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To cite this article: F M Dias and Tsv Popov 2007 J. Phys.: Conf. Ser. 63 012005

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EEDF probe measurements: differentiation methods, noise, and error

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Abstract. An instrumentation approach to electron energy distribution function measurements using Langmuir probes is presented. The noise and error limitations of the most common differentiation techniques are analysed, it is shown how instrumental accuracy can be improved or how acquisition time can be drastically decreased, and a pertinent performance comparison of the harmonic vs. the numerical differentiation schemes is made. In addition, we stress the nasty effects of pink and of coherent noise, and we show how they can be minimised.

1. Introduction

As well known, differentiating twice Langmuir probe characteristics is an irreplaceable way to measure the electron energy distribution function (EEDF) [1], which can be achieved in practice using several techniques, namely, electronic differentiation, intermodulation, and numerical differentiation of digitised data. Although some techniques may be best suited for particular experimental situations, whatever the technique is, results have basically the same quality provided the total acquisition time is the same. Indeed, the main constraint to a perfect differentiation is noise, which can be reduced in an error-free way in steady discharges by increasing the acquisition time. The final noise level depends also on the instrumental function (IF) of the differentiator, but increasing IF’s full width at half maximum (FWHM) value as a noise reduction procedure will unavoidably introduce an increased distortion. As a result, since noise is \textit{a priori} an unknown function of probe bias, the IF of the differentiator should be adjusted at measuring/calculation time in an adaptive way so that noise and distortion are kept within an acceptable balance.

Although the probe current noise can be measured, which unfortunately does not seem to be a common practice, predicting noise contribution to the EEDF uncertainty is far from straightforward because each IF has a sensitivity of its own to different spectral regions, and noise spectra depends on probe bias. In addition, distortion evaluation is complicated since it requires the knowledge of higher order derivatives, which usually may be calculated at processing time only. As a result, it is not possible to suggest \textit{a priori} the best differentiating technique for every particular discharge.

In the current work we suggest several schemes and guidelines intended to improve differentiation accuracy. Although some of the above are rather simple, do not require an increased complexity of the measuring system, and the additional processing time is negligible, the resulting improvement is quite noticeable, namely each one of the above techniques is at least made error-free up to the sixth derivative, instead of the typical forth derivative error of standard symmetrical IFs.
2. Error and noise
As mentioned above, all differentiation techniques are basically equivalent but each one has its own features, which may make it more suitable for some concrete application. Since we shall skip here the equipment price factor, any comparison must be focused on performance, which means taking into consideration the accuracy of the results and its standard deviation (STD), i.e., the uncertainty due to what we shall label as noise. In the current work we shall analyse only harmonic and numerical differentiation because all other techniques are nothing else than variants, in the sense that results are directly valid after applying the pertinent transformation, e.g. a Fourier transform.

First, we shall use a simulated, simple example to show how noise and error should be handled. Let us consider as data a Gaussian function (standard deviation equal to 3 XX units) having superimposed noise, and as IF a rectangular window. In figures 1 we present results for window sizes, \( n \), of 0.2 and 6, which were intentionally chosen for providing either deficient filtering or unacceptable distortion.

Figure 1. Noise and distortion arising from filters having extreme FWHM values (solid). Left: too narrow; right: too wide. Noiseless data is also shown for comparison (dashed).

In figures 2 we present results deduced from more sound approaches. In figure 2-left we used an adaptive window, such that error values are kept equal to noise ones. Although window size values, which are also shown in the above figure as a dotted line, may surprise the reader, note that now the “error-bar” is the noise, which was not the case in figures 1.

Figure 2. Comparison between adaptive filtering (left) and fitting (right): results (solid); FWHM values (dot); noiseless data (dashed).

The solid curve presented in figure 2-right results from fitting data by a Gaussian function, which was the correct choice because we knew \textit{a priori} what the solution was, hence we could also calculate the error \( (9\times10^{-3} \text{ RMS YY units}) \). Conversely, in a practical situation accuracy may only be estimated from the fitting error, which becomes nothing else than noise information once a particular fitting function is chosen.
As a summary of the two approaches presented in figure 2, a fitting is useful to test a model, noise does not need to be evaluated unless it is a function of the abscissa and a $\chi^2$ test is being performed, and, though the fitting error provides a measure of the data-model fitting accuracy, all parameters not included in the model will be sources of noise-like errors. Conversely, in a noise vs. error matching approach a model is not needed a priori but noise and error need to be evaluated.

In the continuation we shall present the equations that enable error and noise sensitivity calculations, whose deduction, though mathematically simple, are a lengthy, boring procedure at times. Yet, first we need to spend a few lines recalling the basics on a subject that everybody would like to forget: noise.

2.1. Noise sources
There are two main rules in instrumentation concerning noise: i) intrinsic noise, which is inherent to all physical processes, cannot possibly be avoided, hence the only way to handle it is by being aware of its presence; ii) external noise, i.e., noise arising from sources that do not belong to the measuring path, should be a concern in careless experiments only. As a result, we shall spend reader’s time on noise that cannot be avoided.

Johnson (thermal) noise [2,3] is produced by every unbiased resistor, $R$. The resulting RMS noise current, $I_{\text{therm}}$, when the resistor terminals are short-circuited is:

$$I_{\text{therm}} = \left( \frac{1}{R} \frac{4kT}{\Delta f} \right)^{1/2},$$

where $k$ is the Boltzmann constant, $T$ the noise temperature, and $\Delta f$ is the bandwidth of the measurement. Although lacking a sound physical background, one may argue that the sheath impedance is electrically an incremental resistor, that plasma electrons will produce noise, and that noise and electron temperatures have a similar value. Under the above assumptions, and in case of a Maxwellian EEDF the resulting noise current in probe measurements becomes:

$$I_{\text{therm}} = \left( 4e | I_e | \Delta f \right)^{1/2},$$

where $e$ is the elementary change, and $I_e$ the electron current collected by the probe.

