QUASI-FREE ACTIONS OF FINITE GROUPS ON THE CUNTZ ALGEBRA $O_\infty$

PAVLE GOLDSTEIN AND MASAKI IZUMI

Abstract. We show that any faithful quasi-free actions of a finite group on the Cuntz algebra $O_\infty$ are mutually conjugate, and that they are asymptotically representable.

1. Introduction

The Cuntz algebra $O_n$, $n = 2, 3, \ldots, \infty$, is the universal $C^*$-algebra generated by isometries $\{s_i\}_{i=1}^n$ with mutually orthogonal ranges, satisfying $\sum_{i=1}^n s_is_i^* = 1$ if $n$ is finite. It is well known that the two algebras $O_2$ and $O_\infty$, among the others, play special roles in the celebrated classification theory of Kirchberg algebras (see [15], [18]).

An action $\alpha$ of a group $G$ on $O_n$ is said to be quasi-free if $\alpha_g(\mathcal{H}_n) = \mathcal{H}_n$ for all $g \in G$, where $\mathcal{H}_n$ is the closed linear span of the generators $\{s_i\}_{i=1}^n$. We restrict our attention to finite $G$ throughout this note. To develop a $G$-equivariant version of the classification theory, it is expected that $G$-actions on $O_2$ with the Rohlin property and the quasi-free $G$-actions on $O_\infty$ would play similar roles as $O_2$ and $O_\infty$ do in the case without group actions. Since we have already had a good understanding of the former thanks to [4], our task in this note is to investigate the latter, the quasi-free $G$-actions on $O_\infty$.

The space $\mathcal{H}_n$ has a Hilbert space structure with inner product $t^*s = \langle s, t \rangle 1$, and a quasi-free $G$-action $\alpha$ gives a unitary representation $(\pi_\alpha, \mathcal{H}_n)$, where $\pi_\alpha(g)$ is the restriction of $\alpha_g$ to $\mathcal{H}_n$. It is known that the association $\alpha \mapsto (\pi_\alpha, \mathcal{H}_n)$ gives a one-to-one correspondence between the quasi-free $G$-actions on $O_n$ and the unitary representations of $G$ in $\mathcal{H}_n$. The conjugacy class of $\alpha$ depends on the unitary equivalence class of $(\pi_\alpha, \mathcal{H}_n)$, at least a priori. Indeed, it really does when $n$ is finite, and this can be seen by computing the $K$-groups of the crossed product (see, for example, [2], [4], [5], [11]). However, when $n = \infty$, the pair $(O_\infty, \alpha)$ is $KK_G$-equivalent to the pair $(\mathbb{C}, \text{id})$, and there is no way to differentiate the quasi-free actions as far as $K$-theory is concerned.

One of the purposes of this note is to show that any two faithful quasi-free $G$-actions on $O_\infty$ are indeed mutually conjugate for every finite group $G$ (Corollary 5.2). Our main technical result is Theorem 1.1, an equivariant version of Lin-Phillips’s result [10, Theorem 3.3], and Corollary 5.2 follows from it via Theorem 5.1, an equivariant version of Kirchberg-Phillips’s $O_\infty$ theorem [7, Theorem 3.15].

Supported in part by the Grant-in-Aid for Scientific Research (B) 22340032, JSPS.
Using Theorem [4.1] we also show that the quasi-free actions are asymptotically representable for any finite group $G$, which is another purpose of this note. The notion of asymptotic representability for group actions was introduced by the second-named author, and it is found to be important in the recent development of the classification of group actions on $C^*$-algebras (see [6], [11]).

The reader is referred to [18] for the basic properties and classification results for Kirchberg algebras. We denote by $\K$ the set of compact operators on a separable infinite dimensional Hilbert space. For a $C^*$-algebra $A$, we denote by $\hat{A}$ and $M(A)$ the unitization and the multiplier algebra of $A$ respectively. When $A$ is unital, we denote by $U(A)$ the unitary group of $A$. For a homomorphism $\rho : A \to B$ between $C^*$-algebras $A$, $B$, we denote by $K_*(\rho)$ the homomorphism from $K_*(A)$ to $K_*(B)$ induced by $\rho$. We denote by $A \otimes B$ the minimal tensor product of $A$ and $B$.

This work originated from the first-named author’s unpublished preprint [3], where the idea of developing an equivariant version of Lin-Phillips’s argument was introduced. Some results in this note are also obtained by N. C. Phillips, and the authors would like to thank him for informing of it.

2. Preliminaries for $G$-$C^*$-algebras

We fix a finite group $G$. By a $G$-$C^*$-algebra $(A, \alpha)$, we mean a $C^*$-algebra $A$ with a fixed $G$-action $\alpha$. We denote by $A^G$ the fixed point algebra

\[
\{a \in A | \alpha_g(a) = a, \forall g \in G\}.
\]

We denote by $\{\lambda^\alpha_g\}_{g \in G}$ the implementing unitary representation of $G$ in the crossed product $A \rtimes_\alpha G$. For a finite dimensional (not necessarily irreducible) unitary representation $(\pi, H_\pi)$ of $G$, we introduce a homomorphism

\[
\hat{\alpha}_\pi : A \rtimes_\alpha G \to (A \rtimes_\alpha G) \otimes B(H_\pi),
\]

which is a part of the dual coaction of $\alpha$, by $\hat{\alpha}_\pi(a) = a \otimes 1$ for $a \in A$, and $\hat{\alpha}_\pi(\lambda^\alpha_g) = \lambda^\alpha_g \otimes \pi(g)$ for $g \in G$. We denote by $\hat{G}$ the unitary dual of $G$, and by $\Z G$ the representation ring of $G$. Then identifying $K_*(A \rtimes_\alpha G)$ with $K_*(((A \rtimes_\alpha G) \otimes B(H_\pi))$, we get a $\Z \hat{G}$-module structure of $K_*(A \rtimes_\alpha G)$ from $K_*(\hat{\alpha}_\pi)$.

Let

\[
e_\alpha = \frac{1}{\#G} \sum_{g \in G} \lambda^\alpha_g,
\]

which is a projection in $(A \rtimes_\alpha G) \cap A^{G'}$. We denote by $j_\alpha$ the homomorphism from $A^G$ into $A \rtimes_\alpha G$ defined by $j_\alpha(x) = xe_\alpha$. When $A$ is simple and $\alpha$ is outer, that is, $\alpha_g$ is outer for every $g \in G \setminus \{e\}$, then $K_*(j_\alpha)$ is an isomorphism from $K_*(A^G)$ onto $K_*(A \rtimes_\alpha G)$. When $A$ is purely infinite and simple, and $\alpha$ is outer, then $A^G$ and $A \rtimes_\alpha G$ are purely infinite and simple.

A $G$-homomorphism $\varphi$ from a $G$-$C^*$-algebra $(A, \alpha)$ into another $G$-$C^*$-algebra $(B, \beta)$ is a homomorphism from $A$ into $B$ intertwining the two $G$-actions $\alpha$ and $\beta$. Such $\varphi$ gives rise to an element in the equivariant $KK$-group $KK_G(A, B)$, which is denoted by $KK_G(\varphi)$. We denote by $\Hom_G(A, B)$ the set of nonzero $G$-homomorphisms from $(A, \alpha)$ into $(B, \beta)$. Two actions $\alpha$ and $\beta$ are said to be conjugate if there exists
an invertible element in Hom$_G(A, B)$. Two $G$-homomorphisms $\varphi, \psi \in$ Hom$_G(A, B)$ are said to be $G$-unitarily equivalent if there exists a unitary $u \in M(B)^G$ satisfying $\varphi(x) = u\psi(x)u^*$ for all $x \in A$. They are said to be $G$-asymptotically unitarily equivalent if there exists a norm continuous family of unitaries $\{u(t)\}_{t \geq 0}$ in $M(B)^G$ satisfying

$$\lim_{t \to \infty} \|\varphi(x) - \text{Ad} u(t) \circ \psi(x)\|, \quad \forall x \in A.$$  

If they satisfy the same condition with a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $M(B)^G$ instead of the continuous family, they are said to be $G$-approximately unitarily equivalent.

For a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and a $G$-$C^*$-algebra $(A, \alpha)$, we use the following notation:

$$c_\omega(A) = \{(x_n) \in \ell^\infty(\mathbb{N}, A)\mid \lim_{n \to \omega} \|x_n\| = 0\},$$  

$$A^\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(A).$$  

As usual, we omit the quotient map from $\ell^\infty(\mathbb{N}, A)$ onto $A^\omega$. We regard $A$ as a $C^*$-subalgebra of $A^\omega$ consisting of the constant sequences, and we set $A_\omega = A^\omega \cap A$. We denote by $\alpha^\omega$ and $\alpha_\omega$ the $G$-actions on $A^\omega$ and $A_\omega$ induced by $\alpha$ respectively, and we regard $(A^\omega, \alpha^\omega)$ and $(A_\omega, \alpha_\omega)$ as $G$-$C^*$-algebras.

**Lemma 2.1.** Let $G$ be a finite group, and let $(A, \alpha)$ be a $G$-$C^*$-algebra. We assume that $A$ is unital, purely infinite, and simple, and $\alpha$ is outer. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

1. $A^\omega$ is purely infinite and simple, and $\alpha^\omega$ is outer.

2. If $A$ is a Kirchberg algebra, $A_\omega$ is purely infinite and simple, and $\alpha_\omega$ is outer.

**Proof.** (1) It is easy to show that $A^\omega$ is purely infinite and simple, and so it suffices to show that if $\theta \in \text{Aut}(A)$ is outer, so is $\theta^\omega \in \text{Aut}(A^\omega)$ induced by $\theta$. Assume that $\theta$ is outer and $\theta^\omega$ is inner. Then there exists $u = (u_n) \in U(A^\omega)$ satisfying $\text{Ad} u = \theta^\omega$. We may assume that $u_n$ is a unitary for all $n \in \mathbb{N}$. Since $A$ is purely infinite, there exist a sequence of nonzero projections $\{p_n\}_{n=1}^\infty$ in $A$ and a sequence of complex numbers $\{c_n\}_{n=1}^\infty$ with $|c_n| = 1$ such that $\{p_n u_n p_n - c_n p_n\}_{n=1}^\infty$ converges to 0. By replacing $u_n$ with $\overline{u_n} u_n$ if necessary, we may assume $c_n = 1$. Since $\theta$ is outer, Kishimoto’s result [8, Lemma 1.1] shows that there exists a sequence of positive elements $a_n \in p_n A p_n$ with $\|a_n\| = 1$ such that $\{a_n \theta(a_n)\}_{n=1}^\infty$ converging to 0. This is contradiction. Indeed, let $a = (a_n) \in A^\omega$, $p = (p_n) \in A^\omega$. On one hand we have $a \theta^\omega(a) = 0$, and on the other hand we have the following

$$a \theta^\omega(a) = a u a^* = a p u a^* = a p u = a^2 u^* \neq 0.$$  

This shows that $\theta^\omega$ is outer.

