RADICALS OF PRINCIPAL IDEALS AND THE CLASS GROUP OF A DEDEKIND DOMAIN

DARIO SPIRITO

Abstract. For a Dedekind domain $D$, let $\mathcal{P}(D)$ be the set of ideals of $D$ that are radical of a principal ideal. We show that, if $D, D'$ are Dedekind domains and there is an order isomorphism between $\mathcal{P}(D)$ and $\mathcal{P}(D')$, then the rank of the class groups of $D$ and $D'$ is the same.

1. Introduction

The class group $\text{Cl}(D)$ of a Dedekind domain $D$ is defined as the quotient between the group of the nonzero fractional ideals of $D$ and the subgroup of the principal ideals of $D$. Since $\text{Cl}(D)$ is trivial if and only if $D$ is a principal ideal domain (equivalently, if and only if it is a unique factorization domain), the class group can be seen as a way to measure how much unique factorization fails in $D$. For this reason, the study of the class group is an important part of the study of Dedekind domains.

It is a non-obvious fact that the class group of $D$ actually depends only on the multiplicative structure of $D$, or, from another point of view, depends only on the set of nonzero principal ideals of $D$. Indeed, the class group of $D^\bullet := D \setminus \{0\}$ as a monoid (where the operation is the product) is isomorphic to the class group of $D$ as a Dedekind domain (see Chapter 2 – in particular, Section 2.10 – of [5]), and thus if $D$ and $D'$ are Dedekind domains whose sets of principal ideals are isomorphic (as monoids) then the class groups of $D$ and $D'$ are isomorphic too.

In this paper, we show that the rank of $\text{Cl}(D)$ can be recovered by considering only the set $\mathcal{P}(D)$ of the ideals that are radical of a principal ideal: that is, we show that if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic as partially ordered sets then the ranks of $\text{Cl}(D)$ and $\text{Cl}(D')$ are equal. The proof of this result can be divided into two steps.

In Section 3 we show that an order isomorphism between $\mathcal{P}(D)$ and $\mathcal{P}(D')$ can always be extended to an isomorphism between the sets $\text{Rad}(D)$ and $\text{Rad}(D')$ of all radical ideals of $D$ (Theorem 3.6): this is accomplished by considering these sets as (non-cancellable) semigroups.

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and characterizing coprimality in $D$ through a version of coprimality in $\mathcal{P}(D)$ (Proposition 3.3).

In Section 4 we link the structure of $\mathcal{P}(D)$ and $\text{Rad}(D)$ with the structure of the tensor product $\text{Cl}(D) \otimes \mathbb{Q}$ as an ordered topological space; in particular, we interpret the set of inverses of a set $\Delta \subseteq \text{Max}(D)$ with respect to $\mathcal{P}(D)$ (see Definition 4.1) as the negative cone generated by the images of $\Delta$ in $\text{Cl}(D) \otimes \mathbb{Q}$ (Proposition 4.2) and use this connection to calculate the rank of $\text{Cl}(D)$ in function of some particular partitions of an “inverse basis” of $\text{Max}(D)$ (Propositions 4.9 and 4.10). As this construction is invariant with respect to isomorphism, we get the main theorem (Theorem 4.11).

In Section 5 we give three examples, showing that some natural extensions of the main result do not hold.

2. Notation and preliminaries

Throughout the paper, $D$ will denote a Dedekind domain, that is, a one-dimensional integrally closed Noetherian integral domain; equivalently, a one-dimensional Noetherian domain such that $D_P$ is a discrete valuation ring for all maximal ideals $P$. For general properties about Dedekind domains, the reader may consult, for example, [2, Chapter 7, §2], [1, Chapter 9] or [7, Chapter 1].

We use $D^*$ to indicate the set $D \setminus \{0\}$. We denote by $\text{Max}(D)$ the set of maximal ideals of $D$. If $I$ is an ideal of $D$, we set $V(I) := \{P \in \text{Spec}(D) \mid I \subseteq D\}$. If $I = xD$ is a principal ideal, we write $V(x)$ for $V(xD)$. If $I \neq (0)$, the set $V(I)$ is always a finite subset of $\text{Max}(D)$. We denote by $\text{rad}(I)$ the radical of the ideal $I$, and we say that $I$ is a radical ideal (or simply that $I$ is radical) if $I = \text{rad}(I)$.

Every nonzero proper ideal $I$ of $D$ can be written uniquely as a product $P_1^{e_1} \cdots P_n^{e_n} = P_1^{e_1} \cap \cdots \cap P_n^{e_n}$, where $P_1, \ldots, P_n$ are distinct maximal ideals and $e_1, \ldots, e_n \geq 1$. In particular, in this case we have $V(I) = \{P_1, \ldots, P_n\}$, and $\text{rad}(I) = P_1 \cdots P_n$. An ideal is radical if and only if $e_1 = \cdots = e_n = 1$. If $P$ is a maximal ideal, the $P$-adic valuation of an element $x$ is the exponent of $P$ in the factorization of $xD$; we denote it by $v_P(x)$. (If $x \notin P$, i.e., if $P$ does not appear in the factorization, then $v_P(x) = 0$.)

If $P_1, \ldots, P_k$ are distinct maximal ideals and $e_1, \ldots, e_k \in \mathbb{N}$, then by the approximation theorem for Dedekind domains (see, e.g., [2, Chapter VII, §2, Proposition 2]) there is an element $x \in D$ such that $v_{P_i}(x) = e_i$ for $i = 1, \ldots, k$.

