Full Counting Statistics in Strongly Interacting Systems: Non-Markovian Effects

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We present a theory of full counting statistics for electron transport through interacting electron systems with non-Markovian dynamics. We illustrate our approach for transport through a single-level quantum dot and a metallic single-electron transistor to second order in the tunnel-coupling strength, and discuss under which circumstances non-Markovian effects appear in the transport properties.

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The study of current fluctuations in mesoscopic systems has become an intense field of research, since it allows to access information about electron correlations that is not contained in the average current. The phenomenon of shot noise \[1, 2, 3\], that dominates the current noise at low temperatures, has been investigated theoretically and experimentally in various contexts. Further understanding in electron transport can be gained from the study of higher moments \[4\] which can be conveniently extracted from the study of full counting statistics (FCS) \[5, 6\]. Up to date FCS has been studied in a variety of cases. Examples include normal-superconductor hybrid structures \[7\], superconducting weak links \[8\], tunnel junctions \[9\], chaotic cavities \[10\], entangled electrons \[11\] and spin-correlated systems \[12\]. Schemes for an experimental measurement of FCS has been put forward in Refs. \[13\].

Electron-electron interaction may strongly influence quantum transport. The effect of electron correlation on FCS has been considered so far in the case of a weakly interacting mesoscopic conductor \[14\], almost open dots \[15, 16, 17\], charge pumping \[15\] and charge shuttles \[15\]. The FCS for Coulomb-blockade devices has been analyzed \[18, 19, 20, 21\] in the framework of a Born-Markov master equation approach. The aim of the present paper is to extend this idea to obtain a general theory of full counting statistics for strongly interacting systems with non-Markovian behavior. In particular, we formulate a perturbative non-Markovian expansion that allows for a systematic study of the relative importance of non-Markovian corrections. We demonstrate for the example of a single-level quantum dot with strong Coulomb interaction that non-Markovian effects become increasingly important for higher moments of the current fluctuations.

Full counting statistics. – Full information about all transport properties of a given system is contained in the probability distribution \(P(N, t)\) that \(N\) charges have passed through the system after time \(t\). This distribution function is related to the cumulant generating function \(S(\chi)\) by

\[
S(\chi) = -\ln \left[ \sum_{N = -\infty}^{\infty} e^{iN\chi} P(N, t) \right],
\]

where \(\chi\) is the counting field. All moments of the current can be obtained from the cumulant generating function by performing derivatives with respect to the counting field \(\langle I \rangle_n = -(-i)^n(e^{\chi t}/t)\partial^2 \chi S(\chi)\big|_{\chi = 0}\). The first four moments are the average current, the (zero-frequency) current noise, the skewness, and the kurtosis.

In this work we consider systems with strong local interactions, such as electrons in a quantum dot, that are coupled to a reservoir of noninteracting degrees of freedom. In these situations transport properties can often be described in terms of few (local) degrees of freedom (the charge of the quantum dot in the previous example). It is then convenient to integrate out the noninteracting degrees of freedom to arrive at an effective description of the reduced system only. Let \(p^N\) be the vector of probabilities to find the system in the corresponding state at the initial time \(t = 0\). The time evolution of the system is described by a generalized master equation,

\[
\frac{d}{dt} p(N, t) = \sum_{N' = -\infty}^{\infty} \int_0^t dt' W(N - N', t, t') \cdot p(N', t'),
\]

where \(p(N, t)\) is the vector of dot occupation probabilities under the condition that \(N\) electrons have passed the system. The cumulant generating function is given by Eq. \[B\] where \(P(N, t) = e^T \cdot p(N, t)\) with \(e^T = (1, 1, \ldots, 1)\). The matrix \(W(N - N', t, t')\) describes transitions during which \(N - N'\) electrons are transferred. The counting field \(\chi\) is introduced by Fourier transforming the master equation \(p(\chi, t) = \sum_N \exp(iN\chi) p(N, t)\) and \(W(\chi, t, t') = \sum_N \exp(iN\chi) W(N, t, t')\).

