ENTITY-ORIENTED SPATIAL CODING AND DISCRETE TOPOLOGICAL SPATIAL RELATIONS

WEINING ZHU

ABSTRACT. Based on a newly proposed spatial data model – spatial chromatic model (SCM), we developed a spatial coding scheme, called full-coded ordinary arranged chromatic diagram (full-OACD). Full-OACD is a type of spatial tessellation, where space is partitioned into a number of subspaces such as cells, edges, and vertexes. These subspaces are called spatial particles and assigned with unique codes – chromatic codes. The generation, structures, computations, and properties of full-OACD are introduced and relations between chromatic codes and particle spatial topology are investigated, indicating that chromatic codes provide a potential useful and meaningful tool not only for spatial analysis in geographical information science, but also for other relevant disciplines such as discrete mathematics, topology, and computer science.

1. Introduction

Coding the objects has been widely used in many scientific and technological fields, such as telecommunications, bioinformatics, and computer cryptography, in which information has been expressed, transferred, and interpreted by various codes in numbers, strings, or symbols. In geographic information science (GIS), there are also some relevant applications of coding. For example, a geographical coordinate systems provides a coding scheme using a single or a series of coordinates to represent a spatial entity or region [1], [2]. Spatial index assigns codes (indexes) to spatial objects so that they can be rapidly retrieved from spatial databases [3], [4]. In geocoding systems, land lots and zip codes allow spatial locations and postal addresses to be readily memorized and exclusively identified [5], [6].

The objective of this study is to do the similar work for coding the pure space itself. Actually a planar Cartesian coordinate system is also a coding scheme where a point in space is coded by such as a coordinate \((x, y)\). Based on a newly proposed GIS data model – spatial chromatic model (SCM) [7], we suggest a spatial coding scheme, called full-coded ordinary arranged chromatic diagram (full-OACD). Full-OACD can be taken as an extension of OACD, which is a standard pattern of SCM. SCM has demonstrated its significant potentials for GIS theories and applications in diverse aspects: the first law of geography, reasoning spatial topology, point pattern recognition, and generalized Voronoi algorithms, etc. [8], [7].

Space in SCM is defined as the object-oriented space where the elementary unit is a cell. A cell is characterized by its chromatic code, typically a string of natural numbers. One problem of OACDs is that only cells are coded, but cellular boundaries and feature nodes, such as edges and vertexes generated from half-plane partitions, have not been coded, and hence we may lose some particular spatial information, for example, the subspaces somewhere that are unable to be assigned.

Key words and phrases. Spatial coding, spatial topology, spatial chromatic model.
to any cell. To solve this problem, we therefore extended OACD to full-OACD, a full-space coding scheme. In full-OACD, all spatial components, including cells, edges and vertexes, are coded in a spatially and mathematically consistent way.

The below sections will introduce, analyze, and discuss the procedures of generating full-OACDs, some important definitions, notations, properties, and theorems (Section 2), topological relations among cells, edges, vertexes, and complexes (Section 3), as well as their spatial implications, notes, and suggested future work (Section 4).

2. Full-coded ordinary arranged chromatic diagram

Let \( P = \{p_1, p_2, \ldots, p_n\} \) is a point set containing \( n \) points associated with an index set \( I = \{1, 2, \ldots, n\} \). The point set is also called the generator set and points in \( P \) are generators, which can be treated as geographical entities or just any general objects. The set \( Q \) is a family of subsets of \( P \) consisting of all unordered point-pairs in \( P \), that is, \( Q = \{\{p_i, p_j\}|p_i, p_j \in P, i \neq j, i, j \in I\} \). The generation of a full-OACD follows the below steps.

**Step (1):** With respect to a point-pair \( q = \{p_i, p_j\} \in Q \), using their perpendicular bisector \( pb(i, j) \) to partition the space into two half-planes \( hp(i, j) \) and \( hp(j, i) \), where a point \( p \) in \( hp(i, j) \) is with Euclidean distance \( d(p, p_i) < d(p, p_j) \), in \( hp(j, i) \) with \( d(p, p_j) < d(p, p_i) \), and in \( pb(i, j) \) with \( d(p, p_i) = d(p, p_j) \).

**Step (2):** Assign two half-planes \( hp(i, j) \) and \( hp(j, i) \) the codes \( (p_1^0, p_2^0, \ldots, p_i^0, \ldots, p_j^0, \ldots, p_n^0) \) and \( (p_1^0, p_2^0, \ldots, p_j^0, \ldots, p_i^0) \), respectively, in which the subscript number corresponds to the index of each point, and the superscript number is the assigned numerical variable \( t(q) \). In this way, only for points \( p_i \) or \( p_j \), \( t(q) = 1 \), but for the others, \( t(q) = 0 \). Similarly, assign the bisector \( pb(i, j) \) with code \( (p_1^0, p_2^0, \ldots, p_i^0, \ldots, p_j^0) \), that is, for both \( p_i \) and \( p_j \), \( t(q) = \frac{1}{2} \), but for the others, \( t(q) = 0 \). See the simplest full-OACD generated from two entities in Fig.1.

**Step (3):** Repeat steps (1) and (2) for all \( k = \frac{1}{2}n(n-1) \) point-pairs in \( Q \), and then overlay the \( 2k \) half-planes so that they generate a spatial tessellation, containing a number of faces, edges, and vertexes.

**Step (4):** The chromatic code of each face, edge, and vertex is the sum of the values \( t(q) \) that are acquired from each half-plane partition, that is,

\[
\left( \sum_{q \in Q} t(q), \sum_{q \in Q} t(q), \ldots, \sum_{q \in Q} t(q), \ldots, \sum_{q \in Q} t(q) \right)
\]

(2.1)

Note that the point set \( P \) could be in any dimensional space \( \mathbb{R}^m \), and hence each partition divides the space into two half-spaces rather than half-planes. This study mainly focuses on the planar full-OACDs in space \( \mathbb{R}^2 \). Fig.2 shows the procedure of generating a full-OACD from 3 points (Fig.2a) in plane, denoted by \( OACD(3, \mathbb{R}^2) \). Through half-plane partitions, we get 6 half-planes in Fig.2b-2d, then we overlay them together into a diagram such that in Fig.2e, and finally we sum the \( t(q) \)'s to compute chromatic code for each subspace in the diagram Fig.2f.

In step (2), if we do not assign \( t(q) = \frac{1}{2} \) to any bisectors, then the obtained diagram is OACD. Therefore edges and vertexes in OACDs are without codes.
This makes the important difference between OACD and full-OACD, where edges and vertexes are with codes.

The subspaces, i.e., faces, edges, and vertexes generated in full-OACD are called *spatial particles* (denoted by \( \Omega \)), and faces are particularly called *cells* (denoted by \( \zeta \)), which has been preliminarily studied in OACD [8], [7]. Chromatic codes of particles are \( n \)-tuples such as \( \Omega(\tau_1, \tau_2, \ldots, \tau_n) \), in which the number \( \tau_i \) is called the *chromatic component* of \( p_i \) in the code, or the component at location \( i \). Easy to know that \( \tau_i \) will be either integer or half-integer. Sometimes, if we are only interested in, say, components of \( p_i \) and \( p_j \), then a chromatic code \( \Omega(\tau_1, \tau_2, \ldots, \tau_i, \ldots, \tau_j, \ldots, \tau_n) \) can be rewritten in a short form such as \( \Omega(\tau_i, \tau_j) \cup (T_{\text{others}}) \), or just \( \Omega(\tau_i, \tau_j) \).

Fig. 3 shows another two examples of full-OACDs. Fig.3a is an original full-coded OACD \((4, \mathbb{R}^2)\) and Fig.3b is a homomorphic part of a full-coded OACD \((6, \mathbb{R}^2)\), where each spatial particles are coded in 6-tuples. Observing particle patterns and codes in these full-OACDs we can find out many interesting properties.

**Definition 1.** Given a particle \( \Omega(\tau_1, \tau_2, \ldots, \tau_n) \), the ascending order of its chromatic components is called the chromatic base of the particle, and denoted by \( \beta(\Omega) = \{\tau_1', \tau_2', \ldots, \tau_n'\} \).

For example, cells \( \zeta_1(0, 2, 3, 1) \) and \( \zeta_2(2, 1, 3, 0) \) both have the same base \( \beta(\zeta_1) = \beta(\zeta_2) = \{0, 1, 2, 3\} \). If two components are equal, their orders are in random. For example, the base of edges \( (3, 2, 0, 3, 2, 2) \) and \( (3, 3, 2, 2, 2, 3) \) are both \( \{0, 3, 3, 3, 3\} \). Chromatic codes are actually the permutations of different bases. In previous studies, chromatic base was also called the primary code of a cell [8].

**Definition 2.** If two particles \( \Omega_1(t_{11}, t_{12}, \ldots, t_{1i}, \ldots, t_{1n}) \) and \( \Omega_2(t_{21}, t_{22}, \ldots, t_{2i}, \ldots, t_{2n}) \) have the same chromatic codes, then they are called equi-color, and denoted by \( \Omega_1 = \Omega_2 \), that is,

\[
\forall i, t_{1i} = t_{2i} \iff \Omega_1 = \Omega_2 \quad (2.2)
\]

otherwise, \( \Omega_1 \neq \Omega_2 \).

If they have the same chromatic bases, then they are called equi-base, denoted by \( \Omega_1 \cong \Omega_2 \), that is, if \( \beta(\Omega_1) = \{t_{11}', t_{12}', \ldots, t_{1i}', \ldots, t_{1n}'\} \) and \( \beta(\Omega_2) = \{t_{21}', t_{22}', \ldots, t_{2i}', \ldots, t_{2n}'\} \), then

\[
\forall i, t_{1i}' = t_{2i}' \iff \Omega_1 \cong \Omega_2 \quad (2.3)
\]

otherwise, \( \Omega_1 \not\cong \Omega_2 \).
Figure 2. The procedure of generating a full-OACD(3, $\mathbb{R}^2$). (a) The generator set consists of three points marked with color R, G, and B; (b)-(d) Half-plane partitions and assignments of chromatic codes with respect to perpendicular bisectors $pb(B,G)$, $pb(G,R)$, and $pb(R,B)$, respectively. (e) Overlapping all the six half-planes in (b)-(d) together; and (f) Adding all chromatic components together to form the chromatic codes.

Property 1. Given two particles $\Omega_1$ and $\Omega_2$,  
\[ \Omega_1 = \Omega_2 \Rightarrow \Omega_1 \cong \Omega_2 \]  
and hence  
\[ \Omega_1 \not\cong \Omega_2 \Rightarrow \Omega_1 \not= \Omega_2 \]  
This property indicates that if two cells are equi-color, they must be equi-base, and if they are not equi-base, they are impossible to be the equi-color.

The number of cells, edges, and vertexes in a full-coded OACD($n, \mathbb{R}^2$) depends on the point pattern of the generator set $P$. This study mainly focuses on the
**Definition 3.** In a general case of the point set $P$, any three point-pairs from three different points generate a vertex, called 3-I vertex (i.e., the intersection of three perpendicular bisectors of a triangle), denoted by $\varphi_3^I$; and any two point-pairs from 4 different points generate a vertex, called 2-I vertex (i.e., the intersection of two perpendicular bisectors), denoted by $\varphi_2^I$.

Therefore vertexes $\varphi$ in full-coded $OACD(n, \mathbb{R}^2)$ are either 2-I or 3-I, see their examples in Fig.3.

**Property 2.** An $OACD(n, \mathbb{R}^2)$ contains $\sum_{i=1}^{C_n^2} i - C_n^3 + 1$ cells, $(C_n^2)^2 - 3C_n^3$ edges, $C_n^3$ 3-I vertexes, and $\frac{1}{2}C_n^2C_{n-2}$ 2-I vertexes.

**Proof.** The proof of the cell number could be referred to [8]. Here we only prove the edge number. Suppose in a plane there are $n$ lines which intersect with each other, then each line is divided into $n$ edges by the other $n-1$ lines, therefore the $n$ lines will generate $n^2$ edges. The total $n$ point will generate $C_n^2$ lines (bisectors) and hence $(C_n^2)^2$ edges. But every three points generate a vertex which will reduce 3 edges, therefore the total edge number will be $(C_n^2)^2 - 3C_n^3$. $\square$

**Property 3.** In an $OACD(n)$, the chromatic base of cells is

$$N = \{0, 1, \ldots, n-1\}$$

(2.6)

This property has been proved by [8]. It implies that all cells are equi-base, and any two components of a cell are not equal. Below we use $N[i,j]$ to denote the integers between $i$ and $j$, and also including $i$ and $j$.

**Property 4.** In an $OACD(n)$, the chromatic bases of edges are

$$\{N\setminus\{z, z+1\}, z + \frac{1}{2}, z + \frac{1}{2}\}$$

(2.7)
for \( z = \mathbb{N}[0, n - 2] \), meaning for each \( z \) from 0 to \( n - 2 \), we obtain a base which removes \( z \) and \( z + 1 \) from \( \mathbb{N} \) and then add two \( z + \frac{1}{2} \).

Particularly, an edge (denote by \( \eta \)) generated by bisector \( pb(i, j) \) bears a code

\[
\eta(x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}})
\]

for \( z = \mathbb{N}[0, n - 2] \).

**Proof.** Suppose \( \eta \) is the edge between two cells \( \zeta_1 \) and \( \zeta_2 \), therefore before the partition of \( pb(i, j) \), \( \zeta_1 \) and \( \zeta_2 \) should be merged into a larger cell \( \zeta \) with code \((x_i^z, x_j^z)\), that is, point \( i \) and \( j \) have the same component \( z \). After the partition, \( \zeta_1 \) and \( \zeta_2 \)’s codes will be \((x_i^{z+1}, x_j^z)\) and \((x_i^z, x_j^{z+1})\), see the proof of Lemma 2 in [8]. With respect to all other bisectors \( pb(i, x) \) or \( pb(j, x) \), \( x \in I \setminus \{i, j\} \), if \( \zeta \) has not gained any components, then minimum of \( z \) could be 0; if \( \zeta \) always gained one component for all the other \( n - 2 \) bisectors, then the maximum of \( z \) could be \( n - 2 \). Therefore \( \eta \)’s chromatic code will be \((x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}})\), and their bases will be \( \{\mathbb{N}\setminus \{z, z + 1\}, z + \frac{1}{2}, z + \frac{1}{2}\} \), for \( z = \mathbb{N}[0, n - 2] \).

**Property 5.** The chromatic bases of 2-I vertexes are

\[
\{\mathbb{N}\setminus \{z_1, z_2, z_1 + 1, z_2 + 1\}, z_1 + \frac{1}{2}, z_1 + \frac{1}{2}, z_2 + 1, z_2 + \frac{1}{2}\} \tag{2.9}
\]

for \( z_1 = \mathbb{N}[0, n - 4] \) and \( z_2 = \mathbb{N}[z_1 + 2, n - 2] \). Particularly, a vertex \( \varphi^{2I} \) generated by two bisectors \( pb(i, j) \) and \( pb(u, v) \) bears a code

\[
\varphi^{2I}(x_i^{z_1+\frac{1}{2}}, x_j^{z_2+\frac{1}{2}}, x_u^{z_1+\frac{1}{2}}, x_v^{z_1+\frac{1}{2}}) \tag{2.10}
\]

or

\[
\varphi^{2I}(x_i^{z_2+\frac{1}{2}}, x_j^{z_2+\frac{1}{2}}, x_u^{z_1+\frac{1}{2}}, x_v^{z_1+\frac{1}{2}}) \tag{2.11}
\]

for \( z_1 = \mathbb{N}[0, n - 4] \) and \( z_2 = \mathbb{N}[z_1 + 2, n - 2] \).

