ASYMPTOTIC PROPERTIES OF THE BOUSSINESQ EQUATIONS WITH
DIRICHLET BOUNDARY CONDITIONS

IGOR KUKAVICA, DAVID MASSATT, AND MOHAMMED ZIANE

Abstract. We address the asymptotic properties for the Boussinesq equations with vanishing thermal
diffusivity in a bounded domain with no-slip boundary conditions. We show the dissipation of the \(L^2\) norm
of the velocity and its gradient, convergence of the \(L^2\) norm of \(Au\), and an \(o(1)\)-type exponential growth
for \(\|A^{3/2}u\|_{L^2}\). We also obtain that in the interior of the domain the gradient of the vorticity is bounded
by a polynomial function of time.

Contents

1. Introduction
2. Main theorems
3. Proofs for the global bounds
4. Interior bounds
Appendix A. Uniform Gronwall inequalities
Acknowledgments
References

1. Introduction

In this paper, we address the asymptotic behavior of the Boussinesq equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p &= \rho e_2 \\
\rho_t + u \cdot \nabla \rho &= 0 \\
\nabla \cdot u &= 0
\end{align*}
\]

(1.1)

with vanishing thermal/density diffusivity, in a smooth bounded domain \(\Omega \subseteq \mathbb{R}^2\) with the Dirichlet boundary
condition

\[ u|_{\partial \Omega} = 0 \]

(1.2)

and subject to the initial condition \((u(0), \rho(0)) = (u_0, \rho_0)\). Here, \(u\) represents the velocity, \(p\) the pressure,
and \(\rho\) the density or the temperature, depending on the physical context. The 2D Boussinesq system of
equations is used in a wide range of physical contexts, from large scale oceanic and atmospheric flows
where rotation and stratification are significant to microfluids and biophysics. It also relates closely to
fundamental models in fluid dynamics. In particular, the vorticity formulation of the incompressible Euler

Date: October 1, 2021.
equations away from the singularity can be described by the 2D Boussinesq equations (cf. [DWZZ]). For simplicity of exposition, we shall refer to the variable $\rho$ as the density, although it may also represent a temperature.

While global existence results have been well-known in the case of positive viscosity and positive thermal diffusivity, i.e., when adding the term $-\kappa \Delta \rho$ in the equation for the density/temperature, we address here the case of vanishing thermal diffusivity. In the case when both viscosity $\nu$ and diffusion coefficients $\kappa$ vanish, the global existence and uniqueness remain open questions, although results on the local existence, blow-up criteria, explicit solutions, and finite time singularities have been proven; cf. the blow-up results in [CH, E1], based on the singularity creation theorem for the Euler equations by Elgindi [E]. The case $\nu > 0$ and $\kappa = 0$, considered here, was initially considered by Chae [C] and Hou and Li [HL]. In particular, Hou and Li obtained the global existence and persistence of regularity in $H^s \times H^{s-1}$ for integer valued $s \geq 3$ in the case of periodic boundary conditions. The paper [LLT] by Lai et al extended the result in [HL] to the Dirichlet boundary conditions. The persistence of regularity for the lower value $s = 2$ in the case of Dirichlet or periodic boundary conditions was addressed in [HKZ1]. Subsequently, Ju obtained in [J] that $C \epsilon^{Ct}$ is an upper bound for the $H^1$ norm for the density, also for the Dirichlet boundary conditions. The bound was lowered to $\epsilon^{Ct}$ in [KW2], where also more precise results were obtained for periodic boundary conditions. In particular, [KW2, Theorem 2.1] contains a uniform in time upper bound for the quantity $\|D^2 u\|_{L^p}$ for all $p \geq 2$ in the periodic case. In a recent paper by Doering et al [DWZZ], the global existence, uniqueness, and regularity for the Boussinesq for the Lions boundary condition on a Lipschitz domain $\Omega$, was proven along with the dissipation of the $L^2$ norm of the velocity and its gradient. For other papers on the global existence and the regularity in Sobolev and Besov spaces, see [ACW, ACS, BFL, BS, BrS, CD, CG, CN, CW, DP, HK1, HK2, HKR, HKZ2, HS, JMWZ, KTW, KW2, KWZ, LPZ, SW].

In this paper, we prove several results on the asymptotic behavior of solutions of the Boussinesq system (1.1) with the Dirichlet boundary conditions (1.2). In our first main theorem, Theorem 2.1, we show that the $H^1$ norm of the velocity dissipates. We also establish a balanced convergence of $Au$, cf. (2.5) below, where $A$ is the Stokes operator. Regarding the growth of the density, we prove that the first Sobolev norm of the density is bounded, up to a constant, by $\epsilon^{Ct}$ for an arbitrarily small $\epsilon > 0$, thus improving a result from [KW2] where the bound of the type $\epsilon^{Ct}$ was proven. Since the growth of the Sobolev norms of the density is controlled by the time integral of $\|\nabla u\|_{L^\infty}$, it is reasonable to expect that the bound was optimal; however, here we prove that the optimal bound is in fact $\epsilon^{Ct}$. It remains an open problem if one can achieve the estimate of the type $\epsilon^{Ct^\alpha}$, where $\alpha \in [0,1)$. The theorem holds under the assumption that $(u_0, \rho_0)$ belongs to $H^2 \times H^1$. The ideas for the proof of Theorems 2.1 draw from the approaches in [DWZZ], [HKZ1], [LLT], [HKZ1], [J], [KW1], and [KW2]. Additionally, in Theorem 3.1, we show that the theorem and the persistence of regularity also hold under the $H^1 \times H^1$ assumption on the initial data.

In the second main theorem, Theorem 2.2, we address the behavior of the solution in a higher regularity norm. We prove that, under the $H^3 \times H^2$ assumption on the initial data, that for every $\epsilon > 0$ the norm of $(u, \rho)$ in the $H^3 \times H^2$ norm is bounded by $\epsilon^{Ct}$, up to a constant depending on $\epsilon > 0$. This holds under the $H^3 \times H^2$ regularity of the initial data $(u_0, \rho_0)$. We point out that, as in Theorem 3.1, the same in fact holds under the $H^2 \times H^2$ assumption on the data.
In the last main theorem, Theorem 2.3, we consider the upper bound for the \( L^p \) norm of the second derivatives of the velocity. As shown in [HKZ1], one may obtain a uniform bound when \( p = 2 \). When \( p > 2 \), this is not known except in the case of periodic boundary condition, which is a result obtained in [KW1]. Here, we prove that we can obtain a polynomial in time bound in the interior of a domain when considering the Dirichlet boundary condition, which is considerably lower than \( e^{\varepsilon t} \) type bound that would result from applying the Gagliardo-Sobolev inequality on the conclusions of Theorem 2.2. The proof is obtained by the change of variable from [KW1] combined with new localization arguments controlling the nonlocal nature of the transformation in [KW1] (see the double cut-off strategy in the proof of Theorem 2.3 below).

We emphasize that all our results extend also in the often-studied problem of the channel with Dirichlet boundary conditions on top and the bottom and periodic boundary conditions on the sides. Also, our proofs are completely self-contained.

2. Main theorems

We consider the asymptotic behavior of the Boussinesq equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p &= \rho e_2, \\
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho &= 0, \\
\nabla \cdot u &= 0
\end{align*}
\]  

(2.1)

and

\[ u \Big|_{\partial \Omega} = 0, \]  

(2.2)

coupling the Navier-Stokes equations [CF, DG, R, K1, K2, T1–T3] for the velocity \( u = (u_1, u_2) \) and the pressure \( p \) with the equation for the density \( \rho \). The system is set on a smooth, bounded, and connected domain \( \Omega \subseteq \mathbb{R}^2 \) and supplemented with the initial condition

\[(u, \rho)(0) = (u_0, \rho_0) \quad \text{in } \Omega.\]

Here, \( u \) denotes the velocity, \( p \) the pressure, and \( \rho \) the density. Note that we set \( \nu = 1 \) for simplicity of exposition; all the results extend to other values of \( \nu \) with constants depending additionally on \( \nu \).

From [CF, T1], we recall the classical spaces

\[ H = \{ u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial \Omega \}, \]

where \( n \) denotes the outward unit normal, and

\[ V = \{ u \in H_0^1(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega \}, \]

utilized in the study of the Navier-Stokes equations. With \( \mathbb{P} : L^2 \to H \) the Leray projector, denote by

\[ A = -\mathbb{P} \Delta, \]

the Stokes operator with the domain \( D(A) = H^2(\Omega) \cap V \).

It is known that for a sufficiently regular initial condition there exists a unique, global in time solution for (2.1)–(2.2) (cf. [C, HL]). In the first theorem, we obtain the asymptotic properties of \( A^{1/2} u \) and \( Au \) in the energy norm.
**Theorem 2.1.** Let \((u_0, \rho_0) \in (H^2(\Omega) \cap V) \times H^1(\Omega)\). Then the solution
\[
(u, \rho) \in (C([0, \infty); H) \cap L^2_{loc}([0, \infty); D(A))) \times L^\infty_{loc}([0, \infty), H^1(\Omega))
\]
of (2.1)--(2.2) satisfies
\[
\|Au\|_{L^2} \leq C,
\]
where \(C\) depends on the size of the initial data, i.e., on the norms \(\|Au_0\|_{L^2}\) and \(\|\rho_0\|_{H^1}\). Moreover,
\[
\|A^{1/2}u\|_{L^2} = \|\nabla u\|_{L^2} \to 0 \quad \text{as} \ t \to \infty,
\]
and
\[
\|Au - P(\rho_2)\|_{L^2} \to 0 \quad \text{as} \ t \to \infty,
\]
and for every \(\epsilon > 0\) we have
\[
\|\rho(t)\|_{H^1} \leq C_\epsilon e^{\epsilon t}, \quad t \geq 0,
\]
where \(C_\epsilon\) is a constant depending on \(\epsilon\) and the size of initial data.

