EHRHART POSITIVITY AND DEMAZURE CHARACTERS

PER ALEXANDERSSON AND ELIE ALHAJJAR

Abstract. Demazure characters, also known as key polynomials, generalize the classical Schur polynomials. In particular, when all variables are set equal to 1, these polynomials count the number of integer points in a certain class of Gelfand–Tsetlin polytopes. This property highlights the interaction between the corresponding polyhedral and combinatorial structures via Ehrhart theory. In this paper, we give an overview of results concerning the interplay between the geometry of Gelfand–Tsetlin polytopes and their Ehrhart polynomials. Motivated by strong computer evidence, we propose several conjectures about the non-negativity of the coefficients of such polynomials.

Demazure characters, key polynomials, Gelfand–Tsetlin polytopes, Ehrhart polynomial

1. Introduction

The theory of Schur polynomials can be seen from two different sets of lenses. On one hand, the traditional approach begins with a definition involving quotients of matrix determinants. This method is mainly useful in representation theory, since it is derived as a special case of the Weyl character formula. On the other hand, the combinatorial approach uses the sum expansion over semi-standard Young tableaux of fixed shape. Note that it is not hard to show explicitly the equivalence of these two approaches.

Demazure characters [Dem74b, Dem74a], also known as key polynomials, generalize the classical Schur polynomials. Key polynomials can be computed recursively via divided difference operators, and are closely related to Schubert polynomials. In particular, each Schubert polynomial can be expressed as a non-negative integer combination of key polynomials.

A combinatorial formula using semi-standard Young tableaux was discovered in [LS90], and this is where the notion of key polynomials come from. Key polynomials are specializations of non-symmetric Macdonald polynomials, [HHL08] so the combinatorial formula of J. Haglund gives an alternative formulation, using skyline fillings. S. Mason [Mas09] explores two variations of skyline fillings, both of which give key polynomials. It is possible to interpolate between the two skyline models, see [Kur16, AS18].

For each a partition $\lambda$, one can construct a Gelfand–Tsetlin polytope. These polytopes play a crucial role in representation theory, algebraic geometry and combinatorics. Their importance stems from the fact that their integer points are in bijection with semi-standard Young tableaux.

R. Stanley and A. Postnikov [PS08] study a certain subfamily of key polynomials, and prove that these are flagged Schur polynomials. This implies that key polynomials in this family can be computed as a certain sum over lattice points in a single face of a Gelfand–Tsetlin polytope.
The result by R. Stanley and A. Postnikov can be extended to the full family
of key polynomials; V. Kirichenko, E. Smirnov and V. Timorin [KST12] show that
key polynomials can be expressed as a sum over lattice points in a certain union
of faces in a Gelfand–Tsetlin polytope. This way of thinking about key polynomials
is not as well-known as the other interpretations, and the purpose of this survey is
to emphasize the polyhedral aspect of key polynomials. A related result appears in
[FM16], where Hall–Littlewood polynomials are expressed as a weighted sum over
lattice points in Gelfand–Tsetlin polytopes.

Recent research has been focused on products involving key polynomials, see
[Pum16]. The main motivation for studying key polynomials is to gain insight
about the expression of products of Schubert polynomials in terms of Schubert
polynomials, a problem of main importance in representation theory. The close
relationship between key and Schubert polynomials is emphasized in [RS05].

The purpose of the current paper is two-fold: on one hand, we aim to collect
some of the main results related to the study of key polynomials. On the other
hand, we propose several conjectures concerning the non-negativity of the coeffi-
cients of the ‘stretched’ version of such polynomials. In Section 2 below, we provide
the basic material and fix the terminology for the remainder of the paper. Section 3
deals with the facial description of GT-polytopes and the formal definition of key
polynomials, where we give several examples to illustrate the main ideas. In Sec-
tion 4 we introduce the connection to Ehrhart theory through Kostka coefficients
and in Section 5 we mention a sample of the computations that lead eventually to
the main conjecture. Finally, we reconstruct a counterexample in Section 6 that
shows the failure of the non-negativity argument in the case of arbitrary faces of
GT-polytopes.

