THE MÖBIUS DISJOINTNESS CONJECTURE
FOR DISTAL FLOWS

JIANYA LIU & PETER SARNAK

1. The M"obius disjointness conjecture

Let $\mathcal{X} = (T, X)$ be a flow, namely $X$ is a compact topological space and $T : X \to X$ a continuous map. The sequence $\xi(n)$ is observed in $\mathcal{X}$ if there is an $f \in C(X)$ and an $x \in X$, such that $\xi(n) = f(T^n x)$. Let $\mu(n)$ be the Möbius function, that is $\mu(n)$ is 0 if $n$ is not square-free, and is $(-1)^t$ if $n$ is a product of $t$ distinct primes. We say that $\mu$ is linearly disjoint from $\mathcal{X}$ if

$$\frac{1}{N} \sum_{n \leq N} \mu(n) \xi(n) \to 0, \quad \text{as } N \to \infty,$$

for every observable $\xi$ of $\mathcal{X}$. The Möbius Disjointness Conjecture of the second author ([16], [17]) states the following.

**Conjecture 1.1** (The Möbius Disjointness Conjecture). The Möbius function $\mu$ is linearly disjoint from every $\mathcal{X}$ whose entropy is 0.

This Conjecture has been established for many flows $\mathcal{X}$ (see [5], [14], [9], [3], [2]) however all of these flows are quasi-regular (or rigid) in the sense that the Birkhoff averages

$$\frac{1}{N} \sum_{n \leq N} \xi(n)$$

exist for every $\xi$ observed in $\mathcal{X}$. In [13] we establish some new cases of the Disjointness Conjecture and in particular for irregular flows $\mathcal{X}$, that is ones for which (1.2) fails. These flows are complicated in terms of the behavior of their individual orbits but they are distal and of zero entropy, so that the disjointness is still expected to hold.

---

*Date: June 30, 2014.*

2000 Mathematics Subject Classification. 11L03, 37A45, 11N37.

Key words and phrases. The Möbius function, distal flow, affine linear map, skew product, nilmanifold.
2. Results

In this section we summarize the results we have established in [13]. The first result in [13] is concerned with certain regular flows, namely affine linear maps of a compact abelian group \( X \). Such a flow \( (T, X) \) is given by

\[
T(x) = Ax + b
\]

(2.1)

where \( A \) is an automorphism of \( X \) and \( b \in X \) (see [10], [11]).

**Theorem 2.1.** Let \( \mathcal{X} = (T, X) \) be an affine linear flow on a compact abelian group which is of zero entropy. Then \( \mu \) is linearly disjoint from \( \mathcal{X} \).

Theorem 2.1 actually holds with a rate of convergence. We first reduce to the torus case and then handle the torus case by Fourier analysis and classical results of Davenport [5] and Hua [12] on exponential sums concerning the Möbius function.

The flows in Theorem 2.1 are distal, and our main result in [13] is concerned with nonlinear distal flows on such spaces. We restrict to \( X = \mathbb{T}^2 \) the two dimensional torus \( \mathbb{R}^2 / \mathbb{Z}^2 \) and consider nonlinear smooth (or even analytic) skew products as discussed in Furstenberg [6]. \( T : \mathbb{T}^2 \to \mathbb{T}^2 \) is given by

\[
T(x, y) = (ax + \alpha, cx + dy + h(x))
\]

(2.2)

where \( a, c, d \in \mathbb{Z}, \) \( ad = \pm 1, \alpha \in \mathbb{R} \) and \( h \) is a smooth periodic function of period 1. The affine linear part is in the form

\[
\begin{bmatrix}
a & 0 \\
c & d
\end{bmatrix} \in GL_2(\mathbb{Z}),
\]

ensuring that \( T \) has zero entropy (and it can always be brought into this form). The flow \( (T, \mathbb{T}^2) \) is distal and this skew product is a basic building block (with \( e(h(x)) \) continuous) in Furstenberg’s classification theory of minimal distal flows [7]. If \( \alpha \) is diophantine, that is

\[
\left| \alpha - \frac{a}{q} \right| \geq \frac{c}{q^B}
\]

for some \( c > 0, B < \infty \) and all \( a/q \) rational, then \( T \) can be conjugated by a smooth map of \( \mathbb{T}^2 \) to its affine linear part

\[
(x, y) \mapsto (ax + \alpha, cx + dy + \beta)
\]