A fractional ideal of $D$ is a $D$-submodule $I$ of the quotient field $K$ of $D$ such that $xI \subseteq D$ (and thus $xI$ is an ideal of $D$) for some $x \in D^*$. The set $\mathcal{F}(D)$ of nonzero fractional ideals of $D$ is a group under multiplication; the inverse of an ideal $I$ is $I^{-1} := (D : I) := \{x \in K \mid xI \subseteq D\}$. A
nonzero fractional ideal $I$ can be written uniquely as $P_1^{e_1} \cdots P_n^{e_n}$, where $P_1, \ldots, P_n$ are distinct maximal ideals and $e_1, \ldots, e_n \in \mathbb{Z} \setminus \{0\}$ (with the empty product being equal to $D$). Thus, $\mathcal{F}(D)$ is isomorphic to the free abelian group over $\text{Max}(D)$. The quotient between this group and its subgroup formed by the principal fractional ideals is called the class group of $D$, and is denoted by $\text{Cl}(D)$.

For a set $S$, we denote by $\mathfrak{P}_{\text{fin}}(S)$ the set of all finite and nonempty subsets of $S$.

3. The two semilattices $\mathcal{P}(D)$ and $\text{Rad}(D)$

Let $(X, \leq)$ be a meet-semilattice, that is, a partially ordered set where every pair of elements has an infimum. Then, the operation $x \wedge y$ associating to $x$ and $y$ their infimum is associative, commutative and idempotent, and it has a unit if and only if $X$ has a maximum. The order of $X$ can also be recovered from the operation: $x \geq y$ if and only if $x$ divides $y$ in $(X, \wedge)$, that is, if and only if there is a $z \in X$ such that $y = x \wedge z$. A join-semilattice is defined in the same way, but using the supremum instead of the infimum.

Let now $D$ be a Dedekind domain. We will be interested in two structures of this kind.

The first one is the semilattice $\text{Rad}(D)$ of all nonzero radical ideals of $D$. In this case, the order $\leq$ is the usual containment order, while the product is equal to

$$I \wedge J := I \cap J = \text{rad}(IJ).$$

The second one is the semilattice $\mathcal{P}(D)$ of the ideals of $D$ that are radical of a nonzero, principal ideal of $D$. This is a subsemilattice of $\text{Rad}(D)$ since

$$\text{rad}(aD) \land \text{rad}(bD) = \text{rad}(abD),$$

i.e., the product of two elements of $\mathcal{P}(D)$ remains inside $\mathcal{P}(D)$.

A nonzero radical ideal $I$ is characterized by the finite set $V(I)$. Hence, the map from $\text{Rad}(D)$ to $\mathfrak{P}_{\text{fin}}(\text{Max}(D))$ sending $I$ to $V(I)$ is an order-reversing isomorphism of partially ordered sets, which becomes an order-reversing isomorphism of semilattices if the operation on the power set is the union. We denote by $\mathcal{V}(D)$ the image of $\mathcal{P}(D)$ under this isomorphism; that is, $\mathcal{V}(D) := \{V(x) \mid x \in D^*\}$. The inverse of this map is the one sending a set $Z$ to the intersection of the prime ideals contained in $Z$.

Those semilattices have neither an absorbing element (which would be the zero ideal) nor a unit (which should be $D$ itself).

**Lemma 3.1.** Let $X,Y \in \mathfrak{P}_{\text{fin}}(\text{Max}(D))$ (resp., $X,Y \in \mathcal{V}(D)$). Then, $X \mid Y$ in $\mathfrak{P}_{\text{fin}}(\text{Max}(D))$ (resp., $X \mid Y$ in $\mathcal{V}(D)$) if and only if $X \subseteq Y$. 

Definition 3.2. Let \( M \) be a commutative semigroup. We say that \( a_1, \ldots, a_n \in M \) are product-coprime if, whenever there is an \( x \in M \) such that \( x = a_1b_1 = a_2b_2 = \cdots = a_nb_n \), then for every \( j \) the element \( a_j \) divides \( \prod_{i \neq j} b_i \).

Proposition 3.3. Let \( D \) be a Dedekind domain, and let \( a_1, \ldots, a_n \in D^\times \). Then, \( a_1, \ldots, a_n \) are coprime in \( D \) if and only if \( V(a_1), \ldots, V(a_n) \) are product-coprime in \( \mathcal{V}(D) \).

Proof. Suppose that \( a_1, \ldots, a_n \) are coprime, and let \( X \in \mathcal{V}(D) \) be such that \( X = V(a_1) \cup B_1 = \cdots = V(a_n) \cup B_n \) for some \( B_1, \ldots, B_n \in \mathcal{V}(D) \). By symmetry, it is enough to prove that \( V(a_1) \) divides \( B_2 \cup \cdots \cup B_n \) in \( \mathcal{V}(D) \), i.e., that \( V(a_1) \subseteq B_2 \cup \cdots \cup B_n \). Take any prime ideal \( P \in V(a_1) \): since \( a_1, \ldots, a_n \) are coprime there is a \( j \) such that \( P \notin V(a_j) \). However, \( P \in V(a_j) \cup B_j \), and thus \( P \in B_j \). Therefore, \( V(a_i) \subseteq B_2 \cup \cdots \cup B_n \), as claimed.

Conversely, suppose \( V(a_1), \ldots, V(a_n) \) are product-coprime, and suppose that \( a_1, \ldots, a_n \) are not coprime. Then, there is a prime ideal \( P \) containing all \( a_i \); passing to powers, without loss of generality we can suppose that the \( P \)-adic valuation of the \( a_i \) is the same, say \( v_P(a_i) = t \) for every \( i \). By prime avoidance, there is a \( b_1 \in D \setminus P \) such that \( v_Q(b_1) \geq v_Q(a_i) \) for all \( i > 1 \) and all \( Q \neq P \). Let \( x := a_1b_1 \). By construction, \( a_i | x \) for each \( i \), and thus we can find \( b_2, \ldots, b_n \in D \) such that \( x = a_1b_1 \). Therefore, \( V(x) = V(a_1) \cup V(b_1) \) for every \( i \); by hypothesis, it follows that \( V(a_1) \) divides \( V(b_2) \cup \cdots \cup V(b_n) \), i.e., that \( V(a_1) \subseteq V(b_2) \cup \cdots \cup V(b_n) \). However, \( v_P(x) = v_P(a_1) + v_P(b_1) = t \), and thus \( v_P(b_i) = 0 \) for every \( i \); in particular, \( P \notin V(b_i) \) for every \( i \). This is a contradiction, and thus \( a_1, \ldots, a_n \) are coprime.

