A SIMPLE REGULARIZATION OF HYPERGRAPHS

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Abstract. We give a simple and natural (probabilistic) construction of hypergraph regularization. It is done just by taking a constant-bounded number of random vertex samplings only one time (thus, iteration-free). It is independent from the definition of quasi-randomness and yields a new elementary proof of a strong hypergraph regularity lemma. Consequently, as an example of its applications, we have a new self-contained proof of Szemerédi’s classic theorem on arithmetic progressions (1975) as well as its multidimensional extension by Furstenberg-Katznelson (1978).

1. Introduction

1.1. Szemerédi-type density theorems. The following is often considered as one of the deepest theorems in combinatorics.

Theorem 1.1 (Multi-dimensional Szemerédi Theorem – Furstenberg-Katznelson (1978) [16]). For any \( \delta > 0, \ r \geq 1, \ \text{and} \ F \subset \mathbb{Z}^r \) with \( |F| < \infty \), if an integer \( N \) is sufficiently large then for any subset \( S \subset \{0,1,\ldots,N-1\}^r \) with \( |S| \geq \delta N^r \) there exist \( a \in \{0,1,\ldots,N-1\}^r \) and \( c \in [N] \) with \( a+cF \subset S \).

Furstenberg and Katznelson (1978) [16] proved this by using ergodic theory. The special case of \( r = 2 \) and \( F = \{(0,0),(0,1),(1,0),(1,1)\} \) was first conjectured by R.L. Graham in 1970 ([1, 11]). The case of \( r = 2 \) and \( F = \{(0,0),(0,1),(1,0)\} \), was investigated initially by Ajtai-Szemerédi (1974) [1].

The following was first conjectured by Erdős and Turán (1936) [12].

Corollary 1.2 (Szemerédi (1975) [41]). For any \( \delta > 0 \) and \( m \geq 1 \), there exists an integer \( N \) such that any subset \( S \subset [N] \) with \( |S| \geq \delta N \) contains an arithmetic progression of length \( m \).

Green and Tao [21] recently proved the existence of arbitrarily long arithmetic progressions in the primes, in which they used Szemerédi’s theorem.

1.2. A brief history of hypergraph regularity. Inspired by the success of the celebrated Graph Regularity Lemma [42], research on quasi-random hypergraphs was initiated independently by at least four groups: Chung or Chung-Graham [5, 6, 7, 8, 9], Frankl-ROdl [13], Haviland-Thomason [23, 24], and Steger [39] (see [32] for its application). For other earlier work, see [4, 10]. Also, Frankl-ROdl (2002) [14] gives a regularity lemma for 3-uniform hypergraphs.

Then ROdl and his collaborators [35, 31] and Gowers [20] independently obtained their hypergraph regularity lemmas. Slightly later, Tao [44] gave another regularity lemma.

It has been noted that unlike the situation for graphs, there are several ways one might define regularity for hypergraphs (ROdl-Skokan [35, pp.1], Tao-Vu [40, pp.455]). (For sparse hypergraphs, an essential difference appears. See [19, §10].) Kohayakawa et al. [31] say that the basic objects involved in the Regularity Lemma and the Counting Lemma are already somewhat technical and that simplifying these lemmas would be of great interest. In this paper we try to meet these requirements. We can naturally obtain strong quasi-random properties not from one basic quasi-random property but from our construction of a certain partition which we will define.

In this paper, we give a new construction of hypergraph regularization. Our regularization is achieved by a quite simple (probabilistic) construction which makes it easy to understand why it works. Note that our construction of regularization is new even if we assume we are working with ordinary graphs. In our construction, the number of random vertex samplings is not a fixed constant and our construction is iteration-free. (In later sections, we will see how different it is from property test more.) But once the statement of our construction is given, its proof may be deduced naturally.

For applications of the main result of this paper, see [27, 26, 28].

2000 Mathematics Subject Classification. 05D40, 05C65.
Key words and phrases. Szemerédi’s theorem, hypergraph regularity lemma, additive number theory.
1.3. Differences from the previous hypergraph regularities. A Regularity lemma works well for applications when its counting lemma accompanies it. All of the previous proofs go as follows.

(i) Define regularity (a basic quasi-random property) for each cell (a $k$-uniform $k$-partite hypergraph),
(ii) Prove the existence of a partition in which most cells satisfy the regularity. [Regularity Lemma]
(iii) Estimate the number of copies of a fixed hypergraph. [Counting Lemma]

Our program will go as follows.

(i') Define the construction of a partition. (Its existence will be clear.)
(ii') Estimate the number of copies of a fixed (colored) hypergraph.

Once the definition of the construction via random samplings is given, the concept of our proof is simple. The most interesting technical part in our proof is to use ‘linearity of expectation.’ All of the previous proofs use the dichotomy (or energy-increment) explicitly and iteratively. (See [20], §6, [45] §1.) Namely, when proving (ii), they define an ‘energy’ (or index) by the supremum (or maximum) of some (energy) function. (For example, see [44] eq. (8)). It corresponds to (23) in this paper. They consider the supremum value of this energy over all subdivisions in each step. If the energy significantly increases by some subdivision, they take the worst subdivision as the base partition of the next step. They then repeat this process. Since the energy is bounded, this operation must stop at some step, in which case there is no quite bad subdivision, and thus, most cells should be quasi-random (dichotomy).

On the other hand, we (implicitly) take an average subdivision instead of the worst one. The definition of our regularization determines the probability space of partitions (subdivisions). We also randomly decide on the number of vertex samples to choose. With these ideas, we can hide the troublesome dichotomy iterations inside linear equations of expectations (22). (Imagine what would happen in (22) if we replaced $E_x$ by $\sup_x$ in (23).) (One of the main reasons why Tao’s [44] proof is relatively shorter than the earlier one may be that he also reduced double-induction concerns by partitioning the previous proofs use the dichotomy (or energy-increment) explicitly and iteratively. (See [20], §6, [45] §1.) Namely, when proving (ii), they define an ‘energy’ (or index) by the supremum (or maximum) of some (energy) function. (For example, see [44] eq. (8)). It corresponds to (23) in this paper. They consider the supremum value of this energy over all subdivisions in each step. If the energy significantly increases by some subdivision, they take the worst subdivision as the base partition of the next step. They then repeat this process. Since the energy is bounded, this operation must stop at some step, in which case there is no quite bad subdivision, and thus, most cells should be quasi-random (dichotomy).

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We have two reasons why we will deal with multi-colored hypergraphs instead of ordinary hypergraphs, even though almost all previous researchers dealt with the usual hypergraphs (with black&white edges). First, our proof of the regularity lemma will be natural. Second, we can naturally combine subgraph (black&invisible) and induced-subgraph (black&white) problems when we apply our result, (of index $I$) := $\dot{\bigcup}_{i \in I} H_i$)

2. Statement of the Main Theorem

In this paper, $\mathbb{P}$ and $\mathbb{E}$ will denote probability and expectation, respectively. We denote conditional probability and expectation by $\mathbb{P}[\cdot | \cdot | \cdot ]$ and $\mathbb{E}[\cdot | \cdot | \cdot ]$.

Setup 2.1. Throughout this paper, we fix a positive integer $r$ and an ‘index’ set $\mathbb{r}$ with $|\mathbb{r}| = r$. Also we fix a probability space $(\Omega_i, \mathcal{B}_i, \mathbb{P})$ for each $i \in \mathbb{r}$. We assume that $\Omega_i$ is finite (but its cardinality will not be a constant in our statements) and that $\mathcal{B}_i = 2^{\Omega_i}$, (for the sake of simplicity). Write $\Omega := (\Omega_i)_{i \in \mathbb{r}}$. 

In order to avoid using measure-theoretic jargon such as measurability or Fubini’s theorem, for the benefit readers who are interested only in applications to discrete mathematics, we assume $\Omega_i$ to be a (non-empty) finite set. However, our arguments should be extendable to a general probability space. For applications, $\Omega_i$ usually would contain a huge number of vertices, though we will not use this assumption in our proof. (Note that this assumption has been actively used by many researchers.)

For an integer $a$, we write $[a] := \{1, 2, \cdots, a\}$, and $\binom{\mathbb{r}}{[a]} := \bigcup_{i \in [a]} \binom{\mathbb{r}}{i} = \bigcup_{i \in [a]} \{I \subset \mathbb{r}||I|| = i\}$. We also use the notation $[a, b] := \{a, a+1, \cdots, b\}$ for integers $a, b$.

Definition 2.1. [\ref{hyper}hyper)graphs] Suppose Setup\ref{setup}. A $k$-bounded ($b_i$)$_{i \in [k]}$-colored ($r$-partite hyper)graph $H$ is an object with the following three ingredients:

- A union $V(H) = \bigcup_{i \in \mathbb{r}} V_i(H)$ of disjoint sets. The sets $V_i(H)$ and their elements are called vertex sets and vertices of $H$, respectively. Write $V_j(H) := \{e \in \bigcup_{i \in J} V_i(H) : e \cap V_j(H) = 1, \forall j \in J\}$ whenever $J \subset \mathbb{r}$. Each element $e \in V_j(H)$ with $I \in \binom{\mathbb{r}}{[k]}$ is called an (index-$I$ size-$|I|$) edge.
- For each $I \in \binom{\mathbb{r}}{[k]}$, a set $C_I(H)$ of exactly $b_I$ elements, where the elements are called (face-)colors (of index $I$ and size $|I|$).
For each \( I \in \binom{k}{r} \), a function from \( V_I(H) \) to \( C_I(H) \). Denote by \( H(e) \) the image of \( e \in V_I(H) \) via the function.

Let \( I \in \binom{k}{r} \) and \( e \in V_I(H) \). For another index \( \emptyset \neq J \subset I \), we denote by \( e|_J \) the index-\( J \) edge \( e \setminus \left( \bigcup_{j \in I \setminus J} X_j \right) \in V_J(H) \). We define the \textbf{frame-color} and \textbf{total-color} of \( e \) by vector \( H(\partial e) := (H(e|_J) \mid \emptyset \neq J \subset I) \) and by vector \( H(e) := (H(e|_J) \mid \emptyset \neq J \subset I) \). Write \( TC_I(H) := \{H(e) \mid e \in X_I\} \), \( TC_s(H) := \bigcup_{I \in \binom{k}{r}} TC_I(H) \), and \( TC(H) := \bigcup_{I \in \binom{k}{r}} TC_s(H) \).

**Example 2.2.** An ordinary \((r\text{-partite})\) graph is a \(2\text{-bounded} \) \((b_1, b_2)\)-colored hypergraph with \( b_1 = 1 \) and \( b_2 = 2 \).

A triple \( e = \{v_1, v_3, v_4\} \) of vertices is an index-\( \{1, 3, 4\} \) edge if and only if \( v_1 \in X_1, v_3 \in X_3 \) and \( v_4 \in X_4 \). In any \( k\text{-bounded} \) \( r\text{-partite} \) hypergraph, any vertex in \( X_i \) is an index-\( \{i\} \) edge (whenever \( k \geq 1 \)). For two \( k\text{-bounded} \) \( r\text{-partite} \) hypergraphs \( H \) and \( H' \) with a common vertex set \( V(H) = V(H') = \bigcup_i X_i \), all the edges of \( H \) are also the edges of \( H' \). In this sense, our definition of the word \textquoteleft edge\textquoteright{} is different from that in the classical (hyper)graph theory. In our setting, the essential structure of a colored hypergraph is determined not by the set of edges but by the map from the edges to the colors.

