LP-based Approximations for Disjoint Bilinear and Two-Stage Adjustable Robust Optimization

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Abstract

We consider the class of disjoint bilinear programs $\max \{x^T y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$ where $\mathcal{X}$ and $\mathcal{Y}$ are packing polytopes. We present an $O\left(\frac{\log \log m_1 \log \log m_2}{\log m_1 \log m_2}\right)$-approximation algorithm for this problem where $m_1$ and $m_2$ are the number of packing constraints in $\mathcal{X}$ and $\mathcal{Y}$ respectively. In particular, we show that there exists a near-optimal solution $(\hat{x}, \hat{y})$ such that $\hat{x}$ and $\hat{y}$ are “near-integral”. We give an LP relaxation of the problem from which we obtain the near-optimal near-integral solution via randomized rounding. We show that our relaxation is tightly related to the widely used reformulation linearization technique (RLT). As an application of our techniques, we present a tight approximation for the two-stage adjustable robust optimization problem with covering constraints and right-hand side uncertainty where the separation problem is a bilinear optimization problem. In particular, based on the ideas above, we give an LP restriction of the two-stage problem that is an $O\left(\frac{\log n \log L}{\log \log n \log \log L}\right)$-approximation where $L$ is the number of constraints in the uncertainty set. This significantly improves over state-of-the-art approximation bounds known for this problem. Furthermore, we show that our LP restriction gives a feasible affine policy for the two-stage robust problem with the same (or better) objective value. As a consequence, affine policies give an $O\left(\frac{\log n \log L}{\log \log n \log \log L}\right)$-approximation of the two-stage problem, significantly generalizing the previously known bounds on their performance.

Keywords: Disjoint bilinear programming; Two-stage robust optimization; Approximation Algorithms; RLT; Affine policies.
1 Introduction

We consider the following class of disjoint bilinear programs,

$$z_{\text{PDB}} = \max_{x,y} \{ x^T y \mid x \in \mathcal{X}, \; y \in \mathcal{Y} \},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are packing polytopes given by an intersection of knapsack constraints. Specifically,

$$\mathcal{X} := \{ x \geq 0 \mid Px \leq p \}$$

and

$$\mathcal{Y} := \{ y \geq 0 \mid Qy \leq q \},$$

where $P \in \mathbb{R}^{m_1 \times n}$, $Q \in \mathbb{R}^{m_2 \times n}$, $p \in \mathbb{R}^{m_1}$, and $q \in \mathbb{R}^{m_2}$. We refer to this problem as a packing disjoint bilinear program $\text{PDB}$. This is a subclass of the well-studied disjoint bilinear problem:

$$\max_{x,y} \{ x^T My \mid x \in \mathcal{X}, \; y \in \mathcal{Y} \},$$

where $M$ is a general $n \times n$ matrix.

Disjoint bilinear programming is NP-hard in general (Chen et al. [14]). We show that it is NP-hard to even approximate within any finite factor. Several heuristics have been studied for this problem including cutting-planes algorithms (Konno et al. [33]), polytope generation methods (Vaish et al. [47]), Benders decomposition (Geoffrion [25]), reduction to concave minimization (Thieu [46]), reformulation linearization techniques (Sherali and Alameddin [41], Adams and Sherali [1], Audet et al. [4], Tawarmalani et al. [45]), mixed integer programming (Gupte et al. [29], Freire et al. [24]) and two-stage robust optimization (Zhen et al. [50]). However, to the best of our knowledge, no approximation algorithms with provable guarantees are known for this problem.

Many important applications can be formulated as a disjoint bilinear program including fixed charge network flows (Rebennack et al. [36]), concave cost facility location (Soland [44]), bilinear assignment problems (Ćučić et al. [16]), non-convex cutting-stock problems (Harjunkoski et al. [32]), multicommodity flow network interdiction problems (Lim and Smith [34]), bimatrix games (Mangasarian and Stone [35], Firouzbakht et al. [23]) pooling problems (Gupte et al. [30]).

One important application closely related to disjoint bilinear optimization that we focus on in this paper, is two-stage adjustable robust optimization. In particular, the separation problem of a two-stage adjustable robust problem can be formulated as a disjoint bilinear optimization problem. More specifically, we consider the following two-stage adjustable robust problem,

$$z_{\text{AR}} = \min_{x,t} \left( c^T x + t \right)$$

$$t \geq Q(x),$$

$$x \in \mathcal{X},$$

(AR)
where for all $x \in \mathcal{X}$,
\[
Q(x) = \max_{h \in \mathcal{U}} \min_{y \geq 0} \{d^T y \mid Ax + By \geq h\}.
\]

Here $A \in \mathbb{R}^{m \times n'}$, $B \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^{n'}$, $d \in \mathbb{R}^{n'}$, $\mathcal{X} \subset \mathbb{R}^{n'}$ is a polyhedral cone, and $\mathcal{U}$ is a polyhedral uncertainty set. The separation problem of $AR$ is the following: given a candidate solution $(x, t)$, decide if it is feasible, i.e., $x \in \mathcal{X}$ and $t \geq Q(x)$ or give a separating hyperplane. This is equivalent to solving $Q(x)$. We will henceforth refer to $Q(x)$ as the separation problem. For ease of notation, we use $Q(x)$ to refer to both the problem and its optimal value. In this two-stage problem, the adversary observes the first-stage decision $x$ and reveals the worst-case scenario of $h \in \mathcal{U}$. Then, the decision maker selects a second-stage recourse decision $y$ such that $By$ covers $h - Ax$. The goal is to select a first-stage decision such that the total cost in the worst-case is minimized. This model has been widely considered in the literature (Dhamdhere et al. [17], Feige et al. [22], Gupta et al. [28], Bertsimas and Goyal [8], Bertsimas and Bidkhori [9], Bertsimas and de Ruiter [12], Xu et al. [48], Zhen et al. [49], El Housni and Goyal [18], El Housni et al. [21, 20]), and has many applications including set cover, capacity planning and network design problems under uncertain demand.

Several uncertainty sets have been considered in the literature including polyhedral uncertainty sets, ellipsoids and norm balls (see Bertsimas et al. [10]). Some of the most important uncertainty sets are budget of uncertainty sets (Bertsimas and Sim [13], Gupta et al. [27], El Housni and Goyal [19]) and intersections of budget of uncertainty sets such as CLT sets (see Bandi and Bertsimas [5]) and inclusion-constrained budgeted sets (see Gounaris et al. [26]). These have been widely used in practice. Following this motivation, we consider in this paper the following uncertainty set
\[
\mathcal{U} := \{h \geq 0 \mid Rh \leq r\},
\]
where $R \in \mathbb{R}^{L \times m}$ and $r \in \mathbb{R}^L$. This is a generalization of the previously mentioned sets. We refer to this as a packing uncertainty set.

Feige et al. [22] show that $AR$ is NP-hard to approximate within any factor better than $\Omega(\log n / \log \log n)$ even in the special case of a single budget of uncertainty set. Bertsimas and Goyal [8] give an $O(\sqrt{m})$-approximation in the case where the first-stage matrix $A$ is non-negative. Recently, El Housni and Goyal [19] give an $O(\log n / \log \log n)$-approximation in the case of a single budget of uncertainty set and an $O(\log^2 n / \log \log n)$-approximation in the case of an intersection of disjoint budgeted sets. In general, they show an $O(\log n / \log \log n)$-approximation in the case of a packing uncertainty set with $L$ constraints. However, this bound scales linearly with $L$. The two-stage robust covering problem was also considered in the discrete case where the variables of the problem are restricted to be in $\{0, 1\}^m$. For this problem, Feige et al. [22] and Gupta et al. [28] give an $O(\log n \log m)$-approximation and
an $O(\log n + \log m)$-approximation respectively in the case where $A = B \in \{0, 1\}^{m \times n}$ and the uncertainty set $U$ is given by a cardinality uncertainty set of the form $U = \{h \in \{0, 1\}^m \mid \sum_{i=1}^m h_i \leq k\}$. Gupta et al. [27] consider a more general uncertainty set, namely, intersection of $p$-system and $q$-knapsack and give an $O(pq \log n)$-approximation of the two-stage problem.

The goal of this paper is to provide LP-based approximation algorithms with provable guarantees for the packing disjoint bilinear program as well as the two-stage adjustable robust problem that improve over the approximation bounds known for these problems.

1.1 Our Contributions

1.1.1 A Polylogarithmic Approximation Algorithm for PDB.

**Algorithm.** We present an LP based randomized approximation algorithm for PDB. Our algorithm relies on a new idea that might be of independent interest. In particular, we show the existence of a near-optimal near-integral solution of this problem. That is, a near-optimal solution $(\tilde{x}, \tilde{y})$ such that $\tilde{x}_i \in \{0, \max \frac{x_i}{\zeta_1} \}$ and $\tilde{y}_i \in \{0, \max \frac{y_i}{\zeta_2} \}$ for some small factors $\zeta_1$ and $\zeta_2$. We give an LP relaxation of PDB, i.e., a linear program whose optimal cost is greater than the optimum of PDB, from which we obtain such $(\tilde{x}, \tilde{y})$ via randomized rounding. More specifically, we show the following theorem,

**Theorem 1.** There exists an LP rounding based randomized algorithm that gives an $O\left(\frac{\log \log m_1 \log m_2}{\log m_1 \log m_2}\right)$-approximation to PDB.

**Relation to the Reformulation Linearization Technique (RLT).** We show that our LP relaxation of PDB is closely related to the reformulation linearization technique (RLT). RLT provides an efficient approximation for non-convex continuous and mixed-integer optimization problems. It was first introduced by Sherali and Adams [1, 2, 3] in the context of binary bilinear problems and has been since then applied to different other problems including general bilinear problems (Sherali and Alameddine [41]), mixed-integer linear problems (Sherali and Adams [39, 40]) and polynomial problems (Sherali and Tuncbilek [43]). We show the existence of a reformulation linearization of PDB that is equivalent to our LP relaxation. This provides a new perspective on our LP relaxation and implies a polylogarithmic approximation guarantee on the performance of tighter relaxations of PDB such as the well studied relaxations of the RLT hierarchy (Sherali and Adams [38, 39]).

**Numerical Experiments.** Our randomized rounding based algorithm gives an approximate solution of PDB that is guaranteed to be within $O\left(\frac{\log m_1 \log m_2}{\log \log m_1 \log \log m_2}\right)$ of the optimum with high probability. We study the empirical performance of our solution by comparing the performance of our algorithm with several benchmarks on randomly generated instances. More specifically, we compare our algorithm with the first level relaxation of the RLT hierarchy, which is a widely
used LP approximation for bilinear programs and that has been observed to be a good empirical approximation (Sherali et al. [42], Sherali and Adams [1, 3], Sherali and Alameddine [41]). This relaxation gives an approximate solution of PDB that we compare with our solution in terms of objective value and running time needed to compute each solution. We also compare our algorithm to the bilinear solver of Gurobi v9.1.2. In particular, we compare the objective value of our solution to the optimal objective computed using the bilinear solver of Gurobi and compare the running time of our algorithm with the running time needed by the bilinear solver of Gurobi to compute a solution that is at least as good as our solution. We show that our solution is significantly faster to compute compared to these benchmarks and gives a good approximation of PDB.

1.1.2 A Polylogarithmic Approximation for the Two-Stage Problem AR.

Algorithm. We present an LP-based approximation for AR. The separation problem for AR is a variant of PDB. However, the objective is a difference of a bilinear and a linear term making it challenging to approximate. Our approach approximates AR directly. In particular, using ideas from our approximation of PDB, we give a compact linear restriction of AR, that is, a linear program whose optimal objective is greater than the optimum of AR, and show that it is a polylogarithmic approximation of AR. In particular, we have the following theorem.

Theorem 2. There exists an LP restriction of AR that gives an $O\left(\frac{\log n}{\log \log n \log \log L} \right)$ approximation to AR.

Our bound improves significantly over the prior approximation bound of $O\left(\frac{L \log n}{\log \log n}\right)$ [19] known for this problem. It also shows a striking contrast between the fractional two-stage robust covering problem and its discrete counter part. In fact, the discrete two-stage robust covering problem under $L$-knapsack uncertainty set considered in [27] is hard to approximate within any factor better than $L^{\frac{1}{2}} - \epsilon$, for any $\epsilon > 0$. This follows from the hardness of the maximum independent set problem.

Relation to Affine Policies. We show that, surprisingly, our LP restriction is tightly related to affine policies. Affine policies are a widely used approximation technique in dynamic robust optimization. They consist of restricting the second-stage variables $y$ to be an affine function of the uncertain right-hand side $h$ (see for example Ben-Tal et al. [6]). It is known that the optimal affine policy can be computed in polynomial time (Ben-Tal et al. [6]). Several approximation bounds are known for affine policies. Bertsimas and Goyal [8] show that affine policies achieve a bound of $O(\sqrt{m})$ under a general polyhedral uncertainty set. They also show that such policies are optimal in the case of a simplex uncertainty set. Recently, El Housni and Goyal [19] show that affine policies achieve the bound of $O(\frac{\log n}{\log \log n})$ in the case of a single budget of uncertainty set. They also show a bound of $O(L \frac{\log n}{\log \log n})$ in the case of a general intersection of $L$ budget of uncertainty sets. In this paper, we prove stronger bounds for affine policies under the general
packing uncertainty set with \( L \) constraints. In particular, we show that our LP restriction of \( AR \) gives a feasible affine policy for the two-stage problem with the same (or better) objective value. This implies the following approximation bound for affine policies.

