Extension the Noether’s theorem to Lagrangian formulation with nonlocality

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Abstract

A Lagrangian formulation with nonlocality is investigated in this paper. The nonlocality of the Lagrangian is introduced by a new nonlocal argument that is defined as a nonlocal residual satisfying the zero mean condition. The nonlocal Euler-Lagrangian equation is derived from the Hamilton’s principle. The Noether’s theorem is extended to this Lagrangian formulation with nonlocality. With the help of the extended Noether’s theorem, the conservation laws relevant to energy, linear momentum, angular momentum and the Eshelby tensor are determined in the nonlocal elasticity associated with the mechanically based constitutive model. The results show that the conservation laws exist only in the form of the integral over the whole domain occupied by body. The localization of the conservation laws is discussed in detail. We demonstrate that not every conservation law corresponds to a local equilibrium equation. Only when the nonlocal residual of conservation current exists, can a conservation law be transformed into a local equilibrium equation by localization.

Key words: nonlocal Euler-Lagrange equation, Noether’s theorem, conservation law, nonlocal elasticity, mechanically based constitutive model

1 Introduction

Nonlocal elasticity has developed into an important branch of continuum mechanics since the first pioneer studies in the last 60–70s [1, 2, 3, 4, 5]. This theory and the extension of it have been applied to various topics in engineering [5, 6, 7, 8]. So far, constitutive models in the nonlocal elasticity can be categorized into three sorts: Eringen’s constitutive model [4, 7], peridynamic constitutive model [9, 10, 11] and mechanically based constitutive model [12, 13]. The postulation of the Eringen’ constitutive model consists in that the stress at a point not only depends on strain of the point but also on strains of all points within body. So stress is equal to the convolution integral of strain over the domain occupied by the body. Due to this fact, the mixed boundary value problem is ill-posed in the nonlocal elasticity associated with the Eringen’s constitutive model [14], except the attenuating kernel being equipped some ad hoc futures, for example, it is assumed to be the Green function of the differential operator \( \nabla^2 \).

Under the latter assumption, the integral-type constitutive model reduces to the so-called implicit gradient model. In the peridynamic constitutive model [9, 10, 11], ones introduce the internal long-range body force to represent the interactions within body, but forsake the conception of stress and strain. An integral operator is used to formulate the constitutive relation between the internal long-range body force and displacements. Since there are no assumptions made on the differentiability of displacements in the motion equation of the peridynamic nonlocal elasticity (peridynamics), this theory is suitable to study phenomena with discontinuities and fragmentation. In the peridynamics, no boundary conditions appear as there are no spatial derivatives. In order to solve the boundary value problems, some unconventional boundary conditions have been prescribed [11, 15]. Unlike the conventional boundary conditions, they are imposed on a boundary layer with non-zero volumetric measure, as opposed to a geometric boundary in the strict mathematical meanings. This inconsistency may give rise to some difficulties in the problems with complicated boundary conditions [12, 13].

The mechanically based constitutive model (MBCM) can be regarded as a fusion between the Eringen’s constitutive model and the peridynamic constitutive model [16]. This model retains the conception of stress and strain but introduces meanwhile the internal long-range body force characterizing the interactions between non-adjacent
particles. The internal long-range body force linearly or nonlinearily depends on the relative displacements between interacting particles within body; while the stress-strain relation is still characterized by the conventional constitutive equation, e.g., the Hooke’s law. Stress and the internal long-range body force are independent on each other. If a nonlocal elasticity is concerned with MBCM, we refer it as to the nonlocal elasticity associated with MBCM. In this theory, all types of boundary conditions are the same as that in the classical elasticity. From the view of this point, it is of advantage to adopt MBCM in nonlocal elasticity.

Paola et al [12, 13] firstly advanced the linear theory of the nonlocal elasticity associated with MBCM. They called it the mechanically based model of nonlocal elasticity. In this theory, the internal long-range body force at a particle is linearly dependent of the relative displacements between the particle and other particles within body. If taking a suitable nonlocal kernel, this internal long-range body force can be also obtained by linearization of the peridynamic constitutive model.

Recently, the Lagrangian formulation has been proposed for the mechanically based model of nonlocal elasticity [17]. The relevant energy-momentum tensor is given under a simplest case in which the nonlocal kernel degrades into a constant equal to the reciprocal of the volume of body. The energy-momentum tensor is of fundamental importance because it represents the configurational force on a defect (e.g., vacancy, inclusion, dislocation and crack etc) in solids; while in the absence of defects, it embodies a conservation law [18]. In order to determine, under a more general case, the energy-momentum tensor and the relevant conservation laws, it is necessary to extend the Noether’s theorem to the nonlocal elasticity associated with MBCM.

The study on the nonlocal form of the Noether’s theorem can be traced back to Edelen [19]. He [19, 20, 21] investigated the reformulation of the Noether theorem in the nonlocal theory based on a general theoretical framework. In this framework, the nonlocal argument specified by Edelen [19] is too general in form to take account of the constraint of physical laws. Later, Edelen [22] simplified the nonlocal argument as a linear integral operator on the field variable. Under this case, Vukobrat and Kuzmanovic [23] addressed the conservation laws in the nonlocal elasticity associated with the Eringen’s constitutive model. Recently, Lazar and Kirchner [24] discussed the energy-momentum tensor and relevant configurational forces in the nonlocal theory. They issued some interesting results on the interaction between dislocation and disclination.

At present, few results are known on the form of the Noether’s theorem and relevant conservation laws in the nonlocal elasticity associated with MBCM. So the objective of current work is to clarify these subjects. The paper is organized as follows: we start in Section 2 by using a nonlinear integral operator to define the nonlocal argument. Based on this nonlocal argument, we develop a Lagrange formulation with nonlocality. The Noether’s theorem is extended into the new Lagrangian formulation in Section 3. According to the extended the Noether’s theorem, in Section 4 we investigate the conservation laws in the nonlocal elasticity associated MBCM in which the internal long-range body force is nonlinearly dependent of the relative displacements between particles. In section 5, the localization of the conservation laws is discussed. The relevant nonlocal residuals are determined. We close this paper in Section 6 by making some concluding remarks.

**Notation:** A compact notation is used, with boldface letters being vectors or tensors. The index rules and summation convention are adopted. Latin indices have range 1, 2, 3; while Greek indices run from 0 to 3. Partial derivatives with respect to coordinates are represented as

\[ \frac{\partial}{\partial x^i} \]

Other symbols will be introduced in the text where they appear for the first time.

