On the Continuity of Strongly Singular Calderón–Zygmund-Type Operators on Hardy Spaces

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Abstract. In this work, we establish results on the continuity of strongly singular Calderón–Zygmund operators of type \( \sigma \) on Hardy spaces \( H^p(\mathbb{R}^n) \) for \( 0 < p \leq 1 \) assuming a weaker \( L^s \)-type Hörmander condition on the kernel. Operators of this type include appropriate classes of pseudodifferential operators \( OpS^{\sigma,b}_{\ge 0}(\mathbb{R}^n) \) and operators associated to standard \( \delta \)-kernels of type \( \sigma \) introduced by Álvarez and Milman. As application, we show that strongly singular Calderón–Zygmund operators are bounded from \( H^p_w(\mathbb{R}^n) \) to \( L^p_w(\mathbb{R}^n) \), where \( w \) belongs to a special class of Muckenhoupt weight.

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1. Introduction

Motivated by the studies of a special class of pseudodifferential operators, Álvarez and Milman [2, Definition 2.1] introduced a new class of Calderón–Zygmund operators associated to \( \delta \)-kernels of type \( \sigma \), which are continuous functions away the diagonal on \( \mathbb{R}^{2n} \) that satisfy

\[
|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y - z|^{\delta}}{|x - z|^{{n+\sigma}}} ,
\]

for all \( |x - z| \geq 2|y - z|^\sigma \) some \( 0 < \delta \leq 1 \) and \( 0 < \sigma \leq 1 \). A linear and bounded operator \( T : \mathscr{S}(\mathbb{R}^n) \rightarrow \mathscr{S}'(\mathbb{R}^n) \) is called a strongly singular
Calderón–Zygmund operator if it is associated to a $\delta$-kernel of type $\sigma$ in the sense
\[
\langle Tf, g \rangle = \int \int K(x, y) f(y) g(x) dy dx, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n)
\]
with disjoint supports, has bounded extension from $L^2(\mathbb{R}^n)$ to itself and in addition $T$ and $T^*$ extend to a continuous operator from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ where
\[
\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n} \text{ for some } (1 - \sigma) \frac{n}{2} \leq \beta < \frac{n}{2}.
\]

These non-convolution operators are a natural extension of weakly-strongly singular integrals of convolution-type which were studied in the works [7,8,10,20]. The assumption $L^q - L^2$ continuity is motivated by convolution operators associated to kernels of type $\sigma$ that satisfy the uniform pointwise control $|\hat{K}(\xi)| \lesssim (1 + |\xi|)^{-\beta}$ for $\beta > 0$ (see for instance [8, Theorem 1] with $\beta = n\sigma/2$). Thus, this assumption can be understood as a suitable correction of the $L^2$ continuity due to action of kernels of type $\sigma$ which are naturally more singular at the diagonal; this justify the nomenclature strongly singular integral operators.

A natural question arises on investigating the boundedness on Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$ of operators associated to weaker $\delta$-kernels of type $\sigma$ given by Hörmander condition
\[
\int_{|x-z| \geq 2|y-z|^\sigma} |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| dx \leq C \ 2^{-\delta}. \quad (1.2)
\]

This question for strongly singular type operators is still open in general, moreover it is known that for non-convolution operators and $\sigma = 1$ the condition (1.2) is not sufficient to obtain boundedness on $H^1(\mathbb{R}^n)$, even provided $L^2(\mathbb{R}^n)$ continuity (see [22, Theorem 2]). Clearly $\delta$-kernels of type $\sigma$ satisfy (1.2).

In this work we continue the program proposed by Álvarez and Milman in [2] on strongly singular Calderón–Zygmund-type operators presenting news perspectives for continuity on Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$ assuming a $L^s$-type Hörmander condition of the associated kernel. We say that a kernel $K(x, y)$ associated to $T$ satisfies the $D_s$ condition for $s \geq 1$ if
\[
\left( \int_{C_j(z, r)} |K(x, y) - K(x, z)|^s + |K(y, x) - K(z, x)|^s dx \right)^{\frac{1}{s}} \lesssim |C_j(z, r)|^{\frac{1}{s} - 1} 2^{-s \delta} \quad (1.3)
\]
for $r > 1$ and
\[
\left( \int_{C_j(z, r^\rho)} |K(x, y) - K(x, z)|^s + |K(y, x) - K(z, x)|^s dx \right)^{\frac{1}{s}} \lesssim |C_j(z, r^\rho)|^{\frac{1}{s} - 1 + \frac{s}{n} \left( \frac{1}{s} - \frac{\sigma}{n} \right) 2^{-s \delta}} \quad (1.4)
\]

\(^{1}\)The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in \mathbb{R}^n$.\]
for $r < 1$, where $C_j(z, r) = \{ x \in \mathbb{R}^n : 2^j r < |x - z| \leq 2^{j+1} r, 0 < \rho \leq \sigma \leq 1, z \in \mathbb{R}^n \text{ and } |y - z| < r \}. \text{ It is easy to verify that } D_{s_1} \text{ condition is stronger than } D_{s_2} \text{ if } s_1 > s_2. \text{ We remark that a simple computation shows that } \delta \text{-kernels of type } \sigma \text{ satisfy } D_s \text{ conditions for every } 1 \leq s < \infty. \text{ By simplicity, we use the nomenclature } D_s \text{ condition omitting the dependence of } \delta. \text{ On the other hand, if necessary we emphasize that the kernel satisfies the } D_s \text{ condition with decay } \delta \text{ (see for instance Proposition 5.3).}

Estimates of this type are slightly different of $D_{s, \alpha}$ conditions presented in [5, Definition 1.1] and they are naturally related to distributional kernels associated to some classes of pseudodifferential operators in the Hörmander class $OpS^m_{\sigma, b}(\mathbb{R}^n)$ with $0 < \sigma \leq 1$ and $0 \leq b < 1$ (see [1, Theorem 5.1]). In particular, it has been shown that if $b \leq \sigma$ and $m \leq -n(1 - \sigma)/2$, the kernel $K(x, y)$ associated to $T \in OpS^m_{\sigma, b}(\mathbb{R}^n)$ satisfies the $D_1$ condition i.e., for $r > 1$

$$
\int_{C_j(z, r)} |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \ dx \lesssim 2^{-j\delta}, \quad \text{for } r > 1,
$$

and for $r < 1$

$$
\int_{C_j(z, r^r)} |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \ dx \lesssim |C_j(z, r^r)| \frac{\delta}{\pi} \left(1 - \frac{1}{p} + \frac{2}{\sigma} \right) 2^{-j\frac{\delta}{\frac{2}{\sigma}}}. \quad (1.6)
$$

The previous $L^1$-type Hörmander conditions (1.5) and (1.6) are slightly stronger then (1.2) and it is still an open question the boundedness on $H^1(\mathbb{R}^n)$ from it. Integral estimates such as (1.3) and (1.4) supply a weaker condition compared to (1.1), which has been extensively studied for convolution and non-convolution standard Calderón–Zygmund operators (see [9, p. 315] and [15, p. 23]).

Our first result is concerning the continuity of strongly singular Calderón–Zygmund operators on $H^p(\mathbb{R}^n)$ satisfying the $D_s$ condition, with emphasis on the relation of the parameters involved (previously omitted for $\delta$-kernels of type $\sigma$). The result is the following:

**Theorem 1.1.** Let $T : \mathscr{S}(\mathbb{R}^n) \rightarrow \mathscr{S}'(\mathbb{R}^n)$ be a linear and continuous operator and suppose that:

(i) $T$ extends to a continuous operator from $L^2(\mathbb{R}^n)$ to itself;

(ii) $T$ is associated to a kernel satisfying the $D_{s_1}$ condition;

(iii) For some $n(1 - \sigma) \left(1 - \frac{1}{s_2}\right) \leq \beta < n \left(1 - \frac{1}{s_2}\right)$, $T$ extends to a continuous operator from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$, where $\frac{1}{q} = \frac{1}{s_2} + \frac{\beta}{n}$ and $s_2 > 1$.

Under such conditions, if $T^\ast(x^\alpha) = 0$ for all $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leq |\delta|$, $1 < s_1 \leq 2$ and $s_1 \leq s_2$, then the operator $T$ can be extended to a bounded
operator from $H^p(\mathbb{R}^n)$ to itself for $p_0 < p \leq 1$, where

$$\frac{1}{p_0} := \frac{1}{s_2} + \frac{\beta}{n \left( \frac{\delta}{\sigma} - \delta + \beta \right)} \left[ \frac{\delta}{\sigma} + n \left( 1 - \frac{1}{s_2} \right) \right]. \quad (1.7)$$

Moreover, if $T^*$ also satisfies (iii) then the conclusion holds for $1 < s_1 < \infty$ and $s_1 \leq s_2$. The case $s_1 = 1$ also holds, however only for the range $p_0 < p < 1$.

This result extends [2, Theorem 2.2] with additional advantage of considering boundedness of operators of type $\sigma$ on $H^p(\mathbb{R}^n)$ associated to kernels with weaker integral conditions. In addition, our approach enables us to include the $D_1$ condition for $p < 1$, which represents the closest of Hörmander condition we can get so far. In contrast to condition (1.1), although any upper bound on $\delta$ is assumed on (1.3) and (1.4), natural examples with $\delta > 1$ are achieved with suitable refinement of weaker integral conditions incorporating derivatives of the kernel (see Sect. 4.2). An application of Theorem 1.1 is presented at Proposition 5.3. A particular weaker class of kernels of type $\sigma$ is presented at Sect. 5.3.

The proof of Theorem 1.1 follows by showing that $T$ maps $(p, \infty)$-atoms into an appropriate notion of molecules associated to (1.3) and (1.4). The molecules will be presented in Sect. 3.

The assumption on the $L^q - L^s$ boundedness of $T^*$ in the Theorem 1.1 guarantees that $T$ is continuous from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$.

**Theorem 1.2.** Let $T$ be an operator satisfying (i) and (iii) as in Theorem 1.1 with kernel satisfying the $D_1$ condition. Assume also that (iii) holds for $T^*$. Then $T$ is continuous from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$.

The statement is analogous to [2, Theorem 2.1] announced for kernels satisfying (1.1). The assumption on $T^*$ satisfying (iii) can be weakened (see Sect. 4.1) in the spirit of [3, Corollary 3.3] for vector valued operators assuming the $D_{1,\alpha}$-condition [5, Definition 1.1]). The next remark will be fundamental in the complement of Theorem 1.1.

**Remark 1.3.** As a direct consequence of Theorem 1.2 and real Interpolation Theorem (see [6, Theorem 6.8]), $T$ is a bounded operator from $L^p(\mathbb{R}^n)$ to itself for $2 \leq p < \infty$.

The critical case $p = p_0$ at Theorem 1.1 remains open, however, the conclusion continues to hold if we replace the target space by $L^p(\mathbb{R}^n)$. Our third result is a weighted continuity version for a special class of Muckenhoupt weight $A_1$.

**Theorem 1.4.** Let $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a linear and continuous operator as in Theorem 1.1. Then, $T$ can be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ for $p_0 \leq p \leq 1$, with $p_0$ given by (1.7), provided $w \in A_1 \cap RH_d$ in which $d = \max \left\{ \frac{s}{s - p}, \frac{s_1}{p(s_1 - 1)} \right\}$ and $s = \min \{2, s_2\}$. Moreover, if $s_1 = 1$ the conclusion holds for $p_0 \leq p < 1$ and $d = \frac{1}{1 - p}$. 
In particular the previous result covers the [13, Theorem 2] due to J. Li and S. Lu taking \( s_1 = s_2 = 2 \), since \( \delta \)-kernels of type \( \sigma \) satisfy the \( D_2 \) condition.

The organization of the paper is as follows. In Sect. 2 we present general aspects of Hardy spaces and some properties on \( T \) associated to kernels satisfying the \( D_s \) condition, in particular the well definition of \( T^* (x^\alpha) \) for \( |\alpha| \leq \lfloor \delta \rfloor \). The Sect. 3 is devoted to present the appropriated molecules that will be used in the proof of Theorem 1.1, in special an atomic decomposition result for these molecules (see Theorem 3.4). In Sect. 4 we present the proof of Theorems 1.1 and 1.2, including some comments and remarks. Also, in Sect. 4.2 we provide a version of Theorem 1.1 assuming some weaker integral derivative conditions. In Sect. 5, we recall some basic definitions on weights considered at Theorem 1.4 and its proof. Lastly, we show that pseudodifferential operators in the Hörmander class \( Op_{S^{m}_{\sigma,b}}(\mathbb{R}^n) \) where \( b \leq \sigma \) and \( m \leq -n(1 - \sigma)/2 \) satisfies the \( D_s \) integral derivative condition, in particular also satisfies (1.3) and (1.4), for all \( 1 \leq s \leq 2 \), extending the [1, Theorem 2.1]. In Sect. 5.3, we provide some generalizations and extensions to operators satisfying Dini-type modulus of continuity.

