Gribov ambiguity and non-trivial vacuum structure of gauge theories on a cylinder

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Abstract

Using the hamiltonian framework, we analyze the Gribov problem for U($N$) and SU($N$) gauge theories on a cylinder (= (1+1) dimensional spacetime with compact space $S^1$). The space of gauge orbits is found to be an orbifold. We show by explicit construction that a proper treatment of the Gribov ambiguity leads to a highly non-trivial structure of all physical states in these quantum field theory models. The especially interesting example of massless QCD is discussed in more detail: There, some of the special static gauge transformations which are responsible for the Gribov ambiguity also lead to a spectral flow, and this implies a chiral condensate in all physical states. We also show that the latter is closely related to the Schwinger term and the chiral anomaly.

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1. The elimination of gauge degrees of freedom is essential for understanding and extracting physical information from Yang-Mills (YM) gauge theories. This is usually done by ‘fixing a gauge’, i.e. requiring that the field configurations obey some gauge condition such as the Coulomb or the Landau gauge (e.g. in the path integral formalism by means of the Faddeev-Popov trick). Though adequate for perturbative calculations, it is well-known that such a procedure is not sufficient for a deeper understanding of non-Abelian YM theories: It has been pointed out already by Gribov [1] that there are many gauge equivalent configurations obeying the Coulomb or the Landau gauge condition, and that the existence of these should play a crucial role for non-perturbative features of these theories such as confinement. Later on it was shown by Singer [2] that in 4 (compact) spacetime dimensions such a Gribov ambiguity arises for any reasonable gauge fixing condition. It has also been found that the lack of a full understanding of Gribov ambiguities and their proper treatment is one major obstacle to a rigorous non-perturbative construction of gauge theories in 4 dimensions [3]. (For a recent discussion of the Gribov problem see [4].)

In 2 spacetime dimensions gauge theories are much simpler. Pure YM theory on a plane $\mathbb{R} \times \mathbb{R}$ is trivial with no propagating degrees of freedom. However, on a manifold with nontrivial topology such as a Riemann surface [5, 6] or (in the hamiltonain approach) a cylinder (spacetime $S^1 \times \mathbb{R}$) [7, 8, 9, 10], it has a finite number of physical degrees of freedom and several non-trivial features which makes it an interesting toy model. Moreover, YM gauge theories with matter (fermions or bosons) on a cylinder allow for a (essentially) rigorous construction on an operator level by means of the theory of quasi-free representations of fermion and boson field algebras [11, 12]
(we are planning to report on that in more detail in future publications; for an alternative approach see [9]). This corresponds to the well-known fact that on a cylinder, normal ordering of the bilinears of the matter fields with respect to the *free* vacuum is sufficient to eliminate all divergences in the fully interacting gauge theory.

In this paper we shall examine the Gribov problem for $U(N)$ and $SU(N)$ YM gauge theories with matter on a cylinder and discuss (some of) its physical consequences. Though the Gribov ambiguity does not allow a gauge *fixing*, we argue that it is possible to maximally *reduce* the gauge freedom. One is left with an infinite discrete non-Abelian group $G'$ of residual static gauge transformations. It is possible to determine the (topologically highly non-trivial) space of all gauge orbits explicitly and to give a general construction of all physical states which are invariant under $G'$.

We discuss some of the implications of this structure for massless QCD(1+1) (i.e. YM theory with massless fermions on a cylinder). The latter is especially interesting due to the interplay between the Gribov ambiguity and the anomaly structure [13].

We note that the existence of a non-trivial vacuum structure in massless QCD(1+1) has already been found in a one-loop calculation by Hetrick and Hosotani [8]. Our construction is without approximation and applies to all physical states (not only the vacuum). Moreover, it is a special case of a more general construction valid for any YM gauge theory on a cylinder.

2. The following discussion is in the hamiltonian framework. The gauge group is $G = SU(N)$ or $U(N)$ and assumed in the fundamental representation.