Definition 3.4. Let \( M \) be a commutative semigroup. We say that \( I \subseteq M \) is product-proper if no finite subset of \( I \) is product-coprime. We denote the set of maximal product-proper subsets of \( M \) by \( \mathfrak{M}(M) \).

Proposition 3.5. Let \( D \) be a Dedekind domain. The maps
\[
\nu: \text{Max}(D) \to \mathfrak{M}(\mathcal{V}(D)),
\]
\[
P \mapsto \{ V(x) \mid x \in P \}
\]
and
\[
\theta: \mathfrak{M}(\mathcal{V}(D)) \to \text{Max}(D),
\]
\[
\mathcal{Y} \mapsto \{ x \in D \mid V(x) \in \mathcal{Y} \}
\]
are bijections, inverse one of each other.

Proof. We first show that \( \nu \) and \( \theta \) are well-defined.
If $P$ is a maximal ideal of $D$, then $P \in X$ for every $X \in \nu(P)$; thus, if $V(a) \in \nu(P)$ then $a \in P$ and $\nu(P)$ is product-proper. If $\nu(P) \subseteq \mathcal{Y} \subseteq \mathcal{V}(D)$, take $Y \in \mathcal{Y} \setminus \nu(P)$; then, $Y = V(b)$ for some $b \notin P$. If $Y = \{Q_1, \ldots, Q_k\}$, by prime avoidance we can find $a \in P \setminus (Q_1 \cup \cdots \cup Q_k)$; then, $a$ and $b$ are coprime and thus $V(a)$ and $V(b)$ are product-coprime. Hence, $\nu(P)$ is a maximal product-proper subset of $\mathcal{V}(D)$.

Conversely, let $\mathcal{Y} \in \mathfrak{M}(\mathcal{V}(D))$. If $\theta(\mathcal{Y})$ is contained in some prime ideal $P$, then $\mathcal{Y} \subseteq \nu(P)$, and thus we must have $\mathcal{Y} = \nu(P)$; in particular, $\theta(\mathcal{Y}) = P \in \text{Max}(D)$. If $\theta(\mathcal{Y})$ is not contained in any prime ideal, let $V(a) = \{Q_1, \ldots, Q_k\} \in \mathcal{Y}$. Since $\theta(\mathcal{Y}) \not\subseteq Q_i$, for every $i$ we can find $b_i \notin Q_i$ such that $V(a_i) \in \mathcal{Y}$; then, $a, b_1, \ldots, b_n$ are coprime and thus $V(a), V(b_1), \ldots, V(b_n)$ are a product-coprime subset of $\mathcal{Y}$, a contradiction. Hence $\mathcal{Y} = \nu(P)$.

The fact that they are inverses one of each other follows similarly. □

**Theorem 3.6.** Let $D, D'$ be Dedekind domains. If there is an order isomorphism $\psi : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$, then there is an order isomorphism $\Psi : \text{Rad}(D) \rightarrow \text{Rad}(D')$ extending $\psi$.

**Proof.** The statement is equivalent to saying that any isomorphism $\phi : \mathcal{V}(D) \rightarrow \mathcal{V}(D')$ can be extended to an isomorphism $\Phi : \mathfrak{P}_{\text{fin}}(\text{Max}(D)) \rightarrow \mathfrak{P}_{\text{fin}}(\text{Max}(D'))$. For simplicity, let $\mathfrak{P} := \mathfrak{P}_{\text{fin}}(\text{Max}(D))$ and $\mathfrak{P}' := \mathfrak{P}_{\text{fin}}(\text{Max}(D'))$.

If $\phi$ is an isomorphism, then it sends product-proper sets into product-proper sets, and thus $\phi$ induces a bijective map $\eta_1 : \mathfrak{M}(\mathcal{V}(D)) \rightarrow \mathfrak{M}(\mathcal{V}(D'))$. Using the map $\theta$ of Proposition 3.3, $\eta_1$ induces a bijection $\eta : \text{Max}(D) \rightarrow \text{Max}(D')$, such that the diagram

$$
\begin{array}{ccc}
\mathfrak{M}(\mathcal{V}(D)) & \xrightarrow{\phi} & \text{Max}(D) \\
\downarrow^{\eta_1} & & \downarrow^{\eta} \\
\mathfrak{M}(\mathcal{V}(D')) & \xrightarrow{\theta'} & \text{Max}(D')
\end{array}
$$

commutes (explicitly, $\eta = \theta' \circ \eta_1 \circ \theta^{-1}$). In particular, $\eta$ induces an order isomorphism $\Phi$ between $\mathfrak{P}$ and $\mathfrak{P}'$, sending $X \subseteq \text{Max}(D)$ to $\eta(X) \subseteq \text{Max}(D')$. To conclude the proof, we need to show that $\Phi$ extends $\phi$.

Let $X = \{P_1, \ldots, P_k\} \in \mathcal{V}(D)$. Then, by definition, $\Phi(X) = \eta(X) = \{\eta(P_1), \ldots, \eta(P_k)\}$. The maximal product-proper subsets of $\mathcal{V}(D)$ containing $X$ are $\mathcal{Y}_i := \nu(P_i)$, for $i = 1, \ldots, k$; since $\phi$ is an isomorphism, the maximal product-proper subsets of $\mathcal{V}(D')$ containing $\phi(X)$ are the sets $\phi(\mathcal{Y}_i)$. By construction, $\phi(\mathcal{Y}_i) = \eta_1(\mathcal{Y}_i)$; however, $\theta'(\eta_1(\mathcal{Y}_i)) = \eta(P_i)$, and thus $\eta(X) = \{\phi(\mathcal{Y}_1), \ldots, \phi(\mathcal{Y}_k)\} = \phi(X)$. Thus, $\Phi$ extends $\phi$, as claimed. □

The following corollary was obtained, with a more ad hoc reasoning, in the proof of [9, Theorem 2.6].
Corollary 3.7. Let $D, D'$ be Dedekind domains such that $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are order-isomorphic. Then, $\text{Cl}(D)$ is torsion if and only if $\text{Cl}(D')$ is torsion.

