Global linear-irreversible principle for optimization in finite-time thermodynamics

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Abstract – There is intense effort into understanding the universal properties of finite-time models of thermal machines —at optimal performance— such as efficiency at maximum power, coefficient of performance at maximum cooling power, and other such criteria. In this letter, a global principle consistent with linear irreversible thermodynamics is proposed for the whole cycle —without considering details of irreversibilities in the individual steps of the cycle. This helps to express the total duration of the cycle as \( \tau \propto \bar{Q}^2 / \Delta_{\text{tot}}S \), where \( \bar{Q} \) models the effective heat transferred through the machine during the cycle, and \( \Delta_{\text{tot}}S \) is the total entropy generated. By taking \( \bar{Q} \) in the form of simple algebraic means (such as arithmetic and geometric means) over the heats exchanged by the reservoirs, the present approach is able to predict various standard expressions for figures of merit at optimal performance, as well as the bounds respected by them. It simplifies the optimization procedure to a one-parameter optimization, and provides a fresh perspective on the issue of universality at optimal performance, for small difference in reservoir temperatures. As an illustration, we compare the performance of a partially optimized four-step endoreversible cycle with the present approach.

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Introduction. – As the demand of human civilization for useful energy grows, it becomes more urgent to understand and improve the performance of our energy-conversion devices. Currently, there is a lot of interest in characterizing the optimal performance of machines operating in finite-time cycles [1–20]. As a paradigmatic model, a heat cycle is studied between two heat reservoirs at temperatures \( T_h \) and \( T_c \) (\(< T_h \)), whose performance may be optimized using a specific objective function, such as: power output [1], cooling power [16], certain trade-off functions between energy gains and losses [19,21], and net entropy generation [22]. An important quantity in this regard is the figure of merit, such as efficiency, in the case of heat engines, and the coefficient of performance (COP) in the case of refrigerators. Notably, the bounds on their values predicted by simple, thermodynamic models, provide a benchmark for the observed performance of real power plants [1,4,8,10,23].

A major focus has been to understand whether these figures of merit display universal properties at optimal performance. For example, the efficiency at maximum power (EMP) is often found to be equal to, or closely approximated by the elegant expression, known as Curzon-Ahlborn (CA) efficiency [1,24–26]

\[ \eta_{\text{CA}} = 1 - \sqrt{1 - \eta_C}, \]

where \( \eta_C = 1 - T_c / T_h \) is the Carnot efficiency. Other expressions for EMP have also been obtained from different models [3,6–8,12,27,28], some of them sharing a common universality with CA efficiency, i.e., for small differences in reservoir temperatures, EMP can be written as \( \eta = \eta_C / 2 + \eta_C^2 / 8 + O(\eta_C^3) \). The first-order term arises with strong coupling under linear irreversible thermodynamics (LIT) [6], while the second-order term is beyond linear response, and has been related to a certain symmetry property in the model [28,29].

The standard analysis often involves solving a two-parameter optimization problem over, say, the pair of intermediate temperatures of the working medium [1,3], or, the time intervals of contacts with reservoirs [7,8]. In this letter, I formulate a simpler optimization problem for

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finite-time machines. While simplicity might lead to a certain loss of predictive power, the generality and unifying power of the proposed framework are remarkable. Here, rather than applying LIT locally, say, at each thermal contact, I assume a global validity of this principle, i.e., for the complete cycle. Accordingly, we need not assume stepwise details of the cycle, but may simply consider the machine as an irreversible channel with an effective (thermal) conductivity $\lambda$. Quite remarkably, we will recover many of the well-known expressions of the figures of merit for both engine and refrigerator modes, indicating a very different origin for them which is independent of the physical model, or the processes assumed in regular models. In this sense, the present approach brings together the results from different models under one general principle consistent with LIT.

