PATH SETS IN ONE-SIDED SYMBOLIC DYNAMICS

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ABSTRACT. Path sets are spaces of one-sided infinite symbol sequences associated to
pointed graphs $(G, v_0)$, which are edge-labeled directed graphs with a distinguished vertex
$v_0$. Such sets arise naturally as address labels in geometric fractal constructions and in
other contexts. The resulting set of symbol sequences need not be closed under the one-
sided shift. This paper establishes basic properties of the structure and symbolic dynamics
of path sets, and shows that they are a strict generalization of one-sided sofic shifts.

1. INTRODUCTION

This paper investigates the concept of a path set, which is a notion in one-sided sym-
bolic dynamics. Path sets form an enlargement of the class of one-sided sofic shifts which
includes certain additional closed sets not invariant under the one-sided shift. Let $\mathcal{A}^\mathbb{N}$
denote the full one-sided shift space on the finite alphabet $\mathcal{A}$, topologized with the product
topology. It is a compact, completely disconnected topological space. Path sets are a distin-
guished class of closed subsets of $\mathcal{A}^\mathbb{N}$ constructed as follows. We are given a finite directed
graph $G$ with edges bearing labels from $\mathcal{A}$ with a marked vertex $v$. A path set prescribed
by the data $(G, v)$ is the collection of one-sided infinite sequences of edge labels assigned
to all infinite paths in a finite directed graph $G$ emanating from the vertex $v$ of $G$. It is easy
to see that path sets are closed subsets of $\mathcal{A}^\mathbb{N}$. We denote the collection of all path sets on
the alphabet $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$.

One reason for studying path sets is that they naturally arise in connection with geo-
metric constructions of fractals and limit sets of discrete groups. Associated to these con-
structions are address maps, given by paths in finite directed graphs, which specify labels
of points in the limit sets of such recursive constructions. Such address maps arise in geo-
metric graph-directed constructions (Mauldin and Williams [32], Edgar [15, Sect. 4.3]),
in iterated function systems (Barnsley [10, Sec. 4.1]), in describing limit sets of various
discrete group actions (Mauldin and Urbanski [31]) and in describing boundaries of frac-
tal tiles (Akiyama and Lorident [4]). Under some conditions addresses are unique, but in
other circumstances multiple addresses label the same geometric point. There are known
conditions, such as the “open set condition” under which almost all point have a unique
address (Bandt et al [8]). The symbolic dynamics objects we study here are the complete
sets of distinct addresses given by an address map, whether or not addresses are unique.

Path sets are not a new concept; they have previously appeared in the symbolic dynam-
ic literature under the name “follower set,” typically as an auxiliary construction. The
usual framework of coding in symbolic dynamics, as given in Lind and Marcus [29], re-
stricts to shift-invariant sets (ones with $\sigma(X) \subset X$) and emphasizes two-sided dynamics;
shift-invariant sets in the one-sided case are considered in Kitchens [23] for shifts of finite

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type. Path sets are objects in one-sided dynamics that are not always invariant under the one-sided shift map; the initial condition imposed by the marked vertex typically breaks shift-invariance. Their distinctive properties related to lack of shift invariance seem not to have been studied in any detail. This paper proposes the terminology path sets, which is consistent with language used in fractal constructions (15, Sec. 4.3), because “follower set” is used in the symbolic dynamics literature with several different meanings, see the discussion in Section 1.2. An alternate descriptive term for path set could be pointed follower set.

The object of this paper is to establish basic properties of path sets in a relatively self-contained manner. It particularly addresses those properties connected to symbolic dynamics and the action on them of the one-sided shift map $\sigma : \mathcal{A}^\mathbb{N} \rightarrow \mathcal{A}^\mathbb{N}$ which sends $\sigma(a_0a_1a_2, \cdots) = (a_1a_2a_3, \cdots)$, with each $a_i \in \mathcal{A}$. We show that path sets form a strict generalization of one-sided sofic shifts on the alphabet $\mathcal{A}$, and characterize sofic shifts as exactly the (one-sided) shift-invariant members of $\mathcal{C}(\mathcal{A})$. Members of $\mathcal{C}(\mathcal{A})$ retain the good set-theoretic and topological entropy properties of one-sided sofic shifts. The enlargement from the class of sofic shifts to the class of path sets gives a class closed under a larger set of operations than for sofic shifts; in particular when the alphabet has $g$ symbols $\mathcal{A} = \{0, 1, 2, \ldots, g-1\}$ the class $\mathcal{C}(\mathcal{A})$ is closed under the $g$-adic arithmetic operations of adding and multiplying by $(g$-integral) rational numbers in the sense of Mahler [30], see our paper [1].

One-sided sofic shifts have two different types of graph presentations, one as a sofic shift graph (represented by paths starting from any vertex) and a second as a path set graph (represented by paths starting from a fixed vertex). There is typically a cost associated with encoding an initial condition in the presentation of a sofic shift. We show that the minimal path set presentation of a sofic system (measured by number of vertices in the graph) requires at most an exponential increase in its size relative to its minimal presentation as a sofic system, and present examples showing that such an exponential increase must sometimes occur. We also relate the notion of path set to several similar concepts, and discuss applications of this concept in fractal constructions.

1.1. Main Results. A pointed graph $(\mathcal{G}, v)$ over a finite alphabet $\mathcal{A}$ comprises a finite edge-labeled directed graph $\mathcal{G} = (\mathcal{G}, \mathcal{E})$ and a distinguished vertex $v$ of the underlying directed graph $\mathcal{G}$. The directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is specified by its vertex and (directed) edge sets $\mathcal{E}$ with edges $e = (v_1, v_2) \in \mathcal{V} \times \mathcal{V}$, and the data $\mathcal{E} \subset \mathcal{E} \times \mathcal{A}$ specifies the set of labeled edges $(e, \ell)$, with labels drawn from the alphabet $\mathcal{A}$. We allow loops and multiple edges, but require that all triples $(e, \ell) = (v_1, v_2, \ell)$ be distinct. The results of this paper regard the alphabet $\mathcal{A}$ as fixed, unless specifically noted otherwise.

Definition 1.1. The path set (or pointed follower set) $\mathcal{P} = X_\mathcal{G}(v)$ specified by a pointed graph $(\mathcal{G}, v)$ is the subset of $\mathcal{A}^\mathbb{N}$ made up of the symbol sequences of successive edge labels of all possible one-sided infinite walks in $\mathcal{G}$ issuing from the distinguished vertex $v$. We let $\mathcal{C}(\mathcal{A})$ denote the collection of all path sets using labels from the alphabet $\mathcal{A}$. Many different $(\mathcal{G}, v)$ may give the same path set $\mathcal{P} \subset \mathcal{A}^\mathbb{N}$, and we call any such $(\mathcal{G}, v)$ a presentation of $\mathcal{P}$.

This definition detects paths without counting multiplicity of occurrence. Therefore without loss of generality we may suppose that the labelled graph $\mathcal{G}$ has the non-redundancy property that each labeled edge datum $((v_1, v_2), \ell)$ occurs at most once in the graph.

We first summarize behavior of path sets under set-theoretic operations.
Theorem 1.2. (Set operations on Path Sets) Path sets on a fixed alphabet $A$ have the following properties.

1. Each path set $P \in \mathcal{C}(A)$ is a closed subset of $A^\mathbb{N}$, carrying the product topology.
2. If $P_1$ and $P_2$ are path sets, then so is $P_1 \cap P_2$.
3. If $P_1$ and $P_2$ are path sets, then so is $P_1 \cup P_2$.

The complement $X \setminus P$ of a path set $P$ inside a given $X = A^\mathbb{N}$ with fixed alphabet $A$ need not be a path set. Thus the collection $\mathcal{C}(A)$ of path sets with fixed alphabet $A$ does not form a Boolean algebra of sets.

Second, we study the action of the one-sided shift operator on path sets, and in doing so relate path sets to sofic shifts in symbolic dynamics. Recall that the one-sided shift map $\sigma : A^\mathbb{N} \to A^\mathbb{N}$ acts on the semi-infinite symbol sequences by

$$\sigma(a_0a_1a_2a_3\cdots) := (a_1a_2a_3\cdots).$$

The notion of a sofic system was originally introduced by Weiss [37] in 1973, in the context of two-sided sequences, and much of the literature treats the two-sided case, particularly Lind and Marcus [29, Chap. 3]. Here we consider a one-sided version of this concept, defined as follows.

Definition 1.3. A one-sided sofic shift is a subset $Y \subset A^\mathbb{N}$ specifying all possible sequences of labels of one-sided infinite walks along a finite edge-labeled directed graph $\mathcal{G}$, starting from any vertex of the graph.

This definition immediately implies that a one-sided sofic shift $Y$ is invariant under the one-sided shift map, i.e. $\sigma(Y) \subseteq Y$.