**Theorem 3.** Affine policies give an \( O\left(\frac{\log n}{\log \log n} \cdot \frac{\log L}{\log \log L}\right) \)-approximation to \( AR \).

Our analysis is constructive and provides a faster algorithm to compute near-optimal affine policies with approximation ratio \( O\left(\frac{\log n}{\log \log n} \cdot \frac{\log L}{\log \log L}\right) \).

**Numerical Experiments.** We compare the performance of our LP restriction of \( AR \) with several benchmarks on randomly generated instances. First, we compare our restriction with the optimal affine policy, which is a widely used approximation for the two-stage problem. Second, we compare our restriction with a generalization of the algorithm of El Housni and Goyal [19] to packing uncertainty sets. This algorithm was shown to have a good empirical performance for the case of a single budget of uncertainty set. We also compare our LP restriction with the lower-bound of Hadjiyiannis et al. [31] who show the bound provides a good empirical approximation of the optimum of \( AR \). We show that our restriction is significantly faster to compute compared to these benchmarks and gives a good approximation of \( AR \).

## 2 A Polylogarithmic Approximation for PDB

In this section, we present an \( O\left(\frac{\log \log m_1}{\log m_1} \cdot \frac{\log \log m_2}{\log m_2}\right) \)-approximation for PDB (Theorem 1). To prove this theorem, we show an interesting structural property of PDB. In particular, we show that there exists a near-optimal solution of PDB that is “near-integral”. Let us define for all \( i \in [n] \),

\[
\theta_i = \max_{x \in X} x_i, \quad \gamma_i = \max_{y \in Y} y_i, \quad \zeta_1 = \frac{3 \log m_1 \log \log m_1}{\log m_1} + 2 \quad \text{and} \quad \zeta_2 = \frac{3 \log m_2 \log \log m_2}{\log m_2} + 2.
\]

We formally state our structural property in the following lemma.

**Lemma 1.** (Structural Property). There exists a feasible solution \((\bar{x}, \bar{y})\) of PDB whose objective value is within \( O\left(\frac{\log \log m_1}{\log m_1} \cdot \frac{\log \log m_2}{\log m_2}\right) \) of the optimum and such that \( \bar{x}_i \in \{0, \frac{\theta_i}{\zeta_1}\} \) and \( \bar{y}_i \in \{0, \frac{\gamma_i}{\zeta_2}\} \) for all \( i \in [n] \), where \( 1 \leq \zeta_1 \leq \zeta_1^* \) and \( 1 \leq \zeta_2 \leq \zeta_2^* \).

We obtain such a solution satisfying the above property using an LP relaxation of PDB via a randomized rounding approach.

**LP Relaxation and Rounding.** We consider the following linear program,
\[ z_{\text{LP-PDB}} = \max_{\omega \geq 0} \left\{ \begin{array}{c} \sum_{i=1}^{n} \theta_i \gamma_i \omega_i \\ \sum_{i=1}^{n} \gamma_i Q_i \omega_i \leq q \\ \sum_{i=1}^{n} \theta_i P_i \omega_i \leq p \end{array} \right\}, \tag{LP-PDB} \]

where \( P_i \) is the \( i \)-th column of \( P \) and \( Q_i \) is the \( i \)-th column of \( Q \).

We first show that \( \text{LP-PDB} \) is a relaxation of \( \text{PDB} \).

**Lemma 2.** \( z_{\text{PDB}} \leq z_{\text{LP-PDB}} \).

**Proof.** Let \((x^*, y^*)\) be an optimal solution of \( \text{PDB} \). Let \( \omega_i^* = \frac{x_i^* y_i^*}{\theta_i \gamma_i} \) for all \( i \in [n] \).

By definition, we have \( x_i^* \leq \theta_i \) and \( y_i^* \leq \gamma_i \) for all \( i \in [n] \). Hence,

\[
\sum_{i=1}^{n} \theta_i P_i \omega_i^* = \sum_{i=1}^{n} \theta_i P_i \frac{x_i^* y_i^*}{\theta_i \gamma_i} \leq \sum_{i=1}^{n} P_i x_i^* \leq p,
\]

and

\[
\sum_{i=1}^{n} \gamma_i Q_i \omega_i^* = \sum_{i=1}^{n} \gamma_i Q_i \frac{x_i^* y_i^*}{\theta_i \gamma_i} \leq \sum_{i=1}^{n} Q_i y_i^* \leq q.
\]

Note that we use the fact that \( P \) and \( Q \) are non-negative in the above inequalities. Therefore, \( \omega^* \) is feasible for \( \text{LP-PDB} \) with objective value

\[
\sum_{i=1}^{n} \theta_i \gamma_i \omega_i^* = \sum_{i=1}^{n} x_i^* y_i^* = z_{\text{PDB}},
\]

which concludes the proof. \( \square \)

Now, to construct our near-optimal near-integral solution, we consider the randomized rounding approach described in Algorithm 1. Note that by definition of \( \theta_i \),

\[
\max_{\omega} \{ \omega_i \mid \sum_{j=1}^{n} \theta_j P_j \omega_j \leq p, \omega \geq 0 \} = 1,
\]

for all \( i \in [n] \). Hence, for all \( i \in [n] \), \( \omega_i^* \) defined in Algorithm 1 is such that \( \omega_i^* \leq 1 \).

In our proof of Lemma 1, we use the following variant of Chernoff bounds.

**Lemma 3.** (Chernoff Bounds [15]).

(a) Let \( \chi_1, \ldots, \chi_r \) be independent Bernoulli trials. Denote \( \Xi := \sum_{i=1}^{r} \epsilon_i \chi_i \) where \( \epsilon_1, \ldots, \epsilon_r \) are reals
Algorithm 1

Input: $\epsilon > 0$.
Output: a feasible solution verifying the structural property in Lemma 1 with probability at least $1 - \epsilon - o(1)$.

1: Let $\omega^*$ be an optimal solution of LP-PDB and let $T = 8[\log \frac{1}{\epsilon}]$.

2: Initialize $\max = 0$, $\hat{x} = 0$ and $\hat{y} = 0$.

3: for $t = 1, \ldots, T$ do

4: let $\tilde{\omega}^*_1, \ldots, \tilde{\omega}^*_n$ be i.i.d. Bernoulli variables with $P(\tilde{\omega}^*_i = 1) = \omega^*_i$ for $i \in [n]$.

5: let $\zeta^{\text{min}}_1 = \min\{\zeta \geq 1 \mid (\theta_1 \tilde{\omega}_1, \ldots, \theta_n \tilde{\omega}_n)/\zeta \in \mathcal{X}\}$ and $\tilde{x} = (\theta_1 \tilde{\omega}_1, \ldots, \theta_n \tilde{\omega}_n)/\zeta^{\text{min}}_1$.

6: let $\zeta^{\text{min}}_2 = \min\{\zeta \geq 1 \mid (\gamma_1 \tilde{\omega}_1, \ldots, \gamma_n \tilde{\omega}_n)/\zeta \in \mathcal{Y}\}$ and $\tilde{y} = (\gamma_1 \tilde{\omega}_1, \ldots, \gamma_n \tilde{\omega}_n)/\zeta^{\text{min}}_2$.

7: if $\tilde{x}^T \tilde{y} \geq \max$ then

8: set $\hat{x}_i = \tilde{x}_i$ and $\hat{y}_i = \tilde{y}_i$ for $i \in [n]$.

9: set $\max = \tilde{x}^T \tilde{y}$.

10: end if

11: end for

12: return $(\hat{x}, \hat{y})$

in $[0,1]$. Let $s > 0$ such that $E(\Xi) \leq s$. Then for any $\delta > 0$ we have,

$$P(\Xi \geq (1 + \delta)s) \leq \left(\frac{e^{\delta}}{(1 + \delta)^{1 + \delta}}\right)^s.$$

(b) Let $\chi_1, \ldots, \chi_r$ be independent Bernoulli trials. Denote $\Xi := \sum_{i=1}^r \epsilon_i \chi_i$ where $\epsilon_1, \ldots, \epsilon_r$ are reals in $\{0,1\}$. Then for any $0 < \delta < 1$,

$$P(\Xi \leq (1 - \delta)E(\Xi)) \leq e^{-\frac{\delta^2 E(\Xi)}{1 - \delta}}.$$

For the sake of completeness, we present the proof for these bounds in Appendix A.

Proof of Lemma 1. We show that at any iteration of Algorithm 1, the vectors $\tilde{x}, \tilde{y}$ verify the structural property Lemma 1 with constant probability. To show this, let us fix an iteration $t$ of the algorithm. It is sufficient to show that the following inequalities hold with constant probability,

$$\sum_{i=1}^n P_i \frac{\theta_i \tilde{\omega}_i}{\zeta^{*}_1} \leq p,$$

$$\sum_{i=1}^n Q_i \frac{\gamma_i \tilde{\omega}_i}{\zeta^{*}_2} \leq q,$$

$$\sum_{i=1}^n \frac{\theta_i \tilde{\omega}_i}{\zeta^{*}_1} \frac{\gamma_i \tilde{\omega}_i}{\zeta^{*}_2} \geq \frac{z_{\text{LP-PDB}}}{2\zeta^*_1 \zeta^*_2}.$$  \hspace{1cm} (1)

In fact, if the inequalities (1) hold with constant probability, then with constant probability, the two vectors $(\theta_1 \tilde{\omega}_1, \ldots, \theta_n \tilde{\omega}_n)/\zeta^{*}_1$ and $(\gamma_1 \tilde{\omega}_1, \ldots, \gamma_n \tilde{\omega}_n)/\zeta^{*}_2$ are feasible for PDB implying that that
$1 \leq \zeta_1^{\min} \leq \zeta_1^*$ and $1 \leq \zeta_2^{\min} \leq \zeta_2^*$ in this iteration. Moreover, these vectors have an objective value at least $\frac{1}{\zeta_1^* \zeta_2^*} z_{LP-PDB}$. This implies that the feasible solution $\tilde{x}, \tilde{y}$ of PDB has an objective value at least $O \left( \frac{\log \log m_1}{\log m_1} \frac{\log \log m_2}{\log m_2} \right)$ of the optimum.

First, we have,

$$\mathbb{P} \left( \sum_{j=1}^{n} \theta_j P_{ji} \frac{\tilde{\omega}_i}{\zeta_1^*} \not\geq p_j \right) \leq \sum_{j=1}^{m_1} \mathbb{P} \left( \sum_{i=1}^{n} \frac{\theta_j P_{ji} \tilde{\omega}_i}{p_j} > \zeta_1^* \right)$$

$$= \sum_{j \in [m_1]: p_j > 0} \mathbb{P} \left( \sum_{i=1}^{n} \frac{\theta_j P_{ji} \tilde{\omega}_i}{p_j} > \zeta_1^* \right)$$

$$\leq \sum_{j \in [m_1]: p_j > 0} \left( e^{\delta - 1} \left( \frac{1}{\zeta_1^*} \right)^{\delta} \right)$$

$$\leq m_1 \left( e^{\delta - 1} \left( \frac{1}{\zeta_1^*} \right)^{\delta} \right),$$

where the first inequality follows from a union bound on $m_1$ constraints. The second equality holds because for all $j \in [m_1]$ such that $p_j = 0$, we have

$$\mathbb{P} \left( \sum_{i=1}^{n} \theta_j P_{ji} \frac{\tilde{\omega}_i}{\zeta_1^*} > p_j \right) = 0.$$

In fact, if $p_j = 0$, by feasibility of $\omega^*$ in LP-PDB, we get

$$\sum_{i=1}^{n} \theta_j P_{ji} \omega^*_i = 0,$$

hence, we almost surely have

$$\sum_{i=1}^{n} \theta_j P_{ji} \frac{\tilde{\omega}_i}{\zeta_1^*} = 0.$$

The second inequality follows from the Chernoff bounds (a) with $\delta = \zeta_1^* - 1$ and $s = 1$. In particular, $\frac{\theta_i P_{ji}}{p_j} \in [0, 1]$ by definition of $\theta_i$ for all $i \in [n]$ and $j \in [m_1]$ such that $p_j > 0$ and for all $j \in [m_1]$ such that $p_j > 0$ we have,

$$\mathbb{E} \left[ \sum_{i=1}^{n} \frac{\theta_j P_{ji} \tilde{\omega}_i}{p_j} \right] = \sum_{i=1}^{n} \frac{\theta_j P_{ji} \omega^*_i}{p_j} \leq 1,$$

which holds by feasibility of $\omega^*$. Next, note that

$$e^{\delta - 1} \left( \frac{1}{\zeta_1^*} \right)^{\delta} = e^{\delta - 1 - \zeta_1^* \log \zeta_1^*}$$

$$= e^{-\zeta_1^* \log \zeta_1^* + o(\zeta_1^* \log \zeta_1^*)}$$
and,
\[
\zeta_1^* \log \zeta_1^* = \left( \frac{3 \log m_1}{\log \log m_1} + 2 \right) \log \left( \frac{3 \log m_1}{\log \log m_1} + 2 \right) = 3 \log m_1 + o(\log m_1)
\]

Hence,
\[
\frac{e^{\zeta_1^*-1}}{(\zeta_1^*)^{\zeta_1^*}} = e^{-3 \log m_1 + o(\log m_1)} = O(e^{-2 \log m_1}) = O\left(\frac{1}{m_1^2}\right)
\]

Therefore, there exists a constant \(c > 0\) such that,
\[
\mathbb{P}\left(\sum_{i=1}^{n} \mathbf{P}_i \frac{\theta_i \tilde{\omega}_i}{\zeta_1^*} \notin \mathbf{p}\right) \leq \frac{c}{m_1}. \tag{2}
\]

