Deformed phase space in a two-dimensional minisuperspace model

H R Sepangi¹, B Shakerin¹ and B Vakili²

¹ Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran
² Department of Physics, Azad University of Chalous, PO Box 46615–397, Chalous, Iran

E-mail: hr-sepangi@sbu.ac.ir and bvakili45@gmail.com

Received 31 October 2008, in final form 8 January 2009
Published 18 February 2009
Online at stacks.iop.org/CQG/26/065003

Abstract
We study the effects of noncommutativity and deformed Heisenberg algebra on the evolution of a two-dimensional minisuperspace cosmological model in classical and quantum regimes. The phase-space variables turn out to correspond to the scale factor of a flat FRW model with a positive cosmological constant and a dilatonic field with which the action of the model is augmented. The exact classical and quantum solutions in commutative and noncommutative cases are presented. We also obtain some approximate analytical solutions for the corresponding classical and quantum cosmology in the presence of the deformed Heisenberg relations between the phase-space variables, in the limit where the minisuperspace variables are small. These results are compared with the standard commutative and noncommutative cases, and similarities and differences of these solutions are discussed.

PACS numbers: 98.80.Qc, 04.60.Ds, 04.60.Kz

(Some figures in this article are in colour only in the electronic version)

1. Introduction
As is well known, standard cosmological models based on classical general relativity have no convincingly precise answer to the question of the initial conditions from which the universe has evolved. This can be traced to the fact that these models suffer from the presence of an initial singularity, the so-called Big Bang singularity. Indeed, there are various forms of singularity theorems in general relativity [1], which show that quite reasonable assumptions lead to at least one consequence which is physically unacceptable. Any hope of dealing with such singularities would be in the development of a concomitant and conducive quantum theory of gravity [2]. On the other hand, one of the most important features of theories which deal with quantum gravity is the existence of a minimal length below which no other length...
can be observed [3]. From perturbative string theory point of view, such a minimal length, of the order of Planck scale, is due to the fact that strings cannot probe distances smaller than the string size. Also, the existence of this minimal length has been suggested in loop quantum gravity [4], quantum geometry [5] and black hole physics [6]. Indeed, at the scale of such a minimum size, i.e. the scales of the order of the Planck length, \( l_p = \sqrt{G \hbar / c^3} \), the quantum effects of gravitation become as important as the electroweak and strong interactions. Clearly, at low energy, these quantum gravity effects are not too important, but in high energy physics, that is, energies of the order of Planck mass \( m_p = \hbar / l_p \) such as the very early universe or in the strong gravitational fields of a black hole, one cannot neglect these effects.

One of the most important features of the existence of a minimal length is that such a length is related to what is known as the generalized uncertainty principle (GUP); the usual Heisenberg uncertainty principle should be reformulated at the Planck scale [7, 8]. In a one-dimensional system, the simplest form of the GUP which shows the appearance of a minimum position uncertainty can be written as [7]

\[
\Delta p \Delta x \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \gamma \right),
\]

(1)

where \( \beta \) and \( \gamma \) are positive and independent of \( \Delta x \) and \( \Delta p \), but may in general depend on the expectation values \( \langle x \rangle \) and \( \langle p \rangle \). If we take \( \gamma = \beta \langle p \rangle^2 \), it is possible to realize equation (1)

from the following commutation relation between the position and momentum operators:

\[
[x, p] = i\hbar (1 + \beta p^2).
\]

(2)

In [7–9], more general GUPs are considered. In more than one dimension, GUP naturally implies a noncommutative geometric generalization of position space [7]. Noncommutativity between spacetime coordinates was first introduced by Snyder [10] which has led to a great deal of interest in this area of research in the recent past [11].

It is generally an accepted practice to introduce GUP or noncommutativity either through the coordinates or fields which may be called geometrical or phase-space deformation respectively [12–17]. Applying GUP or noncommutativity to ordinary quantum field theories where the geometry is considered to obey such deformations is interesting since they could provide an effective theory bridging the gap between ordinary quantum field theory and string theory, currently considered as the most important choice for quantization of gravity. A different approach to GUP and noncommutativity is through their introduction in the phase space constructed by minisuperspace fields and their conjugate momenta [13–17]. Since cosmology provides the ground for testing physics at energies which are much higher than those on Earth, it seems natural to expect the effects of quantum gravity to be observed in this context. Alternatively, in cosmological systems, since the scale factor, matter fields and their conjugate momenta play the role of dynamical variables of the system, introducing GUP and noncommutativity in the corresponding phase space is particularly relevant.

In general, as we mentioned above, GUP and noncommutativity in their original form (see [7]) imply a noncommutative underlying geometry for spacetime. However, formulation of gravity in a non-commutative spacetime is highly nonlinear, rendering the setting up of cosmological models difficult. Here, our aim is to study the aspects related to the application of GUP and noncommutativity in the framework of quantum cosmology, i.e. in the context of a minisuperspace reduction of dynamics. As is well known in the minisuperspace approach to quantum cosmology which is based on the canonical quantization procedure, one first freezes a large number of degrees of freedom by the imposition of symmetries on the spatial metric and then quantizes the remaining ones. Therefore, in the absence of a full theory of quantum gravity, quantum cosmology is a quantum mechanical toy model (finite degrees of freedom) providing a simple arena for testing ideas and constructions which can be introduced.
in quantum general relativity. In this respect, the GUP approach to quantum cosmology appears to be based on physical grounds. In fact, a generalized uncertainty principle can be immediately reproduced by deforming the canonical Heisenberg algebra. In other words, the GUP scheme relies on a modification of the canonical quantization prescriptions and, in this respect, can be reliably applied to any dynamical system. In this sense, one can introduce noncommutativity between different dynamical variables of the corresponding minisuperspace and, of course, get different results. Here, we rely on and use the most common and accepted practices which have been appearing in the literature over the past few years. It is to be noted that our presentation does not claim to clear the role of GUP and noncommutativity in cosmology in a fundamental way since we study the problem in a simple model. However, this may reflect realistic scenarios in similar investigations which deal with such problems in a more fundamental way.

We begin with a flat FRW metric, a positive cosmological constant and a homogeneous scalar dilatonic field. We then write the action in the string frame which leads us to a point-like Lagrangian for the model. We see that the corresponding minisuperspace constructed by the scale factor \( a \) and dilaton field \( \phi \) is curvilinear. Setting up a deformed phase-space formalism in such a minisuperspace is not an easy task. Therefore, we introduce a change of variables \((a, \phi) \rightarrow (u, v)\) which reduces the minisuperspace to a linear (Minkowskian) one. These variables are thus suitable candidates for introducing noncommutativity and GUP in the phase space of the problem at hand and enable us to present exact solutions for the classical and quantum commutative and noncommutative cosmologies studied here. Also in the case when the minisuperspace variables obey the GUP commutation relations, we obtain approximate analytical solutions for the corresponding classical and quantum cosmology. Finally, we compare and contrast these solutions at both classical and quantum levels.

2. The preliminary setup

In the pre-Big Bang scenario, based on the string effective action [18], the birth of the universe is described by a transition from the string perturbative vacuum with weak coupling, low curvature and cold state to the standard radiation-dominated regime, passing through a high curvature and strong coupling phase. This transition is made by the kinetic energy term of the dilaton, a scalar field \( \phi \) to which the Einstein–Hilbert action of general relativity is coupled; see [19] for a more modern review and [20] for some exact solutions of string dilaton cosmology. According to this model, the lowest order gravi-dilaton effective action, in the string frame, can be written as [21]

\[
S = -\frac{1}{2\lambda_s} \int d^4x \sqrt{-g} e^{-\phi} (\mathcal{R} + \partial_\mu \phi \partial^\mu \phi - 2\Lambda),
\]

where \( \phi(t) \) is the dilaton field, \( \lambda_s \) is the fundamental string length \( l_s \) parameter and \( \Lambda \) is a (positive) cosmological constant. We consider a spatially flat FRW spacetime which is specified by the metric

\[
d\xi^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\]

where \( a(t) \) is the scale factor. The Ricci scalar corresponding to metric (4) is

\[
\mathcal{R} = 6 \frac{\ddot{a}^2}{a^2} + 6 \frac{\dot{a}^2}{a},
\]

where a dot represents differentiation with respect to \( t \). By substituting (5) into (3) and integrating over spatial dimensions, we are led to an effective Lagrangian in the minisuperspace \( Q^A = (a, \phi) \):

\[
\mathcal{L} = e^{-\phi} (6\dot{a}^2 a - 6\dot{a} \ddot{a} \dot{\phi} + a^3 \dot{\phi}^2 - 2\Lambda a^3).
\]
The momenta conjugate to the dynamical variables are given by
\[ P_a = \frac{\partial L}{\partial \dot{a}} = e^{-\phi}(12\dot{a}a - 6\sigma^2 \dot{\phi}), \quad P_\phi = \frac{\partial L}{\partial \dot{\phi}} = e^{-\phi}(-6\dot{a}a^2 + 2a^3 \dot{\phi}), \] (7)
leading to the following Hamiltonian:
\[ \mathcal{H} = \frac{1}{2} G^{AB} P_A P_B + \mathcal{U}(Q^A) = e^{\phi} \left( -\frac{1}{12a} P_a^2 - \frac{1}{2a^3} P_\phi^2 + \frac{1}{2a^2} P_a P_\phi \right) + 2\Lambda a^3 e^{-\phi}. \] (8)

