CURVES IN THE COMPLEMENT OF A SMOOTH
PLANE CUBIC WHOSE NORMALIZATIONS ARE $\mathbb{A}^1$

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Abstract. For a smooth plane cubic $B$, we count curves $C$ of degree $d$ such that the normalizations of $C \setminus B$ are isomorphic to $\mathbb{A}^1$, for $d \leq 7$ (for $d = 7$ under some assumption). We also count plane rational quartic curves intersecting $B$ at only one point.

§0 Introduction

Recently, the space of rational curves in a manifold is attracting attention in connection with physics. The number of rational curves of given degree in a Calabi-Yau threefold is studied, inspired by mirror symmetry (see [Ms], [Kn]), and the degree of the variety of rational curves in a fixed homology class in Fano manifolds is computed using the associativity of the quantum product ([KM], [RT]).

In this paper, for a smooth rational surface $X$ (over $\mathbb{C}$) and its smooth anticanonical divisor $B$, we count curves $C$ in a fixed divisor class in $X$ such that the normalizations of $C \setminus B$ are isomorphic to $\mathbb{A}^1$. In this section, we will refer to them, or to $C \setminus B$, as affine lines in $X \setminus B$ for simplicity.

On the one hand, this can be considered as an analog of counting rational curves on a K3 surface or a Calabi-Yau threefold, and it is hoped that they enjoy some good property similar to “mirror phenomena” etc. (while a rational curve is considered as the locus swept by a closed string, an affine line might be seen as that swept by an infinitely long string), although I don’t know what it should be. On the other hand, this problem can be seen as a special case of counting rational curves in rational surfaces. In fact, our computations in this paper use the degree of the variety of rational curves in rational surfaces ([CM]).

The problem has also an arithmetic application ([B], [V]). If $K$ is a number field, $S$ a finite set of places of $K$ containing all infinite ones, $\mathcal{O}_S$ the ring of $S$-integers and $B_S \subset \mathbb{P}^2_{\mathcal{O}_S}$ a cubic curve smooth over $\text{Spec}\mathcal{O}_S$, then an $S$-integral point of $\mathbb{P}^2_{\mathcal{O}_S} \setminus B_S$ is, by definition, its $\mathcal{O}_S$-valued point. If $C_S$ is a curve in $\mathbb{P}^2_{\mathcal{O}_S} \setminus B_S$, it follows from the Siegel-Mahler theorem that there exists a finite extension $K'$ of $K$ and a finite set $S'$ of places of $K'$ containing all infinite ones such that there are infinitely many $S'$-integral points contained in $C_S \otimes \mathcal{O}_{S'}$, if and only if $C_S \otimes \mathbb{C}$ is isomorphic to $\mathbb{A}^1$ or $\mathbb{G}_m$. The number of $S'$-integral points of $C_S \otimes \mathcal{O}_{S'}$ whose heights are less than $N$ is of order $N^a(\log N)^b$ with $a > 0, b \geq 0$ in the former case, while it is of order $N^{a/2}$ in the latter.

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(log $N$)$^b$ with $b > 0$ in the latter case. Thus affine lines in $\mathbb{P}^2 \setminus (B_S \otimes \mathbb{C})$ give many solutions to the corresponding Diophantine equation.

In particular, let $X = \mathbb{P}^2$. If $C$ is of degree $d$, $C \cap B$ is seen to consist of a point of order dividing $3d$ for the addition on $B$ with a flex as zero. We consider mainly the number $n_d$, counted with appropriately defined multiplicity, of such curves with $C \cap B$ a fixed point of order $3d$ w.r.t. some flex. We obtain the answer for $d \leq 6$, and for $d = 7$ under some assumption.

A useful fact is that $\mathbb{P}^2$ has a triple cover $Y$ totally ramified over $B$ and unramified elsewhere: if $B$ is defined by $F(X,Y,Z) = 0$, $Y$ can be written as $F = W^3$ in $\mathbb{P}^3$. In §2, we obtain $n_4 = 16$ using this(Theorem 2.1). We also prove that $n_d > 0$ holds for any $d$ and that, if $B_S$ is a smooth cubic in $\mathbb{P}^2_{O_S}$, where $S$ is a finite set of places of a number field $K$ containing all infinite ones, there exist a finite extension $K'$ of $K$ and a finite set $S'$ of places of $K'$, containing all infinite ones, such that, for any $d$, there exists a morphism $\mathbb{A}^1_{O_{S'}} \to \mathbb{P}^2_{O_{S'}} \setminus (B_S \otimes O_{S'})$ over $O_{S'}$ which is birational onto the image (on the generic fiber) and is of degree larger than $d$ (Proposition 2.3).

For $d = 5$ and $d = 6$, using the same trick, we reduce the problem to counting affine lines in different divisor classes in an affine cubic surface. In §3, as a preparation for the computation of these numbers, we construct a space representing affine lines and some of their degeneration in $X \setminus B$, where $X$ is a smooth rational surface and $B$ is a smooth anti-canonical divisor, and find relations between the numbers of affine lines in $X \setminus B$ for different $X$ and $B$(Theorem 3.8). Then, we obtain $n_5 = 113$ and $n_6 = 948$(Corollary 4.4) from the number of rational plane curves of degree $\leq 4$ which have given multiplicities at given general points.

For $d = 7$, it is necessary to know the number $x$ of rational sextic curves which have multiplicity 2 at 8 general points and pass through another general point, but it seems that this number cannot be obtained in the same way as the number of, say, rational sextic curves which have multiplicity 2 at 7 general points and pass through other 3 general points. By calculating $n_6$ without using the triple cover, we see $x = 90$ under some assumption. Then we obtain $n_7 = 8974$(Corollary 4.5).

Using Theorem 3.8, we also compute the number of quartic curves which meet $B$ at only one point of order 12(Corollary 4.3).

Calculations for the case $d = 3$ are also in [R], [V] and [X]. [V] and [X] are rather interested in the number of rational curves having only one point on the cubic curve, and [V] also contains some explicit equations of such curves for $d = 4$. [X] proves a theorem concerning finiteness of curves in surfaces which has only one point of intersection with a fixed curve, which is more general than Proposition 1.1(with $n = 2$).

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§1 Case $d = 1, 2$ and 3

In this paper, we work over $\mathbb{C}$ unless otherwise stated.

Let $B$ be a smooth cubic curve in $X = \mathbb{P}^2$. We want to count irreducible reduced curves $C$ of degree $d$ in $X$ such that the normalizations of $C \setminus B$ are isomorphic to
$\mathbb{A}^1$ (or, equivalently, rational curves $C$ having only one place on $B$), for, according to a rough dimension counting argument, the expected dimension of the variety of such curves is 0. In fact, we have:

**Proposition 1.1.** Let $X$ be a smooth projective variety and $B \subset X$ a smooth hypersurface such that $|K_X + B| \neq \emptyset$. Then, for any homology class $\beta$ on $X$, the union of rational curves in the class $\beta$ having only one point of intersection with $B$ is contained in a proper closed subset of $X$ (in the Zariski topology).

**Proof.** Assuming the contrary, we have a smooth variety $S$ of dimension $n - 1$, disjoint sections $S_i$ of $p_1: Y = S \times \mathbb{P}^1 \to S$ and a dominant morphism $f: Y \to X$ such that $f^*(B) = \sum a_i S_i$, where $a_i$ are positive integers, and that $f \circ (p_1|_{S_i})^{-1} = f \circ (p_1|_{S_j})^{-1}$ for all $i, j$. Denote this last morphism by $g: S \to B$.

On $X$, there exists a nonzero $n$-form $\omega$, where $n = \dim X$, which is regular outside $B$ and has at most logarithmic pole along $B$. Then the pullback $\omega_Y = f^*\omega$ (as a differential form) is a nonzero $n$-form on $Y$ which has only logarithmic pole along $\cup S_i$, and we have $\text{res}_{S_i} \omega_Y = a_i f^* \text{res}_B \omega$. If we take a non-vanishing $(n - 1)$-form $\omega_S$ on $S$, shrinking $S$ if necessary, we can write $\omega_Y$ as $\omega_{Y/S} f^* \omega_S$ for some relative differential 1-form $\omega_{Y/S}$. Now restricting $\omega_{Y/S}$ to a general fiber of $p_1$, we obtain a nonzero 1-form on $\mathbb{P}^1$ with at most logarithmic poles whose residues are positive integer times some complex number, which is impossible. □

The computations for $d = 1$ and 2 are easy. First, we have

**Lemma 1.2.** Let $B$ be a smooth plane cubic curve and fix a flex $O$ of $B$. Then, for a curve $C$ of degree $d$ meeting $B$ at only one point, $C|_B = 3dP$, where $P$ is a point of order dividing $3d$ with respect to the addition on $B$ with origin $O$. □

So we make the following:

**Definition 1.3.** Let $X$ be a rational surface, $B$ a smooth anti-canonical divisor on $X$ and $P$ a point on $B$. We say a curve $C$ on $X$ satisfies (*) for $(B, P)$ if $C$ is irreducible and reduced, the normalization of $C \setminus B$ is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ and $C \cap B \subseteq \{P\}$.

Using the notation in the lemma above, we have

**Proposition 1.4.** For a flex $P$, there is precisely one line and no conic satisfying (*) for $(B, P)$. For a point $P$ of order 6, there is exactly one conic satisfying (*) for $(B, P)$.

**Proof.** For a point $P$ of order dividing $3d$, $3dP \sim dH|_B$ and therefore the long exact sequence associated to the exact sequence

$$0 \to \mathcal{O}_X(d - 3) \to \mathcal{O}_X(d) \to \mathcal{O}_B(dH|_B) \to 0$$

shows, for $d = 1$ and 2, that there is unique curve of degree $d$ intersecting with $B$ only at $P$. For $d = 1$, this curve is a line and satisfies (*). For $d = 2$, if $P$ is a flex, then this curve is 2 times line and so does not satisfy (*). If $P$ is not a flex, then this curve cannot be reducible or non-reduced, as it would mean that there is a line having triple intersection with $B$ at $P$. □

The case $d = 3$ is a little more difficult.
Proposition 1.5. For a point $P$ of order 9, there are 3 cubic curves satisfying (*) for $(B, P)$, and they have only nodes. For a flex $P$, if $B$ is not isomorphic to the curve $B_0$ defined by $y^2 = x^3 - 1$, there are two cubic curves satisfying (*) for $(B, P)$, and they have only nodes. If $B$ is isomorphic to $B_0$, then there is unique cubic satisfying (*) for $(B, P)$, and it has a cusp outside $B$.

Proof. By the same exact sequence as in the proof of the proposition above, we see that the cubic curves having intersection of multiplicity 9 at $P$ with $B$ together with $B$ itself form a pencil $\Lambda$. Blowing up 9 times the points on the proper transform of $B$ over $P$, we obtain a resolution of $\Lambda$, giving an elliptic fibration $f : X' \to \mathbb{P}^1$.

We see that there is just one member $D$ of $\Lambda$ which is singular at $P$, for being singular at $P$ is a linear condition, and that the proper transform of $D$ and the exceptional curves other than the last one form a set theoretic fiber $F$ of $f$.

If $P$ is not a flex, there is no reducible or non-reduced member in $\Lambda$ by the same argument as in Proposition 1.4. Furthermore, $D$ cannot have a cusp at $P$, for a cusp cannot have a 9-ple intersection with a smooth curve. Therefore $D$ has a node at $P$, and $F$ is a cycle consisting of 9 smooth rational curves. Assume that $\Lambda$ contains a cuspidal cubic $E$. Then for the addition on $E \setminus \{\text{cusp}\}$ with the flex as origin, $P$ is a 9-torsion, and therefore $P$ is the flex of $E$, which is impossible as $P$ is not a flex of $B$ and $B$ and $E$ have 9-ple intersection at $P$. Now the Euler number of $X'$ is 12, and that of $F$ is 9. Consequently, the number of fibers of $f$ isomorphic to a nodal cubic is 3, which is the number of curves satisfying (*) for $(B, P)$.

