The critical region of strong-coupling lattice QCD in different large-\(N\) limits

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Abstract

We study the critical behavior at nonzero temperature phase transitions of an effective Hamiltonian derived from lattice QCD in the strong-coupling expansion. Following studies of related quantum spin systems that have a similar Hamiltonian, we show that for large \(N_c\) and fixed \(g^2 N_c\), mean field scaling is not expected, and that the critical region has a finite width at \(N_c = \infty\). A different behavior rises for \(N_f \to \infty\) and fixed \(N_c\) and \(g^2/N_f\), which we study in two spatial dimensions and for \(N_c = 1\). We find that the width of the critical region is suppressed by \(1/N_f^p\) with \(p = 1/2\), and argue that a generalization to \(N_c > 1\) and to three dimensions will change this only in detail (e.g. the value of \(p > 0\)), but not in principle. We conclude by stating under what conditions this suppression is expected, and remark on possible realizations of this phenomenon in lattice gauge theories in the continuum.

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I. INTRODUCTION AND SUMMARY OF RESULTS

Large-\(N\) expansions are useful to approach nonperturbative dynamics of field theories and are often used to study their phase transitions at nonzero temperature \(T\). Here we focus on the dependence of the critical region of strongly coupled lattice QCD on the number of colors, \(N_c\), and flavors, \(N_f\). This region of temperatures is where mean field (MF) theory fails, and fluctuations, that cannot be ignored, drive the system to a nontrivial fixed point.

What motivates us are the studies of the suppression of this region in the large-\(N\) Gross-Neveu and Yukawa models [1–3]. In particular, the authors in [3], suggest that the same phenomenon occurs in QCD at large-\(N_c\), and discuss its physical implications. In apparent contrast to that, in the letter [4], which presents numerical simulations of strongly-coupled lattice QCD with staggered fermions and large \(N_c \leq 48\), the authors claim to find that MF fails at large-\(N_c\), and as evidence show that the critical region does not shrink with \(N_c\) in their simulations.

To understand this apparent contradiction and when to expect a suppression of the critical region in general, we study an effective Hamiltonian which is derived in second order of the strong-coupling expansion from the lattice QCD Hamiltonian, and that describes the low energy effective excitations of mesons. We work in \(d\) spatial dimensions and with \(N_c\) colors and \(N_f\) flavors of naive fermions. This gives us flexibility to explore a large parameter space, and enables us to show that which large-\(N\) makes a system trivial depends on which \(N\) is large. Following studies of similar Hamiltonians in the condensed matter literature [5–9], we explore the critical behavior of this effective Hamiltonian in three different large-\(N\) limits: (I) the large-\(N_c\) limit, where the ‘t Hooft coupling \(g^2N_c\) is kept fixed. (II) the combined limit of large-\(N_c\) and large-\(N_f\), where a l’á Veneziano, we fix both \(N_c/N_f\) and \(g^2N_c\), and (III) the limit of large \(N_f\) but with fixed \(N_c\) and \(g^2/N_f\). We summarize the results of these studies in the next paragraphs.

In the large-\(N_c\) limit we show that the largeness of \(N_c\) serves to suppress quantum fluctuations, and leads to a classical Hamiltonian. This Hamiltonian is a generalized Heisenberg antiferromagnet, whose ordered ground state corresponds to the chiral broken phase. A MF ansatz in terms of the ‘spins’ is exact only at \(T = 0\), and is not adequate to study the critical behavior near \(T_c\). In particular, the critical exponents are determined by \(N_f\) and \(d\), and are not expected to belong to the Gaussian fixed point. One can, as a first step, analyze the transition in MF, but by definition, this analysis is bound to fail within the critical region, which we show to have a nonzero width at \(N_c = \infty\). Unfortunately, since at \(N_c = \infty\) the temperature \(T_c\) turns out to be infinite, then it is difficult to learn about the behavior of transitions in planar QCD from the ‘t Hooft limit of our effective Hamiltonian, and we therefore use these results only to understand [4], but discuss how they should be improved.

In the combined limit of large \(N_c\) and \(N_f\), the situation is different, and one can solve for a MF ansatz that is exact at \(N_f, N_c \to \infty\) for all \(T\) (\(T_c\) is finite is this limit). Reviewing known results, we show that this solution yields a correlation length that diverges with a MF exponent only for \(d > 4\), and otherwise for \(2 < d \leq 4\). Extending this, we also show that the width of the critical region depends on \(N_f\), and \(N_c\) only through their ratio \(N_c/N_f\), and is nonzero.

In the limit of large-\(N_f\) and fixed \(N_c\), a MF ground state is again exact for all \(T\), but leaves chiral symmetry intact even at \(T = 0\). In \(d = 2\), a global minimum of the system is the “spin-Peierls” state, which breaks lattice translations and rotations [7, 8]. A corresponding
analysis in $d = 3$ includes an extensive search in the space of all possible ansatze for the global ground state, and is out of the scope of this work. Since this limit has a restricted relevance for QCD (for example, asymptotic freedom is lost here) but is crucial for our purpose (see below), we proceed to study the restoration of these lattice symmetries at finite $T$ for the $d = 2$ system, and for simplicity also fix $N_c = 1$. We believe that a generalization to $d = 3$, and $N_c > 1$ will change the results only in detail but not in principle, and find no point to do so. (For example, see the $N_c > 1$ generalization at $T = 0$, and $d = 2$ in [7]).

The lattice symmetries are restored in a second order transition at the finite $T_c$, whose critical behavior is closely related to the behavior seen in [1, 3, 10, 11] for the systems studied there. We show that taking $N_f \to \infty$ before $t \equiv |T - T_c|/T_c \to 0$, makes MF exponents exact. Switching the order of the limits, we find that the Landau-Ginzburg-Wilson (LGW) action for the transition describes scalar fields coupled through $O(1/N_f)$ interactions. As emphasized in [3, 10], this suggests a crossover behavior, which we find to occur when $t \sim 1/\sqrt{N_f}$.

Combining the results of [3, 10, 11], and of our study here, we emphasize that the critical region in large-$N$ phase transitions is suppressed only when the LGW effective action for the transition has an overall factor of $N^\alpha$, $\alpha > 0$. This suppresses thermal fluctuations in the order parameter, and can make MF scaling exact. Our message here is that this does not happen in all large-$N$ treatments, and in particular does not happen for the chiral phase transitions of the effective Hamiltonian that we study here.

We begin this report in Section II where we introduce the effective Hamiltonian of lattice QCD in the strong-coupling limit. We then move to discuss large $N_c$ and fixed $N_f$ in Section III, large $N_c$ and large $N_f$ in Section IV, and the large $N_f$ with fixed $N_c$ in Section V. We conclude in Section VI, and make proposals for future research in Section VII.

II. THE STRONG-COUPLING LIMIT: THE EFFECTIVE HAMILTONIAN

Strong coupling expansions were used in lattice gauge theory since its early days. In particular, they were performed for the gauge group $SU(N_c)$ as well as for $SU(3)$. For example, in the pure $SU(N_c)$ gauge theory, expansions were made for the string tension and the free energy as well as for the beta function [12]. In the case of QCD with $N_c$ colors, and various types of fermions, calculations were made for the effective low energy action of hadrons and for mesons masses [13, 14]. The result of these treatments (which is most relevant to the discussions is Sections III-IV) is that for large values of $N_c$, the natural expansion parameter in the strong coupling series is the inverse 't Hooft coupling $1/(g^2 N_c)$ rather than $1/g^2$ (see also [15, 16]). More precisely, in these expansions one fixes $N_c$ and take $g^2 N_c \gg 1$ to find that quantities which are $O(N_c^0)$ at large-$N_c$, such as $m_{\text{meson}}, m_{\text{baryon}}/N_c, f_\pi/N_c$, etc, can be written as a power series in $(g^2 N_c)^{-1}$ with coefficients that depend on $N_c$ and that become finite at $N_c = \infty$.

The result of this procedure may be different from taking the opposite order of the limits (first $N_c \to \infty$ and only then $g^2 N_c \to \infty$). As discussed in [16, 17], one may find that these limits commute only as long as $(g^2 N_c)_0 < g^2 N_c < \infty$. Indeed for the two-dimensional pure gauge theory with the Wilson action one finds $(g^2 N_c)_0 = 2$ [17]. For higher dimensions the situation is more involved but there still is a finite range of $g^2 N_c$ in which the theory is in a strong coupling phase (For example see early analytical results in [18] and recent numerical results in [19]). Since fermions are quenched in the 't Hooft limit, we take these results to
apply for QCD (with a fixed nonzero mass) as well, but we are not aware of any analogous results in the Veneziano limit of fixed $N_f/N_c$.

