Letter

Family of CV states of definite parity and their metrological power

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Abstract

We introduce a new family of continuous variable (CV) states of definite parity originating from the single mode squeezed vacuum (SMSV) state by subtracting an arbitrary number of photons from it. A beam splitter with arbitrary transmittance and reflectance parameters redirects input photons in an indistinguishable manner to the output and measuring modes followed by probabilistic measurement, thereby converting the initial SMSV photon distribution into a new one after we know the number of registered photons. The family of the measurement-induced CV states is solely determined by the SMSV parameter which inevitably decreases when generating the CV states. We show that the quantum state engineering of CV states of definite parity by subtraction of multiphoton state from input SMSV state can significantly enhance their nonclassical properties (more squeezing, larger value of quantum Fisher information) compared to the initial state from which they originate, which, combined with a significant gain in brightness, makes them attractive for ultra-precise measurements. The potential of the new family of CV states of a certain parity, to which original SMSV, no doubt, belongs, can become decisive for a new push to implementation of optical quantum metrology protocols.

Keywords: single mode squeezed vacuum state, CV states of definite parity, quantum Fisher information, squeezing

(Some figures may appear in colour only in the online journal)

1. Introduction

The measurement process is a cornerstone for obtaining information from both classical and quantum physical systems realized by assigning a measured value to a physical quantity, thus, giving its estimation. Such an estimate cannot be accurate since some uncertainty is attributed to measured quantity. The error in the measurement result is fundamental (although we note that some statistical error can also be imposed by technical problems) that leaves no chance to describe a quantum system by a single point on the phase plane due to Heisenberg uncertainty relation $\delta x \delta p \geq 0.5$, where $\delta x$ and $\delta p$ are the uncertainties of the position and momentum, respectively, of a quantum particle [1, 2]. Although, quantum mechanics imposes fundamental bound on ultimate achievable precision but at the same time it offers quantum resource (quantum states) to be employed in order to overcome the precision limit, that can be, in principle, obtained using only classical resources [3–13]. Different physical systems can be used for performance of metrological tasks [11–13]. Nevertheless, the photons are the most attractive physical
systems for metrological scenarios due to their high mobility and low interaction with environment. It is worth taking into account achievability and practicality of photonic technologies, which includes their generation, manipulation and detection [14–16].

The goal of photonic quantum metrology is to achieve greater sensitivity in the estimation of phase shift by going beyond classical restriction [17–22] known as standard quantum limit (SQL) or shot-noise (SN). The SQL is determined through inverse root of average total number of particles. If quantum probe is allowed, the SQL can be overpassed and the phase uncertainty of estimated unknown phase can be reduced down to the quantum boundary known as the Heisenberg limit which can improve the precision of classical estimations by a square root of the average number of particles [9–13]. To enhance the sensitivity above the SN limit, one can make use of one of two tricks of quantum mechanics, namely, squeezing [23–25] and entanglement [26–30].

We develop a quantum state engineering algorithm of a new family of nonclassical optical bright continuous variable (CV) states of definite parity that can have great potential for solution of photonic quantum metrology problems. A whole family of the measurement-induced CV states of definite parity depends solely on one parameter, which largely determines their quantitative nonclassical properties. Although the parameter decreases by the transmittance coefficient squared compared to the multiplier of the original single mode squeezed vacuum (SMSV) states, the nonclassical properties of the generated CV states can only be enhanced together with an increase in their brightness up to a macroscopic level in terms of the mean number of photons (say, up to 4000). Subtraction of a small number of photons (say, 2) from SMSV can generate nonclassical states with stronger squeezing of the quadrature component compared to one of the input SMSV in a range up to 5 dB. In addition, the measurement-induced CV states can have Fisher quantum information many times greater than the same one in the original SMSV state. This shifts the quantum Cramer–Rao (QCR) bound of the CV states of definite parity below the QCR one of the initial SMSV state allowing the value of the unknown phase to be estimated more accurately than even when using the SMSV state. Our case of reaching sub-SN phase uncertainty is not referred to as an achievement of the Heisenberg limit despite the fact that the sub-Heisenberg sensitivity is present in the input SMSV state.

