JORDAN DERIVATIONS ON A LIE IDEAL OF A SEMIPRIME RING AND THEIR APPLICATIONS IN BANACH ALGEBRAS

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ABSTRACT. Let $R$ be a 3!-torsion free noncommutative semiprime ring, $U$ a Lie ideal of $R$, and let $D : R \to R$ be a Jordan derivation. If $[D(x), x]D(x) = 0$ for all $x \in U$, then $D(x)[D(x), x]y - yD(x)[D(x), x] = 0$ for all $x, y \in U$. And also, if $D(x)[D(x), x] = 0$ for all $x \in U$, then $[D(x), x]D(x)y - y[D(x), x]D(x) = 0$ for all $x, y \in U$. And we shall give their applications in Banach algebras.

1. Introduction

Throughout, $R$ represents an associative ring and $A$ will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for $x, y$ in a ring. Let $\text{rad}(R)$ denote the (Jacobson) radical of a ring $R$. And a ring $R$ is said to be semisimple if its Jacobson radical $\text{rad}(R)$ is zero.

A ring $R$ is called $n$-torsion free if $nx = 0$ implies $x = 0$. Recall that $R$ is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. On the other hand, let $X$ be an element of a normed algebra. Then for every $a \in X$ the spectral radius of $a$, denoted by $r(a)$, is defined by $r(a) = \inf \{ ||a^n||^{\frac{1}{n}} : n \in \mathbb{N} \}$. It is well-known that the following theorem holds: if $a$ is an element of a normed algebra, then $r(a) = \lim_{n \to \infty} ||a^n||^{\frac{1}{n}}$ (see F.F. Bonsall and J. Duncan[1]).

An additive mapping $D$ from $R$ to $R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping $D$ from $R$ to $R$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B.E. Johnson and A.M. Sinclair[12] proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I.M. Singer and J. Wermer[13] states that every continuous linear derivation on a commutative Banach algebra maps the
algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.

M.P. Thomas[14] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

J. Vukman[15] proved the following: let $R$ be a 2-torsion free prime ring. If $D : R \rightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he proved that the following holds: let $A$ be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

B.D. Kim [6] showed the following: let $R$ be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \rightarrow R$ such that $[D(x), x]D(x) = 0$ for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, B.D. Kim[7] showed the following: let $A$ be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \rightarrow A$ such that $D(x)[D(x), x]D(x) \in \text{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \text{rad}(A)$.

In this paper, our first aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

Let $R$ be a 3!-torsion free noncommutative semiprime ring, $U$ a Lie ideal of $R$. And let $D : R \rightarrow R$ be a Jordan derivation on $R$ and $U$ a Lie ideal of $R$. In this case, we show that if $[D(x), x]D(x) = 0$ holds for all $x \in U$, then $D(x)[D(x), x]y - yD(x)[D(x), x] = 0$ for all $x, y \in U$, and also, if $D(x)[D(x), x] = 0$ holds for all $x \in U$, then $[D(x), x]D(x)y - y[D(x), x]D(x) = 0$ for all $x, y \in U$. In particular, when $U = R$, then we conclude that $[D(x), x]D(x) = 0$ is equivalent to $D(x)[D(x), x] = 0$ for all $x \in R$.

Moreover, using the above results, we shall give their applications in Banach algebra as follows.

(i): Suppose there exists a continuous linear Jordan derivation $D : A \rightarrow A$ and $U$ a Lie ideal of $A$. Then

$$[D(x), x]D(x) \in \text{rad}(A) \iff D(x)[D(x), x] \in \text{rad}(A)$$

for all $x \in U$.  

And also, we have their applications in Banach algebras as follows. Of course, the following results are already well-known.

(ii): Suppose there exists a continuous linear Jordan derivation \( D \) on a non-commutative Banach algebra \( A \) with a Lie ideal \( U \) such that

\[
[D(x), x]D(x) \in \text{rad}(A)
\]

for all \( x \in U \).

Then we have \( D(x)[D(x), x]y - yD(x)[D(x), x] \in \text{rad}(A) \) for all \( x, y \in U \). And

(iii): Suppose there exists a continuous linear Jordan derivation \( D \) on a non-commutative Banach algebra \( A \) with a Lie ideal \( U \) such that

\[
D(x)[D(x), x] \in \text{rad}(A)
\]

for all \( x \in U \).

Then \( [D(x), x]D(x) \in \text{rad}(A) \iff D(x)[D(x), x] \in \text{rad}(A) \)

for all \( x \in A \). Moreover, we have \( D(A) \subseteq \text{rad}(A) \).

The following lemma is due to L.O. Chung and J. Luh[4].

Lemma 1.1. Let \( R \) be a \( n! \)-torsion free ring. Suppose there exist elements \( y_1, y_2, \ldots, y_{n-1}, y_n \) in \( R \) such that \( \sum_{k=1}^{n} t^k y_k = 0 \) for all \( t = 1, 2, \ldots, n \). Then we have \( y_k = 0 \) for every positive integer \( k \) with \( 1 \leq k \leq n \).

The following theorem is due to M. Brešar[3].

Theorem 1.2. Let \( R \) be a 2-torsion free semiprime ring and let \( D : R \rightarrow R \) be a Jordan derivation. In this case, \( D \) is a derivation.

We write \( Q(A) \) for the set of all quasinilpotent elements in \( A \). M. Brešar [2] has proved the following theorem.

Theorem 1.3. Let \( D \) be a bounded derivation of a Banach algebra \( A \). Suppose that \( [D(x), x] \in Q(A) \) for every \( x \in A \). Then \( D \) maps \( A \) into \( \text{rad}(A) \).

After this, by \( S_m \) we denote the set \( \{ k \in \mathbb{N} \mid 1 \leq k \leq m \} \) where \( m \) is a positive integer.
We need Theorems 2.4 and 2.5 to obtain the main theorems for Banach algebra theory.

2. Main Results

Lemma 2.1. Let $R$ be a noncommutative semiprime ring, and $U$ a Lie ideal of $R$. And suppose that $aya = 0$ for all $y \in U$ and some $a \in R$. Then $a = 0$.

Proof. By the assumption, we have

$$aya = 0, \ y \in U \quad (2.1)$$

Replacing $[y, z]$ for $y$ in (2.1), we obtain

$$a[y, z]a = 0, \ y \in U, z \in R. \quad (2.2)$$

Writing $zaw$ for $z$ in (2.2), we get

$$a[y, z]awa + az[a[y, w]a + az[y, a]wa = 0, \ y \in U, w, z \in R. \quad (2.3)$$

Combining (2.2) with (2.3),

$$az[y, a]wa = 0, \ y \in U, w, z \in R. \quad (2.4)$$

From (2.4), we have

$$[y, a]waz[y, a]wa = 0, \ y \in U, w, z \in R. \quad (2.5)$$

Since $R$ is semiprime, the relation (2.5) yields

$$[y, a]wa = 0, \ y \in U, w \in R. \quad (2.6)$$

Substituting $aw$ for $w$ in (2.6),

$$ya^2wa - ayawa = 0, \ y \in U, w \in R. \quad (2.7)$$

From (2.1) and (2.7), we have

$$ya^2wa = 0, \ y \in U, w \in R. \quad (2.8)$$

From (2.8), we get

$$ya^2wya^2 = 0, \ y \in U, w \in R. \quad (2.9)$$

Since $R$ is semiprime, the relation (2.9) yields

$$ya^2 = 0, \ y \in U. \quad (2.10)$$

Putting $[x, w]$ instead of $y$ in (2.10), we obtain

$$ywa^2 - wya^2, \ y \in U, w \in R. \quad (2.11)$$
Combining (2.10) with (2.11),

(2.12) \[ ywa^{2} = 0, \; x, y \in U, \; w \in R. \]

From (2.12), we get

(2.13) \[ yw[a^{2}, y] = 0, \; y \in U, \; w \in R. \]

From (2.13), we have

(2.14) \[ [a^{2}, y]w[a^{2}, y] = 0, \; y \in U, \; w \in R. \]

Thus by semiprimeness of \( R \), it follows from (2.14) that

(2.15) \[ [a^{2}, y] = 0, \; y \in U. \]

\[ \square \]

For simplicity, we shall denote the maps \( B : R \times R \rightarrow R, \) \( f, g : R \rightarrow R \) by \( B(x, y) \equiv [D(x), y] + [D(y), x], \) \( f(x) \equiv [D(x), x], \) \( g(x) \equiv [f(x), x] \) for all \( x, y \in R \) respectively. Then we have the basic properties:

\[ B(x, y) = B(y, x), \]
\[ B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z), \]
\[ B(x, x) = 2f(x), \]
\[ B(xy, z) = B(y, z)x + zB(y, x) + D(z)[x, y] + [z, y]D(x), \]
\[ B(x, x^2) = 2(f(x)x + xf(x)), \; x, y, z \in R. \]

After this, we use the above relations without specific reference.

