INTEGRAL POINTS OF FIXED DEGREE AND BOUNDED HEIGHT

MARTIN WIDMER

Abstract. By Northcott's Theorem there are only finitely many algebraic points in affine $n$-space of fixed degree over a given number field and of height at most $X$. For large $X$ the asymptotics of these cardinalities have been investigated by Schanuel, Schmidt, Gao, Masser and Vaaler, and the author. In this paper we study the case where the coordinates of the points are restricted to algebraic integers, and we derive the analogues of Schanuel’s, Schmidt’s, Gao’s and the author’s results.

1. Introduction

In this article we count algebraic points of bounded Weil height with integral coordinates, generating an extension of given degree over a fixed number field $k$. We derive a precise asymptotic formula for their number as the height gets large.

Various related results have appeared in the literature. Schanuel [25] gave asymptotic estimates for the number of points in $k^n$ of bounded height. Franke, Manin and Tschinkel [13] started a program to count rational points on Fano varieties. This program has been further developed by Batyrev, Browning, Derenthal, Heath-Brown, Peyre, Salberger, Thunder, Tschinkel, and many others. Schmidt was the first to study the distribution of algebraic points of fixed degree. In [27] he obtained general upper and lower bounds for the number of points of fixed degree over a fixed number field $k$. Later, in [28] he established the asymptotics for points quadratic over $\mathbb{Q}$, which in turn yield new results in the context of Manin’s program for the symmetric square of $\mathbb{P}^n$. Soon afterwards Gao [17] gave asymptotics for points in $n$ dimensions of degree $e$ over $\mathbb{Q}$, subject to the constraint $n > e$. The case $n = 1$ was treated by Masser and Vaaler in [21], and was generalized in [22] by the same authors to allow arbitrary ground fields $k$. The author [31] has established asymptotic estimates for points in $n$ dimensions of fixed degree $e$ over an arbitrary number field, provided $n > 5e/2 + 5$. However, in general not even the correct order of magnitude of the counting function is known (see also [2, Section 4] for a survey).

Also the distribution of integral points on algebraic varieties has been studied intensively. The classical circle method applies for complete intersections of low degree; see, e.g., [1], [26], and ergodic and spectral methods have been used to handle algebraic groups and homogeneous spaces of semisimple groups; see [9], [11] and [12]. More recently, Chambert-Loir and Tschinkel have extended Manin’s program to treat integral points on partial equivariant compactifications of vector groups [6] and toric varieties [5]. However, all these results apply only when the points are defined over a fixed number field.

Regarding integral points of fixed degree $e > 1$ the subject is less developed. For a number field $k$ let us write $N(O_k(n; e), X)$ for the number of points $\alpha = (\alpha_1, \ldots, \alpha_n)$ of absolute multiplicative Weil height no larger than $X$, whose coordinates are algebraic
integers with \([k(\alpha_1, \ldots, \alpha_n) : k] = e\). In \([13]\) p.81] Lang has stated without proof
\begin{equation}
N(O_K(1; 1), X) = \gamma_K X^{d}(\log X)^{q_K} + O(X^{d}(\log X)^{q_K-1}).
\end{equation}
Here \(d = [K : \mathbb{Q}]\), \(q_K\) is the rank of the group of units and \(\gamma_K\) is an unspecified positive constant depending on \(K\). The formula (1.1) can easily be deduced from a counting principle of Davenport \([5]\). Indeed, we reverse the situation here, and we say something about the

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Here \((1.3)
\begin{equation}
N(O_K(n; 1), X) = B_K^n X^{dn} L_{q_K} (-\log X^{dn}) + O(X^{dn-1}(\log X)^{q_K}).
\end{equation}
Here \(L_{q_K}(x)\) is the \(q_K\)-th Laguerre polynomial, and \(B_K\) is a field invariant defined later on. The somewhat unexpected appearance of the Laguerre polynomial in the main term is another new feature of our result.

However, that the main term is of the form \(X^a P(\log X)\) with some polynomial \(P(x)\)

is a typical phenomenon, and is usually obtained by meromorphic continuation of the corresponding height zeta function, and a suitable Tauberian theorem; see, e.g., \([13]\) Corollary] for the case of rational points on Flag manifolds \(V\). In their case \(\deg P\) is also related to the rank of a group, more precisely, \(\deg P\) is the rank of the Picard group \(\text{Pic}(V)\) minus \(4\). Franke, Manin, and Tschinkel achieved this by expressing the corresponding height zeta function as an Eisenstein series and then using Langland’s work to study its analytic properties. Similar, technically intricate, methods have been used in \([5]\) and \([6]\). We use elementary methods, in particular, our proof makes no use of complex analysis. Indeed, we reverse the situation here, and we say something about the analytic properties of the height zeta function \(\zeta_{k, n, e}(s) = \sum_{\alpha \in O_k(n; e)} H(\alpha)^{-s}\) using our estimates for \(N(O_k(n; e), X)\).

To state our first result we need some notation. Let \(\overline{k}\) be a fixed algebraic closure of the number field \(k\), and let \(H\) be the absolute multiplicative Weil height on \(\overline{k}\) as defined

\[\text{There is a misprint in their Corollary, } t \text{ should read } t - 1.\]
For a subset $S$ of $\mathbb{R}^n$ of uniformly bounded degree and real numbers $X \geq 1$ we define the counting function

$$N(S, X) = |\{\alpha \in S; H(\alpha) \leq X\}|.$$

Thanks to Northcott’s Theorem the quantity above is finite for each $X$. For positive rational integers $e$ and $n$ we define the set of integral points in $n$ dimensions of degree $e$ over the field $k$

$$\mathcal{O}_k(n; e) = \{\alpha \in \mathbb{Z}_k^n; [k(\alpha) : k] = e\}.$$

Here $\mathbb{Z}_k \subset \bar{k}$ denotes the ring of algebraic integers, and $k(\alpha) = k(\alpha_1, \ldots, \alpha_n)$. Let $C_e(k)$ be the collection of all field extensions of $k$ of degree $e$, i.e.,

$$C_e(k) = \{K \subset \bar{k}; [k : k] = e\}.$$

For a number field $K$ we write $\Delta_K$ for the discriminant of $K$, $r_K$ for the number of real, $s_K$ for the number of pairs of complex conjugate embeddings of $K$, and $q_K = r_K + s_K - 1$ for the rank of the group of units. Moreover, we set

$$t_e(k) = \sup\{q_K; K \in C_e(k)\} = e(q_e + 1) - 1,$$

$$B_K = \frac{2^r(2\pi)^{r_K}}{\sqrt{|\Delta_K|}},$$

and for $0 \leq i \leq t_e(k)$ we introduce the formal sum

$$(1.4)\quad D_i = D_i(k, n, e) = \sum_{\substack{\alpha \in C_e(k) \cap \mathbb{Z}_k^n \geq i \\alpha \in C_e(k) \cap \mathbb{Z}_k^n \geq i \\alpha \in C_e(k) \cap \mathbb{Z}_k^n \geq i}} \frac{B_{q_K}^n}{i!} \left(\frac{q_K}{i}\right).$$

For $e > 1$ we define

$$C_{e,m} = \max\{2 + \frac{4}{e - 1} + \frac{1}{m(e - 1)}; 7 - \frac{e}{2} + \frac{2}{me}\} \leq 7.$$

Finally, we put $\log^+ X = \max\{1, \log X\}$. Now we can state our first result.

**Theorem 1.1.** Let $k$ be a number field and $m = [k : \mathbb{Q}]$. Suppose that either $e = 1$ or that $n > e + C_{e,m}$, and set $t = t_e(k)$. Then the sum in (1.4) converges, and for $X \geq 1$ we have

$$(1.5)\quad |N(\mathcal{O}_k(n; e), X) - \sum_{i=0}^{t} D_i X^{men} (\log X)^{men} i | \leq c X^{men-1} (\log X)^t$$

for some positive constant $c = c(n, m, e)$ depending only on $n$, $m$ and $e$.

We remark that the sum in (1.5) can be written as the weighted sum of Laguerre polynomials $X^{men} \sum_{\beta} \beta L_{\beta}(-\log X^{men})$. Here $\beta$ runs over the finite set $\{q_K; K \in C_e(k)\}$, and $\beta_q = \beta_q(k, e, n) = \frac{1}{\sqrt{\Delta_K}} B_{q_K}^n$, the sum taken over all $K \in C_e(k)$ with $q_K = q$.

Note that for $e \geq 9$ the condition $n > e + C_{e,m}$ is equivalent to $n > e + 2$. Unfortunately, this is probably not the sharp bound. However, as $N(\mathcal{O}_k(1; e), X) \leq N(\mathcal{O}_e(n; e), X)$ we see by comparing with (1.2) that if $m = 1$ then (1.5) cannot hold for $n < e$. Borrowing ideas of Masser and Vaaler from [22], Theorem 1.1 combined with standard estimates for the Mahler measure, shows that $N(\mathcal{O}_k(1; e), X) \gg X^{me^2} (\log X)^{ne}$. Hence, (1.5) cannot hold for $n < e$, even if $m > 1$. Note also that for $e = n = 2$ the sums in (1.4) diverge.

Next we exhibit some special cases of Theorem 1.1. First let us choose $e = 1$, so that $k = K$, and set $d = [K : \mathbb{Q}]$. Then we get the formula (1.3) which is a new result, even for $n = 1$. Here the multi-term expansion could probably be worked out from the results in [5], but it is unlikely that the same error term can be obtained.
It is probably not too difficult to extend our theorem to the context of Lipschitz heights as in [22] or even adelic Lipschitz heights as in [31]. These generalizations would have further applications such as asymptotic estimates for \( N(\mathcal{O}_k(1; e), X) \), analogous to the main theorem in [22], or for the number of integral solutions of fixed degree to a system of linear equations, analogous to the main result in [32]. However, to keep the technical difficulties and the required notation at a minimal level, and to emphasize the main ideas and novelties of this work, we decided not to include these generalizations. In fact, Fabrizio Barroero will soon publish results for \( N(\mathcal{O}_k(1; e), X) \).

Let us formally define the height zeta function of \( \mathcal{O}_k(n; e) \) as

\[
\zeta_{k,n,e}(s) = \sum_{\alpha \in \mathcal{O}_k(n; e)} H(\alpha)^{-s}.
\]

The upper bound of order \( X^{\text{men}}(\log X)^t \) implies that \( \zeta_{k,n,e}(s) \) converges in the complex half plane \( \Re(s) > \text{men} \). But Theorem 1.1 implies also that \( \zeta_{k,n,e}(s) \) has a meromorphic continuation to \( \Re(s) > \text{men} - 1 \) with a pole at \( s = \text{men} \) of order \( t + 1 \). More precisely, setting \( D_i + 1 = 0 \), and using summation by parts, we find that the principal part of the Laurent series at \( s = \text{men} \) is given by

\[
\sum_{i=1}^{t+1} \binom{\text{men}}{i}(i-1)!|D_{i-1} + iD_i|.(s - \text{men})^i.
\]

To present our next result we need some more notation. Suppose \( K \) is a field extension of \( k \) of degree \( e = [K : k] \), and put \( [K : \mathbb{Q}] = d \), so that \( d = \text{em} \). We denote by \( \sigma_1, \ldots, \sigma_d \) the embeddings from \( K \) to \( \mathbb{R} \) or \( \mathbb{C} \) respectively, ordered such that \( \sigma_{r+i} = \overline{\sigma_i} \) for \( 1 \leq i \leq s \), i.e., \( \sigma_{r+s+i} \) and \( \sigma_{r+i} \) are complex conjugate. Let \( \mathcal{O} \) be a submodule of the free \( \mathbb{Z} \)-module \( \mathcal{O}_K \) of full rank. Let \( \mathfrak{A}_\mathcal{O} \) be the smallest ideal in \( \mathcal{O}_K \) that contains \( \mathcal{O} \), i.e., \( \mathfrak{A}_\mathcal{O} \) is the intersection over all ideals in \( \mathcal{O}_K \) that contain \( \mathcal{O} \). Set

\[
(1.6) \quad \eta_{\mathcal{O}} = \mathcal{N}(\mathfrak{A}_\mathcal{O})^{1/d} \geq 1,
\]

where \( \mathcal{N}(\mathfrak{A}) = |\mathcal{O}_K/\mathfrak{A}| \) denotes the norm of a nonzero ideal \( \mathfrak{A} \) of \( \mathcal{O}_K \). Furthermore, we define

\[
G(K/k) = \{ [K_0 : k] : k \subset K_0 \subset K \}
\]

if \( K \neq k \), and we put

\[
G(K/k) = \{ 1 \}
\]

if \( K = k \). Then for an integer \( g \in G(K/k) \) we define

\[
\delta_g(K/k) = \inf \{ H(\alpha, \beta) : k(\alpha, \beta) = K, [k(\alpha) : k] = g \},
\]

and we set

\[
(1.7) \quad \mu_g = mn(e - g) - 1.
\]

We remark that \( \delta_g(K/k) \) refines the invariant \( \delta(K) \) introduced by Roy and Thunder [24]. For a point \( \alpha \in \overline{K^n} \setminus \{0\} \) we write \( k(\ldots, \alpha_i/\alpha_j, \ldots) \) for the extension of \( k \) generated by all possible ratios \( \alpha_i/\alpha_j \) \((1 \leq i, j \leq n, \alpha_j \neq 0)\) of the coordinates of \( \alpha \). Next we introduce the set of “projectively primitive” points in \( \mathcal{O}^n \)

\[
\mathcal{O}^n(K/k) = \{ \alpha \in \mathcal{O}^n \setminus \{0\} : K = k(\ldots, \alpha_i/\alpha_j, \ldots) \}.
\]

Note that for \( n = 1 \) the set \( \mathcal{O}_1^n(K/k) \) is empty if \( K \neq k \) and equals \( \mathcal{O} \setminus \{0\} \) if \( K = k \). For a subset \( I \subset \{1, \ldots, r_K + s_K\} \) and \( I^c = \{1, \ldots, r_K + s_K\} \setminus I \) we define

\[
\mathcal{O}^n_I(K/k) = \{ \alpha \in \mathcal{O}^n(K/k) : |\sigma_i(\alpha)|_\infty \geq 1 \text{ for } i \in I, \text{ and } |\sigma_i(\alpha)|_\infty < 1 \text{ for } i \in I^c \},
\]

and
where $|\sigma_i(\alpha)|_\infty = \max \{|\sigma_i(\alpha_1)|, \ldots, |\sigma_i(\alpha_n)|\}$. Finally, let $Z_I(T)$ be the measurable set in Euclidean space, defined in \[(1.1)\], and set $q' = |I| - 1$. In Section \[13\] we will show that for $X \geq 1$

$$\text{Vol}Z_I(X^d) = (2^{r+\kappa_0})^n (-1)^q \left( -1 + X^{dn} \sum_{i=0}^{q'} \frac{(-\log X^{dn})^i}{i!} \right).$$

Recall that $K/k$ is an extension of number fields and $d = [K : \mathbb{Q}]$. We can now state the main result of this article. All our other results mentioned in the introduction will be deduced from this theorem.

