Electric Charges and Magnetic Monopoles in Gravity’s Rainbow

Remo Garattini∗
Università degli Studi di Bergamo, Dipartimento di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy and
I.N.F.N. - sezione di Milano, Milan, Italy.

Barun Majumder†
Indian Institute of Technology Gandhinagar and
Ahmedabad, Gujarat 382424, India

In this work, we explore the possibility that quantum fluctuations induce an electric or magnetic charge or both, in the context of Gravity’s Rainbow. A semi-classical approach is adopted, where the graviton one-loop contribution to a classical energy in a background spacetime is computed through a variational approach with Gaussian trial wave functionals. The energy density of the graviton one-loop contribution, in this context, acts as a source for the electric/magnetic charge. The ultraviolet (UV) divergences, which arise analyzing this procedure, are kept under control with the help of an appropriate choice of the Rainbow’s functions. In this way we avoid the introduction of any regularization/renormalization scheme. A comparison with the observed data lead us to determine the size of the electron and of the magnetic monopole which appear to be of Planckian size. Both results seem to be of the same order for a Schwarzschild and a de Sitter background, respectively. Estimates on the magnetic monopole size have been done with the help of the Dirac quantization procedure. We find that the monopole radius is larger than the electron radius. Even in this case the ratio between the electric and magnetic monopole radius appears to be of the same order for both geometries.

I. INTRODUCTION

It was Andrei Sakharov in 1967[1] who first conjectured the idea of Induced gravity (or emergent gravity), namely space-time background emerges as a mean field approximation of underlying microscopic degrees of freedom, similar to the fluid mechanics approximation of Bose–Einstein condensates. This means that some basic ingredients of General Relativity like the gravitational Newton’s constant can be computed by means of quantum fluctuations of some matter fields. This idea is opposed to the concept of “charge without charge” and “mass without mass” arising from the spacetime foam picture of John A. Wheeler[2], where the matter properties emerge as a geometrical feature of space time. In a foamy spacetime topological fluctuation appear at the Planck scale, meaning that spacetime itself undergoes a deep and rapid transformation in its structure. Wheeler also considered wormhole-type solutions as objects of the spacetime quantum foam connecting different regions of spacetime at the Planck scale. Although the Sakharov approach has the appealing property of being renormalizable “ab initio” because involves only quantum fluctuations of matter fields described by bosons and fermions, the Wheeler picture involves quantum fluctuations of the gravitational field alone and since one of the purposes of Quantum Gravity should be a realization of a theory combining Quantum Field Theory with General Relativity, it appears that spacetime foam is the right candidate for such a description. Unfortunately, every proposal of a Quantum Gravity model except string theory has to face with Ultra Violet (UV) divergences. Recently a proposal which uses a distortion of the gravitational field at the Planck scale, named as Gravity’s Rainbow[3, 4] has been considered to compute Zero Point Energy (ZPE) to one loop[5, 6]. The interesting point is that such a distortion enters into the background metric and activates at Planck’s scale keeping under control UV divergences. Briefly, the situation is the following: one introduces two arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$, denoted as Rainbow’s functions, with the only assumption that

$$\lim_{E/E_P \to 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \to 0} g_2(E/E_P) = 1.$$  \hspace{1cm} (1)

∗Electronic address: Remo.Garattini@unibg.it
†Electronic address: barunbasanta@iitgn.ac.in
On a general spherical symmetric metric such functions come into play in the following manner

\[ ds^2 = -N^2(r) \, \frac{dt^2}{g_1^2(E/E')} + \frac{dr^2}{g_2^2(E/E') \left( 1 - \frac{b(r)}{r} \right)} + \frac{r^2}{g_2^2(E/E')} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

where \( N(r) \) is the lapse function and \( b(r) \) is denoted the shape function. The purpose of this paper is to approach one of the aspects of the Wheeler’s ideas, namely “charge without charge”. In particular, we will investigate if quantum fluctuations of the gravitational field can be considered as a source for the electric/magnetic charge. Note that one tentative to realize a “charge without charge” has been described in Ref.[7]. However, due to UV divergences a regularization/renormalization has been used to obtain finite results and even if a renormalization group-like equation has been used, the final result depends also to the renormalization point scale \( \mu_0 \). Here the renormalization point is fixed since the beginning at the Planck scale because of the Rainbow’s functions. It is clear that, if an electric/magnetic charge can be generated, this information is encoded in the Einstein’s field equations. These equations are simply summarized by

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \]

where

\[ T_{\mu\nu} = \frac{1}{4\pi} \left[ F_{\mu\gamma} F_{\nu}^{\gamma} - \frac{1}{4} g_{\mu\nu} F_{\gamma\delta} F^{\gamma\delta} \right] \]

is the energy-momentum tensor of the electromagnetic field, \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \), \( \kappa = 8\pi G \) with \( G \) the Newton’s constant and where we have neglected the contribution of the cosmological constant \( \Lambda \). The electromagnetic field strength tensor \( F_{\mu\nu} \) can be computed with the help of the electromagnetic potential \( A_{\mu} \) which, in the case of a pure electric field assumes the form \( A_{\mu} = (Q_e/r, 0, 0, 0) \) while in the case of pure magnetic field, the form is \( A_{\mu} = (0, 0, 0, -Q_m \cos \theta) \). \( Q_e \) and \( Q_m \) are the electric and magnetic charge respectively. It is interesting to note that \( Q_e \) and \( Q_m \) contribute in the same way to the electromagnetic hamiltonian density. Indeed for the electric charge, the on-shell contribution of \( T_{\alpha\beta} u^\alpha u^\beta \) is

\[ T_{\mu\nu} u^\mu u^\nu = \frac{1}{8\pi} (F_{01})^2 = \frac{1}{8\pi} \frac{Q_e^2}{r^4} = \rho_e, \]

while when we consider the magnetic charge, we get

\[ T_{\mu\nu} u^\mu u^\nu = \frac{1}{8\pi} (F_{23})^2 = \frac{1}{8\pi} \frac{Q_m^2}{r^4} = \rho_m. \]

