A POWERED GRONWALL-TYPE INEQUALITY AND APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study a powered integral inequality involving a finite sum, which can be used to solve the inequalities with singular kernels. We present that the solution of the inequality is decided by a finite recursion, whose result is proved to be a continuous, bounded or asymptotic function. Meanwhile, in order to overcome an obstacle from powers of integrals, we modify the method of monotonization into the powered monotonization. Furthermore, relying on the result and our technique of concavification, we discuss a generalized stochastic integral inequality, and give an estimate of the mean square. In the end, as applications, we study uniform boundedness and continuous dependence of solutions for a class of stochastic differential equation in mean square.

1. Introduction. Since Gronwall ([8]) and Bellman ([3]) investigated the basic form of integral inequalities, great efforts (see e.g. the monograph [18] and references therein) have been made to develop more complicated forms of integral inequalities separately towards the studies of existence, uniqueness, boundedness and stability of solutions and invariant manifolds for differential equations and integral equations. In 1956 Bihari ([4]) discussed the integral inequality

\[ u(t) \leq \alpha + \int_0^t f(s)\phi(u(s))ds, \quad t \geq 0. \]

In 2000 his work was generalized by Lipovan ([13]) to the delay case, i.e., the time \( t \) is replaced with a differentiable delay \( b(t) \) satisfying \( 0 < b(t) \leq t \). In 2005 Agarwal, Deng and Zhang ([1]) extended the result of [13] to the finite sum of integrals

\[ u(t) \leq a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)\phi_i(u(s))ds, \quad t_0 \leq t < T, \] (1)

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where the monotonicity of $a$ is not required but $\phi_{i+1}$ is stronger monotone than $\phi_i$ for each $i$. Later, by monotonizing the continuous functions $\phi_i$ to build a stronger monotone sequence, Wang ([12]) generalized (1) to the version of two variables, i.e. so-called Wendroff type.

Singular integral inequality is another interesting topic. As indicated in [20], the integral $\int_{t_0}^{t} f(t,s)u(s)\,ds$ is said to be singular if the kernel $f(t,s)$ is singular at least on the line $s = t$; it is referred to be weakly singular if it is singular and $\int_{t_0}^{t} |f(t,s)|\,ds < +\infty$ for all $t_0 \leq t < T \leq +\infty$. The weakly singular integral inequality

$$u(t) \leq a(t) + \beta \int_{0}^{t} (t-s)^{\gamma-1}u(s)\,ds, \quad t \geq 0,$$

where $\beta \geq 0$, $0 < \gamma < 1$ and both $a$ and $u$ are nonnegative and locally integrable, was discussed in Henry’s book [10]. An estimate of the unknown $u$ was given in terms of Mittag-Leffler function ([16]) by an iteration approach. Besides, another approach was given by Medved ([15]) in 1997, where the well-known Hölder’s Inequality (see [9] or the Appendix) was applied to separate the weakly singular kernel $(t-s)^{\gamma-1}$ from the integral. In 2002 Ma and Yang ([14]) improved Medved’s method and gave an estimate for a weakly singular integral inequality with the Erdélyi-Kober kernel $s^{\alpha-1}(t^\beta - s^\beta)^{\gamma-1}$ (see [12, p.105]). Cheung, Ma and Tseng ([5]) extended the result of [14] to Wendroff type.

In this paper we discuss the powered Gronwall-type inequality

$$u(t) \leq a(t) + \sum_{i=1}^{n} \left\{ \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)\phi_i(u(s))\,ds \right\}^{p_i}, \quad t \geq 0,$$

for $t_0 \leq t < +\infty$, where $n \in \mathbb{N}$, $p_i \geq 1$, and all $a$, $b_i$, $f_i$, $\phi_i$ and $u$ are nonnegative continuous functions for $i = 1, \ldots, n$. This inequality contains $n$ powers $p_i$, which enable the inequality to be applicable to the integral inequalities with multiple singular kernels. In [5, 14, 15] integral inequalities with a uniform power were discussed by using the well-known Power Mean Inequality (see [9] or the Appendix), but it does not work in our case of different powers $p_i$. In section 3 of this paper we monotonize the given functions to be a sequence of powered stronger nondecreasing functions rather than a sequence of stronger nondecreasing functions considered in [1], so as to estimate the unknown $u$ by a continuous function which is defined by a finite recursion. In order to investigate the boundedness and asymptotics of the continuous function, we discuss the finite recursion in section 2 as preliminaries. In section 4 we apply the result to a singular integral inequality, in which there are more than one singular kernels and the kernels are of more general form than those in [15] and [14]. We also apply the result in section 5 to a stochastic integral inequality, a nonlinear extension of the inequalities considered in [2, 11]. Unlike [2, 17, 11, 23], we need to concavify some nonlinear functions to deal with the mean square in the stochastic estimation. Finally, in section 6 we apply our results to investigate the uniform boundedness and continuous dependence of solutions on not only initial data but also given functions for a stochastic differential equation in mean square.

2. Finite recursion. Consider the following finite recursion of first order

$$x_i(t) = W_i^{-1}\{W_i(x_{i-1}(t)) + \left\{ \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)\,ds \right\}^{p_i} \}, \quad i = 1, \ldots, n$$

for \( t_0 \leq t < +\infty \), where \( f_i, b_i, p_i \) and \( n \) are given as in (2), each \( b_i \in C([t_0, +\infty), [t_0, +\infty)) \) is nondecreasing such that \( b_i(t) \leq t \) and each \( W_i \in C(\mathbb{R}^+, \mathbb{R}) \) is strictly increasing such that \( W_i(+\infty) = +\infty \). Clearly, \( b_i(t_0) = t_0 \) and the inverse \( W_i^{-1} \) is well defined on \([W_i(t_0), +\infty)\).

Define a mapping \( J_i \) on \( C([t_0, +\infty), \mathbb{R}^+) \) by

\[
J_i x(t) := W_i(x(t)) + \left\{ \int_{t_0}^{b_i(t)} f_i(t, s) ds \right\}^{p_i}, \quad i = 1, \ldots, n.
\]

For each \( x : [t_0, +\infty) \to \mathbb{R}^+ \), we have \( J_i x(t) \in [W_i(0), +\infty) \) for all \( t \in [t_0, +\infty) \), implying that the right hand side of (3), i.e., \( W_i^{-1}(J_i x_{i-1}(t)) \), is well defined for all \( t \in [t_0, +\infty) \). In the following lemma we will use the notation \( C_b([t_0, +\infty), \mathbb{R}) \) to present the set of all bounded functions in \( C([t_0, +\infty), \mathbb{R}) \), which is a Banach space equipped with the norm \( \|x\| := \sup_{t \in [t_0, +\infty)} |x(t)| \). We need the following assumptions:

(P1): \( f_i \in C([t_0, +\infty) \times [t_0, +\infty), \mathbb{R}^+) \) for all \( i = 1, \ldots, n \);

(P2): \( \int_{t_0}^{+\infty} f_i(t, s) ds \) is bounded on \([t_0, +\infty)\) for all \( i = 1, \ldots, n \).

**Lemma 1.** Suppose that \( f_i, b_i, p_i \) and \( n \) are given as in (3). If (P1) holds, then \( J_i(C([t_0, +\infty), \mathbb{R}^+)) \subseteq C([t_0, +\infty), \mathbb{R}) \). If (P1)-(P2) hold, then \( J_i(C_b([t_0, +\infty), \mathbb{R}^+)) \subseteq C_b([t_0, +\infty), \mathbb{R}) \). Moreover, the operator \( J_i \) is continuous on \( C_b([t_0, +\infty), \mathbb{R}^+) \), and if \( W_i \) is Lipschitzian, then \( J_i \) is also Lipschitzian on \( C_b([t_0, +\infty), \mathbb{R}^+) \).

**Proof.** For \( x \in C([t_0, +\infty), \mathbb{R}^+) \), we have \( W_i(x(\cdot)) \in C([t_0, +\infty), \mathbb{R}) \) since \( W_i \in C(\mathbb{R}^+, \mathbb{R}) \). On the other hand, the continuity of both \( b_i \) and \( f_i \) implies the continuity of the function \( F_i(t) := \int_{t_0}^{b_i(t)} f_i(t, s) ds \) in (4). Thus, \( J_i x \in C([t_0, +\infty), \mathbb{R}) \), i.e., the result \( J_i(C([t_0, +\infty), \mathbb{R}^+)) \subseteq C([t_0, +\infty), \mathbb{R}) \) is proved.

For \( x \in C_b([t_0, +\infty), \mathbb{R}^+) \), there exists a constant \( M > 0 \) such that \( 0 \leq x(t) \leq M \) for all \( t \in [t_0, +\infty) \). Since \( W_i \) is strictly increasing such that \( W_i(+\infty) = +\infty \), we have \( W_i(0) \leq W_i(x(t)) \leq W_i(M) < +\infty \) for all \( t \in [t_0, +\infty) \), i.e., \( W_i(x(\cdot)) \in C_b([t_0, +\infty), \mathbb{R}) \). On the other hand, since \( f_i \) is nonnegative, we have \( F_i(t) = \int_{t_0}^{b_i(t)} f_i(t, s) ds \leq \int_{t_0}^{+\infty} f_i(t, s) ds \), implying that \( F_i \) is bounded on \([t_0, +\infty)\) by (P2). Thus, \( J_i x \in C_b([t_0, +\infty), \mathbb{R}) \), i.e., the result \( J_i(C_b([t_0, +\infty), \mathbb{R}^+)) \subseteq C_b([t_0, +\infty), \mathbb{R}) \) is proved.

In order to prove the continuity of the operator \( J_i \), consider an arbitrary function \( x_0 \in C_b([t_0, +\infty), \mathbb{R}^+) \) and an arbitrary \( \varepsilon > 0 \). By the continuity of \( W_i \), there exists \( \delta > 0 \) such that if \( |u - u_0| < \delta \) then

\[
|W_i(u) - W_i(u_0)| < \varepsilon / 2.
\]

Thus, if \( x \in C_b([t_0, +\infty), \mathbb{R}^+) \) such that \( \|x - x_0\| < \delta \), we have \( |x(t) - x_0(t)| < \delta \) for all \( t \in [t_0, +\infty) \), which implies that \( |W_i(x(t)) - W_i(x_0(t))| < \varepsilon / 2 \) for all \( t \in [t_0, +\infty) \). Therefore,

\[
\|J_i x - J_i x_0\| = \|W_i(x) - W_i(x_0)\| \leq \frac{\varepsilon}{2} < \varepsilon,
\]

that is, \( J_i \) is a continuous operator on \( C_b([t_0, +\infty), \mathbb{R}^+) \).
Note that for any \( x \in C_b([t_0, +\infty), \mathbb{R}_+^+) \), we have \( \mathcal{J}_ix \in C_b([t_0, +\infty), \mathbb{R}) \). If \( W_i \) is Lipschitzian associated with constant \( L \), then by (4) we obtain

\[
\|\mathcal{J}_ix - \mathcal{J}_iy\| = \|W_i(x) - W_i(y)\| = \sup_{t \in [t_0, +\infty)} |W_i(x(t)) - W_i(y(t))| \leq L \sup_{t \in [t_0, +\infty)} |x(t) - y(t)| = L\|x - y\|
\]

for all \( x, y \in C_b([t_0, +\infty), \mathbb{R}_+^+) \). That is \( \mathcal{J}_i \) is Lipschitzian on \( C_b([t_0, +\infty), \mathbb{R}_+^+) \).

The notation \( \mathcal{J}_i \) can be employed to simplify the recursion (3).

**Lemma 2.** Suppose that \( f_i, s, i, s \) and \( W_i, s \) and \( n \) are given as in (3). If (P1) holds and \( x_0 \in C([t_0, +\infty), \mathbb{R}_+^+) \), then the general \( x_n \) defined by the recursion (3) is given by

\[
x_n(t) = (W_n^{-1} \circ \mathcal{J}_n) \circ \cdots \circ (W_1^{-1} \circ \mathcal{J}_1)x_0(t) \tag{5}
\]

and \( x_n \in C([t_0, +\infty), \mathbb{R}_+^+) \). If (P2) holds and \( x_0 \in C_b([t_0, +\infty), \mathbb{R}_+^+) \) additionally, then \( x_n \in C_b([t_0, +\infty), \mathbb{R}_+^+) \).

**Proof.** (5) is obvious from (3) and (4). The result \( x_n \in C([t_0, +\infty), \mathbb{R}_+^+) \) comes from the claim that each \( W_i^{-1} \circ \mathcal{J}_i \) in (5) maps \( C([t_0, +\infty), \mathbb{R}_+^+) \) into \( C([t_0, +\infty), \mathbb{R}_+^+) \). In order to prove the claim, for each \( x \in C([t_0, +\infty), \mathbb{R}_+^+) \) we note that \( \mathcal{J}_ix(t) \in [W_i(0), +\infty) \) for all \( t \in [t_0, +\infty) \). Thus, the monotonicity of \( W_i \) implies that \( W_i^{-1} \circ \mathcal{J}_i \subset W_i^{-1}([W_i(0), +\infty)) = [0, +\infty) \). The continuity comes from Lemma 1 and the claim is proved.

