ON PETERSSON'S PARTITION LIMIT FORMULA

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Abstract. For each prime $p \equiv 1 \pmod{4}$ consider the Legendre character $\chi = (\cdot/p)$. Let $p_\pm(n)$ be the number of partitions of $n$ into parts $\lambda > 0$ such that $\chi(\lambda) = \pm 1$. Petersson proved a beautiful limit formula for the ratio of $p_+(n)$ to $p_-(n)$ as $n \to \infty$ expressed in terms of important invariants of the real quadratic field $K = \mathbb{Q}(\sqrt{p})$. But his proof is not illuminating and Grosswald conjectured a more natural proof using a Tauberian converse of the Stolz-Cesàro theorem. In this paper we suggest an approach to address Grosswald’s conjecture. We discuss a monotonicity conjecture which looks quite natural in the context of the monotonicity theorems of Bateman-Erdős.

1. Introduction

Let $K$ be a real quadratic field, $h_K$ its class number, and $\varepsilon_K > 1$ its fundamental unit. Let us assume that the discriminant of $K$ is a prime number $p$, so in particular $p \equiv 1 \pmod{4}$. Consider the Nebentypus cover $X_\chi(p)$ of degree two of the modular curve $X_0(p)$ introduced by Shimura [16, p. 174] in his work towards a theory of “real multiplication”[1]. The Fricke involution $w_p$ of $X_\chi(p)$ is defined over $K$ and the curve $X_\chi(p)$ corresponds to the congruence subgroup

$$\Gamma_\chi(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) : \chi(a) = 1 \right\},$$

where $\chi$ denotes the Legendre character $\chi = (\cdot/p)$ of conductor $p$. To simplify the discussion here, we will assume that $p > 5$. Let $\hat{f}$ be the modular unit on the curve $X_\chi(p)$ introduced by Ogg an Ligozat, as described by Mazur [9, pp. 107–108]. In this paper we define a certain normalization $u$ of $f$ and use its Fourier expansion and that of the composite $\hat{u} = u \circ w_p$ to generalize a limit formula due Schur. (See Proposition[2]) We use this limit formula together with a monotonicity theorem of Bateman and Erdős [24], a consequence of the work of Meinardus [10] (described in the appendix), and a ratio Tauberian theorem due to Sato [14], to prove the following.

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1Shimura determines division points of certain one-dimensional factors of the Jacobian $J_\chi(p)$ of $X_\chi(p)$ that generate abelian extensions of $K$. These one-dimensional factors are cut out by the action of the Hecke algebra and the involution $w_p$ over $K$ on $J_\chi(p)$. 

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Theorem 1. With the above assumptions, for each \( n \in \mathbb{Z}_{\geq 0} \) let \( p_{\pm}(n) \) denote the number of partitions
\[
n = \lambda_1 + \cdots + \lambda_r
\]
with parts \( \lambda_i \in \mathbb{Z}_{>0} \) such that \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r \) and \( \chi(\lambda_i) = \pm 1 \), for \( i = 1, 2, \ldots, r \). Then
\[
\lim_{\nu \to \infty} \frac{\sum_{n=0}^{\nu} p_{+}(n)}{\sum_{n=0}^{\nu} p_{-}(n)} = \varepsilon h_K.
\]

Of course, the above theorem follows directly from the classical Stolz-Cesàro theorem \([12, \text{p. 14}]\) applied to a celebrated partition limit formula due to Petersson \([11]\),
\[
\lim_{n \to \infty} \frac{p_{+}(n)}{p_{-}(n)} = \varepsilon h_K.
\]
(1)

But our proof does not use Eq. (1). In fact, Petersson’s partition limit formula follows directly from Theorem 1 and a converse of the Stolz-Cesàro theorem\(^2\) due to Păltănea \([13]\), provided we assume a special case of Conjecture 1 discussed in Section 4. This is a monotonicity conjecture which looks quite natural in the context of the monotonicity theorems of Bateman and Erdős \([2]\).

