Quantum algorithm for doubling the amplitude of the search problem’s solution states.

Mauro Mezzini
Roma Tre University, Italy

Fernando L. Pelayo and Fernando Cuartero
University of Castilla - La Mancha, Spain

May 17, 2021

Abstract

In this paper we present a quantum algorithm which increases the amplitude of the states corresponding to the solutions of the search problem by a factor of almost two.

1 Introduction and preliminaries

We denote the set of natural number excluding the 0 element by $\mathbb{N}^+$. If $x \in \mathbb{N}$, $0 \leq x < 2^n$ then we say that $|x\rangle$ is a computational state. We denote by $|x_{n-1}x_{n-2}\ldots x_0\rangle$ the binary representation of $|x\rangle$ where $x_{n-1}$ is the most significant bit of the binary representation of $x$. Let also denote by $w(x) = \sum_{j=0}^{n-1} x_j$. If $z \in \mathbb{N}$, $0 \leq z < 2^n$ we denote by $z \cdot x$ the sum $z \cdot x = \sum_{j=0}^{n-1} x_j z_j$. Furthermore, for any $k \in \mathbb{N}$, $0 < k < n$, we denote by $z_k$ a natural number obtained from $z$ by considering only the least $n-k$ significant bits, that is, if the binary representation of $z$ is $z_{n-1}z_{n-2}\ldots z_1z_0$ then the binary representation of $z_k$ is $z_{n-k-1}z_{n-k-2}\ldots z_0$. The $S$ gate for a single qubit is represented by the following matrix:

$$S = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}$$

2 The effect of applying Hadamard, S and Hadamard gates to $|0\rangle^\otimes n$

In this section we want to determine the effect of applying the Hadamard, S and Hadamard gates on quantum state $|0\rangle^\otimes n$, that is, we want to determine a formula for

$$|\alpha\rangle = H^\otimes n S^\otimes n H^\otimes n |0\rangle^\otimes n$$
and we show that in the final superposition \(|\alpha\rangle = \sum_{z=0}^{2^n-1} a_z |z\rangle\) the amplitude \(a_z\) of a single state \(|z\rangle\) depends by \(w(z)\).

It is known \(^2\) that given any computational state \(|x\rangle\), 0 \(\leq x < 2^n\)

\[|\psi\rangle = H^\otimes^n |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{x \cdot z} |z\rangle \tag{3}\]

Now we start with the following Lemma

**Lemma 1.** Let 0 \(\leq x < 2^n\)

\(S^\otimes^n |x\rangle = i^{w(x)} |x_{n-1}x_{n-2}\ldots x_0\rangle\)

**Proof.** By induction on \(n\) being the base case with \(n = 1\) straightforward. So suppose that the statement holds for \(n - 1\). Then

\[S^\otimes^n |x\rangle = S|x_{n-1}\rangle \otimes S|x_{n-2}\rangle \cdots \otimes S|x_0\rangle = S^\otimes^{n-1} |x_{n-1}\ldots x_1\rangle \otimes S|x_0\rangle = \]

\[= i^{\sum_{j=1}^{n-1} x_j} |x_{n-1}\ldots x_1\rangle \otimes S|x_0\rangle = \text{(by induction hypothesis)} \]

\[= i^{\sum_{j=1}^{n-1} x_j |x_{n-1}\ldots x_1\rangle \otimes i^{w(x)} |x_0\rangle = \]

\[= i^{w(x)} |x_{n-1}x_{n-2}\ldots x_0\rangle\]

\(\square\)

Now by Lemma 1 and equation (3), we have that

\[|\psi_1\rangle = S^\otimes^n H^\otimes^n |0\rangle^\otimes^n = S^\otimes^n \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} i^{w(x)} |x\rangle\]

and applying the Hadamard to \(|\psi_1\rangle\), by (3), we have that

\[|\psi_2\rangle = H^\otimes^n |\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} i^{w(x)} \left[ \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{x \cdot z} |z\rangle \right]\]

and reordering the term of the sum we have that

\[|\psi_2\rangle = \frac{1}{2^n} \sum_{z=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{x \cdot z} i^{w(x)} |z\rangle = \frac{1}{2^n} \sum_{z=0}^{2^n-1} \left( \sum_{x=0}^{2^n-1} i^{w(x)} (-1)^{x \cdot z} \right) |z\rangle\]

So in order to compute the amplitudes of \(|\psi_2\rangle\) we need to compute the sum

\[\sum_{x=0}^{2^n-1} i^{w(x)} (-1)^{x \cdot z}\]

for every 0 \(\leq z < 2^n\). We will do this in the following two theorems. First of all we need the following Lemma.

