On the equivalence of Playfair’s axiom
to the parallel postulate

Elizabeth T. Brown∗, Emily Castner †, Stephen Davis ‡, Edwin O’Shea (§), Edouard Seryozhenkov ¶ and AJ Vargas∥ ∗∗

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Abstract

We show that the classical equivalence of Euclid’s parallel postulate and Playfair’s axiom collapses in the absence of triangle congruence. In particular, we construct a non-SAS geometry that models the Playfair axiom but not the parallel postulate.

Introduction

Euclid’s account of the Side-Angle-Side condition for triangle congruence, [3, Proposition 4 of Book I], relies on an appeal to superposition, the application of one triangle to another by “dragging-and-dropping.” Yet he is at pains to avoid this method in Proposition 2 – and in many propositions that follow – which would be easy to prove via the superposition of one line segment or angle to another. Indeed his use of superposition is conservative, used only in Propositions 4 and 8 with latter propositions like Proposition 23 rhyming with Proposition 2 in their restraint.

It is now understood that the presence of the SAS axiom in a geometry is equivalent to that geometry having rigid motions (cf. [4, §17]), so the preservation of distance and angles under motion is equipotent to the ability to declare two triangles congruent. This is but one instance of Klein’s “Erlanger Programm” which proposed that a geometry be completely classified by the invariant theory of its group of transformations [7, Part III, §1.1]. This was a paradigm shift that had as pervasive an influence on mathematics as the discovery of non-Euclidean geometry. In the Euclidean context, Klein’s approach is inescapably tied to triangle congruence, motivating our study of geometries borne of Euclidean axioms but without SAS.

Indeed the failure of SAS has profound consequences for the parallel postulate. Classically, Euclid’s parallel postulate has many equivalents, most notably Playfair’s axiom which was known since antiquity [10] (cf. [5, p.220]). The main result of this paper is as follows:

∗ Department of Mathematics & Statistics, James Madison University, Harrisonburg VA 22807. (brownet@jmu.edu)
† Department of Mathematics & Statistics, Mount Holyoke College, South Hadley, MA 01075. (castn22e@mtholyoke.edu)
‡ Department of Mathematics, University of Wisconsin, Madison, WI 53706. (smdavis7@wisc.edu)
§ Department of Mathematics & Statistics, James Madison University, Harrisonburg VA 22807. (osheaem@jmu.edu)
¶ Department of Mathematics, Willamette University, Salem, OR 97301. (edseryo@gmail.com)
∥ Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010. (avarganjr@brynmawr.edu)
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**Theorem.** In the absence of SAS, Euclid’s parallel postulate is not equivalent to Playfair’s axiom. In particular, there exists a non-SAS geometry $G$ that models the Playfair axiom but not the parallel postulate.

So far as we know, this is the first demonstration of this result. Beeson, in his 2016 monograph “Constructive Geometry and the Parallel Postulate,” [1] shows that the Playfair axiom is strictly weaker than the Parallel Postulate in the setting of constructive logic. His result differs from ours in that the rules of inference in constructive logic are restricted from those of classical first order logic, which is to say, from the rules of inference assumed by most mathematicians. For example, in constructive logic if one shows that the statement “A does not exist” leads to a contradiction, one cannot conclude (without additional argument) that therefore A does exist. Moreover, Beeson’s models are “given without discussing the exact choice of the other axioms of geometry” and thus are not concerned with non-SAS [1 Introduction].

**Axiomatic Context**

In keeping with modern sensibilities, we will use Hilbert’s framework for Euclidean geometry vis-à-vis Foundations of Geometry [6, Chapter I]. His axioms are grouped according to incidence in the plane (Axioms I.1-3), order of points or betweeness (Axioms II.1-4), congruence for segments, angles, and triangles (Axioms III.1-5), and the axiom of parallels (Axiom IV). For reference, we list the central axioms of SAS and parallels.

Hilbert’s version of the SAS axiom (III. 5) is:

\[(\text{SAS}) \quad \text{If two triangles share two sides and the angle in between those two sides then in each triangle there are second angles that are also equal.}\]

This says nothing of the third side and third angle but Hilbert shows [6, §6, Theorem 12] that (classical) Euclidean SAS holds assuming axiom groups I, II, and III. Hilbert’s axiom of parallels, Axiom IV [6, §4], curiously called “Euclid’s Axiom” by Hilbert, states:

\[(h\text{PF}) \quad \text{Let } a \text{ be any line and } A \text{ a point not on it in a common plane. Then there is at most one line in the plane, determined by } a \text{ and } A, \text{ that passes through } A \text{ and does not intersect } a.\]

