MIXTURE MODEL FOR DESIGNS IN HIGH DIMENSIONAL REGRESSION AND THE LASSO

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Abstract. The LASSO is a recent technique for variable selection in the regression model
\[ y = X\beta + z, \]
where \( X \in \mathbb{R}^{n \times p} \) and \( z \) is a centered gaussian i.i.d. noise vector \( \mathcal{N}(0, \sigma^2 I) \). The LASSO has been proved to achieve remarkable properties such as exact support recovery of sparse vectors when the columns are sufficiently incoherent and low prediction error under even less stringent conditions. However, many matrices do not satisfy small coherence in practical applications and the LASSO estimator may thus suffer from what is known as the slow rate regime.

The goal of the present paper is to study the LASSO from a slightly different perspective by proposing a mixture model for the design matrix which is able to capture in a natural way the potentially clustered nature of the columns in many practical situations. In this model, the columns of the design matrix are drawn from a Gaussian mixture model. Instead of requiring incoherence for the design matrix \( X \), we only require incoherence of the much smaller matrix of the mixture’s centers.

Our main result states that \( X\beta \) can be estimated with the same precision as for incoherent designs except for a correction term depending on the maximal variance in the mixture model.

1. Introduction

The goal of the present paper is to study the high dimensional regression problem \( y = X\beta + z \), where \( X \in \mathbb{R}^{n \times p} \), with \( p \gg n \) and \( z \sim \mathcal{N}(0, \sigma^2 I) \). This problem has been the subject of an extensive research activity. This high dimensional setting, where more variables are involved than observations, occurs in many different applications such as image processing and denoising, gene expression analysis, time series (filtering) \[23, 27\], graphical models \[29\], biochemistry \[1\], etc. One very popular approach is the Least Angle Shrinkage and Selection Operator (LASSO) introduced in \[31\] for the purpose of variable selection. The LASSO estimator is given as a solution, for \( \lambda > 0 \), of

\[
\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \frac{1}{2} \|y - Xb\|_2^2 + \lambda \|b\|_1.
\]

(1.1)

The main advantage of the LASSO over more traditional penalized likelihood optimization procedures such as BIC, AIC, etc, is that a solution can be obtained in polynomial time by solving a convex optimisation problem. Very efficient scalable algorithms are available, based on Nesterov’s method \[2\], the Alternating Direction of Method of Multipliers \[29\], the Frank-Wolfe algorithm \[22\] or online versions of them \[24, 18\].

One of the most surprising and important discoveries is that, under appropriate assumptions on the design matrix \( X \), and for at least most regression vectors \( \beta \), the support of \( \beta \) can be recovered exactly when its size is up to the order of \( n/\log(p) \) and the nonzero components are sufficiently large; see \[3, 6, 10, 36\] for instance. Moreover, under similar assumptions, the prediction error can be controlled adaptively as a function of the sparsity of \( \beta \) and the noise variance; see for instance \[10\].

A great amount of work has been devoted to finding error bounds on \( X\beta \) \[10, 13, 37, 15\], etc. Oracle inequalities for this problem are divided into two different classes depending on the so called "regime": the first class describes the slow rate regime and does not require any particular assumption on \( X \), while the second class describes the fast rate regime which does require some structural assumptions on \( X \).

For simplicity, we will assume throughout this paper that the columns of \( X \) have unit \( \ell_2 \)-norm.

*Conditions for uniqueness of the minimizer in this last expression are discussed in \[21, 28, 14\] and \[20\].
Historically, the first is the Restricted Isometry Property [1] [9], which requires that
\[
(1 - \delta)\|\beta_S\|^2 \leq \|X_S\beta_S\|^2 \leq (1 + \delta)\|\beta_S\|^2,
\]
for $S \subset \{1, \ldots, n\}$ with $|S| = s'$ and all $\beta \in \mathbb{R}^p$. This property is satisfied by high probability for most random matrices with i.i.d. entries with variance $1/n$ such as Gaussian or Rademacher variables and for $s' \leq C_{\text{rip}} \, n/\log(p)$, where the constant $C_{\text{rip}}$ depends on the distribution of the individual entries. RIP has been extensively used in signal processing after the emergence of the so-called Compressed Sensing paradigm [3].

A second assumption which is often considered is the Incoherence Condition, which requires that
\[
\mu(X) = \max_{j \neq j'=1, \ldots, p} |\langle X_j, X_{j'} \rangle|<\infty,
\]
is small, e.g. $\mu(X) \leq C_\mu/\log(p)$ as in [10], which is guaranteed for random matrices with i.i.d. gaussian entries with variance $1/n$ in the range $n \geq C_{ic} \log(p)^3$.

The main advantage of the Incoherence Condition over the Restricted Isometry Property is that it can be checked in $p(p-1)/2$ operations, whereas RIP is NP-hard to verify. The main relationship between the Incoherence Condition and RIP is that under the Incoherence Condition, (1.2) holds, not for all, but for most supports $S \subset \{1, \ldots, n\}$ with cardinal $s'$, where $s' \leq C_s \, p/(\|X\| \log(p))$, for some constant $C_s$ controlling the proportion of such supports [10].

The objective of the present paper is to extend the analysis based on the Incoherence Condition to more general situations where $X$ may have a lot of very colinear columns. The main idea is to assume that the columns are drawn from a mixture model of $K$ clusters, and that the set of centers forms a (usually) much smaller matrix for which it is quite realistic to impose the Incoherence Condition.

2. Main results on the LASSO

In this section, we summarise the main results on the LASSO.

2.1. Background. We will study the linear regression model
\[
y = X\beta + z
\]
where $y \in \mathbb{R}^n$ is the data, $X \in \mathbb{R}^{n \times p}$ is the matrix of explanatory variables, $\beta \in \mathbb{R}^p$ is the parameter of interest and $z$ is a centered gaussian i.i.d noise vector $\mathcal{N}(0, \sigma^2 I)$.

The LASSO estimator of $\beta$ is defined by any $\hat{\beta}$ such that
\[
\hat{\beta} = \text{argmin}_{b \in \mathbb{R}^p} \frac{1}{2}\|y - Xb\|^2_2 + \lambda\|b\|_1.
\]

Lemma 2.1. The LASSO estimator obeys
\[
\|X^\dagger (y - X\hat{\beta})\|_\infty \leq \lambda.
\]

2.2. Statistical viewpoint. The LASSO estimator has been the subject of intense research in the recent years in the statistics community. Several results have been obtained about the mean squared error. The first result below is about the case where no specific assumption is required about $X$.

Theorem 2.2. Assume that the linear model (2.3) holds where $z \sim \mathcal{N}(0, \sigma^2)$. Moreover, assume that the columns of $X$ are normalized in such a way that $\max_j \|X_j\|_2 \leq \sqrt{n}$. Then, the Lasso estimator $\hat{\beta}$ with regularization parameter
\[
\lambda = \sigma \sqrt{\frac{2\log(2p)}{n}} + \sigma \sqrt{\frac{2\log(1/\delta)}{n}}
\]
satisfies
\[
\frac{1}{n} \|X\hat{\beta} - X\beta^*\|_2^2 \leq 4\lambda \|\beta^*\|_1 \sigma
\]
with probability at least $1 - \delta$.

\[1\]the $1/n$ assumption on the variance and standard concentration bounds imply that the resulting random matrix has almost normalized columns.
The next theorem states that when $X$ satisfies some incoherence-type assumption, more can be obtained for the LASSO estimator and the mean squared error decreases faster.

**Theorem 2.3.** Fix $n \geq 2$. Assume that the linear model holds where $z \sim \mathcal{N}(0, \sigma^2)$. Moreover, assume that $\|\beta^*\|_0 \leq s$ and that $X$ satisfies assumption INC$(s)$. Then the Lasso estimator $\hat{\beta}$ with regularization parameter defined by

$$
\lambda = 4\sigma \sqrt{\frac{\log(2p)}{n}} + 4\sigma \sqrt{\frac{\log(1/\delta)}{n}}
$$

satisfies

$$
\|\hat{\beta} - \beta^*\|_2^2 \lesssim s\sigma^2 \log(2p/\delta)
$$

with probability at least $1 - \delta$.

In this paper, our goal is to extend this last result to the case where the design matrix has potentially many almost co-linear columns, using a mixture model as a generating model for the columns.

