QUANTUM ERGODICITY FOR SHRINKING BALLS IN ARITHMETIC HYPERBOLIC MANIFOLDS

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Abstract. We study a refinement of the quantum unique ergodicity conjecture for shrinking balls on arithmetic hyperbolic manifolds, with a focus on dimensions 2 and 3. For the Eisenstein series for the modular surface $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$ we prove failure of quantum unique ergodicity close to the Planck-scale and an improved bound for its quantum variance.

For arithmetic 3-manifolds we show that quantum unique ergodicity of Hecke–Maaß forms fails on shrinking balls centred on an arithmetic point and radius $R \approx t^{-\delta}$ with $\delta > 3/4$. For $\text{PSL}_2(\mathbb{O}_K) \setminus \mathbb{H}^3$ with $\mathbb{O}_K$ being the ring of integers of an imaginary quadratic number field of class number one, we prove, conditionally on the Generalized Lindelöf Hypothesis, that equidistribution holds for Hecke–Maass forms if $\delta < 2/5$. Furthermore, we prove that equidistribution holds unconditionally for the Eisenstein series if $\delta < (1 - 2\theta)/(34 + 4\theta)$ where $\theta$ is the exponent towards the Ramanujan–Petersson conjecture. For $\text{PSL}_2(\mathbb{Z}[i])$ we improve the last exponent to $\delta < 2/5$.

Finally, we study massive irregularities for Laplace eigenfunctions on $n$-dimensional compact arithmetic hyperbolic manifolds for $n \geq 4$. We observe that quantum unique ergodicity fails on shrinking balls of radii $R \approx t^{-\delta_n + \epsilon}$ away from the Planck-scale, with $\delta_n = 5/(n + 1)$ for $n \geq 5$.

1. Introduction

1.1. Quantum ergodicity and restriction theorems. A central question in quantum chaos concerns the statistical behaviour of eigenfunctions of the Laplace operator on Riemannian manifolds of negative curvature. Let $\mathcal{M}$ be a compact Riemannian manifold. We denote the volume element of $\mathcal{M}$ by $dv$ and the Laplace–Beltrami operator on $\mathcal{M}$ by $\Delta_{\mathcal{M}}$. Furthermore, let $(\phi_j)_{j \geq 0}$ be an $L^2$-normalized sequence of Laplace eigenfunctions with corresponding eigenvalues $(\lambda_j)_{j \geq 0}$ which we order such that $\lambda_j \to \infty$ as $j \to \infty$. To each of these eigenfunctions we can associate a probability measure via $dv_j := |\phi_j|^2 dv$.

The quantum ergodicity (QE) theorem of Schnirelman [59], Colin de Verdière [10] and Zelditch [72] asserts that, if the geodesic flow on the unit cotangent bundle $S^*\mathcal{M}$ is ergodic, then there exists a density one subsequence $(\lambda_{j_k})_{k \geq 0}$ of $(\lambda_j)_{j \geq 0}$ such that the corresponding measures $dv_{j_k}$ converge weakly to the normalized measure $\frac{1}{\text{vol}(\mathcal{M})} dv$ as $\lambda_{j_k} \to \infty$, i.e.

$$\frac{1}{\text{vol}(B)} \int_B |\phi_{j_k}|^2 dv \to \frac{1}{\text{vol}(\mathcal{M})}$$

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as $k \rightarrow \infty$ for every continuity set $B \subset \mathcal{M}$. In particular, the quantum ergodicity theorem holds for compact manifolds of negative curvature. Zelditch extended this result to the case of non-compact hyperbolic surfaces of finite volume \[73\]. Furthermore, in \[74\] he estimated the order of growth of sums of the form

$$S(\Lambda) := \sum_{\lambda_j \leq \Lambda} \left| \frac{1}{\text{vol}(B)} \int_B |\phi_j|^2 dv - \frac{1}{\text{vol}(\mathcal{M})} \right|^2,$$

$B$ being fixed, and proved the nontrivial bound

$$S(\Lambda) = o_B(N(\Lambda)) \quad (1.2)$$

for the rate of quantum ergodicity. Here $N(\Lambda) = \#\{\lambda_j : \lambda_j \leq \Lambda\}$. Refinements of the ergodicity theorem and related conjectures have been studied in various modifications. We mention the relation of quantum ergodicity to the random wave conjecture of Berry \[3\] (see also \[22\]) which states that eigenfunctions of a classically ergodic system should show Gaussian random behaviour as the eigenvalues goes to infinity, i.e. the eigenfunctions should behave as random waves. Some of the most interesting refinements of the ergodicity theorem are the so-called restriction theorems. Here we are interested in the question whether the quantum unique ergodicity theorem, i.e. (1.1), still holds if the set $B$ is replaced by a sequence of sets whose size is decaying fast. This problem has been extensively studied in various cases, e.g. for the $n$-dimensional sphere $\mathbb{S}^n$ \[17\], the $n$-dimensional flat torus $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ \[24\], \[39\], \[17\] and \[15\], as well as for general compact manifolds of negative sectional curvature \[16\], \[23\] (see subsection 1.2). In the next subsections we will discuss the work of Lester and Rudnick \[39\] in more detail.

In this paper we study quantum ergodicity on shrinking subsets for various arithmetic hyperbolic manifolds.

### 1.2. Quantum unique ergodicity (QUE) for hyperbolic surfaces and shrinking balls.

For hyperbolic manifolds Rudnick and Sarnak \[55\] conjectured that (1.1) holds for all eigenvalues, i.e. that

$$\frac{1}{\text{vol}(B)} \int_B |\phi_j(z)|^2 dv(z) \rightarrow \frac{1}{\text{vol}(\mathcal{M})} \quad (1.3)$$

as $\lambda_j \rightarrow \infty$ for any fixed continuity set $B$ of $\mathcal{M}$. This very deep and strong prediction is called the quantum unique ergodicity (QUE) conjecture. Using ergodic methods Lindenstrauss \[41\] was able to prove the conjecture in the case that the eigenfunctions $(\phi_j)_{j \geq 0}$ are Hecke–Maaß forms on compact hyperbolic surfaces of a certain arithmetic type. In the case of not co-compact but co-finite hyperbolic surfaces he was only able to determine the quantum limit up to a constant $c$. Later Soundararajan \[63\] proved that $c = 1$ so that the quantum unique ergodicity conjecture is now known for Hecke–Maaß forms on hyperbolic surfaces of arithmetic type. Furthermore, Silberman and Venkatesh \[61\], \[62\] established the QUE conjecture for compact quotients of higher rank real Lie groups. For general hyperbolic manifolds the quantum unique ergodicity conjecture is still open.

We now turn to the case of the modular group $\Gamma := \text{PSL}_2(\mathbb{Z})$ which acts on the hyperbolic plane $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}$ by linear fractional transformations. As usual we equip $\mathbb{H}^2$ with the hyperbolic metric and denote the corresponding volume...
element by \( d\mu(z) \). The corresponding Laplace operator is
\[
\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]
Furthermore, let \((u_j)_{j\geq 0}\) be an orthonormalized basis of eigenfunctions of \(-\Delta\) with the property that they are simultaneous eigenfunctions of the Hecke operators \(T_n\), \(n \in \mathbb{N}\). We write the eigenvalue \(\lambda_j\) of \(u_j\) as \(\lambda_j = 1/4 + t_j^2\) where we choose \(\Re(t_j) \geq 0\). Luo and Sarnak [42] studied refined equidistribution results for automorphic forms on the modular surface \(\mathcal{M}_\Gamma := \Gamma \setminus \mathbb{H}^2\), point-wise for the Eisenstein series and on average for Hecke–Maaß forms. For the quantum variance of Hecke–Maaß forms they proved the upper bound
\[
\sum_{|t_j| \leq T} \left| \frac{1}{\text{vol}(B)} \int_B |u_j(z)|^2 d\mu(z) - \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} \right|^2 = O_{B,\epsilon} \left( T^{1+\epsilon} \right) \tag{1.4}
\]
for every fixed continuity set \(B\) (see [42, Theorem 1.4]). Since the Weyl law implies that \(\mathcal{N}(T) := \#\{j \geq 0 : |t_j| \leq T\} \sim T^2/12\), this is a square-root improvement of Zelditch’s bound (1.2). It also implies the bound
\[
\frac{1}{\text{vol}(B)} \int_B |u_j(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} + O_{B,\epsilon} \left( t_j^{-1/2+\epsilon} \right) \tag{1.5}
\]
on average. Luo and Sarnak conjectured that this rate of convergence holds point-wise. This is supported by Watson’s triple product formula (see Subsections 3.2 and 6.1) and if true it is optimal. For the Eisenstein series \(E(z, s)\) which is formally an eigenfunction of the Laplace operator with corresponding eigenvalue \(s(1-s) = 1/4 + t^2 \geq 1/4\) and the Hecke operators but are not in \(L^2(\Gamma \setminus \mathbb{H}^2)\). Luo and Sarnak [42] proved the asymptotic behaviour
\[
\frac{1}{\log \left( \frac{4}{t^2} \right) \text{vol}(B)} \int_B \left| E \left( z, \frac{1}{2} + it \right) \right|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} + O_B \left( \frac{1}{\log \log t} \right) \tag{1.6}
\]
as \(t \to \infty\). Luo and Sarnak’s upper bound in (1.4) was refined by Zhao [75] and Sarnak–Zhao [58] while Huang [26] derived an asymptotic formula for the quantum variance of Eisenstein series for \(\text{PSL}_2(\mathbb{Z})\).

In the situation of the modular group restriction problems now ask whether QUE, i.e. (1.3) and (1.6), still hold if the fixed set \(B\) is replaced by a sequence of balls \(B_R(w) \subset \mathcal{M}_\Gamma\) whose centre \(w \in \mathcal{M}_\Gamma\) is fixed and whose radii \(R = R_j \to 0\) as \(t_j \to \infty\). These small scale equidistribution problems can be understood as an alternative way to quantify the rate of convergence in (1.3). Physical heuristics indicate that we cannot expect equidistribution below the Planck-scale \(1/\sqrt{\lambda_j}\) (also called de-Broglie wavelength) as in this range quantum phenomena disappear and the Laplace eigenfunctions behave like regular functions. For the modular surface this would imply that (1.3) and (1.6) do not hold anymore if \(R \asymp t_j^{-1} \asymp \lambda_j^{-1/2}\). However, Berry’s random wave conjecture [3] (see also Hejhal–Rackner [22] and Lester–Rudnick [39]) implies that one should expect QE or even QUE close to the Planck-scale, i.e. that (1.3) and (1.6) hold if
\[
R \gg t_j^{-1+\epsilon}
\]
for any $\epsilon > 0$. Investigating the topography of Hecke–Maass cusp forms on the modular surface Hejhal and Rackner [22] provided evidence that supports QUE up to the Planck-scale (1.2). Luo and Sarnak [42] were the first to prove that quantum ergodicity holds for shrinking balls on the modular surface if their radii satisfy $R \gg t_j^{-\delta+\epsilon}$ for some small fixed $\delta > 0$. Following Sarnak’s letter to Reznikov [57], Young [69] and Humphries [27] studied the QUE problem on thin sets of $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$. Young investigated ergodicity of Hecke–Maass cusp forms and Eisenstein series on infinite geodesics and on shrinking balls of the modular surface. Applying product formulae he obtained that QUE for shrinking balls holds if the rate of decay satisfies $R \gg t_j^{-\delta+\epsilon}$ for some fixed $\delta > 0$. Since he refers to product formulae his result for Hecke–Maass cusp forms is conditional on the Generalized Lindelöf Hypothesis (GLH) (see [69, Prop. 1.5]) whereas for Eisenstein series his result is unconditional (see [69, Thm. 1.4]). However, for the Eisenstein series the error term is worse than the one given for Hecke–Maass cusp forms. Recently, Humphries improved Young’s result for Eisenstein series (see [27, Thm. 1.16]). Furthermore, he proved that equidistribution for Hecke–Maass cusp forms fails close to the Planck-scale (see [27, Thm. 1.14]). We summarize these results in the following theorems:

**Theorem 1.1** (Young [69], Humphries [27]). Let $w$ be a fixed point on $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$ and $B_R(w)$ a ball of centre $w$ and radius $R$ where $R \to 0$ as $t_j \to \infty$. Then for any $\epsilon > 0$:

(a) Let $(u_j)_j$ be a sequence of $L^2$-normalized Hecke–Maass forms and $R \gg t_j^{-\delta+\epsilon}$ with $\delta \leq 1/3$. Assuming GLH we have, as $t_j \to \infty$,

$$\frac{1}{\text{vol}(B_R(w))} \int_{B_R(w)} |u_j(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} + o_{w,\delta}(1). \tag{1.7}$$

(b) Let $R \gg t^{-\delta+\epsilon}$ with $\delta \leq 1/6$. For the Eisenstein we have unconditionally, as $t \to \infty$,

$$\frac{1}{\log(\frac{1}{4}+t^2) \text{vol}(B_R(w))} \int_{B_R(w)} \left| E \left( z, \frac{1}{2} + it \right) \right|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} + o_{w,\delta}(1). \tag{1.8}$$

**Theorem 1.2** (Humphries [27]). Let $w$ be a fixed Heegner point on $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$ and $f(t)$ a function satisfying $\lim_{t \to \infty} f(t) \to \infty$ and

$$f(t) = o \left( \exp \left( 2 \sqrt{\frac{\log t}{\log \log t}} \left( 1 + O \left( \frac{\log \log \log t}{\log \log t} \right) \right) \right) \right). \tag{1.9}$$

Then for $R \ll t_j^{-1} f(t_j)$ we have

$$\frac{1}{\text{vol}(B_R(w))} \int_{B_R(w)} |u_j(z)|^2 d\mu(z) = \Omega \left( \frac{|u_j(w)|^2}{(f(t_j))^{3/2}} \right) \neq O(1). \tag{1.10}$$

In particular, quantum unique ergodicity (1.3) fails in this range.

**Remark 1.3.** In particular, Theorem 1.2 implies the failure of equidistribution in the range $R \ll t_j^{-1} (\log t_j)^a$ for any $a > 0$. This can be compared to the case of the Euclidean torus $\mathbb{T}^2$ where Granville and Wigman [15] showed that quantum ergodicity holds for $R \gg t_j^{-1} (\log t_j)^{1+\frac{\log 2}{\log t}}$ but fails for $R \ll t_j^{-1} (\log t_j)^{\frac{\log 2}{\log t}}$.

Moreover, Humphries also obtained spatial variance bounds over the surface that support (1.2) for almost all points $w \in \text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$. In the case of Hecke–Maass
cusp forms his results are conditional on the Generalized Lindelöf Hypothesis whereas the results are unconditional in the Eisenstein series case (see [27, Thm. 1.17, 1.18]). Furthermore, Humphries and Khan [28] proved that small scale mass equidistribution holds all the way down to the Planck-scale $R \gg t_j^{-1+\epsilon}$ on surfaces generated by special Hecke congruence subgroups when considering only the sparse subsequence of dihedral Hecke–Maass forms. For general $n$-dimensional negatively curved manifolds Han [16] and Hezari and Rivière [23] proved, using ergodic tools, that quantum ergodicity holds for shrinking balls $B_R$ if

$$R \gg (\log t_j)^{-1/2}.$$  

(1.11)

That means in the general case we know ergodicity only in shrinking balls of very slow (logarithmic) decay.

1.3. Failure of QUE for Eisenstein series close to the Planck-scale. We first prove an analogue of Theorem 1.2 for Eisenstein series.

**Theorem 1.4.** Let $w$ be a fixed Heegner point on the modular surface. There exists a constant $C = C(w)$ such that if $f(t)$ is a function satisfying $\lim_{t \to \infty} f(t) \to \infty$ and

$$f(t) = o\left(\exp\left(C \sqrt{\frac{\log t}{\log \log t}} (\log t)^{-2/9}\right)\right),$$  

(1.12)

we have for $R \ll t^{-1} f(t)$

$$\frac{1}{\log \left(\frac{1}{4} + t^2\right) \text{vol}(B_{R}(w))} \int_{B_{R}(w)} \left|E\left(z, \frac{1}{2} + it\right)\right|^2 d\mu(z) \neq O(1).$$

In particular, quantum unique ergodicity for Eisenstein series fails if $R \ll t^{-1} f(t)$.

Therefore we see that, as in the case of Maass cusp forms, equidistribution of Eisenstein series fails for radii $R \ll t^{-1}(\log t)^a$ for any $a > 0$.

1.4. Quantum variance of Eisenstein series on shrinking balls of the modular surface. Our second result is a uniform upper bound for the quantum variance of Eisenstein series on shrinking balls of the modular surface. As in Young [69] we apply the spectral theorem and product formulae to prove a uniform estimate in $R$ and $T$. This is the shrinking balls analogue of the Luo-Sarnak bound (1.4) and Huang [26]. We improve on average the exponents obtained by Young and Humphries (see Part (b) of Theorem 1.1) and prove the following result:

**Theorem 1.5.** Let $w$ be a fixed point on $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$. The quantum variance of Eisenstein series satisfies the uniform upper bound

$$\int_{T}^{2T} \frac{1}{\log \left(\frac{1}{4} + t^2\right) \text{vol}(B_{R}(w))} \left|E\left(z, \frac{1}{2} + it\right)\right|^2 d\mu(z) \leq \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} \left(\frac{1}{R^3 + R^2 + T} + \frac{T}{R^2 + T} + \frac{T}{(\log \log T)^2}\right).$$  

(1.13)

In particular, quantum ergodicity holds for $R \gg t^{-\delta+\epsilon}$ with $\delta \leq 1/3$. 
1.13. Note that the first term in (1.13) comes from the contribution of the discrete spectrum. The second term, which is strictly smaller than the first one, comes from the contribution of the continuous spectrum. The third term is independent of $R$ and comes from the degenerate contribution in the generalized Plancherel formula.

1.5. Spectral theory and QUE on arithmetic hyperbolic 3-manifolds. The main part of this paper focuses on the case of 3-dimensional arithmetic manifolds. Before stating our results we briefly introduce the notation used in this paper as well as the most important results for the spectral theory of automorphic forms on hyperbolic 3-space following [12] and their notation. Let $\mathbb{H}^3 := \{P = z + rj : z \in \mathbb{C}, j > 0\}$ be the hyperbolic 3-space which we consider as a subset of Hamilton’s quaternions with the standard basis $1$, $i$, $j$ and $k$. As usual $\mathbb{H}^3$ is equipped with the hyperbolic metric and we denote the corresponding volume element by

$$dv = dv(P) = \frac{dxdydr}{r^3}$$

and the corresponding Laplace-Beltrami operator by $\Delta$,

$$\Delta = r^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r}.$$  (1.15)

The group $\text{PSL}_2(\mathbb{C})$ acts on $\mathbb{H}^3$ in a natural way: let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\text{PSL}_2(\mathbb{C})$. Then, if we understand $P \in \mathbb{H}^3$ as a quaternion whose fourth component is zero, $MP$ is given by

$$MP := (aP + b)(cP + d)^{-1}.$$  

Here this inverse is taken in the skew field of quaternions. This action is orientation-preserving isometric. Let $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, be an imaginary quadratic number field of class number $H_K = 1$. There are nine such imaginary quadratic fields, namely for $D = -1, -2, -3, -7, -11, -19, -43, -67$ and $-163$. We denote the ring of integers of $K$ by $\mathcal{O}_K$ and its unit group by $\mathcal{O}_K^*$. If $D = -1$ we have $\mathcal{O}_K^* = \{\pm 1, \pm i\}$, if $D = -3$ then $\mathcal{O}_K^* = \{\pm 1, \pm \rho, \pm \rho^2\}$, $\rho := \frac{1}{2}(-1 + i\sqrt{3})$, otherwise $\mathcal{O}_K^* = \{\pm 1\}$. Furthermore, we denote the discriminant of $K$ by $d_K$ and $N(n) := |n|^2$ is the norm of $n \in \mathcal{O}_K$. The ring $\mathcal{O}_K$ can be viewed as a lattice in $\mathbb{R}^2$ with fundamental parallelogram $F \subset \mathbb{R}^2$. Apart from co-compact groups $\Gamma \subset \text{PSL}_2(\mathbb{C})$ we are interested in the so-called Bianchi groups $\Gamma := \text{PSL}_2(\mathcal{O}_K)$. These are co-finite subgroups of $\text{PSL}_2(\mathbb{C})$. As we only work with imaginary quadratic number fields whose class number is one, we know that up to $\Gamma$-equivalence the group $\Gamma$ has only the cusp $\infty$. The spectral theory of $-\Delta$ on $\mathcal{M}_\Gamma = \Gamma \backslash \mathbb{H}^3$ is well-known (see e.g. [12]). As $\Gamma$ is not co-compact but co-finite the spectrum of $-\Delta$ consists of a discrete part containing the eigenvalues $\lambda_j = 1 + t_j^2$ and an absolutely continuous part spanning $[1, \infty)$ with multiplicity 1. The absolutely continuous part is given by the Eisenstein series. Let $\Gamma_\infty = \{\gamma \in \Gamma : \gamma \infty = \infty\}$ be the stabilizer of the cusp $\infty$. Note that for $D = -1$ and $D = -3$ the stabilizer also contains elliptic elements of $\Gamma$. Therefore, we define $\Gamma_\infty' = \{\gamma \in \Gamma_\infty : |\text{tr } \gamma| = 2\}$ to be the set of all parabolic elements of $\Gamma$ that stabilize $\infty$. Then the Eisenstein series is given by

$$E(P, s) = \sum_{\gamma \in \Gamma_\infty' \backslash \Gamma} r(\gamma P)^s, \quad \Re(s) > 2.$$
The theory of Eisenstein series for the hyperbolic space as we need it in this paper can be found in [12] or [21]. Note that we normalize the Eisenstein series so that the critical line is $\Re(s) = 1$ as it is also done in [36]. A different way to define the Eisenstein series is

$$E_\infty(P, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} r(\gamma P)^s, \quad \Re(s) > 2.$$  

Then $E(P, s) = \frac{|\mathcal{O}_K|}{s} E_\infty(P, s)$ (see e.g. [12], p. 232). In order to give the Fourier expansion of the Eisenstein series let $\zeta_K(s)$ denote the Dedekind zeta function for $K$ and

$$\phi(s) := \frac{2\pi}{s \sqrt{|d_K|}} \frac{\zeta_K(s)}{\zeta_K(1 + s)}$$  

be the scattering matrix for $\text{PSL}_2(\mathcal{O}_K)$. Then the Fourier expansion of $E_\infty(P, 1 + s)$ is given by

$$E_\infty(P, 1 + s) = r^{1+s} + \phi(s) r^{1-s} + \frac{2(2\pi)^{1+s}}{|d_K|^{(1+s)/2} \Gamma(1 + s) \zeta_K(1 + s)} \sum_{0 \neq \omega \in \mathcal{O}_K} |\omega|^s \sigma_s(\omega) r K_s \left( \frac{4\pi |\omega| r}{\sqrt{|d_K|}} \right) e^{2\pi i \left\langle \frac{\omega}{\sqrt{d_K}}, z \right\rangle}$$  

(see e.g. [12], Theorem 2.11, pp. 369–370 or [21], p. 102). Here $\sigma_s(\omega)$ denotes the generalized divisor function

$$\sigma_s(\omega) = \frac{1}{|\mathcal{O}_K|} \sum_{d \in \mathcal{O}, \, d \mid \omega} |d|^{2s}.$$  

In the three-dimensional case the Hecke operators are defined as follows: if $n \in \mathcal{O} \setminus \{0\}$ we define $\mathcal{M}_n$ to be the set of all matrices of the form $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, $ad - bc = 1$. Then for $f$ being a $\Gamma$-invariant function the Hecke operator $T_n$ is given by

$$(T_n f)(P) := \frac{1}{\sqrt{N(n)}} \sum_{\gamma \in \Gamma \setminus \mathcal{M}_n} f(\gamma P).$$  

The theory for Hecke operators for Bianchi groups is developed in [21]. However, in contrast to Heitkamp we have incorporated the factor $1/\sqrt{N(n)}$ in the definition of the Hecke operator.