If $P$ is a flex, $D$ is 3 times the tangent line and the Euler number of $F$ is 10. If $\Lambda$ contains a cuspidal cubic $E$, then $P$ is a flex of $E$. But then, writing as $E : Y^2Z = X^3$ and $P(0 : 1 : 0)$, we see that $B$ must be of the form $Y^2Z = X^3 + aZ^3$ for $a \in \mathbb{C}$, i.e. $B$ is isomorphic to $B_0$. In this case, since the Euler number of a cuspidal cubic is 2, $\Lambda$ contains one cuspidal cubic and no nodal cubic. If $B$ is not isomorphic to $B_0$, $\Lambda$ contains 2 nodal cubics. \qed

§2 Case $d = 4$

In this section, we compute the number for $d = 4$ and $P$ “primitive”.

Theorem 2.1. Let $B$ be a general cubic and $P$ a point of order 12. Then any curve of degree 4 satisfying (*) for $(B, P)$ has only nodes and there are 16 such curves.

Proof. Let $C$ be a curve satisfying (*) for $(B, P)$. Then, since $C$ and $B$ intersect at $P$ with multiplicity 12, $C$ is smooth at $P$. Assume that $C$ has a cusp $Q$ and two nodes $R$, $S$, for example. Denote the normalization by $f : \mathbb{P}^1 \to C$ and the curve obtained by resolving the singularities $R$ and $S$ (resp. $Q$, $Q$ and $R$) of $C$ by $C_1$, $C_2$, $C_3$. Take a coordinate $u$ on $\mathbb{P}^1$ such that $f^{-1}(Q) = \{u = \infty\}$ and that $f^{-1}(R) = \{u = 0, 1\}$. Let $f^{-1}(S) = \{u = a, b\}$ and $f^{-1}(P) = \{u = c\}$. We have $12P \sim 3H_C$. Considering the lines through $R$ and $S$, $S$ and $Q$, $Q$ and $R$, we see that $3([0] + [1] + [a] + [b]) \sim 12[c]$ on $C_1$, $3([a] + [b] + 2[\infty]) \sim 12[c]$ on $C_2$ and $3([0] + [1] + 2[\infty]) \sim 12[c]$ on $C_3$. Thus $3(a + b + 1) = 12c$, $ab/(a - 1)(b - 1) = (c/(c - 1))^3$, and $a(a - 1)/b(b - 1)^3 = ((c - a)/(c - b))^3$. Eliminating $c$ and computing the greatest common divisor of the resulting polynomials, we see that any positive dimensional component of the solution of the equation above satisfies $[0] + [1] + [a] + [b] \sim 4[c]$ on $C_1$, $[a] + [b] + 2[\infty] \sim 4[c]$ on $C_2$ and $[0] + [1] + 2[\infty] \sim 4[c]$ on $C_3$. Then $4[c]$ is linearly equivalent to the pullback of $H$ on $C$, and therefore we have $4[c] \sim 3H_C$. Therefore $C$ has 16 nodes.
C. Writing down a long exact sequence, we see that there exists a line intersecting C with multiplicity 4 at P. This line intersects with B at P with multiplicity 4, which is absurd. Thus there are only finitely many possibilities for C and P modulo projective equivalence. Since B and C intersect with multiplicity 12 at P, B is determined uniquely by C and P. Therefore, for a general B, there does not exist such a curve C. The proof that C cannot have even worse singularity is similar and easier.

Let π : Y → X be the triple cover totally branched along B, defined by t^3 = s in the total space of the line bundle associated to O(1), where t is a local coordinate of that bundle and s is a section of O(3) with B = (s = 0). The line bundle can be embedded in \( \mathbb{P}^3 \), and then Y is represented as a cubic surface: in fact, if B is defined by \( F(X,Y,Z) = 0 \), Y can be written as \( F(X,Y,Z) = W^3 \). Let \( B_Y \) be the reduced inverse image of B, and then it is easy to see that \( B_Y \) is a plane section and maps isomorphically onto B. We identify points on B and \( B_Y \).

Let C be a curve satisfying (*) for \((B, P)\). Since the covering of C by \( \pi^*(C) \) is unbranched outside \( P \), \( \pi^*(C) \) is composed of 3 curves satisfying (*) for \((B_Y, P)\), each mapping birationally to C so that any local analytic branch maps isomorphically onto the corresponding branch. Furthermore, since they are permuted cyclically by the action of \( \text{Gal}(Y/X) \), they have the same degree, i.e., 4. On the other hand, if C is a rational curve of degree 4 on Y which satisfies (*) for \((B_Y, P)\) then \( \pi(C) \) is a curve satisfying (*) for \((B, P)\), and therefore \( \pi : C \to \pi(C) \) is birational and maps any analytic branch isomorphically, from what we have seen above (and therefore C has only nodes). Thus it suffices to prove that there are 48 curves of degree 4 on Y which satisfy (*) for \((B, P)\).

To prove this, let \( g : Y \to Z \cong \mathbb{P}^2 \) be a blow-down. Since the 27 lines on Y are the components of the inverse images of the flex tangent lines to B, g is the blow-down of 6 disjoint lines \( E_1, \ldots, E_6 \) intersecting with \( B_Y \) at different 3-torsions \( P_1, \ldots, P_6 \) (with a flex \( O \) on \( B \subset X \) as zero). Let \( Q_1, Q_2, Q_3 \) be the other 3-torsions, \( B_Z = g_*B_Y \), which is a cubic curve, and \( O_Z \) a flex of \( B_Z \). None of 3 lines on \( Y \) through \( Q_1 \) is contracted by \( g \), and the sum of the degrees of their images is easily seen to be 3. Therefore, they are lines, and any of them meets \( B_Z \) at \( Q_1, P_a \) and \( P_b \) for some \( a \neq b \). Thus we have \( 3O_Z \sim Q_1 + P_a + P_b \) and \( O_Z \) is a 9-torsion relative to \( O \). Now B is a group isomorphic to \( (\mathbb{R}/\mathbb{Z})^2 \). By changing \( O \) if necessary, we can find such an identification with \( Q_1 = (0, 2/3), Q_2 = (1/3, 2/3) \) and \( Q_3 = (2/3, 1/3) \). As any pairwise distinct three of \( P_i \) cannot be on a line in \( Z \), for \( B_Y \) ample, we see that \( Q_3 = (2/3, 2/3) \) and \( 3O_Z = (1/3, 0) \) or \( (2/3, 0) \). We may assume \( O_Z = (1/9, 0) \).

Let C be a curve as above. If \( C \sim eH - \sum a_iE_i \), where \( H \) denotes the pullback of a line in \( Z \), we have \( 0 \leq a_i = E_i.C \leq \pi^*(\text{line}).C = 4 \) and \( e = \deg g_*C = (B_Y + E_1 + \cdots + E_6).C/3 = (4 + \sum a_i)/3 \), and therefore \( 2 \leq d \leq 9 \).

Using furthermore \( p_a(C) = (e - 1)(e - 2)/2 - \sum a_i(a_i - 1)/2 \geq 0 \), we have the following list for \( e, p_a \) and unordered sequences \([a] = [a_1, \ldots, a_6] \):

| e | \( p_a \) | \([a]\) |
|---|---|---|
| 2 | 0 | [0, 0, 0, 1, 1] |
| 3 | 0 | [0, 0, 1, 1, 1, 2] |
| 1 | 1 | [0, 1, 1, 1, 1, 1] |
| 5 | 0 | [1, 1, 2, 2, 2, 3] |
| 4 | 0 | [0, 1, 1, 2, 2, 2] |
| 6 | 0 | [2, 2, 2, 2, 2, 3] |

It can be seen that given such a curve C we can make another choice of \( Z \), say \( Z' \), such that \( e' = 3, \ [a'] = [0, 0, 0, 1, 1] \) for case \( e = 0 \) or that \( e' = 3, \ [a'] = [0, 0, 1, 1, 2] \) for case \( e = 3 \).
[0, 1, 1, 1, 1, 1] for case \(p_a = 1\), where we denote the degrees etc. for \(Z'\) by symbols with ':
for example, in the case \(e = 6\), it suffices to apply two successive quadratic transformations, first with centers \(P_k, P_l, P_m\) with \(a_k = a_l = 3\) and then with the other three points as the centers.

For the case \(p_a = 0\) above, there is one (possibly reducible or nonreduced) conic on \(Z'\) having 4-fold intersection at \(P\) and simple intersections at \(P_k\) and \(P_l\), with \(B_{Z'}\), where \(a'_{k} = a'_{l} = 1, k \neq l\), if and only if \(4P + P_k + P_l \sim 6O_{Z'}\). Let \(V_{H} = \sum a_iE_i \sim 2H' - E'_{k} - E'_{l}\), this is equivalent to say \(4(P - O_{Z}) + \sum a_i(P_i - O_{Z}) \sim 0\). Now we have the following:

**Lemma 2.2.** Let \(D \in \langle eH - \sum a_iE_i \rangle\) be an effective divisor on \(Y\) such that \(D|_{B_Y} = \sum a_iE_i\) and \(E_{k}\) is a point of order 3d w.r.t. a flex \(O \in B \subset X\). Then \(C\) is irreducible and reduced.

**Proof.** If \(C\) is reducible or reduced, we have \(d'P + \sum a'_iP_i \sim e'H\) with \(0 < d' < d\), since \(B_Y\) is ample. This implies \(3d'(P - O) \sim 0\), a contradiction. \(\square\)

Therefore this conic gives a smooth rational quartic curve on \(Y\) which satisfies (*) for \((B_Y, P)\) and is linearly equivalent to \(eH - \sum a_iE_i\).

On the other hand, for the case \(p_a = 1\), there is a linear pencil on \(Z'\) consisting of \(B_{Z'}\) and cubics on \(Z'\) having 4-fold intersection at \(P\) and simple intersection at \(P_1, \ldots, P_k, \ldots, P_6\) with \(B_{Z'}\), where \(a'_i = 0\), if and only if \(4P + P_1 + \cdots + P_{k} + \cdots + P_6 \sim 9O_{Z'}\), which is again equivalent to \(4(P - O_{Z}) + \sum a_i(P_i - O_{Z}) \sim 0\). By Lemma 2.2, all members of the pencil, including \(B_{Z'}\) itself, are irreducible and reduced. As in Proposition 1.5, we see that the member which is singular at \(P\) is nodal and therefore that there are 8 nodal cubics in the pencil with only one place on \(P\), giving nodal rational quartic curves on \(Y\) which satisfy (*) for \((B_Y, P)\) and are linearly equivalent to \(eH - \sum a_iE_i\).

By computation (with a computer), we see that the number of ordered sequences \((a_1, a_2, a_3, a_4, a_5, a_6)\) such that \(4(P - O_{Z}) + \sum a_i(P_i - O_{Z}) \sim 0\) is:

| \(e\) | \([a]\) | if \(4P = (x, 0)\) or \((x, 1/3)\) | if \(4P = (x, 2/3)\) |
|---|---|---|---|
| 2 | [0, 0, 0, 0, 1, 1] | 1 | 3 |
| 2 | [0, 0, 1, 1, 1, 2] | 7 | 6 |
| 4 | [0, 1, 1, 1, 2, 2] | 7 | 6 |
| 5 | [1, 1, 1, 1, 1, 3] | 1 | 0 |
| 5 | [1, 1, 2, 2, 2, 3] | 1 | 3 |
| 6 | [2, 2, 2, 2, 3, 3] | 1 | 3 |

for \(p_a = 0\),

for \(p_a = 1\),

| \(e\) | \([a]\) | if \(4P = (x, 0)\) or \((x, 1/3)\) | if \(4P = (x, 2/3)\) |
|---|---|---|---|
| 3 | [0, 1, 1, 1, 1, 1] | 1 | 0 |
| 4 | [1, 1, 1, 1, 2, 2] | 1 | 3 |
| 5 | [1, 2, 2, 2, 2, 2] | 1 | 0 |

Consequently, for any point \(P\) of order 12, there are 48 quartic rational curves on \(Y\) satisfying (*) for \((B_Y, P)\), as desired. \(\square\)

**Remark.** It may be more natural to study, from the start, the curves in \(Y \setminus B_Y\), where \(Y\) is a general cubic surface and \(B_Y\) is a general plane section, whose normalizations are \(\mathbb{A}^1\).
Proposition 2.3. (1) Let $B$ be a smooth cubic curve and $d$ a positive integer. Then there exists a rational curve $C$ in $\mathbb{P}^2$ of degree $d$ such that the normalization of $C \setminus B$ is isomorphic to $\mathbb{A}^1$ and that the normalization map $\mathbb{P}^1 \to C$ is immersive.

