New families of special numbers for computing negative order Euler numbers

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Abstract

The main purpose of this paper is to construct new families of special numbers with their generating functions. These numbers are related to the many well-known numbers, which are the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers, the Stirling numbers of the second kind and the central factorial numbers. Our other inspiration of this paper is related to the Golombek’s problem [14] “Aufgabe 1088, El. Math. 49 (1994) 126-127”. Our first numbers are not only related to the Golombek’s problem, but also computation of the negative order Euler numbers. We compute a few values of the numbers which are given by some tables. We give some applications in Probability and Statistics. That is, special values of mathematical expectation in the binomial distribution and the Bernstein polynomials give us the value of our numbers. Taking derivative of our generating functions, we give partial differential equations and also functional equations. By using these equations, we derive recurrence relations and some formulas of our numbers. Moreover, we give two algorithms for computation our numbers. We also give some combinatorial applications, further remarks on our numbers and their generating functions.

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1. Introduction

In this section, we consider the following question:
What could be a more basic tools to compute the negative order of the first and the second kind Euler numbers? One of the motivations is associated with this question and its
answer. Especially, in the work of Golombek [14], which is entitled **Aufgabe 1088**, we see the following novel combinatorial sum:

\[
\sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n} \left( e^t + 1 \right)^k \bigg|_{t=0}, \tag{1.1}
\]

where \( n = 1, 2, \ldots \). Golombek [14] also mentioned that this sum is related to the following sequence

\[n2^{n-1}, n(n + 1)2^{n-2}, \ldots\]

The other motivation is to introduce new families of special numbers, which are not only used in counting techniques and problems, but also computing negative order of the first and the second kind Euler numbers and other combinatorial sums. Here, our technique is related to the generating functions. In the historical development of Mathematics, we can observe that the generating functions play a very important role in Pure and Applied Mathematics. These function are powerful tools to solve counting problems and investigate properties of the special numbers and polynomials. In addition, the generating functions are also used in Computer Programming, in Physics, and in other areas. Briefly, in Physics, generating functions, which arise in Hamiltonian mechanics, are quite different from generating functions in mathematics. The generating functions are functions whose partial derivatives generate the differential equations that determine a system’s dynamics. These functions are also related to the partition function of statistical mechanics (cf. [10], [17], [34], [42], [43]). As for mathematics, a generating function is a formal power series in one indeterminate whose coefficients encode information about a sequence of numbers and that is indexed by the natural numbers (cf. [10], [12], [13], [15], [17], [34], [25], [40], [42]). As far as we know that the generating function is firstly discovered by Abraham de Moivre (26 May 1667 - 27 November 1754, French mathematician) (cf. [17], [42]). In order to solve the general linear recurrence problem, Moivre constructed the concept of the generating functions in 1730. In work of Doubilet et al. [13], we also see that Laplace (23 March 1749-5 March 1827, French mathematician, physicist and statistician), discovered the remarkable correspondence between set theoretic operations and operations on formal power series. Their method gives us great success to solve a variety of combinatorial problems. They developed new kinds of algebras of generating functions better suited to combinatorial and probabilistic problems. Their method is depended on group algebra (or semigroup algebra) (see for detail [13]). It is well-known that there are many different ways or approaches to generate a sequence of numbers and polynomials from the series or the generating functions. The purpose of this paper is to construct the generating functions for new families of numbers involving Golombek’s identity in (1.1), the Stirling numbers, the central factorial numbers, the Euler numbers of negative order, the rook numbers and combinatorial sums. Our method and approach provides a way of constructing new special families of numbers and combinatorial sums. We show how several of these numbers and these combinatorial sums relate to each other.

We summarize our paper results as follows:
In Section 2, we briefly review some special numbers and polynomials, which are the Bernoulli numbers, the Euler numbers, the Stirling numbers, the central factorial numbers and the array polynomials.

In Section 3, we give a generating function. By using this function, we define a family of new numbers $y_1(n, k; \lambda)$. We investigate many properties including a recurrence relation of these numbers by using their generating functions. We compute a few values of the numbers $y_1(n, k; \lambda)$, which are given by the tables. We give some remarks and comments related to the Golombek’s identity and the numbers $y_1(n, k; 1)$. Finally, we give a conjecture with two open questions.

In Section 4, we give a generating function for a new family of the other numbers $y_2(n, k; \lambda)$. By using this function, we investigate many properties with a recurrence relation of these numbers. We compute a few values of the numbers $y_2(n, k; \lambda)$, which are given by the tables. We give relations between these numbers, the Fibonacci numbers, the Lucas numbers, and the $\lambda$-Stirling numbers of the second kind. We also give some combinatorial sums.

In Section 5, we define $\lambda$-central factorial numbers $C(n, k; \lambda)$. By using their generating functions, we derive some identities and relations for these numbers and the others.

In Section 6, we give some applications related to the special values of mathematical expectation for the binomial distribution, the Bernstein polynomials and the Bernoulli polynomials.

In Section 7, by using the numbers $y_1(n, k; \lambda)$, we compute the Euler numbers of negative order. In addition, we compute a few values of these numbers, which are given by the tables.

In Section 8, We give two algorithms for our computations.

In Section 9, we give some combinatorial applications, including a rook numbers and polynomials. We also give combinatorial interpretation for the numbers $y_1(n, k)$. Finally in the last section, we give further remarks with conclusion.

**Notations:** Throughout this paper, we use the following standard notations:

$$
\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}
$$

Here, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. The principal value $\ln z$ is the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$. Moreover we also use the following notational conventions:

$$
0^n = \begin{cases} 
1, & (n = 0) \\
0, & (n \in \mathbb{N})
\end{cases}
$$

and

$$
\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{v} = \frac{\lambda(\lambda - 1) \cdots (\lambda - v + 1)}{v!} = \frac{(\lambda)_v}{v!} \quad (n \in \mathbb{N}, \lambda \in \mathbb{C})
$$

(cf. [6], [12], [37]). For combinatorial example, we will use the notations of Bona [4], that is the set $\{1, 2, \ldots, n\}$ is an $n$-element set, that is, $n$ distinct objects. Therefore, Bona introduced the shorter notation $[n]$ for this set. The number $n(n-1)(n-2) \cdots (n-k+1)$ of all $k$-element lists from $[n]$ without repetition occurs so often in combinatorics that there is a symbol for it, namely

$$(n)_k = n(n-1)(n-2) \cdots (n-k+1)$$
Yilmaz Simsek

2. Background

In this section, we give a brief introduction about the Bernoulli numbers, the Euler numbers, the \((\lambda)-\)Stirling numbers and the array polynomials. Because we use these numbers and polynomials in the next sections.

In \([2]-[11]\), we see that there are many known properties and relations involving various kind of the special numbers and polynomials such as the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the Stirling numbers and also the rook polynomials and numbers by making use of some standard techniques based upon generating functions and other known techniques.

The Bernoulli polynomials are defined by means of the following generating function:

\[
\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\]

where \(|t| < 2\pi\) (cf. \([15]-[41]\); see also the references cited in each of these earlier works).

One can observe that

\[B_n = B_n(0),\]

which denotes the Bernoulli numbers (cf. \([15]-[41]\); see also the references cited in each of these earlier works).

The sum of the powers of integers is related to the Bernoulli numbers and polynomials:

\[
\sum_{k=0}^{n} k^r = \frac{1}{r+1} \left( B_{r+1}(n+1) - B_{r+1}(0) \right),
\]

\(2.1\)

(cf. \([15], [40], [37]\)).

