Are the Stieltjes constants irrational?
Some computer experiments

Krzysztof D. Maślanka
1, Marek Wolf
2

1Institute for the History of Science of the Polish Academy of Sciences,
ul. Nowy Świat 72, pok. 9, 00-330 Warsaw, e-mail: krzysiek2357@gmail.com

2Cardinal Stefan Wyszynski University, Faculty of Mathematics and Natural Sciences.
College of Sciences,
ul. Wóycickiego 1/3, PL-01-938 Warsaw, Poland, e-mail: m.wolf@uksw.edu.pl

Abstract
Khinchin’s theorem is a surprising and still relatively little known result.
It can be used as a specific criterion for determining whether or not any
given number is irrational. In this paper we apply this theorem as well as the Gauss–Kuzmin theorem to several thousand high precision
(up to more than 53000 significant digits) initial Stieltjes constants $\gamma_n$,
$n = 0, 1, ..., 5000$ in order to confirm that, as is commonly believed,
they are irrational numbers (and even transcendental). We study also
the normality of these important constants.
1 Introduction

The famous zeta function $\zeta(s)$ discovered by L. Euler in 1737 and published in 1744 [3] as a function of real variable was investigated by G. F. B. Riemann in the complex domain in his famous memoir submitted in 1859 to the Prussian Academy [14]. It is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re(s) > 1. \quad (1)$$

It is divergent in the most interesting area of the complex plane, i.e., in the so called critical strip $0 \leq \Re(s) \leq 1$ where all complex zeros of zeta lie. However, as was shown by Riemann, the definition (1) does contain information about the zeta function on the entire complex plane but the process of analytic continuation must be used in order to reveal global behavior of this function. In fact Riemann in his paper analytically continued (1) to the whole complex plane except $s = 1$ by means of the following contour integral:

$$\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{P} \frac{(-x)^s dx}{e^x - 1} \quad , \quad (2)$$

where the integration is performed along the following path $P$:

\begin{center}
\begin{tikzpicture}
  \draw[thick,->] (0,0) -- (2,0);
  \draw[thick,->] (2,0) -- (2,2);
  \draw[thick,->] (2,2) -- (0,2);
  \draw[thick,->] (0,2) -- (0,0);
  \draw[thick] (1,1) circle (0.5);
\end{tikzpicture}
\end{center}

Till now dozens of integrals and series representing the $\zeta(s)$ function are known, for collection of such formulas see for example the entry Riemann Zeta Function in [17] and references cited therein and [12].

Another representation of this function is given by a power series where appear certain constants $\gamma_n$. These constants are essentially coefficients of the Laurent series expansion of the zeta function around its only simple pole at $s = 1$:

$$\zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s - 1)^n \quad \quad (3)$$

Primary definition of these fundamental constants was found by Th. J. Stieltjes and presented in a letter to Ch. Hermite dated June 23, 1885 [5, letters no. 71–74]

$$\gamma_n = \lim_{m \to \infty} \left[ \left( \sum_{k=1}^{m} \frac{(\ln k)^n}{k} \right) - \frac{(\ln m)^{n+1}}{n+1} \right] \quad . \quad (4)$$

(When $n = 0$ the numerator of the fraction in the first summand in (4) is formally $0^0$ which is taken to be 1.)

Effective numerical computing of the constants $\gamma_n$ is quite a challenge because the formulas (4) converge extremely slowly. Even when $n = 0$, which corresponds to the well-known Euler-Mascheroni constant $\gamma_0$, in order to obtain just 10 accurate digits one has...
to sum up exactly 12366 terms whereas in order to obtain 10000 digits (which is indeed required in some applications) one would have to sum up unrealistically large number of terms: nearly $5 \cdot 10^{4342}$ which is of course far beyond capabilities of the present day computers. However, various fast algorithms were found to efficiently compute specific value of the zeroth Stieltjes constant $\gamma_0$, i.e. the fundamental Mascheroni-Euler constant, see e.g. [16], [2]. For $n > 0$ the situation is still worse. Therefore we have to seek for other faster algorithms. In 1992 J. B. Keiper [7] published an effective algorithm based on numerical quadrature of certain integral representation of the zeta function and alternating series summation using Bernoulli numbers. Keiper’s algorithm was later implemented in widely used program *Mathematica*. An efficient but rather complicated method based on Newton-Cotes quadrature has been proposed by R. Kreminski in 2003 [9]. Quite recently F. Johansson presented particularly efficient method [6].