(2.3)

where

\[
\beta = \int_0^1 h(x)dx
\]

(see [13]). Hence the disjointness of \( \mu \) from \( \mathcal{X} = (T, \mathbb{T}^2) \) for a \( T \) with a diophantine \( \alpha \), follows from Theorem 2.1. However if \( \alpha \) is not diophantine the dynamics of the flow
\( (T, \mathbb{T}^2) \) can be very different from an affine linear flow. For example, as Furstenberg shows it may be irregular (i.e. the limits in (1.2) fail to exist for certain observables). Our main result is that these nonlinear skew products are linearly disjoint from \( \mu \), at least if \( h \) satisfies some further technical hypothesis. Firstly we assume that \( h \) is analytic, namely that
\[
\hat{h}(m) = \sum_{m \in \mathbb{Z}} \hat{h}(m)e(mx) \quad (2.4)
\]
then
\[
\hat{h}(m) \ll e^{-\tau|m|} \quad (2.5)
\]
for some \( \tau > 0 \). Secondly we assume that there is \( \tau_2 < \infty \) such that
\[
|\hat{h}(m)| \gg e^{-\tau_2|m|}. \quad (2.6)
\]
This is not a very natural condition being an artifact of our proof. However it is not too restrictive and the following applies rather generally (and most importantly there is no condition on \( \alpha \)).

**Theorem 2.2.** Let \( \mathcal{X} = (T, \mathbb{T}^2) \) be of the form (2.2), with \( h \) satisfying (2.5) and (2.6). Then \( \mu \) is linearly disjoint from \( \mathcal{X} \).

The assertion of Theorem 2.2 holds for all \( \alpha \), and so we have to consider all diophantine possibilities of \( \alpha \). The tool is the Bourgain-Sarnak-Ziegler [3] finite version of the Vinogradov method, incorporated with various analytic methods such as Poisson’s summation and stationary phase. Furstenberg [6] gives examples of skew product transformations of the form (2.2) which are not regular in the sense of (1.2). Many of the flows \( \mathcal{X} \) in Theorem 2.2 have this property and we show in [13] that Furstenberg’s examples are smoothly conjugate to such \( \mathcal{X} \)’s. In particular his examples are linearly disjoint from \( \mu \).

Theorem 2.1 deals with the affine linear distal flows on the \( n \)-torus. A different source of homogeneous distal flows are the affine linear flows on nilmanifold \( X = G/\Gamma \) where \( G \) is a nilpotent Lie group and \( \Gamma \) a lattice in \( G \). For \( \mathcal{X} = (T, G/\Gamma) \) where \( T(x) = \alpha x \Gamma \) with \( \alpha \in G \), i.e. translation on \( G/\Gamma \), the linear disjointness of \( \mu \) and \( \mathcal{X} \) is proven in [8] and [9]. Using the classification of zero entropy (equivalently distal) affine linear flows on nilmanifolds [4], and Green and Tao’s results we establish in [13] the following.

**Theorem 2.3.** Let \( \mathcal{X} = (T, G/\Gamma) \) where \( T \) is an affine linear map of the nilmanifold \( G/\Gamma \) of zero entropy. Then \( \mu \) is linearly disjoint from \( \mathcal{X} \).

The above results for \( \mu(n) \) can be proved in the same way for similar multiplicative functions such as \( \lambda(n) = (-1)^{\tau(n)} \) where \( \tau(n) \) is the number of prime factors of \( n \).
3. DISJOINTNESS OF $\mu$ FROM FURSTENBERG’S SYSTEMS

As a consequence of Theorem 2.2, it is proved in [13] that $\mu$ is linearly disjoint from Furstenberg’s systems. But no rate of convergence is obtained there since Theorem 2.2 in general offers no rate. In this section we show that, for Furstenberg’s systems, rate of convergence is actually available if we work on these systems directly rather than appeal to Theorem 2.2.

