ASYMPTOTIC OBSERVABLES IN GAPPED QUANTUM SPIN SYSTEMS

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Dedicated to the memory of Rudolf Haag

Abstract. This paper gives a construction of certain asymptotic observables (Araki-Haag detectors) in ground state representations of gapped quantum spin systems. The construction is based on general assumptions which are satisfied e.g. in the Ising model in strong transverse magnetic fields. We do not use the method of propagation estimates, but exploit instead compactness of the relevant propagation observables at any fixed time. Implications for the problem of asymptotic completeness are briefly discussed.

1. Introduction

This paper continues the model-independent discussion of scattering theory for gapped quantum spin systems, initiated in [BDN16]. This recent work gave a construction of wave operators for such systems along the lines of Haag-Ruelle theory [Ha58, Ru62]. In the present paper we consider another classical problem of scattering theory, namely the existence of asymptotic observables. This analysis is a step towards asymptotic completeness for lattice systems, which appears to be an open problem beyond the two body scattering [GS97, AB01]. We build on recent advances in algebraic QFT [DG12, DG13] but in contrast to these two references we will not use the method of propagation estimates [SiSo87] to prove the existence of asymptotic observables. We introduce a different technique, explained in more detailed below, which is based on compactness of the relevant propagation observables at any fixed time. The underlying physical picture resembles the Haag-Swieca compactness condition [HS65], but the mathematical implementation is quite different and does not require any additional phase space assumptions. The method works equally well for relativistic (algebraic) QFT and for quantum spin systems, and gives more satisfactory results than [DG13]. In the case of quantum spin systems additional technical problems arise, in particular involving the Weyl calculus on a lattice, which we treat in Appendix B.

As we will discuss relativistic QFT and lattice systems in parallel in this introduction, the group of space translations $\Gamma$ will stand for $\mathbb{R}^d$ or $\mathbb{Z}^d$ here and $\hat{\Gamma}$ will denote the Pontryagin dual. We consider a $C^*$-dynamical system $(A, \tau)$, where $\tau$ is a representation of the group of spacetime translations $\mathbb{R} \times \Gamma$ in automorphisms of the $C^*$-algebra of observables $A \subset B(H)$. Suppose that the system is in a positive energy representation, that is $\tau$ is implemented by a group of unitaries $U$ and the energy-momentum spectrum $\text{Sp} U$ is contained in $\mathbb{R}_+ \times \hat{\Gamma}$. Suppose furthermore that $A$ contains a norm-dense subalgebra $A_{a-loc}$ of ‘almost local’ observables (see Section 2). An important insight, which emerged in the context of relativistic QFT, is that scattering theory can be studied (in principle) in such general situation [AH67, BPS91, Bu90]. The central concept is the mathematical model of a particle detector, introduced by Araki and Haag in [AH67]

\[ C_t := \int_{\Gamma} d\mu(\mathbf{x}) h(x/t) \tau_{(t,\mathbf{x})}(B^*B), \]

where $d\mu$ is the Haar measure on $\Gamma$, the observable $B \in A_{a-loc}$ is ‘energy decreasing’ (see Section 3) and the function $h$ is supported near the velocity of the particle in question. The integration over whole space $\Gamma$ in [1.1], which compensates for spreading of the wave packet of the particle, allows for non-zero response of such detectors in the limit $t \to \infty$. Further, Araki

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and Haag considered coincidence arrangements of such detectors on vectors $\Psi \in \mathcal{H}$

\begin{equation}
C_{n}^{\text{out}} := \lim_{t \to \infty} \langle \Psi, C_{1,t} \ldots C_{n,t} \Psi \rangle,
\end{equation}

where the respective velocity functions $h_i$ have disjoint supports, and argued that the knowledge of such quantities (and their modifications accounting also for incoming particles) suffices to compute collision cross-sections. However, the difficult and still largely open part of this program is to control the limit in (1.2). This problem is solved in [AH67] under two assumptions: First, $\text{Sp} U$ contains an isolated eigenvalue at $\{0\}$ corresponding to the ground state vector $\Omega$ and and isolated mass shell $\mathfrak{h}$ of single-particle states (see Section 2). Second, $\Psi$ is a Haag-Ruelle scattering state of particles from $\mathfrak{h}$. Due to this second restriction, the analysis of [AH67] merely corroborates the conventional scattering theory based on wave operators and the scattering matrix.

Some progress on this problem of convergence occurred only recently in [DG12, DG13] in the context of algebraic QFT. While in these two references the assumptions on $\text{Sp} U$ are the same as in [AH67], the vector $\Psi$ does not have to be a Haag-Ruelle scattering state. More precisely, given a small bounded region $\Delta \subset \text{Sp} U \setminus \{0\}$, and choosing $\mathfrak{h}_t^*$ to be ‘creation operators’ of particles from $\mathfrak{h}$ whose energies-momenta sum up to a point in $\Delta$, the limit

\begin{equation}
Q_{n}^{\text{out}}(\Delta)\Psi = \lim_{t \to \infty} C_{1,t} \ldots C_{n,t} \Psi
\end{equation}

exists for all $\Psi$ in the spectral subspace of $\Delta$. Interestingly, $Q_{n}^{\text{out}}(\Delta)\Psi$ is always a Haag-Ruelle scattering state of particles from $\mathfrak{h}$ (even if $\Psi$ is not) and any scattering state from the spectral subspace of $\Delta$ can be obtained by this construction. Thus apart from generalizing the discussion of collision cross-sections from [AH67], this result provides a weak variant of asymptotic completeness. Formulated as a relation between asymptotic observables and scattering states, it seems to be sufficient for physical interpretation of experimental data. In fact, between a physical state $\Psi$ and a configuration of particles $Q_{n}^{\text{out}}(\Delta)\Psi$ there is always some intervening apparatus. Furthermore, it appears difficult to get much closer to asymptotic completeness than (1.3), without complete information about physical representations (sectors) of $\mathfrak{A}$: In fact, suppose that $\mathfrak{A}$ has some ‘charged’ representations, disjoint from the vacuum representation, which also carry single-particle states in their energy-momentum spectra. Then a configuration of several such particles with total charge zero gives a state $\Psi$ in the ground state representation. Such $\Psi$ is orthogonal to all Haag-Ruelle scattering states of particles from $\mathfrak{h}$ and thus the conventional asymptotic completeness fails in this situation. But the weaker concept discussed above remains valid, as such $\Psi$ is simply annihilated by the detectors in (1.3).

In this paper we establish (1.3) for gapped quantum spin systems, using a novel method for controlling the convergence of asymptotic observables, which we will now explain. For the purpose of demonstration we consider the quantum-mechanical Hamiltonians $H = \frac{1}{2}p^2 + V(x)$ and $H_0 = \frac{1}{2}p^2$ on $L^2(\mathbb{R}^d)$, where $p = -i\nabla_x$ and $V$ is some rapidly decreasing repulsive potential. The problem of existence of the wave operator

\begin{equation}
W^{\text{out}} = \lim_{t \to \infty} e^{iH}e^{-itH_0}
\end{equation}

is readily solved by applying the Cook’s method to the sequence $W_t := e^{itH}e^{-itH_0}$, i.e. noting that $t \to \|\partial_t W_t \psi\| = \|V(x)e^{-itH_0}\psi\|$ is integrable in $t$ on some dense domain of vectors $\psi \in L^2(\mathbb{R}^d)$. A toy model of an Araki-Haag detector has the form

\begin{equation}
c_t := e^{itH} \chi(p)h(x/t)\chi(p)e^{-itH},
\end{equation}

where $\chi, h \in C_0^\infty(\mathbb{R}^d)$, $0 \notin \text{supp} h$. (For technical reasons specific to our method we also require that $\text{supp} h \subset \text{supp} h$ and $\supp h$ is a convex set.) Here an attempted application of the Cook method gives

\begin{equation}
\partial_t c_t = \frac{1}{i} e^{itH} \chi(p)(\nabla_x h)(x/t)(p-x/t)\chi(p)e^{-itH} + O(t^{-2}),
\end{equation}

which is not manifestly integrable. Certainly in this simple situation we could proceed via the method of propagation estimates which (roughly speaking) amounts to guessing a suitable family of auxiliary sequences $c_t'$, computing $\partial_t c_t'$ and eliminating the problematic terms from the resulting system of inequalities. The underlying physical mechanism is that the difference
between the instantaneous and average velocity $(p - x/t)$ tends to zero along the asymptotic, ballisitic trajectory of the particle [Gr90].

