(2+1)-DIMENSIONAL GRAVITY COUPLED TO A DUST SHELL: QUANTIZATION IN TERMS OF GLOBAL PHASE SPACE VARIABLES

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We perform a canonical analysis of a model in which gravity is coupled to a spherically symmetric dust shell in 2+1 space–time dimensions. The result is a reduced action depending on a finite number of degrees of freedom. We emphasize finding canonical variables supporting a global parameterization for the entire phase space of the model. It turns out that different regions of the momentum space corresponding to different branches of the solution of the Einstein equation form a single manifold in the ADS$_2$ geometry. The Euler angles support a global parameterization of that manifold. Quantization in these variables leads to noncommutativity and also to discreteness in the coordinate space, which allows resolving the central singularity. We also find the map between the ADS$_2$ momentum space obtained here and the momentum space in Kuchař variables, which could be helpful in extending the obtained results to 3+1 dimensions.

Keywords: quantum gravity, thin shell, singularity removal

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1. Introduction

There has long been a hypothesis that a quantum theory of gravity should resolve the singularities of general relativity. The first argument for this goes back to Bronstein [1], who showed that distances less than the Planck length cannot be measured. In the absence of a full theory of gravity, there is still no final answer whether this is indeed the case.

This question can be studied by considering reduced models of general relativity in which all but a few degrees of freedom are frozen. Those models that contain black hole solutions, which are crucial for Bronstein’s argument, present a special interest.

The simplest possible model is a homogenous universe with a scalar matter field. Quantization of this model has been studied extensively, and the general conclusion is that quantum theory does not resolve the singularity in this case [2] unless some exotic matter is added [3].

The next-to-simplest model is a spherically symmetric space–time in which matter is represented by one or more dust shells. In contrast to the homogenous model, this model contains solutions with black holes, which leads to a nontrivial structure of the phase space with branching of the solutions for the constraints.

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There are many various works studying such models on both the classical [4]–[7] and the quantum levels. In some versions of quantum theory, the central singularity is removed [8], [9], but these results do not always agree with each other. In addition to the quantization ambiguity, another possible reason for this is the complicated structure of the phase space of the theory. Different quantum theories can arise in different sectors of such a phase space.

In such a situation, the common wisdom is that the wave function of a quantum theory must be defined on all possible configurations independent of their classical accessibility. This was realized in one of the possible ways in [8], where different sectors of the phase space were combined into one Riemann surface using complex coordinates and the branch point corresponded to the horizon.

Another possibility is to try to find a real global parameterization for the phase space (if it exists). One example where this was realized is (2+1)-dimensional gravity coupled to a point particle [10]. The particle momentum turns out to take values in the Lorentz group manifold, and the different branches of the solution for the constraints result from different ways of intersecting this manifold by a plane. This example is our starting point in trying to unite these two approaches.

In Sec. 2, we repeat the canonical analysis [6], [8] for the gravity+shell system including the Kuchař canonical transformation necessary for it, but now in 2+1 space–time dimensions. The result is very similar to the result in 3+1 dimensions; the only difference is that there is no Newtonian potential, but the branching of the constraint solution is the same.

In Sec. 3, we extend the results in [10] and [11] from a point particle to a spherical shell represented as an ensemble of an infinite number of point particles. The momentum space of the shell turns out to be ADS\(_2\), which in particular results in the noncommutativity of the coordinates. The Hamiltonian constraint in the case of a shell differs from that in the case of a particle, which is explained by the contribution from the energy of particles moving inside the shell to the gravitational field. We also find a relation between the momenta in ADS\(_2\) and the canonical momenta obtained in [6], [8].

In Sec. 4, we perform quantization in the momentum representation on ADS\(_2\). In addition to the noncommutativity of the coordinates, this results in a discreteness of the spectrum of one coordinate (time). The perimeter of the shell has a discrete spectrum if it is timelike and a continuous spectrum separated from zero if it is spacelike. This ensures the disappearance of the central singularity.

Finally, we discuss applying the Hamiltonian constraint and finding the physical Hilbert space. We also discuss a possibility to extend these results to 3+1 space–time dimensions.

2. Canonical analysis of the gravity+shell action and the problem of finding the global phase space variables

In this section, we repeat the derivation of the results in [6], [8] in a space with one fewer dimensions.

2.1. Canonical formalism for spherically symmetric space–times in 2+1 gravity. The Arnowitt–Deser–Misner (ADM) approach starts by slicing the manifold \(M\) into constant-time surfaces \(\Sigma\). Here, \(g_{ij}\) is an induced metric on \(\Sigma\). We consider a spherically symmetric two-dimensional Riemannian space \((\Sigma, g)\). The line element \(d\sigma\) on \(\Sigma\) is characterized by two functions \(\Lambda(r)\) and \(R(r)\),

\[d\sigma^2 = \Lambda^2(r) \, dr^2 + R^2(r) \, d\theta^2.\]  

(1)

We assume that \(\Lambda(r)\) and \(R(r)\) are positive, nonvanishing functions. We note that \(R(r)\) is the curvature radius \(r = \text{const}\) of the shell surface and \(d\sigma = \Lambda(r) \, dr\) is the radial line element. The Dirac–ADM action for the metric in the absence of matter is

\[S_{\Sigma}[g, N, N^a] = \frac{1}{16\pi} \int_M R \sqrt{-g} \, d^3x = \int dt \int_{\Sigma} L_{\Sigma} \, d^2x\]
and is the standard Einstein–Hilbert action for the gravitational field, and the Lagrangian \( L_\Sigma \) is written as

\[
L_\Sigma = \frac{N}{16\pi}(K_{ab}K_{ab} - K^2 + R[g]),
\]

where \( R[g] \) is the scalar curvature of the spatial metric \( g_{ab} = (\Lambda^2, R^2) \) and \( K_{ab} \) is the extrinsic curvature for a constant-time surface. The scalar curvature for line element (1) is

\[
R[g] = -2\Lambda^{-2}R'' + 2\Lambda^{-3}R^{-1}\Lambda'R'.
\]