3.1. The continued fraction expansion of $\alpha$. We assume that $\alpha$ is irrational, and our argument will depend on the continued fraction expansion of $\alpha$. Every real number $\alpha$ has its continued fraction representation

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \quad (3.1)$$

where $a_0 = [\alpha]$ is the integral part of $\alpha$, and $a_1, a_2, \ldots$ are positive integers. The expression (3.1) is infinite since $\alpha \notin \mathbb{Q}$. We write $[a_0; a_1, a_2, \ldots]$ for the expression on the right-hand side of (3.1), which is the limit of the finite continued expressions

$$[a_0; a_1, a_2, \ldots, a_k] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_k}}} \quad (3.2)$$

as $k \to \infty$. Writing

$$\frac{l_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k],$$

we have $l_0 = a_0, l_1 = a_0a_1 + 1, q_0 = 1, q_1 = a_1$, and for $k \geq 2$,

$$l_k = a_k l_{k-1} + l_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}.$$ 

Since $\alpha$ is irrational we have $q_{k+1} \geq q_k + 1$ for all $k \geq 1$. An induction argument gives the stronger assertion that $q_k \geq 2^{(k-1)/2}$ for all $k \geq 2$, and thus $q_k$ increases at least like an exponential function of $k$. The irrationality of $\alpha$ also implies that, for all $k \geq 2$,

$$\frac{1}{2q_k q_{k+1}} < \left| \alpha - \frac{l_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}. \quad (3.3)$$

3.2. Furstenberg’s examples. Furstenberg [6] gave examples of smooth transformation $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that the ergodic averages do not all exist. Let $\alpha$ be as above such that

$$q_{k+1} \geq e^{q_k} \quad (3.4)$$

for all positive $k$. Define $q_{-k} = -q_k$ and set

$$h(x) = \sum_{k \neq 0} \frac{e(q_k \alpha) - 1}{|k|} e(q_k x). \quad (3.5)$$
It follows from (3.3) and (3.4) that \( h(x) \) is a smooth function. We also have \( h(x) = g(x + \alpha) - g(x) \) where

\[
g(x) = \sum_{k \neq 0} \frac{1}{|k|} e(q_k x)
\]

so that \( g(x) \in L^2(0, 1) \) and in particular defines and measurable function. But \( g(x) \) cannot correspond to a continuous function, as shown in Furstenberg [6].

3.3. Disjointness of \( \mu \) from Furstenberg’s systems. In the following we consider slightly more general \( h \)’s with

\[
h(x) = \sum_{k \neq 0} c_k (1 - e(q_k \alpha)) e(q_k x)
\]

where \( \alpha \) satisfies \( \ref{3.4} \) and, for all positive \( k \), the coefficients \( c_k \) satisfy

\[
c_k = c_{-k}, \quad |c_k| \leq C
\]

for some positive constant \( C \) which we may assume to be greater than 1. We note that if \( c_k \) is not bounded then \( g(x) \) will not be \( L^2 \). Now what we want to estimate is essentially

\[
S(N) := \sum_{n \leq N} \mu(n) e\left(\sum_{j=0}^{n-1} h(j\alpha)\right),
\]

and our result is the following.

**Proposition 3.1.** Let \( S(N) \) be as in (3.9). Then

\[
S(N) \ll N \log^{-A} N
\]

where the implied constant depends on \( A \), but is independent of \( \alpha \).

**Proof.** By (3.7) we have

\[
\sum_{j=0}^{n-1} h(j\alpha) = \sum_{k \neq 0} c_k (1 - e(q_k \alpha)) \sum_{j=0}^{n-1} e(q_k j\alpha) = \sum_{k \neq 0} c_k (1 - e(nq_k \alpha)).
\]

We should cut the last sum at some point. Let \( K \) be such that \( q_{K-1} < 2 \log N \leq q_K \). Then for \( k \geq K \) we have

\[
|q_k \alpha - l_k| \leq \frac{1}{q_{k+1}} \leq e^{-q_k},
\]

so that

\[
\sum_{|k| \geq K} |c_k (1 - e(nq_k \alpha))| \ll C \sum_{|k| \geq K} ne^{-q_k} \ll CNe^{-q_K} \ll CN^{-1}
\]
where we the implied constant does not depend on $C$. It follows that

$$S(N) = \sum_{n \leq N} \mu(n) e \left\{ \sum_{1 \leq |k| \leq K-1} c_k (1 - e(n q_k \alpha)) + O \left( \frac{C}{N} \right) \right\}.$$  

