Extending the Set of Quadratic Exponential Vectors

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Abstract

We extend the square of white noise algebra over the step functions on \( \mathbb{R} \) to the test function space \( L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), and we show that in the Fock representation the exponential vectors exist for all test functions bounded by \( \frac{1}{2} \).

1 Introduction

Modulo minor variations in the choice of the test function space, the square of white noise (SWN) algebra has been introduced by Accardi, Lu and Volovich [ALV99] as follows. Let \( \mathcal{L} = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( c > 0 \) a constant. Then the SWN algebra \( \mathcal{A} \) over \( \mathcal{L} \) is the unital \( * \)-algebra generated by symbols \( B_f, N_f \) \( (f \in \mathcal{L}) \) and the commutation relations

\[
[B_f, B_g^*] = 2c\langle f, g \rangle + 4N_{fg}, \quad [N_f, B_g^*] = 2B_{fg},
\]

\( (f, g \in \mathcal{L}) \) and all other commutators 0. Note that by the first relation, \( N_f^* = N_f \).

A Fock representation of \( \mathcal{A} \) is a representation (\( * \), of course) \( \pi \) of \( \mathcal{A} \) on a pre-Hilbert space \( H \) with a unit vector \( \Phi \in H \), fulfilling \( \mathcal{A}\Phi = H \) and \( \pi(B_f)\Phi = \pi(N_f)\Phi = 0 \) for all \( f \in \mathcal{L} \). From the commutation relations it follows that a Fock representation is unique up to unitary equivalence. Existence of a Fock representation has been established by different proofs in [ALV99, AS00a, Sni00, AFS02] for \( d = 1 \). They extend easily to general \( d \in \mathbb{N} \). Henceforth, we speak about the Fock representation. The Fock representation would be faithful, if we require also that the \( N_f \) depend linearly on \( f \). By abuse of notation, we identify \( \mathcal{A} \) with its image \( \pi(\mathcal{A}) \) omitting, henceforth, \( \pi \).

The exponential vector \( \psi(f) \) to an element \( f \in \mathcal{L} \) is defined as

\[
\psi(f) := \sum_{m=0}^{\infty} \frac{B^{*m}_f \Phi}{m!}
\]
whenever the series exists. In Accardi and Skeide [AS00b] is has been shown for \( d = 1 \) that \( \psi(\sigma I_{(0, t)} ) \) exists for \( |\sigma| < \frac{1}{2} \) and that \( \langle \psi(\sigma I_{(0, t)} ), \psi(\rho I_{(0, t)} ) \rangle = e^{-\frac{1}{2} \ln(1 - 4\rho \sigma)} \). As noted in [AS00b], this extends to arbitrary step functions \( f, g \) on \( \mathbb{R} \) with \( \|f\|_\infty < \frac{1}{2} \), with inner product

\[
\langle \psi(f), \psi(g) \rangle = e^{-\frac{1}{2} \int \ln(1 - 4f(t)g(t)) \, dt}.
\]

Our scope is to extend the set of exponential vectors and the formula in (\( \text{(*)} \)) for their inner product to test functions \( f \in \mathcal{L} \) with \( \|f\|_\infty < \frac{1}{2} \).

In the “29th Quantum Probability Conference” in October 2008 in Hammamet, Tunisia, Dhahri explained that the extension can be done for exponential vectors to all elements \( f \) in \( \mathcal{L} \) with \( \|f\|_\infty < \frac{1}{2} \). This a part of the work Accardi and Dhahri [AD08] (in preparation) on the second quantization functor for the square of white noise. Here we give a simple proof of this partial result.

\section{The result}

\subsection{Theorem.} \textit{The exponential vector} \( \psi(f) \) \textit{exists for every} \( f \in \mathcal{L} \) \textit{with} \( \|f\|_\infty < \frac{1}{2} \) \textit{and the inner product of two such exponential vectors is given by (\( \text{(*)} \)).}

\begin{proof}
\begin{enumerate}[label=(\roman*)]
\item We show that the right-hand side of (\( \text{(*)} \)) exists. Indeed, by Taylor expansion we have 
\[ \ln(1 + x) \leq M_\delta |x| \] for \( |x| \leq 1 - \delta \) for every \( \delta \in (0, 1) \), where \( M_\delta \) may depend on \( \delta \) but not on \( x \). Choose \( \delta = 1 - 4 \|f\|_\infty \|g\|_\infty \) in \( (0, 1) \). Then
\[
\left| \ln(1 - 4f(t)g(t)) \right| \leq M_\delta \left| 4f(t)g(t) \right|.
\]

Since \( \left| 4f(t)g(t) \right| \) is integrable, so is \( \ln(1 - 4f(t)g(t)) \).

