ satisfying (5) follows from the relation

$$
\langle \theta \otimes \eta \zeta | \theta' \otimes \eta' \zeta' \rangle_{(H_{\rho_\theta} \otimes \gamma, K)} = \langle \eta \zeta | \tilde{\rho}_{\beta}(L(\theta)^* L(\theta)) \eta' \zeta' \rangle_{K}
$$

valid for all $\theta, \theta' \in D(H_{\rho_\theta} ; \mu^{op})$, $\eta, \eta' \in \beta, \zeta, \zeta' \in H_\mu$, and from the relation $[L(D(H_{\rho_\theta} ; \mu^{op}));\Phi] = H$. Formula (6) follows from the calculation

$$
\Phi(\xi\Lambda_\mu(x) \otimes \eta\zeta) \equiv L(\xi\Lambda_\mu(x)) \zeta \otimes \eta = \xi\pi_\mu(x) \zeta \otimes \eta \equiv \xi\pi_\mu(x) \otimes \eta\zeta.
$$

3 The spatial fiber product of $C^*$-algebras

In this section, we introduce a spatial fiber product of $C^*$-algebras using the $C^*$-relative tensor product discussed above. More precisely, assume that $H, K, \mathfrak{H}_{\mathfrak{B}_1}$, $\alpha, \beta$ are as in (4), so that we can form the $C^*$-relative tensor product $H_{\alpha} \otimes_\beta K$. Moreover, assume that $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ are $C^*$-subalgebras satisfying $[A_{\rho_\alpha}(\mathfrak{B}_1)] \subseteq A$ and $[B_{\rho_\beta}(\mathfrak{B}_1)] \subseteq B$. Then the spatial fiber product of $A$ and $B$ relative to $\alpha, \beta, \mathfrak{H}_{\mathfrak{B}_1}$ will be a $C^*$-algebra

$$
A_\alpha \otimes_\beta B := \text{Ind}_{[\alpha]_1} (B) \cap \text{Ind}_{[\beta]_2} (A) \subseteq \mathcal{L}(H_{\alpha} \otimes_\beta K),
$$

where $\text{Ind}_{[\alpha]_1} (B)$ and $\text{Ind}_{[\beta]_2} (A)$ are obtained by “inducing up” $B$ and $A$, respectively, in a sense explained in Subsection 3.3.

Like the $C^*$-relative tensor product, the fiber product of $C^*$-algebras is an analogue of a classical construction in the setting of von Neumann algebras.

Our spatial fiber product of $C^*$-algebras lacks several desirable properties, e.g., associativity, and raises several natural questions that we can not answer yet. Nevertheless, it serves our main purpose — to describe the target of the comultiplication of a (concrete) Hopf $C^*$-bimodule, in particular for the legs of a regular $C^*$-pseudo-multiplicative unitary. Most importantly, the construction is functorial with respect to a natural class of morphisms and “sufficiently associative” so that we can formulate a coassociativity condition for the comultiplication of a (concrete) Hopf $C^*$-bimodule.

3.1 Induction of $C^*$-algebras via $C^*$-modules

Let $H$ and $K$ be Hilbert spaces, $\Gamma \subseteq \mathcal{L}(H, K)$ a concrete $C^*$-module satisfying $[\Gamma H] = K$, and put $\mathfrak{B} := [\Gamma \Gamma] \subseteq \mathcal{L}(H)$ and $\mathfrak{C} := [\Gamma^* \star] \subseteq \mathcal{L}(K)$.

Let $A \subseteq \mathcal{L}(H)$ be a nondegenerate $C^*$-algebra satisfying $[A \mathfrak{B}] \subseteq A$. Put

$$
\text{Ind}_\Gamma (A) := \{ T \in \mathcal{L}(K) \mid TT^* \Gamma \subseteq [\Gamma A] \} \subseteq \mathcal{L}(K).
$$

Evidently, $\text{Ind}_\Gamma (A)$ is a $C^*$-subalgebra of $\mathcal{L}(K)$. Equivalently, this $C^*$-algebra can be described as follows. Since $[B \mathfrak{A}] \subseteq A$, we can form the internal tensor product $\Gamma \otimes A$. For each $\gamma \in \Gamma$, define $[\gamma]_1 \in \mathcal{L}_A (A, \Gamma \otimes A)$ by $a \mapsto \gamma \otimes a$, and put $[\Gamma]_1 := \{ [\gamma]_1 | \gamma \in \Gamma \} \subseteq \mathcal{L}_A (A, \Gamma \otimes A)$. Finally, let

$$
\text{Ind}_\Gamma (A)' := \{ T \in \mathcal{L}_A (\Gamma \otimes A) \mid T[\Gamma]_1, T^* [\Gamma]_1 \subseteq [\Gamma]_1 A \}.
$$

Since $[\Gamma AH] = K$, we have an isomorphism $\Gamma \otimes A \otimes H \to K$ given by $\gamma \otimes a \otimes \zeta \mapsto \gamma a \zeta$. Using this isomorphism, we define an embedding

$$
i : \mathcal{L}_A (\Gamma \otimes A) \to \mathcal{L}(\Gamma \otimes A \otimes H) \cong \mathcal{L}(K), \ T \mapsto T \otimes \text{id}_H \equiv \iota(T).
$$

Lemma 3.1. $\iota$ restricts to a $*$-isomorphism $\text{Ind}_\Gamma (A)' \to \text{Ind}_\Gamma (A)$. 

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Proof. Clearly, \( \iota \) defines an embedding \( \text{Ind}_\Gamma(A) \to \text{Ind}_\Gamma(A) \). We show that this embedding is surjective. Let \( T \in \text{Ind}_\Gamma(A) \). Since \( \Gamma \otimes A \) embeds into \( \mathcal{L}(H,K) \) via \( \gamma \otimes a \to \gamma a \) and since \( T TA \subseteq [\Gamma A] \), left multiplication by \( T \) on \([\Gamma A]\) corresponds to a map \( \tilde{T} : \Gamma \otimes A \to \Gamma \otimes A \). One easily verifies \( \tilde{T} \in \text{Ind}_\Gamma(A) \) and \( \iota(\tilde{T}) = T \).

Let us give two more equivalent descriptions of the \( C^* \)-algebra \( \text{Ind}_\Gamma(A) \). Denote by \( \tau \) the locally convex topology on \( \mathcal{L}(K) \) induced by the family of semi-norms \( (T \mapsto \|\gamma\|)_\gamma^{\in \Gamma} \) and \( (T \mapsto \|T \gamma\|)_\gamma^{\in \Gamma} \). Given a subset \( X \subseteq \mathcal{L}(K) \), denote by \([X]_\Gamma\), the \( \tau \)-closed linear span of \( X \).

**Lemma 3.2.** For every bounded approximate unit \( (u_i)_i \) of the \( C^* \)-algebra \( E \) and every \( T \in \text{Ind}_\Gamma(A) \), the net \( (u_i T u_i)_i \) converges to \( T \) w.r.t. \( \tau \).

**Proof.** Let \( (u_i)_i \), and \( T \) as above and \( \gamma \in \Gamma \). Then
\[
\|u_i T u_i \gamma - T \gamma\| \leq (\sup_i \|u_i\|) \|T\| \|u_i \gamma - \gamma\| - \|u_i T \gamma - T \gamma\|
\]
for all \( i \), and using \( T \gamma \subseteq [\Gamma A] \) and \( \lim_i u_i \gamma = \gamma' \) for all \( \gamma' \in \Gamma \), we conclude \( \lim_i \|u_i T u_i \gamma - T \gamma\| = 0 \). Similarly, \( \lim_i \|u_i \gamma - \gamma\| = 0 \).

**Lemma 3.3.** \( \text{Ind}_\Gamma(A) = \{[\Gamma A]_\Gamma\} \).

**Proof.** "\( \subseteq \): Let \( T \in \text{Ind}_\Gamma(A) \). Choose an approximate unit \( (u_i)_i \) for \( E \) and put \( T_i := u_i Tu_i \) for each \( i \). Then \( (T_i)_i \) converges to \( T \) w.r.t. \( \tau \), and \( T_i \in [\Gamma^* TT \Gamma^*] \subseteq [\Gamma^* \Gamma^*] = [\Gamma A] \) for all \( i \). Hence, \( T \in [\Gamma A] \).

"\( \supseteq \): If \( T \in [\Gamma^* \Gamma^*] \), then \( TT \subseteq [\Gamma^* \Gamma^*] = [\Gamma A] \) and similarly \( T \Gamma \subseteq [\Gamma A] \), whence \( T \in \text{Ind}_\Gamma(A) \).

**Lemma 3.4.** \( \text{Ind}_\Gamma(A) = \{T \in \text{Ind}_\Gamma(\mathcal{L}(H)) \mid \|T \Gamma\| \leq A\} \).

**Proof.** Clearly, \( \text{Ind}_\Gamma(A) \subseteq \text{Ind}_\Gamma(\mathcal{L}(H)) \). Let \( T \in \text{Ind}_\Gamma(\mathcal{L}(H)) \) such that \( \|T \gamma\| \leq A \). Choose an approximate unit \( (u_i)_i \) for \( E \). Then for each \( \gamma \in \Gamma \), the net \( (u_i T u_i)_i \) converges in norm to \( T \gamma \), and hence \( TT \Gamma \subseteq [\Gamma^* TT \Gamma] = [\Gamma A] \). Similarly, one shows that \( T^* \Gamma \subseteq [\Gamma A] \). Hence, \( T \in \text{Ind}_\Gamma(A) \).

