SIGMA-CONVERGENCE OF SEMILINEAR STOCHASTIC WAVE EQUATIONS

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ABSTRACT. We address the homogenization of a semilinear hyperbolic stochastic partial differential equation with highly oscillating coefficients, in the context of ergodic algebras with mean value. To achieve our goal, we use a suitable variant of the sigma-convergence concept that takes into account both the random and deterministic behaviors of the phenomenon modelled by the underlying problem. We also provide an appropriate scheme for the approximation of the effective coefficients. To illustrate our approach, we work out some concrete problems such as the periodic homogenization problem, the almost periodic and the asymptotically almost periodic ones.

1. Introduction and the main results

The need for taking random fluctuations into account in the study of complex systems and physical phenomena resulting from the modeling to predictions is now widely recognized by scientific community. Wave propagation described by hyperbolic partial differential equations is one of the typical physical phenomena widely observed in the nature, and has been studied over the years and continue to attract the attention of scientists aiming at understanding some physical phenomena such as sonic booms and bottleneck in traffic flows. However due to the presence of turbulence, the more realistic way to model and capture physical features of natural phenomena at large scale is to introduce stochastic models. Stochastic partial differential equations (SPDEs) are the most convenient mathematical models arising from modeling of complex systems undergoing random influences.

Our aim in the current work is to analyze such a model represented by a semilinear stochastic wave equation that can be used to study some problems in nonlinear optics or the ones related to wave motion through the ocean or the atmosphere. To this end, the problem we address is stated as follows.

Let \( Q \) be a Lipschitz domain of \( \mathbb{R}^N \) and \( T \) a positive real number. By \( Q_T \) we denote the cylinder \( Q \times (0,T) \). Let \( \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1}\} \) a stochastic basis, that is a filtered probability space with \( \{W^k\}_{k \geq 1} \) a sequence of independent standard one dimensional Brownian motions relative to \( \mathcal{F}_t \). Fix a separable Hilbert space \( U \) with an associated orthonormal basis \( \{e_k\}_{k \geq 1} \). We may define a cylindrical Wiener process by setting \( W = \sum_{k=1}^{\infty} W^k e_k \) (see [13]). By \( L_2(U,X) \) we denote the space of Hilbert-Schmidt operators from \( U \) to a Hilbert space \( X \). We also define the auxiliary space \( \mathcal{U}_0 \supset \mathcal{U} \) via \( \mathcal{U}_0 = \{v = \sum_{k \geq 1} \alpha_k e_k : \sum_{k \geq 1} \alpha_k^2 k^{-2} < \infty \} \), endowed with the norm \( \|v\|_{\mathcal{U}_0}^2 = \sum_{k=1}^{\infty} \alpha_k^2 k^{-2} \), for \( v = \sum_{k \geq 1} \alpha_k e_k \). It is a well known fact that there exists \( \Omega' \in \mathcal{F} \) with \( \mathbb{P}(\Omega') = 1 \) such that \( W(\omega) \in C(0,T;\mathcal{U}_0) \) for any \( \omega \in \Omega' \) (see [13]).

We consider the following semilinear stochastic hyperbolic initial value problem

\[
\begin{aligned}
&du_\epsilon' - \text{div}(A_0(\epsilon) \nabla u_\epsilon) \, dt = f_\epsilon(\epsilon, x, u_\epsilon) \, dt + g_\epsilon(\epsilon, x, u_\epsilon) \, dW \quad \text{in } Q_T, \\
u_\epsilon = 0 &\quad \text{on } \partial Q \times (0,T), \\
u_\epsilon(x,0) = u^0(x) &\quad \text{and } u_\epsilon'(x,0) = u^1(x) \quad \text{in } Q,
\end{aligned}
\]

where

\[
\begin{aligned}
&d\tilde{u}_\epsilon - \text{div}(A_0(\epsilon) \nabla \tilde{u}_\epsilon) \, dt = f(\tilde{u}_\epsilon, x, \tilde{u}_\epsilon) \, dt + g(\tilde{u}_\epsilon, x, \tilde{u}_\epsilon) \, dW \quad \text{in } Q_T, \\
&\tilde{u}_\epsilon = 0 &\quad \text{on } \partial Q \times (0,T), \\
&\tilde{u}_\epsilon(x,0) = u^0(x) \quad \text{and } \tilde{u}_\epsilon'(x,0) = u^1(x) \quad \text{in } Q.
\end{aligned}
\]

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where $\varepsilon > 0$ is sufficiently small and $u'_\varepsilon$ is the time derivative of $u_\varepsilon$ ($u'_\varepsilon = \frac{\partial u_\varepsilon}{\partial t}$). We assume that the coefficients of (1.1) are constrained as follows:

**(A1) Uniform ellipticity.** The function $A_0^\varepsilon$ defined by $A_0^\varepsilon(x) = A_0(x, x/\varepsilon)$ satisfies $A_0 \in C(\Omega; L^\infty(\mathbb{R}^N_y))^{N \times N}$ and is a $N \times N$ symmetric matrix satisfying the following assumptions:

$$A_0 \eta : \eta \geq \alpha |\eta|^2$$

for all $\eta \in \mathbb{R}^N$ and a.e. in $\mathbb{R}^N$, where $\alpha > 0$ is a given constant not depending on $x, t, y$ and $\eta$.

**(A2) Lipschitz continuity of $f$.** The function $f : (y, \tau, \lambda) \mapsto f(y, \tau, \lambda)$ from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ satisfies the properties:

(i) $f$ is measurable,
(ii) $f(y, \tau, 0) = 0$ for a.e. $y$ and $\tau$,
(iii) there exists a constant $c_1 > 0$ such that

$$|f(y, \tau, \lambda) - f(y, \tau, \mu)| \leq c_1 |\lambda - \mu|$$

for almost $y, \tau$, and for all $\lambda, \mu \in \mathbb{R}$.

From (ii) and (iii) above, we infer that

(iv) there exists a constant $c_2 > 0$ such that $|f(y, \tau, \lambda)| \leq c_2 (1 + |\lambda|)$ for almost $y, \tau$, and for all $\lambda \in \mathbb{R}$.

**(A3) Lipschitz continuity of $g$.** The function $g : (y, \tau, u) \mapsto g(y, \tau, u)$ from $\mathbb{R}^N \times \mathbb{R} \times L^2(Q)$ into $L_2(\mathbb{U}, L^2(Q))$ satisfies:

(i) $g$ is measurable,
(ii) $g(y, \tau, 0) = 0$ for a.e. in $y$ and $\tau$,
(iii) there exists a constant $c_3 > 0$ such that

$$|g(y, \tau, u) - g(y, \tau, v)|_{L_2(\mathbb{U}, L^2(Q))} \leq c_3 |u - v|_{L^2(Q)}$$

for a.e. in $y, \tau$, and for all $u, v \in L^2(Q)$.

Also as above, from (ii) and (iii) above, we infer that

(iv) there exists a constant $c_4 > 0$ such that $|g(y, \tau, u)|_{L_2(\mathbb{U}, L^2(Q))} \leq c_4 \left(1 + |u|_{L^2(Q)}\right)$ for a.e. in $y, \tau$ and for all $u \in L^2(Q)$.

In the following, we introduce the notion of probabilistic strong solution for our problem (1.1).

**Definition 1.1.** A probabilistic strong solution of the problem (1.1) is a stochastic process $u_\varepsilon$ such that:

1) $u_\varepsilon$, $u'_\varepsilon$ are $\mathcal{F}_t$-measurable,
2) $u_\varepsilon \in L^2(\Omega, \mathcal{F}, P; L^\infty(0, T; H^1_0(\Omega)))$, and $u'_\varepsilon \in L^2(\Omega, \mathcal{F}, P; L^\infty(0, T; L^2(\Omega)))$,
3) $u_\varepsilon$ satisfies

$$\left(u_\varepsilon(t), \phi\right) + \int_0^t \left(A_0^\varepsilon \nabla u_\varepsilon(\tau), \nabla \phi\right) d\tau = \left(u^1, \phi\right) + \int_0^t \left(f\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon(\tau)\right), \phi\right) d\tau + \sum_{k=1}^\infty \int_0^t \left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon(\tau)\right)e_k, \phi\right) dW^k(\tau),$$

for all $\phi \in C_0^\infty(\Omega)$ and for a.e. $t \in (0, T)$.
4) $u_\varepsilon(0) = u^0$.

With this in mind, under conditions (A1)-(A3) and provided that $u^0 \in H^1_0(\Omega)$ and $u^1 \in L^2(\Omega)$, the problem (1.1) has a unique strong solution $u_\varepsilon \in L^2(\Omega; C([0, T]; H^1_0(\Omega)))$ with $u'_\varepsilon \in L^2(\Omega; C([0, T]; L^2(\Omega)))$. This existence and uniqueness result has been achieved in [12, Theorem 8.4, p. 189].

To simplify the notations, we set

$$g^\tau(\ldots, u_\varepsilon)(x, t) = g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon(x, t)\right)$$

and

$$g^\tau_k(\ldots, u_\varepsilon) = g^\tau(\ldots, u_\varepsilon)e_k$$

for $k \geq 1$. 


The question in this work is to determine the limit as $\varepsilon \to 0$, of the sequence of processes $u_\varepsilon$ under suitable assumptions on the coefficients of (1.1) (see assumption (A4) below). For a fixed probability space representing the random fluctuations space, we shall assume that the coefficients of (1.1) have various deterministic behaviours ranging from the periodicity to the weak almost periodicity. Our study therefore falls within the scope of the sigma-convergence for stochastic processes that is a generalization of the so-called sigma-convergence concept introduced in 2003 in [34]. One of the chief merits of this work lies in the fact that we do not make use of the concept of the spectrum of an algebra with mean value (viewed as a $C^*$-algebra as always considered before), thereby addressing one of the main concerns of Applied Scientists whom will therefore be able to use results arising from the use of sigma-convergence concept in homogenization theory. Indeed the corrector problem (see (4.20)) is in that case, posed on the numerical space representing by the spectrum of the underlying algebra with mean value as it was always the case in all the previous work dealing with the sigma-convergence concept. In the same direction we refer to the preprint [26], which is the first work in which we have initiated the deterministic homogenization of PDEs without appealing to the spectrum of an algebra with mean value.

Let us clearly state our main results here, in order to fix ideas. Let $A$ be an algebra with mean value on $\mathbb{R}^d$ (integer $d \geq 1$), that is, a closed subalgebra of the $C^*$-algebra of bounded uniformly continuous real-valued functions on $\mathbb{R}^d$, which contains the constants, is translation invariant and is such that any of its elements $u$ possesses a mean value $M(u)$ defined by

$$M(u) = \lim_{R \to \infty} \frac{1}{B_R} \int_{B_R} u(y)dy$$

where $B_R = B(0, R)$ is the open ball in $\mathbb{R}^d$ centered at the origin and of radius $R$. We denote by $B_A^p(\mathbb{R}^d)$ ($1 \leq p < \infty$) the completion of $A$ with respect to the seminorm

$$\|u\|_p = (M(|u|^p))^{\frac{1}{p}} = \left( \limsup_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} |u(y)|^p dy \right)^{\frac{1}{p}}.$$  

Before we may proceed forward, let us note that if $A = C_{per}(Y)$, the algebra of continuous $Y$-periodic functions on $\mathbb{R}^d (Y = (0,1)^d)$ then $B_A^p(\mathbb{R}^d) = L^p_{per}(Y)$, the space of functions in $L^p_{loc}(\mathbb{R}^d)$ that are $Y$-periodic; in that case, $M(u) = \int_Y u(y)dy$ for $u \in L^p_{per}(Y)$. Also, if $A = AP(\mathbb{R}^d)$ (the continuous Bohr almost periodic functions; see e.g., [3]), then $B_A^p(\mathbb{R}^d)$ is exactly the space of Besicovitch almost periodic functions on $\mathbb{R}^d$; see [5].

