On Classical Rational $\mathcal{W}$ Algebras

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Abstract

The structure of classical non-linear $\mathcal{W}$ algebras closing on rational functions is analyzed both for the ordinary and the supersymmetric case. Such algebras appear as a result of a coset construction. Their relevance to physical applications is pointed out.
1 Introduction

The purpose of this talk is to illustrate the basic properties of the so-called classical rational \( \mathcal{W} \) algebras which have been introduced in [1] and further developed in [2].

The name of rational \( \mathcal{W} \) algebra is deserved to a new class of \( \mathcal{W} \) algebras, involving a finite set of generators, among them a stress-energy tensor, while the other fields are primary or quasi-primary with respect to the latter. The algebra is introduced through a Poisson brackets structure among the generators: it satisfies the standard properties of antisymmetry, linearity and Jacobi identities. While standard Kac-Moody algebras close in a linear way, and standard (polynomial) \( \mathcal{W} \) algebras, non-linearly but in terms of polynomials in the generators and their derivatives, rational \( \mathcal{W} \) algebras have Poisson brackets of the following kind

\[
\{ W_i(z), W_j(w) \} = \sum_{k=1}^{N} F_k(W_r(w), \partial^p W_r(w)) \delta^{(k)}(z - w) \tag{1.1}
\]

where \( F_k \) are rational functions (quotient of homogeneous polynomials) in the generators \( W_r \) and their derivatives.

Algebras satisfying the above property appear when considering coset constructions, i.e. the factorization of a spin 1-fields Kac-Moody subalgebra out of a given algebra. Such statement will provide the spin 1 extension of the factorization theorem concerning spin \( \frac{1}{2} \) fields due to Goddard and Schwimmer (see [3]).

Constructions different from ours, leading to algebras satisfying the above (1.1) property have been discussed in [4]. Their quantum extension have been analyzed in [5].

In this talk I will review the basic properties of bosonic and supersymmetric rational \( \mathcal{W} \) algebras. At the end I will discuss their relevance in some applications, mainly concerning the hierarchy of integrable equations.

2 Bosonic Rational \( \mathcal{W} \) Algebras

In this section the construction of rational \( \mathcal{W} \) algebras will be reviewed. For simplicity only the abelian (\( \hat{U}(1) \)) quotients will be considered.

Let \( \hat{\mathcal{G}} \) or \( \mathcal{W} \) denote respectively a Kac-Moody or a \( \mathcal{W} \) algebra admitting a subalgebra generated by a \( \hat{U}(1) \) Kac-Moody current \( J(z) \):

\[
\{ J(z), J(w) \} = \gamma \delta'(z - w) \equiv \gamma \partial_w \delta(z - w) \tag{2.1}
\]

(in the classical case the normalization factor \( \gamma \) can be fixed without loss of generality as will be done in the following). It is possible to express any other element of the algebra in terms of a basis of fields \( W_{i,q_i} \) having a definite charge \( q_i \) with respect to \( J(z) \), namely satisfying:

\[
\{ J(z), W_{i,q_i}(w) \} = q_i W_{i,q_i}(w) \delta(z - w) \tag{2.2}
\]

A derivative \( \mathcal{D} \), covariant with respect to the above relations, can be introduced (see also [6]):

\[
\mathcal{D} W_{i,q_i}(w) = (\partial - \frac{q_i}{\gamma} J(w)) W_{i,q_i}(w) \tag{2.3}
\]
The elements in $Com_J(G, W)$ form the subalgebra of the enveloping algebra commuting with $J(w)$; they are spanned by the vanishing total charge monomials $(D^{n_1}V_{1,q_1})(D^{n_2}V_{2,q_2})\ldots(D^{n_j}V_{j,q_j})$, where the $n_i$'s are non negative integers and the total charge is $q = q_1 + q_2 + \ldots q_j = 0$.

From now on we will concentrate only on invariants produced by bilinear combinations such as

$$D^pV_+ \cdot D^qV_-$$

(with $p, q \geq 0$ and $V_\pm$ have opposite charges), together with of course originally invariant fields. This is still a closed algebra. Let me summarize the basic results of [1], with some extra comments:

i) there exists a linear basis of fields, given by $V^{(p)} = D^pV_+ \cdot V_-$ such that any bilinear invariants of the kind (2.4) is a linear combination of the $V^{(p)}$'s and the derivatives acting on them.

ii) the Poisson brackets algebra of the fields $V^{(p)}$'s among themselves is closed (possibly with the addition of other invariants, in the general case), but never in a finite way (the Poisson brackets of $V^{(p)}$ with $V^{(q)}$ necessarily generates on the right hand side terms depending on $V^{(p')}$, with $p' > p, q$). Moreover it can be explicitly checked that it is a non-linear algebra, so that it has the structure of a non-linear $W_\infty$ algebra.

