Barrenechea, Gabriel R. and Bosy, Michał and Dolean, Victorita (2018)
Numerical assessment of two-level domain decomposition preconditioners for incompressible Stokes and elasticity equations.
ETNA - Electronic Transactions on Numerical Analysis, 49. pp. 41-63.
ISSN 1068-9613 , http://dx.doi.org/10.1553/etna_vol49s41

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Numerical assessment of two-level domain decomposition preconditioners for incompressible Stokes and elasticity equations

Gabriel R. Barrenechea¹, Michal Bosy ª¹, and Victorita Dolean¹

¹Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, G1 1XH
Glasgow, United Kingdom

June 30, 2017

Abstract

Solving the linear elasticity and Stokes equations by an optimal domain decomposition method derived algebraically involves the use of non standard interface conditions. The one-level domain decomposition preconditioners are based on the solution of local problems. This has the undesired consequence that the results are not scalable, it means that the number of iterations needed to reach convergence increases with the number of subdomains. This is the reason why in this work we introduce, and test numerically, two-level preconditioners. Such preconditioners use a coarse space in their construction. We consider the nearly incompressible elasticity problems and Stokes equations, and discretise them by using two finite element methods, namely, the hybrid discontinuous Galerkin and Taylor-Hood discretisations.

Key words. Stokes problem, nearly incompressible elasticity, Taylor-Hood, hybrid discontinuous Galerkin methods, domain decomposition, coarse space, optimized restricted additive Schwarz methods

1 Introduction

In [BBD+16] the one-level domain decomposition methods for Stokes equations were introduced in conjunction with the non standard interface conditions. Although it can be observed there the lack of scalability with respect to the number of subdomains. It means that by splitting the problem in a larger number of subdomains leads to the increase of size of the plateau region in the convergence of an iterative method (see Figure 1) when using the one-level domain decomposition methods. This is caused by the lack of global information, as subdomains can only communicate with their neighbours. Hence, when the number of subdomains increases in one direction, the length of the plateau also increases. Even in cases when the local problems are of the same size, the iteration count grows with the increase of the number of subdomain. This can be also observed in all experiments in this manuscript in case of one-level methods.

The remedy for this is the use of a second level in the preconditioner or a coarse space correction that adds the necessary global information. Two-level algorithms have been analysed for several classes of problems in [TW05]. The key point of these kind of methods is to choose the appropriate coarse space. The classical coarse space introduced by Nicolaides in [Nic87] is defined by vectors that support is in each subdomain. Hence the coarse space has the size equal to a number of

*Corresponding author: E-mail: michal.bosy@strath.ac.uk
subdomains. A coarse space construction, named spectral coarse space, was motivated by the complexity of the problems that classical coarse space performance was not satisfying. This construction allows to enrich a bigger size of the coarse space, but can be also reduced to the classical one. This idea was introduced for the first time in [BHMV99] in the case of multigrid methods. It relies on solving local generalised eigenvalue problems allowing to choose suitable vectors for the coarse space.

For overlapping domain decomposition preconditioners, a similar idea was introduced in the case of Darcy equations in [GE10a, GE10b]. The authors of [NXD10] consider also the heterogeneous Darcy equation and presented a different generalised eigenvalue problem based on local Dirichlet-to-Neumann maps. The method has been analysed in [DNSS12] and proved to be very robust in the case of small overlaps. The same idea was extended numerically to the heterogeneous Helmholtz problem in [CDKN14]. The authors of [LNS15] apply the coarse space associated with low-frequency eigenfunctions of the subdomain Dirichlet-to-Neumann maps for the generalisation of the optimised Schwarz methods, named 2-Lagrange multiplier methods.

The first attempt to extend this spectral approach to general symmetric positive definite problems was made in [EGLW12] as an extension of [GE10a, GE10b]. Since some of the assumptions of the previous framework are hard to fulfil, authors of [SDH14] proposed slightly different approach for symmetric positive definite problems. Their idea of constructing partition of unity operator associated with degrees of freedom allows to work with various finite element spaces. An overview of different kinds of two-level methods can be found in [DJN15, Chapters 5 and 7].

Despite the fact that all these approaches provide satisfying results, there is no universal treatment to build efficient coarse spaces in the case of non definite problems such as Stokes equations. The spectral coarse spaces that we use in this work are inspired by those proposed in [HJN15]. The authors introduced and tested numerically symmetrised two-level preconditioners for overlapping algorithms which use Robin interface conditions between the subdomains (see (5.27) for details). They have applied these preconditioners to solve saddle point problems such as nearly incompressible elasticity and Stokes discretised by Taylor-Hood finite elements. In our case, we use non standard interface conditions. Therefore the use of spectral coarse spaces could lead to an important gain.

In this work, we test this improvement in case of nearly incompressible elasticity and Stokes equations that are discussed in Section 2. As the discretisations we use the Taylor-Hood [GR86, Chapter II, Section 4.2] and hybrid discontinuous Galerkin method [CGL09, CG09] presented in Section 3. In Section 4 we introduce the two-level domain decomposition preconditioners. Sections 5 and 6 present the two and three dimensional numerical experiments, respectively. Finally, a summary is outlined in Section 7.
2 The differential equations

Let $\Omega$ be an open polygon in $\mathbb{R}^2$ or an open Lipschitz polyhedron in $\mathbb{R}^3$, with Lipschitz boundary $\Gamma := \partial \Omega$. We use $d = 2, 3$ to denote the dimension of the space. We use bold for tensor or vector variables. In addition we denote normal and tangential components as follows $u_n := u \cdot n$, and $u_t := u - u_n n$, where $n$ is the outward unit normal vector to the boundary $\Gamma$.

For $D \subset \Omega$, we use the standard $L^2(D)$ space and $C^0(\bar{D})$ denotes the set of all continuous functions on the closure of a set $D$. Let us define the following Sobolev spaces

\begin{align*}
H^m(D) & := \{ v \in L^2(D) : \forall |\alpha| \leq m \partial^\alpha v \in L^2(D) \} \quad \text{for } m \in \mathbb{N}, \\
H^2(\partial D) & := \{ \tilde{v} \in L^2(\partial D) : \exists v \in H^1(D) \quad \tilde{v} = \text{tr}(v) \}, \\
H(\text{div}, D) & := \{ v \in [L^2(D)]^d : \nabla \cdot v \in L^2(D) \},
\end{align*}

where, for $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ we denote $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1} ... \frac{\partial^{\alpha_d}}{\partial x_d}$, and $\text{tr} : H^1(\Omega) \rightarrow H^2(\partial \Omega)$ is the trace operator. In addition, we use the following notation of the space including boundary and average conditions

\begin{align*}
L^2_0(D) & := \{ v \in L^2(D) : \int_D v \, dx = 0 \}, \\
H^1_0(D) & := \{ v \in H^1(D) : v = 0 \text{ on } \bar{\Gamma} \},
\end{align*}

where $\bar{\Gamma} \subset \partial D$. If $\bar{\Gamma} = \partial D$, then $H^1_0(D)$ is denoted $H^1_0(D)$. 

Now we present the two differential problems considered in this work.

2.1 Stokes equation

Let us start with $d$-dimensional, $d = 2, 3$, Stokes problem

\begin{equation}
\begin{aligned}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega,
\end{aligned}
\end{equation}

where $u : \bar{\Omega} \rightarrow \mathbb{R}^d$ is the velocity field, $p : \bar{\Omega} \rightarrow \mathbb{R}$ the pressure, $\nu > 0$ the viscosity which is considered to be constant and $f \in [L^2(\Omega)]^d$ is a given function. We define the stress tensor $\sigma := \nu \nabla u - pI$ and the flux as $\sigma_n := \sigma \cdot n$. For $u_D \in [H^2(\Gamma)]^d$ and $g \in L^2(\Gamma)$ we consider three types of boundary conditions:

- Dirichlet (non-slip)
  \begin{equation}
  u = u_D \text{ on } \Gamma;  
  \end{equation}

- tangential-velocity and normal-flux (TVNF)
  \begin{equation}
  \begin{aligned}
  u_t &= 0 \quad \text{on } \Gamma \\
  \sigma_{nn} &= g \quad \text{on } \Gamma.
  \end{aligned}
  \end{equation}

- normal-velocity and tangential-flux (NVTF)
  \begin{equation}
  \begin{aligned}
  u_n &= 0 \quad \text{on } \Gamma \\
  \sigma_{nt} &= g \quad \text{on } \Gamma.
  \end{aligned}
  \end{equation}

