Renormalization of NN scattering: contact potential

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The renormalization of the $T$-matrix for $NN$ scattering with a contact potential is reexamined in a nonperturbative regime through rigorous nonperturbative solutions. Based on the underlying theory, it is shown that the ultraviolet divergences in the nonperturbative solutions of the $T$-matrix should be subtracted through "endogenous" counter terms, which in turn leads to a nontrivial prescription dependence. Moreover, employing the effective range expansion, the importance of imposing physical boundary conditions to remove the nontrivial prescription dependence, especially before making physical claims, is discussed and highlighted. As byproducts, some relations between the effective range expansion parameters are derived. We also discuss the power counting of the couplings for the nucleon-nucleon interactions and other subtle points present within EFT framework beyond perturbative treatment.

INTRODUCTION

The effective field theory (EFT) method or strategy [1] has become a primary tool for studying a variety of low energy problems in particle and nuclear physics; important examples include chiral perturbation theory ($\chi$PT)[2], heavy quark effective theory (HQET)[3], and non-relativistic quantum chromodynamics (NRQCD)[4]. As an EFT parametrizes the short distance physics in a simple way, severe ultraviolet (UV) divergences appear. Then one must carefully work out pertinent power counting rules and renormalization prescriptions[5]. In nonperturbative regimes (for example, in the application of $\chi$PT to low energy nucleon systems as is advocated by Weinberg[6]), the 'interplay' between power counting schemes and renormalization prescriptions becomes quite complicated[7]. To establish a more reasonable and consistent framework, many proposals have been put forward[7, 8, 9, 10], creating controversies still to be settled. There were once (perhaps are) even doubts about the applicability of the EFT method.

The main difficulties stem from a distinct feature of the nonperturbative formulation, which invalidates the naive use of the perturbative renormalization (through subtraction) programs[11, 12]. In this report, we continue our investigations of the renormalization of the EFT for nucleon-nucleon scattering in the nonperturbative regimes, which we started in Ref.[11]. We work here with a contact potential that allows us to rigorously obtain a closed form of the $T$-matrix through the use of the Lippmann-Schwinger equation (LSE)[13]. In this way, many new features arising in the nonperturbative regime can be explicitly illustrated. In our approach, we utilize the underlying theory to understand the renormalization of an EFT. In this work we will examine the renormalization prescription dependence of the on-shell $T$-matrix together with the observables or parameters coming from the low energy theorems for nucleon-nucleon scattering. The latter are obtained through the effective range expansion[14]. This paper is organized as follows: In Sec. II, we sketch the nonperturbative parametrization of the $T$-matrix proposed in Ref.[11] and its implications for nonperturbative renormalization. In Sec. III, we employ the algebraic method described in Ref.[13] to obtain a rigorous closed form solution of the LSE in the case of contact nucleon-nucleon interactions. Then the regularization and renormalization of the $T$-matrix in the nonperturbative regime are analyzed using the closed form solutions at various chiral orders. Both the prescription dependence and its removal from the observables or parameters obtained via the effective range expansion (low energy theorems) are investigated in Sec. IV. In Sec. V, we study the interplay between the renormalization prescription and the power counting schemes for EFT, and the renormalization group evolution in the nonperturbative regime and the naturalness of the $T$-matrix are also explored. Finally, Sec. VI contains our summary. Our main conclusion is that one should be aware of the nontrivial renormalization prescription dependence in the nonperturbative regime, with emphasis on physical boundaries.
A COMPACT PARAMETRIZATION

Let us start with a standard parametrization for the on-shell partial wave \( T \)-matrix (we consider the diagonal channels for simplicity):

\[
T_{l\text{os}}(p) = -\frac{4\pi}{M_p \cot \delta_l(p)} - ip, \tag{1}
\]

with \( M \) and \( p \) being, respectively, the mass and on-shell momentum of a nucleon, and \( l \) denoting the angular momentum number. Literally, the potential could be systematically constructed or calculated using \( \chi \)PT through counting the powers in terms of \( p^2/\Lambda^2 \) or \( m^2_\pi/\Lambda^2 \) with \( \Lambda \) being the high scale or upper limit for the EFT under consideration (\( \Lambda \sim 500\text{MeV} \)). Then the off-shell \( T \)-matrix for partial wave \( l \) could be found through the solution of LSE,

\[
T_l(p', p; E) = V_l(p', p) + \int \frac{dk k^2}{(2\pi)^2} V_l(p', k) G_0(k; E^+) T_l(k, p; E), \tag{2}
\]

\[
G_0(k; E^+) \equiv \frac{1}{E^+ - k^2/M}, \tag{3}
\]

where \( E^+ \equiv E + i\epsilon \) with \( E \) being the center of mass energy. It should be noted, that Eq.(2) is ill defined, as the potential \( V_l \) calculated using \( \chi \)PT is usually singular.

To highlight the nonperturbative features present in this problem, we previously proposed a tentative nonperturbative parametrization for the \( T \)-matrix based on LSE

\[
\frac{1}{T_l(p, p'; E)} = \frac{1}{V_l(p, p')} - G_l(p, p'; E), \tag{4}
\]

\[
G_l(p, p'; E) \equiv \int \frac{dk k^2}{(2\pi)^2} V_l(p', k) G_0(k; E^+) T_l(k, p; E) / V_l(p, p') T_l(p, p'; E), \tag{5}
\]

where \( G_l \) carries the nonperturbative information of all the quantum processes (all the loop amplitudes in the field theoretic terminology) generated by \( V_l \). We have already shown in [11] that the nonperturbative quantity \( G_l \) could not be renormalized through the introduction of "exogenous" counter terms in the potential [12]. That means \( G_l \), or equivalently \( T_l \), is regularization and renormalization (R/R) prescription dependent. There can be only one renormalization prescription (contrary to perturbative treatment [16]) consistent with physical data or boundary conditions. In the following sections we will demonstrate this point with rigorous solutions of LSE.

CONTACT POTENTIAL: RIGOROUS SOLUTIONS

The specific case of contact potential allows us to transform the integral equation into the algebraic one following Ref.[13]. In other words, the nonlocal pion exchange contributions to the potential are neglected. However, the main conclusions remain qualitatively valid also when pion exchanges are included.

Factorized LSE and its algebraic solutions

Next, to illustrate the role played by the nonperturbative features, we will employ the \( ^1S_0 \) channel. To see how the situation evolves with the chiral orders, we consider the solution of LSE with the following local potentials at three different chiral orders (\( \Delta = 0, 2, 4 \)) (leading order, next-to-leading order, next-to-next-to-leading order):

\[
\Delta = 0 : V^{^1S_0}_{(0)} = C_0; \tag{6}
\]

\[
\Delta = 2 : V^{^1S_0}_{(2)} = C_0 + C_2 (p^2 + p'^2); \tag{7}
\]

\[
\Delta = 4 : V^{^1S_0}_{(4)} = C_0 + C_2 (p^2 + p'^2) + \tilde{C}_4 p^2 p'^2 + C_4 (p^4 + p'^4). \tag{8}
\]
Following Ref. [13], we "factorize" the potentials into matrices: \( V = U^T \lambda U \) with \( U \), and \( U' \) being column vectors and \( \lambda \) an \( n \times n \) matrix. At next-to-next-to-leading order (\( \Delta = 4 \)), they are

\[
\lambda \equiv \begin{pmatrix}
C_0 & C_2 & C_4 \\
C_2 & C_4 & 0 \\
C_4 & 0 & 0
\end{pmatrix}, \quad U \equiv (1, p^2, p^4), \quad U' \equiv (1, p^2, p^4); \quad (9)
\]

whereas at next-to-leading order (\( \Delta = 2 \)) they read,

\[
\lambda \equiv \begin{pmatrix}
C_0 & C_2 \\
C_2 & 0
\end{pmatrix}, \quad U \equiv (1, p^2), \quad U' \equiv (1, p^2). \quad (10)
\]

The off-shell \( T \)-matrix factorizes exactly in the same manner: \( T = U^T \tau U' \), where \( \tau \) is an \( n \times n \) matrix. Generally, the coupling constants \( [C_n] \) come from the chiral expansion of an underlying theory (say, QCD) for nucleons and pions in terms of \( p^2/\Lambda^2 \), and hence they scale like \( C_{2n}/C_0 \sim \Lambda^{-2n} \) in the naive power counting scheme.