Now, according to LIT [30,31], the rate of entropy generation is $\dot{S} = \sum_{\alpha} q_{\alpha} F_{\alpha}$, the sum of products of each flux $q_{\alpha}$ and its associated thermodynamic force, or affinity $F_{\alpha}$. In the simple case of a heat flux $q$ between two heat reservoirs, the corresponding thermodynamic forcing may be defined as $F = T_{c}^{-1} - T_{h}^{-1}$. Then, assuming a linear flux-force relation, $q = \lambda F$, where $\lambda$ is the heat transfer coefficient, and the bilinear form for the rate of entropy generation, we can write $\dot{S} = q F = q^{2}/\lambda$. Now, if the time interval of heat flow is considered long enough, then the flux may be approximated to be constant over this interval. So the amount of heat transferred within time $\tau$ is $Q = q \tau$, which implies $\dot{S} = Q^{2}/\lambda \tau^{2}$. Then, the cyclic operation of the machine can be based on the following two assumptions:

i) There is an effective heat flux ($\bar{q}$) through the machine over one cycle, and the total entropy generation per cycle obeys principles of LIT.

ii) $\bar{q}$ is determined by an effective mean value of the heat $Q$ passing from the hot to the cold reservoir in total cycle time $\tau$. Therefore, $\bar{q} = Q/\tau$. Assumption i) implies, $\dot{S}_{\text{tot}} = \bar{q}^{2}/\lambda$. Then, the total entropy generation per cycle is $\Delta_{\text{tot}} S = \dot{S}_{\text{tot}} \tau = \bar{q}^{2} \tau /\lambda$. From assumption ii), we have $\Delta_{\text{tot}} S = Q^{2}/\lambda \tau$. In other words, we can express the total period as

$$\tau = \frac{Q^{2}}{\lambda \Delta_{\text{tot}} S}. \quad (2)$$

**Optimal power output.** Now, to motivate the optimization procedure, we first optimize the power output of a heat engine. Let $Q_{h}$ and $Q_{c}$ be the amounts of heat exchanged by the working medium with the hot ($h$) and cold ($c$) reservoirs. Let $W = Q_{h} - Q_{c}$ be the total work output in a cycle of duration $\tau$. The total entropy generation per cycle is

$$\Delta_{\text{tot}} S = \frac{Q_{h}}{T_{h}} + \frac{Q_{c}}{T_{c}} > 0. \quad (3)$$

Then the average power output of the cycle is defined as $P = W/\tau$. Using eq. (2), we have

$$P = \lambda (Q_{h} - Q_{c}) \frac{\Delta_{\text{tot}} S}{Q^{2}}. \quad (4)$$

Introducing the parameters $\nu = Q_{c}/Q_{h}$ and $\theta = T_{c}/T_{h}$, and using eq. (3), we can define a dimensionless power function:

$$\tilde{P} \equiv \frac{T_{c} P}{\lambda} = (1 - \nu)(\nu - \theta) \frac{Q_{h}^{2}}{Q^{2}}. \quad (5)$$

For engines, from the positivity of $\Delta_{\text{tot}} S$, we have $\nu > \theta$. Further, knowledge of a specific value of $\lambda$ is not relevant for performing the optimization. However, the above target function is still not in a useful form, until we specify the form of $Q$. We assume $Q$ to be bounded as $Q_{c} \leq Q \leq Q_{h}$. Let us analyze these two limits separately.

$L_{h}$) When $Q \to Q_{h}$, eq. (5) is simplified to $\tilde{P} = (1 - \nu)(\nu - \theta)$, whose optimal value is obtained at $\nu = 2\theta/(1 + \theta)$, or the EMP in this case is $\eta_{h} = \eta c/\nu_{c}$. Again, this formula is obtained as the upper bound for efficiency [3,7,8,12,28], when the dissipation at the cold contact is much more important than at the hot contact [3,7,8,12,28]. In our model, this limit implies that when the effective amount of heat passing through the machine approaches the heat absorbed from the hot reservoir, then the efficiency approaches its lower bound.

