ON PARASTATISTICS DEFINED AS TRIPLE OPERATOR ALGEBRAS

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Abstract

We unify parastatistics, defined as triple operator algebras represented on Fock space, in a simple way using the transition number operators. We express them as a normal ordered expansion of creation and annihilation operators. We discuss several examples of parastatistics, particularly Okubo’s and Palev’s parastatistics connected to many-body Wigner quantum systems. We relate them to the notion of extended Haldane statistics.

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Abstract

We unify parastatistics, defined as triple operator algebras represented on Fock space, in a simple way using the transition number operators. We express them as a normal ordered expansion of creation and annihilation operators. We discuss several examples of parastatistics, particularly Okubo’s and Palev’s parastatistics connected to many-body Wigner quantum systems. We relate them to the notion of extended Haldane statistics.
1. Introduction

Recently, a class of parastatistics (generalizing Bose and Fermi statistics) has been reformulated in terms of Lie supertriple systems. Particularly, Green’s parastatistics as well as new kinds of parastatistics discovered by Palev are reproduced. However, in this approach the positive definite Fock space representations are not treated.

On the other hand, a unified view of all operator algebras represented on Fock spaces has been presented. Furthermore, the permutation invariant statistics are also studied in detail.

Along the lines of Refs.(4,5), in this paper we unify, in a simple way triple operator algebras of Ref.(1), represented on the Fock spaces, as well as Greenberg’s infinite quon statistics and Govorkov’s paraquantization. Particularly, we present and discuss parastatistics which naturally appears in many-body Wigner quantum systems and its (bosonic and supersymmetric) extension. It appears that they are a generalization of Klein-Marshalek algebra, extensively used in nuclear physics. We discuss them in the framework of the Haldane’s definition of statistics. We point out that none of them is an example of the original Haldane exclusion statistics, but can be related to the so-called extended Haldane statistics. For each of them we find the extended Haldane statistics parameters.
2. Operator algebra, Fock space realization and statistics

Let us start with any algebra of \( M \) pairs of creation and annihilation operators \( a_i^\dagger, a_i \), \( i = 1,2,...M \) (\( a_i^\dagger \) is Hermitian conjugated to \( a_i \)). The algebra is defined by a normally ordered expansion \( \Gamma \) (generally no symmetry principle is assumed)

\[
a_i a_j = \Gamma_{ij} (a_i^\dagger; a)_j,
\]

(1)

with the number operators \( N_i \), i.e., \([N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}, [N_i, a_j] = -a_i \delta_{ij}\). In this case no peculiar relations of the type \( a_i^m = a_j^n, i \neq j \) can appear. Then, every monomial in \( \Gamma_{ij} \), Eq.(1), is of the type \((\cdots a_j^\dagger \cdots a_i \cdots)\) and all other indices appear in pairs \((\cdots a_k^\dagger \cdots a_k \cdots)\). The corresponding coefficients of expansion can depend on the total number operator \( N = \sum_{i=1}^{M} N_i \).

We assume that there is a unique vacuum \(|0>\) and the corresponding Fock space representation. The scalar product is uniquely defined by \(<0|0> = 1\), the vacuum condition \( a_i|0> = 0, i = 1,2,...M \), and Eq.(1). A general \( N \)-particle state is a linear combination of the vectors \((a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0>\), \( i_1,\cdots i_N = 1,2,...M \). We consider Fock spaces with no state vectors of negative squared norms. Note that we do not specify any relation between the creation (or annihilation) operators. They appear as a consequence of the norm zero vectors (null-vectors) in Fock space.

