GENERALIZED MINKOWSKI INEQUALITY VIA DEGENERATE HESSIAN EQUATIONS ON EXTERIOR DOMAINS

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ABSTRACT. In this paper, we prove a generalized Minkowski inequality holds for any smooth, \((k-1)\)-convex, starshaped domain \(\Omega\). Our proof relies on the solvability of the degenerate \(k\)-Hessian equation on the exterior domain \(\mathbb{R}^n \setminus \Omega\).

1. INTRODUCTION

In [1], Agostiniani-Fogagnolo-Mazzieri considered the \(p\)-Laplacian equation in an exterior domain \(\mathbb{R}^n \setminus \bar{\Omega}\),

\[
\begin{align*}
\Delta_p u &= \text{div}(|Du|^{p-2}Du) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega} \\
 u &= 1 \quad \text{on} \quad \partial \Omega \\
u(x) &\to 0 \quad \text{as} \quad |x| \to \infty,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded open set with smooth boundary and \(1 < p < n\). The existence, regularity, and asymptotic behavior of (1.1) is clear:

(i) there is a unique solution \(u \in C^{1,\alpha}(\mathbb{R}^n \setminus \Omega)\) and \(u \in C^\infty((\mathbb{R}^n \setminus \Omega) \setminus \{Du \neq 0\})\).

(ii) up to some constant,

\[
u(x) \to \Gamma_p(x) = \frac{p-1}{n-p} \left(\frac{1}{\omega_{n-1}}\right)^{\frac{1}{p-1}} |x|^{\frac{p-n}{p-1}}, \quad \text{as} \quad |x| \to \infty.
\]

Moreover,

\[
\lim_{|x| \to +\infty} \frac{u(x)}{\Gamma_p(x)} = C_p(\Omega)^{-\frac{1}{p-1}},
\]

where

\[
C_p(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |Du|^p dx \left| v \in C^\infty_c(\mathbb{R}^n), v \geq 1 \text{ on } \Omega \right. \right\}.
\]

By applying (i) and (ii), Agostiniani-Fogagnolo-Mazzieri proved the following \(L^p\)-Minkowski inequality

**Theorem 1.1. (Theorem 1.2 of [1])** Let \(\Omega \subset \mathbb{R}^n\) be an open bounded set with smooth boundary. Then, for every \(1 < p < n\) the following inequality holds

\[
C_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|S^{n-1}|} \int_{\partial \Omega} \left| H \right|^p d\sigma,
\]
where $C_p(\Omega)$ is the normalised $p$-capacity of $\Omega$. Moreover, equality holds in (1.2) if and only if $\Omega$ is a ball.

Inspired by [1], in this paper, we study the following Hessian equation in an exterior domain $\mathbb{R}^n \setminus \Omega$,

$$
\begin{cases}
S_k(D^2 u) = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega} \\
u = -1 & \text{on } \partial\Omega \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
$$

(1.3)

where $\Omega$ is a bounded, $(k - 1)$-convex, star-shaped domain and $k < \frac{n}{2}$.

**Remark 1.1.** It is easy to see that when $k = \frac{n}{2}$, the rotationally symmetric solution to equation

$$
\begin{cases}
S_k(D^2 u) = 0 & \text{in } \mathbb{R}^n \setminus \bar{B}_1 \\
u = -1 & \text{on } \partial B_1
\end{cases}
$$

is $u = C \log |x| - 1$ for any $C > 0$. We can see that as $|x| \to \infty$, $u(x) = C \log |x| - 1 \to \infty$. This implies that (1.3) is not solvable when $k = \frac{n}{2}$. Similarly, we can show that when $k > \frac{n}{2}$, equation (1.3) is also not solvable. Therefore, the requirement $k < \frac{n}{2}$ is necessary.

To present the result, we first introduce some notations. In this paper, $D^2 u$ denotes the Hessian of $u$, and the $k$-th elementary symmetric function $S_k(A)$ of a symmetric matrix $A$ is defined by

$$
S_k(A) = S_k(\lambda[A]) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},
$$

where $\lambda[A] = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of $A$. We also have the following definition.

**Definition 1.1.** For any open set $U \subset \mathbb{R}^n$, a function $\nu \in C^{1,1}(U)$ is called $k$-admissible if the eigenvalues $\lambda[D^2 \nu(x)] = (\lambda_1(x), \cdots, \lambda_n(x)) \in \Gamma_k$ for all $x \in U$, where $\Gamma_k$ is the Gårding’s cone

$$
\Gamma_k = \{ \lambda \in \mathbb{R}^n | S_m(\lambda) > 0, m = 1, \cdots, k \}.
$$

A $C^2$ regular hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ is called $k$-convex if its principal curvature vector $\kappa(X) \in \Gamma_k$ for all $X \in \mathcal{M}$.

We prove

**Theorem 1.2.** Let $\Omega$ be a $(k - 1)$-convex, star-shaped domain. Then, there exists a $k$-admissible solution $u \in C^{1,1}(\mathbb{R}^n \setminus \Omega)$ to equation (1.3), such that $\frac{u(x)}{-|x|^\frac{n-1}{k}} \to \gamma$ as $|x| \to \infty$ in $C^2$ topology.

Here, $\gamma > 0$ is a constant depending on $\partial \Omega$. Moreover, for any $s \in [-1, 0)$, the level set $\{ u = s \}$ is regular.

Moreover, we derive

**Corollary 1.1.** Let $\Omega$ be a $(k - 1)$-convex, star-shaped domain. Then we have

$$
\int_{\partial \Omega} |\nabla u|^{k+\beta} \sigma_{k-1}(\partial \Omega) dx \geq |\mathbb{S}^{n-1}| \left( \frac{n-1}{k-1} \right)^{k+\beta} \gamma^{-\frac{\beta}{k-2}},
$$

(1.4)
Here, \( u, \gamma \) are the same as in Theorem 1.2 and \( \beta \) is an arbitrary constant satisfying \( \beta \geq \frac{n-2k}{n-k} \). In particular, when \( \beta = n - 2k \) we get

\[(1.5) \quad \int_{\partial \Omega} |\nabla u|^{n-k} \sigma_{k-1} (\partial \Omega) dx \geq |\mathbb{S}^{n-1}| \left( \frac{n-1}{k-1} \right) \left( \frac{n-2k}{k} \right)^{n-k}.
\]

Moreover, equality holds in (1.4) and (1.5) if and only if \( \Omega \) is a ball.

The classical Minkowski inequality [16] proved in early 20 century states: If \( \Omega \subset \mathbb{R}^3 \) is a convex domain with smooth boundary, then

\[
\frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial \Omega} \frac{H}{n-1} d\sigma \geq \left( \frac{|\partial \Omega|}{|\mathbb{S}^{n-1}|} \right)^{\frac{n-2}{n-1}}.
\]

Since then, the Minkowski inequality has been studied intensively by various authors under different settings (see [2, 7, 9, 10, 17] and references therein). Note that the classical Minkowski inequality gives a lower bound for the total mean curvature of \( \partial \Omega \). Here we generalize this inequality and give a lower bound for the weighted total \( \sigma_{k-1} \) curvature of \( \partial \Omega \).

We now give an outline of this paper. In order to solve equation (1.3) and study the asymptotic behavior of the solution, we constructed the following approximating Dirichlet problem in Section 2,

\[(1.6) \begin{cases}
S_k(D^2 u) = f_\epsilon \text{ in } B_R \setminus \bar{\Omega} \\
u = -1 \text{ on } \partial \Omega \\
u(x) = \phi(x, C_0) \text{ on } \partial B_R.
\end{cases}
\]

Here \( f_\epsilon \to 0 \) as \( \epsilon \to 0 \) and \( \Phi(x, C_0) \) is a supersolution of (1.3) We show the solvability of (1.6) in Sections 3 and 4. The solvability of (1.3) then follows by standard arguments and the first part of Theorem 1.2 is proved.

We want to point out that Section 3 is the most important and innovative part of this paper, where we explicitly constructed a subsolution to equation (1.3) in a large neighborhood of \( \partial \Omega \). Note that, here \( \partial \Omega \) is \((k-1)\)-convex, the usual distance function (that is used for constructing subsolutions) can only be defined in a very small neighborhood of \( \partial \Omega \), which is not sufficient for solving this problem. Therefore, we have to utilize the additional assumption that \( \partial \Omega \) is starshaped, and find a replacement of the distance function. We believe that, the idea developed in this paper will be useful in solving similar problems in warped product spaces.

In Sections 5 and 6, we establish the desired asymptotic behavior of the solution \( u \) of (1.3) as \( |x| \to \infty \). This proves the second half of Theorem 1.2.

In Section 7 we show for the solution \( u \) of (1.3), when \( \beta \geq \frac{n-2k}{n-k} \),

\[
\Phi(\tau) = \int_{\{u = 1/\tau\}} \frac{1}{|Du|^k} s^{ij}_{k} u_i u_j \left( \frac{|Du|}{(-u)^{\frac{k-n}{2k-n}}} \right)^{\beta} dx
\]

is monotone. Combining the monotonicity of \( \Phi(\tau) \) with Theorem 1.2 we obtain Corollary 1.1.
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2. THE APPROXIMATE PROBLEM

Equation (1.3) is a degenerate equation defined on a non-compact domain. It is natural to approach it using a sequence of non-degenerate equations defined on compact domains $\bar{B}_R \setminus \Omega$. The delicacy here is that we want our approximate problem keeping the asymptotic behavior of the solution $u$. More specifically, assume $u_R$ is a solution of the approximate problem defined on $\bar{B}_R \setminus \Omega$, we want $u_R$ satisfying

$$|u_R| = O(R^{2-\frac{n}{k}})$$

on $\partial B_R$.

2.1. Global barriers. Inspired by the rotationally symmetric solution to (1.3), for any fixed small $\epsilon_0 > 0$ and $0 < \epsilon < \epsilon_0$, we let

$$\phi(x, C) = -C(|x| + \epsilon)^{2-\frac{n}{k}}.$$ 

Then a straightforward calculation shows

$$\phi_i = C \left( \frac{n}{k} - 2 \right) (|x| + \epsilon)^{1-\frac{n}{k}} \frac{x_i}{|x|},$$

and

$$\phi_{ij} = C \left( \frac{n}{k} - 2 \right) (|x| + \epsilon)^{-\frac{n}{k}} \left[ \left( |x| + \epsilon \right) \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) + \left( 1 - \frac{n}{k} \right) \frac{x_i x_j}{|x|^2} \right].$$

We can see that the eigenvalues of $D^2 \phi$ are

$$C \left( \frac{n}{k} - 2 \right) (|x| + \epsilon)^{-\frac{n}{k}} \left( 1 - \frac{n}{k}, 1 + \frac{\epsilon}{|x|}, 1 + \frac{\epsilon}{|x|}, \ldots, 1 + \frac{\epsilon}{|x|} \right).$$

Therefore, we get

$$\sigma_k(D^2 \phi) = \left( 1 + \frac{\epsilon}{|x|} \right)^{k-1} \left( \frac{n - 1}{k} \right) \frac{\epsilon}{|x|} \left( \frac{1}{|x| + \epsilon} \right)^n \left( \frac{n}{k} - 2 \right)^k C^k.$$ 

Denote $f_\epsilon = \sigma_k(D^2 \phi(x, 1))$, it is clear that when $n > 2k$, we have $f_\epsilon > 0$. Moreover, consider

$$\begin{align*}
    S_k(D^2 u) = f_\epsilon & \text{ in } \mathbb{R}^n \setminus \bar{\Omega} \\
    u = -1 & \text{ on } \partial \Omega \\
    u(x) & \to 0 \text{ as } |x| \to \infty,
\end{align*}$$

(2.1)

we can see that for $0 < C_0 < 1 < C_1$, $\phi(x, C_0), \phi(x, C_1)$ are super- and sub-solutions of (2.1) respectively. Here, $C_0, C_1$ are chosen such that $\phi(x, C_0) > -1$ on $\partial \Omega$ and $\phi(x, C_1) < -1$ on $\partial \Omega$. In this paper, $C_0$ and $C_1$ are fixed constants.
2.2. The approximate problem. By discussions in subsection 2.1, in the next two sections, we will study the solvability of the following approximate Dirichlet problem:

\[
\begin{cases}
S_k(D^2 u) = f_\epsilon \quad \text{in} \quad B_R \setminus \bar{\Omega} \\
u = -1 \quad \text{on} \quad \partial \Omega \\
u(x) = \phi(x, C_0) \quad \text{on} \quad \partial B_R,
\end{cases}
\]

where \(B_R\) is an \(n\)-dimensional ball with radius \(R\) centered at 0.

It turns out that the existence of the solution to equation (2.2) is highly non-trivial. The biggest challenge here is the \(C^2\) boundary estimates on \(\partial \Omega\). It is well known that in order to obtain \(C^2\) boundary estimates for Dirichlet problems, we need to construct a subsolution of the Dirichlet problem in a neighborhood of its boundary (see [3, 5, 6, 8] for example). However, here \(\partial \Omega\) is \((k - 1)\)-convex, the distance function can only be well defined in a small neighborhood of \(\partial \Omega\). Since we do not have good control on the height of the solution \(u_R\) of (2.2) near \(\partial \Omega\), a distance function defined near \(\partial \Omega\) is not sufficient to construct a desired subsolution. In Section 3 we will address this problem.

3. Construction of the subsolution of (2.2)

This section is the most important section in this paper. We discovered a replacement of the distance function for starshaped domain \(\Omega\). This function behaves like a distance function near the boundary \(\partial \Omega\) and is well defined in \(\mathbb{R}^n \setminus \Omega\).