From an argument due to Besicovitch [3], it is known that $B_A^p(\mathbb{R}^d) \hookrightarrow L^p_{loc}(\mathbb{R}^d)$, so that $B_A^p(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d)$. With this embedding, we view any element of $B_A^p(\mathbb{R}^d)$ as its representative in $L^p_{loc}(\mathbb{R}^d)$, which allows us to define the following space

$$B_A^{1,p}(\mathbb{R}^d) = \{ u \in B_A^p(\mathbb{R}^d) : \nabla_y u \in (B_A^p(\mathbb{R}^d)) \},$$

equipped with the seminorm

$$\|u\|_{1,p} = \left( \|u\|_p + \|
abla_y u\|_p \right)^{\frac{1}{p}} (u \in B_A^p(\mathbb{R}^d)),$$

which is a complete seminormed space. Now, considering two algebras with mean value $A_y$ and $A_\tau$ on $\mathbb{R}^N_y$ and $\mathbb{R}_\tau$, respectively, we define the product algebra with mean value $A = A_y \otimes A_\tau$ as the closure in the sup norm in $\mathbb{R}^{N+1}$, of the tensor product $A_y \otimes A_\tau$, and we assume that the coefficients of (1.1) satisfy:

(A4) For any $\lambda \in \mathbb{R}$ the functions $(y, \tau) \mapsto f(y, \tau, \lambda)$ and $(y, \tau) \mapsto g_k(y, \tau, \lambda)$ belong to $B_A^2(\mathbb{R}^{N+1}_y)$;

for any $x \in \overline{Q}$ the matrix function $y \mapsto A_0(x, y)$ lies $(B_A^2(\mathbb{R}^N_y))^{N \times N}$. Here $g = (g_k)_{1 \leq k \leq m}$.

In what follows, we denote by the same letter $M$ the mean value on each of the algebras $A_y$, $A_\tau$ and $A$ as well. This being so, let $(e_j)_{1 \leq j \leq N}$ denote the canonical basis in $\mathbb{R}^N$. For each fixed
1 \leq j \leq N \text{ and each } x \in Q, \text{ consider the problem }
\begin{align*}
\begin{cases}
\text{Find } \chi_j(x, \cdot) \in B_{\#A^2}(\mathbb{R}^N) \text{ such that:} \\
\text{div}_y (A_0(x, \cdot)(e_j + \nabla_y \chi_j(x, \cdot))) = 0 \text{ in } \mathbb{R}^N.
\end{cases}
\end{align*}
(1.3)

Then we show that (1.3) possesses at least a solution whose gradient is unique in $B^2_{\#A^2}(\mathbb{R}^N)$. With the functions $\chi_j$ at our disposal, we define the so-called homogenized coefficients as follows:
\begin{align*}
\tilde{A}(x) &= M(A_0(x, \cdot)(I + \nabla_y \chi(x, \cdot))), \quad x \in Q \\
\tilde{f}(r) &= M(f(\cdot, \cdot, r)) \text{ and } \tilde{g}(r) = M(g(\cdot, \cdot, r)) \text{ for } r \in \mathbb{R}.
\end{align*}
(1.4)

Here $\chi(x, \cdot) = (\chi_j(x, \cdot))_{1 \leq j \leq N}$ and $I$ is the $N \times N$ identity matrix. As we expect $u_\varepsilon$ to converge strongly, only the direct average of $f$ and $g$ have to be taken, since we will not deal with the product of two weakly convergent sequences at that level.

**Remark 1.1.** It can be easily checked straightforwardly that the functions $\tilde{f}$ and $\tilde{g}$ are Lipschitz, while the matrix $\tilde{A}(x)$ is symmetric and satisfies assumptions similar to those of $A_0$ (see (A1)).

With all this in mind, the first main result of the work is the following theorem.

**Theorem 1.1.** Assume (A1)-(A4) hold. For each $\varepsilon > 0$ let $u_\varepsilon$ be the unique solution to (1.1) on a given stochastic system $(\Omega, \mathcal{F}, \mathbb{P}, W, \mathcal{F}^t)$. Then the sequence $u_\varepsilon$ converges in probability to $u_0$ in $L^2(Q_T)$, where $u_0$ is the unique strong probabilistic solution to
\begin{align*}
\begin{cases}
du_0 - \text{div} \left( \tilde{A}(x) \nabla u_0 \right) dt = \tilde{f}(u_0) dt + \tilde{g}(u_0) dW \\
u_0 = 0 \text{ on } \partial Q \times (0, T) \\
u_0(x, 0) = u^0(x) \text{ and } u_0(x, 0) = u^1(x) \text{ in } Q.
\end{cases}
\end{align*}
(1.5)

The problem (1.3) above that has been used to define the homogenized matrix $\tilde{A}(x)$ is posed on the entire numerical set $\mathbb{R}^N$ and hence the numerical computation of $\tilde{A}(x)$ is somewhat difficult. We overcome this difficulty by considering rather approximate coefficients defined as follows: for each $R > 0$ set
\begin{align*}
\tilde{A}_R(x) = \frac{1}{|B_R|} \int_{B_R} A_0(x, y)(I + \nabla_y \chi_R(x, y)) dy
\end{align*}
(1.6)

where $\chi_R(x, \cdot) = (\chi_{j,R}(x, \cdot))_{1 \leq j \leq N}$ with $\chi_{j,R}(x, \cdot) \equiv u \in H^1_0(B_R)$ being the unique solution to the Dirichlet problem
\begin{align*}
\text{div}_y (A_0(x, \cdot)(e_j + \nabla_y u)) = 0 \text{ in } B_R.
\end{align*}
(1.7)

However in practice, the appropriate computational method used for this kind of problems is the heterogeneous multiscale finite element method arising by choosing a sampling finite subset $\{x_k : 1 \leq k \leq d\}$ of $Q$ allowing to solve (1.6) for a finite family of the macroscopic variable (behaving in (1.10) as a parameter) $x = x_k$. Therefore (1.6) reads
\begin{align*}
\text{div}_y (A_0(x_k, \cdot)(e_j + \nabla_y u)) = 0 \text{ in } B_R.
\end{align*}
(1.8)

Based on (1.7), we assume in the next result that the matrix function $x \mapsto A_0(x, \cdot)$ is constant (with respect to $x$), that is, $A_0(x, y) \equiv A_0(y)$ for any $(x, y) \in Q \times \mathbb{R}^N$. Therefore
\begin{align*}
\begin{cases}
\tilde{A}(x) = \tilde{A} = M(A_0(I + \nabla_y \chi)) \\
\tilde{A}_R(x) = \tilde{A}_R = |B_R|^{-1} \int_{B_R} A_0(y)(I + \nabla_y \chi_R(y)) dy.
\end{cases}
\end{align*}
(1.8)

The second main result of the work reads as follows.

**Theorem 1.2.** Let $\tilde{A}_R$ and $\tilde{A}$ be defined by (1.8). Then $\tilde{A}_R$ converges (as $R \to \infty$) to the homogenized matrix $\tilde{A}$. 

The proof of Theorem 1.1 is given in Section 4 while that of Theorem 1.2 is done in Section 5. Homogenization of stochastic partial differential equations with rapidly oscillating coefficients was studied in [23, 24, 31, 32, 33, 43, 44, 45, 46, 19, 20, 27], to cite a few. The homogenization of hyperbolic stochastic partial differential equations (SPDEs) is at its infancy as evidenced by the very few number of published papers in that direction; see e.g. [31, 32, 33]. In the three references above the authors deal with linear hyperbolic SPDE associated to the operator
\[ u'' - \nabla \cdot \left( A_0 \left( \frac{x}{\varepsilon} \right) \nabla u \right) \]
with periodic coefficients \( A_0 \) depending only on the fast variable \( y = x/\varepsilon \). It is worth recalling that the study undertaken here is the first one dealing with hyperbolic SPDEs beyond the periodic setting.

We emphasize that the use of the concept of sigma convergence allows us, not only to extend the well-known results in the periodic setting to the almost periodic framework and beyond, but also to take into account the microscopic behaviour of the coefficients of the problem studied. This is very important, as far as one deals with problems with strongly oscillating coefficients, as it is the case here.

The rest of the work is organized as follows. Section 2 deals with some useful a priori estimates and the study of the tightness of the sequence of probability laws of the solutions of (1.1). In Section 3, we present the sigma-convergence for stochastic processes revisited. Starting from the notion of algebras with mean value, we end with some properties of the above concept. Finally in Section 6 we present some applications of Theorem 1.1.

2. A priori estimates and tightness property

In this section, we derive some a priori estimates and prove the tightness of the probability measures generated by the solution of problem (1.1). Throughout \( C \) will denote a generic constant independent of \( \varepsilon \) that may vary from line to line.

2.1. A priori estimates. The following result gives some a priori estimates of the solution of problem (1.1).

**Lemma 2.1.** Under the assumptions \((A1)-(A3)\), the solution \( u_\varepsilon \) of problem (1.1) satisfies the following estimates
\[
\mathbb{E} \sup_{0 \leq s \leq T} \| u_\varepsilon(s) \|_{H^1_0(Q)}^4 + \mathbb{E} \sup_{0 \leq s \leq T} \| u'_\varepsilon(s) \|_{L^2(Q)}^4 \leq C, \tag{2.1}
\]
\[
\mathbb{E} \left| u'_\varepsilon(t) - \int_0^t g \left( \frac{x}{\varepsilon}, \tau, u_\varepsilon(\tau) \right) dW(\tau) \right|^2_{W^{1,2}([0,T];L^2(Q))} \leq C, \tag{2.2}
\]
\[
\mathbb{E} \left| \int_0^t g \left( \frac{x}{\varepsilon}, \tau, u_\varepsilon(\tau) \right) dW(\tau) \right|^4_{W^{\alpha,4}([0,T];L^2(Q))} \leq C \tag{2.3}
\]
for \( \alpha \in [0, \frac{1}{2}) \) and for all \( t \geq 0 \).

**Remark 2.1.** See [18] for the definitions and the properties of the spaces \( W^{\alpha,p}([0,T];X) \) and \( W^{1,2}([0,T],X) \) where \( p > 1 \) and \( \alpha \in (0,1) \).