\( u^\mu \) is a time-like unit vector such that \( u \cdot u = -1 \). However, while the electric charge exists, for the magnetic charge or magnetic monopole, there is not experimental evidence of its existence\(^1\). For this reason the magnetic monopole search has a long history in theoretical physics: predicted by Paul Dirac in 1931, he showed that QED allows the existence of point-like magnetic monopole with charge

\[ Q_m = \frac{2\pi}{Q_e} \]

or an integer multiple of it\(^2\). Subsequently this prediction was confirmed also by Gerard ’t Hooft and Alexander Polyakov who showed that magnetic monopoles are predicted by all Grand Unified Theories (GUTs) \(^\text{[10]}\). Although monopoles of grand unified theories would have masses typically of the order of the unification scale \( (m \sim 10^{16} \text{ GeV}) \) but generally there are no tight theoretical constraints on the mass of a monopole. For this reason, the reference value of our calculation will be that of the electric charge. It is important to remark that in a system of units in which \( h = c = k = 1 \), that will be used throughout the paper\(^2\)

\[ e^2 = \frac{1}{137}. \]

\(^1\) Recently, it has been discovered that spin ices, frustrated magnetic systems, have effective quasiparticle excitations with magnetic charges very close to magnetic monopoles\(^3\).

\(^2\) For example in SI units

\[ \frac{e^2}{4\pi\hbar c\epsilon_0} = \frac{1}{137} \]

\(^3\) Recently, it has been discovered that spin ices, frustrated magnetic systems, have effective quasiparticle excitations with magnetic charges very close to magnetic monopoles.
The rest of the paper is organized in the following manner: in Section I we introduce the charge operator, in Section II we introduce the charge operator in presence of Gravity’s Rainbow specified to the Schwarzschild and to the de Sitter metric, in Section IV we will apply the charge operator to the magnetic monopole case and in Section V we will summarize and conclude.

II. THE CHARGE OPERATOR

To build the charge operator, we have to recognize the gravitational field as a fundamental field and see what implications we have on $Q_e$ and $Q_m$. For example, in Ref. [11], the role of $Q_e$ and $Q_m$ has been played by a cosmological constant interpreted as an eigenvalue of an associated Sturm-Liouville problem. To do this, we have introduced the Wheeler-DeWitt equation (WDW) [12] rearranging the Einstein’s field equations, to get:

$$H_\Lambda = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - 2\Lambda_c) = 0,$$

(10)

for the sourceless case and in presence of a cosmological term.

$$H_Q = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - H_M) = 0,$$

(11)

with a matter term and in absence of a cosmological constant. Note the formal similarity between Eqs. (10) and (11).

$G_{ijkl}$ is the supermetric defined as

$$G_{ijkl} = \frac{1}{2\sqrt{g}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl})$$

(12)

and $3R$ is the scalar curvature in three dimensions. $\pi^{ij}$ is called the supermomentum. This is the time-time component of the Einstein’s equations. It represents the invariance under time reparametrization and it works as a constraint at the classical level. Fixing our attention on the constraint (11), the explicit form of $H_M$ is easily obtained with the help of Eqs. (5) and (6), where one finds

$$H_M = 2\kappa T_{\alpha\beta} u^\alpha u^\beta = \frac{\kappa}{4\pi} \frac{Q^2_e + Q^2_m}{r^4},$$

(13)

Thus, the classical constraint $H_Q$ becomes

$$H_Q = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - \frac{\kappa}{4\pi} \frac{Q^2_e + Q^2_m}{r^4}) = 0.$$

(14)

For a spherically symmetric metric described by (2) with $g_1 (E/E_P) = g_2 (E/E_P) = 1$, it is easy to recognize that the classical constraint reduces to

$$3R = 2G \frac{Q^2_e + Q^2_m}{r^4} \quad \Rightarrow \quad b'(r) = G \frac{Q^2_e + Q^2_m}{r^4},$$

(15)

whose solution represents the Reissner-Nordström (RN) metric if

$$N^2 (r) = \left[ 1 - \frac{b(r)}{r} \right]^{-1}$$

(16)

and

$$b(r) = 2MG - \frac{G(Q^2_e + Q^2_m)}{r}.$$  

(17)

On the other hand, if $H_Q$ is promoted to an operator, we obtain the WDW equation in presence of an electromagnetic field

$$H_Q \Psi = \left[ (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - \frac{\kappa}{4\pi} \frac{Q^2_e + Q^2_m}{r^4}) \right] \Psi = 0.$$

(18)
If we multiply Eq. (18) by $\Psi^* \left[ g_{ij} \right]$ and we functionally integrate over the three spatial metric $g_{ij}$, we get

$$\int \mathcal{D} \left[ g_{ij} \right] \Psi^* \left[ g_{ij} \right] \dot{Q}_\Sigma \Psi \left[ g_{ij} \right] = -\frac{\kappa}{4\pi} \int \mathcal{D} \left[ g_{ij} \right] \left( \sqrt{g} \frac{Q^2 + Q^2_m}{r^4} \right) \Psi^* \left[ g_{ij} \right] \Psi \left[ g_{ij} \right]$$

(19)

and after having integrated over the hypersurface $\Sigma$, one can formally re-write the WDW equation as

$$\frac{\langle \Psi \left| \int_{\Sigma} d^3x \dot{Q}_\Sigma \right| \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\kappa}{4\pi} \frac{\int_{\Sigma} d^3x \left( \sqrt{g} \frac{Q^2 + Q^2_m}{r^4} \right) \dot{Q}_\Sigma}{\langle \Psi | \Psi \rangle},$$

