ARTIN REPRESENTATIONS FOR $GSp_4$ ATTACHED TO REAL ANALYTIC SIEGEL CUSP FORMS OF WEIGHT (2,1)

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Abstract. Let $F$ be a vector-valued real analytic Siegel cusp eigenform of weight $(2,1)$ with the eigenvalues $-\frac{5}{12}$ and 0 for the two generators of the center of the algebra consisting of all $Sp_4(\mathbb{R})$-invariant differential operators on the Siegel upper half plane of degree 2. We develop the theory of vector-valued real analytic Siegel modular forms. Under natural assumptions in analogy of holomorphic Siegel cusp forms, we construct a unique symplectically odd Artin representation $\rho_F : \tilde{G}_Q \rightarrow GSp_4(\mathbb{C})$ associated to $F$. Several examples which satisfy these assumptions are given by using various transfers and automorphic induction.

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Key words and phrases. Artin representation, Siegel modular forms.

The first author is partially supported by NSERC. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research No.23740027 and JSPS Postdoctoral Fellowships for Research Abroad No.378.

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1. Introduction

Let $G$ be a quasi-split reductive group over $\mathbb{Q}$ and $L^G$ be the $L$-group. The strong Artin conjecture asserts that an irreducible continuous complex representation $\rho : G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow L^G$ corresponds to an automorphic representation of $G$. We will call such irreducible complex representation, Artin representation. Conversely, the Langlands reciprocity conjecture claims that there exists an $\ell$-adic Galois representation associated to a given automorphic representation. If the Galois representation has the finite image, then it corresponds to an Artin representation.

In [6], Deligne and Serre associated an odd irreducible Artin representation $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$ to any elliptic cusp form of weight one. Conversely, the strong Artin conjecture is now proved for any odd irreducible Artin representation: Solvable cases have been known for a long time [31], [48]. The icosahedral case is proved in [21] and [27]. Note that the automorphy of an even icosahedral Artin representation $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$ is still open.

Contrary to the case $GL_2/\mathbb{Q}$, we can show that there are no holomorphic Siegel cusp forms of weight $(2, 1)$ which give rise to Artin representations (Proposition 9.4). Therefore we have to look for the corresponding objects inside real analytic Siegel modular forms. This fact makes the situation much more difficult as in the case of Maass cusp forms for $GL_2/\mathbb{Q}$.

In this paper we will be concerned with the Langlands reciprocity conjecture for a cuspidal automorphic representation of $GSp_4/\mathbb{Q}$ associated to the real analytic Siegel cusp form of weight $(2, 1)$. More explicitly, let $F$ be such a form of weight $(2, 1)$, level $N$ with central character $\varepsilon$. (See Section 3.1 for the level and a central character of real analytic Siegel modular forms.) Assume that $F$ has the eigenvalues $-\frac{5}{12}$ and 0 for the elements $\Delta_1$ and $\Delta_2$ of degree 2 and 4 respectively which generate the center of the algebra consisting of all $Sp_4(\mathbb{R})$-invariant differential operators on the Siegel upper half plane. (See Section 3.3 and the equation (4.1) for the precise definition of $\Delta_1$ and $\Delta_2$.) We denote by $S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$ the space consisting of any such $F$. Let $t_1 = \text{diag}(1, 1, p, p)$ and $t_2 = \text{diag}(1, p, p^2, p)$ for a prime $p$ and denote by $T_{i,p}$ the Hecke operator corresponding to $t_i$ for $i = 1, 2$ (see Section 3.2). Assume that $F$ is an eigenform for $T_{1,p}$ and...
$T_{2,p}$, $p \nmid N$ with eigenvalues $a_{1,p}$ and $a_{2,p}$ respectively. We define the Hecke polynomial at $p$ by

$$H_p(T) := 1 - a_{1,p}T + \{pa_{2,p} + (1 + p^{-2})\varepsilon(p^{-1})\}T^2 - a_{1,p}\varepsilon(p^{-1})T^3 + \varepsilon(p^{-1})^2T^4.$$ 

Let $\pi_F = \pi_\infty \otimes \otimes_p \pi_p$ be the cuspidal automorphic representation of $GSp_4(\mathbb{A})$ attached to $F$. Then by Theorem 4.2, $\pi_\infty$ is the full induced representation $\text{Ind}_{G}^{G_0} \chi(1, \text{sgn}, \text{sgn})$, which is irreducible. In particular, it is tempered and generic. It is a totally degenerate limit of discrete series in the sense of [5]. It is also denoted as $D_{(1,0)[0]}$ in [35]. We emphasize here that $\pi_\infty$ is not a limit of holomorphic discrete series and it has the minimal $K_\infty$-type $(1,0)$.

We denote by $Q_F = \mathbb{Q}(a_{1,p}, a_{2,p}, \varepsilon(p^{-1}) : p \nmid N)$, the Hecke field of $F$. We also define the Hecke algebra $\mathbb{T}_Q$ over $\mathbb{Q}$ as in Section 3.4. We then make the following assumptions:

1. **(TR)** the validity of the transfer of automorphic representations of $GSp_4$ to $GL_4$ (Section 5);
2. **(Gal)** the existence of mod $\ell$ Galois representation attached to $F$ (Conjecture 7.1);
3. **(Rat)** rationality of the subspace $\langle T_F | T \in T_Q \rangle \subset S(2,1)(\Gamma(N), -\frac{5}{2}\ell, 0)$ (Section 3.4);
4. **(Int)** the integrality of Hecke polynomials $H_p(T)$ of $F$ for all but finitely many prime $p \nmid N$ (Section 6.2).

The assumptions (Gal), (Rat), and (Int) are valid for holomorphic Siegel cusp forms of weight $(k_1, k_2)$, $k_1 \geq k_2 \geq 2$ ([16], [17]). The main purpose of this paper is to associate an Artin representation to $F$. We prove

**Theorem 1.1.** (Main Theorem) Let $F$ be as above, and assume (TR), (Gal), (Rat), and (Int). Then there exists the Artin representation $\rho_F : G_Q \rightarrow GSp_4(\mathbb{C})$ which is unramified outside primes dividing $N$ and symplectically odd, i.e. $\rho_F(c) \sim GSp_4(\mathbb{C}) \cong \text{diag}(1, -1, -1, 1)$ for the complex conjugation $c$ such that $\det(I_4 - \rho_F(\text{Frob}_p)T) = H_p(T)$ for all $p \nmid N$. This representation is irreducible if and only if $\pi_F$ is not an endoscopic representation.

**Remark 1.2.** As we will later (see Lemma 5.1), $F$ cannot be a CAP form.

As a corollary to our main theorem, we see that $\pi_p$ is tempered for all $p$ under our assumptions.

**Remark 1.3.** Let $F$ be a form in Theorem 1.1. Then there exists a real analytic Siegel modular form $F'$ of weight $(1,0)$ such that $\pi_{F'}$ and $\pi_F$ are nearly equivalent. A difference appears in the $L$-functions, namely, $L^N(s, \pi_{F'}) = L^N(s, F) = L^N(s - 1, F') = L^N(s, \pi_{F'})$. In fact, one can get $F$ from $F'$ by first multiplying $\text{det}(Y)$, and then by taking a differential operator, and vice versa. See Remark 4.3 and Section 6.3 for the details. However, only the form of weight $(2,1)$ gives rise to the Artin representation (Remark 6.2).
For the proof of the main theorem, we follow the method of [6], in which the main ingredients are

1. the integrality of Hecke eigenvalues (Hecke polynomials) for all but finitely many primes, (this is related to the fact that the infinity type is a limit of discrete series.)
2. the Rankin-Selberg L-function \( L(s, \pi_f \times \tilde{\pi_f}) \), where \( \pi_f \) is the cuspidal representation attached to a cusp form \( f \) of weight one,
3. the classification of all semisimple subgroup of \( GL_2(\mathbb{F}_\ell) \) for any odd prime \( \ell \),
4. the existence of mod \( \ell \) Galois representations attached to any elliptic modular form of weight one.

The essentially same technique can be applied to the case of Hilbert modular forms (see [40], [37] for the Langlands reciprocity for any Hilbert cusp form with parallel weight one and [42], [19], [11] for partial results of automorphy).

Contrary to the case \( GL_2/\mathbb{Q} \), we cannot use the algebro-geometric techniques for our form \( F \) since \( F \) is non-holomorphic. Therefore we do not have ingredients (1) and (4) in the current situation. With these reasons we make assumptions (Gal), (Rat), and (Int). We will give various examples which satisfy these conditions.

For (3) in \( GSp_4 \) case, we study the classification of certain semisimple subgroups of \( GSp_4(\mathbb{F}_\ell) \) for any odd prime \( \ell \) in Section 8. This part could be simplified in terms of the theory of finite groups for classical groups. But there are no suitable references in the literature. In the upcoming paper [26], we simplify the proof and generalize it to arbitrary semisimple subgroups of \( GL_n(\mathbb{F}_\ell) \).

For (2), we apply the result of Arthur [1] on the transfer from \( GSp_4 \) to \( GL_4 \) to obtain the automorphic representation \( \Pi \) of \( GL_4(\mathbb{A}) \) so that \( L(s, \pi_F) = L(s, \Pi) \). The result of Arthur depends on the twisted stabilization of the trace formula for \( GSp_4 \), which is not done at this moment. We emphasize that we only need the transfer from \( GSp_4 \) to \( GL_4 \). In the upcoming paper [26], we remove this assumption (TR). Let \( \tau \) be an automorphic representation of \( GL_6(\mathbb{A}) \) such that \( \tau_p \simeq \wedge^2 \Pi_p \) for all \( p \neq 2, 3 \) [22]. Then \( \tau = (\Pi_5 \otimes \varepsilon) \boxplus \varepsilon \), where \( \varepsilon \) is the central character of \( \pi_F \), and \( \Pi_5 \) is the automorphic representation of \( GL_5(\mathbb{A}) \) which is a weak transfer of \( \pi_F \) to \( GL_5 \) corresponding to the \( L \)-group homomorphism \( GSp_4(\mathbb{C}) \to GL_5(\mathbb{C}) \), given by the second fundamental weight [23]. Then we can use the Rankin-Selberg L-functions \( L(s, \Pi \times \tilde{\Pi}) \) and \( L(s, \Pi_5 \times \tilde{\Pi}_5) \).

In Section 10, we obtain the existence of the real analytic Siegel cusp form of weight \((2,1)\) attached to the symmetric cube lift of elliptic cusp form of weight 1. More precisely, let \( f \) be an
elliptic cusp form of weight 1 which is a Hecke eigenform. Let $\pi_f$ be the cuspidal representation of $GL_2/\mathbb{Q}$ attached to $f$. Then $\text{Sym}^3(\pi_f)$ be an automorphic representation of $GL_4/\mathbb{Q}$. \[24\] Then by the result of Jacquet, Piatetski-Shapiro, and Shalika (cf. \[25\]), there exists a generic cuspidal representation $\Pi$ of $GSp_4(\mathbb{A})$ whose transfer to $GL_4(\mathbb{A})$ is $\text{Sym}^3(\pi_f)$. We can show that $\Pi_\infty$ is equivalent to $\text{Ind}_{B}^{G}\chi(1, \text{sgn}, \text{sgn})$. Hence we can find a real analytic Siegel cusp form of weight $(2,1)$ with the eigenvalues $-\frac{5}{12}$ and 0 for the generators $\Delta_1$ and $\Delta_2$ such that $\pi_F \simeq \Pi$. This provides infinitely many examples of Siegel cusp forms of weight $(2,1)$ with integral Hecke polynomials. Note that this is an unconditional result. If $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$ is the Artin representation associated to $f$ by Deligne-Serre theorem \[6\], then $\text{Sym}^3(\rho)$ is the Artin representation associated to $F$. Finally we state the strong Artin conjecture related to the automorphy of Artin representations for $GSp_4$:

**Conjecture 1.4.** Let $\rho : G_{\mathbb{Q}} \rightarrow GSp_4(\mathbb{C})$ be a symplectically odd Artin representation whose image does not factor through, up to conjugacy in $GSp_4(\mathbb{C})$, the Levi factor of any parabolic subgroup of $GSp_4(\mathbb{C})$. Then there exists a real analytic Siegel cusp modular form $F$ of weight $(2,1)$ with the eigenvalues $-\frac{5}{12}$ and 0 for the generators $\Delta_1$ and $\Delta_2$ so that $\rho_F \sim \rho$.

As in $GL_2$ case, we consider $\bar{\rho} : G_{\mathbb{Q}} \rightarrow PGS_4(\mathbb{C})$, the composition of $\rho$ and the canonical projection. Then $PGSp_4(\mathbb{C}) \simeq SO_5(\mathbb{C})$ and the finite subgroups of $SO_5(\mathbb{C})$ have been classified (cf. \[32\]). One expects a case by case analysis for different finite subgroups. In fact, K. Martin \[32\] showed the strong Artin conjecture when $\text{Im}(\bar{\rho})$ is a solvable group, $E_{16} \rtimes C_5$, where $E_{16}$ is the elementary abelian group of order 16 and $C_5$ is the cyclic group of order 5. In Section \[11\] we show that K. Martin’s explicit examples gives rise to symplectically odd Artin representations. Hence we obtain examples of real analytic Siegel cusp forms of weight $(2,1)$ with integral Hecke polynomials, which do not come from $GL_2$ forms.

**Acknowledgments.** We would like to thank J. Arthur, K. Gunji, S. Kudla, S. Kuroki, C-P. Mok, T. Moriyama, T. Okazaki, R. Schmidt, and D. Vogan for helpful discussions. In particular, R. Schmidt \[43\] helped us to realize that holomorphic Siegel cusp forms never give rise to Artin representations.

### 2. Preliminaries on $GSp_4$

Let $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, and we realize the algebraic group $G := GSp_4$ over $\mathbb{Q}$ as the subgroup of $GL_4$ consisting of all $g$ such that $^t g J g = \nu(g) J$, for some $\nu(g) \in GL_1$. Let $\nu : G = GSp_4 \rightarrow GL_1$
be the similitude character defined by sending $g$ to $\nu(g)$. Let $S_{P_4} := \text{Ker}(\nu)$. Let $B$ be the Borel subgroup of $GSp_4$, $T$ the maximal torus, and $U$ the unipotent radical of $B$:

$$ T = \{ t = t(t_1, t_2, t_0) = \text{diag}(t_1, t_2, t_0 t_1^{-1}, t_0 t_2^{-1}) | t_0, t_1, t_2 \in GL_1 \}, $$

$$ U = \left\{ \begin{pmatrix} I_2 & A \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix} | A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, B = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, B' = \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \right\} $$

The simple roots are $\alpha(t(t_1, t_2, t_0)) = t_1 t_2^{-1}$ and $\beta(t(t_1, t_2, t_0)) = t_2^2 t_0^{-1}$. The coroots are $\alpha^\vee(x) = t(x, x^{-1}, 1)$ and $\beta^\vee(x) = t(1, x, 1)$. Let

$$ s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} $$

be the representatives of generators of the Weyl group $N_G(T)/T$ which are corresponding to $\alpha$ and $\beta$ respectively.

Note that $Z_G = \{ a I_4 : a \in GL_1 \}$ and $\nu(a I_4) = a^2$. For any place $p \leq \infty$ of $\mathbb{Q}$, any character of $\chi$ of $T(\mathbb{Q}_p)$ is given by $\chi = \chi(\chi_1, \chi_2, \sigma)$ for $\chi_i, \sigma$ with characters of $\mathbb{Q}_p^\times$ so that $\chi(t(t_1, t_2, t_0)) = \chi_1(t_1) \chi_2(t_2) \sigma(t_0)$. Note also that the dual group of $G$ is $\hat{G} = GSp_4(\mathbb{C})$.

Note the Weyl group action; $s_1 : t(t_1, t_2, t_0) \mapsto t(t_2, t_1, t_0)$ and $s_2 : t(t_1, t_2, t_0) \mapsto t(t_1, t_2^{-1} t_0, t_0)$.

Let $P = M_P N_P$ (resp. $Q = M_Q N_Q$) be the Siegel (resp. Klingen) parabolic subgroup of $GSp_4$ containing $B$, where

$$ M_P = \left\{ \begin{pmatrix} A & 0 \\ 0 & u^t A^{-1} \end{pmatrix} : A \in GL_2, u \in GL_1 \right\} \simeq GL_2 \times GL_1, \quad N_P = \left\{ \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix} : S = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\} $$

$$ M_Q = \left\{ \begin{pmatrix} a' \\ 1 \end{pmatrix} u a'^{-1} \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & d \end{pmatrix} \in SL_2 \right\}, $$

$$ N_Q = \left\{ \begin{pmatrix} I_2 & A \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix} | A = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, B' = \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \right\}. $$
3. Vector-valued real analytic Siegel modular forms

In this section we shall discuss the real analytic Siegel modular forms in various settings. Since there are no references in dealing with vector-valued real analytic Siegel modular forms, we will develop the definition and their basic properties by imitating Section 1 to 3 of [16]. We refer [2] for the adelic setting.

3.1. Classical real analytic Siegel modular forms. Let $H_2 = \{Z \in M_2(\mathbb{C}) \, | \, tZ = Z, \text{Im}(Z) > 0 \}$ be the Siegel upper half-plane. For a pair of non-negative integers $\underline{k} = (k_1, k_2)$, $k_1 \geq k_2$ (note that $k_2$ might be negative in real analytic case), we define the algebraic representation $\lambda_{\underline{k}}$ of $GL_2$ by

$$V_{\underline{k}} = \text{Sym}^{k_1-k_2} \text{St}_2 \otimes \det^{k_2} \text{St}_2,$$

where $\text{St}_2$ is the standard representation of dimension 2 with the basis $\{e_1, e_2\}$. More explicitly, if $R$ is any ring, then $V_{\underline{k}}(R) = \bigoplus_{i=0}^{k_1-k_2} Re_{1}^{k_1-k_2-i} \cdot e_2^{i}$ and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$, $\lambda_{\underline{k}}(g)$ acts on $V_{\underline{k}}(R)$ by $g \cdot e_{1}^{k_1-k_2-i} \cdot e_2^{i} := \det(g)^{k_2}(ae_1+be_2)^{k_1-k_2-i} \cdot (ce_1+de_2)^{i}$.