**Proof.** Suppose \( pb(i, j) \) and \( pb(u, v) \) are the last two bisectors partitioning a merged cell, then according to the Lemma 2 in [8], before the two partitions, the cell should be with a code such as \((x_i^{z_1}, x_j^{z_2}, x_u^{z_1}, x_v^{z_2})\). Let \( z_1 \) is the smaller integer, and then \( z_2 = z_1 + \Delta \). After the two partitions by \( pb(i, j) \) and \( pb(u, v) \), four new cells will be generated with codes

\[
(x_i^{z_1}, x_j^{z_1+1}, x_u^{z_1+\Delta+1}, x_v^{z_1+\Delta}) \tag{2.12}
\]

\[
(x_i^{z_1+1}, x_j^{z_1}, x_u^{z_1+\Delta+1}, x_v^{z_1+\Delta}) \tag{2.13}
\]

\[
(x_i^{z_1}, x_j^{z_1+1}, x_u^{z_1+\Delta}, x_v^{z_1+\Delta+1}) \tag{2.14}
\]

\[
(x_i^{z_1+1}, x_j^{z_1}, x_u^{z_1+\Delta}, x_v^{z_1+\Delta+1}) \tag{2.15}
\]

If \( \Delta = 0 \) or 1, then we can always find that in some codes of Eq. 2.12-2.15, two components are equal. For example, if \( \Delta = 0 \), there are two \( z_1 \)'s and two \( z_1 + 1 \)'s in Eq. 2.12, and if \( \Delta = 1 \), there are two \( z_1 + 1 \)'s in Eq. 2.13. But cellular base is \( \mathbb{N} \), meaning any two components are not equal, therefore \( \Delta \geq 2 \). Because \( pb(i, j) \) and \( pb(u, v) \) involve 4 points, then the maximum of \( z_1 \) should be \( n - 4 \), and hence \( z_1 = \mathbb{N}[0, n - 4] \), \( z_2 = \mathbb{N}[z_1 + 2, n - 2] \). The remainder of the proof follows along the line of the proof of Property 4. \( \Box \)
Property 6. The chromatic bases of 3-I vertexes are

\[ \{N \setminus \{z, z + 1, z + 2\}, z + 1, z + 1, z + 1\} \] (2.16)

for \( z = N[0, n - 3] \). Particularly, a vertex \( \varphi^M \) generated by three bisectors \( pb\langle i, j \rangle \), \( pb\langle j, k \rangle \), and \( pb\langle k, i \rangle \) bears a code

\[ \varphi^M(x_i^{z+1}, x_j^{z+1}, x_k^{z+1}) \] (2.17)

for \( z = N[0, n - 3] \).

Proof. Suppose before the partitions of \( pb\langle i, j \rangle \), \( pb\langle j, k \rangle \), and \( pb\langle k, i \rangle \), the merged cell has a code \( (x_i^z, x_j^{z+1}, x_k^{z+1}) \), where \( \Delta_1 \geq 0 \) and \( \Delta_2 \geq 0 \). After the partitions, six new cells will be generated with codes

\[ (x_i^{z+2}, x_j^{z+\Delta_1+1}, x_k^{z+\Delta_2}) \cup (X_{others}) \] (2.18)
\[ (x_i^{z+1}, x_j^{z+\Delta_1}, x_k^{z+\Delta_2+1}) \cup (X_{others}) \] (2.19)
\[ (x_i^{z+1}, x_j^{z+\Delta_1+1}, x_k^{z+\Delta_2}) \cup (X_{others}) \] (2.20)
\[ (x_i^z, x_j^{z+\Delta_1+2}, x_k^{z+\Delta_2}) \cup (X_{others}) \] (2.21)
\[ (x_i^z, x_j^{z+\Delta_1+1}, x_k^{z+\Delta_2+2}) \cup (X_{others}) \] (2.22)
\[ (x_i^z, x_j^{z+\Delta_1+2}, x_k^{z+\Delta_2+1}) \cup (X_{others}) \] (2.23)

We examine the below possible values of \( \Delta_1 \) and \( \Delta_2 \).

(1) \( \Delta_1 = 1 \) or \( \Delta_1 = 2 \), \( \Delta_2 = 1 \) or \( \Delta_2 = 2 \).
   If \( \Delta_1 = 1 \) or \( \Delta_1 = 2 \), for example, in Eq. (2.20) and (2.19) there will be two components equalling \( z + 1 \) or \( z + 2 \); similarly, if \( \Delta_2 = 1 \) or \( \Delta_2 = 2 \), in Eq. (2.21) and (2.18) there will be two components equalling \( z + 1 \) or \( z + 2 \).

(2) \( \Delta_1 \geq 3 \), \( \Delta_2 \geq 3 \).
   According to Eq. (2.22) and (2.23), there must be values \( z + 1 \) and \( z + 2 \) in \( X_{others} \), because they are not in locations \( x_i \), \( x_j \), or \( x_k \). However, according to Eq. (2.18)-(2.21), \( z + 1 \) and \( z + 2 \) are already in \( x_i \), so that they cannot be in \( X_{others} \).
   From the above two cases we know that the only allowed values of \( \Delta_1 \) and \( \Delta_2 \) are both 0, and the merged cell must bear a code

\[ (x_i^z, x_j^z, x_k^z) \] (2.24)

Then at the intersection of the three bisectors, the 3-I vertex acquires components \( \frac{1}{2} \) at \( x_i \) and \( \frac{1}{2} \) at \( x_j \) from \( pb\langle i, j \rangle \), \( \frac{1}{2} \) at \( x_j \) and \( \frac{1}{2} \) at \( x_k \) from \( pb\langle j, k \rangle \), \( \frac{1}{2} \) at \( x_k \) and \( \frac{1}{2} \) at \( x_i \) from \( pb\langle k, i \rangle \), and therefore gain a code

\[ (x_i^{z+\frac{1}{2}}+\frac{1}{2}, x_j^{z+\frac{1}{2}}+\frac{1}{2}, x_k^{z+\frac{1}{2}}+\frac{1}{2}) = (x_i^{z+1}, x_j^{z+1}, x_k^{z+1}) \] (2.25)

From Eq. (2.18)-(2.23), we know that \( X_{others} \) do not contain components \( z \), \( z + 1 \), and \( z + 2 \), thus we know the base of the 3-I vertex is in form of Eq. (2.16). Because the range of \( z \) in a cell is from 0 to \( n - 1 \), the minimum \( z \) should be 0 and maximum \( z \) should be \( z + 2 = n - 1 \Rightarrow z = n - 3 \).

This property indicates that the chromatic codes of 3-I vertexes contain three identical integers which are different from the rest integers in codes. If cancel one \( z + 1 \), Eq. (2.16) can be rewritten as

\[ \{N\setminus\{z, z + 2\}, z + 1, z + 1\} \] (2.26)

for \( z = N[0, n - 3] \).
Theorem 1. Different types of particles in a full-OACD are not equi-base, that is,
\[ \zeta \not\equiv \eta \not\equiv \varphi^{(1)} \not\equiv \varphi^{(3)} \]  
(2.27)

This theorem provides an approach to determine particle types. For example, if we see a particle with chromatic components being all different integers, then it must be a cell; if it contains 2 half-integers, it must be an edge; if it contains 3 equal integers, it must be a 3-I vertex; and if contains 4 half-integers, it must be a 2-I vertex.

Notation 1. The component-counting function \( H(\Omega, m) \) is a function counting the number of \( m \) in the chromatic code of \( \Omega \), that is, the function tells how many components equal to \( m \).

Definition 4. The difference tuple of two particles \( \Omega_1(t_{11}, t_{12}, \ldots, t_{1n}) \) and \( \Omega_2(t_{21}, t_{22}, \ldots, t_{2n}) \) is defined by
\[
\Psi(\Omega_1, \Omega_2) = (\psi_1, \psi_2, \ldots, \psi_n) 
= (|t_{11} - t_{21}|, |t_{12} - t_{22}|, \ldots, |t_{1n} - t_{2n}|) 
\]
where \( \psi_i = |t_{1i} - t_{2i}| \).

Then the chromatic distance between the two particles is defined by
\[
\delta(\Omega_1, \Omega_2) = \sum_{i=1}^{n} \psi_i 
\]
(2.29)
and each \( \psi_i \) is called the chromatic distance at the component \( i \), and denoted by \( \delta(\psi_i) \).

In addition, the code distance between two particles is defined by
\[
\gamma(\Omega_1, \Omega_2) = n - H(\delta(\Omega_1, \Omega_2), 0) 
\]
(2.30)

The chromatic distance is also called transition number \( T \) between two cells in our previous study, and it is actually the Manhattan distance between two particles. The code distance is actually the Hamming distance between two particles if we treat their codes and components as strings rather than numbers.

Definition 5. The union of \( m \) particles \( \Omega_1(t_{11}, t_{12}, \ldots, t_{1n}), \Omega_2(t_{21}, t_{22}, \ldots, t_{2n}), \ldots, \Omega_m(t_{m1}, t_{m2}, \ldots, t_{mn}) \) is called a complex or a \( m \)-complex, denoted by \( \Theta \), and its code is given by
\[
\Theta\{\Omega_1, \Omega_2, \ldots, \Omega_m\} = \sum_{i=1}^{m} \Omega_m 
= \left( \sum_{i=1}^{m} t_{i1}, \sum_{i=1}^{m} t_{i2}, \ldots, \sum_{i=1}^{m} t_{in} \right). 
\]
(2.31)
These \( m \) particles are called the elemental particles of the \( m \)-complex. If the \( m \) particles are all cells, then the \( m \)-complex is also called a \( m \)-cell cluster. One particle could be taken as a 1-complex.

Theorem 2. If \( \zeta_1 \) and \( \zeta_2 \) are two cells in a full-OACD, then
\[ \zeta_1 \neq \zeta_2, \]
(2.32)
This theorem has been proved by [8]. It tells that any two cells are not equi-color – their codes are unique.

Because any vertex in OACD is either 2-I or 3-I, therefore the a full-OACD is tessellated by two types of structural units such as the two in Fig.4: the one containing $\varphi^2I$ is called 2-I unit (Fig.4a), and the other containing $\varphi^3I$ is called 3-I unit (Fig.4b). According to the proofs of Property 5 and 6, particle codes in 2-I/3-I units should be those shown in Fig.4, and then it is easy to calculate and prove the below four properties.
Property 7. A 2-I unit generated by \( pb(i,j) \) and \( pb(u,v) \) contains the following 9 particles.

1. One 2-I vertex with code

\[
\varphi_{ijuv}^{2I}(x_i + \frac{1}{2}, x_j + \frac{1}{2}, x_u + \frac{1}{2}, x_v + \frac{1}{2}) \tag{2.33}
\]

2. Four edges with codes

\[
\eta_{iju}(x_i + \frac{1}{2}, x_j + \frac{1}{2}, x_u + 1, x_v), \eta_{jiv}(x_i + \frac{1}{2}, x_j + \frac{1}{2}, x_u, x_v + 1)
\]

\[
\eta_{uvi}(x_i + 1, x_j, x_u + \frac{1}{2}, x_v + \frac{1}{2}), \eta_{uvf}(x_i, x_j + 1, x_u + \frac{1}{2}, x_v + \frac{1}{2}) \tag{2.34}
\]

3. Four cells with codes

\[
\zeta_{iu}(x_i + 1, x_j, x_u + 1, x_v), \zeta_{iv}(x_i + 1, x_j, x_u, x_v + 1)
\]

\[
\zeta_{ju}(x_i, x_j + 1, x_u + 1, x_v), \zeta_{jv}(x_i, x_j + 1, x_u, x_v + 1) \tag{2.35}
\]

Property 8. A 3-I unit generated by \( pb(i,j) \), \( pb(j,k) \) and \( pb(k,i) \) contains the following 13 particles.

1. One 3-I vertex with code

\[
\varphi_{ijk}^{3I}(x_i + 1, x_j + 1, x_k + 1) \tag{2.36}
\]

2. Six edges with codes

\[
\eta_{jki}(x_i, x_j + \frac{3}{2}, x_k + \frac{3}{2}), \eta_{jik}(x_i + 2, x_j + \frac{1}{2}, x_k + \frac{3}{2}) \tag{2.37}
\]

\[
\eta_{kji}(x_i + \frac{3}{2}, x_j, x_k + \frac{3}{2}), \eta_{kij}(x_i + \frac{1}{2}, x_j + 2, x_k + \frac{3}{2})
\]

\[
\eta_{kji}(x_i + \frac{3}{2}, x_j, x_k + \frac{3}{2}), \eta_{kij}(x_i + \frac{1}{2}, x_j + 2, x_k + \frac{3}{2}) \tag{2.38}
\]

Note, in Eq\[2.37\] the underlined index indicates the perpendicular bisector which makes the edge.

