Reduction, reconstruction, and skew-product decomposition of symmetric stochastic differential equations

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Abstract

We present reduction and reconstruction procedures for the solutions of symmetric stochastic differential equations, similar to those available for ordinary differential equations. Additionally, we use the local tangent-normal decomposition, available when the symmetry group is proper, to construct local skew-product splittings in a neighborhood of any point in the open and dense principal orbit type. The general methods introduced in the first part of the paper are then adapted to the Hamiltonian case, which is studied with special care and illustrated with several examples. The Hamiltonian category deserves a separate study since in that situation the presence of symmetries implies in most cases the existence of conservation laws, mathematically described via momentum maps, that should be taken into account in the analysis.

Keywords: stochastic differential equation, symmetry, symmetry reduction, reconstruction, skew-product decomposition, Hamiltonian stochastic differential equation.

1 Introduction

Symmetries have historically played a role of paramount importance in the study of dynamical systems in general (see [GS85, GS02, ChL00], and references therein) and of physical, mechanical, and Hamiltonian systems in particular (see for instance [AM78, MR99, OR04] for general presentations of the subject, historical overviews, and references). The presence of symmetries in a system usually brings in its wake the occurrence of degeneracies, conservation laws, and invariance properties that can be used to simplify or reduce the system and hence its analysis. In trying to pursue this strategy, researchers have developed powerful mathematical tools that optimize the benefit of this approach in specific situations.

The impressive volume of work that has been done in this field over the centuries does not have a counterpart in the context of stochastic dynamics, probably because most symmetry based mathematical tools are formulated using global analysis and Lie theory in an essential way, and this machinery has been adapted to the stochastic context relatively recently [MS1, MS2, Sch82, ES2, ES9]. As we will show in this paper, most of the symmetry based techniques available for dynamical systems can be formulated and taken advantage of when studying stochastic differential equations.

In a first approach, symmetry based techniques can be roughly grouped into two separate procedures, namely, reduction and reconstruction. Reduction is explicitly implemented by combining the restriction of the system to dynamically invariant submanifolds whose existence is implied by its symmetries and...
by eliminating the remaining symmetry degeneracies through projection to an appropriate orbit space. Even if the space in which the system is originally formulated is Euclidean, the resulting reduced space is most of the time a non-Euclidean manifold hence showing the importance of global analysis in this context. The reduction procedure yields a dimensionally smaller space in which the symmetry degeneracies have been eliminated and that should, in principle, be easier to study; in the stochastic context, reduction has the added value of being able in some instances to isolate the non-stochastic part of the dynamics (see the example on collective motion in Section 7.1).

If once the reduced system has been solved we want to come back to the original one, we need to reconstruct the reduced solutions. In practice, this is obtained by horizontally lifting the reduced motion using a connection and then correcting the result with a curve in the group that satisfies a certain first order differential equation. The strategy of combining reduction and reconstruction in the search for the solutions of a symmetric dynamical system, splits the task into two parts, which most of the time simplifies greatly the problem.

Another approach used to take advantage of the symmetries of a problem consists of using the Slice Theorem \[P61\] and the tangent-normal decomposition \[K90, F91\] available for proper group actions to locally split the dynamics into a direction tangent to the group orbits and another one transversal to them. We will see that this tool, that is used in a standard fashion in the context of deterministic equivariant dynamics and equivariant bifurcation theory, yields in the stochastic case skew-product splittings that have already been extensively studied in the equivariant diffusions literature (see for instance \[PR88, L89, T92\], and references therein) to construct decompositions of the associated second order differential operators.

It must be noticed that the mathematical value of the results obtained with the two approaches that we just briefly discussed, that is, the one based on reduction-reconstruction and the one based on the tangent-normal decomposition, is morally the same. However, there are important technical conditions that make them different and preferable over one another in different specific situations:

(i) The reduction-reconstruction technique uses very strongly the orbit space of the symmetry group in question; this space could be geometrically convoluted and we may need to use only its strata if we want to face regular quotient manifolds where the standard calculus on manifolds is valid.

The main advantage of this technique is that it yields global results.

(ii) The use of the Slice Theorem and the tangent-normal decomposition makes unnecessary the use of quotient manifolds and the entire analysis takes place in the original manifold. However, the results obtained are local and are limited to a tubular neighborhood of the orbits.

In this paper we show how the symmetries of stochastic differential equations can be used by implementing techniques similar to those available for their deterministic counterparts. We start in Section 2 by introducing the notion of group of symmetries of a stochastic differential equation and by studying the associated invariant submanifolds as well as the implied degeneracies in the solutions. The reduction and reconstruction procedures are presented in Section 3; reconstruction is carried out using the horizontal lifts for semimartingales introduced by \[S82, C01\].

The skew-product decomposition of second order differential operators is a factorization technique that has been used in the stochastic processes literature in order to split the semielliptic and, in particular, the diffusion operators, associated to certain stochastic differential equations (see, for instance, \[PR88, L89, T92\], and references therein). This splitting has important consequences as to the properties of the solutions of these equations, like certain factorization properties of their probability laws and of the associated stochastic flows. In Section 4 we show that symmetries are a natural way to obtain this kind of decompositions. Our work extends the existing results in two ways: first, we generalize the notion of skew-product to arbitrary stochastic differential equations by working with the notion of skew-product decomposition of the Stratonovich operator. Obviously, our approach coincides with the
traditional one in the case of diffusions. Second, we use the Slice Theorem [P61] and the tangent-normal decomposition [K90, F91] to construct local skew-product decompositions in the presence of arbitrary proper symmetries (not necessarily free) in a neighborhood of any point in the open and dense principal orbit type. This result generalizes the skew-product decompositions presented in [ELL04] for regular free actions. Section 5 studies stochastic differential equations on associated bundles; in this situation the local skew-product splitting induced by the Slice Theorem is globally available.

Section 6 is dedicated to reduction and reconstruction in the stochastic Hamiltonian category. Stochastic Hamiltonian systems where introduced in [B81] and generalized in [LO07] to accommodate non-Euclidean phase spaces and stochastic components modeled by arbitrary semimartingales and not just Brownian motion. Given the generic non-Euclidean character of reduced spaces, the generalization in [LO07] is in this context of much relevance. It is worth mentioning that, as it was already the case for deterministic Hamiltonian systems, stochastic Hamiltonian systems are stable with respect to symplectic and Poisson reduction; in short, the reduction of a stochastic Hamiltonian system is again a stochastic Hamiltonian system. In Section 7 we present several (Hamiltonian) examples. The first one (Section 7.1) has to do with deterministic systems in which a stochastic perturbation is added using the conserved quantities associated to the symmetry (collective perturbation); such systems share the remarkable feature that symplectic reduction eliminates the stochastic part of the equation making the reduced system deterministic. In Section 7.2 we study the symmetries of stochastic mechanical systems on the cotangent bundles of Lie groups. In this situation, the reduction and reconstruction equations can be written down in a particularly explicit fashion that has to do with the Lie-Poisson structure in the dual of the Lie algebra of the group in question. A particular case of this is presented in Section 7.3 where we analyze two different stochastic perturbations of the free rigid body: one of them models the dynamics of a free rigid body subjected to small random impacts and the other one an ”unbolted” rigid body that is not completely rigid.

2 Symmetries and conservation laws of stochastic differential equations

Let $M$ and $N$ be two finite dimensional manifolds and let $(\Omega, F, \{F_t | t \geq 0\}, P)$ be a filtered probability space. Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semimartingale. Using the conventions in [ES9], a Stratonovich operator from $N$ to $M$ is a family $\{S(x,y)\}_{x,y \in M}$ such that $S(x,y) : T_y N \to T_x M$ is a linear mapping that depends smoothly on its two entries. Let $S^*(x,y) : T_x^* M \to T_y^* N$ be the adjoint of $S(x,y)$.

We recall that a $M$-valued semimartingale $\Gamma$ is a solution of the the Stratonovich stochastic differential equation

$$\delta \Gamma = S(X, \Gamma) \delta X$$

(2.1)

associated to $X$ and $S$, if for any $\alpha \in \Omega(M)$, the following equality between Stratonovich integrals holds:

$$\int \langle \alpha, \delta \Gamma \rangle = \int \langle S^*(X, \Gamma) \alpha, \delta X \rangle.$$

We will refer to $X$ as the noise semimartingale or the stochastic component of the stochastic differential equation (2.1). It can be shown [ES9] Theorem 7.21] that in this setup, given a $\mathcal{F}_0$ measurable random variable $\Gamma_0$, there are a stopping time $\zeta > 0$ and a solution $\Gamma$ of (2.1) with initial condition $\Gamma_0$ defined on the set $\{(t, \omega) \in \mathbb{R}_+ \times \Omega | t \in [0, \zeta(\omega))\}$ that has the following maximality and uniqueness property: if $\zeta'$ is another stopping time such that $\zeta' < \zeta$ and $\Gamma'$ is another solution defined on $\{(t, \omega) \in \mathbb{R}_+ \times \Omega | t \in [0, \zeta'(\omega))\}$, then $\Gamma'$ and $\Gamma$ coincide in this set. If $\zeta$ is finite then $\Gamma$ explodes at time $\zeta$, that is, the path $\Gamma_t$ with $t \in [0, \zeta)$ is not contained in any compact subset of $M$. If the manifold $M$ is
compact then all the solutions of any stochastic differential equation defined on $M$ are defined for all time. Since this is a hypothesis that we are not willing to adopt, the reader should keep in mind that all the solutions that we will work with are defined only up to a maximal stopping time, even if this is not explicitly mentioned.

We also recall that stochastic differential equations can be formulated using Itô integration by associating a natural \textit{Schwartz operator} $S : \tau_N \mapsto \tau_M$ on the second order tangent bundles, to the Stratonovich operator $S$; see \cite{ES9} and references therein for the details.

\textbf{Definition 2.1} Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semimartingale and let $S : TN \times M \to TM$ be a Stratonovich operator. Let $\phi : M \to M$ be a diffeomorphism. We say that $\phi$ is a symmetry of the stochastic differential equation (2.1) if for any $x \in N$ and $y \in M$

$$S(x, \phi(y)) = T_y\phi \circ S(x, y). \tag{2.2}$$

As it was already the case in standard deterministic context, the symmetries of a stochastic differential equation imply degeneracies at the level of its solutions, as we spell out in the following proposition.

\textbf{Proposition 2.2} Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semimartingale, $S : TN \times M \to TM$ a Stratonovich operator, and let $\phi : M \to M$ be a symmetry of the corresponding stochastic differential equation (2.1). If $\Gamma$ is solution of (2.1) then so is $\phi(\Gamma)$.

\textbf{Proof.} Let $\Gamma$ be a solution of (2.1). We need to show that for any $\alpha \in \Omega(M)$,

$$\int \langle \alpha, \delta \phi(\Gamma) \rangle = \int \langle S^*(X, \phi(\Gamma)) \alpha, \delta X \rangle.$$

Since $\phi$ is a diffeomorphism, $\int \langle \alpha, \delta \phi(\Gamma) \rangle = \int \langle \phi^*\alpha, \Gamma \rangle$ (see, for instance, \cite{ES9 §7.5}). Now, since $\Gamma$ is a solution of (2.1), $\int \langle \phi^*\alpha, \Gamma \rangle = \int \langle S^*(X, \Gamma) (\phi^*\alpha), \delta X \rangle$. Since $\phi$ is a symmetry, we have that $S^*(x, \phi(y)) = S^*(x, y) \circ T_y\phi$, for any $x \in N$, $y \in M$ and hence,

$$\int \langle \phi^*\alpha, \Gamma \rangle = \int \langle S^*(X, \Gamma) (\phi^*\alpha), \delta X \rangle = \int \langle S^*(X, \phi(\Gamma)) (\alpha), \delta X \rangle,$$

which shows that $\phi(\Gamma)$ is a solution of (2.1). \hfill \blacksquare

The symmetries that we are mostly interested in are induced by the action of a Lie group $G$ on the manifold $M$ via the map $\Phi : G \times M \to M$. Given $(g, z) \in G \times M$, we will usually write $g \cdot z$ to denote $\Phi(g, z)$. We also introduce the maps

$$\Phi_z : G \longrightarrow M \quad \quad \Psi_g : M \longrightarrow M \quad \quad \quad \quad g \longmapsto g \cdot z \quad \quad \quad \quad z \longmapsto g \cdot z.$$

The Lie algebra of $G$ will be usually denoted by $\mathfrak{g}$ and we will write the tangent space to the orbit $G \cdot m$ that contains $m \in M$ as $\mathfrak{g} \cdot m := T_m(G \cdot m)$.

\textbf{Definition 2.3} We will say that the stochastic differential equation (2.1) is $G$-\textit{invariant} if, for any $g \in G$, the diffeomorphism $\Phi_g : M \to M$ is a symmetry in the sense of Definition 2.1. In this situation we will also say that the Stratonovich operator $S$ is $G$-\textit{invariant}.

\textbf{Remark 2.4} Given a solution $\Gamma$ of a $G$-invariant stochastic differential equation, Proposition 2.2 provides an entire orbit of solutions since for any $g \in G$, the semimartingale $\Phi_g(\Gamma)$ is also a solution. This degeneracy has also a reflection in the probability laws of the solutions in a form that we spell out in the
following lines. Let $\Gamma : \{0 \leq t < \zeta\} \to M$ be a solution of the $G$-invariant system $(M, S, X, N)$ defined up to the explosion time $\zeta$, which may be finite if $M$ is not compact. In such case, $\Gamma$ can be actually understood as a process that takes values in the Alexandroff one-point compactification $\hat{M} := M \cup \{\infty\}$ of $M$ and it is hence defined in the whole space $\mathbb{R}_+ \times \Omega$ ([IW89, Chapter V]). In this picture, the process $\Gamma$ is continuous and with the property that $\Gamma_t(\omega) = \{\infty\}$, for any $(t, \omega) \in \mathbb{R}_+ \times \Omega$ such that $t \geq \zeta(\omega)$.

Let now $\hat{W}(M)$ be the path space defined by

$$\hat{W}(M) = \{w : [0, \infty] \to \hat{M} \text{ continuous such that } w(0) \in M \text{ and } w(t) = \{\infty\} \text{ then } w(t') = \{\infty\} \text{ for any } t' \geq t\}.$$

Let $\{P_z \mid z \in M\}$ be the family of probability measures on $\hat{W}(M)$ defined by the solutions of $(M, S, X, N)$, that is, $P_z$ is the law of the random variable $\Gamma^z : \Omega \to \hat{W}(M)$, where $\Gamma^z$ is the solution of $(M, S, X, N)$ with initial condition $\Gamma^z_{t=0} = z$ a.s.. The action $\Phi : G \times M \to M$ may be extended to $\hat{M}$ just putting $\Phi_g(\{\infty\}) = \{\infty\}$ for any $g \in G$. Since $\Phi_g(\Gamma^z)$ is the unique solution of the system $(M, S, X, N)$ with initial condition $g \cdot z$ by Proposition 2.2 then $P_{\phi g \cdot z} = \Phi^* g P_z$. More explicitly, for any measurable set $A \subset W(M), P_{g}(A) = P_z(\phi g(A))$.

The equivariance property of the probabilities $\{P_z \mid z \in M\}$ can be found in [ELL04] formulated in the context of equivariant diffusions on principal bundles. In that setup, the authors replace the path space $\hat{W}(M)$ by $C(l, r, M) = \{\sigma : [l, r] \to M \mid \sigma \text{ is continuous}, 0 \leq l < r < \infty\}$ and prove [ELL04, Theorem 2.5] that the probability laws $\{P_{l,r} \mid z \in M\}$ admit a factorization through probability kernels $\{P_{l,r}^{H,l,r} \mid z \in M\}$ from $M$ to $C(l, r, M)$ and $\{Q_{l,r}^{l,r} \mid w \in C(l, r, M)\}$ from $C(l, r, M)$ to $C_{c}(l, r, G) = \{\sigma : [l, r] \to G \mid \sigma \text{ is continuous}, \sigma(l) = e\}$ such that

$$P_{l,r}^{l,r}(U) = \int \int 1_U(g \cdot w)Q_{l,r}^{l,r}(dw)P_{l,r}^{H,l,r}(dg)$$

for any Borel set $U \subseteq C(l, r, M)$. The prove of this fact uses a technique very close to the reduction-reconstruction scheme that we will introduce in the next section.

Apart from degeneracies, the presence of symmetry in a stochastic differential equation is also associated with the occurrence of conserved quantities and, more generally, with the appearance of invariant submanifolds.

**Definition 2.5** Let $\Gamma$ be a solution of the stochastic differential equation (2.1) and let $L$ be an immersed submanifold of $M$. Let $\zeta$ be the maximal stopping time of $\Gamma$ and suppose that $\Gamma_0(\omega) = Z_0$, where $Z_0$ is a random variable such that $Z_0(\omega) \in L$, for all $\omega \in \Omega$. We say that $L$ is an invariant submanifold of the stochastic differential equation if for any stopping time $\tau < \zeta$ we have that $\Gamma_\tau \in L$.

**Proposition 2.6** Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semimartingale and let $S : TN \times M \to TM$ be a Stratonovich operator. Let $L$ be an immersed submanifold of $M$ and suppose that the Stratonovich operator $S$ is such that $\text{Im}(S(x,y)) \subset T_y L$, for any $y \in L$ and any $x \in N$. Then, $L$ is an invariant submanifold of the stochastic differential equation (2.1) associated to $X$ and $S$.

**Proof.** By hypothesis, the Stratonovich operator $S : TN \times M \to TM$ induces another Stratonovich operator $S_L : TN \times L \to TL$, obtained from $S$ by restriction. It is clear that if $i : L \hookrightarrow M$ is the inclusion then

$$S_L^i(x,y) \circ T_y i = S^i(x,y),$$

for any $x \in N$ and $y \in L$. Let $\Gamma_L$ be the semimartingale in $L$ that is a solution of the Stratonovich stochastic differential equation

$$\delta \Gamma_L = S_L(X, \Gamma_L) \delta X$$
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with initial condition $\Gamma_0$ in $L$. We now show that $\overline{\Gamma} := i \circ \Gamma_L$ is a solution of

$$\delta \overline{\Gamma} = S(X, \overline{\Gamma}) \delta X.$$  

which proves the statement. Indeed, for any $\alpha \in \Omega(M)$,

$$\int \langle \alpha, \delta \overline{\Gamma} \rangle = \int \langle \alpha, \delta (i \circ \Gamma_L) \rangle = \int \langle i^* \alpha, \delta \Gamma_L \rangle.$$  

Since $\Gamma_L$ satisfies (2.4) and $i^* \alpha \in \Omega(L)$, by (2.3) this equals

$$\int \langle S^*(X, i \circ \Gamma_L)(\alpha), \delta X \rangle = \int \langle S^*(X, \overline{\Gamma})(\alpha), \delta X \rangle,$$

that is, $\delta \overline{\Gamma} = S(X, \overline{\Gamma}) \delta X$, as required. 

We now use Proposition 2.6 to show that the invariant manifolds that can be associated to deterministic symmetric systems are also available in the stochastic context. Let $M$ be a manifold acted properly upon by a Lie group $G$ via the map $\Phi : G \times M \to M$. We recall that the action $\Phi$ is said to be proper when for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n := \Phi(g_n, m_n)\}$ in $M$, there exists a convergent subsequence $\{g_n_k\}$ in $G$. The properness hypothesis on the action implies that most of the useful features that compact group actions have, are still available. For example, proper group actions admit local slices, the isotropy subgroups are always compact, and (the connected components of) the isotropy type submanifolds defined by $M_I := \{z \in M \mid G_z = I\}$, are embedded submanifolds of $M$ for any isotropy subgroup $I \subset G$ of the action.

**Proposition 2.7 (Law of conservation of the isotropy)** Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semi-martingale and let $S : TN \times M \to TM$ be a Stratonovich operator that is invariant with respect to a proper action of the Lie group $G$ on the manifold $M$. Then, for any isotropy subgroup $I \subset G$, the isotropy type submanifolds $M_I$ are invariant submanifolds of the stochastic differential equation associated to $S$ and $X$.

**Proof.** The properness of the action guarantees that for any isotropy subgroup $I \subset G$ and any $z \in M_I$,

$$T_z M_I = (T_z M)^I := \{v \in T_z M \mid T_z \Phi_g \cdot v = v, \text{ for any } g \in I\}. \tag{2.5}$$

Hence, for any $z \in M_I$ and $g \in I$, the $G$-invariance of the Stratonovich operator $S$ implies that

$$T_z \Phi_g \circ S(x, z) = S(x, g \cdot z) = S(x, z),$$

which by (2.5) implies that $\text{Im} (S(x, z)) \subset T_z M_I$. The invariance of the isotropy type manifolds follows then from Proposition 2.6. 

