Euler characteristics of arithmetic groups

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Abstract

We develop a general method for computing the homological Euler characteristic of finite index subgroups $\Gamma$ of $GL_m(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers in a number field $K$. With this method we find, that for large, explicitly computed dimensions $m$, the homological Euler characteristic of finite index subgroups of $GL_m(\mathcal{O}_K)$ vanishes. For other cases, some of them very important for spaces of multiple polylogarithms, we compute non-zero homological Euler characteristic. With the same method we find all the torsion elements in $GL_3\mathbb{Z}$ up to conjugation. Finally, our method allows us to obtain a formula for the Dedekind zeta function at $-1$ in terms of the ideal class set and the multiplicative group of quadratic extensions of the base ring.

0 Introduction

The homological Euler characteristic of a group $\Gamma$ with coefficients in a representation $V$ is defined by

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim H^i(\Gamma, V)$$

(see [S], [B1]). The main theoretical result of our thesis is a general method that allows us to calculate $\chi_h(\Gamma, V)$. Toward the end of the introduction we describe briefly this method. Before that we list the most important results of the thesis which demonstrate the scope of the method. Our first result is about vanishing of homological Euler characteristics.

**Theorem 0.1** Let $\Gamma$ be a finite index subgroup of $GL_m(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in a number field $K$. Let $V$ be a finite dimensional representation of $\Gamma$. Then

(a) For $K = \mathbb{Q}$ if $m > 10$ then $\chi_h(\Gamma, V) = 0$,

(b) For $K = \mathbb{Q}[\sqrt{-d}]$, where $-d$ is the discriminant, if $d = 4$ and $m > 4$ then $\chi_h(\Gamma, V) = 0$;

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if \( d = 3 \) and \( m > 6 \) then \( \chi_h(\Gamma, V) = 0 \);
for the other \( d \)'s and \( m > 2 \) we have \( \chi_h(\Gamma, V) = 0 \),

(c) For the remaining number fields \( K \) we always have \( \chi_h(\Gamma, V) = 0 \).

We obtain a similar result for \( SL_m \) in place of \( GL_m \).

**Theorem 0.2** Let \( \Gamma \) be a finite index subgroup of \( SL_m(\mathcal{O}_K) \), where \( m \geq 2 \) and \( \mathcal{O}_K \) is the ring of integers in a number field \( K \). Let \( V \) be a finite dimensional representation of \( \Gamma \).

(a) For \( K = \mathbb{Q} \) if \( m > 10 \) then \( \chi_h(\Gamma, V) = 0 \),
(b) For \( K = \mathbb{Q}[\sqrt{-d}] \), where \(-d \) is the discriminant,
if \( d = 4 \) and \( m > 4 \) then \( \chi_h(\Gamma, V) = 0 \);
if \( d = 3 \) and \( m > 6 \) then \( \chi_h(\Gamma, V) = 0 \);
for the other \( d \)'s and \( m > 2 \) we have \( \chi_h(\Gamma, V) = 0 \),

(c) For \( K \) totally real field if \( m > 2 \) then \( \chi_h(\Gamma, V) = 0 \),

(d) For the remaining number fields \( K \) we always have \( \chi_h(\Gamma, V) = 0 \).

Parts (a) and (b) follow from the previous theorem while parts are new statements.

We also compute the homological Euler characteristic of the arithmetic subgroups \( \Gamma_1(3, N) \) and \( \Gamma_1(4, N) \) of \( GL_3(\mathbb{Z}) \) and \( GL_4(\mathbb{Z}) \), respectively, where \( \Gamma_1(m, N) \) is the subgroup of \( GL_m(\mathbb{Z}) \) that fixes the vector \([0, \ldots, 0, 1]\) mod \( N \).

**Theorem 0.3** The homological Euler characteristic of \( \Gamma_1(3, N) \) and of \( \Gamma_1(4, N) \) for \( N \) not divisible by \( 2 \) and \( 3 \) is given by

\[
\chi_h(\Gamma_1(3, N), \mathbb{Q}) = -\frac{1}{12} \varphi_2(N) + \frac{1}{2} \varphi(N),
\]

\[
\chi_h(\Gamma_1(4, N), \mathbb{Q}) = \varphi(N),
\]

where \( \varphi(N) \) is the Euler \( \varphi \)-function, and \( \varphi_2(N) \) is the multiplicative arithmetic function generated by \( \varphi_2(p^a) = p^{2a}\left(1 - \frac{1}{p}\right) \) for \( a \geq 1 \).

With the same technique we find all the torsion elements in \( GL_3\mathbb{Z} \) up to conjugation (see proposition 3.2). Using our method we, also, compute \( \chi_h(GL_m(\mathbb{Z}), S^n V_m) \) for \( m = 3 \) and \( m = 4 \). The computation of \( \chi_h(GL_3(\mathbb{Z}), S^n V_3) \) (see theorem 6.4) agrees with the computation of the \( H^i(GL_3(\mathbb{Z}), S^n V_3) \) in [G1], which was used for computation of dimensions of spaces of certain multiple polylogarithms. Also the Euler characteristic of \( \Gamma_1(3, N) \) with trivial coefficients when \( N \) is an odd prime, greater that \( 3 \), agrees with the computation of \( H^i_{\inf}(\Gamma_1(3, N), \mathbb{Q}) \) in the corrected version of [G1]. We compute the homological Euler characteristic of \( GL_2(\mathbb{Z}[i]) \) and \( GL_2(\mathbb{Z}[\xi_3]) \) with coefficients in the symmetric powers of the standard representations. Also we compute the homological Euler characteristic of \( \Gamma_1(2, a) \) for an ideal
a in \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\xi_3] \), respectively. These computations can be used for finding dimensions of spaces of certain elliptic polylogarithms, as it was explained to me by professor A. Goncharov.

**Theorem 0.4** (a) If \( 1 + i \) does not divide \( a \) the homological Euler characteristic of \( \Gamma_1(2, a) \subset GL_2(\mathbb{Z}[i]) \) is given by

\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \frac{1}{2} \varphi_{\mathbb{Z}[i]}(a),
\]

where \( \varphi_{\mathbb{Z}[i]}(a) \) is the multiplicative function defined on the ideals of \( \mathbb{Z}[i] \), generated by

\[
\varphi_{\mathbb{Z}[i]}(p^n) = N_{\mathbb{Q}(i)/\mathbb{Q}}(p)^n \left( 1 - \frac{1}{N_{\mathbb{Q}(i)/\mathbb{Q}}(p)} \right).
\]

(b) If \( 1 + \xi_6 \) does not divide \( a \) the homological Euler characteristic of \( \Gamma_1(2, a) \subset GL_2(\mathbb{Z}[\xi_3]) \) is given by

\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \frac{1}{3} \varphi_{\mathbb{Z}[\xi_3]}(a),
\]

where \( \varphi_{\mathbb{Z}[\xi_3]}(a) \) is the multiplicative function defined on the ideals of \( \mathbb{Z}[\xi_3] \), generated by

\[
\varphi_{\mathbb{Z}[\xi_3]}(p^n) = N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p)^n \left( 1 - \frac{1}{N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p)} \right).
\]

In general the method works for any arithmetic subgroup of \( GL_m(\mathcal{O}_K) \), where \( \mathcal{O}_K \) is the ring of integers in a number field \( K \).

Our approach is the following: we generalize a result of K. Brown [B2] that relates the torsion elements in the group up to conjugation to the homological Euler characteristic of the group. Namely,

\[
\chi_h(\Gamma, V) = \sum_T \chi(C(T)) \text{Tr}(T^{-1}|V),
\]

where the sum is over all torsion elements \( T \) of \( \Gamma \) up to conjugation and \( C(T) \) is the centralizer of \( T \) in \( \Gamma \). Let us recall the definition of orbifold Euler characteristic of \( \Gamma \), denoted by \( \chi(\Gamma) \), which we simply call Euler characteristic. If \( \Gamma \) is a torsion free group then \( \chi(\Gamma) = \chi_h(\Gamma) \). If \( \Gamma \) has torsion consider a finite index torsion free subgroup \( \Gamma' \). Then \( \chi(\Gamma) \) is defined by

\[
\chi(\Gamma) = [\Gamma : \Gamma']^{-1} \chi(\Gamma').
\]

Arithmetic groups do have a finite index torsion free subgroup. So for them the Euler characteristic is defined. The main properties of the Euler characteristic that we are going to use are:

\[
\chi(G) = \frac{1}{|G|} \text{ for a finite group } G,
\]

Given an exact sequence

\[
0 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 0,
\]
we have
\[ \chi(\Gamma) = \chi(\Gamma_1)\chi(\Gamma_2), \]
and
\[ \chi(SL_m(O_K)) = \zeta_K(-1) \cdots \zeta_K(1-m). \]
The first two properties can be found in K. Brown’s book [B1]. And the last one is a difficult result due to Harder [H].

In order to use the Brown’s formula we need to know the torsion elements in the group. We develop a method for finding the torsion elements in $GL_m(O_K)$, and consider another form of the above formula that requires very few of the torsion elements (in general they can be quite a large number). This method involves linear algebra over number rings. More precisely it describes a normal form of matrices over number ring. First we deal with matrices with irreducible characteristic polynomial which lead to a relation to ideal classes (proposition 2.1). Then we examine matrices with reducible characteristic polynomials which leads to resultants (corollary 3.5 and proposition 3.7). Then we examine the relation between the torsion elements in $GL_m(O_K)$ and the torsion elements in an arithmetic subgroup $\Gamma$ of $GL_m(O_K)$. That gives us the homological Euler characteristic of $\Gamma$ with coefficients in a representation.

We obtain another interesting application of our method for computation of the values of the Dedekind zeta functions at $-1$. Now we are going to explain how this is related to our method. First, the Dedekind zeta function at $-1$ vanishes for number fields which are not totally real. For totally real number fields $K$ we consider the arithmetic group $SL_2(O_K)$. By a result of Harder [H] we have that the orbifold Euler characteristic of $SL_2(O_K)$ is the Dedekind zeta function at $-1$, $\zeta_K(-1)$. Also the homological Euler characteristic is always an integer. Let us denote the homological Euler characteristic by an integer $N$, i.e
\[ N = \chi_h(SL_2(O_K)). \]

We also introduce the following notation. Let
\[ Cl(O_K[\xi]/O_K) \]
be the set of ideal classes in $O_K[\xi]$ which are free as $O_K$-modules.

**Theorem 0.5** Let $K$ be a totally real number field. Let $R$ be the ring of integers in $K$. Then the Dedekind zeta function at $-1$ can be expressed as
\[ \zeta_K(-1) = -\frac{1}{4} \sum_{\xi} \sum_{I \in Cl(O_K[\xi]/O_K)} \frac{#O_K^\chi/N_{K(\xi)/K}(R_I^\chi)}{#(R_I^\chi)_{\text{tors}}} + \frac{1}{2} N, \]
where the first sum is taken over all roots of 1 such that $[K(\xi) : K] = 2$, the second sum over all ideal classes in $O_K[\xi]$ that are free as $O_K$-modules, the ring $R_I$ sits between $O_K[\xi]$ and its integral closure $O_{K(\xi)}$, (for precise definition of $R_I$ see theorem 8.1), and $N = \chi_h(SL_2(O_K), \mathbb{Q})$. 


The organization of the thesis is the following. In section 1 we deal with linear algebra over rings of algebraic integers. We give a method for classification of the matrices with integer coefficients for the ones that are diagonalizable over the complex numbers. In section 2 we give a method of computing the centralizer of a matrix which leads to explicit formulas for the homological Euler characteristics of $GL_m\mathbb{Z}$, $GL_m(\mathbb{Z}[i])$, $GL_m(\mathbb{Z}[\xi_3])$ and $GL_2(\mathbb{Q}(\sqrt{-d}))$, see respectively theorems 2.10, 2.11, 2.12 and 2.13. In section 3 we find the torsion elements in $GL_2\mathbb{Z}$ and $GL_3\mathbb{Z}$ up to conjugation. In section 4 is very computational. We compute many resultants and centralizers needed for the homological Euler characteristics of various group. In section 5 we find the homological Euler characteristics of $GL_m(\mathcal{O}_K)$ with coefficients the symmetric powers of the standard representation. In section 6 we find the homological Euler characteristics of arithmetic groups $\Gamma_1(m, N)$ and $\Gamma_1(m, a)$. In section 7 we examine the groups $SL_2(\mathcal{O}_K)$ for totally real number fields $K$ which gives a relation to the Dedekind zeta function of the field. And in section 8 we prove the generalization of Brown’s formula.

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1 Conjugacy classes in $GL_m(\mathcal{O}_K)$

In this section we describe the conjugacy classes of elements in $GL_m(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in a number field $K$. We approach the description of conjugacy classes in the following way. First we deal with matrices in $GL_m\mathbb{K}$. Then we examine matrices in $GL_m(\mathcal{O}_K)$ whose characteristic polynomial is irreducible over $K$. They are described by ideal classes of a larger ring. And, finally, we construct an algorithm for matrices with reducible characteristic polynomial that allows to consider instead matrices of smaller dimension. Using this inductive step we can describe completely conjugacy classes of certain type, knowing the conjugacy classes of matrices of smaller dimension.

1.1 Conjugacy classes in $GL_m\mathbb{K}$

First we recall a few statements about matrices with coefficients in an infinite field $K$.

Proposition 1.1 Let $A$ and $B$ be matrices with coefficients in an infinite field $K$. If $A$ and $B$ are conjugate to each other inside $GL_m\overline{K}$, where $\overline{K}$ is the algebraic closure of $K$, then they are conjugate to each other inside $GL_m\mathbb{K}$.

Proof. Consider the vector space of matrices $M$ over the fields $\overline{K}$ and $K$ such that $AM = MB$. Let $V_{\overline{K}} = \{ M \in Mat_m\overline{K} : AM = MB \}$
and

\[ V_K = \{ M \in \text{Mat}_m K : AM = MB \} \].

Since \( M \) is a solution of a linear system over fields, we have that

\[ \dim_K V_K = \dim_{\bar{K}} V_{\bar{K}}. \]

We know that there is \( Q \in \text{GL}_m \bar{K} \) such that \( AM = MB \). Equivalently, there is \( Q \) such that \( Q \in V_{\bar{K}} \) and \( \det Q \neq 0 \). Let

\[ \text{det}_K^0 = \{ M \in \text{Mat}_n K : \det M = 0 \}. \]

Then the set \( \text{det}_K^0 \) is Zariski closed subset of \( V_K \) that is not the entire space, because \( K \) has infinitely many elements. Let \( P \in V_{\bar{K}} - \text{det}_K^0 \). Then \( AP = PB \), and \( P \in \text{GL}_n K \). Thus, \( A \) and \( B \) are conjugate to each other as elements of \( \text{GL}_n K \).

We are going to use a well known statement.

**Lemma 1.2** The characteristic polynomial of

\[
\begin{pmatrix}
0 & 1 \\
\vdots & 0 & 1 \\
& & \ddots \\
0 & -a_0 & -a_1 & -a_{m-1}
\end{pmatrix}
\]

is \( \lambda^m + a_{m-1} \lambda^{m-1} + \ldots + a_0 \).

For \( 2 \times 2 \)-matrix can checked by direct computation. The one can use induction on the size of the matrix.

Given a polynomial \( f(\lambda) = \lambda^m + a_{m-1} \lambda^{n-1} + \ldots + a_0 \), we denote the above matrix by \( A_f \).

### 1.2 Block-triangular form

This subsection deals with the case when the characteristic polynomial of a matrix is reducible over the rational numbers. The partition of a matrix into blocks will be done in the following way: Given an \( m \times m \) matrix \( A \), let \( m = m_1 + \ldots + m_k \) be a partition of \( n \). Then \( A \) can be thought of as a \( k \times k \) block-matrix whose \( A_{ij} \)-entry,

\[ i, j = 1, \ldots, k \]

is a block (and a matrix) of size \( m_i \times m_j \). This will be the type of block-matrices that we consider. Note that the blocks \( A_{ii} \) are square matrices.

**Theorem 1.3** An \( m \times m \) matrix \( A \) over a principal ideal domain can be conjugated via \( B \in \text{GL}_m (\mathcal{O}_K) \) so that

\[ BAB^{-1} = \begin{bmatrix}
A_{11} & \ldots & A_{1d} \\
0 & \ddots & \vdots \\
0 & 0 & A_{dd}
\end{bmatrix}, \]

where \( A_{ij} \) are matrices such that \( A_{ii} \) has irreducible over \( K \) characteristic polynomial and \( A_{ij} = 0 \) for \( i > j \).
Proof. The plan for the proof is the following: First we consider the invariant subspaces over \( \overline{K} \), where \( \overline{K} \) is the algebraic closure of \( K \), in order to find the invariant spaces over \( K \). Then we construct a good basis over \( \mathbb{R} \).

Let \( A \) be a matrix from \( GL_m(\mathcal{O}_K) \). Let \( \lambda \) be an eigenvalue of \( A \). Let \( v_{1,\lambda} \) be one of the eigenvectors corresponding to \( \lambda \). Let \( v_{i,\lambda}, i = 2, \cdots, k \) be adjoint vectors to \( v_{1,\lambda} \) so that

\[
(A - \lambda I)v_{i+1,\lambda} = v_{i,\lambda}.
\]

We can choose the coefficients of \( v_{1,\lambda} \) to be from the field \( K(\lambda) \), since it is determined from the linear system

\[
(A - \lambda I)v_{1,\lambda} = 0.
\]

Then similarly, all \( v_{i,\lambda} \)'s have coefficients in the field \( K(\lambda) \). Let \( \lambda_1, \cdots, \lambda_d \) be the Galois conjugates to \( \lambda \). And let

\[
L := K(\lambda_1, \cdots, \lambda_d).
\]

We can consider the Galois conjugates of \( v_{1,\lambda} \). If \( \sigma \in Gal(L/K) \) then \( v_{1,\lambda}^\sigma = v_{1,\lambda}\sigma \) is an eigenvector with eigenvalue \( \lambda^\sigma \) and adjoint vectors \( v_{i,\lambda}^\sigma = v_{i,\lambda}\sigma \) for \( i = 2, \cdots, k \).

We can consider invariant subspaces with coefficients in \( L \), namely the vector space over \( L \) spanned by the vectors \( v_{1,\lambda}, \cdots, v_{i,\lambda} \) which we are going to denote by

\[
V_{i,\lambda} := L\{v_{1,\lambda}, \cdots, v_{i,\lambda}\}
\]

for \( i = 1, \cdots, k \). Consider the space

\[
V_i = \bigoplus_{j=1}^d V_{i,\lambda_j} = \text{span}\{V_{i,\lambda_{j_1}}, \cdots, V_{i,\lambda_{j_d}}\}.
\]

Note that

\[
V_{i,\lambda_{j_1}} \cap V_{i,\lambda_{j_2}} = 0.
\]

Therefore the matrix \( A \) and the Galois group \( Gal(L/K) \) send the vector space (over \( L \)) \( V_i \) to itself. Then we can take the Galois invariant subspace of \( V_i \), or equivalently the vectors with coordinates in \( K \). Let

\[
V_{i,K} = V_i \cap K^m.
\]

Now we have a filtration of vector spaces over \( K \), namely,

\[
V_{1,K} \subset V_{2,K} \subset \cdots \subset V_{k,K}.
\]

Let

\[
V_{i,\mathcal{O}_K} := V_{i,K} \cap \mathcal{O}_K^m.
\]

\( V_{i,\mathcal{O}_K} \) is a free \( \mathcal{O}_K \)-module because it is a a finitely generated, torsion free module over a principal ideal domain. Choose a basis

\[
\{e_1, \cdots, e_{i_1}\}
\]

of \( V_{1,\mathcal{O}_K} \). One can extend it to a basis

\[
\{e_1, \cdots, e_{i_1}, e_{i_1+1}, \cdots, e_{i_2}\}
\]
of $V_{2,\mathcal{O}_K}$. In the same way one can extend the basis successively so that at the end we have a basis $\{e_1, \cdots, e_{i_k}\}$ of $V_{k,\mathcal{O}_K}$ whose restriction to $V_{i,\mathcal{O}_K}$ gives again a basis. Finally, take the basis
$$\{e_1, \cdots, e_{i_k}\}$$
and extend it to a basis $\{e_1, \cdots, e_m\}$ of $\mathcal{O}_K^m$. Let $P$ be a matrix whose column vectors are
$$\{e_1, \cdots, e_m\}$$
in the same order. Then $P^{-1}AP$ has the following properties: (1) Consider $P^{-1}AP$ in a block form with blocks $i_k \times i_k$, $(m-i_k) \times i_k$, $i_k \times (m-i_k)$ and $(m-i_k) \times (m-i_k)$. Then the block $(m-i_k) \times i_k$, under the diagonal is zero. (2) In case we have $k > 1$ we can describe the block $i_k \times i_k$. The block $i_k \times i_k$ can be considered as a matrix consisting of $k^2$ square blocks of size $d \times d$. Moreover, under the diagonal the block are zero.

We can apply an induction argument on the block $(m-i_k) \times (m-i_k)$ treating it as a matrix in $GL_{m-i_k}(\mathcal{O}_K)$. When this process is over, we have the matrix $A$ conjugated by a matrix $Q$ so that the resulting matrix $Q^{-1}AQ$ has the following property: each block on the diagonal has irreducible characteristic polynomial, and each block under the diagonal is zero.

If we relax the condition that $\mathcal{O}_K$ is a principal ideal domain then instead of matrices $A_{ij}$ interpreted as homomorphism of free modules, we should have $A_{ij}$ to be a homomorphism of projective modules.

**Definition 1.4** Let $P$ be a finitely generated torsion free module over an integral Noetherian ring $R$ of dimension 1. Let $K$ be the field of fractions of $R$. Let $A$ be an endomorphism of $P$,
$$A \in \text{End}_R(P).$$

Then $A$ can be extended to an endomorphism of a finite dimensional vector space $P \otimes_R K$. Then the characteristic polynomial of $A$ is defined to be the characteristic polynomial of the induced map in $\text{End}_K(P \otimes_R K)$.

The examples of such rings that we consider are orders in a number rings. If the ring is also integrally closed, that is it is a Dedekind domain, we have the following generalization of the previous theorem.

**Theorem 1.5** Let $R$ be a Dedekind domain. And let $P$ be a finitely generated projective module over $R$. Assume that the field of fractions $K$ has characteristic 0. Let, also, $A$ be an endomorphism of $P$. Then in a suitable basis we have

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1d} \\ 0 & \ddots & \vdots \\ 0 & 0 & A_{dd} \end{bmatrix},$$

where $A_{ij} \in \text{Hom}(P_j, P_i)$, with $P = P_1 \oplus \cdots \oplus P_d$, so that $A_{ij} = 0$ for $i > j$ and $A_{ii}$ has irreducible over $K$ characteristic polynomial.
Proof. Consider $A$ as an endomorphism of $P \otimes_R \overline{K}$. Let $\lambda_1$ be an eigenvalue of $A$. Let $\lambda_1, \ldots, \lambda_r$ be the Galois conjugates of $\lambda_1$. Let $v_1$ be an eigenvector in $P \otimes_R \overline{K}$ corresponding to the eigenvalue $\lambda_1$. Let $v_1, \ldots, v_r$ be the Galois conjugates of $v_1$. Let $V = \text{span}(v_1, \ldots, v_r)$. And let $V_K = P \otimes_R K \cap V$ and $V_R = P \cap V$. We have that

$$\text{rank}_R(V_R) = \text{dim}_K(V_K) = \text{dim}_R(V).$$

Note that the endomorphism $A$ sends $V_R$ to itself. We claim that $V_R$ and $P/V_R$ are projective $R$-modules and therefore direct summand of $P$.

Let $\pi : P \to P/V_R$. Suppose $P/V_R$ has torsion. Let $T$ be the torsion submodule of $P/V_R$. Then $V_R$ is of finite index in $\pi^{-1}(T)$. Therefore

$$\pi^{-1}(T) \subseteq (\pi^{-1}(T) \otimes_R K) \cap P = V_K \cap P = V_R.$$ 

Then $R/V_R$ is torsion free. We are working over a Dedekind domain. Thus, the finitely generated torsion free modules are projective. Thus, $P/V_R$ is a direct summand of $P$. Set $P_d = V_R$. Note that $A$ induces the trivial map in $\text{Hom}(P/P_d, P_d)$. Proceed by induction until $P$ is factorized into projective modules $P = P_1 \oplus, \ldots, \oplus P_d$ so that $A$ induces the trivial map in $\text{Hom}(P_j, P_i)$ for $i > j$.

1.3 Irreducible blocks

We have the following characterization of matrices with an irreducible characteristic polynomial. For principal ideal domains the proof is simpler.

Proposition 1.6 Let $O_K$ be the ring of integers in a number field $K$. Assume $O_K$ is a principal ideal domain. Let $f(t)$ be an irreducible over $K$ monic polynomial. Consider the set of all matrices $\{A_i\}$ which have characteristic polynomial $f(t)$ and which are not conjugate to each other via $GL_m(O_K)$ where $m = \text{deg}(f)$. This set is parametrized by the ideal classes in $O_K[t]/(f(t))$.

Proof. The idea of the proof is to consider $calO_K$-endomorphism of ideals in $O_K[\lambda]$, where $\lambda$ is a root of $f(t)$ in an algebraic closure of $K$. Let $I$ be an ideal in

$$O_K[\lambda] \cong O_K[t]/(f(t)).$$

It is a free $O_K$-module because $O_K$ is a principal ideal domain. And let

$$I = O_K\{\alpha_1, \ldots, \alpha_m\}$$

be a basis of $I$. Then the multiplication by $\lambda$ is an endomorphism of the ideal $I$. Let

$$\lambda \cdot \alpha_i = \sum_j a_{ij} \alpha_j,$$

where $a_{ij} \in O_K$. If $A = (a_{ij})$ and

$$\alpha^t = (\alpha_1, \ldots, \alpha_m)^t$$

then $\alpha^t$ is an eigenvector of $A$ with eigenvalue $\lambda$. Also the matrix $A = (a_{ij})$ has characteristic polynomial $f$, because one of the eigenvalues of $A$ is $\lambda$ and because $a_{ij} \in O_K$. 

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If we choose an ideal from the same ideal class we have $I' = \alpha' I$ for some $\alpha' \in K[\lambda]$. Form the basis

$$I' = \mathcal{O}_K\{\alpha'\alpha_1, \cdots, \alpha'\alpha_m\}$$

we obtain the same matrix $A$.

Let

$$I = \mathcal{O}_K\{\beta_1, \cdots, \beta_m\}$$

be a different basis of $I$. Let also

$$\beta_i = \sum_j b_{ij} \alpha_j,$$

and let

$$B = (b_{ij}).$$

Let

$$\beta = (\beta_1, \cdots, \beta_m)$$

and

$$\alpha = (\alpha_1, \cdots, \alpha_m)$$

be vectors. Then

$$BAB^{-1} \cdot \beta = BA \cdot \alpha = B \cdot \lambda \alpha = \lambda B\alpha = \lambda \beta.$$ 

Then the matrix corresponding to the basis

$$I = \mathcal{O}_K\{\beta_1, \cdots, \beta_m\}$$

is $BAB^{-1}$, which is conjugate to $A$. In this way we associate a matrix with integer coefficients to an element of the ideal class.

Conversely, if $A$ is a matrix with coefficients in $\mathcal{O}_K$ and with irreducible over $K$ characteristic polynomial $f$, we want to associate an element of the ideal class of $\mathcal{O}_K[t]/(f(t))$, which is a free $\mathcal{O}_K$-module. If we start with the ideal (1) with $\mathcal{O}_K$-basis

$$\mathcal{O}_K\{1, \lambda, \cdots, \lambda^{m-1}\}$$

then

$$A_0 = \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ -a_0 & \cdots & -a_{m-1} \end{bmatrix}$$

is the corresponding matrix, where $\lambda$ is a root of the characteristic polynomial

$$f(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0.$$

Using proposition 1.1, we can find $P \in GL_m K$ such that $A = PA_0P^{-1}$. We can also assume that $P$ has integer entries otherwise we can consider $N \cdot P$ where $N$ is a suitable integer. Let

$$\alpha_i = \sum_j b_{ij} \lambda^{j-1},$$

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where \((p_{ij}) = P\).

Claim: The \(O_K\)-submodule
\[
I = O_K\{\alpha_1, \cdots, \alpha_m\}
\]
is an ideal in \(O_K[\lambda]\) and its automorphism with respect to the given basis is the matrix \(A\).

Before we prove that let
\[
\vec{\alpha} = (\alpha_1, \cdots, \alpha_m)^t
\]
and
\[
\vec{\lambda} = (1, \lambda, \cdots, \lambda^{m-1})^t
\]
be vectors. Then
\[
\lambda \cdot \vec{\alpha} = \lambda P \cdot \vec{\lambda} = P \cdot A_0 \vec{\lambda} = PA_0P^{-1} \cdot P \vec{\lambda} = PA_0P^{-1} \cdot \vec{\alpha} = A \cdot \vec{\alpha}.
\]

As a consequence the \(O_K\)-module \(I\) is an ideal, since each of its basis elements transforms via
\[
\lambda \alpha_i = \sum_j a_{ij} \alpha_j,
\]
where \(a_{ij} \in O_K\). Also, \(A\) corresponds to the multiplication by \(\lambda\). Thus \(A\) corresponds to the ideal class of \(I\).

**Proposition 1.7** Let \(f(t)\) be an irreducible monic polynomial. And let \(P\) be a projective \(O_K\)-module. Consider the set of all endomorphisms \(\{A_i\}\) of \(P\) which have characteristic polynomial \(f(t)\) and which are not conjugate to each other via an automorphism of \(P\). This set is parametrized by the ideal classes in \(O_K[t]/(f(t))\) which are isomorphic to \(P\) as \(O_K\)-modules.

**Proof.** The idea of the proof is to consider \(O_K\)-endomorphism of ideals in \(O_K[\lambda]\), where \(\lambda\) is a root of \(f(t)\) in an algebraic closure of \(K\). Let \(I\) be an ideal in
\[
O_K[\lambda] \cong O_K[t]/(f(t)),
\]
which is isomorphic to \(P\) as \(O_K\)-module. Let \(A\) be an endomorphism of \(I\) induces by multiplication by \(\lambda\). Tensoring the endomorphism with the field \(K\) we obtain that \(\lambda\) is an eigenvalue. Thus, \(f(t)\) is the characteristic polynomial of \(A\). Also, if \(B\) is an \(O_K\)-automorphism of \(P\), then it is also an \(O_K\)-automorphism of \(I\). Then \(A\) sends \(BI\) to \(BJ\). Therefore, the induced endomorphism on \(I\) is \(BAB^{-1}\). Thus \(A\) is determined up to conjugation. If \(J\) is in the same ideal class as \(I\), then \(J = \alpha I\), for \(\alpha \in K(\lambda)\)\(^*\). That induces an isomorphism between the \(O_K\)-modules \(I\) and \(J\). Thus, an endomorphism on one lead to an endomorphism of the other. This is the one of the direction of the correspondance between ideal classes and endomorphisms.