We note that a dot and a prime respectively denote differentiation with respect to \( t \) and \( r \). The extrinsic curvature is defined by

\[
K^i_j = \text{diag}(K^r_r, K^\theta_\theta), \quad K_{rr} = -N^{-1}\Lambda(\dot{\Lambda} - \Lambda N_r''), \quad K_{\theta\theta} = -N^{-1}R(\dot{R} - R'N').
\]

The ADM action in terms of Eqs. (2)–(4) is

\[
S_\Sigma[R, \Lambda; N, N^r] = \int dt \int_{-\infty}^{\infty} \left[-N^{-1}(\dot{\Lambda} + (\Lambda N^r)')(\dot{R} + R'N') + N(-\Lambda^{-1}R'' + \Lambda^{-2}R'N')\right] dr.
\]

The canonical formalism of the action is derived by differentiating ADM action (5) with respect to the velocities \( \dot{\Lambda} \) and \( \dot{R} \). We obtain

\[
P_\Lambda = -N^{-1}(\dot{R} - R'N'), \quad P_R = -N^{-1}(\dot{\Lambda} - (\Lambda N^r)'),
\]

where the momentum \( P_R \) is a density and the momentum \( P_\Lambda \) is a scalar. The velocities \( \dot{\Lambda} \) and \( \dot{R} \) can be written in terms of \( P_R \) and \( P_\Lambda \) as \( \dot{R} = -NP_\Lambda + R'N' \) and \( \dot{\Lambda} = -NP_R + (\Lambda N^r)' \). The extrinsic curvature as a function of the canonical momenta is \( K_{rr} = \Lambda P_R \) and \( K_{\theta\theta} = R P_\Lambda \).

We can rewrite ADM action (5) in the canonical form by a Legendre transformation as

\[
S_\Sigma[R, \Lambda, P_\Lambda, P_R; N, N^r] = \int dt \int_{-\infty}^{\infty} (P_\Lambda \dot{\Lambda} + P_R \dot{R} - NH^G - N^rH^G_r) dr,
\]

where the Hamiltonian constraint is

\[
H^G = -P_\Lambda P_R + \Lambda^{-1}R'' - \Lambda^{-2}R'N' = 0
\]

and the momentum constraint is

\[
H^G_r = P_R R' - \Lambda P_\Lambda' = 0.
\]

### 2.2. The Kuchař transformation for 2+1 gravity.

In [5], Kuchař proposed a new method for simplifying the Hamiltonian and momentum constraints based on a canonical transformation from the old variables to a new canonical set. In this section, we discuss the Kuchař method in (2+1)-dimensional gravity. The general metric of a spherically symmetric space–time is [5]

\[
ds^2 = -(N^2 - \Lambda^2(N^r)^2)dt^2 + 2\Lambda^2 N^r dt dr + \Lambda^2 dr^2 + R^2 d\theta^2,
\]
where $N$, $N'$, $\Lambda$, and $R$ are continuous functions of only $t$ and $r$. Equation (10) represents the ADM form of the 2+1 decomposition of a space-time metric. The general spherically symmetric (2+1)-dimensional Schwarzschild metric in the curvature coordinates $(T, R)$ is

$$
\text{ds}^2 = -(1 - 2m) \frac{dT^2}{\Delta} + dR^2 + R^2 d\theta^2,
$$

(11)

where the conical singularity has the angle $\sim 2\pi(1 - \sqrt{1 - 2m}) \approx 2\pi M$. We regard the hypersurface as a leaf of a foliation

$$
T = T(t, r), \quad R = R(t, r),
$$

(12)

We substitute Eq. (12) in (11) and obtain

$$
\text{ds}^2 = -[(1 - 2m)T^2 - (1 - 2m)^{-1}\dot{R}^2] dt^2 + 2[-(1 - 2m)\dot{T}\dot{R}' + (1 - 2m)^{-1}\dot{R}R'] dt dr +
+[-(1 - 2m)T'^2 + (1 - 2m)^{-1}R'^2] dr^2 + R^2 d\theta^2.
$$

(13)

Comparing (13) and (10), we obtain

$$
\Lambda^2 = -(1 - 2m)\dot{T}'^2 + (1 - 2m)^{-1}\dot{R}'^2,
$$

(14a)

$$
\Lambda^2 N' = -(1 - 2m)\dot{T}' + (1 - 2m)^{-1}\dot{R}' R',
$$

(14b)

$$
N^2 - \Lambda^2 (N')^2 = (1 - 2m)\dot{T}'^2 - (1 - 2m)^{-1}\dot{R}'^2.
$$

The lapse function and the shift function are given by

$$
N' = \frac{-(1 - 2m)\dot{T}' + (1 - 2m)^{-1}\dot{R}' R'}{-(1 - 2m)\dot{T}'^2 + (1 - 2m)^{-1}\dot{R}'^2},
$$

(15)

$$
N = \frac{\dot{T}' - \dot{R}' R'}{\sqrt{-(1 - 2m)\dot{T}'^2 + (1 - 2m)^{-1}\dot{R}'^2}}.
$$