### 2 Lagrangian formulation with nonlocality

A continuum occupies the domain \( \Omega \) in the three-dimensional Euclidean space. Let every particle in the continuum be referred to by the orthogonal Cartesian coordinates \( x = \{x^1, x^2, x^3\} \) specifying its position in \( \Omega \), and let \( \varphi = \varphi(t, x) \) denote a field variable defined on \( \Omega \). Depending on circumstances, \( \varphi \) is a scalar, vector or tensor. Let \( \langle h|\varphi \rangle \) represent the nonlocal argument of \( \varphi \). We define it as

\[
\langle h|\varphi \rangle = \varphi(t, x) \int_{\Omega} h(x, y, |\varphi(t, x) - \varphi(t, y)|)dv(y) - \int_{\Omega} h(x, y, |\varphi(t, x) - \varphi(t, y)|)\varphi(t, y)dv(y),
\]  

(1)
where \( h(x, y, |\varphi(t, x) - \varphi(t, y)|) \) is called the nonlocal kernel. Let \( r = \varphi(t, x) - \varphi(t, y) \). Eq. (1) can be abbreviated to

\[
\langle h|\varphi \rangle = \varphi(t, x) \int_{\Omega} h(x, y, |r|) \, dv(y) - \int_{\Omega} h(x, y, |r|) \varphi(t, y) \, dv(y)
\]

\[
= \int_{\Omega} h(x, y, |r|) [\varphi(t, x) - \varphi(t, y)] \, dv(y)
\]

\[
= \int_{\Omega} h(x, y, |r|) \, dv(y). \tag{2}
\]

Different from the case recently reported by Huang [17], the nonlocal kernel depends not only on \( x \) and \( y \) but also on the field variables. Therefore, \( \langle h|\varphi \rangle \) is a nonlinear integral operator with respect to \( \varphi \). We enforce the nonlocal kernel to fulfill the symmetry: \( h(x, y, |r|) = h(y, x, |r|) \). Under this symmetry, it is easy to verify that \( \langle h|\varphi \rangle \) satisfies the zero mean condition:

\[
\int_{\Omega} \langle h|\varphi \rangle \, dv(x) = 0. \tag{3}
\]

Due to Eq.(3), the nonlocal argument determined by Eq. (1) is in essence different from the definition given by Edelen [19, 22]. The latter fails to satisfy the zero mean condition.

Let \( L = L(t, x, \varphi, \varphi, \varphi, \varphi, \langle h|\varphi \rangle) \) \((k = 1, 2, 3)\) denote the Lagrangian. So the action functional of \( \varphi \) can be written as

\[
A[\varphi] = \int_{t_0}^{t_1} \int_{\Omega} L(t, x, \varphi, \varphi, \varphi, \varphi, \langle h|\varphi \rangle) \, dv(x) \, dt. \tag{4}
\]

Suppose \( \varphi(t, x), h(x, y, |r|) \) and \( L \) are suitably smooth functions. In order to determine the variation \( \delta A[\varphi] \), we firstly prove the two identities. Using Eq. (2), we have

\[
\delta \langle h|\varphi \rangle = \int_{\Omega} \left\{ [\delta h(x, y, |r|)]|r| + h(x, y, |r|) \delta r \right\} \, dv(y) = \int_{\Omega} (h + |r| \frac{\partial h}{\partial |r|}) \delta r \, dv(y)
\]

\[
= \delta \varphi(t, x) \int_{\Omega} (h + |r| \frac{\partial h}{\partial |r|}) \, dv(y) - \int_{\Omega} (h + |r| \frac{\partial h}{\partial |r|}) \delta \varphi(t, y) \, dv(y)
\]

\[
= \langle g|\delta \varphi \rangle. \tag{5}
\]

\[
\int_{\Omega} \psi(t, x) \langle h|\varphi \rangle \, dv(x) = \int_{\Omega} \int_{\Omega} \psi(t, x) h(x, y, |r|) [\varphi(t, x) - \varphi(t, y)] \, dv(y) \, dv(x)
\]

\[
= \int_{\Omega} \int_{\Omega} \varphi(t, x) h(x, y, |r|) [\psi(t, x) - \psi(t, y)] \, dv(y) \, dv(x)
\]

\[
= \int_{\Omega} \int_{\Omega} \varphi(t, x) \langle h|\psi \rangle \, dv(x). \tag{6}
\]

Eq. (6) is valid for any continuous function \( h, \psi \) and \( \varphi \). For example, we may use \( g \) in Eq. (5) to replace \( h \), where \( g \) reads

\[
g = h + |r| \frac{\partial h}{\partial |r|} = h(x, y, |r|) + |r| \frac{\partial h(x, y, |r|)}{\partial |r|}. \tag{7}
\]

Clearly, \( g \) is symmetric with respect to \( x \) and \( y \). By means of Eq. (5) and Eq.(6), we can calculate the variation of \( A[\varphi] \):

\[
\delta A[\varphi] = \int_{t_0}^{t_1} \int_{\Omega} \left( \frac{\partial L}{\partial \varphi} \delta \varphi + \frac{\partial L}{\partial \varphi} \delta \varphi + \frac{\partial L}{\partial \varphi} \delta \varphi + \frac{\partial L}{\partial \varphi} \delta \varphi \right) \, dv(x) \, dt
\]

\[
= \int_{t_0}^{t_1} \int_{\Omega} \left( \frac{\partial L}{\partial \varphi} \delta \varphi - \frac{\partial L}{\partial \varphi} \delta \varphi \right) \, dv(x) + \int_{t_0}^{t_1} \int_{\partial \Omega} \frac{\partial L}{\partial \varphi} n_k \delta \varphi \, ds(x) \, dt
\]

\[
+ \int_{t_0}^{t_1} \int_{\Omega} \delta \frac{\partial L}{\partial \varphi} \, dv(x) + \int_{t_0}^{t_1} \int_{\partial \Omega} \frac{\partial L}{\partial \varphi} n_k \delta \varphi \, ds(x) \, dt, \tag{8}
\]

where \( \partial \Omega \) is the boundary surface of \( \Omega \) and \( n_k \) denotes the unit normal vector on \( \partial \Omega \). In Eq. (8), a shortened form similar to Eq.(1) is used.