For fixed \( N \) mutually different indices \( i_1,\cdots i_N \), we define the \((N! \times N!)\) hermitian matrix of scalar products between states \((a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0>)\) for all permutations \( \pi \in S_N \). The number of linearly independent states among them is given by \( d_{i_1,\cdots i_N} = rank \mathcal{A}(i_1,\cdots i_N) \). The set of \( d_{i_1,\cdots i_N} \) for all possible \( i_1,\cdots i_N \) and all integers \( N \) completely characterizes the statistics and the thermodynamic properties of a free
system with the corresponding Fock space. If the algebra (1) is permutation invariant, i.e. \( \langle \pi \mu | \pi \nu \rangle = \langle \mu | \nu \rangle \), for all \( \pi, \mu, \nu \in S_N \), all expansion terms in \( \Gamma_{ij} \) of the form (symbolically)

\[
\Gamma_{ij} := \sum_{l-r}(a^\dagger \cdots a^\dagger)(a_j^\dagger \cdots a^\dagger \cdots a_i)(a \cdots a)
\]

have the same coefficient for all \( i, j = 1, 2, \ldots M \). (One single relation, for example \( a_1 a_2^\dagger = \Gamma_{12} \), determines the whole algebra.)

For the permutation invariant algebras there are several important consequences.

**Consequences**

(i) The matrices \( \mathcal{A}(i_1, \cdots i_N) \) and their ranks do not depend on concrete indices \( i_1, \cdots i_N \), but only on the multiplicities \( \lambda_i \) of appearance of the same indices \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M \geq 0, |\lambda| = \sum_{i=1}^{M} \lambda_i = N \), i.e. on the partition \( \lambda \) of \( N \).

(ii) For mutually different indices \( i_1, \cdots i_N \) i.e. \( \lambda_1 = \lambda_2 = \cdots = \lambda_N = 1 \), the generic matrix \( \mathcal{A}_{1N} \) is

\[
\mathcal{A}_{1N} = \sum_{\pi \in S_N} c(\pi)R(\pi),
\]

where \( R \) is the right regular representation of the permutation group \( S_N \) and \( c(\pi) \) are (real) coefficients. In other words, any row (column) of the matrix determines the whole matrix \( \mathcal{A}_{1N} \).

(iii) All matrices \( \mathcal{A}_\lambda \) can be simply obtained from \( \mathcal{A}_{1N} \). To check that the Fock space does not contain states of negative norms, it is sufficient to show that only generic matrices are non-negative.

(iv) For permutation invariant algebras there exist the transition number operators \( N_{ij} \), \( i, j = 1, 2, \ldots M \) with the properties
\[
\begin{align*}
[N_{ij}, a_k^\dagger] &= \delta_{ik} a_j^\dagger, & [N_{ij}, a_k] &= -\delta_{jk} a_i, & N_{ij}^\dagger &= N_{ji}, & N_{ii} \equiv N_i.
\end{align*}
\]

\(N_{ij}\) can be presented similarly as \(\Gamma_{ij}\), i.e. as a normal ordered expansion

\[
N_{ij} = a_j^\dagger a_i + \alpha \sum_l a_l^\dagger a_j a_i a_l + \beta \sum_l (a_j^\dagger a_l a_i a_l + a_j^\dagger a_l a_i a_l) + \gamma \sum_l a_j^\dagger a_l a_i a_l + \cdots,
\]

where \(\alpha, \beta, \gamma\) are constants which do not depend on the indices \(i, j\).

In the next section we show that all permutation invariant statistics considered by Okubo, Palev, Greenberg, Govorkov and Klein and Marshalek can be simply unified in terms of triple-operator algebras

\[
[[a_i, a_j^\dagger]_q, a_k^\dagger] = x \delta_{ij} a_k^\dagger + y \delta_{ik} a_j^\dagger + z \delta_{jk} a_i^\dagger,
\]

for all \(i, j, k = 1, 2, \cdots M\). Here, \(x, y, z \in \mathbb{R}\) are constants, \([, ,]\) denotes the commutator and \([a, b]_q = ab - qba\) is the \(q\)-deformed commutator.