The third type of boundary condition has already been considered for the Stokes problem in [AdDBM+14].
2.2 Nearly incompressible elasticity equation

From a mathematical point of view, the nearly incompressible elasticity problem is very similar to the Stokes equations. The difference is that instead of considering the gradient $\nabla v$, the symmetric gradient $\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla^T v)$ is used. We want to solve the following $d$-dimensional, $d = 2, 3$, problem

\[
\begin{cases}
-2\mu \nabla \cdot \varepsilon(u) + \nabla p = f \quad \text{in } \Omega \\
- \nabla \cdot u = \frac{1}{\lambda} p \quad \text{in } \Omega
\end{cases}
\]

where $u : \bar{\Omega} \to \mathbb{R}^d$ is the displacement field, $p : \Omega \to \mathbb{R}$ the pressure, $f \in [L^2(\Omega)]^d$ is a given function, $\lambda$ and $\mu$ are the Lamé coefficients defined by

$$
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},
$$

where $E$ is the Young modulus and $\nu$ the Poisson ratio. We define the stress tensor as $\sigma^{sym} := 2\mu\varepsilon(u) - pI$ and its normal component as $\sigma^{sym}_n := \sigma^{sym} n$. For $g \in L^2(\Gamma)$ we consider three types of boundary conditions:

- mixed such that $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$

\[
\begin{cases}
u = 0 \text{ on } \Gamma_D \\
\sigma^{sym}_n = 0 \text{ on } \Gamma_N
\end{cases}
\]  (2.6)

- tangential-displacement and normal-normal-stress (TDNNS)

\[
\begin{cases}
\nu_t = 0 \text{ on } \Gamma \\
\sigma^{sym}_{nn} = g \text{ on } \Gamma
\end{cases}
\]  (2.7)

- normal-displacement and tangential-normal-stress (NDTNS)

\[
\begin{cases}
\nu_n = 0 \text{ on } \Gamma \\
\sigma^{sym}_{nt} = g \text{ on } \Gamma
\end{cases}
\]  (2.8)

The second type of boundary condition has already been considered for linear elasticity equation in [PS11].

3 The numerical methods

Let $\{T_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ made of simplices. For each triangulation $T_h$, $E_h$ denotes the set of its facets (edges for $d = 2$, faces for $d = 3$). In addition, for each element $K \in T_h$, $h_K := \text{diam}(K)$, and we denote $h := \max_{K \in T_h} h_K$. We define the following broken Sobolev spaces on the set of all edges in $E_h$ (for $d = 2$)

$$L^2(E_h) := \{ v : v|_E \in L^2(E) \forall E \in E_h \}.$$

Moreover, for $D \subset \Omega$, $\mathbb{P}_k(D)$ denotes the space of polynomials of total degree smaller than, or equal to, $k$ on the set $D$.

We now present the two discretisations to be used in the numerical experiments.

3.1 Taylor-Hood discretisation

We first consider the Taylor-Hood discretisation using the following approximation spaces

$$\mathbf{TH}_h^k := \{ v_h \in [H^1(\Omega)]^d : v_h|_K \in [\mathbb{P}_k(K)]^d \quad \forall K \in T_h \},$$

$$R_{h-1}^k := \{ q_h \in C^0(\Omega) : q_h|_K \in \mathbb{P}_{k-1}(K) \quad \forall K \in T_h \}.$$
where \( k \geq 2 \) (see [GR86, Chapter II, Section 4.2]).

If (2.1) is supplied with the homogeneous boundary conditions (2.2), then the discrete problem reads:

\[
\begin{aligned}
\text{Find } (u_h, p_h) & \in (TH_h^k \cap [H^1_0(\Omega)]^d) \times (R_h^{k-1} \cap L_2^0(\Omega)) \\
\text{s.t. for all } (v_h, q_h) & \in (TH_h^k \cap [H^1_0(\Omega)]^d) \times (R_h^{k-1} \cap L_2^0(\Omega))
\end{aligned}
\]

\[
\int_{\Omega} \nu \nabla u_h : \nabla v_h \, dx - \int_{\Omega} p_h \nabla \cdot v_h \, dx = \int_{\Omega} f v_h \, dx
\]

\[
- \int_{\Omega} \nabla \cdot u_h q_h \, dx = 0.
\]

(3.9)

In case of TVNF boundary conditions (2.3), we define \( V_t := \{ v \in [H^1(\Omega)]^d : v_t = 0 \text{ on } \Gamma \} \), and the discrete problem reads:

\[
\begin{aligned}
\text{Find } (u_h, p_h) & \in (TH_h^k \cap V_t) \times R_h^{k-1} \\
\text{s.t. for all } (v_h, q_h) & \in (TH_h^k \cap V_t) \times R_h^{k-1}
\end{aligned}
\]

\[
\int_{\Omega} \nu \nabla u_h : \nabla v_h \, dx - \int_{\Omega} p_h \nabla \cdot v_h \, dx = \int_{\Omega} f v_h \, dx + \int_{\Gamma} g(v_h)_t \, ds
\]

\[
- \int_{\Omega} \nabla \cdot u_h q_h \, dx = 0.
\]

(3.10)

If NVTF boundary conditions (2.4) are used, then we define the following space \( V_n := \{ v \in [H^1(\Omega)]^d : v_n = 0 \text{ on } \Gamma \} \), and the discrete problem reads:

\[
\begin{aligned}
\text{Find } (u_h, p_h) & \in (TH_h^k \cap V_n) \times (R_h^{k-1} \cap L_2^0(\Omega)) \\
\text{s.t. for all } (v_h, q_h) & \in (TH_h^k \cap V_n) \times (R_h^{k-1} \cap L_2^0(\Omega))
\end{aligned}
\]

\[
\int_{\Omega} \nu \nabla u_h : \nabla v_h \, dx - \int_{\Omega} p_h \nabla \cdot v_h \, dx = \int_{\Omega} f v_h \, dx + \int_{\Gamma} g(v_h)_n \, ds
\]

\[
- \int_{\Omega} \nabla \cdot u_h q_h \, dx = 0.
\]

(3.11)

In similar way, if the problem (2.5) is supplied with the boundary conditions (2.6), then the discrete problem reads

\[
\begin{aligned}
\text{Find } (u_h, p_h) & \in (TH_h^k \cap [H^1_{1,D}(\Omega)]^d) \times R_h^{k-1} \\
\text{s.t. for all } (v_h, q_h) & \in (TH_h^k \cap [H^1_{1,D}(\Omega)]^d) \times R_h^{k-1}
\end{aligned}
\]

\[
\int_{\Omega} 2\mu \varepsilon (u_h) : \varepsilon (v_h) \, dx - \int_{\Omega} p_h \nabla \cdot v_h \, dx = \int_{\Omega} f v_h \, dx
\]

\[
- \int_{\Omega} \nabla \cdot u_h q_h \, dx - \frac{1}{\chi} \int_{\Omega} p_h q_h \, dx = 0.
\]

(3.12)

The rest of the discrete problems associated with (2.5) that is supplied with TDNNS boundary conditions (2.7) or NDTNS boundary conditions (2.8) are similar to (3.10) or (3.11), respectively.
3.2 Hybrid discontinuous Galerkin discretisation

We restrict the discussion of this to two dimensional case $d = 2$. This method has been presented and analysed in [BBD+16]. The velocity is approximated using the Brezzi-Douglas-Marini spaces (see [BBF13, Section 2.3.1]) of degree $k$ given by

$$BDM^k_{h,i} := \left\{ \mathbf{v}_h \in H(div; \Omega) : \mathbf{v}_h|_K \in [P_k(K)]^2 \forall K \in T_h \right\},$$

$$BDM^{k,1}_{h,i} := \left\{ \mathbf{v}_h \in H(div; \Omega) : \mathbf{v}_h|_K \in [P_k(K)]^2 \forall K \in T_h \wedge (\mathbf{v}_h)_n = 0 \text{ on } \bar{\Gamma} \right\},$$

where $\bar{\Gamma} \subset \partial \Omega$. If $\bar{\Gamma} = \partial \Omega$, then $\text{BDM}^{k,1}_{h,i}$ is denoted $\text{BDM}^{k,0}_{h,i}$.