If (P2) holds and \( x_0 \in C_b([t_0, +\infty), \mathbb{R}_+^+) \) additionally, the result \( x_n \in C_b([t_0, +\infty), \mathbb{R}_+^+) \) follows from the assertion that \( W_i^{-1} \circ \mathcal{J}_i \) maps \( C_b([t_0, +\infty), \mathbb{R}_+^+) \) into itself. In order to prove the assertion, for \( x \in C_b([t_0, +\infty), \mathbb{R}_+^+) \), by Lemma 1 we have \( \mathcal{J}_ix \in C_b([t_0, +\infty), \mathbb{R}_+^+) \), that is, there exists \( M > 0 \) such that

\[
W_i(0) \leq \mathcal{J}_ix(t) \leq M, \forall t \in [t_0, +\infty).
\]

Since \( W_i \) is strictly increasing and \( W_i(\infty) = +\infty \), it follows that

\[
0 \leq W_i^{-1} \circ \mathcal{J}_ix(t) \leq W_i^{-1}(M) < +\infty, \forall t \in [t_0, +\infty),
\]

which implies that \( (W_i^{-1} \circ \mathcal{J}_i)x \) is bounded on \( [t_0, +\infty) \). The continuity is the same as above for the case of (P1). The assertion is proved and the proof of the Lemma is completed.

The following lemma gives the asymptoticity of \( x_n \).

**Lemma 3.** Suppose that \( f_i, s, i, s \) and \( W_i, s \) and \( n \) are given as in (3) and (P1) holds. If \( x_0 \in C([t_0, +\infty), \mathbb{R}_+^+) \) satisfies \( \lim_{t \to +\infty} x_0(t) = 0 \) and if

\[
(C1): \text{either } \lim_{t \to +\infty} \int_{t_0}^{t_0+} f_i(s)ds = 0 \text{ for all } i,
\]

\[
(C2): \text{or } \lim_{s \to +\infty} W_i(s) = -\infty \text{ for all } i \text{ and (P2) holds},
\]

then \( \lim_{t \to +\infty} x_n(t) = 0 \).

**Proof.** By definition (4) of \( \mathcal{J}_i \), the recursion (3) implies that

\[
x_i(t) := (W_i^{-1} \circ \mathcal{J}_i)x_{i-1}(t), \quad i = 1, \ldots, n. \tag{6}
\]
In the case (C1), from (6) we have
\[
\limsup_{t \to +\infty} x_i(t) = W_i^{-1}(\limsup_{t \to +\infty} \mathcal{J}_i x_{i-1}(t))
\]
\[
\leq W_i^{-1}\{\limsup_{t \to +\infty} W_i(x_{i-1}(t)) + \limsup_{t \to +\infty} \int_{t_0}^{+\infty} f_i(t,s)ds\}^{p_i}
\]
\[
= W_i^{-1} \circ W_i(\limsup_{t \to +\infty} x_{i-1}(t))
\]
\[
= \limsup_{t \to +\infty} x_i(t),
\]
where we note that \(W_i\) is continuous and strictly increasing and \(\lim_{t \to +\infty} \int_{t_0}^{+\infty} f_i(t,s)ds = 0\). By induction we can prove that \(0 \leq \limsup_{t \to +\infty} x_n(t) \leq \limsup_{t \to +\infty} x_0(t) = 0\).

In the case (C2), we claim that \(\lim_{t \to +\infty} x_i(t) = 0\) for all \(i\). Since \(\lim_{t \to +\infty} x_0(t) = 0\), we assume inductively that \(\lim_{t \to +\infty} x_{i-1}(t) = 0\). From (6) we have
\[
\limsup_{t \to +\infty} x_i(t) = W_i^{-1}(\limsup_{t \to +\infty} \mathcal{J}_i x_{i-1}(t))
\]
\[
\leq W_i^{-1}\{W_i(\limsup_{t \to +\infty} x_{i-1}(t)) + \limsup_{t \to +\infty} \int_{t_0}^{+\infty} f_i(t,s)ds\}^{p_i}
\]
\[
= \lim_{u \to -\infty} W_i^{-1}(u) = 0,
\]
where (P2) guarantees that \(\limsup_{t \to +\infty} \int_{t_0}^{+\infty} f_i(t,s)ds\) is finite and (C2) guarantees that \(\lim_{u \to 0^+} W_i(u) = -\infty\) and \(\lim_{u \to -\infty} W_i^{-1}(u) = 0\). By induction we prove that \(\lim_{t \to +\infty} x_n(t) = 0\).

Further, we give the continuous dependence of \(x_n(t)\) on the initial value \(x_0(t)\) in (5). For convenience, let
\[
\mathcal{V}_n x_0(t) := (W_n^{-1} \circ \mathcal{J}_n) \circ \cdots \circ (W_1^{-1} \circ \mathcal{J}_1)x_0(t).
\]

**Lemma 4.** Suppose that \(f_i, s, b_i, s, p_i, s, W_i\)'s and \(n\) are given as in (3) and (P1)-(P2) hold. If there exist constants \(L_2 \geq L_1 > 0\) such that \(L_1|u_1 - u_2| \leq |W_i(u_1) - W_i(u_2)| \leq L_2|u_1 - u_2|\) for all \(u_1, u_2 \in \mathbb{R}^+\) and all \(i\), then
\[
||\mathcal{V}_n x_0 - \mathcal{V}_n y_0|| \leq (\frac{L_2}{L_1})^n||x_0 - y_0||, \forall x_0, y_0 \in C_b([t_0, +\infty), \mathbb{R}^+). \tag{7}
\]

**Proof.** The assertion given in the case (P2) in the proof of Lemma 2 shows that the composition \(W_i^{-1} \circ \mathcal{J}_i\) maps \(C_b([t_0, +\infty), \mathbb{R}^+)\) into itself. Thus, by Lemma 1, \(\mathcal{J}_i x\) and \(\mathcal{V}_i x\) both lie in \(C_b([t_0, +\infty), \mathbb{R})\) for every \(x \in C_b([t_0, +\infty), \mathbb{R}^+)\). Since \(|W_i(u_1) - W_i(u_2)| \leq L_2|u_1 - u_2|\) for all \(u_1, u_2 \in \mathbb{R}^+\), by Lemma 1 we have
\[
||\mathcal{J}_i x - \mathcal{J}_i y|| \leq L_2||x - y||, \forall x, y \in C_b([t_0, +\infty), \mathbb{R}^+). \tag{8}
\]
On the other hand, since \(|W_i(u_1) - W_i(u_2)| \geq L_1|u_1 - u_2|\) for all \(u_1, u_2 \in \mathbb{R}^+\), we get
\[
|W_i^{-1}(u_1) - W_i^{-1}(u_2)| \leq \frac{1}{L_1}|u_1 - u_2|, \forall u_1, u_2 \in [W_i(0), +\infty). \tag{9}
\]
By (8) and (9) we obtain
\[
||\mathcal{V}_i x_0 - \mathcal{V}_i y_0|| = ||(W_i^{-1} \circ \mathcal{J}_i) \circ \mathcal{V}_{i-1} x_0 - (W_i^{-1} \circ \mathcal{J}_i) \circ \mathcal{V}_{i-1} y_0||
\]
implying that
\[ \frac{1}{L_1} \| \mathcal{J}_i \circ V_{i-1} x_0 - \mathcal{J}_i \circ V_{i-1} y_0 \| \]
\[ \leq \frac{L_2}{L_1} \| V_{i-1} x_0 - V_{i-1} y_0 \|. \]
By induction we can prove (7) and completes the proof. \( \square \)

3. Powered Gronwall-type inequality. In this section, consider inequality (2), where those given functions \( a, b, f, \phi_i \) were assumed to be continuous and nonnegative. We simply suppose that for all \( i = 1, \ldots, n, \)

\( \text{(H1):} \ a \in C([t_0, +\infty), \mathbb{R}_+) \) and \( f_i(t, s) \in C([t_0, +\infty) \times [t_0, +\infty), \mathbb{R}_+) \) are both nondecreasing with respect to \( t, \) and

\( \text{(H2):} \ b_i \in C^1([t_0, +\infty), [t_0, +\infty)) \) are all nondecreasing such that \( b_i(t) \leq t. \)

Otherwise, one can monotonize them as done in [1]. For the functions \( \phi_i \) \( (i = 1, \ldots, n), \) we need not only a monotonization for each but also a reduction to a sequence of strongly nondecreasing functions for them. Define \( w_i s \) \( (i = 1, \ldots, n) \) recursively by

\[ w_i(s) := \begin{cases} \max_{\tau \in [0, s]} \{ \phi_1(\tau) \}, & i = 1, \\
\max_{\tau \in [0, s]} \{ \phi_i(\tau) \} w_{i-1}^{p_i-1/p_i}(s), & i = 2, \ldots, n, \end{cases} \]

where \( p_i \geq 1 \) is given in (2). We can easily check that the sequence \( \{w_i\} \) satisfies:

(i): every \( w_i \) is nondecreasing;

(ii): for every \( i = 1, \ldots, n-1 \) the ratio \( w_{i+1}^{p_{i+1}}/w_i^{p_i} \) is nondecreasing, i.e., \( w_i^{p_i} \propto w_{i+1}^{p_{i+1}} \) as denoted in [19] and [1].

For convenience, such a sequence \( \{w_i\} \) is said to be strongly nondecreasing in power of the sequence \( \{p_i\}. \) The case that \( p_i \equiv 1 \) for all \( i = 1, \ldots, n \) is exactly the strongly nondecreasing one considered in [1, 22]. We will amplify the ‘bad’ \( w_i \)’s to be the ‘good’ \( w_i \)’s in the proof of the main theorem. The procedure of amplification is referred to as a powered strong monotonization.

Let

\[ W_i(x) := \int_{x_0}^{x} \frac{ds}{w_i^p(s)}, \quad i = 1, \ldots, n, \]

where \( x_0 > 0 \) is an arbitrarily given constant. We will remark that the choice of \( x_0 \) does not affect our result just after Theorem 1.

Lemma 5. Suppose that functions \( \phi_i, \) \( i = 1, \ldots, n, \) satisfy

\( \text{(H3):} \ \phi_i \in C([\mathbb{R}_+, \mathbb{R}_+ \setminus \{0\}) \) and \( \phi_i^p(s) \propto s. \)

Then \( W_i(+\infty) = +\infty \) for all \( i = 1, \ldots, n. \)

Note that we assume \( \phi_i(s) > 0 \) for all \( s \geq 0 \) in (H3). Otherwise, we can amplify \( \phi_i \) a little to be

\[ \tilde{\phi}_i(s) := \phi_i(s) + \varepsilon \]

with an arbitrarily chosen constant \( \varepsilon > 0 \) in (2). Clearly,

\[ \tilde{\phi}_i^p(s) = \phi_i^p(s) + \frac{\varepsilon}{s^{p_i}}, \]

implying that \( \tilde{\phi}_i^p(s) \propto s \) if \( \phi_i^p(s) \propto s. \) Thus, the above defreezing process (12) makes the hypothesis (H3) reasonable. Those \( \phi_i \)’s having no zeros guarantee that
$w_i$, appearing in the denominators of (10) and (11), have no zeros and therefore (10) and (11) are meaningful for all $s, x \in \mathbb{R}_+$.

**Proof of Lemma 5.** We claim that

$$w_i^p(s) \propto s, \forall i = 1, \ldots, n. \quad (13)$$

If the claimed (13) is true, then $w_i^p(s)/s$ is nonincreasing in $s \in \mathbb{R}_+$. Clearly, $w_i^p(s)/s > 0$ for all $s \in \mathbb{R}_+$ by (H3) and (10). Then, there exists a constant $C > 0$ such that $w_i^p(s)/s \leq C$ for all $s \geq x_0$, implying that

$$W_i(+\infty) = \int_{x_0}^{+\infty} \frac{ds}{w_i^p(s)} \geq \int_{x_0}^{+\infty} \frac{ds}{Cs} = +\infty.$$  

This proves the lemma.

In order to prove (13), we note that $w_1(s) = \max_{\tau \in [0,s]} \{ \phi_1(\tau) \}$ by (10), which implies that $w_1 \in C(\mathbb{R}_+, \mathbb{R}_+ \setminus \{0\})$ is nondecreasing, as shown in Figure 1. As introduced in [28], an interior point $x_*$ in an interval $I$ is called a fort of a mapping $F: I \rightarrow I$ if $F$ is not strictly monotone in any small neighborhood of $x_*$. Let $S_*(w_1)$ consist of such forts $s_*$ of $w_1$ that $w_1$ is strictly increasing in either a left half-neighborhood $(s_* - \delta, s_*)$ or a right half-neighborhood $(s_*, s_* + \delta)$ with a sufficiently small $\delta > 0$. There are at most countably many such small half neighborhoods because they can be chosen so small not to intersect each other but each of them contains a rational. Hence, $S_*(w_1)$ is a countable set. Let $S_*(w_1) := \{ s_k : k = 1, \ldots, \varsigma \}$, where $\varsigma$ denotes the cardinality of $S_*(w_1)$ and can be equal to $\infty$, and use
it to partition $\mathbb{R}_+$ into

$$
\mathbb{R}_+ = \bigcup_{k=0}^{c} [s_k, s_{k+1}),
$$

where $s_0 := 0$ and $s_{c+1} := +\infty$. Then, for each $k = 0, ..., c$, the restriction $w_1|_{[s_k, s_{k+1})}$ is nondecreasing as shown in Figure 2, i.e., $w_1|_{[s_k, s_{k+1})}$ is either strictly increasing or flat; otherwise, there is a point $\tilde{s} \in S_* \cap (s_k, s_{k+1})$ as shown in Figure 3, a contradiction to the choice of $S_*(w_1)$. If $w_1|_{[s_k, s_{k+1})}$ is strictly increasing, then

$$
w_1(s) = \max_{\tau \in [0, s]} \psi_1(\tau) = \phi_1(s), \ s \in [s_k, s_{k+1}).
$$

(14)

Otherwise, $\max_{\tau \in [0, s]} \phi_1(\tau) > \phi_1(s)$ for some $s \in [s_k, s_{k+1})$. By continuity, $\phi_1(\tau_s) = \max_{\tau \in [0, s]} \phi_1(\tau)$ for some $\tau_s \in [0, s)$. Thus, $w_1(\tau_s) \equiv \phi_1(\tau_s)$ for $\tau \in [\tau_s, s]$, a contradiction to the fact that $w_1|_{[s_k, s_{k+1})}$ is strictly increasing. It follows from (14) that $s/u_1^{p_i}(s) = s/\phi_i^p(s)$, which is nondecreasing in $[s_k, s_{k+1})$ by (H3). It proves (13) for $i = 1$ on $[s_k, s_{k+1})$.