**Theorem 1.2.** Suppose $q' = |I| - 1 \geq 0$, $X \geq 1$ and either $n > 1$ or $K = k$. Then

$$\left| N(O_I^n(K/k), X) - \frac{2^{r+\kappa_0}\text{Vol}Z_I(X^d)}{(\sqrt{\Delta_K}(O_K : O)^n) n} \right| \leq c \sum_{g \in G(K/k)} \frac{X^{dn-1}(\log X)^q'}{\eta_0^{dn-1} \delta_g(K/k)^{\mu_g}},$$

where $c = c(n, d)$ is a positive constant depending only on $n$ and $d$.

Using $P_q(x) = \sum_{i=0}^{q'} \frac{X^i}{i!}$ we can rewrite the main term as

$$\left( \frac{B_K}{O_K : O} \right)^n (-1)^q' \left( X^{dn} P_q(- \log X^{dn}) - 1 \right).$$

Note that this expression depends only on the cardinality of $I$ but not on the particular choice of $I$ itself. Next let us consider some special cases. Again we start with the case $K = k$. Then the statement takes the form

$$\left| N(O_I^n(K/K), X) - \frac{2^{r+\kappa_0}\text{Vol}Z_I(X^d)}{(\sqrt{\Delta_K}(O_K : O)^n) n} \right| \leq c(n, d) \frac{X^{dn-1}(\log X)^q'}{\eta_0^{dn-1} \delta_g(K/k)^{\mu_g}}.$$

Now we take $n = 1$, $O = O_K$, and let us assume $r_K \geq 1$. If we choose $I = \{1\}$ and assume $d > 1$, then $N(O_I^n(K/K), X) = N(O_I, X)$ counts the primitive Pisot numbers in the real field $\sigma_1(K)$. Here the primitivity is induced by the choice of the set $I$. The non-primitive Pisot numbers lie in a strict subfield of $\sigma_1(K)$, and so their number has order of magnitude at most $X^{d/2}$. Thus for the total number of Pisot numbers in $\sigma_1(K)$ of height no larger than $X$ we get

$$B_K X^d + O(X^{d-1}).$$

Still with $K = k$, $O = O_K$, and $n = 1$ we now take $I = \{1, \ldots, r_K + s_K\}$. Then we are counting the nonzero elements $\alpha \in O_K$ with $H(\alpha) = |Nn_K/\mathbb{Q}(\alpha)|^{1/d} \leq X$. Their number is given by

$$\sum_{i=0}^{q_k} (-1)^{q_k} B_K X^d \frac{(-\log X^{d})^i}{i!} + O(X^{d-1}(\log X)^{q_k}).$$

Next note that

$$O^n(K/k) = \bigcup_I O_I^n(K/k),$$

taken over all non-empty subsets of $I$ of $\{1, \ldots, r_K + s_K\}$, is a disjoint union. Thus we may sum the estimate in Theorem \[1.2\] over all non-empty sets $I$ to get estimates for the counting function of $O^n(K/k)$. We even get a geometric interpretation of the main terms. The highest order main term comes from the points in $O_I^n(K/k)$ with maximal $I$, i.e., points satisfying $|\sigma_i(\alpha)|_\infty \geq 1$ for all $i$. For the second order main term there is a negative contribution from $O_I^n(K/k)$ with maximal $I$ and a positive contribution for each $O_I^n(K/k)$ with $|I| = r_K + s_K - 1$, and so forth.

The methods developed for Theorem \[1.2\] can certainly be applied to other counting problems. For instance, ideas of Section \[4\] have been used by Frei in Lemma 7 of \[14\] (cf.
Let $Z(T) = \bigcup_{I} Z_{I}(T)$, where this time $I$ runs over all subsets of $\{1, \ldots, r_{K} + s_{K}\}$, and again this is a disjoint union. In Section \[15\] we will show that for $X \geq 1$
\[
\text{Vol} Z(X^{d}) = (2^{r_{K}} \pi^{s_{K}})^{n} X^{dn} \sum_{i=0}^{q_{K}} \left(\frac{\log X^{dn}}{i!}\right)^{i} = (2^{r_{K}} \pi^{s_{K}})^{n} X^{dn} L_{q_{K}}(- \log X^{dn}).
\]
As $O_{\emptyset}^{n}(K/k) = \emptyset$ we see that the union in (1.8) taken over all subsets remains equals $O^{n}(K/k)$. In Section \[14\] we show that Theorem \[12\] remains valid for $I = \emptyset$, provided $(\log^{+}X)^{q}$ in the error term is replaced by 1. From this and Theorem \[12\] we may deduce the following result.

**Corollary 1.1.** Suppose $X \geq 1$ and either $n > 1$ or $K = k$. Then
\[
\left| N(O^{n}(K/k), X) - \frac{2^{r_{K}n} \text{Vol} Z(X^{d})}{(\sqrt{\Delta_{K}}|O_{K} : \mathcal{O}|)^{n}} \right| \leq c \sum_{g \in G(K/k)} \frac{X^{dn-1}(\log^{+}X)^{q_{K}}}{\eta_{O}^{dn-1} \delta_{g}(K/k)^{\mu_{g}}},
\]
where $c = c(n, d)$ is a positive constant depending only on $n$ and $d$.

Note that here, opposed to in Theorem \[12\] all main terms are positive. Let us briefly explain the strategy of the proof of Theorem \[11\]. To this end we define the set of “non-projectively primitive" points in $O_{K}^{n}$
\[
O_{npp}(K/k) = \{ \alpha \in O_{K}^{n} \setminus O_{K}^{n}(K/k); k(\alpha) = K\}.
\]
Now any $\alpha$ in $O_{k}(e, n)$ lies either in $O_{K}^{n}(K/k)$ or in $O_{npp}(K/k)$, with $K = k(\alpha) \in C_{e}(k)$. Hence we have the following disjoint union
\[
O_{k}(e, n) = \bigcup_{C_{e}(k)} O_{K}^{n}(K/k) \cup O_{npp}(K/k).
\]
Therefore, we just have to sum $N(O_{K}^{n}(K/k), X)$ and $N(O_{npp}(K/k))$ over all $K$ in $C_{e}(k)$. And indeed, we will show that the sum over all main terms as well as the sum over all error terms of $N(O_{K}^{n}(K/k), X)$ converges, provided $n > e + C_{e,m}$, while the sum over $N(O_{npp}(K/k))$ has smaller order of magnitude.

It now is obvious that a crucially important feature of Corollary \[11\] (and so of Theorem \[12\]) is the good dependence of the error term on the extension $K/k$; note that by Northcott’s Theorem $\delta_{g}(K/k)^{-\mu_{g}}$ tends to zero as $K$ runs over the subset $C_{e}^{(g)}(k)$ of those $K \in C_{e}(k)$ with $g \in G(K/k)$. To compare with the discriminant we can apply a well-known inequality of Silverman \[30, Theorem 2\] to get $\delta_{g}(K/k) \geq c_{k}\Delta_{K}^{1/(2me(c-1))}$ for some positive constant $c_{k}$. The reason for using $\delta_{g}(K/k)$ instead of the more common discriminant are the better summatory properties; indeed, we have almost sharp bounds for the number of fields $K \in C_{e}^{(g)}(k)$ with $\delta_{g}(K/k) \leq T$, opposed to the case when we enumerate by the discriminant. Furthermore, as larger $g$ gets, which means as larger the error terms get, the better our upper bounds for the number of $K \in C_{e}^{(g)}(k)$ with $\delta_{g}(K/k) \leq T$ become. These observations have already been used in \[31\].

Our method leads also to asymptotics for more specific sets, e.g., points $\alpha$ of degree $d$ whose coordinates are primitive Pisot numbers of $\mathbb{Q}(\alpha)$, provided $n > d + C_{e,m} + 1$. Here the “+1” is required to exclude the points with some coordinates equal zero.

The special case $K = k$ in Corollary \[11\] yields a generalization of (1.3) (to arbitrary submodules of $O_{K}$ of full rank) with a more precise error term. We have
\[
(1.9) \quad \left| N(O^{n}\setminus \{0\}, X) - \frac{2^{r_{K}n} \text{Vol} Z(X^{d})}{(\sqrt{\Delta_{K}}|O_{K} : \mathcal{O}|)^{n}} \right| \leq c(n, d) \left(\frac{X}{\eta_{O}}\right)^{dn-1} (\log^{+}X)^{q_{K}}.
\]
Now let us choose $\mathcal{O} = \mathfrak{A}$ for a nonzero ideal $\mathfrak{A}$. Then we have $\eta_\mathcal{O} = \mathfrak{R}(\mathfrak{A})^{1/d}$. This allows one to carry out a Möbius inversion to count $\alpha \in \mathfrak{A}^n$ satisfying another type of primitivity, namely $\alpha_1\mathcal{O}_K + \cdots + \alpha_n\mathcal{O}_K = \mathfrak{A}$. Here we need $n \geq 2$ to get for the number of such $\alpha$

\[
\frac{2^{2K}n\text{Vol}(X^d)}{\zeta_K(n)\sqrt{[\mathfrak{R}(\mathfrak{A})^n]}} + O\left(\frac{X^{dn-1}(\log^+ X)^7}{\mathfrak{R}(\mathfrak{A})^{n-1/d}}\right),
\]

where $\overline{q} = q_K$ if $(n, d) \neq (2, 1)$ and $\overline{q} = 1$ if $(n, d) = (2, 1)$.

Let us mention one last example, slightly related to the “quantitative problem” addressed by Fuchs, Tichy, Ziegler in [16]. The latter asks for a given positive integer $M$ how many elements of $O_K$ of bounded height can be written as the sum of $M$ units. Instead one could ask, how many elements of $O_K$ of bounded height can be written as the sum of units. This means one wants to determine $N(O, X)$ for $O = \mathbb{Z}[\gamma; \gamma \in \mathcal{O}_K^*]$. Now either $O = \mathbb{Z}[\gamma; \gamma \in \mathcal{O}_K^*]$ has full rank as a submodule of $\mathcal{O}_K$ or $K$ is a CM-field and $O$ has full rank as a submodule of the ring of integers of the maximal totally real subfield of $K$ ([23], p.107]). In either case one can apply [19].

2. Organisation of the paper and outline of the proofs

The paper is organized as follows. We start with a section on elementary counting principles. Here we recall and provide some simple results on counting lattice points. Then in Section 3 we state a precise estimate (Theorem 4.1) of the quantity $|\Lambda \cap Z_1(T)|$, for lattices $\Lambda$ that satisfy a certain gap principle. Roughly speaking this principle says that the successive minima of $\phi \Lambda$ are uniformly bounded away from zero as $\phi$ runs over all elements of a certain subgroup $T$ of the diagonal endomorphisms with determinant 1.

In Section 4 we introduce some notation and state some simple properties of the sets $Z_1(T)$ and $Z(T)$ which are required for the proof of Theorem 4.1.

Loosely speaking, the counting principles of Section 3 yield nontrivial results only, if the volume of the set is large, the diameter is not too large, and the first successive minimum is not too small. Unfortunately, in the situation of Theorem 4.1 these requirements are not met. To overcome these difficulties we develop a method that builds up on an idea of Schmidt [23]. We shall quickly explain the basic strategy.

The set $Z_1(T)$ can be written as a Cartesian product $A \times B \subset \mathbb{R}^{nd}$ and has volume about $T^n(\log T)^d$. The set $B$ is essentially a cube of edge length 1, but the set $A$ has cusps in various directions, and diameter potentially as large as $T$. In a first step we split the set $A$ in about $(\log T)^d$ subsets $A_i$, and for each of these we find a suitable transformation $\phi_i$ from our group $T$ that sends $A_i$ to a set with small diameter and leaves $B$ invariant. This procedure is carried out in Section 5. However, if $I$ is not maximal, i.e. $q' < q$, then the diameter of the transformed sets are still too large. But we have some space left in the directions of our second component $B$. Therefore, we apply a transformation $\psi$ from $T$ that blows up the component $B$ and shrinks the components $A_i$. This process is performed in Section 6. Now, with $\psi_1 = \psi \circ \phi_1$, the transformed sets $\psi_1(A_i \times B)$ have a nice shape, and, thanks to the gap principle, the successive minima of the transformed lattices $\psi\Lambda$ are still controllable. To apply the simple counting principles we still have to check some technical conditions such as the Lipschitz parameterizability of the boundary, and this is done in Section 7. In Section 8 we are finally in position to apply the counting principles to estimate each component $|\psi_1\Lambda \cap \psi_1(A_i \times B)|$. Then we just have to sum over all about $(\log T)^d$ components, and this concludes the proof of Theorem 4.1.

For the proof of Theorem 1.2 we need all the arguments of Theorem 4.1. Moreover, to get the good dependence on $K$ we need also to utilize the machinery developed in
However, the latter can only be applied to the set \(O^n_{K/K}(K/k)\) of projectively primitive points, and this is exactly why we have to restrict the counting in Theorem 1.2 to these points. Thus, to prove Theorem 1.1 we have to deal with the set \(O^n_{npp}(K/k)\) separately. In Section 10 we show that the lattices coming from embeddings of \(O^n\) satisfy the required gap principles. In fact, to deduce an error term involving the invariants \(δ_3(K/k)\), we need also a refinement of this gap principle in terms of the higher successive minima. The entire Section 10 is heavily based on Section 9 of [33]. In Section 11 we prove an upper bound for the number of lattice points that are not projectively primitive. Using this upper bound we are then in Section 12 in position to prove a precise asymptotic estimate for the number of projectively primitive lattice points in each component \(|ψΛ∩ψι(A_i×B)|\). Section 13 finishes the proof of Theorem 1.2.

