Abstract: The paper announces a family of exact solutions to Navier–Stokes equations describing gradient inhomogeneous unidirectional fluid motions (nonuniform Poiseuille flows). The structure of the fluid motion equations is such that the incompressibility equation enables us to establish the velocity defect law for nonuniform Poiseuille flow. In this case, the velocity field is dependent on two coordinates and time, and it is an arbitrary-degree polynomial relative to the horizontal (longitudinal) coordinate. The polynomial coefficients depend on the vertical (transverse) coordinate and time. The exact solution under consideration was built using the method of indefinite coefficients and the use of such algebraic operations was for addition and multiplication. As a result, to determine the polynomial coefficients, we derived a system of simplest homogeneous and inhomogeneous parabolic partial equations. The order of integration of the resulting system of equations was recurrent. For a special case of steady flows of a viscous fluid, these equations are ordinary differential equations. The article presents an algorithm for their integration. In this case, all components of the velocity field, vorticity vector, and shear stress field are polynomial functions. In addition, it has been noted that even without taking into account the thermohaline convection (creeping current) all these fields have a rather complex structure.

Keywords: exact solution; Poiseuille flow; Navier–Stokes equation; nonuniform flow; unidirectional flow; method of separation of variables; tangential stress; specific kinetic energy; specific helicity

1. Introduction

Poiseuille flow is a gradient unidirectional flow of a viscous incompressible fluid [1] (pp. 961–967); [1] (pp. 1041–1048), [2]. The exact Poiseuille solution has been proven to be an indispensable tool for studying fluid flows in pipelines, in the human or animal circulatory system, in geophysical hydrodynamics, in chemical engineering facilities, in magnetic and applied hydrodynamics, and in many other natural and technogenic processes [3–9].

One cannot overestimate the Poiseuille formula when studying problems of hydrodynamic stability with the change of pressure in the horizontal (longitudinal) direction [10–14]. The study of the stability of the Poiseuille profile is governed by the experimentally observed deviation of the flow velocity from the parabolic profile at large Reynolds numbers for a unidirectional flux in pipes of an arbitrary cross section [15,16]. It is because of the loss of stability of Poiseuille flow with increasing flow velocities that the notion of turbulence appears [17–19]. Note that there is still no rigorous definition of turbulence [20–24].

The scientific community understands turbulence as a synonym for flow with a very complex topology regarding the velocity field caused by the effect of boundary conditions and mass forces. Recently, methods of multiplication (replication) of the exact Poiseuille solution to shear flows of a viscous incompressible fluid (two-dimensional in velocities and three-dimensional in coordinates) have been further developed. Nevertheless, the construction of new exact solutions for nonuniform Poiseuille flows is an important and
urgent problem in nonlinear fluid dynamics [25–31]. Numerous studies on the flows of nonuniform fluids and the motions of microstructural media (the Cosserat continuum) in various force fields have been published lately [29,32–34].

This paper studies the possibility of finding an exact solution describing nonuniform Poiseuille flows. The exact solutions are constructed on a modification of the Lin–Sidorov–Aristov class of exact solutions. The class of exact solutions with the velocity field linearly dependent on two coordinates (horizontal or longitudinal) is known. The coefficients of the linear forms of the velocity fields depend on the third coordinate and time. This paper develops an approach to constructing exact solutions with a nonlinear dependence on the velocity field on the horizontal (longitudinal) coordinates for homogeneous and stratified fluids that was announced in [26,35].

2. Problem Statement

The isothermal flow of a viscous incompressible fluid in an infinite horizontal layer is described by the following equation system [36]:

\[ \frac{\partial V}{\partial t} + (V, \nabla)V = -\nabla P + \nu \Delta V \] (1)

\[ (\nabla, V) = 0 \] (2)

The following notations are used in the Navier–Stokes equation (Equation (1)) and the incompressibility equation (Equation (2)): \( V = (V_x, V_y, V_z) \) is the velocity vector; \( P \) is the deviation from the hydrostatic pressure, normalized to the fluid density; \( \nu \) is the kinematic (molecular) fluid viscosity; \( \nabla = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \) is the Hamilton operator; \( \Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2 \) is the Laplace operator; and the parentheses in Equation (1) define the scalar product.