3.1. Hessian in the spherical coordinate. Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a scalar function, then \(f\) can also be expressed as a function of \((\theta, r) \in S^{n-1} \times \mathbb{R}\). Note that \(g_E = r^2 dz^2 + dr^2\), where \(dz^2\) is the standard metric on \(S^{n-1}\). In the following, we will denote the standard connection by \(D\). Now, we choose a local orthonormal frame \(\{e_1, \cdots, e_{n-1}\}\) on the unit sphere \(S^{n-1}\). Let \(\tau_a = \frac{e_a}{r}\), \(1 \leq a \leq n - 1\), which is the orthonormal frame on the sphere with radius \(r\), and we also let \(\tau_r = \frac{\partial}{\partial r}\). Then we have

\[
Df = f_\tau \tau_r + \sum_{a=1}^{n-1} \frac{1}{r} f_a \tau_a,
\]

where \(f_\tau = \tau_r f, f_a = e_a f\), and

\[
D^2 f = D_\tau (f_\tau \tau_r + \sum_{a=1}^{n-1} \frac{1}{r} f_a \tau_a) \otimes \tau_r + \sum_{b=1}^{n-1} \frac{1}{r} D_{e_b} (f_\tau \tau_r + \sum_{a=1}^{n-1} \frac{1}{r} f_a \tau_a) \otimes \tau_b.
\]

Since \(r \tau_r\) is the position vector and the principal curvature of the radius \(r\) sphere is \(1/r\), we get for \(1 \leq a, b \leq n - 1\),

\[
D_{\tau_r} \tau_r = 0, \quad D_{\tau_a} (r \tau_r) = \tau_a,
\]

\[
D_{\tau_r} \tau_a = -\frac{1}{r^2} e_a + \frac{1}{r} D_{e_a} \tau_r = 0, \quad \text{and} \quad D_{\tau_b} \tau_a = -\frac{1}{r} \tau_r \delta_{ab}.
\]

Thus, the Hessian of \(f\) is

\[
D^2_{ab} f = D^2 f (\tau_a, \tau_b) = \frac{1}{r^2} f_{ab} + \frac{1}{r} f_a \delta_{ab},
\]
(3.3) \[ D^2_{ar}f = D^2f(\tau_a, \tau_r) = \frac{1}{r}f_{ar} - \frac{1}{r^2}f_a, \]

(3.4) \[ D^2_{rr}f = D^2f(\tau_r, \tau_r) = f_{rr}, \]

where \(1 \leq a, b \leq n - 1\), \(f_{ab} = \epsilon_b \epsilon_a f, f_{ar} = \tau_r \epsilon_a f\), and \(f_{rr} = \tau_r \tau_r f\). Below, we will denote \(f_n = \frac{\partial f}{\partial r}\).

3.2. Construction of the subsolution. Recall that we have assumed \(\Gamma := \partial \Omega\) is starshaped and \((k - 1)\)-convex. We can parametrize \(\Gamma\) as a graph of the radial function \(\rho(\theta) : S^{n-1} \to \mathbb{R}\), i.e., \(\Gamma = \{ \rho(\theta) \theta \mid \theta \in S^{n-1} \}\). Same as in Subsection 3.1, suppose \(\{e_1, \cdots, e_{n-1}\}\) is an orthonormal frame on the unit sphere. Then the second fundamental form of \(\Gamma\) is

\[ h_{ij} = \frac{\rho}{w} \left( \delta_{ij} + \frac{\rho_i \rho_j}{\rho^2} - \frac{\rho_i \rho_j}{\rho} \right), \]

where \(w = \sqrt{1 + |\nabla \rho|^2}\), \(\rho_i, j = \nabla_i \rho, \) and \(\nabla\) denotes the Levi-Civita connection on \(S^{n-1}\). Since \(\nabla \epsilon_i e_j = 0\), we get \(\nabla_i \rho = \epsilon_i e_j \rho\). In the following, for any function \(f\) defined on the unit sphere, we denote \(f_i = e_j e_i f\), then we have \(\rho_i, j = \rho_i j\).

We define \(\varphi = \log \rho\), it is clear that the second fundamental form of \(\Gamma\) can be expressed as follows.

\[ h_{ij} = \frac{\rho}{w} (\delta_{ij} + \varphi_i \varphi_j - \varphi_{ij}), \]

where \(w = \sqrt{1 + |\nabla \varphi|^2}\). By a direct calculation, we also obtain

\[ g_{ij} = \rho^2 (\delta_{ij} + \varphi_i \varphi_j), \quad g^{ij} = \frac{1}{\rho^2} \left( \delta_{ij} - \frac{\varphi_i \varphi_j}{w^2} \right), \quad \text{and} \quad \gamma^{ij} = \frac{1}{\rho} \left( \delta_{ij} - \frac{\varphi_i \varphi_j}{w(1 + w)} \right). \]

Here \(\gamma^{ij}\) is the square root of \(g^{ij}\). Let \(a_{ij} = \gamma^{ik} h_{kl} \gamma^{lj}\), then the eigenvalues of \((a_{ij})_{1 \leq i, j \leq n-1}\), denoted by \(\kappa[a_{ij}] = (\kappa_1, \cdots, \kappa_{n-1})\) are the principal curvatures of \(\Gamma\).

Now, at any point \(p \in S^{n-1}\), we may rotate the coordinate such that \(|\nabla \rho| = \rho_1\) and \(\rho_{\alpha \beta} = \rho_{\alpha \beta} \delta_{\alpha \beta}\) for \(2 \leq \alpha, \beta \leq n - 1\). Then at the point \(\hat{p} = \rho(p)p \in \Gamma\) we have

\[
\begin{aligned}
\gamma^{11} &= \frac{1}{\rho} \left( 1 - \frac{w^2 - 1}{w(1 + w)} \right) = \frac{1}{\rho w}, \\
\gamma^{1\alpha} &= 0, \\
\gamma^{\alpha \beta} &= \frac{1}{\rho} \delta_{\alpha \beta},
\end{aligned}
\]

\(2 \leq \alpha, \beta \leq n - 1,\)

and

\[
\begin{aligned}
a_{11} &= \gamma^{1k} h_{kl} \gamma^{l1} = \gamma^{11} h_{11} \gamma^{11} = \frac{h_{11}}{\rho^2 w^2}, \\
a_{1\alpha} &= \gamma^{1k} h_{kl} \gamma^{l\alpha} = \frac{h_{1\alpha}}{\rho^2 w}, \\
a_{\alpha \beta} &= \gamma^{\alpha \alpha} h_{\alpha \beta} \gamma^{\beta \beta} = \frac{1}{\rho^2} h_{\alpha \beta},
\end{aligned}
\]

\(2 \leq \alpha, \beta \leq n - 1.\)

We consider

\[ g = r \rho^{-1}, \]
and we will compute the Hessian of \( g \) at an arbitrary point \((p, r) \in (\mathbb{S}^{n-1} \times \mathbb{R}) \setminus \{0\}\). A straightforward calculation yields

\[
g_1 = e_1 g = -r \rho^{-2} \rho_1, \quad g_{1a} = e_a e_1 g = -r (-2 \rho^{-3} \rho_1 \rho_a + \rho^{-2} \rho_{1a}) \quad \text{for } 1 \leq a \leq n - 1, \\
g_{1\alpha} = -r \rho^{-2} \rho_{1\alpha} \text{ for } 2 \leq \alpha \leq n - 1, \\
g_{\alpha} = e_\alpha g = 0, \quad g_{\alpha\beta} = e_\beta e_\alpha g = -r \rho^{-2} \rho_{\alpha\beta} \text{ for } 2 \leq \alpha, \beta \leq n - 1,
\]

and

\[
g_n = \frac{\partial g}{\partial r} = \rho^{-1}, \quad g_{nn} = \frac{\partial^2 g}{\partial r^2} = 0, \quad g_{1n} = \frac{\partial g_{11}}{\partial r} = -\rho^{-2} \rho_1, \quad g_{an} = \frac{\partial g_{a1}}{\partial r} = 0.
\]

Following the notation in Subsection 3.1, suppose \( \tau_\alpha = \frac{\partial}{\partial \rho} \), \( 1 \leq \alpha \leq n - 1 \) and \( \tau_r = \frac{\partial}{\partial r} \), then \( \{ \tau_1, \cdots, \tau_{n-1}, \tau_r \} \) forms an orthonormal frame at \((p, r)\). By virtue of (3.2), (3.3), and (3.4) we get,

(3.5) \quad D^2_{11} g = D^2 g(\tau_1, \tau_1) = \frac{1}{r^2} g_{11} + \frac{1}{r} \frac{w^3}{\rho} a_{11},

(3.6) \quad D^2_{1\alpha} g = D^2 g(\tau_1, \tau_\alpha) = \frac{1}{r^2} g_{1\alpha} = \frac{w^2}{r} a_{1\alpha}, \quad 2 \leq \alpha \leq n - 1,

(3.7) \quad D^2_{\alpha\beta} g = D^2 g(\tau_\alpha, \tau_\beta) = \frac{1}{r^2} g_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{\rho} \frac{w}{r} a_{\alpha\beta}, \quad 2 \leq \alpha, \beta \leq n - 1, \quad D^2_{ij} g = D^2 g(\tau_i, \tau_j) = 0, \quad 1 \leq i, j \leq n \text{ for all other cases}.

We also notice that at \( p \) we have

\[
h_{\alpha\beta} = \frac{\rho}{w} \left( \delta_{\alpha\beta} - \frac{\rho_{\alpha\beta}}{\rho} \right) = h_{\alpha\alpha} \delta_{\alpha\beta}, \quad 2 \leq \alpha, \beta \leq n - 1,
\]

this implies \( a_{\alpha\beta} = a_{\alpha\alpha} \delta_{\alpha\beta} \) is diagonalized. Therefore, at \((p, r)\) we obtain

\[
\text{Hessian}(g) = \begin{bmatrix}
\frac{w^3}{r} a_{11} & \frac{w^2}{r} a_{12} & \cdots & \frac{w^2}{r} a_{1n-1} & 0 \\
\frac{w^2}{r} a_{12} & \frac{w^2}{r} a_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{w^2}{r} a_{an-1} & 0 & \cdots & \frac{w^2}{r} a_{n-1} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

We want to point out that \( \kappa[a_{ij}] \) are the principal curvatures of \( \Gamma \) at \( \hat{p} = \rho(p)p \).

Let’s consider the function \( \phi = \phi(g) \), then \( \phi \) is also a function defined on \((\mathbb{S}^{n-1} \times \mathbb{R}) \setminus \{0\}\). We will compute the Hessian of \( \phi \) at \((p, r)\). Denote \( \phi'(p, r) = \partial_g \phi = A \), \( \phi''(p, r) = \partial_{gg} \phi = B \), we get at this point, for \( 1 \leq i, j \leq n \),

\[
D^2_{ij} \phi = AD^2_{ij} g + B(\tau_i, \tau_j) g.
\]

We note that \( \rho_{\alpha} = e_{\alpha} \rho \), it is easy to compute

\[
\text{Hessian}(\phi) = \begin{bmatrix}
\frac{Aw^3}{r} a_{11} + B \rho^{-2} \rho_1 & \frac{Aw^2}{r} a_{12} & \cdots & \frac{Aw^2}{r} a_{1n-1} & -B \rho^{-3} \rho_1 \\
\frac{Aw^2}{r} a_{12} & \frac{Aw}{r} a_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{Aw^2}{r} a_{an-1} & 0 & \cdots & \frac{Aw}{r} a_{n-1} & 0 \\
-B \rho^{-3} \rho_1 & 0 & \cdots & 0 & B \rho^{-2}
\end{bmatrix}.
\]
In the following, we will compute $\sigma_k(D^2\phi)$. For our convenience, in the rest of this section we will always assume $2 \leq \alpha, \beta, \gamma \leq n - 1$, lower case letters $2 \leq i, j, m, l, s \leq n$, and $1 \leq i, j \leq n - 1$.

$$\sigma_k(D^2\phi) = \phi_{11}\sigma_{k-1}(\phi_{ij}) + \sigma_k(\phi_{ij}) - \sum_{s=2}^n \phi_{1s}^2 \sigma_{k-2}(\phi_{ij}\phi_{ss})$$

$$= \phi_{11}\left[\sigma_{k-1}(\phi_{ab}) + B\rho^{-2}\sigma_{k-2}(\phi_{ab})\right] + \left[\sigma_k(\phi_{ab}) + B\rho^{-2}\sigma_{k-1}(\phi_{ab})\right]$$

$$- \sum_{\alpha=2}^{n-1} \phi_{1\alpha}^2 \left[\sigma_{k-2}(\phi_{a\beta}\phi_{\alpha\alpha}) + B\rho^{-2}\sigma_{k-3}(\phi_{a\beta}\phi_{\alpha\alpha})\right] - \phi_{1n}^2 \sigma_{k-2}(\phi_{ab})$$

$$= \left(\frac{Aw}{r}\right)^k a_{11} + B\rho^{-4}\rho_1^2 \left[\left(\frac{Aw}{r}\right)^k - \sigma_{k-1}(\phi_{ab}) + B\rho^{-2}\left(\frac{Aw}{r}\right)^k\sigma_{k-2}(\phi_{ab})\right]$$

$$+ \left(\frac{Aw}{r}\right)^k \sigma_k(\phi_{ab}) + B\rho^{-2}\left(\frac{Aw}{r}\right)^k \sigma_{k-1}(\phi_{ab})$$

$$- \sum_{\alpha=2}^{n-1} \left(\frac{Aw}{r}\right) \left(\frac{Aw}{r}\right)^k a_{1\alpha} \sigma_{k-2}(\phi_{a\beta}\phi_{\alpha\alpha})$$

$$- B\rho^{-2} \sum_{\alpha=2}^{n-1} \left(\frac{Aw}{r}\right)^2 \left(\frac{Aw}{r}\right)^k a_{1\alpha} \sigma_{k-3}(\phi_{a\beta}\phi_{\alpha\alpha})$$

$$- B^2 \rho^{-6}\rho_1^2 \left(\frac{Aw}{r}\right)^k \sigma_{k-2}(\phi_{ab})$$

$$= A^k w^{k+2} r^k \left[\sigma_k(\phi_{ij}) - \sigma_k(\phi_{ab})\right] + B\rho^{-2} A^{k-1} w^{k+1} r^{k-1} \sigma_{k-1}(\phi_{ij}) + \left(\frac{Aw}{r}\right)^k \sigma_k(\phi_{ab})$$

$$\geq -c_0 A^k \frac{\rho}{r^k} + c_1 \frac{B A^{k-1} r^{k-1}}{\rho^2 r^k}.$$

Here $c_1 = \min_{\hat{q} \in \Gamma} \sigma_{k-1}(\hat{q}) > 0$ and $c_0 = c_0(|\rho|_{C^2})$ are two positive constants only depending on $\Gamma$. Since $(p, r)$ is an arbitrary point in $(\mathbb{S}^{n-1} \times \mathbb{R}) \setminus \{0\}$, consider $\phi(g) = g^N$, the above calculation gives

$$\sigma_k(D^2\phi) \geq \frac{N k \rho \rho_1^{N-k}}{r^k} \left[\frac{c_1 (N-1)}{\rho} - c_0\right] \text{ in } (\mathbb{S}^{n-1} \times \mathbb{R}) \setminus \{0\}.\quad (3.8)$$

