Unwrapping ADMM: Efficient Distributed Computing via Transpose Reduction

Thomas Goldstein
University of Maryland, College Park

Gavin Taylor
Kawika Barabin
Kent Sayre
United States Naval Academy, Annapolis

Abstract

Recent approaches to distributed model fitting rely heavily on consensus ADMM, where each node solves small sub-problems using only local data. We propose iterative methods that solve global sub-problems over an entire distributed dataset. This is possible using transpose reduction strategies that allow a single node to solve least-squares over massive datasets without putting all the data in one place. This results in simple iterative methods that avoid the expensive inner loops required for consensus methods. To demonstrate the efficiency of this approach, we fit linear classifiers and sparse linear models to datasets over 5 Tb in size using a distributed implementation with over 7000 cores in far less time than previous approaches.

1. Introduction

We study optimization routines for problems of the form

\[
\text{minimize } f(Dx),
\]

where \( D \in \mathbb{R}^{m \times n} \) is a (large) data matrix and \( f \) is a convex function. We are particularly interested in the case that \( D \) is stored in a distributed way across \( N \) nodes of a network or cluster. In this case, the matrix \( D = (D^T_1, D^T_2, \cdots, D^T_N)^T \) is a vertical stack of sub-matrices, each of which is stored on a node. If the function \( f \) decomposes across nodes as well, then problem (1) takes the form

\[
\text{minimize } \sum_{i \leq N} f_i(D_ix),
\]

where the summation is over the \( N \) nodes. Problems of this form include logistic regression, support vector machines, lasso, and virtually all generalized linear models (Hardin & Hilbe, 2001).

Most distributed solvers for equation (2), such as ADMM, are built on the assumption that one cannot solve global optimization problems involving the entire matrix \( D \). Rather, each node alternates between solving small sub-problems involving only local data, and then exchanging information with other nodes.

This work considers methods that solve global optimization problems over the entire distributed dataset on each iteration. This is possible using transpose reduction methods. Such schemes exploit the following simple observation: when \( D \) has many more rows than columns, the matrix \( D^T D \) is considerably smaller than \( D \). The availability of \( D^T D \) enables a single node to solve least-squares problems involving the entire data matrix \( D \). Furthermore, in many applications it is possible (and efficient) to compute \( D^T D \) in a distributed way. This allows our approach to solve extremely large optimization problems much faster than the current state-of-the-art. We support this conclusion both with theoretical results in Section 8 and with experimental results in Section 10.

2. Related Work

Simple first-order methods (methods related to gradient descent) have been available for smooth minimization problems for decades (Tsitsiklis et al., 1986; Rabbat & Nowak, 2004). However, slow performance of gradient schemes for poorly conditioned problems and slow (sublinear) convergence of sub-gradient and stochastic methods for non-differentiable problems has led many data scientists to consider splitting schemes.

Much recent work on solvers for formulation (2) has focused on the Alternating Direction Method of Multipliers (Glowinski & Marroco, 1975; Glowinski & Tallec, 1989;...
Goldstein & Osher, 2009), which has become a staple of the distributed computing and image processing literature. The authors of (Boyd et al., 2010) propose using ADMM for distributed model fitting using the “consensus” formulation. Consensus ADMM has additionally been studied for distributed model fitting (Erseghe et al., 2011), support vector machines (Forero et al., 2010), and numerous domain-specific applications (Chen et al., 2008; He et al., 2012). Many variations of ADMM have subsequently been proposed, including specialized variants for decentralized systems (Mota et al., 2013), asynchronous updates (Zhang & Kwok, 2014; Ouyang et al., 2013), inexact solutions to subproblems (Chang et al., 2015), and online/stochastic updates (Ouyang et al., 2013).

3. Background

The alternating direction method of multipliers (ADMM) is a general method for solving the problem

\[
\begin{align*}
\text{minimize} & \quad g(x) + h(y) \\
\text{subject to} & \quad Ax + By = 0.
\end{align*}
\]

(3)

The ADMM enables each term of problem (3) to be addressed separately. The algorithm in its simplest form begins with estimated solutions \(x^0, z^0\), and a Lagrange multiplier \(\lambda^0\). ADMM then generates the following iterates:

\[
\begin{align*}
x^{k+1} &= \arg \min_x g(x) + \frac{\tau}{2} \|Ax + By + \lambda^k\|^2 \\
y^{k+1} &= \arg \min_y h(y) + \frac{\tau}{2} \|Ax^{k+1} + By + \lambda^k\|^2 \\
\lambda^{k+1} &= \lambda^k + Ax^{k+1} + By^{k+1}
\end{align*}
\]

(4)

with some positive stepsize parameter \(\tau > 0\).

Disparate formulations are achieved by different \(A, B, f\), and \(g\). For example, consensus ADMM (Boyd et al., 2010) addresses the problem

\[
\begin{align*}
\text{minimize} & \quad \sum_i f_i(x_i) \\
\text{subject to} & \quad x_i = z \quad \text{for all } i
\end{align*}
\]

(5)

which corresponds to problem (3) with \(B = (I, I, \cdots I)^T\), \(A = I\), \(f(x) = \sum_i f_i(x_i)\), and \(g = 0\). Rather than solving a single global problem, the consensus ADMM solves many small problems in parallel. On each iteration of consensus ADMM, node \(i\) performs the update

\[
x_i^{k+1} = \arg \min_{x_i} f_i(x_i) + \frac{\tau}{2} \|x_i - z\|^2.
\]

(6)

The shared global variable \(z\) is then updated by the central server. Finally, by updating the Lagrange multipliers \(\{\lambda_i\}\), the solutions to each sub-problem are forced to be progressively more similar on each iteration.

4. Transpose Reduction Made Easy: Regularized Least-Squares

Transpose reduction is most easily described for regularized least-squares problems. Consider the distributed solution of the problem

\[
\begin{align*}
\text{minimize} & \quad J(x) + \frac{1}{2} \|Dx - b\|^2 \\
\end{align*}
\]

(7)

for some penalty term \(J\). When \(J(x) = \mu |x|\) for some scalar \(\mu\), this becomes the lasso regression (Tibshirani, 1994). Typical consensus solvers for problem (7) require each node to compute the solution to equation (6), which is here given by

\[
x_i^{k+1} = \arg \min_{x_i} \frac{1}{2} \|D_i x_i - b_i\|^2 + \frac{\tau}{2} \|x_i - z^k\|^2 \\
= (D_i^T D_i + \tau I)^{-1} (D_i^T b_i + \tau z).
\]

During the setup phase for consensus ADMM, each node forms the matrix \(D_i^T D_i\), and then computes and caches the inverse (or equivalently the factorization) of \((D_i^T D_i + \tau I)\).