Proof. The class group of $D$ is torsion if and only if every prime ideal has a principal power [6, Theorem 3.1], and thus if and only if $\mathcal{P}(D) = \text{Rad}(D)$.

If $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic, then by Theorem 3.6 there is an isomorphism $\Phi : \text{Rad}(D) \rightarrow \text{Rad}(D')$ sending $\mathcal{P}(D)$ to $\mathcal{P}(D')$; hence, $\text{Rad}(D) = \mathcal{P}(D)$ if and only if $\text{Rad}(D') = \mathcal{P}(D')$. Therefore, $\text{Cl}(D)$ is torsion if and only if $\text{Cl}(D')$ is torsion. □

Remark 3.8. Let $\text{Princ}(D)$ be the set of principal ideals of $D$ and $\mathcal{I}(D)$ be the set of all ideals of $D$.

The method used in this section can also be applied to prove the analogous result for non-radical ideals, i.e., to prove that an isomorphism $\phi : \text{Princ}(D) \rightarrow \text{Princ}(D')$ can be extended to an isomorphism $\Phi : \mathcal{I}(D) \rightarrow \mathcal{I}(D')$.

The most obvious analogue of Proposition 3.3 does not hold, since the ideals $(a_1), \ldots, (a_n)$ may be product-coprime in $\text{Princ}(D)$ without $a_1, \ldots, a_n$ being coprime (for example, take $a_1 = y$, $a_2 = y^2$ and $a_3 = y^3$, where $y$ is a prime element of $D$). However, this can be repaired: $a_1, \ldots, a_n \in D^\star$ are coprime if and only if the ideals $(a_1)^{k_1}, \ldots, (a_n)^{k_n}$ are product-coprime in $\text{Princ}(D)$ for every $k_1, \ldots, k_n \in \mathbb{N}$. The proof is essentially analogous to the one given for Proposition 3.3.

Proposition 3.5 carries over without significant changes: the maximal product-proper subsets of $\text{Princ}(D)$ are in bijective correspondence with the maximal ideals of $D$. Theorem 3.6 carries over as well: the only difference is that, instead of the restricted power set $\mathcal{P}_\text{fin}(\text{Max}(D))$ it is necessary to use the free abelian group generated by $\text{Max}(D)$.

In particular, this result directly implies that if $\text{Princ}(D)$ and $\text{Princ}(D')$ are isomorphic as partially ordered sets then the class groups $\text{Cl}(D)$ and $\text{Cl}(D')$ are isomorphic as groups, since the class group depends exactly on which ideals are principal. This result is also a consequence of the theory of monoid factorization (see [5]), of which this reasoning can be seen as a more direct (but less general) version.

4. Calculating the rank

The rank $\text{rk} G$ of an abelian group $G$ is the dimension of the tensor product $G \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$. In particular, the rank of $G$ is 0 if and only if $G$ is a torsion group; therefore, Corollary 3.7 can be rephrased by saying that, if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are order-isomorphic, then the rank of $\text{Cl}(D)$ is 0 if and only if the rank of $\text{Cl}(D')$ is 0. In this section, we want to generalize this result by showing that $\text{rk} \text{Cl}(D)$ is actually determined by $\mathcal{P}(D)$ in every case.
Let $D$ be a Dedekind domain. If $\mathcal{I}(D)$ is the set of proper ideals of $D$, then the quotient from $\mathcal{F}(D)$ to $\text{Cl}(D)$ restricts to a map $\pi : \mathcal{I}(D) \to \text{Cl}(D)$, which is a monoid homomorphism (i.e., $\pi(IJ) = \pi(I) \cdot \pi(J)$). Moreover, $\pi$ is surjective since the class of $I$ coincide with the class of $dI$ for every $d \in D^\times$.

There is also a natural map $\psi_0 : \text{Cl}(D) \to \text{Cl}(D) \otimes \mathbb{Q}$, $g \mapsto \otimes 1$, from the class group to the $\mathbb{Q}$-vector space $\text{Cl}(D) \otimes \mathbb{Q}$; the map $\psi_0$ is a group homomorphism, and its kernel is the torsion subgroup $T$ of $\text{Cl}(D)$. By construction, the image $C$ of $\psi_0$ spans $\text{Cl}(D) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space.

Thus, we have a chain of maps

$$
\mathcal{I}(D) \xrightarrow{\pi} \text{Cl}(D) \xrightarrow{\psi_0} \text{Cl}(D) \otimes \mathbb{Q};
$$

we denote by $\psi$ the composition $\psi_0 \circ \pi$.

**Definition 4.1.** Let $\Delta \subseteq \text{Max}(D)$. A maximal ideal $Q$ is an almost inverse of $\Delta$ if there is a set $\{P_1, \ldots, P_n\} \subseteq \Delta$ (not necessarily nonempty) such that $Q \wedge P_1 \wedge \cdots \wedge P_n$ belongs to $\mathcal{P}(D)$. We denote the set of almost inverses of $\Delta$ as $\text{Inv}(\Delta)$.

