Landau Damping in a weakly collisional regime

Xixia Ma *

Abstract. In this paper, we consider the nonlinear Vlasov-Poisson equations in a weakly collisional regime and study the linear Boltzmann collision operator. We prove that Landau damping still occurs in this case.

0. Introduction

In this paper, it is assumed that the plasma system is weakly collisional, nonrelativistic, hot. The kinetic theory is an effective method of studying the hot plasma particles. Perhaps the most widely used formulation of kinetic theory is the Boltzmann equation, for which the nonrelativistic form for particles of the $s$ species is

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} (E + v \times B) \cdot \nabla_v f_s = \frac{d f_s}{d t} \text{collisions}. \quad (0.1)$$

In Eq. (0.1), $f_s$, $E$, and $B$ may be thought of as the $s$–particle species distribution function $f_s(x,v,t)$, and the electric and magnetic fields in the plasma averaged over a spatial volume that contains many particles. It was A.A.Vlasov (1945) who first pointed out that Eq.(0.1) is dominated by the term on its left-hand side for a hot plasma. And for much of the study of waves in a hot plasma, it suffices to use the set of Vlasov equations in many situations,

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} (E + v \times B) \cdot \nabla_v f_s = 0. \quad (0.2)$$

For Eq.(0.2), it is well known that Mouhot and Villani[28] made a ground-breaking work when $B \equiv 0$. And recent we [26] prove Landau damping on Eq.(0.2) in a uniformly magnetic field case. In this paper, we consider the unmagnetized plasma in the weakly collisional case, that is,

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} E \cdot \nabla_v f_s = \nu \frac{d f_s}{d t} \text{collisions}. \quad (0.3)$$

where $\nu \in (0, \nu_0], \nu_0 > 0$ some small constant.

First we start with the linearized Vlasov equation in an unmagnetized plasma to analyze the effect on between Landau damping and collision. We write the linearized Vlasov equation in collisionless case as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{2}{m} E \cdot \nabla_v f^0 = 0, \\ f(x,v,t_0) = f_0(x,v). \end{cases} \quad (0.4)$$

We can solve Eq.(0.4) with a simple integration,

$$f(x,v,t) = f_0(x - vt, v) - \int_{t_0}^t \frac{q}{m} E(x - v(t-t'), t') \cdot \nabla_v f^0 dt'. \quad (0.5)$$

In order to simplify the analysis, we assume that $E(x,t)$ is known and we represent it as Re$E_1 \exp(i k x - i \omega t)$. Then the integration in Eq.(0.5) becomes

$$f(x,v,t) = f_0(x - vt, v) - \text{Re} \frac{i q E_1}{m} \frac{df^0(v)}{dv} \exp(i k x - i \omega t) \frac{1 - e^{i(\omega - kv)(t-t_0)}}{\omega - kv},$$

$$f(k,v,\omega) = -\text{Re} \frac{i q E_1}{m} \frac{df^0(v)}{dv} \frac{1}{\omega - kv} \quad (0.6)$$

Then from the first equality of Eq.(0.6), we can observe an important feature that $f(x,v,t)$ remains finite even at exact wave-particle resonance, $\omega - kv = 0$. On the other hand, the amplitude of $f(x,v,t)$ at resonance grows linearly with $t - t_0$ and for $t - t_0$ large, $f(x,v,t)$ becomes strongly oscillatory near resonance and displays a large peak exactly at resonance. However, when we consider the collisions among particles, we have to limit the magnitude of $t - t_0$. Now we use a really simple-minded model to simulate the collisional effect on $f(x,v,t)$ as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{2}{m} E \cdot \nabla_v f^0 = -\nu f, \\ f(x,v,t_0) = f_0(x,v). \end{cases} \quad (0.7)$$

*Yau Mathematical Sciences Center, Tsinghua University. E-mail addresses:kfmaxixia@tsinghua.edu.cn
Then we have
\[ f(x, v, t) = f_0(x - vt, v)e^{-\nu(t-t_0)} - \nu \text{Re} \left( \frac{iqE_1}{m} \frac{d f^0(v)}{d v} \right) e^{ikx - i\omega t} e^{-\nu(t-t_0)} \frac{1 - e^{i(\omega - kv)(t-t_0)}}{\omega - kv}. \]  
\hspace{10cm} (0.8)

For Eq.(0.8), we can regard \( e^{-\nu(t-t_0)} \) as the probability that any single particle, now at \( x, v, t \), suffered a collision at \( t_0 \) in the past, here \( \nu \) is the collision frequency. Then setting \( s = t - t_0 \), and averaging over the collision times for all particles that have reached \( x, v, t \), we obtain
\[ \langle f(x, v, t) \rangle = -\nu \text{Re} \left( \frac{iqE_1}{m} \frac{d f^0(v)}{d v} \right) e^{ikx - i\omega t} \int_0^\infty e^{-\nu s} \frac{1 - e^{i(\omega - kv)s}}{\omega - kv} ds \]
\hspace{10cm} (0.9)

From the second line of Eq.(0.9), first of all, we shall observe that the effect of collisions in this model has been to transform the appearance of \( \omega \), namely, \( \omega \rightarrow \omega + i\nu \). The second line of Eq.(0.9) also shows that \( f(x, v, t) \) is the product of two peaking functions, one depending on \( \frac{d f^0(v)}{d v} \) and the other on the resonance denominator, \( \omega - kv + i\nu \). And now we show that a mathematical representation that leads directly to Landau damping is to write the Fourier amplitude for \( f(x, v, t) \) as \( \nu \rightarrow 0 \),
\[ f(\omega, k, v) = \lim_{\nu \rightarrow 0^+} \frac{iqE(\omega, k) \frac{d f^0(v)}{d v}}{\omega - kv + i\nu} \frac{1}{\rho(\frac{1}{\omega - kv})} \frac{i\pi}{|k|} \delta(v - \omega) \]
\hspace{10cm} (0.10)

Although the \( f \) peak is infinitely sharp in Eq.(0.10), the moments of Eqs.(0.6) and (0.10) will be approximately the same provided that \( \frac{d f^0(v)}{d v} \) in Eq.(0.6) does not change appreciably over the range of \( v \) through which \( (\omega - kv)^{-1} \) is large. That is, the collisional model, Eq.(0.6), will lead to the same moments of \( \omega \rightarrow 0 \), and the other on the resonance denominator, \( 1 \) becomes slow comparing with the initial decay rate of the resonance denominator in Eq.(0.6) supplies the dominant damping.

In this paper, based on Mouhot and Villani’s work in [28], we consider the following model,
\[ \begin{align*}
\partial_t f + v \cdot \nabla_x f + \frac{2}{m} E \cdot \nabla_v f &= \nu C(f), \\
E &= \nabla W(x) \ast \rho(t, x), \\
\rho(t, x) &= \int f_0(t, x, v) dv, \\
f(0, x, v) &= f_0(x, v), f^0 = f^0(v),
\end{align*} \]
\hspace{10cm} (0.11)

where \( \nu \in (0, \nu_0] \), \( \nu_0 \) some small constant, and here \( C(f) \) represents the linear Boltzmann collisional case, namely
\[ C(f) = \rho f^0 - f. \]

We recall the related results on Landau damping on weakly collisional plasma as follows. First, if \( \nu = 0 \), some earlier results of the linearized Vlasov-Poisson equation were obtained in [8] by Caglioti and Maffei and in [21] by Hwang and Velázquez. We also refer to the work of Mouhot and Villani [28], they prove Landau damping (linear and nonlinear) in analytic or Gevrey regularity. Later Bedrossian, Masmoudi and Mouhot [6] give a simplified proof in Gevrey norm. Let us mention that lots of literature is devoted to the study of the Vlasov-Poisson-Boltzmann equation with a general Boltzmann collision operator, for example, the paper of Dolbeault and Desvillettes [13] that deals with the large time behavior of solutions and two papers of Guo on the Vlasov-Poisson-Boltzmann equation [18, 19] which is about the large time behaviour of solutions: the first one is in a near-vacuum regime, the second is in a near Maxwellian setting. Meanwhile, there are other many references which are concerned with the large-time behavior of solutions such as Duan and Strain [15], Duan, and Liu [14] and so on.

However, as far as the case of the linear Boltzmann equation, the literature on the stability is very scarce, even in a weakly collisional regime, namely, if \( \nu \rightarrow 0 \), the first paper on Landau damping in this case is by I. Tristani [31] for the linearized Vlasov-Poisson equation. It should be relevant to compare this kind of question with the one studied by Bedrossian, Masmoudi and Vivaldi [3] about the two-dimensional Euler equation where the equivalent of \( \nu \) should be viscosity. I. Tristani [31], Bedrossian [1], Bendrossian and Wang [7] also study this kind of problem of uniform analysis of large time behaviour in a weakly collisional regime through another different form: the Vlasov-Fokker-Planck model.

In this paper, we will consider the nonlinear Vlasov-Poisson equation on weakly collisional plasma. On one hand, different from the linear case, we have to face the resonance that the nonlinear term brings. In order to deal with this difficulty, the method we use is based on the one in [28]. On the other hand, comparing with the collisionless case in which the index of the decay rate becomes smaller, we find that for the linear case in weakly collisional case, the index of the decay rate is the same with the initial data, that is due to the effect of the weak collision. However, for the nonlinear case, the decay rate still becomes slow comparing with the initial
time because of the nonlinear term. In other words, weak collision does not change the dynamical behavior of the plasma.

This paper is organized as follows: Section 1 mainly introduces hybrid analytic norms and the related properties. Section 2 we will prove Landau damping at the linear level. We will state the proof of main theorem at the nonlinear level in section 3. And section 4 will show the deflection estimates of the particles trajectory, section 5 is the key section, it will state the phenomena of plasma echo. We will control the error terms in section 6, and give the iteration in section 7.

Now we state our main result as follows.

**Theorem 0.1** Let \( f^0 : \mathbb{R}^3 \to \mathbb{R}_+ \) be an analytic velocity profile, and let \( W(x) : \mathbb{T}^3 \to \mathbb{R} \) satisfy
\[
\dot{W}(0) = 0, \quad |\dot{W}(k)| \leq \frac{1}{1 + |k|^{\gamma}}, \quad \gamma > 1.
\]
Further assume that, for some constant \( \lambda_0 > 0 \),
\[
\sup_{\eta \in \mathbb{R}^3} e^{2\pi \lambda_0 |v|} |f^0(\eta)| \leq C_0, \quad \sum_{n \in \mathbb{N}_0^3} \frac{\lambda_0^n}{n!} \| \nabla_v f^0 \|_{L_2} \leq C_0 < \infty.
\]
(0.12)
Consider equations (0.11), there is \( \varepsilon = \varepsilon(\lambda_0, \mu_0, \beta, \gamma, \lambda_0', \mu_0') \) with the following property: if \( f_0 = f_0(x, v) \) is an initial data satisfying
\[
\sup_{k \in \mathbb{Z}^3, \eta \in \mathbb{R}^3} e^{2\pi \lambda_0 |v|} e^{2\pi \mu_0 |k|} |f^0 - f_0| + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |f^0 - f_0| e^{\beta |v|} dv dx \leq \varepsilon,
\]
where any \( \beta > 0 \).
At the same time, we also assume that the following stability condition holds:

**Stability condition**: for any velocity \( v \in \mathbb{R}^3 \), there exists some positive constant \( \nu_{Te} \) such that if \( v = \frac{\omega}{\beta} + i \frac{x}{\beta k} \), \( \omega, k \) are frequencies of time and space \( t, x \), respectively, then \( |v| \gg \nu_{Te} \).

Then for any fixed \( \eta, k, \forall r \in \mathbb{N} \), as \( \nu \to 0, |t| \to \infty \), we have
\[
\|E(t, \cdot)\|_{C^r(\mathbb{T}^3)} = O(e^{-2\pi \lambda_0'|t|}),
\]
where \( \rho_0 = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(x, v) dv dx \), for any \( 0 < \lambda_0' < \lambda_0 \).

**Remark 0.2** \( \gamma > 1 \) of Theorem 0.1 can be extended to \( \gamma \geq 1 \), the difference between \( \gamma > 1 \) and \( \gamma = 1 \) is the proof of the growth integral in section 7. The proof of \( \gamma = 1 \) is similar to section 7 in [28], here we omit this case.

**Remark 0.3** First, from the physics viewpoint, for the collisionless case, the condition that the damping occur is that the number of particles that the wave velocity greatly exceeds their velocity is much larger than the number of particles whose velocity is slower than the wave velocity. However, when considering the collision among particles, from the above theorem, we know that if very little energy due to collision loss, then when the stability condition of the collisionless case is satisfied, the damping still occurs. From the dynamical behavior viewpoint, it can also be understood that when the collision is very weak, the electric field play main role on the change of the plasma’ trajectories.

**Remark 0.4** During the proof, it is easily observed that the regularity loss become smaller because of the weak collision, because the collision term provides a regularity \( e^{-\nu t} \).

**Remark 0.5** Combining the results in this paper and the idea in our previous paper on Cyclotron damping in a uniform bounded magnetic field, Landau damping on Vlasov-Maxwell equations in a weakly collisional regime may be proved.

### 1 Notation and Hybrid analytic norm

Now we introduce some notations. We denote \( \mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3 \). For function \( f(x, v) \), we define the Fourier transform as follows.
For a function \( f = f(x) \), \( x \in \mathbb{T}^d \), we define its Fourier transform as follows:
\[
\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ix \cdot k} \, dx, \quad k \in \mathbb{Z}^d.
\]
Similarly, for a function \( f = f(v) \), \( v \in \mathbb{R}^d \), we define its Fourier transform by:
\[
\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(v) e^{-iv \cdot \xi} \, dv, \quad \xi \in \mathbb{R}^d.
\]
Finally, if \( f = f(x, v) \), \( (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \), we define its Fourier transform through the following formula:
\[
\hat{f}(k, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x, v) e^{-ikx - iv \cdot \xi} \, dx \, dv, \quad (k, \xi) \in \mathbb{Z}^d \times \mathbb{R}^d.
\]
We shall also use the Fourier transform in time, if \( f = f(t) \), \( t \in \mathbb{R} \), we denote
\[
\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-it \omega} \, dt, \quad \omega \in \mathbb{C}.
\]

Now we start to introduce the very important tools in our paper. These are time-shift pure and hybrid analytic norms. They are the same with those in the paper [28] written by Mouhot and Villani.

**Definition 1.1 (Hybrid analytic norms)**
\[
\|f\|_{C^{\lambda, \mu}} = \sum_{m,n \in \mathbb{N}_0} \frac{\lambda^n \mu^m}{n! \, m!} \|\nabla_x^m \nabla_v^n f\|_{L^\infty((\mathbb{T}^d \times \mathbb{R}^d)_1)}, \quad \|f\|_{\mathcal{F}^{\lambda, \mu}} = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |\hat{f}(k, \eta)| e^{2\pi \lambda |\eta|} e^{2\pi \mu |k|} \, d\eta,
\]
\[
\|f\|_{Z^{\lambda, \mu}} = \sum_{l \in \mathbb{Z}^3} \sum_{n \in \mathbb{N}_0} \frac{\lambda^n}{n!} e^{2\pi \mu |l|} \|\nabla_v^n \hat{f}(l, v)\|_{L^\infty(\mathbb{R}^2)}.
\]

**Definition 1.2 (Time-shift pure and hybrid analytic norms)** For any \( \lambda, \mu \geq 0, p \in [1, \infty] \), we define
\[
\|f\|_{C^{\lambda, \mu}} = \|f \circ S^0_0(x, v)\|_{C^{\lambda, \mu}} = \sum_{m,n \in \mathbb{N}_0} \frac{\lambda^n \mu^m}{n! \, m!} \|\nabla_x^m (\nabla_v + \tau \nabla_x)^n f\|_{L^\infty((\mathbb{T}^d \times \mathbb{R}^d)_2)},
\]
\[
\|f\|_{\mathcal{F}^{\lambda, \mu}} = \|f \circ S^0_0(x, v)\|_{\mathcal{F}^{\lambda, \mu}} = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |\hat{f}(k, \eta)| e^{2\pi \lambda |\eta + \tau|} e^{2\pi \mu |k|} \, d\eta,
\]
\[
\|f\|_{Z^{\lambda, \mu}} = \|f \circ S^0_0(x, v)\|_{Z^{\lambda, \mu}} = \sum_{l \in \mathbb{Z}^3} \sum_{n \in \mathbb{N}_0} \frac{\lambda^n}{n!} e^{2\pi \mu |l|} \|\nabla_v^n (\nabla_v + 2i \pi \tau \cdot l)^n \hat{f}(l, v)\|_{L^\infty(\mathbb{R}^2)},
\]
\[
\|f\|_{Z^{\lambda, \mu}} \leq \|f\|_{\mathcal{F}^{\lambda, \mu}} \leq \sup_{k \in \mathbb{Z}^d, \eta \in \mathbb{R}} e^{2\pi \mu |k|} e^{2\pi \lambda |\eta + k|} |\hat{f}(k, \eta)|.
\]

From the above definitions, we can state some simple and important propositions, and the related proofs can be found in [26,28], so we remove the proofs.