Using this notation, the LSE can be reduced to the following algebraic equation [13],

\[
\tau(E^+) = \lambda + \lambda \circ \mathcal{I}(E^+) \circ \tau(E^+) \quad (11)
\]

with \( \circ \) denoting the matrix multiplication. For the 3 \( \times \) 3 case, the matrix \( \mathcal{I} \) and the related parametrizations and definitions are listed in Appendix A. With this algebraic parametrization, all the ill defined integrals can be isolated and parametrized in any regularization prescription. The solution to this algebraic equation is easy to obtain:

\[
\tau(E^+) = (1 - \lambda \circ \mathcal{I}(E^+))^{-1} \circ \lambda. \quad (12)
\]

In a similar fashion, the solution of the \( T \)-matrix can be obtained. For the three chiral orders considered so far, they read (on-shell),

\[
\Delta = 0: \quad \frac{1}{T_{os}(p)} = \frac{1}{C_0} + J_0 + \frac{M}{4\pi} ip; \quad (13)
\]

\[
\Delta = 2: \quad \frac{1}{T_{os}(p)} = \frac{(1 - C_2 J_3)^2}{C_0 + C_2^2 J_5 + C_2(2 - C_2 J_3)p^2} + J_0 + \frac{M}{4\pi} ip; \quad (14)
\]

\[
\Delta = 4: \quad \frac{1}{T_{os}(p)} = \frac{N_0 + N_1 p^2 + N_2 p^4}{D_0 + D_1 p^2 + D_2 p^4 + D_3 p^6} + J_0 + \frac{M}{4\pi} ip, \quad (15)
\]

with the ill defined integrals \( [J_\alpha] \) and the coefficients \( [N_n] \) and \( [D_n] \) being defined in Appendix B. The next-to-leading order has been considered in Ref. [13], whereas the \( \Delta = 4 \) result has not been given before. We should note that \( J_0 \) always stands "alone" in the real part of the inverse on-shell \( T \)-matrix. (A rigorous proof of this fact at any order is given in Appendix C.) Here we would like to stress the compact or closed form of the expressions for the \( T \)-matrix in terms of the couplings and the integrals \( [J_n] \). It is this crucial property that distinguishes the nonperturbative solutions from the perturbative ones and complicates the renormalization.

Obviously, the \( T \)-matrix becomes more complicated as higher order interactions are included. Nevertheless, when the nonlocal pion exchanges are included, one can naturally anticipate that the nonperturbative solutions still take compact or closed forms. Thus we expect, that, our conclusions here will also hold in the realistic potentials with nonlocal pion contributions, at least qualitatively.

**Failure of 'exogenous' counter-term renormalization**

Now let us consider the renormalization of the \( T \)-matrix. The leading order case is trivial; one can absorb the only divergence in \( J_0 \) into the inverse coupling: \( 1/C_0 \), similar to the perturbative cases.

However, in the presence of higher order interactions, such operation may not work. For example, at next-to-leading order, in order to renormalize the on-shell \( T \)-matrix in Eq. (14), or to make the fraction
extraction of the EFT vertices or couplings from the UT diagrams. For convenience, let us introduce a projection operator \( \hat{C} \). Moreover, no matter what was done for \( \hat{C}_{1} \) and \( \hat{C}_{2} \), and the three divergent integrals, \( J_{0}, J_{3} \) and \( J_{5} \). The situation differs strikingly from the perturbative case where counter terms are introduced order by order with the higher order terms discarded, as no compact or 'closed' form of expression is involved. To see the first point, consider the diagrams shown in Figs.1 and 2, with the heavy meson exchange diagrams (with \( C_{0} \), \( C_{2} \), \( C_{3} \)) underlying the one with contact interactions. For simplicity, denote \( \hat{C}_{0} \equiv \hat{C}_{2} \equiv \hat{C}_{3} \equiv 0 \). Consider the diagrams in Fig.1, if \( \hat{C}_{0} \) is applied after the loop integration \( \int \frac{d^{4}l}{(2\pi)^{4}} \) has been done \( (\text{the incorrect order}) \), the divergent bubble diagram results. Thus the new divergences generally arise from the incorrect order of computations, as the following commutator does not vanish identically:

\[
\hat{O}_{\text{c.t.}} \equiv \hat{C}_{0} \int \frac{d^{4}l}{(2\pi)^{4}} \neq 0.
\]

Embarrassingly, one has to use EFT either because UT is unavailable or because the calculations in UT are tedious. Combining this procedure with nonperturbative context (infinite iteration or resummation) makes things even worse: the counter terms could not be implemented exogenously.

However, from the underlying theory the solution follows immediately: one should devise some procedures to effectively "recover" the correct order for \( \hat{C}_{0} \) before anything else is done. The clue lies in Eq. (18). Through rearrangement, Eq. (18) is equivalent to the following equation for the integrand of a loop diagram (say, the box diagram integrand \( f_{\text{box}} \)),

\[
\hat{P}_{\text{LE}} \int \frac{d^{4}l}{(2\pi)^{4}} f_{\text{box}} = \int \frac{d^{4}l}{(2\pi)^{4}} \hat{P}_{\text{LE}} f_{\text{box}} + \hat{O}_{\text{c.t.}} f_{\text{box}} = \int \frac{d^{4}l}{(2\pi)^{4}} f_{\text{bubble}} + \hat{O}_{\text{c.t.}} f_{\text{box}}.
\]

That means, in order to recover the correct-order results in EFT, we must introduce a counter term: \( \hat{O}_{\text{c.t.}} f_{\text{box}} \). Therefore, the UT scenario provides a natural interpretation for the counter terms, and more importantly, a rationality for the subtraction at the level of the loop integral \( \text{without} \) any reference to Lagrangian. That is, the counter terms must be endogenous: the divergent integrals must be subtracted \( \text{before} \) the nonperturbative reorganization\( [11] \). Finally, the subtracted integrals (finite) appear in the compact nonperturbative expressions which are no longer compatible with exogenous counter terms. In this logic, the

Nonperturbative renormalization in EFT

It is known that an EFT is often established through certain reorganization of parts of a well defined underlying theory (UT, at least renormalizable). Unfortunately, such reorganization usually (1) brings about new UV divergences and (2) impedes the exogenous counter terms from working. To see the first point, consider the diagrams shown in Figs.1 and 2, with the heavy meson exchange diagrams (with \( g \) and \( m_{h} \) being the coupling constant and the meson mass) underlying the ones with contact interactions. For convenience, let us introduce a projection operator \( \hat{P}_{\text{LE}} \) to symbolize the influences of this heavy meson: the extraction of the EFT vertices or couplings from the UT diagrams.

At tree level (Fig.1),

\[
-i\hat{C}_{0} \equiv \hat{P}_{\text{LE}} \Pi_{\text{tree}}^{(4)} = \hat{P}_{\text{LE}} \left\{ \frac{-ig^{2}}{k^{2} - m_{h}^{2}} \right\} = \frac{g^{2}}{m_{h}},
\]

no divergence appears. The complication comes at the loop diagram level. For example, in the case of the convergent box diagram in Fig.2, if \( \hat{P}_{\text{LE}} \) is applied after the loop integration \( \int \frac{d^{4}l}{(2\pi)^{4}} \) has been done \( (\text{the correct order}) \), one would get a well defined expansion in terms of \( \frac{1}{m_{h}} \). When \( \hat{P}_{\text{LE}} \) is applied before \( \int \frac{d^{4}l}{(2\pi)^{4}} \) \( (\text{the incorrect order}) \), the divergent bubble diagram results. Thus the new divergences generally arise from the incorrect order of computations, as the following commutator does not vanish identically:
formal consistency issue$^{[7]}$ of the Weinberg power counting simply dissolves: in the nonperturbative regime of EFT, there is no point in searching for exogenous counter terms and their counting rules. Of course there might be other approaches that directly renormalize the integrals without the explicit use of counter terms.

Thus the nonperturbative renormalization must be implemented either through endogenous counter terms or other means that effectively "subtract" the EFT integrals (see similar prescriptions in Ref. [17]). For the $T$-matrix considered in this paper, this procedure would be formulated as a simple replacement of the divergent projection and do not directly contribute to EFT renormalization due to scale hierarchy, we will return later the convergent vertices in UT. (Those that are divergent in UT will be renormalized before applying the underlying theory, has already been described in Ref.[19] for renormalizing any EFT.

This argument leads us to the following strategy: One first parametrizes the ill defined integrals in terms of ambiguous constants and then imposes physical boundary conditions. Similar strategy also based on the formal consistency issue$^{[7]}$ of the Weinberg power counting simply dissolves: in the nonperturbative regime of EFT, there is no point in searching for exogenous counter terms and their counting rules. Of course there might be other approaches that directly renormalize the integrals without the explicit use of counter terms.

