TRIVIALITY OF THE HIGHER FORMALITY THEOREM

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Abstract. It is noted that the higher version of M. Kontsevich’s Formality Theorem is much easier than the original one. Namely, we prove that the higher Hochschild-Kostant-Rosenberg map taking values in the $n$-Hochschild complex already respects the natural $E_{n+1}$ operad action whenever $n \geq 2$. To this end we introduce a higher version of the braces operad, which—analogously to the usual braces operad—acts naturally on the higher Hochschild complex, and which is a model of the $E_{n+1}$ operad.

1. Introduction

Let $A$ be any smooth commutative $k$-algebra essentially of finite type. We may consider $A$ as an associative $k$-algebra only, say $A_1$. As such we may form its Hochschild cochain complex

$$C(A_1) = \bigoplus_{k \geq 0} \text{Hom}_k(A^\otimes k, A)[-k]$$

endowed with the Hochschild differential. The cohomology of $C(A_1)$ is computed by the Hochschild-Kostant-Rosenberg Theorem, which states that the Hochschild-Kostant-Rosenberg (HKR) map

$$\Phi_{HKR}: S_A\left(\text{Der}(A)[-1]\right) \to C(A_1)$$

sending a $k$-multiderivation to the obvious map $A^\otimes k \to A$ is a quasi-isomorphism of complexes. Note that $S_A\left(\text{Der}(A)[-1]\right)$ is endowed with the zero differential.

In fact, the degree shifted complexes $S_A\left(\text{Der}(A)[-1]\right)[1]$ and $C(A_1)[1]$ are endowed with differential graded (dg) Lie algebra structures, with the Schouten bracket and the Gerstenhaber bracket, respectively. The central result of deformation quantization is M. Kontsevich’s Formality Theorem [14], stating that there is an $\infty$-quasi-isomorphism of dg Lie algebras

$$S_A\left(\text{Der}(A)[-1]\right)[1] \to C(A_1)[1]$$

extending the HKR map.
Actually, $S_A\left(\text{Der}(A)[-1]\right)$ also carries the structure of a Gerstenhaber algebra (or, $e_2$ algebra). Let Kontsevich’s result has been strengthened by D. Tamarkin [18], who showed that there also exists an $\infty$-quasi-isomorphism of homotopy Gerstenhaber algebras

$$S_A\left(\text{Der}(A)[-1]\right) \longrightarrow C(A_1)$$

extending the HKR map, for some choice of homotopy Gerstenhaber structure on the right hand side.

There is a natural generalization of the objects involved to the higher setting. Denote by $e_n = H_{-\bullet}(E_n)$ the homology operad of the little $n$-disks operad, without zero-ary operations. For $n \geq 2$ it is isomorphic to the $n$-Poisson operad $\text{Pois}_n$ (see [11]) and hence generated by two binary operations, a commutative product of degree 0, and a compatible Lie bracket of degree 1 $-n[2,3]$. We may consider the commutative algebra $A$ as an $e_n$-algebra, say $A_n$, with trivial bracket. We assume that $n \geq 2$. The operad $e_n$ is Koszul [11], and by the standard Koszul theory of operads (see [16, chapter 7], [12]) we may define an $e_n$-deformation complex which we denote by $C(A_n)$. There is a version of the Hochschild-Kostant-Rosenberg Theorem stating that the natural inclusion

$$\Phi^{n}_{\text{HKR}}: S_A\left(\text{Der}(A)[-n]\right) \rightarrow C(A_n)$$

is a quasi-isomorphism of complexes.

There is a natural $e_{n+1}$ algebra structure on $S_A\left(\text{Der}(A)[-n]\right)$, with product being the symmetric product and bracket being a degree-shifted version of the Schouten bracket. Similarly, there is an explicit $\text{hoe}_{n+1}$ structure on $C(A_n)$, constructed by D. Tamarkin [19]. Here $\text{hoe}_{n+1} = \Omega(e^*_n)$ is the minimal resolution of the operad $e_{n+1}$, i.e., the cobar construction of the Koszul dual cooperad $e^*_n \cong e^*_{n+1}\{n+1\}$, cf. [16 sections 6.5, 13.3]. The higher formality conjecture states that the (quasi-isom)orphism $\Phi^{n}_{\text{HKR}}$ may be extended to an $\infty$-(quasi-isom)orphism of $\text{hoe}_{n+1}$ algebras.

The content of the present paper points out that this conjecture is somehow trivial.

**Theorem 1.** For $n \geq 2$ the HKR map $\Phi^{n}_{\text{HKR}}: S_A\left(\text{Der}(A)[-n]\right) \rightarrow C(A_n)$ is already a quasi-isomorphism of $\text{hoe}_{n+1}$ algebras.

This result might be known to the experts, but the authors are unaware of any reference. The proof boils down to a straightforward direct calculation.

**Remark 1.** As will be clear from the proof, the statement of Theorem 1 holds true for $A$ the algebra of smooth functions on a smooth manifold, if one replaces the Hochschild complex by the continuous Hochschild complex, or by the complex of multi-differential operators.