**Proposition 1.3** For any \( \tau \in \mathbb{R}, \lambda, \mu \geq 0 \),
(i) if \( f \) is a function only of \( x \), then \( \|f\|_{C^{\lambda, \mu}} = \|f\|_{C^{\lambda + \tau, \mu}} \), \( \|f\|_{\mathcal{F}^{\lambda, \mu}} = \|f\|_{\mathcal{F}^{\lambda + \tau, \mu}} \), \( \|f\|_{Z^{\lambda, \mu}} = \|f\|_{Z^{\lambda + \tau, \mu}} \);
(ii) if \( f \) is a function only of \( v \), then \( \|f\|_{C^{\lambda, \mu}} = \|f\|_{C^{\lambda, \mu + \tau}} \), \( \|f\|_{\mathcal{F}^{\lambda, \mu}} = \|f\|_{\mathcal{F}^{\lambda, \mu + \tau}} \);
(iii) for any \( \lambda > 0 \), then \( \|f \circ (\text{Id} + G)\|_{\mathcal{F}^{\lambda, \mu}} \leq \|f\|_{\mathcal{F}^{\lambda, \mu}} \);\( \|G\|_{\mathcal{F}^{\lambda, \mu}} \);
(iv) for any \( \lambda > \lambda, p \in [1, \infty] \), \( \|\nabla f\|_{C^{\lambda, \mu}} \leq \frac{1}{\lambda \log(\frac{\lambda}{\lambda})} \|f\|_{C^{\lambda, \mu}} \), \( \|\nabla f\|_{\mathcal{F}^{\lambda, \mu}} \leq \frac{1}{2 \pi \lambda (\lambda - \lambda)} \|f\|_{\mathcal{F}^{\lambda, \mu}} \);
(v) for any \( \lambda > \lambda > 0, \mu > 0 \), then \( \|v\|_{Z^{\lambda, \mu}} \leq \|f\|_{Z^{\lambda, \mu}} \);
(vi) for any \( \lambda > \lambda > \mu > \mu \), \( \|\nabla_v f\|_{Z^{\lambda, \mu}} \leq C \left( \frac{1}{\lambda \log(\frac{\lambda}{\lambda})} \|f\|_{Z^{\lambda, \mu}} + \frac{\tau}{\mu} \|f\|_{Z^{\lambda, \mu}} \right) \);
(vii) for any \( \lambda > \lambda > \lambda \), \( \|\nabla_v + \tau \nabla_x\|_{Z^{\lambda, \mu}} \leq \frac{1}{C \log(\frac{\lambda}{\lambda})} \|f\|_{Z^{\lambda, \mu}} \);
For any $\lambda \geq \rho \geq 0$, $\mu \geq \mu \geq 0$, then $\|f\|_{L^\infty} \leq \|f\|_{L^\infty}$. Moreover, for any $\tau, \tau \in \mathbb{R}$, $p \in [1, \infty]$, we have $\|f\|_{L^{p,\infty}} \leq \|f\|_{L^{p,\infty}}$.

(iii) For any function $f = f(x, v)$, $\|f\|_{L^{p,\infty}} \leq \|f\|_{L^{p,\infty}}$.

Proposition 1.4 For any $X \in \{C, F, Z\}$ and any $t, \tau \in \mathbb{R}$,

$$\|f \circ S_t^0\|_{X^{\lambda,\mu}} = \|f\|_{X^{\lambda,\mu}}.$$  

Lemma 1.5 Let $\lambda, \mu \geq 0$, $t \in \mathbb{R}$, and consider two functions $F, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then there is $\xi \in (0, \frac{1}{2})$ such that if $F, G$ satisfy

$$\|\nabla (F - Id)\|_{L^{\lambda,\mu}} \leq \xi,$$

where $\lambda = \lambda + 2\|F - G\|_{L^{\lambda,\mu}}$, $\mu = \mu + 2(1 + |\tau|)\|F - G\|_{L^{\lambda,\mu}}$, then $F$ is invertible and

$$\|F^{-1} \circ G - Id\|_{L^{\lambda,\mu}} \leq 2\|F - G\|_{L^{\lambda,\mu}}.$$  

Proposition 1.6 For any $\lambda, \mu \geq 0$ and any $p \in [1, \infty], \tau \in \mathbb{R}, \sigma \in \mathbb{R}, \rho \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, we have

$$\|f(x + bv + X, av + V(x, v))\|_{L^{\lambda,\mu}} \leq |a|^2\|f\|_{L^{\lambda,\mu}},$$

where $\alpha = \lambda|a| + \|V\|_{L^{\lambda,\mu}}$, $\beta = \mu + \beta + \tau - \alpha\sigma + \|X - aV\|_{L^{\lambda,\mu}}$.

Lemma 1.7 Let $G = G(x, v)$ and $R = R(x, v)$ be valued in $\mathbb{R}$, and $\beta(x) = \int_{\mathbb{R}^3} (G \cdot R)(x, v)dv$. Then for any $\lambda, \mu, t \geq 0$ and any $b > -1$, we have

$$\|\beta\|_{L^{\lambda,\mu}} \leq 3\|G\|_{L^{\lambda,\mu}, \mu} \|R\|_{L^{\lambda,\mu}}.$$  

2 Linearized Landau damping in weakly collisional plasma

In this section, we consider the linearized Vlasov-Poisson equations in the weakly collisional case as follows:

$$\left\{ \begin{array}{l}
\partial_t f + v \cdot \nabla_x f + \frac{2}{m} E \cdot \nabla_v f^0 = \nu C(f), \\
E = \nabla W(x) \ast \rho(t, x), \\
C(f) = \rho f_0^0 - f, \\
\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v)dv, \\
f(0, x, v) = f_0(x, v), f^0 = f_0^0(v),
\end{array} \right.$$  

for any $\nu \in [0, \nu_0], \nu_0$ some small constant.

Theorem 2.1 Consider equations (2.1). For any $\eta, \nu \in \mathbb{R}^3, k \in \mathbb{N}_0^3$, assume that the following conditions hold:

(i) $\hat{W}(0) = 0$ where $|W(k)| \leq \frac{1}{1 + |k|^2}, \gamma > 1$;

(ii) $\|\nu(\nu + 1)\|_{C^{\lambda,\mu}} \leq C_0$, for some constants $\lambda, C_0 > 0$;

(iii) $\|\nu(t)\|_{L^{\lambda,\mu}} \leq \delta_0$ for some constants $\mu > 0, \delta_0 > 0$;

(iv) For any velocity $v \in \mathbb{R}^3$, there exists some positive constant $v_{Tc} \in \mathbb{R}$ such that if $v = \frac{\hat{k}}{k} + i\frac{\hat{k}}{2\pi}$, then $|v| \geq v_{Tc}$.

Then for any fixed $k$, $\eta$, we have

$$\|f(t, k, \eta) - f_0(k, \eta)\| \leq C(C_0, \delta_0) e^{-\gamma|\nu + b|k|} e^{-\gamma|k|} \max \left\{ e^{-\nu t}, 1 - \frac{1 - e^{-\nu t}}{\nu}, 1 - e^{-\nu t} \right\},$$

$$|\rho(t, k) - \rho_0| \leq C(C_0, \delta_0) e^{-\gamma|\lambda + \mu|k|}$$

$$|\dot{E}(t, k)| \leq C(C_0, \delta_0) e^{-\gamma|\lambda|t} e^{-\gamma|k|}.$$  

(2.2)

where $\rho_0 = \int_{\mathbb{R}^3} f_0(x, v)dvdx.$
Remark 2.2 In the linear case, before proving Theorem 2.1, we recall the result of I. Tristani in [31]. We define
\[
\begin{align*}
K_0^0(t, k) &:= e^{-\nu t} \tilde{f}^0(kt), \\
K_1^0(t, k) &:= -\tilde{W}(k)e^{-\nu t} f^0(kt)|k|^2, \\
K_\nu(t, k) &:= K_0^0(t, k) + K_1^0(t, k), \\
\mathcal{L}_\nu(\eta, k) &= \int_0^\infty e^{2\nu \eta^*|k|^2} K_\nu(t, k) dt,
\end{align*}
\] (2.3)
where $\eta^*$ is the complex conjugate to $\eta$. The stability condition of I. Tristani’s work is as follows: there exists $\varepsilon_0 > 0$ such that $\mathcal{L}_\nu$ satisfies the following condition: for some constant $\kappa > 0$, $\forall \nu \in [0, \nu_0],\$
\[
(H) \quad \inf_{k \in \mathbb{Z} \setminus \{0\}} \inf_{|\eta| \leq 0} |1 - \mathcal{L}_\nu(\eta, k)| \geq \kappa.
\]
Comparing (H) condition, our stability condition in Theorem 2.1 are suitable for the physical intuitive from the energy viewpoint. From the condition of the classical KAM theory, our condition is in correspondence to the Diophantus condition in KAM theory in some sense.

The proof of Theorem 2.1. Without loss of generality, we assume $t \geq 0$. We consider (2.1) as a perturbation of free transport and apply the Duhamel’s formula to get
\[
f(t, x, v) = f_0(x - vt, v)e^{-\nu t} + \nu \int_0^t e^{-\nu (t-s)} \rho(s, x - v(t-s)) f^0(v) ds
\]
\[
- \int_0^t e^{-\nu (t-s)} (E \cdot \nabla_v f^0)(s, x - v(t-s), v) ds.
\]
Then we take the Fourier transform in both variables $(x, v)$,
\[
\hat{f}(t, k, \xi) = e^{-\nu t} \hat{f}_0(k, \xi + kt) + \nu \int_0^t e^{-\nu (t-s)} \rho(s, \xi + k(t-s)) \hat{f}_0(k, \xi + k(t-s)) ds.
\]

and from which we can deduce
\[
\hat{f}(t, k, \xi) = e^{-\nu t} \hat{f}_0(k, \xi + kt) + \nu \int_0^t e^{-\nu (t-s)} \tilde{\rho}(s, k) \hat{f}_0(\xi + k(t-s)) ds
\]
\[
- \int_0^t e^{-\nu (t-s)} k \cdot (\xi + k(t-s)) \tilde{W}(k) \tilde{\rho}(s, k) \hat{f}_0(\xi + k(t-s)) ds.
\]
(2.4)

Then taking $\xi = 0$, we obtain the closed equation on $\tilde{\rho}(t, k) :
\[
\hat{\rho}(t, k) = e^{-\nu t} \hat{f}_0(k, kt) + \nu \int_0^t e^{-\nu (t-s)} \rho(s, k) \hat{f}_0(k, k(t-s)) ds
\]
\[
- \int_0^t e^{-\nu (t-s)} k \cdot (k(t-s)) \tilde{W}(k) \rho(s, k) \hat{f}_0(k(t-s)) ds.
\]
(2.5)

Recall the definition of $K_\nu$, we have
\[
\hat{\rho}(t, k) = e^{-\nu t} \hat{f}_0(k, kt) + \int_0^t K_\nu(t-s, k) \rho(s, k) ds.
\]
(2.6)

First we assume $k \neq 0$, and consider $\lambda > 0$, write
\[
\Phi(t, k) = \hat{\rho}(t, k) e^{2\pi \lambda |k|^2 t} \quad \text{and} \quad A(t, k) = \hat{f}_0(t, k) e^{-\nu t} e^{2\pi \lambda |k|^2 t};
\]
then (0.6) becomes
\[
\Phi(t, k) = A(t, k) + \int_0^t K_\nu(t-s, k) e^{2\pi \lambda |k|(t-s)} \Phi(s, k) ds.
\]
(2.7)

We take the Fourier transform in time variable, after extending $K$, $A$ and $\Phi$ by 0 at negative times. We have,
\[
\tilde{\Phi}(\omega, k) = \tilde{A}(\omega, k) + \mathcal{L}_\nu(\omega, k) \tilde{\Phi}(\omega, k).
\]
By the Stability condition, let \( \eta = \frac{v}{\nu} + i \frac{\nu}{2\pi k} \),

\[
\tilde{\mathcal{L}}_\nu(\omega, k) = -\frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{-\nu t} e^{2\pi \lambda |t|/d} e^{-2\pi i k \cdot \nu} \left| k \hat{W}(k) \partial_\nu f^0 + v f^0(\nu) \right| dv dt
\]

(2.8)

\[
\leq \sup_\omega \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{-\nu t} e^{2\pi i \omega t} e^{-2\pi i k \nu} \left| \sum_n \frac{2\pi i \lambda_n |t|^n}{n!} \left| k \hat{W}(k) \nabla \nabla f^0(\nu) + v f^0(\nu) \right| dv dt \right|
\]

\[
= \sup_\omega \frac{q}{m} \sum_n \frac{\lambda_n}{n!} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{-\nu t} e^{2\pi i k \nu} \left| k \hat{W}(k) \nabla \nabla f^0(\nu) + v \nabla f^0(\nu) \right| dv dt \right|
\]

= \sup_\omega \frac{q}{m} \sum_n \frac{\lambda_n}{n!} \left| \left( -i \left( k \hat{W}(k) \nabla \nabla f^0(\nu) + v \nabla f^0(\nu) \right) \right) \right| \leq \frac{q}{m v T e} e^{-c_0 v t},
\]

where in the last inequality we use the stability condition (iv) that if \( v = \frac{v}{\nu} + i \frac{\nu}{2\pi k} \), then \( \nu \gg v T e \), and the assumption (i) and (iii). Then there exists some constant \( 0 < \kappa < 1 \) such that \( \| \tilde{\mathcal{L}}_\nu(\omega, k) \|_{L^\infty} \leq \kappa \).