Petersson obtained Eq. (1) by first establishing the asymptotic expression for \( p_{+}(n) \) and for \( p_{-}(n) \) separately, after a rather laborious calculation. So given the simplicity of Eq. (1), it seems desirable to have a simpler proof. In fact, Grosswald \([5]\) conjectured that a monotonicity theorem of Bateman and Erdős \([2]\) together with a suitable Tauberian converse to the Stolz-Cesàro theorem, would furnish a nicer proof of Eq. (1). It is hoped that our approach can shed new light on Grosswald’s conjecture. Moreover, the key role played here by the modular unit \( u \) on \( X_\chi(p) \) and the Fricke involution \( w_p \) of \( X_\chi(p) \) may help pave the way towards an explanation why \( h_K \) and \( \varepsilon K \) appear in Eq. (1), a question which was raised by Petersson \([11]\).

The rest of the paper is organized as follows. In Section 2 we use Klein forms to define the modular unit \( u \) on \( X_\chi(p) \). Then we use the class number formula for real quadratic fields to obtain the constant term of the Fourier expansion of \( u \). We express the Fourier expansion of \( u \) as an infinite product and conclude this section with a discussion of the \( p = 5 \) case, where we express the Rogers-Ramanujan continued fraction in terms of \( u \). In Section 3 we use the Fourier expansions of \( u \) and of \( \hat{u} \) to obtain a generalization of a limit formula due to Schur, which we use to prove Theorem 1. In Section 4 we discuss Conjecture 1, including the numerical evidence that supports it, and also suggest an open question. In the appendix Luca shows how Eq. (1) follows from

\(^2\)Păltănea stated his theorem as a converse of L’Hôpital rule for locally integrable functions. But applying his theorem to suitable step functions yields a converse of the Stolz-Cesàro theorem.
the work of Meinardus. The appendix also includes a discussion of the inequality (4), which is used in our proof of Theorem 1.

2. Two Fourier expansions

Following Kubert and Lang [7, p. 27], for each \( z \in \mathbb{C} \) and each lattice \( L \subset \mathbb{C} \) we may define the Klein form

\[
\mathfrak{K}(z, L) = e^{-\frac{1}{2}\eta(z, L)z}\sigma(z, L),
\]

where \( \sigma(z, L) \) is the Weierstraß sigma-function and \( z \mapsto \eta(z, L) \) is the \( \mathbb{R} \)-linear function that gives the quasi-periods of the Weierstraß zeta-function with respect to the lattice \( L \). Put \( k_a(\tau) = \mathfrak{K}(z, L_\tau) \), where the point \( a = (a_1, a_2) \in \mathbb{R}^2 \) is uniquely determined by \( z = a_1\tau + a_2 \) and \( L_\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \), with \( \tau \) lying in the Poincaré upper-half plane

\[ \mathcal{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \}. \]

As before, consider a prime number \( p > 5 \) such that \( p \equiv 1 \pmod{4} \) and define

\[
u(\tau) = \prod_{r=1}^{p-1} \mathfrak{K}(0, r/p)(\tau)^{\chi(r)}.
\]

As we shall see, up to a multiplicative constant this is Ogg and Ligozat’s modular unit \( f \) on the Nebentypus cover \( X_\chi(p) \) described by Mazur [9, pp. 107–108]. The cover \( X_\chi(p) \) has 4 cusps, namely \( \infty \) and \( \overline{\infty} \) conjugate over \( K \), above the cusp \( \infty \) of \( X_0(p) \) and cusps \( o \) and \( \overline{o} \) defined over \( \mathbb{Q} \), above the cusp \( o \) of \( X_0(p) \). Mazur also showed that

\[
(f) = \frac{1}{2} B_{2, \chi} \left((o) - (\overline{o})\right),
\]

where \( B_{n,\chi} \) is the generalized \( n \)-th Bernoulli number attached to \( \chi \) defined by

\[
\sum_{n=0}^{\infty} B_{n,\chi} \frac{X^n}{n!} = \sum_{r=1}^{p} \chi(r) \frac{X e^{rX}}{e^{pX} - 1}.
\]