(20)

where

$$\dot{Q}_\Sigma = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R$$

(21)

is the charge operator. The l.h.s. of Eq. (20) can be interpreted as an expectation value and the r.h.s. can be regarded as the associated eigenvalue with a weight. Thus, after a further reduction, one gets

$$\frac{\langle \Psi \left| \int_{\Sigma} d^3x \dot{Q}_\Sigma \right| \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\kappa}{4\pi} \int_{\Sigma} d^3x \sqrt{g} \frac{Q^2 + Q^2_m}{r^4}.$$  

(22)

Nevertheless the interpretation of Eq. (22) can be put on a different level. For example, if we fix the attention on the Schwarzschild metric, we can see that, on shell, the l.h.s. of Eq. (22) vanishes and this implies that the only compatible solution associated with the Schwarzschild metric at the classical level is $Q_e = Q_m = 0$. The same situation happens for the de Sitter (dS) and Anti-de Sitter (AdS) metric. On the other hand, it is immediate to recognize that the RN metric in Eq. (22) is consistent with the classical constraint (15). However, things can be different if we introduce quantum fluctuations. To this purpose we are interested to see if $Q^2_e \left( Q^2_m \right)$ can be considered as an eigenvalue of the Sturm-Liouville problem for some fixed background. Note that $Q^2_e \left( Q^2_m \right)$ is UV divergent and requires a regularization/renormalization scheme to remove divergences. To avoid such a scheme we compute $Q^2_e \left( Q^2_m \right)$ in presence of Gravity’s Rainbow.

### III. THE CHARGE OPERATOR IN PRESENCE OF GRAVITY’S RAINBOW

To compute the electric/magnetic charge in Gravity’s Rainbow, we begin with the line element (2). The form of the background is such that the shift function

$$N^i = -Nu^i = g^i_0 = 0$$

(23)

vanishes, while $N$ is the previously defined lapse function. Thus the definition of $K_{ij}$ implies

$$K_{ij} = -\frac{\dot{g}_{ij}}{2N} = \frac{g_{1} \left( E \right)}{g_{2} \left( E \right)} \tilde{K}_{ij},$$

(24)

where the dot denotes differentiation with respect to the time $t$ and the tilde indicates the quantity computed in absence of rainbow’s functions $g_1 \left( E \right)$ and $g_2 \left( E \right)$. For simplicity, we have set $E_P = 1$ in $g_1 \left( E/E_P \right)$ and $g_2 \left( E/E_P \right)$ throughout. The trace of the extrinsic curvature, therefore becomes

$$K = g^{ij} K_{ij} = g_1 \left( E \right) \tilde{K}$$

(25)

and the momentum $\pi^{ij}$ conjugate to the three-metric $g_{ij}$ of $\Sigma$ is

$$\pi^{ij} = \frac{\sqrt{g}}{2\kappa} \left( K g^{ij} - K^{ij} \right) = \frac{g_{1} \left( E \right)}{g_{2} \left( E \right)} \pi^{ij}.$$  

(26)

Recalling that $u_\mu = (-N, 0, 0, 0)$, in presence of Gravity’s Rainbow we have the following modification

$$u_\mu = \frac{\tilde{u}_\mu}{g_{1} \left( E \right)} \implies u^\mu = g_{1} \left( E \right) \tilde{u}^\mu$$

(27)
which is useful to compute the distorted electromagnetic energy-momentum tensor. Indeed, from Eqs. (33) and (36), we find

\[
T_{\mu\nu} u^\mu u^\nu = \frac{g^2(E) \tilde{g}_{11}}{8\pi} \left( \hat{F}_{01} \right)^2 \tilde{u}^\mu \tilde{u}^\nu g_1^2(E) = \frac{1}{8\pi} \frac{Q^2}{r^4} g_1^2(E) g_2^2(E) \left( \hat{\rho}_e g_1^2(E) g_2^2(E) \right),
\]

for the electric charge, while when we consider the magnetic charge, we get

\[
T_{\mu\nu} u^\mu u^\nu = \frac{\tilde{g}_{00}}{8\pi g_1^2(E)} \left( \hat{F}_{23} \right)^2 \tilde{u}^\mu \tilde{u}^\nu \tilde{g}^{22} \tilde{g}^{33} g_1^2(E) g_2^2(E) = \frac{1}{8\pi} \frac{Q^2_m}{r^4} g_2^2(E) = \frac{1}{8\pi} \tilde{\rho}_m g_2^2(E).
\]

By defining

\[
\rho_{g_{1g_{2em}}} = \hat{\rho}_e g_1^2(E) + \tilde{\rho}_m g_2^2(E),
\]

we can write the WDW equation for a background described by (2). From Eq. (18) we find that \( \mathcal{H} \Psi = 0 \) becomes

\[
\mathcal{H} \Psi = \left( 2\kappa \frac{g_1^2(E)}{g_2^2(E)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} \right) \tilde{g} \left( \hat{R} - \kappa \frac{4\pi}{\rho_{g_{1g_{2em}}}} \right) \Psi = 0,
\]

where we have used the following property on \( R \)

\[
R = g^{ij} R_{ij} = g_2^2(E) \tilde{R}
\]

and where

\[
G_{ijkl} = \frac{1}{2\sqrt{\tilde{g}}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) = \frac{\tilde{G}_{ijkl}}{g_2(E)}.
\]

On the other hand we can rewrite Eq. (31) as

\[
\frac{\langle \Psi | \int d^3x \tilde{\dot{Q}}_\Sigma | \Psi \rangle}{\langle \Psi | \Psi \rangle} = - \frac{1}{8\pi} \frac{g_1^2(E)}{g_2^2(E)} \frac{\langle \Psi | \int d^3x \sqrt{\tilde{g}} \rho_{g_{1g_{2em}}} | \Psi \rangle}{\langle \Psi | \Psi \rangle},
\]

where we have defined the distorted charge operator

\[
\tilde{Q}_\Sigma = \frac{2\kappa}{\sqrt{\tilde{g}}} \frac{g_1^2(E)}{g_2^2(E)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E)} \tilde{R}.
\]