3.2 Definition and basic properties

Throughout this paragraph, let \( \mathfrak{g} \mathfrak{b} \mathfrak{k} \) be a \( C^* \)-base. We adopt the following terminology:

**Definition 3.5.** Let \( H \) be a Hilbert space, \( \alpha \in \text{C}^* \)-fact\( (H; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \), and \( A \subseteq \mathcal{L}(H) \) a \( C^* \)-algebra. We call \( A \) an \( \alpha \)-module and the triple \( (H,A,\alpha) \) a concrete \( C^* \)-\( \mathfrak{g} \mathfrak{b} \mathfrak{k} \)-\( \alpha \)-algebra if \( \rho_\alpha(\mathfrak{b}^{-1})A \subseteq A \). If \( A \) is nondegenerate, we call \( (H,A,\alpha) \) nondegenerate. We put
\[
\text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) := \{\beta \in \text{C}^* \text{-fact}(H; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \mid A \text{ is an } \beta \text{-module}\}
\]
and, if \( \alpha \in \text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \) and \( \epsilon \in \mathfrak{e}_\mathfrak{k} \) is a second \( C^* \)-base,
\[
\text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) := \{\beta \in \text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \mid \beta \perp \alpha\}.
\]
If \( \alpha \in \text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \) and \( \beta \in \text{C}^* \text{-fact}(A; \mathfrak{g} \mathfrak{b} \mathfrak{k}) \) (and \( A \) is nondegenerate), we call \( (H,A,\alpha,\beta) \) a (nondegenerate) concrete \( C^* \)-\( \mathfrak{g} \mathfrak{b} \mathfrak{k} \)-\( \epsilon \mathfrak{e}_\mathfrak{k} \)-algebra.

We define the fiber product of a concrete \( C^* \)-\( \mathfrak{g} \mathfrak{b} \mathfrak{k} \)-algebra \( (H,A,\alpha) \) and a concrete \( C^* \)-\( \mathfrak{g} \mathfrak{b} \mathfrak{k} \)-algebra \( (K,B,\beta) \) by inducing \( A \) and \( B \) to \( C^* \)-algebras on \( H_\alpha \otimes B K \), using the subspaces
\[
[\beta]_{[1]} \subseteq \mathcal{L}(H,H_\alpha \otimes B K) \quad \text{and} \quad [\alpha]_{[1]} \subseteq \mathcal{L}(K,H_\alpha \otimes B K),
\]
respectively, which were defined in Subsection 2.2.
Definition 3.6. The fiber product of a concrete $C^*$-$\mathfrak{B}$-$\mathfrak{S}_B$-algebra $(H,A,\alpha)$ and a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra $(K,B,\beta)$ is the $C^*$-algebra

$$A_{\alpha \beta}^* B := \text{Ind}_{\alpha|B} (B) \cap \text{Ind}_{\beta|A} (A) \subseteq \mathcal{L}(H_{\alpha} \otimes_{\beta} K).$$

Unfortunately, we can not yet answer the following important question:

Question 3.7. When is $A_{\alpha \beta}^* B \subseteq \mathcal{L}(H_{\alpha} \otimes_{\beta} K)$ non-degenerate?

Note that if $(A,H,\alpha)$ is a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra, then $A' \subseteq \rho_{\alpha}(\mathfrak{B}_B')$.

Lemma 3.8. Let $(H,A,\alpha)$ be a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra and $(K,B,\beta)$ a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra. Then $(A' \otimes \beta) \subseteq (A_{\alpha \beta}^* B)'$.

Proof. Let $T \in A_{\alpha \beta}^* B$ and $S \in A'$. Then for each $\eta \in \beta$,

$$T(S \otimes \text{id})|_{\eta} = T|_{\eta} S = |_{\eta} T(S \otimes \text{id})$$

because $T|_{\eta}|_{\beta} = [\beta]|_{\eta}|_{A}$. Hence, $T(S \otimes \text{id}) = (S \otimes \text{id}) T$. A similar argument shows that $T(\text{id} \otimes R) = (\text{id} \otimes R) T$ for all $R \in B'$.

Lemma 3.9. Let $(H,A,\alpha)$ be a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra, $(K,B,\beta)$ a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra, and $\epsilon \in \mathfrak{R}_C$ a $C^*$-base. Then

$$
\begin{align*}
\gamma \triangleleft \beta & \in C^* \cdot \text{fact}(A_{\alpha \beta}^* B ; \epsilon \in \mathfrak{R}_C) \quad \text{for each } \gamma \in C^* \cdot \text{fact}(A_{\alpha ; \epsilon \in \mathfrak{R}_C}), \\
\alpha \triangleright \delta & \in C^* \cdot \text{fact}(A_{\alpha \beta}^* B ; \epsilon \in \mathfrak{R}_C) \quad \text{for each } \delta \in C^* \cdot \text{fact}(B_{\beta ; \epsilon \in \mathfrak{R}_C}).
\end{align*}
$$

Proof. We only prove the first assertion, the second one follows similarly:

$$
\rho_{(\gamma \triangleleft \beta)}(C^* \cdot (A_{\alpha \beta}^* B)|_{\alpha|B}) \subseteq [\rho_{\beta}(C^* \cdot \alpha)|_{B}] \subseteq [\alpha]|_{B},
$$

$$
\rho_{(\gamma \triangleright \delta)}(C^* \cdot (A_{\alpha \beta}^* B)|_{\beta|B}) \subseteq [\rho_{\beta}(C^* \cdot \alpha)|_{B}] \subseteq [\beta]|_{B}.
$$

The fiber product introduced above seems to fail to be associative and to behave like the functor that associates to two $C^*$-algebras $A$, $B$ the multiplier algebra $M(A \otimes B)$ instead of the tensor product $A \otimes B$.

More precisely, let $\mathfrak{B}_B$, $\epsilon \in \mathfrak{R}_C$ be $C^*$-bases, $(H,A,\alpha)$ a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-algebra, $(K,B,\beta,\gamma)$ a concrete $C^*$-$\mathfrak{B}_B$-$\mathfrak{S}_B$-$\mathfrak{R}_C$-$\mathfrak{S}_C$-algebra, and $(L,C,\delta)$ a concrete $C^*$-$\mathfrak{R}_C$-$\mathfrak{S}_C$-$\mathfrak{S}_C$-algebra. Then we can form the fiber products $A_{\alpha \beta}^* B$ and $B_{\gamma}^* C$, and by Lemma 3.9 also the following iterated fiber products:

$$(A_{\alpha \beta}^* B)_{\alpha \gamma}^* s C \quad \text{and} \quad A_{\alpha \beta}^* s (B_{\gamma} C).$$

We expect that these $C^*$-algebras are not identified by the canonical isomorphism

$$
(H_{\alpha \beta} K)_{\alpha \gamma} \otimes_{\beta} L \cong H_{\alpha \beta} (K_{\gamma} \otimes_{\delta} L)
$$

of Proposition 2.1.

Remark 3.10. Assume that we are given some $n \geq 1$ and

$\bullet$ $C^*$-bases $\mathfrak{B}_1, \mathfrak{B}_2, \ldots , \mathfrak{B}_n$, $\mathfrak{R}_n$.
By Proposition 2.11, we can form an iterated $C^\ast$-relative tensor product
\[(H_1)_{\alpha_1} \otimes \cdots \otimes (H_n)_{\alpha_n}(H_{n+1}),\] (7)
and by Lemma 3.19 we can form various iterated fiber products like
\[(\cdots (A_1 \ast A_2) \ast A_3) \ast \cdots \ast A_{n+1},\quad A_1 \ast (\cdots \ast (A_n \ast A_{n+1})'),\]
which can be identified with $C^\ast$-algebras on the Hilbert space (7). Here, all possible ways (that we want to consider) of forming an iterated fiber product correspond bijectively to all possible ways of completely bracketing a product consisting of $n + 1$ factors or, equivalently, with all binary trees with $n + 1$ leaves. We denote by
\[(A_1)_{\alpha_1} \ast \cdots \ast (A_n)_{\alpha_n}(A_{n+1}) \subseteq \mathcal{L}((H_1)_{\alpha_1} \otimes \cdots \otimes (H_n)_{\alpha_n}(H_{n+1}))\]
the intersection of all $C^\ast$-algebras obtained by iterating the fiber product construction in the ways described above.