Now, it is easy to see that the corresponding minisuperspace has the following minisuper metric:
\[ G_{AB} dq^A dq^B = e^{-\phi} \left( \frac{1}{2} (da^2 + 12a^2 da d\phi + 2a^3 d\phi^2) \right). \] (9)

To apply the deformed commutators to the dynamical variables in a minisuperspace which is represented by a curved manifold with a minisuper metric given by (9), in a natural generalization, one can replace \( p^2 \) in (2) with \( G^{AB} P_A P_B \). In general, this generalization does not provide a suitable expression because of the ambiguity in the ordering of factors \( Q \) and \( PQ \). Therefore, the above minisuperspace does not have the desired form for introducing noncommutativity and GUP among its coordinates. To avoid the physical difficulties and simplify the model, consider the following change of variables
\[ \mathcal{A} \rightarrow \mathcal{A} = (a, \phi) \rightarrow q^A = (u, v): \]
\[ u + v = 4a^2 e^{-\phi/2}, \quad u - v = a^2 e^{-\phi/2}, \] (10)
where \( \alpha \) and \( \beta \) are two constants which satisfy the relations
\[ \alpha + \beta = 3, \quad \alpha \beta = \frac{3}{2}. \]

In terms of these new variables, Lagrangian (6) takes the form
\[ L = \dot{u}^2 - \dot{v}^2 - \omega^2 (u^2 - v^2), \] (11)
with the corresponding Hamiltonian becoming
\[ \mathcal{H} = \frac{1}{4} (\dot{P}_u^2 - \dot{P}_v^2) + \omega^2 (u^2 - v^2), \] (12)
which describes an isotropic oscillator–ghost–oscillator system with frequency \( \omega^2 = \frac{1}{2} \). Thus, in the minisuperspace constructed by \( q^A = (u, v) \), the metric is Minkowskian and represented by
\[ G_{AB} dq^A dq^B = \frac{1}{2} (du^2 - dv^2). \] (13)

Now, we have a set of variables \( (u, v) \) endowing the minisuperspace with a Minkowskian metric and hence this set of dynamical variables are suitable candidates for introducing noncommutativity and GUP in the phase space of the problem at hand. The final remark about the above analysis is that Lagrangian (6) possesses an interesting symmetry, thanks to the presence of the stringy dilaton. This symmetry exhibits itself through the transformation [21]
\[ a(t) \rightarrow \frac{1}{a(t)}, \quad \phi(t) \rightarrow \phi(t) - 6 \ln a(t). \] (14)

It is easy to show that Lagrangian (6) is invariant under this transformation. Such symmetry (duality) is one of the major features of the solutions of equations of motion in string dilaton cosmology [20], so that if the set of variables \( (a, \phi) \) solve the equations of motion, the set \( (1/a, \phi - 6 \ln a) \) is also a solution. On the other hand, in terms of the variables \( (u, v) \) the Lagrangian takes the simple form (11) yielding linear differential equations for the corresponding dynamical equations. Therefore, the duality symmetry is nothing but a suitable
linear combination of $u$ and $v$. Indeed, one can easily show that Lagrangian (11) and also Hamiltonian (12) are invariant under the following transformations:\(^3\)

\[
\begin{align*}
&u \rightarrow \frac{17}{8}u - \frac{15}{8}v, \\
&v \rightarrow \frac{15}{8}u - \frac{17}{8}v,
\end{align*}
\]
\[
\begin{align*}
p_u \rightarrow \frac{17}{8}p_u + \frac{15}{8}p_v, \\
p_v \rightarrow -\frac{15}{8}p_u - \frac{17}{8}p_v.
\end{align*}
\]

(15)

(16)

The preliminary setup for describing the model is now complete. In what follows, we will study the classical and quantum cosmology of the minisuperspace model described by Hamiltonian (12) in noncommutative and GUP frameworks.

3. Classical model

As mentioned above, the dynamical system described by Hamiltonian (12) is a simple isotropic oscillator–ghost–oscillator system and its classical and quantum solutions can be easily obtained. Since our aim here is to study the effects of deformed Poisson brackets on the classical trajectories, in what follows we consider commutative, noncommutative and GUP classical cosmologies and compare the results with each other. In the following section, we shall deal with the quantum cosmology of the model.

3.1. Classical cosmology with ordinary Poisson brackets

As is well known for a dynamical system with phase-space variables $(q_i, p_i)$, the Poisson algebra is described by the following Poisson brackets:

\[
\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij},
\]

(17)

where in our case $q_i(i = 1, 2) = u, v$ and $p_i(i = 1, 2) = p_u, p_v$. Therefore, the equations of motion become

\[
\begin{align*}
\dot{u} &= \{u, \mathcal{H}\} = \frac{1}{2}p_u, \\
\dot{p}_u &= \{p_u, \mathcal{H}\} = -2\omega^2 u,
\end{align*}
\]

(18)

\[
\begin{align*}
\dot{v} &= \{v, \mathcal{H}\} = -\frac{1}{2}p_v, \\
\dot{p}_v &= \{p_v, \mathcal{H}\} = 2\omega^2 v.
\end{align*}
\]

(19)

Integrating the above equations, one is led to

\[
\begin{align*}
u(t) &= \left(\frac{u_0^2 + \dot{u}_0^2}{\omega^2}\right)^{1/2} \sin \left[\omega t + \tan^{-1} \frac{\omega u_0}{\dot{u}_0}\right], \\
v(t) &= \left(\frac{v_0^2 + \dot{v}_0^2}{\omega^2}\right)^{1/2} \sin \left[\omega t + \tan^{-1} \frac{\omega v_0}{\dot{v}_0}\right],
\end{align*}
\]

(20)

(21)

where for the initial conditions we take

\[
\begin{align*}
u(t = 0) &= u_0, \quad \dot{u}(t = 0) = \dot{u}_0, \\
v(t = 0) &= v_0, \quad \dot{v}(t = 0) = \dot{v}_0.
\end{align*}
\]

(22)

\(^3\) In general, Lagrangian (11) is invariant under pseudorotations in a two-dimensional Minkowskian space

\[
u \rightarrow u \cosh \vartheta + v \sinh \vartheta, \quad v \rightarrow u \sinh \vartheta + v \cosh \vartheta,
\]

where $\vartheta$ is the parameter of transformations. In (15) and (16), we take a special choice for $\vartheta$ to recover the duality of the theory represented by (14).
The above solutions must satisfy the Hamiltonian constraint, $\mathcal{H} = 0$. Thus, substitution of equations (20) and (21) into (12) gives the following relation between integration constants:

$$u_0^2 + \frac{\dot{u}_0^2}{\omega^2} = v_0^2 + \frac{\dot{v}_0^2}{\omega^2}. \quad (23)$$

From the above equations, we see that the classical trajectories obey the relation

$$v = \pm \cos \left( \tan^{-1} \frac{\omega v_0}{v_0} - \tan^{-1} \frac{\omega u_0}{u_0} \right) u \pm \sin \left( \tan^{-1} \frac{\omega v_0}{v_0} - \tan^{-1} \frac{\omega u_0}{u_0} \right) \left( u_0^2 + \frac{\dot{u}_0^2}{\omega^2} - u^2 \right)^{1/2}. \quad (24)$$

Note that the minisuperspace of the above model is a two-dimensional manifold which in terms of the old variables $a$ and $\phi$ is represented by $0 < a < \infty$, $-\infty < \phi < +\infty$. Following [22], we may divide its boundary into two: the nonsingular and singular. The nonsingular boundary is the line $a = 0$ with $|\phi| < +\infty$, while at the singular boundary, at least one of the two variables is infinite. In terms of the variables $u$ and $v$, introduced in (10), the minisuperspace is recovered by $u > 0$, $-u < v < u$, and the nonsingular boundary may be represented by $u = v = 0$. This discussion leads us to the imposition of more restrictions on the initial conditions (22) such that the classical trajectories would no longer meet the nonsingular boundary. This condition is achieved when the coefficient of the second term in (24) is nonzero.