**Proof of Lemma 2.1.** The proof of (2.1) follows standard and follows from the application of the Itô’s formula, Doob’s inequality and the Gronwall’s lemma. The proof of (2.2) follows from the relation
\[
u'_\varepsilon(t) - \int_0^t g \left( \frac{x}{\varepsilon}, \tau, u_\varepsilon(\tau) \right) dW(s) = u^1 + \int_0^t \text{div}(A_0 \nabla u_\varepsilon(\tau)) d\tau + \int_0^t f \left( \frac{x}{\varepsilon}, \tau, u_\varepsilon(\tau) \right) d\tau \tag{2.4}
\]
and the estimates (2.1), (2.3) follows from Lemma 2.1 of [18].
2.2. Tightness property of probability measures induced by the solutions. We consider the phase space

\[ S = C(0, T; U_0) \times L^2(0, T; L^2(Q)) \cap C(0, T; H^{-1}(Q)) \times C(0, T; H^{-1}(Q)). \]

We may think of the first component \( S_W = C(0, T; U_0) \) of this phase space as the set where the driving Brownian motion are defined and the second component \( S_u = L^2(0, T; L^2(Q)) \cap C(0, T; H^{-1}(Q)) \) is the set where the solution \( u_\varepsilon \) lives. The third component \( S_{u'} = C(0, T; H^{-1}(Q)) \) is also the set where the solution \( u'_\varepsilon \) lives.

We consider the probability measures

\[ \mu_W(.) = \mathbb{P}(W \in .) \in Pr(C(0, T; U_0), \right \], \]

\[ \mu_u(.) = \mathbb{P}(u_\varepsilon \in .) \in Pr(L^2(0, T; L^2(Q)) \cap C(0, T; H^{-1}(Q))), \]

\[ \mu_{u'}(.) = \mathbb{P}(u'_\varepsilon \in .) \in Pr(C(0, T; H^{-1}(Q))), \]

where \( Pr(A) \) is the set of all probability measures on \((A, \mathcal{B}(A))\) for a complete separable metric space \( A \). This defines a sequence of probability measures

\[ \pi_\varepsilon = \mu_W \times \mu_u \times \mu_{u'}. \]

on the phase space \( S \).

One of the main result of this section is the following theorem.

**Theorem 2.1.** The family of measures \( \{\pi_\varepsilon\} \) is tight over the phase space \( S \).

**Proof.** Let \( Z_1 \) be the space of functions \( \Phi(x, t) \) defined and measurable on \( Q \times [0, T] \) and such that

\[ \sup_{0 \leq t \leq T} \|\Phi(t)\|_{H^0_0(Q)}^2 \leq C; \quad \sup_{0 \leq t \leq T} \|\Phi'(t)\|_{L^2(Q)}^2 \leq C. \]

\( Z_1 \) is a compact subset of \( L^2(0, T; L^2(Q)) \). Let \( B^1_R \) be the ball of radius \( R > 0 \) in \( Z_1 \). Using the estimates \([2.11]\), we get

\[ \mu_u((B^1_R)^\varepsilon) \leq \frac{C}{R^2}. \]  

According to Theorem 2.2 in [18], the following compact embedding holds

\[ W^{1,2}([0, T]; L^2(Q)) \subset C(0, T; H^{-1}(Q)), \]

let \( B^2_R \) the ball of radius \( R \) in \( W^{1,2}([0, T]; L^2(Q)) \). Based on the estimates \([2.11]\), we also get

\[ \mu_u((B^2_R)^\varepsilon) \leq \frac{C}{R^2}. \]

We observe that \( B^1_R \cap B^2_R \) is compact in \( L^2(0, T; L^2(Q)) \cap C(0, T; H^{-1}(Q)) \). and for any \( R > 0 \)

\[ \mu_u((B^1_R \cap B^2_R)^\varepsilon) \leq \frac{C}{R^2}. \]

Using the estimates \([2.2], [2.3]\) and the compact embeddings

\[ W^{1,2}(0, T; L^2(Q)) \subseteq C(0, T; H^{-1}(Q)), \]

\[ W^{\alpha,4}(0, T; L^2(Q)) \subseteq C(0, T; H^{-1}(Q)), \]

where \( \alpha \) is such that \( 4\alpha > 1 \), we can prove as in [18] that \( \mu_{u'}^{\varepsilon} \) is tight in \( C(0, T; H^{-1}(Q)) \). The tightness of \( \mu_W^{\varepsilon} \) in \( C(0, T; U_0) \) is a classical result. This completes the proof of the tightness of \( \{\pi_\varepsilon\} \) in \( S \).

Prokhorov’s compactness result enables us to extract from \( (\pi_\varepsilon) \) a subsequence \( (\pi_\varepsilon^{\varepsilon_n}) \) such that \( \pi_\varepsilon^{\varepsilon_n} \) weakly converges to a probability measure \( \pi \) on \( S \). Skorokhod’s theorem ensures the existence of a complete probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})\) and random variables \((W^{\varepsilon_n}, u_{\varepsilon_n}, u'_{\varepsilon_n})\) and \((\bar{W}, u_0, w)\) defined on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})\) with values in \( S \) such that

\[ \text{the probability law of } (W^{\varepsilon_n}, u_{\varepsilon_n}, u'_{\varepsilon_n}) \text{ is } \pi_\varepsilon^{\varepsilon_n}, \]

\[ \text{the probability law of } (\bar{W}, u_0, w) \text{ is } \pi, \]
\begin{align}
W^n & \to \overline{W} \text{ in } C(0, T; \mathcal{U}_0) \ \overline{\mathbb{P}}\text{-a.s.,} \\
-u^n \to u_0 & \text{ in } L^2(0, T; L^2(\Omega)) \cap C(0, T; H^{-1}(\Omega)) \ \overline{\mathbb{P}}\text{-a.s.,} \\
u'_n & \to w \text{ in } C(0, T; H^{-1}(\Omega)) \ \overline{\mathbb{P}}\text{-a.s.}
\end{align}

In fact let \( u' \) be the time derivative of \( u \) and write formally

\[
u(t) - u(s) = \int_s^t u'(r) dr,
\]

\[
u(t) - u(s) = \int_s^t v^n(r) dr, \quad 0 \leq s \leq t \leq T.
\]

In view of the fact that \((u, u')\) and \((\nu, \nu')\) have the same law, we readily get, thanks to Fubini’s theorem, the relation

\[
\int_s^t \int_A u^n(t) d\overline{\mathbb{P}} = \int_s^t \int_A v^n(r) d\overline{\mathbb{P}}
\]

for any \( A \in \mathcal{F} \). Differentiating with respect to \( t \), we arrive at

\[
\int_s^t \int_A u^n(t) d\overline{\mathbb{P}} = \int_s^t v^n(t) d\overline{\mathbb{P}}
\]

for any \( t \in [0, T] \). Thus \( u'_n = v^n \overline{\mathbb{P}}\text{-a.s.} \).

We can see that \( \{W^n : \varepsilon_n\} \) is a sequence of cylindrical Brownian motions evolving on \( \mathcal{U} \). We let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by \((\overline{W}(s), u_0(s), w(s))\), for \( 0 \leq s \leq t \) and the null sets of \( \mathcal{F} \). We can show by arguing as in [3] that \( \overline{W} \) is an \( \mathcal{F}_t\)-adapted cylindrical Wiener process evolving on \( \mathcal{U} \).

By the same argument as in [4], we can show that

\[
(u'_n, \phi) + \int_0^t A^n_0 \nabla u^n(\tau), \nabla \phi) d\tau = (u^1, \phi) + \int_0^t \left( f \left( \frac{x}{\varepsilon_n}, \frac{\tau}{\varepsilon_n}, u^n(\tau) \right), \phi \right) d\tau
\]

\[
+ \left( \int_0^t g \left( \frac{x}{\varepsilon_n}, \frac{\tau}{\varepsilon_n}, u^n(\tau) \right) dW^n(\tau), \phi \right),
\]

holds for \( \phi \in H^1(\Omega) \) and for almost all \((\omega, t) \in \Omega \times [0, T]\).

Now, we derive a priori estimates for the sequences \( u^n \) and \( u'_n \), obtained from the application of Prokhorov and Skorokhod’s compactness results. We know that they satisfy \( (2.20) \). Therefore they satisfy the a priori estimates corresponding to \( u \) and \( u' \). Namely

\[
\mathbb{E} \sup_{0 \leq s \leq T} \|u^n(s)\|_{L^1(\Omega)}^4 \leq C; \quad \mathbb{E} \sup_{0 \leq s \leq T} \|u'_n(s)\|_{L^2(\Omega)}^4 \leq C,
\]

\[
\mathbb{E} \left| u'_n(t) - \int_0^t g \left( \frac{x}{\varepsilon_n}, \frac{\tau}{\varepsilon_n}, u^n(\tau) \right) dW^n(\tau) \right|_{W^{1, 2}([0, T]; L^2(\Omega))}^2 \leq C,
\]

\[
\mathbb{E} \left| \int_0^t g \left( \frac{x}{\varepsilon_n}, \frac{\tau}{\varepsilon_n}, u^n(\tau) \right) dW^n(\tau) \right|_{W^{\alpha, 4}([0, T]; L^2(\Omega))}^4 \leq C,
\]

for \( \alpha \in [0, \frac{1}{2}) \) and for all \( t \geq 0 \).

Thus modulo extraction of a new subsequence (keeping the same notations) we have

\[
u \to u_0 \text{ weak star in } L^2(\Omega, L^\infty(0, T; H^1(\Omega))),
\]

\[
u' \to w = u'_0 \text{ weak star in } L^2(\Omega, L^\infty(0, T; H^{-1}(\Omega))),
\]

where \( u'_0 \) is the time derivative of \( u_0 \). Next by \( (2.18), (2.19), (2.21) \) and Vitali’s theorem, we have

\[
u \to u_0 \text{ in } L^2(\Omega; L^2(0, T; L^2(\Omega))),
\]

\[
u' \to u'_0 \text{ in } L^2(\Omega; L^2(0, T; H^{-1}(\Omega))).
\]
Hence for almost all \((\omega, t) \in \overline{\Omega} \times [0, T]\), we get
\[
\begin{align*}
    u_{\varepsilon_n} & \to u_0 \text{ in } L^2(Q), \\
    u'_{\varepsilon_n} & \to u'_0 \text{ in } H^{-1}(Q)
\end{align*}
\]  
(2.25)
\(\)  
with respect to the measure \(d\Omega \otimes dt\).

3. Sigma-convergence for stochastic processes

The concept of sigma-convergence relies on the notion of algebra with mean value. Before we can state it, let us first and foremost set some prerequisites about algebras with mean value.

3.1. Algebras with mean value. Let \(\text{BUC}(\mathbb{R}^N)\) denote the Banach algebra of bounded uniformly continuous real-valued functions defined on \(\mathbb{R}^N\). For \(u \in \text{BUC}(\mathbb{R}^N)\) we set
\[
\begin{align*}
    u_R &= \frac{1}{|B_R|} \int_{B_R} u(y)dy
\end{align*}
\]  
where \(B_R\) stands for the open ball in \(\mathbb{R}^N\) of radius \(R\) centered at the origin. We say that the function \(u\) has a \textit{mean value} if the limit \(\lim_{R \to \infty} u_R\) exists in \(\mathbb{R}\). We set
\[
\begin{align*}
    M(u) &= \lim_{R \to \infty} u_R = \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} u(y)dy.
\end{align*}
\]  
(3.1)
\(\)
Let \(u \in \text{BUC}(\mathbb{R}^N)\) and assume that \(M(u)\) exists. Then
\[
\begin{align*}
    u^\varepsilon & \to M(u) \text{ in } L^\infty(\mathbb{R}^N)-\text{weak * as } \varepsilon \to 0
\end{align*}
\]  
(3.2)
where \(u^\varepsilon \in \text{BUC}(\mathbb{R}^N)\) is defined by \(u^\varepsilon(x) = u(x/\varepsilon)\) for \(x \in \mathbb{R}^N\). This is an easy consequence of the fact that the set of finite linear combinations of the characteristic functions of open balls in \(\mathbb{R}^N\) is dense in \(L^1(\mathbb{R}^N)\).