Given an endomorphism of \(P\) with characteristic polynomial \(f(t)\), we want ot associate an ideal class in \(O_K[\lambda]\). Note that the ring \(O_K[A]\) is isomorphic to \(O_K[\lambda]\), since \(f(A) = 0\) and \(f(\lambda) = 0\). The ring \(O_K[A]\) acts on \(P\), because \(P\) is an \(O_K\)-module, and also \(A\) acts on \(P\). Therefore \(P\) is an \(O_K[\lambda]\)-module because \(O_K[\lambda]\) is isomorphic to \(O_K[A]\). It remains to show that torsion free \(O_K[\lambda]\)-modules of rank
1 correspond to ideal classes. Consider the imbedding $P \to P \otimes O_K[\lambda] K(\lambda)$ followed by the isomorphism $\theta : P \otimes O_K[\lambda] K(\lambda) \to K(\lambda)$. The image of the composition is a fractional ideal in $K(\lambda)$. All the choices for the isomorphism are parametrized by $K(\lambda)^\times$. Therefore, the torsion free $O_K[\lambda]$-modules of rank 1 correspond to the set of ideal classes module $K(\lambda)^\times$ which is precisely the ideal classes in $O_K[\lambda]$.

**Corollary 1.8** Let $f(t) \in O_K[t]$ be an irreducible monic polynomial. Assume that $O_K[t]/(f(t))$ is an integrally closed. Then the conjugacy classes of matrices in $GL_m(O_K)$ with characteristic polynomial $f(t)$ are in one-to-one correspondence with the elements in

$$\ker(K_0(O_K[t]/(f(t))) \to K_0(O_K)).$$

**Proof.** It is known that for number rings $O_K$ we have

$$K_0(O_K) = \mathbb{Z} \oplus Cl(K),$$

where $Cl(K)$ is the ideal class group of $O_K$ (see [M]). Then the ideal classes in $O_K[t]/(f(t))$ which are free $O_K$-modules are precisely

$$\ker(K_0(O_K[t]/(f(t))) \to K_0(O_K)).$$

### 1.4 Reducible blocks

Using the theorem 1.5 we obtain the following: If $A$ is an $m \times m$ matrix with coefficients in $O_K$, or more generally, an endomorphism of projective $O_K$-module of rank $m$, having an irreducible characteristic polynomial then $A$ is conjugated by an element of $GL_m(O_K)$, or of $Aut(P)$, to a $2 \times 2$-block endomorphism

$$\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}.$$  

Let $A_{11}$ and $A_{22}$ be automorphisms of projective modules $P_1$ and $P_2$, with

$$P_1 \oplus P_2 = P.$$  

Now we are going to describe a method that simplifies the block $A_{12}$ and leaves $A_{11}$ and $A_{22}$ unchanged. We can assume that

$$A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}.$$  

Conjugate $A$ with an automorphism

$$B = \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix}$$

of the some block form. that is $B_{11} \in Aut(P_1)$, $B_{22} \in Aut(P_2)$ and $B_{12} \in Hom(P_2, P_1)$. We want the conjugation by $P$ to preserve $A_{11}$ and $A_{22}$ that is

$$\begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix} \cdot \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A'_{12} \\
0 & A_{22}
\end{bmatrix} \cdot \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix}.$$
We do have that $A_{12}$ is changed to $A'_{12}$. Then
\[ B_{11}A_{11} = A_{11}B_{11}, \]
\[ B_{22}A_{22} = A_{22}B_{22}, \]
and
\[ A'_{12}B_{22} - B_{11}A_{12} = B_{12}A_{22} - A_{11}B_{12}. \]
Denote by $C(A_{ii})$ the centralizer of $A_{ii}$. Then $B_{11} \in C(A_{11})$ and $B_{22} \in C(A_{22})$.

The block $B_{12}$ could be any map in $\text{Hom}_{O_K}(P_2, P_1)$. Let
\[ P_{A_{11}, A_{22}} : B_{12} \mapsto B_{12}A_{22} - A_{11}B_{12} \]
be a map from the space $\text{Hom}_{O_K}(P_2, P_1)$ to itself. It is a linear. Then the relation
\[ A'_{12}B_{22} - B_{11}A_{12} = B_{12}A_{22} - A_{11}B_{12}, \]
between $A'_{12}$ and $A_{12}$ can be written as
\[ A'_{12}B_{22} \equiv B_{11}A_{12} \mod(\text{Im}P_{A_{11}, A_{22}}). \]

**Lemma 1.9** Let $P_{\text{mod}} = \text{Im}(P_{A_{11}, A_{22}})$ where $P_{A_{11}, A_{22}} : B_{12} \mapsto B_{12}A_{22} - A_{11}B_{12}$

Let also $Q_{\text{mod}} = \text{Hom}_{O_K}(P_2, P_1)/P_{\text{mod}}$. Then $\text{Hom}_{O_K}(P_2, P_1)$, $P_{\text{mod}}$ and $Q_{\text{mod}}$ are $C(A_{11}) \times C(A_{22})$-modules.

**Proof.** Let $B_{11} \in C(A_{11})$ and $B_{22} \in C(A_{22})$. Then
\[ (B_{11}, B_{22}) \cdot P_{A_{11}, A_{22}}(B_{12}) = B_{11} \cdot P_{A_{11}, A_{22}}(B_{12}) \cdot B_{22}^{-1} = B_{11}(B_{12}A_{22} - A_{11}B_{12})B_{22}^{-1} = B_{11}B_{12}A_{22}B_{22}^{-1} - B_{11}A_{11}B_{12}B_{22}^{-1} = (B_{11}B_{12}B_{22}^{-1})A_{22} - A_{11}(B_{11}B_{12}B_{22}^{-1}) = P_{A_{11}, A_{22}}(B_{11}B_{12}B_{22}^{-1}). \]

Obviously, $\text{Hom}_{O_K}(P_2, P_1)$ is a $C(A_{11}) \times C(A_{22})$-module. Thus, the quotient, as abelian group, $Q_{\text{mod}} = \text{Hom}_{O_K}(P_2, P_1)/P_{\text{mod}}$ has the structure of a $C(A_{11}) \times C(A_{22})$-module.

**Proposition 1.10** Let $A_{11} \in \text{Aut}(P_1)$ and $A_{22} \in \text{Aut}(P_2)$. Suppose $A_{11}$ and $A_{22}$ have no common eigenvalues. Then the matrices
\[ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{11} & A'_{12} \\ 0 & A_{22} \end{bmatrix} \]
are conjugate to each other if and only if the projection of $A_{12}$ and $A'_{12}$ onto the finite set $C(A_{11}) \times C(A_{22}) \setminus Q_{\text{mod}}$ coincide.

In order to prove the above proposition, we need the following definition and lemma. Consider the linear space of $n \times m$-matrices, or more generally of homomorphisms. Multiplication on the left and multiplication on the right are linear transformations.
of this space. So this transformations can be written as matrices, or as endomorphisms. Now we are going to set the notation for these matrices, after tensoring with $K$. Let $A = (A_{ij})$ and $B = (b_{kl})$ be two matrices. We shall write the tensor $A \otimes B$ as a matrix in the following way: The matrix can be considered as a block-matrix with each block being the size of $A$, and he $(k, l)$-block being $b_{kl}A$. This tensor product has the following properties: Let $A$ be an $m \times m$ matrix acting by left multiplication on $Mat_{m,n}$, i.e. on the $m \times n$ matrices. Consider $M$ as a vector by arranging the column vectors of $M$ one below the other. Then the left multiplication can be expressed as a matrix. And this matrix is precisely $A \otimes I_n$. If we multiply on the right by an $n \times n$ matrix $B$ then such a linear transform can be expressed by $I_m \otimes B^t$. Another property that is easy to check is $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Lemma 1.11 If $\prod_i(t - \alpha_i)$ and $\prod_j(t - \beta_j)$ are the characteristic polynomials of $A_{11} \in GL_{m_1}K$ and $A_{22} \in GL_{m_2}K$ then the characteristic polynomial of

$$P_{A_{11}, A_{22}} : X \mapsto XA_{22} - A_{11}X$$

is $\prod_{i,j}(t - \beta_j + \alpha_i)$. In particular, If $A_{11}$ and $A_{22}$ have no common eigenvalue then the map $P_{A_{11}, A_{22}}$ is non-singular, and if $R$ is a number ring $Q_{mod}$ is finite, namely the norm of the resultant.

Proof. We write the matrix $X$ as a vector, arranging the columns of $X$ below each other. Then the linear transform $P_{A_{11}, A_{22}}$ becomes

$$I_{m_1} \otimes A_{22}^t - A_{11} \otimes I_{m_2}.$$ If $A_{11}$ and $A_{22}^t$ are diagonal matrices with diagonals

$$(\alpha_1, \cdots, \alpha_{m_1})$$

and

$$(\beta_1, \cdots, \beta_{m_2})$$

then $P_{A_{11}, A_{22}}$ would be diagonal with diagonal entries $\beta_j - \alpha_i$. If $A_{11}$ and $A_{22}$ are arbitrary then we can find $B_1 \in GL_{m_1}K$ and $B_2 \in GL_{m_2}K$ with such that the matrices $B_1A_{11}B_1^{-1}$ and $B_2A_{22}^tB_2^{-1}$ are in Jordan block form. Let

$$(\alpha_1, \cdots, \alpha_{m_1})$$

be the diagonal entries of $B_1A_{11}B_1^{-1}$ and

$$(\beta_1, \cdots, \beta_{m_2})$$

be the diagonal entries of $B_2A_{22}^tB_2^{-1}$. Note that

$$B_1 \otimes B_2(P_{A_{11}, A_{22}})B_1^{-1} \otimes B_2^{-1} = B_1 \otimes B_2(I \otimes A_{22}^t - A_{11} \otimes I)B_1^{-1} \otimes B_2^{-1}$$

$$= I \otimes B_2A_{22}^tB_2^{-1} - B_1A_{11}B_1^{-1} \otimes I$$

is upper triangular. Its diagonal entries are $\beta_j - \alpha_i$. Therefore if $\alpha_i \neq \beta_j$ for all $i$ and $j$ then $P_{A_{11}, A_{22}}$ is non-singular. If $R$ is a number ring then $Im(P_{A_{11}, A_{22}}) = P_{mod}$.
is a sublattice of maximal rank in \( \text{Hom}_{O_K}(P_2, P_1) \) of index the norm \( N_{K/Q} \) of the resultant. Therefore, \( Q_{\text{mod}} = \text{Hom}(P_2, P_1)/P_{\text{mod}} \) is finite.

**Proof.** (of proposition 1.7) If \( A_{12}, A_{12}' \in \text{Hom}_{O_K}(P_2, P_1) \) are such that the endomorphisms

\[
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_{11} & A_{12}' \\
0 & A_{22}
\end{bmatrix}
\]

are conjugate to each other in \( \text{Aut}_{O_K}(P) \), then there exists \( B \in \text{Aut}_{O_K}(P) \) of the same block form such that

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & A_{12}' \\
0 & A_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}.
\]

Then the \((2,1)\)-entry of the product gives

\[B_{21}A_{11} = A_{22}B_{21}.\]

However, the map

\[P_{A_{22},A_{11}} : B_{21} \mapsto B_{21}A_{11} - A_{22}B_{21} = 0\]

is non-singular from the previous lemma. Therefore we have \( B_{21} = 0 \). Now the conditions on the other blocks of \( B \) become simpler:

\[B_{11}A_{11} = A_{11}B_{11},\]
\[B_{22}A_{22} = A_{22}B_{22}\]

and

\[A_{12}'B_{22} - B_{11}A_{12} = B_{12}A_{22} - A_{11}B_{12}\epsilon\text{Im}(P_{A_{11},A_{22}}).\]

Then

\[A_{12}' \equiv B_{11}A_{12}B_{22}^{-1} \mod \text{Im}(P_{A_{11},A_{22}}).\]

Thus, the image of \( A_{12} \) and \( A_{12}' \) in

\[C(A_{11}) \times C(A_{22}) \backslash Q_{\text{mod}}\]

coincide.

Conversely, if \( A_{12} \) and \( A_{12}' \) map to the same element in

\[C(A_{11}) \times C(A_{22}) \backslash Q_{\text{mod}}\]

then

\[A_{12}' \equiv B_{11}A_{12}B_{22}^{-1} \mod \text{Im}(P_{A_{11},A_{22}}),\]

with \( B_{11}\epsilon C(A_{11}) \) and \( B_{22}\epsilon C(A_{22}) \). Equivalently,

\[A_{12}'B_{22} \equiv B_{11}A_{12} \mod \text{Im}(P_{A_{11},A_{22}}).\]
Then there exists $B_{12}$ such that
\[ P_{A_{11}A_{22}}(B_{12}) = A'_{12}B_{22} - B_{11}A_{12}. \]
Equivalently,
\[ B_{12}A_{22} - A_{11}B_{12} = A'_{12}B_{22} - B_{11}A_{12}. \]
Therefore,
\[
\begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & A'_{12} \\
0 & A_{22}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix}.
\]

**Corollary 1.12** If $f(t)$ is monic polynomial of degree $m$ with integer coefficient, and $f(0) = \pm 1$ without repeated roots then there are finitely many matrices $A \in \text{GL}_m(\mathcal{O}_K)$ with characteristic polynomial $f(t)$, where $K$ is a number field.

**Proof.** We will proceed by induction on the number if irreducible factor of the characteristic polynomial. If the characteristic polynomial is irreducible we use the finiteness of class numbers and use corollary 1.4. Suppose we have proven the statement when the characteristic polynomial factors into $n$ irreducible polynomials. Consider a matrix $A$ whose characteristic polynomial factors into $n + 1$ irreducible polynomials. We can assume that $A$ is of the form
\[ A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}, \]
where $A_{11}$ has irreducible characteristic polynomial and the characteristic polynomial of $A_{22}$ factors into $n$ irreducible polynomials. There are finitely many matrices having the same characteristic polynomial as $A_{11}$ and being non-conjugate to $A_{11}$. Also, by assumption we can replace $A_{22}$ only by finitely many matrices in order to obtain new matrices non-conjugate to the initial one and with the same characteristic polynomial. To prove finiteness of matrices that are non-conjugate to $A$ but have the same characteristic polynomial, we use that for fixed $A_{11}$ and $A_{22}$ there are only finitely many options for $A_{12}$ which follow from proposition 1.7.

**2 Centralizers and homological Euler characteristics**

**Lemma 2.1** Let $A_{11}$ and $A_{22}$ be square matrices with coefficients in $\mathcal{O}_K$. Suppose that they have no common eigenvalues. And let
\[ C = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix} \text{ commutes with } A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}. \]

Then the admissible matrices $C$ are determined by the following properties:
\[ C_{21} = 0, \]
\[ C_{11} \in C(A_{11}), \]
\[ C_{22} \in C(A_{22}), \]
\[ C_{11}A_{12}C_{22}^{-1} \equiv A_{12} \mod \text{Im}P_{A_{11}A_{22}}. \]

Also, the matrix \( C_{12} \) is uniquely determined by \( C_{11} \) and \( C_{22} \), and it is given by
\[ C_{12} = P_{A_{11}A_{22}}^{-1}(A_{12}C_{22} - C_{11}A_{12}). \]

In particular,
\[ C\left( \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \right) = C(A_{11}) \times C(A_{22}), \]
where \( C(A) \) denotes the centralizer of \( A \).

**Proof.** We have
\[ \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \]
The (2,1)-entry of the products give
\[ C_{21}A_{11} = A_{22}C_{21}. \]

However the map
\[ P_{A_{22}A_{11}} : C_{21} \mapsto C_{21}A_{11} - A_{22}C_{21} \]
is non-singular by lemma 1.11. Therefore \( C_{21} = 0 \). Considering the (1,1)-entry and the (2,2)-entry of the above matrix product we obtain that \( C_{11} \in C(A_{11}) \) and \( C_{22} \in C(A_{22}) \).

Fix \( C_{11} \) and \( C_{22} \) to be in the centralizers \( C(A_{11}) \) and \( C(A_{22}) \), respectively. Then use that \( A = CAC^{-1} \) in order to determine the block \( C_{12} \). \( C_{11} \in C(A_{11}) \) and \( C_{22} \in C(A_{22}) \). Then
\[ A = CAC^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot \begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1}C_{12}C_{22}^{-1} \\ 0 & C_{22}^{-1} \end{bmatrix} = \]
\[ = \begin{bmatrix} C_{11}A_{11} & C_{11}A_{12} + C_{12}A_{22} \\ 0 & C_{22}A_{22} \end{bmatrix} \cdot \begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1}C_{12}C_{22}^{-1} \\ 0 & C_{22}^{-1} \end{bmatrix} = \]
\[ = \begin{bmatrix} A_{11} & -A_{11}C_{12}C_{22}^{-1} + C_{11}A_{12}C_{22}^{-1} + C_{12}A_{22}C_{22}^{-1} \\ 0 & A_{22} \end{bmatrix}. \]

We have
\[ A_{12} = C_{11}A_{12}C_{22}^{-1} + (C_{12}A_{22} - A_{11}C_{12})C_{22}^{-1} = \]
\[ = C_{11}A_{12}C_{22}^{-1} + P_{A_{11}A_{22}}(C_{12})C_{22}^{-1}. \]

Therefore \( A_{12} \) and \( C_{11}A_{12}C_{22}^{-1} \) coincide as elements in
\[ Q_{\text{mod}} = \text{Hom}_{\mathcal{O}_K}(P_2, P_1)/\text{Im}P_{A_{11}A_{22}}. \]

Therefore \( C_{11} \times C_{22} \) stabilizes \( A_{12} \) as element of \( Q_{\text{mod}} \). Conversely, if \( C_{11} \times C_{22} \) stabilizes \( A_{12} \) as an element of \( Q_{\text{mod}} \) we can find a unique \( C_{12} \) such that
\[ C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \]

commute. Indeed, if

\[ A_{12} - C_{11}A_{12}C_{22}^{-1} \in \text{Im}(P_{A_{11}A_{22}}) \]

then

\[ A_{12}C_{22} = C_{11}A_{12} \in \text{Im}(P_{A_{11}A_{22}}). \]

And we can set

\[ C_{12} = P_{A_{11}A_{22}}^{-1}(A_{12}C_{22} - C_{11}A_{12}). \]

Then the equation

\[ A_{12} = C_{11}A_{12}C_{22}^{-1} + P_{A_{11}A_{22}}(C_{12})C_{22}^{-1} \]

is satisfied.

Let

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \]

We are going to define a map \( i_{A_{12}} : C(A) \to C(A_{11}) \times C(A_{22}) \) and let \( C \) be in the centralizer of \( A \). Consider \( C \) as a block matrix of the same block form as \( A \). Then by lemma 2.1 we have that \( C_{21} = 0 \) and that \( C_{11} \in C(A_{11}) \) and \( C_{22} \in C(A_{22}) \). Let \( i_{A_{12}} \) sends \( C \) to \( C_{11} \times C_{22} \). Then \( i_{A_{12}} \) is a homomorphism.

**Corollary 2.2** With the above definition of \( i_{A_{12}} \) we have that \( i_{A_{12}} \) is always injective and the index of \( \text{Im}(i_{A_{12}}) \) in \( C_{11} \times C_{22} \) is equal to the number of elements in \( C_{11} \times C_{22} \)-orbit of \( A_{12} \) inside \( Q_{\text{mod}} \).

**Proof.** Let \( C \) be in the centralizer of \( A \). Let also

\[ C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \]

be of the same block form as \( A \). From lemma 2.1 we know that the blocks of \( C \) must satisfy \( C_{11} \in C(A_{11}) \) and \( C_{22} \in C(A_{22}) \) with the relation that \( A_{12} \) and

\[ C_{11}A_{12}C_{22}^{-1} \]

are congruent modulo the image of \( P_{A_{11}A_{22}} \). Equivalently, \( A_{12} \) and \( C_{11}A_{12}C_{22}^{-1} \) coincide in

\[ Q_{\text{mod}} = \text{Mat}_{m_1m_2}O_K/\text{Im}P_{A_{11}A_{22}}. \]

Therefore the image \( \text{Im}(i_{A_{12}}) \) consists of such \( C_{11} \) and \( C_{22} \) that the group element \( C_{11} \times C_{22} \) fixes \( A_{12} \) as an element of \( Q_{\text{mod}} \). Therefore the image \( \text{Im}(i_{A_{12}}) \) coincides with the stabilizer of \( A_{12} \in Q_{\text{mod}} \), and the index of \( \text{Im}(i_{A_{12}}) \) in \( C(A_{11}) \times C(A_{22}) \) corresponds to the \( C(A_{11}) \times C(A_{22}) \)-orbit of \( A_{12} \) in \( Q_{\text{mod}} \).
Lemma 2.3 Let $A_{11}$ and $A_{22}$ be invertible matrices with coefficients in $\mathcal{O}_K$, or automorphisms of projective modules having no common eigenvalues. Let $f_1$ and $f_2$ be the characteristic polynomials if $A_{11}$ and $A_{22}$, respectively. Then

$$\sum \chi(C(A)) = |N_{K/Q}(R(f_1, f_2))| \chi(C(A_{11})) \chi(C(A_{22})),$$

where the sum is taken over all non-conjugate torsion elements

$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

with fixed $A_{11}$ and $A_{22}$, and $R(f_1, f_2)$ is the resultant of the two polynomials.

Proof. By Proposition 1.10, taking the sum over all non-conjugate matrices with fixed $A_{11}$ and $A_{22}$ is the same as varying $A_{12}$ through representatives of $\mathcal{C}(A_{11}) \times \mathcal{C}(A_{22}) \mod Q$. For a fixed $A_{12}$ by corollary 2.2, we have that the group $\mathcal{C}(A)$ is a finite index subgroup of $\mathcal{C}(A_{11}) \times \mathcal{C}(A_{22})$, and the index is equal to the number of elements in the $\mathcal{C}(A_{11}) \times \mathcal{C}(A_{22})$-orbit of $A_{12}$ in $Q \mod$. Thus,

$$\chi(C(A)) = \# \text{orbit of } A_{12} \cdot \chi(C(A_{11})) \chi(C(A_{22}))$$

Summing over all orbits, we obtain

$$\sum \chi(C(A)) = \# |Q \mod| \cdot \chi(C(A_{11})) \chi(C(A_{22})).$$

On the other hand, by lemma 1.11

$$\# |Q \mod| = |N_{K/Q}(\det(P_{A_{11}A_{22}}))| = |N_{K/Q}(R(f_1, f_2))|.$$

We are going to define a resultant of $k$ polynomials $f_1, \ldots, f_k$, $k \geq 2$ by

$$R(f_1, \ldots, f_k) = \prod_{i<j} R(f_i, f_j).$$

Proposition 2.4 Let $A_{11}, A_{22}, \ldots, A_{kk}$ be invertible matrices with coefficients in $\mathcal{O}_K$ such that $A_{ii}$ and $A_{jj}$ have no common eigenvalues for $i \neq j$. Let $f_i$ be the characteristic polynomial of $A_{ii}$. Then

$$\sum \chi(C(A)) = |N_{K/Q}(R(f_1, \ldots, f_k))| \cdot \chi(C(A_{11})) \ldots \chi(C(A_{kk})),$$

where the sum is taken over all non-conjugate torsion elements $A$ is a block-diagonal form such that the blocks on the diagonal are $A_{11}, \ldots, A_{kk}$, and the blocks under the diagonal are zero, and $R(f_1, \ldots, f_k)$ is the resultant of $f_1, \ldots, f_k$ defined above.
Note that all $A$’s in the above sum have the same characteristic polynomial; namely $f_1 \cdot \ldots \cdot f_k$.

**Proof.** We are going to prove the statement by induction on $k$. For $k = 2$ it is true by the previous lemma. Assume it is true for $k - 1$. Notice that

$$R(f_1, \ldots, f_k) = R(f_1, \ldots, f_{k-1}) \cdot R((f_1 \cdot \ldots \cdot f_{k-1}), f_k).$$

Consider the matrices $A$ in the sum as $2 \times 2$-block-matrices in the following way:

$$A = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{kk} \end{bmatrix},$$

with $\overline{A}_{11}$ a block-triangular matrix with blocks on the diagonal $A_{11}, \ldots, A_{k-1,k-1}$, and zero blocks under the diagonal. Let $A'$ be another matrix in the above sum. Consider it similarly as a $2 \times 2$-block matrix

$$A' = \begin{bmatrix} \overline{A}'_{11} & \overline{A}'_{12} \\ 0 & \overline{A}_{kk} \end{bmatrix}.$$ We can assume that either $\overline{A}_{11}$ and $\overline{A}'_{11}$ coincide, or that they are non-conjugate. We can make that assumption for the following reason. If they are conjugate say by a matrix $\overline{B}_{11}$ instead of the representative $A'$ one can take $B A' B^{-1}$ where

$$B = \begin{bmatrix} \overline{B}_{11} & 0 \\ 0 & I \end{bmatrix},$$

with the matrix $I$ having the size of $A_{kk}$. Then the new matrix $A'' = B A' B^{-1}$ will have $\overline{A}'_{11} = \overline{A}_{11}$. Thus, we can separate the summation over non-conjugate matrices $A$ into two summations. First fix $\overline{A}_{11}$ and let $\overline{A}_{11}$ Very over all representatives

$$\chi(C(\overline{A}_{11}) \times C(A_{kk})) \chi(C(\overline{A}_{11})) \chi(C(A_{kk})),$$

where $Q_{mod}$ is the cokernel of the inclusion $P_{A_{11}A_{kk}}$. Then sum over non-conjugate $\overline{A}_{11}$ in block-diagonal form with $A_{11}, \ldots, A_{k-1,k-1}$ on the diagonal, and zero under the diagonal. The summation over all The first summation leads to

$$\sum_{\overline{A}_{11}} |N_{K/Q}(R((f_1 \cdot \ldots \cdot f_{k-1}), f_k))| \chi(C(\overline{A}_{11})) \chi(A_{kk}),$$

by the previous lemma because $f_1 \cdot \ldots \cdot f_{k-1}$ is the characteristic polynomial of any of the $\overline{A}_{11}$ matrices. And by induction assumption

$$\sum_{\overline{A}_{11}} \chi(C(\overline{A}_{11})) = |N_{K/Q}(R(f_1, \ldots, f_{k-1}))| \chi(C(\overline{A}_{11})) \ldots \chi(C(A_{k-1,k-1})).$$

We need e few more lemmas on the size of the centralizer.

**Lemma 2.5** Let $A$ and $B$ be matrices in $GL_m \mathcal{O}_K$, or $\text{Aut}(P)$ which are conjugate as elements of $GL_m K$ the $C(A)$ and $C(B)$ are comensurable.
Proof. The group $C(A)$ is an arithmetic subgroup of $C_{GL_m K}(A)$, which is conjugate to $C_{GL_m K}(B)$. Therefore $C_{GL_m O_K}(A)$ and $C_{GL_m O_K}(B)$ are comeasurable.

**Lemma 2.6** Let $T_n \in GL_m(O_K)$ be an $n$-torsion matrix with irreducible characteristic polynomial, or an $n$-torsion automorphisms. Then the centralizer $C(T_n)$ contains $O_K[\xi_n]^\times$, and is comensurable to it where $\xi_n = e^{2\pi i/n}$.

**Proof.** The matrices in $Mat_{m,m}K$ commuting with $T_n$ are precisely

$$K[T_n] \cong K(\xi_n)$$

because this is the maximal abelian sub-Lie algebra commuting with $T_n$; it is of dimension $m$. Let the intersection of $K[T_n]$ with $Mat_{m,m}O_K$ be

$$R \subset O_K(\xi_n),$$

where $R$ and $O_K(\xi_n)$ are comeasurable by the previous lemma. In this intersection $R$ the invertible elements are

$$C(T_n) \cong R^\times \subset O_K^\times[\xi_n].$$

**Lemma 2.7** Let $T_n \in GL_m(Z)$ be an $n$-torsion matrix with irreducible characteristic polynomial. Let $T$ be a $k \times k$-block matrix with blocks on the diagonal $T_n$ and the rest of the blocks being zero. Let $R$ be the ring of endomorphisms that commute with $T_n$. Then $C(T) \cong GL_k R$.

**Proof.** Then the matrices commuting with $T$ are

$$Q[T_n] \otimes Mat_{k,k} K \cong Mat_{k,k}(K(\xi_n)).$$

Among them the ones with integer coefficients are

$$C_{Mat_{km}Z}(T) = C_{Mat_{km}Z}(T_n \otimes I_k) = C_{Mat_{km}Z}(T_n) \otimes Mat_{k,k}Z \cong Mat_{k,k}R,$$

where $R$ is a order in $O_K(\xi_n)$ isomorphic to the ring of matrices (not necessarily with unit determinant) with coefficients in $O_K$ commuting with $T_n$. And the invertible ones with integer coefficients are

$$C(T) \cong GL_k R \subset GL_k(O_K(\xi_n)).$$

The following two propositions give bases for the proof of the vanishing results, namely theorem 0.1 and theorem 0.2.

**Proposition 2.8** Let $A$ be a torsion element of $GL_m O_K$. Then $\chi(C(A)) \neq 0$ if and only if the set of eigenvalues of $A$ is inside the set.
(a) \(\{1, -1, i, -i, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\}\), and the multiplicity of \(1\) and \(-1\) is at most 2 and the multiplicity of the rest of the roots of unity is at most 1 if \(K = \mathbb{Q}\);

(b) \(\{1, -1, i, -i\}\), and the multiplicities are at most 1 if \(K = \mathbb{Q}(i)\);

(c) \(\{1, -1, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\}\) and the multiplicities are at most 1 if \(K = \mathbb{Q}(\xi_3)\);

(d) \(\{1, -1\}\), and the multiplicities are at most 1 if \(K = \mathbb{Q}(\sqrt{-d}), d \neq 3, 4\).

(e) \(\chi(C(A)) = 0\) always when \(K \neq \mathbb{Q}, \mathbb{Q}(\sqrt{-d})\).