Now, let us go back to the old variables $a$ and $\phi$, in terms of which we obtain the corresponding classical cosmology as

$$4a(t)^{a-\beta} = 4a(t)^{a-\beta} = \frac{u + v}{u - v}, \quad (25)$$

$$\phi(t) = 2\beta \ln |a(t)| - 2 \ln |u - v|, \quad (26)$$

leading to the following sets of classical solutions:

$$a_+(t) = a_0 \left[ \tan \omega(t - t_0) \right]^{1/\sqrt{3}},$$

$$\phi_+(t) = (\sqrt{3} - 1) \ln |\tan \omega(t - t_0)| - 2 \ln |\cos \omega(t - t_0)| + \phi_0 \quad (27)$$

and

$$a_-(t) = a_0 \left[ \tan \omega(t - t_0) \right]^{-1/\sqrt{3}},$$

$$\phi_-(t) = (-\sqrt{3} - 1) \ln |\tan \omega(t - t_0)| - 2 \ln |\cos \omega(t - t_0)| + \phi_0, \quad (28)$$

where $a_0$, $t_0$ and $\phi_0$ are some constants which can be written in terms of $u_0$, $v_0$, $v_0$ and $v_0$. These two sets of solutions are related to the duality symmetry (14); indeed, we have

$$a_-(t) = \frac{1}{a_+(t)}, \quad \phi_-(t) = \phi_+(t) - 6 \ln a_+(t). \quad (29)$$

As we mentioned in the previous section, in the minisuperspace $(u, v)$ the duality symmetry is denoted by the linear combination (15) of $u$ and $v$. Therefore, applying the duality transformation (15) we are led to the following set of solutions:

$$U(t) = \left( \frac{u_0^2 + \dot{u}_0^2}{\omega^2} \right)^{1/2} \left\{ \frac{17}{8} \sin \left[ \omega t + \tan^{-1} \frac{\omega u_0}{u_0} \right] - \frac{15}{8} \sin \left[ \omega t + \tan^{-1} \frac{\omega v_0}{v_0} \right] \right\}, \quad (30)$$

$$V(t) = \left( \frac{u_0^2 + \dot{u}_0^2}{\omega^2} \right)^{1/2} \left\{ \frac{15}{8} \sin \left[ \omega t + \tan^{-1} \frac{\omega u_0}{u_0} \right] - \frac{17}{8} \sin \left[ \omega t + \tan^{-1} \frac{\omega v_0}{v_0} \right] \right\}. \quad (31)$$

It is clear that these solutions are essentially a special linear combination of (20) and (21) which obviously solve the classical equations of motion because of their linearity.
3.2. Classical cosmology with noncommutative phase-space variables

Let us now proceed to study the behavior of the above model in a deformed phase-space framework such that the minisuperspace variables do not (Poisson) commute with each other. In general, noncommutativity between phase-space variables can be understood by replacing the usual product with the star product, also known as the Moyal product law between two arbitrary functions of position and momentum, as

\[(f \ast g)(x) = \exp \left[ \frac{i}{2} \alpha^{ab} \frac{d}{d} \right] f(x_1) g(x_2) |_{x_1 = x_2 = x}, \tag{32} \]

where \(\alpha^{ab}\) denote the noncommutative parameters [23]. Here, we consider a noncommutative phase space in which the Poisson algebra is a deformed one given by

\[\{ q_{nc}, p_{nc} \} = \theta \epsilon_{ij}, \quad \{ p_{nc}, p_{nc} \} = 0, \quad \{ q_{nc}, p_{nc} \} = \delta_{ij}, \tag{33} \]

where \(\epsilon_{ij}\) and \(\delta_{ij}\) are Levi-Civita and Kronecker symbols respectively. \(q_{nc}(i = 1, 2) = u_{nc}, v_{nc}\) and \(p_{nc}(i = 1, 2) = p_{u_{nc}}, p_{v_{nc}}\). With the deformed phase space defined above, one may consider the Hamiltonian of the noncommutative model as having the same functional form as (12), but with the dynamical variables satisfying the above-deformed Poisson brackets, that is,

\[H_{nc} = \frac{1}{2} \left( p_{u_{nc}}^2 - p_{v_{nc}}^2 \right) + \alpha^2 \left( u_{nc}^2 - v_{nc}^2 \right). \tag{34} \]

Thus, the dynamics of the system can be described by the following equations of motion:

\[u_{nc}' = \{ u_{nc}, H_{nc} \} = \frac{1}{2} p_{u_{nc}} - 2 \theta \omega^2 v_{nc}, \quad \tag{35} \]
\[v_{nc}' = \{ v_{nc}, H_{nc} \} = -\frac{1}{2} p_{v_{nc}} - 2 \theta \omega^2 u_{nc}. \tag{36} \]

Eliminating the momenta from the above equations, we get

\[u_{nc}'' + \omega^2 u_{nc} + 2 \theta \omega^2 v_{nc} = 0, \tag{37} \]
\[v_{nc}'' + \omega^2 v_{nc} + 2 \theta \omega^2 u_{nc} = 0. \tag{38} \]

We see that the noncommutative parameter appears as a coupling constant between equations of motion for \(u_{nc}\) and \(v_{nc}\). Integrating equations (37) and (38) yields

\[u_{nc}(t) = \mathcal{A} e^{\theta \omega t} \sin[\omega \sqrt{1 - \theta^2 \omega^2 t} + \delta_1] + \mathcal{B} e^{-\theta \omega t} \sin[\omega \sqrt{1 - \theta^2 \omega^2 t} + \delta_2], \tag{39} \]
\[v_{nc}(t) = -\mathcal{A} e^{\theta \omega t} \sin[\omega \sqrt{1 - \theta^2 \omega^2 t} + \delta_1] + \mathcal{B} e^{-\theta \omega t} \sin[\omega \sqrt{1 - \theta^2 \omega^2 t} + \delta_2], \tag{40} \]

where \(\mathcal{A}, \mathcal{B}, \delta_1\) and \(\delta_2\) are integrating constants. The requirement that the noncommutative Hamiltonian constraints should hold during the evolution of the system, that is, \(H_{nc} = 0\), leads to the following relation between the integrating constants:

\[\mathcal{A} \mathcal{B} = 0. \tag{41} \]

This means that in the noncommutative minisuperspace, the system follows one of the trajectories \(u_{nc} = v_{nc}\) (if \(\mathcal{A} = 0\)) or \(u_{nc} = -v_{nc}\) (if \(\mathcal{B} = 0\)). In the case when \(\mathcal{A} = 0\), the two coordinates behave similar to two coupled springs, oscillating back and forth together like \(\rightarrow\rightarrow\) and \(\leftarrow\leftarrow\) with frequency \(\omega \sqrt{1 - \theta^2 \omega^2}\) and an exponentially damping amplitude. Alternatively, if we take \(\mathcal{B} = 0\), the two variables oscillate in opposite directions like \(\rightarrow\leftarrow\) and \(\leftarrow\rightarrow\) with the same frequency \(\omega \sqrt{1 - \theta^2 \omega^2}\) but with an exponentially increasing amplitude.

Before going any further, some remarks are in order. An important ingredient in any model theory related to the quantization of a cosmological setting is the choice of the quantization
procedure used to quantize the system. The most widely used method has traditionally been
the canonical quantization method based on the Wheeler–DeWitt (WD) equation, which is
nothing but the application of the Hamiltonian constraint to the wavefunction of the universe.
A particularly interesting but rarely used approach to study the quantum effects is to introduce
a deformation in the phase space of the system. It is believed that such a deformation of
phase space is an equivalent path to quantization, in par with other methods, namely canonical
and path integral quantization [24]. This method is based on the Wigner quasi-distribution
function and Weyl correspondence between quantum mechanical operators in Hilbert space
and ordinary c-number functions in phase space. The deformation in the usual phase-space
structure is introduced by Moyal brackets which are based on the Moyal product (32) [23].
However, to introduce such deformations it is more convenient to work with Poisson brackets
rather than Moyal brackets.