The Fricke involution \( w_p \) of \( X_\chi(p) \) interchanges the cusps \( o \) and \( \infty \) (resp. \( \overline{o} \) and \( \overline{\infty} \)). So the composite \( \tilde{u} = u \circ w_p \) has a zero of order \( \frac{1}{2} B_{2,\chi} \) at the cusp \( \infty \) of \( X_\chi(p) \). The following proposition provides further details.

**Proposition 1.** We have Fourier expansions

\[
\tilde{u}(\tau) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n)},
\]

and

\[
u(\tau) = \varepsilon_K^{-h_K} (1 - \sqrt{p} q_\tau + \ldots),
\]

where \( q_\tau = e^{2\pi i \tau} \) and \( \tau \) lies in the Poincaré upper-half plane \( \mathcal{H} \). Actually,

\[
u = \varepsilon_K^{-h_K} f,
\]
where \( f \) is the modular unit of Ogg and Ligozat.

**Proof.** Let \( \eta(\tau) \) denote Dedekind’s eta-function

\[
\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).
\]

The Siegel function \( g_a(\tau) = \xi_a(\tau) \eta(\tau)^2 \) has a product expansion

\[
g_a(\tau) = -q^{1/2} B_2(a_1) e^{2\pi i a_1} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^n_z q_z(1 - q^n_z q_z^{-1})),
\]

where \( B_2(X) = X^2 - X + \frac{1}{6} \) is the second Bernoulli polynomial, and \( q_z = e^{2\pi iz} \) with \( z \in \mathbb{C} \). So if we let \( \zeta_p = e^{2\pi i/p} \), then

\[
u(\tau) = \prod_{r=1}^{\frac{p-1}{2}} g(0, r/p)(\tau) \chi(r)
\]

\[
= \prod_{r=1}^{\frac{p-1}{2}} \left( \zeta_p^{-\frac{r}{2}}(1 - \zeta_p^r) \prod_{n=1}^{\infty} (1 - q^n_r \zeta_p^n)(1 - q^n_r \zeta_p^{-r}) \right) \chi(r)
\]

\[
= \left( \prod_{r=1}^{\frac{p-1}{2}} \zeta_p^{-\chi(r)\frac{r}{2}}(1 - \zeta_p^r)^{\chi(r)} \right) \left( \prod_{n=1}^{\infty} \prod_{r=1}^{\frac{p-1}{2}} (1 - q^n_r \zeta_p^n)^{\chi(r)}(1 - q^n_r \zeta_p^{-r})^{\chi(r)} \right)
\]

\[
= \kappa f(\tau),
\]

where

\[
\kappa = \prod_{r=1}^{\frac{p-1}{2}} (\zeta_p^{-\frac{r}{2}} - \zeta_p^{\frac{r}{2}})^{\chi(r)} = e^{-h_K}. \]

The last equality follows from the first equation in Théorème 1 of Borević and Safarević [3, p. 385], namely

\[
h_K = -\frac{1}{\log \varepsilon_K} \sum_{(r,D)=1}^{\infty} \chi(r) \log \sin \frac{\pi r}{D}.
\]

specialized to the positive fundamental discriminant \( D = p \), which is a well-known consequence of the formula

\[
L(1, \chi) = \frac{2h_K}{\sqrt{p}} \log \varepsilon_K.
\]

Here \( L(s, \chi) \) is the Dirichlet L-series attached to \( \chi \)

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \Re(s) > 0.
\]
Therefore \( u = \varepsilon_K^{-hK} f \), which is the third assertion in our proposition.