Proof. From theorem 1.5 we can assume that \(A\) is in block-triangular form with zero under the block-diagonal. We can also assume (by grouping similar blocks together) that the diagonal blocks have no common eigenvalues. Construct a new matrix \(A'\) having the same block diagonal as \(A\) and zeroes both above and below the diagonal. Then \(A\) and \(A'\) are conjugate in \(GL_m K\). By lemma 2.5 the Euler characteristic of \(A\) is a non-zero rational multiple of the Euler characteristic of \(A'\). On the other hand, \(\chi(A')\) can be expressed as the product of the Euler characteristics of the diagonal blocks. Thus, it is enough to examine all possible blocks that cannot be decomposed further. Such block \(B\) has eigenvalues \(n\)-th roots of unity repeated \(k\) times. Also, the centralizer of such a block is comeasurable with the centralizer of a matrix consisting of an \(n\)-torsion matrix \(T_n\) sitting in each diagonal block where \(T_n\) is an \(n\)-torsion matrix in \(GL_l \mathbb{O}_K\). By lemma 2.7

\[
\chi(C(B)) = \chi(GL_l \mathbb{Z}[\xi_n]) = \chi(\mathbb{Z}[\xi_n]^\times) \chi(SL_l \mathbb{Z}[\xi_n]).
\]

The number \(\chi(\mathbb{O}_K[\xi_n]^\times)\) will be zero if the units are infinitely many, because \(\chi(\mathbb{Z}) = 0\).

For \(K = \mathbb{Q}\) we can have only \(n\)-th roots of unity for \(n = 1, 2, 3, 4, 6\) such that

\[
\chi(\mathbb{O}_K[\xi_n]^\times) \neq 0.
\]

Also, we have that

\[
\chi(SL_l \mathbb{Z}) = 0 \text{ for } k \geq 3
\]

and

\[
\chi(SL_l \mathbb{Z}[\xi_n]) = \zeta_{\mathbb{Q}(\xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\xi_n)}(-l + 1) = 0 \text{ for } n = 3, 4, 6, \text{ and } l \geq 2.
\]

Thus, we are left with the roots

\[
\{1, -1, i, -i, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\}
\]

with multiplicities of \(+1\) and \(-1\) at most 2, and multiplicity of \(\pm i, \pm \xi_3\) and \(\pm \xi_6\) at most 1.

For \(K = \mathbb{Q}(i)\), we can have only \(n\)-th roots of unity for \(n = 1, 2, 4\) such that

\[
\chi(\mathbb{Z}[i, \xi_n]^\times) \neq 0.
\]
Also, we have that

\[ \zeta_{\mathbb{Q}(i)}(-1) = 0. \]

Thus,

\[ \chi(SL_l \mathbb{Z}[i, \xi_n]) = \zeta_{\mathbb{Q}(i, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(i, \xi_n)}(-l + 1) = 0 \text{ for } l \geq 2. \]

Thus, we are left with the roots

\[ \{1, -1, i, -i\} \]

with multiplicities at most 1.

For \( K = \mathbb{Q}(\xi_3) \), we can have only \( n \)-th roots of unity for \( n = 1, 2, 3, 6 \) such that

\[ \chi(\mathbb{Z}[\xi_3, \xi_n]^\times) \neq 0. \]

Also, we have that

\[ \zeta_{\mathbb{Q}(\xi_3)}(-1) = 0. \]

Thus,

\[ \chi(SL_l \mathbb{Z}[\xi_3, \xi_n]) = \zeta_{\mathbb{Q}(\xi_3, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\xi_3, \xi_n)}(-l + 1) = 0 \text{ for } l \geq 2. \]

Thus, we are left with the roots

\[ \{1, -1, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\} \]

with multiplicities at most 1.

For \( K = \mathbb{Q}(\sqrt{-d}) \), we can have only \( n \)-th roots of unity for \( n = 1, 2 \) such that

\[ \chi(\mathbb{O}_{\mathbb{Q}(\sqrt{-d}, \xi_n)}^\times) \neq 0, \]

because otherwise the units in the extension will be infinitely many. Also, we have that

\[ \zeta_{\mathbb{Q}(\sqrt{-d})}(-1) = 0. \]

Thus,

\[ \chi(SL_k(\mathbb{O}_{\mathbb{Q}(\sqrt{-d}, \xi_n)})) = \zeta_{\mathbb{Q}(\sqrt{-d}, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\sqrt{-d}, \xi_n)}(1 - l) = 0 \text{ for } k \geq 2. \]

Thus, we are left with the roots

\[ \{1, -1\} \]

with multiplicities at most 1.

If \( K \) is not \( \mathbb{Q} \), not an imaginary quadratic extension of \( \mathbb{Q} \) then

\[ \chi(\mathbb{O}_{K(\xi_n)}^\times) = 0 \]

for any \( n \). Thus

\[ \chi(C(A)) = 0 \]

for such fields \( K \).

For \( SL_m \mathbb{O}_K \) we have a similar statement.
Proposition 2.9 Let $A$ be a torsion element of $SL_mO_K$. Then $\chi(C(A)) \neq 0$ if and only if the set of eigenvalues of $A$ is inside the set

(a) $\{1, -1, i, -i, \xi_3, \xi_3^{-1}, \xi_6, \xi_6^{-1}\}$, and the multiplicity of 1 is at most 2, the multiplicity of $-1$ is 0 or 2 and the multiplicity of the rest of the roots of unity is at most 1 if $K = \mathbb{Q}$;

(b) $\{1, -1, i, -i\}$, and the multiplicities are at most 1 if $K = \mathbb{Q}(i)$;

(c) $\{1, -1, \xi_3, \xi_6, \xi_6^{-1}\}$ and the multiplicities are at most 1 if $K = \mathbb{Q}(\xi_3)$;

(d) $\{\xi, \xi^{-1}\}$ where $\xi$ is a root of 1 and the dimension of the matrix $A$ is at most 2, if $K$ is totally real field different from $\mathbb{Q}$;

(e) $\chi(C(A)) = 0$ always when $K$ is not totally real and different from $\mathbb{Q}(i)$ and $\mathbb{Q}(\xi_3)$.

Proof. Then proof is similar to the one of the previous proposition. Given a torsion matrix $A$ in $SL_mO_K$ we can conjugate it with a matrix from $GL_mK$ to a matrix $A'$. Then $C(A)$ and $C(A')$ will be comeasurable. Consider $A$ as a matrix in $GL_mO_K$. We can assume that $A$ is of block-triangular form, using theorem 1.5. Then we can take $A'$ to be of block-diagonal form with the same blocks on the diagonal as $A$. With another conjugation by a matrix from $GL_mK$, we can modify $A'$ so that blocks with identical eigenvalues will be represented by the same matrices. And finally, we combine the similar matrices into bigger block so that: (1) each new block is zero if it is not on the diagonal; (2) a block on the diagonal consists of smaller blocks such that on the diagonal we have smaller blocks repeated as many times as needed, and off the diagonal the smaller blocks are zero. The centralizer of such a matrix can be computed using lemma 2.6 and 2.7. We need a minor modification of lemma 2.6: We have that $C_{GL_mO_K}(T_n)$ is equal to the group of units in $O_K(\xi_n)$ while $C_{SL_mO_K}(T_n)$ is equal to the group $\text{Ker}(N_{K(\xi_n)/K})$ inside the units in $O_K(\xi_n)$. For the vanishing of the Euler characteristic this does not have an effect since both groups have the same rank.

For $K = \mathbb{Q}$ we can have only $n$-th roots of unity for $n = 1, 2, 3, 4, 6$ such that

$$\chi(\mathbb{Z}[\xi_n]^\times) \neq 0.$$  

Also, we have that

$$\chi(SL_l\mathbb{Z}) = 0 \text{ for } k \geq 3$$

and

$$\chi(SL_l\mathbb{Z}[\xi_n]) = \zeta_{\mathbb{Q}(\xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\xi_n)}(-l + 1) = 0 \text{ for } n = 3, 4, 6, \text{ and } l \geq 2.$$  

Thus, we are left with the roots

$$\{1, -1, i, -i, \xi_3, \xi_3^{-1}, \xi_6, \xi_6^{-1}\}$$
with multiplicities of $+1$ and $-1$ at most 2, and multiplicity of $\pm i$, $\pm \xi_3$ and $\pm \xi_6$ at most 1.

For $K = \mathbb{Q}(i)$, we can have only $n$-th roots of unity for $n = 1, 2, 4$ such that

$$\chi(\mathbb{Z}[i, \xi_n]^\times) \neq 0.$$  

Also, we have that

$$\zeta_{\mathbb{Q}(i)}(-1) = 0.$$  

Thus,

$$\chi(SL_1 \mathbb{Z}[i, \xi_n]) = \zeta_{\mathbb{Q}(i, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(i, \xi_n)}(1 - l) = 0 \text{ for } l \geq 2.$$  

Thus, we are left with the roots

$$\{1, -1, i, -i\}$$

with multiplicities at most 1.

For $K = \mathbb{Q}(\xi_3)$, we can have only $n$-th roots of unity for $n = 1, 2, 3, 6$ such that

$$\chi(\mathbb{Z}[\xi_3, \xi_n]^\times) \neq 0.$$  

Also, we have that

$$\zeta_{\mathbb{Q}(\xi_3)}(-1) = 0.$$  

Thus,

$$\chi(SL_1 \mathbb{Z}[\xi_3, \xi_n]) = \zeta_{\mathbb{Q}(\xi_3, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\xi_3, \xi_n)}(1 - l) = 0 \text{ for } l \geq 2.$$  

Thus, we are left with the roots

$$\{1, -1, \xi_3, \xi_3^{-1}, \xi_6, \xi_6^{-1}\}$$

with multiplicities at most 1.

For $K = \mathbb{Q}(\sqrt{-d})$, we can have only $n$-th roots of unity for $n = 1, 2$ such that

$$\chi(\mathbb{O}_{\mathbb{Q}(\sqrt{-d}, \xi_n)}^\times) \neq 0,$$

because otherwise the units in the extension will be infinitely many. Also, we have that

$$\zeta_{\mathbb{Q}(\sqrt{-d})}(-1) = 0.$$  

Thus,

$$\chi(SL_k \mathbb{O}_{\mathbb{Q}(\sqrt{-d}, \xi_n)}) = \zeta_{\mathbb{Q}(\sqrt{-d}, \xi_n)}(-1) \ldots \zeta_{\mathbb{Q}(\sqrt{-d}, \xi_n)}(1 - l) = 0 \text{ for } k \geq 2.$$  

We are left with the roots

$$\{1, -1\}$$

with multiplicities at most 1. Since $-1$ will lead to a determinant $-1$, we have that the only with the eigenvalues 1 with multiplicity 1, which gives that $A$ is in $SL_1$ and for higher $SL_m$ the Euler characteristic will vanish.

For $K$ a totally real field different from $\mathbb{Q}$, we have that there are infinitely many units and that

$$\zeta_K(-1) \neq 0,$$
which follows from the functional equation for the Dedekind zeta function. Also,

\[ \zeta_K(\xi_n)(-1) = 0 \]

for \( n > 2 \). Thus the eigenvalues 1 or \(-1\) might occur with multiplicity 2, and another root of 1 might occur with multiplicity at most 1.

Suppose \( T_n \) occurs as a block in \( A' \), \( n > 2 \). Recall \( T_n \) is an \( n \)-torsion block that has irreducible over \( K \) characteristic polynomial. By the modification of lemma 2.6 in the beginning of this proof we have that the centralizer of \( A' \) will contain a copy of the group of units \( a \) in \( \mathcal{O}_K \) such that

\[ N_{K(\xi_n)/K}(\alpha) = 1. \]

In order to have non-vanishing Euler characteristic the rank of this group must be zero. This can happen only when the ranks of the groups of units in \( K \) and in \( K(\xi_n) \) coincide. This is the case when \( K(\xi_n) \) is an extension of degree 2 over \( K \). Thus, the block \( T_n \) is of size \( 2 \times 2 \).

We are going to show that \( A' \) consists of at most block \( T_n \). Suppose that the diagonal of \( A' \) consists of \( T_{n_0}, T_{n_1}, \ldots, T_{n_l} \) on the diagonal, where each \( T_{n_i} \) for \( i = 1 \ldots l \) is \( n_i \)-torsion block with irreducible characteristic polynomial of degree 2, and \( T_{n_0} \) is either empty, or \((\pm 1)\), or \( \pm I_2 \). Let \( K_i = K(\xi_{n_i}) \), and \( K_0 = K \).

If \( T_0 \) is empty that is it is not present in \( A' \) then the centralizer of \( A' \) is isomorphic to the group

\[ \{(\alpha_1, \ldots, \alpha_n)|N_{K_i/K}(\alpha_1) \ldots N_{K_i/K}(\alpha_l) = 1\}. \]

The rank of the group is \((l - 1)\) times the rank of units in \( K \). In order to have non-vanishing Euler characteristic we must start with a zero rank group. Thus, \( l = 1 \) and \( A' = T_{n_i} \) for some \( i \).

If \( T_0 = (\pm 1) \) then the centralizer of \( A' \) is isomorphic to the group

\[ \{(\alpha_0, \ldots, \alpha_n)|\alpha_0 N_{K_1/K}(\alpha_1) \ldots N_{K_l/K}(\alpha_l) = 1\}. \]

The rank of the group is \( l \) times the rank of units in \( K \). In order to have non-vanishing Euler characteristic we must start with a zero rank group. Thus, \( l = 0 \) and \( A' = (\pm 1) \).

If \( T_0 = \pm I_2 \) then the Euler characteristic of the centralizer of \( A' \) is isomorphic to the Euler characteristic of

\[ SL_2(\mathcal{O}_K) \times \{(\alpha_0, \ldots, \alpha_n)|\alpha_0 N_{K_1/K}(\alpha_1) \ldots N_{K_l/K}(\alpha_l) = 1\}. \]

The rank of the abelian group is \( l \) times the rank of units in \( K \). In order to have non-vanishing Euler characteristic we must start with a zero rank group. Thus, \( l = 0 \) and \( A' = \pm I_2 \). This completes the case when \( K \) is totally real.

In the rest of the cases we have that the group of units of \( K \) is infinite and that \( \zeta_K(-1) = 0 \). Then the Euler characteristic of the centralizer of \( A' \) always vanishes.

The following theorems are very useful for computational purposes. It expresses the homological Euler characteristic as a sum of very few terms. And each of the terms can be easily computed. We use the following notation:

\[ A = [A_{11}, \ldots A_{ll}] \]
means that the square block $A_{11}$ through $A_{ll}$ are placed on the block-diagonal of $A$ and the blocks of $A$ outside the block-diagonal are zero blocks. Also, let

$$R(A) = R(f_1, \ldots, f_l),$$

where

$$A = [A_{11}, \ldots A_{ll}],$$

and $f_i$ is the characteristic polynomial of $A_{ii}$, and $f_i$ is a power of an irreducible polynomial. As a consequence of proposition 2.4 and 2.8 we obtain theorems 2.10, 2.11, 2.12 and 2.13.

**Theorem 2.10** Let $V$ be a finite dimensional representation of $GL_m \mathbb{Q}$. Then the homological Euler characteristic of $GL_n \mathbb{Z}$ with coefficients in $V$ is given by

$$\chi_h(GL_m \mathbb{Z}, V) = \sum_A |R(A)| \chi(C(A)) \text{Tr}(A^{-1}|V),$$

where the sum is taken over torsion matrices $A$ consisting of blocks $A_{11}, \ldots A_{mm}$ on the block-diagonal and zero blocks off the diagonal. Also the matrices $A_{ii}$ are in the set $\{+1, +I_2, -1, -I_2, T_3, T_4, T_6\}$, where

$$T_3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$  

and the characteristic polynomial $f_i$ of $A_{ii}$ is a power of an irreducible polynomial, and $f_i$ and $f_j$ are relatively prime.

**Theorem 2.11** Let $V$ be a finite dimensional representation of $GL_m \mathbb{Q}(i)$. Then the homological Euler characteristic of $GL_n \mathbb{Z}[i]$ with coefficients in $V$ is given by

$$\chi_h(GL_m \mathbb{Z}[i], V) = \sum_A |N_{\mathbb{Q}(i)/\mathbb{Q}}(R(A))| \chi(C(A)) \text{Tr}(A^{-1}|V),$$

where the sum is taken over torsion matrices $A$ consisting of blocks $A_{11}, \ldots A_{mm}$ on the block-diagonal and zero blocks off the diagonal. Also the matrices $A_{ii}$ are in the set $\{+1, -1, i, -i\}$ and the characteristic polynomials $f_i$ of $A_{ii}$ are relatively prime if $i \neq j$.

**Theorem 2.12** Let $V$ be a finite dimensional representation of $GL_m \mathbb{Q}(\xi_3)$. Then the homological Euler characteristic of $GL_n \mathbb{Z}[\xi_3]$ with coefficients in $V$ is given by

$$\chi_h(GL_m \mathbb{Z}[\xi_3], V) = \sum_A |N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(R(A))| \chi(C(A)) \text{Tr}(A^{-1}|V),$$

where the sum is taken over torsion matrices $A$ consisting of blocks $A_{11}, \ldots A_{mm}$ on the block-diagonal and zero blocks off the diagonal. Also the matrices $A_{ii}$ are in the set $\{+1, -1, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\}$ and the characteristic polynomials $f_i$ of $A_{ii}$ are relatively prime if $i \neq j$.  

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**Theorem 2.13** Let $P$ be a projective module of rank 2 over the ring of integers $\mathcal{O}_K$ in a number field $K = \mathbb{Q}(\sqrt{-d})$ for $d \neq 3, 4$. Let $V$ be a finite dimensional representation of $GL_2(\mathbb{Q}(\sqrt{-d}))$ Then the homological Euler characteristic of $\text{Aut}(P)$ with coefficients in $V$ is given by

$$\chi_h(\text{Aut}(P), V) = n \text{Tr}([1, -1]; V),$$

where $n$ is the number of ways that $P$ can be written as a direct sum of two projective modules, $P = P_1 \oplus P_2$ counting also the order of the summands. In particular if $P$ is free we obtain the homological Euler characteristic of $GL_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$ with coefficients in any representation.

**Proof.** (of theorems 2.10-2.13) Using proposition 2.4, we combine the torsion elements that have a common characteristic polynomial. Using proposition 2.8, we can take the sum only over those torsion elements that lead to a non-zero Euler characteristic of their centralizer. That leads to theorems 2.10-2.12.

For theorem 2.13 we need a little bit more. By proposition 2.8 part (d) we know that the only block-diagonal endomorphisms that will give contribution to the homological Euler characteristic is

$$A = [1, -1],$$

acting as 1 on $P_1$ and as $-1$ on $P_2$. We have to take into account all decompositions of $P$ into a direct sum of rank one projective modules. Note that

$$R([1, -1]) = 2$$

and

$$N_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}(R([1, -1])) = 4.$$  

Also,

$$C(A) = C(1) \times C(-1) = \mathbb{Z}/2 \times \mathbb{Z}/2.$$  

Thus, the formula becomes

$$\chi_h(\text{Aut}(P), V) = N_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}(R([1, -1])) n \chi(C([1, -1]))) \text{Tr}([1, -1]; V) = 4 \cdot \frac{1}{2} n \text{Tr}([1, -1]; V).$$

In the rest of the section we will deal with general statements about the homological Euler characteristics of groups comensurable to $GL_m(\mathcal{O}_K)$ or to $SL_m(\mathcal{O}_K)$ where $\mathcal{O}_K$ is a number ring.

**Proof.** (of theorem 0.1) Let $\Gamma$ be a finite subgroup of $GL_m(\mathcal{O}_K)$. Let $A$ be a torsion element of $\Gamma$. Then then the centralizer of $A$ inside $\Gamma$,

$$C_\Gamma(A)$$

is a finite index subgroup of the centralizer of $A$ inside $GL_m(\mathcal{O}_K)$,

$$C_{GL_m(\mathcal{O}_K)}(A).$$
Thus,
\[ \chi(C \Gamma(A)) = q \cdot \chi(C_{GLmO_K}(A)), \]
where \( q \) is a positive integer. Thus, vanishing of \( \chi(C_{GLmO_K}(A)) \) implies vanishing of \( \chi(C \Gamma(A)) \). We use a generalization of Kenneth Brown’s formula (see section 10, and 132):
\[ \chi(\Gamma) = \sum_{A: \text{torsion}} \text{Tr}(A^{-1}|V) \cdot \chi(C \Gamma(A)), \]
where the summation is over all torsion elements in \( \Gamma \) up to conjugation, and \( V \) is a representation of \( \Gamma \).

Let \( K = \mathbb{Q} \). Let also \( A \in GL_m \mathbb{Z} \).

Suppose that \( \chi(C(A)) \neq 0 \).

Then by proposition 2.9 part (a) we obtain that \( A \) multiplicity of the eigenvalues 1, \(-1\) at most 2, and multiplicity of the eigenvalues \( i, -i, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6 \) at most 1. Thus, the dimension of \( A \) is at most 10. Therefore if the dimension of \( A \) is greater than 10 then the Euler characteristic of the centralizers of \( A \) will vanish and so will the homological Euler characteristic of a finite index subgroup \( \Gamma \) of \( GL_m \mathbb{Z} \) with coefficients in \( V \),
\[ \chi_h(\Gamma, V) = 0. \]

Let \( K = \mathbb{Q}(i) \). Let also \( A \in GL_m(\mathbb{Z}[i]) \).

Suppose that \( \chi(C(A)) \neq 0 \).

Then by proposition 2.9 part (b) we obtain that \( A \) multiplicity of the eigenvalues 1, \(-1, i, -i\) is at most 1. Thus, the dimension of \( A \) is at most 4. Therefore if the dimension of \( A \) is greater than 4 then the Euler characteristic of the centralizers of \( A \) will vanish and so will the homological Euler characteristic of a finite index subgroup \( \Gamma \) of \( GL_m(\mathbb{Z}[i]) \),
\[ \chi_h(\Gamma) = 0. \]

Let \( K = \mathbb{Q}(\xi_3) \). Let also \( A \in GL_m(\mathbb{Z}[\xi_3]) \).

Suppose that \( \chi(C(A)) \neq 0 \).

Then by proposition 2.9 part (c) we obtain that \( A \) multiplicity of the eigenvalues 1, \(-1, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6 \) at most 1. Thus, the dimension of \( A \) is at most 6. Therefore if the dimension of \( A \) is greater than 6 then the Euler characteristic of the centralizers of \( A \) will vanish and so will the homological Euler characteristic of a finite index subgroup \( \Gamma \) of \( GL_m(\mathbb{Z}[\xi_3]) \) with coefficients in \( V \),
\[ \chi_h(\Gamma, V) = 0. \]
Let $K = \mathbb{Q}(\sqrt{-d})$, $d \neq 3, 4$. Let also
\[ A \in GL_m(O_{\mathbb{Q}(\sqrt{-d})}). \]
Suppose that
\[ \chi(C(A)) \neq 0. \]
Then by proposition 2.9 part (d) we obtain that $A$ multiplicity of the eigenvalues 1, $\chi(C(A)) \neq 0$. Thus, the dimension of $A$ is at most 2. Therefore if the dimension of $A$ is greater than 2 then the Euler characteristic of the centralizers of $A$ will vanish and so will the homological Euler characteristic of a finite index subgroup $\Gamma$ of $GL_m(O_{\mathbb{Q}(\sqrt{-d})})$ with coefficients in $V$,
\[ \chi_h(\Gamma, V) = 0. \]

And finally, for all other fields $K$ by proposition 2.9 part (e), we have that always
\[ \chi(C(A)) = 0. \]
Therefore, the homological Euler characteristic of a finite index subgroup $\Gamma$ of $GL_m(O_K)$ with coefficients in $V$,
\[ \chi_h(\Gamma, V) = 0. \]

Similarly we obtain the vanishing result for finite index subgroups of $SL_m(O_K)$.

Proof. (of theorem 0.2) Let $\Gamma$ be a finite subgroup of $SL_m(O_K)$. Let $A$ be a torsion element of $\Gamma$. Then then the centralizer of $A$ inside $\Gamma$,
\[ C_{\Gamma}(A) \]
is a finite index subgroup of the centralizer of $A$ inside $SL_m(O_K)$,
\[ C_{SL_m(O_K)}(A). \]
Thus,
\[ \chi(C_{\Gamma}(A)) = q \cdot \chi(C_{SL_m(O_K)}(A)), \]
where $q$ is a positive integer. Thus, vanishing of $\chi(C_{SL_m(O_K)}(A))$ implies vanishing of $\chi(C_{\Gamma}(A))$. We use a generalization of Kenneth Brown’s formula (see section 10, and [B2]):
\[ \chi(\Gamma) = \sum_{A: \text{torsion}} \text{Tr}(A^{-1}|V) \cdot \chi(C_{\Gamma}(A)), \]
where the summation is over all torsion elements in $\Gamma$ up to conjugation, and $V$ is a representation of $\Gamma$.

Let $K = \mathbb{Q}$. Let also
\[ A \in SL_m\mathbb{Z}. \]
Suppose that
\[ \chi(C(A)) \neq 0. \]
Then by proposition 2.9 part (a) we obtain that $A$ multiplicity of the eigenvalues 1, $-1$ at most 2, and multiplicity of the eigenvalues $i$, $-i$, $\xi_3$, $\xi_6$, $\bar{\xi}_3$, $\bar{\xi}_6$ at most 1. Thus, the dimension of $A$ is at most 10. Therefore if the dimension of $A$ is greater than 10 then the Euler characteristic of the centralizers of $A$ will vanish and so will the homological Euler characteristic of a finite index subgroup $\Gamma$ of $SL_m\mathbb{Z}$ with coefficients in $V$, 

$$\chi_h(\Gamma, V) = 0.$$ 

Let $K = \mathbb{Q}(i)$. Let also $A \in SL_m(\mathbb{Z}[i])$.

Suppose that 

$$\chi(C(A)) \neq 0.$$ 

Then by proposition 2.9 part (b) we obtain that $A$ multiplicity of the eigenvalues 1, $-1$, $i$, $-i$ is at most 1. Thus, the dimension of $A$ is at most 4. Therefore if the dimension of $A$ is greater than 4 then the Euler characteristic of the centralizers of $A$ will vanish and so will the homological Euler characteristic of a finite index subgroup $\Gamma$ of $SL_m(\mathbb{Z}[i])$ with coefficients in $V$, 

$$\chi_h(\Gamma, V) = 0.$$ 

Let $K = \mathbb{Q}(\xi_3)$. Let also $A \in SL_m(\mathbb{Z}[\xi_3])$.

Suppose that 

$$\chi(C(A)) \neq 0.$$ 

Then by proposition 2.9 part (c) we obtain that $A$ multiplicity of the eigenvalues 1, $-1$, $\xi_3$, $\bar{\xi}_3$, $\xi_6$, $\bar{\xi}_6$ at most 1. Thus, the dimension of $A$ is at most 6. Therefore if the dimension of $A$ is greater than 6 then the Euler characteristic of the centralizers of $A$ will vanish and so will the homological Euler characteristic of a finite index subgroup $\Gamma$ of $SL_m(\mathbb{Z}[\xi_3])$ with coefficients in $V$, 

$$\chi_h(\Gamma, V) = 0.$$ 

Let $K$ be totally real field. Let also $A \in GL_m(O_K)$.

Suppose that 

$$\chi(C(A)) \neq 0.$$ 

Then by proposition 2.9 part (d) we obtain that the dimension of $A$ is at most 2. Therefore if the dimension of $A$ is greater than 2 then the Euler characteristic of the centralizers of $A$ will vanish and so will the homological Euler characteristic of a finite index subgroup $\Gamma$ of $GL_m(O_{\mathbb{Q}(\sqrt{-d})})$ with coefficients in $V$, 

$$\chi_h(\Gamma, V) = 0.$$ 

And finally, for all other fields $K$ by proposition 2.9 part (e), we have that always 

$$\chi(C(A)) = 0.$$
Therefore, the homological Euler characteristic of a finite index subgroup $\Gamma$ of $SL_m(\mathcal{O}_K)$ with coefficients in $V$,

$$\chi_h(\Gamma, V) = 0.$$  

### 3 Torsion elements of $GL_2\mathbb{Z}$ and $GL_3\mathbb{Z}$

This section deals with the computational task of finding all the torsion element up to conjugation in $GL_2\mathbb{Z}$ and $GL_3\mathbb{Z}$.

**Proposition 3.1** All torsion elements in $GL_2\mathbb{Z}$ up to conjugation are listed in the table below together with their centralizers and Euler characteristic of the centralizers.

| $A$          | $C(A)$     | $\chi(C(A))$ |
|--------------|------------|---------------|
| (a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | $GL_2\mathbb{Z}$ | $-\frac{1}{24}$ |
| (b) $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ | $GL_2\mathbb{Z}$ | $-\frac{1}{24}$ |
| (c1) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | $C_2 \times C_2$ | $-\frac{1}{7}$ |
| (c2) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | $C_2 \times C_2$ | $-\frac{1}{7}$ |
| (d) $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ | $C_6$ | $\frac{1}{5}$ |
| (e) $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ | $C_6$ | $\frac{1}{5}$ |
| (f) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ | $C_4$ | $\frac{1}{7}$ |

**Proof.** Let $A$ be a torsion element in $GL_2\mathbb{Z}$. If the characteristic polynomial of $A$ is irreducible over the rational numbers then its root belong to a quadratic extension of $\mathbb{Q}$. The only such options occur if the roots are 3-rd, 4-th or 6-th root of unity. The corresponding rings of integers are unique factorization domains. Therefore by proposition 1.3 there is only one matrix $T_n$ up to conjugation having eigenvalues $n$-th roots of unity for $n = 3, 4, \text{ or } 6$. This gives the torsion elements in part (d), (e) and (f). If the characteristic polynomial of $A$ is reducible then its roots are 1 or $-1$. If both roots are 1 or both are $-1$ then we have only one representative for each case which is given in part (a) and (b). It remains to consider the case when the eigenvalues of $A$ are 1 and $-1$. By theorem 1.5 we can assume that $A$ is upper triangular. What remains to be done is to determine which elements in the upper right corner of $A$ give conjugate elements. Consider $P_{A_{11}, A_{22}}$ with $A_{11} = 1$ and $A_{22} = -1$. Then the matrix representing $P_{A_{11}, A_{11}}$ is

$$I \otimes A_{22} - A_{11} \otimes I = 1 \otimes (-1) - 1 \otimes 1 = -2.$$  

Thus,

$$\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$$
are conjugate if and only if \( a \equiv b \mod 2 \). Thus we obtain the two torsion elements in (c1) and (c2) corresponding to even \( a \) and odd \( a \).