In the Appendix at the end of the present paper yet another method of computing Stieltjes constants will be described which is perhaps not as efficient as Johansson’s approach, yet it is by far more simple and it may be easily and quickly used in practical calculations for obtaining $\gamma_n$ up to $n \sim 10000$ with accuracy $\sim 50000$ significant digits.

We proceed as follows. First, we use the algorithm presented in the Appendix to calculate 5000 $\gamma_n$ with accuracies ranging from about 53000 significant digits ($\gamma_0$) to about 24000 digits ($\gamma_{5000}$). Having these numbers we intend to provide an argument in favor of their irrationality. Then we consider the question of their normality, as real expansions in the base equal 10. Finally, in Sect. 3, we develop $\gamma_n$’s into continuous fractions and next use the remarkable theorems due to Khinchin, Lévy and Gauss–Kuzmin. Obtained results support the common opinion that $\gamma_n$ are indeed irrational.

## 2 Normality

Let us recall that a number $r$ is normal in base $b$ if each finite string of $k$ consecutive digits appears in this expansion with asymptotic frequency $b^{-k}$. In the usual decimal base we have that each digit $0, 1, 2, \ldots, 9$ appears in the expansion of the number $r$ with limiting frequency 0.1, each 2–digits string $00, 01, \ldots, 99$ appears with density 0.01. Having the first 5000 Stieltjes constants with accuracies as described earlier we checked that each digit $0, 1, 2, \ldots, 9$ appears almost exactly with frequency 0.1. It is difficult to represent this $5000 \times 10$ data points in one plot. In the Fig. 1 we employed the following artifice: the frequency $h_n(0)$ of appearance of digit 0 in the Stieltjes constant $\gamma_n$ is plotted at $x$–axis value $n$ with the $y$ value $0.1 - h_n(0)$, i.e. the distance from the expected value 0.1, which in this case of $a = 0$ should be around 0.1. In general, the frequency $h_n(a)$ of appearance of digit $a$ in the Stieltjes constant $\gamma_n$ is plotted with the $y$ value $a \times 0.1 + (0.1 - h_n(a))$. We calculated also density of 100 strings of two digits $00, 01, \ldots, 99$ for all 5000 Stieltjes constants $\gamma_n$. Now the result consisted of half a million points, what is impossible to represent on the plot. Instead, in the Table I we present for each pattern of digits $ab$ the maximal difference between calculated frequency of appearance and the expected value of 0.01 and the number $n$ of the Stieltjes constant $\gamma_n$ for which this discrepancy appeared. The difference between the actual computed value of the frequency of two digits patterns and the expected value 0.01 was typically of a few percents.
3 Continued fractions expansions

Continued fractions often reveal various profound and unexpected properties of irrational numbers that are normally hidden in their traditional decimal (or other basis) notation, see e.g. [11].

In this Section we are going to exploit three facts about the continued fractions: the existence of the Khinchin constant, Khinchin–Lévy constant and the Gauss–Kuzmin distribution, see e.g. [8, chapter III, §15], [4, §1.8, §2.17], to support the irrationality of Stieltjes constants $\gamma_n$. The paper [1] presents the regular continued fraction for the Euler’s–Mascheroni constant $\gamma_0$. Let

$$r = [a_0(r); a_1(r), a_2(r), a_3(r), \ldots] = a_0(r) + \frac{1}{a_1(r) + \frac{1}{a_2(r) + \frac{1}{a_3(r) + \ddots}}} \quad (5)$$

be the continued fraction expansion of the real number $r$, where $a_0(r)$ is an integer and all denominators $a_k(r)$ ("partial quotients") with $k \geq 1$ are positive integers. Let us remark that rational numbers have finite number of coefficients $a_k$. Khinchin has proved [8], see
also \cite{15}, that limits of geometrical means of \( a_k(r) \) are the same for almost all real \( r \):

\[
\lim_{l \to \infty} \left( a_1(r) \ldots a_l(r) \right)^{\frac{1}{l}} = \prod_{m=1}^{\infty} \left( 1 + \frac{1}{m(m+2)} \right)^{\log_2 m} \equiv K_0 = 2.685452001 \ldots .
\] (6)

The Lebesgue measure of (all) the exceptions is zero and include rational numbers, quadratic irrationals and some irrational numbers too, like for example the Euler constant \( e = 2.7182818285 \ldots \) for which the limit (6) is infinity.