$L_{c}$) When $Q \to Q_{c}$, then $P \to (1 - \nu)(\nu - \theta)w^{2}$, whose optimal value is obtained at $\nu = 2\theta/(1 + \theta)$, or the EMP in this case is $\eta_{h} = \eta c/(2 - \eta_{c})$. Again, this formula is obtained as the upper bound for efficiency [3,7,8,12,28], when the dissipation at the hot contact approaches the reversible limit, or, is negligible in comparison to dissipation at the hot contact. In our model, $Q \to Q_{c}$ implies that as the effective heat through the machine reaches its lowest possible value $Q_{c}$, the efficiency approaches its upper bound.

Now, it seems natural to assume that, in general, $\tilde{Q} \equiv \dot{Q}(\theta, \nu)$ may be taken as a mean value [32], interpolating between $Q_{c}$ and $Q_{h}$. In the following, we explore consequences of making some simple choices of these mean values. It will be seen that the mean being a homogenous function of the first degree in its arguments, implies the condition $\dot{Q}(\theta, \nu) = Q_{c} \dot{Q}(1, \nu)$, and so the maximization of the power output is reduced to a simple one-parameter optimization problem.

Let $\dot{Q}$ be given by a weighted arithmetic mean: $\dot{Q} = \omega Q_{h} + (1 - \omega)Q_{c}$, where the weight $0 \leq \omega \leq 1$. With this form, eq. (5) is explicitly given by

$$\tilde{P}(\nu) = \frac{(1 - \nu)(\nu - \theta)}{[\omega + (1 - \omega)\nu]^{2}}. \quad (6)$$

Then the optimum of $\tilde{P}$ is obtained at $\nu = [2\theta + \omega (1 - \theta)] / [1 + \theta + \omega(1 - \theta)]$. The maximal nature of the optimum can be ascertainment: $\partial^{2} P / \partial \nu^{2} |_{\nu = \tilde{\nu}} < 0$. As a result, EMP, $\tilde{\eta} = 1 - \tilde{\nu}$, is found to be

$$\tilde{\eta} = \frac{\eta c}{2 - (1 - \omega)\eta_{c}}. \quad (7)$$

The above form has been obtained in refs. [3,7,8,12,28], where the parameter $\omega$ may be determinable, for example,
in terms of the ratio of the dissipation constants or thermal conductivities of the thermal contacts [3,8]. In the present approach, the parameter \( 0 \leq \omega \leq 1 \) is undetermined. In the absence of additional information, one may choose equal weights (\( \omega = 1/2 \)), or the symmetric arithmetic mean \( \bar{Q} = (Q_h + Q_c)/2 \). This results in the so-called Schmiedel-Seifert (SS) efficiency \( \eta_{SS} = 2\eta_C/(4 - \eta_C) \) [7,33].

As an alternative choice, if we set \( \bar{Q} = \sqrt{Q_h Q_c} \), i.e., the geometric mean of \( Q_h \) and \( Q_c \), then we obtain \( \tilde{P} = (1 - \nu)(\nu - \theta)/\nu \), which becomes optimal at CA efficiency, \( \tilde{\eta} = \eta_{CA} \). Further, we may use a generalized, symmetric mean defined as \( \bar{Q} = [(Q_h^r + Q_c^r)/2]^{1/r} \), where \( r \) is a real parameter [32,34]. Special cases with \( r = -1, 0, 1, 2 \) correspond respectively to harmonic, geometric, arithmetic, and quadratic means. The dimensionless power output is then given by

\[
\tilde{P}(\nu) = \frac{(1 - \nu)(\nu - \theta)}{\left(1 + \nu^r/2\right)^{2/r}},
\]

whose optimum is determined by the following condition:

\[
(1 + \theta)^{\nu^r - 2\theta} - 2\theta^{\nu^r - 1} + 2\nu^r - (1 + \theta) = 0.
\]