**Theorem 2.2.** Let \( R \) be a 3!-torsion free noncommutative semiprime ring, \( U \) a Lie ideal of \( R \) with \( [R, U] \neq \{0\} \). Then Let \( D : R \rightarrow R \) be a Jordan derivation on a semiprime ring.

(i): If \( [D(x), x]D(x) = 0 \) holds for all \( x \in U \), then

\[ D(x)[D(x), x]y - yD(x)[D(x), x] = 0 \]

for all \( x, y \in U. \) And

(ii): If \( D(x)[D(x), x] = 0 \) holds for all \( x \in U \), then

\[ [D(x), x]D(x)y - y[D(x), x]D(x) = 0 \]

for all \( x, y \in U. \)

In particular, when \( U = R \), we see that

\[ [D(x), x]D(x) = 0, \; x \in R \iff D(x)[D(x), x] = 0, \; x \in R. \]
Proof. (i)\(\Rightarrow\): By Theorem 1.2, we can see that \(D\) is a derivation on \(R\). By the assumption,
\[
(2.16) \quad f(x)D(x) = 0, \ x \in U
\]
Replacing \(x + ty\) for \(x\) in (2.16), we have
\[
[D(x + ty), x + ty]D(x + ty) \equiv +t\{B(x, y)D(x) + f(x)D(y)\}
\]
\[
(2.17) \quad +t^2H(x, y) + t^3[D(y), y]D(y) = 0, \ x, y \in U, t \in S_3
\]
where \(H\) denotes the term satisfying the identity (2.17).
From (2.16) and (2.17), we obtain
\[
(2.18) \quad t\{B(x, y)D(x) + f(x)D(y)\} + t^2H(x, y) = 0, \ x, y \in U, t \in S_3.
\]
Since \(R\) is \(2\)-torsion free by assumption, by Lemma 2.1 the relation (2.18) yields
\[
(2.19) \quad B(x, y)D(x) + f(x)D(y) = 0, x, y \in U.
\]
Let \(y = x^2\) in (2.19). Then using (2.16), we have
\[
(2.20) \quad 2(f(x)x + xf(x))D(x) + f(x)(D(x)x + xD(x)) = 0, x \in U.
\]
From (2.16) and (2.20),
\[
(2.21) \quad 3f(x)x D(x) + 2xf(x)D(x) + f(x)D(x)x = 3f(x)^2 = 0, x \in U.
\]
Since \(R\) is \(3\)-torsion free, it follows from (2.21) that
\[
(2.22) \quad f(x)^2 = 0, x \in U.
\]
From (2.16), we obtain
\[
(2.23) \quad 0 = [f(x)D(x), x] = g(x)D(x) + f(x)^2, x \in U.
\]
From (2.22) and (2.23), we have
\[
(2.24) \quad g(x)D(x) = 0, x \in U.
\]
Writing \(yx\) for \(y\) in (2.19), we get
\[
(2.25) \quad (B(x, y)x + 2yf(x) + [y, x]D(x))D(x) + f(x)(D(y)x + yD(x)) = 0, x, y \in U.
\]
Right multiplication of (2.19) by \(x\) leads to
\[
(2.26) \quad B(x, y)D(x)x + f(x)D(y)x = 0, x \in U.
\]
Combining (2.25) with (2.26),
\[
(2.27) \quad -B(x, y)f(x) + 2yf(x)D(x) + [y, x]D(x)^2 + f(x)yD(x) = 0, x, y \in U.
\]
From (2.16) and (2.27), we have

\[(2.28) \quad -B(x, y)f(x) + [y, x]D(x)^2 + f(x)yD(x) = 0, x, y \in U.\]

Substituting \(x^2\) for \(y\) in (2.28), we get

\[(2.29) \quad -2(f(x)x + xf(x))f(x) + f(x)x^2D(x) = 0, x \in U.\]

Comparing (2.22) and (2.24), we obtain

\[(2.30) \quad -2f(x)xf(x) - xf(x)^2 - f(x)(f(x)x + xf(x)) = -3f(x)xf(x) = -3g(x)f(x) = 3f(x)g(x) = 0, x \in R.\]

Since \(R\) is \(3!\)-torsion free by assumption, the relation (2.30) yields

\[(2.31) \quad g(x)f(x) = f(x)g(x) = 0, x \in R.\]

Right multiplication of (2.28) by \(D(x)\) leads to

\[(2.32) \quad -B(x, y)f(x)^2 + [y, x]D(x)^2 f(x) + f(x)yD(x)f(x) = 0, x, y \in R.\]

From (2.22) and (2.32), we have

\[(2.33) \quad [y, x]D(x)^2 f(x) + f(x)yD(x)f(x) = 0, x, y \in R.\]

Substituting \(xy\) for \(y\) in (2.33), we get

\[(2.34) \quad x[y, x]D(x)^2 f(x) + f(x)xyD(x)f(x) = 0, x, y \in R.\]

Left multiplication of (2.34) by \(D(x)\) leads to

\[(2.35) \quad -x[y, x]D(x)^2 f(x) + xf(x)yD(x)f(x) = 0, x, y \in R.\]

From (2.34) and (2.35), we have

\[(2.36) \quad g(x)yD(x)f(x) = 0, x, y \in R.\]

Replacing \(yx\) for \(y\) in (2.36), we obtain

\[(2.37) \quad g(x)yxD(x)f(x) = 0, x, y \in R.\]

Right multiplication of (2.36) by \(x\) leads to

\[(2.38) \quad g(x)yD(x)f(x)x = 0, x, y \in R.\]

From (2.22), (2.37) and (2.38), we have

\[(2.39) \quad g(x)yD(x)g(x) = 0, x, y \in R.\]

Left multiplication of (2.39) by \(D(x)\) leads to

\[(2.40) \quad D(x)g(x)yD(x)g(x) = 0, x, y \in R.\]
Thus by semiprimeness of $R$, it is clear that
\begin{equation}
D(x)g(x) = 0, \quad x \in R.
\end{equation}

Putting $xy$ instead of $y$ in (2.28), we get
\begin{equation}
-(xB(x, y) + 2f(x)y + D(x)[y, x])f(x) + x[y, x]D(x)^2 + f(x)xyD(x)
\end{equation}
\begin{equation}
= 0, \quad x, y \in R.
\end{equation}

Left multiplication of (2.28) by $x$ leads to
\begin{equation}
-xB(x, y)f(x) + x[y, x]D(x)^2 + xf(x)yD(x) = 0, \quad x, y \in R.
\end{equation}

Combining (2.42) with (2.43),
\begin{equation}
-2f(x)yf(x) - D(x)[y, x]f(x) + g(x)yD(x) = 0, \quad x, y \in R.
\end{equation}

Right multiplication of (2.44) by $D(x)$ yields
\begin{equation}
-2f(x)yf(x)D(x) - D(x)[y, x]f(x)D(x) + g(x)yD(x)^2 = 0, \quad x \in R.
\end{equation}

Combining (2.16) with (2.45),
\begin{equation}
g(x)yD(x)^2 = 0, \quad x, y \in R.
\end{equation}

Replacing $yD(x)$ for $y$ in (2.45), we get
\begin{equation}
-2f(x)yD(x)f(x) - D(x)[y, x]D(x)f(x) - D(x)yf(x)^2
\end{equation}
\begin{equation}
+ g(x)yD(x)^2 = 0, \quad x \in R.
\end{equation}

From (2.22) and (2.47), we have
\begin{equation}
-2f(x)yD(x)f(x) - D(x)[y, x]D(x)f(x) + g(x)yD(x)^2 = 0, \quad x \in R.
\end{equation}

