IN SEARCH OF PERIODIC SOLUTIONS FOR A REDUCTION OF THE BENNEY CHAIN

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Abstract. We search for smooth periodic solutions for the system of quasi-linear PDEs known as the Lax dispersionless reduction of the Benney moments chain. It is naturally related to the existence of a polynomial in momenta integral for a Classical Hamiltonian system with 1.5 degrees of freedom. For the solution in question it is not known a priori if the system is elliptic or hyperbolic or of mixed type. We consider two possible regimes for the solution. The first is the case of only one real eigenvalue, where we can completely classify the solutions. The second case of strict Hyperbolicity is really a challenge. We find a remarkable 2 by 2 reduction which is strictly Hyperbolic but violates the condition of genuine non-linearity.

1. Motivation and the results

The famous equations of Benney moments are an infinite system of PDEs on the functions $A^k(t, x)$, $k = 0, 1, 2...$

$$A^k_t + A^{k+1}_x + kA^{k-1}_x A^0 = 0, \ k = 0, 1, 2,...$$

(2) (15). It admits many reductions where infinitely many functions $A^n$ become the functions on finitely many field variables $U = (u_1, ..., u_n)$ (see [9, 11, 10, 17]). In this paper we deal with one of the reductions called the dispersionless Lax reduction, where $A_i$ are expressed via $U = (u_1, ..., u_n)$ by the formula:

$$p + \sum_{k=0}^{\infty} \frac{A^k}{p^{k+1}} = (p^{n+1} + (n + 1)u_1 p^{n-1} + ... + (n + 1)u_n)^{\frac{1}{n+1}}.$$

It can be checked that the equations on $(u_1, ..., u_n)$ can be written explicitly as a quasi-linear system of the form:

$$U_t + A(U)U_x = 0, \ A(U) = - \begin{pmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
(n-1)u_1 & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2u_{n-2} & 0 & 0 & \cdots & 0 & -1 \\
u_{n-1} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.$$
for unknown functions $U = (u_1, ..., u_n)^t$.

This is a remarkable Hamiltonian system of the hydrodynamic type related to the $A_n$ singularity (see [12], [13], [20], [9]). Though, local/formal solutions for this system can be studied by the so called generalized hodograph method ([21], [10]), very little is known on the existence of smooth solutions for (1) globally. General belief is that they are very rare. In [4], [5], [6] the global analysis of periodic smooth solution was started for (1) and also in ([7], [8]) for for another Semi-hamiltonian system corresponding to geodesic flows. Here we continue in this direction further.

It was observed in [14] and later in [4], [5], [6] that the question of existence of smooth periodic solution for (1) is ultimately related to the search of polynomial integrals for a Hamiltonian system as we now turn to explain.

Let $H = p^2/2 + u(t, x)$ be a Hamiltonian of a 1,5-degrees of freedom system with the potential $u$ which is assumed throughout this paper to be $C^2$-smooth periodic function in both variables. The only known examples of integrable Hamiltonian of this form with periodic potential functions $u$, are those having the potential $u$ of the form of traveling waves: $u = u(mx + nt)$.

More precisely we want to find all those potential functions $u(t, x)$ for which there exists an additional function $F(p, x, t)$ invariant under the Hamiltonian flow (such an $F$ is called the first integral of motion). Let us stick to the case where $F$ is a polynomial in the variable $p$ of a given degree, say $(n + 1)$, having all the coefficients $C^2$-smooth, periodic in $x$ and $t$. Write

$$F(p, x, t) = u_{-1}p^{n+1} + u_0p^n + u_1p^{n-1} + \cdots + u_n,$$

and substitute to the equation of conservation of $F$

$$(2) \quad F_t + pF_x - u_tF_p = 0.$$ 

Equating to zero the coefficients of various powers of $p$, one easily obtains the following information. The coefficient $u_{-1}$ must be a constant, which can be normalized to be $1/n+1$. Also $u_0$ must be a constant, which we shall assume to be zero (this can be achieved by a linear change of coordinates on the configuration space $T^2$). Moreover the coefficient $u_1$ satisfies $(u_1)_x = (u)_x$. Therefore, $u_1$ and $u$ will be assumed to be equal (the addition of any function of $t$ to the potential $u$ does not change the Hamiltonian equations). Moreover, the functions $U = (u_1, ..., u_n)$ satisfy precisely the system (1).