Ashley, Kitchens and Stafford [7] previously introduced a notion of one-sided sofic shift, defining it as a symbolic dynamical system having the property of finiteness of the collection of all possible finite follower sets (as defined in Section 1.2 below). Appendix B of this paper shows that their definition is equivalent to the definition above.

We show that the notion of a path set is a strict generalization of a one-sided sofic shift.

Theorem 1.4. (One-sided Shift action on Path Sets)

1. For any path set $P$, the shifted set $\sigma(P)$ is also a path set.
2. The (one-sided) shift closure $\overline{P} = \bigcup_{j \in \mathbb{N}} \sigma^j(P)$ of a path set is a path set.
3. Every shift-invariant path set is a one-sided sofic shift, and conversely.

This result raises a realizability problem when treating sofic shifts as path sets. The realizability problem is: Given a sofic shift $Y$ with a labeled graph presentation $\mathcal{G}$, construct a pointed graph presentation $(\mathcal{H}, v')$ for it as a path set. One such presentation can be obtained by a standard construction given in Theorem 1.2, where we show that every path set has a right-resolving presentation. In this construction the new graph presentation $(\mathcal{H}, v')$ obtained may be exponentially larger in size (as measured by the number of vertices) than the original presentation $\mathcal{G}$ of the sofic shift. We show that exponential blow-up in size is sometimes unavoidable in general, in that there exists a family of sofic shifts $Y$ whose minimal right-resolving presentations as a path set $(\mathcal{H}, v_{0b})$ requires an exponentially larger number of states than its minimal right-resolving presentation as a sofic shift, see Example 3.4.

Third, we show closure of path sets under a decimation (or fractionation) operation. Given $j \geq 0$, $m \geq 1$ and define the decimation map $\psi_{j,m} : A^\mathbb{N} \to A^\mathbb{N}$ by

$$\psi_{j,m}(a_0a_1a_2\cdots) := (a_ja_{j+m}a_{j+2m}\cdots).$$
The decimation operation extracts the digits of the path set in a specified infinite arithmetic progression of indices.

**Theorem 1.5.** (Decimation of Path Sets)

1. For any path set $\mathcal{P}$, and any $(j, m)$ with $j \geq 0$ and $m \geq 1$, the decimated set $\mathcal{P}_{j,m} := \psi_{j,m}(\mathcal{P}) = \{ \psi_{j,m}(x) : x \in \mathcal{P} \}$ is a path set.

2. Suppose that the path set $\mathcal{P}$ is shift-invariant. Then every decimated set $\mathcal{P}_{j,m}$ is shift-invariant. In addition, for fixed $m$ all the sets $\mathcal{P}_{j,m}$ for $j \geq 0$ are equal. (We may use the abbreviated notation $\mathcal{P}_m$ in this case.)

We make some further definitions associated with decimation operations.

**Definition 1.6.** For a fixed $m \geq 1$, the $m$-kernel of path set $\mathcal{P}$ is the collection of all path sets $Ker_m(\mathcal{P}) := \{ \psi_{j,m}(\mathcal{P}) : j \geq 0, k \geq 0 \}$.

A priori, the number of distinct path sets in this collection $Ker_m(\mathcal{P})$ could be finite or infinite, depending on $\mathcal{P}$.

**Definition 1.7.** A path set $\mathcal{P}$ is an $m$-automatic path set for a given $m \geq 2$ whenever its $m$-kernel $Ker_m(\mathcal{P})$ is a finite collection of path sets.

This definition is formulated in analogy to that of $m$-automatic sequence, as given in Allouche and Shallit [6]. The property of being an $m$-automatic sequence is a property of a single infinite sequence (e.g. of a single path), while our definition above concerns a property of a set of paths. In our terminology an earlier result of Cobham [12] established: A single infinite sequence is an $m$-automatic sequence (in the sense of [6]) if and only if its $m$-kernel is finite (cf. Allouche and Shallit [5]). Note that in general the shift closure of an $m$-automatic sequence need not be a path set: The Thue-Morse sequence [6, p. 152] is a 2-automatic sequence with $\mathcal{A} = \{0, 1\}$, but its shift closure, the Morse shift (cf. [29] pp. 457–459), is not a path set.$^1$

The full shift on a fixed alphabet is an $m$-automatic path set for every $m \geq 1$. We leave it as an open problem to characterize all $m$-automatic path sets.

Fourth, we characterize path sets in terms of a finiteness property under the shift operation and intersection.

**Definition 1.8.** For an alphabet $\mathcal{A}$ and $j \in \mathcal{A}$, define the prefix set $Z_j := j\mathcal{A}^\infty$ to be the closed set of all sequences whose initial digit is $j$ and whose subsequent digits are arbitrary.

It is easy to see that each prefix set $Z_j$ is a path set. Using these sets we obtain the following structural characterization of path sets.

**Theorem 1.9.** (Structure Theorem for Path Sets) The following are equivalent.

1. $\mathcal{P}$ is a path set in $\mathcal{C}(\mathcal{A})$.
2. $\mathcal{P}$ is a closed subset of $\mathcal{A}^\infty$ that has the property that there is some finite collection of subsets of $\mathcal{P}$ which contains $\mathcal{P}$ and which is closed under the operations:
   (a) apply the one-sided shift,
   (b) intersect with a prefix set $Z_j$, for any $j \in \mathcal{A}$.

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$^1$The Morse shift is known to be a minimal shift and to not contain any periodic word. Each path set contains an eventually periodic path, so a shift-invariant path set always contains a periodic word.
This finiteness property differs from the finiteness property imposed in the definition of $m$-automatic path set above.

Fifth, we study the notion of entropy for path sets. Since these sets need not be shift-invariant, a priori there are two distinct notions of topological entropy, as follows.

**Definition 1.10.** (1) The *path topological entropy* of a path set $\mathcal{P}$ is

$$H_p(\mathcal{P}) := \limsup_{n \to \infty} \frac{1}{n} \log N_n^i(\mathcal{P}),$$

where $N_n^i(\mathcal{P})$ denotes the number of distinct initial blocks of length $n$ from $\mathcal{P}$.

(2) The *topological entropy* of a path set $\mathcal{P}$ is

$$H_{\text{top}}(\mathcal{P}) := \limsup_{n \to \infty} \frac{1}{n} \log N_n(\mathcal{P}),$$

where $N_n(\mathcal{P})$ is the number of distinct blocks of length $n$ occurring anywhere in the symbol sequences in $\mathcal{P}$.

These definitions are independent of the alphabet $\mathcal{A}$ in which the path set $\mathcal{P}$ is viewed as belonging. The definition (2) above corresponds to the definition of topological entropy for shift spaces (Adler and Marcus [3]), given by the shift closure $\overline{\mathcal{P}}$. In fact these two notions of topological entropy agree.

**Theorem 1.11.** (Topological Entropy Equivalence) For a path set $\mathcal{P}$,

$$H_p(\mathcal{P}) = H_{\text{top}}(\mathcal{P}) = H_{\text{top}}(\overline{\mathcal{P}}).$$

All lim sup’s in the definitions are attained as limits.

Since the topological entropy of a sofic shift is well understood, this allows us to compute the path topological entropy $H_p(\mathcal{P})$, using a suitably chosen presentation of $\mathcal{P}$. A presentation $(\mathcal{G}, v)$ of a path set is reachable if each vertex of $\mathcal{G}$ can be reached by a directed path from $v$. We recall a standard definition.

**Definition 1.12.** A labeled directed graph $\mathcal{G}$ is right-resolving if from each vertex of $\mathcal{G}$, all the exit edges have distinct labels.

In Theorem 3.2 in Section 3 we show that every path set has a reachable presentation that is right-resolving. Our final result gives conditions where the standard formula for topological entropy for shifts of finite type or sofic shifts extends to path sets.

**Theorem 1.13.** (Topological Entropy Formula for Path Sets) Let $\mathcal{P}$ be a path set with reachable presentation $(\mathcal{G}, v)$ having a right-resolving labeled graph $\mathcal{G}$. Then

$$H_p(\mathcal{P}) = \log \lambda,$$

where $\lambda$ is the spectral radius of the adjacency matrix $A$ of the underlying directed graph $G$ of $\mathcal{G}$. If in addition $\mathcal{G}$ is irreducible, then so is $A$, and then $\lambda$ is the Perron eigenvalue of $A$.

Here $A$ is the adjacency matrix of the unlabeled, directed graph $G$ underlying $\mathcal{G}$. It takes values in $\mathbb{N} \cup \{0\}$, so if it is irreducible it follows by the Perron-Frobenius Theorem that the Perron eigenvalue $\lambda$ is well defined. It is a positive real number whose magnitude is maximal among the eigenvalues of $A$. 
1.2. Related Work. There is a large literature on concepts similar to path sets in automata theory, semigroups, and descriptive set theory. The book of Perrin and Pin [36] presents many results about infinite paths in these contexts, viewing them as infinite words. The concepts covered typically differ in their level of generality from path sets. We mention some of these below, and also indicate relations to terminology in these areas.