Now we apply the Plancherel’s identity to find (for each \( k \))

\[
\| \Phi \|_{L^2(dt)} \leq \frac{\| A \|_{L^2(dt)}}{\kappa}.
\]

Then we plug this into the equation (2.7) to get

\[
\| \Phi \|_{L^\infty(dt)} \leq \| A \|_{L^\infty(dt)} + \| K e^{2\pi \lambda |t|/d} \|_{L^2(dt)} \| \Phi \|_{L^2(dt)}
\]

\[
\leq \| A \|_{L^\infty(dt)} + \| K e^{2\pi \lambda |t|/d} \|_{L^2(dt)} \| A \|_{L^2(dt)}.
\]

(2.9)

Through simple computation, we can obtain

\[
\sup_{t \geq 0} \left( \sum_{k \in \mathbb{Z} \setminus 0} |\hat{\rho}(t, k)| e^{2\pi (\lambda t + \nu) |k|} \right)
\]

\[
\leq C(\lambda, \kappa) \sup_{t \geq 0} \sum_{k \in \mathbb{Z} \setminus 0} |\tilde{f}(0, k, t)| e^{2\pi (\lambda t + \nu) |k|} e^{-\nu t}
\]

\[
\leq C(\lambda, \kappa) \sup_{t \geq 0} \sum_{k \in \mathbb{Z} \setminus 0} |\tilde{f}(0, k, t)| e^{2\pi (\lambda t + \nu) |k|} e^{-\nu t}.
\]

Equivalently,

\[
\sup_{t \geq 0} \| \rho(t, \cdot) \|_{\mathcal{F}^{\lambda, \nu}} \leq C \sup_{t \geq 0} \int_{\mathbb{R}^d} f_0 \circ S_{-t}^0 dv \| f_0 \|_{\mathcal{F}^{\lambda, \nu}} e^{-\nu t} \leq \sup_{t \geq 0} \| f_0 \circ S_{-t}^0 \|_{\mathcal{F}^{\lambda, \nu}} e^{-\nu t} = \| f_0 \|_{\mathcal{F}^{\lambda, \nu}} \leq \delta_0.
\]

We write

\[
f(t, \cdot) = (f_0 \circ S_{-t}^0) e^{-\nu t} - \int_0^t e^{-(t-s)\nu} \left( (\nabla W * \rho_s) \circ S_{-t}^0 \right) \cdot \nabla f^0 ds + \nu \int_0^t e^{-(t-s)\nu} (\rho_s \circ S_{-t}^0) \cdot \nabla f^0 ds,
\]

where \( \rho_s = \rho(s, \cdot) \). Then we have, for all \( t \geq 0 \),

\[
\| f \|_{\mathcal{F}^{\lambda, \nu}} \leq \| f_0 \circ S_{-t}^0 \|_{\mathcal{F}^{\lambda, \nu}} e^{-\nu t} + \int_0^t e^{-(t-s)\nu} \| (\nabla W * \rho_s) \circ S_{-t}^0 \|_{\mathcal{F}^{\lambda, \nu}} \| \nabla f^0 \|_{\mathcal{F}^{\lambda, \nu}} ds + \nu \int_0^t e^{-(t-s)\nu} \| \nabla f^0 \|_{\mathcal{F}^{\lambda, \nu}} ds
\]

\[
= \| f_0 \|_{\mathcal{F}^{\lambda, \nu}} e^{-\nu t} + \| \nabla f^0 \|_{\mathcal{F}^{\lambda, 1}} \int_0^t e^{-(t-s)\nu} \| \nabla W * \rho_s \|_{\mathcal{F}^{\lambda, 1}} ds + \nu \| \nabla f^0 \|_{\mathcal{F}^{\lambda, 1}} \int_0^t e^{-(t-s)\nu} \| \rho_s \|_{\mathcal{F}^{\lambda, 1}} ds
\]

(2.10)
Now we consider the Vlasov equation in step 
\[ \hat{\rho}(t,0) = e^{-\nu t} \hat{f}_0(0,0) + \nu \int_0^t e^{-\nu(t-s)} \hat{\rho}(s,0) ds, \]
(2.11)
and
\[ \hat{f}(t,0,\xi) = e^{-\nu t} \hat{f}_0(0,\xi) + \nu \int_0^t e^{-\nu(t-s)} \hat{\rho}(s,0) \hat{f}(\xi) ds. \]
(2.12)
If we assume \( f_0 \) a mean-zero distribution, then we have
\[ \hat{\rho}(t,0) \equiv 0, \quad \text{for all} \quad t \geq 0, \]
and
\[ \| \hat{f}(t,0,\xi) \|_{L^1} \leq e^{-\nu t} \| \hat{f}_0(0,\xi) \|_{L^1}. \]
(2.13)

### 3 Nonlinearized picture in weakly collisional plasma

We next give the proof of the main theorem 0.1, stating the primary steps as propositions which are proved in subsections.

#### 3.1 The Newton iteration

First of all, we write a classical Newton iteration: Let
\[ f^n = f^0(v) \quad \text{be given}, \]
and
\[ f^{n+1} = f^0 + h^1 + \ldots + h^n, \]
where
\[ \left\{ \begin{array}{l}
\partial_h h^1 + v \cdot \nabla_x h^1 + E[h^1] \cdot \nabla_v f^0 = \nu (\rho[h^1] f^0 - h^1), \\
h^1(0,x,v) = f_0 - f^0, 
\end{array} \right. \]
(3.1)
and now we consider the Vlasov equation in step \( n+1, n \geq 1, \)
\[ \left\{ \begin{array}{l}
\partial_h h^{n+1} + v \cdot \nabla_x h^{n+1} + E[f^n] \cdot \nabla_v h^{n+1} \\
\equiv - E[h^{n+1}] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n + \nu (\rho[h^{n+1}] f^0 - h^{n+1}), \\
h^{n+1}(0,x,v) = 0, 
\end{array} \right. \]
(3.2)
the corresponding dynamical system is described as follows: for any \( (x,v) \in T^3 \times R^3, \) let \( (X^n_{t,s}, V^n_{t,s}) \) as the solution of the following ordinary differential equations
\[ \left\{ \begin{array}{l}
\frac{d}{dt} X^{n+1}_{t,s}(x,v) = V^{n+1}_{t,s}(x,v), \\
X^{n+1}_{t,s}(x,v) = x, 
\end{array} \right. \]
and
\[ \left\{ \begin{array}{l}
\frac{d}{dt} V^{n+1}_{t,s}(x,v) = E[f^n](t,X^n_{t,s}(x,v)), \\
V^{n+1}_{t,s}(x,v) = v. 
\end{array} \right. \]
(3.3)
At the same time, we consider the corresponding linear dynamics system as follows,
\[ \left\{ \begin{array}{l}
\frac{d}{dt} X^0_{t,s}(x,v) = V^0_{t,s}(x,v), \\
X^0_{t,s}(x,v) = x, \\
X^0_{t,s}(x,v) = x, \\
V^0_{t,s}(x,v) = v. 
\end{array} \right. \]
(3.4)
It is easy to check that
\[ \Omega^n_{t,s} - Id \Delta \left( \delta X^n_{t,s}, \delta V^n_{t,s} \right) \circ (X^0_{t,s}, V^0_{t,s}) = (X^n_{t,s} \circ (X^0_{t,s}, V^0_{t,s}) - Id, V^n_{t,s} \circ (X^0_{t,s}, V^0_{t,s}) - Id). \]

Therefore, in order to estimate \( (X^n_{t,s} \circ (X^0_{t,s}, V^0_{t,s}) - Id, V^n_{t,s} \circ (X^0_{t,s}, V^0_{t,s}) - Id) \), we only need to study \( (\delta X^n_{t,s}, \delta V^n_{t,s}) \circ (X^0_{t,s}, V^0_{t,s}) \).

From Eqs. (3.3) and (3.4),
\[ \left\{ \begin{array}{l}
\frac{d}{dt} X^{n+1}_{s,t}(x,v) = \delta V^{n+1}_{s,t}(x,v), \\
\delta X^{n+1}_{s,t}(x,v) = 0. 
\end{array} \right. \]
Integrating (3.2) in time and \( h^{n+1}(0, x, v) = 0 \), we get

\[
\begin{align*}
\{ \frac{d}{dt} &\delta V_{s,t}^{n+1}(x, v) = E[f^n](t, X_{s,t}^n(x, v)), \\
\delta V_{s,t}^{n+1}(x, v) = 0 &\}
\] (3.5)
\end{align*}
\]

where

\[
\begin{align*}
\Sigma^{n+1}(t, x, v) &= -E[h^{n+1}] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n + \nu \rho[h^{n+1}] f^0.
\end{align*}
\]

By the definition of \((X_{s,t}^n(x, v), V_{s,t}^n(x, v))\), we have

\[
\begin{align*}
h^{n+1}(t, x, v) &= \int_0^t e^{-\nu(t-s)} \Sigma^{n+1}(s, X_{s,t}^n(x, v), V_{s,t}^n(x, v)) ds \\
&= \int_0^t e^{-\nu(t-s)} \Sigma^{n+1}(s, X_{s,t}^n(x, v)) ds \\
&= \int_0^t e^{-\nu(t-s)} \Sigma^{n+1}(s, X_{s,t}^n(x, v)) ds + V_{s,t}^n(x, v) ds.
\end{align*}
\]

Since the unknown \( h^{n+1} \) appears on both sides of (3.6), we hope to get a self-consistent estimate. For this, we have little choice but to integrate in \( t \) and get an integral equation on \( \rho[h^{n+1}] = \int_{\mathbb{R}^3} h^{n+1} dv \), namely

\[
\begin{align*}
\rho[h^{n+1}](t, x) &= \int_0^t \int_{\mathbb{R}^3} e^{-\nu(t-s)} (\Sigma^{n+1} \circ \Omega_{s,t}^n(x, v))(s, X_{s,t}^n(x, v), V_{s,t}^n(x, v)) dv ds \\
&= \int_0^t \int_{\mathbb{R}^3} e^{-\nu(t-s)} \left[ (\mathcal{E}_{s,t}^{n+1}, G_{s,t}^n) - (\mathcal{E}_{s,t}^n \circ H_{s,t}^n) + \nu (\rho[h^{n+1}] f^0) \circ \Omega_{s,t}^n(x, v) \right] ds, \\
&= \int_{\mathbb{R}^3} \rho[h^{n+1}](x, v) ds,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{E}_{s,t}^{n+1} &= E[h^{n+1}] \circ \Omega_{s,t}^n(x, v), \\
\mathcal{E}_{s,t}^n &= E[h^n] \circ \Omega_{s,t}^n(x, v), \\
G_{s,t}^n &= (\nabla_v f^n) \circ \Omega_{s,t}^n(x, v), \\
H_{s,t}^n &= (\nabla_v h^n) \circ \Omega_{s,t}^n(x, v).
\end{align*}
\]

### 3.2 Inductive hypothesis

For \( n=1 \), from (3.1), it is known that (3.1) is a linear Vlasov equation. From section 2, the conclusions of Theorem 0.1 hold.

Now for any \( i \leq n, i \in \mathbb{N}_0 \), we assume that the following estimates hold,

\[
\begin{align*}
\sup_{t \geq 0} \| \rho[h^{i}](t, \cdot) \|_{L^{\lambda_i+\nu_i}}, \\
\sup_{0 \leq s \leq t} \| h^{i} \circ \Omega_{s,t}^{i-1} \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i,
\end{align*}
\]

then we have the following inequalities, denote \((E^n)\):

\[
\begin{align*}
\sup_{t \geq 0} \| E[h^{i}](t, \cdot) \|_{L^{\lambda_i+\nu_i}} \leq \delta_i, \\
\sup_{0 \leq s \leq t} \| \nabla_x(h^{i} \circ \Omega_{s,t}^{i-1}) \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i, \\
\sup_{0 \leq s \leq t} \| (\nabla_x h^{i}) \circ \Omega_{s,t}^{i-1} \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i, \\
\| (\nabla_v + s \nabla x)(h^{i} \circ \Omega_{s,t}^{i-1}) \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i, \\
\| (\nabla_v + s \nabla x) h^{i} \circ \Omega_{s,t}^{i-1} \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i, \\
\frac{1}{(1 + s)^2} \| (\nabla \nabla h^{i}) \circ \Omega_{s,t}^{i-1} \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i, \\
\sup_{0 \leq s \leq t} (1 + s)^2 \| (\nabla_v h^{i}) \circ \Omega_{s,t}^{i-1} - \nabla_v h^{i} \circ \Omega_{s,t}^{i-1} \|_{L^{\lambda_i(1+s)+\nu_i+1}} \leq \delta_i.
\end{align*}
\]

It is easy to check that the first inequality of \((E^n)\) holds since \( E[h^{i}] \) satisfies the Poisson equation, so we only need to show that the other inequalities of \((E^n)\) also hold, the related proofs are found in section 4.
3.3 Local time iteration

Before working out the core of the proof of Theorem 0.1, we shall show a short time estimate, which will play a role as an initial data layer for the Newton scheme. The main tool in this section is given by the following lemma.

**Lemma 3.1** Let $f$ be an analytic function, $\lambda(t) = \lambda - Kt$ and $\mu(t) = \mu - Kt$, $K > 0$, let $T > 0$ be so small that $\lambda(t) > 0, \mu(t) > 0$ for $0 \leq t \leq T$. Then for any $s \in [0, T]$ and any $p \geq 1$,

$$\left. \frac{d^+}{dt} \right|_{t=s} \|f\|_{\tilde{Z}^{\lambda_t, \mu(t)}_{1, p}} \leq -\frac{K}{1 + s} \|\nabla f\|_{\tilde{Z}^{\lambda(t), \mu(t), p}}$$

where $\frac{d^+}{dt}$ stands for the upper right derivative.

**Proposition 3.2** There exists some small constant $T > 0$, such that when all conditions of Theorem 0.1 hold, then for any fixed $\eta, \kappa$, for all $t \in [0, T]$, $0 < \lambda < \lambda_0$, we have

$$|\hat{f}(t, k, n) - \hat{f}_0(k, n)| \leq C(C_0, \delta_0) e^{-2\pi\lambda|n+k|} e^{-2\mu|k|},$$

$$|\rho(t, k) - \rho_0| \leq C(C_0, \delta_0) e^{-2\pi(\lambda+\mu)|k|},$$

$$|\hat{E}(t, k)| \leq C(C_0, \delta_0) e^{-2\pi\lambda|t|} e^{-2\pi\mu|k|}. \tag{3.9}$$

where $\rho_0 = \int_{\mathbb{R}^2} f_0(x, \nu) d\nu dx$.

**Proof.** The first stage of the iteration, namely, $h^1$ was considered in §2. So we only need to care about the higher orders. Recall that $f^k = f^0 + h^1 + \ldots + h^k$. And we define

$$\lambda_k(t) = \lambda - 2Kt \quad \text{and} \quad \mu_k(t) = \mu - Kt,$

where $\{\lambda_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ are decreasing sequences of positive numbers.

We assume inductively that at stage $n$ of the iteration, we have constructed $\{\lambda_k\}_{k=1}^n, \{\mu_k\}_{k=1}^n, \{\delta_k\}_{k=1}^n$ such that

$$\sup_{0 \leq t \leq T} \|h^k(t, \cdot)\|_{\tilde{Z}^{\lambda_k(t), \mu_k(t), 1}} \leq \delta_k \quad \text{for all} \quad 1 \leq k \leq n$$

for some fixed $T > 0$.

In the following we need to show that the induction hypothesis are satisfied at stage $n + 1$. For this, we have to construct $\lambda_{n+1}, \mu_{n+1}, \delta_{n+1}$.

Note that $h_{n+1} = 0$, at $t = 0$. For $n \geq 1$, now let us solve

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} = \hat{S}^{n+1},$$

where

$$\hat{S}^{n+1} = -E[f^n] \cdot \nabla_v h^{n+1} - E[h^{n+1}] \cdot \nabla_v f^n - E[h^{n}] \cdot \nabla_v h^{n+1} + \nu(\partial_t h^{n+1}) f^0 - h^{n+1}. \tag{3.10}$$

Hence,

$$\|h^{n+1}\|_{\tilde{Z}^{\lambda_{n+1}(t), \mu_{n+1}(t), 1}} \leq \int_0^t e^{-\nu(t-s)} \|\hat{S}^{n+1}\|_{Z^{\lambda_{n+1}(t), \mu_{n+1}(t), 1}} ds \leq \int_0^t e^{-\nu(t-s)} \|\hat{S}^{n+1}\|_{Z^{\lambda_{n+1}(t), \mu_{n+1}(t), 1}} ds,$$

where

$$\hat{S}^{n+1} = -E[f^n] \cdot \nabla_v h^{n+1} - E[h^{n+1}] \cdot \nabla_v f^n - E[h^{n}] \cdot \nabla_v h^{n+1} + \nu(\partial_t h^{n+1}) f^0.$$
We gather the above estimates,

\[
\frac{d^+}{dt} \|h^{n+1}\|_{\mathcal{E}_1^{\lambda, n+1, \mu, n+1}(t)} \leq \left( C \sum_{i=1}^n \frac{\delta_i}{\min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\}} - K + \nu C_0 \right) \|\nabla h^{n+1}\|_{\mathcal{E}_1^{\lambda, n+1, \mu, n+1}} \\
+ \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}, \mu_n - \mu_{n+1}\}}.
\]

We may choose

\[
\delta_{n+1} = \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}, \mu_n - \mu_{n+1}\}}
\]

if

\[
C \sum_{i=1}^n \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\} \leq K - \nu C_0
\]

holds.

We choose \(\lambda_i - \lambda_{n+1} = \mu_i - \mu_{n+1} = \frac{\lambda}{\Delta}\), where \(\Delta > 0\) is arbitrarily small. Then for \(i \leq n, \lambda_i - \lambda_{n+1} \geq \frac{\lambda}{\Delta}\), and \(\delta_{n+1} \leq \delta_n^2 / \Lambda\). Next we need to check that \(\sum_{n=1}^{\infty} \delta_n n^2 < \infty\). In fact, we choose \(K\) large enough and \(T\) small enough such that \(\lambda_0 - KT \geq \lambda_*, \mu_0 - KT \geq \mu_*,\) and (3.9) holds, where \(\lambda_0 > \lambda_*, \mu_0 > \mu_*\) are fixed.