The first kind Apostol-Euler polynomials of order \(k\) are defined by means of the following generating function:

\[
F_{P_1}(t, x; k) = \left( \frac{2}{\lambda e^t + 1} \right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x; \lambda) \frac{t^n}{n!},
\]

\(2.2\)

\(|t| < \pi\) when \(\lambda = 1\) and \(|t| < |\ln (-\lambda)|\) when \(\lambda \neq 1\), \(\lambda \in \mathbb{C}, k \in \mathbb{N}\) with, of course,

\[E_n^{(k)}(\lambda) = E_n^{(k)}(0; \lambda),\]

which denotes the first kind Apostol-Euler numbers of order \(k\) (cf. \([19], [15], [23], [24], [26], [40], [41]\)). Substituting \(k = \lambda = 1\) into \(2.2\), we have the first kind Euler numbers \(E_n = E_n^{(1)}(1)\), which are defined by means of the following generating function:

\[
\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},
\]

where \(|t| < \pi\) (cf. \([15]-[41]\); see also the references cited in each of these earlier works).
The Euler numbers of the second kind $E_n^*$ are defined by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!},$$

where $|t| < \frac{\pi}{2}$ (cf. [7], [15], [24], [26], [37], [41]; see also the references cited in each of these earlier works).

The Stirling numbers of the second kind are used in pure and applied mathematics. These numbers occur in combinatorics and in the theory of partitions. The Stirling numbers of the second kind, denoted by $S_2(n, k)$, is the number of ways to partition a set of $n$ objects into $k$ groups ([4], [11], [12], [34], [37]).

The $\lambda$-Stirling numbers of the second kind $S_2(n, v; \lambda)$ are generalized of the Stirling number of the second kind. These numbers $S_2(n, v; \lambda)$ are defined by means of the following generating function:

$$F_{S}(t, v; \lambda) = (\lambda e^t - 1)^v = \sum_{n=0}^{\infty} S_2(n, v; \lambda) \frac{t^n}{n!},$$

(2.3)

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. For further information about these numbers, the reader should consult [21] and ([30], [29], [35]; see also the references cited in each of these earlier works).

Observe that

$$S_2(n, v) = S_2(n, v; 1),$$

which are computing by the following formulas:

$$x^n = \sum_{v=0}^{n} \binom{x}{v} v! S_2(n, v)$$

or

$$S_2(n, v) = \frac{1}{v!} \sum_{j=0}^{v} \binom{v}{j} (-1)^j (v - j)^n$$

(cf. [15]-[41]; see also the references cited in each of these earlier works). A Recurrence relation for these numbers is given by

$$S_2(n, k) = S_2(n - 1, k - 1) + kS_2(n - 1, k),$$

with

$$S_2(n, 0) = 0 \ (n \in \mathbb{N}); \ S_2(n, n) = 1 \ (n \in \mathbb{N}); \ S_2(n, 1) = 1 \ (n \in \mathbb{N})$$

and $S_2(n, k) = 0 \ (n < k)$ or $k < 0$ (cf. [15]-[41]; see also the references cited in each of these earlier works).

In [30], we defined the $\lambda$-array polynomials $S_{v}^\lambda(x; \lambda)$ by means of the following generating function:

$$F_{A}(t, x, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_{v}^{n}(x; \lambda) \frac{t^n}{n!},$$

(2.4)

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [6], [30]).
The array polynomials $S^v_n(x)$ are defined by means of the following generating function:

$$F_A(t, x, v) = \left(\frac{e^t - 1}{v!}\right)^v e^{tx} = \sum_{n=0}^{\infty} S^v_n(x) \frac{t^n}{n!}, \quad (2.5)$$

(cf. [6], [9], [30]; see also the references cited in each of these earlier works). By using the above generating function, we have

$$S^v_n(x) = \frac{1}{v!} \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} (x+j)^n$$

with

$$S^v_0(x) = S^0_n(x) = 1, S^v_n(x) = x^n$$

and for $v > n$,

$$S^v_n(x) = 0$$

(cf. [9], [30], [31]; see also the references cited in each of these earlier works).

Recently, the central factorial numbers $T(n, k)$ have been studied by many authors. These functions have been many applications in theory of Combinatorics and Probability. The central factorial numbers $T(n, k)$ (of the second kind) are defined by means of the following generating function:

$$F_T(t, k) = \frac{1}{(2k)!} \left(\frac{e^t + e^{-t} - 2}{2}\right)^k = \sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!}, \quad (2.6)$$

(cf. [4], [11], [12], [39], [31], [34]; see also the references cited in each of these earlier works).

These numbers have the following relations:

$$x^n = \sum_{k=0}^{n} T(n, k) x(x-1)(x-2^2)(x-3^2) \cdots (x-(k-1)^2).$$

Combining the above equation with (4.1), we also have

$$T(n, k) = T(n-1, k-1) + k^2 T(n-1, k),$$

where $n \geq 1, k \geq 1, (n, k) \neq (1,1)$. For $n, k \in \mathbb{N}$, $T(0, k) = T(n, 0) = 0$ and $T(n, 1) = 1$ (cf. [4], [11], [12], [39], [31], [34]).

3. A FAMILY OF NEW NUMBERS $y_1(n, k; \lambda)$

In this section, we give generating function for the numbers $y_1(n, k; \lambda)$. We give some functional equations and differential equations of this generating function. By using these equations, we derive various new identities and combinatorics relations related to these numbers. Some our observations on these numbers can be briefly expressed as follows: the numbers $y_1(n, k; \lambda)$ are related to the $\lambda$-Stirling numbers of the second kind, the central factorial numbers, the Euler numbers of negative orders and the Golombek's identity.

It is time to give the following generating function for these numbers:

$$F_{y_1}(t, k; \lambda) = \frac{1}{k!} \left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!}, \quad (3.1)$$
where \( k \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{C} \).

Note that there is one generating function for each value of \( k \). This function is an analytic function.

By using (3.1), we get

\[
\sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( t^n \) on both sides of the above equation, we arrive at the following theorem:

**Theorem 1.** Let \( n \) be a positive integer. Then we have

\[
y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j.
\]

(3.2)

We assume that \( \lambda \neq 0 \). For \( k = 0, 1, 2, 3, 4 \) and \( n = 0, 1, 2, 3, 4, 5 \) compute a few values of the numbers \( y_1(n, k; \lambda) \) given by Equation (3.2) as follows:

\[
\begin{array}{cccccccc}
 n \backslash k & 0 & 1 & 2 & 3 & 4 \\
 0 & 1 & \lambda \frac{1}{2} \lambda^2 + \lambda \frac{1}{6} \lambda^3 + \frac{1}{2} \lambda^2 + \frac{1}{2} \lambda & \frac{1}{24} \lambda^4 + \frac{1}{6} \lambda^3 + \frac{1}{6} \lambda^2 + \frac{1}{6} \lambda \\
 1 & 0 & \lambda^2 + \lambda \frac{1}{2} \lambda^3 + \lambda^2 + \frac{1}{2} \lambda & \frac{1}{3} \lambda^4 + \frac{1}{6} \lambda^3 + \frac{1}{2} \lambda^2 + \frac{1}{6} \lambda \\
 2 & 0 & 2 \lambda^2 + \lambda \frac{3}{2} \lambda^3 + 2 \lambda^2 + \frac{1}{2} \lambda & \frac{1}{3} \lambda^4 + \frac{3}{2} \lambda^3 + \lambda^2 + \frac{1}{6} \lambda \\
 3 & 0 & 4 \lambda^2 + \lambda \frac{5}{2} \lambda^3 + 4 \lambda^2 + \frac{1}{2} \lambda & \frac{5}{2} \lambda^4 + \frac{5}{2} \lambda^3 + 2 \lambda^2 + \frac{1}{6} \lambda \\
 4 & 0 & 8 \lambda^2 + \lambda \frac{7}{2} \lambda^3 + 8 \lambda^2 + \frac{1}{2} \lambda & \frac{7}{2} \lambda^4 + \frac{7}{2} \lambda^3 + 4 \lambda^2 + \frac{1}{6} \lambda \\
 5 & 0 & 16 \lambda^2 + \lambda \frac{1}{2} \lambda^3 + 816 \lambda + \frac{1}{2} \lambda & \frac{1}{2} \lambda^4 + \frac{81}{2} \lambda^3 + 8 \lambda^2 + \frac{1}{6} \lambda \\
\end{array}
\]