\[
\langle g|\delta \frac{\partial L}{\partial |\varphi|} \rangle = \frac{\partial L}{\partial |\varphi|} \int_{\Omega} g(x, y, |r|) \, dv(y) - \int_{\Omega} g(x, y, |r|) \frac{\partial L}{\partial |\varphi|} \, dv(y). \tag{9}
\]

\[\text{If necessary, the nonlocal arguments } \langle h|\varphi \rangle \text{ and } \langle h|\varphi \rangle \text{ may be conveniently inserted into } L. \text{ But in this case, the boundary conditions will become too complicated to solve in mathematics.}\]
Let $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2, \partial \Omega_1 \cap \partial \Omega_2 = \emptyset$. On $\partial \Omega_1$, $\varphi$ takes a given value $\bar{\varphi}$. Then, the boundary condition on $\partial \Omega_1$ reads

$$\varphi|_{\partial \Omega_1} = \bar{\varphi}. \quad (10)$$

At the initial and terminal time, we have

$$\varphi|_{t_0} = \bar{\varphi}_0, \quad \varphi|_{t_1} = \bar{\varphi}_1. \quad (11)$$

Due to Eq.(10), $\delta \varphi = 0$ on $\partial \Omega_1$. Similarly, $\delta \varphi = 0$ at the initial and terminal time because of Eq.(11). Thus, Eq.(8) reduces to

$$\delta A[\varphi] = \int_{t_0}^{t_1} \int_{\Omega} \frac{\partial L}{\partial \dot{\varphi}} \, dt \, (\varphi) - \left( \frac{\partial L}{\partial \dot{\varphi}} \right)_k \dot{\varphi} + (g, \frac{\partial L}{\partial \varphi}) \partial \varphi \, dv + \int_{t_0}^{t_1} \int_{\partial \Omega} \frac{\partial L}{\partial \varphi} \, n \delta \varphi \, dx \, dr. \quad (12)$$

In terms of the Hamilton’s principle, we have $\delta A[\varphi] = 0$. So the fundamental lemma of variation yields the below results:

Euler-Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) + \left( \frac{\partial L}{\partial \varphi} \right)_k \dot{\varphi} = \langle g, \frac{\partial L}{\partial \varphi} \rangle. \quad (13)$$

Natural boundary condition:

$$\frac{\partial L}{\partial \varphi} \bigg|_{\partial \Omega_2} = 0. \quad (14)$$

Eq.(13) is also referred to as the nonlocal Euler-Lagrange equation. The right-side term of Eq.(13) is the nonlocal term, called the nonlocal traction. If $h$ is independent of $r$, we have $g = h$, and then Eq.(13) will reduce to the case in [17].

Using the symmetry of $g(x, y, |r|)$, we easily verify the equality below:

$$\int_{\Omega} \frac{\partial L(x)}{\partial \langle h | \varphi \rangle} \int_{\Omega} g(x, y, |r|) \, dv \, dy = \int_{\Omega} \int_{\Omega} g(x, y, |r|) \frac{\partial L(y)}{\partial \langle h | \varphi \rangle} \, dv \, dy \int_{\Omega} \frac{\partial L(x)}{\partial \langle h | \varphi \rangle} \, dv, \quad (15)$$

by interchanging $x$ and $y$. As thus, the integral of Eq.(9) over $\Omega$ leads to

$$\int_{\Omega} \langle g, \frac{\partial L}{\partial \langle h | \varphi \rangle} \rangle \, dv = 0, \quad (16)$$

which shows that the nonlocal traction is a nonlocal residual automatically satisfying the zero mean condition. Due to Eq.(16), the integral of Eq.(13) over $\Omega$ has the same expression as the integral of the Euler-Lagrange equation not concerned with nonlocality.

In physics, if Eq. (13) represents the equation of motion, the nonlocal traction may be interpreted as an internal long-range body force applied on a particle at $x$ by other particles within body. Every particle is subjected to such a force. In terms of the action and reaction law, the sum of all such forces must be zero. Therefore, Eq.(16) is just an embodiment of the action and reaction law.

### 3 Extension of the Noether’s theorem to the Lagrangian formulation with nonlocality

For convenience, time can be treated as a coordinate so as to form a four-dimensional position vector $\hat{x}$ with coordinates $\hat{x} = \{x^\beta \} = \{x^0, x^1, x^2, x^3 \}$ defined on $\hat{\Omega}$, where $x^0 = t$ and $\hat{\Omega} = [t^0, t^1] \cup \Omega$. Thus, the Lagrangian function can be abbreviated as $L(x, \varphi, \varphi_\gamma, \langle \varphi \rangle), \gamma = 0, 1, 2, 3$, and the nonlocal Euler-Lagrange equation (see Eq.(13)) can be rewritten as

$$\left( \frac{\partial L}{\partial \varphi_{\gamma}} \right)_{\gamma} - \frac{\partial L}{\partial \varphi} = \langle g, \frac{\partial L}{\partial \langle h | \varphi \rangle} \rangle. \quad (17)$$

Consider infinitesimal transformations of group

$$\hat{x} \mapsto \hat{x}' = \hat{x} + \delta \hat{x}. \quad (18)$$

$$\varphi(\hat{x}) \mapsto \varphi'(\hat{x}) = \varphi(\hat{x}) + \Delta \varphi(\hat{x}). \quad (19)$$

$$\hat{y} \mapsto \hat{y}' = \hat{y} + \delta \hat{y}. \quad (20)$$

$$\varphi(\hat{y}) \mapsto \varphi'(\hat{y}) = \varphi(\hat{y}) + \Delta \varphi(\hat{y}). \quad (21)$$
Here, \( \Delta \phi \) is different from the variation \( \delta \phi \). The latter is defined as
\[
\delta \phi = \phi'(\hat{x}) - \phi(\hat{x}).
\] (22)

Therefore, between \( \Delta \phi \) and \( \delta \phi \), there exists the relation below:
\[
\Delta \phi = \delta \phi + \delta \phi \gamma. \tag{23}
\]

The action functional \( A[\phi] \) is said to be symmetry if it is form-invariant with respect to the infinitesimal transformations (18) – (21), i.e.,
\[
\int_{\Omega'} L[\hat{x}', \phi'(\hat{x}'), \phi_x'(\hat{x}')] \, dv(\hat{x}') = \int_{\Omega} L[\hat{x}, \phi(\hat{x}), \phi_x(\hat{x}), \langle h|\phi' \rangle] \, dv(\hat{x}). \tag{24}
\]

In the left integral, \( \hat{x}' \) now represents merely a dummy variable of integration and can therefore be relabeled \( \hat{x} \). But there remains a change in the domain of integration, so Eq. (24) becomes
\[
\int_{\Omega'} L[\hat{x}, \phi'(\hat{x}), \phi_x'(\hat{x}), \langle h|\phi' \rangle] \, dv(\hat{x}) = \int_{\Omega} L[\hat{x}, \phi(\hat{x}), \phi_x(\hat{x}), \langle h|\phi \rangle] \, dv(\hat{x}). \tag{25}
\]