By a similar argument there exists a constant \(c' > 0\), such that
\[
\mathbb{P}\left(\sum_{i=1}^{n} \mathbf{Q}_i \frac{\gamma_i \tilde{\omega}_i}{\zeta_2^*} \notin \mathbf{q}\right) \leq \frac{c'}{m_2}. \tag{3}
\]

Finally we have,
\[
\mathbb{P}\left(\sum_{i=1}^{n} \frac{\theta_i \gamma_i \tilde{\omega}_i^2}{\zeta_1^* \zeta_2^*} < \frac{1}{2 \zeta_1^* \zeta_2^*} \sum_{i=1}^{n} \theta_i \gamma_i \omega_i^* \right) = \mathbb{P}\left(\sum_{i=1}^{n} \frac{\theta_i \gamma_i}{\sum_{j=1}^{n} \theta_j \gamma_j \omega_j^*} \tilde{\omega}_i < \frac{1}{2}\right) \leq e^{-\frac{1}{8}}, \tag{4}
\]

where the inequality follows from Chernoff bounds \((b)\) with \(\delta = 1/2\). In particular, for \(i \in [n]\),
\[
\frac{\theta_i \gamma_i}{\sum_{j=1}^{n} \theta_j \gamma_j \omega_j^*} \leq 1.
\]

This is because the unit vector \(\mathbf{e}_i\) is feasible for LP-PDB for all \(i \in [n]\) which implies
\[
\theta_i \gamma_i \leq z_{\text{LP-PDB}} = \sum_{j=1}^{n} \theta_j \gamma_j \omega_j^*,
\]

and we also have,
\[
\mathbb{E}\left[\sum_{i=1}^{n} \frac{\theta_i \gamma_i}{\sum_{j=1}^{n} \theta_j \gamma_j \omega_j^*} \tilde{\omega}_i \right] = 1.
\]

Combining inequalities (2), (3) and (4) we get that properties (1) hold with the probability at least
\[
1 - \frac{c}{m_1} - \frac{c'}{m_2} - e^{-\frac{1}{8}} = 1 - e^{-\frac{1}{8}} - o(1),
\]

which is greater than a constant for \(m_1\) and \(m_2\) large enough. \(\square\)
**Proof of Theorem 1.** Let $(\hat{x}, \hat{y})$ be the output solution of Algorithm 1. Then $(\hat{x}, \hat{y})$ has an objective value at least $O\left(\log \log m_1 \log \log m_2 \log \log m_1 \log \log m_2 \right)$ of optimum of PDB if and only if $(\hat{x}, \hat{y})$ does too at some iteration of the main loop. From our proof of Lemma 1, this happens with probability at least

$$1 - (e^{-\frac{1}{8}} - o(1))^T = 1 - e^{-\frac{T}{8}} - o(1)$$

$$= 1 - e^{-\left\lceil \log \frac{1}{\epsilon} \right\rceil} - o(1)$$

$$\geq 1 - \epsilon - o(1).$$

Therefore, with probability at least $1 - \epsilon - o(1)$, Algorithm 1 outputs a feasible solution of PDB whose objective value is within $O\left(\log \log m_1 \log \log m_2 \right)$ of $z_{PDB}$. $\square$

**Hardness of the General Disjoint Bilinear Program.** Like packing linear programs, the covering linear programs are known to have logarithmic integrality gaps. Hence, a natural question to ask would be whether similar results can be proven for an equivalent covering version of PDB, i.e., a disjoint bilinear program of the form,

$$z_{cdb} = \min_{x,y} \{x^T y \mid Px \geq p, Qy \geq q, x, y \geq 0\}$$

(CDB)

where $P \in \mathbb{R}^{m_1 \times n}$, $Q \in \mathbb{R}^{m_2 \times n}$, $p \in \mathbb{R}^{m_1}$ and $q \in \mathbb{R}^{m_2}$. However, the previous analysis does not extend to the covering case. In particular, we have the following inapproximability result.

**Theorem 4.** The covering disjoint bilinear program CDB is NP-hard to approximate within any finite factor.

The proof of Theorem 4 uses a polynomial time transformation from the Monotone Not-All-Equal 3-Satisfiability (MNAE3SAT) problem and is given in Appendix B.

### 3 Relation to the Reformulation Linearization Technique (RLT).

The reformulation linearization technique (RLT) is a widely used approach to construct tight linear relaxations of non-convex continuous and mixed-integer optimization problems. It was first proposed by Sherali and Adams [1, 2, 3] in the context of binary bilinear programming and has been since then applied to different other problems including general bilinear problems (Sherali and Alameddine [41]), mixed-integer linear problems (Sherali and Adams [39, 40]) and polynomial problems (Sherali and Tuncbilek [43]). RLT constructs an LP relaxation in two phases: a reformulation phase where some valid polynomial constraints are added to the problem. Then a linearization phase where the monomial terms of the resulting problem are linearized. The resulting relaxation can then be combined with a branch-and-bound framework to solve the problem to optimality (see Sherali and Alameddine [41] for example). We show that our LP relaxation LP-PDB is tightly...
related to RLT. In particular, we show the existence of a reformulation linearization of PDB that is equivalent to LP-PDB.

Let us begin by writing PDB in the following equivalent epigraph form,

$$
\begin{align*}
    z_{\text{PDB}} &= \max_{x,y,u} \sum_{i=1}^{n} \theta_i \gamma_i u_{ii} \\
    &\quad u_{ij} \leq x_i y_j, \quad \forall i, j \\
    &\quad \sum_{i=1}^{n} \theta_i P_i x_i \leq p, \quad \sum_{i=1}^{n} \gamma_i Q_i y_i \leq q, \\
    &\quad 0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i.
\end{align*}
$$

**Reformulation Phase.** The reformulation phase consists of adding to the above program a set of valid polynomial constraints that one gets from multiplying the linear constraints by terms of the form $\prod_{i \in S} x_i \prod_{i \in S'} y_i \prod_{i \in T} (1 - x_i) \prod_{i \in T'} (1 - y_i)$ where $S, S', T, T' \subset [n]$. We add the polynomial constraints we get from multiplying the linear constraints involving $x$ by $y_j$ for all $j \in [n]$ and the constraints involving $y$ by $x_j$ for all $j \in [n]$. We get the following equivalent formulation of PDB,

$$
\begin{align*}
    z_{\text{PDB}} &= \max_{x,y,u} \sum_{i=1}^{n} \theta_i \gamma_i u_{ii} \\
    &\quad u_{ij} \leq x_i y_j, \quad \forall i, j \\
    &\quad \sum_{i=1}^{n} \theta_i P_i x_i \leq p, \quad \sum_{i=1}^{n} \theta_i P_i y_j \leq p y_j, \quad \forall j \\
    &\quad \sum_{i=1}^{n} \gamma_i Q_i y_i \leq q, \quad \sum_{i=1}^{n} \gamma_i Q_i x_j \leq q x_j, \quad \forall j \\
    &\quad 0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i, \\
    &\quad 0 \leq x_i y_j \leq y_j, \quad 0 \leq y_i x_j \leq x_j, \quad \forall i, j.
\end{align*}
$$

**Linearization Phase.** In the linearization phase, the bilinear terms $x_i y_j$ in the above LP are replaced with their lower-bounds $u_{ij}$. We get the following linear relaxation of PDB,
We show that the relaxation RLT-PDB above is equivalent to LP-PDB. First, take an optimal solution \(x^*, y^*, u^*\) of RLT-PDB, we have

\[
\sum_{i=1}^{n} \theta_i P_i x_i \leq p, \quad \sum_{i=1}^{n} \theta_i P_i u_{i,j} \leq py_j, \quad \forall j \\
\sum_{i=1}^{n} \gamma_i Q_i y_i \leq q, \quad \sum_{i=1}^{n} \gamma_i Q_i u_{i,j} \leq qx_j, \quad \forall j \\
0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i, \\
0 \leq u_{i,j} \leq y_j, \quad 0 \leq u_{i,j} \leq x_j, \quad \forall i, j.
\]

Moreover, \(0 \leq u_{i,j}^* \leq 1\) for all \(i\). Hence, \(\omega \in [0, 1]^n\) defined as \(\omega_i = u_{i,i}^*\) for all \(i\) is a feasible solution of LP-PDB with value \(z_{RLT-PDB}\). Conversely, let \(\omega^*\) denote an optimal solution of LP-PDB. Then the solution \(x, y, u\) defined as \(x_i = y_i = u_{i,i} = \omega_i^*\) for all \(i\) and \(u_{i,j} = 0\) for all \(i \neq j\) is feasible for RLT-PDB (note that \(\theta_i P_i \leq p\) and \(\gamma_i Q_i \leq q\) for every \(i\) by definition of \(\theta_i\) and \(\gamma_i\)). The value of this solution is \(z_{LP-PDB}\).

One might want to add more polynomial constraints to PDB in the reformulation phase. For instance, one might want to add all the constraints we get from multiplying the linear constraints of PDB by all the first order polynomial terms \(x_i, y_i, (1-x_i)\) and \((1-y_i)\). The resulting relaxation, known as the first level relaxation of the RLT hierarchy, is a widely used LP approximation for bilinear programs and has been observed to be a good empirical approximation (Sherali et al. [42], Sherali and Adams [1, 3], Sherali and Alameddine [41]). We refer the reader to the Appendix C for a derivation of this relaxation and to Section 6 for a numerical comparison of this relaxation to LP-PDB. One might also want to multiply by higher order polynomial constraints of the form 

\[
\Pi(S, S', T, T') = \prod_{i \in S} x_i \prod_{i \in S'} y_i \prod_{i \in T} (1-x_i) \prod_{i \in T'} (1-y_i) \text{ where } S, S', T, T' \subset [n] \text{ at the expense of an exponential increase in the number of constraints of the resulting relaxation.}
\]
we get from multiplying by $\Pi(S, S', T, T')$ for all $S, S', T, T'$ such that $|S| + |S'| + |T| + |T'| \leq t$ is known as the $t$-th level relaxation of the RLT hierarchy.

The equivalence between LP-PDB and RLT-PDB gives a novel perspective on our LP relaxation and shows that the $t$-th level relaxations of the RLT hierarchy are guaranteed to be within $O\left(\frac{\log m_1}{\log \log n} \frac{\log m_2}{\log \log L}\right)$ of the optimum for packing disjoint bilinear programs.

4 From Disjoint Bilinear Optimization to Two-Stage Adjustable Robust Optimization

In this section, we present a polylogarithmic approximation algorithm for $\text{AR}$. In particular, we give a compact linear restriction of $\text{AR}$ that provides near-optimal first-stage solutions with cost that is within a factor of $O\left(\frac{\log n}{\log \log n} \frac{\log L}{\log \log L}\right)$ of $z_{\text{AR}}$. Our proof uses ideas from our approximation of PDB applied to the separation problem $Q(x)$.

In order to simplify the exposition, we make the following assumption in the remainder of the paper,

**Assumption 1.** $Ax \geq 0$ for every $x \in \mathcal{X}$.

Assumption 1 states that first stage solutions can only help cover the whole or part of the uncertain right hand side uncertain parameters $h$. We show in Appendix E that Assumption 1 is without loss of generality. We also give in Appendix E an example of a network design application where Assumption 1 does not hold.

Recall the two-stage adjustable problem $\text{AR},$

$$\min_{x \in \mathcal{X}} \quad c^T x + Q(x),$$

where for all $x \in \mathcal{X},$

$$Q(x) = \max_{h \in \mathcal{H}} \min_{y \geq 0} \{d^T y \mid Ax + By \geq h\}.$$

Let us write $Q(x)$ in its bilinear form. In particular, we take the dual of the inner minimization problem on $y$ to get,

$$Q(x) = \max_{h, z \geq 0} \left\{ h^T z - (Ax)^T z \mid B^T z \leq d, \quad Rh \leq r \right\}.$$

For the special case where $A = 0$, the optimal first-stage solution is $x = 0$ and $\text{AR}$ reduces to an instance of PDB. Therefore, our algorithm for PDB gives an $O\left(\frac{\log n}{\log \log n} \frac{\log L}{\log \log L}\right)$-approximation algorithm of $\text{AR}$ in this special case.
In the general case, the separation problem $Q(x)$ is the difference of a bilinear and a linear term. This makes it challenging to approximate $Q(x)$. Instead, we approximate $AR$ directly. In particular, for any $x \in X$, consider the following linear program:

$$Q_{LP}(x) = \max_{\omega \geq 0} \left\{ \sum_{i=1}^{m} (\theta_i \gamma_i - \theta_i a_i^T x)\omega_i \right\},$$

where for all $i \in [m],$

$$\theta_i := \max_{z} \{ z_i \mid B^T z \leq d, z \geq 0 \}, \quad \gamma_i := \max_{h} \{ h_i \mid Rh \leq r, h \geq 0 \},$$

$a_i$ and $b_i$ are the $i$-th row of $A$ and $B$ respectively and $R_i$ is the $i$-th column of $R$. For ease of notation, we use $Q_{LP}(x)$ to refer to both the problem and its optimal value. Let

$$\eta := \frac{3 \log n}{\log \log n} + 2, \quad \beta := \frac{3 \log L}{\log \log L} + 2.$$

Similar to PDB, we show the following structural property of the separation problem.