2. Preliminaries

Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} \varphi(x) dx \neq 0 \). We define the maximal operator by

\[
\mathcal{M}_\varphi f(x) := \sup_{t > 0} |f \ast \varphi_t(x)|
\]

where \( \varphi_t(x) = t^{-n} \varphi(x/t) \). We say that a tempered distribution \( f \) belongs to the Hardy space \( H^p(\mathbb{R}^n) \) for \( p > 0 \) if there exists \( \varphi \) as before such that \( \mathcal{M}_\varphi f \in L^p(\mathbb{R}^n) \). The functional \( \|u\|_{H^p} \) defines a quasi-norm for \( 0 < p < 1 \) and a norm for \( p \geq 1 \) (we refer as a norm for \( 0 < p < \infty \) by simplicity). In particular \( H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( p > 1 \) with equivalent norms and \( H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \) with continuous inclusion. Moreover \( H^p(\mathbb{R}^n) \) is a complete metric space with the distance \( d(u, v) = \|u - v\|_{H^p} \) with \( u, v \in H^p(\mathbb{R}^n) \) for \( 0 < p \leq 1 \).

For \( 0 < p \leq 1 \) the space \( H^p(\mathbb{R}^n) \) can be decomposed into special functions called atoms, that we present in the sequence.

**Definition 2.1.** Let \( 0 < p \leq 1 \) and \( 1 \leq s \leq \infty \) with \( p < s \). We say that a measurable function \( a(x) \) is a \((p, s)\)-atom in \( H^p(\mathbb{R}^n) \) if there exist a ball \( B := B(z, r) \subset \mathbb{R}^n \) such that

(i) \( \text{supp}(a) \subset B \);
(ii) \( \|a\|_{L^s} \leq |B|^{1/2} \frac{1}{2} \);
(iii) \( \int a(x)x^\alpha dx = 0 \) for all \( |\alpha| \leq N_p \)

where \( N_p := [n(1/p - 1)] \). For the limit case \( s = \infty \), condition (ii) is \( \|a\|_{L^\infty} \leq |B|^{-1/p} \).

**Theorem 2.2.** [9, Theorem 4.10] Let \( 0 < p \leq 1 \), \( 1 \leq s \leq \infty \), \( p < s \) and \( f \in H^p(\mathbb{R}^n) \). Then, there exist a sequence \( \{a_j\}_{j \in \mathbb{N}} \) of \((p, s)\)-atoms and complex
Proof. By simplicity suppose that supp($a$) $\subset B(0,r)$ and $r \geq 1$. Write

$$\int_{\mathbb{R}^n} |x^\alpha Ta(x)|dx = \int_{B(0,2r)} |x^\alpha Ta(x)|dx + \int_{\mathbb{R}^n \setminus B(0,2r)} |x^\alpha Ta(x)|dx.$$ 

From Hölder inequality and boundedness of $T$ on $L^2(\mathbb{R}^n)$ we get

$$\int_{B(0,2r)} |x^\alpha Ta(x)|dx \leq \|x^\alpha\|_{L^\infty(B(0,2r))} \|B(0,2r)\|^{\frac{1}{2}} \|Ta\|_{L^2} \lesssim r^{\alpha + \frac{n}{2}} \|a\|_{L^2} < \infty.$$ 

For the second integral, since $a \in L^2_{\#}(\mathbb{R}^n)$ we may estimate

$$|Ta(x)| = \left| \int_{B(0,r)} [K(x,y) - K(x,0)]a(y)dy \right| \leq \int_{B(0,r)} |K(x,y) - K(x,0)||a(y)||dy.$$ 

Then

$$\int_{\mathbb{R}^n \setminus B(0,2r)} |x^\alpha Ta(x)|dx \leq \int_{B(0,2r)} |a(y)| \int_{\mathbb{R}^n \setminus B(0,2r)} |x|^{|\alpha|} |K(x,y) - K(x,0)|dx \leq \sum_{j=0}^{\infty} (2^jr)^{|\alpha|} \int_{B(0,r)} |a(y)| \int_{C_j(0,r)} |K(x,y) - K(x,0)|dx \leq \sum_{j=0}^{\infty} (2^jr)^{|\alpha|} \|a\|_{L^2} |B(0,r)|^{\frac{1}{2}} 2^{-j\delta} \leq r^{\alpha + \frac{n}{2}} \|a\|_{L^2} \sum_{j=0}^{\infty} (2^j)^{|\alpha| - \delta} < \infty$$

since $|\alpha| < \delta$. For $r < 1$ we may write

$$\int_{\mathbb{R}^n} |x^\alpha Ta(x)|dx = \int_{B(0,2r)} |x^\alpha Ta(x)|dx + \int_{\mathbb{R}^n \setminus B(0,2r)} |x^\alpha Ta(x)|dx,$$
for some $0 < \rho \leq \sigma < 1$. The estimate for the first integral is the same as the previous case and for the second
\[
\int_{\mathbb{R}^n \setminus B(0,2r\rho)} |x^\alpha T a(x)| \, dx \leq \sum_{j=0}^{\infty} (2^j r^\rho)^{|\alpha|} \int_{B(0,r)} |a(y)| \int_{C_j(0,rr)} |K(x,y) - K(x,0)| \, dx dy
\]
\[
\lesssim r^{|\alpha|+\frac{n}{2}+\frac{\delta}{\sigma}} \|a\|_{L^2(x)} \sum_{j=0}^{\infty} (2^j)^{|\alpha|+\frac{n}{s}} < \infty
\]
since $|\alpha| < \delta$ that implies $|\alpha| < \delta/\sigma$. \qed

3. Generalized Molecules

M. Taibleson and G. Weiss introduced in [19] the molecular structure in $H^p(\mathbb{R}^n)$ and a useful method to establish continuity on Hardy spaces for certain classes of linear operators by showing that atoms are mapped into molecules i.e., special measurable functions on $H^p(\mathbb{R}^n)$ satisfying certain size control in $L^s$-norm [19, p. 71]. In particular, this is used to show continuity of standard Calderón–Zygmund operators (see [2, Theorem 1.1]). However, for operators associated to $\delta$-kernels of type $\sigma$ we use a notion of molecules that best fit these types of operators. Such ideas were originally explored by Bordin [4], Álvarez and Milman [2].

Next we define a slight generalization of molecules presented in [2, Definition 2.2].

**Definition 3.1.** Let $0 < p, \rho < 1 < q < s < \infty$, $1 \leq t < \infty$ and $t \leq s$ such that
\[
n \left( \frac{t}{p} - 1 \right) < \lambda \leq n \left( \frac{t}{s} - 1 \right) + \frac{n t}{1 - \rho} \left( \frac{1}{q} - \frac{1}{s} \right).
\]
If $p = 1$, we restrict $1 < t < \infty$. We say that a function $M(x)$ is a $(p, \rho, q, \lambda, s, t)$-molecule if there exist a ball $B(z, r) \subset \mathbb{R}^n$ and a constant $C > 0$ such that for $r > 1$
\[
\text{M1. } \int_{\mathbb{R}^n} |M(x)|^t \, dx \leq C r^{n(1-\frac{t}{p})};
\]
\[
\text{M2. } \int_{\mathbb{R}^n} |M(x)|^t |x-z|^\lambda \, dx \leq C r^{\lambda+n(1-\frac{1}{p})},
\]
and for $r \leq 1$
\[
\text{M3. } \int_{\mathbb{R}^n} |M(x)|^t \, dx \leq C r^{n[\rho(1-\frac{1}{t})+t(\frac{1}{2}-\frac{1}{s})]};
\]
\[
\text{M4. } \int_{\mathbb{R}^n} |M(x)|^t |x-z|^\lambda \, dx \leq C r^{\rho+n[\rho(1-\frac{1}{t})+t(\frac{1}{2}-\frac{1}{p})]},
\]
Besides that, $M(x)$ satisfies for all $r > 0$
\[
\text{M5. } \int_{\mathbb{R}^n} M(x)x^\alpha \, dx = 0 \quad \forall \, |\alpha| \leq N_p.
\]

**Remark 3.2.** (a) If $t = s = 2$ we recover the molecules defined by [2, Definition 2.2];
(b) In order to show that \( M(x) \) satisfies conditions (M1) and (M2) it is sufficient to verify simultaneously

\[
\text{M1a.} \quad \int_{B(z,r)} |M(x)|^t \, dx \leq C \, r^n (1 - \frac{t}{p}) \; ; \\
\text{M2a.} \quad \int_{\mathbb{R}^n \setminus B(z,r)} |M(x)|^t |x - z|^\lambda \, dx \leq C \, r^{\lambda + n (1 - \frac{t}{p})}.
\]

Similar idea applies to (M3) and (M4) integrating on \( B(z,r^p) \) and \( \mathbb{R}^n \setminus B(z,r^p) \) respectively.

Next we verify that condition (M5) is well defined for \( M(x) \) satisfying (M1) to (M4).

**Proposition 3.3.** Suppose \( M(x) \) is a function satisfying (M1) and (M2) or (M3) and (M4). Then \( x^\alpha M(x) \) is an absolutely integrable function.

**Proof.** Suppose \( 1 < t < \infty \). Let \( M \) be a function satisfying (M3) and (M4) for \( r \leq 1 \). Split

\[
\int_{\mathbb{R}^n} |x^\alpha M(x)| \, dx = \int_{B(z,r)} |x^\alpha M(x)| \, dx + \int_{\mathbb{R}^n \setminus B(z,r)} |x^\alpha M(x)| \, dx.
\]

For the first integral, from Hölder inequality and (M3) we have

\[
\int_{B(z,r)} |x^\alpha M(x)| \, dx \leq \|x^\alpha\|_{L^\infty(B(z,r))} |B(z,r)|^{1 - \frac{1}{t}} \left( \int_{\mathbb{R}^n} |M(x)|^t \, dx \right)^{\frac{1}{t}} < \infty
\]

and for the second

\[
\int_{\mathbb{R}^n \setminus B(z,r)} |x^\alpha M(x)| \, dx \leq \sum_{\lambda \geq |\alpha|} C_{\lambda,|\gamma|,|\lambda|} \int_{\mathbb{R}^n \setminus B(z,r)} |x - z|^{|\gamma| - |\lambda|} |M(x)| \, dx \leq \|M\|_{L^\infty} \sum_{\lambda \geq |\alpha|} C_{\lambda,|\gamma|,|\lambda|} \left( \int_{\mathbb{R}^n \setminus B(z,r)} |x - z|^{|\gamma| - |\lambda| + \frac{1}{t}} \right)^{1 - \frac{1}{t}} < \infty
\]

where the integrability of \( |x - z|^{|\gamma| - |\lambda| + \frac{1}{t}} \) on \( \mathbb{R}^n \setminus B(z,r) \) is guaranteed by \( \lambda > n (t/p - 1) \) and \( |\gamma| \leq |\alpha| \leq n (1/p - 1) \). The estimate assuming (M1) and (M2) for \( r > 1 \) follows analogously as well as the case \( t = 1 \). \( \square \)

**Lemma 3.4.** Let \( M(x) \) be a \((p,\rho,q,\lambda,s,t)\)-molecule. Then \( M = \sum_{j=0}^\infty \gamma_j a_j \) in \( \mathcal{H}(\mathbb{R}^n) \) where \( a_j(x) \) is a \((p,t)\)-atom and \( \{\gamma_j\}_{j \in \mathbb{N}} \) a sequence of complex scalars such that \( \sum_j |\gamma_j|^p < \infty \). In particular, there exists a constant \( C > 0 \) independent of \( M \) such that \( \|M\|_{H^p} \leq C \).

**Proof.** This proof is inspired in [9, Theorem 7.16] and [2, Lemma 2.1]. Let \( M(x) \) be a \((p,\rho,q,\lambda,s,t)\)-molecule associated to a ball \( B = B(z,r) \subset \mathbb{R}^n \) and define \( B_j = B(z,2^j r), E_j = B_j \setminus B_{j-1} \) and \( M_j(x) = M(x)|_{E_j}(x) \) for each \( j \in \mathbb{N} \cup \{0\} \). Consider \( \alpha \in \mathbb{Z}_n^\ell \) a multi-index such that \( |\alpha| \leq N_p \) and \( P_{N,j} \) its restriction on the set \( E_j \). By the Gram-Schmidt orthogonalization process on the Hilbert space \( \mathcal{H} := L^2(E_j,|E_j|^{-1} \, dx) \), considering \( P_{N,j} \) as a
subspace of $\mathcal{H}$, with respect to the base $\{x^\beta\}_{|\beta|\leq N_p}$ there exist polynomials $\phi^j_\gamma(x)$ defined on $P_{N,j}$ uniquely determined such that

$$\frac{1}{|E_j|} \int_{E_j} \phi^j_\gamma(x) \ x^\beta \, dx = \delta_{\gamma,\beta} = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \neq \beta. \end{cases} \quad (3.1)$$

In addition, these polynomials satisfy the estimate $(2^j r)^{|\gamma|} |\phi^j_\gamma(x)| \leq C$, $\forall x \in E_j$, uniformly on $j$ (see [9, p. 332]). Set

$$M^j_\gamma = \frac{1}{|E_j|} \int_{E_j} M(x) x^\gamma \, dx, \quad P_j(x) = \sum_{|\gamma| \leq N_p} M^j_\gamma \phi^j_\gamma(x),$$

and split

$$M = \sum_{j=0}^\infty M_j = \sum_{j=0}^\infty (M_j - P_j) + \sum_{j=0}^\infty P_j$$

where the convergence is given in $L^t(\mathbb{R}^n)$. We will show that $(M_j - P_j)$ is a multiple of a $(p, t)$-atom and $P_j$ can be written as a finite linear combination of $(p, \infty)$-atoms for each $j$.