Notice that $g \geq 1$ in $\mathbb{R}^n \setminus \Omega$, choosing $N = N(\Gamma) > 0$ large we have

$$\sigma_k(D^2\phi) \geq \frac{1}{r^k} \text{ in } \mathbb{R}^n \setminus \Omega.\quad (3.9)$$

Moreover,

$$\phi = 1 \text{ on } \Gamma = \partial \Omega.\quad (3.10)$$

In Section 4, we will use $\phi$ to construct the subsolution to equation (2.2), which is crucial for obtaining a priori estimates that are needed for solving equation (2.2).
4. Solvability of the Approximate Problem

Let us consider the following Dirichlet problem (i.e., equation (2.2))

\[
\begin{aligned}
S_k(D^2 u) &= f_\epsilon \text{ in } B_R \setminus \bar{\Omega} \\
u &= -1 \text{ on } \partial \Omega \\
\phi(x, C_0) &= u(x) \text{ on } \partial B_R,
\end{aligned}
\]

(4.1)

where \( f_\epsilon = \left(1 + \frac{\epsilon}{|x|}\right)^{k-1} \left(\frac{n-1}{k}\right) \frac{1}{|x|} \left(\frac{1}{|x|+\epsilon}\right)^n \left(\frac{\epsilon}{R^k} - 2\right)^k \), \( B_R \) is an \( n \)-dimensional ball with radius \( R \) centered at \( 0 \), and \( \Omega \subset \mathbb{R}^n \) is a \((k-1)\)-convex, starshaped domain. For our convenience, in the rest of this paper, we will always denote \( \alpha_0 := \frac{n}{k} - 2 > 0 \). In this section, we prove

**Theorem 4.1.** Given \( \epsilon_0 > 0 \) small and \( R_0 > 0 \) large, then for any \( 0 < \epsilon < \epsilon_0, R > R_0 \) there is a unique, \( k \)-convex solution \( u_R^\epsilon \in C^2(B_R \setminus \bar{\Omega}) \cap C^{1,1}(B_R \setminus \bar{\Omega}) \) satisfying (4.1).

In the following, when there is no confusion, we will drop the superscript \( \epsilon \) and write \( u_R \) instead of \( u_R^\epsilon \). We also want to point out that, in this section we will establish some fairly accurate \( C^1 \) and \( C^2 \) estimates for \( u_R \) on \( \partial B_R \). Later, we will see that we need these accurate estimates for studying the asymptotic behavior of the solution \( u \) of (1.3).

Applying the maximum principle, the \( C^0 \) estimate of \( u_R \) follows from the existence of the global barriers constructed in Subsection 2.1.

**Lemma 4.1.** (\( C^0 \) bounds of \( u_R \)) Let \( u_R \) be the solution of (4.1), then \( u_R \) satisfies

\[
\phi(x, C_1) < u_R < \phi(x, C_0), \text{ in } B_R \setminus \bar{\Omega}.
\]

4.1. \( C^1 \)-estimates.

**Lemma 4.2.** (\( C^1 \) upper bound on \( \partial \Omega \)) Let \( u_R \) be the solution of (4.1), then on the boundary \( \partial \Omega \), \( u_R \) satisfies

\[
\frac{\partial u_R}{\partial \nu} < C,
\]

where \( \nu \) is the outward unit normal of \( \partial \Omega \) (pointing into \( B_R \setminus \bar{\Omega} \)), and \( C = C(\partial \Omega, n, k) \) only depends on \( \partial \Omega, n, \) and \( k \).

**Proof.** Since \( \partial \Omega \) is smooth, we know there exists \( r_0 > 0 \) such that for any \( \xi \in \partial \Omega \) there is \( z_\xi \in \Omega \) satisfying \( B_{r_0}(z_\xi) \cap \partial \Omega = \xi \), where \( B_{r_0}(z_\xi) \) is a ball centered at \( z_\xi \) with radius \( r_0 \). Namely, the sphere \( \partial B_{r_0}(z_\xi) \) and \( \partial \Omega \) are tangent at \( \xi \). We also denote

\[
U_{\delta_0} := \left\{ x \in \mathbb{R}^n \setminus \bar{\Omega} \mid \text{dist}(x, \Omega) < \delta_0 \right\}.
\]

Let \( c_1 := \max_{x \in \partial U_{\delta_0}} \phi(x, C_0) \), then consider

\[
\bar{u}_\xi = -C|x - z_\xi|^{-\alpha_0} + C r_0^{-\alpha_0} - 1.
\]

It’s clear that \( \bar{u}_\xi(\xi) = -1 \) and \( \bar{u}_\xi(x) > -1 \) for any \( x \in \partial \Omega \setminus \{\xi\} \). On \( \partial U_{\delta_0} \) we have

\[
\bar{u}_\xi \geq \frac{C}{r_0^{\alpha_0}} \left[ 1 - \left(1 + \frac{\delta_0}{r_0}\right)^{-\alpha_0} \right] - 1.
\]
Choosing \( C = C(\alpha_0, \delta_0, \partial \Omega) = \frac{(1 + c_1)\alpha_0}{1 - (1 + \delta_0/r_0)^{-\alpha_0}} \), we get \( \bar{u}_\xi \geq c_1 \) on \( \partial U_{\delta_0} \). Since \( S_k(\nabla^2 \bar{u}_\xi) = 0 \), by the maximum principle we obtain \( \bar{u}_\xi > u_R \) in \( U_{\delta_0} \setminus \bar{\Omega} \). This implies at \( \xi \) we have

\[
\frac{\partial u_R}{\partial \nu} < \frac{\partial \bar{u}_\xi}{\partial \nu} = C\alpha_0 r_0^{-\alpha_0 - 1} = \frac{(1 + c_1)\alpha_0 r_0^{-1}}{1 - (1 + \delta_0/r_0)^{-\alpha_0}}.
\]

Since \( \xi \in \partial \Omega \) is arbitrary, we prove this lemma. \( \square \)

**Lemma 4.3.** (\( C^1 \) lower bound on \( \partial \Omega \)) Let \( 0 < \epsilon < \epsilon_0 \) and \( R > (2C_1)^{\frac{1}{\alpha_0}} \) in equation (4.1), where \( \epsilon_0 > 0 \) is a small constant depending on \( \partial \Omega \) and \( C_1 \) is the fixed constant defined in Subsection 2.1. Let \( u_R \) be the solution of (4.1). Then on the boundary \( \partial \Omega \), \( u_R \) satisfies

\[
\frac{\partial u_R}{\partial \nu} > C,
\]

where \( \nu \) is the outward unit normal of \( \partial \Omega \) (pointing into \( B_R \setminus \bar{\Omega} \)), and \( C = C(\partial \Omega, n, k) \) only depends on \( \partial \Omega \), \( n \), and \( k \).

**Proof.** Let \( r_1 = (2C_1)^{\frac{1}{\alpha_0}} < R \), then on \( \partial B_{r_1} \) we get

\[
u = \frac{1}{r_1} (\phi(g) - 1) - 1, \quad \text{where} \quad \phi(g) \text{ is the subsolution constructed in Subsection 3.2 and} \quad A > 0 \text{ is a constant chosen to be sufficiently large such that} \quad \frac{1}{r_1} [\phi(r_1/\rho_0) - 1] < \frac{1}{2}.
\]

By virtue of (3.9) we know that, \( S_k(D^2 \bar{v}) \geq \frac{1}{A^2 r_1^2} \) in \( \mathbb{R}^n \setminus \bar{\Omega} \). It is easy to see that when \( 0 < \epsilon < \epsilon_0 \) and \( \epsilon_0 \) is small, we have \( S_k(D^2 \bar{v}) > f_\epsilon \) in \( B_{r_1} \setminus \bar{\Omega} \). Moreover, \( \bar{v} \) satisfies \( \bar{v} = -1 \) on \( \partial \Omega \) and \( \bar{v} < u_R \) on \( \partial B_{r_1} \). Therefore, the standard maximum principle yields \( \bar{v} < u_R \) in \( B_{r_1} \setminus \Omega \). This implies for any \( \xi \in \partial \Omega \) we have

\[
\frac{\partial u_R}{\partial \nu} > \frac{\partial \bar{v}}{\partial \nu}.
\]

Therefore, we complete the proof of Lemma 4.3. \( \square \)

**Remark 4.1.** By virtue of (3.1) we get \( |D\bar{v}| > c > 0 \) in \( B_{r_1} \setminus \Omega \). We also notice that \( \bar{v} = -1 \) in \( B_{r_1} \setminus \Omega \) and \( \bar{v} = -1 \) on \( \partial \Omega \). Therefore, we obtain \( \frac{\partial u_R}{\partial \nu} > \frac{\partial \bar{v}}{\partial \nu} > c \) on \( \partial \Omega \). Here \( c = c(\partial \Omega) > 0 \) is a positive constant independent of \( R \) and \( \epsilon \).

**Lemma 4.4.** (\( C^1 \) upper and lower bounds on \( \partial B_R \)) Let \( u_R \) be the solution of (4.1) for \( R > R_0 \), then on the boundary \( \partial B_R \), \( u_R \) satisfies

\[
\alpha_0 C_0 (R + \epsilon)^{-\alpha_0 - 1} < |Du_R| < \alpha_0 C_1 (R + \epsilon)^{-\alpha_0 - 1}.
\]

Here, \( R_0 > 0 \) is a constant depending on \( \phi(x, C_1) \) and \( C_0, C_1 \) are constants defined in Subsection 2.1.
Proof. Let \( u = \phi(x, C_1) + C_1(R + \epsilon)^{-\alpha_0} - C_0(R + \epsilon)^{-\alpha_0} \). It is clear that on \( \partial B_R \), \( u = u_R \). When \( R > R_0 > 0 \) is sufficiently large, we have \( u < 1 \) on \( \partial \Omega \). Since \( \sigma_k(D^2 \phi(x, C_1)) > f_\epsilon \), we obtain \( u < u_R < \phi(x, C_0) \) in \( B_R \setminus \Omega \). This implies on \( \partial B_R \) we have
\[
\frac{\partial \phi(x, C_0)}{\partial \nu} < \frac{\partial u_R}{\partial \nu} < \frac{\partial u}{\partial \nu},
\]
where \( \nu \) is the outer norm to \( \partial B_R \) (pointing away from \( B_R \setminus \Omega \)). Therefore, we conclude on \( \partial B_R \)
\[
\alpha_0 C_0(R + \epsilon)^{-\alpha_0 - 1} < |Du_R| < \alpha_0 C_1(R + \epsilon)^{-\alpha_0 - 1}.
\]

\[\Box\]

Lemma 4.5. (C¹ global on \( B_R \setminus \Omega \)) Let \( u_R \) be the solution of \( (4.1) \), then there exists some constant \( A > 0 \) independent of \( R \) and \( \epsilon \) such that
\[
\max_{x \in B_R \setminus \Omega} \{|Du_R| + Au_R\} = \max_{x \in \partial B_R \cup \partial \Omega} \{|Du_R| + Au_R\}.
\]

Proof. Consider \( W := \max_{x \in B_R \setminus \Omega, \xi \in \mathbb{S}^n} \{D_\xi u_R + Au_R\} \). Suppose \( W \) is achieved at an interior point \( (x_0, \xi_0) \in (B_R \setminus \Omega) \times \mathbb{S}^n \). Rotating the coordinate we may assume at \( x_0 \),
\[
D_{\xi_0} u_R = (u_R)_1, \text{ and } (u_R)_{\alpha \beta} = \lambda_\alpha \delta_{\alpha \beta} \text{ for } \alpha, \beta \geq 2.
\]
Then at \( x_0 \), a straightforward calculation gives \( A(u_R)_i + u_{1i} = 0 \), which implies
\[
(u_R)_1 = 0 \text{ for } i \geq 2, \text{ and } (u_R)_{11} = -A(u_R)_1.
\]
Moreover, at \( x_0 \), we also have
\[
0 \geq A S_k^{ii}(u_R)_i + S_k^{i11}(u_R)_{i1i}
\]
\[
= A f_\epsilon + f_\epsilon x_1 \frac{x_1}{|x_0|} \left[ \frac{(k - 1) \epsilon |x_0|^{-2}}{1 + \epsilon |x_0|^{-1}} - |x_0|^{-1} - \frac{n}{|x_0| + \epsilon} \right],
\]
where we have used \( S_k^{i11}(u_R)_{i1i} = k f_\epsilon \) and \( S_k^{i1}(u_R)_{i1} = (f_\epsilon)_{11} \). It’s clear that when \( A = A(n, k, \partial \Omega) > 0 \) large, the right hand side of (4.2) is positive. This leads to a contradiction. Thus \( W \) is achieved on the boundaries.