### 3.3 Functoriality

We shall see that the fiber product is functorial with respect to the following class morphisms:

**Definition 3.11.** Let $\mathfrak{g} \mathfrak{B}_{n}$ be a $C^\ast$-base and $(H, A, \alpha)$, $(K, B, \beta)$ concrete $C^\ast$-$\mathfrak{g} \mathfrak{B}_{n}$-algebras. A morphism from $(H, A, \alpha)$ to $(K, B, \beta)$ is a $*$-homomorphism $\pi: A \rightarrow B$ satisfying the following conditions:

- $i)$ $\pi(\rho(a)(b)) = \pi(a)\rho(b)$ for all $a \in A$ and $b \in \mathfrak{B}_{n}$,
- $ii)$ $\beta = [\mathcal{L}^\ast(H_n, K_{\beta})\alpha]$, where

\[\mathcal{L}^\ast(H_n, K_{\beta}) := \{V \in \mathcal{L}(H_n, K_{\beta}) | \forall a \in A: \pi(a)V = V\alpha\}.

We denote the set of all morphisms from $(H, A, \alpha)$ to $(K, B, \beta)$ by $\text{Mor}(A, B)_{\beta}$.

**Remark 3.12.** Let $\mathfrak{g} \mathfrak{B}_{n}$ be a $C^\ast$-base and $\pi$ a morphism of concrete $C^\ast$-$\mathfrak{g} \mathfrak{B}_{n}$-algebras $(H, A, \alpha)$ and $(K, B, \beta)$. Then
\[K = [\beta\delta] = [\mathcal{L}^\ast(H_n, K_{\beta})\alpha\delta] = [\mathcal{L}^\ast(H_n, K_{\beta})H],\]
and if $A \subseteq \mathcal{L}(H)$ is nondegenerate, then also $\pi(A) \subseteq \mathcal{L}(K)$ is nondegenerate because
\[\pi(A)K = [\pi(A)\mathcal{L}^\ast(H_n, K_{\beta})H] = [\mathcal{L}^\ast(H_n, K_{\beta})AH] = [\mathcal{L}^\ast(H_n, K_{\beta})H] = K.\]

For each $C^\ast$-base $\mathfrak{g} \mathfrak{B}_{n}$, the class of all concrete $C^\ast$-$\mathfrak{g} \mathfrak{B}_{n}$-algebras together with the morphisms defined above evidently forms a category.

**Proposition 3.13.** Let $\mathfrak{g} \mathfrak{B}_{n}$ be a $C^\ast$-base and

- $\phi$ a morphism of nondegenerate concrete $C^\ast$-$\mathfrak{g} \mathfrak{B}_{n}$-algebras $(H, A, \alpha)$ and $(L, \gamma, C)$,
- $\psi$ a morphism of nondegenerate concrete $C^\ast$-$\mathfrak{g} \mathfrak{B}_{n}$-algebras $(K, B, \beta)$ and $(M, \delta, D)$.
Then there exists a unique \(*\)-homomorphism
\[
\phi \ast \psi : A_α \ast_β B \to C_γ \ast_δ D
\]
such that
\[
(\phi \ast \psi)(T) \cdot (X \otimes Y) = (X \otimes Y) \cdot T
\]
whenever \(T \in A_α \ast_β B\) and one of the following conditions holds: i) \(X \in \mathcal{L}^{\psi}(H, L)\) and \(Y \in \mathcal{L}^{\phi}(K_β, D_δ)\) or ii) \(X \in \mathcal{L}^{\phi}(K_α, L_γ)\) and \(Y \in \mathcal{L}^{\psi}(K, D)\).

Proof. Uniqueness follows from the fact that elements of the form (\(\ast\)) \(\in L\)
\[Y\]
whenever \(\phi = \mathrm{id}, \psi = \mathrm{id}\), and constructs \(\phi \ast \psi\) as above, linearly dense in \(L\). Moreover, for each \(L\), we have \(\ast\)-isomorphisms of internal tensor products
\[
H_{\text{id}} \otimes E \cong H_\rho \otimes \beta \cong H_\gamma \otimes \beta K, \quad L_\phi \otimes E \cong L_\rho \otimes \beta \cong L_\gamma \otimes \beta K.
\]

Moreover, we have \(\ast\)-isomorphisms
\[
j_H : \mathcal{L}_A(E) \to \mathcal{L}(H_{\text{id}} \otimes E) \cong \mathcal{L}(H_\rho \otimes \beta K), \quad T \mapsto \text{id}_T \otimes T \equiv j_H(T),
\]
\[
j_L : \mathcal{L}_A(E) \to \mathcal{L}(L_\phi \otimes E) \cong \mathcal{L}(L_\rho \otimes \beta K), \quad T \mapsto \text{id}_L \otimes T \equiv j_L(T).
\]

Since \(j_H\) is injective and \(A_α \ast_β B \subseteq \text{Ind}_{[\beta][\phi]}(\mathcal{L}_A(E))\) (Lemma 3.1), we can define \(\phi \ast \text{id} : A_α \ast_β B \to \mathcal{L}(L_\gamma \otimes \beta K)\) to be the restriction of \(j_L \circ j_H^{-1}\).

Let \(T \in j_H^{-1}(A_α \ast_β B)\) and \(X \in \mathcal{L}^{\phi}(H, L)\). Then the following diagram commutes and shows that \((\phi \ast \text{id})(j_H(T)) \cdot (X \otimes \text{id}) = (X \otimes \text{id}) \cdot j_H(T)\):

\[
\begin{array}{ccc}
H_\rho \otimes \beta K & \overset{j_H(T) \equiv \text{id} \otimes T}{\longrightarrow} & H_{\text{id}} \otimes E \\
\downarrow \text{id}_\rho \otimes \beta & & \downarrow \text{id}_T \otimes \beta \otimes \text{id} \\
L_\gamma \otimes \beta K & \overset{j_L(T) \equiv \text{id} \otimes T}{\longrightarrow} & L_\phi \otimes E
\end{array}
\]

Moreover, for each \(S \in A_α \ast_β B\) and \(Y \in B'\), we have \(\text{S}(\text{id} \otimes Y) = (\text{id} \otimes Y) \text{S}\) by Lemma 3.8. Summarizing, we find that condition [\(\square\)] holds.

Finally, let us show that \((\phi \ast \text{id})(A_α \ast_β B) \subseteq C_γ \ast_δ B\). Let \(T \in A_α \ast_β B\). Using the relation \(\gamma = [L^{\phi}(H, L)\alpha]\), Equation [\(\square\)], and the relation \(T[\alpha]_{[1]} \subseteq [||\alpha||_{[1]}B]\), we find
\[
(\phi \ast \text{id})(T)[\gamma]_{[1]} \subseteq ([(\phi \ast \text{id})(T)]_\mathcal{L}^{\phi}(H, L)\alpha)[[\gamma]_{[1]}]
\]
\[
= [(\mathcal{L}^{\phi}(H, L) \otimes \text{id})T[\alpha]_{[1]}]
\]
\[
\subseteq [(\mathcal{L}^{\phi}(H, L) \otimes \text{id})[\alpha]_{[1]}B] = [\gamma]_{[1]}B.
\]

Another application of Equation [\(\square\)] shows that \((\phi \ast \text{id})(T)[\beta]_{[2]} = T[\beta]_{[2]} \subseteq [\beta]_{[2]}A\). \(\square\)
Corollary 3.14. Let $\mathfrak{B}_{M_1}$, $(H, A, \alpha)$, $(L, \gamma, C)$, $\phi$ and $(K, B, \beta)$, $(M, \delta, D)$, $\psi$ be as in Proposition 3.13, and let $T \in A_{\alpha} \ast_{\beta} B$. Then

$$
\langle \eta[t_{[2]}(\phi \ast \text{id})(T)|\eta']_{[2]} = \phi(\langle \eta[t_{[2]} T|\eta']_{[2]} \rangle) \quad \text{for all } \eta, \eta' \in \beta,
$$

$$
\langle \xi[t_{[1]}(\text{id} \ast \psi)(T)|\xi']_{[1]} = \psi(\langle \xi[t_{[1]} T|\xi']_{[1]} \rangle) \quad \text{for all } \xi, \xi' \in \alpha.
$$

Proof. The equations can be deduced from the proof of Proposition 3.13, but we give an alternative proof. Let $\eta, \eta' \in \beta$ and $X \in \mathcal{L}^{\psi}(H_{\alpha}, L_{\gamma})$. Then

$$
\langle \eta[t_{[2]}(\phi \ast \text{id})(T)|\eta']_{[2]} X = \langle \eta[t_{[2]} (X \otimes \text{id}) T|\eta']_{[2]} = X \langle \eta[t_{[2]} T|\eta']_{[2]} \rangle
$$

by (9), and inserting $X a = \phi(a) X$ for $a := \langle \eta[t_{[2]} T|\eta']_{[2]} \rangle \in A$, we find

$$
\langle \eta[t_{[2]}(\phi \ast \text{id})(T)|\eta']_{[2]} X = \phi(\langle \eta[t_{[2]} T|\eta']_{[2]} \rangle) X.
$$

Since $X \in \mathcal{L}^{\psi}(H_{\alpha}, L_{\gamma})$ was arbitrary and $[\mathcal{L}^{\psi}(H_{\alpha}, L_{\gamma})] = \gamma$, the first equation of the corollary follows. The second one follows similarly.