Under the conventional finite nature of charge carriers, electrical current is due to a discrete collection of charges described by a Poisson distribution. The resulting statistical RMS fluctuation, $I_{\text{shot}}$, called shot noise [4], due to each kind of charge carrier is given by:

$$I_{\text{shot}} = \left( 2q | I | \Delta f \right)^{1/2},$$

where $q$ is the charge, and $I$ the individual current. In probe measurements several charge carriers are involved, hence the total shot noise is the result of all contributions. For instance, if ion charge is $e$, the noise current of a probe at the floating potential will be precisely that given by equation (2).

The spectral densities of Johnson and of shot noise are frequency-independent (white noise), hence their noise power values are proportional to that of the bandwidth, and the resulting noise level after averaging decreases as the square root of the number of samples. Yet, once electrical components are biased, noise spectra usually present a characteristic 1/f frequency variation [5] at low frequencies (pink noise), which is a main concern in instrumentation typically below 1 kHz. When 1/f noise dominates, the value of the noise power becomes proportional to the logarithm of the upper/lower-frequency bandwidth limits ratio, and averaging $n$ samples is equivalent to multiply the above ratio by $n$. As a result, averaging becomes a less-effective filter than in the white noise case.

Although discretisation introduces errors, its effect in numerical differentiation is equivalent to a noise current, $I_{\text{disc}}$, given by:

$$I_{\text{disc}} = \frac{\Delta I_{\text{step}}}{2\sqrt{3}},$$

where $\Delta I_{\text{step}}$ is the resolution of the digitiser. Note that discretisation noise may be considered as white noise only if the characteristic value of the difference between consecutive ADC conversions is much
larger than $\Delta I_{\text{sep}}$. Otherwise, spectra present characteristic peaks at one-half of the sampling frequency and its sub-harmonics, which may lead to ineffective filtering.

Power supplies, either that of the discharge maintaining source or those of instrumentation, are main noise contributors. Power supply noise, namely of mains ill-filtered ones, appears at discrete low frequencies, which may make non-effective bandwidth-decreasing filters as a way to reduce this kind of noise. Although a skilled technician may tangle with the original circuitry of power supplies in order to reduce noise, it is not uncommon that the only way to minimise this noise effects is using deconvolution techniques after measuring time.

Since instrumentation is the most important source of noise in many situations, it requires a special attention. In harmonic differentiation, the frequencies of the measured signals are usually around or above 2 kHz, which means that noise may be considered as white, hence its value will be proportional to the square root of bandwidth independently of what harmonic is being measured. Conversely, numerical differentiation relies on analogue-to-digital conversions, and the anti-aliasing filter sets the value for the bandwidth upper limit only, hence being sensitive to pink noise.

Finally, the plasma itself presents instabilities that are detected by a triple probe arrangement as fluctuations of the active probe current, and as fluctuations of the floating potential of the auxiliary electrode [6].

2.2. Noise evaluation

In the harmonic differentiation case, noise can be evaluated from the STD of the measurements, which will be readily obtained making a statistics from several acquisitions carried out using individual integration times shorter than the expected total time required for achieving the desired signal-to-noise ratio ($S/N$). Indeed, averaging $n$ samples produces, in principle, a result equivalent to a $n$-times longer integration time, and due to current data transfer speeds measuring time is not drastically increased.

Since harmonic differentiation usually relies on lock-in amplifiers for detection, we find appropriate to spend a few lines speaking about how a lock-in handles noise. The output noise power is given by the spectral density $\times$ equivalent noise bandwidth of the output filter (ENB) product, whose value manufacturers often claim to be 1/4$\pi$, in case of a first-order filter having a time-constant $\tau$. Yet, since a lock-in usually detects both side bands after the frequency conversion, the ENB value is actually twice the above. Another common error of lock-in users is to collect the signal from the “R” output instead of the “Rcos” or the “Rsin” outputs and to adjust properly the phase of the reference signal. Indeed, amplitudes are derived from the phase-sensitive ones, whose noises levels are the same since the signal fed to both product-detectors is the same. As a result, the $S/N$ of the “R” output is 3 dB worse than that achievable using, for instance, the “Rcos” output only. Note that phase can be electronically adjusted in an automatic fashion without phase information loss or $S/N$ degradation using the other output.

Statistics, as mentioned above, should be carried out in the numerical differentiation case for each measured quantity, i.e., probe bias, the floating potential at the auxiliary electrode, and the probe current. Although our main concern is the STD of the latter quantity, those from the others provide information on the correct operation of the measuring system.

Statistics, as mentioned above for noise evaluation, are to be carried out for each point of the probe characteristic. Yet, noise may also be evaluated from the RMS deviation of neighbour points from the “expected” probe characteristic, e.g. using equation (5) as we shall do below.

We have been implicitly assuming while speaking about noise and statistics that we are dealing with random variables, and with a large enough number of samples, which often is not the case in practical applications because variables may be correlated, and the number of samples is usually kept small for the sake of measuring speed. As a result, statistics may become unreliable, and those made using neighbour points may differ from those using several samples at the same point. Note that averaging a large enough number of random quantities is the only case when noise decreases as the square root of the number of samples, and that the value of the noise reduction ratio will stay somewhere between 0 and 1 in case of correlated quantities.
In figure 3 we present the noise current as measured by a probe immersed in a SW sustained discharge operated at 500 MHz in Argon, as a function of probe bias. The lower trace in the figure represents shot noise, which is negligible in our case but that may become a concern in very low-density plasmas. A noise contributor about 15 times more important is discretisation (12 bit DAC with auto-ranging, which explains the stepped variation). The two dashed lines in figure 3 represent the noise currents due to the floating and bias potential fluctuations, and were deduced multiplying the values of their STDs (statistics from 201 consecutive probe sweeps) by that of the calculated probe admittance. Accounting for the above fluctuations similarity, they must have a common origin, which then can be only the plasma itself or a ground loop. The minimum value of the probe current STD is ≈10^-7 A, which is imposed by the current measuring electronics. Yet, the combined contributions of the above noise sources cannot justify the actual value of the STD, hence we need to consider also sources whose noise is not evident in the probe characteristic, namely electron density and plasma potential fluctuations. The dotted line in figure 3 represents the best fit, which was reached for electron density fluctuations of ≈2% in-phase with plasma potential fluctuations of ≈0.8%, and for 87% of the above in-phase with those of the floating potential. As can also be seen in figure 3, there is a sharp variation of the noise level in the vicinity of the plasma potential (marked by an arrow).