Our aim is to characterize $\text{Inv}(\Delta)$ in terms of the map $\psi$; to do so, we use the terminology of ordered topological spaces (for which we refer the reader to, e.g., [4]). Given a $\mathbb{Q}$-vector space $V$ and a set $S \subseteq V$, the positive cone spanned by $S$ is

$$
\text{pos}(S) := \left\{ \sum_{i=1}^k \lambda_i v_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}, v_i \in S \right\};
$$

if $C = \text{pos}(S)$, we say that $C$ is positively spanned by $S$. Symmetrically, the negative cone is $\text{neg}(S) := -\text{pos}(S)$.

**Proposition 4.2.** Let $\Delta \subseteq \text{Max}(D)$. Then, $\text{Inv}(\Delta) = \psi^{-1}(\text{neg}(\psi(\Delta)))$.

**Proof.** Let $Q \in \text{Inv}(\Delta)$, and let $P_1, \ldots, P_n \in \Delta$ be such that $L := Q \wedge P_1 \wedge \cdots \wedge P_n \in \mathcal{P}(D)$. Then, there is a principal ideal $I = aD$ with radical $L$; thus, there are positive integers $e, f_1, \ldots, f_n > 0$ such that $I = Q^e P_1^{f_1} \cdots P_n^{f_n}$ (this holds also if $Q = P_i$ for some $i$). Since $I$ is principal, $\psi(I) = 0$; hence,

$$
0 = \psi(I) = \psi(Q^e P_1^{f_1} \cdots P_n^{f_n}) = e\psi(Q) + \sum_{i=1}^n f_i \psi(P_i).
$$

Solving in $\psi(Q)$, we see that $\psi(Q) = \sum_i -\frac{f_i}{e} \psi(P_i) \in \text{neg}(\psi(\Delta))$, as claimed.

Conversely, suppose that $\psi(Q)$ is in the negative cone. Then, either $\psi(Q) = 0$ (in which case $Q \in \text{Inv}(\Delta)$ by taking no $P \in \Delta$ in the definition) or we can find $P_1, \ldots, P_n \in \Delta$ and negative rational numbers
For any $q_1, \ldots, q_n$ such that $\psi(Q) = \sum q_i \psi(P_i)$. By multiplying for the minimum common multiple of the denominators of the $q_i$, we obtain a relation $e\psi(Q) + \sum f_i \psi(P_i) = 0$, with $e, f_i \in \mathbb{N}^+$. If $I := Q^e P_1^{f_1} \cdots P_n^{f_n}$, it follows that $\pi(I)$ is torsion in the class group, i.e., there is an $n > 0$ such that $I^n$ is principal; thus $\text{rad}(I^n) = \text{rad}(I) = Q \wedge P_1 \wedge \cdots \wedge P_n \in \mathcal{P}(D)$, as claimed. □

**Corollary 4.3.** Let $\Delta \subseteq \text{Max}(D)$. Then, $\text{Inv}(\Delta) = \text{Max}(D)$ if and only if $\psi(\Delta)$ positively spans $\text{Cl}(D) \otimes \mathbb{Q}$.

**Proof.** Suppose $\text{Inv}(\Delta) = \text{Max}(D)$, and let $q \in \text{Cl}(D) \otimes \mathbb{Q}$. Since $\mathcal{C}$ of $\psi$ generates $\text{Cl}(D) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space and is a subgroup, there is a $d \in \mathbb{N}^+$ such that $dq \in \mathcal{C}$. Hence, $dq = \psi(I)$ for some $I \in \mathcal{I}(D)$; factorize $I$ as $P_1^{e_1} \cdots P_n^{e_n}$, with $P_i \in \text{Max}(D)$ and $e_i > 0$. By Proposition 4.2, we have

$$\psi(I) = \sum e_i \psi(P_i) \in \sum e_i \text{neg}(\psi(\Delta)) = \text{neg}(\psi(\Delta)), $$

and thus also $q = \frac{1}{d} \psi(I) \in \text{neg}(\psi(\Delta))$. Hence, $\psi(\Delta)$ negatively spans $\text{Cl}(D) \otimes \mathbb{Q}$, and thus it also positively spans $\text{Cl}(D) \otimes \mathbb{Q}$.

Conversely, suppose $\psi(\Delta)$ positively spans $\text{Cl}(D) \otimes \mathbb{Q}$; thus, it also negatively spans $\text{Cl}(D) \otimes \mathbb{Q}$. Let $Q \in \text{Max}(D)$; then, $\psi(Q) \in \text{neg}(\psi(\Delta))$, so that $Q \in \text{Inv}(\Delta)$ by Proposition 4.2. Hence, $\text{Inv}(\Delta) = \text{Max}(D)$. □

We can now characterize when the rank of $\text{Cl}(D)$ is finite.

**Proposition 4.4.** Let $D$ be a Dedekind domain. Then, $\text{rk Cl}(D) < \infty$ if and only if there is a finite set $\Delta \subseteq \text{Max}(D)$ such that $\text{Inv}(\Delta) = \text{Max}(D)$.

**Proof.** Suppose first that $\text{rk Cl}(D) = n < \infty$. Then, $\text{Inv}(\text{Max}(D)) = \text{Max}(D)$, and thus $\psi(\text{Max}(D))$ positively spans $\text{Cl}(D) \otimes \mathbb{Q}$, by Corollary 4.3. Let $\{e_1, \ldots, e_n\}$ be a basis of $\text{Cl}(D) \otimes \mathbb{Q}$; then, each $e_i$ belongs to the positive cone spanned by a finite subset $\Lambda_i$ of $\psi(\text{Max}(D))$. Thus, the union $\Lambda$ of the $\Lambda_i$ is a finite set positively spanning $\text{Cl}(D) \otimes \mathbb{Q}$, so the corresponding subset $\Delta$ of $\text{Max}(D)$ is finite and $\text{Inv}(\Delta) = \text{Max}(D)$ by Corollary 4.3.