(16)

To calculate $P_\Lambda$, we substitute (15) and (16) in Eq. (6) and obtain $-T' = (1 - 2m)^{-1}\Lambda P_\Lambda$. The Schwarzschild mass can be evaluated using Eqs. (14). It is equal to $m = 1/2 + P_\Lambda^2/2 - R^2/2\Lambda^2$. We find that the functions $m(r)$ and $-T'(r)$ are canonically conjugate variables and the dynamical variable $-T'(r)$ can be denoted by $P_m(r)$. The new momentum $P_R(r)$ is written in terms of the old momentum $P_R(r)$ and a dynamical variable $\Phi(r)$: $P_R(r) = P_R(r) + \Phi(r; R, \Lambda, P_\Lambda)$, where $\Phi(r)$ is independent of $P_R(r)$. Finally, the transformation has the form

$$
\Lambda = \sqrt{R^2(1 - 2m)^{-1} - (1 - 2m)P_m^2}, \quad P_\Lambda = \frac{(1 - 2m)P_m}{\sqrt{R^2(1 - 2m)^{-1} - (1 - 2m)P_m^2}},
$$

$$
\dot{R} = R, \quad \dot{P}_R = P_R - \frac{(1 - 2m)^{-1}}{\Lambda^2}[(\Lambda P_\Lambda)' R' - (\Lambda P_\Lambda) R''],
$$

where we use natural units with $G = c = 1$. The Liouville form is

$$
\Theta = \int P_\Lambda \dot{\Lambda} + P_R \dot{R} =
= \int P_m \dot{m} + P_R \dot{R} + \frac{\partial}{\partial t} \left[ \Lambda P_\Lambda + \frac{R'}{2} \log \left| \frac{R' - \Lambda P_\Lambda}{R' + \Lambda P_\Lambda} \right| \right] + \frac{\partial}{\partial t} \left[ \frac{\dot{R}}{2} \log \left| \frac{R' + \Lambda P_\Lambda}{R' - \Lambda P_\Lambda} \right| \right].
$$

Equations (8) and (9) can be represented by a simple set of constraints:

$$
\dot{P}_R = 0, \quad m' = 0.
$$

(17)
2.3. Canonical analysis for (2+1)-dimensional gravity with a thin shell. In this section, we briefly describe the canonical formalism for a spherically symmetric gravitational field in 2+1 dimensions in the presence of a thin shell. The action for this system is

\[ S = S_{\text{gr}} + (\text{boundary terms}) + S_{\text{shell}} = \]

\[ = \frac{1}{16\pi G} \int R \sqrt{-g} \, d^4 x + (\text{boundary terms}) + M \int \Sigma \, d\tau. \quad (18) \]

The first and third term in (18) are the standard Einstein–Hilbert action and the shell action. The shell action has the form

\[ S_{\text{shell}} = -M \int \Sigma \sqrt{\hat{N}^2 - \hat{\Lambda}^2(\hat{\Lambda} r + \hat{r})^2} \, dt, \]

where \( M \) is the shell rest mass and \( \hat{V} \) denotes the on-shell value of a variable \( V \). The explicit form of action (18) in the Hamiltonian formalism is given by

\[ S = \int \hat{\pi} \hat{\dot{r}} \, dt + \int [P_{\Lambda} \hat{\Lambda} + P_r \hat{R} - N(H^s + H^G) - N^r(H^s_r - H^G_r)] \, dr \, dt + \int M_{\text{ADM}} \, dt, \]

where \( M_{\text{ADM}} \) is the total mass of the combined gravity+shell system and \( \hat{\pi} \) is the momentum conjugate to \( \hat{\dot{r}} \): \( \hat{\pi} = m\hat{\Lambda}^2(\hat{\Lambda} r + \hat{r})/\sqrt{\hat{N}^2 - \hat{\Lambda}^2(\hat{\Lambda} r + \hat{r})^2} \). The Hamiltonian constraint of the shell is \( H^s = \sqrt{\hat{\pi}/\hat{\Lambda}}^2 + m^2 \delta(r - \hat{r}) = 0 \), and its momentum constraint is \( H^s_r = \hat{\pi} \delta(r - \hat{r}) = 0 \). The regular contribution to the constraints is the same as in the vacuum and holds in the inner and outer regions of the shell. There is also a delta-functional contribution on the shell that must be combined with the shell Hamiltonian. As a result, we obtain the shell constraints

\[ C^s = \left[ R^s \right] \Lambda + \sqrt{\left( \frac{\hat{\pi}}{\Lambda} \right)^2 + M^2} = 0, \quad (19) \]

\[ C^s_r = \Lambda [P_{\Lambda}] + \hat{\pi} = 0, \quad (20) \]

where the square brackets denote the jump of the field across the shell.

Now we pass to the Kuchař variables, solve the constraints in the inner and outer regions, and substitute the solution in the action. It turns out that the bulk kinetic term vanishes because of bulk constraints (17), the on-shell kinetic term containing \( \hat{\dot{r}} \) vanishes because of constraint (20), and all that remains is the contribution from boundary terms that appear after the Kuchař canonical transformation: \( S = \int dt \left[ m \hat{T} + [P_{\Lambda} \hat{\Lambda} + P_r \hat{R} - N^s C^s] \right] \), where \( \hat{T} \) is the Killing time evaluated at the shell, \( P_{\Lambda} |_{\text{in, out}} = \log |(R' + \Lambda P_{\Lambda})/(R' - \Lambda P_{\Lambda})|_{\text{in, out}} \), and \( C^s \) is constraint (20).

