On Lagrangian algebras in group-theoretical braided fusion categories

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March 16, 2016

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Abstract
We describe Lagrangian algebras in twisted Drinfeld centres for finite groups. Using the full centre construction, we establish a 1-1 correspondence between Lagrangian algebras and module categories over pointed fusion categories.

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0 Introduction

In this paper, we classify Lagrangian algebras in group-theoretical modular categories. This, in particular, gives a classification of physical modular invariants for group-theoretical modular data (a problem raised in [2]). It should be mentioned that the set of labels for physical modular invariants was obtained in [12] (using the language of module categories). What was missing was a way of recovering the modular invariant corresponding to a label. By establishing a correspondence between Lagrangian algebras and module categories, and by computing the characters of Lagrangian algebras, we give a method for determining modular invariants corresponding to module categories.

By a group-theoretical modular category $\mathcal{Z}(G, \alpha)$, we mean the monoidal (or Drinfeld) centre $\mathcal{Z}(\mathcal{V}(G, \alpha))$ of the category of vector spaces $\mathcal{V}(G, \alpha)$ graded by a finite group $G$. Here, $\alpha$ is a 3-cocycle of $G$ with coefficients in the multiplicative group $k^*$ of the ground field, which is used to twist the standard associativity constraint for the tensor product of graded vector spaces. More precisely, we describe objects of $\mathcal{Z}(G, \alpha)$ explicitly as $G$-graded vector spaces with compatible $G$-action (section 2.1), and prove later (section 3.1) that $\mathcal{Z}(G, \alpha)$ is isomorphic to the monoidal centre $\mathcal{Z}(\mathcal{V}(G, \alpha))$.

A commutative algebra $A$ in a braided fusion category $\mathcal{C}$ is Lagrangian if any local $A$-module in $\mathcal{C}$ is a direct sum of copies of $A$ (we recall basic concepts of braided tensor categories in section 1). We classify Lagrangian and more general indecomposable commutative separable (etale for short) algebras in $\mathcal{Z}(G, \alpha)$ in two steps. First, we describe etale algebras with the trivial grading (section 2.3). These are nothing but indecomposable commutative separable algebras with a $G$-action and hence are just function algebras on transitive $G$-sets (we work over an algebraically closed field $k$). Up to isomorphism, they are labelled by (conjugacy classes of) subgroups $H \subset G$. Then we identify the category of local modules $\mathcal{Z}(G, \alpha)^{\text{loc}}_{k(G/H)}$ with the group-theoretical modular category $\mathcal{Z}(H, \alpha|_H)$ (theorem 2.10). A general etale algebra in $\mathcal{Z}(G, \alpha)$ is an extension of its trivial degree component, and hence is an etale algebra in one of $\mathcal{Z}(G, \alpha)^{\text{loc}}_{k(G/H)}$. Considered as an algebra in $\mathcal{Z}(H, \alpha|_H)$, it has the one-dimensional trivial degree component. Our second step (section 2.4) is to classify such algebras (proposition 2.15) and their categories of local modules (theorem 2.17). Then in section 2.5, we combine the results obtaining the description of all etale algebras in $\mathcal{Z}(G, \alpha)$ (theorem 2.19) and their local modules (theorem 2.20). As a corollary, we get a classification of Lagrangian algebras in $\mathcal{Z}(G, \alpha)$ (corollary 2.21). They are parametrised by (conjugacy classes of) subgroups $H \subset G$ together with a coboundary $d(\gamma) = \alpha|_H$ matching with the answer from [12]. In section 3, we explain this matching by identifying Lagrangian algebras with full centres of indecomposable separable algebras in $\mathcal{V}(G, \alpha)$ (theorem 3.2). Finally, after recalling the character theory for objects of $\mathcal{Z}(G, \alpha)$ (section 4.1), we compute characters of Lagrangian algebras (theorem 4.8). We treat the case $G = S_3$ (the symmetric group on 3 symbols) as an example (section 4.3).

This paper extends the results of [3] to the case of $\mathcal{Z}(G, \alpha)$ with a nontrivial cocycle $\alpha$. It could have been titled “Modular invariants for group-theoretical modular data II”. The
scheme of the proof we follow here is very similar to [3]. However, the present paper is not a mere extension of [3]: the presence of a nontrivial cocycle makes all constructions and computations much more elaborate. For example, to identify categories of local modules in section 2.3, we design a recognition tool by looking at more general fusion categories of group theoretic origin (the appendix).

Acknowledgment

The paper was completed while the first author was visiting Max Planck Institut für Mathematik (Bonn, Germany). He thanks the institution for perfect working conditions.

1 Preliminaries

Here, we recall a number of preliminary concepts. Throughout $k$ is the ground field of characteristic zero and $\text{Vect} = \text{Vect}_k$ is the category of (finite dimensional) vector spaces over $k$.

1.1 Algebras and modules

Let $C$ be a monoidal category. An associative unital algebra in $C$ is an object $A \in C$ together with two morphisms $\mu : A \otimes A \to A$ and $\iota : I \to A$ such that

$$\mu(\mu \otimes \text{Id}_A) = \mu(\text{Id}_A \otimes \mu),$$

and

$$\mu(\iota \otimes \text{Id}_A) = \text{Id}_A = \mu(\text{Id}_A \otimes \iota).$$

By an “algebra”, we will mean an associative unital algebra.

Let $A \in C$ be an algebra. A right $A$-module is an object $M \in C$ together with a morphism $\nu : M \otimes A \to M$ such that

$$\nu(\nu \otimes \text{Id}_A) = \nu(\text{Id}_M \otimes \mu).$$

Left $A$-modules are defined similarly. An $A$-$B$-bimodule $M \in C$ is a left $A$-module and a right $B$-module ($B \in C$ is another algebra) such that the diagram of module action maps

$$
\begin{array}{ccc}
A \otimes M \otimes B & \longrightarrow & A \otimes M \\
| & & | \\
M \otimes B & \longrightarrow & M
\end{array}
$$

commutes.

We denote the category of right $A$-modules by $C_A$, and that of $A$-$B$-bimodules by $A_{\cdot}C_B$. 
1.2 Separable algebras

For an algebra $A$ in a spherical category (with the spherical structure $s_X : X \to X^{**}$) denote by $\tau : A \to I$ (the canonical trace function) the composition

$$A \xrightarrow{1 \otimes \ev_A} A \otimes A \otimes A^* \xrightarrow{\mu^1} A \otimes A^* \xrightarrow{s_A^1} A^{**} \otimes A^* \xrightarrow{\ev_A^*} I$$

An algebra $A$ is said to be separable if the following composition is a nondegenerate pairing (denoted $e : A \otimes A \to I$):

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{t_A} I$$

Nondegeneracy of $e$ means that there is a morphism $\kappa : I \to A \otimes A$ such that the compositions

$$A \xrightarrow{1 \otimes \kappa} A^3 \xrightarrow{e \otimes 1} A$$

$$A \xrightarrow{\kappa \otimes 1} A^3 \xrightarrow{1 \otimes e} A$$

are the identity.

If $A \in C$ is a separable algebra in a spherical fusion category, then the category $C_A$ of right $A$-modules is semisimple.

We call an algebra $A$ in a monoidal category $C$ simple if any algebra homomorphism $A \to B$ is a monomorphism.

**Lemma 1.1.** Let $A$ be an indecomposable separable algebra in a spherical fusion category $C$. Then $A$ is simple.

**Proof.** If an algebra $A$ is not simple then there is a surjective (but not bijective) algebra homomorphism $A \to B$. Via the inverse image along this homomorphism the category of $B$-modules $C_B$ becomes a full subcategory of $C_A$. Moreover $C_B$ is a full left $C$-module subcategory of $C_A$.

Recall from [7, Proposition 3.9.] that a semisimple module category over a fusion category is a direct sum of its simple module subcategories. In particular, $C_B$ is a direct summand of $C_A$ as a $C$-module category. Hence the algebra $C(I, A)$ in $Vect$ (which coincides with the algebra $\text{End}_C(Id_{C_A})$ of $C$-module endomorphisms of the identity functor of $C_A$) is a non-trivial direct sum. Thus $A$ is decomposable.

1.3 Commutative algebras and local modules

Let now $C$ be a braided monoidal category with the braiding $c_{X,Y} : X \otimes Y \sim Y \otimes X$.

An algebra $A = \langle A, \mu, \iota \rangle$ in $C$ is commutative if

$$\mu = \mu \circ c_{A,A}.$$ 

Following [5] we call an indecomposable commutative and separable algebra etale.
A right $A$-module $M = \langle M, \nu \rangle$ is said to be local if the following diagram commutes:

\[
\begin{array}{ccc}
M \otimes A & \xrightarrow{\nu} & M \\
\downarrow & & \downarrow \\
A \otimes M & \xrightarrow{\zeta_{A,M}} & M \otimes A
\end{array}
\]

We denote by $C^\text{loc}_A$ the full subcategory of $C_A$ consisting of local right $A$-modules.

Recall from [13] that (in case $C$ has coequalisers) the category $C_A$ is monoidal with respect to the relative tensor product $\otimes_A$ and that the category $C^\text{loc}_A$ is braided.

An algebra $L$ in an additive braided monoidal category $C$ is said to be a Lagrangian algebra if any local $L$-module is a direct sum of copies of the regular module $L$.

Note that in the case when $C$ is a $k$-linear braided monoidal category $L$ is Lagrangian iff the category $C^\text{loc}_L$ is equivalent to the category $\text{Vect}_k$ of $k$-vector spaces.

The following statement was proved in [8]; we state it here without proof.

**Lemma 1.2.** Let $(A, m, i)$ be a commutative algebra in a braided category $C$. Let $(B, \mu, \iota)$ be an algebra in $C_A$. Define $\overline{\mu}$ and $\overline{\iota}$ as compositions

\[
\begin{align*}
B \otimes B & \xrightarrow{\overline{\mu}} B, \\
1 & \xrightarrow{\iota} A \xrightarrow{\iota} B.
\end{align*}
\]

Then $(B, \overline{\mu}, \overline{\iota})$ is an algebra in $C$.

The map $\iota : A \to B$ is a homomorphism of algebras in $C$.

The algebra $(B, \overline{\mu}, \overline{\iota})$ in $C$ is separable or commutative if and only if the algebra $(B, \mu, \iota)$ in $C_A$ is such.

The functor $(C^\text{loc}_A)_B \to C^\text{loc}_B$ given by

\[
(M, m : B \otimes A \to M) \mapsto \left(M, \overline{\mu} : B \otimes M \to B \otimes A \to M \xrightarrow{m} M\right)
\]

is a braided monoidal equivalence.

### 1.4 Full centre

Recall from [4] that the full centre $Z(A)$ of an algebra $A$ in a monoidal category $C$ is an object of the monoidal centre $Z(C)$ together with a morphism $Z(A) \to A$ in $C$, terminal among pairs $(Z, \zeta)$, where $Z \in Z(C)$ and $\zeta : Z \to A$ is a morphism in $C$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A \otimes Z & \xrightarrow{A \zeta} & A \\
\downarrow & & \downarrow \\
Z \otimes A & \xrightarrow{\zeta_A} & A \otimes A
\end{array}
\]
Here $z_A$ is the half-braiding of $Z$ as an object of $\mathcal{Z}(C)$. The terminality condition means that for any such pair $(Z, \xi)$ there is a unique morphism $Z \to Z(A)$ in the monoidal centre $\mathcal{Z}(C)$, which makes the diagram

\[ Z \xrightarrow{\xi} Z(A) \]

commute.