**Remark 2.8** Some of the results that we just stated and others that will appear later on in the paper could be easily proved using their deterministic counterparts and the so called Malliavin’s Transfer Principle [Ma78] which says, roughly speaking, that results from the theory of ordinary differential equations are valid for stochastic differential equations in Stratonovich form. The unavailability of a metatheorem that explicitly proves and shows the range of applicability of this principle makes advisable its use with care.
3 Reduction and reconstruction

This section is the core of the paper. In the preceding paragraphs we explained how the symmetries of a stochastic differential equation imply the existence of certain conservation laws and degeneracies; reduction is a natural procedure to take advantage of the former and eliminate the latter via a combination of restriction and passage to the quotient operations. The end result of this strategy is the formulation of a stochastic differential equation with the same noise semimartingale but whose solutions take values in a manifold that is dimensionally smaller than the original one, which justifies the term reduction when we refer to this process. Smaller dimension and the absence of symmetry induced degeneracies usually make the reduced stochastic differential equation more tractable and easier to solve. The gain is therefore clear if once we have found the solutions of the reduced system, we know how to use them to find the solutions of the original system; that task is feasible and is the reconstruction process that will be explained in the second part of this section.

Theorem 3.1 (Reduction Theorem) Let \( X : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{N} \) be a \( \mathcal{N} \)-valued semimartingale and let \( S : TN \times M \rightarrow TM \) be a Stratonovich operator that is invariant with respect to a proper action of the Lie group \( G \) on the manifold \( M \). Let \( I \subset G \) be an isotropy subgroup of the \( G \)-action on \( M \), \( M_I \) the corresponding isotropy type submanifold, and \( L_I := N(I)/I \), with \( N(I) := \{ g \in G \mid gIg^{-1} = I \} \) the normalizer of \( I \) in \( G \). \( L_I \) acts freely and properly on \( M_I \) and hence the orbit space \( M_I/L_I \) is a regular quotient manifold, that is, the projection \( \pi_I : M_I \rightarrow M_I/L_I \) is a surjective submersion. Moreover, there is a well defined Stratonovich operator \( S_{M_I/L_I} : TN \times M_I/L_I \rightarrow T(M_I/L_I) \) given by

\[
S_{M_I/L_I}(x, \pi_I(z)) = T_z \pi_I(S(x, z)), \quad \text{for any } x \in \mathcal{N} \text{ and } z \in M_I
\]

such that if \( \Gamma \) is a solution semimartingale of the stochastic differential equation associated to \( S \) and \( X \), with initial condition \( \Gamma_0 \subset M_I \), then so is \( \Gamma_{M_I/L_I} := \pi_I(\Gamma) \) with respect to \( S_{M_I/L_I} \) and \( X \), with initial condition \( \pi_I(\Gamma_0) \). We will refer to \( S_{M_I/L_I} \) as the reduced Stratonovich operator and to \( \Gamma_{M_I/L_I} \) as the reduced solution.

Proof. The statement about \( M_I/L_I \) being a regular quotient manifold is a standard fact about proper group actions on manifolds (see for instance [DK99]). Now, observe that \( S_{M_I/L_I} : TN \times M_I/L_I \rightarrow T(M_I/L_I) \) is well defined; if \( z_1, z_2 \in M_I \) are such that \( \pi_I(z_1) = \pi_I(z_2) \), then there exists some \( g \in L_I \) satisfying \( z_2 = \Phi_g(z_1) \) (we use the same symbol \( \Phi \) to denote the \( G \)-action on \( M \) and the induced \( L_I \)-action on \( M_I \)). Hence,

\[
S_{M_I/L_I}(x, \pi_I(z_2)) = T_{z_2} \pi_I \circ S(x, z_2) = T_{z_2} \pi_I \circ T_{z_1} \Phi_g \circ S(x, z_1) = T_{z_1} \pi_I \circ S(x, z_1) = S_{M_I/L_I}(x, \pi_I(z_1)),
\]

where the \( G \)-invariance of \( S \) has been used. Let now \( \Gamma \) be a solution semimartingale of the stochastic differential equation associated to \( S \) and \( X \) with initial condition \( \Gamma_0 \subset M_I \). The \( G \)-invariance of \( S \) implies via Proposition 2.7 that \( \Gamma \subset M_I \) and hence \( \Gamma_{M_I/L_I} := \pi_I(\Gamma) \) is well defined. In order to prove the statement, we have to check that for any one-form \( \alpha \in \Omega(M_I/L_I) \)

\[
\int \langle \alpha, \delta \Gamma_{M_I/L_I} \rangle = \int \langle S^*_{M_I/L_I}(X, \Gamma_{M_I/L_I}) \alpha, \delta X \rangle.
\]

This equality follows in a straightforward manner from (3.1). Indeed,

\[
\int \langle \alpha, \delta \Gamma_{M_I/L_I} \rangle = \int \langle \alpha, \delta (\pi_I \circ \Gamma) \rangle = \int \langle \pi_I^* \alpha, \delta \Gamma \rangle = \int \langle S^*(X, \Gamma) (\pi_I^* \alpha), \delta X \rangle = \int \langle S^*_{M_I/L_I}(X, \Gamma_{M_I/L_I}) \alpha, \delta X \rangle.
\]
as required.

We are now going to carry out the reverse procedure, that is, given an isotropy subgroup $I \subset G$ and a solution semimartingale $\Gamma_{M/I}$ of the reduced stochastic differential equation with Stratonovich operator $S_{M/I}$ we will reconstruct a solution $\Gamma$ of the initial stochastic differential equation with Stratonovich operator $S$. In order to keep the notation not too heavy we will assume in the rest of this section that the $G$-action on $M$ is not only proper but also free, so that the only isotropy subgroup is the identity element $e$ and hence there is only one isotropy type submanifold, namely $M_e = M$. The general case can be obtained by replacing in the following paragraphs $M$ by the isotropy type manifolds $M/I$, and $G$ by the groups $L_I$.

We now make our goal more precise. The freeness of the action $\Phi : G \times M \to M$ guarantees that the canonical projection $\pi : M \to M/G$ is a principal bundle with structural group $G$. We saw in the previous theorem that for any solution $\Gamma$ of a stochastic differential equation associated to a $G$-invariant Stratonovich operator $S$ and a $N$-valued noise semimartingale $X$, we can build a solution $\Gamma_{M/G} = \pi(\Gamma)$ of the reduced stochastic differential equation associated to the projected Stratonovich operator $S_{M/G}$ introduced in (3.1) and to the stochastic component $X$. The main goal of the paragraphs that follow is to show how to reconstruct the dynamics of the initial system from solutions $\Gamma_{M/G}$ of the reduced system. As we will see in Theorem 3.2 any solution $\Gamma$ of the original stochastic differential equation may be written as $\Gamma = \Phi_{g_\pi}(d)$ where $d : \mathbb{R}^+ \times \Omega \to M$ is a semimartingale such that $\pi(d) = \Gamma_{M/G}$ and $g_\pi : \mathbb{R}^+ \times \Omega \to G$ is a $G$-valued semimartingale which satisfies a suitable stochastic differential equation on the group $G$.

We start by picking $A \in \Omega^1(M;\mathfrak{g})$ ($\mathfrak{g}$ is the Lie algebra of $G$) an auxiliary principal connection on the left principal $G$-bundle $\pi : M \to M/G$ and let $TM = \text{Hor} \oplus \text{Ver}$ be the decomposition of the tangent bundle $TM$ into the Whitney sum of the horizontal and vertical bundles associated to $A$. Analogously, the cotangent bundle $T^*M$ admits a decomposition $T^*M = \text{Hor}^* \oplus \text{Ver}^*$ where, by definition, $\text{Hor}_z^* := (\text{Ver}_z)^\circ$ is the annihilator of the vertical subspace $\text{Ver}_z$ at a point $z \in M$ and $\text{Ver}_z^* := (\text{Hor}_z)^\circ$ is the annihilator of the horizontal subspace. Hence, any one form $\alpha \in \Omega(M)$ may be uniquely written as $\alpha = \alpha^H + \alpha^V$ with $\alpha^H \in \text{Hor}^*$ and $\alpha^V \in \text{Ver}^*$. A section of the bundle $\pi_M : T^*M \to M$ taking values in $\text{Hor}^*$ is called a horizontal one form. It is called vertical if $\alpha_z \in \text{Ver}_z^*$ for any $z \in M$.

Let $\Gamma_{M/G} \subset M_{M/G}$ be a solution of the reduced stochastic differential equation associated to the Stratonovich operator $S_{M/G}$, and with stochastic component $X : \mathbb{R}^+ \times \Omega \to V$ as in Theorem 3.1. As we claimed, we are going to find a solution $\Gamma$ to the original $G$-invariant stochastic differential equation associated to $S$, such that $\pi(\Gamma) = \Gamma_{M/G}$ with a given initial condition $\Gamma_0$. We start by horizontally lifting $\Gamma_{M/G}$ to a $M$-valued semimartingale $d$. Indeed, by [SS2, Theorem 2.1] (see also [CO1]), there exists a $M$-valued semimartingale $d : \mathbb{R}^+ \times \Omega \to M$ such that $d_0 = \Gamma_0$, $\pi(d) = \Gamma_{M/G}$ and that satisfies

$$\int \langle A, \delta d \rangle = 0,$$

(3.2)

where (3.2) is a $\mathfrak{g}$-valued integral. More specifically, let $\{\xi_1, ..., \xi_m\}$ be a basis of the Lie algebra $\mathfrak{g}$ and let $A(z) = \sum_{i=1}^m A^i(z) \xi_i$ the expression of $A$ in this basis. Then

$$\int \langle A, \delta d \rangle := \sum_{i=1}^m \int \langle A^i, \delta d \rangle \xi_i.$$

(3.3)

The condition (3.2) is equivalent to $\int \langle \alpha, \delta d \rangle = 0$ for any vertical one-form $\alpha \in \Omega(M)$ (see [CO1, page 1641]) which, in turn, implies

$$\int \langle \theta, \delta d \rangle = 0.$$

(3.4)
for any $T^*M$-valued process $\theta : \mathbb{R}_+ \times \Omega \to \text{Ver}^* \subset T^*M$ over $d$. We want to find a $G$-valued semimartingale $g^\Xi : \Omega \times \mathbb{R}_+ \to G$ such that $\overline{g^\Xi} = e$ a.s. and $\Gamma = g^\Xi \cdot d$ is a solution of the stochastic differential equation associated to the Stratonovich operator $S$ and the $N$-valued noise semimartingale $X$.

Let $g \in G$, $z \in M$. It is easy to see that

$$\ker (T^*_g \Phi_z) = (T^*_{g \cdot z} (G \cdot z))^0 = (\text{Ver}_{g \cdot z})^0 = \text{Hor}^*_g z.$$  \hfill (3.5)

Where $G \cdot z$ denotes the $G$-orbit that contains the point $z \in M$. Therefore, the map

$$\overline{T^*_g \Phi_z} := T^*_g \Phi_z |_{\text{Ver}^*_g z} : T^*_g z \cap \text{Ver}^*_g z \to T^*_g G$$

is an isomorphism. Let

$$\rho (g, z) : T^*_g G \longrightarrow T^*_g z \cap \text{Ver}^*_g z \subset T^*_g M$$

$$\alpha_g \longmapsto (T^*_g \Phi_z)^{-1} (\alpha_g)$$

and define $\psi^* (x, z, g) : T^*_g G \to T^*_x N$ by

$$\psi^* (x, z, g) = S^* (x, g \cdot z) \circ \rho (g, z).$$

Finally, we define a dual Stratonovich operator between the manifolds $G$ and $M \times N$ as

$$K^* ((z, x), g) : T^*_g G \longrightarrow T^*_z M \times T^*_x N$$

$$\alpha_g \longmapsto (0, \psi^* (x, z, g) (\alpha_g)).$$  \hfill (3.7)

**Theorem 3.2 (Reconstruction Theorem)** Let $X : \mathbb{R}_+ \times \Omega \to N$ be a $N$-valued semimartingale and let $S : TN \times M \to TM$ be a Stratonovich operator that is invariant with respect to a free and proper action of the Lie group $G$ on the manifold $M$. If we are given $\Gamma_{M/G}$ a solution semimartingale of the reduced stochastic differential equation then $\Gamma = g^\Xi \cdot d$ is a solution of the original stochastic differential equation such that $\pi (\Gamma) = \Gamma_{M/G}$.

In this statement, $d : \mathbb{R}_+ \times \Omega \to M$ is the horizontal lift of $\Gamma_{M/G}$ using an auxiliary principal connection on $\pi : M \to M/G$ such that $\Gamma_0 = d_0$, and $g^\Xi : \mathbb{R}_+ \times \Omega \to G$ is the semimartingale solution of the stochastic differential equation

$$\delta g^\Xi = K (\Xi, g) \delta \Xi.$$  \hfill (3.8)

with initial condition $g^\Xi_0 = e$, $K$ the Stratonovich operator introduced in [3.4], and stochastic component $\Xi = (X, d)$ We will refer to $d$ as the horizontal lift of $\Gamma_{M/G}$ and to $\Gamma = g^\Xi$ as the stochastic phase of the reconstructed solution.

**Remark 3.3** As we already pointed out, Theorem 3.2 is also valid when the group action is not free. In that situation, one is given a solution of the reduced stochastic differential equation on the quotient $M_I/L_I$, with $I$ an isotropy subgroup of the $G$-action on $M$. The correct statement (and the proof that follows) of the reconstruction theorem in this case can be obtained from the one that we just gave by replacing $M$ by the isotropy type manifold $M_I$ and $G$ by the group $L_I$.

**Proof of Theorem 3.2** In order to check that $\Gamma = g^\Xi \cdot d$ is a solution of the original stochastic differential equation we have to verify that for any $\alpha \in \Omega (M)$,

$$\int \langle \alpha, \delta \Gamma \rangle = \int (S^* (X, \Gamma) \alpha, \delta X).$$  \hfill (3.9)
Since $\Gamma = g^\Xi \cdot d = \Phi(g^\Xi, d)$, the statement in \cite[Lemma 3.4]{SS2} allows us to write
\[
\int \langle \alpha, \delta \Gamma \rangle = \int \langle \Phi^*_g \alpha, \delta d \rangle + \int \langle \Phi^*_g \alpha, \delta g^\Xi \rangle. \tag{3.10}
\]
We split the verification of (3.9) into two cases:

(i) $\alpha \in \Omega (M)$ is horizontal or, equivalently, $\alpha = \pi^*(\eta)$ with $\eta \in \Omega (M/G)$. Since $\alpha$ is horizontal, then $\Phi^*_g \alpha = 0$ by (3.5). Then, using (3.10),
\[
\int \langle \alpha, \delta \Gamma \rangle = \int \langle \Phi^*_g \alpha, \delta d \rangle = \int \langle \Phi^*_g (\pi^*(\eta)), \delta d \rangle = \int \langle (\pi \circ \Phi^*_g)^*(\eta), \delta d \rangle = \int \langle \pi^*(\eta), \delta d \rangle = \int \langle \eta, \delta \Gamma_{M/G} \rangle.
\]
We recall that $\Gamma_{M/G} = \pi(d)$ is a solution of the reduced system, that is,
\[
\int \langle \eta, \delta \Gamma_{M/G} \rangle = \int \langle S^*_{M/G} (X, \Gamma_{M/G}) (\eta), \delta X \rangle
\]
for any $\eta \in \Omega (M/G)$. This implies by (3.1) that
\[
\int \langle \eta, \delta \Gamma_{M/G} \rangle = \int \langle S^*_{M/G} (X, \Gamma_{M/G}) (\eta), \delta X \rangle = \int \langle S^* (X, d) (\pi^*(\eta)), \delta X \rangle.
\]
Now, due to the $G$-invariance of $S$, we know that $S^* (x, g \cdot z) = S^* (x, z) \circ T^*_z \Phi_g$, for any $g \in G$, $x \in N$, $z \in M$. Recall also that $T^*_z \Phi_g$ sends the horizontal space $\text{Hor}_z$ to $\text{Hor}_{g \cdot z}$ and the vertical space $\text{Ver}_z$ to $\text{Ver}_{g \cdot z}$. Moreover, since $\alpha$ is horizontal, $\Phi^*_g \alpha = \alpha$ for any $g \in G$. Therefore,
\[
\int \langle \eta, \delta \Gamma_{M/G} \rangle = \int \langle S^* (X, d) (\alpha), \delta X \rangle = \int \langle S^* (X, d) (\Phi^*_g \alpha), \delta X \rangle
\]
\[
= \int \langle S^* (X, g^\Xi \cdot d) (\alpha), \delta X \rangle = \int \langle S^* (X, \Gamma) (\alpha), \delta X \rangle
\]
and hence (3.9) holds.

(ii) $\alpha \in \Omega (M)$ is vertical. Since $\alpha$ is vertical, so is $\Phi^*_g \alpha$ as a $T^*M$-valued process. Therefore,
\[
\int \langle \Phi^*_g \alpha, \delta d \rangle = 0 \text{ by (3.4). Thus, using (3.10),}
\]
\[
\int \langle \alpha, \delta \Gamma \rangle = \int \langle \Phi^*_g \alpha, \delta g^\Xi \rangle.
\]
Now, as $g^\Xi$ is a solution of the stochastic differential equation (3.8),
\[
\int \langle \Phi^*_g \alpha, \delta g^\Xi \rangle = \int \langle K^* (g^\Xi) (\Phi^*_g \alpha), \delta \Xi \rangle = \int \langle (0, \psi^* (g^\Xi, X, d) (\Phi^*_g \alpha)), \delta \Xi \rangle
\]
\[
= \int \langle \psi^* (g^\Xi, X, d) (\Phi^*_g \alpha), \delta X \rangle. \tag{3.11}
\]
Recall that $\psi^* (x, z, g) = S^* (x, g \cdot z) \circ \rho (g, z)$. Moreover $\rho (g, z) (\gamma_g) = (T^*_g \Phi) (\gamma_g)$ for any $\gamma_g \in T^*_g G$. Hence,
\[
\rho (g, z) \circ T^*_g \Phi_z (\alpha_{g \cdot z}) = (T^*_g \Phi) (\gamma_g) (T^*_g \Phi_z (\alpha_{g \cdot z})) = \alpha_{g \cdot z}
\]
for any \( \alpha_{g,z} \in T_{g,z}^* M \cap \text{Ver}_{g,z}^* \), since in that situation \( T_g^* \Phi_z (\alpha_{g,z}) = \overline{T_g^* \Phi_z (\alpha_{g,z})} \). Therefore, expression (3.11) equals

\[
\int \langle \psi^* (g, X, d) (\Phi_d \alpha), \delta X \rangle = \int \langle S^* (X, g^Z \cdot d) (\alpha), \delta X \rangle = \int \langle S^* (X, \Gamma) (\alpha), \delta X \rangle,
\]

and hence (3.9) also holds whenever \( \alpha \in \Omega (M) \) is vertical, as required. \( \blacksquare \)

The stochastic phase \( g^Z \) introduced in the Reconstruction Theorem admits another characterization that we present in the paragraphs that follow. Let \( \{ \xi_1, \ldots, \xi_m \} \) be a basis of \( \mathfrak{g} \), the Lie algebra of \( G \) and write \( A = \sum_{i=1}^m A^i \xi_i \), where \( A^i \in \Omega (M) \) are the components of the auxiliary connection \( A \in \Omega^1 (M; g) \) in this basis. Consider the \( g \)-valued semimartingale

\[
Y = \sum_{i=1}^m \int \langle S^* (X, d) (A^i), \delta X \rangle \xi_i,
\]

(3.12)

**Proposition 3.4** Let \( Y : \mathbb{R}^+ \times \Omega \rightarrow \mathfrak{g} \) be the \( g \)-valued semimartingale defined in (3.12). Then, the stochastic phase \( g^Z : \mathbb{R}^+ \times \Omega \rightarrow G \) introduced in (3.3) is the unique solution of the stochastic differential equation

\[
\delta g = L (Y, g) \delta Y
\]

(3.13)

associated to the Stratonovich operator \( L \) given by

\[
L (\xi, g) : T^*_g \mathfrak{g} \rightarrow T^*_g G,
\eta \mapsto T_g^* L g (\eta),
\]

with initial condition \( g_0 = e \). The symbol \( L g : G \rightarrow G \) denotes the left translation map by \( g \in G \).