It is remarkable that the system (1) is Hamiltonian and in particular belongs to the class of Semi-hamiltonian or Rich. This means it can be written in terms of Riemann invariants and has infinitely many conservation laws ([21], [19], [18]).

Let us emphasize that the study of smooth solutions of the system is a very challenging one, for the two reasons: the first reason is that the system is of mixed type a-priori, since it depends on the solution in question. Moreover, in the strictly hyperbolic region it fails to be genuinely non-linear, because the sign of the non-linearity for some eigenvalues can change.

It is important to observe that the characteristic polynomial of $A(U)$ has very clear geometric meaning [4], [5], [9]: it coincides with derivative $F_p$ of $F$, thus giving the information on the phase portrait of the system with the Hamiltonian $H$. Namely, the graph of an eigenvalue $p = \lambda(t, x)$ has the
property that invariant torii of the Hamiltonian flow have vertical tangents at the points on the graph.

The purpose of this paper is to study two opposite possible regimes for the system \( \text{(1)} \). In the first part we consider the case when only one eigenvalue of the matrix \( A(U) \) is real and the rest are complex conjugate pairs (not real). In the second part we deal with the strictly hyperbolic regime, when all eigenvalues of the matrix \( A(U) \) are real and distinct.

We turn now to formulation of our result for the first case. Let us remark first that the assumption of one real eigenvalue for the system \( \text{(1)} \) is very natural in view of phase portrait of autonomous Hamiltonian with 1-degree of freedom, where only one chain of separatix islands is present.

Notice that complex eigenvalues are allowed to collide for some \((t,x)\), however we shall assume, in this case, that the characteristic polynomial can be factorized in a continuous way, see remarks below. Our main result in this case is that the only solutions in this regime are autonomous ones. This is formulated in the following theorem:

**Theorem 1.1.** Let \( n = 2l + 1 \) be odd. Assume that periodic solution \( U(t,x) \) be such that the matrix \( A(U) \) has one real and \( l \) complex conjugate pairs of eigenvalues for every \((t,x)\). We assume that the characteristic polynomial of \( A(U) \) can be continuously factorized:

\[
F_p = (p - \mu) \left( (p - \lambda_1)(p - \bar{\lambda}_1) \ldots (p - \lambda_l)(p - \bar{\lambda}_l) \right),
\]

where \( \mu(t,x), \lambda_i(t,x) \) are continuous functions and \( \mu \) is real and \( \lambda_i, \bar{\lambda}_i \) are complex conjugate pairs. Then the solution \( U \) of quasi-linear system \( \text{(1)} \) is the traveling wave solution, where the components \((u_1, \ldots, u_n)\) do not depend on \( t \).

**Remarks.** 1. Let us mention that for the case of even \( n \) when all eigenvalues of \( A(U) \) are complex conjugate pairs (not real), it was proved in \[5\] that \( U \) in this case must be a constant solution. Taking into account the geometric meaning of the eigenvalues mentioned above, one can interpret this result as a reflection of the so called Hopf rigidity phenomenon known in Riemannian geometry.

2. We don’t know if the condition of continuous factorization of \( F_p \) used in Theorem 1.1 is really essential for the result. In smooth 1-parameter family of polynomials, roots can be chosen continuously but in 2-parameter family this does not necessarily hold due to monodromy effect. Continuous factorization obviously holds in the case of all distinct roots.

Our second result deals with the strictly hyperbolic case.