From a cosmological point of view, models are built in a minisuperspace. It is therefore
safe to say that studying such a space in the presence of deformations mentioned above can be
interpreted as studying the quantum effects on cosmological solutions. One should note that
in gravity, the effects of quantization are woven into the existence of a fundamental length [3].
The question then arises as to what form of deformations in phase space is appropriate for
studying quantum effects in a cosmological model? Studies in noncommutative geometry [11]
and GUP [7] have been a source of inspiration for those who have been seeking an answer to
the above question. More precisely, introduction of modifications in the structure of geometry
in the way of noncommutativity has become the basis from which similar modifications in
phase space have been inspired. In this approach, the fields and their conjugate momenta play
the role of coordinate basis in noncommutative geometry [25]. In doing so, an effective model
is constructed whose validity will depend on its power of prediction.

A question worth asking at this stage is: would the noncommutative scheme presented
above really offer a quantum picture of the model at hand and should we refrain from using any
other quantization method simultaneously? It is important to note that equivalence between
the two different approaches of quantization cannot hold true in models where deformation in
phase space is introduced in a Lorentz non-invariant manner, like what we have done here. This
is not hard to understand since the WD equation is a direct consequence of diffeomorphism
invariance and so if a deformation in phase space breaks such an invariance, then the results of
different quantization methods should be different. For models where the Lorentz invariance
deformation is studied, see [26]. Therefore, in the following section when we quantize our
model we also invoke noncommutative quantum cosmology based on the star-product WD
equation.

3.3. Classical cosmology with GUP

In more than one dimension, a natural generalization of equation (2) is defined by the following
commutation relations [7]:

\[ [x_i, p_j] = i(\delta_{ij} + \beta\delta_{ij}p^2 + \beta'p_ip_j), \]

where \( p^2 = \sum p_ip_i \) and \( \beta, \beta' > 0 \) are considered as small quantities of first order. Also,
assuming that

\[ [p_i, p_j] = 0, \]

the commutation relations for the coordinates are obtained as

\[ [x_i, x_j] = i \frac{(2\beta - \beta') + (2\beta + \beta')\beta p^2}{1 + \beta p^2} (p_i x_j - p_j x_i). \]
As is clear from the above expression, the coordinates do not commute. This means that to construct the Hilbert space representations, one cannot work in the position space. It is therefore more convenient to work in momentum space. However, since in quantum cosmology the wavefunction of the universe in momentum space has no suitable interpretation, we restrict ourselves to the special case $\beta' = 2\beta$. As one can see immediately from equation (44), the coordinates commute to first order in $\beta$ and thus a coordinate representation can be defined. Now, it is easy to show that the following representation of the momentum operator in position space satisfies relations (42) and (43) (with $\beta' = 2\beta$) to first order in $\beta$:

$$p_i = -i \left( 1 - \frac{\beta}{3} \frac{\partial^2}{\partial x_i^2} \right) \frac{\partial}{\partial x_i}. \quad (45)$$

Equations (42)–(45) may now be realized from the following commutation relations between the position and momentum operators:

$$[u, p_u] = i(1 + \beta p^2 + 2\beta p_u^2), \quad [v, p_v] = i(1 + \beta p^2 + 2\beta p_v^2), \quad (46)$$

$$[u, p_v] = [v, p_u] = 2i\beta p_u p_v, \quad (47)$$

$$[x_i, x_j] = [p_i, p_j] = 0, \quad x_i(i = 1, 2) = u, v, \quad p_i(i = 1, 2) = p_u, p_v. \quad (48)$$

Before quantizing the model within the GUP framework in the following section, we investigate the effects of the classical version of GUP, i.e. the classical version of commutation relations (46)–(48) on the above cosmology. As is well known, in the classical limit the quantum mechanical commutators should be replaced by the classical Poisson brackets as $[P, Q] \rightarrow i\hbar \{P, Q\}$. Thus, in classical phase space the GUP changes the Poisson algebra (17) according to

$$\{u, p_u\} = 1 + \beta p^2 + 2\beta p_u^2, \quad \{v, p_v\} = 1 + \beta p^2 + 2\beta p_v^2, \quad (49)$$

$$\{u, p_v\} = \{v, p_u\} = 2\beta p_u p_v, \quad (50)$$

$$[x_i, x_j] = [p_i, p_j] = 0, \quad x_i(i = 1, 2) = u, v, \quad p_i(i = 1, 2) = p_u, p_v, \quad (51)$$

where $p^2 = \frac{1}{2}(p_u^2 - p_v^2)$. Such deformed Poisson algebra is used in [27] to investigate the effects of deformations on the classical orbits of particles in a central force field and on the Kepler third law. Also, the stability of planetary circular orbits in the framework of such deformed Poisson brackets is considered in [28]. Note that here we deal with modifications of a classical cosmology that become important only at the Planck scale where the classical description is no longer appropriate and a quantum model is required. However, before quantizing the model we shall provide a deformed classical cosmology. In this classical description of the universe in transition from the commutation relation (2) to its Poisson bracket counterpart, we keep the parameter $\beta$ fixed as $\hbar \rightarrow 0$. In string theory, this means that the string momentum scale is fixed when its length scale approaches zero. Therefore, the equations of motion read as

$$\dot{u} = \{u, \mathcal{H}\} = \frac{1}{2} p_u(1 + 5\beta p^2), \quad \dot{p}_u = \{p_u, \mathcal{H}\} = -2\omega^2 u(1 + \beta p^2 + 2\beta p_u^2) + 4\omega^2 \beta v p_v p_u, \quad (52)$$

$$\dot{v} = \{v, \mathcal{H}\} = -\frac{1}{2} p_v(1 - 3\beta p^2), \quad \dot{p}_v = \{p_v, \mathcal{H}\} = 2\omega^2 v(1 + \beta p^2 + 2\beta p_v^2) - 4\omega^2 \beta u p_u p_v. \quad (53)$$

We see that the deformed classical cosmology forms a system of nonlinear coupled differential equations which unfortunately cannot be solved analytically. In figure 1, employing numerical
methods, we have shown the approximate behavior of $u(t)$ and $v(t)$ for typical values of the parameters and initial conditions respectively. As is clear from the figure, both variables $u$ and $v$ repeat their back and forth oscillatory behavior as the noncommutative case, but here the oscillations are not harmonic. Like the noncommutative case, depending on the initial conditions, the minisuperspace variables behave as $\rightarrow\rightarrow$ and $\leftarrow\leftarrow$ (see the right figure) or $\leftarrow\rightarrow$ and $\rightarrow\leftarrow$ (see the left figure) i.e. they move back and forth either in the same or in opposite directions. A comment on the above results is that although in the limit $\beta \to 0$ one can recover the ordinary classical cosmology described by equations (18) and (19), as this figure shows, taking a nonzero value for $\beta$ may disturb the oscillatory nature of the universe. Also, in the presence of $\beta$ terms, the period of oscillations becomes larger and thus the Big-Crunch in the corresponding cosmological model occurs later in comparison with the usual models where $\beta = 0$. This means that the effects of GUP are important not only in the early but also at late times in the cosmic evolution. In fact, within the GUP framework, the quantum gravitational effects may be detected at large scales as well.

4. The quantum model

We now focus attention on the study of the quantum cosmology of the model described above. Here, as in classical cosmology, for comparison purposes among ordinary commutative, noncommutative and GUP frameworks, we consider the quantum cosmology of the model separately in each case and compare the results. Our starting point is to construct the WD equation from the corresponding Hamiltonian.

4.1. Quantum cosmology with ordinary commutation relations

The quantum version of the model described by relations (17) can be achieved via the canonical quantization procedure which leads to the WD equation, $\nabla \Psi = 0$. Here, $\nabla$ is the operator form of the Hamiltonian (12) which annihilates the wavefunction $\Psi$. By replacing $p_q \rightarrow -i \frac{\partial}{\partial q}$ in (12), we get the WD equation as

$$\left[ -\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 4\omega^2(u^2 - v^2) \right] \Psi(u, v) = 0.$$  (54)