All index-\( I \) edges are colored not only when \( |I| = k \) but also when \( 1 \leq |I| < k \), which is the reason why we call the hypergraph \( k\text{-bounded} \) instead of \( k\text{-uniform} \).

If \( I = \{1, 3, 5\}, J = \{1, 5\} \), \( v_1 \in X_1, v_3 \in X_3, v_5 \in X_5 \) and \( e = \{v_1, v_3, v_5\} \) then \( e|_J = \{v_1, v_5\} \).

Throughout the paper, we will try to embed an \( r\text{-partite} \) graph \( S \) to another larger \( r\text{-partite} \) graph \( G \), where the \( r \) vertex-sets of the larger graph will be always \((\Omega_i)_i \in \mathbb{R} \). And the larger graph and its vertices and edges will be denoted by bold fonts (ex. \( G, v, v', e, \cdots \)) in order to avoid confusing them with those of the smaller graph. The smaller graph will be always a simplicial-complex defined below.

**Definition 2.2.** [Simplicial-complexes] A \((k\text{-bounded})\) simplicial-complex is a \(k\text{-bounded} \) (colored \( r\text{-partite} \) hyper)graph such that for each \( I \in \binom{k}{r} \) there exists at most one index-\( I \) color called \textquoteleft invisible\textquoteright{} and that if (the face-color of) an edge \( e \) is invisible then (the face-color of) any edge \( e^* \subset e \) is invisible. We call an edge invisible when the face-color of the edge is invisible. An edge or its color is \textbf{visible} if it is not invisible.

For a \( k\text{-bounded} \) graph \( G \) on \( \Omega \) and \( s \leq k \), let \( S_{s,h,G} \) be the set of \( s\)-bounded simplicial-complexes \( S \) such that:

1. (1) each of the \( r \) vertex-sets of \( S \) contains exactly \( h \) vertices, and that,
2. (2) for \( I \in \binom{k}{r} \) there is an injection from the index-\( I \) visible colors of \( S \) to the index-\( I \) colors of \( G \).

(When a visible color \( c \) of \( S \) corresponds to another color \( c' \) of \( G \), we simply write \( c = c' \) without presenting the injection explicitly.) For \( S \in S_{s,h,G} \), we denote by \( V_I(S) \) the set of index-\( I \) visible edges. Write \( V_i(S) := \bigcup_{I \in \binom{i}{r}} V_I(S) \) and \( V(S) := \bigcup_i V_i(S) \).

For our purpose of this paper, all of the colors in the larger graph \( G \) can be considered to visible, though we will not use it logically.

**Definition 2.3.** [Partitionwise maps] A \textbf{partitionwise map} \( \varphi \) is a map from \( r \) vertex sets \( W_i, i \in \mathbb{R} \), with \( |W_i| < \infty \), to the \( r \) vertex sets (probability spaces) \( \Omega_i, i \in \mathbb{R} \), such that each \( w \in W_i \) is mapped into \( \Omega_i \). We denote by \( \Phi((W_i)_{i \in \mathbb{R}}) \) or \( \Phi(\bigcup_{i \in \mathbb{R}} W_i) \) the set of partitionwise maps from \( \bigcup_{i \in \mathbb{R}} W_i \). When \( W_i = \{(i, 1), \cdots, (i, h)\} \) or when \( W_i \) are obvious and \( |W_i| = h \), we denote it by \( \Phi(h) \). We write \( \varphi(\mathbb{R}) := \bigcup_{i \in \mathbb{R}} \varphi(W_i) \) for \( \varphi \in \Phi(\bigcup_{i \in \mathbb{R}} W_i) \) (when we want to denote the domain without saying the partition explicitly). A partitionwise map is \textbf{random} if and only if for every \( i \), each \( w \in W_i \) is mutually independently mapped to a point in the probability space \( \Omega_i \).

Define \( \Phi(m_1, \cdots, m_{k-1}) := \Phi(m_1) \times \cdots \times \Phi(m_{k-1}) \).

For two partitionwise maps \( \phi, \phi' \in \Phi((W_i)_{i \in \mathbb{R}}) \) and \( \phi, \phi' \in \Phi((W_i')_{i \in \mathbb{R}}) \), denote by \( \phi \cup \phi' \) the partitionwise map \( \phi^* \in \Phi((W_i \cup W_i')_{i \in \mathbb{R}}) \) such that \( \phi^*(w) = \phi(w) \) and \( \phi^*(w') = \phi'(w') \) for all \( w \in W_i, w' \in W_i', i \in \mathbb{R} \). Here if \( W_i \cap W_i' \neq \emptyset \) for some \( i \) then we consider a copy of \( W_i' \) so that the two domains are disjoint.

**Definition 2.4.** [Regularization] Let \( m \geq 0 \) and \( \varphi \in \Phi(m) \). Let \( G \) be a \( k\text{-bounded} \) graph on \( \Omega \). For an integer \( 1 \leq s < k \), the \textbf{s-regularization} \( G / s^\varphi \) is the \( k\text{-bounded} \) graph on \( \Omega \) obtained from \( G \) by redefining the color of each edge \( e \in \Omega_I \) with \( I \in \binom{s}{r} \) by the \( \left( \sum_{j=0}^{s+1-|I|} \sum_{J \in \binom{s}{r}} m^J \right) \)-dimensional
vector

\[(G/\varphi)(\mathbf{e}) := (G(e \cup f) | J \left( \begin{array}{c} r - I \\ 0, s + 1 - |I| \end{array} \right) f \in \Omega_J \text{ with } f \subset \varphi(\mathbb{D})) \] (1)

In the above, when \( J = \emptyset \), we assume \( f = \emptyset \). (The sets of colors are naturally extended while any edge containing at least \( s + 1 \) vertices (i.e. edge of size at least \( s + 1 \)) does not change its (face-)color.)

When \( s = k - 1 \), we simply write \( G/\varphi := G/k^{-1}\varphi \).

For \( \varphi = (\varphi_i)_{i \in [k-1]} \in \Phi(m_1, \ldots, m_{k-1}) \), we define the regularization of \( G \) by \( \varphi \) by

\[ G/\varphi := ((G/k^{-1}\varphi_{k-1})/k^{-2}\varphi_{k-2}) \cdots /1\varphi_1. \]

When making \( G/\varphi \) from \( G \), a size-\( s \) edge with \( 1 \leq s \leq k \) changes its face-color \( k - s \) times at the operations \( /k^{-1}\varphi_{k-1}, \ldots, /\varphi_s \), depending on \( (m_{k-1} + \cdots + m_s) r \) random vertices in \( \Omega \). It does not change at the operations \( /s^{-1}\varphi_{s-1}, \ldots, /1\varphi_1 \). In particular, any size-\( k \) (full-size) edge never changes its face-color.

\textbf{Definition 2.5.} \textit{[Regularity]} Let \( G \) be a \( k \)-bounded graph on \( \Omega \). For \( \epsilon = (\epsilon_I)_{J \subseteq I} \in TC_I(G), I \in \binom{[k]}{r} \), we define the relative density by

\[ d_G(\epsilon) := \mathbb{P}_{\epsilon \in \Omega_I} |G(\epsilon) = \epsilon_I(G(\partial \epsilon)) = (\epsilon_J)_{J \subseteq I} |. \]

For a positive integer \( h \) and \( \epsilon \geq 0 \), we call \( G \) to be \((\epsilon, k, h)\)-\textit{regular} if and only if there exists a function \( \delta : TC(G) \to [0, \infty) \) such that

\[ \begin{align*}
(1) \quad & \mathbb{P}_{\epsilon \in \Phi(h)} |G(\phi(\epsilon)) = S(\epsilon), \forall \epsilon \in \mathcal{V}(S) | = \prod_{\epsilon \in \mathcal{V}(S)} (d_G(S(\epsilon))^\pm \delta(S(\epsilon))), \forall S \in \mathcal{S}_{k, h, G}, \\
(2) \quad & \mathbb{E}_{\epsilon \in \Omega_I} |\delta(G(\epsilon))| \leq \epsilon/|C_I(G)|, \forall I \in \binom{r}{[k]},
\end{align*} \]

where \( a \pm b \) means a suitable integer \( c \) satisfying \( \max\{0, a - b\} \leq c \leq \min\{1, a + b\} \). Denote by \( \text{reg}_{k,h}(G) \) the minimum value of \( \epsilon \) such that \( G \) is \((\epsilon, k, h)\)-regular. \( \square \)

The minimum value of \( \epsilon \) always exists because inequality (3) includes equality. Note that if \( \delta(\cdot) = 0 \) satisfies the above then the edges of \( G \) are colored uniformly at random.

\textbf{Remark 2.3.} Condition (i) measures how far from random the graph \( G \) is with respect to containing the expected number of copies of the \((\epsilon, k, h)\)-regular subgraphs \( S \in \mathcal{S}_{k, h, G} \). The smaller \( \delta \) is, the closer \( G \) is to being random. When \( \delta = 0 \), then \( G \) behaves exactly like a random graph. On the other hand, if we take \( \delta = 1 \) then (i) is automatically satisfied. Condition (ii) places an upper bound on the size of \( \delta \). Our proof will yield the main theorem even if we replace the right-hand side of (ii) by \( g_I(|C_I(G)|) \) for any fixed functions \( g_I > 0 \), for example, \( g_I(x) = x^{-1/e} \).

\( \square \)

\textbf{Remark 2.4.} In \( \mathbb{P}_{\epsilon \in \Omega_I} [\cdot] \) and \( \mathbb{E}_{\epsilon \in \Omega_I} [\cdot] \), \( \epsilon \) is a random variable, equivalently a sequence of \( |I| \) random vertices. The relative density \( d_G(\epsilon) \) is undefined when \( \mathbb{P}_{\epsilon \in \Omega_I} (G(\partial \epsilon) = (\epsilon_J)_{J \subseteq I}) = 0 \). But this will not cause any trouble later, in particular at (2), since such a relative density will be always multiplied by zero. Here we define \( d_G(\epsilon) \) to be one, if \( \mathbb{P}_{\epsilon \in \Omega_I} (G(\partial \epsilon) = (\epsilon_J)_{J \subseteq I}) = 0 \).

\( \square \)

Our main theorem is as follows.

\textbf{Theorem 2.5 (Main Theorem).} For any \( r \geq k, \bar{b} = (b_i)_{i \in [k]}, \) and \( \epsilon > 0 \), there exist (increasing) functions \( m^{(i)} : \mathbb{N}^{k-i} \to \mathbb{N} \) and \( \bar{n}^{(i)} : \mathbb{N}^{k-1-i} \to \mathbb{N}, i \in [k-1] \) satisfying the following:

If \( G \) is a \( \bar{b} \)-colored \((k\text{-bounded } r\text{-partite hypergraph}) \) on \( \Omega \) then we have

\[ \mathbb{E}_{\bar{n} = (n^{(1)}, \ldots, n^{(k-1)})} \mathbb{E}_{\varphi \in \Phi(\bar{n})} \text{reg}_{k,h}(G/\varphi) \leq \epsilon. \]

In the above probabilistic process, each integer \( n^{(i)} \) (from \( i = k - 1 \) to \( i = 1 \)) is picked uniformly at random from \([0, \bar{n}^{(i)}(n^{(i+1)}, \ldots, n^{(k-1)}) - 1]\). Each \( \varphi_i \in \Phi(m^{(i)}(n^{(i)}, \ldots, n^{(k-1)})) \) is random.

