The topological period–index conjecture

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We prove the topological analogue of the period–index conjecture in each dimension away from a small set of primes.

1. Introduction

We prove the following theorem, partly solving the so-called topological period–index conjecture of [2]. For background, see §2.

Theorem A. Let $X$ be a finite $2d$-dimensional CW complex and let $\alpha \in \text{Br}(X) = H^{3}(X, \mathbb{Z})_{\text{tors}}$ be a Brauer class. Setting $n = \text{per}(\alpha)$, we have

$$\text{ind}(\alpha) | n^{d-1} \prod_{p | n} p^{v_{p}((d-1)!)}$$

where the product ranges over the prime divisors of $n$ and where $v_{p}$ denotes the $p$-adic valuation.

Away from $(d-1)!$, the result simplifies.

Corollary B. If $X$ is a finite $2d$-dimensional CW complex, $\alpha \in \text{Br}(X)$, and $\text{per}(\alpha)$ is prime to $(d-1)!$, then $\text{ind}(\alpha) | \text{per}(\alpha)^{d-1}$.

The corollary is an exact topological analogue, away from some small primes, of the well-known period–index conjecture for division algebras over function fields (see [10, Section 2.4]).

Conjecture (Function field period–index conjecture). Let $k$ be algebraically closed and let $K$ be of transcendence degree $d$ over $k$. If $\alpha \in \text{Br}(K)$, then

$$\text{ind}(\alpha) | \text{per}(\alpha)^{d-1}.$$
conjuncture when \( d = 2 \) for Brauer classes of order relatively prime to the characteristic. Lieblich removed this restriction in [18] thus establishing the \( d = 2 \) case in full generality. There are no other cases known of the conjecture. More precisely, the period–index conjecture for function fields is not known for a single function field of transcendence degree \( d > 2 \) over an algebraically closed field.

Nevertheless, work of de Jong and Starr [20] has reduced the conjecture above to the following special cases.

**Conjecture (Global period–index conjecture).** Let \( k \) be an algebraically closed field and let \( X \) be a smooth projective \( k \)-scheme of dimension \( d \). If \( \alpha \in \text{Br}(X) \subseteq \text{Br}(k(X)) \), then

\[
\text{ind}(\alpha) | \text{per}(\alpha)^{d-1}.
\]

In other words, to prove the period–index conjecture for function fields, it is enough to prove it for unramified classes. We therefore view the period–index conjecture as a global problem. If \( k = \mathbb{C} \), then the space of complex points \( X(\mathbb{C}) \) of a smooth projective \( \mathbb{C} \)-scheme of dimension \( d \) admits the structure of a \( 2d \)-dimensional CW complex. Thus, Corollary B provides evidence for this conjecture.

The topological period–index problem was introduced by the authors in [1] where weak lower bounds were given and where the \( d \leq 2 \) cases were solved. The \( d = 3 \) case of the topological period–index problem was settled in [2], where it was proved moreover that the bound appearing in Theorem A is sharp in that case. The case of \( d = 4 \) was solved by Gu in [16, 17], where the upper bound of Theorem A was found independently (in the \( d = 4 \) case) and where it is shown that the bound appearing in the theorem is sharp for square-free classes. The best possible bound for \( d = 4 \) and \( n = \text{per}(\alpha) \), as proved by Gu, is

\[
\text{ind}(\alpha) \begin{cases} 
 e_3(n)n^3 & \text{if } 4 \nmid n \\
 e_2(n)e_3(n)n^3 & \text{otherwise,}
\end{cases}
\]

where \( e_p(n) = p \) if \( p \nmid n \) and 1 otherwise. In other words, there exist \( 8 \)-dimensional finite CW complexes where these upper bounds on the index are achieved.

