On the structure of formal balls of the balanced quasi-metric domain of words

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Abstract

In “Denotational semantics for programming languages, balanced quasi-metrics and fixed points” (International Journal of Computer Mathematics 85 (2008), 623-630), J. Rodríguez-López, S. Romaguera and O. Valero introduced and studied a balanced quasi-metric on any domain of (finite and infinite) words, denoted by $q_b$. In this paper we show that the poset of formal balls associated to $q_b$ has the structure of a continuous domain.

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1 Introduction and preliminaries

Throughout this paper the symbols $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all non-negative real numbers and the set of all positive integer numbers, respectively.

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Our basic references for quasi-metric spaces are [5, 11], for general topology it is [8] and for domain theory is [9].

In 1998, Edalat and Heckmann [7] established an elegant connection between the theory of metric spaces and domain theory by means of the notion of a (closed) formal ball.

Let us recall that a formal ball for a set $X$ is simply a pair $(x, r)$, where $x \in X$ and $r \in \mathbb{R}^+$. The set of all formal balls for $X$ is denoted by $B_X$.

Edalat and Heckmann observed that, given a metric space $(X, d)$, the relation $\sqsubseteq_d$ defined on $B_X$ as

$$(x, r) \sqsubseteq_d (y, s) \Leftrightarrow d(x, y) \leq r - s,$$

for all $(x, r), (y, s) \in B_X$, is a partial order on $B_X$. Thus $(B_X, \sqsubseteq_d)$ is a poset.

In particular, they proved the following.

**Theorem 1** ([7]). For a metric space $(X, d)$ the following are equivalent:

1. $(X, d)$ is complete.
2. $(B_X, \sqsubseteq_d)$ is a dcpo.
3. $(B_X, \sqsubseteq_d)$ is a continuous domain.

Later on, Aliakbari et al. [1], and Romaguera and Valero [20] studied the extension of Edalat-Heckmann’s theory to the framework of quasi-metric spaces.

Let us recall that a quasi-metric space is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions for all $x, y, z \in X$:

1. $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$;
2. $d(x, y) \leq d(x, z) + d(z, y)$.

The function $d$ is said to be a quasi-metric on $X$.

If the quasi-metric $d$ satisfies for all $x, y \in X$ the condition

1. $x = y \Leftrightarrow d(x, y) = 0$,

then $d$ is called a $T_1$ quasi-metric and the pair $(X, d)$ is said to be a $T_1$ quasi-metric space.

If $d$ is a quasi-metric on a set $X$, then function $d^s$ defined as $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on $X$.

Next we recall some notions and properties of domain theory which will useful later on.
A partially ordered set, or poset for short, is a (non-empty) set \( X \) equipped with a (partial) order \( \sqsubseteq \). It will be denoted by \((X, \sqsubseteq)\) or simply by \( X \) if no confusion arises.

A subset \( D \) of a poset \( X \) is directed provided that it is non-empty and every finite subset of \( D \) has upper bound in \( D \).

A poset \( X \) is said to be directed complete, and is called a dcpo, if every directed subset of \( X \) has a least upper bound. The least upper bound of a subset \( D \) of \( X \) is denoted by \( \sqcup D \) if it exists.

Let \( X \) be a poset and \( x, y \in X \); we say that \( x \) is way below \( y \), in symbols \( x \ll y \), if for each directed subset \( D \) of \( X \) having least upper bound \( \sqcup D \), the relation \( y \sqsubseteq \sqcup D \) implies the existence of some \( u \in D \) with \( x \sqsubseteq u \).

A poset \( X \) is called continuous if for each \( x \in X \), the set \( \downarrow x := \{ y \in X : y \ll x \} \) is directed with least upper bound \( x \).

A continuous poset which is also a dcpo is called a continuous domain or, simply, a domain.

In the sequel we shall denote by \( \Sigma \) a non-empty alphabet and by \( \Sigma^\infty \) the set of all finite and infinite words (or strings) on \( \Sigma \). We assume that the empty word \( \phi \) is an element of \( \Sigma^\infty \), and denote by \( \sqsubseteq \) the prefix order on \( \Sigma^\infty \). In particular, if \( x \sqsubseteq y \) and \( x \neq y \), we write \( x \sqsubset y \). For each \( x, y \in \Sigma^\infty \) we denote by \( x \sqcap y \) the longest common prefix of \( x \) and \( y \), and for each \( x \in \Sigma^\infty \) we denote by \( \ell(x) \) the length of \( x \). In particular, \( \ell(\phi) = 0 \).

It is well known that \( \Sigma^\infty \) endowed with the prefix order has the structure of a domain.