In the above, \( \bar{n}^{(k-1)} \) is read to be a constant integer. When \( k = 1 \), the theorem is read to be true trivially where we do not take \( \bar{n} \) and put \( G/\varphi = G \) while any 1-bounded \( G \) is \((0, 1, h)\)-regular. Thus \( \text{reg}_{1,h}(G) = 0 \).

Note that \( m^{(i)}(\bar{n}^{(i)}) \) depend only on \( r, k, h, \bar{b}, \epsilon \) and are independent of everything else including \( \Omega \). The following immediate consequence is convenient for applications.
Corollary 2.6 (Regularity Lemma (including so-called Counting Lemma)). For any \( r \geq k, h, \tilde{b} = (b_i)_{i \in [k]}, \epsilon > 0 \), there exist integers \( \tilde{m}_1, \ldots, \tilde{m}_{k-1} \) such that if \( G \) is a \( \tilde{b} \)-colored \((k\text{-}bounded \ r\text{-}partite \ hyper)\) graph on \( \Omega \) then for some integers \( m_1, \ldots, m_{k-1} \) with \( m_i \leq \tilde{m}_i, i \in [k-1], \)
\[
\mathbb{E}_{\tilde{\varphi} \in \Phi(m_1, \ldots, m_{k-1})} \left[ \text{reg}_{k,h}(G/\tilde{\varphi}) \right] \leq \epsilon.
\] (4)

In particular, when \( \square \) holds, if we pick a map \( \varphi \notin \Phi(m_1, \ldots, m_{k-1}) \) randomly then with probability at least \( 1 - \sqrt{\epsilon} \), we have \( \text{reg}_{k,h}(G/\varphi) \leq \sqrt{\epsilon} \), thus \( G/\varphi \) is \((\sqrt{\epsilon}, k, h)\)-regular.

Example 2.7. If \( r = k = h = 2 \) and \( (b_1, b_2) = (1, 2) \) then the corollary becomes one of the usual Graph Regularity Lemmas, when \( G \) has black and white edges and \( S \) is an ordinary bipartite graph on \( \{u_1, v_1\} \cup \{u_2, v_2\} \) such that \( u_1, v_1 \) have the same color, say red1, that \( u_2, v_2 \) have the same color, say red2, and that the four edges \( u_1u_2, u_1v_2, v_1u_2, v_1v_2 \) have the same color, say black. (The color red1 may be considered as a sequence of black and white colors.)

Our proof will yield the theorem even if we replace the right-hand side of \( \square \) by \( g_I(|C_I(G)|) \) for any fixed functions \( g_I \geq 0 \), for example, \( g_I(x) = x^{-1/\epsilon} \). If the reader is interested only in applications to Szemerédi’s theorem, then it suffices to consider only the case of \( h = 1 \).

3. Proof of the Main Theorem

3.1. Two lemmas and their proofs.

Definition 3.1. [Notation for the lemmas] Let \( G \) be an \((r\text{-}partite \ (b_i)_{i \in [k]}\text{-}colored) \ k\text{-}bounded \) graph on \( \Omega \). For two edges \( e, e' \in \Omega_I \), we abbreviate \( G(e) = G(e') \) and \( G(\partial e) = G(\partial e') \) by \( e \cong e' \) and \( \partial e \cong \partial e' \), respectively.

An \((s,h)\)-error function of \( G \) is a function \( \delta : \bigcup_{I \in [k]} TC_I(G) \to [0, \infty) \) satisfying (2) for all \( S \in S_{s,h,G} \). We write \( d_G^{(\delta)}(\bar{c}) := d_G(\bar{c}) + \delta(\bar{c}) \) and \( d_G^{-\delta}(\bar{c}) := d_G(\bar{c}) - \delta(\bar{c}) \) for \( \bar{c} \in TC(G) \).

We abbreviate \( \bigcup_{I \in [k-1]} \mathcal{V}_I(S) \) by \( \mathcal{V}_{(k-1)}(S) \).

Denote by \([\cdot]\) the Iverson bracket, i.e., it equals 1 if the statement in the bracket holds, and 0 otherwise.

Lemma 3.1 (Correlation bounds counting error). For a \( k\text{-}bounded \) graph \( G \) and \( S \in S_{k,h,G} \), we have that
\[
\left| \mathbb{P}_{\phi \in \Phi(k)} \left[ G(\phi(e)) = S(e), \forall e \in \mathcal{V}_k(S) \right] \right| - \left| \mathbb{P}_{\phi \in \Phi(k)} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] - d_G(S(e)) \right] \right| \leq \left| \mathbb{P}_{\phi \in \Phi(k)} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] \right] \right|.
\]

Proof: We prove it by induction on \( |\mathcal{V}_k(S)| \). If \( |\mathcal{V}_k(S)| = 0 \) or 1 then it is trivial, in this case, the left-hand side of the inequality is 0. So let us assume that \( |\mathcal{V}_k(S)| \geq 2 \) and that the result holds for all smaller values of \( |\mathcal{V}_k(S)| \). Let \( d_c := d_G(S(e)) \) and let \( \eta \) be the maximum part of the desired right-hand side. Then for \( D := \mathcal{V}_k(S) \) we have
\[
\left| \mathbb{P}_{\phi \in \Phi(h)=\Phi(V(S))} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] - d_G(S(e)) \right] \right| \geq \mathbb{P}_{\phi \in \Phi(h)} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] \right] - \mathbb{P}_{\phi \in \Phi(h)} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] - d_c \right] + \sum_{\phi \notin \mathcal{V}_k(S)} \left( \prod_{e \notin D} (\sum_{\phi \in \Phi(h)} \left[ G(\phi(e)) = S(e) \right] G(\phi(e)) = S(e), \forall e \in \mathcal{V}_{(k-1)}(S) \right) \right),
\]
expanding the product and using the linearity of expectation and the definition of \( d_c \). Now we will focus on second term above. Since the value of \( G(\phi(e)) = S(e) \) is 0 or 1, we can replace \( \mathbb{P} \) by \( \mathbb{F} \), and consequently, apply the induction hypothesis (since \( D \) is nonempty). Consider a complex \( S^- \) with \( \mathcal{V}_k(S^-) = \mathcal{V}_k(S) \setminus D \) by invisualization of the edges in \( D \) of \( S \).
Using the inductive hypothesis for complex \( S^- \) in the place of \( S \), we rewrite the second term and obtain

\[
\mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in \mathcal{V}_k(S)} [G(\phi(e)) = S(e)] \bigg| G(\phi(e)) = S(e), \forall e \in \mathcal{V}_{k-1}(S) \right]^{1,2} \quad \text{L.H.} \quad \sum_{\emptyset \neq D \subset \mathcal{V}_k(S)} \left( \prod_{e \in D} (-d_e) \right) \left( \prod_{e \in \mathcal{V}_k(S) \setminus D} d_e \right) \pm |\mathcal{V}_k(S^-)| \eta \quad \pm \eta
\]

\[
\quad = - \left( \prod_{e \in \mathcal{V}_k(S)} d_e \right) \pm |\mathcal{V}_k(S^-)| \eta \sum_{\emptyset \neq D \subset \mathcal{V}_k(S)} \left( \prod_{e \in D} (-1) \right) \pm \eta \qquad (\because |d_e| \leq 1)
\]

\[
\quad = - \left( \prod_{e \in \mathcal{V}_k(S)} d_e \right) \pm (|\mathcal{V}_k(S)| - 1) \eta \left( 1 - 1^{-|\mathcal{V}_k(S)|} \right) \pm \eta \qquad (\because |\mathcal{V}_k(S)| > |\mathcal{V}_k(S^-)|)
\]

\[
\quad = \left( \prod_{e \in \mathcal{V}_k(S)} d_e \right) \pm |\mathcal{V}_k(S)| \eta.
\]

\[\square\]

We will use the following form of the Cauchy-Schwarz.

**Fact 3.2** (Cauchy-Schwarz inequality). For a random variable \( X \) on a probability space \( \Omega \) if an equivalent relation \( \approx \) on \( \Omega \) is a refinement of another equivalent relation \( \sim \) on \( \Omega \) then

\[
\mathbb{E}_{\omega \in \Omega} (\mathbb{E}_{\omega \in \Omega} |X(\omega)\omega \sim \omega_0|^2) \leq \mathbb{E}_{\omega \in \Omega} (\mathbb{E}_{\omega \in \Omega} |X(\omega)\omega \approx \omega_0|^2).
\]

**Proof:** By the Cauchy-Schwarz (i.e. \( \mathbb{E}[X^2] \mathbb{E}[Y^2] \geq \mathbb{E}[XY]^2 \)), we have \( \mathbb{E}_{\omega \in \Omega} (\mathbb{E}_{\omega \in \Omega} |X(\omega)\omega \approx \omega_0|^2) = \mathbb{E}_{\omega \in \Omega} \mathbb{E}_{\omega' \in \Omega} \left[ (\mathbb{E}_{\omega |X(\omega)\omega \sim \omega'} |\omega' \sim \omega_0|) \mathbb{E}_{\omega' \in \Omega} (\mathbb{E}_{\omega \sim \omega_0} |X(\omega')|)^2 \right] \leq \mathbb{E}_{\omega \in \Omega} \mathbb{E}_{\omega' \in \Omega} (\mathbb{E}_{\omega |X(\omega)\omega \approx \omega_0}|\omega' \sim \omega_0|^2) \cdot \mathbb{E}_{\omega' \in \Omega} (\mathbb{E}_{\omega |X(\omega)\omega \approx \omega_0}|\omega' \sim \omega_0|^2).

With this fact and Definition 3.1, we next tackle

**Lemma 3.3** (Mean square bounds correlation). Let \( k, h, m \) be positive integers and \( G \) a k-bounded graph on vertex sets \( \Omega \). Let \( S \in \mathcal{S}_{k,h,G} \) and let \( F_e : C_I(G) \rightarrow [-1,1] \) be a function for each \( I \in \binom{\emptyset}{k} \) and for each \( e \in \mathcal{V}_I(S) \). If \( \delta \) is a \((k-1,2h)\)-error function of \( G \) then for any \( I \in \binom{\emptyset}{k} \) and \( e_o \in \mathcal{V}_I(S) \), we have that

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in \mathcal{V}_k(S)} F_e(\phi(e)) \prod_{e \in \mathcal{V}_{k-1}(S)} [G(\phi(e)) = S(e)] \right] \right)^2 \leq \mathbb{E}_{\phi,\varphi \in \Phi(mh)} \mathbb{E}_{e^* \in \Omega} \left[ \left( \mathbb{E}_{e \in \Omega} [F_{e_o}(e)[G(\partial e) = S(\partial e_o)] | e^{\partial G/\varphi} \approx e^*] \right)^2 \right] \cdot \left( \prod_{e \in \mathcal{V}_{k-1}(S), e \notin e_o} \mathbb{d}_{G}(S(e)) \right) \left( \prod_{e \in \mathcal{V}_{k-1}(S), e \notin e_o} \mathbb{d}_{G}(S(e)) \right) + \frac{1}{m}
\]

where \( \phi, \varphi \) are random and where we abbreviate \( F_e(G(e)) \) by \( F_e(e) \) (thus, \( F_e(\phi(e)) = F_e(G(\phi(e))) \)).

