Phases of Holographic Superconductors in an External Magnetic Field

Tameem Albash and Clifford V. Johnson

Physics and Astronomy Department, University of Southern California, Los Angeles, CA 90089–0484, USA

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We study a 2+1 dimensional model of superconductors using a 3+1 dimensional gravitational dual theory of a black hole coupled to a scalar field, with negative cosmological constant. In the presence of finite temperature $T$ and a background magnetic field $B$, we use numerical and analytic techniques to solve the full Maxwell–scalar equations of motion in the background geometry, finding non–trivial localized solutions that correspond to condensate droplets, and to vortices. The properties of these solutions enable us to deduce several key features of the $(B,T)$ phase diagram.

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I. INTRODUCTION

For slightly over a decade, there has been a growing branch of string theory research that extracts important physics of strongly coupled systems by computing with a weakly coupled gravitational “dual” system. These “gauge/gravity” dualities —termed thus since the strongly coupled system is a gauge field theory or generalization thereof—are said to be holographic in nature, since the dual gravitational system crucially has at least one extra dimension, and much of the field theory’s properties can be extracted by working on the boundary of thegravitating spacetime. The best understood version of this is the AdS/CFT correspondence[1] between superstring theory on $AdS_5 \times S^5$ (4+1 dimensional anti–de-Sitter times a five–sphere) and 3+1 dimensional $N=4$ supersymmetric $SU(N)$ Yang–Mills theory. The theory can be reduced on the $S^5$ to a gravity theory coupled to a family of fields in the $AdS_5$. The boundary of $AdS_5$ is a copy of 3+1 dimensional Minkowski space, and this is the spacetime where the dual Yang–Mills theory resides. Its ’t Hooft coupling, $\lambda=4\pi g_s N=g^2_{YM} N$, is large in the regime where the dual gravity model is reliable: the large $N$ limit and weak string coupling $g_s$. The asymptotic values of the $AdS_5$ fields supply information about operators in the Yang–Mills theory. Ref.[4] reviews much of this technology, with bibliography. This type of duality has supplied a wealth of information about various strongly coupled systems that can be supplied with a gravitational (or fully stringy) dual along these lines. It has several potentially important applications, ranging from models of the behaviour of the strongly coupled dynamics of quark–gluon plasmas in nuclear physics, to models relevant to condensed matter physics, all of considerable experimental interest. (See refs.[5] [6] [7] for reviews and bibliography.)

A holographic model of some of the key phenomenological attributes of superconductivity in 2+1 dimensions was proposed in ref.[8]. The dual is a simple model of gravity in four dimensions (AdS$_4$) coupled to a $U(1)$ gauge field and a minimally coupled charged complex scalar $\Psi$ with potential $V(|\Psi|) = -2|\Psi|^2/L^2$, where the cosmological constant defines the scale $L$ via $\Lambda = -3/L^2$:

$$S_{\text{bulk}} = \frac{1}{2\kappa_4^2} \int \sqrt{-g} \left( R + \frac{6}{L^2} \right) + \frac{L^2}{4} \left( -\frac{1}{4} F^2 - \partial \Psi - i g A \Psi^2 - V(|\Psi|) \right),$$

where $\kappa_4^2 = 8\pi G_N$ is the gravitational coupling and our signature is $(++++)$. We will use coordinates $(t, z, r, \phi)$ for much of our discussion, with $t$ time, $(r, \phi)$ forming a plane, and $z$ a “radial” coordinate for our asymptotically AdS$_4$ spacetimes such that $z = 0$ is the boundary at infinity. Note that the mass of the scalar $m_h^2 = -2/L^2$ is above the Breitenlohner–Freedman stability bound[9] $m_{BF}^2 = -9/4L^2$ for scalars in AdS$_4$. A black hole (which is planar, its horizon is an $(r, \phi)$ plane at some finite $z = z_h$), with Hawking temperature $T$ and mass per unit horizon area $\epsilon = M/V$, corresponds to the dual 2+1 dimensional system at temperature $T$ and with energy density $\epsilon$. The asymptotic value of $\Psi$ on the boundary sets the vacuum expectation value (vev) of a charged operator $O$ in the system, which is the order parameter. For $T > T_c$, the scalar (and hence $\langle O \rangle$) is zero, and for $T < T_c$ it is non–zero. In the gravitational theory, the high $T$ phase is simply the charged black hole (Reissner– Nordström, or AdS–RN) with $\Psi$ vanishing. Notice that the mass of the scalar is set not just by $V(|\Psi|)$ but by the black hole’s gauge field $A = A_t dt$. $A_t$ does not give an electric field in the dual theory on $(r, \phi)$, but defines instead a $U(1)$ charge density[10], $\rho$. See below.)