**Lemma 2.** Let 0 \(\leq z < 2^{2m+1}\) and 0 \(\leq x < 2^{2m+1}\) and let \(z_{2m}z_{2m-1}\ldots z_0\) and \(x_{2m}x_{2m-1}\ldots x_0\) be the binary representation, respectively, of \(z\) and \(x\). We have that

\[
\sum_{x=2^{2m}}^{2^{2m+1}-1} i^{w(x)} (-1)^{\sum_{j=0}^{2m} z_j x_j} = i(-1)^{z_{2m}} \sum_{x=0}^{2^{2m}-1} i^{w(x)} (-1)^{\sum_{j=0}^{2m-1} z_j x_j} \tag{4}
\]
Proof. We note that, on the left hand of the equation (4), for every element of the sum, we have that $x_{2m} = 1$. Therefore $\sum_{j=0}^{2m} x_j = \sum_{j=0}^{2m-1} z_j + x_{2m}$. Based on this we have that

$$\sum_{x=22m}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m} z_j} x_j = (-1)^{2^{2m}} \sum_{x=22m}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m-1} z_j} x_j$$

Furthermore for the same reason above, if $2^{2m} \leq x < 2^{2m+1}$ and if $0 \leq \bar{x} < 2^{2m}$ then we have that $w(x) = w(\bar{x}) + 1$ and this prove the equation (4).

Theorem 3. Let $0 \leq z < 2^n$, $z \in \mathbb{N}$. If $n = 2m$ is even we have that

$$\sum_{x=0}^{2^n-1} i^w(x)(-1)^{2 \cdot x} = (-1)^w(z)^{m+w(z)} 2^m$$

(5)

Proof. We prove the equation (5) on induction on $m$ being the base case with $m = 1$ easily verifiable for all $z \in \{0,1,2,3\}$. So suppose the statement holds for all $h \leq m$ and for all $0 \leq z < 2^{2m}$. Then, for any $0 \leq z < 2^{2m+2}$ we have

$$\sum_{x=0}^{2^{2m+2}-1} i^w(x)(-1)^{2 \cdot x} = \sum_{x=0}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m} z_j} x_j$$

(6)

$$+ \sum_{x=2^{2m}}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m} z_j} x_j + \sum_{x=2^{2m+1}}^{2^{2m+2}-1} i^w(x)(-1)^{2^{2m+1} \sum_{j=0}^{2m} z_j} x_j$$

(7)

(8)

Now, by equation (4) and by induction hypothesys, we have that (7) is equal to

$$\sum_{x=2^{2m}}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m} z_j} x_j = i(-1)^{2^{2m}} \sum_{x=0}^{2^{2m}-1} i^w(x)(-1)^{2^{2m} \sum_{j=0}^{2m-1} z_j} x_j =$$

(9)

$$= i(-1)^{2^{2m}} (-1)^w(z+2)x^{m+w(z)+2} 2^m$$

Likewise, in the term (8), for each $x$ is the sum, the bit $x_{2m+1}$ is always set to 1, so we have that (8) is, by equation (4), equal to

$$\sum_{x=2^{2m+1}}^{2^{2m+2}-1} i^w(x)(-1)^{2^{2m+1} \sum_{j=0}^{2m} z_j} x_j = i(-1)^{2^{2m+1}} \sum_{x=0}^{2^{2m+1}-1} i^w(x)(-1)^{2^{2m+1} \sum_{j=0}^{2m} z_j} x_j$$

(10)

Now by repeatedly applying equation (4) and the induction hypothesys we have
that the sum in right hand of equation \((10)\) is
\[
\sum_{x=0}^{2m+1-1} i^w(x)(-1)^{\sum_{j=0}^{2m} z_j x_j} = \sum_{x=0}^{2m-1} i^w(x)(-1)^{\sum_{j=0}^{2m-1} z_j x_j} + \sum_{x=2m}^{2m+1-1} i^w(x)(-1)^{\sum_{j=0}^{2m} z_j x_j} =
\]
\[
= (-1)^{w(z)} i^{m+w(z)} 2^m + i(-1)^{2m} \sum_{x=0}^{2m-1} i^w(x)(-1)^{\sum_{j=0}^{2m-1} z_j x_j} =
\]
\[
= (-1)^{w(z)} i^{m+w(z)} 2^m [1 + i(-1)^{2m}]
\]
(12)

