Abstract—Given a plant subject to delayed sensor measurement, there are several approaches to compensate for the delay. An obvious approach is to address this problem in state space, where the n-dimensional plant state is augmented by an N-dimensional (Padé) approximation to the delay, affording (optimal) state estimate feedback vis-à-vis the separation principle. Using this framework, we show: (1) Feedback of the estimated plant states partially inverts the delay; (2) The optimal (Kalman) estimator decomposes into N (Padé) uncontrollable states, and the remaining n eigenvalues are the solution to a reduced-order Kalman filter problem. Further, we show that the tradeoff of estimation error (of the full state estimator) between plant disturbance and measurement noise, only depends on the reduced-order Kalman filter (that can be constructed independently of the delay); (3) A subtly modified version of this state-estimation-based control scheme bears close resemblance to a Smith predictor. This modified state-space approach shares several limitations with its Smith predictor analog (including the inability to stabilize most unstable plants), limitations that are alleviated when using the unmodified state estimation framework.

Index Terms—Riccati equations, state estimation, state space model, Smith predictor, time delay.

I. INTRODUCTION

Time delay is ubiquitous in chemical processes, biological systems, and industrial applications. It is widely known that in biological systems delay is inevitable in the sensory feedback simply because neural transduction of information is slow [1], [2]. For example, visuomotor delays introduce about 110-160 ms into the feedback loop [3]–[5]. However, despite these long latencies, humans and other animals manage smooth and accurate movements with ease. So a fundamental question in neuroscience is how the brain compensates for such delays [1], [5]–[7]. It is believed that our nervous system could provide predictive estimation and control through internal models of the plant, sensor dynamics, and delay, in order to compensate for the delayed feedback.

Two common approaches for model-based delay compensation are Smith-Predictor-like and state-observer-based controllers. Smith Predictors rely on an accurate model of plant and delay and then, save for (most) unstable plants, the controller can be designed without consideration of the time delay [8]. There are many improvements of the basic Smith Predictor architecture, most notably those that could extend to unstable plants [9], [10]. The state observer based method can also compensate for time delay in the feedback loop [7], [11].

Several studies have suggested that the human predictive controller may be modeled as a Smith Predictor [5], [12], [13], while other studies examines state-observer approaches [6], [7], [11]. Indeed, there are structural similarities between these two approaches; for example, Mirkin and Raskin [14] showed that for continuous-time systems every stabilizing time-delay controller has an observer–predictor-based structure. For discrete-time systems with input delay, Mirkin and Zanotto [15] showed that the discrete equivalent of the observer-predictor architecture can be derived via classical state-feedback and observer design. They further characterized the closed loop eigenvalues in the event that the state feedback is LQR optimal by showing that they either lie at the origin or arise from a lower order LQR problem that does not involve the delay.

In the present paper, we study continuous-time systems with a delay at the output that we model using a Padé approximation [16]–[18], and consider a state feedback that depends only on the plant states combined with an estimator for both plant and Padé states. We show in Section III that the poles of the Padé approximation will appear as transmission zeros of the observer-based compensator, thus providing phase lead that partially masks the lag contributed by the delay. We also describe a tradeoff filter between the response of the estimation error to measurement noise, on the one hand, and disturbances and mismatch between the Padé approximation and the time delay, on the other. In Section III we assume that the observer is a steady-state Kalman filter and show that the optimal estimator has uncontrollable eigenvalues equal to those of the Padé approximation. The remaining eigenvalues may be found from a lower order estimation problem that ignores the time delay, and the tradeoff filter is also independent of the delay. We exploit the special structure of the optimal estimator in Section IV by proposing an alternate control architecture together with conditions under which it is stabilizing. This alternate control architecture has properties that are remarkably similar to those of the Smith predictor. We explore these similarities in Section V and discuss further research directions and potential connections to neuroscience in Section VI. Appendix II contains proofs of some of the technical results from the body of the paper. In Appendix III we present the counterpart to the results of Section III for continuous-time systems with a delay at the plant input. We study discrete-time systems with an output delay in Appendices III and IV. We also describe the connections between our work and the work of Mirkin et al. [15] in Appendix IV.
Notation

We let $I_n$ and $0_n$ denote the $n \times n$ identity and zero matrices, respectively, with the subscript suppressed if $n = 1$. Also, we let $0_{n \times m}$ denote an $n \times m$ matrix of zeros, and $I_{n \times m}$ denote a block matrix whose upper left hand block is the identity matrix $I_{\min(m,n)}$ and whose remaining entries are zero. Finally, let $e_i$ denote the $i$th standard basis vector.

II. FEEDBACK OF ESTIMATED PLANT STATES PARTIALLY INVERTS DELAY

In this section, we describe a standard, state-estimate feedback controller in which the observer includes a Padé approximation of the delay. This approach performs delay compensation, in the sense that the estimated plant states partially invert the delay.

Consider the single input, single output linear system

$$\dot{x} = Ax + Bu + Ed, \quad y = Cx, \quad x \in \mathbb{R}^n,$$

(1)

and denote the transfer function from $u$ to $y$ by $G(s) = C(sI_n - A)^{-1}B + D$. Then that from $d$ to $y$ by $G_d(s) = C(sI_n - A)^{-1}E$. Assume that $(A, C)$ is observable and that $(A, B)$ and $(A, E)$ are controllable.

Suppose that the measurement is delayed by $\tau$ seconds, so that only the delayed output

$$w(t) = y(t - \tau)$$

(2)

is available to the controller, and assume the presence of additive measurement noise

$$w^m = w + n.$$  

(3)

To obtain a finite dimensional system, we will approximate the time delay $e^{-s\tau}$ by passing $y(t)$ through an $N$th order Padé approximation [16]–[18] with minimal realization

$$\hat{q}_N = A_N q_N + B_N y, \quad w_N = C_N q_N + D_N y, \quad q_N \in \mathbb{R}^n$$

(4)

and transfer function

$$P_N(s) = C_N(sI_n - A_N)^{-1}B_N + D_N$$

(5)

For example, with $N = 1$, $P_1(s) = (2 - s\tau)/(2 + s\tau)$. Note that $P_1(s)$ has a nonminimum phase zero at $z = 2/\tau$. It is generally true that an $N$th order Padé approximation will have $N$ nonminimum phase zeros $\{z_i, i = 1, \ldots, N\}$, $N$ poles at the arithmetic inverse of these zeros, and is allpass with unity gain: $|P_N(j\omega)| = 1, \forall \omega$.

Denote the system obtained by augmenting the Padé state equations (4) to those of the plant (1) with noisy measurement (3) by

$$\dot{x}_{aug} = A_{aug}x_{aug} + B_{aug}u + E_{aug}d,$$

(6)

$$w^m_N = C_{aug}x_{aug} + n,$$

(7)

where $x_{aug} = [x^T \quad q_N^T]^T$, $A_{aug} = \begin{bmatrix} A & 0_{n \times N} \\ B_N & A_N \end{bmatrix}$, $B_{aug} = \begin{bmatrix} B \\ 0_{n \times N} \end{bmatrix}$, $E_{aug} = \begin{bmatrix} E^T \\ 0_{N \times 1} \end{bmatrix}$ and $C_{aug} = \begin{bmatrix} D_N & C_N \end{bmatrix}$. It is straightforward to show that if $G(s)$ has no zeros at the eigenvalues of $A_N$, then $(A_{aug}, B_{aug})$ is controllable. Similarly, if $P_N(s)$ has no zeros at the eigenvalues of $A$, then $(A_{aug}, C_{aug})$ is observable.

Let the control law be given by state estimate feedback

$$u = -K\hat{x} + Hr,$$

(9)

where $\hat{x}$ must be obtained using the delayed measurement of the output $y$, which we have approximated by passing $y$ through the Padé approximation $P_N(s)$. Hence an observer must estimate both the plant states and the states of (4). Denote the estimator gain for the augmented system by $L_{aug} = [L_1^T \quad L_2^T]^T$, with $L_1 \in \mathbb{R}^n$ and $L_2 \in \mathbb{R}^N$. Then

$$\dot{x}_{aug} = A_{aug}\hat{x}_{aug} + B_{aug}u + L_{aug}\hat{w}^m_N$$

(10)

$$\dot{\hat{w}}_N = C_{aug}\hat{x}_{aug}.$$  

(11)

where $\hat{w}^m_N$, the measured estimation error for the delayed output, is given by

$$\hat{w}^m_N = w^m - \hat{w}_N.$$  

(12)

Substituting the control law $u = -K_{aug}\hat{x}_{aug} + Hr$, where $K_{aug} = [K \quad 0]$, and $r$ is set to zero, and using the fact that $\hat{w}_N$ satisfies (11) yields that the transfer function from $w^m$ to $-u$ is given by $-U(s) = \hat{C}_{obs}(s)W^m(s)$, where $\hat{C}_{obs}(s)$ has the state variable realization

$$\dot{\hat{x}}_{aug} = A_{obs}\hat{x}_{aug} + L_{aug}w^m, \quad -u = K_{aug}\hat{x}_{aug},$$

(13)

and

$$A_{obs} = A_{aug} - B_{aug}K_{aug} - L_{aug}C_{aug}.$$  

(14)

Under mild assumptions, the system (13) is minimal (see Lemma II.1 in the Appendix).

We now study the transmission zeros of the observer based compensator $\hat{C}_{obs}(s)$. Given a state variable system $(A, B, C)$, recall that a zero of the Rosenbrock System Matrix will also be a transmission zero of the associated transfer function if $(A, B)$ is controllable and $(A, C)$ is observable [19]. The following lemma, whose proof is in the Appendix II shows that under very mild assumptions the realization (13) is minimal.

Lemma II.1. Consider the compensator defined by (13).

(i) Assume that the eigenvalues of $A_N$ and $A - BK$ are disjoint, and that $A - BK$ has no eigenvalues that are uncontrollable from $L_1$. Denote the eigenvalues and associated left eigenvectors of $A_N$ by $\lambda_i$ and $\ell_i^T$, $i = 1, \ldots, N$, and assume further that

$$w_i^T B_N C (\lambda_i I_n - A - BK)^{-1} L_1 + w_i^T L_2 \neq 0.\quad (15)$$

Then $(A_{obs}, L_{aug})$ is a controllable pair.

(ii) Assume that the eigenvalues of $A_{aug} - L_{aug}C_{aug}$ and $A - L_1 D_N C$ are disjoint, and that $K (\lambda_i I_n - A + L_1 C)^{-1} L_1 \neq 0$ for any $\lambda$ that is an eigenvalue of $A_{aug} - L_{aug}C_{aug}$. Then $(A_{obs}, K_{aug})$ is an observable pair.