As mentioned above, a YM-field on a cylinder has a finite number of *physical* degrees of freedom. A simple argument identifying these is as follows: the temporal
component \( A_0 \) of the YM-field is not dynamical but only the Lagrange multiplier enforcing Gauss’ law (= invariance under all static gauge transformations), and the only gauge invariant quantities one can construct at some fixed time from the spatial component \( A_1 \) of the YM-field are the eigenvalues of the the Wilson loop over the whole space

\[
W[A_1] = P \exp \left( -i \int_0^{2\pi} dx A_1(x) \right) \in G. \quad (1)
\]

\( W[A_1] \) can always be written as \( h^{-1} Dh \) with \( h \in G \) and \( D \) in the Cartan subgroup of \( G \) (note that this representation is not unique). It is convenient to introduce the basis \( \{ H_i \}_{i=1}^{N-1} \) in the Cartan subalgebra of the Lie algebra \( g \) of \( G = SU(N) \) with \( H_i \equiv e_{ii} - e_{i+1,i+1} \) where \( e_{ij} \) is the \( N \times N \) matrix with the matrix elements \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\). For \( G = U(N) \) we have to add to these \( H_0 = 1 \) (the \( N \times N \) unit matrix). Then we can write \( D \) as

\[
D = \exp \left( -2\pi i \sum_i Y^i H_i \right) \quad (2)
\]

with real \( Y^i \). This suggests that the physical YM degrees of freedom can be represented by these \( Y^i \), and that it should be possible to ‘fix’ the gauge to

\[
A_0(x) = 0, \quad A_1(x) = \sum_i Y^i H_i \quad (3)
\]

which is essentially the Coulomb gauge \( \partial A_1/\partial x = 0 \). Indeed, one can systematically eliminate the gauge degrees of freedom by explicitly solving Gauss’ law and end up with the same effective hamiltonian one obtains by imposing the gauge condition \( (3) \) (see also Ref. [8]).

\(^1P\) is the path-ordering symbol, \( x \in [-\pi, \pi) \) the spatial variable.
After that there are still gauge transformations left which act non-trivially on the $Y^i$ [9, 10], so (3) is rather a reducing than a fixing of the gauge. Firstly, there are gauge transformations permuting the eigenvalues of the diagonalized Wilson loop $D$:

The gauge transformations

$$p_{ij} = \exp \left( \frac{i \pi (e_{ij} e^{i \chi} + e_{ji} e^{-i \chi})}{4} \right)$$

($\chi \in \mathbb{R}$ arbitrary) interchange the $i$-th and the $j$-th eigenvalues and provide a representation $\pi \mapsto \hat{\pi}$ of the permutation group $S_N$,

$$Y^i \rightarrow (\hat{\pi} Y)^i = \sum_j \hat{\pi}^i_j Y^j \quad \forall \pi \in S_N. \quad (5)$$

(Representing the $N$ eigenvalues of $W[A_1]$ as $e^{i Z^i}$ one has $(\hat{\pi} Z)^i = Z^{\pi^{-1}(i)}$, hence by writing $Y^i = \sum_j \alpha^i_j Z^j$ and $Z^i = \sum_j \beta^i_j Y^j$ one can easily work out the matrices $\hat{\pi}^i_j = \sum_k \alpha^i_{\pi(k)} \beta^k_j$ [14]). Secondly, the functions

$$h(x) = \exp \left( i x \sum \nu^j H_j \right), \quad \nu^j \in \mathbb{Z},$$

are gauge transformations [i.e. obey $h(0) = h(2\pi)$]. Though they leave $D$ invariant, they act non-trivially on the $Y^i$:

$$Y^i \rightarrow Y^i + \nu^i, \quad \nu^i \in \mathbb{Z}. \quad (7)$$

Obviously this provides a representation $\nu \mapsto \hat{\nu}$ of the Abelian group $\mathbb{Z}^r$ with $r = N - 1$ for $G = \text{SU}(N)$ and $r = N$ for $G = \text{U}(N)$.

