An Analysis of the Inclusive Decay $\Upsilon(1S) \rightarrow \eta'X$ and Constraints on the $\eta'$-Meson Distribution Amplitudes

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Abstract

We calculate the $\eta'$-meson energy spectrum in the decay $\Upsilon(1S) \rightarrow \eta'ggg \rightarrow \eta'X$ in the leading-order perturbative QCD in the static quark limit for the Orthoquarkonium. Our principal result is the extraction of parameters of the $\eta'g^*g$ effective vertex function (EVF) involving a virtual and a real gluon from the available data on the hard part of the $\eta'$-meson energy spectrum. The perturbative QCD based framework provides a good description of the available CLEO data, allowing to constrain the lowest Gegenbauer coefficients $B_2^{(q)}$ and $B_2^{(g)}$ of the quark-antiquark and gluonic distribution amplitudes of the $\eta'$-meson. The resulting constraints are combined with the existing ones on these coefficients from an analysis of the $\eta - \gamma$ and $\eta' - \gamma$ transition form factors and the requirement of positivity of the EVF, yielding $B_2^{(q)}(\mu_0^2) = -0.008 \pm 0.054$ and $B_2^{(g)}(\mu_0^2) = 4.6 \pm 2.5$ for $\mu_0^2 = 2$ GeV\textsuperscript{2}. This reduces significantly the current uncertainty on these coefficients. The resulting EFV $F_{\eta'g^*g}(p^2, 0, m_{\eta'}^2)$, including the $\eta'$-meson mass effects, is presented.

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1 Introduction

Experiments involving the production and decays of the $\eta$- and $\eta'$-mesons are consistent with the picture that the $\eta$-meson is largely an $SU(3)_F$ flavour-octet state, but the $\eta'$-meson contains a significant amount of a $U(3)_F$ flavour-singlet quark-antiquark ($\bar{q}q$) component in its wave-function \[1\,2\,3\,4\,5]\; ; in addition, the gluonic ($gg$) content of the $\eta'$-meson is substantial \[6\]. This implies that for processes involving gluons and the $\eta'$-meson, in particular in the decays of heavy mesons of current phenomenological interest, such as $b \to sgl\eta'$ inducing the $B \to \eta'X_s$ decay \[7\,8\,9\,10\,11\,12\,13\] and the exclusive $B \to (\eta, \eta')(K, K^*)$ decays \[14\,15\,16\,17\,18\,19\,20\,21\], but also the inclusive decay of the Orthoquarkonium $\Upsilon(1S) \to gg(g^* \to \eta'g) \to \eta'X$ \[22\], the effective $\eta'$-gluon-gluon vertex plays an important role. This effective vertex function (EFV) has to be known sufficiently well to undertake a quantitative analysis of the data involving the $\eta'$-meson. Calling this EFV $F_{\eta'gg}^g(g^* \to \eta'g) \to \eta'X$ \[22\], the effective $\eta'g^*g^{(s)}$ vertex can be calculated in perturbative QCD \[23\,24\,25\,26\].

In this approach, the $\eta'$-meson wave-function is described in terms of two light-cone distribution amplitudes (LCDAs) involving the quark-antiquark $\phi_{\eta'}^{(q)}(x, Q^2)$ and the gluonic $\phi_{\eta'}^{(g)}(x, Q^2)$ components, where $x$ is the scaled energy of one of the partons of the $\eta'$-meson and $Q$ is the typical hard scale in the vertex. These two components mix if the QCD evolution is taken into account. The leading-twist LCDAs of the $\eta'$-meson can be expressed as infinite series of the Gegenbauer polynomials $C_n^{3/2}(x - \bar{x})$ (for the quark-antiquark) and $C_n^{5/2}(x - \bar{x})$ (for the gluonic) components \[27\,28\,29\,30\,31\,32\]

\[\phi_{\eta'}^{(q)}(x, Q^2) = 6x\bar{x} \left[ 1 + \sum_{\text{even } n \geq 2} A_n(Q^2) C_n^{3/2}(x - \bar{x}) \right],\]

\[\phi_{\eta'}^{(g)}(x, Q^2) = x^2\bar{x}^2 \sum_{\text{even } n \geq 2} B_n(Q^2) C_n^{5/2}(x - \bar{x}),\]

where $\bar{x} = 1 - x$, and the following notation is used for the Gegenbauer moments:

\[A_n(Q^2) = B_n^{(q)}(\mu_0^2) \left( \frac{\alpha_s(\mu_0^2)}{\alpha_s(Q^2)} \right) \gamma_n^+ + \rho_n^{(g)} B_n^{(g)}(\mu_0^2) \left( \frac{\alpha_s(\mu_0^2)}{\alpha_s(Q^2)} \right) \gamma_n^- ,\]

\[B_n(Q^2) = \rho_n^{(g)} B_n^{(g)}(\mu_0^2) \left( \frac{\alpha_s(\mu_0^2)}{\alpha_s(Q^2)} \right) \gamma_n^+ + B_n^{(g)}(\mu_0^2) \left( \frac{\alpha_s(\mu_0^2)}{\alpha_s(Q^2)} \right) \gamma_n^- .\]

The constrained parameters $\rho_n^{(q)}$, $\rho_n^{(g)}$, $\gamma_n^+$ and $\gamma_n^-$ are computable and can be found, for example, in the Appendix A of Ref. \[24\]. Usually, one employs an approximate form for the $\eta'$-meson LCDAs in which only the first non-asymptotic terms in both the quark-antiquark and gluonic components are kept. Thus, in this approximation

\[\phi_{\eta'}^{(q)}(x, Q^2) = 6x\bar{x} \left[ 1 + 6(1 - 5x\bar{x}) A_2(Q^2) \right], \quad \phi_{\eta'}^{(g)}(x, Q^2) = 5x^2\bar{x}^2 (x - \bar{x}) B_2(Q^2),\]

where the explicit forms for $C_2^{3/2}(x - \bar{x})$ and $C_1^{5/2}(x - \bar{x})$ have been used. The free parameters $B_2^{(q)}(\mu_0^2)$ and $B_2^{(g)}(\mu_0^2)$, which enter in $A_2(Q^2)$ and $B_2(Q^2)$, are not determined from
first principles, and have to be modeled or extracted from a phenomenological analysis of data. To that end, a fit of the CLEO and L3 data on the $\eta - \gamma$ and $\eta' - \gamma$ transition form factors for $Q^2$ larger than 2 GeV$^2$ was recently undertaken in Ref. [25], yielding:

$$A_2(1 \text{ GeV}^2) = -0.08 \pm 0.04, \quad B_2(1 \text{ GeV}^2) = 9 \pm 12,$$

(6)

where the initial scale of the evolution is taken as $\mu_0^2 = 1 \text{ GeV}^2$. Note, that the coefficients $A_2(1 \text{ GeV}^2)$ and $B_2(1 \text{ GeV}^2)$ are strongly correlated. The estimates can be translated in terms of the universal free parameters $B_2^{(q)}$ and $B_2^{(g)}$, yielding:

$$B_2^{(q)}(1 \text{ GeV}^2) = 0.02 \pm 0.17, \quad B_2^{(g)}(1 \text{ GeV}^2) = 9.0 \pm 11.5.$$

(7)

The current determination of these coefficients, in particular $B_2^{(g)}$, is rather poor, leading to a huge uncertainty in the evaluation of the $\eta'g^*g(\gamma)$ vertex function $F_{\eta'g^*g(\gamma)}(p_1^2, m_{\eta'}^2)$. In this paper, we undertake a perturbative-QCD based analysis of the recent data on the inclusive process $\Upsilon(1S) \rightarrow \eta'X$ in the large $\eta'$-meson energy region published recently by the CLEO collaboration [35], which is expected to be dominated by the process $\Upsilon(1S) \rightarrow \eta'ggg \rightarrow \eta'X$. Moreover, Chen and Kagan [22] have argued that the shape of the $\eta'$-meson energy spectrum in this decay is sensitive to the shape of the $F_{\eta'g^*g(\gamma)}(p^2, 0, m_{\eta'}^2)$ vertex function, involving the $\eta'$-meson, a real and a virtual gluons (see Fig. 1). This sensitivity has already been used by the CLEO collaboration [35] to rule out certain models for this vertex, with the CLEO analysis favouring a rapidly falling $p^2$-dependence of the vertex $F_{\eta'g^*g(\gamma)}(p^2, 0, m_{\eta'}^2)$, in line with the perturbative QCD predictions [23][24]. Motivated by these observations, we undertake a quantitative analysis of the CLEO data to constrain the LCDAs involving the quark-antiquark and the gluonic components of the $\eta'$-meson. The results of this analysis are presented in terms of the coefficients $B_2^{(q)}(2 \text{ GeV}^2)$ and $B_2^{(g)}(2 \text{ GeV}^2)$, where we have taken the initial scale as $\mu_0^2 = m_{\eta'}^2 + p_0^2 = 2 \text{ GeV}^2$, which corresponds to the minimum gluon virtuality $p_0^2 \approx 1 \text{ GeV}^2$. This analysis is then combined with an earlier analysis of the $\eta - \gamma$ and $\eta' - \gamma$ transition form factors [25] to further constrain the two parameters. As the physical interpretation of the function $F_{\eta'g^*g(\gamma)}(p^2, 0, m_{\eta'}^2)$ is that it represents a probability distribution, much the same way as the partonic distributions are in, for example, deep inelastic lepton scattering off nucleons, this function must remain positive definite in the entire $p^2$ range. The requirement of positive definiteness of the function $F_{\eta'g^*g(\gamma)}(p^2, 0, m_{\eta'}^2)$ provides additional constraints on the parameters $B_2^{(q)}(2 \text{ GeV}^2)$ and $B_2^{(g)}(2 \text{ GeV}^2)$, in particular on the latter. The combined analysis leads to the following correlated ranges for these coefficients:

$$B_2^{(q)}(2 \text{ GeV}^2) = -0.008 \pm 0.054, \quad B_2^{(g)}(2 \text{ GeV}^2) = 4.6 \pm 2.5.$$