2. Preliminaries

Given an integer partition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, we can associate a Young
diagram of shape $\lambda$ as a diagram in the plane with $\lambda_i$ left-justified boxes in row $i$.
For example, $\lambda = (5, 3, 2, 2)$ gives rise to the following Young diagram:

```
   * * * * *
   * * * *
   * * *
```

A semi-standard Young tableau of shape $\lambda$ is an assignment of natural numbers to
the boxes of the Young diagram, such that rows are weakly increasing from left to
right, and columns are strictly increasing from top to bottom. Only the first of the
following three assignments is a semi-standard Young tableau:

```
1 1 2 2 4
2 3 3
4 5
5 7
```

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following three assignments is a semi-standard Young tableau:

```
1 1 2 2 4
2 3 3
4 5
5 7
```

2.1. Gelfand–Tsetlin polytopes. There are several families of polytopes which
are referred to as Gelfand–Tsetlin polytopes, see for example [GKT13] and [LM04]
A Gelfand–Tsetlin pattern or GT-pattern for short is a triangular array \((x_{ij})\) visualized as
\[
x_{n1} \quad x_{n2} \quad \cdots \quad x_{nn} \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
x_{21} \quad x_{22} \\
x_{11}
\]
(1)
satisfying the inequalities
\[
x_{i+1,j} \geq x_{ij} \quad \text{and} \quad x_{ij} \geq x_{i+1,j+1}
\]
for all values of \(i, j\) where the indexing is defined. The inequalities simply state that down-right diagonals are weakly decreasing and down-left diagonals are weakly increasing.

Given an integer partition \(\lambda\), the Gelfand–Tsetlin polytope \(\mathcal{GT}(\lambda) \subset \mathbb{R}^{\frac{n(n+1)}{2}}\) is the convex polytope of Gelfand–Tsetlin patterns defined by the inequalities in Equation (2) together with the equalities \(x_{ni} = \lambda_i\) for \(i = 1, 2, \ldots, n\).

The polytope \(\mathcal{GT}(\lambda)\) has integer vertices. In fact, it has a unimodular triangulation, see [Ale17]. Also, note that \(k \cdot \mathcal{GT}(\lambda) = \mathcal{GT}(k\lambda)\) for all \(k \geq 0\).

2.2. Bijection with semi-standard Young tableaux. Note that (2) implies that any two adjacent rows in an integral GT-pattern form a skew Young diagram, see the standard textbook by R. Stanley [Sta01] for terminology.

This property enables us to define a bijection with Young tableaux — the skew shape defined by row \(j\) and \(j + 1\) in an integral GT-pattern \(G\) describes which boxes in a Young tableau \(T\) have content \(j\). In particular, tableaux of shape \(\lambda\) are in bijection with integral GT-patterns with topmost row equal to \(\lambda\). See Figure 1 for an example of this correspondence.

\[
\begin{array}{ccccccc}
5 & 4 & 2 & 1 & 1 & 0 \\
5 & 3 & 2 & 1 & 0 \\
3 & 3 & 2 & 1 \\
3 & 3 & 1 \\
3 & 2 \\
3 \\
\end{array}
\quad \leftrightarrow \quad
\begin{array}{ccccccc}
1 & 1 & 1 & 5 & 5 \\
2 & 2 & 3 & 6 \\
3 & 4 \\
4 \\
6 \\
\end{array}
\]
(3)

Figure 1. The GT-pattern corresponding to a Young tableau. For example, the third row tells us that the shape of the entries \(\leq 3\) in the tableau is \((3, 3, 1)\).

Note that in any integral GT-pattern, \(x_{i+1,j} - x_{ij}\) counts the number of boxes with content \(i\) in row \(j\) in the corresponding tableau. Given a GT-pattern \(G\), we define the weight \(w(G)\) as the vector
\[
w_i(G) := \sum_{j=1}^{i+1} x_{i+1,j} - \sum_{j=1}^{i} x_{ij},
\]
where \(x_{0j} := 0\). Thus, an integral GT-pattern with weight \(w\) is in bijection with a semi-standard Young tableau with \(w_i\) entries equal to \(i\). Hence, integer points in
\( \mathcal{GT}(\lambda) \) are in bijection with \( \text{SSYT}(\lambda, n) \) — the set of semi-standard Young tableaux of shape \( \lambda \) with maximal entry \( n \). Given \( \lambda \) and \( w \), let \( \mathcal{GT}(\lambda, w) \subseteq \mathcal{GT}(\lambda) \) be the intersection of \( \mathcal{GT}(\lambda) \) with the hyperplanes defined by \( (4) \). The lattice points in \( \mathcal{GT}(\lambda, w) \) are enumerated by the Kostka coefficients, see Section 4.1 below.

One can then define the Schur polynomials \( s_\lambda(z_1, \ldots, z_n) \) as
\[
s_\lambda(z_1, \ldots, z_n) := \sum_{G \in \mathcal{GT}(\lambda) \cap \mathbb{Z}^{n+1}} z_1^{w_1(G)} \cdots z_n^{w_n(G)}. \tag{5}\]

In particular, by using the Weyl dimension formula [Sta01, Eq. 7.105] we have that
\[
s_\lambda(1, 1, \ldots, 1) = |\mathcal{GT}(\lambda) \cap \mathbb{Z}^{n+1}| = |\text{SSYT}(\lambda, n)| = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{6}\]

The fact that the right hand side is a polynomial in the \( \lambda_i \) is not a complete surprise. Gelfand–Tsetlin polytopes belong to a larger family of so called marked order polytopes, generalizing the notion of order polytopes introduced by R. Stanley in [Sta86]. K. Jochemko and R. Sanyal proved in [JS14] that marked order polytopes give rise to Ehrhart functions which are piecewise polynomial in the markings, which in our case are the parts of \( \lambda \).

