ON THE GIESEKER HARDER-NARASIMHAN FILTRATION FOR PRINCIPAL BUNDLES

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Abstract. We give an example of an orthogonal bundle where the Harder-Narasimhan filtration, with respect to Gieseker semistability, of its underlying vector bundle does not correspond to any parabolic reduction of the orthogonal bundle. A similar example is given for the symplectic case.

1. Introduction

Mumford and Takemoto (c.f. [GIT] and [Ta]) constructed a moduli space for semistable holomorphic vector bundles over Riemann surfaces, by defining a vector bundle $E$ to be semistable if for every non trivial proper subbundle $F \subset E$ we have
\[
\frac{\deg F}{\text{rk} F} \leq \frac{\deg E}{\text{rk} E}.
\]
Later on, Gieseker and Maruyama (c.f. [Gi] and [Ma]) extended the construction to higher dimensional varieties, by modifying the definition of stability and using Hilbert polynomials instead of degrees.

In [HN], Harder and Narasimhan proved that a holomorphic vector bundle over a Riemann surface which is not stable admits a unique canonical filtration
\[
0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E,
\]
which verifies that the quotients $E^i := E_i/E_{i-1}$ are semistable and the slopes of the quotients are decreasing:
\[
\frac{\deg E^1}{\text{rk} E^1} > \frac{\deg E^2}{\text{rk} E^2} > \cdots > \frac{\deg E^{t+1}}{\text{rk} E^{t+1}}.
\]
This fact can be extended to torsion free sheaves over projective varieties by using Gieseker stability (c.f. [Gi] and [HL, Theorem 1.3.6]).

Ramanathan, in [Ra], constructed a moduli space for semistable principal $G$-bundle over Riemann surfaces, where $G$ is a connected reductive complex algebraic group, by declaring

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that a $G$-bundle $E \to X$ is semistable if for any reduction of structure group $\sigma : X \to E/P$ to any maximal parabolic subgroup $P \subset G$, we have

$$\deg \sigma^* (T_{(E/P)/X}) \geq 0$$

where $T_{(E/P)/X} \to E/P$ is the tangent bundle along the fibers of the projection $E/P \to X$.

To generalize Ramanathan’s construction to higher dimension, a Gieseker-like definition of stability has to be given for principal bundles. It turns out that this definition depends on the choice of a representation $\rho : G \to \text{SL}(V)$. For the classical groups, we can choose the standard representation (c.f. [GS1]). For a connected reductive group, we can take the adjoint representation into the semisimple part of the Lie algebra (c.f. [GS2, GS3]), and if $G$ is semisimple we can take any faithful representation (c.f. [Sc1, Sc2]). See [GLSS1] and [GLSS2] for other cases.

A one parameter subgroup $\lambda : \mathbb{C}^* \to G$ of $G$ defines a parabolic subgroup

$$P(\lambda) = \{ g \in G \mid \lim_{z \to \infty} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \}$$

We remark that all parabolic subgroups are of this form.

Using the representation $\rho : G \to \text{SL}(V)$ chosen above, the one parameter subgroup $\lambda$ gives a filtration of $V$ as follows. Let

$$\gamma_1 < \gamma_2 < \cdots < \gamma_l$$

be the different weights of the action of $\lambda$ on $V$. Let $V^i$ be the subspace where $\lambda$ acts as $t^{\gamma_i}$, and let $V_i = \bigoplus_{j=1}^i$. Therefore, we obtain a filtration

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_l = V \tag{1.1}$$

This filtration is preserved by the action of the parabolic group $P(\lambda)$ on $V$ given by the representation.

We say that an open subset $U \subset X$ is big if the dimension of its complement has codimension at least 2. Consider a reduction of structure group of the principal $G$-bundle $E$ to $P(\lambda)$ over a big open set $\iota : U \subset X$. This reduction, together with the filtration of $V$ (1.1) produces a filtration of vector bundles on $U$

$$0 \subset F_1^U \subset F_2^U \subset \cdots \subset F_l^U = E(V)|_U$$

Extending each vector bundle to a torsion free sheaf on $X$ as $F_i = \iota_* F_i^U \cap E(V)$, we obtain a filtration by torsion free sheaves on $X$

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_l = E(V) \tag{1.2}$$
We say that $E$ is semistable if for any one parameter subgroup which factors through the derived subgroup $[G, G]$ of $G$

$$\rho : \mathbb{C}^* \to [G, G] \to G$$

and for any reduction of structure group of $E$ to the parabolic subgroup $P(\lambda)$ over a big open set $U \subset X$, the associated filtration of torsion free sheaves (1.2) satisfies

$$\sum_{i=1}^{l-1} (\gamma_{i+1} - \gamma_i) \left( \text{rk } F \cdot P_i(m) - \text{rk } F \cdot P_i(m) \right) \leq 0$$

for $m \gg 0$. We say that it is stable if the inequality is strict.