In automata theory a finite directed labeled graph is called a finite automaton. A regular language corresponds to the set of finite labeled paths produced as output by a finite automaton (directed labelled graph) with fixed initial state. This theory goes back to Kleene and Moore; an interesting nonstandard treatment is given in Conway [14]. Infinite paths in graphs have been considered in automata theory; treatments of this topic appear in Eilenberg [16, Chap. 14], Béal and Perrin [11], and Perrin and Pin [36]. The treatment of Eilenberg [16] considers a finite state automaton $A := (Q, I, T)$ in which $Q$ is a finite labeled directed graph, $I$ specifies a set of vertices called initial states, and $T$ specifies a set of vertices called terminal states. Various collections of infinite paths have been studied in this context. In Chapter 14 Eilenberg introduces the set $||A||$ of all successful $\omega$-paths, where a successful $\omega$-path is any infinite path in $Q$ that starts at some state in $I$ and visits some state in $T$ infinitely many times. The notion of path set $(G, v)$ defined in this paper corresponds to the special case of this concept in which $Q = G$, the initial state $I = \{v\}$, and the terminal state set $T$ is the set of all states. But this special case is not singled out in the automata theory literature, to our knowledge.

A more general notion in automata theory of successful $\omega$-paths accepts only infinite paths that visit infinitely often exactly the vertices in one of a specified collection $T = \{T_1, T_2, ..., T_m\}$ of subsets of vertices of $Q$. The totality of sets $||A||$ in this generalized sense were characterized in 1966 by McNaughton [33] (see also [16, Chap. 14], [36, Sec. 3.2]); they form a larger collection of sets which is closed under complement (unlike path sets). In Theorem 1.3 of Chapter 14 Eilenberg classifies all $||A||$ in which both the initial state $I$ and terminal state $T$ are allowed to vary. In this case the set of all $||A||$ is shift-invariant (unlike path sets).

Turning to the symbolic dynamics literature, path sets have appeared under the term “follower set,” for example in Jonowska and Marcus [21]. However the term “follower set” has been used with at least two alternate meanings:

1. One notion of “follower set” is the set of all possible finite paths exiting a fixed vertex, see Lind and Marcus [29, Definition 3.3.7]. We may call this concept a finite follower set. (Note however that a finite follower set may contain infinitely many elements!) A characterization of two-sided sofic shifts is given in terms of the finiteness of the collection of all possible finite (right) follower sets ( [29, Theorem 3.2.10]). In Appendix B we provide a similar characterization for one-sided sofic shifts in terms of finite follower sets.

2. Another notion of “follower set” is the set $\mathcal{F}(W)$ of all possible one-sided (infinite) paths $\alpha_0 \alpha_1 \alpha_2 \ldots$ that can follow a given finite word $W = \alpha_{-k} \cdots \alpha_{-3} \alpha_{-2} \alpha_{-1}$ in a two-sided sofic shift $Y$. We call this concept an infinite follower set. Occurrences of this finite word may terminate in several different vertices, so a priori this notion differs from the notion of path set in this paper. Infinite follower sets appear in the work of Fischer [13], [19] connecting two-sided sofic shifts with graphs. Further work was done by Krieger [24], [25]. Using this concept Fischer [19, Theorem 2] characterized two-sided sofic shifts as those shift-closed subsets of $A^\mathbb{Z}$ having only a finite number of distinct infinite follower sets.

Theorem 6.1 of this paper shows that the concepts of path set and infinite follower set are distinct: for a fixed alphabet $A$ infinite follower sets are always path sets, but the converse
assertion does not always hold. This result also shows that all path sets of an alphabet $A$ are representable as infinite follower sets on a larger label alphabet $A'$, which adds one extra symbol.

### 1.3. Applications

Graph-directed constructions of fractals involve two separate ingredients:

1. A family of geometric maps being iterated;
2. A graph (or automaton) describing the allowable combinations of maps under iteration.

Both ingredients together determine the final geometric fractal object. The same geometric fractal object typically has many quite different constructions, and these constructions may attach different addresses to the points in them. The path set concept captures that part of the geometric construction of fractal objects that occurs purely at the level of the address map. In particular, it facilitates study of situations where the underlying maps used in a geometric graph-directed construction or IFS are held fixed, while the underlying graph giving the address map is varied. For example Theorem 1.2 shows that in this circumstance intersections and unions of fractal sets can be computed purely at the level of address maps.

Path sets arise in $p$-adic versions of fractal constructions. In the paper [1] we introduce a notion of $p$-adic path set fractal, which is the image of a path set interpreting its address as a $p$-adic integer. We show that such sets are constructible by a $p$-adic graph-directed construction with a constant ratio similarity. We also show that the resulting collection of $p$-adic path sets is closed under the $p$-adic arithmetic operations of adding $r$ to a set or multiplying the set by $r$, where $r = \frac{m}{n} \in \mathbb{Q} \cap \mathbb{Z}_p$, e.g. $r$ is a $p$-integral rational number. A need for the path set concept is forced by these operations, because even if one restricts the arithmetic operations to apply only to (one-sided) shift invariant sets, the image path sets under the operations need not be shift-invariant. An interesting feature of the arithmetic constructions in [1] is that the associated graph presentations of the new path sets constructed using arithmetic operations can be far from irreducible, sometimes having nested irreducible subcomponents, cf. examples in the related paper [2].

In special circumstances the Hausdorff dimension of fractal objects may be computed using the topological entropy formulas given in this paper. In graph-directed constructions the Hausdorff dimension of the resulting geometric fractal typically depends in a complicated way on both the geometric maps and the address map. In general these dimensions can be computed using operators that require mixing together both these ingredients, e.g. Mauldin and Williams [32]. In the special case where all the maps are similarities with constant ratio, the Hausdorff dimension is often determinable from the address map data together with only knowledge of the similarity ratio, under extra hypotheses such as variants of the open set condition (Ngai and Wang [34], Bandt [8], Lau and Ngai [27]). These extra hypotheses hold in the $p$-adic path set fractal framework used in [1] and [2], and we use Hausdorff dimension calculations in [2] to answer a question raised in [26, Sec. 4], motivated by a number-theoretic problem of Erdős.

There are several possible directions for further work. First, the decimation operation in Theorem 1.5 of this paper gives a construction of new path sets from old, worthy of further study. Recent papers study various sets closed under decimation operations and determine their Hausdorff dimension ([22], [55]). Second, Feng and Wang [17] have recently considered the question of which iterated function systems (given using self-similar maps) will generate the same limiting geometric object $F$. It asks when a generating IFS family, all giving this object, has a minimal element. This only happens in special situations. An
analogous question in our framework concerns the problem of when does a path set have a
unique minimal presentations, measured by the number of vertices in the underlying graph.
Third, one may consider path sets in the study of general systems of numeration, viewed
for example in the fiber systems model for such systems, as described in Barat, Berthé,
Liardet and Thuswaldner [9].

1.4. Contents. Section 2 presents examples illustrating some distinctive properties of path
sets relative to one-sided sofic shifts. Section 3 treats presentations of path sets and the
question of how efficiently they can be encoded relative to one-sided sofic shifts. Section 4
establishes the set-theoretic properties of path sets given in Theorem 1.2. Section 5 treats
one-sided sofic shifts and gives the proof of Theorem 1.4. Section 6 relates the notions of
path sets and infinite follower sets for two-sided sofic shifts, stated as Theorem 6.1. It also
makes remarks on minimality of presentations of sofic shifts and path sets. Section 7 gives
the proof of Theorem 1.5. Section 8 proves the structural characterization in Theorem 1.9.
Section 9 proves results on topological entropy, giving Theorem 1.11 and Theorem 1.13.
Appendix A (Section 9) gives a proof that Example 3.3 in Section 3 gives a minimal right-
resolving path set presentation of certain sofic shifts. Appendix B establishes a standard
equivalence between different definitions of one-sided sofic shifts.

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2. Examples

We present three simple examples of path sets illustrating some of their distinctive fea-
tures. The first two examples illustrate features coming from the imposition of an initial
condition (the choice of initial vertex v), showing how shift-invariance may fail. The third
example shows that path sets are not closed under the complement operation in a fixed
alphabet.

Example 2.1. For the alphabet \( \mathcal{A} = \{0, 1, \ldots, n-1\} \) and a given symbol \( j \in \mathcal{A} \), consider
the two-state graph \( G \) pictured in Figure 1. It has \( n \) distinct loops at vertex \( v_1 \), labelled
0, 1, \ldots, \( n-1 \), respectively, which are denoted schematically in the figure. The directed
graph \( G \) is connected but not strongly connected, i.e. it is reducible. The path set \( X_G(v_0) \)
is the prefix set \( Z_j = j\mathcal{A}^N \). This set is not shift-invariant, but is of the form \( WY \) where
\( W \) is a finite set, and \( Y \) is shift-invariant, being the full shift. We may call path sets
that are finite unions of sets of the form \( WY \) prefix-shift-invariant. The sets \( Z_j \) are
the simplest examples of path sets that are not shift-invariant. They play an important role in
the structure Theorem 1.9 for path sets.