The above equation cannot be analytically solved for \( \nu = 1 - \nu_r \), with general \( r \). However, it can be shown from (9) that \( \partial\nu^r/\partial r > 0 \), which implies that the EMP \( \tilde{e}_r \) decreases monotonically with increasing \( r \). In particular, when \( r \to +\infty (\rightarrow -\infty) \), then \( \bar{Q} \to Q_h (Q_c) \) and so we have \( \tilde{\eta}_{\infty} \to \eta_{c}/2 \), and \( \tilde{\eta}_{-\infty} \to \eta_{c}/(2 - \eta_C) \), the lower and upper bounds of EMP discussed earlier. Incidentally, this helps to notice that CA efficiency \( (r \to 0) \) is higher than SS efficiency \( (r = 1) \), for a given \( \theta \) (see fig. 1).

For small numerical values of \( r \), we can look into the general form of efficiency by assuming \( \tilde{\eta}_r = a_1\eta_C + a_2\eta_C^2 + a_3\eta_C^3 \), close to equilibrium, where we have \( \eta_C \) as the small parameter. Substituting this form in eq. (9), and keeping terms up to \( \mathcal{O}(\eta_C^3) \), we determine the coefficients \( a_1 \). As a result, for small temperature differences, we have

\[
\tilde{\eta}_r = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{2 - r}{32} \eta_C^3.
\]

Thus, we observe that with some well-known symmetric means, the first two terms in the expansion of EMP have the same universal coefficients as earlier were obtained within LIT [6,29]. Interestingly, the third-order term in eq. (10) also matches with a similar expansion of EMP recently found for Carnot engines within a perturbative approach for open quantum systems [35]. Thereby, the parameter \( r \) is analogous to the frequency scaling exponent of the spectral density of the heat reservoirs.

So, assuming the global validity of LIT for a heat cycle, along with a specific form of the effective heat transferred between the two heat reservoirs, leads to a one-parameter optimization problem, whereby the EMP matches with the well-known expressions predicted by more elaborate models. This is the main result of the present letter. In the following, we extend the analysis to other target functions, and derive various known forms of the figures of merit.
In this case, for small temperature differences, the leading order terms are given as $\eta = 3\eta_C/4 + \eta_C^2/32 + O(\eta_C^3)$. The exact behavior is plotted as the dashed line within the inset, in fig. 1.

Next, we consider the operation of a heat cycle as a refrigerator. For refrigerators, the coefficient of performance (COP) is given by $\xi = Q_c/W = \nu/(1 - \nu)$, where $\nu = Q_c/Q_h$, and the Carnot coefficient is $\xi_C = \theta/(1 - \theta)$.

Here, $Q_c$ is the heat extracted from the cold reservoir and driven into the hot reservoir by an input of $W$ amount of work, in a cycle of time period $\tau$. The total entropy generation per cycle is
\[ \Delta_{\text{tot}}S = \frac{Q_h}{T_h} - \frac{Q_c}{T_c} > 0. \]  

Now, for irreversible refrigerators, the choice of optimization criteria may be analogous to heat engines, is not straightforward. For instance, the cooling power or the rate of refrigeration, $Q_c/\tau$, may not have an optimum for certain models [16,39]. Below, we illustrate the optimization with some of the proposed choices.

**Optimal $\chi$-criterion.** The so-called $\chi$-criterion is defined as [11,39] $\chi = \xi Q_c/\tau$. This criterion simultaneously considers the COP and the cooling power, due to a certain complementarity between the two quantities, i.e., if we maximize one, it minimizes the other. Defining, $\bar{\chi} = T_c\chi/\lambda$, we have
\[ \bar{\chi} = \frac{\nu^2(\theta - \nu)}{(1 - \nu)[\omega + (1 - \omega)]^2}. \]  