Recall that the $\alpha$-induction is the tensor functor

\[ \alpha : \mathcal{Z}(C) \to \mathcal{A}_C, \quad \alpha(Z) = Z \otimes A, \]

with the left $A$-module structure given by

\[ A \otimes Z \otimes A \xrightarrow{z_A \otimes 1} Z \otimes A \otimes A \xrightarrow{1 \otimes \mu} Z \otimes A \]

**Proposition 1.3.** The full centre $Z(A)$ of an indecomposable separable algebra $A$ in a fusion category $C$ coincides with the action internal end $[A, A] \in \mathcal{Z}(C)$ of the trivial bimodule $A \in \mathcal{A}_C$ with respect to the $\mathcal{Z}(C)$-action on $\mathcal{A}_C$ given by $\alpha$-induction. The category of modules $\mathcal{Z}(C)_{Z(A)}$ is equivalent, as a fusion category, to the category $\mathcal{A}_C$ of $A$-bimodules.

**Proof.** The universal property of the action internal end says that $[A, A]$ is the terminal object among $(Z, z)$ where $\xi \in \mathcal{Z}(C)$ and $\xi : \alpha(Z) \to A$ is a morphism of $A$-bimodules. The right $A$-module map $\xi : Z \otimes A \to A$ is completely determined by the morphism $\zeta = \xi(1 \otimes \iota) : Z \to A$ (which is still a left $A$-module map). The left $A$-module property for $\zeta$ is exactly (2).

According to [11] the functors

\[ \mathcal{Z}(C)_{[A, A]} \xrightarrow{- \otimes [A, A]} \mathcal{A}_C \]

are quasi-inverse equivalences. The tensor structure

\[ [A, M] \otimes [A, A] [A, N] \to [A, M \otimes_A N] \quad (3) \]

for the functor $[A, -]$ comes from the universal property of the action internal hom. Indeed the composition

\[ \alpha([A, M] \otimes [A, N] \otimes A) \simeq \alpha([A, M]) \otimes_A A \otimes A \alpha([A, N]) \otimes A A \to M \otimes A N \]

induces the map $[A, M] \otimes [A, N] \to [A, M \otimes_A N]$, which is naturally $[A, A]$-balanced, i.e. factors through $[A, M] \otimes [A, A] [A, N]$ giving rise to (3). \qed
Remark 1.4. Here is a slightly different proof of the second statement of proposition 1.3. The canonical braided equivalence (Morita invariance of the monoidal centre) \( Z(C) \to Z(\mathcal{A}C_A) \) sends the full centre \( Z(A) \) to the full centre \( Z(I) \) of the monoidal unit \( I \in \mathcal{A}C_A \) (which is really the \( A \)-bimodule \( A \)).

For a fusion category \( D (= \mathcal{A}C_A) \), the full centre \( Z(I) \in Z(D) \) coincides with \( L(I) \), where \( L : D \to Z(D) \) is the adjoint to the forgetful functor \( F : Z(D) \to D \). The adjunction is monadic. Moreover, the monad \( T = L \circ F \) on \( Z(D) \) is a \( Z(D) \)-module functor. Thus \( T \)-algebras are the same as \( T(I) \)-modules. Finally, \( T(I) = L(I) = Z(I) \) and the forgetful functor factorises:

\[
\begin{array}{c}
\xymatrix{
Z(D) \\
Z(D)_{Z(I)} \ar[r]^-{F} & D
}
\end{array}
\]

Theorem 1.5. The full centre \( Z(A) \) of an indecomposable separable algebra \( A \) in a fusion category \( C \) is a Lagrangian algebra in \( Z(C) \).

Proof. The tensor equivalence \( Z(C)Z(A) \to \mathcal{A}C_A \) from proposition 1.3 induces a braided tensor equivalence \( Z(C)Z(A) \to Z(\mathcal{A}C_A) \). By Morita invariance of the monoidal centre, \( Z(\mathcal{A}C_A) \simeq Z(C) \).

By [6, Proposition 3.7], we have the decomposition into Deligne product \( Z(C) \boxtimes \overline{Z(C)}^{\text{loc}}_{Z(A)} \simeq Z(Z(C)Z(A)) \).

Combining the above, we get \( Z(C) \boxtimes \overline{Z(C)}^{\text{loc}}_{Z(A)} \simeq Z(C) \), which means that \( Z(C)_{Z(A)} \simeq \text{Vect} \), i.e. \( Z(A) \) is Lagrangian.

\[ \square \]

2 Commutative algebras in group-theoretical categories

2.1 Group-theoretical braided fusion categories

Let \( G \) be a finite group. By \( k^* \) we denote the multiplicative group of the ground field \( k \).

By a 3-cocycle of \( G \) with coefficients in \( k^* \), we mean a normalised 3-cocycle of the standard complex, i.e., a function \( \alpha : G \times G \times G \to k^* \) such that

\[ \alpha(g, h, l)\alpha(f, gh, l)\alpha(f, g, h) = \alpha(fg, h, l)\alpha(f, g, hl), \quad f, g, h, l \in G \]

and such that \( \alpha(f, g, h) = 1 \) each time one of \( f, g, h \) is the identity.

We denote by \( Z^3(G, k^*) \) the group of normalised 3-cocycles.

A vector space \( V \) is \( G \)-\textit{graded} if there given a direct sum decomposition \( V = \bigoplus_{g \in G} V_g \).

The tensor product of graded vector spaces is graded in a natural way \( (V \otimes U)_f = \bigoplus_{gh=f} V_g \otimes U_h \).

The monoidal unit in \( V(G, \alpha) \) is \( I = I_e = k \). A 3-cocycle \( \alpha \in Z^3(G, k^*) \) can be used to twist the standard associativity constraint:

\[
\alpha_{U,V,W}(u \otimes (v \otimes w)) = \alpha(f, g, h)(u \otimes v) \otimes w, \quad u \in U_f, v, w \in V_g, w \in W_h.
\]
Denote by $\mathcal{V}(G, \alpha)$ the category of $G$-graded vector spaces with grading preserving linear maps, equipped with the above structure of a fusion category.

An $\alpha$-projective $G$-action on a $G$-graded vector space $V$ is a collection of automorphisms $f : V \to V, \quad v \mapsto f.v$ for each $f \in G$ such that $f(V_g) = V_{fg^{-1}}$, and

$$ (fg).v = \alpha(f, g|h)f.(g.v), \quad \forall v \in V_h. \quad (5) $$

Here,

$$ \alpha(f, g|h) = \alpha(f, g, h)^{-1} \alpha(f, gh, g) \alpha(fh, f, g)^{-1}. \quad (6) $$

where $fh = fhf^{-1}$. Similarly, define

$$ \alpha(f|g, h) = \alpha(f, g, h)\alpha(fg, f, h)^{-1}\alpha(fg, fh, f), \quad (7) $$

The following identities follow directly from the 3-cocycle condition for normalized $\alpha$:

$$ \alpha(f, gh|u)\alpha(g, h|u) = \alpha(fg, h|u)\alpha(f, gh|uh^{-1}), \quad (8) $$

$$ \alpha(fg|u, v)\alpha(f, g|u)\alpha(f, g|v) = \alpha(fg, uv|u)\alpha(g, u|v)\alpha(f, g|uv^{-1}, gvg^{-1}), $$

$$ \alpha(g, h, u)\alpha(fgh, u)\alpha(f|g, h) = \alpha(f, gh, hu)\alpha(f|h, u)\alpha(fgf^{-1}, fhf^{-1}, fuf^{-1}). $$

Define the category $\mathcal{Z}(G, \alpha)$ as follows. Objects of $\mathcal{Z}(G, \alpha)$ are $G$-graded vector spaces together with $\alpha$-projective $G$-action. Morphisms are grading and action-preserving homomorphisms of vector spaces.

Define the tensor product in $\mathcal{Z}(G, \alpha)$ is the tensor product of $G$-graded vector spaces, with $\alpha$-projective $G$-action defined by

$$ f.(x \otimes y) = \alpha(f|g, h)(f.x \otimes f.y), \quad x \in X_g, \ y \in Y_h. \quad (9) $$

The associativity is given by (4).

The monoidal unit is $I = I_e = k$ with trivial $G$-action.

The braiding is given by

$$ c_{X,Y}(x \otimes y) = f.y \otimes x, \quad x \in X_f, y \in Y. \quad (10) $$

The following is well-known. We leave the proof to the reader (see also proposition 3.1).

**Proposition 2.1.** The category $\mathcal{Z}(G, \alpha)$ defined above is braided monoidal.
Lemma 2.2. The category $Z(G, \alpha)$ is rigid, with dual objects $X^\vee = \oplus_f (X_f)^\vee$ given by

$$(X_f)^\vee = (X_{f^{-1}})^\vee = \text{Hom}_k(X_{f^{-1}}, k),$$

with the $\alpha$-projective action

$$(g.l)(x) = \frac{\alpha(g^{-1}, g|f^{-1})}{\alpha(g|f, f^{-1})} l((g^{-1}.x)), \quad l \in \text{Hom}_k(X_{f^{-1}}, k), \quad x \in X_{g^{-1}g^{-1}}.$$  

Proof. It can be verified that the formula above indeed defines an $\alpha$-projective action on the graded vector space. The evaluation morphism $X^\vee \otimes X \to I$ is given by the canonical pairing $(X_f)^\vee \otimes X_{f^{-1}} \to k$. Similarly the coevaluation morphism $I \to X \otimes X^\vee$ is given by the canonical elements $k \to X_{f^{-1}} \otimes (X_f)^\vee$.  

Lemma 2.3. The category $Z(G, \alpha)$ admits the following decomposition into a direct sum of $k$-linear subcategories:

$$Z(G, \alpha) = \bigoplus_{f \in cl(G)} Z_f(G, \alpha), \quad (11)$$

where the sum is taken over a set $cl(G)$ of representatives of conjugacy classes of elements of $G$, and for $f \in G$, the subcategory $Z_f(G, \alpha)$ is given by

$$Z_f(G, \alpha) = \left\{ Z \in Z(G, \alpha) \mid \text{supp}(Z) = f^G \right\}.$$

Here, $f^G = \{ gfg^{-1} \mid g \in G \}$ denotes the conjugacy class of $f$ in $G$.  

Proof. Clearly, the support of an object of $Z(G, \alpha)$ is a union of conjugacy classes in $G$. It is also straightforward that for $Z \in Z(G, \alpha)$, one has

$$Z = \bigoplus_{c \in cl(G)} Z_c, \quad Z_c = \bigoplus_{f \in c} Z_f$$

is a decomposition into a direct sum of objects in $Z(G, \alpha)$.  

Lemma 2.4. The category $Z_f(G, \alpha)$ is equivalent, as a $k$-linear category, to the category $k [C_G(f), \alpha ( , |f|^{-1})]$-Mod of left modules over the twisted group algebra $k [C_G(f), \alpha ( , |f|^{-1})]$.  

Proof. We show that the functor

$$F : Z_f(G, \alpha) \to k [C_G(f), \alpha ( , |f|^{-1})] \text{-Mod}, \quad F(Z) = Z_f$$

is an equivalence by exhibiting its quasi-inverse $E : k [C_G(f), \alpha ( , |f|^{-1})]$-Mod $\to Z_f(G, \alpha)$ given by

$$E(M) = \left\{ a : G \to M \bigg| a(xy) = \alpha (y^{-1}, x^{-1}|fx^{-1}|) y^{-1}.a(x) \quad \forall x \in G, y \in C_G(f) \right\}.$$  

The $G$-grading on $E(M)$ is defined as follows: a function $a \in E(M)$ is homogeneous if and only if $\text{supp}(a)$ is a single coset modulo $C_G(f)$. More precisely,

$$|a| = xfx^{-1} \iff \text{supp}(a) = xC_G(f).$$
The $\alpha$-projective $G$-action on $E(M)$ is given by
\[(g.a) (x) = \alpha (x^{-1}, g | x f x^{-1}) a (g^{-1} x) .\]

It is straightforward that these definitions make $E(M)$ into an object of $\mathcal{Z}_f (G, \alpha)$.