Example 2.2. In Example 2.1 the path sets are given as a finite prefix followed by a shift-
invartant set. Now consider the three-state graph \( G \) pictured in Figure 2 which uses edge
labels drawn from \( \mathcal{A} := \{0, 1, 2\} \). The graph \( G \) is irreducible. The path sets \( P_i = X_G(v_i) \)
for \( 0 \leq i \leq 2 \) consist of a single infinite path: \( P_0 = (012)^{\infty} \), \( P_1 = (120)^{\infty} \), \( P_2 =
(201)^{\infty} \). These path sets are not shift-invariant. This example cannot be partitioned into
a finite number of path sets of the type \( WY \) of example 2.1 where \( W \) is a finite word, and
\( Y \) is itself a shift-invariant path set.

Example 2.3. (A path set whose complement is not a path set) Consider the two state
graph \( G \) pictured in Figure 3 with edge labels drawn from \( \mathcal{A} = \{0, 1\} \). Here the path set
\[ P = \mathcal{X}_G(v_0) \] is the set of all sequences in \( \mathcal{A}^N \) in which the block 11 does not occur.

We show by contradiction that the complement \( P^c := \mathcal{A}^N \setminus P \) is not a path set. Here \( P^c \) is the set of all sequences in \( \mathcal{A}^N \) in which the sequence 11 occurs at least once. Suppose \( P^c \) were presented as a path set by a pointed graph \( G(v) \). Let \( n = |G| \). Now \( P^c \) contains words with arbitrarily long initial substrings which do not contain the string 11. It follows that we can find an initial segment \( a_0a_1 \cdots a_k \) of a word in \( P^c \) subject to the following constraints:

1. \( a_0a_1 \cdots a_k \) is given by an edge walk \( e_0e_1 \cdots e_k \) originating at the initial vertex \( v \);
2. the string 11 is not contained in \( a_0a_1 \cdots a_k \);
3. \( k > n \) and \( a_k = 0 \);
4. the terminal state of edge \( e_k \) is visited earlier as the initial state of some \( e_j, 0 \leq j < k \).

It follows from (1)-(4) that \( a_0a_1 \cdots a_{j-1}(a_ja_{j+1} \cdots a_k)^\infty \in P^c \). But this path does not contain 11, so it is in \( P \), a contradiction. Thus \( P^c \) cannot be a path set.
3. PRESENTATIONS OF PATH SETS

A given path set $\mathcal{P}$ arises from many different pointed graphs. If we let $G'$ be the labeled graph obtained from $G$ by keeping only the connected component containing the vertex $v$ and iteratively throwing out any stranded states (those vertices with no exit edges) and the edges into or out of them (except $v$), then clearly $X_G(v) = X_{G'}(v)$. Therefore, from a symbol space point of view, it suffices to consider graphs $G$ such that each vertex of $G$ is reachable from $v$ by a path in $G$, and no vertices have out-degree zero. We call such labelled graphs pruned graphs. (Compare [29, Prop. 2.2.10]).

Suppose $\mathcal{P}$ is a path set with presentation $(G, v)$. Given an initial string $a_0a_1a_2\cdots$ belonging to a member of $\mathcal{P}$, it would be useful to know that this string corresponds to a unique finite walk in $G$ originating at the distinguished vertex $v$. This is not true for a general $G$, but is true for a special class of graphs.

**Definition 3.1.** A finite labeled directed graph $G$ is called right-resolving (or deterministic) if for any vertex $w$ of $G$, there is at most one edge originating at $w$ with any given label. A path set with a right-resolving presentation has the uniqueness property given above.

Recall that a presentation $(G, v)$ is termed reachable if each vertex of $G$ can be reached from $v$.

**Theorem 3.2.** Every path set $\mathcal{P}$ has a right-resolving presentation that is reachable.

**Proof.** It suffices to construct a right-resolving presentation, since the reachability condition is then achieved by discarding all vertices of $G$ that are not reachable from $v$.

The existence of a right-resolving presentation $\mathcal{H}$ for (two-sided) sofic shifts, constructed directly from a given presentation $G$, is shown in Lind and Marcus ([29], Theorem 3.3.2, p. 76), using the subset construction described below. In the following Claim, we show this construction yields the desired result for path sets, as well.

The **subset construction** starting from a labelled graph $G$ produces a new labelled graph $\mathcal{H}$ having $2^{|V(G)|} - 1$ vertices which are marked by the nonempty subsets $S$ of the vertex labels of $G$. For each edge label $a \in A$ and a vertex $S$ of $\mathcal{H}$ we assign an exit edge with this label mapping to the vertex $S'$ whose marking is the union of all vertices $w \in V(G)$ such that there is an edge from some $v \in S$ to $w$ with label $a$, unless $S' = \emptyset$, in which case we assign no edge. The labelled graph $\mathcal{H}$ is clearly right-resolving.
Now let \((G, v_0)\) be a presentation of a path set \(P\). We may assume without loss of generality that this presentation is a pruned graph, viewed from vertex \(v_0\).

Claim. The right-resolving construction applied to the underlying graph \(G\) of a pruned pointed graph \((G, v_0)\) has the property that each vertex \(S \in V(H)\) has path set

\[
X_H(S) = \bigcup_{v' \in S} X_G(v').
\]

To prove the claim, let \(B_m(P)\) denote the set of vectors giving in order the first \(m\) symbols of each path in \(P\). It then suffices to show the equality of the sets \(B_m(X_H(S))\) and \(\cup_{v' \in S} B_m(X_G(v'))\), for all \(m \geq 1\). We prove the inclusions of these two sets in both directions by induction on \(m\). The base case \(m = 1\) is clear, using the fact for \(G\) that all one-step paths extend to infinite paths. For the inclusion of \(\cup_{v' \in S} B_m(X_G(v'))\) in \(B_m(X_H(S))\) given \(w \in S\) it suffices to note that a path in \(G\) starting from a vertex \(v\), with a given symbol set lifts to a path in \(H\) starting from \(S_0\) and the same symbol sequence, by definition of the edges of \(H\). For the other inclusion, given a start vertex \(S = S_0\), and proceeding to follow a path to vertices \(S_1, S_2, \ldots, S_n\) in \(H\), and given the symbol sequence, the right-resolving property gives us unique backtracking from \(S_n\) to \(S_0\). A key point is that, although we may not know what state of \(G\) in \(S_n\) we have reached at the \(n\)-th step, from knowledge of the \((n + 1)\)–st step we are guaranteed there exists some state of \(S_n\) giving this symbol that permits backtracking through a series of states \(w_j \in S_j\) to a state \(w_0\) in \(S_0\), since all states of \(S_n\) permit such backtracking. Now the sequence corresponds to a path in \(G\) starting from \(w_0\) that traverses the states \(w_j\) with the correct letters. It corresponds to the initial part of an infinite path in \(X_G(w_0)\), because by our hypothesis on \(G\) all finite paths extend to some infinite path. Thus it belongs to \(\cup_{v' \in S} B_m(X_G(v'))\)). This completes the induction step, proving the claim.

We apply the Claim to the pruned labelled graph \((G, v_0)\) presenting \(P\). The Claim shows that the pointed graph \((H, S_{\{v_0\}})\) with vertex \(S_{\{v_0\}} := \{v_0\}\) is a right-resolving presentation of \(P\), which is the desired result. Note that we may obtain a (possibly smaller) right resolving presentation of \(P\), by taking the induced subgraph of \(H\) obtained by restricting to the set of vertices reachable from \(S_{\{v_0\}}\) by a directed path.

The proof of Theorem \(3.2\) shows that any one-sided sofic shift \(Y\) is a path set. For if \(G\) represents such a shift, then the Claim above applied to \(G\) implies that the pointed graph \((H, S_0)\) with \(S_0 := V(G)\) represents the path set \(X_H(S_0)\), which is the one-sided sofic shift represented by the graph \(G\). We illustrate this on the following example.

Example 3.3. Consider the three-state graph \(G\), pictured in Figure 4, which uses the label alphabet \(A = \{0, 1, 2\}\). Here \(G\) is a presentation of a one-sided sofic shift \(S\), but this presentation is not right-resolving. The subset construction results in the graph \(H\) pictured in Figure 5. The graph \(H\) has 7 vertices, labelled by unions of states in \(G\), and which is right-resolving. The pointed graph \((H, ABC)\) gives a path set presentation of the sofic shift \(S\). In this example the vertices labelled \(A, B, C\) can be pruned from this pointed graph, resulting in a 4-state graph \(H'\) such that the pointed graph \((H', ABC)\) gives also a path set representation of the one-sided sofic shift \(S\).