In general, the kernels \(W(\chi, t', t)\) are nonlocal in time as a consequence of having integrated out the reservoir degrees of freedom. We consider the case in which there is no explicit time dependence of the systems parameters, so that \(W(\chi, t - t')\) can be Laplace transformed,
Performing the inverse Laplace transformation we get
\[ \sum_0^{\infty} \lambda^m \rightcomponent of the transitions \]{\textbf{W}(\chi, z)} are taken into account. Our goal is to describe stationary transport properties in the presence of a memory of the system, described by the full \( z \) -dependence of \( \textbf{W}(\chi, z) \). To solve the master equation without making use of the Markovian approximation, we switch to the Laplace representation,
\[ 2p(z) - p^\text{in} = \textbf{W}(\chi, z) \cdot \textbf{p}(\chi, z), \] which is solved by \( p(z) = \sum_0^{\infty} [\textbf{W}(\chi, z)]^n \cdot p^\text{in} \). By assuming that the kernel \( \textbf{W}(\chi, z) \) decays in time faster than any power law, we can define the Taylor series \( \textbf{W}(\chi, z)^n = \sum_0^{\infty} \frac{\partial^n}{\partial z^n} \textbf{W}(\chi, z)^n \) and substitute it in the previous solution for \( p(z) \). The long-time behavior of \( p(z) \) is determined by its poles in \( z \). Performing the inverse Laplace transformation we get
\[ p(\chi, t) = \sum_0^{\infty} \frac{\partial^n}{\partial z^n} \left( \left[ (\textbf{W}(\chi, z)_n \cdot e^{\textbf{W}(\chi, z)^n} \right) \right) |_{z=0^+} \cdot p^\text{in}, \] for large \( t \). To proceed, we perform a spectral decomposition of the matrix \( \textbf{W}(\chi, z) \). For physical reasonable systems, all eigenvalues have a negative real part. As a consequence of the exponential function in Eq. 4, the long-time behavior will be dominated by the eigenvalue \( \lambda(\chi, z) \) with the smallest absolute value of the real part. Let \( q_0 \) and \( p_0 \) be the corresponding left and right eigenvectors, \( q_0^T \cdot \textbf{W}(\chi, z) = \lambda(\chi, z) q_0^T \) and \( \textbf{W}(\chi, z) \cdot p_0 = \lambda(\chi, z) p_0 \). Unitarity in the absence of counting fields implies \( \lambda(0, z) = 0 \) for all \( z \).

The cumulant generating function becomes
\[ S(\chi) = -\ln \left[ \sum_0^{\infty} \frac{1}{n!} \lambda^n e^{\lambda n} \right] |_{z=0^+}, \] with \( \mu(\chi, z) = \ln[\textbf{q}^T \cdot p_0] \cdot (\textbf{q}^T \cdot p_0) \). By performing the time derivative and making use of the relation
\[ \sum_0^{\infty} \frac{1}{n!} \lambda^n = \sum_0^{\infty} \frac{1}{n!} \lambda^n (ab)^n = \sum_0^{\infty} \frac{1}{n!} \lambda^n (b^n + \lambda b^n) \] that holds for arbitrary functions \( a \) and \( b \), we arrive at the central result of this Letter,
\[ S(\chi) = \sum_0^{\infty} \frac{1}{n!} \lambda^{n+1} |_{z=0^+}. \] The cumulant generating function depends only on the eigenvalue \( \lambda(\chi, z) \). This result can be used as a starting point for a non-Markovian expansion, \( S(\chi) = \sum_0^{\infty} S_n(\chi) \) where \( S_n(\chi) \) contains \( n \) derivatives with respect to \( z \) applied to \( n + 1 \) factors of \( \lambda \). While \( S_0(\chi) \) describes the Markovian limit, \( S_n(\chi) \) is the \( n \)-th non-Markovian correction.