To prove the second assertion note that
\[
f(\tau) = \prod_{n=1}^{\infty} \Psi(q^n),
\]
where
\[
\Psi(X) = \prod_{r=1}^{\frac{p-1}{2}} (1 - X \zeta_p^r)^{\chi(r)} (1 - X \zeta_p^{-r})^{\chi(r)}
\]
\[
= \prod_{r=1}^{\frac{p-1}{2}} (1 - X \zeta_p^r)^{\chi(r)} (1 - X \zeta_p^{-r})^{\chi(-r)}
\]
\[
= \prod_{r=1}^{p-1} (1 - X \zeta_p^r)^{\chi(r)}
\]
\[
\equiv 1 - S_p X \pmod{X^2}.
\]

where \( S_p = \sum_{r=1}^{p} \chi(r) \zeta_p^r \) is the Gauß sum attached to \( \chi \). But we assumed that \( p \equiv 1 \pmod{4} \), so \( S_p = \sqrt{p} \). Hence
\[
f(\tau) = 1 - \sqrt{p} q_\tau + \ldots.
\]

and the third assertion of our proposition implies that
\[
u(\tau) = \varepsilon_K^{-hK} f(\tau) = \varepsilon_K^{-hK} (1 - \sqrt{p} q_\tau + \ldots).
\]

To prove the first assertion of our proposition recall that the Fricke involution \( w_p \) of \( X_{\chi}(p) \) is induced by the Möbius transformation
\[
\tau \mapsto -\frac{1}{p\tau}
\]
acting on the extended upper-half plane \( \mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \), which is the composition of the Möbius transformation attached to
\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})
\]
with the map \( \tau \mapsto pt \). But the basic properties \( K0 \) and \( K1 \) of Kubert and Lang [7, p. 27] imply that for each \( \alpha \in \text{SL}_2(\mathbb{Z}) \) and each \( \tau \in \mathcal{H} \) we have
\[
\mathfrak{e}_{aa}(\tau) = (c\tau + d)\mathfrak{e}_a(\alpha\tau).
\]
Therefore

\[ \tilde{u}(\tau) = \prod_{r=1}^{\infty} \xi_{(-r/p,0)}(p\tau) \chi^{(r)} \]

\[ = \prod_{r=1}^{\infty} \left( \frac{1}{q^{1/2}} B_{2x}(\tau)^{p-1/2} \prod_{n=1}^{\infty} \left( 1 - q^{p^{n+r}} \right) \left( 1 - q^{p^{n-r}} \right) \right)^{\chi^{(r)}} \]

\[ = q^{1/2} B_{2x} \prod_{r=1}^{\infty} (1 - q^{r})^{\chi^{(r)}} \prod_{r=1}^{\infty} \prod_{n=1}^{\infty} (1 - q^{p^{n+r}} \chi^{(r)(1 - q^{p^{n-r}})\chi^{(-r)}}) \]

\[ = q^{1/2} B_{2x} \prod_{n=1}^{\infty} (1 - q^{n})^{\chi^{(n)}}. \]

which is the first assertion of our proposition. \( \square \)

For \( p = 5 \) we have \( \frac{1}{2} B_{2x} = \frac{1}{5} \) and we may see from the above proposition that in this case \( u \) is not an element of the function field of the curve \( X^{\chi}(p) \), but \( u^5 \) is in fact a Hauptmodul for \( X^{\chi}(p) \). Moreover, in this notation the Rogers-Ramanujan continued fraction becomes

\[ \tilde{u}(\tau) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}. \]

3. A LIMIT FORMULA

For each \( n \in \mathbb{Z}_{\geq 0} \) let \( p_A(n) \) denote the number of partitions

\[ n = \lambda_1 + \cdots + \lambda_r \]

with parts \( \lambda_i \in A \) such that \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r \), for \( i = 1, 2, \ldots, r \).