Theorem 3.15. Let $\mathfrak{B}_{M_1}, \mathfrak{C}_{E_1}$ be $C^\ast$-bases, let

- $\phi$ be a morphism of nondegenerate concrete $C^\ast$-$\mathfrak{B}_{M_1}$-algebras $(H, A, \alpha)$ and $(L, \gamma, C)$,
- $\psi$ be a morphism of nondegenerate concrete $C^\ast$-$\mathfrak{B}_{M_2}$-algebras $(K, B, \beta)$ and $(M, \delta, D)$, and assume that $A_{\alpha} \ast_{\beta} B \subseteq \mathcal{L}(H_{\alpha} \otimes_{\beta} K)$ is nondegenerate.

i) If $\alpha' \in \text{C}^\ast$-fact($A_{\alpha}$; $\varepsilon_{\mathfrak{C}_{E_1}}$), $\gamma' \in \text{C}^\ast$-fact($C_{\gamma}$; $\varepsilon_{\mathfrak{C}_{E_1}}$), $\phi \in \text{Mor}(A_{\alpha'}, C_{\gamma'})$, then $\phi \ast \psi \in \text{Mor}((A_{\alpha} \ast_{\beta} B)_{(\alpha' \ast \delta)}, (C_{\gamma} \ast_{D})_{(\gamma' \ast \delta)}).

ii) If $\beta' \in \text{C}^\ast$-fact($B_{\beta}$; $\varepsilon_{\mathfrak{C}_{E_1}}$), $\delta' \in \text{C}^\ast$-fact($D_{\delta}$; $\varepsilon_{\mathfrak{C}_{E_1}}$), $\psi \in \text{Mor}(B_{\beta'}, D_{\delta'})$, then $\phi \ast \psi \in \text{Mor}((A_{\alpha} \ast_{\beta} B)_{(\alpha' \ast \delta')}, (C_{\gamma} \ast_{D})(\gamma' \ast \delta')).$

Proof. We only prove i); assertion ii) follows similarly.

By Lemma 3.7, $(H_{\alpha} \otimes_{\beta} K, A_{\alpha} \ast_{\beta} B, \alpha' \ast \beta)$ is a concrete $C^\ast$-$\mathfrak{B}_{E_1}$-algebra.

We show that $\phi \ast \psi$ satisfies condition i) in Definition 3.11. Fix $T \in A_{\alpha} \ast_{\beta} B$ and $c' \in \mathfrak{C}_1$, and let $X \in \mathcal{L}^{\phi}(H_{\alpha}, L_{\gamma})$, $Y \in \mathcal{L}^{\psi}(K_{\beta}, M_{\delta})$. By (9),

$$
(\phi \ast \psi)(T \rho_{(a \ast \delta)}(c'))(X \otimes Y) = (X \otimes Y)(T \rho_{(a \ast \delta)}(c'))
$$

$$
= (\phi \ast \psi)(T)(X \otimes Y) \rho_{(a \ast \delta)}(c'). \tag{10}
$$

For each $a \in A$, we have $\rho_{\alpha'}((c') a) \in A$, and

$$
X \rho_{\alpha'}([c'] a) = \phi(\rho_{\alpha'}(c') a) X = \rho_{\alpha'}(c') \phi(a) X = \rho_{\alpha'}(c') X a
$$

because $\phi \in \text{Mor}(A_{\alpha'})$ and $X \in \mathcal{L}^{\phi}(H_{\alpha}, L_{\gamma})$. Since $A \subseteq \mathcal{L}(H)$ is nondegenerate, we can conclude $X \rho_{\alpha'}(c') = \rho_{\alpha'}(c') X$. Inserting this equation and the relations $\rho_{(a \ast \delta)} = (\rho_{\alpha'})_{[1]}$, $\rho_{(\gamma' \ast \delta)} = (\rho_{\beta'})_{[1]}$ into (10), we find

$$
(\phi \ast \psi)(T \rho_{(a \ast \delta)}(c'))(X \otimes Y) = (\phi \ast \psi)(T)(\rho_{(a \ast \delta)}(c'))(X \otimes Y).
$$

Since $X \in \mathcal{L}^{\phi}(H_{\alpha}, L_{\gamma})$ and $Y \in \mathcal{L}^{\psi}(K_{\beta}, M_{\delta})$ were arbitrary, we can conclude

$$
(\phi \ast \psi)(T \rho_{(a \ast \delta)}(c')) = (\phi \ast \psi)(T) \rho_{(\gamma' \ast \delta)}(c').
$$

To show that $\phi \ast \psi$ satisfies condition ii) in Definition 3.11, let $X \in \mathcal{L}^{\phi}(H_{\alpha'}, L_{\gamma'})$ and $Y \in \mathcal{L}^{\psi}(K_{\beta'}, M_{\delta'})$. Then a similar argument as above shows that $X \rho_{\alpha'}(b') = \rho_{\alpha'}(b')$.
Remark 3.17. If \( n \) iterated does not matter. Note that for each \( b \) nondegenerate concrete Hopf \( C \)-bimodules. Throughout, we apply the concepts and techniques developed in this section, we introduce \( C^\ast \)-pseudo-multiplicative unitaries and explain their relation to several other generalizations of multiplicative unitaries. Following the treatment of multiplicative unitaries given by Baaj and Skandalis [1], we define the legs of such a \( C^\ast \)-pseudo-multiplicative unitary and show that under a suitable regularity condition, these legs form concrete Hopf \( C^\ast \)-bimodules. Throughout, we apply the concepts and techniques developed in the preceding sections.

4 \( C^\ast \)-pseudo-multiplicative unitaries

\( \frac{3}{2} + 4 \)
4.1 Definition

Recall that a multiplicative unitary [11 Définition 1.1] on a Hilbert space $H$ is a unitary $V$: $H \otimes H \to H \otimes H$ that satisfies the so-called pentagon equation $V_{12}V_{12}V_{23} = V_{23}V_{12}$. Here, $V_{12}, V_{23}$ are operators on $H \otimes H \otimes H$, defined by $V_{12} = V \otimes \text{id}$, $V_{23} = \text{id} \otimes V$. Let $H_{\hat{\beta}, \hat{\beta}}$ be a Hilbert space, $\hat{\beta} \in C^*$-fact($H_0 \hat{\beta}, H_{\hat{\beta}}\hat{\beta}H$), $\beta \in C^*$-fact($H_0 \beta, H_{\beta} \beta H$) pairwise compatible $C^*$-factorizations.

**Lemma 4.1.** Let $V \in \mathcal{L}(H_{\hat{\beta}, \hat{\beta}}H, H_{\beta} \beta H)$ and assume that

$$
V_\alpha(\alpha \circ \alpha) = \alpha \circ \alpha, \quad V_\alpha(\hat{\beta} \circ \hat{\beta}) = \hat{\beta} \circ \hat{\beta},
$$

(12)

Then all operators in the following diagram are well-defined,

$$
\begin{array}{ccc}
H_{\alpha} \otimes_{\alpha} H & \xrightarrow{\sim} & H_{\beta} \otimes_{\beta} H, \\
\xrightarrow{id \otimes V} & & \xleftarrow{id \otimes V} \\
H_{\beta} \otimes_{\alpha} H & \xrightarrow{\sim} & H_{\alpha} \otimes_{\beta} H, \\
\xrightarrow{id \otimes V} & & \xleftarrow{id \otimes V} \\
H_{\beta} \otimes_{\alpha} H & \xrightarrow{\sim} & H_{\alpha} \otimes_{\alpha} H.
\end{array}
$$

(13)

where $\Sigma_{[2]}$ denotes the isomorphism

$$
(H_{\alpha} \otimes_{\beta} H)_{\beta, \alpha} H \cong (H_{\rho \circ \alpha} \otimes_{\rho \circ \alpha} \otimes_{\rho \circ \beta}) \cong (H_{\rho \circ \alpha} \otimes_{\rho \circ \beta}) \cong (H_{\rho \circ \beta} \otimes_{\rho \circ \alpha}) \otimes_{\rho \circ \beta} H, \\
(\xi \otimes \eta) \otimes \xi \mapsto (\xi \otimes \eta) \otimes \xi.
$$

**Proof.** This follows easily from the functoriality of the relative tensor product [2.10] for example, we have operators

$$
H_{\beta \circ \alpha} H_{\beta \circ \alpha} H \cong (H_{\rho \circ \beta} \otimes_{\rho \circ \alpha}) H_{\rho \circ \alpha} \otimes_{\rho \circ \beta} H \cong (H_{\rho \circ \beta} \otimes_{\rho \circ \alpha}) H_{\rho \circ \beta} \otimes_{\rho \circ \alpha} H,
$$

$$
H_{\beta \circ \alpha} (H_{\beta} \otimes_{\alpha} H) \cong H_{\beta} \otimes_{\alpha} (H_{\beta} \otimes_{\alpha} H).
$$

**Definition 4.2.** Let $H$ be a Hilbert space, $\mathfrak{B}_0$ a $C^*$-base, and $\alpha \in C^*$-fact($H_0 \alpha, H_{\alpha} \alpha H$) pairwise compatible. A unitary $V \in \mathcal{L}(H_{\hat{\beta}, \hat{\beta}}H, H_{\beta} \beta H)$ is $C^*$-pseudo-multiplicative if Equations [12] hold and Diagram [13] commutes.