Proposition II.2. Consider the transfer function $\hat{C}_{obs}(s)$ with state variable realization (13), and assume that the hypotheses of Lemma II.1 are satisfied. Then the transmission zeros of $\hat{C}_{obs}(s)$ include zeros at the open left half plane mirror images of the $N$ nonminimum phase zeros of $P_N(s)$.  


Proof. Using definition (4) of \( A_N, B_N, C_N, \) and \( D_N, \) the Rosenbrock System Matrix for this system is given by
\[
RSM(s) = \begin{bmatrix}
sI_n - A_{aug} + B_{aug}K_{aug} + L_{aug}C_{aug} & -L_{aug} \\
K_{aug} & 0
\end{bmatrix}.
\]

Properties of Padé approximations imply that \( A_N \) has \( N \) distinct eigenvalues in the open left half plane located at the arithmetic inverse of the \( N \) nonminimum phase zeros of \( P_N(s) \). Let \( \lambda_i = -z_i \) denote one such eigenvalue and let \( v_i \) denote an associated eigenvector. Then \( RSM(\lambda_i) \) has a nullspace spanned by \([0 \; 1]v_i^T(C_Nv_i)^T\). Since the hypotheses of Lemma (11) are satisfied, we know that the realization of \( \hat{C}_{obs}(s) \) is minimal, and thus the zeros of the Rosenbrock system matrix are also transmission zeros. \( \square \)

A block diagram of the feedback system is in Fig. 1. Note that the zeros at the Padé poles in \( \hat{C}_{obs}(s) \) provide phase lead to partially compensate for the phase lag due to the time delay. This is consistent with the results of Carver et al. [20]; specifically, the zeros in the compensator at the poles of the Padé approximation will partially invert the sensor delay, and thus tend to mask the presence of the delay in the mapping from \( u \) to \( y \).

Denote the estimation error between the delayed output \( w \) and the estimate \( \hat{w}_N \) by
\[
\hat{w}_N = w - \hat{w}_N.
\] (16)

We now show that \( \hat{w}_N \) depends on the disturbance \( d \), the measurement noise \( n \), and the error in the Padé approximation to the time delay.

**Proposition II.3.** Define the tradeoff filter
\[
\hat{F}(s) = (1 + L_{est}^{aug}(s))^{-1}.
\] (17)

where \( L_{est}^{aug}(s) \) denotes the transfer function from \( \hat{w}_N \) to \( \hat{w}_N \) in Fig. 1 given by
\[
L_{est}^{aug}(s) = C_{aug}(sI_n + A_{aug})^{-1}L_{aug} = P_N(s)C(sI_n - A)^{-1}L_1 + C_N(sI_n - A)^{-1}L_2
\] (18)

Then the estimation error (16) satisfies
\[
\hat{W}_N(s) = \hat{F}(s) \left( \Delta(s)G(s)U(s) + e^{-s\tau}G_d(s)D(s) \right) + \left( \hat{F}(s) - 1 \right) N(s),
\] (19)

where \( \Delta(s) = e^{-s\tau} - P_N(s) \).

Proof. It follows from (16–22) that
\[
W(s) = e^{-s\tau}G(s)U(s) + e^{-s\tau}G_d(s)D(s),
\] (20)

and follows from (10)-(11) that
\[
\hat{W}_N(s) = C_{aug}(sI_n + A_{aug})^{-1} (B_{aug}U(s) + L_{aug}\hat{w}_N^{aug}) = P_N(s)G(s)U(s) + L_{est}^{aug}(s)\hat{W}_N^{aug} \tag{21}
\]

Together (3), (12), (20) and (21) imply that
\[
\hat{W}_N^{aug}(s) = \hat{F}(s) \left( \Delta(s)G(s)U(s) + e^{-s\tau}G_d(s)D(s) + N(s) \right) \tag{22},
\]

where \( \hat{F}(s) \) is given by (17). Noting that \( \hat{w}_N^{aug} = \hat{w}_N + n \) together with (22) yields (19). \( \square \)

**Proposition II.3** implies that the filter \( \hat{F}(s) \) describes a tradeoff between the response of the estimation error to the plant disturbance \( D(s) \) and the measurement noise \( N(s) \). In addition, \( \hat{F}(s) \) describes the response of the estimation error to \( \Delta(s) \), the difference between the time delay and the Padé approximation. We note that the first term in (19), which is proportional to \( \Delta(s) \), will also depend on \( D(s) \) and \( N(s) \) through the control input \( U(s) \). However, this first term can be minimized by increasing the degree of the Padé approximation without affecting the tradeoff between \( D(s) \) and \( N(s) \). Using large estimator gains \( L_1 \) and/or \( L_2 \) will force \( \hat{F}(s) \approx 0 \) and the disturbance response will be small at the expense of the noise being passed directly to the estimation error. The same will be true for the Padé approximation error, provided that it is not large enough to destabilize the system. Using small estimator gains has the opposite effect.

Let us now calculate the response of the system output.

**Corollary II.4.** The response of \( y \) to the reference input \( r \), disturbance \( d \), and measurement noise \( n \) is given by
\[
Y(s) = C(sI_n - A + BK)^{-1}BHR(s) + G_d(s)D(s) - C(sI_n - A)^{-1}BK(sI_n - A + BK)^{-1}L_1\hat{W}_N^{aug}(s),
\]

where
\[
\hat{W}_N^{aug}(s) = \hat{F}(s) \left( \Delta(s)G(s)U(s) + e^{-s\tau}G_d(s)D(s) + N(s) \right),
\]

Proof. The definition of \( u \) and \( \hat{x} \) yield
\[
U(s) = -K(sI_n - A + BK)^{-1}L_1\hat{W}_N^{aug}(s) + (1 + K(sI_n - A)^{-1}B)^{-1}HR(s),
\] (23)

and substituting (23) into (1) yields the result. \( \square \)

It follows from Corollary II.4 that, in the absence of disturbances and measurement noise, the response of \( y \) to a command \( r \) is the same with the observer as with state feedback, except for the discrepancy caused by the error in the Padé approximation to the time delay. Specifically, in the
absence of disturbances and noise, the command response is given by
\[ Y(s) = G(s) \left(1 + \hat{\Delta}(s)\right)^{-1} \]
\[ \times (1 + K(sI_n - A)^{-1} B)^{-1} HR(s), \]
where \( \hat{\Delta}(s) = K(sI_n - A + BK)^{-1} L_1 \hat{F}(s) \Delta(s) G(s) \). Hence we see that the bandwidth of the state feedback loop should be limited to the frequency range for which \( \Delta(j\omega) \approx 0 \). Higher order Padé approximations can approximate the delay over a wider frequency range at the expense of the additional zeros in \( C_{obs}(s) \) amplifying sensor noise.

### III. Decomposition of Optimal Estimator with Delay

We now describe special decomposition properties of the estimation problem from Section III that arise when the estimator is optimal. To formulate this problem, we consider the augmented system (8), suppose that \( d \) and \( n \) are zero mean Gaussian white noise processes with covariances \( V \geq 0 \) and \( W > 0 \), respectively, and assume that \( (A_{aug}, C_{aug}) \) is observable and \( (A_{aug}, E_{aug}) \) is controllable. Then the optimal estimator gains satisfy
\[ L_{aug} = \Sigma_{aug} C_{aug}^T W^{-1}, \]
where \( \Sigma_{aug} \) is the unique positive definite solution to the algebraic Riccati equation
\[ 0 = A_{aug} \Sigma_{aug} + \Sigma_{aug} A_{aug}^T + V_{aug} \]
\[ - \Sigma_{aug} C_{aug}^T W^{-1} C_{aug} \Sigma_{aug}, \]
with \( V_{aug} = E_{aug} V E_{aug}^T \).

By substituting (11) into (10) and applying (8), we see that the optimal estimator has state variable description
\[ \hat{x}_{aug} = A_{aug} \hat{x}_{aug} + L_{aug} w^m + B_{aug} u \\
\hat{w}_N = C_{aug} \hat{x}_{aug}, \]
where \( A_{aug} = A_{aug} - L_{aug} C_{aug} \) satisfies
\[ A_{aug} = \begin{bmatrix} A - L_1 D N C & -L_1 C_N \\
B_N C - L_2 D N C & A_N - L_2 C_N \end{bmatrix}. \]

We now characterize the \( N + n \) eigenvalues of (29) when \( L_{aug} \) is the optimal estimator gain given by (25).

**Proposition III.1.** Consider the optimal estimator defined by (25), (29). Define \( L = \Sigma C^T W^{-1} \), where \( \Sigma \) is the unique positive definite solution to the Riccati equation
\[ 0 = A \Sigma + \Sigma A^T - \Sigma C^T W^{-1} C \Sigma + E V E^T. \]

Assume that the eigenvalues of \( A_N, A, \) and \( A - LC \) are disjoint. Then \( A_{aug} \) has

(i) \( N \) eigenvalues identical to those of \( A_N \), and

(ii) \( n \) eigenvalues identical to those of \( A - LC \).

**Proof.** It is well known that the eigenvalues of \( A_{aug} \) are equal to the open left half plane eigenvalues of the Hamiltonian matrix associated with the Riccati equation (29), given by
\[ H_{aug} = \begin{bmatrix} A_{aug}^T & -(C^T D_N^T W^{-1} D_N + C) \\
-E_{aug} V E_{aug}^T & -A_{aug} \end{bmatrix}. \]
Substituting (8) into (31) yields
\[ H_{aug} = \begin{bmatrix} A^T & C^T B_N^T \bar{\Delta}(s) G(s) \]
\[ -E V E^T & -A \end{bmatrix}. \]

Let \( \lambda \) be any eigenvalue of \( H_{aug} \) and let \( \nu \) denote an associated right eigenvector: \( H_{aug} \nu = \lambda \nu \). Partition \( \nu \) as \( \nu = \begin{bmatrix} w^T & v^T & x^T & y^T \end{bmatrix}^T \), where \( w, x \in \mathbb{R}^n \) and \( v, y \in \mathbb{R}^n \). Then (32) implies that
\[ A^T w + C^T B_N^T v + C^T D_N^T W^{-1} D_N C_N y = \lambda w, \]
\[ A^T v + C^T W^{-1} D_N C_N y = \lambda v, \]
\[ -E V E^T w - A x = \lambda x, \]
\[ -B_N C x - A_N y = \lambda y. \]

It follows directly from (35)-(36) that \( x \) and \( y \) must satisfy
\[ x = -\left(\lambda A_n + A\right)^{-1} E V E^T w, \]
\[ y = -\left(\lambda A_N + A_N\right)^{-1} B_N C x, \]
and invoking (5) yields the additional constraint
\[ D_N C x + C_N y = P_N (-\lambda) C x. \]

We now treat the two sets of eigenvalues separately.