For \( GL_3 \mathbb{Z} \) we find all the torsion elements up to conjugation.

**Proposition 3.2** All torsion elements in \( GL_3 \mathbb{Z} \) up to conjugation are listed in the table below together with their centralizers and Euler characteristic of the centralizers.

| \( A \) | \(-A\) | \( C(A) \) | \( \chi(C(A)) \) |
|--------|--------|-----------|----------------|
| (a)    | \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 \\
-1 \\
-1
\end{pmatrix}
\] | \( GL_3 \mathbb{Z} \) | 0 |
| (c1)   | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 & 0 & -1 \\
0 & -1 & 0 \\
0 & -1 & 0
\end{pmatrix}
\] | \( GL_2 \mathbb{Z} \times GL_1 \mathbb{Z} \) | \(-\frac{1}{4} \) |
| (c2)   | \[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
-1 & 1 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 & 0 & -1 \\
0 & -1 & 0 \\
0 & -1 & 0
\end{pmatrix}
\] | \( \Gamma_1(2, 2) \times GL_1 \mathbb{Z} \) | \(-\frac{1}{16} \) |
| (e1)   | \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & 1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\] | \( C_6 \times C_2 \) | \(\frac{1}{12}\) |
| (e2)   | \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & 1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\] | \( C_3 \times C_2 \) | \(\frac{1}{6}\) |
| (g)    | \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & 1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\] | \( C_6 \times C_2 \) | \(\frac{1}{12}\) |
| (i1)   | \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & -1 \\
-1 & 0 & -1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
-1 & 1 & -1
\end{pmatrix}
\] | \( C_4 \times C_2 \) | \(\frac{1}{8}\) |
| (i2)   | \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & -1 \\
-1 & 0 & -1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
-1 & 1 & -1
\end{pmatrix}
\] | \( C_4 \times C_2 \) | \(\frac{1}{8}\) |

**Proof.** If \( A \) is a torsion element in \( GL_3 \mathbb{Z} \) then its eigenvalues are roots of 1. If \( \lambda \) is an eigenvalue then all of its Galois conjugates are eigenvalues. If \( \lambda \) is an \( n \)-th root of 1 then all its Galois conjugates are all the primitive \( n \)-th roots of 1. Their number is \( \varphi(n) \) and we must have at most 3 of them. The inequality \( \varphi(n) \leq 3 \) has solutions \( n = 1, 2, 3, 4, 6 \). In all of these cases we have \( \varphi(n) \leq 2 \). That is, \( \mathbb{Q} (\lambda) \) is at most quadratic extension of \( \mathbb{Q} \). Thus, the remaining eigenvalue must be rational. Therefore the matrix \( A \) must have an eigenvalue \(+1\) or \(-1\).

If all the eigenvalues are 1 then the matrix is either the identity or it contains a non-trivial Jordan block, which cannot be of finite order. That gives case (a). Minus that matrix gives case (b). If \( A \) the characteristic polynomial of \( A \) has a root at \(-1\) and a double root at 1 then we need to use the material that we developed so far. Using theorem 1.6 we know that the matrix \( A \) can be conjugated to a block-triangular matrix with a zero block under the diagonal. The diagonal block must
be $A_{11} = I_2$ and $A_{22} = -1$. Then

$$P_{A_{11}, A_{22}} = I_2 \otimes A_{22} - A_{11} \otimes 1 = -2I_2$$

acts on $Mat_{2,1} \mathbb{Z}$. Therefore

$$P_{\text{mod}} = 2Mat_{2,1} \mathbb{Z}$$

and

$$Q_{\text{mod}} = Mat_{2,1}(\mathbb{Z}/2).$$

The centralizer of $A_{11} = I_2$, $C(A_{11}) = GL_2 \mathbb{Z}$ acts on $Q_{\text{mod}}$ and has two orbits: the zero vector and the non-zero vectors. Therefore we have exactly two non-conjugate matrices up to conjugation with diagonal blocks $I_2$ and $-1$. We assumed that below the diagonal the block is zero. Above the diagonal there are two cases leading to (c1) and (c2). In one of them we have zero above the diagonal, and in the other a non-zero representative of $Q_{\text{mod}}$, for example $[1, 0]^t$.

By lemma 2.1 we obtain the centralizer in the case of (c1). For the case (c2), we have that the centralizer of $A_{22} = -1$ is

$$C(A_{22}) = \pm 1,$$

and $-1$ acts trivially on $[1, 0]^t \epsilon Q_{\text{mod}}$.

On the other hand $C(A_{11}) = GL_2 \mathbb{Z}$ acts on $Q_{\text{mod}}$ as $GL_2(\mathbb{Z}/2)$. And the stabilizer of $[1, 0]^t$ is $\Gamma_1(2, 2)$. We have that

$$\chi(\Gamma_1(2, 2)) = [\Gamma_1(2, 2) : GL_2 \mathbb{Z}] \cdot \chi(GL_2 \mathbb{Z}) = 3 \cdot \left(-\frac{1}{24}\right) = \frac{1}{8}.$$

Considering $-A$, we obtain the cases (d1) and (d2).

In case not all the eigenvalues of $A$ are $\pm 1$, we must have either third, fourth or sixth root of 1. Together with them the third eigenvalue is either 1 or $-1$. That exhausts all cases for the eigenvalues of a torsion element in $GL_3 \mathbb{Z}$.

Suppose the eigenvalues of $A$ are primitive third roots of 1 and 1. We can assume that the matrix $A$ is in block-triangular form. Let the blocks be

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } A_{22} = 1.$$

For $A_{11}$ we could have picked any other matrix with characteristic polynomial

$$t^2 + t + 1.$$

In $\mathbb{Z}[\xi_3]$ all ideals are principal. Now using corollary 1.4, we obtain that any two matrices with characteristic polynomial $t^2 + t + 1$ are conjugate to each other inside $GL_2 \mathbb{Z}$. We assumed that $A_{21} = 0$. For the simplification of $A_{12}$ we use

$$P_{A_{11}, A_{22}} = I_2 \otimes A_{22} - A_{11} \otimes 1.$$

Therefore,
\[ P_{A_{11}A_{22}} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}. \]

Then

\[ \det P_{A_{11}A_{22}} = 3 \]

which means that

\[ \mathcal{Q}_{\text{mod}} \cong C_3. \]

Then \( C(A_{22}) = \pm 1 \) acts on \( \mathcal{Q}_{\text{mod}} \) by exchanging the two nonzero elements. Thus there are two orbits on \( \mathcal{Q}_{\text{mod}} \). A representative of the zero orbit is \([0, 0]^t\) and for the non-zero orbit \([1, 0]^t\). Thus, we obtain the cases (e1) and (e2). By lemma 2.1 we obtain the centralizer of (e1). For the centralizer of (e2), we observe that \(-1\) from \( C(A_{22}) \) and \(-I_2\) from \( C(A_{11}) \) do change independently the element

\[ [1, 0]^t \in \mathcal{Q}_{\text{mod}} \]

to \([ -1, 0]^t\). However, if they act simultaneously, they keep the element \([1, 0]^t\) fixed. This is an element of order 2 that fixes \([1, 0]^t\). An element of order 3 from \( C(A_{11}) \) must fix \([1, 0]^t\). Thus, the centralizer is \( C_3 \times C_2 \). Considering \(-A\) we obtain the six torsion from (f1) and (f2).

If the eigenvalues of \( A \) are primitive third roots of 1 and \(-1\) then we can again assume that \( A \) is in block-diagonal form with block

\[ A_{11} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad A_{22} = -1. \]

Then \( P_{A_{11}A_{22}} = I_2 \otimes A_{22} - A_{11} \otimes 1 \). Therefore,

\[ P_{A_{11}A_{22}} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \]

Then

\[ \det P_{A_{11}A_{22}} = 1 \]

which means that \( \mathcal{Q}_{\text{mod}} = 0 \). Therefore, there is only one conjugacy class with such \( A_{11} \) and \( A_{22} \), and we can take for \( A_{12} \) any vector. So we take \([0, 0]^t\). The centralizer of \( A \) is a product of two centralizers by lemma 2.1. That concludes case (g). Case (h) is taking \(-A\) in the previous case. Thus, we have exhausted all 3-torsions and all 6-torsions in \( GL_3 \mathbb{Z} \).

Suppose we have a matrix \( A \) with eigenvalues \( i, -i \) and 1. (Taking the minus sign we obtain a matrix with eigenvalues \( i, -i \) and \(-1\).) Using theorem 1.3, we can assume that \( A \) is in block-triangular form with zero below the diagonal block, and

\[ A_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad A_{22} = 1. \]

Then

\[ P_{A_{11}A_{22}} = I_2 \otimes A_{22} - A_{11} \otimes 1. \]
Therefore,

\[ P_{A_{11}A_{22}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \]

Then

\[ \det P_{A_{11}A_{22}} = 2 \]

which means that \( Q_{mod} = \mathbb{Z}/2 \). The module \( Q_{mod} \) has no automorphisms therefore each of its elements correspond to a conjugacy class. Thus there are two classes: (i1) and (i2). The centralizer of (i1) can be computed via Lemma 2.1. For (i2) note that every element of \( C(A_{11}) \) or \( C(A_{22}) \) fixes the non-zero element of \( Q_{mod} \). Therefore the centralizer in the case of (i2) is isomorphic as an abstract group to the centralizer of (i1). Taking the minus sign we obtain (j1) and (j2).

## 4 Resultants

This section is computational. It deals with resultants and Euler characteristics of centralizers needed for computation of homological Euler characteristics. Let us recall the notation that we are using. By \( A = [A_{11}, \ldots A_{kk}] \) we mean a matrix \( A \), whose diagonal blocks are \( A_{ii} \) for \( i = 1, \ldots k \). We also assume that the characteristic polynomial \( f_i \) of \( A_{ii} \) is a power of an irreducible polynomial. Also,

\[ R(A) = R(f_1, \ldots, f_k) = \prod_{i<j} R(f_i, f_j), \]

where \( R(f_i, f_j) \) is the resultant of \( f_i \) and \( f_j \).

In the following statements we compute

\[ |R(A)| \cdot \chi(A) \]

for various matrices \( A \) for the need of computation of the homological Euler characteristic of an arithmetic group over \( \mathbb{Z}, \mathbb{Z}[i] \) and \( \mathbb{Z}[\xi_3] \) (see theorem 2.10).

**Lemma 4.1** The resultants needed for the homological Euler characteristic of arithmetic subgroups of \( GL_2 \mathbb{Z} \) are given by:

\[ |R([1, -1])| \cdot \chi(C([1, -1])) = \frac{1}{2}. \]

**Proof.** We have

\[ R([1, -1]) = R(t - 1, t + 1) = 2 \]

and

\[ C([1, -1]) = C([1]) \times C([-1]) = C_2 \times C_2. \]

Then,

\[ \chi(C([1, -1])) = \frac{1}{4}. \]
Lemma 4.2  The resultants needed for the homological Euler characteristic of arithmetic subgroups of $GL_3 \mathbb{Z}$ are given by:

\[(c) \mid R([I_2, -1]) \mid \cdot \chi(C([I_2, -1])) = -\frac{1}{12},\]

\[(d) \mid R([-I_2, 1]) \mid \cdot \chi(C([-I_2, 1])) = \frac{1}{12},\]

\[(e) \mid R([T_3, 1]) \mid \cdot \chi(C([T_3, 1])) = \frac{1}{4},\]

\[(f) \mid R([T_6, 1]) \mid \cdot \chi(C([T_6, 1])) = \frac{1}{12},\]

\[(g) \mid R([T_3, -1]) \mid \cdot \chi(C([T_3, -1])) = \frac{1}{12},\]

\[(h) \mid R([T_6, -1]) \mid \cdot \chi(C([T_6, -1])) = \frac{1}{4},\]

\[(i) \mid R([T_4, 1]) \mid \cdot \chi(C([T_4, 1])) = \frac{1}{4},\]

\[(j) \mid R([T_4, -1]) \mid \cdot \chi(C([T_4, -1])) = \frac{1}{4}.\]

The enumeration follows the one of the table in proposition 3.2.

Proof.  (c) We have

\[R([I_2, -1]) = R((t - 1)^2, t + 1) = 4\]

and

\[C([I_2, -1]) = C([I_2]) \times C([-1]) = GL_2 \mathbb{Z} \times C_2.\]

Then,

\[\chi(C([I_2, -1])) = -\frac{1}{48}.\]

(d) We have

\[R([-I_2, 1]) = R((t + 1)^2, t - 1) = 4\]

and

\[C([-I_2, 1]) = C([-I_2]) \times C([1]) = GL_2 \mathbb{Z} \times C_2.\]

Then,

\[\chi(C([-I_2, 1])) = -\frac{1}{48}.\]

(e) We have

\[R([T_3, 1]) = R((t - \xi_3)(t - \xi_3^{-1}), t - 1) = 3\]

and

\[C([T_3, 1]) = C([T_3]) \times C([1]) = C_6 \times C_2.\]

Then,

\[\chi(C([T_3, 1])) = -\frac{1}{12}.\]

(f) We have

\[R([T_6, 1]) = R((t - \xi_6)(t - \xi_6^{-1}), t - 1) = 1.\]
and 
\[ C([T_6, 1]) = C([T_6]) \times C([1]) = C_6 \times C_2. \]
Then, 
\[ \chi(C([T_6, 1])) = -\frac{1}{12}. \]

(g) We have 
\[ R([T_3, -1]) = R((t - \xi_3)(t - \xi_3^{-1}), t + 1) = 1 \]
and 
\[ C([T_3, -1]) = C([T_3]) \times C([-1]) = C_6 \times C_2. \]
Then, 
\[ \chi(C([T_3, -1])) = -\frac{1}{12}. \]

(h) We have 
\[ R([T_6, -1]) = R((t - \xi_6)(t - \xi_6^{-1}), t + 1) = 3 \]
and 
\[ C([T_6, -1]) = C([T_6]) \times C([1]) = C_6 \times C_2. \]
Then, 
\[ \chi(C([T_6, 1])) = -\frac{1}{12}. \]

(i) We have 
\[ R([T_6, 1]) = R((t - i)(t + i), t - 1) = 2 \]
and 
\[ C([T_4, 1]) = C([T_4]) \times C([1]) = C_4 \times C_2. \]
Then, 
\[ \chi(C([T_4, 1])) = -\frac{1}{8}. \]

(j) We have 
\[ R([T_4, -1]) = R((t - i)(t + i), t + 1) = 2 \]
and 
\[ C([T_4, -1]) = C([T_4]) \times C([-1]) = C_4 \times C_2. \]
Then, 
\[ \chi(C([T_4, -1])) = -\frac{1}{8}. \]

Lemma 4.3 The resultants needed for the homological Euler characteristic of arithmetic subgroups of \( GL_4 \mathbb{Z} \) are given by:

(a) \[ |R([I_2, -I_2])| \cdot \chi(C([I_2, -I_2])) = \frac{1}{36}, \]
(b) \[ |R([I_2, T_3])| \cdot \chi(C([I_2, T_3])) = -\frac{1}{16}, \]
(c) \[ |R([I_2, T_6])| \cdot \chi(C([I_2, T_6])) = -\frac{1}{144}. \]
Proof. (a) We have
\[ R([I_2, -I_2]) = R((t - 1)^2, (t + 1)^2) = 2^4. \]
Also,
\[ C(I_2, -I_2) \cong GL_2 \mathbb{Z} \times GL_2 \mathbb{Z}. \]
Then
\[ \chi(C([I_2, -I_2])) = (-\frac{1}{24})^2. \]
We obtain
\[ |R([I_2, -I_2])| \chi(C([I_2, -I_2])) = 2^4(-\frac{1}{24})^2 = \frac{1}{36}. \]

(b) We have
\[ R([I_2, T_3]) = R((t - 1)^2, (t - \xi_3)(t - \xi_3^{-1})) = 3^2. \]
Also,
\[ C(I_2, T_3) \cong GL_2 \mathbb{Z} \times C_6. \]
Then
\[ \chi(C([I_2, T_3])) = -\frac{1}{24} \cdot \frac{1}{6}. \]
We obtain
\[ |R([I_2, T_3])| \chi(C([I_2, T_3])) = 3^2(-\frac{1}{24})\frac{1}{6} = -\frac{1}{16}. \]

(c) We have
\[ R([I_2, T_6]) = R((t - 1)^2, (t - \xi_6)(t - \xi_6^{-1})) = 1. \]
Also,
\[ C(I_2, T_6) \cong GL_2 \mathbb{Z} \times C_6. \]
Then
\[ \chi(C([I_2, T_6])) = -\frac{1}{24} \cdot \frac{1}{6}. \]

We obtain
\[ |R([I_2, T_6])| \chi(C([I_2, T_6])) = \left(-\frac{1}{24}\right)\frac{1}{6} = -\frac{1}{144}. \]

(d) We have
\[ R([I_2, T_4]) = R((t - 1)^2, (t - i)(t + i)) = 2^2. \]

Also,
\[ C(I_2, T_4) \cong GL_2 \mathbb{Z} \times C_4. \]

Then
\[ \chi(C([I_2, T_4])) = -\frac{1}{24} \cdot \frac{1}{4}. \]

We obtain
\[ |R([I_2, T_4])| \chi(C([I_2, T_4])) = 2^2(-\frac{1}{24})\frac{1}{4} = -\frac{1}{24}. \]

(e) We have
\[ R([-I_2, T_3]) = R((t + 1)^2, (t - \xi_3)(t - \xi_3^{-1})) = 1. \]

Also,
\[ C(-I_2, T_3) \cong GL_2 \mathbb{Z} \times C_6. \]

Then
\[ \chi(C([-I_2, T_3])) = -\frac{1}{24} \cdot \frac{1}{6}. \]

We obtain
\[ |R([-I_2, T_3])| \chi(C([-I_2, T_3])) = -\frac{1}{24} \cdot \frac{1}{6} = -\frac{1}{144}. \]

(f) We have
\[ R([-I_2, T_6]) = R((t + 1)^2, (t - \xi_6)(t - \xi_6^{-1})) = 3^2. \]

Also,
\[ C(-I_2, T_6) \cong GL_2 \mathbb{Z} \times C_6. \]

Then
\[ \chi(C([-I_2, T_6])) = -\frac{1}{24} \cdot \frac{1}{6}. \]

We obtain
\[ |R([-I_2, T_6])| \chi(C([-I_2, T_6])) = 3^2(-\frac{1}{24})\frac{1}{6} = -\frac{1}{16}. \]

(g) We have
\[ R([-I_2, T_4]) = R((t + 1)^2, (t - i)(t + i)) = 2^2. \]

Also,
\[ C(-I_2, T_4) \cong GL_2 \mathbb{Z} \times C_4. \]

Then
\[ \chi(C([-I_2, T_4])) = -\frac{1}{24} \cdot \frac{1}{4}. \]
We obtain
\[ |R([-I_2, T_4])|\chi(C([-I_2, T_4])) = 2^2(-\frac{1}{24})\frac{1}{4} = -\frac{1}{24}. \]

(h) We have
\[ R([1, -1, T_3]) = R(t - 1, t + 1, (t - \xi_3)(t - \xi_3^{-1})) = 2 \cdot 3 \cdot 1. \]

Also,
\[ C(1, -1, T_3) \cong C_2 \times C_2 \times C_6. \]

Then
\[ \chi(C([1, -1, T_3])) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6}. \]

We obtain
\[ |R([-I_2, T_3])|\chi(C([-I_2, T_3])) = 2 \cdot 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{4}. \]

(i) We have
\[ R([1, -1, T_6]) = R(t - 1, t + 1, (t - \xi_6)(t - \xi_6^{-1})) = 2 \cdot 1 \cdot 3. \]

Also,
\[ C(1, -1, T_6) \cong C_2 \times C_2 \times C_6. \]

Then
\[ \chi(C([1, -1, T_6])) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6}. \]

We obtain
\[ |R([1, -1, T_6])|\chi(C([1, -1, T_6])) = 2 \cdot 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{4}. \]

(j) We have
\[ R([1, -1, T_4]) = R(t - 1, t + 1, (t - i)(t + i)) = 2^3. \]

Also,
\[ C(1, -1, T_4) \cong C_2 \times C_2 \times C_4. \]

Then
\[ \chi(C([1, -1, T_4])) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4}. \]

We obtain
\[ |R([1, -1, T_4])|\chi(C([1, -1, T_4])) = 2^3 \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}. \]

(k) We have
\[ R([T_3, T_6]) = R((t - \xi_3)(t - \xi_3^{-1}), (t - \xi_6)(t - \xi_6^{-1})) = 2^2. \]

Also,
\[ C(T_3, T_6) \cong C_6 \times C_6. \]

Then
\[ \chi(C([T_3, T_6])) = \frac{1}{6} \cdot \frac{1}{6}. \]
We obtain
\[ |R([T_3, T_6])|\chi(C([T_3, T_6])) = 2^2 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{9}. \]

(l) We have
\[ R([T_3, T_4]) = R((t - \xi_3)(t - \xi_3^{-1}), (t - i)(t + i)) = 1. \]
Also,
\[ C(T_3, T_4) \cong C_6 \times C_4. \]
Then
\[ \chi(C([T_3, T_4])) = \frac{1}{6} \cdot \frac{1}{4}. \]
We obtain
\[ |R([T_3, T_4])|\chi(C([T_3, T_4])) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}. \]

(m) We have
\[ R([T_6, T_4]) = R((t - \xi_6)(t - \xi_6^{-1}), (t - i)(t + i)) = 1. \]
Also,
\[ C(T_6, T_4) \cong C_6 \times C_4. \]
Then
\[ \chi(C([T_6, T_4])) = \frac{1}{6} \cdot \frac{1}{4}. \]
We obtain
\[ |R([T_6, T_4])|\chi(C([T_6, T_4])) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}. \]

From theorem 2.8 part (b) we have that the only torsion element in \( GL_2 \mathbb{Z}[i] \) are these element whose Euler characteristic of their centralizer is not zero are the ones whose eigenvalues are different and belong to the set
\[ \{ \pm 1, \pm i \}. \]

**Lemma 4.4** This is the computation of \( |N_{\mathbb{Q}(i)/\mathbb{Q}}(R(T))| \cdot \chi(C(T)) \) for all torsion elements up to conjugation \( T \) in \( GL_2 \mathbb{Z}[i] \) such that \( \chi(C(T)) \neq 0 \).

(a) \( R([i^k, i^{k+1}])\chi(C([i^k, i^{k+1}])) = \frac{1}{8} \), for \( k = 0, 1, 2, 3 \),

(b) \( R([i^k, i^{k+2}])\chi(C([i^k, i^{k+2}])) = \frac{1}{4} \), for \( k = 0, 1 \).

**Proof.** If \( k \) is not congruent to \( l \) modulo 4 then
\[ \chi(C([i^k, i^l])) = \chi(C([i^k])) \cdot \chi(C([i^l])) = \frac{1}{16}. \]
It remain to compute \( R(A) \) in the two cases.
\[ R([i^k, i^{k+1}]) = N_{\mathbb{Q}(i)/\mathbb{Q}}(\det(P_{[i^k],[i^{k+1}]})) = N_{\mathbb{Q}(i)/\mathbb{Q}}(i^{k+1} - i^k) = 2. \]
In this section we compute the homological Euler characteristic of $T$ elements up to conjugation.

$5.1$ Homological Euler characteristic of $0.1$ part (c)).

From theorem 2.13. And for all other fields the Euler characteristic vanishes (see theorem Lemma 5.1).

And

$$ R([i^k, i^{k+2}]) = N_{Q(i)/Q}(\det(P_{[i^k], [i^{k+2}]}) = N_{Q(i)/Q}(i^k + 2 - i^k) = 4. $$

$5.2$ Homological Euler characteristic of $GL_2(\mathbb{Z}[\xi_3])$ are these elements whose Euler characteristic of their centralizer is not zero are the ones whose eigenvalues are different and belong to the set

$$ \{\pm 1, \xi_3^{\pm 1}, \xi_6^{\pm 1}\}. $$

**Lemma 4.5** This is the computation of $|N_{Q(i)/Q}(R(T))| \cdot \chi(C(T))$ for all torsion elements up to conjugation $T$ in $GL_2\mathbb{Z}[\xi_3]$ such that $\chi(C(T)) \neq 0$. Let $\xi_6 = e^{\frac{2\pi i}{6}}$.

(a) $R([\xi_6^k, \xi_6^{k+1}])\chi(C([\xi_6^k, \xi_6^{k+1}])) = \frac{1}{36}$, for $k = 0, 1, \ldots, 5$,

(b) $R([\xi_6^k, \xi_6^{k+2}])\chi(C([\xi_6^k, \xi_6^{k+2}])) = \frac{1}{12}$, for $k = 0, 1, \ldots, 5$,

(c) $R([\xi_6^k, \xi_6^{k+3}])\chi(C([\xi_6^k, \xi_6^{k+3}])) = \frac{1}{9}$, for $k = 0, 1, 2$.

**Proof.** If $k$ is not congruent to $l$ modulo 6 then

$$ \chi(C([\xi_6^k, \xi_6^l])) = \chi(C([\xi_6^k])) \cdot \chi(C([\xi_6^l])) = \frac{1}{36}. $$

It remain to compute $R(A)$ in the two cases.

$$ R([\xi_6^k, \xi_6^{k+1}]) = N_{Q(\xi_6)/Q}(\det(P_{[\xi_6^k], [\xi_6^{k+1}]}) = N_{Q(\xi_6)/Q}(\xi_6^{k+1} - \xi_6^k) = 1. $$

And

$$ R([\xi_6^k, \xi_6^{k+2}]) = N_{Q(\xi_6)/Q}(\det(P_{[\xi_6^k], [\xi_6^{k+2}]}) = N_{Q(\xi_6)/Q}(\xi_6^{k+2} - \xi_6^k) = 3. $$

And

$$ R([\xi_6^k, \xi_6^{k+3}]) = N_{Q(\xi_6)/Q}(\det(P_{[\xi_6^k], [\xi_6^{k+3}]}) = N_{Q(\xi_6)/Q}(\xi_6^{k+3} - \xi_6^k) = 4. $$

5 Homological Euler characteristic of $GL_m(\mathcal{O}_K)$ with coefficients symmetric powers

In this section we compute the homological Euler characteristic of $GL_n(\mathcal{O}_K)$ for $K = \mathbb{Q}$, $Q(i)$ and $\mathbb{Q}(\xi_3)$. For other imaginary quadratic fields the formula is given in theorem 2.13. And for all other fields the Euler characteristic vanishes (see theorem 0.1 part (c)).

5.1 Homological Euler characteristic of $GL_m\mathbb{Z}$.

**Lemma 5.1** For the symmetric powers of the standard representation $V_2$ of $GL_2\mathbb{Z}$, we have:

(a) $\text{Tr}(I_2 | S^n V_2) = n + 1$
\( (b) \) \( \text{Tr}(-I_2|S^nV_2) = (-1)^{n+1}(n+1) \)

\( (c) \) \( \text{Tr} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] |S^{2n+k}V_2 \) = \( \begin{cases} 1 & k = 0 \\ 0 & k = 1 \end{cases} \)

\( (d) \) \( \text{Tr}(T_3|S^{3n+k}V_2) \) = \( \begin{cases} 1 & k = 0 \\ -1 & k = 1 \\ 0 & k = 2 \end{cases} \)

\( (e) \) \( \text{Tr}(T_6|S^{6n+k}V_2) \) = \( \begin{cases} 1 & k = 0 \\ 1 & k = 1 \\ 0 & k = 2 \\ -1 & k = 3 \\ -1 & k = 4 \\ 0 & k = 5 \end{cases} \)

\( (f) \) \( \text{Tr}(T_4|S^{4n+k}V_2) \) = \( \begin{cases} 1 & k = 0 \\ 0 & k = 1 \\ -1 & k = 2 \\ 0 & k = 3 \end{cases} \)

**Proof.** If \( \lambda_1 \) and \( \lambda_2 \) are the two eigenvalues of \( A \) acting on \( V_2 \) then

\[
\text{Tr}(A|S^nV_2) = \sum_{i=0}^{n} \lambda_1^i \lambda_2^{n-i}.
\]

Note that \( \text{Tr}(A|S^nV_2) \) depends on the on the eigenvalues of \( A \) on \( V_2 \) not the conjugacy class that \( A \) belongs to. Also,

\[
\text{Tr}(-A|S^nV_2) = (-1)^n \text{tr}(A|S^nV_2).
\]

From

\[
\text{Tr}(I_2|S^nV_2) = \dim(S^nV_2) = n + 1,
\]

we obtain part (a) and (b). If \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \) then

\[
\text{Tr}(A|S^nV_2) = 1 - 1 + 1 - \ldots
\]

and we have \( n + 1 \) summands. Thus \( \text{Tr}(A|S^nV_2) \) is 1 if \( n \) is even and 0 if \( n \) is odd. This proves part (c). For part (d) the eigenvalues of \( T_3 \) on \( V_2 \) are \( \xi_3 \) and \( \xi_3^{-1} \). Thus,

\[
\text{Tr}(T_3|S^nV_2) = \xi_3^n + \xi_3^{n-2} + \ldots + \xi_3^{-n}.
\]

Also the sum of three successive summands is zero, and the total number of summands is \( n + 1 \). Therefore, if \( n \equiv 0 \text{mod} 3 \) then

\[
\text{Tr}(T_3|S^nV_2) = \xi_3^n = 1.
\]
If \( n \equiv 1 \mod 3 \) then
\[
\text{Tr}(T_3|S^nV_2) = \xi_3^n + \xi_3^{n-2} = \xi_3 + \xi_3^{-1} = -1.
\]
And if \( n \equiv 2 \mod 3 \) then
\[
\text{Tr}(T_3|S^nV_2) = 0.
\]
This proves part (d). It also proves part (e) because \(-T_3\) is conjugate to \(T_6\). Similarly, for the 4-torsion \( T_4 \) we have
\[
\text{Tr}(T_4|S^nV_2) = i^n + i^{n-2} + \ldots + i^{-n}.
\]
Also, every two consecutive summands add up to zero. Therefore if \( n \) is odd the trace is zero. If \( n \) is even we have
\[
\text{Tr}(T_4|S^nV_2) = i^n,
\]
which is 1 if \( n \equiv 0 \mod 4 \) and \(-1\) if \( n \equiv 2 \mod 4 \). This proves part (f).