This equation is a quantum isotropic oscillator–ghost–oscillator system with zero energy. Therefore, its solutions belong to a subspace of the Hilbert space spanned by separable
eigenfunctions of a two-dimensional isotropic simple harmonic oscillator Hamiltonian. Separating the eigenfunctions of (54) in the form
\[ \Psi_{n_1, n_2}(u, v) = U_{n_1}(u)V_{n_2}(v) \]
yields
\[ U_{n_1}(u) = \frac{(2\omega)^{1/4}}{\pi} e^{-\omega u^2} \frac{1}{\sqrt{2^n n_1!}} H_{n_1}(\sqrt{2\omega}u), \]
\[ V_{n_2}(v) = \frac{(2\omega)^{1/4}}{\pi} e^{-\omega v^2} \frac{1}{\sqrt{2^n n_2!}} H_{n_2}(\sqrt{2\omega}v), \]
subject to the restriction \( n_1 = n_2 = n \). In (56) and (57), \( H_n(x) \) are Hermite polynomials and the eigenfunctions are normalized according to
\[ \int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) \, dx = 2^n \pi^{1/2} n! \delta_{mn}. \]
Now, we impose the boundary condition on these solutions such that at the nonsingular boundary (at \( u = v = 0 \)) the wavefunction vanishes [29], that is, \( \Psi(u = 0, v = 0) = 0 \), which yields
\[ H_n(u = 0) H_n(v = 0) = 0 \Rightarrow n = \text{odd}. \]
In general, one of the most important features in quantum cosmology is the recovery of classical cosmology from the corresponding quantum model or, in other words, how can the WD wavefunctions predict a classical universe. In this approach, one usually constructs a coherent wavepacket with good asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. Therefore, we may now write the general solution of the WD equation as a superposition of the above eigenfunctions:
\[ \Psi(u, v) = \frac{(2\omega)^{1/2}}{\pi} e^{-\omega(u^2+v^2)} \sum_{n=\text{odd}} c_n \frac{1}{2^n n!} H_n(\sqrt{2\omega}u) H_n(\sqrt{2\omega}v). \]
Figure 2 shows the square of the wavefunction and its contour plot. As we can see from this figure, the peaks follow a path which can be interpreted as the classical trajectories (24). The crests are symmetrically distributed around \( v = 0 \) which correspond to the ± signs in (24). Thus, it is seen that there is an almost good correlation between the quantum patterns and classical trajectories in the \( u-v \) plane.

4.2. Noncommutative quantum model

Now, the quantum version of the noncommutative cosmology is achieved by replacing the Poisson brackets with the corresponding Dirac commutators, \( \{, \} \rightarrow -i \,[,] \). Thus, the Poisson brackets (33) between minisuperspace variables should be modified as follows:
\[ [u_{nc}, v_{nc}] = i \theta, \quad [u_{nc}, p_u] = [v_{nc}, p_v] = i. \]
The corresponding quantum cosmology can be obtained by modification of the operator product in the WD equation \( \hat{H} \Psi = 0 \) with the Moyal-deformed product \( \hat{H}_{nc} \ast \Psi = 0 \), where \( \hat{H}_{nc} \) is the noncommutative Hamiltonian (34). However, there is an alternative expression for the Moyal star product which is given by the shift formula:
\[ u_{nc} = u - \frac{i}{2} \theta p_v, \quad v_{nc} = v + \frac{i}{2} \theta p_u, \]
Figure 2. The figure on the left shows $|\Psi(u, v)|^2$, the square of the commutative wavefunction, while the figure on the right shows the contour plot of $|\Psi(u, v)|^2$. The figures are plotted for numerical value $\omega = 1$ and we have taken a superposition of eight terms in (60) with all $c_k$ up to $c_{15}$ taken to be unity.

$p_{u_c} = p_u, \quad p_{v_c} = p_v$. \hspace{1cm} (63)

It can be easily checked that if the noncommutative variables obey relations (61), then $(u, v, p_u, p_v)$ satisfy the usual Heisenberg algebra:

$[u, v] = 0, \quad [u, p_u] = [v, p_v] = i, \quad [u, p_v] = [v, p_u] = 0. \hspace{1cm} (64)$

In terms of these commutative variables, the Moyal Wei equation $\mathcal{H}_{nc} \Psi = 0$ transforms to the usual WD equation $\mathcal{H} \Psi = 0$ with the Hamiltonian

$\mathcal{H} = \frac{1}{4}(p_u^2 - p_v^2) + \Omega^2(u^2 - v^2) - \theta \omega^2(up_v + vp_u). \hspace{1cm} (65)$

Therefore, the noncommutative version of the WD equation can be written as

$\frac{1}{1 - \omega^2 \theta^2} \mathcal{H} \Psi(u, v) = H \Psi(u, v)$

$= \left[ \frac{1}{4}(p_u^2 - p_v^2) + \Omega^2(u^2 - v^2) - \theta \Omega^2 (up_v + vp_u) \right] \Psi(u, v) = 0, \hspace{1cm} (66)$

where

$\Omega^2 = \frac{\omega^2}{1 - \theta^2 \omega^2}. \hspace{1cm} (67)$

The Hamiltonian operator consists of two parts:

$H = H_{uv} - \theta \Omega^2 L_{uv}, \hspace{1cm} (68)$

where

$H_{uv} = \frac{1}{4}(p_u^2 - p_v^2) + \Omega^2(u^2 - v^2) \quad \text{and} \quad L_{uv} = up_v + vp_u. \hspace{1cm} (69)$

It is easy to see that these two operators commute with each other:

$[H_{uv}, L_{uv}] = 0. \hspace{1cm} (70)$
which means that $H_{uv}$ and $L_{uv}$ have simultaneous eigenfunctions\(^4\). In the coordinate system $(u, v)$, the WD equation (66) is not separable. Thus, to transform it to a separable equation, consider the new variables $(\rho, \varphi)$ defined as

$$u = \rho \cosh \varphi, \quad v = \rho \sinh \varphi,$$

in terms of which we have

$${\cal L}_{uv} = -i u \frac{\partial}{\partial v} - i v \frac{\partial}{\partial u} = -i \frac{\partial}{\partial \varphi}.$$  

$$H_{uv} = -\frac{1}{4} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \Omega^2 (u^2 - v^2) = -\frac{1}{4} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) + \Omega^2 \rho^2.$$  

Therefore, if we separate the solution of the WD equation according to $\Psi(\rho, \varphi) = R(\rho) e^{i n \varphi}$, it is easy to see that this is an eigenfunction of $L_{uv}$ with eigenvalue $v$. The requirement that $\Psi(\rho, \varphi)$ should solve the WD equation yields the following differential equation for $R(\rho)$:

$$\frac{d^2 R}{d \rho^2} + \frac{1}{\rho} \frac{d R}{d \rho} + \left( 4 v \Omega^2 \theta - 4 \Omega^2 \rho^2 + \frac{\mu^2}{\rho^2} \right) R = 0.$$  

This equation, after a change of variable $r = 2 \Omega \rho^2$ and transformation $R = \frac{1}{\rho^{\frac{1}{2}}} W$, becomes

$$\frac{d^2 W}{d r^2} + \left( \frac{1}{4} + \frac{\kappa}{r} + \frac{1/4 - \mu^2}{r^2} \right) W = 0,$$

where $\kappa = \frac{1}{2} v \Omega \theta$ and $\mu = i \frac{\varphi}{2}$. The above equation is the Whittaker differential equation and its solutions can be written in terms of confluent hypergeometric functions $M(a, b; x)$ and $U(a, b; x)$ as

$$W(r) = e^{-r/2} r^{\frac{1}{4} - i \frac{\varphi}{2} \Omega} \left[ e^{U} (\mu - \kappa + \frac{1}{2}, 2 \mu + 1; r) + c' M (\mu - \kappa + \frac{1}{2}, 2 \mu + 1; r) \right].$$  

In view of the asymptotic behavior of $M(a, b; x) \sim e^x / x^{b-a}$ [30], we take $c' = 0$. Thus, going back, the eigenfunctions of the WD equation can be written in terms of variables $(\rho, \varphi)$ as

$$\Psi_n(\rho, \varphi) = \rho^n e^{-\Omega \rho^2} U \left( \frac{1 - \nu \theta \Omega}{2} + i \frac{\varphi}{2} \Omega, 1 + i \nu; 2 \Omega \rho^2 \right) e^{i n \varphi}.$$  

If we demand that $\Psi_n$ should be a single-valued function of $\varphi$, then $\nu$ should be an integer $\nu = n$. Now, we may write the noncommutative wavefunction in the $(\rho, \varphi)$ coordinates as

$$\Psi(\rho, \varphi) = \sum_{n=-\infty}^{\infty} C_n \rho^n e^{-\Omega \rho^2} U \left( \frac{1 - \nu \theta \Omega}{2} + i \frac{n}{2}, 1 + i \nu; 2 \Omega \rho^2 \right) e^{i n \varphi}.$$  

Note that in terms of variables $(\rho, \varphi)$, the boundary condition $\Psi(u = 0, v = 0) = 0$ takes the form $\Psi(\rho = 0, \varphi) = 0$ which for (78) is automatically held.