**Remark 2.** Note that there is a choice in the precise definition of the “Hochschild” complex $C(A_n)$, essentially depending on a choice of cofibrant model for $e_n$. We choose here the minimal model $\text{hoe}_{e_n}$. For some other model $\mathcal{P}$, solving the higher formality conjecture will be “as complicated as” picking a morphism $\mathcal{P} \rightarrow e_n$.

1Deviating from standard notation we will denote by $E_n$ an operad quasi-isomorphic to the chains operad of the little $n$-cubes operad, without zero-ary operations.
The higher formality conjecture for that model can then be recovered by transfer, using Theorem 1.

**Remark 3.** Theorem 1 remains valid for any differential graded algebra \( A \) as soon as one replaces \( \text{Der}(A) \) by its right derived variant \( \text{Der}(A) \). Moreover, functoriality of the HKR map allows us to freely sheafify and get in particular that, for a quasi-projective derived scheme \( X \) and \( n \geq 2 \), the HKR map

\[
\Phi_{HKR}^n: S_{\mathcal{O}_X}(T_X[-n]) \to C((\mathcal{O}_X)_n)
\]

is a quasi-isomorphism of sheaves of \( \text{hoe}_{n+1} \) algebras (the only subtlety is to make \( C((\mathcal{O}_X)_n) \) into a sheaf \(^2\)). Note that this is slightly different from the main result of [20, section 5], also called higher formality, where it is proved that the \( e_n \) Hochschild complex of \( X \) is weakly equivalent to the \( E_n \) Hochschild complex of \( X \) as a \( \text{Lie}_{n+1} \) algebra (in our context this is more or less the content of Remark 2, but then the hard part would be to prove that the \( \text{Lie}_{n+1} \) structures on Hochschild complexes appearing in the present paper and the ones appearing in [20] are the same).

**Remark 4.** All our results and constructions remain valid for every \( n \in \mathbb{Z} \) if one uses \( P_{\text{ois}_n} \) in place of \( e_n \).

**Structure of the paper.** In section 2 we recall some basic definitions and notation. Section 3 contains a rewording of D. Tamarkin’s construction of the \( \text{hoe}_{n+1} \) algebra structure on \( C(A_n) \). The proof of Theorem 1 is a small direct calculation which is presented in section 5.

2. Notation

We will work over a ground field \( K \) of characteristic 0; all algebraic structures should be understood over \( K \). We will use the language of operads throughout. A good introduction can be found in the textbook [16], from which we freely borrow some terminology.

For a (co)augmented (co)operad \( O \) we denote by \( O_o \) the (co)kernel of the (co)augmentation. It is a pseudo-(co)operad: i.e. it does not have a (co)unit.

2.1. **Our favorite operads.**

2.1.1. *The \( e_n \) operad.* We will denote by \( e_n \) the homology of the topological operad \( E_n \), for every \( n \geq 1 \), cf. [3,4]. Note that we work with cohomological gradings (i.e. our differentials have degree +1), so that homology sits in non-positive (cohomological) degree.

As an example, \( e_1 \) is the operad governing non-unital associative algebras. For \( n \geq 2 \), the operad \( e_n \) is isomorphic to an operad obtained by means of a distributive law: \( e_n \cong \text{Com} \circ \text{Lie}_n \), where \( \text{Lie}_n := \text{Lie}\{n-1\} := S^{n-1} \text{Lie} \) is a degree shifted variant of the \( \text{Lie} \) operad.

Let \( O \) be an operad and let \( O_N \) be the free \( O \)-algebra cogenerated by symbols \( X_1, \ldots, X_N \). Then \( O_N \) carries a natural \( \mathbb{Z}^N \) grading by counting the number of occurrences of \( X_1, \ldots, X_N \). Furthermore \( O(N) \) may be identified with the subspace of homogeneous degree \( (1, \ldots, 1) \), \( O(N) \cong O_N^{(1, \ldots, 1)} \). This often provides convenient notation for elements of \( O(N) \).

\(^2\)One shall use the quasi-isomorphic sub-complex of multi-differential operators in \( C(A_n) \), which sheafifies well.
In particular, the space $e_n(N)$ is spanned by formal linear combinations of “Gerstenhaber words”, like

$$[X_1, X_2] \cdot X_4 \cdot [X_3, X_5],$$

in $N$ formal variable $X_1, \ldots, X_N$, each occurring once. We thus have an obvious map $\text{Lie}_n \to e_n$.

2.1.2. The $\text{hoe}_n$ dg operad. The minimal resolution of $e_n$, resp. $\text{Lie}_n$, is denoted by $\text{hoe}_n$, resp. $\text{hoLie}_n$. In particular $\text{hoe}_n = \Omega(e_n^!)$ where $\Omega(\cdot)$ denotes the cobar construction and $e_n^! \cong e_n^*\{n\}$ is the Koszul dual cooperad of $e_n$. Note that $\text{hoe}_n$ and $\text{hoLie}_n$ are dg operads.