**Remark 4.3.** If $V \in \mathcal{L}(H_{\hat{\beta}, \hat{\beta}}H, H_{\beta} \beta H)$ is a $C^*$-pseudo-multiplicative unitary, then also

$$
V^{\text{op}} := \Sigma V^* \Sigma; \quad H_{\beta \circ \alpha} H \xrightarrow{\Sigma} H_{\alpha} \otimes_{\beta} H \xrightarrow{V^*} H_{\beta} \otimes_{\alpha} H \xrightarrow{\Sigma} H_{\beta} \otimes_{\beta} H
$$

is a $C^*$-pseudo-multiplicative unitary; here, the rôles of $\hat{\beta}$ and $\beta$ get reversed.
4.2 The legs of a $C^*$-pseudo-multiplicative unitary

To every multiplicative unitary

$$\begin{align*}
\rho_{\beta}^2 \rightarrow \mathcal{C}(\alpha) \quad \text{and} \quad \rho_{\beta} : \mathcal{B} \rightarrow \mathcal{L}(\alpha),
\end{align*}$$

and the unitary $V$ restricts to a unitary operator on right $C^*$-modules

$$V_{\alpha} : \alpha \otimes_{\rho_{\beta}} \alpha \rightarrow \alpha \otimes_{\rho_{\beta}} \alpha$$

which is a pseudo-multiplicative unitary on $C^*$-modules in the sense of Timmermann [17].

ii) If $\beta = \alpha$, then $\mathcal{B}$ is commutative by Remark 2.6 and the restriction $V_{\alpha}$ is a pseudo-multiplicative unitary in the sense of O'uchi [12]. Similarly, if $\beta = \alpha$, then $V_{\alpha}^{op}$ is a pseudo-multiplicative unitary in the sense of [12].

iii) If $\alpha = \beta = \beta$, then again $\mathcal{B}$ is commutative, and $V_{\alpha}$ is a continuous field of multiplicative unitaries in the sense of Blanchard [2].

iv) Assume that $\gamma_0 \mathcal{B}_1$ is the $C^*$-base associated to a proper KMS-weight $\mu$ on a $C^*$-algebra $B$, that is, $\gamma_0 = H_\mu$ and $\mathcal{B} = \mathcal{B}_\mu(\mathcal{B})$ (see Example 2.2). Put $N := \mathcal{B}^{\alpha} \subset \mathcal{L}(H_\mu)$, denote by $\hat{\mu}$ the extension of $\mu$ to a normal semifinite faithful weight on $N$, and denote by $\hat{\rho}_{\alpha}$, $\hat{\rho}_{\beta}$, and $\hat{\rho}_{\beta}$ the unique normal extensions of $\rho_{\alpha}$, $\rho_{\beta}$, and $\rho_{\beta}$, respectively, to $N$ or $N^{op}$ (see Subsection 2.3). Then we have isomorphisms

$$H_{\beta} \mathcal{B}_\alpha \mathcal{B}_\beta \cong H_{\beta} \mathcal{B}_\alpha \mathcal{B}_\alpha, \quad H_{\alpha} \mathcal{B}_\beta \mathcal{B}_\alpha \cong H_{\alpha} \mathcal{B}_\alpha \mathcal{B}_\beta,$$

and with respect to these isomorphisms, $V$ is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [20].

4.2 The legs of a $C^*$-pseudo-multiplicative unitary

To every multiplicative unitary $V$, Baaj and Skandalis associate two algebras $\hat{A}(V)$, $A(V)$ and two normal $*$-homomorphisms $\Delta_\mathcal{B}$, $\Delta_\mathcal{V}$ such that, under favorable circumstances like regularity, $\hat{A}(V), \hat{A}(V)$ and $A(V), \Delta_\mathcal{V}$ are Hopf $C^*$-algebras [1].

Their construction carries over to $C^*$-pseudo-multiplicative units as follows. Let $V \in \mathcal{L}(H_{\beta} \mathcal{B}_\alpha \mathcal{B}_\beta)$ a pseudo-multiplicative unitary, where $H$, $\mathcal{B}_\alpha$, and $\alpha, \beta, \beta$ are as in Definition 1.2.

The algebras $\hat{A}(V)$ and $A(V)$ We define spaces $\hat{A}(V) \subseteq \mathcal{L}(H)$ and $A(V) \subseteq \mathcal{L}(H)$, using the spaces of ket-bra operators $|\alpha|_{[1]}$, $|\beta|_{[1]} \subseteq \mathcal{L}(H, H_{\beta} \mathcal{B}_\alpha H)$ and $|\alpha|_{[2]}$, $|\beta|_{[2]} \subseteq \mathcal{L}(H_{\alpha} \mathcal{B}_\beta H, H)$ introduced in Subsection 2.2 as follows:

$$\hat{A} := \hat{A}(V) := \{ |\beta|_{[2]} \mathcal{V}|\alpha|_{[2]} \subseteq \mathcal{L}(H), \quad A := A(V) := \{ |\Gamma^* \mathcal{V}||\beta|_{[2]} \subseteq \mathcal{L}(H).$$

The definition of $\hat{A}(V)$ and $A(V)$ is symmetric in the following sense:

**Lemma 4.4.** $\hat{A}(V^{op}) = A(V)^*$ and $A(V^{op}) = \hat{A}(V)^*$. 

**Proof.** Switching from $V$ to $V^{op}$, the roles of $\beta$ and $\beta$ get reversed, so

$$\hat{A}(V) = \{ [\beta|_{[2]} |\alpha|_{[2]}] \mathcal{V} |\beta|_{[2]} \mathcal{V} |\alpha|_{[2]} = [\hat{\beta}_1 |\alpha|_{[1]} = \mathcal{V}|\beta|_{[1]}\}^* = A(V)^*,$$

and similarly $A(V^{op}) = \hat{A}(V)^*$. 

The spaces $\hat{A}(V)$ and $A(V)$ are related to the representations $\rho_\alpha, \rho_\beta, \rho_\beta$ as follows:
Lemma 4.5. We have
\[ \hat{A}\rho_B(B) = [\rho_B(B)\hat{A}] = [\hat{A}\rho_B(B^op)] = [\rho_B(B^op)\hat{A}] = \hat{A} \subseteq \mathcal{L}(H_B) \]
and
\[ A\rho_B(B) = [\rho_B(B)A] = [A\rho_B(B^op)] = [\rho_B(B^op)A] = A \subseteq \mathcal{L}(H_B). \]

Proof. We only prove the assertions concerning \( \hat{A} \): Using Equation (12) and the relations 
\[ [\alpha B] = \alpha = [\rho_B(B)\alpha] \text{ and } [\rho_B(B^op)\beta] = \beta = [\beta B^op], \]
we find
\[ [(\beta)|_2 V|\alpha|_2]\rho_B(B) = [\rho_B(B)\beta|_2 V|\alpha|_2] = \hat{A}, \]
\[ [\rho_B(B)](\beta|_2 V|\alpha|_2) = [(\beta)|_2 \rho_B(B)|_2 V|\alpha|_2] = \hat{A}, \]
\[ [(\beta)|_2 V|\alpha|_2]\rho_B(B^op) = [(\beta)|_2 \rho_B(B^op)|_2 V|\alpha|_2] = \hat{A}, \]
\[ [\rho_B(B^op)](\beta|_2 V|\alpha|_2) = [(\beta)|_2 B^op V|\alpha|_2] = \hat{A}. \]
The inclusion \( \hat{A} \subseteq \mathcal{L}(H_B) \) follows from the inclusions \( |\alpha|_2 \subseteq \mathcal{L}(H_\beta, (H_\beta \otimes_\beta H)_\beta \otimes_\beta), \]
\[ V \in \mathcal{L}(\mathcal{L}(H_\beta \otimes_\beta H)_\beta \otimes_\beta (H_\alpha \otimes_\beta H)_\beta \otimes_\beta), \]
and \( |\beta|_2 \subseteq \mathcal{L}(H_\alpha \otimes_\beta H)_\beta \otimes_\beta, H_B). \)

The spaces \( \hat{A} \) and \( A \) are algebras:

Proposition 4.6. \( [\hat{A}A] = \hat{A} \) and \( [AA] = A \).