For \( k = 0, 1, 2, \ldots, 9 \) and \( n = 0, 1, 2, \ldots, 9 \) we compute a few values of the numbers \( y_1(n, k; 1) \) given by Equation (3.2) as follows:

\[
\begin{array}{cccccccccccc}
 n \backslash k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 2 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 3 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

Some special values of \( y_1(n, k; \lambda) \) are given as follows:

\[
y_1(0, k; \lambda) = \frac{1}{k!} (\lambda + 1)^k,
\]

\[
y_1(n, 0; \lambda) = 0,
\]
and
\[ y_1(n, 1; \lambda) = \lambda. \]

By using (3.1), we derive the following functional equation
\[ \lambda^k e^{kt} = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(t, l; \lambda). \]

Combining (3.1) with the above equation, we get
\[ \lambda^k \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(n, l; \lambda) \right) \frac{t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the following theorem:

**Theorem 2.**
\[ k^n \lambda^k = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(n, l; \lambda). \]

We give a relationship between the numbers \( y_1(n, k; \lambda) \) and the \( \lambda \)-Stirling numbers of the second kind by the following theorem:

**Theorem 3.**
\[ S_2(n, k; \lambda^2) = \frac{k!}{2^n} \sum_{l=0}^{n} \binom{n}{l} S_2(l, k; \lambda) y_1(n - l, k; \lambda). \]

**Proof.** By using (2.3) and (3.1), we derive the following functional equation:
\[ F_S(2t, v; \lambda^2) = k! F_S(t, v; \lambda) y_1(t, k; \lambda). \]

From this equation, we have
\[ \sum_{n=0}^{\infty} 2^n S(n, v; \lambda^2) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S(n, v; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!}. \]

Therefore
\[ \sum_{n=0}^{\infty} 2^n S(n, v; \lambda^2) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( k! \sum_{l=0}^{n} \binom{n}{l} S_2(l, k; \lambda) y_1(n - l, k; \lambda) \right) \frac{t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. \( \square \)

A relationship between the numbers \( y_1(n, k; \lambda) \), \( S_2(n, k; \lambda^3) \) and the array polynomials \( S_k^n(x; \lambda) \) is given by the following theorem:

**Theorem 4.**
\[ S_2(n, k; \lambda^3) = \sum_{l=0}^{n} \sum_{j=0}^{k} \binom{n}{l} \binom{k}{j} \frac{\lambda^{2k-2j} j!}{3^n} y_1(l, j; \lambda) S_{k-l}^{n-l}(2k - 2j; \lambda). \]
Proof. If we combine (2.3), (2.4) and (3.1), we get

\[ F_S(3t, k; \lambda^3) = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \lambda^{2k-2j} F_A(t, 2k-2j, k; \lambda) F_y(t, j; \lambda). \]

By using the above functional equation, we obtain

\[
\sum_{n=0}^{\infty} 3^n S(n, v; \lambda^3) \frac{t^n}{n!} = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \lambda^{2k-2j} \sum_{n=0}^{\infty} S_k^n (n, 2k-2j; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} y_1(n, j; \lambda) \frac{t^n}{n!}.
\]

Therefore

\[
\sum_{n=0}^{\infty} 3^n S(n, v; \lambda^3) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{j=0}^{k} \binom{n}{l} \binom{k}{j} j! \lambda^{2k-2j} y_1(l, j; \lambda) S_k^{n-l}(2k-2j; \lambda) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result.

There are many combinatorics and analysis applications for (3.2). By substituting \( \lambda = 1 \) into (3.2), then we set

\[ B(n, k) = k! y_1(n, k; 1). \]

In [14], Golombek gave the following formula for (3.2):

\[ B(n, k) = \frac{d^n}{dt^n} (e^t + 1)^k \bigg|_{t=0}. \]

Remark 1. If we substitute \( \lambda = -1 \) into (3.2), then we get the Stirling numbers of the second kind:

\[ S_2(n, k) = (-1)^k y_1(n, k; -1) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \]

(cf. [1]-[10]).

Remark 2. We set

\[ B(n, k) = k! y_1(n, k; 1) = \frac{d^n}{dt^n} (e^t + 1)^k \bigg|_{t=0}. \]

These numbers are related to the following numbers:

\[ a_k 2^k \]

where the sequence \( a_k \) is a positive integer depend on \( k \). Consequently, in the work of Spivey [33, Identity 8-Identity 10], we see that

\[ B(0, k) = 2^k, \]
\[ B(1, k) = k 2^{k-1}, \]
\[ B(2, k) = k(k+1) 2^{k-2}, \]

see also [4] P. 56, Exercise 21 and [10] p. 117.
Remark 3. In [33, Identity 12.], Spivey also proved the following novel identity by the falling factorial method:

\[ B(m, n) = \sum_{j=0}^{n} \binom{n}{j} j! 2^{n-j} S_2(m, j). \] (3.4)

The numbers \( B(0, k) \) are given by means of the following well-known generating function: Let \(|x| < \frac{1}{2}\), we have

\[ \sum_{k=0}^{\infty} B(0, k) x^k = \frac{1}{1 - 2x}. \]

The numbers \( B(1, k) \) are given by means of the following well-known generating function: Let \(|x| < \frac{1}{2}\), we have

\[ \sum_{k=1}^{\infty} B(1, k) x^k = \frac{x}{(1 - 2x)^2}. \]

Remark 4. In work of Boyadzhiev [5, p.4, Eq-(7)], we see that

\[ \sum_{j=0}^{k} \binom{k}{j} j^n x^j = \sum_{j=0}^{n} \binom{n}{j} j! 2^{n-j} S_2(m, j) x^j (1 + x)^{n-j}. \]

Substituting \( x = 1 \) into the above equation, we arrive at (3.4).

Theorem 5. Let \( d \) be a positive integer and \( m_0, m_1, m_2, ..., m_d \in \mathbb{Q} \). Let \( m_0 \neq 0 \). Thus we have

\[ \sum_{v=0}^{d-1} m_v B(d - v, k) = 2^{k-d} \left( \begin{array}{c} k \\ d \end{array} \right). \] (3.5)

Proof. It is well-known that

\[ (1 + x) = \sum_{j=0}^{k} \binom{k}{j} x^j. \]

Taking the \( k \)th derivative, with respect to \( x \), we obtain

\[ \left( \begin{array}{c} k \\ d \end{array} \right) (1 + x)^{n-d} = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{d} x^{j-d}. \] (3.6)

Substituting \( x = 1 \) into the above equation, we get

\[ 2^{n-k} \left( \begin{array}{c} k \\ d \end{array} \right) = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{d}. \] (3.7)

In our work of [32], we know that

\[ \binom{j}{d} = m_0 j^d + m_1 j^{d-1} + \cdots + m_{d-1} j, \]
where \(m_0, m_1, \ldots, m_d \in \mathbb{Q}\). Therefore
\[
2^{k-d} \binom{k}{d} = \sum_{j=0}^{k} \binom{k}{j} (m_0 j^d + m_1 j^{d-1} + \cdots + m_d j).
\]
Thus we get
\[
2^{k-d} \binom{k}{d} = \sum_{v=0}^{d-1} m_v \sum_{j=0}^{k} \binom{k}{j} j^{d-v}.
\]
Combining (3.3) with the above equation, we have
\[
2^{k-d} \binom{k}{d} = \sum_{v=0}^{d-1} m_v B(d - v, k).
\]
Thus proof of theorem is completed. \(\square\)