Now let us consider \( g_{ij} = \tilde{g}_{ij} + h_{ij} \) where we want to study the quantum fluctuation \( h_{ij} \) around a background metric \( \tilde{g}_{ij} \). Since \( \tilde{Q}_\Sigma \) has a term quadratic in momenta so the three-scalar curvature term also needs to be expanded up to the quadratic order. The transverse traceless (TT) tensor contribution of Eq. (34) to one loop can be expressed as

\[
\tilde{Q}_\Sigma = \frac{1}{4} \int d^3x \sqrt{\tilde{g}} \tilde{G}_{ijkl} \left[ \frac{2\kappa}{g_2^2(E)} \tilde{K}^{-1\perp}(x, x)_{ijkl} + \frac{1}{2\kappa g_2(E)} \left( \tilde{\Delta}_L^m \tilde{K}^{\perp}(x, x) \right)_{ijkl} \right] .
\]

\( \tilde{\Delta}_L^m \) is the modified Lichnerowicz operator and its operation is expressed as

\[
(\tilde{\Delta}_L^m h^\perp)_{ij} = (\Delta_L h^\perp)_{ij} - 4 R^k_{ij} h^\perp_{kj} + 3 R h^\perp_{ij}.
\]

Now when we consider the eigenvalue equation

\[
(\tilde{\Delta}_L^m h^\perp)_{ij} = E^2 h^\perp_{ij}
\]

we find

\[
(\tilde{\Delta}_L^m h^\perp)_{ij} = \frac{E^2}{g_2^2(E)} h^\perp_{ij}
\]

and the propagator \( K^\perp(x, y)_{ijkl} \) can be represented as

\[
K^\perp(x, y)_{ijkl} = \bar{K}^\perp(x, y)_{ijkl} = \sum_{\tau} \frac{h^{(\tau)\perp}_i(x) \tilde{h}^{(\tau)\perp}_{jl}(y)}{2\lambda(\tau) \ g_2^2(E)}
\]
where $\tilde{h}_{\alpha}^{(\tau)}(\tilde{x})$ are the eigenfunctions of $\tilde{\Delta}_{\alpha}^{T\tau}$. $\tau$ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq. (36). The expectation value of $\tilde{Q}_{\Sigma}^{\lambda}$ is obtained by plugging the propagator in Eq. (39) and minimizing with respect to the variational function $\lambda(\tau)$. Therefore the one-loop charge in Gravity’s Rainbow for the TT tensors is

$$Q_{\Sigma} = -\frac{1}{2} \sum_{\tau} g_1(E)g_2(E) \left[ \sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right],$$  \hspace{1cm} (41)$$

where

$$Q_{\Sigma} = \frac{1}{8\pi} \frac{g_1^2(E)}{g_2(E)} \int_{\Sigma} d^3x \sqrt{-g} \rho_{g_1g_2em}.$$  \hspace{1cm} (42)$$

The expression in Eq. (41) makes sense when $E_2^2(\tau) > 0$. Using the WKB approximation as used by ‘t Hooft in the brick wall problem we can evaluate Eq. (41) explicitly. Extracting the energy density, we can write

$$\frac{1}{2} \frac{g_1^2(E)}{g_2(E)} \rho_{g_1g_2em} = -\frac{1}{3\pi^2} \sum_{i=1}^{2} \int_{E_*}^{\infty} E_i g_1(E)g_2(E) \frac{d}{dE_i} \left[ \frac{E_i^2}{g_2^2(E)} - m_i^2(r) \right] \frac{1}{E_i} dE_i,$$  \hspace{1cm} (43)$$

where $E^*$ is the value which annihilates the argument of the root and where we have defined two $r$-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\left\{ \begin{array}{l}
    m_1^2(r) = \frac{6}{\pi} \left( 1 - \frac{b(r)}{r} \right) + \frac{3}{2\pi} b' (r) - \frac{3}{2\pi} b (r) \\
    m_2^2(r) = \frac{6}{\pi} \left( 1 - \frac{b(r)}{r} \right) + \frac{1}{2\pi} b' (r) + \frac{3}{2\pi} b (r)
\end{array} \right. \hspace{1cm} (44)$$

We have hitherto used a generic form of the background. We now fix the attention on some backgrounds which have the following property

$$m_0^2(r) = m_2^2(r) = -m_1^2(r), \hspace{1cm} \forall r \in (r_t, r_1).$$  \hspace{1cm} (45)$$

For example, the Schwarzschild background represented by the choice $b(r) = r_t = 2MG$ satisfies the property (45) in the range $r \in [r_t, 5r_t/4]$. Similar backgrounds are the Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter. On the other hand other backgrounds, like dS, AdS and Minkowski have the property

$$m_0^2(r) = m_2^2(r) = m_1^2(r), \hspace{1cm} \forall r \in (r_t, \infty).$$  \hspace{1cm} (46)$$

A. The Schwarzschild Case

Before going on, we examine the classical constraint for the Schwarzschild metric. From Eq. (31), the condition $\mathcal{H} = 0$ reduces to

$$\mathcal{R} - \frac{\kappa}{4\pi} \rho_{g_1g_2em} = 0 \hspace{1cm} \Rightarrow \hspace{1cm} \frac{Q^2}{r^4} g_1^2 (E) + \frac{Q^2}{r^4} g_2^2 (E) = 0,$$  \hspace{1cm} (47)$$

leading to the only solution $Q_x = Q_m = 0$. From the quantum point of view, the condition (45) is satisfied by the Schwarzschild background and Eq. (43) becomes

$$\frac{1}{2} \frac{g_1^2(E/E_P)}{g_2(E/E_P)} \rho_{g_1g_2em} = -\frac{1}{3\pi^2} (I_+ + I_-),$$  \hspace{1cm} (48)$$

where

$$I_+ = 3 \int_{E_*}^{\infty} E^2 g_1(E/E_P) \sqrt{\frac{E^2}{g_2^2(E/E_P)} + m_0^2(r)} \left[ \frac{E}{g_2(E/E_P)} \right] dE,$$  \hspace{1cm} (49)$$

