CELLS IN QUANTUM AFFINE $\mathfrak{sl}_n$

KEVIN McGERTY

ABSTRACT. We study Lusztig’s theory of cells for quantum affine $\mathfrak{sl}_n$. Using the geometric construction of the quantum group due to Lusztig and Ginzburg-Vasserot, we describe explicitly the two-sided cells, the number of left cells in a two-sided cell, and the asymptotic algebra, verifying conjectures of Lusztig.

1. INTRODUCTION

Given any algebra with a specified basis it is possible to define a notion of cell from the set of ideals, left, right or two-sided, which are spanned by a subset of the basis. Of course, if one picks the basis arbitrarily, it is unlikely that these objects will contain any interesting information about the algebra in question. However, if the algebra has a natural choice of basis, the situation can be quite different. Examples of this arise in a number of places: The cells attached to the Kazhdan–Lusztig basis of a Hecke algebra associated to a finite Weyl group turn out to be crucial in the classification of the characters of finite groups of Lie type. On the other hand, although the plus part of the quantum group possesses a natural “canonical basis”, the theory of cells there is trivial. If we extend the canonical basis to one for the modified quantum group $\hat{U}$ however, the theory of cells is once again interesting.

In the case of quantum groups of finite type, work of Lusztig [L95] completely describes the cells. In that paper Lusztig also gave a conjectural description of the cell structure in the case of (degenerate) affine quantum groups. In this paper we show that the geometric construction of $\hat{U}$ in [L99] can be used to give complete information about the cell structure of quantum affine $\mathfrak{sl}_n$. Just as in the case of affine Weyl groups, the cells are closely related to the finite dimensional representation theory of the algebra. We will first investigate the structure of cells in the “affine q-Schur algebra” $\mathfrak{A}_D$ and then show how this can be used to obtain the cell structure of $\hat{U}$.

We begin by recalling the definition of cells. Suppose $R$ is a ring, and $A$ an associative algebra over $R$, with an $R$–basis $B$. We say that a left ideal is based if it is the span of a subset of the basis $B$. We define a preorder on the elements of $B$ as follows. Let $x \preceq_L y$ for $x, y \in B$ if $x$ lies in every based left ideal which contains $y$. The equivalence classes of this preorder are precisely the left cells of $A$. If we replace “left ideal” with “right ideal” or “two-sided ideal” we get the corresponding notion of right cells or two-sided cells.

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2. The Affine q-Schur Algebra

We first give a description of the affine q-Schur algebra \( \mathfrak{A}_D \) as a commutator algebra, following the notation of [L99] and [McG]. Thus let \( V \) be a free rank \( D \) module over \( k[\epsilon, \epsilon^{-1}] \), where \( k \) is a finite field of \( q \) elements, and \( \epsilon \) is an indeterminate.

Let \( \mathcal{F}^n \) be the space of \( n \)-step periodic lattices, i.e. sequences \( \mathbf{L} = (L_i)_{i \in \mathbb{Z}} \) of lattices in our free module \( V \) such that \( L_i \subset L_{i+1} \), and \( L_{i-n} = \epsilon L_i \). The group \( G = \text{Aut}(V) \) acts on \( \mathcal{F}^n \) in the natural way. Let \( G_{D,n} \) be the set of nonnegative integer sequences \( (a_i)_{i \in \mathbb{Z}} \), such that the \( i \)-th row of \( \mathbf{L} \) is in the orbit of \( \mathbf{L} \). The group \( G \) acts on \( \mathcal{F}^n \) by \( G \times \mathcal{F}^n \rightarrow \mathcal{F}^n \). Let \( \mathfrak{S}_D \) be the set of \( \mathcal{F}^n \) stable \( \mathcal{F}^n \)-spaces of functions indexed by matrices \( A \in \mathcal{S} \). We have \( \mathcal{F}^n \times \mathcal{F}^n \rightarrow \mathcal{F}^n \) is in the orbit \( \mathcal{O}_A \) corresponding to \( A \).

Let \( D \) be the space of \( n \)-step periodic lattices, i.e. sequences \( \mathbf{L} = (L_i)_{i \in \mathbb{Z}} \) of lattices in our free module \( V \) such that \( L_i \subset L_{i+1} \), and \( L_{i-n} = \epsilon L_i \). The group \( G = \text{Aut}(V) \) acts on \( \mathcal{F}^n \) in the natural way. Let \( G_{D,n} \) be the set of nonnegative integer sequences \( (a_i)_{i \in \mathbb{Z}} \), such that the \( i \)-th row of \( \mathbf{L} \) is in the orbit of \( \mathbf{L} \). The group \( G \) acts on \( \mathcal{F}^n \) by \( G \times \mathcal{F}^n \rightarrow \mathcal{F}^n \). Let \( \mathfrak{S}_D \) be the set of \( \mathcal{F}^n \) stable \( \mathcal{F}^n \)-spaces of functions indexed by matrices \( A \in \mathcal{S} \). We have \( \mathcal{F}^n \times \mathcal{F}^n \rightarrow \mathcal{F}^n \) is in the orbit \( \mathcal{O}_A \) corresponding to \( A \).

For \( A \in \mathfrak{S}_{D,n} \) let \( r(A), c(A) \in \mathfrak{S}_{D,n} \) be given by \( r(A)_i = \sum_j a_{i,j} \) and \( c(A)_i = \sum_j a_{i,j} \).

Similarly let \( B_D \) be the space of complete periodic lattices, that is, sequences of lattices \( \mathbf{L} = (L_i) \) such that \( L_i \subset L_{i+1} \), \( L_{i-D} = \epsilon L_i \), and \( \dim_k(L_i/L_{i-1}) = 1 \) for all \( i \in \mathbb{Z} \). Let \( b_0 = (\ldots, 1, 1, \ldots) \). The orbits of \( G \) on \( B_D \) are indexed by matrices \( A \in \mathcal{S}_{n,n} \) where the matrix \( A \) must have \( r(A) = c(A) = b_0 \).

Let \( \mathfrak{A}_D, \mathfrak{H}_D, \mathfrak{T}_D \) be the span of the characteristic functions of the \( G \)-orbits on \( \mathcal{F}^n \times \mathcal{F}^n \). \( B_D \times \mathcal{F}^n \) and \( \mathcal{F}^n \times B_D \) respectively. Convolution makes \( \mathfrak{A}_D \) and \( \mathfrak{H}_D \) into algebras and \( \mathfrak{T}_D \) into a \( \mathfrak{A}_D \times \mathfrak{H}_D \) bimodule.

Let \( \mathcal{E}_D, \mathcal{F}_D \) be the space of \( \mathcal{F}^n \)-spaces of functions indexed by matrices \( A \in \mathcal{S}_{n,n} \) where the matrix \( A \) must have \( r(A) = c(A) = b_0 \).

Let \( \{ [A] : A \in \mathfrak{S}_{D,n} \} \) be the basis of \( \mathfrak{A}_D \) given by \( q^{-d_A/2} \) times the characteristic function of the orbit corresponding to \( A \). When \( D = n \) an obvious subset of this basis spans \( \mathfrak{H}_D \).