**Remarks**

1. Equation (1), together with the vacuum condition \(a_i |0 >= 0\), uniquely determines all matrices \(\mathcal{A}_{iN}\) and \(\mathcal{A}_\lambda\). However, equation (1) does not imply positive definiteness, which has to be checked separately.

2. All other triple-operator relations follow from Eq.(5) via hermiticity of creation and annihilation operators, a linear combination of Eq.(5) with indices interchanged and, finally, as null-states of matrices \(\mathcal{A}_\lambda\).

3. The algebra with the well defined number operators \(N_i\) imply that \(z = 0\) in (5). If \(z \neq 0\), there exist peculiar relations of the type \(a_i^2 = a_j^2\) for all \(i, j = 1, 2, \cdots M\),
although $a_i^\dagger|0\rangle$ are linearly independent states. Such peculiar algebras are consistent if the Fock space does not contain null-states $^{12}$.

4. We point out that the algebra (5) can be simply written as the normal ordered expansion

$$a_i a_j^\dagger = (1 + xN)\delta_{ij} + qa_j a_i + y N_{ij} + z N_{ji},$$

where $N_{ij}$ are the transition number operator of form (3,4) and $N$ is the total number operator.

3. Examples

**Example 1.** Green’s parastatistics $^2$ can be presented in the form of Eq.(6) with $x = z = 0, y = \frac{2}{p}, p \in N$ and $q = \pm 1$, i.e.

$$a_i a_j^\dagger = \delta_{ij} \mp a_j a_i \pm \frac{2}{p} N_{ij},$$

where the upper (lower) sign corresponds to para-Bose (para-Fermi) statistics.

The transition number operator $N_{ij}$ is, up to the second order, given by $^4$

$$N_{ij} = a_j^\dagger a_i + \frac{p^2}{4(p-1)} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots,$$

where $Y_{il} = a_i a_l - q (\frac{2}{p} - 1) a_l a_i$.

**Example 2.** Govorkov’s new paraquantization $^7$ is given by $x = z = 0, y = \frac{\lambda}{p}, \lambda = \pm 1, p \in N$ and $q = 0$ :

$$a_i a_j^\dagger = \delta_{ij} - \frac{\lambda}{p} N_{ij},$$

$^7$
with the transition number operator, up to the second order,

\[ N_{ij} = a_j^\dagger a_i + \frac{p^2}{p^2 - \lambda^2} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots, \tag{10} \]

where \( Y_{il} = a_i a_l + (\frac{\lambda}{p}) a_l a_i \).

**Example 3.** Greenberg’s infinite quon statistics \(^6\) is given by \( x = y = z = 0 \) and \(-1 < q < 1:\)

\[ a_i a_j^\dagger = \delta_{ij} + q a_j^\dagger a_i \tag{11} \]

with transition number operator, up to the second order,

\[ N_{ij} = a_j^\dagger a_i + \frac{1}{1 - q^2} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots, \tag{12} \]

where \( Y_{il} = a_i a_l - q a_l a_i \). (The closed form for \( N_{ij} \) to all orders and for the general parameter \( q_{ij} \) is presented in Ref.(13).)

**Example 4.** Palev’s A statistics (Fermi case), which appears naturally in the treatment of many-body Wigner quantum systems \(^3\), is described by the following algebra \((i, j, k = 1, 2, \cdots M)\):

\[ [\{a_i, a_j^\dagger\}, a_k^\dagger] = \delta_{ik} a_j^\dagger - \delta_{ij} a_k^\dagger, \]

\[ [\{a_i, a_j^\dagger\}, a_k] = -\delta_{jk} a_i + \delta_{ij} a_k, \tag{13} \]

\[ \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \]

Hereafter, \(\{,\}\) denotes the anticommutator.