In this paper we follow the former approach (which is also the approach of Euclidean treatments such as [4]): we fix $N_c$ and use an expansion in $(g^2 N_c)^{-1} \ll 1$. The result of this expansion in the Hamiltonian formalism is that at second order one obtains the following effective Hamiltonian,

$$H = \frac{J}{N_c} \sum_{\mathbf{n}} \sum_{\mu} \sum_{\eta} Q^n_{\eta} Q^n_{\eta+\mu}.$$  \hspace{1cm} (2.1)

which is derived for naive fermions in [13, 20]. Here $\mathbf{n}$ denotes lattice sites, $\hat{\mu} = \hat{x}_1, \ldots, \hat{x}_d$ denotes the lattice directions, and the coupling $J$ scales with the ‘t Hooft coupling like $(g^2 N_c)^{-1}$ [13, 21]. The operators $Q^n$ generate the $U(4N_f)$ algebra

$$[Q^n_{\eta}, Q^p_{\rho}] = i f^{\eta\rho\sigma} Q^\sigma_{n\delta} \delta_{nm},$$  \hspace{1cm} (2.2)

and are defined as

$$Q^n_{\eta} = \sum_{\alpha\beta, a} \psi^\dagger_{\eta a\alpha} M_{\alpha \beta} Q^\beta_{a, \eta}.$$  \hspace{1cm} (2.3)

Here the fermion indices $\alpha$ and $\beta$ are Dirac-flavor indices that range from 1 to $4N_f$, while $a$ is a color index that ranges from 1 to $N_c$. The matrices $M^n$ are the generators of $U(4N_f)$ in the fundamental representation, and obey

$$\text{tr} M^n M^p = \frac{1}{2} \delta_{np}, \quad \sum_{\eta=1}^{16N_f^2} M^\eta_{\alpha \beta} M^\eta_{\gamma \delta} = \frac{1}{2} \delta_{\alpha \delta} \delta_{\beta \gamma}.$$  \hspace{1cm} (2.4)

The Hilbert space on site $\mathbf{n}$ is an irreducible representation of $U(4N_f)$ that corresponds to a rectangular Young tableau with $N_c$ columns, and $m$ rows, see Fig. 1. The corresponding baryon number on such a site is $B = m - 2N_f$. In this work we restrict to configurations with zero average baryon number, and in particular examine configurations with baryon number $B$ on the even sites and $-B$ on the odd sites. This means that we put conjugate $U(4N_f)$ representations on adjacent sites, represented by Young tableaux with $m$ rows on the even sites, and with $(N - m)$ on the odd sites. Following [5–7] we take $m = 2N_f$ in Sections III, and V, and $m = 1$ in Section IV. The reason is that these cases are the simplest to analyze.

The Hamiltonian Eq. (2.1) is a generalized Heisenberg quantum antiferromagnet, and its Néel ordered state spontaneously breaks the global $U(4N_f)$ symmetry. Adding explicit symmetry breaking terms can reduce $U(4N_f)$ to its subgroup $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$.
and as a result, the Néel phase corresponds to the spontaneous breakdown of chiral symmetry.¹

Note that Eq. (2.1) is a result of Rayleigh-Schrodinger degenerate perturbation theory applied to split the degeneracy of the zero gauge-flux sector of the strong-coupling lattice QCD spectrum. Excitations of strings with a nonzero gauge flux cost an energy that scales like $\sim g^2 N_c \gg 1$ and are neglected here. As a result the chiral phase transition of Eq. (2.1) will not involve a condensation of Polyakov lines of the type discussed in [23]. Adding the effects of these gauge strings will make the description of the transition more realistic, and in particular will relate chiral symmetry restoration to deconfinement. This restriction to the zero flux sector means that the Hamiltonian Eq. (2.1) describes chiral dynamics of QCD only at temperatures $T \lesssim g^2 N_c$. As a result one should be careful when ‘extrapolating’ the behavior of Eq. (2.1) (or its Euclidean counterpart) to QCD at temperatures close to the transition. This is a significant difficulty that exists in the strong coupling approach to finite temperature lattice QCD, which we cannot resolve here, and that the reader should be aware of.

Nonetheless, since the model in Eq. (2.1) shows different critical behaviors in different large-$N_c$ limits, it is a sufficient and useful choice for our study, and can tell what conditions are needed for a large-$N$ transition to have MF critical exponents. Also, the Hamiltonian Eq. (2.1) is sufficient to understand the work of [4], which also neglects excitations of the gauge-flux.

III. THE LARGE-$N_c$ LIMIT : A SIGMA MODEL

As mentioned in the previous section, we now fix the ‘t Hooft coupling $g^2 N_c$ (and hence $J$) and take the large-$N_c$ limit of Eq. (2.1). Writing the partition function of Eq. (2.1) using generalized spin coherent states gives a non-linear sigma model (NLSM) that was first derived in [7]. The $\sigma$ field at site $n$ is a $4N_f \times 4N_f$ hermitian, unitary matrix given by the $U(4N_f)$ rotation, $\sigma_n = U_n \Lambda U_n^\dagger$, of the reference matrix $\Lambda$. In this section we work with $m = 2N_f$ that fixes

$$\Lambda = \begin{pmatrix} 1_{2N_f} & 0 \\ 0 & -1_{2N_f} \end{pmatrix},$$

and makes $\sigma_n$ an element of $U(4N_f)/[U(2N_f) \times U(2N_f)]$. The action of the NLSM is

$$A = \frac{N_c}{2} \int_0^1 dt \left[ -\sum_n \text{Tr} \Lambda U_n^\dagger \partial_t U_n + 2x \sum_{n\mu} \text{Tr} \left( \sigma_n \sigma_{n+\mu} \right) \right],$$

where $x \equiv J/4T$ gives the coupling in units of the temperature $T$. Finally note that the global $U(4N_f)$ transformations are realized here as $U_n \rightarrow VU_n$, or $\sigma_n \rightarrow V\sigma_n V^\dagger$.

For large values of $N_c$, and $x \sim O(1)$, the overall factor of $N_c/2$ suppresses fluctuations, and the MF ansatz

$$\langle \sigma_n \rangle = \begin{cases} +\Lambda & n \in \text{even}, \\ -\Lambda & n \in \text{odd}, \end{cases}$$

which breaks $U(4N_f)$ to $U(2N_f) \times U(2N_f)$, is exact. An expansion around the MF ansatz, results in a chiral theory of mesons with $O(1/N_c)$ interactions [13].

¹ See, however, the remark at the top of Section IV.
In contrast, if $x$ is of $O(1/N_c)$ then fluctuations in the second, ‘interaction’ term, are not suppressed, and many sigma field configurations contribute to the path integral. Nonetheless, the largeness of $N_c$ suppresses quantum fluctuations for any $x$, by making $\partial_t U \simeq 0$ in the first, ‘kinetic’ term. This can also be seen by rescaling the operators $Q^n \rightarrow N_c Q^n$ in Eq. (2.2), to find that the commutation relations vanish at $N_c = \infty$. In fact, this is in analogy with the large-$S$ limit that makes quantum antiferromagnets into classical systems. In our case we get the classical Hamiltonian

$$\mathcal{H}_{\text{classical}} = \frac{N_c J}{4T} \sum_{n\mu} \text{Tr} \left( \sigma_n \sigma_{n+\mu} \right), \quad (3.4)$$

whose critical temperature $T_c \sim O(N_c J)$. This means that $x$ scales like $\frac{1}{N_c} \times \frac{T}{T_c}$ and therefore that at $T \sim O(T_c)$, $x$ is of $O(1/N_c)$. As a result the MF ansatz Eq. (3.3) is not predictive at these temperatures, and treatments such as [27], are expected to fail there. This can be also seen in the following way. The partition function of this classical system looks like

$$Z = \int D\sigma \exp \left[ -a \frac{T_c}{T} \sum_{n\mu} \text{Tr} \left( \sigma_n \sigma_{n+\mu} \right) \right], \quad (3.5)$$

where $a$ is a pure number of $O(1)$ given by $N_c J/(4T_c)$. In terms of $T/T_c$, Eq. (3.5) has no $N_c$ dependence at all, which means that the critical region has a finite width at $N_c = \infty$. Only outside this region will MF be a good description, like it is for ordinary spin systems. MF theory does not fail here since it is not supposed to work at all, and we believe that this is what [4] saw numerically. One might think that MF scaling fails because the ansatz used above is temperature independent, and as emphasized above, is strictly exact only for low $T$. In the next section we show that this expectation is too naive.

Finally we note that the estimate of $T_c \sim N_c$ has been obtained in other strong-coupling expansions as well (for example see [4, 24]). At first sight this is puzzling since there should be only one scale in this theory which is $\Lambda_{QCD}$. This situation is reminiscent of the one discussed in [25], where the author shows that the thermal corrections to $\langle \bar{\psi} \psi \rangle$ in the framework of continuum chiral perturbation theory are of $O(T^2/N_c)$. Naively, this leads to expect $T_c \sim \sqrt{N_c}$. This ‘paradox’ is resolved in [25] by noting that the expansion in $T$ must have a finite radius of convergence of the order of the Hagedorn temperature. The latter is an external notion to the chiral lagrangian, and can be thought of as proliferation of string of nonzero gauge-flux.

Indeed, the action of our nonlinear sigma model Eq. (3.2) is the chiral lagrangian in our context, and is reached only when one neglects flux excitations (see discussion in Section II). As a consequence, it seems that the reason that in strong-coupling $T_c \sim N_c$, and not a function of just $g^2 N_c$, is the discard of flux condensation. In fact the authors in [26] study the interplay of deconfinement and chiral symmetry restoration in strong coupling, and find that an assuming flux condensation gives a chiral symmetry restoration temperature which is $O(N_c^0)$ at large-$N_c$. We thus expect that the inclusion flux condensation will lead to a $T_c \sim O(N_c^0)$ in the Hamiltonian approach as well, and that it will be the critical behavior of the resulting system which may be indicative to what happens in the ‘t Hooft limit in the continuum.

\[ \text{For example, in [4] one finds that the temperature is given by } T_c/N_c = 1.5525(3) + O(1/N_c). \]
The aim of this section is to investigate a case where the MF ansatz is \( T \)-dependent and exact at \( N = \infty \). This can be achieved by taking both \( N_f \) and \( N_c \) to be large, while keeping their ratio fixed (\( g^2 N_c \) is still kept fixed here). This limit was first studied by [6], for \( m = 1 \) with the Schwinger bosons method, and leads to a spontaneous breakdown of \( U(4N_f) \rightarrow U(1) \times U(4N_f - 1) \). This tells us that chiral symmetry may be realized differently than in QCD, and that to break \( U(4N_f) \) in the same pattern as in Section III, one needs to generalize the procedures in [6] to \( m = 2 \). This was done in [9], which found that the simplest ansatz for general \( m \) gives the same MF equations as for \( m = 1 \). This encourages us to choose \( m = 1 \) as well, and to postpone a discussion of a more sophisticated ansatz to possible future research.