(2) Let $K$ be a number field, $S$ a finite set of places of $K$ containing all infinite ones and $B_S$ a divisor of degree 3 in $\mathbb{P}^2_{O_S}$ which is generically smooth. Then there exist a finite extension $K'$ of $K$ and a finite set $S'$ of places of $K'$, containing all infinite ones, which satisfy the following conditions:

(a) $O_{S'} \subseteq O_S$.

(b) Let $B_{S'} := B_S \otimes O_{S'}$. Then, for any $d$, there exists a morphism $\mathbb{A}^1_{O_{S'}} \to \mathbb{P}^2_{O_{S'}} \setminus B_{S'}$ over $O_{S'}$ which is immersive, birational onto the image and of degree larger than $d$ on the generic fiber.

Proof. We use the notations in the proof of Theorem 2.1.

(1) If $d = 2k - 1$, let $D := kH - (k - 1)E_1 - E_2 - E_4$ and $P := ((k - 1)/3d, 1/3d)$, and if $d = 2k$, let $D := kH - (k - 1)E_1 - E_4$ and $P := (k/3d, -1/3d)$. We may assume that $k \geq 2$. Then there exists an effective divisor $C \subseteq |D|$ on $Y$ with $C|B_Y = dP$; in fact, curves $C$ in $Z$ of degree $k$ such that $C|_{B_Z} = dP + (k - 1)P_1 + P_2 + P_4$ (resp. $C|_{B_Z} = dP + (k - 1)P_1 + P_4$) or $C \supset B$ form a linear system of dimension $k - 2$, and there exists such a curve with multiplicity $k - 1$ at $P_1$. Clearly this cannot contain $B_Z$.

Then $C$ is irreducible and reduced by Lemma 2.2, and therefore it is a smooth rational curve.

(2) Take $K'$ and $S'$ such that:

(a) $\alpha$ holds, and $B_{S'}$ is smooth over $\text{Spec} O_{S'}$,

(b) one of the flexes of $B_{S'}$ is defined over $O_{S'}$.

(c) and there exists an open immersion $h : \mathbb{P}^2_{O_{S'}} \setminus B_{S'} \to Y_{S'}$ whose inverse is the contraction of six lines, where $\pi : Y_{S'} \to \mathbb{P}^2_{O_{S'}}$ is the triple cover determined by $B_{S'}'$.

Taking a flex tangent line, we have a morphism of degree 1 as in the assertion.

If we have such a morphism $f$ of degree $d$, then $\pi \circ h \circ f$ is also a morphism as in the assertion. It is of degree $3d$ if the image of the point at infinity by $f$ is not one of the fundamental points of $\pi^{-1}$, $3d - 1$ if it is. \(\square\)

\section{§3 A space representing affine lines}

Let $X$ be a smooth rational surface and $B$ a smooth anti-canonical divisor. In this section, we obtain relations between the number of curves in $X \setminus B$ whose normalizations are $\mathbb{A}^1$ and similar numbers on the blow-up of $X$ at a point on $B$.

Let $d$ be a positive integer. Take two copies of $\mathbb{P}^1$ and denote the second by $S^{(d)}$. In this section, $t \in \mathbb{C} \cup \{\infty\}$ always denotes a inhomogeneous coordinate of $S^{(d)}$, and $s := 1/t$ the coordinate near $\infty$. Consider $C := \mathbb{P}^1 \times S^{(d)}$ with the second projection $p'$, four sections $\sigma' := \{0\} \times S^{(d)}, \sigma_2 := \{1\} \times S^{(d)}, \sigma_4 := \{\infty\} \times S^{(d)}$ and $\sigma_3 := (\text{diagonal})$. Blow up $C'$ successively $d$ times at the intersection of the proper transform of $\sigma_3$ and the fiber over $\infty$, and then blow down the exceptional curves other than the last one. Call this surface $C^{(d)}$ with the projection $p^{(d)} : C^{(d)} \to S^{(d)}$ and sections $\sigma_1, \sigma_2, \sigma_3$ and $\sigma_4$. We write the fibers as $C^{(d)}_t, \sigma_{1,t}$, etc. A point in this construction is that $D^{(d)} := d\sigma_4$ is Cartier.

On the other hand, let $X$ be a projective variety, $B$ a Cartier divisor on $X$, $\beta$ a homology class of degree 2 and $H_1, H_2$ and $H_3$ Cartier divisors. Then let $M$ be the closed subscheme of $\text{Mor}_X(C^{(d)} \times S^{(d)})$ parameterizing morphisms $C^{(d)} \to X$.
with images in the class \( \beta \) by which the pullback of \( H_i \) contains \( \sigma_{i,t} \) for \( i = 1, 2, 3 \) and that of \( B \) contains \( D_t^{(d)} \). Denote by \( M(H_1,H_2,H_3) \) the open subscheme of \( \tilde{M} \), whose geometric morphs represent birational morphisms from \( C_t^{(d)} \) to its images, where \( \sigma_{i,t} \) \( (i = 1, 2, 3, 4) \) are distinct points (i.e. \( t \neq 0, 1 \)), by which the pullback of \( H_i \) is exactly \( \sigma_{i,t} \) in a neighborhood of \( \sigma_{i,t} \) for \( i = 1, 2, 3 \) and that of \( B \) is \( D_t^{(d)} \) in a neighborhood of \( \text{Supp}D_t^{(d)} \).

**Proposition 3.1.** (1) Let \( \tilde{M} \) be a disjoint union of \( M(H_1,H_2,H_3) \) for a finite set of \( (H_1,H_2,H_3) \), \( \tilde{p} : \tilde{C} \to \tilde{M}, \tilde{f} : \tilde{C} \to X \) and \( \tilde{D} \) the representing family. Define \( I \) to be \( \text{Isom}_{\tilde{M} \times \tilde{M}}((\tilde{C}, \tilde{D}, \tilde{f}) \times \tilde{M}, \tilde{M} \times (\tilde{C}, \tilde{D}, \tilde{f})) \), i.e. the scheme parameterizing isomorphisms of different fibers of \( \tilde{p} \) commuting with the restrictions of \( \tilde{f} \) and inducing isomorphisms of the restrictions of \( \tilde{D} \). Then the structure morphism \( \pi : I \to \tilde{M} \times \tilde{M} \) is a closed immersion and defines an étale equivalence relation on \( \tilde{M} \). Therefore it defines a quotient \( M \) of \( \tilde{M} \) as a separated algebraic space, as well as quotients \( C, D \) and \( \tilde{f} \) of \( \tilde{C}, \tilde{D}, \tilde{f} \). For some finite collection of \( H_i \)'s, \( M \) contains any \( M(H_1,H_2,H_3) \) as an étale open set.

(2) Similarly, for \( d = (B, \beta) = 0 \), there exists an algebraic space \( M \) parameterizing birational morphisms \( \mathbb{P}^1 \to X \setminus B \) whose images are in the class \( \beta \) modulo automorphisms of \( \mathbb{P}^1 \).

**Proof.** (1) First, we show that \( \pi \) is a closed immersion. It is easy to see that any geometric fiber of \( \pi \) is empty or consists of one reduced point, since we assumed that \( \tilde{f} \) maps each fiber birationally to the image. Therefore, it suffices to show that \( \pi \) is proper.

Since \( I \) is a locally closed subscheme of \( H := \text{Hilb}_{\tilde{M} \times \tilde{M}}(\tilde{C} \times \tilde{C}) \), which is a disjoint union of projective schemes over \( \tilde{M} \times \tilde{M} \), and is of finite type, if \( \pi \) is not proper, then there exists a curve \( S \) in \( H \) which is not contained in \( I \) but whose generic point is. Now if the pullbacks of \( (\tilde{C}, \tilde{D}, \tilde{f}) \times \tilde{M} \) and \( \tilde{M} \times (\tilde{C}, \tilde{D}, \tilde{f}) \) to the normalization \( \tilde{S} \) of \( S \) are isomorphic over \( \tilde{S} \), then \( S \) is contained in \( I \), a contradiction. Thus it suffices to show the following: let \( S \) be a smooth curve, \( P \) a point of \( S \), \( p_S : C_S \to S \) and \( p'_S : C'_S \to S \) families of curves induced from \( C^{(d)} \) by morphisms \( S \to S^{(d)} \), \( D_S \) and \( D'_S \) the induced Cartier divisors and \( f_S : C_S \to X \times S \) and \( f'_S : C'_S \to X \times S \) morphisms over \( S \) that map any fiber of \( p_S \) and \( p'_S \) birationally. Then an isomorphism of \( (C_S, D_S, f_S) \) and \( (C'_S, D'_S, f'_S) \) over \( S \setminus \{P\} \) extends to an isomorphism over \( S \).

If a general fiber of \( C_S \to S \) is smooth, then \( C_S \) and \( C'_S \) are the normalizations of \( f_S(C_S), f'_S(C'_S) \subset X \times S \), and they coincide as they are irreducible and have the same generic point. By the uniqueness of normalization, \( C_S \) and \( C'_S \) are isomorphic over \( X \times S \). This induces an isomorphism of \( D_S \) and \( D'_S \), since it does over \( S \setminus \{P\} \).

If a general fiber of \( C_S \to S \) is singular, then \( C_S \) and \( C'_S \) are isomorphic to (reducible conic) \( \times S \). As before, there exists a unique isomorphism of the normalizations of \( C_S \) and \( C'_S \) over \( X \times S \), and since the restrictions of this isomorphism to the two components of the inverse image of the double curve commute with the identifications by normalization maps (for they do over \( S \setminus \{P\} \)), it gives an isomorphism of \( C_S \) and \( C'_S \) over \( X \times S \), inducing an isomorphism of \( D_S \) and \( D'_S \). Thus \( \pi \) is a closed immersion.

Clearly, \( I \) defines an equivalence relation. Therefore, what we have to show is that the two projection maps \( I \to M \) are étale, and this is equivalent to the
following: let \( f_S : (C_S, D_S) \to X \times S \) be induced by an \( S \)-valued point of \( \bar{M} \), where \( S \) is the spectrum of an artinian local \( \mathbb{C} \)-algebra \((R, m)\) with \( R/m = \mathbb{C} \), and \( P \) a geometric point of \( M(H_1, H_2, H_3) \), where \( H_1, H_2, H_3 \) are Cartier divisors on \( X \), corresponding to a morphism \( f_P : (C_P, D_P) \to X \) isomorphic to the restriction \( f_0 : (C_0, D_0) \to X \) of \( f_S \) to the central fiber. Then there exists a morphism \( S \to M(H_1, H_2, H_3) \) with image \( P \) which induces a family isomorphic to \((C_S, D_S, f_S)\) over \( S \), and for a constant family over \( S = \text{Spec} \mathbb{C}[u]/(u^2) \) (i.e. one that is induced by a morphism to \( \bar{M} \) whose tangent is zero), such a map is a constant map.

By the unique isomorphism of \( C_0 \) and \( C_P \) commuting with \( f_0 \) and \( f_P \), three points \( P_1, P_2, P_3 \) on \( C_0 \) are given satisfying \( f_0^*H_i = P_i \) in a neighborhood of \( P_i \). Since \( C_S \) has the same support as \( C_0 \), we can take the pullback \( P_{S,i} \) of \( H_i \) in a neighborhood of \( P_i \). By the universal property of \( M(H_1, H_2, H_3) \), it suffices to show the following: there exists a morphism \( g : S \to S^{(d)} \) such that there exists an isomorphism \( h : (C_S, P_{S,1}, P_{S,2}, P_{S,3}, D_S) \to (C^{(d)}, \sigma_1, \sigma_2, \sigma_3, D^{(d)}) \times_{S^{(d)}} S \). Moreover, for \( S = \text{Spec} \mathbb{C}[u]/(u^2) \) and \((C_S, P_{S,1}, P_{S,2}, P_{S,3}, D_S) = (C_0, P_1, P_2, P_3, D_0) \times S \), \( g \) is a constant map with value \( t_0 \) and \( h = h_0 \times S \) where \( h_0 : (C_0, P_1, P_2, P_3, D_0) \to (C^{(d)}_{t_0}, \sigma_{1,t_0}, \sigma_{2,t_0}, \sigma_{3,t_0}, D^{(d)}_{t_0}) \) is the unique isomorphism.