Let $C$ be a cooperad and let $C_N$ be the cofree $C$ coalgebra cogenerated by symbols $X_1, \ldots, X_N$. Then $C_N$ carries a natural $\mathbb{Z}^N$ grading by counting the number of occurrences of $X_1, \ldots, X_N$. Furthermore $C(N)$ may be identified with the subspace of homogeneous degree $(1, \ldots, 1)$, $C(N) \cong C_N^{(1, \ldots, 1)}$. This often provides convenient notation for elements of $C(N)$. In particular, one may understand elements of $e_n^*(N)$ by linear combinations of “co-Gerstenhaber words”, like

$$X_1X_2 \wedge X_4 \wedge X_3X_5X_6,$$

in $N$ formal variable $X_1, \ldots, X_N$, each occurring once. The underline shall indicate that one equates linear combinations that correspond to (signed) sums of shuffle permutations to zero.

We may also consider the extended $e_n$ operad $ue_n = \text{uCom} \circ \text{Lie}_n$, which contains one nullary operation, i.e., $ue_n(0) = \text{uCom}(0) = k$. It governs unital $e_n$-algebras and can be obtained as the homology of the topological little disks operad, which has a nullary operation acting by deleting disks.

2.1.3. The braces operad and the Kontsevich-Soibelman minimal operad. Let $V$ be a dg vector space, and consider the cofree $C$ coalgebra without counit $B(V)$ generated by $V[1]$. A $B_\infty$-algebra structure on $V$ (see [11, section 5.2]) is a dg bialgebra structure on $B(V)$ extending the coalgebra structure. Concretely, since $B(V)$ is cofree such a structure is determined by the projections of the product and differential onto the cogenerators

$$m_{j,k} : (V[1]) \otimes^{j+k} \to V[1], \quad m_k : (V[1]) \otimes^k \to V[1],$$

An algebra over the Kontsevich-Soibelman minimal operad $\widetilde{Br}$ (see [15]) is a $B_\infty$ algebra such that $m_{j,k} = 0$ for $j \geq 1$\footnote{The operad $\widetilde{Br}$ has originally been described by Kontsevich and Soibelman combinatorially as an operad of planar trees. However, it is not hard to check that their description agrees with ours; see for example [21, Corollary 1].} A braces algebra is a $\widetilde{Br}$-algebra such that in addition $m_k = 0$ for $k \geq 3$. We denote the braces operad by $\text{Br}$. It is clearly a quotient of $\widetilde{Br}$, and it is an easy exercise to check that the projection map $\widetilde{Br} \to \text{Br}$ is a quasi-isomorphism [13]. It is well known that $H(\text{Br}) = H(\widetilde{Br}) = e_2$, see, e.g., [8,10,17].

Furthermore, the Hochschild complex of an $A_\infty$ algebra is a $\widetilde{Br}$-algebra. If the $A_\infty$ algebra is in fact an ordinary associative algebra, then its Hochschild complex is also a $\text{Br}$-algebra; see [9,11].
2.1.4. The preLie operad. We will denote by preLie the operad encoding pre-Lie algebras. Following [5], it admits the following combinatorial description. We first introduce the set \( \mathcal{T}(I) \) of rooted trees with vertices labelled by a finite set \( I \), which is constructed via the following inductive process:

- \( \mathcal{T}(\emptyset) \) is empty.
- \( \mathcal{T}(\{i\}) \) consists of a single rooted tree having only one root-vertex labelled by \( i \).
- Let \( I \) be a finite set, \( i \in I \) and a partition \( I_1 \sqcup \cdots \sqcup I_k = I - \{i\} \). Given rooted trees \( t_\alpha \in \mathcal{T}(I_\alpha) \), \( \alpha = 1, \ldots, k \) one can construct a new rooted tree \( B_+(t_1, \ldots, t_k) \in \mathcal{T}(I) \) by grafting the root of each \( t_\alpha \), \( \alpha = 1, \ldots, k \), on a common new root labelled by \( i \):

\[
\begin{array}{c}
\text{\( t_1 \) \hspace{1cm} \cdots \hspace{1cm} \text{\( t_k \)}
\end{array}
\]

\( \begin{array}{c}
\text{\( i \)}}
\end{array} \)

Then preLie\((I)\) is the vector space generated by \( \mathcal{T}(I) \), and the operadic composition can be defined in the following way: if \( J \) is another finite set, \( i \in I \), \( t \in \mathcal{T}(I) \) and \( t' \in \mathcal{T}(J) \), then \( t \circ t' \) is described as a sum over the set of functions \( f \) from incoming edges at the vertex \( i \) of \( t \) to the vertices of \( t' \). For any such \( f \), the corresponding term is obtained by removing the vertex \( i \) from \( t \), reconnecting the outgoing edge to the root of \( t' \) and reconnecting the incoming edges \( e \) to the vertex \( f(e) \). The root of the result is taken to be the root of \( t \) if this is not \( i \), or the root of \( t' \) otherwise.

Note that for any operad \( O \), the vector spaces \( \prod_{n \geq 0} O(n) \) and \( \prod_{n \geq 0} O(n) \) are naturally preLie algebras.