We identify $V_{\underline{k}}(R)$ (resp. $\lambda_{\underline{k}}(g)$) with $R^\oplus(k_1-k_2+1)$ (resp. the represent matrix of $\lambda_{\underline{k}}(g)$ with respect to the above basis). We have the action and the automorphy factor $J$ by

(3.1) \quad $\gamma Z = (AZ+B)(CZ+D)^{-1} \quad J(\gamma, Z) = CZ + D \in GL_2(\mathbb{C}),$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{R})$ and $Z \in H_2$.

For an integer $N \geq 1$, we define a principal congruence subgroup $\Gamma(N)$ to be the group consisting of the elements $g \in Sp_4(\mathbb{Z})$ such that $g \equiv 1 \mod N$. For a parabolic subgroup $H \in \{ B, P, Q \}$, let $\Gamma_H(N)$ be the group consisting of the elements $g \in Sp_4(\mathbb{Z})$ such that $g \mod N \in H(\mathbb{Z}/N\mathbb{Z})$. For a $V_{\underline{k}}(\mathbb{C})$-valued function $f$ on $H_2$, the action of $\gamma \in G(\mathbb{R})^+$ is defined by

(3.2) \quad $F(Z)[[\gamma]_{\underline{k}} := \lambda_{\underline{k}}(\nu(\gamma)J(\gamma, z)^{-1})F(\gamma Z).

The algebra of all $Sp_4(\mathbb{R})$-invariant differential operators on $H_2$ is isomorphic to $\mathbb{C}[\Delta_1, \Delta_2]$, the commutative polynomial ring of two variables (see [16]), where $\Delta_1$ is the degree 2 Casimir element, and $\Delta_2$ is the degree 4 element (see Section 5).
For an arithmetic subgroup \( \Gamma \) of \( \text{Sp}_4(\mathbb{Q}) \) and a finite character \( \chi : \Gamma \to \mathbb{C}^\times \), we say that a function \( F : \mathcal{H}_2 \to V_\mathbf{k}(\mathbb{C}) \) is a real analytic Siegel modular form of weight \((k_1, k_2)\) with the character \( \chi \) with respect to \( \Gamma \) if it satisfies

(i) \( F \) is a \( C^\infty \)-function,
(ii) \( F|\gamma\mathbf{k} = \chi(\gamma)f \) for any \( \gamma \in \Gamma \),
(iii) \( F \) is a common eigenform for \( \Delta_1 \) and \( \Delta_2 \), namely, \( \Delta_i F = c_i F \) for some constants \( c_i, i = 1, 2 \).
(iv) \( F \) satisfies the growth condition: there exist a positive real number \( C \) and \( n \in \mathbb{N} \) such that for any linear functional \( l : V_\mathbf{k}(\mathbb{C}) \to \mathbb{C} \),

\[
|l(F(Z))| \leq C(\sup\{\text{tr}(\text{Im}Z), \text{tr}(\text{Im}Z)^{-1}\})^n.
\]

We denote by \( M_\mathbf{k}(\Gamma, \chi, c_1, c_2) \) the space of such forms. By Harish-Chandra (see Theorem 1.7 of [2]), this space is finite dimensional.

For each parabolic subgroup \( R \) of \( \text{Sp}_4 \) defined over \( \mathbb{Q} \), we denote by \( N_R \) the unipotent radical of \( R \). Then we say \( F \in M_\mathbf{k}(\Gamma, \chi, c_1, c_2) \) is a cusp form if

\[
\int_{(N_{R}(\mathbb{Q})/\Gamma)\setminus N_{R}(\mathbb{R})} F(nZ)dn = 0, \quad \text{for any parabolic subgroup } R \text{ defined over } \mathbb{Q}.
\]

We denote by \( S_\mathbf{k}(\Gamma, \chi, c_1, c_2) \) the space of such cusp forms inside \( M_\mathbf{k}(\Gamma, \chi, c_1, c_2) \).

Similar to the holomorphic case (cf. [10]), we shall define the Hecke operators on \( M_\mathbf{k}(\Gamma(N), c_1, c_2) \).

For any positive integer \( n \) coprime to \( N \), let

\[
\Delta_n(N) := \left\{ g \in M_4(\mathbb{Z}) \cap G\text{Sp}_4(\mathbb{Q}) \mid g \equiv \begin{pmatrix} I_2 & 0 \\ 0 & \nu(g)I_2 \end{pmatrix} \mod N, \nu(g)^{\pm 1} \in \mathbb{Z}[\frac{1}{n}] \right\}.
\]

For \( m \in \Delta_n(N) \), we introduce the action of the Hecke operators on \( M_\mathbf{k}(\Gamma(N), c_1, c_2) \) by

\[
T_m F(Z) := \nu(m)^{k_1 + k_2 - 3 \over 2} \sum_{\alpha \in \Gamma(N) \setminus \Gamma(N)m\Gamma(N)} F(Z)[(\nu(m)^{-1\over 2} \alpha)\mathbf{k}] 
\]

and for any positive integer \( n \), put

\[
T(n) := \sum_{m \in \Gamma(N) \setminus \Delta_n(N)} T_m
\]

We also consider the same actions on \( S_\mathbf{k}(\Gamma(N), c_1, c_2) \). For \( t_1 = \text{diag}(1, 1, p, p) \), \( t_2 = \text{diag}(1, p, p, p^2) \), put \( T_{i,p} := T_{t_i} \) \( i = 1, 2 \) and fix \( \tilde{S}_{p,1}, \tilde{S}_{p,2} \in \text{Sp}_4(\mathbb{Z}) \) so that \( \tilde{S}_{p,1} \equiv \text{diag}(p^{-1}, 1, 1, p) \mod N \) and \( \tilde{S}_{p,2} \equiv \text{diag}(p^{-1}, p^{-1}, p, p) \mod N \). Then we see that

\[
T(p) = T_{1,p}, \quad T_{1,p}^2 - T(p^2) - p^2\tilde{S}_{p,2} = p\{T_{2,p} + (1 + p^2)\tilde{S}_{p,1}\}.
\]
Since the group $\Gamma(N)$ contains the subgroup consists of $\begin{pmatrix} I_2 & NS \\ 0 & I_2 \end{pmatrix}$, $S = tS \in \mathbb{M}_2(\mathbb{Z})$, for a given $F \in \mathbb{M}_k(\Gamma(N), c_1, c_2)$, we have the Fourier expansion

$$F(Z) = \sum_{T \in \mathbb{P}(\mathbb{Z})_{\geq 0}} A_F(T, Y) e^{\frac{2\pi i}{N} \text{tr}(TX)}, \quad Z = X + iY \in \mathcal{H}_2,$$

where $\mathbb{P}(\mathbb{Z})_{\geq 0}$ is the subset of $\mathbb{M}_2(\mathbb{Q})$ consisting of all symmetric matrices $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$, which are semi-positive.

### 3.2. Hecke operators.

The finite group $Sp_4(\mathbb{Z}/N\mathbb{Z}) \simeq Sp_4(\mathbb{Z})/\Gamma(N)$ acts on $\mathbb{M}_k(\Gamma(N), c_1, c_2)$ by $F \mapsto F|\tilde{\gamma}|_k$ if we fix a lift $\tilde{\gamma}$ of $\gamma \in Sp_4(\mathbb{Z}/N\mathbb{Z})$. We denote this action by the same notation $F|\tilde{\gamma}|_k$. This action does not depend on the choice of lifts of $\gamma$. The diagonal subgroup of $Sp_4(\mathbb{Z}/N\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ by sending $S_{a,b} := \text{diag}(a^{-1}, b^{-1}, a, b)$ to $(a, b)$ and it also acts on $\mathbb{M}_k(\Gamma(N))$ factor through the action of $Sp_4(\mathbb{Z}/N\mathbb{Z})$. Then we have the character decomposition

$$\mathbb{M}_k(\Gamma(N), c_1, c_2) = \bigoplus_{\chi_1, \chi_2: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times} \mathbb{M}_k(\Gamma(N), c_1, c_2, \chi_1, \chi_2),$$

where $\mathbb{M}_k(\Gamma(N), c_1, c_2, \chi_1, \chi_2) = \{ F \in \mathbb{M}_k(\Gamma(N), c_1, c_2) \mid F|[S_{a,1}]_k = \chi_1(a)F \text{ and } F|[S_{a,a}]_k = \chi_2(a)F \}$. It is easy to see that the Hecke operators preserve $\mathbb{M}_k(\Gamma(N), c_1, c_2, \chi_1, \chi_2)$ (cf. [11] for holomorphic case). We should remark that if such a $F$ exists, the weight $(k_1, k_2)$ has to satisfy the parity condition

$$\chi_2(-1) = (-1)^{k_1+k_2}.$$

Throughout this paper we assume this parity condition on $F$.

Let

$$F(Z) = \sum_{T \in \mathbb{P}(\mathbb{Z})_{\geq 0}} A_F(T, Y) e^{\frac{2\pi i}{N} \text{tr}(TX)} \in \mathbb{M}_k(\Gamma(N), c_1, c_2, \chi_1, \chi_2), \quad Z = X + iY$$

be an eigenform for all $T(p^i), \ p \nmid N, \ i \in \mathbb{N}$ with eigenvalues $\lambda(p^i)$, i.e.,

$$T(p^i)F = \lambda(p^i)F.$$

We next study the relation between $\lambda(p^i)$ and $A_F(T, Y)$. For a non-negative integer $\beta$, let $R(p^\beta)$ be the set of matrices $\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ of $\Gamma^1(N) := \{ g \in SL_2(\mathbb{Z}) \mid g \equiv I_2 \text{ mod } N \}$ whose first rows
(u_1, u_2) run over a complete set of representatives modulo the equivalence relation:

\[(u_1, u_2) \sim (u_1', u_2') \iff \exists [u_1 : u_2] = [u_1' : u_2'] \in \mathbb{P}^1(\mathbb{Z}/p^\beta \mathbb{Z}).\]

Let \(T(p^j)F = \sum_{T \in P(\mathbb{Z})_{\geq 0}} A_F(p^j; T, Y) e^{\frac{\pi i}{4} \text{tr}(TX)}.\) For simplicity we write \(\rho_j = \text{Sym}^j S'\) for \(j \geq 0\) and \(UTU = \left(\begin{array}{cc} a_U & b_U \\ b_U & c_U \end{array} \right)\) for \(T \in P(\mathbb{Z})_{\geq 0}\) and \(U \in R(p^\beta).\) Put \(D_\beta = \left(\begin{array}{cc} 1 & 0 \\ 0 & p^\beta \end{array} \right).\) By Proposition 3.1 of [10], and the calculations done at p.439-440 of [10], and Section 3.1 of [10], we have

\[
\lambda(p^j)A_F(T, Y) = A_F(p^j; T, Y) := \sum_{\alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0} \chi_1(p^\beta) \chi_2(p^\gamma) p^{\beta(k_2 - 2) + \gamma(k_1 + k_2 - 3)} \times \sum_{U \in R(p^\beta) \atop a_U \equiv 0 \mod p^{\beta + \gamma}, \ b_U \equiv c_U \equiv 0 \mod p^\gamma} \rho_{k_1 - k_2} \left(\begin{array}{cc} 1 & 0 \\ 0 & p^\beta \end{array} \right) U^{-1} A_F \left( p^\alpha \left( \frac{a_U p^{-\beta - \gamma}}{b_U p^{-\gamma}} \right) \frac{b_U p^{-\gamma}}{c_U p^{\beta - \gamma}} \right) , \n
p^{\beta - 2\alpha} D_\beta^{-1} U^{-1} Y U^{-1} D_\beta^{-1} .
\]

3.3. Adelic forms. Let \(\mathbb{A}\) be the adele ring of \(\mathbb{Q}\) and \(\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q}\) the finite adele of \(\mathbb{Q}\).

For a positive integer \(N\) and a parabolic subgroup \(H \in \{B, P, Q\}\), let \(K_H(N)\) be the group consisting of the elements \(g \in GSp_4(\mathbb{Z})\) such that \((g \mod N) \in H(\mathbb{Z}/N\mathbb{Z}).\) It is easy to see that \(K_H(N) \cap Sp_4(\mathbb{Q}) = \Gamma_H(N)\) and \(\nu(K_H(N)) = \hat{\mathbb{Z}}.\)

Let \(K(N)\) be the group consisting of the elements \(g \in GSp_4(\mathbb{Z})\) such that \(g \equiv I_4 \mod N.\) Then we see that \(\Gamma(N) = Sp_4(\mathbb{Q}) \cap K(N)\) and \(\nu(K(N)) = 1 + N\hat{\mathbb{Z}}.\) Then it follows from the strong approximation theorem for \(Sp_4\) that

\[
(3.8) \quad G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ K_H(N) = G(\mathbb{Q})Z_G(\mathbb{R})^+ Sp_4(\mathbb{R})K_H(N)
\]

and

\[
(3.9) \quad G(\mathbb{A}) = \prod_{1 \leq a < N \atop (a, N) = 1} G(\mathbb{Q})G(\mathbb{R})^+ d_a K(N) = \prod_{1 \leq a < N \atop (a, N) = 1} G(\mathbb{Q})Z_G(\mathbb{R})^+ Sp_4(\mathbb{R})d_a K(N)
\]

where \(d_a\) is the diagonal matrix such that \((d_a)_p = \text{diag}(a, a, 1, 1)\) if \(p|N\), \((d_a)_p = I_4\) otherwise.

Let \(I := \sqrt{-1}I_2 \in H_2\) and \(U(2) = \text{Stab}_{Sp_4(\mathbb{R})}(I).\) Let \(\mathfrak{g}_{0, \mathbb{C}}\) be the complexification of \(\mathfrak{g}_0 = \text{Lie}Sp_4(\mathbb{R}).\) We denote by \(Z\) the center of universal enveloping algebra of \(\mathfrak{g}_{0, \mathbb{C}}.\) Under the natural map \(Sp_4(\mathbb{R}) \to H_2, g \mapsto g(I), Z \simeq \mathbb{C}[\Delta_1, \Delta_2]\). Choose \(\tilde{\Delta}_i \in Z\) as in [4.1] in Section 4 and fix \(\Delta_i\) which corresponds to \(\tilde{\Delta}_i\) under this map for \(i = 1, 2.\) For a function \(\phi : GSp_4(\mathbb{Q}) \backslash GSp_4(\mathbb{A}) \to \)
$V_k(\mathbb{C})$ and $D \in \mathfrak{g}_0$, we first define

$$D\phi(g) := \lim_{t \to 0} \frac{d}{dt}\phi(g \exp(tD))$$

and extend this action linearly on $\mathfrak{g}_0, C$. For any open compact subgroup $U$ of $GSp_4(\hat{\mathbb{Z}})$ and complex numbers $c_1, c_2$, we let $A_k(U, c_1, c_2)^\circ$ denote the subspace of functions $\phi : GSp_4(\mathbb{Q}) \backslash GSp_4(\mathbb{A}) \rightarrow V_k(\mathbb{C})$ such that

1. $\phi(guu_\infty) = \lambda_k(J(u_\infty, I)^{-1})\phi(g)$ for all $g \in G(A)$, $u \in U$, and $u_\infty \in U(2)$.
2. For $h \in G(A)$, the function
   $$\phi_h : H_2 \rightarrow V_k(\mathbb{C}), \phi_h(Z) = \phi_h(g_\infty I) := \lambda_k(J(g_\infty, I))\phi(hg_\infty)$$
   is a $C^\infty$ function where $Z = g_\infty I$, $g_\infty \in Sp_4(\mathbb{R})$ (note that this definition is independent of the choice of $g_\infty$),
3. $\hat{\Delta}_i\phi = c_i\phi$ for $i = 1, 2$,
4. for $g \in G(A)$, $\int_{N_*(\mathbb{Q}) \backslash N_*(A)} \phi(ng)dn = 0$ for any parabolic $\mathbb{Q}$-subgroup $\ast$ and $dn$ is the Haar measure on $N_*(\mathbb{Q}) \backslash N_*(A)$.

We define similarly $A_k(U, c_1, c_2)$ by omitting the last condition (4).

Let $\Gamma(N)_a := Sp_4(\mathbb{Q}) \cap d_a^{-1}K(N)d_a$. Note that $\Gamma(N)_a = \Gamma(N)$ for each $a$. Then we have the isomorphism

$$A_k(K(N), c_1, c_2) \sim \bigoplus_{1 \leq a < N \atop (a, N) = 1} M_k(\Gamma(N)_a, c_1, c_2), \quad \phi \mapsto (\phi_{da}).$$

The inverse of this isomorphism is given as follows: Let $F = (F_a)$ be an element of RHS which is a system of real analytic Siegel modular forms such that $F_a \in M_k(\Gamma(N)_a, c_1, c_2)$ for each $a$. For each $g \in G(A)$, there exists a unique $d_a$ such that $g = rz_\infty d_ag_\infty k$ with $r \in G(\mathbb{Q})$, $z_\infty \in Z_G(\mathbb{R})^+$, $g_\infty \in Sp_4(\mathbb{R})$, and $k \in K(N)$. Then we define the function

$$\phi_F(g) = \lambda_k(J(g_\infty, I))^{-1}F_a(g_\infty I).$$

This gives rise to the inverse of the above isomorphism. We also have the isomorphism

$$A_k(K(N), c_1, c_2)^\circ \sim \bigoplus_{1 \leq a < N \atop (a, N) = 1} S_k(\Gamma(N)_a, c_1, c_2)$$

as well (cf. [2] for checking the cuspidality).
Now we restrict the isomorphism \((3.10)\) to a subspace, using the character decomposition \((3.6)\). Given two Dirichlet characters \(\chi_1, \chi_2 \colon (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times\), \(i = 1, 2\), associate the characters
\[\chi'_i : \mathbb{A}_f^\times \rightarrow \mathbb{C}^\times\]
by the natural map \(\mathbb{A}_f^\times \rightarrow \hat{\mathbb{A}}_f^\times /\mathbb{Q} > 0 = \hat{\mathbb{Z}}^\times \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times\).