In the proof of this proposition, we will denote by \( \xi_M (z) := \frac{d}{dt} \big|_{t=0} \exp t \xi \cdot z \) the infinitesimal vector field associated to \( \xi \in \mathfrak{g} \) by the \( G \)-action on \( M \) evaluated at \( z \in M \). Analogously, we will write \( \xi_G \) for the infinitesimal generators of the \( G \)-action on itself by left translations. We recall (see [OR04] for a proof) that for any \( g \in G \), \( \xi \in \mathfrak{g} \), and \( z \in M \),

\[
T_g \Phi_g (\xi_M (z)) = (\text{Ad}_g \xi)_M (g \cdot z).
\]

(3.14)

Moreover, \( T_g \Phi_z (\xi_G (g)) = T_z \Phi_g (\xi_M (z)) \) or, in other words,

\[
\xi_G (g) = \overline{T_g \Phi_z}^{-1} \circ T_z \Phi_g (\xi_M (z)),
\]

(3.15)

where \( \overline{T_g \Phi_z}^{-1} : T_{g \cdot z} G \cap \text{Ver}_{g \cdot z}^* \rightarrow T_g G \) is the isomorphism introduced in (3.6).\[ \]

**Proof of Proposition 3.4** A result in [SS2] shows that in order to prove the statement it suffices to check that \( \int \langle \theta, \delta g^Z \rangle = Y \), where \( \theta \) is the canonical \( \mathfrak{g} \)-valued one form on \( G \) defined by \( \theta_g (\xi_G (g)) = \xi \) for any \( g \in G \) and \( \xi \in \mathfrak{g} \). Indeed, Lemmas 3.2 and 3.3 in [SS2] show that a \( G \)-valued semimartingale \( g^Z \) is such that \( \int \langle \theta, \delta g^Z \rangle = Y \) if and only if \( g^Z \) is a solution of (3.13). Now, suppose that \( g^Z \) is a solution of (3.3),

\[
\int \langle \theta, \delta g^Z \rangle = \int \langle \psi^* (g^Z, X, d) (\theta), \delta X \rangle = \int \langle S^* (X, g^Z \cdot d) \circ \rho (g^Z, d) (\theta), \delta X \rangle.
\]

We are now going to verify that for any \( g \in G \) and \( z \in M \),

\[
\rho (g, z) (\theta) = (\Phi_{g^{-1}} A) (g \cdot z).
\]

(3.16)
First of all notice that as $\rho (g, z) (\gamma ) = \left( T^*_g \Phi^*_z \right)^{-1} (\gamma ) \in T^*_g M \cap \text{Ver}^*_g z$, for any $\gamma \in T^*_g G$ and since $A$ vanishes when acting on horizontal vector fields, it suffices to verify (3.10) when acting on vector fields of the form $\xi M$, for some $\xi \in g$. Using (3.14), the right hand side of (3.10) then reads

$$\left( \Phi^*_{\gamma^{-1} \cdot} A \right) (g \cdot z) (\xi M (g \cdot z)) = A (z) \left( T^*_z \Phi^*_{g^{-1}} (\xi M (g \cdot z)) \right) = A (z) \left( (\text{Ad}_{g^{-1}} \xi)_M (z) \right) = \text{Ad}_{g^{-1}} \xi.$$

As to the left hand side, we can write using (3.14) and (3.15),

$$\rho (g, z) (\theta (g)) (\xi M (g \cdot z)) = \left[ \left( T^*_g \Phi^*_z \right)^{-1} \theta (g) \right] (\xi M (g \cdot z)) = \theta (g) \left[ T^*_g \Phi^*_z \right] (\xi M (g \cdot z))
= \theta (g) \left[ T^*_g \Phi^*_z \circ T^*_g \Phi^*_{g^{-1}} (\xi M (g \cdot z)) \right]
= \theta (g) \left[ (\text{Ad}_{g^{-1}} \xi)_M (z) \right]
= \theta (g) \left[ (\text{Ad}_{g^{-1}} \xi)_G (g) \right] = \text{Ad}_{g^{-1}} \xi.$$

Thus,

$$\int \left\langle S^* (X, g^\Xi \cdot d) \circ \rho (g^\Xi, d) (\theta), \delta X \right\rangle = \int \left\langle S^* (X, g^\Xi \cdot d) \left( \Phi^*_{g^{-1} \cdot} A \right), \delta X \right\rangle.$$

Now, since the Stratonovich operator $S$ is $G$-invariant, we have that $S^* (x, g \cdot z) = S^* (x, z) \circ T^*_z \Phi_g$, for any $x \in N$, $z \in M$, and $g \in G$, and hence

$$S^* (x, g \cdot z) \left( \Phi^*_{g^{-1} \cdot} A \right) (g \cdot z) = S^* (x, z) \circ T^*_z \Phi_g \circ T^*_g \Phi^*_{g^{-1}} (A (z)) = S^* (x, z) (A (z)).$$

Therefore,

$$\int \left\langle \theta, \delta g^\Xi \right\rangle = \int \left\langle S^* (X, g^\Xi \cdot d) \left( \Phi^*_{g^{-1} \cdot} A \right), \delta X \right\rangle = \int \left\langle S^* (X, d) (A), \delta X \right\rangle = Y,$$

and consequently $g^\Xi$ solves (3.13). The argument that we just gave can be easily reversed to prove that if $g^\Xi$ is a solution of (3.13) then it is also a solution of (3.8). \(\blacksquare\)

The combination of the reduction and the reconstruction of the solution semimartingales of a symmetric stochastic differential equation can be seen as a method to split the problem of finding its solutions into three simpler tasks which we summarize as follows:

**Step 1:** Find a solution $\Gamma_{M/G}$ for the reduced stochastic differential equation associated to the reduced Stratonovich operator $S_{M/G}$ on the dimensionally smaller space $M/G$.

**Step 2:** Take an auxiliary principal connection $A \in \Omega^1 (M; g)$ for the principal bundle $\pi : M \to M/G$ and a horizontally lifted semimartingale $d : \mathbb{R}_+ \times \Omega \to M$, that is $\int \langle A, \delta d \rangle = 0$, such that $d_0 = \Gamma_0$ and $\pi (d) = \Gamma_{M/G}$.

**Step 3:** Let $g^\Xi : \mathbb{R}_+ \times \Omega \to G$ be the solution semimartingale of the stochastic differential equation (3.13) on $G$

$$\delta g = L (Y, g) \delta Y$$

with initial condition $g_0 = e$ a.s. and with noise semimartingale $Y = \int \langle S^* (X, d) (A), \delta X \rangle$. The solution of the original stochastic differential equation associated to the Stratonovich operator $S$ with initial condition $\Gamma_0$ is then $\Gamma = \Phi_{g^\Xi} (d)$. 

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**Reduction, reconstruction, and skew-product decomposition of symmetric SDEs**

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Remark 3.5 Theorem 3.2 has as a consequence that the maximal existence times $\zeta$ and $\zeta_{M/G}$ of $\pi$-related solutions $\Gamma$ and $\Gamma_{M/G}$ of the original symmetric and reduced systems, coincide. Indeed, if we write $\Gamma_t = g_t \cdot d_t$, with $d_t$ a horizontal lift of $\Gamma_{M/G}$, then first, $d_t$ is defined up to the same (maybe finite) explosion time $\zeta_{M/G}$ of $\Gamma_{M/G}$. Second, as the semimartingale $g_t$ is the solution of the left-invariant stochastic differential equation (3.13) then it is in principle stochastically complete (see [ES2 Chapter VII §6, Example (i) page 131]) if its stochastic forcing is. Since in our case, the stochastic component $Y$ (3.12) depends on $d_t$, we can conclude that $g_t$ is defined again on the stochastic interval $[0, \zeta_{M/G}]$. We consequently conclude that the maximal existence time of the solutions of the initial symmetric system $(M, S, X, N)$ coincides with that of the corresponding solutions of the reduced system $(M/G, S_{M/G}, X, N)$. Notice that this in particular implies that if the reduced manifold $M/G$ is compact then all the solutions of the original symmetric system are defined for all time, even if $M$ is not compact.

4 Symmetries and skew-product decompositions

The skew-product decomposition of second order differential operators is a factorization technique that has been used in the stochastic processes literature in order to split the semieliptic and, in particular, the diffusion operators, associated to certain stochastic differential equations (see, for instance, [PR88, LS9, L92, and references therein]). This splitting has important consequences as to the properties of the solutions of these equations, like certain factorization properties of their probability laws and of the associated stochastic flows.

Symmetries are a natural way to obtain this kind of decompositions as it has already been exploited in [ELL04]. Our goal in the following pages consists of generalizing the existing results in two ways: first, we will generalize the notion of skew-product to arbitrary stochastic differential equations by working with the notion of skew-product decomposition of the Stratonovich operator; we will indicate below how our approach coincides with the traditional one in the case of diffusions. Second, we will show that the skew-product decompositions presented in [ELL04] for regular free action are also available (at least locally) for singular proper group actions.

Definition 4.1 Let $N$, $M_1$, and $M_2$ be three smooth manifolds and $S(x, m) : T_x N \to T_m (M_1 \times M_2)$, $x \in N$, $m = (m_1, m_2) \in M_1 \times M_2$, a Stratonovich operator from $N$ to the product manifold $M_1 \times M_2$. We will say that $S$ admits a skew-product decomposition if there exists a Stratonovich operator $S_2(x, m_2) : T_x N \to T_{m_2} M_2$ from $N$ to $M_2$ and a $M_2$-dependent Stratonovich operator $S_1(x, m_1, m_2) : T_x N \to T_{m_1} M_1$ such that

$$S(x, m) = (S_1(x, m_1, m_2), S_2(x, m_2)) \in \mathcal{L}(T_x N, T_{m_1} M_1 \times T_{m_2} M_2)$$

for any $m = (m_1, m_2) \in M_1 \times M_2$. The operators $S_1$ and $S_2$ will be called the factors of $S$.

In order to show the relation between this definition and the classical one used in the papers that we just quoted, we first have to briefly recall the relation between the global Stratonovich and Itô formulations for the stochastic differential equations (see [ES9] for a detailed presentation of this subject). Given $M$ and $N$ two manifolds, a Schwartz operator is a family of Schwartz maps (see [ES9 Definition 6.22]) $S(x, z) : T_x N \to \tau M$ between the tangent bundles of second order $\tau N$ and $\tau M$. In this context, the Itô stochastic differential equation defined by the Schwartz operator $S$ with stochastic component a continuous semimartingale $X : \mathbb{R}_+ \times \Omega \to N$ is

$$d\Gamma = S(X, \Gamma) dX. \quad (4.1)$$

Given a Stratonovich operator $S$, there is a unique Schwartz operator $\mathcal{S} : \tau N \times M \to \tau M$ that is an extension of $S$ to the tangent bundles of second order and which makes the Itô and Stratonovich
stochastic differential equations associated to \( S \) and \( S \) equivalent, in the sense that they have the same semimartingale solutions. \( S \) is constructed as follows. Let \( \gamma(t) = (x(t), z(t)) \in N \times M \) be a smooth curve that verifies \( S(x(t), z(t))(\dot{x}(t)) = \dot{z}(t) \), for all \( t \). We define \( S(x(t), z(t))(L_{\dot{x}(t)}) := (L_{\dot{z}(t)}) \), where the second order differential operators \( (L_{\dot{x}(t)}) \in \tau_{x(t)}N \) and \( (L_{\dot{z}(t)}) \in \tau_{z(t)}M \) are defined as \( (L_{\dot{z}(t)})[h] := \frac{d^2}{dt^2}h(x(t)) \) and \( (L_{\dot{z}(t)})[g] := \frac{d^2}{dt^2}g(z(t)) \), for any \( h \in C^\infty(N) \) and \( g \in C^\infty(M) \). This relation completely determines \( S \) since the vectors of the form \( L_{\dot{x}(t)} \) span \( \tau_{x(t)}M \).

It is easy to show that if \( S : TN \times (M_1 \times M_2) \to T(M_1 \times M_2) \) is a Stratonovich operator that admits a skew-product decomposition with factors \( S_1 \) and \( S_2 \) then the equivalent Schwartz operator \( S : \pi N \times (M_1 \times M_2) \to \pi (M_1 \times M_2) \) can be written as

\[
S(x, (m_1, m_2)) = S_1(x, m_1, m_2) + S_2(x, m_2),
\]

for any \( x \in N \) and any \( m = (m_1, m_2) \in M_1 \times M_2 \). In this expression, \( S_1(x, m_1, m_2) : \tau_x N \to \tau_{m_1}(M_1 \times M_2) \) and \( S_2(x, m_2) : \tau_x N \to \tau_{m_2}(M_1 \times M_2) \) are the equivalent Schwartz operators of the Stratonovich operators \( \tilde{S}_1, \tilde{S}_2 : TN \times (M_1 \times M_2) \to T(M_1 \times M_2) \) defined by \( \tilde{S}_1(x, m) := T_{m_1}i_{m_2}(S_1(x, m_1, m_2)) \) and \( \tilde{S}_2(x, m) := T_{m_2}i_{m_1}(S_2(x, m_2)) \). The maps \( i_{m_1} : M_2 \to M_1 \times M_2 \) and \( i_{m_2} : M_1 \to M_1 \times M_2 \) are the natural inclusions obtained by fixing \( m_1 \) and \( m_2 \), respectively.

Now, the notion of skew-product decomposition of a second order differential operator \( L \in \mathfrak{X}_2(M_1 \times M_2) \) on \( M_1 \times M_2 \) that one finds in the literature (see for instance \cite{1992}) consists on the existence of two smooth maps \( L_1 : M_2 \to \mathfrak{X}_2(M_1) \) and \( L_2 : \mathfrak{X}_2(M_1) \to (M_1 \times M_2) \) such that for any \( f \in C^\infty(M_1 \times M_2) \)

\[
L[f](m_1, m_2) = (L_1(m_2)[f, m_2])(m_1) + (L_2[f, m_1])(m_2).
\]

The relation between this notion and the one introduced in Definition 4.1 is very easy to establish for semielliptic diffusions. Indeed, suppose that the Stratonovich operator associated to a semielliptic diffusion admits a skew-product decomposition; we just saw that this implies in general the existence of a skew-product decomposition of the corresponding Schwartz operator, which in turn implies the availability of a skew-product decomposition of the infinitesimal generator associated to \( S \) in the sense of Definition 4.1. See \cite{1992} page 15 for a sketch of the proof of this fact.

In conclusion, since in the cases that have already been studied, the skew-product decompositions of Stratonovich operators carry in their wake the skew-product decompositions as differential operators of the associated infinitesimal generators, we can focus in what follows on the more general situation that consists of adopting Definition 4.1.

### 4.1 Skew-products on principal fiber bundles. Free actions.

Let \( M, N \) be two manifolds, \( G \) a Lie group, and \( \Phi : G \times M \to M \) a proper and free action. We already know that \( M/G \) is a smooth manifold under these hypotheses and that \( \pi_{M/G} : M \to M/G \) is a principal fiber bundle with structural group \( G \). The goal of the following paragraphs is to show that any \( G \)-invariant Stratonovich operator \( S : TN \times M \to TM \) on \( M \) admits a local skew-product decomposition. This result is also true even if the action \( \Phi \) is not free, as we will see in the next section. However, what makes this local decomposition possible in this simpler case is not the fact that the \( G \)-action is free and proper but that \( \pi_{M/G} : M \to M/G \) is a principal fiber bundle. Consequently, in order to keep our exposition as general as possible, we will adopt as the setup for the rest of this subsection a \( G \)-invariant Stratonovich operator \( S : TN \times P \to TP \) on an arbitrary (left) \( G \)-principal fiber bundle \( \pi : P \to Q \). This setup has been studied in detail in \cite{2004} for invariant diffusions. In the following proposition we generalize the vertical-horizontal splitting in that paper to arbitrary Stratonovich operators and we formulate it in terms of skew-products.
Proposition 4.2 Let $N$ be a manifold, $\pi : P \to Q$ a (left) principal bundle with structure group $G$, $S : T\nu \times P \to T\nu$ a $G$-invariant Stratonovich operator, $X : \mathbb{R}_+ \times \Omega \to N$ a $N$-valued semimartingale, and $\sigma : U \to \pi^{-1}(U) \subseteq P$ a local section of $\pi$ defined on an open neighborhood $U \subseteq Q$. Then, $S$ admits a skew-product decomposition on $\pi^{-1}(U)$. More explicitly, there exists a diffeomorphism $F : G \times U \to \pi^{-1}(U)$ and a skew-product split Stratonovich operator $S_{G \times U} : T\nu \times (G \times U) \to T\nu$ such that $F$ establishes a bijection between semimartingales $\Gamma$ starting on $\pi^{-1}(U)$ which are solutions of the stochastic system $(P, S, X, N)$ up to time $\tau = \inf \{ t > 0 \mid \Gamma_t \notin \pi^{-1}(U) \}$ and the $(G \times U)$-valued semimartingales $(\tilde{g}_t, \Gamma_t^Q)$ that solve $(G \times U, S_{G \times U}, X, N)$,

$$\delta(\tilde{g}_t, \Gamma_t^Q) = S_{G \times U}(X, (\tilde{g}_t, \Gamma_t^Q)) \delta X_t. \tag{4.4}$$

Proof. Let $U \subseteq Q$ be an open neighborhood and $\sigma : U \to \pi^{-1}(U) \subseteq P$ a local section of $\pi : P \to Q$. Given that $G$ acts freely on $P$, the map

$$F : G \times U \to \pi^{-1}(U)$$

$$\begin{align*}
(g, q) & \mapsto g \cdot \sigma(q)
\end{align*}$$

is a $G$-equivariant diffeomorphism, where $g \cdot \sigma(q) = \Phi_g(\sigma(q))$ denotes the (left) action of $g \in G$ on $\sigma(q) \in P$ via $\Phi : G \times P \to P$ and the product manifold $G \times U$ is considered as a left $G$-space with the action defined by $g \cdot (h, q) := (g \cdot h, q)$. Thus, we can use $F$ to identify $\pi^{-1}(U) \subseteq P$ with the product manifold $G \times U$.

Now, given $p = g \cdot \sigma(q) \in \pi^{-1}(U)$, define $\text{Hor}_p \subseteq T_p P$ as $\text{Hor}_p := T\nu(g) \circ T_p \sigma(T_p Q)$. It is straightforward to see that the family of horizontal spaces $\{ \text{Hor}_p \mid p \in \pi^{-1}(U) \}$ is invariant by the $G$-action and hence defines a principal connection $A_\sigma \in \Omega^1(\pi^{-1}(U) \backslash G)$ on the open neighborhood $\pi^{-1}(U)$. Moreover, if $\Gamma^Q : \mathbb{R}_+ \times \Omega \to Q$ is a $Q$-valued semimartingale starting at $q$, then $\sigma(\Gamma^Q)$ is the unique horizontal lift on $P$ of $\Gamma^Q$ associated to the connection $A_\sigma$ starting at $\sigma(q) \in \pi^{-1}(q)$ and defined up to time $\tau_U = \inf\{ t > 0 \mid \Gamma_t^Q \notin U \}$.

Consider now the skew-product split Stratonovich operator $S_{G \times U}(x, (g, q)) : T\nu \times (G \times U) \to T\nu \times (G \times U)$ such that, for any $x \in N$, $g \in G$, $q \in U$

$$S_{G \times U}(x, (g, q)) = (K((\sigma(q), x), q), S_{P/G}(x, q)) \in L(T_x N, T_q G \times T_q U),$$

where $K$ is the Stratonovich operator introduced in (3.4) and $S_{P/G}$ the reduced Stratonovich operator constructed out of $S$ as in (3.1). Let $(\tilde{g}_t, \Gamma_t^Q)$ a $(G \times U)$-valued semimartingale solution of the stochastic system (4.4), i.e.

$$\delta(\tilde{g}_t, \Gamma_t^Q) = S_{G \times U}(X, (\tilde{g}_t, \Gamma_t^Q)) \delta X_t,$$

with initial condition $(g, q) \in G \times U$. We claim that $\Gamma_t = F(\tilde{g}_t, \Gamma_t^Q) = \tilde{g}_t \cdot \sigma(\Gamma_t^Q)$ is a solution of the stochastic system $(P, S, X, N)$ with initial condition $g \cdot \sigma(q)$ up to the first exit time $\tau_U = \inf\{ t > 0 \mid \Gamma_t^Q \notin U \}$. This is a consequence of the Reconstruction Theorem 3.2 and the fact that $\sigma(\Gamma_t^Q)$ is the horizontal lift of a solution of the reduced system $(Q, S_{P/G}, X, N)$. Conversely, let $\Gamma$ be a solution of the stochastic system $(P, S, X, N)$ with initial condition $p = g \cdot \sigma(q) \in \pi^{-1}(U)$. By the Reconstruction Theorem 3.2 $\Gamma$ can be written as $\Gamma_t = \tilde{g}_t \cdot \sigma$. We recall that $d_t$ the horizontal lift with respect to an arbitrary connection $A \in \Omega^1(Q; g)$ of the solution $\Gamma_t^Q = \pi(\Gamma_t)$ of the reduced system $(Q, S_{P/G}, X, N)$ (see Theorem 3.2), with initial condition $\sigma(q)$. On the other hand, $\tilde{g}_t$ is the solution of the stochastic system (3.3) with initial condition $g \in G$. If we take in this procedure $A_\sigma \in \Omega^1(\pi^{-1}(U); g)$ as the auxiliary connection, that is, the one given by the local section $\sigma : U \to \pi^{-1}(U)$, then $d_t = \sigma(\Gamma_t^Q)$ and it is straightforward to check that $(\tilde{g}_t, \Gamma_t^Q)$ is a solution of (4.4) with initial condition $(g, q) \in G \times U$. ■
Example 4.3 Let $G$ be a Lie group, $H \subseteq G$ a closed subgroup, and $R$ a smooth manifold. In [PR88], Pauwels and Rogers show several examples of skew-product decompositions of Brownian motions on manifolds of the type $R \times G/H$ which share a common feature, namely, they are obtained from skew-product split Brownian motions on $R \times G$ via the reduction $\pi : R \times G \rightarrow R \times G/H$. The $H$-action on $R \times G$ is $h \cdot (r, g) := (r, gh)$, for any $h \in H$, $r \in R$, and $g \in G$. An important result in this paper is Theorem 2 which reads as follows: suppose that $R \times G/H$ is a Riemannian manifold with Riemannian metric $\eta$ and that the tensor $\pi^{*}\eta$ is $G$-invariant. Furthermore, suppose that the decomposition $T_{[r,g]}(R \times G) = T_{r}R \oplus T_{g}G$ is orthogonal with respect to $\pi^{*}\eta$, for any $r \in R$, $g \in G$, and that the Lie algebra $\mathfrak{g}$ of $G$ admits an $Ad_{H}$-invariant inner product. Under these hypotheses, $R \times G$ admits a $G$-invariant Riemannian metric $\eta$ such that if $\Gamma$ is a Brownian motion on $R \times G$ with respect to $\eta$ then $\Gamma$ has a skew-product decomposition and moreover, $\pi(\Gamma)$ is a Brownian motion on $(R \times G/H, \eta)$. This result is repeatedly used in [PR88] to obtain skew-product decompositions of Brownian motions on various manifolds of matrices.