Alternatively, transpose reduction can solve distributed ADMM on a single machine without moving the entire matrix \(D\) to one place. Using the simple identity

\[
\frac{1}{2} \|Dx - b\|^2 = \frac{1}{2} \langle Dx - b, Dx - b \rangle = \frac{1}{2} x^T (D^T D)x - x^T D^T b + \frac{1}{2} \|b\|^2
\]

we can replace problem (7) with the equivalent problem

\[
\begin{align*}
\text{minimize} & \quad J(x) + \frac{1}{2} x^T (D^T D)x - x^T D^T b.
\end{align*}
\]

(8)

To solve problem (8), the central server needs only the matrix \(D^T D\) and the vector \(D^T b\). When \(D\) is a large “tall” matrix \(D \in \mathbb{R}^{m \times n}\), with \(n \ll m\), \(D^T D\) has only \(n^2\) (rather than \(mn\)) entries, which is practical to store on a single server. Furthermore, because

\[
D^T D = \sum_i D_i^T D_i, \quad \text{and } D^T b = \sum_i D_i^T b_i
\]

the matrix \(D^T D\) can be formed by having each server compute its own local \(D_i^T D_i\), and then aggregating/reducing the results on a central server.

Once \(D^T D\) and \(D^T b\) have been computed in the cloud and cached on a central server, the global problem is solved on a single node. This is done using either a single-node ADMM method for small dense lasso (see (Boyd et al., 2010) Section 6.4) or a forward-backward (proximal) splitting method (Goldstein et al., 2014b). The latter approach only requires the gradient of \(\frac{1}{2} x^T (D^T D)x - x^T D^T b\), which is given by \(D^T D x - D^T b\).
5. Unwrapping ADMM: Transpose Reduction for General Problems

Transpose reduction can be exploited for more complex problems using ADMM. On each iteration of the proposed method, least-squares problems are solved over the entire distributed dataset. In contrast, each step of consensus ADMM relies on sub-problems involving only small subsets of the data.

We aim to solve problem (1). We begin by “unwrapping” the matrix $D$; we remove it from $f$ using the formulation

$$
\text{minimize } f(y) \\
\text{subject to } y = Dx.
$$

Applying the ADMM with $A = D$, $B = -I$, $h = f$, and $g = 0$ yields Algorithm 1.

**Algorithm 1 Unwrapped ADMM**

1: Choose initial $x^0$, $y^0$, $λ^0$, and $τ > 0$
2: while not converged do
3: $x^{k+1} = \arg \min \| Dx - y^k + λ^k \|^2 = D^+(y^k - λ^k)$
4: $y^{k+1} = \arg \min \| y + \frac{τ}{2} \| Dx^{k+1} - y + λ^k \|^2$
5: $λ^{k+1} = λ^k + Dx^{k+1} - y^{k+1}$
6: end while

We will now examine the steps in Algorithm 1. The $x$ update requires the solution of a global least squares problem over the entire dataset. The solution requires the pseudoinverse of $D$, given by

$$D^+ = (D^TD)^{-1}D^T.$$

The $y$ update can be represented using the proximal mapping of $f$, which is given by

$$\text{prox}_f(z, δ) = \arg \min_y f(y) + \frac{1}{2 δ} \| y - z \|^2.$$

Using this notation, Line 4 of Algorithm 1 is written $\text{prox}_f(Dx^{k+1} + λ^k, τ^{-1})$. Provided $f$ is decomposable, the minimization in Line 4 is coordinate-wise decoupled. Each coordinate of $y^{k+1}$ is computed with either an analytical solution, or using a simple 1-dimensional lookup table of pre-computed solutions.

5.1. Distributed Implementation

When $D = (D_1^T, D_2^T, \cdots, D_N^T)^T$ is large and distributed over $N$ nodes, no node has access to the entire matrix $D$ to solve the global least squares problem for $x^{k+1}$. This is where we exploit the transpose reduction.

We can decompose the rows of $y = (y_1^T, y_2^T, \cdots, y_N^T)^T$ and $λ = (λ_1^T, λ_2^T, \cdots, λ_N^T)^T$. The constraint in formulation (9) now becomes $y_i = D_ix$, and the least-squares $x$ update in Algorithm 1 becomes

$$x^{k+1} = D^+(y^k - λ^k) = W \sum_i D_i(y^k - λ^k),$$

where $W = (\sum_i D_i^T D_i)^{-1}$. Each vector $D_i(y_i^k - λ_i^k)$ can be computed locally on node $i$. Multiplication by $W$ must take place on a central server.

Note the massive dimensionality reduction that takes place when $D^T D = \sum_i D_i^T D_i$ is formed. For a data matrix $D \in \mathbb{R}^{m \times n}$ with $n \ll m$, the Gram matrix $D^T D \in \mathbb{R}^{n \times n}$ is small, even in the case that $D$ is far too large to store on the central server. The complete distributed method is listed in Algorithm 2.