Finally, we note that the renormalized $T$-matrix suffers from severe prescription dependence in the nonperturbative regime, which is incompatible with the exogenous counter terms. That means, given specific couplings, only one prescription could yield the physical $T$-matrix; others have to be dropped even though they are finite. So the final resolution boils down to the flexible regularization methods that could facilitate convenient access to physical predictions$^{[11]}$, as was already noted in other nonperturbative contexts$^{[18]}$. This argument leads us to the following strategy: One first parametrizes the ill defined integrals in terms of ambiguous constants and then imposes physical boundary conditions. Similar strategy also based on the underlying theory, has already been described in Ref. [19] for renormalizing any EFT.

LOW ENERGY THEOREMS (LET) AND PRESCRIPTION DEPENDENCE

Effective range expansion

Now let us consider effective range expansion (ERE) defined as follows,

$$\text{Re} \left\{ -\frac{4\pi}{M} T^{-1}_{os}(p) \right\} = p \cot(p) = -\frac{1}{a} + \frac{1}{2} r e p^2 + \sum_{k=2}^{\infty} v_k p^{2k},$$

(25)
with the parameters \( a \) and \( r_e \) being the scattering length and the effective range, which (including \( |v_k| \)) could be extracted from the scattering data. In this sense, we could impose their values as the boundary conditions for the \( T \)-matrix. Performing the expansion for the \( T \)-matrix obtained above, we get:

\[
\Delta = 0 : \quad p \cot(p) = -\frac{4\pi}{M} \left\{ C_0^{-1} + \tilde{J}_0 \right\};
\]

\[
\Delta = 2 : \quad p \cot(p) = -\frac{4\pi}{M} \left\{ \tilde{\nu}_0 \delta_0^{-1} + \tilde{J}_0 - \tilde{\nu}_0 \delta_1 \delta_0^{-2} p^2 + \sum_{k=2}^{\infty} \tilde{\nu}_0 \tilde{\delta}_k \delta_0^{-k-1} (-p^2)^k \right\},
\]

\[
\nu_0 \equiv (1 - C_2 \tilde{J}_3)^2, \quad \tilde{\delta}_0 \equiv C_0 + C_2^2 \tilde{J}_3, \quad \tilde{\delta}_1 \equiv C_2 (2 - C_2 \tilde{J}_3);
\]

\[
\Delta = 4 : \quad p \cot(p) = -\frac{4\pi}{M} \left\{ \tilde{\nu}_0 \tilde{\delta}_0^{-1} + (\tilde{N}_1 \tilde{D}_0 - \tilde{N}_0 \tilde{D}_1) \tilde{D}_0^{-2} p^2 + (\tilde{N}_2 \tilde{D}_0^2 - \tilde{N}_1 \tilde{D}_1 \tilde{D}_0 + \tilde{N}_0 (\tilde{D}_1^2 - \tilde{D}_0 \tilde{D}_2)) \tilde{D}_0^{-3} p^4 + (\tilde{N}_0 (2 \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 - \tilde{D}_3 \tilde{D}_0^2 - \tilde{D}_1^3) + \tilde{N}_1 \tilde{D}_0 (\tilde{D}_1^2 - \tilde{D}_0 \tilde{D}_2) - \tilde{N}_2 \tilde{D}_1 \tilde{D}_0^2) \tilde{D}_0^{-4} p^6 + \cdots \right\},
\]

The scattering length, effective range, and the \( v_k \) can be read from the results above. For the three orders considered so far, we have

\[
\Delta = 0 : \quad a^{-1} = \frac{4\pi}{M} (C_0^{-1} + \tilde{J}_0), \quad r_e = 0, \quad v_k = 0, k \geq 2;
\]

\[
\Delta = 2 : \quad a^{-1} = \frac{4\pi}{M} (\tilde{\nu}_0 \delta_0^{-1} + \tilde{J}_0), \quad r_e = \frac{8\pi}{M} \tilde{\nu}_0 \delta_1 \tilde{\delta}_0^{-2}, \quad v_k = \frac{4\pi}{M} \tilde{\nu}_0 \tilde{\delta}_k \delta_0^{-k-1}, k \geq 2;
\]

\[
\Delta = 4 : \quad a^{-1} = \frac{4\pi}{M} (\tilde{N}_0 \tilde{D}_0^{-1} + \tilde{J}_0), \quad r_e = \frac{8\pi}{M} (\tilde{N}_0 \tilde{D}_1 - \tilde{N}_1 \tilde{D}_0) \tilde{D}_0^{-2},
\]

\[
\nu_2 = \frac{4\pi}{M} [\tilde{N}_0 (\tilde{D}_0 \tilde{D}_2 - \tilde{D}_1^2) + \tilde{N}_1 \tilde{D}_1 \tilde{D}_0 - \tilde{N}_2 \tilde{D}_0^2] \tilde{D}_0^{-3},
\]

\[
\nu_3 = \frac{4\pi}{M} [\tilde{N}_0 (\tilde{D}_3 \tilde{D}_2^2 - 2 \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 + \tilde{D}_0^2) - \tilde{N}_1 \tilde{D}_0 (\tilde{D}_1^2 - \tilde{D}_0 \tilde{D}_2) + \tilde{N}_2 \tilde{D}_1 \tilde{D}_0^2] \tilde{D}_0^{-4},
\]

Note again that, at each order, \( \tilde{J}_0 \) only enters the expression for the scattering length but is "decoupled" with all the other ERE parameters. The reason is clear: \( \tilde{J}_0 \) stands alone in \( T^{-1} \). As we make clear below, this point has very important implications.

Keeping in mind that the parameters \( [J_n] \) are in principle independent of each other, two distinct approaches can be adopted in order to impose physical or reasonable boundary conditions: (1) taking the couplings and the prescription parameters as the fundamental variables, and the ERE parameters as the functions of these variables; or (2) conversely, taking some of the ERE parameters (which should be physical) as fundamental and the others as the functions of them. For convenience, we could also parametrize \( [J_n] \) in terms of a dimensional scale \( \bar{\mu} \) and dimensionless numbers \( [q_{n\ldots}] \):

\[
J_0 \equiv q_0 M \bar{\mu}, \quad J_3 \equiv q_3 M \bar{\mu}^3, \quad J_5 \equiv q_5 M \bar{\mu}^5, \quad J_7 \equiv q_7 M \bar{\mu}^7, \quad J_9 \equiv q_9 M \bar{\mu}^9.
\]

The appearance of \( M \) is easy to see from Appendix A. Generally, the magnitude of \( \bar{\mu} \) could vary from a value on the EFT expansion scale, \( \Lambda \), to the value of the pion decay constant (much smaller than \( M \)): \( \bar{\mu} \in (f_\pi, \Lambda) \). One can also alter the integrals by letting the dimensionless numbers \( [q_{n\ldots}] \) to vary. Thus, a nonperturbative renormalization prescription is parametrized by \( [q_{n\ldots}]; \bar{\mu}] \). However, the magnitude of \( [J_n] \), which also comes from the low energy projection in UT, should not be larger than the naive powers of the chiral symmetry breaking scale \( \Lambda_{\chi SB} \approx M \). In other words, we can safely assume that: \( |J_n| \leq M^{n+1}, n \neq 0, |\tilde{J}_0| \leq M^2 \).

Having made these preparations, we can start to examine the low energy expansions listed above order by order in chiral expansion. The leading order case (\( \Delta = 0 \)) is trivial: we have only one condition, i.e., imposing that \( \frac{1}{C_0} + J_0 = a \), with \( a \) being experimentally measured, is enough since \( r_e = v_k = 0, k \geq 2 \), which is obviously bad theoretical prediction, although it is not prescription dependent. Thus, the situation at this order is physically uninteresting, and most importantly, the distinctive nonperturbative features we wish to expose are not obvious here. Therefore we examine in detail the higher order cases.
LET at next-to-leading order: $\Delta = 2$

Let us start with the next-to-leading order: $\Delta = 2$. As was mentioned before, we shall discuss the problem from two perspectives, respectively.

First point of view

For convenience, let us list the explicit expressions of $a, r_c, \text{ etc.}$ in terms of $[C_n]$ and $[J_n]$, which read,

$$a = \frac{M}{4\pi} \frac{C_0 + C_2^2 J_5}{(1 - C_2 J_3)^2 + J_0(C_0 + C_2^2 J_5)^2},$$

$$r_c = \frac{8\pi}{M} \frac{(2C_2 - C_2^2 J_3)(1 - C_2 J_3)^2}{(C_0 + C_2^2 J_5)^2},$$

$$v_k = \frac{4\pi}{M} \frac{(-)^{k+1}(1 - C_2 J_3)^2 C_5^k (2 - C_2 J_3)^k}{(C_0 + C_2^2 J_5)^{k+1}}, \quad k \geq 2. \quad (33)$$

Before imposing reasonable boundary conditions, we could not make any physical predictions as $\bar{J}_0, \bar{J}_3$ and $\bar{J}_5$ each independently varies. The independent variations of $\bar{J}_0, \bar{J}_3$ and $\bar{J}_5$ could not be easily absorbed into the couplings due to the reasons explained in Sec. III. Thus, unlike the leading order case, we need (more) exogenous constraints to fix the values of $\bar{J}_0, \bar{J}_3$ and $\bar{J}_5$.

From Eq.(33), $\bar{J}_3$ and $\bar{J}_5$ could be solved in terms of $r_c, v_2$ and $C_0, C_2$, the solution might be unique after accounting for a reasonable size. After the insertion of the obtained numbers back into the formula for $a$, $\bar{J}_0$ could be expressed in terms of the physical value of the scattering length. In this sense, imposing two additional boundary conditions could fix the prescription or make the next-to-leading order result unambiguous. Now the $[v_k]$ with $k \geq 3$ are taken to be theoretical predictions, which now shall be better than the leading order ones, as we have more degrees of freedom to work with: $\bar{J}_3$ and $\bar{J}_5$, which come together with the new interactions. The fact, that some of the predictions are still poor can be attributed to the inadequacy of the next-to-leading order potential: even higher order terms should be put in and accordingly more physical boundary conditions are needed. Thus, in spite of the fact, that the procedure for fixing prescription becomes more nontrivial and laborious, the predictions for the ERE parameters improve when the higher order interactions are included (because of the "freedom" brought by the augmented interactions). We must repeat here that the predictions are made using the prescription that is most compatible with the physical boundary conditions.

Employing (33), we could also write the above equations as

$$a = \frac{M}{4\pi} \frac{C_0 + C_2^2 q_3 M \bar{\mu}^5}{(1 - C_2 q_3 M \bar{\mu})^2 + q_0 M \bar{\mu}(C_0 + C_2^2 q_3 M \bar{\mu}^3)^2},$$

$$r_c = \frac{8\pi}{M} \frac{(2C_2 - C_2^2 q_3 M \bar{\mu}^3)(1 - C_2 q_3 M \bar{\mu}^3)^2}{(C_0 + C_2^2 q_3 M \bar{\mu}^3)^2},$$

$$v_k = \frac{4\pi}{M} \frac{(-)^{k+1}(1 - C_2 q_3 M \bar{\mu}^3)^2 C_5^k (2 - C_2 q_3 M \bar{\mu}^3)^k}{(C_0 + C_2^2 q_3 M \bar{\mu}^5)^{k+1}}, \quad k \geq 2. \quad (34)$$

Note that even though the $\bar{\mu}$ scale dependence can be removed (fixed), the prescription dependence remains in terms of the dimensionless parameters $[q_n]$ that are independent of each other, a subtle point that often seems to be overlooked. For example, in a cutoff scheme, the renormalization is usually performed in such a way that the cutoff dependence is removed by letting the couplings to develop a certain cutoff dependence. However, remains residual prescription dependence remains through the cutoff independent but prescription dependent numbers $[q_n]$. Without fully appreciating this point, any fitting procedure that uses tuning of the couplings or even tuning of the cutoff only amounts to fitting along a special orbit in the space $[q_n]$, not in the whole space. The result thus obtained is still prescription dependent.
Taking $a$, $r_e$ and $\{v_2\}$ as the elementary parameters, we can express all the higher order constants $[v_k, k \geq 3]$ as

$$v_k = \frac{M - 4\pi a \tilde{J}_0}{Ma} (-2v_2/r_e)^k, \ k \geq 3. \tag{35}$$

Note that from this perspective the prescription-dependent parameter $\tilde{J}_0$ seems to be an independent constant in addition to the three elementary parameters. At first sight, this ambiguity calls for one more condition: the value for $v_3$. But from the discussion above, we know that this seemingly independent parameter is in fact determined together with $\tilde{J}_3$ and $\tilde{J}_5$ by the equations (33). Of course the nature of the problem remains the same even when it is taken as independent. The most striking point here is that the prescription is "removed" through fixing, i.e., through boundary conditions, which is a nontrivial procedure as articulated above.

One could also put Eq. (35) into the form that contains no explicit prescription dependence:

$$v_k = v_3(-2v_2/r_e)^{k-3}, \ k \geq 4. \tag{36}$$

This relation should hold for any problems (in certain atomic or molecular contexts) with contact potential such as $V(x) \sim C(0)\delta^{(3)}(x) + C(2)\nabla^2\delta^{(3)}(x)$.

So, we may conclude that, no matter what point of view one adopts, the key point is that the compact nonperturbative formulations make the prescription dependence and its removal a very nontrivial problem or procedure. However, the predictions also improve with the use of these formulations, in spite of their technical complexity. This nontrivial procedure will get more involved as more higher order corrections to the potential are included. To verify this, let us turn to the next-to-next-to-leading order.

**LET at next-to-next-to-leading order: $\Delta = 4$**

Again we begin with the first perspective.

**First perspective**

Now we have four couplings $(C_0, C_2, C_4, \tilde{C}_4)$ and five prescription dependent parameters $(\tilde{J}_0, \tilde{J}_3, \tilde{J}_7, \tilde{J}_9, \tilde{J}_9)$ in eight compact expressions: $\hat{N}_0, \hat{N}_1, \hat{N}_2, \hat{D}_0, \hat{D}_1, \hat{D}_2, \hat{D}_3$ and $\tilde{J}_0$, which stays alone. From Eq. (31) it is clear that we need at least five conditions to fix $\tilde{J}_0, \tilde{J}_3, \tilde{J}_7$ and $\tilde{J}_9$, say $a, r_e, v_2, v_3$ and $v_4$. But the compact expressions such as that for the scattering length, $a = \frac{M + \tilde{D}_1 \tilde{D}_0}{\tilde{D}_0 \tilde{N}_0}$ with $\tilde{D}_0$ and $\tilde{N}_0$ given in Appendix B, become more involved. This means that the boundary conditions might be more stringent for $\tilde{J}_0, \tilde{J}_3, \tilde{J}_7$ and $\tilde{J}_9$ and the analytical work more difficult. In the meantime, the predictions for $v_k$ at this order should be better than the leading and next-to-leading orders, as we have more parameters.

Here some remarks are in order. At next-to-leading order, we ignored the possible multiple solutions for the fixing procedure. Here, with more compact expressions being involved, we should be more careful about this multiplicity of solutions. To this end, we note that the multiplicity could be effectively reduced with the limitations on the reasonable magnitudes of $|\tilde{J}_a|$ together with the experimental values of the higher ERE parameters (say, $v_k, k \geq 3$). However, no matter how the multiplicity is removed, the solution is still an approximate (though nonperturbative) one: The equations (29, 30, 31) are obtained from a truncated potential and could not be exact ones. Then, the theoretical predictions based on such equations will be less credible, especially for the ERE parameters that dominate higher and higher energy regions. In other words, the boundary conditions should be given by a procedure similar to fitting the shape of the phase shift within the corresponding ranges at each chiral order. This is actually what most authors have done, though the regularization schemes used vary significantly. Of course our remarks in Sec. IV.B.1 concerning the residual prescription dependence still apply for all the higher order calculations.

Mathematically, the multiplicity of solutions might be generic for nonperturbative renormalization because of the compact expressions involved. Therefore the limit cycles encountered in the Schrödinger approach
of renormalizing singular potentials\cite{20} might just be examples of such multiplicity in certain regularization schemes.