3. Six cells with codes

\[
\zeta_{ijk}(x_i + 2, x_j + 1, x_k), \zeta_{ikj}(x_i + 2, x_j, x_k + 1)
\]

\[
\zeta_{jik}(x_i + 1, x_j + 2, x_k), \zeta_{jki}(x_i, x_j + 2, x_k + 1)
\]

\[
\zeta_{kij}(x_i + 1, x_j, x_k + 2), \zeta_{kji}(x_i, x_j + 1, x_k + 2) \tag{2.39}
\]

Property 9. In 2-I unit space:

1. The codes of the vertex \( \varphi^{2I} \) is the average of (I) two edges which are in the same bisectors, (II) the two cells which are opposite to the vertex, (III) all the four edges, and (IV) all the four cells, that is,

\[
\varphi^{2I} = \frac{1}{2}(\eta_{iju} + \eta_{jiv}) = \frac{1}{2}(\eta_{uvi} + \eta_{uvf}) \tag{2.39}
\]

\[
= \frac{1}{2}(\zeta_{iu} + \zeta_{jv}) = \frac{1}{2}(\zeta_{iv} + \zeta_{jv})
\]

\[
= \frac{1}{4}(\eta_{iju} + \eta_{jiv} + \eta_{uvi} + \eta_{uvf})
\]

\[
= \frac{1}{4}(\zeta_{iu} + \zeta_{iv} + \zeta_{ju} + \zeta_{jv}) \tag{2.40}
\]

(2) The codes of an edge \( \eta \) is the half of the two cells \( \zeta_1 \) and \( \zeta_2 \) which are respectively on the two sides of the edge. If \( \xi = \{\zeta_1, \zeta_2\} \), then,

\[
\eta = \frac{1}{2}(\zeta_1 + \zeta_2) = \frac{1}{2}\xi \tag{2.40}
\]
(3) The two edges are equi-base if they in the same bisector, but not equi-base if they are not in the same bisector, that is,
\[
\eta_{iju} \cong \eta_{ijv}, \eta_{uvi} \cong \eta_{uvj}
\] (2.41)
\[
\eta_{iju} \ncong \eta_{uvi}, \eta_{ijv} \ncong \eta_{uvj}
\]

Property 10. In 3-I unit space:

1. The codes of the vertex \(\varphi^M\) is the average of (I) all the six edges/cells, (II) the three edges/cells which are interval with each other, and (III) the two edges/cells which are opposite to each other (for edges in this case, they are in the same bisector), that is,
\[
\varphi^M = \frac{1}{3} (\varphi_{kij} + \varphi_{jki}) = \frac{1}{3} (\varphi_{jki} + \varphi_{kij}) = \frac{1}{3} (\varphi_{jki} + \varphi_{kij})
\] (2.42)
\[
\varphi^M = \frac{1}{6} (\varphi_{kij} + \varphi_{jki} + \varphi_{kij} + \varphi_{ijk} + \varphi_{kji} + \varphi_{jki})
\]

2. The edge codes have the same property as the Property 9(2).

3. The two edges in the same bisectors are not equi-base, but the three interval edges are equi-base, that is,
\[
\eta_{kij} \ncong \eta_{jki}, \eta_{jki} \ncong \eta_{kij}, \eta_{jki} \ncong \eta_{kij}
\] (2.43)

The spatial relations among particles in 2-I/3-I units have three types: adjacent, interval, and opposite, see Fig. 5. If spatial relations between two particles in 2-I/3-I units are different, their chromatic and code distances are also different, see the below Property 11.

Property 11. Within a 2-I or 3-I unit of a full-coded OACD \(n, I^2\), the chromatic distance \(\delta\) and code distance \(\gamma\) between two particles \(\Omega_1\) and \(\Omega_2\) are listed in Table 1.

An important requirement for full-OACDs is that we expect their particle codes to be unique.

Theorem 3. If \(\eta_1\) and \(\eta_2\) are two edges in a full-OACD, then
\[
\eta_1 \neq \eta_2.
\] (2.44)

Proof. Suppose \(\zeta_{1\text{Left}}\) and \(\zeta_{1\text{Right}}\) are two cells beside \(\eta_1\), and \(\zeta_{2\text{Left}}\) and \(\zeta_{2\text{Right}}\) are two cells beside \(\eta_2\), and then they make two 2-cell clusters \(\xi_1(\zeta_{1\text{Left}}, \zeta_{1\text{Right}})\) and \(\xi_2(\zeta_{2\text{Left}}, \zeta_{2\text{Right}})\), respectively. It has been proved that chromatic codes of any connected 2-cell cluster \(\xi\) are unique \([7]\), that is, \(\xi_1 \neq \xi_2\). Then according to Eq. \(2.40\),
\[
\eta_1 = \frac{1}{2} \xi_1 \neq \frac{1}{2} \xi_2 = \eta_2.
\]

This theorem tells that chromatic codes of edges are unique.

Theorem 4. If \(\varphi_1\) and \(\varphi_2\) are two vertexes in a full-OACD, then
\[
\varphi_1 \neq \varphi_2.
\] (2.45)
Proof. Case (1): One vertex is 2-I and the other vertex is 3-I.

According to Property \([1]\), \(\varphi^2 \neq \varphi^3\), so \(\varphi_1 \neq \varphi_2\).

Case (2): They are both 3-I vertexes.

Suppose \(\varphi_1^M\) and \(\varphi_2^M\) are different 3-I vertexes but with the same code \((x_i^1, x_j^2, x_k^3)\), i.e., they have three chromatic components which are the same integer \(z\) given by three points \(i, j,\) and \(k\). However, the bisectors generated from point can only intersect at one 3-I vertex, so if \(\varphi_1\) and \(\varphi_2\) are different vertexes, their codes are impossible to be the same, i.e., \(\varphi_1^M \neq \varphi_2^M\).

(3) They are both 2-I vertexes.

Suppose \(\varphi_1^F\) and \(\varphi_2^F\) are two different 2-I vertexes. The \(\varphi_1^F\) was generated by \(pb\langle i_1, j_1 \rangle\) and \(pb\langle v_1, v_1 \rangle\), and hence with a code \((x_i^1, x_{j_1}^2, x_{v_1}^3, x_{v_1}^3)\). The \(\varphi_2^F\) was generated by \(pb\langle i_2, j_2 \rangle\) and \(pb\langle u_2, v_2 \rangle\), and hence with a code \((x_{i_2}^1, x_{j_2}^2, x_{u_2}^3, x_{v_2}^3)\).

The only way to make \(\varphi_1^F = \varphi_2^F\) is that \(i_1 = i_2, j_1 = j_2, \) and \(v_1 = v_2,\) but this makes \(\varphi_1^F\) and \(\varphi_2^F\) are the same vertex, because two bisectors can only intersect at one 2-I vertex. Therefore if \(\varphi_1^F\) and \(\varphi_2^F\) are two different 2-I vertexes, their codes are impossible to be equal.

Based on the above three cases, we know that \(\varphi_1 \neq \varphi_2\).

This theorem indicates that chromatic codes of vertexes are also unique. Ultimately, according to Theorems [4], we obtain the below corollary.

**Corollary 1.** Chromatic particle codes in a full-OACD are unique, that is, given two particles \(\Omega_1, \Omega_2 \in \text{OACD}(n, \mathbb{R}^2),\)

\[
\Omega_1 \neq \Omega_2
\]  

(2.46)

3. Spatial particle topology in full-OACD

A planar full-OACD contains three types of particles: vertexes, edges, and cells. Spatial topological relations among these particles are usually similar to those conventional relations for vectorial geometry in GIS, such as equal, adjacent, disjoint, and overlap. Fig. 6 shows the spatial relations between two particles investigated
Figure 6. General spatial topological relations among particles in full-OACD.

in this study, and these relations can be simply represented and calculated by using chromatic codes. In addition, the more complicated spatial relations among \(m\)-complexes can be also reasoned from analyzing their chromatic codes. Below we demonstrate the major spatial topology among particles and complexes, as well as the relations between them and chromatic codes, in which we particularly focus on cells and clusters. Property 11 already gives the conditions from which we know that different particle relations bear different chromatic and code distances, however, those are only necessary conditions. In this section we will give proofs that those conditions are also sufficient so that we can use two distances \(\delta(\Omega_1, \Omega_2)\) and \(\gamma(\Omega_1, \Omega_2)\), and their bases \(\beta(\Omega_1)\) and \(\beta(\Omega_2)\) to determine their topological spatial relations.

3.1. Spatial topology between particles. There are six types of spatial combinations for particles: vertex-vertex (V-V), vertex-edge (V-E), vertex-cell (V-C), edge-edge (E-E), edge-cell (E-C), and cell-cell (C-C), and their relations are typically equal, joint, disjoint, and others, see examples in Fig.6. These particle-particle relations also underlie the further topological analysis of complexes.

3.1.1. Vertex-Vertex (V-V) relations. In terms of theorem, V-V relations between \(\varphi_1\) and \(\varphi_2\) are quite simple – either equal, i.e., \(\cap(\varphi_1, \varphi_2) = \varphi_1\), or disjoint, i.e., \(\cap(\varphi_1, \varphi_2) = \emptyset\), see Fig.6a.
Proposition 1. If \( \varphi_1 \) and \( \varphi_2 \) are two vertexes, then
\[
\cap(\varphi_1, \varphi_2) = \varphi_1 \Leftrightarrow \varphi_1 = \varphi_2
\]
(3.1)
\[
\cap(\varphi_1, \varphi_2) = \emptyset \Leftrightarrow \varphi_1 \neq \varphi_2
\]

Because we have proved that chromatic codes are unique in OACD, the ‘equal’ relation, i.e., two particles are completely overlap, is easy to determine – two particles are topologically equal, if and only if they are equal in codes, that is, \( \Omega_1 \) equal \( \Omega_2 \) \( \Leftrightarrow \Omega_1 = \Omega_2 \). Note, because a full-OACD is a type of spatial tessellation, meaning it contains neither gaps nor overlaps, therefore topologically, it also does not contain any two equal particles.

3.1.2. Vertex-Edge (V-E) relations. Typically an edge contains two ends, and hence a V-E relation is that a vertex is one of the ends of the edge (Fig.6d), that is, \( \cap(\eta, \varphi) = \varphi \); otherwise they are disjoint, that is, \( \cap(\eta, \varphi) = \emptyset \), see Fig.6f.

Proposition 2. Given an edge \( \eta \) and a vertex \( \varphi \),
\[
\cap(\eta, \varphi) = \varphi \Leftrightarrow \delta(\eta, \varphi) \leq 2
\]
(3.2)
\[
\cap(\eta, \varphi) = \emptyset \Leftrightarrow \delta(\eta, \varphi) > 2
\]

Proof. Case(1) \( \varphi \) is a 2-I vertex.

Suppose \( \varphi \) is generated by \( pb(i, j) \) and \( pb(u, v) \), and hence with a code
\[
\varphi = (a_1, x_{a_1}^1 + \frac{1}{2}, a_2, x_{a_2}^2 + \frac{1}{2}) \cup (X_{\text{others}}^A)
\]
(3.3)
Case (1.1). \( \eta \) is an edge generated from bisector \( pb(g, h) \), and \( g, h \notin \{i, j, u, v\} \). The codes of \( \varphi \) and \( \eta \) then can be rewritten to \((x_i + \frac{1}{2}, x_j + \frac{1}{2}, x_u + \frac{1}{2}, x_v + \frac{1}{2}) \cup (X_{\text{others}}^B)\)
and \((x_i^1, x_j^1, x_u^1, x_v^1, x_i^2, x_j^2, x_u^2, x_v^2) \cup (X_{\text{others}}^A)\)
respectively, where \( a_i \) and \( b_i \) are both integers, and \( A \) and \( B \) are both integer tuples. Because for chromatic distance at each component:
\[
\delta(x_i) = |x_i^1 - x_i^2| \geq \frac{1}{2}, \delta(x_j) = |x_j^1 - x_j^2| \geq \frac{1}{2}, \delta(x_u) = |x_u^1 - x_u^2| \geq \frac{1}{2}, \delta(x_v) = |x_v^1 - x_v^2| \geq \frac{1}{2}, \delta(x_g) = |x_g^1 - x_g^2| \geq \frac{1}{2}, \delta(x_h) = |x_h^1 - x_h^2| \geq \frac{1}{2}, \text{ and } \delta(X_{\text{other}}) = |X_{\text{others}}^A - X_{\text{others}}^B| \geq 0,
\]
then we know that \( \delta(\varphi, \eta) \geq 3 \).

Case (1.2). \( \eta \) is an edge generated from the bisector involving one of points \( i, j, u, v \), for example, \( pb(i, k) \), then \( \varphi \) and \( \eta \) can be rewritten as \( \varphi = (x_i, x_j, x_u, x_v, x_k) \cup (X_{\text{others}}^A) \)
and \( \eta = (x_i^1, x_j^1, x_u^1, x_v^1, x_k^1) \cup (X_{\text{others}}^B) \).
Since distances at \( x_j, x_u, x_v, \) and \( x_k \) are all greater than \( \frac{1}{2} \), we know that \( \delta(\varphi, \eta) \geq 2 \).

Case (1.3). \( \eta \) is an edge generated from the bisectors involving two points in \( i, j, u, v \), but not the two who generate the vertex \( \varphi \), for example, \( pb(i, u) \), then \( \eta \) can be rewritten to \((x_i + \frac{1}{2}, x_j + \frac{1}{2}, x_u + \frac{1}{2}, x_v + \frac{1}{2}) \cup (X_{\text{others}}^B)\). Then similar to the above cases (1) and (2), we know that \( \delta(\varphi, \eta) \geq 1 \). However, to reach the minimum \( \delta = 1 \), it must be that \( \delta(x_i) = 0 \) and \( \delta(x_u) = 0 \), so that \( a_1 = b_1 \) and \( a_2 = b_1 \), then we have \( a_1 = a_2 \). But according to the bases of 2-I vertex (Property 3), it is impossible that \( a_1 = a_2 \), so we get \( \delta \) cannot reach the minimum value 1, and hence \( \delta \geq 2 \). To reach the new minimum \( \delta = 2 \), it must be that \( \delta(x_i) = 1 \) and \( \delta(x_u) = 0 \), or vice versa. This will lead to equations \( a_1 = a_2 + 1 \) or \( a_2 = a_1 + 1 \), but they are impossible in terms of Property 3. Consequently, \( \delta \geq 3 \).
Case (1.4) \( \eta \) is an edge generated from the bisectors, for example, \( pb \langle i, j \rangle \), which also make \( \varphi \), then \( \eta \)'s code should be

\[
\eta = (x_1^{b_1+\frac{1}{2}}, x_1^{b_1+\frac{3}{2}}, x_u, x_v) \cup (X_B^{\delta})
\]  

(3.4)

Then comparing Eq. (3.3) and Eq. (3.4), we know the \( \delta(\varphi, \eta) \geq 1 \). To reach the minimum value 1, it must be \( \delta(x_u) = |a_2 + \frac{1}{2} - b_2| = \frac{1}{2} \) and \( \delta(x_v) = |a_2 + \frac{1}{2} - b_3| = \frac{1}{2} \). Then we have two possible solutions \( \{a_2=b_2-1 \} \) and \( \{a_2=b_3-1 \} \). The only allowed solution combinations are either \( a_2=b_2 \) and \( a_2=b_3-1 \), or \( a_2=b_3 \) and \( a_2=b_2-1 \). This indicates that either the case \( b_2=b_3-1 \) or case \( b_3=b_2-1 \). However, the two cases are just the two edges which are on each side of \( pb \langle u, v \rangle \) and joint to \( \varphi \).

And in this case \( \gamma(\varphi, \eta) = 2 \).

Because \( \delta(x_i) = \delta(x_j) \geq 0 \) and when they are both 0, then \( \delta(\varphi, \eta) = 1 \). Therefore if they are both not 0, then \( \delta(x_i, x_j) \geq 2 \rightarrow \delta(\varphi, \eta) \geq 3 \). So if in this case to reach \( \delta(\varphi, \eta) = 2 \), it must be \( \delta(x_u, x_v) = 0 \), \( \delta(x_u, x_v) = 1 \), and \( \delta(X_{others}) = 1 \). Because all components in \( X_{others} \) are integers, therefore \( \delta(X_{others}) \) is either 0 or \( \geq 2 \) but impossible to be 1. As a result, \( \delta(\varphi, \eta) = 2 \) is unable to reach, that is, \( \delta(\varphi, \eta) > 2 \).

Case(2) \( \varphi \) is a 3-I vertex.

Suppose \( \varphi \) is generated by \( pb \langle i, j \rangle \), \( pb \langle j, k \rangle \), \( pb \langle k, i \rangle \), and hence with a code \( \varphi = (x_i^{a_1}, x_j^{a_2}, x_k^{b_3}) \).