If \( C_0 \) is smooth, then \( C_S \cong \mathbb{P}^1 \times S \). After an automorphism over \( S \), we can take homogeneous coordinates \((x : y)\) such that \( P_{S,1} = (y = 0) \), \( P_{S,2} = (y = x) \), \( P_{S,3} = (y = rx) \), where \( r \in R \), and \( D_S = (x^d = 0) \) hold. The morphism \( S \to S^{(d)} \) defined by \( t = r \) is clearly the unique morphism which induces \((C_S, P_{S,1}, P_{S,2}, P_{S,3}, D_S)\), and \( h \) is also unique.

Next, we consider the case where \( C_0 \) is singular. Let \( s \) be the coordinate \( 1/t \) of \( S^{(d)} \) around \( \infty \), as we defined before. Then we have the following description of \( C^{(d)} \): \( C^{(d)} = U_1 \cup U_2 \cup U_3 \), where

\[
U_1 = \text{Spec} \mathbb{C}[w, s],
\]

\[
U_2 = \text{Spec} \mathbb{C}[x, z, s]/((x - s)z - s^d),
\]

\[
U_3 = \text{Spec} \mathbb{C}[y, s],
\]

and \( U_1 \) and \( U_2 \) are patched by \( zw = 1 \) on \( z \neq 0, w \neq 0 \), \( U_2 \) and \( U_3 \) by \( xy = 1 \) on \( x(x - s) \neq 0, y(sy - 1) \neq 0 \), and \( U_1 \) and \( U_3 \) by \( y(s^d w + s) = 1 \) on \( s^d w + s \neq 0, sy \neq 0 \). \( \sigma_1, \sigma_2 \) are defined by \( y = 0, y = 1 \) on \( U_3 \), \( \sigma_3 \) by \( w = 0 \) on \( U_1 \) and \( D^{(d)} \) by \( z + \sum_{i=0}^{d-1} x^i s^{d-1-i} = 0 \). \((C_S, D_S)\) is obtained by replacing \( C \) by \( R \) and \( s \) by an element \( r \in m \).

Consider the family \((\bar{C}, \bar{D}) := (C^{(d)}, D^{(d)}) \times_{S^{(d)}} \bar{S} \) over \( \bar{S} := \{(s, a, b, c) \mid 1/(s^d c + s), a, b \text{ are pairwise distinct and } c \neq -1 \text{ if } d = 1\} \), and let \( \bar{P}_1, \bar{P}_2 \) be defined by \( y = a, y = b \) and \( \bar{P}_3 \) by \( w = c \). Then \((C_S, P_{S,1}, P_{S,2}, P_{S,3}, D_S)\) is induced by a morphism \( S \to \bar{S} \). Thus, to show the existence part, it suffices to show a similar statement for \((\bar{C}, \bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{D})\). We define a morphism \( g : \bar{S} \to S^{(d)} \) by \( g^*s = \bar{D} \) and \( g^*w = \bar{P}_3 \). Then we can find a morphism \( (\bar{C}, \bar{D}) \to C^{(d)} \) such that the above properties hold.
(b - a)(s^d c + r)/(1 - sa - s^d ac) and a morphism \( h : \tilde{C} \to C^{(d)} \) by

\[
\begin{align*}
h^*s &= \frac{(b - a)(s^d c + s)}{1 - sa - s^d ac}, \\
h^*x &= \frac{(b - a)x}{1 - ax}, \\
h^*y &= \frac{y - a}{b - a}, \\
h^*z &= \frac{(z - saz - s^d a)(1 + s^d - 1 c)^d}{1 - cz} \left(\frac{b - a}{1 - sa - s^d ac}\right)^{d - 1}, \\
h^*w &= \frac{w - c}{(1 - sa - s^d aw)(1 + s^d - 1 c)^d} \left(\frac{1 - sa - s^d ac}{b - a}\right)^{d - 1}.
\end{align*}
\]

Then they give the desired cartesian diagram, for they induce isomorphisms of fibers.

For the constant family over \( R = \mathbb{C}[u]/(u^2) \), i.e. that which is induced by \( s = 0 \), there is no such commutative diagram with \( g^*s \neq 0 \): for \( d \geq 2 \), it is easy to see that there are isomorphisms of \( C_S \) and \( C^{(d)} \times S^{(d)} S \), but they do not map \( D_S \) to \( D^{(d)} \times S^{(d)} S \). For \( d = 1 \), \( C_S \) and \( C^{(d)} \times S^{(d)} S \) are not isomorphic. Thus \( g^*s = 0 \), and the claim about \( h \) is easy to see.

Thus \( I \) is an étale equivalence relation and defines \( \tilde{M}/I \) as a separated algebraic space. Patching together pieces of \( \tilde{C}, \tilde{D} \) and \( \tilde{f} \) by the unique isomorphism, we obtain a representing family.

The last assertion is clear from the boundedness of morphisms with images in a fixed homology class.

(2) Similar and easier. \( \square \)

**Lemma 3.2.** Let \( d \) be a positive integer and \( P_1, P_2 \) and \( P_3 \) different points on \( \mathbb{P}^1 \). For Cartier divisors \( H_1, H_2 \) on \( X \), let \( M(H_1, H_2) \) be the scheme of morphisms \( f : \mathbb{P}^1 \to X \) with images in the class \( \beta \) such that \( f^*H_i = P_i \) in a neighborhood of \( P_i (i = 1, 2) \) and that \( f^*B = dP_3 \) in a neighborhood of \( P_3 \). Then there exists a finite collection of \( H_i \)'s so that \( M(H_1, H_2) \)'s form an étale covering of the open subspace \( M_0 \) of \( M \) in Proposition 3.1(1) consisting of morphisms from \( \mathbb{P}^1 \). \( \square \)

**Definition 3.3.** We denote by \( M(X, B, \beta) \) the algebraic space \( M \) in the Proposition 3.1 with \( d = (B, \beta) \), and similarly for \( C, D, f \).

Let \( X \) be a smooth rational projective surface and \( B \) a smooth anti-canonical divisor of \( X \). Then a homology class of degree 2 is the same thing as a divisor class \( D \), and there are only finitely many points \( P \) on \( B \) with \( D|_B \sim dP \) if \( d > 0 \). So, for such a point \( P \) on \( B \), we define \( M(X, B, P, D) \) to be the open and closed subspace of \( M(X, B, D) \) consisting of maps whose image intersect with \( B \) only at \( P \) if \( d > 0 \) and \( M(X, B, P, D) := M(X, B, D) \) if \( d = 0 \).

We denote their open subspaces of morphisms from \( \mathbb{P}^1 \) by adding \( 0 \).

In the situation of the latter half of the definition, we may call \( \deg M_0(X, B, P, D) \) the “number” of curves in \( |D| \) satisfying (*) for \( (B, P) \) by the following lemma.

**Lemma 3.4.** Let \( X \) be a smooth rational projective surface, \( B \) a smooth anti-canonical divisor and \( D \) a divisor class. Then,

(1) Geometric points of \( M_0(X, B, P, D) \) correspond bijectively to curves \( C \) on \( X \) in the class \( D \) which satisfy (*) for \( (B, P) \).
(2) A point of \( M_0(X, B, P, D) \) is reduced if the normalization morphism of the corresponding curve is immersive outside \( B \).

(3) \( M(X, B, D) \) is supported on a finite set.

**Proof.** (1) is easy, for we required morphisms in \( M(X, B, D) \) to be birational.

(3) follows from Proposition 1.1 (for morphisms from reducible conics, we look at components of them).

The following proposition, together with Lemma 3.2 for \( d > 0 \), implies (2).

**Proposition 3.5 (a log version of Proposition 3 in [Mr]).** Let \( X \to S \) be a quasi-projective family of varieties with \( S \) noetherian, \( B_i \) Cartier divisors on \( X \), \( C \to S \) a flat projective family of schemes and \( D_i \) Cartier divisors on \( C \) flat over \( S \). Denote the fibers by \( X_s \) etc. Let \( M \) be the scheme of morphisms \( f : C_s \to X_s \) such that \( f^*B_{i,s} = D_{i,s} \) in a neighborhood of \( \text{Supp}D_{i,s} \) and that \( \sum_{i \in I} B_{i,s} \) is normal crossing over \( S \) at \( f(\cap_{i \in I} \text{Supp}D_{i,s}) \) for any \( I \): in particular, for \( I = \emptyset \), \( X \) is smooth over \( S \) at \( f(C_s) \) and for \( I = \{ i \} \), \( B_{i,s} \) is smooth over \( S \) at \( f(\text{Supp}D_{i,s}) \). For a point \( [f : C_s \to X_s] \) of \( M \), let \( T \) be the locally free sheaf on \( C_s \) which is the pullback of \( T_{X_s}(\log \sum_{i \in I} B_{i,s}) \) near a point \( P \) which is in \( \cap_{i \in I} \text{Supp}D_{i,s} \) and which is not in the support of the other \( D_{i,s} \)'s. Then the completion of \( M \) at \( [f] \) is defined by \( h^1(C_s, T) \) equations in the completion of \( H^0(C_s, T) \times S \).

**Proof.** Similar to the proof of Proposition 3 in [Mr]. \( \square \)

**Definition 3.6.** Let \( l \) and \( m \) be nonnegative integers, \( e \), \((a_i)_{i=1}^l \) and \((b_j)_{j=1}^m \) positive integers with \( d = 3e - \sum a_i - \sum b_j \ge 0 \), \( B \) a smooth cubic in \( \mathbb{P}^2 \), \( P_1, \ldots, P_l \) distinct points on \( B \) and \( P \) a point on \( B \) with \( (d + \sum b_j)P + \sum a_i P_i \sim eH \).

Let \((X((P_i), P, m), B((P_i), P, m; B))\), or \((X, B)\) for short, be the pair obtained by blowing up \( P_1, \ldots, P_l \) and then blowing up \( m \) times the inverse image of \( P \) on the proper transform of \( B \), \( E_i \) the inverse images of \( P_i \) and \( F_i \) the proper transform of the exceptional divisor of the \( i \)-th blow-up at \( P \). In this section and the next, we use these notations and identify \( B \subset \mathbb{P}^2 \) and \( B \subset X \) when it is not confusing.

Let \( B, P \) and \( P \) be general with \( (d + \sum b_j)P + \sum a_i P_i \sim eH \): we also require that \((d + \sum b_j)/gP + \sum a_i/g P_i \sim (e/g)H \mid_B \) does not hold for \( g \) dividing \( e \) and \( a_i \). Let \( (X, B) = (X(e; (a_i); (b_j)), B(e; (a_i); (b_j))) := (X((P_i), P, m), B((P_i), P, m; B)) \) and \( D = D(e; (a_i); (b_j)) := eH - \sum a_i E_i - \sum_{k=1}^m (\sum_{j=1}^k b_j) F_k \). Then we define \( M(e; (a_i); (b_j)) \) to be equal to \( M_0(X, B, P, D) \) and we set \( n(e; (a_i); (b_j)) := \deg M(e; (a_i); (b_j)) \).

Although the variety of \( (B, (P_i), P) \) satisfying the condition above may be reducible, it turns out that these numbers are well defined in the cases we are concerned with.

**Lemma 3.7.** In the situation of Definition 3.6, if \( C \in |D(e; (a_i); (b_j))| \) satisfies \( C_{|B(e; (a_i); (b_j))} = dP \), it is of the form \( A + \sum c_j F_j \) where \( A \) is irreducible and reduced. \( \square \)

Now we have the following relations.

**Theorem 3.8.** (We use the notations in Definition 3.6.)

(1) If \( d(e; (a_i); (b_j))_{j=1}^m > 0 \) and the image of any morphism representing a point in \( M_0(e; (a_i); (b_j)) \) is smooth at \( P \), \( p(e; (a_i); (b_j))_{j=1}^m \) is a point of \( M_0(e; (a_i); (b_j)) \).

(2) The variety of \( (B, (P_i), P) \) satisfying the condition above may be reducible, it turns out that these numbers are well defined in the cases we are concerned with.
(2) If $d(e; (a_i)_{i=1}^{l+1};) = 0$, $n(e; (a_i)_{i=1}^{l+1}) = n(e; (a_i)_{i=1}^{l+1}; a_{l+1})$.

