Laws of Large Numbers for Uncorrelated Set-Valued Random Variables

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Abstract

As the extension of uncorrelated single-valued random variables, set-valued case is studied in this paper. When the underlying space is of finite dimension, by using the support function, We shall prove the weak and strong laws of large numbers for uncorrelated set-valued random variables in the sense of Hausdorff metric $d_H$. Our results generalize weak and strong laws of large numbers for independent identically distributed or independent set-valued random variables.

Keywords: Set-valued random variable, Uncorrelated, Law of large numbers

2020 Mathematics Subject Classification. Primary 65C30, Secondary 26E25, 54C65

1. Introduction

In the real world, since the uncertainty includes not only randomness but also imprecision, it has limitation to describe an event by single-valued random variables. The set-valued random variable is a suitable tool to characterize both randomness and imprecision. For example, the price of a stock within one trading day may change a lot. The single-valued opening price or closing pricing is not enough to describe the uncertainty of market. It is more reasonable to consider the
variation range of stock price, which can be described by a random interval, the special case of set-valued random variable. In the past few decades, the theory of set-valued random variables with a wide range of applications has received a lot of attention. See for example [3, 6, 7, 13, 18, 19] and references therein. Especially, for the applications of set-valued theory to econometrics and finance, we would like to recommend the nice article [20] and references therein.

It is well known that the limit theory plays an important role in classical probability theory, statistical inference and parameter estimation. For set-valued case, the first strong law of large numbers (in short by SLLNs) was given by Artstein and Vitale in 1975 [1], where the set-valued random variables are independent identically distributed and take values in the family of compact subsets of Euclidean space $\mathbb{R}^d$. After that, many other authors such as Giné, Hahn and Zinn [8], Hess [12], Hiai [14], Puri and Ralescu [22] discussed SLLNs under different settings for convex set-valued random variables where the underlying space is a separable Banach space. Artstein and Hansen [2], Hiai [14] independently obtained the SLLNs in Banach space without convexity assumption by using different methods. Taylor and Inoue proved SLLNs for independent (not necessary to be identically distributed) case in Banach space in [23]. Terán and Molchanov [26] studied the law of large numbers in a metric space with a convex combination operation and applied the method in set-valued setting. Guan et al. [10] studied the SLLNs for weighted sums of set-valued random variables in Rademacher type $p$ Banach space. As the extension of set-valued case, Li and Ogura [17] studied the SLLNs for independent (not necessary identically distributed) fuzzy set-valued variables in the sense of extended Hausdorff distance. Guan and Li [9], Guan et al [11] obtained the strong law of large numbers for weighted sum of fuzzy set-valued variables.

The assumption of independence of random variable sequences is a little too strong for some cases. Considering weaker assumption, some researchers studied limit theorems for single-valued random variable sequences. There are dozens of papers studying weak and strong laws of large numbers for single-valued random variables which are not independent. For example, Taylor 1978 in [24] defined uncorrelation of random variables and obtained laws of large numbers. Laws of large numbers also were studied under other weaker assumptions such as positive independence, negative independence [3, 14, 25], dependence [21]. Ko [15] obtained the strong law of large numbers for linear multi-parameter stochastic processes generated by identically distributed and negatively associated random fields.

But for set-valued case, to our knowledge, for the law of large numbers, there
is no result other than the condition of independence. It is also necessary and possible to study the limit behaviour of set-valued random variables which are not independent. The innovation of this work is that we study the law of large numbers for the sequence of set-valued random variables under weaker condition than independence. Firstly, by using the support function, we define the uncorrelated set-valued random variables based on the notion of single-valued case [24]. Uncorrelation is weaker than independence. Secondly, under the assumption of uncorrelation, we shall prove the weak and strong laws of large numbers for the sequence of set-valued random variable in the sense of the Hausdorff metric $d_H$.

This paper is organized as follows. Section 2 is about some definitions and basic results of set-valued random variables. Section 3 contributes to weak and strong laws of large numbers and examples. Section 4 is the concluding remark.

2. Preliminaries

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space. $(X, \| \cdot \|)$ is a real separable Banach space. Let $K(X)$ (resp. $K_k(X)$) denote the family of all nonempty closed (resp. compact) subsets of $X$. And $K_{kc}(X)$ is the family of all nonempty compact convex subsets of $X$. Let $\mathbb{R}$ denote the family of all real numbers and $\mathbb{N}$ the set of all natural numbers.

Let $A$ and $B$ be two nonempty subsets of $X$ and let $\lambda \in \mathbb{R}$ be the set of all real numbers. The Minkowski sum and scalar multiplication are defined by

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$

The Hausdorff metric on $K(X)$ is defined by

$$d_H(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$$

for $A, B \in K(X)$. For an $A$ in $K(X)$, let $||A||_K = d_H(\{0\}, A)$.