Comparing (2.46) and (2.48),
\begin{equation}
2f(x)yD(x)f(x) + D(x)[y, x]D(x)f(x) = 0, \quad x \in R.
\end{equation}

Left multiplication of (2.49) by $D(x)$ leads to
\begin{equation}
2D(x)f(x)yD(x)f(x) + D(x)^2[y, x]D(x)f(x) = 0, \quad x, y \in R.
\end{equation}

Substituting $f(x)y$ for $y$ in (2.49), we obtain
\begin{equation}
2f(x)^2yD(x)f(x) + D(x)f(x)[y, x]D(x)f(x) + D(x)g(x)yD(x)f(x)
\end{equation}
\begin{equation}
= 0, \quad x \in R.
\end{equation}

Combining (2.22), (2.41) with (2.51), we obtain
\begin{equation}
D(x)f(x)[y, x]D(x)f(x) = 0, \quad x \in R.
\end{equation}
Substituting $yD(x)^2z$ for $y$ in (2.52), we get
\begin{equation}
D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)y[D(x)^2, x]zD(x)f(x)
\end{equation}

(2.53)
\begin{equation}
+ D(x)f(x)yD(x)^2[z, x]D(x)f(x) = 0, x, y \in R.
\end{equation}

From (2.16) and (2.53), we have
\begin{equation}
D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)
\end{equation}

(2.54)
\begin{equation}
+ D(x)f(x)yD(x)^2[z, x]D(x)f(x) = 0, x, y \in R.
\end{equation}

Comparing (2.50) and (2.54),
\begin{equation}
D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)
\end{equation}

\begin{equation}
- 2D(x)f(x)yD(x)f(x)zD(x)f(x)
\end{equation}

\begin{equation}
= D(x)f(x)[y, x]D(x)^2zD(x)f(x) - D(x)f(x)yD(x)f(x)zD(x)f(x)
\end{equation}

(2.55)
\begin{equation}
(2.56)
-D(x)f(x)yD(x)f(x))zD(x)f(x) = 0, x, y \in R.
\end{equation}

Thus by semiprimeness of $R$, it is obvious that
\begin{equation}
D(x)f(x)[y, x]D(x)^2 - D(x)f(x)yD(x)f(x) = 0, x, y \in R.
\end{equation}

Replacing $x + tz$ for $x$ in (2.46), we have
\begin{equation}
g(x + tz)\frac{yD(x + tz)^2}{D(x + tz)^2} \equiv g(x)\frac{yD(x)^2}{D(x)^2} + t\{[B(x, z), x] + [f(x), z)]yD(x)^2
\end{equation}

\begin{equation}
+ g(x)y(D(z)D(x) + D(x)D(z))\} + t^2I_1(x, y) + t^3I_2(x, y) + t^4I_3(x, y)
\end{equation}

(2.58)
\begin{equation}
+ t^5g(z)yD(z)^2 = 0, x, y, z \in R, t \in S_4
\end{equation}

where $I_i, 1 \leq i \leq 3$, denotes the term satisfying the identity (2.58).

From (2.46) and (2.58), we obtain
\begin{equation}
t\{[B(x, z), x] + [f(x), z)]yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))\}
\end{equation}

(2.59)
\begin{equation}
+ t^2I_1(x, y) + t^3I_2(x, y) + t^4I_3(x, y) = 0, x, y, z \in R, t \in S_3.
\end{equation}

Since $R$ is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.59) yields
\begin{equation}
([B(x, z), x] + [f(x), z)]yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))
\end{equation}

(2.60)
\begin{equation}
= 0, x, y, z \in R.
\end{equation}
Writing \( u g(x)y \) for \( y \) in (2.60), we get
\[
([B(x, z), x] + [f(x), z]) u g(x)y D(x)^2
\]
\[
+ g(x) u g(x)y(D(z) D(x) + D(x) D(z)) = 0, u, x, y, z \in R. \tag{2.61}
\]
Combining (2.46) with (2.61),
\[
g(x) u g(x)y(D(z) D(x) + D(x) D(z)) = 0, u, x, y, z \in R. \tag{2.62}
\]
Replacing \( y(D(z) D(x) + D(x) D(z))u \) for \( u \) in (2.62), we get
\[
g(x) y(D(z) D(x) + D(x) D(z)) u g(x)y(D(z) D(x) + D(x) D(z)) \]
\[
= 0, u, x, y, z \in R. \tag{2.63}
\]
And so, by semiprimeness of \( R \), it follows that
\[
g(x) y(D(z) D(x) + D(x) D(z)) = 0, x, y, z \in R. \tag{2.64}
\]
Replacing \( x + tw \) for \( x \) in (2.64), we have
\[
g(x + tw) y(D(z) D(x + tw) + D(x + tw) D(z)) = g(x) y(D(z) D(x)
\]
\[
+ D(x) D(z)) + t \{[B(x, w), x] + [f(x), w])y(D(z) D(x) + D(x) D(z))
\]
\[
+ g(x) y(D(z) D(w) + D(w) D(z))\} + t^2 J_1(x, y) + t^3 J_2(x, y)
\]
\[
(2.65)
\]
where \( J_1 \) and \( J_2 \) denote the term satisfying the identity (2.65).

From (2.64) and (2.65), we obtain
\[
t \{[B(x, w), x] + [f(x), w])y(D(z) D(x) + D(x) D(z)) + g(x) y(D(z) D(w)
\]
\[
+ D(w) D(z))\} + t^2 J_1(x, y) + t^3 J_2(x, y) = 0, w, x, y, z \in R, t \in S_4.
\]

Since \( R \) is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.66) yields
\[
([B(x, w), x] + [f(x), w])y(D(z) D(x) + D(x) D(z)) + g(x) y(D(z) D(w)
\]
\[
+ D(w) D(z)) = 0, w, x, y, z \in R. \tag{2.67}
\]
Replacing \( u g(x)y \) for \( y \) in (2.67), we get
\[
([B(x, w), x] + [f(x), w])y(x) y(D(z) D(x) + D(x) D(z))
\]
\[
+ g(x) u g(x)y(D(z) D(w) + D(w) D(z)) = 0, w, x, y, z \in R. \tag{2.68}
\]
Combining (2.64) with (2.68),
\[
g(x) u g(x)y(D(z) D(w) + D(w) D(z)) = 0, w, x, y, z \in R. \tag{2.69}
\]
Replacing $g(x)y(D(z)D(w) + D(w)D(z))u$ for $u$ in (2.69), we obtain
\[
g(x)y(D(z)D(w) + D(w)D(z))ug(x)y(D(z)D(x) + D(x)D(z)) = 0, \quad u, w, x, y, z \in R.
\]
(2.70)
And so, by semiprimeness of $R$, it follows from (2.70) that
\[
g(x)y(D(z)D(w) + D(w)D(z)) = 0, \quad x, y, z \in R.
\]
(2.71)
Let $w = z$ in (2.71). Then we get
\[
g(x)yD(z)^2 = 0, \quad x, y, z \in R.
\]
(2.72)
Replacing $x + tw$ for $x$ in (2.72), we have
\[
g(x + tw)D(z)^2 \equiv g(x)yD(z)^2 + t\{([B(x, w), x] + [f(x), w])yD(z)^2\}
\]
(2.73)
\[+ t^2K(x, y) + t^3g(w)yD(z)^2 = 0, \quad w, x, y, z \in R, \quad t \in S_3\]
where $K$ denotes the term satisfying the identity (2.73).
From (2.72) and (2.73), we obtain
\[
t\{([B(x, w), x] + [f(x), w])yD(z)^2\} + t^2K(x, y)
\]
(2.74)
\[= 0, \quad w, x, y, z \in R, t \in S_3.
\]
Since $R$ is $2!$-torsion free by assumption, by Lemma 2.1 the relation (2.74) yields
\[
([B(x, w), x] + [f(x), w])yD(z)^2 = 0, \quad w, x, y, z \in R.
\]
(2.75)
Replacing $wx$ for $w$ in (2.75), we get
\[
([B(x, w), x]x + 3[w, x]f(x) + 3wg(x) + [[w, x], x]D(x) + [f(x), w]x)yD(x)^2 = 0, \quad w, x, y, z \in R.
\]
(2.76)
From (2.72) and (2.76), we have
\[
{[B(x, w), x]x + 3[w, x]f(x) + [[w, x], x]D(x) + [f(x), w]x}yD(x)^2
\]
(2.77)
\[= 0, \quad w, x, y, z \in R.
\]
Substituting $xy$ for $y$ in (2.75), we get
\[
([B(x, w), x]x + [f(x), w]x)yD(z)^2 = 0, \quad w, x, y, z \in R.
\]
(2.78)
Combining (2.77) with (2.78),
\[
(3[w, x]f(x) + [[w, x], x]D(x))yD(z)^2 = 0, \quad w, x, y, z \in R.
\]
(2.79)
Replacing $D(x)w$ for $w$ in (2.79), we obtain
\[
\{3f(x)wf(x) + 3D(x)[w,x]f(x) + g(x)wD(x) + 2f(x)[w,x]D(x) + D(x)[[w,x],x]D(x)\}yD(z)^2 = 0, w, x, y, z \in R.
\]
(2.80)