All of these spaces of functions are the specialization at \( v = \sqrt{q} \) of modules over \( A = \mathbb{Z}[v^{-1}] \), which we denote by \( \mathfrak{A}_D, \mathfrak{H}_D \) and \( \mathfrak{T}_D \) respectively (here \( v \) is an indeterminate). The \( A \)-algebra \( \mathfrak{A}_D \) is the “affine q-Schur algebra”, it is easy to see that it is the commutator algebra of the right \( \mathcal{H}_D \)-module \( \mathfrak{T}_D \). This is the affine version of the geometric Schur–Weyl duality first described in [SL]. Recall from [L99], section 4 that \( \mathfrak{A}_D \) possesses a canonical basis \( \mathcal{B}_D \) consisting of elements \( \{ A \} \), for \( A \in \mathfrak{S}_{D,n} \). We have

\[
\{ A \} = \sum_{A_1 : A_1 \leq A} \Pi_{A_1} [A_1].
\]

where \( \leq \) is a natural partial order on \( \mathfrak{S}_{D,n} \) and the \( \Pi_{A_1} A \) are certain polynomials in \( \mathbb{Z}[v^{-1}] \). \( \mathfrak{A}_D \) has a natural anti-automorphism \( \Psi \) which sends \( [A] \) to \( [A^t] \), and a related anti-automorphism \( \rho \) such that \( \rho([A]) = v^{d_A - d_A^t} [A^t] \). In [M] a natural inner product \( (\cdot, \cdot) \) is defined on \( \mathfrak{A}_D \) (this is different from the inner product given in [L99]). It has the property that

\[
(xy, z)_D = (y, \rho(x)z)_D; \quad x, y, z \in \mathfrak{A}_D.
\]
Moreover it can be shown that

\[(\{A\},\{B\})_D \in \delta_{A, B} + v^{-1}N[v^{-1}].\]

so that the inner product is positive with respect to the canonical basis, and the canonical basis is “almost orthogonal”.

Let \( \mathcal{H}_D \) be the affine Hecke algebra of \( GL_D \), thus \( \mathcal{H}_D \) is an algebra over \( \mathbb{Z}[v, v^{-1}] \) generated by symbols \( T_i, X_j, X_j^{-1} \), where \( i \in \{1, 2, \ldots, D-1\} \), and \( j \in \{1, 2, \ldots, D\} \), subject to the relations

- \( (T_i - v)(T_i + v) = 0 \), \( T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \), for \( i = 1, 2, \ldots, D-1 \);
- \( T_iT_j = T_jT_i \) if \( |i - j| \geq 2 \);
- \( X_iX_j^{-1} = X_j^{-1}X_i = 1 \), \( X_iX_j = X_jX_i \) for all \( i, j \);
- \( T_i^{-1}X_iT_i^{-1} = X_{i+1} \) for \( i = 1, 2, \ldots, D-1 \); \( T_iX_j = X_jT_i \) for \( j \neq i, i+1 \).

This is the “Bernstein presentation”. Let \( W \) be the affine Weyl group of type \( GL_D \) (when we wish to specify \( D \), we will use the notation \( A_{D-1} \)), that is, \( W \) is the semidirect product of the symmetric group \( S_D \) with \( \mathbb{Z}^D \). It is an extension by \( \mathbb{Z} \) of a Coxeter group, thus the usual yoga can be used to extend the Kazhdan-Lusztig theory. Let the set of simple reflections of the Coxeter group be \( S = \{s_i : i = 0, 1, \ldots, D\} \). The “Iwahori presentation” of \( \mathcal{H}_D \) yields a basis \( \{T_w : w \in W\} \). Lusztig observed that \( W \) has a natural incarnation as a permutation group on the integers, indeed \( W \) is isomorphic to the set of all permutations \( \sigma \) of the integers such that \( \sigma(i + D) = \sigma(i) + D \). See for example [Xi] for more details. Thus an element of \( W \) obviously corresponds to an infinite permutation matrix, which we denote \( A_w \), when we wish to make the distinction between the group element and the matrix. These permutation matrices are precisely the matrices indexing the \( G \)-orbits on \( B^D \times B^D \) described above.

**Proposition 2.1.** The map \( \mathcal{H}_w = \sqrt{\mathfrak{a}} \rightarrow \mathcal{H}_{D, \mathfrak{a}} \) which sends \( T_w \mapsto [A_w] \) is an algebra isomorphism. 

We can describe \( \mathcal{S}_D \) algebraically as follows: To each element \( a \in \mathcal{S}_{D,n} \), we can associate a parabolic subgroup of the symmetric group \( S_n \) — it is the subgroup preserving the subsets \( \{1, 2, \ldots, a\}, \{a_1+1, \ldots, a_1+a+2\}, \ldots, \{D-a_n+1, \ldots, D\} \) of \( \{1, 2, \ldots, D\} \). Set \( T_a = \sum_{w \in S_n} v^{l(w)}T_w \). Then as a module for the Hecke algebra, \( \mathcal{S}_D \) is isomorphic to

\[ \bigoplus_{a \in \mathcal{S}_{D,n}} T_a \mathcal{H}_D \]

Similarly we see that we can describe an element \( [A] \) of \( \mathfrak{A}_D \) uniquely by a triple consisting of an element \( w_A \in W \) together with a pair \( a, b \in \mathcal{S}_{D,n} \). Indeed \( a, b \) are just \( r(A) \) and \( c(A) \) respectively, and \( w_A \) is the element of maximal length in the (finite) double coset of \( S_n \backslash W/S_n \) determined by the matrix \( A \). This also allows us to describe the structure constants for \( \mathfrak{A}_D \) with respect to the basis \( \{[A] : A \in \mathcal{S}_{D,n}\} \) in terms of those for \( \mathcal{H}_D \) with respect to the basis \( \{T_w : w \in W\} \). In fact simple algebraic considerations (or an analogous discussion of the geometry involved) shows that the same holds for the structure constant with respect to the bases coming from intersection cohomology.

More precisely, suppose that we denote the various structure constants for \( \mathcal{H}_D \) and \( \mathfrak{A}_D \) as follows: Let \( A, B \in \mathcal{S}_{D,n} \), let \( v, w \in W \), and let \( \{C_w : w \in W\} \) be the Kazhdan-Lusztig basis of the Hecke algebra.
Lemma 2.2. Let $A, B, C \in \mathfrak{S}_{D,\nu}$, and let $w_A, w_B, w_C \in W$ be the corresponding element of the Weyl group. Suppose that $c(A) = r(B) = c$. Let $w_c$ be the longest element of $S_c$ and let

$$p_c = v^{-((w_c) \cdot x)} \sum_{x \in S_c} v^{2l(x)}$$

be the shifted Poincaré polynomial of $S_c$. We have

$$p_c \eta_{A,B}^{c} = f_{w_A,w_B}^{w_c},$$
$$p_c \nu_{A,B}^{c} = h_{w_A,w_B}^{w_c};$$

Proof. Using the algebraic description of $\Sigma_D$, the first statement can be proved algebraically, by interpreting the basis elements $[A], [B], [C]$ as sums of elements in double cosets of the Hecke algebra. Similarly one can show the second statement entirely algebraically, but it is perhaps more enlightening to use the interpretation of the multiplication in terms of perverse sheaves (see [L99] for a discussion of this). The affine Schubert variety for $w_A$ fibres over the varieties $X^B_i$ with fibre given by a partial flag variety corresponding to $c(A)$, and the polynomial $p_c$ arises from the cohomology of this fibre.

Finally, must describe the connection between our convolution algebras and the modified quantum group of affine type A (in its degenerate, or level zero form). We start with some general definitions.

Definition 2.3. A Cartan datum is a pair $(I, \cdot)$ consisting of a finite set $I$ and a $\mathbb{Z}$-valued symmetric bilinear pairing on the free Abelian group $\mathbb{Z}[I]$, such that

- $i \cdot i \in \{2, 4, 6, \ldots \}$
- $2i(\frac{i-1}{2}) \in \{0, -1, -2, \ldots \}$, for $i \neq j$.

A root datum of type $(I, \cdot)$ is a pair $Y, X$ of finitely-generated free Abelian groups and a perfect pairing $\langle \cdot, \cdot \rangle: Y \times X \to \mathbb{Z}$, together with isomorphisms $I \subset X, (i \mapsto i)$ and $I \subset Y, (i \mapsto i')$ such that $\langle i, j \rangle = 2i(\frac{i-1}{2})$.

Given a root datum, we may define an associated quantum group $U$. Since it is the only case we need, we will assume that our datum is symmetric and simply laced so that $i \cdot i = 2$ for each $i \in I$, and $i \cdot j \in \{0, -1\}$ if $i \neq j$. In this case, $U$ is generated as an algebra over $\mathbb{Q}(v)$ by symbols $E_i, F_i, K^\mu_i, i \in I, \mu \in Y$, subject to the following relations.