We now move to review the Schwinger bosons method, and present some of its results. For a more detailed discussion, we refer to [6, 28, 29] and, in the context of QCD, to [21].

The first step is to write

\[
Q_\eta^n = \begin{cases} 
  b_\alpha^n \cdot M_\eta \cdot b_\alpha^n & n \in \text{even}, \\
  b_\alpha^n \cdot (-M_\eta)^* \cdot b_\alpha^n & n \in \text{odd}, 
\end{cases} \tag{4.1}
\]

where the operators \( b_\alpha^n \) are bosonic fields that live on the lattice sites, and have only Dirac-flavor indices. This is an acceptable representation of \( Q_\eta^n \), since \( [b_\alpha^n, b_\beta^m] = \delta_{\alpha\beta} \delta_{nm} \), and therefore Eq. (2.2) is respected. To set the representation of the operators in Eq. (4.1) one puts \( N_c \) bosons on each site

\[
\sum_{\alpha=1}^{4N_f} b_{\alpha n}^\dagger b_{\alpha n} = N_c. \tag{4.2}
\]

This constraint, together with the fact that the bosonic single-site wave function is symmetric in Dirac-flavor indices, sets the \( U(4N_f) \) representation of all lattice sites to be the tableau of Fig. 1 with \( m = 1 \). Using Eq. (4.1), one can now represent the partition function, \( Z \), with bosonic coherent states, and obtain the path integral

\[
Z = \int \, Db \, Db^* \exp(-A), \tag{4.3}
\]

\[
A = -\int_0^{1/T} d\tau \sum_n \left[ \sum_\alpha b_{\alpha n}^* \partial_\tau b_{\alpha n} + \frac{\bar{J}}{N_f} \sum_\mu_\alpha \mu_\beta \sum_{n=1}^{4N_f} b_{\alpha n}^* b_{\beta n+\mu_\beta} b_{\beta n+\mu_\beta} \right], \tag{4.4}
\]

with \( \bar{J} = JN_f/(2N_c) \). Since we keep \( N_c/N_f \) and \( J \) fixed, then \( \bar{J} \) is fixed as well, and we drop the bar henceforth.

The prime in Eq. (4.3) means that the path integral is constrained to obey Eq. (4.2). Adding a Lagrange multiplier, \( \lambda_n \), to keep this constraint, and a Hubbard-Stratonovich (HS) link field, \( Q_{n \mu} \), to decouple the quartic interaction, one finds

\[
A = \int d\tau \left[ \sum_{n \mu} \frac{N_f}{\bar{J}} |Q_{n \mu}|^2 + i \sum_n N_c \lambda_n - \sum_\alpha b_{\alpha n}^\dagger [\partial_\tau + i \lambda_n] b_{\alpha n} - 2 \text{Re} \sum_{n \mu} Q_{n \mu}^* b_{\alpha n} b_{\alpha n+\mu} \right]. \tag{4.5}
\]
An integration over the bosons, gives the effective action for $Q$ and $\lambda$,

$$A_{\text{eff}} = 4N_f \left\{ \int d\tau \left[ \sum_{n\mu} \frac{1}{4J} |Q_{n\mu}|^2 + i \sum_n \kappa \lambda_n \right] + \text{tr} \log G^{-1} \right\}, \quad (4.6)$$

where $\kappa = N_c/(4N_f)$, and where $G$ is the propagator of a single boson, such that $\log \det G \sim O(1)$. For large values of $N_f$, but $\kappa \sim O(1)$, and $T/J \sim O(1)$, fluctuations are suppressed around any stable MF ansatz. The action of the ansatz $Q_{n\mu}(\tau) = Q, \lambda_n = i\lambda$, is

$$A_{\text{MF}} = \frac{4N_fN_s}{T} \left[ \frac{dQ^2}{4J} - \lambda \left( \kappa + \frac{1}{2} \right) + \frac{2}{N_s\beta} \sum_k \log 2 \sinh \left( \frac{\beta \omega_k}{2} \right) \right], \quad (4.7)$$

where $\beta = 1/T$. Also, $N_s$ is the number of sites, $k$ takes values in the Brillouin zone of the even sublattice. Defining $\gamma = \frac{1}{d} \sum_{i=1}^d \cos(k_i/2)$, one finds that the function $\omega_k = \sqrt{\lambda^2 - 4d^2Q^2\gamma_k^2}$ becomes $\omega_k^2 \simeq \Delta^2 + c^2k^2$ at low $k$, with the mass gap $\Delta = \sqrt{\lambda^2 - 4d^2Q^2}$, and with $c^2 = dQ^2$. A minimization of Eq. (4.7) with respect to $\lambda$ and $Q$, gives the MF equations

$$\frac{4Q}{dJ} = \frac{2}{N_s} \sum_k Q\gamma_k^2 \left( n_B(\omega_k) + \frac{1}{2} \right), \quad (4.8)$$

$$\kappa + \frac{1}{2} = \frac{2}{N_s} \sum_k \lambda \omega_k \left( n_B(\omega_k) + \frac{1}{2} \right). \quad (4.9)$$

with $n_B(\omega) = (e^{\beta \omega} - 1)^{-1}$. These equations are exact at large-$N$, and their solution gives the large-$N$ phase diagram. This is different from Section III where the mean-field ansatz is exact only at low $T$. In fact, we believe that the Schwinger bosons approach is an appropriate candidate to be the ‘finite $T$ mean field theory’ mentioned in [4]. Another candidate is presented in Section V.

The point at which the inverse correlation length $\Delta$ vanishes, signals the condensation of the bosons, which breaks $U(4N_f)$ to $U(1) \times U(4N_f - 1)$. Investigating Eqs. (4.8)–(4.9) shows that this happens for $T = 0$ at $\kappa \geq \kappa_c$,\(^3\) and that the symmetry is restored at $T = T_c(\kappa)$, which is finite [6, 29].

To extract the critical exponents, one expand Eqs. (4.8)–(4.9) around $T_c$. We illustrate this in Appendix A, where we redo the calculation of [29] to show that MF scaling holds only for $d > 4$, and that in terms of the reduced temperature $t = |T - T_c|/T_c$, one finds that $\Delta \sim t^{1/2}$. For $2 < d \leq 4$, infrared (IR) fluctuations change this behavior to $t \sim -\Delta^2 \log \Delta$ for $d = 4$, and to $\Delta \sim t^{1/(d-2)}$ if $2 < d < 4$, which coincides with the behavior of the $CP_N$ model at large-$N$ [30]. Extending these results, we also show that the critical region for $2 < d \leq 4$ is nonzero and depends on $N_f$ and $N_c$ only through $\kappa$.

We conclude this section by noting that the overall factor of $4N_f$ in Eq. (4.6) does not contradict the fact that IR modes are not suppressed. The simple reason is that the fields $Q$ and $\lambda$ are $U(4N_f)$ singlets, and are not the order parameters of the transition. This means that Eq. (4.6) is not the LGW action and that fluctuations in the order parameters need not be suppressed. Large-$N_f$ suppresses fluctuations in $Q$ and $\lambda$, which means that the results one obtained here are exact at $N_f = \infty$. Taking these fluctuations into account leads to corrections of $O(1/N_f)$ to the critical exponents, and to $T_c$.

\(^3\) $\kappa_c$ is $\sim 0.19$ for $d = 2$ [6], and $\sim 0.0778$ for $d = 3$ [28].
V. THE LARGE-$N_f$ LIMIT: FERMIONS

Here we take $N_c = 1$ and work in two spatial dimensions. We begin by using Eq. (2.4) to write Eq. (2.1) as

$$H = -\frac{\bar{J}}{N_f} \sum_{n,\mu} \bar{\psi}_n^{\dagger \alpha} \psi_{n+\mu}^{\alpha} \psi_{n+\mu}^{\beta} \bar{\psi}_n^{\beta} + \frac{\bar{J}}{N_f} \sum_{n,\alpha} \bar{\psi}_n^{\dagger \alpha} \psi_n^{\alpha},$$

(5.1)

where we define $\bar{J} = JN_f/2 \sim N_f/g^2$. Since we take large-$N_f$ with fixed $g^2/N_f$ then $\bar{J}$ is fixed as well, and for brevity, we drop the bar henceforth. To set the representation of the operators in Eq. (2.3) we follow [6, 9] and put $2N_f$ fermions on each site

$$\sum_{\alpha} \bar{\psi}_n^{\dagger \alpha} \psi_n^{\alpha} = 2N_f.$$  

(5.2)

This also means that the second term in Eq. (5.1) is a constant which we drop. Since the wave function of the fermions is anti-symmetric in the Dirac-flavor indices, then Eq. (5.2) puts it in the representation given in Fig. 1 with $m = 2N_f$. The partition function of Eq. (5.1) is [7]

$$Z = \int_{-\pi T}^{\pi T} D\lambda \int DQ DQ^* D\bar{\psi} D\bar{\psi} e^{-A},$$

(5.3)

$$A = \int d\tau \left\{ \sum_n \bar{\psi}_n (\partial_\tau - i\lambda_n) \psi_n \sum_{n,\mu} \left( \bar{\psi}_n Q_{n\mu} \psi_{n+\mu} + h.c. \right) + \frac{N_f}{J} \sum_{n,\mu} |Q_{n\mu}|^2 + i 2N_f \sum_n \lambda_n \right\}.$$ 

(5.4)

As in the previous section, $\lambda_n$ is a constraint field that keeps Eq. (5.2) on each site, and $Q_{n\mu}(\tau)$ is a HS field that decouples the four-Fermi interaction of Eq. (2.1). Apart from the global $U(4N_f)$ symmetry, Eq. (5.4) is also invariant under the $U(1)$ gauge transformation $\psi_n \rightarrow e^{i\Lambda_n(\tau)} \psi_n, \bar{\psi}_n \rightarrow \bar{\psi}_n e^{-i\Lambda_n(\tau)}, Q_{n\mu} \rightarrow e^{i(\Lambda_n(\tau)-\Lambda_{n+\mu}(\tau))} Q_{n\mu}, \lambda_n \rightarrow \lambda_n + \partial_\tau \Lambda_n$. Demanding that the fields $Q, \bar{\psi}, \psi$ will be single-valued, and that $\Lambda_n \in [-\pi T, \pi T]$ remains time-independent, restricts $\Lambda_n(\tau)$ to be time-independent as well.