So if we replace \((12)\) in \((10)\) and if we sum togheter \((6), (9)\) and \((10)\) we obtain

\[
a_z = \begin{cases} 
2P & \text{if } z_{2m} = z_{2m+1} = 0 \\
-2iP & \text{if } z_{2m} \neq z_{2m+1} \\
-2P & \text{if } z_{2m} = z_{2m+1} = 1 
\end{cases}
\]

and it is now easy to verify that
\[
a_z = (-1)^{w(z)} i^{m+w(z)} 2^m
\]
for every \(0 \leq z < 2^{2m+2}\), and this proves the induction step.

\[\square\]

**Theorem 4.** Let \(n = 2m + 1\) an odd natural, \(m \in \mathbb{N}\) and let \(0 \leq z < 2^m\), \(z \in \mathbb{N}\). Then
\[
\sum_{x=0}^{2^n-1} i^w(x)(-1)^{z x} = (-1)^{w(z)} i^{m+w(z)} 2^m [1 + i]
\]
(14)

**Proof.** First of all we note that equation \((14)\) holds if \(m = 0\) and \(z \in \{0, 1\}\). So in the following we suppose that \(m > 1\). We have that
\[
a_z = \sum_{x=0}^{2m+1-1} i^w(x)(-1)^{z x} =
\]
\[
= \sum_{x=0}^{2m-1} i^w(x)(-1)^{\sum_{j=0}^{2m-1} z_j x_j} + \sum_{x=2m}^{2m+1-1} i^w(x)(-1)^{\sum_{j=0}^{2m} z_j x_j}
\]
and, by Theorem 3 and by equation (4), we have
\[
a_z = (-1)^{w(z)} i^{m+w(z)} 2^m + i(-1)^{2m} \sum_{x=0}^{2m-1} i^w(x)(-1)^{\sum_{j=0}^{2m-1} z_j x_j} =
\]
\[
= (-1)^{w(z)} i^{m+w(z)} 2^m + i(-1)^{2m} (-1)^{w(z)} i^{m+w(z)} 2^m =
\]
\[
= (-1)^{w(z)} i^{m+w(z)} 2^m [1 + i(-1)^{2m}]
\]
(15)
\begin{align*}
|z\rangle &= 2 + 2i \\
|000\rangle &= 2 + 2i \\
|010\rangle &= 2 + 2i \\
|100\rangle &= 2 - 2i \\
|110\rangle &= 2 - 2i \\
|111\rangle &= 2 - 2i
\end{align*}

\begin{align*}
|z\rangle &= 2 - 2i \\
|001\rangle &= 2 - 2i \\
|011\rangle &= 2 - 2i \\
|101\rangle &= 2 - 2i \\
|111\rangle &= 2 - 2i \\
|110\rangle &= 2 - 2i \\
|100\rangle &= 2 - 2i \\
|010\rangle &= 2 - 2i \\
|000\rangle &= 2 - 2i
\end{align*}

Table 1: Left the amplitudes of $a_z$ for $n = 3$. Right the amplitudes of $a_z$ for $n = 4$. In order to get the final amplitudes one should multiply them by a suitable normalization factor.

Let $z_{2m}z_{2m-1}\ldots z_0$ be the binary representation of $z$. Suppose first that $z_{2m} = 0$. Then equation (15) become
\begin{equation}
(-1)^w(z)z^{m+w(z)}2^m + (-1)^w(z)z^{m+w(z)+1}2^m
\end{equation}
and the Theorem is therefore proved. So suppose that $z_{2m} = 1$. Then equation (15) become
\begin{equation}
(-1)^w(z)-1z^{m+w(z)-1}2^m + (-1)^w(z)z^{m+w(z)}2^m
\end{equation}
but observing that
\begin{equation}
(-1)^w(z)z^{m+w(z)}1 = (-1)^w(z)-1z^{m+w(z)-1}
\end{equation}
we have that also in this case the Theorem is satisfied.

As an example we have computed the amplitudes $a_z$ (disregarding the normalization factor) for $n \in 3, 4$ and we report them on Table 1.

### 3 Doubling the amplitude of the search problem’s solution states

In this section we consider a quantum circuit for doubling the amplitude of solution’s states of the search problem. For a search problem we refer, in general, to the problem of finding a solution of some NP-complete problem. Like in the Groover algorithm we will use the intrinsic quantum mechanical parallelism and
an oracle \( f(|x\rangle) \in \{0, 1\} \), specifically designed for the specific problem at hand, which return 1 is \( x \) is a solution of the problem and 0 otherwise.