- $K_{\mu} = 1, K_{\mu_1}, K_{\mu_2} = K_{\mu_1+\mu_2}$ for $\mu_1, \mu_2 \in Y$;
- $K_{\mu} E_i K_{\mu}^{-1} = v^{(\mu \cdot i)} E_i, \quad K_{\mu} F_i K_{\mu}^{-1} = v^{-(\mu \cdot i)} F_i$ for all $i \in I, \mu \in Y$;
- $E_i F_j - F_j E_i = \delta_{i,j} K_{\mu} K_{\mu}^{-1}$;
- $E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i$, for $i, j \in I$ with $i \cdot j = 0$;
- $E_i^2 E_j + (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$;
where the category of left $U$-modules.

The forgetful functor to the category of vector spaces has an endomorphism ring $R$. Thus an element of $a$ of $R$ associates to each $V \in \text{Ob}(\text{Mod}_X)$ an endomorphism $a_V$, such that for any morphism $f: V \to W$, $a_W \circ f = f \circ a_V$. Thus any element of $U$ clearly determines an element of $R$. For each $\lambda \in X$, let $\lambda \in R$ be the projection to the $\lambda$ weight space. Then $R$ is isomorphic to the direct product $\prod_{\lambda \in X} U_{1,\lambda}$, and we set

$$U = \bigoplus_{\lambda \in X} U_{1,\lambda}.$$  

To see the connection between our convolution algebra and quantum groups, we will need the following notation. For $a \in D_{D,n}$ let $i_n \in D_{D,n,n}$ be the diagonal matrix with $(i_n)_{i,j} = \delta_{i,j}a_i$. Let $E_{i,j} \in D_{1,1,n,n}$ be the matrix with $(E_{i,j})_{k,l} = 1$ if $k = i + sn$, $l = j + sn$, some $s \in \mathbb{Z}_n$ and 0 otherwise. Let $\mathbb{S}^n$ be the set of all $b = (b_i)_{i \in \mathbb{Z}}$ such that $b_i = b_{i+n}$ for all $i \in \mathbb{Z}$. Let $\mathbb{S}^{n,n}$ denote the set of all matrices $A = (a_{i,j})$, $i, j \in \mathbb{Z}$, with entries in $\mathbb{Z}$ such that

- $a_{i,j} \geq 0$ for all $i \neq j$;
- $a_{i,j} = a_{i+n,j+n}$, for all $i, j \in \mathbb{Z}$;
- For any $i \in \mathbb{Z}$ the set $\{ j \in \mathbb{Z}: a_{i,j} \neq 0 \}$ is finite;
- For any $j \in \mathbb{Z}$ the set $\{ i \in \mathbb{Z}: a_{i,j} \neq 0 \}$ is finite.

Thus we have $\mathbb{S}_{D,n,n} \subset \mathbb{S}^{n,n}$ for all $D$. For $i \in \mathbb{Z}/n\mathbb{Z}$ let $i \in \mathbb{S}^n$ be given by $i_k = 1$ if $k = i$ mod $n$, $i_k = 0$ if $k = i + 1$ mod $n$, and $i_k = 0$ otherwise. We write $a \cup b = a'$ if $a = a' + 1$. For such $a, a'$ set $a e_{a'} \in \mathbb{S}^{n,n}$ to be $i_n = E_{i',i} + E_{i'+1,i}$, and $a f_n \in \mathbb{S}^{n,n}$ to be $i_n = E_{i'+1,i} + E_{i'+1,i}$. Note if $a, a' \in D_{D,n}$ then $a e_{a'} ^{a'} f_n \in \mathbb{S}_{D,n,n}$. For $i \in \mathbb{Z}/n\mathbb{Z}$ set

$$E_i(D) = \sum [a e_{a'}], \quad F_i(D) = \sum [a' f_n],$$

where the sum is taken over all $a, a' \in \mathbb{S}_{D,n}$ such that $a \cup a'$. For $a \in \mathbb{S}^n$ set

$$K_a(D) = \sum_{b \in \mathbb{S}_{D,n}} e^a b [ib],$$

where, for any $a, b \in \mathbb{S}^n$, $a \cdot b = \sum_{i=1}^n a_i b_i \in \mathbb{Z}$. If we let $X' = Y' = \mathbb{S}^n$, and $I = \mathbb{Z}/n\mathbb{Z}$, with the embedding of $I \subset X' = Y'$ and pairing as given above, we obtain a symmetric simply-laced root datum. We call the quantum group associated to it $U(\mathfrak{gl}_n)$. It can be shown [99] that the elements $E_i(D), F_i(D), K_a(D)$, generate a subalgebra $U_D$ which is a quotient of the quantum group $U(\mathfrak{gl}_n)$, via the map the notation suggests. Note that this gives the algebra $A_D$ the structure of a $U(\mathfrak{gl}_n)$-module. In this paper we will consider the slightly smaller algebra $U = U(\mathfrak{sl}_n)$, for which $X = \{ a \in \mathbb{S}^n : \sum_{i=1}^n a_i = 0 \}$, and $Y = \mathbb{S}^n / \mathbb{Z} b_0$, or more precisely its modified form which we will denote by $U$. $U(\mathfrak{sl}_n)$ is a subalgebra of $U(\mathfrak{gl}_n)$, but it is easy to see that its image in $A_D$ is all of $U_D$.  

• $F_i^2 F_j + (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$.  

The other object we need is the modified quantum group $\hat{U}$. Let $\text{Mod}_X$ denote the category of left $U$-modules $V$ with a weight decomposition, that is

$$V = \bigoplus_{\lambda \in X} V_\lambda,$$

where

$$V_\lambda = \{ v \in V : K_\mu v = v^{(\mu,\lambda)} v, \forall \mu \in Y \}.$$
3. Distinguished elements in $\mathfrak{A}_D$

Our first step in understanding the theory of cells in $U$ is to understand the corresponding theory for the affine $q$-Schur algebra. We do this by transferring the information known about the Hecke algebra $H_D$ to our case. The key ingredient in our approach is the use of Lusztig’s notion of distinguished elements.

We begin by defining a somewhat mysterious integer-valued function $a'_D$, which together with certain variants, play a crucial rôle in our study of cells.

For $\{C\} \in \mathfrak{B}_D$ we set $n'_{A,B}$ to be the largest power of $v$ occurring in the structure constant $\nu'_{A,B}$, and for $a \in \mathfrak{S}_{D,n}$ set $|a|^2 = \sum_{i=1}^n a_i^2$.

**Definition 3.1.** For $\{A\} \in \mathfrak{B}_D$, such that $\{A\} = \{A\}[i_a]$, consider the set of positive integers

$$\{n'_{A,B,C} + |b|^2 - |a|^2 : \{B\}, \{C\} \in \mathfrak{B}_D; \{C\} \in \{i_b|\mathfrak{A}_D[i_a]\}\}.$$ 

If it has a largest element $d$ we set $a'_D = d$, otherwise we set $a'_D = \infty$.

At first sight it would seem that we elided by saying that $a'_D$ is “integer-valued” above, however the following lemma shows this is not the case.

**Lemma 3.2.** The function $a'_D$ is finite for every $\{A\} \in \mathfrak{B}_D$.

**Proof.** To show this we use the fact that we can interpret the structure constants in terms of those for the affine Hecke algebra, and then use the result of Lusztig [L83], which shows that the corresponding function on the Kazhdan-Lusztig basis is finite. Indeed, using Lemma 2.2 we see that for $\{A\}, \{B\}, \{C\} \in \mathfrak{B}_D$ we have $\nu_{A,B} = p_{c(A)} h_{x,y}$ where $x, y, z \in W$ are the corresponding element of the affine Weyl group of type $\tilde{A}_{D-1}$. Now by Theorem 7.2 in [L83] we have $v^{-l(u_0)} h_{x,y} \in \mathbb{Z}[v^{-1}]$ for any $x, y, z \in W$, where $u_0$ is the longest element in $S_D$, the finite Weyl group. The result follows.