Returning to the subset construction, the number of vertices \(|V(H)| = 2^{|V(G)|} - 1\), which is exponentially larger than \(|V(G)|\). The following example shows this exponential blowup is sometimes unavoidable using the subset construction to construct a path set realization of a sofic shift \(Y\), even when \(G\) is a minimal right-resolving representation of \(Y\). More is true: in this example the pointed graph \((H, V(G))\) produced by the subset
construction can be shown to give a minimal right-resolving path set presentation of $Y$, see Appendix A. Such a result establishes that minimal right-resolving realizations of a one-sided sofic shift $S$ by a graph $G$ can sometimes be exponentially more efficient than minimal right-resolving presentations as a path set.

**Example 3.4.** Let $G := G_n$ have vertex set $V = \{v_0, \ldots, v_{n-1}\}$, and have edges labelled from the label set $L = \{0, 1, \ldots, 2n - 1\}$, as follows. Each vertex $v_i$ has $n-1$ self-loops, labelled with $n-1$ of the symbols from $(n, n+1, \ldots, 2n-1)$, omitting only the symbol $n+i$. Between distinct $v_i$ and $v_j$ there is a directed edge labelled $j - i \mod n$, taking the least nonnegative residue, noting that this label necessarily falls in the interval $[1, n-1]$. Figure 6 pictures $G_3$. The graph $G$ is right resolving and irreducible (i.e. strongly connected), and has $n(2n-2)$ directed edges. It is therefore an irreducible right-resolving presentation of a one-sided sofic shift $Y = X(G)$.

Applying the subset construction to $G$ yields a pointed graph $(\mathcal{H}, S_G)$ giving a right-resolving presentation of $Y = X_{\mathcal{H}}(S_G)$ as a path set. The graph $\mathcal{H}$ has $2^n - 1$ vertices. We can check that the vertex $S_G$ of $\mathcal{H}$ has directed paths connecting it to every one of the
vertices of $\mathcal{H}$. Namely, for $1 \leq j \leq n$ there is a directed edge labelled $n+j$ connecting it to the vertex labelled $S_{v_j}$ with $\mathcal{G'} := \mathcal{G} \setminus \{v_j\}$. In a similar fashion for $i \neq j$ there is a directed edge labelled $n+i$ connecting the vertex $S_{v_i}$ to $S_{v''}$ with $\mathcal{G''} := \mathcal{G} \setminus \{i, j\}$, and so on. Furthermore all the vertices $S_{v_i}$ for $1 \leq i \leq n$ form a strongly connected component of this graph, reproducing the graph $\mathcal{G}$, and this graph can be reached by a path from every other vertex in $\mathcal{H}$. Thus there are no stranded states, so $(\mathcal{H}, S_{\mathcal{G}})$ is a pruned pointed graph having $2^n - 1$ vertices. The graph $\mathcal{H}$ actually provides a minimal right resolving presentation of $Y$ as a path set, which we establish in Appendix A.

4. SET-THEORETIC PROPERTIES OF PATH SETS

We establish that the collection of path sets are closed under union and intersection, obtaining Theorem 1.2. This collection of sets is not closed under complement by Example 2.3. To establish closure of the collection of path sets under intersections, we will need an appropriate notion of graph product. Lind and Marcus [29] make the following construction:

**Definition 4.1.** Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be directed labeled graphs over the same alphabet $\mathcal{A}$. The **label product** of these graphs, written

$$\mathcal{G}_1 \star \mathcal{G}_2 := (V_1 \times V_2, \mathcal{E})$$

is the labeled directed graph with vertex set $V_1 \times V_2$ and edge set

$$\mathcal{E} := \{(e_1, e_2), \ell) \in E_1 \times E_2 \mid e_1 \text{ and } e_2 \text{ share the same label } \ell \in \mathcal{A}\}.$$

The label assigned to the edge $(e_1, e_2)$ is the common label of the edges $e_1$ and $e_2$.

We adapt this definition to get a pointed label product for our pointed graphs $(\mathcal{G}_1, v_1)$ and $(\mathcal{G}_2, v_2)$ by choosing the pair $(v_1, v_2)$ as the distinguished vertex of $(\mathcal{G}_1 \star \mathcal{G}_2, (v_1, v_2))$, i.e.,

$$(\mathcal{G}_1, v_1) \star (\mathcal{G}_2, v_2) := (\mathcal{G}_1 \star \mathcal{G}_2, (v_1, v_2)).$$
It is then possible to eliminate extraneous components and prune stranded states from the pointed label product to recover an equivalent pointed graph \((\mathcal{G}, v)\) that is irreducible and such that each vertex of \(\mathcal{G}\) is reachable from the distinguished vertex.

**Lemma 4.2.** If \(\mathcal{G}_1\) and \(\mathcal{G}_2\) are labeled directed graphs that are right-resolving, then the label product graph \(\mathcal{G} := \mathcal{G}_1 \ast \mathcal{G}_2\) is also right-resolving.

**Proof.** This is shown in Lind and Marcus ([29], Proposition 3.4.10, p. 89).

This result applies to the pointed label product, since the right-resolving property is determined by the underlying labelled graph. Just as the label product interacts nicely with the structure of sofic shifts (see Lind and Marcus [29, Prop. 3.4.10, p. 89] ) we have the following result concerning the pointed label product of pointed graphs and their associated path sets.

**Proposition 4.3.** If \((\mathcal{G}_1, v_1)\) and \((\mathcal{G}_2, v_2)\) are pointed graphs over the same alphabet \(\mathcal{A}\), then

\[
X_{\mathcal{G}_1}(v_1) \cap X_{\mathcal{G}_2}(v_2) = X_{\mathcal{G}_1 \ast \mathcal{G}_2}((v_1, v_2)).
\]

**Proof.** Suppose \((\mathcal{G}_1, v_1)\) and \((\mathcal{G}_2, v_2)\) are pointed graphs. Let \(\mathcal{G} = \mathcal{G}_1 \ast \mathcal{G}_2\) be the label product, and let \(v = (v_1, v_2)\), a node of \(\mathcal{G}\). Then \((\mathcal{G}, v)\) is also a pointed graph.

We prove now that \(X_{\mathcal{G}}(v) = X_{\mathcal{G}_1}(v_1) \cap X_{\mathcal{G}_2}(v_2)\). Suppose

\[
(e_{11}, e_{21}), (e_{12}, e_{22}), (e_{13}, e_{23}), \ldots
\]
is an infinite walk in \(\mathcal{G}\) originating at \(v\). Then by the definition of the label product \(e_{11}, e_{12}, e_{13}, \ldots\) is an infinite walk in \(\mathcal{G}_1\) originating at \(v_1\) and \(e_{21}, e_{22}, e_{23}, \ldots\) is an infinite walk in \(\mathcal{G}_2\) originating at \(v_2\). The label of \((e_{11}, e_{21}), (e_{12}, e_{22}), (e_{13}, e_{23}), \ldots\) is equal to the label of \(e_{11}, e_{12}, e_{13}, \ldots\) and of \(e_{21}, e_{22}, e_{23}, \ldots\) by definition, so

\[
X_{\mathcal{G}}(v) \subseteq X_{\mathcal{G}_1}(v_1) \cap X_{\mathcal{G}_2}(v_2).
\]

Conversely, suppose \(a_1a_2a_3 \ldots \in X_{\mathcal{G}_1}(v_1) \cap X_{\mathcal{G}_2}(v_2)\). Then there is an infinite edge walk \(e_{11}, e_{12}, e_{13}, \ldots\) in \(\mathcal{G}_1\) originating at \(v_1\) and an edge path \(e_{21}, e_{22}, e_{23}, \ldots\) in \(\mathcal{G}_2\) originating at \(v_2\) such that both of these paths have label \(a_1a_2a_3 \ldots\). Since \(e_{1i}\) and \(e_{2i}\) share the same label for all \(i \in \mathbb{N}\), there is path \((e_{11}, e_{21}), (e_{12}, e_{22}), (e_{13}, e_{23}), \ldots\) in \(\mathcal{G}\) originating at \(v = (v_1, v_2)\) which also has label \(a_1a_2a_3 \ldots\). Thus, we also have

\[
X_{\mathcal{G}_1}(v_1) \cap X_{\mathcal{G}_2}(v_2) \subseteq X_{\mathcal{G}},
\]

and the proposition follows. □

**Proof of Theorem 1.2** Let \(\mathcal{P}\) be a path set in \(\mathcal{C}(\mathcal{A})\).

(1) It is immediate from the definition that \(\mathcal{P}\) is a closed subset of \(\mathcal{A}^\mathbb{N}\).

(2) If follows immediately from Proposition 4.3 that the collection of path sets in \(\mathcal{A}^\mathbb{N}\) is closed under intersections.

(3) Suppose \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are path sets in \(\mathcal{A}^\mathbb{N}\). We must show that \(\mathcal{P}_1 \cup \mathcal{P}_2\) is a path set. Choose presentations \((\mathcal{G}_1, v_1)\) and \((\mathcal{G}_2, v_2)\), and construct \((\mathcal{G}, v)\) as follows: Let \(\mathcal{G}\) be obtained from \(\mathcal{G}_1\) II \(\mathcal{G}_2\) by adding a new vertex \(v\), and, for each edge \(v_i \rightarrow w\) in \(\mathcal{G}_i\) labeled \(a\), an edge \(v \rightarrow w\) in \(\mathcal{G}\) labeled \(a\). We now must show that \(X_{\mathcal{G}}(v) = \mathcal{P}_1 \cup \mathcal{P}_2\).

Indeed, suppose \(a_0a_1a_2 \ldots \in \mathcal{P}_1 \cup \mathcal{P}_2\). Then without loss of generality \(a_0a_1a_2 \ldots \in \mathcal{P}_1\). Let \(v_1 \rightarrow w\) be the first edge in an infinite walk in \(\mathcal{G}_1\) with edge-label \(a_0a_1a_2 \ldots\). Then by assumption there is an edge \(v \rightarrow w\) labeled \(a_0\), and replacing the edge walk \(v_1 \rightarrow w\) with \(v \rightarrow w\) gives an infinite walk in \(\mathcal{G}\) originating at \(v\) with edge-label \(a_0a_1a_2 \ldots\). Thus, \(\mathcal{P}_1 \cup \mathcal{P}_2 \subset X_{\mathcal{G}}(v)\).
Conversely, suppose \(a_0a_1a_2 \cdots \in X_G(v)\). Let \(v \to w \to \cdots\) be an infinite walk in \(G\) with edge label \(a_0a_1a_2 \cdots\). By construction \(G\) has no self-loops at \(v\), so without loss of generality \(w \in P_1\), and in fact there is an edge \(v_1 \to w\) labeled \(a_0\). But there are no edges connecting the \(G_1\) component of \(G\) to the \(G_2\) component of \(G\), so it must be that the remaining edges of our walk remain in \(G_1\). Thus, replacing \(v \to w\) by \(v_1 \to w\) gives an infinite walk in \(G_1\) originating at \(v_1\) with edge-label \(a_0a_1a_2 \cdots\). Therefore \(X_G(v) \subset P_1 \cup P_2\), hence \(X_G(v) = P_1 \cup P_2\), and we conclude that \(P_1 \cup P_2\) is a path set. This proves the theorem. \(\square\)

5. PATH SETS AND ONE-SIDED SOFIC SHIFTS

Proof of Theorem 1.3 (1) Let the path set \(P = X_G(v_0)\). The shifted set \(\sigma(P)\) is a finite union of the path sets \(X_G(v_i)\), where \(v_i\) runs over the vertices reachable from \(v_0\) in one step. Thus \(\sigma(P)\) is a path set using closure of path sets under unions (Theorem 1.2 (3)).

(2) Let \(P\) be a path set, and let \(\overline{P}\) be its one-sided shift closure. Suppose \(G(v)\) is a presentation of \(P\). By the discussion in the introduction we can assume that every vertex of \(G\) is reachable in \(G\) by a path originating at \(v\). Then

\[
\sigma(P) = \bigcup\{w \in G : \exists \text{ an edge } v \to w\} X_G(w),
\]

and hence

\[
\overline{P} = \bigcup_{j \in \mathbb{N}} \sigma^j(P) = \bigcup_{w \in G} X_G(w).
\]

Since \(G\) is a finite graph, this says that \(\overline{P}\) is a finite union of path sets, hence is a path set by Theorem 1.2.

(3) For a path set \(P\) represented by a pruned pointed graph \((G, v_0)\), its shift-closure \(\overline{P}\) is exactly the one-sided sofic shift presented by \(G\), since every vertex of \(G\) is reachable from \(v_0\), and \(\overline{P}\) is a path set by (1). Thus if \(P\) is shift-invariant, i.e. \(P = \overline{P}\), then \(P\) is a one-sided sofic shift.

For the converse, suppose \(Y \subset A^N\) is a one-sided sofic shift. Then \(Y\) is the union of the path sets at all the vertices of \(G\). But the collection of path sets is closed under set union by Theorem 1.2 (3) whence \(Y\) is a path set. (An alternative proof is obtainable by direct construction via the subset construction, as indicated after Theorem 3.2) \(\square\)

6. PATH SETS AND TWO-SIDED SOFIC SHIFTS

We relate path sets to the infinite follower sets defined in Section 1.2 for two-sided sofic shifts. Recall that for a two-sided sofic shift \(Y\) and a finite word \(w := \alpha_{-k} \alpha_{-k+1} \cdots \alpha_{-2} \alpha_{-1}\), the infinite follower set \(F_Y(w) \subset A^\mathbb{N}\) is the set of all possible one-sided (infinite) paths \(\alpha_0 \alpha_1 \alpha_2 \cdots\) in \(Y\) that can directly follow occurrences of the word \(w\) in the sofic shift.

Theorem 6.1. (Path sets and infinite follower sets)

1. Each infinite follower set \(F_Y(w) \subset A^\mathbb{N}\) coincides with a path set in the same alphabet \(A\).

2. Not all path sets \(P \subset A^\mathbb{N}\) are infinite follower sets in the same alphabet \(A\). However any path set \(P\) is representable as an infinite follower set using an enlarged alphabet \(A' := A \cup \{a'\}\) containing one extra symbol.

Proof. (1) Two-sided sofic shifts are characterized among all two-sided subshifts in \(A^\mathbb{Z}\) by the property of having a finite number of distinct infinite (right-) follower sets (Fischer [19] Theorem 2). Given a two-sided sofic shift \(Y \subset A^\mathbb{Z}\), its right Krieger cover (24) is constructed by taking a labelled graph \(G\) whose states have path sets corresponding to
all the possible infinite follower sets $\mathcal{F}_Y(w)$ of $Y$. By construction the path sets in the right Krieger cover coincide with these infinite follower sets. This shows that all infinite follower sets are path sets in the same alphabet.

(2) We claim that the prefix sets $Z_j \subseteq A^N$ with $A = \{0, 1, \ldots, n-1\}$ in Example 2.1 with $n \geq 2$, are path sets that are not\footnote{This fact can be shown by observing that by construction it has the follower-separated property (i.e., every graph vertex has a different follower set); and that its graph is irreducible (strongly connected), which is a consequence of the irreducibility of $Y$. Then one can use the fact that the infinite follower-separated property implies the finite follower-separated property, to show minimality, by invoking [29 Corollary 3.3.19].} infinite follower sets in any two-sided sofic shift using the same alphabet $A$. For simplicity we show this for the prefix set $Z_1 = \{0, 1\}^N$ pictured in Figure 1. In this alphabet any two-sided sofic system in which $Z_1$ is included must contain its two-sided shift closure, and this is the full two-sided shift $Y := \{0, 1\}$\footnote{Sec. 1.4.5].}.

However any infinite follower set $\mathcal{F}_Y(\alpha_{-k-1} \alpha_{-k} \cdots \alpha_{-1})$ for the full shift $Y$ will itself be the full one-sided shift $\{0, 1\}^N$, and this never equals $Z_1$. That is, the "initial condition" data in $Z_1$ is lost when taking the shift closure. Thus $Z_1$ is not an infinite follower set in any two-sided sofic system using the alphabet $A = \{0, 1\}$.

It is easy to see that any path set $P \subseteq A$ can be represented as an infinite follower set in an enlarged alphabet $A' = A \cup \{a'\}$ adding one extra symbol to the label set, call the extra symbol $n$. The extra symbol "$n" will supply the "initial condition" data determining the path set. Take a presentation $(G, v_0)$ for $P$, and create a new graph $G'$ which consists of the graph $G$ augmented with one extra vertex $v^*$, which has two edges, a self-loop labelled with a symbol from $A$, call it "0", and a directed edge from $v^*$ to $v_0$ labelled "$n". We let $Y$ denote the two-sided sofic shift associated to $G'$. The shift $Y$ certainly contains all two-sided infinite paths of the form $0^n s$, where $s \in P$. Now $w = \alpha_{-1} = n$. We assert that the infinite follower set $\mathcal{F}_G(w) = \mathcal{F}_Y$. This is clear, since there is only one edge labelled "$n", which uniquely identifies the vertex $v_0$.

We make a few remarks comparing minimal presentations of these structures.

**Definition 6.2.** Call a presentation of either a two-sided sofic shift or a one-sided sofic shift or a path set minimal if it minimizes the number of vertices in the associated labelled graph $G$, over all presentations of the same type.