We then express constraint (20) in terms of the canonical shell variables \( m \) and \( P_r \):

\[ C^s = \sqrt{1 - 2m} \cosh P_{R_{\text{out}}} - \cosh P_{R_{\text{in}}} + M. \quad (21) \]

From the resulting action, we obtain the equations of motion for \( R \),

\[ \frac{\hat{R}}{N^s} = \sqrt{1 - 2m} \sinh P_{R_{\text{out}}} = \sinh P_{R_{\text{in}}}, \quad (22) \]

which leads to another constraint,

\[ \sqrt{1 - 2m} \sinh P_{R_{\text{out}}} - \sinh P_{R_{\text{in}}} = 0. \quad (23) \]
In contrast to the (3+1)-dimensional situation, constraints (21) and (23) are now first class. Substituting (22) in (21), we recover the Israel equation for the shell:

\[ \sqrt{1 + \frac{\dot{R}^2}{(N^*)^2}} + \sqrt{1 - 2m + \frac{\dot{R}^2}{(N^*)^2}} - M = 0. \]  

(24)

Finally, squaring the two constraints (21) and (23) and adding them, we obtain a single Hamiltonian constraint describing the shell dynamics:

\[ 1 + 1 - 2m - 2\sqrt{1 - 2m \cosh[PR]} - M^2 = 0. \]  

(25)

This constraint is to be used in quantum theory.

We must here note that many of the above equations contain square roots, which are not single-valued functions. Different choices of the signs of the square roots correspond to different sectors of the phase space of the model, which are pictured as different regions in Penrose diagrams. The same \( m \) corresponds to two different points in the momentum space. Moreover, the Killing time \( \vec{T} \) and the radial momentum \( [PR] \) diverge at the gravitational collapse point ("horizon"), where \( 2m = 1 \). We must here note that in (2+1)-gravity with a zero cosmological constant, there is no horizon in the usual sense of being located somewhere in space. The entire space is outside the "horizon" for \( 2m < 0 \) and inside the "horizon" for \( 2m > 1 \). The horizon is therefore located not in the coordinate space but in the phase space at \( 2m = 1 \). For \( 2m > 1 \), Eq. (25) does not have solutions in real variables. In other words, the theory is formulated in variables that do not cover the phase space globally.

One way to circumvent the problem is to introduce a complex phase space and assemble different patches into a Riemann surface [8]. But there is a possibility that there is another set of real variables of the phase space that would cover it globally. We study this possibility in the next section.

3. First-order formalism and anti-De Sitter momentum space

There is an example in the literature where a global parameterization of the phase space was found for a simple model containing gravity and matter. We speak of (2+1)-dimensional gravity coupled to point particles [10], [11]. We here generalize these results to a spherically symmetric shell.

There is a similar result in [12], where the solution for a thin shell coupled to (2+1)-gravity was obtained as a limit of a many-particle solution. The emphasis there was to obtain an explicit classical solution, and the method used was based on cutting and gluing manifolds. Our focus is on preparing the model for quantization by obtaining a reduced canonical action. The method that we use is based on group theory.

3.1. Action principle and the phase space reduction. In this section, we start with a general action (not spherically symmetric) for (2+1)-gravity coupled to a finite number of point particles.

The basic variables are the triad \( e_\mu = e_\mu^a \gamma_a \) and the connection \( \omega^{ab}_\mu \gamma_a \gamma_b \), where \( \gamma^a \) are generators of the sl(2) algebra. The action has the form

\[ S = \int_M \epsilon^{\mu\nu\rho} \text{Tr}(e_\mu R_{\nu\rho}) \, d^3x + S_{\text{shell}}, \]  

(26)

where \( R_{\nu\rho} \) is the curvature of \( \omega_\rho \).

The shell is discretized (consisting of \( N \) particles labeled by the index \( i \)), \( S_{\text{shell}} = \sum_i^N \int_{l_i} \text{Tr}(K_i e_\mu) \, dx^\mu \), where \( l_i \) is the \( i \)th particle worldline and \( K_i = m_i \gamma_0 \) is a fixed element of the sl(2) algebra. We impose the spherical symmetry condition later.
The gravity action is invariant under gauge transformations

\[ \omega_\mu \rightarrow g^{-1}(\partial_\mu + \omega_\mu)g, \quad e_\mu \rightarrow g^{-1}(e_\mu + \partial_\mu \xi)g, \]

where \( g \) is an \( SL(2) \) group element and \( \xi \) is an \( sl(2) \) algebra element.

The shell action is not invariant. The \( i \)th particle term transforms as

\[ \int_{l_i} \text{Tr}(K_i e_\mu) \, dx^\mu \rightarrow \int_{l_i} \text{Tr}(\tilde{K}_i e_\mu) \, dx^\mu + \int_{l_i} \text{Tr}(\tilde{K}_i \xi) \, d\tau, \]

where \( \tilde{K}_i = gK_ig^{-1} \), \( \tau \) is the parameter along the particle worldline, and a dot denotes the derivative with respect to it.

In the last term in the right-hand side of (28), we can recognize the standard particle action because it has the form \( \int p_a \dot{x}^a \), where \( p_a = \text{Tr}(\gamma_0 \tilde{K}_i) \) and \( x^a \text{Tr}(\gamma_0 \tilde{K}_i) \), and \( p^a \) satisfies the constraint equation \( p^a p_a = m^2 \) with the definition of \( K_i \) taken into account. Hence, the particle degrees of freedom are represented as gauge degrees of freedom evaluated at the particle location.