Now we define the adjunction isomorphisms $\phi : \text{Id} \to E \circ F$ and $\psi : F \circ E \to \text{Id}$. For $Z \in \mathcal{Z}_f (G, \alpha)$ and a homogeneous $z \in Z_{xf^{-1}}$ define $\tilde{z} : G \to Z_f$ by
\[\tilde{z} (g) = \begin{cases} g^{-1} z, & g \in x C_G (f) \\ 0, & \text{otherwise} \end{cases}.\]

It is straightforward to see that $\tilde{z} \in E (F (Z))$.

For an object $V \in \mathcal{C} (\{ f \}, C_G (f), \alpha)$, define $\psi_V : F (E (V)) \to V$ by $\psi_V (a) = a (e)$. One can verify directly that $\phi$ and $\psi$ are morphisms and inverse to one another.

Corollary 2.5. Simple objects $Z \in \mathcal{Z} (G, \alpha)$ are labelled by conjugacy classes of pairs $(f, M)$ where $f \in G$ and $M$ is a simple $k \left[ C_G (f), \alpha (\cdot, | f \cdot)^{-1} \right]$-module.

Corollary 2.6. Let $k$ be a field of characteristic zero. The category $\mathcal{Z} (G, \alpha)$ is fusion.

2.2 Algebras in group-theoretical modular categories

We start with expanding the structure of an algebra in the category $\mathcal{Z} (G, \alpha)$ in plain algebraic terms. Recall that a $G$-graded vector space $A = \bigoplus_{g \in G} A_g$ is a $G$-graded algebra if it is equipped with a grading preserving multiplication $A_f A_g \subseteq A_{fg}$. We call a $G$-graded algebra $A$ $\alpha$-associative (for a 3-cocycle $\alpha \in Z^3 (G, k^*)$) if
\[a (bc) = \alpha (f, g, h) (ab)c, \quad \forall a \in A_f, b \in A_g, c \in A_h.\]

Proposition 2.7. An algebra $A$ in the category $\mathcal{Z} (G, \alpha)$ is a $G$-graded $\alpha$-associative algebra together with an $\alpha$-projective $G$-action such that
\[f. (ab) = \alpha (f | g, h) (f.a) (f.b), \quad a \in A_g, b \in A_h. \tag{12}\]

An algebra $A$ in the category $\mathcal{Z} (G, \alpha)$ is commutative iff
\[ab = (f.b) a, \quad \forall a \in A_f, b \in A. \tag{13}\]

Proof. Being a morphism in the category $\mathcal{Z} (G, \alpha)$, the multiplication of an algebra in $\mathcal{Z} (G, \alpha)$ preserves grading and $\alpha$-projective $G$-action (hence the property (12)). Associativity of multiplication in $\mathcal{Z} (G, \alpha)$ is equivalent to $\alpha$-associativity.

The formula (10) for the braiding in $\mathcal{Z} (G, \alpha)$ implies that commutativity for an algebra $A$ in the category $\mathcal{Z} (G, \alpha)$ is equivalent to the condition (13). □

A $G$-algebra is an algebra $A$ (in $\mathcal{V}ect$) together with an action of $G$ on $A$ by algebra homomorphisms.
Corollary 2.8. The degree $e$ part $A_e$ of an algebra $A$ in the category $\mathcal{Z}(G,\alpha)$ is an associative $G$-algebra and $A$ is a module over $A_e$. The algebra $A_e$ is commutative if $A$ is a commutative algebra in the category $\mathcal{Z}(G,\alpha)$.

Proof. The normalization condition for the cocycle $\alpha$, together with $\alpha$-associativity of $A$, implies that $A_e$ is an associative algebra and that $A$ is a $A_e$-module. The same normalization condition, together with $\alpha$-projectivity of the $G$-action and the property (12), implies that the action of $G$ on $A_e$ is genuine and that $G$ acts on $A_e$ by algebra homomorphisms. Commutativity of $A_e$ for a commutative algebra $A \in \mathcal{Z}(G,\alpha)$ follows directly from the identity (13).

Proposition 2.9. An algebra $A$ in the category $\mathcal{Z}(G,\alpha)$ is separable if and only if the composition

$$A_f \otimes A_{f^{-1}} \xrightarrow{\mu} A_e \xrightarrow{\tau} k$$

defines a nondegenerate bilinear pairing for any $f \in G$. In particular, the algebra $A_e$ is separable if $A$ is a separable algebra in the category $\mathcal{Z}(G,\alpha)$.

Proof. Being a homomorphism of graded vector spaces, the standard trace map $\tau : A \rightarrow I$ is zero on $A_f$ for $f \neq e$. Hence the standard bilinear form is zero on $A_f \otimes A_g$ unless $fg = e$. In particular, the restriction of $\tau$ to $A_e$ makes it a separable algebra in the category of vector spaces.

2.3 Etale algebras in trivial degree and their modules

We start with a well known (see for example [10]) description of etale $G$-algebras in $\mathcal{V}ect$. We give (a sketch of) the proof for completeness.

Lemma 2.10. Etale $G$-algebras are function algebras on $G$-sets. Indecomposable $G$-algebras correspond to transitive $G$-sets.

Proof. An etale algebra over an algebraically closed field $k$ is a function algebra $k(X)$ on a finite set $X$ (with elements of $X$ corresponding to minimal idempotents of the algebra). The $G$-action on the algebra amounts to a $G$-action on the set $X$. Obviously, the algebra of functions $k(X \cup Y)$ on the disjoint union of $G$-sets is the direct sum of $G$-algebras $k(X) \oplus k(Y)$ and any direct sum decomposition of $G$-algebras appears in that way.

Let $k(X)$ be an indecomposable $G$-algebra. By choosing an element $p \in X$, we can identify the $G$-set $X$ with the set $G/H$ of cosets modulo the stabiliser subgroup $H = \text{St}_G(p)$.

Let $\mathcal{C}(G,H,\alpha)$ be the category $\mathcal{C}(F,G,\gamma,\beta,\alpha)$ with $(\gamma,\beta,\alpha) = \tau(\alpha)$ as defined in the appendix.

Theorem 2.11. Let $G$ be a finite group, and let $H \subset G$ be a subgroup. The category $\mathcal{Z}(G,\alpha)_{k(G/H)}$ of right modules over the function algebra $k(G/H)$ in the Drinfeld center $\mathcal{Z}(G,\alpha)$ is equivalent, as a monoidal category, to the category $\mathcal{C}(G,H,\alpha)$.

Moreover, the full subcategory $\mathcal{Z}(G,\alpha)_{k(G/H)}^{\text{loc}}$ of local modules is equivalent, as a braided monoidal category, to the Drinfeld center $\mathcal{Z}(H,\alpha|_H)$.
**Proof.** We will exhibit the claimed equivalence of categories by constructing a pair of quasi-inverse functors

![Diagram](https://example.com/diagram.png)

To define the first functor $D$, let us choose a minimal idempotent $p$ in the function algebra $k(G/H)$ to be the indicator function on $H$:

$$p : G \to k, \quad p(x) = \begin{cases} 1, & x \in H \\ 0, & \text{otherwise} \end{cases}$$

Define $D : \mathcal{Z}(G,\alpha)_{k(G/H)} \to \mathcal{C}(G,H,\alpha)$ by $D(M) = Mp$. Since $p$ is of degree zero, $Mp$ is a $G$-graded vector space in a natural way: $(Mp)_g = (M_g)p$. The $G$-action on $M$ reduces to an $H$-action on $Mp$. This makes $D(M)$ an $H$-action on $Mp$. Thus $D(M)$ is a tensor functor.

The second functor $E$ requires more preparation. For $V \in \mathcal{C}(G,H,\alpha)$ let $\text{Map}(G,V)$ be the vector space of set-theoretic maps $G \to V$. Define $G$-grading on $\text{Map}(G,V)$ by

$$|a| = f \in G \iff |a(x)| = x^{-1}fx \quad \forall x \in G, \quad (14)$$

Define an $\alpha$-projective $G$-action on $\text{Map}(G,V)$ as follows. For a homogeneous $a \in \text{Map}(G,V)$ of degree $|a| = f$, define $g.a : G \to V$ by

$$(g.a)(x) = \alpha(x^{-1},g|f)^{-1}a(g^{-1}x).$$

It is straightforward to check that $\text{Map}(G,V)$ is an object of $\mathcal{Z}(G,\alpha)$.

Now consider a subspace of $\text{Map}(G,V)$ given by

$$E(V) = \{ a : G \to V \mid a(xh) = \alpha(h^{-1},x^{-1}|f)h^{-1}a(x), \ h \in H, x \in G, |a| = f \}, \quad (15)$$

for $V \in \mathcal{C}(G,H,\alpha)$. It also is not hard to see that $E(V)$ is an object of $\mathcal{Z}(G,\alpha)$.

Moreover $E(V)$ is a right $k(G/H)$-module, with the action $\nu : E(V) \otimes E(k) \to E(V)$ in $\mathcal{Z}(G,\alpha)$ defined by $\nu(a \otimes b) = ab$, where $(ab)(x) = a(x) \cdot b(x)$ (here we use that $E(k) = k(G/H)$). This makes $E$ a functor $\mathcal{C}(G,H,\alpha) \to \mathcal{Z}(G,\alpha)_{k(G/H)}$.

For $V, W \in \mathcal{C}(G,H,\alpha)$ the universal property of tensor product gives an isomorphism

$$E(V) \otimes_{k(G/H)} E(W) = E(V) \otimes_{E(k)} E(W) \simeq E(V \otimes W),$$

which shows that $E$ is a monoidal functor.
It remains to define the adjunction isomorphisms $\phi : D \circ E \to Id$ and $\psi : Id \to E \circ D$. For $V \in \mathcal{C}(G, H, \alpha)$, define a map $\phi_V : E(V)p \to V$ by $\phi(\alpha p) = a(e)$, where $e \in G$ is the identity element. For $M \in \mathcal{Z}(G, \alpha)_{k(G/H)}$ define a map $\psi_M : M \to E(Mp)$ by $\psi(m)(x) = (x^{-1}.m)p$, $x \in G$. It is a direct task to check that these are natural isomorphisms of functors.

Finally, we address the locality statement. For an object $V \in \mathcal{Z}(G, \alpha)$, denote by $\text{supp}(V) = \{ g \in G | V_g \neq 0 \}$ the support of $V$. Let $M$ be a right $k(G/H)$-module, and $p$ be as above. The support of $D(M) = Mp$ is a subset of $H$. Indeed the locality condition implies (and is equivalent to the fact) that for $m \in M_g$ one has $mp = m(g.p)$. Hence $mp = mp^2 = m((g.p)p)$, which for non-zero $m$ implies that $g.p = p$. Note that the full subcategory of $\mathcal{C}(G, H, \alpha)$ of objects with support in $H$ is $\mathcal{C}(H, H, \alpha) = \mathcal{Z}(H, \alpha|_{H})$. Thus the restriction of $D$ to $\mathcal{Z}(G, \alpha)_{k(G/H)}$ lands in $\mathcal{Z}(H, \alpha|_{H})$. It is straightforward to see that this restriction is braided.

\begin{proof}
\end{proof}

\begin{remark}
\end{remark}

\begin{corollary}
\end{corollary}

\begin{lemma}
\end{lemma}

\begin{section}{Etale algebras trivial in trivial degree}

Here we describe simple etale algebras $B$ in $\mathcal{Z}(H, \beta)$ with $B_e = k$.

\begin{lemma}
\end{lemma}

\begin{proof}
\end{proof}

\end{section}
For non-zero $a, c$, choosing $b$ such that $m(a, b), m(b, c) \neq 0$, we get that $a$ and $c$ are proportional.

Now, it follows from the non-degeneracy of $m : B_g \otimes B_{g^{-1}} \to A_v = k$, that a generator of a non zero $B_f$ is invertible. Thus, for non-zero components $B_f, B_g$ the product $B_f B_g$ is also non-zero.