**Algorithm 2 Unwrapped ADMM with Transpose Reduction**

1: Choose global $x^0$, $τ$, and local $\{y_i^0\}, \{λ_i^0\}$ on each node $i$
2: All nodes: $W_i = D_i^T D_i$, $∀i$
3: Central node: $W = (\sum_i W_i)^{-1}$
4: while not converged do
5: All nodes: $d_i^k = D_i^T(y_i^k - λ_i^k)$, $∀i$
6: Central Node: $x^{k+1} = W \sum_i d_i^k$
7: All nodes: $y_i^{k+1} = \arg \min_{y_i} f_i(y_i) + \frac{τ}{2} \| D_i x^{k+1} - y_i + λ_i^k \|^2$
8: All nodes: $λ_i^{k+1} = λ_i^k + D_i x^{k+1} - y_i^{k+1}$
9: end while

6. Application: Fitting linear classifiers

6.1. Logistic regression

A logistic classifier maps a feature vector $d$ to probabilities using a mapping of the form $d \mapsto \frac{e^{c^T d}}{1 + e^{c^T d}}$ for some row vector $c$ of weights. Given a matrix $D \in \mathbb{R}^{m \times n}$ of feature vectors, and a vector $l \in \mathbb{R}^m$ of labels, the regression weights are found by solving

$$\min_x \sum_{k=1}^m \log(1 + \exp(-l_k(D_k x))) = f_{lr}(Dx),$$

where $f_{lr}$ is the logistic regression loss (negative log-likelihood) function

$$f_{lr}(z) = \sum_{k=1}^m \log(1 + \exp(-l_k z_k)).$$

To apply unwrapped ADMM, we must evaluate the proximal operator

$$\text{prox}_{f_{lr}}(z, τ^{-1}) = \arg \min_y f_{lr}(z) + \frac{τ}{2} \| y - z \|^2.$$
This is a convex, 1-dimensional optimization problem, and the solution is easily computed with a few iterations of Newton’s method. However, because the proximal operator depends on only one input variable, we suggest forming a lookup table of pre-computed solutions.

6.2. Support Vector Machine

A common formulation of the support vector machine (SVM) solves

$$\minimize \frac{1}{2}\|x\|^2 + Ch(Dx) \quad (12)$$

where \(C\) is a regularization parameter, and \(h\) is a simple “hinge loss” function given by

$$h(z) = \sum_{k=1}^{M} \max\{1 - l_k z_k, 0\}.$$

The proximal mapping of \(h\) has the form

$$\text{prox}_h(z, \delta) = z_k + l_k \max\{\min\{1 - l_k z_k, \delta\}, 0\}.$$

Using this proximal operator, the solution to the \(y\) update in Algorithm 1 is simply

$$y^{k+1} = \text{prox}_h \left(D x^{k+1} + \lambda^k C\right).$$

Note that this algorithm is much simpler than the consensus implementation of SVM, which requires each node to solve the sub-problem

$$\minimize Ch(Dx) + \frac{T}{2}\|x - y\|^2. \quad (13)$$

Despite the similarity of this problem to the original SVM (12), this problem form is not supported by available SVM solvers such as LIBSVM (Chang & Lin, 2011) and others, and thus requires a custom solver (see Section A in the Appendix).

7. Handling Sparsity

Variable-selection methods use \(\ell_1\) regularization to obtain sparse solutions. For high dimensional problems, \(\ell_1\) regularization is commonplace to avoid overfitting. Sparse model fitting problems have the form

$$\minimize \mu |x| + f(Dx) \quad (14)$$

for some regularization parameter \(\mu > 0\). Sparse problems can be trivially reduced to the form (1) by defining

$$\hat{D} = \begin{pmatrix} I \\ D \end{pmatrix}, \quad \hat{f}(z)_k = \begin{cases} \mu |z_k|, & \text{for } 1 \leq k \leq n \\ f_k(z_k), & \text{for } k > n \end{cases}$$

and then minimizing \(\hat{f}(\hat{D}x)\). In the distributed setting where \(f\) and \(D\) are computed/stored across \(N\) servers, the problem now has the form (2) with

$$\minimize \sum_{i \leq N+1} f_i(D_i x) \quad (15)$$

where \(f_{N+1}(z) = \mu |z|\), and \(D_{N+1} = I\).

7.1. Splitting Over Columns

When the matrix \(D\) is extremely wide (\(m \ll n\)), it often happens that each server stores a subset of columns of \(D\) rather than rows. Fortunately, such problems can be handled by solving the dual of the original problem. The dual of the sparse problem (14) is given by

$$\minimize_{\alpha} f^*(\alpha) \text{ subject to } \|D^T \alpha\|_{\infty} \leq \mu \quad (16)$$

where \(f^*\) denotes the Fenchel conjugate (Boyd & Vandenberghe, 2004) of \(f\). For example the dual of the lasso problem is simply

$$\minimize_{\alpha} \frac{1}{2}\|\alpha + b\|^2 \text{ subject to } \|D^T \alpha\|_{\infty} \leq \mu.$$  

Problem (16) then reduces to the form (1) with

$$\hat{D} = \begin{pmatrix} I \\ DT \end{pmatrix}, \quad \hat{f}(z)_k = \begin{cases} \frac{1}{2}\|z_k + b_k\|^2, & \text{for } 1 \leq k \leq m \\ \lambda'(z_k), & \text{for } k > m \end{cases}$$

where \(\lambda'(z)\) is the characteristic function of the \(\ell_\infty\) ball of radius \(\mu\). The function \(\lambda'(z)\) is infinite when \(|z_i| > \mu\) for some \(i\), and zero otherwise. The unwrapped ADMM for this problem requires the formation of \(D_i D_i^T\) on each server, rather than \(D_i^T D_i\).

8. Convergence Theory

The convergence of ADMM is in general well understood. Classical results guarantee convergence using monotone operator theory (Eckstein & Bertsekas, 1992), however they do not provide convergence rates. A more recent result due to He and Yuan (He & Yuan, 2012), provides a global convergence rate for the method. Theorem 1 below presents this result, as adapted to the problem form (3).

**Theorem 1.** (He and Yuan) Consider the iterates generated by the ADMM method 4. If \(f\) and \(g\) are proper convex functions, then

$$\|B(y^{k+1} - y^k)\|^2 + \|Ax^{k+1} + By^{k+1}\|^2 \leq \frac{\|B(y^0 - y^*)\|^2 + \|\lambda^0 - \lambda^*\|^2}{k + 1} \quad (17)$$

where \(y^*\) is an optimal solution for (3) and \(\lambda^*\) is the corresponding optimal Lagrange multiplier.
It was observed in (Boyd et al., 2010) that \(\|Ax^{k+1} + By^{k+1}\|^2\) and \(\|A^T B(y^{k+1} - y^k)\|^2\) are measures of primal and dual infeasibility, respectively. Thus, Theorem 1 guarantees the iterates of ADMM approach primal and dual feasibility (and thus optimality) as \(k \to \infty\), and that the infeasibility errors decrease with rate \(O(1/k)\) in the worst case. We can specialize this result to the case of unwrapped ADMM to obtain Corollary 1.