(2) The statement follows from [7, Proposition 3.4] and [13, Lemma 2].

Now we state two results, which are equivariant versions of well-known results in the classification theory of nuclear $C^*$-algebras. We omit their proofs, which are verbatim modifications of the original ones. The first one is an equivariant version of [18, Corollary 2.3.4].
Theorem 2.2. Let $G$ be a finite group, and let $(A, \alpha)$ and $(B, \beta)$ be unital separable $G$-$C^*$-algebras. If there exist $\varphi \in \text{Hom}_G(A, B)$ and $\psi \in \text{Hom}_G(B, A)$ such that $\psi \circ \varphi$ is $G$-approximately unitarily equivalent to $\text{id}_{(A, \alpha)}$ and $\varphi \circ \psi$ is $G$-approximately unitarily equivalent to $\text{id}_{(B, \beta)}$, then the two actions $\alpha$ and $\beta$ are conjugate.

The following result is an equivariant version of [7 Proposition 3.13].

Theorem 2.3. Let $G$ be a finite group, and let $(A, \alpha)$, $(B, \beta)$ be unital separable $G$-$C^*$-algebras. We regard the minimal tensor product $B \otimes B$ as a $G$-$C^*$-algebra with the diagonal action $\alpha \otimes \alpha$, and define $\rho_1, \rho_r \in \text{Hom}_G(B, B \otimes B)$ by $\rho_1(x) = x \otimes 1$ and $\rho_r(x) = 1 \otimes x$ for $x \in B$. We assume that $\rho_1$ and $\rho_r$ are $G$-approximately unitarily equivalent. Then if there exists a unital homomorphism in $\text{Hom}_G(B, A_\omega)$ with $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$, the two $G$-actions $\alpha$ on $A$ and $\alpha \otimes \beta$ on $A \otimes B$ are conjugate.

3. EQUIVARIANT RøRDAM’S THEOREM

The purpose of this section is to show the following theorem, which is an equivariant version of Rørdam’s theorem [17, Theorem 3.6], [18, Theorem 5.1.2].

Theorem 3.1. Let $G$ be a finite group, let $\alpha$ be a quasi-free action of $G$ on $O_n$ with finite $n$, and let $(B, \beta)$ be a $G$-$C^*$-algebra. We assume that $B$ is unital, purely infinite, and simple, and $\beta$ is outer. For two unital $G$-homomorphisms $\varphi, \psi \in \text{Hom}_G(O_n, B)$, we set

$$u_{\psi, \varphi} = \sum_{i=1}^n \psi(s_i)\varphi(s_i)^* \in U(B^G).$$

We introduce an endomorphism $\Lambda_{\varphi} \in \text{End}(B^G)$ by

$$\Lambda_{\varphi}(x) = \sum_{i=1}^n \varphi(s_i)x\varphi(s_i)^*, \quad x \in B^G.$$

Then the following conditions are equivalent.

1. The $G$-homomorphisms $\varphi$ and $\psi$ are $G$-approximately unitarily equivalent.
2. The unitary $u_{\psi, \varphi}$ belongs to the closure of $\{v\Lambda_{\varphi}(v^*) \in U(B^G) | v \in B^G\}$.
3. The $K_1$-class $[u_{\psi, \varphi}] \in K_1(B^G)$ is in the image of $1 - K_1(\Lambda_{\varphi})$.
4. The $K_1$-class $K_1(j_\beta)([u_{\psi, \varphi}]) \in K_1(B \rtimes_\beta G)$ is in the image of $1 - K_1(\beta_{\pi_n})$.
5. The equality $KK_G(\varphi) = KK_G(\psi)$ holds in $KK_G(O_n, B)$.

Proof. The equivalence of (1) and (2) follows from $\psi(s_i) = u_{\psi, \varphi}\varphi(s_i)$ and $v\varphi(s_i)v^* = v\Lambda_{\varphi}(v^*)\varphi(s_i)$.

The implication from (2) to (3) is trivial. In view of the proof of [17, Theorem 3.6], the implication from (3) to (2) is reduced to the Rohlin property of the shift automorphism of $(\bigotimes \mathbb{Z} M_n(\mathbb{C}))^G$, where the $G$-action of the UHF algebra $\bigotimes \mathbb{Z} M_n(\mathbb{C})$ is the product action $\bigotimes \text{Ad} \pi_n(g)$. This follows from Kishimoto’s result [9 Theorem 2.1] (see [4 Lemma 5.5] for details).

The equivalence of (3) and (4) follows from Lemma 5.3 below.

We will show the equivalence of (4) and (5) in Appendix as it follows from a rather lengthy computation, and we do not really require it in the rest of this note. □
To show the equivalence of (3) and (4), we first recall the following well-known fact.

**Lemma 3.2.** Let $A$ be a $C^*$-algebra, and let $\{t_i\}_{i=1}^n \subset M(A)$ be isometries with mutually orthogonal ranges. Let $\{e_{ij}\}_{i,j=1}^n$ be the system of matrix units of the matrix algebra $M_n(\mathbb{C})$. We define two homomorphisms $\rho_1 : A \to A \otimes M_n(\mathbb{C})$ and $\rho_2 : A \otimes M_n(\mathbb{C}) \to A$ by $\rho_1(a) = a \otimes e_{11}$, and $\rho_2(a \otimes e_{ij}) = t_i a t_j^*$. Then $K_*(\rho_2)$ is the inverse of $K_*(\rho_1)$.

**Proof.** Since $K_*(\rho_1)$ is an isomorphism, it suffices to show that the homomorphism $\rho_2 \circ \rho_1(x) = t_1 x t_1^*$ induces the identity on $K_*(A)$. This follows from a standard argument. \qed 

Recall that we regard $K_*(\hat{\beta}_{\pi_\alpha})$ as an element of $\text{End}(K_*(B \rtimes \beta \, G))$ by identifying $K_*(B \rtimes \beta \, G)$ with $K_*((B \rtimes \beta \, G) \otimes B(H_n))$.

**Lemma 3.3.** With the above notation, we have the equality $K_*(j_\beta) \circ K_*(\Lambda_\varphi) = K_*(\hat{\beta}_{\pi_\alpha}) \circ K_*(j_\beta)$.

**Proof.** Identifying $B(H_n)$ with the linear span of $\{s_i s_j^*\}_{i,j=1}^n$ acting on $H_n$ by left multiplication, we have

$$\pi_\alpha(g) = \sum_{i=1}^n \alpha_g(s_i) s_i^*.$$ 

We define a homomorphism $\rho : (B \rtimes \beta \, G) \otimes B(H_n) \to B \rtimes \beta \, G$ by $\rho(x \otimes s_i s_j^*) = \varphi(s_i)x\varphi(s_j)^* \varphi(s_j)^*$, which plays the role of $\rho_2$ in Lemma 3.2 with $A = B \rtimes \beta \, G$ and $t_i = \varphi(s_i)$. Then for $x \in B^G$, we have

$$\rho \circ \hat{\beta}_{\pi_\alpha} \circ j_\beta(x) = \frac{1}{\#G} \sum_{g \in G} \rho \circ \hat{\beta}_{\pi_\alpha}(\lambda_g^\beta x) = \frac{1}{\#G} \sum_{g \in G} \rho(\lambda_g^\beta x \otimes \pi_\alpha(g))$$ 

$$= \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \rho(\lambda_g^\beta x \otimes \alpha_g(s_i) s_i^*) = \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \varphi(\alpha_g(s_i)) \lambda_g^\beta x \varphi(s_i)^*$$ 

$$= \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \lambda_g^\beta \varphi(s_i)x\varphi(s_i)^* = j_\beta \circ \Lambda_\varphi(x),$$

which proves the statement thanks to Lemma 3.2. \qed 

4. **Equivariant Lin-Phillips’s theorem**

The purpose of this section is to show the following theorem, which is an equivariant version of Lin-Phillips’s theorem [10, Theorem 3.3], [18, Proposition 7.2.5].

**Theorem 4.1.** Let $G$ be a finite group, let $\alpha$ be a quasi-free action of $G$ on $O_\infty$, and let $(B, \beta)$ be a unital $G-C^*$-algebra. We assume that $B$ is purely infinite and simple, and $\beta$ is outer. Then any two unital $G$-homomorphisms in $\text{Hom}_G(O_\infty, B)$ are $G$-approximately unitarily equivalent.
Until the end of this section, we assume that $G$, $(\mathcal{O}_\infty, \alpha)$ and $(B, \beta)$ are as in Theorem 4.1. To prove Theorem 4.1, we basically follow Lin-Phillips’s strategy based on Theorem 3.1 in place of [17, Theorem 3.6], though we will take a short cut by using a ultraproduct technique.

Let $n$ be a natural number larger than 2, and let $\mathcal{E}_n$ be the Cuntz-Toeplitz algebra, which is the universal $C^*$-algebra generated by isometries $\{t_i\}_{i=1}^n$ with mutually orthogonal ranges. Note that $p_n = 1 - \sum_{i=1}^n t_i t_i^*$ is a non-zero projection not as in the case of the Cuntz algebras. We denote by $K_n$ the linear span of $\{t_i\}_{i=1}^n$. Quasi-free actions on $\mathcal{E}_n$ are defined as in the case of the Cuntz algebras. For a quasi-free action $\gamma$ of $G$ on $\mathcal{E}_n$, we denote by $(\pi_\gamma, K_n)$ the corresponding unitary representation of $G$ in $K_n$.

