Disorder effect on magneto-transport on the surface of a topological insulator

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We study the magneto-transport properties on the disordered surface of a topological insulator attached with a ferromagnet/ferromagnet junction. Since, in the surface Dirac Hamiltonian, out-of-plane magnetization induces a mass gap, while in-plane magnetization has a role of the effective vector potential, the mechanism of magneto-transport is different between these two cases. The former is similar to the conventional one in ferromagnetic metals, while the latter is due to the shift of Fermi circles in momentum space. Our numerical calculations show that the magnetoconductance in in-plane configuration is robust against disorder compared to that in out-of-plane configuration.

Topological insulators (TIs) are new quantum states of matter, which cannot be adiabatically connected to conventional insulators. A three-dimensional (3D) TI has a finite gap in the bulk but possesses gapless surface modes described by the two-dimensional (2D) massless Dirac Hamiltonian for simple cases.1,2 The surface states are dubbed as the helical surface, which has attracted much interest because of crucial applications in spintronics, where the spin quantization axis is perpendicularly locked to the momentum by spin-orbit coupling. It has been argued that the surface metallic states are stable to any perturbations with time-reversal symmetry. The electrons at the surface of a TI cannot be localized even for strong disorder as long as the bulk energy gap remains intact.3,4 When time-reversal symmetry is broken for example by the exchange interaction with magnetic dopants,5,6 the surface of a TI demonstrates the quantized anomalous Hall effect1,2,7,8 which supports non-trivial effects such as the topological magneto-electric effect.9

In the presence of disorder all surface electrons are localized while the quantized Hall current flows even at strong disorder.9,10

The spin-momentum locking at the surface makes TIs promising for versatile device applications. At the interface of a ferromagnetic insulator and a topological insulator, a variety of unique magneto-transport phenomena has been theoretically proposed11–14 and experimentally examined.15,16 In particular, the magnetoconductance of ferromagnet/ferromagnet junction deposited on the surface of a TI has been studied, which has attracted much interest because of crucial applications of strongly spin-orbit coupled system in spintronics. In the theoretical studies the effect of disorder has not been considered, however, since it is inevitable in experimental situations, the theoretical study is highly desirable.

In this work we study the disorder effects on the magnetoconductance of topological surface attached with a ferromagnet/ferromagnet junction. The ferromagnetism on the surface is induced by the exchange interaction $m \cdot \sigma$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli spin matrix of the surface electrons and $m = (m_x, m_y, m_z)$ is the exchange field which has the direction of the magnetization and the magnitude of the exchange splitting energy. The out-of-plane exchange field generates a mass gap in the surface modes. When the Fermi level is located slightly above the bottom of the conduction band, by projecting into the conduction band, the surface states can be regarded as fully spin-polarized 2D electrons with conventional parabolic dispersion. As in the conventional ferromagnetic metals, the conductance shows a change depending on whether the magnetizations of adjacent ferromagnets are in a parallel or an antiparallel alignment, namely magnetoconductance. On the other hand, the in-plane exchange field acts as an effective vector potential which shifts the Fermi circles in momentum space. A misalignment of the Fermi circles between two regions also gives rise to a magnetoconductance.12–14 With the use of the transfer matrix method,3,17,18 we calculate magnetoconductance in the out-of-plane and the in-plane magnetization configurations, and compare the disorder dependence of them. Our result shows a difference between the two cases, that the in-plane magnetoconductance is relatively robust against disorder, compared with the out-of-plane magnetoconductance.

We consider two ferromagnetic insulators, F1 and F2, deposited on the surface of a TI (Fig. 1). The surface electronic states in this system can be described by the 2D Dirac Hamiltonian

$$H = v \left( p_x \sigma_x - p_y \sigma_y \right) + \sum_{i=x,y,z} m_i \sigma_i + U(x,y),$$

where $v$ is the velocity of the Dirac fermion, $m_x$, $m_y$, and $m_z$ are the exchange fields and $U(x,y)$ is the disorder potential. We note the exchange field in F1 as $m_1 = (m_{1x}, m_{1y}, m_{1z})$ and F2 as $m_2 = (m_{2x}, m_{2y}, m_{2z})$. We assume the TI’s surface as 2D sheet

![Schematic picture of F1/F2 junction on the TI surface. The ferromagnetism on the surface of a TI is induced due to proximity effect by the ferromagnetic insulators deposited on the surface. The current flows on the surface of a TI.](image-url)

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of length \( L \) along longitudinal \( x \) direction and width \( W \) along transversal \( y \) direction. We fixed the aspect ratio \( W/L = 4 \).

