Separable Hilbert space for loop quantization

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We discuss, within the simplified context provided by the polymeric harmonic oscillator, a construction leading to a separable Hilbert space that preserves some of the most important features of the spectrum of the Hamiltonian operator. This construction can be generalized to loop quantum cosmology and is helpful to sidestep some of the issues that appear in that context. In particular those related to superselection and the definition of suitable ensembles for the statistical mechanics of these types of systems.

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I. INTRODUCTION AND PRELIMINARY REMARKS

Loop quantum gravity\textsuperscript{1,2} (LQG) is at present one of the most advanced approaches to address the quantization of general relativity. Recent progress in this field, involving in particular the application of deparametrization techniques with respect to the matter frames\textsuperscript{3,4,5}, has brought LQG to a level which makes it possible to probe its dynamical predictions\textsuperscript{6}.

Despite this progress, some critical problems of the theory remain open. One of the principal issues is the construction of suitable Hilbert space(s). The standard constructions lead to an orthonormal basis labeled by spin-networks –graphs embedded in 3-dimensional differential manifold with colored edges and vertices. Unfortunately, the non-countable number of spin-networks renders the Hilbert space non-separable. This feature creates some difficulties in the development of the formalism both to define unitary evolution and to build suitable statistical ensembles. This problem arises both at the level of the full theory and in its symmetry reduced quantum-mechanical versions\textsuperscript{7}, including in particular the ones applied in loop quantum cosmology (LQC)\textsuperscript{8,9}.

The purpose of this letter is to present a construction of the physical Hilbert space used in loop quantization which avoids the non-separability issues while retaining the correct low energy (large scale) behavior of the resulting framework. The construction is exemplified in the particular case of the polymer quantum harmonic oscillator. This particular system is of critical relevance to inhomogeneous LQC frameworks as the harmonic oscillator it is the main building block of the Fock spaces representing the inhomogeneity modes\textsuperscript{10,11}, thus its loop quantization is a necessary step in going beyond the hybrid quantization scheme\textsuperscript{10,12} (i.e. implementing the loop quantization to all degrees of freedom).

A beneficial side effect of the analysis presented here is the possibility of defining suitable statistical ensembles appropriate for the discussion of thermodynamical problems for these kinds of systems. The construction that we provide here is based on the use of certain “foliations” of the original non-separable Hilbert spaces by means of separable subspaces and a natural Lebesgue measure on it. This construction follows from observations of\textsuperscript{13} and was successfully applied in\textsuperscript{14} to the case of the polymer-quantized scalar field.

We will illustrate the procedure that we suggest in the case of the harmonic oscillator quantized via polymer techniques as specified in\textsuperscript{7}. The main properties of the system are:

1. The Hilbert space is the (non-separable) space of square integrable functions on the Bohr compactification of the real line \( H = L^2(\mathbb{R}_\text{Bohr}, d\mu) \).
2. The spectrum of the Hamiltonian features a (continuous) band structure, however it remains a pure point spectrum.

Classically, the time evolution of the harmonic oscillator is generated by Hamiltonian

$$H(q, p) = \frac{\hbar^2}{2m\ell^2} p^2 + \frac{m\ell^2 \omega^2}{2} q^2, \quad (1.1)$$

where the canonical variables $p, q$ are dimensionless, while $\ell$ and $\omega$ are the oscillator’s characteristic length and frequency respectively.

In loop quantization it is impossible to promote $p$ and $q$ to operators simultaneously. Among the infinite number of non-equivalent representations of the Weyl algebra in non-separable Hilbert spaces, there are two natural—in the context of quantum cosmology— nonequivalent choices: the position representation where the operator $\hat{q}$ is well defined, and the momentum one, where $\hat{p}$ is well defined. In both representations the remaining variable has to be approximated (“regularized”) in terms of other Weyl algebra elements and then promoted to be an operator.

To focus our attention we choose the position representation. A similar procedure works for the momentum one. In order to regularize the momentum we approximate it with use of $V(q) = e^{-ipq}$. Thus the quantum Hamiltonian takes the form

$$\hat{H} = \frac{\hbar^2}{2m(2q_0\ell)^2} \left( 2\hat{q} - \hat{V}(2q_0) - \hat{V}(-2q_0) \right) + \frac{m\ell^2 \omega^2}{2} q^2, \quad (1.2)$$

where $q_0$ is a regularization constant. The quantity $q_0\ell$ can be interpreted as a polymer scale.