Above and in the sequel, we allow all constants to depend on \(\Omega\). We note that in Theorem 2.1 the assumption of \(H^2\) regularity on the initial velocity can be relaxed to \(u_0 \in V\), as shown in Theorem 3.1 below. In the next statement, we obtain the asymptotic behavior of the \(H^3 \times H^2\) norm of the solution \((u, \rho)\). From [LLT, T5], the local existence requires the initial data to satisfy the compatibility condition
\[
(-\Delta u_0 - \nabla p_0 - \rho_0 e_2)|_{\partial \Omega} = 0,
\]
where \(p_0\) denotes the initial pressure, which solves the Neumann boundary problem
\[
\Delta p_0 = \nabla \cdot (\rho_0 e_2 - u_0 \cdot \nabla u_0) \quad \text{in} \ \partial \Omega
\]
\[
\nabla p_0 \cdot n|_{\partial \Omega} = (\Delta u_0 + \rho_0 e_2) \cdot n|_{\partial \Omega}
\]
with \(n\) denoting the outward unit normal.

**Theorem 2.2.** Assume that \((u_0, \rho_0) \in (H^3(\Omega) \cap V) \times H^2(\Omega)\) satisfies the compatibility condition (2.7), and let \((u, \rho)\) be the corresponding solution of (2.1)--(2.2). Then for every \(\epsilon > 0\), we have
\[
\|u(t)\|_{H^3} \leq C_\epsilon e^{\epsilon t}, \quad t \geq 0
\]
and
\[
\|\rho(t)\|_{H^2} \leq C_\epsilon e^{\epsilon t}, \quad t \geq 0,
\]
where \(C_\epsilon\) is a constant depending on \(\epsilon\).

Using the ideas in the proof of Theorem 3.1, the same long time behavior can be obtained with initial data \((u_0, \rho_0) \in D(A) \times H^2(\Omega)\), now without the compatibility condition (2.7).

In the next theorem, we obtain the interior bounds for the \(L^p\) norm of the Hessian \(D^2u\) of the velocity in the interior, for any \(p \geq 2\).

**Theorem 2.3.** Let \((u_0, \rho_0) \in (H^2(\Omega) \cap V) \times H^1(\Omega)\) and \(p \in [2, \infty)\), and suppose that \(\Omega' \subseteq \Omega\) is open and relatively compact. Then for the corresponding solution \((u, \rho)\) of (2.1)--(2.2) and all \(t_0 > 0\) we have a space-time bound
\[
\|D^2u\|_{L^p([t_0, T] ; L^p(\Omega'))} \leq C(T^{1/p} + 1),
\]
(2.9)
for $T \geq t_0 > 0$, while in addition we have a pointwise in time bound
\[
\|D^2 u(t)\|_{L^p(\Omega')} \leq C t^{(p+1)/4}, \quad t \geq t_0,
\] (2.10)
where the constants in (2.9) and (2.10) depend on $t_0$, $p$, and $\text{dist}(\Omega', \partial \Omega)$.

3. PROOFS FOR THE GLOBAL BOUNDS

First, we recall prior results on the $L^2$ norms corresponding to Theorem 2.1. Let $(u_0, \rho_0) \in (H^2(\Omega) \cap V) \times H^1(\Omega)$. Then there exists a unique global solution $(u, \rho)$ such that $u \in L^\infty((0, \infty), H^2(\Omega)) \cap L^2_{\text{loc}}((0, \infty), H^3(\Omega))$ and $\rho \in L^\infty((0, \infty), H^1(\Omega))$ of (2.1)–(2.2). Furthermore, the solution $(u, \rho)$ satisfies
\[
\|u(t)\|_{L^2} + \|\rho(t)\|_{L^2} \lesssim 1, \quad t \geq 0.
\] (3.1)
Here and below, the notation $a \lesssim b$ means $a \leq Cb$, where $C$ is a constant, which is allowed to depend on the size of the initial data in the pertinent norms. We denote by $B(u, v) = P(u \cdot \nabla v)$ the bilinear term corresponding to the Navier-Stokes equations. This allows us to rewrite (2.1) as
\[
\begin{align*}
    &u_t + Au + B(u, u) = P(\rho e_2) \\
    &\rho_t + u \cdot \nabla \rho = 0.
\end{align*}
\] (3.2)

We now turn to the proof of the first theorem.

Proof of Theorem 2.1. We begin by proving that $\|u\|_{L^2}$ dissipates. Inspired by [DWZZ], we shift the density by $x_2$, i.e., introduce
\[
\theta(x_1, x_2, t) = \rho(x_1, x_2, t) - x_2,
\] (3.3)
and compensate with $P = p(x_1, x_2, t) - x_2^2/2$ to derive an equivalent system of equations
\[
\begin{align*}
    &u_t - \Delta u + u \cdot \nabla u + \nabla P = \theta e_2 \\
    &\theta_t + u \cdot \nabla \theta = -u \cdot e_2 \\
    &\nabla \cdot u = 0,
\end{align*}
\] (3.4)
with $u\big|_{\partial \Omega} = 0$. Multiplying the first equation of (3.4) with $u$ and the second by $\theta$, integrating, and applying the Dirichlet boundary conditions and incompressibility, we obtain
\[
\frac{1}{2} \frac{d}{dt}(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 = 0.
\] (3.5)
Observe that the norm $\|\theta\|_{L^2}$ may increase, thus no direct conclusion on decay rates can be reached from (3.5). The identity (3.5) implies $\|u\|_{L^2}^2$ and $\|\theta\|_{L^2}^2$ are uniformly bounded in time and
\[
\int_0^\infty \|\nabla u\|_{L^2}^2 \lesssim 1,
\]
where we allow all constants to depend on $\|u_0\|_{H^2}$ and $\|\rho_0\|_{H^1}$. Utilizing the Poincaré inequality, we also get
\[
\int_0^\infty \|u\|_{L^2}^2 \lesssim 1.
\] (3.6)
To prove the uniform continuity from above of the $L^2$ norm of $u$, we multiply the first equation in (3.4) with $u$ and integrate by parts to find that
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_\Omega \theta u \cdot e_2 \leq \|u\|_{L^2}^2 \|\theta\|_{L^2} \lesssim \|u\|_{L^2},
\]
which, by Poincaré and Young’s inequalities, implies
\[
\frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim 1. \tag{3.7}
\]
It is elementary to show that if a differentiable function $f \colon [0, \infty) \to [0, \infty)$ satisfies $\int_0^\infty f(s) \, ds < \infty$ and $f'(t) \lesssim 1$, then $\lim_{t \to \infty} f(t) = 0$. Applying the statement with $f(t) = \|u\|_{L^2}$, the inequalities (3.6) and (3.7) imply
\[
\|u\|_{L^2} \to 0 \quad \text{as } t \to \infty. \tag{3.8}
\]
Next, we aim to prove that $\|\nabla u\|_{L^2}^2 \to 0$. We take the $L^2$ inner product of (3.2) with $Au$ to find that
\[
\frac{1}{2} \frac{d}{dt} \|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 = -\langle B(u, u), Au \rangle_{L^2} + \langle P(\theta e_2), Au \rangle_{L^2}
\leq \|B(u, u)\|_{L^2} \|Au\|_{L^2} + \|\theta\|_{L^2} \|Au\|_{L^2} \lesssim \|u\|_{L^2}^{1/2} \|A^{1/2}u\|_{L^2} \|Au\|_{L^2}^{3/2} + \|Au\|_{L^2}, \tag{3.9}
\]
where we used
\[
\|B(u, u)\|_{L^2} \lesssim \|u\|_{H^1} \|\nabla u\|_{L^4} \lesssim \|u\|_{L^2}^{1/2} \|Au\|_{L^2} \|Au\|_{H^1}^{1/2} \lesssim \|u\|_{L^2}^{1/2} \|A^{1/2}u\|_{L^2} \|Au\|_{L^2}. \tag{3.10}
\]
In (3.9), we apply Young’s inequality and absorb the factors $\|Au\|_{L^2}$ into the second term on the left side, obtaining
\[
\frac{d}{dt} \|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \lesssim \|u\|_{L^2}^2 \|A^{1/2}u\|_{L^2} + 1 \lesssim \|A^{1/2}u\|_{L^2}^2 + 1.
\]
Utilizing Lemma A.1 in the Appendix, we obtain
\[
\|A^{1/2}u(t)\|_{L^2} \lesssim 1, \quad t \geq 0 \tag{3.11}
\]
and
\[
\|A^{1/2}u(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty,
\]
giving (2.4). In addition, by the same lemma,
\[
\limsup_{t \to \infty} \int_t^{t+\delta} \|Au\|_{L^2}^2 \lesssim \delta, \quad \delta \geq 0. \tag{3.12}
\]
We note in passing, and since it is needed in the proof of Theorem 2.3, that the inequality of type (3.12) also holds with $Au$ replaced with $u_t$. To show that $u_t$ dissipates in the $L^2$ norm, we take the time derivative of (3.4) with $u_t$, multiply by $u_t$, and integrate by parts, to get the equation
\[
\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 = \langle \theta e_2, u_t \rangle_{L^2} - \langle u_t \cdot \nabla u, u_t \rangle_{L^2}. \tag{3.13}
\]
For the first term on the right, we apply (3.4) to obtain
\[
\langle \theta e_2, u_t \rangle_{L^2} = -\int_\Omega (u \cdot \nabla \theta)(\partial_1 u_2) - \int_\Omega u_2 \partial_1 u_2 = \int_\Omega \theta u \cdot \nabla \partial_1 u_2 - \int_\Omega u_2 \partial_1 u_2
\lesssim \|\theta\|_{L^2} \|u_t\|_{L^2}^{1/2} \|A^{1/2}u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|u\|_{L^2} \|u_t\|_{L^2}, \tag{3.14}
\]
\[
\lesssim \|A^{1/2}u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|u\|_{L^2} \|\nabla u_t\|_{L^2},
\]
where we used $||\theta||_{L^t} \lesssim 1$ and $||u_t||_{L^2} \lesssim ||\nabla u_t||_{L^2}$ in the last inequality. For the second term on the right-hand side of (3.13), we write

$$- \langle u_t \cdot \nabla u, u_t \rangle_{L^2} \lesssim ||u_t||_{L^2}^2 ||\nabla u||_{L^2} \lesssim ||u_t||_{L^2} ||\nabla u_t||_{L^2} ||A^{1/2} u||_{L^2}.$$  \hspace{1cm} (3.15)