\(^4\) Like the theory of an ordinary harmonic oscillator, we can also define the creation and annihilation operators

$$\hat{a}_u = \sqrt{i} \rho \frac{\partial}{\partial \rho} + \frac{i}{2 \sqrt{\Omega}} p_u, \quad \hat{a}_u^\dagger = \sqrt{i} \rho \frac{\partial}{\partial \rho} - \frac{i}{2 \sqrt{\Omega}} p_u,$$

and similarly for $\hat{a}_v$ and $\hat{a}_v^\dagger$, satisfying the following commutation relations:

$$[\hat{a}_u, \hat{a}_v^\dagger] = [\hat{a}_u^\dagger, \hat{a}_v] = 1,$$

with other commutators being zero. In terms of these operators $H_{uv}$ and $L_{uv}$, can be viewed as

$$H_{uv} = \Omega (\hat{a}_u^\dagger \hat{a}_u - \hat{a}_u \hat{a}_u^\dagger), \quad L_{uv} = i (\hat{a}_v^\dagger \hat{a}_u^\dagger - \hat{a}_u \hat{a}_v).$$  

Use of the same commutators as above between $\hat{a}$ and $\hat{a}^\dagger$ makes it easy to see that $[H_{uv}, L_{uv}] = 0$.  

In figure 3, we have plotted the wavefunction (78) in the noncommutative quantum model. In the case where $\theta = 0$ (commutative model), although we have analyzed the behavior of the wavefunction in the previous subsection, we have also considered it again in the $(\rho, \phi)$ coordinates. As we can see from this figure, the commutative wavefunction has two dominant peaks in the vicinity of $\rho = 0$ which then follow a path that can be interpreted as the classical trajectory (compare with the similar behavior in figure 2). Therefore, like the commutative wavefunction in the $(u, v)$ coordinates, there is also a correlation between the classical and quantum schemes. On the other hand, the noncommutative wavefunction predicts the emergence of the universe from a state corresponding to one of the two dominant peaks. Although there are some small peaks in this figure, as $\rho$ grows, their amplitudes are suppressed. We see that the correlation with classical trajectories is missed, i.e. noncommutativity implies that the universe escapes the classical trajectories and approaches a stationary state.

4.3. Quantum model with GUP

The study of quantum cosmology of the model presented above in the GUP framework is the goal we shall pursue in this subsection. The Hamiltonian of the model is given by (12) and the corresponding commutation relations between dynamical variables by equations (46)–(48).
In the WD equation, we take the representation (45) for the momenta \( p_u \) and \( p_v \) which leads to the following equation up to first order in \( \beta \):

\[
\left[ \frac{2}{3} \beta \frac{\partial^4}{\partial u^4} - \frac{\partial^2}{\partial u^2} + \frac{2}{3} \beta \frac{\partial^4}{\partial v^4} + \frac{\partial^2}{\partial v^2} + 4 \omega^2 (u^2 - v^2) \right] \psi(u, v) = 0. \tag{79}
\]

We again separate the solutions in the form \( \psi(u, v) = U(u) V(v) \), leading to

\[
\frac{1}{2} \beta \frac{d^4 W_i}{d u^4} + \frac{d^2 W_i}{d u^2} + \left( \lambda - 4 \omega^2 u_i^2 \right) W_i = 0, \tag{80}
\]

where \( W_i(i = 1, 2) = U, V \), \( w_i(i = 1, 2) = u, v \) and \( \lambda \) is the separation constant. The appearance of a fourth-order differential equation to describe a physical phenomenon is interesting, since it requires investigation of the corresponding boundary conditions which is not the goal of our study in this paper. These equations cannot be solved analytically, but since \( \beta \) is a small parameter which appears only in the fourth-order term, we may look for an approximation method which, within its domain of validity, leads us to a second-order differential equation. From figure 2, we note that the dominant peaks of the wavefunction occur in the vicinity of \( u, v \sim 0 \). On the other hand, the effects of \( \beta \) are important at the Planck scales, or in cosmology language in the very early times of the cosmic evolution, which in our model means \( u, v \sim 0 \). When \( \beta = 0 \), the solutions of equation (80) for \( \lambda / 2 \omega = 2n + 1 \) are given by (56) and (57). In the limit \( u, v \rightarrow 0 \), we may take \( e^{-\omega(u^2+v^2)} \sim 1 \) which means that \( W_i \propto H_{2n}(\sqrt{2\omega}u_i) \). Therefore, in this limit, we have

\[
\frac{d^4 W_i}{d u^4} = \left( \frac{2\omega}{\pi} \right) \frac{1}{4} \frac{64\omega^7 n(n - 1)(n - 2)(n - 3)}{\sqrt{2^{2n}} n!} H_{n-4}(\sqrt{2\omega}u_i), \tag{81}
\]

where we have used the relation \( dH_n(x)/dx = 2nH_{n-1}(x) \). Now, using the following series formula for Hermite polynomials:

\[
H_n(x) = \sum_{s=0}^{[n/2]} (-1)^s \frac{n!}{(n - 2s)!s!} (2x)^{n-2s}, \tag{82}
\]

we are led to the approximate formula \( H_{n-4}(x) \propto H_n(x) \) in the case of a small argument and \( n \geq 4 \). Thus, in our approximation, equation (80) becomes

\[
\frac{d^4 W_i}{d u^4} + 2\omega(2n + 1 - 2\beta_0 - 2\omega u_i^2) W_i = 0, \tag{83}
\]

where \( \beta_0 \), up to a numerical factor, is

\[
\beta_0 = \frac{3\omega}{2} \pi \omega n(n - 1)(n - 2)(n - 3). \tag{84}
\]

The solutions of the above equation up to a normalization factor can be written as

\[
W_n(u) = e^{-\omega u^2} H_{n-\beta_0}(\sqrt{2\omega}u). \tag{85}
\]

Finally, the general solutions of the WD equation (79) for small values of \( u \) and \( v \) are as follows:

\[
\psi(u, v) = \sum_{n=odd} C_n e^{-\omega(u^2+v^2)} H_{n-\beta_0}(\sqrt{2\omega}u) H_{n-\beta_0}(\sqrt{2\omega}v), \tag{85}
\]

where to recover solution (60) in the limit \( \beta_0 \rightarrow 0 \), the summation is taken over the odd values of \( n \). Also, to satisfy the boundary condition \( \psi(u = 0, v = 0) = 0 \), we choose \( \beta_0 \) such that \( n - \beta_0 \) is an odd integer for \( n > 4 \).

Figure 4 shows the wavefunction of the corresponding universe when the minisuperspace variables obey the GUP relations for small values of \( u \) and \( v \). As is clear from this figure,
the wavefunction has two single peaks which are symmetrically distributed around \( v = 0 \). Compare to the commutative wavefunction (see figure 2); here we have no wave packet with peaks following the classical trajectories. We see that instead of a series of peaks in the ordinary WD approach, we have only a couple of dominant peaks. This means that, similar to the noncommutative case and within the context of the GUP framework, the wavefunction also shows a stationary behavior. One may then conclude that from the point of view adopted here, noncommutativity and GUP may have close relations with each other.

We should note that the above analysis on the behavior of the GUP wavefunction is achieved in the region \( u, v \sim 0 \). If we relax this approximation, equation (81) is no longer valid and in the perturbation method we should use solutions (56) and (57) in the \( \beta \)-term of equation (80) in order to obtain the GUP wavefunction. However, since for large values of \( u \) and \( v \) these solutions have an exponentially decreasing behavior, we do not expect a major effect on the behavior of the above wavefunction even if we consider the problem more rigorously. We can also rely on the above approximate GUP solutions for a wider range of \( u \) and \( v \).