In figure 4 we present results from noise evaluation using neighbour points (dotted line), which were calculated applying an exponential filter (32-point constant) on values given by:

$$I_{\text{noise}} = \frac{|\Delta I - \Delta V \frac{dI}{dV}|}{\sqrt{2}},$$  \hspace{1cm} (5)

where $\Delta I$ and $\Delta V$ are, respectively, the current and voltage difference between adjacent points.

In the same figure is shown the current STD (solid line), whose values are those already presented in figure 3 divided by the square root of the number of samples. A comparison of the above curves clearly shows that some correlation is tangling with the averaging procedures.

The explanation of the above discrepancy is twofold, and can be found analysing the spectra of the probe current. As an example, in figure 5 we present the spectrum at floating potential conditions (solid line), and we can see that it may be described as pink noise (dashed line) with a large superimposed peak at 50 Hz, which obviously arises from the power supply, and accounts for 46% of the total noise power in the current case. First of all, above we have divided the values of the noise current, measured at each point of the probe characteristic, by the square root of the number of samples, which is valid for white noise only, as already mentioned. Second, if probe sweeping time happens to be close to a multiple of 20 ms, as we intentionally did not try to avoid, then the 50 Hz “noise” is vanishingly small in individual statistics but, conversely, it will be fully noticed in a comparison between neighbour points.
The wishful solutions to avoid the “50 Hz” problem depend on the differentiation method to be used: i) In harmonic differentiation, lock-in output should be sampled synchronously with the mains frequency, and the time-shift must be either always the same or as in the time-resolved measurements, i.e., evenly distributed over the period; in the latter case, the RMS deviations should be calculated after removing the 50 Hz component since it will be eliminated by the averaging process. ii) In numerical differentiation, probe characteristics should be measured as in the time-resolved case.

2.3. Harmonic differentiation

The time-dependent, current-voltage transfer function, \( i(t) \), of a device, e.g. a probe characteristic, in case of a pure sinusoidal applied voltage signal of amplitude \( a \) and frequency \( \omega \), is:

\[
i(t) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \frac{d^n}{dv^{2m}} a^{2m} + \sum_{m=0}^{\infty} \frac{1}{(n+m)!m!} \frac{d^{n+2m}}{dv^{n+2m}} a^{n+2m} \cos(n\omega t),
\]

where \( n \) is the order of the harmonic, and \( \frac{d^n}{dv^n} \) is the \( n \)th current-voltage derivative. Note that the amplitude of the \( n \)th harmonic, \( I(n\omega) \), is a signed, real quantity, and that in equation (6) the second term is valid for \( n \geq 1 \).

As a result, \( I(2\omega) \) and \( I(4\omega) \) are given by:

\[
I(2\omega) = \frac{d^2}{dv^2} a^2 + \frac{a^4}{48} d^4 + \frac{a^6}{1536} d^6 + \frac{a^8}{92160} d^8 + \cdots,
\]

\[
I(4\omega) = \frac{d^8}{dv^8} a^8 + \cdots.
\]
\[
I(4\omega) = \frac{a^4}{192} \frac{d^4 i}{dv^4} + \frac{a^6}{3840} \frac{d^6 i}{dv^6} + \frac{a^8}{184320} \frac{d^8 i}{dv^8} + \cdots .
\]  

(8)

From (7) and (8), we readily get:

\[
\frac{d^2 i}{dv^2} = \frac{4}{a^2} \left[ I(2\omega) - 4 I(4\omega) \right] + \frac{a^4}{640} \frac{d^6 i}{dv^6} + \frac{a^6}{23040} \frac{d^8 i}{dv^8} + \cdots ,
\]  

(9a)

if the same test signal is used to measure the amplitudes of both harmonics, or:

\[
\frac{d^2 i}{dv^2} = \frac{4}{a^2} \left[ I(2\omega) - 4 I(4\omega) \right] + \frac{a^6}{345600} \frac{d^8 i}{dv^8} + \cdots ,
\]  

(9b)

in case the amplitudes, \(a_n\), of the test signals used to measure the \(n\)th harmonic are different, and satisfy the condition:

\[
\frac{a_2}{a_4} = \left( \frac{5}{8} \right)^6 .
\]  

(10)

Equations (7) and (9) enable error calculations using several approaches, which will be analysed below.

The approach suggested by equation (9a) needs one single test signal but requires duplicating either the number of lock-in amplifiers or the measuring time so that amplitudes of both harmonics may be measured. For the time being, we shall let the error to be of the order of the 6th derivative.

Under a white noise assumption, neglecting lock-in noise, and for the same total measuring time for both harmonics, according to equations (9) the total measured noise, \(N_{2+4}\), will be:

\[
N_{2+4} = N_2 \left( 1 + 4^2 \right)^2 ,
\]  

(11)

where \(N_2\) is the noise in 2nd harmonic measurements only. From equations (7) and (9a), the 2nd derivative relative error ratio, \(\varepsilon_{2+4}/\varepsilon_2\), and the signal ratio, \(S_{2+4}/S_2\), of combined 2nd + 4th harmonic measurements to that of 2nd harmonic ones only, are:

\[
\varepsilon_{2+4}/\varepsilon_2 = \frac{3}{160} \frac{a_{2+4}^4}{a_2^4} \frac{d^6 i}{dv^6} ,
\]  

(12)

\[
S_{2+4}/S_2 = \frac{a_{2+4}^2}{a_2^2} ,
\]  

(13)

where \(a_{2+4}\) is the amplitude of the test signal in combined measurements. In order to keep a fixed error to 2nd derivative noise ratio, test signal amplitudes must satisfy the condition:

\[
a_{2+4}^6 = \frac{160\sqrt{17}}{3} \frac{a_2^4}{a_2^4} \frac{d^6 i}{dv^6} ,
\]  

(14)

and S/N ratios are related as:

\[
\frac{S_{2+4}/N_{2+4}}{S_2/N_2} = \left( \frac{160}{51} \frac{1}{a_2^2} \frac{d^6 i}{dv^6} \right)^{1/3} .
\]  