Define \(\tilde{\chi} : T(\mathbb{A}_f) \rightarrow \mathbb{C}^\times\) by
\[\tilde{\chi}(\text{diag}(1, 1, -1, -1)) = \chi'_1(d^{-1}c)\tilde{\chi}'_2(d).
\]
Choose \(F = (F_a)\) from RHS of \((3.10)\) which satisfies \(F|[S_{z, z}]_{\mathbb{A}} = (F_a|[S_{z, z}]_{\mathbb{A}}) = (\chi_2(z)F) = \chi_2(1)\). If we write \(g \in G(\mathbb{A})\) as \(g = rz_\infty d_\infty g_\infty k \in G(\mathbb{A})\) and take \(z_f \in T(\mathbb{A}_f)\), then we define the automorphic function attached to \(F\) by
\[\phi_F(gz_f) = \lambda_{\mathbb{A}}(J(g, 1))^{-1}F_a(g_\infty)\tilde{\chi}(z_f).
\]
Then this gives rise to the isomorphism of the subspaces
\[A_k(K(N), c_1, c_2, \tilde{\chi}) \sim \oplus_{1 \leq a < N} M_k(\Gamma(N)a, c_1, c_2, \chi_1, \chi_2).
\]

We now compute the actions of \(\text{diag}(1, 1, -1, -1) \in GSp_4(\mathbb{R})\) on \(\phi_F \in A_k(K(N), c_1, c_2, \tilde{\chi})\) as follows: Let \(h = (\text{diag}(1, 1, -1, -1), I_{\mathbb{A}_f}) = (\text{diag}(1, 1, -1, -1)(I_{GSp_4(\mathbb{R})}, (\text{diag}(1, 1, -1, -1)))_{\mathbb{P}}) \in G(\mathbb{Q})(GSp_4(\mathbb{R}) \times GSp_4(\mathbb{A}_f)), \)
where \(I_{\mathbb{A}_f}\) (resp. \(I_{GSp_4(\mathbb{R})}\)) is the identity element. Then we have
\[\phi_F(gh) = \phi_F((r \cdot \text{diag}(1, 1, -1, -1))z_\infty d_\infty g_\infty k \cdot \text{diag}(1, 1, -1, -1))p) = \chi_2(-1)\phi_F(g),
\]
since we have assumed \(\chi_2(-1) = (-1)^{k_1 + k_2}\). Hence in the case of \((k_1, k_2) = (2, 1)\), we have
\[\phi_F(\text{diag}(1, 1, -1, -1), I_{\mathbb{A}_f}) = -1
\]
by the parity condition \((3.7)\).

**Remark 3.1.** Note that \(M_k(\Gamma(N), c_1, c_2)\) is embedded diagonally into \(\oplus_{1 \leq a < N} M_k(\Gamma(N)a, c_1, c_2)\).

So given a cusp form \(F \in M_k(\Gamma(N), c_1, c_2)\), we obtain \(\phi_F \in A_k(K(N), c_1, c_2)\) which under the isomorphism \((3.10)\), corresponds to \((F, ..., F)\), and \(\phi_F\) gives rise to a cuspidal representation \(\pi_F\). Conversely, given a cuspidal representation \(\pi\) of \(GSp_4/\mathbb{Q}\), there exists \(N > 0\) and \(\phi \in A_k(K(N), c_1, c_2)\) which spans \(\pi\). Under the isomorphism \((3.10)\), \(\phi\) corresponds to \((F_a)_{1 \leq a < N, (a, N) = 1} \in M_k(\Gamma(N)a, c_1, c_2)\).

For any \(a\), let \(\pi_{F_a}\) be the cuspidal representation associated to \(F_a\). Then \(\pi\) and \(\pi_{F_a}\) have the same Hecke eigenvalues for \(p \nmid N\), and hence in the same \(L\)-packet.
We now study the Hecke operators on $A_{k}(K(N),c_{1},c_{2})$ and its relation to classical Hecke operators. Let $\phi$ be an element of $A_{k}(K(N),c_{1},c_{2})$ and $F = (F_{\alpha})_{\alpha}$ be the corresponding element of RHS via the above isomorphism (3.10). For any prime $p \nmid N$ and $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_{p})$, we define the Hecke action with respect to $\alpha$

$$
\bar{T}_{\alpha}\phi(g) := \int_{G(\mathbb{A}_{f})} ([K(N)_{p}\alpha K(N)p] \otimes 1_{K(N)p}) \phi(gg_{f})dg_{f}
$$

where $dg_{f}$ is the Haar measure on $G(\mathbb{A}_{f})$ so that $\text{vol}(K) = 1$. Here $K(N)_{p}$ is the $p$-component of $K(N)$ and $K(N)^{p}$ is the subgroup of $K(N)$ consists of trivial $p$-component.

Then by using (3.9), we can easily see that

$$(3.12) \quad T_{\alpha}F(Z) = \nu(\alpha)^{-\frac{k_{1}+k_{2}}{2}-3} \bar{T}_{\alpha^{-1}}\phi(g)$$

where $g = rz_{\infty}g_{\infty}g_{\infty}k$ as above and $Z = g_{\infty}I$ (cf. Section 8 of [33]). From this relation, up to the factor of $\nu(\alpha)^{-\frac{k_{1}+k_{2}}{2}-3}$, the isomorphism (3.10) preserves Hecke eigenforms in both sides. We turn to explain the relation to classical Siegel eigenforms. Let $F \in S_{k}(\Gamma(N),c_{1},c_{2})$ be a Siegel cusp form which is a Hecke eigenform. Then it is easy to see that $(F)_{\alpha}$ is an eigenform of $\bigoplus_{1 \leq \alpha < N, (\alpha,N)=1} S_{k}(\Gamma(N)_{\alpha},c_{1},c_{2})$. Hence we have the Hecke eigenform of $A_{k}(K(N),c_{1},c_{2})$ corresponding to $F$. The above things are easy to generalize to all open compact subgroup $U \subset G(\hat{\mathbb{A}})$. We omit the details.

The group $G(\mathbb{A})$ acts on $\lim_{\rightarrow} A_{k}(U,c_{1},c_{2})$ (also on $\lim_{\rightarrow} A_{k}(U,c_{1},c_{2})^{0}$) by right translation:

$$(h \cdot \phi)(g) := \phi(gh), \text{ for } g,h \in G(\mathbb{A}).$$

One would like to have scalar-valued functions lying in the usual $L^{2}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ space. Let $l : V_{k}(\mathbb{C}) \longrightarrow \mathbb{C}$ be any linear functional. Define $\tilde{\phi}(g) = l(\phi(g))$. Since we consider all right translates of $\tilde{\phi}$, the choice of $l$ is irrelevant.

For an open compact subgroup $U \subset G(\hat{\mathbb{A}})$, we say $U$ is of level $N$ if $N$ is the minimum positive integer so that $U$ contains $K(N)$. For such $U$ of level $N$ and $\phi \in A_{k}(U,c_{1},c_{2})$ which is an eigenform for all $T_{\alpha}$, $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_{p})$ and $p \not| N$, we denote by $\pi_{\phi}$, the irreducible direct summand of the representation of $G(\mathbb{A})$ generated by $g \cdot \tilde{\phi}$ for $g \in G(\mathbb{A})$. Then $\pi_{\phi}$ is an automorphic representation in the sense of [2]. Note that if the multiplicity one holds, $\pi_{\phi}$ is the irreducible representation generated by $g \cdot \tilde{\phi}$. Further if $\phi \in A_{k}(U,c_{1},c_{2})^{0}$, then we see that $\pi_{\phi}$ is a cuspidal automorphic representation.
Finally we give a remark on some compatibility related to the compact subgroups $K_*(N), \ast \in \{B, P, Q\}$. The obvious inclusions $M_k(\Gamma_*(N), \bar{c}_1, \bar{c}_2) \subset M_k(\Gamma_B(N), \bar{c}_1, \bar{c}_2) \subset M_k(\Gamma(N), \bar{c}_1, \bar{c}_2), \ast \in \{P, Q\}$ preserve the Hecke actions outside $N$. By (3.8) one has $A_k(K_*(N), \bar{c}_1, \bar{c}_2) \simeq M_k(\Gamma(N), \bar{c}_1, \bar{c}_2)$. Then we have the following commutative diagram which preserves the Hecke actions outside $N$:

$$
\begin{array}{ccc}
A_k(K_*(N), \bar{c}_1, \bar{c}_2) & \longrightarrow & M_k(\Gamma_*(N), \bar{c}_1, \bar{c}_2) \\
\downarrow & & \downarrow \\
A_k(K(N), \bar{c}_1, \bar{c}_2) & \longrightarrow & \bigoplus_{1 \leq a < N \atop (a,N)=1} M_k(\Gamma(N)_a, \bar{c}_1, \bar{c}_2).
\end{array}
$$

Here the left vertical arrow is the natural inclusion and the right vertical arrow is the diagonal embedding (recall that $\Gamma(N)_a = \Gamma(N)$).

### 3.4. Conjectural existence of the rationality.

In this section, let $k = (2, 1)$, $c_1 = -\frac{5}{12}$, $c_2 = 0$, and discuss a conjecture on the existence of some rational structure on $S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$.

Let $T^\text{uni}_Q$ be the Hecke algebra over $\mathbb{Q}$ which is generated by $T_{1,p}, T_{2,p}, \bar{S}_{p,1}$, and $\bar{S}_{p,p}$ for $p | N$. We denote by $T_Q$ its image in $\text{End}_\mathbb{C}(S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0))$. For any $\mathbb{Q}$-algebra $R$, put $T_R = T_Q \otimes_\mathbb{Q} R$.

Let $F_1, \ldots, F_r$ be an orthonormal basis of $S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$ which consists of Hecke eigenforms.

For $T \in T_Q$ and $F_i$ for each $i = 1, \ldots, r$, denote by $a_T(F_i)$, the eigenvalue of $T$ for $F_i$. For $F = \sum_{i=1}^r x_i F_i$, $x_i \in \mathbb{C}$, we define the map

$$
\psi : S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0) \longrightarrow \text{Hom}_{\text{alg}}(T_Q, \mathbb{C}), \quad F \mapsto [T \mapsto \sum_{i=1}^r x_i a_T(F_i)]
$$

which is depending on the choice of the basis $F_1, \ldots, F_r$.

We make the rationality assumption for $F$:

$$
\langle TF \mid T \in T_Q \rangle_\mathbb{C} = \langle TF \mid T \in T_Q \rangle_\mathbb{C} \cap \mathbb{C} \otimes_\mathbb{Q} \psi^{-1}(\text{Hom}_{\text{alg}}(T_Q, \mathbb{Q})).
$$

We denote by $T_{Q,F}$ the image of $T_Q$ in $\text{End}_\mathbb{C}(\langle TF \mid T \in T_Q \rangle_\mathbb{C})$.

### Proposition 3.2.

Let $F \in S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$. Assume (Rat) for $F$. Then the Hecke field $Q_F$ is a finite extension over $\mathbb{Q}$. Furthermore, for any $\tau : Q_F \hookrightarrow \mathbb{C}$, there exists $^\tau F \in S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$ such that $T^\tau F = \tau(a_T(F))^\tau F$ for any $T \in T_Q$. 

Proof. By (Rat), there exists a basis $G_1, \ldots, G_r$ of $\langle TF \mid T \in T_\mathbb{Q} \rangle_\mathbb{C}$ which gives a rational representation $T_{\mathbb{Q}, F} \hookrightarrow \text{End}_\mathbb{Q}(\langle G_1, \ldots, G_r \rangle_\mathbb{Q}) = \text{M}_r(\mathbb{Q})$. This means that $\mathbb{Q}_F$ is a finite extension of $\mathbb{Q}$. Since any Hecke eigenform $F$ in $\langle TF \mid T \in T_\mathbb{Q} \rangle_\mathbb{C}$ can be written as $F = \sum_{i=1}^r x_i G_i$ where $x_i$ is an element of a conjugate field of $\mathbb{Q}_F$ in $\mathbb{C}$. Hence we may set $\tau F = \sum_{i=1}^r \tau(x_i) G_i$. \hfill \Box

Remark 3.3. The rationality assumption holds for any vector-valued holomorphic Siegel cusp forms of arbitrary weight (cf. Lemma 2.1 of [34]). Contrary to the holomorphic case, there is no known general result for (Rat) in our situation due to the lack of algebraic geometry. In the case $U(2,1)$, H. Carayol [3, 4] studied the Griffith-Schmid varieties towards the rationality of cusp forms with the infinity type similar to ours. (See also (cf. [20]) for the case $Sp_4$.) We hope that the Griffith-Schmid varieties can be used to prove the rationality for the real analytic Siegel cusp forms of weight $(2,1)$.

4. Infinity type of the associated automorphic representation of $GSp_4$

Let $F$ be a Hecke eigenform in $S_{(2,1)}(\Gamma(N), -\frac{5}{12}, 0)$ with the associated cuspidal representation $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ of $GSp_4(\mathbb{A})$. Let $\phi_F$ be the function on the adele group $GSp_4(\mathbb{A})$ attached to $F$ as in section 3.3, and let $\tilde{\phi}_F = l(\phi_F)$ for any linear functional $l : V_\mathbb{A}(\mathbb{C}) \to \mathbb{C}$. All the right translates of $\tilde{\phi}_F$ span $\pi_F$. Then $\tilde{\phi}_F = v_\infty \otimes \otimes'_p v_p$ with $v_p \in \pi_p$ and $v_\infty \in \pi_\infty$, and $v_\infty$ inherits the analytic properties from $\tilde{\phi}_F$. For simplicity let us put $G = GSp_4(\mathbb{R})$ and $B = B(\mathbb{R})$ only in this section.

In this section, we determine $\pi_\infty$. We will use the notations from [34]. It is easy to modify the result of [34] for $G$. For simplicity, let $K = K_\infty := \text{Stab}_G(\sqrt{-1}I_2)$. Its identity component $K_0$ has index 2 in $K$, and it is a maximal compact subgroup of $Sp_4(\mathbb{R})$. It is the stabilizer of $I = \sqrt{-1}I_2 \in H_2$ under the action [34], and $u : K_0 \cong U(2)$ via $u : k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto u(k) = A + \sqrt{-1}B$.

We review principal series representations and their $K$-types from [34]: Let $\text{Ind}_B^G \chi$ be the principal series representation which is the space $V$ of all $C^\infty$-functions $f : GSp_4(\mathbb{R}) \to \mathbb{C}$ satisfying

$$f(tug) = \chi(t)|t_0|^{-\frac{3}{2}}|t_1|^2|t_2|f(g), \quad t = \text{diag}(t_1, t_2, t_0t_1^{-1}, t_0t_2^{-1}).$$

Here $B = TU$, $t \in T$, $u \in U$, and $\chi(t) = \prod_{i=0}^2 |e_i(\frac{t_i}{|t_i|})t_i|^{s_i}$ with $e_i = 1$ or $s_i$ and $s = (s_0, s_1, s_2) \in \mathbb{C}^3$. We write $\chi = \chi(e_1|s_1, e_2|s_2, e_0|s_0)$. Note that the infinity component of the
implies that $\epsilon = \epsilon_1 \epsilon_2^2 \epsilon_{s_1} s_{s_2} = \epsilon_1 \epsilon_2 |s_{s_1} + s_{s_2} + 2s_0|$. Since $\epsilon$ is trivial on $Z_G(\mathbb{R})^+$ (It implies that $\epsilon$ is unitary and of finite order), $\epsilon(a) = |a| s_{s_1} s_{s_2} + 2s_0 = 1$ for $aI_4 \in Z_G(\mathbb{R})^+$. Hence $s_1 + s_2 + 2s_0 = 0$. From (3.11), we see that $\epsilon_0 = \text{sgn}$.

If $(\rho, W)$ is a representation of $K$, and $v \in W$ has weight $(l, l')$, then $\rho(\text{diag}(1, 1, -1, -1))v$ has weight $(-l', -l)$. So if we let $\tau_{l, l'}$ be the representation of $K_0$ with dominant weight $(l, l')$, the weight structure of an irreducible representation of $K$ combines that of $\tau_{l, l'}$ and $\tau_{-l', -l}$ for some pair $(l, l')$.

Now let $V_K$ be the subspace of $K$-finite vectors in the representation space $V$. Then

$$V_K = \bigoplus_{\lambda} m(\lambda)\tilde{\tau}_\lambda,$$

where $m(\lambda)$ is the multiplicity and $\tilde{\tau}_\lambda = \begin{cases} \tau_{l, l'} \oplus \tau_{-l', -l}, & \text{if } \lambda = (l, l') \text{, } l > 0, \ l' \neq -l, \\ \tau_{l, -l}, & \text{otherwise}. \end{cases}$

Since $\phi_F$ has weight $(2, 1)$, $\tau_{2, 1}$ occurs in $V_K$. Hence by [34], Proposition 3.2, $\epsilon_1 \epsilon_2 = \text{sgn}$. Without loss of generality, we can assume that $\epsilon_1 = 1$ and $\epsilon_2 = \text{sgn}$.

Let $H_1 = \text{diag}(1, 0, -1, 0), \ H_2 = \text{diag}(0, 1, 0, -1)$; $E_{e_1 - e_2}$ is the matrix with 1 at $(1, 2)$-entry, $-1$ at $(4, 3)$-entry, and zero everywhere else; $E_{e_1 + e_2}$ is the matrix with 1 at $(1, 4)$-entry and $(2, 3)$-entry, and zero everywhere else; $E_{2e_1}$ is the matrix with 1 at $(1, 3)$-entry, and zero everywhere else; $E_{2e_2}$ is the matrix with 1 at $(2, 4)$-entry, and zero everywhere else. Let $E_{-*} := ^tE_*$ for $* \in \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}$. Let

$$M = \begin{pmatrix} H_1 & E_{e_1 - e_2} & 2E_{2e_1} & E_{e_1 + e_2} \\ E_{e_1 + e_2} & H_2 & E_{e_1 + e_2} & 2E_{2e_2} \\ 2E_{-e_1 - e_2} & E_{e_1 - e_2} & -H_1 & -E_{-e_1 + e_2} \\ E_{-e_1 - e_2} & 2E_{-2e_2} & -E_{e_1 - e_2} & -H_2 \end{pmatrix}.$$ 

Then

$$\tilde{\Delta}_1 = \frac{1}{12} \text{trace}(M^2) / 2 \quad \text{and} \quad \tilde{\Delta}_2 = \text{det}(M)$$

give two generators of the center of the universal enveloping algebra. Here $\tilde{\Delta}_1$ is the usual Casimir element as in [23].