Case (2.1) \( \eta \) is an edge generated from the bisector \( pb \langle u, v \rangle \), and \( u, v \notin \{i, j, k\} \), then \( \varphi \) and \( \eta \) can be rewritten as \( \varphi = (x_i^{a_1}, x_j^{a_2}, x_k^{b_3}, x_u^{a_3}, x_v^{a_4}) \cup (X_{others}) \) and \( \eta = (x_i^{b_1+\frac{1}{2}}, x_i^{b_1+\frac{3}{2}}, x_u^{b_1}, x_v^{b_1}) \cup (X_{others}) \). Because \( b_1 \neq b_2 \neq b_3 \), then without loss of generality, assume \( b_2 = b_1 + \Delta_1, \Delta_1 \geq 1, \) and \( b_3 = b_1 + \Delta_2, \Delta_2 \geq 1 \). If assume \( \delta(x_i) = |a_1 - b_1| = \Delta \geq 0 \), then \( \delta(x_j) = |a_1 - b_2| = \Delta + \Delta_1 \geq 1 \), \( \delta(x_k) = |a_1 - b_3| = \Delta + \Delta_2 \geq 1 \), then we have \( \delta(x_i, x_j, x_k) \geq 2 \). In addition, \( \delta(x_u) \geq \frac{1}{2} \) and \( \delta(x_u) \geq \frac{1}{2} \), therefore \( \delta(\varphi, \eta) \geq 3 \).

Case (2.2) \( \eta \) is an edge generated from the bisector \( pb \langle i, u \rangle \), then \( \varphi \) and \( \eta \) can be rewritten as \( \varphi = (x_i^{a_1}, x_j^{a_2}, x_k^{a_3}, x_u^{a_4}) \cup (X_{others}) \) and \( \eta = (x_i^{b_1+\frac{1}{2}}, x_j^{b_2}, x_k^{b_3}, x_u^{b_1+\frac{1}{2}}, x_u^{b_1+\frac{3}{2}}) \cup (X_{others}) \). This means \( X_{removed} \) means some components, such as \( a_1 - 1 \) and \( a_1 + 1 \), are excluded from \( X_{others} \). Then we have four combinations: (1) \( a_1 = b_1 \); (2) \( a_2 = b_1 \); (3) \( a_1 = b_1 + 1 \); (4) \( a_1 = b_1 + 1 \), \( a_2 = b_1 + 1 \), \( a_2 = b_1 + 1 \), \( a_3 = b_1 + 1 \), \( a_2 = b_1 + 1 \), \( a_3 = b_1 + 1 \), \( a_4 = b_1 + 1 \), \( a_3 = b_1 + 1 \), \( a_4 = b_1 + 1 \), \( a_3 = b_1 + 1 \), \( a_4 = b_1 + 1 \), \( a_3 = b_1 + 1 \). However, these combinations are not allowed because they will lead to impossible equations such that \( a_1 = a_2, a_2 = a_1 - 1 \), or \( a_2 = a_1 + 1 \), the components that have been removed. As a result, \( \delta \) is impossible to reach 2 and hence \( \delta \geq 3 \).

Case (2.3) \( \eta \) is an edge generated from the bisector, say \( pb \langle i, j \rangle \), which is one of the three bisectors generating the \( \varphi \), then \( \varphi \) and \( \eta \) can be rewritten as \( \varphi = (x_i^{a_1}, x_j^{a_2}, x_k^{b_3}) \cup (X_{others}) \) and \( \eta = (x_i^{b_1+\frac{1}{2}}, x_j^{b_2}, x_k^{b_3}) \cup (X_{others}) \). Using the similar analysis in Case (2.2), easy to know that \( \delta \geq 2 \), and \( \delta = 2 \) only when case (2.3.1) \( a_1 = b_1 \), \( b_2 = b_1 - 1 \) or case (2.3.2) \( a_1 = b_1 + 1 \), \( b_2 = b_1 + 2 \). For the two cases, the two edges bear codes \( \eta_1 = (x_i^{a_1+\frac{1}{2}}, x_j^{a_1+\frac{1}{2}}, x_k^{a_1-1}) \) and \( \eta_2 = (x_i^{a_1-\frac{1}{2}}, x_j^{a_1+\frac{1}{2}}, x_k^{a_1+1}) \). In terms of Eq. (2.37), the two edges are just the ones that are in bisector \( pb \langle i, j \rangle \) and joint to a \( \varphi \)'s. And in these cases, \( \gamma(\varphi, \eta) = 3 \).
Based on the above cases (1) and (2), we conclude that $\delta = 1$ or $\delta = 2$ are the only cases that an edge contains an either 2-I or 3-I vertex. For other cases, they must be disjoint.

Given an edge, an useful function is to calculate all possible chromatic codes of vertexes contained by the edge. Function $E2V(\eta)$ returns all contained vertexes, and in particular, $E2V(\eta, 2I)$ and $E2V(\eta, 3I)$ return all 2-I and 3-I vertexes, respectively.

**Notation 2.** The procedure of $E2V(\eta, 2I)$:

Let $\eta$ is an edge with code $(x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}}) \cup (X_{\text{others}})$, where $X_{\text{others}} = \mathbb{N}\setminus \{z, z+1\}$.

1. Find the minimum component $w$ in $X_{\text{others}}$, assume it is $x_w^w$; (2) Find $w + 1$:
   - if found, assume it is $x_w^{w+1}$ and $x_w^{w+1}$ both to $w + \frac{1}{2}$ to form a 2-I vertex $(x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}}, x_u^{w+\frac{1}{2}}, x_v^{w+\frac{1}{2}})$; if not found, let $w = w + 1$ and repeat (1) and (2) until $w = n - 2$.

Because $(X_{\text{others}})$ can be partitioned into two parts $N_1 = \mathbb{N}[0, z - 1]$ and $N_2 = \mathbb{N}[z+2, n-1]$, if $z \neq 0$ and $z \neq n - 2$, given any a component pair such as $(x_w^w, x_w^{w+1})$ in $N_1$ or $N_2$, it corresponds an edge with codes $(x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}}, x_u^{w+\frac{1}{2}}, x_v^{w+\frac{1}{2}})$. Because there are $z - 1$ such component pairs in $N_1$, and $n - z - 3$ pairs in $N_2$, therefore total $n - 4$ available pairs. If $z \neq 0$ or $z \neq n - 2$, then either $N_1$ or $N_2$ will be empty and the other will contain $n - 3$ available pairs. For example, for edge $(0, \frac{7}{2}, 5, 1, 2, \frac{7}{2})$ with $z = 3$ (see edge 07A247 in Fig.3b), and hence it has $n - 4 = 2$ available component pairs. We first found $(0, 1)$ and next $(1, 2)$, and then they form two 2-I vertexes $(\frac{1}{2}, \frac{7}{2}, 5, \frac{1}{2}, 2, \frac{7}{2})$ (see vertex 17A147 in Fig.3b) and $(0, \frac{7}{2}, 5, \frac{3}{2}, 2, \frac{7}{2})$

**Notation 3.** The procedure of $E2V(\eta, 3I)$:

Let $\eta$ is an edge with code $(x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}}) \cup (X_{\text{others}})$, where $X_{\text{others}} = \mathbb{N}\setminus \{z, z+1\}$.

1. Find the minimum component $w$ in $X_{\text{others}}$, assume it is $x_w^w$; (2) If $e = (2z + 1 + w) \equiv 0 \pmod{3}$ and $e \notin (X_{\text{others}})$, then change $x_i^{z+\frac{1}{2}}, x_j^{z+\frac{1}{2}}$, and $x_w^w$ to $x_3^3$ to form a 3-I vertex $(x_i^3, x_j^3)$; (3) Let $w = w + 1$ and repeat (2) until $w = n - 1$.

Because $e \equiv 0 \pmod{3}$ and $z = \mathbb{N}[0, n - 2]$, therefore $w = 3m - 2z - 1$, with condition that $w = \mathbb{N}[0, n - 1]\{z, z+1\}$ and $m \in \mathbb{N}[1, \frac{n}{3}(n + 1)]$. For example, for edge $(2, 3, \frac{5}{2}, 0, 1, \frac{3}{2})$ with $z = 4$ (edge 469029 in Fig.3b), only $m = 4$ and $w = 3$ will form a 3-I vertex $(2, 4, 4, 0, 1, 4)$ (see vertex 488028 in Fig.3b).

Therefore the contain relation between a vertex $\varphi$ and an edge $\eta$ can be also determined by checking if $\varphi \in E2V(\eta)$.

Another V-E relation is that two vertexes are exactly the two ends of an edge, called they are segmented (Fig.6e), that is, $\cap(\eta, \varphi_1, \varphi_2) = \{\varphi_1, \varphi_2\}$. This relation is equivalent to two vertexes share an edge.

**Proposition 3.** Given an edge $\eta$ and two vertexes $\varphi_1$ and $\varphi_2$, which could be both 2-I, or 3-I, or one is 2-I and the other is 3-I, then

\[
\cap(\eta, \varphi_1^{2I}, \varphi_2^{2I}) = \{\varphi_1^{2I}, \varphi_2^{2I}\} \iff \delta(\varphi_1^{2I}, \varphi_2^{2I}) = 2
\]

\[
\cap(\eta, \varphi_1^{2I}, \varphi_2^{3I}) = \{\varphi_1^{2I}, \varphi_2^{3I}\} \iff \delta(\varphi_1^{2I}, \varphi_2^{3I}) = 3
\]

\[
\cap(\eta, \varphi_1^{3I}, \varphi_2^{3I}) = \{\varphi_1^{3I}, \varphi_2^{3I}\} \iff \delta(\varphi_1^{3I}, \varphi_2^{3I}) = 4
\]
Proof. Case (1) If \( \varphi_1 \) and \( \varphi_2 \) are both 2-I vertexes, then according to Property \( \varphi_1 \) and \( \varphi_2 \) it is easy to know that \( \delta(\varphi_1, \varphi_2) = 2 \). Below we prove that if \( \delta(\varphi_1, \varphi_2) = 2 \), then they are segmented.

Case (1.1) If \( \varphi_2^{2f} \) is generated by \( pb(i, j) \) and \( pb(u, v) \) and hence with a code 
\[
(x_i^{a_1+1}, x_j^{a_1+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6}) \]
and \( \varphi_2^{2f} \) is generated by \( pb(e, f) \) and \( pb(g, h) \) and hence with a code 
\[
(x_i^{b_1+1}, x_j^{b_2}, x_u^{b_3}, x_v^{b_4}, x_c^{b_5+1}, x_e^{b_6+1}, x_g^{b_7}, x_h^{b_8+1}) .
\]
Apparently their \( \delta \geq 4 \).

Case (1.2) If they share one point, say \( e = i \), then their codes should be 
\[
(x_i^{a_i+1}, x_j^{a_i+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_2}, x_u^{b_3}, x_v^{b_4}, x_c^{b_5+1}, x_e^{b_6+1}, x_g^{b_7}, x_h^{b_8+1}) ,
\]
so \( \delta \geq 3 \).

Case (1.3) If they share two points, say \( e = i \) and \( f = u \), then their codes should be 
\[
(x_i^{a_1+1}, x_j^{a_1+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_2}, x_u^{b_3}, x_v^{b_4}, x_c^{b_5+1}, x_e^{b_6+1}, x_g^{b_7}, x_h^{b_8+1}) ,
\]
so \( \delta \geq 2 \). To make \( \delta = 2 \), it must be \( \delta(x_i) = \delta(x_u) = 0 \), indicating that 
\( a_1 = a_2 \), an impossible case. Therefore \( \delta \geq 3 \).

Case (1.4) If they share two points, say \( e = i \) and \( f = j \), then their codes should be 
\[
(x_i^{a_1+1}, x_j^{a_1+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_2}, x_u^{b_3}, x_v^{b_4}, x_c^{b_5+1}, x_e^{b_6+1}, x_g^{b_7}, x_h^{b_8+1}) .
\]
Theoretically \( \delta \geq 2 \). To reach \( 2 \), it must be \( \delta(x_i) = \delta(x_j) = 0 \Rightarrow \delta(x_u) = \frac{1}{3} \Rightarrow a_1 = b_2 = b_1 \), 
\( \delta(x_v) = \frac{1}{2} \Rightarrow \{ a_2 = b_3 \} \), 
\( \delta(x_c) = \frac{1}{2} \Rightarrow \{ a_3 = b_4 \} \), 
\( \delta(x_e) = \frac{1}{2} \Rightarrow \{ a_4 = b_5 \} \). Then we have 16 combinations for solutions, but because \( a_3 \neq a_4 \), \( b_2 \neq b_3 \), as well as some components have been removed, therefore only the below four solutions are allowed.

\[
\begin{align*}
\{ a_1 = b_2, & a_2 = b_3 - 1, a_3 = b_4, a_4 = b_5 + 1 \\
a_1 = b_2, & a_2 = b_3 - 1, a_3 = b_4 + 1, a_4 = b_5 \\
a_2 = b_2 - 1, & a_3 = b_4, a_4 = b_4 + 1 \\
a_2 = b_2 - 1, & a_3 = b_4 + 1, a_4 = b_5 + 1 
\end{align*}
\]
(3.8)

Also in a 2-I unit, \( \varphi_2^{2f} \) should link to two edges with codes
\[
(x_i^{a_1+1}, x_j^{a_1+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6})
\]
(3.9)
and \( \varphi_2^{2f} \) should link to two edges with codes
\[
(x_i^{b_1+1}, x_j^{b_1+1}, x_u^{b_2}, x_v^{b_3}, x_c^{b_4}, x_e^{b_5+1}, x_g^{b_6}, x_h^{b_7+1})
\]
(3.10)

Given a solution in Eq. (3.8) we can always find two edges, in which one is from \( \varphi_1^{2f} \) (Eq. (3.9)) and the other is from \( \varphi_2^{2f} \) (Eq. (3.10)) are same, and therefore we get \( \varphi_1^{2f} \) and \( \varphi_2^{2f} \) are the two ends of an edge.

Case (1.5) If they share three points, say \( e = i \), \( f = j \), and \( g = u \), then their codes should be 
\[
(x_i^{a_1+1}, x_j^{a_1+1}, x_u^{a_2+1}, x_v^{a_2+1}, x_c^{a_3}, x_e^{a_4}, x_g^{a_5}, x_h^{a_6}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_1+1}, x_u^{b_2}, x_v^{b_3}, x_c^{b_4}, x_e^{b_5+1}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_1+1}, x_u^{b_2}, x_v^{b_3}, x_c^{b_4}, x_e^{b_5+1}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_1+1}, x_u^{b_2}, x_v^{b_3}, x_c^{b_4}, x_e^{b_5+1}) \]
and 
\[
(x_i^{b_1+1}, x_j^{b_1+1}, x_u^{b_2}, x_v^{b_3}, x_c^{b_4}, x_e^{b_5+1}) .
\]
Theoretically \( \delta \geq 1 \). To reach \( 1 \), it must be 
\( a_1 = b_1, a_2 = b_2, \{ a_2 = b_3 - 1 \} \), 
\( a_3 = b_4 \). Because 
\( a_2 = b_3 \Rightarrow b_2 = b_3, a_3 = b_2 \Rightarrow a_2 = a_3, a_2 = b_3 - 1 \Rightarrow b_3 = b_2 + 1, a_3 = b_2 + 1 \Rightarrow \).
\[ a_3 = a_2 + 1 \], but all these equations are impossible because they have been already removed from their codes.