Suppose first $r \leq 1$. Since $M_j$ and $P_j$ are both supported in $E_j \subset B_j$ clearly supp$(M_j - P_j) \subset B_j$ and also for all $|\alpha| \leq N_p$ it has the right cancellation property from

$$\int_{\mathbb{R}^n} (M_j(x) - P_j(x)) \ x^\alpha \, dx = \int_{E_j} \left( M_j(x) - \sum_{|\gamma| \leq N_p} M^j_\gamma \phi^j_\gamma(x) \right) \ x^\alpha \, dx$$

$$= \int_{E_j} M(x) x^\alpha \, dx - \sum_{|\gamma| \leq N_p} \left( \int_{E_j} M(y) y^\gamma \, dy \right) \frac{1}{|E_j|} \int_{E_j} \phi^j_\gamma(x) x^\alpha \, dx = 0. \quad (3.2)$$

In order to estimate the $L^t$-norm of $M_j$, follows from condition (M4) that

$$\|M_j\|_{L^t} \leq (2^j r)^{-\frac{n}{t}} \left( \int_{E_j} |M(x)|^t |x - z|^\lambda \, dx \right)^{\frac{1}{t}}$$

$$\lesssim |B_j|^\frac{1}{1 - \frac{n}{t}} \left( 2^j \right)^{-\frac{n}{t}} + n \left( \frac{1}{2} - \frac{1}{t} \right) r^{-\frac{n}{t}} \left( 1 - \rho \right) + n \left( \frac{1}{2} - \frac{1}{t} \right) \left( 1 - \frac{1}{t} \right)$$

$$\lesssim |B_j|^\frac{1}{1 - \frac{n}{t}} \left( 2^j \right)^{-\frac{n}{t}} + n \left( \frac{1}{2} - \frac{1}{t} \right)$$

since $\lambda \leq n \left( \frac{t}{s} - 1 \right) + \frac{n t}{1 - \rho} \left( \frac{1}{q} - \frac{1}{s} \right)$. On the other hand

$$|P_j(x)| \leq \sum_{|\gamma| \leq d} |\phi^j_\gamma(x)| \frac{1}{|E_j|} \int_{E_j} |M(y)| |y|^{|\gamma|} \, dy$$

$$\leq \left( \sum_{|\gamma| \leq d} (2^j r)^{|\gamma|} |\phi^j_\gamma(x)| \right) \frac{1}{|E_j|} \int_{E_j} |M(y)| \, dy$$

$$\lesssim \frac{1}{|E_j|} \int_{E_j} |M_j(y)| \, dy$$
\[ \leq |E_j|^{-\frac{1}{r}} \|M_j\|_{L^r}. \]

From estimates (3.3) and (3.4) we obtain
\[ \|M_j - P_j\|_{L^r} \leq 2\|M_j\|_{L^r} \lesssim |B_j|^{\frac{1}{r} - \frac{1}{p}} (2^j)^{-\frac{n}{2} + n\left(\frac{1}{p} - \frac{1}{q}\right)}. \]

Finally, write \((M_j - P_j)(x) = d_j A_j(x)\) where \(d_j = \|M_j - P_j\|_{L^r} |B_j|^{\frac{1}{r} - \frac{1}{p}}\) and
\[ A_j(x) = \frac{M_j(x) - P_j(x)}{\|M_j - P_j\|_{L^r}} |B_j|^{\frac{1}{r} - \frac{1}{p}}. \]

By the previous considerations it is clear that \(A_j\) is a \((p,t)\)-atom and in addition
\[ \sum_{j=0}^{\infty} |d_j|^p = \sum_{j=0}^{\infty} \|M_j - P_j\|^p_{L^r} |B_j|^{1 - \frac{q}{r}} \leq \sum_{j=0}^{\infty} \left[ |B_j|^{\frac{1}{r} - \frac{1}{p}} (2^j)^{-\frac{n}{2} + n\left(\frac{1}{p} - \frac{1}{q}\right)} \right]^p |B_j|^{1 - \frac{q}{r}} = \sum_{j=0}^{\infty} (2^j)^{-\frac{n}{2} + n\left(1 - \frac{p}{q}\right)} < \infty \]

since \(\lambda > n \left(\frac{t}{p} - 1\right)\).

We show now that \(P_j\) is a finite linear combination of \((p,\infty)\)-atoms. Define for each \(j \in \mathbb{N} \cup \{0\}\)
\[ N_j^\gamma := |E_k| M_j^\gamma = \sum_{k=\gamma}^{\infty} \int_{E_k} M(x)x^\gamma \, dx \]
and
\[ \psi_j^\gamma(x) := N_j^{\gamma+1} \left[ |E_{j+1}|^{-1} \phi_j^{\gamma+1}(x) - |E_j|^{-1} \phi_j^\gamma(x) \right]. \]

Then, we can represent \(P_j(x)\) in the following way
\[ \sum_{j=0}^{\infty} P_j(x) = \sum_{j=0}^{\infty} \sum_{|\gamma| \leq N_p} M_j^\gamma |E_j| \left( |E_j|^{-1} \phi_j^\gamma(x) \right) \]
\[ = \sum_{|\gamma| \leq N_p} \left\{ \sum_{j=0}^{\infty} \psi_j^\gamma(x) + \sum_{j=0}^{\infty} \left[ N_j^\gamma |E_j|^{-1} \phi_j^\gamma(x) - N_j^{\gamma+1} |E_{j+1}|^{-1} \phi_j^{\gamma+1}(x) \right] \right\} \]
\[ = \sum_{j=0}^{\infty} \sum_{|\gamma| \leq N_p} \psi_j^\gamma(x), \]

since
\[ \sum_{j=0}^{\infty} \left[ N_j^\gamma |E_j|^{-1} \phi_j^\gamma(x) - N_j^{\gamma+1} |E_{j+1}|^{-1} \phi_j^{\gamma+1}(x) \right] = N_0^\gamma = \int_{\mathbb{R}^n} M(x)x^\gamma \, dx = 0 \]
for all \(|\gamma| \leq N_p\). We claim that \(\psi_j^\gamma(x)\) is a multiple of a \((p,\infty)\)-atom. By definition \(\text{supp} (\psi_j^\gamma) \subset E_j \subset B_j\) and moreover
\[ \int_{\mathbb{R}^n} \psi_j^\gamma(x)x^\beta \, dx = N_j^{\gamma+1} \left[ \frac{1}{|E_{j+1}|} \int_{E_{j+1}} \phi_j^{\gamma+1}(x)x^\beta \, dx - \frac{1}{|E_j|} \int_{E_j} \phi_j^\gamma(x)x^\beta \, dx \right] = 0 \]

(3.5)
all $|\beta| \leq N_p$. In order to control $\|\psi_\gamma^j\|_{L^\infty}$ it is enough to estimate $N_{\gamma}^{j+1}$. Then, by Hölder inequality and (3.3) we have

$$|N_{\gamma}^{j+1}| = \left| \sum_{k=j+1}^{\infty} \int_{E_k} M(x)x^\gamma dx \right| \leq \sum_{k=j+1}^{\infty} (2^kr)^{|\gamma|} \int_{E_k} |M_k(x)| dx \leq \sum_{k=j+1}^{\infty} (2^kr)^{|\gamma|} \|M_k\|_{L^r} \|E_k\|^{1-\frac{1}{t}} \lesssim r^{|\gamma|+n(1-\frac{1}{p})} \sum_{k=j+1}^{\infty} (2^k)^{|\gamma|-\frac{t}{p}+n(1-\frac{1}{t})}.$$

Since $|\gamma| \leq n(1/p - 1)$ and $\lambda > n(t/p - 1)$ the series above converges and

$$|N_{\gamma}^{j+1}| \lesssim \sum_{k=j+1}^{\infty} (2^kr)^{|\gamma|+n(1-\frac{1}{p})} (2^k)^{-\frac{t}{p}+n(\frac{1}{p}-\frac{1}{t})} \lesssim |B_j|^{1-\frac{1}{p}} (2^j)^{|\gamma|-\frac{t}{p}+n(\frac{1}{p}-\frac{1}{t})}.$$ 

In that way, using that $(2^k)^{|\gamma|} |\phi_\gamma^j(x)| \leq C$ uniformly in $j$ and the previous control follows

$$|N_{\gamma}^j|_{E_j}^{-1} \phi_\gamma^j(x) \lesssim |B_j|^{-\frac{1}{p}} (2^j)^{-\frac{t}{p}+n(\frac{1}{p}-\frac{1}{t})}.$$ 

Denote $\psi_\gamma^j(x) = h_j B_j \gamma(x)$ where $h_j = (2^j)^{-\frac{t}{p}+n(\frac{1}{p}-\frac{1}{t})}$ and $B_j \gamma(x) = k_j \psi_\gamma^j(x)$ for $k_j = (2^j)^{\frac{t}{p}-n(\frac{1}{p}-\frac{1}{t})}$. It is clear that $B_j \gamma$ is a multiple of a $(p, \infty)$-atom since supp $(B_j \gamma) \subset B_j$, $\|B_j \gamma(x)\|_{L^\infty} \lesssim |B_j|^{-\frac{1}{p}}$ and the moment condition follows immediately from (3.5). Clearly

$$\sum_{j=0}^{\infty} |h_j|^p = \sum_{j=0}^{\infty} (2^j)^{-\frac{t}{p}+n(1-\frac{1}{t})} < \infty.$$ 

The case $r > 1$ follows the same steps with crucial control

$$\|M_j\|_{L^t} \lesssim (2^j)^{-\frac{t}{p}} r^{\frac{t}{p}+n(\frac{1}{t}-\frac{1}{t})} \lesssim |B_j|^{\frac{1}{t}+\frac{1}{p}} (2^j)^{-\frac{1}{p}+n(\frac{1}{p}-\frac{1}{t})}$$

from condition (M2). We point out that the case $t = 1$ follows by the same argument as before with the restriction $0 < p < 1$.

Summarizing we have $M(x) = \sum_{j=0}^{\infty} \gamma_j a_j(x)$ converges in $L^t(\mathbb{R}^n)$, consequently in $\mathcal{S}'(\mathbb{R}^n)$, where $a_j$ are $(p, t)$-atoms and $\{\gamma_j\}_{j \in \mathbb{N}}$ is a sequence of complex scalars such that $\left( \sum_j |\gamma_j|^p \right)^{1/p} \leq C$, with constant independent of $M$. Clearly the series converges in $H^p(\mathbb{R}^n)$ and $\|M\|_{H^p} \leq C$. \(\square\)

4. Proof of Theorems 1.1 and 1.2

We start with a notation that will be useful in the proof of Theorem 1.1.
**Definition 4.1.** Let $\lambda > 0$, $1 \leq s_1 < \infty$, $1 < s_2 < \infty$, $s_1 \leq s_2$, $0 < p, \rho < 1$, $0 < \sigma \leq 1$ and $\beta$ such that $(1 - \sigma) \left(1 - \frac{1}{s_2}\right) n \leq \beta < \left(1 - \frac{1}{s_2}\right) n$. We say that $(p, \rho, \beta, \lambda, s_1, s_2)$ are admissible parameters when the relation

$$n \left(\frac{s_1}{p} - 1\right) < \lambda \leq n \left(\frac{s_1}{s_2} - 1\right) + \frac{s_1 \beta}{(1 - \rho)}$$

holds. For $p = 1$ we restrict $1 < s_1 < \infty$.

**Proof of Theorem 1.1.** Let $a(x)$ be a $(p, \infty)$-atom supported on $B(z, r) \subset \mathbb{R}^n$. From Lemma 3.4, it is enough to show that $Ta$ is a $(p, \rho, q, \lambda, s_2, s_1)$-molecule where

$$\frac{1}{q} = \frac{1}{s_2} + \frac{\beta}{n}$$

and $(p, \rho, \beta, \lambda, s_1, s_2)$ are admissible parameters for some $\rho \leq \sigma$ to be chosen. We will restrict ourselves to the case $0 < p \leq 1$ and $1 < s_1 \leq 2$. The comments for $s_1 = 1$ will be done at the end. Suppose first $r > 1$. From Remark 3.2-(b), we will show that (M1a) and (M2a) holds.