**Lemma 4.2.** Let $\gamma$ be a quasi-free action of $G$ on $\mathcal{E}_n$ with finite $n$, and let $\varphi, \psi \in \text{Hom}_G(\mathcal{E}_n, B)$ be injective $G$-homomorphisms, either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then $\varphi$ and $\psi$ are $G$-approximately unitarily equivalent.

**Proof.** In the same way as in the proof of Lemma 3.3, we can show

$$K_0(j_\beta)([\varphi(p_n)]) = K_0(j_\beta)([\varphi(1)]) - K_0(\hat{\beta}_\pi) \circ K_0(j_\beta)([\varphi(1)]) = 0,$$

in $K_0(B \rtimes G)$. This implies $[\varphi(p_n)] = 0$ in $K_0(B^G)$, and for the same reason, $[\psi(p_n)] = 0$ in $K_0(B^G)$. Thus the statement follows from essentially the same argument as in the proof of [10, Proposition 1.7] by using Theorem 3.1 in place of [17, Theorem 3.6].

Since every quasi-free $G$-action on $\mathcal{O}_\infty$ is the inductive limit of a system of quasi-free actions of the form $\{(\mathcal{E}_n, \gamma(k))\}_{k=1}^\infty$, we get

**Corollary 4.3.** Let $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ be either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then $\varphi$ and $\psi$ are $G$-approximately unitarily equivalent.

Let $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter, and let $\iota_\omega : \mathcal{O}_\infty \to \mathcal{O}_\infty^\omega$ the inclusion map. For $\varphi \in \text{Hom}_G(\mathcal{O}_\infty, B)$, we denote by $\varphi^\omega$ the $G$-homomorphism in $\text{Hom}_G(\mathcal{O}_\infty^\omega, B^\omega)$ induced by $\varphi$. Then it is easy to show the following three conditions for $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ are equivalent:

1. $\varphi$ and $\psi$ are $G$-approximately unitarily equivalent,
2. $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are $G$-approximately unitarily equivalent,
3. $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are $G$-unitarily equivalent.

Note that since $G$ is a finite group, we have $(\mathcal{O}_{\infty^\omega})^G = (\mathcal{O}_\infty^G)^\omega \cap \mathcal{O}_\infty'$ and $(B^\omega)^G = (B^G)^\omega$.

**Proof of Theorem 4.4.** Let $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ be unital. Since $\mathcal{O}_\infty$ is a Kirchberg algebra, the $\omega$-central sequence algebra $\mathcal{O}_{\infty^\omega}$ is purely infinite and simple. Let $H$ be the kernel of $\alpha : G \to \text{Aut}(\mathcal{O}_\infty)$. Since $\alpha$ is quasi-free, we may regard $\alpha$ as an outer action of $G/H$, and so $\alpha_\omega$ is outer as an action of $G/H$. This implies that $(\mathcal{O}_{\infty^\omega})^G$ is purely infinite and simple.
Choosing three nonzero projections \( q_1, q_2, q_3 \in (\mathcal{O}_\infty)^G \) satisfying \( q_1 + q_2 + q_3 = 1 \) and \([1] = [q_1] = [q_2] = -[q_3] \) in \( K_0((\mathcal{O}_\infty)^G) \), we introduce \( \varphi_i, \psi_i \in \text{Hom}_G(\mathcal{O}_\infty, B^\omega) \), \( i = 1, 2, 3 \), by \( \varphi_i(x) = \varphi^\omega(q_i x) \) and \( \psi_i(x) = \psi^\omega(q_i x) \) for \( x \in \mathcal{O}_\infty \). Then we have

\[
\begin{align*}
\varphi(x) &= \varphi_1(x) + \varphi_2(x) + \varphi_3(x), \quad x \in \mathcal{O}_\infty, \\
\psi(x) &= \psi_1(x) + \psi_2(x) + \psi_3(x), \quad x \in \mathcal{O}_\infty,
\end{align*}
\]

\([1] = [\varphi_1(1)] = [\varphi_2(1)] = -[\varphi_3(1)] = [\psi_1(1)] = [\psi_2(1)] = -[\psi_3(1)] \in K_0((B^\omega)^G) \).

Since \([\varphi_2 + \varphi_3](1) = [\psi_2 + \psi_3](1) = 0 \in K_0((B^\omega)^G) \), Corollary 1.3 implies that there exists a unitary \( u \in U((B^\omega)^G) \) satisfying \( u(\varphi_2 + \varphi_3)(x) u^* = (\psi_2 + \psi_3)(x) \) for \( x \in \mathcal{O}_\infty \). We set \( \varphi^u_1(x) = u \varphi_1(x) u^* \). Then \( \varphi^u_1 \) is in \( \text{Hom}_G(\mathcal{O}_\infty, B^\omega) \) satisfying \( \varphi^u_1(1) = \psi_1(1) \), and \( \varphi^u \circ \omega_\gamma \) and \( \varphi^u_1 + \psi_2 + \psi_3 \) are \( G \)-approximately unitarily equivalent. Since \((\varphi^u_1 + \psi_3)(1) = (\psi_1 + \psi_3)(1) \) whose class in \( K_0((B^\omega)^G) \) is 0, Corollary 1.3 again implies that there exists a unitary \( v \in U((B^\omega)^G) \) satisfying \( v \psi_2(1) = \psi_2(1) \) and \( v(\varphi^u_1 + \psi_3)(x) v^* = (\psi_1 + \psi_3)(x) \) for \( x \in \mathcal{O}_\infty \). This shows that \( vuv \varphi(x) v^* v^* = \psi(x) \) for \( x \in \mathcal{O}_\infty \), and so \( \varphi \) and \( \psi \) are \( G \)-approximately unitarily equivalent. \( \square \)

5. Splitting theorem and Uniqueness theorem

Thanks to Theorem 4.1, we can obtain a \( G \)-equivariant version of Kirchberg-Phillips’s \( \mathcal{O}_\infty \) theorem [7, Theorem 3.15], [13, Theorem 7.2.6].

**Theorem 5.1.** Let \( G \) be a finite group, and let \((A, \alpha)\) be a \( G \)-C*-algebra. We assume that \( A \) is a unital Kirchberg algebra and \( \alpha \) is outer. Let \( \{\gamma^{(i)}\}_{i=1}^\infty \) be any sequence of quasi-free actions of \( G \) on \( \mathcal{O}_\infty \). Then \((A, \alpha)\) is conjugate to

\[
(A \otimes \bigotimes_{i=1}^\infty \mathcal{O}_\infty, \alpha \otimes \bigotimes_{i=1}^\infty \gamma^{(i)}).
\]

**Proof.** Let

\[
(B, \beta) = (\bigotimes_{i=1}^\infty \mathcal{O}_\infty, \bigotimes_{i=1}^\infty \gamma^{(i)}),
\]

and let \( \rho_l, \rho_r \in \text{Hom}_G(B, B \otimes B) \) be as in Theorem 2.3 Then Theorem 4.1 implies that \( \rho_r \) and \( \rho_l \) are \( G \)-approximately unitarily equivalent.

To prove the statement applying Theorem 2.3, it suffices to construct a unital embedding of \((B, \beta)\) in \((A_\omega, \alpha_\omega)\). For this, it suffices to construct a unital embedding of \((\mathcal{O}_\infty, \gamma^{(i)})\) into \((A_\omega, \alpha_\omega)\) for each \( i \) because the usual trick of taking subsequences can make the embeddings commute with each other. Let \( \gamma \) be the quasi-free action of \( G \) on \( \mathcal{O}_\infty \) such that \((\pi_\gamma, \mathcal{H}_\infty)\) is unitarily equivalent to the infinite direct sum of the regular representation. Since there is a unital embedding of \((\mathcal{O}_\infty, \gamma^{(i)})\) into \((\mathcal{O}_\infty, \gamma)\), in order to prove the theorem, it only remains to construct a unital embedding of \((\mathcal{O}_\infty, \gamma)\) into \((A_\omega, \alpha_\omega)\).

Thanks to [13, Lemma 3], we can find a nonzero projection \( e \in A_\omega \) satisfying \( e \alpha_\omega(g)(e) = 0 \) for any \( g \in G \setminus \{e\} \). We choose an isometry \( v \in A_\omega \) satisfying \( vv^* \leq e \), and set \( s_{0,g} = \alpha_\omega(g)(v) \). Then \( \{s_{0,g}\}_{g \in G} \) are isometries in \( A_\omega \) with mutually orthogonal ranges satisfying \( \alpha_\omega(s_{0,h}) = s_{0,g} \). Let \( p = \sum_{g \in G} s_{0,g} s_{0,g}^* \), which is a projection in \((A_\omega)^G\). Replacing \( v \) if necessary, we may assume that \( p \neq 1 \). Since \((A_\omega)^G\) is purely
infinite and simple, we can find a sequence of partial isometries \( \{ w_i \}_{i=0}^{\infty} \) in \( (A_\omega)^G \) with \( w_0 = p \) such that \( w_i^* w_i = p \) for all \( i \), and \( \{ w_i w_i^* \}_{i=0}^{\infty} \) are mutually orthogonal. Let \( s_{i,g} = w_i s_{0,g} \). Then \( \{ s_{i,g} \}_{(i,g) \in \mathbb{N} \times G} \) is a countable family of isometries in \( A_\omega \) with mutually orthogonal ranges satisfying \( \alpha_{\omega g}(s_{i,h}) = s_{i,gh} \). Thus we get the desirable embedding of \( (O_\infty, \gamma) \) into \( (A_\omega, \alpha_\omega) \). □

Applying Theorem 5.1 to \( A = O_\infty \) with a faithful quasi-free action \( \alpha \), we obtain

**Corollary 5.2.** Any two faithful quasi-free actions of a finite group on \( O_\infty \) are mutually conjugate.

### 6. Asymptotic Representability

**Definition 6.1.** An action \( \alpha \) of a discrete group \( G \) on a unital \( C^* \)-algebra \( A \) is said to be **asymptotically representable** if there exists a continuous family of unitaries \( \{ u_g(t) \}_{t \geq 0} \) in \( U(A) \) for each \( g \in G \) satisfying

\[
\lim_{t \to \infty} \| u_g(t)xu_g(t)^* - \alpha_g(x) \| = 0, \quad \forall x \in A, \forall g \in G,
\]

\[
\lim_{t \to \infty} \| u_g(t)u_h(t) - u_{gh}(t) \| = 0, \quad \forall g, h \in G,
\]

\[
\lim_{t \to \infty} \| \alpha_g(u_h(t)) - u_{gh^{-1}}(t) \| = 0, \quad \forall g, h \in G.
\]

An action \( \alpha \) is said to be approximately representable if \( \alpha \) satisfies the above condition with a sequence \( \{ u_g(n) \}_{n \in \mathbb{N}} \) in place of the continuous family \( \{ u_g(t) \}_{t \geq 0} \).