3. **Our mixture model and a sketch of our main result**

3.1. **The mixture model.** In order to relax the Incoherence Condition, one needs a model for the design matrix $X$ allowing for a certain amount of almost parallel columns while keeping some of the algebraic structure in the same spirit as in [1,2] for at least most supports indexing a subset of relevant covariates. In what follows, we study such a model, where the columns can be considered as drawn from a finite mixture of $K$ Gaussian distributions.

An important parameter for the theoretical analysis is a separation index for the centers in the mixture model. This separation index we chose to study in this work is simply the coherence of the matrix of centers which is much smaller than the original design matrix $X$.

3.1.1. **Detailed presentation.** Let $K$ be the number of clusters of our model. Consider a matrix $C \in \mathbb{R}^{n \times K}$. The columns $C_k$, $k = 1, \ldots, K$ of the matrix $C$ are the "centers" of each cluster.

In our model,

- the matrix $X$ is obtained as follows.
  - Choose $n_k$, $k = 1, \ldots, K$.
  - Let $X_o \in \mathbb{R}^{n \times p}$ be a random matrix with independent random columns such that the $n_1$ columns follow the distribution $\phi_1$, the $n_2$ next columns follow the distribution $\phi_2$, etc, where
    $$
    \phi_k(x) = \frac{1}{(2\pi s^2)^{\frac{p}{2}}} \exp\left(\frac{-\|x - C_k\|_2^2}{2s^2}\right).
    $$
  - For each $j \in \{1, \ldots, p\}$, $k_j \in \{1, \ldots, K\}$ will denote the index of the Gaussian component from which columns $j$ was drawn, and $\mathcal{J}_k$ will denote the set of indices of the columns drawn from the $k^{th}$ Gaussian component. For any $S \subset \{1, \ldots, p\}$, $k_S$ will denote the subset of $\{1, \ldots, K\}$ indexing the centers of the distributions from which the columns of $X_S$ were drawn.
  - The matrix $X$ is obtained by a random permutation of the columns of $X_o$ and column-wise $\ell_2$-normalization.
- the support of $\beta$ will be drawn at random as follows.
  - We will assume that the support $T$ of the true regression vector $\beta$ is drawn in such a way that $T$ has the uniform distribution on the subsets of $\{1, \ldots, K\}$ with cardinality equal to $s$.

3.1.2. **Best approximation of the class centers and projection of $\beta$.** For any index set $S \subset \{1, \ldots, p\}$, let $\mathcal{K}_S$ denote the list (with possible repetitions)

$$
\mathcal{K}_S = \{k_j \mid j \in S\}.
$$

For each $j \in \{1, \ldots, p\}$, the deviation of column $X_{o,j}$ from center $C_{k_j}$ will be denoted by $\varepsilon_j$:

$$
\varepsilon_j = X_{o,j} - C_{k_j} \sim \mathcal{N}(0, s^2 I).
$$
The matrix $E$ is defined as

$$E = (\varepsilon_{i,j})_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,p\}}.$$ 

For each $k \in \{1,\ldots,K\}$, let $j_k^*$ be the best approximation of the center $C_k$ from the set of columns $X_j$, $j \in J_k$, i.e.

$$j^*_k = \text{Argmin}_{j \in J_k} \|X_j - C_k\|_2.$$ 

Moreover, set

$$T^* = \{j^*_k \mid k = 1,\ldots,K\}.$$ 

Notice that in particular, $|T^*| = s^*$. Let $\beta^*$ be the vector defined as

$$C_{K_T} \beta^*_{T^*} = C_{K_T} \beta_T.$$ 

A simple expression of $\beta^*$ can be obtained by taking

$$\beta^*_j = \sum_{j \in J_k \cap T} \beta_j$$

for all $j^* \in T^*$. Moreover, this expression is unique whenever $X_{T^*}$ has rank equal to $s^*$.\footnote{In Section 4.2, we will show that $X_{T^*}$ is indeed non-singular with high probability under appropriate assumptions on $T$.}

3.2. Main result. The following theorem shows a bound on the prediction error which is a function of the sparsity $s^*$, the number $n$ of observations, the number of columns $p$.

**Theorem 3.1.** (Sketch) Let $\lambda = 2\sigma \sqrt{2 \log(p)}$. Assume that $X$ is drawn from the Gaussian mixture model of Section 3.1. Then, for $p$ sufficiently large, with probability at least $1 - C_{\alpha,n,\rho}(\rho^* + p^{-\alpha})$, we have

$$\frac{1}{2} \|X(\hat{\beta} - \beta)\|_2^2 \leq \frac{3}{2} \lambda s^* \left( \frac{1}{2} - r_{\alpha,n,\rho}(r) \right) \left( \frac{3}{2} \lambda + \sqrt{3} \|X(\beta^* - \beta)\|_2 \right) + \frac{1}{2} \|X(\beta^* - \beta)\|_2^2.$$ 

with

$$r^*_{\alpha,n,\rho}(r) = r \left( \frac{1}{2} + 0.1 \ C_{\alpha,n,\rho} \right) \left( 2 + \frac{1}{2} r + 0.1 \ r \ C_{\alpha,n,\rho} \right)$$ 

where $C_{\alpha,n,\rho} = \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(p-1)}}$.

4. A general result and its proof

Some parts of the proof closely follow the key arguments in the proof of [10, Theorem 1.2], although many details of the needed adaptation are nontrivial. Our Theorem 4.3 below contains the most general statement of our work.

4.1. A more general result. We will require a set of assumptions that are described below.
4.1.1. Assumptions. In the sequel \( \alpha \geq 1 \) and \( r \) will denote a constant in \((0, 1/2)\). The constants \( \vartheta_* \) et \( \nu \) will be specified in Assumptions 4.2 below. The constants \( C_\mu \), \( C_{\text{spar}} \) et \( C_{\text{col}} \) will be used in the Assumptions below:

\[
C_\mu = r/(1 + \alpha), \quad C_{\text{spar}} = r^2/((1 + \alpha)e^2), \quad C_{\text{col}} = \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{(1-r)(1+\alpha)} - (1+r)}\right).
\]

Let \( C_\chi \) denote a positive constant such that

\[
P\left(\frac{\|G\|_2^2}{s^2} \leq u^2\right) \leq C_\chi \left(\frac{u^2}{n}\right)^n
\]

where \( G \) is a \( n \)-dimensional centered and unit-variance i.i.d. gaussian vector.

We will make the following assumptions.

**Assumptions 4.1.** The matrix \( \mathcal{C} \) has a small coherence, i.e. \( \mu(\mathcal{C}) \) should satisfying

\[
\mu(\mathcal{C}) \leq \frac{C_\mu}{\log(K)}
\]

for some positive constant \( C_\mu \).

**Assumptions 4.2.** The clusters must contain sufficiently many points, i.e. there exists a positive real constant \( \vartheta_* \) and a positive integer \( \nu \) such that

\[
\min_{j^* \in \mathcal{T}^*} |J_{k,j^*}| \geq \vartheta_* \log(p)^\nu.
\]

**Assumptions 4.3.** The proxy \( \beta^* \) must be sufficiently sparse, i.e.

\[
s^* \leq K_0 \frac{K}{\log K} \frac{C_{\text{spar}}}{\|\mathcal{C}\|^2}
\]

for some positive constant \( C_{\text{spar}} \) and \( K_0 \leq \rho^{-1} \) for some \( \rho \in (0, 1) \).

**Assumptions 4.4.** The number of columns of \( \mathcal{C} \) satisfying

\[
K \leq C_K \log(p)
\]

for some positive constant \( C_K \).

**Assumptions 4.5.** One must have sufficiently many observations, i.e.

\[
n \geq \frac{\alpha + 1}{c} \log(p)
\]

for some positive constant \( c \).

**Remark 4.1.** The number of observations is both controlled by Assumption 4.4 and Assumption 4.1 on the coherence of \( \mathcal{C} \). For instance, if \( \mathcal{C} \) comes from a Gaussian i.i.d. random matrix, the coherence will be of the order \( \sqrt{\log(K)/n} \) as discussed in [10, Section 1.1] and by Assumption 4.1, \( n \) should be at least of the order \( \log(p)^3/C_\mu^2 \). Notice that this is still less than if \( X \) itself had to satisfy the coherence bound, which would imply that \( n \) be of the order \( \log(p)^3 \). This demonstrates the advantage of using our Gaussian Mixture framework over the standard framework based on incoherence on \( X \).