This being so, a closed subalgebra \(A\) of \(\text{BUC}(\mathbb{R}^N)\) is said to be an algebra with mean value (algebra wmv, in short) on \(\mathbb{R}^N\) if it contains the constants, is translation invariant (\(\tau_{a}u = u(\cdot + a) \in A\) for any \(u \in A\) and \(a \in \mathbb{R}^N\)) and any of its elements possesses a mean value in the sense of (3.1).

To an algebra wmv \(A\) are associated its regular subalgebras \(A^m = \{\psi \in C^m(\mathbb{R}^N) : D_y^\alpha \psi \in A\ \forall \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \leq m\}\) (where \(D_y^\alpha \psi = \frac{\partial^{\alpha} \psi}{\partial y_1^{\alpha_1} ... \partial y_N^{\alpha_N}}\)). Under the norm \(||u||_m = \sup_{|\alpha| \leq m} ||D_y^\alpha \psi||_{\infty}\), \(A^m\) is a Banach space. We also define the space \(A^\infty = \{\psi \in C^\infty(\mathbb{R}^N) : D_y^\alpha \psi \in A \ \forall \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N\}\), a Fréchet space when endowed with the locally convex topology defined by the family of norms \(|||\cdot|||_m\).

The concept of a product algebra wmv will be useful in our study. Let \(A_y\) (resp. \(A_\tau\)) be an algebra wmv on \(\mathbb{R}^N_y\) (resp. \(\mathbb{R}_\tau\)). We define the product algebra wmv \(A_y \otimes A_\tau\) as the closure in \(\text{BUC}(\mathbb{R}^{N+1})\) of the tensor product \(A_y \otimes A_\tau = \{\sum_{\text{finite}} u_i \otimes v_i : u_i \in A_y \text{ and } v_i \in A_\tau\}\). This defines an algebra wmv on \(\mathbb{R}^{N+1}\).

We also define the notion of vector-valued algebra with mean value. Indeed, let \(F\) be a Banach space. We denote by \(\text{BUC}(\mathbb{R}^N; F)\) the Banach space of bounded uniformly continuous functions \(u : \mathbb{R}^N \to F\), endowed with the norm
\[
||u||_F = \sup_{y \in \mathbb{R}^N} ||u(y)||_F
\]  
where \(||\cdot||_F\) stands for the norm in \(F\). Let \(A\) be an algebra with mean value on \(\mathbb{R}^N\). We denote by \(A \otimes F\) the usual space of functions of the form
\[
\sum_{\text{finite}} u_i \otimes c_i \text{ with } u_i \in A \text{ and } c_i \in F
\]  
where \((u_i \otimes c_i)(y) = u_i(y)c_i\) for \(y \in \mathbb{R}^N\). With this in mind, we define the vector-valued algebra wmv \(A(\mathbb{R}^N; F)\) as the closure of \(A \otimes F\) in \(\text{BUC}(\mathbb{R}^N; F)\), and we can check that \(A_y \otimes A_\tau = A_y(\mathbb{R}^N; A_\tau) = A_\tau(\mathbb{R}; A_y)\).
Now, let $f \in A(\mathbb{R}^N; F)$. Then, defining $\|f\|_F$ by $\|f\|_F (y) = \|f(y)\|_F (y \in \mathbb{R}^N)$, we have that $\|f\|_F \in A$. Similarly we can define (for $0 < p < \infty$) the function $\|f\|_{p,F}^p$ and $\|f\|_{p,F}^p \in A$. This allows us to define the Besicovitch seminorm on $A(\mathbb{R}^N; F)$ as follows: for $1 \leq p < \infty$,

$$
\|f\|_{p,F} := \left( \frac{\lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} \|f(y)\|_F^p \, dy}{\|f\|_F^p} \right)^{\frac{1}{p}} \equiv (M(\|f\|_{p,F}^p))^\frac{1}{p}
$$

where $B_R$ is the open ball in $\mathbb{R}^N$ centered at the origin and of radius $R$. Next, we define the Besicovitch space $B^p_{\infty}(\mathbb{R}^N; F)$ as the completion of $A(\mathbb{R}^N; F)$ with respect to $\|\cdot\|_{p,F}$. The space $B^p_{\infty}(\mathbb{R}^N; F)$ is a complete seminormed subspace of $L^p_{loc}(\mathbb{R}^N; F)$, and the following hold true:

1. The space $B^p_{\infty}(\mathbb{R}^N; F) = B^p_{\infty}(\mathbb{R}^N; F)/\mathcal{N}$ (where $\mathcal{N} = \{u \in B^p_{\infty}(\mathbb{R}^N; F) : \|u\|_{p,F} = 0\}$) is a Banach space under the norm $\|\cdot\|_{p,F}$ for $u \in B^p_{\infty}(\mathbb{R}^N; F)$.

2. The mean value $M : A(\mathbb{R}^N; F) \to F$ extends by continuity to a continuous linear mapping (still denoted by $M$) on $B^p_{\infty}(\mathbb{R}^N; F)$ satisfying

$$
L(M(u)) = M(L(u)) \quad \text{for all } L \in F' \text{ and } u \in B^p_{\infty}(\mathbb{R}^N; F).
$$

Moreover, for $u \in B^p_{\infty}(\mathbb{R}^N; F)$ we have

$$
\|u\|_{p,F} = [M(\|u\|_{p,F}^p)]^{1/p} \equiv \left[ \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} \|u(y)\|_F^p \, dy \right]^{1/p},
$$

and for $u \in \mathcal{N}$ one has $M(u) = 0$.

It is to be noted that $B^2_{\infty}(\mathbb{R}^N; H)$ (when $F = H$ is a Hilbert space) is a Hilbert space with inner product

$$(u, v) = M(\langle u, v \rangle_H) \quad \text{for } u, v \in B^2_{\infty}(\mathbb{R}^N; H), \quad (\cdot, \cdot)_H \text{ denoting the inner product in } H.$$  

Let us pay attention to the special case $F = \mathbb{R}$ for which $B^p_{\infty}(\mathbb{R}^N) := B^p_{\infty}(\mathbb{R}^N; \mathbb{R})$ and $B^p(\mathbb{R}^N) := B^p_{\infty}(\mathbb{R}^N; \mathbb{R})$. The mean value extends in a natural way to $B^p_{\infty}(\mathbb{R}^N)$ as follows: for $u = v + \mathcal{N} \in B^p_{\infty}(\mathbb{R}^N)$, we set $M(u) := M(v)$: this is well-defined since $M(v) = 0$ for any $v \in \mathcal{N}$. The Besicovitch seminorm in $B^p_{\infty}(\mathbb{R}^N)$ is merely denoted by $\|\cdot\|_p$, and we have $B^p_{\infty}(\mathbb{R}^N) \subset B^p_{\infty}(\mathbb{R}^N)$ for $1 \leq p < \infty$. From this last property one may naturally define the space $B^\infty_{\infty}(\mathbb{R}^N)$ as follows:

$$
B^\infty_{\infty}(\mathbb{R}^N) = \{f \in \cap_{1 \leq p < \infty} B^p_{\infty}(\mathbb{R}^N) : \sup_{1 \leq p < \infty} \|f\|_p < \infty\}.
$$

We endow $B^\infty_{\infty}(\mathbb{R}^N)$ with the seminorm $[f]_{\infty} = \sup_{1 \leq p < \infty} \|f\|_p$, which makes it a complete seminormed space.

For $u = v + \mathcal{N} \in B^p_{\infty}(\mathbb{R}^N) (1 \leq p \leq \infty)$ and $\mathcal{N} \in \mathbb{R}^N$, we define in a natural way the translate $\tau_y u = v(\cdot + y) + \mathcal{N}$ of $u$, and as it can be seen in [49] [50], this is well defined and induces a strongly continuous $N$-parameter group of isometries $T(y) : B^p_{\infty}(\mathbb{R}^N) \to B^p_{\infty}(\mathbb{R}^N)$ defined by $T(y)u = \tau_y u$. We denote by $\partial_j / \partial y_i (1 \leq i \leq N)$ the infinitesimal generator of $T(y)$ along the $i$th coordinate direction. We refer the reader to [49] [50] for the properties of $\partial_j / \partial y_i$.

Now, let

$$
B^{1,p}_{\infty}(\mathbb{R}^N) = \{u \in B^p_{\infty}(\mathbb{R}^N) : \frac{\partial u}{\partial y_i} \in B^p_{\infty}(\mathbb{R}^N) \ \forall 1 \leq i \leq N\}
$$

and

$$
\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)
$$

where $\varrho$ is the canonical mapping of $B^p_{\infty}(\mathbb{R}^N)$ into $B^p_{\infty}(\mathbb{R}^N) = B^p_{\infty}(\mathbb{R}^N)/\mathcal{N}$ defined by $\varrho(u) = u + \mathcal{N}$ ($\mathcal{N}$ being defined above). It can be shown that $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $B^p_{\infty}(\mathbb{R}^N)$, $1 \leq p < \infty$. We also have that $B^{1,p}_{\infty}(\mathbb{R}^N)$ is a Banach space under the norm

$$
\|u\|_{B^{1,p}_{\infty}} := \left( \|u\|_p^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial y_i} \right\|_p^p \right)^{1/p} \quad (u \in B^{1,p}_{\infty}(\mathbb{R}^N)).
$$
admitting $\mathcal{D}_A(\mathbb{R}^N)$ as a dense subspace.

We end this first part with a further notion which will lead to the definition of the space of correctors. A function $u \in B^1_A(\mathbb{R}^N)$ is said to be invariant if for any $y \in \mathbb{R}^N$, $T(y)u = u$. It is immediate that the above notion of invariance is the well-known one relative to dynamical systems. Therefore, an algebra with mean value will be said to be ergodic if every invariant function $u$ is constant in $B^1_A(\mathbb{R}^N)$. As in [13] one may show that $u \in B^1_A$ is invariant if and only if $\frac{\partial u}{\partial y_i} = 0$ for all $1 \leq i \leq N$. We denote by $\mathcal{I}^p_A(\mathbb{R}^N)$ the set of $u \in B^p_A(\mathbb{R}^N)$ that are invariant. The set $\mathcal{I}^p_A(\mathbb{R}^N)$ is a closed vector subspace of $B^p_A(\mathbb{R}^N)$ satisfying the following important property:

$$u \in \mathcal{I}^p_A(\mathbb{R}^N) \text{ if and only if } \frac{\partial u}{\partial y_i} = 0 \text{ for all } 1 \leq i \leq N. \quad (3.4)$$

We define $B^{1,p}_{\# A}(\mathbb{R}^N)$ to be the completion of $\mathcal{B}^{1,p}_A(\mathbb{R}^N)/\mathcal{I}^p_A(\mathbb{R}^N)$ with respect to the norm of the gradient

$$\|u\|_{\# p} = \|\nabla_y u\|_p := \left( \sum_{i=1}^N \left\| \frac{\partial u}{\partial y_i} \right\|_p^p \right)^{1/p} \quad \text{for } u \in B^{1,p}_{\# A}(\mathbb{R}^N)/\mathcal{I}^p_A(\mathbb{R}^N),$$

which makes it a Banach space. Moreover $B^{1,p}_{\# A}(\mathbb{R}^N)$ is reflexive ($1 < p < \infty$) and further, the space $\{u \in \mathcal{D}_A(\mathbb{R}^N)/\mathcal{I}^p_A(\mathbb{R}^N) : M(u) = 0\}$ is dense in $B^{1,p}_{\# A}(\mathbb{R}^N)$.