**Lemma 4. (Structural Property).** For every $x \in X$, there exists a near-integral solution $(h, z) \in \{0, \frac{\eta}{\beta}\}^m \times \{0, \frac{\theta}{\eta}\}^m$ of $Q(x)$ such that,

$$\sum_{i=1}^{m} b_i z_i \leq d,$$

$$\sum_{i=1}^{m} R_i h_i \leq r,$$

$$\sum_{i=1}^{m} h_i z_i - (a_i^T x) z_i \geq \frac{1}{2\eta \beta} \cdot Q_{LP}(\beta x).$$

We construct such solution following a similar procedure as in Algorithm 1. In particular, let $\omega^*$ be an optimal solution of $Q_{LP}(\beta x)$, consider $\omega_i, \ldots, \omega_m$ i.i.d. Bernoulli random variables such that $\Pr(\omega_i = 1) = \omega_i^*$ for all $i \in [m]$ and let $(h, z)$ such that $h_i = \frac{\beta \omega_i}{\eta}$ and $z_i = \frac{\theta \omega_i}{\eta}$ for all $i \in [m]$. Such $(h, z)$ satisfies the properties (5) with a constant probability. The proof of this fact is similar to the proof of Lemma 1 and is deferred to Appendix D.
(h, z) such that \( h_i = \frac{\omega_i}{\eta} \) and \( z_i = \frac{\theta_i}{\gamma_i} \) for all \( i \in [m] \). The objective value of \((h, z)\) in \( Q(x) \) is now \( \frac{1}{2\eta\beta}(\sum_{i=1}^{m} \theta_i \gamma_i \omega_i - \beta \theta_i a_i^T x \omega_i) \). This cannot be directly related to the objective value of \( \tilde{\omega} \) (and therefore to the objective value of \( \omega^* \)) in \( Q^{LP}(x) \) due to an extra \( \beta \) factor in the linear term. It is therefore unclear how to lower-bound \( Q(x) \) using \( Q^{LP}(x) \). To get around this issue, we use \( Q^{LP}(\beta x) \) to lower-bound \( Q(x) \) instead. We show that this is enough for our purposes given that when \( x \) is feasible for \( AR \) then \( \beta x \) is also feasible and vice versa. The above discussion is formalized in the following lemma.

**Lemma 5.** For \( x \in X \), we have,

\[
\frac{1}{2\eta\beta} Q^{LP}(\beta x) \leq Q(x) \leq Q^{LP}(x).
\]

**Proof.** First, let \((h^*, z^*)\) be an optimal solution of \( Q(x) \). Define \( \omega^* \) such that \( \omega^*_i = \frac{z^*_i}{\gamma_i} h^*_i \) for all \( i \in [m] \). Then \( \omega^* \) is feasible for \( Q^{LP}(x) \) with objective value,

\[
\sum_{i=1}^{m} (\theta_i \gamma_i - \theta_i a_i^T x) \omega^*_i = \sum_{i=1}^{m} h^*_i z^*_i - (a_i^T x) \frac{h^*_i}{\gamma_i} z^*_i \\
\geq \sum_{i=1}^{m} h^*_i z^*_i - (a_i^T x) z^*_i \\
= Q(x)
\]

where the inequality follows from the fact that \( \frac{h^*_i}{\gamma_i} \leq 1 \) for all \( i \in [m] \). Hence \( Q(x) \leq Q^{LP}(x) \).

Now, consider \((h, z) \in \{0, \frac{\omega_i}{\eta}\}^m \times \{0, \frac{\theta_i}{\gamma_i}\}^m\) satisfying properties (5). The first two properties imply that \((h, z)\) is a feasible solution for \( Q(x) \). The third property implies that the objective value of this solution is such that,

\[
\sum_{i=1}^{m} h_i z_i - a_i^T x z_i \geq \frac{1}{2\eta\beta} Q^{LP}(\beta x).
\]

Hence, \( Q(x) \geq \frac{1}{2\eta\beta} Q^{LP}(\beta x) \). \(\square\)

**Our Linear Restriction.** We are now ready to derive our linear restriction of \( AR \). In particular, consider the following problem where \( Q(x) \) is replaced by \( Q^{LP}(x) \) in the expression of \( AR \),

\[
z_{LP-AR} = \min_{x \in X} \left\{ c^T x + Q^{LP}(x) \right\} ,
\]

Note that for any given first stage solution \( x \), \( Q^{LP}(x) \) is a maximization LP. Taking its dual and substituting in (6), we get the following LP:
\[
\begin{align*}
\min_{x, y, \alpha} & \quad c^T x + d^T y + r^T \alpha \\
\text{s.t.} & \quad \theta_i a_i^T x + \theta_i b_i^T y + \gamma_i R_i^T \alpha \geq \theta_i \gamma_i \quad \forall i, \\
& \quad x \in \mathcal{X}, y \geq 0, \alpha \geq 0.
\end{align*}
\]

We claim that LP-AR is a restriction of AR and gives an \( O(\frac{\log n}{\log \log n} \frac{\log L}{\log \log L}) \)-approximation for AR.

**Proof of Theorem 2.** We prove the following:

\[
z_{AR} \leq z_{LP-AR} \leq 3\eta \beta z_{AR}.
\]

Let \( x^*_{LP} \) denote an optimal solution of (6). We have,

\[
\begin{align*}
z_{LP-AR} &= c^T x^*_{LP} + Q^{LP}(x^*_{LP}) \\
&\geq c^T x^*_{LP} + Q(x^*_{LP}) \\
&\geq z_{AR},
\end{align*}
\]

where the first inequality follows from Lemma 5 and the last inequality follows from the feasibility of \( x^*_{LP} \) in AR.

To prove the upper-bound on \( z_{LP-AR} \), let \( x^* \) denote an optimal first-stage solution of AR. We have

\[
\begin{align*}
z_{AR} &\geq Q(x^*) \\
&\geq \frac{1}{2\eta \beta} Q^{LP}(\beta x^*) \\
&= \frac{1}{2\eta \beta} (\beta c^T x^* + Q^{LP}(\beta x^*)) - \frac{1}{2\eta} c^T x^* \\
&\geq \frac{1}{2\eta \beta} (\beta c^T x^* + Q^{LP}(\beta x^*)) - \frac{1}{2} z_{AR} \\
&\geq \frac{1}{2\eta \beta} z_{LP-AR} - \frac{1}{2} z_{AR},
\end{align*}
\]

where the second inequality follows from Lemma 5, the third inequality follows from the fact that \( c^T x^* \leq z_{AR} \) and the fact that \( \eta \geq 1 \). For the last inequality, note that \( \beta x^* \) is a feasible solution for (6). The above implies that \( z_{LP-AR} \leq 3\eta \beta z_{AR} \). \( \square \)
5 Affine Policies and Relation to our LP Restriction

Affine policies are widely used to approximate the two-stage problem AR where the second-stage variables $y$ are restricted to be affine functions of the uncertain right-hand side $h$. In other words, we consider $y(h) = Ph + q$, and optimize over $P$ and $q$. Affine policies were first introduced in Ben-Tal et al. [6] and have been widely considered in the literature. Ben-Tal et al. [6] show that affine policies give a tractable approximation for a large class of dynamic optimization problems. In particular, for a polyhedral uncertainty set $U$, one can find the optimal affine policy by solving a linear program with polynomially many variables and constraints.

Many approximation bounds are known for the worst-case performance of affine policies in many settings. For a simplex uncertainty set, Bertsimas and Goyal [8] show that affine policies are optimal. In the case of a single budget of uncertainty set, El Housni and Goyal [19] show that affine policies achieve an approximation bound of $O(\frac{\log n}{\log \log n})$. They also show an $O(\frac{\log^2 n}{\log \log n})$ approximation bound in the case of an intersection of disjoint budget of uncertainty sets (partition matroid) and an $O(\frac{L \log n}{\log \log n})$ approximation bound in the case of a general intersection of $L$ budget of uncertainty sets. The later results rely on a clever decomposition of the coordinates of $h$ into cheap and expensive coordinates. In this section, we generalize the bound for a single budget of uncertainty set and significantly improve the bounds for partition matroids and general intersections of budget of uncertainty sets. In particular, we show that affine policies give an $O(\frac{\log n}{\log \log n} \cdot \frac{\log L}{\log \log L})$ worst-case approximation to AR under the general packing uncertainty set with $L$ constraints. In contrast with the techniques used in El Housni and Goyal [19], our proof uses our LP restriction for AR. In particular, using an optimal solution of LP-AR, we construct an explicit feasible affine policy with objective value at most $z_{\text{LP-AR}}$.

Proof of Theorem 3. Our construction is as follows: Let $x^*, y^*, \alpha^*$ be an optimal solution of LP-AR. Consider the affine policy with first-stage solution $x^*$ and second-stage policy

$$y_{\text{AFF}}(h) := y^* + \sum_i \frac{R_i^T \alpha^*}{\theta_i} v_i h_i,$$  \hspace{1cm} (7)

for every $h \in U$, where for every $i \in [m]$,

$$v_i \in \arg\min_{y \geq 0} \{d^T y \mid By \geq e_i\}.$$

Let us first show that the above construction gives a feasible affine policy. We know $x^* \in \mathcal{X}$ by construction. We need to verify the feasibility of the second-stage solution $y_{\text{AFF}}(h)$ for every $h \in U$. Consider any $h \in U$, the $i$-th constraint of second-stage problem is given by,
\[ a_i^T x^* + b_i^T y_{\text{AFF}}(h) = a_i^T x^* + b_i^T y^* + \sum_{k=1}^{m} \frac{R_k^T \alpha^*_k}{\theta_k} b_i^T v_k h_k \]
\[ \geq a_i^T x^* + b_i^T y^* + \frac{R_i^T \alpha^*_i}{\theta_i} b_i^T v_i h_i \]
\[ \geq a_i^T x^* + b_i^T y^* + \frac{R_i^T \alpha^*_i}{\theta_i} h_i \]
\[ \geq (a_i^T x^* + b_i^T y^*) h_i + \frac{R_i^T \alpha^*_i}{\theta_i} h_i \]
\[ = \frac{1}{\gamma_i} (\theta_i a_i^T x^* + \theta_i b_i^T y^* + \gamma_i R_i^T \alpha^*) h_i, \]
\[ \geq h_i, \]

where the first inequality follows from the fact that \( R_k^T \alpha^*_k b_i^T v_k h_k \geq 0 \) for every \( k \in [n] \). The second inequality follows from the definition of \( v_i \). The third inequality follows from the non-negativity of \( (a_i^T x^* + b_i^T y^*) \) and because \( h_i \leq \gamma_i \). And the last inequality follows from the feasibility of \( x^*, y^* \) and \( \alpha^* \) in LP-AR.

Next, we show that the objective value of our affine policy is upper-bounded by the \( z_{\text{LP-AR}} \). Consider any \( h \in \mathcal{U} \). The cost of our affine policy under the scenario \( h \) is given by,

\[ c^T x^* + d^T y_{\text{AFF}}(h) = c^T x^* + d^T y^* + \sum_{k=1}^{m} \frac{R_k^T \alpha^*_k}{\theta_k} d^T v_k h_k \]
\[ = c^T x^* + d^T y^* + \sum_{k=1}^{m} R_k^T \alpha^*_k h_k \]
\[ \leq c^T x^* + d^T y^* + r^T \alpha^* \]
\[ = z_{\text{LP-AR}}, \]

where the second equality follows from the fact that \( \theta_k = d^T v_k \) by definition of \( \theta_k \) and \( v_k \) and by strong duality. The inequality follows from the fact that \( \sum_{k=1}^{m} R_k h_k = R h \leq r \). Therefore, the worst-case cost of our affine policy is at most \( z_{\text{LP-AR}} \).

6 Numerical Experiments

In this section, we study the numerical performance of our approximation algorithms for PDB and AR. We compare our algorithms to several state-of-the-art algorithms for these problems and observe that our algorithms are significantly faster and provide good approximate solutions.
6.1 Performance of our Approximation for PDB

Our randomized rounding based algorithm for PDB gives an $O\left(\frac{\log \log m_1 \log \log m_2}{\log m_1 \log m_2}\right)$-approximation with high probability. We would like to note that this is a worst-case guarantee and our goal in this section is to evaluate the empirical performance of our algorithm.

Before presenting the numerical evaluation of our algorithm, we would like to note that our algorithm can be numerically improved in the following natural way.

An Improved Version of Algorithm 1. Let $(\hat{x}, \hat{y})$ denote the solution output by Algorithm 1. We consider a natural heuristic to improve the objective value of our solution which consists of performing an alternating maximization starting from $(\hat{x}, \hat{y})$. In particular, we consider the algorithm which computes a sequence $(x_1^1, y_1^1), (x_2^2, y_2^2), \ldots$ of improving solutions of PDB as follows: Let $(x_1^1, y_1^1) = (\tilde{x}, \tilde{y})$. To compute the improved solution $(x_i^{i+1}, y_i^{i+1})$ from $(x_i^i, y_i^i)$, we fix the variables $x$ in PDB to $x_i^i$ and maximize with respect to $y$ to get $y_i^{i+1}$. Then we fix the variables $y$ to $y_i^{i+1}$ and maximize with respect to $x$ to get $x_i^{i+1}$. Note that each step is a linear maximization problem. The algorithm stops when the change in the objective value between two consecutive solutions $(x_i^i, y_i^i)$ and $(x_{i+1}^{i+1}, y_{i+1}^{i+1})$ is at most $\epsilon$ (where $\epsilon > 0$ is some chosen error tolerance). We refer to this improved version of Algorithm 1 as Algorithm 2.

We evaluate the empirical performance of Algorithm 1 and its improved version Algorithm 2 by comparing these to the following benchmarks.

First Level Relaxation of the RLT Hierarchy. The first level relaxation of the RLT hierarchy is a widely used LP approximation for bilinear programs and has been observed to be a good empirical approximation (Sherali et al. [42], Sherali and Adams [1, 3], Sherali and Alameddine [41]). We denote the relaxation by FL-RLT (we refer the reader to Appendix C for the derivation of this relaxation). An optimal solution $(x, y, u)$ of FL-RLT gives an approximate solution $(x, y)$ for PDB. We compare this solution with the solution given by our algorithms in terms of objective value and running time.