It should be noted that \( \langle h|\phi' \rangle \) in the left-hand term of Eq. (25) is an integral defined on \( \Omega' \), but the integral domain of \( \langle h|\phi \rangle \) in the right-hand term is \( \Omega \). For \( \langle h|\phi' \rangle \), we have
\[
\langle h|\phi' \rangle = \phi'(\hat{x}) \int_{\Omega'} h(x, y, |r'|) \, dv(y) - \int_{\Omega'} h(x, y, |r'|) \phi'(\hat{y}) \, dv(y). \tag{26}
\]

By the transport theorem [25], Eq. (26) leads to
\[
\langle h|\phi' \rangle = \phi'(\hat{x}) \int_{\Omega} h(x, y, |r|) \, dv(y) + \int_{\Omega} \frac{\partial h(x, y, |r|)}{\partial y_k} \, \delta y_k \, dv(y) - \int_{\Omega} h(x, y, |r|) \phi'(\hat{y}) \, dv(y) - \int_{\Omega} \frac{\partial h(x, y, |r|)}{\partial y_k} \phi'(\hat{y}) \, \delta y_k \, dv(y). \tag{27}
\]

Expanding \( h(x, y, |r|) \) at \( r \) to the first-order term yields
\[
h(x, y, |r'|) = h(x, y, |r|) + \frac{\partial h}{\partial r} \frac{r}{|r|} \, dr
\]
\[
= h(x, y, |r|) + \frac{\partial h}{\partial r} \frac{\phi(\hat{x}) - \phi(\hat{y})}{|r|} [\delta \phi(\hat{x}) - \delta \phi(\hat{y})]. \tag{28}
\]

Inserting Eq. (22) and (28) in (27) and omitting the higher-order terms, we have
\[
\langle h|\phi' \rangle = \langle h|\phi \rangle + \langle g|\delta \phi \rangle + \int_{\Omega} \frac{\partial h(x, y, |r|) r \delta y_k}{\partial y_k} \, dv(y). \tag{29}
\]

After Eq. (22) and (29) are substituted into \( L[\hat{x}, \phi'(\hat{x}), \phi_x'(\hat{x}), \langle h|\phi' \rangle] \), it becomes
\[
L[\hat{x}, \phi'(\hat{x}), \phi_x'(\hat{x}), \langle h|\phi' \rangle] = L[\hat{x}, \phi + \delta \phi, \phi_x + (\delta \phi)_x, \langle h|\phi \rangle + \langle g|\delta \phi \rangle + \int_{\Omega} \frac{\partial h(x, y, |r|) r \delta y_k}{\partial y_k} \, dv(y)]
\]
\[
= L(\hat{x}, \phi, \phi_x, \langle h|\phi \rangle) + \delta L, \tag{30}
\]

where
\[
\delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \phi_x} (\delta \phi)_x + \frac{\partial L}{\partial \langle h|\phi \rangle} \langle g|\delta \phi \rangle + \int_{\Omega} \frac{\partial h(x, y, |r|) r \delta y_k}{\partial y_k} \, dv(y). \tag{31}
\]

By Eq. (30), we have
\[
\int_{\Omega'} L[\hat{x}, \phi'(\hat{x}), \phi_x'(\hat{x}), \langle h|\phi' \rangle] \, dv(\hat{x}) = \int_{\Omega} \{ L[\hat{x}, \phi(\hat{x}), \phi_x(\hat{x}), \langle h|\phi \rangle] + \delta L \} \, dv(\hat{x}). \tag{32}
\]

Applying the transport theorem to the right-hand of Eq. (32) and omitting the higher-order terms, we have
\[
\int_{\Omega'} L[\hat{x}, \phi'(\hat{x}), \phi_x'(\hat{x}), \langle h|\phi' \rangle] \, dv(\hat{x}) = \int_{\Omega} \{ L[\hat{x}, \phi(\hat{x}), \phi_x(\hat{x}), \langle h|\phi \rangle] + \delta L \} \, dv(\hat{x}) + \int_{\Omega} \int_{\Omega'} \delta L \, dv(\hat{x}) + \int_{\Omega} \int_{\Omega'} (\delta L x^\gamma) \, dv(\hat{x}). \tag{33}
\]
Substituting Eq. (33) into (25) leads to
\[
\int_{\hat{\Omega}} \delta L d\varepsilon + \int_{\hat{\Omega}} (L\delta x^\gamma)_{,\gamma} d\varepsilon = 0. \tag{34}
\]
Using Eq. (31), we have
\[
\int_{\hat{\Omega}} \delta L d\varepsilon(\hat{x}) = \int_{\Omega_0} dr \int_{\Omega} \delta L d\varepsilon(x) = \int_{\Omega_0} dr \int_{\Omega} \left(\frac{\partial L}{\partial \Phi_\gamma}\delta \Phi + \frac{\partial L}{\partial \Phi_\gamma}(\delta \Phi)_\gamma + \frac{\partial L}{\partial \Phi_\gamma}[(\Phi_{\gamma}]_{,\gamma} + \int_{\hat{\Omega}} \frac{\partial [h(x, y, |r|)r\delta x^\gamma]}{\partial \Phi_\gamma} d\varepsilon(y)]\right) d\varepsilon(\hat{x}).
\]
By Eq. (17), Eq. (35) reduces to
\[
\int_{\hat{\Omega}} \delta L d\varepsilon(\hat{x}) = \int_{\hat{\Omega}} \left(\frac{\partial L}{\partial \Phi_\gamma}\delta \Phi + \frac{\partial L}{\partial \Phi_\gamma}(\delta \Phi)_\gamma + \int_{\hat{\Omega}} \frac{\partial [h(x, y, |r|)r\delta x^\gamma]}{\partial \Phi_\gamma} d\varepsilon(y)]\right) d\varepsilon(\hat{x}). \tag{36}
\]
By Eq. (17), Eq. (35) reduces to
\[
\int_{\hat{\Omega}} \left(\frac{\partial L}{\partial \Phi_\gamma} + L\delta x^\gamma - \delta x^\gamma \int r h(x, y, |r|) \frac{\partial L}{\partial \Phi_\gamma} d\varepsilon(y)]\right) d\varepsilon(\hat{x}) = 0. \tag{37}
\]
If Eq. (18) and (19) belong to a finite Lie group, \(\delta x^\gamma\) and \(\Delta \Phi\) can be represented as [26]
\[\delta x^\gamma = \epsilon^\alpha x^\gamma_\alpha, \quad \Delta \Phi = \epsilon^\alpha \Phi_\alpha, \tag{38}\]
where \(\epsilon^\alpha\) is an infinitesimal parameter independent of the space-time coordinates. \(X^\gamma_\alpha\) and \(\Phi_\alpha\) are the generators of Lie group. By using Eq. (38) and (23), Eq. (37) becomes
\[
\int_{\hat{\Omega}} \left(\frac{\partial L}{\partial \Phi_\gamma} + [(\Phi_{\gamma}]_{,\gamma} + \int_{\hat{\Omega}} \frac{\partial [h(x, y, |r|)r\delta x^\gamma]}{\partial \Phi_\gamma} d\varepsilon(y)]\right) d\varepsilon(\hat{x}) = 0. \tag{39}
\]
Thus, the Noether theorem is extended to the nonlocal Lagrangian formulation with nonlocality. Let
\[
J^\alpha_{\alpha} = \frac{\partial L}{\partial \Phi_\gamma}\Phi_\alpha + \int_{\hat{\Omega}} \left(\frac{\partial L}{\partial \Phi_\gamma} + [(\Phi_{\gamma}]_{,\gamma} + \int_{\hat{\Omega}} \frac{\partial [h(x, y, |r|)r\delta x^\gamma]}{\partial \Phi_\gamma} d\varepsilon(y)]\right) d\varepsilon(\hat{x}) \tag{40}
\]
where \(J^\alpha_{\alpha}\) is called the conservation current (or Noether current). Thus, Eq. (39) can be abbreviated as
\[
\int_{\hat{\Omega}} J^\alpha_{\alpha,\gamma} d\varepsilon(\hat{x}) = 0, \quad \text{or} \quad \int_{\hat{\Omega}} J^\alpha_{\alpha} n d\alpha(\hat{x}) = 0, \tag{41}
\]
where \(n_\gamma\) is an unit normal vector on \(\partial \hat{\Omega}\). Eq.(41) shows that total conservation current \(J^\alpha_{\alpha}\) on \(\hat{\Omega}\) is conserved. However, Eq. (41) will cease to be valid if its integral domain \(\hat{\Omega}\) is replaced by a subdomain of \(\hat{\Omega}\). This is because the nonlocal argument is defined on the whole domain \(\Omega\), and it can not be altered in the deductive process from Eq.(24) to (41). In other words, if we use any \(\tilde{v}(=\tilde{v}_0, \tilde{t}_1) \cup \nu, \nu \subset \Omega\) to replace \(\hat{\Omega}\) in Eq.(24) but keep the integral domain of the nonlocal argument fixed, it will be impossible to derive Eq. (41) from (24) due to needing to change the order of integrals through the interchange of \(x\) and \(y\). Therefore, the conservation current is conserved on \(\hat{\Omega}\) as a whole, and yet when exactly the same statement is made for a subdomain of \(\hat{\Omega}\) it is no longer valid. As a result, we can merely derive \(J^\alpha_{\alpha,\gamma} = R_\alpha(x)\) from the localization of Eq. (41), rather than \(J^\alpha_{\alpha,\gamma} = 0. R_\alpha(x)\) is called the nonlocal residual of the conservation current. It should satisfy the zero mean condition so that the integral of \(J^\alpha_{\alpha,\gamma} = R_\alpha(x)\) over \(\hat{\Omega}\) can return to Eq. (41). Huang put forward a representation of the nonlocal residual automatically satisfying the zero mean condition [27].

