Recursion-Theoretic Ranking and Compression

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Abstract

For which sets \( A \) does there exist a mapping, computed by a total or partial recursive function, such that the mapping, when its domain is restricted to \( A \), is a 1-to-1, onto mapping to \( \Sigma^* \)? And for which sets \( A \) does there exist such a mapping that respects the lexicographical ordering within \( A \)? Both cases are types of perfect, minimal hash functions. The complexity-theoretic versions of these notions are known as compression functions and ranking functions. The present paper defines and studies the recursion-theoretic versions of compression and ranking functions, and in particular studies the question of which sets have, or lack, such functions. Thus, this is a case where, in contrast to the usual direction of notion transferal, notions from complexity theory are inspiring notions, and an investigation, in computability theory.

We show that the rankable and compressible sets broadly populate the 1-truth-table degrees, and we prove that every nonempty coRE cylinder is recursively compressible.

1 Introduction

This paper studies the recursion-theoretic case of how hard it is to squeeze the air (more concretely, the elements of its complement) out of a set \( A \). That is, we want to, by a total recursive function or a partial recursive function, map in a 1-to-1, onto fashion from \( A \) to \( \Sigma^* \). So our function, when viewed as being restricted to the domain \( A \), is a bijection between \( A \) and \( \Sigma^* \). In effect, each string in \( A \) is given a unique “name” (string) from \( \Sigma^* \), and every “name” from \( \Sigma^* \) is used for some string in \( A \). As has been pointed out for the complexity-theoretic analogue (where we are interested not in total and partial recursive functions, but in polynomial-time functions), such functions are the analogues for infinite sets of perfect (i.e., no collisions among elements in \( A \)), minimal (i.e., every element of \( \Sigma^* \) is hit by some element of \( A \)) hash functions, and are called compression functions [GHK92].

A particularly dramatic type of such function would be one that maps from the \( i \)th element of an infinite set \( A \) to the \( i \)th element of \( \Sigma^* \). Such a function—a ranking function—has all the above properties and in addition respects the (lexicographical) ordering of the elements of \( A \). For the case

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of polynomial-time functions, this type of issue was first studied by Allender [All85] and Goldberg and Sipser [GS85, GS91] a quarter of a century ago.

That seminal work of Allender, Goldberg, and Sipser led other researchers to bring a closer lens to the issue of what behavior the ranking function would be required to have on inputs that did not belong to \(A\) [HR90], to study more flexible notions such as the abovementioned compression functions [GHK92] and what are known as scalability [GH96] and semi-ranking [HOZZ05], and to study ranking of extremely simple sets ([Huy90], see also [All85, GS91]). Even the original paper of Goldberg and Sipser already established that there are \(P\) sets whose ranking function is complete for the counting version of \(NP\) (namely \#\(P\)), i.e., that quite simple sets can have quite complex ranking functions.

The present paper studies compression and ranking in their recursion-theoretic analogues. These basically are the same problems as in the complexity-theoretic case, except instead of studying what can (and cannot) be done by polynomial-time functions, we study what can (and cannot) be done by total recursive functions and partial recursive functions. The direction of studying ranking and compression by total recursive functions was previously mentioned as an open direction in the conclusions section of [GHK92], which observed without proof what here are Theorem 5.3 and Corollary 5.7.

Why do we study this? After all, programmers are not clamoring to have recursion-theoretic perfect, minimal hash functions for infinite sets. But our motivation is not about satisfying a programming need. It is about learning more about the structure of sets, and the nature of—and in some cases the impossibility of—compression done by total and partial recursive functions. In particular, what classes of sets can we show to have, or not have, such compression and ranking functions?

Among the results are the following.

- Every 1-truth-table degree except the zero degree contains both sets that are recursively rankable and sets that are not recursively rankable (Theorems 4.1 and 4.3). (So some recursively rankable sets are undecidable, and some even fall outside of the arithmetical hierarchy.)

- Every 1-truth-table degree except the zero degree contains some set that is recursively compressible yet is not recursively rankable (Theorem 4.3).

- Every nonempty coRE cylinder is recursively compressible (Theorem 5.6), and it follows that all coRE-complete sets (see Corollary 5.7) and all nonempty coRE index sets (Corollary 5.8) are recursively compressible. However, no RE-complete set or coRE-complete set is recursively rankable or even partial-recursively rankable (Corollary 4.5 and Corollary 4.8).

- There are infinite \(\Delta^0_2\) sets that are not even partial-recursively compressible (Theorem 5.9).

- Although every recursively compressible RE set is recursive (see Theorem 5.3), each infinite set in \(RE – REC\) is an example of a partial-recursively compressible RE set that is not recursive or recursively compressible (Proposition 3.2 and Corollary 5.4). So although all coRE-complete sets are recursively compressible, no RE-complete set is recursively compressible.
2 Related Work

The most closely related papers are those mentioned in Section 1. Given the importance to this paper of mappings that are onto \( \Sigma^* \), we mention also the line of work, dating back to Brassard, Fortune, and Hopcroft’s early paper on one-way functions [BFH78], that looks at the complexity of inverting functions that map onto \( \Sigma^* \) [BFH78, FFNR03, HRW97, Rot99]. However, both that line and the papers mentioned in Section 1 are about complexity-theoretic functions, while in contrast the current paper is about recursion-theoretic functions.

In fact, this paper is quite the reverse of the typical direction of inspiration. A large number of the core concepts of complexity theory are defined by direct analogy with notions from recursive function theory. As just a few examples, NP, the polynomial hierarchy [MS72, Sto76], most of complexity theory’s reduction notions [LLS75], (complexity-theoretic) creative/simple/immune/bimmune sets [Ber76, BS85, Hom86, Tor86, HM83], and the semi-feasible sets [Sel82] are lifted quite directly from recursive function theory, with, as needed, the appropriate, natural changes to focus on the deterministic and nondeterministic polynomial-time realms. The debt that complexity theory owes to recursive function theory is huge.

Far less common is for notions defined in complexity to then be studied recursion-theoretically. However, this paper is a small example of that, since it is taking the line of ranking/compression work started by Allender, Goldberg, and Sipser in the 1980s and asking the same type of questions in the setting of total and partial recursive functions.

The notions of retraceable sets, regressive sets, and isolic reductions are the closest existing concepts in recursive function theory to the notions of rankable and compressible sets. We now discuss each, pointing out how the notions differ from ours.

A set \( A \subseteq \Sigma^* \) is called regressive if there exists an enumeration (note that the definition does not require that it be a recursive enumeration) of \( A \) without repetitions \( \{a_0, a_1, a_2, \ldots\} \), and a partial recursive function \( f \) such that: \( f(a_{n+1}) = a_n \) and \( f(a_0) = a_0 \) [Dek62]. The set \( A \) is called retraceable if it meets the definition of regressive with respect to a (not necessarily recursive) enumeration that follows the standard lexicographic order [DM58]. Odifreddi [Odi89] comments that there is a “surface analogy” that r.e. is to recursive as regressive is to retraceable. We similarly mention that there is a surface analogy that \( \text{FPR-compressible} \) is to \( \text{FPR-rankable} \) as regressive is to retraceable. We claim (and it is not too hard to see; one basically checks whether the input is \( a_0 \)—which will be hard-coded into the program—and if not tries repeatedly applying \( f \) until, if ever, one reaches \( a_0 \), keeping track of how many applications that took) that each retraceable set is \( \text{FPR-rankable} \) and each infinite regressive set is \( \text{FPR-compressible} \). We further claim that each set that is retraceable under a recursive retracing function \( f \) is \( \text{FREC-rankable} \) (the same approach sketched above works, along with observing that if at any point in the \( f \) application chain starting at a string \( x \) the recursive retracing function maps a string \( y \neq a_0 \) to a string lexicographically greater than or equal to \( y \), then our original string is definitely not in the set and our \( \text{FREC-ranking} \) function can output any value it likes as \( x \)’s purported rank). We claim that the converses of these statements fail rather dramatically; our notions are far more general. For example, there are \( \text{FREC-rankable} \) sets that are not retraceable and indeed that are not even regressive (and recall that the definitions of retraceable and regressive
are with respect to partial recursive retraction functions, so this is a very strong type of separation).\footnote{1}

The notion of rankability is, in fact, so nonrestrictive that, as this paper will establish, every 1-truth-table degree contains an FR\textsubscript{REC}-rankable set. What about the retraceable sets? They are known to populate the truth-table degrees\footnote{2}. However, we can prove that, unlike the FR\textsubscript{REC}-rankable sets, they do not populate the 1-truth-table degrees\footnote{3}.