Finally, a Lagrange multiplier, aimed at approximating the tangential component of the velocity is introduced. The space where this multipliers is sought are given by

$$\text{Q}^{k-1}_h := \{ q_h \in L^2(\Omega) : q_h|_K \in P_{k-1}(K) \forall K \in T_h \}.$$ 

The pressure is approximated in the space

$$\text{M}^{k-1}_h := \{ \tilde{\mathbf{v}}_h \in L^2(\mathcal{E}_h) : \tilde{\mathbf{v}}_h|_E \in P_{k-1}(E) \forall E \in \mathcal{E}_h \},$$

$$\text{M}^{k-1}_{h,i} := \{ \tilde{\mathbf{v}}_h \in M^{k-1}_h : \tilde{\mathbf{v}}_h = 0 \text{ on } \bar{\Gamma} \},$$

where $\bar{\Gamma} \subset \partial \Omega$. If $\bar{\Gamma} = \partial \Omega$, then $M^{k-1}_h$ is denoted $M^{k-1}_{h,0}$. Furthermore, we introduce for all $E \in \mathcal{E}_h$ the $L^2(E)$-projection $\Phi^{k-1}_E : L^2(E) \to P_{k-1}(E)$ defined by

$$\int_E \Phi^{k-1}_E(\tilde{\mathbf{w}}) \tilde{\mathbf{v}}_h \, ds = \int_E \tilde{\mathbf{w}} \tilde{\mathbf{v}}_h \, ds \quad \forall \tilde{\mathbf{v}}_h \in P_{k-1}(E),$$

and we denote $\Phi^{k-1} : L^2(\mathcal{E}_h) \to M^{k-1}_h$ defined as $\Phi^{k-1}|_E := \Phi^{k-1}_E$ for all $E \in \mathcal{E}_h$.

If (2.1) is supplied with the homogeneous boundary conditions (2.2), then the discrete problem reads:

Find $(\mathbf{u}_h, \tilde{\mathbf{u}}_h, p_h) \in \text{BDM}^{k,0}_{h,0} \times M^{k-1}_{h,0} \times (\text{Q}^{k-1}_h \cap L^2(\Omega))$

s.t. for all $(\mathbf{v}_h, \tilde{\mathbf{v}}_h, q_h) \in \text{BDM}^{k,0}_{h,0} \times M^{k-1}_{h,0} \times (\text{Q}^{k-1}_h \cap L^2(\Omega))$

$$(3.14) \quad \left\{ \begin{array}{l}
a ((\mathbf{u}_h, \tilde{\mathbf{u}}_h), (\mathbf{v}_h, \tilde{\mathbf{v}}_h)) + b ((\mathbf{v}_h, \tilde{\mathbf{v}}_h), p_h) = \int_\Omega f \mathbf{v}_h \, dx, \\
b ((\mathbf{u}_h, \tilde{\mathbf{u}}_h), q_h) = \int_0^\Omega \mathbf{v}_h \cdot \mathbf{f} \, dx,
\end{array} \right.$$ 

where

$$a ((\mathbf{w}_h, \tilde{\mathbf{w}}_h), (\mathbf{v}_h, \tilde{\mathbf{v}}_h)) := \sum_{K \in T_h} \left( \int_K \nu \nabla \mathbf{w}_h : \nabla \mathbf{v}_h \, dx \
- \int_{\partial K} \nu (\mathbf{\mathbf{n}} \mathbf{w}_h)_t \left( (\mathbf{v}_h)_t - \tilde{\mathbf{v}}_h \right) \, ds \right.$$

$$\left. + \nu \tau \sum_{K \in T_h} \int_{\partial K} \Phi^{k-1}((\mathbf{w}_h)_t - \tilde{\mathbf{w}}_h) \Phi^{k-1}((\mathbf{v}_h)_t - \tilde{\mathbf{v}}_h) \, ds \right),$$

and $\tau > 0$ is a stabilisation parameter, and

$$b ((\mathbf{v}_h, \tilde{\mathbf{v}}_h), q_h) := - \sum_{K \in T_h} \int_K q_h \nabla \cdot \mathbf{v}_h \, dx.$$ 

If TVNF boundary conditions (2.3) are used, then the discrete problem reads:
Find \((u_h, \tilde{u}_h, p_h) \in BDM_h^k \times M_h^{k-1} \times Q_h^{k-1}\)

s.t. for all \((v_h, \tilde{v}_h, q_h) \in BDM_h^k \times M_h^{k-1} \times Q_h^{k-1}\)

\[
\begin{align*}
\left\{ \begin{array}{ll}
a((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + b((v_h, \tilde{v}_h), p_h) &= \int_{\Omega} f v_h \, dx + \int_{\Gamma} g (v_h)_n \, ds, \\
b((u_h, \tilde{u}_h), q_h) &= 0,
\end{array} \right.
\end{align*}
\]

(3.17)

In case of NVTF boundary conditions (2.4), the discrete problem reads:

Find \((u_h, \tilde{u}_h, p_h) \in BDM_h^k \times M_h^{k-1} \times (Q_h^{k-1} \cap L_0^2(\Omega))\)

s.t. for all \((v_h, \tilde{v}_h, q_h) \in BDM_h^k \times M_h^{k-1} \times (Q_h^{k-1} \cap L_0^2(\Omega))\)

\[
\begin{align*}
\left\{ \begin{array}{ll}
a((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + b((v_h, \tilde{v}_h), p_h) &= \int_{\Omega} f v_h \, dx + \int_{\Gamma} g \tilde{v}_h \, ds, \\
b((u_h, \tilde{u}_h), q_h) &= 0.
\end{array} \right.
\end{align*}
\]

(3.18)

In similar way, if the problem (2.5) is supplied with the mixed boundary conditions (2.6), then the discrete problem reads:

\[
\begin{align*}
\left\{ \begin{array}{ll}
a_s((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + b((v_h, \tilde{v}_h), p_h) &= \int_{\Omega} f v_h \, dx \\
b((u_h, \tilde{u}_h), q_h) + c(p_h, q_h) &= 0,
\end{array} \right.
\end{align*}
\]

(3.19)

where

\[
\begin{align*}
a_s((w_h, \tilde{w}_h), (v_h, \tilde{v}_h)) &:= \sum_{K \in \mathcal{T}_h} \left( \int_{K} 2\mu \varepsilon(w_h) : \varepsilon(v_h) \, dx ight. \\
&\left. - \int_{\partial K} 2\mu \varepsilon_n(w_h)_t \cdot (v_h)_t - \tilde{v}_h \right) ds \\
&\left. - \int_{\partial K} 2\mu (w_h)_t - \tilde{w}_h \cdot (\varepsilon_n(v_h))_t ds \\
&\left. + 2\mu \tau \int_{\partial K} \Phi^{k-1}((w_h)_t - \tilde{w}_h)\Phi^{k-1}((v_h)_t - \tilde{v}_h) ds \right),
\end{align*}
\]

(3.20)

\(b\) is defined by (3.16), and

\[
c(r_h, q_h) := -\frac{1}{\lambda} \int_{\Omega} r_h q_h \, ds.
\]

4 The domain decomposition preconditioners

Let us assume that we have to solve the following linear system \(AU = F\) where \(A\) is the matrix arising from discretisation of the Stokes or linear elasticity equation on the domain \(\Omega\), \(U\) is the vector of unknowns, and \(F\) is the right hand side. To accelerate the performance of an iterative Krylov method [DKN15, Chapter 3] applied to this system we will consider domain decomposition preconditioners which are naturally parallel. They are based on an overlapping decomposition of the computational domain.
Let \( \{ T_{h,i} \}_{i=1}^{N} \) be a partition of the triangulation \( T_h \) (see examples in Figure 2). For an integer value \( l \geq 0 \), we define an overlapping decomposition \( \{ T_{l,h,i} \}_{i=1}^{N} \) such that \( T_{l,h,i} \) is a set of all triangles from \( T_{h,i} \) and all triangles from \( T_h \setminus T_{h,i} \) that have non-empty intersection with \( T_{h,i} \), and \( T_{h,i}^0 = T_{h,i} \). With this definition the width of the overlap will be of \( 2l \). Furthermore, if \( W_h \) stands for the finite element space associated to \( T_h \), \( W_{l,h,i} \) is the local finite element spaces on \( T_{l,h,i} \), which is a triangulation of \( \Omega_i \).