Note that we have shown that $a'_D$ is not only finite, but in fact bounded. We also set $\gamma'_{A,B}$ to be the coefficient of $v^{a'_D(C)-|b|^2+|a|^2}$ in $\nu'_{A,B}$ (which in general may be zero).

**Definition 3.3.** Let $a \in \mathfrak{S}_{D,n}$. For $\{A\} \in \{i_a|\mathfrak{A}_D[i_a]\}$ set $\Delta(A)$ to be the integer $d \geq 0$ such that

$$\{i_a\}, \{A\}\} = a_d v^{-d} + a_{d+1} v^{-d-1} + \ldots,$$

where $a_0 = a_d$. Set $n_A = a_d$.

**Lemma 3.4.** Let $a \in \mathfrak{S}_{D,n}$. For any $\{A\} \in \mathfrak{B}_D$ with $\{i_a\} = \{A\}$ we have $a'_D(A) \leq \Delta(A)$.

**Proof.** This follows an idea of Springer in the Hecke algebra case. Suppose that $\{B\}, \{C\}$ are in $\mathfrak{B}_D$, and consider the product $\{B\} \{C\}$. We may write this as a sum $\sum_{(E) \in \mathfrak{B}_D} \nu'_{B,C}(E)$. Chose $\{B\} \in \{i_b|\mathfrak{A}_D[i_b]\}$, and $\{C\} \in \{i_b|\mathfrak{A}_D[i_a]\}$ so that

$$\nu'_{B,C} = \gamma_{B,C} v^{a'_D(A)-|b|^2+|a|^2} + \ldots,$$

where $\gamma'_{B,C} \neq 0$ and the remaining terms are of lower degree. We have the inner product

$$\{\{B\} \{C\}, [i_a]\} = \sum_{(E) \in \mathfrak{B}_D} \nu'_{B,C}([E], [i_a]) \Delta.$$
All the terms here have nonnegative integer coefficients (using the positivity of the inner product). The properties of the inner product $(\cdot)_D$ show that this is also equal to
\[ v^{|a|^2-|b|^2}(\{C\}, \{B^t\})_D \]
Now since the canonical basis $B_D$ is almost orthogonal with respect to the inner product, we see that all the terms on the left-hand side of Equation 3.1 lie in $v^{|a|^2-|b|^2}N[v^{-1}]$. In particular, taking $\{E\} = \{A\}$ we get that
\[ v^A_{B,C}(\{A\}, [i_a])_D = n_A \gamma^A_{B,C} v^{a'_D(A)-|b|^2+|a|^2-\Delta(\{A\})} + \ldots \in v^{|a|^2-|b|^2}N[|v^{-1}|], \]
and hence the result. \qed

Motivated by this, we define the set of distinguished elements of $B_D$ as follows.

**Definition 3.5.** Let $D_D$ be the set of elements $\{A\}$ in $B_D$ such that there is an $a \in \mathfrak{g}_{D,n}$ with $\{A\} \in [i_a] \mathfrak{a}_D [i_a]$ and $a_D(A) = \Delta(A)$.

The distinguished elements $D_D$ are defined by analogy with the Hecke algebra case due to Lusztig [87]. We note the some consequences of the above proof.

**Corollary 3.6.** We have the following properties:

1. If $\{A\} \in D_D$ and $\{B\}, \{C\} \in B_D$ are such that $\gamma^A_{B,C} \neq 0$ then $\{C\} = \{B^t\}$.
2. For each $\{B\} \in B_D$, there is a unique $\{A\} \in D_D$ with $\gamma^A_{B,B^t} \neq 0$.
3. If $\{A\} \in D_D$, then $\{A\} = \{A^t\}$.

**Proof.** For the first, note that in the above proof, the almost orthogonality of $B_D$ with respect to the inner product implies that it is necessary and sufficient to have $\{C\} = \{B^t\}$. That the product contains just one element of $D_D$ is also immediate. For the last statement, pick $\{B\}, \{C\}$ such that $\gamma^A_{B,C} \neq 0$. By the first statement, we see that $\{C\} = \{B^t\}$. Since the product $\{B\} \{B^t\}$ is preserved by the transpose anti-automorphism $\Psi$, we see that $\gamma^u_{B,B^t} \neq 0$, and so by the second statement, $\{A\} = \{A^t\}$. \qed

Recall that to each element of $B_D$ we have attached an element of the affine Weyl group. We will show that in this way, the distinguished elements of $B_D$ actually correspond to distinguished elements of $W$.

**Lemma 3.7.** Let $\{A\} \in D_D$, then the Weyl group element $w_A$ is distinguished and conversely.

**Proof.** Let $a \in \mathfrak{g}_{D,n}$ be such that $[i_a] \{A\} = \{A\}$. By definition we see that
\[ (\{A\}, [i_a])_D = \Pi_{i_a} E, \]
the stalk of the intersection cohomology sheaf on $\tilde{X}_E$ at the point corresponding to $[i_a]$. But this is equal to $v^l(w_a) p_{1,w_E}$, where $p_{1,w_E}$ is the affine Kazhdan-Lusztig polynomial attached to $1, w_E$, and $w_a$ is the longest element of the parabolic subgroup attached to $a$ (the intersection cohomology sheaves are related by a smooth pullback with fibre dimension $l(w_a)$. Thus we see that if $\Delta(w_E)$ is the lowest power of $v^{-1}$ occurring in $p_{1,w_E}$,
\[ a'_D(E) \leq a'(w_E) - l(w_a) \leq \Delta(w_E) - l(w_a) = \Delta(E). \]
Here the function $a'$ on the Kazhdan-Lusztig basis is the one defined in [87] (there denoted simply $a$). For $z \in W$, we set $a'(z)$ to be the highest power of $v$ appearing
in a structure constant \( h_{x,y}^z \) as \( x,y \) vary over \( W \). The first inequality follows directly from the definitions of \( a'_E, a'_D \), and the second from analog of Lemma 3.4 for \( H_w \).

It follows immediately that \( w_E \) is distinguished if \( \{ A \} \) is. To establish the converse, it is necessary to note that if one picks any element \( x \) of the left cell containing the distinguished element, then by [L87] the structure constant \( h_{x-1,x}^w \) has \( a'(w_E) \) as its highest power of \( v \), and so \( a'_D(E) = a'(w_E) - l(w) \). □

We now show that all the notions of cells for \( \mathfrak{A}_D \) can be deduced from those for the Hecke algebra. More precisely we have the following result.

**Proposition 3.8.** Let \( \{ A \}, \{ B \} \in \mathfrak{B}_D \).

1. \( \{ A \} \sim_L \{ B \} \) if and only if \( w_A \sim_L w_B \) and \( c(A) = c(B) \);
2. \( \{ A \} \sim_R \{ B \} \) if and only if \( w_A \sim_R w_B \) and \( r(A) = r(B) \);
3. \( \{ A \} \sim_{LR} \{ B \} \) if and only if \( w_A \sim_{LR} w_B \);
4. \( a'_D(\{ A \}) = a'(w_A) - l(w) \).
5. Each left cell contains precisely one distinguished element.

**Proof.** For the first claim, note that since the notion of cell in \( \mathfrak{A}_D \) is defined essentially by using a subset of the Kazhdan-Lusztig basis consisting of those elements which are of maximal length in certain double cosets, it is clear that if \( \{ A \} \sim_L \{ B \} \) then \( w_A \sim_L w_B \). Moreover, certainly we have \( c(A) = c(B) \). For the converse, we need to use the distinguished elements. Suppose that \( w_A \sim_L w_B \) and \( c(A) = c(B) \). Then if \( d \) is the unique distinguished element in the left cell \( \Gamma \) containing \( w_A, w_B \) (which exists by [L87]), \( d \) determines a distinguished element of \( \mathfrak{B}_D \), \( \{ E \} \) say, where \( c(E) = c(A) \).

