Conformal Invariance and Finiteness Theorems for Non–Planar $\beta$–deformed $\mathcal{N} = 4$ SYM Theory

Federico Elmetti$^1$, Andrea Mauri$^{1,2}$ and Marco Pirrone$^3$

1 Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
2 Department of Physics, University of Crete, 71003 Heraklion, Greece
3 Dipartimento di Fisica, Università di Milano–Bicocca and INFN, Sezione di Milano–Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy

Abstract

We study the conformal invariance of non–planar $\beta$–deformed $\mathcal{N} = 4$ SYM theory using the coupling constant reduction (CCR) formalism. We show that up to order $g^{10}$, differently from the planar case, we can remove the scheme dependence in the definition of the theory without reducing to the real $\beta$ case. We also compute the gauge beta function up to four loops and see that the generalized finiteness theorems proposed in [hep-th: 0705.1483] still hold.
1 Introduction

Marginal deformations of $\mathcal{N} = 4$ super Yang–Mills theory have recently drawn much attention in the context of conformal generalizations of AdS/CFT correspondence. The so–called $\beta$–deformation is an interesting example of this class of theories thanks to the work of Lunin and Maldacena [1] where its gravity dual description has been found. From the field theory point of view this deformation is realized by enlarging the space of parameters of the original $\mathcal{N} = 4$ theory with the following modification of the superpotential:

$$
ig \text{Tr} (\Phi_1 \Phi_2 \Phi_3 - \Phi_1 \Phi_3 \Phi_2) \rightarrow ih \text{Tr} (e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2)
$$

(1)

where $h$ and $\beta$ are two new complex coupling constants in addition to the gauge coupling $g$, which is chosen to be real. The resulting theory preserves $\mathcal{N} = 1$ supersymmetry and it is expected to become conformally invariant only if a precise relation among the coupling constants exists [2]. Several papers have been devoted to the study of an explicit realization of this condition in the planar case ([3]–[8]). Keeping $\beta$ real, the Leigh–Strassler constraint turns out to be satisfied at all order in perturbation theory by the exact solution $h\bar{h} = g^2$ [3]. The case of complex $\beta$ requires a more careful investigation since the conformal condition gets perturbatively corrected. In order to properly describe the fixed point surface in the space of couplings, the coupling constant reduction (CCR) program has shown to be a powerful tool ([9]–[14]). Using this approach, in [7] it is claimed that conformal invariance and scheme independence of the theory can not be achieved at the same time for the complex $\beta$ deformed case in the planar limit.

The aim of this paper is to achieve a better understanding of the problem by looking at the finite $N$ case (see also [15]–[18]). Working perturbatively we will ask for the chiral and gauge beta functions to vanish in order to define the theory at its conformal point. In Section 2 we will analyze the properties of the two–point chiral correlator. Once again we will make use of the CCR procedure to obtain the vanishing of the anomalous dimension. This amounts to express the chiral couplings as functions of the gauge coupling $g$. As a consequence the perturbation theory is naturally defined in terms of powers of $g$ instead of powers of loops. At order $g^6$ we meet the first non–trivial situation because at this stage different loop diagrams start contributing at the same order in $g$. We will see that up to order $g^{10}$, differently from the planar case, there is enough freedom to remove the scheme dependence without reducing to the real $\beta$ case.

Then we will turn to consider the gauge beta function. As CCR approach allows different loop orders to mix, it is not obvious that standard finiteness theorems [19, 20] should hold. So, having canceled the chiral beta function up to $\mathcal{O}(g^7)$ does not automatically imply the vanishing of the gauge beta function at $\mathcal{O}(g^9)$. The fact that this is still the case is a highly non–trivial check that we will cover in details in Section 3. The same problem was studied in [7] in the planar case where it was shown by an explicit computation that the condition for the vanishing of the anomalous dimension $\gamma$ at $\mathcal{O}(g^8)$ actually ensures the vanishing of the gauge beta function at $\mathcal{O}(g^{11})$. This

\footnote{The possible scheme dependence of the vanishing $\gamma$ condition has been first noted by the authors of [5]. In [7] we explicitly considered this feature and studied its implications.}
result was obtained making use of background field method combined with covariant $\nabla$-algebra. However it is worth noting that the procedure followed in [7] is not the standard one (extensively explained in [21]), which turned out to be too involved. Here, working at a lower order in $g$ but keeping $N$ finite, we will be able to get through the calculation adopting both of the methods and checking explicitly the equivalence of the two.