If $w_1|_{[s_k, s_{k+1})}$ is flat, then

$$
w_1(s) \equiv w_1(s_k), \ s \in [s_k, s_{k+1}).
$$

(15)

Clearly, $s/u_1^{p_i}(s_k)$ is nondecreasing. It follows from (15) that (13) for $i = 1$ also holds on $[s_k, s_{k+1})$.

For an inductive proof, assume that (13) holds for some $i$. Let $\psi_{i+1}(s) := \max_{\tau \in [0, s]} \{ \phi_{i+1}(\tau)/u_i^{p_i/p_{i+1}}(\tau) \}$. Then $w_{i+1}(s) = \psi_{i+1}(s)u_i^{p_i/p_{i+1}}(s)$ by (10). Similarly to $w_i$, we let $S_*(\psi_{i+1}) := \{ \tau_k : k = 1, ..., \zeta \}$ consist of such forts $\tau_k$ that $\psi_{i+1}$ is strictly increasing in either $(\tau_k - \delta, \tau_k]$ or $[\tau_k, \tau_k + \delta)$ with a sufficiently small $\delta > 0$, where $\zeta \leq +\infty$ denotes the cardinality of $S_*(\psi_{i+1})$, and use it to partition $\mathbb{R}_+$ into

$$
\mathbb{R}_+ = \bigcup_{k=0}^{c} [\tau_k, \tau_{k+1}),
$$

where $\tau_0 := 0$ and $\tau_{c+1} := +\infty$. Then $\psi_{i+1}|_{[\tau_k, \tau_{k+1})}$ is either strictly increasing or flat because $\psi_{i+1}$ is nondecreasing. If $\psi_{i+1}|_{[\tau_k, \tau_{k+1})}$ is strictly increasing, using the same arguments as for $w_1|_{[s_k, s_{k+1})}$, we see that $\psi_{i+1}(s) = \phi_{i+1}(s)/u_i^{p_i/p_{i+1}}(s)$ for $s \in [\tau_k, \tau_{k+1})$. It follows that

$$
\frac{s}{w_i^{p_i}(s)} = \frac{s}{\psi_{i+1}^{p_i}(s)} = \frac{s}{(\phi_{i+1}^{p_i}(s)/u_i^{p_i}(s))u_i^{p_i}(s)} = \frac{s}{\phi_{i+1}^{p_i}(s)},
$$

which is nondecreasing on $[\tau_k, \tau_{k+1})$ by (H3). Similarly, if $\psi_{i+1}|_{[\tau_k, \tau_{k+1})}$ is flat, then $\psi_{i+1}(s) \equiv \psi_{i+1}(\tau_k)$ for $s \in [\tau_k, \tau_{k+1})$, implying that

$$
\frac{s}{w_i^{p_i}(s)} = \frac{s}{\psi_{i+1}^{p_i}(s)} = \frac{s}{\phi_{i+1}^{p_i}(\tau_k)},
$$

the right hand side of which is nondecreasing by the inductive assumption that $w_i^{p_i}(s) \propto s$. Both cases prove (13) for $i = 1$ on $[\tau_k, \tau_{k+1})$, from which (13) follows for all $i = 1, ..., n$ by induction. \hfill \Box

The result of Lemma 5, i.e., $W_i(+\infty) = +\infty$, is required in the beginning of section 2. It guarantees that those lemmas given in section 2 is applicable with $W_i$, s defined in (11) by $w_i s$ and therefore by $\phi_i s$. In particular, by Lemma 2, the result $W_i(+\infty) = +\infty$ guarantees that $W_i^{-1} \circ J_i(C([t_0, +\infty), \mathbb{R}_+)) \subset C([t_0, +\infty), \mathbb{R}_+)$ for all $i = 1, ..., n$. So, we are ready to give the following main result.
**Theorem 1.** Suppose that (H1)-(H3) hold and \( u(t) \) satisfies (2) for \( t_0 \leq t < +\infty \). Then
\[
 u(t) \leq (W_n^{-1} \circ \tilde{J}_n) \circ \cdots \circ (W_1^{-1} \circ \tilde{J}_1)a(t), \quad t \in [t_0, +\infty),
\]
where \( \tilde{J}_i \) and \( W_i \) are defined in (4) and (11) respectively for \( i = 1, \ldots, n \).

Remark that, as indicated in [1], different choices of \( x_0 \) in \( W_i \) do not affect the result of Theorem 1. In fact, for positive constant \( y_0 \neq x_0 \), let
\[
 \tilde{W}_i(x) := \int_{y_0}^{x} \frac{ds}{w_i^{p_i}(s)}, \quad \tilde{J}_i x(t) := \tilde{W}_i(x(t)) + \{ \int_{t_0}^{b_i(t)} f_i(t, s) ds \}^{p_i}.
\]
Then \( \tilde{W}_i(x) = W_i(x) + \tilde{W}_i(x_0) \) and \( \tilde{W}_i^{-1}(y) = W_i^{-1}(y - \tilde{W}_i(x_0)) \). It follows that
\[
 (\tilde{W}_i^{-1} \circ \tilde{J}_i)x(t) = W_i^{-1} (\tilde{W}_i(x(t)) + \{ \int_{t_0}^{b_i(t)} f_i(t, s) ds \}^{p_i})
 = W_i^{-1} (W_i(x(t)) + \{ \int_{t_0}^{b_i(t)} f_i(t, s) ds \}^{p_i})
 = (W_i^{-1} \circ J_i)x(t)
\]
and \( \tilde{W}_i(\infty) = W_i(\infty) + \tilde{W}_i(x_0) = +\infty \). Hence, the result (16) is independent of the choice of \( x_0 > 0 \).

**Proof of Theorem 1.** First of all, we monotonize functions \( \phi_i \)s in (2). In view of [1, 22], we employ the procedure of powered strong monotonization, mentioned just before (11), to amplify the functions \( \phi_i \) \((i = 1, \ldots, n)\) into the functions \( v_i \) \((i = 1, \ldots, n)\) defined in (11). Then, if \( u \) satisfies inequality (2), we have
\[
 u(t) \leq a(T) + \sum_{i=1}^{n} \left\{ \int_{t_0}^{b_i(t)} f_i(T, s) w_i(u(s)) ds \right\}^{p_i}, \quad \forall t \leq T,
\]
for an arbitrarily chosen \( T \in [t_0, +\infty) \). In what follows, we solve \( u \) from (17), regarded as the auxiliary inequality of (2). Let
\[
 \tilde{J}_i x(T, t) := W_i(x(T, t)) + \{ \int_{t_0}^{b_i(t)} f_i(T, s) ds \}^{p_i}, \quad i = 1, \ldots, n,
\]
and define \( a_0(T, t) := a(T) \) and \( a_i(T, t) := (W_i^{-1} \circ \tilde{J}_i) \cdots (W_1^{-1} \circ \tilde{J}_1)a_0(T, t) \). We claim that
\[
 u(t) \leq (W_n^{-1} \circ \tilde{J}_n) \circ \cdots \circ (W_1^{-1} \circ \tilde{J}_1)a_0(T, t), \quad \forall t_0 \leq t \leq T.
\]
Clearly, for each nonnegative function \( x(T, t) \) on \( t \in [t_0, +\infty) \) we have \( W_i(0) \leq \tilde{J}_i x(T, t) < +\infty \). By Lemma 5, \( (W_i^{-1} \circ \tilde{J}_i)x(T, t) \in \mathbb{R}_+ \), implying that the right hand side of (19) is well defined for all \( t \in [t_0, +\infty) \) as in Lemma 2.

First, (19) holds for \( n = 1 \). In fact, (17) for \( n = 1 \) is written as
\[
 u(t) \leq a(T) + z(t),
\]
where \( z(t) := \left\{ \int_{t_0}^{b_1(t)} f_1(T, s) w_1(u(s)) ds \right\}^{p_1}, \) a nonnegative and nondecreasing function on \([t_0, +\infty)\). Then
\[
 z'(t) = p_1 \{ \int_{t_0}^{b_1(t)} f_1(T, s) w_1(u(s)) ds \}^{p_1-1} f_1(T, b_1(t)) w_1(u(b_1(t))) b_1'(t).
\]
In the integrand we have \(0 < w_1(u(s)) \leq w_1(z(s) + a(T)) \leq w_1(z(b_1(t)) + a(T)) \leq w_1(z(t) + a(T))\) by (10), (H3) and (20) since \(s \leq b_1(t) \leq t\) and both \(z\) and \(w_1\) are nondecreasing. Thus,

\[
\frac{(z(t) + a(T))'}{w_1'(z(t) + a(T))} \leq \left\{ \frac{p_1 w_1^{p_1-1}(z(b_1(t)) + a(T)) \left( \int_{t_0}^{b_1(t)} f_1(T, s) ds \right)^{p_1-1}}{w_1'(z(b_1(t)) + a(T))} \right\} f_1(T, b_1(t)) w_1(z(b_1(t)) + a(T)) b_1'(t) = p_1 \left( \int_{t_0}^{b_1(t)} f_1(T, s) ds \right)^{p_1-1} f_1(T, b_1(t)) b_1'(t).
\]

Integrating both sides of the above inequality from \(t_0\) to \(t\), we obtain

\[
W_1(z(t) + a(T)) \leq W_1(a(T)) + \int_{t_0}^{t} p_1 \left( \int_{t_0}^{b_1(T)} f_1(T, s) ds \right)^{p_1-1} d \left( \int_{t_0}^{b_1(T)} f_1(T, s) ds \right) = \tilde{J}_1 a_0(T, t), \quad \forall t \in [t_0, T],
\]

where we note that \(z(t_0) = 0\) because \(b_1(t_0) = t_0\) as indicated below (3). By (20) and the monotonicity of \(W_1^{-1}\) we get that \(u(t) \leq a(T) + z(t) \leq W_1^{-1}(\tilde{J}_1 a_0(T, t))\), i.e., (19) is true for \(n = 1\).

In order to prove (19) inductively, assume that (19) holds for \(n = m\). Let \(u\) satisfy the inequality

\[
u(t) \leq a(T) + \sum_{i=1}^{m+1} \int_{t_0}^{b_1(T)} f_i(T, s) w_i(u(s)) ds \right)^{p_i}, \quad \forall t \in [t_0, T].
\]

Then it can be written as

\[
u(t) \leq a(T) + \zeta(t), \tag{21}
\]

where \(\zeta(t) := \sum_{i=1}^{m+1} \int_{t_0}^{b_1(t)} f_i(T, s) w_i(u(s)) ds \right)^{p_i}\), a nonnegative and nondecreasing function on \([t_0, +\infty)\). One can compute the derivative

\[
\zeta'(t) = \sum_{i=1}^{m+1} p_i \int_{t_0}^{b_1(t)} f_i(T, s) w_i(u(s)) ds \right)^{p_i-1} f_i(T, b_1(t)) w_i(u(b_1(t))) b_1'(t),
\]

in the integrand of which we have

\[
0 < w_i(u(s)) \leq w_i(\zeta(s) + a(T)) \leq w_i(\zeta(b_i(t)) + a(T)) \leq w_i(\zeta(t) + a(T))
\]

by (10), (H3) and (21) since \(s \leq b_i(t) \leq t\) and all \(w_i\) and \(\zeta\) are nondecreasing for all \(i = 1, \ldots, m + 1\). Let

\[
\varphi_{i+1}(u) := \frac{w_{i+1}(u)}{w_i^{p_i/p_{i+1}}(u)}, \quad i = 1, \ldots, m.
\]