**Corollary 1.** If formulation (1) is feasible, then the iterates of the unwrapped ADMM I satisfy
\[
\|y^{k+1} - y^k\|^2 + \|Dx^{k+1} - y^{k+1}\|^2 \\
\leq \frac{\|y^0 - Dx^0\|^2 + \|\lambda^0 - \lambda^*\|^2}{k+1}
\]
where \(y^*\) is an optimal solution for (3) and \(\lambda^*\) is the corresponding optimal Lagrange multiplier.

While Theorem 1 and Corollary 1 show convergence in the sense that primal and dual feasibility are attained for large \(k\), it seems more natural to ask for a direct measure of optimality with respect to the objective function (1). In the case that \(f\) is differentiable, we can exploit the simple form of unwrapped ADMM and show the derivative of (1) goes to zero, thus directly showing that \(x^k\) is a good approximate minimizer of (1) for large \(k\).

**Theorem 2.** If the gradient of \(f\) exists and has Lipschitz constant \(L(\nabla f)\), then Algorithm 1 shrinks the gradient of the objective function (1) with global rate
\[
\|\nabla f(Dx^k)\|^2 = \|D^T \nabla f(Dx^k)\|^2 \\
\leq C\frac{\|y^0 - Dx^0\|^2 + \|\lambda^0 - \lambda^*\|^2}{k}
\]
where \(C = (L(\nabla f) + \tau)^2 \rho(D^TD)\) is a constant and \(\rho(D^TD)\) denotes the spectral radius of \(D^TD\).

**Proof.** We begin by writing the optimality condition for the \(x\)-update in Algorithm 1:
\[
D^T(Dx^{k+1} - y^k + \lambda^k) \\
= D^T\lambda^{k+1} + D^T(y^{k+1} - y^k) = 0. \quad (18)
\]
Note we used the definition \(\lambda^{k+1} = \lambda^k + Dx^{k+1} - y^{k+1}\) to simplify (18). Similarly, the optimality condition for the \(y\)-update yields
\[
\nabla f(y^{k+1}) + \tau(y^{k+1} - Dx^{k+1} - \lambda^k) \\
= \nabla f(y^{k+1}) - \tau\lambda^{k+1} = 0,
\]
or equivalently \(\nabla f(y^{k+1}) = \tau\lambda^{k+1}\). Combining this with equation (18) yields
\[
D^T \nabla f(y^k) = \tau D^T(y^k - y^{k+1}).
\]
We now have
\[
\partial_x \{f(Dx^k)\} = D^T \nabla f(Dx^k) = D^T \nabla f(y^k + Dx^k - y^k) \\
= D^T \nabla f(y^k + Dx^k - y^k) - D^T \nabla f(y^k) + \tau D^T(y^k - y^{k+1})
\]
and so \(\|\partial_x \{f(Dx^k)\}\|\) is bounded above by
\[
\|D^T \nabla f(y^k + Dx^k - y^k) - D^T \nabla f(y^k)\| + \tau \|D^T(y^k - y^{k+1})\| \\
\leq L(\nabla f)\|D^T\|_{\text{op}} \|Dx^k - y^k\| + \tau \|D^T\|_{\text{op}} \|y^k - y^{k+1}\|.
\]
where \(\|D^T\|_{\text{op}}\) denotes the operator norm of \(D^T\).
Now, by Corollary (1), we know that both \(\|Dx^k - y^k\|\) and \(||y^k - y^{k+1}\||\) are bounded above by \(\sqrt{\frac{\|y^0 - Dx^0\|^2 + \|\lambda^0 - \lambda^*\|^2}{k}}\). Applying this bound to (8) yields
\[
\|\partial_x \{f(Dx^k)\}\| \leq \sqrt{C\frac{\|y^0 - Dx^0\|^2 + \|\lambda^0 - \lambda^*\|^2}{k}}.
\]
We obtain the result by squaring this inequality and noting that \(\|D\|_{\text{op}}^2 = \rho(D^TD)\).

Note the logistic regression problem (11) satisfies the conditions of Theorem 2 with \(L(\nabla f) = 1/4\).

Finally, we remark that better convergence rates are possible in special cases. For example when \(f\) is strongly convex, accelerated ADMM obtains \(O(1/k^2)\) convergence (Goldstein et al., 2014a), and if we further assume \(D\) has full row rank, R-linear convergence is possible (Deng & Yin, 2012).

### 9. Implementation Details

We compare transpose reduction methods to consensus ADMM using both synthetic and empirical data. We study the transpose reduction scheme for lasso (Section 4) in addition to the unwrapped ADMM (Algorithm 2) for logistic regression and SVM.

We built a distributed implementation of the transpose reduction methods along with consensus optimization, and ran all experiments on Armstrong, a 30,000-core Cray XC30 hosted by the DOD Supercomputing Resource Center. This allows us to study experiments ranging in size from very small to extremely large.

All distributed methods were implemented using MPI. Stopping conditions for both transpose reduction and consensus methods were set using the residuals defined in (Boyd et al., 2010) with \(\epsilon_{\text{rel}} = 10^{-3}\), and \(\epsilon_{\text{abs}} = 10^{-6}\).

Many steps were taken to achieve top performance of the consensus optimization routine. The authors of (Boyd
We use the same synthetic problems used in our experiments, with a convergence number of iterations on a problem instance with $m = 10,000$ data vectors and $n = 100$ features per vector, and then scaled the stepsize parameter up/down to be proportional to $m$. It was found that this scaling made the number of iterations nearly invariant to $n$ and $m$. The iterative solvers for each local logistic regression/SVM problem were warm-started using solutions from the previous iteration.

The logistic regression subproblems were solved using a limited memory BFGS method (with warm start to accelerate performance). The transpose reduced lasso method (Section 4) requires a sparse least-squares method to solve the entire lasso problem on a single node. This was accomplished using the forward-backward splitting implementation FASTA (Goldstein et al., 2014b; 2015).

Note that the SVM problem (13) cannot be solved by conventional SVM solvers (despite its similarity to the classical SVM formulation). For this reason, we built a custom solver using the same coordinate descent techniques as the well-known solver LIBSVM (Chang & Lin, 2011). By using warm starts and exploiting the structure of problem (13), our custom method solves the problem (13) dramatically faster than standard solvers for problem (12). See Appendix A for details.