2. Family of $2m/2m + 1$ heralded CV states and their statistical characteristics

Quantum engineering of nonclassical states is an integral part of quantum information processing. In practice, a number of nonclassical states mainly generated in spontaneous parametric down converter are used. Further development of optical quantum CV state engineering can be based on various modifications of the measurement-induced approach [31–33]. Consider the optical scheme in figure 1 designed to generate measurement-induced CV states of certain parity. It consists of a lossless beam splitter $BS = \left[ \begin{array}{cc} t & -r \\ r & t \end{array} \right]$, with real transmittance $t > 0$ and reflectance $r > 0$ coefficients satisfying the physical condition $t^2 + r^2 = 1$. The BS is considered to be no longer necessarily balanced and its parameter can be arbitrary [31–33]. A single-mode squeezed vacuum state

$$|\text{SMSV}\rangle = \frac{1}{\sqrt{\cosh y}} \sum_{n=0}^{\infty} \frac{y^n}{\sqrt{(2n)!}} \frac{(2n)!}{n!} (2n)(2n+1)$$  

occupies a first mode of the BS, where $y = \tanh s/2$ and $s > 0$ is the squeezing parameter of the SMSV state that provides the following range of its change $0 \leq y \leq 0.5$. Zero value of the squeezing parameter $y = 0$ indicates the absence of the SMSV state at input of the BS, while the value $y = 0.5$ corresponds to the physically unrealizable case of maximally squeezed light with $s \to \infty$. The second mode remains void that is, in a vacuum state $|0\rangle$.

Extraction of a certain number of photons either $2m$ or $2m + 1$ from the SMSV in an indistinguishable manner with loss of all information from which Fock states of the original superposition the photons are subtracted generates $2m, 2m + 1$ measurement-induced CV states leaving the output state with a well-defined parity either even or odd in dependency on the parity of the measurement outcome. Near-unity efficient photon number resolving (PNR) detector is now and experimentally available resource [34–36]. Consider an ideal PNR detector with unit quantum efficiency $\eta = 1$. It follows from the entangled hybrid state in equation (15), present in the section Methods (section 3.1), the following measurement-induced $2m$–CV states

$$|\Psi_{2m}(y_1)\rangle = \frac{1}{\sqrt{2^n(y_1)}} \sum_{n=0}^{\infty} \frac{y_1^n}{\sqrt{(2n)!}} \frac{(2n)!}{n!} (n+m)! (2n)$$  

Figure 1. An optical scheme used to generate CV states of definite parity either even or odd. It consists of the beam splitter with arbitrary transmittance coefficient $t$ through which the original SMSV with squeezing amplitude $s$ passes and PNR detector. Second measurement mode is used to measure number of photons and generate new measurement-induced CV states $|\Psi_n\rangle$, where either $n = 2m$ or $n = 2m + 1$ is used.
are generated provided that even number 2m of photons is detected in auxiliary second mode of the BS, where 2m order derivative \( Z^{(2m)}(y_1) = d^{2m}Z(y_1)/dy_1^{2m} \) of the analytical function \( Z(y_1) = 1/\sqrt{1-4y_1^2} \) with \( y_1 = t^2 \tan \beta /2 = t^2y \) is used. In particular, the normalization factor \( Z(y) \) is derived in section 3.2. The parameter \( y_1 \) changes in the range of \( 0 \leq y_1 \leq 0.5 \), as we adhere to condition \( s > 0 \). The condition \( y_1 = 0 \) formally takes place in the case of either \( s = 0 \) or \( t = 0 \) which is not a case for our consideration. The opposite case \( y_1 = 0.5 \) takes place only in the limiting case \( t = 1 \) and \( s \to \infty \) that going beyond the scope of physical consideration. Assume that the PNR detector in the measuring mode registers an odd 2m + 1 number of photons. Then, the 2m + 1— heralded CV state is generated

\[
|\Psi_{2m+1}(y_1)\rangle = \sqrt{y_1/Z^{(2m+1)}(y_1)} \sum_{n=0}^{\infty} \frac{y_1^n}{(2m+1)!} (2(m+1))^{1/2} \frac{(2(n+m+1))!}{(n+m+1)!} |2(n+1)\rangle.
\]

where the 2m + 1 order derivative of the function \( Z(y_1) \) on \( y_1 \) is used in the normalization factor.