In the present work we initiate the study of quantum equidistribution results for shrinking balls on the 3-dimensional arithmetic hyperbolic manifolds $M_\Gamma = \Gamma \setminus \mathbb{H}^3$. Rudnick and Sarnak [55] conjectured that the sequence of eigenstate measures on higher dimensional manifolds still converges to the volume measure, i.e. (1.3) still holds. However, they noticed that the random wave model (RWM) does not apply to Laplace eigenfunctions on compact arithmetic 3-manifolds since the sup-norm of Laplace eigenfunctions can be significantly large in this case. In fact, the analogue of conjecture (1.3) is still unknown in three dimensions as it does not follow from the work of Silberman and Venkatesh [61], [62]). Quantum ergodicity of Eisenstein series for the hyperbolic 3-space was established by Petridis and Sarnak [50] and Koyama [33] who studied related subconvexity estimates for Rankin-Selberg convolutions. Let $u_j$ be an $L^2$-normalized...
Hecke–Maass forms with corresponding eigenvalue $1 + t_j^2$. As usual we can define a measure $v_j$ on $\mathcal{M}$ via

$$dv_j := |u_j|^2 dv.$$ 

The GLH for the $L$-functions appearing in the corresponding triple product formula (see Theorem 6.1) implies that the expected rate of convergence for a continuity set $B \subset \mathcal{M}$ should be

$$\frac{1}{\text{vol}(\mathcal{B})} \int_B |u_j(Q)|^2 dv(Q) = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H}^3)} + O_B, (t_j^{-1+\epsilon}).$$

(1.18)

Furthermore, the quantum ergodicity result of Koyama for the Eisenstein series reads as follows:

$$\frac{1}{\log(1 + t^2)} \frac{1}{\text{vol}(\mathcal{B})} \int_B |E(Q, 1 + it)|^2 dv(Q) = \frac{|O_K^*| \sqrt{|d_K|}}{4 \text{vol}(\Gamma \backslash \mathbb{H}^3)} + O_{B, \Gamma} \left( \frac{1}{\log \log t} \right)$$

(1.19)

as $t \to \infty$ for $K$ an imaginary quadratic number field of class number one. In fact, Koyama [33, p. 485] derived a wrong main term in (1.19) which was corrected by Laaksonen [36, Rem. 1].

1.6. **QUE on shrinking balls for 3-manifolds.** We study the QUE conjecture for shrinking balls $B_R(P) \subset \mathcal{M}_\Gamma$ with fixed center $P \in \mathcal{M}_\Gamma$ on arithmetic 3-manifolds $\mathcal{M}_\Gamma = \Gamma \backslash \mathbb{H}^3$. The volume of a hyperbolic ball in $\mathbb{H}^3$ of radius $R$ and center $P$ is given by (see [12, Eq. (2.6)])

$$\text{vol}(B_R(P)) = \pi (\sinh(2R) - 2R) \sim R^3$$

(1.20)

as $R \to 0$. By the reasoning of Berry’s conjecture one would also in this case expect quantum (unique) ergodicity close to the Planck-scale $R \asymp t_j^{-1+\epsilon}$. However, our first result shows that for specific arithmetic hyperbolic 3-manifolds we cannot have equidistribution close to the Planck-scale for balls centred on arithmetic points. In the following proposition we use the notion of the QCM points and QCM-manifolds as defined by Miličević [47, p. 1380] as well as the notion of a manifold of Maclachlan–Reid type (see [47, p. 1381]). These points play the role of the classical CM-points on Riemann surfaces. Indeed, for every QCM-point $P \in \Gamma \backslash \mathbb{H}^3$ Miličević proved that $u_j$ admits large values at $P$, thus obtaining his strong lower bound for the sup-norm problem (see [47, Thm. 1]):

$$|u_j(P)| = \Omega \left( t_j^{1+O \left( \frac{1}{\log \log t_j} \right)} \right).$$

(1.21)

**Theorem 1.6.** The following two statements hold:

(a) Let $\Gamma$ be an arithmetic co-finite Kleinian group and $u_j$ be an $L^2$-normalized Hecke–Maass eigenform of $\Gamma$. If $\Gamma \backslash \mathbb{H}^3$ is of Maclachlan–Reid type, $P$ is a fixed QCM-point on $\Gamma \backslash \mathbb{H}^3$ and $R \ll t_j^{-\delta}$ with $\delta > 3/4$ fixed, then

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 dv(Q) = \Omega \left( \frac{|u_j(P)|^2}{t_j^{1-4\delta}} \right) \neq O(1).$$
(b) There exist co-compact arithmetic groups $\Gamma$ and arithmetic points $P \in \Gamma \setminus \mathbb{H}^3$ such that if $R \ll t_j^{-\delta}$ with $\delta > 3/4$ fixed, then
\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 \text{dv}(Q) \neq O(1).
\] (1.22)

In particular, in both cases (a) and (b) QUE fails on these shrinking balls, away from the Planck-scale.

We clarify some points in the statement of Theorem 1.6. Our proof relies on the $\Omega$-results of Rudnick and Sarnak [55] and Miličević [47] for the sup-norm problem on arithmetic 3-manifolds (we are thus indirectly using the existence of explicit theta lifts).

Part (a) of Theorem 1.6 follows from (1.21). Similarly, case (b) follows from the classical lower bound of Rudnick and Sarnak [55] which was the first case of a hyperbolic manifold exhibited with large sup-norms of Laplace eigenfunctions. Although they are less explicit in identifying precise classes of manifolds with power growth of $\|u_j\|_\infty$, their result holds for manifolds where one can construct explicit theta lifts orthogonal to sufficiently many Maaß forms. Other results of this type have been proved by Lapid and Offen [37] using their Waldspurger-type formula. We refer to [47, Subsect. 0.5] and [6, Subsect. 1.3] for detailed discussions on this subject.

On the other hand, for co-finite arithmetic 3-manifolds with one cusp we prove that QUE holds for shrinking balls of some rate. It is remarkable that the study of explicit small scale equidistribution can be better understood in three dimensions than in two.

**Theorem 1.7.** Let $\Gamma \setminus \mathbb{H}^3$ be an arithmetic hyperbolic 3-manifold with $\Gamma = \text{PSL}_2(\mathcal{O}_K)$ being a Bianchi group of class number one and $P \in \Gamma \setminus \mathbb{H}^3$ be a fixed point. Then we obtain:

(a) Let $u_j$ be an $L^2$-normalized Hecke–Maaß eigenform. Assuming the Generalized Lindelöf Hypothesis we get for $R \gg t_j^{-\delta+\epsilon}$ with $\delta \leq 2/5$:

\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 \text{dv}(Q) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} + o_{P,\delta}(1).
\]

(b) Let $\theta$ be an exponent towards the Ramanujan–Petersson conjecture. For $R \gg t^{-\delta+\epsilon}$ with $\delta \leq \frac{1-2\theta}{34+2\theta}$ we have:

\[
\frac{1}{\log(1+t^2) \text{vol}(B_R(P))} \int_{B_R(P)} |E(Q, 1+it)|^2 \text{dv}(Q) = \frac{|\mathcal{O}_K^*| \sqrt{|d_K|}}{4 \text{vol}(\Gamma \setminus \mathbb{H}^3)} + o_{P,\delta}(1).
\]

Assuming the Ramanujan conjecture $\theta = 0$ we have equidistribution up to $R \gg t^{-1/34+\epsilon}$.

(c) For $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$ we improve the exponent for the Eisenstein series to $\delta \leq \frac{1-2\theta}{27+2\theta}$.

**Remark 1.8.** In both cases (b) and (c) of Theorem 1.7, assuming the Generalized Lindelöf Hypothesis we obtain QUE up to scale $R \gg t^{-2/5+\epsilon}$.

Currently, the best known exponent $\theta = 7/64$ is due to Nakasuji [48, Cor. 1.2]. It seems possible that the growth of the sup-norm of Hecke–Maaß forms is the only obstacle causing QUE to fail for lower regimes and that $\delta = 3/4$ is the optimal exponent for QUE of Hecke–Maaß forms for all shrinking balls in the 3-dimensional case. Thus, we expect
that for any center point $P$ and for $R \gg t^{-\delta}$ with $\delta < 3/4$ we have quantum unique ergodicity:
\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 dv(Q) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} + o_{P,\delta}(1).
\]
Since Laplace eigenfunctions with large sup-norm are expected to form a thin subsequence of the discrete spectrum, the question of quantum ergodicity up to the Planck-scale remains open.

1.7. Quantum variance estimates for shrinking balls of the Picard manifold. Combining ideas and methods from the proofs of Theorems 1.5 and 1.7 we can further improve the exponent of the shrinking radius $R \gg t^{-\delta}$ for the Eisenstein series on the Picard manifold $\text{PSL}_2(\mathbb{Z}[i]) \setminus \mathbb{H}^3$. In this case, the slightly better exponent in Theorem 1.7 follows from a large sieve of Watt [66], currently known only for congruence subgroups of the Picard group. Applying again this sieve and a mean Lindelöf estimate for the second integral moment of the Hecke $L$-function $L(s, u_j)$ we obtain a uniform upper bound for the quantum variance of Eisenstein series in shrinking balls of the Picard manifold. This can be considered as an analogue of Theorem 1.5 in dimension 3.

**Theorem 1.9.** Let $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$ and $\alpha \in [0, 1]$ be the parameter related to the twelfth moment of Riemann zeta function as in (9.11). The quantum variance of the Eisenstein series satisfies the uniform upper bound:
\[
\int_T^{2T} \left| \frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |E(Q, 1 + it)|^2 dv(Q) \right|^2 dt \ll_{P, \alpha} \frac{T}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} - \frac{2}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} + \frac{T^{-1+\epsilon}}{R^{5+\epsilon}} + \frac{T^{3a-7+\epsilon}}{R^{2a+12+\epsilon}} + \frac{T}{(\log \log T)^2}.
\]

Thus quantum ergodicity holds for shrinking balls of radii $R \gg t^{-\frac{5}{9}+\epsilon}$.

Note that, as in Theorem 1.5, the first term comes from the discrete spectrum, the second from the continuous spectrum and the last one from the degenerate contribution.

**Remark 1.10.** The unconditional result $\alpha = 1$ follows from Heath-Brown’s result for the twelfth moment of the Riemann zeta function (see [19]).

1.8. Failure of QUE in shrinking sets on manifolds of large dimension. In Section 11 we generalize Theorem 1.6 to higher dimensions. For specific discrete arithmetic groups $\Gamma \subset \text{SO}(n, 1)$ acting on the classical $n$-dimensional hyperbolic space $\mathbb{H}^n$, $n \geq 4$, and $\mathcal{M} = \Gamma \setminus \mathbb{H}^n$ we prove that quantum unique ergodicity in shrinking balls centred at arithmetic points fails for radii
\[
R \asymp t_j^{-\delta_n}
\]
for some $\delta_n < 1$. For $n \geq 5$ we derive (1.24) with the explicit exponent $\delta_n = \frac{5}{n+1}$. This follows from the $\Omega$-result of Donnelly [11] for the sup-norm of Laplace eigenfunction on $n$-dimensional arithmetic hyperbolic manifolds. Thus we use again, though indirectly, specific theta lifts constructed in [11]. The case of $\Omega$-results for the sup-norm problem in dimension $n = 4$ is covered by the general result of Brumley and Marshall [6] with an unspecified exponent $\delta = \delta_4$. Since $\delta_n \to 0$ as $n \to \infty$ we infer that quantum unique ergodicity fails in balls shrinking rapidly in terms of the dimension. An analogous phenomenon was proved by Lester and Rudnick [39] for the $n$-dimensional Euclidean
torus $T^n$, $n \geq 4$, who proved the existence of such ‘massive irregularities’ on Euclidean circles of radii
\[ R \asymp t_j^{-n-1-\epsilon}. \]

The method of Lester and Rudnick [39, Sect. 6] is very arithmetic in nature, relying on lattice counting arguments and estimates for representations of positive definite binary quadratic forms. Our proof is more spectral in nature; one can argue that the arithmetic part of the proof is present in the constructions of theta lifts in [6], [11], [47], [55].