For the convenience, in subsequent sections we call the Eq. (41) the conservation law, while \(J^\alpha_{\alpha,\gamma} = R_\alpha(x)\) is referred to as the local equilibrium equation.
4 Conservation laws in nonlocal elasticity

Consider a linear, homogenous, nonlocal elastic body free of external body forces. For this body, the Lagrangian function is assumed to take the following form:

\[ L = \frac{1}{2} \rho \ddot{u}_i - \frac{1}{2} C_{ijkl} u_i u_j - \frac{1}{2} \langle h | u_i \rangle u_i, \]  

(42)

where \( u_i \) denotes the elastic displacement field. \( \rho \) and \( C_{ijkl} \) are the mass density and elastic tensor. The term \( \langle h | u_i \rangle u_i / 2 \) represents the internal long-range action potential.

Using \( u_i \) instead of \( \varphi \) in (13), and then inserting Eq. (42) in (13), we have

\[ \rho \ddot{u}_i + \langle \kappa | u_i \rangle = C_{ijkl} u_{ik, lj}, \]  

(43)

where the \( \kappa \) is represented as

\[ \kappa = h(x, y, r_i) + \frac{1}{2} |r_i| \frac{\partial h(x, y, r_i)}{\partial |r_i|}, \quad r_i = u_i(x) - u_i(y). \]  

(44)

It is easy to see that Eq. (43) characterizes the motion equation of a nonlocal elasticity associated with MBCM. This theory is an extension of the mechanically based model of nonlocal elasticity \([12, 13, 17]\). If \( h \) is supposed to be independent of \( r_i \), then \( \kappa = h \). Thus, Eq. (43) will reduce to the motion equation in the mechanically based model of nonlocal elasticity, see \([12, 13, 17]\).

In order to find the conservation laws corresponding to Eq. (43), we firstly calculate the conservation current according to Eq. (42). Let \( \varphi = u_k \). Then Eq. (40) becomes

\[ J_{\alpha \gamma} = \frac{\partial L}{\partial \dot{u}_{k \gamma}} \Phi_{k \alpha} + \{ L - \int_{\Omega} r_i h(x, y, r_i) \frac{\partial L}{\partial \langle h | u_i \rangle} \delta_{\mu \alpha} - \frac{\partial L}{\partial \dot{u}_{k \gamma}} u_{k \mu} \} X_{\mu \alpha}. \]  

(45)

It should be remembered that, in the convention of this paper, Latin indices take 1, 2 and 3; while Greek indices run from 0 to 3. Since \( u_0 \) is null, \( \Phi_{k0} = 0 \). Using Eq. (42) and (45), and noticing \( u_{k, 0} = \dot{u}_k = du_k / dt \), we have

\[ \begin{align*}
J_{\alpha \gamma} &= \frac{d}{dt} \rho \ddot{u}_k \Phi_{k \alpha} + \{ L + \frac{1}{2} \int_{\Omega} h(x, y, r_i) r_i u_k(t, y) dv(y) \} X_{\alpha 0} - \rho \dot{u}_k (\dot{u}_k X_{0 \alpha} + u_k X_{\alpha 0}) \\
&- \{ C_{kij} u_{i, j} \Phi_{k \alpha} - \{ L + \frac{1}{2} \int_{\Omega} h(x, y, r_i) r_i u_k(t, y) dv(y) \} X_{\alpha \alpha} - C_{kij} u_{i, j} (\dot{u}_k X_{0 \alpha} + u_k X_{\alpha 0}) \}. \end{align*} \]  