**Table 1**

In the columns A, C, E and G the two digits patterns are given, in the columns B, D, F and H the maximal differences between 0.01 and the frequency that a given pattern \( a, b \), 

\[
a, b = 0, 1, \ldots, 9
\]

appears among the digits of the \( \gamma_n, n = 1, 2, \ldots, 5000 \).

| A  | B                        | C                        | D                        | E                        | F                        | G                        | H                        |
|----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 00 | 2.4914 \times 10^{-3}    | 25                       | 1.7898 \times 10^{-3}    | 50                       | 2.0046 \times 10^{-3}    | 75                       | 1.9586 \times 10^{-3}    |
| 01 | 2.0114 \times 10^{-3}    | 26                       | 2.3187 \times 10^{-3}    | 51                       | 2.1064 \times 10^{-3}    | 76                       | 2.0058 \times 10^{-3}    |
| 02 | 2.0771 \times 10^{-3}    | 27                       | 2.0847 \times 10^{-3}    | 52                       | 2.2251 \times 10^{-3}    | 77                       | 2.1520 \times 10^{-4}    |
| 03 | 2.3235 \times 10^{-3}    | 28                       | 2.5891 \times 10^{-3}    | 53                       | 2.2773 \times 10^{-3}    | 78                       | 2.1413 \times 10^{-3}    |
| 04 | 1.8466 \times 10^{-3}    | 29                       | 2.1732 \times 10^{-3}    | 54                       | 1.9028 \times 10^{-3}    | 79                       | 2.2307 \times 10^{-3}    |
| 05 | 1.9006 \times 10^{-3}    | 30                       | 1.9310 \times 10^{-3}    | 55                       | 2.2080 \times 10^{-3}    | 80                       | 1.8309 \times 10^{-5}    |
| 06 | 1.8525 \times 10^{-3}    | 31                       | 2.0466 \times 10^{-3}    | 56                       | 2.4565 \times 10^{-3}    | 81                       | 2.1083 \times 10^{-3}    |
| 07 | 2.4075 \times 10^{-3}    | 32                       | 2.0625 \times 10^{-3}    | 57                       | 1.8966 \times 10^{-3}    | 82                       | 1.8493 \times 10^{-3}    |
| 08 | 2.4080 \times 10^{-3}    | 33                       | 2.1236 \times 10^{-3}    | 58                       | 1.9259 \times 10^{-3}    | 83                       | 2.1614 \times 10^{-3}    |
| 09 | 2.0118 \times 10^{-3}    | 34                       | 1.9970 \times 10^{-3}    | 59                       | 2.0112 \times 10^{-3}    | 84                       | 2.3112 \times 10^{-5}    |
| 10 | 2.1949 \times 10^{-3}    | 35                       | 2.2988 \times 10^{-3}    | 60                       | 1.9846 \times 10^{-3}    | 85                       | 2.6315 \times 10^{-3}    |
| 11 | 2.3476 \times 10^{-3}    | 36                       | 2.1588 \times 10^{-3}    | 61                       | 1.9017 \times 10^{-3}    | 86                       | 1.9200 \times 10^{-5}    |
| 12 | 1.8161 \times 10^{-3}    | 37                       | 2.2839 \times 10^{-3}    | 62                       | 1.9813 \times 10^{-3}    | 87                       | 2.1604 \times 10^{-5}    |
| 13 | 1.9746 \times 10^{-3}    | 38                       | 1.9860 \times 10^{-3}    | 63                       | 2.3341 \times 10^{-3}    | 88                       | 2.4448 \times 10^{-3}    |
| 14 | 2.3346 \times 10^{-3}    | 39                       | 2.1897 \times 10^{-3}    | 64                       | 2.2752 \times 10^{-3}    | 89                       | 2.3153 \times 10^{-3}    |
| 15 | 2.1317 \times 10^{-3}    | 40                       | 2.1021 \times 10^{-3}    | 65                       | 1.9558 \times 10^{-3}    | 90                       | 1.8766 \times 10^{-5}    |
| 16 | 1.8801 \times 10^{-3}    | 41                       | 2.2182 \times 10^{-3}    | 66                       | 2.3915 \times 10^{-3}    | 91                       | 2.2997 \times 10^{-3}    |
| 17 | 1.8627 \times 10^{-3}    | 42                       | 2.1976 \times 10^{-3}    | 67                       | 2.3017 \times 10^{-3}    | 92                       | 2.1946 \times 10^{-3}    |
| 18 | 2.0085 \times 10^{-3}    | 43                       | 1.9233 \times 10^{-3}    | 68                       | 2.1579 \times 10^{-3}    | 93                       | 1.8714 \times 10^{-3}    |
| 19 | 2.3663 \times 10^{-3}    | 44                       | 2.5452 \times 10^{-3}    | 69                       | 1.8103 \times 10^{-3}    | 94                       | 1.8551 \times 10^{-5}    |
| 20 | 1.8711 \times 10^{-3}    | 45                       | 1.9193 \times 10^{-3}    | 70                       | 2.0240 \times 10^{-3}    | 95                       | 2.7646 \times 10^{-3}    |
| 21 | 2.0741 \times 10^{-3}    | 46                       | 1.9071 \times 10^{-3}    | 71                       | 1.9349 \times 10^{-3}    | 96                       | 1.9379 \times 10^{-3}    |
| 22 | 2.2366 \times 10^{-3}    | 47                       | 2.1403 \times 10^{-3}    | 72                       | 1.9635 \times 10^{-3}    | 97                       | 2.0152 \times 10^{-5}    |
| 23 | 2.2588 \times 10^{-3}    | 48                       | 1.9612 \times 10^{-3}    | 73                       | 1.9174 \times 10^{-3}    | 98                       | 1.9536 \times 10^{-3}    |
| 24 | 2.3669 \times 10^{-3}    | 49                       | 1.9473 \times 10^{-3}    | 74                       | 1.9815 \times 10^{-3}    | 99                       | 2.0863 \times 10^{-3}    |