Example 4.4 (Brownian motion on symmetric spaces) Let $(M, \eta)$ be a Riemannian symmetric space with Riemannian metric $\eta$. We want to define Brownian motions on $(M, \eta)$ by reducing a suitable process defined on the connected component containing the identity of its group of isometries. The notation and most of the results in this example, in addition to a comprehensive exposition on symmetric spaces, can be found in [H78] and [KN69]. The reader is encouraged to check with [ELL98] to learn more about stochastics in the context of homogeneous spaces.

We start by recalling that a $M$-valued process $\Gamma$ is a Brownian motion whenever

$$f(\Gamma) - f(\Gamma_{0}) - \frac{1}{2} \int \Delta(f)(\Gamma_{s}) \, ds$$

is a real valued local semimartingale for any $f \in C^{\infty}(M)$, where $\Delta$ denotes the Laplacian. The Laplacian is defined as the trace of the Hessian associated to the Levi-Civita connection $\nabla$ of $\eta$, that is,

$$\Delta(f)(m) = \sum_{i=1}^{r}(\mathcal{L}_{Y_{i}} \circ \mathcal{L}_{Y_{i}} - \nabla_{Y_{i}Y_{i}})(f)(m)$$

where $\{Y_{1}, ..., Y_{r}\} \subset \mathfrak{X}(M)$ is a family or vector fields such that $\{Y_{1}(m), ..., Y_{r}(m)\}$ is an orthonormal basis of $T_{m}M$, $m \in M$.

Let $G$ be the connected component containing the identity of the isometries group $I(M) \subseteq \text{Diff}(M)$ of $M$. Take $o \in M$ a fixed point and let $s$ be a geodesic symmetry at $o$. The Lie group $G$ acts on $M$ transitively and, if $K$ denotes the isotropy group of $o$, $M$ is diffeomorphic to $G/K$ ([H78] Chapter IV, Theorem 3.3). Denote by $\pi : G \rightarrow G/K$ the canonical projection and suppose that $\dim (G) < \infty$. Let $\sigma : G \rightarrow G$ be the involutive automorphism of $G$ defined by $\sigma(g) = s \circ \Phi_{g} \circ s$ for any $g \in G$, where $\Phi : G \times M \rightarrow G$ denotes as usual the left action of $G$ on $M$. $T_{e}\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ induces an involutive automorphism of $\mathfrak{g}$. That is, $T_{e}\sigma \circ T_{e}\sigma = \text{Id}$ but $T_{e}\sigma \neq \text{Id}$. Let $\mathfrak{t}$ and $\mathfrak{m}$ be the the eigenspaces in $\mathfrak{g}$ associated to the eigenvalues 1 and $-1$ of $T_{e}\sigma$, respectively, such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. It can be checked that $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$ and that (see [KN69] Chapter XI Proposition 2.1))

$$[\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad \text{and} \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{t}.$$ 

Since the infinitesimal generators $\xi^{M} \in \mathfrak{X}(M)$ of the $G$-action $\Phi$ on $M$, with $\xi \in \mathfrak{m}$, span the tangent space at any point $gK \in G/K$, any affine connection is fully characterized by its value on the left-invariant vector fields $\xi^{M}$ with $\xi \in \mathfrak{m}$. In the particular case of the Levi-Civita connection $\nabla$ associated to the metric $\eta$, its $G$-invariance implies via [KN69] Chapter XI, Theorem 3.3 that

$$\nabla_{\xi^{M}}\xi^{M}(gK) = 0$$  \hspace{1cm} (4.5)
for any pair of left-invariant vector fields $\xi^M$ and $\zeta^M$. A consequence of (4.5) is that the Laplacian $\Delta$ takes the expression $\Delta(f)(gK) = \sum_{i=1}^r L_{\xi^M_i} \circ L_{\zeta^M_i}(f)(gK)$, $gK \in G/K$, where $\{\xi^M_1(gK), \ldots, \xi^M_r(gK)\}$ is an orthonormal basis of $T_{gK}(G/K)$.

Let $\{\xi_1, \ldots, \xi_r\}$ be a basis of $\mathfrak{m}$ such that $\{T_\pi \xi_1, \ldots, T_\pi \xi_r\}$ is an orthonormal basis of $T_K(G/K) \simeq T_0M$ with respect to $\eta_0$ and let $\{\xi^G_1, \ldots, \xi^G_r\} \subset \mathfrak{X}(G)$ the corresponding family of right-invariant vector fields built from $\{\xi_1, \ldots, \xi_r\}$. Observe that $\{\xi^G_1, \ldots, \xi^G_r\}$ is an orthonormal basis of the tangent space at any point $gK \in G/K$ due to the transitivity of the $G$-action on $M$ and to the $G$-invariance of the metric $\eta$. Consider now the Stratonovich stochastic differential equation on $G$

$$\delta g_t = \sum_{i=1}^r \xi^G_i(g_t) \delta B^i_t, \quad (4.6)$$

where $(B^1_t, \ldots, B^r_t)$ is a $\mathbb{R}^r$-valued Brownian motion. The equation (4.6) is by construction $K$-invariant with respect to the right action $R : K \times G \to G$, $R_k(g) = gk$. In addition, it is straightforward to check that the projection $\pi : G \to G/K$ sends any right-invariant vector field $\xi^G \in \mathfrak{X}(G)$, $\xi \in \mathfrak{g}$, to the infinitesimal generator $\xi^M \in \mathfrak{X}(M)$ of the $G$-action $\Phi : G \times M \to M$. Indeed, for any $\xi \in \mathfrak{g}$, $g \in G$, and $k \in K$

$$T_{gK} \pi (\xi^G(g)) = T_g \pi \circ T_e R_g (\xi) = \left. \frac{d}{dt} \right|_{t=0} \pi (\exp (t\xi) (g)) = \left. \frac{d}{dt} \right|_{t=0} \Phi (\exp (t\xi), \pi (g)) = \xi^M(gK),$$

and hence (4.6) projects to the stochastic differential equation

$$\delta \Gamma_t = \sum_{i=1}^r \xi^M_i(\Gamma_t) \delta B^i_t \quad (4.7)$$

on $M$ by the Reduction Theorem 3.1. A straightforward computation shows that the solution semimartingales of (4.7) have as infinitesimal generator the Laplacian $\Delta = \sum_{i=1}^r L_{\xi^M_i} \circ L_{\xi^M_i}$ and hence by the Itô formula

$$f(\Gamma) - f(\Gamma_0) = \frac{1}{2} \int \Delta(f)(\Gamma_s) ds = \sum_{i=1}^r \int \xi^M_i[f](\Gamma) dB^i$$

which allows us to conclude that they are Brownian motions. It is worth noticing that since right-invariant systems such that (4.6) are stochastically complete (see [ES2, Chapter VII §6]) and by the Reduction and Reconstruction Theorems 3.1 and 3.2 any solution of (4.7) may be written as $\Gamma_t = \pi (g_t)$ for a suitable solution $g_t$ of (4.6), the Brownian motion on a symmetric space is stochastically complete.

4.2 Skew-products induced by non-free actions. The tangent-normal decomposition

In this section we will show how the results that we just presented for free actions can be generalized to the non-free case by using the notion of slice [Ko53, P61] and a generalization to the context of Stratonovich operators of the so-called tangent-normal decomposition of $G$-equivariant vector fields with respect to proper group actions [K90, F91].

Let $\Phi : G \times M \to M$ be a proper action of the Lie group $G$ on the manifold $M$ and let $M/G$ be the associated orbit space, $M/G$. Observe that as the group action is not necessarily free, the orbit space $M/G$ needs not be a smooth manifold.

In order to introduce the notion of slice we start by considering a subgroup $H \subset G$ of $G$. Suppose that $H$ acts on the left on a certain manifold $A$. The twisted action of $H$ on the product $G \times A$ is defined by

$$h \cdot (g, a) = (gh, h^{-1} \cdot a), \quad h \in H, \; g \in G, \; a \in A.$$
Note that this action is free and proper by the freeness and properness of the action on the $G$-factor. The twisted product $G \times_H A$ is defined as the orbit space $(G \times A)/H$ corresponding to the twisted action. The elements of $G \times_H A$ will be denoted by $[g,a]$, $g \in G$, $a \in A$. The twisted product $G \times_H A$ is a $G$-space relative to the left action defined by $g' \cdot [g,a] := [g'g,a]$. Also, it can be shown that the action of $H$ on $A$ is proper if and only if the $G$-action on $G \times_H A$ just defined is proper (see Proposition 2.3.17).

Let now $m \in M$ and denote $H := G_m$. A tube around the orbit $G \cdot m$ is a $G$-equivariant diffeomorphism

$$\varphi : G \times_H A \rightarrow U,$$

where $U$ is a $G$-invariant neighborhood of the orbit $G \cdot m$ and $A$ is some manifold on which $H$ acts. Note that the $G$-action on the twisted product $G \times_H A$ is proper since by the properness of the $G$-action on $M$, the isotropy subgroup $H$ is compact and, consequently, its action on $A$ is proper.

**Definition 4.5** Let $M$ be a manifold and $G$ a Lie group acting properly on $M$. Let $m \in M$ and denote $G_m := H$. Let $W$ be a submanifold of $M$ such that $m \in W$ and $H \cdot W = W$. We say that $W$ is a slice at $m$ if the $G$-equivariant map

$$\varphi : G \times_H W \rightarrow U$$

$$[g,s] \mapsto g \cdot s$$

is a tube about $G \cdot m$ for some $G$-invariant open neighborhood $U$ of $G \cdot m$. Notice that if $W$ is a slice at $m$ then $\Phi_g(W)$ is a slice at the point $\Phi_g(m)$.

The Slice Theorem of Palais [P61] proves that there exists a slice at any point of a proper $G$-manifold. The following theorem, whose proof can be found in [OR04], provides several equivalent characterizations of the concept of slice that are available in the literature.

**Theorem 4.6** Let $M$ be a manifold and $G$ a Lie group acting properly on $M$. Let $m \in M$, denote $H := G_m$, $\mathfrak{h}$ the Lie algebra of $H$, and let $W$ be a submanifold of $M$ containing $m$. Then the following statements are equivalent:

(i) There is a tube $\varphi : G \times_H A \rightarrow U$ about $G \cdot m$ such that $\varphi(e, A) = W$.

(ii) $W$ is a slice at $m$.

(iii) The submanifold $W$ satisfies the following properties:

(a) The set $G \cdot W$ is an open neighborhood of the orbit $G \cdot m$ and $W$ is closed in $G \cdot W$.

(b) For any $z \in W$ we have that $T_z M = g \cdot z + T_z W$. Moreover, $g \cdot z \cap T_z W = h \cdot z$. In particular, for $z = m$ the sum $g \cdot z + T_z W$ is direct.

(c) $W$ is $H$-invariant. Moreover, if $z \in W$ and $g \in G$ are such that $g \cdot z \in W$, then $g \in H$.

(d) Let $\sigma : U \subset G/H \rightarrow G$ be a local section of the submersion $G \rightarrow G/H$. Then, the map $F : U \times W \rightarrow M$ given by $F(u,z) := \sigma(u) \cdot z$ is a diffeomorphism onto an open subset of $M$.

(iv) $G \cdot W$ is an open neighborhood of $G \cdot m$ and there is an equivariant smooth retraction

$$r : G \cdot W \rightarrow G \cdot m$$

of the injection $G \cdot m \hookrightarrow G \cdot W$ such that $r^{-1}(m) = W$. 

Reduction, reconstruction, and skew-product decomposition of symmetric SDEs

Let now \( S : TN \times M \to TM \) be a \( G \)-invariant Stratonovich operator. The existence of slices for the \( G \)-action allow us to carry out two decompositions of \( S \). The first one, that we will call \textbf{tangent-normal decomposition} is semi-global in the sense that it shares the properties that the Slice Theorem has in this respect, which is global in the orbit directions and local in the directions transversal to the orbits; this decomposition consists of writing \( S \) as the sum of two Stratonovich operators such that, roughly speaking, one is tangent to the orbits of the \( G \)-action and the other one is transversal to them. The second one is purely local and yields a \textbf{skew-product decomposition} of \( S \) in the sense of Definition 4.1 provided that an additional hypothesis on the isotropies in the slice is present. This hypothesis, whose impact will be explained in detail later on, is generically satisfied and hence the following theorem shows that \( S \) admits a skew product decomposition in a neighborhood of most points in \( M \) (those points form an open and dense subset of \( M \)). These statements are rigorously proved in the following theorem.

**Theorem 4.7** Let \( X : \mathbb{R}_+ \times \Omega \to N \) be a \( N \)-valued semimartingale, \( \Phi : G \times M \to M \) a proper Lie group action, and \( S : TN \times M \to TM \) a \( G \)-invariant Stratonovich operator. Let \( m \in M \) and \( W \) a slice at \( m \). Then, there exist two Stratonovich operators \( S_N : TN \times W \to TW \) and \( S_T : TN \times G \cdot W \to T(G \cdot W) \) such that the following statements hold:

(i) Let \( \text{Lie}(N(G_z)) \) denote the Lie algebra of the normalizer \( N(G_z) \) in \( G \) of the isotropy group \( G_z \), \( z \in G \cdot W \). The Stratonovich operator \( S_T \) is \( G \)-invariant and \( S_T(x, z) \in \mathcal{L}(T_z N, \text{Lie}(N(G_z))) \cdot z \) for any \( x \in N \) and any \( z \in G \cdot W \). Moreover, there exists an adjoint \( G \)-equivariant map \( \xi : TN \times G \cdot W \to g \), (that is, \( \xi(x, g \cdot z) = \text{Ad}_g \circ \xi(x, z) \), for any \( g \in G \)) such that \( S_T(x, z) = T_e \Phi_{z} \circ \xi(x, z) \).

(ii) The Stratonovich operator \( S_N : TN \times W \to TW \) is \( G_m \)-invariant.

(iii) If \( z = g \cdot w \in G \cdot W \), with \( g \in G \) and \( w \in W \), then
\[
S(x, z) = S_T(x, z) + T_w \Phi_g \circ S_N(x, w) = T_w \Phi_g \circ (S_T(x, w) + S_N(x, w)).
\]

This sum of Stratonovich operators will be referred to as the \textbf{tangent-normal decomposition} of \( S \).

(iv) Let \( \varphi \) be the flow of the stochastic system \((W, S_N, X, N)\) so that \( \varphi(w) \) denotes the solution of
\[
\delta \Gamma = S_N(X, \Gamma) \delta X
\]
with initial condition \( \Gamma_{t=0} = w \) a.s.. Let \( \Phi_{g \times W} : TN \times (g \times W) \to (g \times W) \) be the Stratonovich operator defined as \( \Phi_{g \times W}(x, (\eta, w)) = \xi(x, w) \times S_N(x, w) \in \mathcal{L}(T_x N, g \times T_w W) \) and let \((\eta^w, \Gamma^w)\) be the solution semimartingale of the stochastic system \((g \times W, \Phi_{g \times W}, X, N)\) with initial condition \( (0, w) \in g \times W \). Finally, let \( \tilde{g} : \{0 \leq t < \tau_x\} \to G \) be the solution of the stochastic system \((G, L, \eta^w, g)\) with initial condition \( g \in G \) and where \( L : Tg \times G \to TG \) is such that \( L(\eta, g)(w) = T_e L_g(\nu) \). Then, the semimartingale
\[
\Gamma_t = \tilde{g}_t \cdot \varphi_t(w)
\]
is a solution up to time \( \tau_x \) of the stochastic system \((M, S, X, N)\) with initial condition \( z = g \cdot w \in G \cdot W \).

(v) Suppose now that \( G_m = G_z \), for any \( w \in W \). Then \( S \) admits a local skew-product decomposition. More specifically, for any point \( m \in M \), there exists an open neighborhood \( V \subseteq G/G_m \) of \( G_m \), a diffeomorphism \( F : V \times W \to U \subseteq M \), and a skew-product split Stratonovich operator \( S_{V \times W} : TN \times (V \times W) \to T(V \times W) \) such that \( F \) establishes a bijection between semimartingales \( \Gamma \) starting on \( U \) which are solution of the stochastic system \((U, S, X, N)\) and semimartingales on \( V \times W \) solution of the stochastic system \((V \times W, S_{V \times W}, X, N)\). Moreover,
\[
S_{V \times W}(x, (gG_m, w)) = T_g \pi_{G_m} \circ T_e L_g(\xi(x, w)) \times S_N(x, w)
\]
for any \( x \in N, gG_m \in V \subseteq G/G_m \), and any \( g \in G \) such that \( \pi_{G_m}(g) = gG_m \).
Remark 4.8 The last point in this theorem shows that proper symmetries of Stratonovich operators imply the availability of skew-products decompositions around most points in the manifold where the solutions take place. Indeed, the Principal Orbit Type Theorem (see for instance [DK99]) shows that there exists an isotropy subgroup $H$ whose associated isotropy type manifold $M_{(H)} := \{ z \in M \mid G_z = kHk^{-1}, k \in G \}$ is open and dense in $M$. Hence, for any point $m \in M_{(H)}$, there exist slice coordinates around the orbit $G \cdot m$ in which the manifold $M$ looks locally like $G \times H W = G \times H W_H \simeq G/H \times W_H$. This local trivialization of the manifold $M$ into two factors and the results in part (v) of the theorem can be used to split the Stratonovich operator $S$, in order to obtain a locally defined skew-product around all the points in the open dense subset $M_{(H)}$ of $M$.

Proof. As we already said, this construction is much inspired by a similar one available in the context of equivariant vector fields [K90, F91]. In this proof we will mimic the strategy for that result followed in [OR04] Theorem 3.3.5.

We start by noting that the properness of the action guarantees that the isotropy subgroup $G_m$ is compact and hence there exists an open $G_m$-invariant neighborhood $V \subseteq G/G_m$ of $G_m$ and a local section $\sigma : V \subseteq G/G_m \to G$ with the following equivariance property [F91]: $\sigma(h \cdot gG_m) = h \sigma(gG_m)h^{-1}$, for any $h \in G_m$ and $gG_m \in V$. If we now construct with this section the map $F : V \times W \to U \subseteq M$ introduced in Theorem 4.8 that is

$$F(gG_m, w) := \sigma(gG_m) \cdot w,$$

we obtain a $G_m$-equivariant map by considering the diagonal $G_m$-action in $V \times W$. Since for any $w \in W$ we have that $F^{-1}(w) = (G_m, \sigma(G_m)^{-1} \cdot w)$,

$$T_wF^{-1} \circ S(x, w) =: S_V(x, w) \times S_W(x, w) \in \mathcal{L}(T_xN, T_{G_m}V \times T_{\sigma(G_m)^{-1} \cdot w}W).$$

Define

$$S_N(x, w) := T_{\sigma(G_m)^{-1} \cdot w} \Phi_{\sigma(G_m)} \circ S_W(x, w) \in T_wW$$

$$S_T(x, g \cdot w) := T_e \Phi_g \circ T_e \Phi_w \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w)$$

$$= T_e \Phi_{g \cdot w} \circ \text{Ad}_g \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w).$$

(i) Let $z = g \cdot w \in G \cdot W$, $g \in G$, $w \in W$, $x \in N$, and define $\xi : TN \times G \cdot W \to g$ by

$$\xi(x, z) = \text{Ad}_g \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w).$$

It can be seen that $\xi(x, z)$ is well defined by reproducing the steps taken in [OR04] Theorem 3.3.5 (i)]. More specifically, it can be shown that if $z$ is written as $z = g' \cdot w'$ for some other $g' \in G$ and $w' \in W$ then

$$\text{Ad}_g \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w) = \text{Ad}_{g'} \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w').$$

Using (4.12b) and (4.13) we have that

$$S_T(x, g \cdot w) = T_w \Phi_{g \cdot w} \circ \xi(x, g \cdot w)$$

It is an exercise to check that $\xi(x, g \cdot w) = \text{Ad}_g \circ \xi(x, w)$, for any $g \in G$, and hence the Stratonovich operator $S_T$ is $G$-invariant. This $G$-invariance implies by Proposition 2.7 that the image of $S_T(x, z)$ is such that $\text{Im}(S_T(x, z)) \subseteq T_zM_{G_z}$. On the other hand, $\text{Im}(S_T(x, z)) = \text{Im}(T_e \Phi_x \circ \xi(x, z)) \subseteq g \cdot z$, therefore

$$\text{Im}(S_T(x, z)) \subseteq T_zM_{G_z} \cap g \cdot z = T_z(N(G_z) \cdot z).$$
by [OR04] Proposition 2.4.5] and hence $\text{Im}(\xi(x, z)) \subset \text{Lie}(N(G_z))$.