The CV states in equation (2) consist exclusively of even Fock states and they are 2m— heralded CV states of even parity. The CV states in equation (3) are odd as they involve only odd number states. Note the CV states in equations (2) and (3) are similar in its form to the original SMSV state from which they originate. The difference lies in the fact that factor \((2n)!/n!\) of the original SMSV state is converted to other multipliers either \((2(n+m))!/(n+m)!\) for even or \((2(n+m+1))!/(n+m+1)!\) for odd CV states involving number \( m \) proportional to the number of extracted photons. Additionally, the parameters \( y \) and \( y_1 \) differ from each other by a decrease factor \( t^2 = y_1/y < 1 \), that means that the BS inevitably lowers squeezing parameter \( y \). The similarity of the CV states in their form allows us to combine the generated CV states into one family (class) of the CV states of definite parity. Undoubtedly, even the SMSV state from which they come must also belong to the family of the CV states of definite parity. The family of the CV states of definite parity can be formally divided into two subgroups even and odd CV states of definite parity. The CV states generated become vacuum state \( |\Psi_{2m}(y_1 = 0)\rangle = |0\rangle \) and single photon \( |\Psi_{2m+1}(y_1 = 0)\rangle = |1\rangle \), respectively, in the case of \( y_1 = 0 \), regardless of the number of photons \( n = 2m,2m+1 \) extracted from original SMSV state. When \( y_1 = y \) and \( m = 0 \), the CV state in equation (2) definitely becomes SMSV state, i.e. \( |\Psi_0(y)\rangle \equiv |\text{SMSV}\rangle \).

The success probabilities to implement the 2m—(even) and 2m + 1—(odd) heralded CV states

\[
P_{2m}(y,y_1) = \sqrt{1-4y^2} (B_{y_1}^{(2m)})^2 Z^{(2m)}(y_1),
\]

\[
P_{2m+1}(y,y_1) = \sqrt{1-4y^2} (B_{y_1}^{(2m+1)})^2 Z^{(2m+1)}(y_1),
\]

respectively, are subject to the normalization condition \( \sum_{m=0}^{\infty} (P_{2m}(y,y_1) + P_{2m+1}(y,y_1)) = 1 \) that is directly verified. Here, we introduce the beam splitter parameter (BSP) \( B = (1 - t^2)/t^2 \). The BSP can act both as an amplifying \( (B > 1 \) performed in the case of \( 1/\sqrt{2} > t \)) \( B > 1, 1/\sqrt{2} > t \) and as a decreasing one \( (B < 1 \) to the generation rate in the case of \( t > 1/\sqrt{2} \) at given value of \( y_1 \). In the case of the use of the balanced BS \( t = 1/\sqrt{2}, B = 1 \), BSP no longer affects the success probability. In general, the parameter \( B(y_1) \) can be transformed into \( B(y_1) = (1 - t^2)/t^2 \) which provides a significant reduction to the probability of observing subtraction of larger number of photons (say, 10 and more), since the power dependence with the base \( B(y_1) < 1 \) can quickly go to zero \( (B(y_1))^{\infty} \to 0 \) with \( n \) growing. In particular, the use of a highly transmissive beam splitter (HTBS) with \( t \to 1 \) can sharply reduce the success probability. Therefore, the strategy of subtracting more photons (say, more than 2) with help of HTBS is impractical as the success probability is greatly reduced with a number of extracted photons increasing \([28]\). Note also that the CV states generated depend solely on one parameter \( y_1 \), while their success probabilities are determined by two parameters \( y \) and \( y_1 \). Dependences of the success probabilities on the squeezing parameter \( s \) for different values of the BS transmittance coefficient \( t \) are shown in figure 2. As can be seen from the plots, the probabilities of extracting (and hence generating CV states of definite parity) a small number of photons (say, 1 and 2 photons) prevail over all other probabilities. The probability \( P_0(y,y_1) \) behaves differently from the others. In case of \( s = 0 \), it acquires a maximum value and then decreases exponentially with increasing squeezing amplitude \( s \). Maximum value of the probabilities with \( m \neq 0 \) shifts towards lower values of the squeezing amplitude \( s \) with a decrease in the transmittance coefficient \( t \) of the BS.

Knowing the exact form of the 2m, 2m + 1 heralded CV states, one can find all their statistical characteristics: average number of photons \( \langle n \rangle \), the average number of photons squared \( n^2 \) and variation \( \delta n \) that are expressed through respective derivatives of the function \( Z(y_1) \)

\[
\langle n \rangle_{2m} = y_1 Z^{(2m)}(y_1)/Z^{(2m+1)}(y_1),
\]

\[
n^2_{2m} = y_1^2 Z^{(2m+2)}(y_1)/Z^{(2m)}(y_1) + \langle n \rangle_{2m},
\]

\[
\delta n_{2m} = n^2_{2m} - \langle n \rangle_{2m}^2 = y_1^2 Z^{(2m+2)}(y_1)/Z^{(2m)}(y_1) + \langle n \rangle_{2m} - \langle n \rangle_{2m}^2,
\]

for even CV states in equation (2) and

\[
\langle n \rangle_{2m+1} = y_1 Z^{(2m+2)}(y_1)/Z^{(2m+1)}(y_1),
\]