Corollary 1.1 is essentially an immediate consequence of Theorem 1.2. However, the present statement requires an analogue of Theorem 1.2 in the case \(q' = -1\). The latter is stated and proved in Section 13. The volumes of the sets \(Z_i(T)\) and \(Z(T)\) are computed in Section 17. In Section 16 we prove that the sum over \(N(O^n_{npp}(K/k))\) taken over all fields \(K ∈ C\) is covered by the error term in Theorem 1.1. Finally, Section 17 is devoted to the proof of Theorem 1.1.

We will use Vinogradov’s notation \(≪\). The implied constants depend only on \(n, m, e\) and \(d\). Throughout this article \(T\) and \(X\) denote real numbers \(≥ 1\).

3. Counting principles

For a vector \(x\) in \(R^D\) we write \(|x|\) for the Euclidean length of \(x\). The closed Euclidean ball centered at \(x\) with radius \(r\) will be denoted by \(B_x(r)\). Let \(Λ\) be a lattice of rank \(D\) in \(R^D\) then we define the successive minima \(λ_1(Λ),...,λ_D(Λ)\) of \(Λ\) as the successive minima in the sense of Minkowski with respect to the unit ball. That is

\[λ_i = \inf\{λ; B_0(λ) ∩ Λ contains i linearly independent vectors\}\]

Definition 1. Let \(M\) and \(D\) be positive integers, and let \(L\) be a non-negative real. We say that a set \(Z\) is in \(Lip(D,M,L)\) if \(Z\) is a subset of \(R^D\), and if there are \(M\) maps \(φ_1,\ldots,φ_M : [0,1]^{D−1} → R^D\) satisfying a Lipschitz condition

\[|φ_i(x) − φ_i(y)| ≤ L|x−y|\]

for \(x, y ∈ [0,1]^{D−1}, i = 1,\ldots,M\) such that \(Z\) is covered by the images of the maps \(φ_i\). For \(D = 1\) this is to be interpreted as the finiteness of the set \(Z\), and the maps \(φ_i\) are considered points in \(R^D\) such that \(Z ⊂ \{φ_i; 1 ≤ i ≤ M}\).

We will apply the following counting result from [33, Theorem 5.4].

Theorem 3.1. Let \(Λ\) be a lattice in \(R^D\) with successive minima \(λ_1,\ldots,λ_D\). Let \(Z\) be a bounded set in \(R^D\) such that the boundary \(∂Z\) of \(Z\) is in \(Lip(D,M,L)\). Then \(Z\) is measurable, and, moreover,

\[|Z ∩ Λ| − \frac{VolZ}{det Λ} ≤ c_0(D)M \max_{0 ≤ i < D} \frac{L^i}{λ_1⋯λ_i}\]

For \(i = 0\) the expression in the maximum is to be understood as 1. Furthermore, one can choose \(c_0(D) = D^{3D^2/2}\).

If \(Λ\) is a lattice in \(R^D\) and \(a\) is an integer with \(1 ≤ a ≤ D\) then we put

\[Λ(a) = \{x ∈ Λ; |x| ≥ λ_a\}\]

(3.1)
Corollary 3.1. Let \( \Lambda \) be a lattice in \( \mathbb{R}^D \) with successive minima \( \lambda_1, \ldots, \lambda_D \). Let \( Z \) be a bounded set in \( \mathbb{R}^D \) such that the boundary \( \partial Z \) of \( Z \) is in \( \text{Lip}(D, M, L) \), and \( Z \subset B_0(\kappa L) \) with \( \kappa \geq 1 \). Then \( Z \) is measurable and we have

\[
\left| Z \cap \Lambda(a) \right| - \frac{\text{Vol}Z}{\det \Lambda} \leq c_1(D) M \frac{(\kappa L)^{D-1}}{\lambda_1^{a-1} \lambda_a^{D-a}}.
\]

One can choose \( c_1(D) = c_0(D)(2\pi D)^D \).

Proof. The measurability comes directly from Theorem 3.1. First suppose \( \kappa L \geq \lambda_a \). By the triangle inequality we get

\[
\left| Z \cap \Lambda(a) \right| - \frac{\text{Vol}Z}{\det \Lambda} \leq \left| Z \cap \Lambda \right| - \frac{\text{Vol}Z}{\det \Lambda} + \left| B_0(\lambda_a) \cap \Lambda \right|.
\]

We apply Theorem 3.1. Since \( \kappa \geq 1 \), we have

\[
\left| Z \cap \Lambda \right| - \frac{\text{Vol}Z}{\det \Lambda} \leq c_0(D) M \max_{0 \leq i < D} \frac{L^i}{\lambda_1 \cdots \lambda_i} \leq c_0(D) M \frac{(\kappa L)^{D-1}}{\lambda_1^{a-1} \lambda_a^{D-a}}.
\]

To estimate \( \left| B_0(\lambda_a) \cap \Lambda \right| \) we observe that \( \partial B_0(\lambda_a) \) lies in \( \text{Lip}(D, 1, 2\pi D \lambda_a) \). Applying Theorem 3.1 gives

\[
\left| B_0(\lambda_a) \cap \Lambda \right| \leq \frac{\text{Vol}B_0(\lambda_a)}{\det \Lambda} + c_0(D) \max_{0 \leq i < D} \frac{(2\pi D \lambda_a)^i}{\lambda_1 \cdots \lambda_i}.
\]

Using Minkowski’s second Theorem we get

\[
\frac{\text{Vol}B_0(\lambda_a)}{\det \Lambda} \leq 2^D \frac{\lambda_2^D}{\lambda_1 \cdots \lambda_D} \leq 2^D \frac{\lambda_a^{D-1}}{\lambda_1 \cdots \lambda_i} \leq 2^D \frac{(\kappa L)^{D-1}}{\lambda_1^{a-1} \lambda_a^{D-a}}.
\]

Moreover,

\[
\max_{0 \leq i < D} \frac{(2\pi D \lambda_a)^i}{\lambda_1 \cdots \lambda_i} \leq (2\pi D)^D \frac{\lambda_2^{D-1}}{\lambda_1 \cdots \lambda_i} \leq (2\pi D)^{D-1} \frac{(\kappa L)^{D-1}}{\lambda_1^{a-1} \lambda_a^{D-a}}.
\]

Next suppose \( \kappa L < \lambda_a \). Then, as \( Z \subset B_0(\kappa L) \), we have \( \left| Z \cap \Lambda(a) \right| = 0 \). Again, by Minkowski’s second Theorem and by \( Z \subset B_0(\kappa L) \) we get

\[
\frac{\text{Vol}Z}{\det \Lambda} \leq \frac{(2\pi L)^D}{\lambda_1 \cdots \lambda_D} \leq 2^D \left( \frac{(\kappa L)}{\lambda_a} \right)^{a-1} \left( \frac{\kappa L}{\lambda_D} \right)^{D-a+1} \leq 2^D \frac{(\kappa L)^{D-1}}{\lambda_1^{a-1} \lambda_a^{D-a}}.
\]

This completes the proof. \( \square \)

4. Counting lattice points using a gap principle

Let \( r \) and \( s \) be non-negative integers not both zero, and put \( d = r + 2s \). For \( 1 \leq i \leq r + s \) we set \( d_i = 1 \) if \( i \leq r \) and \( d_i = 2 \) otherwise. We write \( \mathbf{z}_i = (z_{i1}, \ldots, z_{in}) \) for variables in \( K^n_i \), where \( K_i = \mathbb{R} \) if \( i \leq r \) and \( K_i = \mathbb{C} \) if \( i > r \). Moreover, we write

\[
|z_i|_\infty = \max \{|z_{i1}|, \ldots, |z_{in}|\},
\]

\[
|(1, z_i)|_\infty = \max \{1, |z_{i1}|, \ldots, |z_{in}|\}.
\]

For \( T \geq 1 \) we define the set

\[
Z(T) = \left\{ \left( z_1, \ldots, z_{r+s} \right) \in \prod_{i=1}^{r+s} K_i^n \mid \prod_{i=1}^{r+s} |(1, z_i)|_\infty \leq T \right\}.
\]

For each subset \( I \subset \{1, 2, \ldots, r + s\} \) and \( \Gamma = \{1, 2, \ldots, r + s\} \setminus I \) we define

\[
Z_I = Z_I(T) = \left\{ \left( z_1, \ldots, z_{r+s} \right) \in Z(T) \mid |z_i|_\infty \geq 1 \text{ for } i \in I \text{ and } |z_j|_\infty < 1 \text{ for } j \in \Gamma \right\}.
\]
We put
\[ d' = \sum_i d_i, \]
and
\[ q' = |I| - 1. \]

Let \( \mathcal{T} \) be the group of \( \mathbb{R} \)-linear maps \( \phi \) on \( \prod_{i=1}^{r+s} K_i^n \) of the form
\begin{equation}
\phi(z_1, \ldots, z_{r+s}) = (\xi_1 z_1, \ldots, \xi_{r+s} z_{r+s})
\end{equation}
with positive real \( \xi_i \) satisfying
\begin{equation}
\prod_{i=1}^{r+s} \xi_i^{d_i} = 1,
\end{equation}
so that \( \det \phi = 1 \). The following theorem is an important intermediate step.

**Theorem 4.1.** Suppose \( q' = |I| - 1 \geq 0 \). Let \( \Lambda \) be a lattice in the Euclidean space \( \prod_{i=1}^{r+s} K_i^n \) and suppose there exist positive real numbers \( \eta_1, \ldots, \eta_{nd} \) such that \( \lambda_i(\phi(\Lambda)) \geq \eta_i \) for \( 1 \leq i \leq nd \) and all \( \phi \in \mathcal{T} \). Then, for \( T \geq 1 \), one has
\[
\begin{align*}
|\Lambda \cap Z_I(T)| - \frac{\text{Vol} Z_I(T)}{\det \Lambda} &\leq c(n, d)(\log^+ T)^{q'} \max_{0 \leq j < nd} \frac{T^{j/d}}{\eta_1 \cdots \eta_j}, \\
|\Lambda \cap Z_I(T)| - \frac{\text{Vol} Z_I(T)}{\det \Lambda} &\leq c(n, d)(\log^+ T)^{q'} \frac{T^{n-1/d}}{\eta_1 \cdots \eta_{nd}}.
\end{align*}
\]
For \( j = 0 \) the expression in the maximum is to be understood as 1, and \( c(n, d) \) depends only on \( n \) and \( d \). Moreover, if \( T < (\eta_1/k)^d \) we have
\[ |\Lambda \cap Z_I(T)| = 0, \]
where \( k = \sqrt{dn} \exp(\sqrt{T}) \).

5. Preliminaries

Unless explicitly mentioned otherwise (which will be the case only in Section 14) we always assume \( I \neq \emptyset \). Suppose \( I = \{i_1, \ldots, i_j\} \) with \( i_1 < \cdots < i_j \) then we put \( (z_i)_I = (z_{i_1}, \ldots, z_{i_j}) \). For subsets \( Z_1 \subset \prod_I K_i^n \) and \( Z_2 \subset \prod_I K_i^n \) we identify the Cartesian product \( Z_1 \times Z_2 \) with \( Z_1 \) if \( I^c \) is empty. It is more convenient to group the coordinate vectors according to their maximum norm, and thus we redefine
\begin{equation}
Z_I(T) = \left\{ (z_i)_I \in \prod_I K_i^n; \prod_I |z_i|_{\infty}^{d_i} \leq T; |z_i|_{\infty} \geq 1 \text{ for } i \in I \right\} \\
\times \left\{ (z_i)_{I^c} \in \prod_{I^c} K_i^n; |z_i|_{\infty} < 1 \text{ for } i \in I^c \right\}.
\end{equation}

As we study the cardinality \( |\Lambda \cap Z_I(T)| \) we shall permute the coordinates of \( \Lambda \) in the same manner, and we modify \( \phi \in \mathcal{T} \) accordingly to act on \( \prod_I K_i^n \times \prod_I K_i^n \). Of course, this leaves the volume \( \text{Vol} Z_I(T) \) and the values \( \lambda_i(\phi(\Lambda)) \) invariant. Let \( \Sigma \) be the hyperplane in \( \mathbb{R}^{d'+1} \) defined by \( x_1 + \cdots + x_{d'+1} = 0 \) and
\[ \delta = (d_i/d')_I. \]

Let \( F \) be a set in \( \Sigma \) and put \( F(T) \) for the vector sum
\begin{equation}
F(T) = F + \delta(-\infty, \log T).
\end{equation}
The map $(z_i)_I \rightarrow (d_i \log |z_i|_{\infty})_I$ sends $\prod_I K^n_i \setminus \{0\}$ to $\mathbb{R}^{d'} + 1$. Now we define

\begin{equation}
S_F(T) = \left\{ (z_i)_I \in \prod_I K^n_i \setminus \{0\}; (d_i \log |z_i|_{\infty})_I \in F(T) \right\}.
\end{equation}

Directly from the definition we get

\begin{equation}
S_F(T) = T^{1/d} S_F(1).
\end{equation}

Moreover, if $F$ lies in a ball centered at zero of radius $r_F$, then for any $(z_i)_I \in S_F(T)$

\begin{equation}
|z_i|_{\infty} \leq \exp(r_F)T^{1/d} \quad (i \in I).
\end{equation}

For non-negative reals $a_i \ (i \in I)$ let us write

\begin{equation}
E((a_i)_I) = \left\{ (z_i)_I \in \prod_I K^n_i; |z_i|_{\infty} \geq a_i \text{ for } i \in I \right\}.
\end{equation}

### 6. Partitioning and Transforming $Z_I(T)$

In Section 9 we will prove that for $q' > 0$ we have

\[ Z_I(T) = (S_F(T) \cap E((1)_I)) \times \{(z_i)_I; |z_i|_{\infty} < 1 \text{ for } i \in I^c\} \]

for a certain $F \subset \Sigma$. In this section we focus on the first component $S_F(T) \cap E((1)_I)$ but we will allow arbitrary sets $F \subset \Sigma$. Throughout this section we assume

$q' > 0$.