Recall also that there is a morphism of operads \( \text{Lie} \to \text{preLie} \) which sends the generator of \( \text{Lie} \) to \( \begin{array}{c}
\text{\( 2 \)}
\end{array} \) \( \begin{array}{c}
\text{\( 1 \)}
\end{array} \) \( \begin{array}{c}
\text{\( 2 \)}
\end{array} \). Hence any pre-Lie algebra is also a Lie algebra (obtained by skew-symmetrizing the pre-Lie product).

2.2. The Hochschild complex of a hoe\(_n\) algebra. For a hoe\(_n\) algebra \( B \), we define the “Hochschild” complex as the degree shifted convolution dg Lie algebra\(^4\)

\[
C(B) = \text{Conv}(e_n^*\{n\}, \text{End}_B)[-n]
\]

where \( \text{End}_B \) is the endomorphism operad of \( B \) and the differential is the Lie bracket with the element of \( C(B) \) corresponding to the hoe\(_n\) structure. In particular, if \( B = A_n \) is as in the introduction, then

\[
C(B) \cong A \oplus \text{Conv}(e_n^*\{n\}, \text{End}_A)[-n]
\]

as complexes.

Remark 5. For \( n = 1 \) our definition of the Hochschild complex clearly agrees with the standard definition. More generally, if \( B \) is a \( \mathcal{P}_n \)-algebra, then the definition \( C(B, B) := \mathbb{R} \text{Hom}_{\mathcal{P}_n-B-mod}(B, B) \) is often used. One can show that there is a quasi-isomorphism of complexes \( C(B) \to C(B, B) \). Very shortly, following the notation and convention in [16], we have \( \text{Conv}(\mathcal{P}^1, \text{End}_B) = \text{Hom}(\mathcal{P}^1(B), B) \), which computes \( \mathbb{R} \text{Hom}_{\mathcal{P}_n-B-mod}(B, B) \).

\( ^4\)For the definition of the convolution dg Lie algebra of maps from a cooperad \( C \) to an operad \( O \) we refer the reader to [10] section 6.4, where the notation \( \text{Hom}_\mathbb{R}(C, O) \), or \( \text{Hom}_\mathbb{S}(C, O) \) for the twisted version is used instead.
Note that there is a natural inclusion
\[ \Phi^{n}_{HKR} : S_{A} \left( \text{Der}(A)[-n] \right) \rightarrow C(A_{n}) \]
whose image consists of the elements in
\[ \text{Conv}(u\text{Com}^{*}\{n\}, \text{End}_{A}) \]
that furthermore are (i.e. take values in) derivations in each slot. Analogously to the usual HKR Theorem one may check the following result.

**Theorem 2** (Higher Hochschild-Kostant-Rosenberg Theorem). If \( A \) is a smooth commutative \( \mathbb{k} \)-algebra essentially of finite type, then the map \( \Phi^{n}_{HKR} \) is a quasi-isomorphism of complexes for each \( n \geq 2 \).

**Proof.** One simply observes that, since the bracket on \( A \) is zero,
\[ \text{Conv}(u\text{e}^{*}\{n\}, \text{End}_{A})[-n] = S_{A} \left( \text{Conv}(\text{Lie}^{*}\{1\}, \text{End}_{A})[-n] \right) = S_{A}(\text{Der}(A)[-n]) . \]
Here \( \text{Der} \) is the right derived functor of the derivations functor \( \text{Der} \). If \( A \) is smooth and essentially of finite type, then the canonical map \( \text{Der}(A) \rightarrow \text{Der}(A) \) is a quasi-isomorphism. \( \square \)

3. A VERSION OF D. TAMARKIN’ S PROOF OF THE HIGHER DELIGNE CONJECTURE

The goal of this section is to recall D. Tamarkin’s proof of the following result.

**Theorem 3** (Higher Deligne conjecture; see [19]). For any \( \text{hoe}_{n} \) algebra \( B \), the complex \( C(B) \) carries a natural \( \text{hoe}_{n+1} \) action, given by explicit formulas, for \( n \geq 2 \).

3.1. **Braces for a Hopf cooperad.** A Hopf operad is an operad in the category of counital coalgebras, cf. [16] section 5.3.5. For a Hopf operad \( O \), the tensor product of two \( O \)-algebras is naturally endowed with an \( O \)-algebra structure. Dually, a Hopf cooperad is a cooperad in unital algebras.

For any coaugmented cooperad \( C \) we may define its bar construction \( \Omega(C) \), which is an operad. For example \( \text{hoe}_{n} := \Omega(e_{n}^{*}) \). Here we define a similar construction, the brace construction, which takes a Hopf cooperad \( C \) and returns an operad \( Br_{C} \).