Then

\[
\frac{(\zeta(t) + a(T))'}{w_1'(\zeta(t) + a(T))}.
\]
we know that each $\varphi$ positive on $[0, T]$. Therefore, 

\[
\frac{p_1 \{ \int_0^{b_1(t)} f_1(T, s) w_1(\zeta(s) + a(T)) ds \}^{p_1-1}}{w_1^{p_1}(\zeta(b_1(t)) + a(T))} 
\cdot f_1(T, b_1(t)) w_1(\zeta(b_1(t)) + a(T)) b_1(t) 
\leq \left\{ \frac{p_1 w_1^{p_1-1}(\zeta(b_1(t)) + a(T)) \{ \int_0^{b_1(t)} f_1(T, s) ds \}^{p_1-1}}{w_1^{p_1}(\zeta(b_1(t)) + a(T))} \right\} 
\cdot f_1(T, b_1(t)) w_1(\zeta(b_1(t)) + a(T)) b_1(t) 
+ \sum_{i=1}^m p_i \{ \int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(\zeta(s) + a(T)) ds \}^{p_i-1} f_1(T, b_1(t)) b_1(t) 
+ \sum_{i=1}^m p_i \{ \int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(\zeta(s) + a(T)) ds \}^{p_i-1} \cdot f_i(T, b_1(t)) \varphi_{i+1}(\zeta(b_i(t)) + a(T)) b_i(t). 
\]

Integrating both sides of the above inequality from $t_0$ to $t$, we get 

\[
W_1(\zeta(t) + a(T)) \leq \mathcal{K} a(T) + \sum_{i=1}^m \int_{t_0}^t p_i \{ \int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(\zeta(s)) + a(T) ds \}^{p_i-1} d(\int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(\zeta(s) + a(T)) ds) 
= \mathcal{K} a(T) + \sum_{i=1}^m \{ \int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(\zeta(s) + a(T)) ds \}^{p_i} 
\]

for all $t \in [t_0, T]$. Let $\xi(t) := W_1(\zeta(t) + a(T))$. Then the above inequality can be rewritten as 

\[
\xi(t) \leq \mathcal{K} a(T) + \sum_{i=1}^m \{ \int_0^{b_i(t)} f_1(T, s) \varphi_{i+1}(W_1^{-1}(\xi(s))) ds \}^{p_i}. \quad (22) 
\]

Since 

\[
\varphi_{i+1}(W_1^{-1}(u)) = \left\{ \begin{array}{ll} 
\frac{w_i^{p_i+1}(W_1^{-1}(u))}{w_i^{p_i}(W_1^{-1}(u))} \end{array} \right\}^{p_i}, \quad i = 1, \ldots, m, 
\]

we know that each $\varphi_{i+1}(W_1^{-1}(u))$ $(i = 1, \ldots, m)$ is continuous, nondecreasing and positive on $[0, +\infty)$ and $\varphi_{i+1}(W_1^{-1}(u)) \propto \varphi_{i+1}(W_1^{-1}(u)) (i = 2, \ldots, m)$. It follows that (22) is of the same form as (17) for $n = m$ and fulfills the inductive assumption. Therefore, 

\[
\xi(t) \leq (\Phi_m^{-1} \circ T_m) \circ \cdots \circ (\Phi_1^{-1} \circ T_1) \gamma_0(T, t), \quad t \in [t_0, T], \quad (23) 
\]
where
\[ \gamma_0(T, t) := \tilde{J}_1 a_0(T, t), \]
\[ \gamma_i(T, t) := (\Phi_i^{-1} \circ T_i) \cdots (\Phi_1^{-1} \circ T_1) \gamma_0(T, t), \]
\[ T_i(x(T, t)) := \Phi_i(x(T, t)) + \left\{ \int_{a_i}^{a_{i+1}} f_{i+1}(T, s) ds \right\}^{p_{i+1}}, \quad i = 1, \ldots, m, \]
\[ \Phi_i(u) := \int_{T_i(u)}^{u} \frac{ds}{w_{i+1}^{p_{i+1}}(W_1^{-1}(s))} = \int_{T_i(u)}^{u} \frac{ds}{\tilde{w}_{i+1}^{p_{i+1}}(W_1^{-1}(s))} = \int_{u_{i+1}}^{W_{i+1}^{-1}(u)} \frac{ds}{w_{i+1}^{p_{i+1}}(s)} = W_{i+1}(W_1^{-1}(u)), \quad i = 1, \ldots, m. \]

Thus, we get from (21) and (23) that
\[ u(t) \leq \zeta(t) + a(T) = W_1^{-1}(\zeta(t)) \leq W_1^{-1} \circ \tilde{J}_m \circ (W_1^{-1}(\gamma_{m-1}(T, t))) \quad (24) \]
for \( t \in [t_0, T] \). In order to simplify (24), we claim that
\[ W_1^{-1}(\gamma_i(T, t)) = a_{i+1}(T, t) \]
for \( i = 1, \ldots, m \). It is easy to verify that
\[ W_1^{-1}(\gamma_0(T, t)) = W_1^{-1} \circ \tilde{J}_1 a_0(T, t) = a_1(T, t), \]
i.e., the claim is true for \( i = 0 \). Assume that the claim holds for \( i = k \). Then by the inductive assumption,
\[ W_1^{-1}(\gamma_{k+1}(T, t)) = W_1^{-1}(\Phi_{k+1}^{-1} \circ T_{k+1}(\gamma_k(T, t))) \]
\[ = W_{k+2}^{-1}(\tilde{J}_{k+2} \circ W_1^{-1}(\gamma_k(T, t))) \]
\[ = W_{k+2}^{-1}(\tilde{J}_{k+2} a_{k+1}(T, t)) = a_{k+2}(T, t), \]
which proves the claim by induction. It follows from (24) and the claim that
\[ u(t) \leq (W_1^{-1} \circ \tilde{J}_m) \cdots (W_1^{-1} \circ \tilde{J}_1) a_0(T, t) \]
for all \( t \in [t_0, T] \). Thus, the assertion (19) is proved by induction.

Finally, we observe (19) and (18) and that \( t \leq T \). Letting \( t = T \), we have
\[ u(T) \leq (W_n^{-1} \circ \tilde{J}_n) \cdots (W_1^{-1} \circ \tilde{J}_1) a_0(T, T) \]
\[ = (W_n^{-1} \circ \tilde{J}_n) \cdots (W_1^{-1} \circ \tilde{J}_1) a(T), \quad \forall T \in [t_0, +\infty). \]
Since \( T \) is chosen arbitrarily, (16) is proved and the proof of Theorem 1 is completed.

Our Theorem 1 improves some results obtained in [1, 6, 22, 26]. Look at [1] for instance, where the unknown function \( u \) is estimated by
\[ u(t) \leq W_n^{-1}\{W_n(a_n(t)) + \int_{b_n(t_0)}^{b_n(t)} \max_{t_0 \leq \tau \leq t} f_n(\tau, s) ds\}, \]
but \( a_n(t) \) is determined recursively by
\[ a_1(t) := a(t_0) + \int_{t_0}^{t} |a'(\tau)| d\tau, \]
\[ a_{i+1}(t) := W_i^{-1}\{W_i(a_i(t)) + \int_{b_i(t_0)}^{b_i(t)} \max_{t_0 \leq \tau \leq t} f_i(\tau, s) ds\}, \quad i = 1, \ldots, n - 1. \]
In contrast, the result (16) of our Theorem 1, i.e., \( u(t) \leq (W_n^{-1} \circ \tilde{J}_n) \cdots (W_1^{-1} \circ \tilde{J}_1) a(t) \), is simply dependent on \( a(t) \). One can use Lemmas 2-4, properties of those
operators $J_i$, to give boundedness, asymptotics and dependence for the unknown $u$, as done in section 6 for a stochastic differential equation. Besides, [1] gives the existence interval $[t_0, T_1]$ for the unknown $u$, where $T_1$ is determined by

$$W_i(a_i(T_1)) + \int_{a_i(t_0)}^{b_i(T_1)} \max_{t_0 \leq \tau \leq T_1} f_i(\tau, s) ds \leq \int_{a_i}^{+\infty} \frac{dz}{w_i(z)}, \quad i = 1, \ldots, n,$$

but in contrast our existence interval obtained in Theorem 1 is $[t_0, +\infty)$.

4. Application to singular inequality. Our Theorem 1 can be employed to discuss the generalized singular integral inequality

$$u(t) \leq \hat{a}(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \alpha_i(t, s) f_i(t, s) \hat{\phi}_i(u(s)) ds$$ (25)

for $t_0 \leq t < +\infty$, where $n \in \mathbb{N}$ and functions $\hat{a}$, $\alpha_i$, $f_i$, $\hat{\phi}_i$ and $\hat{u}$ are all nonnegative. Suppose that for each $i = 1, \ldots, n$,

(S1): $\hat{a} \in C([t_0, +\infty), \mathbb{R}_+)$ and $f_i(t, s) \in C([t_0, +\infty) \times [t_0, +\infty), \mathbb{R}_+)$,

(S2): $\hat{\phi}_i \in C(\mathbb{R}_+, \mathbb{R}_+ \setminus \{0\})$ such that $\hat{\phi}_i(s) \propto s$, and

(S3): there exists a constant $p_i > 1$ such that $\alpha_i(t, s)$ is uniformly $L^{p_i}$ integrable on $(t_0, t)$ with respect to $t \in [t_0, +\infty)$, i.e.,

$$G_i(t) := \int_{t_0}^{t} \alpha_i^p(t, s) ds \in C([t_0, +\infty), \mathbb{R}_+).$$

Obviously, the assumption of $\alpha_i(t, s)$ allows the integral in (25) to have a weakly singular kernel such as $(t-s)^{\gamma-1}$ for $\gamma > 0$, which was considered in [10, 15, 27], and $s^{\alpha-1}(t^\beta - s^\beta)^{\gamma-1}$ for $\alpha, \beta, \gamma > 0$, which was discussed in [6, 14, 21]. In fact, for $\gamma > 0$, there exists $p > 1$ such that $p(\gamma - 1) > 1$. It follows that

$$\int_{0}^{t} (t-s)^{p(\gamma-1)} ds = \frac{t^{1+p(\gamma-1)}}{1+p(\gamma-1)} \in C(\mathbb{R}_+, \mathbb{R}_+),$$

implying that the choice $\alpha_i(t, s) = (t-s)^{\gamma-1}$ fulfills (S3). For the same reason, for $[\beta, \gamma, \alpha]$ to be a parameter distribution of type I or II referred to [14], there exists $p > 1$ such that $\theta := p(\beta(\gamma - 1) + \alpha - 1) + 1 \geq 0$. It follows that

$$\int_{0}^{t} s^{p(\alpha-1)}(t^\beta - s^\beta)^{p(\gamma-1)} ds = \frac{t^{\theta}}{\beta} B\left(\frac{p(\alpha - 1)}{\beta} + 1, p(\gamma - 1) + 1\right) \in C(\mathbb{R}_+, \mathbb{R}_+),$$

where $B(\xi, \eta) := \int_{0}^{1} (1-s)^{\eta-1} ds$ is the well-known Beta function. It implies that the choice $\alpha_i(t, s) = \gamma s^{\alpha-1}(t^\beta - s^\beta)^{\gamma-1}$ also fulfills (S3).

Before stating our result, we introduce some notations:

$$\tilde{w}_i(s) := \begin{cases} \max_{\tau \in [0, s]} \left\{ \phi_1^{q_i}(\tau^{1/\lambda}) \right\}, \quad i = 1, \\ \max_{\tau \in [0, s]} \left\{ \phi_i^{q_i}(\tau^{1/\lambda}) \right\} \tilde{w}_i(q_i^{-1} - 1)(s), \quad i = 2, \ldots, n, \end{cases}$$

$$W_i(x) := \int_{x_0}^{x} \frac{ds}{\tilde{w}_i^{x}(s)}, \quad i = 1, \ldots, n,$$

$$\tilde{J}_i(x(t)) := \tilde{W}_i(x(t)) + (2n)^{\lambda - 1} \left( \int_{t_0}^{t} \max_{t_0 \leq \tau \leq t} G_i^{\frac{\tilde{w}}{q_i}}(\tau)f_i^{q_i}(\tau, s) ds \right)^{\frac{1}{q_i}},$$

where $x_0 > 0$ is a given constant, $\lambda \geq \max_{i=1,\ldots,n} (q_i)$ is also a given constant, and $q_i > 1$ is the dual constant of $p_i$, which satisfies $1/p_i + 1/q_i = 1$. Those notations are well defined as functions $\tilde{a}$, $\alpha_i$, $s$, $f_i$ and $\phi_i$ are given.
Corollary 1. Suppose that (S1)-(S3) hold. Then the unknown function $u(t)$ of (25) is estimated as

$$u(t) \leq \{(\tilde{W}_n^{-1} \circ \tilde{J}_n) \circ \cdots \circ (\tilde{W}_1^{-1} \circ \tilde{J}_1)\}^{1/(2^{\lambda-1} \max_{t_0 \leq \tau \leq t} \hat{a}^\lambda(\tau))}$$

for all $t \in [t_0, +\infty)$.