As before, to obtain a reduced action for this model, we must solve the constraints and substitute the solution in the original action. We choose a slicing such that particle worldlines move along the time coordinate and obtain the constraints by varying action (26) with respect to \( \omega_0 \) and \( e_0 \):

\[ \epsilon^{0\mu\nu} \nabla_\mu e_\nu = 0, \quad \epsilon^{0\mu\nu} R_{\mu\nu} = \sum_{i=1}^{N} \tilde{K}_i \delta^2(x, x_i), \]

where \( x_i \) is the location of the \( i \)th particle. The first constraint in (29) generates the first transformation in (27), and the second generates the second.

Using transformations (27), we can simultaneously set one component of \( \omega \) and one component of \( e \) to zero. This automatically linearizes constraints (29). But we cannot choose such a gauge globally, because the model has a nontrivial moduli space, for example, containing the gauge parameter evaluated at the location of one particle with respect to another. Following [13], we divide the spatial slice into regions in each of which the above gauge choice can be made. Each such region can contain no more than one particle. We draw a circle around each particle such that the circles are connected to a common origin but have no common boundaries. Cutting along the circles, we divide the manifold into \( N \) discs, each containing a particle, and a polygon containing no particles but connected to infinity.

For the discs, it is convenient to write the solution in polar coordinates with the origin at the particle locations. We choose a gauge in which the radial components of \( e \) and \( \omega \) are zero, solve the constraints, and substitute the solution in an arbitrary gauge:

\[ \omega_r,i = g_i^{-1} \partial_r g_i, \quad \omega_\phi,i = g_i^{-1} \nabla_\phi g_i, \]
\[ e_r,i = g_i^{-1} \partial_r \xi_i g_i, \quad e_\phi,i = g_i^{-1} \nabla_\phi \xi_i g_i, \]

where \( \nabla_\phi \xi_i = \partial_\phi \xi_i + [\xi_i, K_i] \). And we act similarly for the polygon, for which \( h \) and \( \zeta \) denote the gauge parameters.

We must now substitute these solutions in the kinetic term of the action, which for \( i \)th disc has the form

\[ S_{D_i} = \int_{D_i} \epsilon^{0\mu\nu} \text{Tr}(e_\mu \dot{\omega}_\nu) \, d^3x + \int_{l_i} \text{Tr}(\tilde{K}_i \xi) \, d\tau. \]

Using the identity \( \dot{g}_i^{-1} \nabla_\mu g_i = g_i^{-1} \nabla_\mu (\dot{g}_i g_i^{-1}) g_i \) (we note that \( K_i \) is independent of time and \( \nabla_\phi \) hence commutes with the time derivative), we find that in the first term in (31), there is a \( \delta \)-functional contribution
canceling the second term and another term, which is a total derivative. Hence, the action for the disc collapses to its boundary:

\[ S_{D_i} = \int_{\partial D_i} \text{Tr}(\nabla \phi \xi_i \dot{g}_i g_i^{-1}) \, d^2x. \]  

(32)

The result for the polygon is similar, but its boundary consists of \( N \) edges \( E_i \), and the resulting action is a sum of the contributions from each edge:

\[ S_P = \sum_i \int_{E_i} \text{Tr}(\partial_\phi \xi_i \dot{h}_i h_i^{-1}) \, d^2x. \]  

(33)

The next step is to assemble all the above pieces of the action together and apply the condition for continuity of the metric and the connection across the boundary between discs and the polygon.

We first convert the covariant derivative in (30) into an ordinary derivative by a gauge transformation \( \tilde{g}_i = e^{K\phi} g_i, \tilde{\xi}_i = e^{K\phi} \xi_i e^{-K\phi} \). This condition violates the periodicity, and the boundary of the disc is hence no longer a circle but an interval. Disc action (32) then becomes

\[ S_{D_i} = \int_{\partial D_i} \text{Tr}(\partial_\phi \tilde{\xi}_i \dot{\tilde{g}}_i \tilde{g}_i^{-1}) \, d^2x, \]  

(34)

and continuity conditions (30) for the metric and connection take the simple form

\[ \tilde{g}_i = C_i h|_{E_i}, \quad \tilde{\xi}_i = C_i (\zeta_i|_{E_i} + \chi_i) C_i^{-1}, \]  

(35)

where \( C_i \) and \( \chi_i \) are functions of only time. In what follows, we set \( \chi_i = 0 \), which is possible if the positions of different particles can be related by pure rotation with no translations. This is an implementation of spherical symmetry. Substituting this in (33) and (34) and combining them, we obtain

\[ S_{\text{full}} = S_P + \sum_i S_{D_i} = \sum_i \int_{E_i} \text{Tr}(\partial_\phi \zeta_i C_i^{-1} \dot{C}_i) = - \sum_i \int_{\partial D_i} \text{Tr}(\partial_\phi \tilde{\xi}_i \dot{C}_i C_i^{-1}). \]

The integrands are total derivatives, and the result therefore contains contributions only from the polygon vertices or the endpoints of disc boundaries:

\[ S_{\text{full}} = \sum_i \int_R \text{Tr}((\zeta_{i+1} - \zeta_i) C_i^{-1} \dot{C}_i) = - \sum_i \int_R \text{Tr}((\tilde{\xi}_i(2\pi) - \tilde{\xi}_i(0)) \dot{C}_i C_i^{-1}), \]  

(36)

where \( \zeta_i \) is the value of \( \zeta \) at the \( i \)th vertex of the polygon.