Another concept from recursive function theory that has a similar flavor to the notions we are looking at is the notion of an isolic reduction. A is said to isolic-reduce to B if there exists a one-to-one partial recursive function \(f\) such that \(A = f^{-1}(B)\) (see [Rog67] p. 124). Sets that isolic-reduce to \(\Sigma^1\) thus have a similar definitional flavor to our notion of FR\textsubscript{PR}-compression. However, note that isolic reductions are required to be one-to-one, and so unlike our notions cannot allow even a single element of \(A\) to be mapped to a member of \(B\). That is enough to make them strikingly differ in

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\footnote{1}{Let \(s_0, s_1, s_2, \ldots\) enumerate \(\Sigma^1\) in lexicographic order. Let \(K\) be the RE-complete set, \(\{x \mid x \in L(M_i)\}\). Define \(A = \{s_{3i} \mid i \geq 0\} \cup \{s_{3i+1} \mid i \in K\} \cup \{s_{3i+2} \mid i \in K\}\). Then \(A\) is easily seen to be FR\textsubscript{REC}-rankable. Yet we claim that \(A\) is not retraceable and indeed is not even regressive. (And the definitions of retraceable and regressive are with respect to partial recursive retraction functions, so this is even stronger than the claims that \(A\) is not retraceable or not regressive via some recursive retraction function). Why is \(A\) not regressive? A set is said to be immune (or r.e.-immune) if it is infinite but contains no infinite r.e. (equivalently, recursive) subsets. Every regressive set is either r.e. or immune [Odi89] Prop. II.6.8. Yet \(A\) is not r.e. (\(\Sigma^1\) clearly recursive many-one reduces to \(A\)) and \(A\) is not immune (due to the having the recursive subset \(\{s_{3i} \mid i \geq 0\}\)). Thus \(A\) is not regressive.

\footnote{2}{The literature reference for this is a bit tricky. Odifreddi [Odi89] Proposition II.6.13 proves the result of Dekker and Myhill [DM58] that each Turing degree contains a retraceable set. However, the given proof in fact establishes that each truth-table degree contains a retraceable set, and that fact clearly is known to Odifreddi since at the start of Exercise VI.6.16.b he quietly attributes to Proposition II.6.13 the fact that each truth-table degree other than the zero degree contains an immune, retraceable set [Odi89] p. 600].

\footnote{3}{We state that as the following theorem. The 1-truth-table upward (reducibility) cone of a set \(L\) is \(\{L' \mid L \leq_{tt} L'\}\) (the term is more commonly used for degrees [Odi89] although the difference is inconsequential).

**Theorem 2.1.** There is a 1-truth-table degree (indeed, there is a 1-truth-table upward cone) that contains no retraceable set.

**Proof.** For clarity, we first discuss the 1-truth-table degree case. Recall that a set is said to be immune (or r.e.-immune) if it is infinite but contains no infinite r.e. (equivalently, recursive) subsets. Since every retraceable set is recursive or immune [DM58], it will suffice to find a 1-truth-table degree with no recursive or immune sets. Let \(A\) and \(B\) be a pair of disjoint, r.e. sets that are recursively inseparable. It is well-known that such pairs exist, e.g., \(\{i \mid M(i)\} (i\text{ hails and outputs 1})\) and \(\{i \mid M(i)\} (i\text{ hails and outputs 2})\). We claim that if \(A \leq_{tt} S\), then \(S\) is not immune. Let \(f\) be a 1-truth-table reduction from \(A\) to \(S\) and suppose, seeking a contradiction, that \(S\) is immune. Then the set \(A_L = \{f(x) \mid x \in A\}\) uses the identity truth-table is an r.e. subset of \(S\), so \(L_A\) it must be finite. Similarly, \(L_B = \{f(x) \mid x \in B\}\) and \(f(x)\) uses the negation truth-table must be finite. We can use \(L_A\) and \(L_B\), however, to recursively separate \(A\) and \(B\). Define the function \(g(x)\) as follows:

* If \(f(x)\) uses the identity truth-table, and \(f(x) \in L_A\), then \(g(x) = 1\).
* If \(f(x)\) uses the identity truth-table, and \(f(x) \notin L_A\), then \(g(x) = 0\).
* If \(f(x)\) uses the negation truth-table, and \(f(x) \in L_B\), then \(g(x) = 0\).
* If \(f(x)\) uses the negation truth-table, and \(f(x) \notin L_B\), then \(g(x) = 1\).

For every \(x \in A\), \(g(x) = 1\), and for every \(x \in B\), \(g(x) = 0\). The function \(g\) is recursive because both \(L_A\) and \(L_B\) are finite. However, this contradicts the recursive inseparability of \(A\) and \(B\), so \(S\) must not be immune. Thus the 1-truth-table degree of \(A\) contains no recursive or immune sets, and so contains no retraceable sets.

The proof as given above in fact also shows that the 1-truth-table upward (reducibility) cone of \(A\) contains no retraceable sets.}
behavior from the notions we are studying. In particular, our paper puts recursively- and partial recursively-compressible sets into every 1-truth-table degree, but in contrast the class of sets that isolic-reduce to $\Sigma^*$ is precisely the infinite r.e. sets, and so no sufficiently hard 1-truth-table degrees contain any sets that isolic-reduce to $\Sigma^*$.

## 3 Definitions

Throughout this paper, we fix the alphabet to be the binary alphabet $\Sigma = \{0, 1\}$. So all our notions will involve (total or partial) functions whose input universe is $\Sigma^*$ and whose codomain is $\Sigma^*$, and all classes (e.g., the recursive sets) are viewed as being over sets whose alphabet is $\Sigma$.

Why is it natural to focus just on $\Sigma = \{0, 1\}$? For every two finite alphabets $\Sigma'$ and $\Sigma''$, there is a recursive, order-respecting bijection between $\Sigma''$ and $\Sigma'^*$. So for all natural purposes in the context of recursive function theory, any pair of finite alphabets are essentially computationally interchangeable.

$F_{\text{REC}}$ will denote the class of all total recursive functions from $\Sigma^*$ to $\Sigma^*$. $F_{\text{PR}}$ will denote the class of all partial recursive functions from $\Sigma^*$ to $\Sigma^*$. domain$(f)$ is the set of inputs on which a (potentially partial) function $f$ is not undefined, e.g., if $f$ is a total function, domain$(f) = \Sigma^*$.

$\text{REC}$ and $\text{RE}$ will denote the recursively enumerable sets. As usual, $\text{coRE} = \{A | \overline{A} \in \text{RE}\}$ and $\Delta_2$ the class of all sets $A$ such that there exists a set $B \in \text{RE}$ such that $A$ is recursive in $B$ (i.e., $A$ recursively Turing reduces to $B$). These are low levels of what is known as the arithmetical (or Kleene–Mostowski) hierarchy [Kle43, Mos47]. We will often use r.e. and co-r.e. as adjectival forms of RE and coRE, e.g., “each r.e. set belongs to the class RE,” although at times we will also use the terms RE and coRE themselves as adjectives.