Using (3.14) and (3.15) in (3.13) and then absorbing the factors $||\nabla u_t||_{L^2}$ by Young’s inequality, we get

$$\frac{d}{dt} ||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \lesssim ||A^{1/2} u||_{L^2}^2 + ||u||_{L^2}^2 + ||u_t||_{L^2}^2 ||A^{1/2} u||_{L^2}^2 \lesssim \phi(t)(1 + ||u||_{L^2}^2),$$

where $\phi: [0, \infty) \to [0, \infty)$ is a bounded function, which satisfies $\lim_{t \to \infty} \phi(t) = 0$. By Lemma A.2, we get

$$||u_t||_{L^2} \lesssim 1, \hspace{1cm} t \in [0, \infty)$$

and

$$||u_t(t)||_{L^2} \to 0 \hspace{1cm} \text{as} \hspace{0.2cm} t \to \infty \hspace{1cm} (3.17)$$
as well as

$$\limsup_{t \to \infty} \int_t^{t+\epsilon} ||\nabla u_t||_{L^2}^2 = 0, \hspace{1cm} t_0 \geq 0. \hspace{1cm} (3.18)$$

Next, from (3.2)1, we obtain

$$||Au||_{L^2} \lesssim ||u_t||_{L^2} + ||B(u, u)||_{L^2} + ||\rho||_{L^2} \lesssim ||u_t||_{L^2} + ||u||_{L^2}^{1/2} ||A^{1/2} u||_{L^2} ||Au||_{L^2}^{1/2} + 1.$$ Absorbing the factor $||Au||_{L^2}^{1/2}$ in the left-hand side by using Young’s inequality, we get

$$||Au||_{L^2} \lesssim ||u_t||_{L^2} + ||u||_{L^2} ||A^{1/2} u||_{L^2}^2 + 1,$$

from where, by (2.4) and (3.17), we get (2.3). Note, in passing, that (2.3) and (3.8) imply

$$||u(t)||_{L^\infty} \to 0 \hspace{1cm} \text{as} \hspace{0.2cm} t \to \infty, \hspace{1cm} (3.19)$$

by Agmon’s inequality. From (3.2)1, we get

$$||Au - \mathbb{P}(\rho e_2)||_{L^2} \lesssim ||u_t||_{L^2} + ||B(u, u)||_{L^2} \lesssim ||u_t||_{L^2} + ||u||_{L^2}^{1/2} ||A^{1/2} u||_{L^2} ||Au||_{L^2}^{1/2}. \hspace{1cm} (3.20)$$

By (2.3), (2.4), (3.8), and (3.17), the right-hand side of (3.20) converges to 0 as $t \to \infty$, and we obtain (2.5).

We lastly proceed to prove the $t^{1/2}$-type exponential estimate on the growth of $||\nabla \theta||_{L^2}$. For this, we first need to prove the local in time boundedness of $||\theta||_{H^1}$, which in turn requires us to first bound $\int_0^T ||\nabla u||_{L^\infty}$ for some $T > 0$. As above, we have

$$\int_0^T ||\nabla u_t||_{L^2}^2 \lesssim 1, \hspace{1cm} \text{for all} \hspace{0.2cm} T > 0, \hspace{1cm} \text{where the constant depends on} \hspace{0.2cm} T. \hspace{1cm} (3.21)$$

By [SvW, Theorem 2.7] (see also [GS]) applied with $s = p = 3$, we obtain that for any $\tilde{c} > 0$

$$\int_0^T ||u||_{W^{2,3}}^3 \lesssim ||A_3^{2/3 + \tilde{c}} u_0||_{L^3}^3 + \int_0^T ||u \cdot \nabla u - \rho e_2||_{L^3}^3, \hspace{1cm} (3.21)$$
for all $T > 0$, where the constant depends on $T$ and $\epsilon$. In (3.21), $A_3$ denotes the $L^3$ version of the Stokes operator (cf. [SvW]). For the first term on the right-hand side in (3.21), we use
\[ \|A_3^{2/3+\epsilon} u\|_{L^3} \lesssim \|Au\|_{L^2} \lesssim 1 \]
with $\epsilon = 1/6$ from the embedding property on [SvW, p. 430], while for the second term we estimate
\[ \|u \cdot \nabla u - \rho e_2\|_{L^3} \lesssim \|u\|_{L^6}^3 \|\nabla u\|_{L^6}^3 + \|\rho\|_{L^3}^3 \lesssim \|u\|_{L^2} \|A^{1/2} u\|_{L^6}^6 + 1 \lesssim 1. \]
Applying (3.22) and (3.23) in (3.21), we get
\[ \int_0^T \|D^2 u\|_{L^3}^2 \lesssim 1, \]
where the constant depends on $T$ and consequently
\[ \int_0^T \|\nabla u\|_{L^\infty} \lesssim 1 \]
for all $T > 0$, where the constant depends on $T$, due to the Gagliardo-Nirenberg type inequality
\[ \|v\|_{L^\infty} \lesssim \|v\|_{L^4}^{3/4} \|\nabla v\|_{L^6}^{3/4} + \|v\|_{L^2}. \]
By applying the gradient to (3.4) and taking the inner product with $\nabla \theta$, we find that
\[ \frac{1}{2} \frac{d}{dt} \|\nabla \theta\|_{L^2}^2 = -\langle \nabla (u \cdot \nabla \theta), \nabla \theta \rangle_{L^2} - \langle (\nabla (u \cdot e_2), \nabla \theta) \rangle_{L^2}. \]
The second term is estimated by $C\|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}$, using the Cauchy-Schwarz inequality. The first term is likewise bounded as
\[ -\langle \nabla (u \cdot \nabla \theta), \nabla \theta \rangle_{L^2} = \int_\Omega \partial_j (u_i \partial_i \theta \partial_j \theta) = -\int_\Omega \partial_j u_i \partial_i \theta \partial_j \theta - \frac{1}{2} \int_\Omega u_i \partial_i |\nabla \theta|^2 \lesssim \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^2}, \]
by (3.4) and $u |_{\partial \Omega} = 0$. Thus, estimating the two terms in (3.27) as indicated, we conclude that
\[ \frac{d}{dt} \|\nabla \theta\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} (\|\nabla \theta\|_{L^2} + 1), \]
which implies that the exponential growth of $\|\nabla \theta\|_{L^2}$ is determined by the time integral of $\|\nabla u\|_{L^\infty}$. In particular, applying (3.25) to (3.28) yields
\[ \|\theta\|_{H^1} \lesssim 1, \quad t \in [0, T], \]
for all $T > 0$, where the constant depends on $T$.

Next, we fix $\epsilon \in (0, 1]$ and claim that
\[ \|\theta(t)\|_{H^1} \lesssim e^{\epsilon t}, \quad t \geq 0, \]
where we allow all constants to depend on $\epsilon$. Note that (3.30) directly implies (2.6) by the definition (3.3). To prove (3.30), we need to estimate the time integral of $\|\nabla u\|_{L^\infty}$. Let $0 < t_0 \leq t_1$, where $t_0 \geq 2$ is a large time to be determined based on $\epsilon$. By the Gagliardo-Nirenberg in space and Hölder’s inequalities in time, we have, using (3.26)
\[ \int_{t_1}^{t_1+1} \|\nabla u\|_{L^\infty} \leq \int_{t_1}^{t_1+1} \left( \|\nabla u\|_{L^2}^{1/4} \|\Delta u\|_{L^6}^{3/4} + \|\nabla u\|_{L^2} \right) \]
\[ \leq C \left( \int_{t_1}^{t_1+1} \|\nabla u\|_{L^2}^{1/3} \right)^{3/4} \left( \int_{t_1}^{t_1+1} \|\Delta u\|_{L^3} \right)^{3/4} + \frac{1}{2} \epsilon, \]
where $\Delta u = \Delta (u_1, u_2, u_3)$ denotes the Laplacian of $u$.
provided $t_0$ is sufficiently large. To bound the $L^3$ norm of $\Delta u$, we introduce a smooth cut-off function $\phi: [0, \infty) \to [0, 1]$, where $\phi(t) = 0$ on $[0, t_1 - 1]$ and $\phi(t) = 1$ on $[t_1, \infty]$ with $|\phi'| \lesssim 1$. Now we consider the equation

$$(\phi u)_t - \Delta (\phi u) + \nabla (\phi p) = \phi' u - u \cdot \nabla (\phi u) + \phi \rho e_2$$

which follows from (2.1); note that $\nabla \cdot (\phi u) = 0$ since $\phi$ is a function of time only. Using the $W^{2,3}$ estimate due to Sohr and Von Wahl [SvW] we have, similarly to (3.21)–(3.24),

$$\int_{t_1}^{t_1 + 1} \|D^2 u\|_{L^3}^3 \lesssim \int_{t_1 - 1}^{t_1 + 1} \|u \cdot \nabla (\phi u)\|_{L^3}^3 + \int_{t_1 - 1}^{t_1 + 1} \|\phi' u\|_{L^3}^3 + \int_{t_1 - 1}^{t_1 + 1} \|\rho\|_{L^3}^3$$

$$\lesssim \int_{t_1 - 1}^{t_1 + 1} \|u\|_{L^3}^3 \|\nabla u\|_{L^6}^3 + \int_{t_1 - 1}^{t_1 + 1} \|u\|_{L^3}^3 + 1$$

$$\lesssim \int_{t_1 - 1}^{t_1 + 1} \|u\|_{L^3}^3 \|\nabla u\|_{L^2}^3 \|Au\|_{L^2}^2 + \int_{t_1 - 1}^{t_1 + 1} \|u\|_{L^3}^4 + 1 \lesssim 1, \quad t \geq 0$$

where we used (2.3), (3.1), and (3.11). Also, for the first factor of the first term in (3.31), we use (2.4) to obtain that for any $\epsilon_0 > 0$ there exists $t_0 \geq 1$ sufficiently large so that

$$\left(\int_{t_1 - 1}^{t_1 + 1} \|\nabla u\|_{L^2}^{1/3} \, dt\right)^{3/4} \leq \epsilon_0 \epsilon.$$  

(3.33)

Thus, using (3.32) and (3.33) in (3.31), we obtain

$$\int_{t_1}^{t_1 + 1} \|\nabla u\|_{L^\infty} \, dt \leq C \epsilon_0 \epsilon + \frac{1}{2} \epsilon, \quad t \geq t_0,$$

for $t_0 \geq 1$ sufficiently large, which in turn implies

$$\int_{t_0}^{t} \|\nabla u\|_{L^\infty} \, dt \leq \epsilon(t - t_0), \quad t \geq t_0$$

(3.34)

if we choose $\epsilon_0$ a sufficiently small constant. Note that (3.34) is obtained by adding the integrals of unit length. Returning to (3.28), we find that Gronwall’s inequality implies

$$\|\nabla \theta (t)\|_{L^2} \leq (\|\nabla \theta (t_0)\|_{L^2} + 1) e^{(t - t_0)}.$$

(3.35)

Finally, we use (3.29) implying

$$\|\theta (t_0)\|_{H^1} \lesssim 1,$$

(3.36)

where the constant depends on $t_0$, which in turn only depends on $\epsilon$. Combining (3.35) and (3.36) leads to the claimed inequality (3.30).