Let $F < H$ be a normal subgroup and $\gamma \in C^2(F, k^*)$ be a cochain, such that $d(\gamma) = \alpha|_F$. Denote by $k[F, \gamma]$ an $H$-graded $\alpha$-associative algebra with the basis $e_f, f \in F$, graded as $|e_f| = f$, and with multiplication defined by $e_f e_g = \gamma(f, g) e_{fg}$.

**Proposition 2.15.** An indecomposable separable algebra $B$ in $\mathcal{Z}(H, \alpha|_H)$ with $B_e = k$ has the form $k[F, \gamma]$ with the $\alpha$-projective $H$-action given by:

$$h(e_f) = \varepsilon_h(f)e_{hf^{-1}},$$

for some $\varepsilon : H \times F \to k^*$ satisfying

$$\varepsilon_{gh}(f) = \varepsilon_g(hf^{-1})\varepsilon_h(f)\alpha(f|g, h), \quad g, h \in H, f \in F$$

(16)

$$\gamma(f, g)\varepsilon_h(fg) = \alpha(f, g|h)\varepsilon_h(g)\gamma(hfh^{-1}gh^{-1}), \quad h \in H, f, g \in F$$

(17)

The algebra $k[F, \gamma]$ is commutative (and hence etale) if

$$\gamma(f, g) = \varepsilon_{f}(g)\gamma(fg^{-1}f), \quad f, g \in F.$$  

(18)

**Proof.** Indeed, $\alpha$-projectivity of the action requires that $(gh)(e_f) = \varepsilon_{gh}(f)e_{ghf^{-1}g^{-1}}$ coincides with

$$\alpha(f|g, h)g(h(e_f)) = \alpha(f, g|h)\varepsilon_h(f)\varepsilon_g(hfh^{-1})e_{ghf^{-1}g^{-1}},$$

which gives the first identity. Multiplicativity of the action amounts to the equality between

$$h(e_fe_g) = \gamma(f, g)\varepsilon_h(fg)e_{hfgh^{-1}},$$

and

$$\alpha(f, g|h)h(e_f)e_g = \alpha(f, g|h)\varepsilon_h(f)\varepsilon_h(g)\gamma(hfh^{-1}, gh^{-1})e_{hfgh^{-1}},$$

which gives the second identity. Finally, commutativity implies that $e_f e_g = \gamma(f, g)e_{fg}$ is equal to

$$f(e_g)e_f = \varepsilon_{f}(g)e_{fg^{-1}}e_f = \varepsilon_{f}(g)\gamma(fg^{-1}, f)e_{fg}.$$  

Denote by $k[F, \gamma, \varepsilon]$ the etale algebra in $\mathcal{Z}(H, \alpha|_H)$, defined in proposition 2.15.

**Lemma 2.16.** Two algebras $k[F, \gamma, \varepsilon]$ and $k[F', \gamma', \varepsilon']$ in the category $\mathcal{Z}(H, \alpha|_H)$ are isomorphic iff there is a cochain $c : F \to k^*$ such that

$$c(fg)\gamma(f, g) = \gamma'(f, g)c(f)c(g), \quad \varepsilon_h(f)c(hfh^{-1}) = c(f)\varepsilon'_h(g).$$  

(19)
Proof. Isomorphic algebras in $Z(H, \alpha|_H)$ have to have the same supports. Thus $F = F'$. Since the components of both $k[F, \gamma, \varepsilon]$ and $k[F', \gamma', \varepsilon']$ are all one dimensional, an isomorphism $k[F, \gamma, \varepsilon] \rightarrow k[F', \gamma', \varepsilon']$ has a form $e_f \mapsto c(f)e_f$ for some $c(f) \in k^*$. Finally, multiplicativity of this mapping is equivalent to the first condition, while $H$-equivariance is equivalent to the second.

Note that $\varepsilon$ can be considered as an element of $C^1(H, C^1(F, k^*))$, while $\gamma$ lies naturally in $C^2(F, k^*) = C^0(H, C^2(F, k^*))$. Thus, in the terminology of the appendix, $(\varepsilon, \gamma)$ is a 2-cochain of $\check{C}^*(H, F, k^*)$. The equations (16), (17), together with the condition $d(\gamma) = \alpha|_F$, are equivalent to the coboundary condition $d(\varepsilon, \gamma) = (\alpha_2, \alpha_1, \alpha_0) = \tau(\alpha)$ in $\check{C}^*(H, F, k^*)$. The equations (19) say that $(\varepsilon, \gamma) = d(c)(\varepsilon', \gamma')$ for $c \in C^1(F, k^*) = \check{C}^1(H, F, k^*)$.

Before we describe local modules over the algebras, defined in proposition 2.15, we need to explain how the cochains $\varepsilon, \gamma$ associated with them, allow to reduce the cocycle $\alpha \in Z^3(H, k^*)$ to $\overline{\alpha} \in Z^3(H/F, k^*)$. It will be shown, in the course of the proof of theorem 2.17, that $\overline{\alpha}(x, y, z)$ defined by

$$
\alpha(s(x), s(y), s(y)^{-1}s(x)^{-1}s(xy))\gamma(\tau(y, z), \tau(x, yz))\gamma(\tau(y, z)\tau(x, yz), \tau'(x, y, z)) \\
\times \varepsilon_{s(xyz)^{-1}s(x)s(y)}(\tau(x, y))\gamma(\tau'(x, y, z)^{-1}, \tau'(x, y, z))
$$

(20)
is a 3-cocycle of $H/F$. Here $s : H/F \rightarrow H$ is a section of the quotient map $H \rightarrow H/F$, $\tau(y, z) = s(z)^{-1}s(y)^{-1}s(yz)$ and

$$
\tau'(x, y, z) = s(xy)^{-1}s(x)s(y)\tau(x, y)^{-1}s(y)^{-1}s(x)^{-1}s(xy).
$$

Theorem 2.17. The category $Z(H, \alpha|_H)_k^{loc\{F, \gamma, \varepsilon\}}$, of local right $k[F, \gamma, \varepsilon]$-modules in $Z(H, \alpha|_H)$, is equivalent, as a ribbon category, to $Z(H/F, \overline{\alpha})$.

Proof. The structure of a right $k[F, \gamma, \varepsilon]$-module on an object $M = \oplus_{h \in H} M_h$ of $Z(H, \alpha|_H)$ amounts to a collection of isomorphisms $e_f : M_h \rightarrow M_{hf}$ (right multiplication by $e_f \in k[F, \gamma, \varepsilon]$) such that

$$
e_e = I, \quad e_f e_{f'} = \gamma(f, f') e_{f'f}, \quad h e_f h^{-1} = \varepsilon_h(f) e_{hf^{-1}}, \quad f, f' \in F, h \in H.
$$

Here $h : M_{hf} \rightarrow M_{hfh^{-1}}$ is the $\alpha$-projective $H$-action on $M$. The $k[F, \gamma, \varepsilon]$-module $M$ is local if $e_f = \varepsilon_h(f) hfh^{-1} e_{hf^{-1}}$ on $M_h$. Indeed, the double braiding in $Z(H, \alpha|_H)$ transforms an element $m \otimes e_f \in M \otimes A$ (with $m \in M_h$) as follows

$$
m \otimes e_f \mapsto h(e_f) \otimes m = \varepsilon_h(f) e_{hf^{-1}} \otimes m \mapsto \varepsilon_h(f) hfh^{-1}(m) \otimes e_{hf^{-1}}.
$$

An equivalent way of expressing the locality condition is the following:

$$f = \varepsilon_h(h^{-1}fh)^{-1}\gamma(h^{-1}fh, f^{-1})\gamma(f, f^{-1})^{-1}e_{hf^{-1}} = \varepsilon_h(f) e_{[h^{-1}, f]}.
$$

Now let $s : H/F \rightarrow H$ be a section of the quotient map $H \rightarrow H/F$. For a $k[F, \gamma, \varepsilon]$-module
$M$ define a $H/F$-graded vector space $V$ by $V_x = M_{s(x)}$, where $x \in H/F$. For local $M$ the vector space $V$ can be equipped with a projective $H/F$-action: for $y \in H/F$ define $y : V_x \to V_{yxy^{-1}}$ as the composition

$$V_x = M_{s(x)} \xrightarrow{s(y)} M_{s(y)s(x)s(y)^{-1}} \xrightarrow{e_{f(x,y)}} M_{s(yxy^{-1})} = V_{yxy^{-1}},$$

where

$$f(x,y) = s(y)s(x)^{-1}s(y)^{-1}s(yxy^{-1}) = s(y)s(x)^{-1}s(yx) \in F.$$ 

To compute the projective multiplier one would need to compare $zy$ on $V_x$ with the composition of $y$ and $z$. This can be done with the help of the following diagram:

Here $\sigma(z, y) = s(z)s(y)s(zy)^{-1} \in F$ and

$$g(x, y, z) = s(z)f(x, y)f(y, x, z) = s(z)s(y)s(x)^{-1}s(zyx)$$

so that

$$[s(zy)s(x)^{-1}, \sigma(z, y)]s(z)s(y)s(x)^{-1}s(zyx)$$

coincides with

$$s(y)s(x)^{-1}s(zyx) = f(x, y).$$

The cells of the diagram commute up to multiplication by a scalar (except two top and one bottom cells, which commute on the nose). Carefully gathering the scalars one can write down the multiplier for the projective $H/F$-action on $V$. Fortunately, we do not have to do it. In view of the proposition 4.9 from the appendix, it is enough to know that such a multiplier exists.

The $H/F$-graded vector space $V \otimes U$, corresponding to the tensor product $M \otimes_{k[F, \gamma, \varepsilon]} N$
of local $B = k[F, \gamma, \varepsilon]$-modules, can be identified with the graded tensor product of $V$ and $U$. Indeed, the composition

$$\bigoplus_{y z = x} M_{s(y)} \otimes N_{s(z)} \xrightarrow{1 \otimes e_{\tau(y, x)}} \bigoplus_{y z = x} M_{f} \otimes N_{g} = (M \otimes N)_{s(x)} \xrightarrow{pr} (M \otimes_B N)_{s(x)}$$

defines an isomorphism $\bigoplus_{y z = x} V_y \otimes U_z \rightarrow (V \otimes U)_x$. Here, as before, $\tau(y, z) = s(z)^{-1}s(y)^{-1}s(yz) \in F$. Again we will use a diagrammatic language to prove the compatibility (up to a scalar) of the $H/F$-action on $V \otimes U$ with the $H/F$-actions on $V$ and $U$:

![Diagram](image_url)

Here $h(v, y, g) = s(y)gs(y)^{-1}f(v, y)^{-1}s(y)g^{-1}s(y)^{-1}$. Again, the cells of the diagram commute up to scalars (one for each $v$ and $u$), resulting in an overall factor, rescaling $y \otimes y$ into $y$ on $V \otimes U$. Note that the six vertex cell in the right half of the diagram commutes by
the following property of the projection map: for any $u \in F$ the diagram

\[
\begin{array}{ccc}
(M \otimes_B N) & \xrightarrow{pr} & (M \otimes N) \\
\oplus M_f \otimes N_g & \xrightarrow{\oplus e_u \otimes e_{g-1} \otimes u^{-1} \otimes g} & \oplus M_{f'} \otimes N_{g'} \\
\end{array}
\]

commutes up to multiplication by scalars (one for each $f$ and $g$). So far, their actual form has been unimportant, but it will become crucial in what we are going to do later. To calculate this factor, we start with the identity

\[me_u \otimes n = \varepsilon_{h^{-1}}(u)(m \otimes ne_{h^{-1}u})\]

which follows from the definition of the tensor product over $B$. Multiplying this identity with $e_{h^{-1}u-1}h$ (from the right) we get

\[me_u \otimes ne_{h^{-1}u-1} = \varepsilon_{h^{-1}}(u)\gamma(h^{-1}u, h^{-1}u-1h)(m \otimes n)\]

The last step we need to make is to calculate the associator on $V \otimes (U \otimes W)$ and to prove that it is scalar on $V_x \otimes (U_y \otimes W_z)$. Once again we apply diagrammatic technique:
Here $a$ stands for the multiplication by $\alpha(s(x), s(y), s(y)^{-1}s(x)^{-1}s(xyz))$,

$$* = \varepsilon_{s(xyz)^{-1}s(x)s(y)}(\tau(x, y))\gamma(\tau'(x, y, z)^{-1}, \tau'(x, y, z))(1 \otimes \varepsilon_{\tau(x, y)} \otimes \varepsilon_{\tau'(x, y, z)}),$$

and again

$$\tau'(x, y, z) = s(xyz)^{-1}s(x)s(y)s(y)^{-1}s(x)^{-1}s(xyz).$$

Now, composing the arrows of the top cell and comparing the coefficients gives the formula (20). Finally, by proposition 4.9 from the Appendix, the category $\mathcal{Z}(H, \alpha|_H)_{\text{loc}}^{k[F, \gamma, \varepsilon]}$ is ribbon equivalent to $\mathcal{Z}(H/F, \overline{\alpha})$.