Fix once and for all an orthonormal basis $e_1, \ldots, e_{q'}$ of $\Sigma \subset \mathbb{R}^{q'+1}$. For $i = (i_1, \ldots, i_{q'}) \in \mathbb{Z}^{q'}$ we define the fundamental cell

\[ C_i = i_1 e_1 + [0, 1)e_1 + \cdots + i_{q'} e_{q'} + [0, 1)e_{q'} .\]

For $F \subset \Sigma$ we define

\[ F_i = C_i \cap F. \]

Let $m_F$ be the set of those $i$ that satisfy $F_i \neq \emptyset$. Clearly,

\begin{equation}
F = \bigcup_{m_F} F_i,
\end{equation}

and the latter is a disjoint union.

**Lemma 6.1.** Suppose $F$ is a subset of $\Sigma$ and $F \subset B_0(r_F)$ with $r_F \geq 1$. Then

\[ |m_F| \ll r_F^{q'}. \]

**Proof.** Clearly, $F$ lies in the cube $[-r_F, r_F] e_1 + \cdots + [-r_F, r_F] e_{q'}$ which has non-empty intersection with at most $(2[r_F] + 1)^{q'}$ fundamental cells $C_i$ (here $\lceil r_F \rceil$ denotes the smallest integer not smaller than $r_F$). Since $r_F \geq 1$ the lemma follows. □

Now (6.1) leads to

\begin{equation}
S_F(T) = \bigcup_{m_F} S_{F_i}(T),
\end{equation}

which again is a disjoint union. For each vector $i = (i_1, \ldots, i_{q'}) \in \mathbb{Z}^{q'}$ we define a translation $tr_i$ on $\mathbb{R}^{q'+1}$ by

\[ tr_i(x) = x - \sum_{j=1}^{q'} i_j e_j = x - u(i), \]
where \( u(i) = (u_i)_i = \sum_{j=1}^{d_i} i_j c_j \). This translation sends \( \Sigma \) to \( \Sigma \) and \( C_1 \) to \( C_0 \). For \( i \in I \) set \( \gamma_i = \gamma_i(i) = \exp(-u_i/d_i) \), so that \( \gamma_i > 0 \),

\[
(6.3) \quad \prod_i \gamma_i^{d_i} = 1,
\]

and

\[
(d_i \log |\gamma_i|_\infty)_i = tr_i((d_i \log |z_i|_\infty)_i).
\]

Hence, for the automorphism \( \tau_i \) of \( \prod_I K_i^n \) defined by

\[
\tau_i(z_i)_i = (\gamma_i z_i)_i,
\]

we have

\[
\tau_i S_F(T) = S_{tr_i(F)}(T).
\]

As \( tr_1(F_i) = tr_1(F) \cap C_0 \) we get

\[
(6.4) \quad \tau_i S_{F_i}(T) = S_{tr_1(F) \cap C_0}(T).
\]

Moreover, we have

\[
\tau_i E(1)_i = \left\{ (z_i)_i \in \prod_I K_i^n; |z_i|_\infty \geq \gamma_i \text{ for } i \in I \right\} = E((\gamma_i)_i)_i.
\]

As \( C_0 \subset B_0(\sqrt{q^n}) \) we get from \( \box{red} \) that for any \( (z_i)_i \in S_{C_0}(T) \)

\[
(6.5) \quad |z_i|_\infty \leq \exp(\sqrt{q^n})T^{1/d} \quad (i \in I).
\]

We extend \( \tau_i \) to a diagonal endomorphism \( \phi_i \) on \( \prod_I K_i^n \times \prod_{I^c} K_i^n \) by setting

\[
(6.6) \quad \phi_i((z_i)_i, (z_i)_{I^c}) = (\gamma_i (z_i)_i, (z_i)_{I^c}) = ((\gamma_i z_i)_i, (z_i)_{I^c}).
\]

Next we put

\[
(6.7) \quad Z_{F_i} = (S_{F_i}(T) \cap E((1)_i)) \times \{(z_i)_{I^c}; |z_i|_\infty < 1 \text{ for } i \in I^c \}.
\]

7. Further transforming \( Z_i(T) \)

We define a map

\[
(7.1) \quad \psi: \prod_I K_i^n \times \prod_{I^c} K_i^n \rightarrow \prod_I K_i^n \times \prod_{I^c} K_i^n
\]

by

\[
\psi(((z_i)_i, (z_i)_{I^c})) = (\psi_1((z_i)_i), \psi_2((z_i)_{I^c}))
\]

where

\[
\psi_1((z_i)_i) = ((T^{-1/d_i+1/d}z_i)_i),
\]

\[
\psi_2((z_i)_{I^c}) = ((T^{1/d}z_i)_{I^c}).
\]

For \( q' = q \) (i.e., for \( I^c = \emptyset \)) we interpret, of course, \( \psi = \psi_1 \) as the identity on \( \prod_I K_i^n = \prod_{i=1}^{r+1} K_i^n \). As \( d' = \sum_Id_i \) we see that

\[
(7.2) \quad \det \psi = \prod_I T^{d_i \cdot n(-1/d_i+1/d)} \prod_{I^c} T^{d_i \cdot n/d} = 1.
\]

Therefore, \( \psi \) lies in \( T \).
First suppose \( q' = 0 \), so that \( I = \{ i \} \) is a singleton. Then

\[
\psi Z_i(T) = \left\{ \mathbf{z}_i \in K_1^n; T^{-1/d + 1/d} \leq |\mathbf{z}_i|_\infty \leq T^{1/d} \right\} \times \left\{ \mathbf{z}_j \in K_j^n; |\mathbf{z}_j|_\infty < T^{1/d} \text{ for } j \neq i \right\}.
\]

Now suppose \( q' > 0 \). For \( i \in \mathbb{Z}^q \) we set

\[
Z_1 = \psi_1 (\tau_1 S_{F_1}(T) \cap \tau_1 E((1)_I)) \subset \prod_i K_i^n,
\]

and, with \( \phi_i \) as in (6.6), we define

\[
\psi_1 = \psi \circ \phi_i.
\]

Moreover, we set

\[
Z_2 = \psi_2 \left\{ (\mathbf{z}_i)_I; |\mathbf{z}_i|_\infty < 1 \text{ for } i \in I^c \right\} = \left\{ (\mathbf{z}_i)_I; |\mathbf{z}_i|_\infty < T^{1/d} \text{ for } i \in I^c \right\} \subset \prod_i K_i^n,
\]

so that

\[
Z_1 \times Z_2 = \psi_1 Z_{F_1}.
\]

**Lemma 7.1.** Let \( \kappa = \sqrt{dn} \exp(\sqrt{q}) \) be as in Theorem 4.1. If \( q' = 0 \) then we have

\[
\psi Z_i(T) \subset B_0(\kappa T^{1/d}).
\]

If \( q' > 0 \) and \( i \in \mathbb{Z}^q \) then we have

\[
\psi_1 Z_{F_1} \subset B_0(\kappa T^{1/d}).
\]

In particular,

\[
Z_i \subset B_0(\kappa T^{1/d}) \quad (1 \leq i \leq 2)
\]

for the respective balls \( B_0(\kappa T^{1/d}) \).

**Proof.** As \( \kappa \geq \sqrt{(q + 1)n} \) the claim follows immediately from (7.3). Next suppose \( q' > 0 \). Recall from (6.4) that \( \tau_1 S_{F_1}(T) \subset S_{c_0}(T) \). From (6.5), and not forgetting the effect of \( \psi_1 \), we see that for any \( (\mathbf{z}_i)_I \) in \( Z_1 \) we have \( |\mathbf{z}_i|_\infty \leq \exp(\sqrt{q}) T^{1/d} \) \((i \in I)\). And, obviously, we also have \( |\mathbf{z}_i|_\infty \leq \exp(\sqrt{q}) T^{1/d} \) for any \( (\mathbf{z}_i)_I \) in \( Z_2 \). This proves (7.8). 

8. Lipschitz parameterizations

In this section we shall prove that the sets \( \psi Z_i(T) \) (if \( q' = 0 \)), and \( \psi_1 Z_{F_1} \) (if \( q' > 0 \)) have Lipschitz parameterizable boundaries with Lipschitz constant \( L \ll T^{1/d} \). To this end we need a few simple lemmas. For \( q' > 0 \) we will identify \( \Sigma \) with \( \mathbb{R}^q \) via the basis \( e_1, \ldots, e_q \) from Section 6. For a subset \( Z \) of Euclidean space we write \( \partial Z \) for its topological boundary.

**Lemma 8.1.** Suppose \( q' > 0 \), and let \( F \) be a set in \( \Sigma \) such that \( \partial F \) is in \( \text{Lip}(q', M', L') \), and, moreover, assume \( F \) lies in \( B_0(\tau F) \). Then \( \partial S_F(1) \) is in \( \text{Lip}(d'n, \bar{M}, \bar{L}) \) with \( \bar{M} \) and \( \bar{L} \) depending only on \( n, q', M', L', \tau F \).

**Proof.** The case \( n > 1 \) follows directly from [22, Lemma 3] (see also [33, Lemma 7.1] for a more detailed and completely explicit version). However, for \( n = 1 \) the proof remains correct without change. 

\[ \square \]
Lemma 8.2. Suppose \( q' > 0 \), and recall the definition of \( \text{tr}_1 \) and \( F_1 \) from Section 7. Let \( Y \geq 1 \) be a real number and suppose the boundary of \( \text{tr}_1 F_1 \) lies in \( \text{Lip}(q', M', L') \) with \( M' \ll 1 \) and \( L' \ll 1 \). Then the boundary of \( \tau_i S_{F_1}(Y) \) lies in \( \text{Lip}(d'n, M, L) \) with \( M \ll 1 \) and \( L \ll Y^{-1/d'} \).

Proof. Clearly, \( \text{tr}_1(F_1) = \text{tr}_1(F) \cap C_0 \) is contained in \( B_0(\sqrt{q}) \). Now \( \tau_i S_{F_1}(Y) = S_{\text{tr}_1(F_1)}(Y) \) and thus the lemma follows from (5.4) and Lemma 8.1. □

Lemma 8.3. If \( q' = 0 \) then \( \partial \psi Z_1(T) \) lies in \( \text{Lip}(dn, M, L) \) with \( M \ll 1 \) and \( L \ll T^{1/d} \). If \( q' > 0 \) and \( \partial \text{tr}_1 F_1 \) lies in \( \text{Lip}(q', M', L') \) with \( M' \ll 1 \) and \( L' \ll 1 \) then the set \( \partial \psi_i Z_{F_1} \) lies in \( \text{Lip}(dn, M, L) \) with \( M \ll 1 \) and \( L \ll T^{1/d} \).

Proof. First suppose \( q' = 0 \). The sets in \( K^n_i \) defined by \( |z_i|_\infty = \zeta \) are in \( \text{Lip}(d, n, 2n, \zeta') \) with \( \zeta' \ll \zeta \), e.g., we can take \( 2n \) linear (if \( i \leq r \)) or \( n \) trigonometrical (if \( i > r \)) maps. Then one easily gets a parameterization of the sets \( |z_i|_\infty = \zeta_1, |z_j|_\infty \leq \zeta_2 \) \((j \neq i)\) in \( \prod_i K^n_i \times \prod_i K^n_i \) with \( M \ll 1 \) maps and Lipschitz constants \( L \ll \max\{\zeta_1, \zeta_2\} \). In view of (7.3) this proves the lemma for \( q' = 0 \).

Now suppose \( q' > 0 \). We need to show that \( \partial(Z_1 \times Z_2) \) lies in \( \text{Lip}(dn, M, L) \). Clearly, \( \partial(Z_1 \times Z_2) \) is contained in the union of \( \overline{Z}_1 \times \partial Z_2 \) and \( \partial Z_1 \times \overline{Z}_2 \), where the bar denotes the topological closure. Moreover, by (1.9) we know \( \overline{Z}_1 \) and \( \overline{Z}_2 \) lie both in a ball \( B_0(\kappa T^{1/d}) \).

Therefore, it suffices to show that \( \partial Z_1 \in \text{Lip}(dn, M'', L'') \) and, if \( d - d' > 0 \), also \( \partial Z_2 \in \text{Lip}((d - d')n, M'', L'') \) with some \( M'' \ll 1 \) and some \( L'' \ll T^{1/d} \). Next note that

\[
\begin{align*}
\psi_1 \tau_i (S_{F_1}(T)) &= T^{1/d} S_{\text{tr}_1 F_1}(1), \\
\psi_1 \tau_i (E((1)_I)) &= E((T^{1/d - 1/d'} \gamma_i)_I).
\end{align*}
\]

As \( Z_1 \) is the intersection of these two sets, we see that \( \partial Z_1 \) is covered by the union of \( \partial E((T^{1/d - 1/d'} \gamma_i)_I) \cap \overline{Z}_1 \) and \( \partial T^{1/d} S_{\text{tr}_1 F_1}(1) \). Regarding the latter recall that \( \text{tr}_1 F_1 \subset C_0 \subset B_0(\sqrt{q}) \) and \( \text{tr}_1 F_1 \) lies in \( \text{Lip}(q', M', L') \). Therefore, we can apply Lemma 8.1 to conclude \( \partial T^{1/d} S_{\text{tr}_1 F_1}(1) \) lies in \( \text{Lip}(dn, M'', L'') \) with some \( M'' \ll 1 \) and some \( L'' \ll T^{1/d} \).

And for \( \partial E((T^{1/d - 1/d'} \gamma_i)_I) \cap \overline{Z}_1 \) we use the same argument as for \( q' = 0 \) to see that it is in \( \text{Lip}(dn, M'', L'') \) with an \( M'' \ll 1 \) and an \( L'' \ll T^{1/d} \). And again, the same argument shows that, for \( d > d' \), \( \partial Z_2 \) lies in \( \text{Lip}((d - d')n, M'', L'') \) with an \( M'' \ll 1 \) and an \( L'' \ll T^{1/d} \). This proves the Lemma 8.3. □

9. Proof of Theorem 4.1

First we assume \( q' = 0 \).