We noted that the initial assumptions of Theorem 2.1 can be relaxed, implying the conclusions of Theorem 2.1 for less restrictive initial conditions than those required for (3.1).

**Theorem 3.1.** Let $(u_0, \rho_0) \in V \times H^1(\Omega)$. Then there exists a unique solution $(u, \rho)$ such that $u \in L^2_{\text{loc}}([0, \infty); D(A)) \cap C([0, \infty); H)$ and $\rho \in L^3_{\text{loc}}([0, \infty); H^1(\Omega))$, which moreover satisfies

$$\|Au\|_{L^2} \leq C_\delta, \quad t \geq \delta,$$

(3.37)

where $\delta > 0$ is arbitrary.
Proof of Theorem 3.1. Let \((u, \rho)\) be a solution to (3.4) on \([0, T]\) where \(T \in (0, 1]\). Integrating (3.5) in time, we obtain
\[
\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \lesssim 1, \quad t \in [0, T].
\] (3.38)
We note that all constants are allowed to depend on \(\|u_0\|_{V}\) and \(\|\rho_0\|_{H^1}\). We use this inequality in (3.9) obtaining
\[
\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \lesssim \|A^{1/2}u\|_{L^2}^2 + 1,
\] (3.39)
which implies, along with (3.38), that upon suitably reducing \(T > 0\), we have \(u \in L^\infty([0, T]; V) \cap L^2([0, T]; D(A))\) and
\[
\|A^{1/2}u\|_{L^2}^2 \lesssim 1, \quad t \in [0, T],
\] (3.40)
and then
\[
\int_0^T \|Au\|_{L^2}^2 \lesssim 1, \quad t \in [0, T],
\] (3.41)
on upon returning to (3.39). Note that
\[
\|u_t\|_{L^2}^2 \lesssim \|Au\|_{L^2}^2 + \|B(u, u)\|_{L^2}^2 + \|\rho\|_{L^2}^2 \lesssim \|Au\|_{L^2}^2 + \|u\|_{L^2}^2 \|A^{1/2}u\|_{L^2}^2 \|Au\|_{L^2} + 1
\] \[
\lesssim \|Au\|_{L^2}^2 + \|u\|_{L^2}^2 \|A^{1/2}u\|_{L^2}^2 + 1 \lesssim \|Au\|_{L^2}^2 + 1,
\] (3.42)
by (3.1), (3.10), and (3.11). From (3.41) and (3.42), we obtain \(u_t \in L^2([0, T]; H)\). Thus, we may modify \(u\) on a measure zero subset of \([0, T]\) so that \(u \in C([0, T]; H)\).

In order to prove (3.37), we first need to show uniqueness in the class \(V \times H^1(\Omega)\). Thus, let \((u^{(1)}, \theta^{(1)})\) and \((u^{(2)}, \theta^{(2)})\) be solutions to the Boussinesq equation, and define \(u = u^{(1)} - u^{(2)}\) and \(\rho = \rho^{(1)} - \rho^{(2)}\) with both solutions satisfying the bounds (3.40) and (3.41) on \([0, T]\). Then subtracting the evolution equations (2.1) for \(u^{(1)}\) and \(u^{(2)}\) and testing the equation for the difference with \(Au\), we acquire
\[
\frac{1}{2} \frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \lesssim \|u^{(1)}\|_{L^2}^{1/2} \|A^{1/2}u^{(1)}\|_{L^2}^{1/2} + \|A^{1/2}u\|_{L^2}^{1/2} \|Au\|_{L^2}^{3/2} + \|u\|_{L^2}^{1/2} \|A^{1/2}u\|_{L^2}^{1/2} \|A^{1/2}u^{(2)}\|_{L^2}^{1/2} \|Au^{(2)}\|_{L^2}^{1/2} \|Au\|_{L^2} + \|\rho\|_{L^2} \|Au\|_{L^2},
\] (3.43)
whence, using the bounds on \(u^{(1)}\) and \(u^{(2)}\) and absorbing factors of \(\|Au\|_{L^2}\), we get
\[
\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \lesssim \|A^{1/2}u\|_{L^2}^2 + \|u\|_{L^2} \|A^{1/2}u\|_{L^2} \|Au^{(2)}\|_{L^2} + \|\rho\|_{L^2}^2.
\] On the other hand, from the density equations for \(\rho^{(1)}\) and \(\rho^{(2)}\), we get
\[
\frac{d}{dt}\|\rho\|_{L^2}^2 \lesssim \|u\|_{L^\infty(\Omega)} \|\nabla \rho^{(2)}\|_{L^2} \|\rho\|_{L^2} \lesssim \|u\|_{L^2}^{1/2} \|Au\|_{L^2}^{1/2} \|\nabla \rho^{(2)}\|_{L^2} \|\rho\|_{L^2} \lesssim \epsilon_0 \|Au\|_{L^2}^2 + \|u\|_{L^2}^{2/3} \|\nabla \rho^{(2)}\|_{L^2}^{4/3} \|\rho\|_{L^2}^{4/3}
\] \[
\lesssim \epsilon_0 \|Au\|_{L^2}^2 + (\|A^{1/2}u\|_{L^2}^2 + \|\rho\|_{L^2}^2) \|\nabla \rho^{(2)}\|_{L^2}^{4/3},
\] (3.44)
where \(\epsilon_0\) is a sufficiently small constant to be determined. Adding (3.43) and (3.44), choosing \(\epsilon_0\) sufficiently small and absorbing factors of \(\|Au\|_{L^2}^2\), we obtain
\[
\frac{d}{dt}(\|A^{1/2}u\|_{L^2}^2 + \|\rho\|_{L^2}^2) \lesssim \|A^{1/2}u\|_{L^2}^2 + \|Au^{(2)}\|_{L^2} \|A^{1/2}u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\rho\|_{L^2}^2,
\] (3.45)
where we used \( \| \nabla \rho^{(2)} \|_{L^2} \lesssim 1 \) for \( t \in [0, T] \) on the last term in (3.44), subject to reducing \( T \). Applying a Gronwall argument to (3.45) and using (3.41) for \( u^{(2)} \), we conclude that \( u = 0 \), whence \( u^{(1)} = u^{(2)} \) on \( [0, T] \).

In order to obtain (3.37), we observe that \( u(t) \in D(A) \) for a.e. \( t \in [0, T] \) by (3.41). We choose \( t_0 \in (0, \delta) \) such that \( u(t_0) \in D(A) \). Since \( u \) is unique on \([t_0, \infty)\), we may apply Theorem 2.1 and obtain (3.37) for \( t \geq t_0 \), concluding the proof. \( \square \)

We remark that similar arguments show analogous reduced required regularity for \( u \) in Theorems 2.2 and 2.3.

Next, we address a higher regularity norm.

**Proof of Theorem 2.2.** We start with a priori estimates and at the end of the proof we provide a sketch of the justification. Taking a time derivative of (2.1)\( _1 \), we obtain

\[
u_{tt} - \Delta u_t + u_t \cdot \nabla u + u \cdot \nabla u_t + \nabla p_t = \rho_t e_2,
\]

which, after testing with \( u_{tt} \) gives

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_t \|_{L^2}^2 + \| u_{tt} \|_{L^2}^2 \\
= - \int_{\Omega} u_t \cdot \nabla u_j \partial_t u_j - \int_{\Omega} u \cdot \nabla \partial_t u_j \partial_t u_j + \int_{\Omega} (\rho_t e_2) \cdot u_{tt} \\
\lesssim \| u_t \|_{L^2}^{1/2} \| \nabla u_t \|_{L^2}^{1/2} \| \nabla u \|_{L^2}^{1/2} \| Au \|_{L^2}^{1/2} \| u_{tt} \|_{L^2} + \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \| u_{tt} \|_{L^2} + \| \rho_t \|_{L^2} \| u_{tt} \|_{L^2}.
\]

Now we apply \( \| u \|_{L^\infty} \lesssim \| u \|_{L^2}^{1/2} \| Au \|_{L^2}^{1/2} \) for the second term and

\[
\| \rho_t \|_{L^2} = \| u \cdot \nabla \rho \|_{L^2} \lesssim \| u \|_{L^\infty} \| \nabla \rho \|_{L^2},
\]

by (2.1)\( _2 \), on the last. Absorbing the factors of \( \| u_{tt} \|_{L^2} \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_t \|_{L^2}^2 + \| u_{tt} \|_{L^2}^2 \\
\lesssim \| u_t \|_{L^2} \| \nabla u_t \|_{L^2} \| \nabla u \|_{L^2} \| Au \|_{L^2} + \| u \|_{L^\infty} \| \nabla u_t \|_{L^2}^2 + \| u \|_{L^\infty} \| \nabla \rho \|_{L^2}^2 \\
\lesssim 1 + \| \nabla u_t \|_{L^2}^2 + C_\epsilon e^{2\epsilon t},
\tag{3.46}
\]

where \( \epsilon > 0 \) is arbitrarily small. In (3.46), we also used (2.3). Combining (3.18) and (3.46) with a uniform Gronwall argument, we get

\[
\| \nabla u_t \|_{L^2} \lesssim e^{\epsilon t}, \quad t \geq 0
\tag{3.47}
\]

and

\[
\int_0^t \| u_{tt} \|_{L^2}^2 \lesssim e^{\epsilon t}, \quad t \geq 0,
\]

where we allow constants to depend on \( \epsilon \). Now, consider the stationary, i.e., pointwise in time, Stokes problem

\[
- \Delta u + \nabla p = -u \cdot \nabla u - u_t + \rho e_2 \\
u_t |_{\partial \Omega} = 0.
\tag{3.48}
\]
Note that
\[ \| u \cdot \nabla u + u_t - \rho e_2 \|_{H^1} \lesssim \| D(u \cdot \nabla u - u_t) \|_{L^2} + \| \rho \|_{H^1} \]
\[ \lesssim \| Du \|_{L^4}^2 + \| u \|_{L^\infty} \| D^2 u \|_{L^2} + \| \nabla u_t \|_{L^2} + e^{ct} \]
\[ \lesssim \| A^{1/2} u \|_{L^2} \| Au \|_{L^2} + \| u \|_{L^\infty} \| Au \|_{L^2} + \| \nabla u_t \|_{L^2} + e^{ct} \lesssim e^{ct}, \]
using (3.47) in the last step. Applying the $H^1$ regularity for the Stokes problem (3.48), cf. [T4, Proposition 3.3], leads to
\[ \| u \|_{H^3} + \| \nabla p \|_{H^1} \lesssim e^{ct}. \] (3.49)