**Assumptions 4.6.** The variance inside the clusters must be sufficiently small, so that the clusters are well separated. More precisely, we will require that

\[
s \leq \min\left\{\frac{\alpha}{2\sqrt{n}} \frac{C_{s,n,p}}{\log(\rho^{-1})} \left(\frac{\alpha + 1}{\log(p)}\right)\right\}
\]

for any \( C_{s,n,p} \) such that and

\[
C_{s,n,p} \leq \min\left\{0.1 \cdot \left(\frac{r}{n} \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{2}} \left(\frac{1}{\log(p)^{\nu - 1}}\right)^{\frac{1}{2}} \cdot \frac{1}{2} \sqrt{\log(p)}\right)\right\}.
\]
Assumptions 4.7. The support of $\beta^*_T$ is sufficiently generic. More precisely, we will require that the support of $\beta^*_T$ is random and uniformly distributed among subsets of $\{1, \ldots, p\}$ with cardinal $s^*$. The sign of $\beta^*_T$ is random with uniform distribution on $\{-1, 1\}$.

Remark 4.2. This last assumption is a transposition to the proxy $\beta^*$ of the conditions on $\beta$ in [10].

Assumptions 4.8. Relationships between the constants.

$$C_{col} \geq e^{2(\alpha + 1)} \max\{\sqrt{C_{spar}}, C_{\mu}\}. $$

Assumptions 4.9. Assume that $p \leq 0.01 \cdot \rho^{1-\alpha+1} \log(K)^2$ for the same $\rho$ as in Assumption 4.3.

4.1.2. The general theorem. The main result of this paper is the following theorem.

Theorem 4.3. Let $\lambda = 2\sigma \sqrt{2\alpha \log(p)}$. Assume that $X$ is drawn from the Gaussian mixture model of Section 3.2 with $K$ drawn uniformly at random among all possible index subsets of $\{1, \ldots, K\}$ with cardinal $s^*$. Let Assumptions 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9 hold. Then, for all $\rho \in (0, 0.5)$, with probability at least $1 - C_{a,n,\rho}(\rho^\alpha + p^{1-\alpha})$, we have

$$\|X(\hat{\beta} - \beta)\|_2^2 \leq 3\lambda s^* \frac{1}{1 - r^*_{a,n,\rho}(\rho)} \left(\frac{3}{2} \lambda + \sqrt{1 + r^*_{a,n,\rho}(\rho)} \|X(\beta^* - \beta)\|_2 + \|X(\beta^* - \beta)\|_2^2\right)$$

with

$$r^*_{a,n,\rho}(\rho) = r \left(\frac{1}{2} + 0.1 C_{a,n,\rho}\right) \left(2 + \frac{1}{2} r + 0.1 r C_{a,n,\rho}\right)$$

where $C_{a,n,\rho} = \sqrt{\alpha + \frac{\log(n)}{\log(\rho^{-1})}}$.

Remark 4.4. Notice that our result is of fast rate type but includes new additional terms involving the approximation error $\beta^* - \beta$. More precisely, the right hand side in (4.13) can be decomposed into two parts:

- the term
  $$\frac{9}{2n} \lambda^2 s^* \frac{1}{1 - r^*_{a,n,\rho}(\rho)}$$
  which is similar to the "fast rate term" in the standard incoherent case of Theorem 2.3.
- the term
  $$\frac{3}{2} \frac{\lambda s^*}{1 - r^*_{a,n,\rho}(\rho)} \sqrt{1 + r^*_{a,n,\rho}(\rho)} \|X(\beta^* - \beta)\|_2 + \|X(\beta^* - \beta)\|_2^2$$
  is not present in the standard analysis of the LASSO and depends on how well $\beta$ can be approximated by $\beta^*$, and depends on the model and more precisely $C$ and $\beta$.

Remark 4.5. Notice that the coefficient $C_{a,n,\rho}$ can be made as small as necessary when $\rho$ is sufficiently larger than $n$. Thus, we can always pretend for the ease of the analysis, that $r^*_{a,n,\rho}(\rho)$ is of the same order as $r$.

We now begin the proof of Theorem 4.3.

§$\rho \in (0, 1)$ is introduced in Assumption 4.9.
4.2. Preliminaries: Candès and Plan’s conditions. The following proposition will be much used in the arguments.

Proposition 4.6. We have the following properties:

1. \[ P \left( \| \mathbf{c}_{K_T} \mathbf{c}_{K_T} - \mathbf{I}_{d_s} \| \geq \rho \epsilon_{K_T} \right) \leq \frac{216}{p^\alpha}. \]

2. \[ P \left( \| \mathbf{X}_T^T \mathbf{X}_T - I \| \geq r^*_\alpha, n, p(r) \right) \leq \frac{219}{p^\alpha}. \]

where \( r^*_\alpha, n, p(r) \) is defined by (4.8).

3. \[ P \left( \| \mathbf{X}_T^T \|_\infty \geq \sigma \sqrt{2\alpha \log(p)} \right) \leq \frac{1}{p^\alpha}. \]

4. \[ \| X_{T^c}^T \mathbf{X}_{T^c} (X_{T^c}^T \mathbf{X}_{T^c})^{-1} \|_\infty \leq \sigma C_1 + \lambda C_2 \]

where \( C_1 \) and \( C_2 \) are defined by (5.48) and (5.49).

Proof. See Appendix 5. \( \square \)

4.3. The prediction bound. By definition, the LASSO estimator satisfies

\[ \frac{1}{2} \| y - X \hat{\beta} \|_2^2 + \lambda \| \hat{\beta} \|_1 \leq \frac{1}{2} \| y - X \beta^* \|_2^2 + \lambda \| \beta^* \|_1. \]

One may introduce \( X \beta \) in this expression and obtain

\[ \frac{1}{2} \| y - X \beta + X (\beta - \hat{\beta}) \|_2^2 + \lambda \| \hat{\beta} \|_1 \leq \frac{1}{2} \| y - X \beta + X (\beta - \beta^*) \|_2^2 + \lambda \| \beta^* \|_1, \]

from which we deduce

\[ \frac{1}{2} \| X (\beta - \hat{\beta}) \|_2^2 \leq \langle y - X \beta, X (\hat{\beta} - \beta^*) \rangle - \lambda \left( \| \hat{\beta} \|_1 - \| \beta^* \|_1 \right) + \frac{1}{2} \| X (\beta - \beta^*) \|_2^2. \]

Set \( h^* := \hat{\beta} - \beta^* \). Using sparsity of \( \beta^* \), we obtain that \( h^*_{T^c} = \hat{\beta}_{T^c} - \beta^*_{T^c} = \hat{\beta}_{T^c} \). Thus, we have

\[ \| \hat{\beta} \|_1 - \| \beta^* \|_1 = \| \beta^* + h^* \|_1 - \| \beta^* \|_1 = \| \beta^*_{T^c} + h^*_{T^c} \|_1 + \| \beta^*_{T^c} + h^*_{T^c} \|_1 - \| \beta^*_{T^c} \|_1 \]

\[ = \| \beta^*_{T^c} + h^*_{T^c} \|_1 + \| h^*_{T^c} \|_1. \]

Since, for any \( b \) with no zero component, the gradient of \( \| \cdot \|_1 \) at \( b \) is \( \text{sgn}(b) \), the subgradient inequality gives

\[ \| \beta^*_{T^c} + h^*_{T^c} \|_1 \geq \| \beta^*_{T^c} \|_1 + \langle \text{sgn}(\beta^*_{T^c}), h^*_{T^c} \rangle \]

and combining this latter inequality with (4.19), we obtain

\[ \frac{1}{2} \| X (\beta - \hat{\beta}) \|_2^2 \leq \langle y - X \beta, X h^* \rangle - \lambda \langle \text{sgn}(\beta^*_{T^c}), h^*_{T^c} \rangle - \lambda \| h^*_{T^c} \|_1 + \frac{1}{2} \| X (\beta - \beta^*) \|_2^2. \]

Set \( \gamma := \beta^* - \beta \) and \( h := \hat{\beta} - \beta \). Using these notations, equation (4.20) may be written

\[ \frac{1}{2} \| X h \|_2^2 \leq \langle z, X h^* \rangle - \lambda \langle \text{sgn}(\beta^*_{T^c}), h^*_{T^c} \rangle - \lambda \| h^*_{T^c} \|_1 + \frac{1}{2} \| X \gamma \|_2^2. \]

Using the fact that

\[ \langle X^T z, h^* \rangle = \langle X^T z, h^*_{T^c} \rangle + \langle X^T z, h^*_{T^c} \rangle \]
and the following majorization based on (1.16)

\[
\langle X_{T^*-}^c z, h_{T^*-}^* \rangle \leq ||h_{T^*-}^*||_1 ||X_{T^*-}^c z||_\infty \\
\leq \frac{1}{2} \lambda ||h_{T^*-}^*||_1,
\]

we obtain that

\[
(4.22) \quad \frac{1}{2} \|Xh\|_2^2 \leq \langle v, h_{T^*}^* \rangle - \left( 1 - \frac{1}{2} \lambda \right) ||h_{T^*-}^*||_1 \frac{1}{2} \|X\|_2^2,
\]

where \( v := X_{T^*}^c z - \lambda \operatorname{sgn}(\beta_{T^*}^*) \).

