Expulsion from structurally balanced paradise

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We perform simulations of structural balance evolution on a triangular lattice using the heat-bath algorithm. In contrast to similar approaches — but applied to analysis of complete graphs — the triangular lattice topology successfully prevents the occurrence of even partial Heider’s balance. Starting with the state of Heider’s paradise, it is just a matter of time when the evolution of the system leads to an unbalanced and disordered state. The time of the system relaxation does not depend on the system size. The lack of any signs of balanced state was not observed in earlier investigated systems dealing with structural balance.

Keywords: Heider balance; heat-bath algorithm; zero critical temperature

INTRODUCTION

The structural balance [1] (also termed as the Heider balance [2]) has been attracting attention of physicists for at least the past fifteen years. The considered topologies include triangular lattices [3], complete graphs [4–9], and complex networks [10–13]; for review, see Refs. 14 and 15. In all those cases, the model describes dynamics of negative or positive links which represent hostile or friendly attitudes among actors decorating nodes of an underlying network. The available scenarios of actors’ attitudes in a single triad are presented in Figure 1. The triads shown in Figures 1a and 1c are balanced in Heider sense as they obey the following rules:

• friend of my friend is my friend,
• friend of my enemy is my enemy,
• enemy of my friend is my enemy,
• and enemy of my enemy is my friend.

![Figure 1: Heider’s triads corresponding to balanced (the first and the third from the left) and imbalanced (the second and the fourth) states. Continuous blue lines and dashed red lines represent friendly and hostile relations, respectively.](image)

The configurations presented in Figures 1b and 1d are imbalanced (i.e., do not obey the rules given above) and result in actors’ feelings of discomfort known as cognitive dissonance [16]. In order to relieve this tension actors should change their attitudes to others by switching unfriendly or amicable relations into opposite ones. Such process may be realized for two connected triads using the links dynamics given by

\[ x_{ij}(t+1) = \text{sign} \left[ x_{im}(t)x_{jm}(t) + x_{in}(t)x_{jn}(t) \right], \] (1)

where \( x_{ab} \) symbolizes friendly (\( x_{ab} = +1 \)) or hostile (\( x_{ab} = -1 \)) relation among the actors \( a \) and \( b \) (see Figure 2 for two examples of system evolution towards the Heider balance, from an imbalanced state at time \( t \) to a balanced state at \( t+1 \) after single link flip \( x_{ij}(t+1) = -x_{ij}(t) \)).

![Figure 2: Examples of configuration of signed links \( x_{im}, x_{in}, x_{jm}, x_{jn} \) which influence the link \( x_{ij} \) in the next time step according to Equation (1) in the deterministic case, i.e. for \( T = 0 \).](image)

We note that when the system of \( N \) links is composed only of balanced triads (presented in Figures 1a and 1c) then the system energy defined as

\[ U = -\sum_{i,j,k} x_{ij} x_{jk} x_{ki} \] (2)
is exactly equal to $U = -1$, which allows for easy detection of system balance without the triad-by-triad inspection.

Very recently, the deterministic evolution according to Equation (1) was enriched by introducing the thermal noise simulated by Glauber [7] or heat-bath [8, 9] dynamics. In the latter case, the first order phase transition from (at least partially) balanced (with $-1 \geq U \geq U^* > 0$ [8]) and ordered (+1 $\geq \langle x_{ij} \rangle \geq x^* > 0$ [9]) state to the imbalanced (with energy $U = 0$ [8]) and disordered (with spatially averaged value of links strength $\langle x_{ij} \rangle = 0$ [9]) state was observed for a complete graph. The transition takes place at the critical temperature $T^*$; for $T < T^*$ ordered and balanced system states are observed, while for $T > T^*$ the time evolution drives the system to imbalanced and disordered states.

In this letter we show that keeping the thermal noise, but using the triangular lattice topology instead of a complete graph results in total vanishing of the ordered and balanced phases in the system.

METHODS

The non-deterministic version of Equation (1) reflecting the presence of thermal noise and using the heat-bath method [17] may be written as

$$x_{ij}(t+1) = \begin{cases} +1 & \text{with probability } p, \\ -1 & \text{with probability } (1-p), \end{cases}$$

where $$p = \frac{\exp(c)}{\exp(c) + \exp(-c)}$$

and $$c = \frac{x_{im}(t)x_{jn}(t) + x_{in}(t)x_{jm}(t)}{T},$$

while $m$ and $n$ are the common neighbours of nodes $i$ and $j$ (see Fig. 2), and $T$ stands for temperature.