\[
n^2_{2m+1} = y_1^2 Z^{(2m+3)}(y_1)/Z^{(2m+2)}(y_1) + \langle n \rangle_{2m+1},
\]
cannot be considered to be macroscopic and the peak of its
time in precise measurement. But the increase of average
number of photons
\( n \\sigma \delta \) for different values of the BS parameter
\( s \). So, we have \( \langle n \rangle_0 = 4 \langle n \rangle_1 = 1 + 12 \langle n \rangle_1 \),
\( \delta n_1 = 2 (\langle n \rangle_1 - 1) (\langle n \rangle_1 + 2) / 3 = 2 (\langle n \rangle_1 + \langle n \rangle_1 - 2) / 3 \)
that gives us \( \langle n \rangle_1 (y_1) = 0, \delta n_0 (y_1) = 0 \) and
\( \langle n \rangle_1 (y_1) = 1, \delta n_1 (y_1) = 0 \) = 4/3. By taking \( t = 1 \) in
\( y_1 \) that leads to \( y_1 = y_0 \) one gets \( n_{\text{SMSV}} = \sinh^2 s \) and
\( \delta n_{\text{SMSV}} = 2 (\langle n \rangle_1^2 + \langle n \rangle_1) / 2 (\sinh^2 s + \sinh^2 s) \). In general,
the SMSV state with mean number of photons \( n > 1000 \) could be recognized as macroscopic and may have a serious potential in precise measurement. But the increase of average number of photons \( \langle n \rangle_{\text{SMSV}} \) requires a significant increase of the squeezing parameter \( s \) (for example, \( n(s = 2)_{\text{SMSV}} = 13.18 \),
\( n(s = 3)_{\text{SMSV}} = 100.4, n(s = 4)_{\text{SMSV}} = 744.74, n(s = 5)_{\text{SMSV}} = 5505.6 \)), which can hardly be achieved in practice. As a rule, in experimental setups, the SMSV state cannot be considered to be macroscopic and the peak of its
distribution is shifted towards smaller Fock states. The method of subtraction of \( 2m, 2m + 1 \) photons from the SMSV state makes it possible to generate macroscopic nonclassical CV states with \( \langle n \rangle_{2m}, \langle n \rangle_{2m+1} > 1000 \).

Let us compare statistical characteristics even/odd CV states in equations (2) and (3) with ones of SMSV state. As a comparison, we choose the average number of photons \( \langle n \rangle_n \), the ratio of the average number of photons in \( 2m, 2m + 1 \) heralded and SMSV states i.e. \( Rn_n = \langle n \rangle_n / \langle n \rangle_{\text{SMSV}} \), \( \sqrt{\delta n_n} / \sqrt{\delta n_{\text{SMSV}}} \), square root of the variance or the same photon uncertainty \( \sqrt{\delta n_n} \) and the ratio of the photon uncertainty of the generated CV state of definite parity to one of the SMSV,
i.e. \( Rn_n = \sqrt{\delta n_n} / \sqrt{\delta n_{\text{SMSV}}} = \sqrt{\delta n_n} / \sqrt{2 (\sinh^2 s + \sinh^2 s)} \),
where the subscript \( n \) takes on either even \( n = 2m \) or odd
\( n = 2m + 1 \) values. In figures 3 and 4 we show the dependencies of the \( \langle n \rangle_n, \sqrt{\delta n_n} \) and \( Rn_n \) on the squeezing parameter \( s \) for two parameters \( t = 0.98 \) (figure 3) and \( t = 0.99 \) (figure 4). The figures 3(a) and 4(a) show the mean number of photons \( \langle n \rangle_{\text{SMSV}} \) in SMSV state in black. As can be seen from the plots in the figures 3(a), (b) and 4(a), (b), the

\[
\delta n_{2m+1} = n_{2m+1}^2 - \langle n \rangle_{2m+1}^2 = \frac{\langle n \rangle_1^2}{Z_1(2m+1)} (y_1) + \langle n \rangle_{2m+1}^2 - \langle n \rangle_{2m+1}^2. \quad (11)
\]
condition $\langle n \rangle_n > \langle n \rangle_{\text{SMSV}}$ ($RV_n > 1$) is observed in a wide range of variation of the squeezing parameter for different values of extracted photons $n$. Maximal observed value of the mean number of photons is $\langle n \rangle_{100} \approx 4000$ in the case of the subtraction of 100 photons from original SMSV state with the squeezing amplitude of $s = 3$ and $t = 0.99$ in figure 4(a) can exceed the average number of photons in the initial SMSV by 40 times $\langle n \rangle_{100} \approx 4000 > \langle n \rangle_{\text{SMSV}} = 100.4$ in figure 4(b).