Every asymptotically representable action is approximately representable, but the converse may not be true in general. When \( G \) is a finite abelian group, an action \( \alpha \) is approximately representable if and only if its dual action has the Rohlin property. When \( G \) is a cyclic group of prime power order, approximately representable quasi-free actions on \( O_n \) with finite \( n \) are completely characterized in [5], and there exist quasi-free actions that are not approximately representable.

The purpose of this section is to show the following theorem:

**Theorem 6.2.** Every quasi-free action of a finite group \( G \) on \( O_\infty \) is asymptotically representable.

It is unlikely that one could show Theorem 6.2 directly from the definition of quasi-free actions. Our proof uses the intertwining argument between two model actions; one is obviously quasi-free, and the other is an infinite tensor product action, that can be shown to be asymptotically representable.

We first introduce the notion of \( K \)-trivial embeddings of the group \( C^* \)-algebra. We denote by \( \{ \lambda_g \}_{g \in G} \) the left regular representation of a finite group \( G \). The group \( C^* \)-algebra \( C^*(G) \) is the linear span of \( \{ \lambda_g \}_{g \in G} \).

**Definition 6.3.** Let \( G \) be a finite group, and let \( A \) be a unital \( C^* \)-algebra. An unital injective homomorphism \( \rho : C^*(G) \to A \) is said to be a **\( K \)-trivial embedding** if \( KK(\rho) = KK(C^*(G) \ni \lambda_g \mapsto 1 \in A) \).
For each irreducible representation \((\pi, H_\pi)\) of \(G\), we choose an orthonormal basis \(\{\xi(\pi)_{ij}\}_{i=1}^{n_\pi}\) of \(H_\pi\), where \(n_\pi = \dim \pi\). We set \(\pi(g)_{ij} = \langle \pi(g)\xi(\pi)_{i}, \xi(\pi)_{j}\rangle\), and

\[
e(\pi)_{ij} = \frac{n_\pi}{\# G} \sum_{g \in G} \pi(g)_{ij} \lambda_g.
\]

Then \(\{e(\pi)_{ij}\}_{1 \leq i, j \leq n_\pi}\) is a system of matrix units, and we have

\[
\lambda_g = \sum_{\pi \in G} \sum_{i,j=1}^{n_\pi} \pi(g)_{ij} e(\pi)_{ij}.
\]

Let \(C^*(G)_\pi\) be the linear span of \(\{e(\pi)_{ij}\}_{i,j=1}^{\dim \pi}\). Then \(C^*(G)_\pi\) is isomorphic to the matrix algebra \(M_{n_\pi}(\mathbb{C})\), and \(C^*(G)\) has the direct sum decomposition

\[
C^*(G) = \bigoplus_{\pi \in G} C^*(G)_\pi.
\]

Let \(\chi_\pi(g) = \text{Tr}(\pi(g))\) be the character of \(\pi\). Then

\[
z(\pi) = \frac{n_\pi}{\# G} \sum_{g \in G} \chi_\pi(g) \lambda_g = \sum_{i=1}^{n_\pi} e(\pi)_{ii}
\]

is the unit of \(C^*(G)_\pi\).

It is easy to show the following lemma:

**Lemma 6.4.** Let \(G\) be a finite group, and let \(A, B\) be unital simple purely infinite \(C^*\)-algebras.

1. A unital injective homomorphism \(\rho : C^*(G) \to A\) is a \(K\)-trivial embedding if and only if \([\rho(e(\pi))_{11}] = 0\) in \(K_0(A)\) for any nontrivial irreducible representation \(\pi\). When \(K_0(A)\) is torsion free, it is further equivalent to the condition that \([\rho(z(\pi))] = 0\) in \(K_0(A)\) for any nontrivial irreducible representation \(\pi\).

2. Any two \(K\)-trivial unital embeddings of \(C^*(G)\) into \(A\) are unitarily equivalent.

3. If \(\rho : C^*(G) \to A\) and \(\sigma : C^*(G) \to B\) are \(K\)-trivial embeddings, so is the tensor product embedding \(C^*(G) \ni \lambda_g \mapsto \rho(\lambda_g) \otimes \sigma(\lambda_g) \in A \otimes B\).

We now construct a \(K\)-trivial embedding of \(C^*(G)\) into \(\mathcal{O}_\infty\). We fix a nonzero projection \(p \in \mathcal{O}_\infty\) with \([p] = 0\) in \(K_0(\mathcal{O}_\infty)\), and fix unital embeddings

\[
B(\ell^2(G)) \subset \mathcal{O}_2 \subset p\mathcal{O}_\infty p.
\]

We denote by \(\sigma_0 : C^*(G) \to p\mathcal{O}_\infty p\) the resulting embedding, and set \(u_g = \sigma_0(\lambda_g) + 1 - p\). Then \(\sigma : C^*(G) \ni \lambda_g \mapsto u_g \in \mathcal{O}_\infty\) is a \(K\)-trivial embedding of \(C^*(G)\) into \(\mathcal{O}_\infty\).

Using \(\{u_g\}_{g \in G}\), we introduce a \(G\)-\(C^*\)-algebra \((A, \alpha)\) by

\[
(A, \alpha_g) = \bigotimes_{k=1}^{\infty} (\mathcal{O}_\infty, \text{Ad} u_g).
\]

More precisely, we set

\[
A_n = \bigotimes_{k=1}^{n} \mathcal{O}_\infty, \quad u_g^{(n)} = \bigotimes_{k=1}^{n} u_g,
\]
and $\alpha^{(n)}_g = \text{Ad} \, u^n_g$. Then $(A, \alpha)$ is the inductive limit of the system $\{(A_n, \alpha^{(n)})\}_{n=1}^{\infty}$ with the embedding $\iota_n : A_n \ni x \mapsto x \otimes 1 \in A_{n+1}$. The $C^*$-algebra $A$ is isomorphic to $O_\infty$, and the action $\alpha$ is outer.

**Lemma 6.5.** Let the notation be as above.

(1) The action $\alpha$ is asymptotically representable.

(2) The embedding $\iota_\alpha : C^*(G) \ni \lambda_g \mapsto \lambda_g^\alpha \in A \rtimes_\alpha G$ gives $KK$-equivalence.

**Proof.** (1) It suffices to construct a homotopy $\{v_g(t)\}_{t \in [0,1]}$ of unitary representations of $G$ in $A_3$ satisfying $v_g(0) = u_g \otimes 1 \otimes 1$, $v_g(1) = u_g^{(2)} \otimes 1$, and $\alpha^{(3)}_g(v_h(t)) = v_{gh^{-1}}(t)$. Since $\{u_g \otimes 1\}_{g \in G}$, $\{u_g^{(2)}\}_{g \in G}$, and $\{1 \otimes u_g\}_{g \in G}$ give $K$-trivial embeddings of $C^*(G)$ into $A_2$, there exist unitaries $w_1, w_2 \in U(A_2)$ satisfying $w_1(u_g \otimes 1)w^*_1 = w_2(1 \otimes u_g)w^*_2 = u_g^{(2)}$. Let $w = (w_1 \otimes 1)(1 \otimes w_2^*)$, which is a unitary in $A_3^G = A_3 \cap \{u_g^{(3)}\}_{g \in G}$ satisfying $w(u_g \otimes 1 \otimes 1)w^* = u_g^{(2)} \otimes 1$. Since $A_3^G$ is isomorphic to a finite direct sum of $C^*$-algebras Morita equivalent to $O_\infty$, there exists a homotopy $\{w(t)\}_{t \in [0,1]}$ in $U(A_3^G)$ with $w(0) = 1$ and $w(1) = w$. Thus $v_g(t) = w(t)(u_g \otimes 1 \otimes 1)w(t)^*$ gives the desired homotopy.

(2) We identify $B_n = A_n \rtimes_\alpha(n) G$ with the $C^*$-subalgebra of $A \rtimes_\alpha G$ generated by $A_n$ and $\{\lambda_g^n\}_{g \in G}$, and we denote by $\iota'_n : B_n \to B_{n+1}$ the embedding map. Then $A \rtimes_\alpha G$ is the inductive limit of the system $\{B_n\}_{n=1}^{\infty}$. Let $\iota^{(n)}_\alpha : C^*(G) \ni \lambda_g \mapsto \lambda_g^n \in B_n$. Since we have $\iota'_n \circ \iota^{(n)}_\alpha = \iota^{(n+1)}_\alpha$, in order to prove the statement it suffices to show that $\iota^{(n)}_\alpha$ induces isomorphisms of the $K$-groups for every $n$.

Since $\alpha^{(n)}_g$ is inner, there exists an isomorphism $\theta_n : B_n \to A_n \otimes C^*(G)$ given by $\theta_n(a) = a \otimes 1$ for $a \in A_n$ and $\theta_n(\lambda_g^n) = u_g^{(n)} \otimes \lambda_g$. Thus all we have to show is that the map $\theta_n \circ \iota^{(n)}_\alpha : C^*(G) \ni \lambda_g \mapsto u_g^{(n)} \otimes \lambda_g \in A_n \otimes C^*(G)$ induces isomorphisms of the $K$-groups. This follows from that fact that $A_n$ is isomorphic to $O_\infty$ and $\{u_g^{(n)}\}_{g \in G}$ gives an $K$-trivial embedding of $C^*(G)$ into $A_n$.

**Lemma 6.6.** For the $G$-$C^*$-algebra $(A, \alpha)$ as constructed above, any unital $\varphi \in \text{Hom}_{G}(A, A)$ is $G$-asymptotically unitarily equivalent to id.