**Perturbative non-Markovian expansion.** – For many systems there is a small parameter which allows for a perturbative analysis of all transport properties. In the examples to be discussed below this will be the tunnel-coupling strength between the leads and the interacting region (quantum dot or metallic island). Then, \( \lambda(\chi, z) = \sum_0^{\infty} \lambda^{(1)}(\chi, z) \), where the superscript \( (i) \) indicates the order in the small parameter. The lowest-order transport properties are derived from the lowest-order cumulant generating function \( S^{(1)}(\chi) = -\lambda^{(1)}(\chi, z)|_{z=0^+} \), as found in Ref. [19]. Non-Markovian corrections, signaled by derivatives \( \partial^n \lambda(\chi, z) \), do not enter in this limit. The highest derivative that enters in the evaluation of the \( n \)-th moment in \( m \)-th order perturbation theory is \( \partial^n \lambda(\chi, z)|_{z=0^+} \) with \( k = \min(n, m) - 1 \). As a consequence, non-Markovian behavior is probed only in the second or higher moment combined with second or higher order in perturbation theory. The second-order contribution for example reads \( S^{(2)}(\chi) = -tl^{(2)}(\chi, z) + \lambda^{(1)}(\chi, z) \partial_2 \lambda^{(1)}(\chi, z)|_{z=0^+} \). The appearance of these derivatives in the noise of second-order transport through quantum dots has been also found in Ref. [22]. In the remaining part of this Letter, we illustrate our approach with two examples. We calculate the cumulant generating function for second-order transport through a single-level quantum dot and through a metallic single-electron transistor in the presence of strong Coulomb interaction.

**Single level Q.D.** – The single-level quantum dot is described by the Hamiltonian, \( H = H_L + H_R + H_D + H_T \). The electrons in the noninteracting left and right leads are represented by \( H_L \) and \( H_R \), respectively, \( H_D = \epsilon \sum \sigma \epsilon c_{\sigma}^L c_{\sigma}^R + U n_{\uparrow} n_{\downarrow} \) describes the dot with level energy \( \epsilon \) and charging energy \( U \) for double occupation. Tunneling is modeled by \( H_{T_{\sigma}} = \sum_{\tau} t_{\tau} \epsilon_{\sigma}^L c_{\sigma}^\tau c_{\sigma}^L + \text{c.c.} \), with \( r = \text{L, R} \), where we assume the tunnel matrix element \( t_{\tau} \) to be independent of momentum \( k \) and spin \( \sigma \). The tunnel-coupling strength is characterized by the intrinsic linewidth \( \Gamma = \Gamma_L + \Gamma_R \) with \( \Gamma_r = 2\pi \rho_r |t_r|^2 \) where \( \rho_r \) is the density of states in the leads. An asymmetry of the tunnel couplings is parametrised by \( \gamma = 4\Gamma_L \Gamma_R / \Gamma^2 \).

To derive the kernels \( \textbf{W} \) of the generalized master equation, we make use of a diagrammatic real-time technique for the time evolution of the reduced density matrix formulated on a Keldysh contour. We introduce counting fields \( \chi_r \) for tunneling through barrier \( r \) into lead \( r \) by the replacement \( t_r \rightarrow t_r \exp(i\chi_r) \) for tunnel vertices on the upper and \( t_r \rightarrow t_r \exp(-i\chi_r) \) on the lower branch of the Keldysh contour with \( \chi_L = -\chi_R = \chi/2 \).

We consider the limit \( U \rightarrow \infty \), in which double occupancy of the dot is prohibited, and obtain
\[ S^{(1)}(\chi) = \frac{t t f(\chi)}{2\hbar} \left[ 1 - \sqrt{1 + 2\gamma \sum_k j_k(\chi)(\exp(ik\chi - 1) \right] \frac{1}{|f(\chi)|^2} \]
with \( f_{(+)}(\omega) = [1 - f_L(\omega)]f_R(\omega) \), \( f_{(-)}(\omega) = f_L(\omega)[1 - f_R(\omega)] \), and \( \tilde{f}(\omega) = \sum_r \Gamma_r [1 + f_r(\omega)]/\Gamma_r \), where \( f_r(\omega) \) is the Fermi function for lead \( r \). This result was previously obtained in [10]. The second-order contribution, \( S^{(2)}(\chi) = S^{(2)}_{\text{cot}}(\chi) + S^{(2)}_{\text{ren}}(\chi) \), consists of two terms. The first one,

\[
S^{(2)}_{\text{cot}}(\chi) = -\frac{t\pi^2\Gamma^2}{4\pi\hbar} \sum_{k=\pm} (e^{i\chi k} - 1) \int d\omega f_{(k)}(\omega) R(\omega - \epsilon), \tag{7}
\]

