Lower Bounds on Davenport-Schinzel Sequences via Rectangular Zarankiewicz Matrices

Julian Wellman* \hspace{1cm} Seth Pettie†
Greenhills School \hspace{1cm} University of Michigan
Ann Arbor, MI \hspace{1cm} Ann Arbor, MI

Abstract

An order-$s$ Davenport-Schinzel sequence over an $n$-letter alphabet is one avoiding immediate repetitions and alternating subsequences with length $s + 2$. The main problem is to determine the maximum length of such a sequence, as a function of $n$ and $s$. When $s$ is fixed this problem has been settled (see Agarwal, Sharir, and Shor [1], Nivasch [12] and Pettie [15]) but when $s$ is a function of $n$, very little is known about the extremal function $\lambda(s, n)$ of such sequences.

In this paper we give a new recursive construction of Davenport-Schinzel sequences that is based on dense 0-1 matrices avoiding large all-1 submatrices (aka Zarankiewicz’s Problem.) In particular, we give a simple construction of $n^{2/t} \times n$ matrices containing $n^{1+1/t}$ 1s that avoid $t \times 2$ all-1 submatrices.

Our lower bounds on $\lambda(s, n)$ exhibit three qualitatively different behaviors depending on the size of $s$ relative to $n$. When $s \leq \log \log n$ we show that $\lambda(s, n)/n \geq 2^s$ grows exponentially with $s$. When $s = n^{o(1)}$ we show $\lambda(s, n)/n \geq \left(\frac{s}{2 \log \log n}\right)^{\log \log n}$ grows faster than any polynomial in $s$. Finally, when $s = \Omega(n^{1/t}(t-1)!)$, $\lambda(s, n) = \Omega(n^2s/(t-1)!)$ matches the trivial upper bound $O(n^2s)$ asymptotically, whenever $t$ is constant.

1 Introduction

In 1965 Davenport and Schinzel [4] introduced the problem of bounding the maximum length of a sequence on an alphabet of $n$ symbols that avoids any subsequence of the form $\ldots b \ldots a \ldots b \ldots$ of length $s + 2$. We call any sequence $S$ which does not contain immediate repetitions and which does not contain an alternating subsequence of length $s + 2$ a Davenport-Schinzel (DS) sequence of order $s$. Let $|S|$ be the length of $S$, $\|S\|$ be the number of distinct symbols in $S$, and $DS(s, n)$ be the set of all Davenport-Schinzel sequences of order $s$ on $n$ symbols. We are interested in bounding the extremal function for DS sequences.

$$\lambda(s, n) = \max\{|S| : S \in DS(s, n)\}$$

The behavior of $\lambda(s, n)$ is well understood when $s$ is fixed [7, 1, 12, 15], or when $s \geq n$ [17]. However, very little is known when $s$ is a function of $n$ and $1 \ll s \ll n$.

*Work done as part of the Advanced Research course at Greenhills School, taught by Julie Smith
†Supported by NSF grants CNS-1318294, CCF-1514383, and CCF-1637546.
1.1 Fixed-order Davenport-Schinzel Sequences

Most investigations of DS sequences has focused on the case of fixed \( s \). This is motivated by applications in computational geometry \[18, 19\], where DS sequences are used to bound the complexity of the lower envelope of \( n \) univariate functions, each pair of which cross at most \( s \) times, e.g., a set of \( n \) degree-\( s \) polynomials. The following theorem synthesizes results of Davenport and Schinzel \[4\] (\( s \in \{1, 2\} \)), Agarwal, Sharir, and Shor \[11\] (sharp bounds for \( s = 4 \), lower bounds for even \( s \geq 6 \)), Nivasch \[12\] (lower bounds for \( s = 3 \), upper bounds for even \( s \geq 6 \)), and Pettie \[15\] (upper bounds for all odd \( s \geq 3 \), lower bounds for \( s = 5 \)). Refer to Klazar \[9\] for a history of Davenport-Schinzel sequences from 1965–2002, and Pettie \[15, 14, 16\] for recent developments.