In the case of principal $G$-bundles over Riemann surfaces, this notion of stability coincides with the one in [Ra] (c.f. [GS3, Corollary 5.8], [Sc2, Proposition 5.4], [GLS1, Lemma 3.2.3], [GLSS2, Lemma 2.5.4]) and therefore does not depend on the choice the representation $\rho$.

In the case of the adjoint representation into the semisimple part of the Lie algebra, the filtrations (1.2) that we obtain can be simplified as follows ([GS2, GS3]).

Let $E$ be a principal $G$-bundle over $X$, let $F = E(g')$ be the vector bundle associated to the adjoint representation in the semisimple part $g'$ of the Lie algebra $g = g' \oplus \mathbb{Z}(g')$. Let

$$[\ , ] : F \otimes F \to F$$

be the Lie algebra structure and let

$$\kappa : F \otimes F \to \mathcal{O}_X$$

be the Killing form, inducing an isomorphism $F \cong F^\vee$ and assigning an orthogonal

$$F'^\perp = \ker(F \hookrightarrow F^\vee \twoheadrightarrow F'^\vee)$$

to each subsheaf $F' \subset F$. An orthogonal algebra filtration of $F$ is a filtration

$$(1.3) \quad 0 \subset F_{-t} \subset F_{-t+1} \subset \cdots \subset F_0 \subset \cdots \subset F_{t-1} \subset F_t = F$$

verifying $F_i\perp = F_{i-1}$ and $[F_i, F_j] \subset F_{i+j}$. Define

$$P_{F, \bullet} := \sum_{i} (\text{rk } F \cdot P_i - \text{rk } F \cdot P_i)$$

to be the Hilbert polynomial of a filtration and declare $E$ to be semistable if for every orthogonal algebra filtration $F_\bullet \subset F$ we have

$$P_{F, \bullet} \leq 0.$$

The case of the orthogonal group with the standard representation will be described in more detail in Section 5.

We will now describe in more detail the case of the adjoint representation.
Ramanathan states in [Ra] that every unstable principal $G$-bundle admits a unique canonical reduction $\sigma$ to a parabolic $P \subset G$ satisfying the following two conditions:

1. for every non-trivial character $\chi$ of $P$ given by a nonnegative linear combination of simple roots (with respect to some fixed Borel subgroup of $P$), the associated line bundle $\chi_*\sigma^*E$ has positive degree,
2. for the Levi quotient $P \twoheadrightarrow L$, the associated principal $L$-bundle is semistable.

In [AB], Atiyah and Bott showed that the Harder-Narasimhan filtration in for the adjoint vector bundle is indeed an orthogonal algebra filtration as in (1.3) and the middle term $E_0$ gives a reduction of structure group of the principal $G$-bundle to a parabolic subgroup. In particular, when $G = \text{GL}(n, \mathbb{C})$, the canonical reduction corresponds to the Harder-Narasimhan filtration of the associated vector bundle of rank $n$. In [Be] and [BH] the assertion of Ramanathan is proved and also shown that the reduction in [AB] coincides with the one in [Ra]. Finally, [AAB] generalizes the notion of the canonical reduction to principal $G$-bundles over compact Kähler manifolds $X$ by considering reductions to parabolics over a big open set $U \subset X$ which satisfy properties (1) and (2); the reduction in [AAB] is constructed by following the method in [AB].

Given an orthogonal or symplectic bundle over a projective variety $X$, we can construct the Gieseker Harder-Narasimhan filtration of its underlying vector bundle. On the other hand, we can construct the canonical reduction of the principal bundle. When $X$ is of complex dimension 1, both notions coincide, however they differ for higher dimensional varieties. Here we show an example of this.