Substituting $D(x)y$ for $y$ in (2.80), we have
\[
\{3f(x)wf(x)D(x) + 3D(x)[w,x]f(x)D(x) + g(x)wD(x)^2 + 2f(x)[w,x]D(x)^2 + D(x)[[w,x],x]D(x)^2\}yD(z)^2 = 0, w, x, y, z \in R.
\]
(2.81)

Combining (2.16), (2.66) with (2.81),
\[
(2f(x)[w,x]D(x)^2 + D(x)[[w,x],x]D(x)^2) = 0, w, x, y, z \in R.
\]
(2.82)

Left multiplication of (2.82) by $D(x)$ leads to
\[
(2D(x)f(x)[w,x]D(x)^2 + D(x)[[w,x],x]D(x)^2) = 0, w, x, y, z \in R.
\]
(2.83)

Substituting $yD(x)^2w$ for $y$ in (2.50), we get
\[
2D(x)f(x)yD(x)^2wD(x)f(x) + D(x)^2[y,x]D(x)^2wD(x)f(x)
+ D(x)^2yD(x)f(x)wD(x)f(x) + D(x)^2yf(x)D(x)wD(x)f(x)
+ D(x)^2yD(x)^2[w,x]D(x)f(x) = 0, w, x, y, z \in R.
\]
(2.84)

Combining (2.16) with (2.84), we obtain
\[
2D(x)f(x)yD(x)^2wD(x)f(x) + D(x)^2[y,x]D(x)^2wD(x)f(x)
+ D(x)^2yD(x)f(x)wD(x)f(x) + D(x)^2yD(x)^2[w,x]D(x)f(x)
= 0, w, x, y, z \in R.
\]
(2.85)

From (2.49) and (2.85), we have
\[
(2D(x)f(x)yD(x)^2 + D(x)^2[y,x]D(x)^2 - D(x)^2yD(x)f(x))
\times wD(x)f(x) = 0, w, x, y, z \in R.
\]
(2.86)

Replacing $[y, x]$ for $y$ in (2.86), we get
\[
(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[y,x]D(x)^2 - D(x)^2yD(x)f(x))
\times wD(x)f(x) = 0, w, x, y, z \in R.
\]
(2.87)
Combining (2.49), (2.57) with (2.87),
\[
4D(x)f(x)yD(x)f(x) + D(x)^2[y, x]D(x)^2wD(x)f(x) = 0, w, x, y, z \in R.
\]
(2.88)

From (2.16) and (2.83), we arrive at
\[
(2D(x)f(x)[y, x]D(x)^2 + D(x)^2[y, x]D(x)^2w[D(z)^2, z]
= (2D(x)f(x)[y, x]D(x)^2 + D(x)^2[y, x]D(x)^2wD(z)f(z)
= 0, w, x, y, z \in R.
\]
(2.89)

Let \( z = x \) in (2.89). Then
\[
(2D(x)f(x)[y, x]D(x)^2 + D(x)^2[y, x]D(x)^2wD(x)f(x)
= 0, w, x, y, z \in R.
\]
(2.90)

Combining (2.88) with (2.90),
\[
2(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2wD(x)f(x)
= 0, w, x, y, z \in R.
\]
(2.91)

Since \( R \) is 2!-torsion free by assumption, the relation (2.90) yields
\[
(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2wD(x)f(x)
= 0, w, x, y, z \in R.
\]
(2.92)

From (2.92), we have
\[
(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2w(2D(x)f(x)yD(x)f(x)
= -D(x)f(x)[y, x]D(x)^2 = 0, w, x, y, z \in R.
\]
(2.93)

And so, by semiprimeness of \( R \), we get from (2.93)
\[
2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2 = 0, x, y, z \in R.
\]
(2.94)

Combining (2.57) with (2.94),
\[
D(x)f(x)yD(x)f(x) = 0, x, y \in R.
\]
(2.95)

And so, by semiprimeness of \( R \), it follows from (2.95) that
\[
D(x)f(x) = 0, x \in R.
\]

(ii) \( \iff \): Suppose that
\[
D(x)f(x) = 0, x, y, z \in R.
\]
(2.96)
Replacing $x + ty$ for $x$ in (2.96), we have

\begin{align*}
D(x + ty)[D(x + ty), x + ty] &\equiv +t \{D(y)f(x) + D(x)B(x, y)\} \\
&+ t^2 P(x, y) + t^3 D(y)f(y) = 0, \ x, y \in R, t \in S_3
\end{align*}

(2.97)

where $P$ denotes the term satisfying the identity (2.97).

From (2.96) and (2.97), we obtain

\begin{align*}
t \{D(y)f(x) + D(x)B(x, y)\} + t^2 P(x, y) &= 0, \ x, y \in R, t \in S_3.
\end{align*}

(2.98)

Since $R$ is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.98) yields

\begin{align*}
D(y)f(x) + D(x)B(x, y) &= 0, \ x, y \in R.
\end{align*}

(2.99)

Let $y = x^2$ in (2.213). Then using (2.96), we obtain from (2.99)

\begin{align*}
(D(x)x + xD(x))f(x) + 2D(x)(f(x)x + xf(x)) &= 3D(x)xf(x) + xD(x)f(x) + 2D(x)f(x)x = 0, \ x \in R.
\end{align*}

(2.100)

From (2.96) and (2.100), we get

\begin{align*}
3D(x)xf(x) &= 3f(x)^2 = -3D(x)g(x) = 0, \ x \in R.
\end{align*}

(2.101)

Since $R$ is 3!-torsion free, it follows from (2.101) that

\begin{align*}
f(x)^2 &= 0, \ x \in R,
\end{align*}

(2.102)

and

\begin{align*}
D(x)g(x) &= 0, \ x \in R.
\end{align*}

(2.103)

Writing $xy$ for $y$ in (2.99),

\begin{align*}
(xD(y)f(x) + D(x)g(x) + D(x)xB(x, y) + 2f(x)y + 2D(x)[y, x] = 0, \ x, y \in R.
\end{align*}

(2.104)

Left multiplication of (2.104) by $D(x)$ leads to

\begin{align*}
xD(y)f(x) + xD(x)B(x, y) &= 0, \ x \in R.
\end{align*}

(2.105)

Combining (2.218) with (2.105),

\begin{align*}
D(x)xf(x) + f(x)B(x, y) + 2D(x)f(x)y + D(x)^2[y, x] &= 0, \ x, y \in R.
\end{align*}

(2.106)

From (2.96) and (2.106), we have

\begin{align*}
D(x)xf(x) + f(x)B(x, y) + D(x)^2[y, x] &= 0, \ x, y \in R.
\end{align*}

(2.107)
Left multiplication of (2.107) by $D(x)$ yields

$$(2.108) \quad D(x)^2yf(x) + D(x)f(x)B(x, y) + D(x)^3[y, x] = 0, x, y \in R.$$  

Comparing (2.96) and (2.108), we obtain

$$(2.109) \quad D(x)^2yf(x) + D(x)^3[y, x] = 0, x, y \in R.$$  

Putting $yx$ instead of $y$ in (2.99), we get

$$(2.110) \quad (D(y)x + yD(x))f(x) + D(x)(B(x, y)x + 2yf(x) + [y, x]D(x)) = 0, x, y \in R.$$  