(8)

Finally, we use this information to calculate the $F_{\eta'g^*g(\gamma)}(p^2, 0, m_{\eta'}^2)$ vertex, including the $\eta'$-meson mass effects, relegating the detailed derivation to a subsequent paper [39].

This paper is organized as follows: In section 2, we calculate the branching ratio for the process $\Upsilon(1S) \rightarrow ggg^*(g^* \rightarrow \eta'g) \rightarrow \eta'X$ and the $\eta'$-meson energy spectrum in this decay. Numerical analysis of the CLEO data is carried out in section 3, and the resulting constraints are combined with the analysis of the $\eta' - \gamma$ transition form factor to determine the LCDAs of the $\eta'$-meson. Section 4 contains a brief summary. The expressions for the matrix element squared for the decay $\Upsilon(1S) \rightarrow ggg^*(g^* \rightarrow \eta'g)$ are displayed in the Appendix.
Figure 1: A typical Feynman diagram describing the decay \( \Upsilon(1S) \to ggg^*(g^* \to \eta' g) \to \eta' X \). The directed solid line denotes the quark \( b \) and antiquark \( \bar{b} \) of the \( \Upsilon \)-meson and the double solid line denotes the \( \eta' \)-meson.

2 Branching Ratio and \( E_{\eta'} \)-Distribution in the Decay \( \Upsilon(1S) \to \eta' ggg \to \eta' X \)

Several processes contribute to the inclusive \( \eta' \)-meson production in the \( \Upsilon(1S) \)-meson decay. The two dominant decays are: \( \Upsilon(1S) \to \gamma^* \to q\bar{q} \to \eta' X \) and \( \Upsilon(1S) \to ggg^*(g^* \to \eta' g) \to \eta' X \). The first of these has been estimated from the measured value of the hadronic cross-section \( R_{\text{had}} \) and the branching ratio \( \mathcal{B}(\Upsilon(1S) \to \mu^+\mu^-) \), yielding \( \mathcal{B}(\Upsilon(1S) \to \gamma^* \to q\bar{q}) = (8.8 \pm 0.3)\% \) [35], and the CLEO data has been corrected for the \( q\bar{q} \) component. In addition, there is also the continuum production from the process \( e^+e^- \to \gamma^* \to q\bar{q} \to \eta' X \), which can be estimated from the \( e^+e^- \)-continuum data below the resonance. This data also provides a good profile of the fragmentation in the process \( \gamma^* \to q\bar{q} \to \eta' X \). Typically, the fragmentation processes involving the intermediate \( q\bar{q} \) state populate the low-\( z \) region, where \( z \equiv E_{\eta'}/E_{\text{beam}} = 2E_{\eta'}/M \) is the relative \( \eta' \)-meson energy expressed in terms of the \( \Upsilon(1S) \)-meson mass \( M \). One expects that in the large-\( z \) region, the process \( \Upsilon(1S) \to ggg^*(g^* \to \eta' g) \) dominates. Assuming this, we will concentrate here on the intermediate three-gluon (\( ggg^* \)) state and analyze the \( E_{\eta'} \)-spectrum in the large-\( z \) region alone. The quality of the fits provides a justification of this procedure \textit{a posteriori}.

A typical Feynman diagram describing the decay \( \Upsilon(1S) \to ggg^*(g^* \to \eta' g) \to \eta' X \) is presented in Fig. 1. The other 17 diagrams can be obtained from the above one by the permutations of the gluons in the intermediate (virtual) and final state. For the matrix element calculations, we adopted the static limit for the heavy quark-antiquark pair in the orthoquarkonium \( \Upsilon(1S) \), so that both the quark \( (b) \) and antiquark \( (\bar{b}) \) carry half of the \( \Upsilon(1S) \)-meson four-momentum, and the velocity-dependent corrections are neglected.

The total decay amplitude can be divided into three parts:

\[
\mathcal{M}[\Upsilon(P) \to \eta'(p_{\eta'})g(k_1)g(k_2)g(k_3)] = \sum_{i=1}^{3} \mathcal{M}_i, \tag{9}
\]

where each of the three terms \( \mathcal{M}_i \) collects the contributions from the diagrams with the virtual gluon of the same four-momentum \( p_i = p_{\eta'} + k_i \), with \( p_{\eta'} \) and \( k_i \) being the four-momenta of the \( \eta' \)-meson and the \( i \)th final gluon. The explicit forms of the three
contributions $\mathcal{M}_i$ are as follows:

$$
\mathcal{M}_1 = \frac{d_{ABC} g_s^3 \sqrt{M}}{4\sqrt{N_c}} \psi(0) F_{1'q'}(p_3^2) \left\{ \frac{4}{M^2} (p_1 \tilde{f}_A^* [f_2^B f_3^C] + \tilde{f}_2^B f_3^C) f_V \mathcal{P} \right\} (10)
$$

$$
\mathcal{M}_2 = \frac{d_{ABC} g_s^3 \sqrt{M}}{4\sqrt{N_c}} \psi(0) F_{1'q'}(p_3^2) \left\{ \frac{4}{M^2} (p_2 \tilde{f}_2^B [f_2^B f_3^C] + \tilde{f}_3^C f_1^A) f_V \mathcal{P} \right\} (11)
$$

$$
\mathcal{M}_3 = \frac{d_{ABC} g_s^3 \sqrt{M}}{4\sqrt{N_c}} \psi(0) F_{1'q'}(p_3^2) \left\{ \frac{4}{M^2} (p_3 \tilde{f}_3^C [f_2^B f_3^C] + \tilde{f}_2^B f_1^A) f_V \mathcal{P} \right\} (12)
$$

where $d_{ABC}$ ($A, B, C = 1, \ldots, N_c - 1$ are the colours of the gluons) is the symmetrical constant of the colour $SU(N_c)$ group with $N_c = 3$; $g_s$ is the strong coupling constant; $(f_V)_{\mu\nu} = \mathcal{P}_{\mu} \eta_{\nu} - \mathcal{P}_{\nu} \eta_{\mu}$ is the polarization tensor of the $\Upsilon(1S)$-meson, $M$, $\mathcal{P}_{\mu}$, $\eta_{\mu}$ and $\psi(0)$ are the mass of the quarkonium, the four-momentum, the polarization vector, and the non-relativistic wave-function in the position space evaluated at the origin, respectively; $(f_i^A)_{\mu\nu} = k_{i\nu} e_i^A - k_{i\mu} e_i^A$ and $(\tilde{f}_i^A)_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} k_i^\rho e_i^A$ ($i = 1, 2, 3$) are the gluonic field-strength tensor and its dual involving the $i$th gluon with the polarization vector $e_i^A$ and four-momenta $k_{i\mu}$; $p_i = p_{i'} + k_i$ is the four-momentum of the virtual gluon, and $p_i' = \mathcal{P} - p_i$. In the above equations the notations are as follows: $(f_i^A f_j^B) = (f_i^A)_{\mu\nu} (f_j^B)^{\nu\mu}$, $(p_1 f_1^A f_2^B) p_1 = p_1' (f_1^A)^{\mu} (f_2^B)^{\nu} p_{1\mu}$, etc. It is worth noting that the matrix element satisfies the Bose symmetry under the exchange of gluons, in particular, under the interchange of the first and second gluons $\mathcal{M}_1 \leftrightarrow \mathcal{M}_2$ while the term $\mathcal{M}_3$ remains unchanged.