2.3. Skew GT-polytopes. In analogy with the triangular case, one can define so called skew Gelfand–Tsetlin polytopes using parallelogram arrangements of non-negative numbers,
\[
x_{n1} \quad x_{n2} \quad \cdots \quad \cdots \quad x_{nm} \\
\cdots \quad \cdots \quad \cdots \quad \cdots \\
x_{11} \quad x_{12} \quad \cdots \quad \cdots \quad x_{1m} \\
x_{01} \quad x_{02} \quad \cdots \quad \cdots \quad x_{0m} \tag{7}\]
satisfying the inequalities
\[x_{i+1,j} \geq x_{ij} \text{ and } x_{ij} \geq x_{i+1,j+1}\] for all \( i, j \) where the indexing is defined.

Consider an \((n+1) \times m\) GT-pattern with top row \( \lambda \) and bottom row \( \mu \) — that is, \( x_{ni} = \lambda_i \) and \( x_{0i} = \mu_i \) for \( i = 1, \ldots, m \). These equalities together with the above inequalities define a convex polytope, the skew Gelfand–Tsetlin polytope, \( \mathcal{GT}(\lambda/\mu) \subseteq \mathbb{R}^{(n+1)m} \). Note that the vertices of such polytopes have integer coordinates. In fact, the skew Gelfand–Tsetlin polytopes admit a unimodular triangulation.

The weight of a GT-pattern in the parallelogram form is defined in the same way as in the triangular form \( [4] \). Similar to the bijection described in Section 2.2 the integer points in \( \mathcal{GT}(\lambda/\mu) \) correspond to skew Young tableaux with shape \( \lambda/\mu \), where the entries belong to the set \( \{1, 2, \ldots, n\} \). This allows us to define the skew Schur polynomials as
\[
s_{\lambda/\mu}(z_1, \ldots, z_n) := \sum_{G \in \mathcal{GT}(\lambda/\mu) \cap \mathbb{Z}^{(n+1)m}} z_1^{w_1(G)} \cdots z_n^{w_n(G)}. \tag{9}\]

We note that there is no simple formula for computing the specialization \( s_{\lambda/\mu}(1, 1, \ldots, 1) \) in general.
By specifying a weight vector \( w \), we intersect \( \mathcal{G}T(\lambda/\mu) \) with a set of hyperplanes, and denote the resulting polytope by \( \mathcal{G}T(\lambda/\mu, w) \). As with \( \mathcal{G}T(\lambda, w) \), the polytopes \( \mathcal{G}T(\lambda/\mu, w) \) are also not integral in general.

3. Reduced Kogan faces and key polynomials

By imposing some additional equalities on the coordinates of a GT-polytope, one can obtain faces of the polytope. There is a particular interest with equalities of the form \( x_{ij} = x_{i,j+1} \). By imposing a set of such equalities, we obtain a Kogan face of the GT-polytope. To each equality of the form \( x_{ij} = x_{i,j+1} \), we associate the transposition \( s_{n-i+j-1} \), as shown in Figure 2. We then construct a word from these transpositions by reading the entries from bottom to top, left to right. If this word is reduced, we say that the corresponding Kogan face is reduced. Note that the same word might be constructed from equalities in several different ways. The type of a Kogan face is the permutation obtained from the word. We should really view the equalities that define a Kogan face as some special set of equalities — a point in this face might satisfy some additional equalities present, if it is also a member of some sub-face. This implies that a point in the GT-polytope can be a member of several (reduced) Kogan faces. For example, the point where all equalities are present is the unique face of type \( w_0 \), the longest permutation in \( S_n \). This point is a sub-face of all other Kogan faces. Furthermore, the full GT-polytope has the identity permutation as type.

**Example 1.** Consider the face in Figure 3. All marked equalities are in the southeast direction, so we obtain the word \( s_3 s_1 s_2 s_3 \). It is straightforward to verify that this word is reduced, so this face is a reduced Kogan face.