In [GSZ], the authors explore the connections between the maximal 1-parameter subgroup giving maximal unstability from the GIT point of view in a GIT construction of a moduli space (c.f. [Ke]), and the Harder-Narasimhan filtration. These ideas give a method to construct the Harder-Narasimhan filtration in cases where we do not know it a priori. The paper [Za1] applies this to rank 2 tensors and the method does not work in more general situations (c.f. [Za2, Section 2.5]). It is natural to ask whether there exists a Harder-Narasimhan filtration for principal bundles coming from the construction of the moduli space in [GS3] through tensors, where a Gieseker type stability is used. The present work was motivated by this question.

2. Definitions

Let $X$ be a polarized smooth complex projective variety of dimension $n$ and let $\mathcal{O}_X(1)$ denote the polarization. Denote by $\text{rk } E$ and $\text{deg } E$ the rank and degree of a torsion free coherent sheaf $E$. Recall that we define the degree of a vector bundle $E$ over the polarized
variety \((X, \mathcal{O}_X(1))\) as
\[
\deg E = \int_X c_1(E) \cdot c_1(\mathcal{O}_X(1))^{n-1} = \int_X c_1(E) \cdot H^{n-1},
\]
where \(H\) is a hyperplane divisor of the polarizing line bundle.

**Definition 2.1.** A torsion free coherent sheaf \(E\) over a smooth projective variety is Mumford-Takemoto semistable if for every nontrivial subsheaf \(F \subset E\), with \(\text{rk } F < \text{rk } E\), we have
\[
\frac{\deg F}{\text{rk } F} \leq \frac{\deg E}{\text{rk } E}.
\]
Moreover, if the above inequality is strict, we say \(E\) is Mumford-Takemoto stable. We say that \(F\) Mumford-Takemoto destabilizes \(E\) if
\[
\frac{\deg F}{\text{rk } F} > \frac{\deg E}{\text{rk } E}.
\]

We call \(\deg E / \text{rk } E\) the Mumford-Takemoto slope of \(E\). Let \(\chi(E)\) be the Euler characteristic of a sheaf \(E\). Given a torsionfree coherent sheaf \(E\) over a polarized variety \((X, \mathcal{O}_X(1))\), we define the Hilbert polynomial of \(E\) as
\[
P_E(m) = \chi(E \otimes \mathcal{O}_X(m)) ,
\]
which is a polynomial on \(m\). Given two polynomials \(P(m)\) and \(Q(m)\) we say that \(P(m) \leq Q(m)\) if the inequality holds for \(m \gg 0\).

**Definition 2.2.** A torsion free coherent sheaf \(E\) over a smooth projective variety is Gieseker semistable if for every nontrivial subsheaf \(F \subset E\), with \(\text{rk } F < \text{rk } E\), we have
\[
\frac{P_F(m)}{\text{rk } F} \leq \frac{P_E(m)}{\text{rk } E}.
\]
Moreover, if the above inequality is strict, we say \(E\) is Gieseker stable. We say that \(F\) Gieseker destabilizes \(E\) if
\[
\frac{P_F(m)}{\text{rk } F} > \frac{P_E(m)}{\text{rk } E}.
\]

The above polynomial \(P_E(m) / \text{rk } E\) is called the Gieseker slope of \(E\).

Note that \(F\) Mumford-Takemoto destabilizes \(E\) implies \(F\) Gieseker destabilizes \(E\) but not the other way around.

Given a torsion free sheaf \(E\), there exists a unique filtration
\[
0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E ,
\]
which satisfies the following properties, where \(E^i := E_i / E_{i-1}\):

1. Every \(E^i\) is Gieseker semistable
(2) The Hilbert polynomials verify
\[
\frac{P_{E^1}(m)}{\text{rk } E^1} > \frac{P_{E^2}(m)}{\text{rk } E^2} > \ldots > \frac{P_{E^{t+1}}(m)}{\text{rk } E^{t+1}}
\]
This filtration is called the Harder-Narasimhan filtration of \( E \) (c.f. [HN] or [HL, Theorem 1.3.6]).