Then the next minimum \( \delta \) should be \( 2 \). Because \( \delta(x_i) = \delta(x_j) \geq 0 \), they must be \( = 0 \) in this case, since if they \( = 1 \), then \( \delta \geq 3 \). Because \( \delta(x_v) \) and \( \delta(x_h) \) at least contribute \( 1 \) to \( \delta \), then if \( \delta = 2 \), then another \( \delta(x) \) should be from \( \delta(x_u) \) or \( \delta(x_{\text{others}}) \). But if \( \delta(x_{\text{others}}) = 1 \), then \( \delta(x_u) \) must be \( 0 \), and if \( \delta(x_u) = 0 \), then we have that \( \delta(x_i) = \delta(x_j) = \delta(x_u) = 0 \) and \( \delta(x_v) = \delta(x_h) = \frac{1}{2} \), an impossible case we just proved above. Therefore we have \( \delta(x_u) = 1 \) ⇒ \( \{ a_2 = b_2 + 1 \} \) and also \( \{ a_2 = b_3 \}, \{ a_2 = b_2 + 1 \} \). We then have 8 combinations for these solutions such as

\[
\begin{align*}
    a_2 &= b_2 + 1, a_3 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 + 1, a_2 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 + 1, a_2 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 - 1, a_2 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 - 1, a_2 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 - 1, a_2 = b_3, a_3 = b_2 + 1 \\
    a_2 &= b_2 - 1, a_2 = b_3, a_3 = b_2 + 1 \\
\end{align*}
\]

(3.11)

By checking these solutions we can always find some removed components at \( i \), \( j \), \( u \), \( v \), and \( h \), except for the solution that \( a_2 = b_2 + 1, a_2 = b_3 - 1, a_3 = b_2 \). Also \( \varphi_1^{3 \ell} \) should link to an edge \( \eta_1 = (x_i^{a_1 + \frac{1}{2}}, x_j^{a_1 + \frac{1}{2}}, x_v^{a_2 + 1}, x_h^{a_3}) \) and \( \varphi_2^{3 \ell} \) should link to an edge \( \eta_2 = (x_i^{b_1 + \frac{1}{2}}, x_j^{b_1 + \frac{1}{2}}, x_v^{b_2 + 1}, x_h^{b_2}) \). Since \( (a_3 = b_2, a_2 = b_2 + 1) \) ⇒ \( a_3 = a_2 - 1, a_2 = b_3 - 1 \Rightarrow b_3 = a_2 + 1, a_2 = b_2 + 1 \Rightarrow b_2 = a_2 - 1 \), therefore \( \eta_1 = \eta_2 = (x_i^{a_1 + \frac{1}{2}}, x_j^{a_1 + \frac{1}{2}}, x_v^{a_2 + 1}, x_h^{a_3}) \), indicating that the two vertexes are both the ends of the same edge.

Above we proved that if chromatic distance between two 2-I vertexes is 2, then they must be segmented with an edge. The other two cases Eq.3.6 and 3.7 can be proved in the same way, but they are too long and hence not presented here.

Similarly, we can also use \( E2V \) function to determine if two vertexes and one edge are segmented, that is,

\[
\cap(\varphi_1, \varphi_2) = \{ \varphi_1, \varphi_2 \} \iff \varphi_1, \varphi_2 \in E2V(\eta)
\]

(3.12)

3.1.3. Vertex-Cell (V-C) relations. The relation between a vertex \( \varphi \) and a cell \( \zeta \) is that \( \varphi \) is one of edge ends which are the boundaries of \( \zeta \), called the cell contains \( \varphi^{3 \ell} \), i.e., \( \cap(\zeta, \varphi) = \varphi \), see Fig.6b; otherwise, they are disjoint, i.e., \( \cap(\zeta, \varphi) = \emptyset \), see Fig.6c.

**Proposition 4.** Given a vertex \( \varphi \) and a cell \( \zeta \), then

\[
\cap(\zeta, \varphi) = \varphi \iff \delta(\zeta, \varphi) = 2
\]

(3.13)

\[
\cap(\zeta, \varphi) = \emptyset \iff \delta(\zeta, \varphi) > 2
\]

**Proof.** From table 1 we know that either for \( \varphi^{3 \ell} \) or \( \varphi^{3 \ell} \), \( \delta(\varphi, \zeta) = 2 \).

Case (1): Suppose a vertex \( \varphi^{3 \ell} \) bears a code \( (x_i^{b_1 + \frac{1}{2}}, x_j^{b_2 + \frac{1}{2}}, x_v^{b_3}, x_h^{b_4}) \) \( \cup \) \( X_u \) and a cell \( \zeta \) bears a code \( (x_i^{b_1 + \frac{1}{2}}, x_j^{b_2 + \frac{1}{2}}, x_v^{b_3}, x_h^{b_4}) \cup \emptyset \), then \( \delta(\varphi^{3 \ell}, \zeta) \geq 2 \). To reach 2, it must be \( |a_1 + \frac{1}{2} - b_1| = \frac{1}{2}, |a_1 + \frac{1}{2} - b_2| = \frac{1}{2}, |a_2 + \frac{1}{2} - b_3| = \frac{1}{2}, |a_2 + \frac{1}{2} - b_4| = \frac{1}{2}, \) and \( |X_u - X_v| = 0 \). Therefore for each equation, the
solutions should be the cases that \{a_1=b_1, a_2=b_2\}, \{a_1=b_1, a_2=b_3\}, \{a_2=b_3\}, and \{a_2=b_4\}, respectively. It is also known that \(b_1 \neq b_2 \neq b_3 \neq b_4\), thus only the below four solutions are allowed to give \(\delta(\varphi^{3I}, \zeta) = 2\): (1) \(a_1 = b_1, a_2 = b_2 - 1, b_2 = b_1 - 1\), (2) \(a_1 = b_2, b_2 = b_1 - 1\), (3) \(a_2 = b_3, b_3 = b_4 - 1\), and (4) \(a_2 = b_4, b_4 = b_3 - 1\). It is easy to know that the four solutions are just the four cells around the vertex \(\varphi^{2I}\) in a 2-I unit.

Case (2): suppose a vertex \(\varphi^{3I}\) bears a code \((x_1^{a_1}, x_j^{a_2}, x_k^{a_3})\) and a cell \(\zeta\) bears a code \((x_1^{b_1}, x_j^{b_2}, x_k^{b_3})\), and let \(|a_1 - b_1| = \Delta_1, |a_1 - b_2| = \Delta_2, |a_1 - b_3| = \Delta_3\). Although theoretically \(\delta(\varphi^{3I}, \zeta) = \Delta_1 + \Delta_2 + \Delta_3 \geq 0\), it is impossible to reach 0 or even 1 because \(b_1 \neq b_2 \neq b_3\). Therefore the next minimum \(\delta(\varphi^{3I}, \zeta)\) is 2 and it should be given by two of \(\Delta_1, \Delta_2, \Delta_3\) both = 1, and one of them = 0. Let’s assume \(\Delta_1 = 0\) and hence \(a_1 = b_1\), then we have \(|b_1 - b_2| = 1\) and \(|b_1 - b_3| = 1\). The two equations have solutions \(\{b_1 = b_2 - 1, b_3 = b_1 + 1\}\) and \(\{b_1 = b_2 - 1, b_3 = b_1 + 1\}\), and hence only two solutions are allowed: \(\{b_1 = b_2 - 1, b_3 = b_1 + 1\}\). If \(\Delta_2 = 0\) or \(\Delta_3 = 0\), we can get another four allowed solutions which give \(\delta(\varphi^{3I}, \zeta) = 2\), and in total we get six solutions. Comparing these solutions to those cells (Eq. 2.38) around the \(\varphi^{3I}\) in a 3-I unit, we thus to know that if \(\delta(\varphi^{3I}, \zeta) = 2\), the cell must contain the \(\varphi^{3I}\).

3.1.4. Edge-Edge (E-E) relations. In 2-I and 3-I units there three types of relations between two edges, i.e., adjacent, opposite, and interval, and their chromatic distances could be 2, 3, or 4, respectively, but actually all the three relations are topologically same as two edges share an either 2-I or 3-I vertex as one of their ends. This E-E relation is called the two edges \(\eta_1\) and \(\eta_2\) are joint with a vertex, i.e., \(\cap(\eta_1, \eta_2) = \varphi\), which further has two types: (1) collinear, denoted by \(\ove{\eta_1\eta_2}\) (Fig.6g), and (2) not collinear (Fig.6h). If two edges do not share any particles, they are disjoint (Fig.6i).

The E-E collinear relation is easy to determine by using the below proposition.

**Proposition 5.** Given two edges \(\eta_1(x_1^{z_1+\frac{1}{2}}, x_j^{z_1+\frac{1}{2}})\) and \(\eta_2(x_u^{z_1+\frac{1}{2}}, x_v^{z_1+\frac{1}{2}})\),

\[
\ove{\eta_1\eta_2} \iff i = u, j = v
\] (3.14)

Note that Proposition 5 can only tell if two edges are collinear, but two collinear edges may not be joint. A feasible method to reason topological joint between two edges is using \(E2V(\eta)\) function to calculate all possible vertexes contained by the two edges, and if among them two vertexes are equal, then they are joint with this vertex. This method, however, can only tell if two edges are possible to be joint, but in a real full-OACD, they may not be joint because the joint vertex is hidden in high-dimensional spaces. For example, the edge (36A038) and (25A058) in Fig. 4b both have an end vertex (44A048) and hence joint, but the vertex does not emerge in the \(\mathbb{R}^2\) plane, so that the two edges appear disjoint. Similarly, if we only use the chromatic distances listed in Table 1 to determine E-E relations, then they may also lead to mistakes in \(\mathbb{R}^2\) plane due to the same reason. For the same example, in Fig.3b, \(\delta((36A038), (25A058)) = 2, \gamma((36A038), (25A058)) = 3\), indicating they are joint with a 3-I vertex, i.e., the result (44A048) of \(E2V\), but in the given plane they are not joint. Therefore, the below proposition is true for \(OACD\) at \(\mathbb{R}^{n-1}\) space, which contains all possible edges, cells, and vertexes.

**Proposition 6.** Given two edges \(\eta_1\) and \(\eta_2\) in \(OACD(n, \mathbb{R}^{n-1})\), let \((\delta, \gamma) = (\delta(\eta_1, \eta_2), \gamma(\eta_1, \eta_2))\), then

\[
\cap(\eta_1, \eta_2) = \varphi \iff \begin{cases} \varphi^{2I} \iff (\delta, \gamma) = (2, 2) \lor (2, 4) \\ \varphi^{3I} \iff (\delta, \gamma) = (2, 3) \lor (3, 2) \lor (4, 3) \end{cases}
\] (3.15)
Proof. Suppose \( \eta_1 \) is an edge generated from \( pb \langle i, j \rangle \) and hence with a code \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}) \), and \( \eta_2 \) is an edge generated from \( pb \langle u, v \rangle \) and hence with a code \( (x_u^{b_{uv} + u}, x_v^{b_{uv} + v}) \).

Case (1) If \( i \neq j \neq u \neq v \), then \( \eta_1 \) and \( \eta_2 \) can be rewritten to \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}, x_u^{a_{uv}}, x_v^{a_{uv}}) \) \( \cup \) \( (X^A_{others}) \) and \( (x_i^{b_{ij}}, x_j^{b_{ij}}, x_u^{b_{uv} + u}, x_v^{b_{uv} + v}) \) \( \cup \) \( (X^B_{others}) \), and if they are joint, then the joint vertex should be \( 2-I \). The minimum \( \delta(x_i) = 2 \) if \( \delta(x_i) = \delta(x_j) = \delta(x_u) = \delta(x_v) = \frac{1}{2} \), and \( \delta(X_{others}) = 0 \). Therefore the solutions are \( \{a_{ij} = b_i, a_{ij} = b_j\} \), and \( \{a_{uv} = b_{uv}, a_{uv} = b_{uv} + 1\} \), where only the below four solutions are allowed.

\[
\begin{align*}
\{a_{ij} &= b_i, a_{ij} = b_j - 1, a_u = b_{uv}, a_v = b_{uv} + 1 \\
\{a_{ij} &= b_i, a_{ij} = b_j - 1, a_u = b_{uv} + 1, a_v = b_{uv} \\
\{a_{ij} &= b_j, a_{ij} = b_i - 1, a_u = b_{uv}, a_v = b_{uv} + 1 \\
\{a_{ij} &= b_j, a_{ij} = b_i - 1, a_u = b_{uv} + 1, a_v = b_{uv}
\end{align*}
\]

For each solution, we substitute them into codes of \( \eta_1 \) and \( \eta_2 \), and these substitutions, for example of the first solution, will make their codes being \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}, x_u^{a_{uv}}, x_v^{a_{uv}}) \) \( \cup \) \( (X^A_{others}) \) and \( (x_i^{b_{ij}}, x_j^{b_{ij}}, x_u^{b_{uv} + u}, x_v^{b_{uv} + v}) \) \( \cup \) \( (X^B_{others}) \). Easy to see that the two edges are joint with a \( 2-I \) vertex \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}, x_u^{a_{uv}}, x_v^{a_{uv}}) \) and \( \gamma(\eta_1, \eta_2) = 4, \eta_1 \nsim \eta_2 \).

Since components in \( X_{others} \) are always integers, \( X_{others} \neq 1 \), then to make \( \delta = 3 \), it must be \( \delta(x_i, x_j, x_u, x_v) = 3 \). Suppose \( \delta(x_i) = \frac{1}{2}, \delta(x_j) = \frac{1}{2}, \delta(x_u) = \frac{1}{2}, \delta(x_v) = \frac{1}{2} \), where \( i, j, u, \) and \( v \) are odd numbers. Then \( \frac{i}{2} + \frac{j}{2} + \frac{u}{2} + \frac{v}{2} = 3 \Rightarrow i + j + u + v = 6 \), then the only solution is \( i = j = u = 1, v = 3 \). But this solution is impossible because it leads to \( X_{others} \neq 0 \Rightarrow \delta > 3 \). To make \( \delta = 4 \), it must be two cases (1.1) \( \delta(x_i, x_j, x_u, x_v) = 4 \), \( \delta(X_{others}) = 0 \) or (1.2) \( \delta(x_i, x_j, x_u, x_v) = 2 \), \( \delta(X_{others}) = 2 \). Similarly, only the case (1.2) is possible, and \( \gamma = 6 \). And easy to check that edges in case (1.2) are not joint to a vertex.