From a practical point of view taking into account expressions for the success probability in equations (4) and (5) and plots in figure 2, of interest are those CV states that can be realized with a small number of extracted photons (say, up to 10) and an acceptable value of the squeezing amplitude $s < 1$ of original SMSV state with larger mean number of photons than in the initial SMSV state $\langle n \rangle_n > \langle n \rangle_{\text{SMSV}}$. A strong predominance $\langle n \rangle_n \gg \langle n \rangle_{\text{SMSV}}$ and as consequence $RV_n > 500$, is observed in the practical case with a sufficiently large mean number of photons, although less than the maximum values $\langle n \rangle_n < 1000$. In addition to increasing the average number of photons in generated CV states the photon uncertainty $\sqrt{\delta n}$ can take on a larger values in comparison with one of original SMSV state shown by the black line in figures 3(c) and 4(c), i.e. $\sqrt{\delta n} > \sqrt{\delta n_{\text{SMSV}}}$ is observed in the vast majority of cases. Note that $\sqrt{\delta n}$ with $t = 0.99$ in figure 4(c) is larger of $\sqrt{\delta n}$ with $t = 0.98$ in figure 3(c). The maximum value of the photon uncertainty $\sqrt{\delta n} \approx 400$ observed in figure 4(c) corresponds to the state $|\Psi_{100}\rangle$ at realized from the SMSV state with $s = 3$. An inequality $\sqrt{\delta n} > \sqrt{\delta n_{\text{SMSV}}}$ is observed in the overwhelming majority of cases that provides $RV_n > 1$ for them and can have potential for application in quantum metrology exceeding possibilities of the original SMSV state in estimation of the unknown parameter with sub-SN uncertainty.

The family of the $2m/2m + 1$ heralded states stems from nonclassical SMSV state which possesses pronounced nonclassical properties, namely, the noise in one of the two quadrature components $X_1 = (a + a^\dagger)/2$ and $X_2 = (a - a^\dagger)/2i$, where $a^\dagger, a$ are the creation and annihilation operator, can be less than the quadrature noise of the vacuum state. Using exact forms of the CV states in equations (2) and (3), one can directly derive analytic compact expressions of the
The quadrature variances \((\Delta^2 X_1)_n = X_{1n}^2 - X_{2n}^2\) with zero values of mean quadrature components \(X_{1n} = X_{2n} = 0\), where the number \(n\) is either even \(n = 2m\) or odd \(n = 2m + 1\)

\[
(\Delta^2 X_1)_{2m,2m+1} = \frac{1}{4} + \frac{(n)_{2m,2m+1}}{2} \left[ \frac{y_1}{Z_{2m,2m+1}} \right] (y_1 Z(y_1))^{(2m+1,2m+2)},
\]

(12)

\[
(\Delta^2 X_2)_{2m,2m+1} = \frac{1}{4} + \frac{(n)_{2m,2m+1}}{2} \left[ \frac{y_1}{Z_{2m,2m+1}} \right] (y_1 Z(y_1))^{(2m+1,2m+2)},
\]

(13)