The $O$-term as well as $\sum_{1 \leq |k| \leq K-1} c_k$ above are harmless, and so from now on we concentrate on

$$\tilde{S}(N) := \sum_{n \leq N} \mu(n) e \left\{ \sum_{1 \leq |k| \leq K-1} c_k e(n q_k \alpha) \right\} = \sum_{n \leq N} \mu(n) e \left\{ \sum_{1 \leq |k| \leq K-1} c_k e(n \theta_k) \right\}$$  

(3.12)

on writing $\theta_k = \|q_k \alpha\|$. Note that by (3.11) we have $\theta_k \leq e^{-q_k}$. We have a sequence

$$q_1 < \exp(q_1) \leq q_2 < \exp(q_2) \leq q_3 < \ldots,$$

and therefore $q_k > \exp \cdots \exp(q_2)$ with $k-2$ repeated exp’s. Taking $k = K - 1$ and noting that $q_2 \geq 2$ give

$$\exp \cdots \exp(2) \leq \exp \cdots \exp(q_2) < q_{K-1} \leq 2 \log N$$  

(3.13)

where on the left-hand side there are $K - 3$ exp’s.

Now let

$$\phi(x) = 2c_1 \cos(2\pi x), \quad x \in [\theta_1, \theta_1 N].$$  

(3.14)

Then $e(\phi(x))$ is a smooth periodic function and hence can expanded into Fourier series

$$e(\phi(x)) = \sum_{l \in \mathbb{Z}} a_l(c_1) e(l x),$$  

(3.15)

where

$$a_l(c_1) = \int_0^1 e(\phi(x)) e(-lx) dx.$$  

(3.16)
We must compute the dependence of $a_l(c_1)$ on $c_1$. By partial integration for $l \neq 0$ we have

\[
a_l(c_1) = -\frac{1}{2\pi i l} \int_0^1 e(\phi(x))de(-lx) = -\frac{1}{l} \int_0^1 e(\phi(x))\phi'(x)e(-lx)dx = \frac{1}{2\pi i l^2} \int_0^1 d[e(\phi(x))\phi'(x)]e(-lx)dx.
\]

Since

\[
\phi'(x) = -4\pi c_1 \sin(2\pi x), \quad \phi''(x) = -8\pi^2 c_1 \cos(2\pi x),
\]

we have

\[
\left| \frac{d[e(\phi(x))\phi'(x)]}{dx} \right| = |e(\phi(x))[\phi''(x) + 2\pi i \phi'(x)]| \ll |c_1| + |c_1|^2 \ll C^2
\]

and consequently

\[
a_l(c_1) \ll \frac{C^2}{l^2} \quad \text{(3.17)}
\]

for all $l \neq 0$. The above $\ll$-constant is absolute. Of course the above $l^2$ can be improved to $l^A$ for arbitrary $A > 0$, but we are not going to use this.

The sum $\tilde{S}(N)$ in (3.12) is of the form

\[
\tilde{S}(N) = \sum_{n \leq N} \mu(n)e(\phi(n\theta_1) + F(n)) = \sum_{n \leq N} \mu(n)e(F(n)) \sum_{l \in \mathbb{Z}} a_l(c_1)e(ln\theta_1)
\]

\[
= \sum_{l \in \mathbb{Z}} a_l(c_1) \sum_{n \leq N} \mu(n)e(ln\theta_1 + F(n))
\]
where $F(n)$ stands for the remaining terms in the $e(\cdot)$ in (3.12). It follows from this and (3.17) that

$$|	ilde{S}(N)| \leq \sum_{l \in \mathbb{Z}} |a_l(c_1)| \left| \sum_{n \leq N} \mu(n) e(ln\theta_1 + F(n)) \right|$$

$$\ll C^2 \sum_{l \in \mathbb{Z}} \frac{1}{l^2 + 1} \left| \sum_{n \leq N} \mu(n) e(ln\theta_1 + F(n)) \right|$$

$$\ll C^2 \sup_{l_1} \left| \sum_{n \leq N} \mu(n) e(nl_1\theta_1 + F(n)) \right| .$$