For $n$-dimensional compact Riemannian manifolds Han [18, Thm. 3] recently proved an analogous result for non-equidistribution at shrinking balls centred around points with large eigenfunction values.

**Remark 1.11.** As we emphasized earlier the exponent $\delta$ for the equidistribution of Hecke–Maaß forms on 3-manifolds is better than the exponent obtained for the modular group. This can be roughly justified as follows: for dimension $n = 2, 3$ the exponent follows (under GLH) from an estimate of the form
\[ \frac{T}{R^{2n-1}} = o(T^n) \quad \Rightarrow \quad R \gg T^{-\frac{n-1}{2n-1}+\epsilon}. \] (1.25)

The exact behaviour of the optimal exponent under the Generalized Lindelöf Hypothesis for higher dimension $n \geq 4$ remains open.

**Remark 1.12.** Quantum ergodicity and arithmetic quantum unique ergodicity have also been studied for other rank one cases: Lindenstrauss’s results [41] cover also the case of Hecke–Maaß forms on $(\mathbb{H}^2)^n$, while Truelsen [64] studied the quantum ergodicity of Eisenstein series for $(\mathbb{H}^2)^n$. It is an interesting question to investigate the quantum unique ergodicity problem on shrinking balls for these Hilbert modular surfaces.

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2. **Proof of Theorem 1.4**

In this section we prove Theorem 1.4. For this let $z = \frac{b}{2a} + \frac{i}{2a} \sqrt{|d|}$ be a fixed Heegner point for the modular surface $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$ and denote by $q(x, y) = ax^2 + bxy + cy^2$ the positive definite binary quadratic form of discriminant $d = b^2 - 4ac < 0$ associated to the Heegner point $z$. We now define the positive definite rational binary quadratic form $Q$ by $Q(m, n) := |mz + n|^2$ (cf. [46] or [70]) and consider the Epstein zeta function $Z(s, Q)$ associated to this quadratic form $Q$ which is defined by
\[ Z(s, Q) = \sum_{(m, n) \in \mathbb{Z}^2, \ (m, n) \neq (0, 0)} Q(m, n)^{-s} = \sum_n \frac{r_Q(n)}{n^s}, \quad \Re(s) > 1. \] (2.1)
Here $r_Q(n)$ denotes the number of representations of $n$ by $Q$. The Eisenstein series $E(z, s)$ can be given with the help of this Epstein zeta function, namely we have $\zeta(2s)E(z, s) = 3^s(z)Z(s, Q)$ (cf. e.g. [70]). Selberg made the important discovery that an eigenfunction of the Laplace operator with eigenvalue $\lambda = \frac{1}{4} + t^2$, $t \in \mathbb{C}$ is also an eigenfunction of every invariant integral operator and the corresponding eigenvalue depends only on the original eigenvalue $\lambda$ and the kernel of the integral operator. If this kernel is given by $k \circ \rho$ where $\rho$ incorporates the hyperbolic distance, then the new eigenvalue is $h(t)$. The Selberg transform $h(t)$ can be calculated from the kernel using integral transformations (see e.g. [30, Eq. (1.60), (1.62)]). Applying this to the characteristic kernel

$$k_R(u) = \frac{1}{\text{vol}(B_R)} \cdot \chi_{[0,R]}(u)$$

and keeping in mind that the Eisenstein series $E(z, \frac{1}{2} + it)$ is an eigenfunction of the Laplace operator with eigenvalue $\frac{1}{2} + it$, we obtain

$$\frac{1}{\text{vol}(B_R(w))} \int_{B_R(w)} E(z, 1/2 + it) d\mu(z) = h_R(t) \frac{\Im(z(t))^{1/2 + it}Z(1/2 + it, Q)}{\zeta(1 + 2it)} \zeta(1 + 2it)$$

where $h_R(t)$ is the Selberg transform of $k_R$ defined by [30, Eq. (1.62)]. The right-hand side of (2.2) can now be estimated as follows: first of all, we have $h_R(t) = \Omega((Rt)^{-3/2})$ as $Rt \to \infty$ by [27, Lemm. 4.2]. Furthermore, [13, Thm. 3] implies the $\Omega$-result

$$Z(1/2 + it, Q) = \Omega \left( \exp \left( C' \sqrt{\frac{\log t}{\log \log t}} \right) \right)$$

for some constant $C' > 0$ depending on the quadratic form $Q$, i.e. on the Heegner point $z$. Using as well Vinogradov’s bound

$$\zeta(1 + 2it) \ll (\log t)^{2/3}$$

(see [31, Cor. 8.28]) and the Cauchy–Schwarz inequality, we then infer

$$\frac{1}{\log (1 + t^2) \text{vol}(B_R(w))} \int_{B_R(w)} \left| E \left( z, \frac{1}{2} + it \right) \right|^2 d\mu(z) \gg_w \frac{|h_R(t)|^2}{\log t} \frac{\left| Z(1/2 + it, Q) \right|^2}{\zeta(1 + 2it)} = \Omega_w \left( \frac{\left| Z(1/2 + it, Q) \right|^2}{(Rt)^{3}(\log t)^{7/3}} \right).$$

The statement now follows if we choose $C = 2C'/3$ in (1.12).

Conjecturally we have the stronger bound $\zeta(1 + 2it) \ll \log \log t$ (for instance, this follows from the Riemann Hypothesis) which would allow us to improve the bound (1.12).

3. Young’s machinery for the modular surface and product formulae

Before giving the proof of Theorem 1.5 we describe Young’s approach to the shrinking balls problem on the modular surface and recall some basic background on triple product formulae.
3.1. Young’s method for $\Gamma \setminus \mathbb{H}^2$. Let $\phi = \phi_R$ be a test function that satisfies for every $k \geq 1$

$$\|\Delta^k \phi\|_1 \ll_k R^{-2k}. \tag{3.1}$$

We can consider $\phi$ as a smooth approximation for the characteristic function of $B_R$ and for $R \geq 0$ we pick a family $(\phi_R)_R$ of such test functions with the property $\phi_R \to \chi_{B_R}$ as $R \to 0$. Young’s main result for the case of Hecke–Maaß forms case is summarized in the following proposition:

**Proposition 3.1** (Young [69]). Let $u_j$ be a Hecke–Maaß cusp form on the modular surface with Laplace eigenvalue $\frac{1}{4} + t_j^2$. Furthermore, let $\phi = \phi_R$ be a fixed test function as above with $R \gg t_j^{-\delta}$ for some fixed $0 < \delta < 1$. Assuming the generalized Lindelöf hypothesis (GLH) we have, for any $M \geq 1$,

$$\int_{\Gamma \setminus \mathbb{H}^2} \phi(z) |u_j(z)|^2 d\mu(z) = \int_{\Gamma \setminus \mathbb{H}^2} \phi(z) d\mu(z) + O_\epsilon \left( \|\phi\|_2 R^{-1/2} t_j^{-1/2+\epsilon} \right) + O_M \left( \|\phi\|_1 t_j^{-M} \right). \tag{3.2}$$

Approximating $\chi_{B_R}$ by $\phi_R$, normalizing the appearing integrals and using the asymptotic $\|\phi\|_2 \asymp R$, $\|\phi\|_1 \asymp R^2 \asymp \text{vol}(B_R)$ then allows us to rewrite (3.2) as

$$\frac{1}{\text{vol}(B_R(w))} \int_{B_R(w)} |u_j(z)|^2 d\mu(z) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^2)} + O_\epsilon \left( R^{-3/2} t_j^{-1/2+\epsilon} \right) + O_M \left( t_j^{-M} \right). \tag{3.3}$$

Part (a) of Theorem 1.1 now follows immediately from this identity. The proof of Proposition 3.1 requires spectral theory and triple product formulae.

3.2. Triple product formulae for $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$ and regularization of Eisenstein series integrals. Let $u_j$ and $u$ be two Hecke–Maaß cusp forms for the modular group with corresponding eigenvalues $\frac{1}{4} + t_j^2$ and $\frac{1}{4} + t_j^2$. Based on previous works of Garret, Harris, Kudla, Piatetski-Shapiro, Rallis, to name only a few, Watson [65] proved a formula relating

$$\langle |u_j|^2, u \rangle := \int_{\Gamma \setminus \mathbb{H}^2} u(z)|u_j(z)|^2 d\mu(z)$$

to a triple product of $L$-functions associated to $u_j$ and $u$ and thus relating a priori the QUE conjecture to subconvexity bounds for $L$-functions. More precisely, his formula reads as follows:

$$\left| \langle |u_j|^2, u \rangle \right|^2 = \frac{1}{8} \frac{\Lambda(1/2, \text{sym}^2 u_j \otimes u) \Lambda(1/2, u)}{\Lambda(1, \text{sym}^2 u) \Lambda(1, \text{sym}^2 u_j)^2} \tag{3.4}$$

where $\Lambda$ denotes the completed $L$-functions. Replacing the completed $L$-functions by their definition, we see that the right-hand side of (3.4) can be written as a product of the non-Archimedean parts of the $L$-functions with a product of Gamma factors. In the case $u$ is an even form we get (3.4) is equal to

$$\gamma_2(t_j, t) \frac{L(1/2, \text{sym}^2 u_j \otimes u) L(1/2, u)}{L(1, \text{sym}^2 u) L(1, \text{sym}^2 u_j)^2} \tag{3.5}$$
where $\gamma_2(t_j, t)$ satisfies
\[
\gamma_2(t_j, t) \asymp \frac{|\Gamma\left(\frac{1}{2} + i\frac{t_j}{2}\right)|^4 |\Gamma\left(\frac{1}{2} + i(t_j + \frac{t}{2})\right)|^2 |\Gamma\left(\frac{1}{2} + i(t_j - \frac{t}{2})\right)|^2}{|\Gamma\left(\frac{1}{2} + it_j\right)|^4 |\Gamma\left(\frac{1}{2} + it\right)|^2}
\]
for large $t_j$ and $t$. Using Stirling’s formula we obtain the asymptotic
\[
\gamma_2(t_j, t) \asymp \frac{\exp\left(\frac{3}{2}(Q(t_j, t))\right)}{\mathcal{P}_2(t_j, t)}
\]
as $t_j, t \to \infty$ where
\[
Q(t_j, t) = 4|t_j| - |2t_j + t| - |2t_j - t| = \begin{cases} 0, & \text{if } 2t_j > t > 0, \\ 4t_j - 2t, & \text{if } 2t_j \leq t, \end{cases}
\]
and
\[
\mathcal{P}_2(t_j, t) = (1 + |t|)(1 + |2t_j + t|)^{1/2}(1 + |2t_j - t|)^{1/2}
\]
(see also [69, Eq. (4.2)]). By Hoffstein-Lockhart [25, Thm. 0.1] the $L$-values $L(1, \text{sym}^2 u)$ and $L(1, \text{sym}^2 u_j)$ are of moderate growth, namely we have
\[
t_j^{-\epsilon} \ll L(1, \text{sym}^2 u_j) \ll t_j^\epsilon, \quad t^{-\epsilon} \ll L(1, \text{sym}^2 u) \ll t^\epsilon.
\]
As the convexity bound for $L(1/2, u \otimes \text{sym}^2 u_j)$ is
\[
L(1/2, u \otimes \text{sym}^2 u_j) \ll \epsilon (t_j^2 + t)^{\frac{1}{2} + \epsilon},
\]
we see that any subconvexity bound of the form $o(t_j)$ implies the QUE conjecture. In particular, the generalized Riemann hypothesis (GRH) implies the QUE conjecture with the predicted rate of convergence. A similar product formula to (3.4) holds if we replace the Hecke–Maaß cusp form by an Eisenstein series:
\[
\left| \left\langle |u_j|^2, E(\cdot, 1/2 + it) \right\rangle \right|^2 = \frac{1}{4} \frac{|\Lambda(1/2 + it)|^2 |\Lambda(1/2 + it, \text{sym}^2 u_j)|^2}{|\Lambda(1 + 2it)|^2 |\Lambda(1, \text{sym}^2 u_j)|^2}
\]
(see [42, Eq. (17)] or [27, Prop. 2.8]). The Archimedean part of the product appearing on the right-hand side of this identity is similar to the previous one appearing in (3.5) and has an asymptotic behaviour $\asymp \gamma_2(t_j, t)$.