**Proof.** Let $B = A \rtimes_\alpha G$, and let $\hat{\alpha} : B \to B \otimes C^*(G)$ be the dual coaction of $\alpha$. Then $\varphi$ extends to a unital endomorphism $\hat{\varphi}$ in $\text{End}(B)$ with $\hat{\varphi}(\lambda_g^n) = \lambda_g^n$, which satisfies $\hat{\alpha} \circ \hat{\varphi} = (\hat{\varphi} \otimes \text{id}_{C^*(G)}) \circ \hat{\alpha}$. By Lemma 6.5 (2), we have $KK(\hat{\varphi}) = KK(\text{id}_B)$. Thus Lemma 6.5 (1) and [10], Theorem 4.8 imply that there exists a continuous family of unitaries $\{u(t)\}_{t \geq 0}$ in $A$ satisfying

$$\lim_{t \to \infty} \|u(t)xu(t)^* - \hat{\varphi}(x)\| = 0, \quad \forall x \in B.$$ 

Setting $x = \lambda_g^n$, we know that $\{\alpha_g(u(t)) - u(t)\}_{t \geq 0}$ converges to 1. Since $G$ is a finite group, there exists a conditional expectation from $A$ onto $A^G$, and we can construct a continuous family of unitaries $\{u(t)\}_{t \geq 0}$ in $A^G$ such that $\{u(t) - \bar{u}(t)\}_{t \geq 0}$ converges to 0 by a standard perturbation argument. Therefore $\varphi$ and $\text{id}$ are $G$-asymptotically unitarily equivalent.
Proof of Theorem 6.2. Let $\gamma$ be a faithful quasi-free $G$-action on $O_\infty$. Thanks to Corollary 5.2 we may assume that $O_\infty$ has the canonical generators $\{s_i\}_{i \in J}$ with $G \subset J$ satisfying $\gamma_g(s_h) = s_{gh}$. Since $\alpha$ is asymptotically representable, it suffices to show that $\alpha$ and $\gamma$ are conjugate. Thanks to Theorem 5.1, the action $\alpha$ is conjugate to $\alpha \otimes \gamma$, and so there exists a unital embedding of $(O_\infty, \alpha)$ into $(A, \alpha)$. Thus if there exists a unital embedding of $(A, \alpha)$ into $(O_\infty, \gamma)$, Theorem 2.2, Theorem 4.1 and Lemma 6.6 imply that $\alpha$ and $\gamma$ are conjugate. Since $\gamma$ is conjugate to the infinite tensor product of its copies thanks to Theorem 5.1 again, all we have to show is that there exists a unital embedding of $(O_\infty, \text{Ad } u)$ into $(O_\infty, \gamma)$.

We denote by $O_\infty^\gamma$ the fixed point subalgebra of $O_\infty$ under the $G$-action $\gamma$. Since $O_\infty^\gamma$ is purely infinite and simple, we can choose a nonzero projection $q_0 \in O_\infty^\gamma$ with $[q_0] = 0$ in $K_0(O_\infty^\gamma)$. We set $q_1 = \sum_{g \in G} s_g q_0 s^*_g$. A similar argument as in the proof of Lemma 6.3 implies that $[q_1] = 0$ in $K_0(O_\infty^\gamma)$. We set

$$v_g = \sum_{h \in G} s_{gh} q_0 s^*_h + 1 - q_1.$$

Then $\{v_g\}_{g \in G}$ is a unitary representation of $G$ in $O_\infty$ satisfying $\gamma_g(v_h) = v_{gh^{-1}}$, and so $\{v^*_g\}_{g \in G}$ is a $\gamma$-cocycle. We show that this is a coboundary by using [4, Remark 2.6]. Indeed, we have

$$\frac{1}{\#G} \sum_{g \in G} v^*_g \lambda^\gamma_g = (1 - q_1) e_\gamma + \frac{1}{\#G} \sum_{g \in G} \sum_{h \in G} s_{gh} q_0 s^*_g \lambda^\gamma_g = (1 - q_1) e_\gamma + \sum_{h \in G} s_{gh} q_0 e_h s^*_h.$$

This means that the class of this projection in $K_0(O_\infty \rtimes \gamma G)$ is

$$[(1 - q_1) e_\gamma] + \#G[q_0 e_\gamma] = [e_\gamma],$$

which implies that $\{v^*_g\}_{g \in G}$ is a coboundary. Thus there exists a unitary $v \in O_\infty$ satisfying $v^*_g = v\gamma_g(v^*)$.

We set $w_g = v^* v_g v$, and claim that $\{w_g\}_{g \in G}$ gives a $K$-trivial embedding of $C^*(G)$ into $O_\infty^\gamma$. Indeed,

$$\gamma_g(w_h) = \gamma_g(v^*) \gamma_h(v_h) \gamma_g(v) = v^* v_g v_{gh^{-1}} v_g v = w_h,$$

which shows $w_g \in O_\infty^\gamma$. Let $\rho : C^*(G) \ni \lambda_g \mapsto w_g \in O_\infty^\gamma$. Thanks to Lemma 6.4(1), in order to prove the claim it suffices to show that $[\rho(e(\pi)_{11})] = 0$ in $K_0(O_\infty^\gamma)$ for any nontrivial irreducible representation $(\pi, H_\pi)$ of $G$. Indeed, we have

$$K_0(j_\gamma)([\rho(e(\pi)_{11})]) = [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g w_h] = [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g v^* v_h v]$$

$$= [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g v^* v_{gh^{-1}} v_h v] = [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g v^* v_g v_{gh^{-1}} v_h v]$$

$$= [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g v^* v_h v_{gh^{-1}} v_h v] = [\frac{n_\pi}{\#G^2} \sum_{g, h \in G} \pi(h)_{11}^\gamma \lambda^\gamma_g v^* v_h v_{gh^{-1}} v_h v].$$
Let $\rho_0 : C^*(G) \ni \lambda_g \mapsto v_g \in \mathcal{O}_\infty$. Equation \([6.1]\) implies that this is equal to
\[ [\rho_0(e(\pi)_{11}) \sum_{k \in G} s_k q_0 e_\gamma s_k^*] = n_x[q_0 e_\gamma] = 0. \]
Thus the claim is shown.

We choose a unital embedding $\mu_0 : \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty$. Since both $\{\mu_0(u_g)\}_{g \in G}$ and $
\{w_g\}_{g \in G}$ give $K$-trivial embeddings of $C^*(G)$ into $\mathcal{O}_\infty$, Lemma \([6.4]\)(2) shows that we may assume $\mu_0(u_g) = w_g$ by replacing $\mu_0$ if necessary. Let $\mu(x) = v\mu_0(x)v^*$. Then
\[ \gamma_g \circ \mu(x) = \gamma_g(v)\mu_0(x)\gamma_g(v^*) = v_g\mu_0(x)v^* = v\mu_0(u_gx u_g^*)v^* = \mu \circ \text{Ad } u_g(x). \]
Thus $\mu$ is the desired embedding of $(\mathcal{O}_\infty, \text{Ad } u.)$ into $(\mathcal{O}_\infty, \gamma)$.

From Theorem \([6.2]\) and Lemma \([6.6]\) we get

**Corollary 6.7.** Let $G$ be a finite group, and let $\gamma$ be a quasi-free action of $G$ on $\mathcal{O}_\infty$. Then any unital $\varphi \in \text{Hom}_G(\mathcal{O}_\infty, \mathcal{O}_\infty)$ is $G$-asymptotically unitarily equivalent to id.

### 7. Equivariant Rørdam Group

Let $A$ and $B$ be simple $C^*$-algebras. For simplicity we assume that $A$ and $B$ are unital. Following Rørدام [18, p.40], we denote by $H(A, B)$ the set of the approximately unitary equivalence classes of nonzero $\mathcal{O}_\infty$-homomorphisms from $A$ into $B \otimes \mathbb{K}$. Choosing two isometries $s_1$ and $s_2$ satisfying the $\mathcal{O}_2$ relation in $M(B \otimes \mathbb{K})$, we can define the direct sum $[\varphi] \oplus [\psi]$ of two classes $[\varphi]$ and $[\psi]$ in $H(A, B)$ to be the class of the homomorphism
\[ A \ni x \mapsto s_1\varphi(x)s_1^* + s_2\psi(x)s_2^* \in B \otimes \mathbb{K}. \]
This makes $H(A, B)$ a semigroup. When $A$ is a separable simple nuclear $C^*$-algebra and $B$ is a Kirchberg algebra, the Rørدام semigroup $H(A, B)$ is in fact a group. Moreover, if $A$ satisfies the universal coefficient theorem, it is isomorphic to $KL(A, B)$, a certain quotient of $KK(A, B)$.

Let $G$ be a finite group, and let $\alpha$ and $\beta$ be outer $G$-actions on $A$ and $B$ respectively. We equip $B \otimes \mathbb{K}$ with a $G$-$C^*$-algebra structure by the diagonal action $\beta^*_g = \beta_g \otimes \text{Ad } u_g$, where $\{u_g\}$ is a countable infinite direct sum of the regular representation of $G$. Then we can introduce an equivariant version $H_G(A, B)$ as the set of the $G$-approximately equivalence classes of nonzero $G$-homomorphisms in $\text{Hom}_G(A, B \otimes \mathbb{K})$.

**Theorem 7.1.** Let $(A, \alpha)$ and $(B, \beta)$ be unital $G$-$C^*$-algebras with outer actions $\alpha$ and $\beta$. We assume that $A$ is separable, simple, and nuclear, and $B$ is a Kirchberg algebra. Then $H_G(A, B)$ is a group.

Let $(A, \alpha)$ and $(B, \beta)$ be as above. We say that $\varphi \in \text{Hom}_G(A, B)$ is $\mathcal{O}_2$-absorbing if there exists $\varphi' \in \text{Hom}_G(A \otimes \mathcal{O}_2, B)$ with $\varphi = \varphi' \circ \iota_A$, where $A \otimes \mathcal{O}_2$ is equipped with the $G$-action $\alpha \otimes \text{id}_{\mathcal{O}_2}$, and $\iota_A : A \ni x \mapsto x \otimes 1 \in A \otimes \mathcal{O}_2$ is the inclusion map. We say that $\varphi \in \text{Hom}_G(A, B)$ is $\mathcal{O}_\infty$-absorbing if there exists a unital embedding of $\mathcal{O}_\infty$ in $(\varphi(1)B^G \varphi(1)) \cap \varphi(A)'$. 