(i) Let \( \lambda \) be an eigenvalue of \( A_N \) and let \( \nu \in \mathbb{C}^n \) denote an associated right eigenvector of \( A_N^T \):
\[ A_N^T \nu = \lambda \nu. \]

The fact that the Padé approximation \( P_N(s) \) has right half plane zeros at the mirror images of the eigenvalues of \( A_N \) implies that \( P_N (-\lambda) = 0 \). Hence (24) is satisfied by definition, and (25) will hold provided that
\[ w = (\lambda I_n - A)^{-1} C^T B_N^T v. \]

(ii) Denote the Hamiltonian matrix associated with the Riccati equation (30) by
\[ H = \begin{bmatrix} A^T & -C^T D_N^T W^{-1} D_N C \\
-E V E^T & -A \end{bmatrix}. \]
Let \( \lambda \) be an open left half plane eigenvalue of \( H \) with right eigenvector \( \begin{bmatrix} w^T & x^T \end{bmatrix} \). Then \( w \) and \( x \) must satisfy (35) and
\[ A^T w - C^T D_N^T W^{-1} D_N C x = \lambda w. \]

Define \( y \) as in (38) and
\[ v = \left(\lambda A_N - A\right)^{-1} C^T W^{-1} P_N (-\lambda) C x. \]
Then \( \nu = \begin{bmatrix} w^T & v^T & x^T \end{bmatrix} \) satisfies (34)-(36). To show that \( \nu \) is an eigenvector of \( H_{aug} \) with eigenvalue \( \lambda \) it remains to prove that (33) holds. Substituting (38) and applying (35) and (34) shows that (35) reduces to
\[ C^T B_N^T v - C^T D_N^T W^{-1} D_N y \]
\[ = C^T B_N^T \left(\lambda I_N - A_N^T\right)^{-1} C^T W^{-1} P_N (-\lambda) C x \]
\[ + C^T D_N^T W^{-1} C_N \left(\lambda I_N + A_N\right)^{-1} B_N C x. \]
It follows from (5) that $C_N (\lambda I_N \pm A_N)^{-1} B_N = D_N - P_N(\pm \lambda)$, and thus that
\[
C^T B_N^T v - C^T D_N^T W^{-1} C_N y = (D_N - P_N(\lambda)) W^{-1} P_N(-\lambda) + D_N^T W^{-1} (D_N - P_N(-\lambda)).
\] (46)

The fact that $P_N(s)$ is a Padé approximation implies that $P_N(s) P_N(-s) = 1$ and thus that $D_N = \pm 1$. Together these facts show that (46) reduces to zero, and thus that $\nu$ satisfies (33) and is an eigenvector of $H_{aug}$.

Proposition III.1 implies that if the estimator is optimal then the value of the gain $L$ does not depend on the delay in the feedback measurement. Essential to the proof are the facts that the process noise $d$ directly affects only the plant states upstream of the delay and that the delay and its Padé approximation are allpass.

We next show that the eigenvalues of $A_N$ that are shared between $A_{aug}$ and $A_{aug}^{CL}$ are uncontrollable from the estimator gain $L_{aug}$. To do so, we first present a preliminary result showing that if certain eigenvalues of a system are preserved under state feedback, then these eigenvalues are unobservable in the state feedback.

**Lemma III.2.** Consider the linear system $\dot{x} = Fx + Gu$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, together with the state feedback $u = -Kx$. Assume that and $(F, G)$ is controllable and that $F$ and $F - GK$ have $m$ eigenvalues in common. Then these eigenvalues are unobservable eigenvalues of $(F,K)$.

**Proof.** Assume that $\lambda$ is a common eigenvalue of $F$ and $F - GK$. The latter fact implies there exists a nonzero $v'_1$ such that $(\lambda I - F + GK) v'_1 = 0$; equivalently
\[
\begin{bmatrix} \lambda I - F & G \\ K & 1 \end{bmatrix} v'_1 = 0.
\] (47)

Since $\lambda$ is also an eigenvalue of $F$, $\text{rank} (\lambda I - F) < n$, and $(F, G)$ controllable implies that $\text{rank} (\lambda I - F) G = n$. It follows that $G$ cannot be a linear combination of the columns of $\lambda I - F$, and thus the solution $v'_1$ to (47) must satisfy
\[
(\lambda I - F) v'_1 = 0
\] (48)

\[
K v'_1 = 0.
\] (49)

Together, (48) - (49) imply that
\[
\text{rank} \begin{bmatrix} \lambda I - F \\ K \end{bmatrix} < n,
\] (50)

and hence $\lambda$ is an unobservable eigenvalue of $(F, K)$.

Next suppose that $\lambda$ is a common eigenvalue of $F$ and $F - GK$ with algebraic multiplicity $M > 1$. Then controllability implies that the geometric multiplicity of $\lambda$ is equal to one in both cases. We now show that $\lambda$ is an unobservable eigenvalue of $(F, K)$ with multiplicity $M$. Specifically, if $(F, K)$ denotes the Jordan canonical form of $(F, K)$, then $F$ will have a single $M$-dimensional Jordan block associated with $\lambda$ for which all corresponding entries of $K$ are equal to zero.

First, there exists a chain of generalized eigenvectors $v_1, v_2, ..., v_M$ with
\[
(\lambda I - F) v_1 = 0,
\] (51)
\[
(\lambda I - F) v_{k+1} = v_k, \ k = 1, \ldots, M - 1.
\] (52)

A similarity transformation that places the system in Jordan canonical form is given by $Q = [v_1 \cdots v_M v_{M+1} \cdots v_n]$, where $v_{M+1} \cdots v_n$ represent the eigensubspace of the remaining eigenvalues. Then $F = QFQ^{-1}$ and $K = KQ$, and the result will be proven if we can show that $K v_i = 0, \ i = 1, \ldots, M$.

There also exists a chain of generalized eigenvectors $v'_1, v'_2, ..., v'_M$ with
\[
(\lambda I - F + GK) v'_1 = 0,
\] (53)
\[
(\lambda I - F + GK) v'_{k+1} = v'_k, \ k = 1, \ldots, M - 1.
\] (54)

Since the geometric multiplicity of $\lambda$ is equal to one, it follows from (48) and (51) that
\[
v'_1 = \alpha_1 v_1
\] (55)
for some nonzero $\alpha_1$. Hence (49) and (55) imply that
\[
K v_i = 0.
\] (56)

It remains to show that $K v_i = 0, \ i = 2, \ldots, M$. We shall also need to generalize the intermediate result (55) to apply to the remaining generalized eigenvectors $v'_i$ associated with $\lambda_i$. Hence we assume there exists $\ell < M$ and nonzero scalars $\alpha_1, \ldots, \alpha_\ell$ such that, for $i = 1, \ldots, \ell$,
\[
v'_i = \alpha_1 v_1 + \cdots + \alpha_i v_i
\] (57)
\[
K v_i = 0,
\] (58)

and show that (57) must also hold for $i = \ell + 1$. To do so, we first apply (54) and (57) with $k = \ell$, and (52) with $k = 1, \ldots, \ell$, yielding
\[
(\lambda I - F + GK) v'_{\ell+1} = v'_\ell
\] (59)
\[
= \alpha_1 v_1 + \cdots + \alpha_\ell v_i
\] (60)
\[
= (\lambda I - F) (\alpha_1 v_1 + \cdots + \alpha_\ell v_i)
\] (61)

Rearranging (59) results in
\[
\begin{bmatrix} \lambda I - F & G \\ K & 1 \end{bmatrix} v'_{\ell+1} = 0,
\] (60)

thus implying
\[
(\lambda I - F) (v'_{\ell+1} - \alpha_1 v_{\ell+1} - \cdots - \alpha_\ell v_2) = 0
\] (61)
\[
K v'_{\ell+1} = 0.
\] (62)

The fact that $\lambda$ has geometric multiplicity equal to one, together with (55) and (61), implies that $v'_{\ell+1} - \alpha_1 v_{\ell+1} - \cdots - \alpha_\ell v_2 = \alpha_{\ell+1} v_1$, for some where $\alpha_{\ell+1} \neq 0$, and thus
\[
v'_{\ell+1} = \alpha_1 v_{\ell+1} + \cdots + \alpha_\ell v_2 + \alpha_{\ell+1} v_1.
\] (63)

It then follows from (58) and (62) that
\[
K v_{\ell+1} = 0,
\] (64)
thus completing the induction.

**Proposition III.3.** The N eigenvalues of $A_{aug}^CL_{aug}$ that are identical to those of $A_N$ are uncontrollable eigenvalues of $(A_{aug}, L_{aug})$. Furthermore, a minimal realization of $L_{aug}(s)$ defined in (18) is given by $(A, L, C)$ with transfer function

$$L_{est}(s) = C(sI_n - A)^{-1}L,$$  \hspace{1cm} (65)

with $L$ given by Proposition III.1.

**Proof.** The dual of Lemma III.2 states that if $(F, H)$ is observable and $F$ and $F - LH$ share $m$ eigenvalues, then these eigenvalues are uncontrollable eigenvalues of $(F, L)$. Consider the estimation problem for the augmented system $(A_{aug}, E_{aug}, C_{aug})$ and denote the open loop transfer function from disturbance $d$ to estimation error $w_{est}^d$ by $G_{d}^aug(s) = C_{aug}(sI - A_{aug})^{-1}E_{aug}$. It is straightforward to show that $G_{d}^aug(s) = P_N(s)G_d(s)$, where $P_N(s)P_N(-s) = 1$. Hence the return difference equality [21] associated with the Riccati equation (20) reduces to

$$W + G_d(s)VG_d(-s) = (1 + L_{est}^aug(s)) W (1 + L_{est}^aug(-s)).$$ \hspace{1cm} (66)

The left hand side of (66) must satisfy the return difference equality for the lower order Riccati equation (30)

$$W + G_d(s)VG_d(-s) = (1 + C(sI - A)^{-1}L)W 	imes (1 + C(-sI - A)^{-1}L),$$ \hspace{1cm} (67)

and thus $L_{est}^aug(s)$ reduces to (65).