**Corollary 2.18.** An etale algebra $B = k[F, \gamma, \varepsilon]$ in the category $\mathcal{Z}(H, \alpha|_H)$ is Lagrangian if and only if $F = H$.

**Proof.** By theorem 2.17, an etale algebra $B = k[F, \gamma, \varepsilon]$ in $\mathcal{Z}(H, \alpha|_H)$ is Lagrangian if and only if $\mathcal{Z}(H/F, \overline{\alpha}) \simeq k\text{-Vect}$, i.e. if and only if the quotient group $H/F$ is trivial. □

Note that for $F = H$, $\varepsilon$ is determined by $\gamma$ by the equation (18).

### 2.5 Etale algebras and their local modules

In this section we combine the previous results on etale algebras in group-theoretical modular categories and on their local modules.

Define

$$A(H, F, \gamma, \varepsilon) = E(k[F, \gamma, \varepsilon]),$$

where $E : \mathcal{Z}(H, \alpha|_H) \to \mathcal{Z}(G, \alpha)_{k[G/H]}$ is the functor from the proof of theorem 2.11.

**Theorem 2.19.** An etale algebra in $\mathcal{Z}(G, \alpha)$ has the form $A(H, F, \gamma, \varepsilon)$, where $H \subset G$ is a subgroup, $F \triangleleft H$ is a normal subgroup, $\gamma \in C^2(F, k^*)$ is a coboundary $d(\gamma) = \alpha|_F$ and $\varepsilon : H \times F \to k^*$ satisfies the conditions (16,17,18).

**Proof.** Follows from corollary 2.13 and lemma 1.2. □

**Theorem 2.20.** The category $\mathcal{Z}(G, \alpha)_{A(H, F, \gamma, \varepsilon)}^{\text{loc}}$ of local right $A(H, F, \gamma, \varepsilon)$-modules in $\mathcal{Z}(G, \alpha)$, is equivalent, as a ribbon category, to $\mathcal{Z}(H/F, \overline{\alpha})$.

**Proof.** Follows from theorems 2.11 and 2.17, and lemma 1.2. □

Note that when $F = H$ the function $\varepsilon$ is completely determined by $\gamma$. Indeed, by the commutativity condition (18), one has

$$\varepsilon_f(g) = \frac{\gamma(f, g)}{\gamma(fgf^{-1}, f)}.$$

Denote $A(H, H, \gamma, \varepsilon)$ by $L(H, \gamma)$. Theorems 2.19 and 2.20 allow us to describe Lagrangian algebras in group-theoretical modular categories in purely group-theoretical terms.
Corollary 2.21. A Lagrangian algebra \(L \in \mathcal{Z}(G, \alpha)\) has the form \(L(H, \gamma)\), where \(H \subset G\) is a subgroup and \(\gamma \in C^2(H, k^*)\) is a coboundary \(d(\gamma) = \alpha|_H\).

Proof. Follows from corollary 2.18 and theorems 2.19 and 2.20. \(\square\)

Remark 2.22. A Lagrangian algebra \(L = L(H, \gamma)\) is completely characterised by the conditions
\[
L_e \simeq k(G/H), \quad D(L) = k[H, \gamma] \in \mathcal{C}(G, H, \alpha).
\]
Here \(D : \mathcal{Z}(G, \alpha)_{k(G/H)} \to \mathcal{C}(G, H, \alpha)\) is the functor from the proof of theorem 2.11.

Remark 2.23. It follows from the corollary 2.21 that Lagrangian algebras in \(\mathcal{Z}(G, \alpha) \boxtimes \mathcal{Z}(G, \alpha^{-1})\) correspond to pairs \((U, \gamma)\), where \(U \subset G \times G\) is a subgroup and \(\gamma \in C^2(U, k^*)\) is a coboundary \(d(\gamma) = (\alpha \times \alpha^{-1})|_U\). This coincides with the parametrisation of module categories obtained in [11], which illustrates the fact that the total centre defines a bijection between equivalence classes of indecomposable module categories over \(\mathcal{Z}(G, \alpha)\) and Lagrangian algebras in \(\mathcal{Z}(G, \alpha) \boxtimes \mathcal{Z}(G, \alpha^{-1})\).

3 Full centre

Here we show that Lagrangian algebras in \(\mathcal{Z}(G, \alpha)\) are full centres of separable indecomposable algebras in \(\mathcal{V}(G, \alpha)\).

3.1 Monoidal centre of \(\mathcal{V}(G, \alpha)\)

Denote by \(\mathcal{V}(G, \alpha)\) the category of \(G\)-graded vector spaces with the ordinary tensor product:
\[
(V \otimes U)_g = \bigoplus_{f h = g} V_f \otimes U_h
\]
and the associativity constraint \(\phi_{V,U,W} : V \otimes (U \otimes W) \to (V \otimes U) \otimes W\), twisted by a 3-cocycle \(\alpha \in Z^3(G, k^*)\):
\[
\phi_{V,U,W}(v \otimes (u \otimes w)) = \alpha(f, g, h)(v \otimes u) \otimes w, \quad \forall v \in V_f, u \in U_g, w \in W_h.
\]

Clearly, \(\mathcal{V}(G, \alpha)\) is a fusion category with the set of simple objects \(\text{Irr} (\mathcal{V}(G, \alpha)) = G\). We denote by \(I(g)\) the simple object corresponding to \(g \in G\): the one dimensional graded vector space concentrated in degree \(g\).

Here we describe the monoidal centre of \(\mathcal{V}(G, \alpha)\).

Proposition 3.1. The monoidal centre \(\mathcal{Z}(\mathcal{V}(G, \alpha))\) is isomorphic, as braided monoidal category, to the category \(\mathcal{Z}(G, \alpha)\).

The canonical forgetful functor \(\mathcal{Z}(\mathcal{V}(G, \alpha)) \to \mathcal{V}(G, \alpha)\) corresponds to the functor \(\mathcal{Z}(G, \alpha) \to \mathcal{V}(G, \alpha)\) forgetting the \(G\)-action.
Proof. For an object \((X, x)\) of the centre \(Z(\mathcal{V}(G, \alpha))\), the natural isomorphism
\[
x_V : V \otimes X \to X \otimes V, \quad V \in \mathcal{V}(G, \alpha)
\]
is defined by its evaluations on one-dimensional graded vector spaces. The isomorphism \(x_{I(f)}\) can be seen as an automorphism \(f : X \to X\). The fact, that \(x_{I(f)}\) preserves grading, amounts to the condition \(f(X_g) = X_{fg^{-1}}\):
\[
X_g = (I(f) \otimes X)_{fg} \xrightarrow{x_{I(f)}} (X \otimes I(f))_{fg} = X_{fg^{-1}}.
\]
The coherence condition for \(x\) is equivalent to the equation \((5)\), with the associativity constraints giving rise to \(\alpha(h|f, g)\). The diagram, defining the second component \(x|y\) of the tensor product \((X,x) \otimes (Y,y) = (X \otimes Y,x|y)\), is equivalent to the tensor product of projective actions \((9)\), with the associativity constraints giving rise to \(\alpha(g,h|f)\).
The description of the monoidal unit in a monoidal centre corresponds to the answer for the monoidal unit in \(Z(G, \alpha)\).
Clearly, the braiding \(c_{(X,x),(Y,y)} = y_X\) of the monoidal centre \(Z(\mathcal{V}(G, \alpha))\) corresponds to the braiding \((10)\) of \(Z(G, \alpha)\).

3.2 Full centre

Here we identify Lagrangian algebras in \(Z(G, \alpha)\) with full centres of separable indecomposable algebras in \(\mathcal{V}(G, \alpha)\). Recall that any such algebra is isomorphic to the twisted group algebra \(k[H, \gamma]\) for a subgroup \(H \subset G\) and a coboundary \(d(\gamma) = \alpha|_H\).

**Theorem 3.2.** The full centre \(Z(k[H, \gamma])\) coincides with \(L(H, \gamma)\).

**Proof.** It suffices to construct a homomorphism of algebras \(\zeta : L(H, \gamma) \to k[H, \gamma]\) in \(\mathcal{V}(G, \alpha)\) fitting in the diagram \((2)\). Indeed, such a homomorphism induces a homomorphism of algebras \(\tilde{\zeta} : L(H, \gamma) \to Z(k[H, \gamma])\) in \(Z(G, \alpha)\). Since \(L(H, \gamma)\) is separable and indecomposable in \(Z(G, \alpha)\), by lemma 1.1, \(\zeta\) is a monomorphism. Finally, the dimension of \(Z(G, \alpha)\) is \(|G|\), which coincides with the dimension of the full centre \(Z(k[H, \gamma])\).

Let us define a map \(\zeta : E(k[H, \gamma]) \to k[H, \gamma]\) by \(\zeta(a) = a(e)\). Observe that this definition, effectively, implies that \(\zeta(a) = 0\) if \(|a| \in G \setminus H\). We claim that \(\zeta\) is a homomorphism of algebras in the category \(\mathcal{V}(G, \alpha)\). As \(\zeta\) is an evaluation map, it is automatically multiplicative. It remains to check that \(\zeta\) is \(G\)-graded. Recall from the proof of theorem 2.11 that for a homogeneous \(a \in E(k[H, \gamma])\) the degree of the values are \(|a(x)| = x^{-1}|a|x\) \(\forall x \in G\). Thus \(|\zeta(a)| = |a(e)| = |a|\).

Now we consider the diagram \((2)\). Here, \(A = k[H, \gamma]\) and \(Z = L(H, \gamma)\), so that the
Let \( a \in L(H, \gamma) \), and let \( e_h \in k[H, \gamma] \) for some \( h \in H \). Going the short way, we obtain

\[
e_h \otimes a \mapsto e_h \otimes a(e) \mapsto e_h a(e).
\]

Going the long way, we obtain

\[
e_h \otimes a \mapsto (h.a) \otimes e_h \mapsto ((h.a)(e)) \otimes e_h \mapsto ((h.a)(e)) e_h.
\]

By \( H \)-equivariance of \( a \), the last expression is

\[
((h.a)(e)) e_h = a(h^{-1}) e_h = \left( h \cdot (a(e)) \right) e_h
\]

Finally, since the \( H \)-action on \( k[H, \gamma] \) is inner, one has

\[
\left( h \cdot (a(e)) \right) e_h = e_h a(e) e_h^{-1} e_h.
\]

\[\square\]

4 Characters of Lagrangian algebras

4.1 Characters

Here we recall basic facts about characters of objects of \( Z(G, \alpha) \).
For an object \( Z \in Z(G, \alpha) \), define its character as the function on pairs of commuting elements of \( G \) defined by

\[
\chi_Z(f, g) = \text{Tr}_{Z_f}(g).
\]

Lemma 4.1. The character \( \chi_Z \) of \( Z \in Z(G, \alpha) \) satisfies

\[
\chi_Z(xf^{-1}, xgx^{-1}) = \frac{\alpha(x, g|x|f)}{\alpha(xgx^{-1}, x|f)} \chi_Z(f, g).
\]

(23)
Proof. Let $\rho : G \to \text{Aut}(Z)$ be the homomorphism corresponding to the action of $G$ on the vector space $Z$. Note that by (5) we can write
\[
\rho(x)\rho(g)\rho(x)^{-1} = \frac{\alpha(xgx^{-1}, x|f)}{\alpha(x, g|f)} \rho(xgx^{-1}),
\]
Indeed, $\rho(x)\rho(g) = \alpha(x, g|f)^{-1} \rho(xg)$ together with $\rho(xgx^{-1}) \rho(x) = \alpha(xgx^{-1}, x|f)^{-1} \rho(xg)$ give the desired.