In order to obtain (2.8), we apply $\partial_{ij}$, for $i, j = 1, 2$, to (2.1), test with $\partial_{ij} \rho$, and sum which leads to
\[ \frac{1}{2} \frac{d}{dt} \| \partial_{ij} \rho \|_{L^2}^2 = \langle \partial_{ij}(u \cdot \nabla p), \partial_{ij} \rho \rangle_{L^2} = \int_{\Omega} \partial_{ij} u_k \partial_k \rho \partial_{ij} \rho + 2 \int_{\Omega} \partial_i u_k \partial_{j,k} \partial_{ij} \rho + \int_{\Omega} u_k \partial_{ij,k} \partial_{ij} \rho, \] (3.50)
which holds for all $t \geq 0$. The last term vanishes due to the incompressibility, while the second is bounded by $C \| \nabla u \|_{L^\infty} \| D^2 \rho \|_{L^2}^2$. For the first term on the far right side of (3.50), we write
\[ \int_{\Omega} \partial_{ij} u_k \partial_k \rho \partial_{ij} \rho \lesssim \| \Delta u \|_{L^4} \| \nabla p \|_{L^4} \| D^2 \rho \|_{L^2}, \] (3.51)
where we utilized the Gagliardo-Nirenberg inequalities. Now, we use (2.6) and (3.49) in (3.51), sum in $i$ and $j$, and cancel a factor of $\| D^2 \rho \|_{L^2}$ on both sides to obtain
\[ \frac{1}{2} \frac{d}{dt} \| D^2 \rho \|_{L^2} \lesssim e^{3ct/2} + e^{ct} \| D^2 \rho \|_{L^2}^{1/2} + \| \nabla u \|_{L^\infty} \| D^2 \rho \|_{L^2}, \]
whence, applying Young’s inequality
\[ \frac{1}{2} \frac{d}{dt} \| D^2 \rho \|_{L^2} \leq e^{2ct} + (\epsilon + \| \nabla u \|_{L^\infty}) \| D^2 \rho \|_{L^2}, \]
for all $t \geq 0$. Applying a Gronwall argument and using (3.34), which holds for $t_0 > 0$ sufficiently large depending on $\epsilon$, we get
\[ \| D^2 \rho(t) \|_{L^2} \leq C e^{Ct} (\| D^2 \rho \|_{L^2}(t_0) + 1), \quad t \geq t_0. \] (3.52)
On the other hand, using Gronwall’s argument on $[0, t_0]$ with (3.25) for $T = t_0$, we get
\[ \| D^2 \rho(t) \|_{L^2} \lesssim \| D^2 \rho \|_{L^2}(0) + 1, \quad t \in [0, t_0], \] (3.53)
where the constant depends on $t_0$ and thus on $\epsilon$. Combining (3.52) and (3.53), we finally obtain (2.8) with $Ct$ replacing $\epsilon$.

To justify the a priori bounds above, we consider the sequence of solutions
\[ u_i^{(n+1)} - \Delta u_i^{(n+1)} + u_i^{(n)} \cdot \nabla u_i^{(n+1)} + \nabla P^{(n+1)} = \theta_i^{(n+1)} e_2 \]
\[ \theta_i^{(n+1)} + u_i^{(n)} \cdot \nabla \theta_i^{(n+1)} = -u_i^{(n+1)} . e_2 \]
\[ \nabla \cdot u_i^{(n+1)} = 0, \] (3.54)
with the boundary condition $u_i^{(n+1)} |_{\partial \Omega} = 0$ and with the initial data
\[ (u_i^{(n+1)}(0), \theta_i^{(n+1)}(0)) = (u_0, \rho_0 - x_2), \]
for $n \in \mathbb{N}_0$. For $n = 0$, we define

\[
\begin{align*}
u^{(0)}_t - \Delta u^{(0)} + \nabla P^{(0)} &= \theta^{(0)} e_2 \\
\theta^{(0)} &= -u^{(0)} \cdot e_2 \\
\nabla \cdot u^{(0)} &= 0,
\end{align*}
\]

with the boundary condition $u^{(0)}|_{\partial \Omega} = 0$ and with the initial data

\[
(u^{(0)}(0), \theta^{(0)}(0)) = (u_0, \rho_0 - x_2).
\]

Since the system (3.54) is linear in $(u^{(n+1)}, \theta^{(n+1)})$, it is easy to construct a local solution $(u^{(n+1)}, \theta^{(n+1)})$.

Also, our a priori estimates apply to the sequence and one may pass uniform bounds to the limit. Since the arguments are standard, we omit further details. □

4. Interior bounds

In this section, we establish the final result on the interior regularity of the second order derivatives.

Proof of Theorem 2.3. In the proof, we work in the interior of the domain and thus localize the vorticity equation using a smooth cut-off function. With $\Omega'$ as in the statement, consider a smooth function $\eta : \mathbb{R}^2 \times [0, \infty) \to [0, 1]$ such that $\text{supp } \eta \subseteq \Omega \times [t_0/2, \infty)$ with $\eta = 1$ on $\Omega'' \times [3t_0/4, \infty)$, where $\Omega''$ is an open set such that $\Omega' \Subset \Omega'' \Subset \Omega$.

In order to prove (2.9), we first claim that the vorticity $\omega = \text{curl } u$ satisfies

\[
\| \nabla \omega \|_{L^p([t_0, T] : L^p(\Omega'))} \lesssim T^{1/p} + 1,
\]

where the constant depends on $t_0$, and $\text{dist}(\Omega', \partial \Omega)$. Since (2.9) and (2.10) for $p = 2$ follow from (2.3), we fix $p > 2$. We allow all constants to depend on $p$ and $t_0$, where $t_0 > 0$ should be considered small.

As in [KW2], we introduce the operator

\[
R = \partial_1 (I - \Delta)^{-1}
\]

and a change of variable

\[
\zeta = \omega \eta - R(\rho \eta).
\]

We shall apply $R$ to functions which are compactly supported in $\Omega$, and we consider such functions extended to $\mathbb{R}^2$ by setting them identically to zero on $\Omega^c$. Recalling the vorticity formulation for (2.1),

\[
\omega_t - \Delta \omega + u \cdot \nabla \omega = \partial_1 \rho,
\]

we have, as in [KW2], that

\[
\zeta_t - \Delta \zeta + u \cdot \nabla \zeta = [R, u \cdot \nabla](\rho \eta) - N(\rho \eta) - \rho \partial_1 \eta - 2\partial_j(\omega \partial_j \eta) + \omega(\eta_t + \Delta \eta + u \cdot \nabla \eta) - R(\rho (u \cdot \nabla \eta)),
\]

where

\[
N = ((I - \Delta)^{-1} \Delta - I) \partial_1,
\]

which has the property that $\nabla N$ is in the Calderón-Zygmund class. The equation (4.3) is obtained by a direct computation from

\[
(\omega \eta)_t - \Delta (\omega \eta) + u \cdot \nabla (\omega \eta) = \omega \eta_t + \omega \Delta \eta - 2\partial_j(\omega \partial_j \eta) + \omega u \cdot \nabla \eta + \partial_1 (\rho \eta) - \rho \partial_1 \eta
\]
and

$$(\rho \eta)_t + u \cdot \nabla (\rho \eta) = \rho u \cdot \nabla \eta$$

and then using the identity $\partial_t + \Delta R = -N$. Note that both operators $R$ and $N$ commute with translations (and hence derivatives) and they are smoothing of order one, i.e., they satisfy the heat equation in divergence form, we have

$$\| Rf \|_{L^{1,p}} \leq \| NF \|_{L^{1,p}} \leq \| f \|_{L^p}, \quad f \in L^p(\mathbb{R}^2),$$

for $p \in (1, \infty)$, where the constant depends on $p$; the property (4.4) can be verified by computing the Fourier multiplier symbols corresponding to $R$ and $N$ (or cf. [KW2]). Since $u$ is divergence free, we may rewrite

$$[R, u_j \partial_j](\rho \eta) = R(u_j \partial_j (\rho \eta)) - u_j \partial_j R(\rho \eta) = \partial_j R(u \rho \eta) - \partial_j \partial_j R(\rho \eta).$$

To acquire $L^p$ space-time estimates, we rewrite our solution as $\zeta = \zeta^{(1)} + \zeta^{(2)}$, where $\zeta^{(1)}$ satisfies

$$\zeta^{(1)}_t - \Delta \zeta^{(1)} = f$$

with

$$f = \omega(\eta_t + \Delta \eta + u \cdot \nabla \eta) - R(\rho(u \cdot \nabla \eta)) - N(\rho \eta) - u \cdot \nabla R(\rho \eta) - \rho \partial_1 \eta,$$

while for $\zeta^{(2)}$ we have

$$\zeta^{(2)}_t - \Delta \zeta^{(2)} = \nabla \cdot g$$

$$\zeta^{(2)}|_{t=0} = 0,$$

where

$$g = -u\zeta - 2\omega \nabla \eta + R(\rho \eta).$$

Using the $L^pW^{2,p}$ regularity for the nonhomogeneous heat equation and the Gagliardo-Nirenberg inequality, we have

$$\| D\zeta^{(1)} \|_{L^pL^p(\mathbb{R}^2 \times (0,\infty))} \lesssim \| D^2\zeta^{(1)} \|_{L^pL^2(\mathbb{R}^2 \times (0,\infty))} \lesssim \| f \|_{L^pL^2(\mathbb{R}^2 \times (0,\infty))},$$

observe that $2p/(p+2) > 1$ since $p > 2$. Similarly, using the $L^pW^{1,p}$ regularity for the nonhomogeneous heat equation in divergence form, we have

$$\| D\zeta^{(2)} \|_{L^pL^p(\mathbb{R}^2 \times (0,\infty))} \lesssim \| g \|_{L^pL^p(\mathbb{R}^2 \times (0,\infty))}.$$  