In this case it appears to be a difficult problem to classify possible smooth periodic solutions. The reason for this lies in the fact that genuine non-linearity of the eigenvalues cannot be guarantied in general. To emphasize this fact we introduce in this paper a remarkable 2 by 2 reduction of the system \( \text{(1)} \) for \( n = 4 \) and thus a reduction of the infinite Benney chain as well. We prove:
Theorem 1.2. The following quasi-linear system is a reduction of (1) for $n = 4$:

$$
\begin{align*}
    w_t + (wv)_x &= 0 \\
    v_t + \left(\frac{w^2}{2} - \frac{3v^2}{2}\right)_x &= 0.
\end{align*}
$$

It is strictly hyperbolic outside the origin in the $(u,v)$ plane. Moreover the sign of the nonlinearity changes when $v$ is changing sign.

We believe that understanding of the behaviour of this reduced system can shed light in the search of smooth periodic solutions for Hyperbolic regime in general. We prove Theorem 1.2. in Section 6.

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2. Riemann invariants and critical values of polynomials

We start with the following crucial fact on critical values of smooth family of polynomials, which we believe is of independent interest.

Theorem 2.1. Let $F(p,y)$ be a family of polynomials smoothly depending on the parameters $y = (y_1, \ldots, y_n)$ of the form

$$
F = \frac{1}{n+1} p^{n+1} + u_1(y)p^{n-1} + \ldots + u_n(y).
$$

Assume that $\lambda(y)$ is a (complex) critical point of the polynomial $F$ continuously depending on $y$. Then the corresponding critical value

$$
r(y) := F(\lambda(y), y)
$$

is a $C^1$ function of $y$.

This theorem states that though in a smooth family of polynomials the critical points are not smooth in general (due to collisions) but the corresponding critical values are at least $C^1$.

Coming back to the formulation of Theorem 1.1, we shall denote by $\rho$ and $r_i$ the critical values of the polynomial $F$:

$$
\rho(t,x) := F(\mu(t,x), x, t), \quad r_i(t,x) := F(\lambda_i(t,x), x, t)
$$

It then follows from Theorem 2.1 and the equation (2) that under the conditions of Theorem 1.1 the functions $r_i(t,x)$ and $\rho(t,x)$ are Riemann invariants for system (1):

Corollary 2.2. Under the assumptions of Theorem 1.1 the functions $\rho$, $r_i$ are $C^1$ and satisfy the equations

$$
\rho_t + \mu(t,x)\rho_x = 0, \\
(r_i)_t + \lambda_i(t,x)(r_i)_x = 0.
$$
Proof of Theorem 2.1. One needs to show that partial derivatives of $r$ are continuous at every point $y_0$. If $\lambda(y_0)$ is a simple root of $F_p$, that is $F_{pp}(\lambda(y_0), y_0) \neq 0$ then the statement is obvious because in this case it follows from the implicit function theorem for $F_p(\lambda(y), y) = 0$ that $\lambda(y)$ is continuously differentiable in a neighborhood of $y_0$ and hence also $r(y)$ as a superposition $F(\lambda(y), y)$. Moreover we have

$$\partial_y r(y_0) = F_p(\lambda(y_0), y_0) + \partial_y F(\lambda(y_0), y_0) = \partial_y F(\lambda(y_0), y_0).$$