To clarify this point, we investigate the quantum GUP model in a different representation. To do this, we remember that the WD equation (79) is based on the representation (45) of momenta in the \( u-v \) space. In this sense, we note that the existence of a minimal length means that we cannot have localized physical states. However, as is shown in [7], when there is no minimal uncertainty in momentum one can work with the momentum space wavefunction \( \Phi(\vec{p}) \) with the following representation [7]:

\[
\begin{align*}
p_i \Phi(\vec{p}) &= p_i \Phi(\vec{p}), \\
x_i \Phi(\vec{p}) &= i(1 + \beta p^2) \frac{\partial}{\partial p_i} \Phi(\vec{p}),
\end{align*}
\]

where in our model we have, as before, \( x_i (i = 1, 2) = u, v, p_i (i = 1, 2) = p_u, p_v \) and \( p^2 = \frac{1}{2}(p_u^2 + p_v^2) \). Thus, one can now define the proper physical states with maximal localization and use them to define a ‘quasi-position wavefunction’. In [7], it is shown
that the transition between the momentum space representation of the wavefunction and its quasi-position counterpart is given by a generalized Fourier transformation as
\[
\Psi(x) = \sqrt{\frac{2\sqrt{\beta}}{\pi}} \int_{-\infty}^{+\infty} \frac{dp}{(1 + \beta p^2)^{3/2}} \exp \left( \frac{i x \tan^{-1}(\sqrt{\beta} p)}{\sqrt{\beta}} \right) \Phi(p).
\] (88)

Now, using representations (86) and (87), we get the following form for the WD equation in momentum space:
\[
\left[ \frac{\partial^2}{\partial p_u^2} - \frac{\partial^2}{\partial p_v^2} - \beta \left( p_u \frac{\partial}{\partial p_u} + p_v \frac{\partial}{\partial p_v} \right) + \frac{1}{2\omega^2} \left( p_u^2 - p_v^2 \right) \right] \Phi(p_u, p_v) = 0,
\] (89)

where in contrast to the minisuperspace coordinate representation of the WD equation, it is a second-order differential equation. This equation, up to first order in \(\beta\) and also neglecting \(p^4\) in the last term, takes the form
\[
\left[ \frac{\partial^2}{\partial p_u^2} - \frac{\partial^2}{\partial p_v^2} - \beta \left( p_u \frac{\partial}{\partial p_u} + p_v \frac{\partial}{\partial p_v} \right) + \frac{1}{4\omega^2} \left( p_u^2 - p_v^2 \right) \right] \Phi(p_u, p_v) = 0,
\] (90)

which is a separable equation and its solutions may be written as \(\Phi(p_u, p_v) = \Pi(p_u) \Upsilon(p_v)\), leading to
\[
\left[ \frac{d^2}{dp_u^2} + \beta p_u \frac{d}{dp_u} - \left( \frac{1}{4\omega^2} p_u^2 + E \right) \right] \Pi(p_u) = 0,
\] (91)
\[
\left[ \frac{d^2}{dp_v^2} - \beta p_v \frac{d}{dp_v} - \left( \frac{1}{4\omega^2} p_v^2 + E \right) \right] \Upsilon(p_v) = 0,
\] (92)

where \(E\) is a separation constant. The above equations have exact solutions in terms of confluent hypergeometric functions as
\[
\Pi_E(p_u) = p_u \exp \left[ - \left( \frac{\sqrt{1 + \beta^2 \omega^2}}{4\omega} + \frac{\beta}{4} \right) p_u^2 \right] U \left( \frac{3}{4} + \frac{\omega (2E + \beta)}{4\sqrt{1 + \beta^2 \omega^2}}, \frac{3}{2} ; \frac{\sqrt{1 + \beta^2 \omega^2}}{2\omega}, p_u^2 \right).
\] (93)
\[
\Upsilon_E(p_v) = p_v \exp \left[ - \left( \frac{\sqrt{1 + \beta^2 \omega^2}}{4\omega} - \frac{\beta}{4} \right) p_v^2 \right] U \left( \frac{3}{4} + \frac{\omega (2E - \beta)}{4\sqrt{1 + \beta^2 \omega^2}}, \frac{3}{2} ; \frac{\sqrt{1 + \beta^2 \omega^2}}{2\omega}, p_v^2 \right).
\] (94)

Therefore, up to first order in \(\beta\) the eigenfunctions of equation (90) can be written as
\[
\Phi_E(p_u, p_v) = p_u p_v e^{-\left( \frac{1}{2} + \frac{i}{2} \sqrt{\frac{1 + \beta^2 \omega^2}{4\omega}} \right) p_u^2} e^{-\left( \frac{1}{2} - \frac{i}{2} \sqrt{\frac{1 + \beta^2 \omega^2}{4\omega}} \right) p_v^2} U \left( \frac{3}{4} + \frac{\omega (2E + \beta)}{4\sqrt{1 + \beta^2 \omega^2}}, \frac{3}{2} ; \frac{\sqrt{1 + \beta^2 \omega^2}}{2\omega}, p_u^2 \right)
\] \times \ U \left( \frac{3}{4} + \frac{\omega (2E - \beta)}{4\sqrt{1 + \beta^2 \omega^2}}, \frac{3}{2} ; \frac{\sqrt{1 + \beta^2 \omega^2}}{2\omega}, p_v^2 \right).
\] (95)

5 Our classical analysis (see equations (20)–(23), (39)–(41) and figure 1) shows that the dynamical variables \(a\) and \(v\) have the same order of magnitude. Therefore, the Hamiltonian constraint \(H = 0\) implies that \(p_u\) and \(p_v\) should also have the same order of magnitude. In this sense, neglecting \(p^4\) in equation (89) is quite reasonable.
Figure 5. The figure shows $|\Psi(u, v)|^2$, the square of the GUP wavefunction (97). The figure is plotted for numerical values $\omega = 1/24$, $\beta = 8$ and we have taken a superposition of four terms in (97) with all $C(E)$ taken to be unity.

Now the eigenfunctions in the $u$–$v$ representation can be obtained from (88)

$$
\Psi_E(u, v) = \frac{2\sqrt{\beta}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp_u dp_v}{(1 + \beta p^2)^3} \times \exp\left(\frac{iu \tan^{-1}(\sqrt{\beta} p_u)}{\sqrt{\beta}}\right) \exp\left(\frac{iv \tan^{-1}(\sqrt{\beta} p_v)}{\sqrt{\beta}}\right) \Phi_E(p_u, p_v).
$$

(96)

It is seen that this expression is too complicated for extracting an analytical closed form for the eigenfunctions in terms of the minisuperspace variables $u$ and $v$. Moreover, the wavefunction is a superposition of these eigenfunctions as

$$
\Psi(u, v) = \sum_E C(E) \Psi_E(u, v).
$$

(97)

On the other hand, to construct such a superposition we need to know whether the separating constant $E$ has a quantized or a continuum spectrum. Here, we do not intend to deal with such questions since they are out of the scope of our study. However, in figure 5 we have plotted the square of the wavefunction where we have taken a discrete superposition of four terms in (97), all to first order in $\beta$. It should be noted that we have relaxed the assumption of small values for $u$ and $v$ in this analysis and figure 5 shows the square of the wavefunction for a wide range of these variables. Although this figure does not completely coincide with figure 4, as is clear, the wavefunction has its dominant amplitude in regions represented by the small values of $u$ and $v$, in agreement with our previous analysis of the GUP quantum model based on equation (81).

5. Conclusions

In this paper, we have studied the effects of deformations (noncommutativity and GUP) in phase space on the cosmic evolution of a two-dimensional minisuperspace model. Our starting point was the lowest order gravi-dilaton effective action in the string frame where the Einstein–Hilbert action with a positive cosmological constant is augmented with a scalar field, the dilaton. By considering a flat FRW metric for the space time geometry in this action, we obtained the corresponding effective Hamiltonian in the minisuperspace constructed by the scale factor $a$ and dilaton field $\phi$. We saw that this minisuperspace has a curved metric
and thus does not have the desired form for introducing noncommutativity and GUP between its coordinates. For this reason, we introduced a new set of variables \((u, v)\), in terms of which the minisuper metric took a Minkowskian form and the Hamiltonian of the model described a simple isotropic oscillator–ghost–oscillator system in which the cosmological constant plays the role of frequency. Although this is a simple toy model, these variables are suitable candidates for a phenomenological study of noncommutativity and GUP in the corresponding phase space. Another feature of the model in the \((u, v)\) coordinates is that the duality symmetry of the string dilaton action exhibits itself as a special linear combination of \(u\) and \(v\). In the case of (Poisson) commutative phase space, we saw that both dynamical variables have oscillatory behavior with the same amplitude. Depending on the initial conditions, these oscillations may occur in the same or opposite directions. As for the quantum version of this commutative model, we obtained exact solutions of the WD equation. The wavefunction of the corresponding universe consists of two branches where each may be interpreted as part of the classical trajectory. We saw that since the peaks of the wavefunction follow the classical trajectory, there seems to be good correlations between the corresponding classical and quantum cosmology.