3.1.1. **\( C \)-operads.** In this paragraph we introduce the notion of a \( C \)-operad, for \( C \) a Hopf cooperad as above. A \( C \)-operad is an operad \( \mathcal{O} \) such that each \( \mathcal{O}(n) \) carries an \( S_{n} \)-equivariant right \( C(n) \) module structure. We require that furthermore the right module structures are compatible with the operadic compositions, by which we mean that the following diagram commutes:

\[
\begin{align*}
\mathcal{O}(k) \otimes \mathcal{O}(n_{1}) \otimes \cdots \otimes \mathcal{O}(n_{k}) \otimes C(\sum_{j} n_{j}) & \rightarrow \mathcal{O}(\sum_{j} n_{j}) \otimes C(\sum_{j} n_{j}) \\
\mathcal{O}(k) \otimes \mathcal{O}(n_{1}) \otimes \cdots \otimes \mathcal{O}(n_{k}) \otimes C(k) \otimes C(n_{1}) \otimes \cdots \otimes C(n_{k}) & \\
\mathcal{O}(k) \otimes \mathcal{O}(n_{1}) \otimes \cdots \otimes \mathcal{O}(n_{k}) & \rightarrow \mathcal{O}(\sum_{j} n_{j}).
\end{align*}
\]

Here the two horizontal arrows are the operadic compositions. The upper left vertical arrow is defined using the cooperad structure on \( C \). The remaining two
arrows are defined by using the right action of \( C(n) \) on \( O(n) \). Note that for \( C = uCom^* \) a \( C \)-operad is just an ordinary operad.

**Example 1.** One can check that for an operad \( P \) and a Hopf cooperad \( C \), the convolution operad \( Hom(C\{k\}, P) \) (see [16, section 6.4.2]) is naturally a \( C \)-operad for any \( k \). Here the right action is obtained by composition with the multiplication on \( C(n) \) from the right, i.e.,
\[
(f \cdot c)(x) = f(cx)
\]
for \( f \in Hom(C\{k\}, P)(n), c \in C(n) \) and \( x \in C\{k\}(n) \).

3.1.2. The \( \text{preLie}_C \) operad. In this section we introduce an operad encoding \( C \)-pre-Lie algebras, which are to \( C \)-operads what pre-Lie algebras are to operads. For simplicity, we will assume that the Hopf cooperad \( C \) satisfies:
\[
C(0) \sim C(1) \sim K.
\]
Then one has natural maps
\[
(1) \quad C(j) \to C(j + k) \otimes C(1) \otimes \cdots \otimes C(0) \otimes \cdots C(0) \cong C(j + k)
\]
where the arrow is a cocomposition and the right hand identification uses the canonical identifications \( C(0) \sim C(1) \sim K \) as algebras.

**Example 2.** The most interesting example for us is the Hopf cooperad \( C = uCom^*_n \), whose \( j \)-ary cooperations may be interpreted as the cohomology of the configuration space of \( j \) points in \( \mathbb{R}^n \) for \( j \geq 1 \), and as \( K \) for \( j = 0 \). There are forgetful maps from the configuration space of \( k + j \) points to that of \( j \) points, and in this case the extension map (1) above is just the pull-back of the forgetful map, forgetting the location of the last \( k \) points.

The operad \( \text{preLie}_C \) consists of rooted trees decorated by a Hopf cooperad \( C \). Namely, for every finite set \( I \),
\[
\text{preLie}_C(I) := \bigoplus_{t \in T(I)} \left( \bigotimes_{i \in I} C(t_i) \right),
\]
where \( t_i \) is the set/number of incoming edges at the vertex labelled by \( i \).

The operadic structure on the underlying trees is the one described in section 2.1.4. Let us now explain what happens to the decoration when doing the partial composition \( \circ_i \). Borrowing the notation from section 2.1.4 for every \( f \) we apply a cooperation\(^5\)
\[
C(t_i) \to \bigotimes_{j \in J} C(f^{-1}(j) \cup t'_j).
\]
Then observe that we have natural maps
\[
C(f^{-1}(j) \cup t'_j) \otimes C(t'_j) \to C(f^{-1}(j) \cup t'_j) \otimes C(f^{-1}(j) \cup t'_j) \to C(f^{-1}(j) \cup t'_j) = C((t_0, t'_0)_{j})
\]
where the first map uses the extension map \( \mathbb{1} \) on the second factor and the second map uses the Hopf structure, i.e., it is the multiplication of the algebra \( C(f^{-1}(j) \cup t'_j) \).

\(^5\)Note that possible cooperations that we may apply to elements of \( C(t_i) \) are naturally labelled by rooted trees with leaves labelled by \( t_i \). The cooperation we apply here is the one labelled by the tree \( t' \), with labelled leaves attached according to \( f \) and with the labelling of the vertices of \( t' \) disregarded.
The definition is made such that \text{preLie}_C naturally acts on the convolution “algebra”

\[ \text{Conv}_0(C\{k\}, \mathcal{P}) := \prod_{n \geq 0} \text{Hom}(C\{k\}, \mathcal{P})(n)^{S_n}. \]

More generally, for any \( C \)-operad \( \mathcal{O} \), \( \prod_n \mathcal{O}(n)^{S_n} \) is a \text{preLie}_C algebra (and we have already seen that \( \text{Hom}(C\{k\}, \mathcal{P}) \) is a \( C \)-operad).