Since $T$ is bounded on $L^2(\mathbb{R}^n)$ we have

$$\int_{B(z, 2r)} |Ta(x)|^{p_1} dx \leq |B(z, 2r)|^{1 - \frac{s_1}{2}} \|Ta\|_{L^{p_1}}^{s_1} \lesssim |B(z, 2r)|^{1 - \frac{s_1}{2}} \|a\|_{L^{p_1}}^{s_1} \lesssim r^{n(1 - \frac{s_1}{p})}.$$

To estimate (M2a), the moment condition of the atom allow us to write

$$\int_{\mathbb{R}^n \setminus B(z, 2r)} |Ta(x)|^{s_1} |x - z|^{\lambda} dx = \sum_{j=0}^{\infty} \int_{C_j(z, r)} \left[ \int_{B(z, r)} |K(x, y) - K(x, z)|a(y) dy \right]^{s_1} |x - z|^{\lambda} dx.$$

Then, from Minkowski inequality for integrals, Hölder’s inequality and the assumption (1.3) we may control the previous sum as

$$\sum_{j=0}^{\infty} \left\{ \left[ \int_{C_j(z, r)} \left( \int_{B(z, r)} |K(x, y) - K(x, z)| |a(y)| |x - z|^{\lambda} dy \right)^{s_1} \right]^{\frac{1}{s_1}} \right\}^{s_1}$$

$$\leq \sum_{j=0}^{\infty} \left\{ \int_{B(z, r)} \left[ \int_{C_j(z, r)} |K(x, y) - K(x, z)|^{s_1} |a(y)|^{s_1} |x - z|^{\lambda} dy \right]^{\frac{1}{s_1}} \right\}^{s_1}$$

$$\leq \sum_{j=0}^{\infty} (2^jr)^\lambda \left\{ \int_{B(z, r)} |a(y)| \left[ \int_{C_j(z, r)} |K(x, y) - K(x, z)|^{s_1} dx \right]^{\frac{1}{s_1}} \right\}^{s_1}$$

$$\leq \sum_{j=0}^{\infty} (2^jr)^\lambda (2^jr)^{-n(s_1 - 1)} 2^{-js_1\delta} \left( \int_{B(z, r)} |a(y)| dy \right)^{s_1}$$

$$\leq \sum_{j=0}^{\infty} (2^jr)^\lambda \left( 2^j r \right)^{-n(s_1 - 1)} 2^{-js_1\delta} r^{s_1 n(1 - \frac{1}{s_1})}$$
\begin{align*}
= C \, r^{\lambda+n(1-\frac{d}{p})} \sum_{j=0}^{\infty} 2^j [\lambda-n(s_1-1)-s_1 \delta] \\
= C \, r^{\lambda+n(1-\frac{d}{p})} 
\end{align*}

assuming \( \lambda < n(s_1-1) + s_1 \delta \) and this concludes the proof of (M1a) and (M2a) for \( 1 < s_1 \leq 2 \). In the case \( 2 < s_1 < \infty \), assuming the additional hypothesis (iii) for \( T^* \), we obtain from Remark 1.3 the continuity in \( L^{s_1} (\mathbb{R}^n) \). From this, the condition (M1a) follows by

\[
\int_{B(z,2r)} |Ta(x)|^{s_1} dx \lesssim \|a\|^{s_1}_{L^{s_1}} \lesssim r^{n(1-\frac{d}{p})}
\]

and (M2a) follows analogously.

Now, we deal with the case \( r \leq 1 \). Since \( s_1 \leq s_2 \) and \( T \) is bounded from \( L^q(\mathbb{R}^n) \) to \( L^{s_2}(\mathbb{R}^n) \) we have

\[
\int_{B(z,r')} |Ta(x)|^{s_1} dx \leq |B(z,r')|^{1-\frac{d}{s_2}} \|Ta\|^{s_1}_{L^{s_2}} \lesssim |B(z,r')|^{1-\frac{d}{s_2}} \|a\|^{s_1}_{L^q}
\]

which proves (M3a). For (M4a), using the same argument as before and (1.4) follows

\[
\int_{\mathbb{R}^n \setminus B(z,r')} |Ta(x)|^{s_1} |x-z|^{\lambda} dx
\]

\[
\leq \sum_{j=0}^{\infty} (2^j r')^\lambda \left\{ \int_{B(z,r)} |a(y)| \left[ \int_{C_j(z,r')} |K(x,y) - K(x,z)|^{s_1} dy \right] \right\}^{s_1}
\]

\[
\leq \sum_{j=0}^{\infty} (2^j r')^\lambda \left( |C_j(z,r')|^{\frac{1}{s_1}-1+\frac{d}{s_2} \left( \frac{s_1}{s_2} - \frac{d}{p} \right)} \right)^{s_1} \|a\|^{s_1}_{L^q} \|B(z,r)|^{s_1}
\]

\[
\leq \sum_{j=0}^{\infty} (2^j r')^\lambda \left( |C_j(z,r')|^{\frac{1}{s_1}-1+\frac{d}{s_2} \left( \frac{s_1}{s_2} - \frac{d}{p} \right)} \right)^{s_1} \|
\]

\[
= C \, r^{\rho+n\{s_1+\frac{d}{s_2}-s_1\rho\left(1-\frac{1}{s_1}+\frac{s_1}{n\sigma}\right)-\frac{s_1}{q}\}} \sum_{j=0}^{\infty} 2^j [\lambda-n(s_1-1)-\frac{s_1 \delta}{n\sigma}]
\]

\[
\leq r^{\rho+n\left[\rho\left(1-\frac{1}{s_2}\right)+s_1\left(\frac{1}{s_2}-\frac{d}{s_2}\right)\right]},
\]

where the convergence of the series is evident since \( 0 < \sigma \leq 1 \), \( \lambda < n(s_1-1) + s_1 \delta \) assumed previously and \( \rho \) is chosen such that

\[
s_1 + \frac{s_1 \delta}{n\sigma} \rho \left(1-\frac{1}{s_1}+\frac{s_1}{n\sigma}\right) = \rho \left(1-\frac{s_1}{s_2}\right) + \frac{s_1}{q} \Longleftrightarrow \rho := \frac{n \left(1-\frac{1}{q}\right) + \delta}{n \left(1-\frac{1}{s_2}\right) + \frac{\delta}{\sigma}}.
\]

We point out that \( \rho \leq \sigma \) since \( \beta \geq n(1-\sigma)(1-1/s_2) \) is equivalent to \( 1/q \geq 1 + \sigma / (s_2-1) \) and then

\[
n \left(1-\frac{1}{q}\right) + \delta \leq n\sigma \left(1-\frac{1}{s_2}\right) + \delta \Longleftrightarrow \rho = \frac{n \left(1-\frac{1}{q}\right) + \delta}{n \left(1-\frac{1}{s_2}\right) + \frac{\delta}{\sigma}} \leq \sigma. \quad (4.1)
\]
Summing up, from the admissible parameters it follows that \( n \left( s_1/p - 1 \right) < \lambda < n \left( s_1/s_2 - 1 \right) + s_1 \beta/(1 - \rho) \) and \( \lambda < n(s_1 - 1) + s_1 \delta \). Note that \( s_2 \beta/(1 - \rho) < (s_2 - 1)n + s_2 \delta \) since
\[
\beta < \frac{\delta \left( \frac{1}{\sigma} - 1 \right) \left[ n \left( 1 - \frac{1}{s_2} \right) + \delta \right]}{\delta \left( \frac{1}{\sigma} - 1 \right)} \iff s_2 \beta \left[ n \left( 1 - \frac{1}{s_2} \right) + \delta \right] - \beta + \frac{\delta}{\sigma} - \delta < (s_2 - 1)n + s_2 \delta
\]
and this implies that
\[
n \left( \frac{s_1}{s_2} - 1 \right) + \frac{s_1 \beta}{(1 - \rho)} < n(s_1 - 1) + s_1 \delta. \tag{4.2}
\]
This relation naturally implies a lower bound of \( p \) given by
\[
n \left( \frac{s_1}{p} - 1 \right) < n \left( \frac{s_1}{s_2} - 1 \right) + \frac{s_1 \beta}{1 - \rho} \iff \frac{1}{p} < \frac{1}{s_2} + \frac{\beta}{n} \left( \frac{s_1}{\sigma} - \delta + \beta \right) := \frac{1}{p_0}.
\]
The argument for \( s_1 = 1 \) and \( 0 < p < 1 \) follows in the same way with minor changes.

Clearly \( n/(n + \delta) < p_0 \) thus for \( p_0 < p \leq 1 \) follows \( n(1/p - 1) < \delta \). Since \( T^* (x^\alpha) = 0 \) for \( |\alpha| \leq |\delta| \), then (M5) is trivially valid. \( \square \)

**Proof of Theorem 1.2.** Let \( f \in L^\infty(\mathbb{R}^n) \). In order to prove that \( T \) maps continuously \( L^\infty(\mathbb{R}^n) \) into \( BMO(\mathbb{R}^n) \) we need to show that for any ball \( B \subset \mathbb{R}^n \) there exist a constant \( a_B \) (may depend on \( B \)) such that
\[
\sup_B \frac{1}{|B|} \int_B |Tf(x) - a_B| dx \leq C \|f\|_{L^\infty}
\]
where \( C > 0 \) is a constant independent of \( B \) (the left-hand side of the previous inequality define a norm in \( BMO(\mathbb{R}^n) \)). To do so, let \( B := B(z, r) \subset \mathbb{R}^n \) and suppose \( r \leq 1 \). Split \( f \) into
\[
f = f \chi_{B(z, 2r^\sigma)} + f \chi_{\mathbb{R}^n \setminus B(z, 2r^\sigma)} := f_1 + f_2.
\]
Since \( T^* : \mathcal{L}^q(\mathbb{R}^n) \to \mathcal{L}^{s_2}(\mathbb{R}^n) \) is bounded, then \( T : \mathcal{L}^{s_2'}(\mathbb{R}^n) \to \mathcal{L}^q'(\mathbb{R}^n) \) will also be bounded for \( \frac{1}{s_2'} = 1 - \frac{1}{s_2} \) and \( \frac{1}{q'} = \frac{1}{s_2'} - \frac{\beta}{n} \). In particular, since \( f_1 \in \mathcal{L}^{s_2'}(\mathbb{R}^n) \) then \( Tf_1 \) is well defined, belongs to \( \mathcal{L}^q'(\mathbb{R}^n) \) and
\[
\int_{B(z, r)} |Tf_1(x)| dx \leq |B(z, r)| \frac{1}{q} \|Tf_1\|_{L^{q'}} \lesssim |B(z, r)| \frac{1}{\sigma} \|f_1\|_{\mathcal{L}^{s_2'}} \lesssim |B(z, r)| \frac{1}{\sigma} \left( \frac{1}{q} + \frac{\sigma}{s_2'} \right) \|f\|_{L^\infty}.
\]
\[
\tag{4.3}
\]
Since \( 1/q + \sigma/s_2' - 1 \geq 0 \) and \( r \leq 1 \) we have \( |B(z, r)| \left( \frac{1}{q} + \frac{\sigma}{s_2'} \right) \leq 1 \). Then
\[
\frac{1}{|B(z, r)|} \int_{B(z, r)} |Tf_1(x)| dx \leq C \|f\|_{L^\infty} \tag{4.4}
\]
For \( f_2 \) we use condition (1.6) to show
\[
\int_{\mathbb{R}^n} |K(x, y) - K(z, y)| f_2(y) dy \leq \|f\|_{L^\infty} \int_{|y - z| > 2r^\sigma} |K(x, y) - K(z, y)| dy
\]
\[
= \|f\|_{L^\infty} \sum_{j=0}^{\infty} \int_{C_j(z, r^\sigma)} |K(x, y) - K(z, y)| dy \leq \|f\|_{L^\infty} \sum_{j=0}^{\infty} 2^{-\frac{12}{\sigma}} \lesssim \|f\|_{L^\infty}
\]
and from previous estimate
\[ \frac{1}{|B(z,r)|} \int_{B(z,r)} |Tf_2(x) - Tf_2(z)| \, dx \leq C \|f\|_{L^\infty}. \] (4.5)
Hence, we choose \( a_Q := Tf_2(z) \) and from (4.4) and (4.5) we conclude
\[ \frac{1}{|B(z,r)|} \int_{B(z,r)} |Tf(x) - Tf_2(z)| \, dx \leq \frac{1}{|B(z,r)|} \int_{B(z,r)} |Tf_1(x)| + |Tf_2(x) - Tf_2(z)| \, dx \leq C \|f\|_{L^\infty}. \]
The proof for \( r > 1 \) is analogous if we split \( f \) in \( B(z,2r) \) and \( \mathbb{R}^n \setminus B(z,2r) \). We point out that only (1.5) and \( L^2 \)-boundedness of \( T \) is required for this case. \( \square \)

4.1. Comments and remarks
The conclusion of Theorem 1.1 is still open for \( p = p_0 \), however under assumption \( s = 2 \) and \( D_1 \) condition, the [3, Theorem 3.9] asserts that \( T \) can be extended to a bounded operator from \( H^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for \( p_0 \leq p \leq 1 \). An extension of this result is presented in Corollary 5.2.