Hilbert’s version is slightly weaker than the classical Playfair axiom (cPF), which insists that there is exactly one line rather than merely at most one line. Hilbert’s version allows for, say, the geometry of geodesic lines on the sphere. Euclid’s original parallel postulate [3, Book I, Postulates] asserts:

\[(\text{PP}) \quad \text{That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.}\]

The reader will note that Euclid’s statement involves both lines and angles, his two basic objects of plane geometry, while Playfair refers to lines only. Most authors of the late 19th century chose Playfair over the parallel postulate, a tradition that persists in modern treatments of the foundations of Euclidean geometry like Hartshorne [4]. Dodgson [2] is the most notable holdout, exalting the positive constructive formulation of Euclid’s original postulate over the existence assertion of Playfair (cf. [3, p.313-4]). It is not obvious from the literature why this axiomatic drift to Playfair’s version occurred. In Hilbert’s case it seems reasonable that the success of his non-constructive approach to invariant theory (see [8, Pages xix and 40]) had a significant influence on how he approached axiomatics in geometry. Or he may simply have been scrupulous in applying the aesthetic of parsimony.
Proof of the Main Theorem

We will construct a geometry $G$ that respects Hilbert’s axioms of incidence (I), order (II), and congruence (III) except for the SAS axiom (III.5). The geometry models cPF, and therefore hPF, but not PP. Points and lines of $G$ will be those of the usual Euclidean plane $\mathbb{R}^2$. Angle congruence is manipulated from the usual measure while still maintaining the literal integrity of Hilbert’s axioms.

Hilbert defines angles in terms of rays. Given a line $\overline{h}$ and a point $O$ on $\overline{h}$, there are two rays (or “half lines”), $h$ and $h'$ (well defined by the group of order axioms, Group II) on either side of $O$.

$$\begin{align*}
h' & \quad \overline{h} \\
& \quad O \\
h & \quad \end{align*}$$

An angle $\angle$ is then defined as any two distinct rays $h, k$ with common vertex $O$ and lying on distinct lines $\overline{h}, \overline{k}$ respectively. The pair of rays $h, k$ is called an angle and is denoted by $\angle(h, k)$ or by $\angle(k, h)$ [6, p.11].

Hilbert excludes angles comprised of pairs of rays which together make a line. His definition of the interior of an angle is complicated by a desire to refer only to sides of rays and lines; he then uses axioms of incidence and order to prove that his definition selects the part of the plane that includes any line segment connecting the rays [6, p.11]. Since order of rays is irrelevant and interior of angle is unambiguous, every angle would have, in standard terms, radian measure exceeding 0 but less then $\pi$. Two angles having a vertex and one side in common and whose separate sides form a line are called supplementary angles; angles which are congruent to one of their supplementary angles are called right angles [6, p.13] There is no further restriction on congruence other than that every angle be congruent to itself [6, p.12].

The relation of angle congruence in $G$ will be interpreted as follows. For angle $\angle(h, k)$, let $\mu(\angle(h, k))$ be the standard radian measure without regard to order, so that $\mu(\angle(h, k)) = \mu(\angle(k, h))$. The range of $\mu$ on angles in this context is $(0, \pi)$. For each point $P$ in $\mathbb{R}^2$, choose a bijection

$$\hat{f}_P : (0, \frac{\pi}{2}) \to (0, \frac{\pi}{2}).$$

For each point $P$, we use the function $\hat{f}_P$ to label angles with vertex at $P$ in a way specific to $P$. For $\angle(h, k)$ with vertex $P$, define

$$f_P(\angle(h, k)) = \begin{cases} 
\hat{f}_P(\mu(\angle(h, k))) & \text{if } \mu(\angle(h, k)) < \frac{\pi}{2} \\
\frac{\pi}{2} & \text{if } \mu(\angle(h, k)) = \frac{\pi}{2} \\
\pi - \hat{f}_P(\pi - \mu(\angle(h, k))) & \text{if } \mu(\angle(h, k)) > \frac{\pi}{2}.
\end{cases}$$

\footnote{By contrast, Euclid’s Definition 8 is: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. [3]}
We interpret the congruence relation on angles using vertex labels. Given \( \angle(h, k) \) with vertex \( P \) and \( \angle(h', k') \) with vertex \( P' \),
\[
\angle(h, k) \sim \angle(h', k') \iff f_P(\angle(h, k)) = f_{P'}(\angle(h', k')).
\]
Note that all right angles are in the same equivalence class, since \( f_P(\pi/2) = \pi/2 \) for all \( P \). “Adding angles” is accomplished by adding equivalence classes in the usual way. Suplementarity is also preserved: if two angles \( \angle(h, k) \) and \( \angle(k, l) \) with common vertex \( P \) together make a line, then \( f_P(\angle(h, k)) + f_P(\angle(k, l)) = \pi \).