Remark 6. Note that for \( C = \text{uCom}^* \) we recover the usual \text{preLie} operad, i.e., \( \text{preLie}_{\text{uCom}^*} = \text{preLie} \). Furthermore the construction \text{preLie}_C is functorial in \( C \). Hence from the unit map \( \text{uCom}^* \rightarrow C \) we obtain a map of operads \( \text{preLie} \rightarrow \text{preLie}_C \) for any Hopf cooperad \( C \). In particular, any \text{preLie}_C algebra is a \text{Lie} algebra, and we recover the usual \text{Lie} algebra structure on \( \text{Conv}_0(C\{k\}, \mathcal{P}) \).

Remark 7. Note that the operad \( \text{preLie}_C \) is not the same as the \text{cobar} construction \( \Omega(C) \) of \( C \), not even up to degree shifts. A basis of \( \Omega(C)(I) \) is given by rooted trees whose leaves are decorated by elements of the finite set \( I \), and whose internal nodes are decorated by elements of \( C[I] \). The operadic composition is obtained by gluing the root of one tree to a leaf of another. In contrast, in the trees giving rise to \( \text{preLie}_C(I) \) each node, also each internal node, is decorated by an element of \( I \). The operadic composition allows for inserting one tree at an internal node of another, not just for grafting the root to a leaf.

Next, if we have a morphism \( f : \Omega(C\{k\}) \rightarrow \mathcal{P} \) of dg operads, then it determines a Maurer-Cartan element \( \gamma_f \) in \( \text{Conv}_0(C\{k\}, \mathcal{P}) \), and a new convolution dg \text{Lie} algebra \( \text{Conv}_f(C\{k\}, \mathcal{P}) \) is obtained from the original one by twisting with \( \gamma_f \). Note that we may drop the “\( f \)” from the notation when there is no ambiguity. In general, the action of the operad \( \text{preLie}_C \) will unfortunately not lift to an action on the twisted convolution \text{Lie} algebra \( \text{Conv}_f(C\{k\}, \mathcal{P}) \). However, we now may invoke the formalism of operadic twisting \cite{6}. Given an operad \( \mathcal{P} \) together with a map \( \text{Lie} \rightarrow \mathcal{P} \), operadic twisting produces:

- A dg operad \( \text{twP} \), the twisted operad\(^6\)
- Dg operad maps \( \text{Lie} \rightarrow \text{twP} \rightarrow \mathcal{P} \) whose composition is the given map \( \text{Lie} \rightarrow \mathcal{P} \).
- The dg operad \( \text{twP} \) has the property that if we are given a \( \mathcal{P} \) algebra \( A \), together with a Maurer-Cartan element \( m \) of the \text{Lie} algebra \( A \), then the action of the \( \text{Lie} \) operad on the twisted \text{Lie} algebra \( A^m \) lifts naturally to an action of the dg operad \( \text{twP} \). Moreover, \( \text{twP} \) is universal among such dg operads

In our case we obtain a dg operad\(^7\) \( \text{TwpreLie}_C \), acting naturally on the twisted convolution algebra \( \text{Conv}_f(C\{k\}, \mathcal{P}) \). Concretely the dg operad \( \text{TwpreLie}_C \) is a completed version of the dg operad generated by \( \text{preLie}_C \) and one formal nullary element. The differential on \( \text{TwpreLie}_C \) is defined so that upon replacing the formal nullary element by the Maurer-Cartan element \( \gamma_f \) we obtain an action of \( \text{TwpreLie}_C \) on \( \text{Conv}_f(C\{k\}, \mathcal{P}) \). The formal nullary element we denote in pictures by coloring

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\(^6\) \text{twP} \text{ is defined as follows. First observe that there is a functor from dg operads to 2-colored dg operads which send \( \mathcal{P} \) to the 2-colored operad \( \text{Lie} \mod \mathcal{P} \) encoding a \( \mathcal{P} \) algebra together with a \text{Lie} algebra acting on it by derivations. It has a right adjoint denoted \( tw \). \text{tw} \text{ is then obtained by applying \( tw \) and identifying the two colors. We refer to \cite{10} for more details.}

\(^7\) Which does NOT depend on \( f \).
the appropriate vertices of the tree black. We call these vertices the internal vertices (as opposed to external ones).

Combinatorially, the differential on $Tw_{\text{preLie}}$ splits vertices, either an internal vertex into two internal vertices, or an external vertex into an external and an internal vertex.

We define the brace construction $Br(C) = Tw_{\text{preLie}}^C$ as a synonym for the twisted pre-Lie operad. By construction $Br(\text{ue}_n^*)$ acts on the convolution dg Lie algebra

$$C(B)[n] = \text{Conv}(\text{ue}_n^*\{n\}, \text{End}_B)$$

for any hoe$_n$ algebra $B$. For cosmetic reasons and consistency with the literature we make the following definition.

**Definition 1.** We define the higher braces dg operad $Br_{n+1}$ to be the suboperad

$$Br_{n+1} \subset Br(\text{ue}_n^*)\{n\}$$

formed by operations whose underlying trees contain no internal vertices with less than 2 children.