Suppose we wish to examine the GT-polytope \( \mathcal{G}T(\lambda) \) with \( \lambda = (4, 3, 3, 2) \). The lattice points in the reduced Kogan face with the word \( s_3 s_1 s_2 s_3 \) are the following...
GT-patterns:

\[
\begin{array}{cccc}
4 & 3 & 3 & 2 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 & 3 & 2 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 & 3 & 2 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 & 3 & 2 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 & 3 \\
4 & 3 & 3 \\
4 & 3 \\
4 & 3 \\
4 & 3 \\
\end{array}
\]

It is evident from the bubble-sort algorithm \cite{Knu98} that there is at least one reduced Kogan face of every type \( \sigma \in S_n \).

**Proposition 2** (See \cite{Kog00}). If \( \sigma \in S_n \) avoids the permutation pattern 132 (such permutations are known as Kempf permutations), then there is a unique reduced Kogan face of type \( \sigma \).

See Section 5.1 for more background on this special case.

**Example 3.** There are 11 reduced Kogan faces for \( n = 4 \) that are covered by Proposition 2. These faces are illustrated in Figure 4.

![Figure 4](image_url)

**Figure 4.** The 11 Kogan faces for \( n = 4 \) whose type avoids the pattern 132.

### 3.1. Key polynomials

One possible generalization of Schur polynomials is the so-called Demazure characters, also known as key polynomials. The latter name was introduced by V. Reiner and M. Shimozono in \cite{RS95}, and refers to the combinatorial model for Demazure characters introduced in \cite{LS90}, where they use the so-called key tableaux. In order to define key polynomials, we need some preliminary terminology.

Whenever \( s_i \in S_n \) is a simple transposition, with \( i \in \{1, 2, \ldots, n-1\} \), we let \( s_i \) act on \( \mathbb{C}[z_1, \ldots, z_n] \) by permuting the indices:

\[
s_i \circ f(z_1, \ldots, z_n) = f(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, z_i, z_{i+2}, \ldots, z_n).
\]

Define the **divided difference operator** \( \partial_i \) as

\[
\partial_i(f) = \frac{f - s_i(f)}{z_i - z_{i+1}}.
\]
We now define the operators $\pi_i \in S_n$ that consider the case covered in Proposition 2.

We continue the calculation by applying $\partial_2(z_1^2 z_2^5 z_3^4)$, which is indeed a polynomial, and one can check that $\partial_i(f)$ is symmetric in the variables $z_i$ and $z_{i+1}$. For example,

$$\partial_2(z_1^2 z_2^5 z_3^4) = \frac{z_1^2 z_2^5 z_3^4}{z_2 - z_3} - \frac{z_1^2 z_2^5 z_3^4}{z_2 - z_3} = \frac{z_1^2 z_2^5 z_3^4}{z_2 - z_3} - \frac{z_1^2 z_2^5 z_3^4}{z_2 - z_3} = \frac{z_1^2 z_2^5 z_3^4}{z_2 - z_3}.$$

We now define the operators $\pi_i(f) := \partial_i(z_i z_{i+1})$ for $i = 1, \ldots, n - 1$ whenever $f \in \mathbb{C}[z_1, \ldots, z_n]$. It is straightforward to verify the following properties of the $\pi_i$'s:

- $\pi_i$ preserves the degree,
- $\pi_i^i = \pi_i$ for all $i$,
- $\pi_i \pi_j = \pi_j \pi_i$ whenever $|i - j| > 2$,
- $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for all $i$.

The last two properties allow us to make the following definition: Let $\sigma = s_i s_{i+1} \cdots s_{n}$ be a reduced word of a permutation $\sigma \in S_n$. Then let

$$\pi_{\sigma} := \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_r}.$$ 

The action of $\pi_{\sigma}$ is independent of the choice of reduced word, since we have the relations above.

We are now ready to define the key polynomials. Let $\lambda$ be a partition with at most $n$ parts, and let $\sigma \in S_n$ be a permutation. The key polynomial $\kappa_{\lambda, \sigma}(z)$ is defined as

$$\kappa_{\lambda, \sigma}(z) := \pi_{\sigma}(z_1^{\lambda_1} \cdots z_n^{\lambda_n}).$$

**Example 4.** Let $\lambda = (2, 1, 0, 0)$ and $\sigma = [2, 4, 3, 1] \in S_4$ in one-line notation. The permutation can be expressed as a reduced word as $\sigma = s_2 s_3 s_2 s_1$. We compute the key polynomial as follows:

$$\kappa_{\lambda, \sigma}(z) = \pi_2 \pi_3 \pi_2 \pi_1(z_1^2 z_2) = \pi_2 \pi_3 \pi_2 \partial_1(z_1^2 z_2) = \pi_2 \pi_3 \pi_2 \left(\frac{z_1^3 z_2 - z_1 z_2^2}{z_1 - z_2}\right) = \pi_2 \pi_3 \pi_2(z_1^2 z_2 + z_1 z_2^2).$$

We continue the calculation by applying $\pi_2$ and get

$$\kappa_{\lambda, \sigma}(z) = \pi_2 \pi_3(z_2 z_1^2 + z_3 z_1^2 + z_2 z_1 + z_3 z_1 + z_2 z_3 z_2).$$

Applying $\pi_2 \pi_3$ then finally gives

$$\kappa_{\lambda, \sigma}(z) = z_1^2 z_2 + z_1^2 z_3 + z_1 z_4 + z_1 z_2 + z_1 z_3^2 + z_1 z_3^2 + z_1 z_3^2 + z_1 z_3 z_4. \tag{11}$$

In general, some monomials may appear multiple times.

In [KST12], the following formula for key polynomials using Kogan faces was proved. This generalizes an earlier result by A. Postnikov and R. Stanley [PS08], who considered the case covered in Proposition 2.
Proposition 5. Let $\mathcal{G}T(\lambda, \sigma)$ be defined as the polytopal complex

$$\mathcal{G}T(\lambda, \sigma) := \bigcup_{\mathcal{F} \in \mathcal{G}T(\lambda)} \mathcal{F}_{\text{type}(\mathcal{F})=w_0 \sigma}.$$ 

That is, $\mathcal{G}T(\lambda, \sigma)$ is the union of all reduced Kogan faces of type $w_0 \sigma$ in the polytope $\mathcal{G}T(\lambda)$. The key polynomial $\kappa_{\lambda, \sigma}(z)$ can be computed as

$$\kappa_{\lambda, \sigma}(z_1, \ldots, z_n) = \sum_{G \in \mathcal{G}T(\lambda, \sigma) \cap \mathbb{Z}^n} \frac{n(n+1)}{2} z_1^{w_1(G)} \cdots z_n^{w_n(G)}$$

where we use the same weight for integral Gelfand–Tsetlin patterns as in (4).

As an immediate corollary, it is clear that

$$s_\lambda(z_1, \ldots, z_n) = \kappa_{\lambda, w_0}(z_1, \ldots, z_n).$$

We now recalculate the key polynomial in Example 4 using (12).

Example 6. Let $\lambda = (2, 1, 0, 0)$ and $\sigma = [2, 4, 3, 1] \in \mathcal{S}_4$. We have that $\omega_0 \sigma = [1, 3, 4, 2]$, and we have that

$$[1, 3, 4, 2] = s_3 s_2.$$

There are no other reduced words that give rise to the same permutation. However, there are three reduced Kogan faces that give rise to this particular reduced word (and are hence of type $[1, 3, 4, 2]$):

(A)  
(B)  
(C)

We expect nine lattice points in the union of these faces, as there are nine monomials in (11). These lattice points are given by the following Gelfand–Tsetlin patterns:

\[
\begin{array}{cccccccc}
2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
C & BC & ABC & z_1 z_2 z_3 z_4 & z_1 z_2 z_4 & z_1 z_2 z_4 \\
z_1 z_2^2 & z_1 z_3 z_4 & z_1 z_3 z_4 & z_1 z_3 z_4 & z_1 z_3 z_4 & z_1 z_3 z_4 & z_1 z_3 z_4 & z_1 z_3 z_4
\end{array}
\]
The letters below each pattern indicate which reduced Kogan faces in (13) the pattern is a member of and the monomials represent $z^{w(G)}$ as defined in (4).

**4. Ehrhart Polynomials**

From (6), it follows that the Ehrhart polynomial of $\mathcal{G}T(\lambda)$ is given by

$$i(\mathcal{G}T(\lambda), k) = \prod_{1 \leq i < j \leq n} \frac{k(\lambda_i - \lambda_j) + j - i}{j - i}$$

and it is clear that all coefficients of $k$ are non-negative.

For a skew shape $\lambda/\mu$, the lattice point enumerator is indeed a polynomial $i(\mathcal{G}T(\lambda/\mu), k) = s_{k\lambda/k\mu}(1, 1, \ldots, 1)$. Table 1 below shows a sample computation of such polynomials. Computer evidence suggests the following conjecture:

**Conjecture 7.** Let $\lambda/\mu$ be a skew shape. Then the polynomial $i(\mathcal{G}T(\lambda/\mu), k)$ has non-negative coefficients.