It is well-known that apart from the discrete part the spectrum of the Laplace operator on $L^2(\Gamma \setminus \mathbb{H}^2)$ has also an absolutely continuous part given by the Eisenstein series. As in the case of the Hecke–Maaß forms we define now a measure involving the Eisenstein series as follows:
\[
d\mu_\epsilon(z) = |E(z, 1/2 + it)|^2 d\mu(z).
\]
The inner product of $|E(z, 1/2 + it)|^2$ with the Hecke–Maaß cusp form $u_j$ of Laplace eigenvalue $\frac{1}{4} + \frac{t^2}{4}$ is explicitly given by a product of $L$-functions (see [42, Eq. (17)], [69, Eq. (4.3)]) and we have the product formula
\[
\left| \left\langle |E(\cdot, 1/2 + it)|^2, u_j \right\rangle \right|^2 = \left| \int_{\mathbb{H}^2} u_j(z) d\mu_\epsilon(z) \right|^2 = \frac{1}{2} \frac{|\Lambda(1/2, u_j)|^2 |\Lambda(1/2 + 2it, u_j)|^2}{|\Lambda(1 + 2it)|^4 |\Lambda(1, \text{sym}^2 u_j)|^2}. 
\]
The Gamma factors appearing in (3.10) behave as \( \asymp \gamma_2(t, t_j) \) as \( t_j, t \to \infty \). However, if we replace the Hecke–Maaß cusp form \( u_j \) by an Eisenstein series in (3.10), then the integral does not converge anymore. In order to overcome this technical difficulty we use Zagier’s theory for Rankin–Selberg integrals for functions that are not of not rapid decay but satisfy mild growth conditions [71]. This method was already used in [69] and [27] and consists basically of regularizing the appearing integrals appropriately. Let \( F \) be a \( \Gamma \)-invariant function that satisfies the growth condition

\[
F(z) = \varphi(y) + O(y^{-N})
\]

for any \( N > 0 \) as \( y \to \infty \) where

\[
\varphi(y) = \sum_{i=1}^{m} \frac{c_i}{n_i!} y^{a_i} \log^{n_i} y, \quad a_i, c_i \in \mathbb{C}, \ n_i \geq 0.
\]

Then the regularized integral of \( F \) is defined as

\[
R.N. \int_{\Gamma \backslash \mathbb{H}^2} F(z) d\mu(z) := \int_{\Gamma \backslash \mathbb{H}^2} (F(z) - \mathcal{E}(z)) d\mu(z)
\]

where \( \mathcal{E}(z) \) is a suitable linear combination of Eisenstein series and derivatives of Eisenstein series that can be explicitly given taking into account the \( a_i, c_i \) and \( n_i \), namely we have

\[
\mathcal{E}(z) = \sum_{\alpha_i \geq 1/2} c_i \frac{\partial^{n_i}}{\partial \alpha_i^{n_i}} E(z, \alpha_i)
\]

(see [71], p. 427). If \( F(z) := E(z, 1/2 + it) |E(z, 1/2 + it)|^2 \) we obtain the regularized scalar product

\[
\langle |E(\cdot, 1/2 + it)|^2, E(\cdot, 1/2 + it') \rangle_{\text{reg}} := R.N. \int_{\Gamma \backslash \mathbb{H}^2} (F(z) - \mathcal{E}(z)) d\mu(z).
\]

Zagier’s results [71, Eq. (44)] now give:

**Theorem 3.2.** [71] The regularized triple product integral of Eisenstein series

\[
\langle |E(\cdot, 1/2 + it)|^2, E(\cdot, 1/2 + it') \rangle_{\text{reg}}
\]

is equal to

\[
\frac{\Lambda(1/2 - it')^2 \Lambda(1/2 + i(2t - t')) \Lambda(1/2 - i(2t + t'))}{|\Lambda(1 + 2it)|^2 \Lambda(1 - 2it')}. \tag{3.14}
\]

Squaring and estimating the Gamma factors appearing in the functional equation of the Riemann zeta function we get the following asymptotic behaviour:

\[
\bigg| \langle |E(\cdot, 1/2 + it)|^2, E(\cdot, 1/2 + it') \rangle_{\text{reg}} \bigg|^2 \asymp \gamma_2(t, t') \frac{\zeta(1/2 - it')^4 \zeta(1/2 + i(2t - t'))^2 \zeta(1/2 - i(2t + t'))^2}{|\zeta(1 + 2it)|^4 |\zeta(1 - 2it')|^2}
\]

as \( t, t' \to \infty \). We discuss analogous triple product formulae for arithmetic 3-manifolds in Subsections 6.1 and 6.2.
4. Quantum variance of Eisenstein series for shrinking balls
on the modular surface

For the rest of this section we denote by $\mathcal{B} = \mathcal{B}_{\Gamma}$ a family of non-constant Hecke–Maaß
cusp forms $u_j \in L^2(\Gamma \setminus \mathbb{H}^2)$. The following theorem is an extension of the Plancherel
formula for functions of moderate growth where an extra degenerate contribution appears
naturally.

**Theorem 4.1.** [45, Eq. (4.20), p. 243] If $F$ is a smooth function on the modular surface of
the type $(3.11)$ with $\Re(a_i) \neq 1/2$, $u_0 = \sqrt{3}/\pi$ is the $L^2$-normalized constant eigenfunction
and $G$ is smooth and compactly supported, then

$$
\langle F, G \rangle = \langle F, u_0^2 \rangle \left\langle 1, G \right\rangle + \sum_{u_j \in \mathcal{B}} \langle F, u_j \rangle \langle u_j, G \rangle + \frac{1}{4\pi} \int_{\infty}^{-\infty} \langle F, E(\cdot, 1/2 + it') \rangle \left\langle E(\cdot, 1/2 + it'), G \right\rangle dt' + \langle \mathcal{E}, G \rangle.
$$

(4.1)

Thus in order to bound

$$
\frac{1}{\text{vol}(B_R(w))} \int_{B_R(w)} |E(z, 1/2 + it)|^2 d\mu(z) - \frac{\log(1/4 + t^2)}{\text{vol}(\Gamma \setminus \mathbb{H}^2)}
$$

(4.2)

and prove Theorem 1.5 we use Theorem 4.1 with $G = \phi_R$ and $F(z) = E(z, s_1)E(z, s_2)$,
$s_1 = \overline{s_2} = \frac{1}{2} + it$. This approach using the spectral decomposition allows us to bound
the various terms using the product formulae and the properties of $\phi_R$.

4.1. The contributions of the constant eigenfunction and the degenerate contribution. The contribution of $\langle \mathcal{E}, \phi_R \rangle$ is typically one of the most delicate parts of
the generalized Plancherel formula (4.1) and is related to the constant coefficient of the
Eisenstein series. It follows from [69, pp. 976, 980] that

$$
\left\langle |E(\cdot, 1/2 + it)|^2, u_0^2 \right\rangle \left\langle 1, \phi_R \right\rangle + \langle \mathcal{E}, \phi_R \rangle = \\
\log(1/4 + t^2)\langle \phi_R, u_0^2 \rangle + O \left( \frac{\log t + \|\phi_R\|_1}{\log \log t} \right) + O(t^{-M}).
$$

Hence the contribution of $\langle \mathcal{E}, \phi_R \rangle$ and the constant eigenfunction in (1.13) yield the
leading term and an error term of the size of $O\left( \frac{T}{(\log \log T)^2} \right)$.

4.2. The contribution of the discrete spectrum. If we apply Theorem 4.1 and
use the properties of $\phi_R$ (see (3.1)) and (3.10) to estimate the contribution of those cusp
forms with large eigenvalues, we easily see that the contribution of the discrete spectrum,
i.e. the contribution corresponding to $u_j \in \mathcal{B}$ in the spectral expansion of (4.2), can be
estimated as follows:
\[
\sum_{u_j \in B} \left| \left< E(z, 1/2 + it), u_j \right> \right|^2 \left< u_j, \phi_R \right> \ll \|\phi_R\|_2 \left( \sum_{t_j \leq R^{-1}t^\epsilon} \left| \left< E(z, 1/2 + it), u_j \right> \right|^2 \right)^{1/2} + O_M \left( t^{-M} \right)
\]
\[
\ll \|\phi_R\|_2 \left( \sum_{t_j \leq R^{-1}t^\epsilon} \frac{(1 + |t|)^{\epsilon} |L(1/2, u_j)|^2 |L(1/2 + 2it, u_j)|^2}{(1 + |t_j|)^{1-\epsilon}(1 + |t_j - 2t|)^{1/2}(1 + |t_j + 2t|)^{1/2}} \right)^{1/2} + O_M \left( t^{-M} \right)
\]
(cf. [69], p. 978, (4.23) and (4.24)). We therefore can bound the contribution of the discrete spectrum to the right-hand side of (1.13) by
\[
\int_T^{2T} \frac{1}{R^2} \sum_{t_j \leq R^{-1}t^\epsilon} \frac{(1 + |t|)^{\epsilon} |L(1/2, u_j)|^2 |L(1/2 + 2it, u_j)|^2}{(1 + |t_j|)^{1-\epsilon}(1 + |t_j - 2t|)^{1/2}(1 + |t_j + 2t|)^{1/2}} \, dt
\ll \frac{T^\epsilon}{R^2} \sum_{t_j \leq R^{-1}(2T)^\epsilon} \frac{|L(1/2, u_j)|^2}{(1 + |t_j|)^{1-\epsilon}} \int_T^{2T} \frac{|L(1/2 + 2it, u_j)|^2}{(1 + |t_j - 2t|)^{1/2}(1 + |t_j + 2t|)^{1/2}} \, dt.
\]
Since \( t_j \leq R^{-1}(2T)^\epsilon = o(T) \) we get \( (1 + |t_j - 2t|)^{1/2}(1 + |t_j + 2t|)^{1/2} \approx (1 + |t|) \). Thus the sum is estimated as follows: following Huang [26, Section 3] we infer
\[
\int_T^{2T} \frac{|L(1/2 + 2it, u_j)|^2}{(1 + |t|)} \, dt \ll T^\epsilon (1 + |t_j|)^{\epsilon}.
\]
Thus the mean-subconvexity estimate (see [69, (4.25)])
\[
\sum_{t_j \leq R^{-1}T^\epsilon} |L(1/2, u_j)|^2 \ll R^{-2}T^\epsilon
\]
and summation by parts imply
\[
\sum_{t_j \leq R^{-1}T^\epsilon} \frac{|L(1/2, u_j)|^2}{(1 + |t_j|)^{1-\epsilon}} \ll R^{-1-\epsilon}T^\epsilon.
\]
We finally obtain that the contribution of the discrete spectrum to the right-hand side of (1.13) is bounded by \( R^{-3-\epsilon}T^\epsilon \).