The function $F_{1'q'}(p_3^2)$ is the effective vertex function involving the $\Upsilon'$-meson and two gluons, one of which is on the mass shell ($k_i^2 = 0$). We shall call this interchangeably also the $\eta' - g$ transition form factor. A form of this function motivated by the QCD analysis of the $\eta' g^* g$ loop diagram was suggested by Kagan and Petrov [9]:

$$
F_{1'q'}(p^2) \equiv F_{1'g^*g}(p^2, 0, m_{\eta'}^2) = \frac{m_{\eta'}^2 \mathcal{H}(p^2)}{p^2 - m_{\eta'}^2}. (13)
$$
where the function \( H(p^2) \) was approximated by the constant value \( H(p^2) \approx H_0 \approx 1.7 \text{ GeV}^{-1} \), extracted from the \( J/\psi \rightarrow \eta' \gamma \) decay \([7]\). In a companion paper \([39]\) we argue that the form \([13]\) for the \( \eta' - g \) transition form factor also emerges in the perturbative calculations of this function in the hard-scattering approach by keeping the \( \eta' \)-meson mass. In this approach, the dependence of the function \( H(p^2) \) on the gluon virtuality \( p^2 \) is given by the following expression:

\[
H(p^2) = \frac{4\pi \alpha_s(Q^2)}{m_{\eta'}^2} \sqrt{3} f_{\eta'} \left[ 1 + A_2(Q^2) - \frac{5}{6} B_2(Q^2) G_2^{(g)}(1, \zeta) \right],
\]

where \( f_{\eta'} \approx 2f_\pi/\sqrt{3} \) is the \( \eta' \)-meson decay constant expressed in terms of the \( \pi \)-meson decay constant \( f_\pi \approx 133 \text{ MeV} \), \( \zeta = m_{\eta'}/p^2 \), and the function \( G_2^{(g)}(1, \zeta) \) has the form \([39]\):

\[
G_2^{(g)}(1, \zeta) = \frac{5}{3\zeta} + \frac{2}{\zeta^2} - \frac{4}{\zeta^3} - \frac{1}{\zeta} \left[ 1 - \frac{1}{\zeta} \right] \left[ 1 - \frac{4}{\zeta^2} \right] \ln(1 - \zeta).
\]

In Eq. (13) the scale \( Q^2 \) in the strong coupling \( \alpha_s(Q^2) \), and also in the second Gegenbauer moments \( A_2(Q^2) \) and \( B_2(Q^2) \) of the quark-antiquark and gluonic LCDAs of the \( \eta' \)-meson, is related with the gluon virtuality \( p^2 \), but there is an uncertainty in its precise definition. One of the possibilities is to require that the function \( H(p^2) \) is finite at all values of \( p^2 \) including the singularity point of the EVF \([13]\), \( p^2 = m_{\eta'}^2 \), which can be done, for example, by putting \( Q^2 = |p^2| + m_{\eta'}^2 \).

The dependence of the function \( G_2^{(g)} \) on the momentum squared \( p^2 \) of the virtual gluon is presented in Fig. 2 with the value \( G_2^{(g)}(1, 0) = 1/6 \) in the large-\( |p^2| \) asymptotic region. Since this result is based on the application of perturbation theory, its validity is restricted to the region \( |p^2| > p_0^2 \), where typically \( p_0^2 = 1 \text{ GeV}^2 \) (or somewhat higher). In view of this, we shall set \( \text{Im} \ G_2^{(g)} = 0 \) and ignore the structure in \( \text{Re} \ G_2^{(g)} \) in the low-\( |p^2| \) region seen in Fig. 2.

As already noted, the functions \( A_2(Q^2) \) and \( B_2(Q^2) \) in Eq. (14) are the second Gegenbauer moments of the quark-antiquark and gluonic light-cone distribution amplitudes (LCDAs) of the \( \eta' \)-meson. The scale \( \mu_0 \) appearing in the definitions of these moments is set by the initial value in the evolution of the Gegenbauer moments. As the mass of the \( \eta' \)-meson is large, \( \sim 1 \text{ GeV} \), it is not a good approximation to neglect it. Assuming further, that the evolution of the Gegenbauer moments is valid for the gluons with virtualities larger than \( 1 \text{ GeV}^2 \), we shall take \( \mu_0^2 = Q_0^2 \approx 2 \text{ GeV}^2 \).

The total decay width of the \( \Upsilon(1S) \)-meson into the \( \eta' \)-meson and three gluons, \( \Upsilon \rightarrow \eta'ggg \), averaged over the quarkonium spin states and summed over the polarizations and colours of the final gluons can be written in the form:

\[
\Gamma_{\eta'X} \equiv \Gamma[ \Upsilon \rightarrow \eta'ggg ] = \frac{1}{3!} \frac{1}{(2\pi)^3} \frac{1}{2M} \int \frac{dk_1}{2\omega_1} \frac{dk_2}{2\omega_2} \frac{dk_3}{2\omega_3} \frac{dp_{\eta'}}{2E_{\eta'}} \times \delta^{(4)}(E - k_1 - k_2 - k_3 - p_{\eta'}) \frac{1}{3} \sum |M[ \Upsilon \rightarrow \eta'ggg ]|^2,
\]

where the factor \( 1/3! \) takes into account the identity of the final gluons. The expression for the matrix element squared is rather lengthy and can be found in the Appendix, where we have also discussed some technical details of our Monte Carlo integration.
Figure 2: The real (solid curve) and imaginary (dashed curve) parts of the function $G_2^{(g)}(1, \zeta)$, where $\zeta = m_{\eta'}^2/p^2$, as a function of the momentum squared $p^2$ of the virtual gluon. We use this function only in the $|p^2| > 1$ GeV$^2$ region.

The $\eta'$-meson energy distribution function can be defined as follows:

$$
\frac{dn}{dz} = \frac{1}{\Gamma_{3g}^{(0)}} \frac{d\Gamma_{\eta'} \chi(z)}{dz} = \frac{1}{\Gamma_{3g}^{(0)}} \frac{1}{3!} \left( \frac{1}{2\pi} \right)^8 \frac{1}{2M} \int \frac{dk_1}{2\omega_1} \frac{dk_2}{2\omega_2} \frac{dk_3}{2\omega_3} \frac{dp_{\eta'}}{2E_{\eta'}} \times \delta^{(4)}(\mathcal{P} - k_1 - k_2 - k_3 - p_{\eta'}) \delta(z - 2E_{\eta'}/M) \frac{1}{3} \sum \left| \mathcal{M}[\Upsilon \to \eta'ggg] \right|^2,
$$

where $\Gamma_{3g}^{(0)}$ is the three-gluon decay width of the $\Upsilon(1S)$-meson in the leading order:

$$
\Gamma_{3g}^{(0)} = \frac{16}{9} \left( \pi^2 - 9 \right) C_F B_F \alpha_s^3(M^2) \frac{|\psi(0)|^2}{M^2}.
$$

Here, $C_F = (N_c^2 - 1)/(2N_c)$ and $B_F = (N_c^2 - 4)/(2N_c)$ are the constants of the colour $SU(N_c)$ group, and $\alpha_s(M^2) = g_s^2(M^2)/(4\pi)$ is the strong coupling estimated at the scale of $M$. 


the Υ-meson mass. The $\alpha_s$ corrections to the decay width $\Gamma_{gg}$ are known since a long time and are numerically large. However, we do not take them into account, as we have calculated the decay $\Upsilon(1S) \to ggg^*(g^* \to \eta'g)$ only in the leading order. One anticipates that in the distribution $dn/dz$, a good part of the explicit $O(\alpha_s)$ corrections should drop out, and we are tacitly assuming that the remaining corrections do not greatly influence the energy spectrum derived in the lowest non-trivial order.

3 Numerical Analysis of the $\Upsilon(1S) \to \eta'X$ Data and the $\eta'$-Meson LC Distribution Amplitudes

As the results for the matrix element squared in the decay $\Upsilon(1S) \to ggg^*(g^* \to \eta'g)$ are not yet available in the literature, we shall give their explicit expressions in this paper. In that context we note that it is sufficient to have the expressions for one of the diagonal terms $|M_i|^2$ ($i = 1, 2, 3$) and one of the non-diagonal terms $M_i M^*_j$ ($i \neq j$), as the others can be obtained by using the Bose symmetry. With this, the expressions for the components $(1/3) \sum_i |M_i|^2$, derived from Eq. (10), and $(1/3) \sum_i M_i M^*_i + c.c.$, resulting from the cross terms in Eq. (10) and (11) in $|M|^2$, are given in the Appendix.

We start our numerical calculations by reproducing the already known results for the $\eta'$-meson energy spectrum $dn/dz$ [35][22], as this provides a good consistency check of our calculations. For this purpose, the spectrum in Fig. 3 is calculated with the same set of phenomenological parameters as has been used in Ref. [22] for the following three input forms:

a) A slowly falling EVF: $F_{\eta'g}(p^2) \simeq 2.1 \text{GeV}^{-1} [\alpha_s(p^2)/\alpha_s(m^2_{\eta'})]$ where the two-loop expression is used for the strong coupling $\alpha_s$.

b) A rapidly falling EVF of the form [13] with the function $H(p^2)$ approximated by the constant value $H(p^2) \simeq 1.7 \text{GeV}^{-1}$.

c) An intermediate EVF: $F_{\eta'g}(p^2) \simeq 12.5 \text{GeV}/(p^2 + M_0^2)$ with $M_0 = 2.25 \text{GeV}$.