There are many combinatorial arguments of (3.6). That is, if we substitute \(d = 3\) and \(4\) into (3.6), then we compute \(B(3, k)\) and \(B(4, k)\), respectively, as follows:
\[B(3, k) = k^2(k+3)2^{k-3}\]
and
\[B(4, k) = k(k^3 + 6k^2 + 3k - 2)2^{k-4}.\]
By using (3.5), we derive the following result:
\[B(d, k) = \frac{2^{k-d}}{m_0} \binom{k}{d} - \sum_{v=1}^{d-1} \frac{m_v}{m_0} B(d - v, k).\]
Therefore, we conjecture that
\[B(d, k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-2} k^2 + x_{d-1} k)2^{k-d},\]
where \(x_1, x_2, \ldots, x_{d-1}, d\) are positive integers. Consequently, we arrive at the following open questions:
1-How can we compute the coefficients \(x_1, x_2, \ldots, x_{d-1}\)?
2-We assume that for \(|x| < r\)
\[
\sum_{k=1}^{\infty} B(d, k)x^k = f_d(x).
\]
Is it possible to find \(f_d(x)\) function?

3.1. **Recurrence relation and some identities for the numbers \(y_1(n, k; \lambda)\).** Here, by applying derivative operator to the generating functions (3.1), we give a recurrence relation and other formulas for the numbers \(y_1(n, k; \lambda)\).

**Theorem 6.** Let \(k\) be a positive integer. Then we have
\[y_1(n+1, k; \lambda) = ky_1(n, k; \lambda) - y_1(n, k-1; \lambda).\]
Proof. Taking derivative of (3.1), with respect to $t$, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t} F_{y_1}(t, k; \lambda) = k F_{y_1}(t, k; \lambda) - F_{y_1}(t, k - 1; \lambda).$$

Combining (3.1) with the above equation, we get

$$\sum_{n=1}^{\infty} y_1(n, k; \lambda) \frac{t^{n-1}}{(n-1)!} = k \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} y_1(n, k - 1; \lambda) \frac{t^n}{n!}.$$

After some elementary calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Theorem 7. Let $k$ be a positive integer. Then we have

$$\frac{\partial}{\partial \lambda} y_1(n, k; \lambda) = \sum_{j=0}^{n} \binom{n}{j} y_1(j, k - 1; \lambda).$$

Proof. Taking derivative of (3.1), with respect to $\lambda$, we obtain the following partial differential equation:

$$\frac{\partial}{\partial \lambda} F_{y_1}(t, k; \lambda) = e^t F_{y_1}(t, k - 1; \lambda). \quad (3.8)$$

Combining (3.1) with the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda} y_1(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} y_1(j, k - 1; \lambda) \frac{t^n}{n!}.$$

After some elementary calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Theorem 8. Let $k$ be a positive integer. Then we have

$$\lambda \frac{\partial}{\partial \lambda} y_1(n, k; \lambda) = ky_1(n, k; \lambda) - y_1(n, k - 1; \lambda).$$

Proof. By using (3.8), we obtain the following partial differential equation:

$$\lambda \frac{\partial}{\partial \lambda} F_{y_1}(t, k; \lambda) = kF_{y_1}(t, k; \lambda) - F_{y_1}(t, k - 1; \lambda).$$

Combining (3.1) with the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda} y_1(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} ky_1(n, k; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} y_1(n, k - 1; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$
4. A FAMILY OF NEW NUMBERS \( y_2(n, k; \lambda) \)

In this section, we define a family of new numbers \( y_2(n, k; \lambda) \) by means of the following generating function:

\[
F_{y_2}(t, k; \lambda) = \frac{1}{(2k)!} \left( \lambda e^t + \lambda^{-1} e^{-t} + 2 \right)^k = \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!}, \tag{4.1}
\]

where \( k \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{C} \).

Note that there is one generating function for each value of \( k \).

In this section, by using (4.1) with their functional equation, we derive various identities and relations including our new numbers, the Fibonacci numbers, the Lucas numbers, the Stirling numbers and the central factorial numbers.

By using (4.1), we get the following explicit formula for the numbers \( y_2(n, k; \lambda) \):

**Theorem 9.**

\[
y_2(n, k; \lambda) = \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l-j)^n \lambda^{2l-j}. \tag{4.2}
\]

**Proof.** By (4.1), we have

\[
\sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l-j)^n \lambda^{2l-j} \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result.

For \( k = 0, 1, 2, 3 \) and \( n = 0, 1, 2, 3, 4, 5 \) compute a few values of the numbers \( y_2(n, k; \lambda) \) given by Equation (4.2) as follows:

| \( n \) | \( k \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) |
|---|---|---|---|---|---|
| 0 | 1 | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) |
| 1 | 2 | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) |
| 2 | 3 | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) |
| 3 | 4 | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) |
| 4 | 5 | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) | \( \frac{1}{2\lambda} + \frac{1}{2\lambda} \) |

By using (3.1) and (4.1), we get the following functional equation:

\[
F_{y_2}(t, k; \lambda) = \frac{k!}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} F_{y_1}(t, j; \lambda) F_{y_1}(-t, k-j; \lambda^{-1}).
\]

By combining (3.1) and (4.1) with the above equation, we obtain

\[
\sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} \sum_{j=0}^{k} \left( \sum_{n=0}^{\infty} y_1(n, j; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n y_1(n, k-j; \lambda^{-1}) \frac{t^n}{n!} \right).
\]
Therefore
\[\sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} = \frac{k!}{(2k)!} \sum_{n=0}^{\infty} \sum_{l=0}^{k} \sum_{j=0}^{n} (-1)^{n-l} \binom{n}{l} y_1(l, j; \lambda) y_1(n - l, k - j; \lambda^{-1}) \frac{t^n}{n!}.\]

Comparing the coefficients of \(\frac{t^n}{n!}\) on both sides of the above equation, the numbers \(y_2(n, k; \lambda)\) is given in terms of the numbers \(y_1(n, k; \lambda)\) by the following theorem:

**Theorem 10.**
\[y_2(n, k; \lambda) = \frac{k!}{(2k)!} \sum_{j=0}^{k} \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} y_1(l, j; \lambda) y_1(n - l, k - j; \lambda^{-1}). \quad (4.3)\]

**Theorem 11.**
\[y_1(n, 2k; \lambda) = \lambda^k \sum_{j=0}^{n} \binom{n}{j} k^{n-j} y_2(j, k; \lambda).\]

**Proof.** By using (3.1) and (4.1), we get the following functional equation:
\[\lambda^k e^{kt} F_{y_2}(t, k; \lambda) = F_{y_1}(t, 2k; \lambda).\]

From the above functional equation, we obtain
\[\sum_{n=0}^{\infty} y_1(n, 2k; \lambda) \frac{t^n}{n!} = \lambda^k \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!}.\]

Therefore
\[\sum_{n=0}^{\infty} y_1(n, 2k; \lambda) \frac{t^n}{n!} = \lambda^k \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} k^{n-j} y_2(j, k; \lambda) \right) \frac{t^n}{n!}.\]

Comparing the coefficients of \(\frac{t^n}{n!}\) on both sides of the above equation, we arrive at the desired result.

