The problem of suspension injection into a porous medium within a two-velocity model of deep bed filtration

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Abstract. We consider flows of a suspension through a porous medium with account of the deposition of the suspended particles onto the internal surface of a porous matrix (deep bed filtration). The velocities of the suspended particles and the carrier fluid are plausibly supposed to differ by a known factor that depends on the suspension concentration. We present analytical and numerical solutions to the problem of suspension injection into a porous sample. The main qualitative result of the investigation is that — under certain circumstances — the smooth solutions (i.e., the ones that do not, in contrast to usual assumptions, involve concentration jumps) to the problem may exist. This feature may be of use for experimental data interpretation.

1. Introduction
Flows of suspensions through porous media with gradual contamination of porous matrix [1, 2] arise in water treatment problems, oil industry, etc. The standard way of investigating the parameters of deep bed filtration models is the performance of the laboratory experiments with cylindric porous samples under controlled conditions. In the injection tests, aside from direct measuring the concentration at different positions within the porous sample, the experimenters typically obtain the dependence of the effluent particle concentration on the time (so called breakthrough curve). In such experiments, the data treatment rests upon the exact solutions with jumps in suspended particles concentration that propagate along the porous sample and separate its contaminated and clear parts. In this paper we call special attention to the fact that under certain (practicable) circumstances the solution to the injection problem may be free of jumps, so the analysis of the experimental data should be thoroughly performed with great attention to details.

2. Two-velocity model of deep bed filtration
In this section we briefly present the model of suspension flow through a porous medium that accounts for the retardation of suspended particles relative to the carrier fluid [3].

For one-dimensional flows with plane waves, the mathematical model comprises the continuity equations for suspended particles and carrier fluid

\[ \frac{\partial}{\partial t} (m\alpha) + \frac{\partial}{\partial x} (m\alpha v) = \frac{\partial m}{\partial t}, \quad \frac{\partial}{\partial t} ((1 - \alpha)m) + \frac{\partial}{\partial x} (m(1 - \alpha)w) = 0, \]
the equation for the kinetics of particle deposition (or entrainment) with known right-hand side term
\[ \frac{\partial m}{\partial t} = f(m, \alpha, v, w, \ldots), \]
and the relation between the velocities of the suspended particles and carrier fluid [4]
\[ v = g(w, \alpha, m, \ldots), \]
where \( v \) and \( w \) are the velocities of the particles and the carrier fluid, \( m \) is the porosity, \( \alpha \) is the concentration of the suspended particles, \( t \) and \( x \) are the time and the spatial coordinate along the direction of the flow.

At the discontinuities, the mass flux conditions give the jump conditions
\[
\begin{align*}
[(\alpha - 1)m]D - [\alpha mw] &= 0, \\
[(1 - \alpha)m]D - [(1 - \alpha)mw] &= 0,
\end{align*}
\]
where \( D \) is the velocity of the discontinuity, and brackets denote the jump in the quantity under consideration across the discontinuity. Some additional empirical conditions, e.g., the continuity of the porosity \([m] = 0\), should also be added.

In this paper we consider the simplest kinetic equation with local rate of change in porosity being proportional to the particle flux through a given section of the porous sample
\[ \frac{\partial m}{\partial t} = -\gamma m [v], \]
where \( \gamma \) is a known constant, and a linear dependence of the particles velocity on the carrier fluid velocity
\[ v = \kappa(\alpha) \cdot w. \]

The system of equations is then reduced to the hyperbolic system for \( \alpha(x, t) \) and \( m(x, t) \)
\[ \frac{\partial \alpha}{\partial t} + \frac{qF'(\alpha)}{m} \frac{\partial \alpha}{\partial x} = -\gamma q(1 - \alpha)F(\alpha), \quad \frac{\partial m}{\partial t} = -\gamma qF(\alpha), \]
where \( q = m\alpha v + m(1 - \alpha)w = \text{const}, \ q > 0, \) is the total volumetric flow rate of the suspension (for the sake of simplicity being assumed constant), and
\[ F(\alpha) = \frac{\kappa(\alpha)\alpha}{\kappa(\alpha)\alpha + 1 - \alpha}. \]

We will consider the problem of the injection of the suspension into an uncontaminated \((\alpha(x, 0) = 0, m(x, 0) = m_0 = \text{const})\) porous sample through the inlet section \( x = 0 \) with the boundary condition \( \alpha(0, t) = \alpha_0 = \text{const}, \ t > 0. \)