We do not expect a closed-form formula for $i(\mathcal{G}T(\lambda/\mu), k)$ — the volume of $\mathcal{G}T(\lambda/\mu)$ is related to the number of skew standard Young tableaux for which there are no known closed formulas in general, see [MPP18] and subsequent papers.

| $\lambda/\mu$           | $s_{k\lambda/k\mu}(1, 1, 1)$                           |
|-------------------------|--------------------------------------------------------|
| 221                     | $\frac{1}{2}(k^2 + 3k + 2)$                            |
| 221/21                  | $\frac{1}{2}(k^4 + 6k^3 + 13k^2 + 12k + 4)$           |
| 221/11                  | $k^3 + 3k^2 + 3k + 1$                                  |
| 321                     | $\frac{1}{2}(k^5 + 6k^4 + 14k^3 + 16k^2 + 9k + 2)$    |
| 321/2                   | $\frac{1}{2}(k^6 + 9k^5 + 33k^4 + 63k^3 + 66k^2 + 36k + 8)$ |

**Table 1.** Ehrhart polynomials for $\mathcal{G}T(\lambda/\mu)$.

**4.1. Kostka coefficients.** Kostka coefficients are important numbers appearing in several branches of mathematics such as algebraic combinatorics, symmetric functions, representation theory and algebraic geometry among others. From a representation theoretical point of view, they are defined as the dimension of the weight subspace of the irreducible representation of the Lie algebra $\mathfrak{gl}(C)$. From a combinatorial point of view, they enumerate the number of semi-standard Young tableaux of fixed shape and weight.
Given two partitions $\lambda$ and $\mu$ of $n$, Kostka coefficients arise in the expansion of the Schur polynomial as a linear combination of monomial symmetric polynomials
\[ s_\lambda(z_1, \ldots, z_n) = \sum_{\mu} K_{\lambda\mu} m_\mu(z_1, \ldots, z_n). \tag{15} \]

More generally, Kostka coefficients can be viewed as a special case of so called Littlewood–Richardson coefficients $K_{\lambda\mu} = c_{\tau}^{\lambda,\mu}$, where $\sigma$ and $\tau$ are defined in terms of the partition $\mu$. For more material about the latter coefficients and some standard results, the reader is referred to the book [FH91].

The partitions $\lambda$ and $\mu$ are usually represented as integer vectors, so it makes sense to talk about scaling these vectors by an integer factor $k$. This operation gives rise to what is called “stretched” Kostka coefficients $K_{\lambda\mu}(k) := K_{k\lambda,k\mu}$. Surprisingly at first, $K_{\lambda\mu}(k)$ turns out to be a polynomial function in $k$ with rational coefficients depending on $\lambda$ and $\mu$ — a fact initially shown by A. Kirillov and N. Reshetikhin [KR88].

Expressing Kostka coefficients using the Gelfand–Tsetlin polytopes from Section 2.1 provides a natural geometric interpretation of $K_{\lambda\mu}(k)$. To each Kostka coefficient $K_{\lambda\mu}$, we have the corresponding Gelfand–Tsetlin polytope $\mathcal{GT}(\lambda, \mu)$ in $\mathbb{R}^{\binom{n+1}{2}}$ such that
\[ K_{\lambda\mu} = |\mathcal{GT}(\lambda, \mu) \cap \mathbb{Z}^{\binom{n+1}{2}}| \text{ and } K_{\lambda\mu}(k) = |k \cdot \mathcal{GT}(\lambda, \mu) \cap \mathbb{Z}^{\binom{n+1}{2}}|. \tag{16} \]

The vertices of $\mathcal{GT}(\lambda, \mu)$ have rational coordinates in general ($n \geq 5$). From E. Ehrhart’s fundamental work, it is well-known that $K_{\lambda\mu}(k)$ must be a quasipolynomial. This means that there exist an integer $M$ and polynomials $g_0, g_1, \ldots, g_{M-1}$ such that $K_{\lambda\mu}(k) = g_i(k)$ whenever $k \equiv i \pmod{M}$ — see details in [Sta11]. The “surprising” fact here is that the function $K_{\lambda\mu}(k)$ is indeed a polynomial, which exhibits an example of period collapse in rational polytopes.

In their paper [KTT04], the authors conjectured a formula for the degree of $K_{\lambda\mu}(k)$ which was later proved by T. McAllister [McA08]. In that same paper, they also conjectured that the coefficients of the polynomial $K_{\lambda\mu}(k)$ are non-negative. To the best of our knowledge, this conjecture remains open and provides an instance of a well-known phenomenon called Ehrhart positivity. For a recent survey about this topic, we recommend the article by F. Liu [Liu17].