We shall prove that also in general case, when $\lambda(y_0)$ is a root of multiplicity $m > 1$ the same formula holds true. We use for this the fact that $\lambda(y)$ is Hölder continuous at a neighborhood of $y_0$ of order $1/m$ (see [16]). We have for every coordinate vector $e_i$:

$$\frac{r(y_0 + \varepsilon e_i) - r(y_0)}{\varepsilon} = \frac{F(\lambda(y_0 + \varepsilon e_i), y_0 + \varepsilon e_i) - F(\lambda(y_0), y_0)}{\varepsilon} = A + B,$$

where we write:

$$A := \frac{F(\lambda(y_0 + \varepsilon e_i), y_0 + \varepsilon e_i) - F(\lambda(y_0 + \varepsilon e_i), y_0)}{\varepsilon},$$

$$B := \frac{F(\lambda(y_0), y_0) - F(\lambda(y_0), y_0)}{\varepsilon}.$$

It follows from mean value theorem and the continuity of $\lambda(y)$ that

$$\lim_{\varepsilon \to 0} A = \partial_y F(\lambda(y_0), y_0).$$

Next we need to show that $B$ tends to 0 as $\varepsilon \to 0$. This goes as follows. Since $\lambda(y_0)$ is a root of multiplicity $m > 1$ for $F_p(p, y_0)$ it then follows that the polynomial $F(p, y_0)$ can be written as

$$F(p, y_0) = (p - \lambda(y_0))^m f(p) + r(y_0)$$

for some polynomial $f$. Therefore the term $B$ can be written as:

$$B = \frac{(\lambda(y_0 + \varepsilon e_i) - \lambda(y_0))^m f(\lambda(y_0 + \varepsilon e_i))}{\varepsilon}.$$

Since $\lambda$ is Hölder with exponent $\frac{1}{m}$ then

$$|B| \leq C\varepsilon^{\frac{m+1}{m}} = C\varepsilon^{1/m}.$$

Thus

$$\lim_{\varepsilon \to 0} B = 0.$$

This completes the proof.

3. Maximum principle for complex Riemann invariants

Let $r_j(t, x)$ be a Riemann invariant corresponding to the complex eigenvalue $\lambda_j(t, x)$. We have:

**Theorem 3.1.** The complex Riemann invariants $r_j$ are constant functions on the whole torus $\mathbb{T}^2$ for every $j = 1, \ldots, l$. 

For simplicity we shall omit in this section the index $j$.

Let $r = u(t, x) + iv(t, x)$, $\lambda = a(t, x) + ib(t, x)$, $b > 0$, then the equation $(r)_t + \lambda \cdot (r)_x = 0$ leads to the system of equations:

\[
\begin{aligned}
  u_t + au_x - bv_x &= 0 \\
  v_t + bu_x + av_x &= 0.
\end{aligned}
\]

It is equivalent to the following:

\[
\begin{aligned}
  u_t &= \frac{a}{b} v_t + \frac{a^2 + b^2}{b} v_x \\
  -u_x &= \frac{1}{b} v_t + \frac{a}{b} v_x.
\end{aligned}
\]

This is the famous Beltrami system of equations corresponding to the Riemannian metric

\[ds^2 = (a^2 + b^2) dt^2 - 2adtdx + dx^2.\]

We know that it has continuous coefficients since $\lambda(t, x)$ is continuous by the assumptions. Moreover since $r$ is $C^1$-function (due to Corollary 2.2) we have that $(u, v)$ is a global $C^1$ solution of the system \(3\). Therefore, the mapping $(t, x) \to (u, v)$ is quasi-conformal. Then, it follows from representation theorem for quasi-conformal maps that the maximum principle applies and hence $(u, v)$ are constant functions on the whole torus $\mathbb{T}^2$ (otherwise they have a point of maximum somewhere on the torus). We refer to [3] p. 44 – 48 and [1] for the details on the theory of quasi-conformal maps.

Remark 1. Notice that a-priori it is not true that the functions $r_j$ are functions on the torus, but this can be achieved passing to a suitable finite cover of the torus.