We utilize the transfer matrix method\(^3\) for the 2D Dirac equation \( \hbar \Psi = E \Psi \), where \( \Psi(x,y) \) is the two-component (spinor) wave function. Multiplication of both sides by \( i \sigma_y \) gives

\[
h \partial_t \Psi = \left( \sigma_y p_y + i (U(x,y) - E) \sigma_x + m_1 \sigma_z + i m_y - m \sigma_x \right) \Psi.
\]

(2)

Here we discretize \( x \) at the point \( N \) points \( x_1, x_2, ..., x_N \). The transfer operator \( M \), defined by \( \Psi_k = M \Psi_{k0} \), is given by the operator product

\[
M = P_{Lx} K_N P_{x_1,x_2} K_{N-1} ... K_2 P_{x_1,x_0} K_1 P_{x_1,0},
\]

(3)

\[
P_{x,x'} = \exp \left[ \frac{1}{\hbar} (x - x') \sigma_x \right].
\]

(4)

The operator \( P \) gives the decay of evanescent waves between two scattering events, described by operator \( K_n \),

\[
K_n = \mathcal{V}_n B_{x,n} B_{y,n} B_{z,n},
\]

(5)

where

\[
\mathcal{V}_n = \exp \left[ - \left( i a / \hbar \right) (U_n - E) \sigma_x \right],
\]

(6)

and

\[
B_{x,n} = \exp \left[ (i a / \hbar) m_x \sigma_z \right],
\]

(7)

\[
B_{y,n} = \exp \left[ (i a / \hbar) m_y \sigma_x \right],
\]

(8)

\[
B_{z,n} = \exp \left[ - (a / \hbar) m_x \sigma_y \right],
\]

(9)

where \( a \) is the lattice constant.

To calculate the transfer matrix, we represent the operators in the basis

\[
\psi_{k}^{\pm} = \frac{1}{\sqrt{W}} e^{\pm i \pi k |z|}, \quad q_k = \frac{2 \pi k}{W}, \quad k = 0, \pm 1, \pm 2, ...
\]

(10)

The spinors \( |\pm\rangle \) are eigenvectors of \( -\sigma_y \). By truncating the transverse momenta \( q_x \) at \( |k| = M \), the dimension of the transfer matrix becomes finite. The disorder potential \( U(x,y) = \sum_{n,m} \gamma_{nm} \delta(x-x_n) \delta(y-y_m) \) is given by a collection of isolated impurities distributed uniformly over the scattering region \( 0 < x < L, 0 < y < W \). The strengths \( \gamma_{mn} \) of the scatterers are uniform in the interval \( [-\gamma_0, \gamma_0] \). The disorder strength is quantified by the correlator \( K_0 = \frac{2}{\gamma_{0}^2} \int d\mathbf{r} \langle U(\mathbf{r}) U(\mathbf{r}') \rangle \) which evaluates to \( K_0 = \frac{1}{4} \gamma_{0}^2 (a/\hbar)^2 \), independent of the correlation lengths. The average conductance \( \langle G \rangle \) is obtained by sampling some 200–2000 disorder realizations of the impurity potential.

To formulate the scattering problem, we consider a scattering state \( \Psi_k \) that has unit incident current from the left \( (x=0) \) in mode \( k \) and zero incident current from the right \( (x=L) \). The quantum number \( k \) labels transverse modes. At \( x=0 \), the sum of incoming and reflected waves given by

\[
\Psi_k^{\text{left}} = \phi_k^+ + \sum_{k'} t_{k'k} \phi_{k'}^-.
\]

(11)

while the sum of transmitted waves at \( x=L \) is given by

\[
\Psi_k^{\text{right}} = \sum_{k'} t_{k'k} \phi_{k'}^+.
\]

(12)

The right moving component in mode \( k \) is \( \phi_k^+ \) and left moving component is \( \phi_k^- \). Starting from a mode incident from right, we can similarly obtain the reflection and transmission matrices \( r' \) and \( t' \), which together with \( r \) and \( t \) unitary scattering matrix,

\[
S = \begin{pmatrix} r & t' \\ r' & t \end{pmatrix}.
\]

(13)

As a consequence of unitarity, the matrix product \( t't \) and \( r'r \) have the same eigenvalue called transmission eigenvalues. The conductance \( G \) follows from transmission eigenvalues via Landauer formula \( G = \frac{2}{\pi} \text{Tr} (t't) = \frac{2}{\pi} \text{Tr} (r'r) \).

The information contained in the scattering matrix \( S \) can equivalently be represented by transfer matrix \( M^{\text{left}} \). While the scattering matrix relates outgoing waves to incoming waves, the transfer matrix relates waves at the right to wave at the left.

\[
\Psi_{k}^{\text{right}} = M^{\text{left}} \Psi_{k}.
\]

(14)

We separate the spin degree of freedom of \( M \) into four blocks

\[
M = \begin{pmatrix} M^{++} & M^{+\pi} \\ M^{\pi+} & M^{\pi\pi} \end{pmatrix}.
\]

(15)

As one can verify by substitution into Eq. (14), and comparison Eq. (11) and Eq. (12), the submatrices \( M^{++}, M^{\pi\pi} \) related to the transmission and reflection matrices by

\[
r = - (M^{\pi\pi})^{-1} M^{++}, \quad r' = M^{++} (M^{\pi\pi})^{-1}
\]

(16)

\[
t = M^{\pi+} - M^{\pi\pi} (M^{\pi\pi})^{-1} M^{\pi+}, \quad t' = (M^{\pi\pi})^{-1}
\]

(17)

