A distinguished geometry perspective on multi-time affine quadratic Lagrangians

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Abstract. For a space endowed with a general quadratic multi-time Lagrangian and an associated non-linear connection, the paper constructs the main Riemann-Lagrange distinguished geometric objects (linear connection, torsion and curvature). Some Einstein-like equations for a canonical geometrical abstract multi-time gravitational potential, together with a trivial geometrical abstract electromagnetic-like theory, are derived from the given quadratic affine multi-time Lagrangian and its associated non-linear connection.

M.S.C. 2010: 70S05, 53C60, 53C80.

Key words: 1-jet spaces; quadratic multi-time Lagrangian; nonlinear connection; \(d\)-torsions and \(d\)-curvatures; Einstein-like equations.

1 Introduction

It is notable fact that quadratic multi-time Lagrangians are present in most physical domains. Illustrative examples are present in the theory of elasticity [7], the dynamics of ideal fluids, the magnetohydrodynamics [3], [4] and in the theory of bosonic strings [2]. This fact encourages the natural attempt of geometrization for quadratic multi-time Lagrangians. This framework implies, as can be seen below, the introduction of a corresponding Riemann-Lagrange geometry on 1-jet spaces.

2 The generalized multi-time Lagrange space of a quadratic Lagrangian

Let \( U^{(\alpha)}_{(i)}(t^\gamma, x^k) \) be a \(d\)-tensor (distinguished tensor, in brief) on the 1-jet space \(J^1(T, M)\), and let \( F : T \times M \to \mathbb{R} \) be a smooth function. We further consider the quadratic multi-time Lagrangians \( L : J^1(T, M) \to \mathbb{R} \), of the form

\[
L = G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k) x^i x^j + U^{(\alpha)}_{(i)}(t^\gamma, x^k) x^i + F(t^\gamma, x^k),
\]
whose fundamental vertical metrical $d$-tensor
\[ G^{(\alpha)(\beta)}(t^\gamma, x^k) = \frac{1}{p} \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} \]
is symmetric, of rank $n = \dim M$ and has a constant signature with respect to the indices $i$ and $j$. By using a semi-Riemannian metric $h = (h_{\alpha\beta}(t^\gamma))_{\alpha,\beta=1}^{Tp}$ on $T$ and by considering the canonical Kronecker $h$-regular vertical metrical $d$-tensor attached to the Lagrangian function (2.1), given by
\[ G^{(\alpha)(\beta)}(t^\gamma, x^k) = \frac{1}{p} h^{\alpha\beta}(t^\gamma) h_{\mu\nu}(t^\gamma) G^{(\mu)(\nu)}(t^\gamma, x^k), \]
where $p = \dim T$, we then consider the pair
\[ GL(J) = \left(J^1(T, M), G^{(\alpha)(\beta)}(t^\gamma, x^k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k)\right) \]
which is a generalized multi-time Lagrange space, whose spatial metrical $d$-tensor is given by the formula
\[ g_{ij}(t^\gamma, x^k) = \frac{1}{p} h_{\mu\nu}(t^\gamma) G^{(\mu)(\nu)}(t^\gamma, x^k). \]

**Definition 2.1.** We call the space $GL(J)$ the canonical generalized multi-time Lagrange space associated with the quadratic Lagrangian function given by (2.1).