By substituting \(\lambda = 1\) into (4.1), we have
\[F_{y_2}(t, k) = \frac{1}{(2k)!} (e^t + e^{-t} + 2)^k.\]

The function \(F_{y_2}(t, k)\) is an even function. Consequently, we get the following result:
\[y_2(2n + 1, k; 1) = 0.\]

Thus, we get
\[F_{y_2}(t, k; 1) = \sum_{n=0}^{\infty} y_2(2n, k; 1) \frac{t^{2n}}{(2n)!}. \quad (4.4)\]

By using (4.4), we give the following explicit formula for the numbers \(y_2(n, k)\) (\(= y_2(n, k; 1)\)):

**Corollary 1.**
\[y_2(n, k) = \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l - j)^n. \quad (4.5)\]
For \( k = 0, 1, 2, \ldots, 9 \), we compute a few values of the numbers \( y_2(n, k) \) given by Equation (4.5) as follows:

\[
y_2(0, 0) = 1,
y_2(n, 0) = 0, \ (n \in \mathbb{N})
y_2(n, 1) = (-1)^n + 1,
y_2(n, 2) = \frac{((-1)^n + 1) + 2^{n-1} - (-2)^{n-1}}{3},
y_2(n, 3) = \frac{((-1)^n + 1) + 2^{n-2} + (-2)^{n-2}}{15} + \frac{(-3)^{n-2} + 3^{n-2}}{10},
y_2(n, 4) = \frac{13((-1)^n + 1) + 2^{n-1} - (-2)^{n-1}}{5040} + \frac{2^{n-4} + (-2)^{n-4}}{105},
y_2(n, 5) = \frac{19((-1)^n + 1) + 5^{n-2} + (-5)^{n-2}}{120960} + \frac{2^{n-5} - (-2)^{n-5}}{560} + \frac{2^{n-5} - (-2)^{n-5}}{2835},
y_2(n, 6) = \frac{67((-1)^n + 1) + 5^{n-2} + (-5)^{n-2}}{13305600} + \frac{2^{n-3} - (-2)^{n-3}}{31185} + \frac{2^{n-8} + (-2)^{n-8}}{31185},
y_2(n, 7) = \frac{41((-1)^n + 1) + 7^{n-2} + (-7)^{n-2}}{296524800} + \frac{2^{n-3} - (-2)^{n-3}}{1601600} + \frac{3^{n-5} - (-3)^{n-5}}{1281280} + \frac{2^{n-9} - (-2)^{n-9}}{2027025},
y_2(n, 8) = \frac{53((-1)^n + 1) + 7^{n-2} + (-7)^{n-2}}{20118067200} + \frac{6^{n-3} - (-3)^{n-3}}{63851253575} + \frac{2^{n-7} - (-2)^{n-7}}{18243225} + \frac{3^{n-5} - (-3)^{n-5}}{6081075} + \frac{2^{n-11} - (-2)^{n-11}}{9152000}.
\]
For \( k = 0 \), we have

\[
\begin{align*}
y_2(0, 0) &= 1, \\
y_2(0, 1) &= 2, \\
y_2(0, 2) &= \frac{2}{3}, \\
y_2(0, 3) &= \frac{5}{36}, \\
y_2(0, 4) &= \frac{63}{5292},
\end{align*}
\]

and for \( k = 0, 1, 2, \ldots, 9 \) and \( n = 1, 2, \ldots, 9 \), we compute a few values of the numbers \( y_2(n, k) \) given by Equation (4.5) as follows:

\[
\begin{array}{cccccccccc}
\text{n} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 2 & 2 & 4 & 2 & 15 & 315 & 2835 & 155925 & 6081075 & 638512875 & 10854718875 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 2 & 8 & 8 & 2 & 22 & 84 & 2042 & 155925 & 6081075 & 638512875 & 10854718875 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 1 & 2 & 17 & 47 & 184 & 152 & 454 & 155925 & 6081075 & 638512875 & 10854718875 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 1 & 65 & 338 & 1957 & 2144 & 7984 & 2672 & 41462 & 15206 & 10854718875 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

This function is related to the \( \cosh t \). That is

\[
F_{y_2}(t, k) = \frac{2}{(2k)!} (\cosh t + 1)^k
\]

By using this function, we get the following combinatorial sums:

**Theorem 12.**

\[
y_2(n, k; 1) = \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l-j)^{2n}
\]
and also
\[ \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \left( \binom{j}{l} (2l-j)^{2n+1} = 0. \right. \]

**Proof.** By using (4.4), we have
\[ \sum_{n=0}^{\infty} y_2(n, k) \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \left( \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \left( \binom{j}{l} (2l-j)^{2n} \right) \right) \frac{t^n}{n!}. \]

Comparing the coefficients of \( t^{2n} \) on both sides of the above equation, we arrive at the desired result. \( \square \)

By using (4.4), we obtain
\[ F_{y_1}(t, 2k; 1)e^{-kt} = \frac{k!}{(2k)!} \sum_{v=0}^{k} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{v=0}^{k} \sum_{j=0}^{n} \left( \binom{n}{v} y_2(j, k; 1) y_1(n-j, k-v; 1) \right). \]

By using the above functional equation, we obtain the following theorem:

**Theorem 13.**
\[ \sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} y_1(j, 2k; 1) = \frac{k!}{(2k)!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{v=0}^{k} \sum_{j=0}^{n} \left( \binom{n}{v} y_2(j, k; 1) y_1(n-j, k-v; 1) \right). \]

**Lemma 1.** (24, Lemma 11, Eq-(7))
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n, k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n, n-2k), \]
where \([x]\) denotes the greatest integer function.

**Theorem 14.**
\[ y_1(n, 2k; 1) = \sum_{j=0}^{n \left\lfloor \frac{n}{2j} \right\rfloor} \binom{n}{2j} k^{n-2j} y_2(j, k; 1). \]

**Proof.** By using (4.4), we obtain the following functional equation:
\[ F_{y_1}(t, 2k; 1) = F_{y_2}(t, k)e^{kt} \]

Combining this equation with (3.1), we get
\[ \sum_{n=0}^{\infty} y_1(n, 2k; 1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{(kt)^n}{n!} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} y_2(n, k; 1) \frac{t^{2n}}{(2n)!} \right). \]

Applying Lemma 1 in the above equation, we have
\[ \sum_{n=0}^{\infty} y_1(n, 2k; 1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n \left\lfloor \frac{n}{2j} \right\rfloor} \frac{k^{n-2j}}{(2j)! (n-2j)!} y_2(j, k; 1) \right) t^n. \]
Comparing the coefficients of $t^n$ on both sides of the above equation, we arrive at the desired result.

We now present an explicit relation between the Lucas numbers $L_n$ and the numbers $y_2(n,k;1)$ by the following theorem:

**Theorem 15.** Let $a + b = 1$, $ab = -1$ and $\frac{a-b}{2} = c = \frac{\sqrt{5}}{2}$. Then we have

$$L^{(k)}_n = \sum_{j=0}^{k} \binom{k}{j} t^{n}(2j)!(-2)^{k-j} \sum_{m=0}^{\left\lfloor \frac{n}{2m} \right\rfloor} c^{2m} y_2(m,j;1) \binom{k}{2}^{n-2m},$$

where $L^{(k)}_n$ denotes the Lucas numbers of order $k$.