To proceed, one integrates over the fermions and obtains the action

$$A_{\text{eff}} = 4N_f \left\{ \int d\tau \left[ \frac{1}{4J} \sum_{n,\mu} |Q_{n\mu}|^2 + \frac{i}{2} \sum_n \lambda_n \right] + \text{tr} \log D^{-1}(Q, \lambda) \right\},$$

(5.5)

where $D(Q, \lambda)$ is the Dirac operator of a single fermion, such that $\text{tr} \log D \sim O(1)$, and can be read from Eq. (5.4). For $T/J \sim O(1)$, the overall factor of $4N_f$ in Eq. (5.5) suppresses fluctuations in $Q$ and $\lambda$, and one can exactly solve for the ground state which was shown to be the spin-Peierls state [5, 7, 8]. It has $\lambda_n = 0$ on all sites, and $|Q_{n\mu}(\tau)| = q$ on a subset of the lattice links $B$, but zero otherwise. This ground state is four-fold degenerate, and is shown pictorially in Fig. 2. It is clear that this state breaks lattice translations and rotations, which are then restored at the critical temperature $T_c$. To discuss this transition, one writes the MF effective action for the spin-Peierls state

$$A_{\text{MF}} = \frac{4N_f N_s}{T} \left[ \frac{1}{4J} q^2 \times \frac{1}{4} \times 2 + \frac{1}{2\beta} \log n_F(q)n_F(-q) \right],$$

(5.6)
FIG. 2: The four degenerate spin-Peierls ground states, which break translation and rotation symmetries. The low energy effective action we compute begins from state “A”.

where $\beta = 1/T$, and $n_F(\epsilon) = (e^{\beta \epsilon} + 1)^{-1}$. A minimization of Eq. (5.6) with respect to $q$ yields the MF equation

$$q/2J = \tanh(q/2T),$$

(5.7)

that describes a stable ordered phase with $q > 0$ below $T_c = J$, and a MF scaling $q \sim (T_c - T)^{1/2}$ in its vicinity. Here we reach the same ‘puzzle’ as did [1, 3, 10, 11] for the Gross-Neveu, Yukawa and classical 2D Heisenberg models: the result above is exact at $N_f = \infty$, but for any finite $N_f$ it is wrong. There, the scaling of $q$ is dictated by the symmetry breakdown, and by $d$. The resolution of this puzzle is that the critical region shrinks with $N_f$ [3, 10, 11]. We now turn to show how this happens for our model.

We begin by calculating the effective action for fluctuations by replacing $\lambda_n \rightarrow 1/\sqrt{4N_f} \lambda_n$, and $Q_{n\mu} \rightarrow q + 1/\sqrt{4N_f} Q_{n\mu}$ in $A_{\text{eff}}$ (Here $q$ is the solution of the MF equation, and is nonzero only on $B$). We expand Eq. (5.5) to $O(1/N_f)$ in Appendix B, where we work in Matzubara space and show that the masses $m$ of the zero Matzubara fields $\phi \equiv Q(\omega = 0)$ vanish at $T_c$ like $m^2 \sim t \equiv |T_c - T|/T$. All other fields are massive, and we proceed by integrating them out in Appendix C. The effective action we find is

$$A_{\text{eff}}^0 = \sum_{n\mu} \frac{1}{2} m_{n\mu}^2 |\phi_n|^2 + \sum_{n_1n_2\mu} v_{n_1n_2\mu} \Re \phi_{n_1\mu}^{*} \phi_{n_2\mu} + \frac{1}{\sqrt{N_f}} \sum_{\{n\}, \mu\nu} \left[ V_{\mu\nu}^{(3)}(\{n\}) \phi_{n_1\mu}^{*} \phi_{n_2\mu}^{*} \phi_{n_3\mu} + h.c. \right] + \frac{1}{N_f} \sum_{\{n\}, \mu\nu} \left[ V_{\mu\nu}^{(4)}(\{n\}) \phi_{n_1\mu}^{*} \phi_{n_2\mu}^{*} \phi_{n_3\mu} \phi_{n_4\nu} + h.c. \right],$$

(5.8)

with $m^2, v \sim O(t), V^{(3)} \sim O(t^{1/2})$, and $V^{(4)} \sim O(1)$ (detailed expressions for these are given in Appendix C.)

As emphasized by [3, 10, 11] it is now clear where does the puzzle come from. Taking $N_f \rightarrow \infty$ before $t \rightarrow 0$ gives a Gaussian model. Switching the order of the limits, we get a weakly coupled $\phi^3$ theory, and the four-fold degeneracy of the spin-Peierls state leads one to expect that the universality class of the transition is of a $Z_4$ model in $d = 2$ (see for example
and its references). This points to a crossover behavior, where the susceptibility $\chi$ diverges like

$$\chi \sim t^{-1} f(x), \quad x = tN_f^p,$$

and where $f(x)$ is a scaling function that determines the critical behavior.

A calculation of $f(x)$ is simpler in the high temperature phase, where $q = 0$. There, the effective action has only diagonal and degenerate mass terms, and no cubic interactions. In momentum space the action reads

$$A_0^{\text{eff}} = \sum_{k_i} \frac{1}{2} m^2 |\phi_{k_i}|^2 + \frac{1}{N_f} \sum_{i,j} \sum_{k_1k_2k_3k_4} V_{ijkl}(\{k\}) \phi_{k_1i}^* \phi_{k_2j}^* \phi_{k_3j} \phi_{k_4i} + h.c.,$$

(5.10)

where the momentum $k$ belongs to the first Brillouin zone of the even sublattice, and the index $i = 1, 2, 3, 4$ denotes the four links outgoing from each even site, see Fig. 3. In

FIG. 3: The lattice of even sites, denoted by X, and odd sites, denoted by •. Each even site has four link fields denoted by $\phi_i$, with $i = 1, 2, 3, 4$.

Appendix C we show that $m^2 = (T - T_c)/(4TT_c)$, and give the form of $V \sim O(1)$.

We calculate the susceptibility $\chi_{ij}(k, k') = \langle \phi_i(k) \phi_j(k') \rangle$ to leading order in $1/N_f$ in Appendix D, and show that it coincides with Eq. (5.9) if $x = t = 1/\sqrt{13} \sqrt{N_f}$, and $f(x) = 1-1/x^2$.

This tells us that only if $x = \infty$ then one obtains the MF scaling of $\chi \sim t^{-1}$. As $x$ decreases from infinity, this behavior changes, and will eventually be determined by the nearest fixed point of the $\phi^4$ theory of Eq. (5.10). This crossover behavior will occur when $x \sim O(1)$ or $t \sim O(1/\sqrt{N_f})$.

To make a connection with the results of [3], we recall that a $\phi^4$ theory in $d$ dimensions and a $O(1/N)$ coupling enters its critical region when its mass squared is $m^2 \lesssim (1/N)^{2-d}$. Our result fits here if we put $d = 0$, and gives the largest critical region possible. The reason is that here the bare propagator of $|\phi|$ is momentum independent, and is simply given by $(1/2m^2)^{-1}$. This degeneracy will be removed by higher orders in $1/N_f$, and can be avoided by analyzing other ground states, such as the flux phase [8]. This also means that in a generalization to other values of $N_c$ and $d$, it is reasonable to expect that the width of the critical region will be given by $t \sim 1/N^p$, with $p \geq 1/2$, or by a possible logarithmic modification of that.

4 Or to a possible modification of that by topological effects, induced by the compactness of the gauge field.

These have been shown to be important in the $T = 0$ phase transition between the spin-Peierls and Néel states [31].
In contrast to the combined limit of large-$N_c$ and large-$N_f$, the critical region is suppressed here because the order parameter of the spin-Peierls state is the $Q$ field, whose action has an overall factor of $4N_f$. The latter suppresses the interaction terms in the LGW action given in Eq. (5.10), and can make MF scaling exact.

VI. SUMMARY

We work with an effective Hamiltonian derived from the lattice QCD Hamiltonian in the strong coupling expansion at second order. The large-$N_c$ limit of Eq. (2.1) is the analog of the large-$S$ limit of quantum antiferromagnets. For the latter, quantum fluctuations are suppressed, and one is left with a classical magnet, which is by no means a simple object. In particular, its critical behavior cannot be approached with a MF spin ansatz, because the latter is a good approximation only at sufficiently low temperatures. The spin $S$ determines the energy scale of the system, and for our case of Eq. (2.1), this leads to $T_c \sim N_c$, and to the fact that in terms of $T/T_c$, the critical region of our model is finite at $N_c = \infty$. We suggest that this is what [4] saw numerically in the action formalism, and that MF does not fail there, since it was not supposed to work in the first place.

The combined limit of large $N_c$ and large-$N_f$, but fixed ratio $N_c/N_f$, is more convenient to study the transition. There, the critical temperature is finite, and the MF ansatz, which is exact at $N_f = N_c = \infty$, is temperature dependent, and describes the critical region exactly. For $d > 4$, it gives Gaussian scaling, but for $d \leq 4$, IR modes change this and one can calculate the exponents exactly. Also, the width of the critical region depends on $N_f$ and $N_c$ only through $N_c/N_f$, and is nonzero.