We point out the condition \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq |\delta| \) assumed in Theorem 1.1 can be refined to \( |\alpha| \leq N_{p_0} := \lfloor n(1/p_0 - 1) \rfloor \). Furthermore, this assumption is a necessary condition in Theorem 1.1. In fact, suppose that \( T \) maps continuously \( H^p(\mathbb{R}^n) \) to itself for all \( p_0 < p \leq 1 \) and let \( f \in \dot{L}^2_{\#,N_{p_0}}(\mathbb{R}^n) \).

Since \( f \) is a multiple of a \( (p,2) \)-atom it follows that \( Tf \) satisfies the conditions (M1)-(M4). As a particular case of Proposition 3.3, \( Tf \in L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n) \). Hence, by [17, Sec. 5.4 (c) p.128] it follows that
\[ \int Tf(x)x^\alpha \, dx = 0, \text{ for all } |\alpha| \leq N_p \text{ and } p_0 < p \leq 1. \]
Therefore it will also holds for \( N_{p_0} \), since \( p \setminus p_0 \).

In Theorem 1.2, the crucial hypothesis that \( T^* \) is a bounded operator from \( L^q(\mathbb{R}^n) \) to \( L^{s_2}(\mathbb{R}^n) \) may be weakened by the condition
\[ |B(z,r)|^{-1/s} \int_{B(z,r)} |Tf(x)| \, dx \leq C \|f\|_{L^{s_2}}, \]
i.e. \( T \) maps continuously \( L^{s_2}(\mathbb{R}^n) \) into \( \mathcal{M}_1^1(\mathbb{R}^n) \) where \( 1/\lambda = 1/s_2 - \beta/n \) and
\[ \mathcal{M}_1^1(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \sup_{0<r\leq 1} |B(z,r)|^{1-1/s} \int_{B(z,r)} |f(x)| \, dx < \infty \right\} \]
denotes the local Morrey-space with \( \lambda > 1 \).

4.2. Weaker integral derivative conditions
In this section, we consider strongly singular Calderón–Zygmund operators of type \( \sigma \) associated to kernels satisfying derivative conditions. Let \( \delta > 0 \) and \( K \in C^{1,\delta} \) away the diagonal on \( \mathbb{R}^{2n} \) satisfying
\[ |\partial_y^\gamma K(x,y) - \partial_y^\gamma K(x,z)| + |\partial_y^\gamma K(y,x) - \partial_y^\gamma K(z,x)| \leq C \frac{|y-z|^{\delta-|\gamma|}}{|x-z|^{n+\frac{\delta}{\sigma}}}, \] (4.6)
for \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| = |\delta|, |x-z| \geq 2|y-z|^\sigma \) and \( 0 < \sigma \leq 1 \).
Condition (4.6) is a natural generalization of derivative conditions usually assumed on standard $\delta$-kernels of type $\sigma$ satisfying (1.1) (see [9, p. 320] and [17, p. 117]). In the same way, we may replace the weaker integral $D_s$ condition (1.3) and (1.4) by the derivative $D_s$ condition
\[
\left( \int_{C_j(z,r)} \left| \partial_y^n K(x,y) - \partial_y^n K(x,z) \right|^s + \left| \partial_y^n K(y,x) - \partial_y^n K(z,x) \right|^s dx \right)^{\frac{1}{s}} \lesssim r^{-\lfloor \delta \rfloor} |C_j(z,r)|^{\frac{1}{s} - 1 - \frac{1}{2}} - j\delta
\] (4.7)
if $r > 1$ and
\[
\left( \int_{C_j(z,r^\rho)} \left| \partial_y^n K(x,y) - \partial_y^n K(x,z) \right|^s + \left| \partial_y^n K(y,x) - \partial_y^n K(z,x) \right|^s dx \right)^{\frac{1}{s}} \lesssim r^{-\lfloor \delta \rfloor} |C_j(z,r^\rho)|^{\frac{1}{s} - 1 + \frac{\delta}{\rho} \left( \frac{1}{2} - \frac{1}{s} \right)} 2^{-\frac{j\delta}{\rho}}
\] (4.8)
if $r < 1$.

We announce the following self-improvement of Theorem 1.1:

**Theorem 4.2.** Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a bounded and continuous operator and suppose that $T$ satisfies assumptions (i) and (iii) from Theorem 1.1 and

(ii)’ $T$ is associated to a kernel satisfying the derivative $D_{s_1}$ condition (4.7) and (4.8);

Then, if $T^*(x^\alpha) = 0$ for all $|\alpha| \leq |\delta|$, $1 < s_1 \leq 2$ and $s_1 \leq s_2$, the operator $T$ is bounded from $H^p(\mathbb{R}^n)$ to itself for $p_0 < p \leq 1$, where $p_0$ is given (1.7). Moreover, if $T^*$ also satisfies (iii) then the conclusion holds for $1 < s_1 < \infty$ and $s_1 \leq s_2$. The case $s_1 = 1$ also holds, however only for $p_0 < p < 1$.

The proof of the previous result is analogous of Theorem 1.1 since Taylor’s formula allows us to write $Ta(x) = \int_{B(z,r)} R(x,y)a(y)dy$, in which

\[
R(x,y) = \sum_{|\gamma|=M} \frac{(y-z)^\gamma}{\gamma!} \left[ \partial_y^n K(x,\xi_y) - \partial_y^n K(x,z) \right]
\]
for some $\xi_y$ in the line segment between $y$ and $z$.

Examples of operators satisfying such kernel conditions will be discussed in Sect. 5.2.

5. Applications

5.1. Weighted continuity

In this section we show that, under the hypothesis of Theorem 1.1, strongly singular Calderón–Zygmund operators are bounded from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$, where $w$ belongs to a special class of Muckenhoupt weight.

A non-negative measurable function $w(x)$ belongs to the class $A_1$ if there exists $C > 0$ such that for any ball $B \subset \mathbb{R}^n$ we have

\[
\frac{1}{|B|} \int_B w(y)dy \leq C w(x), \quad \text{for a.e. } x \in B.
\] (5.1)
In comparison with Lebesgue measure, if \( w \in A_1 \) then there exists \( c > 0 \) such that \( |E|w(B) \leq c |B|w(E) \) for any \( E \subseteq B \) in which \( B \subseteq \mathbb{R}^n \) and \( w(B) := \int_B w(x)dx \) (see [9, Chapter IV.2 - Theorem 2.1 (b)])). We say the weight \( w(x) \) satisfies the reverse Hölder inequality for \( 1 < r < \infty \), simply denoted by \( w \in RH_r \), if there exists a constant \( C > 0 \) such that
\[
\left( \frac{1}{|B|} \int_B w^r(x)dx \right)^\frac{1}{r} \leq C \left( \frac{1}{|B|} \int_B w(x)dx \right)
\]
for any ball \( B \subseteq \mathbb{R}^n \). Clearly if \( w \in RH_r \) then \( w \in RH_s \) for all \( 1 < s < r \).

We denote the weighted Lebesgue space \( L^p_w(\mathbb{R}^n) := L^p(\mathbb{R}^n, w(x)dx) \) to be the set of all measurable functions such that
\[
\|f\|_{L^p_w} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^\frac{1}{p} < \infty.
\]
In the same spirit, we define the weighted Hardy space, denoted by \( H^p_w(\mathbb{R}^n) \), to be the set of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that \( M_w f \in L^p_w(\mathbb{R}^n) \) and \( \|f\|_{H^p_w} := \|M_w f\|_{L^p_w} \) denotes its quasi-norm. We refer [18] for further details on the weighted Hardy space.

**Definition 5.1.** [18, p. 112] Let \( 0 < p \leq 1 \) and \( w \in A_1 \). We say that a measurable function \( a(x) \) is a \((w,p,\infty)\)-atom if there exists \( B(z,r) \subseteq \mathbb{R}^n \) such that
\[
\text{supp}(a) \subseteq B(z,r), \quad \|a\|_{L^\infty} \leq w(B(z,r))^{-\frac{1}{p}} \quad \text{and} \quad \int a(x)x^\alpha dx = 0
\]
for any multi-index such that \(|\alpha| \leq N_p\).

If \( w \in A_1 \) and \( f \in H^p_w(\mathbb{R}^n) \), then there exist a sequence of coefficients \( \{\lambda_j\}_j \) and \((w,p,\infty)\)-atoms \( \{a_j\}_j \) such that \( f = \sum \lambda_j a_j \), where the convergence is in \( H^p_w \)-norm. Moreover, \( \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \right\} \approx \|f\|_{H^p_w} \), where the infimum is taken over all such atomic representations of \( f \). In addiction, the converse of this result is also true. For a more general case of this result see [18, Chapter VIII, Theorem 1].

Now we present the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Let \( a(x) \) be a \((p,\infty)\)-atom in \( H^p_w(\mathbb{R}^n) \) supported on \( B(z,r) \). We will show that \( Ta \) is uniformly bounded in \( L^p_w \)-norm. Suppose first \( r > 1 \) and split
\[
\|Ta\|_{L^p_w}^p = \int_{B(z,2r)} |Ta(x)|^p w(x)dx + \sum_{j=1}^\infty \int_{C_j(z,r)} |Ta(x)|^p w(x)dx := I_1 + I_2.
\]

The first integral can be uniformly estimated from Hölder inequality with exponent \( 2/p \), the \( L^2 \)-continuity of \( T \) and from \( w \in RH_{2/(2-p)} \). In fact,
\[
I_1 \leq \left( \int |Ta(x)|^2 dx \right)^\frac{p}{2} \left( \int \frac{1}{|B(z,2r)|} \int_{B(z,2r)} w^{-\frac{p}{2}}(x)dx \right)^{-\frac{1}{p}} \left( |B(z,2r)|^{1-\frac{p}{2}} \right) \leq 1.
\]
For the second integral, note first that since \( w \in RH_{s_1/p(s_1-1)} \) it follows
\[
\int_{C_j(z,r)} |K(x,y) - K(x,z)|w^{\frac{1}{p}}(x)dx \leq \left( \int_{C_j(z,r)} |K(x,y) - K(x,z)|^{s_1}dx \right)^{\frac{1}{s_1}} \times \]
\[
\times \left( \int_{C_j(z,r)} w^{\frac{s_1}{p(s_1-1)}}(x)dx \right)^{1-\frac{1}{s_1}} \]
\[
\lesssim 2^{-j\delta} \left( \frac{1}{|C_j(z,r)|} \int_{C_j(z,r)} w^{\frac{s_1}{p(s_1-1)}}(x)dx \right)^{1-\frac{1}{s_1}} \]
\[
= 2^{-j\delta} \left( \frac{|B_{j+1}(z,r)|}{|C_j(z,r)|} \right)^{1-\frac{1}{s_1}} |B_{j+1}(z,r)|^{-\frac{1}{p}} w(B_{j+1}(z,r))^{\frac{1}{p}}. \tag{5.2} \]

Then,
\[
I_2 \leq \sum_{j=1}^{\infty} \int_{C_j(z,r)} \left( \int_{B(z,r)} |K(x,y) - K(x,z)| |a(y)|dy \right)^p w(x)dx \]
\[
\leq \sum_{j=1}^{\infty} w(B(z,r))^{-1} \left( \int_{C_j(z,r)} \int_{B(z,r)} |K(x,y) - K(x,z)| w^{\frac{1}{p}}(x)dydx \right)^p |C_j(z,r)|^{1-p} \]
\[
= \sum_{j=1}^{\infty} w(B(z,r))^{-1} \left( \int_{B(z,r)} \left( \int_{C_j(z,r)} |K(x,y) - K(x,z)| w^{\frac{1}{p}}(x)dx \right)dy \right)^p |C_j(z,r)|^{1-p} \]
\[
\lesssim \sum_{j=1}^{\infty} \frac{w(B_{j+1}(z,r))}{w(B(z,r))} |C_j(z,r)|^{1-p} |B(z,r)|^p \left( \frac{|B_{j+1}(z,r)|}{|C_j(z,r)|} \right)^{p(\frac{1}{p}-\frac{1}{s_1})} \times \]
\[
\times |B_{j+1}(z,r)|^{-1} 2^{-j\delta} \]
\[
\lesssim \sum_{j=1}^{\infty} |C_j(z,r)|^{-1+p-p(\frac{1}{p}-\frac{1}{s_1})} |B(z,r)|^{p-1} |B_{j+1}(z,r)|^{p(\frac{1}{p}-\frac{1}{s_1})} 2^{-j\delta} \]
\[
\lesssim \sum_{j=1}^{\infty} 2^{j[n-p(n+\delta)]} \lesssim 1 \]

