CLASSIFICATION OF 3-DIMENSIONAL COMPLETE RECTIFIABLE STEADY GRADIENT RICCI SOLITONS

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Abstract. Let \((M, g, f)\) be a 3-dimensional complete steady gradient Ricci soliton. Assume that \(M\) is rectifiable, that is, the potential function can be written as \(f = f(r)\), where \(r\) is a distance function. Then, we prove that \(M\) is isometric to (1) a quotient of \(\mathbb{R}^3\), (2) a quotient of \(\Sigma^2 \times \mathbb{R}\), where \(\Sigma^2\) is the Hamilton’s cigar soliton, or (3) the Bryant soliton. In particular, we show that any 3-dimensional complete rectifiable steady gradient Ricci soliton with positive sectional curvature is isometric to the Bryant soliton.

1. Introduction

A Riemannian manifold \((M^n, g, f)\) is called a gradient steady Ricci soliton if there exists a smooth function \(f\) on \(M\) such that
\[
\text{Ric} + \nabla \Delta f = 0,
\]
where \(\text{Ric}\) is the Ricci tensor on \(M\), and \(\nabla \Delta f\) is the Hessian of \(f\). If the potential function \(f\) is constant, then it is called trivial. It is known that any compact gradient steady Ricci soliton is trivial [13].

In dimension 2, it is well understood. In fact, complete steady gradient Ricci solitons have been completely classified (cf. [2]). In particular, it has been shown that the only complete steady gradient Ricci soliton with positive curvature is Hamilton’s cigar soliton \((\Sigma^2, g = dx^2 + dy^2 + \frac{x^2}{1 + x^2 + y^2})\) (cf. [12]).

However, in dimension 3, the classification of complete steady gradient Ricci solitons is still open. It is known that such examples are (1) a quotient of \(\mathbb{R}^3\), (2) a quotient of \(\Sigma^2 \times \mathbb{R}\), where \(\Sigma^2\) is the Hamilton’s cigar soliton, and (3) the Bryant soliton [3]. It has been shown that a complete steady gradient Ricci soliton is either flat or isometric to the Bryant soliton under (1) locally conformally flat by Cao and Chen [5],

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(2) divergence-free Bach tensor by Cao, Catino, Chen, Mantegazza and Mazzieri [6]. Here we remark that the Bach tensor \( B_{ij} = \nabla_k C_{ijk} \), where \( C \) is the Cotton tensor \( C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik} \), and \( A \) is the Schouten tensor \( A = \text{Ric} - \frac{R}{4} g \), where \( R \) is the scalar curvature on \( M \). Recently, in 2013, S. Brendle [4] proved the Perelman’s conjecture [14], namely “any 3-dimensional complete noncompact \( \kappa \)-noncollapsed gradient steady Ricci soliton with positive curvature is rotationally symmetric, namely the Bryant soliton”.

In this paper, to classify 3-dimensional complete steady gradient Ricci solitons, we consider a more general conjecture (See also page 52 in [10]):

**Conjecture 1.** Any 3-dimensional complete steady gradient Ricci soliton is isometric to (1) a quotient of \( \mathbb{R}^3 \), (2) a quotient of \( \Sigma^2 \times \mathbb{R} \), where \( \Sigma^2 \) is the Hamilton’s cigar soliton, or (3) the Bryant soliton.

G. Catino, P. Mastrolia and D. D. Monticelli showed that (1) Any complete steady gradient Ricci soliton with

\[
\liminf_{s \to +\infty} \frac{1}{s} \int_{B_s(O)} R = 0,
\]

is isometric to either a quotient of \( \mathbb{R}^3 \), or a quotient of \( \Sigma^2 \times \mathbb{R} \) (cf. [7]), (2) Any complete steady gradient Ricci soliton with 3-divergence free cotton tensor \( \text{div}^3(C) = \nabla_i \nabla_j \nabla_k C_{ijk} = 0 \) is isometric to either a quotient of \( \mathbb{R}^3 \), or the Bryant soliton (cf. [3]).