We introduce the new variables

\[ u_i = C_i^{-1} e^{2\pi K} C_i \quad \text{and} \quad \tilde{\xi}_i = C_i^{-1} \tilde{\xi}_i(0) C_i. \]  

(37)

Taking into account that \( \tilde{\xi}_i(2\pi) = e^{2\pi K} \tilde{\xi}_i(0) e^{-2\pi K} \) and

\[ u_i^{-1} \dot{u}_i = C_i^{-1} \dot{C}_i - C_i^{-1} e^{-2\pi K} \dot{C}_i C_i^{-1} e^{2\pi K} C_i, \]

we can then rewrite the second equation in (36) as

\[ S_{\text{full}} = \sum_i \int_R \text{Tr}(\tilde{\xi}_i u_i^{-1} \dot{u}_i). \]  

(38)
To relate the $\xi_i$ for different $i$, we use the second overlap condition in (35), which in particular implies that $\xi_i(0) = C_i\xi_i C_i^{-1}$ and $\xi_i(2\pi) = e^{2\pi K} \xi_i(0) e^{-2\pi K} = C_i \xi_{i+1} C_i^{-1}$, whence using definition (37), we can derive $\xi_{i+1} = u_i \xi_i u_i^{-1}$ and

$$\xi_i = \left( \prod_{j=0}^{i-1} u_j \right) \xi_0 \left( \prod_{j=0}^{i-1} u_j \right)^{-1},$$

where the factors in the product are ordered from right to left.

We now introduce the holonomy around the full shell, which is the product of the holonomies around each particle $U = \prod_{j=0}^{N} u_j$.

Using the obvious identity $U^{-1} \dot{U} = \sum_{i=0}^{N} (\prod_{j=0}^{i-1} u_j)^{-1} u_i (\prod_{j=0}^{i-1} u_j)$ and taking relation (39) into account, we can rewrite the kinetic term of action (38) in the simple form

$$S_{\text{full}} = \int_{R} \text{Tr}(\xi_0 U^{-1} \dot{U}).$$

The action for many particles thus reduces to a term depending on a single variable. This is possible because we set $\chi_i = 0$ in (35), using the spherical symmetry.

3.2. Constraints. To obtain the complete action for the shell, we must find the constraints satisfied by the variables in (40) and add them to the kinetic term. In the definition

$$U = \prod_{i=0}^{N} C_i^{-1} e^{2\pi K_i} C_i,$$

we choose the spherically symmetric ansatz $C_i = e^{\bar{\chi} \gamma_i} e^{2\pi i \gamma_0/N}$, where $i$ (do not confuse it with the imaginary unit) plays the role of an angle variable and $\bar{\chi}$ is a boost parameter independent of $i$. For $K_i = M_\gamma \gamma_0$, spherical symmetry means that $m_i = m/N$, where $M$ is the overall bare mass of the shell. Substituting this in (41), we obtain

$$U = \prod_{i=0}^{N} e^{-2\pi i \gamma_0/N} e^{-2\pi M (\chi + \gamma_2 \sinh \bar{\chi})/N} e^{2\pi i \gamma_0/N} = (e^{2\pi M (\chi + \gamma_2 \sinh \bar{\chi})/N} e^{2\pi i \gamma_0/N})^N.$$

The product of the two exponentials in this equation can be calculated by the Campbell–Hausdorff formula $e^{A} e^{B} = e^{A + B + [A,B]/2 + \ldots}$. But the terms with commutators are of the order of $1/N^2$ and higher and are therefore negligible as $N \to \infty$. Finally, we obtain

$$U = e^{2\pi ((1 - M \cosh \bar{\chi}) \gamma_0 + M \sinh \bar{\chi} \gamma_2)}.$$

The conjugacy class of this holonomy is fixed by the boundary conditions at infinity. It can be evaluated as a Wilson loop of the angular component of the connection, which can be expressed in terms of the ADM variables as $\omega_\phi = R'/\Lambda \gamma_0 + P_A \gamma_1$ and $\text{Tr} U = \text{Tr}(P e^{i \int d\phi \omega_\phi}) = \cos(2\pi \sqrt{\left(R'/\Lambda \right)^2 - P^2_A}) = \cos(2\pi \sqrt{1 - 2m})$, where $m$ is the ADM mass. On the other hand, it follows from (42) that

$$\text{Tr} U = \cos(2\pi \sqrt{(1 - M \cosh \bar{\chi})^2 - (M \sinh \bar{\chi})^2}).$$

Equating the last two expressions for $\text{Tr} U$ and solving for $M$, we obtain

$$M = \sqrt{1 + \sinh^2 \bar{\chi}} + \sqrt{1 - 2m + \sinh^2 \bar{\chi}},$$

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which is Israel equation (24) in the preceding section with $\hat{R}/N^c = \sinh \chi$.

We can now write the relations between the canonical momenta used here and the canonical momenta in the preceding section:

$$\sinh \chi = \sqrt{1 - 2m} \sinh P_{R_{out}} = \sinh P_{R_{in}} \quad \text{and} \quad \text{Tr} U = \cos(2\pi \sqrt{1 - 2m}).$$

It can be seen that $\sinh \chi$ is always real even if $\sqrt{1 - 2m}$ and $P_{R_{out}}$ are complex; $U$ is also always real and elliptic if $1 - 2m > 0$ and hyperbolic if $1 - 2m < 0$.