Gurobi Solver. We also compare the solution given by our algorithms with the optimal solution of PDB computed using the bilinear solver of Gurobi v9.1.2, in terms of both objective value and running time. In particular, we compare the running time of our algorithms with the running time needed by the Gurobi solver to reach a solution that is at least as good as the worst of our two solutions, namely the solution given by Algorithm 1.

Experimental Setup. We consider instances of PDB where $m_1 = m_2$, $p = q = e$, $P = I_m + G_P$ and $Q = I_m + G_Q$ where $I_m$ is the identity matrix and $G_P$ and $G_Q$ are random normalized Gaussian matrices. The results of our experiments were performed on a dual-core laptop with 8GB of RAM and 1.8GHz processor.
Table 1 gives the results of our comparison between the solutions given by Algorithm 1 and its improved version Algorithm 2 and the solutions given by the first level relaxation of the RLT hierarchy and the bilinear solver of Gurobi v9.1.2. In Table 1, $T_{\text{ALG}_1}$ denotes the running time in seconds of Algorithm 1, $T_{\text{ALG}_2}$ the running time in seconds of Algorithm 2, $T_{\text{FL-RLT}}$ the running time in seconds of the first level relaxation of the RLT hierarchy, $T_{\text{GRB}}$ the running time in seconds of Gurobi solver to solve the problem to optimality and $T_{\text{GRB-LB}}$ the running time in seconds needed by the Gurobi solver to reach a solution that is as good as the solution given by Algorithm 1. For the objective values, $z_{\text{SOL-RLT}}$ denotes the objective value of the solution given by the first level relaxation of the RLT hierarchy, $z_{\text{ALG}_1}$ and $z_{\text{ALG}_2}$ the objective value of the solution given by Algorithm 1 and Algorithm 2 respectively and $z_{\text{PDB}}$ denotes the optimal objective.

Note that Algorithm 1 is based on the linear relaxation LP-PDB of PDB. The objective value of this relaxation gives an upper-bound of PDB. The first level relaxation of the RLT hierarchy FL-RLT is also a relaxation of PDB and therefore, also gives an upper-bound of PDB. The bilinear solver of Gurobi gives a sequence of improving upper-bounds of PDB. We compare the upper-bound given by our LP relaxation denoted by $z_{\text{LP-PDB}}$ and the upper-bound given by the first level relaxation of the RLT hierarchy denoted by $z_{\text{FL-RLT}}$. We also compare the running time to compute $z_{\text{LP-PDB}}$, the running time to compute $z_{\text{FL-RLT}}$ and the running time needed by the bilinear solver of Gurobi to reach an upper-bound that is at most $z_{\text{LP-PDB}}$ denoted by $T_{\text{LP-PDB}}$, $T_{\text{FL-RLT}}$ and $T_{\text{GRB-UB}}$ respectively. The results of the comparisons are reported in Table 2.

Discussion. Table 1 describes the results of our comparison between the solutions given by Algorithm 1 and its improved version Algorithm 2 and the solutions given by the first level relaxation of the RLT hierarchy and the bilinear solver of Gurobi v9.1.2 on random instances and for different values of $m_1 = m_2$.

Compared to the first level relaxation of the RLT hierarchy, our algorithms are significantly faster especially for large dimensions. For example, for $m_1 = m_2 \geq 100$, Algorithm 2 is more than 400 times faster while Algorithm 1 is more than 900 times faster. In terms of objective value, the solutions given by our algorithms have higher objective value than the solutions given by the first level relaxation of the RLT hierarchy. Moreover, the gap in the objective value grows with the dimension. For instance, for $m_1 = m_2 \geq 40$, the objective value of the solution given by FL-RLT is less 40% of the objective value of the solution given by Algorithm 1 and less than 34% of the objective value of the solution given by Algorithm 2.

We would like to note that compared to the bilinear solver of Gurobi, the running times of our algorithms are significantly faster. In fact, computing an optimal solution using Gurobi is impractical for higher dimensions (Gurobi solver fails to finish in three hours for $m_1 = m_2 > 40$). We also remark that, as the dimension increases, the bilinear solver of Gurobi takes significantly more time to compute a solution that is comparable to the solutions from our algorithms. For example, for
$m_1 = m_2 \geq 500$, the bilinear solver of Gurobi is at least 1000 times slower. Furthermore, we observe that the objective value of our solutions is a good approximation of the optimal value. For example, for all $m_1 = m_2 \leq 40$, the solution given by Algorithm 1 recovers at least 76% of the optimum while the solution given by Algorithm 2 recovers at least 88% of the optimum. Therefore, our algorithm and its improved version give a faster approach to compute a near-optimal approximate solution of PDB. In particular, this could be used as a more efficient approach to compute a good feasible solution in branch-and-bound based exact algorithms for the problem.

Table 2 gives the results of our comparison between the upper-bounds given by LP-PDB, the first level relaxation of the RLT hierarchy and the bilinear solver of Gurobi v9.1.2 on random instances and for different values of $m_1 = m_2$. Table 2 shows that, as the dimension increases, our upper-bound $z_{\text{LP-PDB}}$ becomes significantly faster to compute than the first level relaxation of the RLT hierarchy. For example, for $m_1 = m_2 \geq 100$ our upper-bound is more than 3700 times faster to compute. Moreover, the ratio between the two upper-bounds is close to 1 and increases with the dimension (our upper-bound is at least 77% of $z_{\text{FL-RLT}}$ for all the instances we consider; it is at least 86% for $m_1 = m_2 = 100$). Compared to Gurobi, and as the dimension increases, the bilinear solver of Gurobi takes significantly more time to compute an upper-bound that is at least $z_{\text{LP-PDB}}$. For example, for $m_1 = m_2 \geq 500$, the bilinear solver of Gurobi is more than 7 times slower. Given the above discussion, we conclude that our relaxation gives a faster to compute near-optimal upper-bound of PDB which can be used in branch-and-bound type of algorithms for the problem.

6.2 Performance of our Approximation for AR

Recall that our LP restriction of AR gives a polylogarithmic approximation and is tightly related to affine policies for AR. In this section, we compare the numerical performance of our LP restriction to several benchmarks including affine policies, a generalization of the algorithm of El Housni and Goyal [19], and the lower-bound of Hadjiyiannis et al. [31]. Let us first briefly discuss the benchmarks.

**Affine Policies.** Affine policies are a commonly used approximation for two-stage adjustable robust problems and have been shown to exhibit good theoretical as well as empirical performance (see for example Bertsimas et al. [11], Bertsimas and Ruiter [12], Ben-Tal et al. [6] and El Housni and Goyal [18]). Affine policies consider a restriction of the second-stage variables in AR to be affine functions of the uncertain parameters, i.e., functions of the form $y(h) = Ph + q$ for $P \in \mathbb{R}^{n \times n}$.
Table 1: Comparison of Algorithm 1 and its improved version Algorithm 2 with the first level relaxation of RLT hierarchy and the bilinear solver of Gurobi v9.1.2 for different values of $m_1 = m_2$. The running times are in seconds and the entries denoted by * exceeded a time limit of three hours.

| $m_1 = m_2$ | $T_{\text{ALG}_1}$ | $T_{\text{ALG}_2}$ | $T_{\text{FL-RLT}}$ | $T_{\text{GRB}}$ | $T_{\text{GRB-LB}}$ |
|-------------|----------------------|----------------------|----------------------|------------------|------------------|
| 10          | 0.464                | 0.686                | 0.223                | 0.356            | 0.219            |
| 20          | 0.323                | 0.474                | 0.715                | 1.993            | 0.271            |
| 30          | 0.185                | 0.342                | 1.729                | 60.667           | 0.228            |
| 40          | 0.210                | 0.369                | 5.080                | 2027.430         | 1.328            |
| 50          | 0.210                | 0.373                | 10.050               | *                | 2.083            |
| 100         | 0.321                | 0.643                | 318.119              | *                | 27.283           |
| 500         | 5.681                | 9.860                | *                    | *                | 7850.564         |
| 1000        | 17.590               | 31.2890              | *                    | *                | *                |

Table 2: Comparison between the upper-bounds given by LP-PDB, by the first level relaxation of the RLT hierarchy and by the bilinear solver of Gurobi v9.1.2 for different values of $m_1 = m_2$. The running times are in seconds and the entries denoted by * exceeded a time limit of three hours.

| $m_1 = m_2$ | $T_{\text{LP-PDB}}$ | $T_{\text{FL-RLT}}$ | $T_{\text{GRB}}$ | $T_{\text{GRB-UB}}$ | $T_{\text{LP-PDB}}$ $T_{\text{FL-RLT}}$ |
|-------------|----------------------|----------------------|------------------|---------------------|------------------|
| 10          | 0.084                | 0.223                | 0.356            | 0.086               | 0.779            |
| 20          | 0.034                | 0.715                | 1.993            | 0.053               | 0.789            |
| 30          | 0.022                | 1.729                | 60.667           | 0.074               | 0.803            |
| 40          | 0.034                | 5.080                | 2027.340         | 0.113               | 0.819            |
| 50          | 0.119                | 34.791               | *                | 0.293               | 0.834            |
| 100         | 0.105                | 394.443              | *                | 0.939               | 0.869            |
| 500         | 2.254                | *                    | *                | 15.570              | *                |
| 1000        | 9.339                | *                    | *                | 325.815             | *                |
and \( q \in \mathbb{R}^n \). The optimal affine policy can be computed using the following restriction of \( AR \),

\[
\begin{align*}
\min_{x,Y} & \quad c^T x + \max_{\zeta \in \hat{U}} d^T Y \zeta \\
\text{s.t.} & \quad Ax + BY \zeta \geq C \zeta, \quad \forall \zeta \in \hat{U}, \\
& \quad Y \zeta \geq 0, \quad \forall \zeta \in \hat{U}, \\
& \quad x \in X
\end{align*}
\]

where \( C \) denotes the matrix \([e_2, \ldots, e_{m+1}]^T\) with \( e_i \) being the \( i \)-th vector of the canonical basis of \( \mathbb{R}^{m+1} \) and

\[
\hat{U} = \{ \zeta \in \mathbb{R}^{m+1} \mid \zeta_1 = 1, C \zeta \in U \}.
\]

Let \( K \) be the cone generated by \( \hat{U} \) given as:

\[
K = \{ (t, h) \in \mathbb{R} \times \mathbb{R}^m \mid Rh \leq rt, \quad t \geq 0, h \geq 0 \},
\]

and let \( K^* \) be its dual cone given by,

\[
K^* = \{ (u, \sigma) \in \mathbb{R} \times \mathbb{R}^m \mid \exists \lambda \in \mathbb{R}^L \text{ s.t. } u \geq r^T \lambda, \quad R^T \lambda \geq -\sigma, \quad u \geq 0, \sigma \geq 0 \}.
\]

For any \( \sigma \in \mathbb{R}^{m+1} \),

\[
\sigma^T \zeta \geq 0 \quad \forall \zeta \in \hat{U} \quad \iff \quad \sigma \in K^*.
\]

The above problem can be expressed as the following equivalent linear program:

\[
\begin{align*}
\min_{x,\alpha} & \quad c^T x + \alpha \\
\text{s.t.} & \quad (\alpha e_1^T - d^T Y) \in K^*, \\
& \quad (Axe_1^T + BY - C)^T \in K^*, \\
& \quad Y \in K^*, x \in X.
\end{align*}
\]

We refer to this LP as the affine restriction of \( AR \). We compare the affine policies given by our LP restriction \( LP-AR \) to the optimal affine policy in terms of worst-case objective value and running time.

**Algorithm of El Housni and Goyal [19].** El Housni and Goyal [19] give an algorithm to efficiently compute near-optimal affine policies that show that achieve a \( O\left(\frac{\log n}{\log \log n}\right) \) approximation guarantee for the case of a single budget of uncertainty set. In particular, they consider affine policies of the following form. Let

\[
v_i \in \arg\min_{y \geq 0} \{d^T y \mid By \geq e_i\}, \quad \forall i \in [m].
\]
El Housni and Goyal [19] consider affine policies of the form:

\[ y(h) = \sum_i \nu_i v_i h_i + q, \]

for single budget of uncertainty. These can be computed using a linear program.

For our comparison benchmark, we consider a generalization of the algorithm in El Housni and Goyal [19] that computes the optimal affine policy of the form \( y(h) = \sum_i \nu_i v_i h_i + q \) for the case of packing uncertainty set given by \( \mathcal{U} = \{h \geq 0 \mid Rh \leq r\} \) (we refer the reader to Appendix F for an expression of the corresponding LP as well as its derivation). We denote the corresponding LP by EG. We compare the affine policies given by our restriction LP-AR to those given by EG in terms of objective value and running time.