Estimates of the potential function played an important role in proving all of the result mentioned above (cf. [5], [6], [4], [7], [8]). Many studies indicate that the potential function is bounded by some functions which are dependent only on a distance function \( r \) on \( M \) (cf. [5], [16]). In particular, Cao and Chen showed the following result:

**Theorem 1.1** ([5]). Let \((M, g, f)\) be a complete steady gradient Ricci soliton with positive Ricci curvature (and Hamilton’s identity \( R + |\nabla f|^2 = c \)). Assume that the scalar curvature attains its maximum at some origin \( O \). Then, there exist some constants \( 0 < c_1 \leq \sqrt{c} \) and \( c_2 > 0 \) such that the potential function \( f \) satisfies the estimates

\[
-\sqrt{c} r(x) - |f(O)| \leq f(x) \leq -c_1 r(x) + c_2.
\]

Therefore, to classify steady gradient Ricci solitons, it is interesting to consider the case that the potential function can be written as \( f = f(r) \), which was introduced by P. Petersen and W. Wylie [15].
**Definition 1.2** ([15]). If the potential function of a gradient Ricci soliton \((M, g, f)\) can be written as \(f = f(r)\), where \(r\) is a distance function on \(M\), then \(M\) is called a rectifiable gradient Ricci soliton.

In this paper, we completely classify 3-dimensional complete rectifiable steady gradient Ricci solitons.

**Theorem 1.3.** Any 3-dimensional complete rectifiable steady gradient Ricci soliton is isometric to (1) a quotient of \(\mathbb{R}^3\), (2) a quotient of \(\Sigma^2 \times \mathbb{R}\), where \(\Sigma^2\) is the Hamilton’s cigar soliton, or (3) the Bryant soliton.

2. **Preliminary**

In this section, we recall some notions and basic facts.

The Riemannian curvature tensor is defined by

\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,
\]

where \(\nabla\) is the Levi-Civita connection on \(M\).

The Weyl tensor \(W\) and the Cotton tensor \(C\) are defined by

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})
+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}),
\]

and

\[
C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik},
\]

where \(A = \text{Ric} - \frac{R}{2(n-1)}g\) is the Schouten tensor, and \(R_{ij} = \text{Ric}_{ij}\). The Cotton tensor is skew-symmetric in the last two indices and totally trace free, that is,

\[
C_{ijk} = -C_{ikj}, \quad C_{iik} = C_{iji} = 0.
\]

As is well known, a Riemannian manifold \((M^n, g)\) is locally conformally flat if and only if (1) for \(n \geq 4\), the Weyl tensor vanishes; (2) for \(n = 3\), the Cotton tensor vanishes. Moreover, for \(n \geq 4\), if the Weyl tensor vanishes, then the Cotton tensor vanishes. We also see that for \(n = 3\), the Weyl tensor always vanishes, but the Cotton tensor does not vanish in general.
To study gradient Ricci solitons, Cao and Chen introduced the tensor $D$ (cf. [5]).

$$D_{ijk} = \frac{1}{n-2} (\nabla_k f R_{ij} - \nabla_j f R_{ik}) + \frac{1}{(n-1)(n-2)} \nabla_t f (R_{ik} g_{ij} - R_{ij} g_{ik})$$

$$- \frac{R}{(n-1)(n-2)} (\nabla_k f g_{ij} - \nabla_j f g_{ik}).$$

In this paper, we call it the Cao-Chen tensor. The Cao-Chen tensor has the same symmetry properties as the Cotton tensor, that is,

$$D_{ijk} = -D_{ikj}, \quad D_{iik} = D_{iji} = 0.$$  

There is a relationship between the Cotton tensor and the Cao-Chen tensor:

$$C_{ijk} + \nabla_t f W_{ijk} = D_{ijk}.$$  

Thus, in dimension 3, we have $D = C$.

3. Proof of Theorem 1.3

In this section, we will show Theorem 1.3.

**Proof.** In general, by B.-L. Chen’s theorem [9], $(M, g, f)$ has nonnegative sectional curvature. By Hamilton’s identity

$$R + |\nabla f|^2 = c,$$

for some constant $c$, $(M, g, f)$ has bounded curvature. Therefore, by Hamilton’s strong maximum principle, $(M, g, f)$ is either (1) flat, or (2) a product $\Sigma^2 \times \mathbb{R}$, where $\Sigma^2$ is the cigar steady soliton, or (3) it has positive sectional curvature.

We only have to consider the case (3). By the soliton equation $\text{Ric} + \nabla \nabla f = 0$, the potential function $f$ is concave. Thus, $f$ has only one critical point $O$. Set $r(x) = \text{dist}(x, O)$, $(x \in \Omega = M \setminus \{C_O \cup \{O\}\})$ is a distance function from the point $O$, where $C_O$ is the cut locus of $O$.