4.3. The contribution of the continuous spectrum. For estimating the contribution of the continuous spectrum, we apply the regularized integral formula from Theorem 3.2. Working as in [69] we bound the contribution of the continuous spectrum by
\[
\int_T^{2T} \left| \frac{1}{R^2} \int_{-\infty}^{\infty} \left< E(z, 1/2 + it), E(z, 1/2 + it') \right>_{\text{reg}} \left< E(z, 1/2 + it'), \phi_R \right> dt' \right|^2 \, dt
\ll \frac{T^\epsilon}{R^{2+\epsilon}} \int_T^{2T} \int_{|t'| \leq R^{-1}T^\epsilon} \left| \zeta(1/2 - it') \right|^4 \left| \zeta(1/2 + it' + 2it) \right|^2 \left| \zeta(1/2 + it' - 2it) \right|^2 \, dt' dt.
\]
together with a small error term $O\left(R^{-1}T^{-M}\right)$ (see [69, Eq. (4.30)]). In order to bound the last integral we use Ingham’s bound for the fourth moment of the Riemann zeta function

$$\int_T^{2T} |\zeta(1/2 + it)|^4 dt \ll T^{1+\epsilon}. \quad (4.3)$$

This bound and an application of the Cauchy–Schwarz inequality imply

$$\frac{T^\epsilon}{R^{2+\epsilon}} \int_{|t'| \leq R^{-1}T^\epsilon} \left| \zeta(1/2 - it') \right|^4 \frac{dt'}{(1 + |t'|)^4} \ll \frac{T^\epsilon}{R^{2+\epsilon}} \int_{|t'| \leq R^{-1}T^\epsilon} \left| \zeta(1/2 + it'/2) \right|^2 \left| \zeta(1/2 + it'/2 - 2it') \right|^2 \frac{dt'}{(1 + |t'|)^4} \ll T^\epsilon R^{-2-\epsilon}.$$  

\[ (4.4) \]

4.4. Proof of Theorem 1.5. Combining the bounds for the various contributions of the spectrum we can now prove Theorem 1.5. Namely, applying Theorem 4.1 we get

$$\int_T^{2T} \left| \int_{B_R(w)} E(z, 1/2 + it) \right|^2 d\mu(z) - \frac{1}{\log(1/4 + t^2) \text{vol}(B_R(w))} \frac{dt}{dt} \ll \frac{T^\epsilon}{R^{2+\epsilon}} + \frac{T^\epsilon}{R^{2+\epsilon}} + \frac{T}{(\log \log T)^2}.$$  

The expression on the right-hand side of the inequality is bounded by $o_w(T)$ if and only if

$$R \gg T^{-1/3 + \epsilon} \ll t^{-1/3 + \epsilon}.$$  

Consequently, we have quantum ergodicity up to this scale.

5. Estimates for the Selberg transform and failure of QUE away from the Planck-scale

In this section we summarize some facts for the Selberg transform for $\Gamma \backslash \mathbb{H}^3$ and give the proof of Theorem 1.6.

5.1. The Selberg transform. For $P = z + rj$ and $Q = z' + r'j \in \mathbb{H}^3$ we set

$$\delta(P, Q) = \frac{|z - z'|^2 + r^2 + r'^2}{2rr'}.$$  

Then the hyperbolic distance $\rho(P, Q)$ of $P$ and $P'$ is given by

$$\cosh \rho(P, Q) = \delta(P, Q)$$  

(see [12, Prop. 1.6]). Furthermore, we define a point-pair invariant $K(P, Q) = k \circ \delta(P, Q)$ by

$$K(\delta(P, Q)) = \frac{1}{\text{vol}(B_R)} \cdot \chi_{[0, R]}(\rho(P, Q)). \quad (5.1)$$
In the situation of the hyperbolic 3-space the Selberg transform (see [12, Ch. 3.5]) of \( k \) is given by

\[
h_R(t) = \frac{a}{\text{vol}(B_R)} \int_{-R}^{R} (\cosh R - \cosh u)e^{itu}du
\] (5.2)

where \( a \) is a constant (see [52, Eq. (4.2)]). Note that \( \chi_{[0,R]} \circ \rho \) denotes the characteristic kernel of the distance \( \rho = \rho(P,Q) \), being 1 if the distance of \( P \) and \( Q \) is less than \( R \) and 0 otherwise.

**Lemma 5.1.** There exists a constant \( c \) such that, as \( R \to 0 \) and \( Rt \to \infty \), we have

\[
h_R(t) = \Omega \left( (Rt)^2 \right).
\] (5.3)

We give a proof of Lemma 5.1 in Section 11 for the general \( n \)-space with \( n \geq 2 \). The estimate (5.3) is useful due to the fact that an eigenfunction of the Laplace operator is also an eigenfunction of the invariant integral operator given by the point-pair invariant (5.1) [12, Ch. 3.5, Thm. 5.3].

**Proposition 5.2.** Let \( u_j \) be a Hecke–Maaß form with eigenvalue \( 1 + t_j^2 \). Then

\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} u_j(Q)dv(Q) = h_R(t_j)u_j(P).
\]

We can now give the proof of Theorem 1.6.

**Proof of Theorem 1.6.** (a) Using the Cauchy–Schwarz inequality and Proposition 5.2 we get the lower bound

\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 dv(Q) \gg |h_R(t_j)|^2 |u_j(P)|^2.
\]

Now assume that \( R \ll t_j^{-\delta} \) for some \( \delta > 0 \). By Lemma 5.1 we have

\[
|h_R(t_j)|^2 |u_j(P)|^2 = \Omega \left( \frac{|u_j(P)|^2}{(Rt_j)^4} \right) = \Omega \left( \frac{|u_j(P)|^2}{t_j^{4-4\delta}} \right).
\]

If \( \mathcal{M} = \Gamma \setminus \mathbb{H}^3 \) is of Maclachlan–Reid type and \( P \) is a fixed QCM-point on \( \Gamma \setminus \mathbb{H}^3 \), then by the lower bound (1.21) we obtain

\[
|u_j(P)| = \Omega \left( t_j^{\frac{4}{3}-\epsilon} \right).
\]

We conclude that

\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |u_j(Q)|^2 dv(Q)
\]

is unbounded for \( \delta > 3/4 \). This completes the proof of part (a) of Theorem 1.6.

(b) Similarly to the lower bound (1.21), Rudnick and Sarnak [55, Thm. 1.2] proved for specific compact manifolds \( \Gamma \setminus \mathbb{H}^3 \) and points \( P \) the bound

\[
|u_j(P)| = \Omega \left( t_j^{1/2} \right).
\]

The proof of part (b) follows. \( \square \)
6. Product formulae on hyperbolic 3-manifolds

6.1. The Watson–Ichino triple formula formula for $\text{PSL}_2(\mathbb{C})$ and other product formulae. Ichino [29] proved a far reaching generalization of Watson’s formula for higher rank reductive groups which was worked out explicitly by Marshall [44] for the case of automorphic representations associated to $\text{PSL}_2(\mathbb{C})$. In our case it simplifies to the following statement:

**Theorem 6.1.** Let $u, u_j$ be two Hecke–Maaß cusp forms for the Bianchi group $\Gamma = \text{PSL}_2(\mathcal{O}_K)$, $\mathcal{O}_K$ being the ring of integers of an imaginary quadratic number field of class number one. Then there exists a constant $C_\Gamma$ such that

\[
\left| \left\langle |u_j|^2, u \right\rangle \right|^2 = \int_{\Gamma \backslash \mathbb{H}^3} u(P) dv_j(P) = C_\Gamma \frac{\Lambda(1/2, u \otimes \text{sym}^2 u_j) \Lambda(1/2, u)}{\Lambda(1, \text{sym}^2 u) \Lambda(1, \text{sym}^2 u_j)^2}. \tag{6.1}
\]

Note that by abuse of notation we denote by $\Lambda(s, f)$, $f = u, \text{sym}^2 u, \text{sym}^2 u_j$ or $u \otimes \text{sym}^2 u_j$, respectively, the completed associated $L$-functions, as in Section 3. These of course are not identical to functions appearing in (3.4). In order to determine the dependence of (6.1) on $u$ and $u_j$ we replace the completed $L$-functions. Then the right-hand side of (6.1) is equal to

\[
C_\Gamma \gamma_3(t_j, t) \frac{L \left(1/2, u \otimes \text{sym}^2 u_j \right) L \left(1/2, u \right)}{L \left(1, \text{sym}^2 u \right) L \left(1, \text{sym}^2 u_j \right)^2}
\]

where the factor $\gamma_3(t_j, t)$ is asymptotic to

\[
\frac{|\Gamma(1 + it)|^4 |\Gamma \left(\frac{1}{2} + i \left(t_j + \frac{t}{2}\right)\right)|^2 |\Gamma \left(\frac{1}{2} + i \left(t_j - \frac{t}{2}\right)\right)|^2}{|\Gamma(1 + it)|^4 |\Gamma(1 + it)|^2}.
\]

Note that by using similar arguments to the ones given in Hoffstein–Lockhart [25] one sees that the $L$-values $L(1, \text{sym}^2 u_j)$ are of moderate growth. Namely, we have

\[
t_j^{-\epsilon} \ll L(1, \text{sym}^2 u_j) \ll t_j^\epsilon
\]

(see for instance [40, Cor. 7]). The factor $\gamma_3(t_j, t)$ is estimated using Stirling’s formula and we see that

\[
\gamma_3(t_j, t) \asymp \frac{\exp \left(\frac{\tau}{2} (Q(t_j, t))\right)}{P_3(t_j, t)} \tag{6.2}
\]

where $Q(t_j, t)$ is defined in (3.6) and

\[
P_3(t_j, t) = (1 + |t|)(1 + |t_j|)^2. \tag{6.3}
\]

Notice here the difference between $P_2$ defined in (3.7) and $P_3$ which allows us to obtain a better exponent $\delta$ for the equidistribution of Hecke–Maaß forms in 3-dimensional shrinking balls.

For reasons of completeness, we also write down explicitly the product formulae for automorphic forms in $L^2(\Gamma \backslash \mathbb{H}^3)$ for the Eisenstein series. We have

\[
\left| \left\langle |u_j|^2, E(\cdot, 1 + it) \right\rangle \right|^2 = \frac{|\mathcal{O}_J|^2}{24} \frac{|\Lambda_K(1/2 + it)|^2 |\Lambda \left(\frac{1}{2} + it, \text{sym}^2 u_j\right)|^2}{|\Lambda_K(1 + it)|^2 |\Lambda \left(1, \text{sym}^2 u_j\right)|^2} \tag{6.4}
\]

where $\Lambda_K(s)$ denotes the completed Dedekind zeta function of $K$ (see e.g. [33, p. 479] or [21], § 17). The Gamma factors of (6.4) are identical to those appearing in (6.1) and...
have therefore the same asymptotic behaviour as $\gamma_3(t_j, t)$ (see (6.2)). From [33, p. 481] or [36, sub. 3.1] we also get
\[
\left|\mathcal{O}^* \sqrt{|d_K|} \Lambda(1/2, u_j)^2 |\Lambda(1/2 + it, u_j)|^2 \right|^{1/2} \nonumber
\]
and the Archimedean part of (6.5) grows like $\gamma_3(t, t_j)$.

6.2. Regularization of Eisenstein integrals for 3-manifolds. In order to handle the problem of the divergence of the integral
\[
\int_{\Gamma \setminus \mathbb{H}^3} E(P, 1 + it') |E(P, 1 + it)|^2 \, dv(P)
\]
we use Zagier’s regularization as in Subsection 3.2. In the 3-dimensional case the renormalized scalar product $\langle \cdot, \cdot \rangle_{\text{reg}}$ is defined analogously to the 2-dimensional case by simply subtracting the divergence causing part in the Fourier expansion. Then using the same arguments as in [71, pp. 429–430] as well as the calculations of [36, p. 8] we obtain:

**Lemma 6.2.** Let $\Gamma = \text{PSL}_2(O_K)$ be a class number one Kleinian group and $E(P, s)$ the Eisenstein series corresponding to the cusp $\infty$. Then
\[
\langle |E(\cdot, 1 + it)|^2, E(\cdot, 1 + it') \rangle_{\text{reg}} = \frac{|\mathcal{O}^*|^3 \sqrt{|d_K|} \Lambda_K(1/2, u_j) \Lambda_K(1 + it') \Lambda_K(1 + it) |\Lambda_K(1 + it)|^2}{\Lambda_K(1 + it') |\Lambda_K(1 + it)|^2} (6.7)
\]

We mention that Zagier’s method in the general $\text{GL}_2$ case has been worked out by Wu [68, Thm. 3.5]. Our case can be worked out directly without recourse to Wu’s work.