Now, observe that

Let us begin by studying \( A_2 \). We have that:

\[
A_2 \geq -\|X_{T^*}^c X_{T^*-}^c (X_{T^*}^c X_{T^*-})^{-1} v\|_\infty \|h_{T^*-}^*\|_1 \\
\geq -\|X_{T^*}^c X_{T^*-}^c (X_{T^*}^c X_{T^*-})^{-1} X_{T^*}^c z\|_\infty \|h_{T^*-}^*\|_1 \\
- \lambda \|X_{T^*}^c X_{T^*-}^c (X_{T^*}^c X_{T^*-})^{-1} \operatorname{sgn}(\beta_{T^*}^*)\|_\infty \|h_{T^*-}^*\|_1 \\
\geq - (\sigma C_1 + \lambda C_2) \|h_{T^*-}^*\|_1
\]

by (4.17). Thus

\[
\langle v, h_{T^*}^* \rangle \leq A_1 + (\sigma C_1 + \lambda C_2) \|h_{T^*-}^*\|_1
\]

and we deduce that

\[
(4.23) \quad \frac{1}{2} \|Xh\|_2^2 \leq A_1 + \left( \sigma C_1 + \lambda C_2 - \frac{1}{2} \lambda \right) \|h_{T^*-}^*\|_1 \frac{1}{2} \|X\|_2^2
\]

Let us now bound \( A_1 \) from above. We have that

\[
A_1 \leq \left\| \frac{X_{T^*}^c X h^*}{B_1} \right\|_\infty \left\| \frac{(X_{T^*}^c X_{T^*-})^{-1} v}{B_2} \right\|_1
\]

Firstly,

\[
B_1 \leq \|X_{T^*}^c (X_{T^*})^* - y\|_\infty + \|X_{T^*}^c (X_{T^*})^* - y\|_\infty \\
\leq \|X_{T^*}^c (X_{T^*})^* - y\|_\infty + \|X_{T^*}^c (y - X_{T^*})\|_\infty \\
\leq \frac{1}{2} \lambda + \|X_{T^*}^c X_{T^*}^c\|_\infty + \lambda
\]

where we used (4.16), and the optimality condition for the LASSO estimator ((2.6)). Secondly,

\[
B_2 \leq \sqrt{s^* \|((X_{T^*}^c X_{T^*-})^{-1} v\|_2 \\
\leq \sqrt{s^* \|((X_{T^*}^c X_{T^*-})^{-1} \|v\|_2 \\
\leq s^* \|((X_{T^*}^c X_{T^*-})^{-1} \|v\|_\infty.
\]

Moreover, (4.15) gives \( \|X_{T^*}^c (X_{T^*})^{-1}\| \leq \frac{1}{1 - r_{n,p}^*} \|X_{T^*}^c z\|_\infty + \lambda \leq \frac{3}{2} \lambda \)

Thus, we obtain that

\[
A_1 \leq \frac{3}{2} \lambda s^* \frac{1}{1 - r_{n,p}^*} \left( \frac{3}{2} \lambda + \|X_{T^*}^c X_{T^*}^c\|_\infty \right)
\]
and thus,
\[
\frac{1}{2} \|Xh\|_2^2 \leq \frac{3}{2} \lambda s^* \frac{1}{1 - r_{\alpha,n,p}(r)} \left( \frac{3}{2} \lambda + \|X_{T^*}^T X\|_\infty \right) + (\sigma C_1 + \lambda C_2 - \frac{1}{2} \lambda) \|h_{T^*}^c\|_1 + \frac{1}{2} \|X\|_2^2.
\]
Since \(\|X_{T^*}^T X\|_\infty \leq \|X_{T^*}^T X\|_2\) and since
\[
\|X_{T^*}^T X\|_2 \leq \sqrt{1 + \frac{1}{2} \|X\|_2^2},
\]
we obtain
\[
\frac{1}{2} \|Xh\|_2^2 \leq \frac{3}{2} \lambda s^* \frac{1}{1 - r_{\alpha,n,p}(r)} \left( \frac{3}{2} \lambda + \sqrt{1 + \frac{1}{2} \|X\|_2^2} \right) + (\sigma C_1 + \lambda C_2 - \frac{1}{2} \lambda) \|h_{T^*}^c\|_1 + \frac{1}{2} \|X\|_2^2.
\]
which completes the proof.

5. Checking the Candes-Plan conditions

The goal of this section is to Proposition 4.6 which gives a version of Candès and Plan’s conditions adapted to our Gaussian mixture model.

5.1. Control of \(\|E_{T^*}\|\).

Consider the matrix \(E_{T^*}\), whose columns are independent. We would like to bound its operator norm.

**Theorem 5.1.** Let the event
\[
E_{\alpha}^* = \bigcap_{j^* \in T^*} \left\{ \|E_{j^*}\|_2 \leq s \sqrt{n \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_X} \right) \frac{1}{\log(p)^{\nu - 1}}} \right\}.
\]
Then, \(P(E_{\alpha}^*) \geq 1 - \rho^a\).

**Proof.** Using the independence of the \(E_j, j \in J_{k_j^*}\), we have
\[
P(\|E_{j^*}\|_2 \geq u) = \prod_{j \in J_{k_j^*}} P(\|E_j\|_2 \geq u),
\]
\[
\leq P(\|E_j\|_2^2 \geq u^2) \min_{j^* \in T^*} |J_{k_j^*}|.
\]
We also have
\[
P(\|E_j\|_2^2 \geq u^2) = 1 - P(\|E_j\|_2^2 \leq u^2).
\]
On the other hand, as is well known, we have
\[
P(\frac{\|E_j\|_2^2}{s^2} \leq u^2) \leq C_X \left( \frac{u^2}{\sigma} \right)^n
\]
for some positive constant \(C_X\). Thus, the union bound gives
\[
P(\max_{j^* \in T^*} \|E_{j^*}\|_2 \geq u) \leq s^* \left( 1 - C_X \left( \frac{u^2}{\sigma^2} \right)^n \right) \min_{j^* \in T^*} |J_{k_j^*}|.
\]
Let us tune \(u\) so that
\[
s^* \left( 1 - C_X \left( \frac{u^2}{\sigma^2} \right)^n \right) \min_{j^* \in T^*} |J_{k_j^*}| \leq \rho^a.
\]
norm of each summand. By Lemma 5.1, on $E$ positive semi-definite random matrices. In order to apply this inequality, we need a bound on the This latter expression is well suited for our problem, since it is the norm of the sum of independent

$$u^2 \geq \frac{n \sigma^2}{C_X^2} \left( 1 - \left( (s^*)^{-1} \rho \min_j \frac{|J_k,j^*|}{|J_k,j^*|} \right)^\frac{1}{2} \right)$$

and since $\min_j \in T^* |J_k,j^*| \geq \theta_* \log(p)^{\nu^*}$ by (4.10),

$$(5.24) u^2 \geq \frac{n \sigma^2}{C_X^2} \left( 1 - \exp \left( - \frac{\alpha}{\theta_* \log(p)^{\nu^*}} - \frac{\log(s^*)}{\theta_* \log(p)^{\nu^*}} \right) \right)^\frac{1}{2}.$$  

On $(0, 1)$, we have

$$\exp(-z) \leq 1 - (1 - e^{-1})z$$

and thus,

$$u^2 \geq n \sigma^2 \left( \frac{\alpha (1 - e^{-1})}{\theta_* C_X} \right)^\frac{1}{2} \left( \frac{1}{\log(p)^{\nu^*}} \right)^\frac{1}{2},$$

from which the desired estimate follows.

\[ \square \]