Let $v_\infty$ be the highest weight vector in the $K$-type $(2, 1)$ in $\pi_\infty$. Then by [34], page 77,

$$\tilde{\Delta}_1 v_\infty = \frac{s_1^2 + s_2^2 - 5}{12} v_\infty, \quad \tilde{\Delta}_2 v_\infty = s_1 s_2 v_\infty.$$
Hence in order that $\Delta_1 F = -\frac{5}{12} F$ and $\Delta_2 F = 0$, one should have $s_1 = s_2 = 0$. Therefore, $\pi_\infty$ is a subquotient of the induced representation $\text{Ind}_B^G \chi$, where $\chi = \chi(1, \text{sgn}, \text{sgn})$. Now under the Weyl group action, for $\chi = \chi(1, \text{sgn}, \text{sgn})$,

$$\{w \chi | w \in W\} = \{\chi(1, \text{sgn}, 1), \chi(\text{sgn}, 1, 1), \chi(1, \text{sgn}, \text{sgn}), \chi(\text{sgn}, 1, \text{sgn})\},$$

and $\text{Ind}_B^G \chi$ and $\text{Ind}_B^G w \chi$ are equivalent. In particular, $\text{Ind}_B^G \chi(1, \text{sgn}, 1)$ and $\text{Ind}_B^G \chi(1, \text{sgn}, \text{sgn})$ are equivalent.

**Lemma 4.1.** The Knapp-Stein $R$-group of $\text{Ind}_B^G \chi(1, \text{sgn}, \text{sgn})$ is trivial. Hence it is irreducible, tempered and generic.

**Proof.** For the definition of the $R$-group, see [29]. If $\chi = \chi(1, \text{sgn}, \text{sgn})$, we can see easily that $W_\chi = \{1, s_1 s_2 s_1\}$. Hence the $R$-group is trivial. \hfill $\square$

Hence by the above lemma, $\pi_\infty \simeq \text{Ind}_B^G \chi(1, \text{sgn}, \text{sgn})$. By [35], page 414, the Langlands parameter of $\text{Ind}_B^G \chi(1, \text{sgn}, 1)$ is

$$\phi : W_R \rightarrow GSp_4(\mathbb{C}); \quad \phi(z) = I_4, \quad \phi(j) = \text{diag}(1, -1, -1, 1),$$

and the Langlands parameter of $\text{Ind}_B^G \chi(1, \text{sgn}, 1)$ is

$$\phi : W_R \rightarrow GSp_4(\mathbb{C}); \quad \phi(z) = I_4, \quad \phi(j) = \text{diag}(-1, 1, 1, -1).$$

In fact, they are conjugate in $GSp_4(\mathbb{C})$, since $s_1 \text{diag}(-1, 1, 1, -1) s_1 = \text{diag}(1, -1, -1, 1)$, where

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We summarize our results as follows:

**Theorem 4.2.** Let $F$ be a real analytic Siegel cusp form of weight $(2, 1)$, level $N$ with the central character $\varepsilon$ such that $F$ has the eigenvalues $-\frac{5}{12}$ and $0$ for the generators $\Delta_1$ and $\Delta_2$. Let $\pi_F = \pi_\infty \otimes \otimes_p \pi_p$ be the associated cuspidal representation of $GSp_4(\mathbb{A})$. Then $\pi_\infty \simeq \text{Ind}_B^G \chi(1, \text{sgn}, \text{sgn})$. The Langlands parameter of $\pi_\infty$ is given by

$$\phi : W_R \rightarrow GSp_4(\mathbb{C}), \quad \phi(z) = I_4 \text{ for } z \in \mathbb{C}, \quad \phi(j) = \text{diag}(1, -1, -1, 1).$$
Note that $\pi_\infty$ is generic and tempered. It is a totally degenerate limit of discrete series in the sense of [5]. It is denoted by $D_{1,0}[0]$ in [35], page 414. which is a limit of large discrete series. Since $\text{Ind}_G^H \chi(1,\text{sgn},\text{sgn})$ is irreducible, the $L$-packet of $\pi_\infty$ is a singleton.

**Remark 4.3.** Note that the minimal $K$-types of $\pi_\infty$ are $(1,0)$ and $(0,-1)$. One can get the highest weight vector in the $K$-type $(2,1)$ from the one in the $K$-type $(1,0)$ by first applying the $\rho(\text{diag}(1,1,-1,-1))$, and then by taking a differential operator. Since the highest weight vectors give rise to Siegel modular forms, we can describe this differential operator in terms of Siegel modular forms explicitly as follows [15]: Let $Z = X + Y \sqrt{-1} = \begin{pmatrix} z_{11} & z_{12} \\
 & z_{12} \\
 & z_{22} \end{pmatrix} \in \mathcal{H}_2$. Let $F$ be a $C^\infty$ Siegel modular form of weight $(0,-1)$, namely, the weight corresponding to $\text{St}_2 \otimes \det^{-1} \text{St}_2$. Define the differential operator $(\partial_{ij})_{1 \leq i \leq j \leq 2}$ for such $F$ by

$$2\sqrt{-1} \det(Z)Z^{-1/2} \left( \frac{\partial}{\partial z_{ij}} \right) Y^{1/2} + \left( \frac{\partial}{\partial z_{ij}} \right) \det(Y^{1/2})^{-1} Y^{1/2},$$

where $(\frac{\partial}{\partial z_{ij}}) = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{1}{2} \frac{\partial}{\partial z_{12}} \\
 \frac{1}{2} \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} \end{pmatrix}$. Then define the differential operator $D$ as

$$D := \det((\partial_{ij})_{1 \leq i \leq j \leq 2}).$$

Then by a minor modification of the calculation in Section 6 of [15], we have

$$D : M_{(0,-1)}(\Gamma(N), c_1, c_2) \longrightarrow M_{(2,1)}(\Gamma(N), c_1, c_2),$$

which commutes with the actions of Hecke operators outside $N$. Hence we have a map

$$M_{(1,0)}(\Gamma(N), c_1, c_2) \xrightarrow{F(Z) \rightarrow \det(Y)F(Z)} M_{(0,-1)}(\Gamma(N), c_1, c_2) \xrightarrow{D} M_{(2,1)}(\Gamma(N), c_1, c_2)$$

which preserves eigenforms. Note that for $F \in M_{(1,0)}(\Gamma(N), c_1, c_2)$, if $T(p^i)F = \alpha(p^i)F$, then one can easily see that $T(p^i) \det(Y)F = p^i \alpha(p^i) \det(Y)F$ which explains $L^N(s-1, F) = L^N(s, D(\det(Y)F))$ as in Remark 7.3. Since $D(\det(Y)F)$ and $F$ create the same automorphic representation, the image of such $F$ under this map is nonvanishing if $F$ is a cusp form.

**Remark 4.4.** The minimal $K$-type of $\text{Ind}_G^H \chi(1,1,\epsilon_0)$, where $\epsilon_0 = 1$ or $\text{sgn}$, is $(0,0)$. The minimal $K$-types of $\text{Ind}_G^H \chi(\text{sgn}, \text{sgn}, \epsilon_0)$, where $\epsilon_0 = 1$ or $\text{sgn}$, are $(1,1)$ and $(-1,-1)$. Also the Langlands parameter of $\text{Ind}_G^H \chi(1,1,1)$ is

$$\phi : W_\mathbb{R} \longrightarrow \text{GSp}_4(\mathbb{C}); \quad \phi(z) = I_4, \quad \phi(j) = I_4.$$
The Langlands parameter of \( \text{Ind}_{B}^{G} \chi(\text{sgn}, \text{sgn}, 1) \) is
\[
\phi : W_{\mathbb{R}} \rightarrow \text{GSp}_{4}(C); \quad \phi(z) = I_{4}, \quad \phi(j) = \text{diag}(1, -1, 1, -1).
\]
So real analytic Siegel cusp forms of weight \((0, 0)\) and \((1, 1)\) with the eigenvalues \(-\frac{5}{12}\) and 0 for the two generators of the algebra of all \(\text{Sp}_{4}(\mathbb{R})\)-invariant differential operators on the Siegel upper half plane, might correspond to symplectically even Artin representations.

5. Correspondence between automorphic representations of \(\text{GSp}_{4}\) and \(\text{GL}_{4}\)

J. Arthur \[1\] described the correspondence between automorphic representations of \(\text{GSp}_{4}(\mathbb{A})\) and \(\text{GL}_{4}(\mathbb{A})\), under the validity of stabilization of the trace formula for \(\text{GSp}_{4}\). We assume his result. In fact, we only need the transfer from the cuspidal representation \(\pi_{F}\) of \(\text{GSp}_{4}(\mathbb{A})\) to an automorphic representation \(\Pi\) of \(\text{GL}_{4}(\mathbb{A})\) so that \(L(s, \pi_{F}) = L(s, \Pi)\).

We summarize his results on \(L^{2}_{\text{disc}}(G(F)\backslash G(\mathbb{A}), \chi)\) as follows. According to its transfer to \(\text{GL}_{4}\), it is divided into 6 families of global \(L\)-packets:

1. stable, semisimple type: its transfer to \(\text{GL}_{4}\) is a cuspidal automorphic representation of \(\text{GL}_{4}(\mathbb{A})\) which is not orthogonal type.
2. unstable, semisimple (Yoshida type): its transfer to \(\text{GL}_{4}\) is an isobaric sum of two distinct cuspidal automorphic representations of \(\text{GL}_{2}(\mathbb{A})\) with the same central character \(\chi\). This is called endoscopic type,
3. stable, mixed (Soudry type): it is a CAP representation from Klingen parabolic subgroup with a cuspidal automorphic representation \(\pi\) of \(\text{GL}_{2}(\mathbb{A})\) of orthogonal type such that \(\omega_{\pi}^{2} = \chi\). In this case, \(L(s, \text{Ad}(\pi) \otimes \eta)\) has a pole at \(s = 1\), where \(\eta\) is determined by \(\pi\), and \(\text{Ad}(\pi)\) is the Gelbart-Jacquet lift to \(\text{GL}_{3}\) \[12\]. Its transfer to \(\text{GL}_{4}\) is the residual automorphic representation which is the Langlands quotient of \(\text{Ind}_{\text{GL}_{2} \times \text{GL}_{2}}^{\text{GL}_{4}} \pi|\det|^{-\frac{1}{2}} \times \pi|\det|^{-\frac{1}{2}}\).
4. unstable, mixed (Saito-Kurokawa type): it is a CAP representation from Siegel parabolic subgroup with a cuspidal automorphic representation \(\pi\) of \(\text{GL}_{2}(\mathbb{A})\) and a character \(\lambda\) such that \(\omega_{\pi} = \lambda^{2} = \chi\). Its transfer to \(\text{GL}_{4}\) is the isobaric representation \(\pi \boxplus \chi(\det_{2})\), where \(\chi(\det_{2})\) is the quotient of \(\text{Ind}_{B}^{\text{GL}_{2}} \chi|\det|^{-\frac{1}{2}} \otimes \chi|\det|^{-\frac{1}{2}}\).
5. unstable, almost unipotent (Howe-Piatetski-Shapiro type): it is a CAP representation from the Borel subgroup with two grössecharacters \(\chi_{1}, \chi_{2}\) such that \(\chi_{1}^{2} = \chi_{2}^{2} = \chi\). Its transfer to \(\text{GL}_{4}\) is the isobaric representation \(\chi_{1}(\det_{2}) \boxplus \chi_{2}(\det_{2})\).
(6) stable, almost unipotent (one dimensional type): Its transfer to $\text{GL}_4$ is $\lambda(\det_4)$ with $\lambda^4 = \chi$, which is the Langlands quotient of $\text{Ind}_{B}^{\text{GL}_4} \lambda|^{\frac{1}{2}} \otimes \lambda|^{\frac{1}{2}} \otimes \lambda|^{-\frac{1}{2}} \otimes \lambda|^{-\frac{3}{2}}$.

Lemma 5.1. Let $F$ be a Hecke eigenform in $S_{2,1}(\Gamma(N),-\frac{\tilde{t}}{12},0)$ with the associated cuspidal representation $\pi_F$ of $\text{GSp}_4(\mathbb{A})$. Then $\pi_F$ is not a CAP representation.

Proof. By Theorem 2.1 of [38], the central character of any CAP representation associated to Borel subgroup or Siegel parabolic subgroup is the square of a character. Hence the parity condition (3.7) implies that $\pi_F$ is not a CAP representation associated to Borel subgroup or Siegel parabolic subgroup. If $\pi_F$ is a CAP associated to Klingen parabolic subgroup, $\pi_F$ comes from a theta lifting from $\text{GO}(2,T)_Q$ where $T$ is an anisotropic quadratic form over $\mathbb{Q}$. Let $D_T$ be the discriminant of $T$ and let $K = \mathbb{Q}(\sqrt{D_T})$. Let $\chi_T : \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ be the quadratic character associated to $K/\mathbb{Q}$. Then by [45], Theorem A and the proof of Lemma 1.4 of [45], there exists a non-trivial map

$$\pi_F \rightarrow \text{Ind}_{Q(\mathbb{A})}^{\text{GSp}_4(\mathbb{A})} | \chi_T^{-1} \times (\sigma \otimes |\det|_{\mathbb{A}}),$$

where $\sigma$ is the automorphic induction to $\text{GL}_2(\mathbb{A})$ from a unitary Hecke character $\varphi$ on $\mathbb{A}_K^\times$. Since $\pi_F$ is irreducible, this map is injective. In particular, $\pi_\infty$ is a subquotient of the principal series $\text{Ind}_{B}^{\text{GL}_2} \chi(|^{-1} \epsilon_1,|^{s_1} \epsilon_2,|^{s_0} \epsilon_0)$ for some $s_1, s_2 \in \mathbb{C}$, and $\epsilon_i = 1$ or $\text{sgn}$. By Theorem 4.2, $\pi_\infty = \pi(1,\text{sgn},\text{sgn})$, and this is a contradiction by Harish-Chandra’s subquotient theorem (cf. [36], Theorem 2.1). □

Hence conjecturally $\pi_F$ falls into the case (1) or (2). So

Theorem 5.2. (TR) Let $\pi_F$ be as above. Then there exists an automorphic representation $\Pi$ of $\text{GL}_4(\mathbb{A})$ which is either cuspidal or an isobaric sum of two distinct cuspidal automorphic representations of $\text{GL}_2$ such that $L(s,\Pi) = L(s,\pi_F)$.

6. Application of Rankin-Selberg method for the associated automorphic representation

6.1. Spinor L-functions. Let $F$ be a real analytic Siegel cusp form of weight $(k_1,k_2)$ so that $F|_{[S_p,1]} = \chi_1(p)$ and $F|_{[S_p,p]} = \chi_2(p)$ for $p \nmid N$. We assume that $F$ is a Hecke eigenform of $T_{i,p}$ for $p \nmid N$ with eigenvalues $a_{i,p}$, $i = 1,2$, i.e., $a_{1,p} = \lambda(p)$ and

$$\lambda(p)^2 - \lambda(p^2) - p^{k-1} \chi_2(p) = pa_{2,p} + p^k (1 + p^{-2}) \chi_2(p),$$
where \( k = k_1 + k_2 - 3 \). Then we define the partial \( L \)-function of \( F \) by 
\[
L_p(s, F) = (1 - a_{1,p}T + \{pa_{2,p} + p^k(1 + p^{-2})\chi_2(p)\}T^2 - \chi_2(p)p^ka_{1,p}T^3 + \chi_2(p)^2p^{2k}T^4)^{-1},
\]
where \( T = p^{-s} \). Let \( \pi_F \) be the cuspidal automorphic representation of \( GSp_4 \) associated to \( F \), and let \( L^N(s, \pi_F) = \prod_{p\nmid N} L_p(s, \pi_p) \) be the partial automorphic \( L \)-function of \( \pi_F \). Then by definition, we have

\[
L^N(s, \pi_F) = L^N(s, F).
\]

The \( L \)-function \( L^N(s, F) \) converges in some half plane \( \text{Re}(s) \gg 0 \), and has a meromorphic continuation to the whole complex plane \( \mathbb{C} \). By Arthur’s conjecture (Theorem 5.2), it satisfies the desired functional equation and is entire. When \( \pi_F \) is globally generic, Moriyama [35] proved the analytic properties of \( L^N(s, F) \).

We now discuss the relation between \( a_{i,p} \) and Satake parameters. Let \( K(N)_p \) be the \( p \)-component of \( K(N) \). Then \( K(N)_p = GSp_4(\mathbb{Z}_p) \) for any \( p \nmid N \). So if \( p \nmid N, \pi_p \) is a spherical unitary representation which can be written as the irreducible quotient of \( \text{Ind}_{B(\mathbb{Q}_p)}^G(\mu_1 \otimes \mu_2 \otimes \eta) \), where \( \mu_1, \mu_2, \eta : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times \) are quasi-characters. Note that the central character of \( \pi_p \) is \( \varepsilon := \mu_1 \mu_2 \eta^2 \).

Let
\[
\alpha_{0p} = \eta(p^{-1}), \quad \alpha_{1p} = \mu_1(p^{-1}), \quad \alpha_{2p} = \mu_2(p^{-1}).
\]

Then under the map \( L^{GSp_4} = GSp_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C}) \), the Satake parameter corresponding to \( \pi_p \) is (see, for example, [35], p 95)

\[
\text{diag}(\alpha_{0p}\alpha_1^{\oplus} \alpha_{2p}, \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p}, \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p}, \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p}).
\]

Then by using adelic form of \( F \) and the relation (3.12) we can easily see that

\[
a_{1,p} = p^{\frac{k_1 + k_2 - 3}{2}} p^\frac{1}{2} (\alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_2^{\oplus} + \alpha_0^{\oplus}) = p^{\frac{k_1 + k_2 - 3}{2}} \alpha_{0p}(1 + \alpha_1^{\oplus})(1 + \alpha_{2p})
\]

and

\[
p_{a_{2,p}} + (p^{k_1 + k_2 - 5} + p^{k_1 + k_2 - 3})\chi_2(p)
\]

\[
= p^{\frac{(k_1 + k_2 - 3)}{2}} (\alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p} + \alpha_{0p}\alpha_1^{\oplus} \alpha_{2p})
\]

\[
= p^{\frac{(k_1 + k_2 - 3)}{2}} \varepsilon(p^{-1}) (\alpha_{1p} + \alpha_{2p} + 2 + \alpha_1^{-1} + \alpha_2^{-1}).
\]

where \( \varepsilon(p^{-1}) = \chi_2(p) \). Here the factor \( p^{\frac{1}{2}} \) in (3.12) is the contribution from the value of \( \delta_B^{-\frac{1}{2}} \) at \( \text{diag}(1,1,p^{-1},p^{-1}) \) where \( \delta_B \) is the modulus character of \( B \) defined by \( \delta_B(h) = |a|^4|b|^2|c|^3 \) for \( h =
diag\((a, b, ca^{-1}, cb^{-1})u \in B(\mathbb{Q}_p) = T(\mathbb{Q}_p)U(\mathbb{Q}_p)\). A reason why the factor\(\delta_B^{-\frac{1}{2}}(\text{diag}(1, 1, p^{-1}, p^{-1}))\) appears there is that the eigenvalues (namely, Satake parameters) of a non-zero spherical vector \((\pi_p)^{GSp_4(\mathbb{Z}_p)}\) are usually computed via the Jacquet module with respect to \(B\) (cf. Proposition 2.3 of [33]).

