Level correlations in disordered superconducting grains

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I study the quasiparticle level correlations in a grain of a weakly disordered d-wave superconductor, and show that, in a wide intermediate energy range, they are characterized by a novel type of universal behavior.

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Level correlations represent a fundamental property of an electron system. They reflect sensitivity of the quasiparticle spectrum to disorder and boundary conditions, and distinguish between extended and localized states. Level correlations in metal grains, mesoscopic wires, quantum dots, SNS junctions and superconducting vortex cores have recently been a subject of much active study. Most simply, level correlations are quantified by the mean square deviation \( \langle \delta N_E^2 \rangle \) of the number \( N_E \) of levels in an energy interval of width \( E \):

\[
\langle \delta N_E^2 \rangle \equiv \langle (N_E - \langle N_E \rangle)^2 \rangle,
\]

where the angular brackets denote disorder average.

In disordered systems, the energy levels of localized states are uncorrelated, and \( \langle \delta N_E^2 \rangle \sim \langle N_E \rangle \). By contrast, extended states are strongly correlated, which is expressed in \( \langle \delta N_E^2 \rangle \) scaling as only a logarithm of \( \langle N_E \rangle \):

\[
\langle \delta N_E^2 \rangle \approx \frac{2 C}{\pi^2} \ln \langle N_E \rangle \ll \langle N_E \rangle ,
\]

where \( C \) is the number of the quasiparticle diffusion modes. Formula \( \langle \delta N_E^2 \rangle \) not only measures the level correlations, but also shows that, in a disordered system, the level number variance is driven by diffusion modes. Formula \( \langle \delta N_E^2 \rangle \) holds for \( E \) smaller than the Thouless energy \( E_T \equiv D/L^2 \), where \( D \) is the diffusion constant, and \( L \) is the system size. At these energies, the level correlations are remarkably universal, with the constant \( C \) being defined solely by the fundamental symmetries of the system. The low energy universality classes of disordered systems have been classified based on the symmetry arguments.

In this article, I study quasiparticle level correlations in a disordered grain of a d-wave superconductor in the presence of both the time reversal \( T \) and the spin rotation invariance \( S \). I show that, in a wide intermediate energy range, the level correlations have the universal form \( \langle \delta N_E^2 \rangle \), yet are different from those in a grain of a metal or of an s-wave superconductor (also invariant under \( T \) and \( S \)) – as well as different from the level correlations in any of the previously charted low energy universality classes. The main result is encapsulated in the number \( C \) of the diffusion modes. In a grain of normal metal (or an s-wave superconductor) with both the time reversal and the spin rotation symmetries present, \( C = 4 \), since both the quasiparticle charge and the three components of the quasiparticle spin are conserved and propagate diffusively. By contrast, I show that, in a d-wave superconducting grain, the charge diffusion mode disappears, which leads to \( C = 3 \), despite the very same set of fundamental symmetries. This is a novel universal type of level correlations.

The reason behind this result is that the impurity scattering leads to the exchange of charge between the quasiparticle subsystem and the condensate. However, this process is sensitive to the momentum anisotropy of the gap, and its rate vanishes for an ideally isotropic gap, as noticed long ago in the context of the branch imbalance relaxation in NS junctions. On the contrary, in a d-wave superconductor, such a charge relaxation occurs at a time scale of order the impurity scattering time. As a result, compared with a normal metal (or an s-wave superconductor), the quasiparticle charge diffusion mode is missing in a d-wave superconductor, which reduces the constant \( C \) from four to three. This reduction is a robust many-body effect of the anisotropic pairing symmetry.

The plan of the paper is as follows. First, I show that, in an s-wave superconducting grain, the level correlations are essentially the same as in the normal state. Then I show how the gap anisotropy eliminates quasiparticle charge diffusion in a d-wave superconductor, and outline the corresponding microscopic calculation. Finally, I describe the applicability range of the result and its relation to the previous findings, and discuss the possible further developments.