The metric space $(K_k(X), d_H)$ is complete and separable. And $K_{kc}(X)$ is a closed subset of $(K_k(X), d_H)$ (cf. [18], Theorems 1.1.2 and 1.1.3).

For each $A \in K(X)$, the support function is defined by

$$s(x^*, A) = \sup_{a \in A} \langle x^*, a \rangle, \; x^* \in X^*,$$

where $X^*$ is the dual space of $X$. Let $S^*$ be the unit sphere in $X^*$, and $C(S^*)$ be the set of all continuous functions $v$ on $S^*$ with respect to the norm $||v||_C = \sup_{x \in S^*} ||v(x)||$. The metric on $C(S^*)$ is defined by

$$d_C(v, w) = \sup_{x \in S^*} ||v(x) - w(x)||$$

for $v, w \in C(S^*)$. For an $A$ in $K(X)$, let $||A||_C = d_C(0, A)$.
sup_{\omega \in \Omega}, |v(\omega)|$. The mapping \( j_0 : A \to s(\cdot, A) \) can embed the space \((K_{k_c}(\mathfrak{x}), d_H)\) into a closed convex cone in \(C(S^*)\) isometrically and isomorphically (cf. [18], Theorem 1.1.12). By using the support function, we have the following equivalent definition of Hausdorff metric. For \( A, B \in K_{k_c}(\mathfrak{x}) \), the Hausdorff metric between \( A \) and \( B \) is
\[
d_H(A, B) = \sup\{d(x^*, A) - d(x^*, B) : x^* \in S^*\}.
\]
Therefore, \( |A|_K = d_H([0], A) = \sup_{a \in A} ||a|| = \sup_{x^* \in S^*} \|s(x^*, A)\| \).

A set-valued mapping \( F : \Omega \to K(\mathfrak{x}) \) is called a set-valued random variable (or a random set, or a multifunction) if, for each open subset \( O \) of \( \mathfrak{x} \), \( F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A} \). The family of all \( K(\mathfrak{x}) \)-valued random variables is denoted by \( M(\Omega; K(\mathfrak{x})) \).

A set-valued random variable \( F \) is called \( L^p \)-integrably bounded \((p \geq 1) \) (cf. [13] or [18]) if \( \int_{\Omega} \|F(\omega)\|_K^p d\mu < \infty \).

Let \( L^p(\Omega, \mathcal{A}, \mu; K_{k_c}(\mathfrak{x})) \) (resp. \( L^p(\Omega, \mathcal{A}, \mu; K_{k_d}(\mathfrak{x})) \)) denote the space of all integrably bounded compact (resp. compact and convex) random variables, which is briefly denoted by \( L^p(\Omega; K_{k_c}(\mathfrak{x})) \) (resp. \( L^p(\Omega; K_{k_d}(\mathfrak{x})) \)). For \( F, G \in L^1(\Omega, \mathcal{A}, \mu; K_{k_c}(\mathfrak{x})) \), \( F = G \) if and only if \( F(\omega) = G(\omega) \) a.s. Regarding the concepts and results of set-valued random variables, readers may refer to nice books [6, 18, 19].

For each set-valued random variable \( F \), the expectation of \( F \), denoted by \( E[F] \), is defined by
\[
E[F] = \{ \int_{\Omega} f d\mu : f \in S_F \},
\]
where \( \int_{\Omega} f d\mu \) is the usual Bochner integral in \( L^1[\Omega, \mathfrak{x}] \) (the family of integrable \( \mathfrak{x} \)-valued random variables), and \( S_F = \{ f \in L^1(\Omega; \mathfrak{x}) : f(\omega) \in F(\omega) \text{ a.s.} \} \).

The following is a known result (Theorem 2.1.12 in [18]), which is needed later.

**Lemma 2.1.** Let \( F \) be a \( K(\mathfrak{x}) \)-valued random variable and \( S_F \neq \emptyset \). Then for any \( x^* \in S^* \), we have
\[
s(x^*, E(F)) = E(s(x^*, F)).
\]

3. Main results

In this section, we shall firstly give the definition of uncorrelated set-valued random variables. Then we will prove the laws of large numbers for uncorrelated set-valued random variable sequences.

\( \mathbb{R} \)-valued random variables \( \xi \) and \( \eta \) are called **uncorrelated** if \( cov(\xi, \eta) = 0 \) (cf. [24]).
**Definition 3.1.** Let $F_1, F_2$ be set-valued random variables. $F_1$ and $F_2$ are called **uncorrelated** if $s(x^*, F_1)$ and $s(x^*, F_2)$ are uncorrelated $\mathbb{R}$-valued random variables for any $x^* \in \mathbb{X}^*$.