$\varepsilon$ will denote the empty string, and we use “lexicographical” in its standard computer science sense, e.g., $\varepsilon \leq_{\text{lex}} 0 \leq_{\text{lex}} 1 \leq_{\text{lex}} 00 \leq_{\text{lex}} \cdots$. successor$(x)$ will denote the lexicographical successor of $x$, e.g., successor$(11) = 000$. For any set $A \subseteq \Sigma^*$ and any string $x \in \Sigma^*$, $A \leq^{\preceq} x$ denotes the set of all strings in $A$ that are lexicographically less than or equal to $x$. $A \leq^{\succeq}$ and $A \geq^{\preceq}$ are defined analogously. We will use these notations even for $\Sigma^*$ itself, e.g., if $x$ is the string 10 then $(\Sigma^*)^{\leq_{\text{lex}}} \Sigma^*$ is the set $\{\varepsilon, 0, 1, 01, 10\}$. For each finite set $A$, $\|A\|$ will denote the cardinality of $A$. The function $\langle \cdot, \cdot \rangle$ will denote a fixed, standard, recursive pairing function, i.e., a recursive bijection between $\Sigma^* \times \Sigma^*$ and $\Sigma^*$.

We say that $A_1$ and $A_2$ are (recursively) isomorphic, denoted $A_1 \equiv_{\text{iso}} A_2$, exactly if there is a 1-to-1, onto, total recursive function $f$ from $\Sigma^*$ to $\Sigma^*$ such that $f(A_1) = A_2$. For any reducibility $\preceq$, we say $A \equiv_{\preceq} B$ if $A \preceq B$ and $B \preceq A$. When $\preceq$ is reflexive and transitive, $\equiv_{\preceq}$ will be an equivalence relation, and each equivalence class with respect to $\equiv_{\preceq}$ is said to be an $\preceq$ degree. The reducibilities whose degrees will be discussed during the rest of this paper are recursive many-one reductions ($\preceq_m$), which we will also refer to simply as many-one reductions, and recursive 1-truth-table reductions ($\preceq_{1tt}$), which we will also refer to simply as 1-truth-table reductions. As is typical, we will refer to the $\preceq_m$ degrees and $\preceq_{1tt}$ degrees as, respectively, many-one degrees and 1-truth-table degrees. (Rather than define here the machinery of truth-table reductions, suffice it to say that the following is a true statement: A 1-truth-table reduces to $B$ exactly if $A$ Turing reduces to $B$ via a recursive transducer that on each input makes at most one query to $B$). All recursive sets belong to a single 1-truth-table degree, which in fact is exactly $\text{REC}$. All many-one degrees except
the somewhat pathological many-one degree \{\emptyset\} contain infinite sets.

The Myhill Isomorphism Theorem [Myh55] (or see [Soa87, pp. 24]) states that \(A \equiv_{\text{iso}} B \iff A \equiv_1 B\), where \(\leq_1\) denotes (recursive) 1-to-1 reductions. Though it is not a standard nickname, for convenience we will use the term Myhill’s Corollary to refer to the result that all RE-complete (with respect to many-one recursive reductions) sets are recursively isomorphic; and we will also refer to as Myhill’s Corollary the fact, semantically identical, that all coRE-complete (with respect to many-one recursive reductions) sets are recursively isomorphic. (Myhill’s Corollary is well-known to with some argumentation follow from the Myhill Isomorphism Theorem, see, e.g., [Soa87, pp. 42–43] or [Odi89, Theorem III.6.6 + Corollary III.7.14].) \(K\) as mentioned earlier will denote the RE-complete set \(\{x \mid x \in L(M_i)\}\), where \(L(M_i)\) denotes the language accepted by \(M_i\), and \(M_1, M_2, M_3, \ldots\) (or the same using strings as the subscripts under the standard correspondence between positive natural numbers and strings) is a fixed, standard enumeration of (language-computing) Turing machines.

A set \(A\) is a cylinder exactly if for some \(B\) it holds that \(A \equiv_{\text{iso}} B \times \Sigma^*\). We say that a set is coRE cylinder exactly if it is in coRE and is a cylinder.

We now define the class of compressible sets. Our definition is the precise analogue of the notion of P-compression of Goldsmith, Hemachandra, and Kunen [GHK92], except since we will be studying the recursion-theoretic case we have removed the requirement that the function be total and polynomial-time computable. Thus we are capturing the notion of a function that when restricted to \(A\) creates a total (on \(A\), 1-to-1, onto mapping to \(\Sigma^*\).

**Definition 3.1 (Compressible sets).**

1. Given a set \(A \subseteq \Sigma^*\), we say that a (possibly partial) function \(f\) is a compression function for \(A\) exactly if
   \begin{enumerate}
   \item \(\text{domain}(f) \supseteq A\),
   \item \(f(A) = \Sigma^*\), and
   \item \((\forall a \in A)(\forall b \in A)[a \neq b \implies f(a) \neq f(b)]\).
   \end{enumerate}

2. Let \(\mathcal{F}\) be any class of (possibly partial) functions mapping from \(\Sigma^*\) to \(\Sigma^*\). A set \(A\) is \(\mathcal{F}\)-compressible exactly if \((\exists f \in \mathcal{F})[f\text{ is a compression function for }A]\).

3. For each \(\mathcal{F}\) as above, \(\mathcal{F}\)-compressible \(= \{A \mid A\text{ is }\mathcal{F}\text{-compressible}\}\).

4. For each \(\mathcal{F}\) as above and each \(\mathcal{C} \subseteq 2^{\Sigma^*}\), we say that \(\mathcal{C}\) is \(\mathcal{F}\)-compressible exactly if \((\forall A \in \mathcal{C})[\text{If } A \text{ is an infinite set, then } A\text{ is }\mathcal{F}\text{-compressible}]\).

In a slight notational overloading, the above definition uses \(\mathcal{F}\)-compressible both as an adjective and to represent the corresponding class of sets. Note that the above definition does not constrain what \(f\) does on elements of \(\overline{A}\). \(f\) can be undefined on some or all of those, and if it is defined on some of those, note that that will make the overall map be non-1-to-1. Of course, \(f\) may otherwise be constrained to be total, e.g., when we speak of \(\mathcal{F}_{\text{REC}}\)-compressible sets, the \(f\) involved must be total due to the definition of \(\mathcal{F}_{\text{REC}}\).

No finite set has a compression function, since a finite set doesn’t have enough strings in it to map \textit{onto} \(\Sigma^*\). This is why part 4 of the above definition defines a class of sets as being compressible
under a certain type of function if all the class’s infinite sets are thus compressible. Though we could rig even parts 2 and 3 of the definition of compression (rather than just part 4) to give finite sets a free pass, the given definition in each of these choices is exactly matching the long-standing, analogous complexity-theoretic definitions \([\text{GHK92}]\). When we do wish to speak of the compressible sets augmented by the finite sets, we will do so explicitly using the following:

\[ \mathcal{F}\text{-compressible}' = \text{def} \mathcal{F}\text{-compressible} \cup \{ A \subseteq \Sigma^* \mid A \text{ is finite} \}. \]

To get a sense of how compression works in a simple case, let us note the following.

**Proposition 3.2.** RE is \(\text{F}_{\text{PR}}\)-compressible.

**Proof.** Let \(A\) be any r.e. set. Since \(A\) is r.e., there is an enumerating Turing machine that enumerates \(A\) without repetitions. Our \(\text{F}_{\text{PR}}\) compression function for \(A\) will map the \(i\)th enumerated string to the \(i\)th string in \(\Sigma^*\), and will be undefined on all strings that are never enumerated (i.e., that belong to \(\overline{A}\)).