**Remarks.**

(1) For two-sided sofic shifts, one can always find a presentation in which the underlying labelled graph $G$ has a property called being "trim" (in the sense of finite automata). A labelled directed graph is trim if every finite path in it can be extended to some doubly infinite path. (For related but not identical concepts of "trim" see Willems [38 Sec. 1.4.5] and Eilenberg [16, p. 23].) Minimal presentations of two-sided sofic shifts are necessarily trim. In contrast, some path sets have no presentation $(G, v)$ in which $G$ is trim; the basic sets $Z_j$ provide an example, as follows from the proof of (2) above in Theorem 6.1.

(2) There has been much study of minimal presentations of two-sided sofic shifts. If a two-sided sofic shift is transitive (i.e., it has a presentation whose graph is strongly connected) then all minimal presentations are isomorphic ([19 Theorem 4]. For reducible two-sided sofic shifts it is known that there can be non-isomorphic minimal presentations. For irreducible two-sided sofic shifts $Y$ the right Krieger cover always gives a minimal right-resolving presentation as a two-sided sofic shift.

(2) A variant of the right Krieger cover construction can be made for path sets. Given a path set $P$, this construction considers the infinite follower sets $P(w)$ consisting of all paths that can follow a given finite initial word $w := \beta_1 \beta_2 \cdots \beta_k$ of some path in $P$.
viewing it as corresponding to \( \alpha_{-k} \cdots \alpha_{-3} \alpha_{-2} \alpha_{-1} \). Here the empty word \( w = \emptyset \) is allowed. As \( w \) varies over all finite words, only a finite number of distinct sets \( \mathcal{P}(w) \) are obtained this way, by virtue of the Structure Theorem \([1.9]\). We then create a graph \((\mathcal{H}, v_0)\) whose vertices correspond to all the distinct \( \mathcal{P}(w) \), with the marked vertex \( v_0 \) corresponding to \( \mathcal{P} \), and with edges added in the obvious fashion (a directed edge labelled \( j \) is added from \( \mathcal{P}(w) \) to \( \mathcal{P}(w, j) \), checking this is well-defined). This variant construction gives a right-resolving presentation of the path set \( \mathcal{P} \). However we do not know if it is always true in the one-sided case that when \( \mathcal{P} \) is irreducible, this construction gives a minimal path set realization. The proof of minimality in the two-sided case does not extend because in the one-sided case the resulting graph \((\mathcal{H}, v_0)\) may be reducible. Example \([3.3]\) gives examples in which \( \mathcal{H} \) is reducible, but we are able to prove minimality of path set presentations in these cases by the specific argument in Appendix A.

7. Decimation of Path Sets

**Proof of Theorem \([1.9]\)** (1) It suffices to prove the result when \( j = 0 \). For we have

\[
\mathcal{P}_{j,m} = \psi_{j,m}(\mathcal{P}) = \psi_{0,m}(\sigma^j(\mathcal{P})).
\]

Now \( \mathcal{P}' := \sigma^j(\mathcal{P}) \) is a path set by Theorem \([1.4]\), so to establish that \( \mathcal{P}_{j,m} \) is a path set, it suffices to show that \( \psi_{0,m}(\mathcal{P}') \) is a path set.

For the case \( j = 0 \), we suppose given a right-resolving presentation \((\mathcal{G}, v)\) of \( \mathcal{P} \). We derive from it a presentation \((\mathcal{G}', v)\) of \( \mathcal{P}_0 \), which is generally not right-resolving, as follows. We make an initial labelled graph \( \mathcal{G}_1 \) which contains vertex \( v \) and all nodes reachable in one step from \( v \), which in the case of self-loops from \( v \) is altered to add a new copy of node \( v \), labelled \( v^{(1)} \), to which all self-loops of \( v \) are required to exit. Secondly, from each new node \( w \) reached after one step, one creates a (large) set of new vertices, corresponding to each \( m \)-step directed path leaving from that node \( w \), with vector of labels \( \ell' := (\ell_1, \ell_2, \ldots, \ell_m) \), say. If \( w' \) is the identifier of the vertex reached in \( \mathcal{G} \) along this path, then the new vertex created is identified by the symbol \( (\ell', w') \). There is now assigned a single directed edge from \( w \) to \( w' \) with label \( \ell_m \). Finally we add edges between \( (\ell', w') \) and \( (\ell'', w'') \) whenever there is an \( m \)-step path in \( \mathcal{G} \) from vertex \( w' \) to \( w'' \) described by the vector \( \ell'' \), and assigned label \( \ell'_m \) the last step in that path. This describes the full graph \( \mathcal{G}' \), and the initial vertex remains \( v \), as before.

It is straightforward to check that the paths from \( v \) in \( \mathcal{G}'' \) produce all paths in \( \mathcal{P}_0 \), and that each such path occurs at least once.

(2) Suppose that \( \mathcal{P} \) is invariant under the one-sided shift. Now we have

\[
\mathcal{P}_{j,m} = \psi_{0,m}(\sigma(\mathcal{P})) = \psi_{0,m}(\mathcal{P}) = \mathcal{P}_{0,m},
\]

so all are equal, to a set denoted \( \mathcal{P}_m \), say. Finally \( \sigma(\mathcal{P}_{j,m}) = \mathcal{P}_{j+m,m} = \mathcal{P}_{j,m} \), showing that \( \mathcal{P}_{j,m} \) is shift-invariant.

8. Structural Characterization of Path Sets

**Proof of Theorem \([1.9]\)** (1) \( \implies \) (2). Let a path set \( \mathcal{P} \in \mathcal{C}(\mathcal{A}) \) be given. We construct a finite set \( S_0 \) closed under the two operations of (2), as follows. Take a right-resolving presentation \( \mathcal{G} \) of the path-set \( \mathcal{P} \), as given by Theorem \([3.2]\). Let \( V(\mathcal{G}) = \{v_k\} \) denote the set of vertices of \( \mathcal{G} \) and let \( \mathcal{P}_k := X^{(k)}(v_k) \) denote the corresponding path set, with \( v_0 \) assigned the path set \( \mathcal{P}_0 = \mathcal{P} \). Now the collection

\[
S_0 := \bigcup_{k \in J} \mathcal{P}_k, \left( \bigcup_{k \in J} \mathcal{P}_k \right) \cap \mathcal{Z}_j : \text{all } J \subset V(\mathcal{G}), \text{all } j \in \mathcal{A}
\]
is a finite collection of closed subsets of $\mathcal{A}^\mathbb{N}$ that contains $\mathcal{P}$. It is manifestly closed under intersection with the prefix sets $\mathcal{Z}_j$, since the sets $\mathcal{Z}_j$’s are pairwise disjoint. Furthermore, since $\sigma(X_k)$ is a finite union of the other $\mathcal{P}_k$, and since each $\sigma(\mathcal{P}_k \cap \mathcal{Z}_j)$ is a single $\mathcal{P}_k$, or the empty set (using the right-resolving property), the set $\mathcal{S}_0$ is also closed under the forward shift operation. Thus (2) follows.

(2) $\implies$ (1). Let $\mathcal{S}$ be the smallest collection of sets containing $\mathcal{P}$ and closed under intersections with all the prefix sets $\mathcal{Z}_j$ and under the one-sided shift. By hypothesis there exists a finite set $\mathcal{S}_0$ containing $\mathcal{P}$ and closed under these operations, and it contains $\mathcal{S}$, therefore $\mathcal{S}$ is finite. We construct a right-resolving presentation for $\mathcal{P}$ using $\mathcal{S}$ as follows. Form a graph whose nodes are labelled with all sets is of $\mathcal{P}$ such that each vertex of $\mathcal{P}$ let $\mathcal{P}_1, j, 1, j \in \mathcal{P_1}$ and make an edge labelled $j$ from $\mathcal{P}$ to the node labelled $X_{1, j}$. Form now $\mathcal{P}_1, j, 1, j \in \mathcal{P_1}$, and make an edge labelled $j_2$ to a node labelled by the set $\mathcal{P}_1, j, 1, j$, and so on. A finite graph results, which is right-resolving by construction. It contains $\mathcal{P}$ as a label of one node. Now one may check that the definitions imply that the set of labelled paths through this graph, starting at $\mathcal{P}$, correspond exactly to the members of $\mathcal{P}$. This certifies that $\mathcal{P}$ is a path set.

9. Topological Entropy

Proof of Theorem 9.1] Let $\mathcal{P}$ be a path set with right-resolving presentation $\mathcal{G}(v)$, so that each vertex of $\mathcal{G}$ is reachable from $v$. Thus $\mathcal{P}$ is a one-sided sofic shift with presentation $\mathcal{G}$. The equality $H_{top}(\mathcal{P}) = H_{top}(\mathcal{P})$ follows directly from the definitions, because $\mathcal{P}$ and $\mathcal{P}$ have the same set of internal blocks, so $N_n(\mathcal{P}) = N_n(\mathcal{P})$.