Assume that $M$ is rectifiable, namely $f = f(r)$, where $r$ is the distance function. Here we remark that $|\nabla r| = 1$ and $\nabla r \nabla r = 0$. Take $\{e_1 = \nabla r, e_2, e_3\}$ be an orthonormal frame on $\Omega$. We use subscripts $a, b = 2, 3$ ($a \neq b$), and denote $\nabla_1 = \nabla_{e_1}$ and $\nabla_a = \nabla_{e_a}$. Since $f$ is rectifiable, we have

$$\nabla_1 f = f'(r) < 0, \quad \nabla_a f = 0.$$  

By Hamilton’s identity $R = c - |\nabla f|^2 = c - (f'(r))^2$,

$$\nabla_a R = 0.$$
A direct computation yields that
\[ R_{ij} = -\nabla_i \nabla_j f = -f''(r)\nabla_i r \nabla_j r - f'(r)\nabla_i \nabla_j r. \]
Since \( \nabla_1 r = 1 \), \( \nabla_a r = \nabla_1 \nabla_1 r = 0 \) and \( \text{Ric}(\nabla f, X) = \frac{1}{2}(\nabla R, X) \), we obtain
\[
R_{11} = -f''(r), \quad R_{1a} = 0, \quad R_{ab} = -f'(r)\nabla_a \nabla_b r. \tag{3.3}
\]
We show that the Cao-Chen tensor \( D \) vanishes. We only have to consider the following 5 cases: \( D_{11a}, D_{1ab}, D_{a1a}, D_{a1b}, D_{aab} \) \((a \neq b)\).

\[
D_{11a} = \nabla_a f R_{11} - \nabla_1 f R_{1a} + \frac{1}{2} \nabla_1 f (R_{1a} g_{11} - R_{11} g_{1a}) - \frac{R}{2} (\nabla_a f g_{11} - \nabla_1 f g_{1a}) = 0,
\]
where we used (3.1) and (3.3). The similar computations show that
\[ D_{1ab} = D_{aab} = 0, \]
\[ D_{a1a} = -\nabla_1 f (R_{aa} + \frac{R_{11}}{2} - \frac{R}{2}), \]
and
\[ D_{a1b} = -\nabla_1 f R_{ab}. \]
Since the dimension of \( M \) is 3, \( D_{ijk} = C_{ijk} \). Hence, we also compute \( C_{ijk} \).