S-wave superconductor. Consider an s-wave superconducting grain. In the approximation of a spatially uniform gap, superconductivity can be described as pairing of the exact time reversed eigenstates of the underlying metal. Thus, the exact quasiparticle energies \( E_n \) in the superconducting state can be expressed via the exact quasiparticle energies \( \varepsilon_n \) in the normal state as per \( E_n \equiv \sqrt{\varepsilon_n^2 + \Delta^2} \). Therefore, the exact density of states \( \nu_S(E) \equiv \sum_n \delta[E - \sqrt{\varepsilon_n^2 + \Delta^2}] \) in the superconducting state is simply related to the exact density of
states $\nu_N(\varepsilon) \equiv \sum_n \delta[\varepsilon - \varepsilon_n]$ in the normal state:

$$\nu_S(E) = \frac{E}{\sqrt{E^2 - \Delta^2}} \sum_n \delta[\sqrt{E^2 - \Delta^2} - \varepsilon_n]$$

$$= \frac{E}{\sqrt{E^2 - \Delta^2}} \nu_N(\sqrt{E^2 - \Delta^2}).$$

As a result, the density of states (DoS) dimensionless autocorrelation function $R_2^S(E, E') = \frac{\langle \nu_S(E) \nu_S(E') \rangle}{\langle \nu_S(E) \rangle \langle \nu_S(E') \rangle} - 1$ in the superconducting state can be expressed through the DoS autocorrelation function in the normal state $R_2^N(\varepsilon, \varepsilon') = \frac{\langle \nu_N(\varepsilon) \nu_N(\varepsilon') \rangle}{\langle \nu_N(\varepsilon) \rangle \langle \nu_N(\varepsilon') \rangle} - 1$ in the form

$$R_2^S(E, E') = R_2^N(\sqrt{E^2 - \Delta^2}, \sqrt{(E')^2 - \Delta^2}).$$

From this simple argument, it follows that the level correlations in an s-wave superconductor are identical (up to the change of variables) to those in the underlying normal metal and that, therefore, the diffusion modes in the two systems are the same.

Before moving further, it is instructive to classify the quasiparticle diffusion modes more precisely. As a two-particle process, diffusion amounts to a coherent propagation of a particle and a hole, each, corresponding to the charge degree of freedom – or a triplet, which corresponds to the three spin degrees of freedom. Both in s- and in d-wave superconductor, the condensate carries no spin, and thus a quasiparticle cannot exchange spin with the condensate. As a result, in a disordered spin-singlet superconductor, the quasiparticle spin does propagate diffusively, as it was recently re-emphasized. By contrast, the situation with the quasiparticle charge is more delicate. Perhaps the simplest way to observe this difference is by inspecting the Bogolyubov quasiparticle creation operator:

$$\gamma_{p\uparrow} = u_p c_{p\uparrow}^\dagger + v_p c_{-p\downarrow},$$

$$u_p^2 = \frac{1}{2}[1 - \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_p^2}}],$$

$$v_p^2 = \frac{1}{2}[1 + \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_p^2}}].$$

Impurity scattering is elastic, i.e. it conserves the quasiparticle energy $E_p = \sqrt{\varepsilon_p^2 + \Delta_p^2}$. In an s-wave superconductor with the uniform gap, $\Delta_p$ is a constant and, in the absence of the Andreev scattering that turns $\varepsilon_p$ into $-\varepsilon_p$, the energy conservation implies conservation of $u_p$ and $v_p$. This means conservation of the particle-hole content of a quasiparticle and leads to the effective charge conservation – even though a Bogolyubov quasiparticle, being a superposition of a particle and a hole, does not have a well defined charge quantum number. The same conclusion can be reached by considering directly the expectation value of the quasiparticle charge $Q_p$:

$$Q_p = u_p^2(+1) + v_p^2(-1) = \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_p^2}}.$$
At the same time, as seen from Eq. 3, the vertex the quasiparticle charge does not propagate diffusively.