Compression of \(A\) implies that in the image of the compression function on \(A\) we leave no holes: \(f(A) = \Sigma^*\). That overall niceness however does not imply that \(f\) will never map any string in \(A\) to a lexicographically larger string. \(f\) certainly can, though the more often it does so, the more often other strings in \(A\) will need to map to lexicographically smaller strings, to prevent any “holes” in the image of \(A\). The more demanding notion called ranking, however, *does* ensure that no string in \(A\) will map to a lexicographically larger string.

Ranking is a particularly nice type of compression—compression that simply maps the \(i\)th string in \(A\) to the \(i\)th string in \(\Sigma^*\). There are three slightly differing versions of ranking, depending on what one requires regarding what happens on inputs that are not in \(A\). The following definition follows the one of those that handles this analogously with the way it is handled in compression, e.g., on inputs that are not in \(A\) we allow the function to map to any strings it wants, or even to be undefined. (Informally put, the compression function can “lie” or can be undefined on inputs \(x \not\in A\).) Hemachandra and Rudich \([\text{HR90}]\) (for the complexity-theoretic case) defined this notion and called it “weak ranking.” However, to keep our notations for compression and ranking in harmony with each other, we will in this paper consistently refer to this simply as “ranking.”

**Definition 3.3** (Rankable sets).

1. Given a set \(A \subseteq \Sigma^*\), we say that a (possibly partial) function \(f\) is a ranking function for \(A\) exactly if

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\[4\]The other two approaches to handling \(\overline{A}\) have a behavior is not too interesting in the recursion-theoretic world. (In the complexity-theory world, due to the work of Goldberg and Sipser \([\text{GS91}]\) and Hemachandra and Rudich \([\text{HR90}]\), it is known that for each of the three notions, one has that all \#P sets are polynomial-time rankable under that notion exactly if all \#P functions—i.e., the counting version of NP—are polynomial-time computable.) The other two notions are (a) to additionally require that on members of \(\overline{A}\) the function either is undefined or states that they are not members of \(A\) (this notion’s analogue is called “ranking” in \([\text{HR90}]\)), or (b) to additionally require that on members of \(\overline{A}\) the function computes and outputs \(|A|\) (this notion’s analogue is called “ranking” in \([\text{GS91}]\) and is called “strong ranking” in \([\text{HR90}]\)). However, under each of these notions, with respect to either of \(\text{F}_{\text{REC}}\) or \(\text{F}_{\text{PR}}\), the class of sets thus rankable is exactly the recursive sets; we include a proof of this in Appendix A. Thus these two notions, though interesting in the complexity-theoretic study of ranking, are not interesting in the recursion-theoretic study of ranking.
(a) \( \text{domain}(f) \supseteq A \), and

(b) if \( x \in A \), then \( f(x) = \|A^x\| \). (That is, if \( x \) is the \( i \)th string in \( A \), then \( f(x) \) is the \( i \)th string in \( \Sigma^* \).)

2. Let \( \mathcal{F} \) be any class of (possibly partial) functions mapping from \( \Sigma^* \) to \( \Sigma^* \). A set \( A \) is \( \mathcal{F} \)-rankable exactly if \((\exists f \in \mathcal{F})[f \text{ is a ranking function for } A]\).

3. For each \( \mathcal{F} \) as above, \( \mathcal{F} \)-rankable = \{\( A \mid A \) is \( \mathcal{F} \)-rankable\}.

4. For each \( \mathcal{F} \) as above and each \( C \subseteq 2^{\Sigma^*} \), we say that \( C \) is \( \mathcal{F} \)-rankable exactly if \((\forall A \in C)[A \text{ is } \mathcal{F} \text{-rankable}]\).

For example, clearly every recursive set is \( \mathcal{F}_{\text{REC}} \)-rankable by brute force. However, we will later see that, in contrast, some infinite r.e. sets are not even \( \mathcal{F}_{\text{REC}} \)-compressible.

Aside from the quirk that finite sets cannot be compressible, rankability clearly implies compressibility. And of course, every total recursive function is a partial recursive function. So we have the following trivial containments.

**Proposition 3.4.**

1. \((\forall \mathcal{F})[\mathcal{F} \text{-rankable} \subseteq \mathcal{F} \text{-compressible}]\).

2. \( \mathcal{F}_{\text{REC}} \text{-rankable} \subseteq \mathcal{F}_{\text{PR}} \text{-rankable} \).

3. \( \mathcal{F}_{\text{REC}} \text{-compressible} \subseteq \mathcal{F}_{\text{PR}} \text{-compressible} \).

**4. Ranking**

In Footnote 4 we noted that, for the ranking variants where the ranking function’s behavior on the complement is constrained, the class of things that can be \( \mathcal{F}_{\text{REC}} \)-ranked, or even \( \mathcal{F}_{\text{PR}} \)-ranked, (in that variant) is precisely REC, the recursive sets.

In contrast with those variants, we now show that arbitrarily complex sets are \( \mathcal{F}_{\text{REC}} \)-rankable. So, certainly, some \( \mathcal{F}_{\text{REC}} \)-rankable sets are not recursive.

**Theorem 4.1.** Every 1-truth-table degree contains an \( \mathcal{F}_{\text{REC}} \)-rankable set.

**Proof.** Let \( \widehat{\oplus} \) be defined by \( A \oplus B = \{x0 \mid x \in A\} \cup \{x1 \mid x \in B\} \), i.e. this is the standard “join” (aka “disjoint union,” aka “marked union”), except the marking bit is the low-order bit rather than as is standard the high-order bit. Note that for any set \( A \subseteq \Sigma^* \), \( A \oplus \overline{A} \) is \( \mathcal{F}_{\text{REC}} \)-rankable (indeed, it is even Logspace-rankable) by the function defined by

\[
    f(\varepsilon) = \varepsilon, \quad f(x0) = z, \quad \text{and} \quad f(x1) = z,
\]

since for each \( x \) exactly one of \( x0 \) and \( x1 \) is in \( A \oplus \overline{A} \), and \( \varepsilon \notin A \oplus \overline{A} \). \( \square \)

**Corollary 4.2.** There exist sets \( A \) that are not in the arithmetical hierarchy yet are \( \mathcal{F}_{\text{REC}} \)-rankable.
However, it follows from Theorem 5.3 of the next section—which establishes that REC = RE ∩ FREC-compressible—that Theorem 4.1 cannot be improved from 1-truth-table degrees to many-one degrees.

Theorem 4.1 shows that FREC-rankable sets occur everywhere. Nonetheless, we show as Theorem 4.3 that the non-FREC-rankable sets also occur everywhere. Theorem 4.4 notes that for the case of r.e. sets, FPR-rankability even implies decidability, thus all sets in RE − REC are non-FREC-rankable.

**Theorem 4.3.** Every 1-truth-table degree except that of the recursive sets contains a set that is FREC-compressible but not FREC-rankable.

**Proof.** Let $A$ be an arbitrary nonrecursive set. Let $s_0, s_1, s_2, \ldots$ enumerate $\Sigma^*$ in lexicographical order. Define

$$B = \{s_{4i} \mid i \geq 0\} \cup \{s_{4i+1} \mid s_i \in A\} \cup \{s_{4i+2} \mid i \geq 0\} \cup \{s_{4i+3} \mid s_i \in \overline{A}\}.$$ 

So $B$ consists of a pattern that repeats every four strings. Namely, the first and third strings are always in, and exactly one of the second and fourth is in. Then $A \equiv_{1-tt} B$ and $B$ is FREC-compressible by the map $f$ defined by

$$f(s_{4i}) = s_{3i},$$

$$f(s_{4i+1}) = s_{3i+1},$$

$$f(s_{4i+2}) = s_{3i+2},$$

and

$$f(s_{4i+3}) = s_{3i+1}.$$ 

However, if $B$ were FREC-rankable, then $A$ would be recursive. Why? If $g$ is an FREC ranking function for $B$, then it holds that

$$s_i \in A \iff g(s_{4i+2}) - g(s_{4i}) = 2.$$ 

Since $A$ is not recursive, $B$ cannot be FREC-rankable. 