and

$$I_- = 3 \int_{E_*}^{\infty} E^2 g_1(E/E_P) \sqrt{\frac{E^2}{g_2^2(E/E_P)} - m_0^2(r)} \left[ \frac{E}{g_2(E/E_P)} \right] dE.$$  \hspace{1cm} (50)$$
For convenience we have reintroduced the Planck energy scale $E_P$ in Eqs. (49) and (50). It is clear that the final result is strongly dependent on the choices we can do about $g_1(E/E_P)$ and $g_2(E/E_P)$. Nevertheless some classes of functions cannot be considered because they do not lead to a finite result. One has to observe that, due to the presence of a minus sign in front of the r.h.s. of Eq. (48), a choice of the form

$$g_2^{-2}(E/E_P) = g_1(E/E_P),$$

is forbidden. Indeed, even if we can calculate exactly $I_+$ and $I_-$, there is no chance to reverse their sign and this means that $Q_x(Q_m^2)$ should be everywhere negative. For the same reason one has to discard another appealing choice, namely

$$g_2^2(E/E_P) = g_1^2(E/E_P),$$

where the l.h.s. of Eq. (48) becomes independent on the rainbow’s functions. In Ref. [5], the following proposal has been considered

$$g_1(E/E_P) = (1 + \beta \frac{E}{E_P}) \exp(-\alpha \frac{E^2}{E_P^2}) \quad g_2(E/E_P) = 1 \quad \alpha, \beta \in \mathbb{R},$$

(53)

to estimate the cosmological constant. The choice of the gaussian was dictated by a comparison with a cosmological constant computation in the framework of Noncommutative theory [13]. By defining

$$x = \sqrt{\frac{m_k^2(r)}{E_P}}$$

(54)

and following the same steps of Ref. [5], one finds

$$\frac{g_1^2(E/E_P)}{2E_P} \rho_{g_1 g_2 cm} = -\frac{1}{2\pi^2} f(\alpha, \beta; x),$$

(55)

where

$$f(\alpha, \beta; x; r) = \left[ \frac{x^2}{\alpha} \cosh \left( \frac{\alpha x^2}{2} \right) K_1 \left( \frac{\alpha x^2}{2} \right) \right]$$

$$+ \beta \left( \frac{3x}{2\alpha^2} - \frac{x^2 \sqrt{\pi}}{\alpha^2} \sinh (\alpha x^2) + \frac{\sqrt{\pi}}{2\alpha} \cos \left( \alpha x^2 \right) + \frac{\sqrt{\pi}}{2\alpha^2} \left( x^2 - \frac{3}{2\alpha} \right) e^{\alpha x^2} \text{erf} \left( \frac{\sqrt{\alpha} x}{2} \right) \right)$$

(56)

and where $K_1(x)$ is the Bessel function of the first kind and $\text{erf}(x)$ is the error function. For the Schwarzschild solution, Eq. (54) becomes

$$x = \sqrt{\frac{m_k^2(r)}{E_P}} = \begin{cases} \frac{3MG}{r^2E_P^2} & r > 2MG \\ \frac{3}{8(MG)^2 E_P^4} & r = 2MG \end{cases}$$

(57)

and its behavior is

$$x \rightarrow \begin{cases} \infty & \text{when} \quad M \rightarrow 0 \quad \text{for} \quad r = 2MG \\ 0 & \text{when} \quad M \rightarrow 0 \quad \text{for} \quad r > 2MG \end{cases}$$

(58)

while

$$x \rightarrow \begin{cases} 0 & \text{when} \quad M \rightarrow \infty \quad \text{for} \quad r = 2MG \\ \infty & \text{when} \quad M \rightarrow \infty \quad \text{for} \quad r > 2MG \end{cases}$$

(59)

The behavior in Eq. (58) will be discarded, because it does not represent a physical realization. Therefore, we fix our attention on Eq. (55). For large $x$, the r.h.s. of Eq. (55) becomes:

$$\frac{g_1^2(E/E_P)}{2E_P} \rho_{g_1 g_2 cm} \simeq -\frac{(2\beta \alpha^{3/2} + \sqrt{\pi} \alpha^2)x}{4\pi^2 \alpha^{1/2}} - \frac{8\beta \alpha^{5/2} + 3\sqrt{\pi} \alpha^3}{16\pi^2 \alpha^{11/2} x} + \frac{3}{128\pi^2} \frac{16\beta \alpha^{7/2} + 5\sqrt{\pi} \alpha^4}{\alpha^{15/2} x^3} + O(x^{-4}),$$