Recall that \( \psi \) lies in \( \mathcal{T} \), and, clearly, we have \( |Z_I \cap \Lambda| = |\psi Z_I \cap \psi \Lambda| \). By (7.7) we have \( \psi(Z_I) \subset B_0(\kappa T^{1/d}) \), and by hypothesis of Theorem 4.1 we have \( \lambda_i(\psi \Lambda) \geq \eta_i \) for \( 1 \leq i \leq dn \). Thanks to Lemma 8.3 we can apply Theorem 8.1 which gives the first inequality of Theorem 4.1. For the second inequality we apply Corollary 3.1 with \( \alpha = 1 \) and note that \( 0 \notin \psi(Z_I) \). And finally, as \( 0 \notin \psi(Z_I) \) and \( \psi(Z_I) \subset B_0(\kappa T^{1/d}) \) we see that \( |\Lambda \cap Z_I| = 0 \) if \( T^{1/d} < (1/\kappa) \eta_1 \). This finishes the proof of Theorem 4.1 for \( q' = 0 \).

For the rest of this section we assume \( q' > 0 \), and, for the rest of the paper, we fix \( F \) as

\[ F = (\mathbb{R}^{q'+1}_0 - \delta \log T) \cap \Sigma \]

Lemma 9.1. We have

\[ Z_I = (S_F(T) \cap E((1)_I)) \times \{z_i \mid |z_i|_\infty < 1 \text{ for } i \in I^c \}. \]
Lemma 9.1. leads to the disjoint union

\[ \text{Proof.} \]

From the definitions (5.3) and (5.6) we see immediately that the right hand-side is contained in the left hand-side for any choice of \( F \subset \Sigma \) whatsoever. Now for the other inclusion note that the left hand-side in (9.2) means

\[ (d_i \log |z_i|_\infty)_t \in \mathbb{R}^{q+1}_\geq \cap (\Sigma + \delta(-\infty, \log T)). \]

Thus we need to show

\[ \mathbb{R}^{q+1}_\geq \cap (\Sigma + \delta(-\infty, \log T)) \subset F(T) = \left( \left( \mathbb{R}^{q+1}_\geq - \delta \log T \right) \cap \Sigma \right) + \delta(-\infty, \log T). \]

Any element in the set on the left hand-side can be written as \( x + \delta t \) with \( x \in \Sigma \) and \( t \in (-\infty, \log T) \). As \( x + \delta t \in \mathbb{R}^{q+1}_\geq \) we get \( x \in \mathbb{R}^{q+1}_\geq - \delta \log T \cap \Sigma \), and therefore

\[ x + \delta t \in \left( \left( \mathbb{R}^{q+1}_\geq - \delta \log T \right) \cap \Sigma \right) + \delta(-\infty, \log T). \]

This concludes the proof. \( \square \)

**Lemma 9.2.** We have

(9.3) \[ F \subset B_0(2 \log T) \]

**Proof.** Suppose \((x_1, \ldots, x_{q+1}) \in F\). As \( x_1 + \cdots + x_{q+1} = 0 \) we see that the sum over the positive coordinates equals minus the sum over the negative coordinates and thus \(|x_1| + \cdots + |x_{q+1}| \leq 2 \sum (d_i/d'_i) \log T = 2 \log T\). This proves the lemma. \( \square \)

Recall the definition of \( Z_{F_1} \) from (6.1). The disjoint union (6.2), in conjunction with Lemma 9.1 leads to the disjoint union

(9.4) \[ Z_I = \bigcup_{m_F} Z_{F_1}, \]

which in turn yields

\[ |Z_I \cap \Lambda| = \sum_{m_F} |Z_{F_1} \cap \Lambda|. \]

As the \( \psi_i \) are automorphisms we conclude

(9.5) \[ |Z_I \cap \Lambda| = \sum_{m_F} \psi_i Z_{F_1} \cap \psi_i \Lambda|. \]

We will apply Lemma 8.3 with our choice of \( F \) given in (9.1). We start off by verifying the necessary conditions.

**Lemma 9.3.** Let \( F \) be as in (9.1). There exist \( M' \ll 1 \) and \( L' \ll 1 \) such that \( \partial \text{tr}_1 F_1 \) lies in \( \text{Lip}(q', M', L') \).

**Proof.** Clearly, \( F \), and therefore also \( \text{tr}_1 F \), is convex. And, clearly, \( C_0 \) is convex and contained in \( B_0(\sqrt{q'}) \). Hence \( \text{tr}_1 F_1 = \text{tr}_1 F \cap C_0 \) is convex and lies in \( B_0(\sqrt{q'}) \). Now if \( q' = 1 \) the lemma is trivial, and if \( q' > 1 \) it follows immediately from [35] Theorem 2.6. \( \square \)

**Lemma 9.4.** The set \( \partial \psi_i(Z_{F_1}) \) lies in \( \text{Lip}(dn, M, L) \) with some \( M \ll 1 \) and some \( L \ll T^{1/d} \).

**Proof.** This is an immediate consequence of Lemma 9.3 and Lemma 8.3. \( \square \)
Lemma 9.5. We have
\[
|\psi_i(Z_{F_i}) \cap \psi_i(\Lambda)| - \frac{\text{Vol}Z_{F_i}}{\text{det} \Lambda} \ll \max_{0 \leq j < dn} \frac{T_j/d}{\eta_1 \cdots \eta_j},
\]
\[
|\psi_i(Z_{F_i}) \cap \psi_i(\Lambda)| - \frac{\text{Vol}Z_{F_i}}{\text{det} \Lambda} \ll \frac{T^{n-1/d}}{\eta_1^{nd-1}},
\]
\[
|\psi_i(Z_{F_i}) \cap \psi_i(\Lambda)| = 0 \text{ if } T^{1/d} < (1/\kappa)\eta_1.
\]

Proof. Again, we want to apply Theorem 3.1 and Corollary 3.1. First recall that \(\psi_i \in T\), in particular, \(\text{Vol} \psi_i Z_{F_i} = \text{Vol} Z_{F_i}\) and \(\text{det} \psi_i(\Lambda) = \text{det}(\Lambda)\). By Lemma 9.4 we know \(\partial \psi_i(Z_{F_i})\) lies in \(\text{Lip}((dn,M,L))\) with some \(M \ll 1\) and some \(L \ll T_1/d\). By (7.8) we have \(\psi_i Z_{F_i} \subset B_0(\kappa T_1/d)\) with \(1 \leq \kappa \ll 1\), and as \(0 \notin Z_I\) we also have \(0 \notin \psi_i Z_{F_i}\). Applying Theorem 3.1 and Corollary 3.1, and using the hypothesis \(\lambda_i(\psi_i(\Lambda)) \geq \eta_i\) yields the inequalities of the lemma. And the last statement follows just as in the case \(q' = 0\).

Lemma 9.6. We have
\[
|m_F| \ll (\log^+ T) q'.
\]

Proof. This follows immediately from (9.3) and Lemma 6.1.

Finally, we use Lemma 9.6 to deduce
\[
\sum_{m_F} 1 \ll (\log^+ T) q'.
\]

This proves Theorem 4.1.

10. Estimates for the successive minima

The results of this section are slight generalizations of those in \([33, \text{Section 9}]\) but they are proved by exactly the same arguments. In particular, we show that the minima of the lattice \(\sigma \mathcal{O}^n\) satisfy the required gap principle in Theorem 4.1. As this is an important point, we prefer to give the full proofs here.

As in the introduction let \(K/k\) be an extension of number fields, and \(d = [K : \mathbb{Q}]\). Recall that \(\sigma_1, \ldots, \sigma_d\) denote the embeddings from \(K\) to \(K_i\), ordered such that \(\sigma_{r+s+i} = \sigma_{r+i}\) for \(1 \leq i \leq s\). We write
\[
\sigma : K \longrightarrow \prod_{i=1}^{r+s} K_i
\]
\[
\sigma(\alpha) = (\sigma_1(\alpha), \ldots, \sigma_{r+s}(\alpha)).
\]

Let \(\phi\) be as in (12). By abuse of notation we may regard \(\phi\) also as an automorphism of \(\mathbb{R}^r \times \mathbb{C}^s\), and from now on, depending on the argument, we view \(\phi\) as an automorphism of
$\mathbb{R}^r \times \mathbb{C}^s$ or $\mathbb{R}^{rn} \times \mathbb{C}^{sn}$. Applying $\phi$ to the lattice $\sigma \mathcal{O}$ gives a new lattice $\phi \sigma \mathcal{O}$ in $\mathbb{R}^r \times \mathbb{C}^s$. As is well-known, see, e.g., [4, Lemma 1], we can choose linearly independent vectors 

$$v_1 = \phi \sigma(\theta_1), \ldots, v_d = \phi \sigma(\theta_d)$$

of the lattice $\phi \sigma \mathcal{O}$ with

$$(10.2) \quad |v_i| = \lambda_i(\phi \sigma \mathcal{O}) \quad (1 \leq i \leq d)$$

for the successive minima $\lambda_i(\phi \sigma \mathcal{O})$. The $v_1, \ldots, v_d$ are $\mathbb{R}$-linearly independent. Hence, $\theta_1, \ldots, \theta_d$ are $\mathbb{Q}$-linearly independent, and therefore $\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_d}{\sigma_1}$ are $\mathbb{Q}$-linearly independent. As $[K : \mathbb{Q}] = d$ we get $K = \mathbb{Q}(\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_d}{\sigma_1}) = k(\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_d}{\sigma_1})$, and this allows the following definition.

**Definition 2.** Let $l \in \{1, \ldots, d\}$ be minimal with $K = k(\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_l}{\sigma_1})$.

We abbreviate

$$(10.3) \quad \lambda_i = \lambda_i(\phi \sigma \mathcal{O})$$

for $1 \leq i \leq d$. Recall the definition of $\eta_{\mathcal{O}}$ from (1.6).

**Lemma 10.1.** We have

$$\lambda_1 \geq \sqrt{d/2} \eta_{\mathcal{O}}.$$  

**Proof.** Observe that by definition

$$\phi \sigma \alpha = (\xi_1 \sigma_1 \alpha, \ldots, \xi_r \sigma_r \alpha),$$

where $\xi_i \in \mathbb{R}_{>0}$ and $\prod_{i=1}^{r+s} \xi_i^{d_i} = 1$. So the squared length of an element $\phi \sigma \alpha$ of $\phi \sigma \mathcal{O}$ is

$$\sum_{i=1}^{r+s} |\xi_i \sigma_i \alpha|^2 \geq \frac{1}{2} \sum_{i=1}^{r+s} d_i |\xi_i \sigma_i \alpha|^2.$$ 

Next we use the inequality between the arithmetic and geometric mean to deduce that this is at least

$$(d/2) \prod_{i=1}^{r+s} |\xi_i \sigma_i \alpha|^{2d_i/d}.$$ 

By (10.3) we see that the latter is $(d/2) \prod_{i=1}^{r+s} |\sigma_i \alpha|^{2d_i/d}$. Here $\prod_{i=1}^{r+s} |\sigma_i \alpha|^{d_i}$ is the absolute value of the norm of $\alpha$ from $K$ to $\mathbb{Q}$ which is at least $\eta_{\mathcal{O}}^d$, provided $\alpha \neq 0$. This proves the lemma.  

**Lemma 10.2.** With $K_0 = k(\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_l}{\sigma_1})$ if $l \geq 2$ and $K_0 = k$ if $l = 1$, and $g = [K_0 : k] \in G(K/k)$ one has

$$\lambda_l \geq \frac{1}{\sqrt{2d e \eta_{\mathcal{O}} \delta_g(K/k)}}.$$ 

**Proof.** First note that $l = 1$ is equivalent to $K = k$. Thus, $l = 1$ implies $k = K$, $g = 1$, $\delta_g(K/k) = 1$, and so the claim follows from Lemma [10.1]. Next suppose $l > 1$. We apply a standard argument (see, e.g., [20, Proof of Theorem 4.6]) to obtain a primitive element $\beta = \sum_{i=1}^{l} m_i \frac{\theta_i}{\sigma_1}$ for the extension $K/k$, where $m_i$ are in $\mathbb{Z}$ and $0 \leq m_i < e$ $(1 \leq i \leq l)$. And once more to get a primitive element $\alpha = \sum_{i=1}^{l-1} m_i' \frac{\theta_i}{\sigma_1}$ for the extension $k(\frac{\theta_1}{\sigma_1}, \ldots, \frac{\theta_l}{\sigma_1})/k$ with $m_i' \in \mathbb{Z}$ and $0 \leq m_i' < e$ $(1 \leq i \leq l-1)$. So $k(\alpha, \beta) = K$ and $[k(\alpha) : k] = g$. For each non-Archimedean place $v$ of $K$ (denoted by $v \uparrow \infty$) let $| \cdot |_v$
be the unique absolute value on $K$ extending the corresponding usual $p$-adic one, and set $d_v = [K_v : \mathbb{Q}_p]$ for the respective local degree. Using the product formula we get

$$\delta_p(K/k)^d \leq H(\alpha, \beta)^d = \prod_{v \mid \infty} \max \left\{ \left| \theta_1|_v \right|, \left| \sum_{i=1}^{l-1} m'_i \theta_i |_v \right|, \left| \sum_{i=1}^{l} m_i \theta_i |_v \right| \right\}^{d_v} \prod_{j=1}^{r+s} \max \left\{ \left| \sigma_j \theta_1 |, \left| \sum_{i=1}^{l-1} m'_i \theta_i |, \left| \sum_{i=1}^{l} m_i \theta_i | \right| \right\}^{d_j}.$$ 

Recall that $\mathfrak{A}_O$ is an ideal containing $O$. Thus $\theta_1, \ldots, \theta_l$ are in $\mathfrak{A}_O$, and therefore, the above is

$$\leq \mathfrak{N}(\mathfrak{A}_O)^{-1} \prod_{j=1}^{r+s} (le)^{d_j} \max \{ |\sigma_j \theta_1|, \ldots, |\sigma_j \theta_l| \}^{d_j}.$$ 

Since $\mathfrak{N}(\mathfrak{A}_O) = \eta_O^d$ and $\prod_{j=1}^{r+s} \zeta_j = 1$ this in turn is

$$= (le)^d \eta_O^{-d} \prod_{j=1}^{r+s} \max \{ |\xi_j |\sigma_j \theta_1|, \ldots, |\xi_j |\sigma_j \theta_l| \}^{d_j}$$

$$= (le)^d \eta_O^{-d} \left( \prod_{j=1}^{r+s} \max \{ |\xi_j |\sigma_j \theta_1|, \ldots, |\xi_j |\sigma_j \theta_l| \}^{2d_j} \right)^{\frac{1}{2}}$$

$$= (le)^d \eta_O^{-d} \left( \prod_{j=1}^{r+s} |w_j|^{2d_j} \right)^{\frac{1}{2}},$$

where $w_j$ is the vector $(\xi_j |\sigma_j \theta_1|, \ldots, |\xi_j |\sigma_j \theta_l|)$ in $\mathbb{R}^l$ if $j \leq r$ and in $\mathbb{C}^l$ if $j > r$, and $| \cdot |_\infty$ denotes the maximum norm. Now using the inequality between the arithmetic and geometric mean and $| \cdot | \geq | \cdot |_\infty$ for the Euclidean norm $| \cdot |$ we may bound the above by

$$(10.4) \quad \leq (le)^d \eta_O^{-d} \left( \frac{1}{d} \sum_{j=1}^{r+s} d_j |w_j|^2 \right)^{\frac{1}{2}} \leq (le)^d (2/d)^{d/2} \eta_O^{-d} \left( \sum_{j=1}^{r+s} |w_j|^2 \right)^{\frac{1}{2}}.$$ 

The vector $(\phi \sigma \theta_1, \ldots, \phi \sigma \theta_l)$ in $(\mathbb{R}^r \times \mathbb{C}^s)^l$ has squared length exactly

$$\sum_{j=1}^{r+s} (|\xi_j |\phi \sigma \theta_1|, \ldots, |\xi_j |\phi \sigma \theta_l|)^2,$$

so that the right-hand side of $(10.4)$ is

$$= (le)^d (2/d)^{d/2} \eta_O^{-d} |(\phi \sigma \theta_1, \ldots, \phi \sigma \theta_l)|^d.$$ 

Moreover, by $(10.2)$ one has

$$| (\phi \sigma \theta_1, \ldots, \phi \sigma \theta_l) | = (|v_1|^2 + \cdots + |v_l|^2)^{\frac{1}{2}} \leq \sqrt{l} \lambda_l.$$ 

Note that by definition $l \leq d$, and thus, combining this estimates the result drops out. □

For the rest of this section we assume $n > 1$.