In particular, if we suppose

\[
\min_{I \in \binom{\emptyset}{k-1}} \min_{e \in \mathcal{V}_I(S)} \left( \frac{1}{2} \mathbb{d}_{G}(S(e)) - \delta(S(e)) \right) > 0 \text{ and } \frac{1}{m} \leq \prod_{e \in \mathcal{V}_{k-1}(S), e \notin e_o} \mathbb{d}_{G}^{-\delta}(S(e))
\]

(i.e. \( \delta \) is small and \( m \) is large) then

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in \mathcal{V}_k(S)} F_e(\phi(e)) \right] \left[ G(\phi(e)) = S(e) \forall e \in \mathcal{V}_{k-1}(S) \right] \right)^2 \leq 2 \cdot 3^{2|\mathcal{V}_{k-1}(S)|} \mathbb{E}_{\phi,\varphi \in \Phi(mh)} \mathbb{E}_{e^* \in \Omega} \left[ \left( \mathbb{E}_{e \in \Omega} [F_{e_o}(e)[e^{\partial G/\varphi} \approx e^*] \right)^2 \right] \left( \mathbb{E}_{e \in \Omega} [F_{e_o}(e)|e^*] \right)^2 (G(\partial e^*)) = S(\partial e_o)].
\]
that implies \( v \) for any

\[
\begin{align*}
e & \approx e' \text{ if and only if } \varphi_i^{(e)}(e) \approx \varphi_i^{(e')}(e), \quad \forall e \in V(S) \setminus \{e_0\}, \forall i \in [m].
\end{align*}
\]

(Note that \( V(S) \setminus e_0 \) is a vertex set while \( V(S) \setminus \{e_0\} \) is an edge set. Since the right-hand side of (9) holds trivially for \( e \) with \( e \cap e_0 = \emptyset \), it is enough to check only for \( e \) with \( 1 \leq |e \cap e_0| \leq k - 1 \).

Let \( S^{(1)}, \ldots, S^{(m)} \) and \( e_0^{(1)}, \ldots, e_0^{(m)} \) be copies of \( S \) and of \( e_0 \). For \( \bar{\varphi} = (\varphi_i)_{i \in [m]} \) with \( \varphi_i \in \Phi(V(S^{(i)})) \setminus e_0^{(i)} \) if \( \varphi^* \in \Phi(mh) = \Phi(V(S^{(1)}) \cup \ldots \cup V(S^{(m)})) \) is an extended function of \( \varphi_i \)'s, i.e., \( \varphi^*(v) = \varphi_i(v) \) for any \( v \in V(S^{(i)}) \setminus e_0^{(i)}, i \in [m] \) then, because of (1)' and (9), it is easily seen that

\[
e \approx e' \text{ implies } e \overset{\text{G/}\varphi^*}{\approx} e'.
\]

where \( \text{G/}\varphi^* = \text{G/}^{k-1}\varphi^* \) is the \((k-1)\)-regularization.

(To see this, observe that \( e \overset{\text{G/}\varphi^*}{\approx} e' \) means that \( \text{G/}\varphi^*(e) \approx \text{G/}\varphi^*(e') \) for all \( J \subseteq I_0 \). By (1), if \( J' \in \{0, k-1\} \), \( f \in \Omega_{J'} \) and \( f \subset \varphi^*(D) \) then \( g_J \approx f \). Since \( |J| = k \), for all \( e \in V(S) \setminus \{e_0\} \), we have \( \varphi_i^{(e)}(e) = \varphi_i^{(e')}(e) \approx \varphi_i^{(e)}(e) = \varphi_i^{(e')}(e) \), where \( \varphi_i^{(e)} \) and \( \varphi_i^{(e')} \) are naturally defined by restricting the domain of \( \varphi_i \) from \( V(S^{(i)}) \) to \( V(S^{(i)}) \setminus e_0^{(i)} \). By (9), \( e \overset{\text{G}}{\approx} e' \).

Let \( F_{\varphi_0}(e) := F_{\varphi_0}(e)[\text{G}(\partial e) = \text{S}(\partial e_0)] \) and let

\[
F^*(\phi) := \prod_{e \in V(S) \setminus \{e_0\}} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [\text{G}(\phi(e)) = \text{S}(e)\].
\]

Note that \( \text{G}(\partial \phi(e_0)) = \text{S}(\partial e_0) \) holds if-and-only-if \( \text{G}(\phi(e)) = \text{S}(e) \) for all \( e \subsetneq e_0 \). Also \([P] \in \{0, 1\}\) implies \([P]^2 = \|P\|^2 \) for any statement \( P \). With the two facts, the left-hand side of (11) equals

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [\text{G}(\phi(e)) = \text{S}(e)] \right] \right)^2
\]

\[
= \left( \mathbb{E}_{\phi \in \Phi(h)} \left[ F_{\varphi_0}(e_0) \prod_{e \in V(S) \setminus \{e_0\}} F_e(\phi(e)) \cdot [\text{G}(\partial \phi(e_0)) = \text{S}(\partial e_0)] \prod_{e \in V(k-1)(S) : e \subsetneq e_0} [\text{G}(\phi(e)) = \text{S}(e)] \right] \right)^2
\]

\[
= \left( \mathbb{E}_{\varphi \in \Phi(h)} \left[ F_{\varphi_0}(e_0) F^*(\phi(e)) \right] \right)^2 \quad \text{(by definitions of } F_{\varphi_0} \text{ and } F^*)
\]

\[
= \left( \mathbb{E}_{\varphi \in \Phi(V(S)) \setminus \{e_0\}} \left[ F_{\varphi_0}(e_0) F^*(\phi(e_0)) \right] \right)^2
\]

\[
= \left( \mathbb{E}_{\varphi \in \Phi(h)} \left[ F_{\varphi_0}(e_0) F^*(\phi(e_0)) \right] \right)^2 \quad \text{(since } \phi \in \Phi(h) = \Phi(V(S)) \text{ consists of } r h = k + (r h - k) \text{ random vertices in } \Omega)\]

\[
= \left( \mathbb{E}_{\varphi \in \Phi(V(S)) \setminus \{e_0\}} \left[ F_{\varphi_0}(e_0) F^*(\phi(e_0)) \right] \right)^2
\]

\[
\leq \text{C.S. } \mathbb{E}_{\varphi \in \Phi(h)} \left[ F_{\varphi_0}(e_0) \left| e \overset{\text{G/}\varphi}{\approx} e_0 \right. \right]^2 \cdot \mathbb{E}_{\varphi \in \Phi(h)} \left[ \left( \mathbb{E}_{\phi \in \Phi(V(S)) \setminus \{e_0\}} \left[ F_{\varphi_0}(e_0) \left| e \overset{\text{G/}\varphi}{\approx} e_0 \right. \right]^2 \right] \right] \]

\[
\leq \mathbb{E}_{\varphi \in \Phi(mh)} \left[ \mathbb{E}_{\varphi \in \Phi(V(S)) \setminus \{e_0\}} \left[ F_{\varphi_0}(e_0) \left| e \overset{\text{G/}\varphi}{\approx} e_0 \right. \right]^2 \right] \cdot \mathbb{E}_{\varphi \in \Phi(h)} \left[ \left( \mathbb{E}_{\phi \in \Phi(V(S)) \setminus \{e_0\}} \left[ F_{\varphi_0}(e_0) \left| e \overset{\text{G/}\varphi}{\approx} e_0 \right. \right]^2 \right] \right].
\]
Looking at the second term first, this can be written as

\[ \frac{1}{m} \mathbb{E}_{\phi \in \Phi(V(S))} \left[ G(\phi(e)) = S(e) \right] \leq \frac{1}{m} \prod_{e \in V(S')} d_G^{(\delta)}(S(e)) \]  

Looking at the second term first, this can be written as

\[ \frac{1}{m} \mathbb{E}_{\phi \in \Phi(V(S))} \left[ G(\phi(e)) = S(e) \right] \leq \frac{1}{m} \prod_{e \in V(S')} d_G^{(\delta)}(S(e)) \]

Applying the assumption that \( \delta \) is a \((k - 1, 2h)\)-error function of \( G \) to an \( S^- \in S_{k-1,h,G} \) with \( V(S^-) := V(k-1)(S) \).

We will interpret the first term by applying the same assumption on \( \delta \) to another complex \( S'' \).

Here \( S'' \in S_{k-1,2h,G} \) is a simplicial-complex obtained from two copies of \( S^- \), say \( S^{(-1)} \) and \( S^{(-2)} \), by identifying any pair of vertices \( v^{(1)} \in e_0^{(1)} \) and \( v^{(2)} \in e_0^{(2)} \) in which \( e_0^{(1)} \) and \( e_0^{(2)} \) are the edges in the copies of \( S^- \) corresponding to \( e_0 \). (Any edge \( e \) containing two vertices \( v^{(1)} \in V(S^{(-1)}) \setminus e_0^{(1)} \) and \( v^{(2)} \in V(S^{(-2)}) \setminus e_0^{(2)} \) is invisible in \( S'' \).) Applying the assumption on \( \delta \) to this \( S'' \), the first term can be rewritten as

\[ \frac{1}{m} \mathbb{E}_{\phi \in \Phi(V(S'))} \left[ G(\phi(e)) = S(e) \right] \]

completing the proof of (B) by (11) and (12).

Next, we show the last sentence of the lemma. The left-hand side of (B) is at most

\[ \left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] / \mathbb{P}_{\phi \in \Phi(h)} [G(\phi(e)) = S(e) \forall e \in V(k-1)(S)] \right)^2 \]

Finally, we show that \( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] / \mathbb{P}_{\phi \in \Phi(h)} [G(\phi(e)) = S(e) \forall e \in V(k-1)(S)] \right)^2 \]

\[ \leq \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right)^2 \]

\[ \mathbb{P}_{e^*} [G(\partial e^*) = S(\partial e_0)] \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right) \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right) + \frac{1}{m} \]

(since \( e \approx G/\phi e^* \) implies \( G(\partial e) = G(\partial e^*) \), thus \( G(\partial e) = S(\partial e_0) \) implies \( G(\partial e^*) = S(\partial e_0) \)).

\[ = \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right)^2 \]

\[ \leq \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right)^2 \]

\[ = \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V(S)} F_e(\phi(e)) \prod_{e \in V(k-1)(S)} [G(\phi(e)) = S(e)] \right] \left( \prod_{e \in V(k-1)(S)} d_G^{(\delta)}(S(e)) \right)^2 \]
\[
\left( \prod_{e \in V_{(k-1)}(S)} d_G^{(\delta)}(S(e)) \right)^2 \left( \prod_{e \in V_{(k-1)}(S)} d_G^{(\delta)}(S(e)) \right) \left( \prod_{e \in V_{(k-1)}(S), e \notin e_0} d_G^{(\delta)}(S(e)) \right) + \frac{1}{m} 
\]

\[
\left( \prod_{e \in V_{(k-1)}(S)} d_G^{(-\delta)}(S(e)) \right)^2
\]

\[
\leq \mathbb{E}_{\varphi \in \Phi(m) \mathbb{E} e^* \in \Omega_I} \left[ \left( \mathbb{E}_{e \in \Omega_I} \left[ F_{e_0}(e) \mid e \right] e^* \right) \right] ^2 \left| G(\partial e^*) = S(\partial e_0) \right|
\]

\[
\left( 1 + \frac{1}{m} \left( \prod_{e \in V_{(k-1)}(S), e \notin e_0} d_G^{(\delta)}(S(e)) \right)^{-1} \right) \cdot \left( \prod_{e \in V_{(k-1)}(S)} \frac{d_G(S(e)) + \delta(S(e))}{d_G(S(e)) - \delta(S(e))} \right)^2.
\]

The assumption (7) completes the proof of (8). \qed

3.2. The body of our proof.

**Definition 3.2.** [Notation for this subsection] Write \( c_i(G) := \max_{I \in \{1\}} |C_I(G)| \) for \( i \in [k] \). For \( \vec{b} = (b_i)_{i \in [k]} \) and an integer \( m \), we write \( \vec{B}(\vec{b}, m) := (B_i(\vec{b}, m))_{i \in [k]} \) where \( B_i(\vec{b}, m) := \prod_{j \in [0, k-i]} b_{i+j}^{(r-1)m_j} \).