Case (2) If \( i = u \) and \( j = v \), then according to Proposition 5, the two edges should be generated from the same bisector \( pb \langle i, j \rangle = pb \langle u, v \rangle \). Therefore their codes can be rewritten to \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}) \) \( \cup \) \( (X^A_{others}) \) and \( (x_i^{b_{ij}}, x_j^{b_{ij}}) \) \( \cup \) \( (X^B_{others}) \). Let us discuss all possible cases of \( \delta(\eta_1, \eta_2) \) for \( D = a_1 - b_1 \).

Case (2.1) If \( D = 0 \), then \( \delta(\eta_1, \eta_2) = |X^A_{others} - X^B_{others}| \), and also \( X^B_{others} \) is a permutation of \( X^A_{others} = \{a_1, a_1 + 1\} \). Therefore the possible values of \( \delta \) will be given by some components in \( X_{others} \) changing their locations. Suppose the number of such components is \( m \).

Case (2.1.1) If \( m = 0 \), then \( \delta = 0 \), so this case can be excluded because it makes the two edges are equal.

Case (2.1.2) If \( m = 2 \), assuming the two components are \( x_a^{a_g} \) and \( x_h^{a_h} \), then \( \delta = 2|a_g - a_h| \). Because \( a_g \neq a_h \), therefore \( \delta \)'s possible values will be even numbers.

If \( \delta = 2 \), then it must be \( |a_g - a_h| = 1 \). The two edges can be rewritten to \( (x_i^{a_{ij} + i}, x_j^{a_{ij} + j}, x_u^{a_{uv}}, x_v^{a_{uv}}) \) \( \cup \) \( (X^A_{others}) \) and \( (x_i^{b_{ij}}, x_j^{b_{ij}}, x_u^{b_{uv} + u}, x_v^{b_{uv} + v}) \) \( \cup \) \( (X^B_{others}) \), indicating they are joint to a \( 2-I \) vertex generated by \( pb \langle i, j \rangle \) and \( pb \langle g, h \rangle \). And in this case, \( \gamma = 2, \eta_1 \nsim \eta_2 \).

If \( \delta = 4 \), then it must be \( |a_g - a_h| = 2 \), and also \( \gamma = 2 \), but in this case, the two edges are not joint.

Case (2.1.3) If \( m = 3 \), indicating that three components involve changing their locations, we can express the three components as \( X_{three} = (a_2, a_2 + \Delta_1, a_2 + \Delta_1 + \Delta_2) \).
$\Delta_2$) with $\Delta_1 \geq 1$ and $\Delta_2 \geq 1$. Easy to know that $X_{three}$ has 6 permutations, but in which only two permutations involve changing all three components, that is, $X^1_{three} = (a_2 + \Delta_1, a_2 + \Delta_2, a_2 + \Delta_2)$ and $X^2_{three} = (a_2 + \Delta_1 + \Delta_2, a_2 + \Delta_2, a_2 + \Delta_2)$. Then $\delta(X_{three}, X^1_{three}) = \delta(X_{three}, X^2_{three}) = 2(\Delta_1 + \Delta_2) \geq 4$, and only when $\Delta_1 = \Delta_2 = 1$, $\delta = 4$, $\gamma = 3$, $\eta_1 \cong \eta_2$, but in this case, the two edges are not joint.

Case (2.1.4) If $m = 4$, $\delta$, then there are four components in $X_{others}$ contributing to $\delta$, but in this case, the two edges are not joint.

Case (2.1.5) If $m > 4$, then $\delta \geq 4$.

Case (2.2) If $D = 1$, then $\delta(x_i) = \delta(x_j) = 1 \Rightarrow \delta \geq 2$. Also $X^A_{others} = \mathbb{N}\{a_1, a_1 + 1\}$ and $X^B_{others} = \mathbb{N}\{a_1 - 1, a_1\}$. For similar cases such as (2.1.1)-(2.1.4), we know that $X^A_{others}$ includes $a_1 - 1$ but excludes $a_1 + 1$, and $X^B_{others}$ includes $a_1 + 1$ but excludes $a_1 - 1$. Therefore we have $|X^A_{others} - X^B_{others}| \geq 2$. The $X^A_{others}$ and $X^B_{others}$ can be further rewritten to $X^A_{others} = (x^A_{g}, x^A_{h})$ and $X^B_{others} = (x^B_{g}, x^B_{h})$.

Case (2.2.1) $g \neq h$. To reach $|X^A_{others} - X^B_{others}| = 2$, it must be $\delta(x_g) = 1$, $\delta(x_h) = 1$, and $\delta(x_{others}) = 0$. Thus the solutions are $\{a_B = a_1\}$ and $\{a_A = a_1 + 2\}$. Because $a_1$ has been already excluded from both $X^A_{others}$ and $X^B_{others}$, the only solution is $a_B = a_1 - 2$ and $a_A = a_1 + 2$. However, this solution will make $X^A_{other}$'s include $a_B$ but exclude $a_A$, while $X^B_{other}$'s include $a_A$ but exclude $a_B$, implying $\delta(x_{other}) > 0$. Therefore, $|X^A_{others} - X^B_{others}|$ is impossible to be 2, and hence $\delta > 4$.

Case (2.2.2) $g = h$. Then $|X^A_{others} - X^B_{others}| = 2$ if $|X^A_{other} - X^B_{other}'| = 0$. In this case, $\delta = 4$ and $\gamma = 3$. The two edges turn to $(x_i, x_j, x_g)$ and $(x_i, x_j, x_g)$. We can find that $x_i + x_j + x_g = 3a_1$, then using $E2V(\eta, 3I)$ we know that they are both joint to a 3-I vertex $(x_i, x_j, x_g)$. Also $D = 1$ always implies $\eta_1 \cong \eta_2$.

Case (2.3) If $D \geq 2$, then $\delta(x_i) \geq 2$, $\delta(x_j) \geq 2$ and hence $\delta \geq 4$. Using the similar analysis in case (2.2), easy to know that $\delta$ is impossible to be 4 and hence $\delta > 4$.

Case (3) If $i = u$ but $j \neq v$, the two edges can be rewritten to $(x_i, x_j, x_g)$ and $(x_i, x_j, x_g)$.

Case (3.1) To reach $\delta = 1$, it must be $\delta(x_i) = \delta(x_{others}) = 0$, $\delta(x_j) = \delta(x_v) = 1$. Therefore we have solutions that $a_1 = b_1$, $\{a_1 = b_2\}$, and $\{a_2 = b_1\}$. Easy to check these solutions will lead to such as $b_2 = b_1$, $b_2 = b_1 + 1$, $a_2 = a_1$, $a_2 = a_1 + 1$ all are impossible since they have been excluded from their codes. As a result, the next minimum $\delta = 2$.

Case (3.2) To reach $\delta = 2$, there two possible cases of chromatic distance at each component.

Case (3.2.1) $\delta(x_i) = 1$, $\delta(x_j) = \delta(x_v) = 1$, and $\delta(X_{others}) = 0$.

This case leads to solutions that $\{a_1 = b_1 + 1\}$, $\{a_1 = b_2\}$, and $\{a_2 = b_1\}$, which correspond to 8 combinations but only 4 of them will lead to the allowed edges $(x_i, x_j, x_g)$ and $(x_i, x_j, x_g)$. Easy to check that the two edges are joint to a 3-I vertex $(x_i, x_j, x_g)$, and in this case $\gamma = 3$, $\eta_1 \cong \eta_2$. 

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Case (3.2.2) $\delta(x_i) = 0, \delta(x_j) = \frac{3}{2}, \delta(x_v) = \frac{1}{2},$ and $\delta(x_{other}) = 0$. These equations give solutions $a_1 = b_1, \left\{ \frac{a_1}{a_1} = b_2 + 1 \right\}$, and $\left\{ \frac{a_2}{a_2} = b_1 + 1 \right\}$. Easy to check these solutions are not allowed.

Case (3.3) To reach $\delta = 3$, there are 4 possible combinations of chromatic distance at each component ($\delta(x_i), \delta(x_j), \delta(x_v), \delta(x_{other})$): (1, $\frac{1}{2}, \frac{1}{2}, 1$), (2, $\frac{1}{2}, \frac{1}{2}, 0$), (0, $\frac{1}{2}, \frac{1}{2}, 0$), and (0, $\frac{1}{2}, \frac{1}{2}, 0$). Easy to check that the first three combinations are impossible and only the fourth are allowed to give edges such as $\left( \frac{x_i^{a_1 + 1}}{a_1}, \frac{x_j^{a_1 + 1}}{a_1}, \frac{x_v^{a_1 - 1}}{a_1}, \frac{x_{other}}{a_1} \right)$ and $\left( \frac{x_i^{a_1 + 1}}{a_1}, \frac{x_j^{a_1 + 1}}{a_1}, \frac{x_v^{a_1 - 1}}{a_1}, \frac{x_{other}}{a_1} \right)$, which also gives $\gamma = 2$ and $\eta_1 \equiv \eta_2$. Using $E2V(\eta, 3I)$ we know the two edges are both joint to a 3-I vertex $\left( \frac{x_i^{a_1}}{a_1}, \frac{x_j^{a_1}}{a_1}, \frac{x_v^{a_1}}{a_1} \right)$.

Case (3.4) To reach $\delta = 4$, there are 8 possible combinations of chromatic distance at each component ($\delta(x_i), \delta(x_j), \delta(x_v), \delta(x_{other})$): (1, $\frac{1}{2}, \frac{1}{2}, 2$), (2, $\frac{1}{2}, \frac{1}{2}, 1$), (3, $\frac{1}{2}, \frac{1}{2}, 0$), (1, $\frac{1}{2}, \frac{1}{2}, 0$), (0, $\frac{1}{2}, \frac{1}{2}, 1$), (1, $\frac{1}{2}, \frac{1}{2}, 0$), (0, $\frac{1}{2}, \frac{1}{2}, 1$), and (0, $\frac{1}{2}, \frac{1}{2}, 0$). Checking these combinations we know that only the first combination is allowed, which gives the edge codes $\left( \frac{x_i^{a_1 + 1}}{a_1}, \frac{x_j^{a_1 + 1}}{a_1}, \frac{x_v^{a_1 - 1}}{a_1}, \frac{x_{other}}{a_1} \right)$ and $\left( \frac{x_i^{a_1 + 1}}{a_1}, \frac{x_j^{a_1 + 1}}{a_1}, \frac{x_v^{a_1 - 1}}{a_1}, \frac{x_{other}}{a_1} \right)$, but they are not joint with the same 3-I vertex and also $\gamma = 5$.

Summarizing the above cases (1), (2.1.2), (2.2.2), (3.2.1), and (3.3) we know that the conditions in Table 1 are also sufficient for reasoning E-E joint relations.

Note that if we do not care the types of the joint vertex, then we can integrate conditions in Table 1 to

$$\cap (\eta_1, \eta_2) = \varphi \iff \left\{ \begin{array}{ll}
\delta \leq 3 \\
(\delta, \gamma) = (4, 3), \eta_1 \not\equiv \eta_2
\end{array} \right. \tag{3.17}$$

3.1.5. Edge-Cell (E-C) relations. There are three types of relations between an edge and a cell: (1) contain: the edge is one boundary of the cell, i.e., $\cap (\zeta, \eta) = \eta$ (Fig.6j), (2) joint: the cell only share a vertex with the edge, i.e., $\cap (\zeta, \eta) = \varphi$ (Fig.6k), and (3) disjoint: they do not share any particles, i.e., $\cap (\zeta, \eta) = \emptyset$ (Fig.6l).

**Proposition 7.** Given a cell $\zeta$ and a edge $\eta$, then

$$\cap (\zeta, \eta) = \eta \iff \delta(\zeta, \eta) = 1 \tag{3.18}$$

**Proof.** A cell $\zeta$ could be taken as the space closed by edges that are from either 2-I or 3-I units, and this gives $\zeta \cap \eta = \eta \Rightarrow \delta(\zeta, \eta) = 1$.

Suppose $\zeta$ bears a code $\left( \frac{x_i^{a_1}}{a_1}, \frac{x_j^{a_2}}{a_2}, \frac{x_v^{a_2}}{a_2} \right) \cup (X_{other}^A)$ and $\eta$ bears a code $\left( \frac{x_i^{b_1 + \frac{1}{2}}}{b_1}, \frac{x_j^{b_1 + \frac{1}{2}}}{b_1}, \frac{x_v^{b_1 + \frac{1}{2}}}{b_1} \right) \cup (X_{other}^B)$. To make $\delta(\zeta, \eta) = 1$, it must be that $|x_i^{a_1} - x_i^{b_1 + \frac{1}{2}}| = \frac{1}{2}, |x_j^{a_2} - x_j^{b_1 + \frac{1}{2}}| = \frac{1}{2},$ and $|X_{other}^A - X_{other}^B| = 0$, so the solutions are $\left\{ \frac{a_1}{a_1} = b_1 - 1 \right\}$ and $\left\{ \frac{a_2}{a_2} = b_1 - 1 \right\}$. Because $a_1 \not\equiv a_2$, then we have only two solutions $\left\{ \frac{a_1}{a_1} = b_1, \frac{a_2}{a_2} = b_1 - 1 \right\}$. It is easy to know that the two solutions are just the two cells who share the edge, that is, $\zeta \cap \eta = \eta$.

Similar to the function $E2V$, a function $C2E$ is used for calculating all edges that bound a cell.

**Notation 4.** The procedure of $C2E(\zeta)$:

Let $\zeta$ is an cell with base $\mathbb{N}[0, n - 1]$. (1) Find the minimum component $z$ in its code, assume it is $x_i^{a_2}$; (2) Find $z + 1$, assume it is $x_j^{a_2 + 1}$, then change $x_i^{a_2}$ and $x_j^{a_2 + 1}$ to $x_i^{a_2 + \frac{1}{2}}$ and $x_j^{a_2 + \frac{1}{2}}$, respectively. (3) Let $z = z + 1$ and repeat (1) and (2) until $z = n - 2$. 

Therefore in theory each cell should be bounded by \( n - 1 \) edges, i.e., they are \( n \)-hedras, but in fact many cells are triangles, quadrilaterals, or polygons with edges much less then \( n - 1 \), indicating that a large number of edges do not emerge in plane.

**Proposition 8.** Given a cell \( \zeta \) and an edge \( \eta \) in \( OACD(n, \mathbb{R}^{n-1}) \), then

\[
\cap (\zeta, \eta) = \varphi \iff 3 \leq \delta(\zeta, \eta) \leq 4 \tag{3.19}
\]

**Proof.** Table [1] shows that in 2-I and 3-I units, if a cell and an edge are joint, then their chromatic distances are either 3 or 4.

Suppose \( \zeta \) bears a code \( (x_1^{a_1}, x_2^{a_2}) \cup (X_{\text{others}}^3) \) and \( \eta \) bears a code \( (x_1^{b_1+\frac{1}{2}}, x_2^{b_1+\frac{1}{2}}) \cup (X_{\text{others}}^B \setminus X_{\text{removed}}^{b_1+1}) \).