By very definition, the operad $Br_{n+1}$ acts naturally on the Hochschild complex $C(B)$.

**Example 3.** The higher braces dg operad $Br_2$ is the same as the Kontsevich-Soibelman minimal operad $\tilde{Br}$, cf. section 2.1.3 or [15].

3.2. Tamarkin’s morphism. D. Tamarkin proved Theorem 4 by noting that for $n \geq 2$ there is a quite simple but very remarkable explicit map

$$T: \text{hoe}_{n+1} \to Br_{n+1}.$$

It is defined on generators by the following prescription:

- Generators of the form $X_1 \cdots X_k \in e^i_{n+1}(k)$ are mapped to a corolla of the form

$$1 \quad \cdots \quad k$$

(2)

decorated by $X_1 \cdots X_k \in e^i_n(k)$.

- Generators of the form $X_0 \wedge X_1 \cdots X_k \in e^i_{n+1}(k+1)$ are mapped to a corolla of the form

$$1 \quad \cdots \quad k$$

(3)

decorated by $X_1 \cdots X_k \in e^i_n(k)$. In the special case $k = 1$ one takes the (anti-)symmetric combination of the two possible choices.

- All other generators are mapped to zero.
This prescription indeed gives a morphism $\text{hoe}_{n+1} \rightarrow \text{Br}_{n+1}$ of the underlying graded operads (because, as such, $\text{hoe}_{n+1}$ is free). This leaves us with the task of verifying that the map $T : \text{hoe}_{n+1} \rightarrow \text{Br}_{n+1}$ commutes with the differentials. It suffices to check this on the generators. Furthermore it suffices to check the statement on generators of one of the forms

$$X_1 \cdots X_k \quad X_1 \cdots X_k \wedge X_{k+1} \cdots X_{k+l} \quad X_1 \wedge X_2 \cdots X_{k+1} \wedge X_{k+2} \cdots X_{k+l+1}$$

since in all other cases the differential of the generator and the generator itself are mapped to zero, so that the map $T$ trivially commutes with the differentials. One has to check each of the three types of generators above in turn. The calculation is a bit lengthy, due to several special cases that need to be considered. Since the construction of $T$ is essentially the result of D. Tamarkin [19] we will only show how to handle a few cases in Appendix A as an illustration.

4. Br$_n$ is an $E_n$ operad

Theorem 4. The above map $T : \text{hoe}_{n+1} \rightarrow \text{Br}_{n+1}$ is a quasi-isomorphism of operads for all $n = 2, 3, 4, \ldots$, so in particular $H(\text{Br}_{n+1}) \cong e_{n+1}$.

In the case $n = 1$ it is still true that there is a quasi-isomorphism $\text{hoe}_2 \rightarrow \text{Br}_2$, but this morphism is much more complicated to construct than the Tamarkin quasi-isomorphism $T$ we described above. It can be obtained by combining a quasi-isomorphism from $\text{Br}_2$ to the chains of the little disks operad [15] with a choice of formality morphism of the little disks operad.

Theorem 4 is not used in this note, so we only sketch the proof.

**Sketch of proof.** First one checks that $H(\text{Br}_{n+1}) \cong e_{n+1}$. The proof of this statement follows along the lines of the proof of the $n = 1$ case in [7]. The only point where the proof in loc. cit. has to be adapted is that in [7] section 4.2 one has to compute the Hochschild cohomology of a free $e_{n+1}$ algebra, considered as an $e_n$ algebra, instead of computing the Hochschild cohomology of a free $e_2$ algebra, considered as an $e_1$ algebra. The answer is provided by the higher Hochschild-Kostant-Rosenberg Theorem, i.e., Theorem 2 above, instead of the usual one.

Once one knows that $H(\text{Br}_{n+1}) \cong e_{n+1}$, the statement of the theorem is shown by checking that the induced map in cohomology

$$e_{n+1} \cong H(\text{hoe}_{n+1}) \rightarrow H(\text{Br}_{n+1}) \cong e_{n+1}$$

is the identity, which amounts to checking that it is the identity on the two generators. □

The brace construction $\text{Br}_{n+1}$ is intuitively similar to taking a product of $\Omega(e_1)$ with an $E_1$ operad. So the above theorem shall be understood as a version of the statement that the product of an $E_1$ operad with an $E_n$ operad is an $E_{n+1}$ operad.

5. Proof of Theorem 11

One shall verify that the action of all generators of $\text{hoe}_{n+1}$ commutes with the HKR map $\Phi_{HKR}^n$. Namely, for any $g \in e_1^1(N)$ we shall prove that $(g \cdot) \circ (\Phi_{HKR}^n)^{\otimes N} = \Phi_{HKR}^n \circ g$.

First of all let $g = X_1 \wedge X_2$ (i.e., the Lie bracket). Then it is a simple verification that $g \cdot (\Phi_{HKR}^n(u), \Phi_{HKR}^n(v)) = [\Phi_{HKR}^n(u), \Phi_{HKR}^n(v)] = \Phi_{HKR}^n([u, v]) = \Phi_{HKR}^n(g \cdot (u, v))$. 