(In the original algebra, the operators depend on two indices, \(a_i \mapsto a_{\alpha i}\), but the structure of the algebra depends on the single index. One recovers the original algebra with \(\delta_{\alpha i, \beta j} = \delta_{\alpha \beta} \delta_{ij}\).) The vacuum conditions are \(a_i |0\rangle = 0, a_i a_j^\dagger |0\rangle = p \delta_{ij} |0\rangle\).
for \( p \in \mathbb{N} \). Upon the redefinition of the operators \((a_i, a_i^\dagger) \mapsto (\sqrt{p}a_i, \sqrt{p}a_i^\dagger)\), we write the above algebra as normal ordered expansion with \( x = -\frac{1}{p} \), \( y = \frac{1}{p} \), \( z = 0 \) and \( q = -1 \)

\[
a_i a_j^\dagger = (1 - \frac{N}{p})\delta_{ij} - a_j^\dagger a_i + \frac{1}{p} N_{ij}. \tag{14}
\]

The action of the annihilation operators \( a_i \) on the Fock states is obtained from the above relation, Eq.(14). For example,

\[
a_i a_j^\dagger a_k^\dagger |0\rangle = (1 - \frac{1}{p})(\delta_{ij} a_k^\dagger - \delta_{ik} a_j^\dagger)|0\rangle.
\]

It follows that

\[
a_i(a_i^\dagger)^2|0\rangle = 0, \quad \forall i
\]

\[
a_i a_i^\dagger a_k^\dagger |0\rangle = -a_i a_k^\dagger a_i|0\rangle = (1 - \frac{1}{p}) a_k^\dagger |0\rangle, \quad i \neq k. \tag{15}
\]

Hence, we obtain \(\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0\).

Generally, for mutually different indices \(i_1, \cdots i_N\), we find

\[
a_{i_1} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_N}^\dagger |0\rangle = (1 - \frac{N - 1}{p}) a_{i_2}^\dagger \cdots a_{i_N}^\dagger |0\rangle, \tag{16}
\]

in accordance with Ref.(3). The Fock space does not contain negative norm states if \( p \in \mathbb{N} \). The above equation (16) implies that the allowed states are only those with \( N \leq p \), and the states with \( N > p \) are null-states.

The transition number operator \( N_{ij} \), up to the second order, is:

\[
N_{ij} = a_j^\dagger a_i + \frac{1}{(p - 1)} \sum_l a_l^\dagger a_j^\dagger a_i a_l + \frac{2}{(p - 1)(p - 2)} \sum_{l_1,l_2} a_{l_1}^\dagger a_i^\dagger a_j^\dagger a_i a_l a_{l_2} + \cdots \tag{17}
\]

and terminates with \( p \) creation and \( p \) annihilation operator terms. For example, if \( p = 2 \), the terms with \( (p - 2) \) appearing in the denominator do not appear at all.
The $p \to \infty$ reproduces the Fermi algebra. We note that case $p = 1$ reproduces the Klein-Marshalek algebra, namely only the one-particle states are allowed:

$$a_i a_j^\dagger = (1 - N) \delta_{ij}, \quad N = \sum_l a_l^\dagger a_l. \quad (18)$$

In this sense, algebra (14) generalizes the Klein-Marshalek algebra.

It is interesting that the Fock space generated by the algebra (14) is equivalent to the Fock space generated by the algebra (with the same vacuum condition imposed)

$$a_i a_j^\dagger = (1 - \frac{N}{p}) (\delta_{ij} - a_j^\dagger a_i), \quad (19)$$

with the same $N_{ij}$ and $N$ as given by Eq.(17).

Furthermore, there are infinitely many algebras leading to different generic matrices, but with the same statistics. They can be represented by

$$a_i a_j^\dagger = f(N)(\delta_{ij} - a_j^\dagger a_i), \quad (20)$$

with $f(n) > 0$, $n < p$ and $f(p) = 0$. The simplest choice is the step function $f(N) = \Theta(p - N)$ ( $\Theta(x) = 0$, $x \leq 0$ and $\Theta(x) = 1$, $x > 0$).