We also studied the noncommutative cosmology where the commutator between the minisuperspace variables is deformed through a noncommutative parameter \(\theta\). In the classical noncommutative model, this parameter plays the role of a coupling constant between equations of motion for dynamical variables \(u\) and \(v\). We solved the equations of motion exactly and showed that the variables \(u\) and \(v\) oscillate in the same or opposite directions with an exponentially damping or increasing amplitude respectively. In the noncommutative quantum cosmology, we found that the corresponding Hamiltonian is constructed out of two operators which commute with each other and thus have the same eigenfunctions. Although the ensuing WD equation in this case is not separable in the \((u, v)\) coordinates, we introduced a new set of variables \((\rho, \phi)\) in terms of which the WD equation was amenable to exact solutions in terms of confluent hypergeometric functions. Finally, when the phase-space variables obey the GUP relations we constructed the classical equations of motion and seen that they form a system of nonlinear differential equations which cannot be solved analytically. We showed that the oscillatory behavior of \(u\) and \(v\) in the same or opposite directions is again repeated but in this case the oscillations are not harmonic. We also found that the GUP parameter \(\beta\) causes a larger period in comparison to the ordinary model where \(\beta = 0\). This can be interpreted as the importance of quantum gravitational effects not only at early times but also at late times of the cosmic evolution. The resulting quantum cosmology and the corresponding WD equation in the GUP framework were also studied and approximate analytical expressions for the wavefunctions of the universe were presented in the limit of small \(u\) and \(v\) variables. These solutions show only one possible state with no classical correlation at early times from which our universe can emerge. We saw that such behavior also occurs in the noncommutative quantum model, showing that from the point of view adopted here, noncommutativity and GUP may be considered as similar concepts.

References

[1] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
Landau L D and Lifshitz E M 1975 The Classical Theory of Fields (Oxford: Pergamon)
Ryan M P and Shepley L C 1975 Homogeneous Relativistic Cosmologies (Princeton, NJ: Princeton University Press)
Islam J N 2001 An Introduction to Mathematical Cosmology (Cambridge: Cambridge University Press)

[2] DeWitt B S 1967 Phys. Rev. 160 1113
Misner C W 1969 Phys. Rev. 186 1319
Hartle J B and Hawking S W 1983 Phys. Rev. D 28 2960
Hawking S W 1984 Nucl. Phys. B 239 257
Hawking S W 1988 Phys. Rev. D 37 904
Hawking S W and Page D 1986 Nucl. Phys. B 264 185
[3] Gross D J and Mende P F 1988 Nucl. Phys. B 303 407
Amati D, Ciafaloni M and Veneziano G 1989 Phys. Lett. B 216 41
Kato M 1990 Phys. Lett. B 245 43
de Haro S 1998 J. High Energy Phys. JHEP10(1998)023
Garay L G 1995 Int. J. Mod. Phys. A 10 145 (arXiv:gr-qc/9403008)
Konishi K, Paffuti G and Provero P 1990 Phys. Lett. B 234 276
[4] Ashtekar A and Lewandowski J 2004 Class. Quantum Grav. 21 R53
Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[5] Capozziello S, Lambiase G and Scarpetta G 2000 Int. J. Theor. Phys. 39 15
[6] Scardigli F 1999 Phys. Lett. B 452 39 (arXiv:hep-th/9904025)
Nozari K and Fazlpour B 2007 Mod. Phys. Lett. A 23 113 (arXiv:hep-th/0605109)
Nozari K and Fazlpour B 2008 Acta Phys. Pol. B 39 1363 (arXiv:gr-qc/0608077)
Nozari K and Mehdipour S H 2006 Failure of standard thermodynamics in Planck scale black hole system arXiv:hep-th/0610076
[7] Kempf A, Mangano G and Mann R B 1995 Phys. Rev. D 52 1108 (arXiv:hep-th/9412167)
[8] Kempf A and Mangano G 1997 Phys. Rev. D 55 7909 (arXiv:hep-th/9612084)
[9] Maggiore M 1993 Phys. Lett. B 319 83
Kempf A 1994 J. Math. Phys. 35 4483
Kempf A and Niemeyer J C 2001 Phys. Rev. D 64 103501
Chang L N, Minic D, Okamura N and Takeuchi T 2002 Phys. Rev. D 65 125027
[10] Snyder H 1947 Phys. Rev. 71 38
[11] Connes A 1994 Noncommutative Geometry (New York: Academic)
Connes A 2000 J. Math. Phys. (NY) 41 3832
Varilly J C 1997 An introduction to noncommutative geometry (arXiv:physics/9709045)
Douglas M R and Nekrasov N A 2001 Rev. Mod. Phys. 73 977
[12] Hassan S F and Sloth M S 2002 Nucl. Phys. B 674 434 (arXiv:hep-th/0204110)
[13] Vakili B, Khosravi N and Sepangi H R 2007 Class. Quantum Grav. 24 931 (arXiv:gr-qc/0701075)
Khosravi N, Sepangi H R and Sheikh-Jabbari M M 2007 Phys. Lett. B 647 219 (arXiv:hep-th/0611236)
Khosravi N, Jalalzadeh S and Sepangi H R 2007 Gen. Rel. Grav. 39 899 (arXiv:gr-qc/0702067)
[14] Guzman W, Sabido M and Socorro J 2008 On noncommutative minisuperspace and the Friedmann equations arXiv:0812.4251
Garcia-Compean H, Obregon O and Ramirez C 2002 Phys. Rev. Lett. 88 161301 (arXiv:hep-th/0107250)
Cai Y-F and Piao Y-S 2007 Phys. Lett. B 657 1 (arXiv:gr-qc/0701114)
Bastos C, Bertolami O, Dias N C and Prata J N 2008 Phys. Rev. D 78 023516 (arXiv:0712.4122)
[15] Battisti M V and Montani G 2007 Phys. Lett. B 656 96 (arXiv:gr-qc/0703025)
Battisti M V and Montani G 2008 Phys. Rev. D 77 023518 (arXiv:0707.2726)
Bina A, Atazadeh K and Jalalzadeh S 2008 Int. J. Theor. Phys. 47 1354 (arXiv:0709.3623)
Battisti M V and Montani G 2008 AIP Conf. Proc. 966 219 (arXiv:0709.4610)
Battisti M V and Montani G 2008 Int. J. Mod. Phys. A 23 1257 (arXiv:0802.0688)
[16] Vakili B and Sepangi H R 2007 Phys. Lett. B 651 79 (arXiv:0706.0273)
[17] Vakili B 2008 Phys. Rev. D 77 044023 (arXiv:0801.2438)
[18] Veneziano G 1991 Phys. Lett. B 265 287
Gasperini M and Veneziano G 1993 Astropart. Phys. 1 317
Bozza V, Gasperini M, Giovannini M and Veneziano G 2002 Phys. Lett. B 543 14
Bozza V, Gasperini M, Giovannini M and Veneziano G 2003 Phys. Rev. D 67 063514
[19] Gasperini M 2007 Dilaton cosmology and phenomenology arXiv:hep-th/0702166
[20] Capozziello S and de Ritis R 1993 Int. J. Mod. Phys. D 2 373
Copeland E J, Lahiri A and Wands D 1994 Phys. Rev. D 50 4868
Capozziello S, Lambiase G and Capaldo R 1999 Int. J. Mod. Phys. D 8 213 (arXiv:gr-qc/9805046)
[21] Gasperini M, Maharana J and Veneziano G 1996 Nucl. Phys. B 472 349
Danielsson U H 2005 Class. Quantum Grav. 22 S1–S40 (arXiv:hep-th/0409274)
[22] Vilenkin A 1988 Phys. Rev. D 37 888
[23] Moyal J E 1949 Proc. Camb. Phil. Soc. 45 99
Hirschfeld A C and Henselder P 2002 Am. J. Phys. 70 5 (arXiv:quant-ph/0208163)
[24] Zachos C 2002 Int. J. Mod. Phys. A 17 297 (arXiv:hep-th/0110114)
[25] Carmona J M, Cortes J L, Gamboa J and Mendez F 2003 J. High Energy Phys. JHEP03(2003)058
(arXiv:hep-th/0301248)
Carmona J M, Cortes J L, Gamboa J and Mendez F 2003 Phys. Lett. B 565 222 (arXiv:hep-th/0207158)
[26] Romero J M, Vergara J D and Santiago J A 2007 Phys. Rev. D 75 065008 (arXiv:hep-th/0702113)
Khosravi N and Sepangi H R 2008 Phys. Lett. A 372 3356 (arXiv:0802.0767)
Khosravi N and Sepangi H R 2008 J. Cosmol. Astropart. Phys. JCAP04(2008)011 (arXiv:0803.1714)
[27] Benczik S et al 2002 Phys. Rev. D 66 026003 (arXiv:hep-th/0204049)
Benczik S et al 2002 Classical implications of the minimal length uncertainty relation (arXiv:hep-th/0209119)
[28] Nozari K and Akhshabi S 2006 On the stability of planetary circular orbits in noncommutative spaces
arXiv:gr-qc/0608076
Nozari K and Akhshabi S 2007 Europhys. Lett. 80 20002 (arXiv:0708.3714)
[29] Vilenkin A 1988 Phys. Rev. D 37 888
[30] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover)