Two disjoint cycles in digraphs

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Abstract
Bermond and Thomassen conjectured that every digraph with minimum outdegree at least $2k - 1$ contains $k$ vertex disjoint cycles. So far the conjecture was verified for $k \leq 3$. Here we generalise the question asking for all outdegree sequences which force $k$ vertex disjoint cycles and give the full answer for $k \leq 2$.

KEYWORDS
cycles, digraphs, extremal graph theory

MATHEMATICAL SUBJECT CLASSIFICATION 2020
Primary: 05C35

1 | INTRODUCTION

In 1963, Corrádi and Hajnal [5] proved that every undirected graph with at least $3k$ vertices and minimum degree at least $2k$ contains $k$ vertex disjoint cycles. In 1981, Bermond and Thomassen [2] proposed an analogous conjecture for digraphs.

Conjecture 1.1. For every positive integer $k$ every digraph with minimum outdegree at least $2k - 1$ contains $k$ vertex disjoint cycles.

They also noted that the complete digraph on $2k - 1$ vertices shows that the bound offered by Conjecture 1.1 is optimal. For $k = 1$ the problem is easy and the case $k = 2$ was solved in 1983 by Thomassen [8]. More than two decades later Lichiardopol et al. [7] managed to solve the case $k = 3$ and for all $k > 3$ the problem is wide open. It is known, however, that Conjecture 1.1 holds for tournaments [3, 4].

The existence of some finite integer $f(k)$ such that every digraph of minimum outdegree at least $f(k)$ contains $k$ vertex disjoint cycles was established by Thomassen [8]. Later Alon [1] proved that it suffices to take $f(k) = 64k$.

In this paper, we generalise the question and ask for all degree sequences which force the existence of $k$ vertex disjoint cycles. For instance, our main result shows that degree sequences
of the form \((1, 3, 3, 3, 4, ...)\) force two disjoint cycles, whereas sequences of the form \((1, 3, 3, 3, 4, 4, ...)\) (with a sequence of fours at the end) do not.

**Definition 1.2.** For a nonempty and nondecreasing sequence \((d_1, ..., d_n)\) of nonnegative integers and a positive integer \(k\) the relation

\[(d_1, ..., d_n) \rightarrow k\]

means that every directed graph on \(n\) vertices with outdegree sequence \((d_1, ..., d_n)\) contains \(k\) vertex disjoint cycles. The failure of this statement is indicated by

\[(d_1, ..., d_n) \not\rightarrow k.\]

For example, Conjecture 1.1 asserts that \(d_1 \geq 2k - 1\) implies \((d_1, ..., d_n) \rightarrow k\). In analogy with Chvatal’s well-known graph Hamiltonicity theorem [6] one would hope that given \(k\) there is a somewhat satisfactory characterisation of all outdegree sequences forcing \(k\) disjoint cycles. Since minimum degree conditions are sometimes difficult to maintain in inductive proofs, one may even speculate whether such a strengthened form of Conjecture 1.1 could be easier to solve than the original question. For \(k = 1\) such a characterisation is easily obtained (see Lemma 2.1). The case \(k = 2\) requires the following concept.

**Definition 1.3.** Let integers \(1 \leq r \leq s \leq n\) be given. A nondecreasing sequence \((d_1, ..., d_n)\) satisfying

(a) \(d_r \geq r, d_s \geq s + 1,\) and

(b) if \(n \geq 2s - r + 2\) and \(d_{2s-r+2} = s + 1,\) then there is an integer \(j \in [2s - r + 3, n]\) such that \(d_j \geq j\)

is called \((r, s)\)-large. We say that \((d_1, ..., d_n)\) is large if it is \((r, s)\)-large for some two integers \(r \leq s\) in \([n]\).

Our main theorem says that being large and forcing two vertex disjoint cycles are equivalent properties of outdegree sequences.

**Theorem 1.4.** Let \((d_1, ..., d_n)\) be a nonempty, nondecreasing sequence of nonnegative integers. The relation

\[(d_1, ..., d_n) \rightarrow 2\]

holds if and only if the sequence \((d_1, ..., d_n)\) is large.

Note that every nondecreasing outdegree sequence \((d_1, ..., d_n)\) with \(d_1 \geq 3 = 2 \cdot 2 - 1\) is \((1, 1)\)-large. So the case \(k = 2\) of Conjecture 1.1 is a straightforward consequence of Theorem 1.4. It would be very interesting to find a similar result for three cycles.
1.1 Notation

A directed graph $D$, also called a digraph, is a pair $(V(D), E(D))$, where $V(D)$ is a set of vertices and $E(D)$ is a set of ordered pairs of vertices called arcs. Digraphs considered here may have loops and 2-cycles, but no parallel arcs. If an arc $x \rightarrow y$ is present, we say that $x$ dominates $y$, and $x$ dominates a set $W \subseteq V(D)$ if it dominates each vertex of $W$. The in-neighbourhood of $x \in V(D)$ is the set of all vertices dominating $x$ and the number such vertices is called the indegree of $x$. Similarly, the out-neighbourhood and outdegree of $x$ are defined to be, respectively, the set of all vertices dominated by $x$ and the number of them.

If $U \subseteq V(D)$ we write $D[U]$ for the subdigraph of $D$ induced by $U$ and $D - U$ for the digraph obtained by deleting $U$ (and all arcs starting or ending in $U$).

By a cycle we always mean a directed cycle, that is an oriented path starting and ending at the same vertex. A cycle of length $\ell$ is an $\ell$-cycle. A 1-cycle is a loop. In what follows, the outdegree sequence of every digraph is nondecreasing.

A transitive tournament $T_n$ is a digraph whose vertex set can be enumerated such that $V(T_n) = \{v_1, v_2, ..., v_n\}$ and $E(T_n) = \{v_i \rightarrow v_j : v_i, v_j \in V(T_n) \text{ and } i > j\}$.

2 Preliminary Results

We begin by describing all outdegree sequences of vertices of a digraph that force the existence of a cycle.

**Lemma 2.1.** The statement $(d_1, ..., d_n) \rightarrow 1$ is true if and only if for some $j \in [n]$ the inequality $d_j \geq j$ holds.

**Proof.** The transitive tournament $T_n$ exemplifies $(0, 1, ..., n - 1) \not\rightarrow 1$. So deleting some arcs from $T_n$ one can show $(d_1, ..., d_n) \not\rightarrow 1$ whenever $d_j < j$ holds for all $j \in [n]$. In other words, the positive relation $(d_1, ..., d_n) \rightarrow 1$ entails $d_j \geq j$ for some $j \in [n]$.

If, conversely, $d_j \geq j$ holds for some $j \in [n]$, we delete the vertices with outdegrees $d_1, ..., d_{j-1}$. The minimum outdegree of the remaining digraph is at least 1 and, hence, it contains a cycle. □

Next we show that deleting one term from a sequence cannot destroy its largeness.

**Fact 2.2.** Let integers $1 \leq r \leq s < n$ be given. If a nondecreasing sequence $\bar{d} = (d_1, ..., d_n)$ with $d_n < n$ is $(r, s)$-large, then every sequence $\bar{e} = (e_1, ..., e_{n-1})$ obtained from $\bar{d}$ by deleting one arbitrary term is also $(r, s)$-large.

**Proof.** Since $s \leq n - 1$ and $e_i \geq d_i$ for all $i \in [n - 1]$, the sequence $\bar{e}$ satisfies (a). To show that it also satisfies (b), assume $n - 1 \geq 2s - r + 2$ and $e_{2s-r+2} = s + 1$. This implies $s + 1 \leq d_i \leq d_{2s-r+2} \leq e_{2s-r+2} = s + 1$, and therefore, in view of (b) applied to $\bar{d}$, there is an integer $j \in [2s - r + 3, n]$ such that $d_j \geq j$. Because of $d_n < n$ we infer $j \leq n - 1$, which yields $e_j \geq d_j \geq j$, as required. □
3 | PROOF OF THEOREM 1.4

Our goal in this section is to establish Theorem 1.4. We split the proof into two parts, one for each direction of the equivalence.

3.1 | The forward implication

Suppose first that \((d_1, \ldots, d_n) \to 2\). We are to prove that the sequence \((d_1, \ldots, d_n)\) is \((r, s)\)-large for appropriate natural numbers \(r\) and \(s\). The digraph \(T^+_n\) depicted in Figure 1 shows \((1, \ldots, n) \to 2\).

Therefore for every sequence \((d_1, \ldots, d_n)\) satisfying \(d_j \leq j\) for each \(j \in [n]\), one can construct a digraph with the outdegree sequence \((d_1, \ldots, d_n)\), which additionally does not contain two disjoint cycles, by removing some arcs from \(T^+_n\). Hence we may assume that this is not the case and thus there is a smallest number \(s \in [n]\) such that \(d_s \geq s + 1\). Now the set

\[X = \{i \in [s] : d_i \geq i\}\]

cannot be empty and there exists \(r = \min(X)\). The numbers \(r\) and \(s\) obey condition (a) of Definition 1.3.

Now the only possibility for \((d_1, \ldots, d_n)\) of not being large is that the numbers \(r\) and \(s\) fail the condition (b), namely that \(n \geq 2s - r + 2, d_{2s-r+2} = s + 1, \) and \(d_j < j\) for every \(j \in [2s - r + 3, n]\). The digraph \(D\) shown in Figure 2 exemplifies

\[(0, 1, \ldots, r - 2, r, r + 1, \ldots, s - 1, s + 1, \ldots, s + 1, 2s - r + 2, \ldots, n - 1) \to 2,\]

\[s-r+3\]

**Figure 1** A transitive tournament \(T_{n-1}\) joined by 2-cycles to a single vertex \(v\) with a loop. An arc from/to a box goes from/to every vertex of the box. Each directed cycle in this digraph goes through \(v\).

**Figure 2** A digraph \(D\) obtained from three transitive tournaments, \(T_{s-r}\), \(T_{r-1}\), \(T_{n-2s+r-2}\), one directed cycle \(C_{s-r+2}\) and one single vertex \(v\). An arc from/to a box goes from/to every vertex of the box. Each directed cycle different from \(C_{s-r+2}\) in \(D\) goes through \(V(C_{s-r+2})\) and \(v\). The outdegree sequence of vertices of \(D\) is described.
and thus again we can construct a digraph with the outdegree sequence \((d_1, ..., d_n)\) and without two disjoint cycles, by removing some arcs from \(D\). This final contradiction ends the first part of the proof.

### 3.2 The backwards implication

In this section, we shall show that if \((d_1, ..., d_n)\) is large, then \((d_1, ..., d_n) \to 2\). Arguing indirectly, we consider a counterexample, given by a directed graph \(D = (V, E)\) on \(n\) vertices whose outdegree sequence \((d_1, ..., d_n)\) is \((r, s)\)-large for two integers \(r \leq s\) in \([n]\), but which fails to contain two disjoint cycles. We may assume that the triple \((D, r, s)\) has been chosen in such a way that \(|V| + |E| + s\) is as small as possible. The desired contradiction will emerge after nine preliminary claims.

**Claim 3.1.** We have \(r = 1\).

**Proof.** Otherwise delete the vertices with degrees \(d_1, ..., d_{r-1}\) from \(D\), thus obtaining a smaller digraph \(D^*\). Observe that for every \(r \leq j \leq n\) there are at least \(n - (j - 1)\) vertices in \(D^*\) with outdegree at least \(d_j - (r - 1)\). Therefore, the nondecreasing outdegree sequence \((e_1, ..., e_{n-r+1})\) of \(D^*\) has the property \(e_k \geq d_{k+r-1} - (r - 1)\) holds for every \(k \in [n - r + 1]\). Consequently, \((D^*, 1, s - r + 1)\) is another counterexample to our claim that contradicts the minimality of \((D, r, s)\). □

Notice that now our conditions (a) and (b) read

(i) \(d_1 \geq 1, d_s \geq s + 1\), and

(ii) if \(n \geq 2s + 1\) and \(d_{2s+1} = s + 1\), then there is an integer \(j \in [2s + 2, n]\) such that \(d_j \geq j\).

The pointwise minimal possibilities for an outdegree sequence with these properties are illustrated in Figure 3.

The minimality of \(|E|\) tells us that \((d_1, ..., d_n)\) is in one of these three cases, whence

(iii) If \(i \in [s - 1]\), then \(d_i = 1\).

(iv) If \(i \in [s, \min(2s, n)]\), then \(d_i = s + 1\).

**Claim 3.2.** There are no loops in \(D\). In particular, \(d_n < n\).

\[
\begin{align*}
(1, \ldots, 1, s + 1, \ldots, s + 1); & \text{ if } n \leq 2s; \\
(1, \ldots, 1, s + 1, \ldots, s + 1, d_{2s+1}, \ldots, d_n); & \text{ if } n > 2s \text{ and } d_{2s+1} = s + 1; \\
(1, \ldots, 1, s + 1, \ldots, s + 1, d_{2s+1}, \ldots, d_n); & \text{ if } n > 2s \text{ and } d_{2s+1} \neq s + 1.
\end{align*}
\]

**Figure 3** All the possible outdegree sequences when \(r = 1\).
Proof. Assume that $D$ possesses a loop at some vertex $x$. Then by (i) the directed graph $D^* = D - x$ has at most $s - 1$ vertices whose outdegree is smaller than $s$ and, moreover, as $n \geq d_n \geq s + 1$, it has a further vertex whose outdegree is at least $s$. So by Lemma 2.1 $D^*$ contains a cycle, which together with the loop at $x$ yields two disjoint cycles in $D$. 

Combining the above claim with (i) one gets $s + 1 \leq d_n < n$, and thereby

$$n \geq s + 2. \quad (1)$$

Claim 3.3. Every 2-cycle of $D$ is dominated by a vertex of outdegree $s + 1$.

Proof. Note that (1) and (i) tell us that $D$ has at most $s - 1$ vertices of outdegree smaller than $s + 1$ and at least three vertices whose outdegrees are at least $s + 1$. Therefore, if $x \leftrightarrow y$ is a 2-cycle in $D$ not dominated by any vertex of outdegree $s + 1$, then the digraph $D^* = D - \{x, y\}$ has at most $s - 1$ vertices whose outdegree is smaller than $s$ and at least one further vertex whose outdegree is at least $s$. So by Lemma 2.1 $D^*$ contains a cycle, which together with the 2-cycle $x \leftrightarrow y$ yields two disjoint cycles in $D$. \hfill \Box

Claim 3.4. Suppose that an arc $x \rightarrow y$ of $D$ does not appear in a 2-cycle.

(1) There is some vertex $a \not\in \{x, y\}$ dominating $x$ and $y$.
(2) If the outdegree of $x$ is 1, then at least $s + 1$ vertices distinct from $y$ and having outdegree $s + 1$ dominate $xy$.

Proof. We construct a digraph $D^*$ from $D - x$ by adding all arcs of the form $z \rightarrow y$, $z \in V(D^*) \setminus \{y\}$, whenever $z \rightarrow y \not\in E(D)$ and $z \rightarrow x \in E(D)$ (see Figure 4). Plainly $D^*$ cannot contain two disjoint cycles. Observe that the only vertices whose outdegree is smaller in $D^*$ than in $D$ are those dominating $\{x, y\}$. Their outdegree dropped by one.

If (1) fails, then the outdegree sequence of $D^*$ is the same as that of $D$ with the term corresponding to $x$ removed and thus, in view of Fact 2.2 and Claim 3.2, it is $(r, s)$-large, meaning that $(D^*, r, s)$ contradicts the supposed minimality of $(D, r, s)$.

Let us now suppose that $x$ has outdegree 1. Due to (iv) this is only possible if $s > 1$. By (iii) the digraph $D^*$ has $s - 2$ vertices whose outdegree is 1, while (iv) shows that each other vertex has the outdegree at least $s$. Now the only possibility for the triple $(D^*, 1, s - 1)$ not to contradict the supposed minimality of $(D, 1, s)$ is that clause (ii) fails, which means, in particular, that $n - 1 \geq 2(s - 1) + 1$, that is, $n \geq 2s$, and that $d_{2s-1} = s$ meaning $D^*$ has $(2s - 1) - (s - 2) = s + 1$ vertices of outdegree $s$. In $D$ these vertices need to have outdegree $s + 1$ and thus they dominate $\{x, y\}$. This proves (2). \hfill \Box

\textsc{Figure 4} New arcs in $D^*$. 
Claim 3.5. The inneighbourhood of every vertex of $D$ contains a cycle.