Combining Lemma 4.4 with Lemma 4.5, we conclude

Proposition 4.2. Let \( u_R \) be the solution of \( (4.1) \), then there exists some constant \( C \) independent of \( R \) and \( \epsilon \) such that
\[
|Du_R| \leq C \text{ in } B_R \setminus \Omega.
\]

4.2. C² boundary estimates. Let \( x_0 \in \partial \Omega \) be an arbitrary point on \( \partial \Omega \). Without loss of generality, we may choose a local coordinate \( \{\tilde{x}_1, \ldots, \tilde{x}_n\} \) such that \( x_0 \) is the origin. Let \( \tilde{x}_n \) axis be the exterior normal of \( \partial \Omega \) and the boundary near the origin is represented by
\[
\tilde{x}_n = \rho(\tilde{x}') = -\frac{1}{2} \sum_{\alpha=1}^{n} \kappa_{\alpha} \tilde{x}_\alpha^2 + O(|\tilde{x}'|^3),
\]
where \( \kappa_1, \kappa_2, \ldots, \kappa_{n-1} \) are the principal curvatures of \( \partial \Omega \) at the origin and \( \tilde{x}' = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}) \).

Let \( u_R \) be the solution of \( (4.1) \), then
\[
u_R(\tilde{x}', \rho(\tilde{x}')) = -1 \text{ on } \partial \Omega.
\]

(4.3)
First, differentiating (4.3) with respect to tangential directions at $x_0$ we obtain,

$$(u_R)_\alpha + (u_R)_n \rho_\alpha = 0 \quad \text{and} \quad (u_R)_{\alpha\beta} + (u_R)_n \rho_{\alpha\beta} = 0.$$ 

This gives,

$$(4.4) \quad (u_R)_{\alpha\beta} = (u_R)_n \kappa_\alpha \delta_{\alpha\beta} \quad \text{for} \quad \alpha, \beta < n.$$ 

Next, we estimate $|(u_R)_{\alpha n}(0)|$.

**Lemma 4.6. (C\(^2\) bound on \(\partial \Omega\) in mixed directions)** Let $u_R$ be the solution of (4.1), then on $\partial \Omega$, $u_R$ satisfies

$$|(u_R)_{\tau \nu}| < C,$$

where $\tau$ is an arbitrary unit tangential vector of $\partial \Omega$, $\nu$ is the outward unit normal of $\partial \Omega$ (pointing into $B_R \setminus \bar{\Omega}$), and $C = C(\partial \Omega, n, k)$ is a constant only depending on $\partial \Omega$, $n$, and $k$.

**Proof.** In this proof, let $\varphi$ be the subsolution of (4.1) constructed in the proof of Lemma 4.3. Let $x_0 \in \partial \Omega$ be an arbitrary point on $\partial \Omega$. Without loss of generality, we may choose a local coordinate \{\(\tilde{x}_1, \cdots, \tilde{x}_n\)\} such that $x_0$ is the origin. Let $T = \partial_\alpha - \kappa_\alpha (\tilde{x}_\alpha \partial_n - \tilde{x}_n \partial_\alpha)$, then on $\partial \Omega$ near $x_0$ we have

$$T u_R = (\partial_\alpha + \rho_\alpha \partial_n) u_R + O(|\tilde{x}|^2) = O(|\tilde{x}|^2).$$

Denote $\mathcal{L} := S^{ij}_k \partial_{ij}$ and $\bar{B}_{\delta_0} = B_{\delta_0}(x_0) \setminus \bar{\Omega}$, where $\delta_0 > 0$ is a fixed constant chosen to be so small that $\bar{B}_{\delta_0} \subset B_{r_1} \setminus \bar{\Omega}$. Note that $r_1 = (2C_1)^{1/\gamma_0}$ is a fixed constant defined in Lemma 4.5 and $B_{r_1}$ is the ball centered at 0 (under the original coordinate, not $x_0$) with radius $r_1$. A straightforward calculation yields

$$|\mathcal{L} T u_R| < c_0 f_\epsilon \text{ in } \bar{B}_{\delta_0}$$

for some $c_0 > 0$ independent of $R$ and $\epsilon$. Notice that $\varphi$ is strictly $k$-convex, i.e., $\lambda[D^2 \varphi] \in \Gamma_k$. Moreover, without loss of generality we may also assume $S_k(D^2 \varphi) > 4f_\epsilon$ in $B_{r_1} \setminus \bar{\Omega}$. It is easy to see that there exists some constant $\theta = \theta(\partial \Omega) > 0$ satisfying

$$\lambda[D^2(\varphi - \theta|\tilde{x}|^2)] \in \Gamma_k$$

and

$$S_k[D^2(\varphi - \theta|\tilde{x}|^2)] > 2f_\epsilon \text{ in } B_{r_1} \setminus \bar{\Omega}.$$ 

Consider the barrier $h = u_R - \varphi + \theta|\tilde{x}|^2$, then it is easy to see that

$$h = \theta|\tilde{x}|^2 \text{ on } \partial \bar{B}_{\delta_0} \cap \partial \Omega, \text{ and } h \geq \theta \delta_0^2 \text{ on } \partial \bar{B}_{\delta_0} \setminus \partial \Omega.$$ 

Moreover, by the concavity of $S^{1/k}_k$, we have

$$\mathcal{L} h = \mathcal{L} u_R - \mathcal{L}(\varphi - \theta|\tilde{x}|^2) < (1 - 2^{1/k}) k f_\epsilon := -c_1 f_\epsilon,$$

where $c_1 = k(2^{1/k} - 1)$. Now let $\omega = T u_R + Bh$, where the constant $B$ is chosen such that

$$c_1 B > c_0, \quad B \theta|\tilde{x}|^2 > |T u_R| \text{ on } \partial \Omega \cap \bar{B}_{\delta_0}, \text{ and } B \theta \delta_0^2 > |T u_R| \text{ on } \partial \bar{B}_{\delta_0} \setminus \partial \Omega.$$ 

Then we obtain

$$\mathcal{L} \omega < 0 \text{ in } \bar{B}_{\delta_0}, \text{ and } \omega \geq 0 \text{ on } \partial \bar{B}_{\delta_0}.$$ 

Therefore, we conclude $T u_R > -Bh$ in $\bar{B}_{\delta_0}$. This implies $(T u_R)_n > -Bh_n$ at $x_0$. Similarly, by considering $T u_R - Bh$ we get $(T u_R)_n < Bh_n$ at $x_0$. Hence, our lemma is proved. \(\square\)
Finally, we estimate \(|(u_R)_{n\alpha}(0)|\). Let \(\kappa = (\kappa_1, \ldots, \kappa_{n-1})\) be the principal curvatures of \(\partial \Omega\). In view of Remark 4.1 we know on \(\partial \Omega\), \(D_\nu u_R > 0\). Then by \(S_k(D^2 u_R) = f_\epsilon\) and equation (4.4), we have

\[
|D_\nu u_R|^k S_k(\kappa) + |D_\nu u_R|^{k-1} S_{k-1}(\kappa)(u_R)_\nu - \sum_{\alpha=1}^{n-1} |D_\nu u_R|^{k-2} S_{k-2}(\kappa)\alpha (u_R)^2_{\alpha\nu} = f_\epsilon.
\]

Combining the above equality with Lemma 4.2 and 4.6, we conclude

\[\text{Lemma 4.7. (C}^2\text{ bound on }\partial \Omega\text{ in double normal directions)}\]

Let \(u_R\) be the solution of (4.1), then on \(\partial \Omega\), \(u\) satisfies

\[|(u_R)_{\nu\nu}| < C,
\]

where \(\nu\) is the outward unit normal of \(\partial \Omega\) (pointing into \(B_R \setminus \hat{\Omega}\)), and the constant \(C = C(\partial \Omega, n, k)\) only depends on \(\partial \Omega\), \(n\), and \(k\).

In the rest of this subsection, we will consider the \(C^2\) boundary estimates on \(\partial B_R\).

Let \(x_0 \in \partial B_R\) be an arbitrary point on \(\partial B_R\). Without loss of generality, we may assume \(x_0 = (0, \ldots, 0, R)\). Choose a local coordinate \(\{x_1, \ldots, \tilde{x}_{n-1}\}\) around \(x_0\) such that \(x_0\) is the origin, and the \(\tilde{x}_n\) axis is the interior normal of \(\partial B_R\) (pointing into \(B_R \setminus \hat{\Omega}\)). Then the half sphere containing \(x_0\), which will be denoted by \(\partial B_{R^+}\), can be written as

\[
\bar{x}_n = \rho(\bar{x}') = R - \sqrt{R^2 - |\bar{x}'|^2}, \quad |\bar{x}'| < R,
\]

where \(\bar{x}' = (\bar{x}_1, \ldots, \tilde{x}_{n-1})\). Note that on \(\partial B_R\), we have \(u_R(\bar{x}', \rho(\bar{x}')) = b_R\), where \(b_R = -C_0(R + \epsilon)^{-\alpha_0}\). Therefore, on \(\partial B_{R^+}\) for \(\alpha, \beta < n\), we have

\[\begin{align*}
(u_R)_\alpha + (u_R)_n \rho_\alpha &= 0, \\
(u_R)_\alpha + (u_R)_\beta \rho_\beta + (u_R)_\alpha \rho_\beta + (u_R)_n \rho_{\alpha\beta} &= 0.
\end{align*}
\]

In particular, at \(x_0\) we get

\[\text{(4.5)}\]

\[\begin{align*}
(u_R)_\alpha &= -(u_R)_n \rho_\alpha = -\frac{(u_R)_n}{R} \delta_\alpha
\end{align*}
\]

Combining Lemma 4.4 with (4.5) we conclude,

\[\alpha_0 C_0 (R + \epsilon)^{-\alpha_0} R^{-1} \delta_{\alpha\beta} < |(u_R)_{\alpha\beta}(x_0)| < \alpha_0 C_1 (R + \epsilon)^{-\alpha_0} R^{-1} \delta_{\alpha\beta}
\]

for \(\alpha, \beta < n\).

\[\text{Lemma 4.8. (C}^2\text{ bound on }\partial B_R\text{ in mixed directions)}\]

Let \(u_R\) be the solution of (4.1), then on \(\partial B_R\), \(u\) satisfies

\[\text{for } \nu \text{ an arbitrary unit tangential vector of } \partial \Omega, \ \nu \text{ is the outward unit normal of } \partial B_R \text{ (pointing away from } B_R \setminus \hat{\Omega}), \text{ and the constant } C = C(\partial \Omega, n, k) \text{ only depends on } \partial \Omega, n, \text{ and } k.
\]

\[\text{Proof. We use the coordinate } \{\tilde{x}_1, \ldots, \tilde{x}_{n}\} \text{ chosen above. Let}
\]

\[
Tu_R = (u_R)_\alpha + \frac{1}{R}(\tilde{x}_\alpha (u_R)_n - \tilde{x}_n (u_R)_\alpha),
\]

\[\text{where } \tilde{x}_n = \rho(\tilde{x}') = R - \sqrt{R^2 - |\tilde{x}'|^2}, \quad |\tilde{x}'| < R,
\]

\[\text{and}
\]

\[\text{where } \tilde{x}' = (\tilde{x}_1, \ldots, \tilde{x}_{n-1})\].
then it is clear that on $\partial B_{R+}$ we have $Tu_R = 0$. Denote

$$Q_R = \left\{ x \in B_R \setminus \bar{\Omega} : \left| \tilde{x}' \right| < \frac{R}{2}, \rho(\tilde{x}') < \tilde{x}_n < \left( 1 - \frac{\sqrt{3}}{2} \right) R \right\}.$$ 

By Lemma 4.5 we can choose $A > 0$ such that $|Du_R| + Au_R$ achieves its maximum on $\partial B_R \cup \partial \Omega$. Moreover, in view of the Dirichlet boundary condition, we can see that if $A, R$ are sufficiently large, then we get

$$|Du_R|_{\partial \Omega} - A \leq |Du_R|_{\partial B_R} + Ab_R.$$ 

Thus, the maximum of $|Du_R| + Au_R$ is achieved on $\partial B_R$. By Lemma 4.4 we derive

$$|Tu_R| \leq 3 \left[ A(b_R - u_R) + \alpha_0 C_1(R + \epsilon)^{-\alpha_0 - 1} \right] \text{ in } Q_R.$$ 

Since when $R > R_0$ large,

$$\underline{u} = \phi(x, C_1) + C_1(R + \epsilon)^{-\alpha_0} + b_R$$

is a subsolution of (4.1), we have in $Q_R$

$$|Tu_R| \leq 3 \left[ A(b_R - \underline{u}) + \alpha_0 C_1(R + \epsilon)^{-\alpha_0 - 1} \right].$$

Let $h = 3A(b_R - \underline{u}) + C_2(R + \epsilon)^{-\alpha_0 - 1} \sin \left( \frac{\tilde{x}_R}{R} \right)$, then we can choose $C_2$ so large that $\omega > |Tu_R|$ on $\partial Q_R$. Notice that the choice of $C_2$ is independent of $R$ and $\epsilon$. Moreover, denote $\mathcal{L} = S_i^j \partial_{i,j}$, then it is easy to see that

$$\mathcal{L}h < -3Akf_\epsilon,$$

and

$$|\mathcal{L}Tu_R| = \left| (f_\epsilon)_\alpha + \frac{\tilde{x}_\alpha}{R} (f_\epsilon)_n - \frac{\tilde{x}}{R} (f_\epsilon)_\alpha \right| \leq cf_\epsilon \text{ in } Q_R,$$

where $c > 0$ is a constant independent of $R$ and $\epsilon$. Therefore, when $R > R_0$ and $A > 0$ large we have $|\mathcal{L}Tu_R| < |\mathcal{L}h|$. Following the argument in the proof of Lemma 4.6 we obtain $|(Tu_R)_n| < h_n$ at $x_0$. The proof of this lemma is completed.