iii) due to the properties of the covariant derivative, the fields $V^{(p)}$, which are linearly independent, satisfy algebraic relations like the following quadratic ones

$$V^{(p+1)} \cdot V^{(0)} = V^{(0)} \cdot \partial V^{(p)} + V^{(p)} \cdot (V^{(1)} - \partial V^{(0)})$$

Such relations allow to express algebraically the fields $V^{(p)}$, for $p \geq 2$ in terms of the fundamental fields $V^{(0)}$ and $V^{(1)}$. The above derived non-linear $W_\infty$ algebra has therefore the structure of a rational $W$ algebra. Notice that relations like (2.5) contain no informations if the fields $V_\pm$ are fermionics. (2.5) can still be applied to superalgebras if $V_\pm$ are bosonic superfields.

iv) if $Com_J(G, W)$ contains a field $T(w)$ (the stress-energy tensor) whose Poisson brackets are the Virasoro algebra with non-vanishing central charge, and moreover $V^{(0)}$ is primary with conformal dimension $h$, then there exists a one-to-one correspondence between the fields $V^{(p)}$ of the basis, and an infinite tower of uniquely determined fields $W_{h+p}$, primary with respect to $T$, with conformal dimension $h + p$. This relation should be understood as follows: $V^{(p)}$ is the leading term in the associated primary field. The remaining terms are fixed without ambiguity, some of them just requiring $W_{h+p}$ being primary, some others once a specific scheme to determine them is adopted (as an analogy, one should think to the choice of the renormalization scheme when dealing with renormalizable quantum field theories).

As we will see, the condition of having a non-vanishing central charge drops for the $sl(2)/U(1)$ coset model: therefore no infinite tower of primary fields associated to each $V^{(p)}$ can be generated (we have an infinite tower of “almost” primary fields associated to them). An infinite number of primary fields can still be produced, but they are of a

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1 It is better conceptually to understand the derivative operator $\partial = \frac{d}{dw}$ as an element of the original algebra, as this is the case for Kac-Moody algebras.
trivial type, being just products of lower order primary fields. Anyway the structure of a rational algebra with two primary fields is maintained even in this case. The next simplest model admitting an infinite tower of invariant primary fields associated to $V^{(p)}$ is based on the coset $\frac{sl(2)_2 \times U(1)}{U(1)}$, since now there exists another $U(1)$ current, commuting with $J(w)$, which allows to define an invariant stress-energy tensor with non-vanishing charge.

To be definite let us discuss here the simplest example, given by the coset $\frac{sl(2)}{U(1)}$.

The classical $sl(2)$ algebra is given by the following Poisson brackets:

\[
\{ J_+(z), J_-(w) \} = \delta' (z - w) - 2J_0(w)\delta(z - w) \equiv D(z)\delta(z - w) \\
\{ J_0(z), J_+(w) \} = \pm J_+(w)\delta(z - w) \\
\{ J_0(z), J_0(w) \} = -\frac{1}{2}\delta'(z - w) \\
\{ J_\pm(z), J_\pm(w) \} = 0 \tag{2.6}
\]

$J_\pm$ play the role here of the fields $V_\pm$; they have conformal dimension 1 (here and in the following, the symbol $\delta'(z - w)$ is understood as $\partial_w \delta(z - w)$).

The rational coset algebra of the commutant with respect to the $J_0$ current is given by the following Poisson brackets:

\[
\{ W_2(z), W_2(w) \} = 2W_2(w)\delta'(z - w) + \partial W_2(w)\delta(z - w) \\
\{ W_2(z), W_3(w) \} = 3W_3(w)\delta'(z - w) + \partial W_3(w)\delta(z - w) \\
\{ W_3(z), W_3(w) \} = 2W_2(w)\delta''(z - w) + 3\partial W_2(w)\delta(z - w)'' + [16V^{(2)} - 8\partial W_3 + 8W_2^2 - 3\partial^2 W_2](w)\delta'(z - w) + \partial_w [8V^{(2)} - 4\partial W_3 + 4W_2^2 - 2\partial^2 W_2](w)\delta(z - w) \tag{2.7}
\]

We have preferred to express the above algebra in the basis of (uniquely determined) primary fields $W_2 = J_+ \cdot J_-$ and $W_3 = D J_+ \cdot J_+ - J_+ \cdot D J_-$. They have dimension 2, 3 respectively, while

\[
V^{(2)} = D^2 J_+ \cdot J_- = \frac{1}{4W_2}[W_3^2 + 2W_2\partial W_3 + 2W_2\partial^2 W_2 - \partial W_2\partial W_2]. \tag{2.8}
\]

The second equality follows from the relation (2.3).