Set-valued random sequence $F_1, F_2, \cdots$ are called **uncorrelated** if the sequence $s(x^*, F_1), s(x^*, F_2), \cdots$ are pairwise uncorrelated for any $x^* \in \mathbb{X}^*$.

If $F_1, F_2$ are uncorrelated, we denote it as $Cov(F_1, F_2) = 0$. Specially, if $\{F_1\} = \{f\}, \{F_2\} = \{g\}, f, g$ are real-valued random variables, then it reduces to the classical case for real-valued random variables.

To judge the uncorrelation of two interval-valued random variables, it reduces to consider the uncorrelation of endpoints according to the following result.

**Theorem 3.1.** For interval-valued random variables $F = [f_-, f_+], G = [g_-, g_+] \in \mathcal{M}(\Omega; K_2(\mathbb{R}))$, $F$ and $G$ are uncorrelated if and only if $f_-$ and $g_-$ are uncorrelated, $f_+$ and $g_+$ are uncorrelated.

**Proof.** For the underlying space $\mathbb{R}$, we have $S^* = \{Id, -Id\}$, where $Id$ is the identity function.

**step 1:** Assume $F$ and $G$ are uncorrelated. By Definition 3.1, then for any $x^* \in \{Id, -Id\}$, $s(x^*, F)$ and $s(x^*, G)$ are uncorrelated.

Since

$$s(x^*, F) = \sup_{f \in F} < x^*, f > = \begin{cases} f_+ & \text{if } x^* = Id \\ -f_- & \text{if } x^* = -Id \end{cases}$$

and

$$s(x^*, G) = \sup_{g \in G} < x^*, g > = \begin{cases} g_+ & \text{if } x^* = Id \\ -g_- & \text{if } x^* = -Id \end{cases},$$

this implies that $f_+$ and $g_+$ are uncorrelated, so are $f_-$ and $g_-.$

**Step 2:** Conversely, assume $f_+$ and $g_+$ are uncorrelated, so are $f_-$ and $g_-.$ Then for $x^* = Id$, $s(x^*, F)$ and $s(x^*, G)$ are uncorrelated. For any $x^* \in \mathbb{R}^*$, there exists an $a \in \mathbb{R}$, such that $x^* = a \times Id$. Then

$$s(x^*, F) = \sup_{f \in F} < x^*, f > = \begin{cases} af_+ & \text{if } a \geq 0 \\ af_- & \text{if } a < 0 \end{cases}$$

and

$$s(x^*, G) = \sup_{g \in G} < x^*, f > = \begin{cases} ag_+ & \text{if } a \geq 0 \\ ag_- & \text{if } a < 0 \end{cases}$$

Then $s(x^*, F)$ and $s(x^*, G)$ are uncorrelated since $af_+, ag_+$ and $af_-, ag_-$ are uncorrelated. By Definition 3.1 $F, G$ are uncorrelated. \hfill \Box
Example 3.1. Assume the real-valued vector \((\xi, \eta)\) are uniformly distributed in the ellipse
\[
\left\{(x, y) \in \mathbb{R}^2 \mid \frac{(x - 2)^2}{2^2} + \frac{(y - 3)^2}{3^2} \leq 1\right\}.
\]
By simple calculation, we know that \(\xi\) and \(\eta\) are not independent with different distributions. But the correlation \(\rho(\xi, \eta) = 0\). Then by Theorem 3.1, the two interval-valued random sets \([0, \xi]\) and \([0, \eta]\) are uncorrelated while \([0, \xi]\) and \([0, \eta]\) are not independent with different distributions.

Furthermore, we assume that the non-negative real-valued random sequence \(\{f_1, \cdots, f_n, \cdots\}\) are not independent with different distributions but uncorrelated, then so do \(\{[0, f_1], \cdots, [0, f_n], \cdots\}\).

Now we give a weak law of large numbers for uncorrelated \(K_{kc}(\mathbb{R})\)-valued random variables.