For the right-hand side of (4.5), we use (4.4) to obtain

$$\| f \|_{L^{2p/(p+2)}} \lesssim \| \omega \|_{L^2} (\| \eta_t \|_{L^p} + \| \Delta \eta \|_{L^p} + \| u \|_{L^\infty} \| \nabla \eta \|_{L^p}) + \| \rho \|_{L^2} \| u \|_{L^\infty} \| \nabla \eta \|_{L^p} + \| \rho \|_{L^2} \| \partial_1 \eta \|_{L^p}$$

$$\lesssim \| \omega \|_{L^2} + 1 \lesssim 1,$$

for every $t \geq 0$, where the domains are understood to be $\mathbb{R}^2$. For the right-hand side in (4.6), we determine that

$$\| g \|_{L^p} \lesssim \| u \|_{L^{2p}} + \| \omega \|_{L^{2p}} \| \nabla \eta \|_{L^{2p}} + \| u \|_{L^{\infty}} \| \rho \|_{L^p} \| \eta \|_{L^{\infty}}$$

$$\lesssim \| \omega \|_{L^{2p}} + \| u \|_{L^{2p}} + 1$$

for every $t \geq 0$, by (4.4). To bound the right-hand side of (4.8), we write

$$\| \zeta \|_{L^q} \lesssim \| \omega \|_{L^q} + \| R(\rho \eta) \|_{L^q} \lesssim \| \omega \|_{L^2} + \| \rho \|_{L^2} \lesssim 1, \quad q \in [2, \infty).$$

(4.9)
Therefore, we obtain \( \|g\|_{L^p} \lesssim 1 \) for all \( t \geq 0 \). This fact and (4.7) imply by integration that the left-hand sides of (4.5) and (4.6) are bounded by \( T^{1/p} \) for \( T \geq t_0 \), from where
\[
\|D\zeta\|_{L^p L^p(\mathbb{R}^2 \times (0, \infty))} \lesssim T^{1/p}
\]
and thus
\[
\|\nabla (\omega \eta)\|_{L^p L^p(\mathbb{R}^2 \times (0, \infty))} \lesssim \|\nabla \zeta\|_{L^p L^p(\mathbb{R}^2 \times (0, \infty))} + \|R\nabla (\rho \eta)\|_{L^p L^p(\mathbb{R}^2 \times (0, \infty))} \lesssim T^{1/p} + 1,
\]
which proves (4.1). The bound (2.9) then follows by a simple application of the interior elliptic estimate connecting \( u \) and \( \omega \).

The pointwise in time bound in (2.10) follows once we obtain
\[
\|\nabla \omega(t)\|_{L^p(\Omega')} \lesssim t^{1/4+2/p+1/p^2}, \quad t \geq t_0,
\]
where the constant depends on \( t_0, p, \) and \( \text{dist}(\Omega', \partial \Omega) \). To prove (4.11), we begin by introducing a second smooth cut-off function \( \phi: \mathbb{R}^2 \times [0, \infty) \to [0, 1] \) for which
\[
\text{supp } \phi \subseteq \{ \eta = 1 \} = \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : \eta(x, t) = 1 \}
\]
and is such that \( \phi = 1 \) on \( \Omega' \times [t_0, \infty) \). Denote
\[
\tilde{\zeta} = \zeta \phi.
\]
Using (4.3), we find that
\[
\tilde{\zeta}_t - \Delta \tilde{\zeta} + u \cdot \nabla \tilde{\zeta} = ([R, u \cdot \nabla] (\rho \eta) - N(\rho \eta)) \phi - R(\rho (u \cdot \nabla) \eta) \phi - 2\nabla \zeta \cdot \nabla \phi + \zeta(\phi_t - \Delta \phi + u \cdot \nabla \phi); \tag{4.12}
\]
note that the terms in (4.3) containing derivatives of \( \eta \) vanish after multiplication with \( \phi \), except for the term involving \( R \), which is a non-local operator. The main reason for introducing the second cut-off function \( \phi \) is that \( \zeta \) does not vanish on the boundary \( \partial \Omega \) due to nonlocality of \( R \); cf. the definition (4.2). In order to estimate \( \nabla \tilde{\zeta} \), we apply \( \partial_k \) to (4.12) for \( k = 1, 2 \), multiply by \( |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} \), integrate, and sum in \( k \) to acquire
\[
\frac{1}{2p} \frac{d}{dt} \sum_k \|\partial_k \tilde{\zeta}\|_{L^p}^{2p} - \sum_k \left[ \int \Delta \partial_k \tilde{\zeta} |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} + \frac{1}{2} \int \partial_k (\phi_t \partial_k \zeta) |\partial_k \tilde{\zeta}|^{p-2} \partial_k \tilde{\zeta} \right]
\]
\[
= - \sum_k \left[ \int \partial_k (u \partial_j \partial_k \tilde{\zeta}) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} + \sum_k \int \partial_k (\phi R(u \cdot \nabla \eta)) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} \right]
\]
\[
- \sum_k \int \partial_k (\phi N(\rho \eta)) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta}
\]
\[
- \sum_k \int \partial_k (\phi R(\rho (u \cdot \nabla) \eta)) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} - \sum_k \int \partial_k (\phi R(u \cdot \nabla \rho)) |\partial_k \tilde{\zeta}|^{p-2} \partial_k \tilde{\zeta}
\]
\[
+ 2\sum_k \int \partial_k (\phi (\partial_j \partial_k \tilde{\zeta}) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta})
\]
\[
+ \sum_k \int \partial_k (\phi_t (\partial_k \tilde{\zeta}) - \Delta \phi + u \cdot \nabla \phi) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta}.
\tag{4.13}
\]
The second term on the left-hand side of (4.13) is estimated as
\[
- \sum_k \int \Delta \partial_k \tilde{\zeta} |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \tilde{\zeta} = \frac{2p-1}{p^2} \sum_k \int \partial_j (|\partial_k \tilde{\zeta}|^p |\partial_j (|\partial_k \tilde{\zeta}|^p) \geq \frac{1}{p} \sum_k \|\nabla (|\partial_k \tilde{\zeta}|^p)\|_{L^2}^2 = \frac{1}{p} \tilde{D}, \tag{4.14}
\]
where
where we denoted \( D = \sum_k \| \nabla((\partial_k \tilde{\zeta})^p) \|_{L^2}^2 \). For the first term on the right-hand side of (4.13), we use the incompressibility of \( u \) to determine that

\[
- \sum_k \int \partial_k(u_j \partial_j \tilde{\zeta}) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} = - \sum_k \int \partial_k(u_j \partial_j \tilde{\zeta}) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \lesssim \| \nabla u \|_{L^2} \| \nabla \tilde{\zeta} \|_{L^4} \sum_k \| \partial_k \tilde{\zeta}^{2p-1} \|_{L^{4p/(2p-1)}} \lesssim o(1) \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^p, \tag{4.15}
\]

where \( o(1) \) denotes a function which is bounded on \([0, \infty)\) and converges to 0 as \( t \to \infty \). Applying the estimate

\[
\| \partial_k \tilde{\zeta} \|_{L^{2p}}^2 = \| \partial_k \tilde{\zeta}^p \|_{L^2}^2 \lesssim \| \partial_k \tilde{\zeta}^p \|_{L^2} \| \nabla((\partial_k \tilde{\zeta})^p) \|_{L^2} \lesssim \tilde{D}^{1/2} \| \partial_k \tilde{\zeta} \|_{L^{2p}}^p
\]

in (4.15), we obtain

\[
- \sum_k \int \partial_k(u_j \partial_j \tilde{\zeta}) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \leq o(1) \tilde{D}^{1/2} \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^p \leq \frac{\tilde{D}}{8} + o(1) \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^p. \tag{4.16}
\]

For the second term on the right-hand side of (4.13), we use integration by parts and write

\[
\sum_k \int \partial_k(\phi[R, u \cdot \nabla](\rho \eta)) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} = -(2p-1) \sum_k \int \partial_k(\phi[R, u \cdot \nabla](\rho \eta)) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \partial_k(|\partial_k \tilde{\zeta}|^p) \lesssim \| \phi[R, u \cdot \nabla](\rho \eta) \|_{L^{2p}} \sum_k \| \partial_k \tilde{\zeta}^{p-1} \|_{L^{2p/(p-1)}} \| \nabla(|\partial_k \tilde{\zeta}|^p) \|_{L^2} \lesssim \tilde{D}^{1/2} \| \phi[R, u \cdot \nabla](\rho \eta) \|_{L^{2p}} \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^p. \tag{4.17}
\]

For the second factor in the last expression, we have

\[
\| \phi[R, u \cdot \nabla](\rho \eta) \|_{L^{2p}} \lesssim \| \phi \|_{L^\infty} \| (R(u_j \partial_j(\rho \eta)) - u_j \partial_j R(\rho \eta) \|_{L^{2p}} \lesssim \| \partial_j R(u_j(\rho \eta)) \|_{L^{2p}} + \| u \|_{L^\infty} \| \partial_j R(\rho \eta) \|_{L^{2p}} \lesssim \| \rho \|_{L^{2p}} \lesssim 1, \tag{4.18}
\]

using the incompressibility of \( u \) and

\[
\| \rho(t) \|_{L^{2p}} \lesssim 1, \tag{4.19}
\]

which follows from \( \| \rho_0 \|_{L^{2p}} \lesssim \| \rho_0 \|_{H^{1}} \lesssim 1 \) and the \( L^p \) conservation for \( \rho \). (Recall that all constants depend on \( p \).) Thus, by (4.17)–(4.18), we have

\[
\sum_k \int \partial_k(\phi[R, u \cdot \nabla](\rho \eta)) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \leq \tilde{D}^{1/2} \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^{p-1} \leq \frac{\tilde{D}}{8} + C \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}}^{2p-2}. \tag{4.20}
\]
For the third term on the right-hand side of (4.13), we obtain

\[ -\sum_k \int \partial_k (\phi N(\rho)) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \lesssim \sum_k \| \partial_k (\phi N(\rho)) \|_{L^2} \| \partial_k \tilde{\zeta}^{2p-1} \|_{L^{2p/(2p-1)}} \]

\[ \lesssim (\| \nabla \phi \|_{L^\infty} N(\rho) \|_{L^2} + \| \phi \|_{L^\infty} \| \nabla N(\rho) \|_{L^2}) \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \lesssim \| \rho \|_{L^2} \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \]

\[ \lesssim \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \cdot . \]