Repeating this procedure in (3.12), we get

$$|\tilde{S}(N)| \ll C^{2(K-1)} \sup_{l_1, \ldots, l_{K-1}} \left| \sum_{n \leq N} \mu(n) e(nl_1\theta_1 + \ldots + nl_{K-1}\theta_{K-1}) \right| .$$

The inner sum involving $\mu$ can be estimated by classical results of Davenport [5] and Hua [12] as

$$\ll N \log^{-A} N$$

where $A > 0$ is arbitrary, and the implied constant depends at most on $A$, i.e. it does not depend on the coefficients $l_1, l_2, \ldots, l_{K-1}$ or $\theta_1, \theta_2, \ldots, \theta_{K-1}$. By (3.13) we have $C^{2(K-1)} \leq \log N$. The proposition is proved.

Acknowledgements. The first author is supported by the 973 Program, NSFC grant 11031004, and IRT1264 from the Ministry of Education. The second author is supported by an NSF grant.

References

[1] N. Aoki, Topological entropy of distal affine transformations on compact abelian groups, J. Math. Soc. Japan 23(1971), 11-17.
[2] J. Bourgain, On the correlation of the M"obius function with random rank one systems (2011), arXiv:1112.1031.
[3] J. Bourgain, P. Sarnak, and Ziegler, Disjointness of M"obius from horocycle flows, From Fourier analysis and number theory to radon transforms and geometry, 67-83, Dev. Math. 28, Springer, New York, 2013.
[4] S. G. Dani, Dynamical systems on homogeneous spaces, in Chapter 10, Dynamical Systems, Ergodic Theory and Applications, Encyclopedia of Mathematical Sciences, Vol. 100, Springer.
[5] H. Davenport, On some infinite series involving arithmetical functions II, Quart. J. Math. 8(1937), 313-350.
[6] H. Furstenberg, Strict ergodicity and transformation of the torus, Amer. J. Math. 83(1961), 573-601.
[7] H. Furstenberg, The structure of distal flows, Amer. J. Math. 85(1963), 477-515.
[8] B. Green and T. Tao, The quantitative behaviour of polynomial obits on nilmanifolds, Ann. Math. (2), 175(2012), 465-540.
[9] B. Green and T. Tao, The Möbius function is strongly orthorgonal to nilsequences, Ann. Math. (2), 175 (2012), 541-566.
[10] F. J. Hahn, On affine transformations of compact abelian groups, Amer. J. Math. 85(1963), 428-446; Errata: Amer. J. Math. 86(1964), 463-464.
[11] H. Hoare and W. Parry, Affine transformations with quasi-discrete spectrum I, J. London Math. Soc. 41(1966), 88-96.
[12] L. K. Hua, Additive theory of prime numbers, AMS Translations of Mathematical Monographs, Vol. 13, Providence, R.I. 1965.
[13] J. Liu and P. Sarnak, The Möbius function and distal flows, arXiv:1303.4957.
[14] C. Mauduit and J. Rivat, Sur un problème de Gelfond: la somme des chiffres des nombres premiers, Ann. Math. (2) 171(2010), 1591-1646.
[15] N. M. dos Santos and R. Urzúa-Luz, Minimal homeomorphisms on low-dimensional tori, Ergodic Theory Dynam. Systems 29(2009), 1515-1528.
[16] P. Sarnak, Three lectures on the Möbius function, randomness and dynamics, IAS Lecture Notes, 2009; http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures(2).pdf.
[17] P. Sarnak, Möbius randomness and dynamics, Not. S. Afr. Math. Soc. 43 (2012), 89-97.
[18] P. Sarnak and A. Ubis, The horocycle at prime times, arXiv:1110.0777v2.

School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: jyliu@sdu.edu.cn

Department of Mathematics, Princeton University & Institute for Advanced Study, Princeton, NJ 08544-1000, USA
E-mail address: sarnak@math.princeton.edu