2 Chiral Beta Function and Conformal Condition

Let us consider the $\mathcal{N} = 1$ $\beta$–deformed action written in terms of the superfield strength $W_\alpha = i\bar{D}^2(e^{-gV} D_\alpha e^{gV})$, where $V$ is a real prepotential, and three chiral superfields $\Phi_i$ with $i = 1, 2, 3$, all in the adjoint representation of the $SU(N)$ gauge group. With notations as in [22] we have

$$S = \int d^8z \, \text{Tr}(e^{-gV}\Phi_1 e^{gV}\Phi^1) + \frac{1}{2g^2} \int d^6z \, \text{Tr}(W_\alpha W_\alpha)$$

$$+ \bar{h} \int d^6z \, \text{Tr}(q \, \Phi_1 \Phi_2 \Phi_3 - \frac{1}{q} \, \Phi_1 \Phi_3 \Phi_2 )$$

$$+ \bar{i} \, \bar{h} \int d^6\bar{z} \, \text{Tr}(\frac{1}{q} \, \bar{\Phi}_1 \bar{\Phi}_2 \bar{\Phi}_3 - \bar{q} \, \bar{\Phi}_1 \bar{\Phi}_3 \bar{\Phi}_2 ) \quad q \equiv e^{i\pi\beta}$$

Here $h$ and $\beta$ are complex couplings and $g$ is the real gauge coupling constant. In the undeformed $\mathcal{N} = 4$ SYM theory one has $h = g$ and $q = 1$. From now on we will be considering ’t Hooft rescaled quantities

$$h \to \frac{h}{\sqrt{N}} \quad g \to \frac{g}{\sqrt{N}}$$

(3)

in order to easily make contact with the planar limit. Moreover we notice that the phase of $h$ can always be reabsorbed by a field redefinition, so that the effective number of independent real parameters in the superpotential is actually three. For later convenience we choose them to be $|h_1|^2$, $|h_2|^2$ and $|h_3|^2$, where

$$h_1 \equiv h \, q \quad h_2 \equiv \frac{h}{q} \quad h_3 \equiv h \, q - \frac{h}{q}$$

(4)

In the spirit of [2] the idea is to find a surface of renormalization group fixed points in the space of the coupling constants. To this end one can consider the coupling constant reduction program ([9]-[14]) and express the renormalized Yukawa couplings in terms of the gauge one.
This operation has an immediate consequence: we are forced to work perturbatively in powers of $g$ instead of powers of loops. To single out a conformal theory we will ask for the chiral and gauge beta functions to vanish. In this section we will concentrate on $\beta_h$ and adopt dimensional regularization within minimal subtraction scheme. The chiral beta function is proportional to the anomalous dimension $\gamma$ of the elementary fields and the condition $\beta_h = 0$ can be conveniently traded with $\gamma = 0$. Even working in a generic scheme, one can easily convince oneself that at a given order in $g^2$ the proportionality relation between $\beta_h$ and $\gamma$ gets affected only by terms proportional to lower order contributions to $\gamma$. Therefore, if we set $\gamma = 0$ order by order in the coupling, we are guaranteed that $\beta_h$ vanishes as well \[23\]. So the object we will be mainly interested in is the two–point chiral correlator.

In \[7\] this issue has been analyzed by considering the planar limit where only two independent real constants enter the color factors, namely $|h_1|^2$ and $|h_2|^2$. As a result the definition of the conformal theory was found to be scheme dependent as long as $\beta$ was complex. In the non–planar case all of the three parameters enter the calculation of the two–point chiral correlator. We will see that this difference will be important in the definition of the fixed point surface.

The idea is to proceed perturbatively in superspace. Supergraphs will be evaluated performing the $D$–algebra inside the loops and the corresponding divergent integrals will be computed using dimensional regularization in $n = 4 - 2\epsilon$. In this framework one could allow the coefficients $a_i, b_i, c_i$ in (5) to be expanded in power series of $\epsilon$ \[8\]. Doing this, evanescent terms are introduced \textit{ad hoc} in order to deal with the $1/\epsilon$ poles and ensure the complete finiteness of the theory. However, after sending $\epsilon \to 0$, they do not enter the relation between renormalized coupling constants so we will neglect their possible presence hereafter.