since \( p > \frac{n}{n+\delta} \). Let us consider now the case \( 0 < r \leq 1 \). In the same way, we split
\[
||Ta||_{L_w^p}^p = \int_{B(z,2r^\rho)} |Ta(x)|^p w(x)dx + \sum_{j=1}^{\infty} \int_{C_j(z,r^\rho)} |Ta(x)|^p w(x)dx := I_3 + I_4 \]
for some \( 0 < \rho \leq \sigma \) that will be chosen conveniently later. For the first integral, using Hölder inequality with exponent \( s_2/p \), the \( L^s - L^{s_2} \) continuity of \( T \) and \( w \in RH_{s_2/(s_2-1)} \) implies
\[
\int_{B(z,2r^\rho)} |Ta(x)|^p w(x)dx \]
\[
\leq ||Ta||_{L_w^{s_2}}^p \left( \frac{1}{|B(z,2r^\rho)|} \int_{B(z,2r^\rho)} w^{\frac{s_2}{s_2-p}}(x)dx \right)^{1-\frac{p}{s_2}} |B(z,2r^\rho)|^{1-\frac{p}{s_2}} \]
\[
\lesssim \frac{w(B(z,2r^\rho))}{w(B(z,r))} |B(z,r)|^{\frac{p}{s_2}} |B(z,2r^\rho)|^{-\frac{p}{s_2}} \]
\[
\lesssim |B(z,r)|^{\frac{p}{s_2}-1} |B(z,2r^\rho)|^{1-\frac{p}{s_2}} \]
\[
\lesssim r^{\frac{p}{s_2}-1+\rho \left( 1-\frac{p}{s_2} \right)} \lesssim 1 \]
for $\rho \geq \rho_1 := \frac{1-p\left(\frac{1}{p}+\frac{\rho}{n}\right)}{1-\frac{\rho}{n}}$. For the second integral, proceeding just like in (5.2), it follows from $w \in RH_{s_1/p(s_1-1)}$ and $D_{s_1}$ condition that

$$
\int_{C_j(z,r^\rho)} |K(x,y) - K(x,z)|w^{\frac{1}{p}}(x)dx
\lesssim |C_j(z,r^\rho)|^{\frac{1}{p} - 1} \lesssim B_{j+1}(z,r^\rho)|^{1-\frac{1}{p}} \lesssim B_{j+1}(z,r^\rho)^{1-\frac{1}{p}} w(B_{j+1}(z,r^\rho))^{\frac{1}{p}} 2^{-j\frac{\rho}{n}}.
$$

Then,

$$
I_4 \lesssim \sum_{j=1}^{\infty} \left( \frac{w(B_{j+1}(z,r^\rho))}{w(B(z,r))} \right) |B(z,r)|^p |C_j(z,r^\rho)|^{\frac{p}{n} - 1} |B_{j+1}(z,r^\rho)|^{1-\frac{1}{p}} \rho - \frac{\rho}{n} - 2^{-j\frac{\rho}{n}} \times
$$

$$
\lesssim \sum_{j=0}^{\infty} |B(z,r)|^p |C_j(z,r^\rho)|^{\frac{p}{n} - 1} \rho - \frac{\rho}{n} - 2^{-j\frac{\rho}{n}} \left( \frac{|B_{j+1}(z,r^\rho)|}{|C_j(z,r^\rho)|} \right) \rho^{1-\frac{1}{s_1}} 2^{-j\frac{\rho}{n}}
\lesssim r^{-\rho(n(p-1)+\frac{n\rho}{p})+p\delta+n(p-1)} \sum_{j=1}^{\infty} 2^j |n-p(n+\frac{\rho}{p})| \lesssim 1
$$

in which $\rho \leq \rho_2 := \frac{p(n+\delta)-n}{p(n+\frac{\rho}{p})-n} \leq \sigma$. The restriction $\rho_1 \leq \rho_2$ implies that uniform estimate holds for every $p \geq p_0$. Hence, given $f \in H^p_w(\mathbb{R}^n)$, by standard arguments one has

$$
\|Tf\|_{L^p_w} \leq \sum_{j \in \mathbb{N}} |\lambda_j|^p \|Ta\|_{L^p_w} \lesssim \|f\|_{H^p_w},
$$

which concludes the proof.

We remark that since $p_0 \leq p \leq 1$, we may replace the assumption $w \in A_1 \cap RH_{d}$ for $d = \max \left\{ \frac{s}{s-1}, \frac{s_1}{p(s_1-1)} \right\}$ at Theorem 1.4 by the stronger condition $w \in A_1 \cap RH_{d_0}$ for $d_0 = \max \left\{ \frac{s}{s-1}, \frac{s_1}{p_0(s_1-1)} \right\}$, where $s = \min \{2, s_2\}$.

A direct consequence of Theorem 1.4 is the following:

**Corollary 5.2.** Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear and continuous operator as in Theorem 1.1. Then, $T$ can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p_0 \leq p \leq 1$, with $p_0$ given by (1.7).

### 5.2. A special class of pseudodifferential operators

It is well understood that pseudodifferential operators in the class $\text{Op}S^m_{\sigma,b}(\mathbb{R}^n)$ for certain parameters $m, \sigma$ and $b$ have distributional kernels satisfying the pointwise estimate (1.1) for $\delta = 1$. This can be easily seen for operators $\text{Op}S^{-n(1-\sigma)}_{\sigma,b}(\mathbb{R}^n)$ from derivative estimate of the kernel presented in [1, Theorem 1.1 (d)]. However, integral estimates are more suitable in dealing with this type of operators and using them we have the advantage of finding a wider set of examples. In [2, Section 3], the authors have shown that $\text{Op}S^{-n}_{\sigma,b}(\mathbb{R}^n)$
for $0 < b < 1$ and $n(1 - \sigma)/2 \leq m < n/2$ satisfies a Hormander type condition instead the pointwise condition (1.1), i.e.,
\[
\int_{|x| \geq 2r^\sigma} |K(x + z, x - y) - K(x + z, x)| dx \\
+ \int_{|x| \geq 2r^\sigma} |K(x - y, x + z) - K(x, x + z)| dx \leq C
\]
for all $z \in \mathbb{R}^n$, $|y| \leq r$ and $r \geq 0$. This represents the weakest condition known so far, but unfortunately it is still an open question to prove continuity on $H^p(\mathbb{R}^n)$ from it (see for instance the counterexample in [22]).

In this section, we present classes of pseudodifferential operators satisfying the hypothesis of Theorems 1.1 and 4.2. We start verifying the derivative $D_{s_1}$ condition for $1 \leq s_1 \leq 2$, extending the case $s_1 = 1$ and $|\gamma| = 0$ proved in [1, Theorem 2.1].

**Proposition 5.3.** Let $\delta > 0$ and $T \in OpS^{m}_d(\mathbb{R}^n)$ with $0 < \sigma \leq 1$, $0 < b < 1$, $b \leq \sigma$ and $m \leq -n(1 - \sigma)/2$. If $1 \leq s_1 \leq 2$, then $T$ satisfies the derivative $D_{s_1}$ condition with decay $|\delta| + 1$. In particular, when $0 < \delta < 1$ it satisfies integral conditions (1.3) and (1.4) with decay 1.

It follows from [1, Theorem 3.5] that $T$ maps continuously $L^q(\mathbb{R}^n)$ into $L^{s_2}(\mathbb{R}^n)$ where $\frac{1}{q} = \frac{1}{s_2} + \frac{\beta}{n}$ and $n(1 - \sigma) \left( 1 - \frac{1}{s_2} \right) \leq \beta < n \left( 1 - \frac{1}{s_2} \right)$ since:

(a1) $m \leq -\beta - n(1 - \sigma) \left( \frac{1}{s_2} - \frac{1}{2} \right)$, if $1 < q \leq s_2 \leq 2$;

(a2) $m \leq -\beta$, if $1 < q \leq 2 \leq s_2$;

(a3) $m \leq -n(1 - \sigma)/2$, if $2 \leq q \leq s_2$.

Note that $m \leq -n(1 - \sigma)/2$ in all the cases and since $0 \leq b \leq \sigma < 1$ we have that $T \in OpS^{m}_d(\mathbb{R}^n)$ is bounded from $L^2(\mathbb{R}^n)$ to itself. Now we present the proof of Proposition 5.3.

**Proof.** Let $T \in OpS^{m}_d(\mathbb{R}^n)$, $K$ its distributional kernel and we denote by $\tilde{K}(x, y) = \partial_y^n K(x, y)$ for $|\gamma| = |\delta|$. In order to obtain the derivative $D_{s_1}$ condition for $1 \leq s_1 < 2$ is suffices to prove it for $s_1 = 2$. We claim that under the restriction $m \leq -n[(1 - \sigma)/2 + \lambda]$ in which $\lambda = \max\{0, (b - \sigma)/2\}$ it follows for $r \geq 1$

$$
\sup_{|y-z| \leq r} \left( \int_{C_r(z, r)} |\tilde{K}(x, y) - \tilde{K}(x, z)|^2 dx \right)^{\frac{1}{2}} \lesssim r^{-|\delta|} |C_2(z, r)|^{-\frac{1}{2}} 2^{-j(|\delta|+1)},
$$

and for $r < 1$

$$
\sup_{|y-z| \leq r} \left( \int_{C_r(z, r^\rho)} |\tilde{K}(x, y) - \tilde{K}(x, z)|^2 dx \right)^{\frac{1}{2}} \lesssim r^{-|\delta|} |C_2(z, r^\rho)|^{-\frac{1}{2}} 2^{-\frac{1}{2}(|\delta|+1)} 2^{-\lambda(|\delta|+1)}.
$$

The same estimate for the adjoint $\tilde{K}(y, x)$ will be treated in the end assuming $m \leq -n(1 - \sigma)/2$. 
The proof consists an adaptation of [1, Theorem 2.1]. Assume without loss of generality that the symbol \( p(x, \xi) \) associated to the operator \( T \) vanishes for \( |\xi| \leq 1 \) and consider \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}) \) a non-negative function such that \( \text{supp}(\psi) \subset [1/2, 1] \) and

\[
\int_0^\infty \psi \left( \frac{1}{t} \right) \frac{1}{t} dt = \int_1^2 \psi \left( \frac{1}{t} \right) \frac{1}{t} dt = 1. \tag{5.3}
\]

Define \( K(x, y, t) = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} p(x, \xi) \psi \left( \frac{|\xi|}{t} \right) d\xi \) and consequently

\[
\tilde{K}(x, y, t) = (-i)^{|\delta|} (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} p(x, \xi) \psi \left( \frac{|\xi|}{t} \right) \xi^\gamma d\xi.
\]

By the standard representation of the kernel of a pseudodifferential operator and from (5.3) we may write

\[
\tilde{K}(x, y) = \int_0^\infty \tilde{K}(x, y, t) \frac{dt}{t} = \int_1^\infty \tilde{K}(x, y, t) \frac{dt}{t}. \tag{5.4}
\]

Consider first \( 0 < r < 1 \). From Minkowski inequality for integrals

\[
\left( \int_{C_j(z, r^r)} |\tilde{K}(x, y) - \tilde{K}(x, z)|^2 dx \right)^{\frac{1}{2}} \leq \int_{C_j(z, r^r)} \left( \int_1^\infty |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)| \frac{dt}{t} \right)^2 dx
\]

\[
= \left\{ \left( \int_{C_j(z, r^r)} \left( \int_1^\infty |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)| \frac{dt}{t} \right)^2 dx \right)^{\frac{1}{2}} \right\}^2
\]

\[
\leq \left\{ \int_1^\infty \left( \int_{C_j(z, r^r)} |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)|^2 dx \right)^{\frac{1}{2}} \frac{dt}{t} \right\}^2.
\]

Let \( \Gamma(t) = \|\tilde{K}(\cdot, y, t) - \tilde{K}(\cdot, z, t)\|_{L^2[C_j(z, r^r)]} \) and then

\[
\left( \int_{C_j(z, r^r)} |\tilde{K}(x, y) - \tilde{K}(x, z)|^2 dx \right)^{\frac{1}{2}} \leq \int_1^{r^{-1}} \Gamma(t) \frac{dt}{t} + \int_{r^{-1}}^\infty \Gamma(t) \frac{dt}{t} = I_1 + I_2. \tag{5.5}
\]

Let’s deal first with \( I_1 \), in which the estimate relies strongly on the assumption \( tr < 1 \). Throughout this proof we consider \( N \in \mathbb{Z}_+ \) a constant that will be chosen conveniently after. Clearly

\[
\Gamma(t) \leq \left( \int |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)|^2 (1 + t^{2\sigma} |x - z|^2)^N dx \right)^{\frac{1}{2}}
\]

\[
\times \sup_{x \in C_j(z, r^r)} (1 + t^{2\sigma} |x - z|^2)^{-\frac{N}{2}}.
\]