4. USING CRITICAL VALUES AS A SYSTEM OF LOCAL COORDINATES

Consider the space $\mathbb{C}^n$ of all polynomials of degree $(n + 1)$ of the form

\[F = \frac{1}{n + 1} p^{n+1} + u_1 p^{n-1} + \ldots + u_n,\]

with (complex) coefficients $u_i$. Denote by $\Lambda_1 \subset \mathbb{C}^n$ the subset of polynomials with Morse critical points. The complement to $\Lambda_1$ consists of those polynomials that the derivative has a multiple root. We define the strata

\[\Lambda_{m_1 \ldots m_k} \subset \mathbb{C}^n, \quad \sum_{i=1}^{k} m_i = n, \quad m_1 \geq \ldots \geq m_k \geq 1\]

to be the subset of those polynomials $F$ such that the derivative $F_p$ has precisely $k$ distinct roots of the multiplicities $m_i$. The following lemma used in many papers (see for instance [19], [20]) We could not find however a proof in the literature. The proof below was communicated to us by Eugene Shustin.

Lemma 4.1. The critical values $r_i = F(\lambda_i), i = 1, \ldots, k$ form a system of local coordinates on $\Lambda_{m_1 \ldots m_k}$. 

Proof. First let us remark that the claim is immediate if all the roots of the derivative are of multiplicity 1.

In the general case proof goes as follows. First, notice the relation
\[ m_1 \lambda_1 + \ldots + m_k \lambda_k = 0, \]
and denote by
\[ D : \mathbb{C}^n \to \mathbb{C}^{n-1} \]
the linear projection given by the derivation. In the space \( \mathbb{C}^{n-1} \) of polynomials of degree \( n \), consider the germ \( V = D(\Lambda_{m_1 \ldots m_k}) \) at
\[ G_0 = \prod_{i=1}^{k} (p - \lambda_i)^{m_i} \]
of the family of polynomials having \( k \) roots of multiplicities \( m_1, \ldots, m_k \) respectively. It follows that \( V \) is smooth and its (affine-linear) tangent space consisting of the polynomials \( G \) which can be defined by the affine-linear equations on \( u_1, \ldots, u_{n-1} \):
\[ G^{(j)}(\lambda_i) = 0, 0 \leq j \leq m_i - 2, i = 1, \ldots, k, \]
(here \( \lambda \)'s are fixed and the equation appears only when \( m_i \geq 2 \)). These \( n-k \) linear equations are transversal in \( \mathbb{C}^{n-1} \), since adding \( k-1 \) more equations
\[ G^{(m_i-1)}(\lambda_i) = 0, i = 1, \ldots, k-1, \]
we obtain a system of \( (n-1) \) linear equations evidently having the unique solution \( G_0 \).

Hence, \( \Lambda_{m_1 \ldots m_k} = D^{-1}(V) \) is a smooth germ of a \( k \)-dimensional subvariety in \( \mathbb{C}^n \), and its (affine-linear) tangent space \( W \) at each point \( F_0 \in D^{-1}(G_0) \) is given by \( (n-k) \) (transversal) linear equations
\[ W = \{ F^{(j)}(\lambda_i) = 0, 1 \leq j \leq m_i - 1, i = 1, \ldots, k \}. \]
Consider the map \( R \) taking \( F \in D^{-1}(V) \) to the set \( S \in \mathbb{C}^k \) of critical values of \( F \). The differential of \( R \) at \( F_0 \) acts from \( W \) to \( \mathbb{C}^k \) by formula
\[ W \ni F \mapsto (F(\lambda_1), \ldots, F(\lambda_k)). \]
This is an affine-linear isomorphism, since the preimage of each point is unique: the difference \( H \) of two preimages of the same point must be a polynomial of degree \( \leq n - 1 \) satisfying equations
\[ H^{(j)}(\lambda_i) = 0, 0 \leq j \leq m_i - 1, i = 1, \ldots, k, \]
hence \( H = 0. \)

This lemma implies the following property.