Let \( N \) be the set of indices of degrees of freedom of \( W_h \) and \( N_{l,i} \) the set of indices of degrees of freedom of \( W_{l,h,i} \) for \( l \geq 0 \). Moreover, we define the restriction operator \( R_i : W_h \rightarrow W_{l,h,i} \) as a rectangular matrix \( |N_i| \times |N| \) such that if \( V \) is the vector of degrees of freedom of \( v_h \in W_h \), then \( R_i V \) is the vector of degrees of freedom of \( v_{l,i} \in W_{l,h,i} \) in \( \Omega_i \). The extension operator from \( W_{l,h,i} \) to \( W_h \) and its associated matrix are both given by \( R_i^T \). In addition we introduce a partition of unity \( D_i \) as a diagonal matrix \( |N_{l,i}| \times |N_{l,i}| \) such that

\[
\text{Id} = \sum_{i=1}^{N} R_i^T D_i R_i,
\]

where \( \text{Id} \in \mathbb{R}^{|N| \times |N|} \) is the identity matrix.

We first recall the Modified Restricted Additive Schwarz (MRAS) preconditioner introduced in [BBD+16] for the Stokes equation. This preconditioner is given by

\[
M_{MRAS}^{-1} = \sum_{i=1}^{N} R_i^T B_i^{-1} R_i,
\]

where \( B_i \) is the matrix associated to a discretisation of Stokes equation (2.1) in \( \Omega_i \) where we impose either TVNF (2.3) or NVTF (2.4) boundary conditions in \( \Omega_i \). In case of a discretisation of elasticity equation (2.5) in \( \Omega_i \), we impose either TDNNS (2.7) or NDTNS (2.8) boundary conditions in \( \Omega_i \).

We new introduce a symmetrised variant of (4.22) called Symmetrised Modified Restricted Additive Schwarz (SMRAS), given by

\[
M_{SMRAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i.
\]
4.1 Two-level methods

A two-level version of the SMRAS and MRAS preconditioners will be based on a spectral coarse space obtained by solving the following local generalised eigenvalue problems

\[
\text{Find } (V_{jk}, \lambda_{jk}) \in \mathbb{R}^{|\mathcal{N}_j|} \setminus \{0\} \times \mathbb{R} \text{ s.t.}
\]

\[
\tilde{A}_j V_{jk} = \lambda_{jk} B_j V_{jk},
\]

where \(\tilde{A}_j\) are local matrices associated to a discretisation of local Neumann boundary value problem in \(\Omega_j\). Let \(\theta > 0\) be a user-defined threshold. We define \(Z_{\text{GenEO}} \subset \mathbb{R}^{|\mathcal{N}|}\) as the vector space spanned by the family of vectors \((R_j^T D_j V_{jk})_{\lambda_{jk} < \theta}, 1 \leq j \leq N\), corresponding to eigenvalues smaller than \(\theta\). The value of \(\theta\) is chosen such that for a given problem and preconditioner, the behaviour of the method should be robust in the sense that, its convergence should not depend, or depends very weakly, on the number of subdomains.

We are now ready to introduce the two-level method with coarse space \(Z_{\text{GenEO}}\). Let \(P_0\) be the \(A\)-orthogonal projection onto the coarse space \(Z_{\text{GenEO}}\). The two-level SMRAS preconditioner is defined as

\[
M_{\text{SMRAS},2}^{-1} = P_0 A^{-1} + (\text{Id} - P_0) M_{\text{SMRAS}}^{-1} (\text{Id} - P_0^T).
\]

Furthermore, if \(R_0\) is a matrix whose rows are a basis of the coarse space \(Z_{\text{GenEO}}\), then

\[
P_0 A^{-1} = R_0^T \left( R_0 A R_0^T \right)^{-1} R_0.
\]

In similar way, we can introduce the two-level MRAS preconditioner

\[
M_{\text{MRAS},2}^{-1} = P_0 A^{-1} + (\text{Id} - P_0) M_{\text{MRAS}}^{-1} (\text{Id} - P_0^T).
\]

5 Numerical results for two dimensional problems

In this section we assess the performance of the preconditioners defined in Section 4.1. We will compare the newly introduced ones with that of ORAS and SORAS introduced in [HJN15]. These kind of preconditioners are associated with the Robin interface conditions and require an optimised parameter as it can be seen in (5.27) below. The big advantage of SMRAS and MRAS preconditioners from the previous section is that they are parameter-free. We consider the partial differential equation model for nearly incompressible elasticity and Stokes flow as problems of similar mixed formulation. Each of these problems is discretised by using the Taylor-Hood methods from Section 3.1 and the hdG discretisation from Section 3.2.

Our experiments will be based on the classical weak scaling test. This test is built as follows. A domain \(\Omega\) is split into a triangulation \(T_h\). For each of element \(K \in T_h\), \(h_K = \text{diam}(K)\), and we denote the mesh size by \(h := \max_{K \in T_h} h_K\). Then, this triangulation is split into overlapping subdomains of size \(H\), in such a way \(\frac{H}{h}\) remains constant. In the absence of a second level in the preconditioner, if the number of subdomains grows, then the convergence gets slower. A coarse space provides a global information and leads to a more robust behaviour.

The simplest way to build a coarse space is to consider the zero energy modes. More precisely they are the eigenvectors associated with the zero eigenvalues of (4.24) on a floating subdomains. Hence, by a floating subdomain we mean a subdomain without Dirichlet boundary condition on any part of the boundary. Then the matrix on the left hand side of (4.24) is singular and there are zero eigenvalues. These zero energy modes are the rigid body motions (three in two dimensions, six in three dimensions) for the elasticity problem, and the constants (two in two dimensions, three in three dimensions) for the Stokes equations. Unfortunately for some cases, this choice is not sufficient, so we have collected the smallest \(M\) eigenvalues for each subdomain and build a coarse
space by including the eigenvectors associated to them. The different values of $M$ are presented in the table in brackets.

All experiments have been made by using FreeFem++ [Hec12], which is a free software specialised in variational discretisations of partial differential equations. We use GMRES [SS86] as an iterative solver. Generalized eigenvalue problems to generate the coarse space are solved using ARPACK [LSY98]. The overlapping decomposition into subdomains can be uniform (Unif) or generated by METIS (MTS) [KK98]. In each of the examples in this section we consider decomposition with two layers of mesh size $h$ in the overlap. Tables show the number of iterations needed to achieve a relative $l^2$ norm of the error smaller than $10^{-6}$, $\frac{\|U - U_m\|_{l^2}}{\|U_0\|_{l^2}} < 10^{-6}$, where $U$ is the solution of global problem given by direct solver and $U_m$ denotes $m$-th iteration of the iterative solver. In addition, DOF stands for number of degrees of freedom and $N$ for the number of subdomains in all tables.

5.1 Taylor-Hood discretisation

In this section we consider the Taylor-Hood discretisation from Section 3.1 with different values of $k \geq 2$ for nearly incompressible elasticity and Stokes equations.

5.1.1 Nearly incompressible elasticity

Since we consider the preconditioners with various interface conditions we need to comment the way of imposing them. ORAS and SORAS preconditioners follow [HJN15] and use Robin interface conditions. This means, the weak formulation of the linear elasticity problem contains the following term

$\int_{\partial \Omega \setminus \Gamma} \sigma_n^{ym} \cdot (v_h)_n \, ds + \int_{\partial \Omega \setminus \Gamma} 2\alpha \frac{\mu (2\mu + \lambda)}{\lambda + 3\mu} \mu_h v_h \, ds$

where again, following [HJN15] we choose $\alpha = 10$. Fortunately, the MRAS and SMRAS preconditioners are parameter-free. For all numerical experiments associated in this section we use zero as an initial guess for the GMRES iterative solver. Moreover, the overlapping decomposition into subdomains is generated by METIS.

![Figure 3: L-shaped domain problem.](image)

**Test case 1** (The L-shaped domain problem). We consider the L-shaped domain $\Omega = (-1,1)^2 \setminus \{(0,1) \times (-1,0)\}$ clamped on the left side and partly from a top and bottom as it is depicted in
Figure 3a. This example is similar to the one in [CS15]. The associated boundary value problem is
\[
\begin{aligned}
-2\mu \nabla \cdot \varepsilon(u) + \nabla p &= (0,-1)^T \quad \text{in } \Omega \\
-\nabla \cdot u &= \frac{1}{\lambda} p \quad \text{in } \Omega \\
\mathbf{u}(x,y) &= (0,0)^T \quad \text{on } \Gamma_D \\
\sigma_{\text{sym}}^N(x,y) &= (0,0)^T \quad \text{on } \Gamma_N
\end{aligned}
\]

The physical parameters are \(E = 10^5\) and \(\nu = 0.4999\) (nearly incompressible). In Figure 3b we plot the mesh of the bent domain.