We point out that the corresponding statistics is Fermi statistics restricted up to $N \leq p$ N-particle states. Hence, the counting rule is simply $D^F(M, N) = \binom{M}{N}$, $N \leq p$ and $D^F(M, N) = 0$ if $N > p$. Recall that Haldane introduced the statistics parameter $g$ through the change of the single-particle Hilbert space dimension $d_n$

$$g_{n \to n + \Delta n} = \frac{d_n - d_{n + \Delta n}}{\Delta n},$$

where $n$ is the number of particles and $d_n$ is the dimension of the one-particle Hilbert space obtained by keeping the quantum numbers of $(n - 1)$ particles fixed. In the
similar way we define the extended statistics parameter through the change of the available one-particle Fock-subspace dimension. Therefore, the above statistics is characterized by the Haldane statistical parameter $g = 1$

\[
g_{n\rightarrow n+k} = \frac{d_n - d_{n+k}}{k} = \frac{(M - n + 1) - (M - n - k + 1)}{k} = 1, \quad (21)
\]

if $n + k \leq p$. If $n + k = p + 1$, then $g_{n\rightarrow n+k} = \frac{(M-n+1)}{(p-n+1)}$, $n = 1, 2, \ldots p$ is fractional but $g$ is not constant any more. Hence, this is not an example for the original Haldane statistics for which the statistics parameter is $g = \text{const}$. Moreover, the above statistics is also not the statistics of the Karabali-Nair type, where $a_i^p \neq 0$, $a_i^{p+1} = 0$, and for any $N \leq Mp$ N-particle state is allowed, since from the Eq.(15) we already have $a_i^2 = 0$ and $N \leq p$.

**Example 5.** Palev’s A statistics (Bose case) is the counterpart of the algebra (14), namely:

\[
[[a_i, a_j^\dagger], a_k^\dagger] = -\delta_{ik} a_j^\dagger - \delta_{ij} a_k^\dagger, \quad (22)
\]

\[
[[a_i, a_j^\dagger], a_k] = \delta_{jk} a_i + \delta_{ij} a_k,
\]

\[
[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i, j, k = 1, 2, \ldots M.
\]

and the vacuum condition $a_i a_j^\dagger |0\rangle = p \delta_{ij} |0\rangle$. After the redefinition of the operators $(a_i, a_i^\dagger) \mapsto (\sqrt{p}a_i, \sqrt{p}a_i^\dagger)$, we write the normal ordered expansion of $a_i a_j^\dagger$ as

\[
(x = y = -\frac{1}{p}, \ z = 0, \ q = -1)
\]

\[
a_i a_j^\dagger = (1 - \frac{N}{p})\delta_{ij} + a_j^\dagger a_i - \frac{1}{p} N_{ij}. \quad (23)
\]

The action of the annihilation operators $a_i$ on the Fock states is obtained from
Eq.(23). For example,

\[ a_i a_j^\dagger a_k^\dagger |0\rangle = (1 - \frac{1}{p}) (\delta_{ij} a_k^\dagger + \delta_{ik} a_j^\dagger) |0\rangle. \]

Hence, we obtain \([a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.\]

Generally, we find

\[ a_i (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \cdots (a_M^\dagger)^{n_M} |0\rangle = N_i (1 - \frac{N - 1}{p}) (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_i^\dagger)^{n_i-1} \cdots a_M^\dagger)^{n_M} |0\rangle \]

(24)

where \( N = \sum_{i=1}^{M} n_i. \) The Fock space does not contain negative norm states if \( p \in \mathbb{N}. \) The above equation (24) implies that the states with \( N \leq p \) are allowed and the states with \( N > p \) are null-states. The transition number operator \( N_{ij} \) has the same form, with the same coefficients as in the Fermi case, Eq.(17), and terminates with p-annihilation and p-creation operator terms. The limit \( p \to \infty \) reproduces the Bose algebra. We note that if \( p = 1 \) the above algebra (23) reproduces the Klein-Marshalek algebra. Hence, this algebra is the Bose generalization of the Klein-Marshalek algebra.