10. Numerical Experiments

To illustrate the behavior of transpose reduction methods on various data corpus sizes and multiple optimization problems, we applied consensus and transpose solvers to both synthetic and empirical datasets. We recorded both the total compute time and the wallclock time. We define total compute time as the total amount of time computing cores spend performing calculations. Total compute time does not include communication and extra diagnostic computation not necessary for optimization (such as per-iteration computation of the objective function); wall time includes all calculation and communication.

10.1. Synthetic Data

Synthetic data was constructed as follows:

**Lasso problems** We use the same synthetic problems used to study consensus ADMM in (Boyd et al., 2010). The data matrix $D$ is a random Gaussian matrix. The true solution $x_{true}$ contains 10 active features with unit magnitude, and the remaining entries are zero. The $\ell_1$ penalty $\mu$ is chosen as suggested in (Boyd et al., 2010) — i.e., the penalty is $10\%$ the magnitude of the penalty for which the solution to (7) becomes zero. The observation vector is $b = Dx_{true} + \eta$, where $\eta$ is a standard Gaussian noise vector with $\sigma = 1$.

**Classification problems** We generated two random Gaussian matrices, one for each class. The first class consists of zero-mean Gaussian entries. The first 5 columns of the second matrix were random Gaussian with mean 1, and the remaining columns were mean zero. Note the data classes generated by this process are not perfectly linearly separable. The $\ell_1$ penalty was set using the “10\%” rule used in (Boyd et al., 2010).

In addition to these “standard” test problem, we also study the important (and often more realistic) case where data is heterogeneous across nodes.

**Heterogeneous Data** To create data heterogeneity, we chose one random Gaussian scalar for each node, and added it to the matrix $D$. This has the effect of making the behavior of the sub-problems on each node different. Note that, with standard Gaussian test problems, each consensus node is essentially solving the same problem (i.e., data matrices have perfectly identical statistical properties), and thus they arrive at a consensus quickly. In practical applications, data on different nodes may represent data from different sources, or else not be identically distributed. The behavior of consensus in such situations is radically different than when the data is identically distributed. The heterogeneous synthetic data models this scenario.

We performed experiments using logistic regression, SVM, and Lasso. On different trial runs, we vary the number of cores used, the length of the feature vectors, and the number of data points per core. Three examples of our results are illustrated in Figure 1, while more complete tables of results appear Appendix B. In addition, convergence curves for experiments on both homogeneous and heterogeneous data can be seen in Figures 2a and 2b.

10.2. Empirical Case Study: Classifying Guide Stars

We study the behavior of transpose reduction and consensus methods using the Second Generation Guide Star Catalog (GCS-II) (Lasker et al., 2008), an astronomical database containing 950 million stars and other astronomical objects. Intensity measurements in different spectral bands are recorded for each object, in addition to geometric properties such as size and eccentricity. The GSC-II also contains a binary classification that labels every astronomical body as “star” or “not a star.” We train a sparse logistic classifier to discern this classification using only spectral and geometric features.

A data matrix was compiled by selecting the 17 spectral and geometric measurements reported in the catalog, and also “interaction features” made of all second-order products of the 17 measurements; all features were then normalized. After the addition of a bias feature, the resulting
matrix has 307 features per object, and requires 1.8 TB of space to store in binary form.

We ran experiments observing the decrease of the global objective function as a function of wallclock time; these experiments showed that transpose ADMM converged far more quickly than consensus. Convergence curves for this experiment are shown in Figure 2c.

We also experiment with the effect of storing the data matrix across different numbers of cores (i.e. increasing the number of nodes). These experiments illustrated that this variable had little effect on the relative performance of the two optimization strategies; transpose methods remained far more efficient regardless of the load on individual cores. Table 1 reports runtimes with varying numbers of cores.

Note the strong advantage of transpose reduction over consensus ADMM in Figure 2c. This confirms the results of Section 10.1, where it was observed that transpose reduction methods are particularly powerful for the heterogeneous data observed in realistic datasets, as opposed to the identically distributed matrices used in many synthetic experiments.

11. Discussion

In all experiments, transpose reduction methods required substantially less computation time than consensus methods. As seen in Figure 1, Figure 2, and Appendix B, performance of both transpose reduction and consensus methods scales nearly linearly with the number of cores and amount of data used for classification problems. For the lasso problem (Figure 1c), the runtime of the transpose reduction method appears to grows sub-linearly with number of cores, in contrast to consensus methods that have a strong dependence on this parameter. This is largely because the transpose reduction method does all iteration on a single machine, whereas consensus optimization requires progressively more communication with larger numbers of cores.

Note that transpose reduction methods require more startup time for some problems than consensus methods because the local Gram matrices \( D_i^T D_i \) must be sent to the central node, aggregated, and the result inverted; this is not true for the lasso problem, for which consensus solvers must also invert a local Gram matrix on each node, though this at least saves startup communication costs. This startup time is particularly noticeable when overall solve time is short, as in Figure 2b. Note, however, that even for this problem total computation time and wall time was still shorter with transpose reduction than with consensus methods.
11.1. Effect of Heterogeneous Data

An important property of the proposed transpose reduction methods is that they solve global subproblems over the entire data corpus (as opposed to consensus methods where subproblems involve a small portion of the distributed data). The power of solving global problems is very apparent when data is heterogeneous across nodes. Heterogeneity means that data on different nodes has different statistical properties (as opposed to homogeneous experiments where every data matrix has identical properties).

Heterogeneity has a strong effect on consensus methods. When data is heterogeneous across nodes it causes local subproblems to differ, and thus the nodes have a stronger tendency to “disagree” on the solution, taking longer to reach a consensus. This effect is illustrated by a comparison between Figures 2a and 2b, where consensus methods took much longer to converge on heterogeneous data sets, while the transpose method was not affected.

In contrast, because transpose reduction solves global subproblems across the entire distributed data corpus, it is relatively insensitive to data heterogeneity across nodes. Data heterogeneity explains the strong advantage of transpose reduction on the GSC-II dataset (Figure 2c, Table 1), which contains empirical data and is thus not uniformly distributed.