\item The function \( x \mapsto \ln x \) is increasing on the whole half line \( (0, \infty) \). It follows that also the function \( x \mapsto -\ln(1 - x) \) is increasing on \( (-1, 1) \). We conclude that \( \frac{1}{2} > |f| \geq |g| \) implies
\[ -\ln(1 - 4|f(t)|^2) \geq -\ln(1 - 4|g(t)|^2). \]

Choose for \( f \) an \( L^2 \)-approximating sequence of step functions \( (f_n)_{n \in \mathbb{N}} \) in such a way that \( |f| \geq |f_n| \) for all \( n \in \mathbb{N} \). By the dominated convergence theorem,
\[
\lim_{n \to \infty} e^{-\frac{1}{2} \int \ln(1 - 4f_n(t)g(t)) \, dt} = e^{-\frac{1}{2} \int \ln(1 - 4f(t)g(t)) \, dt}.
\]

\item In precisely the same way as in [AS00b], one shows that (\( \text{(*)} \)) is true for all step functions strictly bounded by \( \frac{1}{2} \). It follows that
\[
\lim_{n \to \infty} \|\psi(f_n)\|^2 = e^{-\frac{1}{2} \int \ln(1 - 4f(t)^2) \, dt}.
\]

\item Since \( \langle B^m \Phi, B^m \Phi \rangle \) is a polynomial (of degree \( m \)) in \( \langle f, f \rangle \), it depends continuously in
\end{enumerate}
\end{proof}

\footnote{The correlation kernel on the right-hand side coincides, modulo scaling, with the correlation kernel in Boukas’ representation [Bou91] of Feinsilver’s finite difference algebra [Fei87]. In [AS00b], this observation gave rise to the discovery of an intimate relation between the SWN algebra and the finite difference algebra.}
$L^2$–norm on $f$. So, for every $M \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$\left\langle \sum_{m=0}^{M} \frac{B^m_f \Phi}{m!}, \sum_{m=0}^{M} \frac{B^m_f \Phi}{m!} \right\rangle \leq \left\langle \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!}, \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!} \right\rangle + 1$$

$$\leq \left\langle \sum_{m=0}^{\infty} \frac{B^m_{f_n} \Phi}{m!}, \sum_{m=0}^{\infty} \frac{B^m_{f_n} \Phi}{m!} \right\rangle + 1 = \|\psi(f_n)\|^2 + 1 \leq e^{-\frac{1}{2} \int \ln(1-4|f(t)|^2) dt} + 1.$$

By the theorem on exchange of limits under domination, it follows that

$$\lim_{M \to \infty} \left\langle \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!}, \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!} \right\rangle = \lim_{M \to \infty} \lim_{n \to \infty} \left\langle \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!}, \sum_{m=0}^{M} \frac{B^m_{f_n} \Phi}{m!} \right\rangle = \lim_{n \to \infty} \|\psi(f_n)\|^2 = e^{-\frac{1}{2} \int \ln(1-4|f(t)|^2) dt}.$$

From this we conclude that $\psi(f)$ exists and that $\|\psi(f)\|^2 = e^{-\frac{1}{2} \int \ln(1-4|f(t)|^2) dt}$.

(v) Doing the same sort of computation for the difference $\psi(f) - \psi(f_n)$, it follows that

$$\lim_{n \to \infty} \psi(f_n) = \psi(f).$$

Approximating also $g$ by a sequence of step functions $g_n$ with $|g| \geq |g_n|$, we find $\lim_{n \to \infty} \langle \psi(f_n), \psi(g_n) \rangle = \langle \psi(f), \psi(g) \rangle$ (continuity of the inner product), and

$$\lim_{n \to \infty} e^{-\frac{1}{2} \int \ln(1-4|f_n(t)|^2) dt} = e^{-\frac{1}{2} \int \ln(1-4|f(t)|^2) dt}$$

(once more, by dominated convergence for $|f_n g_n| \leq |f g|$) on the other side. This shows (v) for all $f, g$ as specified.

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