The shapes of the $\eta'$-meson energy spectrum resulting from these EVFs are presented in Fig. 3. The experimental points in the decay $\Upsilon(1S) \to \eta'X$ shown in this figure are taken from Ref. [35]. Our results are in agreement with the ones in Ref. [22], and we confirm the previous observation that the curve (b) corresponding to the rapidly falling EVF is in reasonable agreement with the experimental data in the hard-energy region $z \geq 0.7$ [35].

Note that the allowed kinematic domain of the variable $z$ is $z_0 < z < 1 + z_0^2/4$, where $z_0 = 2m_{\eta'}/M$. The region $1 < z < 1 + z_0^2/4$ is too small to be resolved experimentally, given the statistics of the CLEO data [35], and hence the energy distribution above $z = 1$ is included in the last energy bin $0.9 < z < 1.0$.

Let us now turn to the analysis of the perturbative-QCD based form of the effective vertex presented in Eqs. (13)-(15). To fit the parameters $B_2^{(0)}(\mu_0^2)$ and $B_2^{(0)}(\mu_3^2)$ from the existing experimental data on the $\Upsilon(1S) \to \eta'ggg \to \eta'X$ decay, it is convenient to first
Figure 3: The energy spectrum of the $\eta'$-meson in the decay $\Upsilon(1S) \to \eta'ggg \to \eta'X$ for the three assumed phenomenological models for the $\eta'g^*g$ vertex given in the text, in comparison with the data from the CLEO collaboration [35]. Note that only the data points for $z = 2E_{\eta'}/M \geq 0.7$ are relevant for comparison with the theoretical models.

derive an approximate numerical formula for the $\eta'$-meson energy spectrum. This expression will come in handy for subsequent comparison with other independent calculations and will also facilitate undertaking analysis of the data in future by the experimental groups themselves. As the amplitude of the process considered is linear in $B_2^{(q)}$ and $B_2^{(g)}$, the energy spectrum $dn/dz$ is quadratic in these parameters. The general form for the energy spectrum in the decay $\Upsilon(1S) \to \eta'ggg \to \eta'X$ can be written as follows (suppressing in this equation the scale-dependence of the coefficients for ease of writing):

$$
\frac{d\tilde{n}}{dz}(z, B_2^{(q)}, B_2^{(g)}) = C_{00}(z) + C_{0q}(z)B_2^{(q)} + C_{0g}(z)B_2^{(g)}
+ C_{qq}(z)[B_2^{(q)}]^2 + C_{qg}(z)B_2^{(q)}B_2^{(g)} + C_{gg}(z)[B_2^{(g)}]^2.
$$

We have generated the theoretical $E_{\eta'}$-spectrum, using a high statistics Monte Carlo phase space programme, with fixed values of the coefficients $B_2^{(q)}(\mu_0^2)$ and $B_2^{(g)}(\mu_0^2)$, and have varied their values over a certain range to numerically determine the dependence of the
Table 1: Input parameters used in the numerical analysis.

\[
\begin{align*}
M &= 9.46 \text{ GeV} & m_c &= 1.3 \text{ GeV} \\
m_{q'} &= 958 \text{ MeV} & m_b &= 4.3 \text{ GeV} \\
f_\pi &= 133 \text{ MeV} & \Lambda_{\text{QCD}}^{(4)} &= 305 \text{ MeV} \\
\mu_0^2 &= 2 \text{ GeV}^2
\end{align*}
\]

Table 2: The coefficients in the interpolating function \(d\tilde{n}/dz\) defined in the text for the decay \(\Upsilon(1S) \to \eta'X\). The numbers in the brackets are from the Monte Carlo statistical errors.

| \(z\) | \(C_{00}(z)\) | \(C_{0q}(z)\) | \(C_{qq}(z)\) | \(C_{gg}(z)\) | \(C_{gg}(z)\) | \(C_{gg}(z)\) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.6 \div 0.7 | 1.9172(27) | 2.5187(36) | -0.3108(13) | 0.8744(19) | -0.1888(07) | 0.0146(06) |
| 0.7 \div 0.8 | 1.6750(25) | 2.2368(34) | -0.2583(12) | 0.7953(18) | -0.1560(07) | 0.0122(06) |
| 0.8 \div 0.9 | 1.2768(21) | 1.7344(29) | -0.1855(10) | 0.6343(15) | -0.1097(06) | 0.0089(05) |
| 0.9 \div 1.0 | 0.6475(17) | 0.8945(17) | -0.0866(07) | 0.3396(09) | -0.0484(05) | 0.0044(05) |

spectrum on these coefficients. Other parameters in our numerical analysis are listed in Table 1 which are the same as the ones used in the analysis of the \(\eta' - \gamma\) and \(\eta - \gamma\) transition form factors \[^{25}\], for the sake of consistency, as we are going to make use of this analysis. We have not included any errors on \(m_c, m_b\) and \(\Lambda_{\text{QCD}}^{(4)}\), as the parametric uncertainties from the Gegenbauer coefficients are by far the largest, which we study. The coefficients \(C_{ab}(z)[a, b = 0, q, g]\) for the four bins of the \(\eta'\)-meson energy spectrum are presented in Table 2. The numbers presented in the parentheses represent the statistical error of our Monte Carlo calculations, for which we have used the Monte Carlo phase space generator FOWL from the CERN Library of programmes. With the help of the program MINUIT \[^{11}\], the following best fit values of the parameters \(B_2^{(g)}(\mu_0^2)\) and \(B_2^{(g)}(\mu_0^2)\) are obtained:

\[
\begin{align*}
B_2^{(g)}(\mu_0^2) &= -0.89^{+1.32}_{-1.58}, \quad B_2^{(g)}(\mu_0^2) = -2.86^{+20.04}_{-5.80}, \quad \chi^2 = 2.45, \\
B_2^{(q)}(\mu_0^2) &= -1.09^{+1.51}_{-1.36}, \quad B_2^{(q)}(\mu_0^2) = 11.53^{+5.55}_{-20.09}, \quad \chi^2 = 2.37,
\end{align*}
\]

for the last three experimental bins with \(z > 0.7\) (the stated \(\chi^2\) corresponds to three degrees of freedom), and

\[
\begin{align*}
B_2^{(q)}(\mu_0^2) &= -0.77^{+0.73}_{-0.78}, \quad B_2^{(q)}(\mu_0^2) = -4.36^{+6.28}_{-4.46}, \quad \chi^2 = 24.13, \\
B_2^{(g)}(\mu_0^2) &= -1.29^{+0.76}_{-0.73}, \quad B_2^{(g)}(\mu_0^2) = 12.51^{+4.53}_{-6.33}, \quad \chi^2 = 23.69,
\end{align*}
\]

for the four experimental data points in the large-\(z\) region (\(z > 0.6\)). The minimum \(\chi^2\) of the fit in the last case, namely \(\chi^2 \approx 24\) for four degrees of freedom, is unacceptably large. Thus, as already observed in Ref. \[^{33}\], only the last three bins in the energy spectrum are dominated by the \(\Upsilon(1S) \to \eta'ggg \to \eta'X\) decay. Following this, we concentrate on the analysis of the last three bins with \(z \geq 0.7\). The \(1\sigma\) contours both in the Gegenbauer coefficients \((B_2^{(g)}(\mu_0^2), B_2^{(g)}(\mu_0^2))\) and in the Gegenbauer moments \((A_2(\mu_0^2), B_2(\mu_0^2))\) are
The additional constraint that follows from demanding that the vertex function \( F_{\eta'g}(p^2) \) remains positive for all values of \( p^2 > m_{\eta'}^2 \), allowing only the regions below these lines. Note that all the contours correspond to using \( \mu_0^2 = 2 \text{ GeV}^2 \).