Proof. Applying Hölder’s Inequality (see [9] or the Appendix) to (25), we obtain

$$u(t) \leq \hat{a}(t) + \sum_{i=1}^{n} G_i^{\frac{1}{\lambda}}(t)(\int_{t_0}^{t} \tilde{f}_i^{\mu}(t, s) \tilde{\phi}_i^{\mu}(u(s))ds)^{\frac{1}{\lambda}},$$

where $G_i(t) = \int_{t_0}^{t} \alpha_i^{\mu}(t, s)ds \in C([t_0, +\infty), \mathbb{R}_+)$ by (S3). Then, giving a power $\lambda$ to both sides of (27) and using the Power Mean Inequality (see [9] or the Appendix), we obtain

$$u^\lambda(t) \leq 2^{\lambda-1} \hat{a}^\lambda(t) + (2n)^{\lambda-1} \sum_{i=1}^{n} G_i^{\frac{1}{\lambda}}(t)(\int_{t_0}^{t} \tilde{f}_i^{\mu}(t, s) \tilde{\phi}_i^{\mu}(u(s))ds)^{\frac{\lambda}{\lambda}},$$

that is, the function $v(t) := u^\lambda(t)$ satisfies the powered integral inequality (2), where

$$a(t) := 2^{\lambda-1} \max_{t_0 \leq \tau \leq t} \hat{a}^\lambda(\tau), \quad \phi_i(s) := \tilde{\phi}_i^{\mu}(s^{\frac{1}{\lambda}}),$$

$$f_i(t, s) := (2n)^{\frac{(\lambda-1)\mu}{\lambda}} \max_{t_0 \leq \tau \leq t} G_i^{\frac{\mu}{\lambda}}(\tau) \tilde{f}_i^{\mu}(\tau, s).$$

One can verify that $a$ and $f_i$s satisfy (H1) by (S1) and (S3). In addition, $\phi_i$s satisfy (H3) by (S2). In fact, by (S2) we see that $s/\tilde{\phi}_i(s)$ is nondecreasing, implying that $s^{1/\lambda}/\tilde{\phi}_i(s^{1/\lambda})$ is also nondecreasing. Since

$$\frac{s}{(\phi_i^{\mu}(s^{1/\lambda}))^{\lambda/q_i}} = \frac{s}{\phi_i^{\lambda}(s^{\frac{1}{\lambda}})} = \frac{(s^{1/\lambda})^{\lambda}}{\phi_i^{\lambda}(s^{1/\lambda})},$$

the function on the left hand side of the above is also nondecreasing. It follows by (29) that $\phi_i^{\lambda/q_i}(s) \propto s$, i.e., (H3) holds for inequality (28). By Theorem 1 we obtain

$$v(t) \leq \{(\tilde{W}_n^{-1} \circ \tilde{J}_n) \circ \cdots \circ (\tilde{W}_1^{-1} \circ \tilde{J}_1)a(t),$$

which implies the estimate (26) by (29), that is, Corollary 1 is proved. \qed

Remark that the method used in [5, 6, 14, 15, 21] for a single power does not work in our case. In fact, when $p_i \equiv p$ for $i = 1, \ldots, n$, the inequality (27) is of a single power, i.e., $q_i \equiv q$ for $i = 1, \ldots, n$, which includes those powered inequalities considered in [5, 6, 14, 15, 21]. In this case, $\lambda = q$ in (28). Thus, the power $q$ outside the integrals in (27) is eliminated and the method of monotonization used in [1, 22] is applicable. However, this method is not efficient to our case of different powers.

5. Application to stochastic inequality. The development of stochastic differential equations demands results on stochastic integral inequalities (see e.g. [2, 11, 23]). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a nonempty set $\Omega$, a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ consisting of subsets of $\Omega$, and a probability measure $P$. A random variable $X(\omega)$ on $(\Omega, \mathcal{F}, P)$ is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$, i.e., $X^{-1}(U) := \{\omega \in \Omega | X(\omega) \in U\} \in \mathcal{F}$ for each Borel set $U \subset \mathbb{R}^n$. The expectation of a random variable $X(\omega)$ is defined by the integral

$$EX(\omega) := \int_{\Omega} X(\omega)dP(\omega)$$
provided that \( \int_{\Omega} |X(\omega)| dP(\omega) < +\infty \). Let \( \{ B_t(\omega) : t \geq 0 \} \) be a 1-dimensional standard Brownian motion in \( \Omega \). Suppose that \( \alpha(\omega) \) is a random variable independent of the \( \sigma \)-algebra generated by all \( B_s \) \((s \leq t)\) and satisfies that \( E[\alpha(\omega)^2] < +\infty \), \( F_t \) is the \( \sigma \)-algebra generated by all \( B_s \) \((0 \leq s \leq t)\) and \( \alpha(\omega) \), and \( M^2_{\omega}[t_0, T] \) consists of all measurable and \( F_t \)-adapted stochastic processes \( G \) such that \( \int_{t_0}^T EG^2(s, \omega) ds < +\infty \). In 1946 Itô ([11]) considered the stochastic integral inequality system

\[
\begin{aligned}
|u(t, \omega)| &\leq |\alpha(\omega)| + \beta \int_{t_0}^t |u(t, \omega)| ds + \int_{t_0}^t G(s, \omega) dB_s, \\
|G(t, \omega)| &\leq \gamma |u(t, \omega)|, \quad t \in [t_0, T],
\end{aligned}
\]

(30)

where \( \beta, \gamma > 0 \) are constant and \( G \in M^2_{\omega}[t_0, T] \). He obtained the estimate of mean square \( u(t, \omega) \). It is worthy mentioning that, unlike deterministic ones, one cannot amplify the integrand \( G \) in the first inequality of (30) with the second inequality of (30) because the Itô integral \( \int_{t_0}^t G(s, \omega) dB_s \), being a stochastic process, does not satisfy the triangle inequality with respect to the absolute value, as indicated in many monographs e.g. [7, 17]. In 2005 Amano ([2]) discussed an extension of (30), where \( \alpha(\omega) \) was replaced by a given stochastic process.

In this section we consider the inequality

\[
\begin{aligned}
|u(t, \omega)| &\leq |\hat{a}(t, \omega)| + \beta \int_{b(t_0)}^{b(t)} \hat{f}_i(t, s) \hat{\phi}_i(|u(s, \omega)|) ds \\
&\quad + \sum_{j=1}^n \int_{c_j(t_0)}^{c_j(t)} \hat{G}_j(t, s, \omega) dB_s, \\
|\hat{G}_j(t, s, \omega)| &\leq \hat{g}_j(t, s) \hat{\psi}_j(|u(s, \omega)|), \quad t \in [t_0, T],
\end{aligned}
\]

(31)

a generalization of both (30) and the one discussed in [2], where \( t_0 \leq T < +\infty \), \( n, m \in \mathbb{N} \) and all \( b_i, c_j, \hat{f}_i, \hat{g}_j, \hat{\phi}_i \) and \( \hat{\psi}_j \) are nonnegative functions for \( i = 1, \ldots, m; j = 1, \ldots, n \). For all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), suppose that

(A1): \( \hat{a}, \hat{G}_j(t, \cdot, \cdot) \in M^2_{\omega}[t_0, T] \) and \( \hat{a}(t, \cdot) \) is nondecreasing,

(A2): \( b_i, c_j \in C^1([t_0, T], [t_0, T]) \) are nondecreasing and \( b_i(t), c_j(t) \leq t \),

(A3): \( \hat{f}_i(t, s), \hat{g}_j(t, s) \in C([t_0, T] \times [t_0, T], \mathbb{R}_+) \) are both nondecreasing with respect to \( t \), and

(A4): \( \hat{\phi}_i, \hat{\psi}_j \in C^1(\mathbb{R}_+, \mathbb{R}_+ \setminus \{0\}) \) satisfy \( \hat{\phi}_i(s) \propto s \) and \( \hat{\psi}_j(s) \propto s \).

Note that if \( \hat{a}(t, \cdot), \hat{f}_i(t, \cdot) \) or \( \hat{g}_j(t, \cdot) \) is not nondecreasing in the above (A1) and (A3) then we can monotonize them as in [1].

Before we state our result, we need the notations

\[
\hat{\phi}_i(s) := \hat{\phi}_i^2(\sqrt{s}), \quad \hat{\psi}_j(s) := \hat{\psi}_j^2(\sqrt{s}), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.
\]

(32)

For amplification in inequalities, we need to modify \( \hat{\phi}_i \) and \( \hat{\psi}_j \) to be concave functions, which are defined by

\[
\hat{\phi}_i(s) := \hat{\phi}_i(0) + \int_0^s \sup_{\mu < \eta < +\infty} \frac{\hat{\phi}'_i(\eta)}{\mu} d\mu, \quad i = 1, \ldots, m,
\]

(33)

\[
\hat{\psi}_j(s) := \hat{\psi}_j(0) + \int_0^s \sup_{\mu < \eta < +\infty} \frac{\hat{\psi}'_j(\eta)}{\mu} d\mu, \quad j = 1, \ldots, n.
\]

(34)

Obviously, \( \hat{\phi}_i(s) \leq \hat{\phi}_i(s) \leq \hat{\psi}_j(s) \) and both \( \hat{\phi}_i \) and \( \hat{\psi}_j \) are concave, \( i = 1, \ldots, m, \quad j = 1, \ldots, n \), because \( \hat{\phi}_i(s) = \sup_{s < \eta < +\infty} \frac{\hat{\phi}'_i(\eta)}{\eta} \) and \( \hat{\psi}_j(s) = \sup_{s < \eta < +\infty} \frac{\hat{\psi}'_j(\eta)}{\eta} \) and the function of sup is nonincreasing, which implies that \( \hat{\phi}_i(s) \) and \( \hat{\psi}_j(s) \) are both nonincreasing. The above modification procedure for \( \hat{\phi}_i \) and \( \hat{\psi}_j \) can be regarded
as a concavification of $\phi$ and $\psi$. From (33) or (34) we see that function $\phi$ is concavifiable if $\sup_{s, \eta < +\infty} \phi'(\eta)$ is integrable on $[0, s)$ for each $s > 0$, i.e., on each closed subset $[u, s] \subset (0, s]$, the limit

$$\lim_{u \to 0^+} \int_u^s \sup_{\mu < \eta < +\infty} \phi'(\eta) d\mu$$

exists. Clearly, if either $\phi'(s)$ is bounded on $\mathbb{R} \setminus \{0\}$ or, in a weaker sense,

$$| \sup_{s, \eta < +\infty} \phi'(\eta) | \leq \frac{C}{s^p}$$

for all $s \in \mathbb{R} \setminus \{0\}$, where $C > 0$ and $p < 1$ are constants, then $\phi$ is concavifiable on $\mathbb{R} \setminus \{0\}$.

Further, define

$$\hat{W}_i(x) := \int_{x_0}^x \frac{ds}{\hat{w}_i(s)}, \ i = 1, \ldots, m + n,$$

$$\mathcal{J}_i(x(t)) := \begin{cases} \hat{W}_i(x(t)) + 3m \int_{t_0}^{b_i(t)} \hat{f}_i(t, s) ds^2, & i = 1, \ldots, m, \\ \hat{W}_i(x(t)) + 3n \int_{t_0}^{c_i(t)} \hat{g}_{i-1}(t, s) ds, & i = m + 1, \ldots, m + n \end{cases}$$

for a given constant $x_0 > 0$, where

$$\hat{w}_i(s) := \begin{cases} \max_{\tau \in [0, s]} \{ \hat{\phi}_i(\tau) \}, & i = 1, \\ \max_{\tau \in [0, s]} \{ \hat{\phi}_i(\tau) \hat{w}_{i-1}(\tau) \} \hat{w}_{i-1}(s), & i = 2, \ldots, m, \\ \max_{\tau \in [0, s]} \{ \hat{\phi}_i(\tau) \hat{w}_{i-1}(\tau) \} \hat{w}_{i-1}(s), & i = m + 1, \ldots, m + n. \end{cases}$$

Corollary 2. Suppose that (A1)-(A4) hold and $u(t, \omega)$ satisfies (31). Then the mean square $EU^2(t, \omega)$ has the estimate

$$EU^2(t, \omega) \leq (\hat{W}_{m+n}^{-1} \circ \mathcal{J}_{m+n}) \circ \cdots \circ (\hat{W}_1^{-1} \circ \mathcal{J}_1)(3E\hat{a}^2(t, \omega))$$

(36)

for all $t \in [t_0, T]$.