There are again infinitely many algebras leading to different generic matrices but of the same ranks, i.e. statistics. They can be represented by

\[ a_i a_j^\dagger = f(N) (\delta_{ij} + a_j^\dagger a_i), \]

(25)

with \( f(n) > 0, \ n < p \) and \( f(p) = 0. \) The simplest choice is the step function mentioned after Eq.(20) or \( f(N) = 1 - \frac{N}{p}. \) The corresponding statistics is Bose statistics restricted to N-particle states with \( N \leq p. \) Hence, the counting rule is simply \( D^B(M, N) = \binom{M + N - 1}{N}, \ N \leq p \) and \( D^B(M, N) = 0 \) if \( N > p. \) Therefore, the above statistics is characterized by the Haldane statistics parameter \( g = 0. \)
\[ g_{n\rightarrow n+k} = \frac{d_n - d_{n+k}}{k} = \frac{M - M}{k} = 0, \quad (26) \]

if \( n + k \leq p \). If \( n + k = p + 1 \), then \( g_{n\rightarrow n+k} = \frac{M}{(p-n+1)} \), \( n = 1, 2, \cdots p \), is fractional but not constant. Hence, this is not an example for the original Haldane exclusion statistics for which \( g \) should be constant. The above statistics is also not of the Karabali-Nair type \(^{14}\), since \( a_i^p \neq 0 \), \( a_i^{p+1} = 0 \) but \( N \leq p \). This would be equivalent only for the single-mode oscillator, \( M = 1 \).

**Example 6.** The Bose and Fermi restricted algebra of Refs. (1,3) (the super-triple system) can be defined as

\[
[a_I, a_J^\dagger]_q = (1 - \frac{N}{p})\delta_{IJ} - \frac{(-)^{\sigma(I)\sigma(J)}}{p} N_{IJ},
\]

\[ q = (-)^{\sigma(I)\sigma(J)}, \]

\[ \sigma(I) = \begin{cases} 
0 & \text{if } I = i \quad \text{(Bose)} \\
1 & \text{if } I = \alpha \quad \text{(Fermi)}
\end{cases} \]

where the index \( I \doteq (i = 1, 2, \cdots M_B; \alpha = 1, 2, \cdots M_F.) \) denotes bosonic (fermionic) oscillator and \( N = N_B + N_F \) is the total number operator.

Explicitly,

\[
[a_i, a_j^\dagger] = (1 - \frac{N}{p})\delta_{ij} - \frac{1}{p}N_{ij},
\]

\[
\{a_\alpha, a_\beta^\dagger\} = (1 - \frac{N}{p})\delta_{\alpha\beta} + \frac{1}{p}N_{\alpha\beta}
\]

\[
[a_i, a_\alpha^\dagger] = -\frac{1}{p}N_{i\alpha}
\]

\[
[a_\alpha, a_i^\dagger] = -\frac{1}{p}N_{\alpha i}.
\]

The consistency condition for the algebra (27) reads:
\[ N_{IJ}a_K^\dagger - (-)^{(\sigma(I)+\sigma(J))\sigma(K)}a_K^\dagger N_{IJ} = \delta_{IK}a_J^\dagger. \]  

(28)

For example,

\[ \{N_{i\alpha}, a_\gamma\} = +\delta_{\alpha\gamma}a_i, \]

\[ [N_{i\alpha}, a_j^\dagger] = \delta_{ij}a_\alpha^\dagger. \]

Notice that \((N_{i\alpha})^2 = 0\). Thus, \(N_{i\alpha}\) plays the role of supersymmetric charge. Furthermore, it follows that

\[ [a_i, a_j] = \{a_\alpha, a_\beta\} = [a_i, a_\alpha] = 0. \]

The action of the annihilation operators \(a_i, a_\alpha\) on the Fock states is obtained by combining Eqs.(27) and (28). The N-particle states are allowed only if \(N \leq p\), with \(p\) being an integer.