$W_2$ plays the role of a stress-energy tensor having no central charge. As already stated, in this simple example there exists no infinite tower of primary fields associated to the $V^{(p)}$'s fields, the only primary ones being $W_{2,3}$ and their products $W_m W_3^n$ for $m, n$ non-negative integers.

The algebra of the fields $V^{(p)} = D^p J_+ \cdot J_-$ is a non-linear $\mathcal{W}_\infty$ algebra: if we let from the very beginning identify $J_0 \equiv 0$, then $J_\pm$ can be identified with the fields $\partial \beta$ and $\gamma$ of a bosonic $\beta - \gamma$ system, the covariant derivative in $V^{(p)}$ must be replaced by the ordinary derivative and the non-linear $\mathcal{W}_\infty$ algebra is reduced to the standard linear $w_\infty$ algebra.

3 The $N = 1$ supersymmetric case

In this section I will extend the definition of rational $\mathcal{W}$ algebras to the $N = 1$ supersymmetric case, which presents as already pointed out some peculiarities with respect to
the bosonic case. Examples concerning the supersymmetric case can be found also in [7]. For simplicity the discussion will be limited to the abelian coset.

The $N = 1$ superspace is introduced through the supercoordinate $X \equiv x, \theta$, with $x$ and $\theta$ real, respectively bosonic and grassmann, variables. The supersymmetric spinor derivative is given by

$$D \equiv D_X = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \quad (3.9)$$

(therefore $D_X^2 = \frac{\partial}{\partial x}$).

The supersymmetric delta-function $\Delta(X, Y)$ is a fermionic object

$$\Delta(X, Y) = \delta(x - y)(\theta - \eta) \quad (3.10)$$

In order to produce covariant derivatives we have therefore to look for a fermionic spin $\frac{1}{2}$ superfield which is the $N = 1$ counterpart of the $U(1) - \mathcal{K}\mathcal{M}$ current $J_0(z)$. It can be introduced as $\Psi_0(X) = \psi_0(x) + \theta J_0(x)$, satisfying the super-Poisson brackets relation\footnote{we recall that super-Poisson brackets are symmetric when taken between odd elements, antisymmetric otherwise.}

$$\{\Psi_0(X), \Psi_0(Y)\} = D_Y \Delta(X, Y) \quad (3.11)$$

which implies, at the level of components

$$\{\psi_0(x), \psi_0(y)\} = -\delta(x - y)$$
$$\{J_0(x), J_0(y)\} = -\partial_0 \delta(x - y) \quad (3.12)$$

Super-covariant fields and the supercovariant derivative can now be defined through the relations

$$\{\Psi_0(X), \Phi_q(Y)\} = q \Delta(X, Y) \Phi_q(Y)$$
$$\mathcal{D} \Phi_q = D \Phi_q + q \Psi_0 \Phi_q \quad (3.13)$$

$\Phi_q$ is a covariant superfield (either bosonic or fermionic).

Let us specialize now our discussion to the simplest example of supersymmetric algebra giving rise to a rational coset: it is introduced in terms of two spin $\frac{1}{2}$ superfields $\Psi_{\pm}$ superfields which can be assumed to be either fermionic or bosonic (fermionic in the following discussion). The Poisson brackets are given by the relations

$$\{\Psi_0(X), \Psi_{\pm}(Y)\} = \pm \Delta(X, Y) \Psi_{\pm}(Y)$$
$$\{\Psi_{\pm}(X), \Psi_{\mp}(Y)\} = \mathcal{D}_Y \Delta(X, Y) = D_Y \Delta(X, Y) + \Delta(X, Y) \Psi_0(Y)$$
$$\{\Psi_{\pm}(X), \Psi_{\pm}(Y)\} = 0 \quad (3.14)$$

The linear generators of the commutants are the composite superfields $V_n(X)$,

$$V_n = \Psi_{-} \mathcal{D}^n \Psi_{+} \quad n = 0, 1, 2, ... \quad (3.15)$$
which have vanishing Poisson brackets with respect to $\Psi_0$:

$$\{\Psi_0(X), V_n(Y)\} = 0$$  \hfill (3.16)