6.2. Application of Rankin-Selberg method. Let \(F\) be a Hecke eigenform in \(S_{2,1}(\Gamma(N), -\frac{\pi}{\text{Tr}}, 0)\) with the associated cuspidal representation \(\pi_F\) of \(GSp_4(\mathbb{A})\). Let \(\Pi\) be the transfer of \(\pi_F\) to \(GL_4/\mathbb{Q}\) by our assumption (TR). Then by (6.1),

\[
L(s, \Pi) = L(s, \pi_F) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

where \(a(p) = \lambda(p)\).

Now consider the Rankin-Selberg \(L\)-function \(L(s, \Pi \times \hat{\Pi})\) [17]. Write

\[
L(s, \Pi \times \hat{\Pi}) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},
\]

where \(b(p) = |\lambda(p)|^2\). Note that either \(\Pi\) is cuspidal or \(\Pi = \pi_1 \boxplus \pi_2\), where \(\pi_i\)'s are cuspidal representation of \(GL_2/\mathbb{Q}\) with \(\pi_1 \not\cong \pi_2\). Hence \(L(s, \Pi \times \hat{\Pi})\) has a pole of order 1 at \(s = 1\) when \(\Pi\) is cuspidal, and of order 2 when \(\Pi = \pi_1 \boxplus \pi_2\). Since \(L(s, \Pi \times \hat{\Pi}) = \sum_p \frac{\lambda(p)^2}{p^s} + g_1(s)\) for a holomorphic function \(g_1(s)\) near \(s = 1\), as \(s \to 1^+\),

\[
\sum_p \frac{\lambda(p)^2}{p^s} = \begin{cases} 
\log \frac{1}{s-1} + O(1), & \text{if } \Pi \text{ is cuspidal}, \\
2 \log \frac{1}{s-1} + O(1), & \text{if } \Pi = \pi_1 \boxplus \pi_2.
\end{cases}
\]

Now let \(\Pi_5\) be the transfer of \(\pi_F\) to \(GL_5/\mathbb{Q}\). It is obtained as follows: Let \(\tau\) be an automorphic representation of \(GL_6\) such that \(\tau_p \simeq \Lambda^2(\Pi_p)\) for \(p \neq 2, 3\) [22]. Since \(\Pi\) is the transfer of \(\pi_F\), \(L(s, \Pi, \Lambda^2 \otimes \varepsilon^{-1})\) has a pole at \(s = 1\) [25]. Hence \(\tau\) is an isobaric automorphic representation given by

\[
\tau = (\Pi_5 \otimes \varepsilon) \boxplus \varepsilon,
\]

where \(\Pi_5\) is an automorphic representation of \(GL_5\). It is easy to see that \(\Pi_5\) is a weak transfer of \(\pi_F\) to \(GL_5\) corresponding to the \(L\)-group homomorphism \(GSp_4(\mathbb{C}) \longrightarrow GL_5(\mathbb{C})\), given by the second fundamental weight [23].
The Satake parameter for $\wedge^2(\Pi)_p$ is

$$\text{diag}(\alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0^2\alpha_1\alpha_2, \alpha_0^2\alpha_1\alpha_2, \alpha_0^2\alpha_1\alpha_2, \alpha_0^2\alpha_1\alpha_2) = \text{diag}(\varepsilon(p^{-1})\alpha_1, \varepsilon(p^{-1})\alpha_2, \varepsilon(p^{-1})\alpha_1\alpha_2, \varepsilon(p^{-1})\alpha_1\alpha_2).$$

Hence the unramified factor of the standard $L$-function of $\Pi_5$ is given by

$$(1 - \alpha_1 T)(1 - \alpha_2 T)(1 - T)(1 - \alpha_1^{-1} T)(1 - \alpha_2^{-1} T).$$

Let

$$L(s, \Pi_5) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

Then $c(p) = \alpha_1 + \alpha_2 + 1 + \alpha_1^{-1} + \alpha_2^{-1}$. This is called the standard $L$-function of $F$.

Consider the Rankin-Selberg $L$-function $L(s, \Pi_5 \times \hat{\Pi}_5)$. Write

$$L(s, \Pi_5 \times \hat{\Pi}_5) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

where $d(p) = |c(p)|^2$. Note that $L(s, \Pi_5 \times \hat{\Pi}_5)$ has a pole of at least order 1, and at most order 5 at $s = 1$. Since $L(s, \Pi_5 \times \hat{\Pi}_5) = \sum_p \frac{|c(p)|^2}{p^s} + g_2(s)$ for a holomorphic function $g_2(s)$ near $s = 1$, $\sum_p \frac{|c(p)|^2}{p^s} \leq 5 \log \frac{1}{s-1} + O(1)$ as $s \to 1^+$.

Now from (6.3),

$$\lambda(p)^2 - \lambda(p^2)^{-1} = \alpha_0^2\alpha_1 + \alpha_0^2\alpha_2 + \alpha_0^2\alpha_1\alpha_2 + \alpha_0^2\alpha_1\alpha_2 + \alpha_0^2\alpha_1\alpha_2 + \alpha_0^2\alpha_1\alpha_2.$$  

Hence $\lambda(p^2) = \lambda(p)^2 - \varepsilon(p^{-1})p^{-1} - \varepsilon(p^{-1})(c(p) + 1)$.

Let $L$ be the Galois closure of the Hecke field $\mathbb{Q}_F = \mathbb{Q}(\lambda(p), c(p), \varepsilon(p^{-1}), p \nmid N)$ and $\mathcal{O}_L$ be the ring of integers of $L$. Here $\mathbb{Q}_F = \mathbb{Q}(\lambda(p), \lambda(p^2), \varepsilon(p^{-1}), p \nmid N)$. We make the integrality assumption:

(INT) \hspace{1cm} \lambda(p), \lambda(p^2) - \lambda(p^2) - \varepsilon(p^{-1})p^{-1} \in \mathcal{O}_L \left[ \frac{1}{N} \right].

It is equivalent to $\lambda(p), c(p) \in \mathcal{O}_L[\frac{1}{N}]$. Now we prove

**Proposition 6.1.** Under (TR) and (Int), for any positive integer $\eta$, there exists a set $X_\eta$ of rational primes such that $\text{den} \sup X_\eta \leq \eta$, and the set $\{ (\lambda(p), \lambda(p^2)) \mid p \notin X_\eta \}$ is a finite set, or equivalently, $\{ \text{Satake parameters at } p \mid p \notin X_\eta \}$ is finite.
Here \( \text{den.sup} X_\eta \) is defined by
\[
\lim_{s \to 1^+} \sup \frac{\sum_{p \in X_\eta} p^{-s}}{\log \frac{1}{s-1}}.
\]
We also define the Dirichlet density \( \text{den}(X_\eta) \) by
\[
\lim_{s \to 1^+} \frac{\sum_{p \in X_\eta} p^{-s}}{\log \frac{1}{s-1}}.
\]

**Proof.** For \( c > 0 \), consider two sets:

\[
Y(c) = \{ a \in \mathcal{O}_L \mid |\sigma(a)|^2 < c \text{ for any } \sigma \in \text{Gal}(L/\mathbb{Q}) \},
\]
\[
X(c) = \{ p \mid \lambda(p) \text{ or } c(p) \text{ does not belong to } Y(c) \}.
\]

Note that since \( \mathcal{O}_L \) is a lattice, \( Y(c) \) is a finite set. By the assumption (Int), \( \lambda(p), c(p) \in \mathcal{O}_L[\frac{1}{N}] \).

If \( p \notin X(c), N\lambda(p), Nc(p) \in Y(N^2c) \). Hence the set \( \{(N\lambda(p), Nc(p)) \mid p \notin X(c)\} \) is finite, and so the set \( \{(\lambda(p), c(p)) \mid p \notin X(c)\} \) is finite since \( \lambda(p^2) = \lambda(p)^2 - \varepsilon(p^{-1})p^{-1} - \varepsilon(p^{-1})c(p) + 1 \), the set \( \{(\lambda(p), \lambda(p^2)) \mid p \notin X(c)\} \) is finite.

For each \( \sigma \in \text{Gal}(L/\mathbb{Q}) \), \( \sigma F \) is a cuspidal eigenform with \( T(p)^\sigma F = \sigma(\lambda(p))^\sigma F \). Hence

\[
\sum_{\sigma} \sum_p \frac{\sigma(\lambda(p))^2}{p^s} \leq 2r \log \frac{1}{s-1} + O(1), \text{ as } s \to 1^+
\]

where \( r = [L : \mathbb{Q}] \). Also

\[
\sum_{\sigma} \sum_p \frac{\sigma(c(p))^2}{p^s} \leq 5r \log \frac{1}{s-1} + O(1), \text{ as } s \to 1^+
\]

If \( p \in X(c) \), \( |\sigma_0(\lambda(p))|^2 > c \) or \( |\sigma_1(c(p))|^2 > c \) for some \( \sigma_0, \sigma_1 \in \text{Gal}(L/\mathbb{Q}) \). Therefore

\[
c \sum_{p \in X(c)} p^{-s} \leq \sum_{\sigma} \sum_p \frac{|\sigma(\lambda(p))|^2}{p^s} + \sum_{\sigma} \sum_p \frac{|\sigma(c(p))|^2}{p^s} \leq 7r \log \frac{1}{s-1} + O(1), \text{ as } s \to 1^+.
\]

Hence, \( \text{den.sup} X(c) \leq \frac{7r}{c} \). Take \( c \) such that \( c \geq \frac{7r}{\eta} \), and \( X_\eta = X(c) \). This proves Proposition 6.1.

\[\square\]

**Remark 6.2.** If \( F \) is a Siegel cusp form of weight \((k_1, k_2)\), then by (6.1), \( L(s - \frac{k_1 + k_2 - 3}{2}, \pi_F) = L(s, F) \). So

\[
\sum_p \frac{|\lambda(p)|^2}{p^s} = a \log \frac{1}{s - (k_1 + k_2 - 2)} + O(1), \text{ as } s \to k_1 + k_2 - 2
\]

where \( a = 1 \) or \( 2 \). Hence only when \((k_1, k_2) = (2, 1)\), we can use the above argument.
In this section we formulate a conjecture on the existence of the mod $\ell$ Galois representations attached to a real analytic Siegel modular form $F$ of weight $(2, 1)$, in analogy with holomorphic Siegel cusp forms.

Let $F \in S_{(2, 1)}(\Gamma(N), -\frac{5}{12}, 0)$ be a Hecke eigenform with eigenvalues $\lambda(p^i)$ for $T(p^i)$, $F|\mathbb{S}_{p,1} = \chi_1(p)F$, and $F|\mathbb{S}_{p, p} = \chi_2(p)F$ $(p \nmid N)$. Let $\pi_F = \pi_\infty \otimes \otimes_p \pi_p$ be the cuspidal automorphic representation attached to $F$. Recall that $\pi_F$ is not a CAP representation (Lemma 5.1).

**Conjecture 7.1.** Assume (Rat) and (Int) for $F$. Let $\ell$ be an odd prime which is coprime to $N$.

Then for each finite place $\lambda$ of $\mathbb{Q}_F$ with the residue field $\mathbb{F}_\lambda$, there exists a continuous semi-simple representation $\rho_{F, \lambda} : G_{\mathbb{Q}} \rightarrow GSp_4(\mathbb{F}_\lambda)$, which is unramified outside of $\ell N$ so that

$$\det(I_4 - \rho_{F, \lambda}(\text{Frob}_p)T) \equiv 1 - \lambda(p)T + \{\lambda(p)^2 - \lambda(p^2) - p^{-1}\chi_2(p)\}T^2 - \chi_2(p)\lambda(p)T^3 + \chi_2(p^2)T^4 \mod \lambda,$$

for any $p \nmid \ell N$. Furthermore, $\rho_{F, \lambda}$ is symplectically odd, i.e. $\rho_{F, \lambda}(c)$ has eigenvalues $1, 1, -1, -1$ and $\rho_{F, \lambda}(c) \sim_{GSp_4} \text{diag}(1, -1, -1, 1)$ for the complex conjugation $c$.

**Lemma 7.2.** The property of being symplectically odd is equivalent to $\nu(\rho_{\lambda}(c)) = -1$, where $\nu$ is the similitude character in Section 2.

**Proof.** One implication is clear. So we assume that $\nu(\rho_{F, \lambda}(c)) = -1$. Let $V$ be the representation space of $\rho_{F, \lambda}$. It is easy to see that $\rho_{F, \lambda}(c)$ has eigenvalues $1, 1, -1, -1$. (See Lemma 2.3 and Lemma 2.5 of [7].) Let $v$ be an eigenvector in $V$ for the eigenvalue $1$ and $\{e_1, e_2, f_1, f_2\}$ be the symplectic basis with respect to $J$. As explained in the proof of Proposition [7,4], there exists a matrix $P \in GSp_4(\mathbb{F}_\lambda)$ such that $Pv = e_1$. Then we may assume that

$$\rho_{F, \lambda}(c) \sim_{GSp_4(\mathbb{F}_\lambda)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & t & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_3 & x_2 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_1 & 1 \end{pmatrix} \in M_Q(\mathbb{F}_\lambda).$$
Since $\rho_{F,\lambda}(c)$ is of order 2 and has eigenvalues $1, 1, -1, -1$, one has $t = ad - bc = -1$. The unipotent part of RHS is preserved by the conjugation of the matrix of the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & x & 0 & y \\
0 & 0 & 1 & 0 \\
0 & z & 0 & w
\end{pmatrix}
$$

with $xw - yz = 1$. Hence we have

$$\rho_{F,\lambda}(c) GSp_4(\mathbb{F}_\lambda) \sim \begin{pmatrix} 1 & x_1 & x_2 \\
0 & 1 & x_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & -x_1 & 1 \end{pmatrix} := A \in M_Q(\mathbb{F}_\lambda).$$

The condition $\nu(\rho_{F,\lambda}(c))^2 = I_4$ implies that $x_1 = 0$. Let $P = \begin{pmatrix} 1 & 0 & -\frac{s_2}{2} & -\frac{s_2}{2} \\
0 & 1 & -\frac{s_2}{2} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \in M_Q(\mathbb{F}_\lambda).$ Then one has

$$\rho_{F,\lambda}(c) GSp_4(\mathbb{F}_\lambda) A GSp_4(\mathbb{F}_\lambda) P^{-1} A P = \text{diag}(1, 1, -1, -1)$$

$$GSp_4(\mathbb{F}_\lambda) s_2^{-1} \text{diag}(1, 1, -1, -1)s_2 = \text{diag}(1, -1, 1, -1).$$

\[\square\]

**Remark 7.3.** If $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ is endoscopic (i.e., its transfer to $GL_4$ is not cuspidal), then by [39], $\pi_F$ is associated to a pair $(\pi_1, \pi_2)$ of two automorphic cuspidal representations of $GL_2(\mathbb{A})$ with the same central character $\varepsilon$ via theta lifting. Since the $L$-packet of $\pi_\infty$ is a singleton, by Proposition 4.2-(2) of [39], $(\pi_i)_\infty$ should be tempered, but not essentially square integrable. Hence one has $(\pi_i)_\infty = \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(|\cdot| s_1^{(i)} \varepsilon_1^{(i)}, |\cdot| s_2^{(i)} \varepsilon_2^{(i)})$, $i = 1, 2$ where $s_j^{(i)} \in \mathbb{C}$ and $\varepsilon_i$ is 1 or $\text{sgn}$.

Comparing Langlands parameters, one can see that $\pi_i$ has to correspond to an elliptic newform $f_i$ of weight one. Thus there exists a finite set $S$ of rational primes which includes all ramified prime of $\pi_F$, $\pi_{f_1}$, and $\pi_{f_2}$ so that

$$L(s, \pi_\infty) = L(s, \pi_{f_1,p})L(s, \pi_{f_2,p}), \text{ for any } p \notin S.$$
By Deligne-Serre [6], each \( f_i \) gives rise to a unique Artin representation \( \rho_{f_i} : G_G \to GL_2(\mathbb{C}) \).

Hence we may put \( \rho_F := \rho_{f_1} \oplus \rho_{f_2} \). We define the endoscopic subgroup of \( GSp_4 \) by

\[
H^{en} := \left\{ g = \begin{pmatrix}
a & 0 & b & 0 \\
0 & x & 0 & y \\
c & 0 & d & 0 \\
0 & z & 0 & w \\
\end{pmatrix} \in GSp_4 \right\} \simeq \{(A, B) \in GL_2 \times GL_2 \mid \det A = \det B\},
\]

where the isomorphism is given by \( g \mapsto \left(\begin{pmatrix}a & b \\ c & d\end{pmatrix}, \begin{pmatrix}x & y \\ z & w\end{pmatrix}\right)\). Since the central characters of \( f_i \) are the same, we have \( \det(\rho_{f_1}) = \det(\rho_{f_2}) \). Hence the image of \( \rho_F \) is actually in \( H^{en}(\mathbb{C}) \).

Further it is easy to see that \( \rho_F \) is symplectically odd.

Let \( \iota : GSp_4 \hookrightarrow GL_4 \) be the natural embedding. In what follows, we describe the image of semisimplification of \( \iota \circ \rho_{F,\lambda} : G_G \to GL_4(\mathbb{F}_\lambda) \).