\[ \text{Theorem 1.1. When } s \text{ is fixed, the asymptotic behavior of } \lambda(s, n), \text{ as a function of } n, \text{ is as follows.} \]

\[
\lambda(s, n) = \begin{cases} 
    n & s = 1 \\
    2n - 1 & s = 2 \\
    2n\alpha(n) + O(n) & s = 3 \\
    \Theta(n2^{\alpha(n)}) & s = 4 \\
    \Theta(n\alpha(n)2^{\alpha(n)}) & s = 5 \\
    n \cdot 2^{(1+o(1))\alpha(n)/t!} & \text{for both even and odd } s \geq 6, \ t = \left\lfloor \frac{s-2}{2} \right\rfloor.
\end{cases}
\]

Here \( \alpha(n) \) is the slowly growing inverse-Ackermann function. Observe that if we regard \( \alpha(n) \) as a constant, the dependence of \( \lambda(s, n) \) on \( s \) is \textit{doubly exponential}. This doubly exponential growth can be extended to non-constant \( s \), but the constructions of \[11, 12, 15\] only work when \( s = O(\alpha(n)) \). When \( s = \Omega(\alpha(n)) \) the existing lower bounds break down, but the upper bounds of \[11, 12, 15\] continue to give non-trivial upper bounds for \( s = o(\log n) \). They imply, for example, that \( \lambda(s, n) = O(n(\log^{s-2}(n))) \), for any fixed number of stars.\(^1\)

1.2 Large-order Davenport-Schinzel Sequences

A straightforward pigeonhole argument (see \[9\] p. 3)) gives the following upper bound on \( \lambda(s, n) \).

\[
\lambda(s, n) \leq \left( \frac{n}{2} \right) s + 1 \tag{1}
\]

For fixed \( s \) this bound is off by nearly a factor \( n \), but for fixed \( n \) this bound is quite tight as a function of \( s \). In fact, Roselle and Stanton \[17\] showed that for \( s = \Omega(n) \), \( \lambda(s, n) = \Theta(n^2s) \), and that for fixed \( n \), \( \lim_{s \to \infty} \lambda(s, n)/s = \left( \frac{n}{2} \right) \), i.e., Eqn. (1) is sharp up to the leading constant \( \left( \frac{n}{2} \right) \). Let us give a brief description of Roselle and Stanton’s construction. The sequence \( RS(s, n)[a_1, a_2, \ldots, a_n] \) is a DS(\( s, n \)) sequence constructed from the alphabet \( \{a_1, \ldots, a_n\} \) in which the first occurrences of each symbol are in the order \( a_1a_2\cdots a_n \). If omitted, take the

\(^1\)This result is not stated explicitly in \[11, 12, 15\], but it is straightforward to cobble together, e.g., from Pettie \[15\] Lemma 3.1(2,4) and Recurrence 3.3]. The \( \ast \) operator is defined for any \( f \) that is strictly decreasing on \( \mathbb{N}\setminus\{0\} \). By definition \( f^\ast(n) = \min\{i \mid f^{(i)}(n) \leq 1\} \), where \( f^{(i)} \) is the \( i \)-fold iteration of \( f \).
alphabet to be $[1, 2, \ldots, n]$. The construction is recursive, and bottoms out in one of two base cases, depending on whether $s > n$ or $s \leq n$ initially.

$$RS(2, n) = 121314 \cdots 1(n - 1)1n1$$
$$RS(s, 2) = 1212 \cdots \text{(length } s + 1\text{)}$$

When $s, n > 2$ we construct $RS(s, n)$ inductively.

$$RS(s, n) = \text{Alt}(s, n) \cdot RS(s - 1, n - 1)[n, n - 1, \ldots, 2]$$

where $\text{Alt}(s, n) = 12121313141n1n1\cdots 1n1$

In other words, with $\text{Alt}(s, n)$ we introduce the maximum number of alternations between 1 and each $k \in \{2, \ldots, n\}$, then “retire” the symbol 1 and append a copy of $RS(s - 1, n - 1)$ on the alphabet $\{2, \ldots, n\}$. Observe that it is crucial that the remaining alphabet be ‘reversed’ in the recursive invocation of $RS(s - 1, n - 1)$. In $\text{Alt}(s, n)$ the symbols 2, 3, $\ldots, n$ appeared in this order, so to minimize the number of alternations the symbols in $RS(s - 1, n - 1)$ should make their first appearances in the order $n, n - 1, \ldots, 2$. It is easily seen that $|\text{Alt}(s, n)| = \Theta(sn)$ and $|RS(s, n)| = \Theta(\min\{n^2s, ns^2\})$, depending on whether $s > n$ or $s \leq n$. See [17] for a careful analysis of the leading constant and lower order terms.