Note that for an irreversible refrigerator, we have $\theta > \nu$. Then, setting $\partial \bar{\chi}/\partial \nu = 0$, we can obtain the COP at optimal performance:
\[ \bar{\xi} = \frac{\sqrt{\omega}}{2} \left( \frac{\sqrt{9\omega + 8\xi_C} - 3\sqrt{\omega}}{1 + \sqrt{\omega}} \right). \]  

As $0 \leq \omega \leq 1$, it implies the bounds on COP as
\[ 0 \leq \bar{\xi} \leq \frac{1}{2} \left( \sqrt{9 + 8\xi_C} - 3 \right), \]
which have been earlier obtained in the literature, from a two-parameter optimization procedure. Note that, here we obtain a simple closed form for $\bar{\xi}$, which has not been possible in the low-dissipation approach of ref. [11].

**Optimal cooling power.** The target function for a refrigerator may be chosen as the cooling power $Z = Q_c/\tau$, or in dimensionless form
\[ \bar{Z} = \frac{T_c Z}{\lambda} = \frac{\nu(\theta - \nu)}{[\omega + (1 - \omega)]^2}. \]  

Then, setting $\partial \bar{Z}/\partial \nu = 0$, we obtain the COP at optimal $Z$ to be
\[ \bar{\xi} = \frac{\omega \xi_C}{2\omega + \xi_C}. \]  

The above expression is exactly the one derived in ref. [16] for the so-called exoreversible refrigerator, with the consequent bounds on COP as $0 \leq \xi \leq \xi_C/(2 + \xi_C)$.

When the ecological or the trade-off criterion for the refrigerator [37] is implemented as $\Omega_R = (2\xi - \xi_C)W/\tau$, with $Q$ as a geometric mean, then we obtain the COP at optimal $\Omega_R$ [37,40], as
\[ \bar{\xi} = \frac{\xi_C}{\sqrt{(1 + \xi_C)(2 + \xi_C)^2} - \xi_C}. \]  

Similarly, it can be seen that for a refrigerator and with geometric mean, we obtain $\bar{\xi} = \nu(\theta - \nu)/(1 - \nu)$, whose optimal value is obtained at $\xi = \sqrt{1 + \xi_C} - 1$ [10]. On the other hand, the cooling power does not have an optimum if the geometric mean is used.

**An illustration.** In the following, we compare the above effective approach with the optimization of power output in a four-step cycle within the so-called endoreversible approximation [1,39]. The latter implies that the only sources of irreversibilities during the cycle happen to be the thermal contacts with the heat reservoirs, when heat is exchanged between reservoirs and the working medium across a finite heat conductance. Consequently, there are two such thermal steps, occurring with time intervals $t_h$ and $t_c$. Further, the working medium is assumed to maintain a fixed temperature during a specific thermal contact, which we assume to be $T_1$ and $T_2$ during hot and cold contact, respectively ($T_h > T_1 > T_2 > T_c$). The other two steps of the cycle are adiabatic in nature—preserving the entropy of the working medium—and are assumed to occur with a negligible time interval.

Now, the problem of power optimization for the above model involves two variables, which may be conveniently chosen as the two temperatures of the working medium. Equivalently, we may choose another pair of variables as discussed below. Consider the heat flux between the working medium and the respective reservoir to be given by [3]
\[ q_h = \alpha_h (T_1^{-1} - T_h^{-1}), \]
\[ q_c = \alpha_c (T_2^{-1} - T_c^{-1}), \]
where $\alpha_j > 0$, with $j = c,h$ are the heat transfer coefficients. As the flux is assumed to be constant during the thermal contact, so the amounts of heat transferred during the times $t_h$ and $t_c$, respectively, are $Q_h = qLt_h$ and $Q_c = qLt_c$. Further, the cyclicality within the working medium implies $Q_h/T_1 = Q_c/T_2$, also known as the endoreversibility condition.