The fact that for \( d = 3 \) there exist \( 6 \)-dimensional finite CW complexes with Brauer classes \( \alpha \) having \( \text{per}(\alpha) = 2 \) and \( \text{ind}(\alpha) = 8 \) leads to the natural question in [2] of whether the global period–index conjecture might be false.
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at the prime 2 for 3-folds over the complex numbers. However, recent work of Crowley–Grant [11] proves (a) that these topological examples can be found among closed orientable 6-manifolds but that (b) these examples cannot be found among closed 6-dimensional Spin$^c$-manifolds and hence they cannot be found among closed 6-manifolds of the form $X(C)$ for a smooth projective complex 3-fold $X$. The question of what happens for $d = 4$ is the subject of ongoing work of Crowley–Gu–Haesemeyer who prove in [12] that for closed orientable 8-manifolds $X$ one has $\text{ind}(\alpha)|\text{per}(\alpha)^3$ for $\alpha \in \text{Br}(X)$ unless $\text{per}(\alpha) \equiv 2 \pmod{4}$ in which case $\text{ind}(\alpha)|2\text{per}(\alpha)^3$.

Note that Gu’s bound $\text{ind}(\alpha)|e_3(n)n^3$ if $4|n$ is better than the bound $\text{ind}(\alpha)|e_2(n)e_3(n)n^3$ arising from Theorem A. We do not further address in this paper the sharpness of the bounds in Theorem A except to make the following conjecture.

**Conjecture C.** The bounds of Corollary A are the best possible. That is, for every $d \geq 1$ and every natural number $n$ prime to $(d−1)!$, there exists a finite $2d$-dimensional CW complex $X$ and a Brauer class $\alpha \in \text{Br}(X)$ such that $\text{per}(\alpha) = n$ and $\text{ind}(\alpha) = n^{d−1}$.

The bounds in the period–index conjecture for function fields are known to be sharp, for example by Gabber’s appendix to [9].

**2. Background and strategy**

We quickly review the period–index problem in three settings.

**Period–index for fields.** The period–index problem originated in the domain of division algebras over fields. Specifically, for a field $K$, we have the Brauer group $\text{Br}(K) = H^2_{\text{ét}}(\text{Spec } K, G_m)$. This group is isomorphic to the set of isomorphism classes of finite dimensional division $K$-algebras with center exactly $K$. Given $\alpha \in \text{Br}(K)$, we have two numbers: $\text{per}(\alpha)$, which is the order of $\alpha$ in the torsion abelian group $\text{Br}(K)$, and $\text{ind}(\alpha)$ which is the unique positive integer such that $\text{ind}(\alpha)^2 = \dim_K D$ where $D$ is a division algebra with Brauer class $[D] = \alpha$. It is not hard to see that

$$\text{per}(\alpha)|\text{ind}(\alpha)$$

and Noether proved that these two numbers have the same prime divisors. It follows that there is some integer $e_\alpha$ such that $\text{ind}(\alpha)|\text{per}(\alpha)^{e_\alpha}$.

The period–index problem is to find for a fixed field $K$ a number $e$ such that

$$\text{ind}(\alpha)|\text{per}(\alpha)^e$$
for all $\alpha \in \text{Br}(K)$ and, this done, to find the smallest such number. For example, the Albert–Brauer–Hasse–Noether theorem says that if $K$ is a number field, then $\text{ind}(\alpha) = \text{per}(\alpha)$ so that $e = 1$ works (see [13] Remark 6.5.6).

The period–index conjecture for function fields can be rephrased as saying that if $K$ has transcendence degree $d$ over an algebraically closed field, then $e = d - 1$ is the solution. For $d \geq 3$, it is not yet known that there is any $e$ that works for all Brauer classes, but some results are known one prime at a time [19].