It is easy to see that the group $\mathcal{G}'$ generated by all these gauge transformations (3) and (7) is a semidirect product of $\mathbb{Z}^r$ by $S_N$ [14]. This residual gauge group gives a full
characterization of the Gribov ambiguity on a cylinder and allows us to determine the space of gauge orbits explicitly: the set of all YM fields $A_1(x)$ obeying our gauge condition (3) can be identified with $\mathbb{R}^r$ ($r$ as above), and $G'$ is the group of all gauge transformations compatible with (3). Therefore $G'$ generates all Gribov copies of a given YM field configuration obeying (3), and the space of gauge orbits is $\mathbb{R}^r/G'$. It is interesting to note that (for $G \neq U(1)$) this is not a manifold but only an orbifold [13]. From this we can also easily see that (for $G \neq U(1)$) there is no gauge fixing on the cylinder avoiding the Gribov ambiguity: the space $A$ of all YM fields $A_1(x)$ is contractible but $\mathbb{R}^r/G'$ is not. Hence the fibre bundle $A \to \mathbb{R}^r/G'$ is non-trivial and does not allow for a global cross section.

3. To discuss the implications of this in gauge theories with matter, we refer to the construction of these models mentioned above. By means of the theory of quasi-free representations of fermion and boson field algebras [11, 12] one can construct a Hilbert space $H$ so that the (Fourier components of the) YM field $A_1(x)$, the matter fields and the matter currents are represented by (closeable) operators on $H$ (with a common, dense, invariant domain [12]), the hamiltonian $H$ is a hermitean operator on $H$, and there is a unitary representation $\Gamma$ of the group $G$ of all static gauge transformations on $H$ so that $\Gamma(U)H = H\Gamma(U)$ $\forall U \in G$.

The physical Hilbert space of the model is of course

$$H_{\text{phys}} = \{ \Psi \in H | \Gamma(U)\Psi = \Psi \quad \forall U \in G \},$$

and it can be explicitly constructed in two steps: After imposing the gauge condition (3), $H$ is reduced to a subspace $H'_{\text{phys}}$. $H'_{\text{phys}}$ is a tensor product of the Hilbert space $L^2(\mathbb{R}^r)$ of the YM variables $Y^i$ and a Hilbert space $F'_M$ of the matter fields. On $F'_M$
we have a unitary representation \( \Gamma_M \) of the residual gauge group \( G' \), and eqs. (8) and (7) provide a representation of \( G' \) on \( L^2(\mathbb{R}^r) \). The subspace \( \mathcal{H}''_{\text{phys}} \) of \( \mathcal{H}'_{\text{phys}} \) invariant under the gauge transformations (6) is obviously spanned by the states

\[
\Psi(\vec{\theta}) = \sum_{\nu_i \in \mathbb{Z}} \prod_i \exp \left( i \theta_i (Y^i + \nu^i) \right) \Gamma_M(h_i)^{\nu^i} \Psi
\]  

\( [\vec{\theta} \in \mathbb{R}^r; \text{strictly speaking, we should 'smear these out' by appropriate test functions } \ f(\vec{\theta}) \] \) where \( h_i(x) \equiv \exp(ixH_i) \) and \( \Psi \in \mathcal{F}'_M \). On \( \mathcal{H}''_{\text{phys}} \) we have then a unitary representation \( \hat{\Gamma} \) of the \( S_N \) subgroup of \( G' \),

\[
\hat{\Gamma}(\pi)\Psi(\vec{\theta}) = \Gamma_M(\pi) \sum_{\nu_i \in \mathbb{Z}} \prod_i \exp \left( i \theta_i (\hat{\pi}(Y^i) + \nu^i) \right) \Gamma_M(h_i)^{\nu^i} \Psi, \quad \forall \pi \in S_N.
\]  