The extracted work per cycle is $W = Q_h - Q_c$, with thermal efficiency equal to
\[ \eta = \frac{W}{Q_h} = 1 - \frac{T_2}{T_1} \equiv 1 - \nu. \]  

Then, the average power per cycle is defined as
\[ P = \frac{Q_h - Q_c}{t_h + t_c}. \]
which can be expressed as a function of $T_1$ and $T_2$, or equivalently (from eq. (24)), as a function of $T_1$ and $\nu$: $P \equiv P(T_1, \nu)$. Rather than optimizing the power with respect to the two variables, let us consider a partial optimization, by setting $(\partial P/\partial T_1)_\nu = 0$, which yields the optimum value of $T_1$:

$$\tilde{T}_1 = \frac{K + \nu^{-1}}{K T_1^{-1} + T_c^{-1}},$$

(26)

where $K = \sqrt{\alpha_h/\alpha_c}$. Using the above value, we obtain $P(\nu) = P(\tilde{T}_1, \nu)$, as

$$P(\nu) = \frac{\alpha_h(1 - \nu)(\nu - \theta)}{T_c \left[1 + (K^\nu)^2\right]},$$

(27)

which can be rewritten as

$$P(\nu) = \frac{\alpha_h \alpha_c}{T_c (\sqrt{\alpha_h} + \sqrt{\alpha_c})^2} \frac{(1 - \nu)(\nu - \theta)}{\omega + (1 - \omega)^2},$$

(28)

where $\omega = (1 + K)^{-1}$. We identify $\lambda = \alpha_h \alpha_c (\sqrt{\alpha_h} + \sqrt{\alpha_c})^{-2}$, and so we can compare $T_c P(\nu)/\lambda$ in the above with the reduced power in eq. (6). Thus we note that the partially optimized power output in the endoreversible model with the inverse-temperature law, is equivalent to our effective model that employs a weighted arithmetic mean for the effective transferred heat. Similarly, it can be shown that within the endoreversible model under the assumption of Newtonian heat transfer [1,3],

$$q_h = \alpha_h' (T_h - T_1),$$

(29)

$$q_c = \alpha_c' (T_2 - T_c),$$

(30)

where $\alpha_i' > 0$ are thermal conductivities, the partially optimized power is obtained in a form that implies the effective heat as a geometric mean $\bar{Q} = \sqrt{q_h q_c}$, with the effective value of $\lambda = T_h (T_h - T_1) (\sqrt{\alpha_h} + \sqrt{\alpha_c})^{-2}$.

**Concluding remarks and outlook.**—An effective framework has been proposed for a class of thermal machines based on LIT, that makes a novel use of algebraic means ([41] and references therein) to model the effective heat transferred in a cycle. Quite surprisingly, the method reproduces well-known bounds and expressions for figures of merit —both for the engines and for the refrigerators, without incorporating details of a specific heat cycle. This has been illustrated for various optimization criteria. These general results arising from a simple framework, thereby suggest a universal character of these figures of merit.

The approach also provides a fresh perspective on the issue of universality of EMP near equilibrium [6,28,29], as in eq. (10). The first-order term $(\nu c_2/2)$ is universal, for any mean $\bar{Q}$, with extremal values of $Q_h$ and $Q_c$. Then, the universality of the second-order term can be related to the property of the mean $\bar{Q}$ being symmetric. Further, an agreement up to the third-order term has been seen within an open quantum systems framework. Thus, the present approach could be relevant for the optimal regimes of quantum heat engines, where such efficiencies and bounds have been recently derived [35].

It is desirable that the simple approach proposed above can be generalized to include more realistic scenarios, such as allowing for heat leaks between the reservoirs, finite sizes of heat source/sink, and the nonlinear regime. In particular, it is important to understand the physical reason as to the use of different means leading to varied expressions for the figures of merit. Here, the comparison with the partially optimized endoreversible cycle can provide a useful insight. Finally, it will be good if the proposed approach encourages a more global perspective while regarding the operation of machines and their impact on our environment.

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