The proof of Theorem 7.1 follows from essentially the same argument as in [18, Lemma 8.2.5] with the following lemma.

**Lemma 7.2.** Let the notation be as above.

1. Let \( \varphi, \psi \in \text{Hom}_G(A, B) \) be \( \mathcal{O}_2 \)-absorbing \( G \)-homomorphisms, either both unital or both nonunital. Then \( \varphi \) and \( \psi \) are \( G \)-approximately unitarily equivalent.

2. Any element in \( \text{Hom}_G(A, B) \) is \( G \)-approximately unitarily equivalent to a \( \mathcal{O}_\infty \)-absorbing one in \( \text{Hom}_G(A, B) \).

**Proof.** (1) When \( \varphi \) and \( \psi \) are nonunital, the two projections \( \varphi(1) \) and \( \psi(1) \) are equivalent in \( B^G \), and we may assume \( \varphi(1) = \psi(1) \). Replacing \( B \) with \( \varphi(1)B\varphi(1) \), we may assume that \( \varphi \) and \( \psi \) are unital.

Let \( \gamma \) be a faithful quasi-free action of \( G \) on \( \mathcal{O}_\infty \). Since \((A \otimes \mathcal{O}_2, \alpha \otimes \text{id}_{\mathcal{O}_2})\) is conjugate to \((\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \text{id}_{\mathcal{O}_2})\) thanks to [4, Corollary 4.3], it suffices to show that any unital \( \varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty \otimes \mathcal{O}_2, B) \) are \( G \)-approximately unitarily equivalent. Theorem 4.1 implies that there exists \( u \in U((B^\omega)^G) \) satisfying \( u\varphi(x \otimes 1)u^* = \psi(x \otimes 1) \) for any \( x \in \mathcal{O}_\infty \), where \( \omega \in \beta \mathbb{N} \setminus \mathbb{N} \) is a free ultrafilter. Let \( D = (B^\omega)^G \cap \psi(\mathcal{O}_\infty \otimes 1)' \). Then it suffices to show that the two unital homomorphisms \( \rho, \sigma \in \text{Hom}(\mathcal{O}_2, D) \) defined by \( \rho(y) = u\varphi(1 \otimes y)u^* \), \( \sigma(y) = \psi(1 \otimes y) \) for \( y \in \mathcal{O}_2 \), are approximately unitarily equivalent. Indeed, since \((B^\omega)^G \cap B' = (B_\omega)^G \) is purely infinite and simple, for any separable \( C^* \)-subalgebra \( C \) of \( D \) there exists a unital embedding of \( \mathcal{O}_\infty \) in \( D \cap C' \). Thus essentially the same proof of [15, Lemma 2.17] shows that \( \text{cel}(D) \) is finite (see [15, Lemma 2.1.1] for the definition). Therefore \( \rho \) and \( \sigma \) are approximately unitarily equivalent thanks to [17, Theorem 3.6].

(2) Since \((B, \beta)\) is conjugate to \((B \otimes \mathcal{O}_\infty, \beta \otimes \text{id}_{\mathcal{O}_\infty})\) thanks to [4, Corollary 2.10], the statement follows from the same argument as in the proof of [18, Lemma 8.2.5,(i)]. \( \square \)

**Remark 7.3.** There are two natural homomorphisms

\[ \mu : H_G(A, B) \to H(A, B), \]
\[ \nu : H_G(A, B) \to H(A \rtimes_\alpha G, B \rtimes_\beta G). \]

The first one is the forgetful functor. Every \( \varphi \in \text{Hom}_G(A, B) \) extends to \( \tilde{\varphi} \in \text{Hom}(A \rtimes_\alpha G, B \rtimes_\beta G) \) by \( \tilde{\varphi}(\lambda_g^\alpha) = \lambda_g^\beta \), and the second one is given by associating \( [\varphi] \in H(A \rtimes_\alpha G, B \rtimes_\beta G) \) with \( [\varphi] \in H_G(A, B) \). The following hold for the two maps (see [6, Section 4] for more general treatment):

1. If \( \beta \) has the Rohlin property, then \( \mu \) is injective, and the image of \( \mu \) is
   \[ \{[\rho] \in H(A, B) | [\beta^* \circ \rho] = [\rho \circ \alpha_g], \forall g \in G \}. \]

2. If \( \beta \) is approximately representable, then \( \nu \) is injective, and the image of \( \nu \) is
   \[ \{[\rho] \in H(A \rtimes_\alpha G, B \rtimes_\beta G) | \tilde{\beta}^* \circ \rho = [(\rho \otimes \text{id}_{C^*_\beta(G)}) \circ \tilde{\alpha}] \}. \]

**Remark 7.4.** Let \( \hat{H}_G(A, B) \) be the set of the \( G \)-asymptotically equivalence classes of nonzero \( G \)-homomorphisms in \( \text{Hom}_G(A, B \otimes \mathbb{K}) \). It is tempting to conjecture that the natural map from \( \hat{H}_G(A, B) \) to the equivariant \( KK \)-group \( KK_G(A, B) \) is an isomorphism, as it is the case for trivial \( G \) (see [15]).
8. Appendix

In this appendix, we show the equivalence of (4) and (5) in Theorem 3.1. Since our argument works for a compact group \( G \), we assume that \( G \) is compact in what follows. Our proof is new even for trivial \( G \). Let \( \alpha \) be a quasi-free action of \( G \) on \( \mathcal{O}_n \) with finite \( n \), and let \((B, \beta)\) be a unital \( G\)-\( C^* \)-algebra. Now the definition of the projection \( e_\beta \in B \rtimes_\beta G \) should be modified to \( e_\beta = \int_G \lambda \beta_g dg \), where \( dg \) is the normalized Haar measure of \( G \). For two unital \( \varphi, \psi \in \text{Hom}_G(\mathcal{O}_n, B) \), we define \( u_{\varphi, \psi} \in U(B^G) \) as in Theorem 3.1.

Let \( \mathcal{E}_n \) be the Cuntz-Toeplitz algebra with the canonical generators \( \{t_i\}_{i=1}^n \). We denote by \( q_n : \mathcal{E}_n \to \mathcal{O}_n \) sending \( t_i \) to \( s_i \) for \( i = 1, 2, \ldots, n \). Then the kernel \( J_n \) of \( q_n \) is the ideal generated by \( p_n = 1 - \sum_{i=1}^n t_i t_i^* \), and is isomorphic to the compact operators \( \mathbb{K} \). We denote by \( i_n : J_n \to \mathcal{E}_n \) the inclusion map. Since \( \mathcal{O}_n \) is nuclear, the exact sequence

\[
0 \longrightarrow J_n \xrightarrow{i_n} \mathcal{E}_n \xrightarrow{q_n} \mathcal{O}_n \longrightarrow 0,
\]

is semisplit, that is, there exists a unital completely positive lifting \( l_n : \mathcal{O}_n \to \mathcal{E}_n \) of \( q_n \). We denote by \( \tilde{\alpha} \) the quasi-free action of \( G \) on \( \mathcal{E}_n \) that is a lift of \( \alpha \). By replacing \( l_n \) with \( l_n^G \) given by

\[
l_n^G(x) = \int_G \tilde{\alpha}_g \circ l_n \circ \alpha_{g^{-1}}(x) dg, \quad x \in \mathcal{O}_n,
\]

we see that (8.1) is a semisplit exact sequence of \( G\)-\( C^* \)-algebras. Thus it induces the following 6-term exact sequence of \( KK_G \)-groups:

\[
\begin{array}{ccc}
KK^0_G(J_n, B) & \xleftarrow{i_n^*} & KK^0_G(\mathcal{E}_n, B) \xleftarrow{q_n^*} \quad KK^0_G(\mathcal{O}_n, B) \\
\delta & \downarrow & \delta \\
KK^1_G(\mathcal{O}_n, B) & \xrightarrow{q_n} & KK^1_G(\mathcal{E}_n, B) \xrightarrow{i_n^*} \quad KK^1_G(J_n, B)
\end{array}
\]

Let \( H_n \) be the \( n \)-dimensional Hilbert space \( \mathbb{C}^n \) with the canonical orthonormal basis \( \{e_i\}_{i=1}^n \). We regard \( H_n \) as a \( \mathbb{C} - \mathbb{C} \) bimodule with a \( G \)-action given by \( \pi_\alpha \). We denote by \( \mathcal{F}_n \) the full Fock space

\[
\mathcal{F}_n = \bigoplus_{m=0}^\infty H_n^\otimes m,
\]

with a unitary representation \( \pi_{\mathcal{F}_n} \) of \( G \) coming from \( \pi_\alpha \). Identifying \( t_i \) with the creation operator of \( e_i \) acting on \( \mathcal{F}_n \), we regard \( \mathcal{E}_n \) as a \( C^* \)-subalgebra of \( \mathcal{B}(\mathcal{F}_n) \). With this identification, we have \( J_n = \mathbb{K}(\mathcal{F}_n) \), and \( p_n \) is the projection onto \( H_n^\otimes 0 \). We regard \( \mathcal{F}_n \) as \( J_n - \mathbb{C} \) bimodule, which gives the \( KK_G \)-equivalence of \( J_n \) and \( \mathbb{C} \).
Pimsner’s computation [16, Theorem 4.9] yields the following 6-term exact sequence:

\[
\begin{array}{ccccc}
KK_G^0(\mathbb{C}, B) & \xleftarrow{1-[H_n]_{\hat{\otimes}}} & KK_G^0(\mathbb{C}, B) & \xleftarrow{\delta'} & KK_G^0(O_n, B) \\
\delta'' & & & & \delta''
\end{array}
\]

where \([H_n]_{\hat{\otimes}}\) denote the left multiplication of the class \([H_n] \in KK_G(\mathbb{C}, \mathbb{C})\). Note that the identification of \(KK_G^*(J_n, B)\) and \(KK_G^*(\mathbb{C}, B)\) is given by \([\mathcal{F}_n] \in KK_G(J_n, \mathbb{C})\), and so \(\delta'' = \delta \circ ([\mathcal{F}_n]_{\hat{\otimes}})\).