**Lemma 5.2.** We have

$$\mathbb{P} \left( \| E_{T^*} \| \geq s K_{n,s^*} \mid \mathcal{E}_\alpha^* \right) \leq \frac{2}{p^8}$$

where

$$(5.25) K_{n,s^*} = \sqrt{n (\alpha \log(p^{-1}) + \log(n)) \left( \frac{\alpha (1 - e^{-1})}{\theta_* C_X} \right)^\frac{1}{2} \left( \frac{1}{\log(p)^{\nu^*}} \right)^\frac{1}{2}}.$$  

**Proof.** Let us first notice that since $\| E_{T^*} \| = \| E_{T^*}^t \|$, we can write

$$\| E_{T^*} \| = \sqrt{\| E_{T^*}^t E_{T^*}^t \|} = \sqrt{\sum_{j^* \in T^*} E_{j^*}^t E_{j^*}^t}$$

This latter expression is well suited for our problem, since it is the norm of the sum of independent positive semi-definite random matrices. In order to apply this inequality, we need a bound on the norm of each summand. By Lemma [5.1] on $\mathcal{E}^*$, we have

$$\| E_{j^*}^t E_{j^*}^t \|_2 = \| E_{j^*} \|_2^2 \leq s^2 n \left( \frac{\alpha (1 - e^{-1})}{\theta_* C_X} \right)^\frac{1}{2} \left( \frac{1}{\log(p)^{\nu^*}} \right)^\frac{1}{2}.$$  

We also need a bound on the norm of the expectation. We have

$$\left\| \mathbb{E} \left[ \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \mid \mathcal{E}_\alpha^* \right] \right\| = \left\| \sum_{j^* \in T^*} \mathbb{E} \left[ E_{j^*} E_{j^*}^t \mid \mathcal{E}_\alpha^* \right] \right\|.$$  

Due to rotational invariance, we have that the law of $E_{j^*}$ is the same as the law of $D(\zeta) E_{j^*}$, where $\zeta_1, \ldots, \zeta_n$ are i.i.d. Rademacher $\pm 1$ random variables independent from $E_{j^*}$. Thus, for $i \neq i'$,

$$\mathbb{E} \left[ \zeta_i E_{i,j^*} \zeta_i' E_{i',j^*} \mid \mathcal{E}_\alpha^* \right] = \mathbb{E} \left[ \mathbb{E} \left[ \zeta_i E_{i,j^*} \zeta_i' E_{i',j^*} \mid E_{i,j^*}, E_{i',j^*} \right] \mid \mathcal{E}_\alpha^* \right]$$

$$(5.26) = 0.$$

On the other hand, we have the following result.
Lemma 5.3. We have

\[ \mathbb{E} \left[ E_{i,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right] \leq s^2 \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}. \]

Proof. Due to rotational invariance of the law of \( E_{j^*} \) and the event \( \mathcal{E}_{\alpha}^* \), we have

\[ \mathbb{E} \left[ E_{i,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right] = \cdots = \mathbb{E} \left[ E_{n,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right]. \]

Therefore,

\[ \mathbb{E} \left[ E_{i,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right] \leq \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} E_{i,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right] \]

and by the definition of \( \mathcal{E}_{\alpha}^* \),

\[ \mathbb{E} \left[ E_{i,j^*}^2 \mid \mathcal{E}_{\alpha}^* \right] \leq s^2 \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}. \]

\[ \square \]

Based on this lemma, and the fact that the matrix

\[ \mathbb{E} \left[ \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \mid \mathcal{E}_{\alpha}^* \right], \]

is diagonal by (5.20), we obviously obtain that

\[ \left\| \mathbb{E} \left[ \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \mid \mathcal{E}_{\alpha}^* \right] \right\| \leq s^2 \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}. \]

With the bound on the norm of the expectation and on the variance in hand and obtain

\[ \mathbb{P} \left( \left\| \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \right\| \geq u \mid \mathcal{E}_{\alpha}^* \right) \]

\[ \leq n \left( \frac{s^2 \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}}{u} \right)^{1/\pi} \cdot S^2 n \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}. \]

Let us finally tune \( u \) so that the right hand side term is less than \( \rho^\alpha \), i.e.

\[ \log \left( \frac{s^2 \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}}{u} \right) \]

\[ \leq - \frac{s^2 n \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}}{u} \cdot (\alpha \log(\rho^{-1}) + \log(n)). \]

Take

\[ u = s^2 n \left( \alpha \log(\rho^{-1}) + \log(n) \right) \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}. \]

Moreover, the value of \( u \) given by (5.28) is less than or equal to \( s^2 K_{n,s^*}^2 \) with

\[ K_{n,s^*} = \sqrt{n \left( \alpha \log(\rho^{-1}) + \log(n) \right) \left( \frac{\alpha (1 - e^{-1})}{\vartheta_s C_\chi} \right)^{1/\pi} \left( \frac{1}{\log(p)^{\nu-1}} \right)^{1/\pi}}. \]

This completes the proof. \( \square \)
5.2. Important properties of \( C \). The invertibility condition for \( (4.14) \) is a direct consequence of \[ 5.32 \]. An alternative approach, based on the Matrix Chernoff inequality is proposed in \[ 16 \], with improved constants. We have in particular

**Theorem 5.4.** \[ 16 \] Theorem 1 Let \( r \in (0, 1/2), \alpha \geq 1 \). Let Assumptions \[ 4.1 \] and \[ 4.3 \] hold with

\[
C_{\text{spar}} = \frac{r^2}{4(1 + \alpha)e^2}.
\]

With \( K \subset \{1, \ldots, K\} \) chosen randomly from the uniform distribution among subsets with cardinality \( s^* \), the following bound holds:

\[
P \left( \|C_k^* C_K - \text{Id}_K\|_2 \geq r \right) \leq \frac{216}{r^\alpha}.
\]

Moreover, the following property will also be very useful.

**Lemma 5.5.** (Adapted from \[ 16 \] Lemma 5.3.) If \( v^2 \geq e s^* \|C\|/K_o, \) we have

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{v}{1 - r} \right) \leq K_o \left( e^{s^* \|C\|^2} \left( \frac{v^2}{K_o \|C\|^2} \right)^{\frac{1}{e^2(\alpha + 1)}} \log(\rho^{-1}) \log(K) \right)^2.
\]

Based on this lemma, we easily get the following bound.

**Lemma 5.6.** Take \( C_{\text{col}} \geq \sqrt{e^2(\alpha + 1)} \max \{ \sqrt{C_{\text{spar}}}, C_\mu \} \). Then, we have

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq \frac{1}{\rho^{1 - (\alpha + 1) \log(K)}}.
\]

**Proof.** Taking \( v = C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})} \), we obtain from Lemma 5.5

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})} \log(K)}{K_o \log(K) C_{\text{col}}^2 \cdot \log(\rho^{-1})}.
\]

Using Assumption \[ 4.3 \] this gives

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq K_o \left( e K_o \log(K) C_{\text{col}}^2 \cdot \log(\rho^{-1}) \right)^2.
\]

and using Assumption \[ 4.4 \] we have

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq K_o \left( e \frac{C_{\text{spar}} C_K}{K_o \log(K) C_{\text{col}}^2} \right)^2.
\]

Since \( C_{\text{col}} \geq \sqrt{e^2(\alpha + 1)} \max \{ \sqrt{C_{\text{spar}}}, C_\mu \} \), we get

\[
K_o \left( e \frac{C_{\text{spar}} C_K}{K_o \log(K) C_{\text{col}}^2} \right)^2 \leq K_o \left( e \frac{C_K}{K_o \log(K) e^{2(\alpha + 1)}} \right)^2.
\]

Thus, we have

\[
P \left( \max_{k \in K^c} \|C_k^* C_k\|_2 \geq \frac{C_{\text{col}} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq K_o \left( \frac{1}{\rho^{1 - (\alpha + 1) \log(K)}} \cdot \frac{1}{\frac{1}{K_o \log(K) \log(\rho^{-1}) \log(K)} \right).
\]
and since $K_0 \leq \rho^{-1}$ by Assumption 4.3 we obtain
\[
\mathbb{P}\left( \max_{k \in K} \left\| \mathbf{e}_{k} \right\|_2 \geq \frac{C_{cd} \cdot \sqrt{\log(\rho^{-1})}}{1 - r} \right) \leq \frac{1}{\rho^{1-(\alpha+1)\log(K)^2}}.
\]

\[\square\]