Second perspective

To discuss the problem from the second perspective, we need to express everything in terms of the first five ERE parameters. Then the arguments go as in the preceding subsection. We shall not, however, repeat such complicated technical details here. One could also find the relations like Eq.(35) or (36), which hold true independently of the prescriptions, by repeatedly using the following recursive relations:

\[
\begin{align*}
\bar{v}_n &= -\sum_{k=1}^{3} \frac{D_k}{D_0} \bar{v}_{n-k}, n \geq 5; \\
\bar{v}_3 &= \frac{D_3}{D_0} \bar{a}^{-1} - \frac{D_2}{D_0} \bar{r}_e - \frac{D_1}{D_0} \bar{v}_2; \\
\bar{v}_4 &= -\frac{D_3}{D_0} \bar{r}_e - \frac{D_2}{D_0} \bar{v}_2 - \frac{D_1}{D_0} \bar{v}_3;
\end{align*}
\]

(37)

\[
\bar{a}^{-1} \equiv \frac{M}{4\pi a} - \bar{J}_0, \quad \bar{r}_e \equiv \frac{M}{8\pi r_e}, \quad \bar{v}_n \equiv \frac{M}{4\pi} \bar{v}_n, n \geq 2,
\]

(38)

where the coefficients $\frac{D_k}{D_0}$ could be solved in terms of $M, a, r_e, v_2, v_3$ and $v_4$.

Lessons from nonperturbative solutions

Now it is clear that things get more complicated as more higher order terms are included in the potential. Given this, we should not take the lower order results too seriously. For instance, the low energy theorems at leading order are too simple to be true in practice: $r_e = v_k = 0, k \geq 2$. This implies the necessity to include higher order terms, which, however, will bring us both favorable and unfavorable consequences. On one hand, more severe prescription dependence will show up and make this analysis more difficult. On the other hand, more prescription ambiguities also provide us with more chances to access the measured values of the ERE parameters. Although our calculations were done for the case of contact potentials, the core feature of our analysis–more ambiguities or more divergences at higher orders–holds true also for realistic potentials. At this stage, we shall mention that the freedoms in the prescription are in fact limited: $[\bar{J}_n]$ must satisfy certain requirements as presented in the discussion following Eq.(32). Moreover, the coupling constants should generally follow certain rules of EFT power counting. Then, after putting all these theoretical aspects into consideration, the EFT predictions must lie in certain region of the ‘space’ of observables.

Now we provide another way to see the virtue of the fitting procedure. Let us examine the variation of the functional form of the scattering length $a$ in terms of the couplings and $[\bar{J}_n]$ for different chiral orders:

\[
\begin{align*}
\Delta = 0 : \quad a &= \frac{M}{4\pi} \frac{C_0}{1 + C_0 J_0}, \\
\Delta = 2 : \quad a &= \frac{M}{4\pi} \frac{C_0 + C_2 \bar{J}_5}{(1 - C_2 J_3)^2 + J_0(C_0 + C_2 J_3)}, \\
\Delta = 4 : \quad a &= \frac{M}{4\pi} \frac{\bar{D}_0(C_0, C_2, C_4, \tilde{C}_4; J_5, J_7, J_9) + J_0 \bar{D}_0(C_0, C_2, C_4, \tilde{C}_4; J_5, J_7, J_9)}{N_0(C_0, C_2, C_4, \tilde{C}_4; J_5, J_7, J_9)}.
\end{align*}
\]

It is obvious that the theoretical form of the scattering length varies with the chiral order quite significantly! So the scattering length calculated at lower orders should not be directly identified with the experimental value in order to accommodate the higher order terms. Thus a more reasonable way for fixing the renormalization prescription will be the one that avoids the direct identification of physical parameters in order to accommodate higher order contributions in a consistent way. To this end, again a procedure like fitting the empirical curve over appropriate low energy regions might be more plausible.

POWER COUNTING AND RENORMALIZATION IN THE NONPERTURBATIVE REGIME

In all the discussions above, we have left out the power counting of the couplings. Since they constitute the basis for the EFT methods, it is necessary to see what the nonperturbative renormalization procedure
described above means for the power counting rules. In fact, as was stressed in Ref. [11], the parametrization in the nonperturbative regime given in Eq. (11) implies that, in order for a power counting scheme for couplings to be meaningful, the corresponding prescription for the constants $[\bar{J}_n]$ must be appropriately chosen. Otherwise, one could not obtain the physical $T$-matrix.

From the standpoint of UT, both $[C_n]$ and $[\bar{J}_n]$ come from the well-defined low energy projection ($\bar{P}_{LR}$) applied to UT amplitudes. So both $[C_n]$ and $[\bar{J}_n]$ serve as the elementary parameters for parametrizing the $T$-matrix for the low energy nucleon-nucleon scattering. In the EFT treatment without knowledge of the details from UT, we are forced to employ $[C_n]$ as the elementary couplings according to certain counting rules, while the constants $[\bar{J}_n]$ appear as the divergent pieces in the EFT loops constructed with the use of $[C_n]$. Thus it is the EFT treatment that makes $[C_n]$ and $[\bar{J}_n]$ look disparate. In UT they are organized and derived together according to more elementary rules. Therefore, changing any of them (each single parameter in $[C_n]\cup[\bar{J}_n]$) alone would alter the physical behavior of $T$. Thus they must be considered together.

One could also understand it from the Wilsonian definition of EFT through successive decimation of the higher scales, where different EFT expansion point would lead to both different couplings and different $[J_n]$, as long as the expansions are compatible with the chiral power counting.

To be specific, the variations of $[C_n]$ and $[\bar{J}_n]$ (from now on $\bar{J}_0$ is excluded from $[\bar{J}_\ldots]$ for the reasons to be given below) must not alter the functional form (shape) of the $T$-matrix:

$$\text{Re} \left\{ \frac{1}{T_{\alpha\beta}(p)} - \bar{J}_0 \right\} = \sum_i \tilde{N}_i([C_{\ldots}]; [\bar{J}_{\ldots}]) p^{2i} = \sum_i \bar{N}_i([C_{\ldots}]; [\bar{J}_{\ldots}]) p^{2i} = \sum_i N_i^{(\text{phys})} p^{2i} = \sum_i D_j^{(\text{phys})} p^{2j}.$$  \hspace{1cm} (39)

Here we use the superscript 'phys' to indicate that the parameters in the last fraction are physically determined, for example, from a genuine UT. To see why $\bar{J}_0$ is excluded from $[\bar{J}_n]$, consider the physical parametrization of $T$-matrix (independent of the variations of $[C_{\ldots}]$ and $[\bar{J}_n]$), which has the following form

$$\frac{1}{T^{(\text{phys})}} = \sum_i N_i^{(\text{phys})} p^{2i} = \frac{M}{4\pi} p \bar{J}_0 + M\gamma + \frac{M}{4\pi} p,$$  \hspace{1cm} (40)

where $\gamma$ must be a physical scale, just like the nucleon mass $M$ and the on-shell momentum $p$. Now it is clear that $\bar{J}_0$ alone corresponds to the physical parameter $M\gamma$, which should therefore be independent of prescriptions. If we were not the case, or if $\bar{J}_0$ could vary with prescriptions, we would have to alter $\bar{N}_0$ and $D_0$ to compensate for such variation in $\bar{J}_0$. Now, to keep the proportionality between $\bar{N}_0$, $D_0$ and $\bar{N}_0 p^{2i}$, $D_j p^{2j}$ invariant, all the rest of $[\bar{N}_{\ldots}, D_{\ldots}]$ must be accordingly altered, which in turn leads to an overall factor for $(\sum N_i^{(\text{phys})} p^{2i})/(\sum D_j^{(\text{phys})} p^{2j})$. Then, the functional dependence of the $T$-matrix upon $p$ would be altered, since its imaginary part, $M 4\pi p$, remains intact. Hence $\bar{J}_0$ must stay independent of prescriptions, i.e., physical. One could also verify this by examining the consequences on the ERE parameters.

As $p$ is arbitrary in the supposed range, the 'invariance' discussed above leads to the following nontrivial equations for $[C_n]$ and $[\bar{J}_n]$ with the crucial presence of the physical parameters $[N_i^{(\text{phys})}]$ and $[D_j^{(\text{phys})}]$ for the on-shell $T$-matrix,

$$\tilde{N}_i([C_{\ldots}]; [\bar{J}_{\ldots}]) = N_i^{(\text{phys})}, \hspace{1cm} \bar{D}_j([C_{\ldots}]; [\bar{J}_{\ldots}]) = D_j^{(\text{phys})}, \hspace{1cm} \forall i, j.$$  \hspace{1cm} (41)

These equations have dual implications: they could be used either (1) to fix the prescription $(\bar{J}_{\ldots})$ in terms of the couplings $([C_{\ldots}])$ and the physical parameters $(N_i^{(\text{phys})}; D_j^{(\text{phys})})$ or conversely (2) to examine the influence of prescription upon the couplings with the help of the physical parameters. The first use just parallels what we have done in Sec. IV.B and C.