The constant \( K_0 \) is called the Khinchin constant, see e.g. \cite{4} §1.8. If the quantities

\[
K(r; l) = (a_1(r)a_2(r) \ldots a_l(r))^{\frac{1}{l}}
\]

for a given number \( r \) are close to \( K_0 \) we can regard it as an indication that \( r \) is irrational.

We developed the fractional parts of Stieltjes constants (in Sect.2, investigating the normality, we used the whole number, e.g. \( \gamma_{62} = 111670.9578149410793387893 \ldots \) and we
Fig. 2 The plot of maximal $a_k(n)$ for $n = 1, 2, 3, \ldots, 5000$.

use in this section only digits after the decimal dot) using built in PARI/GP [13] the function $\text{contfrac}(r, \{n_{\text{max}}\})$ which creates the row vector $a(r)$ whose components are the denominators $a_k(r)$ of the continued fraction expansion of $r$, i.e. $a = [a_0(r); a_1(r), \ldots, a_l(r)]$ means that

$$r \approx a_0(r) + \frac{1}{a_1(r) + \frac{1}{a_2(r) + \frac{1}{\ddots + \frac{1}{a_l(r)}}}}$$ (8)

The parameter $n_{\text{max}}$ limits the number of terms $a_{n_{\text{max}}}(r)$; if it is omitted the expansion stops with a declared precision of representation of real number $r$ at the last significant partial quotient: the values of the convergents $P_k(r)/Q_k(r)$

$$\frac{P_k(r)}{Q_k(r)} = a_0(r) + \frac{1}{a_1(r) + \frac{1}{a_2(r) + \frac{1}{\ddots + \frac{1}{a_l(r)}}}}$$ (9)

approximate the value of $r$ with accuracy at least $1/Q_k^2$ [8, Theorem 9, p.9]:

$$\left| r - \frac{P_k(r)}{Q_k(r)} \right| < \frac{1}{Q_k^2(r)},$$ (10)
hence when $1/Q_k^2$ is smaller than the accuracy of the number $r$ the process stops.

We checked that the PARI precision set to $\sqrt{p} \approx 120000$ digits is sufficient in the sense that scripts with larger precision generated exactly the same results: the rows $a(\gamma_n)$ obtained with accuracy 140000 digits were the same for all $n$ as those obtained for accuracy 120000 and the continued fractions with accuracy set to 100000 digits had different denominators $a_k(\gamma_n)$. The number of partial quotients $a_k$ varied from over 110000 for initial Stieltjes constants to 48027 for $\gamma_{5000}$, i.e. the value of $l(n)$ was roughly 2 times the number of digits in the expansion of $\gamma_n$. However, there have been cases of extremely large values of partial quotients. The largest was $a_{13034} = 17399017050$ for $\gamma_{2366}$, marked by the red arrow at the top in Fig. 2.