**Theorem 5.2** Let \( S^nV_2 \) be the \( n \)-th symmetric power of the standard representation of \( GL_2 \). Then

\[
\chi_h(GL_2\mathbb{Z}, S^{12n+k}V_2) = \left\{ \begin{array}{l}
-n + 1 \\ -n \\ -n \\ -n \\ -n \\ -n - 1 \\ 0 \\
\end{array} \right. \begin{array}{l}
k = 0 \\ k = 2 \\ k = 4 \\ k = 6 \\ k = 8 \\ k = 10 \\ k = odd, \\
\end{array}
\]

and

\[
\chi_h(GL_2\mathbb{Z}, S^{12n+k}V_2 \otimes \text{det}) = \left\{ \begin{array}{l}
-n \\ -n - 1 \\ -n - 1 \\ -n - 1 \\ -n - 2 \\ 0 \\
\end{array} \right. \begin{array}{l}
k = 0 \\ k = 2 \\ k = 4 \\ k = 6 \\ k = 8 \\ k = 10 \\ k = odd.
\]

**Proof.** If \( k \) is odd then \(-I_2\) acts non-trivially on the symmetric power. Therefore the cohomology groups of \( GL_2\mathbb{Z} \) with coefficient that symmetric power vanishes. Consequently, the homological Euler characteristic vanishes. If \( k = 0 \), using lemma 5.1, we obtain
\[
\text{Tr}(I_2|S^{12n}V_2) = 12n + 1,
\]
\[
\text{Tr}(-I_2|S^{12n}V_2) = 12n + 1,
\]
\[
\text{Tr}([1,-1]|S^{12n}V_2) = 1,
\]
\[
\text{Tr}(T_3|S^{12n}V_2) = 1,
\]
\[
\text{Tr}(T_6|S^{12n}V_2) = 1, \\
\text{Tr}(T_4|S^{12n}V_2) = 1.
\]

Using the Euler characteristic of the centralizers of the torsion elements in \( GL_2\mathbb{Z} \) from proposition 3.1, we obtain

\[
\chi_h(GL_2\mathbb{Z}, S^{12n}V_2) = \chi(GL_2\mathbb{Z})(12n + 1) + \chi(GL_2\mathbb{Z})(12n + 1) + \\
+2 \cdot \chi(C([1, -1])) \cdot 1 + \chi(C(T_3)) \cdot 1 + \\
+\chi(C(T_6)) \cdot 1 + \chi(C(T_4)) \cdot 1 = \\
= -\frac{1}{24}(12n + 1) - \frac{1}{24}(12n + 1) + \\
+2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{4} \cdot 1 = \\
= -n + 1.
\]

For \( k = 2 \) from lemma 5.1 we have

\[
\text{Tr}(I_2|S^{12n+2}V_2) = 12n + 3, \\
\text{Tr}(-I_2|S^{12n+2}V_2) = 12n + 3, \\
\text{Tr}([1, -1]|S^{12n+2}V_2) = 1, \\
\text{Tr}(T_3|S^{12n+2}V_2) = 0, \\
\text{Tr}(T_6|S^{12n+2}V_2) = 0, \\
\text{Tr}(T_4|S^{12n+2}V_2) = -1.
\]

Using proposition 3.1 we obtain

\[
\chi_h(GL_2\mathbb{Z}, S^{12n+2}V_2) = -\frac{1}{24}(12n + 3) - \frac{1}{24}(12n + 3) + \\
+2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 0 + \frac{1}{4} \cdot (-1) = \\
= -n.
\]

For \( k = 4 \) from lemma 5.1 we have

\[
\text{Tr}(I_2|S^{12n+4}V_2) = 12n + 5, \\
\text{Tr}(-I_2|S^{12n+4}V_2) = 12n + 5, \\
\text{Tr}([1, -1]|S^{12n+4}V_2) = 1, \\
\text{Tr}(T_3|S^{12n+4}V_2) = -1, \\
\text{Tr}(T_6|S^{12n+4}V_2) = -1, \\
\text{Tr}(T_4|S^{12n+4}V_2) = 1.
\]
Using proposition 3.1 we obtain
\[ \chi_h(\GL_2, S^{12n+4} V_2) = -\frac{1}{24} (12n + 5) - \frac{1}{24} (12n + 5) + 
\]
\[ + 2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot (1) + \frac{1}{6} \cdot (-1) + \frac{1}{4} \cdot (-1) = 
\]
\[ = -n. \]

For \( k = 6 \) from lemma 5.1 we have
\[ \text{Tr}(I_2 | S^{12n+6} V_2) = 12n + 7, \]
\[ \text{Tr}(-I_2 | S^{12n+6} V_2) = 12n + 7, \]
\[ \text{Tr}([1, -1] | S^{12n+6} V_2) = 1, \]
\[ \text{Tr}(T_3 | S^{12n+6} V_2) = 1, \]
\[ \text{Tr}(T_6 | S^{12n+6} V_2) = 1, \]
\[ \text{Tr}(T_4 | S^{12n+6} V_2) = -1. \]

Using proposition 3.1 we obtain
\[ \chi_h(\GL_2, S^{12n+6} V_2) = -\frac{1}{24} (12n + 7) - \frac{1}{24} (12n + 7) + 
\]
\[ + 2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{4} \cdot (-1) = 
\]
\[ = -n. \]

For \( k = 8 \) from lemma 5.1 we have
\[ \text{Tr}(I_2 | S^{12n+8} V_2) = 12n + 9, \]
\[ \text{Tr}(-I_2 | S^{12n+8} V_2) = 12n + 9, \]
\[ \text{Tr}([1, -1] | S^{12n+8} V_2) = 1, \]
\[ \text{Tr}(T_3 | S^{12n+8} V_2) = 0, \]
\[ \text{Tr}(T_6 | S^{12n+8} V_2) = 0, \]
\[ \text{Tr}(T_4 | S^{12n+8} V_2) = 1. \]

Using proposition 3.1 we obtain
\[ \chi_h(\GL_2, S^{12n+8} V_2) = -\frac{1}{24} (12n + 9) - \frac{1}{24} (12n + 9) + 
\]
\[ + 2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 0 + \frac{1}{4} \cdot 1 = 
\]
\[ = -n. \]

For \( k = 10 \) from lemma 5.1 we have
\[ \text{Tr}(I_2 | S^{12n+10} V_2) = 12n + 11, \]

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\[
\begin{align*}
\text{Tr}(-I_2|S^{12n+10}V_2) &= 12n + 11, \\
\text{Tr}([1, -1]|S^{12n+10}V_2) &= 1, \\
\text{Tr}(T_3|S^{12n+10}V_2) &= -1, \\
\text{Tr}(T_6|S^{12n+2}V_2) &= -1, \\
\text{Tr}(T_4|S^{12n+2}V_2) &= -1.
\end{align*}
\]

Using proposition 4.1 we obtain
\[
\chi_h(GL_2\mathbb{Z}, S^{12n+10}V_2) = -\frac{1}{24}(12n + 11) - \frac{1}{24}(12n + 11) + \\
+ 2 \cdot \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot (-1) + \frac{1}{6} \cdot (-1) + \frac{1}{4} \cdot (-1) = \\
= -n - 1.
\]

Note that
\[
\text{tr}(-A|S^nV_3) = (-1)^n \text{tr}(A|S^nV_3).
\]

Therefore we need only formulas for half of the torsion elements. One other remark: we need only to examine the trace for torsion elements whose Euler characteristic of the centralizer is not zero. For \(GL_3\mathbb{Z}\) these elements are \(I_3\) and \(-I_3\).

**Lemma 5.3** The trace of the torsion elements in \(GL_3\mathbb{Z}\) acting on the symmetric power of the standard representation are given by:

\[
\begin{align*}
(c) \quad & \text{Tr}([I_2, -1]|S^{2n+k}V_3) = \begin{cases} 
 n + 1 & k = 0 \\
 n + 1 & k = 1
\end{cases} \\
(d) \quad & \text{Tr}([-I_2, 1]|S^{2n+k}V_3) = \begin{cases} 
 n + 1 & k = 0 \\
 -n - 1 & k = 1
\end{cases} \\
(e) \quad & \text{Tr}([T_3, 1]|S^{3n+k}V_3) = \begin{cases} 
 1 & k = 0 \\
 0 & k = 1 \\
 0 & k = 2 \\
 1 & k = 0 \\
 2 & k = 1 \\
 2 & k = 2 \\
 1 & k = 3 \\
 0 & k = 4 \\
 0 & k = 5
\end{cases} \\
(f) \quad & \text{Tr}([T_6, 1]|S^{6n+k}V_3) = \begin{cases} 
 1 & k = 0 \\
 -2 & k = 1 \\
 2 & k = 2 \\
 -1 & k = 3 \\
 0 & k = 4 \\
 0 & k = 5
\end{cases} \\
(g) \quad & \text{Tr}([T_3, -1]|S^{6n+k}V_3) = \begin{cases} 
 1 & k = 0 \\
 -2 & k = 1 \\
 2 & k = 2 \\
 -1 & k = 3 \\
 0 & k = 4 \\
 0 & k = 5
\end{cases}
\end{align*}
\]
\( (h) \quad \text{Tr}(T_6, -1|S^{6n+k}V_3) = \begin{cases} 
1 & k = 0 \\
0 & k = 1 \\
0 & k = 2 \\
-1 & k = 3 \\
0 & k = 4 \\
0 & k = 5
\end{cases} \)

\( (i) \quad \text{Tr}(T_4, 1|S^{4n+k}V_3) = \begin{cases} 
1 & k = 0 \\
1 & k = 1 \\
0 & k = 2 \\
0 & k = 3
\end{cases} \)

\( (j) \quad \text{Tr}(T_4, -1|S^{4n+k}V_3) = \begin{cases} 
1 & k = 0 \\
-1 & k = 1 \\
0 & k = 2 \\
0 & k = 3
\end{cases} \)

We have omitted part (a) and (b) corresponding to \( I_3 \) and \(-I_3\) because they give no contribution to the homological Euler characteristic since

\[ \chi(C(\pm I_3)) = \chi(GL_3\mathbb{Z}) = 0. \]

**Proof.** The above formulas can be derived from the following fact: let \( A \in GL_k\mathbb{Z} \) and \( B \in GL_l\mathbb{Z} \). Set

\[ f(n) = \text{Tr}([A, B]|S^n(V_k \oplus V_l)), \]

\[ g(n) = \text{Tr}(A|S^nV_k) \]

and

\[ h(n) = \text{Tr}(B|S^nV_k). \]

Then

\[ f(n) = \sum_{i=0}^{n} g(i)h(n - i) = (g \ast h)(n). \]

Then

\[ \text{Tr}(I_2, -1|S^{2n}V_3) = \text{Tr}(I_2|S^{2n}V_2) - \text{Tr}(I_2|S^{2n-1}V_2) + \ldots = \]

\[ = 2n - (2n - 1) + (2n - 2) - \ldots = \]

\[ = n + 1. \]

And

\[ \text{Tr}(I_2, -1|S^{2n+1}V_3) = \text{Tr}(I_2|S^{2n+1}V_2) - \text{Tr}(I_2|S^{2n}V_2) + \ldots = \]

\[ = (2n + 1) - 2n + (2n - 1) - \ldots = \]

\[ = n + 1. \]
This proves part (c) and (d).

For part (e) we use lemma 5.1 part(d) and

\[ \text{Tr}([T_3, 1]|S^n V_3) = \sum_{i=0}^{n} \text{Tr}([T_3]|S^i V_2) \text{Tr}([1]|S^{n-i} V_1). \]

Then

\[ \text{Tr}([T_3, 1]|S^{3n} V_3) = 1 - 1 + 0 + \ldots 1 - 1 + 0 + 1 = 1. \]

Similarly,

\[ \text{Tr}([T_3, 1]|S^{3n+1} V_3) = 1 - 1 + 0 + \ldots 1 - 1 + 0 + 1 - 1 = 0. \]

And

\[ \text{Tr}([T_3, 1]|S^{3n+2} V_3) = 1 - 1 + 0 + \ldots 1 - 1 + 0 = 0. \]

This proves part (e) and (h).

For part (f) we use lemma 5.1 part (e) and

\[ \text{Tr}([T_6, 1]|S^n V_3) = \sum_{i=0}^{n} \text{Tr}([T_6]|S^i V_2) \text{Tr}([1]|S^{n-i} V_1). \]

Then

\[ \text{Tr}([T_6, 1]|S^{6n} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 1 = 1. \]

Similarly,

\[ \text{Tr}([T_6, 1]|S^{6n+1} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 1 = 2, \]

\[ \text{Tr}([T_6, 1]|S^{6n+2} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 1 + 0 = 2, \]

\[ \text{Tr}([T_6, 1]|S^{6n+3} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 1 + 0 - 1 - 1 = 1, \]

\[ \text{Tr}([T_6, 1]|S^{6n+4} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 1 + 0 - 1 - 1 = 0, \]

\[ \text{Tr}([T_6, 1]|S^{6n+5} V_3) = 1 + 1 + 0 - 1 - 1 + 0 \ldots 1 + 1 + 0 - 1 - 1 + 1 + 0 = 0. \]

This proves part (f) and part (g).

For part (e) we use lemma 5.1 part (f) and

\[ \text{Tr}([T_4, 1]|S^n V_3) = \sum_{i=0}^{n} \text{Tr}([T_4]|S^i V_2) \text{Tr}([1]|S^{n-i} V_1). \]

Then

\[ \text{Tr}([T_4, 1]|S^{4n} V_3) = 1 + 0 - 1 + 0 + \ldots 1 + 0 - 1 + 0 + 1 = 1. \]

Similarly,

\[ \text{Tr}([T_4, 1]|S^{4n+1} V_3) = 1 + 0 - 1 + 0 + \ldots 1 + 0 - 1 + 0 + 1 + 0 = 1, \]

\[ \text{Tr}([T_4, 1]|S^{4n+2} V_3) = 1 + 0 - 1 + 0 + \ldots 1 + 0 - 1 + 0 + 1 + 0 - 1 = 0, \]

\[ \text{Tr}([T_4, 1]|S^{4n+3} V_3) = 1 + 0 - 1 + 0 + \ldots 1 + 0 - 1 + 0 = 0. \]
This proves part (i) and (j).

**Theorem 5.4**  \( \chi_h(GL_3 \mathbb{Z}, S^n V_3) = \chi_h(GL_2 \mathbb{Z}, S^n V_2) \),

**Proof.** We are going to use theorem 2.10, lemma 4.2 and lemma 5.3. Also, for \( \chi_h(GL_3 \mathbb{Z}, S^n V_3) \) we only need to consider even \( n \) because if \( n \) is odd then \( -I_3 \) acts non-trivially on the representation. Another observation that we need to make is that if \( A \) is a torsion element in \( GL_3 \mathbb{Z} \) then the torsion element \( -A \) is non-conjugate to \( A \). However

\[
\text{Tr}(-A|S^{2n}V_3) = \text{Tr}(A|S^{2n}V_3).
\]

Thus we only need to consider half of the torsion elements, and just multiply by 2 in order to incorporate the other half of the torsion elements. For the representation \( S^{12n}V_3 \) we have

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n} V_3) = 2 \sum R(A) \chi(C(A)) \text{Tr}(A^{-1}|S^{n}V_3),
\]

where the sum is taken over \( A \) varying in the set

\[ \{[I_2, -1], [T_3, 1], [T_6, 1], [T_4, 1] \} . \]

Then

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n} V_3) = 2(4(-\frac{1}{48})(6n + 1) + 3 \cdot \frac{1}{12} \cdot 1 + 1 \cdot \frac{1}{12} \cdot 1 + 2 \cdot \frac{1}{8} \cdot 1) = -n + 1,
\]

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n+2} V_3) = 2(4(-\frac{1}{48})(6n + 2) + 3 \cdot \frac{1}{12} \cdot 0 + 1 \cdot \frac{1}{12} \cdot 2 + 2 \cdot \frac{1}{8} \cdot 0) = -n,
\]

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n+4} V_3) = 2(4(-\frac{1}{48})(6n + 3) + 3 \cdot \frac{1}{12} \cdot 0 + 1 \cdot \frac{1}{12} \cdot 0 + 2 \cdot \frac{1}{8} \cdot 1) = -n,
\]

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n+6} V_3) = 2(4(-\frac{1}{48})(6n + 4) + 3 \cdot \frac{1}{12} \cdot 1 + 1 \cdot \frac{1}{12} \cdot 1 + 2 \cdot \frac{1}{8} \cdot 0) = -n,
\]

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n+8} V_3) = 2(4(-\frac{1}{48})(6n + 5) + 3 \cdot \frac{1}{12} \cdot 0 + 1 \cdot \frac{1}{12} \cdot 2 + 2 \cdot \frac{1}{8} \cdot 1) = -n,
\]

\[
\chi_h(GL_3 \mathbb{Z}, S^{12n+10} V_3) = 2(4(-\frac{1}{48})(6n + 6) + 3 \cdot \frac{1}{12} \cdot 0 + 1 \cdot \frac{1}{12} \cdot 0 + 2 \cdot \frac{1}{8} \cdot 0) = -n - 1.
\]

Thus,

\[
\chi_h(GL_3 \mathbb{Z}, S^n V_3) = \chi_h(GL_2 \mathbb{Z}, S^n V_2).
\]
Lemma 5.5 The trace of the torsion elements of $GL_4\mathbb{Z}$ acting on the symmetric powers of the standard representation are given by

(a) $\text{Tr}([I_2, -I_2]|S^{2n+k}V_4) = \begin{cases} n + 1 & k = 0 \\ 0 & k = 1 \end{cases}$

(b) $\text{Tr}([I_2, T_3]|S^{3n+k}V_4) = \begin{cases} n + 1 & k = 0 \\ n + 1 & k = 1 \\ n + 1 & k = 2 \end{cases}$

(c) $\text{Tr}([I_2, T_6]|S^{6n+k}V_4) = \begin{cases} 6n + 1 & k = 0 \\ 6n + 3 & k = 1 \\ 6n + 5 & k = 2 \\ 6n + 6 & k = 3 \\ 6n + 6 & k = 4 \\ 6n + 6 & k = 5 \end{cases}$

(d) $\text{Tr}([I_2, T_4]|S^{4n+k}V_4) = \begin{cases} 2n + 1 & k = 0 \\ 2n + 2 & k = 1, 2, 3 \end{cases}$

(e) $\text{Tr}([-I_2, T_3]|S^{6n+k}V_4) = \begin{cases} 6n + 1 & k = 0 \\ -(6n + 3) & k = 1 \\ 6n + 5 & k = 2 \\ -(6n + 6) & k = 3 \\ 6n + 6 & k = 4 \\ -(6n + 6) & k = 5 \end{cases}$

(f) $\text{Tr}([-I_2, T_6]|S^{6n+k}V_4) = \begin{cases} 2n + 1 & k = 0 \\ -(2n + 1) & k = 1 \\ 2n + 1 & k = 2 \\ -(2n + 2) & k = 3 \\ 2n + 2 & k = 4 \\ -(2n + 2) & k = 5 \end{cases}$

(g) $\text{Tr}([-I_2, T_4]|S^{4n+k}V_4) = \begin{cases} 2n + 1 & k = 0 \\ -(2n + 2) & k = 1 \\ 2n + 2 & k = 2 \\ -(2n + 2) & k = 3 \end{cases}$

(h) $\text{Tr}([1, -1, T_3]|S^{6n+k}V_4) = \begin{cases} 1 & k = 0 \\ -1 & k = 1 \\ 1 & k = 2 \\ 0 & k = 3 \\ 0 & k = 4 \\ 0 & k = 5 \end{cases}$
(i) \[ \text{Tr}([1, -1, T_6]|S^{6n+k}V_4) = \begin{cases} 
1 & k = 0 \\
1 & k = 1 \\
1 & k = 2 \\
0 & k = 3 \\
0 & k = 4 \\
0 & k = 5 
\end{cases} \]

(j) \[ \text{Tr}([1, -1, T_4]|S^{4n+k}V_4) = \begin{cases} 
1 & k = 0 \\
0 & k = 1 \\
0 & k = 2 \\
0 & k = 3 
\end{cases} \]

(k) \[ \text{Tr}([T_3, T_6]|S^{6n+k}V_4) = \begin{cases} 
1 & k = 0 \ 
0 & k = 1 \ 
-1 & k = 2 \ 
0 & k = 3 \\
0 & k = 4 \\
0 & k = 5 
\end{cases} \]

(l) \[ \text{Tr}([T_3, T_4]|S^{12n+k}V_4) = \begin{cases} 
1 & k = 0, \ k = 6 \\
-1 & k = 1, \ k = 7 \\
-1 & k = 2, \ -1 & k = 8 \\
2 & k = 3, \ 0 & k = 9 \\
0 & k = 4, \ 0 & k = 10 \\
-2 & k = 5, \ 0 & k = 11 
\end{cases} \]

(m) \[ \text{Tr}([T_6, T_4]|S^{12n+k}V_4) = \begin{cases} 
1 & k = 0, \ k = 6 \\
1 & k = 1, \ -1 & k = 7 \\
-1 & k = 2, \ -1 & k = 8 \\
-2 & k = 3, \ 0 & k = 9 \\
0 & k = 4, \ 0 & k = 10 \\
2 & k = 5, \ 0 & k = 11 
\end{cases} \]

Remark: In the above lemma we have listed only the torsion elements whose Euler characteristic of their centralizer is non zero. That is only these torsion elements will have a contribution toward the homological Euler characteristic of \( GL_4\mathbb{Z} \) or of arithmetic subgroup.

Proof. We are going to use that

\[ \text{Tr}(-A|S^nV_4) = (-1)^n \text{Tr}(A|S^nV_4). \]

For part (a) we use lemma 6.3 part(c) and

\[ \text{Tr}([I_2, -I_2]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([-I_2, 1]|S^iV_3)\text{Tr}([1]|S^{n-i}V_1). \]
Then
\[
\text{Tr}([I_2, -I_2]|S^{2n}V_4) = n + 1 - n + n + \ldots - 1 + 1 = n + 1.
\]
\[
\text{Tr}([I_2, -I_2]|S^{2n+1}V_4) = -(n + 1) + (n + 1) - n + n + \ldots - 1 + 1 = 0.
\]
For part (b) we use lemma 5.3 part(e) and
\[
\text{Tr}([I_2, T_3]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_3, 1]|S^iV_3)\text{Tr}([1]|S^{n-i}V_1).
\]
Then
\[
\text{Tr}([I_2, T_3]|S^{3n}V_4) = 1 + 0 + 0 \ldots 1 + 0 + 0 + 1 = n + 1
\]
\[
\text{Tr}([I_2, T_3]|S^{3n+1}V_4) = 1 + 0 + 0 \ldots 1 + 0 + 0 + 1 + 0 = n + 1
\]
\[
\text{Tr}([I_2, T_3]|S^{3n+2}V_4) = 1 + 0 + 0 \ldots 1 + 0 + 0 + 1 + 0 + 0 = n + 1
\]
This proves part (b) and part (f).
For part (d) we use lemma 5.3 part(i) and
\[
\text{Tr}([I_2, T_4]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_4, 1]|S^iV_3)\text{Tr}([1]|S^{n-i}V_1).
\]
Then
\[
\text{Tr}([I_2, T_3]|S^{4n}V_4) = 1 + 1 + 0 + 0 \ldots 1 + 1 + 0 + 0 + 1 = 2n + 1,
\]
\[
\text{Tr}([I_2, T_3]|S^{4n+1}V_4) = 1 + 1 + 0 + 0 \ldots 1 + 1 + 0 + 0 + 1 + 1 = 2n + 2,
\]
\[
\text{Tr}([I_2, T_3]|S^{4n+2}V_4) = 1 + 1 + 0 + 0 \ldots 1 + 1 + 0 + 0 + 1 + 1 = 2n + 2,
\]
\[
\text{Tr}([I_2, T_3]|S^{4n+3}V_4) = 1 + 1 + 0 + 0 \ldots 1 + 1 + 0 + 0 + 1 + 0 + 1 = 2n + 2.
\]
This proves part (d) and (g).

For part (f) we use lemma 5.3 part(f) and

$$\text{Tr}(I_2, T_6)|S^nV_4) = \sum_{i=0}^{n} \text{Tr}(T_6, 1)|S^nV_3)\text{Tr}(1)|S^{n-i}V_i).$$

Then

$$\begin{align*}
\text{Tr}(I_2, T_6)|S^{6n}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 + 1 + 0 + 0 + 1 = \\
&= 6n + 1, \\
\text{Tr}(I_2, T_6)|S^{6n+1}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 + 1 + 0 + 0 + 1 + 2 = \\
&= 6n + 3, \\
\text{Tr}(I_2, T_6)|S^{6n+2}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 = \\
&= 6n + 5, \\
\text{Tr}(I_2, T_6)|S^{6n+3}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 + 1 + 0 = \\
&= 6n + 6, \\
\text{Tr}(I_2, T_6)|S^{6n+4}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 + 1 + 0 = \\
&= 6n + 6, \\
\text{Tr}(I_2, T_6)|S^{6n+5}V_4) &= 1 + 2 + 2 + 1 + 0 + 0 + \ldots + 1 + 2 + 2 + 1 + 0 + 0 = \\
&= 6n + 6
\end{align*}$$

This proves parts (c) and (e).

For part (h) we use lemma 5.3 part(g) and

$$\text{Tr}([1, -1, T_3]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_3, -1]|S^nV_3)\text{Tr}([1]|S^{n-i}V_i).$$

Then

$$\begin{align*}
\text{Tr}([1, -1, T_3]|S^{6n}V_4) &= 1 - 2 + 2 - 1 + 0 + 0 + \ldots - 1 - 2 - 1 + 0 + 0 + 1 = \\
&= 1, \\
\text{Tr}([1, -1, T_3]|S^{6n+1}V_4) &= 1 - 2 + 2 - 1 + 0 + 0 + \ldots - 1 - 2 = \\
&= -1, \\
\text{Tr}([1, -1, T_3]|S^{6n+2}V_4) &= 1 - 2 + 2 - 1 + 0 + 0 + \ldots - 1 - 2 + 2 = \\
&= 1,
\end{align*}$$

55
\[\text{Tr}([1, -1, T_3]|S^{6n+3}V_4) = 1 - 2 + 2 - 1 + 0 + 0 \ldots 1 - 2 + 2 - 1 = 0,\]
\[\text{Tr}([1, -1, T_3]|S^{6n+4}V_4) = 1 - 2 + 2 - 1 + 0 + 0 \ldots 1 - 2 + 2 - 1 + 0 = 0,\]
\[\text{Tr}([1, -1, T_3]|S^{6n+5}V_4) = 1 - 2 + 2 - 1 + 0 + 0 \ldots 1 - 2 + 2 - 1 + 0 + 0 = 0.\]

This proves parts (h) and (i).

For part (j) we use lemma 5.3 part(j) and
\[\text{Tr}([1, -1, T_4]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_4, -1]|S^iV_3)\text{Tr}([1]|S^{n-i}V_1).\]

Then
\[\text{Tr}([1, -1, T_4]|S^{4n}V_4) = 1 - 1 + 0 + 0 \ldots 1 - 1 + 0 + 0 + 1 = 1,\]
\[\text{Tr}([1, -1, T_4]|S^{4n+1}V_4) = 1 - 1 + 0 + 0 \ldots 1 - 1 + 0 + 0 + 1 - 1 = 0,\]
\[\text{Tr}([1, -1, T_4]|S^{4n+2}V_4) = 1 - 1 + 0 + 0 \ldots 1 - 1 + 0 + 0 + 1 - 1 + 0 = 0,\]
\[\text{Tr}([1, -1, T_4]|S^{4n+3}V_4) = 1 - 1 + 0 + 0 \ldots 1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 = 0.\]

For part (k) we use lemma 5.1 parts (d) and (e), and
\[\text{Tr}([T_3, T_6]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_3]|S^iV_2)\text{Tr}([T_6]|S^{n-i}V_2).\]

Then
\[\text{Tr}([T_3, T_6]|S^{6n}V_4) = n(1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1) +
+ 1 \cdot 1 = 1,\]
\[
\text{Tr}([T_3, T_6]|S^{6n+1}V_4) = n(1 \cdot 1 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot (-1) + 0 \cdot 0) + \\
+1 \cdot 1 - 1 \cdot 1 = \\
= 0, \\
\text{Tr}([T_3, T_6]|S^{6n+2}V_4) = n(1 \cdot 0 - 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot (-1)) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 1 = \\
= -1, \\
\text{Tr}([T_3, T_6]|S^{6n+3}V_4) = n(1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1)) + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = \\
= 0, \\
\text{Tr}([T_3, T_6]|S^{6n+4}V_4) = n(1 \cdot (-1) - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot 0) + \\
+1 \cdot (-1) - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 1 = \\
= 0, \\
\text{Tr}([T_3, T_6]|S^{6n+5}V_4) = (n + 1)(1 \cdot 0 - 1 \cdot (-1) + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 1) + \\
= 0.
\]

For parts (l) and (m) we use lemma 5.1 parts (d) and (f), and

\[
\text{Tr}([T_3, T_4]|S^nV_4) = \sum_{i=0}^{n} \text{Tr}([T_3]|S^iV_2)\text{Tr}([T_4]|S^{n-i}V_2).
\]

Then
\[
\text{Tr}([T_3, T_4]|S^{12n}V_4) = n(1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0) + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0) + \\
+1 \cdot 1 = \\
= 1, \\
\text{Tr}([T_3, T_4]|S^{12n+1}V_4) = n(1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1)) + \\
+1 \cdot 0 - 1 \cdot 1 = \\
= -1,
\]
\[
\text{Tr}(T_3 T_4 | S_{12n+2} V_4) = n(1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0) + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 = \\
= -1,
\]
\[
\text{Tr}(T_3 T_4 | S_{12n+3} V_4) = n(1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1) + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 = \\
= 2,
\]
\[
\text{Tr}(T_3 T_4 | S_{12n+4} V_4) = n(1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0) + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 = \\
= 0,
\]
\[
\text{Tr}(T_3 T_4 | S_{12n+5} V_4) = n(1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1)) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 = \\
= -2,
\]
\[
\text{Tr}(T_3 T_4 | S_{12n+6} V_4) = n(1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0) + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + \\
+1 \cdot 1 = \\
= 1,
\]
\[
\text{Tr}([T_3, T_4]|S^{12n+7}V_4) = n(1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1) + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + \\
+1 \cdot 0 - 1 \cdot 1 = \\
= 1,
\]

\[
\text{Tr}([T_3, T_4]|S^{12n+8}V_4) = n(1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0) + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 = \\
= -1,
\]

\[
\text{Tr}([T_3, T_4]|S^{12n+9}V_4) = n(1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1)) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + \\
+1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 = \\
= 0,
\]

\[
\text{Tr}([T_3, T_4]|S^{12n+10}V_4) = n(1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0) + \\
+1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + \\
+1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 0 - 1 \cdot 1 = \\
= 0,
\]

\[
\text{Tr}([T_3, T_4]|S^{12n+11}V_4) = (n + 1)(1 \cdot 0 - 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 - 1 \cdot 0 + 0 \cdot (-1) + \\
+1 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) - 1 \cdot 0 + 0 \cdot 1) = \\
= 0.
\]

**Theorem 5.6** \( \chi_h(GL_4\mathbb{Z}, S^nV_4) = \chi_h(GL_2\mathbb{Z}, S^nV_2) \)
Proof. If we consider odd symmetric powers then \(-I_4\) acts non-trivially. Therefore the cohomology \(H^i(GL_4\mathbb{Z}, S^{2n+1}V_4)\) will vanish. And so will the homological Euler characteristic. In order to commute the homological Euler characteristic with coefficients in the even symmetric powers we use theorem 2.10, lemma 5.5 and lemma 4.3. Another observation is that for torsion matrices \(A\) in \(GL_m\mathbb{Z}\) we have that \(A^{-1}\) and \(A\) have the same eigenvalues. Thus, we can sum traces of the type \(\text{Tr}(A|V)\).