Proof. Taking a mean square on both sides of the first inequality of (31) and applying the Power Mean Inequality (see [9] or the Appendix), we obtain

$${\text{Eu}}^2(t, \omega) \leq E \left\{ |\hat{a}(t, \omega)| + \sum_{i=1}^m \int_{t_0}^{b_i(t)} \hat{f}_i(t, s) \hat{\phi}_i(|u(s, \omega)|) ds + \sum_{j=1}^n \int_{t_0}^{c_j(t)} \hat{G}_j(t, s, \omega) dB_s \right\}^2$$

$$\leq E \left\{ 3\hat{a}^2(t, \omega) + 3 \left( \sum_{i=1}^m \int_{t_0}^{b_i(t)} \hat{f}_i(t, s) \hat{\phi}_i(|u(s, \omega)|) ds \right)^2 \\ + 3 \left( \sum_{j=1}^n \int_{t_0}^{c_j(t)} \hat{G}_j(t, s, \omega) dB_s \right)^2 \right\}$$

$$\leq 3E\hat{a}^2(t, \omega) + 3m \sum_{i=1}^m E \left( \int_{t_0}^{b_i(t)} \hat{f}_i(t, s) \hat{\phi}_i(|u(s, \omega)|) ds \right)^2 \\ + 3n \sum_{j=1}^n E \left( \int_{t_0}^{c_j(t)} \hat{G}_j(t, s, \omega) dB_s \right)^2.$$

(37)
Note that
\[
E\left(\int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) \tilde{\phi}_i(|u(s, \omega)|) ds\right)^2
\]
\[
\leq E\left(\int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) ds \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) \tilde{\phi}_i^2(|u(s, \omega)|) ds\right)
\]
\[
\leq \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) ds \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) E\tilde{\phi}_i^2(|u(s, \omega)|) ds
\]  
(38)
by Hölder’s Inequality (see [9] and the Appendix). Using the Itô isometry ([17, p.29]) and the second inequality of (31), we obtain
\[
E\left|\int_{t_0}^{c_j(t)} \tilde{G}_j(t, s, \omega) dB_s\right|^2 = \int_{t_0}^{c_j(t)} E\tilde{G}_j^2(t, s, \omega) ds
\]
\[
\leq \int_{t_0}^{c_j(t)} \tilde{g}_j^2(t, s) E\tilde{\psi}_j^2(|u(s, \omega)|) ds.
\]  
(39)
Thus, substituting (38) and (39) into (37), we get
\[
Eu^2(t, \omega) \leq 3E\tilde{u}^2(t, \omega) + 3m \sum_{i=1}^{m} \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) ds \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) E\tilde{\phi}_i(u^2(s, \omega)) ds
\]
\[
+ 3n \sum_{j=1}^{n} \int_{t_0}^{c_j(t)} \tilde{g}_j^2(t, s) E\tilde{\psi}_j(u^2(s, \omega)) ds, \]
(40)
where \(\tilde{\phi}_i\) and \(\tilde{\psi}_j\) are defined in (32). We concavify the functions \(\tilde{\phi}_i\) and \(\tilde{\psi}_j\) to be \(\hat{\phi}_i\) and \(\hat{\psi}_j\) as in (33) and (34) respectively. Then both \(-\hat{\phi}_i\) and \(-\hat{\psi}_j\) are convex. By Jensen’s Inequality (see [9] or the Appendix) and the inequalities just below (34), we have
\[
E\hat{\phi}_i(u^2) \leq E\tilde{\phi}_i(u^2) \leq \hat{\phi}_i(\E u^2), \ E\hat{\psi}_j(u^2) \leq E\tilde{\psi}_j(u^2) \leq \hat{\psi}_j(\E u^2).
\]  
(41)
Substituting (41) into (40), we get
\[
Eu^2(t, \omega) \leq 3E\tilde{u}^2(t, \omega) + 3m \sum_{i=1}^{m} \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) ds \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) \hat{\phi}_i(\E u^2(s, \omega)) ds
\]
\[
+ 3n \sum_{j=1}^{n} \int_{t_0}^{c_j(t)} \tilde{g}_j^2(t, s) \hat{\psi}_j(\E u^2(s, \omega)) ds, \ t \in [t_0, T],
\]  
(42)
that is, the function \(v(t) := Eu^2(t, \omega)\) satisfies the powered integral inequality (2), where
\[
a(t) := 3E\tilde{u}^2(t, \omega), \ \phi_i(s) := \begin{cases} \tilde{\phi}_i(s), & i = 1, \ldots, m, \\ \tilde{\psi}_{i-m}(s), & i = m + 1, \ldots, m+n, \end{cases}
\]
\[
f_i(t, s) := \begin{cases} 3m \int_{t_0}^{b_i(t)} \tilde{f}_i(t, s) ds \tilde{f}_i(t, s), & i = 1, \ldots, m, \\ 3n \tilde{g}_i^2(t, s), & i = m + 1, \ldots, m+n. \end{cases}
\]
One can verify that \(a\) and \(f_i\) satisfy (H1) for \(t, s \in [t_0, T]\) by (A1)-(A3). Clearly, (H2) follows (A2). In order to verify (H3), we need to prove
\[
\hat{W}_i(+\infty) = +\infty, \ \forall i = 1, \ldots, m+n.
\]  
(43)
We first claim that
\[
\limsup_{s \to +\infty} \varphi_i'(s) < +\infty, \quad \limsup_{s \to +\infty} \psi_j'(s) < +\infty. \tag{44}
\]
In fact, \(\hat{\varphi}_i, \hat{\psi}_j \in C^1(\mathbb{R}_+, \mathbb{R}_+\setminus\{0\})\) as known in (A4), which implies that \(\varphi_i, \psi_j \in C^1(\mathbb{R}_+, \mathbb{R}_+\setminus\{0\})\) and further implies that \(\varphi_i', \psi_j' \in C(\mathbb{R}_\cup\{0\}, \mathbb{R})\). Moreover, \(\hat{\varphi}_i(s)/s\) is nonincreasing, as known in (A4). It follows that
\[
\hat{\varphi}_i'(s) s - \hat{\varphi}_i(s) s^2 = \left(\frac{\hat{\varphi}_i(s)}{s}\right)' \leq 0,
\]
which gives \(\hat{\varphi}_i'(s) \leq \hat{\varphi}_i(s)/s\). Thus, we have
\[
\varphi_i'(s) s = \frac{\hat{\phi}_i(\sqrt{s})}{\sqrt{s}} \frac{\hat{\phi}_i'(\sqrt{s})}{\sqrt{s}} \leq \left(\frac{\hat{\phi}_i(\sqrt{s})}{\sqrt{s}}\right)^2,
\]
where the definition (32) of \(\varphi_i(s)\) is used. Since \(\hat{\phi}_i(\sqrt{s})/\sqrt{s}\) is nonincreasing by (A4), we see from (45) that
\[
\limsup_{s \to +\infty} \varphi_i'(s) s \leq \limsup_{s \to +\infty} \left(\frac{\hat{\phi}_i(\sqrt{s})}{\sqrt{s}}\right)^2 < +\infty,
\]
which proves the first inequality of claimed (44). Similarly, we can prove the second inequality and complete the proof of the claimed (44).

Next, we need the following lemma but leave its proof to the end of this section.

**Lemma 6.** \(\lim_{s \to +\infty} \hat{w}_i(s)/s = C_i, \quad i = 1, ..., m + n, \) where \(C_i\) s are nonnegative constants.

By Lemma 6, there exists \(x_1 \geq x_0\) such that \(\hat{w}_i(s) \leq (C_i + 1)s\) for all \(s \geq x_1\). Then
\[
\hat{W}_i(\infty) = \int_{x_0}^{x_1} \frac{ds}{\hat{w}_i(s)} + \int_{x_1}^{+\infty} \frac{ds}{\hat{w}_i(s)} \geq \int_{x_0}^{x_1} \frac{ds}{\hat{w}_i(s)} + \int_{x_1}^{+\infty} \frac{ds}{(C_i + 1)s} = +\infty,
\]
implying that the assertion (43) is true.

Therefore, applying Theorem 1 to (42), we obtain (36), which proves Corollary 2.

We end this section with the following proof.

**Proof of Lemma 6.** Since \(\hat{w}_i\) s are defined inductively in (35), we first prove Lemma 6 for \(i = 1, 2\). We need to discuss for bounded \(\hat{\varphi}_1\) and unbounded \(\hat{\varphi}_1\). If \(\hat{\varphi}_1\) is bounded, then there is a constant \(M > 0\) such that \(\hat{\varphi}_1(s) \leq M\) for all \(s \geq 0\). By (35) we have
\[
0 < \hat{w}_1(s) = \max_{\tau \in [0, s]} \hat{\varphi}_1(\tau) \leq M.
\]
It follows that
\[
0 \leq \limsup_{s \to +\infty} \frac{\hat{w}_1(s)}{s} \leq \limsup_{s \to +\infty} \frac{M}{s} = 0. \tag{46}
\]
If \( \tilde{\phi}_1 \) is unbounded, then \( \tilde{\phi}'_1 \geq 0 \); otherwise, by (33) the derivative given below there exists \( s_1 > 0 \) such that \( \sup_{s_1 < \mu < +\infty} \tilde{\phi}'_1(\mu) < 0 \), which implies that

\[
\tilde{\phi}_1(s) = \tilde{\phi}_1(0) + \int_0^{s_1} \sup_{\mu < \eta < +\infty} \tilde{\phi}'_1(\eta) d\mu + \int_{s_1}^{s} \sup_{\mu < \eta < +\infty} \tilde{\phi}'_1(\eta) d\mu < \tilde{\phi}_1(0) + \int_0^{s_1} \sup_{\mu < \eta < +\infty} \tilde{\phi}'_1(\eta) d\mu < +\infty,
\]
a contradiction to the unboundedness. Thus, by the L'Hôpital rule, we obtain

\[
0 \leq \lim_{s \to +\infty} \frac{\tilde{w}_1(s)}{s} = \lim_{s \to +\infty} \frac{\sup_{\tau \in [0,s]} \tilde{\phi}_1(\tau)}{s} = \lim_{s \to +\infty} \frac{\tilde{\phi}_1(s)}{s} = \lim_{s \to +\infty} \sup_{s < \mu < +\infty} \tilde{\phi}_1(\mu) =: C_1 < +\infty,
\]
where (33) and (44) are employed. Thus, (46) and (47) prove the result of Lemma 6 for \( i = 1 \).

For an inductive proof, we assume that \( \lim_{s \to +\infty} \tilde{w}_k(s)/s = C_k \geq 0 \). Let

\[
\rho_{k+1}(s) := \left\{ \begin{array}{ll}
\tilde{\phi}_{k+1}(s)/\tilde{w}_k(s), & k = 1, \ldots, m - 1, \\
\tilde{\psi}_{k+1-m}(s)/\tilde{w}_k(s), & k = m, \ldots, m + n - 1
\end{array} \right.
\]
in (35). Without loss of generality, we only consider \( 1 \leq k \leq m - 1 \), and the case of \( m \leq k \leq m + n - 1 \) can be proved similarly. We discuss for bounded \( \rho_{k+1} \) and unbounded \( \rho_{k+1} \). If \( \rho_{k+1} \) is bounded, there exists a constant \( N > 0 \) such that \( \lim_{s \to +\infty} \sup_{\tau \in [0,s]} \rho_{k+1}(\tau) = N \) by the monotonicity of \( \max_{\tau \in [0,s]} \rho_{k+1}(\tau) \). It follows from (35) and the inductive assumption that

\[
\lim_{s \to +\infty} \frac{\tilde{w}_{k+1}(s)}{s} = \lim_{s \to +\infty} \sup_{\tau \in [0,s]} \rho_{k+1}(\tau) \frac{\tilde{w}_k(s)}{s} = NC_k =: C_{k+1}.
\]
If \( \rho_{k+1} \) is unbounded, then for each \( l = 1, 2, \ldots \) there exists \( s_l \in [0,l] \) such that

\[
\rho_{k+1}(s_l) := \max_{\tau \in [0,l]} \rho_{k+1}(\tau)
\]
by the continuity. Without loss of generality, let \( s_l \) denote the greatest one satisfying (48). Then \( s_l \leq s_{l+1} \) because

\[
\max_{\tau \in [0,l+1]} \rho_{k+1}(\tau) = \max \{ \max_{\tau \in [0,l]} \rho_{k+1}(\tau), \max_{\tau \in (l,l+1]} \rho_{k+1}(\tau) \},
\]
implies that \( s_{l+1} = s_l \) if \( \max_{\tau \in (l,l+1]} \rho_{k+1}(\tau) < \max_{\tau \in [0,l]} \rho_{k+1}(\tau) \) or that \( s_{l+1} > l \) if \( \max_{\tau \in (l,l+1]} \rho_{k+1}(\tau) \geq \max_{\tau \in [0,l]} \rho_{k+1}(\tau) \). Moreover, \( \lim_{l \to +\infty} s_l = +\infty \); otherwise, \( \lim_{l \to +\infty} s_l = C_1 \), a constant, which implies that \( \lim_{l \to +\infty} \max_{\tau \in [0,l]} \rho_{k+1}(\tau) = \lim_{l \to +\infty} \rho_{k+1}(s_l) < +\infty \), a contradiction to the assumption that \( \rho_{k+1} \) is unbounded. Thus, by (48), the definition of \( \rho_{k+1} \) and the inductive assumption, we get

\[
\lim_{s \to +\infty} \frac{\tilde{w}_{k+1}(s)}{s} = \lim_{s \to +\infty} \max_{\tau \in [0,s]} \rho_{k+1}(\tau) \frac{\tilde{w}_k(s)}{s} = \lim_{l \to +\infty} \max_{\tau \in [0,l]} \rho_{k+1}(\tau) \frac{\tilde{w}_k(s_l)}{s_l} = \lim_{l \to +\infty} \rho_{k+1}(s_l) \frac{\tilde{w}_k(s_l)}{s_l}.
\]
On the other hand, if $\tilde{\phi}_{k+1}$ is bounded, we obtain
\[
\lim_{s \to +\infty} \frac{\tilde{\phi}_{k+1}(s)}{s} = 0;
\]
if $\tilde{\phi}_{k+1}$ is unbounded, by the L'Hôpital rule, (33) and (44), we obtain
\[
\lim_{s \to +\infty} \frac{\tilde{\phi}_{k+1}(s)}{s} = \lim_{s \to +\infty} \frac{\tilde{\phi}'_{k+1}(s)}{s} = \lim_{s \to +\infty} \sup_{s < \mu < +\infty} \tilde{\phi}'_{k+1}(\mu) = \lim_{s \to +\infty} \sup_{s < \mu < +\infty} \tilde{\phi}'_{k+1}(\mu) = C_{k+1} < +\infty.
\]
It follows from (49) that
\[
0 \leq \lim_{s \to +\infty} \frac{\tilde{\phi}_{k+1}(s)}{s} = \lim_{i \to +\infty} \frac{\tilde{\phi}_{k+1}(s_i)}{s_i} = \lim_{s \to +\infty} \frac{\tilde{\phi}_{k+1}(s)}{s} = C_{k+1},
\]
which shows that the result of Lemma 6 is true for $i = k + 1$. Thus, Lemma 6 is proved by induction.