**Theorem 4.4.** Every r.e. FPR-rankable set is recursive. (Equivalently, REC = RE ∩ FPR-rankable = RE ∩ FREC-rankable.)

**Proof.** This proof is similar in flavor to the proof that comprises the whole of Appendix A except in that proof but not here one has a model in which the ranker is not allowed to output “lies” as to the rank of nonmembers of the set, and here but not there we have the assumption that the set is r.e. Let $A$ be an r.e. FPR-rankable set. Let $f$ be an FPR-ranking function for $A$. Since $A$ is r.e., there exists an enumerating Turing machine, $E$, for $A$, and without loss of generality, we assume that $E$ enumerates the elements of $A$ without repetition. If $A$ is finite, then $A$ is recursive, so we in the following consider just the case that $A$ is infinite. Here is our algorithm, which will always halt, to decide membership in $A$. On arbitrary input $x$, for which we wish to test whether $x \in A$, start running the enumerating machine $E$. Each time the machine outputs an element, run $f$ on that element to determine the correct rank of that element (since the elements output by $E$ all belong to $A$, $f$ halts on each and outputs the correct rank value, e.g., if the string is the seventh string in $A$, the function
f will output the lexicographically seventh string in $\Sigma^*$). Each time we thus obtain a rank, check to see if either: (a) the string just output by $E$ is $x$, in which case accept as $x \in A$, (b) the ranker has mapped some string $y$ output by $E$ and satisfying $y \geq_{\text{lex}} x$ to the string $\varepsilon$ (i.e., has declared it to be the lexicographically least string in $A$), in which case reject as $x \notin A$, or (c) the ranker has mapped some two strings, $y$ and $y'$—such that $y <_{\text{lex}} x <_{\text{lex}} y'$ and both $y$ and $y'$ have by now have been output by $E$—to outputs $f(y)$ and $f(y')$ such that $f(y')$ is the lexicographical successor in $\Sigma^*$ of $f(y)$, in which case reject as $x \notin A$. 

**Corollary 4.5.** No RE-complete set is $F_{PR}$-rankable.

Let us now turn to seeing how the co-r.e. sets—especially the coRE-complete sets—interact with $F_{REC}$-rankability and $F_{PR}$-rankability.

First, though, let us notice that for the co-r.e. sets, $F_{REC}$-rankability and $F_{PR}$-rankability precisely coincide (though unlike the case—see Theorem 4.4—of the RE sets, that as shown by Corollary 4.8 is not due to them both collapsing to the recursive sets).

**Theorem 4.6.** $\text{coRE} \cap F_{REC}$-rankable = $\text{coRE} \cap F_{PR}$-rankable.

**Proof.** Let $A$ be a set in $\text{coRE} \cap F_{PR}$-rankable. We give an $F_{REC}$-ranker for $A$. Namely, on input $x$, run both the $F_{PR}$-ranker $f$ and an enumerator for $A$, dovetailed, until we either get a value for $f(x)$ from the ranker or we see the enumerator state that $x \notin A$. If the former, output that value, and if the latter, output any fixed string, e.g., 101010. 

Next we give the following theorem, which implies us our desired result about coRE-complete sets, and more.

**Theorem 4.7.** If $A$ is an $F_{PR}$-rankable co-r.e. set that has an infinite r.e. subset, then $A \in \text{REC}$.

**Proof.** Let $A$ be as in the theorem’s hypothesis. Let $s_0,s_1,s_2...$ enumerate $\Sigma^*$ in lexicographical order. Let $E$ be an enumerating Turing machine without repetitions for $A$ and let $F$ be an enumerating Turing machine for an infinite r.e. subset of $A$. Suppose $g$ is a ranking function for $A$. In light of Theorem 10, $A$ is $F_{PR}$-rankable $\iff$ $A$ is $F_{REC}$-rankable, so w.l.o.g. we assume that $g \in F_{REC}$.

Then the following procedure decides whether $x \in A$. Run $F$ until it enumerates some string $s_n >_{\text{lex}} x$. Compute $g(s_n)$. Since $s_n \in A$, $(\Sigma^*)^{\leq s_n}$ is composed of $g(s_n)$ members of $A$ and $n - g(s_n)$ members of $\overline{A}$. Run $E$ until it enumerates $n - g(s_n)$ strings in $(\Sigma^*)^{\leq s_n}$. If $x$ is one of those $n - g(s_n)$ strings, then we know that $x \notin A$, and otherwise we know that $x \in A$. 

**Corollary 4.8.** No coRE-complete set is $F_{PR}$-rankable.

**Proof.** This follows directly from Theorem 4.7, in light of Post’s [Pos44] early result that every coRE-complete set has an infinite r.e. subset. 

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5Post’s result is trivial to see these days, using the fact (what we are calling Myhill’s Corollary) that all coRE-complete sets are isomorphic, but Post didn’t have the benefit of Myhill’s Corollary and thus proved his result directly. We note in passing the following slight extension of Post’s result, since we could not find it in the existing literature: Every coRE-complete set has an infinite $\text{RE} - \text{REC}$ subset. To see this, just note that the coRE-complete set $\overline{K} \oplus \Sigma^*$ has the infinite $\text{RE} - \text{REC}$ subset $\{1y \mid y \in K\}$, and so by Myhill’s Corollary the claim follows.
Corollary 4.9. Every $F_{PR}$-rankable co-r.e. cylinder is recursive.

Proof. Each finite co-r.e. set is recursive. Each infinite co-r.e. cylinder has an infinite recursive subset (if the cylinder is recursively isomorphic to $B \times \Sigma^*$ via recursive isomorphism function $h$, then for any fixed $x \in B$, we have that the set $h^{-1}(\{(x, y) \mid y \in \Sigma^*\})$ is such an infinite recursive set), and so we are done by Theorem 4.7.

Though by Theorem 4.1 there are $F_{REC}$-rankable sets in the 1-truth-table degree of $K$, we also know that none of those sets can be RE-complete or coRE-complete. The impossibility of them being coRE-complete follows from Corollary 4.8 which indeed precludes even $F_{PR}$-rankability. The impossibility of them being RE-complete will follow from the coming Corollary 5.4 which indeed precludes even $F_{REC}$-compressibility. We state this as the following corollary.

Corollary 4.10. Although the 1-truth-table degree of the RE-complete sets contains $F_{REC}$-rankable sets, no RE-complete or coRE-complete sets are $F_{REC}$-rankable.

5 Compression

Proposition 3.2 shows that every infinite r.e. set is $F_{PR}$-compressible. We note in passing that from that and Theorem 4.4 we immediately have the following.

Proposition 5.1. There exist r.e. sets—in fact, all of $RE - REC$—that are $F_{PR}$-compressible yet are not $F_{REC}$-rankable or even $F_{PR}$-rankable.

We will soon see that in that proposition $F_{PR}$-compressible cannot be improved to $F_{REC}$-compressible.

The following result shows that for $F_{REC}$ compression (and even for Logspace compression, if one looks inside the proof of Theorem 4.1), compressible sets exist in every 1-truth-table degree. (This result is a corollary to the proof of Theorem 4.1—it follows, in light of Proposition 3.4’s part 1 from the fact that the sets constructed in the proof of Theorem 4.1 are infinite.)