If \(\delta_1 = \delta\), then \(\delta_n = n^2 \delta (2^2)^{n^2-2} (4^2)^{n^2-2} \ldots ((n-1)^2)^{2^n-2}\). To prove the sequence convergence for \(\delta\) small enough, by induction that \(\delta_n \leq z^n\), where \(z\) small enough and \(a \in (1, 2)\). We claim that the conclusion holds for \(n+1\). Indeed, \(\delta_{n+1} \leq \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}, \mu_n - \mu_{n+1}\}} \leq \frac{z^n}{\min\{\lambda_n - \lambda_{n+1}, \mu_n - \mu_{n+1}\}} \leq z^{n+1} \leq z^{a^n + 1}\). If \(z\) is so small that \(z^{(2-a)\alpha} \leq \frac{\lambda}{\Delta}\) for all \(n \in \mathbb{N}\), then \(\delta_{n+1} \leq z^{a^n} + 1\), this concludes the local-time argument.

### 3.4 Global time iteration

Based on the estimates of the local-time iteration, without loss of generality, sometimes we only consider the case \(s \geq \frac{bT}{b+t}\), where \(b\) is small enough.

First, we give deflection estimates that compare the free evolution with the true evolution for the particles trajectories.

**Proposition 3.3** Assume for any \(i \in \mathbb{N}, 0 < i \leq n\),

\[
\sup_{t \geq 0} \|E[h^i(t, \cdot)]\|_{\mathcal{F}_1^{\lambda, i+1, \mu, i+1}} < \delta_i.
\]

And there exist constants \(\lambda_0 > 0, \mu_0 > 0\) such that \(\lambda_0 > \lambda_0 > \lambda_1 > \lambda_1 > \ldots > \lambda_i > \lambda_i > \ldots > \lambda_*, \mu_0 > \mu_1 > \mu_1 > \ldots > \mu_i > \mu_i > \ldots > \mu_0\).

Then we have

\[
\|\delta X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0)\|_{\mathcal{E}_1^{\lambda, i+1, \mu, i+1}} \leq C \sum_{i=1}^n \delta_i e^{-\pi(\lambda_i - \lambda_i')s} \min\left\{\frac{(t-s)^2}{2}, \frac{1}{2\pi(\lambda_i - \lambda_i')^2}\right\},
\]

\[
\|\delta V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0)\|_{\mathcal{E}_1^{\lambda, i+1, \mu, i+1}} \leq C \sum_{i=1}^n \delta_i e^{-\pi(\lambda_i - \lambda_i')s} \min\left\{\frac{(t-s)^2}{2}, \frac{1}{2\pi(\lambda_i - \lambda_i')^2}\right\},
\]

for \(0 < s < t, b = b(t, s)\) sufficiently small.

**Remark 3.4** From the above proposition, we know that weak collision has little impact on the trajectory of the plasma particles.

**Proposition 3.5** Under the assumptions of Proposition 3.3, then

\[
\left\|\nabla \Omega^{n+1} X_{t,s} = (Id, 0)\right\|_{\mathcal{E}_1^{\lambda, i+1, \mu, i+1}} < C_1^n, \quad \left\|\nabla \Omega^{n+1} V_{t,s} = (0, Id)\right\|_{\mathcal{E}_1^{\lambda, i+1, \mu, i+1}} < C_2^n,
\]

where \(C_1^n = C \sum_{i=1}^n e^{-\pi(\lambda_i - \lambda_i')s} \min \left\{\frac{(t-s)^2}{2}, 1\right\}, C_2^n = C \sum_{i=1}^n e^{-\pi(\lambda_i - \lambda_i')s} \min \left\{t-s, 1\right\}\).
Remark 3.7 Note that $C_{i,n}^{1,2}$ decay fast as $s \to \infty, i \to \infty$, and uniformly in $n \geq i$, since the sequence \( \{\delta_n\}_{n=1}^{\infty} \) has fast convergence. Hence, if $r \in \mathbb{N}$ given, we shall have
\[
C_{i,n}^{1} \leq \omega_{i,n}^{-1}, \quad \text{and} \quad C_{i,n}^{2} \leq \omega_{i,n}^{-2}, \quad \text{all} \quad r \geq 1,
\]
with $\omega_{i,n}^{-1} = C_{\omega}^{r} \sum_{j=1}^{n} \frac{d_j}{2\pi(n_j - n_j)^{r+1}}$ and $\omega_{i,n}^{-2} = C_{\omega}^{r} \sum_{j=1}^{n} \frac{d_j}{2\pi(n_j - n_j)^{r+2}}$, for some absolute constant $C_{\omega}^{r}$ depending only on $r$.

Proposition 3.8 Under the assumptions of Proposition 3.3, then
\[
\left\| (\Omega_{i,t,s}^{-1})^{-1} \circ \Omega_{t,s}^{n} - Id \right\|_{L^{\lambda_{i,n}(t-s),\mu_{n}}^{1}} < C_{1,n}^{i} + C_{2,n}^{i}.
\]

To give a self-consistent estimate, we have to control each term of Eq.(3.7): I,II,III. And the most difficult term is I, because there is some resonance phenomena occurring in this term that makes the propagated wave away from equilibrium.

Let us first consider the first term $I$.
\[
I^{n+1,n}(t,x) = \int_{0}^{t} \int_{\mathbb{R}^3} e^{-\nu(t-s)}(E_{s,t}^{n+1} \cdot G_{s,t}^{n})(s, x - v(t-s), v)dvds.
\]

To handle this term, we start by introducing
\[
\tilde{G}_{s,t}^{n} = \nabla v f^{0} + \sum_{i=1}^{n} \nabla v (h^{i} \circ \Omega_{s,t}^{n-1}),
\]
and the error terms $R_{0}, \tilde{R}_{0}$ are defined by
\[
R_{0} = \int_{0}^{t} \int_{\mathbb{R}^3} e^{-\nu(t-s)}((E[h^{n+1}] \circ \Omega_{s,t}^{n}(x,v) - E[h^{n+1}]) \cdot G_{s,t}^{n})(s, x - v(t-s), v)dvds,
\]
\[
\tilde{R}_{0} = \int_{0}^{t} \int_{\mathbb{R}^3} e^{-\nu(t-s)}((E[h^{n+1}] \cdot (G_{s,t}^{n} - \tilde{G}_{s,t}^{n}))(s, x - v(t-s), v)dvds,
\]
then we can decompose
\[
I^{n+1,n} = \tilde{I}^{n+1,n} + R_{0} + \tilde{R}_{0}.
\]

We decompose as
\[
\tilde{I}^{n+1,n} = \tilde{I}_{0}^{n+1,n} + \sum_{i=1}^{n} \tilde{I}_{i}^{n+1,n},
\]
where
\[
\tilde{I}_{0}^{n+1,n}(t,x) = \int_{0}^{t} \int_{\mathbb{R}^3} E[h^{n+1}](s, x - v(t-s)) \cdot \nabla v f^{0}(v)dvdr,
\]
\[
\tilde{I}_{i}^{n+1,n}(t,x) = \int_{0}^{t} \int_{\mathbb{R}^3} E[h^{n+1}](s, x - v(t-s)) \cdot (\nabla v h^{i} \circ \Omega_{s,t}^{n-1})(s, x - v(t-s), v))dvds,
\]

Proposition 3.9 (Main term I) Assume $b(t,s) \geq 0$ small. And there exist constants $\lambda_{\ast} > 0, \mu_{\ast} > 0$ such that $\lambda_{0} > \lambda_{0} > \lambda_{1} > \lambda_{1} > \ldots > \lambda_{i} > \lambda_{i} > \ldots > \lambda_{i} > \lambda_{j} > \ldots > \lambda_{j}$. We have
\[
\left\| \tilde{I}_{i}^{n+1,n}(t,\cdot) \right\|_{L^{\lambda_{i,n}^{1}+\mu_{n}^{1}}} \leq C \int_{0}^{t} e^{-(s-t)^{\nu}} k^{n+1}(t,s) \left\| \nabla v (h^{i} \circ \Omega_{s,t}^{n-1}) - (\nabla v (h^{i} \circ \Omega_{s,t}^{n-1})) \right\|_{L^{\lambda_{i,n}^{1}+\mu_{n}^{1}}^{1}} ds
\]
\[
\left\| E[h^{n+1}] \right\|_{L^{\nu}} ds + \int_{0}^{t} e^{-(s-t)^{\nu}} K_{i}^{n+1}(t,s) \left\\| (\nabla v (h^{i} \circ \Omega_{s,t}^{n-1})) \right\\|_{L^{\lambda_{i,n}^{1}+\mu_{n}^{1}}^{1}} ds,
\]
where
\[
\nu = \max \left\{ \lambda_{i}^{\prime} s + \mu_{i}^{\prime} - \frac{1}{2} \lambda_{i}^{\prime} b(t,s), 0 \right\},
\]
\[
K_{i}^{n+1}(t,s) = e^{-\pi(\lambda_{i,n} - \lambda_{i}'')(t-s)},
\]
\[
K_{i}^{n+1}(t,s) = \sup_{k,l \in \mathbb{Z}^{3}} e^{-2\pi(\mu_{i,n}^{l} - \mu_{i,n}^{l})} e^{-\pi(\lambda_{i,n} - \lambda_{i}')(k(t-s)+ls)} e^{-2\pi(\lambda_{i,n} - \lambda_{i}')(k-l)}.\]
Corollary 3.10 From the above statement, we have

\[ \|I_{i}^{n+1,n}(t, \cdot)\|_{X^{\lambda_{n+1}, \nu_{n}}} \leq \int_{0}^{t} e^{-(t-s)\nu} K_{0}^{n+1}(t, s) \delta_{i} \|\rho[h^{n+1}]\|_{X^{\lambda_{i}+\nu_{s}}} ds \]

\[ + \int_{0}^{t} e^{-(t-s)\nu} K_{1}^{n+1}(t, s)(1 + s) \delta_{i} \|\rho[h^{n+1}]\|_{X^{\lambda_{i}+\nu_{s}}} ds, \]

where \( K_{0}^{n}(t, s) = e^{-\pi(\lambda_{i}' - \lambda_{n})}(t-s) \), and

\( K_{1}^{n+1}(t, s) = e^{-\pi(\mu_{n}' - \mu_{s}')}(t-s) + |s| e^{-2\pi(\lambda_{i}'(s-s') + \mu_{n}' - \mu_{s}')(k-1)} \).

Proposition 3.11 (Error term \( R_{0} \))

\[ \|R_{0}(t, \cdot)\|_{X^{\lambda_{n+1}, \nu_{s}}} \leq C \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{i=1}^{n} \frac{\delta_{i}}{\lambda_{i} - \lambda_{i}'} \right) \int_{0}^{t} \|\rho[h^{n+1}]\|_{X^{\lambda_{i}+\nu_{s}}} \frac{ds}{(1 + s)^{2}}. \]

Proposition 3.12 (Error term \( \tilde{R}_{0} \))

\[ \|\tilde{R}_{0}(t, \cdot)\|_{X^{\lambda_{n+1}, \nu_{s}}} \leq \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{i=1}^{n} \frac{\delta_{i}}{2\pi(\lambda_{i} - \lambda_{i}')} \right) \int_{0}^{t} \|\rho[h^{n+1}]\|_{X^{\lambda_{i}+\nu_{s}}} \frac{1}{(1 + s)^{2}} ds. \]

3.5 The proof of main theorem

Step 2. Note from the definition of \( \delta_{n+1} \) in (7.17), more smaller \( \nu \) is, more larger the coefficient of \( \delta_{n}^{2} \) is. Therefore, without loss of generality, we assume that \( \nu \) is small enough, up to slightly lowering \( \lambda_{1} \), we may choose all parameters in such a way that \( \lambda_{k}, \lambda_{k}' \to \lambda_{\infty} > \frac{\Delta}{\mu_{s}} \) and \( \mu_{k}, \mu_{k}' \to \mu_{\infty} > \frac{\mu_{s}}{\pi} \) as \( k \to \infty \); then we pick up \( B > 0 \) such that \( \mu_{\infty} - \lambda_{\infty}(1 + B) \geq \mu_{\infty} > \mu_{s} \), and we let \( b(t) = \frac{B}{1+t} \). From the iteration, we have, for all \( k \geq 2 \),

\[ \sup_{0 \leq s \leq t} \|h_{k}' \circ \Omega_{t, s}^{\lambda_{\infty}(1+s), \mu_{\infty}} \|_{Z^{\lambda_{\infty}(1+s), \mu_{\infty}}} \leq \delta_{k}, \] (16.3)

where \( \sum_{k=2}^{\infty} \delta_{k} \leq C \delta \). Choosing \( t = s \) in (3.15) yields

\[ \sup_{0 \leq s \leq t} \|h_{k}'\|_{Z^{\lambda_{\infty}(1+s), \mu_{\infty}}} \leq \delta_{k}. \]

This implies that

\[ \sup_{0 \leq s \leq t} \|h_{k}'\|_{Z^{\lambda_{\infty}(1+s), \mu_{\infty}}} \leq \delta_{k}. \]

In particular, we have a uniform estimate on \( h_{k}' \) in \( Z_{t}^{\lambda_{\infty}, \mu_{\infty}} \). Summing up over \( k \) yields for \( f = f_{0} + \sum_{k=1}^{\infty} h_{k}' \), the estimate

\[ \sup_{t \geq 0} \|f(t, \cdot) - f_{0}\|_{Z^{\lambda_{\infty}, \mu_{\infty}}} \leq C \delta. \] (17.3)

From (viii) of Proposition 1.3, we can deduce from (3.21) that

\[ \sup_{t \geq 0} \|F(t, \cdot) - f_{0}\|_{Y^{\lambda_{\infty}}} \leq C \delta. \] (18.3)

Moreover, \( \rho = \int_{\mathbb{R}^{3}} f dv \) satisfies similarly \( \sup_{t \geq 0} \|\rho(t, \cdot)\|_{L^{\infty}} \leq C \delta \). It follows that \( |\rho(t, k)| \leq C \delta e^{-2\pi \lambda_{\infty} |k| t} e^{-2\pi \mu_{\infty} |k|} \) for any \( k \neq 0 \). On the one hand, by Sobolev embedding, we deduce that for any \( r \in \mathbb{N} \),

\[ \|\rho(t, \cdot) - \langle \rho \rangle_{C_{r}(\mathbb{R}^{3})}\| \leq C_{r} \delta e^{-2\pi \lambda_{\infty} |k|}; \]

on the other hand, multiplying \( \hat{\rho} \) by the Fourier transform of \( W \), we see that the electric field \( E \) satisfies

\[ |\hat{E}(t, k)| \leq C \delta e^{-2\pi \lambda_{\infty} |k| t} e^{-2\pi \mu_{\infty} |k|}; \] (19.3)

for some \( \lambda_{0} > \lambda' > \frac{\Delta}{\mu_{0}} > \mu' > \mu \).

Now, from (3.15), we have, for any \( (k, \eta) \in \mathbb{Z}^{3} \times \mathbb{R}^{3} \) and any \( t \geq 0 \),

\[ |f_{0}(t, k, \eta + kt)| \leq C \delta e^{-2\pi \mu_{\infty} |k| t} e^{-2\pi \lambda_{\infty} |\eta|}; \] (20.3)

this finishes the proof of Theorem 0.1.
4 Dynamics of the particles’ trajectory

Because the proof of Proposition 3.3 can be found in [26,28] here we sketch the key steps in the proof. To prove Proposition 3.3, the idea is to use the classical Picard iteration, we only need to consider the following equations

\[
\begin{align*}
\frac{d}{dt}\delta X_{t,s}^{n+1}(x,v) &= \delta V_{t,s}^{n+1}(x,v), \\
\frac{d}{dt}\delta V_{t,s}^{n+1}(x,v) &= E[f^n](t,\delta X_{t,s}^{n+1}(x,v) + X_{t,s}^0(x,v)), \\
\delta X_{t,s}^{n+1}(x,v) &= 0, \delta V_{t,s}^{n+1}(x,v) = 0.
\end{align*}
\]  

(4.1)

It is easy to check that

\[
\Omega_{t,s}^{n+1} - Id \triangleq (\delta X_{t,s}^{n+1}, \delta V_{t,s}^{n+1}) \circ (X_{t,s}^0, V_{t,s}^0) = (X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0) - Id, V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0) - Id).
\]

Therefore, in order to estimate \((X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0) - Id, V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0) - Id)\), we only need to study \((\delta X_{t,s}^{n+1}, \delta V_{t,s}^{n+1}) \circ (X_{s,t}^0, V_{s,t}^0)\).