Proof. Let $y$ denote any vertex of $D$ and let $N$ be its inneighbourhood.

Assume first that $N = \emptyset$. Then the outdegree sequence of $D^* = D - y$ is obtained from that of $D$ by removing the term corresponding to $y$ and, therefore, in view of Fact 2.2 and Claim 3.2, is $(1, s)$-large. Thus, the triple $(D^*, 1, s)$ contradicts the minimality of $(D, 1, s)$. Thereby we have shown $N \neq \emptyset$.

If each vertex of $D[N]$ has indegree at least one, then Lemma 2.1 yields the existence of a cycle in $D[N]$, proving the claim. Therefore, we can assume that there is a vertex $x \in N$ whose indegree in $D[N]$ is zero.

By Claim 3.3 this is only possible if there is no arc from $y$ to $x$, but this contradicts Claim 3.4(1).

□

Claim 3.6. There is a 2-cycle in $D$.

Proof. Otherwise, in view of Claim 3.5, each vertex of $D$ has indegree at least 3. Let $D^* = (V, E^*)$ be a directed graph arising from $D$ by first reversing all arrows and then deleting an arbitrary arc. Its outdegree sequence $(e_1, \ldots, e_n)$ has the properties $e_1 \geq 2$ and $e_3 \geq e_2 \geq 3$ and thus it is $(1, 1)$-large. Since $|V| + |E^*| + 1 < |V| + |E| + s$, the minimality of $(D, r, s)$ tells us that $D^*$ and thus also $D$ contains two disjoint cycles, which is a contradiction.

□

Claim 3.7. In $D$ there is a directed cycle $C$ such that all its vertices have outdegree $s + 1$ and there is a vertex $x \notin V(C)$ connected to every vertex of $C$ by a 2-cycle.

Proof. Let us first assume that $D$ has a vertex $x$ belonging to all 2-cycles. Define

$$A = \{z \in V : z \neq x \text{ and } x \leftrightarrow z \text{ is a 2-cycle}\}$$

and

$$B = \{z \in A : \text{the outdegree of } z \text{ is } s + 1\}.$$

Claim 3.6 informs us that $A \neq \emptyset$. Consider any $a \in A$. By Claim 3.3 the 2-cycle $a \leftrightarrow x$ is dominated by some vertex $b \notin \{a, x\}$ whose outdegree is $s + 1$. The inneighbourhood of $b$ contains some cycle by Claim 3.5. This cycle cannot be disjoint to $a \leftrightarrow x$ and thus there is an arc from $a$ or $x$ to $b$. The former is impossible, for then the 2-cycle $a \leftrightarrow b$ would not pass through $x$ and for this reason we must have $b \in B$ (see Figure 5).

We have thus shown that every member of $A$ has an inneighbour in $B \subseteq A$. In particular, $B \neq \emptyset$ and in the restriction $D[B]$ every vertex has indegree at least 1. So, in view of Lemma 2.1, there is some cycle $C$ in $B$ and the claim follows in this case.

![Figure 5](image-url) A 2-cycle $a \leftrightarrow x$ and a vertex $b$ dominating $[a, x]$. 
From now on we may assume that $D$ does not have a vertex belonging to all of its 2-cycles. This means that the undirected graph $G = (V, E_G)$ the edges of which correspond to the 2-cycles of $D$ is not the disjoint union of a (possibly degenerate) star and isolated vertices. As $G$ cannot have two independent edges either, it has to be the disjoint union of a triangle and isolated vertices. In other words, there are three distinct vertices $x$, $y$, and $z$ of $D$ such that $x \leftrightarrow y$, $x \leftrightarrow z$, and $y \leftrightarrow z$ are 2-cycles.