The transition number operators, up to the second order, are basically similar to (17) and read

\[
N_{IJ} = a_j^\dagger a_I + \frac{1}{(p-1)} \sum_L (-)^{\sigma(L)(\sigma(I)+\sigma(J))}a_L^\dagger a_j^\dagger a_I a_L +
\]

\[
+ \frac{2}{(p-1)(p-2)} \sum_{L_1, L_2} (-)^{\sigma(L_1)+\sigma(L_2)}(-)^{\sigma(I)+\sigma(J)}a_{L_2}^\dagger a_{L_1}^\dagger a_J a_L a_{L_1} a_{L_2} + \cdots,
\]

where the sum over \(L\) runs over bosonic \((i = 1, 2, \cdots M_B)\) and fermionic \((\alpha = 1, 2, \cdots M_F)\) indices.

In the limit \(p \to \infty\), the above algebra reduces to the ordinary Bose and Fermi algebra. If \(p = 1\), the above algebra reduces to the Klein - Marshalek algebra with \(M_B + M_F\) oscillators.
Example 7. Okubo’s triple operator algebra (Example 4. in Ref.(1)) is defined for the fermionic operators \( a_i \) as

\[
\{a_i, a_j^\dagger, a_k^\dagger\} = \left( \frac{2}{p} \right) (-\delta_{ij}a_k^\dagger - \delta_{jk}a_i^\dagger + \delta_{ik}a_j^\dagger).
\]  

(30)

The normal ordered expansion of \( a_i a_j^\dagger \) is given by \( (x = z = -\frac{2}{p}, y = \frac{2}{p}, q = -1) \)

\[
a_i a_j^\dagger = (1 - \frac{2N}{p})\delta_{ij} - \frac{a_j^\dagger a_i}{2} + (\frac{2}{p}) (N_{ij} - N_{ji}).
\]  

(31)

In the limit \( p \to \infty \), it becomes the Fermi algebra.

From (31) it follows that

\[
a_i (a_j^\dagger)^2 |0\rangle = -(\frac{2}{p}) a_j^\dagger |0\rangle, \quad \forall i, j
\]

\[
a_i a_j^\dagger a_k^\dagger |0\rangle = -a_i a_k^\dagger a_j^\dagger |0\rangle = (1 - \frac{2}{p}) a_k^\dagger |0\rangle \quad i \neq k.
\]  

(32)

Therefore,

\[
\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \quad i \neq j,
\]

\[
(a_i)^2 = A, \quad [a_i, A] = 0, \quad [a_i, A^\dagger] = -(\frac{2}{p}) a_i^\dagger, \quad \forall i,
\]

\[
(a_i)^p \neq 0, \quad (a_i)^p+1 = 0.
\]  

(33)

However, in the Fock space there are negative norm states since \( \langle 0|(a_i)^2(a_i^\dagger)^2|0\rangle = -(\frac{2}{p}) < 0 \). The necessary condition for absence of such states is \( z \geq 0 \). The algebra similar to the algebra described by Eqs.(31-33) but with the positive definite Fock representations has been called peculiar algebra and was studied in Ref.(12).

Finally, let us mention that all Lie (super) algebras are triple systems ( since \( [a_i, a_j^\dagger]_\pm = \delta_{ij}(c_i + d_iN_i) \) ) and for a irreducible representations characterized with highest (lowest) weight state \( \Lambda \) (“vacuum”) one can find the following normal ordered expansion

\[
a_i a_i^\dagger = \Gamma_i (a_i^\dagger, a; \Lambda), \quad a_i a_j^\dagger = \pm a_j a_i
\]

However, these systems are not permutation invariant in the sense we defined in this paper.
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