**Proof.** In [20, pp. 232-233] and [7], the Lucas numbers $L_n$ are defined by means of the following generating function:

$$e^{at} + e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!}.$$  

From the above, we have

$$F_L(t, k; a, b) = (e^{at} + e^{bt})^k = \sum_{n=0}^{\infty} L^{(k)}_n \frac{t^n}{n!}.$$  

(4.6)

where

$$L^{(k)}_n = \sum_{j=0}^{n} \binom{n}{j} L^{(m)}_n L^{(k-m)}_n.$$  

By combining (4.6) with (4.1), we obtain the following functional equation

$$F_L(t, k; a, b) = e^{tk} \sum_{j=0}^{k} \binom{k}{j} (-2)^{k-j} (2j)! F_{y_2}(ct, j; 1).$$

Since $F_{y_2}(ct, j; \lambda)$ is an even function, we have

$$\sum_{n=0}^{\infty} L^{(k)}_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{k}{2}^{n} \frac{t^n}{n!} \sum_{j=0}^{k} \binom{k}{j} (-2)^{k-j} (2j)! \sum_{m=0}^{\infty} y_2(m,j;1) e^{m} \frac{t^{2m}}{(2m)!}.$$  

Applying Lemma [1] in the above equation, we get

$$\sum_{n=0}^{\infty} L^{(k)}_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (2j)! (-2)^{k-j} \sum_{m=0}^{\left\lfloor \frac{n}{2m} \right\rfloor} c^{2m} y_2(m,j;1) \binom{k}{2}^{n-2m} \frac{t^n}{n!}.$$  

Comparing the coefficients of $t^n$ on both sides of the above equation, we arrive at the desired result.  

We also present an identity including the Fibonacci numbers $f_n$, the Lucas numbers $L_n$ and the numbers $y_1(n,k;1)$ by the following theorem:
Theorem 16. Let $a + b = 1$, $ab = -1$ and $\frac{a - b}{2} = c = \frac{\sqrt{5}}{2}$. Then we have

$$L_n^{(k)} = k! \sum_{j=0}^{n} \binom{n}{j} (2c)^{n-j} y_1(n-j, k; 1) \left( f_j (a - 2ck^j) + f_{j-1} \right).$$

Proof. We set

$$F_f(t, a, b) = \frac{e^{at} - e^{bt}}{a - b} = \sum_{n=0}^{\infty} \frac{f_n t^n}{n!}$$

(cf. [20] p. 232, [7]). By combining (4.6) and (3.1) with the above equation, we obtain the following functional equation

$$F_L(t, k; a, b) = k! F_y_1(2ct, k; 1) \left( e^{akt} - 2c F_f(kt, a, b) \right).$$

Therefore

$$\sum_{n=0}^{\infty} \frac{L_n^{(k)} t^n}{n!} = k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} a^j (2c)^{n-j} y_1(n-j, k; 1) \frac{t^n}{n!}$$

$$- k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (2c)^{n-j+1} y_1(n-j, k; 1) k^j f_j \frac{t^n}{n!}.$$  

After some elementary calculations and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

\[ \square \]

4.1. Recurrence relation for the numbers $y_2(n, k; \lambda)$. Here, taking derivative of (4.1), with respect to $t$, we give a recurrence relation for the numbers $y_2(n, k; \lambda)$.

Theorem 17. Let $k$ be a positive integer. Then we have

$$y_2(n+1, k; \lambda) = ky_2(n, k; \lambda) - y_2(n, k-1; \lambda) - \lambda^{-1} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} y_2(j, k-1; \lambda).$$

Proof. Taking derivative of (4.1), with respect to $t$, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t} F_{y_2}(t, k; \lambda) = k F_{y_2}(t, k; \lambda) - F_{y_2}(t, k-1; \lambda) - \lambda^{-1} e^{-t} F_{y_2}(t, k-1; \lambda).$$

Combining (4.1) with the above equation, we obtain

$$\sum_{n=1}^{\infty} y_2(n, k; \lambda) \frac{t^{n-1}}{(n-1)!} = k \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} y_2(n, k-1; \lambda) \frac{t^n}{n!}$$

$$- \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} y_2(j, k-1; \lambda) \frac{t^n}{n!}.$$  

After some elementary calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.  

\[ \square \]
Theorem 18. Let \( k \) be a positive integer. Then we have
\[
\frac{\partial}{\partial t} y_2(n, k; \lambda) = \lambda y_2(n, k; \lambda) - \frac{\lambda}{k(2k-1)} y_2(n, k-1; \lambda).
\]

Proof. Taking derivative of (4.1), with respect to \( \lambda \), we obtain the following partial differential equation:
\[
\frac{\partial}{\partial \lambda} F_{y_2}(t, k; \lambda) = \lambda F_{y_2}(t, k; \lambda) - \frac{\lambda}{k(2k-1)} F_{y_2}(t, k-1; \lambda).
\]
Combining (4.1) with the above equation, we get
\[
\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda} y_2(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \lambda y_1(n, k; \lambda) \frac{t^n}{n!} - \frac{\lambda}{k(2k-1)} \sum_{n=0}^{\infty} y_1(n, k-1; \lambda) \frac{t^n}{n!}.
\]
After some elementary calculation, comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result. \( \square \)

5. \( \lambda \)-central factorial numbers \( C(n, k; \lambda) \)

In this section, we define \( \lambda \)-central factorial numbers \( C(n, k; \lambda) \) by means of the following generating function:
\[
F_{C}(t, k; \lambda) = \frac{1}{(2k)!} (\lambda e^t + \lambda^{-1} e^{-t} - 2)^k = \sum_{n=0}^{\infty} C(n, k; \lambda) \frac{t^n}{n!}
\]
where \( k \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{C} \).
Note that there is one generating function for each value of \( k \).
For \( \lambda = 1 \), we have the central factorial numbers
\[
T(n, k) = C(n, k; 1)
\]
(cf. [2], [11], [16], [31], [39]).

Theorem 19.
\[
T(n, k; \lambda^2) = 2^{-n} (2k)! \sum_{j=0}^{n} \binom{n}{j} C(j, k; \lambda) y_2(n-j, k; \lambda).
\]

Proof. By using (4.1) and (5.1), we get the following functional equation:
\[
F_{C}(2t, k; \lambda^2) = (2k)! F_{C}(t, k; \lambda) F_{y_2}(t, k; \lambda).
\]
From this equation, we get
\[
\sum_{n=0}^{\infty} C(n, k; \lambda^2) \frac{(2t)^n}{n!} = (2k)! \sum_{n=0}^{\infty} C(n, k; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!}.
\]
Therefore
\[
\sum_{n=0}^{\infty} C(n, k; \lambda^2) \frac{2n^2 t^n}{n!} = \sum_{n=0}^{\infty} (2k)! \sum_{j=0}^{n} \binom{n}{j} C(j, k; \lambda) y_2(n-j, k; \lambda) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result.

By using (2.6) and (4.4), we obtain the following functional equation:

\[
F_T(t, k)F_{y^2}(t, k) = \frac{1}{(2k)!} F_T(2t, k).
\]

Combining the above equation with (2.6) and (4.4), we get

\[
\frac{1}{(2k)!} \sum_{n=0}^{\infty} T(n, k) \frac{2^n t^n}{n!} = \sum_{n=0}^{\infty} y_2(j, k) \frac{t^n}{n!} \sum_{n=0}^{\infty} T(n, k) \frac{t^n}{n!}.
\]

Therefore

\[
\frac{1}{(2k)!} \sum_{n=0}^{\infty} T(n, k) \frac{2^n t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} y_2(j, k; 1) T(n - j, k) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we obtain a relationship between the central factorial numbers \( T(n, k) \) and the numbers \( y_2(j, k) \) by the following theorem:

**Theorem 20.**

\[
T(n, k) = 2^{-n} (2k)! \sum_{j=0}^{n} \binom{n}{j} y_2(j, k; 1) T(n - j, k).
\]

**Remark 5.** In [3], Alayont et al. have studied the rook polynomials, which count the number of ways of placing non-attacking rooks on a chess board. By using generalization of these polynomials, they gave the rook number interpretations of generalized central factorial and the Genocchi numbers.

In [2], Alayont and Krzywonos gave the following results for the classical central factorial numbers:

The number of ways to place \( k \) rooks on a size \( m \) triangle board in three dimensions is equal to \( T(m + 1, m + 1 - k) \), where \( 0 \leq k \leq m \).