From (6.7) we conclude
\[
\left| \langle |E(\cdot, 1 + it)|^2, E(\cdot, 1 + it') \rangle \right|^2 \propto \gamma_3(t, t') \frac{|\zeta_K((1 + it')/2)|^4 |\zeta_K(1/2 + it + t'/2)|^2 |\zeta_K(1/2 + it - t'/2)|^2}{|\zeta_K(1 + it)|^4 |\zeta_K(1 + it')|^2}. (6.8)
\]

7. Equidistribution of Hecke–Maass forms on $\Gamma \setminus \mathbb{H}^3$

In this section we give the proof of Part (a) of Theorem 1.7. The key tool of the proof is the triple product formula of Ichino. Since we follow the steps of the proof for the case of the modular surface we omit standard technicalities trying instead to concentrate on the new ingredients coming into the play.

We start by adjusting Young’s machinery for the shrinking balls problem in the hyperbolic 3-space. Let $\phi = \phi_R$ be a smooth approximation for the characteristic function of $B_R$, i.e. we take a test function that for every $k \geq 1$ satisfies
\[
\|\Delta^k \phi\|_1 \ll_k R^{-2k}. (7.1)
\]

As before, we denote by $\mathcal{B} = \mathcal{B}_\Gamma$ a family of non-constant Maass cusp forms for the Kleinian group $\Gamma$. We obtain the following estimate:
Proposition 7.1. Let \( \Gamma = \text{PSL}_2(\mathcal{O}_K) \) be a Bianchi group and assume that the class number of \( K \) is one. Furthermore, let \( \phi = \phi_R \) be a fixed test function as in (7.1) and \( u_j \) a Hecke–Maass cusp form on the arithmetic 3-manifold \( \Gamma \backslash \mathbb{H}^3 \). Then, assuming \( R \gg t_j^{-\delta} \) for some fixed \( 0 < \delta < 1 \) as well as \( GLH \), we have for any \( M \gg 1 \)

\[
\int_{\Gamma \backslash \mathbb{H}^3} \phi_R(P) |u_j(P)|^2 dv(P) = \int_{\Gamma \backslash \mathbb{H}^3} \phi_R(P) dv(P) + O_{\epsilon} \left( \|\phi\|_2 R^{1-\epsilon} t_j^{1+\epsilon} \right) + O \left( \|\phi\|_1 t_j^{-M} \right). \tag{7.2}
\]

Proof. Let \( u_j \in \mathcal{B} \). In the class number one case the spectral theorem ([12, Sect. 6.3, Thm. 3.4]) implies

\[
|u_j|^2 = \frac{\langle |u_j|^2, 1 \rangle}{\text{vol}(\Gamma \backslash \mathbb{H}^3)} + \sum_{u_m \in \mathcal{B}} \langle |u_j|^2, u_m \rangle u_m
\]

\[
+ \frac{1}{\pi \sqrt{|d_K||\mathcal{O}_K^*|}} \int_{-\infty}^{+\infty} \langle |u_j|^2, E(\cdot, 1 + it) \rangle E(\cdot, 1 + it) dt.
\]

Since \( \phi_R \) is a smooth compactly supported function on \( \mathbb{H}^3 \) we get

\[
\langle \phi_R, |u_j|^2 \rangle = \frac{\langle \phi_R, 1 \rangle}{\text{vol}(\Gamma \backslash \mathbb{H}^3)} + \sum_{u_m \in \mathcal{B}} \langle |u_j|^2, u_m \rangle \langle \phi_R, u_m \rangle
\]

\[
+ \frac{1}{\pi \sqrt{|d_K||\mathcal{O}_K^*|}} \int_{-\infty}^{+\infty} \langle |u_j|^2, E(\cdot, 1 + it) \rangle \langle \phi_R, E(\cdot, 1 + it) \rangle dt.
\]

In order to estimate \( \langle |u_j|^2, u_m \rangle \) we use the Watson–Ichino formula (see Theorem 6.1). Using Stirling’s formula as well as the standard bound \( |u_m|_\infty \ll (1 + t_m^2)^{1/2} \) for the sup-norm of \( u_m \) we cut the sum and the range of integration to conclude

\[
\langle \phi_R, |u_j|^2 \rangle = \frac{\langle \phi_R, 1 \rangle}{\text{vol}(\Gamma \backslash \mathbb{H}^3)} \ll \sum_{u_m \in \mathcal{B}, t_m \leq t_j + c \log t_j} \langle |u_j|^2, u_m \rangle \langle \phi_R, u_m \rangle
\]

\[
+ \frac{1}{\pi \sqrt{|d_K||\mathcal{O}_K^*|}} \int_{|t| \leq t_j + C \log t_j} \langle |u_j|^2, E(\cdot, 1 + it) \rangle \langle \phi_R, E(\cdot, 1 + it) \rangle dt
\]

\[
+ O_M \left( \|\phi\|_1 t_j^{-M} \right) \tag{7.3}
\]

for sufficiently large constants \( c, C \gg 1 \) and \( M \gg 1 \). The initial assumption (7.1) for the test function \( \phi_R \) allows us to restrict the sum and the range of integration in (7.3) further as (7.1) implies

\[
\langle \phi, u_m \rangle \ll \frac{\|\Delta^k \phi\|_1 u_m\|_\infty}{(1 + t_m^2)^k} \ll \frac{R^{-2k}}{(1 + t_m^2)^{k-1/2}}.
\]
We assume $R \gg t_j^{-\delta}$ hence, up to the cost of a small error term, we can cut the sum and the integration in (7.3) at $t_m \leq R^{-1}t_j^\epsilon$ so that

$$
\langle \phi_R, |u_j|^2 \rangle - \frac{\langle \phi_R, 1 \rangle}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} \ll \sum_{u_m \in B_j, t_m \leq R^{-1}t_j^\epsilon} \langle |u_j|^2, u_m \rangle \langle \phi_R, u_m \rangle \ll \frac{\text{vol}(\Gamma \setminus \mathbb{H}^3)}{t_j^{\delta} \pi \sqrt{|d_K| O_K^*}} \int_{|t| \leq R^{-1}t_j^\epsilon} \langle |u_j|^2, E(\cdot, 1 + it) \rangle \langle \phi_R, E(\cdot, 1 + it) \rangle dt + O\left(\|\phi\|_{1_tM}^{-1}t_j^M\right).
$$

In order to bound the discrete contribution note that the Cauchy–Schwarz inequality implies

$$
\sum_{u_m \in B_j, t_m \leq R^{-1}t_j^\epsilon} \langle |u_j|^2, u_m \rangle \langle \phi_R, u_m \rangle \ll \left( \sum_{u_m \in B_j, t_m \leq R^{-1}t_j^\epsilon} |\langle |u_j|^2, u_m \rangle|^2 \right)^{1/2} \left( \sum_{u_m \in B_j, t_m \leq R^{-1}t_j^\epsilon} |\langle \phi_R, u_m \rangle|^2 \right)^{1/2}.
$$

The second term is bounded by $\|\phi_R\|_2$. The first one is estimated using (6.1). Using Stirling’s formula and assuming GLH we have

$$
\sum_{u_m \in B_j, t_m < R^{-1}t_j^\epsilon} |\langle |u_j|^2, u_m \rangle|^2 \ll \sum_{u_m \in B_j, t_m < R^{-1}t_j^\epsilon} t_j^{2+\epsilon} t_m^{1+\epsilon} \ll R^{-2-\epsilon} t_j^{2+\epsilon}.
$$

Similarly, using (6.4) and assuming GLH we get

$$
\left( \int_{|t| \leq R^{-1}t_j^\epsilon} \langle |u_j|^2, E(\cdot, 1 + it) \rangle \langle \phi_R, E(\cdot, 1 + it) \rangle dt \right)^2 \ll \|\phi_R\|_2^2 \int_{|t| \leq R^{-1}t_j^\epsilon} |\langle |u_j|^2, E(\cdot, 1 + it) \rangle|^2 dt \ll \|\phi_R\|_2^2 R^{-\epsilon} t_j^{2+\epsilon} \int_{|t| \leq R^{-1}t_j^\epsilon} \frac{1}{1 + |t|} dt \ll \|\phi_R\|_2^2 R^{-\epsilon} t_j^{2+\epsilon}.
$$

Hence the contribution of the continuous spectrum is bounded by $\|\phi_R\|_2^2 R^{-\epsilon} t_j^{-1+\epsilon}$ and we finally conclude

$$
\langle \phi, |u_j|^2 \rangle - \langle \phi, 1 \rangle = O_M, \epsilon \left( \|\phi\|_2^2 R^{-1-\epsilon} t_j^{-1+\epsilon} + \|\phi\|_1 t_j^{-M} \right).
$$

This proves (7.2). \qed

Now let us prove Part (a) of Theorem 1.7:

**Proof of Part (a) of Theorem 1.7.** Taking $\phi$ to be an approximation of the characteristic function of a ball of radius $R$ and normalizing $\langle \phi, 1 \rangle \approx \text{vol}(B_R) \approx R^3$ and $\|\phi\|_2 \approx R^{\frac{3}{2}}$ we get

$$
\frac{1}{\text{vol}(B_R(Q))} \int_{B_R(Q)} |u_j(P)|^2 dv(P) = \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} + O_\epsilon \left( R^{-5/2-\epsilon} t_j^{-1+\epsilon} \right) + O_M(t_j^{-M}).
$$
The error term is smaller than the main term if and only if
\[ R^{-5/2-\epsilon} t_j^{-1+\epsilon} = o(1), \]
i.e. if and only if \( R \gg t_j^{-\epsilon} + \epsilon \).

\[ \square \]

8. Subconvexity and mean Lindelöf estimates

For the study of QUE for Eisenstein series we need explicit subconvexity estimates for the \( L \)-functions appearing in the product formulae (6.5) and (6.7). In particular, we need good estimates for
\[ \left| L(1/2, u_j) \right|^2 \ll \epsilon C(t, t_j)^{1/4+\epsilon}, \]
and
\[ \sum_{u_j \in S, \ t_j \leq R^{-1} T} \frac{\left| L(1/2, u_j) \right|^2}{(1 + |t_j|)^2} \int_T^{2T} \frac{\left| L(1/2 + it, u_j) \right|^2}{(1 + |t|)^2} dt. \]

Let
\[ C(t, t_j) := \left( 1 + \left| t + \frac{t_j}{2} \right|^2 \right) \left( 1 + \left| t - \frac{t_j}{2} \right|^2 \right) \]
be the analytic conductor for \( L(s, u_j) \) as defined in [32]. Then we have the convexity bound
\[ L(1/2 + it, u_j) \ll \epsilon C(t, t_j)^{1/4+\epsilon}. \]

Petridis and Sarnak [50] were the first to prove a subconvexity result for \( L(1/2 + 2it, u_j) \) in the \( t \)-aspect. For point-wise estimates we refer to the following recent hybrid estimate of Wu [67, Cor. 1.6]:

**Theorem 8.1.** [67, Cor. 1.6] Let \( \theta \) be the exponent towards the Ramanujan-Petersson conjecture. Then the following estimate holds:
\[ L(1/2 + it, u_j) \ll \epsilon C(t, t_j)^{1/4+\epsilon}. \]

By Nakasuji’s work (see [48, Cor. 1.2]) we can take \( \theta = 7/64 \). Hence we can use the following unconditional upper bound
\[ L(1/2 + it, u_j) \ll \epsilon C(t, t_j)^{231/1024+\epsilon}. \]

8.1. Second moment estimates for central \( L \)-values. For \( \Gamma = \operatorname{PSL}_2(\mathbb{Z}[i]) \) we can improve the subconvexity exponent on average and get the following mean Lindelöf estimate:

**Proposition 8.2.** For \( \Gamma = \operatorname{PSL}_2(\mathbb{Z}[i]) \) we have:
\[ \sum_{t_j \leq T} \left| L(1/2, u_j) \right|^2 \ll T^{3+\epsilon}. \]
Proof. The proof of Proposition 8.2 uses the standard approach of the approximate functional equation: the $L$-function

$$L(s, u_j) = \sum_{n \in \mathbb{Z}[i] \setminus \{0\}} \frac{\lambda_j(n)}{N(n)^s}$$

(8.6)

satisfies the functional equation

$$\gamma(s, t_j) L(s, u_j) = \gamma(1 - s, t_j) L(1 - s, u_j)$$

(8.7)

where the gamma factor is given by

$$\gamma(s, t_j) = \pi^{-2s} \Gamma\left(s + \frac{it_j}{2}\right) \Gamma\left(s - \frac{it_j}{2}\right)$$