We saw in Proposition III.3 that the optimal estimation error is governed by the tradeoff filter $F(s)$ defined in (17). With optimal estimation, Proposition III.3 implies that $F(s)$ reduces to

$$F(s) = (1 + L_{est}(s))^{-1} = (1 + C(sI_n - A)^{-1}L)^{-1},$$ \hspace{1cm} (68)

and thus is independent of the time delay. The optimal estimation error thus has the form

$$\tilde{W}_N(s) = [F(s)G(s)U(s)] \cdot \Delta(s) \quad \text{(Padé error)}$$

$$+ [F(s)e^{-sT}G_d(s)] \cdot D(s) \quad \text{(disturbance)}$$

$$\quad \quad \quad \quad \quad + [F(s) - 1] \cdot N(s), \quad \text{(noise)}$$ \hspace{1cm} (69)

and depends on the time delay only through the Padé approximation error $\Delta(s)$.

**IV. AN ALTERNATE CONTROL ARCHITECTURE**

We now exploit the result of Proposition III.3 by proposing an alternate control architecture for the compensator that has several properties remarkably similar to those of the Smith Predictor [8].

The state equations (10) and (11) imply that the estimator for $\tilde{w}_N$ in Fig. 1 satisfies

$$\tilde{W}_N(s) = C_{aug}(sI_n + N - A_{aug})^{-1}L_{aug}\tilde{W}_N^m(s) + V_N(s),$$ \hspace{1cm} (70)

where $V_N(s) = P_N(s)G(s)U(s)$. Suppose, as shown in Fig. 2 that we implement this estimator using separate subsystems for the responses to the measured estimation error $\tilde{w}_N^m$ and to the input $u$. Doing so may seem counterintuitive as it leads to a compensator with higher dynamical order than that depicted in Fig. 1. Indeed, the additional dynamics may imply that the compensator is no longer stabilizing. Nevertheless, we shall show that if the estimator is optimal, then this control architecture has several interesting features, and we shall state conditions under which it does stabilize the plant. Then in Section V we describe similarities between the architecture of Fig. 2 and the Smith Predictor.

**Fig. 2: Rearranged block diagram depicting the tradeoff filter $F(s)$.**

We have seen in Section III that if the estimator gain is optimal, then (70) reduces to

$$\tilde{W}_N(s) = C(sI_n - A)^{-1}L\tilde{W}_N^m(s) + V_N(s).$$ \hspace{1cm} (71)

A derivation similar to that used to prove Proposition II.3 shows that the tradeoff between response to noise and response to disturbances and Padé approximation error continues to hold, with $F(s)$ defined by (17) replaced by (65)

$$F(s) = (1 + L_{est}(s))^{-1}. $$

Using this fact, the block diagram in Fig. 2 simplifies to that in Fig. 3 whose properties we now explore.

To begin, we note that the observer based compensator $C_{obs}(s)$ has a state variable description

$$\dot{x}_C = A_Cx_C + B_Cw^m, \quad u = -C_Cx_C, $$ \hspace{1cm} (72)

where $x_C = [x^T \quad p^T \quad \beta^T]^T$, $A_C = \begin{bmatrix} A - LC & -LD_N & -LC_N \\ -BK & A - BK & 0_{n \times N} \\ 0_{N \times n} & B_N & A_N \end{bmatrix}$, $B_C = \begin{bmatrix} 0_{n \times 1} \\ 0_{N \times 1} \end{bmatrix}$,  \hspace{1cm} (73)

and $C_C = K \begin{bmatrix} K & 0_{1 \times N} \end{bmatrix}$. The formula for the inverse of a block matrix and some algebra shows that

$$C_{obs}(s) = C_C(sI_{2n + N} - A_C)^{-1}B_C = \frac{1 + G(s)(sI_n - A + BK + LC)^{-1}L(1 - P_N(s))}{1 + G(s)(sI_n - A - BK)^{-1}L},$$ \hspace{1cm} (74)

Unlike the original architecture in Fig. 1 the estimator gain $L$ may be chosen without regard for the delay, and has dimension determined only by that of the plant. On the other hand, the compensator $C_{obs}(s)$ now has dynamical order
2n + N, which is larger than that of the compensator in Fig. 1. Finally, note that the estimator gain L in Fig. 3 will generally differ from the gain L1 in Fig. 2 and thus the former estimator cannot be obtained from the latter simply by setting L2 = 0.

The next lemma, whose proof is in Appendix I, shows that under mild assumptions the realization (72)-(73) will be minimal, with the exception that if \( \lambda = 0 \) is an eigenvalue of A, then zero will be an unobservable eigenvalue of \((A_C, C_C)\).

**Lemma IV.1.** Consider the compensator defined by (72), (73).

(i) Assume that \( A_N \) has no eigenvalues that are also zeros of \( G(s) \), that \( A_N \) and \( A - BK \) have no eigenvalues that are zeros of \( K(sI_n - A)^{-1}L \) and that the eigenvalues of \( A_N \), \( A \), and \( A - BK \) are mutually disjoint. Then \((A_C, C_C)\) is a controllable pair.

(ii) Assume that the eigenvalues of \( A \), \( A_N \), and \( A - LC \) are mutually disjoint, and that \( (A - LC, K) \) is an observable pair. Then if \( A \) has no eigenvalues at the origin, the pair \((A_C, C_C)\) is observable. If \( A \) has an eigenvalue equal to zero, then \( \lambda = 0 \) is an unobservable eigenvalue of \((A_C, C_C)\).

The following result shows that, as in Proposition I.2 in Appendix I, the eigenvalues of \( A_N \) will appear as zeros of \( C_{obs}(s) \). Furthermore, with one important exception, \( C_{obs}(s) \) will also have zeros at the eigenvalues of \( A \).

**Proposition IV.2.** Consider \( C_{obs}(s) \) defined by (72), (73), and assume that the hypotheses of Lemma IV.1 hold.

(i) Let \( \lambda \) be an eigenvalue of \( A_N \). Then \( \lambda \) is a transmission zero of \( C_{obs}(s) \).

(ii) Assume that \( \lambda \) is a nonzero eigenvalue of \( A \). Then \( \lambda \) is a transmission zero of \( C_{obs}(s) \).

(iii) Assume that \( \lambda = 0 \) is an eigenvalue of \( A \) with multiplicity \( m \geq 1 \). Then \( \lambda \) is a transmission zero of \( C_{obs}(s) \) with multiplicity \( m - 1 \).

**Proof.** Consider \( C_{obs}(s) \) defined in (74), introduce coprime factorizations \( K(sI_n - A + BK + LC)^{-1}L = N_C(s)/D_C(s), G(s) = N_G(s)/D_G(s), \) and \( P_S(s) = N_P(s)/D_P(s) \), and note that \( D_G(s) \) may be factored as \( D_G(s) = s^m D_{\bar{G}}(s) \), where \( D_{\bar{G}}(0) \neq 0 \). Further note that

\[
\frac{N_P(s)}{D_P(s)} = \prod_{i=1}^{N} \frac{s_i - s}{s_i + s},
\]

from which it follows that

\[
D_P(s) - N_P(s) = 2s \sum_{i=1}^{N} z_j + \text{higher order terms in } s.
\]

Hence we may factor \( D_P(s) - N_P(s) = s(\Delta(s)) \), where \( \Delta(0) \neq 0 \).

Substituting these factorizations into (74) and rearranging results in

\[
C_{obs}(s) = \frac{N_C(s) \bar{D}_G(s) D_P(s)}{D_G(s) D_C(s) D_P(s) + N_G(s) N_C(s) \Delta(s)}. \tag{75}
\]

The assumption that \( C_{obs}(s) \) is stabilizing implies that \( N_C(0) \neq 0 \), and the fact that Padé approximations are stable implies that \( D_P(0) \neq 0 \). Hence if \( m = 1 \) then \( C_{obs}(0) \neq 0 \); otherwise, \( C_{obs}(s) \) will have a zero at the origin of order \( m - 1 \).

We now analyze stability of the feedback system in Fig. 3 with the time delay replaced by a Padé approximation, so that the system has a state variable description

\[
\dot{x}_{fb} = A_{fb} x_{fb} + B_{fb} \begin{bmatrix} r \\ d \\ n \end{bmatrix}, \quad \begin{bmatrix} u \\ y \end{bmatrix} = C_{fb} x_{fb} + \begin{bmatrix} H \\ 0 \end{bmatrix} r, \tag{76}
\]

where \( x_{fb} = \begin{bmatrix} x^T \\ q_N^T \\ z^T \\ p^T \\ \beta^T \end{bmatrix}^T \), and

\[
A_{fb} = \begin{bmatrix} A_{aug} & -B_{aug} C_C \\ B_C C_{aug} & A_C \end{bmatrix}, \tag{77}
\]

\[
B_{fb} = \begin{bmatrix} B H \\ 0_{n \times 1} \\ 0_{n \times 1} \\ L \\ 0_{n \times 1} \end{bmatrix}, \tag{78}
\]

\[
C_{fb} = \begin{bmatrix} 0_{1 \times N} \\ 0_{1 \times N} \\ -K \\ 0_{1 \times N} \end{bmatrix}, \tag{79}
\]

**Proposition IV.3.** The eigenvalues of \( A_{fb} \) are the union of the eigenvalues of \( A, A - BK, A - LC, \) and \( A_N \), with the latter having multiplicity two.

**Proof.** Substituting (68) and (73) into \( A_{fb} \) yields a block 5 \times 5 matrix. Adding column 2 to column 5 and column 1 to column 4 and subtracting row 4 from row 1 and row 5 from row 2 on \( A_{fb} - \lambda I_{3n+2N} \) yields a lower block triangular matrix with the same determinant \( \det(A_{fb} - \lambda I_{3n+2N}) = \det(A - \lambda I_n) \det(A - BK - \lambda I_n) \det(A - LC - \lambda I_n) \det(A - \lambda I_n)^2 \).

In the event that \( A \) has an eigenvalue \( \lambda = 0 \) with multiplicity \( m \geq 1 \), then there exists a realization \((A_{fb}^*, B_{fb}^*, C_{fb}^*)\) of the system in Fig. 3 that has one less state than that given by (77)-(79). Specifically, the multiplicity of \( \lambda = 0 \) as an eigenvalue of \( A_{fb}^* \) will be equal to \( m - 1 \); for details see Lemma I.1 and Proposition I.2 in Appendix I.
V. RELATION TO THE SMITH PREDICTOR

The Smith predictor

\[ \mathcal{C}_p(s) = \frac{C(s)}{1 + C(s)G(s)(1 - e^{-s\tau})}. \]

shown in the dashed box in Fig. 4 is a commonly proposed architecture for compensating systems with a time delay. This architecture has several key features:

S-(a) The predictor \( \mathcal{C}_p(s) \) contains a model of both the time delay as well as the plant \( G(s) \).