We choose \(k = 3\) for the Taylor-Hood discretisation. In Figure 4 we plot the eigenvalues of one floating subdomain. The clustering of small eigenvalues of the generalised eigenvalue problem defined in (4.24) suggests the number of eigenvectors to be added to the coarse space. The three zero eigenvalues correspond to the zero energy modes.

Table 1: Comparison of preconditioners for Taylor-Hood discretisation \((TH^3_h, R^2_h)\) - the L-shaped domain problem.

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 124 109 | 4  | 26   | 60    | 26         | 60          | 30         | 59          |
| 478 027 | 16 | 57   | 131   | 69         | 143         | 65         | 140         |
| 933 087 | 32 | 84   | 180   | 109        | 221         | 104        | 211         |
| 1 899 125 | 64 | 130  | 293   | 181        | 362         | 161        | 312         |
| 3 750 823 | 128| 209  | 412   | 302        | 568         | 251        | 510         |

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 124 109 | 4  | 18   | 40    | 19         | 36          | 24         | 41          |
| 478 027 | 16 | 37   | 52    | 40         | 57          | 46         | 56          |
| 933 087 | 32 | 49   | 57    | 56         | 67          | 53         | 66          |
| 1 899 125 | 64 | 65   | 64    | 70         | 75          | 61         | 74          |
| 3 750 823 | 128| 83   | 64    | 74         | 77          | 75         | 72          |

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 124 109 | 4  | 15   | 32    | 17         | 35          | 24         | 37          |
| 478 027 | 16 | 31   | 41    | 31         | 47          | 42         | 47          |
| 933 087 | 32 | 40   | 48    | 38         | 52          | 53         | 51          |
| 1 899 125 | 64 | 49   | 51    | 45         | 53          | 64         | 56          |
| 3 750 823 | 128| 69   | 54    | 49         | 54          | 70         | 53          |

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 124 109 | 4  | 14   | 33    | 16         | 30          | 24         | 35          |
| 478 027 | 16 | 26   | 41    | 25         | 38          | 42         | 44          |
| 933 087 | 32 | 31   | 43    | 25         | 42          | 49         | 46          |
| 1 899 125 | 64 | 39   | 47    | 30         | 39          | 59         | 50          |
| 3 750 823 | 128| 58   | 49    | 30         | 43          | 61         | 50          |

The results of Table 1 show a clear improvement in the scalability of the two-level preconditioners over the one-level ones. In fact, using five eigenvectors per subdomain, the number of iterations is virtually unaffected by the number of subdomains. All two-level preconditioners show a comparable performance. For this case, increasing the dimension of the coarse space beyond \(5 \times N\) eigenvectors does not seem to improve the results dramatically.

**Test case 2** (The heterogeneous beam problem). We consider a heterogeneous beam with ten layers of steel and rubber. Five layers are made from steel with the physical parameters \(E = 210 \cdot 10^9\) and \(\nu = 0.3\), and other five are made from rubber with the physical parameters \(E = 10^8\) and \(\nu = 0.4999\) as it is depicted in Figure 5a. A similar example was considered in [HJN15]. The computational domain is the rectangle \(\Omega = (0, 5) \times (0, 1)\). The beam is clamped on its left side,
Figure 4: Eigenvalues on one of the floating subdomains in case of uniform decomposition and Taylor-Hood discretisation \((TH^3_h, R^2_h)\) - the L-shaped domain problem.

Figure 5: Heterogeneous beam.

hence we consider the following problem

\[
\begin{cases}
-2\mu \nabla \cdot \varepsilon(u) + \nabla p = (0, -1)^T & \text{in } \Omega \\
- \nabla \cdot u = \nabla p & \text{in } \Omega \\
u(x, y) = (0, 0)^T & \text{on } \partial\Omega \cap \{x = 0\} \\
\sigma_{sym}(x, y) = (0, 0)^T & \text{on } \partial\Omega \setminus \{x = 0\}
\end{cases}
\]

(5.29)

In Figure 5b we plot the mesh of the bent beam. Because of the heterogeneous nature of the problem, we do not notice a clear clustering of the eigenvalues (see Figure 6). In such case it is well known that coarse space including only three zero energy modes is not sufficient [DNSS12]. That is why we consider a coarse space built using 5 or 7 eigenvectors per subdomain.

Figure 6: Eigenvalues on one of the floating subdomains in case of METIS decomposition and Taylor-Hood discretisation \((TH^3_h, R^2_h)\) - the heterogeneous beam.

As in the previous example, the introduction of a coarse space provides a significant improvement in the number of iterations needed for convergence. Due to the high heterogeneity of this problem, more eigenvectors per subdomain are needed to obtain scalable results. We notice an important improvement of the convergence when using two-level methods (see Table 2). Although we get a stable number of iterations only when considering a coarse space which is sufficiently big,
Table 2: Comparison of preconditioners for Taylor-Hood discretisation ($TH^3_h, R^2_h$) - the heterogeneous beam.

| DOF  | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|----|------|-------|------------|-------------|------------|-------------|
| 44 963 | 8  | 109  | 160   | 147        | 148         | 136        |             |
| 87 587 | 16 | 136  | 192   | 200        | 181         | 184        |             |
| 177 923 | 32 | 193  | 296   | 275        | 326         | 276        |             |
| 347 651 | 64 | 260  | 363   | 282        | 491         | 299        |             |
| 707 843 | 128| 412  | 420   | 369        | 601         | 346        |             |
| 1 385 219 | 256| 379  | 448   | 400        | 711         | 317        |             |

Two-level (5 eigenvectors)

| DOF  | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|----|------|-------|------------|-------------|------------|-------------|
| 44 963 | 8  | 76   | 118   | 124        | 113         | 103        |             |
| 87 587 | 16 | 106  | 146   | 166        | 138         | 123        |             |
| 177 923 | 32 | 157  | 202   | 203        | 185         | 214        |             |
| 347 651 | 64 | 178  | 191   | 225        | 170         | 182        |             |
| 707 843 | 128| 140  | 114   | 153        | 112         | 122        |             |
| 1 385 219 | 256| 119  | 86    | 118        | 77          | 94         |             |

5.1.2 Stokes equation

We now turn to the Stokes discrete problem given in Sections 3.1. Once again in case of ORAS and SORAS we choose $\alpha = 10$ as in [HJN15] for the Robin interface conditions (5.27). In the first case we consider a random initial guess for the GMRES iterative solver. Later with the second example we use zero as an initial guess.

Figure 7: Numerical solution of the driven cavity problem - the driven cavity problem.

(a) Velocity field  (b) Pressure

Test case 3 (The driven cavity problem). The test case is the driven cavity. We consider the
following problem on the unit square $\Omega = (0,1)^2$

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega \\
- \nabla \cdot u &= 0 \quad \text{in } \Omega \\
\begin{array}{ll}
\mathbf{u}(x, y) &= (1, 0)^T \quad \text{on } \partial \Omega \cap \{y = 1\} \\
\mathbf{u}(x, y) &= (0, 0)^T \quad \text{on } \partial \Omega \setminus \{y = 1\}
\end{array}
\end{align*}
\] (5.30)

In Figure 7 we plot the vector field and pressure, after solving numerically the problem.

We start with the two energy modes only (see Figure 8). This already provides some improvement. Then, we add more eigenvectors to see if they bring improvement.

Figure 8: Eigenvalues on one of the floating subdomains in case of uniform decomposition and Taylor-Hood discretisation ($TH^2_h, R^1_h$) - the driven cavity problem.