Recall (1). The \((k-1)\)-regularization \( G/\varphi \) is the \( k \)-bounded graph on \( \Omega \) obtained from \( G \) by redefining the color of each edge \( e \in \Omega_I \) with \( I \in \binom{[r]}{k} \) by the \( \left( \sum_{j=0}^{k-i} \binom{r-i}{j} \right) \)-dimensional vector

\[
(G/\varphi)(e) := \left( G(e \cup f) \right)_{I \in \binom{[r]}{k-i}, f \in \Omega_J \text{ with } f \subset \varphi(\Omega)}.
\]

Thus obviously if \( G \) is a \( k \)-bounded \( \vec{b} \)-colored graph then

\[
c_i(G/\varphi) \leq B_i(\vec{b}, m), \quad \forall i \in [k], \forall \varphi \in \Phi(m).
\]

(For example, \( B_k(\vec{b}, m) = b_k \) and \( B_{k-1}(\vec{b}, m) = b_{k-1} b_k^{(r-1)m_k} \).)

Fix \( 0 < \epsilon < 1 \) and \( \vec{b} \). We proceed by induction on \( k \). When \( k = 1 \), it is trivial as the remark after Theorem 2.3. Let \( k \geq 2 \).

- **[Definition of the sample-size functions]** Let \( m_{k,h,b,e}^{(0)}(0) := 0 \) and \( m_{k,h,b,e}^{(i)}(n_i, \cdots, n_{k-2}, 0) := n_{k-1, h, b, e}^{(i)}(n_i, \cdots, n_{k-2}, \forall i \in [k-2]) \), which is defined by the induction hypothesis on \( k-1 \) of the theorem. Define \( \tilde{n}_{k,h,b,e}^{(k-1)} = \tilde{n}_{k-1}^{(k-1)} \) to be large enough so that

\[
Cb_k \sqrt{\frac{b_k}{\tilde{n}_{k-1}^{(k-1)}}} \leq \frac{\epsilon}{4\binom{k}{k}}
\]

where

\[
C := \sqrt{2} \left( \binom{r}{k} \right) k^h \left( \frac{b_k}{2^{\sqrt{\epsilon}}} \right)^{\binom{k}{h}+1} \sum_{j \in [k-1]} \binom{j}{h} \text{ and } \epsilon_1 := \left( \frac{\epsilon}{12 \cdot 2b_k^{\binom{k}{k}}} \right)^2.
\]

(These expressions will appear in (31) and (33).) Also let \( \tilde{n}_{k,h,b,e}^{(j)}(n_j+1, \cdots, n_{k-2}, 0) := \tilde{n}_{k-1, h, b, e}^{(j)}(n_j+1, \cdots, n_{k-2}) \) for all \( j \in [k-2] \).

Given \( n_{k-1} \geq 0 \), we will inductively define functions \( m_{k,h,b,e}^{(i)}(\bullet, \cdots, \bullet, n_{k-1} + 1) \), \( \forall i \in [k-1] \), and functions \( \tilde{n}_{k,h,b,e}^{(j)}(\bullet, \cdots, \bullet, n_{k-1} + 1) \), \( \forall j \in [k-2] \), by using \( m_{k,h,b,e}^{(\bullet)}(\bullet, \cdots, \bullet, n_{k-1}) \), \( m_{k,h,b,e}^{(\bullet)}(\bullet, \cdots, \bullet, n_{k-1}), m_{k,h,b,e}^{(\bullet)}(\bullet, \cdots, \bullet, \bullet), \cdots, m_{k,h,b,e}^{(\bullet)}(\bullet, \cdots, \bullet, \bullet), \cdots, \bullet, n_{k-1} \), and \( \tilde{n}_{k-1, h, b, e}^{(\bullet)}(\bullet, \cdots, \bullet, n_{k-1}) \), as follows. Let

\[
m := \prod_{i \in [k-1]} \left( B_i(\vec{b}, \sum_{j=1}^{k-1} m_{k,h,b,e}^{(j)}(\tilde{n}_{k-1}^{(j)}, \cdots, \tilde{n}_{k-2}^{(j)}, n_{k-1})) \right)^{\binom{k}{i}}
\]

(16)
Thus, there exists a function $\delta$ and that (ii) for each fixed $k$.

Next, we define the remaining $k-2$ functions so that

$$m_{k,h,b,e}(n_{k-1} + 1) := \sum_{j=1}^{k-1} m_{k,h,b,e}(n_{j}, \ldots, n_{k-2}, n_{k-1}) + mh. \quad (17)$$

Finally we define

$$\tilde{m}_{k,h,b,e}(n_{j+1}, \ldots, n_{k-2}, n_{k-1}) := \tilde{m}_{k,h,b,e}(n_{j}, \ldots, n_{k-2}), \quad \forall j \in [k-2]. \quad (19)$$

(It will be easily seen that the three equalities $\tilde{\eta} := \eta$ can be replaced by $\geq$.)

• [Definition of the error function] For $\tilde{\eta}_k = (n_{1}, \ldots, n_{k})$ and for $\tilde{\varphi}_j \in \Phi(\tilde{m}_{k,h,b,e}(n_{k-1}))$,

we write $G^* := G/\tilde{\varphi}$ and we define a $(k,h)$-error function $\delta = \delta_{k,h,e,G^*}$ inductively as follows.

Since $13$ implies $c_{1}(G/\varphi_{k-1}) \leq B_{1}(\tilde{b}_{k}, m_{k,h,b,e}(n_{k-1}))$ and $G^* = (G/\varphi_{k-1})/\varphi_{j} \in [k-2]$, we apply the induction hypothesis on $k$ with $18$ and $19$ for $G/\varphi_{k-1}$ and see that for the $e_{1} > 0$ of $18$, 

$$E_{n_{k} = (n_{1}, \ldots, n_{k-1})} E_{\tilde{\varphi}_{j} = (\varphi_{j})_{j} \in [k-2]} \left[ \text{reg}_{k-1,2h}(G^*) \right] \leq c_{1}.$$ 

Thus, there exists a function $\delta = \delta_{k-1,2h,e,G^*} : TC(G^*) = \bigcup_{k \leq k} TC_{k}(G^*) \to [0, \infty)$ with the two property that (i) for any $S \in S_{k-1,2h,G^*}$,

$$P_{\tilde{\varphi} \in \Phi(\tilde{b}_{k})} \left[ G^*(\tilde{\varphi}(e)) = S(e), \forall e \in V_{k-1}(S) \right] = \prod_{e \in V_{k-1}(S)} d_{G^*}^{(\tilde{\varphi})}(S(e)) \quad (20)$$

and that (ii) for each fixed $\varphi_{k-1} \in \Phi(\tilde{m}_{k,h,b,e}(n_{k-1}))$

$$E_{n_{k} = (n_{1}, \ldots, n_{k-1})} E_{\varphi_{j} = (\varphi_{j})_{j} \in [k-2]} \left[ \max_{I \in \left[ n_{k-1} \right]} |C_{I}(G^*)| E_{e \in \Omega_{k}} \left[ \delta(G^*(e)) \right] \right] \leq c_{1} \leq c_{2}. \quad (21)$$

(This $\delta_{k-1,2h,e,G^*}$ depends not only on $\varphi_{k-1}$ but also on $n_{k}$ and $\varphi_{j}$.) Define $\delta_{k,h,e,G^*}^{(\tilde{c})} := \delta_{k-1,2h,e,G^*}(\tilde{c})$ for any $\tilde{c} \in TC_{k}(G^*)$, $I \in \left[ n_{k-1} \right]$. Before defining $\delta(\tilde{c})$ for $\tilde{c} \in TC_{k}(G^*)$, we define ‘bad colors’ $BAD \subset TC(G^*)$. For $I \in \left[ n_{k-1} \right]$,

we define BAD $I := \bigcup_{I \in \left[ n_{k-1} \right]} BAD^{I}$. For $\tilde{c} \in (c_{j})_{j \in I} \in BAD^{I}$, we define, using $m$ and $C$ of $13$ and $10$,

$$\eta_{k,h}(\tilde{c}) := E_{\tilde{\varphi} \in \Phi(\tilde{b}_{k})} E_{e \in \Omega_{k}} \left[ \left( P_{e \in \Omega_{k}} \left[ G^*(e) = e_{1} | e_{1} \partial G^* \supset e_{1} \right] - d_{G^*}(\tilde{c}) \right)^{2} \right] | G^*(\partial e_{1}) = (c_{j})_{j \leq I} \quad (23)$$

$$\delta_{k,h}(\tilde{c}) := \begin{cases} 1/C \sqrt{\eta_{k,h}(\tilde{c})}, & \text{if } \tilde{c} \in BAD^{I}, \\ \text{otherwise,} & \end{cases} \quad (24)$$

• [The qualification as an error function] Because of $20$ and $21$, it is enough for the first requirement $2$ to show that

$$P_{\tilde{\varphi} \in \Phi(\tilde{b}_{k})} \left[ G^*(\tilde{\varphi}(e)) = S(e), \forall e \in V(S) \right] = \prod_{e \in V(S)} \left( d_{G^*}(S(e)) \pm \delta(S(e)) \right) \quad (25)$$

or

$$P_{\tilde{\varphi} \in \Phi(\tilde{b}_{k})} \left[ G^*(\tilde{\varphi}(e)) = S(e), \forall e \in \bigcup_{k} V(S) \right] = \prod_{e \in \bigcup_{k} V(S)} d_{G^*}^{(\tilde{\varphi})}(S(e)) \quad (26)$$

for any $S \in S_{k,h,G^*}$. Furthermore without loss of generality, we can assume the property that

$$S(e) \notin BAD \forall e \in V(S). \quad (27)$$

(Indeed, we can show the case of $27$ suffices by the induction on the number of bad edges in $S$. Let a complex $S$ be given where $S$ contains a bad edge $e_{1}$. Without loss of generality, assume that any visible edge $e \in V(S)$ is not bad if $|e_{1} | < |e_{1} |$. We construct a new complex $S^{*}$ from $S$ by recoloring
all (bad) edges containing \( e^* \) in the invisible color. By the induction hypothesis, (25) holds for \( S^* \). Equality (26) means that the real number the left hand side suggests belongs to the interval which the right-hand side suggests. Denote by \([p^-, p^+]\) this interval. Again we reconstruct \( S \) from \( S^* \) by recoloring some invisible edges in ‘original’ bad colors. By this process from \( S^* \) to \( S \), the left hand side will not increase (probably decrease because of added visible edges \( e \)) and the right-hand side will suggest interval \([0, p^+]\) because, for bad edges \( e \), \( d_{G^*}(S(e)) - \delta(S(e)) = [0, 1] \) by (24). Then (26) holds not only for \( S^* \) but also for \( S \).