Consider unidirectional fluid flows with the following velocity field, as follows:

\[ V = (V_x(x, y, z, t), 0, 0) \] (3)

The substitution of the velocity vector (3) into Equations (1) and (2) results in the following equation system:

\[ \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \] (4)

\[ \frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 0 \] (5)

\[ \frac{\partial V_x}{\partial x} = 0 \] (6)

Equation (4) is a parabolic heat conduction equation of dimensionality \((3 + 1)\). The equations of System (5) suggest that the pressure \( P \) depends only on one spatial coordinate (the \( x \)-coordinate) and on time \( t \). On the contrary, it follows from the incompressibility Equation (6) that the fluid velocity \( V \) is independent of this coordinate. Thus, the unidirectional fluid flow is nonuniform; the velocity field and the pressure field have the following respective forms:

\[ P = P(x, t), \quad V_x = V_x(y, z, t) \] (7)

The set of functions Equation (7) describing the unidirectional nonuniform Couette–Poiseuille flow satisfies the following equation of dimensionality \((2 + 1)\):

\[ \frac{\partial V_x}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \] (8)
In the following, we analyze the obtained equation and present new exact solutions to Equation (8), which satisfy the Navier–Stokes equation (Equation (1)) and the incompressibility equation (Equation (2)). Note that the steady-state fluid flows \( \mathbf{V} = (V_x(y,z), 0, 0) \), according to Equation (8), distinct from the classical Couette flow [35], depend on the viscosity. However, in general terms, the function \( V_x \) is not harmonic, as, by virtue of Equation (8), it does not satisfy the Poisson equation:

\[
\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} = \frac{1}{\nu} \frac{\partial P}{\partial x}
\]

3. Exact Solution Class

We started constructing exact solutions to Equation (8), equivalent to the Navier–Stokes equation and the incompressibility equation, from the structurally simplest dependences. We then complicated the structure of the exact solutions using the method of the separation of variables and its modification in the construction of solutions [36–38].

Let us first consider the classical Couette–Poiseuille flow assuming that the velocity profile is determined by the following dependence:

\[
V_x = U(z, t)
\]

By substituting solution (9) into Equation (8), we arrive at the following equation:

\[
\frac{\partial U}{\partial t} - \nu \frac{\partial^2 U}{\partial z^2} = -\frac{\partial P}{\partial x}
\]

The left-hand side of this equation depends on the spatial \( z \)-coordinate, the right-hand side being independent of it. Herewith, the right-hand side is not necessarily dependent on the spatial \( x \)-coordinate, although \( P = P(x, t) \).

Consider the following possible situation: suppose the right-hand side of Equation (10) depends on the variable \( x \), i.e., \( \frac{\partial P}{\partial x} = P_1(t) \). Then, because of the independence of the variables \( x, y, z, \) and \( t \), Equation (10) splits into two independent equations as follows:

\[
\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial z^2}, \quad \frac{\partial P}{\partial x} = 0
\]

(11)

It directly follows from the second equation in (11) that \( P_1(x, t) = 0 \), i.e., that the derivative \( \frac{\partial P}{\partial x} \) is independent of \( x \). This yields a contradiction. Consequently, \( \frac{\partial P}{\partial x} = P_1(t) \).

Then, Equation (10) does not split into independent equations but is an inhomogeneous heat conduction equation of the dimensionality \((1 + 1)\). Simultaneously, the condition \( \frac{\partial P}{\partial x} = P_1(t) \) determines the pressure distribution in the flow region under study:

\[
P = P_1(t)x + P_0(t)
\]

(12)

If we assume that, besides the foregoing, the viscous incompressible fluid flow under study is steady-state \((V_x = U(z), P = P_1x + P_2)\), the velocity profile is determined from the second-order inhomogeneous ordinary differential equation, and it has the following form:

\[
U = \frac{P_1}{2\nu}z^2 + C_1z + C_2
\]

here, \( C_1 \) and \( C_2 \) are integration constants. Particularly, this solution describes the profile of the classical Poiseuille flow [1] (pp. 961–967); [1] (pp. 1041–1048), [2].