11.2. Communication & Computation

The transpose reduction ADMM leverages a tradeoff between communication and computation. When \( N \) nodes are used to solve a problem with a distributed data matrix \( D \in \mathbb{R}^{m \times n} \), each node in consensus ADMM transmits \( x_i \in \mathbb{R}^n \) to the central server, which totals to \( O(Nn) \) communication. Unwrapped ADMM requires \( O(m) \) communication per iteration, which is often somewhat more. Despite this difference, transpose reduction methods are still highly efficient because (a) they require dramatically less computation, and (b) communication with the server is substantially more efficient, even when more information is sent. We discuss these two issues in detail below.

First, unwrapped ADMM requires substantially less computation than consensus methods. Consensus ADMM requires inner iterations to solve expensive subproblems on each node. Because logistic regression and SVM can be very expensive for large data sets, consensus optimization is often CPU bound rather than communication bound. In contrast, the sub-problems of unwrapped ADMM are coordinate-wise separable and generally available in closed form.

Second, transpose reduction methods stay synchronized better than consensus ADMM, which makes communication more efficient on synchronous architectures. The iterative methods used by consensus ADMM for logistic regression and SVM subproblems do not terminate at the same time on every machine, especially when the data is heterogeneous across nodes. Because all client nodes must complete their inner iterations before an iteration is complete, most of the communication overhead of ADMM is caused by calls to MPI routines that block until all nodes become synchronized. In contrast, Algorithm 2 requires the same computations on each server, allowing nodes to stay synchronized naturally.

12. Conclusion

We introduce transpose reduction ADMM — an iterative method that solves model fitting problems using global least-squares subproblems over an entire distributed dataset. Numerical experiments using both synthetic and empirical data suggest that the transpose reduction is substantially more efficient than consensus methods. When the distributed dataset has heterogeneous properties across
nodes, the global properties of transpose reduction are particularly advantageous compared to consensus methods, which solve many small sub-problems using different datasets.

References

Boyd, S., Parikh, N., Chu, E., Peleato, B., and Eckstein, J. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. Foundations and Trends in Machine Learning, 2010.

Boyd, Stephen and Vandenberghe, Lieven. Convex Optimization. Cambridge University Press, 2004.

Chang, Chih C. and Lin, Chih J. LIBSVM: A Library for Support Vector Machines. ACM Trans. Intell. Syst. Technol., 2(3), May 2011. ISSN 2157-6904.

Chang, T.-H., Hong, M., and Wang, X. Multi-agent distributed optimization via inexact consensus admm. Signal Processing, IEEE Transactions on, 63(2):482–497, Jan 2015.

Chen, Ruiliang, Park, Jung-Min, and Bian, Kaigui. Robust distributed spectrum sensing in cognitive radio networks. In INFOCOM 2008. The 27th Conference on Computer Communications. IEEE, April 2008.

Deng, Wei and Yin, Wotao. On the global and linear convergence of the generalized alternating direction method of multipliers. UCLA CAM technical report, 12-52, 2012.

Eckstein, Jonathan and Bertsekas, Dimitri P. On the douglas-rachford splitting method and the proximal point algorithm for maximal monotone operators. Mathematical Programming, 55:293–318, 1992.

Erseghe, T., Zennaro, D., Dall’Anese, E., and Vangelista, L. Fast consensus by the alternating direction multipliers method. Signal Processing, IEEE Transactions on, 59 (11):5523–5537, Nov 2011.

Forero, Pedro A., Cano, Alfonso, and Giamakis, Georgios B. Consensus-based distributed support vector machines. J. Mach. Learn. Res., 11:1663–1707, August 2010. ISSN 1532-4435.

Glowinski, R. and Marroco, A. Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité d’une classe de problèmes de Dirichlet non linéaires. Rev. Française d’Automat. Inf. Recherche Opérationelle, 9(2):41–76, 1975.

Glowinski, Roland and Tallec, Patrick Le. Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

Goldstein, T., O’Donoghue, B., Setzer, S., and Baraniuk, R. Fast alternating direction optimization methods. SIAM Journal on Imaging Sciences, 7(3):1588–1623, 2014a.

Goldstein, Tom and Osher, Stanley. The Split Bregman method for $\ell_1$ regularized problems. SIAM J. Img. Sci., 2(2):323–343, April 2009.

Goldstein, Tom, Studer, Christoph, and Baraniuk, Richard. A field guide to forward-backward splitting with a FASTA implementation. arXiv eprint, abs/1411.3406, 2014b.

Goldstein, Tom, Studer, Christoph, and Baraniuk, Richard. FASTA: A generalized implementation of forward-backward splitting, January 2015. http://arxiv.org/abs/1501.04979.

Hardin, James W. and Hilbe, Joseph. Generalized Linear Models and Extensions. College Station, Texas: Stata Press, 2001.

He, B., Tao, M., and Yuan, X. Alternating direction method with gaussian back substitution for separable convex programming. SIAM Journal on Optimization, 22(2):313–340, 2012.

He, Bingsheng and Yuan, Xiaoming. On non-ergodic convergence rate of Douglas-Rachford alternating direction method of multipliers. Optimization Online, January 2012.

Lasker, Barry M, Lattanzi, Mario G, McLean, Brian J, Bucciarelli, Beatrice, Drimmel, Ronald, Garcia, Jorge, Greene, Gretchen, Guglielmetti, Fabrizia, Hanley, Christopher, Hawkins, George, et al. The second-generation guide star catalog: description and properties. The Astronomical Journal, 136(2):735, 2008.

Mota, J.F.C., Xavier, J.M.F., Aguiar, P.M.Q., and Puschel, M. D-admm: A communication-efficient distributed algorithm for separable optimization. Signal Processing, IEEE Transactions on, 61(10):2718–2723, May 2013.

Ouyang, Hua, He, Niao, Tran, Long, and Gray, Alexander G. Stochastic alternating direction method of multipliers. In Dasgupta, Sanjoy and Mcallester, David (eds.), Proceedings of the 30th International Conference on Machine Learning (ICML-13), volume 28, pp. 80–88. JMLR Workshop and Conference Proceedings, 2013.

Rabbat, M. and Nowak, R. Distributed optimization in sensor networks. In Information Processing in Sensor Networks, 2004. IPSN 2004. Third International Symposium on, pp. 20–27, April 2004.