(15)

For a Maxwellian EEDF of temperature \(T_e\), the above equation becomes:

\[
\frac{S_{2+4}/N_{2+4}}{S_2/N_2} = 1.464 \left( \frac{kT_e}{\epsilon a_2} \right)^{2/3} .
\]  

(16)

As a numerical example, if acceptable EEDFs were measured by harmonic differentiation in the past using a test signal amplitude \(a_2 e/kT_e = 0.1\), the current scheme provides results whose noise and error values are simultaneously 6.8 times better. Yet, a more striking way to show the improvement is through the measuring time required to achieve results of the same previous quality, which will be reduced by more than 46 times. In the current example, the test signal amplitude must be increased by 5.29 times.
Analysing now the approach suggested by equation (9b), the error here is of the order of the 8th derivative, and equations (12), and (14)-(16) become:

\[
\varepsilon_{2+4}^8 = \frac{1}{28800} \frac{a_{2+4}^6 d^8 i / dv^8}{a_2^5 d^4 i / dv^4}
\]

(17)

\[
a_{2+4}^8 = 28800 \sqrt{17} a_2^4 d^4 i / dv^4 \frac{d^8 i / dv^8}{d^5 i / dv^8}
\]

(18)

\[
\frac{S_{2+4} / N_{2+4}}{S_2 / N_2} = \left(411 - \frac{1}{a_2^4 d^8 i / dv^8}\right)^{1/4}
\]

(19)

\[
\frac{S_{2+4} / N_{2+4}}{S_2 / N_2} = 4.5 \frac{kT_e}{ea_2}
\]

(20)

Under the conditions of the above Maxwellian example, the current scheme provides results whose noise and error values are simultaneously 45 times better or, concerning speed, measurements can be carried out much faster than 2027 times. The new 2nd harmonic test signal amplitude, \(a_{2+4}\), must be 13.6 times larger, and that for 4th harmonic measurements must satisfy equation (10).

At a first glance, the situation just considered seems very attractive, but the actual error is extremely sensitive to deviations of the value of the \(a_4/a_2\) ratio from that stated by equation (10). For instance, using again a Maxwellian EEDF as an example, a 1.9x10^-4 \((a_4/e kT)\) relative error in the above ratio makes the error arising from the 8th derivative to be equal to that from the 8th derivative. Since the error sensitivity on the above ratio decreases fast as test signal amplitudes increase, this approach may be considered in case of large test signals, e.g. in noisy conditions.

Note that if the 2nd and 4th harmonics are to be measured simultaneously using the above approach, test signal frequencies must be different, and equation (6) cannot be used since the pure sinusoidal test signal assumption is not valid. In addition to the need to include intermodulation products, which is a lengthy, boring procedure, also the test signal frequencies must be chosen carefully. Otherwise, intermodulation products due to higher-order derivatives will be detected. A theoretical evaluation of the above problem is very complex, and may not be carried out a priori since it must take into account the actual value of higher-order derivatives, and the frequency roll-off of lock-in filters.

In the above numerical examples we have evaluated performance doing a comparison with standard harmonic differentiation. Yet, under our point of view a good differentiator must be a standalone system, hence able to find by itself the correct value of test signal amplitude, which means having the ability to evaluate noise and error. Since the former was already analysed in section 2.2, we still need to focus on error evaluation, which means evaluating the value of higher order derivatives. Excluding immediately higher harmonic measurements for economical reasons, the obvious solution is additional numerical differentiation, whose practical implementation is given in section 2.7.

2.4. Numerical differentiation

Numerical differentiation is a straightforward application of the convolution theorem, which leads to:

\[
f \star \frac{d^n g}{dx^n} = g \star \frac{d^n f}{dx^n},
\]

(21)

where \(f(x)\) is the function to be differentiated, \(g\) is whatever function the user may like, which we shall call filter, and the “*” symbol stands for convolution. We shall assume in the continuation that averaging (zero-order) filters are “1” normalised, i.e.:

\[
\int_{-\infty}^{+\infty} g \, dx = 1.
\]

(22)
The three filter parameters of interest here are the error, \( \varepsilon \), the RMS response to noise, which we shall call filter’s noise factor, \( N_g \), and filter’s frequency response, \( G \). Unless otherwise stated, noise will be considered as white.

\[
\varepsilon(x_0) = g \ast f - f(x_0)
\]

\[
N_g = \left( \frac{\int g^2 dx}{\int dx} \right)^{1/2}
\]  

(23)  

(24)

Whenever possible in the continuation we shall make an analytical analysis to avoid having to include the actual number of filter points as an additional parameter, and we choose Gaussian filters for their ability to be indefinitely differentiated. For simplicity, the order of the derivatives will be included as an index. Hence, the filter, \( g_0 \), the noise factor, \( N_{g0} \), and the frequency response, \( G_0(1/L) \), where \( L \) is the period, are:

\[
g_0 = \frac{1}{\sigma \sqrt{\pi}} e^{-\left( \frac{x}{\sigma} \right)^2}
\]

\[
\sigma = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{2^n n!} dx^{2n}
\]

\[
N_{g0} = \sigma^{-1/2} \pi^{-1/4} 2^{-1/4}
\]

\[
G_0(1/L) = \exp \left[ -\frac{\pi \sigma}{L} \right]
\]

(25)  

(26)  

(27)  

(28)

At this point we find appropriate to compare equations (27) and (28). Indeed, the former shows that white noise will be eliminated proportionally to the square root of filter’s characteristic length, \( \sigma \), while the latter shows that noise at discrete frequencies is filtered in a completely different way, and that effective filtering is possible only when the period is small enough as compared to the filter’s characteristic length. As an immediate result, a fast digitiser is the worst possible approach as a source of data for numerical differentiation in case of detectable low-frequency noise, e.g. at 50 Hz.

Although filter’s FWHM values are commonly used as the order of magnitude of the achievable x-resolution after filtering, hence independent on actual data, we also wish to stress that error is extremely dependent on actual derivative values, as equation (26) clearly suggests.