![Figure 3](image)

Fig.3 The plot of $K_l(n(l))$ for $n = 1, 2, 3, \ldots, 5000$. There are 384 points closer to $K_0$ than 0.001 and 30 points closer to $K_0$ than 0.0001. The largest value of $|K_0 - K_n(l(n))|$ is $4.47 \times 10^{-2}$ and it occurred for the Stieltjes constant number $n = 3235$ (marked with the red arrow), the smallest value of $|K_0 - K_n(l(n))|$ is $1.02 \times 10^{-5}$ and it occurred for $\gamma_{1563}$.

With the precision set to 120000 digits we have expanded each $\gamma_n$, $n = 1, 2, \ldots, 5000$ into its the continued fractions ($\doteq$ means “approximately equal”)

$$\gamma_n \doteq [a_0(n); a_1(n), a_2(n), a_3(n), \ldots, a_{l(n)}(n)] \equiv a(n) \quad (11)$$

without specifying the parameter $n_{max}$, thus the length of the vector $a(n)$ depended on $\gamma_n$ and it turns out that the number $l(n)$ of denominators was contained between 53000 for Stieltjes constants with index around 5000 and 110000 for gammas with smallest index $n$. The value of the product $a_1 a_2 \ldots a_{l(n)}$ was typically of the order $10^{47000}$ for beginning Stieltjes constants to $10^{23000}$ for the last $\gamma_n$’s. It means that, if these Stieltjes constants...
are rational numbers $P/Q$ then $Q$ are larger then those big numbers, for justification see e.g. [8] Theorems 16, 17]. Next for each $n$ we have calculated the geometrical means:

$$K_n(l(n)) = \left( \prod_{k=1}^{l(n)} a_k(n) \right)^{1/l(n)}.$$  \hspace{1cm} (12)

The results are presented in the Fig.3. Values of $K_n(l(n))$ are scattered around the red line representing $K_0$. To gain some insight into the rate of convergence of $K_n(l(n))$ we have plotted in the Fig. 4 the number of sign changes $S_K(n)$ of $K_n(m) - K_0$ for each $n$ when $m = 100, 101, \ldots l(n)$, i.e.

$$S_K(n) = \text{number of such } m \text{ that } (K_n(m + 1) - K_0)(K_n(m) - K_0) < 0.$$  \hspace{1cm} (13)

The largest $S_K(n)$ was 961 and it occurred for the $\gamma_{1175}$ and for 124 gammas there were no sign changes at all. It is well known that the convergence to Khinchin’s constant is very slow. In the Fig.4 for each $\gamma_n$ we present the closest to the Khinchin constant $K_0$ value of the “running” geometrical means

$$K_n(m) = \left( \prod_{k=1}^{m} a_k(n) \right)^{1/m}, \quad m = 100, 101, \ldots, l(n).$$  \hspace{1cm} (14)

Fig.4 The number of sign changes $S_K(n)$ for each $n$, i.e. the number of such $m$ that $(K_n(m + 1) - K_0)(K_n(m) - K_0) < 0$ (the initial transient values of $m$ were skipped—sign changes were detected for $m = 100, 101, \ldots l(n)$).
Fig. 5 The plot of the closest to the Khinchin constant $K_0$ values of the “running” geometrical means $K_n(m)$.

Let the rational $P_k/Q_k$ be the $n$-th partial convergent of the continued fraction:

$$
\frac{P_k}{Q_k} = [a_0; a_1, a_2, a_3, \ldots, a_k].
$$

For almost all real numbers $r$ the denominators of the finite continued fraction approximations fulfill \[\text{[8, chapter III, §15]}:\]

$$
\lim_{k \to \infty} \left( Q_k(r) \right)^{1/k} = e^{\pi^2/12\ln^2} \equiv L_0 = 3.275822918721811 \ldots
$$

where $L_0$ is called the Khinchin—Lévy’s constant \[\text{[11, §1.8]}\]. Again the set of exceptions to the above limit is of the Lebesgue measure zero and it includes rational numbers, quadratic irrational etc.