(3) For $d(e; (a_i)_{i=1}^{l+1};) > 0$, assume the following:
   
   (a) Any curve in $M(e; (a_i)_{i=1}^{l+1}; a_{l+1} + 1)$ is transversal to $F_1$ at $P$ if $d(e; (a_i)_{i=1}^{l+1};) > 1$, and is transversal to $F_1$ if $d(e; (a_i)_{i=1}^{l+1};) = 1$.

   (b) Any curve in $M(e; (a_i)_{i=1}^{l+1}; a_{l+1} + k)$ is transversal to $F_1$ for $k \geq 2$.

Then,

$$n(e; (a_i)_{i=1}^{l+1};) = n(e; (a_i)_{i=1}^{l+1}; a_{l+1}) + (1 + \delta_{d(e; (a_i)_{i=1}^{l+1};), 1} a_{l+1}) n(e; (a_i)_{i=1}^{l+1}; a_{l+1} + 1).$$

**Proof.** For (1), it suffices to show the following:

**Lemma 3.9.** Let $X$ be a smooth rational surface, $B \subset X$ a smooth anti-canonical divisor, $S$ the spectrum of an artinian local $\mathbb{C}$-algebra $(R, m)$ with $R/m = \mathbb{C}$ and $C_S$ a projective flat scheme over $S$ such that $C := C_S \otimes R/m$ is a reduced curve. Let $\sigma_S \subset C_S$ be a Cartier divisor mapping isomorphically onto $S$ (and hence lying on the smooth locus) and let $D_S = d\sigma_S$. Let $f_S : C_S \to X \times S$ be a morphism over $S$ such that $f_S^*(B \times S) = D_S$ and that the restriction $f := f_S|_C$ is birational onto its image and immersive at singular points. Then the composition of $f_S|_C$ and the projection to $X$ is a constant map.

**Proof.** Let $\mathcal{I}_S$ denote the kernel of the homomorphism $f_S^\# : O_{X \times S} \to f_*O_{C_S}$ and $C'_S$ the subscheme of $X \times S$ defined by $\mathcal{I}_S$.

Then $\mathcal{I}_S$ is locally free: first, for $Q \in f(C)$ and $Q' \in f^{-1}(Q)$, we look for a formal function in $I_S := \ker(g_S : \hat{O}_{X \times S, Q} \to \hat{O}_{C_S, Q'})$ which generates $I := \ker(g : \hat{O}_{X, Q} \to \hat{O}_{C, Q'})$.

If $Q'$ is singular, $g$ is surjective by assumption, hence $g_S$ is also surjective. Then, writing down exact sequences, we see that the claim holds.

If $Q'$ is smooth, take coordinate functions $u \in \hat{O}_{C_S, Q'}$ and $x, y \in \hat{O}_{X \times S, Q}$ over $R$. Replacing $u$ by another parameter in $(u)$, we have first $y \mod m = u^a \mod m$ and then $y = u^a + \sum_{i=0}^{a-1} b_i u^i$, where $b_i \in m$. If $x = \sum_{i=0}^{\infty} a_i u^i$, the formal power series obtained from $\prod_{j=1}^{a} \sum_{i=0}^{\infty} a_i u_j^i$ by substituting $-b_{a-1}$ for $\sum u_j, (-1)^a (b_0 - y)$ for $\prod u_j$, etc., is an element of $I_S$ and generates $I$.

Taking the product over the points of $f^{-1}(Q)$, we see that at any point $Q$ there exists a local section $\phi$ of $\mathcal{I}_S$ which generates $I := \ker(f^\# : O_X \to f_*O_C)$. For a small neighborhood $U \subset X_S$ of $Q$, $f_S$ is a closed immersion on $U \setminus \{Q\}$. Since $C_S$ is flat over $S$, $\ker(\mathcal{I}_S \to \mathcal{I}) = m\mathcal{I}_S$ holds on $U \setminus \{Q\}$, and we see that $\mathcal{I}_S$ is generated by $\phi$ on $U \setminus \{Q\}$. Therefore, for a section $\psi$ of $\mathcal{I}_S$ on $U$, there exists a (unique) section $\alpha \in \Gamma(U \setminus \{Q\}, \mathcal{O}_{X \times S})$ such that $\psi = \alpha \phi$ (for $\phi$ is not a zero divisor). Then $\alpha$ extends to a section $\hat{\alpha} \in \Gamma(U, \mathcal{O}_{X \times S})$ and $\psi = \hat{\alpha} \phi$ holds, since $X$ satisfies (S2). Thus $\mathcal{I}_S$ is invertible.

Take a local analytic coordinate $u$ on $C_S$ with $\sigma_S = (u = 0)$ and local parameters $(x, y)$ on $X$ such that $B = (y = 0)$ and $P := f(\sigma) = (x = y = 0)(\sigma := \sigma_S|_C)$. Then we have $f_S^\# y = u^d A$, where $A \in R[[u]]^\times$. By changing $u$ as above, we may assume $f_S^\# y = u^d$. If $f_S^\# x = \sum_{i=0}^{\infty} a_i u^i$, where $a_i \in R$ and $a_0 \in m$, we see that $\mathcal{I}_S|_B$ is generated by $(x - a_0)^d$ at $P$. What we have to show is that $a_0$ is zero.

Now $\mathcal{I}_S|_{B \times S}$ is a line bundle on $B \times S$ with a section defining $C'_S \cap (B \times S)$, which is given by $(x - a_0)^d = 0$. Therefore, if we define a morphism $\alpha : S \to B$ by $x = a_0$ and $\beta : B \to \operatorname{Pic}^d(B)$ by the line bundle $O_{B, \alpha}(\Delta)$, where $\Delta$ is the diagonal, we have $\beta \circ \alpha = \alpha_0$. Then $\alpha$ is surjective onto $B$ and $\alpha_0 = 0$, hence $a_0 = 0$.
\( \mathcal{I}_S \mid_{B \times S} \) is isomorphic to the pullback of the universal bundle on \( B \times \Pic^d(B) \) by \( \beta \circ \alpha \). As \( X \) is rational, there exists an isomorphism \( \mathcal{I}_S \cong \mathcal{O}_X(-f_*C \times S) \), and therefore \( \beta \circ \alpha \) is a constant map. Since \( \beta \) is étale, \( \alpha \) is also constant, that is, \( a_0 = 0 \). □

(2) is clear.

To prove (3), we look at the behavior of \( M \) when \( P_{l+1} \) approaches \( P \).

Let \( B' \) be a general cubic curve in \( \mathbb{P}^2 \), \( \mathcal{P}'_1, \ldots, \mathcal{P}'_{l+1}, \mathcal{P}' \subset B' \times \Delta \subset \mathbb{P}^2 \times \Delta \) be sections over a germ \((0 \in \Delta) \) of a smooth curve and \( e \) and \( a_i \) positive integers such that \( \mathcal{P}'_i \) are pairwise disjoint, that \( \mathcal{P}' \) does not intersect \( \mathcal{P}'_i \) for \( i = 1, \ldots, l \) and possibly intersects \( \mathcal{P}'_{l+1} \) only over 0 and that \( d \mathcal{P}' + \sum a_i \mathcal{P}'_i \sim eH \mid_{B' \times \Delta} \), where \( d = d(e; (a_i);) > 0 \).

Blow up \( \mathbb{P}^2 \times \Delta \) at \( \bigcup \mathcal{P}'_i \) and call this \( X' \), and let \( B \) be the proper transform of \( B' \times \Delta \), \( \mathcal{E}_i \) the exceptional divisors, \( \mathcal{P}'_i \) their intersections with \( B \), \( \mathcal{P} \) the proper transform of \( \mathcal{P}' \) and \( \mathcal{H} \) the pullback of \( H \times \Delta \). Denote their fibers over 0 by \( X, B \), etc.

Now the construction of \( M(X, B, P, D) \) can be relativized for \((X, B, \mathcal{P}, e\mathcal{H} - \sum a_i \mathcal{E}_i) \to \Delta \). Denote the resulting algebraic space by \( \mathcal{M} \), and the universal family by \( \tilde{p} : C \to \mathcal{M}, \mathcal{D} \subset C \) and \( \tilde{f} : C \to X \).

**Lemma 3.10.** (1) \( \mathcal{M} \) is flat over \( 0 \in \Delta \).

(2) Assume that any element \( C \in [e\mathcal{H} - \sum a_i \mathcal{E}_i] \) with \( C \mid_B = d \mathcal{P} \) is irreducible and reduced if \( P \neq P_{l+1} \), and is of the form \( A + kE_{l+1} \) if \( P = P_{l+1} \), where \( A \) is irreducible and reduced. \( \mathcal{P} \) reduces to a curve in \( M(X, B, P, e\mathcal{H} - \sum i=1^l a_i \mathcal{E}_i - (a_{l+1} + k)E_{l+1}) \) is transversal to \( E_{l+1} \) for \( k \geq 2 \). Then \( \mathcal{M} \) is finite over \( \Delta \).

**Proof.** (1) Let \( f : (C, D) \to (X, B) \) be a morphism corresponding to a point \([f] \) of \( \mathcal{M} \). Take general hypersurfaces \( \mathcal{H}_i \) \((i = 1, 2, 3) \), and define \( \mathcal{N} := \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \) in the same way as \( M(H_1, H_2, H_3) \). Take a point of \( \mathcal{N} \) over \([f] \) representing \( f : (C, D, P_1, P_2, P_3) \to (X, B, H_1, H_2, H_3) \). Then \( \mathcal{N} \) is an étale neighborhood of \([f] \) in \( \mathcal{M} \).

Applying Proposition 3.5 to \( \mathcal{N} \to S^{(d)} \times \Delta \), the completion of \( \mathcal{N} \) at \([f] \) is defined by \( h^1(C, \mathcal{T}) \) equations in the completion of \( H^0(C, \mathcal{T}) \times S^{(d)} \times \Delta \), where \( \mathcal{T} \) is defined as in Proposition 3.5. By Lemma 3.4, the fiber over 0 is 0-dimensional, and by \( \chi(C, \mathcal{T}) = -1 \), we see that the equations above restrict on \( H^0(C, \mathcal{T}) \times S^{(d)} \times \{0\} \) to the ones defining a complete intersection. Therefore, \( \mathcal{M} \) is flat over \( \Delta \).

(2) Let \( R \) be a discrete valuation ring, \( K \) the quotient field of \( R \), \( \Spec R \to \Delta \) a morphism which maps the closed point to 0 and \( \Spec K \to \mathcal{M} \) a morphism commuting with natural morphisms. Then what we have to show is that there exists a finite extension \((R', K') \) such that there exists a morphism \( \Spec R' \to \mathcal{M} \) commuting with natural morphisms.

First, there exists a finite extension \( K' \) of \( K \) with a morphism \( i : \Spec K' \to \mathcal{N} \) commuting with other morphisms, where \( \mathcal{N} \) is an étale neighborhood constructed as in the proof of (1) for divisors \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \). Since \( \mathcal{N} \) is quasi-finite over \( \Delta \) and since the assertion is clear if \( \text{im}(i) \) is a geometric point, we may assume that the image of \( i \) is the generic point of a curve dominating \( \Delta \). So, it suffices to show the following: let \( S \) be a smooth curve with a dominant morphism \( S \to \Delta, P \) a point on \( S \) over 0, \( \alpha : S \to \Spec S \) \( \{P\} \) a morphism. Then, by replacing \( S \) by an étale}
neighborhood of $P$, there exists a morphism $S \to \mathcal{M}$ that commute with natural morphisms.