Obviously, the quadrature variances \((\Delta^2 X_2)_n\) take a smaller value compared to \((\Delta^2 X_1)_n\), where \(n\) can be either even \(n = 2m\) or odd \(n = 2m + 1\) provided that the derivatives \((y_1 Z)_{2m+1,2m+2}\) take positive values. Thus, the presence of the last term with sign \(-\) in equation (13) may lead to the observation of the quadrature squeezing, i.e. quadrature noise reduction in comparison with vacuum one \((\delta X)_n \equiv (\Delta X)_n = \sqrt{\langle \Delta^2 X \rangle_n} < 0.5\). Indeed, the quadrature variances for the SMSV directly follow from equations (12) and (13), i.e. \((\Delta X)_n^\text{SMSV} = \exp(2\delta s)/4 \equiv 1/4\) (SMSV desqueezing) and \((\Delta X)_n^\text{SMSV} = \exp(-2\delta s)/4 \leq 1/4\) (SMSV squeezing) if we take \(m = 0\) and \(y_1 = y_1 = \tanh(s/2)\), where Product of two quadrature uncertainties is \((\delta X)_n^\text{SMSV} (\delta X)_n^\text{SMSV} = 1/4\), where \((\delta X)_n^\text{SMSV} = \sqrt{\langle \Delta^2 X \rangle_n^\text{SMSV}}\) and \((\delta X)_n^\text{SMSV} = \sqrt{\langle \Delta^2 X \rangle_n^\text{SMSV}}\). In the case of \(y_1 \neq y_1\), quadrature variances of even \(0\)-heralded state are given by \((\Delta^2 X_2)_0 = 1/4 - y_1/(1 + 2y_1)\) and \((\Delta^2 X_1)_0 = 1/4 + y_1/(1 + 2y_1)\). The quadrature uncertainty for even CV states takes on the value \((\delta X)_{2m} (y_1 = 0) = 0.5\) as \((n)_{2m} (y_1 = 0) = 0\), while the quadrature uncertainty of odd CV states in equation (3) changes \((\delta X)_{2m+1} (y_1 = 0) = 0.75\) \((\langle \Delta^2 X \rangle_{2m+1} = 0.75\) as \((n)_{2m+1} (y_1 = 0) = 1\) which is related to the aforementioned approximation of \(2m + 1\)-heralded states by a single photon in the case of \(y_1 = 0\).
We show dependencies of the quadrature squeezing \((\delta X)_{2m}\) for \(2m \rightarrow (\delta X)_{2m+1}\) heralded CV states (figures 5(a) and 6(a)) and \((\delta X)_{2m+1}\) for \(2m+1 \rightarrow (\delta X)_{2m+1}\) heralded CV states as well as quadrature squeezing \((\delta X)_{SMSV}\) (blue line) for the SMSV itself on the squeezing amplitude \(s\) of the original SMSV for the BS parameters \(t = 0.9\) and \(t = 0.99\) in figures 5(a), (b) and 6(a), (b), respectively. As can be seen from the figures, the generated CV states can exhibit the quadrature squeezing \((\delta X)_{a} < 0.5\). So, the quadrature squeezing of \(2m \rightarrow\) heralded CV states (figures 5(a) and 6(a)) is observed for any value of \(m\) in the entire range of the squeezing amplitude \(s\). There is also a range of values of \(s\) at which quadrature squeezing is observed for the odd CV states, i.e. \((\delta X)_{2m+1} < 0.5\). But here it is worth distinguishing simply squeezing of the quadrature component from comparing the generated quadrature component with one in original SMSV state expressed by the ratio \(RS_{2m} = (\delta X)_{2m}/(\delta X)_{SMSV} = (\delta X)_{2m}/\sqrt{\exp(-2s)/4} = 2\exp(s)(\delta X)_{2m}\) (figures 5(c) and 6(c)) and \(RS_{2m+1} = (\delta X)_{2m+1}/(\delta X)_{SMSV} = (\delta X)_{2m+1}/\sqrt{\exp(-2s)/4} = 2\exp(s)(\delta X)_{2m+1}\) (figures 5(d) and 6(d)) in dependency on the squeezing amplitude \(s\). The value \(RS_{2m} < 1\) or the same \((\delta X)_{2m} < (\delta X)_{SMSV}\) observed in a range \(s < 0.6\) means that a more squeezed CV state than the original SMSV state can be generated at the same value of the squeezing amplitude \(s\). Note that for all values of the squeezing amplitude \(s\), the condition \((\delta X)_{2m+1} > (\delta X)_{SMSV}\) is satisfied, which guarantees \(RS_{2m+1} > 1\).

Engineering of macroscopic nonclassical states that can have large quantum Fisher information for a particular observable is the cornerstone for ultra-precise estimation of an unknown parameter [8–10, 18–21], in our case, unknown phase \(\varphi\). In general, quantum Fisher information is difficult to compute under general scenario with arbitrary quantum channel \(\Phi_{\varphi}(\rho) = \rho(\varphi)\) that transforms an initial state \(\rho\) into...
\[ \rho(\varphi) \] already carrying information about an unknown parameter. Unitary encoding \( |\Psi_\varphi = U_\varphi|\Psi \rangle \) over pure state, where \( U_\varphi = \exp(-i\varphi G) \) is a unitary operator providing the parameter encoding and \( G \) is a Hermitian operator often named the generator of the unitary transformation allows for one to calculate Fisher quantum information in simpler way \( F(\Psi_n) = 4(\partial \Psi_\varphi / \partial \varphi - |\Psi_\varphi| \partial^2 \Psi_\varphi / \partial \varphi^2) = 4\Delta^2 G \), where symbolic notation \( \partial \Psi_\varphi \) denotes the derivative of the phase modulated state \( |\Psi_\varphi \rangle \) with respect to the unknown parameter \( \varphi \). Let us consider unitary encoding \( U_\varphi = \exp(-i\varphi n/2) \) which causes clockwise rotation by angle \( \varphi/2 \) on phase plane that induces phase change \( |\Psi_\varphi \rangle \) relative to the initial state \( |\Psi \rangle \). Then, the Fisher quantum information is related to the dispersion of the generator as \( F(\Psi_n) = 4G \) [18–21] which, as applied to CV states of definite parity in equations (2) and (3), gives