Consider another known particular solution to Equation (8) in the assumption that the velocity field is described by the following function:

\[
V_x = yu_1(z, t)
\]

(13)
The dependence described by Equation (13) is substituted into Equation (8):

\[ y \frac{\partial u_1}{\partial t} = -\frac{\partial P}{\partial x} + y \nu \frac{\partial^2 u_1}{\partial z^2}. \]

This equation can be represented as the condition that the linear form with nonlinear coefficients is equal to zero,

\[ \frac{\partial P}{\partial x} + y \left( \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial z^2} \right) = 0. \] (14)

Distinct from the previous case, Equation (14) splits into the following system of two independent equations:

\[ \frac{\partial P}{\partial x} = 0, \quad \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial z^2} = 0. \]

In other words, at each instant the pressure \( P \) is distributed uniformly \( (P = P(t)) \), and the function \( u_1 \) satisfies the simplest parabolic heat conduction equation of the dimensionality \( 1 + 1 \), whose solution in the steady-state case has the following form:

\[ u_1 = C_1 z + C_2. \]

As Equation (8) is linear, the sum of the particular solutions (9) and (13)

\[ V_x = U(z, t) + yu_1(z, t) \] (15)

is also a solution. Expression (15) is substituted into Equation (8),

\[ \frac{\partial U}{\partial t} + y \frac{\partial u_1}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 U}{\partial z^2} + y \frac{\partial^2 u_1}{\partial z^2} \right) \]

Making some transformations, we represent this equation as

\[ \left( \frac{\partial U}{\partial t} - \nu \frac{\partial^2 U}{\partial z^2} + \frac{\partial P}{\partial x} \right) + y \left( \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial z^2} \right) = 0. \]

Taking into account the linear independence of the spatial coordinates, we arrive at the following system of equations consisting of two isolated parabolic equations \( 1 + 1 \) for the determination of velocity \( U \) and spatial acceleration \( u_1 \):

\[ \frac{\partial U}{\partial t} - \nu \frac{\partial^2 U}{\partial z^2} = -\frac{\partial P}{\partial x}, \quad \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial z^2} = 0. \]

Herewith, the pressure \( P \) is distributed according to the law represented by Equation (12). The solution for each of these equations for steady-state flows is given above.

We make a nonlinear addition to the above solution (Equation (15)) as follows:

\[ V_x = U(z, t) + yu_1(z, t) + \frac{y^2}{2} u_2(z, t). \] (16)

The sum represented by Equation (16) is substituted into Equation (8), and simple transformations result in the following equation:

\[ \frac{\partial U}{\partial t} + y \frac{\partial u_1}{\partial t} + \frac{y^2}{2} \frac{\partial u_2}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( u_2 + \frac{\partial^2 U}{\partial z^2} + y \frac{\partial^2 u_1}{\partial z^2} + \frac{y^2}{2} \frac{\partial^2 u_2}{\partial z^2} \right). \]
By virtue of the method of undetermined coefficients at the \( y \)-coordinate, this equation splits into a number of equations,

\[
\frac{\partial U}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( u_2 + \frac{\partial^2 U}{\partial z^2} \right), \quad \frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial z^2}, \quad \frac{\partial u_2}{\partial t} = \nu \frac{\partial^2 u_2}{\partial z^2}.
\] (17)

Note that in the resulting system (Equation (17)), some of the heat conduction equations of dimensionality \((1 + 1)\), used for the determination of velocity field components, stop being isolated. This result is attributed exactly to the nonlinearity of the last term in solution (16) with respect to the variable \( y \) and the contribution of the term \( \partial^2 V_x / \partial y^2 \) in the right-hand side of Equation (8). In other words, solution (16) is no longer a superposition of the previously presented solutions, i.e., it inherits the nonlinear properties of the Navier–Stokes equations. This trend continues with a further increase in the power of the \( y \)-coordinate in the terms determining the form of velocity \( V_x \).

4. Arbitrary-Order Polynomial Exact Solutions

Now, consider the exact solution of Equation (8) as the polynomial sums of a special form,

\[ V_x = U(z, t) + \sum_{k=1}^{n} \frac{y^k}{k!} u_k(z, t). \] (18)

The factor \( k! \) in the second term in Equation (18) denotes taking the factorial of the natural number \( k \), and coefficients \( U \) and \( u_k \) depend on the vertical \( z \)-coordinate and time \( t \). The form of the exact solution can be treated as the application of modified separation of variables [39–44].