(ii) and (iii) It is immediate to see that the Stratonovich operator $S_N : TN \times W \to TW$ defined in \((4.12a)\) is $G_m$-invariant. Let $w \in W$; using \((4.11)\) and \((4.10)\)
\[
S(x, w) = T_{(G_m,\sigma(G_m)^{-1}, w)}F \circ (S_V(x, w) \times S_W(x, w))
= T_\epsilon \Phi_w \circ T_{\sigma(G_m)}R_{\sigma(G_m)^{-1}} \circ T_{G_m} \sigma \circ S_V(x, w) + T_{\sigma(G_m)^{-1}, w} \Phi_{\sigma(G_m)} \circ S_W(x, w)
= S_T(x, w) + S_N(x, w),
\]
where \((4.12a)\) and \((4.12b)\) have been used. The equality \((4.8)\) then follows from the $G$-invariance of $S$ and $S_T$.

(iv) First of all observe that if $(\eta^w, \Gamma^w)$ is the $g \times W$-valued semimartingale solution of the stochastic system $(g \times W, S_g \times W, X, N)$ with constant initial condition $(0, w) \in g \times W$, then
\[
(\mu, \eta^w) = \int \langle \xi (X, \varphi_t(w))^* (\mu), \delta X \rangle
\]
for any $\mu \in g^*$. In other words, $\eta^w$ may be regarded as the solution of the stochastic differential equation
\[
\delta \eta^w = \xi (X, \varphi_t(w)) \delta X
\]
with initial condition $\eta^w_{t=0} = 0$ a.s. Notice that $\eta^w$ is defined up to time $\tau_{\varphi(w)}$, that is, the time of existence of the solution $\varphi(w)$. Let now $\Gamma_t = g_{\varphi_t(w)}$ be the M-valued semimartingale in the statement. Applying the rules of Stratonovich differential calculus and the Leibniz rule we obtain
\[
\delta \Gamma_t = T_{\tilde{g}_t} \Phi_{\varphi_t(w)}(\delta \tilde{g}_t) + T_{\varphi_t(w)} \Phi_{\varphi_t(w)}(\delta \varphi_t(w))
\]
We rewrite the first summand in this expression as
\[
T_{\tilde{g}_t} \Phi_{\varphi_t(w)}(\delta \tilde{g}_t) = T_{\varphi_t(w)} \Phi_{\tilde{g}_t} \circ T_{\varphi_t(w)} \Phi_{\varphi_t(w)} \circ T_{\tilde{g}_t} L_{\tilde{g}_t^{-1}}(\delta \tilde{g}_t)
= T_{\varphi_t(w)} \Phi_{\tilde{g}_t} \circ T_{\varphi_t(w)} (\delta \eta^w_{\tilde{g}_t})
= T_{\varphi_t(w)} \Phi_{\tilde{g}_t} \circ T_{\varphi_t(w)} \xi (X_t, \varphi_t(w)) \delta X_t
= T_{\varphi_t(w)} \Phi_{\tilde{g}_t} \circ S_T(X_t, \varphi_t(w)) \delta X_t,
\]
where in the second and third line we have used that $\tilde{g}_t$ is a solution of $(G, L, \eta^w, g)$ and equation \((4.14)\), respectively. The second summand of \((4.15)\) can be written as
\[
T_{\varphi_t(w)} \Phi_{\varphi_t(w)}(\delta \varphi_t(w)) = T_{\varphi_t(w)} \Phi_{\varphi_t(w)} \circ S_N(X, \varphi_t(w)) \delta X_t
\]
because $\varphi_t(w)$ is a solution of \((4.9)\). Therefore, using \((4.8)\) we can conclude that
\[
\delta \Gamma_t = T_{\varphi_t(w)} \Phi_{\varphi_t(w)} \circ (S_N(X, \varphi_t(w)) + S_T(X, \varphi_t(w))) \delta X_t
= S(x, \tilde{g}_t \cdot \varphi_t(w)) \delta X_t = S(x, \Gamma_t) \delta X_t
\]
which shows that $\Gamma_t$ is a solution up to time $\tau_{\varphi}$ of the stochastic system $(M, S, X, N)$ with initial condition $z = g \cdot w \in G \cdot W$.

(v) Let $w \in W$ and $h \in G_m = G_w$. Let $\Psi$ be the twisted action of $G_m$ on $W$, that is, $\Psi : G_m \times (G \times W) \to (G \times W)$ defined as $\Psi_h(g, w) := (gh, h^{-1} \cdot w)$, and whose orbit space is the twisted product
Consider now the Stratonovich operator defined by

$$\Phi(x,w) = S_{GW}(x,(gh,w)) = T_{g}L_{h} \circ \xi(x,w) \times S_{N}(x,w)$$

We are going to show that $S_{GW}$ is $G$-invariant under the action defined by $\Psi$. Indeed, given that $G_{w} = G_{m}$, $\Psi_{h}(g,w) = (gh,w)$ for any $h \in G_{m}$, $g \in G$, and $w \in W$, we have

$$S_{GW}(x,\Psi_{h}(g,w)) = S_{GW}(x,(gh,w)) = T_{h}L_{g} \circ \xi(x,w) \times S_{N}(x,w)$$

which shows that $S_{GW}$ is $G_{m}$-invariant.

We can therefore apply the Reduction Theorem 3.1 to conclude that $S_{GW}$ projects onto a stochastic system $(G/G_{m} \times W, S_{GW \times W}, X, N)$ on $G \times G_{m} \times W \simeq G/G_{m} \times W$ with Stratonovich operator

$$S_{GW \times W}(x,(g,G_{m},w)) := T_{g}S_{GW}(x,(g,h,w)) = T_{g}S_{G_{m}} \circ T_{g}L_{h} \circ \xi(x,w) \times S_{N}(x,w), \quad (4.17)$$

where $x \in X$, $w \in W$, and $g \in G$ is any element such that $\pi_{G_{m}}(g) = gG_{m}$. Notice that by (4.8), expression (4.17) proves that the Stratonovich operator $S_{GW \times W}$ is a local skew-product decomposition of $S$ on $G/G_{m} \times W$.

Concerning the solutions, by (iv) any solution of the stochastic system $(M,S,X,N)$ starting at some point $z = g \cdot w \in U \subseteq G \cdot W$ can be written as the image by the action $\Phi$ of the solution $(g_{t}, \varphi_{t}(w))$ of the stochastic system $(G \times W, S_{GW}, X, N)$ starting at $(g,w) \in G \times W$ and defined up to time $T_{g}(w)$. Then, the Reduction Theorem 3.1 guarantees that this solution can be projected to a solution of $(G/G_{m} \times W, S_{GW \times W}, X, N)$ starting at $(gG_{m},w) \in G/G_{m} \times W$, also defined up to time $T_{g}(w)$. Conversely, in order to recover a solution of the original system from a solution $((g_{t},w_{t}))$ of $(G/G_{m} \times W, S_{GW \times W}, X, N)$ we need to invoke the Reconstruction Theorem 3.2 by choosing an auxiliary connection $A \in \Omega^{1}(G,g_{m})$. This will yield a solution $(g_{t},w_{t})$ of $(G \times W, S_{GW}, X, N)$ with $g_{t}$ a $G$-valued semimartingale that can be written as

$$g_{t} = d_{t}h_{t},$$

where $d_{t} : \mathbb{R}_{+} \times \Omega \rightarrow G$ is the horizontal lift of $(gG_{m})_{t}$ with respect to $A$ and $h_{t} : \mathbb{R}_{+} \times \Omega \rightarrow G_{m}$ is a suitable semimartingale on $G_{m}$. The key point is that the image by the action $\Phi$ of the solution $(g_{t},w_{t})$ of $(G \times W, S_{GW}, X, N)$, that is,

$$\Phi(g_{t},w_{t}) = g_{t} \cdot w_{t} = d_{t}h_{t} \cdot w_{t}$$

yields a solution of $(M,S,X,N)$. Notice that the semimartingale $h_{t}$ plays no role. Indeed, let $\sigma : V \subseteq G/G_{m} \rightarrow G$ be the local $G_{m}$-equivariant section introduced in the beginning of the proof. We
already saw in Proposition 4.2 that if \((gG_m)_t : \mathbb{R}_+ \times \Omega \rightarrow G/G_m\) is a \(G/G_m\)-valued semimartingale then \(\sigma ((gG_m)_t)\) is the horizontal lift with respect to the connection \(A_\sigma \in \Omega^1 (\pi_{G_m}^{-1} (V) ; g_m)\) induced by the local section \(\sigma\). Consequently, any solution \(\Gamma_t\) of the initial stochastic system \((M,S,X,N)\) with initial condition \(\Gamma_{t=0} = g \cdot w \in U \subset G \cdot W\) can be locally expressed as \(\sigma ((gG_m)_t) \cdot w_t\) where \(((gG_m)_t, w_t)\) is a solution of the stochastic system \((G/G_m \times W, \sigma G/G_m \times W, X, N)\) with initial condition \((\pi_{G_m} (g), w) \in G/G_m \times W\).  

**Example 4.9 (Liao decomposition of Markov processes)** The possibility of decomposing stochastic processes using a group invariance property has been used beyond the context of stochastic differential equations. For example, Liao [L07] has used what he calls the angular-radial submanifolds of a compact group action to carry out an angular-radial decomposition of the Markov processes that are equivariant with respect to those actions. To be more specific, let \(M\) be a manifold acted upon by a Lie group \(G\) and let \(\Gamma : \mathbb{R}_+ \times \Omega \rightarrow M\) be a \(M\)-valued Markov process with transition semigroup \(P_t\); that is, \(\Gamma\) is a process with càdlàg paths that satisfies the simple Markov property 

\[
E [f (\Gamma_{t+s}) | \mathcal{F}_t] = P_s f (\Gamma_t)
\]
a.s., for \(s < t\) and \(f \in C^\infty_b (M)\), where \(C^\infty_b (M)\) is the space of bounded smooth functions on \(M\), and \(\mathcal{F}_t\) is the natural filtration induced by \(\Gamma\). Furthermore, suppose that the Markov process \(\Gamma\) or, equivalently, its transition semigroup \(P_t\) is \(G\)-equivariant in the sense that 

\[
P_t (f \circ \Phi_g) = (P_t f) \circ \Phi_g
\]

for any \(g \in G\). Additionally, in [L07] it is assumed the existence of a submanifold \(W \subseteq M\) which is **globally transversal** to the \(G\)-action. This means that \(W\) intersects each \(G\)-orbit at exactly one point, that is, for any \(w \in W\), \(G \cdot w \cap W = \{w\}\) and \(M = \bigcup_{w \in W} G \cdot w\). The existence of such global transversal section is a strong hypothesis that only a limited number of actions satisfy. A larger range of applicability of the results in [L07] can be obtained if one is willing to work locally using the slices introduced in this section. Indeed, suppose now that the group \(G\) is not compact but just that the group action is proper; let \(m \in M\) and \(\varphi : G \times G_m W \rightarrow U \subseteq M\) a tube around the orbit \(G \cdot m\) where, additionally, we assume that \(G_m = G_m\) for any \(w \in W\). With this hypothesis which, incidentally is the same one that in part (v) of Theorem 4.7 allowed us to obtain a skew-product decomposition of the invariant Stratonovich operator, the slice \(W\) is a local transversal manifold in the sense of [L07]. 

Let now \(J : U \subseteq M \rightarrow W\) be the projection that associates to each point, the unique element in its orbit that intersects \(W\). Liao proves [L07, Theorem 1] that the radial part \(y := J(\Gamma)\) of the Markov process \(\Gamma\) is also a Markov process with transition semigroup \(Q_t := J^* P_t\). Moreover, if the group \(G\) is compact and \(\Gamma\) is Feller then so is \(y\) and its generator is fully determined by that of \(\Gamma\). 

Let now \(\pi_{G_m} : G \rightarrow G/G_m\) be the canonical projection and let \(\phi : V \times W \rightarrow U\) be the diffeomorphism associated to the local section \(\sigma : V \rightarrow \pi_{G_m}^{-1} (V) \subseteq G\) such that \(\phi (gG_m, w) = \sigma (gG_m) \cdot w\). Let \(\Gamma\) be \(U\)-valued Markov process starting at \(m\) and \(y = J (\Gamma)\) its radial part. Let \(\hat{\Gamma} : \{0 \leq t < \tau_U\} \rightarrow V \subseteq G/G_m\) be the process such that \(\Gamma_t = \sigma (\hat{\Gamma}_t) \cdot y_t\), where \(\tau_U = \inf \{ t > 0 \mid \Gamma_t \notin U \}\). \(\hat{\Gamma}\) is called the angular part of \(\Gamma\). Liao shows (see [L07, Theorem 3]) that the angular process \(\hat{\Gamma}\) is a nonhomogeneous Lévy process under the conditional probability built by conditioning with respect to the \(\sigma\)-algebra generated by the radial process. The reader is encouraged to check with [L07] for precise definitions and statements (see also [L04]).

### 5 Projectable stochastic differential equations on associated bundles

In the previous section we saw how the availability of the slices associated to a proper group action allows the local splitting of the invariant Stratonovich operators using what we called the tangent-normal
decomposition. Additionally, this decomposition yields generically a local skew-product splitting of the invariant Stratonovich operator in question. The key idea behind these splittings was the possibility of locally modeling the manifold where the solutions of the stochastic differential equation take place as a twisted product. A natural setup that we could consider are the manifolds \( M \) where this product structure is global, that is \( M = P \times_G W \), with \( P \) and \( W \) two \( G \)-manifolds. The most standard situation where such manifolds are encountered is when \( M \) is the associated bundle to the \( G \)-principal bundle \( \pi : P \to Q \): let \( W \) be an effective left \( G \)-space and \( \tilde{\pi} : P \times_G W \to Q \), \( \tilde{\pi}([p, w]) = \pi(p) \). A classical theorem in bundle theory shows that such construction is a principal \( G \)-bundle with fiber \( W \) and it is usually referred to as the bundle associated to \( \pi : P \to Q \) with fiber \( W \). To be more specific, consider the commutative diagram that defines \( \tilde{\pi} \):

\[
\begin{array}{ccc}
P \times W & \xrightarrow{\kappa} & P \times_G W \\
\downarrow_{pr_1} & \searrow & \downarrow_{\tilde{\pi}} \\
P & \xrightarrow{\pi} & Q.
\end{array}
\]  

(5.1)

In this diagram, \( \kappa_p : \{p\} \times W \to \tilde{\pi}^{-1}(\pi(p)) =: (P \times_H W)_{\pi(p)} \) is a diffeomorphism (see for instance [KMS98] 10.7]). Hence, the correspondence \( p \to \kappa_p, p \in P \), allows us to consider the elements of \( P \) as diffeomorphisms from the typical fiber \( W \) of \( P \times_G W \) to \( \tilde{\pi}^{-1}(q) \), with \( q = \pi(p) \).

Stochastic processes and diffusions on associated bundles have deserved certain attention in the literature (see [L89] for example) because, as we will see in the following paragraphs, the available geometric structure makes possible a Reduction-Reconstruction procedure that in some cases implies the existence of a global skew-product decomposition. In this context, the notion of invariance is replaced by what we will call \( \tilde{\pi} \)-projectability: if \( N \) is a manifold and \( S : TN \times M \to TM \) a Stratonovich operator from \( N \) to \( M \), we say that \( S \) is \( \tilde{\pi} \)-projectable if the Stratonovich operator \( S_Q \) from \( N \) to \( Q \)

\[
S_Q (x, q) := T_{[p,w]} \tilde{\pi} \circ S (x, [p, w]) \in L(T_xN, T_{[p,w]}M)
\]

is well defined, where \([p, w] \in M\) is any point such that \( \tilde{\pi} ([p, w]) = q \in Q \).

**Theorem 5.1** Let \( \tilde{\pi} : M = P \times_G W \to Q \) be the associated bundle introduced in the previous discussion. Let \( N \) be a manifold, \( S : TN \times M \to TM \) a \( \tilde{\pi} \)-projectable Stratonovich operator onto \( Q \), and \( X : \mathbb{R}_+ \times \Omega \to N \) a \( N \)-valued semimartingale. Then there exist a Stratonovich operator \( S_{P \times W} : TN \times (P \times W) \to TP \times TW \) with the property that if \((p_t, w_t)\) is any solution of the stochastic system \((P \times W, S_{P \times W}, X, N)\) with initial condition \((p, w) \in P \times W\), then \( \Gamma_t := \kappa(p_t, w_t) \) is the solution of \((M, S, X, N)\) starting at \([p, w]\). Furthermore, \( p_t \) can be written as the horizontal lift of \( \tilde{\pi}(\Gamma_t) \) with respect to an auxiliary connection \( A \in \Omega^1 (P; \mathfrak{g}) \). Conversely, if \( \Gamma_t \) is a solution of \((M, S, X, N)\) and \( p_t \) the horizontal lift of \( \tilde{\pi}(\Gamma_t) \) with respect to \( A \), then \((p_t, \kappa_{\mu}^{-1}(\Gamma_t))\) is a solution of \((P \times W, S_{P \times W}, X, N)\).

**Proof.** Let \( A \in \Omega^1 (P; \mathfrak{g}) \) be an auxiliary principal connection for \( \pi : P \to Q \) and let \( \hat{A}_p : T_{\pi(p)}Q \to \text{Hor}_p P \subseteq T_p P \) be the inclusion of the tangent space \( T_S Q \) at \( q = \pi(p) \) into the horizontal space \( \text{Hor}_p P \) at \( p \in P \) defined by \( A \). Consider the family of linear maps \( \hat{A}_{[p,w]} : T_{\tilde{\pi}([p,w])}Q \to T_{[p,w]}M \) for any \([p, w] \in P \times_G W\) as

\[
\hat{A}_{[p,w]} = T_p \kappa_w \circ \hat{A}_p,
\]

(5.2)

where \( \kappa_w(p) := \kappa(p, w) \) for any \( w \in W \). The family of maps \( \{\hat{A}_{[p,w]} \mid [p, w] \in M\} \) define what is called the induced connection \( A \) ([KMS98] 11.8]) on \( P \times_G W \) by \( A \in \Omega^1 (P; \mathfrak{g}) \). It can be easily checked that \( A \) is well-defined, that is, the expression (5.2) does not depend on the particular choice of \( p \in P \) and \( w \in W \) in the class \([p, w] \in P \times_G W\) used to define it. Indeed, if \([p, w] = [p', w']\) then there exists some \( g \in G \) such that \( p' = g \cdot p \) and \( w' = g^{-1} \cdot w \). Since the connection \( A \) is principal,
\[ \hat{A}_{p'} = T_p R_g \circ \hat{A}_p, \] where \( R : G \times P \to P \) denotes the \( G \)-right action on \( P \). On the other hand, since \( \kappa (p' = p \cdot g, w') = \kappa (p, g, w) \), we have
\[ T_{p'} \kappa_{w'} \circ T_p R_g = T_p \kappa_{g \cdot w'} \quad \text{or, equivalently,} \quad T_{p'} \kappa_{w'} = T_p \kappa_{g \cdot w'} \circ T_{p'} R_{\pi \cdot R_{g^{-1}}}. \] (5.3)

Therefore, \( \hat{A}_{p', w'} = T_{\pi' \kappa_{w'}} \circ \hat{A}_{p'} = T_{\pi' \kappa_{g \cdot w'}} \circ T_{p'} R_{g^{-1}} \circ T_p R_g \circ \hat{A}_p = T_{\pi' \kappa_{w'}} \circ \hat{A}_p = \hat{A}_{[p, w]} \).