In the elliptic case, $U = g^{-1} e^{\gamma_0 \phi^b} g$, and we have $0 < \phi < \pi$ if $\sqrt{1 - 2m} > 0$ and $\pi < \phi < 2\pi$ if $\sqrt{1 - 2m} < 0$. Similarly, in the hyperbolic case, $U = g^{-1} e^{\gamma_0 \chi} g$, and we have $\chi > 0$ if $i\sqrt{1 - 2m} > 0$ and $\chi < 0$ if $i\sqrt{1 - 2m} < 0$. In other words, $U$ provides a real global parameterization of the momentum space of the model.

### 3.3. Poisson brackets.

By a change of variables, we can bring the kinetic term in action (40) to the standard canonical form $S_{\text{full}} = -\int_R p_a q^a$, where $a = 0, 1$ labels the temporal and radial components of a coordinate (momentum). The explicit form of this transformation is $p_{-1} = \text{Tr} U$, $p_a = \text{Tr} (\gamma_a U)$, and

$$q^a = (p_{-1} \eta^{ab} - p_{-1}^b p^b) \xi_b,$$  \hspace{1cm} (44)

satisfying the standard commutation relations $\{p_a, p^b\} = 0$, $\{q_a, q^b\} = 0$, and $\{q_a, p^b\} = \delta^b_a$. But we do not use these variables in the subsequent quantization. The reason is that the momenta satisfy the relation $p_{-1}^2 - p_a p^a = 1$ because $U$ is an $SL(2)$ group element. This means that the momentum space of the model is a one-sheet hyperboloid or the two-dimensional anti-de Sitter space AdS$_2$, where $p_{-1}$ and $p_a$ are coordinates of the three-dimensional space in which AdS$_2$ is embedded. The momenta $p_a$ do not form a global parameterization on AdS$_2$. The global parameterization on AdS$_2$ is formed by the Euler angles, which parameterize the group element $U$ as

$$U = e^{\rho \gamma_0 / 2} e^{\chi \gamma_1} e^{\rho \gamma_0 / 2}.$$  \hspace{1cm} (45)

They are related to the $p$ as $p_{-1} = \cos \rho \cosh \chi$, $p_0 = \sin \rho \sinh \chi$, and $p_1 = \sinh \chi$. It is clear that the Euler angles together with coordinates canonically conjugate to them cannot form the standard canonical set of variables, because the Euler angles parameterize a curved space, which cannot be mapped onto a flat space, and translations in such a space do not commute.

In fact, the coordinates conjugate to the Euler angles are the original translation parameters $\xi^a$ in (40). To show this, we can calculate the Poisson brackets for them using the inverse of (44):

$$\{\xi^a, p_b\} = p_{-1}^{-1} (\delta^a_b + p^a p_b), \quad \{\xi^a, p_{-1}^{-1}\} = p^a, \quad \{\xi^a, \xi^b\} = p_{-1} (\xi^a p^b - \xi^b p^a).$$

It hence follows that

$$\{U, \xi^a\} = U \gamma^a,$$  \hspace{1cm} (46)

and we have the expected noncommutativity of the coordinates.

### 4. Quantization

In this section, we briefly describe quantization of the considered model using the momentum (Euler angle) representation. The exposition basically follows [11], but we take into account that there is one fewer degree of freedom because of the spherical symmetry and that the dynamical constraints are now different.
4.1. Quantum kinematics. We define the kinematical states of the model as functions of $U$ given by (45), $|\Psi\rangle = \Psi(U) = \Psi(\rho, \chi)$, which are single-valued functions on the entire momentum space. The periodicity $\Psi(\rho + 2\pi, \chi) = \Psi(\rho, \chi)$ follows immediately from the requirement that $\Psi$ is single-valued, and this has an important consequence for the coordinate spectrum.

Next, to define the scalar product, we need a Lorenz-invariant measure on our momentum space, which can be inferred from the Haar measure on $SL(2)$, $dU = (1/\pi) \sin 2\chi \, d\rho \, d\chi$. The scalar product is hence $\langle \Phi, \Psi \rangle = (1/\pi) \int \sin 2\chi \, d\rho \, d\chi \, \Phi(\rho, \chi)^* \Psi(\rho, \chi)$.

It is easiest to calculate the spectrum of the time coordinate $\xi^0$, which is canonically conjugate to $\rho$, and the corresponding operator and its eigenstates are

$$\hat{T}|\rho, \chi\rangle = \hbar \frac{\partial|\rho, \chi\rangle}{\partial \rho}, \quad |t; \psi\rangle = \frac{1}{\pi} \int \sinh 2\chi e^{it\rho} \psi(\chi)|\rho, \chi\rangle \, d\rho \, d\chi,$$

where $t$ is an integer. The time operator hence has a discrete spectrum: $\hat{T}|t; \psi\rangle = \hbar t |t; \psi\rangle$. We note that it is quantized in units of the Planck length because the above equation also contains the Newton constant, which is set to unity in this paper.

A more interesting observable is the perimeter $R^2 = 2\pi \xi_a \xi^a$, which is a Lorenz-invariant quantity defining the size of the shell. Because $\xi_a$ in (46) is defined as the left-invariant derivative on the group, its square is the Beltrami–Laplace operator on our momentum space: $\hat{R}^2|t; \psi\rangle = 2\pi |t; \Delta \psi\rangle$, where

$$\Delta = \hbar^2 \left( \frac{1}{\sinh 2\chi} \frac{\partial}{\partial \chi} \sinh 2\chi \frac{\partial}{\partial \chi} + \frac{t^2}{\cosh^2 2\chi} \right).$$

It was shown in [11] that this operator has two series of eigenvalues. One is continuous but separated from zero and corresponds to positive (i.e., spacelike) $R^2$: $\hat{R}^2|t, \lambda\rangle = 2\pi (\lambda^2 + 1) \hbar^2 |t, \lambda\rangle$, where $\lambda$ is real. The other is discrete but contains zero and corresponds to negative (i.e., timelike) $R^2$: $\hat{R}^2|t, l\rangle = -2\pi l(l+2) \hbar^2 |t, l\rangle$, where $l$ is a nonnegative integer satisfying the condition $l \leq t$.