Thus the formula from theorem 2.10 can be simplified to

\[
\chi_h(GL_4\mathbb{Z}, S^nV_4) = \sum R(A)\chi(C(A))\text{Tr}(A|V).
\]

Then

\[
\chi_h(GL_4\mathbb{Z}, S^{12n}V_4) = \frac{1}{36}(6n+1) - \frac{1}{16}(4n+1) - \frac{1}{144}(12n+1) - \frac{1}{24}(6n+1) - \frac{1}{144}(12n+1) - \frac{1}{16}(4n+1) - \frac{1}{24}(6n+1) + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{24} \cdot 1 + \frac{1}{24} \cdot 1 = 1 - n,
\]

\[
\chi_h(GL_4\mathbb{Z}, S^{12n+2}V_4) = \frac{1}{36}(6n+2) - \frac{1}{16}(4n+1) - \frac{1}{144}(12n+5) - \frac{1}{24}(6n+2) - \frac{1}{144}(12n+5) - \frac{1}{16}(4n+1) - \frac{1}{24}(6n+2) + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 0 + \frac{1}{6} \cdot (-1) + \frac{1}{24} \cdot (-1) + \frac{1}{24} \cdot (-1) = -n,
\]

\[
\chi_h(GL_4\mathbb{Z}, S^{12n+4}V_4) = \frac{1}{36}(6n+3) - \frac{1}{16}(4n+2) - \frac{1}{144}(12n+6) - \frac{1}{24}(6n+3) - \frac{1}{144}(12n+6) - \frac{1}{16}(4n+2) - \frac{1}{24}(6n+3) + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{6} \cdot 0 + \frac{1}{24} \cdot 0 + \frac{1}{24} \cdot 0 = -n,
\]

\[
\chi_h(GL_4\mathbb{Z}, S^{12n+6}V_4) = \frac{1}{36}(6n+4) - \frac{1}{16}(4n+3) - \frac{1}{144}(12n+7) - \frac{1}{24}(6n+4) - \frac{1}{144}(12n+7) - \frac{1}{16}(4n+3) - \frac{1}{24}(6n+4) + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 + \frac{1}{6} \cdot 1 + \frac{1}{24} \cdot 1 + \frac{1}{24} \cdot 1 = -n,
\]

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\[\chi_h(GL_4\mathbb{Z}, S^{12n+8}V_4) = \frac{1}{36}(6n + 5) - \frac{1}{16}(4n + 3) - \frac{1}{144}(12n + 11) - \frac{1}{72}(6n + 5) - \frac{1}{144}(12n + 11) - \frac{1}{16}(4n + 3) - \frac{1}{24}(6n + 5) + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot (-1) + \frac{1}{24} \cdot (-1) + \frac{1}{24} \cdot (-1) = -n,\]

\[\chi_h(GL_4\mathbb{Z}, S^{12n+10}V_4) = \frac{1}{36}(6n + 6) - \frac{1}{16}(4n + 4) - \frac{1}{144}(12n + 12) - \frac{1}{24}(6n + 6) - \frac{1}{144}(12n + 12) - \frac{1}{16}(4n + 4) - \frac{1}{24}(6n + 6) + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{24} \cdot 0 + \frac{1}{24} \cdot 0 = -1 - n,\]

### 5.2 Homological Euler characteristic of \(GL_m(\mathbb{Z}[i])\)

**Lemma 5.7** For the symmetric power of the standard representation of \(GL_2(\mathbb{Z}[i])\) we have

\[\text{Tr}([i^k, i^l]|S^{4n}V_2) = 1,\]

for \(k\) and \(l\) not congruent to each other modulo 4.

**Proof.** First, consider the problem in dimension 1. We have

\[\text{Tr}([i^k]|S^nV_1) = i^{kn},\]

To obtain the result in dimension 2 we use that following observation: If

\[g(n) = \text{Tr}(A|S^nV_1),\]
\[h(n) = \text{Tr}(B|S^nV_2),\]

and

\[f(n) = \text{Tr}([A, B]|S^n(V_1 \oplus V_2)),\]

then

\[f(n) = (g \ast h)(n).\]

Therefore,

\[\text{Tr}([1, i]|S^{4n}V_2) = i^{4n} + i^{4n-1} + \ldots + 1 = 1.\]

Because we take 4-th symmetric power we have

\[\text{Tr}([i^k, i^{k+1}]|S^{4n}V_2) = \text{Tr}([1, i]|S^{4n}V_2) = 1.\]

Also,

\[\text{Tr}([1, -1]|S^{4n}V_2) = (-1)^{4n} + (-1)^{4n-1} + \ldots + 1 = 1.\]

And similarly,

\[\text{Tr}([i^k, i^{k+2}]|S^{4n}V_2) = \text{Tr}([1, -1]|S^{4n}V_2) = 1.\]
Therefore,
\[ \text{Tr}([i^k, i^l]|S^{4n}V_2) = 1, \]
for \( k \) and \( l \) not congruent to each other modulo 4.

**Theorem 5.8** The homological Euler characteristic of \( GL_2(\mathbb{Z}[i]) \) with coefficients the symmetric powers of the standard representation are given by
\[
\chi_h(GL_2(\mathbb{Z}[i]), S^nV_2) = \begin{cases} 
1 & n \equiv 0 \mod 4, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** If in the symmetric power \( n \) is not divisible by 4 then the central element \([i, i]\) acts non-trivially on the cohomology \( H^i(GL_2(\mathbb{Z}[i]), S^nV_2) \).

Therefore the cohomology and the homological Euler characteristic vanish. If the symmetric power is \( 4n \) then we can use lemma 4.4 and 5.7, and we obtain
\[
\chi_h(GL_2(\mathbb{Z}[i]), S^{4n}V_2) = \sum R(A)\chi(C(A))\text{Tr}(A|S^{4n}V_2) = 4 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} = 1.
\]

**5.3 Homological Euler characteristic of \( GL_m(\mathbb{Z}[\xi]) \)**

**Lemma 5.9** For the symmetric power of the standard representation of \( GL_2(\mathbb{Z}[\xi]) \), we have
\[ \text{Tr}([\xi^k, \xi^l]|S^{6n}V_2) = 1, \]
for \( k \) and \( l \) not congruent to each other modulo 6.

**Proof.** First, consider the problem in dimension 1. We have
\[ \text{Tr}([\xi^k]|S^nV_1) = \xi^{kn}. \]

Then
\[ \text{Tr}([1, \xi]|S^{6n}V_2) = \xi^{6n} + \xi^{6n-1} + \ldots + 1 = 1. \]

Since \( \xi^6 = 1 \) we have
\[ \text{Tr}([\xi^k, \xi^6]|S^{6n}V_2) = \text{Tr}([1, \xi]|S^{6n}V_2) = 1. \]

Similarly,
\[ \text{Tr}([1, \xi^2]|S^{6n}V_2) = \xi^{6n} + \xi^{6n-1} + \ldots + 1 = 1. \]

Again,
\[ \text{Tr}([\xi^k, \xi^6]|S^{6n}V_2) = \text{Tr}([1, \xi^2]|S^{6n}V_2) = 1. \]

And
\[ \text{Tr}([1, -1]|S^{6n}V_2) = (-1)^{6n} + (-1)^{6n-1} + \ldots + 1 = 1. \]

Also,
\[ \text{Tr}([\xi^k, \xi^6]|S^{6n}V_2) = \text{Tr}([1, -1]|S^{6n}V_2) = 1. \]

Thus,
\[ \text{Tr}([\xi^k, \xi^l]|S^{6n}V_2) = 1, \]
for \( k \) and \( l \) not congruent to each other modulo 6.
Theorem 5.10 The homological Euler characteristic of $\text{GL}_2 \mathbb{Z}[\xi_3]$ with coefficients the symmetric powers of the standard representation are given by

$$\chi_h(\text{GL}_2(\mathbb{Z}[\xi_3]), S^n V_2) = \begin{cases} 1 & n \equiv 0 \mod 6, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. If in the symmetric power $n$ is not divisible by 6 then the central element $[\xi_6, \xi_6]$ acts non-trivially on the cohomology $H^i(\text{GL}_2(\mathbb{Z}[\xi_3]), S^n V_2)$. Therefore the cohomology and the homological Euler characteristic vanish. If the symmetric power is $6n$ then we can use lemma 4.5 and 5.9, and we obtain

$$\chi_h(\text{GL}_2(\mathbb{Z}[\xi_3]), S^{6n} V_2) = \sum R(A) \chi(C(A)) \text{Tr}(A|S^{6n} V_2) = 6 \cdot \frac{1}{36} + 6 \cdot \frac{1}{12} + 3 \cdot \frac{1}{9} = 1.$$ 

6 Homological Euler characteristic of $\Gamma_1(m, N)$ and $\Gamma_1(m, a)$.

We are going to compute the homological Euler characteristic of $\Gamma_1(2, N)$, $\Gamma_1(3, N)$ and $\Gamma_1(4, N)$ using the torsion elements in the groups. Recall $\Gamma_1(m, N)$ is a subgroup of $GL_m \mathbb{Z}$ stabilizing the covector $[0, \cdots, 0, 1]$ modulo $N$. The same notation is used in [G1]. Recall, also, the homological Euler characteristic of a group $G$ with coefficients a finite dimensional representation $V$ over a field $k$ over is

$$\chi_h(G, V) = \sum (-1)^i \dim_k (H^i(G, V)).$$

We are going to use a generalization of a formula due to K. Brown (see [B2])

$$\chi_h(G, V) = \sum_{A \in \mathcal{C}} \chi(C(A)) \text{Tr}(A^{-1}|V),$$

where $\mathcal{C}$ consists of representatives of the torsion elements of $G$ up to conjugation, and $C(A)$ is the centralizer of $A$. In order to make use of this formula, we have to find the torsion elements of $\Gamma_1(m, N)$ using the torsion elements in $GL_m \mathbb{Z}$. In a more general setting let $G$ be a group and $\Gamma$ be a subgroup. It is possible to find two elements of $\Gamma$ which are conjugate to each other in $G$ but not in $\Gamma$. In the case of $\Gamma = \Gamma_1(m, N)$ and $G = GL_m \mathbb{Z}$, we need to find all element of $\Gamma$ which are not conjugate to any of the others however they become conjugate when considered as elements of the bigger group $G$. To do that we construct a set $N^\Gamma_G(A)$ with the property that the double quotient

$$\Gamma \backslash N^\Gamma_G(A)/C_G(A)$$

parametrizes the elements in $\Gamma$ that are non-conjugate to each other but all become conjugate to $A$ when considered as elements of $G$. The same set $N^\Gamma_G(A)$ is used in K. Brown paper [B2].
Lemma 6.1 Let $G$ be a group and $\Gamma$ be a subgroup. Given an element $A \in \Gamma$, the set of elements in which are conjugate to $A$ in $G$ but not in $\Gamma$ is parametrized by the elements of the double quotient

$$\Gamma\backslash N^\Gamma_G(A)/C_G(A),$$

where $N^\Gamma_G(A) = \{X \in G : XAX^{-1} \in \Gamma\}$ and $C_G(A)$ is the centralizer of $A$ inside the group $G$.

Proof. Let $A_1$ and $A_2$ be two elements of $\Gamma$ both conjugate to $A$ in $G$ and Conjugate to each other in $\Gamma$. Then there exist $X_1, X_2$ in the bigger group $G$ such that

$$A_1 = X_1AX_1^{-1}$$

and

$$A_2 = X_2AX_2^{-1}.$$

Since $A_1$ and $A_2$ are conjugate in the smaller group $\Gamma$, there exists $Y \in \Gamma$ such that

$$YA_1Y^{-1} = A_2.$$

Then

$$YX_1AX_1^{-1}Y^{-1} = X_2AX_2^{-1}$$

and, equivalently,

$$(X_2^{-1}YX_1)A = A(X_2^{-1}YX_1).$$

We obtain that

$$X_2^{-1}YX_1 \in C_G(A).$$

Therefore $YX_1 \in X_2C_G(A)$ and $X_1 \in \Gamma X_2C_G(A)$ We obtain that $X_1$ and $X_2$ belong to the same double quotient in

$$\Gamma\backslash N^\Gamma_G(A)/C_G(A).$$

Conversely, suppose $X_1$ and $X_2$ belong to the same double quotient. We are going to show that both elements $X_1AX_1^{-1}$ and $X_2AX_2^{-1}$ belong to $\Gamma$, and also that they are conjugated to each other in $\Gamma$. By definition of $N^\Gamma_G(A)$ we have that $X_1AX_1^{-1}$ and $X_2AX_2^{-1}$ belong to $\Gamma$. Since the two elements belong to the same double quotient, we have that $X_2 = YX_1C$, where $Y \in \Gamma$ and $C \in C_G(A)$. Then

$$X_2AX_2^{-1} = YX_1CAC^{-1}X_1^{-1}Y^{-1} = Y(X_1AX_1^{-1})Y^{-1}.$$

Thus, the elements $X_1AX_1^{-1}$ and $X_2AX_2^{-1}$ are conjugate to each other in $\Gamma$.

6.1 Homological Euler characteristic of $\Gamma_1(m, N)$

Lemma 6.2 Let $A$ be an $k$-torsion element of $\Gamma_1(m, N)$. If $k$ and $N$ are relatively prime then $A$ has an eigenvalue $+1$. 

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Proof. Let $L$ be the field obtained by adjoining all eigenvalues of $A$ to $\mathbb{Q}$. Let $p$ be a prime dividing $N$. And let $p$ be a prime ideal in $L$ sitting above $p$. Denote, also by $\mu(L)$ the roots of 1 in $L$. And let $\pi : L \to \mathbb{F}_q$, where $\mathbb{F}_q = L/p$.

Then $\pi$ maps $\mu(L)$ onto $\mathbb{F}_q$ because there are no $p$-roots of 1 in $L$ since $p$ does not divide $k$. So the eigenvalues on $A$ in $L$ are mapped onto $\mathbb{F}_q$. However, 1 is an eigenvalue of $A$ mod $p$ in $\mathbb{F}_q$. Therefore 1 is an eigenvalue of $A$ in $L$.

By proposition 2.8 part (a) we know that for a torsion element $A$, we have $\chi(C(A)) \neq 0$ only when the eigenvalues of $A$ are among

$$\{1, -1, i, -i, \xi, \bar{\xi}, \xi_3, \bar{\xi}_3, \xi_6, \bar{\xi}_6\}.$$
Consider the $(2,1)$-block of the product. The left hand side is

\[ X_{21} A_{11} \]

and the right hand side is

\[ 1 \cdot X_{21} \bmod N. \]

Examine the map

\[ P_{B_{22}A_{11}} : X_{21} \mapsto X_{21} A_{11} - B_{22} X_{21} \]

with $B_{22} = 1$. The eigenvalues of $A_{11}$ and $B_{22}$ are different. From lemma 1.11 follows that the eigenvalues of

\[ P_{B_{22}A_{11}} \]

are $\lambda_i - 1$ where $\lambda_i$ runs through the eigenvalues of $A_{11}$. From the previous lemma we have that $\lambda_i - 1$ is invertible mod $N$. Thus,

\[ P_{B_{22}A_{11}} \]

is non-singular mod $N$. Therefore,

\[ P_{B_{22}A_{11}}(X_{21}) = X_{21} A_{11} - B_{22} X_{21} \equiv 0 \bmod N \]

implies that

\[ X_{21} \equiv 0 \bmod N. \]

Therefore,

\[ N_G^\Gamma(A) \subset \Gamma_0(m, N), \]

where $\Gamma_0(m, N)$ is defined as the subgroup of $GL_m \mathbb{Z}$ that sends $[0, \ldots, 0, 1]$ to $[0, \ldots, 0, a]$ modulo $N$ for any $a \neq 0 \bmod N$.

On the other hand, $\Gamma_0(m, N)$ lies inside the normalizer of $\Gamma_1(m, N)$. Also,

\[ \Gamma_0(m, N) \subset N_G^\Gamma(A) \]

from the definition of $N_G^\Gamma(A)$. We obtain that

\[ N_G^\Gamma(A) = \Gamma_0(m, N). \]

Let $X$ be an element in $\Gamma_0(m, N)$ of the same block form as before. Inside the quotient

\[ \Gamma_1(m, N) \backslash \Gamma_0(m, N) \]

the element $X$ is determined by

\[ X_{22} \bmod N. \]

Note that there are $\varphi(N)$ options for $X_{22}$ because it has to be invertible modulo $N$. From the block triangular theorem (theorem 2.3) we know that we can choose $A$ so that $A_{21} = 0$. If $C$ is in the centralizer of $A$ inside $GL_m \mathbb{Z}$, by lemma 3.2 we know
that $C_{21} = 0$. Therefore the centralizer of $A$ modulo $\Gamma_1(m, N)$ is determined by $C_{22}$ which could be $+1$ or $-1$. Therefore the double quotient consist of

$$\frac{1}{2}\varphi(N)$$

elements.

(b) If $n = 2$ then $A = I_2$

$$N_G^\Gamma(A) = GL_2\mathbb{Z}$$

and

$$C_{GL_2\mathbb{Z}}(A) = GL_2\mathbb{Z}.$$ 

Therefore, the double quotient has one element. Assume that $m > 2$. By the block triangular theorem we can conjugate $A$ to a matrix of the block form

$$\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

Note that the block $A_{11}$ is an $(m-2) \times (m-2)$ matrix and it does not have an eigenvalue 1. We can assume that $A$ is of the above form. Let $X$ be an element of $N_G^\Gamma(A)$, and let $B = XAX^{-1}$. If we write this equation with respect to the block form of $A$ we obtain:

$$\begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{bmatrix} \cdot \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{bmatrix} \mod N$$

Consider the $(3,1)$-block of the product. From the left hand side we obtain

$$X_{31}A_{11}$$

and from the right hand side we obtain $1 \cdot X_{31}$. We need to examine the map

$$X_{31} \mapsto X_{31}A_{11} - 1 \cdot X_{31} \mod N.$$ 

Set

$$B_{33} = 1.$$ 

We have to consider the map

$$P_{B_{33}A_{11}} : X_{31} \mapsto X_{31}A_{11} - 1 \cdot X_{31}.$$ 

If $\lambda_k$ are the eigenvalues of $A_{11}$ then $\lambda_k - 1$ are the eigenvalues of $P_{B_{33}A_{11}}$ from the previous lemma it follows that $\lambda_k - 1$ is not divisible by $p$ for any $p$ dividing $N$. This, the map

$$P_{B_{33}A_{11}}$$

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is non-singular modulo $N$. We have

$$P_{B_{33}A_{11}}(X_{31}) \equiv 0 \mod N.$$ 

Therefore

$$X_{31} \equiv 0 \mod N.$$ 

Let

$$N_{31} = \{(X_{ij}) \in GL_m \mathbb{Z} : 1 \leq i, j \leq 3, X_{31} \equiv 0 \mod N\}.$$ 

Then

$$N_G^\Gamma(A) \subset N_{31}.$$ 

We are going to show that the inclusion is in fact an equality. We are going to prove the following inclusions modulo $N$:

$$N_G^\Gamma(A) \subset N_{31} \subset \Gamma_1(m, N) \cdot H' \subset \Gamma_1(m, N) \cdot C_{GL_m \mathbb{Z}}(A) \subset N_G^\Gamma(A),$$

where $H'$ is a subgroup of $C_{GL_m \mathbb{Z}}(A)$. The last inclusion always holds, not only modulo $N$. Modulo any prime number $q$ relatively prime to $N$, we have

$$\Gamma_1(m, N) = SL_m^\pm(\mathbb{Z}/q\mathbb{Z}) = N_G^\Gamma(A),$$

where $SL_m^\pm(\mathbb{Z}/q\mathbb{Z})$ is the group of $m \times m$ matrices with coefficients in $\mathbb{Z}/q\mathbb{Z}$ whose determinant is $\pm 1$. Therefore, the only restriction on $N_G^\Gamma(A)$ become appearant modulo $N$. From these inclusions it will follow that

$$N_G^\Gamma(A) = \Gamma_1(m, N) \cdot C_{GL_m \mathbb{Z}}(A).$$

Therefore the double quotient

$$\Gamma_1(m, N) \backslash N_G^\Gamma(A) / C_{GL_m \mathbb{Z}}(A)$$

will have one element.

We have proved the first inclusion. The last inclusion follows directly from the definition of $N_G^\Gamma(A)$. And the second to the last inclusion holds because $H'$ is a subgroup of $C_{GL_m \mathbb{Z}}(A)$. Now we are going to prove the second inclusion. Consider $A$ as a $2 \times 2$-block matrix

$$A = \begin{bmatrix} A_{11} & \overline{A}_{12} \\ 0 & I_2 \end{bmatrix},$$

with

$$\overline{A}_{12} = [A_{12}, A_{13}].$$

Let

$$C \in C_{GL_m \mathbb{Z}}(A)$$

be in the centralizer of $A$. Consider $C$ as a $2 \times 2$ block matrix. Namely,

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}.$$
with \( C_{22} \) a \( 2 \times 2 \) matrix. The admissible matrices \( C \) are such that \( C_{11}\overline{A}_{12}C_{22}^{-1} \) and \( \overline{A}_{12} \) map to the same element in \( Q_{\text{mod}} \), where

\[
Q_{\text{mod}} = \text{Mat}_{n-2,2}\mathbb{Z}/\text{Im} \ P_{A_{11},I_2},
\]

the matrix \( C_{12} \) is determined uniquely from \( C_{11} \) and \( C_{22} \). Let

\[
H = \{ C \in C(A) : C_{11} = I_{m-2} \}
\]

be a subgroup of \( C(A) \). We are going to show that

\[
N_{31} \subset \Gamma_1(m, N) \cdot H',
\]

where

\[
H' \subset H \subset C(A)
\]

are subgroups. In this setting \( C_{12} \) is uniquely determined by \( C_{11} \); and we are going to write it as

\[
C_{12}(C_{11}).
\]

Let

\[
M = \text{det}(P_{A_{11},I_2}).
\]

Then we have that

\[
M \cdot P_{A_{11},I_2}^{-1}
\]

has integer entries. Therefore,

\[
\text{Im} \ (P_{A_{11},I_2}) \subset M \cdot \text{Mat}_{m-2,2}\mathbb{Z}.
\]

If \( C_{22} \) is in \( \Gamma(2,M) \) then

\[
\overline{A}_{12}C_{22}^{-1} \equiv \overline{A}_{12} \mod M \cdot \text{Mat}_{m-2,2}\mathbb{Z}.
\]

Therefore

\[
\overline{A}_{12}C_{22}^{-1} \equiv \overline{A}_{12} \mod \text{Im} \ (P_{A_{11},I_2}).
\]

Set

\[
H' = \{ C \in C(A) : C_{11} = I_{m-2}, C_{22} \in \Gamma(2,M), C_{12} = C_{12}(C_{22}) \}.
\]

Then

\[
H' \cong \Gamma(2,M).
\]

However, we need \( H' \) for comparison of groups modulo \( N \). Note that \( M \) and \( N \) are relatively prime because

\[
M = \text{det}(P_{A_{11},I_2})
\]

which has prime factors only 2 and 3. And by assumption \( N \) is relatively prime to 2 and 3. Then

\[
\Gamma(2,M) \equiv SL_2\mathbb{Z} \mod N.
\]

We need to show that

\[
N_{31} \subset \Gamma_1(n, N) \cdot H'
\]
modulo $N$. Let $X \in N_{31}$. We have that $X_{32}$ and $X_{33}$ are relatively prime modulo $N$. Otherwise their common factor will divide $\det(X)$, which is $\pm 1$. Then we can find $Y$ and $Z$ such that
\[ YX_{33} - ZX_{32} = 1. \]
Let
\[ C_{22} = \begin{bmatrix} Y & Z \\ X_{32} & X_{33} \end{bmatrix}. \]
Then Let $C$ be a corresponding element in $H'$ with $C_{22}$ congruent to the above matrix modulo $M$ and
\[ C_{11} = I_{n-2}. \]
Then
\[ XC^{-1} \in \Gamma_1(m, N), \]
and we are done.

**Theorem 6.4** Let $V$ be a representation of $GL_m\mathbb{Z}$, and let $N$ be an integer relatively prime to 2 and 3. Then
\[
\chi_h(\Gamma_1(m, N), V) = \varphi(N) \sum_{A=[A_1]} R(A) \chi(C_{GL_m-I\mathbb{Z}}(A_1)) \text{Tr}(A^{-1}|V) + \\
+ \varphi_2(N) \sum_{A=[A_2,I_2]} R(A) \chi(C_{GL_m-I\mathbb{Z}}(A_2)) \text{Tr}(A^{-1}|V),
\]
where $A_1$ and $A_2$ are block-diagonal matrices with zero blocks off the diagonal, and with block on the diagonal $A_{ii}$ varying through
\[ \{-1, -I_2, T_3, T_4, T_6\} \]
and $A_{ii}$ and $A_{jj}$ have no common eigenvalues. We set
\[ T_3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \ T_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ T_6 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}. \]
Also $\varphi(N)$ is the Euler function of $N$, and $\varphi_2(N)$ is the arithmetic function generated by
\[ \varphi_2(p^n) = p^{2k}(1 - \frac{1}{p^2}). \]

**Proof.** Let $A$ be a torsion element of
\[ \Gamma = \Gamma_1(m, N) \]
such that
\[ \chi(C_{\Gamma}(A)) \neq 0. \]
Then
\[ \chi(C_{GL_m\mathbb{Z}}(A)) \neq 0. \]
By lemma 6.2 we have that 1 is an eigenvalue of $A$. Also, if
\[ \chi(C\Gamma(A)) \neq 0 \]
then the multiplicity of the eigenvalue 1 is at most 2. We are going to prove that if the eigenvalue has multiplicity 1 then
\[ \chi(C\Gamma(A)) = 2 \cdot \chi(C_{GL_mZ}(A)), \]
and if the eigenvalue 1 has multiplicity 2 then
\[ \chi(C\Gamma(A)) = \varphi_2(N) \cdot \chi(C_{GL_mZ}(A)). \]
Assume that we have proven these two formulas. From the generalization of Brown’s formula, we have
\[ \chi_h(\Gamma_1(m, N), V) = \sum_{A: \text{torsion}} \chi(C_{\Gamma_1}(m, N)) Tr(A^{-1}|V). \]
In the summation it is enough to sum over torsion elements $A$ that have eigenvalue 1 either with multiplicity 1 or with multiplicity 2. First take the sum over elements $A$ that become conjugate to each other in $GL_mZ$. Let $A$ has eigenvalue 1 with multiplicity 1. By lemma 6.3
\[ \sum \chi(C_{\Gamma_1}(m, N)) Tr(A''^{-1}|V) = \varphi(N) \chi(C_{GL_mZ}(A')) Tr(A'^{-1}|V), \]
where the sum is taken over all non-conjugate $A''$ in $\Gamma_1(m, N)$ that become conjugate to $A'$ in $GL_mZ$. Now sum over the torsion elements $A'$ in $GL_mZ$ that become conjugate to each other in $GL_m\mathbb{C}$.