Case (1) To make \( \delta(\zeta, \eta) = 3 \), all possible combinations of \( (\delta(x_1), \delta(x_2), \delta(x_{\text{others}})) \) are \((\frac{1}{2}, \frac{1}{2}, 2), (\frac{1}{2}, \frac{3}{2}, 1), (\frac{3}{2}, \frac{3}{2}, 0), \) and \((\frac{3}{2}, \frac{3}{2}, 0) \). Checking these combinations we know that only the cases \((\frac{1}{2}, \frac{1}{2}, 2) \) and \((\frac{1}{2}, \frac{3}{2}, 1) \) are allowed, in which \((\frac{1}{2}, \frac{1}{2}, 2) \) gives two solutions \( \{ a_1 = b_1 \} \) and \( \{ a_1 = b_1 + 1 \} \), namely, two cells are opposite to an edge in a 2-I unit, and \((\frac{1}{2}, \frac{3}{2}, 1) \) gives two solutions \( \{ a_2 = b_1 + 2 \} \) and \( \{ a_2 = b_1 + 1 \} \), namely, two edges are interval to a cell in a 3-I unit. And easy to know that their code distances are 4 and 3, respectively.

Case (2) To make \( \delta(\zeta, \eta) = 4 \), all possible combinations of \( (\delta(x_1), \delta(x_2), \delta(x_{\text{others}})) \) are \((\frac{1}{2}, \frac{1}{2}, 3), (\frac{1}{2}, \frac{3}{2}, 2), (\frac{3}{2}, \frac{3}{2}, 1), (\frac{3}{2}, \frac{3}{2}, 0), \) and \((\frac{3}{2}, \frac{3}{2}, 0) \). Similarly, only \((\frac{1}{2}, \frac{3}{2}, 2) \) is allowed to give two solutions \( \{ a_1 = b_1 \} \) and \( \{ a_1 = b_1 + 1 \} \) that correspond to the two cells being opposite to an edge in a 3-I unit, with \( \gamma = 3 \).

Based on the above two propositions, the disjoint E-C relation can be determined by the below corollary.

**Corollary 2.** Given a cell \( \zeta \) and an edge \( \eta \), then

\[
\cap (\zeta, \eta) = \emptyset \leftrightarrow \delta(\zeta, \eta) > 4 \tag{3.20}
\]

We can use \( C2E \) and then \( E2V \) function, that is, \( E2V(C2E(\zeta)) \), to obtain all vertexes contained by a cell. Also we can define a new function \( C2V(\zeta) \) to directly find out all of such that vertexes, using the similar procedures in \( E2V(\eta) \). Therefore using functions \( C2E \) and \( C2V \), E-C relations can be also expressed by

\[
\cap(\zeta, \eta) = \eta \iff \eta \in C2E(\zeta) \\
\cap(\zeta, \eta) = \varphi \iff \eta \notin C2E(\zeta) \land C2V(\zeta) \cap E2V(\eta) \neq \emptyset \\
\cap(\zeta, \eta) = \emptyset \iff C2V(\zeta) \cap E2V(\eta) = \emptyset \tag{3.21}
\]

Note that because some joint vertexes may be hidden in high dimensional spaces, therefore some joint E-C relations may appear to be disjoint in \( \mathbb{R}^2 \) plane.

### 3.1.6. Cell-Cell (C-C) relations.

The C-C relations can also be three types: (1) **connected**: two cells share a common edge, i.e., \( \cap(\zeta_1, \zeta_2) = \eta \) (Fig.6m), (2) **joint**: they share a common vertex, i.e., \( \cap(\zeta_1, \zeta_2) = \varphi \) (Fig.6n), and (3) **disjoint**: they do not share any particles, i.e., \( \cap(\zeta_1, \zeta_2) = \emptyset \) (Fig.6o). The connected and joint relations actually correspond to the five C-C relations occurred in 2-I or 3-I units (Table 1), where the adjacent corresponds to the connected, and the opposite and interval both correspond to the joint.
Proposition 9. Given two cells $\zeta_1$ and $\zeta_2$,

$$ \cap (\zeta_1, \zeta_2) = \eta \iff \delta(\zeta_1, \zeta_2) = 2 $$

(3.22)

Proof. We have proved that if two cell are adjacent, then their chromatic distance is 2. Now we need to prove if $\delta(\zeta_1, \zeta_2) = 2$, then they must be adjacent. Suppose $\zeta_1$ and $\zeta_2$ are with codes $(x_i^{a_1}, x_j^{a_2}) \cup (X_i^A, X_j^B)$ and $(x_i^{b_1}, x_j^{b_2}) \cup (X_i^B, X_j^B)$. The only way giving $\delta = 2$ is that $|a_i - b_i| = 1$, $|a_j - b_j| = 1$, and $|X_i^A - X_i^B| = 0$. This leads to solutions \( \{a_1 = b_1, a_2 = b_2, \} \) \( \{a_1 = b_1 + 1, a_2 = b_2 - 1, \} \) \( \{a_1 = b_1 - 1, a_2 = b_2 + 1, \} \) \( \{a_1 = b_1 + 1, a_2 = b_2 - 1, \} \) then we can obtain the allowed two solutions that (1) $a_1 = b_2, a_2 = b_1, a_1 = b_1 - 1, a_2 = b_2 + 1$ and (2) $a_1 = b_2, a_2 = b_1, a_1 = b_1 - 1, a_2 = b_2 - 1$. Substituting the two solutions back to chromatic codes of $\zeta_1$ and $\zeta_2$, then they turn to such as $\zeta_1 = (x_i^{a_1}, x_j^{a_2})$, $\zeta_2 = (x_i^{a_1 + 1}, x_j^{a_2})$, or $\zeta_1 = (x_i^{a_1 - 1}, x_j^{a_2})$, $\zeta_2 = (x_i^{a_1 - 1}, x_j^{a_2})$, or similar codes. Using C2E($\zeta$) we know that they both joint with an edge $(x_i^{a_1 + 1}, x_j^{a_2})$ or $(x_i^{a_1 - 1}, x_j^{a_2})$, or similar codes. \( \square \)

Proposition 10. Given two cells $\zeta_1$ and $\zeta_2$,

$$ \cap (\zeta_1, \zeta_2) = \varphi \iff \delta(\zeta_1, \zeta_2) = 4 $$

(3.23)

Proof. The properties [11] shows that if two cells are joint to a vertex, then their chromatic distance is 4. Now let us explore all possible cases that make chromatic distance is 4.

First we rewrite $\zeta_1$ and $\zeta_2$ to $(X_{\text{diff}1}^{D_1}) \cup (X_{\text{same}}^S)$ and $(X_{\text{diff}2}^{D_2}) \cup (X_{\text{same}}^S)$, where $X_{\text{same}}^S = X_{\text{same}}^S$. This indicates that $\delta(\zeta_1, \zeta_2)$ is only contributed by $X_{\text{diff}}$, where for each component, $x_{\text{diff}}^{D_1} \neq x_{\text{diff}}^{D_2}$. Suppose $X_{\text{diff}}$ contains $m$ components, then

Case (1) $m = 2$, namely, $\gamma(\zeta_1, \zeta_2) = 2$.

Let $X_{\text{diff}}^{D_1} = (x_i^{a_1}, x_j^{a_2})$ and $X_{\text{diff}}^{D_2} = (x_i^{b_1}, x_j^{b_2})$, then we have $a_1 \neq b_1, a_2 \neq b_2$.

Also all cells are equi-base, $\zeta_1 \equiv \zeta_2$, and $X_{\text{same}}^S = X_{\text{same}}^S \Rightarrow S_1 \equiv S_2$, therefore we always have $X_{\text{diff}}^{D_1} \equiv X_{\text{diff}}^{D_2}$ \( \{a_1 = b_1, a_2 = b_2, \} \). Because $x_{\text{diff}}^{D_1} \neq x_{\text{diff}}^{D_2}$, therefore we have $a_1 = b_2$ and $a_2 = b_1$, and then $\delta = 2|a_1 - a_2| = 4 \Rightarrow a_1 = a_2 = 2$ or $a_1 = a_2 = 2$. We thus further rewrite $X_{\text{diff}}^{D_1}$ and $X_{\text{diff}}^{D_2}$ to $(x_i^{a_2 + 2}, x_j^{b_2}) \cup (x_k^{a_2 + 2}, x_j^{b_2})$ and $(x_i^{a_2}, x_j^{b_2 + 2}) \cup (x_k^{a_2}, x_j^{b_2 + 2})$, because $a_2 + 1$ must be somewhere in $X_{\text{same}}$, assuming it is $k$. Then $\zeta_1$ should be bounded by two edges $\eta_{11} = (x_i^{a_2 + 2}, x_j^{b_2 + 2}, x_k^{b_2 + 2}), \eta_{12} = (x_i^{a_2 + 2}, x_j^{b_2 + 2}, x_k^{b_2 + 2})$, and $\zeta_2$ should be bounded by two edges $\eta_{21} = (x_i^{a_2 + 2}, x_j^{b_2 + 2}, x_k^{b_2 + 2})$, $\eta_{22} = (x_i^{a_2 + 2}, x_j^{b_2 + 2}, x_k^{b_2 + 2})$.

From the properties of 3-I unit we know that $\eta_{11}$ and $\eta_{22}$ are joint to a $\varphi^3$ and both in bisector $pb(j, k)$, and in the same way, $\eta_{12}$ and $\eta_{21}$ are joint to the same $\varphi^3$ and also both in bisector $pb(i, k)$ – this is just the case that the two cells are in a 3-I units and with an opposite relation.

Case (2) $m = 3$, namely, $\gamma(\zeta_1, \zeta_2) = 3$.

We rewrite $X_{\text{diff}}^{D_1}$ and $X_{\text{diff}}^{D_2}$ to $(x_i^{a_1}, x_j^{a_2}, x_k^{a_3})$ and $(x_i^{b_1}, x_j^{b_2}, x_k^{b_3})$. Because $X_{\text{diff}}^{D_1} \equiv X_{\text{diff}}^{D_2}$, and $x_{\text{diff}}^{D_1} \neq x_{\text{diff}}^{D_2}$, then we have $a_1 = b_3, a_2 = b_1, a_3 = b_2$, or $a_1 = b_2, a_2 = b_3, a_3 = b_1$. Suppose $a_2 = a_1 + \Delta_1$ and $a_3 = a_1 + \Delta_1 + \Delta_2$, with $\Delta_1 \geq 1$ and $\Delta_2 \geq 1$, then easy to obtain that $\delta = 2\Delta_1 + 2\Delta_2$. Therefore $\delta = 4$, only if $\Delta_1 = \Delta_2 = 1$. The $X_{\text{diff}}^{D_1} = (x_i^{a_1}, x_j^{a_2 + 1}, x_k^{a_3 + 2})$ and $X_{\text{diff}}^{D_2} = (x_i^{a_1 + 1}, x_j^{a_2}, x_k^{a_3})$, or $X_{\text{diff}}^{D_1} = (x_i^{a_1 + 1}, x_j^{a_2}, x_k^{a_3})$, or their corresponding permutations. Using C2E($\zeta$), we can get $\zeta_1$ has two edges $\eta_{11} = (x_i^{a_1 + 1}, x_j^{a_2}), \eta_{12} = (x_i^{a_1 + 1}, x_j^{a_2})$, and $\eta_{21} = (x_i^{a_1 + 1}, x_j^{a_2})$, or case $X_{\text{diff}}^{D_1}$, $\zeta_2$ has two edges $\eta_{211} = (x_i^{a_1 + 1}, x_j^{a_2})$.
\(\eta_{212} = (x_i^{a_1 + \frac{1}{2}}, x_k^{a_1 + \frac{1}{2}})\); and for case \(X_{diff_2}\), \(\zeta_{222}\) has two edges \(\eta_{221} = (x_j^{a_1 + \frac{1}{2}}, x_k^{a_1 + \frac{1}{2}})\), \(\eta_{222} = (x_i^{a_1 + \frac{3}{2}}, x_k^{a_1 + \frac{3}{2}})\); Using \(E2V(\eta)\) we know that all the six edges are joint to the same \(\varphi^{m} = (x_i^{a_1+1}, x_j^{a_1+1}, x_k^{a_1+1})\). Also according to the Proposition 5, the edges in pairs \((\eta_{11}, \eta_{11})\), \((\eta_{12}, \eta_{221})\), \((\eta_{212}, \eta_{222})\) are both in the same bisector. These cell-edge-vertex relations are just the case that two cells \(\zeta_{21}\) and \(\zeta_{22}\) are in interval relations to the cell \(\zeta_{1}\).

Case (3) \(m = 4\), namely, \(\gamma(\zeta_{1}, \zeta_{2}) = 4\).

We rewrite \(X_{diff_1}\) and \(X_{diff_2}\) to \((x_i^{a_1}, x_j^{a_2}, x_u^{a_3}, x_v^{a_4})\) and \((x_i^{b_1}, x_j^{b_2}, x_u^{b_3}, x_v^{b_4})\). Because each component in \(X_{diff}\) at least contributes 1 chromatic distance to \(\delta\), therefore to make \(\delta = 4\), it must be \(\delta(x_i) = \delta(x_j) = \delta(x_u) = \delta(x_v) = 1\). Then the solutions are \(\{a_1 = b_1 + 1\}, \{a_2 = b_2 + 1\}, \{a_3 = b_3 + 1\}, \{a_4 = b_4 + 1\}\). Also, it is needed that \(X_{diff_1} \cong X_{diff_2}\). Assume \(a_2 = a_1 + \Delta_1, a_3 = a_1 + \Delta_1 + \Delta_2, a_4 = a_1 + \Delta_1 + \Delta_2 + \Delta_3\), then the only solution for \(X_{diff_2}\) is \((a_1 + \Delta_1, a_1 + \Delta_1 + \Delta_2 + \Delta_3, a_1 + \Delta_1 + \Delta_2)\) with \(\Delta_1 = 1, \Delta_2 = 1\), corresponding to the opposite cell in 2-I unit.

Case (4) \(m \geq 5\).

Because each components in \(X_{diff}\) at least contribute 1 chromatic distance to \(\delta\), therefore \(\delta \geq 5\). \(\square\)

Another approach to reason C-C relations is using \(C2E(\zeta)\) function to calculate all edges of two cells, and if among them two edges are equal, then the two cells must be connected. We can also use \(C2V(\zeta)\) to calculate all vertexes contained by two cells and hence determine if they are joint with a vertex, but here we will encounter the same problem that the joint vertex is hidden.

3.2. Spatial topology between complexes. Real objects or geographical entities in space usually occupy massive spatial particles. Topological relations and computations among complexes are hence much more complicated than those among particles. As the union set of particles, a complex may contain different types of particles, for example, containing two cells, two edges, and three vertexes, such complexes are called \textit{mixed complexes}; or it contains only a single type of particles, for example, containing only vertexes, edges, or cells, such complexes are called \textit{uniform complexes}. In addition, there are also two information scenarios: for a given complex, (1) we know its code as well as its all elemental particles, and (2) we only know its code but do not know its elemental particles. This section demonstrates an tentative study of spatial complex topology, particularly focusing on the most important uniform complex – cluster, as well as the scenario (1) that we know each elemental cell.