However, it turns out that for our particular asymptotic observable there is a more direct way to proceed. It starts from a simple observation that for $\psi \in \text{Ran } W^{\text{out}}$ the limit $\lim_{t \to \infty} e^{itH} \psi$ is readily computed, thus it suffices to consider $\psi \in (\text{Ran } W^{\text{out}})^{\perp}$. For such vectors we can write

$$e^{itH} \psi = e^{itH} e^{-itH} \chi(p) h(x/t + p) \chi(p) e^{itH} e^{-itH} \psi$$

$$= e^{itH} e^{-itH} \chi(p) h(x/t + p) \chi(p) (W^{\text{out}}_\ast - (W^{\text{out}})^\ast) \psi$$

$$= e^{itH} e^{-itH} \chi(p) h(x/t + p) \chi(p) \int_{t}^{\infty} ds (-\partial_s W^{\ast}_s) \psi$$

$$= e^{itH} e^{-itH} \chi(p) h(x/t + p) \chi(p) \int_{t}^{\infty} ds \chi e^{isH} V(x) e^{-isH} \psi.$$  

(1.7)

Here in the second step we used that $(W^{\text{out}})^\ast \psi = 0$, in the third step we exploited compactness of $\chi(p) h(x/t + p) \chi(p)$ and in the last step we obtained an expression which tends to zero in norm. This follows from the decay of the potential, the assumptions on the supports of $\chi, h$ and the equality $g(x/t) \leq g(x/s)$ valid for $s \geq t$ if $g$ is the characteristic function of a convex set containing zero. Some similarity with existing methods of scattering theory has to be admitted (see the proofs of Theorems XI.7 and XI.112 of [RS3]). Nevertheless the strategy of controlling asymptotic observables presented in (1.7) seems to be new.

Of course in the above quantum mechanical situation the method does not give any new information, since asymptotic completeness of such Hamiltonians is well known from the outset. But for systems of $n > 2$ interacting particles with non-quadratic dispersion relations, as for example relativistic QFT or spin systems, it has advantages. In such systems only observables which keep the average and instantaneous velocity of a particle very close together could (so far) be handled by the method of propagation estimates [DG13]. The alternative method explained above only requires that the set of average velocities contains the set of instantaneous velocities (see Theorem 14.2 below for precise assumptions). To cover the case when the two sets are disjoint it seems necessary to combine the method with Mourre theory, which is so far an open problem. Another important open problem, which probably requires a different approach, is to control convergence (or existence of non-zero limit points) of Araki-Haag detectors without assuming mass shells in the energy-momentum spectrum.

This paper is organized as follows: Sections 2 and 3 are devoted to our framework and preliminaries. In section 4 we state and prove our main result. Some auxiliary results are postponed to Appendix A. Appendix B concerns pseudo-differential calculus on a lattice.

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2. Framework

We work in a framework outlined in the introduction of [BDN16] which is suitable for gapped quantum spin systems in irreducible ground state representations. Let $\Gamma = \mathbb{Z}^d$ be the abelian group of space translations and $\hat{\Gamma}$ its Pontryagin dual which is in our case $S^d_\mathbb{Z}$ i.e. the Brillouin zone. We will often use the parametrisation $S^d_\mathbb{Z} = [-\pi, \pi]^d$. Concerning the Schwartz classes $S(\mathbb{R} \times \Gamma), S(\Gamma)$ and Fourier transforms we follow the conventions and notation of Appendix D of [BDN16].

We consider a $C^\ast$-dynamical system $(\mathfrak{A}, \tau)$, where $\tau$ is a strongly continuous representation of space-time translations $\mathbb{R} \times \Gamma$ in automorphisms of a concrete $C^\ast$-algebra $\mathfrak{A}$, which acts irreducibly on a Hilbert space $\mathcal{H}$. We denote

$$\tau_f(a) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R} \times \Gamma} dt \mu(x) \tau_{(t,x)}(a) f(t,x), \quad f \in L^1(\mathbb{R} \times \Gamma),$$

$$\tau_g^{(d)}(a) := (2\pi)^{-\frac{d}{2}} \int_{\Gamma} d\mu(x) \tau_{x}(a) g(x), \quad g \in L^1(\Gamma),$$

(2.1)
The existence of such subalgebra \( \mathcal{A}_{\text{a-lo}} \) with the help of the Lieb-Robinson bounds.

As indicated in \[BDN16\], all our assumptions hold in the Ising model in strong transverse magnetic fields in any space dimension.

The set \( \text{Vel}(\text{supp } \tau) \) where \( \Delta \) supports. Then there exist \( \tau_{\tau_1}(A_1), \tau_{\tau_2}(A_2) = O((x_1 - x_2)^{-\infty}) \).

The existence of such subalgebra \( \mathcal{A}_{\text{a-lo}} \) in gapped quantum spin systems was shown in \[BDN16\] with the help of the Lieb-Robinson bounds.

Furthermore, we assume that \( \tau \) is implemented by a strongly continuous group of unitaries \( U \), i.e.

\[
\tau_{(t,x)}(A) = U(t,x)AU(t,x)^*, \quad A \in \mathcal{A}.
\]

We denote by \( P(\cdot) \) the spectral measure of \( U \) given by the SNAG theorem and denote its support by \( \text{Sp } U \subset \mathbb{R} \times \hat{\Gamma} \). We impose several restrictions on this energy-momentum spectrum:

(1) **Positivity of energy.** \( \text{Sp } U \subset \mathbb{R}_+ \times \hat{\Gamma} \).

(2) **Ground state vector.** \( (0,0) \) belongs to \( \text{Sp } U \) and is an isolated, simple eigenvalue corresponding to the eigenvector \( \Omega \).

(3) **Single-particle states.** There is an isolated mass-shell \( \h \subset \text{Sp } U \) which is a graph of a function \( \Sigma \in C^\infty(\hat{\Gamma}) \) whose Hessian matrix vanishes at most on a subset of Lebesgue measure zero. Moreover, \( (\h - \h) \cap \text{Sp } U = \emptyset \).

As indicated in \[BDN16\], all our assumptions hold in the Ising model in strong transverse magnetic fields in any space dimension.

3. **Preliminaries**

The Arveson spectrum (or the energy-momentum transfer) of an observable \( B \in \mathcal{A} \) is denoted \( \text{Sp } B \). We recall that \( (E,p) \in \text{Sp } B \) if for any neighbourhood \( V \) of this point there is \( f \in L^1(\mathbb{R} \times \Gamma) \), whose Fourier transform \( \hat{f} \) is supported in \( V \) and \( \tau_f(A) \neq 0 \). There holds

\[
\text{Sp } B = -\text{Sp } B, \quad \text{Sp } B \subset P(\Delta + \text{Sp } B)H,
\]

where \( \Delta \subset \mathbb{R} \times \hat{\Gamma} \) is a Borel subset. Clearly, for \( A \in \mathcal{A}_{\text{a-lo}} \) and \( f \in \text{Sp } S(\mathbb{R} \times \Gamma) \) we have \( \tau_f(A) \in \mathcal{A}_{\text{a-lo}} \) and \( \text{Sp } \tau_f(A) \subset \text{supp } \hat{f} \). For a comprehensive discussion of the Arveson spectrum in the context of spin systems we refer to \[BDN16\].