Hence the states invariant under the whole residual gauge group \( G' \), thus spanning \( \mathcal{H}_{\text{phys}} \), are given by

\[
\frac{1}{n!} \sum_{\pi \in S_N} \hat{\Gamma}(\pi)\Psi(\vec{\theta}), \quad \Psi \in \mathcal{F}'_M, \vec{\theta} \in \mathbb{R}^r.
\]  

We can write \( \mathcal{H}_{\text{phys}} = S\mathcal{H}''_{\text{phys}} \) where \( S = \frac{1}{n!} \sum_{\pi \in S_N} \hat{\Gamma}(\pi) \) is an orthogonal projection. It is worth pointing out that for this construction to work the semidirect product structure of \( G' \) is crucial, i.e. that \( \pi^{-1}\nu\pi \in \mathbb{Z}^r \) for all \( \nu \in \mathbb{Z}^r \) and \( \pi \in S_N \).

4. As an illustration and an especially interesting example, we discuss in more detail massless QCD(1+1). Here the very gauge transformations \( h(x) \) also cause a spectral flow of the fermions. Due to this, the YM variables \( Y^i \) can combine with fermionic degrees of freedom \( Q_5^i \) resulting in variables invariant under the \( h(x) \), and this leads to an interesting additional structure of the physical states. To show this, we refer to a (essentially) rigorous construction of massless QCD(1+1) by means the
non-trivial quasi-free representation of the fermion field operators $\psi^{(s)}(x) \equiv \psi_{\sigma,A}^{(s)}(x)$ naturally associated with the free fermion hamiltonian

$$H_0 = \int_0^{2\pi} dx : \psi^*(x) \gamma_5 \frac{1}{i} \partial_x \psi(x) :,$$

with $\gamma_5 \equiv (\gamma_5)_{\sigma \sigma'} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ \[1\] and $: \cdots :$ normal ordering with respect to the free fermion vacuum $\Omega$ (see below). This representation is on the fermion Fock space $\mathcal{F}_F$ generated from a state $\Omega (= \text{free Dirac sea})$ obeying

$$\hat{\psi}_{1,A}(n)\Omega = \hat{\psi}_{2,A}(-n-1)\Omega = 0, \ \forall A, n \geq 0,$$

(11)

where $\hat{\psi}_{\sigma,A}(n) = \int_0^{2\pi} dx \exp(-inx)\psi_{\sigma,A}(x)/\sqrt{2\pi}$ and $n \in \mathbb{Z}$. After reducing the gauge to (3), the remnant of the Gauss' law is $\hat{\rho}_i(0) \equiv \int_0^{2\pi} dx : \psi(x) H^i \psi(x) : \simeq 0$ \[10\] with the $r N \times N$ matrices $H^i$ obeying $\text{tr}(H^i H^j) = \delta^i_j$. Hence the reduced Hilbert space $\mathcal{H}'_{\text{phys}}$ of the model is a tensor product of the space $\mathcal{F}_F' = \{ \Psi \in \mathcal{F}_F | \hat{\rho}_i(0)\Psi = 0, \ \forall i \}$ and the Hilbert space $L^2(\mathbb{R}^r)$ for the YM variables $Y^i$, and there is a unitary representation of the residual gauge group $\mathcal{G}'$ on $\mathcal{H}'_{\text{phys}}$. Especially, the special gauge transformations $h(x)$ \[3\] are implemented by unitary operators $\Gamma_F(h)$ on $\mathcal{F}_F' \ [10, 11]$, and they act non-trivially on the axial ‘charges’ $Q_5^i \equiv \int_0^{2\pi} dx : \psi^*(x) \gamma_5 H^i \psi(x) :$ (the $H^i$ as above):

$$Q_5^i \rightarrow Q_5^i + 2\nu^i \ \forall i.$$

(12)