\[ C_{11a} = C_{1ab} = 0, \]
\[ C_{a1a} = -\nabla_1 (R_{aa} - \frac{R}{4}), \]
\[ C_{a1b} = -\nabla_1 R_{ab}, \]
and
\[ C_{aab} = \nabla_b R_{aa} - \nabla_a R_{ab}. \]
Since \( D_{ijk} = C_{ijk} \), we have
\[
\nabla_1 (R_{aa} - \frac{R}{4}) = \nabla_1 f (R_{aa} + \frac{R_{11}}{2} - \frac{R}{2}), \tag{3.4}
\]
\[ \nabla_1 f R_{ab} = \nabla_1 R_{ab}, \tag{3.5} \]
\[ \nabla_b R_{aa} = \nabla_a R_{ab}. \tag{3.6} \]
By (3.4), we obtain
\[ \nabla_1(R_{22} + R_{33} - \frac{R}{2}) = \nabla_1 f(R_{22} + R_{33} + R_{11} - R) = 0. \]
Thus we have \( \nabla_1(R - 2R_{11}) = 0 \), and hence \( R - 2R_{11} = h \), for some function \( h \) which is independent of \( r \). However, since \( \nabla_a(R - 2R_{11}) = 0 \), \( h \) is a constant \( C \). From this and (3.4) again, one has
\[ \nabla_1(R_{aa} - \frac{R}{4} - \frac{C}{4}) = 0. \]
Assume that \( R_{aa} - \frac{R}{4} - \frac{C}{4} \neq 0 \) on some open set \( \Omega' \). We may assume that \( R_{aa} - \frac{R}{4} - \frac{C}{4} > 0 \) on \( \Omega' \). (The same argument yields a contradiction in the other case). We have
\[ R_{aa} - \frac{R}{4} - \frac{C}{4} = e^{f+c_a}, \]
where \( c_a \) is some function which is independent of \( r \). Substituting them into \( R = R_{11} + R_{22} + R_{33} \), we obtain
\[ e^{f+c_a} + e^{f+c_a} = 0, \]
which is a contradiction. Thus, \( R_{aa} - \frac{R}{4} - \frac{C}{4} = 0 \) on all of \( \Omega \). Therefore, we obtain
(3.7) \[ R_{11} = \frac{R - C}{2}, \quad R_{22} = R_{33} = \frac{R + C}{4}, \]
and \[ D_{al1} = 0. \]
By (3.7) and (3.2), one has \( \nabla_2R_{22} = \nabla_3R_{22} = \nabla_2R_{33} = \nabla_3R_{33} = 0 \). From this and (3.6), we obtain \( \nabla_2R_{23} = \nabla_3R_{23} = 0 \).
We will show that \( R_{23} \equiv 0 \). We use a useful formula of Catino-Mastrolia-Monticelli (cf. [8]):
\[ R_{kl}C_{kli} = \nabla_k\nabla_l D_{itk}. \]
Assume that \( R_{23} \neq 0 \) at a point \( p \in \Omega \). Note that, in the following argument, we consider the Ricci soliton on some small neighborhood \( \tilde{\Omega} \) of \( p \). Let \( \varphi \) be a distance function from \( p \). For a small \( s \), assume that \( \phi : \mathbb{R} \to \mathbb{R} \) is a nonnegative \( C^3 \) function such that \( \phi(u) = 1 \ (u \in [0, s]) \), \( \phi(u) = 0 \ (u \in [2s, \infty)) \) and \( \phi'(u) \leq 0 \ (x \in [s, 2s]) \). Then, we can take a cut off function \( \eta(\varphi) = \phi(\varphi^2) \) with compact support (See also [8]). We have
\[ \int_{\tilde{\Omega}} \nabla_k \nabla_l D_{itk} \eta(\varphi) = -\int_{\tilde{\Omega}} \nabla_l D_{itk} \nabla_k \varphi^2 \phi'(\varphi^2) \]
\[ = \int_{\tilde{\Omega}} D_{itk}(\nabla_l \nabla_k \varphi^2 \phi'(\varphi^2) + \nabla_k \varphi^2 \nabla_l \varphi^2 \phi''(\varphi^2)), \]
where, we used that $\phi' (r^2)$ is $C^2$ with compact support. Therefore we have
\begin{equation}
(3.8) \quad \int_{\tilde{\Omega}} R_{kt} C_{kti} \phi (r^2) = \int_{\tilde{\Omega}} D_{itk} (\nabla_t \nabla_k r^2 \phi' (r^2) + \nabla_k r^2 \nabla_t r^2 \phi'' (r^2)) .
\end{equation}
Notice that $s$ is small enough. Take $i = 1$. Since $D_{1tk} = 0$, the right hand side of (3.8) vanishes. On the other hand, one has
\[ R_{kt} C_{kt1} = R_{23} (C_{231} + C_{321}) = \nabla_1 f (R_{23})^2 , \]
where we used $R_{1a} = C_{a1a} = 0$, and $C_{ab1} = -C_{a1b} = -D_{a1b} = \nabla_1 f R_{ab}$. Therefore,
\begin{equation}
0 \geq \int_{\tilde{\Omega}} \nabla_1 f (R_{23})^2 \phi (r^2) = 0 ,
\end{equation}
hence $\nabla_1 f (R_{23})^2 = 0$ on some set including $p$, which is a contradiction. Thus, we obtain $D_{a1b} = 0$.

Therefore, the Cao-Chen tensor vanishes identically on $\Omega$. In particular, $|D| \equiv 0$ on $\Omega$. Since every complete Ricci soliton is analytic [1], $|D| \equiv 0$ on all of $M$. Therefore, $M$ is rotationally symmetric [5], and hence it is isometric to the Bryant soliton. 

**Remark 3.1.** To prove $D \equiv 0$, we used only the assumption that the Ricci curvature is positive.

We can consider the similar problem to expanding gradient Ricci solitons, namely $\text{Ric} + \nabla \nabla f = \lambda g$ for $\lambda < 0$.

**Conjecture 2.** Any 3-dimensional complete expanding gradient Ricci soliton with positive sectional curvature is rotationally symmetric.

By using the same argument as in the proof of Theorem 1.3 we can show the following.

**Theorem 3.2.** Any 3-dimensional complete rectifiable expanding gradient Ricci soliton with positive Ricci curvature is rotationally symmetric.

In fact, the same argument as in the proof of Theorem 1.3 shows that $D = C \equiv 0$ on $M$, that is, $M$ is locally conformally flat, and the result follows from [5].

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