The polytopes $\mathcal{GT}(\lambda/\mu, w)$ also exhibit a period collapse, and have a polynomial Ehrhart function. This follows from the work of E. Rassart in [Ras04]. Integrality and the integer decomposition property of the family $\mathcal{GT}(\lambda/\mu, w)$ is studied in the work of P. Alexandersson [Ale16], where several questions are left unanswered. The skew Kostka coefficients $K_{\lambda/\mu,\nu}$ are defined analogously to the usual Kostka coefficients, via the identity
\[ s_{\lambda/\mu}(z_1, \ldots, z_n) = \sum_{\nu} K_{\lambda/\mu,\nu} m_\nu(z_1, \ldots, z_n). \tag{17} \]

Scaling by an integer factor $k$ gives rise to the polynomial $i(\mathcal{GT}(\lambda/\mu, \nu), k) = K_{k\lambda/k\mu,k\nu} \in \mathbb{Q}[k]$. The conjecture by R.C. King et al. seems to extend to the skew case:

**Conjecture 8.** Let $\lambda/\mu$ be a skew shape and $\nu$ be a weight. Then the polynomial $i(\mathcal{GT}(\lambda/\mu, \nu), k)$ has non-negative coefficients.
5. A conjecture on key polynomials

Given the positivity phenomena above, it is reasonable to ask if we have positive Ehrhart coefficients for the polytopal complexes in Proposition 5. In other words, for a partition \( \lambda \) and permutation \( \sigma \in S_n \), do

\[
i(\mathcal{GT}(\lambda, \sigma), k) = \kappa_{k, \lambda, \sigma}(1, 1, \ldots, 1) \in \mathbb{Q}[k]
\]

have non-negative coefficients?

By using the divided difference operators in Equation (10), it is possible to compute the Ehrhart polynomials for small values of \( n \). Note first that for all \( \lambda, \sigma \) and \( m \geq 0 \), we have that

\[
\kappa_{m+\lambda, \sigma}(z_1, \ldots, z_n) = (z_1 z_2 \cdots z_n)^m \kappa_{\lambda, \sigma}(z_1, \ldots, z_n),
\]

where \( m + \lambda \coloneqq (m + \lambda_1, m + \lambda_2, \ldots, m + \lambda_n) \). It follows that

\[
\kappa_{m+\lambda, \sigma}(1, \ldots, 1) = \kappa_{\lambda, \sigma}(1, \ldots, 1) \text{ and } i(\mathcal{GT}(m + \lambda, \sigma), k) = i(\mathcal{GT}(\lambda, \sigma), k).
\]

Example 9. Let \( \lambda = (a + b, a, 0) \), with \( a, b \geq 0 \), since we can assume that the last part is equal to zero, due to (18). We have the following table for \( \kappa_{\lambda, \sigma}(z_1, z_2, z_3) \), for different values of \( \sigma \in S_3 \):

| \( \sigma \) | \( \kappa_{\lambda, \sigma}(z_1, z_2, z_3) \) |
|----------------|---------------------------------------------|
| \([1, 2, 3]\) | \( z_1^{a+b} z_2^a \) |
| \([2, 1, 3]\) | \( z_2^a z_1^a (z_1 + z_2)^{b+1} \) |
| \([1, 3, 2]\) | \( z_1^a z_2^b z_3^{a+b} \) |
| \([3, 1, 2]\) | \( z_2^a (z_1 - z_3)^{b+1} + (z_1 - z_2)^{b+1} - (z_1 - z_2)^a (z_1 - z_3)^{b+1} + (z_1 - z_2)^a (z_1 - z_3)^{b+1} \) |
| \([2, 3, 1]\) | \( z_1^a (z_2^{a+b} z_3 + z_2^{a+b+2} z_1 + z_2^{a+1} z_3) \) |
| \([3, 2, 1]\) | \( z_1^a (z_2^{a+b} z_3 + z_2^{a+b+2} z_1 + z_2^{a+1} z_3) \) |

Table 2. Key polynomials computed for all \( \sigma \in S_3 \).

By taking the limit \( z_i \to 1 \) in Example 9 and multiplying \( a \) and \( b \) with \( k \), we get the Ehrhart polynomials \( i(\mathcal{GT}(\lambda, \sigma), k) \), which we present in Table 3. The last polynomial for \( \sigma = [3, 2, 1] \) agrees with Equation (14) with \( \lambda_1 = a + b \), \( \lambda_2 = a \) and \( \lambda_3 = 0 \), since we get

\[
\kappa_{k, \lambda, [3, 2, 1]}(1, 1, 1) = \frac{k(a + b - a) + (2 - 1) k(a - 0) + (3 - 2) k(a + b - 0) + (3 - 1) k(a + b - 0)}{2 - 1}.
\]

We are then fairly confident in the following conjecture:

Conjecture 10. The Ehrhart polynomial \( i(\mathcal{GT}(\lambda, \sigma), k) \in \mathbb{Q}[k] \) has only non-negative coefficients for all \( \lambda, \sigma \). Furthermore, in the new variables variables \( \lambda_i = a_1 + a_2 + \cdots + a_i \), we have

\[
i(\mathcal{GT}(\lambda, \sigma), k) \in \mathbb{Q}[k, a_1, a_2, \ldots, a_n]
\]