Finally
\[
\chi_Z(f, g) = \text{Tr}_{Z_f} (\rho(g)) = \text{Tr}_{Z_{xf^{-1}}} (\rho(x)\rho(g)\rho(x)^{-1}) = \frac{\alpha(xgx^{-1}, x|f)}{\alpha(x, g|f)} \chi_Z(xfx^{-1}, xgx^{-1}).
\]

By a character of $\mathcal{Z}(G, \alpha)$, we will mean a function on pairs of commuting elements of $G$ satisfying the projective class function property (23).

**Remark 4.2.** Equation (23) implies that a character of $\mathcal{Z}(G, \alpha)$ can be non-zero only on those commuting pairs $(f, g)$ for which
\[
\frac{\alpha(x, f, g)\alpha(g, x, f)\alpha(f, g, x)}{\alpha(x, g, f)\alpha(f, x, g)\alpha(g, f, x)} = 1
\]
for any $x$ from the centraliser $C_G(f, g)$.

For characters $\chi$ and $\xi$ of $\mathcal{Z}(G, \alpha)$, define their product by
\[
(\chi \xi)(f, g) = \sum_{f_1, f_2 = f} \alpha(g|f_1, f_2) \chi(f_1, g) \xi(f_2, g).
\]
It can be checked that the product of characters is a character.

**Lemma 4.3.** Let $Z, W \in \mathcal{Z}(G, \alpha)$, then
\[
\chi_Z \chi_W = \chi_{Z \otimes W}.
\]

**Proof.** Write
\[
\chi_{Z \otimes W}(f, g) = \text{Tr}_{(Z \otimes W)_f} (g) = \sum_{f_1, f_2 = f} \text{Tr}_{Z_{f_1} \otimes W_{f_2}} (g).
\]
Using (9), we get
\[
\sum_{f_1, f_2 = f} \text{Tr}_{Z_{f_1} \otimes W_{f_2}} (g) = \sum_{f_1, f_2 = f} \alpha(g|f_1, f_2) \text{Tr}_{Z_{f_1}} (g) \text{Tr}_{W_{f_2}} (g) = \sum_{f_1, f_2 = f} \alpha(g|f_1, f_2) \chi_Z(f_1, g) \chi_W(f_2, g) = (\chi_Z \chi_W)(f, g).
\]
\[\square\]
Remark 4.4. Using lemma 2.2, one finds that the character of the dual object (the dual character) has the form

$$
\chi_{Z^\vee}(f, g) = \frac{\alpha(g^{-1}, g|f^{-1})}{\alpha(g|f, f^{-1})} \chi_Z(f^{-1}, g^{-1}).
$$

Define the scalar product of characters of $Z(G, \alpha)$ (see [1]):

$$
(\chi, \psi) = \frac{1}{|G|} \sum_{f, g \in G, \text{fg} = \text{gf}} \alpha(g^{-1}, g|f) \chi(f^{-1}, g^{-1}) \psi(f, g),
$$

The scalar product calculates dimensions of corresponding Hom-spaces in $Z(G, \alpha)$.

Lemma 4.5. Let $Z, W \in Z(G, \alpha)$. Then

$$
(\chi_Z, \chi_W) = \dim(Z(G, \alpha)(Z, W)).
$$

Proof. Note first that the Hom-space $Z(G, \alpha)(I, W)$ coincides with the vector space of $G$-invariants $W^G_e$, so that

$$
\dim(Z(G, \alpha)(I, W)) = \dim_k(W^G_e) = \frac{1}{|G|} \sum_g \chi_W(e, g).
$$

In the general case, $Z(G, \alpha)(Z, W) \simeq Z(G, \alpha)(I, Z^\vee \otimes W)$, and

$$
\dim(Z(G, \alpha)(Z, W)) = \dim(Z(G, \alpha)(I, Z^\vee \otimes W)) = \frac{1}{|G|} \sum_g \chi_{Z^\vee \otimes W}(e, g) =
$$

$$
= \frac{1}{|G|} \sum_{f, g \in G, \text{fg} = \text{gf}} \alpha(g|f^{-1}, f) \chi_{Z^\vee}(f^{-1}, g) \chi_W(f, g) = \frac{1}{|G|} \sum_{f, g \in G, \text{fg} = \text{gf}} \alpha(g|f^{-1}, f) \frac{\alpha(g^{-1}, g|f^{-1})}{\alpha(g|f, f^{-1})} \chi_Z(f, g^{-1}) \chi_W(f, g) =
$$

$$
= \frac{1}{|G|} \sum_{f, g \in G, \text{fg} = \text{gf}} \alpha(g^{-1}, g|f) \chi_Z(f, g^{-1}) \chi_W(f, g).
$$

In particular, for simple $Z, W \in Z(G, \alpha)$ the scalar product $(\chi_Z, \chi_W) = 1$ iff $Z \simeq W$ and zero otherwise.

4.2 Characters of Lagrangian algebras

Here we compute the characters of Lagrangian algebras in $Z(G, \alpha)$.

Recall from the proof of theorem 2.11 the functor $E: Z(H, \alpha|_H) \to Z(G, \alpha)$ given by (15).
Lemma 4.6. For \( V \in \mathcal{Z}(H, \alpha|_H) \), the character \( \chi_E(V) \) has the form

\[
\chi_E(V)(f,g) = \sum_{y \in Y} \frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)} \chi_V(y^{-1}fy, y^{-1}gy), \quad f, g \in G,
\]

where \( Y \) is a set of representatives in \( G \) of the cosets from

\[
\{ y \in G \mid y^{-1}fy \in H, \ y^{-1}gy \in H \}/H \subset G/H.
\]

Proof. Let \( V \in \mathcal{Z}(H, \alpha|_H) \). It follows from the defining condition that functions from \( E(V) \) are supported by unions of right \( H \)-cosets of \( G \). Clearly, any function from \( E(V) \) is (a unique) sum of functions, each of which is supported on a single coset. For a coset \( yH \), a function \( a \in E(V) \) with the support \( \text{supp}(a) = yH \) is uniquely determined by its value \( a(y) = v \in V \). Denote such a function by \( a_{y,v} \). The space \( E(V) \) is spanned by \( a_{y,v} \) for \( y \in G \) and \( v \in V \).

By (14) the degree of \( a_{y,v} \) is \( f \) iff \( v \in V_{y^{-1}fy} \). Note that for \( v \in V_{h^{-1}y^{-1}fy} \) one has

\[
\alpha(h^{-1}, y^{-1}|f) \alpha(h, h^{-1}|y^{-1}fy) a_{y,h,v} = a_{y,h,v}, \quad h \in H.
\]

Indeed, \( v = a_{y,h,v}(y) = (h^{-1}, y^{-1}|f) h^{-1}. a_{y,v}(y) \) gives

\[
a_{y,h,v}(y) = h.v = \alpha(h^{-1}, y^{-1}|f) \alpha(h, h^{-1}|y^{-1}fy) a_{y,h,v}(y).
\]

Thus we can write

\[
E(V)_f = \bigoplus_{y} \langle a_{y,v} \mid v \in V_{y^{-1}fy} \rangle \simeq \bigoplus_{y} V_{y^{-1}fy},
\]

where the sum is taken over representatives of the cosets \( G/H \).

For \( g \in C_G(f) \), consider the linear operator \( g : E(V)_f \to E(V)_f \). Recall (from the proof of theorem 2.11) that the action of \( g \) on \( E(V) \) is given by

\[
(g.a)(x) = \alpha(x^{-1}, g|f)^{-1} a(g^{-1}x), \quad x \in G, \ a \in E(V)_f.
\]

In particular \( \text{supp}(g.a) = g \text{ supp}(a) \). Thus \( g.a_{y,v} \) is supported by the coset \( gyH \) and hence can be written as \( a_{gy,v'} \) for some \( v' \in V \). We can say that the action of \( g \) on \( E(V)_f \) permutes the direct summands of (25) according to the left action of \( g \) on the cosets \( G/H \).

Thus the trace \( \text{Tr}_{E(V)_f}(g) \) is a sum

\[
\sum_{y} \text{Tr}_{\langle a_{y,v} \mid v \in V_{y^{-1}fy} \rangle}(g), \quad \text{where \( y \) runs through representatives of those cosets \( yH \) for which \( gyH = yH \).}
\]

Let now \( y \in G \) be such that \( gyH = yH \) (or equivalently \( y^{-1}gy \in H \)). To compute the trace \( \text{Tr}_{\langle a_{y,v} \mid v \in V_{y^{-1}fy} \rangle}(g) \) note that \( g.a_{y,v} \) (with \( v \in V_{y^{-1}fy} \)) can be written as \( a_{y,v'} \) for some \( v' \in V \). More explicitly

\[
v' = g.a_{y,v}(y) = (y^{-1}, g|f)^{-1} a_{y,v}(g^{-1}y) = \alpha(y^{-1}, g|f)^{-1} a_{y,v}(yy^{-1}g^{-1}y) = \]

\[
= \frac{\alpha(y^{-1}gy, y^{-1}|f)}{\alpha(y^{-1}, g|f)} (y^{-1}gy). a_{y,v}(y) = \frac{\alpha(y^{-1}gy, y^{-1}|f)}{\alpha(y^{-1}, g|f)} (y^{-1}gy). v.
\]
Thus the trace \( \text{Tr}_{\langle a_y, v \rangle \in V_{y^{-1}f_y}} (g) \) coincides with \( \frac{\alpha(y^{-1}g_y, y^{-1}|f)}{\alpha(y^{-1}, g|f)} \text{Tr}_{V_{y^{-1}f_y}} (y^{-1}g_y) \).

Finally
\[
\chi_{E(V)} (f, g) = \text{Tr}_{E(V)} (g) = \sum_{y \in Y} \frac{\alpha(y^{-1}g_y, y^{-1}|f)}{\alpha(y^{-1}, g|f)} \text{Tr}_{V_{y^{-1}f_y}} (y^{-1}g_y) = \\
= \sum_{y \in Y} \frac{\alpha(y^{-1}g_y, y^{-1}|f)}{\alpha(y^{-1}, g|f)} \chi_V (y^{-1}f_y, y^{-1}g_y).
\]

Lemma 4.7. Let \( \gamma \in C^2(H, k^*) \) be a coboundary for \( \alpha|_H \). The \( Z(H, \alpha|_H) \) character \( \chi_{k[H, \gamma]} \) of the twisted group algebra \( k[H, \gamma] \) is given by

\[
\chi_{k[H, \gamma]} (h, u) = \frac{\gamma(u, h)}{\gamma(h, u)}.
\]

Proof. For any \( h \in H \), the degree-\( h \) component \( k[H, \gamma]_h \) is one-dimensional and has the form

\[
k[H, \gamma]_h = ke_h.
\]

According to (22)
\[
u.e_h = \frac{\gamma(u, h)}{\gamma(uh^{-1}, h)} e_uh^{-1}, \quad u \in H.
\]

For \( u \in C_H (h) \), this becomes
\[
u.e_h = \frac{\gamma(u, h)}{\gamma(h, u)} e_h.
\]

Now we are ready to prove the main result of this section.