**Theorem 3.2.** Let \(\{V_n : n \in \mathbb{N}\}\) be a sequence of uncorrelated \(K_{kc}(\mathbb{R})\)-valued random variables such that for each \(n\), \(\text{Var}(s(x^*, V_k))\) exists and for any \(x^* \in \mathbb{R}^* (= \mathbb{R})\),
\[
\frac{1}{n^2} \sum_{k=1}^{n} \text{Var}(s(x^*, V_k)) \longrightarrow 0 \text{ as } n \rightarrow \infty.
\]
Then
\[
P\left(d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) > \varepsilon\right) \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{1}
\]

**Proof.** For any \(\varepsilon > 0\). By the Markov inequality and the equivalent definition of Hausdorff metric, we have
\[
P\left(d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) > \varepsilon\right) \leq \frac{1}{(\varepsilon n)^2} E\left[d_H\left(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} E[V_k]\right)^2\right]
\]
\[
= \frac{1}{(\varepsilon n)^2} E\left[\sup_{x^* \in S^*} |s(x^*, \sum_{k=1}^{n} V_k) - s(x^*, \sum_{k=1}^{n} E[V_k])|^2\right]
\]
\[
= \frac{1}{(\varepsilon n)^2} E\left[\sup_{x^* \in S^*} |s(x^*, \sum_{k=1}^{n} V_k) - s(x^*, \sum_{k=1}^{n} E[V_k])|^2\right]\tag{2}
\]

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For the real space $\mathbb{R}$, $S^* = \{-Id, Id\}$. Denote $s(Id, V_k) = y_k$, $s(-Id, V_k) = z_k$. Then by Definition 3.1, both $\{y_k : k \geq 1\}$ and $\{z_k : k \geq 1\}$ are uncorrelated real-valued random variable sequences. Therefore

$$\frac{1}{(en)^2}E\left[\sup_{x^* \in S^*} |s(x^*, \sum_{k=1}^{n} V_k) - s(x^*, \sum_{k=1}^{n} E[V_k])|^2\right]$$

$$\leq \frac{1}{(en)^2}E\left[\|\sum_{k=1}^{n} s(Id, V_k) - \sum_{k=1}^{n} s(Id, E[V_k])\|^2 + \|\sum_{k=1}^{n} s(-Id, V_k) - \sum_{k=1}^{n} s(-Id, E[V_k])\|^2\right]$$

$$= \frac{1}{(en)^2}E\left[\sum_{k=1}^{n} (y_k - E[y_k])^2 + \sum_{k=1}^{n} (z_k - E[z_k])^2\right]$$

$$+ \sum_{k=1}^{n} (z_k - E[z_k])(y_l - E[y_l])$$

$$+ \sum_{k=1}^{n} (z_k - E[z_k])(z_l - E[z_l])$$

$$= \frac{1}{(en)^2}E\left[\sum_{k=1}^{n} (y_k - E[y_k])^2 + \sum_{k=1}^{n} (z_k - E[z_k])^2\right]$$

$$= \frac{1}{(en)^2}\left[\sum_{k=1}^{n} Var(y_k) + \sum_{k=1}^{n} Var(z_k)\right]$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty,$$

which together with (2) yields the result (1). □

The following is an example of weak law of large numbers, in which the sequence is uncorrelated but is not independent and has no identical distribution. That means the condition is weaker than the existing results of laws of large numbers such as in [1, 14, 23] etc.

**Example 3.2.** Let the real-valued random vector $(X_1, X_2, \cdots, X_n)$ be uniformly distributed in the following $n$-dimensional ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1.$$
We assume that $a_i > 0$ for each $i$ and the equality $a_1 = a_2 = \cdots = a_n$ does not hold. The joint density function is

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} \frac{\Gamma(n+1)}{\pi^n \prod_{i=1}^{n} a_i} x_i \in (-a_i, a_i), i = 1, 2, \cdots, n; \\
0, \quad \text{otherwise.} \end{cases}$$

We know that $E[X_i] = 0$, for each $i$ and $cov[X_i, X_j] = 0$ for $i \neq j$. Therefore the sequence $(X_1, X_2, \cdots, X_n)$ are pairwise uncorrelated, and are not identically distributed. For any pair $(i, j)$, we know that $X_i$ and $X_j$ are not independent. Then the sequence $X_1, X_2, \cdots, X_n$ are not independent.

For each $i$, we have $Var(X_i) = E(X_i^2) = \frac{a_i^2}{n+2}$.

Set $(Y_1, Y_2, \cdots, Y_n) = (X_1 + a_1, X_2 + a_2, \cdots, X_n + a_n)$. Then $(Y_1, Y_2, \cdots, Y_n)$ are non-negative and uncorrelated with $E[Y_i] = a_i$ for each $i$. $(Y_1, Y_2, \cdots, Y_n)$ are neither independent nor identically distributed. $(Y_1, Y_2, \cdots, Y_n)$ are uniformly distributed in the ellipsoid

$$\frac{(y_1 - a_1)^2}{a_1^2} + \frac{(y_2 - a_2)^2}{a_2^2} + \cdots + \frac{(y_n - a_n)^2}{a_n^2} \leq 1.$$ 

Clearly, $Var(Y_i) = Var(X_i) = \frac{a_i^2}{n+2}$.