The above Hamiltonian, when acting on the physical states, represented respectively via the wave functions $\tilde{\Psi} \in L^2(\mathbb{R}_{\text{Bohr}}, dq)$, or respectively in terms of its Fourier-Bohr transform $\hat{\Psi} \in \ell^2(\mathbb{R})$ (where $\ell^2(\mathbb{R})$ is the space of square summable functions on $\mathbb{R}$) can be written as a difference operator in $q$ and differential one in $p$

$$[\hat{H}\Psi](q) = \frac{\hbar^2}{2m(2q_0\ell)^2} \left( 2\Psi(q) - \Psi(q + 2q_0) - \Psi(q - 2q_0) \right) + \frac{m\ell^2 \omega^2}{2} q^2 \Psi(q), \quad (1.3a)$$

$$[\hat{H}\hat{\Psi}](p) = -\frac{m\ell^2 \omega^2}{2} \hat{\Psi}''(p) + \frac{\hbar^2}{2m(2q_0\ell)^2} \sin^2(q_0p) \hat{\Psi}(p). \quad (1.3b)$$

The eigenvalue problem involving the form $\hat{H}\hat{\Psi} = E\hat{\Psi}$

$$\hat{H}\hat{\Psi} = E\hat{\Psi} \quad (1.4)$$

takes the form of the Mathieu equation and the differential symbols appearing in the Hamiltonian have the same form of the ones describing a particle in periodic potential—a case well studied in the literature (see [12] for relevant mathematical details). We have to remember, however, that here the Hilbert space is different (in particular non separable).

If we consider the form of Hamiltonian specified via $\hat{H}\hat{\Psi} = E\hat{\Psi}$ it is a difference operator coupling the points separated by $2q_0$. One can thus divide the domain of $\hat{\Psi}(q)$ onto the set of uniform lattices—sets preserved by the action of $\hat{H}$

$$\mathbb{R} = \bigcup_{\epsilon \in [0,1]} \mathcal{L}_\epsilon, \quad \mathcal{L}_\epsilon := 2q_0(\epsilon + \mathbb{Z}). \quad (1.5)$$

This observation has led to the solution presented in [16, 17]. Since the lattices are preserved by the time evolution we can treat the subspaces $\mathcal{H}_\epsilon$ spanned by the cut-off of the wave function support to a single $\mathcal{L}_\epsilon$ as “superselection” sectors. The customary way to proceed in such case is to select the single sector (represented by a single value of $\epsilon$) and work just with it. This approach has been applied, for example, in LQC [18, 19].

Under this choice, the Hilbert space $\mathcal{H}$ gets restricted to a subspace $\mathcal{H}_\epsilon$ defined by the projection $\mathcal{H} \ni \tilde{\Psi} \mapsto \hat{\Psi}_\epsilon = \tilde{\Psi}|_{\mathcal{L}_\epsilon} \in \mathcal{H}_\epsilon$. The subspace $\mathcal{H}_\epsilon$ is then a space of quasi-periodic functions of $p$ satisfying

$$\hat{\Psi}_\epsilon(p + \pi/q_0) = e^{-2\pi i \epsilon} \hat{\Psi}_\epsilon(p). \quad (1.6)$$

Such subspace is homeomorphic to a space of square integrable functions (in momentum representation) on a unit sphere $L^2(S^1, dp)$ with the gluing (boundary) conditions depending on $\epsilon$. In particular, the case $\epsilon = 0$ corresponds to periodic conditions, whereas $\epsilon = 1/2$ corresponds to the antiperiodic ones. The spectrum $S$ of the Hamiltonian $\hat{H}$ is a point spectrum and can be written as the union $S = \cup \mathcal{S}_\epsilon$. 
On the other hand, the reasoning presented in Sec. 4 of [7] and the references therein shows that a similar approach—in the context of LQC—based on working within the subspaces $S_\epsilon$ may be problematic because the dynamics may connect different sectors. To avoid this kind of problem one should take into account all the sectors. In the case of the polymer harmonic oscillator this means that all the points of the bands describing the spectrum must be considered as for the particle in periodic potential in standard (Schrödinger) quantization. Notice, however, that the spectrum remains a pure point one despite having an uncountable number of elements. This immediately implies the non-separability of the physical Hilbert space (constructed through the spectral decomposition of $\hat{H}$) which is not a surprise as it should be equivalent to a nonseparable $L^2(\mathbb{R}_{\text{Bohr}},d\mu)$. This severely hinders the application of this construction to analyze the physical properties of the loop quantum harmonic oscillator, in particular the statistical mechanics of the system as explained in detail in [7].