Right multiplication of (2.110) by $x$ leads to

$$(2.111) \quad D(y)f(x)x + D(x)B(x, y)x = 0, x \in R.$$  

Combining (2.110) with (2.111),

$$(2.112) \quad -D(y)g(x) + 2D(x)yf(x) + D(x)[y, x]D(x) = 0, x, y \in R.$$  

From (2.96) and (2.112), we have

$$(2.113) \quad -D(y)g(x) + 2D(x)yf(x) + D(x)[y, x]D(x) = 0, x, y \in R.$$  

Writing $xy$ for $y$ in (2.113), we get

$$(2.114) \quad -xD(y)g(x) - D(x)yg(x) + 2D(x)xyf(x) + D(x)x[y, x]D(x) = 0, x, y \in R.$$  

Left multiplication of (2.113) by $x$ leads to

$$(2.115) \quad -xD(y)g(x) + 2xD(x)yf(x) + xD(x)[y, x]D(x) = 0, x, y \in R.$$  

Combining (2.114) with (2.115),

$$(2.116) \quad -D(x)yg(x) + 2f(x)yf(x) + f(x)[y, x]D(x) = 0, x, y \in R.$$  

Left multiplication of (2.116) by $D(x)$ gives

$$(2.117) \quad -D(x)^2yg(x) + 2D(x)f(x)yg(x) + (D(x)f(x)[y, x]D(x) = 0, x, y \in R.$$  

Comparing (2.96) and (2.117), we obtain

$$(2.118) \quad D(x)^2yg(x) = 0, x, y \in R.$$  

Let $y = D(x)$ in (2.113). Then we get

$$(2.119) \quad -D^2(x)g(x) + 2D(x)^2f(x) + D(x)f(x)D(x) = 0, x, y \in R.$$  

Combining (2.96) with (2.119),

\[(2.120)\quad D^2(x)g(x) = 0, x \in R.\]

Writing \(yD(x)\) for \(y\) in (2.113), we have

\[-D(y)D(x)g(x) - yD^2(x)g(x) + 2D(x)yD(x)f(x) + D(x)[y, x]D(x)^2\]

\[(2.121)\quad + D(x)yf(x)D(x) = 0, x, y \in R.\]

Combining (2.96), (2.103), (2.120) with (2.121), we arrive at

\[(2.122)\quad D(x)[y, x]D(x)^2 + D(x)yf(x)D(x) = 0, x, y \in R.\]

Left multiplication of (2.122) by \(f(x)\) leads to

\[(2.123)\quad f(x)D(x)[y, x]D(x)^2 + f(x)D(x)yf(x)D(x) = 0, x, y \in R.\]

Writing \(yD(x)\) for \(y\) in (2.116), we get

\[-D(x)yD(x)g(x) + 2f(x)yD(x)f(x) + f(x)[y, x]D(x)^2 + f(x)yf(x)D(x)\]

\[(2.124)\quad = 0, x, y \in R.\]

Comparing (2.96) and (2.103), we obtain from (2.124)

\[(2.125)\quad f(x)[y, x]D(x)^2 + f(x)yf(x)D(x) = 0, x, y \in R.\]

Substituting \(D(x)y\) for \(y\) in (2.116),

\[-D(x)^2yg(x) + 2f(x)D(x)yf(x) + f(x)D(x)[y, x]D(x) + f(x)^2yD(x)\]

\[(2.126)\quad = 0, x, y \in R.\]

Comparing (2.102) and (2.118) and (2.126), we obtain

\[(2.127)\quad 2f(x)D(x)yf(x) + f(x)D(x)[y, x]D(x) = 0, x, y \in R.\]

Right multiplication of (2.127) by \(D(x)\) leads to

\[(2.128)\quad 2f(x)D(x)yf(x)D(x) + f(x)D(x)[y, x]D(x)^2 = 0, x, y \in R.\]

From (2.237) and (2.128), we have

\[f(x)D(x) = 0, x \in R.\]

\[\square\]

**Corollary 2.3.** Let \(R\) be a 3!-torsion free noncommutative semiprime ring and \(D : R \rightarrow R\) a Jordan derivation. Then

\[[D(x), x]D(x) = 0 \iff D(x)[D(x), x] = 0\]
for all \(x \in R\).

**Theorem 2.4.** Let \(R\) be a 3!-torsion free noncommutative semiprime ring, \(U\) a Lie ideal of \(R\), and let \(D : R \rightarrow R\) be a Jordan derivation on \(R\). And suppose that 
\([D(x), x]D(x) = 0\) \(\forall x \in U\). Then \([D(x), x]^2 = 0\) \(\forall x \in U\).

**Proof.** By Theorem 2.2, we can see that \(D\) is a derivation on \(R\). By assumption, 
\([D(x), x]D(x) = 0\), \(x \in U\) \((2.129)\)
Replacing \(x + ty\) for \(x\) in \((2.129)\), we have 
\([D(x + ty), x + ty]D(x + ty) \equiv +t\{B(x, y)D(x) + f(x)D(y)\}\)
\((2.130)\)
where \(H\) denotes the term satisfying the identity \((2.130)\).
From \((2.129)\) and \((2.130)\), we obtain 
\(t\{B(x, y)D(x) + f(x)D(y)\} + t^2H(x, y) = 0, \ x, y \in U, t \in S_3\)
Since \(R\) is 2!-torsion free by assumption, by Lemma 2.1 the relation \((2.131)\) yields 
\(B(x, y)D(x) + f(x)D(y) = 0, x, y \in U\).
\((2.132)\)
Let \(y = x^2\) in \((2.132)\). Then using \((2.129)\), we get 
\(2(f(x)x + xf(x))D(x) + f(x)(D(x)x + xD(x)) = 0, x \in U\).
\((2.133)\)
From \((2.129)\) and \((2.133)\), we arrive at 
\(3f(x)xD(x) + 2xf(x)D(x) + f(x)D(x)x = 3f(x)^2 = 0, x \in U\).
\((2.134)\)
Since \(R\) is 3!-torsion free, it follows from \((2.134)\) that 
\(f(x)^2 = 0, x \in U\).
\((2.135)\)
From \((2.129)\), we obtain 
\(0 = [f(x)D(x), x] = g(x)D(x) + f(x)^2, x \in U\).
\((2.136)\)
From \((2.135)\) and \((2.136)\), we have 
\(g(x)D(x) = 0, x \in R\).
\((2.137)\)
Writing \(yx\) for \(y\) in \((2.132)\), we get 
\((B(x, y)x + 2yf(x) + [y, x]D(x))D(x) + f(x)(D(y)x + yD(x))\)
\((2.138)\)
\(= 0, x, y \in U.\)
Right multiplication of (2.132) by $x$ leads to
\[(2.139) \quad B(x, y)D(x)x + f(x)D(y)x = 0, x, y \in U.\]
Combining (2.138) with (2.139),
\[(2.140) \quad -B(x, y)f(x) + 2yf(x)D(x) + [y, x]D(x)^2 + f(x)yD(x) = 0, x, y \in R.\]
From (2.129) and (2.140), we have
\[(2.141) \quad -B(x, y)f(x) + [y, x]D(x)^2 + f(x)yD(x) = 0, x, y \in U.\]
Substituting $x^2$ for $y$ in (2.141), we get
\[(2.142) \quad -2(f(x)x + xf(x))f(x) + f(x)x^2D(x) = 0, x \in R.\]
Comparing (2.135) and (2.137), we obtain
\[(2.143) \quad -2f(x)xf(x) - xf(x)^2 - f(x)(f(x)x + xf(x)) = 0, x \in R.\]
Since $R$ is 3!-torsion free by assumption, the relation (2.143) yields
\[(2.144) \quad g(x)f(x) = f(x)g(x) = 0, x \in U.\]
Right multiplication of (2.141) by $D(x)$ leads to
\[(2.145) \quad -B(x, y)f(x)^2 + [y, x]D(x)^2f(x) + f(x)yD(x)f(x) = 0, x, y \in U.\]
From (2.135) and (2.145), we have
\[(2.146) \quad [y, x]D(x)^2f(x) + f(x)yD(x)f(x) = 0, x, y \in U.\]
Substituting $xy$ for $y$ in (2.146), we get
\[(2.147) \quad x[y, x]D(x)^2f(x) + f(x)xyD(x)f(x) = 0, x, y \in U.\]
Left multiplication of (2.147) by $D(x)$ gives
\[(2.148) \quad -x[y, x]D(x)^2f(x) + xf(x)yD(x)f(x) = 0, x, y \in U.\]
From (2.147) and (2.148), we have
\[(2.149) \quad g(x)yD(x)f(x) = 0, x, y \in U.\]
Replacing $yx$ for $y$ in (2.149), we get
\[(2.150) \quad g(x)yxD(x)f(x) = 0, x, y \in U.\]
Right multiplication of (2.149) by $x$ yields
\[(2.151) \quad g(x)yD(x)f(x)x = 0, x, y \in U.\]
From (2.135), (2.150) and (2.151), we have

\[(2.152) \quad g(x)yD(x)g(x) = 0, x, y \in U.\]