Define $V_1 = [0, Y_1], \cdots, V_n = [0, Y_n]$. We have $E[V_i] = [0, a_i]$ for each $i$. By Theorem 3.1, the interval-valued random sequence $V_1, \cdots, V_n$ are neither independent nor identical distributed but uncorrelated. By simple calculation, we have

$$\frac{1}{n^2} \sum_{i=1}^{n} Var(Y_i) = \frac{1}{n^2} \sum_{i=1}^{n} \frac{a_i^2}{n+2} = \frac{1}{n^2(n+2)} \sum_{i=1}^{n} a_i^2.$$ 

For fixed $n$, we can confine $a_1, \cdots, a_n$ such that $a_i \leq \sqrt{n}$ for each $i$. Then we obtain

$$\frac{1}{n^2(n+2)} \sum_{i=1}^{n} a_i^2 \leq \frac{1}{n+2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ 

Moreover, for any $\epsilon > 0$

$$P\left(d_H\left(\frac{1}{n} \sum_{i=1}^{n} V_i, \frac{1}{n} \sum_{i=1}^{n} [0, a_i]\right) > \epsilon\right) \leq \frac{2 \sum_{i=1}^{n} Var(Y_i)}{(en)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implied the weak law of large numbers.
Theorem 3.3. Assume the dimension $\dim \mathfrak{x} < \infty$. Let $\{V_n : n \in \mathbb{N}\}$ be a sequence of uncorrelated set-valued random variables such that for all $n, V_n \in L^2(\Omega; K_{k_n}(\mathfrak{x}))$, and for any $x^* \in S^*$

$$\frac{1}{n^2} \sum_{k=1}^{n} \text{Var}(s(x^*, V_k)) \longrightarrow 0 \text{ as } n \to \infty. \number{4}$$

Then

$$P\{d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) > \varepsilon\} \longrightarrow 0.$$ 

Proof. Step 1. For each $i \in \mathbb{N}$, $\|V_i\|_k \in L^2(\Omega; \mathbb{R})$ since $V_i \in L^2(\Omega; K_{k_n}(\mathfrak{x}))$ and for any $n \in \mathbb{N}$,

$$d_H\left(\sum_{i=1}^{n} V_i, E[\sum_{i=1}^{n} V_i]\right) \leq \|\sum_{i=1}^{n} V_i\|_k + E\left[\|\sum_{i=1}^{n} V_i\|_k\right]$$

$$\leq \sum_{i=1}^{n} \|V_i\|_k + \sum_{i=1}^{n} E\left[\|V_i\|_k\right] \in L^2(\Omega; \mathbb{R}). \number{5}$$

Since the bounded linear functional $x^*$ is continuous, by Lemma 2.1 and Definition 3.1, for $k \neq l$, we have

$$E\left[\left\|s(x^*, V_k) - s(x^*, E[V_k])\right\| s(x^*, V_l) - s(x^*, E[V_l])\right]$$

$$= E\left[\left\|s(x^*, V_k) - E[s(x^*, V_k)]\right\| s(x^*, V_l) - E[s(x^*, V_l)]\right] = 0. \number{6}$$

Step 2. By (5), $d_H \in L^2(\Omega; \mathbb{R})$. Then for any $\varepsilon > 0$, we can use Markov inequality and obtain

$$P\{d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) > \varepsilon\} \leq \frac{1}{(\varepsilon n)^2} E\left[d_H\left(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} E[V_k]\right)^2\right]$$

$$= \frac{1}{(\varepsilon n)^2} E\left[\sup_{x^* \in S^*} \left\|s(x^*, \sum_{k=1}^{n} V_k) - s(x^*, \sum_{k=1}^{n} E[V_k])\right\|^2\right].$$

By the continuity of $s(\cdot, A)$ for fixed $A \in K(\mathfrak{x})$ (cf. [18]) and the compactness of $S^*$ (since $S^*$ is a bounded closed subset in the finite dimensional space), by the property of a continuous operator defined in a compact set (cf. [4]), there exists an $x^*_0 \in S^*$ (It may depend on $n$), such that

$$\sup_{x^* \in S^*} \left|s(x^*, \sum_{k=1}^{n} V_k) - s(x^*, \sum_{k=1}^{n} E[V_k])\right| = \left|s(x^*_0, \sum_{k=1}^{n} V_k) - s(x^*_0, \sum_{k=1}^{n} E[V_k])\right| \text{ a.s.} \number{8}$$
By the property of Hausdorff metric together with (7) and (8), we have

\[ P\left\{ d_H\left( \frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k] \right) > \varepsilon \right\} = P\left\{ \frac{1}{n} \left| s(x_0^*, \sum_{k=1}^{n} V_k) - s(x_0^*, \sum_{k=1}^{n} E[V_k]) \right| > \varepsilon \right\} \]

\[ \leq \frac{1}{(\varepsilon n)^2} E\left[ \left| s(x_0^*, \sum_{k=1}^{n} V_k) - s(x_0^*, \sum_{k=1}^{n} E[V_k]) \right|^2 \right] \]  