With the Green-Julg isomorphism \(h_* : KK_G^*(\mathbb{C}, B) \to K_*(B \rtimes_{\beta} G)\) ([1, Theorem 11.7.1]), we have the commutative diagram

\[
\begin{array}{ccccc}
KK_G^*(\mathbb{C}, B) & \xrightarrow{[H_n]_{\hat{\otimes}}} & KK_G^*(\mathbb{C}, B) & \xrightarrow{h_*} & K_*(B \rtimes_{\beta} G) \\
h_* & & & & \downarrow h_*
\end{array}
\]

and so we get the following 6-term exact sequence

\[
\begin{array}{ccccc}
K_0(B \rtimes_{\beta} G) & \xleftarrow{1-K_0(\beta_{\alpha})} & K_0(B \rtimes_{\beta} G) & \xleftarrow{\delta''} & KK_G^0(O_n, B) \\
\delta'' & & & & \delta''
\end{array}
\]

with \(\delta'' = \delta \circ ([\mathcal{F}_n]_{\hat{\otimes}}) \circ h_*^{-1}\). Now the proof of the equivalence of (4) and (5) in Theorem 3.1 follows from the next theorem.

**Theorem 8.1.** With the above notation, we have

\[
\delta''(K_1(j_{\beta}([u_{\psi, \varphi}]))) = KK_G(\psi) - KK_G(\varphi).
\]

The proof of Theorem 8.1 follows from a standard and rather tedious computation below. In what follows, we freely use the notation in Blackadar’s book [1] for KK-theory. We regard \(\mathbb{C}_1, C = C_0[0, 1]\), and \(S = C_0(0, 1)\) as \(G\)-C*-algebras with trivial \(G\)-actions.

[1, Theorem 19.5.7] shows that \(\delta''\) is given by the left multiplication of the class \(\delta_{\alpha}\) of the extension [8, 1] in \(KK_G^1(O_n, \mathbb{C}) = KK_G(O_n, \mathbb{C})\), whose Kasparov module \((\hat{E}_1, \phi_1, F_1) \in \mathcal{E}_G(O_n, \mathbb{C})\) is given as follows. By the Stinespring dilation of the G-equivariant lifting \(l_{\beta}^G : O_n \to \mathcal{E}_n \subseteq \mathbb{B}(\mathcal{F}_n)\), we get a Hilbert space \(H\) including \(\mathcal{F}_n\), with a unitary representation \(\pi_H\) of \(G\) extending \(\pi_{\mathcal{F}_n}\), satisfying the following condition: there is a unital \(G\)-homomorphism \(\Phi : O_n \to \mathbb{B}(H)\) such that if \(P\) is the projection from \(H\) onto \(\mathcal{F}_n\), then \(l_{\beta}^G(x) = P\Phi(x)P\) for any \(x \in O_n\). Now we have

\[
(E_1, \phi_1, F_1) = (H \otimes \mathbb{C}_1, \Phi \otimes 1, (2P - 1) \otimes \varepsilon),
\]

where \(\varepsilon = 1 \oplus -1\) is the generator of \(\mathbb{C}_1 \cong C^*(\mathbb{Z}_2)\).
Let \( z(t) = e^{2\pi it} \), and let \( \theta \) be the element in \( \text{Hom}_G(C_\mathbb{C}(0, 1), B) \) determined by \( \theta(z - 1) = u_{\psi, \varphi} - 1 \). Then \( h_1^{-1} \circ K_1(j_\beta)([u_{\psi, \varphi}]) \) is given by
\[
KK_G(\theta) \in KK_G(C_\mathbb{C}(0, 1), B) \cong KK_G(\mathbb{C}, B).
\]

In order to compute the Kasparov product of \( \delta_{q_n} \in KK_G(O_n, \mathbb{C}_1) \) and \( KK_G(\theta) \in KK_G(S, B) \), we need to identify \( KK_G(S, B) \) with \( KK_G(\mathbb{C}_1, B) \) explicitly, and we need the invertible element \( x \in KK_G(\mathbb{C}_1, S) \) defined in \([1, \text{Section 19.2}]\). By the extension
\[
0 \longrightarrow S \longrightarrow C \longrightarrow \mathbb{C} \longrightarrow 0,
\]
we get an invertible element in \( KK_G(\mathbb{C}, S \hat{\otimes} \mathbb{C}_1) \). Then \( x \) is the image of this element by the isomorphism
\[
\tau_{\mathbb{C}_1}: KK_G(\mathbb{C}, S \hat{\otimes} \mathbb{C}_1) \rightarrow KK_G(\mathbb{C} \hat{\otimes} \mathbb{C}_1, S \hat{\otimes} \mathbb{C}_1 \hat{\otimes} \mathbb{C}_1)
= KK_G(\mathbb{C}_1, S \hat{\otimes} M_2(\mathbb{C})) = KK_G(\mathbb{C}_1, S).
\]

For the identification of \( \mathbb{C}_1 \hat{\otimes} \mathbb{C}_1 \) and \( M_2(\mathbb{C}) \) with standard even grading, we follow the convention in the proof of \([1, \text{Theorem 18.10.12}] \) (our computation really depends on it). A direct computation shows that \( x \) is given by the Kasparov module \( (E_2, \phi_2, F_2) \in E_G(\mathbb{C}_1, S) \) with \( E_2 = \mathbb{C}^2 \hat{\otimes} (S \oplus S) \),
\[
F_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
\[
\phi_2(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Q, \quad \phi_2(\varepsilon) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes Q,
\]
where the projection \( Q \in M_2(M(S)) \) is given by
\[
Q(t) = \begin{pmatrix} 1 - t & \sqrt{t(1 - t)} \\ \sqrt{t(1 - t)} & t \end{pmatrix},
\]
and the grading of \( E_2 \) is given by
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

With this \( x \), we have
\[
\delta''(K_1(j_\beta)([u_{\psi, \varphi}])) = \delta_{q_n} \hat{\otimes} \mathbb{C}_1 x \hat{\otimes} S KK_G(\theta) = \theta_s(\delta_{q_n} \hat{\otimes} \mathbb{C}_1 x),
\]
and so our task now is to compute \( \delta_{q_n} \hat{\otimes} \mathbb{C}_1 x \) explicitly.

**Lemma 8.2.** The class \( \delta_{q_n} \hat{\otimes} \mathbb{C}_1 x \in KK_G(O_n, S) \) is given by the quasi-homomorphism \( \rho = (\rho^{(0)}, \rho^{(1)}) \) from \( O_n \) to \( S \) such that \( \rho^{(0)} \) and \( \rho^{(1)} \) are unital homomorphisms from \( O_n \) to \( \mathbb{B}(H \hat{\otimes} S) \) with \( \rho^{(0)}(x) = \Phi(x) \otimes 1 \) and
\[
\rho^{(1)}(x) = (P \otimes 1 + (1 - P) \otimes z)(\Phi(x) \otimes 1)(P \otimes 1 + (1 - P) \otimes z)^*.
\]

**Proof.** We regard \( H \hat{\otimes} S \) as an \( O_n \)-\( S \) bimodule with trivial grading, and we set \( E = (H \hat{\otimes} S) \oplus (H \hat{\otimes} S)^{op} \). We denote by \( \Psi: S \rightarrow Q(S \oplus S) \) a Hilbert \( S \)-module isomorphism given by
\[
\Phi(f)(t) = (\sqrt{1 - tf(t)}, \sqrt{tf(t)}).
\]
Then $E_1 \hat{\otimes}_{C_1} E_2$ is identified with $E$ via the identification of $(\xi_1 \hat{\otimes} f_1, \xi_2 \hat{\otimes} f_2) \in E$ and
\[
\xi_1 \hat{\otimes} 1 \hat{\otimes} C_1 (1, 0) \hat{\otimes} \Psi(f_1) + \xi_2 \hat{\otimes} 1 \hat{\otimes} C_1 (0, 1) \hat{\otimes} \Psi(f_2) \in H \hat{\otimes} C_1 \hat{\otimes} C_1 \mathbb{C}^2 \hat{\otimes} (S \oplus S).
\]

We claim that $\delta_{q_n} \hat{\otimes}_{C_i} x$ is given by the Kasparov module $(E, \phi, F) \in \mathcal{E}_G(O_n, S)$ with
\[
\phi(x) = \text{diag}(\Phi(x) \otimes 1, \Phi(x) \otimes 1),
\]
\[
F = \begin{pmatrix}
0 & 1 \otimes c \\
1 \otimes c & 0
\end{pmatrix} + \begin{pmatrix}
0 & -i(2P - 1) \otimes s \\
-i(2P - 1) \otimes s & 0
\end{pmatrix},
\]
where $c(t) = \cos(\pi t)$, $s(t) = \sin(\pi t)$. Indeed, it is easy to show that $(E, \phi, F)$ is a Kasparov module, and the graded commutator $[F_1 \hat{\otimes} 1_{E_2}, F]$ is positive. We show that $F$ is a $F_2$-connection (see [1, Definition 18.3.1] for the definition). Let $\xi \in H$, $x = (x_1, x_2) \in \mathbb{C}^2$, and $f = (f_1, f_2) \in S \oplus S$. Then we have
\[
T_{\xi \hat{\otimes} 1}(x \hat{\otimes} f) = (x_1 \xi \hat{\otimes} (\sqrt{1 - t} f_1 + \sqrt{t} f_2), x_2 \xi \hat{\otimes} (\sqrt{1 - t} f_1 + \sqrt{t} f_2)) \in E,
\]
\[
T_{\xi \hat{\otimes} x}(x \hat{\otimes} f) = (-ix_2 \xi \hat{\otimes} (\sqrt{1 - t} f_1 + \sqrt{t} f_2), ix_1 \xi \hat{\otimes} (\sqrt{1 - t} f_1 + \sqrt{t} f_2)) \in E.
\]
A direct computation shows that $T_{\xi \hat{\otimes} 1} \circ F_2 - F \circ T_{\xi \hat{\otimes} 1}$ and $T_{\xi \hat{\otimes} x} \circ F_2 + F \circ T_{\xi \hat{\otimes} x}$ are in $\mathbb{K}(E_2, E)$. Since $F_2$ and $F$ are self-adjoint, we see that $F$ is a $F_2$-connection. Therefore $(E, \phi, F)$ gives the Kasparov product $\delta_{q_n} \hat{\otimes}_{C_i} x$.