**Period–index for schemes.** We introduced the period–index problem in other settings in [1]. For instance, if $X$ is a quasicompact scheme, then the Brauer group

$$\text{Br}(X) \subseteq H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}} \subseteq H^2_{\text{ét}}(X, \mathbb{G}_m)$$

of Azumaya algebras of Grothendieck [15] is a torsion abelian group. Given $\alpha \in \text{Br}(X)$, we again let $\text{per}(\alpha)$ be the order of $\alpha$ in $\text{Br}(X)$. We define

$$\text{ind}(\alpha) = \gcd\{\deg(\mathcal{A}) : \mathcal{A} \text{ is an Azumaya algebra with } [\mathcal{A}] = \alpha\}.$$ 

In [3], we showed that even on smooth schemes over the complex numbers, it is necessary to take the greatest common divisor to obtain a good theory. In this setting, we have $\text{per}(\alpha)|\text{ind}(\alpha)$ and the numbers have the same prime divisors by [4]. Thus, one can formulate the period–index problem for $X$.

**Period–index for topological spaces.** Finally, if $X$ is a topological space, we have the Brauer group $\text{Br}(X) \subseteq H^3(X, \mathbb{Z})_{\text{tors}} \subseteq H^3(X, \mathbb{Z})$ of topological Azumaya algebras, also introduced in [15]. We define $\text{per}(\alpha)$ and $\text{ind}(\alpha)$ as for schemes. Again, $\text{per}(\alpha)|\text{ind}(\alpha)$ and we proved that these have the same prime divisors if $X$ is a finite CW complex in [1] and in general in [4].

The topological period–index problem of [1] asks the following. Given $d \geq 1$, find bounds $e$ such that if $X$ is a finite $2d$-dimensional CW complex and $\alpha \in \text{Br}(X)$, then $\text{ind}(\alpha)|\text{per}(\alpha)^e$. We proposed $e = d - 1$ as a straw man in [2], where we immediately proved that for $d = 3$ this fails in general when $2|\text{per}(\alpha)$. Gu has proved this fails for $d = 4$ if 2 or 3 divides $\text{per}(\alpha)$. But, in these low-dimensional cases, these small primes are the only obstruction. We prove in Theorem [A] that this pattern continues in higher dimensions.

The topological results reveal a pattern which has not yet been discovered in algebra: a dependence on the prime divisors of the period and their relationship to $d$. This dependence comes, as we will see, from the ‘jumps’ in the cohomology of the Eilenberg–MacLane spaces $K(\mathbb{Z}/(n), 2)$. 
Strategy. If $X$ is a topological space and $\alpha \in H^3(X, \mathbb{Z})$, Atiyah and Segal constructed in [6] an $\alpha$-twisted form of complex $K$-theory $KU(X)_\alpha$ and in [7] an $\alpha$-twisted Atiyah–Hirzebruch spectral sequence $E^{s,t}_2 = H^s(X, \mathbb{Z}(\frac{t}{2})) \Rightarrow KU^{s+t}(X)_\alpha$, where $\mathbb{Z}(\frac{t}{2}) \cong \mathbb{Z}$ if $t$ is even and $\mathbb{Z}(\frac{t}{2}) = 0$ if $t$ is odd. The differentials $d^\alpha_r$ have bidegree $(r, 1-r)$. We proved in [1] that if $X$ is a connected finite CW complex, then $\text{ind}(\alpha)$ generates the group $E^{0,0}_\infty \subseteq H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ of permanent cycles. Thus, to bound the index, one attempts to bound the orders of the differentials $d_{2r+1} \colon E^{0,0}_r \to E^{2r+1,-2r}_r$.

There is a universal case to consider for all order $m$ topological Brauer classes, namely the space $K(\mathbb{Z}/(m), 2)$ and a generator $\alpha$ of $H^3(K(\mathbb{Z}/(m), 2), \mathbb{Z}) \cong \mathbb{Z}/(m)$. By studying the orders of the differentials in this particular case, we prove Theorem A.

3. The cohomology of $K(\mathbb{Z}/(n), 2)$

We recall some results of Cartan [8] on the cohomology of Eilenberg–MacLane spaces in the special case of $K(\mathbb{Z}/(n), 2)$, which we will use in the next section to give upper bounds on the index of period $n$ classes. We claim no originality in our presentation here, but we hope its inclusion will be useful to the reader.