Proof. We only prove the first equation. The following diagram commutes and shows that
\[ [\hat{A}A] = [(\beta)|_2|\alpha|_2 V|\alpha|_2|\alpha|_2|\beta|_2]. \]

Indeed, the cells labeled by (D) commute by definition, cell (C) commutes because
\[ |\xi|_2 < |\eta|_2 (\zeta \otimes \eta) = \rho_\alpha(\eta'^* \eta)(\zeta \otimes \xi) = \rho_\alpha(\eta'^* \eta)(\zeta \otimes \xi) = (\eta'|_3|\xi|_2)(\zeta \otimes \eta) \]
for all \( \xi \in \alpha, \eta, \eta' \in \beta, \zeta \in H \), cell (P) is just Diagram (13), and the remaining cells commute because of (12).
The following commutative diagram shows that \( ((\beta)_2|\alpha|_3 V_{[1]2}|\alpha)_3|\alpha|_2) = \hat{A} \) and completes the proof:

The algebras \( \hat{A} \) and \( A \) are nondegenerate in the following strong sense:

**Proposition 4.7.** \( \hat{A}\beta = \beta = [\hat{A}^* \beta] \) and \( A\beta = \hat{\beta} = [A^* \hat{\beta}] \).

**Proof.** We only prove the first equation, the others follow similarly: Since \( V_* (\beta \angle \alpha) = \beta \angle \beta \), \( \hat{A}\beta = \left( \langle (\beta)_2|\alpha|_3 V_{[1]2}|\alpha\rangle_3|\alpha\rangle_2 \right) = \hat{A} \beta = \left( \langle (\beta)_2|\alpha|_3 V_{[1]2}|\alpha\rangle_3|\alpha\rangle_2 \right) = \beta. \)

**Corollary 4.8.** \( \hat{A}H = H = [\hat{A}^* H] \) and \( AH = H = [A^* H] \).

**The comultiplications \( \hat{\Delta}_V \) and \( \Delta_V \)** We define maps

\[ \hat{\Delta} = \hat{\Delta}_V : \rho_{\beta}(\mathfrak{B})' \to \mathcal{L}(H_{\beta_\alpha} \otimes \alpha H), \ y \mapsto V^*(1 \otimes y)V \]

and

\[ \Delta = \Delta_V : \rho_{\beta}(\mathfrak{B})' \to \mathcal{L}(H_{\alpha} \otimes \beta H), \ z \mapsto V(z \otimes 1)V^*. \]

**Proposition 4.9.** We have

\[ \hat{\Delta} \in \text{Mor} \left( \rho_{\beta}(\mathfrak{B})', \mathcal{L}(H_{\beta_\alpha} \otimes \alpha H)_{\alpha\alpha} \right) \cap \text{Mor} \left( \rho_{\beta}(\mathfrak{B})', \mathcal{L}(H_{\beta_\alpha} \otimes \alpha H)_{\beta_\beta} \right) \]

and

\[ \Delta \in \text{Mor} \left( \rho_{\beta}(\mathfrak{B})', \mathcal{L}(H_{\alpha} \otimes \beta H)_{\beta\beta} \right) \cap \text{Mor} \left( \rho_{\beta}(\mathfrak{B})', \mathcal{L}(H_{\alpha} \otimes \beta H)_{\beta\beta} \right). \]

**Proof.** We only prove the assertion concerning \( \hat{\Delta} \). The intertwining relations (12) immediately imply that for each \( b \in \mathfrak{B} \) and \( b^! \in \mathfrak{B}^! \),

\[ \hat{\Delta}(\rho_{\alpha}(b)) = \rho_{\alpha\alpha}(b), \quad \hat{\Delta}(\rho_{\beta}(b)) = \rho_{\beta\beta}(b), \quad \Delta(\rho_{\beta}(b)) = \rho_{\beta\beta}(b), \quad \Delta(\rho_{\alpha}(b^!)) = \rho_{\alpha\alpha}(b^!). \]

We claim that \( V^*|\alpha|_1 \subseteq \mathcal{L}^\Delta(H_{\alpha}, (H_{\beta_\alpha} \otimes \alpha H)) \). Indeed, for each \( y \in \rho_{\beta}(\mathfrak{B})' \) and \( \xi \in \alpha \),

\[ \hat{\Delta}(y)V^*|\xi|_1 = V^*(1 \otimes y)VV^*|\xi|_1 = V^*(1 \otimes y)|\xi|_1 = V^*|\xi|_1|\gamma. \]

Therefore, \( \alpha \angle \alpha \subseteq V^*|\alpha|_1|\alpha| \subseteq \left[ \mathcal{L}^\Delta(H_{\alpha}, (H_{\beta_\alpha} \otimes \alpha H)_{\alpha\alpha}) \right] \alpha \), and a similar argument shows that \( \beta \circ \beta \subseteq \left[ \mathcal{L}^\Delta(H_{\beta}, (H_{\beta_\alpha} \otimes \alpha H)_{\beta\beta}) \right] \beta \). The assertion follows.
4.3 Regularity

The regularity condition for multiplicative unitaries, introduced by Baaj and Skandalis [1] and generalized to pseudo-multiplicative unitaries by Enock [4], carries over to $C^*$-pseudo-multiplicative unitaries as follows:

**Definition 4.10.** Let $V \in \mathcal{L}(H_β \otimes α H, H_α \otimes β H)$ a pseudo-multiplicative unitary, where $H, α, β$ are as in Definition 4.2. A $C^*$-pseudo-multiplicative unitary $V : H_β \otimes α H \rightarrow H_α \otimes β H$ is regular if the composition

$$[\langle α | [1] V | α | [2] \rangle] : H → H_β \otimes α H \overset{V}{\rightarrow} H_α \otimes β H \overset{\langle α | [1] \rangle}{\rightarrow} H$$

is equal to $[α α^*]$.

**Remark 4.11.** Evidently, $V$ is regular if and only if $V^{op}$ is regular.

An example of a regular $C^*$-pseudo-multiplicative unitary is given in Section 5.

We shall show that for a regular $C^*$-pseudo-multiplicative unitary, the associated legs form concrete Hopf $C^*$-bimodules. The first step in this direction is the following:

**Proposition 4.12.** Let $V : H_β \otimes α H \rightarrow H_α \otimes β H$ be a regular $C^*$-pseudo-multiplicative unitary. Then $bA(V)$ and $A(V)$ are $C^*$-algebras.

**Proof.** We only prove the assertion concerning $bA = \tilde{A}(V)$. The following diagram commutes and shows that $[\tilde{A} \tilde{A}] = [(α | [2] | β | [3] V_{[12]} | β | [3] | β | [2])]$:

Indeed, the cells labeled by (D) commute by definition, cell (R) commutes because $V$ is regular, cell (P) is just Diagram 13, and the remaining cells commute because of 12.
The following commutative diagram shows that \[ \tilde{\langle\alpha|\beta}\rho|\beta\tilde{\langle\alpha|\beta}\tilde{\rangle} = \tilde{\alpha}^* \] and completes the proof:

![Diagram](https://via.placeholder.com/150)

**Lemma 4.13.** Let \( V : H_{\beta} \otimes_0 H_{\beta} \rightarrow H_{\alpha} \otimes_0 H_{\beta} \) be a \( C^* \)-pseudo-multiplicative unitary. Then \( \tilde{\Delta}(\tilde{\alpha}) \) is equal to the composition

\[
H_{\beta} \otimes_0 H_{\beta} \xrightarrow{(\alpha)[3]} H_{\beta} \otimes_0 H_{\beta} \xrightarrow{V_{[13]}V_{[23]}} (H_{\beta} \otimes_0 H_{\beta})_{\alpha \otimes_0 H_{\beta}} \xrightarrow{\langle\beta|\beta\rangle} H_{\beta} \otimes_0 H_{\beta}.
\]

**Proof.** This follows from the fact that the following diagram commutes:

![Diagram](https://via.placeholder.com/150)

Again, the cells labeled by (D) commute by definition, cell (P) is just Diagram [13], and the other cells commute because of [12].

The main result of this article is the following:

**Theorem 4.14.** Let \( V : H_{\beta} \otimes_0 H_{\beta} \rightarrow H_{\alpha} \otimes_0 H_{\beta} \) be a regular \( C^* \)-pseudo-multiplicative unitary, where \( H, \mathcal{S}_{\beta}, \alpha, \beta, \delta \) are as in Definition 4.2. Then

\[
(\mathcal{S}_{\beta}, H, \tilde{\alpha}, \beta, \Delta) \quad \text{and} \quad (\mathcal{S}_{\delta}, A(V), \alpha, \beta, \Delta)
\]

are concrete Hopf \( C^* \)-bimodules.

**Proof.** First, we show that \( \tilde{\Delta}(\tilde{\alpha}) \subseteq \tilde{\alpha}_{\beta} \tilde{\Delta}(\tilde{\alpha}) \). The following diagram commutes and shows
that \([\hat{\Delta}(\tilde{\alpha})\alpha_{\{2\}}] = [\alpha\{2\} \tilde{\alpha}\]:

Here, cell (D) commutes by definition, cell (C) commutes by a similar calculation as in [13], cell (L) commutes by Lemma 4.13, and the remaining cells commute because of (12). A similar commutative diagram shows that \([\hat{\Delta}(\tilde{\alpha})\alpha_{\{1\}} = [\beta\{1\}]\tilde{\alpha}\] Thus, \(\hat{\Delta}(\tilde{\alpha}) \subseteq \hat{\alpha}_{\{1\}} \tilde{\alpha}\).