Let us now separately calculate the partial derivatives to be substituted into the heat conduction Equation (8), i.e.,

\[
\frac{\partial V_x}{\partial t} = \frac{\partial U}{\partial t} + \sum_{k=1}^{n} \frac{y^k}{k!} \frac{\partial u_k}{\partial t},
\]

\[
\frac{\partial^2 V_x}{\partial z^2} = \frac{\partial^2 U}{\partial z^2} + \sum_{k=1}^{n} \frac{y^k}{k!} \frac{\partial^2 u_k}{\partial z^2},
\]

\[
\frac{\partial^2 V_x}{\partial y^2} = \sum_{k=2}^{n} \frac{y^{k-2}}{(k-2)!} u_k.
\]

In view of the obtained expressions, Equation (8) becomes the following:

\[
\frac{\partial U}{\partial t} + \sum_{k=1}^{n} \frac{y^k}{k!} \frac{\partial u_k}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left[ \sum_{k=2}^{n} \frac{y^{k-2}}{(k-2)!} u_k + \frac{\partial^2 U}{\partial z^2} + \sum_{k=1}^{n} \frac{y^k}{k!} \frac{\partial^2 u_k}{\partial z^2} \right],
\]

\[
\frac{\partial U}{\partial t} + \sum_{k=1}^{n} \frac{1}{k!} \frac{\partial u_k}{\partial t} y^k = -\frac{\partial P}{\partial x} + \nu \left[ \frac{\partial^2 U}{\partial z^2} + \sum_{k=2}^{n} \frac{u_k}{(k-2)!} y^{k-2} + \sum_{k=1}^{n} \frac{1}{k!} \frac{\partial^2 u_k}{\partial z^2} y^k \right].
\]

This equation can be rewritten as a system of differential equations if the method of undetermined coefficients is applied,

\[
\frac{\partial U}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 U}{\partial z^2} + u_2 \right),
\]

\[
\frac{\partial u_1}{\partial t} = \nu \left( u_3 + \frac{\partial^2 u_1}{\partial z^2} \right),
\]

\[
\frac{\partial u_2}{\partial t} = \nu \left( u_4 + \frac{\partial^2 u_2}{\partial z^2} \right).
\]
\[
\begin{align*}
\frac{\partial u_{n-2}}{\partial t} &= \nu \left( u_n + \frac{\partial^2 u_{n-2}}{\partial z^2} \right), \\
\frac{\partial u_{n-1}}{\partial t} &= \nu \frac{\partial^2 u_{n-1}}{\partial z^2}, \\
\frac{\partial u_n}{\partial t} &= \nu \frac{\partial^2 u_n}{\partial z^2}.
\end{align*}
\]

(19)

Here, in parallel with the case represented by Equation (9), the pressure \( P \) is distributed according to Equation (12).

Note that in the resulting system, there are only two isolated equations (the equations for the coefficient \( u_{n-1} \) and the coefficient \( u_n \)). System (19) is integrated inversely, from the last equation to the first one. Each equation in system (19) is a nonstationary equation of the heat conduction type. Herewith, the majority of the equations are inhomogeneous, with their inhomogeneity being determined by solving the preceding equations. Thus, recurrent integration of the equations of system (19) occurs.

If \( n = 2 \) is set in the expansion represented by Equation (18), i.e., if we restrict ourselves to the terms quadratic with respect to \( y \), system (19) becomes

\[
\begin{align*}
\frac{d U}{dt} &= \frac{\partial P}{\partial x} + \nu \left( u_2 + \frac{\partial^2 U}{\partial z^2} \right), \\
\frac{d u_1}{dt} &= \nu \frac{\partial^2 u_1}{\partial z^2}, \\
\frac{d u_2}{dt} &= \nu \frac{\partial^2 u_2}{\partial z^2}.
\end{align*}
\]

The resulting system completely coincides with system (17), determining the components of solution (16).

Let us now illustrate recurrent integration for the description of steady-state flows for the velocity field (16), which has the following form:

\[
\begin{align*}
\frac{d^2 U}{dz^2} &= \frac{p_1}{\nu} - u_2, \\
\frac{d^2 u_1}{dz^2} &= 0, \\
\frac{d^2 u_2}{dz^2} &= 0.
\end{align*}
\]

The exact solution of the latter system has the following form:

\[
\begin{align*}
u_2 &= C_1 z + C_2, \\
u_1 &= C_3 z + C_4,
\end{align*}
\]

\[
U = -C_1 \frac{z^3}{6} + \left( \frac{p_1}{\nu} - C_2 \right) \frac{z^2}{2} + C_5 z + C_6.
\]

The last coefficient \( u_2 \) and the last but one coefficient \( u_1 \) are linear functions, and the homogeneous term \( U \) is described by a third-degree polynomial.