Write

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

and pick a generator $u_i$ of the subgroup $\mathbb{Z}/(p_i^{e_i})$ of $\mathbb{Z}/(n)$ for $1 \leq i \leq k$. Let $v_i = p_i^{e_i-1} u_i$. Cartan [8, Théorème 3] gives a recipe for computing the integral homology (Pontryagin) ring of $K(\mathbb{Z}/(n), 2)$.

For each prime $p$ and positive integer $f$, consider certain words in the 4 symbols:

$$\sigma, \quad \gamma_p, \quad \phi_p, \quad \psi_{p^f}.$$  

The symbol $\psi_{p^f}$, if it appears, is the last symbol in a word. The height of a word $\alpha$ is the total number of $\sigma, \phi_p$ and $\psi_{p^f}$ appearing. The degree of $\alpha$ is
defined recursively by letting \(\deg(\emptyset) = 0\) and
\[
\begin{align*}
\deg(\sigma \alpha) &= 1 + \deg(\alpha), \\
\deg(\phi_p \alpha) &= 2 + p \deg(\alpha), \\
\deg(\gamma_p \alpha) &= p \deg(\alpha), \\
\deg(\psi_p) &= 2.
\end{align*}
\]

For each prime \(p\), an \textit{admissible} \(p\)-word \(\alpha\) is a word on 3 symbols \(\sigma, \gamma_p, \) and \(\phi_p\) such that \(\alpha\) is non-empty, the first and last letters of \(\alpha\) are \(\sigma\) or \(\phi_p\), and for each letter \(\gamma_p\) or \(\phi_p\), the number of letters \(\sigma\) appearing to the right in \(\alpha\) is even. In addition to the admissible words, we will use the auxiliary words \(\sigma^{h-1} \psi_p\), of height \(h\) and degree \(h + 1\).

Let \(E(x, 2q - 1)\) denote the exterior graded algebra over \(\mathbb{Z}\) with generator \(x\) of degree \(2q - 1\) endowed with the trivial dg-algebra structure \(dx = 0\). Let \(P(x, 2q)\) be the divided power polynomial algebra over \(\mathbb{Z}\) with generator \(x\) of degree \(2q\) and given the trivial dg algebra structure with \(dx = 0\). Cartan calls \(E(x, 2q - 1)\) and \(P(x, 2q)\) \textit{elementary complexes of the first type}. Define tensor dg algebras \(E(x, 2q - 1) \otimes \mathbb{Z} P(y, 2q)\) by \(dx = 0\) and \(dy = hx\) for some integer \(h\) and \(P(x, 2q) \otimes \mathbb{Z} E(y, 2q + 1)\) by \(dx = 0\) and \(dy = hx\) (the integer \(h\) is part of the data, even though it is not specified in the notation). These are the \textit{elementary complexes of the second type}. The positive-degree homology groups of \(E(x, 2q - 1) \otimes \mathbb{Z} P(x, 2q)\) are
\[
H_{2q-1+2qk}(E(x, 2q - 1) \otimes \mathbb{Z} P(x, 2q)) = \mathbb{Z}/h \cdot x \gamma_k(y)
\]
for \(k \geq 0\) and \(0\) otherwise, where \(\gamma_k\) is the \(k\)th divided power operation.

For \(P(x, 2q) \otimes \mathbb{Z} E(y, 2q + 1)\), we get
\[
H_{2qk}(P(x, 2q) \otimes \mathbb{Z} E(y, 2q + 1)) = \mathbb{Z}/hk \cdot \gamma_k(x)
\]
for \(k \geq 0\) and all other homology groups are 0.