1.3 Summary and New Results

Suppose we fix $n$ at some very large value and let $s$ increase. Theorem 1.1 (and a close inspection of the constructions of [1, 12, 15]) shows that $\lambda(s, n)/n$ grows doubly exponentially with $s$, but only up to $s = O(\alpha(n))$. For somewhat larger $s$ the best lower bounds on $\lambda(s, n)/n$ are quadratic ($\Omega(s^2)$) [11, 17] and best upper bounds exponential ($((\log^{s-2}(n))^{s-2})$).

Eventually $s \geq n$ and $\lambda(s, n)/s$ is known to be $\Theta(n^2)$, tending to $\binom{n}{2}$ in the limit [17]. Thus, when $1 \ll s \ll n$ we know very little about the true behavior of the extremal function $\lambda(s, n)$.

In this paper we present a new construction of Davenport-Schinzel sequences that bridges the gap between the small-order ($s = O(\alpha(n))$) and large-order ($s = \Omega(n)$) regimes. It exhibits three new qualitatively different lower bounds on $\lambda(s, n)/n$.

- When $s \leq \log \log n$, $\lambda(s, n)/n = \Omega(2^s)$ grows at least (singly) exponentially in $s$, which improves on [1, 12, 15] when $s \geq 2^{\Omega(\alpha(n))}$.

- When $s > \log \log n$ we have $\lambda(s, n)/n = \Omega((\frac{s}{2 \log \log n})^{\log \log n})$. For example, $\lambda(\log n, n)/n > 2^{\Omega((\log \log n)^2)}$ is quasi-polylogarithmic in $n$.

- Suppose that $s \geq n^{1/t}(t - 1)!$ for an integer $t$. In this case we obtain asymptotically sharp lower bounds on $\lambda(s, n) = \Omega(n^2s/(t - 1)!)$ whenever $t$ is constant.
1.4 Overview of the Paper

In Section 2 we give a simple construction showing that \( \lambda(s, n)/n = \Omega(2^s) \), for \( s \) up to \( \log \log n \). In Section 3 we construct an \( n^{2/t} \times n \) Zarankiewicz matrix with \( n^{1+1/t} \) 1s which avoids \( t \times 2 \) all-1 submatrices. Zarankiewicz matrices are used in Section 4 to construct Davenport-Schinzel sequences of length \( \Omega(n^2 s/(t - 1)!) \) when \( s \geq n^{1/t}(t - 1)! \). The space where \( \log \log n \ll s \ll n^{o(1)} \) is addressed in Section 5. We conclude with some remarks and open problems in Section 6.

2 A Simple Construction for Small Orders

In this section we present a simple construction for the special case \( s = \log \log n + 2 \), which can easily be scaled down to the case when \( s \leq \log \log n + 1 \). The sequence \( S(k) \) is an order-\( s(k) \) DS sequence over an \( n(k) \)-letter alphabet in which each symbol occurs \( \mu(k) \) times. We will construct \( S(k + 1) \) inductively from \( S(k) \) and thereby obtain recursive definitions for \( n(k + 1), s(k + 1), \mu(k + 1) \). Let \( S(k)[a_1, \ldots, a_{\mu(k)}] \) denote a copy of \( S(k) \) in which the letters \( a_1, \ldots, a_{\mu(k)} \) make their first appearance in that order, and let \( S \) be the reversal of \( S \). If left unspecified, the alphabet is \([1, \ldots, n(k)]\).