Table 3: Comparison of preconditioners for Taylor-Hood discretisation ($TH^2_h, R^1_h$) - the driven cavity problem.

| DOF N | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-------|------|-------|-----------|------------|-----------|------------|
|       | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS |
| 91 003 4 | 12 | 17 | 24 | 34 | 22 | 22 | 34 | 40 | 22 | 25 | 50 | 40 |
| 362 003 16 | 28 | 35 | 56 | 67 | 52 | 53 | 90 | 106 | 54 | 53 | 70 | 84 |
| 813 003 36 | 39 | 75 | 92 | 103 | 85 | 91 | 165 | 185 | 91 | 88 | 118 | 136 |
| 1 444 003 64 | 53 | 91 | 120 | 144 | 120 | 135 | 254 | 283 | 132 | 132 | 169 | 206 |
| 2 728 003 121 | 80 | 278 | 180 | 212 | 182 | 280 | 412 | 580 | 199 | 213 | 251 | 439 |
| 5 768 003 256 | >1000 | >1000 | 271 | 317 | 303 | 452 | 917 | 955 | 322 | 319 | 397 | 695 |
|       | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS |
| 91 003 4 | 10 | 14 | 18 | 22 | 19 | 17 | 26 | 30 | 27 | 20 | 21 | 26 |
| 362 003 16 | 20 | 25 | 32 | 37 | 33 | 34 | 50 | 62 | 60 | 40 | 42 | 51 |
| 813 003 36 | 27 | 33 | 36 | 44 | 47 | 49 | 62 | 86 | 79 | 53 | 59 | 63 |
| 1 444 003 64 | 31 | 42 | 38 | 53 | 104 | 66 | 85 | 114 | 85 | 52 | 62 | 79 |
| 2 728 003 121 | 39 | 103 | 39 | 51 | 74 | 81 | 85 | 133 | 92 | 86 | 62 | 93 |
| 5 768 003 256 | 300 | 849 | 46 | 54 | 109 | 108 | 146 | 132 | 91 | 78 | 63 | 90 |
|       | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS | Unif MTS |
| 91 003 4 | 9 | 12 | 13 | 16 | 16 | 15 | 18 | 20 | 25 | 20 | 16 | 18 |
| 362 003 16 | 16 | 20 | 21 | 24 | 27 | 22 | 28 | 37 | 56 | 37 | 26 | 35 |
| 813 003 36 | 23 | 37 | 25 | 26 | 33 | 30 | 39 | 40 | 65 | 41 | 28 | 37 |
| 1 444 003 64 | 26 | 36 | 27 | 29 | 29 | 34 | 35 | 45 | 77 | 45 | 28 | 42 |
| 2 728 003 121 | 35 | 41 | 29 | 32 | 43 | 38 | 34 | 48 | 84 | 72 | 29 | 47 |
| 5 768 003 256 | 66 | 60 | 32 | 33 | 56 | 41 | 60 | 49 | 88 | 61 | 29 | 44 |

The conclusions remain the same as for the L-shaped domain problem for the nearly incompressible elasticity equation discretised by Taylor-Hood method ($TH^3_h, R^1_h$) since Tables 3 and 1 show similar results.
**Test case 4** (The T-shaped domain problem). Finally, we consider a T-shaped domain $\Omega = (0, 1.5) \times (0, 1) \cup (0.5, 1) \times (-1, 1)$, and we impose mixed boundary conditions given by

\[
\mathbf{u}(x, y) = \begin{cases} 
(4y(1- y), 0)^T & \text{if } x = 0 \text{ or } x = 1.5 \\
(0, 0)^T & \text{otherwise}.
\end{cases}
\]

The numerical solution of this problem is depicted in Figure 9. The overlapping decomposition into subdomains is generated by METIS.

Once again a clustering of small eigenvalues of generalised eigenvalue problem defined in (4.24) is a motivation of the size of the coarse space (see Figure 10).

![Figure 9: Numerical solution - the T-shaped problem.](image)

![Figure 10: Eigenvalues on one of the floating subdomains in case of METIS decomposition and Taylor-Hood discretisation ($TH^3_h, R^2_h$) - the T-shaped problem.](image)

The same as in all examples for Taylor-Hood discretisation we notice an important improvement of the convergence when using two-level methods. Although from Table 4 we can see that the coarse spaces containing five eigenvectors seem to be sufficient.
Table 4: Comparison of preconditioners for Taylor-Hood discretisation ($TH^3_h, R^2_h$) - the T-shaped problem.

| DOF    | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|--------|----|------|-------|-----------|------------|-----------|------------|
| 33 269 | 4  | 13   | 20    | 12        | 19         | 13        | 19         |
| 138 316| 16 | 36   | 51    | 33        | 52         | 31        | 45         |
| 269 567| 32 | 59   | 85    | 52        | 85         | 49        | 75         |
| 553 103| 64 | 92   | 132   | 83        | 136        | 78        | 115        |
| 1 134 314| 128| 146  | 208   | 132       | 223        | 117       | 188        |
| 2 201 908| 256| 232  | 328   | 209       | 357        | 189       | 293        |

| DOF    | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|--------|----|------|-------|-----------|------------|-----------|------------|
| 33 269 | 4  | 10   | 14    | 9         | 15         | 12        | 15         |
| 138 316| 16 | 21   | 27    | 19        | 24         | 22        | 24         |
| 269 567| 32 | 29   | 35    | 30        | 38         | 25        | 30         |
| 553 103| 64 | 35   | 45    | 34        | 43         | 33        | 35         |
| 1 134 314| 128| 42   | 52    | 47        | 58         | 34        | 41         |
| 2 201 908| 256| 47   | 56    | 69        | 76         | 38        | 45         |

| DOF    | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|--------|----|------|-------|-----------|------------|-----------|------------|
| 33 269 | 4  | 8    | 13    | 8         | 13         | 12        | 14         |
| 138 316| 16 | 15   | 16    | 14        | 16         | 20        | 18         |
| 269 567| 32 | 14   | 19    | 20        | 22         | 24        | 19         |
| 553 103| 64 | 16   | 20    | 18        | 19         | 29        | 20         |
| 1 134 314| 128| 17   | 22    | 23        | 24         | 30        | 22         |
| 2 201 908| 256| 16   | 21    | 34        | 37         | 35        | 24         |

5.2 hdG discretisation

In this section we discretise nearly incompressible elasticity equation and the Stokes flow by using the lowest order hdG discretisation introduced in Section 3.2.

5.2.1 Nearly incompressible elasticity

In case of ORAS and SORAS we consider the Robin interface conditions as in [HJN15] with $\alpha = 10$. For all numerical experiments in this section we use zero as an initial guess for the GMRES iterative solver. Moreover, the overlapping decomposition into subdomains is generated by METIS.

Test case 5 (The L-shaped domain problem). We consider the L-shaped domain discrete problem (5.28).

Table 5 shows an important improvement of the convergence that is brought by the two-level methods. We cannot conclude that SMRAS preconditioners are much better than SORAS. Although we can note that coarse space improvement is visible for MRAS preconditioners and not for ORAS. For symmetric preconditioners (SMRAS and SORAS) five eigenvectors seem to lead already to satisfying results. While for the non-symmetric ones a bigger coarse space is required. On the other hand, we state the fact that the new preconditioners are parameter-free, which makes them easier to use as no parameter is required.
Table 5: Comparison of preconditioners for hdG discretisation - the L-shaped domain problem.

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 238  | 8   | 61   | 158   | 64         | 174         | 77         | 177        |
| 466  | 16  | 123  | 232   | 101        | 259         | 109        | 306        |
| 948  | 32  | 267  | 331   | 160        | 415         | 179        | 473        |
| 1 874| 64  | 622  | 477   | 243        | 685         | 254        | 657        |
| 3 856| 128 | >1000| 752   | 479        | >1000       | 523        | >1000      |

Two-level (3 eigenvectors)

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 238  | 8   | 48   | 98    | 52         | 96          | 46         | 116        |
| 466  | 16  | 89   | 99    | 71         | 123         | 75         | 148        |
| 948  | 32  | 250  | 130   | 110        | 158         | 118        | 173        |
| 1 874| 64  | 535  | 135   | 135        | 155         | 129        | 159        |
| 3 856| 128 | >1000| 152   | 172        | 176         | 181        | 192        |

Two-level (5 eigenvectors)

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 238  | 8   | 43   | 81    | 44         | 74          | 61         | 94         |
| 466  | 16  | 77   | 82    | 51         | 92          | 63         | 103        |
| 948  | 32  | 197  | 100   | 79         | 119         | 96         | 121        |
| 1 874| 64  | 429  | 103   | 102        | 122         | 110        | 138        |
| 3 856| 128 | >1000| 118   | 122        | 129         | 141        | 167        |

Two-level (7 eigenvectors)

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 238  | 8   | 35   | 67    | 38         | 71          | 44         | 82         |
| 466  | 16  | 77   | 82    | 51         | 92          | 63         | 103        |
| 948  | 32  | 153  | 90    | 95         | 95          | 74         | 115        |
| 1 874| 64  | 423  | 95    | 71         | 93          | 72         | 111        |
| 3 856| 128 | >1000| 118   | 122        | 129         | 141        | 167        |

Test case 6 (The heterogeneous beam problem). We consider the heterogeneous beam with ten layers of steel and rubber that is defined as a problem (5.29).

Table 6: Comparison of preconditioners for hdG discretisation - the heterogeneous beam.