We claim that for \( m \leq -n[(1 - \sigma)/2 + \lambda] \)

\[
\left( \int_{\mathbb{R}^n} |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)|^2 (1 + t^{2\sigma} |x - z|^2)^N dx \right)^{1/2}
\]

\[
\lesssim (tr)^{\frac{m}{2} + [\delta]} \quad \text{for} \quad tr \leq 1 \tag{5.6}
\]
and \( \sup_{x \in C_j(z,r^\rho)} (1 + t^{2\sigma} |x - z|^2)^{-N/2} \leq [1 + t^{2\sigma} (2^j r^\rho)^2]^{-N/2} \). Using these estimates and the change of variables \( \omega = t^{\sigma} 2^j r^\rho \) we obtain

\[
\int_1^{r^{-1}} \Gamma(t) \frac{dt}{t} \lesssim \int_1^{r^{-1}} r t^{\frac{\sigma n}{2} + |\delta|} [1 + t^{2\sigma} (2^j r^\rho)^2]^{-\frac{N}{2}} dt
\]

\[
\lesssim r^{1 - \frac{\sigma n}{2} - \frac{\delta}{n}(1 + |\delta|)} (2^j)^{-\frac{n}{2} - 1 - |\delta|} \int_{2^j r^\rho}^{2^2 r^\rho} \frac{\omega^{n-1 + |\delta|}}{(1 + \omega^2)^{\frac{N}{2}}} d\omega
\]

\[
\lesssim r^{-|\delta|} |C_j(z, r^\rho)|^{-\frac{1}{2} + |\delta| + |\frac{1}{n} - \frac{1}{2}|} 2^{-\frac{1}{2} (|\delta| + 1)}, \quad (5.7)
\]

since \( \int_0^\infty \frac{\omega^{n-1 + |\delta|}}{(1 + \omega^2)^{\frac{N}{2}}} d\omega < \infty \) for \( N > \frac{n}{2} + \frac{1 + |\delta|}{\sigma} \). Let us give an idea of the proof of (5.6). Using integration by parts, for \( \alpha \in \mathbb{Z}_+^n \) such that \( |\alpha| \leq N \) we may write

\[
t^{\sigma |\alpha|} (x - z)^\alpha [\tilde{K}(x, y, t) - \tilde{K}(y, z, t)]
\]

\[
= \sum_{|\beta| \leq |\alpha|} C_{\alpha, \beta} t^{\sigma |\alpha| + |\beta|} \int e^{i(x-y) \cdot \xi} |\xi|^{n(1-\sigma)/2 + |\beta|} \partial_\xi^\beta \left[ (e^{i(z-y) \cdot \xi} - 1)p(x, \xi) \right]
\]

\[
\times |\xi|^{-n(1-\sigma)/2 - |\beta|} \partial_\xi^{-\beta} \left[ \psi \left( \frac{|\xi|}{t} \right) \left( \frac{\xi}{t} \right)^\gamma \right] d\xi. \quad (5.8)
\]

Since \( |e^{i(z-y) \cdot \xi} - 1| \leq tr \) and \( |\partial_\xi^\beta e^{i(z-y) \cdot \xi} | \lesssim |\xi|^{-|\beta|} (tr)^{|\beta|} \) one can show that if \( \chi \in C_0^\infty(\mathbb{R}^n) \) is a function such that \( \chi = \psi \) on the support of \( \psi \), then

\[
\left\{ |\xi|^{n(1-\sigma)/2 + |\beta|} \partial_\xi^\beta \left[ (e^{i(z-y) \cdot \xi} - 1)p(x + z, \xi) \right] \chi(|\xi|/t) : |y - z| < r, \ z \in \mathbb{R}^n \right\}
\]

is a bounded subset of \( S_{\sigma,b}^{m-n(1-\sigma)/2}(\mathbb{R}^n) \) with bounds being less than or equal to \( Ctr \). Therefore, since \( m \leq -n \lambda \), the family of symbols above defines pseudodifferential operators that are bounded in \( L^2(\mathbb{R}^n) \) with norm proportional to \( Ctr \) (see [11, Theorem 1]). Therefore from this consideration and (5.8) we have

\[
\left( \int_{\mathbb{R}^n} |\tilde{K}(x, y, t) - \tilde{K}(x, z, t)|^2 (1 + t^{2\sigma} |x - z|^2)^N dx \right)^{1/2}
\]

\[
\lesssim tr \sum_{|\alpha| \leq N} \sum_{|\beta| \leq |\alpha|} C_{\alpha} \left| t^{\sigma |\alpha| + |\beta|} |\xi|^{-n(1-\sigma)/2 - |\beta|} \partial_\xi^{-\beta} \left[ \psi \left( \frac{|\xi|}{t} \right) \left( \frac{\xi}{t} \right)^\gamma \right] \right|_{L^2}
\]

\[
\lesssim (tr) t^{\frac{\sigma n}{2} + |\delta|}.
\]

On the other hand, to control \( I_2 \) we split

\[
\Gamma(t) \leq \| \tilde{K}(\cdot, y, t) \|_{L^2[C_j(z,r^\rho)]} + \| \tilde{K}(\cdot, z, t) \|_{L^2[C_j(z,r^\rho)]}.
\]

If \( x \in C_j(z,r^\rho) \) and \( |y - z| < r < 1 \), then \( |x - y| \geq |x - z| - |y - z| \geq 2^{j-1} r^\rho \) and

\[
\| \tilde{K}(\cdot, y, t) \|_{L^2[C_j(z,r^\rho)]} \leq \left( \int_{\mathbb{R}^n} |\tilde{K}(x, y, t)|^2 (t^{2\sigma} |x - y|^2)^N dx \right)^{1/2} \times
\]

\[
\sup_{|x - y| > 2^{j-1} r^\rho} (t^{2\sigma} |x - y|^2)^{-N/2}.
\]
We claim that

$$\left( \int_{\mathbb{R}^n} |\tilde{K}(x, y, t)|^2 (t^{2\sigma}|x - y|^2)^N \, dx \right)^{\frac{1}{2}} \lesssim t^{\frac{n}{2} + |\delta|} \tag{5.9}$$

and the second term is clearly estimated by \((t^{\sigma} 2^{j-1} r^\rho)^{-N}\). Thus \(\|\tilde{K}(\cdot, y, t)\|_{L^2[C_j(z, r^\rho)]} \lesssim t^{\sigma n/2 + |\delta|} (t^{\sigma} 2^{j-1} r^\rho)^{-N}\). Analogously \(\|\tilde{K}(\cdot, z, t)\|_{L^2[C_j(z, r^\rho)]} \lesssim t^{\sigma n/2 + |\delta|} (t^{\sigma} 2^{j-1} r^\rho)^{-N}\). Using these estimates and assuming \(N > \frac{n}{2} + \frac{|\delta| + 1}{\sigma}\) we obtain

$$\int_{r-1}^{\infty} \Gamma(t) \frac{dt}{t} \lesssim \int_{r-1}^{\infty} t^{\frac{n}{2} + |\delta| - \sigma N - 1} (2^j r^\rho)^{-N} dt \lesssim (2^j r^\rho)^{-\frac{n}{2} - \frac{1 + |\delta|}{\sigma}} r$$

$$\lesssim r^{-|\delta|} |C_j(z, r^\rho)|^{-\frac{1}{2} + \frac{|\delta| + 1}{\sigma} \left( \frac{1}{2} - \frac{1}{b} \right) 2^{-\frac{1}{2}(|\delta| + 1)}. \tag{5.10}$$

It just remains to show now (5.9). In the same spirit as previously, taking \(|\alpha| = N\) we may write

$$t^{\sigma |\alpha|} (x - y) \tilde{K}(x, y, t) \approx \sum_{|\beta| \leq |\alpha|} t^{\sigma |\alpha| + |\delta|} \int e^{i(x-y) \cdot \xi} \partial_x^\beta \partial_\xi^\alpha p(x + y, \xi) \partial_\xi^\gamma \left[ \psi \left( \frac{|\xi|}{t} \right) \frac{\xi}{t} \right] d\xi.$$ 

Since the class of symbols \(\left\{ |\xi|^{n(1 - \sigma)/2 + |\alpha|} |\beta| \partial_x^\beta p(x + y, \xi) : y \in \mathbb{R}^n \right\}\) are a bounded subset of \(S_{\sigma, b}^{m+n(1 - \sigma)/2}(\mathbb{R}^n)\), follows directly that the family of pseudodifferential associated is uniformly bounded on \(L^2(\mathbb{R}^n)\). Therefore

$$\left( \int_{\mathbb{R}^n} |\tilde{K}(x, y, t)|^2 (t^{2\sigma}|x - y|^2)^N \, dx \right)^{\frac{1}{2}} \lesssim \sum_{|\beta| \leq |\alpha|} t^{\sigma |\alpha| + |\delta|} \left\| \left| \xi \right|^{n(1 - \sigma)/2} |\beta| \partial_x^\beta \partial_\xi^\alpha \left[ \psi \left( \frac{|\xi|}{t} \right) \frac{\xi}{t} \right] \right\|_{L^2}$$

Now we are moving on to the case \(r > 1\). Since we can estimate \(\|\tilde{K}(\cdot, y, t)\|_{L^2[C_j(z, r)]}\) and \(\|\tilde{K}(\cdot, z, t)\|_{L^2[C_j(z, r)]}\) in the same way as before, we obtain for

\(N > \max \left\{ \frac{n}{2} + \frac{|\delta|}{\sigma}, \frac{n}{2} + |\delta| + 1 \right\}\),

$$\left( \int_{C_j(z, r)} |\tilde{K}(x, y) - \tilde{K}(x, z)|^2 \, dx \right)^{\frac{1}{2}} \leq \int_1^\infty \left( \|\tilde{K}(\cdot, y, t)\|_{L^2[C_j(z, r)]} + \|\tilde{K}(\cdot, z, t)\|_{L^2[C_j(z, r)]} \right) \frac{dt}{t}$$

$$\lesssim (2^j r)^{-N} \int_{r-1}^{\infty} t^{\frac{n}{2} + |\delta| - \sigma N - 1} dt \lesssim r^{-\frac{n}{2} - |\delta| - (1 - \sigma) - |\delta| (1 - \sigma)} (2^j)^{-\frac{n}{2} - |\delta| - 1}$$