Suppose $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ is the principal curvature vector of $\partial B_R$. On $\partial B_R$, by Lemma 4.4 we know $|D_{\nu}u| \sim R^{-\alpha_0 - 1}$. Moreover, it is clear that on $\partial B_R$ we have $\kappa = O(R^{-1}), f_\epsilon = O(R^{-(n+1)}),$ and

$$|D_{\nu}u_R|^k S_k(\kappa) + |D_{\nu}u_R|^{k-1} S_{k-1}(\kappa)(u_R)_{\nu\nu} - \sum_{\alpha=1}^{n-1} |D_{\nu}u_R|^{k-2} S_{k-2}(\kappa)(\alpha)(u_R)_{\alpha\nu}^2 = f_\epsilon.$$ 

Applying Lemma 4.8 we conclude the following lemma:

**Lemma 4.9.** (*C^2 bound on \partial B_R in double normal directions*) Let $u_R$ be the solution of (4.1), then on $\partial \Omega$, $u$ satisfies

$$|(u_R)_\nu| < CR^{-\alpha_0},$$

where $\nu$ is the outward unit normal of $\partial \Omega$ and the constant $C = C(\partial \Omega, n, k)$ only depends on $\partial \Omega$, $n$, and $k.$
4.3. **C^2 global estimates.** Differentiating \( f_\varepsilon \), we get

\[
(f_\varepsilon)_i = f_\varepsilon \frac{x_i}{r} \left( -r^{-1} - \frac{n}{r + \varepsilon} - \frac{(k - 1)\varepsilon}{r^2 + \varepsilon r} \right),
\]

and

\[
(f_\varepsilon)_{ii} = (f_\varepsilon)_i \frac{x_i}{r} \left[ -r^{-1} - n(r + \varepsilon)^{-1} - (k - 1)\varepsilon(r^2 + \varepsilon r)^{-1} \right]
+ f_\varepsilon \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right) \left[ -r^{-1} - n(r + \varepsilon)^{-1} - (k - 1)\varepsilon(r^2 + \varepsilon r)^{-1} \right]
+ f_\varepsilon \frac{x_i^2}{r^2} [r^{-2} + n(r + \varepsilon)^{-2} + (k - 1)\varepsilon(r^2 + \varepsilon r)^{-2}(2r + \varepsilon)].
\]

It is easy to see that \( f_\varepsilon \) satisfies

\[
|Df_\varepsilon| \leq A f_\varepsilon, \quad \Delta f_\varepsilon \geq -A f_\varepsilon \text{ in } B_R \setminus \bar{\Omega},
\]

where \( A > 0 \) is a constant independent of \( R \) and \( \varepsilon \).

**Lemma 4.10.** (\( C^2 \) global bound) Let \( u_R \) be the solution of (4.1), then \( u_R \) satisfies

\[
|D^2 u_R| < C(1 + \sup_{\partial B_R \cup \partial \Omega} |D^2 u_R|),
\]

where \( C = C(n, k, \partial \Omega) \) is a positive constant depending on \( \partial \Omega, n, k \).

**Proof.** The proof here is the same as the proof in [12]. For the completeness, we include it here. In this proof, we will drop the subscript \( R \) and write \( u \) instead of \( u_R \). We consider the following test function

\[
H = \Delta u + \frac{1}{2} |Du|^2.
\]

Differentiating the equation (4.1) twice, we get

\[
S_k^{ij} u_{ijm} = (f_\varepsilon)_m,
\]

and

\[
S_k^{ij} (\Delta u)_{ij} + \sum_i S_k^{pq,rs} u_{pqij} u_{rsi} = \Delta f_\varepsilon.
\]

This gives

\[
S_k^{ij} H_{ij} = -\sum_i S_k^{pq,rs} u_{pqij} u_{rsi} + \sum_m S_k^{ij} u_{mij} + \sum_m u_m (f_\varepsilon)_m + \Delta f_\varepsilon.
\]

Without loss of generality, we assume that \( H \) attains its maximum at an interior point \( x_0 \in B_R \setminus \bar{\Omega} \). Then at \( x_0 \) we have

\[
H_i = (\Delta u)_i + \sum_m u_m u_{mi} = 0,
\]

and

\[
0 \geq -\sum_i S_k^{pq,rs} u_{pqij} u_{rsi} + c_0 f_\varepsilon \Delta u + \langle Du, Df_\varepsilon \rangle + \Delta f_\varepsilon.
\]
Here we have used the Newton-Maclaurin inequality to get
\[ \sum_{m} s^{ij}_{m} u_{mi} u_{mj} \geq c_0 f_{\epsilon} \Delta u, \]
for some constant \( c_0 \) only depends on \( n, k \). Using the concavity of \( s^{1/k}_{ij} \), we have
\[ -\sum_{i} s^{pq,rs}_{k} u_{pq} u_{rsi} \geq -\frac{|Df_{\epsilon}|^2}{f_{\epsilon}}. \]
(4.14)

Combining (4.9), (4.13), (4.14), and Proposition 4.2, we get
\[ 0 \geq -\frac{|Df_{\epsilon}|^2}{f_{\epsilon}} + c_0 f_{\epsilon} \Delta u + \langle Du, Df_{\epsilon} \rangle + \Delta f_{\epsilon} \]
(4.15)
\[ \geq A^2 f_{\epsilon} + c_0 f_{\epsilon} \Delta u - AC f_{\epsilon} - Af_{\epsilon} \]
(4.16)
where \( C \) is some constant only depending on \( \partial \Omega, n, k \). Thus, we can see that if \( H \) obtains its maximum at an interior point, then \( H \) is bounded by a constant \( C = C(n, k, \partial \Omega) \). This implies the desired result.

Combing Lemmas 4.6–4.10, we obtain the uniform \( C^2 \) bound for \( u_R \) on \( \bar{B}_R \setminus \Omega \).

**Proposition 4.3.** Let \( u_R \) be the solution of (4.1), then there exists some constant \( C \) independent of \( R \) and \( \epsilon \) such that
\[ |D^2 u_R| \leq C, \text{ in } \bar{B}_R \setminus \Omega. \]

The solvability of the approximate Dirichlet problem (4.1) follows from Lemma 4.1, Proposition 4.2, and Proposition 4.3. Therefore, we proved Theorem 4.1.

Now, let \( \{u_{R_i}^{\epsilon_i}\}_{i=1}^{\infty} \) be a sequence of solutions to (4.1). We also assume as \( i \to \infty, R_i \to \infty \) and \( \epsilon_i \to 0 \). Notice that the \( C^0 - C^2 \) estimates we obtained in this section are independent of \( R \) and \( \epsilon \). By the standard convergence theorem, we conclude that there exists a subsequence of \( \{u_{R_i}^{\epsilon_i}\}_{i=1}^{\infty} \) converges to a function \( u \) which satisfies equation (1.3). This completes the proof of the first part of Theorem 1.2.

5. THE DECAY ESTIMATES

For the prototype function \( |x|^{-\alpha_0} \), its \( C^0, C^1, \) and \( C^2 \) decay orders are \( \alpha_0, \alpha_0 + 1, \alpha_0 + 2 \). In this section, we would like to prove that \( u_R \) the solution of (4.1) has the same decay order as \( |x|^{-\alpha_0} \).

In view of Lemma 4.1, the \( C^0 \) decay estimate follows directly.

**Lemma 5.1.** (**\( C^0 \) decay estimates**) For any \( x_0 \in \bar{B}_R \setminus \Omega \), denote \( |x_0| = r_0 \). Let \( u_R \) be a solution to equation (4.1). Then we have
\[ |u_R(x_0)| \leq B|x_0|^{-\alpha_0} \]
for some constant \( B > 0 \) independent of \( r_0, \epsilon, \) and \( R \).

**Lemma 5.2.** (**\( C^1 \) decay estimates**) For any \( x_0 \in \mathbb{R}^n \setminus \Omega \), denote \( |x_0| = r_0 \). Let \( u_R \) be a solution to equation (4.1) with \( R > 10r_0 \). Then we have
\[ |Du_R(x_0)| \leq B|x_0|^{-\alpha_0 - 1} \]
for some constant \( B > 0 \) independent of \( r_0, \epsilon, \) and \( R \).
\textbf{Proof.} For our convenience, in this proof we drop the subscript \(R\) and write \(u\) instead of \(u_R\). Denote 
\[ \rho_1 := \max_{x \in \partial \Omega} |x| \text{ and let } \alpha = \frac{3}{2} \rho_1. \]
Then for any \(x \in \mathbb{R}^n \setminus B_\alpha(0)\), we have 
\[ \text{dist}(x, \partial \Omega) + \rho_1 > |x|, \]
which implies 
\[ \text{dist}(x, \partial \Omega) > |x| - \rho_1 > \frac{|x|}{2}. \]

When \(x_0 \in B_\alpha(0) \setminus \Omega\), we can see that \(|x_0|^{-\alpha_0 - 1} \geq \alpha^{-\alpha_0 - 1}\), and by Proposition 4.2 we also know \(|Du|\) has an uniform upper bound in \(B_\alpha(0) \setminus \Omega\). This gives when \(x_0 \in B_\alpha(0) \setminus \Omega\), (5.2) is satisfied for some constant \(B = B(n, k, \partial \Omega) > 0\).

In the following, we will prove (5.2) for the case when \(x_0 \in \mathbb{R}^n \setminus B_\alpha(0)\). Choose any \(x_0 \in \mathbb{R}^n \setminus B_\alpha(0)\) and denote \(\tilde{r} = \frac{1}{8}|x_0|\). Then in \(B_{\tilde{r}}(x_0)\) we have \(\frac{7}{8}|x_0| \leq |x| \leq \frac{9}{8}|x_0|\). Moreover, by Lemma 4.4 we also have 
\[ C_0 \left( \frac{9}{8}|x_0| \right)^{-\alpha_0} \leq |u(x)| \leq C_1 \left( \frac{7}{8}|x_0| \right)^{-\alpha_0}. \]

Here and in the rest of this paper we will always assume \(\epsilon > 0\) is very small. We will let \(M = 4C_1 \left( \frac{7}{8}|x_0| \right)^{-\alpha_0}\). Following the argument of Chou-Wang [4], we consider the test function 
\[ G(x) = u_\xi(x) \varphi(u) \rho(x), \]
where \(\xi\) is some unit vector field, \(\varphi(\tau) = (M - \tau)^{-\frac{3}{4}}\) and \(\rho(x) = 1 - \frac{|x-x_0|^2}{\tilde{r}^2}\). We will show 
\[ |Du(x_0)| \leq C_2 \frac{M}{\tilde{r}}. \]
Suppose \(G\) attains its maximum at \(x = \hat{x}\) and \(\xi = e_1\). We rotate the coordinate so that \(u_{\alpha \beta} = u_{\alpha 0} \delta_{\alpha \beta}\) for \(\alpha, \beta \geq 2\). Then at \(\hat{x}\), we have \(G_i = 0\) and \(\{G_{ij}\} \leq 0\), \(i, j \geq 1\). That is
\[ u_{1i} = \frac{-u_1}{\varphi \rho} (u_i \varphi' \rho + \varphi \rho_i), \]
and
\[ 0 \geq \varphi \rho (f\varphi - \frac{\varphi}{\varphi}) - \frac{S_{ij}^{ij}}{\rho} \left( \varphi' \rho - \frac{\varphi'}{\varphi} \right) u_{1i} \rho + \varphi \rho (f\varphi - \frac{\varphi'}{\varphi}) \rho_i \rho_j u_{1j} \frac{S_{ij}^{ij}}{\rho} \rho_i \rho_j \rho_j \rho_i \rho_j. \]
(5.4)

By our choice of \(\varphi\) we have 
\[ \varphi'(u) = \frac{1}{2} (M - u)^{-\frac{3}{4}}, \varphi''(u) = \frac{3}{4} (M - u)^{-\frac{5}{4}}. \]
Therefore, \( \varphi'' - \frac{2\varphi^2}{\varphi} = \frac{1}{4}(M - u)^{-\frac{5}{2}} > \frac{1}{16}M^{-\frac{5}{2}} \). Moreover, by virtue of (4.7) we can see (5.4) implies

\[
0 \geq - \frac{C\varphi_1 f_\epsilon}{\rho} + \rho S_k^{11} u_1^3 \frac{1}{16} M^{-\frac{5}{2}} - \frac{2u_1 \varphi}{\rho^2} \sum_i S_k^{ii} - \left[ \varphi' u_1^2 |D\varphi| + \frac{\varphi u_1}{\rho} |D\varphi|^2 \right] C(n) \sum_i S_k^{ii}.
\]

(5.5)

This yields

\[
0 \geq \rho S_k^{11} u_1^3 - \frac{2u_1 M^{-\frac{5}{2}}}{\rho^2} 16M^{\frac{5}{2}} \sum_i S_k^{ii} - \left( M^{-\frac{5}{2}} u_1^2 + \frac{4u_1 M^{-\frac{5}{2}}}{\rho^2} \right) C(n) 16M^{\frac{5}{2}} \sum_i S_k^{ii}
\]

(5.6)

\[
- \frac{CM^{-\frac{5}{2}} f_\epsilon}{\bar{r}^2} 16M^{\frac{5}{2}}.
\]

where \( C \) is independent of \( M \) and \( \bar{r} \).