Returning to the study of the velocity field (18), we consider the case of a steady-state flow. In view of Equation (12) it can be reduced to the integration of system (19) of the following form:

\[
\begin{align*}
\frac{d^2 U}{dz^2} &= \frac{p_1}{\nu} - u_2, \\
\frac{d^2 u_1}{dz^2} &= -u_3, \\
\frac{d^2 u_2}{dz^2} &= -u_4, \\
\ldots, \\
\frac{d^2 u_{n-2}}{dz^2} &= -u_{n-1}, \\
\frac{d^2 u_{n-1}}{dz^2} &= 0,
\end{align*}
\]
\[
\frac{d^2 u_n}{dz^2} = 0.
\]

The system is integrated as in the particular case for \( n = 2 \). Coefficients \( u_n \) and \( u_{n-1} \) are the linear functions, and the coefficients determined directly through \( u_n \) and \( u_{n-1} \) are cubic polynomials. The degree of polynomials describing the other coefficients in the representation of the velocity field will gradually increase by two each time \( V_x \).

5. Discussion of Boundary Conditions

After the exact solution of the form (18) is obtained (the general solution of the system of Equation (19) or its special cases) through direct substitution, one can verify that it satisfies the original system (1) and (2). However, at the same time, the constructed solution describes a set of flows similar in a structure, but with different properties, as different boundary conditions act on the boundaries of the flow region. To use the solution of the form (18) (or its particular cases) to determine the characteristics of a particular flow, it is necessary to calculate all the integration constants that arise when solving the system of Equation (19). In other words, it is necessary to stipulate the boundary conditions corresponding to this particular flow.

Let us consider, for definiteness, the flow in an extended horizontal layer with non-deformable boundaries, as is considered in the classical Couette [45] and Poiseuille [1] (pp. 961–967); [1] (pp. 1041–1048), [2] flows.

The condition on the lower boundary of the fluid is the condition on the flow velocity (and hence on the components \( U_k, u_k \) in the representation (18)). It depends on the type of contact surface. If the contact solid surface is hydrophilic, then the sticking condition is used; if the contact solid surface is hydrophobic, then the ideal slip condition is used. In other cases, the most appropriate for the physics of the process is the slip condition (for example, in the form of the Navier slip condition [25,30]). On the upper boundary of the layer, in addition to the specified types of conditions (if the layer is limited from above by a solid surface), for example, the wind action (velocity field distribution) or the wind shear stress (shear stress field distribution) in the case of the “fluid–gas” boundary can also be set.

The issue with the choice of boundary conditions becomes even more difficult if the fluid is density stratified (multilayer fluid, each layer of which has its own characteristics—density, viscosity, and others). In this case, it is necessary to stipulate the conditions at the common boundaries of two neighboring layers. By virtue of the continuity hypothesis, the choice of the velocity field continuity condition on these “internal” boundaries is obvious. However, these conditions alone are not enough. The next idea, typical of disciplines such as the mathematical and functional analysis, is to use the smoothness condition for the velocity field, i.e., the condition of continuity of the derivatives of the velocity field with respect to spatial coordinates. However, in the hydrodynamics of Newtonian multilayer fluids, this approach will not work, as it will lead to a rupture of the first kind in the shear stress field in the transverse direction in view, for example, of Newton's law on the relationship between the shear stress field and the flow velocity field. The problem with choosing boundary conditions at the “internal” boundaries of multilayer fluids is discussed in more detail, for example, in [46].