**Remark 3.1.** Assume that the algebra wmv $A$ is ergodic. Then $\mathcal{I}^p_A(\mathbb{R}^N) = \mathbb{R}$. Hence

1. $\mathcal{B}^{1,p}_A(\mathbb{R}^N)/\mathcal{I}^p_A(\mathbb{R}^N) = \{u \in \mathcal{B}^{1,p}_A(\mathbb{R}^N) : M(u) = 0\}$;
2. we may, instead of $\mathcal{B}^{1,p}_{\# A}(\mathbb{R}^N)$, rather consider its dense subspace $B^{1,p}_{\# A}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : M(\nabla u) = 0\}$ equipped with the gradient seminorm $\|u\|_{B^{1,p}_{\# A}(\mathbb{R}^N)} = \|\nabla_y u\|_p$.

It can be seen from [11, Theorem 3.12] that any function in $B^{1,p}_{\# A}(\mathbb{R}^N)$ is an equivalence class of an element in $B^{1,p}_{\# A}(\mathbb{R}^N)$ in the sense that two elements in $B^{1,p}_{\# A}(\mathbb{R}^N)$ are identified by their gradients: $u = v$ in $B^{1,p}_{\# A}(\mathbb{R}^N)$ if and only if $\nabla u = \nabla v$ in $B^p_{\# A}(\mathbb{R}^N)$, i.e., $\|\nabla_y (u - v)\|_p = 0$. As we shall see later on, the latter space will be more convenient in practice.

For $u \in B^p_A(\mathbb{R}^N)$ (resp. $v = (v_1, \ldots, v_N) \in (B^p_A(\mathbb{R}^N))^N$), we define the gradient operator $\nabla_y u$ and the divergence operator $\nabla_y v$ by

$$\nabla_y u := \left( \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_N} \right) \quad \text{and} \quad \nabla_y v := \sum_{i=1}^N \frac{\partial v_i}{\partial y_i}.\,$$

Then the divergence operator sends continuously and linearly $(\mathcal{B}^p_A(\mathbb{R}^N))^N$ into $(\mathcal{B}^{1,p}_A(\mathbb{R}^N))^\prime$ and satisfies

$$\langle \nabla_y u, v \rangle = -\langle u, \nabla_y v \rangle \quad \text{for } v \in \mathcal{B}^{1,p}_A(\mathbb{R}^N) \text{ and } u = (u_i) \in (\mathcal{B}^p_A(\mathbb{R}^N))^N, \quad (3.5)$$

where $\langle u, \nabla_y v \rangle := \sum_{i=1}^N M \left( u_i \nabla_y v_i \right)$.

**3.2. Sigma-convergence for stochastic processes.** We follow here the presentation made in [14]. For the results stated here, the reader is referred to [14] for the proofs and other comments. However, for the sake of completeness, we recall some facts that have been presented in the above mentioned work.

In all that follows, $Q$ is an open subset of $\mathbb{R}^N$ (integer $N \geq 1$), $T$ is a positive real number and $Q_T = Q \times (0, T)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ will throughout be denoted by $\mathbb{E}$. Let us first recall the definition of the Banach space of bounded $\mathcal{F}$-measurable functions. Denoting by $F(\Omega)$ the Banach space of all bounded functions $f : \Omega \rightarrow \mathbb{R}$ (with the sup norm), we define $B(\Omega)$ as the closure in $F(\Omega)$ of the vector space $H(\Omega)$ consisting
of all finite linear combinations of the characteristic functions $1_X$ of sets $X \in \mathcal{F}$. Since $\mathcal{F}$ is an $\sigma$-algebra, $B(\Omega)$ is the Banach space of all bounded $\mathcal{F}$-measurable functions. Likewise we define the space $B(\Omega; Z)$ of all bounded $(\mathcal{F}, B_Z)$-measurable functions $f : \Omega \to Z$, where $Z$ is a Banach space endowed with the $\sigma$-algebra of Borelians $B_Z$. The tensor product $B(\Omega) \otimes Z$ is a dense subspace of $B(\Omega; Z)$: this follows from the obvious fact that $B(\Omega)$ can be viewed as a space of continuous functions over the gamma-compactification \cite{gamma} of the measurable space $(\Omega, \mathcal{F})$, which is a compact topological space. Next, for $X$ a Banach space, we denote by $L^p(\Omega, \mathcal{F}; \mathbb{P}; X)$ the space of $X$-valued random variables $u$ such that $\|u\|_X = L^p(\Omega, \mathcal{F}; \mathbb{P})$-integrable.

Now, let $A_y$ and $A_\tau$ be two algebras wmv on $\mathbb{R}^N$ and $\mathbb{R}$ respectively, and let $A = A_y \otimes A_\tau$ be their product. We know that $A$ is the closure in $\text{BUC}(\mathbb{R}^{N+1}_y)$ of the tensor product $A_y \otimes A_\tau$. Points in $\Omega$ are as usual denoted by $\omega$. The generic element of $Q_T$ is denoted by $(x, t)$ while any function in $A_y$ (resp. $A_\tau$ and $A$) is of variable $y \in \mathbb{R}^N$ (resp. $\tau \in \mathbb{R}$ and $(y, \tau) \in \mathbb{R}^{N+1}$). The mean value over $A_y$, $A_\tau$ and $A$ is denoted by the same letter $M$. However, we shall often write $M_y$ (resp. $M_\tau$ and $M_{y, \tau}$) to differentiate them if there is any danger of confusion. For a function $u \in L^p(\Omega, \mathcal{F}; \mathbb{P}; L^p(Q_T; B^p_{y, \tau}(\mathbb{R}^{N+1}))$ we denote by $u(x, t, \cdot, \omega)$ (for any fixed $(x, t, \omega) \in Q_T \times \Omega$) the function defined by

$$u(x, t, \cdot, \omega)(y, \tau) = u(x, t, y, \tau, \omega) \text{ for } (y, \tau) \in \mathbb{R}^{N+1}.$$ 

Then $u(x, t, \cdot, \omega) \in B^p_{y, \tau}(\mathbb{R}^{N+1})$, so that the mean value of $u(x, t, \cdot, \omega)$ is defined accordingly.

Unless otherwise stated, random variables will always be considered on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, the letter $E$ will throughout denote exclusively an ordinary sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In what follows, the notations are those of the preceding section.

**Definition 3.1.** A sequence of random variables $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega, \mathcal{F}; \mathbb{P}; L^p(Q_T))$ $(1 < p < \infty)$ is said to weakly $\Sigma$-converge in $L^p(Q_T \times \Omega)$ to some random variable $u_0 \in L^p(\Omega, \mathcal{F}; \mathbb{P}; L^p(Q_T; B^p_{y, \tau}(\mathbb{R}^{N+1})))$ if as $\varepsilon \to 0$, we have

$$\int_{Q_T \times \Omega} u_\varepsilon(x, t, \omega)v(x, t, y, \tau, \omega) \, dx \, dt \, d\mathbb{P} \to \int_{Q_T \times \Omega} M(u_0(x, t, \cdot, \omega)f(x, t, \cdot, \omega)) \, dx \, dt \, d\mathbb{P}$$

for every $v \in L^p'(\Omega, \mathcal{F}; \mathbb{P}; L^p(Q_T; A^))$ ($1/p' = 1 - 1/p$). We express this by writing $u_\varepsilon \rightharpoondown u_0$ in $L^p(Q_T \times \Omega)$-weak $\Sigma$.

**Remark 3.2.** The above weak $\Sigma$-convergence in $L^p(Q_T \times \Omega)$ implies the weak convergence in $L^p(Q_T \times \Omega)$. One can show as in the usual setting of $\Sigma$-convergence method \cite{gamma} that each $v \in L^p(\Omega, \mathcal{F}; \mathbb{P}; L^p(Q_T; A))$ weakly $\Sigma$-converges to $\varphi \circ v$.

In order to simplify the notation, we will henceforth denote $L^p(\Omega, \mathcal{F}; \mathbb{P}; X)$ merely by $L^p(\Omega; X)$ if it is understood from the context and there is no danger of confusion.

The following results can be found in \cite{gamma} (see especially Theorems 2, 3 and 4 therein).

**Theorem 3.1.** Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega; L^p(Q_T))$ be a sequence of random variables verifying the following boundedness condition:

$$\sup_{\varepsilon \in E} \mathbb{E} \left\| u_\varepsilon \right\|_{L^p(Q_T)}^p < \infty.$$ 

Then there exists a subsequence $E'$ from $E$ such that the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ is weakly $\Sigma$-convergent in $L^p(Q_T \times \Omega)$.

**Theorem 3.2.** Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega; L^p(0, T; W^{1,p}_0(Q)))$ be a sequence of random variables which satisfies the following estimate:

$$\sup_{\varepsilon \in E} \mathbb{E} \left\| u_\varepsilon \right\|_{L^p(0, T; W^{1,p}_0(Q))}^p < \infty.$$ 

Then there exist a subsequence $E'$ of $E$ and a couple of random variables $(u_0, u_1)$ with $u_0 \in L^p(\Omega; L^p(0, T; W^{1,p}_0(Q); B^p_{y, \tau}(\mathbb{R}^{N+1})))$ and $u_1 \in L^p(\Omega; L^p(Q_T; B^p_{y, \tau}(\mathbb{R}^{N+1})))$ such that, as $E' \ni \varepsilon \to 0$.
\[ \varepsilon \to 0, \quad u_\varepsilon \to u_0 \quad \text{in } L^p(Q_T \times \Omega)\text{-weak } \Sigma \]

and
\[ \frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad \text{in } L^p(Q_T \times \Omega)\text{-weak } \Sigma, \quad 1 \leq i \leq N. \quad (3.7) \]

The following modified version of Theorem 3.2 will be used below.

**Theorem 3.3.** Assume that the hypotheses of Theorem 3.2 are satisfied. Assume further that \( A_y \) is ergodic. Finally suppose further that \( p \geq 2 \) and that there exist a subsequence of \( E' \) from \( E \) and a random variable \( u_0 \in L^p(\Omega; L^p(0,T;W_0^{1,p}(Q))) \) such that, as \( E' \ni \varepsilon \to 0, \)
\[ u_\varepsilon \to u_0 \quad \text{in } L^2(Q_T \times \Omega). \quad (3.8) \]

Then there exist a subsequence of \( E' \) (not relabeled) and a \( B_{A_p}^p(\mathbb{R}_r;B_{\#A_y}^1(\mathbb{R}^N)) \)-valued stochastic process \( u_1 \in L^p(\Omega; L^p(Q_T;B_{A_p}^p(\mathbb{R}^N_{y,T})) \) such that \( (3.7) \) holds when \( E' \ni \varepsilon \to 0. \)

We will also deal with the product of sequences. For that reason, we give one further

**Definition 3.2.** A sequence \((u_\varepsilon)_{\varepsilon > 0}\) of \( L^p(Q_T \times \Omega; \cdot) \) is said to strongly \( \Sigma \)-converge to \( u_0 \) in \( L^p(Q_T \times \Omega; \cdot) \) if for any \( f \in L^q(Q_T \times \Omega; \cdot) \) the sequence \((f(u_\varepsilon))_{\varepsilon > 0}\) is strongly \( \Sigma \)-convergent to \( f(u_0) \) in \( L^p(Q_T \times \Omega; \cdot) \).