S-(b) The controller \( C(s) \) may be designed to stabilize the plant \( G(s) \), ignoring the time delay, and the resulting command response is given by \( Y(s) = G(s)C(s)/(1 + G(s)C(s)) \).

S-(c) The only input to the compensator is the error signal, \( e = r - w \), processed in a One Degree of Freedom (1DOF) control architecture.

S-(d) Only an input-output model of the plant is assumed, as opposed to an internal state variable description.

S-(e) The poles of \( G(s) \) appear as zeros of \( \mathcal{C}_p(s) \), and thus the Smith predictor cannot be used to stabilize unstable systems. The one exception to this rule is if \( G(s) \) has a single pole at \( s = 0 \). In that case \( \mathcal{C}_{obs}(0) \neq 0 \), and it follows that the architecture can be used to stabilize a plant with a single integrator.

First consider properties (a) and (b). Taking the limit as \( N \rightarrow \infty \) allows us to replace the Padé approximation \( P_N(s) \) by an exact copy of the time delay. It follows that the approximation error \( \Delta(s) = 0 \) and the command response in (24) no longer depends on the time delay. With regard to property (e), it is possible to rearrange the block diagram in Fig. 3 by subtracting the reference signal from the summing junction at the upper left, so that both systems have 1DOF control architectures. Note that, with no disturbance or noise present, and assuming perfect models of the plant and delay, the signal \( e^m - v = -r \), and thus the system output \( y \) will not depend on the delay despite the rearranged control architecture. The remaining difference between the two architectures, property (d), is needed in order to give more detailed modeling of the effect of the disturbance \( d \) on the plant dynamics, to apply state estimate feedback, and to use optimal estimation to update these state estimates based on the output measurement.

VI. DISCUSSION

The cerebellum is critical for coordinating movement, balance, and motor learning [22]. How does the cerebellum compensate for delays? It is proposed that the cerebellum could function through the state estimation process to compensate the delay [7], [23], [24].

In this paper, we focus on the delay-masking problem in the state estimation process, which must be understood to properly interpret experimental data that investigates delay compensation. First we show the delay is “masked”—in the sense described by Carver et al. [20]—during delay compensation. That is, the feedback of estimated plant states partially inverts the delay by introducing zeros at the poles of the Padé approximation (and zeros at the origin in the discrete-time case). This pole–zero cancellation was found to mask both the sensory feedback delay and how it is regulated in the input–output relationship from plant measurement \( y \) to the computed motor commands \( u \). Second, we found that optimal estimation leads to a decomposition of the Kalman filter; specifically, we found that the \( N \) Padé states were uncontrollable while the remaining eigenvalues were identical to those found by solving a lower order estimation problem that ignores the time delay. The optimal reduction identified above may give insights into how the brain processes the sensory information for state estimation. Furthermore, the tradeoff filter in estimation error, which modulates the balance between process disturbance and measurement noise, could be obtained by ignoring the time delay, as long as the Padé approximation error is sufficiently small.

We explore further how to exploit this decomposition to produce structural simplifications in control. Specifically, we found that an optimal observer-based compensator, where only the plant (and not the delay) states are used for the controller, can be subtly modified so that it acts surprisingly similarly to a Smith predictor, sharing a number of a Smith predictor’s shortcomings. Specifically, both the Smith predictor and our
new “Smith-like” predictor fail to stabilize (most) unstable plants.

It has been hypothesized that the cerebellum might act like a Smith predictor for such behaviors as reaching [5], [12], but it cannot be used to explain the cerebellum’s role in postural balance [25], because the latter involves stabilization of right-half-plane poles. While there are many studies working on generalizing the Smith predictor for cases such as unstable plants [10], the simple formulation we use in this paper of an observer-based controller based on a Padé approximation can stabilize unstable plants and does not suffer the drawbacks of a Smith predictor. Nevertheless, our calculations suggest that Smith predictors serve as simplified (though limited) near-optimal solutions to the delay compensation problem, and the minimal modifications needed to transform a state-estimation-based controller into a Smith predictor suggest that it may be infeasible to experimentally differentiate the two approaches, while at the same time suggesting that either may be a useful model, depending on the goal of any given study.

In our analysis we included the measurement noise $n$ after the time delay; see Equation (3). This is the equivalent condition as adding noise before the time delay as follows:

$$w^n(t) = w(t) + n(t - \tau),$$

(80)

This works since the noise is serially uncorrelated, and Equations (3) and (80) are statistically equivalent.

Future work will include applying these results to multiple sensory systems—such as haptic and visual feedback—with differing delays. In addition, robustness is important for systems with time delays [26], especially in the face of errors in delay approximation. We are interested in exploring this and other controller design issues as future work.

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APPENDIX I

Supplementary Results for Continuous System with Output Delay

Proof of Lemma III.1

Proof. (i) We know that $\lambda$ is an uncontrollable eigenvalue of $(A_{obs}, L_{aug})$ if and only if there exists a nonzero vector $\begin{bmatrix} x^T & w^T \end{bmatrix}$ such that

$$x^T \begin{bmatrix} I_{n+N} - A_{obs} & L_{aug} \end{bmatrix} w = 0.$$  

(81)

Substituting $w$ and the definitions of $K_{aug}$ and $L_{aug}$ into (81) yields

$$x^T (\lambda I_n - A + BK) + w^T (-B_N C) = 0$$  

(82)

and using (82) implies

$$x^T (\lambda I_n - A + BK) + w^T (-B_N C) = 0$$  

(83)

from which it follows that either $w = 0$ or that $\lambda$ must be an eigenvalue of $A_N$. In the former case (83) implies that $\lambda$ must be an eigenvalue of $A - BK$ that is uncontrollable from $L_1$, which is ruled out by assumption. In the latter case, it follows from (83) that $w^T B_N C (\lambda I_n - A + BK)^{-1} L_1 + w^T L_2 = 0$. (84)

Since (84) must hold for each eigenvalue of $A_N$, the result follows. (ii) We know that $\lambda$ is an unobservable eigenvalue of $(A_{obs}, K_{aug})$ if and only if there exists a nonzero vector $\begin{bmatrix} x^T & w^T \end{bmatrix}$ such that

$$\begin{bmatrix} \lambda I_{n+N} - A_{obs} & K_{aug} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0.$$  

(85)

Using (8) in (85) yields

$$\begin{cases} (\lambda I_n - A + BK + L_1 D_N C) x + L_1 C_N w = 0 & (86) \\ (L_2 D_N C) x + (\lambda I_n - A + L_2 C_N) w = 0 & (87) \\ K x = 0. & (88) \end{cases}$$

Note that if $x = 0$ then (86) and (87) imply that $\lambda$ must be an unobservable eigenvalue of $A_N$, which is ruled out by the assumption that (4) is minimal. Using (85) in (88) yields

$$\begin{cases} (\lambda I_n - A + L_1 D_N C) x + L_1 C_N w = 0 & (89) \\ K (\lambda I_n - A + L_1 D_N C)^{-1} L_1 C_N w = 0, & (90) \end{cases}$$

which is ruled out by hypothesis, and the result follows. □

Proof of Lemma IV.1

Proof. (i) We know that $\lambda$ is an uncontrollable eigenvalue of $(A_C, B_C)$ if and only if there exists a nonzero vector $\begin{bmatrix} x^T & w^T \end{bmatrix}$ such that

$$\begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} I_{2n+N} - A_C & B_C \end{bmatrix} = 0.$$  

(90)

Substituting (23) into (90) and rearranging yields $x^T L = 0$ and

$$x^T (\lambda I_n - A + BK) + w^T B_N C = 0$$  

(91)

$$w^T (\lambda I_n - A + BK) - u^T B_N C = 0$$  

(92)

$$w^T (\lambda I_n - A) = 0$$  

(93)

and thus (93) $\lambda$ must be an eigenvalue of $A_N$ or $u^T = 0$.

In the latter case, it follows from (92) that $\lambda$ must be an eigenvalue of $A - BK$ and from (91) and $x^T L = 0$ that $w^T B (\lambda I_n - A)^{-1} L = 0$. However, our assumptions imply that $w^T B \neq 0$ (otherwise $\lambda$ would be an uncontrollable eigenvalue of $A$) and that $K (\lambda I_n - A)^{-1} L \neq 0$. □
In the former case, (91) and (92) imply that
\[ x^T = -w^T BK (\lambda \pi_n - A)^{-1} \]  
and substituting (95) into (94) and rearranging yields
\[ u^T B_N C (A_N - A + BK) (\lambda \pi_n - A)^{-1} L = 0. \]  
(96)

The assumption that \((A_N, B_N)\) is minimal implies that \(u^T B_N \neq 0\), and \(K (\lambda \pi_n - A)^{-1} L\) is also nonzero by assumption. The identity \(C (\lambda \pi_n - A + BK)^{-1} B = G(s)/(1 + K(s I - A)^{-1} B)\) together with the assumption that \(G(\lambda) \neq 0\) imply that (96) cannot hold and thus that \((A_C, B_C)\) is controllable.

(ii) We know that \(\lambda\) is an unobservable eigenvalue of \((A_C, C)\) if and only if there exists a nonzero vector 
\[ [x^T \ v^T \ w^T]^T \]

such that
\[ \begin{bmatrix} \lambda I_{2n+N} - A_C \\ C_C \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = 0. \]  
(97)

Substituting (15) into (97) yields the four equations
\[ (\lambda I_N - A + LC)x + LD_N Cv + LC_N w = 0 \]  
(98)
\[ BK x + (\lambda I_N - A + BK) v = 0 \]  
(99)
\[ -B_N Cv + (\lambda I_N - A) w = 0 \]  
(100)
\[ K x + K v = 0. \]  
(101)

If \(A - LC\) has an eigenvector \(x\) satisfying \(K x = 0\) then a solution to (97) is obtained by setting \(v\) and \(w\) equal to 0; this scenario is ruled out by the hypothesis that \((A - LC, K)\) is observable. Hence, substituting (101) into (99) yields \((\lambda I_N - A) w = 0\) and thus \(\lambda\) must be an eigenvalue of \(A\). It follows from (100) that \(w = (\lambda I_N - A)^{-1} B_N C v\), and substituting into (98) and rearranging yields \((\lambda I_N - A + LC)x + LP_N (\lambda) C v = 0\). The assumption that \(\lambda\) is not an eigenvalue of \(A - LC\) implies that \(x = -(\lambda I_N - A + LC)^{-1} LP_N (\lambda) C v\), and the fact that \(v\) is an eigenvector of \(A\) implies that \(x + v = (\lambda I_N - A + LC)(LC_N - LP_N (\lambda) C v)\), which is equal to zero if and only if \(\lambda = 0\) so that \(P_N(0) = 1\), and the result follows.