(9)

Noticing that

\[ \frac{1}{\varepsilon^2} E\left\{ \frac{1}{n^2} \sum_{k=1}^{n} \left[ s(x_0^*, V_k) - s(x_0^*, E[V_k]) \right]^2 \right\} + \frac{1}{\varepsilon^2 n^2} E\left\{ \sum_{k \neq l} \left[ s(x_0^*, V_k) - s(x_0^*, E[V_k]) \right] \left[ s(x_j^*(n), V_l) - s(x_j^*(n), E[V_l]) \right] \right\} = \frac{1}{\varepsilon^2} E\left\{ \frac{1}{n^2} \sum_{k=1}^{n} \left[ s(x_0^*, V_k) - s(x_0^*, E[V_k]) \right]^2 \right\} \quad \text{(By (6))} \]

(10)

\[ \leq \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^{n} Var(s(x_0^*, V_k)) \]

\[ \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{(by (4))}. \]

The result was proved. \( \square \)

An obvious corollary of Theorem 3.3 is the weak laws of large numbers for uncorrelated identically distributed \( L^2(\Omega, K, (\mathcal{X})) \)-valued random variables. In fact, for any \( x^* \in S^* \),

\[ \frac{1}{n} \sum_{k=1}^{n} Var(s(x^*, V_k)) = Var(s(x^*, V_1)) \leq Var(\sup_{x^* \in S^*} s(x^*, V_1)) = Var(\|V_1\|_{K}) < \infty, \]

which implies

\[ \frac{1}{n^2} \sum_{k=1}^{n} Var(s(x^*, V_k)) \leq \frac{1}{n} Var(\|V_1\|_{K}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

In the following, we shall prove the strong laws of large numbers for uncorrelated set-valued random variables.
Theorem 3.4. Assume $\dim \tilde{x} < \infty$ and let $\{V_n, n \in \mathbb{N}\}$ be a sequence of uncorrelated set-valued random variables such that for all $n$, $V_n \in L^2(\Omega; K_{k_c}(\tilde{x}))$. And for any $x^* \in S^*$, $\text{Var}(s(x^*, V_n)) \leq M$ ($M$ is a positive constant). Then

$$d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) \longrightarrow 0 \text{ as } n \rightarrow \infty \text{ a.s.}$$

**Proof.** Set $S_n = d_H(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} E[V_k]) \geq 0$. It remains to prove $\frac{S_n}{n} \rightarrow 0$ as $n \rightarrow \infty$ a.s..

*step 1:* At first, we show $\frac{S_n^2}{n^2} \rightarrow 0$ as $n \rightarrow \infty$ a.s.

For any $\epsilon > 0$, by the triangle inequality, Markov inequality and the definition of uncorrelation, it holds that

$$\sum_{n=1}^{\infty} P\left\{\frac{S_n^2}{n^2} > n^2 \epsilon\right\}$$

$$= \sum_{n=1}^{\infty} P\left\{d_H(\sum_{k=1}^{n^2} V_k, \sum_{k=1}^{n^2} E[V_k]) > n^2 \epsilon\right\}$$

$$\leq \sum_{n=1}^{\infty} E\left[\frac{d_H^2(\sum_{k=1}^{n^2} V_k, \sum_{k=1}^{n^2} E[V_k])}{n^4 \epsilon^2}\right]$$

$$= \sum_{n=1}^{\infty} E\left[\sup_{x^* \in S^*} |s(x^*, \sum_{k=1}^{n^2} V_k) - s(x^*, \sum_{k=1}^{n^2} E[V_k])|^2\right]\left/ n^4 \epsilon^2\right.\right.$$ (There exists some $x_0^* \in S^*$, which may depend on $n$)

$$= \sum_{n=1}^{\infty} \frac{\text{Var}(s(x_0^*, V_k))}{n^4 \epsilon^2} + \sum_{n=1}^{\infty} \frac{\text{Cov}(s(x_0^*, V_k), s(x_0^*, V_l))}{n^4 \epsilon^2}$$

$$\leq \sum_{n=1}^{\infty} n^2 \frac{M}{n^4 \epsilon^2} < \infty.$$

By the Borel-Cantelli lemma, it holds that $\frac{S_n^2}{n^2} \rightarrow 0$, a.s.
step 2: For $k \in \mathbb{N}$ and $n^2 < k < (n+1)^2$, we have

$$\mathbb{P}\left\{ \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}| > n^2 \varepsilon \right\} \leq \frac{E\left[ \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|^2 \right]}{\varepsilon^2 n^4}.$$ 