Usually it is defined a distinguished complete metric \( d_B \) on \( \Sigma^\infty \), the so-called Baire metric (or Baire distance), which is given by

\[
d_B(x, x) = 0 \text{ for all } x \in \Sigma^\infty, \quad \text{and} \quad d_B(x, y) = 2^{-\ell(x \sqcap y)} \text{ for all } x, y \in \Sigma^\infty \text{ with } x \neq y.
\]

(We adopt the convention that \( 2^{-\infty} = 0 \)).

Observe that \((B\Sigma^\infty, \sqsubseteq d_B)\) is also a domain by Theorem 1 above.

Recall that the classical Baire metric (or Baire distance) provides a suitable framework to obtain denotational models for programming languages and parallel computation \[2, 3, 4, 10\] as well as to study the representation of real numbers by means of regular languages \[13\]. However, the Baire metric is not able to decide if a word \( x \) is a prefix of another word \( y \), or not, in general. In order to avoid this disadvantage, some interesting and useful quasi-metric modifications of the Baire metric has been constructed. For instance:
(A) The quasi-metric $d_w$ defined on $\Sigma^\infty$ as (compare \cite{12, 15, 18, 20, etc.})
$$d_w(x, y) = 2^{-\ell(x \cap y)} - 2^{-\ell(x)} \text{ for all } x, y \in \Sigma^\infty.$$

(B) The quasi-metric $d_0$ defined on $\Sigma^\infty$ as (compare \cite{12, 16, 20, 21, etc.})
$$d_0(x, y) = 0 \text{ if } x \text{ is a prefix of } y,$$
$$d_0(x, y) = 2^{-\ell(x \cap y)} \text{ otherwise.}$$

(C) The $T_1$ quasi-metric $q_b$ defined on $\Sigma^\infty$ as (compare \cite{16})
$$q_b(x, y) = 2^{-\ell(x)} - 2^{-\ell(y)} \text{ if } x \text{ is a prefix of } y,$$
$$q_b(x, y) = 1 \text{ otherwise.}$$

Observe that in Examples (A) and (B) above, the fact that a word $x$ is a prefix of another word $y$ is equivalent to say that the distance from $x$ to $y$ is exactly zero, so this condition can be used to distinguish between the case that $x$ is a prefix of $y$ and the remaining cases for $x, y \in \Sigma^\infty$. Observe also that $(d_0)^s$ coincides with the Baire metric while $(d_w)^s$ does not.

Nevertheless, if $x, y, z \in \Sigma^\infty$ satisfy $x \sqsubseteq y \sqsubseteq z$, one obtains $d_w(x, z) = d_w(y, z) = d_0(x, z) = d_0(y, z) = 0$, and it is not possible to decide which word of the two, $x$ or $y$, provides a better approximation to $z$. The quasi-metric $q_b$ as constructed in (C) saves this inconvenience because if $x \sqsubseteq y \sqsubseteq z$, it follows that $\ell(x) < \ell(y) < \ell(z)$, and thus $q_b(y, z) < q_b(x, z)$. Moreover, for $x \neq \phi$, $x$ is a prefix of $y$ if and only if $q_b(x, y) < 1$, so this condition also allows us to distinguish between the case that $x$ is a prefix of $y$ and the rest of cases (see \cite{16} Remark 3). We also point out that, contrarily to $d_w$ and $d_0$, the quasi-metric $q_b$ has rich topological and distance properties; in particular, it is a balanced quasi-metric in the sense of Doitchinov \cite{6}, and consequently its induced topology is Hausdorff and completely regular \cite{16} Theorem 1 and Remark 4.

By \cite{19} Theorem 3.1 (see also \cite{20} p. 461), $(\mathcal{B}\Sigma^\infty, \sqsubseteq_{d_w})$ is a domain. On the other hand, it was shown in \cite{20} Example 3.1 that $(\mathcal{B}\Sigma^\infty, \sqsubseteq_{d_0})$ is a domain. In the light of these results, it seems natural to wonder if $(\mathcal{B}\Sigma^\infty, \sqsubseteq_{q_b})$ is also a domain. Here we show that, indeed, this is the case.

2 The results

In the rest of the paper, given a quasi-metric space $(X, d)$, the way below relation associated to $\sqsubseteq_d$ will be denoted by $\ll_d$. 