Lemma 1.1. Assume that \(A\) has an eigenvalue \(\lambda = 0\) with eigenvector \(v\). Then \((A_C, C_C)\) has an unobservable eigenvalue \(\lambda = 0\) with eigenvector \(v_C = [v^T \ -e^T \ (A_N^{-1} B_N e)^T]^T\).

Proof. The result follows immediately from the proof of Lemma [V.1.1].

Define \(\zeta = A_N^{-1} B_N C v\) and \(\Theta = (N - (\zeta - e_N) e_N^T / e_N^T \zeta)\), and let \(A_{N \times N - 1}, C_{N \times N - 1}, \) and \(I_{N - 1 \times N}\) denote the first \(N - 1\) columns of \(A_N, C_N\), and the first \(N - 1\) rows of \(I_N\), respectively.

Proposition 1.2. Assume that \(A\) has an eigenvalue \(\lambda = 0\) with multiplicity \(m \geq 1\). Then a minimal realization of \(C_{obs}(s)\) defined in (72) is given by
\[ A_C = \begin{bmatrix} A - LC - LD_N C & A_C(1, 3) & A_C(1, 2) & A_C(3, 1) \\ -BK & A_C(2, 2) & v^T L A_{N \times N - 1} & 0_{N \times 1} \\ 0_{N-1 \times n} & I_{N-1 \times N} \Theta B_N C & I_{N-1 \times N} \Theta A_{N \times N - 1} \end{bmatrix} \]
and \(x^T = Q^{-1} x_C, A_C = Q^{-1} A C_Q, B_C = Q^{-1} B C, \) and the result follows.

\[ A_C = \begin{bmatrix} A - LC - LD_N C & A_C(1, 3) & A_C(1, 2) & A_C(3, 1) \\ -BK & A_C(2, 2) & v^T L A_{N \times N - 1} & 0_{N \times 1} \\ 0_{N \times n} & I_{N \times N} \Theta B_N C & I_{N \times N} \Theta A_{N \times N - 1} \end{bmatrix} \]

(102)

APPENDIX II

Optimal Lower Order Property for Continuous System with Input Delay

The lower order property of optimal state feedback design for the continuous-time system with input delay is dual to the output delay condition as above. Since the proof follows similarly, we state the results below without proof and point out their dual relationship to the output delay case.

Consider the single input, single output linear system
\[ \dot{x} = Ax + Bu(t) + Ed, \quad y = Cx, \quad x \in \mathbb{R}^n, \]  
(103)

Suppose that the control input is delayed by \(\tau\) seconds, so that only the delayed input
\[ w(t) = u(t - \tau) \]
is available to the controller. And denote the transfer function from \(w(t)\) to \(y\) by \(G(s) = \tilde{C}(s I_n - A)^{-1} B\) and that from \(d\) to \(y\) by \(G_d(s) = \tilde{C}(s I_n - A)^{-1} E\). Assume that \((A, C)\) is observable and that \((A, B)\) and \((A, E)\) are controllable. Assume the presence of additive measurement noise
\[ y^m = y + n. \]  
(104)
To obtain a finite dimensional system, we will approximate the time delay $e^{-st}$ by passing $u(t)$ through an $N$’th order Padé approximation with minimal realization
\[ q_N = \tilde{A}_N q_N + \tilde{B}_N u, \quad w_N = \tilde{C}_N q_N + \tilde{D}_N u, \quad q_N \in \mathbb{R}^N \] (104)
and transfer function $P_N(s) = \tilde{C}_N (sI_N - \tilde{A}_N)^{-1} \tilde{B}_N + \tilde{D}_N$.

Denote the system obtained by augmenting the Padé state equations (104) to those of the plant (102) with noisy measurement (103) by
\[
\begin{align*}
x_{aug} &= \bar{A}_{aug} x_{aug} + \bar{B}_{aug} u + \bar{E}_{aug} d, \\
y_m &= \bar{C}_{aug} x_{aug} + n,
\end{align*}
\] (105) (106)
where $x_{aug} = [x^T \ q_N^T ]^T$, 
\[
\bar{A}_{aug} = \begin{bmatrix} A & \bar{B} \bar{C}_N \\ 0_{N \times N} & \bar{A}_N \end{bmatrix}, \quad \bar{B}_{aug} = \begin{bmatrix} \bar{B} \bar{D}_N \\ \bar{B}_N \end{bmatrix},
\] (107)
\[
\bar{E}_{aug} = [E^T \ 0_{N \times 1}]^T \quad \text{and} \quad \bar{C}_{aug} = [C \ 0].
\]
It is straightforward to show that if $G(s)$ has no zeros at the eigenvalues of $A_N$, then $(\bar{A}_{aug}, \bar{C}_{aug})$ is observable. Similarly, if $P_N(s)$ has no zeros at the eigenvalues of $A$, then $(\bar{A}_{aug}, \bar{B}_{aug})$ is controllable.

Let the control law be given by state estimate feedback
\[ u = -K_{aug} x_{aug} + H r \]

$K_{aug}$ is the optimal feedback gain given by LQR design as $K_{aug} = R^{-1} \bar{B}_{aug}^T \Sigma_{aug}$, and $\Sigma_{aug}$ is the solution to $(n + N)$ dimensional Riccati equation
\[
\bar{A}_{aug}^T \Sigma_{aug} + \Sigma_{aug} \bar{A}_{aug} + Q - \Sigma_{aug} \bar{B}_{aug} R^{-1} \bar{B}_{aug}^T \Sigma_{aug} = 0
\]

where $Q_{aug} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$ and $Q$ is the cost on plant states.

We now characterize the $N + n$ eigenvalues of state feedback $\bar{A}_{aug} - \bar{B}_{aug} K_{aug}$.

**Proposition B.1.** Define $K = R^{-1} \bar{B}^T \Sigma$, $\Sigma$ is the solution to $n$ dimensional Riccati equation
\[
\bar{A}_d \Sigma + \Sigma \bar{A} + Q - \Sigma \bar{B} R^{-1} \bar{B}^T \Sigma = 0
\]
Assume that the eigenvalues of $A_N$, $\bar{A}$, and $\bar{A} - \bar{B} K$ are disjoint. Then $\bar{A}_{aug} - \bar{B}_{aug} K_{aug}$ has
(i) $N$ eigenvalues identical to those of $A_N$, and 
(ii) $n$ eigenvalues identical to those of $\bar{A} - \bar{B} K$.

**Proof.** Note that $(A_N, B_N, C_N, D_N)$ is an $N$’th order Padé approximation as in the main manuscript, then $(A_N, B_N, C_N, D_N) = (A_N, C_N T_N, B_N T_N, D_N)$ is also an $N$’th order Padé approximation. Consider the plant dynamics $(\bar{A}, \bar{B}, \bar{C}) = (A_f, C_f T, B_f T)$, which are dual to the output delay condition as above. Then the augmented system would also show duality as $(\bar{A}_{aug}, \bar{B}_{aug}, \bar{C}_{aug}) = (A_{aug}, C_{aug} T, B_{aug} T)$. $Q_{aug}$ has a block structure similar to $V_{aug}$ in (86). Essential to the proof are the facts that the cost only takes into account the plant states and does not penalize the delay states. So this partial pole placement property follows.

**Appendix III**

**Discrete System: Feedback of Estimated Plant States Partially Inverts Delay**

Carver et al. [20] showed that if the dynamics of a sensor were included in the model of the system used for estimator design, but that the state-estimate feedback only included nonzero gains for the plant states, then the transfer function of the resulting observer-based compensator would place zeros at sensor poles. While their proofs were performed for continuous-time systems, their results suggest that, in the discrete-time setting, a one-step delay in the sensor (which introduces a pole at the origin) will be canceled by a zero at the origin in the observer-based compensator (under similar assumptions as in the present paper). Their result does not trivially extend to the multi-step delay case (which leads to repeated poles at $z = 0$). In this appendix, we generalize to the multi-step delay.

Consider the single input, single output linear system
\[
x_{k+1} = Ax_k + Bu_k + Ed_k, \quad x \in \mathbb{R}^n,
\]
\[y_k = Cx_k
\]
and define the plant transfer function $G(z) = C(zI_n - A)^{-1} B$. Assume there exists an $M$-step delay in the measurement of the output,
\[w_k = y_{k-M},
\]
as well as additive measurement noise
\[w_k = w_k + n_k.
\]
For later reference, we adopt a state variable model of the delay, where
\[q_{k+1} = A_q q_k + B_q y_k, \quad q \in \mathbb{R}^m,
\]
\[w_k = C_q q_k,
\]
\[z^{-M} = C_q (zI_M - A_q)^{-1} B_q,
\]
\[A_q = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{M \times M}, \quad B_q = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{M \times 1}
\]
\[C_q = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{1 \times M}.
\]
The control law is given by state estimate feedback,
\[u_k = -K \hat{x}_{k|k-1} + H r_k,
\]
where the state estimates must be obtained from the delayed measurement of the output $y_k$. Hence the observer must estimate both the plant states and the delay states:
\[
\begin{bmatrix} \hat{x}_{k+1|k} \\ q_{k+1|k} \end{bmatrix} = \begin{bmatrix} A - BK & L_1 \\ B_q C & A_q - L_2 C_q \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k-1} \\ q_{k|k-1} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} w^m_k + \begin{bmatrix} BH \\ 0 \end{bmatrix} r_k.
\]
(111)

Substituting the control law (110) into (111) yields
\[
\begin{align*}
\begin{bmatrix} \hat{x}_{k+1|k} \\ q_{k+1|k} \end{bmatrix} &= \begin{bmatrix} A - BK & L_1 \\ B_q C & A_q - L_2 C_q \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k-1} \\ q_{k|k-1} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \hat{w}^m_k + \begin{bmatrix} BH \\ 0 \end{bmatrix} r_k.
\end{align*}
\] (112)
Denote the transfer function of the observer based compensator mapping \( w_{k}^m \) to \(-u_k\) by \( C_{\text{obs}}(z) \). It follows from (112) that \( C_{\text{obs}}(z) \) has the state variable description

\[
\begin{bmatrix}
\dot{x}_{k+1|k} \\
\dot{q}_{k+1|k}
\end{bmatrix}
= \begin{bmatrix}
A - BK & -L_1 C_q \\
B_q C & A_q - L_2 C_q
\end{bmatrix}
\begin{bmatrix}
\dot{x}_{k|k-1} \\
\dot{q}_{k|k-1}
\end{bmatrix}
+ \begin{bmatrix}
L_1 \\
L_2
\end{bmatrix}
w_{k}^m,
\]