The formal solution to the problem [3] involves the jump in concentration propagating with the velocity
\[ D = \frac{q}{m} \frac{|F(\alpha)|}{[\alpha]}, \]
the time-independent concentration distribution behind the jump being determined by the equation
\[ \gamma x = G(\alpha(x)), \quad G(\alpha) = \int_{\alpha_0}^{\alpha} \frac{F'(\alpha_1)\,d\alpha_1}{(1 - \alpha_1)F(\alpha_1)}. \]
The jump conditions allow determining the implicit dependence of the concentration behind the jump on the time \( \alpha_1(t) \)
\[ \gamma q t = -\int_{\alpha_0}^{\alpha_1} m_0 \alpha F'(\alpha) \frac{d\alpha}{(1 - \alpha)F^2(\alpha)}, \]
and the porosity distribution behind the jump is determined by a simple integration.
3. Jumpless solutions. Numerical solutions to the problem of the injection into a porous sample

The presented formal solution may fail to exist if the concentration jump is not evolutionary [3], so the actual solution turns out to be smooth. In order to find the form of the solution, we use the second-order accuracy finite difference scheme. The discontinuous solutions are evaluated through the use of artificial viscosity [5] and quite simple smoothing procedure.

![Figure 1. Smooth dependence of concentration $\alpha$ on dimensionless coordinate $x_1 = \gamma x$ for $\beta = 5$, $m_0 = 0.3$, $\alpha_0 = 0.1$. Curves 1 to 6 correspond to $t = 0.06, 0.135, 0.21, 0.285, 0.39, 0.72$.](image1)

In numerical calculations, we consider the velocities ratio $\varkappa$ to be of the form $\varkappa(\alpha) = 1 - \beta\alpha$, $\beta = \text{const}$, which has a simple and clear physical meaning: the more the concentration of suspended particles is, the more the retardation of particles reveals itself, and, on the other hand, for small concentrations, $\alpha \ll 1$, the velocities tend to coincide.

For positive $\beta$ considered, the distribution of $\alpha$ consists of an uncontaminated zone $\alpha \equiv 0$ ahead of the wave, an expanding wave with gradual change in concentration, and a time-independent zone immediately adjacent to the inlet section. An example of temporal evolution of $\alpha$ for $\beta = 5$ is given in figure 1.

![Figure 2. Discontinuous solution for $\beta = -5$, $m_0 = 0.3$, $\alpha_0 = 0.1$. Curves 1 to 4 correspond to the dimensionless time $t_1 \equiv \gamma qt = 0.0307, 0.107, 0.184, 0.261$. Solid lines represent the numerical calculations, dash lines, exact analytical solution.](image2)

The leftmost zone for $\beta \neq 1$ can be analytically described as

$$(\beta - 1)\gamma x = \beta \ln \frac{(\beta\alpha_0 - 1)(\beta\alpha^2 - 1)}{(\beta\alpha - 1)(\beta\alpha_0^2 - 1)} + 2\sqrt{\beta}(\arctanh(\sqrt{\beta}\alpha_0) - \arctanh(\sqrt{\beta}\alpha))$$

$$+ (\beta - 1) \ln \frac{\alpha_0}{\alpha} + \ln \frac{\alpha_0 - 1}{\alpha - 1}.$$
It is notable that the intermediate zone, as opposed to, for example, analogous Riemann problem in two-phase Backley–Leverett model, has not the self-similar form $\alpha(x/t)$, as it follows from the consideration of the sole second-order equation for $\alpha$

$$
-\gamma q F(\alpha) \left( \frac{\partial \alpha}{\partial t} \right)^2 + H(\alpha) \frac{\partial^2 \alpha}{\partial t^2} = \frac{\partial \alpha}{\partial t} \frac{\partial H(\alpha)}{\partial t}, \quad H(\alpha) = -\gamma q (1 - \alpha) F(\alpha) - q F'(\alpha) \frac{\partial \alpha}{\partial x}.
$$

For negative $\beta$ the function $F(\alpha)$ may be, for certain combinations of governing parameters, convex downwards, so the jump in analytical solution appears to be evolutionary, and the discontinuous solution is realizable. It is worth noting that in this case $\kappa > 1$, and this solution corresponds to the velocity of the particles being greater than the carrier fluid velocity.

An example of temporal evolution of $\alpha$ for $\beta = -5$ is presented in figure 2. The minor oscillations at the jump arise, as is well known, due to the numerical nature of calculations, and are not physical. For discontinuous solutions, the concentration jump decelerates, as follows from graphs in figure 3.

Figure 3. The dependence of the dimensionless jump position $x_1 = \gamma x$ on the dimensionless time $t_1 = \gamma q t$ for $m_0 = 0.3$, $a_0 = 0.1$. Curve 1 corresponds to $\beta = -5$; curve 2, to $\beta = 0$ (standard one-velocity model); dash curve 3, to formal (unrealizable) discontinuous solution for $\beta = 5$.

4. Conclusion
In this paper we numerically investigate the problem of suspension injection of into a porous sample. Although the vast majority of investigations of the problem is based on the solutions with strong discontinuities in concentration of suspended particles, we show that even for simple realistic relations between the velocities of the particles and the carrier fluid it is possible that the solutions without concentration jump exist. This fact may affect the interpretation of typical laboratory experiments and should therefore be borne in mind.

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