Let the rational $P_l(n)/(\gamma_n)/Q_l(n)/(\gamma_n)$ be the $l$-th partial convergent of the continued fractions \[\text{[11]}\] of $\gamma_n$:

$$
\frac{P_l(n)/(\gamma_n)}{Q_l(n)/(\gamma_n)} = a(n) \div \gamma_n.
$$

For each Stieltjes constant $\gamma_n$ we calculated the partial convergents $P_l(n)/(\gamma_n)/Q_l(n)/(\gamma_n)$ using the recurrence:

$$
P_0 = a_1, \quad Q_0 = 1, \quad P_1 = 1 + a_1 a_2, \quad Q_1 = a_1
$$

$$
P_k = a_k P_{k-1} + P_{k-2}, \quad Q_k = a_k Q_{k-1} + Q_{k-2}, \quad k \geq 2.
$$
Next from these denominators $Q_{l(n)}(\gamma_n)$ we have calculated the quantities $L_n(l(n))$:

$$L_n(l(n)) = (Q_{l(n)})^{1/l(n)}, \quad n = 1, 2, \ldots, 5000. \tag{19}$$

The obtained values of $L_n(l(n))$ are presented in the Fig.6. These values scatter around the red line representing the Khinchin—Lévy’s constant $L_0$ and are contained in the interval $(L_0 - 0.053, L_0 + 0.053)$. Again this plot is somehow misleading because there are Stieltjes constant $\gamma(n)$ for which there appear sign changes of $L_0 - L_n(m), \ m = 1, 2, \ldots, l(n)$. As in the case of $K_n(m)$ Fig.7 presents the number of sign changes of the difference $L_n(m) - L_0$ of the denominator of the $m$-th convergent $P_m/Q_m$

$$S_L(n) = \text{number of such } m \text{ that } (L_n(m + 1) - L_0)(L_n(m) - L_0) < 0. \tag{20}$$

The maximal number of sign changes was 922 and appeared for the Stieltjes constant $\gamma_{771}$ and there were 117 gammas without sign changes.

Finally we looked into the distribution of the values of partial quotients $a_l(n)$. The Gauss–Kuzmin theorem [8, chapter III, §15] asserts that the density $d(k)$ of the denominators $a_m, \ m = 1, 2, \ldots, l$, with the value $k$ is given by

$$\lim_{l \to \infty} \frac{\sharp \{m : a_m = k\}}{l} = \log_2 \left( \frac{1 + \frac{1}{k}}{1 + \frac{1}{1+k}} \right) \tag{21}$$

for almost all real numbers. In the Fig. 9 the results are presented for the Stieltjes constants.

4 Appendix: Obtaining high precision numerical values of Stieltjes constants

In 1997 it was shown by one of the authors of the present note [10] (M.K.) that the Riemann zeta function may be expressed as

$$\zeta(s) = \frac{1}{s-1} \left[ A_0 + \left( 1 - \frac{s}{2} \right) A_1 + \left( 1 - \frac{s}{2} \right) \left( 2 - \frac{s}{2} \right) \frac{A_2}{2!} + \ldots \right] = \tag{22}$$

$$= \frac{1}{s-1} \sum_{k=0}^{\infty} A_k \prod_{i=1}^{k} \left( i - \frac{s}{2} \right) = \tag{23}$$

$$= \frac{1}{s-1} \sum_{k=0}^{\infty} \frac{\Gamma \left( k + 1 - \frac{s}{2} \right) \Gamma \left( 1 - \frac{s}{2} \right)}{k!} A_k \quad s \in \mathbb{C}\{1\} \tag{24}$$

where

$$A_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j + 1) \zeta(2j + 2) = \tag{25}$$

$$= \frac{1}{2} \sum_{j=0}^{k} \binom{k}{j} (2j + 1) \frac{(2\pi)^{2j+2} B_{2j+2}}{(2j + 2)!} \tag{26}$$

10
Fig.6 The plot of $L_n(l(n))$ for $n = 1, 2, 3, \ldots, 5000$. There are 352 points closer to $L_0$ than 0.001 and 38 closer to $L_0$ than 0.0001. The largest value of $|L_0 - L_n(l(n))|$ is $4.503 \times 10^{-2}$ and it occurred for the Stieltjes constant number $l = 3235$ (marked with the red arrow), the smallest value of $|L_0 - L_n(l(n))|$ is $2.336 \times 10^{-6}$ and it occurred for the Stieltjes constant number $l = 3226$.