In particular, in order to present in the detail the results of this paper, we will use a quantum oracle for the Partition Problem (PP). In the PP we have a finite set of elements \( E \) and a function \( s : E \to \mathbb{N}^+ \). We want to find a subset \( E' \subseteq E \) such that \( \sum_{e \in E'} s(e) = \sum_{e \in E \setminus E'} s(e) \). From now on we do not lose generality if we consider the set \( E \) equal to the set of the first \( |E| \) naturals, that is we always consider \( E = \{0, 1, \ldots, n-1\} \). Furthermore we note that if PP has a solution \( E' \) then \( E - E' \) is also a solution of the PP:

The partition problem (PP) is well known to be an NP-complete problem [1].

We describe, in the following, an application of the gates described in the previous section in a quantum circuit to deal with PP (see Figure 1). While the following results apply specifically to the PP they can be applied to any other search problem.

Denote by \( S = \sum_{e \in E} s(e)/2 \). Recall that PP problem has a solution only if \( S \) is an integer. We use the two’s complements representation of \(-S\) requiring \( m = \lceil \log_2 S \rceil + 1 \) qubits. Then for each \( e \in E \), we use \( k_e = \lceil \log_2 s(e) \rceil + 1 \) qubits to encode \( s(e) \). These qubits will remain constant in every phase of the circuit and therefore we will not consider them in the reasoning that follows. We use \( n \) qubits to encode a subset \( E' \) of \( E \). If \( |x_{n-1}x_{n-2} \ldots x_0\rangle \) is the state of those \( n \) qubits, then an element \( e, 0 \leq e < n \), is included in the set \( E' \) if and only if \( x_e = 1 \). We will use \( m \) qubits, denoted in the following by \( |\sigma\rangle \), to store the sum \( \sigma = -S + \sum_{e \in E'} s(e) \) for the elements selected in \( |x\rangle \). In this way \( |\sigma\rangle = |0\rangle^\otimes m \) for a solution \( |x\rangle \) of the PP. We also use a control qubit \( |c\rangle \).

So we have four groups of bits: \( |x\rangle \), \( |\sigma\rangle \), \( |c\rangle \) and the sets of qubits used to represent the constants \( s(e) \) for each element of \( E \). Note that the number of qubits of the circuit, \( n + m + 1 + \sum_{e \in E} k_e \), is polynomial in the size of a concise specification of the PP.

At the beginning of the circuit we have the following superposition:

\[
|\varphi_0\rangle = |0\rangle^\otimes n |\sigma\rangle |c\rangle
\]

where \( \sigma \) is set to the two’s complement of \(-S\) and \( |c\rangle \) is set to \( |1\rangle \). Then, we apply the Hadamard gate to the first \( n \) qubits obtaining

\[
|\varphi_1\rangle = \left(H^\otimes n \otimes I^{m+1}\right) |\varphi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |\sigma\rangle |c\rangle
\]

Next, we use each qubit \( x_e \) to conditionally sum the element \( s(e) \) to \( |\sigma\rangle \). If there exist a solution to the PP then, in the final superposition of \( |\sigma\rangle \), the amplitude of the state \( |x\rangle|0\rangle^\otimes m |c\rangle \) will be not 0. The states \( |x\rangle \) for which \( |\sigma\rangle \) is zero will be referred as the solutions states of the PP. The control qubit \( |c\rangle \) will be set to zero exactly for those states for which \( |\sigma\rangle = |0\rangle^\otimes m \). At this point we apply an uncomputational step in order to set \( |\sigma\rangle = |-S\rangle \). Now if we apply the \( S \) gate to the first \( n \) qubits we obtain, by Lemma [2],

\[
|\varphi_2\rangle = \left(S^\otimes n \otimes I^{m+1}\right) |\varphi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} s^{(x)} |x\rangle |\sigma\rangle |c\rangle
\]
After this we apply again the Hadamard gate to the first $n$ qubits. This operation is controlled by the control qubit in a way that the Hadamard port is applied only to non solution states. For the sake of simplicity we suppose in the following, that the PP has only two solution whose numeric representation are $y$ and its bitwise complement $\overline{y}$. By equation (19), we obtain

$$
|\varphi_2\rangle \xrightarrow{\text{contr. } H^\otimes n} |\varphi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{y, \overline{y}\}} i^{w(z)}|z\rangle|\sigma\rangle|0\rangle + 
$$

$$
+ \frac{1}{2^n} \sum_{z=0}^{2^n-1} \sum_{x \notin \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot z}|z\rangle|\sigma\rangle|1\rangle = 
$$

$$
= \frac{1}{2^n} \left[ \sqrt{2^n} \sum_{z \in \{y, \overline{y}\}} i^{w(z)}|z\rangle|\sigma\rangle|0\rangle + \sum_{z=0}^{2^n-1} \sum_{x \notin \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot z}|z\rangle|\sigma\rangle|1\rangle \right] 
$$

(19)