Physically, the shell perimeter would be spacelike outside the horizon and timelike inside. In the (2+1)-dimensional gravity with no cosmological constant considered here, there is no event horizon. We can therefore say that only the continuous series is relevant in our model. Thinking about the entire universe being inside an event horizon is not compatible with any sensible boundary conditions at infinity.

On the other hand, we could consider a more complicated model with several concentric shells that admits space–times with no boundary. In such a situation, a shell perimeter could be timelike, and the discrete series is relevant.

Nevertheless, the result for the spectrum of $R^2$ provides a regularization near the $R=0$ singularity in both cases.

4.2. Physical states. We now try to solve the Hamiltonian constraint to obtain physical states. In contrast to the particle case, momentum (42) has a complicated dependence on the external parameter of the model, the bare mass $M$. In view of this complication, we use implicit expressions in this section.

The analogue of the Hamiltonian constraint for a particle is (43), which in terms of the Euler angles is $\cos \rho \cosh \chi = \cos(2\pi \sqrt{1 + M^2 - 2M \cosh \chi}) \equiv p_{-1}(\bar{\chi})$. This expression contains the additional parameter $\bar{\chi}$, of which the wave function of the kinematical Hilbert space is independent. To fix it, we can use any other equation in system (43), for example,

$$\sin \rho \cosh \chi = \frac{(1 - M) \cosh \bar{\chi} \sin(2\pi \sqrt{1 + M^2 - 2M \cosh \chi})}{\sqrt{1 + M^2 - 2M \cosh \chi}} \equiv p_0(\bar{\chi})$$

or

$$\sinh \chi = \frac{M \sinh \bar{\chi} \sin(2\pi \sqrt{1 + M^2 - 2M \cosh \chi})}{\sqrt{1 + M^2 - 2M \cosh \chi}} \equiv p_1(\bar{\chi}),$$

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where $p_{-1}(\bar{\chi})^2 + p_0(\chi)^2 - p_1(\bar{\chi})^2 = 1$. These constraints canonically commute with each other, i.e., they are first class. Hence, we can start with the kinematical Hilbert space as a space of functions of three parameters, $\Psi(\rho, \chi, \bar{\chi})$, and then impose two of the three constraints listed above.

We can write the solution as

$$\Psi(\rho, \chi, \bar{\chi}) = \delta(\cos \rho \cosh \chi - p_{-1}(\bar{\chi})) \delta(\sin \rho \cosh \chi - p_0(\bar{\chi})) \Psi(\chi),$$

where $\Psi(\chi)$ is an arbitrary function. As usual, it is not normalizable with respect to the kinematical Hilbert space.

The scalar product on the physical Hilbert space can be defined in terms of the functions $\Psi(\chi)$ in (47) as

$$\langle \Psi, \Phi \rangle_{\text{phys}} = \frac{1}{\pi} \int \sinh 2\chi d\chi d\rho d(\cosh \bar{\chi}) \delta(\cos \rho \cosh \chi - p_{-1}(\bar{\chi})) \delta(\sin \rho \cosh \chi - p_0(\bar{\chi})) \Psi(\chi)^* \Phi(\chi).$$

It is easy to show that in the limit $M \ll 1$ and $\chi \ll 1$, it reduces to the standard scalar product for the states of relativistic particles in 1+1 dimensions,

$$\langle \Psi, \Phi \rangle_{\text{phys}} = \frac{1}{2\pi} \int \frac{d\chi}{\sqrt{\chi^2 + M^2}} \Psi(\chi)^* \Phi(\chi),$$

as expected. In particular, it hence follows that in contrast to the (3+1)-dimensional situation, the spectrum of the energy–momentum of the model is fully continuous. This is unsurprising: in (2+1)-dimensional gravity, there is no Newtonian potential and therefore no potential well to hold the shell in a bounded region.

5. Conclusion

We do not yet have a full quantum theory of the model studied here. It remains to describe the dynamics, in particular, to calculate the transition amplitudes between different eigenvalues of the shell perimeter.

We have shown that the spectrum of the shell perimeter in the case of a timelike motion of the shell does not reach zero. This means that the naked singularity existing in the classical theory does not arise in the quantum theory. Regarding the singularity beyond an event horizon, which is classically attained by spacelike motion of the shell, it belongs to a discrete spectrum of the perimeter and is therefore also regularized.

The most interesting question is whether our results above can be generalized to (3+1)-dimensional gravity. There are some results for the quantum kinematics of a Schwarzschild black hole in the frame of a test particle [14], where it was shown that it also has the properties of noncommutativity and discreteness of the coordinates. But as it turns out, generalizing these results to a spherical shell dynamics as was done in Sec. 3 above is impossible, because the many-body problem in (3+1)-dimensional gravity is unsolvable.

On the other hand, almost all the works on a spherical shell mentioned in the introduction were considered in the (3+1)-dimensional case. Even the expression for the Hamiltonian constraint was found (although not in global coordinates). The map between the coordinates used in the phase space and the global coordinates should be analogous to the map found at the end of Sec. 3.3. The only difference is the presence of a Newtonian potential. Based on this, we can conjecture the form of the Hamiltonian constraint in global coordinates.

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