Tibshirani, Robert. Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society, Series B, 58:267–288, 1994.
Tsitsiklis, J.N., Bertsekas, D.P., and Athans, M. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *Automatic Control, IEEE Transactions on*, 31(9):803–812, Sep 1986.

Zhang, Ruiliang and Kwok, James. Asynchronous distributed admm for consensus optimization. In Jebara, Tony and Xing, Eric P. (eds.), *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pp. 1701–1709. JMLR Workshop and Conference Proceedings, 2014.
Appendices

A. SVM Sub-steps for Consensus Optimization

The consensus SVM requires a solution to the problem (13). Despite the apparent similarity of this proximal-regularized problem to the original SVM (12), problem (13) cannot be put into a form that is solvable to popular solvers for (12). However, techniques for the classical SVM problem can be easily adapted to solve (13).

A common numerical approach to solving (12) is the attack its dual, which is

\[
\min_{\alpha_i \in [0,C]} \frac{1}{2} \|A^T L \alpha\|^2 - \alpha^T 1 = \sum_{i,j} \alpha_i \alpha_j l_{ij} A_i A_j^T - \sum_i \alpha_i. \tag{19}
\]

Once (19) is solved to obtain \(\alpha^*\), the solution to (12) is simply given by \(w^* = L^T \alpha\). The dual formulation (19) is advantageous because the constraints on \(\alpha\) act separately on each coordinate. The dual is therefore solved efficiently by coordinate descent, which is the approach used by the popular solver LIBSVM (Chang & Lin, 2011). This method is particularly powerful when the number of support vectors in the solution is small, in which case most of the entries in \(\alpha\) assume the value 0 or \(C\).

In the context of consensus ADMM, we must solve

\[
\min_{\alpha_i \in [0,C]} \frac{1}{2} \|w\|^2 + Ch(Aw,l) + \frac{\tau}{2} \|w - z\|^2. \tag{20}
\]

Following the classical SVM literature, we dualize this problem to obtain

\[
\min_{\alpha_i \in [0,C]} \frac{1}{2} \|A^T L \alpha\|^2 - \alpha^T ((1 + \tau) 1 - \tau L z). \tag{21}
\]

We then solve (21) for \(\alpha^*\), and recover the solution via

\[
w^* = \frac{A^T L \alpha^* + \tau z}{1 + \tau}.
\]

We solve (21) using a dual coordinate descent method inspired by (Chang & Lin, 2011). The implementation has \(O(M)\) complexity per iteration. Also following (Chang & Lin, 2011) we optimize the convergence by updating coordinates with the largest residual (derivative) on each pass.

Because our solver does not need to handle a “bias” variable (in consensus optimization, only the central server treats the bias variables differently from other unknowns), and by using a warm start to accelerate solve time across iterations, our coordinate descent method significantly outperforms even LIBSVM for each sub-problem. On a desktop computer with a Core i5 processor, LIBSVM solves the synthetic data test problem with \(m = 100\) datapoints and \(n = 200\) features in 3.4 seconds (excluding “setup” time), as opposed to our custom solver which solves each SVM sub-problem for the consensus SVM with the same dimensions (on a single processor) in 0.17 seconds (averaged over all iterations). When \(m = 10000\) and \(n = 20\), LIBSVM requires over 20 seconds, while the average solve time for the custom solver embedded in the consensus method is only 2.3 seconds.

B. Tables of Results

In the following tables, we use these labels:

- **N**: Number of data points per core
- **F**: Number of features per data point
- **Cores**: Number of compute cores used in computation
- **Space**: Total size of data corpus in GB (truncated at GB)
Logistic regression with homogeneous data

| N   | F    | Cores | Space(GB) | TWalltime | TCompute | CWalltime | CCompute |
|-----|------|-------|-----------|-----------|----------|-----------|----------|
| 50000 | 2000 | 800   | 596       | 0:00:53   | 6:19:14  | 0:01:36   | 17:25:18 |
| 50000 | 2000 | 1600  | 1192      | 0:00:58   | 12:40:24 | 0:01:51   | 1 day 10:51:33 |
| 50000 | 2000 | 2400  | 1788      | 0:01:00   | 19:05:13 | 0:01:52   | 2 days 4:21:25  |
| 50000 | 2000 | 3200  | 2384      | 0:01:00   | 1 day 1:30:18 | 0:01:41 | 2 days 21:46:28 |
| 50000 | 2000 | 4000  | 2980      | 0:00:58   | 1 day 7:58:24 | 0:01:39 | 3 days 15:17:51 |
| 50000 | 2000 | 4800  | 3576      | 0:00:58   | 1 day 14:27:31 | 0:02:31 | 4 days 8:49:58  |
| 50000 | 2000 | 5600  | 4172      | 0:01:00   | 1 day 21:10:38 | 0:02:13 | 5 days 2:16:56  |
| 50000 | 2000 | 6400  | 4768      | 0:01:03   | 2 days 3:46:42 | 0:02:08 | 5 days 19:39:40 |
| 50000 | 2000 | 7200  | 5364      | 0:01:21   | 2 days 10:36:38 | 0:01:47 | 6 days 13:12:59 |

Logistic regression with heterogeneous data

| N   | F    | Cores | Space(GB) | TWalltime | TCompute | CWalltime | CCompute |
|-----|------|-------|-----------|-----------|----------|-----------|----------|
| 20000 | 500  | 4800  | 357       | 0:00:33   | 4:04:11  | 0:00:26   | 21:01:22 |
| 20000 | 1000 | 4800  | 715       | 0:00:26   | 7:51:06  | 0:01:22   | 1 day 21:44:7  |
| 20000 | 1500 | 4800  | 1072      | 0:00:38   | 11:23:22 | 0:01:37   | 2 days 19:42:30 |
| 20000 | 2000 | 4800  | 1430      | 0:00:42   | 15:15:01 | 0:01:30   | 3 days 19:27:24 |
| 20000 | 2500 | 4800  | 1788      | 0:00:42   | 18:59:04 | 0:01:48   | 4 days 17:24:59 |
| 30000 | 2000 | 4800  | 2145      | 0:00:47   | 22:53:25 | 0:02:04   | 5 days 16:30:28 |
| 35000 | 2000 | 4800  | 2503      | 0:00:57   | 1 day 2:43:48 | 0:02:46 | 6 days 15:10:40 |
| 40000 | 2000 | 4800  | 2861      | 0:00:54   | 1 day 6:22:51 | 0:02:47 | 7 days 14:58:02 |
| 45000 | 2000 | 4800  | 3218      | 0:00:57   | 1 day 10:05:17 | 0:03:02 | 8 days 15:11:42 |
| 50000 | 2000 | 4800  | 3576      | 0:01:02   | 1 day 14:28:30 | 0:03:24 | 9 days 15:51:21 |