II. INTEGRAL HILBERT STRUCTURE

Non-separability is a source of problems for some systems of physical interest related to LQC, in particular in the polymer quantization of the scalar field as discussed in [14]. There, the time dependence of the “lattice gap” causes a mixing of the putative “superselection” sectors during the time evolution, thus preventing one from working with just one superselected subspace. This feature seems to be a generic one in LQC models beyond the isotropic ones. On the other hand, in case of the flat anisotropic Bianchi I universe with massless scalar field one can show using the spectral properties of the evolution operator for the model that the single sector Hilbert spaces do not admit a mixing of the putative “superselection” sectors during the time evolution, thus preventing one from working with a semiclassical sector [20].

In the context of LQC a possible solution to the problem has been presented in Appendix C of [21]. There, one described the action of the evolution operator (playing the role of a Hamiltonian) via an action of its adjoint on a (bigger) dual to the original Hilbert space next projected to a single superselection spaces (the so called shadow states technique [22]). The dual space was then modified via postulating as its inner product the one corresponding to a Schrödinger quantization.

Here we present a systematic construction of a separable Hilbert space as certain integral of superselection sector Hilbert spaces: “$\hat{H} = \int_{[0,1]} \mathcal{H}_\epsilon \, d\epsilon$” with the induced scalar product making it separable. The specific construction is inspired by the Hilbert space structures observed in LQC in the presence of a positive cosmological constant: more precisely the dependence of these structures on the lapse function [13]. The goal of that work was to construct the physical Hilbert space generated by the Hamiltonian constraint (isotropic and flat Friedmann-Lemaître-Robertson-Walker background with a scalar field source) through group averaging for various choices of the lapse $N$. Two examples, leading to distinct results, were considered:

1. $N = a^3$ (where $a$ is a scale factor): The Hamiltonian constraint admits then a 1-parameter family of self-adjoint extensions. Each extension has a discrete spectrum consisting of isolated points.

2. $N = 1$: The Hamiltonian constraint admits a unique extension, its spectrum is purely continuous (well defined Lebesgue measure). As a set, the spectrum is the union of the spectra of all the extensions found in the case $N = a^3$.

By comparing the inner product structures in the Hilbert space $\hat{H}$ constructed in the case $N = a^3$ with the ones that appear in each extension $\mathcal{H}_\beta$ of the case $N = 1$ we notice that

$$\hat{H} = \int \mathcal{H}_\beta \, d\sigma(\beta), \quad \langle \Psi|\chi \rangle_{\hat{H}} = \int \langle \Psi_\beta|\chi_\beta \rangle_{\mathcal{H}_\beta} \, d\sigma(\beta), \quad \Psi_\beta(\omega) := \Psi(\omega)|_{\omega^2 \in \text{supp}(\mathcal{H})}, \quad (2.1)$$

where the measure $d\sigma$ is induced by the Lebesgue measure on the spectrum of the constraint for $N = 1$.

Following the previous observation, and noticing that the set of the $\epsilon$-lattice labels is Lebesgue measurable. As a set, the spectrum is the union of the spectra of all the extensions found in the case $N = a^3$.

$$\hat{H} := \int_{[0,1]} \mathcal{H}_\epsilon \, d\epsilon, \quad \langle \Psi|\chi \rangle_{\hat{H}} = \int_{[0,1]} \langle \Psi_\epsilon|\chi_\epsilon \rangle_{\mathcal{H}_\epsilon} \, d\epsilon, \quad \Psi_\epsilon = \Psi|_{\mathcal{L}_\epsilon}. \quad (2.2)$$

Notice that the measure $d\epsilon$ can be replaced by $f(\epsilon)d\epsilon$ (that takes into account the density of states in a suitable way) giving a unitarily equivalent Hilbert space structure.