For \( x,y,z \in W \) let \( \gamma_{x,y}^z \) denote the coefficient of \( v^{a'(z)} \) in \( h_{x,y}^z \). Then by [L87, Theorem 1.8] we know that \( \gamma_{x,y}^z = \gamma_{y,z}^{-1} = \gamma_{z^{-1},x}^{-1} \), and so as \( \gamma_{w_A^{-1},w_A}^{-1} \neq 0 \) we see that

\[
\gamma_{w_A^{-1},w_A}^{-1} \gamma_{w_B^{-1},w_B}^{-1} \gamma_{w_A^{-1},w_B}^{-1} \gamma_{w_B^{-1},w_A}^{-1}
\]

are all nonzero, and hence the same is true of

\[
\nu_{A,1}^{E,A}, \nu_{A,E}^{V_{A,E}^{A}}, \nu_{B,1}^{E,B}, \nu_{B,E}^{V_{B,E}^{B}}.
\]

It follows that \( \{ A \} \sim_L \{ E \} \) and \( \{ B \} \sim_L \{ E \} \), and hence \( \{ A \} \sim_L \{ B \} \).

The second claim either follows in the same way, or by taking inverses in \( W \), which corresponds to applying the transpose map \( \Psi \) in \( \mathfrak{A}_D \).

For the third, the forward implication is again clear. If \( w_A \sim_{LR} w_B \), then it follows that the left cell containing \( w_A \) and the right cell containing \( w_B \) intersect (since this is true of any left and right cell in the same two-sided cell of an affine Hecke algebra). As any element in this intersection will give rise to an element \( \{ C \} \) of the q-Schur algebra with \( r(C) = r(B) \) and \( c(C) = c(A) \), we obtain the result using the first two parts of the proposition. The fourth claim follows from Corollary 3.6 (see the end of the proof of Lemma 3.7). Since each left cell of the Hecke algebra contains a unique distinguished element, the fifth claim follows from the Lemma 5.3 and the first claim. □

4. **Cells in \( \mathfrak{A}_D \)**

We saw at the end of the last section that the theory of cells of the affine q-Schur algebra is determined by that for the type A affine Hecke algebra. This allows us to
describe explicitly the number of two-sided cells in the affine q-Schur algebra, and also the number of left cells (and hence right cells) in a given two-sided cell. To do this we recall the combinatorial definitions which describe the bijection between cells for the Hecke algebra and partitions.

**Definition 4.1.** Suppose \( w \in W \) the affine Weyl group of type \( \tilde{A}_{D-1} \). Then we may view \( w \) as a permutation of \( \mathbb{Z} \). A sequence \( (i_1, i_2, \ldots, i_r) \) is called a \( d \)-chain if \( i_1 < i_2 < \ldots < i_r \) and \((i_1)w > (i_2)w > \ldots > (i_r)w \), and the \( \{i_j, j = 1, \ldots, r\} \) are all incongruent modulo \( D \).

Let \( \mathcal{P}_D \) be the set of partitions of \( D \). We define a map \( \sigma : W \to \mathcal{P}_D \) as follows. For \( w \in W \), let \( d_j \) be the maximal size of a set of \( j \) \( d \)-chains, the union of whose elements are all incongruent modulo \( D \). Then it is known that \( \lambda = (d_1, d_2 - d_1, d_3 - d_2, \ldots, d_{D-1} - d_D) \) is a partition of \( D \). Set \( \sigma(w) = \lambda \).

The following result is due to Lusztig, based on the work of Shi, see [L85a], [Sh].

**Theorem 4.2.** The fibres of \( \sigma \) are precisely the two-sided cells of \( W \). \( \square \)

We may use this to give a description of the two-sided cells in the affine q-Schur algebra as follows:

**Definition 4.3.** Let \( A \in \mathcal{S}_{D,n} \). An anti-diagonal path in \( A \) is an infinite strip of entries \( a_{i_k,j_k} : k \in \mathbb{Z} \) such that \((i_k,j_k)\) is either equal to \((i_k-1,j_k-1)\) or \((i_k-1,j_k+1)\) with the latter being the case for all but finitely many \( k \). Thus visually if you draw the matrix with rows increasing from top to bottom, and columns from left to right, (as we will do) then path starts and ends with infinite vertical strips, and takes finitely many right or vertical turns.

Let \( d_j \) be the maximal size of the sum of entries in the union of \( j \) anti-diagonal paths. Then we define a map \( \rho : \mathcal{S}_{D,n} \to \mathcal{P}_D^j \) where \( \mathcal{P}_D^j \) is the set of partitions of \( D \) with at most \( n \) parts, by setting \( \rho(A) = (d_1, d_2 - d_1, \ldots, d_n - d_{n-1}) \). As above, it follows from general results on posets that \( \rho(A) \) is indeed a partition. The fact that it can have at most \( n \) parts is obvious. We will sometimes view \( \rho \) as a map from \( \mathcal{B}_D \) in the obvious way.

**Proposition 4.4.** The fibres of the map \( \rho : \mathcal{B}_D \to \mathcal{P}_D^j \) are the two-sided cells of \( \mathcal{A}_D \).

**Proof.** This is a simple combination of the statements of Proposition 3 and Theorem 1.3. \( \square \)

**Example 4.5.** Suppose that \( n = 2 \) and \( D = 5 \). Consider the element \( \{A\} \) of \( \mathcal{B}_5 \) corresponding to

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 1 & 1 & 0 & 1 \textbf{1} & 0 \cdots \\
\cdots & 0 & 1 & 0 & 1 & 0 \cdots \\
\cdots & 0 & 0 & 1 & 1 & 0 \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

where the top left entry shown is in the \((1, 1)\) entry of \( A \). Then \( \{A\} \) lies in the two-sided cell corresponding to the partition \((4, 1)\). The boxed entries give part of an anti-diagonal path which has entry sum 4. Note that it is not unique.
Given a partition $\lambda \in \mathcal{P}_D^n$, we denote the two-sided cell $\rho^{-1}(\lambda)$ by $c_\lambda$. We will often use the same notation for a partition in $\mathcal{P}_D$ and a two-sided cell of $\mathcal{H}_D$.

Somewhat more elaborate is a description of the number of left cells in a two-sided cell of the affine $q$-Schur algebra. Notice that Proposition $3$ shows that each left cell of the affine Hecke algebra gives rise to a number of left cells of the affine $q$-Schur algebra, with the number depending on the set of simple reflections of the symmetric group $S_D$, which decrease the length of any element of the left cell when multiplied on the right.

For $w \in W$ let $R(w) = \{ s \in S : l(ws) < l(w) \}$ and $L(w) = \{ s \in S : l(sw) < l(w) \}$. It is known that the functions $R, L$ are constant on right and left cells respectively. Thus for $\Gamma$ a left cell, we may write $R(\Gamma)$ for the set $R(w)$, where $w$ is any element of $\Gamma$. The left cells of $\mathcal{H}_D$ have been described by Shi ([Sh], chapter 14) as the fibres of a map to a set of tableaux, such that the shape of the tableau associated to a left cell is given by the partition of the two-sided cell it lies in, and the entries must increase down the columns.

In order to describe this map in more detail we need some to make some definitions. We use the description of $W$ as a group of permutations of the integers, and in particular the associated infinite matrices. Let $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ be such a matrix. A block is a set of consecutive rows of $A$. For a block of $m$ rows $i+1, i+2, \ldots, i+m$, let the nonzero entries be $\{a_{i+1,j_1}, a_{i+2,j_2}, \ldots, a_{i+m,j_m}\}$. We say the block is a descending chain if $j_1 > j_2 > \ldots > j_m$. A block is a maximal descending chain (MDC) if it cannot be imbedded in a larger such block.