(60)
while for small $x$ we obtain
\[
\frac{g^2(E/E_P)}{2E_P^4} \rho_{g_{zz}} \approx -\frac{4\alpha^{5/2} + 3\sqrt{\pi} \beta_d^2}{4\pi^2 \alpha^{9/2}} - \left(2\sqrt{\alpha} \gamma + 2\sqrt{\alpha} \ln\left(\frac{\pi \alpha \sqrt{e}}{2\sqrt{\pi} \beta_d}ight) - 2\sqrt{\pi} \beta_d\right) x^4 \frac{2\beta_d}{15\pi^2} x^5 + O(x^7),
\]
where $\gamma$ is the Euler's constant. It is straightforward to see that if we set
\[
\beta = -\frac{\sqrt{\alpha \pi}}{2},
\]
then the linear divergent term of the asymptotic expansion (60) disappears and Eq. (65) vanishes for large $x$, namely when $r = r_t = 2MG$ and $M \to 0$. Therefore on the throat $r_t$ one gets
\[
Q_\infty^2(\alpha, \beta, r_t) = -\frac{r_t^4 E_P^4}{\pi^2} \cdot f\left(\alpha, -\frac{\sqrt{\alpha \pi}}{2}, 3 \frac{3}{2r_t^2 E_P^2}\right),
\]
where we have set $Q_m^2 = 0$ and where we have kept only the cis-planckian region which is not suppressed by the Rainbow's function in the l.h.s. of Eq.(63). If we impose that,
\[
Q_\infty^2(\alpha, \beta, \bar{r}_t) = \frac{1}{137} = 0.73 \times 10^{-2},
\]
then we find
\[
\bar{r}_t = 0.295/E_P,
\]
where we have fixed $\alpha = 1/4$. The situation is not much different if we choose
\[
\beta = -\frac{4}{3} \frac{\sqrt{\alpha}}{\pi}
\]
Indeed, Eq.(65) becomes
\[
Q_\infty^2(\alpha, \beta, r_t) = -\frac{r_t^4 E_P^4}{\pi^2} \cdot f\left(\alpha, -\frac{4}{3} \frac{\sqrt{\alpha}}{\pi}, 3 \frac{3}{2r_t^2 E_P^2}\right)
\]
and fixing again $Q_\infty^2$ like in Eq.(64), one finds
\[
\bar{r}_t = 0.571/E_P.
\]
Both the solution require a sub-planckian throat. It is clear that the comparison of the fine structure constant $1/137$ with $Q_\infty^2$ is not possible. However in Section we will discuss how the charge operator $Q$ can give information about the magnetic monopole. If me move to the region where $5r_t/4 > r > r_t$, we introduce a dependence on the radius $r$, which can be eliminated with the computation of
\[
\frac{dQ_\infty^2}{dr} = 0.
\]
However when we choose the parameterization (62), the solution of Eq.(69) is imaginary and therefore will be discarded. On the other hand, when we choose parametrization (65), we find that the expression (61) reduces to
\[
Q_\infty^2 = -\frac{1}{4\pi^2} \ln\left(\frac{3MG}{4\pi^3 E_P^2} \alpha \gamma + 11/6\right) \left(\frac{3MG}{r^2}\right)^2 + O\left((2MG)^{5/2}\right)
\]
and the computation of Eq.(69) in this case leads to the following relationship
\[
r^3 = \frac{3\alpha \gamma + 10}{8E_P^2} \exp\left(\frac{\gamma}{3}\right).
\]
Since $r \in [r_t, 5r_t/4]$, this implies that
\[
r \in \left[\sqrt[3]{\frac{3\alpha \gamma + 10}{8E_P^2} \exp\left(\frac{\gamma}{3}\right)} \cdot \frac{5}{4} \sqrt[3]{\frac{3\alpha \gamma + 10}{8E_P^2} \exp\left(\frac{\gamma}{3}\right)}\right].
\]
then we find the following bound
\[
2.963 \times 10^{-2} = \frac{3\pi}{64\pi^2} \geq Q_\infty^2(\alpha, \bar{r}) \geq \frac{3\pi}{64\pi^2} \sqrt[3]{\frac{5}{4}} = 3.990 \times 10^{-2}.
\]
B. The de Sitter (Anti-de Sitter) Case

Even in this case, we examine the classical constraint for the dS and AdS metric, respectively. For the AdS metric is immediate to verify that the condition \( \mathcal{H} = 0 \) reduces to

\[
\dot{R} - \frac{\kappa}{4\pi} \rho_{g_{12\text{em}}} = 0 \quad \implies \quad G \left( \frac{Q^2}{r^4} g_1^2 (E) + \frac{Q_m^2}{r^4} g_2^2 (E) \right) = -\Lambda_{\text{AdS}},
\]

which is never satisfied, while for the dS metric, we find

\[
\dot{R} - \frac{\kappa}{4\pi} \rho_{g_{12\text{em}}} = 0 \quad \implies \quad G \left( \frac{Q^2}{r^4} g_1^2 (E) + \frac{Q_m^2}{r^4} g_2^2 (E) \right) = \Lambda_{\text{dS}}.
\]

Moreover, if we fix the radius to the value \( r = \sqrt{3/\Lambda_{\text{dS}}} \), we find

\[
G \Lambda_{\text{dS}} \left( Q^2 g_1^2 (E) + Q_m^2 g_2^2 (E) \right) = 9,
\]

which fixes the values of \( Q_e, Q_m \), and \( \Lambda_{\text{dS}} \) to values incompatible with observation. However, things can be different from the quantum point of view. Since in the dS and AdS cases, the condition \( \mathcal{H} = 0 \) holds, Eq.(43) becomes

\[
\frac{1}{2} g_1^2 (E/E_P) \rho_{g_{12\text{em}}} = -\frac{2}{3\pi^2} I_-,
\]

where \( I_- \) is given by Eq.(50). Choosing the Rainbow’s functions like in the Schwarzschild case, one finds

\[
\frac{g_1^2 (E/E_P)}{2E_P^3} \rho_{g_{12\text{em}}} = -\frac{\beta}{4\alpha \pi^2} (3 + 2\alpha x^2)e^{-\alpha x^2} - \frac{x^2}{2\alpha \pi^2} e^{-\alpha x^2} K_1 \left( \frac{\alpha x^2}{2} \right),
\]

where \( x \) is expressed by Eq.(51), but with a different \( m_0^2 (r) \). Indeed, we have

\[
x = \sqrt{\frac{m_0^2 (r)}{E_P^2}} = \frac{1}{E_P} \begin{cases} \sqrt{6 - \Lambda_{\text{AdS}} r^2} & \text{de Sitter} \quad \beta (r) = \Lambda_{\text{dS}} r^3 \frac{3}{4} \\ \sqrt{6 + \Lambda_{\text{AdS}} r^2} & \text{Anti-de Sitter} \quad \beta (r) = \Lambda_{\text{dS}} r^3 \frac{3}{4} \end{cases}.
\]