(46)

By Hooke’s law \( \sigma_{ks} = C_{kij} u_{i, j} \), Eq. (46) is rewritten as

\[ \begin{align*}
J_{\alpha \gamma} &= \frac{d}{dt} \rho \ddot{u}_k \Phi_{k \alpha} + \{ L + \frac{1}{2} \int_{\Omega} h(x, y, r_i) r_i u_k(t, y) dv(y) \} X_{\alpha 0} - \rho \dot{u}_k (\dot{u}_k X_{0 \alpha} + u_k X_{\alpha 0}) \\
&- \{ \sigma_{ks} u_{k \alpha} - \{ L + \frac{1}{2} \int_{\Omega} h(x, y, r_i) r_i u_k(t, y) dv(y) \} X_{\alpha \alpha} - \sigma_{ks} (\dot{u}_k X_{0 \alpha} + u_k X_{\alpha 0}) \}. \end{align*} \]  

(47)

In the classical elasticity, Fletcher proved the completeness of conservation laws under the infinitesimal transformations below \([28]\):

\[ \begin{align*}
t &\mapsto t' = t + \varepsilon (vt + c_0) , \\
x_i &\mapsto x_i' = x_i + \varepsilon (v x_i + e_{ijk} x_j y_k + c_i) , \\
u_i &\mapsto u_i' = u_i + \varepsilon (-v u_i + e_{ijk} u_j y_k + e_{ijk} c_i + d_i) ,
\end{align*} \]  

(48)

where \( \varepsilon \) is an infinitesimal parameter. \( v, a_i, b_i, c_i \) and \( d_i \) are arbitrary real constants. In terms of Eq. (48), we turn now to investigating the concrete forms of the conservation laws under four typical transformations.

### 4.1 Case 1: \( t' = t + \varepsilon, \quad x'_k = x_k, \quad u'_k = u_k \)

The transformations above are equivalent to taking \( X_{00} = 1, X_{\alpha 0} = 0, \Phi_{k \alpha} = 0 \) in Eq. (38). Under this case, Eq. (47) reduces to

\[ J_{\gamma \gamma} = \frac{d}{dt} \left[ \frac{1}{2} \int_{\Omega} h(x, y, |r_i|) r_i u_k(t, y) dv(y) - \rho \dot{u}_k \dot{u}_k \right] + \{ \sigma_{ks} \dot{u}_k \}. \]  

(49)

Substituting Eq. (49) into (41) yields

\[ \int_{\Omega} \left[ \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} h(x, y, |r_i|) r_i u_k(t, y) dv(y) - \rho \dot{u}_k \dot{u}_k \right) + \{ \sigma_{ks} \dot{u}_k \} \right] dv(x) = 0. \]  

(50)
In terms of Eq. (3), it is easy to verify that
\[
\int_{\Omega} \int_{\Omega} h(x, y, |r_1|) r_k u_k(t, y) dv(y) dv(x) = \int_{\Omega} u_k(t, y) \int_{\Omega} h(x, y, |r_1|) r_k dv(x) dv(y)
\]
\[
= \int_{\Omega} u_k(t, y) \int_{\Omega} (|h|u_k) dv(x) dv(y) = 0.
\] (51)

Substituting Eq. (51) into (50) yields
\[
\int_{\Omega} \frac{d}{dt} (L - \rho \ddot{u}_k + (\sigma_{k\alpha} u_k)_\alpha) dv(x) = 0.
\] (52)

By Eq. (54) and the divergence theorem, Eq. (52) is rewritten as
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho \ddot{u}_k + \frac{1}{2} C_{ijkl} u_i u_j u_{k,l} + \frac{1}{2} (|h|u_k) dv(x) \right) - \int_{\partial \Omega} \sigma_{k\alpha} n_\alpha \dot{a}(x) = 0.
\] (53)

Eq. (52) corresponds to the conservation of energy that shows total energy on \( \Omega \), including the sum of kinetic energy, elastic potential energy and the internal long-range action potential energy, is equal to work done by external traction. However, because the internal long-range interactions give rise to energy transferring among different parts of body, the same form as Eq. (52) or (53) is no longer valid for the subdomain of \( \Omega \).—This is just a intrinsic character solely processed by the nonlocal theory.

**4.2 Case 2:** \( t' = t \), \( x'_k = x_k \), \( u'_k = u_k + \varepsilon_k \), \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \)

The transformations above represent the rigid body translations. To satisfy these transformations, we set \( X_{\alpha\beta} = 0, \Phi_{i\alpha} = \delta_{i\alpha} \). Thus, Eq. (47) reduces to
\[
J_{k\gamma,\gamma} = \frac{d}{dt} (\rho \ddot{u}_k) - (\sigma_{k\gamma})_\gamma.
\] (54)

Substituting Eq. (54) into (41) leads to
\[
\frac{d}{dt} \int_{\Omega} \rho \ddot{u}_k dv(x) - \int_{\partial \Omega} \sigma_{k\gamma} n_\gamma dv(x) = 0,
\] (55)

which is the integral representation of the conservation law of linear momentum. Although Eq. (55) has the same form as the relevant conservation law in the classical elasticity, it cannot be transformed into the same differential equation due to the nonlocality.

**4.3 Case 3:** \( t' = t \), \( x'_k = x_k \), \( u'_k = u_k + \varepsilon_{ij} x_i \varepsilon_j \), \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \)

The transformations above characterize the rigid body rotations. Under this case, we have \( X_{\alpha\beta} = 0, \Phi_{i\alpha} = \varepsilon_{ij} x_i \). So Eq. (47) leads to
\[
J_{k\gamma,\gamma} = \frac{d}{dt} (\rho \ddot{u}_k - (\varepsilon_{ij} x_i \sigma_{j\gamma})_\gamma).
\] (56)

Substituting Eq. (56) into (41) gives
\[
\frac{d}{dt} \int_{\Omega} \rho \varepsilon_{ij} x_i \dot{u}_j dv(x) - \int_{\partial \Omega} \varepsilon_{ij} x_i \sigma_{j\gamma} n_\gamma dv(x) = 0.
\] (57)

Therefore, the conservation of total angular momentum on \( \Omega \) is associated with the invariance of the action functional under the rigid body rotation.