6. **Application in Statistics: In the binomial distribution and the Bernstein polynomials**

Let \( n \) be a nonnegative integer. For every function \( f : [0, 1] \to \mathbb{R} \) and the \( n^{th} \) Bernstein polynomial of \( f \) is defined by

\[
B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{n}{k} \right) B_k^n(x),
\]

where \( B_k^n(x) \) denotes the Bernstein basis functions:

\[
B_k^n(x) = \binom{n}{k} x^k (1 - x)^{n-k}
\]
and \( x \in [0, 1] \). Let \((U_k)_{k \geq 1}\) be a sequence of independent distributed random variable having the uniform distribution on \([0, 1]\) and defined by Adel et al. [1]:

\[
S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(U_k).
\]

In [1], it is well-know that, \( S_n(x) \) is a binomial random variable. That is the theory of Probability and Statistics, the binomial distribution is very useful. This distribution, with parameters \( n \) and probability \( x \), is the discrete probability distribution. This distribution is defined as follows:

\[
P(S_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k},
\]

where \( k = 0, 1, 2, \ldots, n \). \( E \) denotes mathematical expectation, than

\[
Ef \left( \frac{S_n(x)}{n} \right) = B_n(f, x)
\]

(cf. [1]). For any \( x \in (0, 1) \), \( n \geq 2 \), and \( r > 1 \), Adel et al. [1] defined

\[
E (S_n(x))^r = \sum_{k=0}^{n} \binom{n}{k} k^r x^k (1-x)^{n-k}.
\]

(6.1)

Substituting \( x = \frac{1}{2} \) into (6.1), we get

\[
E \left( S_n \left( \frac{1}{2} \right) \right)^r = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} k^r.
\]

(6.2)

By combining (3.2) with (6.2), we arrive at the following theorem:

**Theorem 21.** Let \( n \geq 2 \). Let \( r \) be a positive integer with \( r > 1 \). Then we have

\[
y_1(r, n) = \frac{2^n}{n!} E \left( S_n \left( \frac{1}{2} \right) \right)^r.
\]

Integrating (6.1) from 0 to 1, we get

\[
\int_{0}^{1} E (S_n(x))^r \, dx = \frac{1}{n+1} \sum_{k=0}^{n} k^r.
\]

By substituting (2.1) into the above equation, we arrive at the following theorem:

**Theorem 22.**

\[
\int_{0}^{1} E (S_n(x))^r \, dx = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{(n+1)(r+1)}.
\]
7. Computation of the Euler numbers of negative order

In this section, we not only give elementary properties of the first and the second kind Euler polynomials and numbers, but also compute the first kind of Apostol type Euler numbers associated with the numbers \( y_1(n, k; \lambda) \) and \( y_2(n, k; \lambda) \).

We define the second kind Apostol type Euler polynomials of order \( k \), with \( k \geq 0 \), \( E_n^{(k)}(x; \lambda) \) by means of the following generating functions:

\[
F_P(t, x; k, \lambda) = \left( \frac{2}{\lambda e^t + \lambda^{-1} e^{-t}} \right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x; \lambda) \frac{t^n}{n!}.
\]

Substituting \( x = 0 \) into the above equation, we get the second kind Apostol type Euler numbers of order \( k \), \( E_n^{(k)}(\lambda) \) by means of the following generating function:

\[
F_N(t; k, \lambda) = \left( \frac{2}{\lambda e^t + \lambda^{-1} e^{-t}} \right)^k = \sum_{n=0}^{\infty} E_n^{(k)}(\lambda) \frac{t^n}{n!}.
\]

If we substitute \( k = \lambda = 1 \) into the above generating function, then we have

\[
E_n^* = E_n^{(1)}(1).
\]

The first kind Apostol-Euler numbers of order \( -k \) are defined by means of the following generating function:

\[
G_E(t, -k; \lambda) = \left( \frac{\lambda e^t + 1}{2} \right)^k = \sum_{n=0}^{\infty} E_n^{(-k)}(\lambda) \frac{t^n}{n!}. \tag{7.1}
\]

The second kind Apostol type Euler numbers of order \( -k \) are defined by means of the following generating function:

\[
F_N(t; -k, \lambda) = \left( \frac{\lambda e^t + \lambda^{-1} e^{-t}}{2} \right)^k = \sum_{n=0}^{\infty} E_n^{(-k)}(\lambda) \frac{t^n}{n!}. \tag{7.2}
\]

The numbers \( E_n^{(-k)}(\lambda) \) are related to the numbers \( E_n^{(-k)}(\lambda) \) and the Apostol Bernoulli numbers \( B_n^{(-k)}(\lambda) \) of the negative order. By using (7.2), we get the following functional equation:

\[
F_N(t; -k, \lambda) = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} (-1)^j G_E(t, -j; \lambda) H_B(-t, -k + j; \lambda^{-1}),
\]

where

\[
H_B(t, -k; \lambda) = \left( \frac{\lambda e^t - 1}{t} \right)^k = \sum_{n=0}^{\infty} B_n^{(-k)}(\lambda) \frac{t^n}{n!} \quad (c.f. \ [21], \ [22], \ [35]).
\]

By using this equation, we get

\[
\sum_{n=0}^{\infty} E_n^{(-k)}(\lambda) \frac{t^n}{n!} = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} (-1)^j \sum_{n=0}^{\infty} E_n^{(-j)}(\lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n^{(-k+j)}(\lambda^{-1}) \frac{(-t)^n}{n!}.
\]
Therefore

\[
\sum_{n=0}^{\infty} E_n^{(-k)}(\lambda) \frac{t_n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \sum_{l=0}^{n-k+j} (-1)^{n+j-k-l} \binom{n-k+j}{l} 2^{j-k} (n)_{k-j} E_{j}^{(-j)}(\lambda) B_{n+j-k-l}(\lambda^{-1}) \frac{t_n}{n!}.
\]

Comparing the coefficients of \( \frac{t_n}{n!} \) on both sides of the above equation, we arrive at the following theorem:

**Theorem 23.**

\[
E_n^{(-k)}(\lambda) \frac{t_n}{n!} = \sum_{j=0}^{k} \binom{k}{j} \sum_{l=0}^{n-k+j} (-1)^{n+j-k-l} \binom{n-k+j}{l} 2^{j-k} (n)_{k-j} E_{j}^{(-j)}(\lambda) B_{n+j-k-l}(\lambda^{-1}).
\]

After the results of the preceding sections, we are ready to compute the Euler numbers of negative order.

Combining (3.1) and (7.1), we get

\[
k!2^k \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t_n}{n!} = \sum_{n=0}^{\infty} E_n^{(-k)}(\lambda) \frac{t_n}{n!}.
\]

Comparing the coefficients of \( \frac{t_n}{n!} \) on both sides of the above equation, we arrive at the following theorem:

**Theorem 24.** Let \( k \) be nonnegative integer. Then we have

\[
E_n^{(-k)}(\lambda) = k!2^{-k} y_1(n, k; \lambda).
\]  

(7.3)

**Remark 6.** Substituting \( \lambda = 1 \) into (7.3), we obtain the following explicit formula for the first kind Euler numbers of order \(-k\) as follows:

\[
E_n^{(-k)} = 2^{-k} \sum_{j=0}^{k} \binom{k}{j} j^n
\]  