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 46   | 8   | 196  | 440   | 159        | 402         | 186        | 463        |
| 88   | 16  | 317  | 602   | 330        | 582         | 326        | 666        |
| 179  | 32  | 537  | >1000 | 574        | >1000       | 587        | >1000      |
| 353  | 64  | 899  | >1000 | 847        | >1000       | 846        | >1000      |
| 704  | 128 | >1000| >1000 | >1000      | >1000       | >1000      | >1000      |
| 1 410| 256 | >1000| >1000 | >1000      | >1000       | >1000      | >1000      |

Two-level (5 eigenvectors)

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 46   | 8   | 188  | 255   | 182        | 230         | 161        | 275        |
| 88   | 16  | 244  | 313   | 273        | 299         | 262        | 346        |
| 179  | 32  | 385  | 525   | 442        | 458         | 469        | 587        |
| 353  | 64  | 514  | 444   | 551        | 526         | 590        | 558        |
| 704  | 128 | >1000| >1000 | >1000      | >1000       | >1000      | >1000      |
| 1 410| 256 | >1000| >1000 | >1000      | >1000       | >1000      | >1000      |

Two-level (7 eigenvectors)

| DOF  | N   | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|------|-----|------|-------|------------|-------------|------------|------------|
| 46   | 8   | 148  | 197   | 149        | 192         | 158        | 231        |
| 88   | 16  | 205  | 201   | 286        | 187         | 283        | 273        |
| 179  | 32  | 318  | 337   | 385        | 301         | 433        | 419        |
| 353  | 64  | 403  | 262   | 397        | 247         | 460        | 389        |
| 704  | 128 | 490  | 168   | 447        | 182         | 558        | 443        |
| 1 410| 256 | >1000| 116   | 397        | 138         | 473        | 298        |

We notice an improvement only when using a coarse space which is sufficiently big (see Table 6).
Furthermore, we get a stable number of iterations only for the symmetric preconditioners (SMRAS and SORAS), and the coarse space improvement in case of ORAS preconditioner is much less visible than in case of MRAS preconditioners. This may be due to the fact we have not chosen an optimized parameter in the Robin interface conditions (5.27).

5.2.2 Stokes equation

We now turn to the Stokes discrete problem given in 3.2. Once again in case of ORAS and SORAS we choose $\alpha = 10$ as in [HJN15] for the Robin interface conditions (5.27). In the first case we consider a random initial guess for the GMRES iterative solver. Later with the second example we use zero as an initial guess.

Test case 7 (The driven cavity problem). We consider the driven cavity defined as a problem (5.30).

Table 7: Comparison of preconditioners for hdG discretisation - the driven cavity problem.

| DOF | N  | ORAS Unif MTS | SORAS Unif MTS | NVTF-MRAS Unif MTS | NVTF-SMRAS Unif MTS | TVNF-MRAS Unif MTS | TVNF-SMRAS Unif MTS |
|-----|----|---------------|---------------|-------------------|-------------------|-------------------|-------------------|
| 93 656 | 4  | 17 | 18 | 37 | 38 | 24 | 22 | 44 | 44 | 32 | 25 | 48 | 50 |
| 373 520 | 16 | 76 | 122 | 75 | 84 | 52 | 54 | 107 | 111 | 68 | 67 | 122 | 126 |
| 839 592 | 36 | 152 | 327 | 120 | 133 | 91 | 96 | 194 | 200 | 112 | 115 | 266 | 210 |
| 1 491 872 | 64 | 261 | 587 | 162 | 176 | 130 | 143 | 294 | 303 | 159 | 158 | 292 | 286 |
| 2 819 432 | 121 | 364 | >1000 | 229 | 256 | 199 | 213 | 504 | 649 | 238 | 251 | 628 | 643 |
| 5 963 072 | 256 | 592 | >1000 | 367 | 398 | 326 | 477 | >1000 | >1000 | 392 | 404 | 995 | 740 |

Two-level (2 eigenvectors)

| DOF | N  | ORAS Unif MTS | SORAS Unif MTS | NVTF-MRAS Unif MTS | NVTF-SMRAS Unif MTS | TVNF-MRAS Unif MTS | TVNF-SMRAS Unif MTS |
|-----|----|---------------|---------------|-------------------|-------------------|-------------------|-------------------|
| 93 656 | 4  | 12 | 14 | 30 | 28 | 18 | 18 | 33 | 32 | 40 | 23 | 38 | 37 |
| 373 520 | 16 | 81 | 80 | 47 | 57 | 36 | 40 | 61 | 73 | 100 | 49 | 85 | 82 |
| 839 592 | 36 | 236 | 228 | 61 | 60 | 57 | 65 | 97 | 104 | 132 | 66 | 112 | 107 |
| 1 491 872 | 64 | 395 | 463 | 67 | 71 | 79 | 85 | 139 | 129 | 142 | 70 | 128 | 122 |
| 2 819 432 | 121 | 840 | >1000 | 73 | 86 | 113 | 127 | 188 | 178 | 157 | 86 | 127 | 139 |
| 5 963 072 | 256 | 592 | >1000 | 80 | 87 | 171 | 179 | 283 | 287 | 167 | 108 | 132 | 148 |

Two-level (5 eigenvectors)

| DOF | N  | ORAS Unif MTS | SORAS Unif MTS | NVTF-MRAS Unif MTS | NVTF-SMRAS Unif MTS | TVNF-MRAS Unif MTS | TVNF-SMRAS Unif MTS |
|-----|----|---------------|---------------|-------------------|-------------------|-------------------|-------------------|
| 93 656 | 4  | 10 | 12 | 28 | 24 | 14 | 16 | 24 | 22 | 12 | 22 | 29 | 26 |
| 373 520 | 16 | 27 | 35 | 38 | 37 | 27 | 29 | 38 | 41 | 117 | 39 | 53 | 53 |
| 839 592 | 36 | 135 | 84 | 45 | 41 | 35 | 37 | 51 | 50 | 145 | 49 | 64 | 61 |
| 1 491 872 | 64 | 278 | 212 | 49 | 45 | 44 | 42 | 58 | 55 | 157 | 59 | 64 | 64 |
| 2 819 432 | 121 | 607 | 584 | 56 | 49 | 46 | 56 | 58 | 62 | 162 | 81 | 65 | 75 |
| 5 963 072 | 256 | >1000 | >1000 | 62 | 55 | 52 | 64 | 57 | 69 | 166 | 75 | 65 | 75 |

The conclusions remain the same as in the case of nearly incompressible elasticity equation for the L-shaped domain. Although Table 7 shows that coarse spaces containing five eigenvectors seem to decrease the number of iteration even in the case of MRAS preconditioners that are not fully scalable.

Test case 8 (The T-shaped domain problem). Finally, we consider a T-shaped domain $\Omega = (0,1.5) \times (0,1) \cup (0.5,1) \times (-1,1)$, and we impose mixed boundary conditions (5.31). The numerical solution of this problem is depicted in Figure 9.

In this case scalable results can be only observed for the preconditioners associated with the non standard interface conditions (MRAS and SMRAS), and when using a coarse space which is sufficiently big (see Table 8). In the case of ORAS or SORAS, one possibility is to choose a different parameter $\alpha$, but the proof of this, and even the question of whether this would have a positive impact, are open problems.
Table 8: Comparison of preconditioners for hdG discretisation - the T-shaped problem.

| DOF | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-----|----|------|-------|-----------|-----------|-----------|-----------|
| 38 803   | 4  | 22   | 45    | 36        | 49        | 22        | 51        |
| 154 606  | 16 | 111  | 98    | 83        | 172       | 83        | 182       |
| 311 369  | 32 | 265  | 144   | 133       | 262       | 130       | 266       |
| 616 772  | 64 | 568  | 238   | 212       | 410       | 195       | 412       |
| 1 246 136| 128| >1000| 494   | 333       | 665       | 313       | 602       |
| 2 451 365| 256| >1000| 712   | 464       | 477       | 889       |           |

| DOF | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-----|----|------|-------|-----------|-----------|-----------|-----------|
| 38 803   | 4  | 16   | 35    | 31        | 37        | 21        | 38        |
| 154 606  | 16 | 113  | 69    | 73        | 75        | 38        | 75        |
| 311 369  | 32 | 254  | 99    | 103       | 176       | 93        | 162       |
| 616 772  | 64 | 510  | 153   | 171       | 273       | 121       | 140       |
| 1 246 136| 128| >1000| 221   | 242       | 252       | 155       | 138       |
| 2 451 365| 256| >1000| 286   | 343       | 515       | 189       | 231       |

6 Numerical results for three dimensional problems

In this section we again assess the performance of the preconditioners as in Section 5, but this time in case of three dimensional problems. We consider the partial differential equation model for nearly incompressible elasticity and Stokes flow as three dimensional problems of similar mixed formulation. Each of these problems is discretised by using the Taylor-Hood methods from Section 3.1. In addition, we use the same tools as in Section 5. For both test cases we use zero as an initial guess.