| N   | F    | Cores | Space(GB) | TWalltime | TCompute | CWalltime | CCompute |
|-----|------|-------|-----------|-----------|----------|-----------|----------|
| 20000 | 5000 | 4800  | 357       | 0:00:05   | 2:18:21  | 0:00:35   | 20:51:20 |
| 20000 | 1000 | 4800  | 715       | 0:00:12   | 5:33:31  | 0:01:40   | 1 day 18:43:21 |
| 20000 | 1500 | 4800  | 1072      | 0:00:25   | 9:44:07  | 0:01:08   | 2 days 20:08:20 |
| 20000 | 2000 | 4800  | 1430      | 0:00:31   | 15:10:01 | 0:01:29   | 3 days 19:28:56 |
| 20000 | 2500 | 4800  | 1788      | 0:01:23   | 1 day 12:24:25 | 0:03:30 | 4 days 20:53:53 |
| 20000 | 3000 | 4800  | 2145      | 0:01:50   | 1 day 20:29:59 | 0:03:44 | 5 days 19:45:31 |
| 20000 | 3500 | 4800  | 2503      | 0:02:27   | 2 days 5:40:09 | 0:03:56 | 6 days 19:44:54 |
| 20000 | 4000 | 4800  | 2861      | 0:03:03   | 2 days 16:50:51 | 0:03:46 | 7 days 18:17:21 |
| 20000 | 4500 | 4800  | 3218      | 0:04:00   | 3 days 3:35:02 | 0:04:28 | 8 days 19:49:26 |
| 20000 | 5000 | 4800  | 3576      | 0:04:52   | 3 days 16:50:21 | 0:04:44 | 9 days 23:56:16 |

Logistic regression with heterogeneous data
Lasso with heterogeneous data

| N  | F   | Cores | Space(GB) | TWalltime | TCompute | CWalltime | CCompute |
|----|-----|-------|-----------|-----------|----------|-----------|----------|
| 50000 | 200 | 800   | 59        | 0:00:12   | 0:01:45  | 0:00:37   | 0:04:55  |
| 50000 | 200 | 1600  | 119       | 0:00:02   | 0:03:31  | 0:00:47   | 0:10:56  |
| 50000 | 200 | 2400  | 178       | 0:00:02   | 0:05:14  | 0:01:14   | 0:17:50  |
| 50000 | 200 | 3200  | 238       | 0:00:00   | 0:07:00  | 0:01:22   | 0:25:24  |
| 50000 | 200 | 4000  | 298       | 0:00:04   | 0:09:00  | 0:01:36   | 0:33:49  |
| 50000 | 200 | 4800  | 357       | 0:00:11   | 0:10:25  | 0:01:57   | 0:43:29  |
| 50000 | 200 | 5600  | 417       | 0:00:10   | 0:12:09  | 0:02:07   | 0:55:47  |
| 50000 | 200 | 6400  | 476       | 0:00:07   | 0:13:48  | 0:02:19   | 1:04:51  |
| 50000 | 200 | 7200  | 536       | 0:00:09   | 0:15:31  | 0:02:39   | 1:19:22  |

SVM with homogeneous data

| N  | F  | Cores | Space(GB) | TWalltime | TCompute | CWalltime | CCompute |
|----|----|-------|-----------|-----------|----------|-----------|----------|
| 50000 | 20 | 48    | 0         | 0:00:01   | 0:00:46  | 0:02:45   | 2:01:12  |
| 50000 | 20 | 96    | 0         | 0:00:01   | 0:01:32  | 0:02:47   | 4:03:05  |
| 50000 | 20 | 144   | 1         | 0:00:02   | 0:02:19  | 0:02:49   | 5:58:08  |
| 50000 | 20 | 192   | 1         | 0:00:02   | 0:03:06  | 0:02:45   | 7:56:14  |
| 50000 | 20 | 240   | 1         | 0:00:02   | 0:03:53  | 0:02:51   | 9:54:05  |
| 50000 | 50 | 48    | 0         | 0:00:03   | 0:01:23  | 0:05:14   | 3:44:06  |
| 50000 | 50 | 96    | 1         | 0:00:03   | 0:02:47  | 0:05:19   | 7:26:30  |
| 50000 | 50 | 144   | 2         | 0:00:03   | 0:04:11  | 0:05:25   | 11:07:51 |
| 50000 | 50 | 192   | 3         | 0:00:07   | 0:05:38  | 0:05:25   | 14:54:03 |
| 50000 | 50 | 240   | 4         | 0:00:03   | 0:07:00  | 0:05:25   | 18:26:10 |
| 50000 | 100| 48    | 1         | 0:00:05   | 0:02:20  | 0:09:28   | 6:25:55  |
| 50000 | 100| 96    | 3         | 0:00:05   | 0:04:40  | 0:09:56   | 12:49:20 |
| 50000 | 100| 144   | 5         | 0:00:05   | 0:07:04  | 0:09:45   | 19:09:22 |
| 50000 | 100| 192   | 7         | 0:00:06   | 0:09:25  | 0:09:53   | 1 day 1:27:47 |
| 50000 | 100| 240   | 8         | 0:00:05   | 0:11:46  | 0:10:06   | 1 day 8:08:18 |

Star data

| Cores | TWalltime | TCompute | CWalltime | CCompute |
|-------|-----------|----------|-----------|----------|
| 2500  | 0:01:06   | 11:35:25 | 0:24:39   | 31 days 19:59:13 |
| 3000  | 0:00:49   | 12:10:33 | 0:21:43   | 32 days 2:44:11 |
| 3500  | 0:00:50   | 12:17:27 | 0:17:01   | 30 days 7:56:19 |
| 4000  | 0:00:45   | 12:38:24 | 0:29:53   | 40 days 13:38:19 |