In the base case \( k = 0 \) we let \( S(0) = 12 \). Thus,

\[
S(0) = 2, \quad \mu(0) = 1, \quad s(0) = 1.
\]

Now we construct \( S(k + 1) \) from \( S(k) \). Arrange \( n(k)^2 \) distinct symbols in an \( n(k) \times n(k) \) matrix. Let \( C_i \) (and \( R_i \)) be the sequences of symbols in column \( i \) (and row \( i \)), \( 1 \leq i \leq n(k) \), listed in increasing order of row index (and column index). The sequence \( S(k + 1) \) is constructed as follows:

\[
S(k + 1) = \overline{S(k)[C_1]} \overline{S(k)[C_2]} \cdots \overline{S(k)[C_{\mu(k)}]} \overline{S(k)[R_1]} S(k)[R_2] \cdots S(k)[R_{\mu(k)}]
\]

It follows that \( S(k + 1) \) has the following parameters.

\[
n(k + 1) = n(k)^2, \quad \mu(k + 1) = 2\mu(k), \quad s(k + 1) = \max\{3, s(k + 1)\}
\]

The expression for \( n(k + 1) \) is by construction and the expression for \( \mu(k + 1) \) follows from the fact that each symbol appears in one row and one column. The claim that \( s(k + 1) = \max\{3, s(k + 1)\} \) requires a more careful argument. Consider two symbols \( a, b \) at positions \((i, j)\) and \((i', j')\) in the \( n(k) \times n(k) \) symbol matrix. If \( i \neq i' \) and \( j \neq j' \) then we may see the subsequence \( abab \) in \( S(k + 1) \), but never \( ababa \). Suppose that \( i = i' \) and \( j < j' \). In the first half of \( S(k + 1) \), all \( a \)s (in \( S(k)[C_j] \)) precede all \( b \)s (in \( S(k)[C_{j'}] \)) and in the second half of \( S(k + 1) \), all occurrences of \( a \) and \( b \) appear in \( S(k)[R_i] \). Moreover, because \( a \) precedes \( b \) in \( R_i \), the first occurrence of \( b \) precedes the first occurrence of \( a \) in \( S(k)[R_i] \). Symmetric observations hold when \( i < i' \) and \( j = j' \). Thus, for any two symbols \( a, b \), either \( ababa \) does not appear in \( S(k + 1) \) or \( S(k + 1) \) introduces one more alternation than \( S(k) \). We conclude that \( s(k + 1) = \max\{3, s(k + 1)\} \).
By induction on $k$, we have the following closed form bounds on the parameters of $S(k)$.

$$n(k) = 2^{2^k}$$
$$s(k) = k + 2$$
$$\mu(k) = 2^k$$

As constructed $S(k + 1)$ contains immediate repetitions: the last symbol of $S(k)[C_{n(k)}]$ is identical to the first symbol of $S(k)[R_1]$. In order to make $S(k + 1)$ a proper order-$s(k + 1)$ DS sequence we must remove one of these copies, and apply the procedure recursively to each copy of $S(k)$. The fraction of occurrences removed is slightly more than $\frac{1}{8}$.

**Theorem 2.1.** For any $s \leq \log \log n + 2$, $\lambda(s, n) = \Omega(n \cdot 2^s)$.

**Proof.** Partition the alphabet $[n]$ into subsets of size $n' = 2^{2^s - 2}$ and concatenate $\lfloor n/n' \rfloor$ copies of $S(s - 2)$, one on each part of the alphabet. Each part has length $\Omega(n'2^{s-2})$, so the whole sequence has length $\Omega(n 2^{s-2})$.

In the case of $s = \log \log n + 2$, we can get a sequence of length $\Omega(n \log n)$, which is not known from prior constructions. The longest sequences that can be generated using [12, 15, 16] have length $O(n2^{o(n)})$.

## 3 Rectangular Zarankiewicz Matrices

The construction of the previous section is limited by the fact that each letter of $S(k + 1)$ appears in only two copies of $S(k)$ (corresponding to the letter’s row and column). In order to bound $s(k + 1) \leq s(k) + 1$, it was crucial that each pair of symbols appeared in only one common copy of $S(k)$. In general, one could imagine generalized constructions of $S(k + 1)$ over an $n$-letter alphabet that are formed by concatenating $m$ copies of $S(k)$, each over a subset of the alphabet, with the property that two symbols do not appear in too many common subsets. Designing such a system of subsets is an instance of Zarankiewicz’s problem.