In particular, consider the set of quadratic residues \( S_+ \) and the set of quadratic non-residues \( S_- \) modulo \( p \),

\[ S_\pm = \{ m \in \mathbb{Z}_{>0} : \chi(m) = \pm 1 \}, \]

so that \( p_{\pm}(n) = p_{S_\pm}(n) \). Here as before, \( \chi = (\cdot/p) \) is the Legendre character attached to \( p \).

**Proposition 2.** As before consider a prime \( p \equiv 1 \pmod{4} \). We have the limit

\[ \lim_{t \to 0^{+}} \sum_{n=0}^{\infty} p_{\pm}(n) e^{-2\pi n t} = \varepsilon_K^h. \]
Proof. From the first part of Proposition 1 we have
\[ \frac{1}{\bar{u}} = q^{-\frac{1}{2}B_{2,\chi}} \prod_{m \in S_+} \frac{1}{1-q^m} = q^{-\frac{1}{2}B_{2,\chi}} \sum_{n=0}^{\infty} p_+(n)q^n \sum_{n=0}^{\infty} p_-(n)q^n. \]
So the second part of Proposition 1 yields
\[ \lim_{t \to 0} \bar{u}(it) = \lim_{t \to 0} u(w_p(it)) = \lim_{t \to \infty} u(it) = \varepsilon^{-h_K} \]
and the proposition follows. \( \Box \)

The above proposition is a generalization of a limit formula due to Schur\[15, p. 321\] for \( p = 5 \). It may be regarded as a consequence of the Rogers-Ramanujan continued fraction, since the right-hand side of Eq. (2) tends to
\[ \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = -1 + \sqrt{5} \] as \( t \to 0 \), and we know that \( \varepsilon_K^{-1} = -\frac{1+\sqrt{5}}{2} \) and that \( h_K = 1 \) for the real quadratic field \( K = \mathbb{Q}(\sqrt{5}) \).

Remark 1. From the first two terms of the Fourier expansion of \( u(\tau) \) we may see that the real-analytic function \( h : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) defined by \( t \mapsto \bar{u}(it) \) is a monotone concave function in a neighbourhood of \( t = 0 \), as depicted in Figure 1 for \( p = 13 \). Moreover, the function \( h \) is log-concave on \( \mathbb{R}_{>0} \). This is an easy consequence of the fact that the logarithmic derivative of \( u(\tau) \) is, up to a positive scalar multiple, the well-known Eisenstein series \( G_{2,\chi}(\tau) \) of weight 2 attached to the character \( \chi \). (Cf. Lang \[8, p. 250\].)

Fix \( k \in \mathbb{Z} \) and define \( p^{(k)}(n) = p^{(k)}_A(n) \) by the formal power series equality
\[ \sum_{n=0}^{\infty} p^{(k)}(n)X^n = (1-X)^k \prod_{a \in A} \frac{1}{1-X^a} \]
Following Bateman and Erdős [2], we say that a subset \( A \subset \mathbb{Z}_{>0} \) satisfies property \( P_k \) if \( |A| > k \) and if \( \gcd(A \setminus S) = 1 \), for each \( S \subset A \) such that \( |S| = k \). Note that \( p^{(k)}(n) \) is the \( k \)-th difference of \( p(n) \) if \( k > 0 \), the \(-k\)-th order summatory function of \( p(n) \), and \( p^{(0)}(n) = p(n) \).

**Lemma 1.** If \( A \) is an infinite subset of \( \mathbb{Z}_{>0} \) such that \( \gcd(A) = 1 \), then for each positive integer \( h \) we have the limit

\[
\lim_{n \to \infty} \frac{p(0) + \cdots + p(n + h)}{p(0) + \cdots + p(n)} = 1.
\]

**Proof.** From Bateman and Erdős [2, p. 10], the corollary after Theorem 6 says that for each positive integer \( h \) we have

\[
\frac{p^{(k-1)}(n + h) - p^{(k-1)}(n)}{h} = (1 + o(1)) p^{(k)}(n).
\]