Let \( S_Q : T[\pi \circ Q] \to TQ \) be the Stratonovich operator defined as
\[ S_Q (x, q) := T_{[p, w] \pi} \circ S (x, [p, w]), \] (5.4)

where \([p, w] \in P \times G W\) is any point such that \( \pi ([p, w]) = q \), \( x \in N \), and \( w \in W \). This Stratonovich operator is well-defined because \( S \) is by hypothesis \( \pi \)-projectable. Let \( \hat{H}_{[p, w]} : T_{[p, w]} M \to \text{Hor}_{[p, w]} M \subseteq T_{[p, w]} M \) and \( \hat{V}_{[p, w]} : T_{[p, w]} M \to \text{Ver}_{[p, w]} M \subseteq T_{[p, w]} M \) be the projections onto the horizontal and vertical spaces associated to \( A \), respectively, at \([p, w] \in P \times G W\). Define the Stratonovich operator \( S_{p \times W} : T[\pi \times W] \to T(P \times W) \) as
\[ S_{p \times W} (x, (p, w)) = \hat{A}_p \circ S_Q (x, \pi (p)) \times (T_w \kappa_p)^{-1} \circ \hat{V}_{[p, w]} \circ S (x, [p, w]) \in L (T_x N, T_{(p, w)} (P \times W)) \] (5.5)
for any \( x \in N \), \( w \in W \), and \( p \in P \). Recall from (5.1) that \( \kappa_p : W \to M_{\pi(p)} \) is a diffeomorphism for any \( p \in P \) and hence \((T_w \kappa_p)^{-1}\) exists as a map. Now, we claim that if \((p_t, w_t)\) is a \((P \times W)\)-valued semimartingale solution of the stochastic system \((P \times W, S_{p \times W}, X, N)\) then \( \Gamma_t := \kappa_p (w_t) \) is a solution of \((M, S, X, N)\). Indeed, applying the Stratonovich rules for differential calculus,
\[ \delta \Gamma_t = T_{w_t \kappa_{p_t}} (\delta w_t) + T_{p_t} \kappa_{w_t} (\delta p_t) \]
\[ = T_{w_t \kappa_{p_t}} \circ (T_{w_t} \kappa_{p_t})^{-1} \circ \hat{V}_{[p_t, w_t]} \circ S (X_t, [p_t, w_t]) \delta X_t + T_{p_t} \kappa_{w_t} \circ \hat{A}_{p_t} \circ S_Q (X_t, \pi (p_t)) \delta X_t \]
\[ = \hat{V}_{[p_t, w_t]} \circ S (X_t, [p_t, w_t]) \delta X_t + \hat{A}_{[p_t, w_t]} \circ S_Q (X_t, \pi (p_t)) \delta X_t \]
\[ = \hat{V}_{[p_t, w_t]} \circ S (X_t, [p_t, w_t]) \delta X_t + \hat{H}_{[p_t, w_t]} \circ S (X_t, [p_t, w_t]) \delta X_t = S (X_t, [p_t, w_t]) \delta X_t = S (X_t, \Gamma_t) \delta X_t, \]
and hence \( \Gamma_t \) is a solution of \((M, S, X, N)\).

Conversely, let \( \Gamma_t \) be a solution of \((M, S, X, N)\) such that \( \Gamma_{t=0} = [p, m] \) a.s. and let \( p_t \) be the horizontal lift of \( \pi (\Gamma_t) \) with respect to the auxiliary connection \( A \in \Omega^1 (P, g) \) starting at some \( p_0 \in \pi^{-1} (\pi ([p, w])) \). Define \( \tilde{w}_t := \kappa_{p_t}^{-1} (\Gamma_t) \). Observe that \( \tilde{w}_{t=0} = \tilde{w}_0 \) is such that \([p_0, \tilde{w}_0] = [p, m] \). Since \( \kappa_p : W \to M_{\pi(p)} \) is a diffeomorphism, \( \tilde{w}_t \) is uniquely determined a.s. by \( \Gamma_t \) once \( p_t \) is fixed. Indeed, \( \tilde{w}_t \) is the unique semimartingale such that \( \kappa_{p_t} (\tilde{w}_t) = \Gamma_t \). But we have already seen that the solution of \((P \times W, S_{p \times W}, X, N)\) starting at \((p_0, \tilde{w}_0) \in P \times W\) may be expressed as \((p_t, w_t)\), with \( p_t \) the fixed horizontal lift of \( \pi (\Gamma_t) \) that we have been using all along. Therefore \( w_t = \tilde{w}_t \) a.s. necessarily and \( w_t = \kappa_{p_t}^{-1} (\tilde{w}_t) = \kappa_{p_t}^{-1} (\Gamma_t) \).

**Corollary 5.2** Using the same notation as in the proof of Theorem 5.1, suppose that \((T_w \kappa_p)^{-1} \circ \hat{V}_{[p, w]} \circ S (x, [p, w])\) in (5.3) does not depend on \( p \in P \). In such case there exists a unique \( G \)-invariant Stratonovich operator \( S_W : T[\pi \circ W] \to TW \) from \( N \) to \( W \) determined by the relation
\[ T_{w \kappa} \circ S_W (x, w) = \hat{V}_{[p, w]} \circ S (x, [p, w]) \] (5.6)
for any \( x \in N \), \( w \in W \), and \( p \in P \). Moreover, \( S_{p \times W} \) in (5.5) admits the skew-product decomposition
\[ S_{p \times W} (x, (p, w)) = \hat{A}_p \circ S_Q (x, \pi (p)) \times S_W (x, w). \]
Proof. First of all notice that as \((T_w\kappa_p)^{-1} \circ \hat{V}_{[p,w]} \circ S(x,[p,w])\) does not depend on \(p \in P\), the expression (5.6) is a good definition that uniquely determines \(S_W\). The only non-trivial point in the statement that needs proof is the \(G\)-invariance of \(S_W\): let \(q \in G\) and \((p',w')\), \((p,w) \in P \times W\) such that \(p' = p \cdot g\) and \(w' = g^{-1} \cdot w\). Since \(\hat{V}_{[p,w]} \circ S(x,[p,w]) = \hat{V}_{[p',w']} \circ S(x,[p',w'])\), we necessarily have

\[
T_{w'} \kappa_{p'} \circ S_W(x,w') = T_w \kappa_p \circ S_W(x,w).
\]

As \(\kappa(p,g,w) = \kappa(p,g \cdot w)\), we have that \(T_{g \cdot w} \kappa_p \circ T_w l_g = T_w \kappa_{p \cdot g}\), where \(l : G \times W \rightarrow W\) is the \(G\)-action on \(W\). Thus,

\[
T_{w'} \kappa_{p'} \circ S_W(x,w) = T_w \kappa_p \circ T_{g^{-1} \cdot w} l_g \circ S_W(x,g^{-1} \cdot w).
\]

Since \(T_w \kappa_p : T_w W \rightarrow T_{[p,w]} (P \times_G W)\) is an isomorphism, we conclude comparing the two previous relations that

\[
S_W(x,w) = T_{g^{-1} \cdot w} l_g \circ S_W(x,g^{-1} \cdot w),
\]

necessarily, which amounts to \(S_W\) being \(G\)-invariant. \(\blacksquare\)

Remark 5.3 It is worth noticing that, under the hypotheses of Corollary 5.2 and unlike Theorem 4.7, the skew-product decomposition of \(S_{P \times W}\) is now global.

Remark 5.4 If the hypotheses of Corollary 5.2 hold, we can solve a stochastic system \((M,S,X,N)\) on the associated bundle \(\bar{\pi} : M = P \times_G W \rightarrow Q\) with \(\bar{\pi}\)-projectable Stratonovich operator \(S\) using the following reduction-reconstruction scheme. On one hand, we find the solution starting at \(\bar{\pi}([p,w])\) on the base space system \((Q,SQ,X,N)\), where \(SQ\) was given in (5.3). We lift then this solution to the principal bundle \(P\) using an auxiliary connection \(\Lambda \in \Omega^1 (P;g)\). We choose the lift \(p_t\) starting at some \(p_0 \in \pi^{-1} (\bar{\pi}([p,w]))\). On the other hand, we find the solution \(w_t\) of the independent stochastic system \((W,S_W,X,N)\) with initial condition \(w_0\) such that \(\kappa(p_0,w_0) = [p,w]\). Then \(\kappa_{p_t}(w_t)\) is the solution of \((M,S,X,N)\) starting at \([p,w]\).

Example 5.5 Projectable SDEs and the horizontal-vertical factorization of diffusion operators. In this example we show how some of the results in [LS9] on the factorization of certain semielliptic differential operators on associated bundles can be rethought in the light of the results in Theorem 5.1 and Corollary 5.2. We recall that a second order differential operator \(L_Q \in \mathfrak{X}_2(Q)\) on a manifold \(Q\) is called semielliptic if any point \(q \in Q\) has an open neighborhood \(U\) where \(L_Q\) can be locally written as

\[
L_Q|_U = \sum_{i=1}^n \mathcal{L}_{Y_i} \mathcal{L}_{Y_i} + \mathcal{L}_{Y_0}
\]

for some \(Y_0, Y_i \in \mathfrak{X}(U), i = 1, ..., s\). Such a semielliptic operator can be seen as the infinitesimal generator for the laws of the solution semimartingales of the following stochastic system \((Q,S_Q,X,R \times R^s)\) (see for instance [IW89], Theorem 1.2, page 238): let \(X : R_+ \times \Omega \rightarrow R \times R^s\) be the semimartingale

\[
X_t(\omega) = (t, B^1_t(\omega), ..., B^s_t(\omega)),
\]

where \((B^1, ..., B^s)\) is a \(s\)-dimensional Brownian motion and consider the Stratonovich operator

\[
S_Q(x,q) : T_x (R \times R^s) \rightarrow T_q U \subseteq T_q Q \quad (u,v^1, ..., v^s) \mapsto u Y_0 + \sum_{i=1}^s v^i Y_i.
\]

Let now \(G\) be a Lie group, \(\pi : P \rightarrow Q\) a principal \(G\)-bundle, and consider a manifold \(W\) acted upon by \(G\) via the map \(l : G \times W \rightarrow W\). Let \(L_W \in \mathfrak{X}_2(W)\) be the semielliptic differential operator on \(W\) given by

\[
L_W = \sum_{i=1}^n \mathcal{L}_{Z_i} \mathcal{L}_{Z_i} + \mathcal{L}_{Z_0}
\]
where \( Z_0, Z_1, \ldots, Z_n \in \mathcal{X}(V) \) on some \( V \subseteq W \). As we just did, we will consider \( L_W \) as the generator for the laws of the solutions of the stochastic system \( (W, S_W, X', \mathbb{R}^{n+1}) \), where \( X' : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n+1} \) is a noise semimartingale constructed using the time process \( t \) and \( n \) independent Brownian motions, and \( S_W \) is the Stratonovich operator given by

\[
S_W (x, w) : T_x (\mathbb{R} \times \mathbb{R}^n) \to T_w V \subseteq T_w W
\]

\[
(u, v^1, \ldots, v^n) \mapsto uZ_0 + \sum_{i=1}^n v^iZ_i.
\]

In addition, we will assume that both \( L_W \) and \( S_W \) are \( G \)-invariant. Let \( \tilde{A} \) be a connection on the associated bundle \( M = P \times_G Q \) and define the Stratonovich operator \( S : T\mathbb{R}^{n+1} \times M \to TM \) as

\[
S(x, [p, w]) = T_w\kappa_p \circ S_W (x, w) + \tilde{A}_{[p, w]} \circ S_Q (x, \pi (p))
\]

consistently with the notation introduced so far. Taking \((B^1_t, \ldots, B^{n+s}_t)\) a \((n+s)\)-dimensional Brownian motion, the stochastic system \((M, S, \tilde{X}, \mathbb{R}^{n+s+1})\) with stochastic component \( \tilde{X} : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n+s+1} \) given by \( \tilde{X}_t (\omega) = (t, B^1_t (\omega), \ldots, B^{n+s}_t (\omega)) \) satisfies by construction the hypotheses of Theorem 5.1 and Corollary 5.2. The projected stochastic system of \((M, S, \tilde{X}, \mathbb{R}^{n+s+1})\) onto \( Q \) is obviously \((Q, S_Q, X, \mathbb{R}^{n+1})\) and the one induced in the typical fiber \( W \) is \((W, S_W, X', \mathbb{R}^{n+1})\). It is straightforward to check that the probability laws of the solutions of \((M, S, \tilde{X}, \mathbb{R}^{n+s+1})\) have as infinitesimal generator

\[
L_M = \tilde{L}_Q + L_W,
\]

where \( \tilde{L}_Q \) is what Liao [L89] calls the horizontal lift of \( L_Q \) and \( L_W \) the vertical operator induced by \( L_W \).

Many of the results presented in [L89] about the factorization [5.8] of semielliptic operators on associated bundles and their related diffusions can be understood from the perspective of stochastic systems and stochastic differential equations that we have adopted here using Theorem 5.1 and Corollary 5.2. In order to illustrate this point consider the following result in Liao’s article about Riemannian submersions [see also [EK85] and [BB82, Remark 1.10, page 185]]: let \((M, \eta)\) be a complete Riemannian space with Riemannian metric tensor \( \eta \) and let \( \tilde{\pi} : M \to Q \) be a Riemannian submersion with totally geodesic fibers. In this setup, \( \tilde{\pi} : M \to Q \) is an associated bundle whose structure group \( G \) is the group of isometries of the standard fiber \( W := \pi^{-1} (q_0) \) for some \( q_0 \in Q \) [H60]. Indeed, it can be checked that all the fibers of \( \tilde{\pi} : M \to Q \) are isometric, so we can take any of them as a standard fiber, and that \( G \) has finite dimension [BB82, Remark 1.10, page 185]. Let \( \pi : P \to Q \) be the corresponding principal bundle. Additionally, since \( \kappa_p : W \to \pi^{-1} (q) \) is an isometry for any \( p \in P \), the restriction \( \eta_{\pi^{-1} (q)} \) of the metric \( \eta \) to \( \pi^{-1} (q) \) may be considered as induced from the metric \( \eta_{\tilde{\pi}^{-1} (q_0)} \) of \( W \) by \( \kappa_p \), which, in addition, is invariant by \( G \). Then,

\[
\Delta_M = \tilde{\Delta}_Q + \Delta_W,
\]

where \( \Delta_Q \) is the Laplacian on \( Q \) and \( \Delta_W \) the Laplacian on \( W \) ([L89, Proposition 3]). As a consequence of (5.4), if \( \Gamma_t \) is a \( M \)-valued Brownian motion associated to the Laplacian \( \Delta_M \) on \( M \) then \( \tilde{\pi} (\Gamma_t) \) is a Brownian motion on \( Q \) with generator \( \Delta_Q \) ([LS82, Theorem 10E]). Let now \( \Lambda \in \Omega^1 (P, g) \) be the principal connection on \( P \) whose associated connection \( A \) on \( \tilde{\pi} : M \to Q \) is such that \( \text{Hor}_m = \text{Ver}_m \) for any \( m \in M \), that is, the horizontal subspace \( \text{Hor}_m \subset T_m M \) of \( A \) is the orthogonal complement of \( \text{Ver}_m \), \( m \in M \). Then, if \( \pi_t \) denotes the horizontal lift of \( \pi (\Gamma_t) \) to \( P \) with respect to \( A \) then \( \kappa_{\pi_t} (\Gamma_t) \) is a Brownian motion on \( W \) with generator \( \Delta_W \) [L89, Proposition 6].

6 The Hamiltonian case

Hamiltonian dynamical systems are a class of differential equations in the non-stochastic deterministic context in which reduction techniques have been much developed. This is mainly due to their central
role in mechanics and applications to physics and also to the added value that symmetries usually have in this category. As we saw in Proposition 2.7, the symmetries of a stochastic differential equation bring in their wake certain invariance properties of its flow that have to do with the preservation of the isotropy type submanifolds. Symmetric Hamiltonian deterministic systems also preserve isotropy type submanifolds but they usually exhibit additional invariance features caused by the presence of symmetry induced first integrals or constants of motion, usually encoded as components of a momentum map.

The goal in this section is to show that the reduction and reconstruction techniques that have been developed for deterministic Hamiltonian dynamical systems can be extended to the stochastic Hamiltonian systems that have been introduced in [LO07] as a generalization of those in [BS11] and that we now briefly review. The reader is encouraged to look at the original references [AM78, OR04] and references therein.

Let \((M, \{\cdot, \cdot\})\) be a finite dimensional Poisson manifold, \(X : \mathbb{R}_+ \times \Omega \rightarrow V\) a continuous semimartingale that takes values on the vector space \(V\) with \(X_0 = 0\), and let \(h : M \rightarrow V^*\) be a smooth function with values in \(V^*\), the dual of \(V\). Let \(\{\epsilon^1, \ldots, \epsilon^r\}\) be a basis of \(V^*\) and let \(h_1, \ldots, h_r \in C^\infty(M)\) be such that \(h = \sum_{i=1}^r h_i \epsilon^i\). The stochastic Hamiltonian system associated to \(h\) with stochastic component \(X\) is the stochastic differential equation

\[
\delta \Gamma^h = H(X, \Gamma)\delta X
\]

defined by the Stratonovich operator \(H(v, z) : T_v V \rightarrow T_z M\) defined by

\[
H(v, z)(u) := \sum_{i=1}^r \langle \epsilon^i, u \rangle X_{h_i}(z),
\]

where \(X_{h_i}\) is the Hamiltonian vector field associated to \(h_i \in C^\infty(M)\). In this case, the dual Stratonovich operator \(H^*(v, z) : T^*_z M \rightarrow T^*_v V\) of \(H(v, z)\) is given by \(H^*(v, z)(\alpha_z) = -dh(z) \cdot B(z)(\alpha_z)\), where \(B^* : T^* M \rightarrow TM\) is the vector bundle map naturally associated to the Poisson tensor \(B \in \Lambda^2(M)\) of \(\{\cdot, \cdot\}\) and \(dh = \sum_{i=1}^r dh_i \otimes \epsilon^i\). We will usually summarize this construction by saying that \((M, \{\cdot, \cdot\}, h, X)\) is a stochastic Hamiltonian system. A case in which we will dedicate particular attention is the one in which the Poisson manifold \((M, \{\cdot, \cdot\})\) is actually symplectic with symplectic form \(\omega\) and the bracket \(\{\cdot, \cdot\}\) is obtained from \(\omega\) via the expression \(\{f, h\} = \omega(X_f, X_h)\), \(f, h \in C^\infty(M)\).

### 6.1 Invariant manifolds and conserved quantities of a stochastic Hamiltonian system

As we already said, the presence of symmetries in a Hamiltonian system forces the appearance of invariance properties that did not use to occur for arbitrary symmetrical dynamical systems. Before we proceed with the study of those conservation laws in the stochastic Hamiltonian case, we extract some conclusions on invariant manifolds that can be obtained from Proposition 2.7 in that situation.

**Proposition 6.1** Let \((M, \{\cdot, \cdot\}, h : M \rightarrow V^*, X)\) be a stochastic Hamiltonian system. Let \(\{\epsilon^1, \ldots, \epsilon^r\}\) be a basis of \(V^*\) and write \(h = \sum_{i=1}^r h_i \epsilon^i\). Consider the following situations:

- **(i)** Suppose that \(M\) is symplectic (respectively, Poisson) and let \(z \in M\) be such that \(dh(z) = 0\) (respectively, \(X_{h_i}(z) = 0\), for all \(i \in \{1, \ldots, r\}\)). Then, the Hamiltonian semimartingale \(\Gamma^h\) with constant initial condition \(\Gamma_0(\omega) = z\), for all \(\omega \in \Omega\), is an equilibrium, that is \(\Gamma^h = \Gamma_0\).

- **(ii)** Let \(S_1, \ldots, S_r\) be submanifolds of \(M\) with transverse intersection \(S := S_1 \cap \ldots \cap S_r\), such that \(X_{h_i}(z_i) \in T_{z_i} S_i\), for all \(z_i \in S_i\) and \(i \in \{1, \ldots, r\}\). Then \(S\) is an invariant submanifold of the stochastic Hamiltonian system \((M, \{\cdot, \cdot\}, h : M \rightarrow V^*, X)\).
(iii) The symplectic leaves of \((M,\{\cdot,\cdot\})\) are invariant submanifolds of the stochastic Hamiltonian system \((M,\{\cdot,\cdot\},h:M \to V^*,X)\).

**Proof.** It is a direct consequence of Proposition 2.6 and of the fact that the Stratonovich operator is given by \(H(v,z)(u) := \sum_{i=1}^r \langle e^i, u \rangle X_{h_i}(z)\). In (i) the hypothesis \(dh(z) = 0\) implies in the symplectic case that \(X_{h_i}(z) = 0\), for all \(i \in \{1, \ldots, r\}\). Hence, both in the symplectic and in the Poisson cases \(H(v,z) = 0\) and hence by Proposition 2.6 the point \(z\) is an invariant submanifold and consequently an equilibrium. For (ii) it suffices to recall that the transversality hypothesis implies that \(T_z S = T_z S_1 \cap \ldots \cap T_z S_r\), for any \(z \in S\). (iii) follows from the fact that the tangent space to the symplectic leaves is spanned by the Hamiltonian vector fields and hence \(\text{Im}(H(v,z)) \subset T_z L_z\), for any \(z \in M\) and any \(v \in V\), with \(L_z\) the symplectic leaf that contains the point \(z\). ■

In the Hamiltonian case, most of the invariant manifolds of a system come as the level sets of a conserved quantity (also called first integral) of the motion. In the next definition we come back for a second to the case of general stochastic differential equations.

**Definition 6.2** Let \(M\) and \(N\) be two manifolds, let \(X: \mathbb{R}_+ \times \Omega \to N\) be a \(N\)-valued semimartingale, and let \(S: TN \times M \to TM\) be a Stratonovich operator. A function \(f \in C^\infty(M)\) is said to be a conserved quantity (respectively strongly conserved quantity) of the stochastic differential equation associated to \(X\) and \(S\) when for any solution semimartingale \(\Gamma\) we have that \(f(\Gamma) = f(\Gamma_0)\) (respectively, when \(S^*(x,z)(df(z)) = 0\), for any \(x \in N\), \(z \in M\)).