(7.4)
For \( k = 0, -1, -2, \ldots, -7 \), we compute a few values of the numbers \( E_n^{(-k)} \) given by Equation (7.4) as follows:

\[
\begin{align*}
E_0^{(0)} &= 1, \\
E_1^{(0)} &= 0, (n \neq 0) \\
E_2^{(-1)} &= \frac{1}{2}, \\
E_3^{(-2)} &= 2^{n-2} + \frac{1}{2}, \\
E_4^{(-3)} &= \frac{3n}{8} + 3.2^{n-3} + \frac{3}{8}, \\
E_5^{(-4)} &= \frac{3n^2}{4} + 4^{n-2} + 3.2^{n-3} + \frac{1}{4}, \\
E_6^{(-5)} &= \frac{5n^3}{32} + \frac{5.3n^2}{16} + \frac{5.4n^2}{2} + 5.2^{n-4} + \frac{5}{32}, \\
E_7^{(-6)} &= \frac{6n^4}{64} + \frac{3.5n^3}{32} + \frac{5.3n^3}{16} + 15.4^{n-3} + 15.2^{n-6} + \frac{3}{32}, \\
E_8^{(-7)} &= \frac{7n^5}{128} + \frac{7.6n^4}{128} + \frac{21.5n^4}{128} + \frac{35.3n^4}{128} + \frac{35.4^{n-3}}{2} + 21.2^{n-7} + \frac{7}{128}, \ldots
\end{align*}
\]

That is for \( n = 0, 1, 2, \ldots, 9 \) and \( k = 0, -1, -2, \ldots, -9 \), we compute a few values of the numbers \( E_n^{(-k)} \), given by the above relations, as follows:

\[
\begin{array}{cccccccccccc}
n \backslash k & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots & \ldots \\
1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
2 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
3 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots
\end{array}
\]

Theorem 25. Let \( k \) be nonnegative integer. Then we have

\[
y_2(n, k; \lambda) = \frac{2^k}{(2k)!} \sum_{l=0}^{k} \binom{k}{l} E_n^{*(l)}(\lambda).
\]

Proof. By using (4.1) and (7.2), we get the following functional equation:

\[
F_C(t, k; \lambda) = \frac{1}{(2k)!} \sum_{l=0}^{k} \binom{k}{l} 2^k F_N(t, -l, \lambda).
\]
From this equation, we obtain
\[ \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{k} \binom{k}{l} 2^k E_{n-l}^{(-1)}(\lambda) \right) \frac{t^n}{n!}. \]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result.

**Theorem 26.** Let \( k \) be nonnegative integer. Then we have
\[ y_2(n, k; \lambda) = \frac{(-1)^n 2^{n+k}}{(2k)!} \sum_{l=0}^{k} \binom{k}{l} \lambda^{-l} E_{n-l}^{(-1)} \left( -\frac{l}{2}; \lambda^2 \right). \]

**Proof.** By using (4.1) and (2.2), we get the following functional equation:
\[ F_C(t, k; \lambda) = 2^k \sum_{l=0}^{k} \binom{k}{l} \lambda^{-l} F_{P_1} \left( 2t, -\frac{l}{2}; -k, \lambda^2 \right). \]
From this equation, we obtain
\[ \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{k} \binom{k}{l} 2^k \frac{n!}{E_{n-l}^{(-1)}(\lambda) \lambda^{-l}} \right) \frac{t^n}{n!}. \]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the desired result.

By applying derivative operator to the generating function in (3.1), we give a relationship between the numbers \( y_1(n, k; \lambda) \) and \( E_{n-1}^{(-1)}(\lambda) \) by the following theorem:

**Theorem 27.** Let \( k \geq 2 \). Then we have
\[ y_1(n + 2, k; \lambda) = ky_1(n, k; \lambda) + y_1(n, k - 2; \lambda) - y_1(n, k - 1; \lambda) + 2k \sum_{l=0}^{n} \binom{n}{l} E_{l}^{(-1)}(\lambda) y_1(n - l, k; \lambda). \]

**Proof.** By applying derivative operator to (3.1) with respect to \( t \), we obtain the following partial differential equation:
\[ \frac{\partial^2}{\partial t^2} F_{y_1}(t, k; \lambda) = k (F_{y_1}(t, k; \lambda) + 2G_E(t, 1; \lambda)) + F_{y_1}(t, k - 2; \lambda) - F_{y_1}(t, k - 1; \lambda). \]
Combining (3.1) and (7.1) with the above equation, we get
\[ \sum_{n=2}^{\infty} y_1(n, k; \lambda) \frac{t^{n-2}}{(n-2)!} = k \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} y_1(n, k - 2; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} y_1(n, k - 1; \lambda) \frac{t^n}{n!} + 2k \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} E_l^{(-1)}(\lambda) y_1(n - l, k; \lambda) \frac{t^n}{n!}. \]
We make arrangement of the series and then compare the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, and we obtain the desired result. \( \square \)

8. Algorithms and Computation

Computer science and Applied Mathematics have study information and computation and also their theoretical foundations. In these areas practical techniques are very important (cf. [44]). Therefore algorithmic processes play a very important role in both areas. Thus, in this section, we give two algorithms for the computation of the numbers \( y_1(n, k; \lambda) \) and \( y_2(n, k; \lambda) \).

9. Combinatorial applications and further remarks

In this section, we discuss some combinatorial interpretations of these numbers, as well as the generalization of the central factorial numbers given in Section 3-5. These interpretations includes the rook numbers and polynomials and combinatorial interpretation for the numbers \( y_1(n, k) \). We see that our numbers are associated with known counting problems. By using counting techniques and generating functions techniques, Bona [4] rederived several known properties and novel relations involving enumerative combinatorics and related areas. A very interesting further special cases of the numbers \( B(n, k) \) is worthy of note by the work of Bona [4]. That is, in [4, P. 46, Exercise 3-4], Bona gave the following two exercises which are associated with the numbers \( B(n, k) \):

**Exercise 3.** Find the number of ways to place \( n \) rooks on an \( n \times n \) chess board so that no two of them attack each other.

**Exercise 4.** How many ways are there to place some rooks on an \( n \times n \) chess board so that no two of them attack each other?

**Remark 7.** Our numbers occur in combinatorics applications. In [4, P. 46, Exercise 3-4], Bona gave detailed and descriptive solution of these two exercises, which are related to the numbers \( B(n, k) \), respectively as follows:

There has to be one rook in each column. The first rook can be anywhere in its column (\( n \) possibilities). The second rook can be anywhere in its column except in the same row where the first rook is, which leaves \( n - 1 \) possibilities. The third rook can be anywhere in its column, except in the rows taken by the first and second rook, which leaves \( n - 2 \) possibilities, and so on, leading to \( n(n - 1) \cdots 2.1 = n! \) possibilities.

Exercise 4. If we place \( k \) rooks, then we first need to choose the \( k \) columns in which these rooks will be placed. We can do that in \( \binom{n}{k} \) ways. Continuing the line of thought of the solution of the previous exercise, we can then place our \( k \) rooks into the chosen columns in \( (n)_k \) ways. Therefore, the total number of possibilities is

\[
\sum_{k=1}^{n} \binom{n}{k} (n)_k.
\]

**Remark 8.** In (3.7), for \( j < d \), it is well-known that \( \binom{j}{d} = 0 \).
Therefore, we arrive at solutions of Exercise 16 (a) in [4, p. 55, Exercise 16(a)] and also Exercise 10 [10, p. 126] as follows:

\[ 2^{n-k} \binom{k}{d} = \sum_{j=d}^{k} \binom{j}{d} \binom{k}{j} \]

10. Conclusions

In this paper, we have constructed some new families of special numbers with their generating functions. We give many properties of these numbers. These numbers are related to the many well-known numbers, which are the Bernoulli numbers, the Stirling numbers of the second kind, the central factorial numbers and also related to the Golombek’s problem [14] “Aufgabe 1088”. We have discussed some combinatorial interpretations of these numbers. Besides, we give some applications about not only rook polynomials and numbers, but also combinatorial sum.

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