(113)

\[-u_k = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix}
\dot{x}_{k|k-1} \\
\dot{q}_{k|k-1}
\end{bmatrix}.
\]

(114)

The Rosenbrock System Matrix associated with (113) is given by

\[
RSM(z) = \begin{bmatrix}
z I_n - A + BK & L_1 C_q \\
-B_q C & z I_M - A_q + L_2 C_q & -L_1 \\
0 & 0 & -L_2
\end{bmatrix}.
\]

(115)

Let \( 0_{n \times 1} \) denote an \( n \) dimensional column vector of zeros, and let \( e_1 \) denote the first standard basis vector. Then it may be verified from (115) and the structure of \( A_q \) and \( C_q \) that \( RSM(0) \) has a nontrivial nullspace spanned by the vector

\[
\begin{bmatrix} 0_{n \times 1} \\ e_1 \\ 1 \end{bmatrix},
\]

and thus that \( C_{\text{obs}}(z) \) has at least one zero at \( z = 0 \). Indeed, the transmission zeros of \( C_{\text{obs}}(z) \) include zeros at \( z = 0 \) with multiplicity \( M \). To do so, assume \( M > 1 \) and use the Rosenbrock matrix and induction to show that \( C_{\text{obs}}(z)/z, \ldots, C_{\text{obs}}(z)/z^{M-1} \) each have zeros at \( z = 0 \), but that \( C_{\text{obs}}(z)/z^M \) has no such zero. A block diagram of the feedback system with compensator \( C_{\text{obs}}(z) \) is in Fig. 5 the dashed box in this diagram contains \(-C_{\text{obs}}(z)\). The additional zeros at \( z = 0 \) in \( C_{\text{obs}}(z) \) will lower its relative degree and thus reduce the delay in its impulse response, partially compensating for the effect of the time delay in the feedback path.

![Fig. 5: Discrete-time feedback system with M-step delay.](image)

**Corollary C.1.** Define the tradeoff filter

\[
F(z) = \left( 1 + C_{\text{aug}} (z) I_{n} - A_{\text{aug}} \right)^{-1}
\]

(118)

The z-transform of the estimation error (117) satisfies

\[
\tilde{W}(z) = F(z) z^{-M} C(z I_n - A)^{-1} E D(z) + (F(z) - 1) N(z).
\]

(119)

**Proof.** First, \((108)-(109)\) imply that

\[
W(z) = -z^{-M} G(z) U(z) + z^{-M} C(z I_n - A)^{-1} E D(z).
\]

(120)

Taking \( z \)-transforms and applying (110) yields

\[
\tilde{X}(z) = (z I_n - A)^{-1} B U(z) + (z I_n - A)^{-1} L_1 \tilde{W}(z)
\]

\[
= (I + (z I_n - A)^{-1} B K)^{-1} (z I_n - A)^{-1}
\]

\[
\times \left( L_1 \tilde{W}(z) + B H R(z) \right).
\]

(121)

Next, it follows from (113) that

\[
\tilde{w}_{k+1} = B_q \dot{x}_{k+1|k} + A_q \tilde{q}_{k+1|k} + L_2 \tilde{w}_k^m,
\]

(122)

\[
\tilde{w}_{k+1} = C_q \tilde{q}_{k+1|k},
\]

which implies

\[
\tilde{W}(z) = C_q (z I_M - A_q)^{-1} B_q C \tilde{X}(z) + C_q (z I_M - A_q)^{-1} L_2 \tilde{W}(z).
\]

(123)

Together, \((116)\) and \((122)\) imply

\[
\tilde{W}(z) = W_m(z) - C_q (z I_M - A_q)^{-1} \left( B_q C \tilde{X}(z) + L_2 \tilde{W}(z) \right).
\]

(124)

Substituting (121) yields

\[
\tilde{W}(z) = \left( 1 + z^{-M} C(z I_n - A)^{-1} L_1 + C_q (z I_M - A_q)^{-1} L_2 \right)^{-1}
\]

\[
\times \left( W_m(z) - z^{-M} G(z) U(z) \right)
\]

\[
= F(z) \left( W_m(z) - z^{-M} G(z) U(z) \right).
\]

(125)

It follows from Corollary C.1 that \( F(z) \) describes the tradeoff between the response of the estimation error to the plant disturbance \( D(z) \) and the measurement noise \( N(z) \). Using large estimator gains \( L_1 \) and/or \( L_2 \) will force \( F(z) \approx 0 \) and the disturbance response will be small at the expense of the noise being passed directly to the estimation error. Using small estimator gains has the opposite effect.

Let us now calculate the response of the system output \( y_k \). The definition of \( u_k \) and (121) yield

\[
U(z) = -K(z I_n - A + BK)^{-1} L_1 \tilde{W}(z)
\]

\[
+ (1 + K(s I_n - A)^{-1} B)^{-1} H R(z).
\]

(126)

It follows from \((108)\) and \((125)\) that

\[
Y(z) = C(z I_n - A + BK)^{-1} B H R(z) + C(z I_n - A)^{-1} E D(z)
\]

\[
- C(z I_n - A)^{-1} B K(z I_n - A + BK)^{-1} L_1 \tilde{W}(z),
\]

(127)
and from \( \eqref{124} \) that 
\[
\bar{W}^m(z) = F(z)z^{-M}C(zI_n - A)^{-1}ED(z) + F(z)N(z).
\]
Hence in the absence of disturbances and measurement noise the response of \( \hat{y}_k \) to a command \( r_k \) is the same with the observer as with state feedback, even with the \( M \)-step delay in the feedback loop.

**APPENDIX IV**

**Discrete System: Decomposition of Optimal Estimator with Delay**

We now characterize the closed loop eigenvalues for a discrete-time estimator with the structure given in Appendix III when the estimator is optimal, and present counterparts to the results for continuous-time in Section III. After doing so we will describe connections to the work of [15], who study discrete-time systems with a delay at the plant input.

To formulate the optimal estimation problem, we consider the augmented system
\[
\dot{x}_{aug}(k+1) = A_{aug}\hat{x}_{aug}(k) + B_{aug}u_k + L_{aug}(w_k^m - \hat{w}_{k|k-1}), \quad (126)
\]
\[
\hat{w}_{k|k-1} = C_{aug}\hat{x}_{aug}(k),
\]
where \( \hat{x}_{aug}(k) = \begin{bmatrix} x_{k|k-1} \\ \hat{x}_{k|k-1} \end{bmatrix}, A_{aug} = \begin{bmatrix} A & 0 \\ B_{aug} & C_{aug} \end{bmatrix}, B_{aug} = \begin{bmatrix} B \\ 0 \end{bmatrix}, L_{aug} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, C_{aug} = \begin{bmatrix} 0 & C_q \end{bmatrix} \). Suppose that \( d_k \) and \( n_k \) are zero mean Gaussian white noise processes with covariances \( V \geq 0 \) and \( W > 0 \), respectively, and assume that \( (A_{aug}, C_{aug}) \) is observable and \( (A_{aug}, E_{aug}) \) is controllable. Then the optimal estimator gain satisfies
\[
L_{aug} = A_{aug}^{-1}\Sigma_{aug}C_{aug}^{T}(C_{aug}\Sigma_{aug}C_{aug}^{T} + W)^{-1}, \quad (127)
\]
where \( \Sigma_{aug} \) is the unique positive semidefinite solution to the algebraic Riccati equation
\[
\Sigma_{aug} = A_{aug}\Sigma_{aug}A_{aug}^{T} + V_{aug} - A_{aug}\Sigma_{aug}C_{aug}^{T}(C_{aug}\Sigma_{aug}C_{aug}^{T} + W)^{-1}C_{aug}\Sigma_{aug}A_{aug}^{T} \quad (128)
\]
with \( V_{aug} = E_{aug}VE_{aug}^{T} \).

The eigenvalues of the optimal estimator are those of the closed loop matrix \( A_{CL} = A_{aug} - L_{aug}C_{aug} \), and will be characterized in Propositions D.2 and D.3 below, after the following lemma. Define \( L = \Sigma C_{CL}^{T}(C_{CL}^{T} + W)^{-1} \), where \( \Sigma \) is the unique positive semidefinite solution to the Riccati equation
\[
\Sigma = A\Sigma A^{T} - A\Sigma C_{CL}^{T}(C_{CL}^{T} + W)^{-1}C\Sigma A^{T} + EV^{T}E^{T}.
\]

**Lemma D.1.** Suppose that \( A \) has no eigenvalues equal to zero. Then \( A_{CL} = A - LC \) also has no eigenvalues equal to zero. If \( A \) has an eigenvalue equal to zero with multiplicity \( m \geq 1 \), then \( A_{CL} \) also has an eigenvalue equal to zero but with multiplicity one. Furthermore, this zero eigenvalue is an uncontrollable eigenvalue of \( (A, L) \).

**Proof.** The results of \([27]\) imply that the eigenvalues of \( A_{CL} \) may be found from the generalized eigenvalue problem \( M_{\ell}z_1 = \lambda M_{r}z_1, \quad (129) \) where \( M_{\ell} = \begin{bmatrix} I_n & CTW^{-1}A \\ 0_{n \times n} & A \end{bmatrix} \) and \( M_{r} = \begin{bmatrix} A^{T} & 0_{n \times n} \\ -EW^{T} & I_n \end{bmatrix} \). If \( A \) has no zero eigenvalues, then it is clear that \( M_{r} \) is nonsingular and thus zero cannot be an eigenvalue of \( A_{CL} \). If \( A \) has an eigenvalue equal to zero with multiplicity \( m \geq 1 \), then observability of \((A, C)\) implies that its geometric multiplicity must be equal to one. Hence there exist nonzero vectors \( v_1, \ldots, v_n \) such that \( Av_1 = 0 \) and \( Av_k = Av_{k-1}, \quad k = 2, \ldots, m \). Note that the vector \( z_1 = \begin{bmatrix} -CTW^{-1}Cv_1 \\ v_1 \end{bmatrix} \) lies in the nullspace of \( M_{r} \) and thus \( \lambda = 0 \) is an eigenvalue of \( A_{CL} \) with multiplicity at least one. For the multiplicity be greater than one, it is necessary that there exists \( z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} \) such that \( M_{r}z_2 = M_{r}z_1 \) from which it follows that \( Az_2 = EV^{T}CTW^{-1}Cv_1 + v_1 \). Properties of generalized eigenvectors imply that \( v_1 = Av_2 \), and thus that
\[
A(z_2 - v_2) = EV^{T}CTW^{-1}Cv_1. \quad (129)
\]

The fact that \((A, E)\) is controllable implies that \( E \) does not lie in the column space of \( A \), and thus that \( \eqref{129} \) has no nonzero solution. It follows that \( \lambda = 0 \) is an eigenvalue of \( A_{CL} \) with multiplicity \( m = 1 \). The dual of Lemma III.2 states that if \((A, C)\) is observable and \( A - LC \) share eigenvalues at zero, then these zero eigenvalues are uncontrollable eigenvalues of \((A, L)\).