**Theorem 4.2.** Suppose \( F(p, t) = \frac{1}{n+1} p^{n+1} + u_1(t)p^n + \ldots + u_n(t), t \in I \) be a smooth curve in the space of polynomials such that the set of critical values of \( F(\cdot, t) \) does not depend on \( t \). Then \( F \) is a polynomial with constant coefficients.
Proof. First let us factorize the derivative $F(p, t)$:

$$F_p(t) = \prod_{i} (p - \lambda_i(t)),$$

where $\lambda_i(t)$ are continuous (and not necessarily distinct). Denote by $r_i$ the corresponding critical values. Suppose that for some open sub interval $J \subset I$ the polynomial $F(\cdot, t)$, $t \in J$ lies on one stratum $\Lambda_{m_1...m_k}$. Then it follows from the Lemma that $F(p, t)$ is a polynomial with constant coefficients for all $t \in J$. Indeed, this holds locally by Lemma 4.1 and therefore on the whole $J$. So the derivative $F_t \equiv 0$ on $J$. Denote by $U$ the open subset of $I$ which is the union of all those maximal sub-intervals for which $F(\cdot, t)$ belongs to one stratum. As we explained above on $U$ we have $F_t \equiv 0$. Moreover, it follows that the complement $I \setminus U$ has no interior points and thus every point of $I \setminus U$ can be approached by a sequence of points from $U$ and hence by continuity $F_t$ must vanish on the whole $I$. \qed

5. Traveling wave solutions for the system

Now we are in position to apply the result of the previous section to the proof of Theorem 1.1. The first step is the following

**Theorem 5.1.** Let $U$ be a non-constant periodic solution of (1), then the real eigenvalue $\mu(t, x)$ is a constant number on the whole torus and the solution $U$ is a traveling wave solution for the system (1), $U = U(x - \mu t)$.

**Proof.** Denote by $\rho$ the Riemann invariant corresponding to the real eigenvalue $\mu(t, x)$, i.e. $\rho(t, x) = F(\mu(t, x), x, t)$. Function $\rho$ satisfies

$$\rho_t + \mu(t, x)\rho_x = 0.$$

Notice this equation means that $\rho$ must have constant values along characteristic curves. These are integral curves of the equation

$$\dot{x} = \mu(t, x).$$

Therefore, using Theorem 3.1, we conclude that along every characteristic curve all Riemann invariants preserve constant values. It then follows from Theorem 4.2 of the previous section that all the coefficients $u_i(t, x), i = 1, ..., n$ are constants along every characteristic curve. Then $\mu$ as a function of $u_i$-s is also constant along every characteristic. But $\mu$ is the slope of the tangent line to the characteristic, so every characteristic curve must be a straight line. Moreover all these lines must be parallel, since otherwise they intersect, which is impossible for solutions of ODE. So $\mu$ is a constant. This completes the proof. \qed

The next step is to show that the only traveling wave solutions for system (1) are autonomous:

**Theorem 5.2.** Let $U = U(x - \mu t)$ be a traveling wave solution of (1). Then $\mu \equiv 0$, and the polynomial $F$ is a function of the Hamiltonian $H$. 



Proof. Let $U(x - \mu t)$ be a traveling wave solution for the system (1). Then $U'(x)$ must be an eigenvector of the matrix $A(U)$ and we have the following system of ordinary differential on the components $u_i$:

$$
\begin{align*}
\mu u_1' &= u_2' \\
\mu u_2' &= -(n-1)u_1 u_1' + u_3' \\
&\ldots \\
\mu u_{n-1}' &= -2u_{n-2}u_1' + u_n' \\
\mu u_n' &= -u_{n-1}u_1'
\end{align*}
$$

Notice, that first $(n-1)$ equations of this system can be integrated step by step starting from the first one and all functions $u_2, \ldots, u_n$ become polynomial expressions on $u_1$. We write an additional equation using the fact that $\mu$ is an eigenvalue of the matrix $A(U)$, so:

$$
\mu^n + (n-1)u_1 \mu^{n-2} + \ldots + u_{n-1} = 0.
$$

Substituting polynomial expressions of $u_2, \ldots, u_{n-1}$ into the last equation we get the following alternative: either $\mu = 0$, or $u_1$ satisfies certain polynomial equation with constant coefficients and therefore must be a constant. In the second case all the components of the solution are constants. In the first case we get that the Hamiltonian $H$ is autonomous, and $F$ turns out to be a polynomial function of $H$. This completes the proof. □