It remains to show $H_{top}(\mathcal{P}) = H_{top}(\mathcal{P})$. For this it suffices to show that $N_n^I(\mathcal{P})$ is asymptotic to $N_n(\mathcal{P})$. Clearly, $N_n^I(\mathcal{P}) \leq N_n(\mathcal{P})$. Now we will show that there is a fixed $k$ such that $N_n^I(\mathcal{P}) \geq N_n(\mathcal{P})$ for all $n$. $\mathcal{G}$ is by definition a finite graph, and by assumption each vertex of $\mathcal{G}$ is reachable from $v$. Thus, there is a finite integer $k$ such that each vertex of $\mathcal{G}$ is reachable by a path originating at $v$ of length no longer than $k$. For a vertex $w$ of $\mathcal{G}$ let $\mathcal{P}(w, n)$ denote the set of all paths of length $n$ in $\mathcal{G}$ originating at $w$. Then

(9.1) \[ N_n^I(\mathcal{P}) = |\mathcal{P}(v, n)| \]

and

(9.2) \[ N_n(\mathcal{P}) = \left| \bigcup_{w \text{ a vertex of } \mathcal{G}} \mathcal{P}(w, n) \right|. \]

We construct an injective maps $f_w : \mathcal{P}(w, n) \rightarrow \mathcal{P}(v, n + w)$ for each vertex $w$ of $\mathcal{G}$, where $w$ is an integer less than or equal to $k$, such that the images of the maps $f_w$ are non-intersecting.

For a vertex $w \in \mathcal{G}$, choose some path $e_1 e_2 \ldots e_{r_w}$ from $v$ to $w$ for some $r \leq k$. Then for a path $e'_1 e'_2 \ldots e'_n$ of length $n$ originating $w$, let

(9.3) \[ f_w(e'_1 e'_2 \ldots e'_n) = e_1 e_2 \ldots e_{r_w} e'_1 e'_2 \ldots e'_n \in \mathcal{P}(v, n + r) \]

be obtained by concatenating on the left by our fixed path $e_1 e_2 \ldots e_{r_w}$. The maps $f_w$ are clearly injective, and since the $(n - 1)$st to last vertex of each path in the image of $f_w$ must be $w$, there images are nonintersecting.

Composing the pasting $\mathcal{P}(w, n + w) \rightarrow \mathcal{P}(v, n + k)$ (which we can do since $\mathcal{G}$ has no stranded states) gives an injection $\mathcal{P}(w, n) \rightarrow \mathcal{P}(v, n + k)$, which by (9.1) and (9.2) gives

(9.4) \[ N_n(\mathcal{P}) \leq N_n^I(\mathcal{P}) \]
The final level \( \leq 1 \) unique omitted value in omit an exit edge having value \( 1 \leq \) to equal any one of \( k \) the collection of all vertices reached after exactly \( n \) vertices at step \( n \). We consider those paths emanating from the pointed vertex \( v_0 \) that have successive labels \( \sigma := (n + i_1, ..., n + i_{n-1}) \) in which \( (i_1, ..., i_{n-1}) \) is a subset of \( n - 1 \) distinct values of \( (0, 1, ..., n - 1) \); there are \( n! \) such paths, and they are all realized in the one-sided sofic shift \( Y \). We term vertices reached at level \( k \) the collection of all vertices reached after exactly \( k \) steps along any of these paths, for \( 1 \leq k \leq n - 1 \). We observe that such paths never extend to the value \( i_n \) or \( n \), where \( i_n \) is the unique omitted value in \( (0, 1, ..., n - 1) \), but they do extend one step further, allowing \( i_n \) to equal any one of \( i_1, i_2, ..., i_{n-1} \). It follows that these paths must reach at least \( n \) distinct vertices at step \( n - 1 \), which are distinguished from each other by the fact that they each omit an exit edge having value \( i_n \) and do not omit an exit edge with any other value \( n + ij \), \( 1 \leq i \leq n - 1 \). Thus we have shown that least \( n = \binom{n}{n-1} \) distinct vertices are reached at the final level \( n - 1 \). We also note that all paths of length \( n \) starting from \( v \) having labels \( (n + i_1, ..., n + i_n) \) with each \( 1 \leq i_j \leq n \) and with at least two repeated values are legal paths occurring in the graph and are unique paths by the right-resolving property.

We now study the vertices reached at step \( j \geq 0 \) for all these paths, for \( 0 \leq j \leq n - 1 \), and show by induction on \( j \) that at the \( j \)-th step there are at least \( \binom{n}{j} \) such vertices not
reached at earlier steps. This result implies
\[ |V(H')| \geq \sum_{j=0}^{n-1} \binom{n}{j} = 2^n - 1, \]
which proves minimality. The base case \( j = 0 \) of the induction holds, since at step 0 there is the initial vertex \( v_0 \). For the induction step, assume it is true at stage \( j - 1 \), and consider all the vertices reached at stage \( j \). Note that for \( j \leq n - 2 \) all these vertices have exit edges for all \( n \) possible labels \( n, \ldots, 2n - 1 \), a property which distinguishes these vertices from final level vertices. These vertices also cannot coincide with those from any of the earlier levels, because if in those cases there is a shorter path reaching these vertices \( v_1 \), which then extends to a path at the final level vertex using \( n - 2 \) or fewer letters. But such a path is automatically extendible one more step with an arbitrary choice of label \( n + i_n, 0 \leq i \leq n - 1 \), which contradicts the path having reached a vertex at which extension by one of these letters is forbidden. This extendability property holds because the right-resolving property requires all such other paths be unique from the start vertex \( v_0 \). It follows that there must be at least \( \binom{n}{j} \) distinct such vertices \( w \), corresponding to each possible of \( j \) distinct (unordered) labels \( i_1, \ldots, i_j \) of \((0, 1, \ldots, n - 1)\). For if two such vertices coincided, we could extend a path to a final level vertex, and at least one of the two extensions would contain a repeated letter, showing that the final level vertex has an exit edge labelled \( n + i \) for \( 0 \leq i \leq n - 1 \), a contradiction. This completes the induction step, and the proof.

APPENDIX B: CHARACTERIZATIONS OF ONE-SIDED SOFIC SHIFTS

This appendix establishes the equivalence of our definition\([1,3]\) of one-sided sofic shift, to that used in Ashley, Kitchens and Stafford [7].

Theorem B-1. The following are equivalent, for a one-sided shift invariant set \( S \subset A^\mathbb{N} \)

(1) \( S \) is a one-sided sofic shift, i.e. it is the set of labels of all one-sided infinite walks along a finite labelled directed graph \( \mathcal{G} \), starting from any vertex \( \mathcal{G} \).

(2) \( S \) is closed one-sided subshift that has finitely many different infinite follower sets.

(3) \( S \) is a closed one-sided subshift that has finitely many different finite follower sets.

Proof. Here condition (3) is the definition used in [7]. For a closed one-sided subshift, given a finite word \( W \), we let \( F(W) \) denote the finite follower set determined by \( W \), consisting of all finite words that directly follow an occurrence of \( W \). We let \( \mathcal{F}(W) \) denote the infinite follower set, consisting of all one-sided infinite words that follow an occurrence of \( W \).

(1) \( \Rightarrow \) (3). By hypothesis the one-sided sofic shift \( S \) has a labelled graph presentation \( \mathcal{G} \), and this may be regarded as the presentation of a two-sided sofic shift as well. Finite follower sets with prefix \( W \) are the set of all allowable finite paths that follow a path with the given (finite) prefix, and they are the same for the one-sided shift as for the two-sided shift. The result follows since the follower set property (3) is established for two-sided shifts ([29, Theorem 3.2.10]).

(3) \( \Rightarrow \) (2) By hypothesis \( S \) is closed and shift-invariant. Then each finite follower set \( F(W) \) associated to a finite word \( W \) determines a unique infinite follower set \( \mathcal{F}(W) \), which is the set of one-sided infinite words that are limits of words in the finite follower set. Indeed any infinite string in \( \mathcal{F}(W) \) has all its finite prefixes in \( F(W) \) and is therefore in the limit set; conversely any limit word of \( F(W) \) will be in the infinite follower set, since
$S$ is closed and shift-invariant. Because the shift $S$ is one-sided, each infinite follower set in it is determined by some finite prefix $W$. But there are only finitely different such sets $F(W)$ by (2).

(2) $\Rightarrow$ (1). We construct a directed labelled graph $\mathcal{G}$ whose vertices correspond to the distinct infinite follower sets. Given such a set $F(W)$, we can add one letter $a \in A$ to it on the right $W' := Wa$ and then draw an edge in the graph $\mathcal{G}$, labelled $a$ to the vertex corresponding to the follower set $F(W')$. One now checks that the graph $\mathcal{G}$ generates the one-sided shift $S$.

Ashley, Kitchens and Stafford [7, Lemma 2.8] characterize one-sided sofic shifts $S$ as those closed shift-invariant sets that avoid an infinite set of forbidden blocks describable as the complete set of finite paths of a certain finite automaton between a given marked initial node $I$ and a given marked terminal node $T$.

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