3. **A Gieseker unstable bundle which is Mumford-Takemoto semistable**

Let \( X \) be a \( K3 \) surface, which means that \( X \) is a smooth complex projective surface with irregularity \( q(X) = h^1(\mathcal{O}_X) = 0 \) and trivial canonical bundle. Since \( X \) is Kähler, there exists a Kähler-Einstein metric on \( X \) (c.f. [Ya]), hence \( TX \) is polystable (c.f. [Ko]). Moreover, \( TX \) is indecomposable, therefore \( TX \) is Mumford-Takemoto stable. The first Chern class of \( TX \) vanishes, so \( \deg(TX) = 0 \).

We consider the sheaves \( TX \) and \( \mathcal{O}_X \) and calculate their Hilbert polynomials. Let us calculate \( \chi(TX \otimes \mathcal{O}_X(m)) \) by using the Hirzebruch-Riemann-Roch theorem:
\[
\chi(TX \otimes \mathcal{O}_X(m)) = \text{deg}(\text{ch}(TX \otimes \mathcal{O}_X(m)) \cdot \text{td}(TX))_2,
\]
where the subscript 2 refers to the component of the cohomology in \( H^2(X, \mathbb{Q}) \).

The Chern class of a vector bundle is (c.f. [Ha, Appendix A.4]) \( \text{ch}(E) = \sum_{i=1}^r e^{a_i} \), with \( c_i(E) = \Pi_{i=1}^r (1 + a_i t) \) being the Chern polynomial, while \( \text{td}(TX) \) denotes the Todd character of the tangent bundle. Given that \( \text{ch}(E \otimes F) = \text{ch}(E) \cdot \text{ch}(F) \), and \( \mathcal{O}_X(m) \) is a line bundle,
\[
\text{ch}(\mathcal{O}_X(m)) = e^{\mathcal{O}_X(m)} = e^{(mH)} = 1 + mH + \frac{(mH)^2}{2!} + \frac{(mH)^3}{3!} + \ldots,
\]
where \( H \) is the class of a hyperplane divisor of \( \mathcal{O}_X(1) \). Therefore, in our case,
\[
\chi(TX \otimes \mathcal{O}_X(m)) = \text{deg}(\text{ch}(TX \otimes \mathcal{O}_X(m)) \cdot \text{td}(TX))_2 = \text{deg}(\text{ch}(TX) \cdot \text{td}(TX) \cdot \text{ch}(\mathcal{O}_X(m)))_2.
\]

The Chern and Todd classes of a vector bundle \( E \) are given by
\[
\text{ch}(E) = \text{rk } E + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \ldots
\]
\[
\text{td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots,
\]
then, in our case,
\[
\text{ch}(TX) = \text{rk}(TX) + c_1(TX) + \frac{1}{2}(c_1(TX)^2 - 2c_2(TX)) = 2 - c_2(TX)
\]
\[
\text{td}(TX) = 1 + \frac{1}{2}c_1(TX) + \frac{1}{12}(c_1(TX)^2 + c_2(TX)) = 1 + \frac{c_2(TX)}{12}
\]
\[ \text{ch}(\mathcal{O}_X(m)) = 1 + mH + \frac{m^2H^2}{2}. \]

Hence we have,

\[ \chi(TX \otimes \mathcal{O}_X(m)) = \]

\[ [(2 - c_2(TX)) \cdot (1 + \frac{1}{12}c_2(TX)) \cdot (1 + mH + \frac{m^2H^2}{2})]_2 = \]

\[ [(2 - \frac{5}{6}c_2(TX)) \cdot (1 + mH + \frac{m^2H^2}{2})]_2 = \]

\[ m^2H^2 - \frac{5}{6}c_2(TX). \]

Using the same arguments, let us calculate \( \chi(\mathcal{O}_X(m)) \),

\[ \chi(\mathcal{O}_X(m)) = \text{deg}(\text{ch}(\mathcal{O}_X(m)) \cdot \text{td}(TX))_2 = \]

\[ [(1 + mH + \frac{m^2H^2}{2}) \cdot (1 + \frac{c_2(TX)}{12})] = \]

\[ \frac{m^2H^2}{2} + \frac{c_2(TX)}{12}. \]

The Euler characteristic of \( \mathcal{O}_X \) is also given by

\[ \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = \]

\[ 2h^0(\mathcal{O}_X) = 2 \]

because on a \( K3 \) surface we have \( h^1(\mathcal{O}_X) = 0 \) and, by Serre duality and triviality of the canonical bundle, \( h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X) \). On the other hand, Hirzebruch-Riemann-Roch theorem gives

\[ \chi(\mathcal{O}_X) = \frac{1}{12}c_2(TX), \]

from which we get \( c_2(TX) = 24 \).