Conversely, suppose there is a finite set $\Delta = \{P_1, \ldots, P_k\} \subseteq \text{Max}(D)$ such that $\text{Inv}(\Delta) = \text{Max}(D)$. For every $Q \in \text{Max}(D)$, there are $i_1, \ldots, i_r$ such that $Q \wedge P_{i_1} \wedge \cdots \wedge P_{i_r} \in \mathcal{P}(D)$; as in the proof of Proposition 4.2, it follows that there are $r, f_1, \ldots, f_r > 0$ such that $Q^e P_{i_1}^{f_1} \cdots P_{i_r}^{f_r}$ is principal. It follows that $[Q] \otimes 1$ belongs to the $\mathbb{Q}$-vector subspace of $\text{Cl}(D) \otimes \mathbb{Q}$ generated by $P_{i_1} \otimes 1, \ldots, P_{i_r} \otimes 1$. Since $Q$ was arbitrary, the set $\{P_1 \otimes 1, \ldots, P_k \otimes 1\}$ is a basis of $\text{Cl}(D) \otimes \mathbb{Q}$. In particular, $\text{rk Cl}(D) = \dim_{\mathbb{Q}} \text{Cl}(D) \otimes \mathbb{Q} \leq k < \infty$. □

We will also need a criterion to understand when $\text{Inv}(\Delta)$ correspond to a linear subspace.
Proposition 4.5. Let \( \Delta \subseteq \text{Max}(D) \). Then, \( \text{neg}(\psi(\Delta)) \) is a linear subspace of \( \text{Cl}(D) \otimes \mathbb{Q} \) if and only if \( \Delta \subseteq \text{Inv}(\Delta) \).

Proof. Suppose \( \text{neg}(\psi(\Delta)) \) is a linear subspace, and let \( Q \in \Delta \). Then, there are \( P_i \in \Delta \), \( \lambda_i \in \mathbb{Q}^\times \) such that \( \psi(Q) = \sum \lambda_i \psi(P_i) \); multiplying by the minimum common multiple of the denominators we get an equality \( e\psi(Q) + \sum f_i \psi(P_i) = 0 \) where \( e, f_i \in \mathbb{N}^+ \). Let \( I := Q^e P_1^{f_1} \cdots P_n^{f_n} \); then, \( \psi(I) = 0 \), so that \( \pi(I) \) is torsion in \( \text{Cl}(D) \), i.e., \( I^n \) is principal for some \( n \). Thus, \( Q \wedge P_1 \wedge \cdots \wedge P_n \in \mathcal{P}(D) \), and \( Q \in \text{Inv}(\Delta) \).

Conversely, suppose \( \Delta \subseteq \text{Inv}(\Delta) \), and let \( q \) be an element of the linear subspace generated by \( \psi(\Delta) \). Then, there are \( P_i, Q_j \in \Delta \), \( \theta_i \in \mathbb{Q}^+ \) and \( \mu_j \in \mathbb{Q}^- \) such that

\[
q = \sum \theta_i \psi(P_i) + \sum \mu_j \psi(Q_j).
\]

By construction, each \( \theta_i \psi(P_i) \) belongs to \( \text{pos}(\psi(\Delta)) \). Furthermore, each \( \psi(Q_j) \) is in \( \text{neg}(\psi(\Delta)) \) by Proposition 4.4 and thus \( \mu_j \psi(Q_j) \in \text{pos}(\psi(\Delta)) \) for every \( j \). Therefore, \( q \in \text{pos}(\psi(\Delta)) \), so the positive cone of \( \psi(\Delta) \) is a linear subspace and \( \text{neg}(\psi(\Delta)) = \text{pos}(\psi(\Delta)) \) is a subspace too. \( \square \)

Proposition 4.5 can be interpreted by saying that \( \text{rk} \text{Cl}(D) \) is finite if and only if \( \text{Max}(D) \) is “negatively generated” by a finite set. In the case of finite rank, we need a way to link the dimension of \( \text{Cl}(D) \otimes \mathbb{Q} \) with the cardinality of the sets spanning it as a positive cone; that is, we need to consider a notion analogue to the basis of a vector space.

Since we need only to consider the case of finite rank, from now on we suppose that \( n := \text{rk} \text{Cl}(D) < \infty \), and we identify \( \text{Cl}(D) \otimes \mathbb{Q} \) with \( \mathbb{Q}^n \).

Definition 4.6. A set \( X \subseteq \mathbb{Q}^n \) is positive basis of \( \mathbb{Q}^n \) if \( \text{pos}(X) = \mathbb{Q}^n \) and if \( \text{pos}(X \setminus \{x\}) \neq \mathbb{Q}^n \) for every \( x \in X \).

Definition 4.7. A subset \( \Delta \subseteq \text{Max}(D) \) is an inverse basis of \( \text{Max}(D) \) if \( \text{Inv}(\Delta) = \text{Max}(D) \) and \( \text{Inv}(\Delta') \neq \text{Max}(D) \) for every \( \Delta' \subsetneq \Delta \).

These two notions are naturally connected.

Proposition 4.8. Let \( \Delta \subseteq \text{Max}(D) \). Then, \( \Delta \) is an inverse basis of \( \text{Max}(D) \) if and only if \( \psi(\Delta) \) is a positive basis of \( \mathbb{Q}^n \).

Proof. If \( \Delta \) is an inverse basis, then \( \psi(\Delta) \) positively spans \( \mathbb{Q}^n \) by Corollary 4.3, while \( \psi(\Delta') \) does not for every \( \Delta' \subsetneq \Delta \) (again by the corollary). Hence, \( \psi(\Delta) \) is a positive basis. The converse follows in the same way. \( \square \)

Given a positive basis \( X \) of \( \mathbb{Q}^n \), we call a partition \( \{X_1, \ldots, X_s\} \) of \( X \) a weak Reay partition if, for every \( j \), the positive cone of \( X_1 \cup \cdots \cup X_i \) is a linear subspace of \( \mathbb{Q}^n \). The following is a variant of [8, Theorem 2].