This completes the description of our numerical method. We now turn to the result. In the case of out-of-plane exchange fields, \( m_1 = (0, m_{1z}) \) and \( m_2 = (0, m_{2z}) \), and we fix \( m_{1z} = 0.2 i \hbar /a \) and vary \( m_{2z} \), from \( -m_{2z} \) to \( +m_{2z} \), with fixing \( m_{2} = 0.2 i \hbar /a \). The aspect ratio and the energy are fixed at \( W/L = 4 \) and \( E = 1.35 m_{2z} \).
the case where \( |m_z| \gg v \hbar k \), and focus on the positive energy band on the surface of a TI. In this case, the positive energy band has only one spin \( \sigma_z \), and the dispersion approximated as \( E = \sqrt{(\hbar k)^2 + m_z^2} \sim \frac{\hbar k^2}{2m_1} + |m_z| \), regarded as fully spin polarized Schrödinger electrons. For this reason, the mechanism of this magnetoconductance induced by out-of-plane exchange field corresponds to that in a conventional ferromagnetic metal.

Next, we consider the influence of disorder on the surface of a TI. Figure. 2 shows that the \( m_z \) dependence of the conductance changes abruptly when weak disorder is introduced. The conductance \( G(m_z) \) does not become maximum but minimum at \( m_z = +m_{2z} \) (parallel configuration), in contrast to the clean limit case. In the disordered case the conductance takes its maximum at \( m_z = 0 \). At strong disorder, \( K_0 = 1.0 \), \( m_z \) dependence of the conductance is nearly symmetric around \( m_z = 0 \), indicating that the parallel/antiparallel configuration does not matter.

When the exchange field is applied in the \( x \) direction, \( k_x \sigma_x \) term in the original Dirac Hamiltonian is replaced by \( k_x + m_z / \hbar v \sigma_x \), indicating that the Fermi circle is shifted by \( -m_z / \hbar v \) in momentum space, while the size of Fermi circle remains unchanged. When \( m_{1x} \neq m_{2x} \), the position of the Fermi circles in F1 and F2 are different. This misalignment of the Fermi circles causes a change of conductance (magnetoconductance) with qualitatively different mechanism from conventional one. Namely, as \( |m_{2z} - m_{1z}| \) increases, the overlap region of the Fermi circles between F1 and F2 is reduced, and thus the number of evanescent modes increases. The in-plane exchange field induced magnetoconductance is characteristic to the spin-momentum locking and qualitatively different from the conventional magnetoconductance.

The normalized magnetoconductance in the in-plane configuration, \( \mathbf{m}_1 = (m_{1x}, 0, 0) \) and \( \mathbf{m}_2 = (m_{2x}, 0, 0) \), is shown in Fig. 3. We fix the exchange field in F2 at \( m_{2z} (a / \hbar v) = 0.2 \), and vary the exchange field \( m_{1z} \) from \(-m_{2z}\) to \( +m_{2z} \). In the clean limit \( (K_0 = 0) \), the conductance increases for the parallel configuration \( (m_{1x} / m_{2x} > 0) \), and decreases for the antiparallel configuration \( (m_{1x} / m_{2x} < 0) \). As mentioned above, the conductance is influenced by the relative positions of the Fermi circles, it takes maximum when there is no misalignment of Fermi circles between F1 and F2, and decreases with introducing the misalignment.

As Fig. 3 shows, the dependence of magnetoconductance on the in-plane exchange field becomes gradually weak as the disorder strength increases, in contrast to the case of out-of-plane dependence. Even in the presence of disorder, the magnetoconductance remains positive for the parallel configuration \( (m_{1x} / m_{2x} > 0) \), while negative for the antiparallel configuration \( (m_{1x} / m_{2x} < 0) \).

To compare disorder dependence of the magnetoconductance with the out-of-plane exchange field and that with the in-plane, we plot the normalized conductance differences \( \Delta G_{\text{out/in}}(K_0) / \Delta G_{\text{out/in}}(0) \) in Fig. 4, where \( \Delta G_{\text{out/in}}(K_0) = \langle G(m_{1x} = m_{2z}/2z, 0) \rangle - \langle G(m_{1x} = -m_{2z}/2z, 0) \rangle \). These results clearly indicate that the magnetoconductance with the in-plane field is robust against disorder compared to that with the out-of-plane field.

In conclusion, we have studied the disorder effect on the magnetoconductance of the ferromagnet/ferromagnet junction on the surface of a TI. With the use of the transfer matrix method, we calculated the magnetoconductance in both the out-of-plane and the in-plane exchange field configurations. In the out-of-plane field, the Dirac electrons are regarded as fully spin-polarized Schrödinger electrons when the Fermi level is located slightly above the bottom of the conduction band. The mechanism of the magnetoconductance in this regime corresponds to that in a conventional ferromagnetic metal. On the other hand, the in-plane field induced magnetoconductance is characteristic to the surface of a TI. These two cases show different disorder dependence. These results are consistent to the fact that all wave functions are localized in the presence of a mass gap, while in the in-plane fields massless Dirac fermion systems belong to the critical point of the quantum Hall transition, and thus wave functions are extended. Since the latter is robust against disorder, it is an advantage of TI based devices.
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