If the 2\(^{\text{nd}}\) derivative, \( g_2(v) \), of a Gaussian filter, \( g_0(v) \), is convoluted with the probe characteristic, \( i(v) \), we get:

\[
i \ast g_2 = \frac{d^2 i}{dv^2} + \frac{\sigma^2}{4} \frac{d^4 i}{dv^4} + \frac{\sigma^4}{32} \frac{d^6 i}{dv^6} + \frac{\sigma^6}{384} \frac{d^8 i}{dv^8} + \cdots
\]

\[
N_{g2} = \frac{\sqrt{3}}{\sigma^2} N_{g0}.
\]

(29)  

(30)

Equations (27) and (30) provide another immediate, interesting result. Indeed, while in the harmonic differentiation case \( S/N \) was proportional to the square of test signal amplitude, in the numerical differentiation case it increases as a power 5/2 of filter’s characteristic length. As a result, though due to pink noise the digitiser’s output may be noisier than that of a lock-in, numerical differentiation has a greater ability to reduce noise.

As equation (29) suggests, the 4\(^{\text{th}}\) order term of the error will be eliminated if we use

\[
g_{0+2} = g_0 - \frac{\sigma^2}{4} g_2 = \left( \frac{3}{2} - \frac{v^2}{\sigma^2} \right) \frac{1}{\sigma \sqrt{\pi}} \exp \left[ -\frac{(v/\sigma)^2}{2} \right]
\]

as a filter instead of \( g_0 \). Making the convolution of data with \( g_{2+4} \), i.e., the 2\(^{\text{nd}}\) derivative of \( g_{0+2} \), we get:
\[ i \ast g_{2+4} = \frac{d^2 i}{dv^2} - \frac{\alpha^4}{32} \frac{d^6 i}{dv^6} - \frac{\alpha^6}{192} \frac{d^8 i}{dv^8} - \ldots \]  \hspace{1cm} (32)

\[ N_{g_{2+4}} = \frac{\sqrt{91}}{4} N_{g_{2+4}}. \]  \hspace{1cm} (33)

As before, we shall analyse results for a constant error-to-noise ratio, which happens when the characteristic lengths of \( g_2 \) and \( g_{2+4} \), respectively \( \sigma_2 \) and \( \sigma_{2+4} \), are related as:

\[ \sigma_{2+4}^{13/2} = 2\sqrt{91}\sigma_2^{12/2} \frac{d^4 i}{dv^4} \frac{d^4 i}{dv^6}. \]  \hspace{1cm} (34)

Under the above condition, the error ratio becomes:

\[ \frac{\varepsilon_{2+4}}{\varepsilon_2} = \left( \frac{\sigma_{2+4}^{12/2}}{\sigma_2} \right)^{5/13} \left( \frac{d^4 i}{dv^6} \frac{d^4 i}{dv^4} \right)^{5/13}, \]  \hspace{1cm} (35)

or:

\[ \frac{\varepsilon_{2+4}}{\varepsilon_2} = 0.767 \left( \frac{e\sigma_2}{kT_e} \right)^{10/13} \]  \hspace{1cm} (36)

in case of a Maxwellian EEDF. Although equations (35) and (16) are not too different, we wish to stress that the inclusion of the 4th derivative in numerical differentiation produces improvements that are more sensitive to the value of the characteristic filter length (power 10/13) than those in harmonic differentiation are concerning the amplitude of the test signal (power 2/3).

As a numerical example, if acceptable EEDFs were obtained by the convolution of data with \( g_2 \) when \( \alpha_2 e/kT_e = 0.1 \), the convolution with \( g_{2+4} \) for \( \alpha_{2+4} = 3.2 \alpha_2 \) provides results whose noise and error values are simultaneously 7.7 times better. As an alternative, measurements can be made much faster than 59 times without quality lost.

A further improvement can be expected if

\[ g_{0+2+4} = g_0 - \frac{\sigma_2^2}{4} g_2 + \frac{\sigma_4^4}{32} g_4 \]  \hspace{1cm} (37)

is used as a filter. Then we get:

\[ i \ast g_{2+4+6} = \frac{d^2 i}{dv^2} + \frac{\alpha^6}{384} \frac{d^8 i}{dv^8} - \ldots \]  \hspace{1cm} (38)

\[ N_{g_{2+4+6}} = \frac{\sqrt{49707}}{32} N_{g_{2+4}}. \]  \hspace{1cm} (39)

The above performance-improving scheme can be indefinitely iterated either for the same \( \sigma \) value for all \( g_n \), as we have done, or for different \( \sigma_i \) values in a similar way as that carried out in section 2.3. Yet, all the above approaches rely on an increased filter’s characteristic length in order to keep noise value under control, which means that truncating a series expansion at a given order may become unacceptable in case of steep probe characteristic variations, e.g. as happens in the vicinity of the plasma potential. On the other hand, the numerical differentiation method, for being carried out by a convolution whose inverse is also a convolution, has the unique theoretical ability to provide results that can be either further integrated or differentiated without a net loss of information. As a result, we consider that a 2nd derivative calculated with a 6th order error is a good preliminary approach, and that further corrections may always be carried out at later stage.

2.4.1. The 1/f noise problem. To our knowledge, the physics of 1/f noise is not yet understood, its mathematical handling is complex, and the required amount of data so that statistics may become close to meaningful is not acceptable under most experimental conditions. As a result, all the information presented below concerning discrete approaches should be understood as indicative only.
In case of 1/f noise, and under a continuum approach, the noise, \( N \), within a bandwidth \([f_1, f_2]\) is given by:

\[
N = S_0 \sqrt{f_0} \sqrt{\ln(f_2 / f_1)},
\]

(40)

where \( S_0 \) is the value of the noise density at a frequency \( f_0 \).

When we are sampling data, the anti-aliasing filter sets the value of the upper cut-off frequency, while the lower cut-off frequency value is, in principle, given by the inverse of the total acquisition time. Indeed, the latter is valid only if an average is a meaningful concept, which is not the case in the presence of 1/f noise since averages never reach steady values, i.e., there is some kind of “DC noise”. Hence, the data presented as a solid line in figure 4 in the vicinity of the floating potential is underestimated by a factor of the order of \( (\ln(1024 \times 201/2))^{1/2} \approx 3.4 \) while that presented as a dotted line is slightly overestimated due to correlations between the statistics at the two considered points.