**Definition 3.1.** (Zarankiewicz’s Problem) Define $z(m, n; s, t)$ to be the maximum number of 1s in an $m \times n$ 0-1 matrix that contains no all-1 $s \times t$ submatrix. Define $z(n, t)$ to be short for $z(n, n; t, t)$.

The Kővári, Sós, and Turán theorem [11], explicitly proven in [8], gives the following general upper bound on $z(m, n; s, t)$.

$$z(m, n; s, t) \leq (s - 1)^{1/t}(n - t + 1)m^{1 - 1/t} + (t - 1)m$$

It is generally believed that the Kővári-Sós-Turán upper bound on $z(n, t) = O(n^{2-1/t})$ is asymptotically sharp, but this has only been established for $t \in \{2, 3\}$ [3]. Kollár, Rónyai, and Szabó [10] gave sharp bounds on $z(n, n, t! + 1, t) = \Omega(n^{2-1/t})$, where the forbidden

---

2It is dominated by the occurrences removed in copies of $S(1)$, which has length 8 originally and length 7 afterward.
submatrix is highly skewed. In this paper we need bounds on Zarankiewicz’s problem in which both the \( m \times n \) matrix and forbidden pattern are rectangular. The following theorem may be folklore in some quarters; nonetheless, it is not mentioned in a recent survey [5]. The only existing construction avoiding \( t \times 2 \) all-1 submatrices is tailored to square matrices [5].

**Theorem 3.1.** For any fixed integer \( t \geq 2 \) and large enough \( n \),

\[
z(n^{2/t}, n, t, 2) = \Theta(n^{1+1/t}).
\]

**Proof.** Let \( q \) a prime power and \( \mathbb{F} \) be the Galois field of order \( q \). We will show that \( z(q^2, q^t, t, 2) \geq q^{t+1} \). By [11] this bound is asymptotically sharp. It is straightforward to extend this to any \( n \) (not of the form \( q^k \)) with a constant factor loss.

We will construct a matrix \( A \in \{0, 1\}^{q^2 \times q^t} \) as follows. The columns of \( A \) are indexed by all degree-(\( t-1 \)) polynomials over \( \mathbb{F} \). A polynomial \( f_c \) is identified with its coefficient vector \( c = (c_0, c_1, \ldots, c_{t-1}) \in \mathbb{F}^t \), where

\[
f_c(x) = \sum_{i=0}^{t-1} c_i x^i.
\]

The rows of \( A \) are indexed by *evaluations* \( (x, v) \in \mathbb{F}^2 \). The matrix \( A \) is generated by putting a 1 wherever we see a correct evaluation:

\[
A((x, v), c) = \begin{cases} 
1 & \text{if } f_c(x) = v \\
0 & \text{otherwise.}
\end{cases}
\]

Suppose \( A \) actually contains a \( t \times 2 \) all-1 submatrix defined by rows \( \{(x_i, v_i)\}_{i \in [0,t]} \) and columns \( c, c' \). Clearly \( x_0, x_1, \ldots, x_{t-1} \) are distinct field elements. It follows from the definition of \( A \) that \( f_c(x_i) - f_{c'}(x_i) = 0 \) for each \( i \in [0, t) \). However \( (f_c - f_{c'})(x) = \sum_{i=0}^{t-1} (c_i - c'_i) x^i \) is a degree-(\( t-1 \)) polynomial over \( \mathbb{F} \) and therefore has at most \( t-1 \) roots. It is impossible for \( f_c - f_{c'} \) to have \( t \) distinct roots, namely \( x_0, \ldots, x_{t-1} \).