Now we recall the Haag-Ruelle scattering theory for spin systems developed in \[BDN16\]: Let \( B^*_i \in \mathcal{A}_{\text{a-lo}}, \ i = 1, \ldots, n, \) be s.t. \( \text{Sp } B^*_i \cap \text{Sp } U \subset \h \) and consider positive-energy wave packets of particles from \( \h \)

\[
g_{i,t}(x) := (2\pi)^{-\frac{d}{2}} \int_{\hat{\Gamma}} dp e^{-i\Sigma(p)t + ip \cdot x} \hat{g}_i(p), \quad \hat{g}_i \in C^\infty(\hat{\Gamma}).
\]

The set \( \text{Vel}(\text{supp } \hat{g}) := \{ \nabla \Sigma(p) \mid p \in \text{supp } \hat{g} \} \) is called the velocity support of \( \text{supp } \hat{g} \). The Haag-Ruelle creation operators are defined as

\[
B^*_i, t(g_{i,t}) := (2\pi)^{-\frac{d}{2}} \int_{\hat{\Gamma}} dp \mu(x) B^*_i(x)g_{i,t}(x),
\]

where \( B^*_i, t(x) := B^*_i(t,x) := U(t,x)B^*_i(t,x)^* \).

**Theorem 3.1.** \[BDN16\] Let \( B^*_i, \hat{g}_i \) be as above and s.t. \( \text{supp } \hat{g}_i \) have mutually disjoint velocity supports. Then there exist \( n \)-particle scattering states given by

\[
\Psi^{\text{out}} = \lim_{i \to \infty} B^*_1, t(g_{1,t}) \ldots B^*_n, t(g_{n,t})\Omega.
\]

Moreover, such states for \( n = 0, 1, 2 \ldots \) span a subspace \( \mathcal{H}^{\text{out}} \subset \mathcal{H} \) which is naturally isomorphic to the symmetric Fock space over the single-particle space \( \mathcal{H}_{\h} := P(\h)\mathcal{H} \).
This theorem belongs to a long list of results adapting the relativistic Haag-Ruelle theory to various non-relativistic models and frameworks, see e.g. [BF91, A173, NRT83]. We refer to [BDN16] for a more thorough discussion and comparison with the literature.

Next, we denote by $\mathcal{L}_0$ the subspace of all observables which are almost local and energy decreasing i.e. $B \in \mathfrak{A}_{\text{loc}}$ and $\text{Sp}_B \tau \subset (-\infty, 0) \times \hat{\Gamma}$. For such operators there holds an important result from [Bu90] which was adapted to the lattice case in [BDN16]:

**Theorem 3.2.** Let $\Delta \subset \mathbb{R} \times \hat{\Gamma}$ be compact and $B \in \mathcal{L}_0$. Then, for any finite set $K \subset \Gamma$,\n\n\begin{equation}
\| P(\Delta) \int_K \mu(x) (B^* B)(x) P(\Delta) \| \leq c_{\Delta},
\end{equation}

where $c_{\Delta}$ is independent of $K$.\n
In view of Theorem 3.2, the following maps are well defined for any $B_1, \ldots, B_n \in \mathcal{L}_0$:\n
\begin{equation}
\begin{array}{l}
a_{B_n, \ldots, B_1} : \mathcal{H} \to \mathcal{H} \otimes L^2(\Gamma^n), \\
(a_{B_n, \ldots, B_1}) (x_1, \ldots, x_n) = B_n(x_n) \cdots B_1(x_1) \Psi,
\end{array}
\end{equation}

where $\mathcal{H}$ is the domain of vectors of bounded energy, i.e. such $\Psi \in \mathcal{H}$ that $\Psi = P(\Delta) \Psi$ for some compact $\Delta$. For brevity we will write $a_\Gamma := a_{B_n, \ldots, B_1}$.

The maps $a_\Gamma$ where introduced and thoroughly studied in [DG12, DG13] in the context of local relativistic QFT and this discussion is easy to adapt to the lattice framework. Here we merely point out that $a_\Gamma P(\Delta) : \mathcal{H} \to \mathcal{H} \otimes L^2(\Gamma^n)$ is a bounded map for any compact set $\Delta$ and therefore

\begin{equation}
\| P(\Delta) a_\Gamma (A \otimes b) a_\Gamma P(\Delta) \| \leq c_{\Delta} \| A \| \| b \|,
\end{equation}

where $A \in B(\mathcal{H})$, $b \in B(L^2(\Gamma^n))$ and $c_{\Delta}$ is independent of $A, b$.

4. Main result

For $h \in C_0^\infty(\mathbb{R}^d)$, $B \in \mathcal{L}_0$ we define the approximating sequence of an Araki-Haag detector:

\begin{equation}
C_t := \int_{\Gamma} \mu(x) h(x/t)(B^* B) (t, x) = a_{B_1}(1 \otimes h_t) a_{B_1},
\end{equation}

where the second equality holds on $\mathcal{H}$ and $h_t(x) := h(x/t)$ is understood as a multiplication operator on $L^2(\Gamma)$. Our main result, stated in Theorem 4.2 below, concerns the strong convergence as $t \to \infty$ of products of such operators $C_{i,t}$ on the range of $P(\Delta)$, where $\Delta \subset \mathbb{R} \times \hat{\Gamma}$ is an open bounded set. Similarly as in [DG13], the set $\Delta$ and Arveson spectra of $B_t$ are linked by the following condition:

**Definition 4.1.** Let $\Delta \subset \mathbb{R} \times \hat{\Gamma}$ be an open bounded set and $B_1, \ldots, B_n \in \mathcal{L}_0$. We say that $B = (B_n, \ldots, B_1)$ is $\Delta$-admissible if

\begin{equation}
\text{Sp}_{B_i^*} \tau \cap \text{Sp} U \subset \mathfrak{h}, \ i = 1, \ldots, n,
\end{equation}

\begin{equation}
\text{Sp}_{B_1^*} \tau + \cdots + \text{Sp}_{B_n^*} \tau \subset \Delta,
\end{equation}

\begin{equation}
(\Delta - (\text{Sp}_{B_1^*} \tau + \cdots + \text{Sp}_{B_n^*} \tau)) \cap \text{Sp} U \subset \{0\},
\end{equation}

and addition is meant in the abelian group $\mathbb{R} \times \hat{\Gamma}$.

This condition essentially says that $B_i^*$ are ‘creation operators’ of particles with energies-momenta near some $(E_i, p_i) \in \mathfrak{h}$, which add up to a point in $\Delta$. In other words, if we used these $B_i^*$ to construct the scattering state $\Psi^\text{out}$ given by (3.3), then $\Psi^\text{out} \in P(\Delta) \mathcal{H}$. On the other hand, if for a given set $\Delta$ there is no $\Delta$-admissible $B$, then also $P(\Delta) \mathcal{H}^\text{out} = \{0\}$.