Figure 4: The ±1 σ contours (long-dashed curves) in the parameter space \( B_2^{(q)}(\mu_0^2) - B_2^{(q)}(\mu_0^2) \) (left frame) and for the Gegenbauer moments \( B_2(\mu_0^2) - A_2(\mu_0^2) \) (right frame) resulting from the analysis of the last three experimental bins in the \( \eta' \)-meson energy spectrum in the process \( \Upsilon(1S) \rightarrow \eta'X \) measured by the CLEO collaboration [35]. The crosses (×) represent the two solutions with the minimum \( \chi^2 \) given in the text. The narrow solid contours result from the analysis of the data on the \( \eta' - \gamma \) transition form factor, scaled from Ref. [25]. The short-dashed lines in both the frames result by demanding that the vertex function \( F_{\eta'g}(p^2) \) remains positive for all values of \( p^2 > m_{\eta'}^2 \), allowing only the regions below these lines. Note that all the contours correspond to using \( \mu_0^2 = 2 \text{ GeV}^2 \).

presented as long-dashed curves in Fig. 4. The best fit values [20] are denoted by the crosses (×). The narrow contours (solid curves) also shown in these figures result from the analysis of the data on the \( \eta' - \gamma \) transition form factor [25]. As we are using \( \mu_0^2 = 2 \text{ GeV}^2 \) in our analysis, the results from Ref. [25] have been rescaled from \( \mu_0^2 = 1 \text{ GeV}^2 \) to \( \mu_0^2 = 2 \text{ GeV}^2 \) with the help of Eqs. 3 and 11 for \( n = 2 \). We note that the analysis of the \( \eta' - \gamma \) data, being more sensitive to the quark-antiquark LCDA of the \( \eta' \)-meson, provides a much more precise determination of the parameter \( B_2^{(q)}(\mu_0^2) \) and hence of the Gegenbauer moment \( A_2(\mu_0^2) \), than what can be determined at present from the CLEO data on \( \Upsilon(1S) \rightarrow \eta'X \). However, what concerns the coefficient \( B_2^{(g)}(\mu_0^2) \), and also the Gegenbauer moment \( B_2(\mu_0^2) \), the CLEO data on \( \Upsilon(1S) \rightarrow \eta'X \), despite its statistical limitations, has cut out some of the allowed parameter region from the \( \eta' - \gamma \) analysis. The additional constraint that follows from demanding that the vertex function \( F_{\eta'g}(p^2) \) remains positive definite for all values of \( p^2 > m_{\eta'}^2 \) is shown through the short-dashed lines in these figures, admitting only the parameter space below these lines. We will discuss this constraint in more detail below.

A blow up of the overlapping region in the parameter space \((B_2^{(q)}(\mu_0^2), B_2^{(g)}(\mu_0^2))\) is shown in Fig. 5. Here, the 1σ contours following from the \( \eta' - \gamma \) transition form factor analysis and the \( \Upsilon(1S) \rightarrow \eta'X \) decay fit are shown through the solid and dashed curves, respectively. In drawing the allowed parameter space from the \( \Upsilon(1S) \rightarrow \eta'X \) decay fit, we
Figure 5: The combined fit for the parameters $B_2^{(q)}(\mu_0^2 = 2 \text{ GeV}^2)$ and $B_2^{(g)}(\mu_0^2 = 2 \text{ GeV}^2)$ from the data on the $\eta' - \gamma$ transition form factor (solid curve) and $\Upsilon(1S) \to \eta'X$ decay (long-dashed and short-dashed curves) with the requirement that the vertex $F_{\eta' g}(p^2)$ remains positive definite in the entire $p^2$ region. The resulting $1\sigma$ contour (combined best fit) is shown by the yellow (shaded) region. The point denoted by $\otimes$ corresponds to the asymptotic light-cone distribution amplitude.

have imposed the additional condition that the $\eta' g^* g$ vertex function $F_{\eta' g}(p^2)$ for $p^2 > m_{\eta'}^2$ has the same sign as the corresponding function calculated with the asymptotic forms of the $\eta'$-meson quark-antiquark and gluonic LCDAs which is defined as positive-definite in the entire $p^2$ region. With Eq. (13) taken into account, this condition implies $H(p^2) > 0$ and results [with the help of the explicit from (14) of the function $H(p^2)$] in the following inequalities:

$$B_2(\mu_0^2) \leq \frac{36}{5} \left[ 1 + A_2(\mu_0^2) \right],$$

$$B_2^{(g)}(\mu_0^2) \leq \frac{36 + (36 - 5\rho_2^{(g)}) B_2^{(g)}(\mu_0^2)}{5 - 36\rho_2^{(g)}},$$

where the function $G_2^{(g)}(1, \zeta)$ (15) is approximated by its value $1/6$ in the large-$|p^2|$ asymptotics. The values $\rho_2^{(g)} = 2.86$ and $\rho_2^{(q)} = -0.01$ were taken for the constrained parameters in the numerical analysis. The positivity constraint removes the larger values of $B_2^{(g)}(\mu_0^2)$ above the short-dashed curve in Fig. 4 which would otherwise force the vertex function to cross the zero at some value of $p^2$ and become negative. This is exemplified below.
Figure 6: The $\eta'$-meson quark-antiquark and gluonic LCDAs $\phi_{\eta'}^{(q,g)}(x,\mu_0^2)$ as a function of $x$ (left frame), and the resulting $\eta' - g$ transition form factor $F_{\eta'g}(p^2)$ (right frame). The solid curves correspond to the asymptotic quark-antiquark LCDA, while the LCDAs for the combined best fit values of the Gegenbauer coefficients given in Eq. (23) are drawn as dotted curves. The LCDAs with the values $B_2^{(q)} = 0.15$ and $B_2^{(g)} = 13.5$, which are allowed within $1\sigma$ from the analysis of the data on the $\eta' - \gamma$ transition form factor [25], are presented as the dashed curves. Note, that for this case, the function $F_{\eta'g}(p^2)$ is no longer positive-definite, as shown in the right frame.

for specific values of the Gegenbauer coefficients. The resulting combined best fit of the parameters $B_2^{(q)}(\mu_0^2)$ and $B_2^{(g)}(\mu_0^2)$ is shown as the coloured (shaded) region. This yields the following correlated values:

$$
B_2^{(q)}(\mu_0^2) = -0.008 \pm 0.054, \quad B_2^{(g)}(\mu_0^2) = 4.6 \pm 2.5,
$$

$$
A_2(\mu_0^2) = -0.054 \pm 0.029, \quad B_2(\mu_0^2) = 4.6 \pm 2.7,
$$

with the central values having $\chi^2 = 2.66$ for three degrees of freedom. For comparison, the point shown as $\bigotimes$, corresponding to the asymptotic LCDA, has $\chi^2 = 8.41$ for three degrees of freedom. Thus, the asymptotic form of the $\eta'$-meson LCDA is not favoured by our analysis.

The shapes of the quark-antiquark and gluonic LCDAs are presented in Fig. 6 (left frame); the resulting $\eta' - g$ transition form factor $F_{\eta'g}(p^2)$ corresponding to these LCDAs is also shown in this figure (right frame). The solid and dotted curves in this figure correspond to the asymptotic LCDA and the combined best fit values of the Gegenbauer coefficients [23], respectively. We also show, for the sake of illustration, representative LCDAs with the Gegenbauer coefficients having the values $B_2^{(q)}(2 \text{ GeV}^2) = 0.15$ and
The various curves correspond to the input values of the Gegenbauer coefficients \( B_{2(q)}(\mu_0^2) \) and \( B_{2(g)}(\mu_0^2) \) resulting from the combined best fit shown in Fig. 5. The solid curves correspond to the central values of the fit, while the dashed curves are drawn for the parameters corresponding to the uppermost and lowermost allowed values of the combined best fit contour shown in Fig. 5.

\[ B_{2(g)}(2 \text{ GeV}^2) = 13.5. \] These parametric values are taken from the analysis in Ref. [25], but are in conflict with the positive definiteness of the vertex function \( F_{\eta'g}(p^2) \), as shown by the dashed curve for \( F_{\eta'g}(p^2) \). In fact, the constraint of positivity on the effective vertex function cuts out regions in the parameter space \( B_{2(q)}(2 \text{ GeV}^2) > 0.045 \) and \( B_{2(g)}(2 \text{ GeV}^2) > 7.1, \) as otherwise the contribution from the gluonic LCDA starts to dominate, which makes the vertex function \( F_{\eta'g}(p^2) \) negative for some range of \( p^2 \) considered here and in Ref. [25]. Thus, positivity criterion provides an effective constraint on the magnitude of the coefficient \( B_{2(g)}(\mu_0^2) \), reducing significantly the otherwise allowed range in Eq. (7). In the context of our analysis, we note that the asymptotic \( \eta' \)-meson LCDA provides a fair description (though not the best fit) of the current data on \( \Upsilon(1S) \to ggg^*(g^* \to \eta'g) \to \eta'X \), and hence one anticipates that the subasymptotic corrections in the LCDA, and the vertex function \( F_{\eta'g}(p^2) \), while important in the analysis of data, are not dominant.\(^4\)

The EVF \( F_{\eta'g}(p^2) \) and the function \( m_{\eta'}^2 H(p^2) \) for the combined best fit values \(^2\) are presented in Fig. 7. The \( \eta' \)-meson energy spectrum for the EVF motivated by the perturbative QCD analysis is presented in Fig. 8 in comparison with the experimental data \(^3\) and the spectrum corresponding to the “rapidly falling” form of the EVF \(^2\) labeled by the constant value of the function \( H(p^2) = H_0 \simeq 1.7 \text{ GeV}^{-1} \). The second dotted line labeled by \( H_{\text{as}}(p^2) \) corresponds to the EVF when only the asymptotic form of the \( \eta' \)-