Once again we can obtain an interesting result. Indeed, conversely to white noise situations, the way as data is sampled is not irrelevant under a 1/f noise case since it has an effect on the \( f_2/f_1 \) ratio. For instance, should each point of the probe characteristic be sampled consecutively before probe bias is changed, instead of having been sampled once per probe characteristic, the noise value would be reduced to 63%.

Fortunately, since differentiation is equivalent to multiplying the Fourier transform by the angular frequency, derivatives’ spectra remain finite, and noise calculations can be carried out analytically in case of 1/f noise. As an example, the noise of a 2\textsuperscript{nd} derivative calculated using a Gaussian filter, \( N_2 \), is:

\[
N_2 = \frac{2S_0 \sqrt{f_0}}{\sigma^2}.
\]

(41)

Since noise is actually pink, and since there is always a non-zero low-frequency limit, equation (41) represents the worst possible situation for numerical differentiation, which enables us to state that in numerical differentiation S/N is never a slower changing function of filter’s characteristic length than it is of the test signal amplitude in harmonic differentiation.

2.5. Harmonic vs. numerical differentiation

To our knowledge, a comparison of harmonic vs. numerical differentiation has never been presented for a real, practical case, i.e., keeping the same total acquisition time, considering actual noise levels, and matching noise vs. error. Accounting for the number of parameters involved, we shall make here such a comparison for the simple case of a Maxwellian EEDF, and assuming that: lock-in and anti-aliasing filters are ideal, i.e., their response is 1 or 0, respectively, below or above cut-off frequencies; the test signal frequency is set at 1 kHz; the digitiser contributes with 50% of the total noise power; noise is pink, and the relative importance of white noise is set by the 1/f–white noise intersect frequency, hence noise will be 100% white when the above frequency is 0.

| \( S_0 \) @ 1 kHz (A Hz\(^{-1}\)) | Harmonic differentiation | Numerical differentiation |
|----------------------------------|--------------------------|--------------------------|
| \( \frac{d^2i}{dv^2} \) relative error | 1.6×10\(^{-4}\) | 5.8×10\(^{-2}\) | 2.2×10\(^{-2}\) | 1.4×10\(^{-4}\) |
| \( a \) or \( \sigma (e/kT_0) \) | 0.044 | 0.836 | 0.109 | 0.274 |
| Intersect frequency (kHz) | 1 | 0 | 1 | 50 |
| Sampling rate (Hz) | 1 | 50 | 50 | 50 |

Table 1. Performance of the 2\textsuperscript{nd} harmonic vs. numerical differentiation by a Gaussian filter.

A summary of the comparison under the above conditions is presented in Table 1. Note that though values of the noise spectral density in the three rightmost columns may be different, the actual white noise density is precisely the same. We can see from the above columns that 2\textsuperscript{nd} derivative results from numerical differentiation are more accurate than those from harmonic differentiation in case the noise is purely white. Conversely, harmonic differentiation is a more accurate technique in the pink
noise case under the current conditions. Yet, both methods would provide a similar accuracy if the white noise level is increased by about one order of magnitude.

Note that these results are independent of the actual value of the electron current as long as noise is proportional to the current.

We wish to call reader's attention to test signal amplitude values, which we consider optimum, i.e., which provide the same value for error and noise. Indeed, much smaller amplitudes are commonly used, which means that precious measuring time has been wasted in eliminating noise. Even more striking is the data presented in leftmost column for harmonic differentiation in table 1, where only lock-in noise was considered, and we got 44 mV as the optimum value for the test signal amplitude (1 eV Maxwellian EEDF, 1 point per second, and an electron current of 1.7x10^-6 A), which means that much larger amplitudes should be used in a practical situation under the above conditions.

2.6. The noise-induced error problem.

We have been speaking mainly about noise that can be easily measured, and that basically introduces a "noise-like" uncertainty on results, i.e., averages of the measurements asymptotically converge to correct values as the number of points increase. Yet, due to the non-linearity, which in fact is what we want to measure, noise also produces a real error, whose value obviously depends on its actual level. Although the solution to this problem should start by the probe circuitry, i.e., by probe compensation, such techniques depend on the spectra of the disturbing signals, they are often incompatible, and may not be completely effective. In addition, since users may not be aware of all noisy contributions, the outcome may prove to be a pitfall. Hence, we find pertinent to suggest supplementary techniques and correction procedures.

A full, theoretical analysis of noise-induced error is lengthy, and requires the knowledge of a vector spectrum of the disturbance as well as the frequency response of the probe & plasma circuit. Hence, for shortness without a net lost of generality, we shall consider here the simple cases of coherent, single-frequency noise under numerical differentiation, namely 50 Hz from the mains, and broadband noise under harmonic differentiation.

Let us recall that making averaged probe measurements, <i>, concurrently with an applied periodic signal produces distorted results, and that the resulting DC shift is given by the first term on the right side of equation (6) in the sinusoidal case. Yet, results can readily be corrected using the 2nd and 4th harmonic amplitudes as:

\[
<i> - I(2\omega) + I(4\omega) = i + \frac{a^6}{120 \times 192} \frac{d^6}{dv^6} + \frac{a^8}{1024 \times 720} \frac{d^8}{dv^8} + \cdots,
\]

where \( i \) above stands for the undisturbed probe current. Equation (32) then becomes

\[
<i> - I(2\omega) + I(4\omega) \ast g_{2+4} = \frac{d^2}{dv^2} \frac{a^4}{32} \frac{d^4}{dv^4} \frac{1}{192} \left( \sigma^6 - \frac{a^6}{120} \right) \frac{d^6}{dv^6} - \cdots,
\]

which, in comparison to the noiseless case, shows that accuracy does not decrease while higher order harmonics and derivatives must be taken into consideration.