\[
F(\Psi_n) = \delta n_n = y_1 Z^{n+2}(y_1) + y_1 Z^{n+1}(y_1) Z^{(n)}(y_1)
- \left( y_1 Z^{(n+1)}(y_1) Z^{(n)}(y_1) \right)^2, \tag{14}
\]

where the subscript \( n \) can take both even \( n = 2m \) and odd \( n = 2m + 1 \) integers. It allows us to estimate the lower bound (QCR bound) [18–21] on the estimation of the unknown parameter \( \varphi \) as \( \Delta \varphi_{QCR,n} = 1/\sqrt{F(\Psi_n)} \) and compare it with QCR bound of the SMSV state \( \Delta \varphi_{QCR,SMSV} = 1/\sqrt{F(\Psi)} \) in wide range of the squeezing amplitude \( s \) of the initial SMSV state. As can be seen from the dependencies, the condition \( \Delta \varphi_{QCR,n} < \Delta \varphi_{QCR,SMSV} \) (\( g_n > 1 \)) is observed for the overwhelming number \( n \) in wide range of the squeezing amplitude \( s \) including case of small subtraction of photons (say up to \( s < 1 \)) from the initial SMSV state with a squeezing amplitude \( s < 1 \) achievable in practice. The QCR bound can reach the values of \( \Delta \varphi_{QCR,n} \approx 0.05 \) in the case of \( s < 1 \) and can decrease to \( \Delta \varphi_{QCR,n} \approx 0.01 \) in the case of the squeezing amplitude \( s = 3 \). As result, the sensitivity gain \( g_n \) can take values from \( g_n > 5 \) for \( s < 1 \). Note that here we do not talk about reaching the Heisenberg limit, since the average number of photons

Figure 6. (a)–(d) The same dependences of the quadrature components \((\delta X)_n\) and squeezing ratios \(RS_n\) as in figure 5 on the squeezing amplitude \( s \) of the original SMSV state with BS parameter \( t = 0.99 \). Blue lines in (a) and (c) describe SMSV quadrature squeezing.
squared exceeds the dispersion of the number of photons, i.e. 
\[ \langle n \rangle_n^2 > \delta n_n > \langle n \rangle_n \] for \( n > 0 \).

3. Methods

3.1. Derivation of the CV states in equations (2,3)

The analytical derivation of the \( 2m/2m + 1 \) heralded states in equations (2) and (5) starts with transformations of the creation operators imposed by the BS: \( a_1^+ \rightarrow \text{ta}_1^+ - \text{ra}_2^+ \), \( a_2^+ \rightarrow \text{ra}_1^+ + \text{ta}_2^+ \) that realize the following unitary transformation over input SMSV due to linearity of the BS operator

\[ BS_{12} (|\text{SMSV}\rangle_1|0\rangle_2) = \frac{1}{\sqrt{\cosh s}} \sum_{n=0}^{\infty} C_n |\Psi_{n1}|n\rangle_2, \quad (15) \]

where the amplitudes \( C_n \) are determined as

\[ C_n = (-1)^n \frac{(By)^2}{\sqrt{n!}} \left\{ \begin{array}{ll} \sqrt{Z_{2m}(y_1)}, & \text{if } n = 2m \\ \sqrt{Z_{2m+1}(y_1)}, & \text{if } n = 2m + 1 \end{array} \right. \quad (16) \]

The success probabilities to realize the \( 2m/2m + 1 \) heralded CV states are the following \( P_n = |C_n|^2 / \cosh s \).

3.2. Formula derivation for \( Z(y) \)

Mathematical inference of the function \( Z(y) \), which is the basis for the derivation of the normalized factors of the \( 2m/2m + 1 \) heralded states as well as for their statistical characteristics, can start with inner product of the squeezed even state \( |SS\rangle = \sum_{n=0}^{\infty} a_{2n} |2n\rangle \) with normalization condition \( \sum_{n=0}^{\infty} |a_{2n}|^2 = 1 \) written in the \( x \) (coordinate) − representation \( \Psi_{SS} (x) = xSS \) and the coherent state

\[ BS_{12} (|\text{SMSV}\rangle_1|0\rangle_2) = \frac{1}{\sqrt{\cosh s}} \sum_{n=0}^{\infty} C_n |\Psi_{n1}|n\rangle_2, \quad (15) \]
\[ |\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \left(\alpha^n/\sqrt{n!}\right) |n\rangle \] with real amplitude \( \alpha \) also presented in \( x^- \) representation \( \Psi_\alpha(x) = x^\alpha \).