It is immediate to check that any strongly conserved quantity is a conserved quantity. The concept of strongly conserved quantity can be equally defined for Schwartz operators. Indeed, it can be shown that if \(S(x,z): T_zN \to T_zM\) is a Stratonovich operator and \(S(x,z): \tau_2 N \to \tau_2 M\) is the Schwartz operator induced by \(S\), then \(f \in C^\infty(M)\) is a strongly conserved quantity for \(S\) if and only if

\[
S^*(x,z)(d_2 f(z)) = 0.
\]

(6.3)

for any \(z \in M\) and any \(x \in N\). We recall that the second order one-form \(d_2 f \in \Omega_2(M)\) is defined as \(d_2 f(L)(z) = L[f](z)\), for any \(L \in \tau_2 M\).

We now go back to the Hamiltonian category. Hamiltonian conserved quantities have an interesting partial characterization in terms of Poisson commutation relations with the components of the Hamiltonian function that the reader can find as Proposition 2.11 of [LO07]. In the case of strongly conserved quantities the situation is much simpler, as the next proposition shows.

**Proposition 6.3** Let \((M,\{\cdot,\cdot\},h:M \to V^*,X)\) be a stochastic Hamiltonian system. Let \(\{e^1, \ldots, e^r\}\) be a basis of \(V^*\) and write \(h = \sum_{i=1}^r h_i e^i\). Consider the Stratonovich operator \(H\) given by (6.3). Then, \(f \in C^\infty(M)\) is a strongly conserved quantity of \(H\) if and only if \(\{f, h_i\} = 0\) for all \(i = 1, \ldots, r\). ([LO07]).

**Proof.** Let \(f \in C^\infty(M)\), \(v \in V\), \(z \in M\), and \(u \in T_z V\). By (6.2),

\[
\langle H^*(v,z)(df(z)), u \rangle = \langle df(z), H(v,z)(u) \rangle = \sum_{i=1}^r \{f, h_i\}(z) \langle u, e^i \rangle.
\]

Since \(u \in T_z V\) is arbitrary, \(H^*(v,z)(df(z)) = 0\) if and only if \(\{f, h_i\}(z) = 0\). ■

We now concentrate on the conserved quantities that one can associate to the invariance of a Hamiltonian system with respect to a group action. We recall that given a Lie group \(G\) acting on the Poisson manifold \((M,\{\cdot,\cdot\})\) (respectively, symplectic \((M,\omega)\)) via the map \(\Phi: G \times M \to M\), we will say that the action is canonical when for any \(g \in G\) and \(f,h \in C^\infty(M)\), \(\{f,h\} \circ \Phi_g = \{\Phi^*_g f, \Phi^*_g h\}\) (respectively, \(\Phi^*_g \omega = \omega\)). In this context, we will say that the Hamiltonian system \((M,\{\cdot,\cdot\},h:M \to V^*,X)\)
is $G$-invariant whenever the $G$-action on $M$ is canonical and the Hamiltonian function $h : M \to V^*$ is $G$-invariant. Notice that the invariance of $h$ and the canonical character of the action imply that the associated Stratonovich operator $H$ is also $G$-invariant. Indeed, Let $\{e^1, \ldots, e^r\}$ be a basis of $V^*$ and write $h = \sum_{i=1}^r h_i e^i$; if $h$ is $G$-invariant, then so are the components $h_i$, $i \in \{1, \ldots, r\}$, that is $h_i \in C^\infty(M)^G$, and hence, for any $g \in G$ we have that $T\Phi_g \circ X_{h_i} = X_{h_i} \circ \Phi_g$, which implies that $H(v, z)(u) := \sum_{i=1}^r \langle e^i, u \rangle X_{h_i}(z)$ is $G$-invariant.

Now suppose that $M$ is a Poisson manifold $(M, \{\cdot, \cdot\})$ acted properly and canonically upon by a Lie group $G$. We also recall that the optimal momentum map $\Omega : M \to M/D_G$ of the $G$-action on $(M, \{\cdot, \cdot\})$ is the projection onto the leaf space of the integrable distribution $D_G \subset TM$ (in the generalized sense of Stefan-Sussmann) given by $D_G := \{X_f \mid f \in C^\infty(M)^G\}$.

**Proposition 6.4** Let $(M, h, X, V)$ be a standard Hamiltonian system acted properly and canonically upon by a Lie group $G$ via the map $\Phi : G \times M \to M$. Suppose that $h : M \to V^*$ is a $G$-invariant function.

(i) **Law of conservation of the isotropy:** The isotropy type submanifolds $M_I$ are invariant submanifolds of the stochastic Hamiltonian system associated to $h$ and $X$, for any isotropy subgroup $I \subset G$.

(ii) **Noether’s Theorem:** If the $G$-action on $(M, \{\cdot, \cdot\})$ has a momentum map associated $J : M \to g^*$ then its level sets are left invariant by the stochastic Hamiltonian system associated to $h$ and $X$. Moreover, its components are conserved quantities.

(iii) **Optimal Noether’s Theorem:** The level sets of the optimal momentum map $\tilde{J} : M \to M/D_G$ are left invariant by the stochastic Hamiltonian system associated to $h$ and $X$.

**Proof.** (i) As we already saw, the $G$-invariance of $h$ implies that $H(v, z)(u) := \sum_{i=1}^r \langle e^i, u \rangle X_{h_i}(z)$ is $G$-invariant. The statement follows from Proposition 2.7. (ii) Let $\xi \in g$ be arbitrary and let $J^\xi := \langle J, \xi \rangle \in C^\infty(M)$ be the corresponding component. The $G$-invariance of the components $h_i$ of the Hamiltonian implies that $\{J^\xi, h_i\} = -d h_i \cdot \xi_M = 0$, where $\xi_M \in \mathfrak{X}(M)$ is the infinitesimal generator associated to the element $\xi$. By formula (2.8) in [LO07] we have that

$$J^\xi(G_\Gamma) - J^\xi(G_0) = \sum_{j=1}^r \int \{J^\xi, h_j\} \delta X^j = 0,$$

where $X_j$, $j \in \{1, \ldots, r\}$, are the components of $X$ in the basis $\{e_1, \ldots, e_r\}$ of $V$ dual to the basis $\{e^1, \ldots, e^r\}$ of $V^*$. Since this equality holds for any $\xi \in g$, we have that $J(G^h) = J(G_0)$ and the result follows. (iii) It is a straightforward consequence of the construction of the optimal momentum map and Proposition 2.6.
6.2 Stochastic Hamiltonian reduction and reconstruction

The goal of this section is showing that stochastic Hamiltonian systems share with their deterministic counterpart a good behavior with respect to symmetry reduction. The main idea that our following theorem tries to convey to the reader is that the symmetry reduction of a stochastic Hamiltonian system yields a stochastic Hamiltonian system, that is, the stochastic Hamiltonian category is stable under reduction.

The following theorem spells out, in the simplest possible case, how to reduce symmetric Hamiltonian stochastic systems. In a remark below we give the necessary prescriptions to carry this procedure out in more general situations. The main simplifying hypothesis is the freeness of the action. We recall that stochastic systems. In a remark below we give the necessary prescriptions to carry this procedure out reduction.

Theorem 6.7 Let \((M, \{\cdot , \cdot \}, h : M \rightarrow V^*, X)\) be a stochastic Hamiltonian system that is invariant with respect to the canonical, free, and proper action \(\Phi : G \times M \rightarrow M\) of the Lie group \(G\) on \(M\).

(i) Poisson reduction: The projection \(h_{M/G}\) of the Hamiltonian function \(h\) onto \(M/G\), uniquely determined by \(h_{M/G} \circ \pi = h\), with \(\pi : M \rightarrow M/G\) the orbit projection, induces a stochastic Hamiltonian system on the Poisson manifold \((M/G, \{\cdot , \cdot \}_{M/G})\) with stochastic component \(X\) and whose Stratonovich operator \(H_{M/G} : TV \times M/G \rightarrow T(M/G)\) is given by

\[
H_{M/G}(v, \pi(z))(u) = T_z \pi (H(v, z)(u)) = \sum_{i=1}^{r} \langle \epsilon^i, u \rangle X_{h_{M/G}^i}(\pi(z)), \quad u, v \in V \text{ and } z \in M. \tag{6.4}
\]

In the previous expression \(\{\epsilon^1, \ldots, \epsilon^r\}\) is a basis of \(V^*\), \(h_{M/G} = \sum_{i=1}^{r} h_i^{M/G} \epsilon^i\), and \(h = \sum_{i=1}^{r} h_i \epsilon^i\); notice that the functions \(h_i^{M/G} \in C^\infty(M/G)\) are the projections of the components \(h_i \in C^\infty(M)^G\), that is \(h_i^{M/G} \circ \pi = h_i\). Moreover, if \(\Gamma\) is a solution semimartingale of the Hamiltonian system associated to \(H\) with initial condition \(\Gamma_0\), then so is \(\Gamma_{M/G} := \pi(\Gamma)\) with respect to \(H_{M/G}\), with initial condition \(\pi(\Gamma_0)\).

(ii) Symplectic reduction: Suppose that \(M\) is now symplectic and that the group action has a coadjoint equivariant momentum map \(J : M \rightarrow g^*\) associated. Then for any \(\mu \in g^*\), the function \(h_{\mu} : M_\mu := J^{-1}(\mu)/G_\mu \rightarrow V^*\) uniquely determined by the equality \(h_{\mu} \circ \pi_\mu = h \circ \iota_\mu\), induces a stochastic Hamiltonian system on the symplectic reduced space \((M_\mu := J^{-1}(\mu)/G_\mu, \omega_\mu)\) with stochastic component \(X\) and whose Stratonovich operator \(H_\mu : TV \times M_\mu \rightarrow TM_\mu\) is given by

\[
H_\mu(v, \pi_\mu(z))(u) = T_z \pi_\mu (H(v, i_\mu(z))(u)) = \sum_{i=1}^{r} \langle \epsilon^i, u \rangle X_{h_\mu^i}(\pi_\mu(z)), \quad u, v \in V \text{ and } z \in J^{-1}(\mu), \tag{6.5}
\]

where Remark 6.6 has been implicitly used. In the previous expression, the functions \(h_\mu^i \in C^\infty(J^{-1}(\mu)/G_\mu)\) are the coefficient functions in the linear combination \(h_\mu = \sum_{i=1}^{r} h_\mu^i \epsilon^i\) and are related to the components \(h_i \in C^\infty(M)^G\) of \(h\) via the relation \(h_\mu^i \circ \pi_\mu = h_i \circ \iota_\mu\). Moreover, if \(\Gamma\) is a solution semimartingale of the Hamiltonian system associated to \(H\) with initial condition \(\Gamma_0 \subset J^{-1}(\mu)\), then so is \(\Gamma_\mu := \pi_\mu(\Gamma)\) with respect to \(H_\mu\), with initial condition \(\pi_\mu(\Gamma_0)\).
Remark 6.8 In the absence of freeness of the action the orbit spaces \( M/G \) and \( J^{-1}(\mu)/G_\mu \) cease to be regular quotient manifolds. Moreover, it could be that (even for free actions) there is no standard momentum map available (this is generically the case for Poisson manifolds). This situation can be handled by using the so called optimal momentum map \([OR02]\) and its associated reduction procedure \([O02]\). Given that by part (iii) of Proposition 6.4 the fibers of the optimal momentum map are preserved by the Hamiltonian semimartingales associated to invariant Hamiltonians one can formulate, for any proper group action on a Poisson manifold, a theorem identical to part (ii) of Theorem 6.7 with the standard momentum map replaced by the optimal momentum map. In the particular case of a (non-necessarily free) symplectic proper action that has a standard momentum map associated, such result guarantees the good behavior of the symmetric stochastic Hamiltonian systems with respect to the singular reduced spaces in \([SL91]\); see also \([OR06, OR06a]\) for the symplectic case without a standard momentum map.

Proof of Theorem 6.7. (i) can be proved by mimicking the proof of Theorem 3.1 by simply taking into account the fact that the \( G \)-invariance of \( h \) implies that of \( H \) and that for any \( i \in \{1, \ldots, r\} \), one has that \( T\pi \circ X_{h_i} = X_{h_i/M/G} \circ \pi \).

(ii) Expression (6.5) is guaranteed by the fact that \( X_{h_i/\mu} \circ \pi_{\mu} = T\pi_{\mu} \circ X_{h_i} \circ \mu, \) for any \( i \in \{1, \ldots, r\} \) (see for instance \([OR04, Theorem 6.1.1]\)). Let now \( \Gamma \) be a solution semimartingale of the Hamiltonian system associated to \( H \) with initial condition \( \Gamma_0 \subset J^{-1}(\mu) \). Notice first that by part (ii) in Proposition 6.4 \( \Gamma \subset J^{-1}(\mu) \) and hence the expression \( \Gamma_{\mu} := \pi_{\mu}(\Gamma) \) is well defined. In order to prove the statement, we have to check that for any one-form \( \alpha_{\mu} \in \Omega(M_\mu) \)

\[
\int \langle \alpha_{\mu}, \delta \Gamma_{\mu} \rangle = \int \langle H^*_{\mu}(X, \Gamma_{\mu}) \alpha_{\mu}, \delta X \rangle.
\]

This equality follows in a straightforward manner from (6.5). Indeed,

\[
\int \langle \alpha_{\mu}, \delta \Gamma_{\mu} \rangle = \int \langle \alpha_{\mu}, \delta (\pi_{\mu} \circ \Gamma) \rangle = \int \langle \pi_{\mu}^* \alpha_{\mu}, \delta \Gamma \rangle = \int \langle H^*(X, \Gamma) \pi_{\mu}^* \alpha_{\mu}, \delta X \rangle = \int \langle H^*_{\mu}(X, \Gamma_{\mu}) \alpha_{\mu}, \delta X \rangle,
\]

as required. ■

As to the reconstruction problem of solutions of a symmetric stochastic differential equation starting from a solution of the Poisson or symplectic reduced stochastic differential equation, Theorem 5.2 can be trivially modified to handle this situation. In the Poisson reduction case the theorem works without modification and when working with a solution of the symplectic reduced space it suffices to change the principal fiber bundle \( \pi : M \to M/G \) by \( \pi_{\mu} : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu \) all over.

7 Examples

7.1 Stochastic collective Hamiltonian motion

Our first example shows a situation in which the symplectic reduction of a symmetric stochastic Hamiltonian system offers, not only the advantage of cutting its dimension, but also of making it into a deterministic system. From the point of view of obtaining the solutions of the system, the procedures introduced in the previous section allow in this case the splitting of the problem into two parts: first, the solution of a standard ordinary differential equation for the reduced system and second, the solution of a stochastic differential equation in the group at the time of the reconstruction.

Let \( (M, \omega) \) be a symplectic manifold, \( G \) a Lie group and \( \Phi : G \times M \to M \) a free, proper, and canonical action. Additionally, suppose that this action has a coadjoint equivariant momentum map
\[ J : M \to g^* \] associated. Let \( h_0 \in C^\infty(M)^G \) be a \( G \)-invariant function and consider the deterministic Hamiltonian system with Hamiltonian function \( h_0 \).

A function of the form \( f \circ J \in C^\infty(M) \), for some \( f \in C^\infty(g^*) \), is called \textbf{collective}. We recall that by the Collective Hamiltonian Theorem (see for instance \cite{MR99})

\[ X_{f \circ J}(z) = \left( \frac{\delta f}{\delta \mu} \right)_M (z), \quad z \in M, \mu = J(z), \quad (7.1) \]

where the functional derivative \( \frac{\delta f}{\delta \mu} \in g \) is the unique element such that for any \( \nu \in g^* \), \( Df(\mu) \cdot \nu = \langle \nu, \frac{\delta f}{\delta \mu} \rangle \).

A straightforward consequence of (7.1) is that the \( G \)-invariant functions, in particular \( h_0 \), commute with the collective functions. Indeed, if \( h \in C^\infty(M)^G \), then for any \( z \in M \),

\[ \{h, f \circ J\}(z) = d h(z) \cdot X_{f \circ J}(z) = d h(z) \cdot \left( \frac{\delta f}{\delta \mu} \right)_M (z) = 0. \]

Collective functions play an important role to prove the complete integrability of certain dynamical systems (see \cite{GS83}). Moreover, some relevant physical systems may be described using collective Hamiltonian functions. In that case, the (deterministic) equations of motion exhibit special features and, in some favorable cases, may be partially integrated using geometrical arguments (see \cite{GS80}). The aim of this example is to study stochastic perturbations of deterministic symmetric mechanical systems introduced by means of collective Hamiltonians.

Let \( Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}^r \) be a \( \mathbb{R}^r \)-valued continuous semimartingale and \( \{f_1, \ldots, f_r\} \subset C^\infty(g^*) \) a finite family of \( \text{Ad}_{g^*}^G \)-invariant functions on \( g^* \). The coadjoint equivariance of the momentum map and the \( \text{Ad}_{g^*}^G \)-invariance of the functions allows us to construct the following \( G \)-invariant Hamiltonian function

\[ h : M \longrightarrow \mathbb{R} \times \mathbb{R}^r \]

\[ m \longmapsto (h_0(m), (f_1(J(m))), \ldots, f_r(J(m)))) . \]

Let \( X \) be the continuous semimartingale

\[ X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}_+ \times \mathbb{R}^r \]

\[ (t, \omega) \longmapsto (t, Y_t(\omega)) . \]

Consider the stochastic Hamiltonian system \((M, \omega, h, X)\) which is, by construction, \( G \)-invariant. Noether’s theorem (Proposition 6.2 (ii)) guarantees that the level sets of \( J \) are left invariant by the solution semimartingales of \((M, \omega, h, X)\). As to the reduction of this system, its main feature is that if we apply to it the reduction scheme introduced in Theorem 6.7 (ii), for any \( \mu \in g^* \), the reduced stochastic Hamiltonian system \((M_\mu, \omega_\mu, h_\mu, X)\) is such that

\[ h_\mu \circ \pi_\mu = h_0 \circ i_\mu , \]

since \( J \), and hence the functions \( f_i \circ J \), are constant on the level sets \( J^{-1}(\mu) \), for any \( i = 1, \ldots, r \). Consequently, the reduced system \((M_\mu, \omega_\mu, h_\mu, X)\) is equivalent to the deterministic Hamiltonian system \((M_\mu, \omega_\mu, h_\mu)\). In other words, the reduced system obtained from \((M, \omega, h, X)\) coincides with the one obtained in deterministic mechanics by symplectic reduction of \((M, h_0, t, \mathbb{R}_+)\). Thus, we have \( \text{perturbed stochastically a symmetric mechanical system preserving its symmetries and without changing the deterministic behavior of its corresponding reduced system.} \)

\textbf{Remark 7.1} If we want to perturb the deterministic Hamiltonian system associated to \( h_0 \) with the only prescription that the level set \( J^{-1}(\mu) \) is left invariant, for a given value \( \mu \in g^* \), we can weaken the requirement on the \( \text{Ad}_{g^*}^G \)-invariance of the functions \( f_i \in C^\infty(g^*) \), \( i = 1, \ldots, r \). Indeed, if we just ask that \( \delta f_i / \delta \mu \in g_\mu \), we then have that \( X_{g_\mu}(z), X_{f_\circ J}(z), \ldots, X_{f_\circ J}(z) \in T_zJ^{-1}(\mu) \), for any \( z \in J^{-1}(\mu) \). The required invariance property follows then from (7.1) and Proposition 2.6.
Remark 7.2 In this example, the reduction-reconstruction scheme provides a global decomposition of the system \((M, \omega, h, X)\) into its deterministic and stochastic parts. If one is willing to work only locally, this splitting could be carried out without reduction in the neighborhood of any point in phase space, given that as \(\{h_0, f_i \circ J\} = 0\), for any \(i \in \{1, \ldots, r\}\), then \([X_{h_0}, X_{f_i \circ J}] = 0\).

7.2 Stochastic mechanics on Lie groups

The presence of mechanical systems whose phase space is the cotangent bundle of a Lie group is widespread. Besides the importance that this general case has in specific applications it is also very useful at the time of illustrating some of the theoretical developments in this paper since most of the constructions that we presented admit very explicit characterizations. We start by recalling the main features of (deterministic) Hamiltonian systems over Lie groups. The reader interested in further details is encouraged to check with [AM78, MR99] and references therein.