6.1 Taylor-Hood discretisation

In this section we consider the Taylor-Hood discretisation from Section 3.1 with \( k = 2 \) for nearly incompressible elasticity and Stokes equations.

6.1.1 Nearly incompressible elasticity

In three dimensional space, ORAS and SORAS preconditioners also require an optimized parameter. We follow [HJN15] and use Robin interface conditions (5.27) with \( \alpha = 10 \).

Test case 9 (The homogeneous beam problem). We consider a homogeneous beam with the physical parameters \( E = 10^8 \) and \( \nu = 0.4999 \). The computational domain is the rectangle \( \Omega = (0,5) \times (0,1) \times (0,1) \). The beam is clamped on one side, hence we consider the following problem

\[
\begin{align*}
-2\mu \nabla \cdot \varepsilon(u) + \nabla p &= (0,0,-1)^T \quad &\text{in } \Omega \\
\nabla \cdot u &= \frac{1}{\lambda} p \quad &\text{in } \Omega \\
\u(x,y) &= (0,0,0)^T \quad &\text{on } \partial \Omega \cap \{ x = 0 \} \\
\sigma_{nn}^{sym}(x,y) &= (0,0,0)^T \quad &\text{on } \partial \Omega \setminus \{ x = 0 \}
\end{align*}
\]

(6.32)

The results of Table 9 show a clear improvement in the scalability of the two-level preconditioners over the one-level ones. In fact, using only zero energy modes, the number of iterations
Table 9: Comparison of preconditioners for Taylor-Hood discretisation ($TH_h^2, R_h^1$) - the homogeneous beam.

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 32 446 | 8 | 21   | 45    | 29         | 36          | 27         | 37          |
| 73 548 | 16 | 31   | 70    | 38         | 64          | 26         | 67          |
| 139 794 | 32 | 43   | 99    | 74         | 94          | 66         | 91          |
| 299 433 | 64 | 55   | 143   | 161        | 140         | 149        | 139         |
| 549 396 | 128 | 78   | 192   | 314        | 192         | 229        | 199         |

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 32 446 | 8 | 10   | 17    | 13         | 18          | 12         | 17          |
| 73 548 | 16 | 11   | 22    | 16         | 25          | 16         | 22          |
| 139 794 | 32 | 13   | 26    | 25         | 28          | 17         | 26          |
| 299 433 | 64 | 15   | 27    | 19         | 27          | 24         | 28          |
| 549 396 | 128 | 17  | 28    | 20         | 25          | 21         | 26          |

| DOF | N  | ORAS | SORAS | NDTNS-MRAS | NDTNS-SMRAS | TDNNS-MRAS | TDNNS-SMRAS |
|-----|----|------|-------|------------|-------------|------------|-------------|
| 32 446 | 8 | 9     | 16    | 12         | 17          | 12         | 16          |
| 73 548 | 16 | 10   | 19    | 15         | 24          | 14         | 20          |
| 139 794 | 32 | 11   | 21    | 17         | 23          | 17         | 21          |
| 299 433 | 64 | 14   | 24    | 17         | 24          | 21         | 23          |
| 549 396 | 128 | 16  | 27    | 18         | 23          | 20         | 22          |

is virtually unaffected by the number of subdomains. All two-level preconditioners show a comparable performance. For this case, increasing the dimension of the coarse space beyond $6 \times N$ eigenvectors does not seem to improve the results dramatically.

### 6.1.2 Stokes equation

We now turn to the Stokes discrete problem given in 3.1. Once again in case of ORAS and SORAS we choose $\alpha = 10$ as in [HJN15] for the Robin interface conditions (5.27).

**Test case 10** (The driven cavity problem). The test case is the three-dimensional version of the driven cavity problem. We consider the following problem on the unit cube $\Omega = (0, 1)^3$

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega \\
- \nabla \cdot u &= 0 \quad \text{in } \Omega \\
\mathbf{u}(x, y) &= (1, 0, 0)^T \quad \text{on } \partial \Omega \cap \{y = 1\} \\
\mathbf{u}(x, y) &= (0, 0, 0)^T \quad \text{on } \partial \Omega \setminus \{y = 1\}.
\end{align*}
\]

The conclusions remain the same as for the homogeneous beam example for the nearly incompressible elasticity equation discretised by Taylor-Hood method ($TH_h^2, R_h^2$) since Tables 10 and 9 show similar results.
Table 10: Comparison of preconditioners for Taylor-Hood discretisation \((TH^2_h, R^1_h)\) - the driven cavity problem.

| DOF | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-----|----|------|-------|-----------|------------|-----------|------------|
| 38  | 229| 12   | 24    | 12        | 22         | 11        | 23         |
| 76  | 542| 18   | 34    | 18        | 31         | 15        | 31         |
| 158 | 818| 23   | 45    | 20        | 45         | 19        | 45         |
| 325 | 293| 28   | 60    | 36        | 64         | 25        | 60         |
| 643 | 137| 37   | 79    | 64        | 91         | 33        | 88         |

Two-level (3 eigenvectors)

| DOF | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-----|----|------|-------|-----------|------------|-----------|------------|
| 38  | 229| 10   | 17    | 10        | 18         | 11        | 18         |
| 76  | 542| 11   | 20    | 11        | 19         | 14        | 19         |
| 158 | 818| 13   | 24    | 13        | 24         | 16        | 23         |
| 325 | 293| 15   | 27    | 15        | 27         | 19        | 26         |
| 643 | 137| 18   | 31    | 17        | 32         | 22        | 31         |

Two-level (7 eigenvectors)

| DOF | N  | ORAS | SORAS | NVTF-MRAS | NVTF-SMRAS | TVNF-MRAS | TVNF-SMRAS |
|-----|----|------|-------|-----------|------------|-----------|------------|
| 38  | 229| 9    | 16    | 9         | 16         | 12        | 17         |
| 76  | 542| 10   | 17    | 10        | 18         | 15        | 17         |
| 158 | 818| 11   | 19    | 11        | 20         | 17        | 20         |
| 325 | 293| 13   | 19    | 13        | 21         | 20        | 20         |
| 643 | 137| 15   | 21    | 16        | 22         | 22        | 22         |

7 Conclusion

We tested numerically two-level preconditioners with spectral coarse spaces for nearly incompressible elasticity and Stokes equations. We considered two finite element methods, namely, Taylor-Hood (Section 3.1) and the hdG (Section 3.2) discretisations.

In the case of the homogeneous nearly incompressible elasticity the two-level methods coupled with SORAS preconditioner defined in [HJN15] and SMRAS preconditioner defined by (4.23) allowed us to achieve good scalability results for both discretisations. Furthermore, for these symmetric preconditioners coarse spaces containing only zero energy modes seem to be enough for two and three dimensional problems. For the heterogeneous problem we also achieved scalability for two-level SORAS and SMRAS preconditioners, but, as expected, only in the case when the size of the coarse space is sufficiently big.

The improvement of the convergence in the case of the Stokes flow is visible only when the coarse space contains more eigenvectors than only constants. For the Taylor-Hood discretisation, taking sufficient big coarse space we were able to achieve good scalability for all preconditioners. It is remarkable that these good results occur even when using the hdG discretisation, despite the fact the optimized parameter to be used in SORAS and ORAS is not available.

We can conclude that the two-level preconditioners associated with non standard interface conditions are at least as good as the two-level ones in conjunction with Robin interface conditions using optimised parameters. It shows an important advantage of newly introduced preconditioners as they are parameter-free.

Numerical tests have shown that the coarse spaces bring an important improvement in the convergence, but the size of the coarse space depends on the problem. Building as small as possible coarse spaces is important from computational point of view. Thus, it is necessary to investigate what could be an optimal criterion for choosing the eigenvectors for a coarse space.

As we mentioned before, the theoretical foundation of the two-level preconditioners has not been extended to saddle point problems. Hence, future research will be devoted to this topic.
Acknowledgements

This research was supported by the Centre for Numerical Analysis and Intelligent Software (NAIS). We thank Frédéric Nataf and Pierre-Henri Tournier for many helpful discussions and insightful comments, and Ryadh Haerssas and Frédéric Hecht for their assistance with the FreeFem++ codes.

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