Therefore, by the proof of step 1, we obtain

$$E\left\{ \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|^2 \right\}$$

$$= E\left[ \max_{n^2 < k < (n+1)^2} \left( d_H(\sum_{i=1}^{k} V_i, \sum_{i=1}^{k} E[V_i]) - d_H(\sum_{i=1}^{n^2} V_i, \sum_{i=1}^{n^2} E[V_i]) \right)^2 \right]$$

$$= E\left[ \max_{n^2 < k < (n+1)^2} \left( \sup_{x^* \in S^*} |s(x^*, \sum_{i=1}^{k} V_i) - s(x^*, \sum_{i=1}^{k} E[V_i])| \right. \right.$$

$$- \left. \sup_{x^* \in S^*} |s(x^*, \sum_{i=1}^{n^2} V_i) - s(x^*, \sum_{i=1}^{n^2} E[V_i])| \right)^2 \right]$$

$$\leq E\left[ \max_{n^2 < k < (n+1)^2} \sup_{x^* \in S^*} \left( |s(x^*, \sum_{i=1}^{k} V_i) - s(x^*, \sum_{i=1}^{k} E[V_i])| \right.$$

$$- |s(x^*, \sum_{i=1}^{n^2} V_i) - s(x^*, \sum_{i=1}^{n^2} E[V_i])| \right)^2 \right]$$
Let $Y_i(\cdot) = s(\cdot, V_i) - s(\cdot, E[V_i])$, then

$$E\left\{ \max_{n^2 < k < (n+1)^2} \left| S_k - S_{n^2} \right|^2 \right\}$$

$$\leq E\left\{ \max_{n^2 < k < (n+1)^2} \left( \left| \sum_{i=1}^{k} Y_i(x^*) - \sum_{i=1}^{n^2} Y_i(x^*) \right|^2 \right) \right\}$$

$$\leq \sum_{k=n^2+1}^{(n+1)^2-1} E\left[ \sup_{x^* \in S^*} \left( \left| \sum_{i=n^2+1}^{k} Y_i(x^*) \right|^2 \right) \right]$$

$$= \sum_{k=n^2+1}^{(n+1)^2-1} E\left[ \left( \sum_{i=n^2+1}^{k} Y_i(x^*) \right)^2 \right]$$

$$= \sum_{k=n^2+1}^{(n+1)^2-1} \sum_{i=n^2+1}^{k} \text{Var}(s(x^*_0, V_i))$$

$$\leq \sum_{k=n^2+1}^{(n+1)^2-1} (k - n^2) M = (2n + 1)nM$$

which means

$$\sum_{n=1}^{\infty} P\left\{ \max_{n^2 < k < (n+1)^2} \left| S_k - S_{n^2} \right| > n^2 \epsilon \right\} < \infty.$$  

So by Borel-Cantelli lemma,

$$\frac{\max_{n^2 < k < (n+1)^2} \left| S_k - S_{n^2} \right|}{n^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ a.s.}$$
Hence, for $n^2 < k < (n + 1)^2$, we have

\[
\frac{S_k}{k} = \frac{S_k + S_{n^2} - S_{n^2}}{k} \\
\leq \frac{S_{n^2} + \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{k} \\
\leq \frac{S_{n^2} + \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{n^2} \\
= \frac{S_{n^2}}{n^2} + \frac{\max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{n^2} \to 0 \text{ as } k \to \infty \text{ a.s.}
\]

So, we obtain

\[
d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} EV_k\right) = \frac{1}{n} d_H\left(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} EV_k\right) \to 0 \text{ as } n \to \infty \text{ a.s.}
\]

The strong law of large numbers was obtained. □

Furthermore, the following stronger result can be obtained.

**Theorem 3.5.** Assume $\dim \mathcal{X} < \infty$. Let $\{V_n, n \in \mathbb{N}\}$ be a sequence of uncorrelated set-valued random variables such that for all $n$, $V_n \in L^2(\Omega; \mathbf{K}_{k})$ and for any $x^* \in S^*$,

\[
\sum_{n=1}^{\infty} \frac{\text{Var}(s(x^*, V_n))}{n^2} \ln^2 n < \infty. \quad (11)
\]

Then

\[
d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) \to 0. \text{ a.s.}
\]