Proposition 4.9 and Remark 3.12 imply that \(\hat{\alpha}_{\{1\}} \tilde{\alpha} \subseteq \mathcal{L}(H_{\beta\alpha} H)\) is nondegenerate. Moreover, by Proposition 4.9

\[
\hat{\Delta} \in \text{Mor}(\hat{\alpha}_{\alpha}, (\hat{\alpha}_{\beta\alpha} \tilde{\alpha})_{\alpha\alpha}) \cap \text{Mor}(\hat{\alpha}_{\beta}, (\hat{\alpha}_{\beta\alpha\alpha} \tilde{\alpha})_{\alpha\alpha\alpha}).
\]

The following commutative diagram shows that \((\hat{\Delta} \ast \text{id})(\hat{\Delta}(\tilde{\alpha})) = (\text{id} \ast \hat{\Delta})(\hat{\Delta}(\tilde{\alpha}))\) for each \(\tilde{\alpha} \in \tilde{\alpha}\) and thus completes the proof:

Here, the upper and lower cells commute by (13), and the other cells commute by definition of \(\Delta\) or trivially. Hence,

\[
(\hat{\Delta} \ast \text{id})(\hat{\Delta}(\tilde{\alpha})) = V_{\{12\}}[(\text{id} \ast \hat{\Delta})(\hat{\Delta}(\tilde{\alpha}))] V_{\{12\}}
\]

\[
= V_{\{12\}}(\hat{\Delta}(\tilde{\alpha}) \otimes \text{id}) \hat{\Delta}(\tilde{\alpha}) V_{\{12\}} = (\hat{\Delta} \ast \text{id})(\hat{\Delta}(\tilde{\alpha})).
\]

For completeness, we include the following additional result:

**Proposition 4.15.** Let \(V : H_{\beta\alpha} H \to H_{\alpha\beta} H\) be a \(C^*\)-pseudo-multiplicative unitary. Then the space

\[
C := [(\alpha\{1\} V\alpha\{2\}] \subseteq \mathcal{L}(H)
\]

is an algebra. If \(V\) is regular, then \(C\) is a \(C^*\)-algebra.
Proof. The first assertion follows by combining the following commutative diagrams:

To prove the second assertion, we combine the following two commutative diagrams:
5 Locally compact groupoids

The prototypical example of a C*-pseudo-multiplicative unitary is the unitary associated to a locally compact groupoid. The underlying pseudo-multiplicative unitary was introduced by Vallin [23], and associated unitaries on C*-modules were discussed in [15, 17]. We construct the C*-pseudo-multiplicative unitary, prove that it is regular, and show that the associated legs are just the function algebra of the groupoid on one side and the reduced groupoid C*-algebra on the other side. For background on groupoids, Haar measures, and quasi-invariant measures, see [16] or [15].

Let $G$ be a locally compact, Hausdorff, second countable groupoid. We denote its unit space by $G^0$, its range map by $r_G$, its source map by $s_G$, and put $G^u := r_G^{-1}(\{u\})$, $G_u := s_G^{-1}(u)$ for each $u \in G^0$.

We assume that $G$ has a left Haar system $\lambda$, and denote the associated right Haar system by $\lambda^{-1}$. Let $\mu$ be a measure on $G^0$ and denote by $\nu$ the measure on $G$ given by

$$\int_G f \, d\nu := \int_{G^0} \int_{G^u} f(x) \, d\lambda^u(x) \, d\mu(u) \quad \text{for all } f \in C_c(G).$$

The push-forward of $\nu$ via the inversion map $G \to G$, $x \mapsto x^{-1}$, is denoted by $\nu^{-1}$; evidently,

$$\int_G f \, d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \, d\lambda_u^{-1}(x) \, d\mu(u).$$

We assume that the measure $\mu$ is quasi-invariant, i.e., that $\nu$ and $\nu^{-1}$ are equivalent. Note that there always exist sufficiently many quasi-invariant measures [15]. We denote by $D := dv/d\nu^{-1}$ the Radon-Nikodym derivative.

The measure $\mu$ defines a tracial proper weight on the C*-algebra $C_0(G^0)$, which we denote by $\mu$ again. Moreover, by the first, the C*-base associated to $\mu$ as in Example 2.2, thus, $\delta = L^2(G^0, \mu)$. Note that $\delta \oplus \delta \cong \delta \oplus \delta$ because $C_0(G^0)$ is commutative.

Put $H := L^2(G, \nu)$ and define representations $r, s: C_0(G^0) \to \mathcal{L}(L^2(G, \nu))$ by

$$(r(f)\xi)(x) := f(r_G(x))\xi(x), \quad (s(f)\xi)(x) := f(s_G(x))\xi(x)$$

for all $x \in G$, $\xi \in C_c(G)$, and $f \in C_0(G^0)$.

The space $C_c(G)$ forms a pre-C*-module over $C_0(G^0)$ with respect to the structure maps

$$\langle \xi | \xi \rangle(u) = \int_{G_u} \xi^*(x)\xi(x) \, d\lambda^u(x), \quad (\xi f)(x) = \xi(x) f(r_G(x)),$$

and also with respect to the structure maps

$$\langle \xi | \xi \rangle(u) = \int_{G_u} \xi^*(x)\xi(x) \, d\lambda_u^{-1}(x), \quad (\xi f)(x) = \xi(x) f(s_G(x)).$$
We denote the completions of these pre-$C^*$-modules by $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$, respectively.

**Proposition 5.1.** There exist isometric embeddings

$$ j : L^2(G, \lambda) \rightarrow \mathcal{L}(\mathcal{H}, H) \quad \text{and} \quad \tilde{j} : L^2(G, \lambda^{-1}) \rightarrow \mathcal{L}(\mathcal{H}, H) $$

such that for all $\xi \in C_c(G)$, $\zeta \in L^2(G^0, \mu)$, $x \in G$,

$$ (j(\xi)\zeta)(x) = \xi(x)\zeta(r_G(x)), \quad (\tilde{j}(\xi)\zeta)(x) = \xi(x)D^{-1/2}(x)\zeta(s_G(x)). $$

The images $\alpha := j(L^2(G, \lambda))$ and $\tilde{\beta} := \tilde{j}(L^2(G, \lambda^{-1}))$ are compatible $C^*$-factorizations of $H$ with respect to $\mathfrak{c}_{\mathcal{H}^*}$. We have $\rho_\alpha = r$ and $\rho_{\tilde{\beta}} = s$. Finally, $j$ and $\tilde{j}$ are unitary maps of $C^*$-modules over $C_0(G^0) \cong \mathfrak{c}$.

**Proof.** Let $\xi, \xi' \in C_c(G)$ and $\zeta, \zeta' \in C_c(G^0)$. Then

$$ \langle j(\xi')\zeta | j(\xi)\zeta \rangle_H = \int_{G^0} \int_{G^0} \overline{\xi'(x)}\xi(r_G(x))\zeta(x)\zeta'(s_G(x))d\lambda^u(x)d\mu(u) $$

and

$$ \langle \tilde{j}(\xi')\zeta | \tilde{j}(\xi)\zeta \rangle_H = \int_{G^0} \int_{G^0} \overline{\xi'(x)}\xi(s_G(x))\zeta(x)\zeta'(s_G(x))D^{-1}(x)d\nu(x) $$

These calculations prove the existence of the isometric embeddings $j$ and $\tilde{j}$. Straightforward arguments show the images of these embeddings are $C^*$-factorizations, and the calculations above show that these embeddings are unitary maps of $C^*$-modules. The assertions $\rho_\alpha = r$, $\rho_{\tilde{\beta}} = s$, and $\alpha \perp \beta$ follow from routine calculations and arguments.

Put $\beta := \alpha$.

Denote by $\tilde{\mu}$ the extension of the weight $\mu$ to the von Neumann algebra $\pi_\mu(C_0(G^0))'' \cong L^\infty(G^0, \mu)$, and by $\tilde{f}$ and $\tilde{s}$ the extensions of $r = \rho_\alpha$ and $s = \rho_{\tilde{\beta}}$ to $L^\infty(G^0, \mu)$. From now on, we identify

$$ H_{\tilde{f}} \cong H_{\tilde{h}} \cong H_{\tilde{g}} \cong H_{\tilde{h}} \cong H_{\tilde{f}} $$

as in Proposition 2.1. By [20], the Hilbert spaces above can be described as follows. Define a measure $\nu^2_{s,r}$ on $G_{s,r} := \{(x, y) \in G \times G \mid s(x) = r(y)\}$ by

$$ \int_{G_{s,r}} f \, d\nu^2_{s,r} := \int_{G^0} \int_{G^0} \int_{G^0} f(x, y) d\lambda^o(x) d\lambda^u(x) d\mu(u), $$

and a measure $\nu^2_{r,s}$ on $G_{r,s} := \{(x, y) \in G^2 \mid r_G(x) = s_G(y)\}$ by

$$ \int_{G_{r,s}} g \, d\nu^2_{r,s} := \int_{G^0} \int_{G^0} \int_{G^0} g(x, y) d\lambda^o(y) d\lambda^u(x) d\mu(u), $$

where $f \in C_c(G_{s,r}^2)$ and $g \in C_c(G_{r,s}^2)$. Then

$$ H_{\tilde{f}} \cong L^2(G_{s,r}, \nu^2_{s,r}), \quad H_{\tilde{h}} \cong L^2(G_{r,s}, \nu^2_{r,s}). $$

(16)
By [29], there exists a pseudo-multiplicative unitary

\[ V : H_\beta \otimes \alpha H \to H_\alpha \otimes \beta H \]

such that, with respect to the isomorphisms (16) and (17),

\[ (V \zeta)(x, y) = \zeta(x, x^{-1}y) \quad \text{for all } \zeta \in L^2(G^2_{s,r}, \nu^2_{s,r}), (x, y) \in G^2_{r,r}. \]

Using the identifications (16), we consider \( V \) as a unitary \( H_\beta \otimes \alpha H \to H_\alpha \otimes \beta H \).