Each row \( (x, v) \) of \( A \) has precisely \( q^{t-1} \) 1s, since for any partial coefficient vector \( (c_1, \ldots, c_{t-1}) \), there is some \( c_0 \) for which \( f_{(c_0,\ldots,c_{t-1})}(x) = v \). Similarly, each column \( c \) of \( A \) has precisely \( q \) 1s since there is one value \( v \) for which \( f_c(x) = v \). Thus, \( A \) contains precisely \( q^{t+1} \) 1s. \qed

## 4 Polynomial Order Davenport-Schinzel Sequences

Let \( q \) be a prime power and \( \hat{s} \geq q \) be a parameter. For each integer \( t \geq 1 \) we will construct an order-\( O((\hat{s}t-1)!)/t! \) sequence \( S_t(\hat{s}, q) \) over an alphabet of size \( q^t \) with length \( \Omega(q^{2t}\hat{s}) \). Phrased in terms of \( n = q^t \) and \( s = O((\hat{s}t-1)!)/t! \), this shows that \( \lambda(s, n) = \Omega(n^2 s/(t-1)! \). The construction is inductive. In the base case \( t = 1 \) we revert to Roselle and Stanton’s construction. (See Section 1.2)

\[
S_1(\hat{s}, q) = RS(\hat{s}, q^t).
\]

Now suppose that \( t \geq 2 \). Let \( A \) be the \( q^t \times q^t \) 0–1 matrix from Theorem 3.1. Each column of \( A \) is identified with a symbol in the alphabet of \( S_t(\hat{s}, q) \) and each row is identified with a
subset of its alphabet. In particular, let \( C_i, i \in [1, q^2] \), be the list of columns (symbols) in which \( A(i, *) = 1 \). We form \( S_t(\hat{s}, q) \) as follows:

\[
S_t(\hat{s}, q) = S_{t-1}(\hat{s}, q)[C_1] \cdot S_{t-1}(\hat{s}, q)[C_2] \cdots S_{t-1}(\hat{s}, q)[C_{q^2}],
\]

where \( S_{t-1}(\hat{s}, q)[X] \) is a copy of \( S_{t-1}(\hat{s}, q) \) over the alphabet \( X \). According to the proof of Theorem 3.1, \( |C_i| = q^{t-1} \), so the alphabets have the requisite cardinality. By construction we have

\[
|S_t(\hat{s}, q)| = q^2 \cdot |S_{t-1}(\hat{s}, q)|
\]

inductive hypothesis

\[
= q^2 \cdot \Omega(q^{2(t-1)}\hat{s})
\]

\[
= \Omega(q^{2t}\hat{s}).
\]

Let \( s_t = s(t, \hat{s}, q) \) be the length of the longest alternating subsequence in \( S_t(\hat{s}, q) \), which would make it an order-\((s_t - 1)\) DS sequence. We want to bound \( s_t \) in terms of \( s_{t-1} \). Pick two arbitrary symbols \( a, b \). Because \( A \) avoids all-1 \( t \times 2 \) submatrices, \( a \) and \( b \) appear in up to \( t - 1 \) common subsets among \( \{C_i\} \) and therefore at least \( q - (t - 1) \) subsets in which the other does not appear. Each subset of the first type contributes \( s_{t-1} \) alternations between \( a \) and \( b \) and each subset of the second type contributes 1, in the worst case where they happen to be interleaved. Thus, we have the following recursive expression for \( s_t \).

\[
s_1 = \hat{s} + 1
\]

\[
s_t \leq (t - 1)s_{t-1} + 2(q - t + 1)
\]

Since \( \hat{s} \geq q \), \( s_t = O((t-1)!\hat{s}) \).

**Theorem 4.1.** When \( s = \Omega(n^{1/t}(t-1)!), \lambda(s, n) \) is \( \Omega(n^2s/(t-1)!) \) and \( O(n^2s) \).

5 Medium Order Davenport-Schinzel Sequences

The construction of Theorem 4.1 is asymptotically sharp when \( t \) is constant (and \( s \) polynomial in \( n \)), but becomes trivial when \( t = \omega(\log n / \log \log n) \). In this section we design a simpler construction that works well when \( \log \log n < s = n^{o(1)} \).