In order to construct the main Riemann-Lagrange geometric objects of the space $GL(J)$, i.e., its $d$-linear connection, torsions and curvatures, one needs a nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on $J^1(T, M)$. The fundamental vertical metrical $d$-tensor $G^{(\alpha)(\beta)}(t^\gamma, x^k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k)$, where $g_{ij}$ is given by (2.2), produces the following natural nonlinear connection [6, p. 88]:
\[ M^{(i)}_{(\alpha)\beta} = -H^\gamma_{\alpha\beta} x^i_{\gamma}, \quad N^{(i)}_{(\alpha)j} = \Gamma^i_{jm} x^m + \frac{g^{im}}{2} \frac{\partial g_{jm}}{\partial x^\alpha}, \]
where $H^\gamma_{\alpha\beta}$ are the Christoffel symbols of the temporal semi-Riemannian metric $h_{\alpha\beta}$,
\[ \Gamma^i_{jk}(t^\mu, x^m) = \frac{g^{ir}}{2} \left( \frac{\partial g_{jr}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r} \right) \]
are the generalized Christoffel symbols of the spatial metric $g_{ij}(t^\gamma, x^k)$. Let
\[ \left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^\alpha} \right\} \subset \mathcal{X}(J^1(T, M)) \text{ and } \{ dt^\alpha, dx^i, \delta x^i \} \subset \mathcal{X}^*(J^1(T, M)) \]
be the adapted bases of the nonlinear connection $\Gamma$, where
\[ \delta = \frac{\partial}{\partial t^\alpha}, \quad M^{(i)}_{(\beta)\alpha} \frac{\partial}{\partial x^\beta}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(i)}_{(\beta)\alpha} \frac{\partial}{\partial x^\beta}. \]
\[ \delta x^i = dx^i + M^{(i)}_{(\alpha)\beta} dt^\beta + N^{(i)}_{(\alpha)j} dx^j. \]

\[ ^{1}\text{Throughout the rest of the paper, the constructed geometrical objects will be expressed in local adapted components with respect to previous adapted bases.} \]
3 The Riemann-Lagrange geometry of the space $GL(J)$

We further adopt the formalism introduced and developed in core seminal research studies ([1],[6],[5]), and accordingly provide the main results of the Riemann-Lagrange geometry of the generalized multi-time Lagrange space $GL(J)$.

**Theorem 3.1** (the Cartan linear connection). The canonical Cartan linear connection of the space $GL(J)$ is given by its adapted components

$$C_T = \left( H^\gamma_{\alpha\beta}, G^k_{\gamma j}, L^i_{jk} = \Gamma^i_{jk}, C_{j(k)}^{(\gamma)} = 0 \right),$$

where $G^k_{\gamma j} = \frac{g^{km}}{2} \frac{\partial g_{mj}}{\partial t^\gamma}$.

**Proof.** The formulas which describe the adapted coefficients of the Cartan canonical connection are given by

$$G^k_{\gamma j} = \frac{g^{km}}{2} \frac{\delta g_{mj}}{\delta t^\gamma} = \frac{g^{km}}{2} \frac{\partial g_{mj}}{\partial t^\gamma}, \quad L^i_{jk} = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\partial x^k} + \frac{\delta g_{km}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^m} \right) = \Gamma^i_{jk},$$

$$C_{j(k)}^{(\gamma)} = \frac{g^{im}}{2} \left( \frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) = 0.$$

□

**Remark 3.1.** The generalized Cartan canonical connection $C_T$ of the space $GL(J)$ satisfies the metricity conditions

$$h_{\alpha\beta|\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta}^{(\gamma)}(k) = 0, \quad g_{ij|\gamma} = g_{ij|k} = g_{ij}^{(\gamma)}(k) = 0,$$

where " $\beta_j$", " $|k$" and " $^{(\gamma)}(k)$" are the local covariant derivatives produced by the Cartan connection $C_T$.