**Proposition 7.4.** Let \( \rho_{F,\lambda} : G_G \to GSp_4(\mathbb{F}_\lambda) \) be as in Conjecture 7.1. Then one of the following holds:

1. \( \iota \circ \rho_{F,\lambda} \) is absolutely irreducible and \( \text{Im} \rho_{F,\lambda} \) is contained in \( GSp_4(\mathbb{F}_\lambda) \),
2. \( \iota \circ \rho_{F,\lambda} \) is irreducible but not absolutely irreducible and there exists a finite extension \( \mathbb{F}'/\mathbb{F}_\lambda \), and an absolutely irreducible representation \( \sigma : G_G \to GL_n(\mathbb{F}'_\lambda) \) with \( 4 = n[\mathbb{F}'_\lambda : \mathbb{F}_\lambda], n \neq 4 \) so that \( \iota \circ \rho_\lambda = \prod_{\tau \in \text{Gal}(\mathbb{F}'_\lambda/\mathbb{F}_\lambda)} \tau \sigma \), where \( \tau \sigma(g) = \tau(\sigma(g)) \) for \( g \in G_G \),
3. \( \text{Im} \rho_{F,\lambda} \) is contained in \( M_\ast(\kappa), \ast \in \{B, P\} \) where \( \kappa \) is a finite extension over \( \mathbb{F}_\lambda \) with the degree at most 4.
4. \( \text{Im} \rho_{F,\lambda} \) is contained in \( M_{Q}(\kappa') \) or \( H^{en}(\kappa') \) where \( \kappa' \) is a finite extension over \( \mathbb{F}_\lambda \) with the degree at most 2.

**Proof.** There exists a finite extension \( \mathbb{F}'_\lambda/\mathbb{F}_\lambda \) such that \( \rho_{F,\lambda} : G_G \to GSp_4(\mathbb{F}'_\lambda) \). If \( \iota \circ \rho_{F,\lambda} \) is irreducible, then so is \( \rho_{F,\lambda} \). Let \( \varepsilon = \chi_2^{-1} \). Then by the symplectic pairing furnished on \( \rho_{F,\lambda} \) by Conjecture 7.1, we have an isomorphism

\[
\rho_{F,\lambda}^\vee \simeq \rho_{F,\lambda} \otimes \varepsilon^{-2}.
\]

By Lemma 6.13 of [6], \( \iota \circ \rho_{F,\lambda} \) is isomorphic to an irreducible representation \( \Phi : G_G \to GL_4(\mathbb{F}_\lambda) \). By Chebotarev density theorem, we have an isomorphism between \( \Phi^\vee \) and \( \Phi \otimes \varepsilon^{-2} \) as \( \mathbb{F}_\lambda[G_G] \)-modules. We now divide into two cases. If \( \Phi \) is not absolutely irreducible, this corresponds to
the second claim and it is easy to prove it. So we assume that $\Phi$ is absolutely irreducible. Then by Schur’s lemma, one has (dropping the action of the character in notation for simplicity)

$$F_\lambda = \text{End}_{F_\lambda[G_Q]}(\Phi) = (\Phi^\vee \otimes \Phi)^{G_Q} = (\Phi \otimes \Phi)^{G_Q} = (\text{Sym}^2 \Phi)^{G_Q} \oplus (\wedge^2 \Phi)^{G_Q}.$$ 

Hence

$$(\text{Sym}^2 \Phi)^{G_Q} = \text{Bil}^{\text{sym}}_{F_\lambda[G_Q]}(\Phi \times \Phi, F_\lambda) = F_\lambda,$$

where $\text{Bil}^{\text{sym}}_{F_\lambda[G_Q]}(\Phi \times \Phi, F_\lambda)$ (resp. $\text{Bil}^{\text{anti-sym}}_{F_\lambda[G_Q]}(\Phi \times \Phi, F_\lambda)$) is the space consisting of all symmetric (resp. anti-symmetric) bilinear forms which commute with the Galois action. This means that there exists the symmetric or symplectic structure on $\Phi$. On the other hand, there exists a matrix $A \in \text{GL}_4(F'_\lambda)$ such that $\Phi = A^{-1} \rho_{F_\lambda} A$. Since the conjugate by an element of $GL_4(F'_\lambda)$ preserves the symmetric or symplectic structure, we have $\text{Bil}^{\text{anti-sym}}_{F_\lambda[G_Q]}(\Phi \times \Phi, F_\lambda) = F_\lambda$.

Next we consider the reducible cases. Let $\{e_1, e_2, f_1, f_2\}$ be the standard symplectic basis corresponding to $J$. We assume that $\rho_{F_\lambda}$ has an one dimensional subspace $V_1$ which is stable under the action of $G_\lambda$. Fix a non-zero $v \in V_1$. Then it is easy to see that there exists a matrix $P \in GSp_4(F_\lambda)$ so that $P e_1 = v$. Hence we may assume that $(\rho_{F_\lambda})^{ss} = \epsilon_1 \oplus \rho' \oplus \epsilon_2 \subset M_P(F'_\lambda)$ where $\epsilon_i : G_Q \rightarrow F'_\lambda \times (i = 1, 2)$ is a character and $\rho'$ is a 2-dimensional mod $\ell$ representation of $G_Q$. Let $F_\lambda'' = F_\lambda(\epsilon_1, \epsilon_2)$ and $\kappa = F_\lambda'' \cap F_\lambda$. Then $\kappa$ is of degree at most 4 over $F_\lambda$. Applying Lemma 6.13 of [3] to $\rho'$, there exists a matrix $P \in GSp_4(F_\lambda)$ so that $P^{-1}(\rho_{F_\lambda})^{ss} P = \epsilon_1 \oplus \rho'' \oplus \epsilon_2 \subset M_P(\kappa)$ where $\rho''$ is a 2-dimensional mod $\ell$ representation of $G_Q$ over $\kappa$. It is the same in the case that $\rho_{F_\lambda}$ has a three dimensional subspace $V_3$ which is stable under the action of $G_Q$ by taking the duality with respect to the symplectic pairing on $\rho_{F_\lambda}$ into account.

Finally we consider the case that $\rho_{F_\lambda}$ has an 2-dimensional irreducible subspace $V_2$ which is stable under the action of $G_Q$. Let $r$ be the dimension of the kernel of the linear map $V_2 \rightarrow V_2^*, v \mapsto \langle *, v \rangle$. It is easy to see that $r = 1$ or 2.

First we assume $r = 2$. Fix a basis $\{v_1, v_2\}$ of $V_2$. One can easily find vectors $w_1, w_2 \in V$ so that $\langle v_1, w_1 \rangle = \langle v_2, w_2 \rangle = 1$ and $\langle w_1, w_2 \rangle = 0$ since $V_2 \rightarrow V_2^*$. Then we may assume that $V_2 = \langle e_1, e_2 \rangle$ or $V_2 = \langle f_1, f_2 \rangle$. In this case, we may have $(\rho_{F_\lambda})^{ss} \subset M_P(\kappa')$ giving the claim by Lemma 6.13 of [3] again. Here $\kappa'$ is a finite extension of $F_\lambda$ with the degree at most 2.

Next we assume $r = 1$. Take a non-zero vector $v$ in the kernel of the map and denote by $v^*$ the dual basis of $v$ which is identified a vector in $V$ by the pairing. Then $\langle v, v^* \rangle = 1$. Take $w \in V_2$ (and denote by $w^*$ the dual vector of $w$) so that $\langle v, w \rangle = 0$. This gives us that $\langle v^*, w^* \rangle = 0$. Hence $\{v, w, v^*, w^*\}$ makes the standard symplectic basis. Therefore one may have that $V_2 = \langle e_1, f_1 \rangle$.
or $V_2 = \langle e_2, f_2 \rangle$. In this case, one has $(\rho_{F, \lambda})^{ss} \subset H^{en}(\kappa')$ giving the claim by Lemma 6.13 of [4] again.

\[ \square \]

8. Bounds of certain subgroups of $GSp_4(\mathbb{F}_\ell^n)$

In this section, we will study the bounds of certain subgroups of $GSp_4(\mathbb{F}_\ell^n)$ for odd rational prime $\ell$ and $n \geq 1$. For a finite set $X$, we denote by $|X|$, the cardinality of $X$.

By imitating the strategy of [4] for $GL_2(\mathbb{F}_\ell)$, we consider the following property of a subgroup $G$ of $GSp_4(\mathbb{F}_\ell^n)$.

**Definition 8.1.** Let $M$ and $\eta$ ($0 < \eta < 1$) be positive constants.

\[ C(\eta, M) : \text{there exists a subset } H \text{ of } G \text{ such that} \]

\[ \begin{align*}
(i) & \ |H| \geq (1 - \eta)|G|, \\
(ii) & \ |\{\det(1 - hT) \in \mathbb{F}_\ell[T] | h \in H\}| \leq M.
\end{align*} \]

Then the following lemma is easy to prove.

**Lemma 8.2.** (cf. the proof of Proposition 7.2 in [4]) Let $G$ be a finite group with a subgroup $G'$ of index 2. Then if $G$ satisfies $C(\eta, M)$, then $G'$ satisfies $C(2\eta, M)$.

**Proof.** Let $H$ be a subset of $G$ which satisfies the property $C(\eta, M)$. Let $H' = H \cap G'$. Then $|H'| \geq (1 - \eta)|G| = (2 - 2\eta)|G'| \geq (1 - 2\eta)|G'|$. The second condition is obvious. \[ \square \]

We denote by $M_*$, the Levi factor of the parabolic subgroup $* \in \{B, P, Q\}$. Recall $M_B(\mathbb{F}_\ell^n)$, $M_P(\mathbb{F}_\ell^n)$, and $M_Q(\mathbb{F}_\ell^n)$ from Section 2. Recall also $H^{en}(\mathbb{F}_\ell^n)$ from Section 7.

For a subgroup $G$ of $GSp_4(\mathbb{F}_\ell)$, we say $G$ is semisimple if the identity representation $G \hookrightarrow GSp_4(\mathbb{F}_\ell^n) \hookrightarrow GL_4(\mathbb{F}_\ell^n)$ is semisimple.

We need the classification of all semisimple subgroups of $GSp_4(\mathbb{F}_\ell^n)$, $n \geq 1$. All of them are the semisimple parts of groups taken from [7] and [8] though some of explicit forms are not given there.

**Lemma 8.3.** Let $G$ be a semisimple subgroup of $GSp_4(\mathbb{F}_\ell^n)$, $n \geq 1$. Then up to conjugacy, $G$ is one of the following:

(reducible cases)

1. $G$ is contained in $M_*(\mathbb{F}_\ell^n)$ for some $* \in \{B, P, Q\}$.

(2) $G$ is contained in $H^{en}(\mathbb{F}_\ell^n)$.

(irreducible cases and $n = 1$)
(3) $G$ contains $Sp_4(\mathbb{F}_\ell)$, 

(4) $G$ is contained in $\text{Sym}^3 GL_2(\mathbb{F}_\ell)$, 

(5) $G$ is contained in $H_\ell := \langle M_\ell(\mathbb{F}_\ell), \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \rangle$, but $G \not\subset M_\ell(\mathbb{F}_\ell)$. 

(6) $G$ is contained in $H_\ell := \langle H_{\text{en}}(\mathbb{F}_\ell), \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \rangle$, but $G \not\subset H_{\text{en}}(\mathbb{F}_\ell)$. 

(7) Fix a quadratic non-residue $u \in \mathbb{F}_\ell$ and a square root $\sqrt{u} \in \mathbb{F}_{\ell^2}$ of $u$. Choose a solution $(a, b) \in \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$ so that $a^2 + b^2 = u$. Then for $a_i = x_i + y_i\sqrt{u} \in \mathbb{F}_{\ell^2}$, $x_i, y_i \in \mathbb{F}_\ell$ ($i = 1, \ldots, 4$), let $S(a_i) = \begin{pmatrix} x_i + ay_i & by_i \\ by_i & x_i - ay_i \end{pmatrix}$. Note that $t^4 S(a_i) = S(a_i)$ and 

$$
\begin{pmatrix} S(a_1) & S(a_2) \\ S(a_3) & S(a_4) \end{pmatrix} \in GSp_4(\mathbb{F}_\ell) \text{ if and only if } a_1a_4 - a_2a_3 \in \mathbb{F}_\ell^\times.
$$

Then $G$ is contained in 

$$
\left\{ \begin{pmatrix} S(a_1) & S(a_2) \\ S(a_3) & S(a_4) \end{pmatrix} \in GSp_4(\mathbb{F}_\ell) \mid a_i \in \mathbb{F}_{\ell^2} \right\}, \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \simeq S_\ell \rtimes \{\pm 1\},
$$

where $S_\ell := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \right\} = \{ g \in GL_2(\mathbb{F}_{\ell^2}) \mid \det(g) \in \mathbb{F}_\ell^\times \}$. 

(8) Fix a quadratic non-residue $u \in \mathbb{F}_\ell$ and choose a solution $\lambda \in \mathbb{F}_{\ell^2}^{\times}$ so that $\lambda^2 = u$. Then $G$ is contained in 

$$
\left\{ u(A, B) = \begin{pmatrix} A & B \\ uB & A \end{pmatrix} \in GSp_4(\mathbb{F}_\ell) \right\}, \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \simeq GU_2(\mathbb{F}_\ell) \rtimes \{\pm 1\}
$$

where $GU_2(\mathbb{F}_\ell) = \{ g \in GL_2(\mathbb{F}_{\ell^2}) \mid \sigma(g) = \nu I_2, \nu \in \mathbb{F}_\ell^\times \}$ and $\sigma$ is the generator of $\text{Gal}(\mathbb{F}_{\ell^2}/\mathbb{F}_\ell)$. The matrices $A, B \in M_2(\mathbb{F}_\ell)$ satisfy $A^t A - uB^t B = \nu I_2$, $\nu \in \mathbb{F}_\ell^\times$ and $A^t B - B^t A = 0$. Then the above isomorphism is given by $u(A, B) \mapsto A + \lambda B$.

(9) $G$ is contained in 

$$
\left\{ \begin{pmatrix} av & 0 & bv & 0 \\ 0 & az & 0 & bz \\ cv & 0 & dv & 0 \\ 0 & cz & 0 & dz \end{pmatrix} \in GSp_4(\mathbb{F}_\ell) \right\} \cup \left\{ \begin{pmatrix} 0 & av & 0 & bv \\ az & 0 & bz & 0 \\ 0 & cv & 0 & dv \\ cz & 0 & dz & 0 \end{pmatrix} \in GSp_4(\mathbb{F}_\ell) \right\},
$$

which is realized by taking the tensor product of $GL_2(\mathbb{F}_\ell)$ and a dihedral subgroup $D = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cup \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ of $GL_2(\mathbb{F}_\ell)$. 


(10) We denote by $\overline{G}$, the image of $G$ in $\text{PGSp}_4(\mathbb{F}_\ell)$. Then $\overline{G}$ is isomorphic to $A_6$, $S_6$, or $A_7$, or there exists a normal abelian subgroup $E$ of $\overline{G}$ with order 16 so that $\overline{G}/E \simeq A_5$ or $S_5$.

We prove the following key proposition by brute force with the help of the above Lemma.

**Proposition 8.4.** For positive constants $M$ and $\eta$, $(0 < \eta < \frac{1}{2})$, there exists a constant $A = A(\eta, M)$ such that for every rational odd prime $\ell$ and every semisimple subgroup $G$ of $M_*(\mathbb{F}_\ell)$, $* \in \{B, P\}$, $M_Q(\mathbb{F}_\ell)$, $H^\eta(\mathbb{F}_\ell)$, or $\text{GSp}_4(\mathbb{F}_\ell)$, or $\prod_{\tau \in \text{Gal}(\mathbb{F}_{\ell^m}/\mathbb{F}_\ell)} \tau(GL_n(\mathbb{F}_{\ell^m})) \subset GL_4(\mathbb{F}_\ell)$ with $nm = 4, n \neq 4$, satisfying $C(\eta, M)$, we have $|G| < A$.

**Proof.** Case (1)-$M_B(\mathbb{F}_\ell)$: At most 8 ($= |W_G|$ where $W_G$ is the Weyl group of $G$) elements of $M_B(\mathbb{F}_\ell)$ have a given characteristic polynomial. The hypothesis $C(\eta, M)$ (with $0 < \eta < 1$) gives $(1 - \eta)|G| \leq |H| \leq 8\{|\det(1 - hT) \in \mathbb{F}_\ell[T] \mid h \in H\}| \leq 8M$.

giving a bound $|G| < \frac{8M}{1 - \eta}$. Case (1)-$M_P(\mathbb{F}_\ell)$: In this case, the conjugacy classes are isomorphic to the product of the conjugacy classes of $GL_2(\mathbb{F}_{\ell^n})$ and $\mathbb{F}_{\ell^n}^\times$. Hence the similitude does not essentially affect the result. It is easy to generalize Proposition 7.2 of [6] to the case $GL_2(\mathbb{F}_{\ell^n})$ for any $n \geq 1$. Let $A$ be the analogous constant of Proposition 7.2 of [6] in the case $GL_2(\mathbb{F}_\ell)$.

Then we have $|G| \leq 2A$ by taking the action of the element $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ into account.

Case (1)-$M_Q(\mathbb{F}_\ell)$ is the same as well.

Case (2): Let $\text{pr}_i : H^\eta(\mathbb{F}_\ell) \simeq \{(A, B) \in GL_2(\mathbb{F}_\ell) \times GL_2(\mathbb{F}_\ell) \mid \det A = \det B\} \overset{\text{pr}_i}{\longrightarrow} GL_2(\mathbb{F}_\ell)$ be the $i$-th projection for $i = 1, 2$. Note that there is an exact sequence

\begin{equation}
1 \longrightarrow SL_2(\mathbb{F}_\ell) \times SL_2(\mathbb{F}_\ell) \longrightarrow H^\eta(\mathbb{F}_\ell) \longrightarrow \mathbb{F}_{\ell^2}^\times \longrightarrow 1
\end{equation}

by the obvious way. Then $\text{pr}_i(G)$ satisfies one of the conditions (a), (b), (c), or (d) of Proposition 7.2 of [6]. So essentially there are at most 6 possibilities of $G$. For simplicity, we say $(\ast_1)$-$(\ast_2)$ case for $\ast_1, \ast_2 \in \{a, b, c, d\}$ if $\text{pr}_1(G)$ satisfies $(\ast_1)$ and $\text{pr}_2(G)$ satisfies $(\ast_2)$. We recall the following fact which is easy to prove:

\begin{equation}
\text{There are at most } \ell^{2n} + \ell^4 \text{ elements of } GL_2(\mathbb{F}_{\ell^n}) \text{ which have a given characteristic polynomial. (see the proof of Proposition 7.2 of [6] for } n = 1).}
\end{equation}
(a)-(a) case: Let $r := [G : SL_2(\mathbb{F}_{\ell^2}) \times SL_2(\mathbb{F}_{\ell^2})]$. Then we have $|G| = r\ell^4(\ell^4 - 1)^2$. It is easy to see that the characteristic polynomial of any element $g$ of $G$ which corresponds to $(A, B) \in GL_2(\mathbb{F}_{\ell^2}) \times GL_2(\mathbb{F}_{\ell^2})$, det $A = \det B$ is of form

$$\Phi_g(T) = \Phi_A(T)\Phi_B(T).$$

By (8.2), at most $8(\ell^4 + \ell^2)^2$ elements of $G$ have a given characteristic polynomial. If $G$ satisfies $C(\eta, M)$, one has

$$(1 - \eta)r\ell^4(\ell^4 - 1)^2 \leq 8(\ell^4 + \ell^2)^2M,$$

giving

$$(1 - \eta)r\ell \leq (1 - \eta)r(\ell^2 - 1)^2 \leq 8M \text{ and } \ell \leq \frac{8M}{1 - \eta}.$$ 

Hence we have the bound of $|G|$ which is depending only on $M$ and $\eta$. 