Similarly, in the case when the eigenvalue 1 has multiplicity 2, we have
\[ \varphi_2(N) = [\Gamma_1(m, N) : GL_mZ] \]
instead of $\varphi(N)$. Thus, it remains to prove that if the eigenvalue 1 has multiplicity 1 then
\[ \chi(C\Gamma(A)) = 2 \cdot \chi(C_{GL_mZ}(A)), \]
and if the eigenvalue 1 has multiplicity 2 then
\[ \chi(C\Gamma(A)) = \varphi_2(N) \cdot \chi(C_{GL_mZ}(A)). \]

Suppose $A$ has eigenvalue 1 with multiplicity one. We can assume that $A$ is in block-triangular
\[ \begin{bmatrix} A_{11} & A_{22} \\ 0 & 1 \end{bmatrix} \]
with $A_{22} = 1$ and $A_{21} = 0$. If $C$ is a matrix in $GL_m\mathbb{Z}$ commuting with $A$ then $C$ is if the same block form with $C_{21} = 0$ and $C_{22} = \pm 1$. Exactly one of the matrices $C$
and $-C$ belongs to $\Gamma_1(m, N)$ because of the $C_{22}$ entry. Thus the centralizer of $A$ in $GL_m\mathbb{Z}$ contains twice as many elements as the centralizer inside $\Gamma(m, N)$. Therefore,

$$\chi(C_\Gamma(A)) = 2 \cdot \chi(C_{GL_m\mathbb{Z}}(A)).$$

Suppose $A$ has eigenvalue 1 with multiplicity two. We can assume that $A$ is of the form

$$\begin{bmatrix}
A_{11} & A_{22} \\
0 & I_2
\end{bmatrix}$$

with

$$A_{22} = I_2$$

and

$$A_{21} = 0.$$ 

If $C$ commutes with $A$ then it is of the same block type with

$$C_{21} = 0.$$ 

The possibilities of $C_{22}$ are determined by relations modulo the image of

$$P_{A_{11}I_2}.$$ 

Note that if

$$\chi(C(A)) \neq 0$$

then the eigenvalues of $A$ are among

$$\{\pm 1, \pm i, \pm \xi_3\}.$$ 

Then

$$\det(P_{A_{11}, I_2})$$

has only prime factors 2 and 3. Therefore the conditions on $C_{22}$ imposed by

$$P_{A_{11}, I_2}$$

are independent from the condition $Ce\Gamma_1(n, N)$, because that last condition is a congruence modulo $N$ but $N$ is relatively prime 2 and 3. Therefore

$$\chi(C_{\Gamma_1(m, N)}(A)) = [C_{\Gamma_1(m, N)}(A) : C_{GL_m\mathbb{Z}}(A)] \chi(C_{GL_m\mathbb{Z}}(A)) = \varphi_2(N) \cdot \chi(C_{GL_m\mathbb{Z}}(A)).$$

**Corollary 6.5** For $N$ not divisible by 2 and 3 the homological Euler characteristics of $\Gamma_1(2, N)$ with coefficients the symmetric powers of $V_2$ are given by

$$\chi_h(\Gamma_1(2, N), S^{2n+k}V_2) = \begin{cases}
-\frac{1}{24}\varphi_2(N)(2n + 1) + \frac{1}{2}\varphi(N) & k = 0 \\
-\frac{1}{24}\varphi_2(N)(2n + 2) & k = 1
\end{cases}$$

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where $\varphi(N)$ is the multiplicative Euler function generated by

$$\varphi(p^n) = p^n(1 - \frac{1}{p})$$

and $\varphi_2(N)$ is the multiplicative function generated by

$$\varphi_2(p^n) = p^{2n}(1 - \frac{1}{p^2}).$$

**Proof.** From theorem 6.4, we have that

$$\chi_h(\Gamma_1(2, N), S^{2n}V_2) = \varphi(N)R([-1, 1])\chi(C([-1, 1]))\text{Tr}([-1, 1]|S^{2n}V_2) +$$

$$\varphi_2(N)\chi(\Gamma_1(2, N))\text{Tr}(I_2|S^{2n}V_2) =$$

$$= \varphi(N) \cdot 2 \cdot \frac{1}{4} \cdot 1 + \varphi_2(N)(-\frac{1}{24})(2n + 1) =$$

$$= -\frac{1}{24} \varphi_2(N)(2n + 1) + \frac{1}{2} \varphi(N).$$

Similarly,

$$\chi_h(\Gamma_1(2, N), S^{2n+1}V_2) = \varphi(N)R([-1, 1])\chi(C([-1, 1]))\text{Tr}([-1, 1]|S^{2n+1}V_2) +$$

$$+ \varphi_2(N)\chi(\Gamma_1(2, N))\text{Tr}(I_2|S^{2n+1}V_2) =$$

$$= \varphi(N) \cdot 2 \cdot \frac{1}{4} \cdot 0 + \varphi_2(N)(-\frac{1}{24})(2n + 2).$$

**Theorem 6.6** The torsion elements of $\Gamma_1(3, N)$, for $N$ not divisible by 2 or 3, are given in the following table together with the Euler characteristic of their centralizers. In the table

$$\varphi_2(N) = [\Gamma_1(2, N) : GL_2\mathbb{Z}]$$

is the multiplicative function generated by

$$\varphi_2(p^n) = p^{2n}(1 - \frac{1}{p^2}).$$

For abbreviation set

$$\Gamma = \Gamma_1(3, N) \text{ and } G = GL_3\mathbb{Z}.$$
\[
\begin{array}{|c|c|c|c|}
\hline
A & |\Gamma \backslash N_G^T(A)/C_G(A)| & C_T(A) & \chi(C_T(A)) \\
\hline
(a) & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & 1 & \Gamma_1(3, N) & 0 \\
\hline
(c1) & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & 1 & C_2 \times \Gamma_1(2, N) & -\frac{1}{48} \varphi_2(N) \\
\hline
(c2) & \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & 1 & C_2 \times \Gamma_1(2, 2N) & -\frac{1}{16} \varphi_2(N) \\
\hline
(d1) & \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & GL_3 \mathbb{Z} & -\frac{1}{24} \\
\hline
(d2) & \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & \Gamma_1(2, 2) & -\frac{1}{5} \\
\hline
(e1) & \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & C_6 & \frac{1}{6} \\
\hline
(e2) & \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & C_3 & \frac{1}{3} \\
\hline
(h) & \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ 1 & 0 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & C_6 & \frac{1}{6} \\
\hline
(i1) & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & C_4 & \frac{1}{4} \\
\hline
(i2) & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\
\hline -1 & 0 & 1 \\ 0 & 1 & 0 \\
\hline 1 & 0 & 1 \end{bmatrix} & \frac{1}{2} \varphi(N) & C_4 & \frac{1}{4} \\
\hline
\end{array}
\]

**Proof.** This table follows from the table in proposition 3.2 and from the previous two lemmas on the calculation of \( \Gamma \backslash N_G^T(A)/C_G(A) \).

The only part that requires a proof given two elements \( A \) and \( B \) in \( \Gamma_1(3, N) \) such that both are conjugate in \( GL_3 \mathbb{Z} \) to the same element form the above list, we have

\[
C_{T_\Gamma(3, N)}(A) \cong C_{T_\Gamma(3, N)}(B).
\]

Indeed, if \( A \) and \( B \) are conjugate to each other in \( \Gamma_1(3, N) \) then we are done. If they are not conjugate to each other in \( \Gamma_1(3, n) \) then the double quotient \( \Gamma \backslash N_G^T(A)/C_G(A) \) has more than one element. From lemma 5.3 we have that

\[
N_G^T(A) = \Gamma_0(3, N)
\]
which is the normalizer of $\Gamma_1(3, N)$ in $GL_3\mathbb{Z}$. Let
\[ \text{P} \in N^G_{\Gamma}(A) \]
such that
\[ B = \text{P} \cdot \text{A} \cdot \text{P}^{-1} \]
then
\[ C_{\Gamma_1(3, N)}(B) = \text{P} \cdot C_{\Gamma_1(3, N)}(A) \cdot \text{P}^{-1} \]
because $\text{P}$ is in the normalizer of $\Gamma_1(3, N)$ in $GL_3\mathbb{Z}$.

**Corollary 6.7** For $N$ not divisible by 2 and 3 the homological Euler characteristic of $\Gamma_1(3, N)$ is given by
\[ \chi_h(\Gamma_1(3, N)) = -\frac{1}{12}\varphi_2(N) + \frac{1}{2}\varphi(N). \]
where $\varphi(N)$ is the multiplicative Euler function generated by
\[ \varphi(p^n) = p^n(1 - \frac{1}{p}) \]
and $\varphi_2(N)$ is the multiplicative function generated by
\[ \varphi_2(p^n) = p^{2n}(1 - \frac{1}{p^2}). \]

**Proof.** Brown’s formula gives the homological Euler characteristic as a sum of the usual (orbifold) Euler characteristic of the centralizers of the torsion elements. From the previous proposition we obtain
\[ \chi_h(\Gamma_1(3, N)) = -\frac{1}{36}\varphi_2(N) - \frac{1}{16}\varphi_2(N) - \frac{1}{36}\varphi(N) - \frac{1}{16}\varphi(N) + \frac{1}{144}\varphi(N) + \frac{1}{8}\varphi(N) + \frac{1}{8}\varphi(N) + \frac{1}{8}\varphi(N) = \]
\[ = -\frac{1}{12}\varphi_2(N) + \frac{1}{2}\varphi(N). \]

**Corollary 6.8** For $N$ not divisible by 2 and 3 the homological Euler characteristic of $\Gamma_1(4, N)$ is given by
\[ \chi_h(\Gamma_1(4, N)) = \varphi(N). \]

**Proof.** Using theorem 6.4 and lemma 4.3 we obtain
\[ \chi_h(\Gamma_1(4, N)) = \varphi_2(N)(\frac{1}{36} - \frac{1}{16} - \frac{1}{144} - \frac{1}{8}) + \]
\[ + \varphi(N)(\frac{1}{8} + \frac{1}{8} + \frac{1}{2}) = \]
\[ = \varphi(N). \]
6.2 Homological Euler characteristic of $\Gamma_1(m, a) \subset GL_m(\mathbb{Z}[i])$.

**Lemma 6.9** Let $A$ be an $k$-torsion element of $\Gamma_1(m, a)$. If $2k$ and $a$ are relatively prime then $A$ has an eigenvalue $+1$.

**Proof.** Let $L$ be the field obtained by adjoining all eigenvalues of $A$ to $\mathbb{Q}(i)$. Let $p$ be a prime ideal dividing $a$. And let $\mathfrak{P}$ be a prime ideal in $L$ sitting above $p$. Denote, also by $\mu(L)$ the roots of 1 in $L$. And let

$$\pi : L \to \mathbb{F}_q,$$

where

$$\mathbb{F}_q = L/\mathfrak{P}.$$ Let $p$ be the rational prime sitting below $p$. Then $\pi$ maps $\mu(L)$ onto $\mathbb{F}_q$ because there are no $p$-roots of 1, in $L$ since $p$ does not divide $k$. So the eigenvalues on $A$ in $L$ are mapped onto $\mathbb{F}_q$. However, 1 is an eigenvalue of $A$ mod $p$ in $\mathbb{F}_q$. Therefore 1 is an eigenvalue of $A$ in $L$.

By proposition 2.8 part (b) we know that for a torsion element $A$, we have $\chi(C(A)) \neq 0$ only when the eigenvalues of $A$ are among

$$\{1, -1, i, -i\}.$$  

**Lemma 6.10** Let $A \in \Gamma_1(m, a)$ be a torsion element with $\chi(C(A)) \neq 0$, let $a$ be relatively prime with $(1 + i)$, and let $f$ be the characteristic polynomial of $A$. Then $+1$ is a root of $f$ and it has multiplicity 1. And the set

$$\Gamma_1(m, N) \backslash N_{GL_m(\mathbb{Z}[i])}^{\Gamma_1(m, a)}(A)/C_{GL_m(\mathbb{Z}[i])}(A)$$

has $\frac{1}{4}\varphi(\mathbb{Z}[i])(a)$ elements, where $\varphi(\mathbb{Z}[i])(a)$ is a multiplicative function on the ideals of $\mathbb{Z}[i]$ generated by

$$\varphi(\mathbb{Z}[i])(\mathfrak{p})^n = N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{p})^n(1 - \frac{1}{N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{p})}).$$

**Proof.** In order to save some space set

$$\Gamma = \Gamma_1(m, a)$$

and

$$G = CL_m(\mathbb{Z}[i]).$$

Let $X \in N^G_\Gamma(A)$ and let $B = XAX^{-1}$. Then $B \in \Gamma$. We can write the matrices $A$, $B$ and $X$ in $2 \times 2$ block form with diagonal blocks $A_{11}$, $B_{11}$ and $X_{11}$ of size $(m - 1) \times (m - 1)$ and $A_{22}$, $B_{22}$ and $X_{22}$ of size $1 \times 1$. Then

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{12} \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} B_{11} & B_{12} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \mod a.$$
Consider the (2,1)-block of the product. The left hand side is
\[ X_{21}A_{11} \]
and the right hand side is
\[ 1 \cdot X_{21} \mod a. \]
Examine the map
\[ P_{B_{22}A_{11}} : X_{21} \mapsto X_{21}A_{11} - B_{22}X_{21} \]
with \( B_{22} = 1 \). The eigenvalues of \( A_{11} \) and \( B_{22} \) are different. From lemma 1.8 follows that the eigenvalues of
\[ P_{B_{22}A_{11}} \]
are \( \lambda_i - 1 \) where \( \lambda_i \) runs through the eigenvalues of \( A_{11} \). From the previous lemma we have that \( \lambda_i - 1 \) is invertible mod \( a \). Thus,
\[ P_{B_{22}A_{11}} \]
is non-singular mod \( a \). Therefore,
\[ P_{B_{22}A_{11}}(X_{21}) = X_{21}A_{11} - B_{22}X_{21} \equiv 0 \mod a \]
implies that
\[ X_{21} \equiv 0 \mod a. \]
Therefore,
\[ N^\Gamma_G(A) \subset \Gamma_0(m, a), \]
where \( \Gamma_0(m, a) \) is defined as the subgroup of \( GL_m(\mathbb{Z}[i]) \) that sends \([0, \ldots, 0, 1]\) to \([0, \ldots, 0, \alpha]\) modulo \( a \) for any \( \alpha \neq 0 \mod a \).

On the other hand, \( \Gamma_0(m, a) \) lies inside the normalizer of \( \Gamma_1(m, a) \). Also,
\[ \Gamma_0(m, a) \subset N^\Gamma_G(A) \]
from the definition of \( N^\Gamma_G(A) \). We obtain that
\[ N^\Gamma_G(A) = \Gamma_0(m, a). \]
Let \( X \) be an element in \( \Gamma_0(m, a) \) of the same block form as before. Inside the quotient
\[ \Gamma_1(m, a) \backslash \Gamma_0(m, a) \]
the element \( X \) is determined by
\[ X_{22} \mod a. \]
Note that there are \( \varphi_{\mathbb{Z}[i]}(a) \) options for \( X_{22} \) because it has to be invertible modulo \( a \). From the block-triangular theorem (theorem 2.3) we know that we can choose \( A \) so that \( A_{21} = 0 \). If \( C \) is in the centralizer of \( A \) inside \( GL_m(\mathbb{Z}[i]) \), by lemma 3.2 we know that \( C_{21} = 0 \). Therefore the centralizer of \( A \) modulo \( \Gamma_1(m, a) \) is determined by \( C_{22} \) which could be +1, −1, \( i \) or −\( i \). Therefore the double quotient consist of
\[ \frac{1}{4} \varphi_{\mathbb{Z}[i]}(a) \]
elements.
Theorem 6.11  Let $V$ be a representation of $GL_m(\mathbb{Z}[i])$, and let $a$ be an ideal relatively prime to $(1 + i)$. Then

$$
\chi_h(\Gamma_1(m, a), V) = \varphi_{\mathbb{Z}[i]}(a) \sum_{A = [A_0, 1]} R(A) \chi(C_{GL_{m-1}\mathbb{Z}[i]}(A_0)) \text{Tr}(A^{-1}|V),
$$

where

$$
A_0 = [A_{11}, \ldots A_{kk}],
$$

$A_{ii}$ vary through

$$
\{ -1, i, -i \}
$$

and $A_{ii}$ and $A_{jj}$ are distinct, $\varphi_{\mathbb{Z}[i]}(a)$ is the arithmetic function defined on the ideals of $\mathbb{Z}[i]$ generated by

$$
\varphi_{\mathbb{Z}[i]}(p^k) = N_{\mathbb{Q}(i)/\mathbb{Q}}(p)^{k}(1 - \frac{1}{N_{\mathbb{Q}(i)/\mathbb{Q}}(p)}).
$$

Proof. Let $A$ be a torsion element of

$$
\Gamma = \Gamma_1(m, a)
$$

such that

$$
\chi(C_\Gamma(A)) \neq 0.
$$

Then

$$
\chi(C_{GL_m\mathbb{Z}[i]}(A)) \neq 0.
$$

By lemma 6.9 we have that 1 is an eigenvalue of $A$. Also, if

$$
\chi(C_\Gamma(A)) \neq 0
$$

then the multiplicity of the eigenvalue 1 is at most 1. We are going to prove that

$$
\chi(C_\Gamma(A)) = 4 \cdot \chi(C_{GL_m\mathbb{Z}[i]}(A)),
$$

Assume that we have proven this formula. From the generalization of Brown’s formula, we have

$$
\chi_h(\Gamma_1(m, a), V) = \sum_{A \text{ torsion}} \chi(C_{\Gamma_1(m, a)}(A)) \text{Tr}(A^{-1}|V).
$$

In the summation it is enough to sum over torsion elements $A$ that have eigenvalue 1 either with multiplicity 1. First take the sum over elements $A$ that become conjugate to each other in $GL_m(\mathbb{Z}[i])$. Let $A$ has eigenvalue 1 with multiplicity 1. By lemma 6.9

$$
\sum \chi(C_{\Gamma_1(m, a)}(A'')) \text{Tr}(A''^{-1}|V) = \varphi_{\mathbb{Z}[i]}(a) \chi(C_{GL_n\mathbb{Z}[i]}(A')) \text{Tr}(A'^{-1}|V),
$$

where the sum is taken over all non-conjugate $A''$ in $\Gamma_1(m, a)$ that become conjugate to $A'$ in $GL_n(\mathbb{Z}[i])$. 
We can assume that $A$ is in block-triangular
\[
\begin{bmatrix}
A_{11} & A_{22} \\
0 & 1
\end{bmatrix}
\]
with $A_{22} = 1$ and $A_{21} = 0$. If $C$ is a matrix in $GL_m(\mathbb{Z}[i])$ commuting with $A$ then $C$ is if the same block form with $C_{21} = 0$ and $C_{22} = i^k$. Exactly one of the matrices $C$, $-C$, $iC$ and $-iC$ belongs to $\Gamma_1(m, a)$ because of the $C_{22}$ entry. Thus the centralizer of $A$ in $GL_m(\mathbb{Z}[i])$ contains 4 times as many element as the centralizer inside $\Gamma(m, a)$. Therefore,
\[
\chi(C_{\Gamma}(A)) = 4 \cdot \chi(C_{GL_m(\mathbb{Z}[i])}(A)).
\]
By the previous lemma we have that the number of non-conjugate matrices in $\Gamma$ that become conjugate in $G$ is $\varphi_{\mathbb{Z}[i]}(a)/4$. This proves the theorem.

**Theorem 6.12** The homological Euler characteristic of $\Gamma_1(2, a) \subset GL_2(\mathbb{Z}[i])$ is given by
\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \frac{1}{2} \varphi_{\mathbb{Z}[i]}(a),
\]
where $\varphi_{\mathbb{Z}[i]}(a)$ is the multiplicative function defined on the ideals of $\mathbb{Z}[i]$, generated by
\[
\varphi_{\mathbb{Z}[i]}(p^n) = N_{\mathbb{Q}(i)/\mathbb{Q}}(p^n)(1 - \frac{1}{N_{\mathbb{Q}(i)/\mathbb{Q}}(p)}).
\]

**Proof.** From theorem 6.11 and lemma 4.4 we have
\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \varphi_{\mathbb{Z}[i]}(a)(|N_{\mathbb{Q}(i)/\mathbb{Q}}(R([1,i]))|\chi(C([1,i])) + |N_{\mathbb{Q}(i)/\mathbb{Q}}(R([1,-i]))|\chi(C([1,-i]))) + |N_{\mathbb{Q}(i)/\mathbb{Q}}(R([1,-1]))|\chi(C([1,-1])) =
\]
\[
= \varphi_{\mathbb{Z}[i]}(a)(\frac{1}{8} + \frac{1}{8} + \frac{1}{4}) = \frac{1}{2} \varphi_{\mathbb{Z}[i]}(a).
\]

### 6.3 Homological Euler characteristic of $\Gamma_1(m, a) \subset GL_m(\mathbb{Z}[\xi_3])$

**Lemma 6.13** Let $A$ be an $k$-torsion element of $\Gamma_1(m, a)$. If $6k$ and $a$ are relatively prime then $A$ has an eigenvalue $+1$.

**Proof.** Let $L$ be the field obtained by adjoining all eigenvalues of $A$ to $\mathbb{Q}(\xi_3)$. Let $p$ be a prime ideal dividing $a$. And let $\mathfrak{P}$ be a prime ideal in $L$ sitting above $p$. Denote, also by $\mu(L)$ the roots of 1 in $L$. And let
\[
\pi : L \to \mathbb{F}_q,
\]
where
\[
\mathbb{F}_q = L/\mathfrak{P}.
\]

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Let \( p \) be the rational prime sitting below \( p \). Then \( \pi \) maps \( \mu(L) \) onto \( \mathbb{F}_q \) because there are no \( p \)-roots of 1, in \( L \) since \( p \) does not divide \( k \). So the eigenvalues on \( A \) in \( L \) are mapped onto \( \mathbb{F}_q \). However, 1 is an eigenvalue of \( A \) mod \( p \) in \( \mathbb{F}_q \). Therefore 1 is an eigenvalue of \( A \) in \( L \).

By proposition 2.8 part (b) we know that for a torsion element \( A \), we have \( \chi(C(A)) \neq 0 \) only when the eigenvalues of \( A \) are among \( \{1, -1, \xi_3, -\xi_3, \xi_6, -\xi_6\} \).

**Lemma 6.14** Let \( A \in \Gamma_1(m, a) \) be a torsion element with \( \chi(C(A)) \neq 0 \), let \( a \) be relatively prime with \( (1 - \xi_3) \), and let \( f \) be the characteristic polynomial of \( A \). Then \( +1 \) is a root of \( f \) and it has multiplicity 1. And the set

\[
\Gamma_1(m, N) \backslash N_{\text{GL}_m(\mathbb{Z}[\xi_3])}^G(A) / \text{GL}_m(\mathbb{Z}[\xi_3])(A)
\]

has \( \frac{1}{6} \phi_{\mathbb{Z}[\xi_3]}(a) \) elements, where \( \phi_{\mathbb{Z}[\xi_3]}(a) \) is a multiplicative function on the ideals of \( \mathbb{Z}[\xi_3] \) generated by

\[
\phi_{\mathbb{Z}[\xi_3]}((p))^n = N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p)^n(1 - \frac{1}{N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p)}).
\]

**Proof.** In order to save some space set

\[
\Gamma = \Gamma_1(m, a)
\]

and

\[
G = \text{GL}_m(\mathbb{Z}[\xi_3]).
\]

Let \( X \in N_G^\Gamma(A) \) and let \( B = XAX^{-1} \). Then \( B \in \Gamma \). We can write the matrices \( A, B \) and \( X \) in \( 2 \times 2 \) block form with diagonal blocks \( A_{11}, B_{11} \) and \( X_{11} \) of size \((m - 1) \times (m - 1)\) and \( A_{22}, B_{22} \) and \( X_{22} \) of size \( 1 \times 1 \). Then

\[
\begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix} \cdot \begin{bmatrix}
A_{11} & A_{12} \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
B_{11} & B_{12} \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix} \mod a
\]

Consider the \((2,1)\)-block of the product. The left hand side is

\[
X_{21}A_{11}
\]

and the right hand side is

\[
1 \cdot X_{21} \mod a.
\]

Examine the map

\[
P_{B_{22}A_{11}} : X_{21} \mapsto X_{21}A_{11} - B_{22}X_{21}
\]

with \( B_{22} = 1 \). The eigenvalues of \( A_{11} \) and \( B_{22} \) are different. From lemma 1.8 follows that the eigenvalues of

\[
P_{B_{22}A_{11}}
\]
are $\lambda_i - 1$ where $\lambda_i$ runs through the eigenvalues of $A_{11}$. From the previous lemma we have that $\lambda_i - 1$ is invertible mod $a$. Thus,

$$P_{B_{22}A_{11}}$$

is non-singular mod $a$. Therefore,

$$P_{B_{22}A_{11}}(X_{21}) = X_{21}A_{11} - B_{22}X_{21} \equiv 0 \text{ mod } a$$

implies that

$$X_{21} \equiv 0 \text{ mod } a.$$ 

Therefore,

$$N_G^\Gamma(A) \subset \Gamma_0(m, a),$$

where $\Gamma_0(m, a)$ is defined as the subgroup of $GL_m(\mathbb{Z}[\xi_3])$ that sends $[0, \ldots, 0, 1]$ to $[0, \ldots, 0, \alpha]$ modulo $a$ for any $\alpha \neq 0 \mod a$.

On the other hand, $\Gamma_0(m, a)$ lies inside the normalizer of $\Gamma_1(m, a)$. Also,

$$\Gamma_0(m, a) \subset N_G^\Gamma(A)$$

from the definition of $N_G^\Gamma(A)$. We obtain that

$$N_G^\Gamma(A) = \Gamma_0(m, a).$$

Let $X$ be an element in $\Gamma_0(m, a)$ of the same block form as before. Inside the quotient

$$\Gamma_1(m, a) \backslash \Gamma_0(m, a)$$

the element $X$ is determined by

$$X_{22} \mod a.$$ 

Note that there are $\varphi_{\mathbb{Z}[\xi_3]}(a)$ options for $X_{22}$ because it has to be invertible modulo $a$. From the block-triangular theorem (theorem 2.3) we know that we can choose $A$ so that $A_{21} = 0$. If $C$ is in the centralizer of $A$ inside $GL_m(\mathbb{Z}[\xi_3])$, by lemma 3.2 we know that $C_{21} = 0$. Therefore the centralizer of $A$ modulo $\Gamma_1(m, a)$ is determined by $C_{22}$ which could be $+1$, $-1$, $i$ or $-i$. Therefore the double quotient consist of

$$\frac{1}{4} \varphi_{\mathbb{Z}[\xi_3]}(a)$$

elements.

**Theorem 6.15** Let $V$ be a representation of $GL_m(\mathbb{Z}[\xi_3])$, and let $a$ be an ideal relatively prime to $(1 - \xi_3)$. Then

$$\chi_h(\Gamma_1(m, a), V) = \varphi_{\mathbb{Z}[\xi_3]}(a) \sum_{A = [A_{01}, A_{11}] \subset [A_{00}, 1]} R(A)\chi(C_{GL_{m-1}\mathbb{Z}[\xi_3]}(A_0))\text{Tr}(A^{-1}|V),$$

where

$$A_0 = [A_{11}, \ldots, A_{kk}],$$
$A_{ii}$ vary through
\[\{-1, \xi_3, -\xi_3, \xi_6, -\xi_6\}\]
and $A_{ii}$ and $A_{jj}$ are distinct, $\varphi_{\mathbb{Z}[\xi_3]}(a)$ is the arithmetic function defined on the ideals of $\mathbb{Z}[\xi_3]$ generated by
\[
\varphi_{\mathbb{Z}[\xi_3]}(p^n) = N_{Q(\xi_3)/Q}(p)^k \left(1 - \frac{1}{N_{Q(\xi_3)/Q}(p)}\right).
\]

Proof. Let $A$ be a torsion element of
\[\Gamma = \Gamma_1(m, a)\]
such that
\[\chi(C_T(A)) \neq 0.\]
Then
\[\chi(C_{GL_n,\mathbb{Z}[\xi_3]}(A)) \neq 0.
\]
By lemma 6.9 we have that 1 is an eigenvalue of $A$. Also, if
\[\chi(C_T(A)) \neq 0\]
then the multiplicity of the eigenvalue 1 is at most 1. We are going to prove that
\[\chi(C_T(A)) = 6 \cdot \chi(C_{GL_n,\mathbb{Z}[\xi_3]}(A)),\]
Assume that we have proven this formula. From the generalization of Brown’s formula, we have
\[
\chi_h(\Gamma_1(m, a), V) = \sum_{A: \text{torsion}} \chi(C_{\Gamma_1(m, a)}(A)) \text{Tr}(A^{-1}|V).
\]
In the summation it is enough to sum over torsion elements $A$ that have eigenvalue 1 either with multiplicity 1. First take the sum over elements $A$ that become conjugate to each other in $GL_m(\mathbb{Z}[\xi_3])$. Let $A$ has eigenvalue 1 with multiplicity 1. By lemma 6.9
\[
\sum \chi(C_{\Gamma_1(m, a)}(A'')) \text{Tr}(A''^{-1}|V) = \varphi_{\mathbb{Z}[\xi_3]}(a) \chi(C_{GL_n,\mathbb{Z}[\xi_3]}(A')) \text{Tr}(A'^{-1}|V),
\]
where the sum is taken over all non-conjugate $A''$ in $\Gamma_1(m, a)$ that become conjugate to $A'$ in $GL_n(\mathbb{Z}[\xi_3])$.

We can assume that $A$ is in block-triangular
\[
\begin{bmatrix}
A_{11} & A_{22} \\
0 & 1
\end{bmatrix}
\]
with $A_{22} = 1$ and $A_{21} = 0$. If $C$ is a matrix in $GL_m(\mathbb{Z}[\xi_3])$ commuting with $A$ then $C$ is if the same block form with $C_{21} = 0$ and $C_{22} = i^k$. Exactly one of the matrices $\xi_k^k C, k = 0, 1, \ldots, 5$ belongs to $\Gamma_1(m, a)$ because of the $C_{22}$ entry. Thus the centralizer of $A$ in $GL_m(\mathbb{Z}[\xi_3])$ contains 6 times as many element as the centralizer inside $\Gamma(m, a)$. Therefore,
\[
\chi(C_T(A)) = 6 \cdot \chi(C_{GL_n,\mathbb{Z}[\xi_3]}(A)).
\]
By the previous lemma we have that the number of non-conjugate matrices in $\Gamma$ that become conjugate in $G$ is $\varphi_{\mathbb{Z}[\xi_3]}(a)/6$. This proves the theorem.
Theorem 6.16 The homological Euler characteristic of \( \Gamma_1(2, a) \subset GL_2(\mathbb{Z}[\xi_3]) \) is given by
\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \frac{1}{3} \varphi_{\mathbb{Z}[\xi_3]}(a),
\]
where \( \varphi_{\mathbb{Z}[\xi_3]}(a) \) is the multiplicative function defined on the ideals of \( \mathbb{Z}[\xi_3] \), generated by
\[
\varphi_{\mathbb{Z}[\xi_3]}(p^n) = N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p^n)(1 - \frac{1}{N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}(p)}).
\]

Proof. From theorem 6.15 and lemma 4.5 we have
\[
\chi_h(\Gamma_1(2, a), \mathbb{Q}) = \varphi_{\mathbb{Z}[\xi_3]}(a)\left(\frac{1}{36} + \frac{1}{12} + \frac{1}{9} + \frac{1}{12} + \frac{1}{36}\right) = \frac{1}{3} \varphi_{\mathbb{Z}[\xi_3]}(a).
\]

7 Dedekind zeta function at \(-1\) and \( \chi_h(SL_2(\mathcal{O}_K)) \) for totally real number fields

In this section we examine the homological Euler characteristic of \( SL_2(\mathcal{O}_K) \) for \( K \) a totally real number field. At the end of the section we consider two examples for the fields \( \mathbb{Q} \) and \( \mathbb{Q}(\sqrt{3}) \). The relation to the Dedekind zeta function is
\[
\chi(SL_2(\mathcal{O}_K)) = \zeta_K(-1).
\]

We obtain the following theorem.