The good news is that the above scheme can be implemented in practice without complex circuitry requirements, and alternatives depend on the 4th harmonic value: if it falls inside the bandwidth of the anti-aliasing filter then the user simply needs to connect two narrow, band-pass filters (tuned at 2\omega, and at 4\omega) to the output of the transimpedance amplifier, otherwise an additional amplifier, e.g. inductively coupled, is needed. Additional simple circuitry can detect signed amplitudes, which should be combined electronically in order neither to waste acquisition time nor increase the number of acquisition channels. A simpler hardware solution is possible in case time-resolved measurements can be made, but then data storage, software, and timing requirements need to be changed in order to deduce amplitudes.
Considering that we are now combining three quantities, hence we also have three noise sources, some increase of the random noise level is expected. The resulting level depends on the hardware approach, namely on the amplitude detector used. Yet, since the increase can be kept low, we do not consider this issue as a concern.

Unfortunately, disturbances arising from the mains may not be considered as due to a sinusoidal signal, which means that the correct solution to such problem may be much more complex. Anyway, the above results already enable the authors to improve their simple, previous approach on how to handle distortions due to ill-filtered power supplies [7].

In the next simple example we shall go one step forward by considering a general, periodic disturbance, which we shall use as a starting point towards the continuum spectrum case.

Let us assume now harmonic differentiation, and that some time-periodic disturbance, of fundamental angular frequency \( \omega_0 \), is superimposed over the user applied signal, i.e., the actual test signal is:

\[
\nu(t) = a \cos(\omega_0 t) + \sum_{n=-\infty}^{\infty} a_n \cos(n \omega_0 t - \varphi_n).
\]  

Under the above conditions, the order of magnitude of the absolute error, \( \varepsilon_{nj} \), which appears at lock-in output due to the \( n^{th} \) and \( j^{th} \) harmonics of the disturbing signal, is:

\[
\varepsilon_{nj} = O\left( a^2 a_n^j a_{jxn} \cos(\varphi_{jxn} - j \varphi_n) \frac{d^{j+3} i}{dv^{j+3}} \right) \quad j \geq 1,
\]

which enables us to extract two very important conclusions: i) the relative error value does not depend on the main test signal amplitude value, \( a \); ii) the lock-in bandwidth cannot be used to reduce error because the above intermodulation products are exactly at the frequency \( 2\omega_0 \). These conclusions can be readily understood if we keep in mind that \( \varepsilon_{nj} \) results from the intermodulation of the 2\( \text{nd} \) harmonic of the test signal with the DC components generated by the disturbance. Note that the major error contribution usually arises from \( j = 1 \), hence due to the 4\( \text{th} \) derivative.

Although a vector spectrum of the disturbance needs to be known in order to integrate equation (45) in the general case, the solution for random signals is much simpler. Assuming white noise, and that the probe & plasma circuit response is frequency-independent, we get:

\[
\varepsilon = O\left( a^2 A_N^2 \frac{d^4 i}{dv^4} \right),
\]

where \( A_N \) is the RMS value of the disturbance.

We find imperative to aware readers that \( A_N \) can be a huge quantity, hence the same can happen to the noise-induced error. Indeed, the bandwidth that must be considered for the evaluation of \( A_N \) is not whatsoever related to the instrumentation one, which means that error value given by equation (46) can be orders of magnitude larger than that of instrumentation noise.

As we have already stressed, the solution to minimise noise-induced error requires the knowledge of the spectrum of the disturbing signal. Yet, since \( a_e \) values (or \( A_N \)) can seldom be measured directly, we must rely on the probe current to derive information, which can be a bonus instead of an extra burden. Indeed, provided the probe circuitry is well characterised, a measured spectrum of the current accounts for derivatives and for probe & plasma frequency response in a way not too different from how the disturbance produces error (see below).

The way the spectra of the probe currents can be measured as a function of probe bias depends on the actual spectra: individual product detectors for discrete cases; dedicated equipment for general, more complex situations. Whatever the case, the above measurements can be made concurrently with the acquisition of data intended for the differentiation purpose.

Unfortunately, an approach as simple as that described by equation (43) is seldom acceptable in a general case due to a complex blending of spectral contributions, i.e., derivatives and spectral components do not contribute in the same way for the spectra as they do for error, which means that
we have to solve an inverse integral problem (IIP). The good news is that runtime is not wasted because calculations are carried out a posteriori. Although we do not intend to impose any particular way on how to solve IIPs, we shall assume that most readers will appreciate being aware that 2nd derivatives with the correction from equation (45) or (46) can be readily calculated iteratively, namely in case of small errors.

2.7. Error evaluation
In sections 2.3 and 2.4 we have shown that the evaluation of the error introduced by a non-ideal IF requires the knowledge of derivatives of order higher than that intended for the differentiation itself. As the differentiation procedure should account for all derivatives that can be deduced accurately, only estimations of higher order ones should be considered for error evaluation.

Whatever the differentiation method is, numerical differentiation is the only way to evaluate higher order derivatives at measuring-time without the need for additional hardware. Although a superior differentiation method can be achieved combining harmonic and numerical differentiation, for shortness we shall focus here the individual solutions described by equations (9a) and (32), which enable the evaluation of 2nd derivatives with an error O(d^6i/dv^6). Note that modifying the following approach for different orders is a simple exercise.

Error evaluation is simpler in the numerical differentiation case for not being carried out at measuring time. Conversely, in the harmonic case the amplitude of the test signal should be adjusted at a time when the data required for the derivative is not yet available. Although trial-and-error would be the easiest way, such approach would waste measuring time. Hence, we prefer to use a look-ahead procedure to guess 6th derivatives at measuring time.