Say that an element of $w \in W$ has full MDC form at $i$, if there exist consecutive MDC blocks $(A_i, A_{i-1}, \ldots, A_1)$ of $A_w$ of size $m_t$, for $t = 1, \ldots, l$, with $\sum_{t=1}^l m_t = D$, and $i+1$ the first row of $A_i$ (so $i + \sum_{j=1}^k m_j + 1$ is the first row of $A_{i-k}$). Suppose a full MDC form has blocks which are of (weakly) increasing size (so $A_i$ has at most as many rows as $A_{i-1}$). Let $j_i^u$ be the column of the nonzero entry in the $u$-th row of $A_i$. Then we say the form is normal if $j_i^u - n < j_i^v < j_i^{u-1} < \ldots < j_i^1$, for each $u$ (where we ignore terms in this sequence which do not exist).

For $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$ a partition of $D$, let $N_\lambda$ be the set of the entries of the two-sided cell $c_\lambda$ corresponding to $\lambda$ which have normal MDC form $(A_r, A_{r-1}, \ldots, A_1)$ at $i$ for some $i \in \mathbb{Z}$, where $\lambda_j$ is the number of rows in $A_j$.

**Theorem 4.6.** [Sh] Let $w \in c_\lambda$. Then there is a $y \in N_\lambda$ with $y \sim_L w$. 

Let $C_\lambda$ be the set of Young diagrams of shape $\lambda$ with entries $\{1, 2, \ldots, D\}$ which decrease down columns. In [Sh] chapter 14 Shi defines a map $T$ from the left cells in $c_\lambda$ to $C_\lambda$. Let $\Gamma$ be a left cell in $c_\lambda$. Choose $y \in N_\lambda \cap \Gamma$ and then set the entries of column $u$ of $T(\Gamma)$ to be the residues modulo $D$ of the numbers $\{j_i^u : 1 \leq t \leq \mu_u\}$ where $\mu$ is the partition dual to $\lambda$. Shi shows this is independent of the choice of $y$, and that it gives a bijection.

**Example 4.7.** Consider the matrix $A_w$ in $\hat{A}_4$ given as follows

$$
\begin{pmatrix}
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 1 & 0 & 0 & 0 & 0 & \cdots 
\end{pmatrix}
$$
where the first column shown is column 1. Then it is easy check that \( w \in N_{(3,2)} \) and the associated tableau is given below.

\[
\begin{array}{ccc}
5 & 4 & 1 \\
3 & 2 \\
\end{array}
\]

It follows directly from this construction (though this is not explicitly described in [SH]) that the set of simple reflections in \( R(\Gamma) \) is determined by this tableau. Indeed since \( R(\Gamma) \) is given by \( R(w) \) for any \( w \in \Gamma \), we may use an element of \( N_\lambda \) as above. Then it is easy to see that the simple reflection \( s_i \) is in \( R(\Gamma) \) precisely when \( i \) appears to the right of \( i+1 \) in the tableau (where one reads modulo \( D \) for \( s_0 \)).

We now consider the left cells of \( A_D \). Each such cell correspond to a left cell \( \Gamma \) of \( H_D \) and an element \( a \) of \( S_{D,n} \) such that the simple reflections \( J \) of \( S_a \) are a subset of \( R(\Gamma) \setminus \{ s_0 \} \).

**Definition 4.8.** For \( \lambda \in \mathcal{P}_D \), let \( C_\lambda^n \) be the the set of tableaux of shape \( \lambda \) with entries from \( \{1, 2, \ldots, n\} \) strictly decreasing down columns. Thus \( C_\lambda^n \) is empty if \( \lambda \) has more than \( n \) parts.

If we fix a two-sided cell \( c_\lambda \), where \( \lambda \in \mathcal{P}_D^n \), using the description of \( R(\Gamma) \) in terms of the tableau \( T(\Gamma) \), it is easy to see that the left cells in \( c_\lambda \) are indexed by the elements of \( C_\lambda^n \). Indeed to each tableau \( T \in C_\lambda^n \) there is a well-defined element \( h(T) \) of \( C_\lambda \) given as follows. Order the boxes of \( T \) by listing those labelled 1 first, then 2, and so on, always reading from right to left. Then construct \( h(T) \in C_\lambda \) by labelling each box with its position in the order just described. This gives the left cell of \( W \). The element \( a \) is determined by letting \( a_i \) be the number of boxes of \( T \) labelled \( i \), for \( i \in \{1, 2, \ldots, n\} \). The following example makes the correspondence clear.

**Example 4.9.** Let \( D = 5 \) and \( n = 3 \). Suppose that we consider the tableau

\[
\begin{array}{cc}
3 & 2 \\
2 & 1 \\
1 \\
\end{array}
\]

in \( C_{(2,2,1)}^3 \). Then the tableau corresponding to it in \( C_{(2,2,1)} \) is

\[
\begin{array}{ccc}
5 & 3 & \\
4 & 1 & \\
2 & & \\
\end{array}
\]

and the sequence \( a \) is \( (2, 2, 1) \) (repeated periodically).

This allows us to count the number of left cells in a two-sided cell of \( A_D \).

**Proposition 4.10.** Let \( c \) be a two-sided cell of \( A_D \). If \( \lambda \) is the partition of \( D \) associated to \( c \), and \( \lambda(i) := \lambda_i - \lambda_{i+1} \), then the number of left cells in \( c \) is

\[
\prod_{i=1}^{n-1} \binom{\lambda(i)}{i}
\]

\( \square \)
Finally we wish to construct the asymptotic algebra associated to a two-sided cell. We will need a variant of the function $a_D$.

**Definition 4.11.** Let $\{A\} \in \mathcal{B}_D$. If there is an integer $d \geq 0$ such that $v^{-d}\nu_{A_{\ell}}^C \in \mathbb{Z}[v^{-1}]$ for all $\{B\}, \{C\} \in \mathcal{B}_D$ then let $a(A)$ be the smallest such. Otherwise set $a(A) = \infty$.

Note that the proof of Lemma 3.2 shows that $a_D$ is always finite. More interestingly we have the following result.

**Lemma 4.12.** The functions $a_D'$ and $a_D$ agree. Moreover the function $a_D$ is constant on $c[\mathfrak{i}_n]$ for any two-sided cell $c$ and any $\mathfrak{i}_n \in \mathfrak{S}_{D,n}$.

**Proof.** Both of these follow from facts about the Hecke algebra: If $\gamma^{\ell}_{x,y}$ denotes the coefficient of $v^{a'(x)}$ in $h^{\ell}_{x,y}$ then it follows from the results above that for $\{A\}, \{B\}, \{C\} \in \mathcal{B}_D$ we have $\gamma^{\ell}_{A_{\ell}B} = \gamma^{w_{A_{\ell},w_{B}}}_{\ell}$, Moreover, by the results of [1,87] (see Lemma 3) we know that $\gamma^{\ell}_{x,y} = \gamma^{x^{-1}}_{y^{-1},x} = \gamma^{x^{-1}}_{z^{-1},x}$ and hence $\gamma^{C}_{A_{\ell}B} = \gamma^C_{B,C} = \gamma^{C}_{G\ell,A}$. It is now easy to see that $a_D' = a_D$. Since $a'$ is constant on two-sided cells of the Hecke algebra, the second statement is clear. 

We now rescale the canonical basis of $\mathfrak{A}_D$. Set $\langle A \rangle = v^{-a_D(A)}\{A\}$. Let $\mathfrak{A}_c$ denote the span of the elements in $c$ a two-sided cell of $\mathfrak{B}_D$. This becomes an algebra by identifying it with a subquotient of $\mathfrak{A}_D$ in the obvious way. The structure constants of $\mathfrak{A}_c$ are the coefficients $v^{-a_D(A)}\nu_{A_{\ell}}^C$, for all $\{A\}, \{B\}, \{C\} \in \mathcal{B}_D$ and $a_D(A) = a_D(C)$, and the co-efficients $v^{-a_D(A)}\nu_{A_{\ell}B}^C$ all lie in $\mathbb{Z}[v^{-1}]$. Thus if $\mathcal{L}_c$ is the $\mathbb{Z}[v^{-1}]$ span of the $\{\langle A \rangle : \{A\} \in c\}$, $\mathcal{L}_c$ has the structure of a $\mathbb{Z}[v^{-1}]$ algebra. The quotient $\mathcal{J}_c = \mathcal{L}_c/v^{-1}\mathcal{L}_c$ is then a $\mathbb{Z}$ algebra, where if $t_A$ is the image of $\langle A \rangle$, the multiplication in $\mathcal{J}_c$ is given by

$$t_A t_B = \sum_C \gamma_{A_{\ell}B}^C t_C.$$ 

Let $\mathcal{D}_c = \mathcal{D}_D \cap c$. It follows from the above that the set $\{t_E : \langle E \rangle \in \mathcal{D}_c\}$ gives a decomposition of the identity into orthogonal idempotents.