**Interplay between Power counting and prescription: next-to-leading order**

Let us illustrate the interplay between power counting and prescription at next-to-leading order; that is, we try to solve the following equations for couplings:

$$\frac{C_0 + C_3^2 \bar{J}_0}{(1 - C_2 \bar{J}_3)^2} = 0 = \frac{D_0^{(\text{phys})}}{N_0^{(\text{phys})}}; \hspace{1cm} \frac{2C_2 - C_3^2 \bar{J}_3}{(1 - C_2 \bar{J}_3)^2} = 0 = \frac{D_2^{(\text{phys})}}{N_0^{(\text{phys})}}.$$  \hspace{1cm} (42)
The solutions are easy to find as,
\begin{align*}
C_2^{(\pm)} &= J_3^{-1} \{1 \pm (1 + \alpha_2 J_3)^{-\frac{1}{2}}\}, \\
C_0^{(\pm)} &= \frac{\alpha_0}{1 + \alpha_2 J_3} \frac{J_5}{J_3} \{1 \pm (1 + \alpha_2 J_3)^{-\frac{1}{2}}\}^2.
\end{align*}

Taking into account the natural boundary condition for \(C_2\): \(C_2|_{\mu \to 0} = \alpha_2/2\), we are left with unique solution: \(C_2^{(-)} (\text{and } C_0^{(-)})\). Thus assigning a power counting to \(C_0\) and \(C_2\) means assigning the sophisticated scaling for \(J_3\) and \(J_5\). Conversely, one can come up with an alternative interpretation: The power counting for the couplings could only be preserved in some particular prescription in order to obtain the expected physical behavior from the T-matrix. Note that here we have deliberately not mentioned \(J_0\), it will be exclusively discussed below. Equations (41) or (42) now formalize our discussions concerning the interplay between power counting and prescription.

More interestingly, these equations have a further utility: they could be used to describe the evolution of the couplings in terms of a sliding scale (\(\mu\)) in \([J_n]|(= [q_n M \mu^n, n \neq 0])\). Since the exogenous counter terms are incompatible with the closed form of the T-matrix, the conventional route to the evolution described by renormalization group equation does not exist. But we could take the evolution implied by Eqs. (41) or (42) as a nonperturbative "renormalization group" evolution. We discuss this point in the next subsection.

**Nonperturbative ‘renormalization group’ (RG) evolution**

To proceed, let us choose the prescription with \([J_n] \equiv q_n M \mu^n\) to examine the evolution of the couplings enforced by (42). Let us assume that there exist enough boundary conditions to obtain the ‘physical’ solutions for the couplings from the equations in (41):
\begin{equation}
C_i = F_i([N^{(\text{phys})}_-, D^{(\text{phys})}_-, M]; [q_-]; \mu), \forall i.
\end{equation}

With such nonperturbative solutions, the complete evolution of the couplings are determined and both the IR and the UV fixed points can be identified. For example, at next-to-leading order, we have from Eqs. (41, 42, 44, 45),
\begin{align*}
C_0(\alpha_0, \alpha_2, M, q_3, q_5; \mu) &= \frac{\alpha_0}{1 + q_3 \alpha_2 M \mu^3} - \frac{q_5}{q_3^2 M \mu^2} \{1 - (1 + q_3 \alpha_2 M \mu^3)^{-\frac{1}{2}}\}^2, \\
C_2(\alpha_0, \alpha_2, M, q_3, q_5; \mu) &= (q_3 M \mu^3)^{-1} \{1 - (1 + q_3 \alpha_2 M \mu^3)^{-\frac{1}{2}}\}^2.
\end{align*}

It is easy to see that they have both IR and UV fixed points:
\begin{align*}
\text{IR fixed point} (\mu \Rightarrow 0): \quad &C_{0}^{(\text{IR})} = \alpha_0, \quad C_{2}^{(\text{IR})} = \alpha_2/2; \\
\text{UV fixed point} (\mu \Rightarrow \infty): \quad &C_{0}^{(\text{UV})} = C_{2}^{(\text{UV})} = 0.
\end{align*}

Note that the prescription dependence is obvious in Eq. (45) with the presence of \([q_-]\), but the UV and IR fixed points are prescription independent. While the IR fixed points are realistic as the couplings were defined in the low energy limit, the UV fixed points seem not to be realistic. But such UV behavior of the EFT couplings is compatible with the fact that the EFT couplings would be dominated by the UT couplings at high energy, and therefore ‘vanish’. Of course we should bear in mind that, what we obtained are only approximate answers, though nonperturbative.

Note that the Eqs. (41, 44, 45) contain the full dependence upon the prescription parameters. So one could also derive the equations a la Stückelberg and Petermann [21] that describe the laws for transitions from one prescription to another which are not related by running the renormalization scale:
\begin{equation}
\frac{d}{d[J_-]} \{\tilde{N}_i, \tilde{D}_j\} = 0, \forall i, j.
\end{equation}

In terms of \([q_-; \mu]\) they become
\begin{equation}
\frac{d}{dq_-} \{\tilde{N}_i, \tilde{D}_j\} = 0, \forall i, j.
\end{equation}
In the foregoing discussions the physical requirements are imposed on the functional shape of the $T$-matrix. Alternatively, we could also employ the physically determined ERE parameters (scattering length, effective range, etc) instead of $[N_i^{(\text{phys})}, D_j^{(\text{phys})}]$ to solve the couplings in terms of $[J_n]$. In principle, the two approaches should lead to the same evolution behavior, but the ERE approach is more involved than the shape approach as is clear from the comparison between the Eqs. 30-31 and Eqs. 33-34.

**Determinant for the natural or unnatural scattering length: $\tilde{J}_0$**

Now we discuss the determinant(s) of the size of physical parameters $[N_i^{(\text{phys})}]$ and $[D_j^{(\text{phys})}]$ or $a, r_\varepsilon, v_k, \forall k \geq 2$. As argued above, a complete parametrization of the $T$-matrix is given by $[C_{\varepsilon}]$ supplemented with $[\tilde{J}_{\varepsilon}]$, then $[C_{\varepsilon}]$ and $[\tilde{J}_{\varepsilon}]$ together determine whether the physical parameters are of natural size or not. We could have four rough scenarios, listed in Table I where by a natural $C_n$ we mean that the scale $\Lambda$ in its parametrization $C_n \sim 1/(\Lambda \Lambda_{n+1})$ is of the size of the expansion scale (unnatural if $\Lambda \sim p, m_\pi$), while for $[\tilde{J}_{\varepsilon} = q_\varepsilon M \mu^{-1}]$ the situation is reversed: the natural size of $\mu$ should be $\sim p, m_\pi$. A natural $T$-matrix is parametrized by $[N_i^{(\text{phys})}, D_j^{(\text{phys})}]$ (or for $a, r_\varepsilon, v_k, \forall k \geq 2$) such that the dimensional parameters are of the same magnitudes as the natural couplings.

Examining the concrete expressions of the $T$-matrix, we find that whether the $T$-matrix is natural or not is determined by both the sizes of the couplings and the magnitudes of the dimensionless combinations like $\prod_{n,m} C_n^{-1} \tilde{J}_n$ (dim $\prod_{n,m} C_n^{1+} \tilde{J}_n = 0$). Now suppose we have natural couplings, i.e., $C_n \sim 1/(\Lambda \Lambda_{n+1})$, $\forall n$. If $[\tilde{J}_{\varepsilon}]$ are also natural, we should have $|\prod_{n,m} C_n^{-1} \tilde{J}_n| \ll 1$ for all the dimensionless combinations. Then, given our experience at next-to-leading and next-to-next-to-leading orders, we can anticipate that:

$$N_0^{(\text{phys})} \sim 1, \quad N_i^{(\text{phys})} \sim \frac{1}{\Lambda^{2i}}, \quad D_j^{(\text{phys})} \sim \frac{1}{\Lambda^{2j+1}}(\sim C_j), \forall i, j.$$  (52)

In this case we shall obtain a natural $T$-matrix, or ERE parameters $(a, r_\varepsilon, v_k, \forall k \geq 2)$, of natural sizes. If $[\tilde{J}_{\varepsilon}]$ are unnatural, then in general, we could have $|\prod_{n,m} C_n^{-1} \tilde{J}_n| \ll 1$ for the dimensionless combinations. Therefore, we have,

$$N_0^{(\text{phys})} \gg 1(\ll 1), \quad N_i^{(\text{phys})} \gg (\ll) \frac{1}{\Lambda^{2i}}, \quad D_j^{(\text{phys})} \gg (\ll) \frac{1}{\Lambda^{2j+1}}(\sim C_j), \forall i, j.$$  (53)