Left multiplication of (2.152) by \(D(x)\) leads to

\[(2.153) \quad D(x)g(x)yD(x)g(x) = 0, x, y \in U.\]

Thus by semiprimeness of \(R\), it is clear that

\[(2.154) \quad D(x)g(x) = 0, x \in U.\]

Putting \(xy\) instead of \(y\) in (2.141), we get

\[(2.155) \quad -xB(x,y)f(x) + x[y,x]D(x)f(x) + x[f(x)xyD(x) + x[y,x]D(x)]f(x) + f(x)xyD(x) = 0, x, y \in U.\]

Left multiplication of (2.141) by \(x\) yields

\[(2.156) \quad -xB(x,y)f(x) + x[y,x]D(x)f(x) + f(x)xyD(x) = 0, x, y \in U.\]

Combining (2.155) with (2.156),

\[(2.157) \quad -2f(x)xyD(x) + g(x)yD(x) = 0, x, y \in U.\]

Right multiplication of (2.157) by \(D(x)\) leads to

\[(2.158) \quad -2f(x)xyD(x) - D(x)[y,x]D(x)f(x) + g(x)yD(x)^2 = 0, x \in U.\]

Combining (2.129) with (2.158),

\[(2.159) \quad g(x)yD(x)^2 = 0, x, y \in U.\]

Replacing \(yD(x)\) for \(y\) in (2.158), we get

\[(2.160) \quad -2f(x)xyD(x)f(x) - D(x)[y,x]D(x)f(x) - D(x)yf(x)^2 + g(x)yD(x)^2 = 0, x \in U.\]

From (2.135) and (2.160), we have

\[(2.161) \quad -2f(x)xyD(x)f(x) - D(x)[y,x]D(x)f(x) + g(x)yD(x)^2 = 0, x, y \in U.\]

Comparing (2.159) and (2.161),

\[(2.162) \quad 2f(x)xyD(x)f(x) + D(x)[y,x]D(x)f(x) = 0, x, y \in U.\]

Left multiplication of (2.162) by \(D(x)\) gives

\[(2.163) \quad 2D(x)f(x)xyD(x)f(x) + D(x)^2[y,x]D(x)f(x) = 0, x, y \in U.\]
Substituting $f(x)y$ for $y$ in (2.162), we get
\[2f(x)^2gD(x)f(x) + D(x)f(x)[y, x]D(x)f(x) + D(x)g(x)yD(x)f(x)\]
(2.164) \[= 0, x, y \in U.\]

Combining (2.135), (2.154) with (2.164), we obtain
\[(2.165) \quad D(x)f(x)[y, x]D(x)f(x) = 0, x, y \in U.\]

Substituting $yD(x)^2z$ for $y$ in (2.165),
\[D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)\]
(2.166) \[+ D(x)f(x)yD(x)^2[z, x]D(x)f(x) = 0, x, y \in U.\]

From (2.129) and (2.166), we have
\[(2.167) \quad D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)\]
Comparing (2.163) and (2.167),
\[D(x)f(x)[y, x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)\]
\[-2D(x)f(x)yD(x)f(x)zD(x)f(x)\]
\[= D(x)f(x)[y, x]D(x)^2zD(x)f(x) - D(x)f(x)yD(x)f(x)zD(x)f(x)\]
(2.168) \[= \{D(x)f(x)[y, x]D(x)^2 - D(x)f(x)yD(x)f(x)\}zD(x)f(x) = 0, x, y \in U.\]

From (2.168), we obtain
\[(2.169) \quad -D(x)f(x)yD(x)f(x) = 0, x, y \in U.\]

Thus by semiprimeness of $R$, it is obvious that
\[(2.170) \quad D(x)f(x)[y, x]D(x)^2 - D(x)f(x)yD(x)f(x) = 0, x, y \in U.\]

Replacing $x + tz$ for $x$ in (2.159), we have
\[g(x + tz)yD(x + tz)^2 \equiv g(x)yD(x)^2 + t\{[B(x, z), x] + [f(x), z]yD(x)^2\}
+ g(x)g(D(z)D(x) + D(x)D(z)) + t^2L_1(x, y) + t^3L_2(x, y) + t^4L_3(x, y)\]
(2.171) \[+ t^5g(z)yD(z)^2 \equiv 0, \quad x, y, z \in U, t \in S_4.\]
where \( L_i, 1 \leq i \leq 3 \), denotes the term satisfying the identity (2.171).

From (2.159) and (2.171), we obtain

\[
t\{(B(x, z), x) + [f(x), z])yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))\}
\]

\[+ t^2L_1(x, y) + t^3L_2(x, y) + t^4L_3(x, y) = 0, \ x, y, z \in U, t \in S_3.\]

(2.172)

Since \( R \) is \( 3! \)-torsion free by assumption, by Lemma 2.1 the relation (2.172) yields

\[
([B(x, z), x] + [f(x), z])yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))
\]

\[= 0, \ x, y, z \in U.\]

Writing \( ug(x)y \) for \( y \) in (2.173), we get

\[
([B(x, z), x] + [f(x), z])ug(x)yD(x)^2 + g(x)ug(x)y(D(z)D(x) + D(x)D(z))
\]

\[= 0, \ u, x, y, z \in U.\]

(2.174)

Combining (2.159) with (2.174),

\[
g(x)ug(x)y(D(z)D(x) + D(x)D(z)) = 0, \ u, x, y, z \in U.
\]

(2.175)

Replacing \( y(D(z)D(x) + D(x)D(z))u \) for \( u \) in (2.175), we obtain

\[
g(x)y(D(z)D(x) + D(x)D(z))ug(x)y(D(z)D(x) + D(x)D(z))
\]

\[= 0, \ u, x, y, z \in U.\]

(2.176)

And so, by semiprimeness of \( R \), it follows that

\[
g(x)y(D(z)D(x) + D(x)D(z)) = 0, \ x, y, z \in U.
\]

(2.177)

Replacing \( x + tw \) for \( x \) in (2.177), we have

\[
g(x + tw)y(D(z)D(x + tw) + D(x + tw)D(z)
\]

\[\equiv g(x)y(D(z)D(x) + D(x)D(z)) + t\{(B(x, w), x) + [f(x), w])yD(z)D(x) + D(x)D(z)
\]

\[+ t^2M_1(x, y) + t^3M_2(x, y)\}

\[= 0, \ w, x, y, z \in U, t \in S_4\]

(2.178) + \( t^4g(w)y(D(z)D(w) + D(w)D(z)) = 0, \ w, x, y, z \in U, t \in S_4 \)

where \( M_1 \) and \( M_2 \) denote the term satisfying the identity (2.178).