Let us start at order $g^2$. As first proposed in \[17\] it is convenient to consider the difference between divergent diagrams in the $\beta$–deformed and in the $\mathcal{N} = 4$ theory. This amounts to the evaluation of the chiral bubbles in Fig.1 which give the following divergent contribution to the chiral propagator

\[
\frac{1}{(4\pi)^2} \left[ |h_1|^2 + |h_2|^2 - \frac{2}{N^2} |h_3|^2 - 2g^2 \right] \frac{1}{\epsilon} \left( \frac{\mu^2}{p^2} \right)^\epsilon \tag{6}
\]

where we have explicitly indicated the factors coming from dimensionally regulated integral (here $p$ is the external momentum and $\mu$ is the standard renormalization mass).

At this stage, in order to obtain a vanishing chiral beta function, the following condition has to be imposed

\[
|h_1|^2 = a_1 g^2 + a_2 g^4 + a_3 g^6 + \ldots \\
|h_2|^2 = b_1 g^2 + b_2 g^4 + b_3 g^6 + \ldots \\
|h_3|^2 = c_1 g^2 + c_2 g^4 + c_3 g^6 + \ldots
\]
Moreover, it is well known that
\[ |h_1|^2 + |h_2|^2 - \frac{2}{N^2}|h_3|^2 = 2g^2 \] (8)
ensures \( \gamma = 0 \) up to two loops [16]. So, looking at the chiral two–point contribution (6) at order \( g^4 \), we have the following additional requirement
\[ \mathcal{O}(g^4) : \quad a_2 + b_2 - \frac{2}{N^2}c_2 = 0 \] (9)
It is easy to see that equations (7) and (9) reduce to the ones found in [7] in the large \( N \) limit. When we move up to the next order the situation becomes more involved with respect to the planar case. In fact, working with finite \( N \) we need to consider the non–planar graph in Fig.2, whose contribution is:
\[ \frac{1}{(4\pi)^6} \frac{2\zeta(3)}{6} \mathcal{F} \frac{1}{\epsilon} \left( \frac{\mu^2}{p^2} \right)^{3\epsilon} \] (10)
where \( \mathcal{F} \equiv \mathcal{F}(|h_1|^2, |h_2|^2, |h_3|^2, N^2) \) reads [17, 18]
\[ \mathcal{F} = \frac{N^2 - 4}{N^4} |h_3|^2 \left[ \frac{N^2 + 5}{N^2} |h_3|^4 - 3|h_3|^2(|h_1|^2 + |h_2|^2) + 3(|h_1|^2 - |h_2|^2)^2 \right] \] (11)
Notice that the color factor in (11) is suppressed as \( 1/N^2 \) for large \( N \). Due to the expansion in (5) both the one loop (6) and three loops (10) structures contribute to the evaluation of \( \gamma \) at \( \mathcal{O}(g^6) \). The final result can be recast as
\[ \frac{1}{\epsilon} \left[ A \left( \frac{\mu^2}{p^2} \right)^\epsilon + B \left( \frac{\mu^2}{N^2} \right)^{3\epsilon} \right] \] (12)
where we have defined for concision

\[ A \equiv \frac{1}{(4\pi)^2} (a_3 + b_3 - \frac{2}{N^2} c_3) \] (13)

\[ B \equiv \frac{2\zeta(3)}{(4\pi)^6} \frac{N^2 - 4}{N^2 c_1} \left[ \frac{N^2 + 5}{N^2} c_1^2 - 3c_1(a_1 + b_1) + 3(a_1 - b_1)^2 \right] \] (14)

The vanishing condition of the anomalous dimension at order \( g^6 \) can be read directly from the finite \( \log \) term in (12):

\[ \mathcal{O}(g^6) : \quad A + \frac{3B}{N^2} = 0 \] (15)