In the particular case of the polymeric harmonic oscillator the resulting Hilbert space $\hat{H}$ is mathematically equivalent to the one appearing in the Schrödinger quantization of a particle in a periodic potential. The main difference is that we have the standard band structure and the spectrum of $\hat{H}$ is purely continuous. In consequence all the standard
quantum mechanical tools can be used and, in particular, the quantum statistical mechanics of the system can be studied by following the usual approach.

It is worth noting, that in the Schrödinger quantization of the particle in periodic potential present in (1.3b), the Hilbert space admits a natural fiber bundle decomposition \[ \mathcal{H}_\epsilon \] of which fibers are exactly the spaces \( \mathcal{H}_\epsilon \) specified earlier. In polymer quantization considered here the fiber structure is not present. Our construction can be seen as a recomposition of the (original) Hilbert space so that \( \mathcal{H}_\epsilon \) are its fibers, using the natural Lebesgue measure on the space of superselection sector labels. When applied to an example of the flat FRW universe with massless scalar field this method will give the result equivalent to construction specified in Appendix C of [21].

A word of caution is necessary here. While in order to deal with the mixing of lattices by time evolution described in [14] it was convenient to introduce a separable Hilbert space as above, it is not clear why we cannot just consider a single superselection sector in the context of the polymer harmonic oscillator. This question is of particular relevance for LQC as the latter approach is, precisely, the one that has been followed there. Technically, as long as the “polymerization scale” \( q_0 \) is constant in time, restricting the quantum dynamics to a single superselection sector is both correct and consistent. The situation changes if we allow \( q_0 \) to be time-dependent. In such a case, similarly to what happens in [14], one expects to have the phenomenon of “sector mixing” and then something should be done in order to avoid the problems associated with non-separability, for instance, using the construction described before. This observation may be relevant for the loop quantization of the inhomogeneity modes in LQC as an explicit time dependence naturally arises there [12]. If one wants to have a uniform treatment for all the relevant cases, one should also follow the same approach when the polymerization scale is a constant. One has to understand, however, how the different choices can affect the physical results. In the studies of isotropic universes in LQC the dependence on the choice of the superselection sector has been systematically analyzed (see for example [19, 23]). The differences in exact physical predictions appeared to be minor (confined to dispersion differences in the scattering picture [24] and the fine details of the near-bounce dynamics [25]), especially when appropriate quantization prescriptions were chosen [25]. Furthermore, for models with noncompact spatial slices the discrepancies vanished in the infrared regulator removal limit (see the discussion in [26]), whereas for the compact ones the differences became relevant only for “very quantum”, physically uninteresting universes [27]. Since the sectors (the “fibers” in our approach) are orthogonal to each other, these features transfer directly to the theory arising from the construction proposed here. Thus, at least in the case of models studied so far, one can safely work with just one superselection sector without introducing significant errors in physical predictions as long as that choice does not violate the consistency of the model.

The construction presented here is of relevance not only for the simplified cosmological models. Indeed LQG itself suffers from the very same problem of nonseparability of the (diffeo-invariant) Hilbert space due to uncountable number of spin networks labeling its basis. Finding a suitable space is still an open problem. Our construction appears to be applicable at least in some of the approaches to the theory featuring spin network graphs of fixed topology like, for example, in algebraic quantum gravity [28].

Finally, it is important to point out one relevant feature of the construction introduced here. In principle, instead of following the loop quantization program strictly, one could regularize the Hamiltonian at the classical level (by introducing by hand a periodic potential) and quantize it in the standard Schrödinger representation. The final result would be identical to the one resulting from the point of view presented here. Whether such approach should be taken depends of the goals of the program. The alternative mentioned here gives rise to a consistent treatment deviating from LQG more than the standard polymeric quantization but still incorporating some of its central features. While without a direct reference to loop quantization the regularization of the Hamiltonian would not be justified, the present approach may be interesting at a phenomenological level. When exploring the consequences of the polymer quantization no such shortcut should be permitted and the precise construction of the separable Hilbert space must be provided. Skipping this step may lead to an incorrect description of the dynamical sector of the theory, as discussed in [14].