**Lemma 10.3.** Assume $a \in \{1, \ldots, d\}$ and $\mu_1, \ldots, \mu_a$ in $\mathbb{R}$ with $\mu_a \neq 0$ are such that $w = \mu_1 v_1 + \cdots + \mu_a v_a$ lies in $\phi \sigma O$. Then we have

$$|w| \geq \lambda_a.$$
Proof. For $a = 1$ it is clear. For $a > 1$ we apply [33] Lemma 4.1 with $V = \mathbb{R}v_1 + \cdots + \mathbb{R}v_a$.

Lemma 10.4. Assume $l \geq 2$, and let $(\omega_1, \ldots, \omega_n)$ be in $K^n \setminus \{0\}$ with $k(\ldots, \omega_i/\omega_j, \ldots) = k$. Then not all of the $\omega_1, \ldots, \omega_n$ are in $k\theta_1 + \cdots + k\theta_{l-1}$.

Proof. Set $K_0 = k(\frac{\theta_1}{\theta_2}, \ldots, \frac{\theta_{l-1}}{\theta_1})$. By definition of $l$ we have $K_0 \subseteq K$. Let $a, b$ be in $\{1, \ldots, n\}$ with $\omega_b \neq 0$. Suppose $\omega_a, \omega_b$ are in $k\theta_1 + \cdots + k\theta_{l-1}$. Then there are $\alpha_j, \beta_j$ ($1 \leq j \leq l-1$) in $k$ such that

$$\frac{\omega_a}{\omega_b} = \frac{\sum_{j=1}^{l-1} \alpha_j \theta_j}{\sum_{j=1}^{l-1} \beta_j \theta_j} = \frac{\sum_{j=1}^{l-1} \alpha_j \frac{\theta_j}{\theta_1}}{\sum_{j=1}^{l-1} \beta_j \frac{\theta_j}{\theta_1}}.$$

But numerator and denominator of the last fraction are in $K_0$, and so $\frac{\omega_a}{\omega_b}$ is in $K_0$. So if all $\omega_1, \ldots, \omega_n$ are in $k\theta_1 + \cdots + k\theta_{l-1}$ then $k(\ldots, \omega_i/\omega_j, \ldots) \subseteq K_0$.

Lemma 10.5. Let $(\omega_1, \ldots, \omega_n)$ be in $\mathcal{O}^n \setminus \{0\}$ with $k(\ldots, \omega_i/\omega_j, \ldots) = K$. Then for $v = (\phi \sigma \omega_1, \ldots, \phi \sigma \omega_n)$ we have

$$|v| \geq \lambda_l.$$

Proof. Each of the $\phi \sigma \omega_1, \ldots, \phi \sigma \omega_n$ lies in the lattice $\phi \sigma(\mathcal{O})$. The sublattice generated by $v_1, \ldots, v_d$ has finite index in $\phi \sigma \mathcal{O}$. Hence, there are $\mu_j^{(i)} \in \mathbb{Q}$ such that

$$v = \left( \prod_{j=1}^{d} \mu_j^{(1)} v_j, \ldots, \prod_{j=1}^{d} \mu_j^{(n)} v_j \right).$$

Lemma 10.4 and the condition $k(\ldots, \omega_i/\omega_j, \ldots) = K$ imply at least one of the numbers $\mu_j^{(i)}$ for $l \leq j \leq d$, $1 \leq i \leq n$ is nonzero, and so the result follows by Lemma 10.3.

We remind the reader that $[K : k] = e$, $[k : \mathbb{Q}] = m$, and $d = em$.

Lemma 10.6. If $l \geq 2$ then

$$\frac{l-1}{m} \leq [k \left( \frac{\theta_1}{\theta_1}, \ldots, \frac{\theta_{l-1}}{\theta_1} \right) : k] \leq \max\{1, e/2\}.$$

Proof. The $l-1$ numbers $\frac{\theta_1}{\theta_1}, \ldots, \frac{\theta_{l-1}}{\theta_1}$ are $\mathbb{Q}$-linearly independent. Hence, $[K_0 : \mathbb{Q}] \geq l-1$ for $K_0 = k(\frac{\theta_1}{\theta_1}, \ldots, \frac{\theta_{l-1}}{\theta_1})$. The first inequality follows at once, since $m = [k : \mathbb{Q}]$. The second one follows immediately from the definition of $l$ since $[K : k] = e$.

11. Upper bounds for the projectively non-primitive points

We extend the embeddings $\sigma_i$ from Lemma 10.1 componentwise to get an embedding of $K^n$

$$\sigma : K^n \hookrightarrow \prod_{i=1}^{r+s} K_i^n.$$  

Depending on the argument we either see $\sigma$ as a map on $K$ or on $K^n$. Again, let $\phi$ be as in (12). In this section we prove an upper bound for the number of nonzero points in $\phi \sigma \mathcal{O}^n$ that (as projective points) do not generate $K/k$ and lie in some ball. For brevity we write

$$\Lambda' = \phi \sigma \mathcal{O}^n \setminus (\phi \sigma \mathcal{O}^n(K/k) \cup \{0\}).$$

Lemma 11.1. Suppose $n > 1$, let $B_0(R)$ be the zero centered ball in the Euclidean space $\mathbb{R}^{nr} \times \mathbb{C}^n$ of radius $R$, and let $\lambda_i$ be as in (10.3). Then

$$|\Lambda' \cap B_0(R)| \ll \max_{0 \leq i \leq d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \left( \max_{0 \leq i \leq d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \right)^{n-1}.$$
Proof. We follow the lines of proof in [33, Proposition 10.1]. For \((\phi \sigma \omega_1, \ldots, \phi \sigma \omega_n)\) in \(\Lambda'\) the field \(k(\ldots, \omega_i/\omega_j, \ldots)\) lies in a strict subfield, say \(K_1\), of \(K\). Hence, there exist two different embeddings \(\sigma_a, \sigma_b\) of \(K\) with
\[
\sigma_a \alpha = \sigma_b \alpha
\]
for all \(\alpha\) in \(K_1\). Now \((\phi \sigma \omega_1, \ldots, \phi \sigma \omega_n) \neq 0\), and thus, at least one of the numbers \(\omega_1, \ldots, \omega_n\) is nonzero. By symmetry we lose only a factor \(n\) if we assume \(\omega_1 \neq 0\). So let us temporarily regard \(\omega_1 \neq 0\) as fixed; then for \(2 \leq j \leq n\) every \(\omega_j\) satisfies
\[
\sigma_a \omega_j = \sigma_b \omega_j = \omega_1.
\]
Therefore, all these \(\sigma \omega_j\) lie in a hyperplane \(\mathcal{P}(\omega_1)\) of \(\mathbb{R}^d\), and so all these \(\phi \sigma \omega_j\) lie in the hyperplane \(\phi \mathcal{P}(\omega_1)\). As \((\phi \sigma \omega_1, \ldots, \phi \sigma \omega_n) \in B_0(R)\) we have \(|\phi \sigma \omega_j| \leq R\). The intersection of a ball with radius \(R\) and a hyperplane in \(\mathbb{R}^d\) is a ball in some \(\mathbb{R}^{d-1}\) with radius \(R' \leq R\) and thus, lies in a cube of edge length \(2R\). Thus, this set belongs to the class Lip\((d, 1, 2R)\). Moreover, its \(d\)-dimensional volume is zero. Hence, by Theorem 3.1 we obtain the upper bound
\[
\ll \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i}
\]
for the number of \(\phi \sigma \omega_j\) for each \(j\) satisfying \(2 \leq j \leq n\).

Next we have to estimate the number of \(\phi \sigma \omega_1\). Again, we have \(|\phi \sigma \omega_1| \leq R\). Now by virtue of Theorem 3.1 we deduce the following upper bound
\[
\ll \frac{R^d}{\det \phi \sigma \mathcal{O}} + \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i}
\]
for the number of \(\phi \sigma \omega_1\). Going right up to the last minimum, we see that this is bounded by
\[
\ll \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i}.
\]
Multiplying the bounds for the number of \(\phi \sigma \omega_1\) and \(\phi \sigma \omega_j\), and then summing over all (of the at most \(2^d\)) strict subfields \(K_1\) of \(K\) leads to
\[
|\Lambda'| \ll \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \left( \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \right)^{n-1}.
\]
This completes the proof. \(\Box\)

12. Counting Projectively Primitive Points

The height of an element \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}^n \subset \mathcal{O}_K^n\) is given by
\[
H(\alpha) = \prod_{i=1}^{r+s} |(1, \sigma_i(\alpha))|_{\infty}^{d_i/d}.
\]
Therefore, and by the definition (4.1) of \(Z_f(X^d)\), we have
\[
(12.1) \quad N(\mathcal{O}_K^n(K/k), X) = |Z_f(X^d) \cap \sigma \mathcal{O}^n(K/k)|.
\]
Recall the definitions of \(Z_{F_1}, \psi, \psi_1\) and \(F\) from (6.1), (7.1), (7.5) and (9.1). Also recall that \(q' = |I| - 1\). We permute the coordinates of \(\sigma \mathcal{O}^n\) and \(\sigma \mathcal{O}^n(K/k)\) as in (5.1), so that they become subsets of \(\prod_{I} K_{F_1}^\circ \times \prod_{F} K_{I}^\circ\). Just as in (9.5) we conclude
\[
(12.2) \quad |\sigma \mathcal{O}^n(K/k) \cap Z_f(T)| = \begin{cases} |\psi Z_f(T) \cap \psi \sigma \mathcal{O}^n(K/k)| & \text{if } q' = 0 \\ \sum_{m_F} |\psi Z_{F_1} \cap \psi \sigma \mathcal{O}^n(K/k)| & \text{if } q' > 0. \end{cases}
\]
Of course, the first equation in (12.2) holds always, although we use it only for $q’ = 0$. It is well known that $\sigma O^n$ is a lattice of determinant
\[
\det \sigma O^n = (2^{-s} \sqrt{\Delta_K|O_K : O|})^n.
\]

**Proposition 12.1.** Suppose $T \geq 1$ and $n > 1$. If $q’ = 0$ then we have
\[
\left| \psi \sigma O^n(K/k) \cap \psi Z_1(T) \right| - \frac{2^{s\kappa n} \text{Vol}(Z_1(T))}{(\sqrt{\Delta_K|O_K : O|})^n} \leq \frac{T^{n-1/d}}{\lambda_1^{n(l-1)} \lambda_1^{n(d-l+1)-1}},
\]
where $\lambda_i = \lambda_i(\sigma O)$. If $q’ > 0$ then we have
\[
\left| \psi \sigma O^n(K/k) \cap \psi Z_1(T) \right| - \frac{2^{s\kappa n} \text{Vol}(Z_1(T))}{(\sqrt{\Delta_K|O_K : O|})^n} \leq \frac{T^{n-1/d}}{\lambda_1^{n(l-1)} \lambda_1^{n(d-l+1)-1}}.
\]
where $\lambda_i = \lambda_i(\psi \sigma O)$.

**Proof.** As the case $q’ = 0$ can be proven by exactly the same arguments we restrict ourselves to the case $q’ > 0$. Let us write $R = \kappa T^{1/d}$, where $\kappa$ is as in Lemma 9.1 and thus $R \ll T^{1/d}$, and
\[
\psi Z_1 \subset B_0(R).
\]
Put $\Lambda = \psi \sigma O^n$, and recall that $\psi \in T$. The proof splits in two cases. First we assume
\[
R < \lambda_1.
\]
By Lemma 10.5 and recalling the definition (3.1), we conclude $\psi \sigma O^n(K/k) \subset \Lambda(l)$. As $\psi Z_1 \subset B_0(R)$ we get in particular $0 = |\Lambda(l) \cap \psi Z_1| = |\psi \sigma O^n(K/k) \cap \psi Z_1|$. Using Lemma 9.4, det $\psi = 1$, and applying Corollary 3.1 proves the proposition in the first case. Now we assume
\[
R \geq \lambda_1.
\]
First we ignore the primitivity condition defining $O^n(K/k)$ and we count all points in $\Lambda(l) \supset \psi \sigma O^n(K/k)$. Again, using Lemma 9.4 and applying Corollary 3.1 yields
\[
\left| \Lambda(l) \cap \psi(Z_1(T)) \right| - \frac{2^{s\kappa n} \text{Vol}(Z_1(T))}{(\sqrt{\Delta_K|O_K : O|})^n} \leq \frac{T^{n-1/d}}{\lambda_1^{n(l-1)} \lambda_1^{n(d-l+1)-1}}.
\]
Next we estimate the number of points in $\Lambda(l) \cap \psi(Z_1(F))$ that do not generate $K/k$ (in the projective sense), i.e., that do not lie in $\psi \sigma O^n(K/k)$. To this end we apply Lemma 11.1 Using $R \geq \lambda_1$ we get the following upper bound for these
\[
\ll \max_{0 \leq i \leq d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \left( \max_{0 \leq i < d} \frac{R^i}{\lambda_1 \cdots \lambda_i} \right)^{n-1} \leq \frac{R^d}{\lambda_1^{l-1} \lambda_1^{d-l+1}} \left( \frac{R^{d-1}}{\lambda_1^{l-1} \lambda_1^{d-l}} \right)^{n-1}.
\]
As $n > 1$ we see that the latter is
\[
\leq \frac{R^{dn-1}}{\lambda_1^{n(l-1)} \lambda_1^{n(d-l+1)-1}} \leq \frac{T^{n-1/d}}{\lambda_1^{n(l-1)} \lambda_1^{n(d-l+1)-1}}.
\]
This concludes the proof of the proposition. \hfill \Box

Recall the definitions of $\eta_0$ and $\mu_g$ from (1.6) and (1.7) respectively.