Corollary 5.2. Every 1-truth-table degree contains an $F_{REC}$-compressible set.

Can we improve Proposition 5.1’s claim from $F_{PR}$-compressible to $F_{REC}$-compressible? Can we improve Corollary 5.2’s claim from 1-truth-table degrees to many-one degrees (to avoid this being trivially impossible due to the pathological many-one degree that contains only the empty set, what we actually are asking is whether we can change Corollary 5.2 to “every many-one degree other than $\{\emptyset\}$ contains an $F_{REC}$-compressible set”) or, and this would not be an improvement but rather would be an incomparable claim, can we change the claim as just mentioned to all many-one degrees other than $\{\emptyset\}$ if we in addition restrict our attention just to the r.e. degrees? Or can we perhaps hope to show that $RE \subsetneq F_{REC}$-compressible’? The following result, observed without proof in the conclusions section of [GHK92], implies that the answer to each of these questions is “no”; $F_{REC}$ compression is impossible for sets in $RE - REC$.

Theorem 5.3 ([GHK92]). $REC = RE \cap F_{REC}$-compressible'.

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Proof. The \( \subseteq \) direction is immediate. Let us show the \( \supseteq \) direction. Let \( A \in \text{RE} \cap \text{F}_{\text{REC}}\text{-compressible}' \). If \( A \) is finite then certainly \( A \in \text{REC} \), so only the case of infinite \( A \) remains. \( A \in \text{coRE} \), since, where \( f \) is the \( \text{F}_{\text{REC}}\text{-compressor function for our infinite r.e. set} \)
\[
A = \{ x \mid (\exists y)[y \in A \land y \neq x \land f(y) = f(x)] \}.
\]
So \( A \) is r.e. and co-r.e., and thus is recursive.

\[\square\]

**Corollary 5.4.** No set in \( \text{RE} - \text{REC} \) is \( \text{F}_{\text{REC}}\text{-compressible} \). In particular, no \( \text{RE}\text{-complete set is} \text{ F}_{\text{REC}}\text{-compressible}. \)

Corollary 5.4 follows immediately from Theorem 5.3 and so needs no proof. Corollary 5.4 brings out a clear asymmetry, regarding compression, between \( \text{RE} \) and \( \text{coRE} \): no \( \text{RE}\text{-complete set is} \text{ F}_{\text{REC}}\text{-compressible}, but as we will soon see Corollary 5.7 all \( \text{coRE}\text{-complete sets are} \text{ F}_{\text{REC}}\text{-compressible}. \)

Does Theorem 5.3 remain true if we change \( \text{F}_{\text{REC}} \) to \( \text{F}_{\text{PR}} \)? We already know that answer is “no”; in fact, from Proposition 3.2 not only do we have that \( \text{RE} \cap \text{F}_{\text{PR}}\text{-compressible} \not\subseteq \text{REC} \), we indeed have that \( \text{RE} \cap \text{F}_{\text{PR}}\text{-compressible}' = \text{RE} \).

As promised above, although the \( \text{RE}\text{-complete sets—indeed, all sets in} \text{ RE} - \text{REC} \)—are not \( \text{F}_{\text{REC}}\text{-compressible}, we will now establish that all \( \text{coRE}\text{-complete sets are} \text{ F}_{\text{REC}}\text{-compressible}. \)

**Proposition 5.5.** \( \text{F}_{\text{REC}}\text{-compressible and} \text{ F}_{\text{PR}}\text{-compressible are each closed under recursive isomorphisms}. \)

Proposition 5.5 is immediate and needs no proof.

**Theorem 5.6.** Every \( \text{coRE} \) cylinder except \( \emptyset \) is \( \text{F}_{\text{REC}}\text{-compressible}. \)

**Proof.** Let \( A \) be co-r.e. and a nonempty cylinder. Let \( s_0, s_1, s_2, \ldots \) enumerate \( \Sigma^* \) in lexicographical order. Let \( E \) be an enumerating Turing machine without repetitions for \( \overline{A} \). Define \( L_A = \{ \langle x, \varepsilon \rangle \mid x \in A \} \cup \{ \langle x, s_i \rangle \mid i \geq 1 \land x \text{ is the } i^{\text{th}} \text{ string enumerated by} E \} \). Then \( L_A \) is \( \text{F}_{\text{REC}}\text{-compressible by projection onto the first coordinate.} \)

We claim that \( A \) is recursively isomorphic to \( L_A \). Why?

Clearly \( A \leq_1 L_A \) by mapping \( x \) to \( \langle x, \varepsilon \rangle \). Now let us show that \( L_A \leq_1 A \). Fix strings \( x_0 \in A \) and \( x_1 \notin A \). Then the following gives a many-one reduction from \( L_A \) to \( A \):

- On input \( \langle x, s_i \rangle \), if \( s_i = \varepsilon \), output \( x \). Otherwise, check if \( x \) is the \( i^{\text{th}} \) string enumerated by \( E \). If so, output \( x_0 \). Otherwise, output \( x_1 \).

\[\footnote{For the reader who might like a direct, simple proof of Corollary 5.7, we include here such a construction. Consider the set \( A = \{ \langle x, \varepsilon \rangle \mid x \in K \} \cup \{ \langle x, \text{successor}(y) \rangle \mid M_x(x) \text{ accepts in exactly } y \text{ steps} \} \), where \( \text{successor}(y) \) denotes the string immediately after \( y \) in lexicographical order. \( A \) clearly is coRE-complete and, since for each \( x \) there is exactly one \( y \) such that \( \langle x, y \rangle \in A \), \( A \) is \( \text{F}_{\text{REC}}\text{-compressible via the function} f(\langle x, y \rangle) = x \). In fact, the proof of Theorem 5.6 is simply a more flexible version of this idea.} \]
So \( L_A \leq_m A \). Since \( A \) is a cylinder, it follows (by \cite[p. 89]{Rog67}) that \( L_A \leq_1 A \). Since \( A \leq_1 L_A \) and \( L_A \leq_1 A \), by the Myhill Isomorphism Theorem \( L_A \) is recursively isomorphic to \( A \), so by Proposition \ref{prop:compression}\( A \) is \( \text{F}_{\text{REC}} \)-compressible.

Note that the set \( \mathbb{K}_{\text{cyl}} = \{ (a, b) \mid a \in \mathbb{K} \land b \in \Sigma^* \} \) is clearly a \( \text{coRE} \)-complete cylinder (since it is trivially recursively isomorphic to the two-dimensional set \( \mathbb{K} \times \Sigma^* \) via \( (a, b) \mapsto (a, b) \)). So by Theorem \ref{thm:all-coRE-cylinders} we have that that \( \text{coRE} \)-complete set is \( \text{F}_{\text{REC}} \)-compressible. But since \( \text{F}_{\text{REC}} \)-compressible is, as Proposition \ref{prop:compression} notes, clearly closed under recursive isomorphisms (as already has been analogously noted before for the case of polynomial-time compressibility and polynomial-time isomorphisms \cite{GHK92}), and since by Myhill’s Corollary all \( \text{coRE} \)-complete sets are recursively isomorphic to \( \mathbb{K}_{\text{cyl}} \), we have that all \( \text{coRE} \)-complete sets are \( \text{F}_{\text{REC}} \)-compressible. We summarize this as the following corollary, whose claim appeared without proof in \cite{GHK92}.

**Corollary 5.7** (Stated without proof in \cite{GHK92}). All \( \text{coRE} \)-complete sets are \( \text{F}_{\text{REC}} \)-compressible.