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Lemma 1 ([1]). For any quasi-metric space \((X, d)\) the following holds:

\[(x, r) \ll_d (y, s) \Rightarrow d(x, y) < r - s.\]

Lemma 2. Let \((X, d)\) be a quasi-metric space. If there is \((x, r) \in BX\) such that \((x, r + s) \ll_d (x, r)\) for all \(s > 0\), then \(\downarrow (x, r)\) is directed and \((x, r) = \sqcup \downarrow (x, r)\).

Proof. Obviously \(\downarrow (x, r) \neq \emptyset\). Now let \((y, s), (z, t) \in BX\) such that \((y, s) \ll_d (x, r)\) and \((z, t) \ll_d (x, r)\). By Lemma 1, \(d(y, x) < s - r - \varepsilon\) and \(d(z, x) < t - r - \varepsilon\) for some \(\varepsilon > 0\). Thus \((y, s) \sqsubseteq_d (x, r + \varepsilon)\) and \((z, t) \sqsubseteq_d (x, r + \varepsilon)\). Since \((x, r + \varepsilon) \in \downarrow (x, r)\), we conclude that \(\downarrow (x, r)\) is directed.

Finally, let \((z, t)\) be an upper bound of \(\downarrow (x, r)\). In particular, we have that \((x, r + 1/n) \sqsubseteq_d (z, t)\) for all \(n\), so \(d(x, z) \leq r - t + 1/n\) for all \(n\). Hence \(d(x, z) \leq r - t\), i.e., \((x, r) \sqsubseteq_d (z, t)\). Consequently \((x, r) = \sqcup \downarrow (x, r)\).

A net \((x_\alpha)_{\alpha \in \Lambda}\) in a quasi-metric space \((X, d)\) is called left K-Cauchy ([17, 22] (or simply, Cauchy [13]) if for each \(\varepsilon > 0\) there is \(\alpha_\varepsilon \in \Lambda\) such that \(d(x_\alpha, x_\beta) < \varepsilon\) whenever \(\alpha_\varepsilon \leq \alpha \leq \beta\). The notion of a left K-Cauchy sequence is defined in the obvious manner.

Let \((X, d)\) be a quasi-metric space. An element \(x \in X\) is said to be a Yoneda-limit of a net \((x_\alpha)_{\alpha \in \Lambda}\) in \(X\) if for each \(y \in X\), we have \(d(x, y) = \inf_{\alpha} \sup_{\beta \geq \alpha} d(x_\beta, y)\). Recall that the Yoneda-limit of a net is unique if it exists.

A quasi-metric space \((X, d)\) is called Yoneda-complete if every left K-Cauchy net in \((X, d)\) has a Yoneda-limit, and it is called sequentially Yoneda-complete if every left K-Cauchy sequence in \((X, d)\) has a Yoneda-limit.

Lemma 3 ([20, Proposition 2.2]). A \(T_1\) quasi-metric space is Yoneda-complete if and only if it is sequentially Yoneda-complete.

Proposition 1. The quasi-metric space \((\Sigma^\infty, q_b)\) is Yoneda-complete.

Proof. Since \((\Sigma^\infty, q_b)\) is a \(T_1\) quasi-metric space it suffices to show, by Lemma 3, that it is sequentially Yoneda-complete. To this end, let \((x_n)_{n \in \mathbb{N}}\) be a left K-Cauchy sequence in \((\Sigma^\infty, q_b)\). Then, there is \(n_1 \in \mathbb{N}\) such that \(q_b(x_n, x_m) < 1\) whenever \(n_1 \leq n \leq m\). So, \(x_n\) is a prefix of \(x_m\), i.e., \(x_n \sqsubseteq x_m\), whenever \(n_1 \leq n \leq m\).
Now we distinguish two cases.

Case 1. There exists \( n_0 \geq n_1 \) such that \( x_n = x_{n_0} \) for all \( n \geq n_0 \). Then, it is clear that
\[
q_b(x_{n_0}, y) = \inf_n \sup_{m \geq n} q_b(x_m, y).
\]
for all \( y \in X \).

Case 2. For each \( n \geq n_1 \) there exists \( m > n \) such that \( x_n \sqsubseteq x_m \). In this case, there exists \( x = \sqcup \{ x_n : n \geq n_1 \} \), and \( \ell(x) = \infty \). We shall show that \( x \) is the Yoneda-limit of the sequence \( (x_n)_{n \in \mathbb{N}} \).