\(^4\)The approximate validity of the asymptotic transition form factors involving the \( \pi- \), \( \eta- \) and \( \eta' \)-mesons compared to the data, observed in Refs. [37,38], also suggests that the coefficient \( B_{2(g)}(\mu_0^2) \) in the \( \eta' \)-meson is bounded by this data. The gluonic components of the \( \pi- \) and the \( \eta \)-meson LCDA are small in any case.
meson LCDA ($B_2^{(q)} = 0$ and $B_2^{(g)} = 0$) is taken into account. The yellow (shaded) region demarcates the spectrum with the Gegenbauer coefficients having values in the range of the combined best fit [23]. The blue (solid) curve lying inside this region corresponds to the best fit values [20] obtained from the analysis of the $\Upsilon(1S) \to \eta'X$ decay only. From Fig. 8 it is seen that the rapidly falling phenomenological EVF with $H_0 = 1.7$ GeV$^{-1}$ gives a harder $\eta'$-meson energy spectrum for the large energy $z$ region compared to the CLEO data. The spectrum with the asymptotic form of the $\eta'$-meson LCDA is well correlated with the experimental point in the bin $0.7 \leq z \leq 0.8$, but overestimates the data in the other two bins. It should be noted that the last two bins (especially the bin $0.8 \leq z \leq 0.9$) are very uncertain in the current data, which has to be statistically improved to draw sharper conclusions.

Figure 8: Energy spectrum of the $\eta'$-meson in the decay $\Upsilon(1S) \to \eta'X$. The upper dotted curve corresponds to a constant value of the function $H(p^2) = H_0 \simeq 1.7$ GeV$^{-1}$, and the curve marked as $H_{as}(p^2)$ corresponds to the asymptotic form of the $\eta'$-meson LCDA (i.e., $B_2^{(q)} = 0$ and $B_2^{(g)} = 0$). The spectrum with the Gegenbauer coefficients in the combined best-fit range of these parameters is shown by the yellow (shaded) region. The blue (solid) curve inside this region corresponds to the best fit values of the Gegenbauer coefficients from the analysis of the $\Upsilon(1S) \to \eta'X$ data alone.
4 Summary

We have calculated the $\eta'$-meson energy spectrum in the decay $\Upsilon(1S) \to \eta'ggg \to \eta'X$ in leading order perturbative QCD in the static quark limit for the Orthoquarkonium. Assuming some phenomenological vertex functions, our results are in agreement with the ones obtained earlier in Ref. [22]. In our analysis, the $\eta'$-meson is described in the leading-twist (twist-two) accuracy in terms of the quark-antiquark and gluonic LCDAs for which the asymptotic and the first non-asymptotic terms are taken into account. In this approximation, the $\eta'g^*g$ EVF depends essentially on the Gegenbauer coefficients $B_2^{(q)}(\mu_0^2)$ and $B_2^{(g)}(\mu_0^2)$. They are determined from the CLEO data on $\Upsilon(1S) \to \eta'X$ in the large-$z$ region ($z \geq 0.7$) of the $\eta'$-meson energy spectrum [35], which is well explained by our perturbative QCD approach. However, the resulting $1\sigma$ contour in the Gegenbauer coefficients have a large dispersion. Combining the analysis of the $\Upsilon(1S) \to \eta'X$ data with an earlier analysis of the $(\eta', \eta) - \gamma$ transition factors in a similar theoretical framework [25], and requiring additionally that the vertex function $F_{\eta'g}(p^2)$ remains positive definite in the entire $p^2 > m_{\eta'}^2$ region, yield much improved determination of the Gegenbauer coefficients, yielding $B_2^{(q)}(2\text{ GeV}^2) = -0.008 \pm 0.054$ and $B_2^{(g)}(2\text{ GeV}^2) = 4.6 \pm 2.5$. Our analysis improves the phenomenological profile of the LCDAs in the $\eta'$-meson, and in turn yields a better determination of the vertex function $F_{\eta'g}(p^2)$ compared to the earlier estimates of the same. The resulting function (the $\eta'$ - gluon transition form factor) is presented in this paper including the $\eta'$-meson mass effects.

5 Acknowledgments

We would like to thank Alex Kagan, Christoph Greub, Peter Minkowski and Sheldon Stone for numerous helpful discussions. We also thank Gustav Kramer for his comments on the earlier version of this manuscript. The work of A.Ya.P. has been supported by the Schweizerischer Nationalfonds.

A Matrix Element Squared for the Decay $\Upsilon(1S) \to ggg^*(g^* \to \eta'g)$

In Sec. 2 we have displayed explicitly the terms $\mathcal{M}_i$ ($i = 1, 2, 3$) in the decomposition of the total amplitude of the decay $\Upsilon(1S) \to \eta'ggg$ in the leading-order perturbative QCD [Eqs. (10)–(12)]. Squaring this amplitude results in very lengthy expressions. Nevertheless, it is possible to present compact analytical expressions for some parts of the matrix element squared. In particular, the square of the term $\mathcal{M}_1$, summed over the polarizations and colours of the final gluons and averaged over the polarization states of the $\Upsilon$-meson, can be written as follows:

$$\frac{1}{3} \sum |\mathcal{M}_1|^2 = \frac{512}{3} C_F B_F \alpha_s^3(M^2) \frac{|\psi(0)|^2}{M} \left| F_{\eta'g}(p_1^2) \right|^2 \left( \frac{J_{11}(\mathcal{P}, k_i; p_i)}{(\mathcal{P}k_2)^2(\mathcal{P}k_3)^2(p_1^0p_1^1)^2(p_1^2)^2} \right) \right. \tag{24}$$
where the well-known property of the symmetrical constants $d_{ABC}$ of the colour $SU(N_c)$-group has been used:

$$ \sum_{C,D=1}^{N_c^2-1} d_{ACD} d_{BCD} = \frac{N_c^2 - 4}{N_c} \delta_{AB} \equiv 2B_F \delta_{AB}, \quad (25) $$

and the dynamical function $J_{11}(P, k_i, p_i)$ has the form:

$$ J_{11}(P, k_i, p_i) = 4M^2(k_1p_1)^2 \left[(k_2k_3)^2(Pp_1)^2 + (k_2p_1)^2(Pk_3)^2 + (k_3p_1)^2(Pk_2)^2\right] $$

$$ + 4M^2(p_1^2)^2(k_2k_3)^2 \left[(k_1k_2)^2 + (k_1k_3)^2\right] $$

$$ + 8M^2p_1^2(p_1k_1)^2(k_2k_3) \left[(k_2k_3)^2 - (k_2p_1)(k_3p_1)\right] $$

$$ - 8M^2p_1^2(p_1k_1)(k_2k_3)^2 \left[(k_1k_2)(k_2p_1) + (k_1k_3)(k_3p_1)\right] $$

$$ + 2M^2p_1^2(M^2 - p_1^2) \left\{ \left[(k_1k_2)(k_3p_1) - (k_1k_3)(k_2p_1)\right]^2 - (p_1k_1)^2(k_2k_3)^2\right\} $$

$$ + 2p_1^2(M^2 - p_1^2) \left[M^2(k_2k_3) + 2(Pk_2)(Pp_1)\right] $$

$$ \times \left[p_1^2(k_1k_2)(k_2k_3) - (k_2p_1)(k_1k_2)(k_3p_1) - (k_1p_1)(k_1k_3)(k_2p_1)\right]. \quad (26) $$

It is easy to check that this function is invariant under the interchange $(k_2, p_2) \leftrightarrow (k_3, p_3)$, which reflects the Bose symmetry for the set of the Feynman diagrams considered. The other two dynamical functions $J_{22}(P, k_i, p_i)$ and $J_{33}(P, k_i, p_i)$ originating from the squares of the terms $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively, can be obtained from $J_{11}(P, k_i, p_i)$ by the obvious replacements: $(k_1, p_1) \leftrightarrow (k_2, p_2)$ and $(k_1, p_1) \leftrightarrow (k_3, p_3)$.

The differential width of the $1 \to 4$ decay has a non-trivial dependence on five variables (three angles can be integrated out trivially) in the rest frame of the decaying particle. In the Monte Carlo generator, the matrix element squared is rewritten in terms of the following dimensionless variables:

$$ x_i = \frac{2(Pk_i)}{M^2}, \quad y_i = \frac{p_i^2}{M^2}, \quad i = 1, 2, 3, \quad (27) $$

which satisfy the relation:

$$ x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 2 + z_0^2/4, \quad (28) $$

where $z_0 = 2m_{\eta'}/M$ is the relative mass of the $\eta'$-meson.

The experimentally measured quantity is the $\eta'$-meson energy spectrum – the number of $\Upsilon \to \eta'X$ events $\Delta n_i$ inside the energy bin with the central value $z_i$ and the width $\Delta z$.