Second of all let \( g = X_1X_2 \) (i.e., the product). Then we obviously have
\[
g \cdot (\Phi^n_{HKR}(u), \Phi^n_{HKR}(v)) = \Phi^n_{HKR}(u)\Phi^n_{HKR}(v) = \Phi^n_{HKR}(u) = \Phi^n_{HKR}(g \cdot (u, v)).
\]
Finally, all other generators \( g \) act trivially on the domain of \( \Phi^n_{HKR} \). Thus it suffices to check that the action of the generators \( X_1 \cdots X_k \) \((k \geq 3)\) and \( X_0 \wedge X_1 \cdots X_k \) \((k \geq 2)\) on the image of \( \Phi^n_{HKR} \) is trivial, which we do now.

Generators \( X_1 \cdots X_k \) act using the corresponding components of the \( \text{hoe}_n \) structure on \( A_n \), which vanish. Hence they act trivially (as long as \( k \geq 3 \)).

Note also that the image of \( \Phi^n_{HKR} \) has only “\( \text{hoLie}_n \) components”, i.e., the corresponding maps \( e^A_i(N) \to \text{End}(V)(N) \) factor through \( e^A_i(N) \to \text{Lie}^i_n(N) \). However, the prescription for the action of the component \( X_0 \wedge X_1 \cdots X_k \) advises us to evaluate the arguments on components \( X_1 \cdots X_k \), which are sent to zero under the projection \( e^A_i(k) \to \text{Lie}^i_n(k) \). Hence the action of the components \( X_0 \wedge X_1 \cdots X_k \) vanishes (as long as \( k \geq 2 \)) on the image of the HKR map. \( \square \)

**Appendix A. The map \( T \) commutes with the differentials**

**A.1. The generator \( X_1 \cdots X_k \).** In this case the differential of the generator consists of
\[
\sum_{j,r} \pm X_1 \cdots X_{j-1} * X_{j+r+1} \cdots X_k \circ_* X_j \cdots X_{j+r},
\]
where the notation \( A \circ_* B \) shall mean the operadic composition in \( \text{hoe}_{n+1} \) of the operations \( A \) and \( B \) in \( \text{hoe}_{n+1} \), with \( B \) being “inserted in the slot” of \( A \) labelled by \(*\). The map \( T \) sends the above to
\[
\sum_{j,r} \pm \begin{array}{c}
\cdots \\
1 \\
\cdots \\
\end{array}^{j} \cdots ^{j+r} \cdots \begin{array}{c}
1 \\
\cdots \\
\end{array}^{k}
\]
This is precisely the differential of the tree \( (2) \), which is the image of \( X_1 \cdots X_k \) by \( T \). Decorations are obvious.

**A.2. The generator \( X_0 \wedge X_1 \cdots X_k \).** In this case the differential of the generator consists of the following terms:
\[
(5) \quad \sum_j \sum_{r \geq 1} \pm X_0 \wedge X_1 \cdots X_{j-1} * X_{j+r+1} \cdots X_k \circ_* X_j \cdots X_{j+r} + \sum_j \sum_{r \geq 1} \pm X_1 \cdots X_{j-1} * X_{j+r+1} \cdots X_k \circ_* \left( X_0 \wedge X_j \cdots X_{j+r} \right)
\]
\[
\quad \pm \left( X_0 \wedge * \right) \circ_* X_1 \cdots X_k + \sum_j \pm X_1 \cdots X_{j-1} * X_{j+2} \cdots X_k \circ_* \left( X_0 \wedge X_j \right).
\]
This is mapped under \( T \) to a linear combination of trees of the following form:
\[
\sum_{j \geq 1} \sum_{r \geq 1} \pm \begin{array}{c}
\cdots \\
1 \\
\cdots \\
\end{array}^{j} \cdots ^{j+r} \cdots \begin{array}{c}
\cdots \\
0 \\
\cdots \\
\end{array} \quad + \quad \sum_{j \geq 0} \sum_{r \geq 0} \pm \begin{array}{c}
\cdots \\
1 \\
\cdots \\
\end{array}^{j} \cdots ^{j+r} \cdots \begin{array}{c}
\cdots \\
0 \\
\cdots \\
\end{array} \quad \pm \begin{array}{c}
\cdots \\
1 \\
\cdots \\
\end{array} \quad \end{array} \cdots ^{k}
\]
Here all corollas are decorated by the top degree elements of \( \text{ue}^*_n \), except for the last tree, where the decoration is by the element \( X_0 \wedge X_1 \cdots X_k \).
One checks that this linear combination of trees is exactly the differential of \( \mathcal{B} \), which is the image of the generator we considered by \( T \). Note also that the trees of the form

\[
\begin{tikzpicture}
  \draw (0,0) node[dot] (0) {} -- ++(1,0) node[dot] (1) {} -- ++(1,0) node[dot] (2) {};
  \end{tikzpicture}
\]

occur twice, with the two contributions from the third and fourth term of \( \mathcal{B} \) cancelling each other.

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