[Proof: We first consider the operators $Q_{5,AB} = \int_0^{2\pi} dx : \psi^*(x) \gamma_5 e_{AB} \psi(x) :$ (with $e_{AB}

\sigma, \sigma' \in \{1, 2\}$ and $A, B \in \{1, \ldots, N\}$ are the spin and color indices, respectively

\[3\] i.e. $H^i = \sum_j (b^{-1})^{ij} H_j$ with $(b^{-1})^{ij}$ the inverse matrix to $b_{ij} = \text{tr}(H_i H_j)$

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as above) which can be written as

\[ Q_{5,AB} = \sum_{n \geq 0} \left( \hat{\psi}^*_1(n) \hat{\psi}_1(1-n) - \hat{\psi}_1(1-n) \hat{\psi}^*_1(n) \right) \cdot \left( \begin{array}{c} 1 \leftrightarrow 2 \\ n \leftrightarrow 1-n \end{array} \right) \].

Obviously under the transformation \( \psi(x) \rightarrow \exp(i\epsilon \xi) \psi(x) \) we have \( \hat{\psi}^{(s)}_{\sigma,A}(n) \rightarrow \hat{\psi}^{(s)}_{\sigma,A}(n - \delta_{i,A}) \), hence by the canonical anticommutator relations of the fermion field operators,

\[ Q_{5,AB} \rightarrow Q_{5,AB} + 2\delta_{AB}\delta_{iA} = Q_{5,AB} + 2(e_{ii})_{AB}. \]

From this we readily deduce that under \( \psi(x) \rightarrow h(x)\psi(x) \), \( Q_{5,AB} \rightarrow Q_{5,AB} + 2\sum_{j} \nu^j(H_j)_{AB} \), hence as \( Q_5^i = \sum_{A,B}(H^i)_{BA}Q_{5,AB} \),

\[ Q_5^i \rightarrow Q_5^i + 2\sum_{A,B}(H^i)_{BA} \sum_{j} \nu^j(H_j)_{AB} \]

identical with eq. (12).

From this proof it is clear that the implementers \( \Gamma_F(h_i) \) create particle-antiparticle pairs in the fermion Fock space (this can also be seen from the explicit formulas for these derived in [13]), hence all physical states \( \Psi(\bar{\theta}) \) (including, of course, the groundstate=vacuum of the model) contain a condensate of fermion-anti-fermion pairs. Moreover, on \( \mathcal{H}_{\text{phys}} \) the YM-degrees of freedom \( Y^i \) can be combined with the axial ‘charges’ \( Q_5^i \) to the gauge invariant \( \theta \)-variables

\[ \theta_i = \frac{1}{i} \frac{\partial}{\partial Y^i}, \quad p^i \equiv \frac{1}{i} \frac{\partial}{\partial \theta_i} = Y^i - \frac{1}{2} Q_5^i. \]  \hspace{1cm} (13)

This can be understood also as a result of the anomalies in this model [13]: The naive fermion currents\(^4\)

\[ \rho^a(x) =: \psi^*(x)T^a\psi(x) : \quad \text{and} \quad j^a(x) =: \psi^*(x)\gamma_5T^a\psi(x) : \]