with \( R(\omega) = \text{Re}[1/(\omega + i0^+)^2] \), describes cotunneling processes [23], and is in agreement with previous work about noise in cotunneling regime [24]. The counting-field dependence corresponds to a bidirectional Poisson statistics of a single barrier where the transition rates are substituted by the cotunneling rates of the quantum dot. However, \( S^{(2)}(\chi) \) contains a second contribution,

\[
S^{(2)}_{\text{ren}}(\chi) = \partial_\chi \left[ S^{(1)}(\chi) \text{ Re}[\sigma(\epsilon)] \right] \tag{8}
\]

with \( \sigma(\epsilon) = -\sum_r (\Gamma_r/2\pi) \int d\omega f_r(\omega)/(\omega - \epsilon + i0^+) \). In the previous formulas an high energy cut-off \( E_\text{c} \), of the order of charging energy, has to be introduced in order to cure spurious divergences related to the fact that we restricted the charge states to 0,1 [27]. The contribution \( S^{(2)}_{\text{ren}}(\chi) \) is also of second order in the tunnel-coupling strength but obeys the same statistics as the first-order (sequential-tunneling) result, Eq. (6). This suggests that there are two different types of second-order contributions to transport. In addition to the usual cotunneling processes, there are corrections to sequential tunneling due to quantum-fluctuation induced renormalization of the system parameters. From the form of Eq. (8) we deduce a renormalization of level position and coupling strength given by \( \tilde{\epsilon} = \epsilon + \text{Re} \sigma(\epsilon) \) and \( \tilde{\Gamma} = \Gamma + \partial_\epsilon \text{Re} \sigma(\epsilon) \).

In Fig. 1 we plot the first four moments (current, noise, skewness, and kurtosis) as a function of level position \( \epsilon \). The solid lines represent the full first- plus second-order result, as compared to the first-order contribution (dashed line).

The relative importance of the non-Markovian contributions is illustrated in Fig. 2. While for the current only Markovian contributions enter (see discussion above), non-Markovian corrections become increasingly important for higher moments.

**Metallic QD.** A similar analysis can be performed for a metallic single-electron transistor, which accommodates a continuum of states on the dot and includes a large number of transverse channels. Following the notation of Refs. [27] we characterize the tunnel-coupling strength by the dimensionless conductance \( G_0 = h/(4\pi e^2R) \) where \( R \) is the resistance of barrier \( r = \text{L, R} \). We concentrate on the low-temperature regime, in which only two charge states of the metallic island have to be taken into account. This requires, again, the introduction of an high-energy cut-off to regularize the integrals.

\[
S^{(2)}_{\text{cot}}(\chi) = -\frac{2\pi t}{h} \sum_{k=\pm} (e^{i\chi k} - 1) \int d\omega \alpha_{(k)}(\omega) R(\omega - \Delta), \tag{10}
\]

and a term

\[
S^{(2)}_{\text{ren}}(\chi) = \partial_\Delta \left[ S^{(1)}(\chi) \text{ Re}[\sigma(\Delta)] \right] \tag{11}
\]

associated with sequential-tunneling processes with renormalized system parameters, where \( \sigma(\Delta) = -\sum_r \int d\omega \alpha_r(\omega)/(\omega - \Delta + i0^+) \). This interpretation of the
different types of second-order contributions is consistent with the analysis of the second-order current [28] and of the FCS within a drone Majorana fermion representation [29].

Conclusions. – We present a theory of FCS for interacting systems with non-Markovian dynamics. A general expression for the cumulant generating function is derived that provides the starting point for a perturbative non-Markovian expansion. As examples we study transport through a single-level quantum dot and a metallic single-electron transistor to second order in the tunnel-coupling strength. From our formulation we could identify two different types of contributions to second-order transport, namely cotunneling and corrections to sequential tunneling due to renormalization of the system parameters. Furthermore, we demonstrate the increasing importance of non-Markovian effects for higher moments and higher orders in the tunnel-coupling strength.

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![FIG. 2: Relative contribution of the non-Markovian part $\langle I \rangle_n - \langle I \rangle_n^{\text{Markov}}$ to the $n$-th moment $\langle I \rangle_n$ in first-plus second-order in tunneling. The full, dashed, and dot-dashed line corresponds to noise ($n = 2$), skewness ($n = 3$), and kurtosis ($n = 4$) for the same parameters as in Fig. 1.](image)

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