The next result is of capital interest in the sequel (see the proof of Proposition 4.1). Its proof is copied on that of \([42, \text{Theorem } 6] \).

**Theorem 3.4.** Let \( 1 < p < q < \infty \) and \( r \geq 1 \) be such that \( 1/r = 1/p + 1/q \leq 1. \) Assume \((u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q_T \times \Omega) \) is weakly \( \Sigma \)-convergent to \( u_0 \) in \( L^q(Q_T \times \Omega; \cdot) \) to some \( u_0 \in L^p(Q_T \times \Omega; B_{A_p}^p(\mathbb{R}^N_{y,T})) \), and \((v_\varepsilon)_{\varepsilon \in E} \subset L^p(Q_T \times \Omega) \) is strongly \( \Sigma \)-convergent to \( v_0 \) in \( L^p(Q_T \times \Omega; B_{A_p}^p(\mathbb{R}^N_{y,T})) \). Then the sequence \((u_\varepsilon, v_\varepsilon)_{\varepsilon \in E} \) is weakly \( \Sigma \)-convergent in \( L^r(Q_T \times \Omega) \) to \( u_0 v_0 \).

**4. Homogenization results and proof of Theorem 3.1**

### 4.1 Preliminaries

The notations are those of the preceding sections. We remark that property (3.0) in Definition 3.1 still holds true for \( v \in B(\Omega; C(Q_T; B_{A_p}^{p,\infty}(\mathbb{R}^N_{y,T})) \) where \( B_{A_p}^{p,\infty}(\mathbb{R}^N_{y,T}) = B_{A_p}^p(\mathbb{R}^N_{y,T}) \cap L^\infty(\mathbb{R}^N_{y,T}) \) and as usual, \( p' = p/(p-1). \)

With this in mind, use of the sigma-convergence method to solve the homogenization problem for \( (1.1) \) will be possible provided that the assumption (A4) stated in Section 1 holds true. We recall it here for explicitness.

(A4) For any \( \lambda \in \mathbb{R} \) the functions \( f(\cdot, \cdot, \lambda) \) and \( g_k(\cdot, \cdot, \lambda) \) belong to \( B_{A_p}(\mathbb{R}^N_{y,T}); \) the matrix \( A_0(x, \cdot) \in (B_{A_y}^2(\mathbb{R}^N_{y,T}))^{N \times N} \) for any \( x \in Q \) and \( k \geq 1. \)

**Remark 4.1.** Hypothesis (A4) includes a variety of behaviours, ranging from the periodicity to the weak almost periodicity.

The following important result is needed in order to pass to the limit in the stochastic term.

**Lemma 4.1.** Let \((u_\varepsilon)_{\varepsilon} \) be a sequence in \( L^2(Q_T \times \Omega) \) such that \( u_\varepsilon \to u_0 \) in \( L^2(Q_T \times \Omega) \) as \( \varepsilon \to 0 \) where \( u_0 \in L^2(Q_T \times \Omega). \) Then for each positive integer \( k, \) we have
\[ g_k(\cdot, \cdot, u_\varepsilon) \to g_k(\cdot, \cdot, u_0) \quad \text{in } L^2(Q_T \times \Omega)\text{-weak } \Sigma \quad \text{as } \varepsilon \to 0. \quad (4.1) \]

We also have
\[ f^*(\cdot, \cdot, u_\varepsilon) \to f^*(\cdot, \cdot, u_0) \quad \text{in } L^2(Q_T \times \Omega)\text{-weak } \Sigma \quad \text{as } \varepsilon \to 0. \quad (4.2) \]
Proof. Let us first check (4.1). For \( u \in B(\Omega; C(\overline{Q_T}))^N \), the function \((x, t, y, \tau, \omega) \mapsto g_k(y, \tau, u(x, t, \omega))\) lies in \( B(\Omega; C(\overline{Q_T}; B^2_{\infty}(\mathbb{R}^N))) \), so that we have \( g_k^\varepsilon(\cdot, u) \rightarrow g_k(\cdot, u) \) in \( L^2(\Omega \times \Omega) \)-weak \( \Sigma \) as \( \varepsilon \rightarrow 0 \). Next, since \( B(\Omega; C(\overline{Q_T})) \) is dense in \( L^2(Q_T \times \Omega) \), it can be easily shown that
\[
 g_k^\varepsilon(\cdot, u_0) \rightarrow g_k(\cdot, u_0) \text{ in } L^2(Q_T \times \Omega)-\text{weak } \Sigma \text{ as } \varepsilon \rightarrow 0. \tag{4.3}
\]
Now, let \( v \in L^2(\Omega; L^2(Q_T; A)) \); then
\[
\int_{Q_T \times \Omega} \int_{Q_T \times \Omega} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} - \frac{\partial u_0}{\partial y_i} \frac{\partial v}{\partial y_i} \ dx dt \leq C \int_{Q_T \times \Omega} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} - \frac{\partial u_0}{\partial y_i} \frac{\partial v}{\partial y_i} \ dx dt \]
associated to (4.3) therefore stems from the inequality
\[
\int_{Q_T \times \Omega} (g_k^\varepsilon(\cdot, u_0) - g_k(\cdot, u_0))^2 \ dx dt \]
4.2. Passage to the limit. Let \((u_n)\) be the sequence determined in the Subsection and satisfying Eq. (4.20). In view of (4.20) the sequence \((u_n)\) also satisfies the a priori estimates (4.21). In view of (4.21) and by a diagonal process, one can find a subsequence of \((u_n)\) (not relabeled) which weakly converges in \( L^2(\Omega; L^2(0, T; H_0^1(Q))) \) to the function \( u_0 \), so that \( u_0 \in L^2(\Omega; L^2(0, T; H_0^1(Q))) \). From Theorem 3.3 we infer the existence of a function \( u_1 \in L^2(Q_T \times \overline{\Gamma}; B^2_{\infty}(\mathbb{R}^N)/A_y) \) such that the convergence result
\[
\int_{Q_T \times \Omega} \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} - \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} \ dx dt \leq C \int_{Q_T \times \Omega} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} - \frac{\partial u_0}{\partial y_i} \frac{\partial v}{\partial y_i} \ dx dt \]
holds when \( \varepsilon_n \rightarrow 0 \). Next observe that \((u_0, u_1) \in F_0^1 \) where
\[
F_0^1 = L^2(\Omega \times (0, T); H_0^1(Q)) \times L^2(Q_T \times \overline{\Gamma}; B^2_{\infty}(\mathbb{R}^N)/A_y) \}
\]
where \((\psi_0, \psi_1 = \varrho_\varepsilon(\psi)) \in F_{0}^\infty\) with \(\psi\) being a representative of \(\psi_1\) and \(\varrho_\varepsilon\) the canonical surjection of \(B_{A_0}^2(\mathbb{R}_y^N)\) onto \(B_{A_0}^2(\mathbb{R}_y^N)\). We recall that \(\psi \in B(\Omega) \otimes C_0^\infty(Q_T) \otimes [A^\infty \otimes (A^\infty_0 / \mathbb{R})], A^\infty_0 / \mathbb{R} = \{ \phi \in A_y^\infty : M_y(\phi) = 0 \}\). Then \(\Phi_\varepsilon \in B(\Omega) \otimes C_0^\infty(Q_T)\), and taking \(\Phi_\varepsilon\) as a test function in the variational formulation of (4.1), we get
\[
- \int_{Q_T \times \Omega} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt + \int_{Q_T \times \Omega} A_0^\varepsilon \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt = 0 
\tag{4.6}
\]

Our aim is to pass to the limit in (4.6). We shall consider each term separately. But before we proceed forward, let us first observe that:
\[
\frac{\partial \Phi_\varepsilon}{\partial t} = \frac{\partial \psi_0}{\partial t} + \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon + \varepsilon \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon 
\]
\[
\nabla \Phi_\varepsilon = \nabla \psi_0 + \left( \nabla \psi \right)^\varepsilon + \varepsilon (\nabla \psi)^\varepsilon ,
\]
so that (up to a subsequence \(\varepsilon \to 0\))
\[
\frac{\partial \Phi_\varepsilon}{\partial t} \to \frac{\partial \psi_0}{\partial t} \text{ in } L^2(\Omega \times (0, T); H^1_0(Q))\)-weak \tag{4.7}
\]
\[
\nabla \Phi_\varepsilon \to \nabla \psi_0 + \nabla \psi \text{ in } L^2(Q_T \times \Omega)^N\)-strong \(\Sigma\) (see Remark 3.3). \tag{4.8}
\]

Next, from (4.10) we deduce that
\[
g^\varepsilon(\cdot, \cdot, u_\varepsilon) \to M(g(\cdot, \cdot, u_0)) \text{ in } L^2(Q_T \times \Omega)\)-weak as \(\varepsilon \to 0\). \tag{4.10}
\]
Combining (4.10) with (4.14), we get
\[
g^\varepsilon(\cdot, \cdot, u_\varepsilon) \Phi_\varepsilon \to M(g(\cdot, \cdot, u_0)) \psi_0 \text{ in } L^2(Q_T \times \Omega)\)-weak \tag{4.11}
\]
Similarly, we have
\[
f^\varepsilon(\cdot, \cdot, u_\varepsilon) \Phi_\varepsilon \to M(f(\cdot, \cdot, u_0)) \psi_0 \text{ in } L^2(Q_T \times \Omega)\)-weak . \tag{4.12}
\]
Now combining (4.11) with (2.13) and arguing as in [3], we get
\[
\int_{Q_T \times \Omega} g^\varepsilon(\cdot, \cdot, u_\varepsilon) \Phi_\varepsilon dxW^\varepsilon d\mathbb{P} \to \int_{Q_T \times \Omega} M(g(\cdot, \cdot, u_0)) \psi_0 dxW d\mathbb{P}. \tag{4.13}
\]
Now, coming back to (4.6) and considering there the second term of the left-hand side, we note that we may use \(A_0\) as test function for the sigma-convergence (since it belongs to \(C(\overline{Q}_T; B_{A_0}^\infty(\mathbb{R}_y^N))\), which is contained in \(B(\Omega) \otimes C(\overline{Q}_T; B_{A_0}^\infty(\mathbb{R}_y^N + 1))\) and therefore get
\[
\int_{Q_T \times \Omega} A_0^\varepsilon \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt \to \int_{Q_T \times \Omega} M(A_0 \mathbb{D} u_\varepsilon \cdot \mathbb{D} \Phi_\varepsilon) dxdt. \tag{4.14}
\]
Indeed, we have (4.4) and (4.8), so that, by Theorem 3.3
\[
\nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon \to \mathbb{D} u \cdot \mathbb{D} \Phi \text{ in } L^1(Q_T \times \Omega)\)-weak \(\Sigma\).
\]
Hence, using the convergence results (4.7), (2.14) (for the first term of the left-hand side of (4.6) and (4.11), (4.12), (4.13) we are led at once to (4.5). \(\square\)