In the large-$N_f$ limit, we have studied the case of $d = 2$, and $N_c = 1$. Here the MF ansatz is also exact for all $T$, and breaks lattice symmetries. This transition is characterized by MF critical exponents if $N_f$ is sent to infinity before sending $t = |T - T_c|/T_c$ to zero. In contrast, one can take $N_f \to \infty$, and $t \to 0$ with fixed $x \sim t\sqrt{N_f}$. This leads to a crossover from a Gaussian fixed point (at $x = \infty$) to a nontrivial fixed point of a weakly coupled scalar field theory. A schematic view of a possible renormalization group flow for this case is given in Fig. 4.

To conclude, it is clear that the critical region in large-$N$ phase transitions is suppressed when the Landau-Ginzburg-Wilson action has an overall factor of $N^\alpha$ with $\alpha > 0$. Taking the $N \to \infty$ can then remove IR fluctuations in the order parameter, and makes Gaussian scaling exact. We stress that this does not happen for all large-$N$ theories, and for our model Hamiltonian of Eq. (2.1), it happens only in the limit of large-$N_f$ and fixed $N_c$.

VII. FUTURE PROSPECTS

A generalization of the large-$N_f$ discussion to $d = 3$, and $N_c > 1$ is straight forward, but relatively costly. For example, one has to find the true vacuum out of all possible ansatze (although an effective action for fluctuation around a metastable vacuum can also be calculated). We believe that the results of such a generalization will differ from the case

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5 $\alpha = 1$ in conventional cases where $N$ is related to a global symmetry, while for deconfinement in a pure $SU(N)$ gauge theory one expects $\alpha = 2$. 
FIG. 4: A cartoon of a possible renormalization group (RG) flow in the \((t, 1/N_f)\) space \((t)\) is the reduced temperature \(\frac{|T_c - T|}{T_c}\). RG drives the system away from the Gaussian fixed point denoted by the ‘\(\bullet\)’. This happens along the \(1/N_f\) axis towards a possible nontrivial fixed point denoted by the ‘X’ (which for simplicity we placed at \(N_f = 1\)). The critical region is to the left of the dashed line, \(1/N_f \sim t^2\).

studied in Section V only in detail, and not in principle.\(^6\) In view of that, and the limited relevance of the large-\(N_f\) limit for QCD in our context, we find no point to do so analytically. A numerical check of our predictions, and of the possible generalization mentioned above, can be made with similar methods to the one of [4]. Ideally this should be complemented with an analytical study in the action formalism at large-\(N_f\), in similar lines to Section V.

As suggested by [25] and [3] the critical region may be suppressed in planar QCD (large-\(N_c\) with fixed \(g^2N_c\) and fixed \(N_f\)). While [3] suggests that the outside the critical region, one observes a MF behavior, then [25] expects that the chiral condensate will not change as a function of \(T\) at all until the critical region is reached. It will be interesting to check these theoretical expectations, but unfortunately the current study of our model Hamiltonian cannot (and is not intended to) accomplish this. The reason is two-fold; First we are very far from the continuum limit. Second, the Hamiltonian Eq. (2.1) is not designed to explore temperatures which are close to the deconfinement temperature. While the latter problem can be in principle tackled by including gauge-flux condensation (see the discussion at the end of Section III), it is hard to see how to overcome the former. Approaching continuum physics with numerical simulations at large-\(N_c\) and/or large-\(N_f\) is very costly. While simulating dynamical quarks at large-\(N_c\) is unrealistic, then quenching them or using dynamical fermions at large-\(N_f\) is expensive.

This leads us to seek phase transitions in other related models in which the critical region may be suppressed as well. An obvious choice, which is numerically ‘cheap’, is deconfinement phase transitions in large-\(N_c\), where the role of the LGW action is played by a Polyakov loop action, in the spirit of [32]. The fact that the pure gauge theory has a first order deconfinement transition at large-\(N_c\) (e.g. see [33]) means that to study such a phenomenon in pure gauge, one has to approach a hidden Hagedorn transition, where the mass of the loop

\(^6\) For example, one expects that both \(x\) and \(f(x)\), depend on \(t\) and \(N_f\), in a way that changes with the dimension \(d\).
vanishes. This was done in [34], but the results are not accurate enough to unambiguously
determine the critical exponents.\footnote{The reason is the high calculational cost, and the
fact that the field configurations used there are metastable rather than stable.}

Adding bosonic matter to the pure gauge theory may change the deconfinement transition
into second order. In particular, using bosons in a bi-fundamental representation with $N_c$
even, leaves a $Z_2$ subgroup of the $Z_N$ center intact. As a result, the deconfinement transition
(if second order) is in the Ising model universality class, and by adding a flavor structure to
the bosons, one can study the chiral symmetry restoration as well [35]. Also, these scalar
theories are interesting on their own right, as they may serve to check the large-$N_c$
relation
between supersymmetric and non-supersymmetric gauge theories discussed in [36].

**APPENDIX A: THE CRITICAL EXPONENTS AND THE WIDTH OF THE CRITICAL REGION IN THE COMBINED LARGE $N_c$ AND LARGE $N_f$ LIMIT**

To extract the critical exponents from Eqs. (4.8)–(4.9), we restrict to the disordered
phase, where there are no Bose condensates, and where one can replace the momentum
sums in Eqs. (4.8)–(4.9) with the integrals

\[
\Lambda = \int \left( \frac{dk}{2\pi} \right)^d \frac{\gamma^2 \coth(y \sqrt{1 - \gamma^2 + \gamma^2 \Delta^2})}{\sqrt{1 - \gamma^2 + \gamma^2 \Delta^2}}.
\]

\[
2\kappa + 1 = \int \left( \frac{dk}{2\pi} \right)^d \frac{\coth(y \sqrt{1 - \gamma^2 + \gamma^2 \Delta^2})}{\sqrt{1 - \gamma^2 + \gamma^2 \Delta^2}}.
\]

Here we also divided Eq. (4.8) by $Q/\lambda$, and defined $y = \beta \lambda/2$, and $\Lambda = 8\lambda/Jd$. Let us now
concentrate on Eq. (A2), and on values of $\kappa$ where the $T=0$ ground state is ordered. For
each such $\kappa$ there exists $y = y_c(\kappa)$, where Eq. (A2) gives $\Delta = 0$, signaling the transition.

Following [29] we denote the integral on the right hand side of Eq. (A2) by $I(y, \Delta)$, and
write Eq. (A2) as

\[
I(y, 0) - I(y_c, 0) = \int \left( \frac{dk}{2\pi} \right)^d \left[ \frac{\coth(y \sqrt{1 - \gamma^2}) - \coth(y \sqrt{1 - \gamma^2 + \gamma^2 \Delta^2})}{\sqrt{1 - \gamma^2 + \gamma^2 \Delta^2}} \right].
\]

For $I' \equiv -\partial I(y, 0)/\partial y$ we find

\[
I' = \int \left( \frac{dk}{2\pi} \right)^d \frac{1}{\sinh^2 y \sqrt{1 - \gamma^2}} > 0,
\]

which means that for $y \simeq y_c$, the left hand side scales like $(y_c - y) > 0$. Next, we expand
the right hand side of Eq. (A3) in $\Delta$, and note that sufficiently close to $T_c$ the integral is
controlled by IR momenta. For these we write $(1 - \gamma^2) = k^2/4d$ to find that the IR part of the integral is

$$f(\Delta) = \frac{(4d\Delta)^2}{y} \int_0^R \left( \frac{dk}{2\pi} \right)^d \frac{1}{k^2(k^2 + 4d\Delta^2)}. \tag{A5}$$

Here $R$ separates between the IR modes and rest, and we take $1 \gg R \gg 2\Delta \sqrt{d}$. As a result Eq. (A3) becomes

$$(y_c - y)/y_c = A_d \left[ \frac{\Delta^2}{\Delta^{d-2}} \log \frac{R}{\Delta} \right] \quad \text{for} \quad d > 4,
\frac{\Delta^2}{\Delta^{d-2}} \quad \text{for} \quad d = 4,
2 < d < 4, \tag{A6}$$

with the finite $A_d$ given by

$$A_d = \frac{(4d)^2}{y_c^2} \left[ \int \left( \frac{dk}{2\pi} \right)^d \frac{1}{k^4} \right] \quad \text{for} \quad d > 4,
\frac{1}{(2\pi)^d} \quad \text{for} \quad d = 4,
\int_0^{R/\Delta} \left( \frac{dx}{2\pi} \right)^d \frac{1}{x^2(x^2 + 4d)} \quad 2 < d < 4, \tag{A7}$$

We proceed to write Eq. (A1) as

$$\Lambda(1 - \Delta^2) = 2\kappa + 1 - \int \left( \frac{dk}{2\pi} \right)^d \sqrt{1 - \gamma^2 + \Delta^2 \gamma^2} \times \coth(y \sqrt{1 - \gamma^2 + \Delta^2 \gamma^2}), \tag{A8}$$

which at $T = T_c(\kappa)$ gives

$$\Lambda_c = 2\kappa + 1 - \int \left( \frac{dk}{2\pi} \right)^d \sqrt{1 - \gamma^2} \coth(y_c \sqrt{1 - \gamma^2}). \tag{A9}$$

This means that

$$\Lambda_c - \Lambda = -\frac{1}{2} \Delta^2(y_c I'' + \Lambda_c) + (y_c - y) I' , \tag{A10}$$

with

$$I'' = \int \left( \frac{dk}{2\pi} \right)^d \frac{1 - \gamma^2}{\sinh^2(y_c \sqrt{1 - \gamma^2})} , \tag{A11}$$

$$I''' = I' - I''. \tag{A12}$$

Together with $T = \Lambda Jd/(16y)$, this gives

$$t \equiv (T - T_c)/T_c = B_d \Delta^2 + A_d' \times \left[ \frac{\Delta^2}{\Delta^{d-2}} \log \frac{R}{4\Delta} \right] \quad \text{for} \quad d > 4,
\frac{\Delta^2}{\Delta^{d-2}} \quad \text{for} \quad d = 4,
2 < d < 4, \tag{A13}$$
with
\[ B_d = \frac{1}{2}(1 + \frac{Jd}{16T_c} I') \]
\[ A_d' = A_d(1 - \frac{Jd}{16T_c} I') \]
(A14)
(A15)

We choose to work with values of \( \kappa \), where both \( A_d' \) and \( B_d \) are positive.\(^8\) Also note the \( B_d \) should include other \( O(1) \) coefficients that come from momenta \( k^2 > R^2 \) in the left hand side of Eq. (A3).