Proposition 4.9. Let \( X \) be a positive basis of \( \mathbb{Q}^n \). Then:
(a) every weak Reay partition of $X$ has cardinality at most $|X| - n$;
(b) there is a weak Reay partition of cardinality $|X| - n$.

Proof. Let $\{X_1, \ldots, X_s\}$ be a weak Reay partition, and let $V_i$ be the linear space spanned by $X_1, \ldots, X_i$ (with $V_0 := (0)$). We claim that $\dim V_i - \dim V_{i-1} \leq |X_i| - 1$. Indeed, let $X_i := \{z_1, \ldots, z_t\}$: then, $-z_t$ belongs to the positive cone generated by $V_{i-1}$ and $X_i$, and we can write $-z_t = y + \sum_j \lambda_j z_j$ for some $y \in V_{i-1}$ and $\lambda_j \geq 0$. Thus, $-(1 + \lambda_t)z_t = y + \lambda_1 z_1 + \cdots + \lambda_{t-1} z_{t-1}$, and since $\lambda_t \neq -1$ we have that $z_t$ is linearly dependent from $X_1 \cup \cdots \cup X_{i-1} \cup \{z_1, \ldots, z_{t-1}\}$. Hence, $\dim V_i \leq \dim V_{i-1} + t - 1$, as claimed.

Therefore,

$$n = \dim Q^n = (\dim V_s - \dim V_{s-1}) + \cdots + \dim V_i \leq (|X_s| - 1) + \cdots + (|X_1| - 1) = |X| - s,$$

and thus $s \leq |X| - n$, and (a) is proved. (b) is a direct consequence of [8, Theorem 2]. \hfill $\Box$

Similarly, if $\Delta \subseteq \Max(D)$ is an inverse basis of $\Max(D)$, we call a partition $\{\Delta_1, \ldots, \Delta_s\}$ a weak Reay partition if $\Delta_1 \cup \cdots \cup \Delta_i \subseteq \Inv(\Delta_1 \cup \cdots \cup \Delta_i)$ for every $i$.

Proposition 4.10. Let $\Delta \subseteq \Max(D)$ be an inverse basis of $\Max(D)$, and let $\{\Delta_1, \ldots, \Delta_s\}$ be a partition of $\Delta$. Then, $\{\Delta_1, \ldots, \Delta_s\}$ is a weak Reay partition of $\Delta$ if and only if $\{\psi(\Delta_1), \ldots, \psi(\Delta_s)\}$ is a weak Reay partition of $\psi(\Delta)$.

Proof. By Proposition 4.5, $\Delta_1 \cup \cdots \cup \Delta_i \subseteq \Inv(\Delta_1 \cup \cdots \cup \Delta_i)$ if and only if the positive cone of $\psi(\Delta_1 \cup \cdots \cup \Delta_i) = \psi(\Delta_1) \cup \cdots \cup \psi(\Delta_i)$ is a linear subspace of $Q^n$. The claim now follows from the definition. \hfill $\Box$

Theorem 4.11. Let $D, D'$ be Dedekind domains such that $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic. Then, $\rk \Cl(D) = \rk \Cl(D')$.

Proof. Let $\phi : \mathcal{P}(D) \rightarrow \mathcal{P}(D')$ be an isomorphism; by Theorem 3.6 we can find an isomorphism $\Phi : \Rad(D) \rightarrow \Rad(D')$ sending $\mathcal{P}(D)$ to $\mathcal{P}(D')$. In particular, $\Phi(\Max(D)) = \Max(D')$.

Since $\Inv(\Delta)$ is defined only through $\mathcal{P}(D)$ and $\Rad(D)$, $\Phi$ respects the inverse construction, in the sense that $\Phi(\Inv(\Delta)) = \Inv(\Phi(\Delta))$ for every $\Delta \subseteq \Max(D)$. In particular, $\Inv(\Delta) = \Max(D)$ if and only if $\Inv(\Phi(\Delta)) = \Max(D')$; by Proposition 4.4 it follows that $\rk \Cl(D) = \infty$ if and only if $\rk \Cl(D') = \infty$.

Suppose now that the two ranks are finite, say equal to $n$ and $n'$ respectively. Let $\Delta \subseteq \Max(D)$ be an inverse basis of $\Max(D)$. Let $\{\Delta_1, \ldots, \Delta_s\}$ be a weak Reay partition of $\Delta$ of maximum cardinality; by Propositions 4.10 and 4.9 $s = |\Delta| - n$.

Every weak Reay partition of $\Delta$ gets mapped by $\Phi$ into a weak Reay partition of $\Delta' := \psi(\Delta)$, and conversely; therefore, the maximum
cardinality of the weak Reay partitions of $\Delta'$ is again $|\Delta| - n$. However, applying Propositions 4.10 and 4.9 to $\Delta'$ we see that this quantity is $|\Delta'| - n'$; since $|\Delta| = |\Delta'|$, we get $n = n'$, as claimed.

\begin{corollary}
Let $D, D'$ be Dedekind domains, and let $T(D)$ (respectively, $T(D')$) be the torsion subgroup of $\text{Cl}(D)$ (resp., $\text{Cl}(D')$). If $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic and if $\text{Cl}(D)$ and $\text{Cl}(D')$ are finitely generated, then $\text{Cl}(D)/T(D) \simeq \text{Cl}(D')/T(D')$.

\begin{proof}
Since $\text{Cl}(D)$ is finitely generated, it has finite rank $n$ and $\text{Cl}(D)/T(D) \simeq \mathbb{Z}^n$; analogously, $\text{Cl}(D')/T(D') \simeq \mathbb{Z}^m$, where $m := \text{rk}\ Cl(D')$. By Theorem 4.11, $n = m$, and in particular $\text{Cl}(D)/T(D) \simeq \text{Cl}(D')/T(D')$.
\end{proof}
\end{corollary}

5. Counterexamples

In this section, we collect some examples showing that Theorem 4.11 is, in many ways, the best possible.

\begin{example}
It is not possible to improve the conclusion of Theorem 4.11 from “$\text{rk}\ Cl(D) = \text{rk}\ Cl(D')$” to “$\text{Cl}(D) \simeq \text{Cl}(D')$”. Indeed, if $\text{rk}\ Cl(D) = 0$ (i.e., if $\text{Cl}(D)$ is torsion) then $\mathcal{P}(D) = \text{Rad}(D)$, and thus whenever $\text{rk}\ Cl(D) = \text{rk}\ Cl(D') = 0$ the posets $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic.