For the fourth term on the right-hand side of (4.13), we observe that

\[ -\sum_k \int \partial_k (\phi R(\rho(u \cdot \nabla \eta))) \partial_k \tilde{\zeta}^{2p-2} \partial_k \tilde{\zeta} \lesssim \sum_k \| \partial_k (\phi R(\rho(u \cdot \nabla \eta))) \|_{L^2} \| \partial_k \tilde{\zeta}^{2p-1} \|_{L^{2p/(2p-1)}} \]

\[ \lesssim (\| \phi \|_{L^\infty} \| \nabla R(\rho(u \cdot \nabla \eta)) \|_{L^2} + \| \nabla \phi \|_{L^\infty} \| R(\rho(u \cdot \nabla \eta)) \|_{L^2}) \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \]

\[ \lesssim \| \rho \|_{L^2} \| u \|_{L^\infty} \| \nabla \eta \|_{L^\infty} \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \lesssim \sum_k \| \partial_k \tilde{\zeta} \|_{L^{2p}} \cdot , \]

where we used (4.19). For the fifth term on the right-hand side of (4.13), we determine that

\[ -2 \sum_k \int \partial_k (\partial_j \zeta \phi) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \zeta = -\frac{2p-1}{p} \sum_k \int \partial_j \zeta \phi |\partial_k \tilde{\zeta}|^{p-2} \partial_k \zeta \phi (|\partial_k \tilde{\zeta}|^p) \]

\[ \lesssim \| \partial_j \zeta \|_{L^4} \sum_k \| |\partial_k \tilde{\zeta}|^{p-1} \|_{L^\infty} \| \nabla (|\partial_k \tilde{\zeta}|^p) \|_{L^2} \lesssim D^{1/2} \| \nabla \zeta \|_{L^4} \sum_k \| |\partial_k \tilde{\zeta}|^p \|_{L^{4(p-1)/p}} . \]

By the Gagliardo-Nirenberg inequality, we have for the last factor

\[ \| |\partial_k \tilde{\zeta}|^p \|_{L^{4(p-1)/p}} \lesssim \left( \| |\partial_k \tilde{\zeta}|^p \|_{L^2}^{(2p-2)} \| \nabla (|\partial_k \tilde{\zeta}|^p) \|_{L^2}^{(p-2)/(2p-2)} \right)^{(p-1)/p} \]

\[ \lesssim \| |\partial_k \tilde{\zeta}|^p \|_{L^2} \| \nabla (|\partial_k \tilde{\zeta}|^p) \|_{L^2} \lesssim \tilde{D}^{(p-2)/4p} \| |\partial_k \tilde{\zeta}|^p \|_{L^{2p}} , \]

for $k = 1, 2$. Therefore, by Young’s inequality, we conclude that

\[ -2 \sum_k \int \partial_k (\partial_j \zeta \phi) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \zeta \lesssim \tilde{D}^{(3p-2)/4p} \| \nabla \zeta \|_{L^4} \sum_k \| |\partial_k \tilde{\zeta}|^{p/2} \|_{L^{2p}} \]

\[ \lesssim \frac{\tilde{D}}{8} + C \| \nabla \zeta \|_{L^4}^{4p/(p+2)} \sum_k \| |\partial_k \tilde{\zeta}|^{2p/(p+2)} \|_{L^{2p}} . \]

For the final term of (4.13), we integrate by parts and obtain

\[ \sum_k \int \partial_k (\zeta \phi_t - \Delta \phi + u \cdot \nabla \phi)) |\partial_k \tilde{\zeta}|^{2p-2} \partial_k \zeta \]

\[ = -\frac{2p-1}{p} \sum_k \int \zeta \phi_t - \Delta \phi + u \cdot \nabla \phi)) |\partial_k \tilde{\zeta}|^{p-2} \partial_k \tilde{\zeta} \partial_k (|\partial_k \tilde{\zeta}|^p) \]

\[ \lesssim \| \zeta \|_{L^2} \| \phi_t - \Delta \phi + u \cdot \nabla \phi \|_{L^\infty} \sum_k \| |\partial_k \tilde{\zeta}|^{p-1} \|_{L^{2p/(p-1)}} \| \nabla (|\partial_k \tilde{\zeta}|^p) \|_{L^2} \]

\[ \lesssim \tilde{D}^{1/2} \sum_k \| |\partial_k \tilde{\zeta}|^{p-1} \|_{L^{2p}} , \]
using (3.19) and (4.9). Therefore, we have
\[ \sum_k \int \partial_k (\zeta (\phi_t - \Delta \phi + u \cdot \nabla \phi)) |\partial_k \zeta|^{2p-2} \partial_k \zeta \leq \frac{D}{8} + C \sum_k |\partial_k \zeta|^{2p-2}. \]  
(4.21)

Introducing
\[ \psi(t) = \sum_k \int |\partial_k \zeta|^{2p}, \]
we may rewrite (4.13) by applying (4.14), (4.16), (4.20)–(4.21) as
\[ (1 + \psi)' + \frac{\tilde{D}}{2} \lesssim o(1)(1 + \psi) + (1 + \psi)^{(p-1)/p} + (1 + \psi)^{(2p-1)/2p} + \|\nabla \zeta\|^{4p/(p+2)}(1 + \psi)^{p/(p+2)}. \]
(4.22)

It may seem that the first term in (4.22) causes an exponential increase of \( \psi \), but importantly we have the property
\[ \int_0^t (1 + \psi) \lesssim t + \|\nabla \zeta\|^{2p}_{L^{2p}(\mathbb{R}; L^{2p})} \lesssim t, \quad t \geq 0, \]
(4.23)
where we used (4.10) in the second step. Now we show that the inequality (4.23) implies that the growth is algebraic. We divide the inequality (4.22) by \((1 + \psi)^{p/(p+2)}\), obtaining
\[ ((1 + \psi)^{2/(p+2)})' \lesssim o(1)(1 + \psi)^{2/(p+2)} + (1 + \psi)^{(p^2-2)/(2p+2)} + (1 + \psi)^{(3p-2)/(2p+2)} + \|\nabla \zeta\|^{4p/(p+2)}, \]

which upon integration and applying Jensen’s (or Hölder’s) inequality yields for \( t \geq 0 \),
\[ (1 + \psi)^{2/(p+2)} \lesssim 1 + o(1) \int_0^t (1 + \psi)^{2/(p+2)} + \int_0^t (1 + \psi)^{(p^2-2)/(p(p+2))} + \int_0^t (1 + \psi)^{(3p-2)/(2p(p+2))} \]
\[ + \int_0^t \|\nabla \zeta\|^{4p/(p+2)} \]
\[ \lesssim 1 + o(1) t^{1-2/(p+2)} \left( \int_0^t (1 + \psi)^{2/(p+2)} \right) + t^{1-(p^2-2)/(p(p+2))} \left( \int_0^t (1 + \psi)^{(p^2-2)/(p(p+2))} \right) \]
\[ + t^{1-(3p-2)/(2p(p+2))} \left( \int_0^t (1 + \psi)^{(3p-2)/(2p(p+2))} \right) \]
\[ + \left( \int_0^t \|\nabla \zeta\|^{4p/(p+2)} \right)^{p/(p+2)} t^{1-\eta/(p+2)}, \]

where we also used \( \psi(0) = 0 \) since \( \phi \) vanishes in a neighborhood of \( \{ t = 0 \} \). Therefore, recalling (4.23), we have for \( t \geq t_0 \) the inequality
\[ (1 + \psi)^{2/(p+2)} \lesssim t, \]
where we used \( \int_0^t \|\nabla \zeta\|^{4p}_{L^{2p}} \lesssim_t \delta > 0 \) on the last term in (4.24). Raising the resulting inequality to \( (p + 2)/2 \), we obtain
\[ 1 + \psi \leq C_p t^{(p+2)/2}. \]

By the support properties of \( \phi \) and \( \eta \), we get for \( p \geq 1 \)
\[ \|\nabla \omega\|_{L^{2p}(\Omega')} \lesssim \|\nabla \zeta\|_{L^{2p}(\Omega')} + p^{3/2} \lesssim \|\nabla \zeta\|_{L^{2p}} + 1 \lesssim \psi^{1/2} + 1 \lesssim t^{(p+2)/2}, \]
concluding the proof. \( \square \)
In the appendix, we state and prove two Gronwall inequalities needed in the proof of Theorem 2.1. The following lemma is used to show (3.8).

**Lemma A.1.** Assume that \( x, y : [0, \infty) \to [0, \infty) \) are measurable functions with \( x \) differentiable, which satisfy
\[
\dot{x} + y \leq C(x^2 + 1) \tag{A.1}
\]
and
\[
x \leq Cy, \tag{A.2}
\]
for some positive constant \( C \). If
\[
\int_0^\infty x(s) \, ds < \infty, \tag{A.3}
\]
then \( x(t) \leq C \) for \( t \geq 0 \) and
\[
\lim_{t \to \infty} x(t) = 0. \tag{A.4}
\]
Moreover,
\[
\limsup_{t \to \infty} \int_t^{t+a} y(s) \, ds \leq Ca \tag{A.5}
\]
for every \( a > 0 \), where the constant in (A.5) depends on the constants in (A.1) and (A.2).

**Proof of Lemma A.1.** Let \( \epsilon \in (0,1] \), and denote \( b = \sqrt{\epsilon} \). Based on (A.3), there exists \( t_0 > 0 \) such that
\[
\int_t^{t+2b} x(s) \, ds \leq \epsilon, \quad t \geq t_0. \tag{A.6}
\]
Integrating the inequality \( \dot{x} \lesssim x^2 + 1 \) and using (A.6), we obtain
\[
x(t_2) \leq e^{\epsilon t} (x(t_1) + Cb) \lesssim x(t_1) + b \tag{A.7}
\]
for all \( t_1 \) and \( t_2 \) such that \( t_1 \leq t_2 \leq t_1 + 2b \). By (A.6), for every \( t \geq t_0 \), there exists \( \tilde{t} \in [t, t+b] \) such that
\[
x(\tilde{t}) \lesssim \frac{\epsilon}{b},
\]
and thus applying (A.7) with \( t_1 = \tilde{t} \) leads to
\[
x(t_2) \lesssim \frac{\epsilon}{b} + b \lesssim \sqrt{\epsilon}, \quad \tilde{t} \leq t_2 \leq t_1 + 2b, \tag{A.8}
\]
where we used \( b = \sqrt{\epsilon} \) in the last step. The inequality (A.8) holds for all \( t_2 \geq t_0 + a \), and since \( \epsilon > 0 \) is arbitrarily small, (A.4) follows. The inequality (A.5) is obtained by integrating \( y \lesssim x^2 + 1 \) and using (A.4). \( \square \)

The next Gronwall-type lemma is needed to establish (3.16) and (3.17), which are necessary for the proofs of (2.3) and (2.5).