6. Applications to a stochastic equation. In this section we end this paper with applications to the stochastic differential equation
\[
\begin{aligned}
&dx(t, \omega) = F(t, x(t, \omega))dt + G(t, x(t, \omega))dB_t, \\
x(0, \omega) = \alpha(\omega), \\
&0 \leq t \leq T,
\end{aligned}
\]
where $F, G \in C_b([0, T] \times \mathbb{R}, \mathbb{R})$, the set of all bounded functions in $C([0, T] \times \mathbb{R}, \mathbb{R})$, satisfy
\[
|F(t, x) - F(t, y)| \leq f(t)\hat{\phi}(|x - y|),
\]
\[
|G(t, x) - G(t, y)| \leq g(t)\hat{\psi}(|x - y|)
\]
for all $t \in [0, T]$, $f, g \in C([0, T], \mathbb{R}_+)$ and $\hat{\phi}, \hat{\psi} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha, B_t, \mathcal{F}_t$ and $M^2_\alpha[0, T]$ are defined in the beginning of Section 5. An earlier work was given by Itô ([11]) in 1946 for the existence and uniqueness for solutions of equation (50) in the special case that $f, g$ are both constant and $\hat{\phi}(x) = \hat{\psi}(x) = |x|$, which is equivalent to the Lipschitz conditions for $F$ and $G$. Later, in 1971 Yamada and Watanabe ([24, 25]) generally considered the condition (51), where $f, g$ are both constant, $\hat{\phi}$ is a strictly increasing and concave nonnegative function and $\hat{\psi}$ is a strictly increasing positive function such that $\int_0^\infty dx/\hat{\psi}^2(x) = +\infty$ for some $x_0 > 0$ and gave the uniqueness for solutions of equation (50).

In what follows, we suppose that
\begin{itemize}
\item[(G1):] $\hat{\phi}(s) \propto s$ and $\hat{\psi}(s) \propto s$;
\item[(G2):] both $\hat{\phi}'(s)$ and $\hat{\psi}'(s)$ are nonincreasing;
\item[(G3):] $\begin{bmatrix}
\hat{\phi}(s) & \hat{\psi}(s) \\
\hat{\phi}'(s) & \hat{\psi}'(s)
\end{bmatrix} \succeq 0$.
\end{itemize}
The following corollary gives an estimate for solutions of equation (50).

**Corollary 3.** Assume that (G1)-(G3) hold. Then the solution of $x(t, \omega)$ of the initial problem (50) satisfies
\[
E \sup_{0 \leq t \leq T} x^2(t, \omega) \leq \left(\hat{W}_2^{-1} \circ \hat{J}_2\right) \circ \left(\hat{W}_1^{-1} \circ \hat{J}_1\right) \{4E\alpha^2(\omega) + 4\left(\int_0^T |F(s, 0)| ds\right)^2 + 32 \int_0^T G^2(s, 0) ds\},
\]
(52)
where $\hat{W}_i$ and $\hat{F}_i$ ($i = 1, 2$) are defined by
\[
\hat{W}_1(x) := \int_{x_0}^{x} \frac{ds}{\phi^2(\sqrt{s})}, \quad \hat{F}_1(x) := \hat{W}_1(x(T)) + 4\int_0^T f(s)ds^2,
\]
\[
\hat{W}_2(x) := \int_{x_0}^{x} \frac{ds}{\psi^2(\sqrt{s})}, \quad \hat{F}_2(x) := \hat{W}_2(x(T)) + 32\int_0^T g^2(s)ds.
\]

**Proof.** For convenience, we use the following notations:
\[
\phi(s) := \hat{\phi}^2(\sqrt{s}), \quad \tilde{\phi}(s) := \phi(0) + \int_0^s \sup_{\mu \leq \eta < +\infty} \phi'(\eta)d\mu,
\]
\[
\psi(s) := \hat{\psi}^2(\sqrt{s}), \quad \tilde{\psi}(s) := \psi(0) + \int_0^s \sup_{\mu \leq \eta < +\infty} \psi'(\eta)d\mu.
\]

We claim that
\begin{align*}
\text{(G'2):} & \quad \hat{\phi} \equiv \phi \text{ and } \tilde{\psi} \equiv \psi, \text{ which are both nondecreasing and concave;} \\
\text{(G'3):} & \quad \tilde{\phi}(s) \preceq \psi(s).
\end{align*}

In fact, since $\hat{\phi}, \tilde{\phi} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, we have $\hat{\phi}, \tilde{\phi} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$. By (G1) and (G2), $\hat{\phi}'(\sqrt{s})/\sqrt{s}$ and $\tilde{\phi}'(\sqrt{s})$ are nonincreasing. Recalling (53), we have
\[
\tilde{\phi}'(s) = \frac{\hat{\phi}'(\sqrt{s})}{\sqrt{s}} \tilde{\phi}'(\sqrt{s}).
\]

Thus, $\tilde{\phi}'(s)$ is also nonincreasing and, therefore, $\tilde{\phi}$ is concave. If $\phi$ is not nondecreasing, then there exists $s_1 > 0$ such that $C := \hat{\phi}'(s_1) < 0$. Further, by the monotonicity of $\tilde{\phi}'$ we obtain $\tilde{\phi}'(s_2) \leq C$ for all $s_2 \geq s_1$. It implies that
\[
\lim_{s \to +\infty} \tilde{\phi}(s) = \hat{\phi}(0) + \int_0^{s_1} \tilde{\phi}'(\mu)d\mu + \int_{s_1}^{+\infty} \tilde{\phi}'(\mu)d\mu 
\]
\[
\leq \hat{\phi}(s_1) + \int_{s_1}^{+\infty} C\mu = -\infty,
\]
a contradiction to the fact $\tilde{\phi}(s) = \hat{\phi}^2(\sqrt{s}) \geq 0$. By (53) we obtain
\[
\tilde{\phi}(s) = \hat{\phi}(0) + \int_0^s \sup_{\mu < \eta < +\infty} \phi'(\eta)d\mu = \hat{\phi}(0) + \int_0^s \phi'(\mu)d\mu = \phi(s),
\]

implying that $\tilde{\phi} \in C(\mathbb{R}_+, \mathbb{R}_+)$. $\tilde{\psi}$ is similarly discussed and (G'2) is true. On the other hand, it follows from (G3) that
\[
\left( \frac{\psi(s)}{\phi(s)} \right)' = \frac{\psi'(s) \phi(s) - \psi(s) \phi'(s)}{\phi^2(s)} = \frac{\hat{\psi}'(\sqrt{s})\hat{\phi}'(\sqrt{s})\hat{\phi}^2(\sqrt{s}) - \hat{\psi}'(\sqrt{s})\hat{\phi}(\sqrt{s})\hat{\phi}'(\sqrt{s})}{\phi^2(s)\sqrt{s}}
\]
\[
= \frac{\hat{\phi}'(\sqrt{s})\hat{\phi}'(\sqrt{s})}{\phi^2(s)\sqrt{s}} \left( \frac{\phi(s)}{\phi'(s)} \right) \left( \frac{\psi(s)}{\psi'(s)} \right) \geq 0,
\]
implying that $\psi(s)/\phi(s)$ is nondecreasing, i.e., (G'3) holds and the claim is proved.
The equivalent integral equation of (50) is
\[ x(t, \omega) = \alpha(\omega) + \int_0^t F(s, x(s, \omega))ds + \int_0^t G(s, x(s, \omega))dB_s. \] (54)

By (54) we obtain
\[
|x(t, \omega)| \leq |\alpha(\omega)| + \int_0^t |F(s, x(s, \omega))|ds + \int_0^t |G(s, x(s, \omega))|dB_s |
\]
\[ \leq |\alpha(\omega)| + \int_0^t |F(s, x(s, \omega)) - F(s, 0)|ds + \int_0^t |F(s, 0)|ds 
+ \int_0^t |G(s, x(s, \omega))|dB_s |,
\]
which implies by the Power Mean Inequality (see [9] and the Appendix) that
\[
\sup_{t \in [0,T]} x^2(t, \omega) \leq 4\alpha^2(\omega) + 4\int_0^T |F(s, x(s, \omega)) - F(s, 0)|ds^2 + 4\int_0^T |F(s, 0)|ds^2 
+ 4 \sup_{t \in [0,T]} |\int_0^t G(s, x(s, \omega))dB_s|^2. \] (55)

By Doob’s Submartingale Theory ([7, p.71]),
\[
E \sup_{t \in [0,T]} |\int_0^t G(s, x(s, \omega))dB_s|^2 \leq 4E |\int_0^T G(s, x(s, \omega))dB_s|^2.
\]

Further, by the Itô isometry ([17, p.29]) and the Power Mean Inequality (see [9]
and the Appendix),
\[
E \int_0^T |G(s, x(s, \omega))dB_s|^2 = \int_0^T E|G(s, x(s, \omega))|^2 ds 
= \int_0^T E|G(s, x(s, \omega)) - G(s, 0) + G(s, 0)|^2 ds 
\leq 2 \int_0^T E|G(s, x(s, \omega)) - G(s, 0)|^2 ds + 2 \int_0^T E|G(s, 0)|^2 ds.
\]

By (51),
\[
E(\int_0^T |F(s, x(s, \omega)) - F(s, 0)|ds)^2 \leq E(\int_0^T f(s)\hat{\phi}(|x(s, \omega)|)ds)^2 
\leq E(\int_0^T f(s)\hat{\phi}^2(|x(s, \omega)|)ds)^2(\int_0^T f(s)ds)^2 
\leq \int_0^T f(s)ds \int_0^T f(s)\hat{\phi}(Ex^2(s, \omega))ds 
\leq \int_0^T f(t)dt \int_0^T f(t)\hat{\phi}(E \sup_{s \in [0,t]} x^2(s, \omega))dt,
\]
where Hölder’s Inequality and Jensen’s Inequality (see [9] or the Appendix) are employed because \( \hat{\phi} \) is nondecreasing and concave by \( G'2 \). Similarly,
\[
\int_0^T E|G(s, x(s, \omega)) - G(s, 0)|^2 ds \leq \int_0^T g^2(t)\psi(E \sup_{s \in [0,t]} x^2(s, \omega))dt.
\]
It follows from (55) that
\[
E \sup_{t \in [0, T]} x^2(t, \omega) \leq 4E\alpha^2(\omega) + 4(\int_0^T |F(t, 0)|dt)^2 + 32 \int_0^T G^2(t, 0)dt \\
+ 4 \int_0^T f(t)dt \int_0^T f(t)\bar{\phi}(E \sup_{s \in [0, t]} x^2(s, \omega))dt \\
+ 32 \int_0^T g^2(t)\bar{\psi}(E \sup_{s \in [0, t]} x^2(s, \omega))dt.
\]

It follows from (56) that the function \( u(T) := E \sup_{t \in [0, T]} x^2(t, \omega) \) satisfies the inequality (2), where \( n = 2, p_1 = p_2 = 1, b_1(t) = b_2(t) = t \) and
\[
a(T) = 4\{E\alpha^2(\omega) + (\int_0^T |F(s, 0)|ds)^2 + 8 \int_0^T G^2(s, 0)ds\},
\]
\[
f_1(T, t) = 4f(t) \int_0^T f(s)ds, \quad f_2(T, t) = 32g^2(t), \quad \phi_1(s) = \bar{\phi}(s), \quad \phi_2(s) = \bar{\psi}(s).
\]

Recalling \( \phi(s) \sim \psi(s) \) in (G3), we see that the sequence \( \{\phi, \psi\} \) is strongly nondecreasing. One can verify that (H1)-(H3) hold for \( t \in [0, T] \). By Theorem 1 and definition (53) we obtain the estimate (52). The proof is complete. \( \square \)

In the case that \( \hat{\phi} = \hat{\psi} = \text{id} \) (the identity function) in Corollary 3, as required by Itô ([11]), (51) is a pair of nonuniform Lipschitz conditions. In this case estimate \( (52) \) can be simplified as
\[
E \sup_{t \in [0, T]} x^2(t, \omega) \leq \{4E\alpha^2(\omega) + 4(\int_0^T |F(s, 0)|ds)^2 + 32 \int_0^T G^2(s, 0)ds\} \\
\times \exp\{4(\int_0^T f(s)ds)^2 + 32 \int_0^T g^2(s)ds\}.
\]

It implies that solutions \( x(t, \omega) \) of the system are bounded on \( \mathbb{R}_+ \) in the sense of mean square if \( F, G \) and \( f, g \) satisfy
\[
\int_0^\infty |F(s, 0)|ds < \infty, \quad \int_0^\infty G^2(s, 0)ds < \infty, \quad \int_0^\infty f(s)ds < \infty, \quad \int_0^\infty g^2(s)ds < \infty.
\]

The next corollary gives the continuous dependence for solutions \( x \) of system (50) on the given functions \( F \) and \( G \) and initial data \( \alpha \) in mean square.