If both $y$ and $z$ have outdegree $s + 1$ the claim still holds with $y \leftrightarrow z$ playing the rôle of $C$. So let us finally assume that the outdegree of $y$ is not $s + 1$. Applying Claim 3.3 to $x \leftrightarrow z$ we get a vertex $a \not\in \{x, y, z\}$ dominating $x$ and $z$. Since the cycle contained in the inneighbourhood of $a$ has to intersect $\{x, z\}$ we may further suppose that $x$ dominates $a$. But now $a \leftrightarrow x$ and $y \leftrightarrow z$ are two disjoint 2-cycles in $D$ (see Figure 6).

From now on, we fix $C$ and $x$ as obtained by the previous claim. Observe that the cycle $C$ has to be induced, for otherwise $D$ would contain two disjoint cycles.

**Claim 3.8.** We have $s > 1$.

**Proof:** Assume $s = 1$. If $D$ has no vertices besides $x$ and those on $C$, then the outdegree sequence of $D$ has to be

\[
(2, \ldots, 2, n - 1)
\]

for some $n \geq 3$, contrary to (ii). So there is a vertex $y \neq x$ not lying on $C$. By Claim 3.5 the inneighbourhood of $y$ contains a cycle. This cycle needs to have a common vertex $z$ with $C$. But now $x$, $y$ and the successor of $z$ on $C$ are three distinct outneighbours of $z$, contrary to the fact that the members of $C$ have outdegree $s + 1 = 2$ (see Figure 7).

**Claim 3.9.** If $n \geq 2s + 1$, then $d_{2s+1} \geq s + 2$. In particular, $D$ has at most $s + 1$ vertices of outdegree $s + 1$.

**FIGURE 6** 2-cycles $x \leftrightarrow y$, $y \leftrightarrow z$, $y \leftrightarrow x$ in $D$ and a vertex $a$ dominating $\{x, z\}$.

**FIGURE 7** The cycle $C$ and the vertices $x$ and $y$. 
Proof. Assume for the sake of contradiction that \( n \geq 2s + 1 \) and \( d_{2s+1} < s + 2 \). Since \( d_{2s+1} \geq d_j = s + 1 \), we have \( d_{2s+1} = s + 1 \), meaning that (ii) yields the existence of some \( j \in [2s + 2, n] \) with \( d_j \geq j \). Let \( D^* \) be the digraph that arises from \( D \) if its \( s - 1 \) vertices of outdegree 1 are removed. The nondecreasing outdegree sequence \( (e_1, \ldots, e_{n-s+1}) \) of \( D^* \) satisfies \( e_1 \geq 2 \) and \( e_{j-s+1} \geq j - s + 1 \). Since \( j - s + 1 \geq s + 3 \geq 4 \) this tells us that the triple \((D^*, 1, 1)\) violates the minimality of \( (D, 1, s) \). \( \square \)

Notice that the graph \( D - V(C) \) cannot contain a cycle. Since it is not empty, it needs to contain some vertex \( z \) without any outneighbours by Lemma 2.1.

If \( z \) had outdegree 1 in \( D \), then \( z \neq x \) and its unique outneighbour \( a \) had to lie on \( C \). In particular, the outdegree of \( a \) had to be \( s + 1 \).

Moreover observe that an arc \( z \to a \) does not appear in a 2-cycle because otherwise we could find two disjoint 2-cycles in \( D \), namely \( z \leftarrow a \) and \( x \leftarrow b \), where \( b \in V(C) \setminus \{a\} \). Therefore, in view of Claim 3.4(2), \( D \) had at least \( s + 2 \) vertices the outdegree of which is \( s + 1 \), contrary to Claim 3.9.

By (iii) and (iv), this tells us that the outdegree \( \ell \) of \( z \) in \( D \) is consequently at least \( s + 1 \). Since \( z \) has no outneighbours in \( D - V(C) \), the length of \( C \) needs to be at least \( \ell \) and possibly together with \( z \) itself this yields \( s + 2 \) vertices of \( D \) whose outdegree is \( s + 1 \). We have thereby reached the final contradiction that concludes the proof of Theorem 1.4.

ACKNOWLEDGEMENT

We would like to thank both referees for reading this article very carefully and for their helpful suggestions.

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How to cite this article: M. Lewandowski, J. Polcyn and C. Reiher, Two disjoint cycles in digraphs, J. Graph Theory. 2023;104:461–469. https://doi.org/10.1002/jgt.22972