The height-2 admissible or auxiliary \(p\)-words are
\[
\sigma^2, \quad \sigma^{\gamma_p^k} \phi_p, \quad \phi_p^{\gamma_p^k} \phi_p, \quad \sigma \psi_p
\]
for \(k \geq 0\) and \(f \geq 1\). These are of degrees 2, \(1 + 2p^k\), \(2 + 2p^{k+1}\), and 3, respectively. Below, the symbols \(u_i\) and \(v_i\), are just formal indeterminates to keep track of generators for different dg algebras.
For each \( p_i \), we define a dg algebra \( X_{p_i} \) as the following tensor product of elementary complexes of the second type
\[
X_{p_i} = P(\sigma^2 u_i, 2) \otimes E(\sigma \psi_{p_i}^k u_i, 3) \bigotimes \limits_{k=0}^{\infty} E(\sigma \gamma_{p_i}^{k+1} \phi_{p_i} v_i, 1 + 2p_i^{k+1}) \otimes P(\phi_{p_i} \gamma_{p_i}^k \phi_{p_i} v_i, 2 + 2p_i^{k+1}).
\]

The differentials are
\[
d(\sigma \psi_{p_i}^k u_i) = p_i^r \sigma^2 u_i \quad \text{and} \quad d(\phi_{p_i} \gamma_{p_i}^k \phi_{p_i} v_i) = p_i \sigma \gamma_{p_i}^{k+1} \phi_{p_i} v_i,
\]
i.e., \( h = p_i^r \) or \( h = p_i \), respectively. (Here the \( r_i \) are the exponents appearing in the prime decomposition of \( n \).) Let
\[
X = X_{p_1} \otimes \cdots \otimes X_{p_k}.
\]

Then, Cartan \[8, Théorème 1\] gives a surjection, which depends on the choice of the \( u_i \),
\[
H_k(X) \to H_k(K(\mathbb{Z}/(n), 2), \mathbb{Z}),
\]
and the kernel is described. This map induces for each \( i \) a surjection
\[
H_k(X_{p_i}) \to H_k(K(\mathbb{Z}/(n), 2), \mathbb{Z})\{p_i\},
\]
the \( p_i \)-primary part of homology. The largest possible torsion in \( H_{2k}(X_{p_i}) \) comes from the first term \( P(\sigma^2 u_2, 2) \otimes E(\sigma \psi_{p_i}^k u_i, 3) \). Specifically, by using the Künneth theorem, the exponent of \( H_{2k}(X_{p_i}) \) is the same as the exponent of \( H_{2k}(P(\sigma^2 u_2, 2) \otimes E(\sigma \psi_{p_i}^k u_i, 3)) \), namely
\[
p_i^r k.
\]
It follows that the exponent of \( H_{2k}(X) \) is at most \( nk \). Write \( k = k'a \), where \( a \) is largest integer dividing \( k \) and prime to \( m \). Since \( H_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z}) \) is \( n \)-primary torsion, we see that \( nk' \) kills all of \( H_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z}) \); in fact, the exponent is exactly \( nk' \), as follows from Cartan’s description of the kernel of \( H_{2k}(X) \to H_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z}) \), but we will not need this fact.

Since we will be able to argue prime-by-prime in a moment, it is helpful to record the \( p_i \)-primary part of the exponent of \( H_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z}) \) for a given prime \( p_i \). If \( \xi \in H_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z}) \) is an element of \( p_i \)-primary order, then the order of \( \xi \) divides \( p_i^{r_i + v_{p_i}(k)} \).
4. Proof of the main theorem

If \( X \) is a topological space with finitely many connected components, then the topological Brauer group \( \text{Br}(X) \) is a subgroup of \( \text{Br}'(X) = H^3(X, \mathbb{Z})_{\text{tors}} \). Serre showed that if \( X \) is compact, then \( \text{Br}(X) = \text{Br}'(X) \) (see [15]). This will be the main setting of this paper. We want to make a general remark on a different version of the index to which our methods apply whether or not \( X \) is compact.