By using the results of [13] or [10] we can also give an explicit description of this asymptotic algebra. For $\lambda \in \mathcal{P}_n$ and $i \in \{1, 2, \ldots, n\}$, let $\lambda(i) = \lambda_i - \lambda_{i+1}$, (where $\lambda_{n+1} = 0$). Let $G_{\lambda}$ be the reductive group $\prod_{i=1}^n GL_{\lambda(i)}(\mathbb{C})$ and let $R_{\lambda}$ be the $K$-group of its representations, so that the irreducible representations $\bar{G}_\lambda$ form a $\mathbb{Z}$-basis of $R_{\lambda}$. Let $T_{\lambda}$ be the set of triples $(E_1, E_2, \kappa)$ where $\{E_1\}, \{E_2\} \in \mathcal{D}_\lambda$, and $\kappa \in \bar{G}_\lambda$. Let $\mathcal{J}_\lambda$ be the free Abelian group on $T_{\lambda}$. Define a ring structure on $\mathcal{J}_\lambda$ by

$$(E_1, E_2, \kappa)(E_1', E_2', \kappa') = \sum c_{\kappa,\kappa'}^{\kappa''} \delta_{E_2,E_2'} (E_1, E_2', \kappa'')$$

where the sum is over $\kappa'' \in \bar{G}_\lambda$ and $c_{\kappa,\kappa'}^{\kappa''}$ is the multiplicity of $\kappa''$ in the $G_{\lambda}$-module $\kappa \otimes \kappa'$. Thus $\mathcal{J}_\lambda$ is a matrix ring of rank $N$ over the representation ring $R_{\lambda}$, where $N$ is the number of left cells in $c_\lambda$ given in Proposition 4.10.

**Proposition 4.13.**

(1) There is a ring isomorphism $\mathcal{J}_c \rightarrow \mathcal{J}_\lambda$ which restricts to a bijection between the canonical basis of $\mathcal{J}_c$ and $T_{\lambda}$.

(2) For any $\{E\} \in \mathcal{D}_c$, the subset of $c_\lambda$ corresponding to $\{(E_1, E_2, \kappa) \in \mathcal{J}_\lambda : E_2 = E\}$ under the bijection is a left cell.
For any $\{E\} \in D_{c_{\lambda}}$, the subset of $c_{\lambda}$ corresponding to $\{(E_1, E_2, \kappa) \in T_{\lambda}: E_1 = E\}$ under the bijection is a right cell.

□

This shows that all the simple modules of the $\mathbb{C}$-algebra $\mathbb{C} \otimes J_{c_{\lambda}}$ are $N$-dimensional and that the set of isomorphism classes of such modules is in bijection with the semisimple conjugacy classes of $G_{\lambda}$.

The asymptotic algebra also receives a homomorphism from the original algebra, once we tensor with $\mathbb{Q}(v)$. Define a map $\Phi_{c_{\lambda}}: A_{D} \to \mathbb{Q}(v) \otimes J_{c_{\lambda}}$ as follows:

$$\Phi_{c_{\lambda}}(\{A\}) = \sum_{\{E\} \in D_{c_{\lambda}}, \{B\} \in c_{\lambda}} \nu_{A,B}^{E} t_{E}.$$ 

One shows it is a homomorphism as in [L95, Proposition 1.9], where the property of the structure constants which is needed follows from the Hecke algebra case. This allows one to pull back representations of $J_{c_{\lambda}}$ to representations of $A_{D}$.

5. Cells in $\hat{U}$

Let $\hat{U}$ be the modified quantum group of affine $\mathfrak{sl}_n$, and let $\hat{B}$ be its canonical basis (see [L93]). We now show how we can lift information about the cell structure of the affine q-Schur algebra to the modified quantum group. If $\phi_D$ were surjective this would be a straightforward consequence of the previous section. Indeed in the finite type case (see [BLM], [L99a]), the analogue of $\phi_D$ is surjective, and the results of [L95] in the case of $\mathfrak{sl}_n$ can be recovered in this way, as was essentially done by Du in [Du] (note however that [L95] is much more general, classifying the cell structure for any finite type quantum group).

In the affine case it is no longer true that the homomorphism from the quantum group is surjective. Thus we need to be more careful in lifting information from $A_{D}$ to $\hat{U}$. The following theorem relating the canonical bases $\hat{B}$ and $B_{D}$ was conjectured by Lusztig, and proved in [ScV]. A more geometric proof can be found in [M].

Theorem 5.1. For all $b \in \hat{B}$ we have $\phi_D(b) \in \{0\} \cup B_D$. Moreover the kernel of $\phi_D$ is spanned by the elements $b \in B$ such that $\phi_D(b) = 0$.

□

It follows that the image $U_D$ is a union of two-sided cells of $\hat{U}$. Moreover the injectivity result of [L99a] (see also [M]) shows that any two-sided cell will eventually lie in some $U_D$. Given $A \in \mathcal{S}_{D,n}$, we say that $A$ is aperiodic if, for any integer $k \neq 0$ there is an integer $p$ with $a_{p,p+k} = 0$. Thus only the main diagonal of $A$ can consist entirely of nonzero entries. In [L99], Lusztig showed that $U_D$ is spanned by a subset $B_D$ of $B_D$ consisting of those $\{A\}$ for which $A$ is aperiodic.

We now define an analogue of the $a_D$ function, following [L93]. Let $c_{b,b'}^{''}$ be the structure constants of $\hat{U}$ with respect to $\hat{B}$. For a two-sided cell $c$ in $\hat{U}$ let $U_c$ be the subspace of $\hat{U}$ spanned by the elements of $c$. We endow $U_c$ with an algebra structure by identifying it with a subquotient of $\hat{U}$, so that for $b,b' \in c$ the product is given by

$$bb' = \sum_{b'' \in c} c_{b,b'}^{''} b''.$$
**Definition 5.2.** Let $b \in \hat{B}$. If there is an integer $n \geq 0$ such that $v^{-n}c_{b,b}^{b'} \in \mathbb{Z}[v^{-1}]$ for all $b', b'' \in c$ then let $a(b)$ be the smallest such. Otherwise set $a(b) = \infty$.

The following observation simple observation tells us about the left cells in $\hat{U}$.

**Lemma 5.3.** Let $\Gamma$ be a left cell of $A_{D}$, and let $\{E\} \in D_{D}$ be the unique distinguished element in $\Gamma$. Then if $B_{D} \cap \Gamma \neq \emptyset$ we must have $\{E\} \in B_{D}$. Moreover $B_{D} \cap \Gamma$ is a single left cell of $U_{D}$.

**Proof.** Pick $\{A\} \in B_{D} \cap \Gamma$. Then we know that
\[
\{A\} = \nu_{A', A}^{E} \{E\} + \ldots,
\]
where $\nu_{A', A}^{E} \neq 0$ since $\gamma_{A', A}^{E} \neq 0$ (the unique distinguished element for which $\gamma_{A', A}^{E} \neq 0$ must clearly be the one in the left cell containing $\{A\}$). This implies that $\{E\} \in B_{D}$. By arguing as in the proof of the first claim in Proposition 3) we see that the intersection $B_{D} \cap \Gamma$ is a single left cell. 

This has some important corollaries which we now record.