Finally, we introduce some notation used in the following theorem and its proof: For any $X \subset \hat{\Gamma}$ we define the velocity support $\text{Vel}(X) := \{ \nabla \Sigma(p') \mid p' \in X \} \subset \mathbb{R}^d$. For $Y \subset \mathbb{R} \times \hat{\Gamma}$, understood as a subset of the energy-momentum spectrum, we write $\pi_p(Y) := \{ p' \in \hat{\Gamma} \mid (E', p') \in Y \}$. For $Y \subset \mathbb{R}^d \times \hat{\Gamma}$, understood as a subset of the phase space (cf. Appendix B), we write $\pi_x(Y) := \{ x' \in \mathbb{R}^d \mid (x', \xi') \in Y \}$. The convex hull of a set $Z \subset \mathbb{R}^d$ will be denoted by $Z^\text{cv}$. We will denote by $Z^\delta$ a slightly larger set than $Z$, namely $Z^\delta := Z + B(0, \delta)$, where $B(0, \delta)$ is a closed ball of radius $\delta$ centered at zero.
Theorem 4.2. Let \( \Delta \subset \mathbb{R} \times \hat{\Gamma} \) be an open bounded set and \( B_i \in \mathbb{L}_0, i = 1, \ldots, n \), be \( \Delta \)-admissible in the sense of Definition 4.1 and s.t. \( \pi_\tau (\text{Sp}_{B_i}) \) are disjoint sets in the interior of \([-\pi, \pi]^d\). Let \( h_i \in C^\infty_0(\mathbb{R}^d) \) be s.t. \( \text{supp} h_i \supset \text{Vel}(\pi_\tau (\text{Sp}_{B_i})) \) and \( \text{(supp} h_i)^c \) are disjoint sets. Then, for \( C_i,t \) given by (4.4) and any \( \Psi \in P(\Delta)\mathcal{H} \) there exists the limit
\[
Q_n(\Delta) \Psi := s- \lim_{t \to \infty} C_{1,t} \cdots C_{n,t} \Psi,
\]
and belongs to \( P(\Delta)\mathcal{H}^{\text{out}} \). Moreover, \( P(\Delta)\mathcal{H}^{\text{out}} \) is the closed span of vectors of the form \( Q_n(\Delta) \Psi_2 \) for \( n = 1, 2, 3, \ldots \) provided that \( (\Sigma - \Sigma') \cap \text{Sp} U = \{ 0 \} \) and \( 0 \notin \Delta \).

Proof. We first note that for \( \Psi_2 \in P(\Delta)\mathcal{H}^{\text{out}} \) one can show the existence of \( Q_n(\Delta) \Psi_2 \) by standard arguments [AH67, DG13], which will not be repeated here. For similar reasons we skip the proof of the last statement of the theorem, which follows closely the reasoning from Section 7 of [DG13] (with some input from Subsection 4.4 of [BDN16]). We focus here on the most essential part of the proof, which consists in showing that for \( \Psi_2 \in P(\Delta)\mathcal{H}^{\text{out}} \), the sequence
\[
t \mapsto \langle \Psi_1, C_{1,t}, \ldots C_{n,t} \Psi_2 \rangle,
\]
tends to zero uniformly in \( \Psi_1 \in \mathcal{H}_c, \| \Psi_1 \| \leq 1 \). Due to almost locality of \( B_i \), disjointness of supports of \( h_i \) and \( \Delta \)-admissibility, expression (4.10) equals, up to a term of order \( \| \Psi_1 \| \| \Psi_2 \| O(t^{-\infty}) \),
\[
I_t := \langle \Psi_1, t^{\alpha_{\text{out}}}(\Omega) \langle \Omega \rangle \otimes h_{1,t} \cdots h_{n,t} \rangle a_{\text{out}}^{\Psi_2},
\]
Now by a partition argument, it suffices to consider
\[
I_t' := \langle \Psi_1, t^{\alpha_{\text{out}}}(\Omega) \langle \Omega \rangle \otimes h_{1,t} \cdots h_{n,t} \rangle a_{\text{out}}^{\Psi_2},
\]
where \( \text{Sp}_{B_i} \subset \text{Sp}_{B_i^c} \tau \) are sufficiently small, say contained in balls of radii \( 0 < \delta \ll 1 \). Before we proceed, let us introduce some notation:
\[
\tilde{x} := (x_1, \ldots, x_n),
\]
\[
\tilde{k} := (k_1, \ldots, k_n),
\]
\[
D \tilde{x} := (D x_1, \ldots, D x_n),
\]
\[
\tilde{\Sigma}(k) := \Sigma(k_1) + \cdots + \Sigma(k_n),
\]
\[
\tilde{g}(k) := \tilde{g}_1(k_1) \cdots \tilde{g}_n(k_n),
\]
\[
H(\tilde{x}) := h_1(x_1) \cdots h_n(x_n),
\]
\[
D := \{ \tilde{x} \in \mathbb{R}^{nd} | x_i = x_j \text{ for some } i \neq j \},
\]
where the lattice momentum operators \( D_{x_i} \) are defined in Appendix [3] and \( \tilde{g}_i \in C^\infty(\hat{\Gamma}) \) are equal to one on \( \pi_\tau (\text{Sp}_{B_i^c}) \) and supported in slightly larger sets \( (\pi_\tau (\text{Sp}_{B_i^c}))^\delta \) which are in the interior of \([-\pi, \pi]^d\). We note that by the disjointness of supports of \( h_i \) we have \( \text{supp} \text{H} \cap D = \emptyset \). Moreover, by Lemma [3] we can write
\[
I_t' := \langle \Psi_1, t^{\alpha_{\text{out}}}(\Omega) \langle \Omega \rangle \otimes H_t(\tilde{x}) \tilde{g}(D_{\tilde{x}}) \rangle a_{\text{out}}^{\Psi_2},
\]
Now we choose \( \tilde{g}_i' \in C^\infty(\hat{\Gamma}) \) s.t. \( \tilde{g}_i' \tilde{g}_i = \tilde{g}_i \) and \( \tilde{g}_i' \) are supported in \( (\pi_\tau (\text{Sp}_{B_i^c}))^\delta \). Setting \( \tilde{g}(D_{\tilde{x}}) := \tilde{g}(D_{x_1}) \cdots \tilde{g}(D_{x_n}) \), we have by the pseudo-differential calculus (see Appendix [3])
\[
H_t(\tilde{x}) \tilde{g}(D_{\tilde{x}}) = (H_t \tilde{g}(D_{\tilde{x}}))^2 + O(t^{-1})
\]
\[
= e^{-i\tilde{E}(D_{\tilde{x}})}(H_t \tilde{g}(D_{\tilde{x}})) e^{i\tilde{E}(D_{\tilde{x}})} + O(t^{-1})
\]
\[
= e^{-i\tilde{E}(D_{\tilde{x}})}(H_t \tilde{g}(D_{\tilde{x}})) e^{i\tilde{E}(D_{\tilde{x}})} + O(t^{-1}),
\]
where \( H_t(\tilde{x}, \xi) = H(\tilde{x} + \nabla \tilde{\Sigma}(\xi)) \) and \( H_0 \in C^\infty_0(\mathbb{R}^{nd}) \), \( 0 \leq H_0 \leq 1 \), is equal to one on \( \text{supp}(H_t \tilde{g}(\tilde{x})) \) and supported in \( (\pi_\tau \text{supp}(H_t \tilde{g}))^\delta \). Decomposing \( I_t' = I_t'' + O(t^{-1}) \) in accordance with (4.11)
Now we recall that
\(0^{{(4.18)}}\)
\(\pi h^{{(4.19)}}\) and contains zero, (here we use the assumption that \(\text{supp } I_g \in (4.15)\)). (The analysis of terms involving \(\dot{\partial} \partial \) using Lemma A.4 and sum back the series w.r.t. to \((4.12)\) and then obtain convergence to zero using locality and the fact that \(\text{supp } H^{{(4.13)}}\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.14)}}\)
\(\tilde{g}^\delta_i\) are wave packets introduced in Lemma A.3. By our assumptions, \(\text{Vel}(\pi_p(\text{Sp } B^\ast \tau))\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.15)}}\)
\(\tilde{g}^\delta_i\) are wave packets introduced in Lemma A.3. By our assumptions, \(\text{Vel}(\pi_p(\text{Sp } B^\ast \tau))\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.16)}}\)
\(\tilde{g}^\delta_i\) are wave packets introduced in Lemma A.3. By our assumptions, \(\text{Vel}(\pi_p(\text{Sp } B^\ast \tau))\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.17)}}\)
\(\tilde{g}^\delta_i\) are wave packets introduced in Lemma A.3. By our assumptions, \(\text{Vel}(\pi_p(\text{Sp } B^\ast \tau))\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.18)}}\)
\(\tilde{g}^\delta_i\) are wave packets introduced in Lemma A.3. By our assumptions, \(\text{Vel}(\pi_p(\text{Sp } B^\ast \tau))\) are disjoint sets, thus velocity supports of \(\text{supp } \tilde{g}^\delta_i\) are disjoint for \(\delta \) small. We obtain from Lemma A.2 that
\(0^{{(4.19)}}\)
where $H_0^t \in C_0^\infty(\mathbb{R}^n)$ is equal to one on $(\pi_\delta \text{supp } (H_0 \hat{g}'))^{t,\text{cv}}$ and supported in $(\pi_\delta \text{supp } (H_1 \hat{g}'))^{t,\text{cv},\delta}$. Thus we can write using Lemmas B.1, B.3
\[
\|H_0(\tilde{z}/t) e^{i \tilde{\Sigma}(D_x) s}\hat{g}(D_x) F_{2,s}\|_2 \leq \|H_0(\tilde{z}/s) e^{i \tilde{\Sigma}(D_x) s}\hat{g}(D_x) F_{2,s}\|_2
\]
(4.20)
\[
\leq \|(H_0^t g) w e^{i \tilde{\Sigma}(D_x) s} F_{2,s}\|_2 + \frac{1}{2s}\|\langle \nabla H_0^t \cdot \nabla \hat{g}\rangle^w e^{i \tilde{\Sigma}(D_x) s} F_{2,s}\|_2 + O(s^{-2})
\]
\[
= \|(H_0^t g) F_{2,s}\|_2 + \frac{1}{2s}\|\langle \nabla H_0^t \cdot \nabla \hat{g}\rangle^w F_{2,s}\|_2 + O(s^{-2}),
\]
where $H'(\tilde{z}, \xi) := H_0^t(\tilde{z} - \nabla \tilde{\Sigma}(\tilde{\xi}))$. As shown in Lemma A.3, $\pi_\delta \text{supp } (H' \hat{g}) \cap D = \emptyset$ for $\delta$ sufficiently small. Thus for such $\delta$ we can choose $G \in C_0^\infty(\mathbb{R}^n)$ which is equal to one on $\pi_\delta \text{supp } (H' \hat{g})$, supported in $(\pi_\delta \text{supp } (H' \hat{g}))^t$ and s.t. $\text{supp } G \cap D = \emptyset$. Then, making use of Lemma B.3.
(4.21)
\[
\|(H_0^t g) F_{2,s}\|_2 = \|(H_0^t g) G(\tilde{z}) F_{2,s}\|_2 + O(s^{-\infty}) = O(s^{-\infty}),
\]
where in the last step we exploited almost locality and the fact that $F_{2,s}$ contains a commutator. As the term involving $\nabla H_0^t$ can be treated analogously, this concludes the proof. $\square$