Theorem 4.8. Let \( L = L(H, \gamma) \) be a Lagrangian algebra in \( Z(G, \alpha) \). Then the character \( \chi_L \) has the form

\[
\chi_L (f, g) = \sum_{y \in Y} \frac{\alpha(y^{-1}g_y, y^{-1}|f)}{\alpha(y^{-1}, g|f)} \gamma(y^{-1}f_y, y^{-1}g_y), \quad f, g \in G, \quad (26)
\]

where \( Y \) is a set of representatives in \( G \) of the cosets from

\[
\{ y \in G | y^{-1}f_y \in H, \ y^{-1}g_y \in H \} / H \subset G / H.
\]

Proof. Follows from lemmas 4.6 and 4.7.
4.3 Example: \( \mathcal{Z}(D_3, \alpha) \)

Here \( D_3 \) is the dihedral group of order 6: \( D_3 = \langle r, s \mid r^3 = s^2 = e; srs = r^{-1} \rangle \).

It is known that the third cohomology group \( H^3(D_3, \mathbb{C}^*) \) is cyclic of order 6 (see e.g. [2, section 6.3]). More explicitly, the cohomology group \( H^3(D_3, \mathbb{C}^*) \) is generated by the class of the 3-cocycle \( \theta \) given by

\[
\theta(s^{m_1}r^{n_1}, s^{m_2}r^{n_2}, s^{m_3}r^{n_3}) =
\exp \left( \frac{2\pi i}{9} \left( (-1)^{m_2+m_3}n_1((-1)^{m_3}n_2 + n_3 - [(-1)^{m_3}n_2 + n_3]_3) + \frac{9}{2}m_1m_2m_3 \right) \right),
\]

where \([ \ ]_3\) denotes taking the residue modulo 3.

Recall that the second cohomology group \( H^2(D_3, \mathbb{C}^*) \) is trivial. This implies that a coboundary for the restriction \( \alpha|_H \) of a 3-cocycle \( \alpha \in Z^3(D_3, \mathbb{C}^*) \) to a subgroup \( H \subset G \) is unique if exists. Thus Lagrangian algebras in \( \mathcal{Z}(D_3, \alpha) \) are labelled just by (conjugacy classes of) subgroups on which the restriction of \( \alpha \) is cohomologically trivial (admissible subgroups).

In terms of the cocycle \( \alpha \), there are four distinct cases, depending on the order of the class of \( \alpha \) in \( H^3(D_3, \mathbb{C}^*) \).

**The case of trivial \( \alpha \)**

In this case \( \mathcal{Z}(D_3, \alpha) = \mathcal{Z}(D_3) \). The character table for \( \mathcal{Z}(D_3) \) is

| \( \chi \) | \( (e, e) \) | \( (e, r) \) | \( (e, s) \) | \( (r, e) \) | \( (r, r) \) | \( (r, r^2) \) | \( (s, e) \) | \( (s, s) \) |
|---|---|---|---|---|---|---|---|---|
| \( \chi_0 \) | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_1 \) | 1 | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_2 \) | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_3 \) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| \( \chi_4 \) | 0 | 0 | 0 | 1 | \( \omega \) | \( \omega^{-1} \) | 0 | 0 |
| \( \chi_5 \) | 0 | 0 | 0 | 1 | \( \omega^{-1} \) | \( \omega \) | 0 | 0 |
| \( \chi_6 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| \( \chi_7 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 |

Here, \( \omega \) is a primitive third root of unity.

All subgroups are admissible. Up to conjugation there are four subgroups

\( \{ e \}, \quad C_2 = \langle s \rangle, \quad C_3 = \langle r \rangle, \quad D_3. \)

The characters of the corresponding Lagrangian algebras are

| \( \chi_{L(e)} \) | \( (e, e) \) | \( (e, r) \) | \( (e, s) \) | \( (r, e) \) | \( (r, r) \) | \( (r, r^2) \) | \( (s, e) \) | \( (s, s) \) |
|---|---|---|---|---|---|---|---|---|
| \( \chi_{L(e)} \) | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{L(C_2)} \) | 3 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| \( \chi_{L(C_3)} \) | 2 | 2 | 0 | 2 | 2 | 2 | 0 | 0 |
| \( \chi_{L(D_3)} \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
which give the decompositions into irreducible characters
\[ \chi_L(\{e\}) = \chi_0 + \chi_1 + 2\chi_2 , \]
\[ \chi_L(C_2) = \chi_0 + \chi_2 + \chi_6 , \]
\[ \chi_L(C_3) = \chi_0 + \chi_1 + 2\chi_3 , \]
\[ \chi_L(D_3) = \chi_0 + \chi_3 + \chi_6 . \]

**The case of \( \alpha \) of order two.**
In this case, \( \alpha = \theta^3 \). The character table for \( \mathbb{Z}(D_3, \theta^3) \) is

|       | \((e, e)\) | \((e, r)\) | \((e, s)\) | \((r, e)\) | \((r, r)\) | \((r, r^2)\) | \((s, e)\) | \((s, s)\) |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|
| \(\chi_0\) | 1          | 1          | 1          | 0          | 0          | 0          | 0          | 0          |
| \(\chi_1\) | 1          | 1          | -1         | 0          | 0          | 0          | 0          | 0          |
| \(\chi_2\) | 2          | -1         | 0          | 0          | 0          | 0          | 0          | 0          |
| \(\chi_3\) | 0          | 0          | 0          | 1          | 1          | 1          | 0          | 0          |
| \(\chi_4\) | 0          | 0          | 0          | 1          | \(\omega\) | \(\omega^{-1}\) | 0          | 0          |
| \(\chi_5\) | 0          | 0          | 0          | 1          | \(\omega^{-1}\) | \(\omega\) | 0          | 0          |
| \(\chi_6\) | 0          | 0          | 0          | 0          | 0          | 0          | 1          | \(\varepsilon\) |
| \(\chi_7\) | 0          | 0          | 0          | 0          | 0          | 0          | 1          | \(-\varepsilon\) |

Here, \(\omega\) and \(\varepsilon\) are primitive third and fourth roots of unity, respectively.
The admissible subgroups are \(\{e\}\) and \(C_3\). The characters of the corresponding Lagrangian algebras are

|       | \((e, e)\) | \((e, r)\) | \((e, s)\) | \((r, e)\) | \((r, r)\) | \((r, r^2)\) | \((s, e)\) | \((s, s)\) |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|
| \(\chi_L(\{e\})\) | 4          | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
| \(\chi_L(C_3)\) | 2          | 2          | 0          | 2          | 1 + \(\omega^{-1}\) | 1 + \(\omega\) | 0          | 0          |

which give the decompositions into irreducible characters
\[ \chi_L(\{e\}) = \chi_0 + \chi_1 + 2\chi_2 , \]
\[ \chi_L(C_3) = \chi_0 + \chi_1 + \chi_3 + \chi_5 , \]

**The case of \( \alpha \) of order three.**
In this case, \( \alpha = \theta^2 \). The character table for \( \mathbb{Z}(D_3, \theta^2) \) is

|       | \((e, e)\) | \((e, r)\) | \((e, s)\) | \((r, e)\) | \((r, r)\) | \((r, r^2)\) | \((s, e)\) | \((s, s)\) |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|
| \(\chi_0\) | 1          | 1          | 1          | 0          | 0          | 0          | 0          | 0          |
| \(\chi_1\) | 1          | 1          | -1         | 0          | 0          | 0          | 0          | 0          |
| \(\chi_2\) | 2          | -1         | 0          | 0          | 0          | 0          | 0          | 0          |
| \(\chi_3\) | 0          | 0          | 0          | 1          | \(\eta^1\) | \(\eta^2\) | 0          | 0          |
| \(\chi_4\) | 0          | 0          | 0          | 1          | \(\eta^4\) | \(\eta^8\) | 0          | 0          |
| \(\chi_5\) | 0          | 0          | 0          | 1          | \(\eta^7\) | \(\eta^3\) | 0          | 0          |
| \(\chi_6\) | 0          | 0          | 0          | 0          | 0          | 0          | 1          | 1          |
| \(\chi_7\) | 0          | 0          | 0          | 0          | 0          | 0          | 1          | \(-1\)     |
Here \( \eta \) is a primitive ninth root of unity. The admissible subgroups are \( \{ e \} \) and \( C_2 \). The characters of the corresponding Lagrangian algebras are

\[
\begin{array}{c|cccccccc}
\chi_{L(\{ e \})} & (e, e) & (e, r) & (e, s) & (r, e) & (r, r) & (r, r^2) & (s, e) & (s, s) \\
\hline
\chi_{L(\{ e \})} & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\chi_{L(C_2)} & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

which give the decompositions into irreducible characters

\[
\begin{align*}
\chi_{L(\{ e \})} &= \chi_0 + \chi_1 + 2\chi_2 \\
\chi_{L(C_2)} &= \chi_0 + \chi_2 + \chi_6
\end{align*}
\]

The case of \( \alpha \) of order six.

In this case, \( \alpha = \theta \). The character table for \( \mathcal{Z}(D_3, \theta) \) is

\[
\begin{array}{c|cccccccc}
\mathcal{Z}(D_3, \theta) & (e, e) & (e, r) & (e, s) & (r, e) & (r, r) & (r, r^2) & (s, e) & (s, s) \\
\hline
\chi_0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\chi_1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\chi_2 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\chi_3 & 0 & 0 & 0 & 1 & \eta & \eta^2 & 0 & 0 \\
\chi_4 & 0 & 0 & 0 & 1 & \eta^4 & \eta^8 & 0 & 0 \\
\chi_5 & 0 & 0 & 0 & 1 & \eta^7 & \eta^3 & 0 & 0 \\
\chi_6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \varepsilon \\
\chi_7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\varepsilon \\
\end{array}
\]

Here again, \( \varepsilon \) and \( \eta \) are primitive fourth and ninth roots of unity, respectively. Only the trivial subgroup \( \{ e \} \) is admissible. The character of the corresponding Lagrangian algebra is

\[
\begin{array}{c|cccccccc}
\chi_{L(\{ e \})} & (e, e) & (e, r) & (e, s) & (r, e) & (r, r) & (r, r^2) & (s, e) & (s, s) \\
\hline
\chi_{L(\{ e \})} & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

with the decomposition into irreducible characters

\[
\chi_{L(\{ e \})} = \chi_0 + \chi_1 + 2\chi_2
\]

Appendix A. Certain monoidal categories associated with finite groups and Hochschild-Serre spectral sequence.

Let \( G \) be a finite group and let \( F \) be another finite group, acting on \( G \) by group automorphisms. Let \( \gamma : F \times F \times G \to k^* \) be a function, satisfying

\[
\gamma(f, gh|x)\gamma(g, h|x) = \gamma(f, h|x)\gamma(f, g|h(x)), \quad f, g, h \in F, \; x \in G
\] (27)
and the normalisation condition:

\[ \gamma(e, g|x) = \gamma(f, e|x) = \gamma(f, g|e) = 1. \]

Define the category \( \mathcal{C}(F, G, \gamma) \), whose objects are (finite-dimensional) \( G \)-graded vector spaces \( V = \bigoplus_{x \in G} V_x \), equipped with \( F \)-action

\[ f : V \to V, \ f(V_x) = V_{f(x)}, \]

which is \( \gamma \)-projective

\[ (fg)(v) = \gamma(f, g|x)f(g(v)), \quad x \in V_x. \]

Morphisms are graded, action preserving maps.

Now let \( \beta : F \times G \times G \to k^* \) be a normalised function, satisfying

\[ \beta(fg|x, y)\gamma(f, g|x)\gamma(f, g|y) = \gamma(f, g|xy)\beta(g|y, y)\beta(f|g(x), g(y)). \] (28)

Define a tensor product in the category \( \mathcal{C}(F, G, \gamma) \) by

\[ (U \otimes V)_z = \bigoplus_{xy=z} U_x \otimes V_y, \]

with the \( F \)-action

\[ f(u \otimes v) = \beta(f|x, y)f(u) \otimes f(v). \]

The condition (28) implies that this action is \( \gamma \)-projective.