From (2.177) and (2.178), we arrive at

\[
t\{(B(x, w), x) + [f(x), w])yD(z)D(x) + D(x)D(z)
\]

\[+ g(x)y(D(z)D(w) + D(w)D(z)) + t^2M_1(x, y) + t^3M_2(x, y)
\]

\[= 0, \ w, x, y, z \in U, t \in S_3.\]

(2.179)
Since \( R \) is 3\!-torsion free by assumption, by Lemma 2.1 the relation (2.179) yields
\[
([B(x, w), x] + [f(x), w]) y(D(z)D(x) + D(x)D(z))
+ g(x)y(D(z)D(w) + D(w)D(z)) = 0, w, x, y, z \in U.
\]
(2.180)

Replacing \( u g(x)y \) for \( y \) in (2.180), we get
\[
([B(x, w), x] + [f(x), w]) y g(x) y(D(z)D(x) + D(x)D(z))
+ g(x)u g(x)y(D(z)D(w) + D(w)D(z)) = 0, w, x, y, z \in U.
\]
(2.181)

Combining (2.177) with (2.181),
\[
g(x)u g(x)y(D(z)D(w) + D(w)D(z)) = 0, w, x, y, z \in U.
\]
(2.182)

Replacing \( y(D(z)D(w) + D(w)D(z)) \) for \( u \) in (2.182), we obtain
\[
g(x)y(D(z)D(w) + D(w)D(z))u g(x)y(D(z)D(x) + D(x)D(z))
= 0, u, w, x, y, z \in U.
\]
(2.183)

And so, by semiprimeness of \( R \), it follows from (2.183) that
\[
g(x)y(D(z)D(w) + D(w)D(z)) = 0, x, y, z \in U.
\]
(2.184)

Let \( w = z \) in (2.184). Then we get
\[
g(x)yD(z)^2 = 0, x, y, z \in U.
\]
(2.185)

Replacing \( x + tw \) for \( x \) in (2.185), we have
\[
g(x + tw)yD(z)^2 = g(x)yD(z)^2 + t\{([B(x, w), x] + [f(x), w])yD(z)^2\}
+ t^2P(x, y) + t^3g(w)yD(z)^2 = 0, w, x, y, z \in U, t \in S_3
\]
(2.186)

where \( P \) denotes the term satisfying the identity (2.186).

From (2.185) and (2.186), we obtain
\[
t\{([B(x, w), x] + [f(x), w])yD(z)^2\} + t^2P(x, y)
= 0, w, x, y, z \in U, t \in S_3.
\]
(2.187)

Since \( R \) is 2\!-torsion free by assumption, by Lemma 2.1 the relation (2.187) yields
\[
([B(x, w), x] + [f(x), w])yD(z)^2 = 0, w, x, y, z \in U.
\]
(2.188)

Replacing \( w x \) for \( w \) in (2.188), we get
\[
([B(x, w), x] + 3[w, x]f(x) + 3wg(x) + [w, x], x]D(x) + [f(x), w]x)D(x)^2
= 0, w, x, y, z \in U.
\]
(2.189)
From (2.185) and (2.189), we have
\[(B(x, w), x)x + 3[w, x]f(x) + [w, x]D(x) + [f(x), w]x)yD(x)^2\]
(2.190) = 0, w, x, y, z ∈ U.

Substituting xy for y in (2.188),
\[(B(x, w), x)x + [f(x), w]x)yD(x)^2 = 0, w, x, y, z ∈ U.\]
(2.191)

Combining (2.129), (2.179) with (2.194),
Replacing \(D(x)w\) for \(w\) in (2.192), we obtain
\[(3w, x)f(x) + [[w, x], x]D(x))yD(z) = 0, w, x, y, z ∈ U.\]
(2.192)

Replacing \(D(x)w\) for \(w\) in (2.193), we have
\[(3f(x)w, x)f(x)D(x) + 3D(x)[w, x]f(x) + g(x)wD(x) + 2f(x)[w, x]D(x)\]
\[+D(x)[[w, x], x]D(x))yD(z)^2 = 0, w, x, y, z ∈ U.\]
(2.193)

Combining (2.129), (2.179) with (2.194),
\[(2f(x)[w, x]D(x)^2 + D(x)[[w, x], x]D(x)^2)yD(z)^2 = 0, w, x, y, z ∈ U.\]
(2.195)

Left multiplication of (2.195) by \(D(x)\) leads to
\[(2D(x)f(x)[w, x]D(x)^2 + D(x)^2[[w, x], x]D(x)^2)yD(z)^2\]
\[= 0, w, x, y, z ∈ U.\]
(2.196)

Left multiplication of (2.162) by \(D(x)\) yields
\[2D(x)f(x)yD(x)f(x) + D(x)^2[y, x]D(x)f(x) = 0, x, y ∈ U.\]
(2.197)

Substituting \(yD(x)^2w\) for \(y\) in (2.197), we get
\[2D(x)f(x)yD(x)^2wD(x)f(x) + D(x)^2[y, x]D(x)^2wD(x)f(x)\]
\[+D(x)^2yD(x)^2, x]wD(x)f(x) + D(x)^2yD(x)^2[w, x]D(x)f(x)\]
\[= 0, w, x, y, z ∈ U.\]
(2.198)

Combining (2.129) with (2.198), we obtain
\[2D(x)f(x)yD(x)^2wD(x)f(x) + D(x)^2[y, x]D(x)^2wD(x)f(x)\]
\[+D(x)^2yD(x)^2f(x)wD(x)f(x) + D(x)^2yD(x)^2[w, x]D(x)f(x)\]
\[= 0, w, x, y, z ∈ U.\]
(2.199)
From (2.162) and (2.199),

\[
(2D(x)f(x)yD(x)^2 + D(x)^2[y, x]D(x)^2 - D(x)^2yD(x)f(x))
\]

(2.200) \quad wD(x)f(x) = 0, w, x, y, z \in U.

Replacing \([y, x]\) for \(y\) in (2.200), we have

\[
(2D(x)f(x)[y, x]D(x)^2 + D(x)^2[[y, x], x]D(x)^2 - D(x)^2yD(x)f(x))
\]

(2.201) \quad wD(x)f(x) = 0, w, x, y, z \in U.

Combining (2.162), (2.170) with (2.201),

\[
(4D(x)f(x)yD(x)f(x) + D(x)^2[[y, x], x]D(x)^2wD(x)f(x)
\]

(2.202) \quad = 0, w, x, y, z \in U.

From (2.129) and (2.196),

\[
(2D(x)f(x)[y, x]D(x)^2 + D(x)^2[[y, x], x]D(x)^2w[D(z)^2, z]
\]

(2.203) \quad = 0, w, x, y, z \in U.

Let \(z = x\) in (2.203). Then

\[
(2D(x)f(x)[y, x]D(x)^2 + D(x)^2[[y, x], x]D(x)^2wD(x)f(x)
\]

(2.204) \quad = 0, w, x, y, z \in U.

Combining (2.203) with (2.204),

\[
2(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2wD(x)f(x)
\]

(2.205) \quad = 0, w, x, y, z \in U.

Since \(R\) is 2!-torsion free by assumption, the relation (2.204) yields

\[
(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2wD(x)f(x)
\]

(2.206) \quad = 0, w, x, y, z \in U.

From (2.206), we have

\[
(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2wD(x)f(x) - D(x)f(x)[y, x]D(x)^2)
\]

(2.207) \quad = 0, w, x, y, z \in U.

And so, by semiprimeness of \(R\), it follows from (2.207) that

\[
2D(x)f(x)yD(x)f(x) - D(x)f(x)[y, x]D(x)^2 = 0, x, y, z \in U.
\]
Combining (2.170) with (2.208),
\[(2.209) \quad D(x)f(x)yD(x)f(x) = 0, \ x, y \in U.\]
And so, by semiprimeness of \(R\), obtain from (2.209)
\[D(x)f(x) = 0, \ x \in U.\]
\[\iff:\] Suppose that
\[(2.210) \quad D(x)f(x) = 0, \ x \in R.\]
Replacing \(x + ty\) for \(x\) in (2.210), we have
\[(2.211) \quad t^2Q(x, y) + t^3D(y)f(y) = 0, \ x, y \in R, \ t \in S_3\]
where \(Q\) denotes the term satisfying the identity (2.211).
From (2.210) and (2.211), we get
\[(2.212) \quad t\{D(y)f(x) + D(x)B(x, y)\} + t^2Q(x, y) = 0, \ x, y \in R, \ t \in S_3.\]
Since \(R\) is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.212) yields
\[(2.213) \quad D(y)f(x) + D(x)B(x, y) = 0, \ x, y \in R.\]
Let \(y = x^2\) in (2.213). Then using (2.210), we obtain from (2.213)
\[(2.214) \quad (D(x)x + xD(x))f(x) + 2D(x)(f(x)x + xf(x)) = 3D(x)xf(x) + xD(x)f(x) + 2D(x)f(x)x = 0, \ x \in R.\]
From (2.210) and (2.214), we get
\[(2.215) \quad 3D(x)xf(x) = 3f(x)^2 = -3D(x)g(x) = 0, \ x \in R.\]
Since \(R\) is 3!-torsion free, it follows from (2.215) that
\[(2.216) \quad f(x)^2 = 0, \ x \in R,\]
and
\[(2.217) \quad D(x)g(x) = 0, \ x \in R.\]
Writing \(xy\) for \(y\) in (2.213), we obtain
\[(2.218) \quad (xD(y)f(x) + D(x)g(x) + D(x)(xB(x, y) + 2f(x)y + D(x)[y, x]) = 0, \ x, y \in R.\]
Left multiplication of (2.218) by $D(x)$ leads to

$$(2.219) \quad xD(y)f(x) + xD(x)B(x, y) = 0, x \in R.$$ 

Combining (2.218) with (2.219),

$$(2.220) \quad D(x)yf(x) + f(x)B(x, y) + 2D(x)f(x)y + D(x)^2[y, x] = 0, x, y \in R.$$ 