5.3. Similar properties for $X_T$.

5.3.1. Control of $\|X_T^e X_T - I\|$. We have
\[
\sigma_{\min}(X_T^e X_T^e) = \sigma_{\min}\left((\mathbf{e}_{K_T} + E_T^e)^T D_s^2 (\mathbf{e}_{K_T} + E_T^e)\right)
\]
where $D_s$ is a diagonal matrix whose diagonal elements are indexed by $T^*$ and are defined by
\[
D_{s,j^*} = \frac{1}{\|\mathbf{e}_{j^*} + E_T^e\|_2^2}
\]
for $j^* \in T^*$. By the definition of $E_T^e$, we have
\[
\sigma_{\min}(D_s) \geq \frac{1}{1 + \sqrt{n \left( \sigma_{\max}(K_T^e) \right)^{\frac{1}{\pi}} \left( \frac{1}{\log(p^{\nu-1})} \right)^{\frac{1}{2}}}}
\]
and
\[
\sigma_{\max}(D_s) \leq \frac{1}{1 - \sqrt{n \left( \sigma_{\max}(K_T^e) \right)^{\frac{1}{\pi}} \left( \frac{1}{\log(p^{\nu-1})} \right)^{\frac{1}{2}}}}.
\]

By the triangular inequality,
\[
\sigma_{\min}(X_T^e X_T^e) \geq \sigma_{\min}\left(\mathbf{e}_{K_T}^T D_s^2 \mathbf{e}_{K_T} - 2 \left\| \mathbf{e}_{K_T}^T D_s^2 E_T^e \right\| - \left\| E_T^e D_s^2 E_T^e \right\|\right)
\]
\[
\geq \frac{1 - \log(\rho^{-1})^{\frac{1}{\pi}}}{\left( 1 + \sqrt{n \left( \sigma_{\max}(K_T^e) \right)^{\frac{1}{\pi}} \left( \frac{1}{\log(p^{\nu-1})} \right)^{\frac{1}{2}}} \right)^{\frac{1}{2}}}
\]
\[
- 2\sqrt{1 + \log(\rho^{-1})^{\frac{1}{\pi}} \left( \frac{1}{\log(p^{\nu-1})} \right)^{\frac{1}{2}}}.
\]

and
\[
\sigma_{\max}(X_T^e X_T^e) \leq \left\| \mathbf{e}_{K_T}^T D_s^2 \mathbf{e}_{K_T} \right\| + 2 \left\| \mathbf{e}_{K_T}^T D_s^2 E_T^e \right\| + \left\| E_T^e D_s^2 E_T^e \right\|
\]
\[
\leq \frac{1 + \log(\rho^{-1})^{\frac{1}{\pi}}}{\left( 1 - \sqrt{n \left( \sigma_{\max}(K_T^e) \right)^{\frac{1}{\pi}} \left( \frac{1}{\log(p^{\nu-1})} \right)^{\frac{1}{2}}} \right)^{\frac{1}{2}}}
\]

Moreover, using Theorem 5.1 and Lemma 5.2 we obtain
\[
\mathbb{P}\left( \|X_T^e X_T - I\| \geq r^* | E_T^e \right) \leq \frac{218}{p^2}.
\]
with \( r^* \) given by

\[
r^* = \max \left\{ \frac{(1 + r) + 2\sqrt{1 + r} \, sK_{n,s^*} + s^2 K_{n,s^*}^2}{\left(1 - s\sqrt{n} \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha} \right)^2} - 1; \right. \\
\left. 1 - \left( \frac{1 - r}{1 + s\sqrt{n} \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha} \right)^2 \right) \right\}.
\]

(5.33)

Using (5.25) and Assumption (4.6), we have

\[
s K_{n,s^*} \leq C_{s,n,p} \sqrt{\frac{\alpha \left( \frac{1 - e^{-1}}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}{\sqrt{\log(n) \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}}} \leq 0.1 \, r \left( \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(\rho - 1)}} \right)
\]

and thus,

\[
s K_{n,s^*} \leq C_{s,n,p} \sqrt{\frac{\alpha \left( \frac{1 - e^{-1}}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}{\sqrt{\log(n) \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}}} \leq 0.1 \, r \left( \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(\rho - 1)}} \right)
\]

On the other hand,

\[
\sqrt{n} \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha} \leq C_{s,n,p} \frac{\sqrt{\frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}}{\sqrt{\log(\rho - 1)}} \leq C_{s,n,p} \frac{\sqrt{\frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha}}}{\sqrt{\log(\rho - 1)}} \leq 0.1 \, r
\]

which, by Assumption (4.6) gives

\[
\sqrt{n} \left( \frac{\alpha(1 - e^{-1})}{\vartheta_s C_X} \right)^{1 \over \alpha} \left( \frac{1}{\log(p)\nu - 1} \right)^{1 \over \alpha} \leq 0.1 \, r
\]

(5.34)
Summing up, we get
\[ r^* \leq (1 + r) + 2\sqrt{1 + r} \cdot 0.1 \cdot r \left( \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(p - 1)}} \right) + 0.01 \cdot r^2 \left( \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(p - 1)}} \right)^2 - 1 \]
(5.35) \[ = \left( \sqrt{1 + r} + 0.1 \cdot r \cdot C_{\alpha,n,p} \right)^2 - 1 \]
\leq r_{a,n,p}^* (r)
with
\[ r_{a,n,p}^* (r) = r \left( \frac{1}{2} + 0.1 \cdot C_{\alpha,n,p} \right) \left( 2 + \frac{1}{2} \cdot r + 0.1 \cdot r \cdot C_{\alpha,n,p} \right) \]
where \( C_{\alpha,n,p} = \sqrt{\alpha} + \sqrt{\frac{\log(n)}{\log(p - 1)}} \). Thus, using (5.43) and Lemma 5.1
(5.37) \[ \mathbb{P} \left( \| X_{T^*}^T X_{T^*} - I \| \geq r_{a,n,p}^* (r) \right) \leq \frac{218 + 1}{p^a}, \]

5.3.2. Control of \( \max_{k \in T^*} \| X_{T^*}^T X_k \|_2 \). By the triangular inequality, we have that
\[ \| X_{T^*}^T X_k \|_2 = \max_{k \in T^*} \left( \| (C_k + E_{T^*})^* D_k^* (C_k + E_k) \|_2 \right) \]
(5.38) \[ \leq \left( \max_{k \in T^*} \| C_k^* C_k \|_2 + \| C_k \| \max_{k \in T^*} \| E_k \|_2 + \| E_{T^*} \| \max_{k \in T^*} \| E_k \|_2 \right) \| D_k^* \|_2. \]

On the other hand, we have
\[ \mathbb{P} \left( \max_{k \in \{1, \ldots, n\}} \| E_k \|_2 \geq s \left( \sqrt{n} + \sqrt{\frac{\alpha + 1}{c} \log(p)} \right) \right) \leq \frac{C}{p^a}. \]
(5.39) Thus, using Lemma 5.3 and Lemma 5.2, we obtain
\[ \mathbb{P} \left( \max_{k \in T^*} \| X_{T^*}^T X_k \|_2 \geq \left( \frac{C_{col} \cdot \sqrt{\log(p - 1)}}{1 - r} + \frac{1}{\sqrt{1 - r}} + sK_{n,s^*} \right) \left( \sqrt{n} + \sqrt{\frac{\alpha + 1}{c} \log(p)} \right) \right. \]
\[ + \frac{sK_{n,s^*} \cdot C_{col} \cdot \sqrt{\log(p - 1)}}{1 - r} \left( \frac{1}{sK_{n,s^*}} \frac{1}{\sqrt{n} \left( \frac{\alpha + 1}{c} \log(p) \right)^{1/n}} \right) \]
\[ \leq \frac{C + 2}{p^a} + \frac{2}{\rho^{-(\alpha + 1) \log(K)^2}}. \]
(5.40)

Using the fact
\[ \mathbb{P} (A) \leq \mathbb{P} (A \cap E) + \mathbb{P} (E^c) \]
with \( A = \max_{k \in T^*} \| X_{T^*}^T X_k \|_2 \) and
\[ E = \left( \max_{k \in T^*} \| C_k^* C_k \|_2 + \| C_k \| \max_{k \in T^*} \| E_k \|_2 + \| E_{T^*} \| \max_{k \in T^*} \| E_k \|_2 \right) \| D_k^* \|_2. \]
Furthermore, since \( A \subset E^c \) and \( E = E_1 \cap E_2 \cap E_3 \cap E_4 \), we have by union bound,
(5.41) \[ \mathbb{P} (A) \leq \mathbb{P} (E_1^c \cup E_2^c \cup E_3^c \cup E_4^c) \leq \mathbb{P} (E_1^c) + \mathbb{P} (E_2^c) + \mathbb{P} (E_3^c) + \mathbb{P} (E_4^c). \]