Finally, a normalised function \( \alpha : G \times G \times G \to k^* \), satisfying

\[ \alpha(x, y, z)\beta(f|xy, z)\beta(f|x, y) = \beta(f|x, yz)\beta(f|y, z)\alpha(f(x), f(y), f(z)) \] (29)

and a 3-cocycle condition

\[ \alpha(y, z, w)\alpha(x, yz, w)\alpha(x, y, z) = \alpha(x, y, zw)\alpha(xy, z, w). \] (30)

Then the map \( \alpha : U \otimes (V \otimes W) \to (U \otimes V) \otimes W \)

\[ u \otimes (v \otimes w) \mapsto \alpha(x, y, z)(u \otimes v) \otimes w, \quad u \in U_x, v \in V_y, w \in W_z \]

is a morphism in the category \( \mathcal{C}(F, G, \gamma) \) (with condition (29 implying its \( F \)-equivariance), satisfying the pentagon axiom (by condition (30)).

We denote by \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) the category \( \mathcal{C}(F, G, \gamma) \) with the monoidal structure defined by \( \beta \) and \( \alpha \).

Representation categories and categories of graded vector spaces fit in the series of categories \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) as extreme cases, with (categories of modules in) group-theoretical modular categories appearing as intermediate examples. Indeed, for \( G = \{e\}, \gamma = \beta = \alpha = 1 \), the category \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) is the category \( \text{Rep}(G) \) of representations of \( G \). If \( F = \{e\}, \gamma = \beta = 1, \mathcal{C}(F, G, \gamma, \beta, \alpha) \) is the monoidal category \( \mathcal{C}(G, \alpha) \) of \( G \)-graded vector spaces with the associator given by \( \alpha \). For \( F = G \) with conjugation action and with \( \gamma \) and
β defined by (7) and (6), the category \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) coincides with \( \mathcal{Z}(G, \alpha) \). Finally, for \( H \) being a subgroup of \( G \) again with conjugation action and with \( \gamma \) and \( \beta \) defined by (7) and (6), the category \( \mathcal{C}(H, G, \gamma, \beta, \alpha) \) coincides with the category \( \mathcal{Z}(G, \alpha)_{k(G/H)} \) of modules of the algebra \( k(G/H) \) in \( \mathcal{Z}(G, \alpha) \).

To compare different categories of the form \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) we introduce certain monoidal equivalences between them, which we call for short gauge equivalences. Let \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) and \( \mathcal{C}(F, G, \gamma', \beta', \alpha') \) be two (monoidal) categories. Let \( b : F \times G \to k^* \) be a normalised function, satisfying

\[
\beta(f|g)x \gamma(f, g|x) = \gamma'(f, g| x)b(f|x)b(f|g(x)). \tag{31}
\]

Define a monoidal structure on the functor \( T(b) : \mathcal{C}(F, G, \gamma) \to \mathcal{C}(F, G, \gamma') \), which preserves the \( G \)-grading on an object \( V \) and modifies the \( F \)-action:

\[
\tilde{f}(v) = b(f|x)f(v), \quad v \in V_x.
\]

The condition (31) implies that the new action is \( \gamma' \)-projective. Now let \( a : G \times G \to k^* \) be a normalised function such that

\[
\beta(f|xy)x \gamma(f|x,y) = \beta'(f|x,y)b(f|x)b(f|y)a(f(x), f(y)), \tag{32}
\]

\[
\alpha(x, y, z)a(x, y)a(xy, z) = a(y, z,a(x,yz)a'(x, y, z). \tag{33}
\]

Define a monoidal structure on the functor \( T(b) \):

\[
u \otimes v \mapsto b(x, y)u \otimes v, \quad u \in U_x, v \in V_y.
\]

The condition (32) implies that this map is \( F \)-equivariant, while (33) is equivalent to the coherence axiom for a monoidal functor. Denote the resulting monoidal equivalence by \( T(b, a) : \mathcal{C}(F, G, \gamma, \beta, \alpha) \to \mathcal{C}(F, G, \gamma', \beta', \alpha') \).

To describe gauge equivalence classes of categories \( \mathcal{C}(F, G, \gamma, \beta, \alpha) \) consider a double complex \( C^{p,q}(F, G, k^*) = C^p(F, C^q(G, k^*)) \). Elements of \( C^{p,q}(F, G, k^*) \) are normalised functions

\[
c : F^{\times p} \times G^{\times q} \to k^*, \quad (f_1, ..., f_p, g_1, ..., g_q) \mapsto c(f_1, ..., f_p|g_1, ..., g_q).
\]

The vertical differential \( \partial : C^{p,q} \to C^{p,q+1} \) is induced by the standard differential \( C^q(G, k^*) \to C^{q+1}(G, k^*) \):

\[
\partial(c)(f_1, ..., f_p|g_1, ..., g_{q+1}) =
\]

\[
\sum_{i=1}^{q} c(f_1, ..., f_i, f_{i+1}, ..., f_p|g_1, ..., g_{q+1}, g_i)(-1)^i c(f_1, ..., f_p|g_1, ..., g_{q+1})^{(-1)^{q+1}}.
\]

The horizontal differential \( d : C^{p,q} \to C^{p+1,q} \) is the standard differential itself:

\[
d(c)(f_1, ..., f_p+1|g_1, ..., g_q) =
\]

\[
\sum_{i=1}^{p} c(f_1, ..., f_i + 1, ..., f_p|g_1, ..., g_q)(-1)^i c(f_1, ..., f_p|f_p(g_1), ..., f_p(g_q))^{(-1)^{p+1}}.
\]
The conditions (27)-(30) can be rewritten as
\[ d(\gamma) = 1, \quad \partial(\gamma) = d(\beta), \quad \partial(\beta) = d(\alpha), \quad \partial(\alpha) = 1. \]

In other words, the collection \((\gamma, \beta, \alpha)\) is a 3-cocycle of the truncated total complex \(\tilde{C}^n(F, G, k^*) = \oplus_{p=0}^{n-1} C^{p,n-p}(F, G, k^*)\). The conditions (31)-33) say that
\[ \gamma' \cdot d(b) = \gamma, \quad \beta' \cdot d(a) = \beta \partial(b), \quad \alpha' \cdot \partial(a) = \alpha. \]

Hence \((b, c)\) is a 2-coboundary in \(\tilde{C}^*\) for the collection \((\gamma, \beta, \alpha)(\gamma', \beta', \alpha')^{-1}\). Thus gauge equivalence classes of categories of the form \(\mathcal{V}(F, G, \gamma, \beta, \alpha)\) correspond to 3-cohomology classes of \(\tilde{C}^*\). Note that \(\tilde{C}^*\) is a direct summand of the (untruncated) total complex \(C^n = \oplus_{p=0}^{n} C^{p,n-p}\), which is quasi-isomorphic to the standard complex \(C^*(F \ltimes G, k^*)\) of the crossed product of groups \(F \ltimes G\) [9]. Since the kernel of the projection \(C^* \to \tilde{C}^*\) coincides with \(C^*(F, k^*)\) we have an isomorphism
\[ H^n(\tilde{C}) \oplus H^n(F, k^*) \cong H^n(F \ltimes G, k^*). \]

In particular, when \(F = G\), acting on itself by conjugation, the crossed product of groups \(G \ltimes G\) is isomorphic to the product \(G \times G\). In this case the cohomology of \(C^*\) and \(\tilde{C}^*\) are
\[ H^n(C) = \oplus_{p=0}^{n} H^p(G, H^{n-p}(G, k^*)), \quad H^n(\tilde{C}) = \oplus_{p=0}^{n-1} H^p(G, H^{n-p}(G, k^*)). \]

A cochain map \(\tau : C^n(G, k^*) = C^0(G, C^n(G, k^*)) \to \tilde{C}^n\) giving the splitting in cohomology \(H^n(G, k^*) = H^0(G, H^n(G, k^*)) \to H^n(\tilde{C})\) was constructed in [9] and has the following form:
\[ \tau(\alpha) = \sum_{p=0}^{n-1} \alpha_p, \]
where \(\alpha_p \in C^p(G, C^{n-p}(G, k^*))\) is
\[ \alpha_p(f_1, \ldots, f_p | g_1, \ldots, g_{n-p}) = \prod_{\sigma} \alpha(x_1, \ldots, x_n)(-1)^{\sigma}. \]

Here the product is over all \((p, n-p)\)-shuffles and \(x_i = h_i g_{\sigma(i)} h_i^{-1}\) with \(h_i\) being the ordered product \(\prod_{\sigma(j) > \sigma(i)} f_j\). For example, for \(\alpha \in C^3(G, k^*)\)
\[ \alpha_1(f | g, h) = \alpha(f, g, h) \alpha(f g f^{-1}, f, h)^{-1} \alpha(f g f^{-1}, f h f^{-1}, f) = \alpha(f | g, h), \]
\[ \alpha_2(f, g | h) = \alpha(f, g, h) \alpha(f, g h g^{-1}, g)^{-1} \alpha(f g h g^{-1} f^{-1}, g h g^{-1} g, g) = \alpha(f, g | h). \]

Thus the monoidal category \(\mathcal{Z}(G, \alpha)\) coincides with \(\mathcal{C}(F, G, \gamma, \beta, \alpha)\), where \((\gamma, \beta, \alpha) = \tau(\alpha)\).

Now we return to the categories \(\mathcal{C}(F, G, \gamma', \beta', \alpha')\). We assume that \(F = G\) with the conjugation action on \(G\). In this case we can ask when the functorial isomorphism, defined by
\[ c_{U,V}(u \otimes v) = c(x, y) x(v) \otimes u, \quad u \in U_x, v \in V_y. \]
is a braiding in \( \mathcal{C}(F, G, \gamma', \beta', \alpha') \). The hexagon axioms are equivalent to the equations

\[
\beta(x|y, z)c(x, yz) = c(x, y)c(x, z)\alpha(x|y, z), \tag{35}
\]

\[
c(xy, z)\gamma(z|xy) = c(x, z)c(x, yz^{-1})\alpha(x, y|z). \tag{36}
\]

Denote by \( \mathcal{C}(G, G, \gamma, \beta, \alpha, c) \) a braided category \( \mathcal{C}(G, G, \gamma, \beta, \alpha) \) with the braiding (34). A gauge equivalence \( T(b, a) : \mathcal{C}(F, G, \gamma, \beta, \alpha, c) \rightarrow \mathcal{C}(F, G, \gamma', \beta', \alpha', c') \) is braided iff

\[
b(x, y)c'(x, y)a(x|y) = b(xy^{-1}, x)c(x, y). \tag{37}
\]

The equations (35),(36) can be interpreted as a coboundary condition \( d(c) = (\gamma, \beta, \alpha)\tau(\alpha)^{-1} \) (here \( c \) is considered as an element of \( C^1(G, C^1(G, k^*)) \)). Thus, at least at the level of monoidal categories, any \( \mathcal{C}(G, G, \gamma, \beta, \alpha) \) with braiding of the form (34) is equivalent to \( \mathcal{Z}(G, \alpha) \). Note that in the assumption \( (\gamma, \beta, \alpha) = \tau(\alpha) \) the equations (35),(36) turn into bi-multiplicativity condition for \( c \):

\[
c(x, yz) = c(x, y)c(x, z), \quad c(xy, z) = c(x, z)c(y, z).
\]

Thus by (37) the gauge equivalence \( T(1, c) \) is a braided equivalence between \( \mathcal{C}(G, G, \gamma, \beta, \alpha, c) \) (with \( (\gamma, \beta, \alpha) = \tau(\alpha) \)) and \( \mathcal{Z}(G, \alpha) \). In particular we have proved the following.

**Proposition 4.9.** Braided monoidal category of the form \( \mathcal{C}(G, G, \gamma, \beta, \alpha, c) \) is braided equivalent to \( \mathcal{Z}(G, \alpha) \).

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