**4.4 Case 4:** \( t' = t \), \( x'_k = x_k + \varepsilon_k \), \( u'_k = u_k \), \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \)

The transformations above correspond to the coordinate translations that are identical to setting \( X_{ij} = \delta_{ij}, X_{0\alpha} = 0, X_{\alpha i} = 0, \Phi_{x\alpha} = 0 \). Thus, Eq. (47) reduces to
\[
J_{k\gamma,\gamma} = - \frac{d}{dt} (\rho \ddot{u}_k + L + \frac{1}{2} \int_{\Omega} h(x, y, |r_1|) r_k u_k(t, y) dv(y)) \delta_{jk} + (\sigma_{ij} u_{i,k})_j.
\] (58)

Substituting Eq. (58) into (41) and using (51), we have
\[
\frac{d}{dt} \int_{\Omega} \rho \ddot{u}_k u_k dv(x) - \int_{\partial \Omega} (L n_k + \sigma_{ij} u_{i,k} n_j) dv(x) = 0.
\] (59)
Let $T_{jk} = L \delta_{jk} + \sigma_{j \mu} u_{\mu,k}$. $T_{jk}$ is the so-called Eshelby tensor. In the classical elasticity, the Eshelby tensor is independent of the integral path. However, the same conclusion is no longer available in nonlocal elasticity. This is because Eq. (59) holds only on $\Omega$ as a whole. For any $\nu \subset \Omega$, it ceases to be valid. Eq. (59) represents the conservation laws relevant to the Eshelby tensor.

In the formal four-dimensional space of $\hat{\Omega}$, Eq. (53) and (59) can be combined in a unified expression with the help of the energy-momentum tensor. This expression has been given in [17].

## 5 Localization of conservation laws and nonlocal residuals

In previous sections, we have pointed out that, if the nonlocality is concerned, the conservation laws are valid only on whole domain occupied by body but fail on the local domain of the body. Therefore, the localization of a conservation law is bound to result in the nonlocal residual of conservation current appearing in the local equilibrium equation, in which, the divergence of the conservation current is equal to the nonlocal residual rather than zero. Thus, by localization we can give the local residual of the Eshelby tensor. All nonlocal residuals should be satisfy the zero-mean codition, i.e.,

$$ \int_{\Omega} (\hat{E}, \hat{P}_k, \hat{M}_k, \hat{J}_k) dv(x) = 0, $$

so that the integrals of Eq. (60), (61), (62) and (63) over $\Omega$ can return to Eq. (52), (55), (57) and (59), respectively. Now, we turn to determining the nonlocal residuals and the existence of Eq. (60) – (63).

### 5.1 Nonlocal residual of energy

The conservation laws are firmly associated with the characters of the Lagrangian $L(t, x, u_i, \dot{u}_i, u_{i,k}, \langle h | u_i \rangle)$. In fact, we have noticed that if $L$ does not depend explicitly upon $t$, Eq. (52) can be determined by calculating total derivation of $L$ with respect to $t$ in the integral sign of the action functional $A[\phi]$. Thus, if directly evaluating the derivation of $L$ with respect to $t$, we have

$$ \frac{dL}{dt} = \frac{\partial L}{\partial u_i} \dot{u}_i + \frac{\partial L}{\partial \dot{u}_i} \ddot{u}_i + \frac{\partial L}{\partial u_{i,k}} \dot{u}_{i,k} + \frac{\partial L}{\partial \langle h | u_i \rangle} \frac{d}{dt} \langle h | u_i \rangle $$

$$ = \frac{\partial L}{\partial u_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{u}_i} \right) + \frac{\partial L}{\partial \dot{u}_{i,k}} \dot{u}_{i,k} + \frac{\partial L}{\partial \langle h | u_i \rangle} \frac{d}{dt} \langle h | u_i \rangle. $$

(65)

Applying Eq. (13) to (65) gives

$$ \frac{dL}{dt} = \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{u}_i} \right) + \frac{\partial L}{\partial \dot{u}_{i,k}} \dot{u}_{i,k} + \frac{\partial L}{\partial \langle h | u_i \rangle} \frac{d}{dt} \langle h | u_i \rangle. $$

(66)

Eq. (66) can be rewritten as

$$ \frac{dL}{dt} - \frac{\partial L}{\partial \ddot{u}_i} \ddot{u}_i = \frac{\partial L}{\partial \dot{u}_{i,k}} \dot{u}_{i,k} + \frac{\partial L}{\partial \langle h | u_i \rangle} \frac{d}{dt} \langle h | u_i \rangle. $$

(67)

By using Eq. (42), Eq. (67) leads to

$$ \frac{d}{dt} (L - \rho \dot{u}_i \dot{u}_i + \sigma_{j \mu} \epsilon_{\mu \nu} u_{\nu,k}) = \dot{u}_i(t, x) \int_\Omega g(x,y, |r_i|) u_i(t,y) dv(y) - u_i(t, x) \int_\Omega g(x,y, |r_i|) \dot{u}_i(t,y) dv(y). $$

(68)

Compared Eq. (68) with (60), we have

$$ \hat{E} = \dot{u}_i(t, x) \int_\Omega g(x,y, |r_i|) u_i(t,y) dv(y) - u_i(t, x) \int_\Omega g(x,y, |r_i|) \dot{u}_i(t,y) dv(y). $$

(69)

So far, $\hat{E}$ has been determined. Clearly, it satisfies Eq. (64).
5.2 Nonlocal residuals of linear and angular momentum

Eq. (61) characterizes the equilibrium of linear momentum. So it is identical to Eq. (43). Due to this fact, it is easy to see that \( \hat{P}_k = -\langle \kappa | u_k \rangle \), i.e.,

\[
\hat{P}_k = \int_\Omega \kappa(x, y, |r_i|) u_k(t, y) dv(y) - u_k(t, x) \int_\Omega \kappa(x, y, |r_i|) dv(x). \tag{70}
\]

\( \hat{P}_k \) can be used to represent \( \hat{M}_k \). In order to verify this point, we firstly rewritten Eq. (62) as follows:

\[
\hat{M}_k = \epsilon_{k ij} x_i \hat{P}_j. \tag{72}
\]

In a general case, it is easy to see that \( \epsilon_{k ij} x_i \hat{P}_j \) always violates the zero-mean condition. As a result, \( \hat{M}_k \) given by Eq. (72) does not satisfy Eq. (64). On the other hand, \( \hat{M}_k \) should follow Eq. (64) so as to ensure the integrals of Eq. (62) over \( \Omega \) returning to Eq. (57). The paradox caused by Eq. (72) shows that it is impossible, under a general case, to transform Eq. (57) into the form of Eq. (62) by localization. Eq. (62) is inaccessible.