By $g$, we have a (smooth) ruled surface $C_{S_0} \to S_0$ (since the curves are irreducible by assumption), a Cartier divisor $D_{S_0}$ whose support is a section and a morphism $f_{S_0} : C_{S_0} \to \mathcal{X}$ such that $f_{S_0}^*B = D_{S_0}$. By resolution of indeterminacy and Stein factorization, we can extend $C_{S_0}$ and $f_{S_0}$ to a morphism $p_S : C_S \to S$, where $C_S$ is a normal surface, and a morphism $f_S : C_S \to \mathcal{X}$ which contracts no component of $C_P := p_S^{-1}(P)$. Considering $f_S(C_S)|_B$, we see that $f_P(C_P)|_B = dP$, where $f_P := f_S|_{C_P}$. By assumption, $f_P \ast C_P = A + kE_{l+1}$, where $A$ is irreducible, reduced and different from $E_{l+1}$. Since $D_S := f_S^*(B)$ is supported on a section and does not contain any component of $C_P$, $f_P^{-1}(B)$ consists of one point. From this, we see that $A$ is a curve corresponding to a point $M(X, B, P, eH - \sum_{i=1}^{l+1} a_i E_i - (a_{l+1} + k) E_{l+1})$ and that the component of $C_P$ mapping to $A$ intersects with other components at one point (if $k > 0$). By the latter fact and the assumption, $f_P^{-1}(E_{l+1})$ contains $a_{l+1} + k - 1$ isolated points if $k > 0$. Since $(f_S^{-1}(E_{l+1})$ is generically of degree $a_{l+1}$ over $\Delta$, $k$ is 0 or 1. Thus, $C_P$ is reduced, with one or two components, $f_P$ maps each fiber birationally, and one component maps isomorphically to $E_{l+1}$ if $C_P$ is reducible.

Take general divisors $\mathcal{H}_1', \mathcal{H}_2', \mathcal{H}_3'$. By taking an étale neighborhood of $P$ in $S$, we may assume that $p_S$ has disjoint sections $s_1, s_2, s_3$ disjoint from the singular point of $C_P$ that are also disjoint from $D_S$ such that the pullback of $\mathcal{H}_i'$ is equal to $s_i$ in a neighborhood of $s_i$. When $C_P$ is reducible, we may assume that $s_3$ is on the component $C_2$ mapped to $E_{l+1}$ and that $s_1$ and $s_2$ are on the other component $C_1$.

We show that there is a morphism $S \to S^{(d)}$ which induces $(C_S, s_1, s_2, s_3, D_S)$. If $C_P$ is smooth, the assertion is clear. Assume that $C_P$ is reducible. Now $C_2$ can be contracted, and if we denote the contraction by $h : C_S \to C_S$, $C_S$ is smooth. Let $s_i' = h(s_i)(i = 1, 2, 3)$ and $s_4' = h(D_S)_{red}$. Now it is easy to see the (shortest) resolution of the indeterminacy of $h$ by blowing up smooth points is given by blowing up $d(s_3', s_4')$ times the intersection of the proper transform of $s_3'$ and the fiber over $P$. Thus, $(C_S, s_1, s_2, s_3, D_S)$ is isomorphic to the base change of $(C^{(d)}_d, \sigma_1, \sigma_2, \sigma_3, D^{(d)})$ by the morphism $S \to \mathbb{P}^1$ defined by $t = (s_2' - s_4')(s_3' - s_1')/(s_2' - s_1')(s_3' - s_4')$.

Finally, we obtain a morphism from $S$ to $\mathcal{M}(\mathcal{H}_1', \mathcal{H}_2', \mathcal{H}_3')$ by the universality, and we are done. \hfill $\square$.

Taking $P_i (i = 1, \ldots, l)$ to be general and $P_{l+1} = P$, the following lemma proves the theorem.

**Lemma 3.11.** For $e, a_1, \ldots, a_1, b$, let the assumption be as in Theorem 3.8 (3) with $a_{l+1} = b$, and let $(X, B) = (X(e; (a_i); b), B(e; (a_i); b))$, $F_1$, etc., be as in Definition 3.6. Let $M'$ be equal to $M_0(X, B, P, eH - \sum a_i E_i - (b+1) F_1)$ if $d := d(e; (a_i); b) > 1$ and the scheme of incidence in $M_0(X, B, P, eH - \sum a_i E_i - (b+1) F_1) \times F_1$ if $d = 1$. Then there is an isomorphism

$$j : M' \to M'' := M(X, B, P, eH - \sum a_i E_i - b F_1) \setminus M_0(X, B, P, eH - \sum a_i E_i - b F_1).$$

**Proof.** We construct $j$ as follows.

Case $d > 1$. Let $f_S : (C_S, D_S) \to (X, B)$ be a morphism corresponding to a connected component $S$ of $M'$, which is the spectrum of an artinian local $\mathbb{C}$-algebra. Then we construct a morphism from $C^{(d)}_d \times S$ to $X$ by mapping the first component

$$j_1 : C^{(d)}_d \times S \to X.$$
$C_1 \times S$ by an isomorphism of $(C_1 \times S, (D^{(d)}_{\infty}|_{C_1}) \times S)$ and $(C_S, D_S)$ and the second component $C_2 \times S$ by an isomorphism to $F_1 \times S$ that maps $(C_1 \cap C_2) \times S$ to $P$: by Lemma 3.9, they can be patched. For some choice of the latter isomorphism, the pullback of $B$ is $D^{(d)}_{\infty} \times S$, and this gives a morphism $S \to M''$.

Case $d = 1$. Let $(f_S : C_S \to (X, B), Q_S \subset F_1 \times S)$ be a pair corresponding to a connected component $S$ of $M'$. Then we construct a morphism from $C^{(1)}_S \times S$ to $X$ by mapping the first component $C_1 \times S$ by an isomorphism of $C_1$ and $C_S$ so that $(C_1 \cap C_2) \times S$ is mapped to $Q_S$ and the second component $C_2 \times S$ by an isomorphism to $F_1 \times S$ that maps $(C_1 \cap C_2) \times S$ to $Q_S$ and $D^{(1)}_S$ to $B$. This gives a point of $M''$.

This map $j$ is bijective by Lemma 3.7.

Next, we prove the following: let $S$ be the spectrum of an artinian local $\mathbb{C}$-algebra $(R, m)$ and let $(C_S, D_S)$ be the family induced by the morphism $S \to S^{(d)}; s = \epsilon$, where $s = 1/t$, for some $\epsilon \in m$. If there exists a morphism $f_S : C_S \to X$ such that $f^{\#}_SB = D_S$, then $\epsilon = 0$.

We may assume that $\epsilon m = 0$. $C_S$ is described by replacing $\mathbb{C}$ by $R$ and substituting $\epsilon$ for $s$ in the description of $C^{(d)}$ in the proof of Proposition 3.1. Denote the central fiber by $C_0$, the restriction of $f_S$ by $f_0$, $(z = 0) \subset C_0$ by $A_0$, $(x = 0) \subset C_0$ by $B_0$ and the point $x = z = 0$ by $Q$.

For $d \geq 2$, this is isomorphic to $S$ times reducible conic: in fact, by the coordinate change $x' = x - \epsilon$ and $y' = y + \epsilon y^2$, $C_S$ is given by $C_S = U_1 \cup U_2 \cup U_3$, where

$$U_1 = \text{Spec}R[w],$$

$$U_2 = \text{Spec}R[x', z]/(x'z),$$

$$U_3 = \text{Spec}R[y],$$

and $U_1$ and $U_2$ are patched by $zw = 1$ and $U_2$ and $U_3$ by $x'y' = 1$. Let $A_S$ denote the closed subscheme defined by $z = 0$, $B_S$ the one defined by $x' = 0$, each isomorphic to $\mathbb{P}_S^1$. $D_S|_{A_S}$ is defined by $x'^{d-1} + dx^{d-2} = 0$. If $d \geq 2$, we have $f_0(B_0) = F_1$. If $d = 2$, $D_S|_{B_S}$ is defined by $z + 2\epsilon = 0$ and we may assume $f_0(B_0) = F_1$ by interchanging $A_S$ and $B_S$.

Take functions $u$ and $v$ on a neighborhood of $P$ in $X$ such that $B = (v = 0)$ and $F_1 = (u = 0)$ hold. Since $F_1$ is a $(-1)$-curve, $f_S|_{B_S}$ factors through $F_1$, and therefore $f^{\#}_S u$ divides $x'$. As $f_0(A_0)$ is transversal to $F_1$ at $P$, we have $(f_S|_{A_S})^{\#} u \in x'O^*_{A_S, Q}$. For $\bar{x} := x' + (d/(d-1))\epsilon = x - (1/(d-1))\epsilon$ and $\bar{y} := 1/\bar{x}$, $(f_S|_{A_S})^{\#} v$ divides $\bar{x}^{d-1}$ and $(f_S|_{A_S})^{\#} u \in (\bar{x} - (d/(d-1))\epsilon)O^*_{A_S, Q}$. By Lemma 3.9, we have $\epsilon = 0$.

For $d = 1$, we can describe $C_S$ as $U_1 \cup U_2 \cup U_3$, where

$$U_1 = \text{Spec}R[w],$$

$$U_2 = \text{Spec}R[x, z]/(xz - \epsilon),$$

$$U_3 = \text{Spec}R[y],$$

and $U_1$ and $U_2$ are patched by $zw = 1$, $U_2$ and $U_3$ by $xy = 1$, and $D_S$ as $w = 0$. We have $x|_{U_1 \cap U_2} = \epsilon w$.

Let $g : X \to X'$ be the contraction of $F_1$, and take functions $u, v$ on a neighborhood of $g(F_1)$ such that $g(B) = (v = 0)$ and $(u = 0)$ is a smooth curve tangent to the branch of $g^*(f(C_1))$ corresponding to $Q$. They define functions $u$ and $v$ in a
neighborhood of $F_1$ in $X$, $u_1$ near $F_1 \backslash B$ and $v_1$ near $F_1 \backslash \{f_0(Q)\}$ with $u = u_1 v$ and $v = v_1 u$.

Since $F_1$ is a $(-1)$-curve, the restriction of $f_S$ to the closed subscheme defined by $x = \epsilon = 0$ factors through $F_1$. Therefore, we have $(f_S|_{U_1})^#u \in (\epsilon)$ and $(f_S|_{U_2})^#v \in (x, \epsilon)$, where $U_2' = U_2 \backslash \{\text{some points on } U_3\}$. By $f_S^*B = D_S$ and Lemma 3.9, we have $(f_S|_{U_1})^#u \in (w)$, and we have $(f_S|_{U_1})^#u = \epsilon w a(w)$ with $a(w) \in \mathbb{C}[w]$. Since $f_0|_{A_0}$ is transversal to $F_1$, we have $(f_0|_{A_0})^#v \in xO_{A_0,Q}^*$.

For positive integers $e, a_i$ with $d(e; (a_i);) = 1$, $n(e; (a_i);)$ is equal to the number of rational curves in the blow up of $\mathbb{P}^2$ at general points $P_i$ in the divisor class $eH - \sum a_i E_i$, where $E_i$ are inverse images of $P_i$.

Proof. If $P_i, P$ are as in Definition 3.6, then $M_0(X((P_i, P, 0), B((P_i), P, 0; B), P, eH - \sum a_i E_i)$ is the same thing as the space of rational curves in $X((P_i), P, 0)$ in the divisor class $eH - \sum a_i E_i$. Now move $P_i$ to general points in $\mathbb{P}^2$ over a parameter space $\Delta$. Then the space of rational curves as above is flat, by an argument similar to that in the proof of Lemma 3.10, and proper, since the limit is irreducible and reduced by Lemma 3.7. □

Remark. From the proof of (1) and (3) and Proposition 2.3, we see that, for a general point $P$ of $B$, there exists a curve $C$ through $P$ which is immersible and is of degree $e$ such that the normalization of $C \backslash B$ is isomorphic to $\mathbb{C}^\times$.

§4 Applications of Theorem 3.8

Now we give some results as corollaries of Theorem 3.8.

Lemma 4.1([KI]). For positive integers $e, a_i$ with $d(e; (a_i);) = 1$, $n(e; (a_i);)$ is equal to the number of rational curves in the blow up of $\mathbb{P}^2$ at general points $P_i$ in the divisor class $eH - \sum a_i E_i$, where $E_i$ are inverse images of $P_i$.

Proof. If $P_i, P$ are as in Definition 3.6, then $M_0(X((P_i, P, 0), B((P_i), P, 0; B), P, eH - \sum a_i E_i)$ is the same thing as the space of rational curves in $X((P_i), P, 0)$ in the divisor class $eH - \sum a_i E_i$. Now move $P_i$ to general points in $\mathbb{P}^2$ over a parameter space $\Delta$. Then the space of rational curves as above is flat, by an argument similar to that in the proof of Lemma 3.10, and proper, since the limit is irreducible and reduced by Lemma 3.7. □

Corollary 4.2. (1) For $e \leq 4$, any curve in $M(e; (a_i); (b_j))$ has only nodes and transversal to $\sum F_j$.