6. Vorticity and Tangential Stresses

It is known that, for the classical unidirectional Couette flow, the tangential stress

\[
\tau_{xz} = \eta \frac{\partial V_x}{\partial z}
\]

(\( \eta \) is the dynamic viscosity coefficient for the fluid) is constant, i.e., stresses of the same sign are detected in the fluid [45,47–55]. Let us now study the behavior of the stress tensor components if the velocity field is representable in the form of Equation (18).
According to [36], the stress tensor in the fluid is computed as follows:

\[
\tau = \begin{pmatrix}
-p + 2\eta \frac{\partial V_y}{\partial x} & \eta \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_x}{\partial z} \right) & \eta \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_y}{\partial z} \right) \\
\eta \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_x}{\partial z} \right) & -p + 2\eta \frac{\partial V_x}{\partial y} & \eta \left( \frac{\partial V_y}{\partial x} + \frac{\partial V_z}{\partial y} \right) \\
\eta \left( \frac{\partial V_x}{\partial z} + \frac{\partial V_y}{\partial y} \right) & \eta \left( \frac{\partial V_y}{\partial x} + \frac{\partial V_z}{\partial y} \right) & -p + 2\eta \frac{\partial V_z}{\partial x}
\end{pmatrix}
\] (18)

Where \( \eta = \frac{\eta}{\xi} \) is the value of hydrostatic pressure. For comparison, the tangential stress \( \tau_{xz} \) is computed as:

\[
\tau_{xz} = \eta \frac{\partial V_x}{\partial z} = \eta \left( \frac{\partial U}{\partial z} + \sum_{k=1}^{n} \frac{\partial u_k}{\partial z} y^k \right).
\]

With due regard for the spatial inhomogeneity of the velocity of unidirectional flows, the tangential stress field has a complex topology regarding the zones of tensile and compressive stresses. The zero value boundaries of the tangential stress \( \tau_{xz} \) for each instant are determined by the following algebraic equation:

\[
\frac{\partial U}{\partial z} + \sum_{k=1}^{n} \frac{\partial u_k}{\partial z} y^k = 0.
\]

It is obvious from Equation (20) that the inhomogeneous velocity distribution entails the appearance of tangential stress

\[
\tau_{xy} = \eta \frac{\partial V_y}{\partial y} = \eta \sum_{k=1}^{n} u_k \frac{y^{k-1}}{(k-1)!}
\]

in the fluid.

We now analyze flow vorticity for the exact solution of the form represented by Equation (18). By definition, vorticity is determined by the following determinant:

\[
\Omega = \text{rot} V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \frac{\partial V_z}{\partial y} i - \frac{\partial V_y}{\partial z} k = \frac{\partial V_x}{\partial z} j - \frac{\partial V_z}{\partial x} k.
\] (21)

The obtained expression enables us to state that the unidirectional flow represented by Equation (18) is a vortex flow everywhere (possibly, except for several points). The appearance of the vortex is attributable to a strong nonlinear \( y \)-dependence of the structure on solution (18).

Potential flow zones appear when both coefficients in the vector representation of vorticity (21) simultaneously become zero. That is, the potential flow zones are determined by the following condition

\[
\tau_{xz}^2 + \tau_{xy}^2 = 0.
\]

In this case, the tensor in Equation (21) becomes diagonal. Its form formally coincides with the stress tensor for a perfect fluid.

Helicity is another flow characteristic that is directly associated with the notion of vorticity. Under certain conditions, helicity is an invariant of the Euler equations, which characterizes the extent of the linkage of vortex lines in a flow. This was independently proven by Moreau and Moffatt.

The presence of helicity generates large-scale vortex structures in multiphase and shear flows. Helicity estimation is used to study the stability of a uniform turbulent shear flow relative to large-scale velocity perturbations. According to [56], helicity can be calculated as follows:

\[
\chi = V \cdot \text{rot} V = V \cdot \Omega.
\]
Taking into account the structure of the velocity field in Equation (18) and the obtained expression for vorticity (22), we have the following:

\[ \chi = \left( U(z, t) + \sum_{k=1}^{n} \frac{\eta}{k!} u_k(z, t) \right) \cdot \begin{pmatrix} 0 \\ \frac{\tau_{xy}}{\eta} \\ \frac{-\tau_{xz}}{\eta} \end{pmatrix} = 0. \]

It was reported in [57,58] that, in the absence of helicity, large-scale turbulent structures decay.

Finally, we discuss specific kinetic energy as being dependent on the components of velocity representation (18),

\[ \frac{2E_k}{\rho} = V_s^2 = \left( U + \sum_{k=1}^{n} \frac{\eta}{k!} u_k \right)^2. \]

As the study deals with a unidirectional flow, we obtained a result that any stagnation (zero) point of the velocity field (flow stop point) automatically becomes the bending point of specific kinetic energy. Note that, in general terms, the finding of stagnation points (first of all, in the exact statement) is a nontrivial problem, as the coefficients of the polynomial representation (18) depend arbitrarily both on the spatial z-coordinate and on time t.