This can be calculated theoretically in the Monte Carlo approach, which we have used. In particular, as an input for the calculation of the contribution coming from $|\mathcal{M}_1|^2$, the following equation was used:

$$ \frac{\Delta n_i^{(11)}}{\Delta z} = \int_{z_i - \Delta z/2}^{z_i + \Delta z/2} dz \delta(z - 2E_{\eta'}/M) \int \frac{dk_1}{2\omega_1} \frac{dk_2}{2\omega_2} \frac{dk_3}{2\omega_3} \frac{dp_{\eta'}}{2E_{\eta'}} \delta^{(4)}(P - k_1 - k_2 - k_3 - p_{\eta'}) $$

$$ \times \left[ MF_{\eta'g}(y_1) \right]^2 \frac{\mathcal{F}_{11}(x_i, y_i, \mu)}{64\pi^5(\pi^2 - 9)M^4 \left[y_1x_2x_3(2 - 2y_1 - x_2 - x_3)\right]^2}, \quad (29) $$
where the representation \( |\mathcal{M}_1|^2 \) was put into the definition of \( dn/dz \) \((17)\). The dimensionless functions \( F_{11}(x, y, \mu) \) is related to the functions \( J_{11}(P, k_i, p_i) \) as follows:

\[
J_{11}(P, k_i, p_i) = \frac{M^{14}}{128} F_{11}(x, y, \mu) . \tag{30}
\]

The first integral in Eq. \((29)\) serves for a selection of the generated Monte Carlo events with the \( \eta' \)-meson energy being inside the bin \( z_i - \Delta z/2 < z < z_i + \Delta z/2 \).

In the same way all the other contributions from the matrix element squared were implemented in the numerical analysis. In particular, the contribution to the energy spectrum coming from the product \((1/3) \sum \mathcal{M}_1^* \mathcal{M}_2 + \text{c.c.} \) can be written as:

\[
\frac{\Delta n_{12}^{(12)}}{\Delta z} = \int_{z_i - \Delta z/2}^{z_i + \Delta z/2} dz \, \delta(z - 2E_{\eta'}/M) \int \frac{dk_1}{2\omega_1} \frac{dk_2}{2\omega_2} \frac{dk_3}{2\omega_3} \frac{dP_{\eta'}}{2E_{\eta'}} \delta^{(4)}(P - k_1 - k_2 - k_3 - p_{\eta'})
\times \frac{M^2 F_{\eta'g}(y_1) F_{\eta'g}(y_2)}{64\pi^5(\pi^2 - 9)M^4} \frac{2F_{12}(x, y, \mu)}{y_1 y_2 x_1 x_2 x_3^3 (2 - 2y_1 - x_2 - x_3)(2 - 2y_2 - x_1 - x_3)}. \tag{31}
\]

However, we have not been able to find compact expressions for the non-diagonal terms \((1/3) \sum \mathcal{M}_i \mathcal{M}_j^* (i \neq j)\). We present here instead the dimensionless function \( F_{12}(x, y, \mu) \) in a form of a series in powers of \( z_0^2/4 = m_{\eta'}/M^2 \):

\[
F_{12}(x, y, z_0) = \sum_{k=0}^5 f_{12}^k(x, y) \left( \frac{z_0^2}{4} \right)^k . \tag{32}
\]

where functions \( f_{12}^k(x, y) \) are as follows:

\[
f_{12}^0 = \frac{1}{4} \left\{ (\Delta x_{12})^4 (\Delta y_{12})^2 - 4(\Delta x_{12})^3 (\Delta y_{12})^3 + 4(\Delta x_{12})^3 \Delta y_{12} y_3^2 \right. \tag{33}
+ 8(\Delta x_{12})^3 \Delta y_{12} y_3 (x_{12} + x_3 - 2) + 4(\Delta x_{12})^2 \Delta y_{12} (x_{12}^2 + 2x_{12} x_3 - 4x_{12} + x_3^2 - 4x_3 + 4)
+ 5(\Delta x_{12})^2 (\Delta y_{12})^2 + 2(\Delta x_{12}) (\Delta y_{12})^2 y_3(-2x_{12} - 4x_3 + 7)
+ 2(\Delta x_{12})^2 (\Delta y_{12})^2 (-x_{12}^2 - 6x_{12} x_3 + 7x_{12} - 2x_3^2 + 9x_3 - 8)
- 2\Delta x_{12} (\Delta y_{12})^5 + 4\Delta x_{12} (\Delta y_{12})^3 y_3^2 (-x_3 - 2)
+ 4\Delta x_{12} (\Delta y_{12})^3 y_3 (-x_{12} x_3 - 2x_{12} + 2x_3) + 4\Delta x_{12} (\Delta y_{12})^3 x_{12}(-x_{12} + x_3 + 1)
+ 2\Delta x_{12} (\Delta y_{12})^3 (2x_3 + 5) + 4\Delta x_{12} (\Delta y_{12})^3 (3x_{12} + 2x_3 + 2x_3^2 - 12x_3 + 12)
+ 4\Delta x_{12} (\Delta y_{12})^3 (3x_{12} + 4x_3 x_3^2 - 17x_{12} x_3 + 19x_{12} + x_3^3 + 11x_3^3 - 36x_3 + 36)
+ 4\Delta x_{12} (\Delta y_{12})^3 (3x_{12} x_3 + 2x_3 x_3^2 - 12x_3 x_3 + 14x_3^2 + 12x_3 x_3 + 16x_3 x_3^2
+ 54x_{12} x_3 - 52x_{12} - 4x_3^3 + 28x_3^2 - 64x_3 + 48)
+ 2\Delta x_{12} (\Delta y_{12})^3 (-x_{12}^4 - 6x_{12}^3 + 14x_{12}^2 - 8x_{12} x_3^2 + 44x_{12}^2 x_3 - 56x_{12}^2 - 2x_{12} x_3^3
+ 30x_{12} x_3^2 - 96x_{12} x_3 + 88x_{12} + x_3^4 - 24x_3^2 + 64x_3 - 48)
+ 4(\Delta y_{12})^3 y_3 + 2(\Delta y_{12})^3 x_3 (3x_{12} - 1)
+ (\Delta y_{12})^4 y_3^2 + 2(\Delta y_{12})^4 x_3 (3x_{12} - 1)
+ (\Delta y_{12})^4 (3x_{12} + 2x_{12} x_3 - 6x_{12} + 2x_3^2 - 6x_3 + 4)
\right\}.
\[-8(\Delta y_{12})^2 y_3^4 + 4(\Delta y_{12})^2 y_3^3(x_{12} x_3 - 4x_{12} + 2x_3^2 + 2x_3 - 7) + 4(\Delta y_{12})^2 y_3^2(2x_{12} x_3 - 2x_{12}^2 + 3x_{12} x_3^2 - 4x_{12} x_3 - 7x_{12} - 9x_3^2 - 3x_3 + 20) + 2(\Delta y_{12})^2 y_3(2x_{12} x_3 - 2x_{12}^3 + 2x_{12}^2 x_3^2 - 10x_{12} x_3^3 - 7x_{12}^2 - 20x_{12} x_3^2 - 4x_{12} x_3 + 48x_{12} + 10x_3^2 + 32x_3 - 52) + (\Delta y_{12})^2(x_{12}^4 - 14x_{12}^3 - 12x_{12}^2 x_3^2 - 10x_{12} x_3^3 + 56x_{12}^2 - 8x_{12} x_3^3 + 36x_{12} x_3^2 + 32x_{12} x_3 - 88x_{12} - 4x_3^4 + 20x_{12}^3 - 32x_3^2 - 24x_3 + 48) - (\Delta x_{12})^4 y_3^2 + 2(\Delta x_{12})^4 y_3(-x_{12} - x_3 + 2) + (\Delta x_{12})^4(-x_{12}^2 - 2x_{12} x_3 + 4x_{12} - x_3^2 + 4x_3 - 4) + 7(\Delta x_{12})^2 y_3^4 + 2(\Delta x_{12})^2 y_3^2(8x_{12} + 10x_3 - 19) + 2(\Delta x_{12})^2 y_3^2(6x_{12} + 20x_{12} x_3 - 33x_{12} + 11x_3^2 - 43x_3 + 40) + 2(\Delta x_{12})^2 y_3^2(3x_{12} + 16x_{12} x_3 - 21x_{12}^2 + 20x_{12} x_3^2 - 70x_{12} x_3 + 56x_{12} + 6x_3^3 - 37x_3^2 + 72x_3 - 44) + \Delta x_{12}^2(12x_{12}^4 + 12x_{12} x_3^3 - 14x_{12}^3 + 24x_{12}^2 x_3^2 - 78x_{12} x_3 + 56x_{12}^3 + 16x_{12} x_3^3 - 90x_{12} x_3^2 + 160x_{12} x_3 - 88x_3 + 3x_3^4 - 26x_3^2 + 80x_3^2 - 104x_3 + 48) + 4y_3^4 - 2y_3^5(-2x_{12} x_3 + 5x_3 - 4x_3^2 - 4x_3 + 15) + y_3^4(-16x_{12}^2 + 9x_3^2 - 36x_{12} x_3^2 + 38x_{12} x_3 + 50x_{12} - 16x_3^2 + 82x_3^2 - 14x_3 - 84) + 2y_3^4(-12x_{12} x_3 + 2x_{12}^3 - 30x_{12} x_3^2 + 56x_{12} x_3 + 15x_{12}^2 - 22x_{12} x_3 + 126x_{12} x_3^2 - 68x_{12} x_3 - 64x_{12} - 4x_3^4 + 52x_3^2 - 126x_3^2 + 24x_3 + 52) + y_3^4(-16x_{12}^4 + x_3^4 - 44x_{12}^2 x_3^2 + 104x_{12} x_3 + 10x_{12}^3 - 40x_{12}^2 x_3^2 + 298x_{12} x_3^2 + 250x_{12} x_3 - 64x_{12}^3 - 12x_{12} x_3^3 + 228x_{12} x_3^2 - 644x_{12} x_3 + 288x_{12} x_3 + 104x_{12} + 40x_3^4 - 260x_3^3 + 456x_3^2 - 152x_3 - 48) + 2y_3 x_3(-2x_{12}^5 - 6x_{12} x_3^2 + 19x_{12}^4 - 6x_{12} x_3^2 + 66x_{12} x_3 - 74x_{12}^3 - 2x_{12} x_3^3 - 68x_{12} x_3^2 - 227x_{12} x_3 + 144x_{12}^2 + 21x_{12} x_3^2 - 156x_{12} x_3^2 + 304x_{12} x_3 - 136x_{12} - 17x_3^3 + 88x_3^2 - 132x_3 + 48) + x_3^2(7x_4^4 + 16x_4 x_3^2 - 46x_{12} x_3^2 + 13x_{12} x_3^2 - 78x_{12} x_3 + 112x_{12}^3 + 6x_4 x_3^2 - 46x_{12} x_3 + 120x_{12} + 2x_3 - 14x_3^2 + 44x_3^2 - 72x_3 + 48) \right) ,