**Lemma A.2.** Assume that \( x, y : [0, \infty) \to [0, \infty) \) are measurable functions with \( x \) differentiable, which satisfy
\[
\dot{x} + y \leq \phi(t)(x + 1) \tag{A.9}
\]
and
\[
x \leq Cy, \tag{A.10}
\]
where \( \phi: [0, \infty) \to [0, \infty) \) is such that \( \phi(t) \leq C \) for \( t \in [0, \infty) \) and \( \phi(t) \to 0 \), as \( t \to \infty \). If also \( x(0) \leq C \), then

\[
x(t) \lesssim 1, \quad t \in [0, \infty)
\]

and

\[
\lim_{t \to \infty} x(t) = 0
\]  

(A.11)

as well as

\[
\limsup_{t \to \infty} \int_t^{t+a} y(s) \, ds = 0,
\]

(A.12)

for every \( a > 0 \).

Proof of Lemma A.2. First, by the boundedness of \( \phi \), we have

\[
x(t) \lesssim 1, \quad t \in [0, T],
\]

for every \( T > 0 \), where the constant depends on \( T \). Next, there exists \( t_0 > 0 \) such that

\[
\dot{x} + \frac{1}{2}y \leq \phi(t), \quad t \geq t_0,
\]

which is obtained by choosing \( t_0 \) so large that the term containing \( x \) on the right-hand side of (A.9) is absorbed in the half of the second term on the left-hand side, cf. (A.10).

Let \( \epsilon > 0 \). Then there exists \( t_1 \geq t_0 \) such that

\[
\dot{x} + \frac{1}{C} x \leq \frac{\epsilon}{2}, \quad t \geq t_1,
\]

which shows that as long as \( x \geq \epsilon \), we have \( \dot{x} + (1/C)x \leq 0 \), implying an exponential decay of \( x \). Therefore, by increasing \( t_1 \), we can assume that

\[
x(t) \leq \epsilon, \quad t \geq t_1.
\]  

(A.13)

Since \( \epsilon > 0 \) was arbitrary, we obtain (A.11). To prove (A.12), note that we may assume

\[
\dot{x} + \frac{1}{2}y \leq \frac{\epsilon}{2}, \quad t \geq t_1,
\]  

(A.14)

by increasing \( t_1 \) if necessary. Integrating (A.14) between \( t \) and \( t + a \), we get

\[
\int_t^{t+a} y(s) \, ds \lesssim x(t) + \epsilon a \lesssim \epsilon(1 + a), \quad t \geq t_1,
\]

where we used (A.13) in the last step. Since \( \epsilon > 0 \) was arbitrary, we obtain (A.12).  

\[\square\]

Acknowledgments

IK and DM were supported in part by the NSF grant DMS-1907992.
ASYMPTOTIC PROPERTIES OF THE BOUSSINESQ EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

References

[ACW] D. Adhikari, C. Cao, and J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, J. Differential Equations 251 (2011), no. 6, 1637–1655.

[ACS..] D. Adhikari, C. Cao, H. Shang, J. Wu, X. Xu, and Z. Ye, Global regularity results for the 2D Boussinesq equations with partial dissipation, J. Differential Equations 260 (2016), no. 2, 1893–1917.

[BS] L.C. Berselli and S. Spirito, On the Boussinesq system: regularity criteria and singular limits, Methods Appl. Anal. 18 (2011), no. 4, 391–416.

[BFL] A. Biswas, C. Foias, and A. Larios, On the attractor for the semi-dissipative Boussinesq equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), no. 2, 381–405.

[BrS] L. Brandolese and M.E. Schonbek, Large time decay and growth for solutions of a viscous Boussinesq system, Trans. Amer. Math. Soc. 364 (2012), no. 10, 5057–5090.

[C] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math. 203 (2006), no. 2, 497–513.

[CD] J.R. Cannon and E. DiBenedetto, The initial value problem for the Boussinesq equations with data in $L^p$, Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), Lecture Notes in Math., vol. 771, Springer, Berlin, 1980, pp. 129–144.

[CF] P. Constantin and C. Foias, Navier-Stokes equations, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[CG] M. Chen and O. Goubet, Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. B 2 (2009), no. 1, 37–53.

[CH] J. Chen and T.Y. Hou, Finite time blowup of 2D Boussinesq and 3D Euler equations with $C^{1,\alpha}$ velocity and boundary, Comm. Math. Phys. 383 (2021), no. 3, 1559–1667.

[CN] D. Chae and H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), no. 5, 935–946.

[CW] C. Cao and J. Wu, Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation, Arch. Ration. Mech. Anal. 208 (2013), no. 3, 985–1004.

[DG] C.R. Doering and J.D. Gibbon, Applied analysis of the Navier-Stokes equations, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.

[DP] R. Dauchin and M. Paicu, Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux, Bull. Soc. Math. France 136 (2008), no. 2, 261–309.

[DWZZ] C.R. Doering, J. Wu, K. Zhao, and X. Zheng, Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion, Phys. D 376/377 (2018), 144–159.

[E] T.M. Elgindi, Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$, arXiv:1904.04795, 2019.

[EJ] T.M. Elgindi and I.-J. Jeong, Finite-time singularity formation for strong solutions to the Boussinesq system, Ann. PDE 6 (2020), no. 1, Paper No. 5, 50.

[GS] Y. Giga and H. Sohr, On the Stokes operator in exterior domains, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (1989), no. 1, 103–130.

[HK1] T. Hmidi and S. Keraani, On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity, Adv. Differential Equations 12 (2007), no. 4, 461–480.

[HK2] T. Hmidi and S. Keraani, On the global well-posedness of the Boussinesq system with zero viscosity, Indiana Univ. Math. J. 58 (2009), no. 4, 1591–1618.

[HKR] T. Hmidi, S. Keraani, and F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, Comm. Partial Differential Equations 36 (2011), no. 3, 420–445.

[HL] T.Y. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, Discrete Contin. Dyn. Syst. 12 (2005), no. 1, 1–12.

[HKZ1] W. Hu, I. Kukavica, and M. Ziane, On the regularity for the Boussinesq equations in a bounded domain, J. Math. Phys. 54 (2013), no. 8, 081507, 10.

[HKZ2] W. Hu, I. Kukavica, and M. Ziane, Persistence of regularity for the viscous Boussinesq equations with zero diffusivity, Asymptot. Anal. 91 (2015), no. 2, 111–124.

[HS] F. Hadadifard and A. Stefanov, On the global regularity of the 2D critical Boussinesq system with $\alpha > 2/3$, Comm. Math. Sci. 15 (2017), no. 5, 1325–1351.
Q. Jiu, C. Miao, J. Wu, and Z. Zhang, The two-dimensional incompressible Boussinesq equations with general critical dissipation, SIAM J. Math. Anal. 46 (2014), no. 5, 3426–3454.

N. Ju, Global regularity and long-time behavior of the solutions to the 2D Boussinesq equations without diffusivity in a bounded domain, J. Math. Fluid Mech. 19 (2017), no. 1, 105–121.

J.P. Kelliher, R. Temam, and X. Wang, Boundary layer associated with the Darcy-Brinkman-Boussinesq model for convection in porous media, Phys. D 240 (2011), no. 7, 619–628.

I. Kukavica, On the dissipative scale for the Navier-Stokes equation, Indiana Univ. Math. J. 48 (1999), no. 3, 1057–1081.

I. Kukavica, Interior gradient bounds for the 2D Navier-Stokes system, Discrete Contin. Dynam. Systems 7 (2001), no. 4, 873–882.

I. Kukavica and W. Wang, Global Sobolev persistence for the fractional Boussinesq equations with zero diffusivity, Pure Appl. Funct. Anal. 5 (2020), no. 1, 27–45.

I. Kukavica and W. Wang, Long time behavior of solutions to the 2D Boussinesq equations with zero diffusivity, J. Dynam. Differential Equations 32 (2020), no. 4, 2061–2077.

I. Kukavica, F. Wang and M. Ziane, Persistence of regularity for solutions of the Boussinesq equations in Sobolev spaces, Adv. Differential Equations 21 (2016), no. 1/2, 85–108.

A. Larios, E. Lunasin, and E.S. Titi, Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion, J. Differential Equations 255 (2013), no. 9, 2636–2654.

M.-J. Lai, R. Pan, and K. Zhao, Initial boundary value problem for two-dimensional viscous Boussinesq equations, Arch. Ration. Mech. Anal. 199 (2011), no. 3, 739–760.

J.C. Robinson, Infinite-dimensional dynamical systems, An introduction to dissipative parabolic PDEs and the theory of global attractors, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

H. Sohr and W. von Wahl, On the regularity of the pressure of weak solutions of Navier-Stokes equations, Arch. Math. (Basel) 46 (1986), no. 5, 428–439.

A. Stefanov and J. Wu, A global regularity result for the 2D Boussinesq equations with critical dissipation, J. Anal. Math. 137 (2019), no. 1, 269–290.

R. Temam, Navier-Stokes equations, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.

R. Temam, Navier-Stokes equations and nonlinear functional analysis, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, second ed., Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

R. Temam, Navier-Stokes equations, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.

R. Temam, Behaviour at time $t = 0$ of the solutions of semilinear evolution equations, J. Differential Equations 43 (1982), no. 1, 73–92.

Department of Mathematics, University of Southern California, Los Angeles, CA 90089
Email address: kukavica@usc.edu

Department of Mathematics, University of Southern California, Los Angeles, CA 90089
Email address: dmassatt@usc.edu

Department of Mathematics, University of Southern California, Los Angeles, CA 90089
Email address: ziane@usc.edu