**Theorem 3.2.** The generalized multi-time Lagrange space $GL(J)$ is characterized by the following adapted torsion $d$-tensors:

$$T^m_{\alpha j} = -G^m_{\alpha j}, \quad P^{(\beta)}_{i(j)} = C^{(\beta)}_{i(j)} = 0, \quad P^{(m)}_{i(j)}(\beta) = \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^j_\beta} - \delta^\beta_\mu L^m_{ij} = 0,$$

$$P^{(m)}_{(\mu)\alpha(j)} = \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j_\beta} - \delta^\beta_\mu G^m_{\alpha j} + \delta^m_j H^\alpha_{\mu \beta} = -\delta^\alpha_\mu C^{m}_{\alpha j},$$

$$R^{(m)}_{(\mu)\alpha\beta} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta t^\beta} - \frac{\delta M^{(m)}_{(\mu)\beta}}{\delta t^\alpha}, \quad R^{(m)}_{(\mu)\alpha j} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta x^j_\beta} - \frac{\delta N^{(m)}_{(\mu)\beta}}{\delta x^j_\alpha},$$

$$R^{(m)}_{(\mu)ij} = \frac{\delta N^{(m)}_{(\mu)ij}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)ij}}{\delta x^i}, \quad S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)} = \delta^\alpha_\mu C^{m(i)(\beta)}_{j(i)} - \delta^\beta_\mu C^{m(\alpha)(i)}_{j(i)} = 0.$$

Using the general formulas from [6], which provide the curvature $d$-tensors of a generalized multi-time Lagrange space, we obtain the following result:
Theorem 3.3. The generalized multi-time Lagrange space $GL(J)$ is characterized by the following adapted curvature d-tensors:

$$H^\alpha_{\eta\beta\gamma} = \frac{\partial H^\alpha_{\eta\beta}}{\partial t^\gamma} - \frac{\partial H^\alpha_{\eta\gamma}}{\partial t^\beta} + H^\mu_{\eta\gamma} H^\alpha_{\mu\beta} - H^\mu_{\eta\beta} H^\alpha_{\mu\gamma},$$

$$R^l_{i\beta\gamma} = \frac{\partial G^l_{i\beta}}{\partial t^\gamma} - \frac{\partial G^l_{i\gamma}}{\partial t^\beta} + G^m_{i\gamma} G^l_{m\beta} - G^m_{i\beta} G^l_{m\gamma},$$

$$R^l_{i\tilde{k}\beta} = \frac{\partial \Gamma^l_{ik}}{\partial x^\beta} - \frac{\partial \Gamma^l_{ik}}{\partial x^\beta} + \Gamma^m_{ik} \Gamma^l_{m\beta} - \Gamma^m_{i\beta} \Gamma^l_{m\gamma},$$

$$R^l_{i\beta j} = \frac{\partial \Gamma^l_{ij}}{\partial x^\beta} - \frac{\partial \Gamma^l_{ij}}{\partial x^\beta} + \Gamma^m_{ij} \Gamma^l_{m\beta} - \Gamma^m_{i\beta} \Gamma^l_{m\gamma},$$

$$P^l_{i\beta(k)} = 0, \quad P^l_{ij(k)} = 0, \quad S^l_{i(j)(k)} = 0.$$

4 Generalized multi-time field theories on the space $GL(J)$

4.1 Multi-time gravitational field

The fundamental vertical metrical d-tensor of the space $GL(J)$ naturally induces the multi-time gravitational h-potential $G$, defined by

$$G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h_{\alpha\beta} g_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta.$$

We postulate that the generalized Einstein-like equations corresponding to the multi-time gravitational h-potential of the space $GL(J)$, are of the form

$$(4.1) \quad \text{Ric}(CT) - \frac{Sc(CT)}{2} G = \mathcal{K} \mathcal{T},$$

where $\text{Ric}(CT)$ represents the Ricci d-tensor associated with the generalized Cartan connection, $Sc(CT)$ is the scalar curvature, $\mathcal{K}$ is the Einstein curvature scalar and $\mathcal{T}$ is the stress-energy d-tensor of matter.