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REFERENCES

[1] T. Thiemann, Modern canonical quantum general relativity (Cambridge University Press, London, 2007)
[2] C. Rovelli, Quantum gravity (Cambridge University Press, London, 2004)
[3] K. Giesel and T. Thiemann, Class.Quant.Grav. 27, 175009 (2010), arXiv:0711.0119 [gr-qc]
[4] M. Domagala, K. Giesel, W. Kamiński, and J. Lewandowski, Phys. Rev. D82, 104038 (2010), arXiv:1009.2445 [gr-qc]
[5] V. Husain and T. Pawlowski, Phys.Rev.Lett. 108, 141301 (2012), arXiv:1108.1145 [gr-qc]
[6] K. Giesel and T. Thiemann(2012), arXiv:1206.3807 [gr-qc]
[7] J. F. Barbero G., J. Prieto, and E. J. Villaseñor, Class.Quant.Grav. 30, 165011 (2013), arXiv:1305.5406 [gr-qc]
[8] M. Bojowald, Liv. Rev. Rel. 11, 4 (2008)
[9] A. Ashtekar and P. Singh, Class. Quant. Grav. 28, 213001 (2011), arXiv:1108.0893 [gr-qc]
[10] L. Garay, M. Martin-Benito, and G. Mena Marugan, Phys.Rev. D82, 044048 (2010), arXiv:1005.5654 [gr-qc]
[11] I. Agullo, A. Ashtekar, and W. Nelson, Phys.Rev. D87, 043507 (2013), arXiv:1211.1354 [gr-qc]
[12] M. Fernandez-Mendez, G. A. Mena Marugan, and J. Olmedo, Phys.Rev. D86, 024003 (2012), arXiv:1205.1917 [gr-qc]
[13] W. Kamiński, J. Lewandowski, and T. Pawlowski, Class.Quant.Grav. 26, 245016 (2009), arXiv:0907.4322 [gr-qc]
[14] A. Kreienbuehl and T. Pawlowski, Phys.Rev. D88, 043504 (2013), arXiv:1302.6566 [gr-qc]
[15] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol. II (Academic Press, New York, 1978)
[16] A. Corichi, T. Vukasinac, and J. A. Zapata, Phys.Rev. D76, 044016 (2007), arXiv:0704.0007 [gr-qc]
[17] A. Corichi, T. Vukasinac, and J. A. Zapata, Class.Quant.Grav. 24, 1495 (2007), arXiv:gr-qc/0610072 [gr-qc]
[18] A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. 7, 233 (2003), arXiv:gr-qc/0304074
[19] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D 74, 084003 (2006), arXiv:gr-qc/0607039
[20] A. Henderson and T. Pawlowski, “Bianchi I Universe Dynamics in LQC,” (2011), in preparation
[21] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D 73, 124038 (2006), arXiv:gr-qc/0604013
[22] A. Ashtekar, S. Fairhurst, and J. L. Willis, Class.Quant.Grav. 20, 1031 (2003), arXiv:gr-qc/0207106 [gr-qc]
[23] E. Bentivegna and T. Pawlowski, Phys. Rev. D 77, 124025 (2008), arXiv:0803.4446 [gr-qc]
[24] W. Kamiński and T. Pawlowski, Phys. Rev. D 81, 084027 (2010), arXiv:1001.2663 [gr-qc]
[25] G. A. Mena Marugan, J. Olmedo, and T. Pawlowski, Phys.Rev. D84, 064012 (2011), arXiv:1108.0829 [gr-qc]
[26] A. Corichi and E. Montoya, Int.J.Mod.Phys. D21, 1250076 (2012), arXiv:1105.2804 [gr-qc]
[27] A. Ashtekar, T. Pawlowski, P. Singh, and K. Vandersloot, Phys. Rev. D 75, 024035 (2007), arXiv:gr-qc/0612104
[28] K. Giesel and T. Thiemann, Class.Quant.Grav. 24, 2465 (2007), arXiv:gr-qc/0607099 [gr-qc]