3.2.1. Spatial connectivity of a cluster. The spatial connectivity is an important issue for analyzing complexes and clusters. In a general sense, the connectivity of clusters in OACD and SCM is similar to those in graph theory, complex network, algebraic geometry, and point set topology. A disconnected cluster is usually treated as a number of connected clusters rather than a single cluster.

Let us define the connectivity of a cluster. If a cluster \(\xi\) contains two cells \(\zeta_1\) and \(\zeta_2\) and they are connected as in Proposition 9, namely, \(\delta(\zeta_1, \zeta_2) = 2\), then there is a \textit{path} linking them, denoted by \(\rho(\zeta_1, \zeta_2)\). If a cluster contains three cells \(\zeta_1, \zeta_2,\) and \(\zeta_3,\) and there is a path \(\rho(\zeta_1, \zeta_2),\) and another path \(\rho(\zeta_2, \zeta_3),\) then we define a path
\( \rho(\zeta_1, \zeta_3) \) between \( \zeta_1 \) and \( \zeta_3 \), and call them path-connected by a path-cell \( \zeta_2 \), that is, \( \rho(\zeta_1, \zeta_3) = (\zeta_2) \). Similarly, any two cells are path-connected if they are linked by a series of path-cells.

**Definition 6.** Given a cluster \( \xi\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \), it is connected if it meets two conditions: (1) any two cells are path-connected, and (2) all path-cells are the elements of the cluster.

**Notation 5.** The function \( \text{Conn}(\xi) \) returns the connectivity of \( \xi \). It can be carried out by steps (1) select any one cell from \( \xi \) as the seed of the connected set \( C_c \) and the other cells remain as the waiting-list set \( C_w \), (2) search cells in \( C_w \) to find out the cell \( \zeta_w \), which is connected to any cell \( \zeta_c \) in \( C_c \), that is, \( \delta(\zeta_c, \zeta_w) = 2 \). (3) If found, then move \( \zeta_w \) from \( C_w \) to \( C_c \), and repeat step (2) until \( C_w \) becomes empty, and then return \( \text{Conn}(\xi) = 1 \), meaning \( \xi \) is connected; If not found, return \( \text{Conn}(\xi) = 0 \), meaning \( \xi \) is disconnected.

### 3.2.2. Types and reasoning of cluster-cluster topological relations.

Given two clusters \( \xi_1 \) and \( \xi_2 \), their cluster-cluster (Cs-Cs) topological relations are demonstrated in Fig.7, such as equal, contain, touch, and overlap. Because clusters are union set of cells and if their elemental cells are known, say, \( \xi_1 = \{\zeta_{11}, \zeta_{12}, \ldots, \zeta_{1n}\} \) and \( \xi_2 = \{\zeta_{21}, \zeta_{22}, \ldots, \zeta_{2m}\} \), then some of Cs-Cs relations are easy to determine by using below set operations.

\[
\begin{align*}
\xi_1 \text{ equals } \xi_2 & \iff \xi_1 \cap \xi_2 = \xi_1 = \xi_2 & (3.24) \\
\xi_1 \text{ contains } \xi_2 & \iff \xi_1 \supset \xi_2 \\
\xi_1 \text{ disjoints } \xi_2 & \iff \xi_1 \cap \xi_2 = \emptyset \\
\xi_1 \text{ overlaps } \xi_2 & \iff \xi_1 \cap \xi_2 \neq \emptyset \neq \xi_1 \neq \xi_2
\end{align*}
\]

Another usual topological relation between two clusters is adjacency (Fig.7d), which can be determined by

\[
\xi_1 \text{ touch } \xi_2 \iff \xi_1 \cap \xi_2 = \emptyset \land \text{Conn}(\xi_1 \cup \xi_2) = 1 & (3.25)
\]

Or we can use \( C2E \) and \( C2V \) function to compare their edges and vertexes, that is,

\[
\begin{align*}
\xi_1 \text{ touch } \xi_2 & \iff \xi_1 \cap \xi_2 = \emptyset \land C2E(\xi_1) \cap C2E(\xi_2) \neq \emptyset & (3.26) \\
\xi_1 \text{ joint } \xi_2 & \iff C2E(\xi_1) \cap C2E(\xi_2) = \emptyset \land C2V(\xi_1) \cap C2V(\xi_2) \neq \emptyset \\
\end{align*}
\]

where \( C2E(\xi) = \bigcup_{\zeta \in \xi} C2E(\zeta) \) and \( C2V(\xi) = \bigcup_{\zeta \in \xi} C2V(\zeta) \).

A more comprehensive but perhaps more complicated method to explore Cs-Cs relations is examining all C-C relations among their elemental cells. Given two complexes \( \Theta_1(\Omega_{11}, \Omega_{12}, \ldots, \Omega_{1n}) \) and \( \Theta_2(\Omega_{21}, \Omega_{22}, \ldots, \Omega_{2m}) \), their chromatic-distance matrix is defined by \( dM(\Theta_1, \Theta_2) = [\delta_{ij}]_{n \times m} \), where \( \delta_{ij} = \delta(\Omega_{1i}, \Omega_{2j}) \), that is,

\[
dM(\Theta_1, \Theta_2) = \begin{bmatrix}
\delta(\Omega_{11}, \Omega_{21}) & \delta(\Omega_{11}, \Omega_{22}) & \cdots & \delta(\Omega_{11}, \Omega_{2m}) \\
\delta(\Omega_{12}, \Omega_{21}) & \delta(\Omega_{12}, \Omega_{22}) & \cdots & \delta(\Omega_{12}, \Omega_{2m}) \\
\vdots & \vdots & \ddots & \vdots \\
\delta(\Omega_{1n}, \Omega_{21}) & \delta(\Omega_{1n}, \Omega_{22}) & \cdots & \delta(\Omega_{1n}, \Omega_{2m})
\end{bmatrix}
\]

Given a complex \( \Theta \), \( dM(\Theta, \Theta) \) is also called the internal matrix of \( \Theta \), and denoted by \( iM(\Theta) \), which can be used to determine the connectivity of a cluster.
Replacing all $\delta = 2$ in $iM(\xi)$ by 1 and all others by 0, then we get an adjacency matrix $aM(\xi)$, the same one used in graph theory. The $aM(\xi)$ can be transferred to a reachability matrix $rM(\xi) = aM(\xi) + aM(\xi)^2 + \cdots + aM(\xi)^n$, or by such as Floyd-Warshall, Thorup, or Kameda’s algorithms [9], [10], [11]. If $rM(\xi) = 1$, then $\xi$ is an connected cluster.

By using Proposition 9 and 10, we can determine Cs-Cs relations by whether some particular chromatic distances are found in $dM$. For example, if we found 0 or 2 in $dM$, then it means a cell in one cluster is equal or connected to a cell in the other cluster.

**Notation 6.** Function $cdn(dM,k)$ returns the number of $\delta(\Omega_1, \Omega_2) = k$ in a chromatic-distance matrix $dM(\Theta_1, \Theta_2)$. $k$ also can be some conditions such as $> 0$, or $\neq 2$.

We then can determine Cs-Cs relations by using $dM$, $cdn$ function, and the below rules.

\[
\begin{align*}
\xi_1 & \text{ equals } \xi_2 \Leftrightarrow |\xi_2| = \frac{1}{2}cdn(dM, 0) = |\xi_1| \quad (3.28) \\
\xi_1 & \text{ contains } \xi_2 \Leftrightarrow |\xi_2| = \frac{1}{2}cdn(dM, 0) < |\xi_1| \\
\xi_1 & \text{ overlaps } \xi_2 \Leftrightarrow 1 \leq \frac{1}{2}cdn(dM, 0) < \min(|\xi_1|, |\xi_2|) \\
\xi_1 & \text{ joint } \xi_2 \Leftrightarrow cdn(dM, \leq 2) = 0 \land cdn(dM, 4) > 0 \\
\xi_1 & \text{ touch } \xi_2 \Leftrightarrow cdn(dM, 0) = 0 \land cdn(dM, 2) > 0 \\
\xi_1 & \text{ disjoints } \xi_2 \Leftrightarrow cdn(dM, \leq 4) = 0
\end{align*}
\]

where $dM = dM(\xi_1, \xi_2)$, and $|\xi|$ is the cardinal number of $\xi$, that is, if $\xi$ is a $m$-cell cluster, then $|\xi| = m$. 

**Figure 7.** Six types of complex topological relations in full-OACD.
The third methods to determine Cs-Cs relations is directly using their chromatic codes and distance $\delta(\xi_1, \xi_2)$, for example, using the codes in Fig.7. Through tentative studies we found the general rule that ‘the closer the chromatic distance, the closer the spatial topology’ [8]. However, the full investigation and more rigorous mathematics remain for future work.

4. Discussions and summary

As one type of spatial chromatic tessellations, full-OACDs provide a scheme to partition and encode space. Technically a full-OACD is an irregular discrete spatial data model based necessarily on the given object sets. If we have enough perception, we should see that the similar schemes provide a new approach to study discrete geometry. In SCT, as the generator number increases, the cell number and neighborhood number of each cell will become larger and larger, and the size of each cell will become smaller and smaller. This property is different from other discrete tessellation models, such as raster model and Voronoi diagrams. For example, the neighborhood number of a pixel in a raster model is always 4 or 8, even though its spatial resolution may be very high. When the generator number turns to be the infinite, the space represented by SCT turns to be a continuous space, and its topology turns to be the classic point-set topology.

Chromatic codes are the keys for characterization, computation and analysis of spatial particles and their complexes. Spatial coding is a new topic in GIS. As a scheme of spatial coding, full-OACD is still in its immature stage, where many problems and directions remain unanswered and unexplored. Below we discuss some issues that might be worth further investigation.

**Vertex types.** There are two types of vertexes in full-OACD: 2-I and 3-I vertexes, with quite different code bases. The 2-I’s codes contain half-integers but the 3-I’s do not. 3-I vertexes actually are those degenerated cells, also called singular cells in pervious studies [8]. Instead of using perpendicular bisectors, if we use weighted perpendicular bisectors, then 3-I vertexes will change their faces to real cells, accompanying some new edges and 2-I vertexes, see an example in Fig.8a. In such diagrams, particle bases are quite different from those in full-OACDs.

In addition, there are no more other types of vertexes in full-OACDs, such as 4-I or 5-I, since we have excluded them by assuming the generators being in general cases. For non-general cases, for example, if 4 points are concyclic, then their six perpendicular bisectors will intersect at the center of a circle, and hence generate a 6-I vertex with code such as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, see Fig.8b. Although we can exclude 4 concyclic points from the plane, in 3d space, 4 points are generally always on a sphere, expect they are all in a plane, implying that many 6-I vertexes will be found in $OACD(n, \mathbb{R}^3)$. This is similar to that 3 points are generally always concyclic in a plane, except they are all in a line.

**Hidden spatial particles.** In terms of the Propositions [9] and [10] as well as the results of functions $C2E$ and $E2V$, many spatial particles should exist in full-OACDs but in one real space they are hidden. Checking the codes of the hidden particles, we could not find any structural and permutation differences from those emerged particles. However, the hidden particles will indeed emerge in full-OACDs if we change point patterns of the generator set, but if we did so, some previous emerged particles will be hidden again. Therefore a big challenge of full-OACD is to determine what cells are hidden by given a kind of generator pattern.
Hidden particles and complex codes can be also applied to analyze generator patterns. For example, the complex of all 3-I vertexes in Fig.3b has a code \((t_{21}, t_{26}, t_{34}, t_{43}, t_{46}, t_{64})\), indicating this complex is much closer to the generator \(t_{3} = 43\) than to \(t_{4} = 6\). Note that in a SCM or full-OACD\((n, \mathbb{R}^{n-1})\), all particles are emerged, but when it is mapped into the lower and lower dimensional spaces, more and more particles will be hidden.

**From coded space to real space.** If we compare the belonging relationships of different models proposed in SCM, the result should be that \(\text{OACD} \subseteq \text{full-OACD} \subseteq \text{SCT} \subseteq \text{SCM}\). Full-OACD is generated from half-space partitions, so it is still a type of spatial chromatic tessellations (SCT), namely, a mapping from SCM spaces, which are usually in higher dimensions, to real spaces, which are usually in lower dimensions. From the perspective of SCM, there is a question that how to understand the half-integers in edge codes. If we are allowed to use half-integers to make space, then how about if we use other numbers such as \(\frac{1}{4}\)-integers. It seems that edges and vertexes are not real spaces, just as we often say that lines are only with lengths but no areas. When we use a pen to draw a line to partition a piece of paper, it appears that we get three parts: half at right, half at left, and the middle line that cannot be assigned to any half. When we use a scissors to cut a piece of paper, however, we can only get two pieces of paper but never three parts. Therefore, we would like to emphasize that in full-OACD and SCM, the cell is the elementary subspace, whereas edges, vertexes and other lower dimensional subspaces are only boundaries. Therefore, in order to use coded spaces to represent the real-world spaces, we suggest only use cells. It is like in raster model, any spatial entities and objects are always represented by pixels, no matter it is a point, line, or area.

**References**

[1] Longley, P. A., Goodchild, M. F., Maguire, D. J., Rhind, D. W., 2001, Geographic information systems and science, John Wiley & Sons.
[2] Frank Hardisty, F., Robinson, A. C., 2010, The geoviz toolkit: using component-oriented coordination methods for geographic visualization and analysis, International Journal of Geographical Information Science, 25(2), 191-210.

[3] Schubert, E., Zimek, A., Kriegel, H. P., 2013, Geodetic distance queries on R-trees for indexing geographic data, Advances in Spatial and Temporal Databases, Lecture Notes in Computer Science, 8098, 146-164.

[4] Halaoui, H. F., 2008, A spatio-temporal indexing structure for efficient retrieval and manipulation of discretely changing spatial data, Journal of Spatial Science, 53(2), 1-12.

[5] Curry, M., 1998, Digital Places: Living with Geographic Information Technologies, Routledge.

[6] Davis, C. A., Fonseca, F. T., 2007, Assessing the certainty of locations produced by an address geocoding system, Geoinformatica, 11(1), 103-129.

[7] Zhu, W. N., 2015, Spatial chromatic model in high-dimensional spaces and the uniqueness of chromatic code: a new perspective of geographic entity-space relationship, International Journal of Geographical Information Science, 29(1), 28-45.

[8] Zhu, W. N. and Yu, Q., 2010, Spatial chromatic tessellation: conception, interpretation, and implication, Annals of GIS, 16, 237-254.

[9] Cormen, T. H., Leiserson, C. E., Rivest, R. L., Stein, C., 2001, Transitive closure of a directed graph, Introduction to Algorithms (2nd ed.), MIT Press and McGraw-Hill, 632-634.

[10] Thorup, M., 2004, Compact oracles for reachability and approximate distances in planar digraphs, Journal of the ACM, 51(6), 993–1024.

[11] Kameda, T., 1975, On the vector representation of the reachability in planar directed graphs, Information Processing Letters, 3(3), 75–77.

(Zhu, W. N.) Laboratory of Geospatial Information, Zhejiang University

E-mail address: zhuwn@zju.edu.cn