We emphasize that at this order the condition for the vanishing of \( \gamma \) and \( \beta_h \) is completely scheme independent. However, from now on we will have to care about the scheme dependence in the definition of the fixed points. To see this, let us consider the counterterm needed at this stage to properly renormalize the propagator in an arbitrary scheme:

\[ g^6 \left( A + \frac{B}{N^2} \right) \left( \frac{1}{\epsilon} + \rho \right) \] (16)

where \( \rho \) is a constant related to the choice of finite renormalization. In fact, if we were to push the conformal invariance condition one order higher we should compute the chiral beta function at order \( g^9 \). We expect to have several sources of nontrivial contributions to \( \gamma \) at this order: one coming from the one–loop bubble proportional to \((a_4 + b_4 - \frac{2}{N^2} c_4)\), then from two–loop, three–loop and four–loop diagrams. All of the diagrams containing subdivergences, namely the two and four loop contributions, will be subtracted making use of the appropriate counterterms. To be specific, a term like

\[ g^8 \left( A + \frac{B}{N^2} \right) \left( \frac{1}{\epsilon} + \rho \right) \frac{1}{\epsilon} \left( \frac{\mu^2}{p^2} \right)^\epsilon \] (17)
will appear in the calculation of $\gamma$. Therefore the request for vanishing anomalous dimension depends unavoidably on the arbitrary constant $\rho$ which appears in the form

$$(A + \frac{B}{N^2})\rho$$

If we wanted to kill the scheme dependence of the result we would also need to impose the vanishing of the combination $A + B/N^2$ which together with (15) would lead immediately to $A = B = 0$. The crucial observation is that in the non–planar case we deal with three parameters and at this stage we have enough freedom to eliminate the scheme dependence from the conformal condition without reducing to the real $\beta$ case. In fact, the constraint $A = 0$ gives

$$a_3 + b_3 - \frac{2}{N^2}c_3 = 0$$

while the condition $B = 0$ combined with equation (7) yields

$$\left\{ \begin{aligned} a_1 + b_1 &= 2 \\ c_1 &= 0 \end{aligned} \right.$$  

or, if $c_1 \neq 0$

$$\left\{ \begin{aligned} a_1 + b_1 &= 2 \left( 1 + \frac{c_1}{N^2} \right) \\ a_1 - b_1 &= \pm \sqrt{2c_1 \left( 1 - \frac{N^2 - 1}{6N^2}c_1 \right)} \end{aligned} \right.$$  

These solutions allow for a non vanishing imaginary part of $\beta$ (which is proportional to the combination $|h_1|^2 - |h_2|^2$). At the same time, they define the surface of renormalization fixed points without any ambiguity related to the choice of regularization scheme. It is clear that in the planar limit only the condition coming from $A = 0$ survives as the $B = 0$ condition is subleading. So we are left with $a_3 + b_3 = 0$, in complete agreement with the result found in [7].

If we move to the next order, a new scenario will show up. Having imposed (20) or (21) only three graphs will contribute to the anomalous dimension at order $g^8$ (Fig.3). Since these diagrams are primitively divergent (no subdivergences are present) the condition for $\gamma = 0$ at this order turns out to be completely scheme independent. In fact we have to consider the following expression:

$$\frac{1}{\epsilon} \left[ A' \left( \frac{\mu^2}{p^2} \right)^{\epsilon} + B' \left( \frac{\mu^2}{N^2p^2} \right)^{3\epsilon} + H \left( \frac{\mu^2}{p^2} \right)^{4\epsilon} \right]$$

where we have denoted
Figure 3: Diagrams contributing to $\gamma$ at order $g^8$

\[ A' \equiv \frac{1}{(4\pi)^2} \left( a_4 + b_4 - \frac{2}{N^2} c_4 \right) \quad (23) \]
\[ B' \equiv \frac{6 \zeta(3) N^2 - 4}{(4\pi)^6} \left[ (a_1 - b_1)^2 c_2 + c_1 \left( \frac{N^2 - 1}{N^2} c_1 c_2 + 4 (a_1 - b_1) (a_2 - \frac{c_2}{N^2}) - 4 c_2 \right) \right] \quad (24) \]
\[ H \equiv -\frac{5 \zeta(5)}{2(4\pi)^8} \left[ (a_1 - b_1)^4 + (a_1 + b_1)^4 + \frac{1}{N^2} f \left( a_1, b_1, c_1, \frac{1}{N^2} \right) - \frac{16 (N^2 + 12)}{N^2} \right] \quad (25) \]

where $f$ can be read from Appendix A and we have used the relations (7) and (9).