The problem (4.5) is called the global homogenized problem for (1.1).
4.3. Homogenized problem. The goal here is to derive the problem arising from the passage to limit (as $\varepsilon \to 0$) whose $u_0 \in L^2(\Omega \times (0, T); H^1_0(Q))$ is the solution. For that, we first observe that (4.19) is equivalent to the system made of (4.15) and (4.16) below:

$$\int_{Q_T \times \Omega} M(A_0 \mathbb{D}u \cdot \nabla_y \psi_1) dxdtd\mathcal{F} = 0, \text{ all } \psi_1 \in B(\Omega) \otimes C_0^\infty(Q_T) \otimes \mathcal{E}$$  \quad \text{(4.15)}

$$\begin{cases}
- \int_{Q_T \times \Omega} u_0 \psi_0 dxdtd\mathcal{F} + \int_{Q_T \times \Omega} M(A_0 \mathbb{D}u) \cdot \nabla \psi_0 dxdtd\mathcal{F} \\
= \int_{Q_T \times \Omega} M(f(\cdot, \cdot; u_0) \psi_0 \nabla \psi_0 dxd\mathcal{W} d\mathcal{F}
\end{cases}
$$

for all $\psi_0 \in B(\Omega) \otimes C_0^\infty(Q_T)$. Let us first consider the problem (4.15) and there choose $\psi_1 = \rho(\psi)$ with $\psi(x, t, y, \tau, \omega) = \varphi(x, t)\varphi(y)\chi(\tau)\eta(\omega)$ with $\varphi \in C_0^\infty(Q_T)$, $\phi \in A_y^\infty/\mathbb{R}$, $\chi \in A_N^\infty$ and $\eta \in B(\Omega)$. Then (4.15) becomes

$$M_y(A_0(x, \cdot) \mathbb{D}u \cdot \nabla_y \varphi) = 0, \text{ all } \varphi \in A_y^\infty/\mathbb{R}. \quad \text{(4.18)}$$

So for $\xi \in \mathbb{R}^N$ be freely fixed, consider the cell problem:

$$\begin{cases}
\text{Find } \pi(\xi) \in B_{1,2,1}^{1,2}(\mathbb{R}_y^N) \text{ such that:} \\
- \text{div}_y \left( A_0(x, \cdot) (\xi + \nabla_y \pi(\xi)) \right) = 0 \text{ in } \mathbb{R}_y^N
\end{cases} \quad \text{(4.19)}$$

Instead of (4.19) and in view of (4.18), we may rather consider the more convenient problem

$$\begin{cases}
\text{Find } \pi(\xi) \in B_{\# A_y}^{1,2}(\mathbb{R}_y^N) \text{ (see Remark 3.1-(2)) such that:} \\
- \text{div}_y \left( A_0(x, \cdot) (\xi + \nabla_y \pi(\xi)) \right) = 0 \text{ in } \mathbb{R}_y^N
\end{cases} \quad \text{(4.20)}$$

Then it can be easily shown (using property (3.5) and the assumption (A1) on $A_0$) that (4.20) possesses at least a solution $\pi(\xi)$ whose the equivalence class $\pi(\xi)$ in $B_{\# A_y}^{1,2}(\mathbb{R}_y^N)$ and solves (4.19). Now, taking $\xi = \nabla u_0(x, t, \omega)$ (for a.e. $(x, t, \omega) \in Q_T \times \Omega$) in (4.19) and testing the resulting equation with $\psi$ as in (4.17), and next integrating over $Q_T \times \Omega$, we get (by the uniqueness of the solution to (4.19)) that

$$u_1(x, t, y, \cdot, \omega) = \pi(\nabla u_0(x, t, \omega))(y) \text{ for a.e. } (x, t, \omega) \in Q_T \times \Omega. \quad \text{(4.21)}$$

From which the uniqueness of $u_1$ defined as above and belonging to $L^2(Q_T \times \Omega; B_{A_y}^1(\mathbb{R}; B_{\# A_y}^{1,2}(\mathbb{R}_y^N)))$. Next for fixed $\xi \in \mathbb{R}^N$ and $r \in \mathbb{R}$ define the homogenized coefficients as follows:

$$\tilde{\Lambda}(x)\xi = M(A_0(x, \cdot) (\xi + \nabla_y \pi(\xi)), x \in Q$$

$$\tilde{f}(r) = M(f(\cdot, \cdot; r)) \text{ and } \tilde{g}(r) = M(g(\cdot, \cdot; r)).$$

It is important to note that in view of the equality $\nabla_y \pi(\xi) = \nabla_y \pi_1(\xi)$, if we take in the above definition of $\tilde{\Lambda}(x)\xi$ the special $\xi = e_j$ ($1 \leq j \leq N$) then we get the exact definition of $\tilde{\Lambda}(x)$ given in Section $\Pi$ (see (1.4) therein). With this in mind, the next result holds.

**Proposition 4.2.** The function $u_0$ solves the boundary value problem

$$\begin{cases}
da_t^\varepsilon - \text{div} \left( \tilde{\Lambda}(x) \nabla u_0 \right) dt = \tilde{f}(u_0) dt + \tilde{g}(u_0) d\mathcal{W} \text{ in } Q_T \\
u_0 = 0 \text{ on } \partial Q \times (0, T) \\
u_0(x, 0) = u_0^1(x) \text{ and } \tilde{u}_0(x, 0) = u_1(x) \text{ in } Q.
\end{cases} \quad \text{(4.22)}$$

**Proof.** If in (4.16) we replace $u_1$ by its expression in (4.21) and take therein $\psi_0(x, t, \omega) = \varphi(x, t)\varphi(\omega)$ with $\phi \in B(\Omega)$ and $\varphi \in C_0^\infty(Q_T)$, we get readily the variational formulation of (4.22). The initial conditions are getting accordingly. \hfill \square

**Proposition 4.3.** Let $u_0$ and $\tilde{u}_0$ be two solutions of (4.22) on the same probabilistic system $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{W}, \mathcal{F})$ with the same initial conditions $u^0$ and $u^1$. Then $u_0 = \tilde{u}_0$ $\mathbb{P}$-almost surely.
Proof. The functions \( \tilde{f} \) and \( \tilde{g} \) are Lipschitz. The matrix \( \tilde{A}(x) \) is symmetric and satisfies assumptions similar to those of \( A_0 \) (see (A1)). We can then apply [12] Theorem 8.4 of p. 189) to prove the existence and uniqueness of strong the solution of problem (1.1). \( \square \)

The proof of Theorem 1.1 that will follow shortly, combines the pathwise uniqueness of of the solution of equation (1.1) and the Gyöngy-Krylov characterization of convergence in probability introduced in [20]. We recall here the precise result.

Lemma 4.2. Let \( X \) be a Polish space equipped with the Borel \( \sigma \)-algebra. A sequence of \( X \)-valued random variables \( \{Y_n, n \in \mathbb{N}\} \) converges in probability if and only if every subsequence of joint laws \( \{\mu_{n_k,m_k}, k \in \mathbb{N}\} \), there exists a further subsequence which converges weakly to a probability measure \( \mu \) such that

\[
\mu((x, y) \in X \times X : x = y) = 1.
\]

Let us set \( X = L^2(0, T; L^2(Q)) \cap C(0, T; H^{-1}(Q)) \times C(0, T; H^{-1}(Q)); X_1 = C(0, T; \mu_0) \). For \( S \in B(X) \), we set \( \pi^S(u, v) = \mathbb{P}(u \in S) \). For \( S \in B(X) \), we set \( \pi^W = \mathbb{P}(W \in S) \).

Next, we define the joint probability laws:

\[
\pi^{\varepsilon, \varepsilon'} = \pi^\varepsilon \times \pi^{\varepsilon'},
\]

\[
\nu^{\varepsilon, \varepsilon'} = \pi^\varepsilon \times \pi^{\varepsilon'} \times \pi^W.
\]

The following tightness property is satisfied.

Lemma 4.3. The collection \( \{\nu^{\varepsilon, \varepsilon'} : \varepsilon, \varepsilon' > 0\} \) is tight on \( (X_2, B(X_2)) \).

Proof. The proof is similar to the one of Theorem 1.1 \( \square \)

4.4. Proof of Theorem 1.1. Lemma 4.3 implies that there exists a subsequence from \( \{\nu^{\varepsilon, \varepsilon'}\} \) still denoted by \( \{\nu^{\varepsilon, \varepsilon'}\} \) which converges to a probability measure \( \nu \) on \( (X_2, B(X_2)) \). By Skorokhod’s theorem, there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) on which a sequence \( \left((u_{\varepsilon_j}, u'_{\varepsilon_j}), (u_{\varepsilon_j}, u'_{\varepsilon_j}), W_j\right) \) is defined and converges almost surely in \( X_2 \) to a couple of random variables \( ((0, 0_0), (v_0, v_0), W) \). Furthermore, we have

\[
\mathcal{L}\left((u_{\varepsilon_j}, u'_{\varepsilon_j}), (u_{\varepsilon_j}, u'_{\varepsilon_j}), W_j\right) = \nu^{\varepsilon_j, \varepsilon'_j},
\]

\[
\mathcal{L}\left((0, 0_0), (v_0, v_0), W\right) = \nu.
\]

Now let

\[
Z_{j}^{u_{\varepsilon_j}, u'_{\varepsilon_j}} = \left(u_{\varepsilon_j}, u'_{\varepsilon_j}, W_j\right) \text{ and } Z_{j}^{\varepsilon_j, \varepsilon'_j} = \left(u_{\varepsilon_j}, u'_{\varepsilon_j}, W_j\right),
\]

\[
Z^{(u_0, v_0)} = (u_0, v_0, W) \text{ and } Z^{(v_0, v_0)} = (v_0, v_0, W).
\]

We can infer from the above argument that \( \pi^{\varepsilon_j, \varepsilon'_j} \) converges to a measure \( \pi \) such that

\[
\pi(\cdot) = \mathbb{P}(\cdot((0, 0_0), (v_0, v_0)) \in \cdot).
\]

As above, we can show that \( Z^{u_{\varepsilon_j}, u'_{\varepsilon_j}} \) and \( Z_{j}^{\varepsilon_j, \varepsilon'_j} \) satisfy (2.24) and that \( Z^{(u_0, v_0)} \) and \( Z^{(v_0, v_0)} \) satisfy (1.1) on the same stochastic system \( (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_1, \mathbb{F}_1) \) where \( \mathcal{F}_1 \) is the filtration generated by the couple \( ((0, 0_0), (v_0, v_0), W) \). Since we have the uniqueness result above, then we conclude that \( u_0 = v_0 \) in \( L^2(Q_T); u_0 = v_0 \) in \( L^2(0, T; H^{-1}(Q)) \). Therefore

\[
\pi(((x, y), (x', y')) \in X \times X : (x, y) = (x', y')) = \mathbb{P}((0, 0_0) = (v_0, v_0) \text{ in } X) = 1.
\]

This fact together with Lemma 4.2 imply that the original sequence \( (u_{\varepsilon_j}, u'_{\varepsilon_j}) \) defined on the original probability \( (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_1, W \) converges in probability to an element \( (u_0, v_0) \) in the topology of \( X \).
This implies that the sequence \((u_ε)\) converges in probability to \(u_0\) in \(L^2(Q_T)\) and \(u'_ε\) converges in probability to \(u'_0\) in \(L^2(0,T;H^{-1}(Q))\). By the passage to the limit as in the previous subsection, it is not difficult to show that \(u_0\) is the unique strong solution of (4.22). This ends the proof of the theorem.