Note that in the proof, in order to obtain

\[
\|\delta X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0)\|_{Z_{t,s}^{n+1}(+; n, \lambda)} \leq C_1^n,
\]

we need the following assumptions:

If \(s \geq \frac{bt}{1+\theta^n}\), then

\[
\nu_n' \leq \lambda_n s + \mu_n + C_1^n \leq \lambda_n s + \mu - (\lambda_n - \lambda_n s)
\]

as soon as

\[
C_1^n \leq \frac{\lambda_n b(t-s)}{2} \quad (I);
\]

If \(s \leq \frac{bt}{1+\theta^n}\), then

\[
\nu_n' \leq \lambda_n' b(t) + \mu_n' - \lambda_n'(1+b) s + C_1^n \leq \lambda_n' B + \mu_n - (\lambda_n' - \lambda_n'(1+b) s + C_1^n \leq \mu_0 - (\lambda_n - \lambda_n') s
\]

as soon as

\[
C_1^n \leq \frac{\mu_0 - \mu_n'}{2} \quad (II).
\]

In order to the feasibility of the conditions \((I)\) and \((II)\), we only need to check that the following assumption \((I)\) holds

\[
2C_1^n \left( \sum_{i=1}^n \frac{\delta_{i}}{2(\pi(\lambda_i - \lambda_n'))^2} \right) \leq \min \left\{ \frac{\lambda_n b(t-s)}{6}, \frac{\mu_0 - \mu_n'}{2} \right\}.
\]

Combining (4.1) and (I), we can obtain the following conclusion

\[
\|\delta V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0)\|_{Z_{t,s}^{n+1}(+; n, \lambda)} \leq C \sum_{i=1}^n \delta_i \int_{s}^{t} e^{-2\pi(\lambda_i - \lambda_n')s} ds \leq C \sum_{i=1}^n \delta_i e^{-2\pi(\lambda_i - \lambda_n')s} m \left\{ \frac{(t-s)}{2}, \frac{1}{2\pi(\lambda_i - \lambda_n')} \right\},
\]

then we have

\[
\|\delta X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0)\|_{Z_{t,s}^{n+1}(+; n, \lambda)} \leq C \sum_{i=1}^n \delta_i \int_{s}^{t} e^{-2\pi(\lambda_i - \lambda_n')s} ds \leq C \sum_{i=1}^n \delta_i e^{-2\pi(\lambda_i - \lambda_n')s} m \left\{ \frac{(t-s)^2}{2}, \frac{1}{2\pi(\lambda_i - \lambda_n')} \right\}.
\]

We finish the proof of Proposition 3.3.

In the following we estimate \(\Omega_{t,s}^{n+1} - Id\). In fact, we write \((\Omega_{t,s}^{n+1} - Id)(x,v) = (\delta X_{t,s}^{n+1}, \delta V_{t,s}^{n+1}) \circ (X_{s,t}^0, V_{s,t}^0)\), and get by differentiation \(\nabla_x \Omega_{t,s}^{n+1} - (0,0) = \nabla_x \delta X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0), \delta V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0), \nabla_v \Omega_{t,s}^{n+1} - (0,0) = (\nabla_v + (t-s)\nabla_x)(\delta X_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0), \delta V_{t,s}^{n+1} \circ (X_{s,t}^0, V_{s,t}^0))\).

\[
\begin{align*}
\frac{d}{dt}\nabla_x \delta X_{t,s}^{n+1}(x,v) &= \nabla_x \delta V_{t,s}^{n+1}(x,v), \\
\frac{d}{dt}\nabla_v \delta V_{t,s}^{n+1}(x,v) &= \nabla_v E[f^n](t,\delta X_{t,s}^{n+1}(x,v) + X_{t,s}^0(x,v)), \\
\delta X_{t,s}^{n+1}(x,v) &= 0, \delta V_{t,s}^{n+1}(x,v) = 0.
\end{align*}
\]  

(4.2)

Using the same process in the proof of Proposition 3.3, we can obtain Proposition 3.5.
To establish a control of $\Omega_{t,s}^i - \Omega_{t,s}^n$ in norm $Z_{s - \frac{1}{n+1}}^{2\lambda_{n}^{(1+b)}\mu_{n}^i}$, we start again from the differential equation satisfied by $\delta V_{t,s}^i$ and $\delta V_{t,s}^n$:

\[
\begin{align*}
\frac{d}{dt}(\delta X_{t,s}^i - \delta X_{t,s}^n)(x,v) &= \delta V_{t,s}^i(x,v) - \delta V_{t,s}^n(x,v), \\
\frac{d}{dt}(\delta V_{t,s}^i - \delta V_{t,s}^n)(x,v) &= E[f^{i-1}](t, \delta X_{t,s}^{i-1}(x,v) + X_{t,s}^0(x,v)) - E[f^{n-1}](t, \delta X_{t,s}^{n-1}(x,v) + X_{t,s}^0(x,v)).
\end{align*}
\]  

(4.3)

So from (4.3), $\delta V_{t,s}^i - \delta V_{t,s}^n$ satisfies the equation:

\[
\frac{d}{dt}(\delta V_{t,s}^i - \delta V_{t,s}^n)(x,v) = E[f^{i-1}](t, \delta X_{t,s}^{i-1}(x,v) + X_{t,s}^0(x,v)) - E[f^{n-1}](t, \delta X_{t,s}^{n-1}(x,v) + X_{t,s}^0(x,v)).
\]

Under the assumption (I), we can use the similar proof of Proposition 3.3 to finish Proposition 3.6.

Let $\varepsilon$ be the small constant appearing in Lemma 1.7. If

\[3C_1^i + C_2^i \leq \varepsilon, \text{ for all } i \geq 1, \text{ (II)};\]

if in addition

\[2(1+s)(1+B)(3C_1^i + C_2^i)(s,t) \leq \max\{\lambda_i - \lambda_n^i, \mu_i - \mu_n^i\} \text{ (III)}\]

for all $i \in \{1, \ldots, n-1\}$ and all $t \geq s$, then

\[
\begin{align*}
\lambda_n^i(1+b) + 2\|\Omega^n - \Omega^i\|_{Z_{s - \frac{1}{n+1}}^{\lambda_n^i(1+b),\mu_n^i}} &\leq \lambda_n^i(1+b), \\
\mu_i' + 2(1 + |s - \frac{b}{n+1}|)\|\Omega^n - \Omega^i\|_{Z_{s - \frac{1}{n+1}}^{\lambda_n^i(1+b),\mu_n^i}} &\leq \mu_i'.
\end{align*}
\]

(4.4)

Then Lemma 1.7 and (4.6) yield Proposition 3.7.

As a corollary of Proposition 3.7 and Proposition 1.8, under the assumption (IV):

\[4(1+s)(C_1^i + C_2^i) \leq \min\{\lambda_i - \lambda_n^i, \mu_i - \mu_n^i\} \text{ for all } i \in \{1, \ldots, n\} \text{ and all } s \in [0, t],\]

we have

Corollary 4.1 under the assumption (3.8), we have

\[
\begin{align*}
\|h_i^s \circ \Omega_{t,s}^{n} - h_i^s \circ \Omega_{t,s}^{n}||_{Z_{s - \frac{1}{n+1}}^{\lambda_n^i(1+b),\mu_n^i}} &\leq \bar{\delta}_i, \\
\|((\nabla_v + s\nabla_x)h_i^s \circ \Omega_{t,s}^{n}||_{Z_{s - \frac{1}{n+1}}^{\lambda_n^i(1+b),\mu_n^i}} &\leq \bar{\delta}_i.
\end{align*}
\]

5 The estimates of main terms

In order to estimate that term $I$, we have to make good understanding of plasma echo. First of all, one of the key steps is that we need to translate the physical phenomenon into the mathematical language. In the following we give the corresponding mathematical analysis.

5.1 Plasma echoes: Mathematical expression

From the above physical point of view, under the assumption of the stability condition, we have known that, echoes occurring at distinct frequencies are asymptotically well separated. In the following, through complicate computation, we give a detailed description by using mathematical tool. And the proof is simple, so we omit.

Theorem 5.1 Let $\lambda, \bar{\lambda}, \mu, \bar{\mu}$ be such that $2\lambda \geq \bar{\lambda} > \lambda > 0$, $\bar{\mu} \geq \mu > 0$, and let $b = b(t,s) > 0$, $R = R(t,x), G = G(t,x,v)$ and assume $G(t,0,0) = 0$, we have, if

\[
\sigma(t,x,v) = \int_0^t R(s,x + (t-s)v)G(s,x + (t-s)v,v)ds,
\]

then

\[
\|\sigma(t,\cdot)\|_{Z_{s - \frac{1}{n+1}}^{\lambda,\mu}} \leq \int_0^t K(t,s)\|R\|_{Z_{s - \frac{1}{n+1}}^{\lambda(1+b),\mu}} \frac{\|G\|_{Z_{s - \frac{1}{n+1}}^{\lambda(1+b),\mu}}}{1+s}ds, \tag{5.1}
\]

where $K(t,s) = (1+s)\sup_{k,l \in \mathbb{Z}_+^2} e^{-\pi(\bar{\mu} - \mu)} e^{-\pi(\bar{\lambda} - \lambda)k(t-s)+is} e^{-2\pi[\mu' - \mu + \lambda(b(t-s))]} e^{k|l|}$. 


Now we try to explain this above theorem from the two aspects: mathematical and physical, respectively. First, the inequality (5.1) is vital for the Vlasov-Poisson equations (0.11). Now we assume that the function $G(t, x, v)$ is known and is good enough, then in some sense, if the kernel is “good”, (5.1) is considered as a “monotone” energy formula. However, in order to prove the inverse Hölder inequality or the “monotone” energy formula holds, we have to check whether the kernel $K_{t,s}$ is good or not.

Indeed, let $\mu' - \mu = \sigma$, assuming that $b = \frac{1}{t_s}$ with $B$ so small that $(\mu' - \mu) - \lambda b(t - s) \geq \frac{1}{2}$, then $K$ is bounded by

$$K^\alpha(t, s) = (1 + s) \sup_{k, l \in \mathbb{Z}^d} e^{-\pi \alpha |l| |e^{-\pi \alpha |k(t - s) + ls|} e^{-2\pi \alpha |k - l|}},$$

where $\alpha = \frac{1}{2} \min\{\lambda - \bar{\lambda}, \mu - \bar{\mu}, \sigma\}$. Through simple computation, it is easy to find that when $s \leq \frac{1}{2}$, $K^\alpha(t, s)$ is “good”; however, when $\frac{1}{2} t < s \leq t$, whenever $s/t$ is a rational number distinct from 0 or 1, there are $k, l \in \mathbb{Z}^d$ such that $|k(t - s) + ls| = 0$, $K^\alpha(t, s)$ is “bad”, that is, as $t \to \infty$, $K^\alpha_{t,s}$ maybe cannot be controlled. From the physical point of view, one can consider $l, k - l$ as frequencies of two different waves, and start a wave at frequency $l$ at time 0, and force it at time $s$ by a wave of frequency $k - l$, a strong response is obtained at time $t$ and frequency $k$ such that $k(t - s) + ls = 0$. And the corresponding strong response is called plasma echo in plasma physics. It is worthy mentioned that the condition $x \in T^d$ guarantees the asymptotically well separated behavior of echoes occurring at distinct frequencies, which was discovered by Mouhot and Villani. The detailed computation is found in the following section 7.2 (also see the paper [26,28]).

5.2 Estimates of main terms

In the following we estimate $\overline{P}_{t,s}^{n+1, n}(t, x)$. Note that their zero modes vanish. For any $n \geq i \geq 1$,

$$\overline{P}_{t,s}^{n+1, n}(t, k) = \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-(t-s)\nu} e^{-2\pi ik \cdot x} \left(E[h^{n+1}] \cdot (\nabla_v (h^i \circ \Omega_i^{t,s})) \right) (s, x - v(t - s), v) dv dx ds,$$

$$|\overline{P}_{t,s}^{n+1, n}(t, k)| \leq \int_0^t \left( \sum_{l \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^3} e^{-(t-s)\nu} e^{-2\pi ik \cdot (v(t-s))} \left(\nabla_v (h^i \circ \Omega_i^{t,s})) \right) (s, l, v) dv \right| \right) ds$$

$$= \int_0^t \left( \sum_{l \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^3} e^{-(t-s)\nu} e^{-2\pi ik \cdot (v(t-s))} \left(\nabla_v (h^i \circ \Omega_i^{t,s})) \right) (s, l, v) dv \right| \right) ds$$

(5.2)

From (5.1) of Theorem 5.1 and (5.2), we can get Proposition 3.9.

6 Estimates of error terms

In the following we estimate one of the error terms $R_0$.

Recall

$$R_0(t, x) = \int_0^t \int_{\mathbb{R}^3} e^{-(t-s)\nu} \left( E[h^{n+1}] \circ \Omega^n_{t,s}(x, v) - E[h^{n+1}] \right) \cdot G^n_{t,s} \right) (s, x_t^0(x, v), V_t^0(x, v)) dv ds.$$

First,

$$\|R_0(t, \cdot)\|_{Z^{(1+\theta), \nu'}_{x, t}} \leq \int_0^t e^{-(t-s)\nu} \|E[h^{n+1}] \circ \Omega^n_{t,s}(x, v) - E[h^{n+1}]\|_{Z^{(1+\theta), \nu'}_{x, t}} \|G^n_{t,s}\|_{Z^{(1+\theta), \nu'}_{x, t}} ds.$$

Next,

$$\|E[h^{n+1}] \circ \Omega^n_{t,s}(x, v) - E[h^{n+1}]\|_{Z^{(1+\theta), \nu'}_{x, t}} \leq \|\Omega^n_{t,s} - I\|_{Z^{(1+\theta), \nu'}_{x, t}} \int_0^1 \|\nabla E[h^{n+1}]((1 - \theta)I + \theta \Omega^n_{t,s})\|_{Z^{(1+\theta), \nu'}_{x, t}} d\theta$$

$$\leq \|\nabla E[h^{n+1}]\|_{Z^{(1+\theta), \nu'}_{x, t}} \|\Omega^n_{t,s} - I\|_{Z^{(1+\theta), \nu'}_{x, t}},$$

(6.1)
where \( \nu'_n = \lambda'_n (1 + b) |s - \frac{\lambda'_n}{1 + t}| + \mu'_n + \| \Omega^n X_{t,s} - Id \|_{F^{\lambda'_n(1+b),\nu'_n}} \).

Here we only focus on the case \( s \geq \frac{1}{1 + t} \), then we need to show \( \| \nabla E[h^{n+1}] \|_{F^{\nu'_n}} \leq \| \rho [h^{n+1}] \|_{F^{\lambda'_n,\nu'_n}} \). For that, we need to use the fact \( E[h^{n+1}] = \nabla_x W(x) * \rho([h^{n+1}]) \) and prove \( \nu'_n < \lambda'_n + s, \mu'_n < t, \) for some constant \( t > 0 \) sufficiently small.