**Lemma 12.2.** Suppose $X \geq 1$ and $n > 1$. If $q’ = 0$ then
\[
\left| \psi \sigma O^n(K/k) \cap \psi Z_1(T) \right| - \frac{2^{s\kappa n} \text{Vol}(Z_1(T))}{(\sqrt{\Delta_K|O_K : O|})^n} \ll \sum_{g \in G(K/k)} \eta_0^{dn-1} \delta_g(K/k)^{\mu_g}.
\]
If $q' > 0$ then
\[
|\psi_1 \sigma \mathcal{O}^n(K/k) \cap \psi_1 Z_f| - \frac{2^{s_K n} \text{Vol} Z_f}{(\sqrt{\Delta_K} ||\mathcal{O}_K : \mathcal{O}||)^n} \leq \sum_{g \in G(K/k)} \frac{T^{n-1/d}}{\eta g^{d n-1} \delta g(K/k)^{\mu g}}.
\]

**Proof.** Recall that $\psi$ and $\psi_1$ are in $T$, and thus, to estimate the successive minima we can apply the results from Section 10 with $\phi = \psi_1$ and $\phi = \psi$ respectively. Let $K_0 = k(\theta_1, \ldots, \theta_l)$ if $l \geq 2$, and let $K_0 = k$ if $l = 1$, and put $g = [K_0 : k]$. In particular, we have $g \in G(K/k)$. Therefore, and by Proposition 12.1 it suffices to show
\[
\left| \lambda^{n(l-1)} l^{n(d-l+1)-1} \right| \gg \eta g^{d n-1} \delta g(K/k)^{\mu g}.
\]

First suppose $l = l(\phi) \geq 2$. Then by Lemma 10.6 we have $n(d - l + 1) - 1 \geq \mu g$, and thus, (12.3) follows immediately from Lemma 10.2. Now suppose $l = 1$. Then $\delta g(K/k) = 1$ and thus, (12.3) follows again from Lemma 10.2. This proves the lemma. \(\square\)

### 13. **Proof of Theorem 1.2**

We start with the case $n = 1$. Hence, by hypothesis, we have $k = K$. From (12.1) and since $0 \not\in Z_f(X^d)$ we obtain
\[
N(\mathcal{O}_f(K/K), X) = |\sigma \mathcal{O}(K/K) \cap Z_f(X^d)| = |\sigma \mathcal{O} \cap Z_f(X^d)|.
\]

Applying Theorem 4.1 with $\Lambda = \sigma \mathcal{O}$ and using Lemma 10.1 yields
\[
N(\mathcal{O}_f(K/K), X) \sim \frac{2^{s_K n} \text{Vol} Z_f(X^d)}{(\sqrt{\Delta_K} ||\mathcal{O}_K : \mathcal{O}||)^n} \leq c(1, d) \frac{(\log X)^q X^{d-1}}{\eta g^{d n-1}}.
\]

This proves Theorem 1.2 for $n = 1$.

Now we assume $n > 1$. Combining Lemma 12.2, 12.1 and 12.2 yields for $q' = 0$
\[
N(\mathcal{O}_f^n(K/k), X) \sim \frac{2^{s_K n} \text{Vol} Z_f(X^d)}{(\sqrt{\Delta_K} ||\mathcal{O}_K : \mathcal{O}||)^n} \leq \sum_{g \in G(K/k)} \frac{X^{d n-1}}{\eta g^{d n-1} \delta g(K/k)^{\mu g}}.
\]

For $q' > 0$ we additionally use (9.4) to get
\[
N(\mathcal{O}_f^n(K/k), X) \sim \frac{2^{s_K n} \text{Vol} Z_f(X^d)}{(\sqrt{\Delta_K} ||\mathcal{O}_K : \mathcal{O}||)^n} \leq \sum_{m_F g \in G(K/k)} \sum \frac{X^{d n-1}}{\eta g^{d n-1} \delta g(K/k)^{\mu g}}.
\]

By Lemma 9.6 we know $|m_F| \ll (\log X)^q$, and this completes the proof of Theorem 1.2.

### 14. **Proof of Corollary 1.1**

Recall that $\mathcal{O}^n_0(K/k) = \emptyset$, and thus $N(\mathcal{O}^n(K/k), X) = \sum_f N(\mathcal{O}_f^n(K/k), X)$, where the sum runs over all subsets of $\{1, \ldots, r_K + s_K\}$. Also recall the definition of $Z_0(X^d)$ from (11). As the $2^{r+s}$ sets $Z_f(T)$ define a partition of $Z(T)$ we see that Corollary 1.1 follows immediately from Theorem 1.2 and the following lemma.

**Lemma 14.1.** Suppose $X \geq 1$ and either $n > 1$ or $K = k$. Then
\[
N(\mathcal{O}^n_0(K/k), X) \sim \frac{2^{s_K n} \text{Vol} Z_0(X^d)}{(\sqrt{\Delta_K} ||\mathcal{O}_K : \mathcal{O}||)^n} \leq \sum_{g \in G(K/k)} \frac{1}{\eta g^{d n-1} \delta g(K/k)^{\mu g}}.
\]
Lemma 15.1. \(\text{Suppose } l_{10.1}. \text{ If }\) of the set \(\{V\}\), then \(\text{Let }\)
\[\sqrt{|\Delta_K| |O_K : O|} \gg n_0^d \delta_g(K/k)^{m(e-g)}.\]

(14.1)

Let \(\phi\) be the identity on \(\mathbb{R}^r \times \mathbb{C}^s\), let \(\lambda_i\) be as in (10.2), and let \(l\) be as in Definition 2. Then
\[\sqrt{|\Delta_K| |O_K : O|} \gg \lambda_1 \cdots \lambda_d \geq \lambda_i^{l-1} \lambda_i^{l-t+1}.\]

If \(l = 1\) then \(K = k\) and \(\delta_g(K/k) = 1\), so that (14.1) follows from the above and Lemma 10.1. If \(l \geq 2\) we take \(g = [k(\theta_1/\theta_1, \ldots, \theta_{l-1}/\theta_1) : k] \in G(K/k)\). Applying Lemma 10.1, Lemma 10.2 and Lemma 10.6 yields (14.1), and thereby proves the lemma. \(\square\)

15. Volume computations

Lemma 15.1. \(\text{Suppose } q' \geq 0 \text{ and } T \geq 1. \text{ Then we have}\)
\[\text{Vol}Z(T) = 2^{rn} \pi^{sn} (-1)^q \left( -1 + T^n \sum_{i=0}^{q} \frac{(-\log T^n)!}{i!} \right).\]

Proof. Put \(r' = |I \cap \{1, \ldots, r\}|\) and \(s' = |I \cap \{r+1, \ldots, r+s\}|\). From (5.1) we see that \(\text{Vol}Z(T)\) is given by the product of \(2^{(r-r')n} \pi^{(s-s')n}\) and the \(d'n\)-dimensional volume of the set \(\{z_i\}_{i \in I} \prod_{i \in I} |z_i|_\infty \leq T, |z_i|_\infty \geq 1 \text{ for } i \in I\). Denote the latter by \(V_{r', s'}(T)\). Clearly, we have \(V_{0,1}(T) = \pi^n(T^n - 1)\), and Fubini’s Theorem implies
\[V_{0, s'}(T) = \int_{1 \leq |z_s|_\infty \leq \sqrt{T}} V_{0, s'-1}(T/|z_s|_\infty^2) dz_s\]
\[= n \int_{1 \leq |z_s|_\infty \leq \sqrt{T}} \int_{0 \leq |z_s|_\infty^2 \leq |z_s|_\infty} \cdots \int_{0 \leq |z_s|_\infty \leq |z_s|_\infty} V_{0, s'-1}(T/|z_s|_\infty^2) dz_{s1} \cdots dz_{s1}\]
\[= n \int_{1 \leq |z_s|_\infty \leq \sqrt{T}} (\pi |z_s|_\infty^2)^{n-1} V_{0, s'-1}(T/|z_s|_\infty^2) dz_{s1}\]
\[= n \int_{1}^{\sqrt{T}} \int_{0}^{2\pi} (\pi \theta^2)^{n-1} V_{0, s'-1}(T/\theta^2) d\theta d\theta\]
\[= 2\pi^n \int_{1}^{\sqrt{T}} \theta^{2n-1} V_{0, s'-1}(T/\theta^2) d\theta.\]
By induction we conclude
\[V_{0, s'}(T) = \pi^{s'} \left( (-1)^{s'} + \sum_{i=0}^{s'-1} \frac{(-1)^{s'-1-i} n_i}{i!} T^n \log T \right).\]

Again, by Fubini’s Theorem we find
\[V_{r', s'}(T) = \int_{1 \leq |z_s|_\infty \leq T} V_{r'-1, s'}(T/|z_s|_\infty) dz_r\]
\[= 2^n \int_{1}^{T} z_r^{n-1} V_{r'-1, s'}(T/z_r) dz_r.\]
Once more a simple induction argument shows
\[ V_{r',s'}(T) = 2^{-n\pi s'}n \left( (-1)^{q'-1} + \sum_{i=0}^{q'} \frac{(-1)^{q'-i}n^i}{i!} T^n (\log T)^i \right) \]
\[ = 2^{-n\pi s'}n (-1)^{q'} \left( -1 + T^n \sum_{i=0}^{q'} \frac{(-\log T^n)^i}{i!} \right). \]
As \( \text{Vol}_{Z} (T) = 2^{(r-r')n}n^{(s-s')}n V_{r',s'}(T) \) the lemma is proved.

**Lemma 15.2.** Suppose \( T \geq 1 \). Then we have
\[ \text{Vol}_{Z} (T) = \sum_{i=0}^{q} c_i T^n (\log T)^i, \]
where
\[ c_i = \frac{2^{-n\pi s}n^i}{i!} \binom{q}{i}. \]

**Proof.** Clearly, we have \( \text{Vol}_{Z} (T) = \sum_{I} \text{Vol}_{Z} (T) \), where the sum runs over all subsets \( I \) of \( \{1, \ldots, r+s\} \). Now in order to compute the coefficient \( c_i \) we have to sum the contribution from each \( \text{Vol}_{Z} (T) \). First note that
\[ 2^{-n\pi s}n \sum_{i} (-1)^{q'+1} = 2^{-n\pi s}n \sum_{j=0}^{q'+1} (-1)^j \binom{q+1}{j} = 0. \]
It remains to compute the coefficients \( c_i \). The contribution of \( \text{Vol}_{Z} (I) \) is zero if \( q' = |I| - 1 < i \), and
\[ 2^{-n\pi s}n (-1)^{q'+i} \]
if \( q' \geq i \). As we have \( \binom{q'+1}{i} \) sets \( I \) of cardinality \( q' + 1 \) we conclude
\[ c_i = \frac{2^{-n\pi s}n}{i!} \sum_{q'=i}^{q} (-1)^{i+q'} \binom{q+1}{q'+1} = \frac{2^{-n\pi s}n}{i!} \binom{q}{i}. \]
This concludes the proof of the lemma.

\[ \square \]

**16. Upper bounds for the non-projectively primitive points**

Recall the definition of the set of non-projectively primitive points in \( \mathcal{O}_K^n \)
\[ \mathcal{O}_{npp}(K/k) = \{ \alpha \in \mathcal{O}_K^n \setminus \mathcal{O}_K^n(K/k); k(\alpha) = K \}. \]
Let \( k(n;e) \) be the subset of \( \mathcal{O}_K^n \) of points \( \alpha \) with \( [k(\alpha) : k] = e \). Schmidt [27] Theorem has shown the following estimate:

\[ (16.1) \quad N(k(n;e), X) \leq c_2 (m, e, n) X^{me(n+e)}, \]
where \( c_2 (m, e, n) = 2^{me(n+e+3)+e^2+n^2+10e+10n} \).