From (2.210) and (2.220), we have

$$(2.221) \quad D(x)yf(x) + f(x)B(x, y) + 2D(x)f(x)y + D(x)^2[y, x] = 0, x, y \in R.$$ 

Left multiplication of (2.221) by $D(x)$ yields

$$(2.222) \quad D(x)^2yf(x) + D(x)f(x)B(x, y) + D(x)^3[y, x] = 0, x, y \in R.$$ 

Comparing (2.210) and (2.222),

$$(2.223) \quad D(x)^2yf(x) + D(x)^3[y, x] = 0, x, y \in R.$$ 

Putting $yx$ instead of $y$ in (2.213), we get

$$(D(y)x + yD(x))f(x) + D(x)(B(x, y)x + 2yf(x) + [y, x]D(x)) = 0, x, y \in R.$$ 

Right multiplication of (2.224) by $x$ gives

$$(2.225) \quad D(y)f(x)x + D(x)B(x, y)x = 0, x \in R.$$ 

Combining (2.224) with (2.225),

$$(2.226) \quad -D(y)g(x) + yD(x)f(x) + 2D(x)yg(x) + D(x)[y, x]D(x) = 0, x, y \in R.$$ 

From (2.210) and (2.226), we have

$$(2.227) \quad -D(y)g(x) + 2D(x)yg(x) + D(x)[y, x]D(x) = 0, x, y \in R.$$ 

Writing $xy$ for $y$ in (2.227), we get

$$-xD(y)g(x) - D(x)ygy(x) + 2D(x)xyf(x) + D(x)x[y, x]D(x) = 0, x, y \in R.$$ 

Left multiplication of (2.227) by $x$ leads to

$$(2.229) \quad -xD(y)g(x) + 2xD(x)yg(x) + xD(x)[y, x]D(x) = 0, x, y \in R.$$ 

Combining (2.228) with (2.229),

$$(2.230) \quad -D(x)ygy(x) + 2f(x)yg(x) + f(x)[y, x]D(x) = 0, x, y \in R.$$
Left multiplication of (2.230) by \( D(x) \) yields
\[
-\frac{D(x)^2 y g(x) + 2D(x) f(x) yf(x) + D(x) f(x)[y, x] D(x)}{0, x, y \in R.}
\]
Comparing (2.210) and (2.231), we obtain
\[
D(x)^2 y g(x) = 0, x, y \in R.
\]
Let \( y = D(x) \) in (2.227). Then we get
\[
-\frac{D^2(x) g(x) + 2D(x)^2 f(x) + D(x) f(x) D(x)}{0, x, y \in R.}
\]
Combining (2.210) with (2.233),
\[
D^2(x) g(x) = 0, x \in R.
\]
Writing \( yD(x) \) for \( y \) in (2.227), we have
\[
\begin{align*}
-D(y) D(x) g(x) - yD^2(x) g(x) + 2D(x) y D(x) f(x) + D(x)[y, x] D(x)^2 \\
+D(x) y f(x) D(x) = 0, x, y \in R.
\end{align*}
\]
Combining (2.210), (2.217), (2.234) with (2.235),
\[
\begin{align*}
D(x)[y, x] D(x)^2 + D(x) y f(x) D(x) = 0, x, y \in R.
\end{align*}
\]
Left multiplication of (2.236) by \( f(x) \) leads to
\[
\begin{align*}
\frac{f(x) D(x)[y, x] D(x)^2 + f(x) D(x) y f(x) D(x)}{0, x, y \in R.}
\end{align*}
\]
Writing \( yD(x) \) for \( y \) in (2.230), we get
\[
\begin{align*}
-\frac{D(x) y D(x) g(x) + 2f(x) y D(x) f(x) + f(x)[y, x] D(x)^2 + f(x) y f(x) D(x)}{0, x, y \in R.}
\end{align*}
\]
Comparing (2.210) and (2.217), we obtain from (2.238)
\[
\begin{align*}
f(x)[y, x] D(x)^2 + f(x) y f(x) D(x) = 0, x, y \in R.
\end{align*}
\]
Substituting \( D(x) y \) for \( y \) in (2.230), we have
\[
\begin{align*}
-\frac{D(x)^2 y g(x) + 2f(x) D(x) y f(x) + f(x) D(x)[y, x] D(x) + f(x)^2 y D(x)}{0, x, y \in R.}
\end{align*}
\]
From (2.216), (2.232) and (2.240), we obtain
\[
\begin{align*}
2f(x) D(x) y f(x) + f(x) D(x)[y, x] D(x) = 0, x, y \in R.
\end{align*}
\]
Right multiplication of (2.241) by \( D(x) \) leads to
\[
\begin{align*}
2f(x) D(x) y f(x) D(x) + f(x) D(x)[y, x] D(x)^2 = 0, x, y \in R.
\end{align*}
\]
From (2.237) and (2.242),

\[ f(x)D(x) = 0, x \in R. \]

□

**Theorem 2.5.** Let \( A \) be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation \( D : A \longrightarrow A \). Then we have

\[ [D(x), x]D(x) \in \text{rad}(A) \iff D(x)[D(x), x] \in \text{rad}(A) \]

for all \( x \in A \).

**Proof.** By the result of B.E. Johnson and A.M. Sinclair[5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair[12] proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of \( A \) invariant. Hence for any primitive ideal \( P \subseteq A \) one can introduce a linear Jordan derivation \( D_P : A/P \longrightarrow A/P \), where \( A/P \) is a prime and factor Banach algebra, by \( D_P(\hat{x}) = D(x) + P \), \( \hat{x} = x + P \). Then \( D \) is a derivation on \( A/P \). By the assumption that \( D(x)^2f(x) \in \text{rad}(A) \), \( x \in A \), we obtain \( (D_P(\hat{x}))^2[D_P(\hat{x}), \hat{x}] = 0 \), \( \hat{x} \in A/P \), since all the assumptions of Theorem 2.4 are fulfilled. And since the prime and factor algebra \( A/P \) is noncommutative, from Theorem 2.4 we have \( [D_P(\hat{x}), \hat{x}]^7 = 0 \), \( \hat{x} \in A/P \). And for each \( P \), by the elementary properties of the spectral radius \( r_P \) in a Banach algebra \( A/P \), it follows that \( r_P([D_P(\hat{x}), \hat{x}]^7) = r_P([D_P(\hat{x}), \hat{x}]^7) = 0 \) for all \( \hat{x} \in A/P \). Hence we obtain \( r_P([D_P(\hat{x}), \hat{x}]) = 0 \) for all \( \hat{x} \in A/P \). Thus \( [D_P(\hat{x}), \hat{x}] \in Q(A/P) \) for all \( \hat{x} \in A/P \). On the one hand, since \( D \) is continuous, we see that \( D_P \) is also continuous. Thus by Theorem 2.3, we obtain \( D_P(A/P) \subseteq \text{rad}(A/P) \). But since \( A/P \) is semisimple, \( D_P(A/P) = \{0\} \) for all primitive ideals of \( A \). Hence we see that \( D(A) \subseteq P \) for all primitive ideals of \( A \). And so, \( D(A) \subseteq \text{rad}(A) \). On the other hand, In case \( A/P \) is a commutative Banach algebra, one can conclude that \( D_P = 0 \) as well, since \( A/P \) is semisimple and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebra. In other words, \( D(x) \in P \) for all primitive ideals of \( A \) and all \( x \in A \), i.e. we get \( D(A) \subseteq \text{rad}(A) \). Therefore in any case we have \( D(A) \subseteq \text{rad}(A) \). □

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