Indeed,

\[
\nu'_n \leq \lambda'_n s + \mu'_n - \lambda'_n b(t - s) + C \sum_{i=1}^{n} \delta_i e^{-\pi |k| (\lambda_i - (\lambda'_i)) t} \cdot \min \left\{ \frac{(t-s)^2}{2}, \frac{1}{\pi (\lambda_i - \lambda'_i)^2} \right\}
\]

\[
\leq \lambda'_n s + \mu'_n - \lambda'_n b(t - s) + C \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \min \{ t - s, 1 \}.
\]

Note that \( \frac{\min(t-s,1)}{1+s} \leq 3\frac{t-s}{1+s} \). In the following we also need to show that

\[
C \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \leq \frac{\lambda^*}{3} - \iota, \quad (VI)
\]

From Proposition 3.3, then

\[\| E[h^{n+1}] \circ \Omega^n_{t,s} (v, t) - E[h^{n+1}] \|_{F^{\lambda'_n(1+b),\nu'_n}} \leq C \left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \frac{1}{(1+s)^{\frac{1}{2}}} \| \rho [h^{n+1}] \|_{F^{\lambda'_n,\nu'_n}} \]

Since \( G^n_{t,s} = \nabla_v f^n \circ \Omega^n_{t,s} \),

\[
\| G^n_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} \leq \| (\nabla_v f^0) \circ \Omega^n_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} + \sum_{i=1}^{n} \| \nabla_v h^i \circ \Omega^n_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}}
\]

\[
\leq C_0 + \left( \sum_{i=1}^{n} \delta_i \right) (1 + s).
\]

We can conclude

\[\| R_0 (t, \cdot) \|_{F^{\lambda'_n,\nu'_n}} \leq C \left( C_0 + \sum_{i=1}^{n} \delta_i \right) \left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \int_0^t e^{-(t-s)^{\beta \nu'_n}} \| \rho [h^{n+1}] \|_{F^{\lambda'_n,\nu'_n}} \frac{ds}{(1+s)^{\frac{1}{2}}} \]

In order to finish the control of \( R_0 \), we still need the estimate of \( \| G^n_{t,s} - \hat{G}^n_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} \). In fact,

\[
\| G^n_{t,s} - \hat{G}^n_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} \leq \| \nabla_v f^0 \circ \Omega^n_{t,s} - \nabla_v f^0 \|_{F^{\lambda'_n(1+b),\nu'_n}} + \sum_{i=1}^{n} \| \nabla_v h^i \circ \Omega^n_{t,s} - \nabla_v h^i \circ \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}}
\]

\[
+ \sum_{i=1}^{n} \| (\nabla_v h^i) \circ \Omega_{t,s} - (\nabla_v h^i) \circ \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}}.
\]

Now on the one hand, we treat the second term

\[
\sum_{i=1}^{n} \| (\nabla_v h^i) \circ \Omega^n_{t,s} - (\nabla_v h^i) \circ \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}}
\]

\[
\leq \int_0^1 \| \nabla_v h^i (1 - \theta) \Omega^n_{t,s} + \theta \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} \cdot \| \Omega^n_{t,s} - \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} d\theta
\]

\[
\leq 2 \| \nabla_v h^i \circ \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}} \| \Omega^n_{t,s} - \Omega_{t,s} \|_{F^{\lambda'_n(1+b),\nu'_n}}
\]

17
\[
\leq 4C_2 \left( \sum_{j=1}^{n} \delta_j \left| \frac{1}{(1 + s)^2} \right| \right) \frac{1}{(1 + s)^2},
\]
where we need to prove
\[
\|\Omega^n X_{t,s} - \Omega^{-1}_{t,s} X_{t,s} \|_{z_j^{(t,s)}, \nu'_{(t,s)}} \leq 2R_1^{i-1,n}(t,s), \]
\[
\|\Omega^n V_{t,s} - \Omega^{-1}_{t,s} V_{t,s} \|_{z_j^{(t,s)}, \nu'_{(t,s)}} \leq R_2^{i-1,n}(t,s) + R_2^{i-1,n}(t,s)
\]
with
\[
R_1^{i-1,n}(t,s) = \left( \sum_{j=1}^{n} \frac{\delta_j e^{-2\pi(\lambda_j - \lambda'_j)s}}{2\pi(\lambda_j - \lambda'_j)^2} \right) \min\{t-s, 1\}, \quad R_2^{i-1,n}(t,s) = \left( \sum_{j=1}^{n} \frac{\delta_j e^{-2\pi(\lambda_j - \lambda'_j)s}}{2\pi(\lambda_j - \lambda'_j)^2} \right) \min\{(t-s)^2/2, 1\}.
\]

On the other hand, by the induction hypothesis, since \(Z^{\lambda_\mu}_\nu\) norm is increasing as a function of \(\lambda\) and \(\mu\), then
\[
\sum_{i=1}^{n} \|(\nabla h^i_s) \circ \Omega^{-1}_{t,s} - \nabla h^i_s \circ \Omega^{-1}_{t,s}\|_{z_j^{(t,s)}, \nu'_{(t,s)}} \leq \left( \sum_{i=1}^{n} \delta_i \right) \frac{1}{(1 + s)^2}.
\]
So we have
\[
\|\tilde{\mathcal{R}}_0(t,\cdot)\|_{z_j^{\lambda_\mu}_\nu} \leq \left( C^0 \left( C^0_0 + \sum_{i=1}^{n} \delta_i \right) \left( \sum_{j=1}^{n} \frac{\delta_j e^{-2\pi(\lambda_j - \lambda'_j)s}}{2\pi(\lambda_j - \lambda'_j)^2} \right) + \sum_{i=1}^{n} \delta_i \right) \int_0^t e^{-(t-s)\nu} \|\rho\|_{z_j^{\lambda_\mu}_\nu} \frac{1}{(1 + s)^2} ds,
\]
\[
= \int_0^t e^{-(t-s)\nu} \tilde{\mathcal{R}}_1^{n+1}(t,s) \|\rho\|_{z_j^{\lambda_\mu}_\nu} \frac{1}{(1 + s)^2} ds.
\]
(6.2)

Up to now we finish the estimates of error terms.

7 Iteration

Now let us first deal with the source term
\[
\int \tilde{\mathcal{R}}^{n,n}(t,k) = - \int_0^t \int \int \int \mathbb{R}^3 e^{-(t-s)\nu} e^{-2\pi i k \cdot x} (\mathcal{E}^{n}_t \cdot H^{n}_{t,s})(s, X_{t,s}(x, v), V_{t,s}(x, v)) dv dx ds,
\]
then
\[
\|\mathcal{II}(t,\cdot)\|_{z_j^{\lambda_\mu}_\nu} \leq \int_0^t e^{-(t-s)\nu} \|\mathcal{E}^{n}_t\|_{z_j^{(t,s), \nu_\mu}} \|H^{n}_{t,s}\|_{z_j^{(t,s), \nu_\mu}} ds \leq \int_0^t e^{-(t-s)\nu} \|\rho^n\|_{z_j^{(t,s), \nu_\mu}} ds \leq \int_0^t e^{-(t-s)\nu} \|\rho^n\|_{z_j^{(t,s), \nu_\mu}} d\tau \leq \frac{C^2_2}{(\lambda_n - \lambda'_n - 2\pi)^2},
\]
(7.2)
and
\[
\|\mathcal{III}^{n+1,0}(t,\cdot)\|_{z_j^{\lambda_\mu}_\nu} \leq \nu \int_0^t e^{-(t-s)\nu} \|\rho^n\|_{z_j^{(t,s), \nu_\mu}} \|f^0\|_{z_j^{(t,s), \nu_\mu}} ds
\]
(7.3)
From Propositions 3.5-3.11, combining (3.10), we conclude
\[
\|\rho^n(t,\cdot)\|_{z_j^{\lambda_\mu}_\nu} \leq \frac{C_2^2}{(\lambda_n - \lambda'_n - 2\pi)^2} + \int_0^t e^{-(t-s)\nu} \|K^n_0(t, s)\|_{z_j^{(t,s), \nu_\mu}} ds + 2 \int_0^t e^{-(t-s)\nu} |K^n_0(t, s)|
\]
18
By symmetry we may also assume that $k > C$, then for any $z$ where $z = \sup_{x} xe^{-x} = e^{-1}$.

**Proposition 7.2** (Exponential moments of the kernel) Let $\gamma \in (1, \infty)$ be given. For any $\alpha, \gamma \in (0, 1)$, let $K^{(\alpha), \gamma}(t,s)$ be defined

$$K^{(\alpha), \gamma}(t,s) = (1 + s)^{-\frac{1}{\alpha}} e^{-\alpha (t-s) + |s| + s}$$

Then for any $\gamma < \infty$, there is $\bar{\alpha} = \bar{\alpha}(\gamma) > 0$ such that if $\alpha \leq \bar{\alpha}$ and $\nu \in (0, \nu_{0})$, then for any $t > 0$,

$$e^{-\nu t} \int_{0}^{t} K^{(\alpha), \gamma}(t,s) e^{\nu s} ds \leq C \left( \frac{1}{\alpha \nu^{\gamma}} \log \frac{1}{\alpha} + \frac{1}{\alpha^{2} \nu^{\gamma + \frac{1}{\gamma}}} + \left( \frac{1}{\alpha^{2}} + \frac{1}{\alpha^{2} \nu \log \frac{1}{\alpha}} \right) \right) e^{-\nu t + \frac{\nu^{\gamma}}{\alpha^{3}}},$$

where $C = C(\gamma)$.

In particular, if $\nu \leq \alpha$, then $e^{-\nu t} \int_{0}^{t} K^{(\alpha), \gamma}(t,s) e^{\nu s} ds \leq C(\gamma) \frac{\alpha^{\nu}}{\alpha^{\nu} + \nu^{-\gamma}}$.

Proof. Without loss of generality, we shall set $d = 1$ and first consider $s \leq \frac{1}{\nu} t$. We can write

$$K^{(\alpha), \gamma}(t,s) \leq (1 + s)^{-\frac{1}{\alpha}} e^{-\alpha |k - l|/2} e^{-\alpha (t-s) + s}.$$
Using the bounds (for $\alpha \sim 0^+$)
\[
\sum_{l \in \mathbb{Z}} e^{-\alpha l} = O\left(\frac{1}{\alpha}\right), \quad \sum_{l \in \mathbb{Z}} e^{-\alpha l} l = O\left(\log \frac{1}{\alpha}\right), \quad \sum_{l \in \mathbb{Z}} e^{-\alpha l} l^2 = O(1),
\]
we end up, for $\alpha \leq \frac{1}{4}$, with a bound like
\[
e^{-\nu t} \int_0^t K^{(\alpha)}(t, s)e^{\nu s} ds \leq C \left[ e^{-\frac{\nu t}{\alpha^3}} \left(\frac{1}{\alpha^3} + \frac{1}{\alpha^2 \nu} + \frac{e^{-\alpha t/2}}{\alpha^3}\right) \right].
\]
Next we turn to the more delicate contribution of $s \geq \frac{1}{2} t$. We write
\[
K^{(\alpha)}(t, s) \leq (1 + s) \sup_{k \in \mathbb{Z}} e^{-\alpha |(k-t)+ls|} 1 + |k-l|^{\gamma}.
\] (7.6)

Without loss of generality, we restrict the supremum $l > 0$. The function $x \to (1 + |x - l|^{\gamma} - e^{-\alpha |x(t-s)+ls|}$ is decreasing for $x \geq l$, increasing for $x \leq -ls/(t-s)$; and on the interval $[-ls/(t-s), l]$, its logarithmic derivative goes from $\left( -\alpha + \frac{\gamma l}{1+(t-s)/l} \right) (t-s)$ to $-\alpha (t-s)$. It is easy to check that a given integer $k$ occurs in the supremum only for some times $s$ satisfying $k-1 < -ls/(t-s) < k+1$. We can assume $k \geq 0$, then $k - 1 < \frac{\nu t}{\gamma} < k + 1$ holds, and it is equivalent to $\frac{k}{k+1} t < s < \frac{k+1}{k+1} t$. More importantly, $\tau > \frac{1}{2} t$ implies that $k \geq 1$. Thus, for $t \geq \frac{1}{2} t$, we have
\[
e^{-\nu t} \int_0^t K^{(\alpha)}(t, s)e^{\nu s} ds \leq e^{-\nu t} \int_{1}^{\frac{k}{k+1} t} (1 + s) e^{-\alpha |(k-t)+ls|} e^{\nu s} ds \leq e^{-\frac{\nu t}{\alpha}} \left( \frac{1}{\alpha (k+l)} + \frac{k t}{\alpha (k+l)^2} \right),
\]
\[
e^{-\nu t} \int_{\frac{k}{k+1} t}^{\frac{k+1}{k+1} t} (1 + s) e^{-\alpha |(k-t)+ls|} e^{\nu s} ds \leq e^{-\frac{\nu t}{\alpha}} \left( \frac{1}{\alpha (k+l)} + \frac{k t}{\alpha (k+l)^2} \right).
\]
Hence, (7.6) is bounded above by
\[
\int_{1}^{\frac{k}{k+1} t} (1 + s) e^{-\alpha |(k-t)+ls|} e^{\nu s} ds \leq e^{-\frac{\nu t}{\alpha}} \left( \frac{1}{\alpha (k+l)} + \frac{k t}{\alpha (k+l)^2} \right).
\]
We consider the first term $I(t)$ of (7.7) and change variables $(x, y) \mapsto (x, u)$, where $u(x, y) = \frac{\nu t}{\gamma}$, then we can find that
\[
I(t) = \int_{1}^{\frac{k}{k+1} t} e^{-\alpha x} e^{\nu u^2} dx \int_{0}^{\nu t/2} e^{-\nu u^2} du = O\left( \frac{1}{\alpha^2 \nu^{1+\gamma} t^{1+\gamma}} \right).
\]
The same computation for the second integral of (7.7) yields
\[
\int_{1}^{\frac{k}{k+1} t} e^{-\alpha x} e^{\nu u^2} dx \int_{0}^{\nu t/2} e^{-\nu u^2} du = O\left( \alpha^{1/\gamma} \right).
\]
Finally, we estimate the last term of (7.7) that is the worst. It yields a contribution $\frac{1}{\alpha^2} \sum_{l=1}^{\infty} e^{-\alpha l} \sum_{k=1}^{\infty} e^{-\alpha (k+l)/t^{1+\gamma}}$. We compare this with the integral $\frac{1}{\alpha^2} \sum_{l=1}^{\infty} e^{-\alpha l} \int_{x}^{\infty} e^{-\alpha u/(x+y)} dy dx$, and the same change of variables as before equates this with
\[
\int_{1}^{\infty} e^{-\alpha x} dx \int_{0}^{\nu t/2} e^{-\nu u^2} du - \int_{1}^{\infty} e^{-\alpha x} dx \int_{0}^{\nu t/2} e^{-\nu u^2} du = O\left( \alpha^{1/\gamma} \right).
\]
The proof of Proposition 7.2 follows by collecting all these bounds and keeping only the worst one. To finish the growth control, we have to prove the following result.

**Proposition 7.3** With the same notation as in Proposition 7.2, for any $\gamma > 1$, we have
\[
\sup_{s \geq \nu t} e^{\nu s} \int_{s}^{\infty} e^{-\nu t} K^{(\alpha)}(t, s) dt \leq C(\gamma) \left( \frac{1}{\alpha^2 \nu} + \frac{1}{\alpha \nu^2} \right),
\] (7.9)
Proof. We first still reduce to $d = 1$, and split the integral as

$$e^{\nu s} \int_s^\infty e^{-\nu t} K^{(\alpha)}(t, s) dt = e^{\nu s} \int_s^\infty e^{-\nu t} K^{(\alpha)}(t, s) dt + e^{\nu s} \int_s^2 e^{-\nu t} K^{(\alpha)}(t, s) dt = I_1 + I_2.$$  

For the first term $I_1$, we have $K^{(\alpha)}(t, s) \leq (1 + s) \sum_{k=0}^\infty t \sum_{l \in \mathbb{N}_0} e^{-\alpha l} \leq \frac{C(1 + s)}{\alpha^2}$, and thus $e^{\nu s} \int_s^\infty e^{-\nu t} K^{(\alpha)}(t, s) dt \leq \frac{C}{\alpha \nu^2}$.

We treat the second term $I_2$ as in the proof of Proposition 7.2:

$$e^{\nu s} \int_s^\infty e^{-\nu t} K^{(\alpha)}(t, s) dt \leq e^{\nu s}(1 + s) \sum_{l=1}^\infty e^{-\alpha l} \int_{(l+1)s}^{(l+1)s} e^{-\alpha |k(t) - l|} \frac{1}{1 + (k + 1)^\gamma} e^{-\nu t} dt \leq \frac{C}{\alpha \nu^\gamma},$$

where the last inequality is obtained by the same method in Proposition 7.2 with the change of variable $u = \frac{t - s}{\nu}$.

### 7.2 Growth control

From now on, we will state the main result of this section that is the same with section 7.4 in [26,28], the detailed proof can be found in appendix (also see [26,28]). We define $\|\Phi(t)\|_{\Lambda} = \sum_{k \in \mathbb{N}_0} |\Phi(k, t)| e^{2\pi \lambda|k|}$.