$T$ also depends on $\rho$. In fact $m_h^2$ decreases with $T$ until at $T_c$ it goes below $m_{BF}^2$, becoming tachyonic. The theory seeks a new solution, in which the black hole is no longer AdS–RN, but one that has a non–trivial profile for $\Psi$ (i.e., it has “scalar hair” — for a discussion of violations of non–hair theorems in this context, see ref.[11]). The $U(1)$ is broken by $\langle \bar{O} \rangle$. These solutions can be found by solving equations of motion in certain limits (explored below), and the transport properties of the low temperature phase were examined in refs.[8][12], using linear response theory, with the result that the DC conductivity

*Electronic address: talbash [at] usc.edu
†Electronic address: johnson1 [at] usc.edu
diverges in a way consistent with expectations that the phase is superconducting. (Strictly speaking, the $U(1)$ that was broken by $\langle \mathcal{O} \rangle$ is global on the boundary, but it can be gauged in a number of ways without affecting the conclusions. See e.g. ref.\cite{12}.)

It is clearly of interest to study this system further, since it could well open the door to a whole new phenomenology of superconductivity in a wide range of physical systems. We report here on our study of the system in an external magnetic field, continuing the work we began in ref.\cite{13}. The magnetic field $B$ (which fills the two dimensions of the superconducting theory), also contributes to $m_4^2$, via its square, but contributes with opposite sign to the electric contribution of the back-ground. It therefore lowers the temperature $T_c$ at which $m_4^2$ falls below $m_{4F}^2$, triggering the phase transition. On these grounds alone one then expects a critical line in the ($B, T$) plane of a form like that of a full curve from fig. 3, but it is important to determine exactly what physics lies on either side of the line. Generically, for non–zero $B$, it is inconsistent to have non–trivial spatially independent solutions on the boundary, and we study two classes of localized solutions in some detail. The first is a “droplet” solution, the prototype of which was found in our earlier work\cite{13} as a strip in 2D (straightforwardly generalized to circular symmetry in ref.\cite{12}), and the second is a vortex solution, with integer winding number $\xi \in \mathbb{Z}$, which is entirely new. We obtain these as full solutions of the Maxwell–scalar sector in a limit, and their properties allow us to determine key features of the ($B, T$) phase diagram (correcting statements made in ref.\cite{12}).

II. SPATIALLY INDEPENDENT SOLUTIONS

This section reviews the spatially independent condensate solution of ref.\cite{12} that corresponds to the superconducting (symmetry breaking) phase below some $T_c$. It is found in a certain decoupling limit. Under a redefinition $A_\mu \to A_\mu/g$, $\Psi \to \Psi/g$, the Maxwell–scalar part of the action (1) gets a prefactor of $1/g^2$, and the $g$ disappears from the $A\Psi$ coupling. This means that in the limit $g \to \infty$, this sector decouples from gravity. We can therefore take as background black hole the (planar) Schwarzschild solution:

$$ds^2 = \frac{L^2 \alpha^2}{z^2} (-f(z)dt^2 + dr^2 + r^2 d\phi^2) + \frac{L^2}{z^2 f(z)} dz^2,$$

(2)

with $f(z)=1 - z^{-3}$. It has an horizon at $z=1$, temperature $T=3\alpha/4\pi$, and mass density $\varepsilon=L^2\alpha^3/\kappa_4^2$. Let us write our complex scalar as $\Psi=\tilde{\rho}\exp(i\theta)/\sqrt{2L}$ and write $A_t=\alpha \dot{\theta} t$. The equations of motion allow an ansatz: $\theta=\zeta + \xi \phi$, and $\tilde{\rho}=\tilde{\rho}(\tilde{r}, z)$, $\dot{\tilde{A}}_{\tilde{t}}=\dot{A}_t(\tilde{r}, z)$, and $\dot{A}_{\phi}=0$. Near the boundary $z=0$ we have, for constants $\tilde{\rho}_1, \tilde{\rho}_2, \rho, \mu$:

$$\tilde{\rho} \to \tilde{\rho}_1 z + \tilde{\rho}_2 z^2, \quad \dot{A}_t \to -\mu - \rho z.$$

(3)

$\tilde{\rho}_i$ ($i=1, 2$) sets the vev of a $\Delta=i$ operator $\mathcal{O}_i$\cite{14}. Only one of these vevs can be non–zero at a time, and we will choose to study the case of $i=1$, for brevity. $\mu$ sets a chemical potential, and $\rho$ sets a charge density for the $U(1)$ symmetry that we are studying here. To find a full solution of the equations, we solve the equations of motion numerically, using a shooting method starting at the horizon ($z=1$). There we set $\tilde{\rho}(1)$ to a constant, and for regularity of the gauge field, we have $\dot{A}_t(1)=0$.