We may weaken the hypothesis on the finiteness of \( X \) in Theorem A at the expense of using the \( K \)-theoretic index rather than the topological index. Recall from [1] that the \( K \)-theoretic index \( \text{ind}_K(\alpha) \) is defined as the (positive) generator of the image of the rank map \( \text{KU}_0(X) \alpha \rightarrow \mathbb{Z} \), where \( \text{KU}_0(X) \alpha \) denotes the \( \alpha \)-twisted \( K \)-theory group. When \( X \) is finite-dimensional (and connected), one computes \( \text{ind}_K(\alpha) \) as the generator of the group \( E^0,0 \subseteq E^2,0 \approx \mathbb{Z} \) by the convergence of the \( \alpha \)-twisted Atiyah–Hirzebruch spectral sequence; hence the differentials \( d_{2k+1}: E^{0,0}_{2k+1} \rightarrow E^{2k+1,-2k}_{2k+1} \) of the twisted Atiyah–Hirzebruch spectral sequence control the \( K \)-theoretic index. It is this crucial fact we exploit below.

In general, \( \text{per}(\alpha) \mid \text{ind}_K(\alpha) \mid \text{ind}(\alpha) \) and if \( X \) is compact, then \( \text{ind}_K(\alpha) = \text{ind}(\alpha) \). Thus, Theorem A follows from the following theorem.

**Theorem 4.1.** Let \( X \) be a \( 2d \)-dimensional CW complex, and let \( \alpha \in \text{Br}'(X) \) have period \( m = p_1^{r_1} \cdots p_k^{r_k} \) where the \( p_i \) are distinct primes. Then,

\[
\text{ind}_K(\alpha) \prod_{i=1}^{k} p_i^{(d-1)r_i + v_{p_i}(2) + \cdots + v_{p_i}(d-1)} = m^{d-1} \prod_{i=1}^{k} p_i^{v_{p_i}((d-1)!)},
\]

where \( v_{p_i} \) is the \( p_i \)-adic valuation.

**Proof.** According to the main result of [5], it is enough to prove the theorem when \( m \) is a prime power. So, suppose that \( m = p_1^{r_1} = p^r \). Let \( \beta \) be a generator of \( H^3(K(\mathbb{Z}/(p^r), 2), \mathbb{Z}) \cong \mathbb{Z}/(p^r) \). There is some map \( \sigma: X \rightarrow K(\mathbb{Z}/(p^r), 2) \) such that \( \sigma^* \beta = \alpha \). The twisted Atiyah–Hirzebruch spectral sequence is functorial, so we obtain—in particular—a map on the \((0,0)\)-terms in each page:

\[
E^0_{j,0}(K(\mathbb{Z}/(p^r), 2)) \rightarrow E^0_{j,0}(X).
\]
An elementary induction argument shows that this map is a monomorphism on each page, and so by cellular approximation, \( \text{ind}_K(\alpha) \) is bounded above by \( \text{ind}_K \) for the restriction of \( \beta \) to a 2\( d \)-skeleton of \( K(\mathbb{Z}/(n), 2) \).

There is an isomorphism

\[
\tilde{H}^{2j+1}(K(\mathbb{Z}/(p^r), 2), \mathbb{Z}) \cong \tilde{H}_{2j}(K(\mathbb{Z}/(p^r), 2), \mathbb{Z}).
\]

We have seen in Section 3 that the latter group is \( p^r p^{v_p(j)} \)-torsion. In the Atiyah–Hirzebruch spectral sequence for the Bockstein \( \beta \) on a 2\( d \)-skeleton of \( K(\mathbb{Z}/(p^r), 2) \), only the differentials \( d_{2j+1}^\beta \) for \( 1 \leq j \leq d - 1 \) are possibly non-zero—in particular, the cohomology group in degree 2\( d \) of the skeleton, which may differ from that of \( K(\mathbb{Z}/(p^r), 2) \), plays no part in the calculation. We are interested in the order of the image of

\[
d_{2j+1}^\beta : E^{0,0}_{2j+1} \to E^{2j+1, -2j}_{2j+1},
\]

where the latter group is a subquotient of \( H^{2j+1}(K(\mathbb{Z}/(p^r), 2), \mathbb{Z}) \), and hence the order of the image divides \( p^r p^{v_p(j)} \). This gives the result. \( \square \)

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