The above-discussed classes for the velocity field in the unidirectional flow under study are applicable to the description of unidirectional flows of stratified fluids. Many chemical engineering facilities are based on the use of stratified fluids [59–61], and can thus be used for finding solutions to problems of astrophysics, atmospheric physics, and geophysical hydrodynamics [62,63]. The stratified structure of isothermal flows of viscous incompressible fluids is first of all due to density change. Experiments show that the variation of density values is the most pronounced with the change of the vertical (transverse) coordinate. In view of the continuity hypothesis, the density changes continuously in terms of time and the vertical coordinate. However, this dependence is as a rule unknown, and only the values of this function at some points are known. This results in using an approximate representation of the dependence of density on time and the vertical coordinate in calculations. The simplest approximate representation of this kind is the model with a step function of density, according to which, for each fluid flow layer, its own (constant) density and, strictly speaking, its own (constant) dynamic viscosity coefficient are specified.

Then, the description of the flows of such stratified fluids is reduced to solving an equation system of the form, as in Equation (8),

\[ \frac{\partial V_s^{(i)}}{\partial t} = -\frac{\partial p^{(i)}}{\partial x} + \nu_i \left( \frac{\partial^2 V_s^{(i)}}{\partial y^2} + \frac{\partial^2 V_s^{(i)}}{\partial z^2} \right), \quad i = 1, 2, \ldots, n. \]  

Here, index i indicates a specific layer from n layers constituting the isothermal unidirectional flow of a stratified flow.

The velocity of each i-th layer is described by the class of the form of Equation (18),

\[ V_s^{(i)} = U^{(i)}(z, t) + \sum_{k=1}^{n} \frac{\eta}{k!} u_k^{(i)}(z, t), \quad i = 1, 2, \ldots, n. \]  

For each velocity of the form, as in Equation (23), all of the inferences made in the foregoing analysis of the velocity \( V_s \) of the unidirectional flow of a single-layer fluid are true. However, taking into account stratification makes it necessary to study additional issues. The fact is that, despite the stratified structure, it is still a unified flow; consequently, the continuity hypothesis must hold. In other words, there must be no hydrodynamic field discontinuities on the interface between two layers (at the transition from one layer to another). This necessitates the formulation of additional conditions (besides the boundary
conditions specified at the boundary of the flow region). An obvious solution is the requirement of velocity field continuity at the interlayer transition (at the common boundary $\Gamma_{i,i+1}$):

$$V_x^{(i)}|_{\Gamma_{i,i+1}} = V_x^{(i+1)}|_{\Gamma_{i,i+1}}, \ i = 1, 2, \ldots, n - 1.$$ 

However, these $(n - 1)$ conditions prove to be insufficient (at the specified boundary conditions) for the determination of all of the integration constants appearing in the solution of system (22) in the class represented by Equation (23). More details about the features of modeling stratified flows on the basis of classes of the form as in Equation (23) and some other (more general) classes can be found, e.g., in [26].

7. Conclusions

The study proposed a new nonlinear (in terms of spatial coordinates) class of solutions to the Navier–Stokes equations describing a nonstationary isothermal unidirectional vertical vortex fluid flow. This class enables many classical flows to be generalized, e.g., the Couette–Poiseuille flow. A distinctive feature of this class is the arbitrary-order polynomial dependence of velocity on one of the horizontal coordinates, the coefficients of this polynomial representation being arbitrarily dependent on the vertical coordinate and time. The paper shows that solutions of this form, starting from second-order polynomials, cannot be obtained by the superposition of lower-power solutions. An equation system allowing one to determine the variable coefficients in the velocity representation is presented. The pressure remains a linear function of time and the longitudinal coordinate changing along the flow. The solution of the system has been shown to be reducible to the successive integration of heat conduction nonstationary inhomogeneous equations. It has also been noted that the presented class of solutions is applicable to the description of stratified fluids.

Author Contributions: Conceptualization, N.B. and E.P.; methodology, N.B. and E.P.; writing—original draft preparation, N.B. and E.P.; writing—review and editing, N.B. and E.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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