\[ \psi_{SS} = \int_{-\infty}^{\infty} \rho_\alpha(x) \rho_{SS}(x) dx = \sqrt{2R/1+R^2} \exp\left(-R^2 \alpha^2/1+R^2\right), \] (17)

where

\[ \psi_{SS}(x) = \frac{\sqrt{R}}{\pi^{1/4}} \exp\left(-R^2 x^2/2\right), \] (18)

with the real parameter \( R \) responsible for state’s ‘squeezing’ properties and

\[ \psi_\alpha(x) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{(x-\sqrt{2}\alpha)^2}{2}\right). \] (19)

Expanding the right expression in a Taylor series in equation (17), one obtains

\[ a_{2n} = 2nSS = \sqrt{\frac{2R}{1+R^2}} \left(1 - R^2\right)^n \frac{\sqrt{2n!}}{n!} \] (20)

which can be rewritten in terms of new parameter \( y = (1 - R^2)/(2(1 + R^2)) \) as

\[ a_{2n} = \frac{y^n \sqrt{2n!}}{\sqrt{Z(y)n!}} \] (21)

where the normalization factor

\[ Z(y) = \frac{1 + R^2}{2R} = \frac{1}{\sqrt{1 - 4y^2}} \] (22)

is introduced. In particular, if we choose \( R = \exp(-s) \), then we have \( y = \tanh s/2 \) that gives SMSV amplitudes: \( a_{2n} = b_{2n}/\sqrt{\cosh s} \).

4. Conclusion

We have introduced into consideration a new family of the CV states of definite parity originating from the SMSV state by redirecting a number of indistinguishable photons into output and measurement modes of the BS and measuring them in measuring mode. The photon redistribution eliminates lower-photon states contribution, involving vacuum, which can have a larger weight in the initial SMSV distribution and shifts the initial SMSV photon distribution towards higher multiphoton states in generated CV states. All CV states of the family exclusively depends on only one parameter \( y \) which inevitably decreases by BS transmittance coefficient squared when they are generated. Despite the decrease of the initial parameter \( y \), which at least, determines the degree of squeezing in the quadrature component of the initial SMSV state, the generated CV states can have non-classical properties that make them attractive for optical quantum information processing.

So, the mean number of photons in the generated CV states can exceed tenfold the average number of photons in the initial SMSV, especially in the case of a practically used SMSV state with squeezing amplitude \( s < 1 \) even with a small number (<10) of extracted photons. The gain in brightness of the nonclassical CV states is feasible and can only grow with an increase in the number of photons subtracted. We have also showed the possibility of increasing the degree of squeezing of the quadrature component in comparison with the initial SMSV state. SMSV state with small squeezing amplitude could be transformed into the CV state of certain parity with larger squeezing. So, subtraction of two photons from input SMSV with squeezing degree of \( S_{SMSV} = 2.5 \, \text{dB} \) (\( S = -10 \log(\exp(-2s)) \)) allows for one to squeeze quadrature noise up to \( S_2 = (\delta X)_2 = 5 \, \text{dB} \). We have shown that QCR bound of the CV states of certain parity can take smaller values in comparison with the QCR bound of the SMSV state even despite its sub-Heisenberg sensitivity, when observable measuring the number of photons in the optical mode is chosen. The sensitivity gain can even reach values of several tens in the experimentally implemented case.

In general, the photon subtraction approach is feasible in practice, at least with a small number of photons \( n = 0 - 10 \) [36]. Its extension to a larger number of subtracted photons expands the number of new CV states of certain parity with their superior nonclassical properties compared to the original SMSV state. This superiority can be represented at almost any value of the squeezing amplitude of original SMSV, including a light with 15 dB (\( s \approx 1.7 \)) squeezing implemented in practice [25]. A decrease in the success probability in generating the CV state of certain parity with an increase of the number of extracted photons can be compensated for by using photonic or other CV (coherent, another SMSV state with different squeezing amplitude) states in the second auxiliary mode of BS, which is already a topic for a separate study. Another way to increase the success probability of success of quantum engineering of CV states of certain parity is to use the procedure of demultiplexing a multiphoton state by splitting it with a system of BSs standing one behind the other and multiplexing PNR outputs.

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References

[1] Heisenberg W 1927 Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik Z. Phys. 43 172
[2] Sen D 2014 The uncertainty relations in quantum mechanics Curr. Sci. 107 203
[3] Caves C M 1981 Quantum-mechanical noise in an interferometer Phys. Rev. D 23 1693
[4] Braunstein S L and Caves C M 1994 Statistical distance and the geometry of quantum states Phys. Rev. Lett. 72 3439