Note that $F$ satisfies $F = F^*, F^2 = 1$. Let
\[
U = \begin{pmatrix}
1 & 0 \\
0 & 1 \otimes c + i(2P - 1) \otimes s
\end{pmatrix},
\]
which is a unitary in $\mathbb{B}(E)$. Then we have
\[
U^* FU = \begin{pmatrix}
0 & 1 \otimes 1 \\
1 \otimes 1 & 0
\end{pmatrix},
\]
\[
U^* \phi(x) U = \begin{pmatrix}
\rho^{(0)}(x) & 0 \\
0 & \rho^{(1)}(x)
\end{pmatrix},
\]
which finish the proof. $\square$

To continue the proof, we need more detailed information of the homomorphism $\Phi$.

**Lemma 8.3.** Let the notation be as above.

1. We can choose $\Phi$ so that it has the following form with respect to the orthogonal decomposition $H = F_n \oplus F_n^\perp$:
   \[
   \Phi(s_i) = \begin{pmatrix}
t_i & r_i \\
0 & v_i
\end{pmatrix}.
   \]

2. For $\Phi$ as in (1), the quasi-homomorphism $\rho = (\rho^{(0)}, \rho^{(1)})$ in Lemma 8.2 is expressed as
   \[
   \rho^{(0)}(s_i) = \begin{pmatrix}
t_i \hat{\otimes} 1 & r_i \hat{\otimes} 1 \\
0 & v_i \hat{\otimes} 1
\end{pmatrix},
   \rho^{(1)}(s_i) = \begin{pmatrix}
t_i \hat{\otimes} 1 & r_i \hat{\otimes} z^* \\
0 & v_i \hat{\otimes} 1
\end{pmatrix}.
   \]
In particular, we have
\[ \sum_{i=1}^{n} \rho_{G}^{(1)}(s_i) \rho_{G}^{(0)}(s_i) = (1 - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} z^*. \]

Proof. (1) We first construct \( l_n^G : O_n \to E_n \) explicitly. Ignoring the \( G \) actions, we can find a representation \( \Phi' \) of \( O_n \) on \( F_n \otimes F_n \) of the form
\[
\Phi'(s_i) = \begin{pmatrix} t_1 & p_n \\ 0 & w_1 \end{pmatrix},
\]
and \( \Phi'(s_i) = \begin{pmatrix} t_i & 0 \\ 0 & w_i \end{pmatrix}, \quad 2 \leq i \leq n. \)

Using \( \Phi' \), we define \( l_n \) by
\[
\begin{pmatrix} l_n(x) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi'(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
and \( l_n^G \) by \( l_n^G(x) = \int_G \tilde{\alpha}_g^{-1} \circ l_n \circ \alpha_g(x) dg \). We have \( l_n^G(s_i) = t_i \) for all \( 1 \leq i \leq n \) by construction.

We show that the Stinespring dilation \( (\Phi, H) \) of this \( l_n^G \) has the desired property. Recall that \( H \) is the closure of the algebraic tensor product \( O_n \otimes F_n \) with respect to the inner product
\[ \langle x \odot \xi, y \odot \eta \rangle = \langle l_n^G(y^* x) \xi, \eta \rangle, \]
and \( \Phi \) is given by the left multiplication of \( O_n \). The space \( F_n \) is identified with \( 1 \otimes F_n \), and the unitary representation \( \pi_H \) is given by \( \pi_H(g)(x \odot \xi) = \alpha_g(x) \odot \pi_{F_n}(g) \xi \). To show that \( \Phi \) has the desired property, it suffices to show \( \|s_i \odot \xi - 1 \odot t_i \xi\| = 0 \) for all \( \xi \in F_n \). Indeed,
\[
\|s_i \odot \xi - 1 \odot t_i \xi\|^2 = \langle l_n^G(s_i s_i) \xi, \xi \rangle - \langle l_n^G(s_i) \xi, t_i \xi \rangle - \langle l_n^G(s_i^*) t_i \xi, \xi \rangle + \langle t_i \xi, t_i \xi \rangle = 0.
\]

(2) The first statement follows from (1) and Lemma 8.2. The Cuntz algebra relation implies
\[
p_n r_i = r_i, \quad r_j^* r_i + v_j^* v_i = \delta_{ij},
\]
\[
\sum_{i=1}^{n} r_i r_i^* = p_n, \quad \sum_{i=1}^{n} r_i v_i^* = 0, \quad \sum_{i=1}^{n} v_i v_i^* = 1.
\]

These relations and the first statement imply the second statement. \( \square \)

Proof of Theorem 8.4. Thanks to the previous lemma, we may assume that the class \( \theta_\ast(\delta_{q_n} \otimes \psi_1) \in KK_G(O_n, B) \) is given by a quasi-homomorphism \( \sigma = (\sigma^{(0)}, \sigma^{(1)}) \) from \( O_n \) to \( B \) of the form
\[
\sigma^{(0)}(s_i) = \begin{pmatrix} t_i \otimes 1 & r_i \otimes 1 \\ 0 & v_i \otimes 1 \end{pmatrix}, \quad \sigma^{(1)}(s_i) = \begin{pmatrix} t_i \otimes 1 & r_i \otimes u_{\psi, \varphi}^* \\ 0 & v_i \otimes 1 \end{pmatrix},
\]
and they satisfy
\[ \sum_{i=1}^{n} \sigma^{(1)}(s_i) \sigma^{(0)}(s_i)^* = (1_H - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} u_{\psi, \varphi}^*. \]

We set \( \sigma^{(0)} = \sigma^{(0)} \oplus \varphi, \sigma^{(1)} = \sigma^{(1)} \oplus \psi \), which are unital homomorphisms from \( \mathcal{O}_n \) to \( \mathcal{B}(H \oplus \mathbb{C}) \otimes B \). Then \( \tilde{\sigma} = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(1)}) \) is a quasi-homomorphism with
\[ \sum_{i=1}^{n} \tilde{\sigma}^{(1)}(s_i) \tilde{\sigma}^{(0)}(s_i)^* = (1_H - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} u_{\psi, \varphi}^* \oplus (1_C \hat{\otimes} u_{\psi, \varphi}), \]
which is denoted by \( u \). Then we can construct a norm continuous path \( \{u_t\}_{t \in [0,1]} \) of unitaries in \( \mathbb{C}1 + \mathcal{K}(H \oplus \mathbb{C}) \otimes B^G \) satisfying \( u(0) = u \) and \( u(1) = 1 \). Let \( \tilde{\sigma}_t^{(0)} = \tilde{\sigma}^{(0)} \), and let \( \tilde{\sigma}_t^{(1)} \) be the homomorphism from \( \mathcal{O}_n \) to \( \mathcal{B}(H \oplus \mathbb{C}) \otimes B \) determined by \( \tilde{\sigma}_t^{(1)}(s_i) = u(t)\tilde{\sigma}^{(0)}(s_i) \). Then \( \tilde{\sigma}_t = (\tilde{\sigma}_t^{(0)}, \tilde{\sigma}_t^{(1)}) \) gives a homotopy of quasi-homomorphisms connecting \( \tilde{\sigma} \) and \( \tilde{\sigma}_1 = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(0)}) \). This shows \( [\tilde{\sigma}] = 0 \) in \( KK_G(\mathcal{O}_n, B) \), and so \( \theta_s(\delta_{\pi_n} \hat{\otimes} x) = KK_G(\psi) - KK_G(\varphi) \).

**Remark 8.4.** The above argument shows that there exists a short exact sequence
\[ 0 \rightarrow \text{Coker}(1 - K_{\pi_\alpha}) \rightarrow KK^0_G(\mathcal{O}_n, B) \rightarrow \text{Ker}(1 - K_{\pi}(\hat{\beta}_{\pi_\alpha})) \rightarrow 0. \]

**Remark 8.5.** From (8.1), we obtain the 6-term exact sequence (see [16, Theorem 4.9]),
\[ \begin{array}{ccc}
KK^0_G(B, \mathbb{C}) & \xrightarrow{1-\hat{\otimes}[H_n]} & KK^0_G(B, \mathbb{C}) \\
\uparrow & & \downarrow \\
KK^1_G(B, \mathcal{O}_n) & \xleftarrow{1-\hat{\otimes}[H_n]} & KK^1_G(B, \mathbb{C})
\end{array} \]
In particular, we have the following exact sequence by setting \( B = \mathbb{C} \):
\[ 0 \rightarrow K_1(\mathcal{O}_n \rtimes \alpha G) \rightarrow K^G_0(\mathbb{C}) \xrightarrow{1-\hat{\otimes}[H_n]} K^G_0(\mathbb{C}) \rightarrow K_0(\mathcal{O}_n \rtimes \alpha G) \rightarrow 0. \]
Let \( \iota_\alpha : C^*(G) \rightarrow \mathcal{O}_n \rtimes \alpha G \) be the embedding map, let \( (\pi, H_\pi) \) be an irreducible representation of \( G \), and let
\[ e(\pi)_{ij} = \dim \pi \int_G \overline{\pi(g)_{ij}} \lambda_g dg \in C^*(G). \]
Then the canonical isomorphism from \( K^G_0(\mathbb{C}) \) onto \( K_0(C^*(G)) \) sends the class of \((\pi, H_\pi)\) in \( K^G_0(\mathbb{C}) \) to \([e(\pi)_{11}] \in K_0(C^*(G))\). Thus we have the exact sequence
\[ 0 \rightarrow K_1(\mathcal{O}_n \rtimes \alpha G) \rightarrow \mathbb{Z} \hat{G} \xrightarrow{1-[\pi]} \mathbb{Z} \hat{G} \rightarrow K_0(\mathcal{O}_n \rtimes \alpha G) \rightarrow 0, \]
where \([\pi] \in \mathbb{Z} \hat{G}\) is sent to \( K_0(\iota_\alpha)([e(\pi)_{11}]) \in K_0(\mathcal{O}_n \rtimes G) \). With the identification of \( K_*(\mathcal{O}_n \rtimes \alpha G) \) and \( K_*(\mathcal{O}_n^G) \), this recovers the formula of \( K_*(\mathcal{O}_n^G) \) obtained in [11, 13].
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Department of Mathematics, University of Zagreb, 41000 Zagreb, Croatia
E-mail address: payo@math.hr

Department of Mathematics, Graduate School of Science, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan
E-mail address: izumi@math.kyoto-u.ac.jp