Since, by Assumption 4.10,
\[ \left( \frac{1}{\sqrt{1 - r}} + sK_{n,s^*} \right) \left( \sqrt{n} + \sqrt{\frac{\alpha + 1}{c} \log(p)} \right) \leq \left( \frac{1}{\sqrt{1 - r}} + 0.1 \cdot r \cdot C_{\alpha,n,p} \right) \frac{C_{s,n,p}}{\sqrt{\log(p - 1)}}, \]

we obtain
\[
\mathbb{P}
\left(\max_{k \in T^c} \|X^k_{T^*}X_k\|_2 \geq \left(C_{col} \cdot \frac{\sqrt{\alpha \log(\rho^{-1})}}{1 - r} + \left(\frac{1}{\sqrt{1 - r}} + 0.1 \cdot r \cdot C_{a,n,p} \cdot \frac{C_s,n,p}{\sqrt{\log(\rho^{-1})}}\right) \times \frac{1}{1 - s \sqrt{n \left(\frac{\alpha(1 - e^{-1})}{\theta_\ast C_\chi}\right)^{\frac{1}{2}} \left(\frac{1}{\log(p)^{\nu - 1}}\right)^{\frac{1}{2}}}^2\right) \right) \mid \mathcal{E}_\alpha\right)
\]
\[
\leq \frac{C + 2}{p^\alpha} + 2\rho^{(\alpha + 1)\log(K)^2 - 1}.
\]
Moreover, for any event \(A\),
\[
\mathbb{P}(A) \leq \mathbb{P}(A \mid \mathcal{E}_\alpha) + \mathbb{P}(\mathcal{E}_\alpha),
\]
and Lemma \ref{lem:main} we obtain
\[
\mathbb{P}
\left(\max_{k \in T^c} \|X^k_{T^*}X_k\|_2 \geq \left(C_{col} \cdot \frac{\sqrt{\alpha \log(\rho^{-1})}}{1 - r} + \left(\frac{1}{\sqrt{1 - r}} + 0.1 \cdot r \cdot C_{a,n,p} \cdot \frac{C_s,n,p}{\sqrt{\log(\rho^{-1})}}\right) \times \frac{1}{1 - s \sqrt{n \left(\frac{\alpha(1 - e^{-1})}{\theta_\ast C_\chi}\right)^{\frac{1}{2}} \left(\frac{1}{\log(p)^{\nu - 1}}\right)^{\frac{1}{2}}}^2\right) \right) \leq \frac{C + 3}{p^\alpha} + 2\rho^{(\alpha + 1)\log(K)^2 - 1}.
\]

5.4. The last two inequalities. The proof of \eqref{ineq:main} is standard and, under Assumption \ref{ass:main}, the proof of \eqref{ineq:main} can be proved using the ideas of \cite[Section 3.3]{AO}. We give the proofs for the sake of completeness.

5.4.1. Control of \(\|X^k_{T^*}X_k(X^k_{T^*}X_k)^{-1}X^k_{T^*}z\|_\infty\). For any \(j \in T^c\), by the results of section 5.3.2 we have
\[
\|X^k_{T^*}(X^k_{T^*}X_k)^{-1}X^k_{T^*}X_j\|_2 \leq \frac{\sqrt{1 + r^{*}_{a,n,p}(r)}}{(1 - r^{*}_{a,n,p}(r))^{\frac{1}{2}}} \left(1 - s \sqrt{n \left(\frac{\alpha(1 - e^{-1})}{\theta_\ast C_\chi}\right)^{\frac{1}{2}} \left(\frac{1}{\log(p)^{\nu - 1}}\right)^{\frac{1}{2}}}^2\right) \times \left(0.1 \cdot r \cdot C_{a,n,p} \cdot \frac{C_{col} \cdot \sqrt{\log(\rho^{-1})}}{(1 - r)^{3/2}} + C_{col} \cdot \sqrt{\log(\rho^{-1})} \right)
\]
with probability at least \(1 - \left(\frac{C + 3}{p^\alpha} + 2\rho^{(\alpha + 1)\log(K)^2 - 1}\right)\), we get
\[
\mathbb{P}
\left(X^k_{T^*}(X^k_{T^*}X_k)^{-1}X^k_{T^*}z \geq u\right) \leq \frac{1}{2} \exp\left(-\frac{u^2}{2 \sigma^2 \left(1 + r^{*}_{a,n,p}(r)\right)^{\frac{1}{2}}} \left(0.1 \cdot r \cdot C_{a,n,p} \cdot \frac{C_{col} \cdot \sqrt{\log(\rho^{-1})}}{(1 - r)^{3/2}} + C_{col} \cdot \sqrt{\log(\rho^{-1})} \right)^2 + \left(\frac{1}{\sqrt{1 - r}} + 0.1 \cdot r \cdot C_{a,n,p} \cdot \frac{C_s,n,p}{\sqrt{\log(\rho^{-1})}}\right) \right) + \frac{C + 3}{p^\alpha} + \rho^{(\alpha + 1)\log(K)^2 - 1}.
\]
Taking $u$ such that

\[
\frac{1}{2} \exp \left( - \frac{u^2}{2 \sigma^2 \sqrt{1 + r_{n, p}(r)}} \right) \leq \left( \left( \frac{0.1 r \ C_{n, p} \ \sqrt{\log(\rho^{-1})}}{(1-r)^{3/2}} + \frac{C_{n, p} \ \sqrt{\log(\rho^{-1})}}{1-r} \right) \right)
\]

i.e.

\[
u = \sqrt{(\alpha \log(\rho^{-1}) - \log(2))}
\]

Using the union bound, we finally obtain

\[
\mathbb{P} \left( \frac{\left\| X_T^{t_s} X_T^{*} (X_T^{t_s} X_T^{*})^{-1} X_T^{t_s} z \right\|_\infty}{\sqrt{(\alpha \log(\rho^{-1}) - \log(2))}} \leq C + 4 \frac{1}{p^{\alpha-1}} + p \ \rho^{(\alpha+1) \log(K)^2 - 1} \right)
\]

(5.45)

5.4.2. Control of $\left\| X_T^{t_s} X_T^{*} (X_T^{t_s} X_T^{*})^{-1} \text{sgn}(\beta_T^{*}) \right\|_\infty$. For any $j \in T^{*c}$, again by the results of section (C.2.2), we have

\[
\left\| X_T^{t_s} X_T^{*} (X_T^{t_s} X_T^{*})^{-1} X_T^{t_s} \right\|_2 \leq \frac{\left( \left( \frac{0.1 r \ C_{n, p} \ \sqrt{\log(\rho^{-1})}}{(1-r)^{3/2}} + \frac{C_{n, p} \ \sqrt{\log(\rho^{-1})}}{1-r} \right) \right)}{(1 - r_{n, p}(r)) \left( 1 - \frac{\left\| \alpha \left( \frac{1}{\log(p^{\alpha-1})} \right) \right\|_\infty}{\sqrt{\log(\rho^{-1})}} \right)^2}
\]
with probability at least 

\[ \Pr \left( X_j^T X_T \left( X_T^* \right)^{-1} \operatorname{sgn}(\beta_{T^*}) \geq u \right) \leq \frac{1}{2} \exp \left( -\frac{u^2}{2 \left\| X_T^* X_T \right\|} \right) \]

Choosing

\[ u = \sqrt{\frac{2(\alpha \log(\rho) - \log(2))}{1 - r_{\alpha, n, \rho}(r)}} \left( 0.1 r C_{\alpha, n, \rho} \cdot \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{(1-r)^{3/2}} + \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{1-r} \right) \]

and applying the union bound, we obtain

\[ \Pr \left( \left\| X_T^* X_T \left( X_T^* \right)^{-1} \operatorname{sgn}(\beta_{T^*}) \right\|_{\infty} \right) \]

\[ \leq \frac{C_{\alpha, n, \rho}^2}{p^2} + 2p^{(\alpha+1)} \log(K)^2 - 1. \]