For the next examples, we need to use a construction of Claborn [3]. Let $G := \sum_i x_i \mathbb{Z}$ be the free abelian group on the countable set $\{x_i\}_{i \in \mathbb{N}}$. Let $I$ be a subset of $G$ satisfying the following two properties:

- all coefficients of the elements of $I$ (with respect to the $x_i$) are nonnegative;
- for every finite set $x_{i_1}, \ldots, x_{i_k}$ and every $n_1, \ldots, n_k \in \mathbb{N}$ there is an element $y$ of $I$ such that the component of $y$ relative to $x_{i_t}$ is $n_t$.

Then, [3, Theorem 2.1] says that there is an integral domain $D$ with countably many maximal ideals $\{P_i\}_{i \in \mathbb{N}}$ such that the map sending the ideal $P_1^{n_1} \cdots P_k^{n_k}$ to $n_1x_1 + \cdots + n_kx_k$ sends principal ideals to elements of the subgroup $H$ generated by $I$. In particular, $\text{Cl}(D) \simeq G/H$.

\begin{example}
Corollary 4.12 does not hold without the hypothesis that $\text{Cl}(D)$ and $\text{Cl}(D')$ are finitely generated.

For example, let $H_1$ be the subgroup of $G$ generated by $x_n + x_{n+1}$, as $n$ ranges in $\mathbb{N}$, and $I_1$ to be the subset of the elements of $H_1$ having all coefficients nonnegative. Then, $I_1$ satisfies the above conditions; the corresponding domain $D_1$ has a class group isomorphic to $\mathbb{Z}$, and its prime ideals are concentrated in two classes: if $n$ is even $P_n$ is equivalent to $P_0$, if $n$ is odd $P_n$ is equivalent to $P_1$, and $P_0P_1$ is principal. (This is exactly Example 3-2 of [3].) In particular, $\mathcal{P}(D_1)$ is equal to the members of $\text{Rad}(D_1)$ that are contained both in some $P_n$ with $n$ even and in some $P_m$ with $m$ odd.
Let now $H_2$ to be the subgroup of $G$ generated by $x_n + 2x_{n+1}$, as $n$ ranges in $\mathbb{N}$, and let $I_2$ be the subset of the elements of $H_2$ having all coefficients nonnegative. Then, $I_2$ too satisfies the condition above. Let $D_2$ be the corresponding Dedekind domain. Then, $\text{Cl}(D_2)$ is isomorphic to the quotient $G/H_2$, which is isomorphic to the subgroup $\mathbb{Z}(2^\infty)$ of $\mathbb{Q}$ generated by $1, \frac{1}{2}, \frac{1}{4}, \ldots$ (that is, to the Prüfer 2-group): this can be seen by noting that the map

\[ G \to \mathbb{Q}, \]

\[ P_n \mapsto (-1)^n \frac{1}{2^n} \]

is a group homomorphism with kernel $H_2$ and range $\mathbb{Z}(2^\infty)$. In this isomorphism, the prime ideals $Q_n$ with $n$ even are mapped to positive elements of $\mathbb{Z}(2^\infty)$, while the prime ideals $Q_m$ with $m$ odd are mapped to the negative elements. Hence, $\mathcal{P}(D_2)$ is equal to the member of $\text{Rad}(D_2)$ that are contained in both an “even” and an “odd” prime.

Therefore, the map $\text{Rad}(D_1) \to \text{Rad}(D_2)$ sending $P_{i_1} \cap \cdots \cap P_{i_k}$ to $Q_{i_1} \cap \cdots \cap Q_{i_k}$ is an isomorphism sending $\mathcal{P}(D_1)$ to $\mathcal{P}(D_2)$. However, the class groups of $D_1$ and $D_2$ are both torsionfree (i.e., $T(D_1) = T(D_2) = 0$) but not isomorphic.

**Example 5.3.** The converse of Theorem 4.11 does not hold; that is, it is possible that $\text{rk Cl}(D) = \text{rk Cl}(D')$ even if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are not isomorphic.

Take $H_1$ and $D_1$ as in the previous example.

Take $H_3$ to be the subgroup of $G$ generated by $x_0$ and by $x_n + x_{n+1}$ for $n > 0$, and let $I_3$ be the subset of the elements of $H_3$ having all coefficients nonnegative. Then, $I_3$ satisfies Claborn’s conditions, and the corresponding domain $D_3$ satisfies $\text{Cl}(D_3) \simeq \mathbb{Z}$ (in particular, $\text{rk Cl}(D_3) = 1$), so $\text{Cl}(D_1)$ and $\text{Cl}(D_3)$ are isomorphic.

However, $D_3$ has a principal maximal ideal (the one corresponding to $x_0$), while $D_1$ does not. Therefore, there is no isomorphism $\text{Rad}(D_1) \to \text{Rad}(D_3)$ sending $\mathcal{P}(D_1)$ to $\mathcal{P}(D_3)$; by Theorem 3.6 it follows that $\mathcal{P}(D_1)$ and $\mathcal{P}(D_3)$ cannot be isomorphic.

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**Dipartimento di Matematica e Fisica, Università degli Studi “Roma Tre”, Roma, Italy**

*E-mail address*: spirito@mat.uniroma3.it