The mean square deviation of the number of levels in an energy interval of width $\Delta E$ is given by

$$
\langle \delta N^2 \rangle = \int_E d\varepsilon d\varepsilon' K(\varepsilon, \varepsilon').
$$

The calculation of $\langle \delta N^2 \rangle$ amounts to evaluating the Feynman diagram on Fig. 2, as done in [3], and has to be performed in the four-spinor Balian-Werthamer space. The technical difference with regards to a metal being that now the Green functions reside in the matrix space, whereas the impurity ladder resides in the space of direct products of the two matrices. The calculation for nodal quasiparticles in a d-wave superconductor leads to Eq. 4, with $C = 3$, and its details will be published elsewhere.

The validity range. The applicability range of this calculation is set by the possibility to treat the gap as spatially uniform [3]. Hence the energy interval $E$ should be much wider than the fluctuations of the gap. The latter scale is set by the perturbation of the gap due to the impurity potential. Neglecting the Coulomb interaction, the perturbation of the gap due to a single impurity has the form

$$
\langle \tau_i \rangle = \frac{\tau_0}{R A} + \frac{\tau_0}{R A} + \cdots
$$

FIG. 1. The ladder series for the renormalization of the vertices $\langle \tau_i \rangle$.

where $\Delta$ is the value of the gap far from the impurity, $[uN(0)]$ is the dimensionless strength of the impurity potential, $[\lambda N(0)]$ is the dimensionless BCS coupling constant, and $F(R)$ is a function decaying to zero at the lengthscale of order the coherence length. Scattering off $\delta \Delta(R)$ is the Andreev reflection off the inhomogeneities of the gap, with the rate $\pi A^{-1} \sim \tau^{-1}[\Delta/(\varepsilon F)^2][\lambda N(0)]^2 \ll \tau^{-1}$, where $\tau^{-1}$ is the impurity scattering rate. Thus, the main result of this paper holds for $E$ much greater than $\pi A^{-1}$, and for the levels separated from the Fermi energy by more than $\pi A^{-1}$. At the same time, $E$ must be much smaller than the Thouless energy $E_T = D/L^2$ which requires $\pi A^{-1} \ll D/L^2$. The latter inequality bounds the grain size by

$$
L \ll \frac{k_F l_\xi}{[\lambda N(0)]^{1/2}},
$$

which is much greater than the coherence length. Note that, for nodal quasiparticles in a d-wave superconductor, both $\varepsilon$ and $\xi$ are functions of the quasiparticle energy $\varepsilon$ and scale as $\varepsilon(\xi) \sim l \Delta/\varepsilon$, which further increases the upper bound on the grain size.

Before closing, it is instructive to put the main result of this work, Eq. (4) with $C = 3$, in context. Equation (4) (with different values of $C$) is commonly associated with the random matrix theory (RMT) [12] which furnishes very general and powerful phenomenological framework for treating random systems, and allows a symmetry classification [13] of possible universality classes. However, a crucial underlying assumption of the RMT is that all the matrix elements of the Hamiltonian (including the matrix elements of the gap) be random and drawn independently from a broad distribution. This requirement automatically rules out the possibility to distinguish superconductors of different pairing symmetry.

By contrast, the present work studies the intermediate energy limit when the matrix elements of the gap may be treated as completely non-random, the randomness being restricted to the diagonal of the Bogolyubov-de
Gennes Hamiltonian. In this limit, the Hamiltonian is only a “partly random” matrix, with the matrix elements of the gap fixed by the pairing symmetry. As shown above, the level correlations in such a “partly random” ensemble are sensitive to the momentum anisotropy of the pairing and are qualitatively different e.g. for s- and d-wave superconducting grains.

More importantly, this work shows that, in anisotropic superconductors, the condensate assumes the role of a “charge reservoir” coupled to the quasiparticle subsystem, and affects the level correlations. It would be interesting to study this problem further by explicitly including the dynamics of the condensate, especially in view of the cuprate superconductors as an obvious experimental object.

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