5.4.3. **Summing up.** We obtain that

\[ \left\| X_T^* X_T \left( X_T^* \right)^{-1} X_T^* \right\|_{\infty} + \lambda \left\| X_T^* X_T \left( X_T^* \right)^{-1} \operatorname{sgn}(\beta_{T^*}) \right\|_{\infty} \leq \sigma C_1 + \lambda C_2 \]

where

\[ C_1 = \sqrt{(\alpha \log(\rho) - \log(2))} \left( \frac{0.1 r C_{\alpha, n, \rho} \cdot \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{(1-r)^{3/2}} + \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{1-r} \right) \]

and

\[ (5.48) \]

\[ C_1 = \sqrt{(\alpha \log(\rho) - \log(2))} \left( \frac{0.1 r C_{\alpha, n, \rho} \cdot \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{(1-r)^{3/2}} + \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{1-r} \right) \]

\[ + \left( \frac{1}{\sqrt{1-r}} + 0.1 r C_{\alpha, n, \rho} \frac{C_{\text{col}} \cdot \sqrt{\log(\rho)}}{1-r} \right) \]
The goal of this paper is to propose a sound study of the behavior of the LASSO algorithm for the linear model in the case where the design matrix is not satisfying the usual non-collinearity conditions that are enforced in standard analysis. We introduce a new model for the design matrix. In this new model, the columns are assumed to be drawn from a Gaussian mixture model where the centers of the interesting example of applying the LASSO to a non-incoherent matrix and we obtain a performance bound. The price to pay for such a generality is that our prediction bounds hold with arbitrarily high but fixed probability, as compared with the incoherent setting where the the probability goes to one as $p$ tends to $+\infty$.

6. Conclusion

The goal of this paper is to propose a sound study of the behavior of the LASSO algorithm for the linear model in the case where the design matrix is not satisfying the usual non-collinearity conditions that are enforced in standard analysis. We introduce a new model for the design matrix. In this new model, the columns are assumed to be drawn from a Gaussian mixture model where the centers of the interesting example of applying the LASSO to a non-incoherent matrix and we obtain a performance bound. The price to pay for such a generality is that our prediction bounds hold with arbitrarily high but fixed probability, as compared with the incoherent setting where the the probability goes to one as $p$ tends to $+\infty$.

References

1. AlQuraishi, M. and McAdams, H., Direct inference of protein–DNA interactions using compressed sensing methods, PNAS 108, 14819 (2011).
2. Becker, S., Bobin, J. and Candès, E. J., Nesta: a fast and accurate first-order method for sparse recovery. In press SIAM J. on Imaging Science.
3. Bickel, P. J., Ritov, Y., Tsybakov, A. B. Simultaneous analysis of lasso and Dantzig selector. Ann. Statist. 37 (2009), no. 4, 1705–1732.
4. Bock, M.E, Judge, G.G, Yancey, T.A. A simple form for the inverse moments of non-central $\chi^2$ and $F$ random variable and certain confluent hypergeometric functions. Journal of Econometrics 25 (1984); 217–234. North-Holland.
5. Bousquet, O., A Bennett concentration inequality and its application to suprema of empirical processes, Comptes Rendus Mathematique, 334 (2002), no. 6, 495–500.
6. Bunea, F., Tsybakov, A., and Wegkamp, M. (2007a). Sparsity oracle inequalities for the Lasso. Electronic Journal of Statistics, (2008) Vol. 2, 1153-1194.
7. Bunea, F., Honest variable selection in linear and logistic regression models via penalization , the Electronic Journal of Statistics, (2008) Vol. 2, 1153-1194.
8. Candès, E., Compressive sampling, (2006) 3, International Congress of Mathematics, 1433–1452, EMS.
9. Candès, E. J. The restricted isometry property and its implications for compressed sensing. C. R. Math. Acad. Sci. Paris 346 (2008), no. 6, 495–500.
10. Candès, E. J. and Plan, Yaniv. Near-ideal model selection by $\ell_1$ minimization. Ann. Statist. 37 (2009), no. 5A, 285–2177.
11. Candès, E. and Tao T., Decoding by linear programming. IEEE Information Theory, 51 (2005) no. 12, 4203-4215.
12. Candès, E. J. and Tao, T., The Dantzig Selector: statistical estimation when $p$ is much larger than $n$. Ann. Stat.
13. Chandrasekaran, V., Recht, B., Parrilo, P.A. and Willsky, A.S., 2012. The convex geometry of linear inverse problems. Foundations of Computational mathematics, 12(6), pp.805-849.
14. Chretien, S. and Darses, S., The LASSO for generic design matrices as a function of the relaxation parameter, arXiv preprint [arXiv:1105.1430] (2011).
15. Chretien, S. and Darses, Sparse recovery with unknown variance: a lasso-type approach, IEEE Transactions on Information Theory 60.7 (2014): 3970-3988.
16. Chretien, S. and Darses, S. Invertibility of random submatrix via tail decoupling and a matrix Chernoff inequality, Stat. and Prob. Lett. 82 (2012), no. 7, 1479-1487.
17. Chretien, S., On prediction with the LASSO when the design is not incoherent, arXiv preprint [arXiv:1203.5223] (2012).
18. Chretien, S., Gibberd, A. and Roy, S., 2018. Hedging parameter selection for basis pursuit. arXiv preprint [arXiv:1805.01870]
19. D. L. Donoho, A. Maleki, and A. Montanari, Constructing message passing algorithms for compressed sensing, submitted to IEEE Trans. Inf. Theory.
20. Dossal, C. A necessary and sufficient condition for exact recovery by $\ell_1$ minimization. http://hal.archives-ouvertes.fr/docs/00/16/47/38/PDF/DossalMinimisationL1.pdf
21. Fuchs, J.J., On sparse representations in arbitrary redundant bases. IEEE Trans. Info. Th., 2002.
22. Jaggi, M., 2013, February. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. In International Conference on Machine Learning (pp. 427-435). PMLR.
23. Kim, S.-J., Koh, K., Boyd, S. and Gorinevsky, D., SIAM Review, problems and techniques section, 51 , (2009), no. 2, 339–360.
24. Lafond, J., Wai, H.T. and Moulines, E., 2015. On the online Frank-Wolfe algorithms for convex and non-convex optimizations. arXiv preprint [arXiv:1510.01171]
25. Massart, P., Concentration inequalities and model selection. Lectures from the 33rd Summer school on Probability Theory in Saint Flour. Lecture Notes in Mathematics, 1896. Springer Verlag (2007).
26. Meinshausen, N. and Bühlmann, P., High-dimensional graphs and variable selection with the Lasso, Ann. Statist. 34 (2006), no. 3, 1436-1462.
27. Neto, D., Sardy, S. and Tseng, P., l1-Penalized Likelihood Smoothing and Segmentation of Volatility Processes allowing for Abrupt Changes, Journal of Computational and Graphical Statistics, 21 (2012), no. 1, 217–233.
28. Osborne, M.R., Presnell, B. and Turlach, B.A., A new approach to variable selection in least squares problems, IMA J. Numer. Anal. 20 (2000), no. 3, 389–403.
29. Parikh, N. and Boyd, S., 2014. Proximal algorithms. Foundations and Trends in optimization, 1(3), pp.127-239.
30. Rudelson, M. and Vershynin, R. Smallest singular value of a random rectangular matrix. Comm. Pure Appl. Math. 62 (2009), no. 12, 170720131739.
31. Tibshirani, R. Regression shrinkage and selection via the LASSO, J.R.S.S. Ser. B, 58, no. 1 (1996), 267–288.
32. Tropp, J. A. Norms of random submatrices and sparse approximation. C. R. Math. Acad. Sci. Paris 346 (2008), no. 23-24, 1271–1274.
33. Tropp, J. A., User friendly tail bounds for sums of random matrices, http://arxiv.org/abs/1004.4389 (2010).
34. van de Geer, S., High-dimensional generalized linear models and the Lasso. The Annals of Statistics 36, 614-645.
35. Vershynin, R., Introduction to the non-asymptotic analysis of random matrices, Chapter 5 of the book Compressed Sensing, Theory and Applications, ed. Y. Eldar and G. Kutyniok. Cambridge University Press, 2012. pp. 210–268. [arXiv:1011.3027] Aug 2010.
36. Wainwright, Martin J., Sharp thresholds for high-dimensional and noisy sparsity recovery using \(\ell_1\)-constrained quadratic programming (Lasso). IEEE Trans. Inform. Theory 55 (2009), no. 5, 2183–2202.
37. Zhang, T., 2009. Some sharp performance bounds for least squares regression with \(l_1\) regularization. The Annals of Statistics, 37(5A), pp.2109-2144.

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