**Lower-Bound of Hadjiyiannis et al. [31].** Hadjiyiannis et al. [31] propose a scenario based lower-bound for AR. In particular, they consider a finite set of scenarios \( \Delta \subset \mathcal{U} \) referred to as a critical set, and solve a relaxation of the two-stage problem for these scenarios. We denote such relaxation as \( AR(\Delta) \). Hadjiyiannis et al. [31] propose an efficient algorithm to compute one such critical set that relies on the conic dual of the affine restriction AFF given as:

\[
\begin{align*}
\max_{\lambda, \Lambda, \Theta} & \quad \sum_{i=1}^{m} \Lambda_{i+1,i} \\
\text{s.t} \quad & 1 - e_i^T \lambda = 0 \\
& \Lambda B + \Theta = \lambda d^T \\
& e_i^T \Lambda A \leq c^T \\
& \lambda \in \mathcal{K} \\
& \Lambda_j \in \mathcal{K}, \quad \Theta_j \in \mathcal{K}, \quad \forall j.
\end{align*}
\]

(D-AFF)

Here \( \Lambda_j \) and \( \Theta_j \) denote the \( j \)-th column of \( \Lambda \) and \( \Theta \) respectively where \( \lambda, \Lambda \) and \( \Theta \) are the dual variables corresponding the first, second and third conic constraints in \( AFF \) respectively. In particular, given an optimal dual solution \( \lambda^*, \Lambda^*, \Theta^* \) of D-AFF, Hadjiyiannis et al. [31] show that the critical set given by \( \Delta_{AFF} = \{\lambda^*\} \cup \{\Lambda_j^* \mid \Lambda_j^* \neq 0\} \cup \{\Theta_j^* \mid \Theta_j^* \neq 0\} \) is such that under \( \Delta_{AFF} \) the relaxation \( AR(\Delta_{AFF}) \) gives an empirically good lower-bound for AR. We compare the worst-case cost of our affine policies to the lower bound \( AR(\Delta_{AFF}) \).

**Experimental Setup.** Following Ben-Tal et al. [7], we consider instances of AR where \( n = m \), \( c = d = e \) and \( A = B = I_m + G \), where \( I_m \) is the identity matrix and \( G \) is a random normalized Gaussian matrix. We consider the case where \( \mathcal{X} = \mathbb{R}_+^m \) and \( \mathcal{U} \) is an intersection of \( L \) budget of
uncertainty sets of the form:

\[ \mathcal{U} = \left\{ h \in [0, 1]^m \mid \omega_l^T h \leq 1 \forall l \in [L] \right\}, \]

where the weight vectors \( \omega_l \) are normalized Gaussian vectors, i.e., \( \omega_{l,i} = \frac{|G_{l,i}|}{\sqrt{\sum_i (G_{l,i})^2}} \) for \( G_{l,i} \) i.i.d. standard Gaussian variables. We use a dual-core laptop with 8GB of RAM and 1.8GHz processor for the experiments and present the results in Tables 3, 4 and 5. Here \( z_{\text{AFF}} \) denotes the optimum of the affine restriction \( \text{AFF} \), \( z_{\text{LP-AR}} \) the optimum of our restriction \( \text{LP-AR} \), \( z_{\text{EG}} \) the optimal value of \( \text{EG} \) and \( z_{\text{LB}} \) the lower-bound given by \( \text{AR}(\Delta_{\text{AFF}}) \) as defined above. In terms of running time, \( T_{\text{AFF}} \) denotes the running time in seconds of the affine restriction, \( T_{\text{LP-AR}} \) the running time in seconds of our restriction and \( T_{\text{EG}} \) the running time in seconds of \( \text{EG} \).

**Discussion.** Tables 3, 4 and 5 show that our restriction \( \text{LP-AR} \) is significantly faster than both the generalization of El Housni and Goyal [19] \( \text{EG} \) and the optimal affine policy especially for higher dimensions. For instance, for \( n = n' = 100 \) and \( L = 20 \), our restriction is more than 30000 times faster to compute than the optimal affine policy. For \( n = n' = 1000 \) and \( L = 20 \), our restriction is more than 85 times faster than \( \text{EG} \). Furthermore, for all values of \( n \) we consider, our restriction has nearly the same objective value as \( \text{EG} \) (the ratio is less than \( 10^{-4} \) in all instances we consider). Note that \( \text{EG} \) optimizes over a richer class of affine policies than the affine policies we construct using \( \text{LP-AR} \) (\( \text{EG} \) optimizes over all affine policies of the form \( y(h) = \sum_i \nu_i v_i h_i + q \), while the affine policies we construct using \( \text{LP-AR} \) are such that there exists an \( \alpha \in \mathbb{R}_+^L \) such that \( \nu_i = \frac{R_i^T \alpha}{b_i} \) for all \( i \in [m] \) and it is a priori not clear that the two linear programs are equivalent. However, given the numerical findings, we conjecture that the optimal affine policies given by our restriction \( \text{LP-PDB} \) are optimal among the set of affine policies of the form of the form \( y(h) = \sum_i \nu_i v_i h_i + q \).

Compared to the optimal affine policy, our restriction gives a good approximation that scales well with the dimension. For example, for \( L = 20 \), our affine policy is within at most 20% for all \( n = n' \leq 100 \). Finally, compared to the optimum of \( \text{AR} \), note that affine policies are less than 31% from the Hadjiyiannis et al. [31] lower-bound for all \( n = n' \leq 100 \) and \( L = 20 \) implying that our restriction is within less than 69% from the Hadjiyiannis et al. [31] lower-bound for all \( n = n' \leq 100 \) and \( L = 20 \). However, this comparison is with respect to a lower-bound of the optimum and not with respect to the optimum itself and includes the gap between the lower-bound and the optimum as well. Our computational study demonstrates that our restriction gives a good approximation of the two-stage adjustable robust problem that is significantly faster to compute than state-of-the-art approximation methods of this problem.
Table 3: Comparison of the objective value and the running time in seconds between our restriction LP-AR, the generalization of El Housni and Goyal [19] EG, and the optimal affine policy for different values of \( n = n' \) and \( L = 20 \). Entries denoted by a star exceeded a time limit of three hours.

| \( n = n' \) | \( T_{LP-AR} \) | \( T_{EG} \) | \( T_{AFF} \) | \( \frac{z_{LP-AR}}{z_{EG}} \) | \( \frac{z_{LP-AR}}{z_{AFF}} \) | \( \frac{z_{AFF}}{z_{LB}} \) |
|------------|----------------|------------|------------|----------------|----------------|----------------|
| 20         | 0.052          | 0.080      | 0.412      | 1.000          | 1.304          | 1.197          |
| 40         | 0.074          | 0.273      | 21.866     | 1.000          | 1.226          | 1.303          |
| 60         | 0.104          | 0.609      | 202.732    | 1.000          | 1.248          | 1.310          |
| 80         | 0.309          | 1.726      | 822.509    | 1.000          | 1.207          | 1.298          |
| 100        | 0.153          | 3.227      | 4627.206   | 1.000          | 1.196          | 1.299          |
| 500        | 0.540          | 22.064     | *          | 1.000          | *              | *              |
| 1000       | 1.607          | 137.646    | *          | 1.000          | *              | *              |

Table 4: Comparison of the objective value and the running time in seconds between our restriction LP-AR, the generalization of El Housni and Goyal [19] EG, and the optimal affine policy for different values of \( n = n' \) and \( L = 50 \). Entries denoted by a star exceeded a time limit of three hours.

| \( n = n' \) | \( T_{LP-AR} \) | \( T_{EG} \) | \( T_{AFF} \) | \( \frac{z_{LP-AR}}{z_{EG}} \) | \( \frac{z_{LP-AR}}{z_{AFF}} \) | \( \frac{z_{AFF}}{z_{LB}} \) |
|------------|----------------|------------|------------|----------------|----------------|----------------|
| 20         | 0.038          | 0.072      | 0.507      | 1.000          | 1.389          | 1.347          |
| 40         | 0.055          | 0.444      | 17.815     | 1.000          | 1.263          | 1.436          |
| 60         | 0.095          | 1.171      | 193.404    | 1.000          | 1.227          | 1.461          |
| 80         | 0.087          | 1.426      | 978.970    | 1.000          | 1.121          | 1.472          |
| 100        | 0.068          | 2.212      | 5649.392   | 1.000          | 1.190          | 1.420          |
| 500        | 0.580          | 104.684    | *          | 1.000          | *              | *              |
| 1000       | 1.756          | 2062.449   | *          | 1.000          | *              | *              |

Table 5: Comparison of the objective value and the running time in seconds between our restriction LP-AR, the generalization of El Housni and Goyal [19] EG, and the optimal affine policy for different values of \( n = n' \) and \( L = 100 \). Entries denoted by a star exceeded a time limit of three hours.

| \( n = n' \) | \( T_{LP-AR} \) | \( T_{EG} \) | \( T_{AFF} \) | \( \frac{z_{LP-AR}}{z_{EG}} \) | \( \frac{z_{LP-AR}}{z_{AFF}} \) | \( \frac{z_{AFF}}{z_{LB}} \) |
|------------|----------------|------------|------------|----------------|----------------|----------------|
| 20         | 0.073          | 0.179      | 1.166      | 1.000          | 1.369          | 1.381          |
| 40         | 0.046          | 0.605      | 18.804     | 1.000          | 1.285          | 1.521          |
| 60         | 0.119          | 1.894      | 188.047    | 1.000          | 1.230          | 1.588          |
| 80         | 0.138          | 2.230      | 1112.060   | 1.000          | 1.200          | 1.561          |
| 100        | 0.122          | 5.606      | 2812.385   | 1.000          | 1.186          | 1.634          |
| 500        | 0.997          | 4986.632   | *          | 1.000          | *              | *              |
| 1000       | 1.189          | 9734.252   | *          | 1.000          | *              | *              |
7 Conclusion

In this paper, we consider the class of packing disjoint bilinear programs $PDB$ and present an LP rounding based randomized approximation algorithm for this problem. In particular, we show the existence of a near-optimal near-integral solution for $PDB$. We give an LP relaxation from which we obtain such solution using a randomized rounding of an optimal solution. We show that out relaxation is closely related to the reformulation linearization technique (RLT). We apply our ideas to the two-stage adjustable problem $AR$ whose separation problem is a variant of $PDB$. While a direct application of the approximation algorithm for $PDB$ does not work for $AR$, we derive an LP restriction of $AR$, based on similar ideas, that gives a polylogarithmic approximation of $AR$. We relate our LP restriction to affine policies. In particular, using an optimal solution of the LP restriction, we construct a near-optimal affine policy whose objective value is smaller than the optimal cost of the LP restriction. This proves that affine policies give a polylogarithmic approximation of $AR$ and gives a new algorithm to compute near-optimal affine policies. We evaluate the numerical performance of our algorithms for $PDB$ and $AR$ and show that they are significantly faster and provide good empirical solutions.

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Appendices

A Chernoff bounds

Proof of Chernoff bounds (a). From Markov’s inequality we have for all $t > 0$,

$$
P(\xi \geq (1 + \delta)s) = P(e^{t\xi} \geq e^{t(1+\delta)s}) \leq \frac{E(e^{t\xi})}{e^{t(1+\delta)s}}.
$$

Denote $p_i$ the parameter of the Bernoulli $\chi_i$. By independence, we have

$$
E(e^{t\xi}) = \prod_{i=1}^r E(e^{t\epsilon_i \chi_i}) = \prod_{i=1}^r \left( p_i e^{t\epsilon_i} + 1 - p_i \right) \leq \prod_{i=1}^r \exp \left( p_i (e^{t\epsilon_i} - 1) \right)
$$

where the inequality holds because $1 + x \leq e^x$ for all $x \in \mathbb{R}$. By taking $t = \ln(1 + \delta) > 0$, the right hand side becomes

$$
\prod_{i=1}^r \exp \left( p_i ((1 + \delta)^{\epsilon_i} - 1) \right) \leq \prod_{i=1}^r \exp \left( p_i \delta \epsilon_i \right) = \exp \left( \delta \cdot E(\xi) \right) \leq e^{\delta s},
$$

where the first inequality holds because $(1 + x)^t \leq 1 + \alpha x$ for any $x \geq 0$ and $\epsilon \in [0, 1]$ and the second one because $s \geq E(\xi) = \sum_{i=1}^r \epsilon_i p_i$. Hence, we have

$$
E(e^{t\xi}) \leq e^{\delta s}.
$$

On the other hand,

$$
e^{t(1+\delta)s} = (1 + \delta)^{(1+\delta)s}.
$$

Therefore,

$$
P(\xi \geq (1 + \delta)s) \leq \left( \frac{e^{\delta s}}{(1 + \delta)^{(1+\delta)s}} \right) = \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^s.
$$

Proof of Chernoff bounds (b). From Markov’s inequality we have for all $t < 0$,

$$
P(\xi \leq (1 - \delta)E(\xi)) = P(e^{t\xi} \geq e^{t(1-\delta)E(\xi)}) \leq \frac{E(e^{t\xi})}{e^{t(1-\delta)E(\xi)}}.
$$

Denote $p_i$ the parameter of the Bernoulli $\chi_i$. By independence, we have

$$
E(e^{t\xi}) = \prod_{i=1}^r E(e^{t\epsilon_i \chi_i}) = \prod_{i=1}^r \left( p_i e^{t\epsilon_i} + 1 - p_i \right) \leq \prod_{i=1}^r \exp \left( p_i (e^{t\epsilon_i} - 1) \right),
$$

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where the inequality holds because $1 + x \leq e^x$ for all $x \in \mathbb{R}$. We take $t = \ln(1 - \delta) < 0$. We have $t \leq -\delta$, hence,

$$\prod_{i=1}^{r} \exp \left( p_i (e^{t \epsilon_i} - 1) \right) = \prod_{i=1}^{r} \exp \left( p_i ((1 - \delta)^{\epsilon_i} - 1) \right) \leq \prod_{i=1}^{r} \exp (-p_i \delta \epsilon_i),$$

where the inequality holds because $(1 - x)^\epsilon \leq 1 - \epsilon x$ for any $0 < x < 1$ and $\epsilon \in [0, 1]$. Therefore,

$$\mathbb{E}(e^{t \Xi}) \leq \prod_{i=1}^{r} \exp (-p_i \delta \epsilon_i) = e^{-\delta \mathbb{E}(\Xi)}.$$

On the other hand,

$$e^{t(1-\delta)\mathbb{E}(\Xi)} = (1 - \delta)^{(1-\delta)\mathbb{E}(\Xi)}.$$

Therefore,

$$\mathbb{P}(\Xi \leq (1 - \delta)\mathbb{E}(\Xi)) \leq \left( \frac{e^{-\delta \mathbb{E}(\Xi)}}{(1 - \delta)^{(1-\delta)\mathbb{E}(\Xi)}} \right)^{\mathbb{E}(\Xi)} = \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^{\mathbb{E}(\Xi)}.$$

Finally, we have for any $0 < \delta < 1$,

$$\ln(1 - \delta) \geq -\delta + \frac{\delta^2}{2}$$

which implies

$$(1 - \delta) \cdot \ln(1 - \delta) \geq -\delta + \frac{\delta^2}{2}$$

and consequently

$$\left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^{\mathbb{E}(\Xi)} \leq e^{-\frac{\delta^2 \mathbb{E}(\Xi)}{4}}.$$

\[\square\]

**B Proof of Theorem 4**

Our proof uses a polynomial time transformation from the *Monotone Not-All-Equal 3-satisfiability* (MNAE3SAT) NP-complete problem (Schaefer [37]). In the (MNAE3SAT), we are given a collection of Boolean variables and a collection of clauses, each of which combines three variables. (MNAE3SAT) is the problem of determining if there exists a truth assignment where each clause has at least one true and one false literal. This is a subclass of the *Not-All-Equal 3-satisfiability* problem where the variables are never negated.