**Proposition D.2.** Consider the optimal estimator defined by \( \eqref{126} - \eqref{128} \). Then \( A_{aug}^{-1}A_{aug} - L_{aug}C_{aug} \) has

(i) \( M \)-zero eigenvalues,

(ii) \( n \)-eigenvalues identical to those of \( A_{CL} \).

**Proof.** To prove part (i), we note the results of \([27]\) imply that the optimal closed loop eigenvalues corresponding to the discrete Riccati equation \( \eqref{128} \) may be found by solving the generalized eigenvalue problem
\[
M_{\ell}z_1 = \lambda M_{r}z_1, \quad (129)
\]
where \( M_{\ell} = \begin{bmatrix} I_{n + M} & CTW^{-1}C_{aug} \\ 0_{(n + M) \times (n + M)} & A_{aug} \end{bmatrix} \) and \( M_{r} = \begin{bmatrix} A^{T} & 0_{(n + M) \times (n + M)} \\ -V_{aug} & I_{n + M} \end{bmatrix} \). Using the structure of \( A_{q} \) and \( C_{q} \) it is straightforward to verify that
\[
z_1 = \begin{bmatrix} 0^{T}_{(n + M - 1) \times 1} \\ 1 \\ 0^{T}_{(n + M - 1) \times 1} \end{bmatrix} - W^{T} \]
lies in the nullspace of \( M_{r} \), and thus that \( \lambda = 0 \) is an eigenvalue of \( A_{CL} \). Furthermore, it may be shown by induction that the sequence of vectors
\[
z_2 = \begin{bmatrix} 0^{T}_{(n + M - 2) \times 1} \\ 1 \\ 0^{T}_{(n + M - 1) \times 1} \end{bmatrix} - W^{T} \]
\[
\ldots 
\]
\[
z_M = \begin{bmatrix} 0^{T}_{n \times 1} \\ 1 \\ 0^{T}_{(n + M - 1) \times 1} \end{bmatrix} - W^{T} \begin{bmatrix} 0^{T}_{(M - 1) \times 1} \end{bmatrix}^{T}
\]
satisfies \( M_{\ell}z_k = M_{r}z_{k-1}, \quad k = 2, \ldots, M \) and thus, from the results of \([27]\), the multiplicity of \( \lambda = 0 \) as an eigenvalue of \( A_{CL}^{aug} \) must be at least \( M \) due to the zeros introduced by the \( M \)-step delay.
To prove part (ii), let $\lambda$ be a nonzero eigenvalue of $A^{CL}$. Then there exists a nonzero vector $\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$ such that
\[
\begin{bmatrix} I_n & C^T W^{-1} C \\ 0_{n \times n} & A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} A^T \\ 0_{n \times n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]
Define $a_1 = -W^{-1} C v_2 \begin{bmatrix} \lambda^{-M} \\ \lambda^{-(M-1)} \\ \cdots \\ \lambda^{-1} \end{bmatrix}^T$, $a_2 = -W a_1$, and $\bar{v} = \begin{bmatrix} v_1^T \\ a_1^T \\ v_2^T \\ a_2^T \end{bmatrix}^T$. Then direct calculation shows that $M \bar{v} = \lambda M \bar{v}$, and thus $\lambda$ is also an eigenvalue of $A^{aug}$.

It follows from Lemma D.1 that $A^{CL}$ can have an eigenvalue equal to zero with multiplicity at most one. In this case there exists a nonzero vector $\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$ such that
\[
\begin{bmatrix} I_n & C^T W^{-1} C \\ 0_{n \times n} & A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0_{2n \times 1}.
\]
Note next that $v_2$ must be nonzero, and define
\[
z_{M+1} = \begin{bmatrix} C & 0_{n \times 1} \\ 0_{1 \times n} & 1 \\ W v_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} = 0_{(M+1) \times 1}.
\]
Then $M z_{M+1} = M z_{M+1}$, and thus by the same reasoning as in the proof of part (i), $\lambda = 0$ is an eigenvalue of $A^{aug}$ with multiplicity $M + 1$.

**Proposition D.3.** The zero eigenvalues of $A^{CL}$ are uncontrollable eigenvalues of $(A^{aug}, L^{aug})$.

**Proof.** From the dual of Lemma III.2, we see that since $(A^{aug}, C^{aug})$ is observable and that $A^{aug}$ and $A^{CL}$ share zero eigenvalues with $A^{aug}$, then these zero eigenvalues are uncontrollable eigenvalues of $(A^{aug}, L^{aug})$.

Let us now relate the preceding results to those of [15]. We showed in Proposition D.3 that an optimal estimation problem with a delay at the plant output results in closed loop estimator eigenvalues that are either at the origin or that arise from a lower order estimation problem that does not involve the delay. The authors of [15], on the other hand, consider an optimal regulator problem with a delay at the plant input and show that the optimal eigenvalues are either at the origin or arise from a lower optimal regulator problem that ignores the delay. Although these are dual results, the proof techniques used in [15] are different than those above.

**REFERENCES**

[1] H. L. More and J. M. Donelan, “Scaling of sensorimotor delays in terrestrial mammals,” *Proc R Soc B*, vol. 285, no. 1885, p. 20180613, 2018.

[2] M. S. Madhav and N. J. Cowan, “The synergy between neuroscience and control theory: the nervous system as inspiration for hard control challenges,” *Annu Rev Control Robot Auto Syst*, vol. 3, pp. 243–267, 2020.

[3] D. W. Franklin and D. M. Wolpert, “Specificity of reflex adaptation for task-relevant variability,” *J Neurosci*, vol. 28, no. 52, pp. 14 165–14 175, 2008.

[4] A. M. Haith, J. Pakpoor, and J. W. Krakauer, “Independence of movement preparation and movement initiation,” *J Neurosci*, vol. 36, no. 10, pp. 3007–3015, 2016.

[5] A. M. Zimmet, D. Cao, A. J. Bastian, and N. J. Cowan, “Cerebellar patients have intact feedback control that can be leveraged to improve reaching,” *eLife*, vol. 9, p. e53246, 2020.

[6] D. Susilardayya, W. Xu, T. M. Hall, F. Galán, K. Alter, and A. Jackson, “Extrinsic and intrinsic dynamics in movement intermittency,” *eLife*, vol. 8, p. e40145, Apr. 2019.

[7] F. Crevecoeur and M. Gevers, “Filtering compensation for delays and prediction errors during sensorimotor control,” *Neural Comput*, vol. 31, no. 4, pp. 738–764, 2019.

[8] O. J. M. Smith, “Closer control of loops with dead time,” *Chem. Eng. Prog.*, vol. 53, no. 5, pp. 217–219, 1957.

[9] P. García, P. Albertos, and T. Hägglund, “Control of unstable non-minimum-phase delayed systems,” *Journal of Process Control*, vol. 16, no. 10, pp. 1099–1111, 2006.

[10] R. Sanz, P. García, and P. Albertos, “A generalized Smith predictor for unstable time-delay siso systems,” *ISA transactions*, vol. 72, pp. 197–204, 2018.

[11] B. M. Lima, D. M. Lima, and J. E. Normey-Rico, “A robust predictor for dead-time systems based on the Kalman filter,” *IFAC-PapersOnLine*, vol. 51, no. 25, pp. 24–29, 2018.

[12] R. C. Miall, D. J. Weir, D. M. Wolpert, and J. Stein, “Is the cerebellum a Smith predictor?” *J Motor Behav*, vol. 25, no. 3, pp. 203–216, 1993.

[13] S. Tolou, M. C. Capolei, L. Vannucci, C. Laschi, E. Falotico, and M. V. Hernández, “A cerebellum-inspired learning approach for adaptive and anticipatory control,” *International Journal of neural systems*, vol. 30, no. 01, p. 1950028, 2020.

[14] L. Mirkin and N. Raskin, “Every stabilizing dead-time controller has an observer–predictor-based structure,” *automatica*, vol. 39, no. 10, pp. 1747–1754, 2003.

[15] L. Mirkin and D. Zanutto, “Dead-time compensation as an observer-based design,” *IEEE Control Systems Letters*, vol. 6, pp. 1604–1609, 2021.

[16] K. Natori, “A design method of time-delay systems with communication disturbance observer by using Padé approximation,” in 2012 12th *IEEE International Workshop on Advanced Motion Control (AMC)*, IEEE, 2012, pp. 1–6.

[17] A. Probst, M. Magana, and O. Sawodny, “Using a Kalman filter and a Padé approximation to estimate random time delays in a networked feedback control system,” *IEEE Control Theory Appl*, vol. 4, no. 11, pp. 2263–2272, 2010.

[18] G. Franklin, J. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 5th ed. *Reading, Mass.*: Addison–Wesley, 2006.

[19] P. J. Antsaklis and A. N. Michel, *Linear Systems*. *New York: McGraw–Hill*, 1997.

[20] S. G. Carver, T. Kiemen, N. J. Cowan, and J. J. Jeka, “Optimal motor control may mask sensory dynamics,” *Biol Cybern*, vol. 101, no. 1, pp. 35–42, July 2009.

[21] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. *New York NY: Wiley-Interscience*, 1972.

[22] A. J. Bastian, “Learning to predict the future: the cerebellum adapts feedforward movement control,” *Curr Opin Neurobiol*, vol. 16, no. 6, pp. 645–649, 2006.

[23] R. C. Miall and D. M. Wolpert, “Forward models for physiological motor control,” *Neural Netw*, vol. 9, no. 8, pp. 1265–1279, 1996.

[24] M. Paulin, “A Kalman filter theory of the cerebellum,” in *Dynamic interactions in neural networks: Models and data*. *Springer*, 1989, pp. 239–259.

[25] T. Kiemen, Y. Zhang, and J. J. Jeka, “Identification of neural feedback for upright stance in humans: stabilization rather than sway minimization,” *J Neurosci*, vol. 31, no. 42, pp. 15 144–15 153, Oct. 2011.

[26] Q.-C. Zhong, *Robust control of time-delay systems*. *Springer Science & Business Media*, 2006.

[27] T. Pappas, A. Laub, and N. Sandell, “On the numerical solution of the discrete-time algebraic riccati equation,” *IEEE Transactions on Automatic Control*, vol. 25, no. 4, pp. 631–641, 1980.