For \( k = 0 \) the assumption \( \gcd(A) = 1 \) yields

\[
\frac{p(n + 1) + \cdots + p(n + h)}{p(n)} = (1 + o(1)) h,
\]

for each positive integer \( h \). But from Theorem 5 of Bateman and Erdős [2, p. 7] we know that

\[
\lim_{n \to \infty} \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = 0,
\]

provided \( A \) is infinite and such that property \( P_k \) holds. In particular, for \( k = -1 \) we see that property \( P_k \) is trivially satisfied and we thus have the limit

\[
\lim_{n \to \infty} \frac{p(n)}{p(0) + \cdots + p(n)} = 0
\]

This limit together with Eq. (3) yield

\[
\lim_{n \to \infty} \frac{p(n + 1) + \cdots + p(n + h)}{p(0) + \cdots + p(n)} = 0,
\]

which gives

\[
\lim_{n \to \infty} \frac{p(0) + \cdots + p(n + h)}{p(0) + \cdots + p(n)} = 1 + \lim_{n \to \infty} \frac{p(n + 1) + \cdots + p(n + h)}{p(0) + \cdots + p(n)} = 1
\]

and the proposition follows. \( \square \)

Now we shall prove Theorem 1. From the appendix, for all large enough \( n \) we have

\[
p_{+}(n) < p_{-}(n).
\]

But Lemma 1 yields

\[
\frac{\sum_{m=0}^{\mu} p_{+}(m)}{\sum_{n=0}^{\nu} p_{+}(n)} \to 1 \quad \text{as} \quad \mu, \nu \to \infty \quad \text{with} \quad \frac{\mu}{\nu} \to 1.
\]

Hence the limit formula of Proposition 2 satisfies all the hypothesis of Theorem 3.2 of Sato [14, p. 85] and Theorem 1 follows.
Remark 2. If we replace Inequality 4 by the weaker condition
\[ p_-(n) = O(p_+(n)), \]
then Sato’s ratio Tauberian theorem still applies here.

4. A conjecture and some open questions

Given \( k \in \mathbb{Z} \), let \( p^{(k)}(n) \) be as in Section 3 and for each \( n \in \mathbb{Z}_{\geq 0} \) define
\[ \rho^{(k)}(n) = \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = \frac{p^{(k)}(n) - p^{(k)}(n-1)}{p^{(k)}(n)}. \]
Note that the corollary of Theorem 5 of Bateman and Erdős [2, p. 9] says that if \( A \) has property \( P_k \) then
\[ p^{(k)}(n) \to \infty \]
and
\[ \rho^{(k)}(n) \to 0, \]
as \( n \to \infty \). They also show that if \( A \) satisfies property \( P_{k+1} \), then \( p^{(k)}(n) \) is eventually strictly increasing. But the question of the monotonicity of \( \rho^{(k)}(n) \) has not been raised before. This is an interesting question, as the eventual monotonicity of \( \rho^{(k)}(n) \) is a natural generalization of the log-concavity of \( p(n) \) for all large enough \( n \). Indeed, we have
\[ \rho^{(k)}(n) > \rho^{(k)}(n+1) \]
if and only if
\[ 0 < p^{(k)}(n)^2 - p^{(k)}(n-1)p^{(k)}(n+1). \]
This monotonicity question has been settled for the case \( A = \mathbb{Z}_{>0} \) and \( k = 0 \) by DeSalvo and Pak [4], as they proved that the classical partition function \( p(n) \) is log-concave for all \( n > 25 \). Moreover, we may also see that the monotonicity of \( \rho^{(k-1)}(n) \) for all large enough \( n \) is equivalent to having the sequence
\[ \left\{ \frac{1}{p(\nu)} \sum_{n=0}^{\nu} p(n) \right\}_{\nu=0}^{\infty} \]
eventually strictly increasing. Eq. [5] is the key condition of the converse of Stolz-Cesàro theorem due to Păltănea [13] which (together with Theorem [1]) yields Petersson’s partition limit formula. Considering the Tauberian condition \( T_2 \) of the conjecture due to Grosswald [5, pp. 55-56], we propose the following.