Now, we prove by contradiction and assume \( |Du(x_0)| > C_2 \frac{M}{\bar{r}} \) for some undetermined constant \( C_2 > 0 \). Then since \( G(\bar{x}) \geq G(x_0) \), we have

\[
C_0 \frac{7}{9} \cdot C_1 \frac{M}{\bar{r}}.
\]

Suppose \( \lambda' = (u_2, \ldots, u_n) \). Since \( S_k = S_{k-1}(\lambda') u_1 + S_k(\lambda) - \sum_{i \geq 2} S_{k-2}(\lambda') u_1 u_i \), we get \( S_k^{11} = S_{k-1}(\lambda') \) and \( S_k^{ii} = S_{k-2}(\lambda') u_1 u_i + S_{k-1}(\lambda') u_i \), \( i \geq 2 \). It is clear that from (5.3) we have, \( u_1 \leq -\frac{\varphi'}{2\varphi} u_1^2 < 0 \). Therefore, \( \sum_{i \geq 2} S_k^{ii} < (n - k) S_k^{11} \), which yields \( S_k^{11} > \frac{1}{n-k+1} \sum_i S_k^{ii} \). On the other hand, since \( S_k^{11} \) is concave, we know \( \frac{1}{n-k+1} \sum_i S_k^{ii} > \left( C_n \right)^{\frac{1}{2}} \). Thus, \( \sum_i S_k^{ii} > C(n, k) f_\epsilon^{\frac{1}{2} - \frac{1}{k}} \).

Substituting the above inequalities into (5.6), we obtain

\[
0 \geq \frac{1}{n-k+1} \rho u_1^3 - \frac{32M^2}{\bar{r}^2} u_1 - 16C(n) \frac{M}{\bar{r}} u_1^2
\]

\[
- 64C(n) \frac{M^2 u_1}{\bar{r}^2} - \frac{CM^2}{\bar{r}} f_\epsilon^{\frac{1}{2}}.
\]

(5.7)

Note that \( f_\epsilon \sim \bar{r}^{-(n+1)} \), it’s easy to see that \( f_\epsilon^{\frac{1}{2}} < \frac{M}{\bar{r}} \). Therefore, when \( u_1 \rho(\bar{x}) > \frac{C_1}{\epsilon_0} \left( \frac{7}{9} \right)^{\alpha_0} \cdot C_2 \frac{M}{\bar{r}} \) and \( C_2 > C(n, k, \partial \Omega) \) is sufficiently large, the right hand side of (5.7) would be positive. This leads to a contradiction.

**Lemma 5.3.** \((C^2 \text{ decay estimates})\) For any \( x_0 \in \mathbb{R}^n \setminus \Omega \), denote \( |x_0| = r_0 \). Let \( u_R \) be a solution to equation (4.1) with \( R \gg r_0^{\frac{1}{1-\frac{2\alpha}{\alpha_0}}} \). Then we have

\[
|D^2 u_R(x_0)| \leq B|x_0|^{-\alpha_0 - 2}
\]

(5.8)

for some constant \( B > 0 \) independent of \( r_0, \epsilon, \) and \( R \).
Let \( \phi \) be a function in \( \Omega \) such that \( \phi \geq 0 \) and \( \phi = 0 \) on \( \partial \Omega \). We consider the domain \( \Omega_u := \{ x \in B_R : |u| < |u_0| \} \).

**Step 1.** In this step, we want to show when \( x \in \Omega_u \), \( |Du(x)| < c_1 r_0^{-\alpha_0} \). By Lemma 4.1 and Lemma 5.2 we know on \( \partial \Omega_u \setminus \partial B_R \) we have \( |Du(x)| < c_2 r_0^{-\alpha_0} \). By Lemma 4.4 we also know on \( \partial B_R \) \( |Du(x)| < c_3 r_0^{-\alpha_0} \). Applying Lemma 4.5 in the domain \( \Omega_u \) and choose \( A = \frac{\alpha}{\alpha_0} \) we obtain
\[
|Du| < c_5 r_0^{-\alpha_0} \text{ in } \Omega_u.
\]
Here \( c_1, c_2, c_3, c_4, c_5 \) are uniform constants independent of \( r_0, \epsilon, \) and \( R \).

**Step 2.** In this step we will prove (5.8). Denote \( V := |Du|^2 \) and \( M := 4 \max_{x \in \Omega_u} V \sim r_0^{-2(\alpha_0+1)} \).

Let \( \varphi(V) = (M - V)^{-\beta} \), where \( \beta = \frac{\alpha_0+2}{2\alpha_0+2} \). By our choice of \( \beta \), we can see that \( \varphi(V) \sim r_0^{\alpha_0+2} \).

Consider \( G = \rho^4 u_{\xi \xi} \varphi(V) \), where \( \rho = 1 - \frac{u}{2u_0} \) and \( \xi \) is some unit vector field. Applying inequality (4.6), Lemma 4.8, and Lemma 4.7 we know when \( x \in \Omega_u \setminus \partial B_R \), \( G \leq CR^{-\alpha_0+\alpha_0+2} \leq C \) on \( \partial B_R \). Moreover, on \( \partial \Omega_u \setminus \partial B_R \), we have \( G = 0 \). Our goal is to show \( \max_{\Omega_u} G \leq C \), for some \( C \) independent of \( r_0, R, \) and \( \epsilon \).

Following the argument of Chou-Wang [4], we assume \( G_{\text{max}} \) is achieved at an interior point \( \hat{x} \). We may also rotate the coordinate such that at \( \hat{x} \), we have \( u_{ij}(\hat{x}) = u_{ii}(\hat{x})\delta_{ij} = \lambda_i \delta_{ij} \) and \( u_{11} \geq \cdots \geq u_{nn} \). Then at this point, we have
\[
0 = G_i = 4 \frac{\rho_i}{\rho} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}},
\]
and
\[
0 \geq 4 F^{ii} \left( \frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2} \right) + F^{ii} \left( \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} \right) + F^{ii} \left( \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \right),
\]
where \( F = \frac{S_k}{r_0} \) and \( F^{ii} = \frac{\partial F}{\partial u_{ii}} \). We will analyze (5.10) in two cases.

Case 1. \( u_{kk} \geq \theta_0 u_{11} \) at \( \hat{x} \), where \( \theta_0 > 0 \) is some fixed small constant to be determined. By (5.9) we get for any \( \gamma > 0 \)
\[
\left( \frac{u_{11i}}{u_{11}} \right)^2 \leq (1 + \gamma) \frac{\varphi_i^2}{\varphi^2} + 16 \left( 1 + \frac{\rho_i^2}{\rho^2} \right).
\]
Plugging it into (5.10), we obtain
\[
0 \geq 4 F^{ii} \left( \frac{\rho_{ii}}{\rho} - \left[ 1 + 4 \left( 1 + \frac{1}{\gamma} \right) \frac{\rho_i^2}{\rho^2} \right] \right) + F^{ii} \left[ \frac{\varphi_{ii}}{\varphi} - (2 + \gamma) \frac{\varphi_i^2}{\varphi^2} \right] + F^{ii} \frac{u_{11ii}}{u_{11}},
\]
Note that
\[
F^{ii} \left[ \frac{\varphi_{ii}}{\varphi} - (2 + \gamma) \frac{\varphi_i^2}{\varphi^2} \right] = \frac{\varphi''}{\varphi} - (2 + \gamma) \frac{\varphi'^2}{\varphi^2} F^{ii} V_i V_i + \frac{\varphi'}{\varphi} F^{ii} V_i.
\]
By a straightforward calculation we can see that, in order for \( \frac{\varphi''}{\varphi} - (2 + \gamma) \frac{\varphi'^2}{\varphi^2} \) to be nonegative, we need to choose \( \gamma \leq \frac{\alpha_0}{\alpha_0+2} \). In this case we have
\[
F^{ii} \left[ \frac{\varphi_{ii}}{\varphi} - (2 + \gamma) \frac{\varphi_i^2}{\varphi^2} \right] \geq \frac{\varphi'}{\varphi} F^{ii} (2u_{ii}^2 + 2u_{kk} u_{kk}).
\]
Denote $\hat{f} = f^{\frac{1}{k}}_e$, then when $\gamma \leq \frac{\alpha_0}{\alpha_0 + 2}$, (5.11) becomes

\begin{equation}
0 \geq \frac{4}{2|u_0|\rho} \frac{4[4(1 + \gamma^{-1}) + 1]|Du|^2}{4\rho^2 u_0^2} \sum_i F^{ii} + 2\frac{\varphi'}{\varphi} F^{ii} u_{ii}^2 + 2\frac{\varphi'}{\varphi} Du \cdot D\hat{f} + \frac{\hat{f}_{11}}{u_{11}}
\end{equation}

By formula (3.2) in [4], we get

\begin{equation}
\sum_{i=1}^n F^{ii} u_{ii}^2 \geq F^{kk} u_{kk}^2 \geq c_0 u_{11}^2 \sum_i F^{ii}
\end{equation}

for some constant $c_0 = c_0(n, k, \theta_0)$. Recall (4.7) and (4.8) we obtain

\begin{equation}
\hat{f}_i = \frac{1}{k} f^{\frac{1}{k}-1}_e (f_e)_i \sim \hat{f} \frac{1}{r},
\end{equation}

and

\begin{equation}
\hat{f}_{11} = \frac{1}{k} f^{\frac{1}{k}-1}_e (f_e)_{11} + \frac{1}{k} \left( \frac{1}{k} - 1 \right) f^{\frac{1}{k}-1}_e (f_e)^2 \sim \hat{f} \frac{r}{r^2}.
\end{equation}

Therefore, (5.12) implies

\begin{equation}
0 \geq -\frac{C|Du|^2}{4u_0^2 \rho^2} \sum_i F^{ii} + 2\frac{\varphi'}{\varphi} c_0 u_{11}^2 \sum_i F^{ii} - C\frac{\varphi'}{\varphi} |Du|\hat{f}_r - \frac{C\hat{f}}{r^2 u_{11}}.
\end{equation}

Since $\sum F^{ii} \geq (C_n^k)^\frac{2}{k}$, we get

\begin{equation}
2\frac{\varphi'}{\varphi} c_0 u_{11}^2 < \frac{C|Du|^2}{4u_0^2 \rho^2} + \frac{C\varphi'}{\varphi} |Du|\hat{f}_r + \frac{C\hat{f}}{r^2 u_{11}}.
\end{equation}

Denote $u_{11}\rho(\hat{x}) := X$ then (5.16) implies

\begin{equation}
r_0^{2(\alpha_0+1)} X^2 \leq C r_0^{-2} + C r_0^{-2 - \frac{1}{k}} + C r_0^{-2 - \frac{1}{k}} X.
\end{equation}

It’s easy to see that if $X > Br_0^{-2(\alpha_0+2)}$ for some $B > 0$ large, then inequality (5.17) would not hold, which leads to a contradiction.

Case 2. $u_{kk} < \theta_0 u_{11}$ at $\hat{x}$. By (5.9) we have

\begin{equation}
\frac{u_{11}^2}{u_1^2} \leq (1 + \gamma) \frac{\varphi_i^2}{\varphi^2} + 16 (1 + \gamma^{-1}) \frac{\rho_i^2}{\rho^2},
\end{equation}

and for $i \geq 2$,

\begin{equation}
4\frac{\rho_i^2}{\rho^2} = \frac{1}{4} \left( \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}} \right)^2 \leq \frac{1}{2} \left( \frac{\varphi_i}{\varphi} \right)^2 + \frac{1}{2} \frac{u_{11i}^2}{u_{11}^2}.
\end{equation}
Here, $\gamma$ takes the same value as in Case 1. Equation (5.10) can be written as

$$0 \geq \left\{ \sum_{i=1}^{n} \left[ 4 F_{ii} \frac{D_{ii}}{\rho} + F_{ii} \left( \frac{\phi_{ii}}{\phi} - (2 + \gamma) \frac{\phi_{ii}^{2}}{\phi^{2}} \right) \right] - 4[1 + 4(1 + \gamma^{-1})] F_{11} \rho \right\}$$

$$+ \left\{ \sum_{i=1}^{n} \frac{F_{ii} u_{11}}{u_{11}} - \frac{3}{2} \sum_{i=2}^{n} \frac{F_{ii} u_{11}}{u_{11}} \right\}$$

$$:= I_{1} + I_{2}$$

(5.18)

A straightforward calculation implies

$$I_{1} \geq \frac{N_{f}}{2|u_{0}|\rho} + 2 \frac{\phi_{f}'}{\phi_{f}} F_{ii} u_{11}^{2} + 2 \frac{\phi_{f}'}{\phi_{f}} D_{f} \cdot D_{f} - C \frac{F_{11}^{11} |D_{f}|^{2}}{4u_{0}^{2}\rho^{2}}$$

(5.19)

It’s well known that $F$ is concave. Therefore,

$$F_{pq,rs} u_{pq} u_{rs} \leq 2 \sum_{p>1} \frac{F_{pp} - F_{11}}{\lambda_{p} - \lambda_{1}} u_{p1}^{2}$$

$$\frac{2}{k} f_{f}^{k-1} - \frac{1}{k} \sum_{p>1} \frac{S_{k-1}(\lambda|p) - S_{k-1}(\lambda|1)}{\lambda_{p} - \lambda_{1}} u_{11}^{2} p$$

$$= - \frac{2}{k} f_{f}^{k-1} \sum_{p>1} S_{k-2}(\lambda|1) u_{11}^{2} p.$$ 

(5.20)