In this case, we obtain an unnatural $T$-matrix, or unnatural ERE parameters $(a, r_\varepsilon, v_k, \forall k \geq 2)$ with natural couplings. For example, at next-to-leading order, we have,

**natural $[\tilde{J}_{\varepsilon}]$:**

$|C_2 J_3| \ll 1, |C_2^2 C_0^{-1} J_5| \ll 1,$

$$\Rightarrow \text{natural } T: \quad \frac{(1 - C_2 J_3)^2}{C_0 + C_2^2 J_3 + C_2(2 - C_2 J_3)p^2} \simeq \frac{1}{C_0 + 2C_2 p^2};$$  (54)

**unnatural $[\tilde{J}_{\varepsilon}]$:**

$|C_2 J_3| \sim 1, |C_2^2 C_0^{-1} J_5| \sim 1,$

$$\Rightarrow \text{unnatural } T: \quad \frac{(1 - C_2 J_3)^2}{C_0 + C_2^2 J_3 + C_2(2 - C_2 J_3)p^2} = \frac{\zeta_1}{\zeta_2 C_0 + 2\zeta_3 C_2 p^2},$$  (55)

where each of $\zeta_\varepsilon$ can be either pretty small or pretty large and therefore the $T$-matrix could not be a natural one.

Now let us consider $\tilde{J}_0$ and the scattering length in particular. As argued above, $\tilde{J}_0$ should be viewed as an independent physical parameter, not as a common prescription parameter. From the parametrization of $T$ and the formulae in the preceding sections, $\tilde{J}_0$ will only contribute to the scattering length:

$$a^{-1} = \frac{4\pi}{M} \left\{ \tilde{J}_0 + \frac{N_0([C_{\varepsilon}, \tilde{J}_{\varepsilon}])}{D_0([C_{\varepsilon}, \tilde{J}_{\varepsilon}])} \right\},$$

$$r_\varepsilon = \frac{8\pi}{M} \left\{ \frac{N_0([C_{\varepsilon}, \tilde{J}_{\varepsilon}])}{D_0([C_{\varepsilon}, \tilde{J}_{\varepsilon}])} - \frac{N_1([C_{\varepsilon}, \tilde{J}_{\varepsilon}])}{D_0([C_{\varepsilon}, \tilde{J}_{\varepsilon}])} \right\}, \ldots.$$

$$\Rightarrow \quad \frac{\partial a^{-1}}{\partial \tilde{J}_0} = \frac{4\pi}{M}, \quad \frac{\partial r_\varepsilon}{\partial \tilde{J}_0} = \frac{\partial v_k}{\partial \tilde{J}_0} = 0, \forall k \geq 2.$$  (56)
Now we can see that, even when both $[C_{..}]$ and $[J_{..}]$ are of natural sizes, the scattering length could be unnaturally large once $\tilde{J}_0$ is unnatural ($\sim M\Lambda$),

\begin{align}
\text{natural} \tilde{J}_0(\sim M\mu) & : a^{-1} \simeq -O(\Lambda) + O(\mu) \sim -O(\Lambda); \\
\text{unnatural} \tilde{J}_0(\sim M\Lambda) & : a^{-1} \simeq -O(\Lambda) + O(\Lambda) \sim -O(\mu). 
\end{align}

That is, in the $^1S_0$ channel, there theoretically exists such a scenario that the scattering length could be unnaturally large while all the rest ERE parameters are naturally sized. Then the first situation in Table 1 should be amended as follows: even when all the couplings and all the rest $[\tilde{J}_{..}]$ are natural, we would get an unnatural scattering length as long as $\tilde{J}_0$ is unnatural.

For an unnatural power counting of the couplings, the discussion would be more difficult and we refrain from exploring such situations here. As we have shown, that both the natural and the unnatural physical parameters could be explained with the natural couplings (provided the nontrivial nonperturbative prescription dependence is fully explored), we feel that it is more reasonable to work with natural or conventional power counting of EFT couplings.

**DISCUSSIONS AND SUMMARY**

Now it is time for us to address some theoretical aspects that have been omitted or not fully discussed so far. Let us start with the relation between the UT renormalization and the EFT renormalization that is involved in Sec. III.C. Generally, in an EFT one deals with the new divergences in the diagrams that are induced by some low energy expansion (with the "wrong" order of operations, C.f. Sec. III.C) or similar operation in UT, whereas the diagrams that need renormalization in UT are usually hidden in the EFT couplings. Note, that the diagrams that renormalize UT dominate the quantum fluctuations at short distances, while the ones that are divergent in EFT dominate those at long distances. So the two renormalization do not interfere with (affect) each other due to the large scale hierarchy between UT and EFT, i.e., they work at two widely separated scales. Thus, the renormalization in UT does not affect the renormalization in EFT. This supplements our remarks after Eq. (22) in Sec. III.C.

Next, let us address the effect of the potential truncation on the nonperturbative renormalization group evolution. At any fixed chiral order, the nonperturbative evolution behaviors of the coupling of the highest chiral dimension should be less trustworthy. This is because once the next order interactions are included, the coefficients for the term with the highest power of $p$ would suffer the largest changes in the functional forms; the coefficients for the lower power terms receive smaller changes from the new couplings. That means that due to the truncation of the potential, the nonperturbative evolution behaviors of the EFT couplings with lower chiral dimensions should be more trustworthy than those with higher chiral dimensions. One could see this point by noting how the forms of $[N_i([C_{..}]; [J_{..}]), D_i([C_{..}]; [J_{..}])]$ (as functions of the couplings $[C_{..}]$) change with the inclusion of higher order interactions.

In Sec. V.C., we have shown that a natural (or conventional) chiral power counting of the EFT couplings does allow the $T$-matrix to have unnatural parameters, or unnatural scattering length, etc. In particular, there is a possibility that only the scattering length is unnatural while the rest of the parameters are natural. This seems to be just the realistic situation with the $^1S_0$ channel nucleon-nucleon scattering at low energy. This scenario is clearly different from the one discussed in the literature where unusual power counting of the couplings was employed [22]. Here the key role is played by the nonperturbative renormalization prescription.

Although our conclusions or remarks have been reached with contact interactions, we feel that the conclusions or scenarios depicted here should remain qualitatively true even in a realistic situation because the crucial features of the nonperturbative renormalization remain unchanged: (1) More ill defined pieces in the loop integrals appear at higher chiral orders; (2) The nonperturbative solution of the $T$-matrix takes a closed form that can only be renormalized via endogenous counter terms. Alternatively, one could also take the rational function form as a Padé approximant to the realistic $T$-matrix.

Now let us comment on the literature. In Ref. [23], a subtraction similar to the endogenous one described in the present paper is employed: the counter term is introduced before the $T$-matrix is calculated, a procedure that parallels the loop integrations. However, it is not clear if the subtraction described in some papers is equivalent to the endogenous one or not. For example, the subtraction procedure described in Ref. [24] does
not appear to be an endogenous one. Thus it may be flawed, as was already noted in Ref.\cite{25}. In Ref.\cite{25}, the whole investigation is made in the nonperturbative formulation (compact) of the $T$-matrix, a positive aspect of this study. However, a special regularization (cutoff regularization) exclusively used in Ref.\cite{25} unfortunately makes their analysis inevitably prescription dependent. In a contrast, the strategy employed in Ref.\cite{9} for parametrizing and fixing the nonperturbative renormalization prescription dependence is closer to the one used in the present paper. The importance of boundary conditions has already been stressed in Ref.\cite{10}, where the physical observables, such as phase shifts, were parametrized without involving explicit divergences.

Obviously, we just explored some convenient scenarios of the nonperturbative solutions. Our arguments have been unable to exclude many other possible scenarios. The only point in favor of the scenarios discussed in this paper is that they are relatively simple, whereas the rest possibilities seem rather sophisticated, and often use fine tuning or similar arguments.

In our opinion, a better way to work with the renormalization in the nonperturbative regime is to appreciate the presence of a well-defined theory underlying an EFT, as illustrated in this paper. In this sense, the renormalization of singular potentials in quantum mechanics\cite{26}, or equivalently the self-adjoint extension of singular operators in Hilbert space, should also be embedded in the underlying theory background. This is plausible since quantum mechanics IS an effective theory of quantum field theory.