(19)
\[ \kappa_{k\lambda,\sigma}(1,1,1) = i(\mathcal{G}\mathcal{T}(\lambda, \sigma), k) \]

| \sigma | \kappa_{k\lambda,\sigma}(1,1,1) |
|--------|-------------------------------|
| [1, 2, 3] | 1 |
| [2, 1, 3] | 1 + bk |
| [1, 3, 2] | 1 + ak |
| [3, 1, 2] | \( \frac{1}{2}(bk + 1)(2ak + bk + 2) \) |
| [2, 3, 1] | \( \frac{1}{2}(ak + 1)(2bk + ak + 2) \) |
| [3, 2, 1] | \( \frac{1}{2}(ak + 1)(bk + 1)(2 + ak + bk) \) |

Table 3. Values of \( \kappa_{k\lambda,\sigma}(z_1, z_2, z_3) \) after taking the limit \( z_i \to 1 \).

Note that \( a \) and \( b \) have been multiplied by \( k \).

with non-negative coefficients.

Above, we verified this for \( \sigma \in S_3 \), and we have also verified this for \( \sigma \in S_4 \).

The symbolic computations done in Mathematica become tedious for larger values of \( n \).

5.1. A determinant formula. An interesting special case was considered in [PS08], where \( \sigma \) is 231-avoiding and \( i(\mathcal{G}\mathcal{T}(\lambda, \sigma), k) \) reduces to the Ehrhart polynomial of a single reduced Kogan face of \( \mathcal{G}\mathcal{T}(\lambda) \), with type \( w_0\sigma \). The number of 231-avoiding permutations in \( \mathfrak{S}_n \) is given by the Catalan numbers, \( \frac{1}{n+1} \binom{2n}{n} \). For an excellent survey on Catalan numbers, see the book by R. Stanley [Sta15].

From [PS08, Corollary 14.6], it follows that the Ehrhart polynomial \( i(\mathcal{G}\mathcal{T}(\lambda, \sigma), k) \) for a 231-avoiding permutation \( \sigma \in \mathfrak{S}_n \) can be obtained as the determinant

\[ i(\mathcal{G}\mathcal{T}(\lambda, \sigma), k) = \det \left( \begin{array}{c} k_{\lambda i} + b_i - i \\ b_i - j \end{array} \right)_{1 \leq i, j \leq n} \]  

(20)

where \( b_1, b_2, \ldots, b_n \) form a sequence of non-negative integers (determined by \( \sigma \)) satisfying

\[ b_1 \leq b_2 \leq \cdots \leq b_n \leq n \quad \text{and} \quad b_i \geq i \quad \text{for} \quad i = 1, 2, \ldots, n. \]

Such sequences are in bijection with 231-avoiding permutations in \( \mathfrak{S}_n \). Conjecture [10] thus states that for every integer partition \( \lambda \), the determinant in (20) has non-negative coefficients (as a polynomial in \( k \)).

The reason (20) exists is due to a Jacobi–Trudi identity for flagged Schur polynomials, as key polynomials are certain flagged Schur polynomials in this special setting.

6. Faces of Gelfand–Tsetlin polytopes

As a consequence of Conjecture [10] and Proposition [2], it would follow that certain faces of Gelfand–Tsetlin polytopes have non-negative coefficients in the Ehrhart polynomial. It is therefore appropriate to investigate Ehrhart polynomials of general faces of \( \mathcal{G}\mathcal{T}(\lambda) \).

However, in this generality there are certain faces where negative coefficients appear. The following example was constructed in [Ale17].
Example 11. Consider the following face $F_\ell$ of $GT(\lambda)$ where $\lambda = (1^\ell, 0^\ell+1)$:

![Diagram of a face](image)

All entries in each respective region are forced to be equal by additional constraints, thus giving rise to a face of a Gelfand–Tsetlin polytope. One can construct a bijection between lattice points in this face, and lattice points in a certain order polytope. It is then straightforward to prove that

$$i(F_\ell, k) = \sum_{j=1}^{k+1} j^\ell \in \mathbb{Q}[k]$$

and this polynomial has a negative coefficient for $\ell = 20$.

For more background on negative Ehrhart coefficients of order polytopes, we refer to [LT18].

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Department of Mathematics, KTH, SE-100 44 Stockholm, Sweden

*E-mail address: per.w.alexandersson@gmail.com*

Department of Mathematics, USMA, West Point, New York, USA

*E-mail address: eliealhajjar@gmail.com*