**Theorem 7.4** Assume that $f^0(v), W = W(x)$ satisfy the conditions of Theorem 0.1, and the Stability condition holds. Let $A \geq 0, \mu \geq 0$ and $\lambda \in (0, \lambda^*)$ with $0 < \lambda^* < \lambda_0$. Let $(\Phi(k, t))_{k \in \mathbb{N}_0, t \geq 0}$ be a continuous functions of $t \geq 0$, valued in $\mathcal{C}^2$, such that for all $t \geq 0$,

$$\|\Phi(t) - \int_0^t e^{-(t-s)\nu} K^0(t-s)\Phi(s) ds\|_{\Lambda + \mu} \leq A + \int_0^t e^{-(t-s)\nu} (K_0(t,s) + K_1(t,s) + \frac{c_0}{(1 + s)^m}) \|\Phi(s)\|_{\Lambda + \mu} ds, \tag{7.10}$$

where $c_0 > 0$, $m > 1$, and $K_0(t,s), K_1(t,s)$ are non-negative kernels. Let $\varphi(t) = \|\Phi(t)\|_{\Lambda + \mu}$. Then we have the following:

(i) Assume that $\gamma > 1$ and $K_1 = c K^{(\alpha)}$ for some $c > 0, \alpha \in (0, \alpha(\gamma))$, where $K^{(\alpha)}$ is the same with that defined by Proposition 7.2. Then there are positive constants $C, \chi$, depending only on $\gamma, \lambda^*, \lambda_0, \kappa, c_0, C_W$ and $m$, uniform as $\gamma \to 1$, such that if $\sup_{t \geq 0} \int_0^t K_0(t,s) ds \leq \chi$ and $\sup_{t \geq 0} \left( \int_0^t K_0(t,s)^2 ds \right)^{\frac{1}{2}} \leq 1$, then for any $\nu \in (0, \alpha)$, for all $t \geq 0$,

$$\varphi(t) \leq C A \frac{1 + c_0^2}{\sqrt{\nu}} e^{C \alpha \nu} \left( 1 + \frac{c_0}{\alpha \nu} \right) e^{CT} e^{C(1 + T^2)} e^{\nu t}, \tag{7.11}$$

where $T_\nu = C \max \left\{ \left( \frac{c_0^2}{\alpha \nu} \right)^{1+\gamma}, \left( \frac{c_0}{\alpha \nu} \right)^{1+\gamma}, \left( \frac{c_0}{\nu} \right)^{1+\gamma} \right\}$.

(ii) Assume that $K_1 = \sum_{j=1}^N c_j K^{(\alpha_j)}$ for some $\alpha_j \in (0, \alpha(\gamma))$, where $\alpha(\gamma)$ also appears in proposition 7.2; then there is a numeric constant $\Gamma > 0$ such that whenever $1 \geq \nu \geq \Gamma \sum_{j=1}^N \frac{c_j}{c_j}$, with the same notation as in (i), for all $t \geq 0$, one has,

$$\varphi(t) \leq C A \frac{1 + c_0^2}{\sqrt{\nu}} e^{C \alpha \nu} \left( 1 + \frac{c_0}{\alpha \nu} \right) e^{CT} e^{C(1 + T^2)} e^{\nu t}, \tag{7.12}$$

where $c = \sum_{j=1}^N c_j$ and $T = \max \left\{ \frac{1}{\nu}, \sum_{j=1}^N \frac{c_j}{c_j}, \left( \frac{c_0}{\nu} \right)^{1+\gamma} \right\}$.

**Corollary 7.5** Assume that $f^0 = f^0(v)$, under the assumptions of Theorem 0.1, we pick up $\nu_n$ such that $\lim_{n \to \infty} \nu_n = \nu$; recalling that $\hat{\rho}(t,0) = 0$, and that our conditions imply an upper bound on $c_n$ and $c_n^0$, we have the uniform control,

$$\|\hat{\rho}[h_n^{(1)}(t,\cdot)]\|_{\mathcal{F}_{\nu_n + \nu_n}} \leq C \delta_n^2 \left( 1 + \frac{c_n}{\nu_n} \right)^2 \left( 1 + \frac{1}{\alpha_n \nu_n} \right) e^{CT_n^2},$$

where $T_n = C \left( \frac{1}{\alpha_n \nu_n} \right)^{1+\gamma}$. 

21
Proof. From Propositions 7.1-7.3, we know that
\[ \int_{0}^{t} K_{0}^{n}(t,s)ds \leq C_{W} \sum_{i=1}^{n} \frac{\delta_{i}}{\pi(\lambda_{i} - \lambda_{n}^{i})}, \quad \int_{s}^{\infty} K_{0}^{n}(t,s)ds \leq C_{W} \sum_{i=1}^{n} \frac{\delta_{i}}{\pi(\lambda_{i} - \lambda_{n}^{i})}. \]
\[ \left( \int_{0}^{t} K_{0}^{n}(t,s)^{2}ds \right)^{\frac{1}{2}} \leq C_{W} \sum_{i=1}^{n} \frac{\delta_{i}}{\sqrt{2\pi(\lambda_{i} - \lambda_{n}^{i})}}. \]

Here \( \alpha_{n} = \pi \min\{\mu_{n} - \mu_{n}^{i}\}, (\lambda_{n} - \lambda_{n}^{i}) \}, and assume \( \alpha_{n} \) is smaller than \( \alpha(\gamma) \) in Theorem 7.4, and that
\[ \left( C_{\alpha} \left( \sum_{i=1}^{n} \delta_{i} + 1 \right) \left( \int_{0}^{t} K_{0}^{n}(t,s)^{2}ds \right) \right)^{\frac{1}{2}} \leq \frac{1}{8} \] (VII)
\[ C_{W} \sum_{i=1}^{n} \frac{\delta_{i}}{\sqrt{2\pi(\lambda_{i} - \lambda_{n}^{i})}} \leq \frac{1}{4}, \quad \sum_{i=1}^{n} \frac{\delta_{i}}{\pi(\lambda_{i} - \lambda_{n}^{i})} \leq \max\{\chi_{i}, \frac{1}{8}\}. \] (VIII)

Applying Theorem 7.4, we can deduce that for any \( \nu_{s} \in (0, \alpha_{n}) \) and \( t \geq 0 \),
\[ \|\rho[h^{n+1}](t, \cdot)\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} \leq C_{\alpha}^{2} \left( 1 + c_{0}^{n} \right) \frac{1}{\sqrt{\nu(\lambda_{n} - \lambda_{n}^{i})}} \left( 1 + \frac{1}{\alpha_{n} \nu_{s}^{i}} \right) e^{CT^{2}}, \]

where \( T_{n} = C \left( \frac{1}{\alpha_{n} \nu_{s}^{i}} \right)^{\frac{1}{\nu_{s}^{i}}}. \)

### 7.3 Estimates related to \( h^{n+1}(t, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v)) \)

To finish Proposition 3.11 and Proposition 3.12, we shall again use the Vlasov equation. We rewrite it as
\[ h^{n+1}(t, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v)) = \int_{0}^{t} e^{-(t-s)^{\nu} \sum_{i=1}^{n} \rho^{n+1}(s, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))} ds. \]

Then we get
\[ \|h^{n+1}(t, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} \leq \int_{0}^{t} e^{-(t-s)^{\nu} \sum_{i=1}^{n} \rho^{n+1}(s, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))} ds. \]
\[ = \int_{0}^{t} e^{-(t-s)^{\nu} \sum_{i=1}^{n} \rho^{n+1}(s, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))} ds + \int_{0}^{t} e^{-(t-s)^{\nu} \sum_{i=1}^{n} \rho^{n+1}(s, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))} ds. \]

therefore, from the induction assumptions, we obtain
\[ \|h^{n+1}(t, X_{s,t}^{n}(x,v), V_{s,t}^{n}(x,v))\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} \leq \sum_{i=1}^{n} \delta_{i} \frac{\delta_{i}^{2}}{\frac{\nu_{s}^{i}}{1 + \nu_{s}^{i}}} + \left( \sum_{i=1}^{n} \delta_{i} \right) \sup_{0 \leq s \leq t} \|\rho^{n+1}\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})}. \] (7.13)

this is the conclusion of Proposition 3.11. Next we show the control on \( h^{i} \).

**Lemma 7.6** For any \( n \geq i \geq 1 \),
\[ \|\nabla_{v}(h_{s}^{i} \circ \Omega_{s,t}^{i-1}) - (\nabla_{v}(h_{s}^{i} \circ \Omega_{s,t}^{i-1}))\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} \leq (1 + s)\delta_{i}. \]

**Proof.** First, we consider \( i = 1 \).

In fact,
\[ \|\nabla_{v}h_{s}^{1}(t, x - v(t - s), v) - (\nabla_{v}h_{s}^{1}(t, x))\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} \]
\[ \leq \|\nabla_{v}h_{s}^{1}\|_{L^{\infty}_{t}(\nu_{s}, \beta_{n}^{i})} + \left( \int_{T}^{3} \|\nabla_{v}h_{s}^{1}\| ds \right). \]
\[
\leq \| \nabla h_s^1 \|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu';\mu)} \leq \| h_s^1 \|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)} \leq (1+s)\delta_1,
\]
where we use the property (v) of Proposition 1.5.
\[
\left\| \int_{\Omega} \nabla v \cdot h_s^1 dx \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)} = \| (\nabla v \cdot h_s^1) \|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)}
\]
\[
= \| (\nabla v + s\nabla x) h_s^1 \|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)} \leq \| (\nabla v + s\nabla x) h_s^1 \|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)} \leq \delta_1,
\]
where we use (vi) of Proposition 1.5.

Suppose that \( i = k \), the conclusion holds, that is,
\[
\left\| \nabla v (h_s^k \circ \Omega_{t,s}^{k-1}) - (\nabla v (h_s^k \circ \Omega_{t,s}^{k-1})) \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu)} \leq (1+s)\delta_k.
\]

**We need to show** the conclusion still holds for \( i = k+1 \). We can get the estimate for \( h^{k+1}(t, X_{t,s}(x, v), V_{t,s}(x, v)) \) from (3.11).

Note that
\[
\begin{cases}
(\nabla h) \circ \Omega = (\nabla \Omega)^{-1} (\nabla h) \\
(\nabla^2 h) \circ \Omega = (\nabla \Omega)^{-1} (\nabla^2 \Omega) (\nabla \Omega)^{-1} (\nabla h) \circ \Omega.
\end{cases}
\]

Therefore, from (7.14), we get
\[
\left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq C \left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \\
\leq \frac{C(1+s)}{\min \{ \lambda_n - \lambda'_n, \mu'_n - \mu_n \}} \left\| h^{n+1}_s \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq \frac{C(1+s)}{\min \{ \lambda_n - \lambda'_n, \mu'_n - \mu_n \}} \left\| h^{n+1}_s \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')}.
\]

and
\[
\left\| (\nabla^2 h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq C \left[ \left\| (\nabla^2 h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \\
+ \left\| (\nabla^2 \Omega_{t,s}) \left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \right]\right].
\]

We first write
\[
(\nabla h^{n+1}_s) \circ \Omega_{t,s}^n - (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n = \nabla (\Omega_{t,s}^n) - I_d \cdot [(\nabla h^{n+1}_s) \circ \Omega_{t,s}^n],
\]
and we get
\[
\left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n - (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq \left\| (\nabla \Omega_{t,s}^n) - I_d \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq C \left( \frac{1+s}{\min \{ \lambda_n - \lambda'_n, \mu'_n - \mu_n \}} \right)^2 \left\| \nabla \Omega_{t,s}^n - I_d \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \left\| (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \right\|_{\mathcal{L}^2_{\lambda}((\lambda+\delta),\mu')} \leq C \left( \frac{1+s}{\min \{ \lambda_n - \lambda'_n, \mu'_n - \mu_n \}} \right)^2 \leq \frac{C C^4}{\min \{ \lambda_n - \lambda'_n, \mu'_n - \mu_n \}} \left( \sum_{k=1}^d \left( \frac{\delta_k}{2\pi (\lambda_k - \lambda'_k)^2} \right) (1+s)^{-2} \right). \]

the above inequality implies \( (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \simeq (\nabla h^{n+1}_s) \circ \Omega_{t,s}^n \) as \( s \to \infty \).
\[ \|\nabla(h_n^{t+1} \circ \Omega_{t,s})\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} \leq C \left( \frac{1 + s}{\min\{\lambda_n - \lambda_n', \mu_n' - \mu_n\}} \right) \|h_n^{t+1} \circ \Omega_{t,s}^n\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} \]

and

\[ \|\nabla_x(h_n^{t+1} \circ \Omega_{t,s})\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} + \|((\nabla_x + s \nabla_v)(h_n^{t+1} \circ \Omega_{t,s}))\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} \]

\[ \leq \frac{C}{\min\{\lambda_n - \lambda_n', \mu_n' - \mu_n\}} \|h_n^{t+1} \circ \Omega_{t,s}^n\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} \]

we have

\[ \|((\nabla_x h_n^{t+1} \circ \Omega_{t,s})\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} + \|((\nabla_x + s \nabla_v)(h_n^{t+1} \circ \Omega_{t,s}))\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}} \]

\[ \leq C \left( \frac{C^2}{\min\{\lambda_n - \lambda_n', \mu_n' - \mu_n\}} \right) \left( \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k'))^n} \right) \left( \frac{1}{\min\{\lambda_n - \lambda_n', \mu_n' - \mu_n\}} \right) \|h_n^{t+1} \circ \Omega_{t,s}^n\|_{Z^{(1+\delta)}_\infty, \mu_n^{t+1}}. \]

### 7.4 Conclusion

If we define

\[ \lambda_{n+1} = \lambda_n^1, \quad \mu_{n+1} = \mu_n^1, \]

then we see that the \( n \) th step of the inductive hypothesis have all been established with

\[ \delta_{n+1} = \frac{C_F(1 + C_F)(1 + C_W^2) e^{CT_n^2}}{\min\{\lambda_n - \lambda_n', \mu_n' - \mu_n\}} \max\left\{ 1, \sum_{i=1}^n \delta_i \right\} \left( 1 + \sum_{i=1}^n \frac{\delta_i}{1 + s} \right) \delta_n^2. \]  

(7.17)

For any \( n \geq 1 \), we set \( \lambda_n - \lambda_n' = \lambda_n - \lambda_{n-1} = \mu_n - \mu_{n-1} = \mu_n' - \mu_{n-1} = \frac{\lambda_n}{\lambda_n'} \) for some \( \Lambda > 0 \). By choosing \( \Lambda \) small enough, we can make sure that the conditions \( 2\pi(\lambda_k - \lambda_k') < 1 \) and \( 2\pi(\mu_k - \mu_k') < 1 \) are satisfied for all \( k \), as well as the other smallness assumptions made throughout this section. We also have \( \lambda_k - \lambda_k' \geq \frac{\lambda_n}{\lambda_n'} \). \( \Phi(1) \) and \( \Phi(2) \) will be satisfied if we choose constants \( \Lambda, \omega \) such that \( \sum_{n=1}^\infty \frac{1}{n^\omega} \leq \Lambda \omega. \)

Then we have that \( T_n \leq C_\gamma(n^2/\Lambda) \) \( \gamma \), so the induction relation on \( \delta_n \) gives \( \delta_n \leq C_\delta \) and \( \delta_{n+1} = C(\frac{\lambda_n^2}{\lambda_n'} e^{C(n^2/\Lambda)^{1/4+\gamma}})^{1/2} \delta_n^2. \)

To make this relation hold, we also assumed that \( \delta_n \) is bounded below by \( C_\delta \), the error coming from the short-time iteration; but this follows easily by construction, since the constraints imposed on \( \delta_n \) are much worse than those on \( \zeta_n \).