Let us add a remark about this proof. Since Lemma A.3 gives $|F(t)| \leq C_0 t^{-N}$, it may seem possible to compensate the polynomial growth with $t$ of the bound in (4.13) and conclude more directly that (4.12) tends to zero. Unfortunately, this strategy does not seem to work: with the resulting dependence of $C_0$ on $\alpha, \beta$ we were not able to control the sum in (4.12). Therefore, we followed a different route above.

**APPENDIX A. SOME AUXILIARY LEMMAS**

**Lemma A.1.** Let $B_1, \ldots, B_n \in \mathcal{L}_0$, $\Psi \in \mathcal{H}_c$ and $F_i(x_1, \ldots, x_n) := \langle \Omega, B_1(t, x_1) \ldots B_n(t, x_n) \Psi \rangle$. Then $F_i \in L^2(\Gamma^n)$ and for any $\tilde{g}_i \in C_0^\infty(\tilde{\Gamma})_\mathbb{R}$ which are equal to one on $\pi_\delta(\text{Sp } B_i \tau)$
\[
F_i = \tilde{g}_i(D_{x_1}) \ldots \tilde{g}_n(D_{x_n}) F_i.
\]

**Proof.** It follows from [Bos00] that $F_i \in L^2(\Gamma^n)$. By the same token, for $B \in \mathcal{L}_0$ and $\Phi, \Psi \in \mathcal{H}_c$ we have $\langle \Phi, B(t, \cdot) \Psi \rangle \in L^2(\Gamma)$. Moreover, for $g$ as in the statement of the lemma
\[
\langle \Phi, B(t, x) \Psi \rangle = \langle \tau_0^{(d)}(B^*(t, x)) \Phi, \Psi \rangle
\]
(4.2)
\[
= (2\pi)^{-d/2} \int_{\Gamma} d\mu(y) \bar{g}(y) \langle B^*(t, x + y) \Phi, \Psi \rangle
\]
\[
= (2\pi)^{-d/2} \int_{\Gamma} d\mu(y) \bar{g}(y) e^{i D_x y} \langle \Phi, B(t, x) \Psi \rangle = \tilde{g}(D_x) \langle \Phi, B(t, x) \Psi \rangle.
\]

By iterating this argument we conclude the proof. $\square$

The following lemma is a lattice variant of a result from [Bos00].

**Lemma A.2.** Let $\hat{g} \in C_0^\infty(\tilde{\Gamma})_\mathbb{R}$, $H_0 \in C_0^\infty(\mathbb{R}^n)$. Then
\[
\tilde{g}(D_x) H_0(t, \tilde{x}) \hat{g}(D_x) = \sum_{\alpha, \beta \in \mathbb{N}_0^n} \frac{\tilde{H}_{0,\alpha,\beta}^{(\alpha, \beta)}}{\alpha! \beta!} \langle \hat{g}(k) \hat{g}(k) \rangle |\hat{g}(k)\hat{g}(k)| + O(t^{-\infty}),
\]
where
\[
\tilde{H}_{0,\alpha,\beta}^{(\alpha, \beta)} := (2\pi)^{-d} \langle -1 \rangle^{d} |\alpha|! |\alpha|! + n^d \sum_{\ell \in \Gamma^n} \hat{H}_{0,\alpha,\beta}^{(\alpha, \beta)} (2\pi \ell t),
\]
\[
\hat{H}_{0,\alpha,\beta}^{(\alpha, \beta)}
\]

**Proof.** As in this proof we have to consider Fourier transforms both on $S(\mathbb{R}^n)$ and $S(\Gamma^n)$, we will treat $\Gamma^n = [-\pi, \pi]^n$ as a subset of $\mathbb{R}^n$. Let us consider the kernel of the operator on the l.h.s. of (A.3) in momentum space
\[
\langle \tilde{g}(D_x) H_0(t, \tilde{x}) \hat{g}(D_x) \rangle (\tilde{k}_1, \tilde{k}_2) = (2\pi)^{-nd/2} \hat{g}(\tilde{k}_2) \sum_{\ell \in \Gamma^n} \hat{H}_{0}(t(k_2 - k_1 + 2\pi \ell)) \hat{g}(k_1),
\]
(4.5)
where we made use of property \[\text{[B.11]}\] below. We decompose $\Gamma^n = \Gamma^n_1 + \Gamma^n_2$ s.t. $\Gamma^n_1$ is finite and for $\ell \in \Gamma^n_2$ we have $|k_2 - k_1 + 2\pi \ell| \geq 1$ for all $k_1, k_2 \in \hat{\Gamma}$. We analyse the following kernels:
\begin{equation}
\label{eq:A.6}
(g(D_{x})H_{0,\ell}(\hat{x})g(D_{x}))_{\ell}(\hat{k}_1, \hat{k}_2) := (2\pi)^{-nd/2}t^{nd}g(\hat{k}_2) \sum_{\ell \in \Gamma^n_{\ell}} \hat{H}_0(t(\hat{k}_2 - \hat{k}_1 + 2\pi \ell))g(\hat{k}_1).
\end{equation}

We note that due to the rapid decay of $\hat{H}_0$
\begin{equation}
\label{eq:A.7}
|\langle g(D_{x})H_{0,\ell}(\hat{x})g(D_{x}) \rangle_{\ell}(\hat{k}_1, \hat{k}_2)| \leq (2\pi)^{-nd/2}t^{nd} \frac{C_{2N}}{t^{N \ell}} \sum_{\ell \in \Gamma^n_{\ell}} \frac{1}{(e^t + |2\pi \ell|)^N},
\end{equation}
thus by Lemma \[\text{[3.3]}\] the corresponding operator gives the error term in \[\text{[A.3]}\].