Consider an instance $I$ of (MNAE3SAT), let $\mathcal{V}$ be the set of variables of $I$ and let $\mathcal{C}$ be the set of clauses. Let $\mathbf{A} \in \{0, 1\}^{\mathcal{C} \times \mathcal{V}}$ be the variable-clause incidence matrix such that for every variable $v \in \mathcal{V}$ and clause $c \in \mathcal{C}$ we have $A_{cv} = 1$ if and only if the variable $v$ belongs to the clause $c$. We
consider the following instance of CDB denoted by $I'$,

$$\min_{x,y} \sum_{v \in V} x_v y_v$$

s.t

$$Ax \geq e, \quad x \geq 0$$

$$Ay \geq e, \quad y \geq 0.$$

$(I')$

Let $z_{I'}$ denote the optimum of $I'$. We show that $I$ has a truth assignment where each clause has at least one true and one false literal if and only if $z_{I'} = 0$.

First, suppose $z_{I'} = 0$. Let $(x, y)$ be an optimal solution of $I'$. Let $\tilde{x}$ such that $\tilde{x}_v = 1$ if $x_v > 0$ for all $v \in V$. We claim that $(\tilde{x}, e - \tilde{x})$ is also an optimal solution of $I$. In fact, for all $c \in C$, we have,

$$\sum_{v \in V} A_{cv} x_v \geq 1,$$

hence,

$$\sum_{v \in V} A_{cv} \min(x_v, 1) \geq 1,$$

since the entries of $A$ are in $\{0, 1\}$. Therefore,

$$\sum_{v \in V} A_{cv} \tilde{x}_v \geq \sum_{v \in V} A_{cv} \min(x_v, 1) \geq 1.$$

Finally, $A\tilde{x} \geq e$. Similarly, for all $c \in C$ we have,

$$\sum_{v \in V} A_{cv} y_v \geq 1,$$

hence,

$$\sum_{v \in V} A_{cv} \min(y_v, 1) \geq 1,$$

since the entries of $A$ are in $\{0, 1\}$. Therefore,

$$\sum_{v \in V} A_{cv} (1 - \tilde{x}_v) \geq \sum_{v \in V} A_{cv} (1 - \tilde{x}_v) \min(y_v, 1) = \sum_{v \in V} A_{cv} \min(y_v, 1) \geq 1.$$

where the equality follows from the fact that $y_v = 0$ for all $v$ such that $\tilde{x}_v = 1$. Note that $\sum_{v \in V} x_v y_v = z_{I'} = 0$. This implies that $A(e - \tilde{x}) \geq e$. Therefore, $(\tilde{x}, e - \tilde{x})$ is a feasible solution of $I'$ with objective value $\sum_{v \in V} \tilde{x}_v (1 - \tilde{x}_v) = 0 = z_{I'}$. It is therefore an optimal solution. Consider now the truth assignment where a variable $v$ is set to be true if and only if $\tilde{x}_v = 1$. We show that each clause in such assignment has at least one true and one false literal. In particular, consider a
clause \( c \in \mathcal{C} \), since,

\[
\sum_{v \in \mathcal{V}} A_{cv} \tilde{x}_v \geq 1,
\]

there must be a variable \( v \) belonging to the clause \( c \) such that \( \tilde{x}_v = 1 \). Similarly, since

\[
\sum_{v \in \mathcal{V}} A_{cv}(1 - \tilde{x}_v) \geq 1,
\]

there must be a variable \( v \) belonging to the clause \( c \) such that \( \tilde{x}_v = 0 \). This implies that our assignment is such that the clause \( c \) has at least one true and one false literal.

Conversely, suppose there exists a truth assignment of \( I \) where each clause has at least one false and one true literal. We show that \( z_{I'} = 0 \). In particular, define \( x \) such that for all \( v \in \mathcal{V} \) we have \( x_v = 1 \) if and only if \( v \) is assigned true. We have for all \( c \in \mathcal{C} \),

\[
\sum_{v \in \mathcal{V}} A_{cv} x_v \geq 1,
\]

since at least one of the variables of \( c \) is assigned true. And,

\[
\sum_{v \in \mathcal{V}} A_{cv}(1 - x_v) \geq 1,
\]

since at least one of the variables of \( c \) is assigned false. Hence, \((x, e - x)\) is feasible for \( I' \) with objective value \( \sum_{v \in \mathcal{V}} x_v(1 - x_v) = 0 \). It is therefore an optimal solution and \( z_{I'} = 0 \).

If there exists a polynomial time algorithm approximating \( \text{CDB} \) to some finite factor, such an algorithm can be used to decide in polynomial time whether \( z_{I'} = 0 \) or not, which is equivalent to solving \( I \) in polynomial time; a contradiction. \( \square \)

C Derivation of the first level relaxation of the RLT hierarchy.

Let us begin by writing \( \text{PDB} \) in the following equivalent epigraph form,
\[
\max_{x, y, u, v, w} \sum_{i=1}^{n} \theta_i \gamma_i u_{ii} \\
\quad u_{ij} \leq x_i y_j, \quad \forall i, j \\
\quad v_{i,j} \leq x_i x_j, \quad \forall i, j \\
\quad w_{i,j} \leq y_i y_j, \quad \forall i, j \\
\quad \sum_{i=1}^{n} \theta_i P_i x_i \leq p, \\
\quad \sum_{i=1}^{n} \gamma_i Q_i y_i \leq q, \\
\quad 0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i.
\]

**Reformulation phase.** In the reformulation phase we add to the above program the polynomial constraints we get from multiplying all the linear constraints by linear constraints by \(x_i, y_i, (1-x_i)\) and \((1-y_i)\) for all \(i \in [n]\). We get the following equivalent formulation of PDB,
\[
\begin{align*}
\max_{x,y,u,v,w} & \quad \sum_{i=1}^{n} \theta_i \gamma_j u_{ij} \\
\text{s.t.} & \quad u_{ij} \leq x_i y_j, \quad \forall i,j, \\
& \quad v_{i,j} \leq x_i x_j, \quad \forall i,j, \\
& \quad w_{i,j} \leq y_i y_j, \quad \forall i,j, \\
& \quad \sum_{j=1}^{n} \theta_j P_j x_j \leq p, \\
& \quad \sum_{j=1}^{n} \theta_j P_j x_j x_i \leq px_i, \\
& \quad \sum_{j=1}^{n} \theta_j P_j (x_j - x_j x_i) \leq p(1 - x_i) \\
& \quad \sum_{j=1}^{n} \theta_j P_j x_j y_i \leq py_i, \\
& \quad \sum_{j=1}^{n} \theta_j P_j (x_j - y_i x_j) \leq p(1 - y_i) \\
& \quad \sum_{j=1}^{n} \gamma_j Q_j y_j \leq q, \\
& \quad \sum_{j=1}^{n} \gamma_j Q_j y_j x_i \leq qx_i, \\
& \quad \sum_{j=1}^{n} \gamma_j Q_j (y_j - y_j x_i) \leq p(1 - x_i) \\
& \quad \sum_{j=1}^{n} \gamma_j Q_j y_j y_i \leq qy_i, \\
& \quad \sum_{j=1}^{n} \gamma_j Q_j (y_j - y_j y_i) \leq p(1 - y_i) \\
& \quad 0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i \\
& \quad x_i x_j \leq x_i, \quad x_i x_j \leq x_j, \\
& \quad y_i y_j \leq y_i, \quad y_i y_j \leq y_j, \\
& \quad x_i y_j \leq x_i, \quad x_i y_j \leq y_j,
\end{align*}
\]

**Linearization phase.** We replace the bilinear terms \(x_i y_j, x_i x_j\) and \(y_i y_j\) in the above LP with their respective lower-bounds \(u_{ij}, v_{i,j}, w_{i,j}\). We get the following linear relaxation of \(\text{PDB}\),
The above LP is known as the first level relaxation of the RLT hierarchy. The above LP can be further simplified as follows: the constraints in $v$ and $w$ in the above are redundant and can be removed without loss of generality. In fact, for any feasible solution $x, y, u$ of the resulting LP, one can take $v_{i,j} = x_i x_j$ and $w_{i,j} = y_i y_j$ to get a feasible solution of the above LP with same cost, and conversely for every feasible solution $x, y, u, v, w$ of the above LP, the solution $x, y, u$ is a feasible solution of the resulting LP with same cost. Also, the first (resp. sixth) set of constraints in the above LP can be obtained by summing the fourth and fifth (resp. seventh and eighth) set of constraints and can therefore be removed as well. We get the following equivalent LP relaxation.
of PDB that we refer to in this paper as the first level relaxation of the RLT hierarchy,

\[
\max_{x,y,u} \sum_{i=1}^{n} \theta_i \gamma_j u_{ii} \\
\sum_{j=1}^{n} \theta_j p_j u_{ji} \leq p y_i, \\
\sum_{j=1}^{n} \theta_i p_j (x_j - u_{ji}) \leq p(1 - y_i) \\
\sum_{j=1}^{n} \gamma_j q_j u_{ij} \leq q x_i, \\
\sum_{j=1}^{n} \gamma_j q_j (y_j - u_{ij}) \leq q(1 - x_i) \\
0 \leq u_{ij} \leq x_i \leq 1, \quad 0 \leq u_{ij} \leq y_j \leq 1.
\]

**(FL-RLT)**

**D Proof of Lemma 4.**

Let \( \omega^* \) be an optimal solution of \( Q^{LP}(\beta x) \), consider \( \tilde{\omega}_1, \ldots, \tilde{\omega}_m \) i.i.d. Bernoulli random variables such that \( P(\tilde{\omega}_i = 1) = \omega^*_i \) for all \( i \in [m] \) and let \( (h, z) \) such that \( h_i = \frac{\gamma_i \tilde{\omega}_i}{\beta} \) and \( z_i = \frac{\theta_i \tilde{\omega}_i}{\eta} \) for all \( i \in [m] \). We show that \( (h, z) \) satisfies the properties (5) with a constant probability. In particular, we have,

\[
P(\sum_{i=1}^{m} b_i z_i \not\leq d) = P\left( \sum_{i=1}^{m} \frac{\theta_i b_i \tilde{\omega}_i}{\eta} \not\leq d \right) \\
\leq \sum_{j=1}^{n} P\left( \sum_{i=1}^{m} \frac{\theta_i B_{ij}}{d_j} \tilde{\omega}_i > \eta \right) \\
= \sum_{j \in [n]: d_j > 0} P\left( \sum_{i=1}^{m} \frac{\theta_i B_{ij}}{d_j} \tilde{\omega}_i > \eta \right) \\
\leq \sum_{j \in [n]: d_j > 0} \left( e^{\eta - 1} / \eta^n \right) \\
\leq n e^{\eta - 1} / \eta^n,
\]

where the first inequality follows from a union bound on \( n \) constraints. The second equality holds because for all \( j \in [n] \) such that \( d_j = 0 \) we have