(2) Non-zero $n(e; (a_i); (b_j))$ for $e \leq 4$ with $(b_j) \neq (1, 1, \ldots)$ are as follows:

The followings are 1:

\begin{align*}
n(1; 1^1); & (i \leq 3), \\
n(2; 1^1); & (i \leq 5), \\
n(3; 2, 1^1); & n(3; 1^1; 2) (i \leq 7), \\
n(4; 3, 1^1); & n(4; 1^1; 3) (i \leq 9), \\
n(4; 2^3, 1^1); & n(4; 2^3, 1; 2), n(4; 1^1; 2^2), n(4; 1^1; 2^3) (i \leq 6).
\end{align*}
Proof. (1) We prove the assertion for \( e = 4 \). The cases \( e \leq 3 \) are similar.

(a) Any curve in \( M(4; 1^i; 3) \) is smooth and is transversal to \( F_1 \).

Let \( C \) be the curve corresponding to the unique point of \( M(4; 1^i; 3) \). It is clearly smooth. \( C \cap F_1 \) cannot be one point since \( C \) and \( B \) intersect at \( P \) with multiplicity 9. Let \( C' \) denote the image of \( C \) in \( \mathbb{P}^2 \). The possibility left is that \( C' \) has two analytic branches at \( P \), one smooth, intersecting with \( B \) with multiplicity 10, and the other an ordinary cusp. Then we see that \( C' \) can be parameterized as \((x, y) = (u^2/(1 + au^3 + u^4), t = u/(1 + au^3 + u^4))\), where \((x, y)\) is a coordinate on \( \mathbb{P}^2 \), by looking at the pullbacks of two tangent lines at \( P \). Now substituting them in the equation of \( B \), we have \( a = 0 \) and \( B : y^2 = -x^3 + x \), which contradicts the assumption that \( P \) is of order 12.

For \( 0 < i \leq 9 \), consider a family whose generic fiber is \( X(4; 1^i; 3) \) and whose central fiber is \( X(4; 1^{i-1}; 3, 1) \), where \( P \) is blown up first, and then \( P_i \). We have a Cartier divisor \( F_1 \) whose generic fiber is \( F_1 \) and whose central fiber is \( F_1 + F_2 \). Then the limit \( C_0 \) of the curve \( C \) representing the unique point in \( M(4; 1^i; 3) \) lies in \( M(4; 1^{i-1}; 3, 1) \) by Lemma 3.7, and this comes from the curve \( C'_0 \) representing the unique point of \( M(4; 1^{i-1}; 3) \). Since \( C \) is clearly smooth, the transversality is equivalent to reducedness of the pullback of \( F_1 \), and this holds as the pullback of \( F_1 + F_2 \) to \( C_0 \), which is the same thing as the pullback of \( F_1 \) to \( C'_0 \), is reduced by induction.

\[
\begin{array}{cccc}
  i & n(3; 1^i;) & n(4; 2^2, 1^i;) & n(4; 2, 1^i; 2) & n(4; 1^i; 2^2) \\
  0 & 3 & 4 & 4 & 2 \\
  1 & 4 & 5 & 5 & 3 \\
  2 & 5 & 6 & 6 & 4 \\
  3 & 6 & 7 & 7 & 5 \\
  4 & 7 & 8 & 8 & 6 \\
  5 & 8 & 9 & 9 & 7 \\
  6 & 9 & 10 & 10 & 8 \\
  7 & 10 & 12 & 12 & 10 \\
  8 & 12 & 12 & 12 & 10 \\
  9 & 12 & & & \\
\end{array}
\]

\[
\begin{array}{ccc}
  i & n(4; 2^i; 1^i;) & n(4; 1^i; 2) & n(4; 1^i;) \\
  0 & 11 & 10 & 16 \\
  1 & 15 & 14 & 26 \\
  2 & 20 & 19 & 40 \\
  3 & 26 & 25 & 59 \\
  4 & 33 & 32 & 84 \\
  5 & 41 & 40 & 116 \\
  6 & 50 & 49 & 156 \\
  7 & 60 & 59 & 205 \\
  8 & 72 & 71 & 264 \\
  9 & 96 & 93 & 335 \\
  10 & 96 & 96 & 428 \\
  11 & & & 620 \\
  12 & & & 620 \\
\end{array}
\]

(Thus we have \( n(4; ;) = 16 \) again.)
(b) Any curve in $M(4; 1^i; 2^3)$ is smooth and is transversal to $F_1 + F_2 + F_3$.

Let $C$ be the curve corresponding to the unique point of $M(4; 2^3)$. $C$ is clearly smooth. Since $C$ and $B$ intersect at $P$ with multiplicity 6, $C$ intersects with $F_3$ in two points, which shows the assertion for $i = 0$.

For $0 < i \leq 6$, we have a family in which the limit of the curve representing the point in $M(4; 1^i; 2^3)$ lies in $M(4; 1^{i-1}; 2^3, 1)$, and we have the assertion inductively as in (a).

(c) Any curve in $M(4; 1^i; 2^2)$ has one node and is transversal to $F_1 + F_2$.

Let $C$ be the curve corresponding to a point of $M(4; 2^2)$. Then, since $C$ and $B$ intersect with multiplicity 8 at $P$, $C$ is transversal to $F_2$, and it is disjoint from $F_1$. Assume that $C$ has a cusp. Let $C'$ be the image of $C$ in $\mathbb{P}^2$ and take the quadratic transformation defined by blowing up $P$ (on the proper transform of $B$) three times and then blowing down the proper transform of the tangent line at $P$ and the first and second exceptional curves. Then we have a smooth cubic $B_1$ and a cubic $C_1$ with a cusp outside of $B_1$ such that $C_1|_{B_1} = 7P_1 + 2Q_1$, where $Q_1$ is a point of order 12 on $B_1$ and $P_1$ is a point with $P_1 + 2Q_1 \sim H|_{B_1}$. Since $P_1$ is a point of order 6, there exists a conic $D$ such that $D|_{B_1} = 6P_1$. Then we have $D|_{C_1} = 6P_1$, and $P_1$ is a flex of $C_1$ for the group of smooth points on $C_1$ forms an additive group. Thus $P_1$ is a flex of $B_1$, a contradiction. Hence $C$ has only one node.

For $0 < i \leq 8$, we consider a smooth family $\mathcal{X}$ over a germ $\Delta$ of smooth curve whose general fiber is $X(4; 1^i; 2^2)$ and whose central fiber is $X(4; 1^{i-1}; 2^2, 1)$ with Cartier divisors $\mathcal{F}_1$ and $\mathcal{F}_2$ whose general fibers are $F_1$ and $F_2$ and whose central fibers are $F_1$ and $F_2 + F_3$. Changing the base if necessary, we have a morphism from a ruled surface to $\mathcal{X}$ over $\Delta$ whose generic fiber corresponds to a point in $M(4; 1^i; 2^2)$, with image $C$. Then the image $C_0$ of the central fiber is of the form $A + \sum a_i F_i$, where $A$ is irreducible and reduced, by Lemma 3.7. By $C_0 \sim 4H - 2F_1 - 4F_2 - 5F_3$ and $a_i \geq 0$, we see that $C_0 = A$ with $[A] \in M(4; 1^{i-1}; 2^2, 1)$, $C_0 = A + F_1$ with $[A] \in M(4; 1^{i-1}; 3, 1^2)$ or $C_0 = A + F_3$ with $[A] \in M(4; 1^{i-1}; 2^3)$ (only when $d \leq 7$). We can exclude the second one noting that the inverse image of $\mathcal{F}_1$ would contain a isolated point, by (a), while $C \cap F_1 = \phi$. By (b) and induction, $C$ has only nodes. Also, the inverse image of $F_1 + F_2$ to the normalization of $C$ is reduced since the limit contains a reduced point and it is of degree 2. For $i \leq 7$, this shows that $C$ is transversal to $F_1 + F_2$, since one of the intersection is on $B$, which cannot be a node of $C$. For $i = 8$, the limit is a point of $M(4; 1^7; 2^2, 1)$, and we see that $C$ is transversal to $F_1 + F_2$.

(d) Any curve in $M(4; 1^i; 2)$ has two nodes and is transversal to $F_1$.

Let $C$ be the curve representing a point of $M(4; 2)$. Then, since $C$ and $B$ intersect at $P$ with multiplicity 10, $C$ is transversal to $F_1$. Assume that $C$ has a cusp $Q$ and a node $R$, for example. Let $C'$ be the image of $C$ in $\mathbb{P}^2$. Denote the normalization by $f : C'' \to C'$ and the curve obtained by resolving the singularities $P$ and $Q$ (resp. $P$ and $R$) of $C'$ by $C_1$ (resp. $C_2$). Take a coordinate $u$ on $C''$ such that $f^{-1}(P) = \{u = 0, 1\}$, with $f*B = 11[0] + [1]$, and that $f^{-1}(Q) = \{u = \infty\}$. Let $f^{-1}(R) = \{u = a, b\}$. Then, taking the line through $P$ and $R$, we have $3([0] + [1] + [a] + [b]) \sim 11[0] + [1]$ on $C_2$, and taking the line through $P$ and $Q$ we have $3([0] + [1] + 2[\infty]) \sim 11[0] + [1]$ on $C_1$. These are equivalent to $3(a+b) + 2 = 0$ and $(a-1)^2 b^8 = (b-1)^2 a^8$. Thus there are finitely many possibilities for $a$ and $b$, hence for $C$ and $P$ modulo projective equivalence. Since $B$ and one of the branches of $C$ at $P$ intersect with multiplicity 11, $B$ is determined uniquely by $C$ and $P$. Therefore, for a general $B$, there does not exist such a curve $C$. The case when $C$
Corollary 4.4. Any quartic rational curves intersecting with a fixed general cubic $B$ only at a fixed point $P$ of order 12 is nodal outside of $B$, and the number of such curves is as follows:
(a) 16: smooth at $P$,
(b) 10: nodal at $P$,
(c) 2: tacnodal at $P$,
(d) 1: with two smooth branches with triple contact at $P$ and
(e) 1: with ordinary triple point at $P$. □

Corollary 4.4. (1) $n(5; ; ) = 113$.
(2) $n(6; ; ) = 948$.

Proof. (1) Let $P$ be a point of order 15 on $B$. As in the proof of Theorem 2.1, we list up the candidates for $d$ and $[a]$, with the notation there, by requiring that $p_a \geq 0$, that the sum of two of $a_i$ does not exceed $d$ and that the sum of five of $a_i$ does not exceed $2d$. Then we have $n(5; ; ) = 16x + 8y + z$ with
\[ x := \deg M(Y, B_Y, P, 2H - E_i), \]
\[ y := \deg M(Y, B_Y, P, 3H - \sum_{j \neq i} E_j), \]
\[ z := \deg M(Y, B_Y, P, 4H - 2E_i - \sum_{j \neq i} E_j), \]
where we take $i$ such that $(2H-E_i)|_{B_Y} \sim 5P$ for $x$, etc., assuming that they are well defined. By Lemma 2.2, Lemma 3.10 and Corollary 4.2, we have $z = n(4; 2, 1^5; ) = 41$, $y = n(3; 1^4; ) = 7$ and $x = n(2; 1; ) = 1$, hence the assertion.

(2) In the same way, we have $n(6; ; ) = 21n(2; ; ) + 27n(3; 1^3; ) + 9n(4; 2, 1^4; ) + 3n(4; 1^6; ) = 948$. □
Corollary 4.5. Assume that any curve in $M(e; (a_i); b)$ is immersed and transversal to $F_1$ for $e \leq 6$. Then $n(7; ; ) = 8974$.