To look for solutions we tune $\partial_z \dot{A}_t(1)$ such that at the other boundary, $\tilde{\rho}(0)$ has the required Neumann condition $\partial_z \tilde{\rho}(0)=0$. From that we read off $\tilde{\rho}(z=0) = \sqrt{2\kappa_4} \langle \mathcal{O}_1 \rangle/(L\alpha)$, and $\rho$. Since the only scale in the theory besides $T$ is set by the charge density $\rho$, the value of $T_c$ is given in terms of $\rho$. This can be determined by noting when a non–zero $\langle \mathcal{O}_1 \rangle$ develops, and there we find $T_c=0.226\alpha \sqrt{\rho}$. Fig. 1 shows the result for the vev as a function of temperature, showing the low and high temperature phases separated by a second order phase transition at $T_c$ below which the $U(1)$ is spontaneously broken.

III. SPATIALLY DEPENDENT SOLUTIONS

We can seek non–trivial solutions with $\theta=\zeta + \xi \phi$, and

$$\tilde{\rho} = \tilde{\rho}(\tilde{r}, z), \quad \dot{A}_{\tilde{t}} = \dot{A}_t(\tilde{r}, z), \quad \dot{A}_{\phi} = \dot{A}_{\phi}(\tilde{r}, z),$$

(4)

where $\tilde{r}=\alpha r$ and ($\zeta, \xi$) are constants, with $\xi$ integer. Regularity of the equations of motion require that near $\tilde{r}=0$, we must have $\tilde{\rho} \sim \tilde{r}^\kappa$. This feeds into the behaviour of all the other fields near $\tilde{r}=0$, and determining this near the horizon is important in order to seed the numerical search in a manner that will carefully find solutions. We report on this in a longer publication\cite{15}. The solutions that we find using our analysis break into two broad classes. There are those for which $\tilde{\rho} \to 0$ for $\tilde{r} \to \infty$ which are called droplets. The prototype localized solution of this type was found in our earlier work\cite{13}, and also studied in ref.\cite{12}, but in a “probe” limit of small fields and finite $g$. Here we have the full solutions in the large $g$ decoupling limit. Those solutions for which $\tilde{\rho} \to \text{const.}$ for $\tilde{r} \to \infty$ we call vortex solutions ($\xi \in \mathbb{Z}^+$). They behave exactly as expected of vortices in this context. Note that $\xi$ defines a non–trivial topological winding number. $A_{\phi}$ becomes constant at infinity so gauge symmetry cannot be used to unwind $\theta$. Gauge symmetry is unbroken.
at infinity for the droplets, so $\xi$ is not a winding number for them.

A. The Droplet

A sample droplet solution is presented in fig. 2 (for $\hat{\rho}$ and $\hat{B}_z$). The current density and magnetic field are read off via the $z \rightarrow 0$ asymptotic $\hat{A}_\phi \rightarrow a_\phi + J_\phi(\hat{r}) z$, with $\hat{B}_z = \partial_\phi a_\phi / \hat{r}$. The constant $\gamma$ appears in the small $\hat{r}$ expansion of the function $\tilde{R}$ defined by $\tilde{\rho} = z\hat{r}^\xi \tilde{R}$, where $\tilde{R} \rightarrow R_0 (1 - \gamma \hat{r}^2/4 + \cdots)$ at the horizon. We see that the magnetic field fills the plane, asymptoting to a constant value at $\hat{r} \rightarrow \infty$, and approaching a non-zero value in the core. We studied several solutions for a range of $T$ and $B \equiv B_z$, noticing that for high temperatures $B$ is decreased somewhat in the core as compared to the asymptotic value, while for low temperatures $B$ is enhanced there.