$$\lesssim r^{-|\delta|} |C_j(z, r)|^{-\frac{1}{2} - j(1 + |\delta|)}. \tag{5.11}$$
Now we deal with estimates of the adjoint. Suppose first $0 < r < 1$. Since
\[
(-i)^{-|\gamma|}(2\pi)^n \tilde{K}(y, x, t) - \tilde{K}(z, x, t) = \int e^{-i(x-y)\cdot\xi} [p(y, \xi) - p(z, \xi)] \psi \left( \frac{\xi}{t} \right) \xi^\gamma d\xi
\]
+ \int e^{ix\cdot\xi} (e^{iy\cdot\xi} - e^{iz\cdot\xi}) p(z, \xi) \psi \left( \frac{\xi}{t} \right) \xi^\gamma d\xi
\]
= f(x - y, y, z, t) + g(x, y, z, t),
\]
then
\[
\left( \int_{C_j(z, r^\rho)} |\tilde{K}(y, x, t) - \tilde{K}(z, x, t)|^2 dx \right)^{\frac{1}{2}} \lesssim \|g(\cdot, y, z, t)\|_{L^2[C_j(z, r^\rho)]} + \|f(\cdot, y, y, z, t)\|_{L^2[C_j(z, r^\rho)]}
\]
and we will obtain analogous estimates for the $L^2$-norm as presented before. Suppose first $tr < 1$ and note that
\[
|g(x, y, z, t)|^2 (1 + t^{2\sigma} |x|^2)^N = \sum_{|\alpha| \leq N} \left[ |g(x, y, z, t)| (t^{\sigma} |x|^{|\alpha|})^2 \right]. \tag{5.12}
\]
Considering $G(\xi, y, z, t) = (e^{iy\cdot\xi} - e^{iz\cdot\xi}) p(z, \xi) \psi (|\xi|/t) \xi^\gamma$ and taking the Fourier transform in the first variable we have the identity $\tilde{G}(x, y, z, t) = (2\pi)^{-n} g(x, y, z, t)$. In addition, from mean value inequality it follows for $|y - z| \leq r$ and $tr < 1$ that
\[
|\partial_{\xi}^\beta [(e^{iy\cdot\xi} - e^{iz\cdot\xi}) p(z, \xi)]| \lesssim (tr)^{t^{m-\sigma|\beta|}}. \tag{5.13}
\]
Then, from (5.12) and (5.13)
\[
\left( \int_{R^n} |g(x, y, z, t)|^2 (1 + t^{2\sigma} |x|^2)^N dx \right)^{\frac{1}{2}} \lesssim \sum_{|\alpha| \leq N} t^{\sigma|\alpha|} \|\tilde{G}(\cdot, y, z, t)|_{x^{|\alpha|}}\|_{L^2}
\]
\[
\lesssim \sum_{|\alpha| \leq N} t^{\sigma|\alpha|} \|\partial_{\xi}^\beta G(\cdot, y, z, t)\|_{L^2}
\]
\[
\leq \sum_{|\alpha| \leq N} t^{\sigma|\alpha|+|\beta|} \|\partial_{\xi}^\beta [(e^{iy\cdot\xi} - e^{iz\cdot\xi}) p(z, \xi)] \partial_{\xi}^{-\beta} \left[ \psi \left( \frac{\xi}{t} \right) \left( \frac{\xi}{t} \right) \right] \|_{L^2}
\]
\[
\lesssim \sum_{|\alpha| \leq N} t^{\sigma|\alpha|+|\beta|} (tr)^{t^{m-\sigma|\beta|} |t| - |\alpha|} t^{\frac{N}{2}}
\]
\[
\lesssim (tr)^{t^{m-\sigma|\alpha|}}
\]
since $m \leq -n(1 - \sigma)/2$. The estimate for $f$ follows by the same steepe as presented for $g$. We proceed as before replacing $G$ by $G'(\xi, y, z, t) = [p(y, \xi) - p(z, \xi)] \psi (|\xi|/t) \xi^\gamma$ and using the estimate
\[
|\partial_{\xi}^\beta [p(y, \xi) - p(z, \xi)]| \leq (tr)^{t^{m-\sigma|\alpha|}},
\]
Thus, the conclusion follows in the same way did in (5.7). If we drop the assumption $tr < 1$ we will proceed as following. Write
\[
g(x, y, z, t) = \int e^{-i(x-y)\cdot\xi} p(z, \xi) \psi \left( \frac{\xi}{t} \right) \xi^\gamma d\xi - \int e^{-i(x-z)\cdot\xi} p(z, \xi) \psi \left( \frac{\xi}{t} \right) \xi^\gamma d\xi
\]
\[
=: g_1(x, y, z, t) - g_2(x, y, z, t).
\]
Thus, we will obtain the $L^2$-norm estimate for $g_1$ and $g_2$. In the same way as before
\[
\|g_2(\cdot, y, z, t)\|_{L^2[C_j(z, r^\rho)]} \leq \left( \int \|g_2(x, y, z, t)\|^2 \left[ (x - z)^2 t^{2\sigma} \right] dx \right)^{\frac{1}{2}} \sup_{x \in C_j(z, r^\rho)} \left[ (x - z)t^{\sigma} \right]^{-N}
\]
and consider $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| = N$. Integration by parts gives us
\[
g_2(x, y, z, t)(x - z)^{\alpha}t^{\sigma|\alpha|} = C t^{\sigma|\alpha|} \partial^{\alpha}_\xi G(x - z, y, z, t),
\]
where $G(\xi, y, z, t) = p(z, \xi) \psi(|\xi|/t) \xi^\gamma$. Using that
\[
|\partial^{\alpha}_\xi G(\xi, y, z, t)| \leq t^{[\delta]} \sum_{|\beta| \leq |\alpha|} |\partial^\beta \psi(z, \xi)| \left| \partial^{\alpha - \beta}_\xi \left( \psi \left( \frac{|\xi|}{t} \right) \left( \frac{\xi}{t} \right) \right) \right|
\]
we get
\[
\|g_2(\cdot, y, z, t)(x - z)^{\alpha}t^{\sigma|\alpha|}\|_{L^2[C_j(z, r^\rho)]} \leq \|t^{\sigma|\alpha|} \partial^{\alpha}_\xi G(\cdot - z, y, z, t)\|_{L^2}
\]
\[
= \|t^{\sigma|\alpha|} \partial^{\alpha}_\xi G(\cdot - z, y, z, t)\|_{L^2}
\]
\[
\lesssim t^{m - \sigma|\alpha| + |\delta|} t^{(1 - \sigma)(|\beta| - |\alpha|)} \lesssim t^{m - \sigma|\alpha| + |\delta|}
\]
since $m \leq -(1 - \sigma)n/2$. On the other hand, the same estimate for $g_1$ is valid.
Indeed
\[
\|g_1(\cdot, y, z, t)\|_{L^2[C_j(z, r^\rho)]} \leq \left( \int \|g_1(x, y, z, t)\|^2 \left[ (x - y)^2 t^{2\sigma} \right] dx \right)^{\frac{1}{2}} \sup_{x \in C_j(z, r^\rho)} \left[ (x - y)t^{\sigma} \right]^{-N}.
\]
The control of the integral is analogous as in the previous case and for supremum term note that since $r < 1$, $x \in C_j(z, r^\rho)$ and $|y - z| < r$ we get $|x - y| > 2^{j - 1}r^\rho$ and thus
\[
\sup_{x \in C_j(z, r^\rho)} \left[ (x - y)t^{\sigma} \right]^{-N} \leq (2^{j - 1}r^\rho t^{\sigma})^{-N}.
\]
From that point we proceed as in (5.10) and obtain the desired estimates for $g$. The same argument applies to $f$ if we split
\[
f(x - y, y, z, t) = \int e^{-i(x - y) \cdot \xi} p(y, \xi) \psi \left( \frac{|\xi|}{t} \right) \xi^{\gamma} d\xi
\]
\[
- \int e^{-i(x - y) \cdot \xi} p(z, \xi) \psi \left( \frac{|\xi|}{t} \right) \xi^{\gamma} d\xi
\]
and the estimate follows exactly in the same way as did for $g$.

The case $r \geq 1$ is analogous as the previous and we obtain that
\[
\|g(\cdot, y, z, t)\|_{L^2[C_j(z, r)]} \lesssim t^{\frac{m}{2} + |\delta|} (2^j r t^{\sigma})^{-N}
\]
and
\[ \| f(\cdot - y, y, z, t) \|_{L^2[C_j(z, r)]} \lesssim t^{\frac{n}{2} + [\delta]} (2^j r t^\sigma)^{-N}. \]

Thus, the desired estimate follows as in (5.11).

5.3. Strongly singular $\theta$-Calderón–Zygmund operator of type $\sigma$

K. Yabuta considered in [21, Definition 2.1] a generalization of classical Calderón–Zygmund operators assuming a $\theta$-modulus of continuity of the kernel and a complete study on boundedness ($L^p - L^p$ for $1 < p < \infty$, $L^\infty - \text{BMO}$ and $H^1 - L^1$) of this type of operators. Kernels satisfying $\theta$-modulus of continuity are related with general classes of pseudodifferential operators (beyond Hörmander class), see for instance [21, Theorems 3.1 and 3.2].

In this short subsection we introduce a generalization of strongly singular Calderón–Zygmund operators of type $\sigma$ assuming an analogous $\theta$-modulus of continuity of the kernel. This has its own interests and can lead to new paths in connection to pseudo-differential operators associated to rough symbols.

**Definition 5.4.** Let $\theta : (0, \infty) \to (0, \infty)$ be an increasing function and $0 < \sigma \leq 1$. We say that a continuous function $K(x, y)$ defined on $\mathbb{R}^{2n}$ away the diagonal is a $\theta$-kernel of type $\sigma$ if
\[ |K(x, y)| \lesssim \frac{1}{|x - y|^n} \quad \forall \ x \neq y \] (5.14)
and
\[ |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \lesssim \theta \left( \frac{|y - z|}{|x - z|^{\frac{1}{\sigma}}} \right) |y - z|^{-n} \] (5.15)
for all $|x - z| \geq 2|y - z|^{\sigma}$. A linear and continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is called a strongly singular $\theta$-Calderón–Zygmund operator of type $\sigma$ if it satisfies conditions (i) and (iii) of Theorem 1.1.

Clearly if $\theta(t) = t^\delta$ for some $0 < \delta \leq 1$, then we recover (1.1) and if one consider the Dini-condition
\[ \int_0^1 \frac{\theta(t)}{t} dt < \infty \]
then we recover (1.2) with $\delta = 1$.

**Theorem 5.5.** Let $0 < p \leq 1$ and $T$ a strongly singular $\theta$-Calderón–Zygmund operator of type $\sigma$. If $T^*(x^\alpha) = 0$ for every $|\alpha| \leq [\delta]$ and
\[ \int_0^1 \frac{\theta(t)^{s_1}}{t^{1+8s_1}} \ dt < \infty \] (5.16)
for some $\delta > 0$ and $1 \leq s_1 < \infty$ with $p < s_1$, then $T$ is a bounded operator on $H^p(\mathbb{R}^n)$ to itself for $p_0 < p \leq 1$, in which $p_0$ is given by (1.7).
Conditions like (5.16) have already been considered in the literature to obtain boundedness of standard $\theta$-Calderón–Zygmund operators. For instance, see [12, Theorem 1.2], where the same condition with $\sigma = s_1 = 1$ has been used in the setting of weighted Hardy spaces and also [16, Theorems 8 and 9], for a similar one in weak-Hardy spaces. Conditions like $\int_0^1 \frac{[\theta(t)]^a}{t} dt < \infty$ for $a > 0$ have also been considered in the literature (see [14] and their cited papers) and is usually referred as $a$-Dini condition.

**Proof.** Let $a(x)$ be a $(p, \infty)$-atom supported on $B(z, r)$. We show that $Ta$ is a $(p, \rho, q, \lambda, s_2, s_1)$-molecule like in the proof of Theorem 1.1. Since conditions (M1) and (M3) rely only on the continuity properties of $T$, and we show (M2). Since $\theta$ is increasing and by (5.15) it follows that

$$|Ta(x)| \leq \int_{B(z, r)} |K(x, y) - K(x, z)| |a(y)|dy$$

$$\lesssim r^{n(1 - \frac{1}{p})}\theta \left( \frac{r}{|x - z|^\frac{1}{\beta}} \right) |x - z|^{-n}. $$

Therefore

$$\int_{\mathbb{R}^n \setminus B(z, 2r)} |Ta(x)|^{s_1} |x - z|^{\lambda} dx$$

$$\leq r^{s_1 n(1 - \frac{1}{p})} \int_{\mathbb{R}^n \setminus B(z, 2r)} \left[ \theta \left( \frac{2r}{|x - z|^{\frac{1}{\beta}}} \right) \right]^{s_1} |x - z|^{\lambda - s_1 n} dx$$

$$= r^{\lambda + n(1 - \frac{1}{p})} \int_{|w| > 1} \left[ \theta \left( \frac{|w|^{\frac{1}{\beta}}}{} \right) \right]^{s_1} |w|^{\lambda - s_1 n} dw$$

$$= r^{\lambda + n(1 - \frac{1}{p})} \int_{\mathbb{R}^n \setminus B(z, 2r)} \left[ \theta \left( \frac{|w|^{\frac{1}{\beta}}}{} \right) \right]^{s_1} |w|^{\lambda - n(s_1 - 1)} dw$$

$$\lesssim r^{\lambda + n(1 - \frac{2}{p})} \int_0^1 \frac{[\theta(t)]^{s_1}}{t^{1+\delta s_1}} |t^{\sigma \lambda + \sigma n(s_1 - 1) + \delta s_1} dt$$

$$\lesssim r^{\lambda + n(1 - \frac{2}{p})},$$

using condition (5.16) and $\lambda < n(s_1 - 1) + \frac{\delta s_1}{\sigma}$, which is valid and was already pointed out in (4.2). For $r < 1$,

$$\int_{\mathbb{R}^n \setminus B(z, 2r^p)} |Ta(x)|^{s_1} |x - z|^{\lambda} dx \lesssim r^{s_1 n(1 - \frac{1}{p})}$$

$$\int_{\mathbb{R}^n \setminus B(z, 2r^p)} \left[ \theta \left( \frac{r}{|x - z|^{\frac{1}{\beta}}} \right) \right]^{s_1} |x - z|^{\lambda - s_1 n} dx$$

$$= r^{s_1 n(1 - \frac{1}{p}) + \rho[\lambda - n(s_1 - 1)]]} \int_{|w| > 1} \left[ \theta \left( \frac{r^{1 - \frac{1}{\beta}}}{|w|^{\frac{1}{\beta}}} \right) \right]^{s_1} |w|^{\lambda - s_1 n} dw$$

$$= r^{s_1 n(1 - \frac{1}{p}) + \sigma(\lambda - n s_1 + n) + \rho - 1} \int_0^{r^{1 - \frac{1}{\beta}}} \frac{[\theta(t)]^{s_1}}{t^{1+\delta s_1}} t^{-\sigma \lambda + \sigma n(s_1 - 1) + \delta s_1} dt$$

$$\lesssim r^{\rho \lambda + n \left[ \rho \left( 1 - \frac{s_1}{\sigma} \right) + s_1 \left( \frac{1}{\beta} \right) \right]},$$
where in the last integral we estimate $t \leq r^{1-\frac{\sigma}{2}}$ and we choose $\rho$ as in (4.1).

\[\square\]

Remark 5.6. Condition (5.16) can be refined for one related to (1.3) and (1.4). Let $I = (2^{-\frac{1}{2}}, 1)$, $I^\rho_j = r^{1-\frac{\sigma}{2}}2^{-\frac{1}{2}} \times I$ and $I_j = r^{1-\frac{1}{2}}2^{-\frac{1}{2}} \times I$. Then $\theta$-kernels of type $\sigma$ such that

\[
\left( \int_{I^\rho_j} \left[ \frac{\theta(t)}{t} \right]^{s_1} dt \right)^{\frac{1}{s_1}} \lesssim |I^\rho_j|^{\delta} \text{ if } r < 1 \text{ and } \left( \int_{I_j} \left[ \frac{\theta(t)}{t} \right]^{s_1} dt \right)^{\frac{1}{s_1}} \lesssim (2^j)^{-\delta} \text{ if } r > 1.
\]

satisfy the $D_{s_1}$ condition.

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