**Corollary 5.4.** We have the following properties of $a$ functions.

(1) The functions $a_{D}, a'_{D}$ coincide with the analogous functions defined in terms of $U_{D}$ instead of $A_{D}$.

(2) For $b \in B$ if $\phi_{D}(b) \neq 0$ then $a(b) = a_{D}(\phi_{D}(b))$. In particular, $a(b)$ is finite.

**Proof.** The claims are easy consequences of the above lemma, using distinguished elements. 

We now know that each left cell in $\hat{U}_{D}$ contains a unique distinguished element. We wish to show that the notion of distinguished elements lifts to $\hat{U}$. Since any left cell will occur as a left cell of $U_{D}$ for sufficiently large $D$, it suffices to show that the distinguished element we obtain is independent of $D$. Since the distinguished element is characterized as the idempotent element in the asymptotic algebra, which is determined by the two-sided cell, it is independent of the algebra $U_{D}$ we choose. Moreover one of the main results of [M] is that the inner product on $\hat{U}$ is obtained as a limit from those on $A_{D}$, thus we may give an intrinsic characterization of the set $D_{c}$ of distinguished elements in a two-sided cell $c$.

Recall from [L95, 3.7] that $\hat{U}$ possesses an anti-automorphism $\#: \hat{U} \rightarrow \hat{U}$, which is such that $\phi_{D}(x^{\#}) = \Psi(\phi_{D}(x))$, for any $x \in \hat{U}$.

**Proposition 5.5.** Let $b \in c$ and $\lambda \in X$ be such that $b \in \hat{U}_{1} \lambda$. Then $v^{a(b)}(1, b) \in \mathbb{Z}[v^{-1}]$ with nonzero constant term for $b \in D_{c}$ and $v^{a(b)}(1, b) \in v^{-1} \mathbb{Z}[v^{-1}]$ otherwise. Moreover $b = b^{\#}$. 

It remains to investigate the structure of the two-sided cells of $U_{D}$. This again lifts from $A_{D}$, as the following simple observations show: The transfer map $\psi_{D}: U_{D} \rightarrow U_{D-n}$ is such that $\psi_{D}(\{A\}) = \{A - I\}$ if the entries of $A - I$ are nonnegative and $\psi_{D}(\{A\}) = 0$ otherwise ($I = (\delta_{ij})$ is the identity matrix). Using this along with our combinatorial description of two-sided cells in $A_{D}$, we obtain the following statement. Let $\lambda = (\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{n} \geq 0)$ be in $P_{D}^{\circ}$, and let $k_{\lambda}$ denote the intersection $B_{D} \cap c_{\lambda}$. Then $k_{\lambda}$ is a union of two-sided cells of $U_{D}$ and moreover it follows from the above discussion that $k_{\lambda}$ maps to $0$ under $\psi_{D}$ unless $\lambda_{n} > 0$.
when it maps to $k_{\lambda}$ in $U_{D-n}$ where $\lambda' = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1)$ (i.e. $\lambda'$ is obtained by removing the first column of the Young diagram for $\lambda$).

However, the following observation which follows easily from Proposition 4.4 now shows that we are almost done.

**Lemma 5.6.** Let $A$ be an element of $\hat{S}_{D,n}$. Then $\rho(A)$ has strictly less than $n$ parts, precisely when $A$ has no completely nonzero diagonal (i.e. for each $k \in \mathbb{Z}$ there is some $p \in \mathbb{Z}$ with $a_{p,p+k} = 0$). In particular, if $\lambda \in P^+_D$ has fewer than $n$ parts $c_\lambda$ consists entirely of aperiodic elements. □

Thus for such $\lambda$ we see that $k_\lambda = c_\lambda$, and it consists of a single two-sided cell of $U_D$, or $\hat{U}$.

Recall the group $X$ of the root datum of $\hat{U}$ from section 8. For convenience, here we will view it as a quotient of $\mathbb{Z}^n$ (by taking the entries $a_1, a_2, \ldots, a_n$). We define $X^+$ to be the “dominant weights” in $X$. Let $I_0 = \{ i \in \mathbb{Z}/n\mathbb{Z} : i \neq 0 \mod n \}$. The set $X^+$ consists of those $\mu \in X$ with $\mu(i) = \langle i, \mu \rangle \geq 0$ for $i \in I_0$.

**Proposition 5.7.** The two-sided cells of $\hat{U}$ are naturally parameterized by $X^+$.

*Proof.* First note that each partition $\lambda$ with at most $n$ parts determines an element $\underline{\mu}$ in $X^+$ by taking the coset of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $X$, and the previous paragraph shows that this gives a natural bijection between $X^+$ and the two-sided cells of $\hat{U}$. Indeed each $\mu$ in $X^+$ has a unique representative $\hat{\mu}$ in $\mathbb{Z}^n$ with final entry 0. The cell corresponding to $\mu$ is $c_\mu$, thought of as a cell of $\hat{U}$. (It is actually a cell of $\hat{U}$, $U_D$, and $\mathfrak{A}_D$!) □

Note that this classification has an interesting consequence: The number of left cells in a two-sided cell $c_\xi$ of $\mathfrak{A}_D$ depends only on the element of $X^+$ it determines, as can be seen from the formula in Proposition 4.10. Thus since each left cell of $\mathfrak{A}_D$ intersects $U_D$ in at most one left cell, and the two algebras have the same number of left cells, this intersection is always nonempty.

We may also give an explicit formula for the value of the $a$ function, using the fact that we know the value of the corresponding function on the Hecke algebra. Indeed if $w \in \check{A}_{D-1}$ lies in the cell $c_\xi$ then $a(w) = (D - \sum \lambda_i^2)/2$.

**Lemma 5.8.** Let $\mu \in X^+$ and let $\hat{\mu} \in \mathbb{Z}^n$ be its representative with 0 in the final entry. Suppose $b \in \hat{B}$ lies in the cell $c_\mu$ corresponding to $\mu$, and $b_\nu = b$, for some $\nu \in X$. Pick the unique representative $v$ of $\nu$ in $\mathbb{Z}^n$ such that $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n v_i$. Then we have we have $a(b) = \sum_{i=1}^n (\lambda_i^2 - v_i^2)$.

We have also already constructed the asymptotic algebra $A_\mu$ for each $\mu \in X^+$. This is just the ring $\mathfrak{J}_{\xi_\mu}$ constructed in the previous section, where $\xi_\mu$ is the representative of $\mu$ described above.

Let $G_\mu = \prod_{i=1}^{n-1} GL_{\mu(i)}(\mathbb{C})$, and let $R_\mu$ be its representation ring. Combining the above with Proposition 4.13 we find that the asymptotic ring $A_\mu$ is isomorphic to a matrix ring over $R_\mu$ of size $\prod_{i=1}^{n-1} (\lambda_i)_{\mu(i)}$. Thus we may pull-back modules of this matrix ring to obtain modules for $\hat{U}$. These are the “extremal weight modules” of Kashiwara [K02], which are in turn related to the universal standard modules defined by Nakjima in his geometric classification of simple modules for quantum affine algebras. Indeed Kashiwara has a number of conjectures about the structure of these modules which he suggests should be closely related to the conjectures.
of Lusztig that we establish here for \( \hat{sl}_n \) (see the remark below). Kashiwara’s conjectures have recently been proved in the simply-laced case by Nakajima \[N\] and Beck \[B\], and using them it is easy to show that the modules obtained from the asymptotic algebra are indeed the extremal weight modules. This observation can be used to give another proof of the formula for the number of left cells in a two-sided cell. It should be possible to give another approach to the results of this paper using these techniques, which would work for all the simply-laced cases.

**Remark 5.9.** The results of this section establish (in the case of \( \hat{sl}_n \)) all the conjectures in \[L95, section 5\]. It should be noted that paragraph 5.4 of that section contains a misprint. Given \( \lambda \in X^+ \) the numbers \( \lambda(i) \), for \( i \in I_0 \), should be given by the formula \( \lambda(i) = \langle i, \lambda \rangle \).

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