The vanishing of $\gamma$ reads

\[ \mathcal{O}(g^8) : \quad A' + \frac{3B'}{N^2} + 4H = 0 \quad (26) \]

Again, in order to remove scheme dependence from the $\mathcal{O}(g^{10})$ conformal condition we have to impose:

\[ A' + \frac{B'}{N^2} + H = 0 \quad (27) \]

At this stage, independently of the choice (20) or (21), we have enough parameters to solve both equations without restricting to the real $\beta$ case as in the planar theory. On the other hand, if one sends $N \to \infty$, equations (26) and (27) reduce to the ones found in [7]. This large $N$ limit turns out to be smooth and does not present any sort of singularity, so there is no contradiction between our results and those found in [7]. We observe that a scheme-independent definition of the complex $\beta$ conformal theory can be achieved only thanks to subleading coefficients which are projected out by the planar limit.

### 3 Gauge Beta Function and Finiteness Theorems

Now we turn to consider the gauge beta function. Standard finiteness theorems [19, 20] ensure the vanishing of $\beta_g$ at $L+1$–loops once $\beta_h$ has been set to zero at $L$–loops. Here, as
a consequence of coupling constant reduction, we are forced to work order by order in $g^2$ instead of loop by loop and it is not obvious that such theorems still hold. Nevertheless in [7] it was shown that in the planar $\beta$-deformed theory the vanishing condition for $\beta_h$ at $\mathcal{O}(g^9)$ was sufficient to have vanishing $\beta_g$ at $\mathcal{O}(g^{11})$. This result was a strong indication that finiteness theorems could be generalized as follows: if the matter chiral beta function vanishes up to order $g^{2n+1}$ then the gauge beta function vanishes as well up to order $g^{2n+3}$. Here we are going to check this result at finite $N$ and for $n = 3$. In order to do this, we take advantage of covariant supergraph techniques combined with background field method [21]. The standard procedure consists in looking at vacuum diagrams at a given perturbative order and performing covariant $\nabla$–algebra. Then by expanding propagators one extracts tadpole type contributions with vector connections as external legs. Moreover one only selects diagrams containing at least a $1/\epsilon^2$ pole (see [20] for details). In the present case, contributions to the gauge beta function at $\mathcal{O}(g^9)$ come from two and four loop vacuum diagrams (Fig.4).

The analysis of the two loop diagram is straightforward and completely analogous to the one in [20]. Expanding the covariant propagators one obtains three times the diagram in Fig.5 which corresponds to the term

$$\frac{1}{2} \text{Tr} (\Gamma^a \Gamma_a) \int \frac{d^n k \ d^n q}{(2\pi)^{2n}} \frac{1}{q^2 (q + k)^2 k^4}$$

(28)

where $\Gamma_a$ is the vector connection.

Figure 4: Two and four loop vacuum diagrams

Figure 5: Two loops tadpole diagram
This integral contains a one–loop ultraviolet subdivergence and it is infrared divergent. It is convenient to remove the IR divergence using the $R^*$ subtraction procedure of [24]. After UV and IR subtractions one isolates the $1/\epsilon^2$ term and rewrites the result in a covariant form, obtaining the following contribution to the two loop effective action:

$$
\frac{1}{(4\pi)^2} \frac{3(N^2-1)}{4N} A \frac{1}{\epsilon} \text{Tr} \int d^4x \, d^2\theta \, W^\alpha W_\alpha
$$

where we have inserted the A factor defined in (13).

Figure 6: $\nabla$–algebra operations on four–loop vacuum diagram

Now we turn to consider the four loop contributions. In this case the computation is much more involved because we need to perform very non trivial $\nabla$–algebra operations. In [7]
an analogous problem was solved by using an alternative procedure, though different from the one just described which turned out to be too hard to deal with. Here we want to consider both methods and show that they indeed give the same result. Let us start with the standard procedure. A detailed explanation of $\nabla$--algebra operations can be found in Fig.6. Starting from the top vacuum diagram and performing integration by parts we end up with three different graphs. Each of them gives rise to a single bosonic diagram: Fig.6 (a), (b), (c), where we have denoted

\[
\equiv \frac{1}{2} \frac{\nabla^a \nabla_a}{\Box} \rightarrow 1 - \frac{1}{2} \frac{\Gamma^a \Gamma_a}{\Box} \quad \equiv \frac{1}{2} \partial^a \partial_a \quad \equiv \nabla_a = \partial_a - i \Gamma_a \quad (30)
\]

Now we are ready to expand the covariant propagators to extract tadpole--type contributions. It is easy to see that (a) and (b) diagrams are equivalent and give rise to the tadpole graphs shown in Fig.7.