The construction is parameterized by a prime power \( q \) and parameter \( \hat{s} \leq q \). The sequence \( S_t(\hat{s}, q) \) will be a sequence over an alphabet of size \( q^{2t} \). In the base case \( t = 0 \) we have

\[
S_0(\hat{s}, q) = RS(\hat{s}, q)
\]

so \( |S_0(\hat{s}, q)| = \Theta(q\hat{s}^2) \). When \( t \geq 1 \) we build \( S_t(\hat{s}, q) \) using a truncated version of the Zarankiewicz matrix from Theorem 3.1. Let \( \hat{q} = q^{2t-1} \) and \( A \) be the \( \hat{q}^2 \times q^2 \) 0-1 matrix avoiding \( 2 \times 2 \) all-1 submatrices. Let \( A' \) consist of the first \( \hat{q}\hat{s} \) rows of \( A \); i.e., each row of \( A' \) has \( \hat{q} \) 1s and each column of \( A' \) has \( \hat{s} \) 1s. This particular matrix could have been constructed using mutually orthogonal Latin squares, still generated based on finite fields as in [5] or [2]. As in Section 4 we identify the columns with symbols and the rows with sequences of symbols \( C_1, \ldots, C_{\hat{q}\hat{s}} \). The sequence \( S_t(\hat{s}, q) \) is formed as follows:

\[
S_t(\hat{s}, q) = S_{t-1}(\hat{s}, q)[C_1] \cdot S_{t-1}(\hat{s}, q)[C_2] \cdots S_{t-1}(\hat{s}, q)[C_{\hat{q}\hat{s}}]
\]
Assuming inductively that $|S_{t-1}(\hat{s}, q)| = \Omega(q^{2^{t-1}\hat{s}t+1})$, we have

$$|S_t(\hat{s}, q)| = q^{2^{t-1}\hat{s}}|S_{t-1}(\hat{s}, q)|$$
$$= q^{2^{t-1}\hat{s}} \cdot \Theta(q^{2^{t-1}\hat{s}t+1})$$
$$= \Theta(q^{2^t\hat{s}t+2})$$

Each symbol appears in exactly $\hat{s}$ distinct sequences among $\{C_i\}$ and any two symbols appear in at most one common sequence among $\{C_i\}$. Thus, if $s_t$ is the length of the longest alternating sequence in $S_t(\hat{s}, q)$, we have

$$s_0 = \hat{s} + 1$$
$$s_t = s_{t-1} + 2(\hat{s} - 1)$$

Clearly $s_t = (2t+1)(\hat{s} - 1) + 2$. In terms of the alphabet size $n = q^t$, $t = \log \log_q n$. In terms of $s = s_t$ and $n$, the length of $S_t(\hat{s}, q)$ is

$$\Theta(ns^{t+2}) = \Omega\left(n \left(\frac{s}{2 \log \log_q n + 1}\right)^{\log \log_q n + 2}\right) = \Omega\left(n \left(\frac{s}{2 \log \log_q n}\right)^{\log \log_q n}\right).$$

**Theorem 5.1.** For any $s = \Omega(\log \log n)$, $\lambda(s, n) = \Omega(n(\frac{s}{2 \log \log_q n + 1})^{\log \log_q n+1})$. For example, $\lambda(\log n, n)/n = 2^{\Omega((\log \log n)^2)}$ is quasi-polylogarithmic in $n$.

### 6 Conclusion and Open Problems

We have attained asymptotically tight bounds on $\lambda(s, n)$ when $s = n^\epsilon$. Specifically, the trivial upper bound $\lambda(s, n) = O(n^2 \cdot s)$ can be achieved asymptotically, with the leading constant depending on $\epsilon$. Even when $s = n$ the true leading constant of $\lambda(n, n)$ is only known approximately; it is in the interval $[1/3, 1/2]$ [17, 9]. Several interesting open problems remain, among them:

- Our lower bounds on $\lambda(s, n)$ when $1 \ll s \ll n^{o(1)}$ are quite far from the best upper bounds in this range [12, 15]. It is still consistent with all published results that $\lambda(s, n)/n$ grows (at least) exponentially in $s$ for all $s \leq \log n$, and that $\lambda(s, n) = \Theta(n^2s)$ for $s \geq \log n$.