This gives

$$u_{11} I_{2} \geq \hat{f}_{11} + \frac{1}{k} f_{f}^{k-1} \left[ 2 \sum_{p>1} S_{k-2}(\lambda|1) - \frac{3 S_{k-1}(\lambda|p)}{2 \lambda_{1}} \right] u_{11}^{2}.$$ 

(5.21)

Note that we assumed $u_{kk} < \theta_{0} u_{11}$. Since $\sum_{j=k}^{n} u_{jj} \geq 0$ we can derive $|u_{jj}| \leq C(n, k) \theta_{0} u_{11}, j \geq k$. Applying Lemma 3.1 of [4], we know when $\theta_{0}$ is sufficiently small, we have $2 \lambda_{1} S_{k-2}(\lambda|1) > \frac{1}{2} S_{k-1}(\lambda|1)$. Substituting (5.19) and (5.21) into (5.18), we obtain

$$0 \geq \frac{2 \hat{f}}{|u_{0}|\rho} + 2 \frac{\phi_{f}'}{\phi_{f}} F_{ii} u_{11}^{2} + 2 \frac{\phi_{f}'}{\phi_{f}} D_{f} \cdot D_{f} - C \frac{F_{11}^{11} |D_{f}|^{2}}{4u_{0}^{2}\rho^{2}} + \hat{f}_{11}.$$ 

(5.22)

Combining with (5.13) and (5.14), we get at $\hat{x}$

$$0 \geq \frac{2 \hat{f}}{|u_{0}|\rho} + 2 \frac{\phi_{f}'}{\phi_{f}} F_{ii} u_{11}^{2} - C \frac{\phi_{f}'}{\phi_{f}} r_{0}^{-\alpha_{0}-2} \hat{f} - \frac{C F_{11}^{11} r_{0}^{-2}}{\rho^{2}}.$$ 

(5.23)

Since $\frac{\phi_{f}'}{\phi_{f}} = \beta(M - V)^{-1} \geq \beta M^{-1} \sim r_{0}^{2(\alpha_{0}+1)}$ and when $G(\hat{x})$ is very large, we have $u_{11} \rho(\hat{x}) > C r_{0}^{-\alpha_{0}-2}$, which implies $\frac{\phi_{f}'}{\phi_{f}} F_{ii} u_{11}^{2} > \frac{C F_{11}^{11} r_{0}^{-2}}{\rho^{2}}$. Therefore, in this case, (5.22) becomes

$$0 \geq \frac{b \hat{f}}{2|u_{0}|\rho} + 2 \frac{\phi_{f}'}{\phi_{f}} F_{ii} u_{11}^{2} - C \frac{\phi_{f}'}{\phi_{f}} r_{0}^{-\alpha_{0}-2} \hat{f} - \frac{C \hat{f}}{u_{11} r_{0}^{2}}.$$ 

(5.23)
It is easy to see that \( \sum P_{ii} u_{ii}^2 > c_0 \hat{f} \lambda_1 \). Same as before assume \( u_{11} \rho(\hat{x}) := X > Br_0^{-\alpha_0 - 2} \) for some sufficiently large constant \( B > 0 \), then by (5.23), we get
\[
\frac{2^{(\alpha_0 + 1)}}{r_0} Br_0^{-\alpha_0 - 2} \leq (\frac{C r_0^{(\alpha_0 + 1) - \alpha_0 - 2}}{X})^{\rho^2}.
\]
When \( B > 0 \) is very large, this leads to a contradiction. Therefore, we have proved Lemma 5.3.

\[\square\]

5.1. **Convergence.** In this subsection, we will show that there exists a subsequence of \( \{u_R\} \) converging to the desired solution \( u \) of (1.3).

In the following, for any fixed \( R > R_0 \) very large, let \( \epsilon_R = c_0 R^{-k(\alpha_0 + 3)} \), where \( c_0 = c_0(\partial \Omega, n, k) > 0 \) is a fixed small constant. Let \( u_R^\epsilon \) be the solution of (4.1) with \( \epsilon = \epsilon_R \), then we have the following lemma.

**Lemma 5.4.** For any \( R > R_0 \) very large, let
\[
\Psi_R = |Du_R^\epsilon| - b_0(-u_R^\epsilon)^A - \theta_0 \frac{|x|^2}{R^{\alpha_0 + 3}},
\]
where \( A = \frac{\alpha_0 + 1}{\alpha_0} \), and \( b_0 = b_0(\partial \Omega, n, k) \), \( \theta_0 = \theta_0(\partial \Omega, n, k) > 0 \) small such that \( \Psi_R > 0 \) on \( \partial(B_R \setminus \hat{\Omega}) \). Then
\[
\Psi_R > 0 \text{ in } B_R \setminus \hat{\Omega}.
\]

**Proof.** Before proving this lemma, we want to point out that by Lemma 4.3 and 4.4, we know there always exist \( b_0 = b_0(\partial \Omega, n, k) \), \( \theta_0 = \theta_0(\partial \Omega, n, k) > 0 \) small such that \( \Psi_R > 0 \) on \( \partial(B_R \setminus \hat{\Omega}) \).

Now, we will prove this lemma by a contradiction argument. We assume \( \Psi \) achieves its non-positive minimum point at \( x_0 \). Then at this point we may rotate the coordinate such that \( |Du_R^\epsilon| = (u_R^\epsilon)_1 \). A straightforward calculation yields at \( x_0 \)
\[
S_{k}^{ij}(\Psi_R)_{ij} \leq (f_{\epsilon R})_1 + Ab_0|u_R^\epsilon|^A - 2\frac{\theta_0}{R^{\alpha_0 + 3}} \sum S_i^k.
\]
Since \( S_{k}^{1/k} \) is concave, we get \( \sum S_i^k > c_1 f_{\epsilon R}^{1-k} \) for some \( c_1 = c_1(n, k) > 0 \) only depending on \( n, k \). By a proper choice of \( c_0 \) that is independent of \( x_0 \) and \( R \), we can see that \( S_{k}^{ij}(\Psi_R)_{ij} < 0 \) at \( x_0 \). This leads to a contradiction.

Combining Lemma 5.1, 5.2, 5.3, and 5.4 with the standard convergence theorem we conclude

**Theorem 5.1.** There exists a \( k \)-admissible solution \( u \) of equation (1.3) satisfying
\[
|u(x)| < B|x|^{-\alpha_0}, \quad |Du(x)| < B|x|^{-\alpha_0 - 1}, \quad \text{and} \quad |D^2u(x)| < B|x|^{-\alpha_0 - 2},
\]
for any \( x \in \mathbb{R}^n \setminus \Omega \). Here \( B > 0 \) is a constant depending on \( n, k, \) and \( \Omega \). Moreover, we also have
\[
|Du| - b_0|u|^{\frac{\alpha_0 + 1}{\alpha_0}} \geq 0 \text{ in } \mathbb{R}^n \setminus \Omega
\]
for some \( b_0 = b_0(\partial \Omega) > 0 \).
6. ASYMPTOTIC BEHAVIOR AT \( \infty \)

In this section, we will consider the asymptotic behavior of the solution \( u \) of equation (1.3). By using the maximum principle, we obtain the lemma below.

**Lemma 6.1.** Let \( G \) be a connected, bounded, open subset in \( \mathbb{R}^n \setminus \{0\} \) and \( \mu = -|x|^2 - \frac{1}{|x|} \). Assume \( u \) is a \( k \)-admissible solution satisfies \( S_k(D^2 u) = 0 \). If \( \frac{u}{\mu} \) achieves its maximum (minimum) in \( G \), then \( \frac{u}{\mu} \) is a constant.

**Lemma 6.2.** Let \( u \) be a \( k \)-admissible solution of (1.3). Then there exists some constant \( C_0 \leq \gamma \leq C_1 \) such that \( u \to \gamma \mu \) in \( C^2 \) topology as \( |x| \to \infty \).

**Proof.** This proof follows the idea of [15]. Without loss of generality, in the following, we assume \( \partial \Omega \subset B_1(0) \). By Lemma 4.1, we can define \( \gamma^+ \) and \( \gamma^- \) by

\[
\gamma^+ = \limsup_{x \to \infty} \frac{u(x)}{\mu(x)}, \quad \gamma^- = \liminf_{x \to \infty} \frac{u(x)}{\mu(x)}.
\]

It is clear that \( C_0 \leq \gamma^+, \gamma^- \leq C_1 \). We can also assume \( \max_{x \in \mathbb{R}^{n-1}} |u(x)| = 1 \). Otherwise, we use \( \frac{u(x)}{\max_{x \in \mathbb{R}^{n-1}} u(x)} \) to replace \( u(x) \). If \( \gamma^+ = \gamma^- = 1 \) we would have \( \lim_{x \to \infty} \frac{u(x)}{\mu(x)} = 1 \). Thus we will assume \( \gamma^+ > 1 \) (or similarly \( \gamma^- < 1 \)). Let \( \tilde{\gamma}(r) = \sup_{1 \leq |x| \leq r} \frac{u(x)}{\mu(x)} \), then it’s easy to see that \( \tilde{\gamma}(r) \geq 1 \) is nondecreasing. Therefore we have \( \lim_{r \to \infty} \tilde{\gamma}(r) = \gamma^+ \).

Define \( u_r \) on \( \Lambda_r = \{ \xi \mid |\xi| > \frac{1}{r} \} \) by \( u_r(\xi) = -\frac{u(r\xi)}{\mu(r\xi)} \). By Theorem 5.1, we have

\[
|u_r(\xi)| \leq C|\xi|^{-\alpha_0},
\]

\[
|D u_r(\xi)| \leq C|\xi|^{-\alpha_0 - 1},
\]

and

\[
|D^2 u_r(\xi)| \leq C|\xi|^{-\alpha_0 - 2}.
\]

This implies there exists a function \( v \) satisfying \( S_k(D^2 v) = 0 \) such that \( u_r(\xi) \to v(\xi) \) in \( C^2 \) topology on any compact subset of \( \mathbb{R}^n \setminus \{0\} \). Moreover, we have

\[
\frac{u_r(\xi)}{\mu(\xi)} = -\frac{u(r\xi)}{\mu(r\xi)} \cdot \frac{\mu(r\xi)}{\mu(\xi)} = \frac{u(r\xi)}{\mu(r\xi)}.
\]

Suppose \( \tilde{\gamma}(r) = \frac{u(r\xi)}{\mu(r\xi)} \), by the monotonicity of \( \tilde{\gamma} \), we know \( |x_r| = r \). Let \( \xi_r = \frac{x_r}{r} \), then we have \( \frac{u_r(\xi)}{\mu(\xi)} = \tilde{\gamma}(r) \). Choose a subsequence of \( \{ \xi_r \} \), denote by \( \{ \xi_{r_n} \} \), and assume \( \{ \xi_{r_n} \} \to \xi_0 \). Then we have

\[
\frac{v(\xi_0)}{\mu(\xi_0)} = \gamma^+ \text{ and } \frac{u(\xi)}{\mu(\xi)} \leq \gamma^+ \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\}.
\]

By Lemma 6.1, we know \( v(\xi) = \gamma^+ \mu(\xi) = \lim_{r \to \infty} u_r(\xi) \) uniformly on every compact subset of \( \mathbb{R}^n \setminus \{0\} \) in \( C^2 \) topology. This gives \( \lim_{|x| \to \infty} \frac{u(x)}{\mu(x)} = \gamma \) and Lemma 6.2 is proved. \( \square \)
7. THE MONOTONICITY QUANTITY

In this section, we will derive a monotonicity quantity satisfied by the solution $u$ of (1.3). To start, let us consider the radial function

$$\mu = -|x|^{2-\frac{2}{k}}.$$ 

One can verify that for $n > 2k$, $\lambda[D^2 \mu]$ belongs to $\bar{\Gamma}^+_k$ and $\mu$ satisfies the Hessian equation

$$S_k(D^2u) = 0.$$ 

Here $\lambda[D^2 \mu] = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of the Hessian of $\mu$. In polar coordinates, a straightforward calculation gives

$$D_r \mu = (\frac{n}{k} - 2)|x|^{1-\frac{2}{k}}, \quad D_\theta \mu = 0,$n \quad D^2 \mu(\partial_r, \partial_r) = -\left(\frac{n}{k} - 2\right)\left(\frac{n}{k} - 1\right)|x|^{-\frac{2}{k}}, \quad D^2 \mu(\partial_r, \partial_\theta) = 0,$$

$$\Delta \mu = \left(\frac{n}{k} - 2\right)(n - \frac{n}{k})|x|^{-\frac{2}{k}}.$$ 

One sees on a level set of $\mu$,

(7.1) $$\frac{1}{|D\mu|} S_{ij}^k \mu_i \mu_j = S^r_k \mu_r = \left(\frac{n}{k} - 1\right)\left(\frac{n}{k} - 2\right)^k |x|^{1-n}.$$ 

(7.2) $$\frac{|D\mu|}{(-\mu)^{\frac{1}{2-\frac{2}{k}}}} = \frac{n}{k} - 2.$$ 

Thus the quantity $\Phi : (-\infty, -1] \to \mathbb{R}$,

(7.3) $$\Phi(\tau) = \int_{\{u = \frac{1}{\tau}\}} \frac{1}{|Du|} S_{ij}^k \mu_i \mu_j \left(\frac{|Du|}{(-u)^{\frac{2n-n}{2k-2}}}\right)^{\beta} dx$$

is a constant independent of $\tau$ for the radial solution $\mu$ with any constant $\beta \in \mathbb{R}$.

We show in the following that for the $k$-admissible solution $u$ of (1.3), when $\beta \geq \frac{n-2k}{n-k}$, $\Phi$ is monotone.