It is important to determine exactly where in the $(B, T)$ plane these solutions can appear. We find our full solutions at a variety of values of $B$, but none below certain values of $B$, which is suggestive. To determine if there is a minimum value of $B$ to form droplets, we can use the probe limit, where $\tilde{\rho}$ is small. Here we can take $\hat{A}_\phi = \gamma \hat{r}^2/2$ (which will determine the magnetic field in this limit) and have $\hat{A}_t = \hat{A}_t(z)$. The equation of motion for $\tilde{R}$ on the horizon reduces to:

$$\left[ \partial_t^2 \tilde{R} + \frac{1}{\hat{r}} \partial_{\hat{r}} \tilde{R} - \frac{1}{4} \hat{r}^2 \gamma^2 \tilde{R} + \gamma \tilde{R} \right]_{\hat{r}=1} = 0 , \quad (5)$$

and has as solution: $\tilde{R}(\hat{r}) = R_0(1) \exp \left( -\gamma \hat{r}^2/4 \right)$. We use this as the seed to solve the coupled equations for $\hat{A}_t$ and $\tilde{R}$ for the full $z$ dependence in this limit, seeking the appropriate boundary conditions at $z=0$. This extracts, as in section II, the $T$ of the solution for our magnetic field. This procedure sets the critical magnetic field at which the droplet solutions first form for a given $T/T_c$. Below that $B$ the droplets simply vanish since in this probe limit they are already at zero size. Our result is the top curve in fig. 3. Large fields will begin to back-react on the geometry, and our analysis breaks down, as can be seen in our curve for low $T/T_c$. To proceed further, we can carry out the probe computation again, but not in the $g \rightarrow \infty$ limit. Since the Maxwell sector is not decoupled from the geometry for finite $g$ we use as background a dyonic black hole. This is the analysis of our earlier work[13]. The metric is of the form given in eqn. (2), but with a Maxwell field $F = 2h \omega^2 r dr \wedge d\phi + 2q \omega dz \wedge dt$ and $f(z) = (1 - z) \left( z^2 + z + 1 - (h^2 + q^2) z^3 \right)$. Now, $T$ and $\rho$ are determined by the background (we do not display them here). We note that the equations of motion have a separable solution $\tilde{\rho} = zZ(z)R(\hat{r})$, and $R(\hat{r})$ satisfies an equation of the same form as (5), but with $\gamma \rightarrow 2gh$, giving the Gaussian profile. We can then solve for $Z(z)$ numerically using the same shooting techniques as before. This gives a complete $(B, T)$ curve for a given $g$. In fact, these curves connect to the $g \rightarrow \infty$ probe computation of above, as can be established by careful comparison of the different definitions of the temperatures and charge densities in the two limits. We carry this out in our longer paper[15], and simply show here how the two limits connect by rescaling our curves (with the appropriate factor of $g$) and superposing. See fig. 3.

B. The Vortex

We display the $\xi = 1$ vortex solution in fig. 4 (for $\hat{\rho}$ and $\hat{B}_z$). As noted before, there is one unit of winding, and the scalar field runs to a constant at infinity. So does the charge density, and so we are able to read off the value of the temperature for these solutions. The magnetic field drops to zero at large $\hat{r}$, and $\hat{A}_\phi$ becomes constant there. That constant is $\xi$ in general, as is consistent with the

FIG. 3: Scaled $(B, T)$ diagram connecting to decoupling limit.
IV. CONCLUSION

We have studied the prototype holographic superconductor of ref. [8] in the presence of magnetic field, continuing our earlier work [13], examining two important classes of solution, the droplet and the vortex. The latter is completely new, and this is the first time the droplet has been fully constructed. Our analysis in various connected limits shows where these solutions can exist in the \((B,T)\) plane. There is a critical line below which droplets are not found, while vortices can be found there. Our interpretation is that this region is the superconducting phase, and that for non-zero \(B\), the vortices develop, trapping the magnetic flux into filaments, as is familiar in type II superconductors. Above the critical line, the system leaves the superconducting phase, and either forms droplets of condensate or simply reverts to the normal phase (dual to a dyonic black hole with zero scalar everywhere, which may well yield lower action than the droplets if we had back-reacting solutions to work with). We have performed a stability analysis of these solutions [15]. Both classes are stable for the examples and modes that we study (this is expected for the vortices given their conserved winding). Finally, note that we disagree with the suggestion made in ref. [12]. The authors find the critical line, but state (similarly to our ref. [13]) that the droplets exist below the line, and are superconducting. As they did not have the full droplet solutions, nor the vortex solutions, their analyses are not sufficient to make these determinations. Our work here gives a much stronger picture of the phase diagram.

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