- Our constructions are not very robust to slight variants in the definition of the extremal function $\lambda(s, n)$. For example, if we insist that the sequence be 3-sparse (every three consecutive symbols must be distinct) rather than 2-sparse (merely avoiding immediate repetitions), the Roselle-Stanton construction no longer works and we cannot claim that when $s > n^\epsilon$, $\lambda(s, n) = \Omega(n^2s)$ is witnessed by some 3-sparse sequence. This is in sharp contrast to the fixed-$s$ world [12, 15], which are highly robust to different notions of sparseness.
• A popular way to constrain DS sequences is to specify the number blocks \([12, 15]\). A block is a sequence of distinct symbols. Let \(\lambda(s, n, m)\) be the length of an order-\(s\) DS sequence over an \(n\)-letter alphabet that is partitioned into \(m\) blocks. In the fixed-\(s\) world \([12, 15]\), \(\lambda(s, n)\) is roughly \(\lambda(s, n, n)\); see, e.g., [15, Lemma 3.1]. Our constructions for \(s\) in the “small” and “medium” range do give non-trivial bounds on \(\lambda(s, n, n)\), but say nothing interesting when \(s = n^\epsilon\). Bounding \(\lambda(s, n, n)\) is essentially identical [13] to bounding the number of 1s in an \(n \times n\) 0-1 matrix avoiding \(2 \times (s + 1)\) alternating submatrices of the following form.

\[
\begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\end{pmatrix}
\]

Clearly the extremal function \(\lambda(s, n, n)\) tends to \(n^2\) as \(s \to n\), but we know very little about the rate of convergence. For example, how large must \(s\) be in order for \(\lambda(s, n, n) = \Omega(n^{2-o(1)})\)?

References

[1] P. Agarwal, M. Sharir, and P. Shor. Sharp upper and lower bounds on the length of general Davenport-Schinzel sequences. *J. Combin. Theory Ser. A*, 52:228–274, 1989.

[2] R.C. Bose. On the application of the properties of galois fields to the problem of construction of hyper-græco-latin squares. *Sankhyā: The Indian Journal of Statistics (1933-1960)*, 3(4):323–338, 1938.

[3] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.*, 9:281–285, 1966.

[4] H. Davenport and A. Schinzel. A combinatorial problem connected with differential equations. *American J. Mathematics*, 87:684–694, 1965.

[5] Z. Füredi. New asymptotics for bipartite Turán numbers. *J. Combin. Theory Ser. A*, 75(1):141–144, 1996.

[6] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*, pages 169–264. 2015.

[7] S. Hart and M. Sharir. Nonlinearity of Davenport-Schinzel sequences and of generalized path compression schemes. *Combinatorica*, 6(2):151–177, 1986.

[8] C. Hyltén-Cavallius. On a combinatorial problem. *Colloq. Math. 6*, pages 59–65, 1958.

[9] M. Klazar. Generalized Davenport-Schinzel sequences: results, problems, and applications. *Integers*, 2:A11, 2002.

[10] J. Kollár, L. Rónyai, and T. Szabó. Norm-graphs and bipartite Turán numbers. *Combinatorica*, 16(3):399–406, 1996.
[11] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[12] G. Nivasch. Improved bounds and new techniques for Davenport-Schinzel sequences and their generalizations. *J. ACM*, 57(3), 2010.

[13] S. Pettie. Degrees of nonlinearity in forbidden 0-1 matrix problems. *Discrete Mathematics*, 311:2396–2410, 2011.

[14] S. Pettie. Generalized Davenport-Schinzel sequences and their 0-1 matrix counterparts. *J. Comb. Theory Ser. A*, 118(6):1863–1895, 2011.

[15] S. Pettie. Sharp bounds on Davenport-Schinzel sequences of every order. *J. ACM*, 62(5):36, 2015.

[16] S. Pettie. Three generalizations of Davenport-Schinzel sequences. *SIAM J. Discrete Mathematics*, 29(4):2189–2238, 2015.

[17] D. P. Roselle and R. G. Stanton. Some properties of Davenport-Schinzel sequences. *Acta Arithmetica*, XVII:355–362, 1971.

[18] M. Sharir and P. Agarwal. *Davenport-Schinzel Sequences and their Geometric Applications*. Cambridge University Press, 1995.

[19] M. Sharir, R. Cole, K. Kedem, D. Leven, R. Pollack, and S. Sifrony. Geometric applications of Davenport-Schinzel sequences. In *Proceedings 27th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 77–86, 1986.