**Proposition 7.1.** Let $n > 2k$ and $\beta \geq \frac{n-2k}{n-k}$. Let $u$ be a $k$-admissible solution of

(7.4) $$\begin{cases}
    S_k(D^2u) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega} \\
    u = -1 \quad \text{on} \quad \partial \Omega \\
    u(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases}$$

Then $\Phi(\tau)$ defined by (7.3) is monotone non-decreasing for regular value $\frac{1}{\tau}$, i.e., for $\tau$ such that $\{u = \frac{1}{\tau}\}$ is a regular level set.
Proof. In this proof, we denote \( l = \frac{n - k}{n - 2k} > 0 \) and we also note that \( \partial_j(S_k^{ij} u_i) = 0 \). Using the Divergence theorem and the co-area formula we have

\[
\Phi(\tau) - \Phi(-\infty) = \int_{\{u = 1/\tau\}} S_k^{ij} u_i u_j \left( \frac{|Du|}{(u)^{\frac{\beta}{2k - n}}} \right)^\beta dx - \int_{\{u = 0\}} \frac{1}{|Du|} S_k^{ij} u_i u_j \left( \frac{|Du|}{(u)^{\frac{\beta}{2k - n}}} \right)^\beta dx
\]

\[
= - \int_{\{u > 1/\tau\}} \partial_j \left[ (-u)^{-l\beta} |Du|^\beta S_k^{ij} u_i \right]
\]

\[
= - \int_{\{u > 1/\tau\}} \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] S_k^{ij} u_i
\]

\[
= - \int_{1/\tau}^0 \int_{\{u = s\}} \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] S_k^{ij} u_i \frac{1}{|Du|} dx
\]

Taking the derivative of \( \Phi \), we get

\[
\Phi'(\tau) = - \int_{\{u = 1/\tau\}} u^2 \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] S_k^{ij} u_i \frac{1}{|Du|}
\]

\[
= \int_{\{u > 1/\tau\}} \partial_i \left\{ u^2 \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] S_k^{ij} \right\}
\]

\[
= \int_{\{u > 1/\tau\}} u^2 S_k^{ij} \partial_i \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] + 2u \partial_j \left[ (-u)^{-l\beta} |Du|^\beta \right] S_k^{ij} u_i,
\]

(7.5)

where we have used \( \partial_j S_k^{ij} = 0 \). Since \( S_k(D^2 u) = 0 \), we have

\[
S_k^{ij} u_{ijk} = 0,
\]

which yields

\[
S_k^{ij} (|Du|^\beta)_{ij} = \beta |Du|^\beta - 2 S_k^{ij} u_{ik} u_{jk} + \beta(\beta - 2) |Du|^\beta - 2 S_k^{ij} \partial_i |Du| \partial_j |Du|.
\]
Therefore, we deduce
\begin{align*}
&u^2 S_k^{ij} \partial_i \partial_j \left[ (-u)^{-l \beta} |Du|^{\beta} \right] + 2u \partial_j \left[ (-u)^{-l \beta} |Du|^{\beta} \right] S_k^{ij} u_i \\
&= (-\beta)(-\beta-1)(-u)^{-l \beta} S_k^{ij} u_i u_j |Du|^{\beta} + 2(\beta)(\beta) (-u)^{-l \beta+1} S_k^{ij} u_i |Du|^{\beta-1} \partial_j |Du| \\
&+ \beta(-u)^{-l \beta+2} |Du|^{\beta-2} \left[ S_k^{ij} u_k u_j + (\beta - 2) S_k^{ij} \partial_i |Du| \partial_j |Du| \right] \\
&- 2\beta(-u)^{-l \beta+1} |Du|^{\beta-1} S_k^{ij} u_i \partial_j |Du| - 2\beta(-u)^{-l \beta} S_k^{ij} u_i u_j |Du|^{\beta} \\
&= \beta(-u)^{-l \beta+2} |Du|^{\beta-2} \left[ S_k^{ij} u_k u_j + \left( \frac{1}{l} - 2 \right) S_k^{ij} \partial_i |Du| \partial_j |Du| \right] \\
&+ 2\beta(-l \beta-1)(-u)^{-l \beta+1} |Du|^{\beta-1} S_k^{ij} u_i \partial_j |Du| + l\beta(-1)(-u)^{-l \beta} |Du|^{\beta} S_k^{ij} u_i u_j \\
&= \beta(-u)^{-l \beta+2} |Du|^{\beta-2} \left[ S_k^{ij} u_k u_j + \left( \frac{1}{l} - 2 \right) S_k^{ij} \partial_i |Du| \partial_j |Du| \right] \\
&+ \frac{1}{l^2} \beta(-l \beta-1)(-u)^{-l \beta+2} |Du|^{\beta-2} S_k^{ij} \left( \partial_i |Du| - \frac{|Du|}{u_i} \right) \left( \partial_j |Du| - \frac{|Du|}{u_j} \right).
\end{align*}

By our choice of \( l \) we can see that \( \frac{1}{l} - 2 = -\frac{n-n}{n-l} \) and \( \beta \geq \frac{1}{l} \), applying the Kato’s inequality (see Proposition \( \ref{prop:katoineq} \)), we obtain that
\[ u^2 S_k^{ij} \partial_i \partial_j \left[ (-u)^{-l \beta} |Du|^{\beta} \right] + 2u \partial_j \left[ (-u)^{-l \beta} |Du|^{\beta} \right] S_k^{ij} u_i \geq 0. \]

In view of \( \ref{prop:katoineq} \), we have shown that \( \Phi'(\tau) \geq 0 \). This completes the proof of Proposition \( \ref{prop:katoineq} \). \( \square \)

In view of Proposition \( \ref{prop:katoineq} \) we obtain

**Corollary 7.1.** Let \( n > 2k \) and \( \beta \geq \frac{n-2k}{n-k} \). Let \( u \) be a k-admissible solution of \( \ref{eq:kadmissible} \) constructed in Subsection \( \ref{subsec:construction} \). Then \( \Phi(\tau) \) defined by \( \ref{eq:phi} \) satisfies
\[ \Phi(-1) \geq \Phi(-\infty). \]

Below, we prove the Kato’s inequality.

**Proposition 7.2 (Kato’s inequality).** Let \( u \) be a k-admissible function satisfying \( S_k(D^2 u) = 0 \). Then at any regular point, i.e., \( |Du| \neq 0 \), we have
\begin{equation}
\label{eq:katoineq}
\sum_m S_k^{ij} u_{im} u_{mj} - \frac{n}{n-k} S_k^{ij} D_i |Du| D_j |Du| \geq 0.
\end{equation}

**Proof.** In this proof, the index range are \( 1 \leq \alpha, \beta, \ldots \leq n-1 \) and \( 1 \leq i,j,m, \ldots \leq n \). At any point \( p \) satisfying \( Du(p) \neq 0 \), choose \( \{ e_\alpha \} \) such that
\[ u_{\alpha \beta}(p) = \lambda_{\alpha} \delta_{\alpha \beta}, \quad e_n = \frac{Du}{|Du|}(p), \]
and denote \( \lambda' = (\lambda_1, \ldots, \lambda_{n-1}) \). Note also that in the chosen coordinate we have \( D_i |Du| = u_{in} \). Therefore at this point we have
\begin{equation}
0 = S_k(D^2 u) = S_{k-1}(\lambda') u_{nn} + S_k(\lambda') - \sum_\alpha S_{k-2}(\lambda'|\alpha) u_{in}^2.
\end{equation}
Similarly, we can derive
\[ S_{k+1}[u] = S_k(\lambda')u_{nn} + S_{k+1}(\lambda') - \sum_{\alpha} S_{k-1}(\lambda'|\alpha)u_{\alpha n}^2. \]  
(7.8)

A direct calculation shows that
\[ S_{k}^{\alpha\alpha} = S_{k-2}(\lambda'|\alpha)u_{nn} + S_{k-1}(\lambda'|\alpha) - \sum_{\beta \neq \alpha} S_{k-3}(\lambda'|\alpha\beta)u_{\beta n}^2, \]
and
\[ S_{k}^{\alpha n} = -S_{k-2}(\lambda'|\alpha)u_{\alpha n}. \]
(7.9)
(7.10)

It is easy to see that for \( \alpha \neq \beta, S_{k}^{\alpha\beta} = 0. \) Notice that
\[ kS_k(\lambda') = \sum_{\alpha} S_{k-1}(\lambda'|\alpha)\lambda_{\alpha}, \]
(7.11)
\[ kS_k(\lambda'|\beta) = \sum_{\alpha} S_{k-1}(\lambda'|\alpha\beta)\lambda_{\alpha}. \]
(7.12)

Next, computing \( S_{k}^{ij} u_{in}u_{jn} \) we obtain
\[ S_{k}^{ij} u_{in}u_{jn} = S_{k}^{\alpha\alpha} u_{\alpha n}^2 + 2S_{k}^{\alpha n} u_{\alpha n}u_{nn} + S_{k}^{nn} u_{nn}^2 \]
\[ = \sum_{\alpha} (S_{k-2}(\lambda'|\alpha)u_{nn} + S_{k-1}(\lambda'|\alpha) - \sum_{\beta \neq \alpha} S_{k-3}(\lambda'|\alpha\beta)u_{\beta n}^2)u_{\alpha n}^2 \]
\[ - 2\sum_{\alpha} S_{k-2}(\lambda'|\alpha)u_{\alpha n}u_{nn} + S_{k-1}(\lambda')u_{nn}^2 \]
(7.13)
\[ \leq -S_k(\lambda')u_{nn} + \sum_{\alpha} S_{k-1}(\lambda'|\alpha)u_{\alpha n}^2. \]

In the last inequality, we have used (7.17) and \( S_{k-3}(\lambda'|\alpha\beta) \geq 0. \)

We also have
\[ S_{k}^{ij} u_{ik}u_{kj} = S_1(D^2 u)S_k(D^2 u) - (k + 1)S_{k+1}(D^2 u) = -(k + 1)S_{k+1}(D^2 u). \]
(7.14)

Since \( \lambda[D^2 u] \in \bar{\Gamma}_k, \) it is easy to deduce that \( \lambda' \in \bar{\Gamma}_{k-1}. \)

**Case 1:** When at the point \( p \) we have \( S_{k-1}(\lambda') = 0, \) by (7.14) we can see
\[ S_k(\lambda') = \sum_{\alpha} S_{k-2}(\lambda'|\alpha)u_{\alpha n}^2 \geq 0. \]

This yields \( S_k(\lambda') = 0 \) and for any \( \alpha \) we have
\[ S_{k-2}(\lambda'|\alpha)u_{\alpha n}^2 = 0, \]
which in turn implies for any \( \alpha \)
\[ S_{k-1}(\lambda'|\alpha)u_{\alpha n}^2 = 0. \]
Combining the above equalities with (7.13) we conclude, in this case

\[ S^i_k u_{in} u_{jn} \leq 0. \]

Therefore, we have

\[ S^i_k u_{ik} u_{kj} - \frac{n}{n-k} S^i_k u_{in} u_{jn} \geq 0. \]

**Case 2:** In the following we will assume \( S_{k-1}(\lambda') > 0 \) at the point under consideration. Applying (7.11)-(7.14), we get

\[
S^i_k u_{ik} u_{kj} - \frac{n}{n-k} S^i_k u_{in} u_{jn} \\
= - (k+1) S_{k+1}(D^2 u) - \frac{n}{n-k} S^i_k u_{in} u_{jn} \\
\geq - (k+1) \left[ S_k(\lambda') u_{nn} + S_{k+1}(\lambda') - \sum_\alpha S_{k-1}(\lambda|\alpha) u^{2}_{\alpha n} \right] \\
- \frac{n}{n-k} \left[ - S_k(\lambda') u_{nn} + \sum_\alpha S_{k-1}(\lambda|\alpha) u^{2}_{\alpha n} \right] \\
= - (k+1) S_{k+1}(\lambda') - \frac{k(n-k-1)}{n-k} \left[ S_k(\lambda') u_{nn} - \sum_\alpha S_{k-1}(\lambda|\alpha) u^{2}_{\alpha n} \right] \\
= - (k+1) S_{k+1}(\lambda') - \frac{k(n-k-1)}{n-k} \left[ S_k(\lambda') - \sum_\alpha S_{k-2}(\lambda|\alpha) u^{2}_{\alpha n} \right] \\
- \frac{k(n-k-1)}{n-k} \sum_\alpha \left( S_k(\lambda') \frac{S_{k-2}(\lambda|\alpha)}{S_{k-1}(\lambda')} - S_{k-1}(\lambda|\alpha) \right) u^{2}_{\alpha n}.
\]

Finally, the Newton-Maclaurin inequality implies

\[
(7.16) \quad \left( \frac{k(n-k-1)}{n-k} \right) \frac{S^2_k(\lambda')}{S_{k-1}(\lambda')} - (k+1) S_{k+1}(\lambda') \geq 0,
\]

and

\[
(7.17) \quad S_{k-1}(\lambda|\alpha) - \frac{S_k(\lambda') S_{k-2}(\lambda|\alpha)}{S_{k-1}(\lambda')} = \frac{S^2_{k-1}(\lambda|\alpha)}{S_{k-1}(\lambda')} - S_k(\lambda'|\alpha) S_{k-2}(\lambda'|\alpha) \geq 0.
\]

Inserting (7.16) and (7.17) into (7.15), we conclude that

\[ S^i_k u_{ik} u_{kj} - \frac{n}{n-k} S^i_k u_{in} u_{jn} \geq 0. \]

Thus we finish the proof. \(\square\)

Finally, applying Lemma 2.2 in [11], Theorem 1.2 and Corollary 7.1, Corollary 1.1 follows directly.
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