Now we analyse the leading term. We write for $\ell \in \Gamma^n_1$, exploiting analyticity of $\hat{H}_0$,
\begin{equation}
\label{eq:A.8}
\hat{H}_0(t(\hat{k}_2 - \hat{k}_1 + 2\pi \ell)) = \sum_{\kappa \in N^n_0} \frac{i^{\kappa}}{\kappa !} \hat{H}_0^{(\kappa)}((2\pi t)) (\hat{k}_2 - \hat{k}_1)^{\kappa}
\end{equation}
which concludes the proof. $\square$

Lemma A.3. Let $\hat{g}_i$, $i = 1, \ldots, n$, be as in Lemma \[\text{[A.7]}\] and supported in the interior of $]-\pi, \pi[^d$. Then
\begin{equation}
\label{eq:A.9}
\langle \psi, a_{\Gamma}(\Omega) \otimes e^{-i\Sigma(D_{x})} \hat{g}(\hat{k}) \hat{g}(\hat{l}) \rangle = \langle \psi, B_{1, \ell}(g_{1, \ell}(1)) \ldots B_{n, \ell}(g_{n, \ell}(n)) \Omega \rangle,
\end{equation}
where
\begin{equation}
\label{eq:A.10}
g_{i, \ell}(x) = (2\pi)^{-d/2} \int_{\Gamma} dk e^{-i\Sigma(k)t + ik \cdot x} g_i(k) \hat{g}_i(k)^{\beta_i}, \quad i = 1, \ldots, n,
\end{equation}
and $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in N^n_0$.

Proof. The observation that $\langle \hat{x} | e^{-i\Sigma(D_{x})} \hat{g}(\hat{k}) \hat{g}(\hat{l}) \rangle = g_{i, \ell}(x_1) \ldots g_{n, \ell}(x_n)$ proves \[\text{[A.9]}\]. Due to the assumption that $\hat{g}_i$ are supported in the interior of $]-\pi, \pi[^d$, we have that $\hat{g}_i(k) \hat{g}_i(k^\beta) \in C_{\infty}(\hat{\Gamma})$ and \[\text{[A.10]}\] is a wave packet in the sense of definition \[\text{[3.2]}\]. $\square$

Lemma A.4. Let $g_{i, \ell}$ be as in Lemma \[\text{[A.3]}\] and s.t. $\text{Vel}(\text{supp} \hat{g}_i)$ are disjoint. Then, for any compact set $\Delta$ we have
\begin{equation}
\label{eq:A.11}
||B^*_{1, \ell}(g_{1, \ell}) P(\Delta)|| \leq c^{\beta_1},
\end{equation}
\begin{equation}
\label{eq:A.12}
||P(\Delta) [B^*_{1, \ell}(g_{1, \ell}), B^*_{2, \ell}(g_{2, \ell})] P(\Delta)|| \leq c_N s^{-N} c^{\beta_1 + \beta_2},
\end{equation}
where $c$ is independent of $\beta_1$ and $s$. The bound remains true if $B^*_{1, \ell}$ is replaced with $\partial_{\lambda}(B^*_{1, \ell}(g_{1, \ell}^{\lambda})))$.

Proof. By Theorem \[\text{[5.2]}\] compact energy-momentum transfer of $B_i$ and the Cauchy-Schwarz inequality we immediately get
\begin{equation}
\label{eq:A.13}
||B^*_{1, \ell}(g_{1, \ell}) P(\Delta)|| \leq c ||g_{1, \ell}||_2 \leq c^{\beta_1},
\end{equation}
where in the last step we used that $\hat{\Gamma}$ is a compact set.

Now let $\chi_i \in C_{\infty}(\mathbb{R}^d)$ be approximate characteristic functions of the velocity supports $\text{Vel}(\text{supp} \hat{g}_i)$ and $\chi'_i := 1 - \chi_i$. Clearly, we can choose $\chi_i$ with disjoint supports. Thus setting $\chi_i(x) := \chi_i(x/^s)$ we have by almost locality of $B_i$
\begin{equation}
\label{eq:A.14}
||B^*_{1, \ell}(\chi_1 g_{1, \ell}^{\beta_1}), B^*_{2, \ell}(\chi_2 g_{2, \ell}^{\beta_2})|| \leq c_N s^{-N} ||g_{1, \ell}^{\beta_1}||_2 \leq c_N s^{-N} c^{\beta_1 + \beta_2}.
\end{equation}
Now terms involving $\chi'_i$ can be estimated as follows using \[\text{[A.11]}\]
\begin{equation}
\label{eq:A.15}
||B^*_{1, \ell}(\chi_{1, \ell}^{\beta_1}, \chi_{2, \ell}^{\beta_2})|| \leq C ||\chi'_i(1 - g_{1, \ell}^{\beta_1} ||_2 ||g_{2, \ell}^{\beta_2}||_2.
\end{equation}
By a standard application of the non-stationary phase method (Theorem XI.14 of \[\text{[RS3]}\]) we obtain
\begin{equation}
\label{eq:A.16}
||\chi'_i g_{1, \ell}^{\beta_1}||_2 \leq c_N s^{-N} c^{\beta_1},
\end{equation}
which concludes the proof. $\square$
Lemma A.5. With definitions as in the proof of Theorem 4.2, $\pi_2\text{supp}(H\hat{g}) \cap D = \emptyset$ for $\delta$ sufficiently small.

Proof. Recall that

\begin{align}
H'(\tilde{x}, \tilde{\xi}) &= H'_0(\tilde{x} - \nabla \tilde{\Sigma}(\tilde{\xi})), \quad \text{supp } H'_0 \subset (\pi_2 \text{supp}(H_1\hat{g}'))^{\delta, \text{cv}, \delta}, \\
H_1(\tilde{x}, \tilde{\xi}) &= H(\tilde{x} + \nabla \tilde{\Sigma}(\tilde{\xi})), \quad \text{supp } H^{\text{cv}} \cap D = \emptyset.
\end{align}

Making use of the facts that for $X, Y, Z \subset \mathbb{R}^4$ we have $(X+Y)^{\text{cv}} = X^{\text{cv}}+Y^{\text{cv}}$, $(X+Y)^{\delta} \subset X^{\delta}+Y^{\delta}$, $Z^{\delta, \text{cv}} = Z^{\text{cv}, \delta}$, we obtain

\begin{align}
\pi_2\text{supp}(H'\hat{g}) &\subset \text{supp } H'_0 + \text{Vel}(\text{supp } \hat{g}) \\
&\subset (\pi_2 \text{supp}(H_1\hat{g}'))^{\delta, \text{cv}, \delta} + \text{Vel}(\text{supp } \hat{g})
\end{align}