### 1. Appendix

**Proof of Theorem 7.4.** Here we only prove (i), the proof of (ii) is similar. We decompose the proof into three steps.

**Step 1.** Crude pointwise bounds. From (7.9), we have

\[ \varphi(t) = \sum_{k \in \mathbb{Z}^3} |\Phi(k, t, e^{2\pi(\lambda_0 + \mu_0)|k|})| \leq A + \sum_{k \in \mathbb{Z}^3} \int_0^t |K^0(k, t - s)| e^{2\pi(\lambda_0 + \mu_0)|k| \varphi(t, s)| ds \]

\[ + \int_0^t (K_0(t, s) + K_1(t, s) + \frac{c_0}{1 + s^\delta}) \varphi(s) ds \]

\[ \leq A + \int_0^t (K_0(t, s) + K_1(t, s) + \frac{c_0}{1 + s^\delta}) + \sup_{k \in \mathbb{Z}^3} |K^0(k, t - s)| e^{2\pi(\lambda_0(1-s))|k| \varphi(t, s)| ds. \]

We note that for any \( k \in \mathbb{Z}^3 \) and \( t \geq 0 \),

\[ |K^0(k, t - s)| e^{2\pi\lambda|k|(t-s)} \leq 4\pi^2 |W(k)| C_0 e^{2\pi(\lambda_0 - \lambda)|k| |k|^2 t} \leq 4 C_0^2 C_\lambda \frac{C_\lambda}{\lambda_0 - \lambda}, \]

where (here and below) \( C \) stands for a numeric constant which may change from line to line. Assuming that \( \int_0^t K_0(t, \tau) d\tau \leq \frac{1}{2} \), we deduce that

\[ \varphi(t) \leq A + \frac{1}{2} \sup_{0 \leq s \leq t} \varphi(s) + C \int_0^t \left( \frac{C_0 C_\lambda}{\lambda_0 - \lambda} + c(1 + s) + \frac{c_0}{1 + s^\delta} \right) \varphi(s) ds, \]
and, by Grönwall’s lemma,
\[ \varphi(t) \leq 2Ae^{C(C_0C_0T/(\lambda_0 \lambda) + t(t^2 + e_\alpha C_0)),} \]

(7.18)

where \( C_m = \int_0^\infty (1 + s)^{-d} \). 

Step 2. \( L^2 \) bound. For all \( k \in \mathbb{Z}_n^2 \) and \( t \geq 0 \), we define \( \Psi_k(t) = e^{-ezt} \Phi(k, t) \Phi(\lambda t + \mu) |k|, C_k^0(t) = e^{-ezt} K^0(k, t) e^{2 \pi (\lambda t + \mu) |k|} \), \( R_k(t) = e^{-ezt} \Phi(k, t) - \int_0^T K^0(k, t - s) \Phi(s) ds \) \( e^{2 \pi (\lambda t + \mu) |k|} = (\Psi_k - \Psi_k * C_k^0) \), and we extend all these functions by 0 for negative values of \( t \). Taking Fourier transform in the time-variable yields \( \hat{R}_k = (1 - \hat{C_k}^0) \hat{\Psi}_k \). Since the Stability condition implies that \( |1 - \hat{C_k}^0| \geq \kappa \), we can deduce that \( \|
\phi_k\|_{L^2} \leq \kappa^{-1} \|
\hat{R_k}\|_{L^2} \), i.e., \( \|
\Psi_k\|_{L^2} \leq \kappa^{-1} \|
\hat{R_k}\|_{L^2} \). So we have
\[ \|
\Psi_k - R_k\|_{L^2(dt)} \leq \kappa^{-1} \|
C_k^0\|_{L^1(dt)} \|
R_k\|_{L^2(dt)} \text{ for all } k \in \mathbb{Z}_n^2. \]

(7.19)

Then
\[ \|
\varphi(t)e^{-zt}\|_{L^2(dt)} = \| \sum_{k \in \mathbb{Z}_n^2} \Psi_k \|_{L^2(dt)} \leq \| \sum_{k \in \mathbb{Z}_n^2} R_k \|_{L^2(dt)} + \| \sum_{k \in \mathbb{Z}_n^2} \Psi_k - R_k \|_{L^2(dt)} \]

(7.20)

Next, we note that
\[ \|C_k^0\|_{L^1(dt)} \leq 4\pi^2 |\hat{W}(k)| \int_0^\infty C_0 e^{2 \pi |\lambda_0 - \lambda| |k|^2} |k|^2 dt \leq 4\pi |\hat{W}(k)| \frac{C_0}{(\lambda_0 - \lambda)^2}, \]

so \( \sum_{k \in \mathbb{Z}_n^2} \|
C_k^0\|_{L^1(dt)} \leq 4\pi (\sum_{k \in \mathbb{Z}_n^2} |\hat{W}(k)|) \frac{C_0}{(\lambda_0 - \lambda)^2} \). Furthermore, we get
\[ \|\varphi(t)e^{-zt}\|_{L^2(dt)} \leq \left( 1 + \frac{CC_0 C_n}{\kappa(\lambda_0 - \lambda)^2} \right) \| \sum_{k \in \mathbb{Z}_n^2} \|_{L^2(dt)} \]

(7.21)

By Minkowski’s inequality, we separate (7.15) into various contributions which we estimate separately. First,\[ \left( \int_0^T e^{-2zt} A^2 dt \right)^\frac{1}{2} = \frac{A}{\sqrt{2\pi}}. \]

Next, for any \( T \geq 1 \), by Step 1 and \( \int_0^t K_1(t, s) ds \leq \frac{C(1 + t)}{2} \), we have
\[ \left( \int_0^T e^{-2zt} \left( \int_0^t K_1(t, s) \varphi(s) \right) ds \right)^\frac{1}{2} \leq \left( \sup_{0 \leq t \leq T} \varphi(t) \right) \left( \int_0^T e^{-2zt} \left( \int_0^t K_1(t, s) \right)^2 ds \right)^\frac{1}{2} \]

(7.22)

Invoking Jensen’s inequality and Fubini’s theorem, we also have
\[ \int_T^\infty e^{-2zt} \left( \int_0^t K_1(t, s) \varphi(s) \right)^2 ds \right)^\frac{1}{2} \leq \int_T^\infty \left( \int_0^t K_1(t, s) e^{-2zt(s)} e^{-2z_\alpha} \varphi(s) ds \right)^2 dt \]

\[ \leq \left( \sup_{t \geq T} \int_0^t K_1(t, s) e^{-z(t-s)} ds \right)^\frac{1}{2} \left( \int_T^\infty \left( \int_0^t K_1(t, s) e^{-z(t-s)} e^{-2z_\alpha} \varphi(s) ds \right)^2 dt \right)^\frac{1}{2} \]

(7.23)
Similarly,
\[
\int_T^\infty e^{-2ct} \left( \int_0^t K_0(t,s) \varphi(s) ds \right)^2 dt \leq \left( \sup_{t \geq T} \int_0^t K_0(t,s) ds \right)^2 \left( \sup_{s \geq 0} \int_\tau^\infty K_0(t,s) dt \right)^{1/2} \left( \int_0^\infty \varphi(s)^2 ds \right)^{1/2}
\] (7.24)

The last term is also split, this time according to \( \tau \leq T \) or \( \tau > T \):
\[
\left( \int_0^\infty e^{-2ct} \left( \int_0^T \frac{c_0 \varphi(s)}{(1 + s)^m} ds \right)^2 dt \right)^{1/2} \leq c_0 \left( \sup_{0 \leq s \leq T} \varphi(s) \right) \left( \int_0^\infty e^{-2ct} \left( \int_0^T \frac{ds}{(1 + s)^m} \right)^2 dt \right)^{1/2}
\leq c_0 \frac{CA}{\sqrt{\varepsilon}} e^{C(C_0C_W(T/\lambda_0^2) + c(T + T^2))} C_m,
\] (7.25)

and
\[
\left( \int_0^\infty e^{-2ct} \left( \int_T^t \frac{c_0 \varphi(s)}{(1 + s)^m} ds \right)^2 dt \right)^{1/2} \leq c_0 \left( \int_0^\infty e^{-2ct} \varphi(t)^2 \right)^{1/2} \left( \int_0^\infty \int_T^t \frac{e^{-2c(t-s)}}{(1 + s)^{2m}} ds dt \right)^{1/2}
= c_0 \left( \int_0^\infty e^{-2ct} \varphi(t)^2 \right)^{1/2} \left( \int_0^\infty \int_T^t \frac{ds}{(1 + s)^{2m}} \right)^{1/2} \left( \int_0^\infty e^{-2ct} \varphi(t)^2 \right)^{1/2}.
\] (7.26)

Gathering estimates (7.16)-(7.20), we deduce from (7.15) that
\[
\| \varphi(t)e^{-ct} \|_{L^2(dt)} \leq \left( 1 + \frac{CC_0C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \frac{CA}{\sqrt{\varepsilon}} \left( 1 + \frac{c}{\alpha \varepsilon} + c_0 C_m \right) e^{C(C_0C_W(T/\lambda_0^2) + c(T + T^2))} + a \| \varphi(t)e^{-ct} \|_{L^2(dt)},
\] (7.27)

where
\[
a = \left( 1 + \frac{CC_0C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \left( \sup_{t \geq T} \int_0^t e^{-ct} K_1(t,s) e^{\varepsilon s} ds \right)^{1/2} \left( \sup_{s \geq 0} \int_0^\infty e^{\varepsilon s} K_1(t,s) e^{-ct} dt \right)^{1/2} + \left( \sup_{t \geq T} \int_0^t K_0(t,s) ds \right)^{1/2} \left( \sup_{s \geq 0} \int_0^\infty K_0(t,s) dt \right)^{1/2} + \frac{C^{3}_{2m}C_0}{T^{m-\frac{1}{2}}/\sqrt{\varepsilon}}.
\]

Using Proposition 7.2 and 7.3, together with the assumptions of Theorem 7.4, we see that \( a \leq \frac{1}{2} \) for \( \chi \) sufficiently small. Then we have
\[
\| \varphi(t)e^{-ct} \|_{L^2(dt)} \leq \left( 1 + \frac{CC_0C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \frac{CA}{\sqrt{\varepsilon}} \left( 1 + \frac{c}{\alpha \varepsilon} + c_0 C_m \right) e^{C(C_0C_W(T/\lambda_0^2) + c(T + T^2))}.
\]

**Step 3.** For \( t \geq T \), using (7.9) we get
\[
e^{-ct} \varphi(t) \leq Ae^{-ct} + \left[ \int_0^t \left( \sup_{k \in \mathbb{Z}_2} |K_0(k,t-s)| e^{2\pi \lambda t} \right)^2 ds \right]^{1/2} + \left( \int_0^T K_0(t,\tau)^2 d\tau \right)^{1/2} + \left( \int_0^\infty \int_0^\infty \frac{\sigma^2}{(1 + s)^{2m}} ds dt \right)^{1/2} + \left( \int_0^t e^{-2ct} K_1(t,\tau)^2 e^{2c\tau} d\tau \right)^{1/2} \left( \int_0^\infty \varphi(s) e^{\varepsilon s} ds \right)^{1/2}.
\] (7.28)

We note that, for any \( k \in \mathbb{Z}_2 \), \( |K_0(k,t)|e^{2\pi \lambda t} \leq C \pi^4 |\hat{W}(k)|^2 |\hat{\varphi}(kt)|^2 |\hat{\varphi}(kt)|^2 \leq \frac{CC_0}{(\lambda_0 - \lambda)^2} C^2_{\chi} e^{-2\pi(\lambda_0 - \lambda)t} \), so we get\[
\int_0^t \left( \sup_{k \in \mathbb{Z}_2} |K_0(k,t-s)| e^{2\pi \lambda t} \right)^2 ds \leq \frac{CC_0^2 C^2_{\chi}}{(\lambda_0 - \lambda)^4}.
\]

From Proposition 7.2(7.22), the conditions of Theorem 7.4 and Step 2, the conclusion is finished. Having fixed \( A \), we will check that for \( \delta \) small enough, the above relation hold and the fast convergence of \( \delta_i \) for \( i = 1 \). The details are similar to that of the local-time case, and it can be also found in [21], here we omit it.

Acknowledgements: The author is grateful for the comfortable and superior academic environment of Yau Mathematical Science Center, Tsinghua University. I also thank professor Yifei Wu in Tianjin University for providing the financial support and hospitality.
References

[1] J.Bedrossian, Nonlinear echoes and Landau damping with insufficient regularity. arXiv:1605.06841v2.
[2] J.Bedrossian, N.Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes tudes Sci. 122 (2015), 195-300.
[3] J.Bedrossian, N. Masmoudi, V. Vicol, Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow. Arch. Ration. Mech. Anal. 219 (2016), no. 3, 1087-1159.
[4] J.Bedrossian, N. Masmoudi, C. Mouhot, Landau damping: paraproducts and Gevrey regularity, Ann. PDE 2 (2016), no. 1, Art. 4, 71 pp.
[5] J.Bedrossian, P. Germain, N. Masmoudi, On the stability threshold for the 3D Couette flow in Sobolev regularity, Ann. Math. 185 (2017), 541-608.
[6] J.Bedrossian, N. Masmoudi, C. Mouhot, Landau damping in finite regularity for unconfined systems with screened interaction. Comm. Pure Appl. Math. 71 (2018), no. 3, 537-576.
[7] J.Bedrossian, Fei Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field. arXiv:1805.10756v1.
[8] E. Caglioti, C. Maffei, Time Asymptotics for Solutions of Vlasov-Poisson Equation in a Circle, Journal of Statistical Physics. Vol. 92 (1998). Nos. 1/2.
[9] R. Davidson, Theory of Nonneutral Plasmas, W.A. Benjamin, INC. Advanced book program reading Massachusetts, 1974.
[10] P. Degond, Spectral theory of the linearized Vlasov-Poisson equation. Trans. Amer. Math. Soc. 294 (1986), 435-453.
[11] R. Dendy, Plasma Dynamics, Clarendon Press · Oxford, 1990.
[12] Y. Deng, N. Masmoudi, Long time instability of the Couette flow in low Gevrey spaces. arXiv:1803.01246v1[math.AP] 3 Mar 2018.
[13] L. Desvillettes, J. Dolbeault, On long time asymptotics of the Vlasov-Poisson-Boltzmann equation, Commun. Partial Differ. Equ. 16 (2-3), 451-489, 1991.
[14] R. Duan, S. Liu, Stability of the rarefaction wave of the Vlasov-Poisson-Boltzmann system, SIAM J. Math. Anal. 47 (5), 3585-3647, 2015.
[15] R. Duan, M. R. Strain, Optimal time decay of the Vlasov-Poisson-Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal. 219 (2), 887-902, 2016.
[16] E. Faou, F. Rousset, Landau damping in Sobolev spaces for the Vlasov-HMF model. Arch. Ration. Mech. Anal., 219 (2): 887-902, 2016.
[17] B. Fernandez, D. Gerard-Varet, G. Giacomin, Landau damping in the Kuramoto model. Ann. Institute Poincaré-Analysis nonlinéaire.
[18] Y. Guo, The Vlasov-Poisson-Boltzmann system near vacuum. Commun. Math. Phys. 218 (2), 293-313, 2001.
[19] Y. Guo, The Vlasov-Poisson-Boltzmann system near Maxwellians. Commun. Pure Appl. Math. 55 (9), 1104-1135, 2002.
[20] L. He, L. Xu, P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvé waves, Ann. PDE 4 (2018), no. 1, Art. 5, 105 pp.
[21] J. Hwang, L. J. J. Velázquez, On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem, Indiana Univ. Math. J. 58 (6), 2623-2660, 2009.
[22] R. Glassy, J. Schaeffer, Time decay for solutions to the linearized Vlasov equation, Transport Theory Statist. Phys., 23 (1994). 411-453.
[23] J. Hwang, L. Velázquez, On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem. Indiana Univ. Math. J., 58:6 (2009), 2623-2660.
[24] H. Li, T. Yang, M. Zhong, Spectrum analysis for the Vlasov-Poisson-Boltzmann system. preprint arXiv:1402.3633.
[25] E.Lifshitz, L.Pitaevskii, Physical Kinetics Course of Theoretical Physics Volume 10, Elsevier (Singapore) Pte Ltd, 2008.

[26] X.Ma, Cyclotron damping along a uniform magnetic, preprint arXiv:1807.05254.

[27] P.Maslov, V.Fedoryuk, The linear theory of Landau damping. Mat.Sb., 127(169),445-475,559(Russian),1985; English translation in Math.USSR-Sb.,55,437-465,1986.

[28] C.Mouhot, C.Villani, On Landau damping, Acta Math., 207 , 29-201,2011.

[29] T.Stix, Waves in Plasmas, American Insitute of Physics,1992.

[30] W.Sáenz, Long-time behavior of the electric potential and stability in the linearized Vlasov theory. J.Math.Phys.,6, 859-875,1965.

[31] I.Tristani, Landau damping for the Linearized Vlasov Poisson equation in a weakly collisional regime, J.Stat.Phys.169:107-125,2017.

[32] D.Wei, Z.Zhang, W.Zhao, Linear inviscid damping for a class of monotone shear flow in Sobolev spaces, Comm.Pure Appl.Math.71, no.4, 617-687,2018.