Indeed, we first note that \( q_b(x_n, x) = 2^{-\ell(x_n)} \) for all \( n \geq n_1 \), and hence
\[
\sup_{m \geq n} q_b(x_m, x) = \sup_{m \geq n} 2^{-\ell(x_m)} = 2^{-\ell(x_n)},
\]
whenever \( n \geq n_1 \). Therefore
\[
\inf_n \sup_{m \geq n} q_b(x_m, x) = \inf_n 2^{-\ell(x_n)} = 0 = q_b(x, x).
\]

Finally, let \( y \in \Sigma^\infty \) such that \( y \not= x \). Since \( \ell(x) = \infty \) it follows that \( x \) is not a prefix of \( y \), and thus for each \( n \in \mathbb{N} \) there exists \( m \geq \max\{n, n_1\} \) such that \( x_m \) is not a prefix of \( y \), so \( q_b(x_m, y) = 1 \). We conclude that
\[
\inf_n \sup_{m \geq n} q_b(x_m, y) = 1 = q_b(x, y).
\]

This finishes the proof.

**Lemma 4** \( (\mathbb{P}) \). Let \( (X, d) \) be a quasi-metric space.

(a) If \( D \) is a directed subset of \( BX \), then \( (y(y, r))(y, r) \in D \) is a left \( K \)-Cauchy net in \( (X, d) \).

(b) If \( BX \) is a dcpo and \( D \) is a directed subset of \( BX \) having least upper bound \( (z, s) \), then \( s = \inf\{r : (y, r) \in D\} \) and \( z \) is the Yoneda-limit of the net \( (y(y, r))(y, r) \in D \).

(c) If \( (X, d) \) is Yoneda-complete, the poset \( (BX, \sqsubseteq_d) \) is a dcpo.

**Proposition 2.** For each \( x \in \Sigma^\infty \) such that \( \ell(x) < \infty \), each \( u \in \mathbb{R}^+ \) and each \( v > 0 \), we have
\[
(x, u + v) \ll_{q_b} (x, u).
\]

**Proof.** Let \( x \in \Sigma^\infty \) with \( \ell(x) < \infty \), \( u \in \mathbb{R}^+ \) and \( v > 0 \), and let \( D \) be a directed subset of \( (BS^\infty, \sqsubseteq_{q_b}) \) whose least upper bound \( (z, s) \) satisfies
\[(x, u) \sqsubseteq q_b \ (z, s).\] (The existence of least upper bound is guaranteed by Proposition 1 and Lemma 4(c)). We shall show that there exists \((y, r) \in D\) such that \((x, u + v) \sqsubseteq q\ (y, r)\).

We first note that, by Lemma 4 (a), there exists \((y_1, r_1) \in D\) such that \(q_b(y_{(y,r)}, y_{(y',r')} < 1\) whenever \((y, r), (y', r') \in D\) with \((y_1, r_1) \sqsubseteq q_b \ (y, r) \sqsubseteq q_b \ (y', r').\) Therefore, by the definition of \(q_b\), we deduce that \(y_{(y,r)}\) is a prefix of \(y_{(y',r')}\) whenever \((y, r) \sqsubseteq q_b \ (y, r) \sqsubseteq q_b \ (y', r').\)

Furthermore, by Lemma 4 (b), we have \(s = \inf \{r : (y, r) \in D\}\), and there exists \((y_0, r_0) \in D\), with \((y_1, r_1) \sqsubseteq q_b \ (y_0, r_0)\), such that \(y_{(y,r)}\) is a prefix of \(z\) whenever \((y_0, r_0) \sqsubseteq q_b \ (y, r)\).

Now we distinguish two cases.

Case 1. \(x\) is a prefix of \(z\). Since, by assumption, \(\ell(x) < \infty\), there exists \((y, r) \in D\) such that \((y_0, r_0) \sqsubseteq q_b \ (y, r)\), \(r < s + v\), and \(x\) is a prefix of \(y_{(y,r)}\). Then

\[q_b(x, y_{(y,r)}) = 2^{-\ell(x)} - 2^{-\ell(y_{(y,r)})} \leq 2^{-\ell(x)} - 2^{-\ell(z)} = q_b(x, z) \leq u - s < u + v - r,\]

and hence \((x, u + v) \sqsubseteq q_b \ (y, r)\).

Case 2. \(x\) is not a prefix of \(z\). Since, by assumption, \((x, u) \sqsubseteq q_b \ (z, s)\), we deduce that \(q_b(u, z) = 1 \leq u - s\). Choose \((y, r) \in D\) such that \(r < s + v\). Then

\[q_b(x, y_{(y,r)}) \leq 1 \leq u - s < u + v - r,\]

and hence \((x, u + v) \sqsubseteq q_b \ (y, r)\). The proof is complete.

**Theorem.** The poset of formal balls \((B^{\Sigma^\infty}, \sqsubseteq q_b)\) is a domain.

**Proof.** From Proposition 1 and Lemma 4 (c) it follows that the poset \((B^{\Sigma^\infty}, \sqsubseteq q_b)\) is a dcpo, so it is only necessary to prove that is also a continuous poset.