(A.18)

where $R(\delta) \to 0$ as $\delta \to 0$. Here we denoted

\begin{align}
\text{Vel}(\text{supp } \hat{g}) := \{ \nabla \tilde{\Sigma}(\tilde{\xi}) \mid \tilde{\xi} \in \text{supp } \hat{g} \} = \text{Vel}(\text{supp } \hat{g}_1) \times \cdots \times \text{Vel}(\text{supp } \hat{g}_n),
\end{align}

and analogously for $\text{Vel}(\text{supp } \hat{g}')$. In the last step of (A.18) we used that $\text{supp } \hat{g}_i$, $\text{supp } \hat{g}_i'$ are contained in some balls $\mathcal{B}(k_i, 3\delta)$ and therefore, by continuity of $\nabla \Sigma$, the velocity supports $\text{Vel}(\text{supp } \hat{g}_i)$, $\text{Vel}(\text{supp } \hat{g}_i')$ are contained in some balls $\mathcal{B}(v_i, r(\delta))$, where $r(\delta) \to 0$ with $\delta \to 0$. This gives

\begin{align}
\text{Vel}(\text{supp } \hat{g}) - \text{Vel}(\text{supp } \hat{g}')^{\text{cv}, \delta} \subset \mathcal{B}(\bar{v}, \sqrt{\text{nr}(\delta)}) - \mathcal{B}(\bar{v}, \sqrt{\text{nr}(\delta)} + 2\delta) \\
&\subset \mathcal{B}(0, 2\sqrt{\text{nr}(\delta)} + 4\delta),
\end{align}

(A.20)

and therefore the last line of (A.18). Since $(\text{supp } H)^{\text{cv}} \cap D = \emptyset$, we conclude that $\pi_2\text{supp}(H'\hat{g}) \cap D = \emptyset$ for sufficiently small $\delta$.\qed

Appendix B. Pseudo-differential calculus on a lattice

A systematic exposition of pseudo-differential calculus on a lattice can be found e.g. in Appendix B of [LMR]. However, for the reader’s convenience, we derive the particular properties needed in this paper, as they are not easy to extract from the literature. Lemmas B.1, B.2 are well known in the continuum case and the proofs are obvious generalizations, so we can be brief. Lemma B.3 is less standard and the lattice causes some complications, so we give more details. We start with some definitions:

1. We say that $a \in S(\mathbb{R}^{d} \times \hat{\Gamma})$ if $a \in C^\infty(\mathbb{R}^{d} \times \hat{\Gamma})$ and $\sup_{x \in \mathbb{R}^{d}, \xi \in \hat{\Gamma}} |x^a \partial_x^2 \partial_\xi^2 a(x, \xi)| < \infty$.

2. We define a family of bounded self-adjoint operators $D_\xi^a$, $i = 1, \ldots, d$, on $L^2(\Gamma)$ by their action in motion space

\begin{align}
\hat{D}^a_\xi \phi(p) &= p^i \tilde{\phi}(p),
\end{align}

where we use the parametrization $\hat{\Gamma} = [-\pi, \pi]^d$. It immediately follows that $(e^{ip \cdot y} \tilde{\phi})(x) = \phi(x + y)$, where $\phi \in L^2(\Gamma)$, $y \in \Gamma$.

Now we define the Weyl quantization of a symbol $a \in S(\mathbb{R}^{d} \times \hat{\Gamma})$: For $\phi, \psi \in L^2(\Gamma)$ we write

\begin{align}
(a, a^w)(\phi, \psi) &= (2\pi)^{-d} \int_{\Gamma} d\mu(x) \int_{\Gamma} d\mu(y) \int_{\Gamma} d\xi \ a \left( \frac{x + y}{2}, \xi \right) \tilde{\phi}(x) \psi(y) e^{i(x \cdot y) \cdot \xi} \\
&= (2\pi)^{-d/2} \int_{\Gamma} d\mu(x) \int_{\Gamma} d\mu(y) \tilde{a} \left( \frac{x + y}{2}, x - y \right) \tilde{\phi}(x) \psi(y),
\end{align}

where check denotes here the inverse Fourier transform in the second variable.

Lemma B.1. Let $h \in S(\mathbb{R}^d)$, $g \in S(\Gamma)$. Then, with $h_t(x) := h(x/t)$,

\begin{align}
h_t(x)\hat{g}(D_x) = (h_t\hat{g})^w + \frac{i}{2t}((\nabla h)_t \cdot \nabla \hat{g})^w + O(t^{-2}),
\end{align}

where the l.h.s. is defined via functional calculus.
Proof. By formula \((B.2)\) and standard considerations we determine the kernels of \((h_t \hat{g})^w\) and \(h_t(x) \hat{g}(D_x)\)

\[
(h_t \hat{g})^w(x, y) = (2\pi)^{-d/2}h\left(\frac{x + y}{2t}\right)g(x - y),
\]

\[
(h_t(x) \hat{g}(D_x))(x, y) = (2\pi)^{-d/2}h(x/t)g(x - y).
\]

Next we write

\[
(h_t(x)g(D_x)\psi)(x, y) = (2\pi)^{-d/2}h((x + y)/(2t) + (x - y)/(2t))g(x - y).
\]

We set \(z := (x + y)/2, w := (x - y),\) denote

\[
K_s(x, y) := h(z/t + sw/(2t))g(w)
\]

and expand \(K_s\) into the Taylor series w.r.t. \(s:\)

\[
K_1(x, y) = h(z/t)g(w) + \frac{1}{2t}(\nabla h)(z/t) \cdot wg(w)
\]

\[
+ \frac{1}{(2t)^2} \int_0^1 ds' (1 - s') \partial_i \partial_j h(z/t + s'w/(2t))w^i w^j g(w).
\]

It is clear that the first two terms on the r.h.s. of \((B.7)\) are kernels of \((h_t \hat{g})^w\) and \(\frac{1}{2t}((\nabla h)_t, \nabla \hat{g})^w\). The norm of the last term is estimated using Lemma \(B.3\) below. \(\square\)

**Lemma B.2.** Let \(a \in S(\mathbb{R}^d \times \hat{\Gamma})\) and \(h \in S(\mathbb{R}^d)\) be s.t. \(h = 1\) on \(\pi_x \text{supp} a\). Then

\[
(1 - h_t(x))a_t^w = O(t^{-\infty}).
\]

**Proof.** Taylor expansion of kernels, analogous to the proof of Lemma \(B.3\). \(\square\)

**Lemma B.3.** Let \(a \in S(\mathbb{R}^d \times \hat{\Gamma})\) and \(\Sigma \in C^\infty(\hat{\Gamma})\). Then

\[
e^{i\Sigma(D_x)t}a_t^we^{-i\Sigma(D_x)t} = a_1^w + O(t^{-2}),
\]

where \(a_1(x, \xi) = a(x + \nabla \Sigma(\xi), \xi)\).

**Proof.** As in this proof we have to consider Fourier transforms both on \(S(\mathbb{R}^d)\) and \(S(\Gamma)\), it will be at times convenient to treat \(\hat{\Gamma} = [-\pi, \pi]^d\) as a subset of \(\mathbb{R}^d\) and functions on \(\hat{\Gamma}\) as periodic functions on \(\mathbb{R}^d\).

For \(\varepsilon \in \{0, 1\}\) we define the operator \(K_{\varepsilon, t}\) on \(L^2(\Gamma)\) given by the kernel

\[
K_{\varepsilon, t}(x, y) = (2\pi)^{-d} \int_{\hat{\Gamma}} d\xi \ a\left(\frac{x + y}{2t} + \varepsilon \nabla \Sigma(\xi), \xi\right) e^{i\xi(x - y)}
\]

\[
= (2\pi)^{-3d/2}t^d \int_{\mathbb{R}^d} d\xi \ a'(t\xi', \xi)e^{i\xi \cdot \nabla \Sigma(\xi)}e^{i\xi (x - y)}e^{i\xi' \cdot (x + y)},
\]

tilde denotes the Fourier transform (on \(S(\mathbb{R}^d)\)) in the first variable. Now using the Parseval theorem and the fact that for \(p \in \mathbb{R}^d\)

\[
(2\pi)^{-d} \int_{\Gamma} d\mu(x) e^{ip \cdot x} = \sum_{\ell \in \mathbb{Z}^d} \delta(p - 2\pi \ell),
\]

as an equality on \(S(\mathbb{R}^d)\), we obtain the kernel of \(K_{\varepsilon, t}\) in momentum space

\[
K_{\varepsilon, t}(p, q) = (2\pi)^{-d} \int_{\Gamma^2} d\mu(x) d\mu(y) e^{-ip \cdot x + iq \cdot y} K_{\varepsilon, t}(x, y)
\]