**Theorem 5.2.** The unitary \( V \) is a \( C^* \)-pseudo-multiplicative unitary.

**Proof.** We only need to prove

\[ V_\alpha(\alpha \lhd \alpha) = \alpha \triangleright \alpha, \quad V_\beta(\beta \triangleright \beta) = \beta \lhd \beta, \quad \text{and } V_\beta(\beta \triangleright \alpha) = \beta \lhd \beta. \quad (18) \]

By Lemma 1.2 and Proposition 5.1, we can identify the factorizations \( \alpha \triangleleft \alpha = \beta \triangleleft \alpha, \beta \triangleright \beta, \beta \triangleright \beta \) of \( H_\beta \otimes \alpha H \) as \( C^* \)-modules with the internal tensor products

\[ L^2(G, \lambda) \otimes L^2(G, \lambda), \quad L^2(G, \lambda^{-1}) \otimes_r L^2(G, \lambda), \quad L^2(G, \lambda^{-1}) \otimes_r L^2(G, \lambda), \quad (19) \]

and the factorizations \( \alpha \triangleright \alpha, \beta \triangleright \beta, \alpha \triangleright \beta, \beta \triangleright \beta \) of \( H_\alpha \otimes \beta H \) as \( C^* \)-modules with the internal tensor products

\[ L^2(G, \lambda) \otimes_r L^2(G, \lambda), \quad L^2(G, \lambda^{-1}) \otimes_r L^2(G, \lambda), \quad L^2(G, \lambda^{-1}) \otimes_r L^2(G, \lambda), \quad (20) \]

Note that \( \alpha \triangleright \alpha = \beta \triangleright \beta \). The internal tensor products in (19) can be identified with certain completions of \( C_c(G^2_{s,r}) \), and the internal tensor products in (20) can be identified with certain completions of \( C_c(G^2_{r,s}) \). In each of the equations in (18), the unitary \( V \) maps the subspace \( C_c(G^2_{s,r}) \) of the \( C^* \)-factorization of the right hand side to the subspace \( C_c(G^2_{r,s}) \) of the \( C^* \)-factorization of the left hand side because \( V \) is the transpose of a homeomorphism \( G^2_{r,s} \to G^2_{s,r} \). The claim follows.

**Proposition 5.3.** The unitary \( V \) is regular.

**Proof.** For each \( \xi, \xi' \in C_c(G), \zeta \in L^2(G, \nu) \), and \( y \in G \),

\[ \langle (j(\xi'))|_1 V|(j(\xi))|_2 \rangle(y) = \int_{G \cap \alpha(x)} \overline{\xi'}(x) \xi(x) \xi(x^{-1}y) d\lambda^{\alpha \cap \nu}(y)(x). \]

\[ \langle (j(\xi'))(j(\xi)') \rangle(y) = \xi'(y) \int_{G \cap \alpha(x)} \overline{\xi'}(x) \xi(x) d\lambda^{\alpha \cap \nu}(y)(x). \]

These equations and standard arguments show that \( [(\alpha)|_1 V|\alpha)]_{(2)} \) and \( [\alpha \alpha^*] \) coincide with the closed span of operators on \( L^2(G, \nu) \) of the form

\[ \zeta \mapsto \left( y \mapsto \int_{G \cap \alpha(x)} f(x, y) \xi(x) d\lambda^{\alpha \cap \nu}(x) \right), \]

where \( f \in C_c(G^2_{s,r}) \).
By Theorem [4.14] the regular $C^*$-pseudo-multiplicative unitary $V$ gives rise to two concrete Hopf $C^*$-bimodules

$$(a, b, A, H, \tilde{A}(V), \tilde{\beta}, \alpha, \tilde{\Delta}) \quad \text{and} \quad (a, b, A(V), \alpha, \beta, \Delta).$$

Denote by $m: C_0(G) \to \mathcal{L}(L^2(G, \nu))$ the representation given by multiplication operators. Recall that for each $g \in C_c(G)$, there exists a unique operator $L(g) \in \mathcal{L}(L^2(G, \nu))$ such that

$$(L(g)\zeta)(y) = \int_{G \times G} g(x) D^{-1/2}(x) \zeta(x^{-1}y) d\lambda^G(x) \quad \text{for all} \ g \in G, \ z \in L^2(G, \nu),$$

and that the reduced groupoid $C^*$-algebra $C^*_r(G)$ is the closed linear span of all operators of the $L(g)$, where $g \in C_c(G)$ [10].

**Theorem 5.4.**

i) $\hat{A}(V) = m(C_0(G)) \cong C_0(G)$, and for each $f \in C_0(G)$, $\zeta \in H_{\alpha} \otimes \beta H \cong L^2(G^2_{r,r}, \nu^2_{r,r})$, $(x, y) \in G^2_{r,r}$,

$$(\tilde{\Delta}(m(f))\zeta)(x, y) = f(xy)\zeta(x, y).$$

ii) $A(V) = C^*_r(G)$, and for each $g \in C_c(G)$, $\zeta \in H_{\alpha} \otimes \beta H \cong L^2(G^2_{r,r}, \nu^2_{r,r})$, $(x, y) \in G^2_{r,r}$,

$$(\Delta(L(g))\zeta)(x, y) = \int_{G \times G} g(z) D^{-1/2}(z) \zeta(z^{-1}x, z^{-1}y) d\lambda^G(z).$$

**Proof.** i) Let $\xi, \xi' \in C_c(G)$ and put $\tilde{a}_{\xi, \xi'} := (\langle j(\xi)|[2] V | j(\xi')\rangle|2)$. For all $\zeta, \zeta' \in L^2(G, \nu)$,

$$\langle \zeta | \tilde{a}_{\xi, \xi'} \zeta' \rangle = \langle \zeta \otimes j(\xi) | V(\zeta' \otimes j(\xi')) \rangle = \int_G \int_{G \times G} \overline{\zeta(x)} \xi(y) \zeta'(x^{-1}y) d\lambda^G(x) d\nu(x) = \langle \zeta | m(f) \zeta' \rangle,$$

where $f \in C_c(G)$ is given by

$$x \mapsto \int_{G \times G} \overline{\zeta(y)} \xi'(x^{-1}y) d\lambda^G(x).$$

Standard arguments show that the closed linear span of all operators $m(f)$ with $f \in C_c(G)$ as above is equal to $m(C_0(G))$. Hence, $\hat{A}(V) = m(C_0(G))$.

For each $f \in C_0(G), \zeta \in L^2(G^2_{r,r}, \nu^2_{r,r})$, and $(x, y) \in G^2_{r,r}$,

$$(\tilde{\Delta}(m(f))\zeta)(x, y) = (V^* (\id \otimes m(f)) V) \zeta)(x, y)$$

$$= ((\id \otimes m(f)) V) \zeta)(x, xy) \in \mathcal{L}(L^2(G, \nu)).$$

ii) Let $\xi, \xi' \in C_c(G)$ and put $a_{\xi, \xi'} := (\langle j(\xi)|[1] V | j(\xi')\rangle|1)$. For all $\zeta, \zeta' \in L^2(G, \nu)$,

$$\langle \zeta | a_{\xi, \xi'} \zeta' \rangle = \langle j(\xi) \otimes \zeta | V(j(\xi') \otimes \zeta') \rangle = \int_G \int_{G \times G} \overline{\xi(x)} \zeta(y) \xi'(x^{-1}y) D^{-1/2}(x) \zeta'(x^{-1}y) d\lambda^G(x) d\nu(y) = \langle \zeta | L(g) \zeta' \rangle,$$

where $g \in C_c(G)$ is given by

$$x \mapsto \int_{G \times G} \overline{\xi(x)} \xi'(x).$$

By definition, the closed linear span of all operators $L(g)$ with $g \in C_c(G)$ is equal to $C^*_r(G)$, so $A(V) = C^*_r(G)$.
Finally, for each \( g \in C_c(G) \), \( \zeta \in L^2(G^2_{r,r}, \nu^2_{r,r}) \), and \((x,y) \in G^2_{r,r}\),

\[
\Delta(L(g))\zeta(x,y) = (V(L(g) \otimes \text{id})V^*\zeta)(x,y) = (L(g) \otimes \text{id})V^*\zeta(x, x^{-1}y) = \int_{G^2_{r(r)}} g(z)D^{-1/2}(z)(V^*\zeta)(z^{-1}x, x^{-1}y)d\lambda^{r(r)}(z).
\]

\( \square \)

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