**Lemma 16.1.** Suppose \( e > 1 \). Then we have
\[ \sum_{C_k} N(\mathcal{O}_{npp}(K/k), X) \ll \sup_{g|e} X^{m(g^2+gn+e^2)/g+e}, \]
where the supremum runs over all positive divisors $g < e$ of $e$. Moreover, for $e = 1$ (and $X \geq 1$) we have

$$\sum_{C_{\ell}(k)} N(\mathcal{O}_{npp}^m(K/k), X) = 1.$$

**Proof.** If $e = 1$ then $C_{\ell}(k) = \{k\}$ and $\mathcal{O}_{npp}^m(k/k) = \emptyset$. As $X \geq 1$ the lemma holds. From now on we assume $e > 1$. Then the left-hand side counts points $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathcal{O}_k(e; n)$ with $k(\ldots, \alpha_i/\alpha_j, \ldots) \leq k(\alpha)$ and $H(\alpha) \leq X$. First suppose $n = 1$. Then the left-hand side simply counts algebraic integers of degree $e$ over $k$ and height no larger than $X$. The number of these is by (16.1)

$$\leq c_2(m, e, 1)X^{me(e+1)} \ll \sup_{g|e} X^{m(g^2+gn+e^2/g+e)}.$$

This proves the lemma for $n = 1$. Now we assume $n > 1$. As $e > 1$ each $\alpha$ is nonzero, and so we loose only a factor $n$ if we assume $\alpha_1 \neq 0$. Under this assumption $\alpha$ has the form $\alpha = (\theta, \theta\beta_2, \ldots, \theta\beta_n)$ such that with $F = k(\ldots, \alpha_i/\alpha_j, \ldots)$ one has: $k(\alpha) = F(\theta)$ and $k(\beta_2, \ldots, \beta_n) = F$. Furthermore, we have

$$X \geq H(\alpha) = H(\theta, \theta\beta_2, \ldots, \theta\beta_n) = H(1/\theta, \beta_2, \ldots, \beta_n) \geq \max\{H(\theta), H(\beta_2, \ldots, \beta_n)\}.$$

Therefore, it suffices to give an upper bound for the number of $((\beta_2, \ldots, \beta_n, \theta) \in \mathbb{F}^n$ with

$$\begin{align*}
|k(\beta_2, \ldots, \beta_n) : k| &= g \leq e/2, \\
|k(\theta, \beta_2, \ldots, \beta_n) : k(\beta_2, \ldots, \beta_n)| &= e/g, \\
H(\beta_2, \ldots, \beta_n), H(\theta) &\leq X.
\end{align*}$$

Let us fix a $g$ as above. From (16.1) we obtain the upper bound

$$c_2(m, g, n-1)X^{mg(g+n)}$$

for the number of such vectors $(\beta_2, \ldots, \beta_n)$. Next for each $(\beta_2, \ldots, \beta_n)$ we count the number of $\theta$. Now we have $|k(\theta, \beta_2, \ldots, \beta_n) : k(\beta_2, \ldots, \beta_n)| = e/g$, and, moreover, $H(\theta) \leq X$. Applying (16.1) once more yields the upper bound

$$c_2(mg, e/g, 1)X^{kg(\beta_2, \ldots, \beta_n) : \mathbb{Q}[e/g]}(c/g)X^{me(e/g+1)} \ll X^{me(e/g+1)}$$

for the number of $\theta$, provided $(\beta_2, \ldots, \beta_n)$ is fixed. Multiplying the bound (16.2) for the number of $(\beta_2, \ldots, \beta_n)$ and (16.3) for the number of $\theta$ gives the upper bound

$$\ll X^{mg(e^2/g+e)}$$

for the number of tuples $(\beta_2, \ldots, \beta_n, \theta)$. Taking the supremum over all possible values of $g$ proves the lemma. \qed

### 17. Proof of Theorem 1.1

We start with a simple lemma. Put

$$\gamma_g = m(g^2 + g + e^2/g + e).$$

We remind the reader that $\mu_g = mn(e - g) - 1$ and $C_{e,m} = \max\{2 + \frac{4}{e-1} + \frac{1}{m(e-1)}, 7 - \frac{e}{2} + \frac{2}{m(e-1)}\}$.

**Lemma 17.1.** Suppose $e > 1$, $n > e + C_{e,m}$ and $1 \leq g \leq e/2$. Then we have

$$\begin{align*}
\gamma_g - \mu_g &\leq -2/e, \\
m(g^2 + gn + e^2/g + e) &\leq men - 1, \\
(e + 2)/4 - n/2 &\leq -C_{e,m}/2.
\end{align*}$$
Proof. Let us write (17.2) as

\[ m(g^2 + g + e^2/g + e) - mn(e - g) + 1 + 2/e \leq 0. \]

With

\[ F(g) = \frac{g^2 + g + e^2/g + e}{e - g} + \frac{1}{m(e - g)}, \]

this means

(17.5)

\[ n \geq F(g) + \frac{2}{me(e - g)}. \]

As \( F(g) \) is a fraction with denominator dividing \( mg(e - g) \) we conclude that \( n > F(g) \) implies \( n \geq F(g) + \frac{1}{mg(e - g)} \geq F(g) + \frac{2}{me(e - g)} \). Hence, it suffices to check \( n > F(g) \).

Using that \( (e - g)e^2/g^3 \geq e^2/g^2 \) for \( 1 \leq g \leq e/2 \), one sees that the second derivative \( F''(g) \) is positive for \( 1 \leq g \leq e/2 \). Hence, \( F(g) \) is here concave, and so it suffices to check that \( n > F(1) \) and \( n > F(e/2) \), which is equivalent to our hypothesis \( n > e + C_{e,m} \). The claim (17.3) is equivalent to

(17.6)

\[ n \geq F(g) - \frac{g}{e - g}. \]

But we have just seen that (17.5) holds and thus (17.6) holds as well. And, finally, (17.4) follows from the assumptions \( n > e + C_{e,m} \) and \( e > 1 \). This proves the lemma. \( \square \)

We have the following disjoint union

\[ \mathcal{O}_k(n; e) = \bigcup_{C_e(k)} \mathcal{O}_K^n(K/k) \cup \mathcal{O}_{npp}^n(K/k). \]

Therefore,

(17.7)

\[ N(\mathcal{O}_k(n; e), X) = \sum_{C_e(k)} N(\mathcal{O}_K^n(K/k), X) + \sum_{C_e(k)} N(\mathcal{O}_{npp}^n(K/k), X). \]

Combining Lemma [16.1] and Lemma [17.1] shows that for \( e > 1 \) and \( n > e + C_{e,m} \)

\[ \sum_{C_e(k)} N(\mathcal{O}_{npp}^n(K/k), X) \ll X^{men^{-1}}. \]

But by Lemma [16.1] the latter remains trivially true for \( e = 1 \). Therefore, we may focus on the first sum in (17.7). By virtue of Corollary [1.1] and Lemma [15.2] it suffices to show that the following sums converge

(17.8)

\[ \sum_{C_e(k)} |\Delta_K|^{-n/2}, \]

(17.9)

\[ \sum_{C_e(k)} \sum_{g \in G(K/k)} \delta_g(K/k)^{-\mu_g}. \]

First suppose \( e = 1 \). Then \( C_e(k) = \{k\} \) consists of a single field, and, hence, both sums converge. Next we assume \( e > 1 \), and thus by hypothesis \( n > e + C_{e,m} \). Let us start with the sum in (17.8). Let

\[ N_\Delta(C_e(k), T) = |\{K \in C_e(k); |\Delta_K| \leq T\}| \]

be the number of fields in \( C_e(k) \) with discriminant no larger than \( T \) in absolute value. Schmidt [29] has shown that

(17.10)

\[ N_\Delta(C_e(k), T) \leq c(k, e)T^{(e+2)/4}. \]
Ellenberg and Venkatesh [10] have established a better bound for large values of $e$. However, for our purpose Schmidt’s bound is good enough. A simple dyadic summation argument proves the desired convergence. More precisely,

$$
\sum_{\mathcal{C}_e(k)} |\Delta_K|^{n/2} = \sum_{i=1}^{\infty} \sum_{k \in \mathcal{C}_e(k)} |\Delta_K|^{n/2} \leq \sum_{i=1}^{\infty} \frac{N\Delta(C_e(k), 2^i)}{2^{(i-1)n/2}}
$$

$$
\leq c(k, e) \sum_{i=1}^{\infty} \frac{2^{i(e+2)/4}}{2^{(i-1)n/2}} = c(k, e)2^{n/2} \sum_{i=1}^{\infty} 2^{i(e+2)/4 - n/2}.
$$

By (17.4) we have $(e + 2)/4 - n/2 \leq -C_{e,m}/2 < 0$. Therefore, the last sum converges, and this proves the convergence of (17.8).

To deal with the sum (17.9) we need some more notation and an analogue of (17.10) for the counting function associated to $\delta_g$. We define

$$
G_u = \bigcup_{\mathcal{C}_e(k)} G(K/k).
$$

Clearly, $G_u \subset \{1, \ldots, \lceil e/2 \rceil\}$. Now for any $g \in G_u$ we define

$$
\mathcal{C}_e^{(g)}(k) = \{K \in \mathcal{C}_e(k); g \in G(K/k)\}
$$

and its counting function

$$
N_{\delta_g}(\mathcal{C}_e^{(g)}(k), T) = |\{K \in \mathcal{C}_e^{(g)}(k); \delta_g(K/k) \leq T\}|.
$$

**Lemma 17.2.** For $g$ in $G_u$, and $\gamma_g$ as in (17.1) we have

$$
N_{\delta_g}(\mathcal{C}_e^{(g)}(k), T) \ll T^{\gamma_g}.
$$

**Proof.** Since $H(\alpha_1, \alpha_2) \geq \max\{H(\alpha_1), H(\alpha_2)\}$ it suffices to show that the number of tuples $(\alpha_1, \alpha_2) \in \R^2$ with

$$
[k(\alpha_1) : k] = g,
$$

$$
[k(\alpha_1, \alpha_2) : k(\alpha_1)] = e/g,
$$

$$
H(\alpha_1), H(\alpha_2) \leq T
$$

is $\ll T^{\gamma_g}$. But the latter can be shown exactly in the same manner as in the proof of Lemma 16.1. \hfill \Box

Now we can show the convergence of (17.9). We proceed similar as for (17.8).

$$
\sum_{K \in \mathcal{C}_e(k)} \sum_{g \in G(K/k)} \delta_g(K/k)^{-\mu_g} = \sum_{g \in G_u} \sum_{K \in \mathcal{C}_e^{(g)}(k)} \delta_g(K/k)^{-\mu_g}
$$

$$
= \sum_{g \in G_u} \sum_{i=1}^{\infty} \sum_{K \in \mathcal{C}_e^{(g)}(k)} \delta_g(K/k)^{-\mu_g}
$$

$$
\leq \sum_{g \in G_u} \sum_{i=1}^{\infty} N_{\delta_g}(\mathcal{C}_e^{(g)}(k), 2^i)
$$

$$
\ll \sum_{g \in G_u} \sum_{i=1}^{\infty} 2^{i(\gamma_g - \mu_g)}.
$$

By (17.2) we have $\gamma_g - \mu_g \leq -2/e$, and this proves the convergence of (17.9). Therefore, the proof of Theorem 1.1 is complete.
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REFERENCES

1. B. J. Birch, *Forms in many variables*, Proc. London Math. Soc. 265A (1962), 245–263.
2. E. Bombieri, *Problems and results on the distribution of algebraic points on algebraic varieties*, J. Théor. Nombres Bordeaux 21 (2009), 41–57.
3. E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge University Press, 2006.
4. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer, 1997.
5. A. Chambert-Loir and Y. Tschinkel, *Integral points of bounded height on toric varieties*, preprint (2010).
6. Integral points of bounded height on partial equivariant compactifications of vector groups, Duke Math. J. 161 (2012), 2799–2836.
7. S-J. Chern and J. D. Vaaler, *The distribution of values of Mahler’s measure*, J. reine angew. Math. 540 (2001), 1–47.
8. H. Davenport, *On a principle of Lipschitz*, J. London Math. Soc. 26 (1951), 179–183.
9. W. Duke, Z. Rudnick, and P. Sarnak, *Density of integer points on affine homogeneous varieties*, Duke Math. J. 71 (1993), 143–179.
10. J. Ellenberg and A. Venkatesh, *The number of extensions of a number field with fixed degree and bounded discriminant*, Ann. of Math. 163 (2006), 723–741.
11. A. Eskin and C. McMullen, *Mixing, counting and equidistribution on Lie groups*, Duke Math. J. 71 (1993), 181–209.
12. A. Eskin, S. Mozes, and N. Shah, *Unipotent flows and counting lattice points on homogeneous varieties*, Ann. of Math. 143 (1996), 253–299.
13. J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. 95 (1989), 421–435.
14. C. Frei, *On rings of integers generated by their units*, Bull. London Math. Soc. 44 (2012), 167–182.
15. C. Frei and M. Widmer, *Schanuel’s theorem for heights defined via extension fields*, submitted (2013), 33 pages.
16. C. Fuchs, R. Tichy, and V. Ziegler, *On quantitative aspects of the unit sum number problem*, Arch. Math. 93 (2009), 259–268.
17. X. Gao, *On Northcott’s Theorem*, Ph.D. Thesis, University of Colorado (1995).
18. M. Hindry and J. H. Silverman, *Diophantine Geometry An Introduction*, Springer, 2000.
19. S. Lang, *Fundamentals of Diophantine Geometry*, Springer, 1983.
20. , *Algebra*, Springer, 2002.
21. D. W. Masser and J. D. Vaaler, *Counting algebraic numbers with large height I*, Diophantine Approximation - Festschrift für Wolfgang Schmidt (eds. H. P. Schlickewei, K. Schmidt, R. F. Tichy), Developments in Mathematics 16, Springer 2008, (pp.237–243).
22. *Counting algebraic numbers with large height II*, Trans. Amer. Math. Soc. 359 (2007), 427–445.
23. W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, Springer, 1990.
24. D. Roy and J. L. Thunder, *A note on Siegel’s lemma over number fields*, Monatsh. Math. 120 (1995), 307–318.
25. S. H. Schanuel, *Heights in number fields*, Bull. Soc. Math. France 107 (1979), 433–449.
26. W. M. Schmidt, *Small zeros of additives forms in many variables. II*, Acta Math. 143 (1979), 219–232.
27. , *Northcott’s Theorem on heights I. A general estimate*, Monatsh. Math. 115 (1993), 169–183.
28. , *Northcott’s Theorem on heights II. The quadratic case*, Acta Arith. 70 (1995), 343–375.
29. , *Number fields of given degree and bounded discriminant*, Astérisque 228 (1995), 189–195.
30. J. Silverman, *Lower bounds for height functions*, Duke Math. J. 51 (1984), 395–403.
31. M. Widmer, *Counting points of fixed degree and bounded height*, Acta Arith. 140.2 (2009), 145–168.
32. , *Counting points of fixed degree and bounded height on linear varieties*, J. Number Theory 130 (2010), 1763–1784.
33. , *Counting primitive points of bounded height*, Trans. Amer. Math. Soc. 362 (2010), 4793–4829.
34. _____, *Asymmetric inequalities for inhomogeneous algebraic linear forms*, in preparation (2012).
35. _____, *Lipschitz class, narrow class, and counting lattice points*, Proc. Amer. Math. Soc. **140** (2012), 677–689.

**Department of Mathematics, Royal Holloway, University of London, TW20 0EX Egham, UK**

*E-mail address: martin.widmer@rhul.ac.uk*