**Corollary 4.** Suppose that (G1)-(G3) hold and that
\[
\lim_{x \to 0^+} \int_{x_0}^x \frac{ds}{\phi^2(\sqrt{s})} = -\infty, \quad \lim_{x \to 0^+} \int_{x_0}^x \frac{ds}{\psi^2(\sqrt{s})} = -\infty.
\]
Then each solution \( x(t) \) of (50) is continuously dependent on \( F, G \) and \( \alpha \) in mean square.

**Proof.** We still used the notations \( \phi, \psi \) as defined in (53) and consider the Banach space \( C_b([0, T] \times \mathbb{R}, \mathbb{R}) \) endowed with the norm \( \|F\| := \sup_{(t, x) \in [0, T] \times \mathbb{R}} |F(t, x)| \).

Consider a variation of (50)
\[
\begin{align*}
\begin{cases}
    dx(t, \omega) = \hat{F}(t, x(t, \omega))dt + \hat{G}(t, x(t, \omega))dB_t, \\
x(0, \omega) = \beta(\omega), \quad t \in [0, T],
\end{cases}
\end{align*}
\]
where $\hat{F}, \hat{G} \in C_b([0,T] \times \mathbb{R}, \mathbb{R})$ are in the same class as $F, G$ used in (50) and satisfy (51). Let $x$ and $y$ be solutions of (50) and (57) respectively, and suppose that

$$E(\alpha(\omega) - \beta(\omega))^2 < \epsilon, \ E\|F - \hat{F}\|^2 < \epsilon, \ E\|G - \hat{G}\|^2 < \epsilon,$$

where $\epsilon > 0$ is a small constant. By the equivalent integral equation (54), we have

$$|x(t, \omega) - y(t, \omega)| \leq |\alpha(\omega) - \beta(\omega)| + \int_0^t |F(s, x(s, \omega)) - \hat{F}(s, y(s, \omega))| ds$$

$$+ |\int_0^t G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|$$

$$\leq |\alpha(\omega) - \beta(\omega)| + \int_0^t |F(s, x(s, \omega)) - F(s, y(s, \omega))| ds$$

$$+ \int_0^t |F(s, y(s, \omega)) - \hat{F}(s, y(s, \omega))| ds$$

$$+ |\int_0^t G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|,$$

where the triangle inequality cannot be applied to the third integral, an Itô integral, as indicated in e.g. [7, 17]. Applying the Power Mean Inequality (see [9] and the Appendix), we get

$$\sup_{t \in [0,T]} |x(t, \omega) - y(t, \omega)|^2$$

$$\leq 4|\alpha(\omega) - \beta(\omega)|^2 + 4(\int_0^T |F(s, x(s, \omega)) - F(s, y(s, \omega))| ds)^2$$

$$+ 4(\int_0^T |F(s, y(s, \omega)) - \hat{F}(s, y(s, \omega))| ds)^2$$

$$+ 4 \sup_{t \in [0,T]} |\int_0^t G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|^2.$$

By Doob’s Submartingale Theory ([7, p.71]),

$$E \sup_{t \in [0,T]} |\int_0^t G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|^2 \leq 4E |\int_0^T G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|^2.$$

Further, by the Itô isometry ([17, p.29]) and the Power Mean Inequality (see [9] and the Appendix),

$$E|\int_0^T G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega)) dB_s|^2 \leq \int_0^T E|G(s, x(s, \omega)) - \hat{G}(s, y(s, \omega))|^2 ds$$

$$\leq \int_0^T E|G(s, x(s, \omega)) - G(s, y(s, \omega)) + G(s, y(s, \omega)) - \hat{G}(s, y(s, \omega))|^2 ds$$

$$\leq 2 \int_0^T E|G(s, x(s, \omega)) - G(s, y(s, \omega))|^2 ds$$

$$+ 2 \int_0^T E|G(s, y(s, \omega)) - \hat{G}(s, y(s, \omega))|^2 ds.$$
It follows from (58) that

$$
E \sup_{t \in [0,T]} |x(t, \omega) - y(t, \omega)|^2 \leq 4E[|\alpha(\omega) - \beta(\omega)|^2 + 4E(\int_{0}^{T}|F(s, x(s, \omega)) - F(s, y(s, \omega))|ds)^2 + 4E(\int_{0}^{T}|\tilde{F}(s, x(s, \omega)) - \tilde{F}(s, y(s, \omega))|^2ds)^{\frac{1}{2}}(\int_{0}^{T}1ds)^{\frac{1}{2}}] \\
+ 32 \int_{0}^{T} E|G(s, x(s, \omega)) - G(s, y(s, \omega))|^2ds \\
+ 32 \int_{0}^{T} E|G(s, y(s, \omega)) - \tilde{G}(s, y(s, \omega))|^2ds \\
\leq 4(T^2 + 8T + 1)\epsilon + 4E(\int_{0}^{T}|F(s, x(s, \omega)) - F(s, y(s, \omega))|ds)^2 + 32 \int_{0}^{T} E|G(s, x(s, \omega)) - G(s, y(s, \omega))|^2ds,
$$

(59)

where Hölder’s Inequality (see [9] or the Appendix) is used. By (51),

$$
E(\int_{0}^{T}|F(s, x(s, \omega)) - F(s, y(s, \omega))|ds)^2 \leq E(\int_{0}^{T} f(s)\phi(|x(s, \omega) - y(s, \omega)|)ds)^2 \\
\leq E\left(\int_{0}^{T} f(s)\phi^2(|x(s, \omega) - y(s, \omega)|)ds\right)^{\frac{1}{2}}(\int_{0}^{T} f(s)ds)^{\frac{1}{2}} \\
\leq \int_{0}^{T} f(s)ds \int_{0}^{T} f(s)\phi(E|x(s, \omega) - y(s, \omega)|)^2ds \\
\leq \int_{0}^{T} f(t)dt \int_{0}^{T} f(t)\phi(E \sup_{s \in [0,t]} |x(s, \omega) - y(s, \omega)|^2)dt,
$$

where Hölder’s Inequality and Jensen’s Inequality (see [9] or the Appendix) are employed because \(\phi\) is nondecreasing and concave by (G’2). Similarly,

$$
\int_{0}^{T} E|G(s, x(s, \omega)) - G(s, y(s, \omega))|^2ds \leq \int_{0}^{T} g^2(t)E(\sup_{s \in [0,t]} |x(s, \omega) - y(s, \omega)|^2)dt.
$$

It follows from (59) that the function \(u(T) := E \sup_{t \in [0,T]} |x(t, \omega) - y(t, \omega)|^2\) satisfies the inequality (2), where \(n = 2, p_1 = p_2 = 1, b_1(t) = b_2(t) = t\) and

$$
\alpha(T) = 4(T^2 + 8T + 1)\epsilon, \quad f_1(T, t) = 4f(t) \int_{0}^{T} f(t)dt, \quad f_2(T, t) = 32g^2(t), \\
\phi_1(s) = \phi(s), \quad \phi_2(s) = \psi(s), \quad \tilde{W}_1(x) = \int_{x_0}^{x} \frac{ds}{\phi(s)}, \quad \tilde{W}_2(x) = \int_{x_0}^{x} \frac{ds}{\psi(s)}, \\
\tilde{J}_1x(T) = \tilde{W}_1(x(T)) + 4(\int_{0}^{T} f(s)ds)^{2}, \quad \tilde{J}_2x(T) = \tilde{W}_2(x(T)) + 32 \int_{0}^{T} g^2(s)ds.
$$

It follows from (G’3) that the sequence \(\{\phi, \psi\}\) is strongly nondecreasing, i.e., both \(\phi\) and \(\psi\) are nondecreasing and \(\psi/\phi\) is also nondecreasing. One can verify that (H1)-(H3) hold for \(t \in [0, T]\). By Theorem 1 we obtain for all \(\epsilon > 0\),

$$
E \sup_{t \in [0,T]} |x(t, \omega) - y(t, \omega)|^2 \leq (W_2^{-1} \circ \tilde{J}_2) \circ (W_1^{-1} \circ \tilde{J}_1)x_0(T, K),
$$

(60)
where \( x_0(T, K) := 4(T^2 + 8T + 1)/K \) and \( K := 1/\epsilon \). Extend \( f, g \in C([0, T], \mathbb{R}+) \) to \( \hat{f}, \hat{g} \in C(\mathbb{R}^+, \mathbb{R}^+) \) as follows

\[
\hat{f}(t) := \begin{cases} 
  f(t), & t \in [0, T], \\
  -f(t)t + (T + 1)f(T), & t \in (T, T + 1], \\
  0, & t \in (T + 1, +\infty),
\end{cases}
\]

and

\[
\hat{g}(t) := \begin{cases} 
  g(t), & t \in [0, T], \\
  -g(t)t + (T + 1)g(T), & t \in (T, T + 1], \\
  0, & t \in (T + 1, +\infty). 
\end{cases}
\]

Let

\[
I_1 \vartheta(K) := \hat{W}_1(\vartheta(K)) + 4 \int_0^K (\hat{f}(s)ds)^2, \quad I_2 \vartheta(K) := \hat{W}_2(\vartheta(K)) + 32 \int_0^K \hat{g}^2(s)ds.
\]

One can choose \( \epsilon \) so small that \( K \geq T \). Thus, for any continuous function \( \vartheta : [t_0, +\infty) \times [t_0, +\infty) \to \mathbb{R}+ \),

\[
\hat{J}_1 \vartheta(T, K) = \hat{W}_1(\vartheta(T, K)) + 4 \int_0^T (f(s)ds)^2 
\leq \hat{W}_1(\vartheta(T, K)) + 4 \int_0^K (\hat{f}(s)ds)^2 = I_1 \vartheta(T, K), \tag{61}
\]

and similarly

\[
\hat{J}_2 \vartheta(T, K) \leq I_2 \vartheta(T, K). \tag{62}
\]

Noting that \( x_0(T, K) \to 0 \) as \( K \to +\infty \) and

\[
\lim_{x \to +\infty} \hat{W}_1(x) = \lim_{x \to +\infty} \int_{x_0}^x \frac{ds}{\vartheta(s)} = -\infty, \quad \lim_{x \to +\infty} \hat{W}_2(x) = \lim_{x \to +\infty} \int_{x_0}^x \frac{ds}{\vartheta(s)} = -\infty,
\]

which verifies all assumptions and especially the condition (C2) in Lemma 3, we conclude that

\[
\lim_{K \to +\infty} (\hat{W}_2^{-1} \circ I_2) \circ (\hat{W}_1^{-1} \circ I_1)x_0(T, K) = 0.
\]

Hence, by (60), (61), (62) and the monotonicity and continuity of functions \( \hat{W}_1^{-1} \) and \( \hat{W}_2^{-1} \),

\[
E \sup_{t \in [0, T]} |x(t, \omega) - y(t, \omega)|^2 \leq \lim_{K \to +\infty} (\hat{W}_2^{-1} \circ \hat{J}_2) \circ (\hat{W}_1^{-1} \circ \hat{J}_1)x_0(T, K) 
\leq \lim_{K \to +\infty} (\hat{W}_2^{-1} \circ I_2) \circ (\hat{W}_1^{-1} \circ I_1)x_0(T, K) = 0.
\]

This completes the proof of continuous dependence. \( \square \)

Corollary 4 gives a continuous dependence of solutions on initial data \( \alpha \), which actually implies the uniqueness of solutions under condition \( (51) \) together with \( (G1)-(G3) \), where condition \( (51) \) is weaker than the one of Yamada and Watanabe ([24, 25]).
Appendix: Useful inequalities. 1. Power Mean Inequality

\[ \frac{C_1 + \cdots + C_n}{n} \leq \left( \frac{C_1^p + \cdots + C_n^p}{n} \right)^{\frac{1}{p}}, \]

where \( C_i \geq 0 \) and \( p \geq 1 \) for \( i = 1, \ldots, n, n \in \mathbb{N} \).

2. Hölder’s Inequality

\[ \int_{t_0}^{t} |f(s)g(s)|ds \leq \left( \int_{t_0}^{t} |f(s)|^p ds \right)^{\frac{1}{p}} \left( \int_{t_0}^{t} |g(s)|^q ds \right)^{\frac{1}{q}}, \]

where \( f \in L^p([t_0, +\infty), \mathbb{R}), g \in L^q([t_0, +\infty), \mathbb{R}), \) and \( p, q \geq 1 \) are constants such that \( \frac{1}{p} + \frac{1}{q} = 1 \).

3. Jensen’s Inequality

\[ \varphi(E(X)) \leq E(\varphi(X)) \]

for any real-valued random variable \( X \) such that \( E(X) < \infty \) and \( E(\varphi(X)) < \infty \), where \( E \) denotes the expectation and \( \varphi \) is a convex real-valued function, i.e., \( \lambda \varphi(x) + (1 - \lambda) \varphi(y) \geq \varphi(\lambda x + (1 - \lambda)y) \) for all \( \lambda \in (0, 1) \) and \( x, y \in \mathbb{R} \).

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