**Conjecture 1.** If \( A \subset \mathbb{Z}_{>0} \) is such that \( P_{k+1}, P_{k+2}, \ldots \), then \( \rho^{(k)}(n) \) is eventually strictly decreasing.
With the help of PARI/GP [17] we obtained strong numerical evidence supporting Conjecture 1 for \( p(n) = p_\pm(n) \) in the range \( p \leq 1987 \), with \( |k| \leq 5 \) and \( n \leq 10000 \) and also for the classical partition function \( p(n) \), with \( |k| \leq 10 \) and \( n \leq 100000 \).

As described by Iwasawa [6, p. 61], there is a remarkable non-archimedean analogue of 

\[
L(1, \chi) = \frac{S_p}{p} \sum_{r=1}^{p-1} \chi(r) \log|1 - \zeta_p^r|
\]

known as Leopoldt’s formula. Moreover, Siegel functions have natural rigid-analytic avatars. It seems to be an interesting open question whether there are analogues of Proposition 2 (which is a generalization of a limit formula due to Schur) and of Petersson’s limit partition formula within this realm.

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Appendix A. By Florian Luca

Here, we show how equation (1) follows from Meinardus’ scheme [10] (see also [1]).

Let us recall Meinardus’ scheme. Let \( \mathcal{A} \subseteq \mathbb{N} \) be a set of positive integers. Put

\[
p_\mathcal{A}(n) = \#\{(\lambda_1, \ldots, \lambda_k) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1, \lambda_1 + \cdots + \lambda_k = n, \lambda_i \in \mathcal{A}, i = 1, \ldots, k\}
\]

for the number of partitions of \( n \) with parts from \( \mathcal{A} \). Writing \( \{a_n\}_{n \geq 1} \) for the characteristic function of \( \mathcal{A} \); that is, \( a_n = 1 \) if \( n \in \mathcal{A} \) and \( a_n = 0 \) otherwise, the generating function of \( p_\mathcal{A} \) is

\[
\prod_{n \geq 1} (1 - e^{-n\tau})^{-a_n} = 1 + \sum_{n \geq 1} p_\mathcal{A}(n)e^{-n\tau}, \quad \text{with } \text{Re}(\tau) > 0.
\]

Meinardus, in his 1954 paper [10], makes the following assumptions:

(i) Let

\[
D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{where } s = \sigma + it.
\]
Assume that $D(s)$ is convergent for $\sigma > \alpha > 0$. Assume further that $D(s)$ can be analytically continued up to $\sigma = -c_0$, where $0 < c_0 < 1$. Assume that for $\sigma \geq -c_0$, $D(s)$ is holomorphic except for $s = \alpha$ where it has a pole of order 1 with residue $A$. Assume further that in this region, we have

$$D(s) = O(|t|^{c_1})$$

as $t \to \infty$ for some $c_1 > 0$.

(ii) For $\tau = y + 2\pi ix$ with $y > 0$ put

$$g(\tau) = \sum_{n \geq 0} a_n e^{-n\tau}.$$

Assume that for $|\arg(\tau)| > \pi/4$, $|x| \leq 1/2$, one has

$$\text{Re}(g(\tau) - g(y)) \leq -c_2 y^{-\epsilon}$$

for $y$ sufficiently small, where $c_2 > 0$ and $\epsilon > 0$ are some positive real numbers.