In summary, we reconsidered the renormalization of the EFT for nucleon-nucleon scattering in the nonperturbative regime using contact potentials that facilitate rigorous solutions of LSE. Detailed analysis reveals that the $T$-matrix in the nonperturbative regime should be renormalized through the endogenous counter terms whose net effects are to remove the divergences in the loop integrals, or through means that could yield the same results. The rationality for the subtractions at loop integral level could be naturally explained from the underlying theory, with the UV divergences being shown to come from the "incorrect" order of operations in the construction of EFT. Then, using the effective range expansion, we demonstrated that the nontrivial renormalization prescription dependence in the nonperturbative regime must be "removed" by imposing appropriate boundary conditions. We also argued that when imposing boundary conditions, the full "space" for renormalization prescriptions should be explored in order to be able to remove any residual prescription dependence. It is also important to impose the boundary conditions in such a way that higher order terms in the potential could be consistently incorporated. Finally, the nontrivial relation between the power counting of the couplings and the renormalization prescription was highlighted in the nonperturbative regime. As byproducts, (1) the nonperturbative 'renormalization group' evolution was described; (2) the naturalness of the scattering length, etc., were shown to be compatible with the natural or conventional power counting of the couplings because of the nontrivial prescription dependence. That is, the nontrivial prescription dependence becomes a virtue in such a case. Obviously, much work remains to be done.

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Appendix A

\[
\mathcal{I}(E^+) \equiv \left( \int \frac{d^3k}{(2\pi)^3} \frac{1}{E^+ - \frac{k^2}{M}} \quad \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E^+ - \frac{k^2}{M}} \quad \int \frac{d^3k}{(2\pi)^3} \frac{k^4}{E^+ - \frac{k^2}{M}} \quad \int \frac{d^3k}{(2\pi)^3} \frac{k^4}{E^+ - \frac{k^2}{M}} \right).
\]  
(59)

The entries of this matrix can be parametrized as follows \((p \equiv \sqrt{ME})\):

\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{E^+ - \frac{k^2}{M}} \equiv I_0 = -J_0 - \frac{iM}{4\pi};
\]  
(60)

\[
\int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E^+ - \frac{k^2}{M}} = J_3 + i0p^2;
\]  
(61)

\[
\int \frac{d^3k}{(2\pi)^3} \frac{k^4}{E^+ - \frac{k^2}{M}} = J_5 + J_3p^2 + i0p^4;
\]  
(62)

\[
\int \frac{d^3k}{(2\pi)^3} \frac{k^6}{E^+ - \frac{k^2}{M}} = J_7 + J_5p^2 + J_3p^4 + i0p^6;
\]  
(63)

\[
\int \frac{d^3k}{(2\pi)^3} \frac{k^8}{E^+ - \frac{k^2}{M}} = J_9 + J_7p^2 + J_5p^4 + J_3p^6 + i0p^8.
\]  
(64)

Here \({J_n}\) with \(n = 0, 3, 5, 7, 9\) are regularization and renormalization prescription dependent constants.

Appendix B

\[
N_0 = (1 - C_2J_3 - C_4J_5)^2 - C_0 \tilde{C}_4J_3^2 \tilde{C}_4J_5 + 2\tilde{C}_4C_4J_5^2 - \tilde{C}_4C_4J_3^3 - 2\tilde{C}_4C_4J_3J_7
\]

\[-\tilde{C}_4C_4J_3J_9 + 2\tilde{C}_4C_4J_3J_9J_7\];

\[
N_1 = -2C_4J_3 - \tilde{C}_4J_5 + 2C_4C_4J_3^2 + 2\tilde{C}_4C_4J_3J_5 + 2C_4J_3J_5 - \tilde{C}_4C_4J_3J_5 + \tilde{C}_4C_4J_3J_7;
\]

\[
N_2 = C_4^2J_3^2;
\]

\[
D_0 = C_0 + C_4J_3 + C_4J_5 - \tilde{C}_4C_4J_3 + C_4^2J_5 + 2C_4C_4J_7 - C_4^2\tilde{C}_4J_3 + 0;
\]

\[
D_1 = 2C_2 - C_4J_3 + C_4J_5 + C_4J_7 + 2\tilde{C}_4C_4J_7 - C_4^2\tilde{C}_4J_3 + C_4C_4J_3J_7;
\]

\[
D_2 = 2C_4 + \tilde{C}_4 - 2C_4C_4J_3 - 2\tilde{C}_4C_4J_5 - C_4^2J_3 - C_4^2C_4J_3J_7;
\]

\[
D_3 = -C_4^2J_3.
\]  
(65)

\[
J_{0}^{(UT)}(M, m_h) = \frac{m_h^4}{(4\pi)^2M^2} \int_0^1 dx \int_0^x dy \frac{(y + 3 - 2x)^2 + 8(x - 1)}{[(y + 1 - 2x)^2 + y\frac{m_h^2}{M^2}]^2}.
\]  
(66)

Appendix C

Consider the contact potential given at any chiral order. In the matrix form defined in Sec. III.A, we have \(V = UTU^T, T = U^T\tau U\), with \(U(p) \equiv (1, p^2, p^4, \ldots)\) being a column vector and \(U^T\) being the transposed vector. Then the convolution in LSE could be factorized as \(VG_\tau T = U^T(p)\mathcal{I}rU(q)\), with the matrix \(\mathcal{I}\) being defined as follows,

\[
\mathcal{I} \equiv \int \frac{kdk^2}{(2\pi)^2} \frac{U(k)U^T(k)}{E - k^2/M + i\epsilon}.
\]  
(67)
The 3×3 case of $\mathcal{I}$ is given in Appendix A. It is easy to see that we could rewrite $\mathcal{I}$ as follows,

$$\mathcal{I} = I_0 U (\sqrt{ME}) U^T (\sqrt{ME}) + \tilde{T}(J_m), ME, m \neq 0,$$

where $I_0$ and $J_m$ with $m \neq 0$ are defined in Appendix A. Here $\tilde{T}$ is a real matrix independent of $I_0$. From Eq. (68) it follows that,

$$V G_0 T = U^T \lambda (I_0 U U^T + \tilde{T}) \tau U = I_0 V T + U^T \lambda \tilde{T} \tau U$$

Then, using the parametrization in Eq. (41), we find that, for on-shell momentum:

$$T^{-1} = V^{-1} - \mathcal{G} = V^{-1} - I_0 - \tilde{\mathcal{G}},$$

with $\tilde{\mathcal{G}} \equiv \frac{U^T \tilde{T} \tau U}{V T} = \mathcal{G} - I_0$. Now comparing this with the following representation of $T$ derived in Ref. [11] using the relation between the on-shell $T$-matrix and on-shell $K$-matrix, $T^{-1} = K^{-1} + \frac{M}{4\pi} \imath p$, we could find that,

$$\tilde{\mathcal{G}} = V^{-1} - K^{-1} + J_0,$$

that is, $\tilde{\mathcal{G}}$ must be a real number. But this real quantity is constructed with a complex $T$ that contains the infinite iterations of the complex number $I_0$ as given in (70). That means $I_0$ must cancel out in the infinite iteration, and hence must disappear in the real quantity $\tilde{\mathcal{G}}$. This in turn implies that, $J_0$, as the real part of $I_0$, does not appear in $\tilde{\mathcal{G}}$. Finally these facts will lead to following from of $T$-matrix constructed with local potential:

$$T^{-1} = \sum_i \frac{N_i([C_i], [J_i]) p^{2i}}{\sum_i D_i([C_i], [J_i]) p^{2j}} - I_0 = \sum_i \frac{N_i([C_i], [J_i]) p^{2i}}{\sum_i D_i([C_i], [J_i]) p^{2j}} + J_0 + \frac{M}{4\pi} \imath p,$$

with $[N_i, D_j]$ being independent of $J_0$. QED.
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\[
\begin{align*}
\sim \frac{-ig^2}{k^2 - m_h^2} & \quad \tilde{\mathcal{P}}_{\text{LE}} & \sim -i\mathcal{C}_0 = \frac{ig^2}{m_h^2}
\end{align*}
\]

Fig. 1 Tree vertex for 4-nucleon in UT (left) and in EFT (right).

\[
\begin{align*}
\tilde{\mathcal{P}}_{\text{LE}}
\end{align*}
\]

Fig. 2 Box diagram in UT (left) and bubble diagram in EFT (right).