Figure 7: Tadpole contributions from propagator expansions of diagrams (a) and (b)

Analogously the (c) diagram can be expanded to give the relevant tadpole contributions as indicated in Fig.8. The latter integrals are much harder to compute because of the presence of four derivatives, indicated by the black arrows. However, after some proper integrations by parts, they can be reduced to simpler scalar integrals, as depicted in Fig.9. Notice that in the whole procedure we have neglected all tadpole graphs with $1/\epsilon$ divergences, which do not contribute to the four--loop effective action.

Now we just need to sum up the various contributions generated by (a), (b) and (c) diagrams. Actually there is no need to compute all these integrals explicitly thanks to a beautiful diagrammatic cancellation. In fact, the only surviving terms sum up to give nine times the same diagram, shown in Fig.10. The corresponding bosonic integral is:
Figure 8: Tadpole contributions from relevant propagator expansions of diagram (c)

\[
\frac{1}{2} \text{Tr}(\Gamma^a \Gamma_a) \int \frac{d^4k}{(2\pi)^4} \frac{d^nq}{d^nq} \frac{d^4r}{d^4r} \frac{d^4t}{d^4t} \frac{1}{k^4 q^2 t^2 (r-q)^2 (r+t)^2 (t+q)^2 (r+k)^2} \tag{31}
\]

So the total four-loop contribution to the effective action, after inserting color and combinatorial factors and subtracting IR and UV subdivergences is given by:

\[
\frac{1}{(4\pi)^2} \frac{9(N^2-1)}{8N^3} B \frac{1}{\epsilon} \text{Tr} \int d^4x \ d^2\theta \ W^\alpha W_\alpha \tag{32}
\]

with B defined as in (14). This completes the computation of the four loops contribution with the standard method.

Had we followed the alternative procedure developed in [7] we would have first expanded each of the nine propagators of the four-loop vacuum diagram in Fig. 4 and then performed \(\nabla\)–algebra. In this case, the only possible contributions would come from two types of diagrams:

I. the ones with flat \(D^2\) and \(\bar{D}^2\) factors at the vertices, flat propagators and one tadpole insertion, for which now standard \(D\)–algebra can be performed

and

II. the vacuum diagrams with flat propagators but \(\nabla^2\) and \(\bar{\nabla}^2\) at the chiral vertices in which the tadpole insertion will have to appear after completion of the \(\nabla\)–algebra.

Analogously to [7], it is easy to see that only type I diagrams contribute. The computation is now straightforward. As the vacuum diagram is completely symmetric we have nine equivalent choices for the propagator to expand. Once a choice has been made the
standard D–algebra gives rise to a unique contribution, producing precisely the result depicted in Fig.10. We have therefore checked that as expected the two methods actually give the same answer.

Now we come back to the computation of the gauge beta function and combine (29) and (32). We can easily read the vanishing condition at order $g^9$:

$$A + \frac{3B}{N^2} = 0 \quad (33)$$

which is exactly the one obtained by requiring the vanishing of $\beta_h$ at order $g^7$. Thus we provide one more confirmation that finiteness theorems for the gauge beta functions hold even in the CCR context.
4 Conclusions

In this paper we have considered the $\mathcal{N} = 1$ $SU(N)$ super Yang–Mills theory obtained as a marginal deformation of the $\mathcal{N} = 4$ theory. In particular, we have focused on the superconformal condition working perturbatively with a complex deformation parameter $\beta$ at finite $N$.

We have addressed the issue of finding a surface of renormalization fixed points by requiring the theory to have vanishing beta functions and using the coupling constant reduction (CCR) procedure. In the CCR prescription the renormalized chiral couplings are expressed in terms of a power expansion in the real gauge coupling constant $g$ and this amounts to face loop mixing at a given order of $g$.