Fix such an \( S \in \mathcal{S}_{k,h,G^*}. \) For any \( e \in \mathcal{V}_J(S), J \subset \tau \), it follows from (27) and (22) that

\[
d_{G^*}^{-\delta}(S(e)) > \frac{\sqrt{c_G}}{|\mathcal{J}(G^*)|} > 0 (\text{if } |J| < k) \text{ and } \delta(S(e)) < \frac{1}{2} d_{G^*}^{-\delta}(S(e)) (\text{if } |J| < k). \tag{28}
\]

Clearly, \( c_k(G^*) = c_k(G/\varphi) \leq c_k(G/k-1(\varphi_{k-1} \cup \cdots \cup \varphi_1)) \) and \( |\mathcal{V}_I(S)| \leq \binom{1}{h}. \) Thus, it follows from (16) and (13) that

\[
\frac{1}{m} \leq \prod_{i \in [k-1]} \left( \frac{\sqrt{c_G}}{|\mathcal{J}(G^*)|} \right)^{|\mathcal{V}_I(S)|} \leq \prod_{i \in [k-1]} \prod_{e \in \mathcal{V}_I(S)} d_{G^*}^{-\delta}(S(e)) \leq \prod_{e \in \mathcal{V}_I(S), e \in \mathcal{E}} d_{G^*}^{-\delta}(S(e)) \tag{29}
\]

for any \( e_0 \in \mathcal{V}_k(S). \) Let \( F_e(e) := [G^*(e) = S(e)] - d_{G^*}(S(e)). \) For any \( \emptyset \neq D \subset \mathcal{V}_k(S), \) we apply Lemma 3.3 (where \( G := G^* \)) with any \( S' \in \mathcal{S}_{k,h,G^*} \) with \( \mathcal{V}_I(S') = D \) and \( \mathcal{V}_I(S') = \mathcal{V}_I(k-1)(S') \), and see that

\[
\left( E_{\phi \in \Phi} \left[ \prod_{e \in D} (|G^*(\phi(e)) = S(e)| - d_{G^*}(S(e))) \right] \right)^2 = \left( E_{\phi \in \Phi} \left[ \prod_{e \in D} F_e(e) \right] \right)^2 \leq 2 \cdot 3! |\mathcal{V}_I(k-1)(S)| \cdot E_{\phi \in \Phi(h)} E_{e^* \in \Omega} \left( E_{e \in \Omega} |F_e(e) = e^* \approx e^*| G^* = S(e) \right) \tag{30}
\]

Taking an edge \( e_0 \in \mathcal{V}_k(S) \) which maximizes \( \eta_{k,h}(S(e_0)) \). Then it follows from Lemma 3.1 that

\[
\sum_{e \in \mathcal{V}_k(S)} d_{G^*}(S(e)) \geq \frac{\sqrt{c_G}}{\eta_{k,h}(S(e_0)) \sqrt{|\mathcal{V}_k(S)|}} \frac{3! |\mathcal{V}_I(k-1)(S)|}{\eta_{k,h}(S(e_0))} \tag{31}
\]

where for the last equality we use the fact that \( c_k(G^*) = c_k(G) \leq b_k \) (cf. (13)).

**[Bounding the average error size]** With the abbreviation \( a_n := m_{k,h,b,c}(n) \), for any \( I \in \binom{1}{h} \), the linearity of expectation gives us that

\[
\left( E_{\tilde{e}, \tilde{e}} E_{e \in \Omega} \left[ \sqrt{\eta_{k,h}(G^*(\tilde{e}))} \right] \right)^2 \leq E_{\tilde{e}, \tilde{e}} E_{e \in \Omega} \left[ \eta_{k,h}(G^*(\tilde{e})) \right] \left( E_{e \in \Omega} [G^*(\tilde{e}) = G^*(\tilde{e})] \right)^2 \left( E_{e \in \Omega} [G^*(\tilde{e}) \approx e^*] \right)^2 \left( E_{e \approx \tilde{e}} \right)^2 \tag{28}
\]

where \( \eta_{k,h}(G^*(\tilde{e})) \) is the expected value of the indicator function \( \mathbb{I}_{G^*(\tilde{e}) = G^*(\tilde{e})} \).

\[
\sum_{e \in G} d_{G^*}(S(e)) \geq \frac{\sqrt{c_G}}{\eta_{k,h}(S(e_0)) \sqrt{|\mathcal{V}_k(S)|}} \frac{3! |\mathcal{V}_I(k-1)(S)|}{\eta_{k,h}(S(e_0))} \tag{31}
\]

where for the last equality we use the fact that \( c_k(G^*) = c_k(G) \leq b_k \) (cf. (13)).
Thus we obtain that

\[ \sum_{\epsilon \in \mathcal{G}(G)} E_{\epsilon, \varphi} \left[ E_{\varphi, \epsilon'} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \right] \]

\[ = |C \cap(G)| E_{\epsilon \in \mathcal{G}(G)} E_{\varphi, \epsilon} \left[ E_{\varphi' \in \Phi(\mathrm{mh})} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \right] \]

\[ \leq b_k E_{\Omega \leq n \leq \tilde{n}(k-1)} E_{\epsilon, \Omega} \left[ E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \right] \]

\[ \leq b_k E_{\Omega \leq n \leq \tilde{n}(k-1)} E_{\epsilon, \Omega} \left[ E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \right] \]

\[ \leq b_k E_{\Omega \leq n \leq \tilde{n}(k-1)} E_{\epsilon, \Omega} \left[ E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \right] \]

(32)

where in the above (*) we use the property that, after \( n = n(k-1) \) is chosen, it follows from (17) that \( a_{n+1} \geq m+1 \sum_{j=1}^{k-2} m_j \) (for all possible \( n(1), \ldots, n(k-2) \)) (cf. definition of \( \tilde{n}(k-2), \ldots, \tilde{n}(1) \)) just after (10) and that \( \varphi_0(D) \supseteq (\bigcup_{i \in [k-1]} \varphi_i(D)) \cup \varphi(D) \) then \( e \overset{\partial G/\varphi}{\sim} \tilde{\epsilon} \) implies \( e \overset{\partial G/\varphi}{\sim} \tilde{\epsilon} \) (thus \( E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \) \[ \leq E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \]

and further, that \( e \overset{\partial G/\varphi}{\sim} \tilde{\epsilon} \) implies \( e \overset{\partial G/\varphi}{\sim} \tilde{\epsilon} \) where \( \varphi_0 = \varphi_{k-1} \) (thus \( E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \) \[ \leq E_{\varphi \in \Phi(\epsilon)} \left( \left( P_{e \in \Omega} | G(e) = \epsilon | e \overset{\partial G/\varphi}{\sim} \epsilon' \right)^2 \right) \]

Thus, for any \( I \in \binom{\tilde{n}(k-1)}{k} \), we see that

\[ E_{\tilde{n}, \varphi} | C \cap(G/\varphi) | E_{\epsilon, \Omega} \left[ \delta_{k,h}(G^*(e)) \right] \]

\[ \leq b_k E_{\tilde{n}, \varphi} \left[ C P_{e \in \Omega} \left( \frac{b_k}{\tilde{n}(k-1)} \right) + E_{\tilde{n}, \varphi} \left[ \sum_{j \in I} \sum_{\epsilon \in \mathcal{G}(G)} P_{e \in \Omega} \left[ \delta_i(G^*(e)) \right] \right] \right] \]

\[ \leq b_k \left( C \sqrt{\frac{b_k}{\tilde{n}(k-1)}} + \frac{\sum_{\epsilon \in \mathcal{G}(G)} P_{e \in \Omega} \left[ \delta_i(G^*(e)) \right]}{C \sqrt{\frac{b_k}{\tilde{n}(k-1)}}} \right) \]

\[ \leq b_k \left( C \sqrt{\frac{b_k}{\tilde{n}(k-1)}} + 2 \sqrt{\epsilon \cdot \epsilon - 2 \epsilon} + 2 \sqrt{\epsilon \cdot \epsilon} \right) \]

\[ \leq b_k \left( C \sqrt{\frac{b_k}{\tilde{n}(k-1)}} + 2 \sqrt{\epsilon \cdot \epsilon} + 2 \sqrt{\epsilon \cdot \epsilon} \right) \]

\[ \leq \frac{\epsilon}{2} \left( k \right) \]

(33)

where in the above (**) we use (43) and the fact that

\[ P_{e \in \Omega} \left[ \delta_i(G^*(e)) \right] \]

\[ = \sum_{\epsilon_j \in \mathcal{G}(G^*)} P_{e \in \Omega} \left[ G^*(e) = \epsilon_j \right] \]

\[ \leq \sum_{\epsilon_j \in \mathcal{G}(G^*)} 1 \cdot P_{e \in \Omega} \left[ G^*(e) = \epsilon_j \right] \]

\[ = \sum_{\epsilon_j \in \mathcal{G}(G^*)} \delta_i(G^*(e)) \]

\[ \leq \frac{2 \sqrt{\epsilon \cdot \epsilon}}{C \sqrt{\frac{b_k}{\tilde{n}(k-1)}}} \]

Thus we obtain that

\[ E_{\tilde{n}, \varphi} \left[ \max_{I \in \binom{\tilde{n}(k-1)}{k}} C \cap(G/\varphi) \right] \]

\[ \leq E_{\tilde{n}, \varphi} \left[ \left( \frac{b_k}{\tilde{n}(k-1)} \right) \right] \]

\[ \left( \frac{b_k}{\tilde{n}(k-1)} \right) \]

(34)
\[ \leq \mathbb{E}_{\bar{r}, \bar{q}} \left[ \max_{I \in \langle r, k \rangle} \left| C_I(G/\bar{q}) \right| \mathbb{E}_{e \in \Omega_I} [\delta(G^*(e))] \right] + \mathbb{E}_{\bar{r}, \bar{q}} \left[ \sum_{I \in \langle r, k \rangle} \left| C_I(G/\bar{q}) \right| \mathbb{E}_{e \in \Omega_I} [\delta(G^*(e))] \right] \]

\[ \leq \frac{\varepsilon}{2} + \sum_{I \in \langle r, k \rangle} \frac{\varepsilon}{2^{\langle r, k \rangle}} = \varepsilon. \]

It shows the second requirement \[ \mathcal{M} \] for function \( \delta \), completing the proof of the main theorem. \( \square \)

### 4. The Removal Lemma and Proof of Theorem \ref{thm:main}

While there had been known that some strong versions of hypergraph regularity lemmas imply Szemerédi’s theorem \( \langle 13 \rangle \) before they were proven, Solymosi \( \langle 37 \rangle \) inspired by Erdős and Graham showed that they also yield a combinatorial proof of Theorem \ref{thm:main}. We will describe his argument for completeness and for seeing the length of the entire proof of Theorem \ref{thm:main}.

**Definition 4.1.** \( k \)-uniform graphs] A \( k \)-uniform \( b \)-colored \( (r \text{-partite hyper}) \)graph is a \( k \)-bounded \( (b_i)_{i \in [k]} \)-colored graph such that (1) if \( i < k \) then \( b_i = 1 \) and the unique color is called invisible and (2) for each \( I \) with \( |I| = k \), there is at most one index-\( I \) color which is called invisible. Denote by \( \mathcal{V}(F) \) the set of visible edges of a \( k \)-uniform graph \( F \), where a visible edge means an edge whose color is not invisible. Such a graph is called \( h \)-vertex if each partite set contains exactly \( h \) vertices. \( \square \)

**Theorem 4.1** (The Removal Lemma). For any \( r \geq k, h, \tilde{b} = (b_i)_{i \in [k]} \), and for any \( \varepsilon > 0 \), there exists a constant \( c = c_{\varepsilon, r, k, \tilde{b}, \varepsilon} > 0 \) with the following property.