6. Hiperbolic 2 by 2 reduction for Benney chain

In this section we derive a remarkable reduction of system (1) for $n = 4$. It was shown in [4] that for strictly hyperbolic case of the system (1) the smallest and the largest eigenvalues are genuinely non-linear. This implies that the corresponding Riemann invariants are constants. This motivates the following construction:

We are looking for the function

$$
F = \frac{1}{5}p^5 + u_1 p^3 + u_2 p^2 + u_3 p + u_4,
$$

polynomial of degree 5 satisfying system (2), where $u_1 = u$ is the potential. We write $F$ in the form:

$$
F = \frac{1}{5}(p-f)^2(p-g)^2(p-a),
$$

where $f, g, a$ are some functions. Equating the coefficients of $p^4$ and $p^3$ of the polynomials in (4) and (5) we get

$$
a = -(2f + 2g),$$
$$
u = -\frac{1}{5}(3f^2 + 4fg + 3g^2).
$$

In order to write the equations on $f, g$ notice that $p = f$ and $p = g$ are level sets of $F$ thus are invariant tori of the Hamiltonian system. Therefore the following equations hold:

$$
\begin{align*}
f_t + ff_x + (u)_x &= 0 \\
g_t + gg_x + (u)_x &= 0.
\end{align*}
$$
Introduce

\[ \frac{f - g}{2} = h, \quad \frac{f + g}{2} = q. \]

We get the following equations on \( h, q \)

\[
\begin{align*}
  h_t + (hq)_x &= 0 \\
  q_t + \left( \frac{h^2}{2} + \frac{q^2}{2} - \frac{1}{5} (3f^2 + 4fg + 3g^2) \right)_x &= 0.
\end{align*}
\]

Rewriting the last equation in terms of \( h \) and \( q \) we come to the system:

\[
\begin{align*}
  h_t + (hq)_x &= 0 \\
  q_t + \left( \frac{h^2}{2} - \frac{3q^2}{2} \right)_x &= 0.
\end{align*}
\]

Finally changing \((h, q) \rightarrow (w, v)\), \( w = \frac{h}{\sqrt{3}}, \ v = q \), we come to the system:

\[
\begin{align*}
  w_t + (wv)_x &= 0 \\
  v_t + \left( \frac{w^2}{2} - \frac{3v^2}{2} \right)_x &= 0.
\end{align*}
\]

The matrix of this system reads

\[ A(w, v) = \begin{pmatrix} v & w \\ w & -3v \end{pmatrix}. \]

The matrix is strictly hyperbolic away of the origin on the \((w, v)\)-plane. The eigenvalues are given by the following formula:

\[ \lambda_{1,2} = -v \pm R, \quad R := \sqrt{4v^2 + w^2}. \]

In order to check type of non-linearity one needs to check the sign of the derivative \( \frac{\partial \lambda_1}{\partial \xi_1} \). The eigenvector of \( \lambda_1 \) is given by \( \xi_1 = (w, R-2v) \). Therefore we compute:

\[
\frac{\partial \lambda_1}{\partial \xi_1} = \frac{d\lambda_1(\xi_1)}{d\xi_1} = \partial_w(\lambda_1)w + \partial_v(\lambda_1)(R-2v) =
\]

\[
= \frac{w^2}{R} + \left(-1 + \frac{4v}{R}\right)(R-2v) = \frac{6v(R-2v)}{R}.
\]

So we have the sign of the derivative \( \frac{\partial \lambda_1}{\partial \xi_1} \) equals that of \( v \). This proves Theorem 1.2.

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