We can gain more information by evaluating the r.h.s. of Eq.(58) for small and large \( x \). For large \( x \), one gets

\[
\frac{g_1^2 (E/E_P)}{2E_P^3} \rho_{g_{12\text{em}}} \simeq e^{-\alpha x^2} \left[ -\frac{\beta}{2\pi^2 \alpha^3/2} x^2 - \frac{1}{2\pi^2 \alpha^3/2} x^2 - \frac{3\beta}{4\pi^2 \alpha^5/2} x - \frac{3}{8\pi^2 \alpha^5/2} x^3 + \frac{15}{64\pi^2 \alpha^7/2} x^3 + O(x^{-5}) \right],
\]

while for small \( x \), we get

\[
\frac{g_1^2 (E/E_P)}{2E_P^3} \rho_{g_{12\text{em}}} \simeq -\left( \frac{4\sqrt{\alpha} + 3\beta \sqrt{\alpha}}{4\pi^2 \alpha^{3/2}} \right) x^2 + \left( 2\sqrt{\alpha} + \beta \sqrt{\alpha} \right) x^2 - \left[ \sqrt{\alpha} \ln \left( \frac{4\sqrt{\alpha} + 3\beta \sqrt{\alpha}}{4\pi^2 \alpha^{3/2}} \right) + \gamma \sqrt{\alpha} - \beta \sqrt{\alpha} \right] \frac{x^4}{8\pi^2 \sqrt{\alpha}} + O(x^6).
\]

It is interesting to note that the expression is finite for every \( x \). Beginning with the dS case, we observe that the range of the radius \( r \) is \( \left[ 0, \sqrt{3/\Lambda_{\text{dS}}} \right] \) and when \( r \to 0, x \to \infty \) which is vanishing because of behavior \( \mathcal{O}(x^{-3}) \). On the other hand, when \( r \to \sqrt{3/\Lambda_{\text{dS}}, x \to 0} \). However \( r = \sqrt{3/\Lambda_{\text{dS}}} \) corresponds to a region external to the dS horizon which is unphysical and therefore will be discarded. Rather when

\[
r = \sqrt{\frac{3}{\Lambda_{\text{dS}}} \frac{1}{E_P}} \quad \implies \quad x = \sqrt{\frac{\Lambda_{\text{dS}}}{E_P}}.
\]

Therefore keeping the same parametrization that allows a vanishing contribution for small \( x \), Eq.(58) becomes

\[
Q_e^2 \left( \alpha_e - \frac{4}{3} \sqrt{\frac{\alpha_{\text{AdS}}}{E_P}} \right) = \frac{9E_P^3}{\Lambda_{\text{dS}}^2} \left[ \frac{2}{3\alpha \pi^2} \left( 3 + \alpha_{\text{AdS}} E_P^2 \right) \exp \left( -\frac{\alpha_{\text{AdS}} E_P^2}{2E_P} \right) - \frac{\Lambda_{\text{dS}}}{\alpha \pi^2 E_P^2} \exp \left( -\frac{\alpha_{\text{AdS}} E_P^2}{2E_P} \right) K_1 \left( \frac{\alpha_{\text{AdS}} E_P^2}{2E_P} \right) \right],
\]

where we have excluded the trans-planckian region which suppresses the charge contribution. By imposing that

\[
Q_e^2 \left( \frac{1}{4} - \frac{2}{3\sqrt{\pi}} \sqrt{\frac{\Lambda_{\text{dS}}}{E_P}} \right) = \frac{1}{137},
\]

\[
\text{REFERENCES}
\]

\[
\text{ACKNOWLEDGMENTS}
\]

\[
\text{SUPPLEMENTARY MATERIAL}
\]

\[
\text{APPENDICES}
\]
we find that
\[ \Lambda_{dS} \simeq 16E_P^2 \quad (85) \]
and the corresponding “Cosmological radius” becomes
\[ r^{Q_m}_\Lambda = \sqrt{\frac{3}{\Lambda_{dS}}} \cdot \frac{0.43301}{E_P}. \quad (86) \]
Concerning the AdS case, we observe that since \( r \in [0, +\infty) \), when
\[ r + \infty \rightarrow x = \frac{\sqrt{\Lambda_{AdS}}}{E_P}. \quad (87) \]
and \( Q^2_e (1/4, -2/(3\sqrt{\pi}), \sqrt{\Lambda_{AdS}/E_P}) \rightarrow \infty \) and therefore will be discarded.

IV. MAGNETIC MONOPOLES

As introduced in Section II, our calculation applies also to magnetic monopoles. However, since we have no experimental evidence in high energy physics, we need to use the Dirac proposal between the magnetic monopole and the electric charge described by the relationship (7) to fix numbers. Therefore, it is immediate to see that
\[ Q_m = \frac{2\pi}{Q_e} = 73.543 \quad \Rightarrow \quad Q^2_m = 5408.6. \quad (88) \]
Since the value of \( Q^2_m \) is quite large, for the Schwarzschild metric we can use parametrization (62) which keeps under control large values of \( x \), while the parametrization (66) will be discarded. Setting
\[ -r^4E_P \frac{d}{dr} f \left( \frac{1}{4} \sqrt{\frac{\pi}{4}}, \frac{3}{2r^4E_P^2} \right) = 5408.6, \quad (89) \]
we find
\[ r^{M}_M = \frac{6.6}{E_P}. \quad (90) \]
Note that when we compare \( r^{Q_m}_M \) with \( r^{Q_e}_M \), we find
\[ \frac{r^{Q_m}_M}{r^{Q_e}_M} = \frac{6.6}{0.295} = 22.373. \quad (91) \]
On the other hand, when we use the dS metric, we find
\[ Q^2_m \left( \frac{1}{4}, -\frac{2}{3\sqrt{\pi}}, \frac{\sqrt{\Lambda_{dS}}}{E_P} \right) = 5408.6, \quad (92) \]
which implies
\[ \Lambda_{dS} \simeq 0.024E_P^2 \quad (93) \]
and the corresponding “Cosmological radius” becomes
\[ r^{Q_m}_\Lambda = \sqrt{\frac{3}{\Lambda_{dS}}} = \frac{11.18}{E_P}. \quad (94) \]
Once again, when we compare \( r^{Q_m}_\Lambda \) with \( r^{Q_e}_\Lambda \) we find
\[ \frac{r^{Q_m}_\Lambda}{r^{Q_e}_\Lambda} = \frac{11.18}{0.43301} = 25.819. \quad (95) \]
V. CONCLUSIONS