To this end we distinguish two cases.

Case 1. Let \((x, r) \in B^{\Sigma^\infty}\) such that \(\ell(x) < \infty\). By Proposition 2 and Lemma 2, \(\sqsubseteq (x, r)\) is a directed subset of \((B^{\Sigma^\infty}, \sqsubseteq q_b)\) for which \((x, r)\) is its least upper bound.

Case 2. Let \((x, r) \in B^{\Sigma^\infty}\) be such that \(\ell(x) = \infty\). Choose a sequence \((x_n)_{n \in \mathbb{N}}\) of elements of \(\Sigma^\infty\) such that \(\ell(x_n) = n\), \(x_n \sqsubseteq x_{n + 1}\) and \(x_n \sqsubseteq x\) for all \(n \in \mathbb{N}\). By Lemma 4 (a), \((x_n)_{n \in \mathbb{N}}\) is a left K-Cauchy sequence, of distinct elements, in \((\Sigma^\infty, q_b)\), and, by Lemma 4 (b), \(x\) is its Yoneda-limit.
Similarly to the proof of Proposition 2 we shall show that \((x, 2^{-n} + r) \ll_{q_b} (x, r)\) for all \(n \in \mathbb{N}\), which implies, in particular, that \(\downarrow (x, r) \neq \emptyset\).

Indeed, let \(D\) be a directed subset of \((\mathbf{B}^{\Sigma^\infty}, \sqsubseteq_{q_b})\) with least upper bound \((z, t)\) such that \((x, r) \sqsubseteq_{q_b} (z, t)\). Then \(t \leq r\), and, by Lemma 4 (b), \(t = \inf\{s : (y, s) \in D\}\), and there exists \((y_0, s_0) \in D\) such that \(y_{(y,s)}\) is a prefix of \(z\) whenever \((y_0, s_0) \sqsubseteq_{q_b} (y, s)\).

If \(x = z\), from the fact that \(x_n\) is a prefix of \(x\) we deduce the existence of some \((y, s) \in D\) such that \((y_0, s_0) \sqsubseteq_{q_b} (y, s), s < t + 2^{-\ell(y_{(y,s)})}\), and \(x\) is a prefix of \(y_{(y,s)}\). Therefore

\[
q_b(x_n, y_{(y,s)}) = 2^{-n} - 2^{-\ell(y_{(y,s)})} \leq 2^{-n} + t - s \leq 2^{-n} + r - s,
\]

so that \((x, 2^{-n} + r) \sqsubseteq_{q_b} (y, s)\).

If \(x \neq z\) we have \(q_b(x, z) = 1 \leq r - t\). Let \((y, s) \in D\) such that \(s < t + 2^{-n}\). Then

\[
q_b(x_n, y_{(y,s)}) \leq 1 \leq r - t < r + 2^{-n} - s,
\]

so that \((x, 2^{-n} + r) \sqsubseteq_{q_b} (y, s)\).

Next we show that \(\downarrow (x, r)\) is directed. Indeed, let \((y, s), (z, t) \in \mathbf{B}^{\Sigma^\infty}\) be such that \((y, s) \ll_{q_b} (x, r)\) and \((z, t) \ll_{q_b} (x, r)\). Since \((x, 2^{-n} + r)_{n \in \mathbb{N}}\) is an ascending sequence in \((\mathbf{B}^{\Sigma^\infty}, \sqsubseteq_{q_b})\) with least upper bound \((x, r)\), there exists \(k \in \mathbb{N}\) such that \((x_k, 2^{-k} + r)\) is an upper bound of \((y, s)\) and \((z, t)\). From the fact, proved above, that \((x_k, 2^{-k} + r) \ll_{q_b} (x, r)\), we deduce that \(\downarrow (x, r)\) is directed.

Finally, let \((z, t)\) be an upper bound of \(\downarrow (x, r)\). Then \(q_b(x_n, z) \leq 2^{-n} + r - t\) for all \(n \in \mathbb{N}\). Since \(q_b(x, z) = \inf_n \sup_{m \geq n} q_b(x_m, z)\), we deduce that \(q_b(x, z) \leq r - t\), and thus \((x, r) \sqsubseteq_{q_b} (z, t)\). Therefore \((x, r)\) is the least upper bound of \(\downarrow (x, r)\).

We conclude that \((\mathbf{B}^{\Sigma^\infty}, \sqsubseteq_{q_b})\) is a domain.

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