Under (i) and (ii), Meinardus proves that

$$(6) \quad p_\chi(n) = C n^\chi e^{n \frac{2\pi i}{p} (1+\frac{1}{p})(A\Gamma(\alpha+1)\zeta(\alpha+1))^{1-2D(0)}} (1 + O(n^{-\chi_1})) \quad \text{as } n \to \infty,$$

where

$$C = e^{D(0)} (2\pi(1+\alpha))^{-1/2} (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{1-2D(0)} (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{1/2} \zeta(s)(1 - p^{-s} + L(s, \chi)),$$

$$\chi = \frac{2D(0) - 2 - \alpha}{2(1 + \alpha)}.$$

He also gives some estimates for $\chi_1$ which we don’t need. Well, let us apply it to our case. For us, $a_n = \chi(n)$ in the case of $p_+$ and $a_n = -\chi(n)$ in the case of $p_-$, where $\chi(n) = \left( \frac{n}{p} \right)$ is the Legendre character modulo $p$. Putting again

$$L(s, \chi) = \sum_{n \geq 1} \chi(n) n^{-s},$$

one sees easily that

$$D_+(s) = \sum_{\chi(n)=1, n \geq 1} n^{-s} = \frac{1}{2} \sum_{n \geq 1} \frac{1 + \chi(n)}{n^s} = \frac{1}{2} \left( \zeta(s)(1 - p^{-s}) + L(s, \chi) \right),$$

and similarly

$$D_-(s) = \frac{1}{2} \left( \zeta(s)(1 - p^{-s}) - L(s, \chi) \right).$$

So, we see that hypothesis (i) of Meinardus’ scheme is fulfilled for both $D_+(s)$ and $D_-(s)$ with $\alpha = 1$, $A = (1 - p^{-1})$ since for $\sigma > -1/2$, $\zeta(s)$ is holomorphic except for a single pole at $s = 1$ with residue 1 and $L(s, \chi)$ is holomorphic. Condition (ii) is also fulfilled by standard
results about vertical growth of $\zeta(s)$ and $L(s, \chi)$. Furthermore, since $\zeta(0) = -1/12$, and $L(0, \chi) = 0$ (because $p \equiv 1$ (mod 4)), we get that
\[ D_+(0) = \frac{1}{2} \left( (1/12)(1 - p^{-0}) + L(0, \chi) \right) = 0, \]
and similarly $D_-(0) = 0$. So, the “main” terms of $p_+(n)$ and $p_-(n)$ in (6) coincide up to the constants $C_+$ and $C_-$, that is
\[ \chi_+ = \frac{2D_+(0) - 3}{4} = \frac{5}{4} \quad \text{and} \quad \chi_- = \frac{2D_-(0) - 3}{4} = \frac{3}{4}, \]
so $p_+(n) > p_-(n)$ holds for all $n > n_0(p)$.

One may wonder if the fact that the inequality $p_+(n) > p_-(n)$ holds might be due to the fact that 1 is a quadratic residue and being the smallest positive integer it likely contributes to a lot of elements counted by $p_+(n)$. Well, let us test it. Let $p_{1, +}(n)$ be the number of partitions of $n$ with parts that are $> 1$ but quadratic residues modulo $p$. Then
\[ D_{1, +}(s) = D_+(s) - 1, \]
so $D_{1, +}(0) = D_+(0) - 1 = -1$. It thus follows that
\[ p_{1, +}(n) = (1 + o(1))c_3n^{-1/2} \quad \text{as} \quad n \to \infty, \]
where \( c_3 := C_{1,+}/C_- \). The above asymptotic shows that the inequality \( p_{1,+}(n) < p_-(n) \) holds for large \( n \). So, indeed, if we eliminate the 1’s from the partitions of \( p_+(n) \) we get a number of partitions much smaller (in fact, of a smaller order of magnitude asymptotically) than \( p_-(n) \), whereas \( p_+(n) \) and \( p_-(n) \) are of the same order or magnitude, which can be indeed interpreted by saying that the fact that the inequality \( p_+(n) > p_-(n) \) holds for large \( n \) is driven by the contribution of the 1’s in the \( p_+(n) \) side.

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