For sufficiently small \( \Delta \), Eq. (A13) leads to \( \Delta \sim t^{1/2} \) for \( d > 4 \), while for \( d = 4 \) and \( 2 < d < 4 \), one finds that \( -t \sim \Delta^2 \log \Delta \), and \( \Delta \sim t^{1/(d-2)} \), respectively. The width of the critical region is given by comparing the two terms in the right hand side of Eq. (A13), and we find that MF behavior will fail when
\[ \Delta/R \ll \frac{1}{4} e^{-B_d/A_d'} \]
for \( d = 4 \), and when
\[ \Delta \ll \left( \frac{1}{B_d/A_d'} \right)^{1/(4-d)} \]
(A16)
(A17)

for \( 2 < d < 4 \). Since \( B_d/A_d' \) depends only on \( \kappa \) we conclude that the critical region of the phase transition in the large \( N_c \) and large \( N_f \) limit does not shrink with \( N_f \) or \( N_c \), and is nonzero (Here we ignore scenarios where one tunes \( \kappa \) to special values where \( B_d/A_d' \gg 1 \), and concentrate in generic values of \( \kappa \) where \( B_d/A_d' \sim O(1) \).)

**APPENDIX B: CALCULATION OF \( A_{\text{eff}} \) IN THE LARGE-\( N_f \) LIMIT**

In this section we calculate the large-\( N_f \) effective action for the auxiliary fields \( \bar{Q} \) and \( \lambda \) around \( T_c \), where the spin-Peierls state dissolves. We begin by replacing \( Q_{n\mu} \to q + \frac{1}{\sqrt{4N_f}} Q_{n\mu} \), and \( \lambda_n \to \frac{1}{\sqrt{4N_f}} \lambda_n \) in Eq. (5.5), and move to Matzubara space, where we denote the frequencies of the fermions by \( \epsilon \), and the external bosonic frequencies by \( \omega \). The action we get is
\[
A = A_{MF} + \sqrt{4N_f} \left[ \frac{\sqrt{\beta}}{2J} \sum_{n\mu \in B} q \text{ Re } Q_{n\mu}(\omega = 0) + \beta \frac{i}{2} \sum_n \lambda_n \right] + \frac{1}{4J} \sum_{n\mu} \sum_{\omega} |Q_{n\mu}(\omega)|^2 + \delta A,
\]
(B1)

\[
\exp(-\delta A) = \langle \exp \left\{ \frac{i}{\sqrt{4N_f}} \sum_n \lambda_n \sum_{\epsilon} \bar{\psi}_n(\epsilon) \psi_n(\epsilon) - \frac{1}{\sqrt{4N_f} \beta} \sum_{n\mu} \sum_{\epsilon, \omega} [\bar{\psi}_n(\epsilon + \omega) Q_{n\mu}(\omega) \psi_{n+\mu}(\epsilon) + h.c.] \right\} \rangle_0.
\]
(B2)

\(^8\) Numerically, we find that for \( \kappa \gtrsim 0.21 \) then \( A_d' \) is negative for \( d = 3 \). This leaves us with the window \( 0.0778 < \kappa < 0.21 \) to explore the symmetry breakdown. We are not aware of any discussions in the literature with regards to the cases where \( A_d' < 0 \).
The $\langle \cdot \rangle_0$ in Eq. (B2) means an average with respect to the free fermionic action given by $A_F = \bar{\psi} D_0 \psi$, with $D_0$ connecting fermions that reside on edges of a link on $\mathcal{B}$, and is given by

$$D_0^{-1} = \frac{1}{\epsilon^2 + q^2} \begin{pmatrix} -i \epsilon & q \\ q & -i \epsilon \end{pmatrix}. \tag{B3}$$

The calculation of $\delta A$ is straightforward and the result is given pictorially in Fig. 5, where the external legs represent the fields $Q$ and $\lambda$. We denote the contributions to $\delta A$ as

$$\exp(- \delta A) = \exp \left( \sum_{n,m=1}^4 \delta A_{Q^n \lambda^m} + O(1/N_f^3/2) \right),$$

where $N_{\tau}$ is the number of Euclidean time slices. Using the MF equation $\tanh(q/2T) = q/2J$, it is easy to show that Eqs. (B4–B5) cancel the $O(\sqrt{N_f})$ contributions in Eq. (B1). This is guaranteed since the mean-field solution is a stationary point of $A_{\text{eff}}$. 

1. $O(\sqrt{N_f})$ terms

The first term in Fig. 5 contributes

$$\delta A_Q = - \sum_{\mu \in \mathcal{B}} (Q_{n\mu}(\omega = 0) + h.c.) \sum_{\epsilon} \frac{q}{\epsilon^2 + q^2} \frac{1}{\sqrt{4N_f \beta}} \times 4N_f$$

$$= - \sqrt{4N_f \beta} \tanh(q/2T) \sum_{\mu \in \mathcal{B}} \text{Re} Q_{n\mu}(\omega = 0), \tag{B4}$$

$$\delta A_{\lambda} = - \sum_n \lambda_n \sum_{\epsilon} \frac{1}{\beta} \left[ -i \epsilon - \frac{1}{2} \frac{\epsilon^2 \beta/N_{\tau}}{\epsilon^2 + q^2} + O(1/N_{\tau}^2) \right] \times$$

$$\frac{i}{\sqrt{4N_f}} \times 4N_f \beta = - \frac{i}{2} \beta \sqrt{4N_f} \sum_n \lambda_n, \tag{B5}$$

where $N_{\tau}$ is the number of Euclidean time slices.
2. O(1) terms

The second diagram in Fig. 5 and the quadratic term in Eq. (B1), give the mass terms of $\lambda$ and $Q$. For $Q$ we find

$$
\delta A_{Q^2} = \sum_\omega \left[ \sum_{n \mu \in B} \left( \frac{1}{2} m^2 |Q_{\mu}(\omega)|^2 + v \text{Re} Q_{\mu}(\omega) Q_{\mu}(-\omega) \right) \right. \\
+ \left. \sum_{n \mu \notin B} \frac{1}{2} m^2 |Q_{\mu}(\omega)|^2 + \sum_{n_1, \mu \notin B'} 2v \text{Re} Q_{n_1 \mu}(\omega) Q_{n_2 \mu}(\omega) \right],
$$

(B6)

where the links $n_1 \mu$, and $n_2 \mu$ in $B'$ are connected by two links in $B$. The form of $m, v$ (see also the related [37]) is

$$
\frac{1}{2} m^2 = \frac{1}{4J} - \frac{1}{\beta} \sum_{\epsilon} \frac{\epsilon(\epsilon + \epsilon)}{(\epsilon^2 + q^2)((\epsilon + \omega)^2 + q^2)},
$$

(B7)

$$
v = \frac{1}{\beta} \sum_{\epsilon} \frac{q^2}{(\epsilon^2 + q^2)((\epsilon + \omega)^2 + q^2)}.
$$

(B8)

At $\omega = 0$, and $T < T_c$ we get

$$
m^2 = 2v = \frac{1}{4} \left[ \frac{1}{J} - \frac{\beta}{\cosh^2(\beta q/2)} \right],
$$

(B9)

which vanishes like $T_c - T$ near $T_c = J$. In the disordered phase, where $q = 0$, then $v = 0$, and $m^2(\omega = 0) = (T - T_c)/(4TT_c)$. For $\omega \neq 0$, we find that close to $T_c$ (where $q = 0$) the difference $\delta m^2 = m^2(\omega) - m^2(0)$ is given by

$$
\delta m^2 = \frac{2\omega}{\beta} \sum_{\epsilon} \frac{1}{\epsilon^2(\epsilon + \omega)} = \frac{2\omega}{\beta} \sum_{\epsilon} \frac{1}{(\epsilon - \omega)^2 \epsilon} \\
= \frac{\omega}{\beta} \left[ \sum_{\epsilon} \frac{1}{(\epsilon - \omega)^2 \epsilon} - \frac{1}{(\epsilon + \omega)^2 \epsilon} \right] \\
= \frac{4\omega^2}{\beta} \sum_{\epsilon} \frac{1}{(\epsilon - \omega)^2(\epsilon + \omega)^2} > 0.
$$

(B10)

Finally, the $\lambda$ fields are all massive

$$
\delta A_{\lambda^2} = \frac{1}{2} \sum_n \sum_{\epsilon} \lambda_n^2 \frac{\epsilon^2}{(\epsilon^2 + q^2)^2} \\
- \sum_{n \mu \in B} \lambda_n \lambda_{n + \mu} \sum_{\epsilon} \frac{q^2}{(\epsilon^2 + q^2)^2},
$$

(B11)

and $\delta A_{\lambda Q} = 0.$

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3. $O(1/\sqrt{N_f})$ terms

The cubic terms come from the third diagram in Fig. 5. There are contributions of the type $Q^3, Q^2 \lambda, Q \lambda^2,$ and $\lambda^3$. Provided that we are interested only in the regime $q^2 \sim 1/\sqrt{N_f}$, one needs to keep track of terms which are of $O(q^2/\sqrt{N_f})$. This leaves us with the following

$$\delta A_{Q^3}^- = \frac{1}{\sqrt{4N_f}} \sum_{n_{\mu} \in B} \sum_{\omega_{1,2}} \text{Re} \left( Q_{n\mu}(\omega_1) Q_{n\mu}(\omega_2) \right) \cdot Q_{n\mu}^*(\omega_1 + \omega_2) \times \left( V^{(3)} + O(q^3) \right),$$