(a)-(b) case: There exists a subgroup $K$ of $\mathbb{F}_{\ell^2}^\times$ such that

$$G = \left\{ g = \begin{pmatrix} A, & \begin{pmatrix} a & 0 \\ 0 & a^{-1}\det A \end{pmatrix} \end{pmatrix} \bigg| A \in \text{pr}_1(G), \ a \in K \right\}.$$ 

Let $r = [\text{pr}_1(G) : SL_2(\mathbb{F}_{\ell^2})]$. Then we have $|G| = |K|r\ell^2(\ell^4 - 1)$. The characteristic polynomial of any element $g$ of $G$ is of form

$$\Phi_g(T) = \Phi_A(T)(T - a)(T - a^{-1}\det A).$$

Then by (8.2), at most $2(\ell^4 + \ell^2)$ elements of $G$ have a given characteristic polynomial. If $G$ satisfies $C(\eta, M)$, one has

$$(1 - \eta)|K|r\ell^2(\ell^4 - 1) \leq 2(\ell^4 + \ell^2)M,$$

giving

$$(1 - \eta)r|K|\ell \leq (1 - \eta)r|K|(\ell^2 - 1) \leq 2M \text{ and } \ell \leq \frac{2M}{1 - \eta}.$$ 

Hence we have the bound of $|G|$ which is depending only on $M$ and $\eta$. 

(a)-(c) case: This case reduced to the case (a)-(b) by Lemma S.2. 

(a)-(d) case: In this case, the group $K = \text{pr}_2(G) \cap SL_2(\mathbb{F}_{\ell^2})$ is of order at most 120, whence has at most 120 elements of the given determinant. Let $r = [\text{pr}_1(G) : SL_2(\mathbb{F}_{\ell^2})]$. Then we have $|G| = |K|r\ell^2(\ell^4 - 1)$ by (S.1). Then by (S.2), at most $120(\ell^4 + \ell^2)$ elements of $G$ have a given characteristic polynomial. If $G$ satisfies $C(\eta, M)$, one has

$$(1 - \eta)|K|r\ell^2(\ell^4 - 1) \leq 120(\ell^4 + \ell^2)M,$$
(1 - η)r|K|ℓ ≤ (1 - η)r|K|(ℓ^2 - 1) ≤ 120M and ℓ ≤ \frac{120M}{1 - η}.

Hence we have the bound of |G| which is depending only on M and η.

(b)-(b) case: Any element of G is of form \begin{pmatrix} a & 0 \\ 0 & a^{-1}c \\ 0 & b^{-1}c \end{pmatrix}. Hence at most 8 elements of G have a given characteristic polynomial. Then one has |G| ≤ \frac{8M}{1 - η}.

(b)-(c) case: This case is reduced to the case (b)-(b) by Lemma 8.2.

(b)-(d) case: By the analysis of (a)-(d) case, we see that at most 120 elements of G have a given characteristic polynomial. Hence we have |G| ≤ \frac{240M}{1 - η}.

(c)-(d) case: This case reduced to the case (b)-(d) by Lemma 8.2.

For the case \prod_{τ ∈ \text{Gal}(\mathbb{F}_{ℓm}/\mathbb{F}_ℓ)} T(\text{GL}_n(\mathbb{F}_{ℓm})) ⊂ \text{GL}_4(\mathbb{F}_ℓ) with nm = 4, n ≠ 4, it is reduced to the case (1)-M_P(\mathbb{F}_ℓ). So we omit the proof.

Case (3): Let r := [G : \text{Sp}_4(\mathbb{F}_ℓ)]. Then we have |G| = r(ℓ^4(ℓ^4 - 1)(ℓ^2 - 1)). By Table 1 and Table 2 of [44], one can compute the number of elements of G which have a given characteristic polynomial. As a result, such number is at most Cℓ^8 for some positive constant C which is independent of ℓ. For instance, if the semi-simple part of g ∈ G is diag(a, a, a, a), a ∈ \mathbb{F}_ℓ, from the centralizer of the elements of types A_0, A_1, A_{21}, A_{22}, and A_3 of Table 2 in [44], the number of elements of G with the characteristic polynomial (T - a)^4 is computed as the sum of orbits of each types:

\begin{align*}
\frac{|G_{SP_4}(F_ℓ)|}{|G_{SP_4}(F_ℓ)|} + \frac{|G_{SP_4}(F_ℓ)|}{2ℓ^3(ℓ - 1)^2} + \frac{|G_{SP_4}(F_ℓ)|}{2ℓ^3(ℓ^2 - 1)} + \frac{|G_{SP_4}(F_ℓ)|}{ℓ^2(ℓ - 1)} = ℓ^8 - \frac{1}{2}ℓ^6 + \frac{1}{2}ℓ^4 + \frac{1}{2}ℓ^2 - ℓ + \frac{1}{2}.
\end{align*}

If G satisfies C(η, M), one has

\begin{align*}
(1 - η)rℓ^4(ℓ^4 - 1)(ℓ^2 - 1) ≤ Cℓ^8 M,
\end{align*}

giving

\begin{align*}
(1 - η)rℓ ≤ (1 - η)r(ℓ^2 - 1) ≤ CM \text{ and } ℓ ≤ \frac{CM}{1 - η}.
\end{align*}

Hence we have the bound of |G| which is depending only on M and η. Case (4): It is reduced to the case GL_2(\mathbb{F}_ℓ). Cases (5) and (6): These cases are reduced to the cases (1)-M_P and (2) for n = 1 by Lemma 8.2.

Case (7): Let S(A) := \begin{pmatrix} S(a_1) & S(a_2) \\ S(a_3) & S(a_4) \end{pmatrix} for A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} ∈ S_ℓ. Then it is easy to see that Φ_{S(A)}(T) = Φ_A(T)σ(Φ_A(T)) where Φ means the characteristic polynomial.
of $\ast$ and $\sigma$ is the generator of $\text{Gal}(\mathbb{F}_{\ell^2}/\mathbb{F}_\ell)$. As in the proof of the reducible case (2) (replacing the base field by $\mathbb{F}_{\ell^2}$) we have three possibilities for $G$. We give a proof for the case when $G \cap S_\ell$ contains $SL_2(\mathbb{F}_{\ell^2})$. The other cases are the same as well.

Let $r = [G : SL_2(\mathbb{F}_{\ell^2})]$ so that $|G| = r\ell^2(\ell^4 - 1)$. Then by (8.2), at most $4(\ell^4 + \ell^2)$ elements of $G$ have a given characteristic polynomial. Here the factor 4 comes from the orders of $\{\pm 1\}$ and $\text{Gal}(\mathbb{F}_{\ell^2}/\mathbb{F}_\ell)$. If $G$ satisfies $C(\eta, M)$, one has
\[(1 - \eta)r\ell^2(\ell^4 - 1) \leq 4(\ell^4 + \ell^2)M,\]
giving
\[(1 - \eta)r\ell \leq (1 - \eta)r(\ell^2 - 1) \leq 8M \text{ and } \ell \leq \frac{8M}{1 - \eta}.\]
Hence we have the bound of $G$ which is depending only on $M$ and $\eta$.

Case (8): Let $U := A + \lambda B$. Then it is easy to see that $\Phi_{g(A,B)}(T) = \Phi_U(T)\sigma(\Phi_U(T)) = \Phi_U(T)\Phi_{\sigma(U)}(T)$.

Since $G$ is irreducible, the composition $G \cap GU_2(\mathbb{F}_\ell) \hookrightarrow GL_2(\mathbb{F}_{\ell^2})$ is also irreducible and it has three possibilities as in case (2). We give a proof for the case when $G \cap GU_2(\mathbb{F}_\ell)$ contains $SL_2(\mathbb{F}_{\ell^2}) \cap GU_2(\mathbb{F}_\ell) = SU_2(\mathbb{F}_\ell)$. The other cases are the same as well. Note that $SU_2(\mathbb{F}_\ell) \simeq SL_2(\mathbb{F}_\ell)$ (the isomorphism is considered in $GL_2(\mathbb{F}_\ell)$). and hence $|SL_2(\mathbb{F}_{\ell^2}) \cap GU_2(\mathbb{F}_\ell)| = 2(\ell^2 - 1)$.
As in the case $GL_2(\mathbb{F}_\ell)$, it is not so hard to show that the number of elements in $GU_2(\mathbb{F}_\ell)$ with the given polynomial is $\ell^2 + \ell, \ell^2, \text{ or } 2\ell^2 + \ell$ as the polynomial in question has 2, 1, or 0 roots in $\mathbb{F}_\ell$. Let $r = [G : SU_2(\mathbb{F}_\ell)]$ so that $|G| = r\ell(\ell^2 - 1)$. Then at most $4(\ell^4 + \ell^2)$ elements of $G$ have a given characteristic polynomial. Here the factor 4 comes from the orders of $\{\pm 1\}$ and $\text{Gal}(\mathbb{F}_{\ell^2}/\mathbb{F}_\ell)$. If $G$ satisfies $C(\eta, M)$, one has
\[(1 - \eta)r\ell(\ell^2 - 1) \leq 4(\ell^2 + \ell)M,\]
giving
\[(1 - \eta)r\ell \leq 8M \text{ and } \ell \leq \frac{8M}{1 - \eta}.\]
Hence we have the bound of $G$ which is depending only on $M$ and $\eta$.

Case (9): Since $G$ is contained in the tensor representation $GL_2(\mathbb{F}_\ell) \otimes D$, where $D$ is a dihedral subgroup of $GL_2(\mathbb{F}_\ell)$, it is reduced to the case $GL_2(\mathbb{F}_\ell)$ by Lemma 8.2. So we omit the proof.

Case (10): Among the finite groups appearing in case (10), $A_7$ is the largest: $|A_7| = 2520$. The group $G \cap SL_4(\mathbb{F}_\ell)$ is of order at most $4 \times 2520$, whence $G$ has at most 10080 elements with the
given characteristic polynomial. If $G$ satisfies $C(\eta, M)$, one has

$$(1 - \eta)|G| \leq 10080M,$$

giving the bound of $G$. This completes the proof. □

9. Proof of Main Theorem

In this section we give a proof of the main theorem (Theorem 1.1). Let $\pi_F = \pi_\infty \otimes \otimes_p \pi_p$ be the cuspidal automorphic representation of $GSp_4(\mathbb{A})$ attached to the real analytic Siegel cusp form of weight $(2,1)$. By Lemma 5.1, such $\pi_F$ is not a CAP representation. Let $Q_F$ be the Hecke field of $F$, and let $L$ be the Galois closure of $Q_F$. By the assumption (Rat), $L$ is a finite extension of $\mathbb{Q}$. We denote by $S_\pi$ the set of rational primes consisting of primes $p$ so that $\pi_p$ is ramified. Let $P_L$ be the set prime numbers $\ell$ which splits completely in $L$. For each $\ell \in P_L$, choose a finite place $\lambda_\ell$ of $L$ dividing $\ell$. By Conjecture 7.1, there exists a continuous semi-simple representation $\rho_\ell := \rho_{\lambda_\ell} : G_{\mathbb{Q}} \rightarrow GSp_4(F_\ell)$ which is unramified outside $S_\pi \cup \{\ell\}$, and

$$\det(I_4 - \rho_\ell(Frob_p)T) \equiv H_p(T) \bmod \lambda_\ell,$$

where $H_p(T) = 1 - a_1 T + (p a_2 + (1 + p^2 - \varepsilon p^{-1}) T^2 - \varepsilon (1 + p^2 - \varepsilon p^{-1}) a_1 T^3 + \varepsilon (1 + p^2 - \varepsilon p^{-1})^2 T^4$. Let $G_\ell := \text{Im} \rho_\ell$.

Lemma 9.1. For any $\eta, 0 < \eta < 1$, there exists a constant $M$ such that $G_\ell$ satisfies $C(\eta, M)$ for every $\ell \in P_L$.

Proof. By Proposition 6.1 if we let $M := \{H_p(T) \mid p \notin X_\eta\}$, then $M$ is a finite set. Let $M := |M|$ which will be a desired constant. Let us consider the subset of $G_\ell$ defined by

$$H_\ell := \{g \in G_\ell \mid g \sim \rho_\ell(Frob_p) \text{ for some } p \notin X_\eta\}.$$

By Chebotarev density theorem, one has

$$1 = \frac{|H_\ell|}{|G_\ell|} + \text{den}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \text{den.sup}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \eta,$$

giving $(1 - \eta)|G_\ell| \leq |H_\ell|$.

The eigen polynomial of each element of $H_\ell$ is the reduction of some element of $M$. Therefore one has

$$|\{\det(I_4 - hT) \mid h \in H_\ell\}| \leq M.$$

□
By Lemma 9.1 together with Proposition 8.4, there exists a constant $A$ such that $|G_\ell| \leq A$ for any $\ell \in P_L$. Let $Y$ be the set of polynomials $(1 - \alpha T)(1 - \beta T)(1 - \gamma T)(1 - \delta T)$, where $\alpha, \beta, \gamma,$ and $\delta$ are roots of unity of order less than $A$. If $p \not\in S_\pi$, for all $\ell \in P_L$ with $\ell \neq p$, there exists $R(T) \in Y$ such that
\[ H_p(T) \equiv R(T) \mod \lambda_\ell. \]
Since $Y$ is finite, there exists $R(T) \in Y$ such that for infinitely many $\ell \in P_L$,
\[ H_p(T) = R(T). \]
Let $P'_L$ be the set of $\ell \in P_L$ such that $\ell > A$ and for $R, S \in Y$, $R \not\equiv S \mod \lambda_\ell$. Then it is easy to see that $P'_L$ is infinite. For each $\ell \in P'_L$, $\ell$ does not divide $|G_\ell|$, since $\ell > A \geq |G_\ell|$. Let $P := \text{Ker}(GSp_4(O_{\lambda_\ell}) \mod \lambda_\ell \to GSp_4(F_\ell))$ which is a pro-$\ell$-group. By applying Schur-Zassenhaus’ theorem (cf. [9], page 829) to the projection $\pi: GSp_4(O_{\lambda_\ell}) \to G_\ell$, there exists a subgroup $H \subset GSp_4(O_{\lambda_\ell})$ such that $\pi$ induces an isomorphism
\[ H \sim \to G_\ell = \text{Im} \rho_\ell. \]
Hence we have a lift $\rho'_\ell: G_\mathbb{Q} \to GSp_4(O_{\lambda_\ell})$ of $\rho_\ell$. Since the coefficient of $\rho'_\ell$ is of characteristic zero and its image is finite, for $p \nmid N\ell$, one has $\det(I_4 - \rho'_\ell(Frob_p)T) \in Y$. On the other hand, we have
\[ \det(I_4 - \rho'_\ell(Frob_p)T) \equiv H_p(T) \mod \lambda_\ell. \]
Since $\ell \in P'_L$, the above congruence relation implies the equality
\[ \det(I_4 - \rho'_\ell(Frob_p)T) = H_p(T). \]
for all $p \nmid N\ell$. Now we replace $\ell$ with another prime $\ell' \in P'_L$. Then one has $\rho'_{\ell'}: G_\mathbb{Q} \to GSp_4(O_{\lambda_{\ell'}})$ such that
\[ \det(I_4 - \rho'_{\ell'}(Frob_p)T) = \det(I_4 - \rho'_{\ell'}(Frob_p)T) \]
for all $p \nmid N\ell\ell'$. By Chebotarev density theorem, one has $\iota \circ \rho'_{\ell'} \sim \iota \circ \rho'_\ell$ and this means that $\rho'_{\ell'}$ is unramified at $\ell$. Hence we have the desired representation
\[ \rho_F := \rho'_\ell: G_\mathbb{Q} \to GSp_4(O_{\lambda_\ell}) \hookrightarrow GSp_4(\mathbb{C}), \]
where the second map comes from a fixed embedding $O_{\lambda_\ell} \hookrightarrow \mathbb{C}$. Since $\nu(\rho_F(c)) \equiv -1 \mod \ell$ and $\nu(\rho_F(c))$ has eigenvalues $1, 1, -1, -1$ and $-1 \mod \ell$, by Conjecture [4] for all but finitely many
ℓ, one has \( \nu(\rho_F(c)) = -1 \) and \( \nu(\rho_F(c)) \) has eigenvalues 1, 1, -1, and -1. This implies \( \rho_F \) is symplectically odd by Lemma 7.2.

It remains to show that \( \rho_F \) is reducible if and only if \( F \) is of endoscopic type. If \( \rho_F \) is reducible, then we have the following four cases:

1. \( \text{Im} \rho_F \) is contained in \( M_B(\mathbb{C}) \);
2. \( \text{Im} \rho_F \) is contained in \( M_Q(\mathbb{C}) \), but not in \( M_B(\mathbb{C}) \);
3. \( \text{Im} \rho_F \) is contained in \( M_P(\mathbb{C}) \), but not in \( M_B(\mathbb{C}) \);
4. \( \text{Im} \rho_F \) is contained in \( H_{\text{en}}(\mathbb{C}) \), but not in \( M_B(\mathbb{C}) \).

We will prove that only the case (4) occurs and further it is the case only when \( F \) is of endoscopic type.