Now we want to quantify the amplitude of the state $|y\rangle|\sigma\rangle|1\rangle$ and $|\overline{y}\rangle|\sigma\rangle|1\rangle$ of equation (19). We consider only the state $|y\rangle|\sigma\rangle|1\rangle$ since the same arguments can be applied to state $|\overline{y}\rangle|\sigma\rangle|1\rangle$. The amplitude $b_y$ fr the state $|y\rangle|\sigma\rangle|1\rangle$ (in the following we disregard the normalization factor $1/2^n$) is given by the following formula

$$
b_y = \sum_{x \notin \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot y} 
$$

(20)

We may write the above sum as

$$
b_y = \sum_{x \notin \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot y} = \sum_{x=0}^{2^n} i^{w(x)}(-1)^{x \cdot y} - \sum_{x \in \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot y} 
$$

(21)

We have that

$$
\sum_{x \in \{y, \overline{y}\}} i^{w(x)}(-1)^{x \cdot y} = i^{w(y)}(-1)^{y \cdot y} + i^{w(\overline{y})}(-1)^{\overline{y} \cdot y} = 
$$

$$
= i^{w(y)}(-1)^{y \cdot y} + i^{n-w(y)} = 
$$

(22)

Then, recalling that $i^x = i^{-x}$ when $x$ is even and $i^{-x} = -i^x$ when $x$ is odd, we have two cases: $w(y)$ is even and then

$$
i^{w(y)}(-1)^{w(y)} + i^{n-w(y)} = i^{w(y)}(-1)^{w(y)} + i^{n+w(y)} = 
$$

$$
= i^{w(y)}(1 + i^n) 
$$

(23)

while if $w(y)$ is odd

$$
i^{w(y)}(-1)^{w(y)} + i^{n-w(y)} = i^{w(y)}(-1)^{w(y)} - i^{n+w(y)} = 
$$

$$
= -i^{w(y)}(1 + i^n) 
$$

(24)

For simplicity of notation in the following we denote $w(y)$ as simply $\overline{w}$. We have that if $n = 2m$ is even then, by Theorem [3] $b_y$ is

$$
b_y = \begin{cases} 
(\overline{w})^{m+\overline{w}}2^m - i^{\overline{w}}(1 + i^{2m}) & \text{if } \overline{w} \text{ is even} \\
(\overline{w})^{m+\overline{w}}2^m + i^{\overline{w}}(1 + i^{2m}) & \text{if } \overline{w} \text{ is odd} 
\end{cases} 
$$

(25)
while if $n = 2m + 1$ is odd, by Theorem 4 $b_y$ is

$$b_y = \begin{cases} 
(−1)^{w}i^{m}+w2^m(1 + i) - i^w(1 + i^{2m+1}) & \text{if } w \text{ is even} \\
(−1)^{w}i^{m}+w2^m(1 + i) + i^w(1 + i^{2m+1}) & \text{if } w \text{ is odd} 
\end{cases}$$

(26)

It is immediate to check that in the above equations (25) and (26) the term $i^w(1 + i^{n})$ become trascurable, with respect to the other term in the equation, as $m$ become bigger. We conclude that the amplitude of the state $|y⟩|σ⟩|1⟩$ is almost the same of the amplitude of state $|y⟩|σ⟩|0⟩$, thus effectively duplicating the chances of state $|y⟩$ at the end of the circuit. For example if $n = 2m + 1 = 3$ and $|y⟩ = |011⟩$ we have that $b_y = 3 - 3i$, so that the probability of getting $|y⟩$ is, by (19), $\frac{1}{64} [2\sqrt{2}^2 + |3 - 3i|^2] = \frac{26}{64} = .40625$ which is exactly the output of the quirk simulator.

![Figure 1: The circuit exploiting S gates.](image)

4 Conclusion and future work

We presented here a quantum algorithm for doubling the amplitude of the state correspondig to the solution of the partition problem. This is interesting because if we would be able to iterate such a doubling we could provide a polynomial quantum algorithm for solving an NP-complete problem. Possible future works
should focus on: generalizing the mathematical results to instance of search problems where there are more than 2 solutions, find out if this algorithm can be combined to the Goover algorithm in order to seed up the latter of a factor of $p$ where $p \geq 2$ and, more important, to check if and how it is possible to iterate the doubling of the amplitude in order to get some polynomial time algorithm for solving the search problem.
References

[1] R. Karp. Reducibility among combinatorial problems. In R. Miller and J. Thatcher, editors, Complexity of Computer Computations, pages 85–103. Plenum Press, 1972.

[2] M. A. Nielsen and I. L. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, USA, 10th edition, 2011.