Therefore, we can write the Hilbert polynomials of \( TX \otimes \mathcal{O}_X \) and \( \mathcal{O}_X \):

\[ P_{TX}(m) = \chi(TX \otimes \mathcal{O}_X(m)) = \]

\[ m^2H^2 - \frac{5}{6}c_2(TX) = m^2H^2 - 20, \]

\[ P_{\mathcal{O}_X}(m) = \chi(\mathcal{O}_X(m)) = \]

\[ \frac{m^2H^2}{2} + \frac{c_2(TX)}{12} = \frac{m^2H^2}{2} + 2. \]

**Proposition 3.1.** The Gieseker slope of the line bundle \( \mathcal{O}_X \) is strictly bigger than the Gieseker slope of \( TX \).
Proof. From the above calculations we have
\[
\frac{P_{\mathcal{O}_X}(m)}{\text{rk } \mathcal{O}_X} = \frac{m^2H^2}{2} + 2 > \frac{P_{TX}(m)}{\text{rk } TX} = \frac{m^2H^2 - 20}{2} = m^2H^2 - 10,
\]
proving the proposition. \qed

4. Extensions and Harder-Narasimhan filtration

First, we construct a stable bundle as an extension of \( \mathcal{O}_X \) by \( TX \).

Proposition 4.1. Let \( V \) be a nontrivial extension
\[
0 \to TX \to V \to \mathcal{O}_X \to 0.
\]
Then \( V \) is Gieseker semistable.

Proof. First, we see that there exists such a nontrivial extension. Indeed,
\[
\dim \text{Ext}^1(\mathcal{O}_X, TX) = h^1(\mathcal{O}_X^\vee \otimes TX) = h^1(TX),
\]
and
\[
\chi(TX) = h^0(TX) - h^1(TX) + h^2(TX) = -20
\]
as calculated before. Hence \( h^1(TX) > 0 \), so and there exist nontrivial extensions.

Suppose \( V \) is not Gieseker semistable. Let \( W \subset V \) be the maximal Gieseker destabilizer (first nonzero term of the Harder-Narasimhan filtration (see (2.1))) of \( V \). Consider the short exact sequence
\[
0 \to W_0 := W \cap TX \to W \to W_1 := W/W_0 \to 0.
\]
Note that \( W_1 \subset V/TX = \mathcal{O}_X \).

First assume that \( W_1 = 0 \). This implies that \( W \subset TX \). But \( TX \) is Gieseker stable and the Gieseker slope of \( TX \) is strictly smaller than the Gieseker slope of \( \mathcal{O}_X \) (Proposition 3.1), so the Gieseker slope of \( W \) is not bigger than the Gieseker slope of \( V \). This contradicts the assumption that \( W \) is destabilizes \( V \).

Now assume that \( W_1 \neq 0 \). So \( \text{rk } W_1 > 0 \). Since \( \text{rk } W < 3 \), this implies that \( \text{rk } W_0 < 2 \). As \( TX \) is Gieseker stable (because it is Mumford-Takemoto stable), if \( W_0 \neq 0 \) the Gieseker slope of \( W_0 \) is strictly smaller than the Gieseker slope of \( TX \). Also, The Gieseker slope of \( W_1 \) is not bigger than the Gieseker slope of \( \mathcal{O}_X \). Therefore, the Gieseker slope of \( W \) is not bigger than the Gieseker slope of \( V \). This again contradicts the assumption that \( W \) is destabilizes \( V \). Therefore, \( W_0 = 0 \) and \( W \) would split the extension, hence, \( V \) is semistable. \qed
Let $V$ be a non-split extension
\[ 0 \to TX \to V \to \mathcal{O}_X \to 0. \]
Dualizing the above sequence we obtain
\[ 0 \to \mathcal{O}_X \to V^\vee \to TX \to 0 \]
because $\Omega^1(X) = (TX)^\vee \simeq TX$ with the isomorphism given by a trivialization of the canonical line bundle on the K3 surface $X$.