Case (1): One can see that \( \rho_F = \text{diag}(\chi_1, \chi_2, \chi_1^{-1} \varepsilon, \chi_2^{-1} \varepsilon) \) where \( \chi_1, \chi_2 : G_\mathbb{Q} \rightarrow \mathbb{C}^\times, i = 1, 2 \) are grössencharacters of finite order and

\[
\lambda(p) = \chi_1(p) + \chi_2(p) + \chi_1^{-1}(p) \varepsilon(p) + \chi_2^{-1}(p) \varepsilon(p),
\]

for any \( p \nmid N \). Then we have

\[
|\lambda(p)|^2 = 4 + 2(\chi_1 \chi_2(p) + \chi_1 \chi_2(p) + \chi_1 \chi_2(p) + \chi_1 \chi_2(p)) + 2(\chi_1 \chi_2(p) + \chi_1 \chi_2(p)) + 2(\chi_1 \chi_2(p) + \chi_1 \chi_2(p)) + 2(\chi_1 \chi_2(p) + \chi_1 \chi_2(p)).
\]

One then has

\[
\lim_{s \to 1^+} \frac{1}{\log s} \sum_{p \mid N} \frac{|\lambda(p)|^2}{p^s} \geq 4
\]

which contradicts to Section 6.2. Hence this case does not occur.

Case (2): One can see that \( \rho_F = \chi_1 \oplus \rho \oplus \chi_2 \) where \( \chi_1, \chi_2 : G_\mathbb{Q} \rightarrow \mathbb{C}^\times, i = 1, 2 \) are grössencharacters of finite order and \( \rho : G_\mathbb{Q} \rightarrow GL_2(\mathbb{C}) \) is an odd irreducible Artin representation. By Corollary 0.4 of [27], \( \rho \) is modular, i.e., there exists an elliptic cusp form \( f \) attached to \( \rho \). Let \( \tilde{\rho} \) be the complex conjugate of \( \rho \), i.e., the composite of \( \rho \) and the complex conjugate \( GL_2(\mathbb{C}) \rightarrow GL_2(\mathbb{C}) \). Since

\[
\lambda(p) = \chi_1(p) + \chi_2(p) + \text{tr}(\rho(Frob_p))
\]

for any \( p \nmid N \), one has

\[
|\lambda(p)|^2 = 2 + \text{tr}(\rho \otimes \tilde{\rho}(Frob_p)) + 2(\chi_1 \chi_2(p) + \chi_1 \chi_2(p)) + (\chi_1(p) + \chi_2(p)) \text{tr}(\rho(Frob_p)) + (\chi_1(p) + \chi_2(p)) \text{tr}(\rho(Frob_p)).
\]
A standard argument on Rankin-Selberg convolution of $f$ shows that
\[
\lim_{s \to 1^+} \frac{1}{\log s} \sum_{p \nmid N} \frac{\text{tr}(\rho \otimes \overline{\rho}(\text{Frob}_p))}{p^s} = 1.
\]
Then one has
\[
\lim_{s \to 1^+} \frac{1}{\log s} \sum_{p \nmid N} \frac{\lambda(p)^2}{p^s} \geq 3
\]
which contradicts to Section 6.2. Hence this case does not occur.

Case (3): One can see that $\rho_F = \rho' \otimes \rho \otimes \chi = \chi \oplus \text{Ad}(\rho) \otimes \chi$ where $\chi : G_\mathbb{Q} \to \mathbb{C}^\times$ is a grössencharacter of finite order and $\rho : G_\mathbb{Q} \to GL_2(\mathbb{C})$ is an odd irreducible Artin representation. Let $f$ be the elliptic cusp form attached to $\rho$ explained as above, and let $Ad(\pi_f)$ be the Gelbart-Jacquet lift of $\pi_f$. Then the transfer of $\pi_F$ to $GL_4$ is of the form $\chi \oplus Ad(\pi_f) \otimes \chi$. This contradicts to the fact that $\Pi$ is either cuspidal or $\pi_1 \boxplus \pi_2$.

Case (4): One can see that $\rho_F = \rho_1 \oplus \rho_2$ where $\rho_i : G_\mathbb{Q} \to GL_2(\mathbb{C})$, $i = 1, 2$ are odd irreducible Artin representations. Let $f_i$ be the elliptic cusp form attached to $\rho_i$ explained as above. Then $L(s, \pi_p) = L_p(s, \pi_{f_1})L_p(s, \pi_{f_2})$ for all $p \nmid N$ and hence $\pi_F$ is of endoscopic type. Conversely, if $\pi_F$ is of endoscopic type, then there exist elliptic cusp forms $f_1, f_2$ such that $L(s, \pi_p) = L_p(s, \pi_{f_1})L_p(s, \pi_{f_2})$. It follows from the coefficients of $p^{-4s}$ of the local L-factors that each $f_i$ is of weight one. Hence we have $\rho_F = \rho_{f_1} \oplus \rho_{f_2}$ where $\rho_{f_i}$ is the Artin representation attached to $f_i$.

Finally we remark that the independence of the once fixed embedding $O_{\lambda_\ell} \hookrightarrow \mathbb{C}$ to our $\rho_F$ in (9.1) follows from the proof of Proposition 7.4 and Chebotarev density theorem. This proves the main theorem.

**Corollary 9.2.** (Ramanujan conjecture) Let $\pi_F = \pi_\infty \otimes' \pi_p$ be the cuspidal representation of $GSp_4$ attached to the real analytic Siegel cusp form of weight $(2, 1)$ with the eigenvalues $-\frac{5}{12}$ and 0 for the generators $\Delta_1$ and $\Delta_2$. Then under the assumptions in Theorem 1.1, $\pi_p$ is tempered for all $p$.

**Proof.** By Theorem 1.1 there exists the Artin representation $\rho_F : G_\mathbb{Q} \to GSp_4(\mathbb{C})$ such that $L(s, \rho_p) = L(s, \pi_p)$ for almost all $p$. By Proposition A.1 of [32], $L(s, \rho_p) = L(s, \pi_p)$ for all $p$. Hence $\pi_p$ is tempered for all $p$. \qed

The following proposition is due to R. Schmidt [33], Corollary 3.2.3.
Proposition 9.3. Let $F$ be a holomorphic Siegel cusp form of weight $(k_1, k_2)$, and $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ be the associated cuspidal representation of $GSp_4/\mathbb{Q}$. Then $\pi_\infty$ is a subquotient of $\text{Ind}_B^G \chi_1 \otimes \chi_2 \rtimes \sigma$, where $\chi_1, \chi_2, \sigma$ are characters of $\mathbb{R}^\times$ such that $\sigma(x) = x^{\frac{3-k_1-k_2}{2}}$ for $x > 0$, and

$$\chi_1(x) = \begin{cases} |x|^{k_2-2}, & \text{if } k_2 \text{ even} \\ |x|^{k_2-2} \operatorname{sgn}(x), & \text{if } k_2 \text{ odd} \end{cases}, \quad \chi_2(x) = \begin{cases} |x|^{k_1-1}, & \text{if } k_2 \text{ even} \\ |x|^{k_1-1} \operatorname{sgn}(x), & \text{if } k_2 \text{ odd} \end{cases}.$$

Using it, we can prove

Proposition 9.4. Let $(k_1, k_2)$ be a pair of integers $k_1 \geq k_2 \geq 0$. Then there are no holomorphic Siegel cusp forms of weight $(k_1, k_2)$ which give rise to the Artin representations.

Proof. Let $F$ be a holomorphic Siegel cusp form of weight $(k_1, k_2)$ which is a Hecke eigenform. We may assume that $k_2 > 0$ by the holomorphy. Let $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ be the associated cuspidal representation of $GSp_4/\mathbb{Q}$. Then by the above proposition, $\pi_\infty$ is a subquotient of $\text{Ind}_B^G \chi_1 \otimes \chi_2 \rtimes \sigma$, where $\chi_1, \chi_2, \sigma$ are as in the above proposition. If $\pi_F$ corresponds to an Artin representation $\rho : G_\mathbb{Q} \rightarrow GSp_4(\mathbb{C})$, then $L(s, \rho_p) = L(s, \rho_\infty)$ for almost all $p$. By [32], Proposition A.1, $L(s, \pi_\infty) = L(s, \rho_\infty)$. So the Langlands parameter of $\pi_\infty$ is $\phi : \text{Ind}_B^G \chi(\epsilon_1, \epsilon_2, \epsilon_0)$, where $\epsilon_i = 1$ or $\operatorname{sgn}$ from the discussion in Section [3]. This is a contradiction. □

10. Symmetric cube of elliptic cusp forms of weight 1

Let $\pi$ be a cuspidal representation of $GL_2/\mathbb{Q}$ which corresponds to the weight 1 new form with respect to $\Gamma_0(N)$ with the central character $\epsilon$. Let $\rho$ be the Galois representation $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$ which corresponds to $\pi$ by Deligne-Serre theorem [6]. Then $\text{Sym}^3(\pi)$ be an automorphic representation of $GL_4/\mathbb{Q}$ with the central character $\epsilon^3$ [24].

If $\pi$ is of dihedral or tetrahedral type, $\text{Sym}^3(\pi) = \pi_1 \boxplus \pi_2$ for cuspidal representations $\pi_1, \pi_2$ of $GL_2/\mathbb{Q}$. Otherwise, i.e., if $\pi$ is of octahedral or icosahedral type, $\text{Sym}^3(\pi)$ is cuspidal. In all cases, since $L(s, \text{Sym}^3(\pi), \lambda^2 \otimes \epsilon^{-3})$ has a pole at $s = 1$, by the result of Jacquet, Piatetski-Shapiro and Shalika (cf. [25]), there exists a generic cuspidal representation $\tau$ of $GSp_4(\mathbb{A})$ with the central character $\epsilon^3$ whose transfer to $GL_4(\mathbb{A})$ is $\text{Sym}^3(\pi)$. The Langlands parameter of $\pi_\infty$ is

$$\phi : W_\mathbb{R} \rightarrow GL_2(\mathbb{C}), \quad \phi(z) = I_2, \quad \phi(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Hence the Langlands parameter of $\text{Sym}^3(\pi_\infty)$ is

$$\text{Sym}^3(\phi) : W_\mathbb{R} \rightarrow GL_4(\mathbb{C}), \quad \text{Sym}^3(\phi)(z) = I_4, \quad \text{Sym}^3(\phi)(j) = J',$$

where $J' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Let $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} P^{-1} J' P = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then $P^{-1} = t P, P^{-1} J' P = \text{diag}(1, -1, -1, 1)$ in $GSp_4(\mathbb{C})$. Note that $L(s, \text{Sym}^3(\pi_\infty)) = \Gamma_C(s)^2$, where $\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s)$. Since the Langlands parameter of $\tau_\infty$ is $\text{Sym}^3(\phi)$, and from the discussion in Section 5, $\tau_\infty = \text{Ind}_{G}^{B} \chi(1, \text{sgn}, \text{sgn})$.

Now choose the highest weight vector in the $K_\infty$-type $(2,1)$ in $\tau_\infty$, and take an automorphic form $\phi$ whose archimedean component is the highest weight vector. Then $\phi$ corresponds to a real analytic Siegel modular form $F$ on the upper half-space taking values in some two-dimensional vector space (namely, a model for the $K_\infty$-type $(2,1)$) so that $\pi_F$ is in the same $L$-packet as in $\tau$ (cf. Remark 3.1). This provides infinitely many examples of real analytic Siegel cusp forms of weight $(2,1)$ with integral Hecke polynomials. Note that this is an unconditional result. We summarize our result as follows:

**Theorem 10.1.** Let $f$ be a cusp form of weight one with respect to $\Gamma_0(N)$ with the central character $\epsilon$. Suppose $f$ is a Hecke eigenform. Then there exists a real analytic Siegel cusp form $F$ of weight $(2,1)$ with the eigenvalues $-\frac{5}{12}$ and $0$ for the generators $\Delta_1$ and $\Delta_2$, and with integral Hecke polynomials such that $\text{Sym}^3(\pi_F)$ is the transfer of $\pi_F$.

Note that the image of $\text{Sym}^3(\rho) : G_\mathbb{Q} \rightarrow GL_4(\mathbb{C})$, is in $GSp_4(\mathbb{C})$ (cf. [14], page 244), and the parameter $\text{Sym}^3(\rho) : G_\mathbb{Q} \rightarrow GSp_4(\mathbb{C})$ corresponds to $\pi_F$.

### 11. Siegel cusp forms of solvable type

Let $\rho : G_\mathbb{Q} \rightarrow GSp_4(\mathbb{C})$ and \(\bar{\rho} : G_\mathbb{Q} \rightarrow PGSp_4(\mathbb{C})\) be as in the introduction. In this section, we recall K. Martin’s result on the strong Artin conjecture of $\rho$ [32]. He showed the strong Artin conjecture when $\text{Im}(\bar{\rho})$ is a solvable group, $E_{16} \rtimes C_5$, where $E_{16} \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ is the elementary abelian group of order 16 and $C_5$ is the cyclic group of order 5. We denote by $Q_8$ (resp. $D_8$) the quaternion group of order 8 (resp. dihedral group of order 8). Note that $D_8$ here is denoted as $D_4$ in [18], page 35.
We will give an explicit example of such \( \rho \) which is taken from Section 5 of [32], but we make a slight change for the reader’s convenience.

Let \( \zeta_{11} \) be a primitive 11-th root of unity, and \( \alpha_i = \zeta_{11} + \zeta_{11}^{-i} \). Let \( E = \mathbb{Q}(\alpha_1) \) be a quintic extension over \( \mathbb{Q} \). Let

\[
K = E(\sqrt{u}), \quad M = E(\sqrt[4]{\alpha_1}, \sqrt{-1}),
\]

where \( u = (1 + \frac{1}{\sqrt{a}})(1 + \frac{1}{\sqrt{v}}), \ u = 1 + \alpha_3^2, \ v = 1 + \alpha_1^2 + \alpha_1^3 \alpha_3^2 \). Then \( K/E, M/E \) are Galois extensions with \( \text{Gal}(K/E) \simeq \mathbb{Q}_8, \ \text{Gal}(M/E) \simeq D_8 \). (For \( K/E \), let \( (\alpha, \beta, \gamma) = (\alpha_3, 0, \alpha_1) \) in Remark of [18], page 135, and for \( M/E \), see Theorem 2.2.7 of [18], page 35.)

Let \( L = E(\sqrt{u}, \sqrt{v}, \sqrt[4]{\alpha_1}, \sqrt{-1}) \) and \( L_0 = E(\sqrt{u}, \sqrt{v}, \sqrt[4]{\alpha_1}, \sqrt{-1})\). Then \( L \) is a subextension of \( KM \) of index 2 which corresponds to a subgroup of the central product \( \mathbb{Q}_8 D_8 = \text{Gal}(KM/E) \) of \( \mathbb{Q}_8 \) and \( D_8 \). Then

\[
\text{Gal}(L/\mathbb{Q}) \simeq ((\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \times D_8) \rtimes C_5, \quad \text{Gal}(L_0/\mathbb{Q}) \simeq ((\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \times D_8/\{\pm 1\}) \rtimes C_5 \simeq E_{16} \rtimes C_5.
\]

Note that \( D_8/\{\pm 1\} \) splits, and hence it is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \).

Therefore one has \( \text{Gal}(L/\mathbb{Q}) \hookrightarrow \text{GSp}_4(\mathbb{C}) \) by Section 5 of [32], and it gives rise to an Artin representation \( \rho : G_{\mathbb{Q}} \rightarrow \text{GSp}_4(\mathbb{C}) \). Further \( \bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{PGSp}_4(\mathbb{C}) \) gives \( \text{Gal}(L_0/\mathbb{Q}) \hookrightarrow \text{PGSp}_4(\mathbb{C}) \).

An explicit description of \( \rho \) is given as follows. Let \( J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Let

\[
A_1 = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \sqrt{-1}J_2 & 0 \\ 0 & \sqrt{-1}J_2 \end{pmatrix},
\]

\[
A_4 = \begin{pmatrix} 0 & \sqrt{-1}J_2 \\ \sqrt{-1}J_2 & 0 \end{pmatrix}, \quad A_5 = \text{diag}(1, -1, -1, 1),
\]

\[
T = -\frac{1 + \sqrt{-1}}{2} \begin{pmatrix} -\sqrt{-1} & 0 & 0 & \sqrt{-1} \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -\sqrt{-1} & \sqrt{-1} & 0 \end{pmatrix}.
\]

Then \( \langle A_1, A_2, A_3, A_4, A_5 \rangle \simeq \text{Gal}(L/E) \) and \( \langle T \rangle \simeq C_5 \) acts on \( \text{Gal}(L/E) \) by conjugation. The Galois action \( \sqrt{-1} \mapsto -\sqrt{-1} \) on \( L \) (and also on \( L_0 \)) corresponds to \( A_5 \). Clearly, \( \langle A_1, A_2, A_3, A_4, A_5 \rangle / \{ \pm I_4 \} \simeq \text{E}_{16} \).
Since the complex conjugate acts on $L_0$ non-trivially, $\rho(c) \neq \pm I_4$. Hence $\rho : G_Q \to GSp_4(\mathbb{C})$ is symplectically odd. K. Martin showed that $\rho$ is modular. So it corresponds to a cuspidal automorphic representation $\tilde{\Pi}$ of $GL_4(\mathbb{A})$, and descends to a cuspidal representation $\Pi$ of $GSp_4(\mathbb{A})$. Since $L(s, \Pi_p) = L(s, \rho_p)$ for almost all $p$, by [32], Appendix, $L(s, \rho_\infty) = L(s, \Pi_\infty)$. Since $L(s, \rho_\infty) = \Gamma_C(s)^2$, the Langlands parameter of $\Pi_\infty$ is $\phi : W_R \to GSp_4(\mathbb{C})$, which is the composition of $i : W_R \to G_R \hookrightarrow G_Q$ and $\rho$. Hence $\phi(z) = I_4$ and $\phi(j) \sim diag(1, -1, -1, 1)$. So $\Pi_\infty = Ind_B^G \chi(1, sgn, sgn)$. As in Section [10] there exists a real analytic Siegel cusp form of weight $(2,1)$ which corresponds to the Galois representation $\rho$. This gives an example of Siegel cusp form of weight $(2,1)$ with the eigenvalues $-\frac{5}{12}$ and 0 for the generators $\Delta_1$ and $\Delta_2$ and with integral Hecke polynomials, which does not come from $GL_2$ form.

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