However, if the nonlocal kernel takes the form of the central pair potential, \( \epsilon_{k ij} x_i \hat{P}_j \) can be consistent with the zero-mean condition. Under this circumstance, \( \langle h | u_k \rangle \) is simplified into \( \langle h | x_k \rangle \), and \( \hat{P}_k \) can be represented as

\[
\hat{P}_k = \int_\Omega \kappa(x, y, |r_i|) y_i dv(y) - x_k \int_\Omega \kappa(x, y, |r_i|) dv(x). \tag{73}
\]

Substituting Eq. (73) into (72) yields

\[
\hat{M}_k = \epsilon_{k ij} x_i \int_\Omega \kappa(x, y, |r_i|) y_i dv(y), \tag{74}
\]

which is equivalent to the expression below:

\[
\hat{M} = x \times \int_\Omega \kappa(x, y, |r_i|) y dv(y). \tag{75}
\]

Since both \( x \) and \( y \) are the dummy variables of integration in the below, we have

\[
\int_\Omega \hat{M} dv(x) = \int_\Omega \int_\Omega \kappa(x, y, |r_i|) x \times y dv(y) dv(x) = \int_\Omega \int_\Omega \kappa(y, x, |r_i|) y \times x dv(x) dv(y) = - \int_\Omega \hat{M} dv(x). \tag{76}
\]

Eq. (76) leads to

\[
\int_\Omega \hat{M} dv(x) = 0, \quad \text{i.e.,} \quad \int_\Omega \hat{M} dv(x) = 0. \tag{77}
\]

So far, we have demonstrated that \( \hat{M}_k \) satisfies the zero-mean condition when the long-range body force is governed by a central pair potential.

5.3 Nonlocal residual of the Eshelby tensor

The conservation law relevant to the Eshelby tensor attributes to the invariance of the action functional under the coordinate translations. Therefore, both the Lagrangian \( L \) and the nonlocal kernel \( h \) are bound to be explicitly independent of \( x_k \). As thus, we have

\[
L_{ik} = \frac{\partial L}{\partial u_i} u_{ik} + \frac{\partial L}{\partial u_{ik}} u_{ik} + \frac{\partial L}{\partial (h | u_i)} (h | u_i)_{ik} = \frac{\partial L}{\partial u_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial u_i} \right) + \frac{\partial L}{\partial (h | u_i)} (h | u_i)_{ik} + \frac{\partial L}{\partial (h | u_{ik})} (h | u_{ik})_{ik} + \frac{\partial L}{\partial (h | u_i)_{ik}} (h | u_{ik})_{ik}. \tag{78}
\]
Applying Eq. (13) to (78) leads to
\[
L_k = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_i} u_{i,k} \right) + \left( \frac{\partial L}{\partial u_{i,j}} \right) u_{i,j} + \frac{\partial L}{\partial \langle h \rangle} \langle h \rangle_{i,k} - \langle g \rangle \frac{\partial L}{\partial \langle h \rangle} u_{i,k}.
\] (79)

Substituting Eq. (42) into (79) yields
\[
L_k = \frac{d}{dt} \left( \rho \dot{u}_i u_{i,k} \right) - \left( \sigma_{ij} u_{i,j} \right)_{i,j} + u_i \langle h \rangle_{i,k} - \langle g \rangle u_{i,k}.
\] (80)

Eq. (80) is rewritten as
\[
\frac{d}{dt} \left( \rho \dot{u}_i u_{i,k} \right) - \left( L \delta_{i,j} + \sigma_{ij} u_{i,j} \right)_{i,j} = \langle g \rangle u_{i,k} - u_i \langle h \rangle_{i,k}.
\] (81)

Comparison between Eq. (80) and Eq. (63) will gives
\[
\dot{J}_k = \langle g \rangle u_{i,k} - u_i \langle h \rangle_{i,k}.
\] (82)

Clearly, \(J_k\) determined by Eq. (82) fails to agree with the zero-mean condition. – This means that the nonlocal residual of the Eshelby tensor does not exist. Therefore, Eq. (81) is not a local equilibrium equation corresponding to Eq. (59). In other word, Eq. (59) can not be transformed into a local form similar to Eq. (63). The conservation law relevant to the Eshelby tensor exists only in the form of integration.

6 Conclusions

In this paper, we extend the definition of the nonlocal argument by introducing a nonlinear nonlocal kernel depending not only on the spatial coordinates but also on the field variables. The extended definition retains the zero mean character of the nonlocal argument. It is this character to distinguish the nonlocal Lagrangian formulation developed in this paper from other nonlocal variational theories.

On the basis of the extended nonlocal argument, a Lagrangian formulation with nonlocality is established. The nonlocal Euler-Lagrange equation is derived from the Hamilton’s principle. Accompanied with this equation, the nonlocal traction appears in the form of the nonlocal residual satisfying the zero mean condition automatically. – This is an obvious difference between the new theory and the other theories. Physically, the nonlocal traction represents the long-range interactions within body. Therefore, the zero-mean character of the nonlocal traction is just a embodiment of the action and reaction law.

The Noether’s theorem is extended to the Lagrangian formulation with nonlocality. The result shows that conservation law exists only in the form of the integral over the whole domain occupied by body. The local equilibrium equation derived from the localization of the conservation law is equal to the nonlocal residual of the conservation current, provided it exists, rather than zero like the case in the variational theories without nonlocality. In physics, the presence of the nonlocal residual attributes to the conservation current transferring, caused by the internal long-range interactions, among different parts within body.

A Lagrangian including the nonlocal argument is advanced in the quadratic form. The motion equation derived from this Lagrangian is consistent with the nonlocal elasticity associated with MBCM. In this theoretical framework, we use the extended Noether’s theorem to determine the conservation laws relevant to energy, linear momentum, angular momentum and the Eshelby tensor. The localization of these conservation laws are discussed in detail. We demonstrate that local equilibrium equation of energy and of linear momentum can be respectively derived from the corresponding conservation laws by localization, but the conservation law relevant to the Eshelby tensor can not be transformed into a local form by localization. So no local equilibrium equation relevant to the Eshelby tensor exists in the nonlocal elasticity associated with MBCM. The nonlocal residual of energy and nonlocal residual of linear momentum have been determined, respectively. They are consistent in mathematical form with the representation of nonlocal residual given in [27].

In general, there is no local equilibrium equation corresponding to the conservation law of angular momentum. However, if the nonlocal kernel takes the form of the central pair potential, the local equilibrium equation of angular momentum will occur in the nonlocal elasticity associated with MBCM. Under this case, the internal long-range interaction manifests itself as a central force field.

We therefore conclude that, in the nonlocal elasticity associated with MBCM, not every conservation law corresponds to a local equilibrium equation. Only when the nonlocal residual of conservation current exists, can a
conservation law be transformed into a local equilibrium equation by localization. The results in this paper imply that the existence of the local equilibrium equation is to some degree influenced by the nonlocal kernel. However, that is a problem needing further exploration.

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