**Lemma 4.2.** The above vector bundle $V^\vee$ is not Gieseker semistable.

**Proof.** From Proposition 3.1 it follows that the subbundle $\mathcal{O}_X$ of $V^\vee$ destabilizes it. \qed

**Proposition 4.3.** The Gieseker Harder-Narasimhan filtration of $V \oplus V^\vee$ is
\[ (4.1) \quad 0 \subset \mathcal{O}_X \subset V \oplus \mathcal{O}_X \subset V \oplus V^\vee. \]

**Proof.** Let us check that the filtration in (4.1) satisfies the two properties which characterize the Harder-Narasimhan filtration in (2.1). The successive quotients for the filtration in (4.1) are: $\mathcal{O}_X$, $(V \oplus \mathcal{O}_X)/\mathcal{O}_X = V$, and $(V \oplus V^\vee)/(V \oplus \mathcal{O}_X) = TX$. All these three vector bundles $\mathcal{O}_X$, $V$ and $TX$ are Gieseker semistable; $V$ is semistable by Proposition 4.1 and $TX$ is semistable because it is Mumford-Takemoto stable.

Concerning the Hilbert polynomials of the above three vector bundles,
\[ \frac{P_{E_1}(m)}{\text{rk } E_1} = \frac{P_{\mathcal{O}_X}(m)}{\text{rk } \mathcal{O}_X} = \frac{m^2H^2}{2} + 2 \]
\[ \frac{P_{E_2}(m)}{\text{rk } E_2} = \frac{P_V(m)}{\text{rk } V} = \frac{m^2H^2}{2} - 6 \]
\[ \frac{P_{E_3}(m)}{\text{rk } E_3} = \frac{P_{TX}(m)}{\text{rk } TX} = \frac{m^2H^2}{2} - 10, \]
which are, clearly, decreasing. \qed

5. **Constructing a Principal Bundle**

Consider the vector bundle $V \oplus V^\vee$ in Proposition 4.3 which is self dual. The duality gives an orthogonal structure on $V \oplus V^\vee$, i.e., a symmetric nondegenerate homomorphism
\[ \varphi : (V \oplus V^\vee) \otimes (V \oplus V^\vee) \to \mathcal{O}_X; \]
nondegeneracy means that $\varphi$ induces an isomorphism between $(V \oplus V^\vee)$ and its dual. In other words, we get a reduction of the structure group of the $\text{GL}(6, \mathbb{C})$-bundle to $\text{O}(6, \mathbb{C})$. 
Given \((E, \varphi)\) an orthogonal bundle, we say that \(F \subset E\) is *isotropic* if \(\varphi|_{F \otimes F} = 0\). For a subsheaf \(F \subset E\), by using \(\varphi\) we can associate the *annihilator* subsheaf
\[
F^\perp := \ker(E \xrightarrow{\varphi} E^\vee \to F^\vee).
\]

If we apply the general definition of stability in the introduction to the case of the orthogonal bundle and the standard representation, we obtain the following definition.

**Definition 5.1** ([GSI, Definition 5.4]). An orthogonal bundle \((E, \varphi)\) is called *semistable* (as an orthogonal bundle) if for every isotropic subbundle \(F \subset E\),
\[
P_F(m) + P_{F^\perp}(m) \leq P_E(m).
\]

We say that \((E, \varphi)\) is *stable* if the above inequality is strict.

**Lemma 5.2.** The Harder-Narasimhan filtration of \(V \bigoplus V^\vee\) in Proposition 4.3 does not correspond to any parabolic reduction of the principal \(O(6, \mathbb{C})\)-bundle.

**Proof.** The annihilator of the subbundle \(O_X \subset V \bigoplus V^\vee\) in (4.1) is \(TX \bigoplus V^\vee\). The lemma follows immediately from this. \(\Box\)

5.1. **Symplectic case.** The natural duality pairing between \(V\) and \(V^\vee\) also gives a symplectic structure on \(V \bigoplus V^\vee\). The annihilator of the subbundle \(O_X \subset V \bigoplus V^\vee\) in (4.1) for the symplectic structure on \(V \bigoplus V^\vee\) is again \(TX \bigoplus V^\vee\). Therefore, the Harder-Narasimhan filtration of \(V \bigoplus V^\vee\) in Proposition 4.3 does not correspond to any parabolic reduction of this principal \(Sp(6, \mathbb{C})\)-bundle.

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