Let \(G\) be a Lie group. The tangent bundle \(TG\) of \(G\) is trivial since it is isomorphic to the product \(G \times g\), where \(g = T_1 G\) is the Lie algebra of \(G\) and \(e \in G\) is the identity element. The identification \(TG = G \times g\) is usually carried out by means of two isomorphisms, denoted by \(\lambda\) and \(\rho\) and induced by left and right translations on \(G\), respectively. More specifically, let \(\lambda: TG \to G \times g\) be the map given by \(\lambda(v) = (g, T_g L_g^{-1}(v))\), where \(g = \tau_G(v)\) with \(\tau_G : TG \to G\) the natural projection. On the other hand, \(\rho: TG \to G \times g\) is defined by \(\rho(v) = (g, T_g R_g^{-1}(v))\). We refer to the image of \(\lambda\) as **body coordinates** and to the image of \(\rho\) as **space coordinates**. The cotangent bundle \(T^*G\) is also trivial and isomorphic to \(G \times g^*\). We can introduce **body coordinates** and **space coordinates** on \(T^*G\) by \(\tilde{\lambda}(\alpha) = (g, T^*_g L_g^* (\alpha)) \in G \times g^*\) and \(\tilde{\rho}(\alpha) = (g, T^*_g R_g (\alpha))\) respectively, where \(g = \pi_G(\alpha)\) and \(\pi_G : T^*G \to G\) is the canonical projection. The transition from body to space coordinates is as follows:

\[
(\rho \circ \tilde{\lambda}^{-1})(g, \xi) = (g, T_g R_g^{-1} \circ T_g L_g(\xi)) = (g, \text{Ad}_g(\xi))
\]

\[
(\tilde{\rho} \circ \tilde{\lambda}^{-1})(g, \mu) = (g, T^*_g R_g^{-1} \circ T^*_g L_g^{-1}(\mu)) = (g, \text{Ad}^*_g(\mu)),
\]

for any \((g, \xi) \in G \times g\) and any \((g, \mu) \in G \times g^*\). The group action of \(G\) by left or right translations can be lifted to both \(TG\) and \(T^*G\). We will denote by \(\Phi_L: G \times TG \to TG\) and \(\Phi_R: G \times T^*G \to T^*G\) the lifted action of left translations on the tangent and cotangent bundle respectively, and by \(\Phi_L: G \times TG \to TG\) and \(\Phi_R: G \times T^*G \to T^*G\) the lifted actions of right translations. The lifted actions have particularly simple expressions in suitable body or space coordinates. Indeed, it is more convenient to express \(\Phi_L\) and \(\Phi_R\) in body coordinates, where for any \(g, h \in G\), \(\xi \in g\), and \(\mu \in g^*\),

\[
(\Phi_L)_g(h, \xi) = (\lambda \circ T L_g \circ \lambda^{-1})(h, \xi) = (gh, \xi),
\]

\[
(\Phi_R)_g(h, \mu) = (\tilde{\lambda} \circ T^* L_g^{-1} \circ \tilde{\lambda}^{-1})(h, \mu) = (g^{-1}h, \mu).
\]

As to \(\Phi_R\) and \(\Phi_R\), space coordinates are particularly convenient; for any \(g, h \in G\), \(\zeta \in g\), and \(\alpha \in g^*\),

\[
(\Phi_R)_g(h, \zeta) = (\rho \circ T R_g \circ \rho^{-1})(h, \zeta) = (hg, \zeta)
\]

\[
(\Phi_R)_g(h, \alpha) = (\tilde{\rho} \circ T^* R_g^{-1} \circ \tilde{\rho}^{-1})(h, \alpha) = (h g^{-1}, \alpha).
\]

The actions \(\Phi_L\) and \(\Phi_R\), being the cotangent lifted actions to \(T^*G\) of an action on \(G\), have canonical momentum maps \(J_L: T^*G \to g^*\) and \(J_R: T^*G \to g^*\), respectively, when we endow \(T^*G\) with its canonical symplectic form. Let \(\theta \in \Omega^1(T^*G)\) be the Liouville canonical one-form on \(T^*G\). Then, \(J_L\) and \(J_R\) are given by

\[
\langle J_L(z_g), \xi \rangle = \langle z_g, [\xi]_L^L(g) \rangle, \quad \langle J_R(z_g), \xi \rangle = \langle z_g, [\xi]_R^R(g) \rangle,
\]

for any \(z_g \in T^*_g G\) and any \(\xi \in g\). Here \([\xi]_L^L \in \mathfrak{X}(G)\) (respectively \([\xi]_R^R \in \mathfrak{X}(G)\)) denotes the infinitesimal generator associated to \(\xi \in g\) by the left (respectively right) action of \(G\) on itself. This expression clearly
shows that $J_L$ is right-invariant and $J_R$ left-invariant. Observe that $J_L = \text{Ad}^{-1}_g \circ J_R$. For example, in body coordinates, these momentum maps have the following expressions ([AM78 Theorem 4.4.3])

$$(J_L)_B((g, \mu)) = \text{Ad}^{-1}_g(\mu) \quad \text{and} \quad (J_R)_B((g, \mu)) = \mu.$$  \hspace{1cm} (7.2)

In this context, the classical results on symplectic and Poisson reduction that we have described in the previous section admit a particularly explicit formulation. In all that follows we will suppose that the action with respect to which we are reducing is lifted left translations. Using body coordinates, it is easy to see that in this case the Poisson reduced space $T^*G/G$ coincides with the dual of the Lie algebra $g^*$ endowed with the Lie-Poisson structure given by

$$\{f_1, f_2\}_{g^*}(\mu) = -\left\langle \mu, \frac{\delta f_1}{\delta \mu} \frac{\delta f_2}{\delta \mu} \right\rangle,$$

for any $\mu \in g^*$ and $f_1, f_2 \in C^\infty(g^*)$. The symplectic reduced spaces $J_L^{-1}(\mu)/G_\mu$ are naturally symplectomorphic to the symplectic leaves of the Lie-Poisson structure on $g^*$, that is, the coadjoint orbits endowed with the so-called Kostant-Kirillov-Souriau symplectic form $\omega_\mu^*$:

$$\omega_\mu^*(\xi^\mu, \eta^\mu) = \omega_\mu(\mu)(-\text{ad}^*_\mu \xi, -\text{ad}^*_\mu \eta) = -\langle \mu, [\xi, \eta] \rangle.$$

Let now $V$ be a vector space, $X : \mathbb{R}_+ \times \Omega \to V$ a continuous semimartingale, and $h : T^*G \to V^*$ a smooth map invariant under the lifted left translations of $G$ on $T^*G$. If we use body coordinates and we visualize $T^*G$ as the product $G \times g^*$, the invariance of $h : G \times g^* \to V^*$ allows us to write it as $h = \sum_{i=1}^r h_i e_i^*$, where $\{e^1, \ldots, e^r\}$ is a basis of $V^*$ and $h_1, \ldots, h_r \in C^\infty(g^*)$. Let $\{e_1, \ldots, e_r\}$ be the dual basis of $V$ and write $X = \sum_{i=1}^r X^i e_i$. Using the left trivialized expression of the Hamiltonian vector fields in the deterministic case (see [OR03 Theorem 6.2.5]) it is easy to see that the stochastic Hamiltonian equations in this setup are

$$\delta \Gamma^h = \sum_{i=1}^r \left( T_e L_{\Gamma^G} \left( \frac{\delta h_i}{\delta \Gamma^G} \right) , \text{ad}^{\ast}_{\delta h_i} \Gamma^g \right) \delta X^i$$  \hspace{1cm} (7.3)

where $\Gamma^G$ and $\Gamma^g$ are the $G$ and $g^*$ components of $\Gamma^h$, respectively, that is, $\Gamma^h := (\Gamma^G, \Gamma^g)$. In the left trivialized representation, the reduced Poisson and symplectic Hamiltonians are simply the restrictions $h^g$ and $h^G$ of $h$ to $g^*$ and to the coadjoint orbits $O_\mu \subset g^*$, respectively. Additionally, the reduced stochastic Hamiltonian equations on $g^*$ and $O_\mu$ are given by

$$\delta \Gamma^g = \sum_{i=1}^r \text{ad}^{\ast}_{\delta h_i} \Gamma^g \delta X^i \quad \text{and} \quad \delta \Gamma^O = \sum_{i=1}^r \text{ad}^{\ast}_{\delta h_i} \Gamma^O \delta X^i$$  \hspace{1cm} (7.4)

where $h^g = \sum_{i=1}^r h_i e^i$ and $h^O = \sum_{i=1}^r h_i^O e^i$.

The combination of expressions (7.3) and (7.4) shows that in this setup, the dynamical reconstruction of reduced solutions is particularly simple to write down. Indeed, suppose that we are given a solution $\Gamma^h$ of, say, the Poisson reduced system. In order to obtain the solution $\Gamma^h_0 = (\Gamma^G_0, \Gamma^g_0)$ and $\pi(\Gamma^h_0) = \Gamma^g_0$, with $\pi : T^*G \simeq G \times g^* \to T^*G/G \simeq g^*$ the Poisson reduction projection, it suffices to solve the stochastic differential equation in $G$

$$\delta \Gamma^G = \sum_{i=1}^r T_e L_{\pi^G} \left( \frac{\delta h_i}{\delta \Gamma^G} \right) \delta X^i,$$

with the initial condition $\Gamma^G_0$. The reconstructed solution that we are looking for is then $\Gamma^h = (\Gamma^G, \Gamma^g)$. 

Reduction, reconstruction, and skew-product decomposition of symmetric SDEs
7.3 Stochastic perturbations of the free rigid body

The free rigid body, also referred to as Euler top, is a particular case of systems introduced in the previous section where the group $G$ is $SO(3, \mathbb{R})$. We recall that in the context of mechanical systems on groups, a Hamiltonian system is called free when the energy of the system is purely kinetic and there is no potential term. Let $(\cdot, \cdot)$ be a left invariant Riemannian metric on $G$; the kinetic energy $E$ associated to $(\cdot, \cdot)$ is $E(v) = \frac{1}{2} \langle v, v \rangle$, $v \in TG$. Then, using the left invariance of the metric, we can write in body coordinates

$$E(g, \xi) = \frac{1}{2} \langle \xi, \xi \rangle_c = \frac{1}{2} \langle I(\xi), \xi \rangle,$$

for any $(g, \xi) \in G \times g$, where $(\langle \cdot, \cdot \rangle)$ is the natural pairing between elements of $g^*$ and $g$, and $I : g \rightarrow g^*$ is the map given by $\xi \mapsto (\xi, \cdot)_c$ and usually known as the inertia tensor associated to the metric $(\cdot, \cdot)$. The Legendre transformation associated to $E$ can be used to define a Hamiltonian function $h : T^*G \rightarrow \mathbb{R}$ that, in body coordinates, can be written as

$$h(g, \mu) = \frac{1}{2} \langle \mu, \Lambda(\mu) \rangle,$$  \hspace{1cm} (7.6)

where $\Lambda = I^{-1}$. Notice that as the kinetic energy is left invariant (invariant with respect to the lifted $G$-action to $T^*G$ of the action of $G$ on itself by left translations), then the components of $J_L$ are conserved quantities of the corresponding Hamiltonian system. In order to connect with example in Section 7.4 let $f \in C^\infty(g^*)$ be the function $f : g^* \rightarrow \mathbb{R}$ given by $\mu \mapsto \frac{1}{2} \langle \mu, \Lambda(\mu) \rangle$. By (7.2), the Hamiltonian function $h$ may be expressed as $h = f \circ J_R$. Therefore $h$ is collective with respect to $J_R$.

We now go back to the free rigid body case, that is, $G = SO(3, \mathbb{R})$. We recall that the Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is the vector space of three dimensional skew-symmetric real matrices whose bracket is just the commutator of two matrices. As a Lie algebra, $(\mathfrak{so}(3), [\cdot, \cdot])$ is naturally isomorphic to $(\mathbb{R}^3 \times \times)$, where $\times$ denotes the cross product of vectors in $\mathbb{R}^3$. Under this isomorphism, the adjoint representation of $SO(3, \mathbb{R})$ on its Lie algebra is simply the action of matrices on vectors of $\mathbb{R}^3$ and the Lie-Poisson structure on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is given by $\{f, g\}(v) = -v \cdot (\nabla f \times \nabla g)$, for any $f, g \in C^\infty(\mathbb{R}^3)$, where $\nabla$ is the usual Euclidean gradient and $\times$ denotes the Euclidean inner product.

Given a free rigid body with inertia tensor $I : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, since $\delta h_B/\delta \mu = \Lambda(\mu)$, for any $\mu \in \mathbb{R}^3$, the left-trivialized equations of motion of the system are

$$(\dot{A}, \dot{\mu}) = \left( A \cdot \Lambda(\mu), \mu \times \Lambda(\mu) \right),$$  \hspace{1cm} (7.7)

where the dot in the right hand side of (7.7) stands for matrix multiplication and $\Lambda(\mu)$ is the skew-symmetric matrix associated to $\Lambda(\mu) \in \mathbb{R}^3$ via the mapping that implements the Lie algebra isomorphism between $(\mathfrak{so}(3), [\cdot, \cdot])$ and $(\mathbb{R}^3 \times \times)$. In the context of the free rigid body motion the momentum map $J_L$ (respectively, $J_R$) is called spatial angular momentum (respectively, body angular momentum). The second component of (7.7), that is,

$$\dot{\mu} = \mu \times \Lambda(\mu)$$  \hspace{1cm} (7.8)

are the well-known Euler equations for the free rigid body.

Random perturbations of the body angular momentum. We now introduce stochastic perturbations of the free rigid body by using some of the geometrical tools that we have introduced above. Later on we will compare this example with the model of the randomly perturbed rigid body studied in [L97] and [LW05], whose physical justification, as we will briefly discuss, involves the same ideas as ours.

Let $V = \mathbb{R} \times \mathfrak{so}(3) \simeq \mathbb{R}^+ \times \mathbb{R}^3$ and let $h$ be the Hamiltonian function $h : T^*SO(3) \rightarrow V^* = \mathbb{R} \times \mathfrak{so}(3)^*$ defined as $h = (h_0, J_R)$, where $h_0$ is the Hamiltonian function of the free (deterministic) rigid body.
Observe that $h$ is a left-invariant function because so is $J_R$. Let $Y : \mathbb{R}^+ \times \Omega \to \mathfrak{g}$ be a continuous semimartingale which we may suppose, for the sake of simplicity, that it is a $\mathfrak{g}$-valued Brownian motion and then $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \times \mathfrak{g}$ be a semimartingale defined as $X_t(\omega) = (t, Y_t(\omega))$ for any $(t, \omega) \in \mathbb{R} \times \Omega$. Consider the stochastic Hamiltonian system on $T^*G$ associated to $h$ and $X$. Since $h$ is left invariant, the momentum map $J_L$ is preserved by the solution semimartingales of this system and moreover, we can apply the reduction scheme introduced in the previous sections. For example, if we carry out Poisson reduction we have a reduced Hamiltonian function $h^{\Omega} : \mathfrak{g}^* \to \mathbb{R}$ given by $h^{\Omega}(\mu) = \langle \frac{1}{2} (\mu, \Lambda(\mu)), \mu \rangle$. Let $\{\xi_1, \xi_2, \xi_3\}$ a basis of the Lie algebra $\mathfrak{g}$ and $\{e^1, e^2, e^3\} \subset \mathfrak{g}^*$ its dual basis. Observe that if we write $J_R(\mu) = \sum_{i=1}^3 (\mu, \xi_i) e^i$ and $Y = \sum_{i=1}^3 Y^i \xi_i$, the reduced stochastic Lie-Poisson equations can be expressed as
\[
\delta \mu_t = \mu_t \times \Lambda(\mu_t) \delta t + \sum_{i=1}^3 \left( \mu_t \times \xi_i \right) \delta Y_t^i.
\tag{7.9}
\]
Regarding the reconstruction of the reduced dynamics, one has to solve the stochastic differential equation on the rotations group $SO(3)$ given by (7.5) that, in this particular case, is given by
\[
\delta A_t = A_t \cdot \Lambda(\mu_t) \delta t + \sum_{i=1}^3 \left( A_t \cdot \Lambda(\mu_t) \right) \delta Y_t^i.
\tag{7.10}
\]
A physical model whose description fits well in a stochastic Hamiltonian differential equation like the one associated to $h$ and $X$ is that of a free rigid body subjected to small random impacts. Each impact causes a small and instantaneous change in the body angular momenta $\mu_t$ at time $t$ that justifies the extra term in (7.9), when compared to the Euler equations (7.8).

Our model is very similar to the one proposed in (L97) where, instead of introducing the random perturbation by means of a Hamiltonian function, a stochastic differential equation on the group $G$ is introduced. This equation, also studied in detail in (LW05), is
\[
\delta A_t = A_t \cdot \Lambda \circ \text{Ad}_{A_t}^* (\alpha) \delta t + \sum_{i=1}^3 \left( A_t \cdot \Lambda \circ \text{Ad}_{A_t}^* (e^i) \right) \delta Y_t^i,
\tag{7.11}
\]
where $\alpha \in \mathfrak{g}^*$ is a constant vector. It important to note that the drift terms of equations (7.10) and (7.11) coincide. Indeed, for any $(g, \mu) \in G \times \mathfrak{g}^*$ we can write
\[
\mu = \text{Ad}_{g}^* \circ \text{Ad}_{g^{-1}}^* (\mu) = \text{Ad}_{J_L(\mu)}^* (J_L(\mu, \mu)).
\]
Since in our model the spatial angular momentum is conserved, $\Lambda(\mu_t) = \Lambda \left( \text{Ad}_{A_t}^* \left( \text{Ad}_{A_t^{-1}}^* \mu_t \right) \right) = \Lambda \left( \text{Ad}_{A_t}^* (\alpha) \right)$, where $\alpha = J_L(\mu_t)$ is the preserved value of the spatial angular momentum of a solution $(A_t, \mu_t)$ of (7.9) and (7.10). The difference between (7.10) and (7.11) lies in the stochastic terms. The justification given by the author in (L97) for the equation (7.11) is the following: since in the (deterministic) rigid body the spatial angular momentum $J_L$ is conserved, once we have fixed the value of this conserved quantity, we can simply study the dynamics of the free rigid body by looking at the first component of the ordinary differential equation (7.7), now rewritten as
\[
\dot{A} = A \left( \Lambda (\text{Ad}_{A}^* (\alpha)) \right)
\tag{7.12}
\]
where $\alpha \in \mathfrak{g}^*$ is the $J_L$-value of the solution. Under random impacts, the spatial angular momentum $\alpha$, which was preserved in the deterministic case, is now randomly modified. The idea is then to replace $\alpha dt$ in (7.12) by $\alpha \delta t + \sum_{i=1}^3 e^i \delta Y_t^i$. Unlike our model, where the random perturbation is introduced in the cotangent bundle respecting the underlying symmetries of the deterministic system, there is no preservation of $J_L$ in the stochastic model of (L97).
One advantage of working on $T^*G$ is that, even in the stochastic context, classical quantities such as the angular momentum, are still well defined. These objects do not have a clear counterpart if one follows the configuration space based approach in [L97] (see for instance [AW05] for a non-trivial definition of angular velocity in the stochastic context).

**Not so rigid rigid bodies. Random perturbation of the inertia tensor.** In this example we want to write the equations that describe a rigid body some of whose parts are slightly loose, that is, the body is not a true rigid body and hence its mass distribution is constantly changing in a random way. This will be modelled by stochastically perturbing the tensor of inertia.

For the sake of simplicity, we will write $G = SO(3,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{so}(3)$. Let $\mathcal{L}(\mathfrak{g}^*, \mathfrak{g})$ be the vector space of linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$. As we know $(\mathfrak{so}(3), [\cdot, \cdot]) \simeq (\mathbb{R}^3, \times)$. Furthermore, we can establish an isomorphism $\mathbb{R}^3 \simeq (\mathbb{R}^3)^*$ using the Euclidean inner product and hence we can write $\mathfrak{g} \simeq \mathfrak{g}^*$. Let $V = \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g}) = \{ A \in \mathcal{L}(\mathfrak{g}^*, \mathfrak{g}) \mid A^* = A \}$ be the vector space of selfadjoint linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$. Define the Hamiltonian $h : T^*G \rightarrow V^*$ in body coordinates as

$$h : T^*G \simeq G \times \mathfrak{g}^* \rightarrow V^*$$

$$(g, \mu) \mapsto \bar{\mu},$$

where $\bar{\mu}$ is such that

$$\bar{\mu} : \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g}) \rightarrow \mathbb{R}$$

$$A \mapsto \frac{1}{2} \langle \mu, A(\mu) \rangle.$$

Observe that in body coordinates the Hamiltonian $h$ does not depend on $G$, so the Hamiltonian is $G$-invariant by the action $\Phi_L$ on $T^*G$. On the other hand, consider some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ and introduce a stochastic component $X : \mathbb{R}^+ \times \Omega \rightarrow V$ in the following way:

$$X : \mathbb{R}^+ \times \Omega \rightarrow \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g})$$

$$(t, \omega) \mapsto \Lambda t + \varepsilon A_t(\omega),$$

where $\Lambda \in \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g})$ plays the role of the inverse of the tensor of inertia given by the deterministic (rigid) description of the body, $\varepsilon$ is a small parameter, and $A$ is an arbitrary $\mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g})$-valued semimartingale.

In order to show how the stochastic Hamiltonian system on $T^*G$ associated to $h$ and $X$ models a free rigid body whose inertia tensor undergoes random perturbations, we write down the associated stochastic reduced Lie-Poisson equations in Stratonovich form

$$\delta \mu_t = \mu_t \times \Lambda(\mu_t) dt + \varepsilon \mu_t \times \delta A_t(\mu_t).$$

Thus we see that these Lie-Poisson equations consist in changing $\Lambda(\mu_t) dt$ in the Euler equations (7.3) by $\Lambda(\mu_t) dt + \varepsilon \delta A_t(\mu_t)$, which accounts for the stochastic perturbation of the inertia tensor.

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