Let \( G \) be a \( k \)-bounded \( \tilde{b} \)-colored \( (r \text{-partite hyper}) \)graph on \( \Omega = (\Omega_i)_{i \in [r]} \). Let \( F \) be an \( h \)-vertex \( k \)-uniform \( (b_i)_{i \in [k]} \)-colored \( (r \text{-partite hyper}) \)graph. Then at least one of the following two holds.

(i) There exists a \( k \)-bounded \( \tilde{b} \)-colored \( (r \text{-partite hyper}) \)graph \( G' \) on \( \Omega \) such that

\[ \mathbb{P}_{e \in \Omega_I} [G'(e) \neq G(e)] \leq \varepsilon, \quad \forall I \in \langle r, k \rangle \quad \text{and} \quad \mathbb{P}_{\phi \in \Phi(h)} [\phi(e) = F(e), \forall e \in \mathcal{V}(F)] = 0. \]

(ii) \( \mathbb{P}_{\phi \in \Phi(h)} [\phi(e) = F(e), \forall e \in \mathcal{V}(F)] \geq c. \)

**Proof.** \( \square \) Let \( \varepsilon \leq \left( \frac{1}{2^k} \right)^2 \), which is different from \( \varepsilon \). Corollary \ref{cor:1} gives constants \( m_1, \ldots, m_{k-1} \) such that, given \( G \), there exist constants \( \bar{m}_1, \ldots, \bar{m}_{k-1} \) together with \( \bar{q} \in \Phi(m_1, \ldots, m_{k-1}) \) and with a \( (k, h) \)-error function \( \delta = \delta_{\bar{q}} \) of \( G^* := G/\bar{q} \) for which

\[ \mathbb{E}_{e \in \Omega_I} [\delta(G^*(e))] \leq \varepsilon / |C_I(G^*)|, \quad \forall I \in \langle r, k \rangle. \]

(35)

For \( I \in \langle \tilde{k} \rangle \), define \( \text{BAD}_I \subset \text{TC}_I(G^*) \) by the relation that \( \tilde{c} = (c_J)_{J \subseteq I} \in \text{BAD}_I \) if and only if there exists an \( I' \subset I \) such that \( \text{d}_{G^*}((c_J)_{J \subset I'}) \leq \frac{2 \sqrt{\varepsilon}}{|C_{I'}(G^*)|} \) or that \( \delta((c_J)_{J \subset I'}) \geq \frac{\sqrt{\varepsilon}}{|C_{I'}(G^*)|} \). For each \( I \in \langle \tilde{k} \rangle \), there exists a color \( c^*_I \in \text{C}(G) \setminus C_I(F) \) since \( F \) is \( (b_k) \)-colored. We replace each \( \tilde{c} = (c_J)_{J \subseteq I} \in \text{BAD}_I \) by \( c^* = (c^*_J)_{J \subseteq I} \) where \( c^*_J := c_J \) for any \( J \subseteq I \). Denote the resulting graph by \( G' \). Then for each \( I \in \langle \tilde{k} \rangle \) the same argument as in \ref{33} and \ref{34} gives that

\[ \mathbb{P}_{e \in \Omega_I} [G'(e) \neq G(e)] = \mathbb{P}_{e \in \Omega_I} [G(e) \in \text{BAD}_I] \leq \sum_{I' \subseteq I} \left( \mathbb{P}_{e \in \Omega_{I'}} [\text{d}_{G^*}(G^*(e)) \leq \frac{2 \sqrt{\varepsilon}}{|C_{I'}(G^*)|}] + \mathbb{P}_{e \in \Omega_{I'}} [\delta(G^*(e)) \geq \frac{\sqrt{\varepsilon}}{|C_{I'}(G^*)|}] \right) \]

\[ \leq \frac{2^k}{\sqrt{\varepsilon}} = \mathbb{P}_{\phi \in \Phi(h)} [\phi(e) = F(e), \forall e \in \mathcal{V}(F)] \geq c. \]

For each \( I \in \langle \tilde{k} \rangle \), define \( S \in \mathcal{S}_{k,h,G} \) such that \( \mathcal{V}_k(S) = \mathcal{V}(F) \) and such that \( \mathcal{S}(e) = F(e) \forall e \in \mathcal{V}_k(S) \). Denote by \( S^* \) the set of such \( S \) with the additional property that \( S(e) \notin \mathcal{V}_I \text{BAD}_I \) for any \( e \in \mathcal{V}_k(S) \). Then our way of recoloring gives that

\[ \mathbb{P}_{\phi \in \Phi(h)} [\phi(e) = F(e), \forall e \in \mathcal{V}(F)] \geq \mathbb{P}_{\phi \in \Phi(h)} [G'(e) = F(e) \forall e \in \mathcal{V}(F)] \]

\[ = \sum_{S \in S^*} \mathbb{P}_{\phi \in \Phi(h)} [G^*(e) = S(e) \forall e \in \mathcal{V}(S)] \]

\[ = \sum_{S \in S^*} \prod_{e \in \mathcal{V}(S)} (\text{d}_{G^*}(S(e)) + \delta(S(e))) \]
\[ \geq \sum_{S \in S^*} \prod_{I \in \binom{[k]}{1}} \prod_{e \in V_I(S)} \frac{2\sqrt{\varepsilon} - \sqrt{\varepsilon}}{|C_I(G^*)|} \]
\[ \geq |S^*| \prod_{I \in \binom{[k]}{1}} \left( \frac{\sqrt{\varepsilon}}{(b_{|I|}^{(m_{|I|}^k)} + b_{|I|}^{(k-1)})} \right)^{h|I|}. \]

Therefore if \( S^* = \emptyset \) then the first equality in the above with \( m \) gives the first condition. Otherwise the second condition holds. \( \square \)

For an integer \( m \), we write \([m]_0 := [0, m-1] = \{0, 1, \ldots, m-1\}\). Write \( E_r := \{(0, \ldots, 0, 1, \ldots, 0) \in \mathbb{Z}^r \mid i \in [r]\} \).

**Lemma 4.2.** For any \( \delta > 0 \) and \( k \geq 1 \), there exists an \( \epsilon > 0 \) satisfying the following. If an integer \( N \) is sufficiently large then for any subset \( S \subset T(N,k) := \{x = (x_0, \ldots, x_k) \in [N]^{k+1} \mid r_0 + \ldots + x_k = N - 1\} \) with \( |S| \geq \delta N^k \), there exists \( a = (a_0, \ldots, a_k) \in \mathbb{Z}^{k+1} \setminus T(N,k) \) with \( a + cE_{k+1} \subset S \) where \( c := N - 1 - \sum_{i=0}^k a_i \neq 0 \). Furthermore, there are at least \( \epsilon N^{k+1} \) of such vectors \( a \).

**Proof.** [Tool: Theorem 4.1] Let \( S \subset T(N,k) \). Let \( r := \{0, \ldots, k\} \) and \((\Omega_i, B_i, \mu_i) := ([N]_0, 2[N]_0, \mu_i(\bullet) = 1/|\bullet|) \) for \( i \in r \). Define \((b_1 = 1, \ldots, b_{k-1} = 1, b_k = 2)\)-colored \( k \)-bounded \( r \)-partite hypergraph \( G \) with vertex sets \( \Omega = (\Omega_i)_{i \in r} \) so that for each \( I \in \binom{r}{k} \) and for each \( k \)-tuple \( e = (x_i) \in [N]_{0}^{i \in I} \), \( e \) is red if and only if there exists \( v = (v_i) \in [N]_{0}^{i \in I} \subset S \) such that \( v_i = x_i \) for \( i \) in \( I \).

Let \( F \) be a \( 1 \)-vertex \( k \)-uniform \( 1 \)-colored graph on vertices \( V(F) := (\{i\})_{i \in e} \) such that all the \( k + 1 \) visible edges of \( F \) are red. We say that \( \phi \in \Phi(1) = \Phi(0,\emptyset) \) is red (in \( G \)) if and only if \( G(\phi) \) is red for any \( e \in \cup V(F) \). We also say that a red \( \phi \in \Phi(1) \) is degenerate if and only if \( (\phi(i))_{i \in [0,k]} \subset S \). Suppose that there exists a graph \( G' \) such that \( \mathbb{P}_{\mathcal{E} \in \Omega_i}[G'(e) \neq G(e)] \leq 0.99\delta/(k+1) \) for any \( I \in \binom{r}{k} \) and \( \mathbb{P}_{\phi \in \Phi(1)}[\phi \text{ is red in } G'] = 0 \). Then \( |S| \geq |\{ \phi \in \Phi(1) \mid \phi \text{ is degenerate} \}| \geq \sum_{I \in \binom{r}{k}} |\{ e \in \Omega_i : G'(e) \neq G(e) \}| \leq 0.99\delta N^k \leq |S| \), where (in the first inequality) we use the fact that one cannot delete two distinct degenerate \( \phi \)'s by recoloring one red edge in \( G \). Therefore, such a graph \( G' \) does not exist and Theorem 4.1 gives a constant \( c^* = \epsilon N^{k+1}N^{k+1} = \epsilon N^{k+1} \).

Thus, if \( N \geq 1/0.9c^* \) then there exist \( 0.1c^*N^{k+1} \) non-degenerate red \( \phi \in \Phi(1) \). Observe that a non-degenerate red \( \phi \) yields the desired \( a + cE_{k+1} \subset S \) with \( a := (\phi(i))_{i \in [0,k]} \) and \( c := N - 1 - \sum_{i=0}^k \phi(i) \neq 0 \) since \( c = 0 \) then it is degenerate. \( \square \)

**Proof of Theorem 1.1** [Tool: Lemma 4.2] First we show that it is sufficient to prove the existence of an integer \( c \in [-N,N] \setminus \{0\} \) instead of \( c \in [N] \). Observe that it is true if there exists a subset \( T \subset S \subset [N]_0 \) with \( |T| \geq \delta rN^r \) such that \( T \) is symmetric with respect to some \( x_T = x \in \frac{1}{2}[2N]_0 \) and \( x_T = x \in (\frac{1}{2}[2N]_0)^r : \{ \frac{1}{2}|z \in [2N]_0|^r \} \). Hence, for any \( t \in T \) there is a \( t' \in T \) with \( \frac{1}{2}(t + t') = x \) where \( \delta r > 0 \) is a constant independent of \( N \). Randomly picking a point \( x \in (\frac{1}{2}[2N]_0)^r \), the expected number of pairs \( s, s' \in S \) with \( s + s' = 2x \) is \( \left( \frac{|S|}{2} \right)^r\). Thus there exists the desired \( T \) with \( |T| \geq 0.98\delta^2 N^r \).

By the above remark, it easily follows from [Lemma 4.2] that the theorem holds when \( F \subset B_r := E_r \cup \{0 = (0, \ldots, 0)\} \), by ignoring the \( 0\)th coordinate.