Using now the general formulas from [6], we infer the following:

Proposition 4.1. The Ricci d-tensor $\text{Ric}(CT)$ of the space $GL(J)$ has the following adapted components:

$$R_{\alpha\beta} := H_{\alpha\beta} = H^\mu_{\alpha\beta\mu}, \quad R_{i(j)}^{(a)} := P_{i(j)}^{(a)} = -P_{m(j)}^{m(a)} = 0,$$

$$R_{(i)(j)}^{(a)} := P_{(i)(j)}^{(a)} = P_{m(j)}^{m(a)} = 0, \quad R_{(i)}^{(a)} := P_{(i)}^{(a)} = P_{m(i)}^{m(a)} = 0,$$

$$R_{(i)(j)}^{(a)(\beta)} := S_{(i)(j)}^{(a)(\beta)} = S_{m(j)(m)}^{m(\beta)(a)} = 0, \quad R_{(i)}^{a} = R_{m(i)}^{a}, \quad R_{ij} = R_{mij}^{m}.$$

Corollary 4.2. The scalar curvature $Sc(CT)$ of the space $GL(J)$ is given by

$$Sc(CT) = H + R,$$

where $H = h^{\alpha\beta} H_{\alpha\beta}$ and $R = g^{ij} R_{ij}$. 

Then we can state the main result of the generalized Riemann-Lagrange geometry of the multi-time gravitational field:

**Theorem 4.3.** The global generalized Einstein-like equations (4.1) of the space $GL(J)$ have the local form

\[
\begin{cases}
H_{\alpha\beta} - \frac{H + R}{2} h_{\alpha\beta} = \kappa T_{\alpha\beta} \\
R_{ij} - \frac{H + R}{2} g_{ij} = \kappa T_{ij} \\
- \frac{H + R}{2} h^{\alpha\beta} g_{ij} = \kappa T^{(\alpha)(\beta)}_{(i)(j)} \\
0 = T_{\alpha i}, \quad R_{\alpha\alpha} = \kappa T_{\alpha\alpha}, \quad 0 = T^{(\alpha)}_{(i)\beta} \\
0 = T^{(\beta)}_{(\alpha)i}, \quad 0 = T^{(\alpha)}_{(i)j},
\end{cases}
\]

where $T_{AB}, A, B \in \{\alpha, i, (\alpha), (i)\}$ are the adapted components of the stress-energy $d$-tensor $T$.

### 4.2 The multi-time electromagnetism

The multi-time electromagnetic theory of the space $GL(J)$ relies on the metrical deflection $d$-tensors

\[
D^{(\alpha)}_{(ij)} = \left[ G^{(\alpha)(\mu)}_{(i)(m)} x^m \right]_{ij} = - \frac{h^{\alpha\mu}}{2} \frac{\partial g_{ij}}{\partial t^\mu},
\]

\[
d^{(\alpha)(\beta)}_{(i)(j)} = \left[ G^{(\alpha)(\mu)}_{(i)(m)} x^m \right]_{(i)}^{(\beta)} = h^{\alpha\beta} g_{ij}.
\]

This yields the electromagnetic-like 2-form of the space $GL(J)$, via:

\[
F = F^{(\alpha)}_{(i)j} \delta x^i_\alpha \wedge dx^j + f^{(\alpha)(\beta)}_{(i)(j)} \delta x^i_\alpha \wedge \delta x^j_\beta,
\]

where

\[
F^{(\alpha)}_{(i)j} = \frac{1}{2} \left[ D^{(\alpha)}_{(i)j} - D^{(\alpha)}_{(j)i} \right] = 0, \quad f^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \left[ d^{(\alpha)(\beta)}_{(i)(j)} - d^{(\alpha)(\beta)}_{(j)(i)} \right] = 0.
\]

Since $F = 0$, we infer that the multi-time electromagnetic theory of the space $GL(J)$ is formally trivial.

**Acknowledgements.** A version of this paper was presented at the XIV-th International Conference "Differential Geometry and Dynamical Systems" (DGDS-2020), 27 - 29 August 2020 *ONLINE* *[Bucharest, Romania]*.

Many thanks go to Professor Vladimir Balan, whose useful advice helped to improve this paper.
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