In this paper we have explored the possibility that quantum fluctuations of the gravitational field be considered as a source for the electric/magnetic charge. The idea is not new, because it has its origin in the Wheeler’s proposal of “charge without charge” and “mass without mass” arising from the spacetime foam picture[2]. Moreover, a first approach has been proposed by one of us in Ref.[6]. What is new in this paper is that the UV divergences are kept under control by Gravity’s Rainbow which is a distortion of space time activating at the Planck’s scale. This distortion avoids the introduction of any regularization/renormalization process, like in Noncommutative theory approaches[13]. Note that the Rainbow’s functions $g_1(E/E_P)$ and $g_2(E/E_P)$ are constrained only by the low energy limit[11] and by the request that the one loop integrals be UV finite[3, 6, 15]. Nevertheless in Ref.[16] we have adopted different proposal to discuss an inflationary scenario governed by Gravity’s Rainbow. Note also that the electric and magnetic charges appear as a quantum effect of the gravitational field. Indeed, the classical contribution related to the specific proposal to discuss an inflationary scenario governed by Gravity’s Rainbow. Note also that the electric and magnetic charges be considered as a good tool for probing the geometries hitherto examined leads to $Q_e = Q_m = 0$. Three basic geometries have been examined. One of these, the AdS background leads to inconsistent solutions and therefore has been discarded. On the other hand, the Schwarzschild and the de Sitter background show that the computed particle radius of the electron is of the Planckian order. This has been obtained by fixing the value of the electric charge to the fine structure constant that, in the units we have adopted, is coincident with the square of the electron charge. As regards the magnetic monopole, since no direct observation at very high energies has ever been announced, we have used the Dirac quantization rule to obtain information about the magnetic charge and therefore recover its own particle radius. It is interesting to note that the ratio between the magnetic monopole radius and the electron radius $r_e/r_m$ is of the same order for both the Schwarzschild and the de Sitter background. Recently another result relating Gravity’s Rainbow and its influence on topology change has been obtained[17]. This seems to suggest that Gravity’s Rainbow can be considered as a good tool for probing the spacetime foam picture suggested by Wheeler.

Appendix A: The electromagnetic energy-momentum tensor in SI Units

The electromagnetic energy-momentum tensor in free space and in SI Units is defined as

$$T_{\mu\nu} = \frac{1}{\mu_0} \left[ F_{\mu\gamma} F^\gamma_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\gamma\delta} F^{\gamma\delta} \right], \quad (A1)$$

where $\mu_0$ is the vacuum permeability, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. For a pure electric field, the electromagnetic potential $A_\mu$ assumes the form

$$A_\mu = \left( \frac{Q_e}{4\pi\epsilon_0 c}, 0, 0, 0 \right) \quad \Rightarrow \quad F_{\mu\nu} = F_{01} = -\frac{Q_e}{4\pi\epsilon_0 r^2 c}, \quad (A2)$$

where $\epsilon_0$ is the vacuum permittivity, $c$ is the speed of light and $Q_e$ is the electric charge. On the other hand, for the pure magnetic field, the form is

$$A_\mu = \left( 0, 0, 0, -\mu_0 \frac{Q_m}{4\pi} \cos \theta \right) \quad \Rightarrow \quad F_{\mu\nu} = F_{23} = \mu_0 \frac{Q_m}{4\pi} \sin \theta, \quad (A3)$$

where $Q_m$ is the magnetic charge measured in Ampère-meter (A-m). Thus the $T_{00}$ component of the energy-momentum tensor for the electromagnetic charges becomes

$$T_{00} = \frac{1}{\mu_0} \left\{ \frac{1}{2} g^{11} (F_{01})^2 - \frac{1}{2} g_{00} \left( g^{22} g^{33} \right) (F_{23})^2 \right\} = \frac{1}{2\mu_0 (4\pi r^2)^2} \left\{ g^{11} \frac{Q_e^2}{\epsilon_0^2 r^4} - g_{00} \mu_0^2 Q_m^2 \right\}$$

$$= \frac{1}{2 (4\pi r^2)^2} \left\{ g^{11} \frac{Q_e^2}{\epsilon_0} - g_{00} \mu_0 Q_m^2 \right\}, \quad (A4)$$

where we have used the following relationship $c^2 \epsilon_0 \mu_0 = 1$. With the help of the time-like vector $u^\mu$, we obtain

$$T_{\mu\nu} u^\mu u^\nu = \frac{1}{2 (4\pi r^2)^2} \left\{ \frac{Q_e^2}{\epsilon_0} + \mu_0 Q_m^2 \right\}. \quad (A5)$$
For a spherically symmetric metric described by (2) with \(g_1 (E/E_P) = g_2 (E/E_P) = 1\), it is easy to recognize that the classical constraint (11) reduces to

\[
3R = \frac{2\kappa}{c^4} T_{\alpha\beta} u^\alpha u^\beta \implies b'(r) = \frac{G}{4\pi r^3 c^4} \left\{ \frac{Q^2}{\epsilon_0} + \mu_0 Q_m^2 \right\},
\]

(A6)

whose solution represents the Reissner-Nordström (RN) metric if

\[
N^2 (r) = \left[ 1 - \frac{b(r)}{r} \right]^{-1}
\]

(A7)

and

\[
b(r) = \frac{2MG}{c^4} - \frac{G}{4\pi r c^4} \left\{ \frac{Q^2}{\epsilon_0} + \mu_0 Q_m^2 \right\}.
\]

(A8)

Note that in CGS units, one defines \(\epsilon_0 = (4\pi)^{-1}\) and \(\mu_0 = 4\pi\) and the energy-momentum tensor is in agreement with the expression in (4).

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