Fix a standard, nice indexing (naming scheme)—\( \phi_1, \phi_2, \phi_3, … \)—for the partial recursive functions. A set \( A \) is an index set exactly if there exists a (possibly empty) collection \( \mathcal{F} \) of partial recursive functions such that \( A = \{ i \mid \phi_i \in \mathcal{F} \} \). (Since all our sets are over \( \Sigma^* \), we are implicitly associating the \( i \)th positive natural number with the \( i \)th string in \( \Sigma^* \), so that our index sets are type-correct.)

Since all index sets are cylinders (see \cite[p. 23]{Soa87}; \cite{Soa87}’s definition of cylinders is well-known—see \cite[Theorem VIII(c)]{Rog67}—to be equivalent to the definition given in our Section \ref{section:cylinders}), and thus all \( \text{co-} \text{r.e.} \) index sets are \( \text{co-} \text{r.e.} \) cylinders, Theorem \ref{thm:all-coRE-cylinders} implies that all \( \text{co-} \text{r.e.} \) index sets except (the finite, and thus not compressible index set) \( \emptyset \) are \( \text{F}_{\text{REC}} \)-compressible.

**Corollary 5.8.** All \( \text{coRE} \) index sets except \( \emptyset \) are \( \text{F}_{\text{REC}} \)-compressible.

On the other hand, by diagonalization we can build a set, even one recursive in \( K \), that is not compressible even by any \( \text{F}_{\text{PR}} \) function.

**Theorem 5.9.** \( \Delta^0_2 \not\subseteq \text{F}_{\text{PR}} \)-compressible'.

*Proof.* Fix a standard enumeration of Turing machines, \( M_1, M_2, M_3, … \), with each machine viewed as computing a partial recursive function. \( \phi_i \) will denote the partial recursive function computed by \( M_i \). We will explicitly construct a set \( A \) that belongs to \( \Delta^0_2 \) but that is not \( \text{F}_{\text{PR}} \)-compressible. This will be done by a stage construction. At stage \( i \), we will define a set \( A_i \) and a string \( w_i \). \( A \) will be defined as \( \bigcup_{i \geq 0} A_i \). We will ensure that \( A_i \subseteq A_{i+1} \) and \( A^{<w_i} = A_i^{<w_i} \). That is, after stage \( i \), all strings lexicographically preceding \( w_i \) will be fixed—their membership/nonmembership in \( A \) will not be changed by later stages. At each stage \( i \), \( i \geq 1 \), at least one string will be added to \( A \), in order to ensure that \( A \) ultimately becomes infinite, and \( \phi_i \) will be eliminated as a \( \text{F}_{\text{PR}} \)-compressor for \( A \).

We start by setting \( A_0 \) to be \( \emptyset \) and \( w_0 \) to be \( \varepsilon \). We then do stage 1, then stage 2, etc.

At stage \( i \), \( i \geq 1 \), first check whether \( \phi_i \) is injective when restricted to \( A_{i-1} \cup (\Sigma^*)^{\geq w_{i-1}} \). This is an r.e. condition, since we are looking for a pair \( (x, y) \) with \( x, y \in \text{domain}(\phi_i) \), \( x \neq y \), \( x, y \in A_{i-1} \cup (\Sigma^*)^{\geq w_{i-1}} \), and \( \phi_i(x) = \phi_i(y) \). If such a pair exists (which our \( \Delta^0_2 \) process can easily test), then
we set \( A_i \) to be \( A_{i-1} \cup \{ x, y, w_{i-1} \} \), we set \( w_i \) to be successor(\( \max(x, y, w_{i-1}) \)), and we go to stage \( i + 1 \). (We added \( w_{i-1} \) to ensure that we always add a string—even in the case that \( x, y \in A_{i-1} \).) Since \( x, y \in A_i \) (and thus \( x, y \in A \)) and \( x \neq y \), we have ensured that for two strings in \( A \), namely \( x \) and \( y \), \( \phi_i \) maps to the same output. So \( \phi_i \) has been eliminated as a potential compressor for \( A \).

However, if we cannot find such a pair \( (x, y) \), then we will freeze a string out of \( A \) in such a way as to permanently ensure that \( \phi_i \) is not surjective. In particular, check whether \( \phi_i((\Sigma^*)^{\geq w_{i-1}}) \neq \emptyset \). This is again an r.e. test. If the test determines that \( \phi_i((\Sigma^*)^{\geq w_{i-1}}) \neq \emptyset \), then there is an \( x \in (\Sigma^*)^{\geq w_{i-1}} \) such that \( \phi_i(x) \) is defined. Let \( x \) be the lexicographically smallest string in \((\Sigma^*)^{\geq w_{i-1}} \) such that this holds. Such an \( x \) can be found by further r.e. queries (within our \( \Delta^0_2 \) process). Then set \( A_i \) to be \( A_{i-1} \cup \{ \text{successor}(x) \} \), set \( w_i \) to be successor(successor(\( x \))), and go to stage \( i + 1 \). Note that we have ensured that \( \phi_i(x) \neq \phi(A) \) (the reason we know that no string in \( A_{i-1} \cup (\Sigma^*)^{\geq w_{i-1}} \) can map to \( \phi_i(x) \) is that if so we would have had a pair \( (x, y) \) of the sort sought above), and so \( \phi_i(A) \neq \Sigma^* \), and so we have ensured that \( \phi_i \) is not a compressor for \( A \).

In the last remaining case, we must have \( \phi_i((\Sigma^*)^{\geq w_{i-1}}) = \emptyset \). So \( \phi_i \) is only defined for finitely many strings, and thus cannot be a compressor function for \( A \). So set \( A_i \) to \( A_{i-1} \cup \{ w_{i-1} \} \), set \( w_i \) to successor(\( w_{i-1} \)), and go to stage \( i + 1 \).

Note that in all three cases, \( A_i \) has at least one more element than \( A_{i-1} \), so \( A \) will be infinite. And at stage \( i \), we ensured that \( \phi_i \) will not be a compressor for \( A \). So \( A \) is not \( \text{FPR} \)-compressible, since no partial recursive function is a compressor for it.

\section{Conclusions and Open Problems}

This paper defined and studied the recursion-theoretic analogues of the complexity-theoretic notions of ranking and compression. We particularly sought to determine where rankable and compressible sets could be found. For example, we found that all \( \text{coRE} \)-complete sets are recursively compressible but no \( \text{RE} \)-complete set is recursively compressible, and that no \( \text{RE} \)-complete or \( \text{coRE} \)-complete set is recursively (or even partial-recursively) rankable. Nonetheless, we showed that every 1-truth-table degree—even the one containing the \( \text{RE} \)-complete and the \( \text{coRE} \)-complete sets—contains recursively rankable sets. And we also showed that every nonempty \( \text{coRE} \) cylinder is recursively compressible.

We mention some open issues that we commend to the interested reader. We conjecture that there exist infinite, \( \text{co-re} \). sets that are not partial-recursively compressible, although this paper establishes that only for the larger class \( \Delta^0_2 \). Also, can one construct a set that is \( \text{FPR} \)-rankable but not \( \text{FREC} \)-rankable, and if so, what is the smallest class in which such a set can be constructed? Note that by Theorems \ref{thm:rankable} and \ref{thm:compressible}, separating \( \text{FPR} \)-rankable from \( \text{FREC} \)-rankable on any set in \( \text{RE} \cup \text{coRE} \) is impossible. Some additional research directions and ideas for work building on the notions of the present paper can be found in \cite{HR16}.
References

[ÁJ93] C. Álvarez and B. Jenner. A very hard log-space counting class. *Theoretical Computer Science*, 107:3–30, 1993.

[All85] E. Allender. Invertible functions, 1985. PhD thesis, Georgia Institute of Technology.