Proof. We have

$$n(7; ) = 16n(3; 2; ) + 40n(3; 1^2; ) + 40n(4; 2, 1^3; ) + 16n(4; 1^5; ) + n(5; 3, 1^5; ) + 8n(5; 2^2, 1^4; ) + n(6; 2^5, 1; ).$$

By using associativity of quantum cohomology, we have the numbers $n(e; (a_i);)$ in Table with $d(e; (a_i);) = 1$, except for $n(6; 2^8, 1; )$. Also, it is easy to see that $m(6; 2^5; 2) = 12$. From these, Theorem 3.8 gives Table, where $x := n(6; 2^8, 1; )$. (We obtain $n(5; ; ) = 113$ again.) In particular, $n(6; ; ) = 81n(6; 2^8, 1; ) + 6342$. Since this is equal to 948, we have $n(6; 2^8, 1; ) = 90$ and therefore $n(6; 2^5, 1; ) = 2419 - 7n(6; 2^8, 1; ) = 1789$. Thus $n(7; ; ) = 8974$. □

Table. $n(d; (a_i); b)$ for $d = 5, 6$ under the assumption of Corollary 4.5. $n(e; (a_i); b)$ is 0 or 1 for other values of $e, (a_i), b$. We set $x := n(6; 2^8, 1; )$, and this turns out to be 90 in the proof of Corollary 4.5.

$d = 5$.

$$n(5; 3, 2^2, 1^i; ) = n(5; 3, 2, 1^i; 2) = n(5; 2^2, 1^i; 3) = n(3; 1^{i+1};),$$

$$n(5; 2^5, 1^i; ) = n(5; 2^4, 1^i; 2) = n(3; 1^{i+4};),$$

$$n(5; 3, 2, 1^i; ) = n(5; 3, 1^i; 2) = n(5; 2, 1^i; 3) = n(4; 2, 1^i; ),$$

$$n(5; 2^4, 1^i; ) = n(4; 2, 1^{i+3};),$$

$$n(5; 2^3, 1^i; 2) = n(4; 1^{i+3}; 2),$$

$$n(5; 2^3, 1^i; ) = n(4; 1^{i+4};).$$

| $i$ | $n(5; 2^2, 1^i; 2)$ | $n(5; 3, 1^i; )$ | $n(5; 1^i; 3)$ | $n(5; 2^2, 1^i; )$ | $n(5; 2, 1^i; 2)$ |
|-----|--------------------|-----------------|----------------|-----------------|-----------------|
| 0   | 55                 | 24              | 23             | 100             | 89              |
| 1   | 79                 | 35              | 34             | 155             | 140             |
| 2   | 110                | 50              | 49             | 234             | 214             |
| 3   | 149                | 70              | 69             | 344             | 318             |
| 4   | 197                | 96              | 95             | 493             | 460             |
| 5   | 255                | 129             | 128            | 690             | 649             |
| 6   | 325                | 170             | 169            | 945             | 895             |
| 7   | 416                | 220             | 219            | 1270            | 1210            |
| 8   | 584                | 280             | 279            | 1686            | 1614            |
| 9   | 620                | 352             | 351            | 2270            | 2174            |
| 10  | 640                | 448             | 447            | 3510            | 3222            |
| 11  | 640                | 636             | 3510           |                 |                 |
| 12  | 640                |                 |                |                 |                 |
\( d = 6. \)

\[
n(6; 4, 2^3, 1^i; ) = n(6; 4, 2^2, 1^i; 2) = n(6; 2^3, 1^i; 4) = n(3; 1^{i+1}),
\]

\[
n(6; 3^3, 1^i; ) = n(6; 3^2, 1^i; 3) = n(3; 1^i),
\]

\[
n(6; 3^2, 2^3, 1^i; ) = n(6; 3^2, 2^2, 1^i; 2) = n(6; 3, 2^3, 1^i; 3) = n(3; 1^{i+3}),
\]

\[
n(6; 3, 2^6, 1^i; ) = n(6; 3, 2^5, 1^i; 2) = n(6; 2^6, 1^i; 3) = n(3; 1^{i+6}),
\]

\[
n(6; 2^9, i) = n(6; 2^8; 2) = 12,
\]

\[
n(6; 4, 2^2, 1^i; ) = n(6; 4, 2, 1^i; 2) = n(6; 2^2, 1^i; 4) = n(4; 2, 1^i),
\]

\[
n(6; 3^2, 2^2, 1^i; ) = n(6; 3, 2^2, 1^i; 3) = n(4; 2, 1^{i+2}),
\]

\[
n(6; 3^2, 2, 1^i; 2) = n(4; 1^{i+2}; 2),
\]

\[
n(6; 3, 2^5, 1^i; ) = n(6; 2^5, 1^i; 3) = n(4; 2, 1^{i+5}),
\]

\[
n(6; 3, 2^4, 1^i; 2) = n(4; 1^{i+5}; 2),
\]

\[
n(6; 4, 2, 1^i; ) = n(6; 4, 1^i; 2) = n(6; 2, 1^i; 4) = n(5; 3, 1^i),
\]

\[
n(6; 3^2, 2, 1^i; ) = n(6; 3, 2, 1^i; 3) = n(4; 1^{i+2}),
\]

\[
n(6; 3^2, 1^i; 2) = n(5; 2^2, 1^{i-1}; 2)(i > 0),
\]

\[
n(6; 3^2, 2) = 37,
\]

\[
n(6; 3, 2^4, 1^i) = n(4; 1^{i+5}),
\]
\[ n(6; 3^2, 1^i; ) = n(6; 3, 1^i; 3) = n(5; 2^2, 1^{i-1})(i > 0), \]
\[ n(6; 3^2; ) = n(6; 3; 3) = 63, \]
\[ n(6; 3, 2^3, 1^i; ) = n(5; 2^2, 1^{i+2}; ), \]
\[ n(6; 3, 2^2, 1^i; 2) = n(5; 2, 1^{i+2}; 2), \]

| \(i\) | \(n(6; 4, 1^i)\) | \(n(6; 1^i; 4)\) | \(n(6; 2^3, 1^i; 3)\) | \(n(6; 2^6, 1^i; )\) | \(n(6; 2^5, 1^i; 2)\) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0   | 46              | 45              | 230             | -180 + 9x       | -221 + 9x       |
| 1   | 70              | 69              | 339             | 415 + 5x        | 365 + 5x        |
| 2   | 105             | 104             | 487             | 984 + 2x        | 924 + 2x        |
| 3   | 155             | 154             | 683             | 1548            | 1476            |
| 4   | 225             | 224             | 937             | 2088            | 1992            |
| 5   | 321             | 320             | 1261            | 3240            | 2952            |
| 6   | 450             | 449             | 1676            | 3240            | 3240            |
| 7   | 620             | 619             | 2258            |                 |                 |
| 8   | 840             | 839             | 3462            |                 |                 |
| 9   | 1120            | 1119            | 3510            |                 |                 |
| 10  | 1472            |                 |                 |                 |                 |
| 11  | 1920            |                 |                 |                 |                 |
| 12  | 2560            |                 |                 |                 |                 |
| 13  | 3840            |                 |                 |                 |                 |
| 14  | 3840            |                 |                 |                 |                 |

\[ n(6; 3, 2^2, 1^i; ) = n(5; 2, 1^{i+2}), \]

| \(i\) | \(n(6; 3, 2^1, 1^i; 2)\) | \(n(6; 2^2, 1^i; 3)\) | \(n(6; 2^5, 1^i; )\) | \(n(6; 2^4, 1^i; 2)\) |
|-----|-----------------|-----------------|-----------------|-----------------|
| 0   | 316             | 345             | 2640 - 16x      | 2525 - 16x      |
| 1   | 511             | 555             | 2419 - 7x       | 2264 - 7x       |
| 2   | 804             | 868             | 2784 - 2x       | 2580 - 2x       |
| 3   | 1232            | 1322            | 3708            | 3445            |
| 4   | 1841            | 1964            | 5184            | 4850            |
| 5   | 2687            | 2851            | 7176            | 6749            |
| 6   | 3838            | 4052            | 10128           | 9512            |
| 7   | 5381            | 5656            | 16608           | 14748           |
| 8   | 7462            | 7818            | 16608           | 16608           |
| 9   | 10492           |                 |                 |                 |
| 10  | 16272           |                 |                 |                 |
| 11  | 18132           |                 |                 |                 |
| $i$ | $n(6; 3, 2, 1^i; )$ | $n(6; 3, 1^i; 2)$ | $n(6; 2, 1^i; 3)$ | $n(6; 2^4, 1^i; )$ | $n(6; 2^3, 1^i; 2)$ |
|-----|-------------------|-----------------|-----------------|-----------------|-----------------|
| 0   | 444               | 381             | 420             | $-473 + 25x$    | $-703 + 25x$    |
| 1   | 760               | 660             | 725             | $2052 + 9x$     | $1713 + 9x$     |
| 2   | 1271              | 1116            | 1221            | $4316 + 2x$     | $3829 + 2x$     |
| 3   | 2075              | 1841            | 2005            | 6896            | 6213            |
| 4   | 3307              | 2963            | 3211            | 10341           | 9404            |
| 5   | 5148              | 4655            | 5019            | 15191           | 13930           |
| 6   | 7835              | 7145            | 7665            | 21940           | 20264           |
| 7   | 11673             | 10728           | 11453           | 31452           | 29194           |
| 8   | 17054             | 15784           | 16774           | 46200           | 42738           |
| 9   | 24516             | 22830           | 24164           | 79416           | 68886           |
| 10  | 35008             | 32738           | 34560           | 79416           | 79416           |
| 11  | 51280             | 47770           | 50640           |                 |                 |
| 12  | 87544             | 77014           | 84984           |                 |                 |
| 13  | 87544             | 87544           | 87544           |                 |                 |

| $i$ | $n(6; 3, 1^i; )$ | $n(6; 1^i; 3)$ | $n(6; 2^3, 1^i; )$ | $n(6; 2^2, 1^i; 2)$ |
|-----|-----------------|----------------|-------------------|-------------------|
| 0   | 459             | 414            | $5340 - 36x$      | $4995 - 36x$      |
| 1   | 840             | 771            | $4637 - 11x$      | $4082 - 11x$      |
| 2   | 1500            | 1396           | $6350 - 2x$       | $5482 - 2x$       |
| 3   | 2616            | 2462           | 10179             | 8857             |
| 4   | 4457            | 4233           | 16392             | 14428            |
| 5   | 7420            | 7100           | 25796             | 22945            |
| 6   | 12075           | 11626          | $39726$           | 35674            |
| 7   | 19220           | 18601          | $59990$           | 54334            |
| 8   | 29948           | 29109          | 89184             | 81366            |
| 9   | 45732           | 44613          | $131922$          | 120906           |
| 10  | 68562           | 67091          | $200808$          | 183060           |
| 11  | 101300          | 99381          | $359640$          | $305244$         |
| 12  | 149070          | 146511         | $359640$          | $359640$         |
| 13  | 226084          | 222249         |                 |                 |
| 14  | 401172          | 385812         |                 |                 |
| 15  | 401172          | 441072         |                 |                 |
\[\begin{array}{cccccc}
i& n(6; 2^2, 1^i; ) & n(6; 2, 1^i; 2) & n(6; 2, 1^i; ) & n(6; 1^i; 2) & n(6; 1^i; ) \\
0 & -2381 + 49x & -2801 + 49x & 7336 - 64x & 6922 - 64x & -6342 + 81x \\
1 & 2614 + 13x & 1889 + 13x & 4535 - 15x & 3764 - 15x & 580 + 17x \\
2 & 6696 + 2x & 5475 + 2x & 6424 - 2x & 5028 - 2x & 4344 + 2x \\
3 & 12178 & 10173 & 11899 & 9437 & 9372 \\
4 & 21035 & 17824 & 20372 & 17839 & 18809 \\
5 & 35463 & 30444 & 39896 & 32796 & 36648 \\
6 & 58408 & 50743 & 70340 & 69444 & \\
7 & 94082 & 82629 & 121083 & 102482 & 128158 \\
8 & 148416 & 131642 & 203712 & 174603 & 230640 \\
9 & 229782 & 205618 & 335354 & 290741 & 405243 \\
10 & 350688 & 316128 & 540972 & 473881 & 695984 \\
11 & 533748 & 483108 & 857100 & 757719 & 1169865 \\
12 & 838992 & 754008 & 1340208 & 1193697 & 1927584 \\
13 & 1558272 & 1295640 & 2094216 & 1871967 & 3121281 \\
14 & 1558272 & 1558272 & 3389856 & 3004044 & 4993248 \\
15 & 6506400 & 5302884 & 7997292 & & \\
16 & 6506400 & 6506400 & 13300176 & & \\
17 & 26312976 & & & & \\
18 & 26312976 & & & & \\
\end{array}\]

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