This relation respects Hilbert’s angle congruence axiom, III.4: Each \( \angle(h, k) \) is equivalent to itself. Since \( f_P \) and \( f_{P'} \) are bijections, given a fixed \( h, k \) with common vertex \( P \) and a fixed \( h' \) with vertex \( P' \) and a given side of the line that contains \( h' \), there is a unique \( k' \) with vertex at \( P' \) on the given side of the line that contains \( h' \) such that \( f_P(\angle(h, k)) = f_{P'}(\angle(h', k')) \). Since Hilbert’s axioms before III.4 do not concern angles – only points and lines – the geometry \( G \) satisfies Hilbert’s axiom groups I, II, and III with the exception of III.5, the SAS axiom. SAS fails in \( G \) since rigid motions of a triangle do not preserve the angles of that triangle.

To complete the construction fix values of the \( f_P \) functions at the origin and at the point \((1, 1)\), with lines \( \ell \) given by \( y = 0 \), \( \ell' \) given by \( y = 1 \), transversal \( t \) given by \( y = x \), and angles \( \theta, \theta', \theta'' \) as below.

Fix bijections \( f_{(0,0)} \) and \( f_{(1,1)} \) so that \( f_{(0,0)}(\pi/4) = \pi/4 \) and \( f_{(1,1)}(\pi/4) = 7\pi/16 \). Then \( \theta \in [\pi/4] \), and \( \theta'' \in [7\pi/16] \). Then \( \theta', \theta'' \)'s supplement, satisfies \( \theta' \in [9\pi/16] \). So \( [\theta] + [\theta'] = [13\pi/16] \), which is less than two right angles. The geometry \( G \)'s angle congruency \( \sim \) has not changed lines or intersections, so Playfair's axiom is still satisfied. But the parallel postulate fails, since the congruency class \( [\theta] + [\theta'] \) is less than two right angles, but lines \( y = 0 \) and \( y = 1 \) do not intersect. This completes the proof.

Although the assignments \( f_{(0,0)} \) and \( f_{(1,1)} \) were specific, the \( f_P \) functions are quite flexible. To create a non-SAS geometry with this labeling scheme requires only one point \( P \) at which angle congruence classes are assigned in an unorthodox but axiomatically legitimate way. It is also possible to insist that \( G \) preserve order on angles; to each point \( P \) in the plane at which nonstandard labeling is desired, assign a distinct \( r \in (1, \infty) \). Then let
\[
\hat{f}_P(x) = x \left( \frac{2\pi}{16} \right)^r \quad \text{for} \quad x \in (0, \frac{\pi}{2})
\]
This assignment for every point \( P \) in \( G \) would preserve Common Notion 5 of Euclid, that “the whole is greater than the part” and the manner in which it is invoked in the angle ordering of Propositions 16 and 20 of Book I.
Closing Remarks

The question of the converse is already settled. Hilbert (and others) showed that in the presence of all his axioms, Playfair and the parallel postulate are equivalent. So, a model of PF and \( \neg \text{PP} \) must fail at least one of the other axioms. Without further ado, then, we can conclude that, if Hilbert’s other axioms are assumed,

\[(\text{PF} \land \neg \text{PP}) \rightarrow \neg \text{SAS}\]

by Hilbert’s own arguments. Our contribution lies in showing that the statement is not vacuously true since the antecedent is satisfiable.

Playfair is the most-used but by no means only equivalent of the parallel postulate (cf. \cite{5} p.220). The behavior of these equivalencies in absolute geometry (see Pambuccian \cite{9}) and other weakened forms of Euclidean geometry is of ongoing interest. For example, it would be natural to suppose that in our geometry \( \mathcal{G} \) those equivalences which involve angles ought not to hold. This is false. Legendre’s equivalence, “there exists one triangle whose angle sum is two right angles,” is based on angles only but \( \mathcal{G} \) has enough flexibility to model Legendre’s condition: Take any three points \( Q, R, S \) and let \( f_Q = f_R = f_S = \text{id} \), the identity map. Then Legendre’s condition holds on the triangle \( \Delta QRS \) in \( \mathcal{G} \).

We close by asking for statements weaker than, or simply other than, SAS that are strong enough to recover all of the classical equivalencies of the parallel postulate. For example, an intuitive property of angles that holds in the standard model of Euclidean geometry yet is stronger than Common Notion 5, is that a whole angle ought to be the sum of its parts. Namely, given rays \( h, k, r \) with common vertex and with \( r \) “inside” the angle \( \angle(h,k) \) then \( \angle(h,k) = \angle(h,r) + \angle(r,k) \). Does a non-SAS geometry with this provision necessarily satisfy all the traditional equivalencies of PP? If so, does this provision form an intermediate axiom between non-SAS and SAS in the presence of Hilbert’s other axioms?

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