**Proof.** Let $S_n = d_H\left(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} E[V_k]\right) \geq 0$. 

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step 1: For any $\varepsilon > 0$, by the triangle inequality, Markov inequality and the definition of uncorrelated set-valued random variables, it holds that

$$
\sum_{n=1}^{\infty} P\left\{ S_{n^2} > \frac{n^2 \varepsilon}{\log n} \right\}
$$

$$
= \sum_{n=1}^{\infty} P\left\{ d_{H}(\sum_{k=1}^{n^2} V_k, \sum_{k=1}^{n^2} E[V_k]) > \frac{n^2 \varepsilon}{\log n} \right\}
$$

$$
\leq \sum_{n=1}^{\infty} \frac{E\left[ d_{H}^2(\sum_{k=1}^{n^2} V_k, \sum_{k=1}^{n^2} E[V_k]) \right]}{n^4 \varepsilon^2 / \log^2 n}
$$

$$
= \sum_{n=1}^{\infty} \frac{E\left[ \sup_{x^* \in S^*} |s(x^*, \sum_{k=1}^{n^2} V_k) - s(x^*, \sum_{k=1}^{n^2} E[V_k])|^2 \right]}{n^4 \varepsilon^2 / \log^2 n}
$$

$$
= \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n^2} \text{Var}(s(x_0^*, V_k))}{n^2 \varepsilon^2 / \log^2 n} + \sum_{n=1}^{\infty} \frac{\sum_{k \neq l} \text{Cov}(s(x_0^*, V_k), s(x_0^*, V_l))}{n^4 \varepsilon^2 / \log^2 n}
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{n^2 \varepsilon^2} \sum_{k=1}^{n^2} \text{Var}(s(x_0^*, V_k)) \log^2 n
$$

$$
\leq \sum_{n=1}^{\infty} \frac{\log^2 n}{n^2 \varepsilon^2} \sum_{k=3}^{n^2} \text{Var}(s(x_0^*, V_k)) \frac{\log^2 k}{k^2} + \sum_{n=1}^{\infty} \frac{\sum_{k=3}^{n^2} \text{Var}(s(x_0^*, V_k)) \log^2 k + (\text{Var}(s(x_0^*, V_1) + \text{Var}(s(x_0^*, V_2)))}{n^4 \varepsilon^2}
$$

$$
< \infty \text{ (by condition (11) and } \sum_{n=1}^{\infty} \frac{\log^2 n}{n^2} < \infty).}
$$

By the Borel-Cantelli lemma, we obtain

$$
\frac{S_n}{n^2 / \log n} \rightarrow 0 \text{ as } n \rightarrow \infty. \ a.s.
$$
Then
\[ \frac{S_n}{n^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \ a.s.. \]

**step 2:** For \( n^2 \leq k < (n+1)^2 \), by using the same method as the proof in theorem 3.4, it has
\[
\max_{n^2 < k < (n+1)^2} \frac{|S_k - S_{n^2}|}{n^2/ \log n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \ a.s.
\]

Further
\[
\max_{n^2 < k < (n+1)^2} \frac{|S_k - S_{n^2}|}{n^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \ a.s.
\]

Hence, for \( n^2 < k < (n+1)^2 \), we have
\[
\frac{S_k}{k} = \frac{S_k + S_{n^2} - S_{n^2}}{k}
\leq \frac{S_{n^2} + \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{k}
\leq \frac{S_{n^2} + \max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{n^2}
= \frac{S_{n^2}}{n^2} + \frac{\max_{n^2 < k < (n+1)^2} |S_k - S_{n^2}|}{n^2}
\rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \ a.s.
\]

Then
\[
d_H\left(\frac{1}{n} \sum_{k=1}^{n} V_k, \frac{1}{n} \sum_{k=1}^{n} E[V_k]\right) = \frac{1}{n} d_H\left(\sum_{k=1}^{n} V_k, \sum_{k=1}^{n} E[V_k]\right)
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \ a.s..
\]

\[ \Box \]

**4. Concluding remark**

There are some known results about weak and strong laws of large numbers for independent identically distributed or independent (not necessarily identically
distributions) set-valued random variables. The innovation of this paper is to consider laws of large numbers for set-valued random variables under the weaker condition: uncorrelation. Since the hyperspace $K(\mathcal{X})$ is not linear, there is no nice subtraction and multiplication between two sets, no suitable set-valued covariance for set-valued random variables. Here we firstly give the definition of uncorrelated set-valued random variables by using support function and discuss its properties. Uncorrelation is more reasonable than independence to describe the complexity of actual data. Then, we obtain the weak and strong laws of large numbers for uncorrelated set-valued random variables in the sense of Hausdorff metric $d_H$ under different additional conditions. These results are also the extension of single-valued uncorrelated random variables, which are expected to be used in set-valued especially interval-valued statistical modeling and analysis.

**Acknowledgment**

This work is partly supported by Beijing Municipal Natural Science Foundation No.1192015 (Jinping Zhang) and National Social Science Fund of China No.19BTJ017 (Li Guan).

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