[Ber76] L. Berman. On the structure of complete sets. In *Proceedings of the 17th IEEE Symposium on Foundations of Computer Science*, pages 76–80. IEEE Computer Society, October 1976.

[BFH78] G. Brassard, S. Fortune, and J. Hopcroft. A note on cryptography and NP∩coNP = P. Technical Report TR-338, Department of Computer Science, Cornell University, Ithaca, NY, April 1978.

[BS85] J. Balcazar and U. Schoning. Bi-immune sets for complexity classes. *Mathematical Systems Theory*, 18(1):1–10, 1985.

[Dek62] J. Dekker. Infinite series of isols. In *Proceedings of the 5th Symposium in Pure Mathematics*, pages 77–96. American Mathematical Society, 1962.

[DM58] J. Dekker and J. Myhill. Retraceable sets. *Canadian Journal of Mathematics*, 10:357–373, 1958.

[FFNR03] S. Fenner, L. Fortnow, A. Naik, and J. Rogers. Inverting onto functions. *Information and Computation*, 186(1):90–103, 2003.

[GH96] J. Goldsmith and S. Homer. Scalability and the isomorphism problem. *Information Processing Letters*, 57(3):137–143, 1996.

[GHK92] J. Goldsmith, L. Hemachandra, and K. Kunen. Polynomial-time compression. *Computational Complexity*, 2(1):18–39, 1992.

[GS85] A. Goldberg and M. Sipser. Compression and ranking. In *Proceedings of the 17th ACM Symposium on Theory of Computing*, pages 440–448. ACM Press, May 1985.

[GS91] A. Goldberg and M. Sipser. Compression and ranking. *SIAM Journal on Computing*, 20(3):524–536, 1991.

[HM83] S. Homer and W. Maass. Oracle dependent properties of the lattice of NP sets. *Theoretical Computer Science*, 24(3):279–289, 1983.

[Hom86] S. Homer. On simple and creative sets in NP. *Theoretical Computer Science*, 47(2):169–180, 1986.

[HOZZ06] L. Hemaspaandra, M. Ogihara, M. Zaki, and M. Zimand. The complexity of finding top-Toda-equivalence-class members. *Theory of Computing Systems*, 39(5):669–684, 2006.
[HR90] L. Hemachandra and S. Rudich. On the complexity of ranking. *Journal of Computer and System Sciences*, 41(2):251–271, 1990.

[HR16] L. Hemaspaandra and D. Rubery. More on compression and ranking. Technical Report arXiv:1611.01696[cs.LO], Computing Research Repository, arXiv.org/cond-mat, November 2016. Revised, December 2017.

[HRW97] L. Hemaspaandra, J. Rothe, and G. Wechsung. Easy sets and hard certificate schemes. *Acta Informatica*, 34(11):859–879, 1997.

[Huy90] D. Huynh. The complexity of ranking simple languages. *Mathematical Systems Theory*, 23(1):1–20, 1990.

[Kle43] S. Kleene. Recursive predicates and quantifiers. *Transactions of the AMS*, 53:41–73, 1943.

[LLS75] R. Ladner, N. Lynch, and A. Selman. A comparison of polynomial time reducibilities. *Theoretical Computer Science*, 1(2):103–124, 1975.

[Mos47] A. Mostowski. On definable sets of positive integers. *Fundamenta Mathematicae*, 34:81–112, 1947.

[MS72] A. Meyer and L. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential space. In *Proceedings of the 13th IEEE Symposium on Switching and Automata Theory*, pages 125–129. IEEE Press, October 1972.

[Myh55] J. Myhill. Creative sets. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 1:97–108, 1955.

[Odi89] P. Odifreddi. *Classical Recursion Theory*. North-Holland/Elsevier, 1989.

[Pos44] E. Post. Recursively enumerable sets of integers and their decision problems. *Bulletin of the AMS*, 50:284–316, 1944.

[Rog67] H. Rogers, Jr. *The Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.

[Rot99] J. Rothe. Complexity of certificates, heuristics, and counting types, with applications to cryptography and circuit theory. Habilitation thesis, Friedrich-Schiller-Universität Jena, Institut für Informatik, Jena, Germany, June 1999.

[Sel82] A. Selman. Analogues of semirecursive sets and effective reducibilities to the study of NP complexity. *Information and Control*, 52(1):36–51, 1982.

[Soa87] R. Soare. *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*. Perspectives in Mathematical Logic. Springer-Verlag, 1987.
A Deferred Proof from Section 3

In Footnote 4 in Section 3 we claimed that under both the “a” and “b” variants of ranking mentioned in that footnote, and for each of those under both $\mathcal{F}_{\text{REC}}$ and $\mathcal{F}_{\text{PR}}$ ranking functions, the class of sets thus ranked is exactly the recursive sets. We now prove that. It is immediately obvious that each recursive set is $\mathcal{F}_{\text{REC}}$ (the more restrictive of the two function classes) rankable even under the “b” variant, which is the more restrictive of the two variants. So all that remains is to show that each set that is $\mathcal{F}_{\text{PR}}$ rankable under the “a” variant is recursive. Let $A$ be a set that is $\mathcal{F}_{\text{PR}}$ rankable under the “a” variant. If $A$ is finite, then trivially $A \in \text{REC}$. So let us consider the case where $A$ is infinite. Let $f$ be an $\mathcal{F}_{\text{PR}}$ ranking function for $A$ of the variant “a” sort. Let us quickly make clear what variant “a” means, especially in the context of $\mathcal{F}_{\text{PR}}$ functions. If on an input $f$ halts in an accepting state we view the string that is at that moment on its output tape (namely, from the left end of the output tape up to but not including the leftmost blank cell) as the output of $f$, and if $f$ halts in a rejecting state we view it as stating that the input is not in the set. On inputs $x \in A$, $f$ must output the string whose rank order within $\Sigma^*$ is the same as the rank order of $x$ within $A$. On inputs $x \not\in A$, $f$ can either halt in a rejecting state or run forever (but it cannot halt in an accepting state, i.e., it cannot output some string; this contrasts with Definition 3.3, which allows $f$ to even “lie” on inputs $x \not\in A$). Here is the description of a procedure, which halts on every input, for testing whether $x \in A$. In a standard dovetailing manner (i.e., interleaved, e.g., running on the first string in $\Sigma^*$ for one step, then running on the first two strings in $\Sigma^*$ for two steps each, then running on the first three strings in $\Sigma^*$ for three steps each, and so on), run $f$ on every string in $\Sigma^*$. If $f(x)$ is ever computed in that process, we reject $x$ if $f(x)$ declares that $x$ is not in $A$ (recall that as noted above in variant “a” the ranker can declare the string to not be in $A$, in particular by halting in a rejecting state), and we accept $x$ otherwise. Also, as the process goes on, if any string $y$ such that $y >_{\text{lex}} x$ evaluates to the lexicographically first string in $\Sigma^*$, namely $\epsilon$, then we reject $x$. Also, as the process goes on, if for some strings $w$ and $y$ with $w <_{\text{lex}} x <_{\text{lex}} y$ and such that $f(y)$ and $f(w)$ have both evaluated, it holds that $f(y)$ evaluates to the lexicographical successor of $f(w)$, then reject $x$.

At least one of these cases must eventually occur. Why? If $x \in A$, eventually, $f$ will compute $f(x)$ and we will correctly accept. If $x \not\in A$, then there are two cases. If $x$ is lexicographically strictly less than the lexicographically first string $z$ in $A$, then eventually we will evaluate $f(z)$ to be $\epsilon$ and will correctly reject $x$. Otherwise, eventually the strings in $A$ that are most closely lexicographically greater than (recall that we are here handling the case that $A$ is infinite, so such a string must exist) and less than $x$ will evaluate under $f$, at which point we will correctly reject $x$.  

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