First, we have concentrated on the chiral beta function ($\beta_h$) up to $O(g^7)$. To this end we have fixed the arbitrary coefficients which appear in the power expansions of the chiral couplings (5) by requiring $\gamma = 0$ order by order. If we want to work with a well-defined and a physically meaningful quantum field theory, we believe that the condition $\beta_h = 0$ should not be affected by scheme dependence. Scheme independence of the conformal definition of the theory introduces a further constraint on the couplings. Here comes the novelty with respect to the planar case studied in [7]. The planar limit involves only two of the three independent constants in (4) and scheme independence of the theory forces $\beta$ to be real. On the other hand, keeping $N$ finite, all of the three parameters $|h_1|^2, |h_2|^2, |h_3|^2$ enter the superconformal condition allowing for a complex deformed theory which is scheme–independent at least at $O(g^{10})$. We expect this pattern should hold even for higher orders.

Then we have considered the gauge beta function $\beta_g$. Working in the CCR context we are not guaranteed that standard finiteness theorems [19, 20] are valid. In [7] a generalization of these theorems was proposed: if $\beta_h = 0$ up to $O(g^{2n+1})$ then $\beta_g = 0$ up to $O(g^{2n+3})$. This statement was checked in the planar limit for $n = 4$ using an alternative procedure for covariant $\nabla$–algebra. Here we have provided another highly non–trivial confirmation of this proposal in the non–planar theory for $n = 3$. Moreover, we have explicitly checked that the simplified $\nabla$–algebra technique used in [7] is equivalent to the standard one.
Appendix A

In this Appendix we report the full expression for the color of the four loop diagram depicted in Fig.3:

\[
K_4 = \frac{1}{2} \left[ (|h_1|^2 + |h_2|^2)^4 + (|h_1|^2 - |h_2|^2)^4 \right] + \\
+ \frac{4}{N^2} \left[ |h_3|^8 - 4|h_3|^6(|h_1|^2 + |h_2|^2) + 2|h_3|^4(3|h_1|^4 + 4|h_1|^2|h_2|^2 + 3|h_2|^4) + \\
- 2|h_3|^2(3|h_1|^6 + 5|h_1|^4|h_2|^2 + 5|h_1|^2|h_2|^4 + 3|h_2|^6) + \\
+ (|h_1|^8 + 8|h_1|^6|h_2|^2 + 6|h_1|^4|h_2|^4 + 8|h_1|^2|h_2|^6 + |h_2|^8) \right] + \\
- \frac{4}{N^4} \left[ 5|h_3|^8 - 20|h_3|^6(|h_1|^2 + |h_2|^2) + 12|h_3|^4(|h_1|^4 + |h_1|^2|h_2|^2 + |h_2|^4) + \\
- 8|h_3|^2(|h_1|^6 - |h_1|^4|h_2|^2 - |h_1|^2|h_2|^4 + |h_2|^6) \right] + \\
- \frac{4}{N^6} \left[ 10|h_3|^8 + 32|h_3|^6(|h_1|^2 + |h_2|^2) - 8|h_3|^4|h_1|^2|h_2|^2 \right] + \frac{256}{N^8} |h_3|^8
\]

From this formula one can easily obtain the explicit value of the \( f \) function in (25):

\[
f = 8 \left[ a_1^4 + 8a_1^3b_1 + 6a_1^2b_1^2 + 8a_1b_1^3 + b_1^4 - 2(a_1 + b_1)(3a_1^2 + 2a_1b_1 + 3b_1^2)c_1 + \\
+ 2(3a_1^2 + 4a_1b_1 + 3b_1^2)c_1^2 - 4(a_1 + b_1)c_1^3 + c_1^4 \right] + \frac{8}{N^2} \left[ 8(a_1 - b_1)^2(a_1 + b_1)c_1 + \\
- 12(a_1^2 + a_1b_1 + b_1^2)c_1^2 + 20(a_1 + b_1)c_1^3 - 5c_1^4 \right] + \frac{8}{N^4} \left[ 8a_1b_1c_1^2 - 32(a_1 + b_1)c_1^3 - 10c_1^4 \right] + \\
+ \frac{512}{N^6} c_1^4
\]

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