In order to keep 6th derivative noise at a low level, some simplifying assumption must be included. One possibility is to calculate the 2nd-to-4th derivative ratio, which is the same for any n-th-to-(n+2)th derivative ratios in case derivatives may be acceptably described by exponential functions of the probe bias within a vicinity of the order of the test signal amplitude. From equations (7) and (8) we get:

\[
\frac{I(2\omega)-4I(4\omega)}{I(4\omega)} = \frac{48}{a^2} \frac{d^2i/dv^2}{d^4i/dv^4} \left(1 - \frac{a^2}{20} \frac{d^6i/dv^6}{d^4i/dv^4} \right),
\]

and we can calculate an equivalent local temperature, T, as:

\[
\frac{kT}{ea} \approx \frac{I(2\omega)}{48I(4\omega)} = \frac{1}{30},
\]

which enables estimating the value of the 6th derivative as:

\[
\frac{d^6i}{dv^6} = \frac{4}{a^2} I(2\omega) \left[1 + \frac{1}{12} \left(\frac{ea}{kT}\right)^2 \right]^{-1} \left(\frac{e}{kT}\right)^4 (49)
\]

\[
N_6 = 100 \left(\frac{d^6i}{dv^6} \frac{N_2}{I(2\omega)} \frac{kT}{ea}\right)^2.
\]

Yet, as the “100” constant in equation (50) suggests, 6th derivatives evaluated in this way have a very high relative noise, and additional filtering is required. A well-suited choice under a local temperature approach is an exponential filter (the RC circuit numerical equivalent) since it introduces a shift without affecting the value of T. The standard discrete exponential filter having a constant \(\tau\) is:

\[
g(i) = \frac{1}{m+1} \left(\frac{m}{m+1}\right)^i
\]

\[
N_g = (2m + 1)^{1/2},
\]

where \(m = \tau/\Delta\). Yet, the above shift can be eliminated if \(g(i)\) is multiplied by
\[ K = m \left( 1 - e^{-\frac{\Delta}{T}} \right) + 1, \] (53)

and the resulting noise, \( N_k \), on \( K \) evaluation is:

\[ N_k = \frac{N_h \, mK^2}{I(2\omega)} \left( 2m+1 \right)^{-1} + e^{-\frac{2\Delta}{T}} \left( (m+1)^2 - m^2 e^{-\frac{2\Delta}{T}} \right)^{-\frac{1}{2}}. \] (54)

Conversely, an exponential filter with the correction given by equation (53) is a \( T \)-fitting procedure, \( i.e., \)

\[ T = -\frac{\Delta}{\ln \left( 1 - \frac{1}{m} \left( \frac{I(2\omega)}{I(2\omega) g^{-1}} \right) \right)} \], (55)

and the resulting relative noise is:

\[ \frac{N_T}{T} = \frac{N_k \, T \, e^{-\frac{\Delta}{T}}}{m \Delta}. \] (56)

Equations (49) and (55) provide \( n^{th} \) derivative values having a noise level,

\[ N_n = \left| \frac{d^n i}{dv^n} \right| \left( \frac{N_h}{I(2\omega)} \right)^2 + \left[ (n-2) \frac{N_T}{T} \right]^2, \] (57)

which is usually rather acceptable. As an example, for \( T/\Delta = 100, m = 99, \) and a \( 2^{nd} \) harmonic amplitude measured with a relative noise level of 2%, that of the \( 6^{th} \) derivative is about 3.4%.

2.8. Less-common instrumentation

We have shown that differentiation accuracy can be readily increased using wide IFs and accounting for higher-order derivatives, which is an approach that only requires some additional equipment under numerical differentiation if noise-induced error tangles with accuracy. The situation is not too different under harmonic differentiation because the \( 2^{nd} \) and \( 4^{th} \) harmonic amplitudes can be measured by one single lock-in within the same measuring-time, and an increased accuracy will be always noticed. Nevertheless, we wish to recall the reader that a wider IF also decreases the apparent instrumentation noise level, which means that the current approach also turns low-price instruments into an option to be considered.

Excluding here tokamaks, RF discharges, and other situations that require fast, time-resolved probe measurements (see \( e.g. \) the “low-price” instrumentation approaches suggested in [8]), a now-a-days data acquisition board, able to supersede the most common requirements for EEDF measurements, is one of the least expensive devices in any laboratory. Hence, we shall focus on less-common instrumentation intended for harmonic-like differentiation. In addition, far from intending to minimise those who have been contributing towards the upgrade of differentiation schemes, for shortness we shall mention here only two alternatives.

As an alternative to sinusoidal probing and to lock-in detection, a DC coupled stepped signal and the appropriate detection circuitry has been suggested [9]. Basically, a \( \pm \Delta V \) signal is superimposed on the probe bias for 50% of the time, which allows measuring the undisturbed as well as the resulting upper- and lower-values of the probe current from which EEDFs can be deduced and visualised at measuring time. Since undisturbed probe currents can actually be measured, \( i.e., \) data is free from errors due to non-vanishing test signal amplitudes, \( 2^{nd} \) derivatives can also be accurately deduced using numerical differentiation, which is a redundancy check. Although data statistics were performed at measuring-time, adjusting test signal amplitudes required user intervention due to hardware
limitations. Anyway, to our knowledge this is the earliest work where a numerical-intermodulation combined differentiation scheme having an adaptive IF was presented.

In a recent paper [10] distortion meters were suggested as a convenient alternative to lock-in detection. Indeed, there are superb distortion meters, which fashion committed to surplus, and which are currently available by a small faction of its original value. Since distortion is a measure of RMS contributions from the 2nd and higher-order derivatives, the IF of a distortion meter is sharper than that of a lock-in. In addition, the relative weight of higher-order derivatives can be readily tailored by filtering, which means that users have a somewhat extended possibility to adjust the IF at will.

As a bottom line, we wish to state that differentiation accuracy can be further improved using non-sinusoidal test signals, which allow tailoring the IF under lock-in detection also. Yet, as we have shown using simple examples in section 2.6, the mathematical handling becomes extremely complex.

3. Conclusions
We have devoted this work to foster the accuracy of EEDF measurements from an instrumentation point of view. Since our aim is to achieve measurements having similar error and noise values, which we consider to be the condition for minimum uncertainty, we have shown how error and noise can be measured at a time when the IF of the differentiator can be adaptively adjusted, hence avoiding time-consuming, trial-and-error approaches. In order to reach the above results we made a comprehensive analysis of harmonic and of numerical differentiation, which has enabled us to present guidelines on how to avoid common pitfalls that less-experienced users are not aware of.

Although this paper is the result of an expertise gathered from a long experience dealing with probe measurements, authors consider the above results as being still far from ultimate, and are currently working on a combined harmonic-numerical differentiation scheme with automatic, adaptive IF capability, intended either to improve accuracy above that of an 8th derivative or to increase measuring speed by a factor far above one thousand.

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