Here $B_n$ denotes the $n$th Bernoulli numbers. However, the particular choice of nodes in $s = 2, 4, 6, \ldots$, albeit the most natural, is by no means the only one. One only requires that the prescribed points be strictly equally spaced. For the purpose of present calculations we choose the following sequence of points:

$$1 + \varepsilon, 1 + 2\varepsilon, 1 + 3\varepsilon, \ldots$$

where $\varepsilon$ is certain real, not necessarily small number.

More precisely, define certain entire function $\varphi$ as:

$$\varphi(s) := (s - 1)\zeta(s) \quad s \neq 1$$

together with $\varphi(1) = 1$ which stems from the appropriate limit. Then, instead of (22), we have

$$\varphi(s) = \sum_{k=0}^{\infty} \frac{\Gamma \left( k - \frac{s-1}{\varepsilon} \right) \alpha_k}{\Gamma \left( - \frac{s-1}{\varepsilon} \right) k!}$$

with

$$\alpha_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \varphi(1 + j\varepsilon)$$

(27)

Note that coefficients $\alpha_k$ depend on $\varepsilon$ but we shall for simplicity drop temporarily this dependence in notation.
Fig. 7 The number of sign changes $S_L(n)$ for each $n$, i.e. the number of such $m$ that $(L_n(m+1) - L_0)(L_n(m) - L_0) < 0$ (the initial transient values of $m$ were skipped—sign changes were detected for $m = 100, 101, \ldots l(n)$).

As mentioned in the Introduction the Stieltjes constants are essentially coefficients of the Laurent series expansion of the zeta function around its only simple pole at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n$$

(28)

Now directly from (3) we have:

$$\gamma_n = \left. (-1)^n \frac{d^{n+1}}{ds^{n+1}} \varphi(s) \right|_{s-1}.$$  

Then, after some elementary calculations, we get the following useful result:

$$\gamma_n = \frac{(-1)^n n! \varepsilon^{n+1}}{n+1} \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \alpha_k S(k, n+1)$$

(29)

where $S(k, i)$ are signed Stirling numbers of the first kind. Note that in the literature there are different conventions concerning denotation and indices of Stirling numbers which can be confusing. Here we shall adopt the following convention involving the Pochhammer symbol:

$$(x)_k \equiv \frac{\Gamma(k+x)}{\Gamma(x)} = \prod_{i=0}^{k-1} (x+i) = (-1)^k \sum_{i=0}^{k} (-1)^i S(k, i)x^i$$

Denoting

$$\beta_{nk} \equiv (-1)^{n+k} \frac{n! S(k, n+1)}{\varepsilon^{n+1}}$$

12
we can rewrite (29) as formally an infinite matrix product

\[ \gamma_n = \sum_{k=n+1}^{\infty} \beta_{nk} \alpha_k \]  

(30)

The summation over \( k \) starts from \( n + 1 \) since \( \beta_{nk} \equiv 0 \) for \( k \leq n \). Accuracy of \( \alpha_1 \) is equal to accuracy of precomputed values of \( \varphi(s) \) in equidistant nodes. When \( k \) grows the accuracy of consecutive \( \alpha_k \) quickly tends do zero. Thus there always exists certain cut-off value of \( k = k_0 \). Therefore the summation in (30) may be performed to this value:

\[ \gamma_n = \sum_{k=n+1}^{k_0} \beta_{nk} \alpha_k \]  

(31)

(Numerical experiment confirm that adding more terms do not affect the value of the sum (31).) As pointed earlier \( \varepsilon \) need not to be small, however, choosing smaller \( \varepsilon \) greatly accelerates convergence of the series. However, it also turns out that smaller \( \varepsilon \) implies smaller \( k_0 \). What is really important: All significant digits of \( \gamma_n \) obtained from the finite sum (31) are correct.

Of course, \( \gamma_n \) eventually does not depend on \( \varepsilon \) although \( \alpha_k \) as well as the rate of convergence of (29) does. In fact series (29) converges for any value of \( \varepsilon > 0 \) but the rate of convergence becomes terribly small for \( \varepsilon \gg 1 \). On the other hand, the smaller \( \varepsilon \) the faster the rate of convergence. However, since \( \alpha_k \) also depends on \( \varepsilon \), choosing smaller
Fig. 9 The plot of the density of partial quotients $a_k$ equal to $k = 1, 2, \ldots, 10$ from top to bottom for first 5000 Stieltjes constants. In red are the values of (21) plotted. The $y$ axis is logarithmic to move the plots apart.

value for $\varepsilon$ requires higher accuracy of precalculated values of $\varphi(s)$ which in turn may be very time consuming. Hence, an appropriate compromise in choosing $\varepsilon$ is needed.