\[
= (2\pi)^{-d/2}t^d \sum_{\ell_1, \ell_2 \in \mathbb{Z}^d} \chi(\ell_1, \ell_2 \in \hat{\Gamma}) e^{i\ell \cdot \xi} e^{i\xi \cdot \nabla \Sigma(\xi)} a(t\xi', \xi),
\]

where \(p, q \in \hat{\Gamma}, \chi\) is the characteristic function and in the last step we set for \(\ell = (\ell_1, \ell_2)\):

\[
\xi_\ell := (p + q)/2 + \pi(\ell_2 - \ell_1), \quad \zeta_\ell := p - q - 2\pi(\ell_1 + \ell_2).
\]
We will study the kernel of the operator on the l.h.s. of (B.39), which is $e^{it\Sigma(p)}K_{0,t}(p,q)e^{-it\Sigma(q)}$, and show that its leading part gives the kernel of the first term on the r.h.s. of (B.39), which is $K_{1,t}(p,q)$. Using periodicity of $\Sigma$, we obtain for any $\ell$

\begin{equation}
\Sigma(p) = \Sigma \left( \xi_{\ell} + \frac{1}{2} \xi_{\ell}' \right), \quad \Sigma(q) = \Sigma \left( \xi_{\ell} - \frac{1}{2} \xi_{\ell}' \right).
\end{equation}

Consequently, we can write

\begin{equation}
e^{it\Sigma(p)}K_{0,t}(p,q)e^{-it\Sigma(q)} = (2\pi)^{-d/2}d^d \sum_{\xi_{\ell}, \xi_{\ell}' \in \Gamma} \chi(\xi_{\ell} \in \hat{\Gamma}) \tilde{a}(t(\xi_{\ell}'), \xi_{\ell}) e^{it(\Sigma(\xi_{\ell}+\xi_{\ell}')-\Sigma(\xi_{\ell}-\xi_{\ell}')).}
\end{equation}

We define the function of $\lambda \in \mathbb{R}$

\begin{equation}
f_{\xi_{\ell}, \xi_{\ell}'}(\lambda) := \Sigma(\xi_{\ell} + \lambda \xi_{\ell}'/2) - \Sigma(\xi_{\ell} - \lambda \xi_{\ell}'/2),
\end{equation}

and consider the Taylor expansion

\begin{equation}
f_{\xi_{\ell}, \xi_{\ell}'}(1) = \nabla \Sigma(\xi_{\ell}) \cdot \xi_{\ell}' + R_{\xi_{\ell}, \xi_{\ell}'};
\end{equation}

\begin{equation}
\Sigma_{\alpha, \ell'}(\xi_{\ell}, \xi_{\ell}') := \frac{1}{2(2\pi)^d} \left( \partial^\alpha \Sigma \right)(\xi_{\ell} + \lambda \xi_{\ell}'/2) - \left( \partial^\alpha \Sigma \right)(\xi_{\ell} - \lambda \xi_{\ell}'/2)).
\end{equation}

Here we exploited that $f_{\xi_{\ell}, \xi_{\ell}'}(0) = f_{\xi_{\ell}, \xi_{\ell}'}'(0) = 0$. (Due to this fact the error term in (B.39) is only $O(t^{-\frac{3}{2}})$. The above considerations give

\begin{equation}
e^{it(\Sigma(p)-\Sigma(q))} = e^{it\nabla \Sigma(\xi_{\ell}) \cdot \xi_{\ell}'} e^{itR_{\xi_{\ell}, \xi_{\ell}'}} = e^{it\nabla \Sigma(\xi_{\ell}) \cdot \xi_{\ell}'} + e^{it\nabla \Sigma(\xi_{\ell}) \cdot \xi_{\ell}'} \int_0^1 d\lambda' e^{it\lambda' R_{\xi_{\ell}, \xi_{\ell}'}}.
\end{equation}

First term on the r.h.s. of (B.39), substituted to (B.39), gives indeed $K_{1,t}(p,q)$. We estimate the remainder using Lemma [B.3] Thus we are interested in the norm of the operator $r$ given by the kernel

\begin{equation}
r(p, q) := (2\pi)^{-d/2}d^d \sum_{\xi_{\ell}, \xi_{\ell}' \in \Gamma} \tilde{a}(t(\xi_{\ell}'), \xi_{\ell}) e^{it(\xi_{\ell}') \cdot \nabla \Sigma(\xi_{\ell})} \chi(\xi_{\ell} \in \hat{\Gamma}) \int_0^1 d\lambda' e^{it\lambda' R_{\xi_{\ell}, \xi_{\ell}'}}.
\end{equation}

We note the estimate:

\begin{equation}
|r(p, q)| \leq ct^{d+1} \sum_{\xi_{\ell}, \xi_{\ell}'} |\tilde{a}(t(\xi_{\ell}'), \xi_{\ell})||\xi_{\ell}|^3 \chi(\xi_{\ell} \in \hat{\Gamma}).
\end{equation}

We verify the assumptions of Lemma [B.3] We set $\xi_{\pm} = \xi_{\pm} \pm 1$ and compute

\begin{equation}
\sup_{p \in \Gamma} \int_{\mathbb{R}^d} dq |r(p, q)| \leq ct^{d+1} \sup_{p \in \Gamma} \sum_{\xi_{\ell}, \xi_{\ell}'} \int_{\hat{\Gamma}} dq |\tilde{a}(t(\xi_{\ell}'), \xi_{\ell})| |\xi_{\ell}|^3 \chi(\xi_{\ell} \in \hat{\Gamma})
\end{equation}

\begin{equation}
\leq ct^{d+1} \sup_{p \in \Gamma} \sum_{\xi_{\ell}, \xi_{\ell}'} \int_{\hat{\Gamma}} dq |\tilde{a}(t(p - q - 2\pi \ell_{-}), (p + q)/2 + \pi \ell_{-})| 
\end{equation}

\begin{equation}
\times |p - q - 2\pi \ell_{-}|^{\alpha} \chi((p + q)/2 + \pi \ell_{-})
\end{equation}

\begin{equation}
\leq cN^{d+1} \sup_{p \in \Gamma} \sum_{\xi_{\ell}, \xi_{\ell}'} \int_{\mathbb{R}^d} dq (t(p - q - 2\pi \ell_{-}))^{-N} |p - q - 2\pi \ell_{+}|^{3}
\end{equation}

\begin{equation}
\leq cN^{d+1} \int_{\mathbb{R}^d} dq |tq|^{-N} |q|^{3} \leq cN^{-2} \int_{\mathbb{R}^d} dq |q|^{-N} |q|^{3}.
\end{equation}

Here in the third step we used the rapid decay of $\tilde{a}$ in the first variable and fact that $\chi((p + q)/2 + \pi \ell_{-})$ together with the condition $p, q \in \Gamma$ restrict the range of $\ell_{-}$ to a bounded set. In the fourth step we used that the union of $\hat{\Gamma} + 2\pi \ell_{+}$ over all $\ell_{+} \in \mathbb{Z}^d$ is $\mathbb{R}^d$. The second bound from Lemma [B.3] is verified analogously. \(\square\)

The following fact is known as the Schur lemma. Its proof is elementary (see e.g. [DG37]).
Lemma B.4. Let $Y, Y'$ be spaces with measures $d\mu$, $d\mu'$. Let $k(\cdot, \cdot)$ be a measurable function on $Y \times Y'$ s.t.

\begin{equation}
\operatorname{essup}_Y \int |k(y, y')| d\mu' \leq C, \quad \operatorname{essup}_{Y'} \int |k(y, y')| d\mu \leq C'.
\end{equation}

Then the operator $K : L^2(Y, d\mu') \to L^2(Y, d\mu)$ given by the kernel $k$ is bounded and $\|K\| \leq (CC')^{1/2}$.

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