(B12)

$$\delta A_{Q^3}^{--} = \frac{1}{\sqrt{4N_f}} \sum_{n_{\mu} \in B, \nu} \sum_{\omega_{1,2}} \text{Re} \left( Q_{n\mu}(\omega_1) Q_{n\nu}(\omega_2) \right) \cdot Q_{n\mu}^*(\omega_1 + \omega_2) \times V^{(3)},$$

(B13)

$$\delta A_{Q^3}^\top = \frac{1}{\sqrt{4N_f}} \sum_{n_{\mu} \in B, \mu} \sum_{\omega_{1,2}} \text{Re} \left( Q_{n+\mu,\mu}(\omega_1) Q_{n\mu}(\omega_2) \right) \cdot Q_{n+\mu,\mu}(\omega_1 - \omega_2) \times V^{(3)}.\]$$

(B14)

where by “−−” we mean paths along two links, with one link in $B$, and by “⊓” we mean staples that begin and end on the edges of a link in $B$. The cubic coupling $V^{(3)}$ depends on the external frequencies, and is given by

$$V^{(3)}(\omega_1, \omega_2) = \frac{2}{\beta^{3/2}} \sum_\epsilon \frac{q}{\epsilon^2 + q^2} \times \frac{\epsilon + \omega_2}{(\epsilon + \omega_2)^2 + q^2} \times \frac{\epsilon - \omega_1}{(\epsilon - \omega_1)^2 + q^2},$$

(B15)

Next we find that $\delta A_{Q^2 \lambda}$ is given by

$$\delta A_{Q^2 \lambda} = \frac{1}{\sqrt{4N_f}} \sum_\omega \left[ \sum_{n_{\mu}} |Q_{n\mu}(\omega)|^2 (\lambda_n + \lambda_{n+\mu}) \right] \times \frac{1}{\beta^{3/2}} \sum_\epsilon \frac{\epsilon^2 (\omega - \epsilon)}{(\epsilon^2 + q^2)((\omega - \epsilon)^2 + q^2)} + \sum_{n_{\mu} \in B} |Q_{n\mu}(\omega)|^2 (\lambda_n + \lambda_{n+\mu}) \times \frac{1}{\beta^{3/2}} \sum_\epsilon \frac{q^2 (\epsilon - \omega)}{(\epsilon^2 + q^2)((\omega - \epsilon)^2 + q^2)}.$$

(B16)

Here note that both contributions of Eq. (B16) do not involve the zero modes, since for $\omega = 0$, the sums over $\epsilon$ vanish. Next we find

$$\delta A_{\lambda^2 Q} = -\frac{1}{\sqrt{4N_f}} \sum_{n_{\mu} \in B} \text{Re} Q_{n\mu}(\omega = 0) \left( \lambda_n^2 + \lambda_n \lambda_{n+\mu} + \lambda_{n+\mu}^2 \right) \times \frac{2}{\beta^{3/2}} \sum_\epsilon \frac{q \epsilon^2}{(\epsilon^2 + q^2)^3},$$

(B17)

and that $\delta A_{\lambda^3} = 0.$
4. $O(1/N_f)$ terms

Here we have five type of interactions terms: $\delta A_{Q^4}$, $\delta A_{Q^2\lambda}$, $\delta A_{Q^2\lambda^2}$, $\delta A_{Q\lambda^3}$, $\delta A_{\lambda^4}$. Out of these, we present only $\delta A_{Q^4}$. The reason is that $\delta A_{Q^2\lambda}$ and $\delta A_{Q\lambda^3}$ are $O(q/N_f)$, and that the $O(1/N_f)$ terms, $\delta A_{\lambda^4}$ and $\delta A_{\lambda^2Q^2}$, contribute negligible corrections to the effective action of the $Q$ fields (see Appendix C). What we find is

$$\delta A_{Q^4}^- = \frac{1}{4N_f} \sum_{\mu, \nu} \text{Re} \left( Q_{n\mu}(\omega_1) Q^*_{n\mu}(\omega_2) Q_{n\mu}(\omega_3) \right) \times \frac{1}{2} V^{(4)},$$

$$\delta A_{Q^4}^+ = \frac{1}{4N_f} \sum_{\mu, \nu} \text{Re} \left( Q_{n\mu}(\omega_1) Q^*_{n\mu}(\omega_2) Q_{n\nu}(\omega_3) \right) \times V^{(4)},$$

$$\delta A_{Q^4}^0 = \frac{1}{4N_f} \sum_{\mu, \nu} \text{Re} \left( Q_{n\mu}(\omega_1) Q^*_{n+\mu,\nu}(\omega_2) Q_{n+\mu,\nu}(\omega_3) \right) \times 2V^{(4)},$$

where “$-$” we again mean all self-avoiding paths of length two, while the $\delta A_P$ term includes self-avoiding closed paths of length four, i.e. plaquettes. The quartic coupling is given by

$$V^{(4)}(\omega_1, \omega_2, \omega_3) = \frac{1}{\beta^2} \sum_{\epsilon} \frac{\epsilon}{\epsilon^2 + q^2} \times \frac{\epsilon + \omega_1}{(\epsilon + \omega_1 - \omega_2)^2 + q^2} \times \frac{\epsilon + \omega_1 - \omega_2}{(\epsilon + \omega_1 + \omega_3 - \omega_2)^2 + q^2} \times \frac{\epsilon + \omega_1 + \omega_3 - \omega_2}{(\epsilon + \omega_1 + \omega_1 - \omega_2)^2 + q^2}.$$

APPENDIX C: CALCULATION OF $A_{\text{eff}}^0$

In this section we integrate over the massive fields $Q(\omega \neq 0)$ and $\lambda$. (The gauge transformations are of course still massless, but do not cause any divergences in the path integral, since the gauge group is compact). To integrate over $\lambda$ we can assume that it has only a mass term, as close to $T_c$ the second term in Eq. (B11) is negligible. The effective action for the link fields $A_{\text{eff}}(Q)$ is defined by

$$\exp(-A_{\text{eff}}(Q)) = \exp\left(-\sum_n \delta A_{Q^n\lambda^0}\right) \times \langle e^{-\delta A_{Q\lambda^2} - \delta A_{Q^2\lambda} - \delta A_{Q^2\lambda^2}} \rangle,$$

where by $\langle \cdot \rangle$ we mean an average with respect to $\delta A_{\lambda^2}$, which we evaluate by expanding the exponential in $1/N_f$. Here we have neglected $\delta A_{\lambda^4}$, which is of $O(1/N_f)$, and will contribute to $A_{\text{eff}}(Q)$ only through $\langle \delta A_{\lambda^4} \delta A_{Q\lambda^2} \rangle$ which is of $O(q/N_f^{3/2})$.

Close to $T_c$, only the first term in Eq. (B11) is important, which means that only even terms in this expansion will survive. This leaves us with the following contributions. From
\[ \langle \delta A_{Q^2} \rangle \] we get a term of \( O(q/\sqrt{N_f}) \), which is linear in \( \text{Re}Q(\omega = 0) \). Together with Eq. (B4) and the first \( O(\sqrt{N_f}) \) term in Eq. (B1), this yields a \( 1/N_f \) correction for the MF equation, which can be absorbed in a \( O(1/N_f) \) correction to \( T_c \). From \( \langle \delta A_{Q^2\lambda^2} \rangle \) we get a \( 1/N_f \) contribution to the square masses of the \( Q \) fields. Since the masses of the \( \omega \neq 0 \) fields, and of the zero modes are of \( O(1) \), and \( O(q^2) \), respectively, we neglect this contribution as well. In fact, since the square mass of the zero modes scales like \( |\omega|^2 \), this contribution can also be considered as an \( O(1/N_f) \) correction to \( T_c \). Finally from \( \langle (\delta A_{Q^2})^2 \rangle \sim O(1/N_f) \) we get an \( O(1/N_f) \) contribution to the quartic interactions of the \textit{massive} modes, \( Q(\omega \neq 0) \).

We now proceed to integrate out the fields \( \tilde{Q} \equiv Q(\omega \neq 0) \). The starting point is the action

\[
A_{\text{eff}}(Q) = \sum_n \delta A_{Q^2\lambda^2} - \frac{1}{2} \langle (\delta A_{Q^2})^2 \rangle. \tag{C2}
\]

Writing the first expression on the right hand side in terms of the zero modes \( \phi = Q(\omega = 0) \), and the massive fields \( \tilde{Q} \), one finds that interactions of the type \( \tilde{Q}^2\phi \sim O(q/\sqrt{N_f}) \), and \( \tilde{Q}^2\phi^2, \tilde{Q}^3\phi \sim O(1/N_f) \). Repeating the steps we took to integrate over \( \lambda \), one finds almost the same conclusions; there are \( O(1/N_f) \) corrections to the square mass of the zero modes and to the MF equation, but there are no corrections to the quartic interactions of the zero modes.

In light of the above, the effective action of the zero modes, \( A_{\text{eff}}^0(\phi) \), is given by \( A_{\text{eff}}(Q) \) when setting \( \tilde{Q} = 0 \). This gives Eq. (5.8), with the masses \( m_{n\mu} = m(\omega = 0) \), and with the cubic and quartic terms, at zero external frequencies \( \omega_1 = \omega_2 = \omega_3 = 0 \). For our discussion in Section V, it is sufficient to write the effective action for the zero modes \textit{above} \( T_c \). This is simpler, since there \( q = 0 \), which makes all the mass differences and the cubic interactions vanish. The result is

\[
A_{\text{eff}}^0 = \sum_{n\mu} \frac{1}{2} m_n^2 |\phi_{n\mu}|^2 + \frac{V(4)}{4N_f} \left\{ \sum_{n\mu} \frac{1}{2} |\phi_{n\mu}|^4 + \sum_n |\phi_{n\mu}|^2 |\phi_{n\mu}|^2 \right\}

+ 2 \text{Re} \left( \phi_{n\mu} \phi_{n\mu}^* \phi_{n\mu}^* \phi_{n\mu}^* \right). \tag{C3}
\]

Here the second and third quartic interactions are between adjacent links lattice, and links on a common plaquette. Also we have \( m^2 = (T - T_c)/(4TT_c) \), and \( V(4) = \frac{1}{16} \sum_e e^{-1} = 1/(48T^2) \).