- Let \( \delta, r, F, S \) be given as in the theorem. Without loss of generality, \( F \) can be written as \([m]_0^r = \{(x_1, \ldots, x_r) \mid x_i \in [m]_0\} \) for a constant \( m \). Let \( r' := |F| - 1 = m^r - 1 \). Take a linear map \( \phi: \mathbb{R}^{r'} \to \mathbb{R}^r \) such that the restriction \( \phi|_{B_{r'}} \) is a bijection from \( B_{r'} \) to \( F \) with \( \phi(0) = 0 \). Define \( S' \subset [N]_0^{r'} \) by \( S' := \phi^{-1}(S) \cap [N]_0^{r'} = \{ z \mid (\phi(z) \in S) \} \). Clearly \( \phi^{-1}(s) \) forms an \((r' - r)\)-dimensional linear subspace of \( \mathbb{R}^{r'} \) for any \( s \in S \), by observing the rank of an \( r \times r' \)-matrix. Then it is straightforward to see that there exists a constant \( \delta' = \delta'(\delta, r, m) \) such that \( |S'| \geq \delta' N^{r'} \). Taking \( N \) large, the last paragraph yields \( a \in [N]_0^{r'} \) and \( c > 0 \) such that \( a + cB_{r'} \subset S' \). Thus \( S \supset \phi(a + cB_{r'}) = \phi(a) + c\phi(B_{r'}) = \phi(a) + cF \), completing the proof. \( \square \)

5. Remarks

Let \( F \) be a \( k \)-uniform (2-colored: black and invisible) hypergraph. Denote by \( \text{ex}^{(k)}(n,F) \) the maximum number of black edges of a \( k \)-uniform (2-colored: black and white) hypergraph on exactly \( n \)
vertices with no copy of $F$ as a subgraph. By an easy modification of the proof of our removal lemma, we can easily show a hypergraph version of the Erdős-Stone theorem.

**Proposition 5.1** (A hypergraph version of the Erdős-Stone theorem). Let $F, F_0$ be any $k$-uniform hypergraphs such that $F$ is a ‘blow-up’ of $F_0$ (i.e., there exists a map from the vertex set $V(F)$ to $V(F_0)$ such that each (black) edge of $F$ is mapped to a (black) edge of $F_0$). Then $\text{ex}^{(k)}(n, F) \leq \text{ex}^{(k)}(n, F_0) + o(n^k)$.

Rödl and Skokan [36] have already shown the above for black-only $F_0$ (i.e., $F_0 = K^{(k)}$) by adding extra arguments to a removal lemma. Although it should not be hard to obtain the above by previously known techniques, ours is a direct and shorter proof.

It is worthwhile to note that not only the way of regularizing but also the construction of the error function [24] is quite simple and clear in our proof. It is easy to find a simple $O(1)$-time random algorithm by which we can approximately grasp the entire hypergraph $G$.

Alon et al. [3] discussed the relation between Regularity Lemma and Property Testing for ordinary graphs. Although their proof is conceptually clear, many of their technical details may come from their problem setting (non-partiteness). In order to understand the essential relation between Regularization (Regularity Lemma) and Property Testing, it may be even easier and more natural to consider them on partite hypergraphs rather than on nonpartite ordinary graphs. Property Testing and Regularization are essentially equivalent. They are all about random samplings. If there exists a difference between the two, it is whether the number of random vertex samplings is (PT) a fixed constant or (R) bounded by a constant but chosen randomly. The above difference is essentially insignificant, as far as we do not consider the sizes of constants. Property Testing is stronger than Regularization in the sense that a (non-canonical) property tester can ignore some random number of vertex samples after choosing the vertices [1]. On the other hand, Regularization is stronger than Property Testing in the sense that Regularization ‘knows’ the number of copies of all fixed-sized subgraphs approximately. (If there is another difference, the Property Tester outputs one of only two choices (YES/NO), while Regularization can output some of a constant number of choices; also see [27]).

Therefore our result on hypergraph regularization is not a simple extension of graph regularization. It helps our understanding of regularization (and property testing) both for graphs and hypergraphs.

**REFERENCES**

[1] M. Ajtai and E. Szemerédi, Sets of lattice points that form no squares, Stud. Sci. Math. Hungar. 9 (1974), 9-11.
[2] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, Proc. 40th FOCS, New York, NY, IEEE (1999), 656-666. Also: Combinatorica 20 (2000), 451-476.
[3] N. Alon, E. Fischer, I. Newman and A. Shapira, A combinatorial characterization of the testable graph properties: it’s all about regularity, STOC ’06; May 21-23, (2006) Seattle, Washington, USA.
[4] L. Babai, N. Nisan, and M. Szegedy, Multiparty protocols, pseudorandom generators for logspace, and time-space tradeoffs, in “Twenty-first Symposium on the Theory of Computing (Seattle, WA, 1989)”, J. Comput. System Sci. 45 (1992), 204-232.
[5] F.R.K. Chung, Quasi-random classes of hypergraphs, Random Structures Algorithms 1, No.4 (1990), 363-382.
[6] F.R.K. Chung, Regularity lemmas for hypergraphs and quasi-randomness, Random Structures and Algorithms 2 (1991), 241-252.
[7] F.R.K. Chung and R.L. Graham, Quasi-random hypergraphs, Random Structures and Algorithms 1 No.1 (1990), 105-124.
[8] F.R.K. Chung and R.L. Graham, Quasi-random set systems, J. Amer. Math. Soc. 4 No.1 (1991), 151-196.
[9] F.R.K. Chung and R.L. Graham, On hypergraphs having evenly distributed subhypergraphs, Disc. Math. 111 (1993), 125-129.
[10] F.R.K. Chung and P. Tetali, Communication complexity and quasi randomness, SIAM J. Discrete Math. 6 No.1 (1993), 110-123.
[11] P. Erdős, Problems and results on combinatorial number theory. In A Survey of Combinatorial Theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), North-Holland, Amsterdam, pp.117-138.
[12] P. Erdős and P. Turán, On some sequences of integers, J. London Math. Soc. 11 (1936), 261-264.
[13] P. Frankl ad V. Rödl, The uniformity lemma for hypergraphs, Graphs and Combinatorics, 8 (1992), 309-312.
[14] P. Frankl ad V. Rödl, Extremal problems on set systems, Random Structures and Algorithms, 20(2) (2002)
[15] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, J. Analyse Math. 31 (1977), 204-256.
[16] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for commuting transformations, J. Analyse Math. 34 (1978), 275-291.

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1 Canonical property testing chooses (a fixed number of) vertices at random, but once the vertices are chosen, it outputs its answer deterministically. Therefore, at first sight, canonical property testing may be weaker. However as seen in [2][13 Th.2], for any given non-canonical property test, there exists a canonical property test which is equivalent to it. (Its derandomizing process is easy, since the sampling size of a non-canonical tester is a constant. The canonical tester repeats the samplings many (but a constant number of) times. Then it computes the probability that the noncanonical tester accepts for each sampling. The canonical tester accepts if the average of the probabilities is at least 1/2.)
[17] H. Furstenberg, Y. Katznelson, and D. Ornstein, The ergodic theoretical proof of Szemerédi’s theorem, *Bull. Amer. Math. Soc.* 7 (1982), 527-552.

[18] O. Goldreich and L. Trevisan, Three theorems regarding testing graph properties, *Random Structures and Algorithms* 23 (2003), 23-57.

[19] W.T. Gowers, Quasirandomness, counting, and regularity for 3-uniform hypergraphs, *Comb. Probab. Comput.* 15 (1-2), (2006), 143-184.

[20] W.T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, *Annals of Mathematics.* 166 (2007), 897-946.

[21] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, *Annals of Mathematics.* 167 (2008), 481-547. ([arXiv:math/0404188](https://arxiv.org/abs/math/0404188))

[22] B. Green and T. Tao, New bounds for Szemerédi’s theorem, II: a new bound for $r_4(N)$, 26 pages, preprint (2006.10)

[23] J. Haviland and A.G. Thomason, Pseudo-random hypergraphs, in “Graph Theory and Combinatorics(Cambridge, 1988)” *Discrete Math.* 75, No.1-3 (1989), 255-278.

[24] J. Haviland and A.G. Thomason, On testing the “pseudo-randomness”of a hypergraph, *Discrete Math.* 103, No.3 (1992), 321-327.

[25] Y. Ishigami, A simple regularization of hypergraphs, *arXiv:math/0612838* [math.CO].

[26] Y. Ishigami, Linear Ramsey numbers for bounded-degree hypergraphs, *arXiv:math/0612601* [math.CO].

[27] Y. Ishigami, Removal lemma for infinitely-many forbidden hypergraphs and property testing, *arXiv:math/0612669* [math.CO].

[28] Y. Ishigami, The number of hypergraphs and colored Hypergraphs with hereditary properties, *arXiv:0712.0425* [math.CO].

[29] Y. Kohayakawa, V. Rödl and J. Skokan, Hypergraphs, quasi-randomness, and conditions for regularity, *J. Combin. Theory A* 97 (2002) no.2, 307-352.

[30] Y. Kohayakawa, B. Nagle and V. Rödl, Hereditary properties of triple systems, *Combinatorics, Probability and Computing, (2003)* 12, 155-189.

[31] B. Nagle, V. Rödl and M. Schacht, The counting lemma for regular $k$-uniform hypergraphs, *Random Structures and Algorithms, 28* (2006), no.2, 113-179.

[32] H.J. Prömel and A. Steger, Excluding induced subgraphs III. A general asymptotic, *Random Structures and Algorithms 3* (1992), no.1, 19-31.

[33] V. Rödl and M. Schacht, Regular partitions of hypergraphs: Regularity Lemmas, *Combinatorics, Probability & Computing, 16*(2007), no.6, 833-885.

[34] V. Rödl and M. Schacht, Regular partitions of hypergraphs: Counting Lemmas, *Combinatorics, Probability & Computing, 16*(2007), no.6, 887-901.

[35] V. Rödl and J. Skokan, Regularity lemma for $k$-uniform hypergraphs, *Random Structures and Algorithms 25* (2004) (1), 1-42.

[36] V. Rödl and J. Skokan, Applications of the regularity lemma for $k$-uniform hypergraphs, *Random Structures and Algorithms 26* (2006), 180-194.

[37] J. Solymosi, Note on a generalization of Roth’s theorem, *Discrete and Computational Geometry, 825-827, Algorithms Combin.* 25, Springer, Berlin 2003.

[38] J. Solymosi, A note on a question of Erdős and Graham, *Combin. Probab. Comput.* 13 (2004), 263-267.

[39] A. Steger, Die Kleitman-Rothschild Methode, Dissertation, Universität Bonn, March 1990.

[40] E. Szemerédi, On sets of integers containing no four elements in arithmetic progression, *Acta Math. Acad. Sci. Hungar.* 20 (1969), 89-104.

[41] E. Szemerédi, On sets of integers containing no $k$ elements in arithmetic progression, *Acta Arithmetica 27* (1975), 199-245. [Collection of articles in memory of Jurii Vladimirovič Linnik.]

[42] E. Szemerédi, Regular partitions of graphs in *Problèmes combinatoires et théorie des graphes*, Orsay 1976, J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, D. Sotteau, eds., Colloq. Internat. CNRS 260, Paris, 1978, 399-401.

[43] E. Szemerédi, Integer sets containing no arithmetic progressions, *Acta Math. Hungar.* 56 (1990), 155-158.

[44] T. Tao, A variant of the hypergraph removal lemma, *J. Combin. Theory A* 113 (2006), no.7, 1257-1280.

[45] T. Tao, The dichotomy between structure and randomness, arithmetic progressions, and the primes, (ICM2006 lecture) *arXiv:math/0512114v2* [math.NT]

[46] T. Tao and V.H. Vu, *Additive Combinatorics*, Cambridge University Press, (2006) 512 pages.

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