5. Approximation of homogenized coefficients and proof of Theorem 1.2

We assume that the notation is as in the preceding sections. In the preceding section, we saw that the corrector problem is posed on the whole of \(\mathbb{R}^N\). However, if the coefficients of our problem are locally periodic (say the function \(y \mapsto A_0(x,y)\) is \(Y\)-periodic for each fixed \(x, Y = (−1/2, 1/2)^N\)), then this problem reduces to another one posed on the bounded subset \(Y\) of \(\mathbb{R}^N\), and this yields coefficients that are computable when \(x\) is fixed. Contrasting with the periodic setting, the corrector problem in the general deterministic framework cannot be reduced to a problem on a bounded domain. Therefore, truncations must be considered, particularly on large domains like \(B_R\) (or \((-R,R)^N\) in practice) with appropriate boundary conditions. In that case the homogenized coefficients are captured in the asymptotic regime. We proceed exactly as in the random setting (see [9]).

We make a truncation on the ball \(B_R\) \((R > 0)\) and impose linear Dirichlet boundary condition on \(\partial B_R\):

\[-\nabla_y \cdot (A_0(x,\cdot)(ε_j + \nabla_y χ_{j,R}(x,\cdot))) = 0\] in \(B_R\), \(χ_{j,R}(x,\cdot) \in H^1_0(B_R)\).

(5.1)

The following result is classical and the proof is omitted.

**Lemma 5.1.** Problem (5.1) possesses a unique solution \(u\) that satisfies the estimate

\[\left(\frac{1}{|B_R|} \int_{B_R} |\nabla_y χ_{j,R}(x,\cdot)|^2 dy\right)^{1/2} \leq C\] for any \(R \geq 1\)

(5.2)

where \(C\) is a positive constant independent of \(R\).

Let \(χ_{j,R}(x,\cdot)\) be the solution to (5.1). As we saw in Section 1 we may assume here that the matrix \(A_0\) does not depend on the macroscopic variable \(x\), so that the functions \(χ_{j,R}(x,\cdot)\) are constant with respect to \(x \in Q\), that is, \(χ_{j,R}(x,y) \equiv χ_{j,R}(y)\). We define therefore the effective and approximate effective matrices \(A\) and \(A_R\) respectively, as in (1.8) (see Section 1). Here below, we restate and prove Theorem 1.2

**Theorem 5.1.** The generalized sequence of matrices \(A_R\) converges, as \(R \to \infty\), to the homogenized matrix \(A\).

**Proof.** We set \(w_j^R(y) = \frac{1}{R}χ_{j,R}(R y)\) for \(y \in B_1\) and consider the rescaled version of (5.1) whose \(w_j^R\) is solution. It reads as

\[-\nabla_y \cdot (A_0(ε_j + \nabla_y w_j^R)) = 0\] in \(B_1\), \(w_j^R = 0\) on \(\partial B_1\).

(5.3)

Then (5.3) possesses a unique solution \(w_j^R \in H^1_0(B_1)\) satisfying the estimate

\[\|\nabla_y w_j^R\|_{L^2(B_1)} \leq C\] (1 ≤ \(j \leq N\)

(5.4)

where \(C > 0\) is independent of \(R > 0\). Based on (5.4) and for a fixed \(1 \leq j \leq N\), let \(w_j \in H^1_0(B_1)\) be the weak limit in \(H^1_0(B_1)\) of a weakly convergent subnet \((w_j^{R'})\) of \((w_j^R)_R\). Then proceeding as in the proof of Theorem 1.1 (see especially the proof of (4.14) therein), it is an easy exercise to see that \(w_j\) solves the equation

\[-\nabla_y \cdot (A(ε_j + \nabla_y w_j)) = 0\] in \(B_1\),

(5.5)

and further thanks to [28] Theorem 5.2, the convergence result (as \(R' \to \infty\))

\[A_0(ε_j + \nabla_y w_j^{R'}) \to A(ε_j + \nabla_y w_j)\] in \(L^2(B_1)^N\)-weak

(5.6)

is satisfied. From the ellipticity property of \(A_0\) and the uniqueness of the solution to (5.5) in \(H^1_0(B_1)\), we deduce that \(w_j = 0\), so that \(w = (w_1,...,w_N) = 0\). We infer that the whole sequence
continuous almost periodic functions on $\mathbb{R}$, and $\mathcal{AP}$.

6.2. Problem 2 (Stochastic almost periodic homogenization).

Proof. The above result stems from the characterization of the mean value in the periodic case: if

\[ u \] converges in probability to the solution of (1.1) with

\[ A(x, y)(I + \nabla_y \chi(x, y))dy \]

\[ f(r) = \int_{Y \times T} f(y, \tau, r)dyd\tau \text{ and } g(r) = \int_{Y \times T} g(y, \tau, r)dyd\tau, \]

\[ \chi(x, \cdot) = (\chi_j(x, \cdot))_{1 \leq j \leq N} \in H^1_\#(Y)^N \text{ being defined as the solution of the problem} \]

\[ \nabla_y \cdot (A_0(x, \cdot) (e_j + \nabla_y \chi_j(x, \cdot))) = 0 \text{ in } Y. \]

\[ \int_{Y \times T} u(y, \tau)d\tau. \]

6.2. Problem 2 (Stochastic almost periodic homogenization). The functions $A_0(x, \cdot), f(\cdot, \cdot, \lambda)$ and $g(\cdot, \cdot, \lambda)$ are assumed to be Besicovitch almost periodic [5]. We then get (A4) with $A_y = \mathcal{AP}(\mathbb{R}^N), A_r = \mathcal{AP}(\mathbb{R})$ and so $A = \mathcal{AP}(\mathbb{R}^{N+1}),$ where $\mathcal{AP}(\mathbb{R}^N)$ is the algebra of Bohr continuous almost periodic functions on $\mathbb{R}^N$. In this case the mean value of a function $u \in \mathcal{AP}(\mathbb{R}^N)$ can be obtained as the unique constant belonging to the closed convex hull of the family of the translates $(u(\cdot + a))_{a \in \mathbb{R}^N};$ see e.g. [25].
6.3. Problem 3. Let $F$ be a Banach space. Let $B_\infty(\mathbb{R}^d; F)$ denote the space of all continuous functions $\psi \in C(\mathbb{R}^d; F)$ such that $\psi(\zeta)$ has a limit in $F$ as $|\zeta| \to \infty$. When $F = \mathbb{R}$ we set $B_\infty(\mathbb{R}^d; \mathbb{R}) \equiv B_\infty(\mathbb{R}^d)$. It is known that $B_\infty(\mathbb{R}^d)$ is an algebra with mean value on $\mathbb{R}^d$ for which the mean value of any function $u \in B_\infty(\mathbb{R}^d)$ is obtained as the limit at infinity:

$$M(u) = \lim_{|y| \to \infty} u(y);$$

see [34].

With this in mind, our goal here is to homogenize Problem (1.1) under the hypothesis

$$(A4)_2 A_0(x, \cdot) \in (L^2_{per}(Y))^{N \times N} \text{ for any } x \in \overline{\mathcal{Q}}; f(\cdot, \cdot, \lambda), g_k(\cdot, \cdot, \lambda) \in B_\infty(\mathcal{R}_\tau; L^2_{per}(Y)) \text{ for all } \lambda \in \mathbb{R} \text{ and } k \geq 1.$$

It is an easy task to see that the appropriate algebras with mean value here are $A_y = C_{per}(Y)$ and $A_\tau = B_\infty(\mathcal{R}_\tau)$, so that $(A4)$ holds true with $A = B_\infty(\mathcal{R}_\tau) \otimes C_{per}(Y) = B_\infty(\mathcal{R}_\tau; C_{per}(Y))$.

6.4. Problem 4. With the notations of Problem 3, we replace here $L^2_{per}(Y)$ by $B^2_{AP}(\mathbb{R}^N)$, the space of Besicovitch almost periodic functions. Then $(A4)$ is verified with $A_y = AP(\mathbb{R}^N)$ and $A_\tau = B_\infty(\mathcal{R}_\tau)$, and hence $A = B_\infty(\mathcal{R}_\tau; AP(\mathbb{R}^N))$.

6.5. Problem 5 (Stochastic asymptotic almost periodic homogenization). Let $B_{\infty, AP}(\mathbb{R}^N) = B_{\infty}(\mathbb{R}^N) + AP(\mathbb{R}^N)$. We know that $B_{\infty, AP}(\mathbb{R}^N)$ is an algebra with mean value on $\mathbb{R}^N$ with the property that $B_{\infty, AP}(\mathbb{R}^N) = C_0(\mathbb{R}^N) \oplus AP(\mathbb{R}^N)$ (direct and topological sum; see e.g. [40]) where $C_0(\mathbb{R}^N)$ stands for the space of those $u \in BUC(\mathbb{R}^N)$ that vanish at infinity. Since $\lim_{|y| \to \infty} u(y) = 0$ for any $u \in C_0(\mathbb{R}^N)$, any element in $B_{\infty, AP}(\mathbb{R}^N)$ is asymptotically an almost periodic function.

Bearing all this in mind, we aim at studying the homogenization problem for (1.1) under the hypothesis:

$$(A4)_3 A_0(x, \cdot) \in B^2_{\infty, AP}(\mathbb{R}^N) \text{ and } f(\cdot, \cdot, \cdot, \lambda), g(\cdot, \cdot, \cdot, \lambda) \in B^2_{AP}(\mathcal{R}_\tau; B^2_{\infty, AP}(\mathbb{R}^N)), \text{ all } x \in \overline{\mathcal{Q}} \text{ and } \lambda \in \mathbb{R}.$$

We recall that here $B^2_{\infty, AP}(\mathbb{R}^N)$ denotes the Besicovitch space associated to the algebra $\mathbb{R}^N$.

Assumption $(A4)_3$ leads to $(A4)$ with $A_y = B_{\infty, AP}(\mathbb{R}^N)$, $A_\tau = AP(\mathcal{R}_\tau)$ and hence $A = AP(\mathcal{R}_\tau; B_{\infty, AP}(\mathbb{R}^N))$.

Remark 6.1. 1) Some other assumptions leading to $(A4)$ are in order; 2) The assumption of Problem 5 includes the special case of asymptotic periodic homogenization in which $A_y = B_\infty(\mathbb{R}^N) + C_{per}(Y)$ and $A_\tau = C_{per}(\mathcal{R}_\tau)$, a self-contained problem; 3) It is worth noticing that $B_{\infty}(\mathcal{R}_\tau; AP(\mathbb{R}^N)) \neq B_{\infty}(\mathbb{R}^N) + AP(\mathbb{R}^N)$.

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