The action Eq. (C3) has a manifest \( U(1) \) gauge symmetry, which means that some of the fluctuations will have an identically zero mass. This action is also invariant under lattice translations and rotations, which are the symmetries that the spin-Peierls state breaks. Using the notations of Fig. 3 we introduce the Fourier transform

\[
\phi_{k_i} = \sqrt{\frac{2}{N_s}} \sum_N \phi_{N_i} e^{i N k} \times \begin{cases} 1 & i = 1 \\ e^{i k_3/2} & i = 2 \\ e^{i (k_3 + k_2)/2} & i = 3 \\ e^{i k_2/2} & i = 4 \end{cases}, \tag{C4}
\]

Here \( N \) denotes a site on the even sublattice, and \( k \) denotes a momentum in this lattice’s first Brillouin zone. In momentum space the action looks like

\[
A = \sum_{k_i} \frac{1}{2} m_n^2 |\phi_{k_i}|^2 + \frac{1}{48T^2} \frac{1}{4N_f} \sum_{k_i} \left\{ \frac{1}{2} \sum_i \phi_{k_i}^* \phi_{k_i}^* \phi_{k_i}^* \phi_{k_i}^* \right\}
\]

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+ \sum_{i>j} V_{ij}(k_3 - k_4) \phi_{k_i} \phi_{k_j} + V_{P}(\{k\}) \Re \left( \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \right) \delta_{k_1-k_2+k_3-k_4},

\tag{C5}

where \( k \) belongs to the Brillouin zone of the even sublattice, and
\[ V_{12} = V_{34} = 2 \cos(\hat{y}(k_3 - k_4)/2), \]
\[ V_{14} = V_{23} = 2 \cos((\hat{x} + \hat{y})(k_3 - k_4)/2), \]
\[ V_{24} = 2 \cos((\hat{x} - \hat{y})(k_3 - k_4)/2), \]
and \[ V_{P} = 4 \cos((k_2(\hat{x} - \hat{y}) + k_3 \hat{y} + k_4 \hat{x})/2). \]

**APPENDIX D: CALCULATION OF THE SCALING FUNCTION \( f(x) \)**

To see the crossover behavior of Eq. (5.10) we study the susceptibility in similar lines to the discussion in [10, 11].

\[
\chi_{ij}(\mathbf{k}, \mathbf{k}') = \frac{\int D\phi D\phi^* \exp(-A_{\text{eff}}^0) \phi_{k_i} \phi_{k_j}}{\int D\phi D\phi^* \exp(-A_{\text{eff}}^0)}.
\]

\tag{D1}

If \( N_f = \infty \) then \( A_{\text{eff}}^0 \) contains only quadratic terms, and \( \chi \sim t^{-1} \), where \( t \) is the reduced temperature \((T - T_c)/T_c\). To see deviations from this we expand the exponential in \( 1/N_f \).

The result is given pictorially in Fig. 6, and we find that \( \chi_{ij}(\mathbf{k}, \mathbf{k}') = \chi \delta_{ij} \delta_{k,k'} \), while \( \chi \) is

\[
\chi = \frac{1}{2} m^2 - \frac{1}{2} m^2 \times \left[ \frac{1}{4\pi^2 4N_f} \left( \frac{1}{2} + 2 \cdot 3 \right) \right] \times \frac{1}{2} m^2.
\]

\tag{D2}

Here the term in the brackets comes from the vertices Eq. (C5), and also counts the number of fields that run in the loop. In terms of \( t \) we find that the susceptibility is

\[
\chi = 8T_c t^{-1} \left[ 1 - \frac{13}{6} \frac{1}{N_f t^2} \right].
\]

\tag{D3}

Defining \( x = t \sqrt{\frac{4}{13} N_f} \), and comparing Eq. (D3) with Eq. (5.9), we identify \( f(x) = 1 - 1/x^2 \) and \( p = 1/2 \). More important, Eq. (D3) means that fluctuations become significant when \( x \sim O(1) \), indicating a crossover from MF (at \( x = \infty \)) to a nontrivial fixed point of the effective theory in Eqs. (5.8),(5.10). This occurs at \((T - T_c)/T_c \sim 1/\sqrt{N_f}\).

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[1] B. Rosenstein, A. D. Speliotopoulos and H. L. Yu, Phys. Rev. D 49, 6822 (1994).
[2] A. Kocic and J. B. Kogut, Phys. Rev. Lett. 74, 3109 (1995) [arXiv:hep-lat/9407021].
[3] J. B. Kogut, M. A. Stephanov and C. G. Strouthos, Phys. Rev. D 58, 096001 (1998) [arXiv:hep-lat/9805023].
[4] S. Chandrasekharan and C. G. Strouthos, Phys. Rev. Lett. 94, 061601 (2005) [arXiv:hep-lat/0410036].
[5] I. Affleck and J. B. Martson, Phys. Rev. B. 37, 3774 (1988).
[6] D. P. Arovas and A. Auerbach, Phys. Rev. B 38, 316 (1988).
[7] N. Read and S. Sachdev, Nucl. Phys. B 316, 609 (1989).
[8] J. B. Martson, I. Affleck, Phys. Rev. B 39, 11538 (1989).
[9] N. Read and S. Sachdev, Phys. Rev. B. 42, 4568 (1990).
[10] S. Caracciolo, B. M. Mognetti and A. Pelissetto, Nucl. Phys. B 707, 458 (2005) [arXiv:cond-mat/0409536].
[11] S. Caracciolo, B. M. Mognetti and A. Pelissetto, arXiv:hep-lat/0500963.
[12] J. M. Drouffe and J. B. Zuber, Phys. Rept. 102, 1 (1983). J. B. Kogut and J. Shigemitsu, Phys. Rev. Lett. 45, 410 (1980) [Erratum-ibid. 45, 1217 (1980)].
[13] J. Smit, Nucl. Phys. B 175, 307 (1980).
[14] N. Kawamoto and J. Smit, Nucl. Phys. B 192, 100 (1981). I. Ichinose, Nucl. Phys. B 249, 715 (1985). S. Aoki, Phys. Rev. D 34, 3170 (1986). F. Berruto, G. Grignani and P. Sodano, Phys. Rev. D 62, 054510 (2000) [arXiv:hep-lat/9912038]. G. Grignani, D. Marmottini and P. Sodano, Phys. Rev. D 68, 076003 (2003) [arXiv:hep-lat/0306004].
[15] G. ’t Hooft, arXiv:hep-th/0204069.
[16] I. Bars and F. Green, Phys. Rev. D 20, 3311 (1979).
[17] D. J. Gross and E. Witten, Phys. Rev. D 21, 446 (1980).
[18] F. Green and S. Samuel, Nucl. Phys. B 194, 107 (1982).
[19] M. Campopriini, Nucl. Phys. Proc. Suppl. 73 (1999) 724 [arXiv:hep-lat/9809072]. B. Lucini and M. Teper, JHEP 0106, 050 (2001) [arXiv:hep-lat/0103027]. F. Bursa and M. Teper, arXiv:hep-th/0511081.
[20] B. Svetitsky, S. D. Drell, H. R. Quinn, and M. Weinstein, Phys. Rev. D 22, 490 (1980);
[21] B. Bringoltz, arXiv:hep-lat/0407018.
[22] B. Bringoltz and B. Svetitsky, Phys. Rev. D 69, 014502 (2004) [arXiv:hep-lat/0310032].
[23] A. M. Polyakov, Phys. Lett. B 72, 477 (1978).
[24] P. H. Damgaard, N. Kawamoto and K. Shigemoto, Phys. Rev. Lett. 53, 2211 (1984).
[25] A. V. Smilga, Phys. Rept. 291, 1 (1997) [arXiv:hep-ph/9612347].
[26] A. Gocksch and M. Ogilvie, Phys. Rev. D 31, 877 (1985).
[27] Y. Umino, arXiv:hep-lat/0510063.
[28] A. Auerbach, *Interacting electrons and quantum magnetism* (Springer-Verlag, Berlin, 1994).
[29] S. Sarker, C. Jayaprakash, H. R. Krishnamurthy and M. Ma, Phys. Rev. B 40, 5028 (1989).
[30] V. Yu. Irkhin, A. A. Katanin, and M. I. Katsnelson Phys. Rev. B 54 11953 (1996). [arxiv: cond-mat/9703011].
[31] Michael Levin, T. Senthil arXiv:cond-mat/0405702.
[32] P. H. Damgaard and A. Patkos, Nucl. Phys. B 272, 701 (1986). A. Dumitru, J. Lenaghan and
R. D. Pisarski, Phys. Rev. D 71 (2005) 074004 [arXiv:hep-ph/0410294].

[33] B. Lucini, M. Teper and U. Wenger, JHEP 0502, 033 (2005) [arXiv:hep-lat/0502003].

[34] B. Bringoltz and M. Teper, Phys. Rev. D 73, 014517 (2006) [arXiv:hep-lat/0508021].

[35] F. Sannino, arXiv:hep-th/0507251; private communications.

[36] A. Armoni, M. Shifman and G. Veneziano, Nucl. Phys. B 667, 170 (2003) [arXiv:hep-th/0302163].

[37] M. Di Stasio, E. Ercolessi, G. Morandi, Phys. Rev. B 45, 1939 (1992)