\[
P\left( \sum_{i=1}^{m} \frac{\theta_i B_{ij}}{d_j} \tilde{\omega}_i > d_j \right) = 0.
\]
Note that $d_j = 0$ implies
\[ \sum_{i=1}^{m} \theta_i B_{ij} \frac{\omega_i^*}{\eta} = 0, \]
by feasibility of $\omega^*$ in $Q^{\text{LP}}(\beta x)$. Therefore,
\[ \sum_{i=1}^{m} \theta_i B_{ij} \bar{\omega}_i = 0, \]
almost surely. The second inequality follows from the Chernoff bounds (a) with $\delta = \eta - 1$ and $s = 1$. In particular, $\frac{\theta_i B_{ij}}{d_j} \in [0, 1]$ by definition of $\theta_i$ for all $i \in [m]$ and $j \in [n]$ such that $d_j > 0$ and for all $j \in [n]$ such that $d_j > 0$ we have,
\[ E \left[ \sum_{i=1}^{m} \frac{\theta_i B_{ij}}{d_j} \bar{\omega}_i \right] = \sum_{i=1}^{m} \frac{\theta_i B_{ij}}{d_j} \omega_i^* \leq 1, \]
by feasibility of $\omega^*$. Next, note that
\[ \frac{e^{\eta - 1}}{\eta^\eta} = O(\frac{1}{n^2}). \]
Therefore, there exists a constant $c > 0$ such that,
\[ P\left( \sum_{i=1}^{m} b_i z_i \not\in d \right) \leq \frac{c}{n}. \quad (8) \]
By a similar argument there exists a constant $c' > 0$,
\[ P\left( \sum_{i=1}^{m} R_i h_i \not\in q \right) \leq \frac{c'}{L}. \quad (9) \]
Finally we have,
\[ P\left( \sum_{i=1}^{m} h_i z_i - (a_i^T x) z_i < \frac{1}{2\eta \beta} Q^{\text{LP}}(\beta x) \right) \]
\[ = P\left( \sum_{i=1}^{m} \frac{\theta_i \gamma_i \bar{\omega}_i^2}{\eta \beta} - \theta_i (a_i^T x) \frac{\bar{\omega}_i}{\eta} < \frac{1}{2\eta \beta} Q^{\text{LP}}(\beta x) \right) \]
\[ = P\left( \frac{1}{\eta \beta} \sum_{i=1}^{m} (\theta_i \gamma_i - \theta_i a_i^T (\beta x)) \bar{\omega}_i < \frac{1}{2\eta \beta} Q^{\text{LP}}(\beta x) \right). \]
Let $I$ denote the subset of indices $i \in [m]$ such that
\[ \theta_i \gamma_i - \theta_i a_i^T (\beta x) \geq 0. \]
Since $\omega^*$ is the optimal solution of the maximization problem $Q^{LP}(\beta x)$ we can suppose without loss of generality that $\omega^*_i = 0$ for all $i \notin \mathcal{I}$. In fact, the packing constraints of $Q^{LP}(\beta x)$ are down-closed (i.e., for every $\omega \leq \omega'$, if $\omega'$ is feasible then $\omega$ is also feasible), hence setting $\omega^*_i = 0$ for all $i \notin \mathcal{I}$ still gives a feasible solution of $Q^{LP}(\beta x)$ and can only increase the objective value. Hence, $\tilde{\omega}_i = 0$ almost surely for all $i \notin \mathcal{I}$ and we have,

$$
P \left( \sum_{i=1}^{m} h_i z_i - (a_i^T x) z_i < \frac{1}{2\eta\beta} Q^{LP}(\beta x) \right) = P \left( \frac{1}{\eta\beta} \sum_{i \in \mathcal{I}} (\theta_i \gamma_i - \theta_i a_i^T (\beta x)) \tilde{\omega}_i < \frac{1}{2} Q^{LP}(\beta x) \right) \leq e^{-\frac{1}{4}}, \tag{10}
$$

where the last inequality follows from Chernoff bounds (b) with $\delta = 1/2$. In particular we have for all $i \in \mathcal{I}$

$$
\frac{(\theta_i \gamma_i - \theta_i a_i^T (\beta x))}{Q^{LP}(\beta x)} \leq 1.
$$

This is because the unit vector $e_i$ is feasible for $Q^{LP}(\beta x)$ for all $i \in \mathcal{I}$ which implies

$$
(\theta_i \gamma_i - \theta_i a_i^T (\beta x)) \leq Q^{LP}(\beta x).
$$

We also have,

$$
E \left[ \sum_{i \in \mathcal{I}} \frac{(\theta_i \gamma_i - \theta_i a_i^T (\beta x))}{Q^{LP}(\beta x)} \tilde{\omega}_i \right] = 1.
$$

Combining inequalities (8), (9) and (10) we get that $(h, z)$ verifies the properties (5) with probability at least

$$
1 - \frac{c}{n} - \frac{c'}{L} - e^{-\frac{1}{4}},
$$

which is greater than a constant for $n$ and $L$ large enough. Which concludes the proof of the structural property. \qed

\section{On Assumption 1.}

Assumption 1 can be made without loss of generality. To show this, we construct for every instance $I$ of AR a new instance $\tilde{I}$ such that Assumption 1 holds under $\tilde{I}$ and the optimal value of $\tilde{I}$ is within a factor 2 of the value of $I$. This implies that our LP approximation from Section 4 under $\tilde{I}$ is an $O\left(\frac{\log n}{\log \log n} \frac{\log L}{\log \log L}\right)$ approximation for $I$. In particular, consider an instance $I$ of AR given by,

$$
z_I = \min_{x \in \mathcal{X}} c^T x + \max_{h \in \mathcal{H}} \min_{y \geq 0} \{ d^T y \mid Ax + By \geq h \},
$$

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To $I$ we associate the following modified instance $\tilde{I}$,

$$z_{\tilde{I}} = \min_{(x,y_0) \in \tilde{X}} (c^T d^T) \begin{pmatrix} x \\ y_0 \end{pmatrix} + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ \begin{pmatrix} d^T y \\ (A B) \begin{pmatrix} x \\ y_0 \end{pmatrix} \right\} + By \geq h \right\},$$

where

$$\tilde{X} = \left\{ (x,y_0) \left| \begin{array}{c} x \in \mathcal{X} \\ y_0 \geq 0 \\ Ax + By_0 \geq 0 \end{array} \right. \right\}.$$

Note that $\tilde{I}$ is indeed an instance of AR as the first and second-stage cost vectors $\begin{pmatrix} c \\ d \end{pmatrix}$ and $d$ and the second-stage matrix $B$ all have non-negative coefficients and the first stage feasible set $\tilde{X}$ is still a polyhedral cone. Assumption 1 is verified under $\tilde{I}$ by definition of $\tilde{X}$. We now show that $\tilde{I}$ gives a 2-approximation of $I$. In particular, we prove the following lemma,

**Lemma 6.** $z_I \leq z_{\tilde{I}} \leq 2z_I$

**Proof.** First of all, let $\tilde{x}^*, \tilde{y}_0^*$ be an optimal solution of $\tilde{I}$. For every $h \in \mathcal{U}$ we have,

$$d^T \tilde{y}_0^* + \min_{y \geq 0} \left\{ d^T y \left| A \tilde{x}^* + B(y + \tilde{y}_0^*) \geq h \right. \right\} = \min_{y \geq \tilde{y}_0} \left\{ d^T y \left| A \tilde{x}^* + B(y + \tilde{y}_0^*) \geq h \right. \right\} \geq \min_{y \geq 0} \left\{ d^T y \left| A \tilde{x}^* + By \geq h \right. \right\}.$$

Hence,

$$z_{\tilde{I}} = c^T \tilde{x}^* + d^T \tilde{y}_0^* + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ d^T y \left| A \tilde{x}^* + B(y + \tilde{y}_0^*) \geq h \right. \right\} \geq c^T \tilde{x}^* + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ d^T y \left| A \tilde{x}^* + By \geq h \right. \right\} \geq z_I.$$

where the last inequality follows by feasibility of $\tilde{x}^*$ for $I$. For the inverse inequality, let $x^*$ be an optimal solution of $I$, and let $y(0) \in \arg\min_{y \geq 0} \left\{ d^T y \left| Ax^* + By \geq 0 \right. \right\}$. We have,

$$z_I = c^T x^* + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ d^T y \left| Ax^* + By \geq h \right. \right\} \geq c^T x^* + \frac{1}{2} \left( d^T y(0) + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ d^T y \left| Ax^* + By \geq h \right. \right\} \right) \geq c^T x^* + \frac{1}{2} \left( d^T y(0) + \max_{h \in \mathcal{U}} \min_{y \geq 0} \left\{ d^T y \left| Ax^* + By + By(0) \geq h \right. \right\} \right) \geq \frac{1}{2} z_{\tilde{I}}.$$

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where the first inequality follows from the fact that 0 is a feasible scenario of the uncertainty set and the last inequality follows by feasibility of \((x^*, y(0))\) for \(\tilde{I}\) and because \(c^T x^* \geq 0\).

Remark 1. Note that for every feasible affine policy of \(\tilde{I}\) given by the first stage solution \((\tilde{x}, \tilde{y}_0)\) and the second-stage affine function \(\tilde{y}_{AFF}\), the affine policy given by the first stage solution \(\tilde{x}\) and the second-stage affine function \(y_{AFF} = \tilde{y}_{AFF} + \tilde{y}_0\) is a feasible affine policy of \(I\) with same worst-case cost. This implies that the results of Section 5 also hold in the general case. In particular, the affine policies constructed in Section 5 under \(\tilde{I}\) can be used to construct affine policies for \(I\) that are an \(O\left(\frac{\log n}{\log \log n} \frac{\log L}{\log \log L}\right)\) approximation for \(I\).

An example of an application of the two-stage problem where Assumption 1 does not hold is the following two-stage network design problem: consider a directed graph \(D(V, A)\) where each arc \(a \in A\) is associated with a first-stage cost \(c_a \geq 0\) and each node \(v \in V\) is associated with a second stage cost \(d_v \geq 0\) and receives an uncertain demand \(h_v \geq 0\). The decision maker chooses a flow vector \(f \in \mathbb{R}^A_+\) where \(f_{vw}\) represents the quantity of supply to node \(w\) coming from node \(v\) and incurs the first stage cost \(\sum_{vw} c_{vw} f_{vw}\), the demands are then revealed and the decision maker incurs a second-stage cost \(d_v \cdot (h_v + \sum_{wvw \in A} f_{vw} - \sum_{wvw' \in A} f_{vw'})^+\) for each node \(v\) where the flow balance \(\sum_{wvw \in A} f_{vw} - \sum_{wvw' \in A} f_{vw'}\) is lower than the demand \(h_v\). This example can be modeled as an instance of \(AR\) where \(B = I\) and \(A\) is the incidence matrix of the considered directed graph. The flow vector \(f\) such that \(f_{vw} = 1\) for some arc \(vw\) and \(f_{v'w'} = 0\) for every other arc \(v'w'\) is such that \((Af)_v = -1 < 0\).

F Derivation of the LP formulation corresponding to the generalization of Algorithm 2 of El Housni and Goyal [19] to packing uncertainty sets.

When the second-stage variable \(y\) is restricted to affine policies of the form \(y(h) = \sum_i \nu_i v_i h_i + q\), for some \(\nu_1, \ldots, \nu_m \in \mathbb{R}\) and \(q \in \mathbb{R}^n\), the two-stage problem \(AR\) becomes,

\[
\begin{align*}
\min \quad & c^T x + z \\
\quad & z \geq d^T (\sum_i \nu_i v_i h_i + q), \quad \forall h \in U \\
\quad & Ax + B (\sum_i \nu_i v_i h_i + q) \geq h, \quad \forall h \in U \\
\quad & \sum_i \nu_i v_i h_i + q \geq 0, \quad \forall h \in U \\
\quad & x \in \mathcal{X}, \quad q \in \mathbb{R}^n, \quad \nu_i \in \mathbb{R}, \quad z \in \mathbb{R},
\end{align*}
\]
We use standard duality techniques to derive formulation EG. The first constraint is equivalent to
\[ z - d^T q \geq \max_{R^T v \geq (Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d} \min_{v \geq 0} r^T v. \]
By taking the dual of the maximization problem, the constraint is equivalent to
\[ z - d^T q \geq r^T v \]
\[ R^T v \geq (Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d \]
\[ v \in R^L_+. \]
Where \( Y := [v_1, \ldots, v_m] \). We then drop the min and introduce \( v \) as a variable to obtain the following linear constraints,
\[ z - d^T q \geq r^T v \]
\[ R^T v \geq (Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d \]
\[ v \in R^L_+. \]
We use the same technique for the second sets of constraints, i.e.,
\[ A x + B q \geq \max_{R^T v \geq (I_m - B Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d} \min_{v \geq 0} r^T v. \]
By taking the dual of the maximization problem for each row and dropping the min we get the following formulation of these constraints
\[ A x + B q \geq V^T r \]
\[ V^T R \geq I_m - B Y \cdot \text{diag}(\nu_1, \ldots, \nu_m) \]
\[ V \in R^{L \times m}_+. \]
Similarly, the last constraint
\[ q \geq \max_{R^T v \geq (I_m - B Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d} \min_{v \geq 0} r^T v. \]
is equivalent to
\[ q \geq U^T r \]
\[ U^T R + Y \cdot \text{diag}(\nu_1, \ldots, \nu_m) \geq 0 \]
\[ U \in R^{L \times n}_+. \]
Putting all together, we get the following formulation,

\[
z_{EG} = \min c^T x + z \\
\quad \quad \quad \quad \quad z - d^T q \geq r^T v \\
\quad \quad \quad \quad \quad R^T v \geq (Y \cdot \text{diag}(\nu_1, \ldots, \nu_m))^T d \\
\quad \quad \quad \quad \quad Ax + Bq \geq V^T r \\
\quad \quad \quad \quad \quad V^T R \geq I_m - BY \cdot \text{diag}(\nu_1, \ldots, \nu_m) \tag{EG} \\
\quad \quad \quad \quad \quad q \geq U^T r \\
\quad \quad \quad \quad \quad U^T R + Y \cdot \text{diag}(\nu_1, \ldots, \nu_m) \geq 0 \\
\quad \quad \quad \quad \quad x \in \mathcal{X}, \quad v \in \mathbb{R}^L_+, \quad U \in \mathbb{R}^{L \times n}_+, \quad V \in \mathbb{R}^{L \times m}_+ \\
\quad \quad \quad \quad \quad q \in \mathbb{R}^n, \quad \nu_1, \ldots, \nu_m \in \mathbb{R}, \quad z \in \mathbb{R},
\]

\[\square\]