The superfields $V_n$ are respectively bosonic for even values of $n$ and fermionic for odd values. The set $\{V_0, V_1\}$ constitutes a finite super-algebra, given by the Poisson brackets

$$\begin{align*}
\{V_0(X), V_0(Y)\} &= -\Delta(X,Y)(DV_0 + 2V_1)(Y) \\
\{V_0(X), V_1(Y)\} &= \Delta^{(2)}(X,Y)V_0(Y) + \Delta^{(1)}(X,Y)V_1(Y) - \Delta(X,Y)DV_1(Y) \\
\{V_1(X), V_1(Y)\} &= -2\Delta^{(2)}(X,Y)V_1(Y) - \Delta(X,Y)Dy^2V_1(Y)
\end{align*}$$

(3.17)

In terms of component fields it is given by two bosons of spin 1 and 2 respectively, and two spin $\frac{3}{2}$ fermions. It is the maximal finite subalgebra of the coset superalgebra: as soon as any other superfield is added to $V_0, V_1$, the whole set of fields $V_n$ is needed to close the algebra, giving to the coset the structure of a super-$W_\infty$ algebra. Moreover such algebra closes in non-linear way. Such a superalgebra is associated to the $N=1$ supersymmetric non-linear Schrödinger equation.

Let us make now some comments concerning the rational character of the above defined super-$W_\infty$ algebra: the whole set of algebraic relations can be expressed just in terms of closed rational super-$W$ algebra involving 4 superfields as the following reasoning shows: let us introduce the superfields

$$\Lambda_p = \text{def} \ D\Psi_\cdot D^{(p+1)}\Psi_+$$

then

$$\Lambda_p = DV_{p+1} - V_{p+2}$$

Due to standard properties of the covariant derivative we can write down for the superfields $\Lambda_p$ the analogue of the relation (2.5) of the bosonic case:

$$\Lambda_0\Lambda_{p+1} = \Lambda_0D\Lambda_p + (\Lambda_1 - D\Lambda_0)\Lambda_p$$  \hfill (3.18)

which implies that $\Lambda_p$ are rational functions of $\Lambda_{0,1}$, which in their turns are determined by $V_i$, $i = 0, 1, 2, 3$.

Inverting the relation (3.18) we can express any higher field $V_{p+1}$ in terms of $V_p, \Lambda_{p-1}$. As a consequence of this we have the (rational) closure of the superalgebra on the superfields $V_0, V_1, V_2, V_3$.

We remark that now it is not possible, like in the bosonic case, to determine higher order superfields $V_p$ from the formula (3.18) by simply inserting $V_0, V_p$ in place of $\Lambda_0, \Lambda_p$: this is due to the fact that any product $V_0 \cdot V_{p+1}$ identically vanishes since it is proportional to a squared fermion ($\Psi_-^2 = 0$). That is the reason why four superfields are necessary to produce a finite rational algebra and not just two as one would have naively expected.
Conclusions

In this talk I have analyzed the mathematical structure of rational $\mathcal{W}$ algebras, both for the bosonic and the supersymmetric case, and I have provided a general framework to construct such kind of algebras.

In this conclusion I feel necessary to provide some insights about why such structures are relevant for physical applications. The first problems which could be mentioned are those, like the quantum Hall effect or the black hole problem which involve a $\mathcal{W}_\infty$ algebra. Even if rational algebras have not been explicitly exploited in such problems, it is very likely they play a role. This is certainly sure for the black hole problem, where the appearance of a rational $\mathcal{W}$ algebra is a well-established fact. There is however another field of applications of rational $\mathcal{W}$ algebras which is particularly interesting and well understood now. It concerns the hierarchies of integrable equations and their relations to matrix models.

Such hierarchies can be regarded as consistent reductions of KP and super-KP flows. In the physical literature appeared recently non-standard reductions of the KP-hierarchy, called multi-field reductions, which have different properties with respect to the standard Drinfeld-Sokolov type of reductions. In particular they do not lead to a purely differential Lax operator. It turns out that, while standard Drinfeld-Sokolov reductions are related to polynomial $\mathcal{W}$ algebras, multifield reductions are related to rational (coset) $\mathcal{W}$ algebras. Better stated, there exists a Poisson brackets structure for the reduced KP hierarchy, such that the hamiltonian densities of the infinite tower of hamiltonians in involution belong all to the coset algebra and therefore generate a rational $\mathcal{W}$ algebra.

The simplest example of such kind hierarchy is given by the Non-Linear-Schrödinger equation, which is associated to the previously analyzed $\hat{\mathfrak{sl}}(2)/\hat{\mathfrak{u}}(1)$ coset. The supersymmetric rational $\mathcal{W}$ algebra discussed above is related to the $\mathcal{N} = 1$ extension of the above equation. More complicated cosets provide new integrable hierarchies and KP reductions.

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