We apply Equation (3) to find the time-evolution of $N = 3L^2$ links on a triangular lattice of $L^2$ nodes assuming the periodic boundary conditions. A single Monte Carlo time step takes $3L^2$ attempts to modification of $x_{ij}$ performed synchronously on the whole system.

RESULTS

In Figure 3 we present the time evolution of the average value of links strengths $\langle x_{ij} \rangle$ and the system energy $U$ for various values of the social temperate $T$ and the system size $L$. The angle brackets $\langle \cdot \cdot \cdot \rangle$ indicate the averaging procedure performed over all $3L^2$ links, and all the results are also averaged over one hundred simulations. Initially, the Heider paradise is assumed, i.e., at $t = 0$ every link is set to $x_{ij} = +1$. Neither the system size $L$, nor the assumed temperature $T$ prevents reaching the unbalanced and disordered state with $\langle x_{ij} \rangle = 0$ and $U = 0$. However, the time between the start of simulation and the first flip of any link value from $+1$ to $-1$ is equal to the reciprocal of probability that one link breaks out of the paradise state and changes the energy by $\Delta U = 4$.

Such probability is proportional to the number of links as the changes may happen independently and at any place, which means that

$$\tau_0 = \frac{1 + \exp(4/T)}{3L^2}$$

and tends to infinity in the limit of very low temperatures, $T \to 0^+$, as shown in Figure 4 where both the above dependence (lines) and the results of simulation (points) are presented. Thus for $T = 0.1$ and $L = 100$ no link switching from $+1$ to $-1$ was observed up to $t = 10^8$ (i.e. until $Nt = 3 \cdot 10^{12}$ trials). To analyze the system evolution in the low temperature limit we repeat our simulations with the initial state where $5\%$ of the links are set to $x_{ij} = -1$. For such starting point also systems kept in low temperature ($T = 0.1$) reach unbalanced $U = 0$ and disordered $\langle x_{ij} \rangle = 0$ state (see Figure 5).

We assume that the system reaches the stationary state when $U$ and $\langle |x_{ij}| \rangle$ are smaller than $\varepsilon = 10^{-2}$. The time $\tau$ needed for reaching the stationary state increases with decreasing the temperature $T$ independently on the assumed system size $L$ (see upper part of Figure 4).
Figure 4: (a) Time $\tau$ of reaching the stationary state and (b) time $\tau_0$ of the first link switching from $x_{ij} = +1$ to $x_{ij} = -1$. The results are averaged over $R = 100$ simulations. The error bars are smaller than size of the symbols and they are of the order of $1/\sqrt{R}$. Lines indicate the theoretical dependence $\tau_0 = [1 + \exp(4/T)]/(3L^2)$.

deviations from this statement are observed only for very small system sizes ($L < 50$) and only in low temperatures ($T < 0.5$).

To confirm the system transition to a disordered phase we look into the time evolution (see Figure 6) and final probability distribution of triads presented in Figure 1. The obtained probabilities are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$ for triads presented in Figures 1a to 1d, respectively. The results prove to be independent on system size $L$ and temperature $T$, and we will discuss this issue shortly in Conclusions.

CONCLUSIONS

In contrast to the stochastic evolution of the system with hostile and friendly attitudes among actors on a complete graph [9] we show that the triangular lattice topology successfully prevents the occurrence of even partial Heider balance.

Starting at the state of paradise ($\forall i, j : x_{ij}(t = 0) = +1$) it is just a matter of time $\tau$, when the thermally driven evolution (governed by the heat-bath algorithm) of the system leads to an unbalanced and disordered state. In contrast to the previously obtained results [18, 19] the relaxation time $\tau$ does not depend on the system size.

The probabilities distribution of various triads presented in Figures 1a to 1d changes from $(1, 0, 0, 0)$ at $t = 0$ to $(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$ at $t \to \infty$. The latter is in agreement with probabilities

$$\binom{3}{k} p^k (1-p)^{3-k}$$

of $k = 3, 2, 1,$ and 0 successes in three Bernoulli trials.
when the probability of success is equal to $p = \frac{1}{2}$. The result accentuates the randomness of the final link values distribution and emphasizes the system disorder for any positive temperature.

In summary, we found that introducing noise to the system of hostile and friendly attitudes on triangular lattice leads to the disordered and imbalanced state independently on the system size and for any positive temperature. In other words, the critical temperature of the system is zero. None of the earlier studies devoted to the problem of structural balance, and conducted for various topologies and different schemes of updating the link values, have shown complete lack of any signs of balanced state.

Our results prove that at least in certain cases the behavior of the system does not have to follow what was so far believed to be a general tendency towards a global structural balance. Signatures of such possibility were already provided, for instance in studies of bilayer networks [13].

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