Formula (29) is particularly suited for numerical calculations. As already pointed above, one has to choose parameter $\varepsilon$ in order to optimally perform the calculations. Typically the algorithm has three simple steps:

1. Tabulating $\varphi(1 + j\varepsilon), j = 0, 1, 2, \ldots$. This requires appropriate choosing of parameter $\varepsilon$ (see below) and is most time consuming. The most convenient for this seems small but extremely efficient program PARI/GP which has implemented particularly optimal zeta procedure. The first of authors used Cyfronet ZEUS computer in Cracow, where calculating single value of $\varphi(s)$ with 51000 significant digits requires about 13 minutes. Since this procedure may easily be parallelized therefore in order to compute 10000 values of $\varphi$ 20 independent routines were performed (each calculating 500 values of $\varphi$) which took nearly one week.

2. Calculating $\alpha_k$ using (27) and the precomputed values.

3. Calculating Stieltjes constants using (29).

(Contrary to the above step 1 which requires a powerful computer, steps 2 and 3 can be quickly performed on a typical PC.) Several properties concerning accuracies may be obtained experimentally. It should be stressed out that given $\alpha_k$ calculating single $\gamma_n$ with accuracy of about 50000 digits requires several minutes on a very modest PC machine.

**Acknowledgement:** One of the authors (KM) would like to express his gratitude to
the Academic Computer Center Cyfronet, AGH, Cracow, for the computational grant of 1000 hours under the PL-Grid project (Polish Infrastructure for Supporting Computational Science in the European Research Space).

References

[1] R. P. Brent. Computation of the regular continued fraction for Euler’s constant. *Mathematics of Computation*, 31(139):771–777, Jul 1977.

[2] R. P. Brent and E. M. McMillan. Some New Algorithms for High-Precision Computation of Euler’s Constant. *Mathematics of Computation*, 34(149):305–312, 1980.

[3] L. Euler. Variae observationes circa series infinitas. *Commentarii academiae scientiarum Petropolitanae*, 9:160–188, 1744.

[4] S. R. Finch. *Mathematical Constants*. Cambridge University Press, 2003.

[5] C. Hermite and T. J. Stieltjes. Correspondance d’Hermite et de Stieltjes. vol. 1, (8 novembre 1882 - 22 juillet 1889), 1905.

[6] F. Johansson. Rigorous high-precision computation of the Hurwitz zeta function and its derivatives. *Numerical Algorithms*, 69(2):253–270, Jul 2014.

[7] J. B. Keiper. Power series expansions of Riemann’s ζ function. *Mathematics of Computation*, 58(198):765–765, May 1992.

[8] A. Y. Khinchin. *Continued Fractions*. Dover Publications, New York, 1997.

[9] R. Kreminski. Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants. *Mathematics of Computation*, 72(243), Dec 2002.

[10] K. Maślanka. A hypergeometric-like Representation of Zeta-function of Riemann, Cracow Observatory preprint no. 1997/60, 1997. *posted at arXiv: math-ph/0105007*, 2001. *“http://xxx.lanl.gov/abs/math/0105007”*.

[11] G. Martin. The unreasonable effectualness of continued function expansions. *Journal of the Australian Mathematical Society*, 77(3):305–320, 2004.

[12] M. Milgram. Integral and Series Representations of Riemann’s Zeta Function and Dirichlet’s Eta Function and a Medley of Related Results. *Journal of Mathematics*, 2013:Article ID 181724, 2013.

[13] PARI/GP version 2.11.2, 64 bits, 2019. available from *http://pari.math.u-bordeaux.fr/*

[14] B. Riemann. Übe die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, pages 671–680, November 1859. english translation available at *http://www.maths.tcd.ie/pub/HistMath/People/Riemann*
[15] C. Ryll-Nardzewski. On the ergodic theorems II (Ergodic theory of continued fractions). *Studia Mathematica*, 12:74–79, 1951.

[16] D. W. Sweeney. On the Computation of Euler’s Constant. *Mathematics of Computation*, 17(82):170, Apr 1963.

[17] E. W. Weisstein. *CRC Concise Encyclopedia of Mathematics*. Chapman & Hall/CRC, 2009.