\(^4\)the \( T^a \) are the generators of the Lie algebra \( g \) of \( G \).
constructed by means of normal ordering: \( \cdots \) with respect to the free fermion vacuum \( \Omega \). Due to the well-known Schwinger term in the \([\rho, j]\)-commutator [17, 11] the currents \( j^a(x) \) do not transform covariantly under static gauge transformations. However, there is a freedom in the normal ordering prescription [11] which can be used to fix this problem: one can show that the (essentially unique) gauge covariant currents are given by \( \tilde{\rho}(x) = \rho(x) \) and \( \tilde{j}^a_i(x) = j^a_i(x) - A^a_i(x)/\pi \) (one can think of this as gauge-covariant normal ordering). The gauge covariant axial ‘charges’ are therefore \( \tilde{Q}_5 = \int_0^{2\pi} dA \tilde{j}(x) = Q_5 - \int_0^{2\pi} dA A_1(x)/\pi \). By direct calculation one can show that these give covariantly conserved vector currents \( \tilde{\nu} = (\tilde{\rho}(x), \tilde{j}(x)) \): \( D_\nu \tilde{\nu}(x) = 0 \), whereas the gauge covariant axial current \( \tilde{j}_5^i(x) = (\tilde{j}(x), \tilde{\rho}(x)) \) has an anomaly: \( D_\nu \tilde{j}_5^i(x) = -E(x)/\pi \) (with \( E = F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1] \) the YM field strength as usual). In the gauge (3) this implies \( \partial_\nu \tilde{j}_5^i(x) = -\sum_i \partial_0 Y^i H_i/\pi \) and gauge invariance (under the residual gauge transformations in \( G' \)) and non-conservation of the axial charges \( \tilde{Q}_5^i \) identical with \(-2p^i \) (\( p^i \) cf. eq. (13)).

5. Usually (especially in path integral approaches) Gribov ambiguities are accounted for by restricting the YM-field to one fundamental domain [1, 4]. In our case this would correspond to restricting the \( Y^i (3) \) to some appropriate compact subset \( FD \) of \( \mathbb{R}^r \) and forgetting about the special gauge transformations (5), (7). [E.g. for

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5 we write \( \rho(x) = \sum_a \rho^a(x) T^a \) and similarly for \( j, A_1 \) etc.

6 \( \nu \in \{0, 1\} \) is a spacetime index

7 \( D_\nu \) is the covariant derivative in the adjoint representation

8 \( \partial_\nu = \partial/\partial x^\nu, x^0 = t \) (time) and \( x^1 = x \)
\( G = U(N), \ N \geq 2, \) such an appropriate \( FD \) would be given by

\[
0 \leq Z^1 \leq Z^2 \leq \ldots \leq Z^N < 1
\]

where \( Z^1 = Y^0 + Y^1, \ Z^i = Y^0 + Y^i - Y^{i-1} \) for \( i = 2, \ldots, N-1, \) and \( Z^N = Y^0 - Y^{N-1}. \]

Though in principle possible, we think that (at least on a cylinder) our approach is much more natural.

We finally want to stress that our explicit treatment is restricted in an essential way to \((1 + 1)\) dimensions. In higher dimensions a YM-field has an infinite number of degrees of freedom allowing for a lot of unitarily inequivalent representations one has to worry about (only for a finite number of quantum degrees of freedom, all (reasonable) Hilbert space representation are unitarily equivalent \[18\]). Moreover, it is only in \((1 + 1)\) dimensions that the physical relevant representations of the fermion or boson field algebra \[18\] of a gauge theory model is unitarily equivalent to the one of the corresponding free model, and this is crucial for being able to construct the interacting hamiltonian on the ‘free’ fermion or boson Fock space, respectively. Nevertheless we think that our results give an example of a highly non-trivial interplay between the special gauge transformations implying the Gribov ambiguity and matter fields in gauge theories, especially when fermions are involved. It is tempting to conjecture a similar mechanism also in 4 dimensional QCD which might be a key to understanding confinement. We do not know, however, whether a similar approach to ours can be used there. As a prerequisite this would require a \textit{full} understanding of the Gribov ambiguity in 4 dimensions which has not been achieved.
yet [4].

**Note added:** After submission of this paper we received Ref. [19] where also the space of gauge orbits for SU(N) Yang-Mills fields on a cylinder is found.

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[14] e.g. for $G = SU(3)$ one gets $(\hat{\pi}_1)_j^i = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $(\hat{\pi}_2)_j^i = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, and

$(\hat{\pi}_3)_j^i = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ representing the $S_3$ elements $\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, and $\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, respectively.

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