Theorem 7.1 Let \( K \) be a totally real number field. Then
\[
\zeta_K(-1) = -\frac{1}{4} \sum_{\xi} \sum_{I \in \text{Cl}(\mathcal{O}_K[\xi]/\mathcal{O}_K)} \#\mathcal{O}_K^\times/N_{\mathcal{O}_K[\xi]/\mathcal{O}_K}(p_I) + \frac{1}{2} N,
\]
where the first summation is taken over all roots \( \xi \) of 1 such that \([K(\xi) : K] = 2\), \( \text{Cl}(\mathcal{O}_K[\xi]/\mathcal{O}_K) \) is the set of ideal classes in \( \mathcal{O}_K[\xi] \) that are free as \( \mathcal{O}_K \)-modules, the ring \( p_I \) is an order inside \( \mathcal{O}_K[\xi] \) that contains \( \mathcal{O}_K[\xi] \) which is isomorphic to the ring of matrices with coefficients in \( \mathcal{O}_K \) that commute with a matrix \( A_I \) corresponding to the ideal class \( I \) (see theorem 1.3). And \( N = \chi_h(SL_2(\mathcal{O}_K)) \).
Proof. From Brown’s theorem we have

\[ N = \chi_h(SL_2(O_K)) = \sum_A \chi(C(A)), \]

where the sum is taken over the torsion elements of \( SL_2(O_K) \) counted up to conjugation. Note that besides \( \pm I_2 \) the rest of the torsion elements have irreducible characteristic polynomial. Their eigenvalues are \( \xi \) and \( \xi^{-1} \), where \( \xi \) is a root of 1 such that \( [K(\xi) : K] = 2 \).

The centralizers of \( I_2 \) and of \( -I_2 \) are the entire group. Therefore the Euler characteristic of their centralizer gives two copies of \( \zeta_K(-1) \).

We need to examine closely the rest of the torsion elements \( A \) in \( SL_2(O_K) \). We have that the eigenvalues of \( A \) are \( \xi \) and \( \xi^{-1} \), where

\[ [K(\xi) : K] = 2. \]

For that reason, in the theorem we sum over such \( \xi \). By proposition 1.7 we have that the non-conjugate matrices with the same characteristic polynomial as \( A \) are parametrized by

\[ Cl(O_K[\xi]/O_K), \]

where the above set denotes the set of ideal classes in \( O_K[\xi] \) that are free as \( O_K \)-modules. For that reason, in the theorem, we are summing over ideal classes from this set. Fix such a torsion matrix \( A \). Let \( I \) be the corresponding ideal class from \( Cl(O_K[\xi]/O_K) \), and \( \xi \) and \( \xi^{-1} \) be its eigenvalues. We are going to define \( R_I \).

Consider the matrices with coefficients in \( K \) that commute with \( A \). Denote their centralizer by \( C_{Mat_2}(A) \). We have that \( C_{Mat_2}(A) \cong K(\xi) \). By sending \( A \) to \( \xi \), this isomorphism \( \psi \) is an isomorphism of \( K \)-algebras. Consider the ring of matrices with coefficients in \( O_K \) that commute with \( A \). Denote it by \( C_{Mat_2}(A) \). Define \( R_I = \psi(C_{Mat_2}(A)) \). Then for the centralizers in \( GL_2(O_K) \) and \( SL_2(O_K) \) we have

\[ R_I^\chi = \psi(C_{GL_2}(A)), \]

and

\[ (R_I^\chi)_{tors} = \psi(C_{SL_2}(A)). \]

The second equality holds because the determinant of a matrix \( B \) in the centralizer \( C_{GL_2}(A) \) corresponds to the norm \( N_{K(\xi)/K}(\beta) \), where \( \beta = \psi(B) \). Thus, \( \det(B) = 1 \) is equivalent to \( N_{K(\xi)/K}(\beta) = 1 \). The last equality holds only when \( \beta \) is a root of 1 because the ranks of the groups of units in \( K \) and in \( K(\xi) \) coincide. In Brown’s formula we need to compute \( \chi(C(A)) \), which we have done

\[ \chi(C_{SL_2}(A)) = \chi((R_I^\chi)_{tors}) = \frac{1}{\#(R_I^\chi)_{tors}}. \]

Proposition 1.3 classifies non-conjugate matrices in \( GL_2(O_K) \). In order to pass to \( SL_2(O_K) \), we use lemma 6.1. It gives that the non-conjugate matrices in \( SL_2(O_K) \) that become conjugate to \( A \) in \( GL_2(O_K) \) are parametrized by

\[ SL_2(O_K) \setminus N_{GL_2(O_K)}(A)/C_{GL_2(O_K)}(A). \]
Note that
\[ N_{SL_2(O_K)}^{GL_2(O_K)}(A) = GL_2(O_K). \]
Thus, the left quotient \( SL_2(O_K)(A) \backslash GL_2(O_K) \) is parametrized by the determinant of the matrices, namely, by \( O_K^\times \). The group by which we quotient from the right is
\[ \text{C}_{GL_2(O_K)}(A) \cong R_I^\times. \]
Its determinant leads to \( N_{K(\xi)/K}(R_I^\times) \). Thus, the double quotient is isomorphic to \( O_K^\times / N_{K(\xi)/K}(R_I^\times) \), which gives the last ingredient in the theorem. Thus, we have
\[ N = 2 \cdot \zeta_K(-1) + \frac{1}{2} \sum_{\xi} \sum_{I \in \text{Cl}(O_K[\xi]/O_K)} \frac{\#O_K^\times / N_{K(\xi)/K}(R_I^\times)}{\#(R_I^\times)_{\text{tors}}}, \]
where the \( 1/2 \) occurs because we are summing over all roots of unity \( \xi \) in
\[ \sum_{\xi} \sum_{I \in \text{Cl}(O_K[\xi]/O_K)}, \]
while when we are summing over the torsion matrices \( A \) we count the eigenvalues \( \xi \) and \( \xi^{-1} \) at the same time.

**Corollary 7.2** Let \( K \) be a totally real number field. Suppose that for each root of 1 \( \xi \) such that \( [K(\xi):K] = 2 \), we have that \( \text{cal}O_K[\xi] \) is integrally closed. Then
\[ \chi_h(SL_2(O_K), \mathbb{Q}) = 2 \zeta_K(-1) + \frac{1}{2} \sum_{\xi} C_{\xi}, \]
where the sum is taken over all roots of 1 \( \xi \) such that \( [K(\xi):K] = 2 \), and
\[ C_{\xi} = \frac{\#\text{Ker}(K_0(O_K(\xi)) \to K_0(O_K)) \#\text{Coker}(K_1(O_K(\xi)) \to K_1(O_K))}{\#\text{Ker}(K_1(O_K(\xi)) \to K_1(O_K))}, \]
where all the maps are norm maps.

**Proof.** If \( R \) is a number ring we have that \( K_0(R) = Cl(R) \oplus \mathbb{Z} \) and \( K_1(R) = R^\times \). If \( O_K[\xi] \) is integrally closed then it coincides with \( O_K(\xi) \). Then the ideal classes in \( O_K[\xi] \) that are free \( O_K \)-modules are precisely
\[ \text{Ker}(K_0(O_K(\xi)) \to K_0(O_K)). \]
Also, the rings \( R_I \) from the previous theorem are precisely \( O_K(\xi) \). Then
\[ \#O_K^\times / N_{K(\xi)/K}(O_K^\times) = \#\text{Coker}(K_1(O_K(\xi)) \to K_1(O_K)), \]
and
\[ \#(O_K^\times)_{\text{tors}} = \#\text{Ker}(K_1(O_K(\xi)) \to K_1(O_K)). \]
From the previous theorem the statement follows.
In the remaining portion of the section we give a simple method for determining what \( R_I \) is. And we end the section with two examples.

Let \( A \) be a matrix in \( SL_2(\mathcal{O}_K) \) that has an irreducible over \( K \) characteristic polynomial. Let \( \xi \) and \( \xi^{-1} \) be its eigenvalues. By theorem 1.3 we have a correspondence between the ideal classes in \( \mathcal{O}_K[\xi] \) that are free as \( \mathcal{O}_K \)-modules. We give a way of determining what is the order \( R_I \). Let \( \psi : C_{Mat_{2,2}O_K}(A) \to K(\xi) \)

that send \( A \) to \( \xi \). Then \( R_I \) is defined by

\[
R_I = \psi(C_{Mat_{2,2}O_K}(A)).
\]

Every element \( \lambda \) in \( \mathcal{O}_{K(\xi)} \) can be written as a linear function \( f(t) \) with coefficients in \( K \), so that \( \lambda = f(\xi) \).

The procedure of determining \( R_I \) is the following: Let \( \lambda \in \mathcal{O}_{K(\xi)} \) and let \( \lambda = f(\xi) \) where \( f(t) \) is a linear polynomial with coefficients in \( K \). Then \( \lambda \in R_I \) if and only if \( f(A) = \psi^{-1}(\lambda) \) has coefficients in \( \mathcal{O}_K \).

**Example 7.3** Let \( K = \mathbb{Q} \). The root \( \xi \) of 1 that give a quadratic extensions are \( i^{\pm 1}, \xi_3^{\pm 1} \) and \( \xi_6^{\pm 1} \). For each of the extensions \( Z[\xi] \) is integrally closed. Also ideal class in all the cases are trivial. Thus, for each \( \xi \) we have only one \( R_I \) and \( R_I = \mathbb{Z}[\xi] \). Then

\[
\#\mathbb{Z}^*/N_{\mathbb{Q}(\xi)/\mathbb{Q}}(R_I^*) = 2.
\]

The torsion elements in \( \mathbb{Z}[\xi] \) are 4 if \( \xi = i^{\pm 1} \) and 6 if \( \xi = \pm \xi_3^{\pm 1} \). Then

\[
\chi_h(SL_2\mathbb{Z}, \mathbb{Q}) = 2 \cdot \zeta(-1) + \frac{1}{2}(2 \cdot \frac{2}{4} + 2 \cdot \frac{2}{6} + 2 \cdot \frac{2}{6}) = 2 \cdot \zeta(-1) + \frac{7}{6}.
\]

Since

\[
\zeta(-1) = \frac{-1}{12},
\]

we obtain

\[
\chi_h(SL_2\mathbb{Z}) = 1.
\]

**Example 7.4** Let \( K = \mathbb{Q}(\sqrt{5}) \). Then \( \xi \) can be 3-rd, 6-th, 4-th, 5-th or 10-th root of unity.

If \( \xi = \xi_{10}^k \) for \( k = 1, \ldots, 4, 6, \ldots, 9 \) then

\[
\mathcal{O}_K[\xi] = \mathbb{Z}[\xi_{10}],
\]

which is integrally closed with class number 1 (see [W] theorem 11.1). Let \( N = N_{\mathbb{Q}(\xi_{10})/\mathbb{Q}(\sqrt{5})} \) be the norm map. And let \( \xi = \xi_{10} \). We want to find the image of the norm map \( N \) on the units. This leads to solving

\[
N(\alpha) = \frac{1 + \sqrt{5}}{2}.
\]
We have
\[ N(a + b\xi + c\xi^2 + d\xi^3) = \]
\[ = (c - a/2)^2 + (b - d/2)^2 + \frac{3}{4}(a + 2d/3)^2 + \frac{5}{12}d^2 + (\frac{1 + \sqrt{5}}{2}) \]

If the norm is \((1 + \sqrt{5})/2\) then all the constants \(a, b, c,\) and \(d\) must be zero. Therefore \((1 + \sqrt{5})/2\) is not in the image of the norm map and the cokernel of \(N\) is

\[ Z[(1 + \sqrt{5})/2]^\times /N(Z[\xi_{10}]^\times) = C_2 \times C_2 \]

The kernel of \(N\) is

\[ Z[\xi_{10}]_{tors}^\times = C_{10} \]

Thus the contribution towards the homological Euler characteristic is

\[ \frac{1}{2} \cdot \frac{8}{4} \cdot \frac{4}{10} = \frac{8}{5} \]

If \(\xi = \xi_6^k\) for \(k = 1, 2, 4, 5\) then

\[ \mathcal{O}_K[\xi] = Z[(1 + \sqrt{5})/2, (1 + \sqrt{-3})/2], \]

which is integrally closed. We are going to show that it has class number 1. First, we need to examine the units in the ring. Let \(N\) be the norm map from \(\mathbb{Q}(\sqrt{5}, \sqrt{-3})\) to \(\mathbb{Q}(\sqrt{5})\). We try to solve

\[ N(\alpha) = \frac{1 + \sqrt{5}}{2}. \]

We have

\[ N(a + b\frac{1 + \sqrt{-3}}{2} + c\frac{1 + \sqrt{5}}{2} + d\frac{1 + \sqrt{5}}{2} \cdot \frac{1 + \sqrt{-3}}{2}) = \]

\[ = a^2 + ab + b^2 + c^2 + cd + d^2 + (\frac{1 + \sqrt{5}}{2}). \]

Thus

\[ a^2 + ab + b^2 + c^2 + cd + d^2 = 0. \]

It can happen only when \(a = b = c = d = 0\). Thus, the cokernel of \(N\) is isomorphic to \(C_2 \times C_2\). Also the regulator for the field \(\mathbb{Q}(\sqrt{5}, \sqrt{-3})\) is

\[ R = \log((1 + \sqrt{5})/2). \]

We are going to use the formula for the leading coefficient of the Dedekind zeta function at 1. It is given by

\[ \frac{2^{r_1} (2\pi)^{r_2} h R}{w \sqrt{|d|}}, \]

where \(r_1\) and \(r_2\) are the number of the real and the complex imbeddings if the field \(h\) is the class number, \(R\) is the regulator, \(w\) is the number of roots of unity in the field, and \(d\) is the discriminant of the field.
The Dedekind zeta function of \( \mathbb{Q}(\sqrt{5}, \sqrt{-3}) \) decomposes into a product of the Riemann zeta function and three \( L \)-functions corresponding to the quadratic extensions \( \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-3}) \) and \( \mathbb{Q}(\sqrt{-15}) \). The first two of the quadratic field have class number 1 and the last one class number 2. Let \( h \) be the class number of \( \mathbb{Q}(\sqrt{5}, \sqrt{-3}) \).

Then
\[
\frac{(2\pi)^2 h \log((1 + \sqrt{5})/2)}{6 \cdot 15} = \frac{2^4 \log((1 + \sqrt{5})/2)}{2 \cdot \sqrt{5}} \cdot \frac{(2\pi)}{6 \cdot \sqrt{3}} \cdot \frac{(2\pi)}{2 \cdot \sqrt{15}},
\]
which gives \( h = 1 \). The cokernel of \( N \) acting on the units is isomorphic to \( C_2 \times C_2 \) and the kernel is isomorphic to \( C_6 \). Thus the contribution to the homological Euler characteristic is
\[
\frac{1}{2} \cdot 4 \cdot \frac{4}{6} = \frac{4}{3}.
\]

If \( \xi = \pm i \) then \( \mathcal{O}_K[\xi] = \mathbb{Z}[(1 + \sqrt{5}, i)] \), which is integrally closed and of class number 1. Let \( N \) be the norm map from \( \mathbb{Q}(\sqrt{5}, i) \) to \( \mathbb{Q}(\sqrt{5}) \). We want to solve
\[
N(\alpha) = \frac{1 + \sqrt{5}}{2}.
\]

We have
\[
N(a + ib + c\frac{1 + \sqrt{5}}{2} + di\frac{1 + \sqrt{5}}{2}) =
\]
\[
= a^2 + b^2 + c^2 + d^2 + (\frac{1 + \sqrt{5}}{2})
\]
Thus,
\[
a^2 + b^2 + c^2 + d^2 = 0.
\]
Therefore \( (1 + \sqrt{5})/2 \) is not in the image of the norm map. Thus, the cokernel of \( N \) acting on the units in \( \mathbb{Q}(\sqrt{5}, i) \) is isomorphic to \( C_2 \times C_2 \). And the kernel is isomorphic to \( C_4 \). Then the contribution to the homological Euler characteristic is
\[
\frac{1}{2} \cdot 2 \cdot \frac{4}{4} = 1.
\]

Finally,
\[
\chi_h(SL_2(\mathbb{Z}[(1 + \sqrt{5})/2])) = 2\zeta_{\mathbb{Q}(\sqrt{5})}(-1) + \frac{8}{5} + \frac{4}{3} + 1 = 2\zeta_{\mathbb{Q}(\sqrt{5})}(-1) + \frac{14}{15}.
\]

We obtain that
\[
\zeta_{\mathbb{Q}(\sqrt{5})}(-1) = \frac{1}{30} + \frac{k}{2},
\]
for some integer \( k \). In fact
\[
\zeta_{\mathbb{Q}(\sqrt{5})}(-1) = \frac{1}{30},
\]
It follows from \([I]\) theorem 1, which expresses values of \( L \)-functions in terms of generalized Bernoulli numbers. Therefore
\[
\chi_h(SL_2(\mathbb{Z}[(1 + \sqrt{5})/2]), \mathbb{Q}) = 4.
\]

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8 Generalization of Brown’s formula

The main result of this section is to prove a generalization of Brown’s formula relating the homological Euler characteristic to the Euler characteristics of certain centralizers. Given an arithmetic group $\Gamma$ and a finite dimensional representation $V$ over a field $K$ of characteristic zero. We prove the following theorem.

**Theorem 8.1** The homological Euler characteristic of $\Gamma$ with coefficients in $V$ is given by

$$\chi_h(\Gamma, V) = \sum \chi(C(A)) \cdot \text{Tr}(A^{-1}|V),$$

where the sum is taken over all torsion elements of $\Gamma$ counted up to conjugation, $C(A)$ is the centralizer of $A$ inside $\Gamma$ and $\text{Tr}(A^{-1}|V)$ is the trace of the action of $A^{-1}$ on the finite dimensional vector space $V$.

Before we start with the proof we recall what is complete Euler characteristic, and what are some of its properties. For more detailed treatment of generalized Euler characteristic we refer to Bass’ article [Ba] and Brown’s book [B1].

Let $R$ be a unital non necessarily commutative ring. Let $F$ be a free $R$-module. And let

$$f : F \to F$$

be an endomorphism of $R$-modules. We can choose a basis for $F$ and then $f$ can be written as a matrix $(a_{ij})$. We define the trace of $f$ to be

$$\text{Tr}_R(f) = \sum_i \bar{a}_{ii},$$

where $\bar{a}$ is the projection of $a$ from $R$ to $T(R) = R/[R,R]$, and $[R,R]$ is the additive group generated by $ab - ba$ for $a,b \in R$. Note that $[R,R]$ is not an ideal so $T(R)$ is only an additive group.

The trace can be extended to endomorphisms of a finitely generated projective module $P$. Such a module can be imbedded in a free module of finite rank $F$. Denote by $i$ the imbedding, and by $\pi$ the projection from $F$ to $P$. The the trace of

$$f : P \to P$$

if defined by

$$\text{Tr}_R(f) := \text{Tr}_R(i \circ f \circ \pi).$$

Note that $i \circ f \circ \pi$ is an endomorphism of $F$.

The trace can be also extended to modules $M$ that admit finite length projective resolution by finitely generated projective modules. Then an endomorphism

$$f : M \to M$$

extends to an endomorphism of its projective resolution $P_*$

$$f_i : P_i \to P_i.$$
Then the trace of \( f \) is defined by
\[
\text{Tr}_R(f) := \sum_i (-1)^i \text{Tr}_R(f_i).
\]

Now setting \( R \) to be the group ring \( \mathbb{Q}\Gamma \) the complete Euler characteristic is defined by the trace of \( \text{id}_\mathbb{Q} : \mathbb{Q} \to \mathbb{Q} \), namely,
\[
E(\Gamma) := \text{Tr}_{\mathbb{Q}\Gamma}(\text{id}_\mathbb{Q}).
\]

Note that the complete Euler characteristic takes values in
\[
T(\mathbb{Q}\Gamma) = \mathbb{Q}\Gamma/\mathbb{Q}\Gamma = \{ \sum_{(A)} c(A) \cdot (A) \},
\]
where \((A)\) denotes the conjugacy class of \( A \in \Gamma \).

Let
\[
E(\Gamma) = \sum_{(A)} c(A) \cdot (A).
\]

Let \( \Gamma \) be an arithmetic group. The relation between \( E(\Gamma) \), \( \chi(\Gamma) \) and \( \chi_h(\Gamma) \) is the following
\[
\chi(\Gamma) = c(I),
\]
\[
\chi_h(\Gamma) = \sum_{(A)} c(A),
\]
where the sum is taken over all conjugacy classes. In fact this sum is finite which follows from Brown’s theorem

**Theorem 8.2** If \( \Gamma \) is an arithmetic group and
\[
E(\Gamma) = \sum_{(A)} c(A) \cdot (A).
\]

Then
\[
a_{(A)} = \chi(C(A)).
\]

We generalize the complete Euler characteristic in the following way. For a finite dimensional representation \( V \) of \( \Gamma \), let
\[
E(\Gamma, V) = \text{Tr}_{\mathbb{Q}\Gamma}(\text{id}_V).
\]

In order to pass to homological Euler characteristic we use a lemma due to Chiswell (see [Ch], lemma 12)

**Lemma 8.3** If
\[
E(\Gamma, V) = \sum_{(A)} c(A)\cdot (A)
\]

then
\[
\chi_h(\Gamma, V) = \sum_{(A)} c(A).
\]
Proof. Let

\[ \ldots \to P_1 \to P_0 \to V \to 0 \]

be a projective resolution of \( V \) of finite length by finitely generated projective \( \mathbb{Q}\Gamma \)-modules. Then

\[ E(\Gamma, V) = \sum_i (-1)^i E(\Gamma, P_i). \]

tensor the projective resolution with \( \mathbb{Q} \otimes_{\mathbb{Q}\Gamma} \). Then the augmentation map \( \mathbb{Q}\Gamma \to \mathbb{Q} \) induces a map between \( T(\mathbb{Q}\Gamma) \) and \( T(\mathbb{Q}) \). This map sends

\[ E(\Gamma, V) = \text{Tr}_{\mathbb{Q}\Gamma}(id_V) \]

to

\[ \text{Tr}_{\mathbb{Q}}(id_{\mathbb{Q}\otimes_{\mathbb{Q}\Gamma} V}) = \sum_i (-1)^i \text{Tr}(id_{\mathbb{Q}\otimes_{\mathbb{Q}\Gamma} P_i}) = \sum_i (-1)^i H^i(\Gamma, V) \]

The augmentation map induced on \( T(\mathbb{Q}\Gamma) \) sends \( \sum (A) c(A)(A) \) to \( \sum (A) c(A) \). And that proves the lemma.

We are going to use another lemma by Chiswell (see [Ch] lemma 2).

**Lemma 8.4** If \( f : R_1 \to R_2 \) is a unital ring homomorphism. Then \( f \) induces a map \( f : T(R_1) \to T(R_2) \). If \( V \) is a finitely generated projective \( R_1 \)-module then

\[ f(\text{Tr}(id_V)) = \text{Tr}(id_{(R_2 \otimes R_1) V}). \]

Now we proceed with the proof of theorem 9.1.

**Proof.** (of theorem 9.1) Let \( \Gamma \) be an arithmetic group, and let \( V \) be a finite dimensional representation of \( \Gamma \) over a field \( K \) of characteristic 0. In stead of the group ring \( \mathbb{Q}\Gamma \) we shall consider a much bigger ring \( R \). We are going to compute traces with respect to this ring \( R \). And at the very end we shall compare the final result to traces over the group ring \( \mathbb{Q}\Gamma \).

Let

\[ R_0 = \left\{ \sum_A c_A \cdot A | c_a \in K, A \in \Gamma \right\}, \]

where sum over infinitely many elements \( A \) is allowed, and \( K \) is a field of characteristic zero. We define formal variables that correspond to infinite summation. Let

\[ x = \sum_{A \in \Gamma} 1. \]

Denote by \((A)\) the conjugacy class of \( A \). Denote by

\[ x_{(A)} = \sum_{B \in (A)} 1. \]

Let \( x \) and \( x_{(A)} \) be variables that commute with the elements of \( R_0 \). Let

\[ R := R_0[x, x^{-1}, x_{(A)}, \ldots] \]

be the ring obtained from \( R_0 \) by adjoining central variables \( x, \) \( x^{-1} \) and \( x_{(A)} \) for all conjugacy classes \((A)\). Note that we allow to use only finitely many of these formulas
at a time. Using infinitely many of them would lead to a contradiction. Another remark is that if any of the summation is finite, we can use the finite number. Let

\[ V_R = R \otimes_{K \Gamma} V. \]

Let

\[ F = R \otimes_K V. \]

We are going to define an inclusion

\[ i: V_R \to F, \]

and a projection

\[ \pi: F \to V_R. \]

Let \( v \) be an element of \( V \). We define

\[ i(v) = \frac{1}{x} \sum_{A \in \Gamma} A \cdot (A^{-1} v), \]

And extend it by \( R \)-linearity to \( V_R \). Let

\[ \pi(r \otimes v) = rv, \]

where \( r \in R \) and \( v \in V \) Then

\[ \pi \circ i = id_{V_R}. \]

Note that \( F \) is a free \( R \)-module of finite rank and \( V_R \) is a projective module. By the definition of the trace, we have

\[ \text{Tr}_R(id_{V_R}) = \text{Tr}_R(i \circ \pi) = \sum_{(A)} \frac{x(A)}{x} \text{Tr}(A^{-1}|V) \cdot (A). \]

In order to find the right regularization of \( x(A)/x \), we compare this result to the the Brown formula for the trivial representation. We have a natural inclusion

\[ \mathbb{Q} \Gamma \to R. \]

Then this inclusion induces equality between \( \text{Tr}_{\mathbb{Q}\Gamma}(id_{\mathbb{Q}}) \) and \( \text{Tr}_R(id_{\mathbb{Q}R}) \). Using Brown’s formula, we obtain that

\[ \frac{x(A)}{x} = \chi(C(A)). \]

We also have that \( \chi(C(A)) = 0 \) for non-torsion elements \( A \). Thus, the formula for the trace over \( R \) leads to

\[ E(\Gamma, V) = \sum_{A: \text{torsion}} \chi(C(A)) \text{Tr}(A^{-1}|V) \cdot (A). \]

Using Chiswell’s lemma we obtain the formula for the homological Euler characteristic.
Proof. (second proof of theorem 8.1) This proof follows more closely the paper of Brown [B2]. And it turns out that the formula we need is a consequence of a more general formula of Brown. For a finite length chain $C^\bullet$ of $\mathbb{Q}\Gamma$-modules that admit generalized Euler characteristic, define

$$E(\Gamma, C^\bullet) = \sum_i (-1)^i E(\Gamma, C^i).$$

Let

$$E(\Gamma, C^\bullet) = \sum_{(A)} c(A) \cdot (A).$$

Define

$$e(\Gamma, C^\bullet) = c(I),$$

where $I$ is the identity element in $\Gamma$. We have that

$$e(\Gamma, \mathbb{Q}) = \chi(\Gamma).$$

The general formula that Brown obtains is that $c(A)$ coincides with the coefficient next to $(I)$ in $\text{Tr}_{\mathbb{Q}\mathbb{F}}(M_A)$, where $M_A : (C^\bullet)^A \rightarrow (C^\bullet)^A$ is multiplication by $A$. Now, let $X$ be a contractable CW complex on which $\Gamma$ act properly and discontinuously. One can assume also that a cell is mapped to itself by an element of $\Gamma$ then the entire cell is fixed by that element pointwise. One can achieve that by subdivision of the cells. Then

$$E(\Gamma, V) = E(\Gamma, C^\bullet(X, V)),$$

where $C^\bullet(X, V)$ is the cochain complex of $X$ with coefficients in $V$. Let $f \in C^i(X, V)$ and $A \in \Gamma$, let also $\sigma$ be an $i$-th cell and $v \in V$ so that $f(\sigma) = v$. Then $A$ acts on $f$ by $(A \cdot f)(A\sigma) = A \cdot v$. Since we want to find the trace we consider the fixed cells under the action of $A$. Consider a the following basis for $C^i(X, V)$. Let $v_1, \ldots, v_n$ be a basis for $V$. Let $f_{\sigma, v_i}$ be the function that sends the cell $\sigma$ to the vector $v_i$, and sends all other cell to zero. Consider the contribution of $f_{\sigma, v_i}$ to the trace. We have that $A \cdot f_{\sigma, v_i}$ maps $\sigma$ to $a_{ii} \cdot v_i$, where $a_{ii}$ is a constant. But $A \cdot f_{\sigma, v_i}$ maps $A\sigma$ to $Av_i$. Thus, we have a contribution to the trace if

$$A\sigma = \sigma.$$

Consider the space $X^A$ of cells which are fixed under the action of $A$. On $X^A$ the centralizer $C(A)$ acts properly and discontinuously. Also $X^A$ is contractable. Thus

$$c(A) = e(C(A), C(X^A, V)).$$

We have

$$(A \cdot f)(\sigma) = A^{-1}(f(A\sigma)) = A^{-1}(f(\sigma)).$$

When we take the trace the last equation leads to $\text{Tr}(A^{-1}|V)$. Thus, we obtain

$$c(A) = e(C(A), C^\bullet(X^A)) \cdot \text{Tr}(A^{-1}|V) = \chi(C(A)) \cdot \text{Tr}(A^{-1}|V).$$
References

[Ba] Bass, H.: Euler Characteristic and Characters of Discrete Groups, Inventiones math. 35, 155-196 (1976).

[B1] Brown, K.: Cohomology of Groups, Graduate Text in Mathematics, Springer-Verlag: New York, 1982.

[B2] Brown, K.: Complete Euler characteristics and fixed-point theory, J. Pure Appl. Algebra 24 (1982), 103-121.

[Ch] Chiswell, I.: Euler characteristics of groups, Math. Z. 147, 1-11 (1976).

[G1] Goncharov, A.: The dihedral Lie algebras and the Galois symmetries of $\pi_1(P^1 - \{0, \infty\} \cup \mu_n)$, Duke Math. J. vol. 110, No. 3 (2001), 397-487.

[H] Harder, G.: A Gauss-Bonnet formula for discrete arithmetically defined groups, Ann. Sci. École Norm. Sup. (4) 4 (1971), 409-455.

[I] Iwasawa, K.: Lectures on p-adic L-functions, Annals of Math Studies no, 74. Princeton Univ. Press: Princeton, N.J., 1972.

[M] Milnor, J.: Introduction to algebraic K-theory Ann. of Math Studies 72, Princeton University Press, Princeton, 1971.

[S] Serre, J.-P.: Cohomologie des groupes discretes, Ann. of Math. Studies 70 (1971), 77-169.

[W] Washington, L.: Introduction to cyclotomic fields Graduate Text in Mathematics, Springer-Verlag: New York, 1982.