f_{12}^1 = \frac{1}{2} \left\{ -8(\Delta x_{12})^3 \Delta y_{12} y_3 + 4(\Delta x_{12})^3 \Delta y_{12} (-x_{12} - x_3 + 2) + 2(\Delta x_{12})^2 (\Delta y_{12})^2 y_3 (x_3 + 7) + (\Delta x_{12})^2 (\Delta y_{12})^2 (2x_{12} x_3 + 4x_{12} - 2x_3 - 9) - 6(\Delta x_{12}) (\Delta y_{12})^3 y_3 + 2(\Delta x_{12}) (\Delta y_{12})^3 (x_3 + 2) - 2(\Delta x_{12}) (\Delta y_{12}) y_3^2 (2x_3 + 3) + 2(\Delta x_{12}) y_3 y_3^2 (-4x_{12} x_3 - 2x_3 - 2x_3^3 + 7x_3 - 10) + 2(\Delta x_{12}) y_3 y_3^2 (-2x_{12} x_3 - 3x_3^2 - 2x_{12} x_3^2 + 2x_{12} x_3 + 2x_{12} - x_3^2 + 8x_3) + 2(\Delta x_{12}) y_3 (-2x_{12}^2 + x_{12} x_3 + 8x_3 - 4x_{12} x_3^2 - 2x_{12} x_3 - 12x_{12} + x_3 - 2x_3^2 - 4x_3 + 8) + (\Delta y_{12})^4 (x_{12} + 1) + 8(\Delta y_{12})^2 y_3^3 + 2(\Delta y_{12})^2 y_3^2 (-x_{12} x_3 + 9x_{12} + 2x_3 + 5) + 4(\Delta y_{12})^2 y_3^2 (-x_{12}^2 x_3 + 2x_3^3 + 2x_{12} x_3 - 2x_{12} - 6x_3) + (\Delta y_{12})^2 (-2x_{12} x_3 + 4x_{12}^2 + 6x_{12} x_3 - 15x_{12} - 6x_{12} x_3^2 + 16x_{12} - 4x_3^3)}
\[ f_{12}^2 = \frac{1}{4} \left\{ - 4 (\Delta x_{12})^3 y_{12} - 4 (\Delta x_{12})^2 (\Delta y_{12})^2 + 4 \Delta x_{12} (\Delta y_{12})^3 \right\}^{(35)} \\
+ 4 \Delta x_{12} \Delta y_{12} y_3 (x_3 - 2) + 4 \Delta x_{12} \Delta y_{12} y_3 (x_{12} - 2 x_{12} - 6 x_3 + 8) \\
+ 4 \Delta x_{12} \Delta y_{12} (3 x_{12}^2 - x_{12} x_3 - x_{12} + 4 x_3 - 4) - 8 (\Delta y_{12})^2 y_3^2 - 12 (\Delta y_{12})^2 y_3 (2 x_{12} + 1) \\
+ 4 (\Delta y_{12})^2 (2 x_{12}^2 - 8 x_{12} - 3 x_3 + 5 x_3) + 3 (\Delta x_{12})^4 + 2 (\Delta x_{12})^2 y_3^2 (4 x_3 + 23) \\
+ 2 (\Delta x_{12})^2 y_3 (8 x_{12} x_3 + 20 x_{12} + 4 x_3 - 6 x_3 - 17) \\
+ 2 (\Delta x_{12})^2 (4 x_{12}^2 + 4 x_{12} x_3 - 9 x_{12} + 5 x_3 - 27 x_3 + 16) \\
+ 24 y_3^4 + 4 y_3^2 (-3 x_{12} x_3 + 17 x_{12} - 3 x_3 + 23) \\
+ 2 y_3^2 (-16 x_{12} x_3 + 27 x_{12} - 10 x_{12} x_3 + 46 x_{12} x_3 + 22 x_{12} + 22 x_3 - 50 x_3 - 52) \\
+ 2 y_3 (-14 x_{12} x_3 + 6 x_{12}^2 - 10 x_{12} x_3 + 44 x_{12} x_3 - 7 x_{12} + 18 x_{12} x_3 - 12 x_{12} x_3 \\
- 32 x_{12} + 10 x_3 - 56 x_3 + 44) \\
- 8 x_{12}^3 x_3 + x_{12}^2 - 8 x_{12}^3 x_3 + 8 x_{12} x_3 + 10 x_3^3 + 18 x_{12} x_3 + 86 x_{12} x_3 - 64 x_{12} \\
+ 28 x_{12} x_3^3 - 52 x_{12} x_3^2 - 176 x_{12} x_3 + 104 x_{12} + 12 x_3^4 - 68 x_3^3 + 88 x_3^3 + 72 x_3 - 34) \\
\]

\[ f_{12}^3 = \frac{1}{2} \left\{ 2 (\Delta y_{12})^2 (x_{12} + 1) + 2 \Delta y_{12} \Delta x_{12} (3 y_3 + 2 x_{12} - x_3 - 2) \right\}^{(36)} \\
- 2 (\Delta x_{12})^2 y_3 (x_3 + 3) + (\Delta x_{12})^2 (-2 x_{12} x_3 - 6 x_{12} + 8 x_3 + 1) \\
- 8 y_3 + 2 y_3 (x_{12} x_3 - 13 x_{12} - 2 x_3 - 11) + 2 y_3 (2 x_{12}^2 x_3 - 9 x_{12} - 8 x_{12} x_3 + 20 x_3 + 4) \\
+ 2 x_{12}^3 x_3 - 2 x_{12}^2 - 12 x_{12} x_3 + 11 x_{12}^2 + 6 x_{12} x_3 + 24 x_{12} x_3 - 16 x_{12} + 4 x_3^3 - 22 x_3^3 + 4 \right\} \\

\[ f_{12}^4 = \frac{1}{4} \left\{ 3 (\Delta x_{12})^2 - 2 \Delta y_{12} \Delta x_{12} + 4 y_3^2 + 2 y_3 (9 x_{12} + 7) \right\}^{(37)} \\
+ 9 x_{12}^2 + 2 x_{12} x_3 + 2 x_{12} + 2 x_3^3 - 14 x_3 - 4 \right\} \\

19
\[ f_{12}^5 = -\frac{1}{2}(x_{12} + 1), \]  

where \( x_{12} = x_1 + x_2 \), \( \Delta x_{12} = x_1 - x_2 \), and \( \Delta y_{12} = y_1 - y_2 \). It is easy to see that the equations for \( f_{12}^5 \) are symmetric under the interchange \((x_1, y_1) \leftrightarrow (x_2, y_2)\).

The expressions for the other \( F_{ij}(x_i, y_i, \mu) \) can be derived from the ones given above by using the Bose symmetry.

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