(see e.g. [21], Satz 16.4, p. 115). Thus the approximate functional equation implies that we can write

$$L\left(\frac{1}{2}, u_j\right) = \sum_n \frac{\lambda_j(n)}{\sqrt{N(n)}} V_{t_j}(N(n))$$

where

$$V_{t_j}(y) = \int_{(\sigma)} y^{-u} G(u) \frac{\gamma(1/2 + u, t_j)}{\gamma(1/2, t_j)} \frac{du}{u},$$

$G(u)$ is holomorphic, even and bounded on vertical strips and satisfies $G(0) = 1$. We adapt the proof of [2, Thm. 3] for the central point $s = 1/2$ and as in the proof of [2, Thm. 3] we use the mean Ramanujan bound due to Koyama [34, Thm. 2.1]

$$\sum_{N(n) \leq N} |\lambda_j(n)|^2 = O((1 + |t_j|^\epsilon)N).$$

(8.8)

Note that we have the factor $1/\sqrt{N(n)}$ in the definition of the Hecke operators. This factor is not present in Koyama’s definition. Then we can mostly follow the arguments of [2]. However, there is a difference concerning the estimation of the finite series part

$$\sum_{t_j \leq T} \left| \sum_{N(n) \leq N} \frac{\lambda_j(n)}{\sqrt{N(n)}} V_{t_j}(N(n)) \right|^2.$$  

(8.9)

The asymptotics of

$$\frac{\gamma(1/2 + u, t_j)}{\gamma(1/2, t_j)}$$

(8.10)

allow us to cut the sum (8.9) to $N(n) \approx t_j$. Thus we get

$$\sum_{t_j \leq T} |L(1/2, u_j)|^2 \ll \sum_{t_j \leq T} \left| \sum_{N(n) \leq N} \frac{\lambda_j(n)}{\sqrt{N(n)}} \right|^2$$

(8.11)

with $N \approx T$. For $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$ the spectral large sieve of Watt [66, Thm. 1] reads as

$$\sum_{t_j \leq T} \left| \sum_{N(n) \leq N} a_n \lambda_j(n) \right|^2 \ll (T^3 + T^{3/2}N^{1+\epsilon}) \sum_{N(n) \leq N} |a_n|^2$$

(8.12)

which implies (8.5). \[\square\]
Remark 8.3. Note that a difference between the situation of dimension 2 and dimension 3 is the available spectral large sieve. This is optimal for the case of the modular surface due to the work of Deshouillers and Iwaniec [9]. For the Picard manifold $\text{PSL}_2(\mathbb{Z}[i]) \setminus \mathbb{H}^3$ Watt’s sieve [66] is not optimal as the large sieve constant in (8.12) is expected to be $T^3 + N^{1+\epsilon}$. This causes an important difference when bounding the $L$-functions attached to higher symmetric powers of $u_j$ (see [2]). However, this does not affect the second moment upper bound in our case where we only have to consider bounds for $L(s, u_j)$.

8.2. The integral second moment on the critical line. In this section we prove a mean Lindelöf estimate for the integral second moment of $L(s, u_j)$ on the critical line as a consequence of the approximate functional equation.

**Proposition 8.4.** For $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$ we have the following estimate for the integral second moment of $L(s, u_j)$:

$$
\int_{T}^{2T} |L(1/2 + it, u_j)|^2 dt \ll (1 + T)^{1+\epsilon} (1 + |t_j|)^\epsilon.
$$

(8.13)

Proof. Let $G(u)$ be a function that is even, holomorphic and bounded for $|\Re(u)| < 4$ and satisfies $G(0) = 1$. The approximate functional equation for $L(s, u_j)$ implies that

$$
L(1/2 + it, u_j) = \sum_{n \in \mathbb{Z}[i] \setminus \{0\}} \frac{\lambda_j(n)}{N(n)^{1/2+it}} V_{i_j}(N(n), t) + \frac{\gamma(1/2 - it, t_j)}{\gamma(1/2 + it, t_j)} \sum_{n \in \mathbb{Z}[i] \setminus \{0\}} \frac{\lambda_j(n)}{N(n)^{1/2-it}} V_{i_j}(N(n), -t),
$$

(8.14)

where

$$
V_{i_j}(y, t) = \int_{(\sigma)} y^{-u} G(u) \frac{\gamma(1/2 + it + u, t_j)}{\gamma(1/2 + it, t_j)} \frac{du}{u}.
$$

Hence we infer that $L(1/2 + it, u_j)L(1/2 - it, u_j)$ can be written as the sum of four terms that have the following form:

$$
\sum_{n, m \neq 0} \frac{\lambda_j(n)\lambda_j(m)}{\sqrt{N(n)N(m)}} N(n)^{-it} N(m)^{\pm it} V_{i_j}(N(n), t)V_{i_j}(N(m), \mp t).
$$

(8.15)

We bound the first of them, the remaining terms can be treated in the same way. We want to estimate

$$
\int_{T}^{2T} \left(\frac{N(m)}{N(n)}\right)^{it} V_{i_j}(N(n), t)V_{i_j}(N(m), -t) dt.
$$

Estimating the $\Gamma$-factors we can shift the integral to $\sigma_1 = \sigma_2 = \epsilon$. Since we only care for a crude bound without any explicit asymptotics we bound

$$
\int_{T}^{2T} |L(1/2 + it, u_j)|^2 dt
$$

using (8.8) by

$$
\sum_{n \neq 0} \frac{|\lambda_j(n)|^2}{N(n)} \int_{T}^{2T} \frac{|t|^{4\epsilon}}{N(n)\epsilon} dt \ll (1 + T)^{1+4\epsilon} (1 + |t_j|)^\epsilon
$$

and the statement follows. \qed
Note that, using his trace formula, Kuznetsov [35] gave an explicit asymptotic result for the second moment in the case of the modular group with an error term of $O(c_j T + T^{6/7 + \epsilon})$ for some constant $c_j$. In our case the crude bound (8.13) suffices.

9. Equidistribution of Eisenstein series on $\Gamma \setminus \mathbb{H}^3$

In this section we prove parts (b) and (c) of Theorem 1.7, i.e. the unconditional equidistribution of Eisenstein series in shrinking balls. It follows from [45] that Theorem 4.1 holds in some generality. In particular, it allows us to estimate the spectral expansion of

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |E(Q, 1 + it)|^2 dv(Q) - \frac{|O_K^*| \sqrt{|d_K|} \log(1 + t^2)}{4 \text{vol}(\Gamma \setminus \mathbb{H}^3)}$$

for $F = E(P, s_1)E(P, s_2)$ and $G = \phi_R$. For the Eisenstein series we get the following result:

**Proposition 9.1.** Let $\Gamma$ be a Bianchi group of class number one and $\phi = \phi_R$ be a fixed test function as in (7.1). Assume that $R \gg t^{-\delta}$ for some fixed $0 < \delta < 1$. Then we have:

$$\int_{\Gamma \setminus \mathbb{H}^3} \phi_R(P) |E(P, 1 + it)|^2 dv(P) = \log(1 + t^2) \frac{|O_K^*| \sqrt{|d_K|}}{4 \text{vol}(\Gamma \setminus \mathbb{H}^3)} \int_{\Gamma \setminus \mathbb{H}^3} \phi_R(P) dv(P)$$

$$+ O_\epsilon \left( \|\phi\|_2 (1 + |t|)^{-\frac{1}{2} + 2\epsilon} R^{-\frac{11 + 2\theta}{4}} \right)$$

$$+ O_\epsilon \left( \|\phi\|_2 (1 + |t|)^{-1/3 + \epsilon} R^{-\frac{14 + 3\theta}{15}} \right)$$

$$+ O \left( \frac{\log t}{\log \log t} \right) + O (\|\phi\|_{1-M}) + O_M (\|\phi\|_2 t^{-M})$$

(9.1)

for any $M \gg 1$. Here $\theta$ is an exponent towards the Ramanujan–Petersson conjecture and the parameter $0 \leq a \leq 1$ is related to the subconvexity exponent for the twelfth moment of the Riemann zeta function as in (9.11).

For $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$ the first error term can be improved to

$$O_\epsilon \left( \|\phi\|_2 (1 + |t|)^{-\frac{1}{2} + 2\epsilon} R^{-\frac{15 + 2\theta}{26}} \right).$$

(9.2)

**Remark 9.2.** If we assume the Generalized Lindelöf Hypothesis, then for any $\Gamma = \text{PSL}_2(O_K)$, with $K$ an imaginary quadratic number field of class number one, the first error term appearing in (9.1) is bounded by

$$O_\epsilon \left( \|\phi\|_2 (1 + |t|)^{-1+\epsilon} R^{-1-\epsilon} \right)$$

and the second term by

$$O_\epsilon \left( \|\phi\|_2 (1 + |t|)^{-1+\epsilon} R^{-\epsilon} \right).$$

We split the proof of Proposition 9.1 in three parts: first we determine the contribution of the various elements coming from the spectral decomposition (cf. Theorem 4.1) and in the end we combine these results to prove the proposition.
9.1. **The contributions of** \( \lambda_0 = 0 \) **and** \( \langle \mathcal{E}, \phi_R \rangle \). The main term of

\[
\left\langle |E(\cdot, 1 + it)|^2, \phi_R \right\rangle
\]

was already determined by Koyama [33, p. 485] who also gave an error term. However, there is an error in his proof as he does not take the term coming from the double pole of \( \zeta_K^2(s/2) \) at \( s = 1 \) into account. This error has subsequently been corrected by Laaksonen [36, Rem. 1]. Using the spectral theorem and Zagier’s Rankin–Selberg method of functions that are not of rapid decay we see that the main term of the asymptotic behaviour comes from the following terms in (4.1):

\[
\left\langle |E(P, 1 + it)|^2, u_0 \right\rangle_{\text{reg}} \langle u_0, \phi_R \rangle + \langle \mathcal{E}, \phi_R \rangle
\]

More precisely, as \( \left\langle |E(P, 1 + it)|^2, 1 \right\rangle_{\text{reg}} \) vanishes the term \( \langle \mathcal{E}, \phi_R \rangle \) is responsible for the main term. For determining its asymptotic behaviour we can adapt the approach of [69, pp. 976, 980]. First of all, we note that the divergence causing term of the product \( E(P, 1 + s_1)E(P, 1 + s_2) \) is given by

\[
\frac{|O_K^s|^2}{4} (r^{2+s_1+s_2} + \phi(s_2)r^{2+s_1-s_2} + \phi(s_1)r^{2-s_1+s_2} + \phi(s_1)\phi(s_2)r^{2-s_1-s_2}).
\]

Eventually, we will set \( s_1 = \alpha + it \) and \( s_2 = -it \) and consider the limit \( \alpha \to 0 \). Thus we have

\[
\mathcal{E} = \frac{|O_K^s|}{2} (E(P, 2 + s_1 + s_2) + \phi(s_2)E(P, 2 + s_1 - s_2)
\]

\[
+ \phi(s_1)E(P, 2 - s_1 + s_2) + \phi(s_1)\phi(s_2)E(P, 2 - s_1 - s_2)).
\]

The second and the third term appearing on the right-hand side do not contribute to the main term. Using the decaying properties of \( \phi_R \) as in [69, p. 980] their contribution can be bounded by

\[
\langle E(P, 2 \pm s_1 \pm s_2), \phi_R \rangle = O \left( t^{-M} \right).
\]

Hence it remains to treat

\[
\frac{|O_K^s|}{2} \lim_{\alpha \to 0} \langle E(P, 2 + \alpha) + \phi(\alpha + it)\phi(-it)E(P, 2 - \alpha), \phi_R \rangle.
\]

In order to determine this limit note that \( \phi(s)\phi(-s) = 1 \) and that

\[
\text{res}_{s=2} E(P, s) = \frac{2\pi^2}{|d_K| \zeta_K(2)} = \frac{\sqrt{|d_K|}}{2\text{vol}(\Gamma \backslash \mathbb{H}^3)}
\]

(see [12], Theorem 1.11, pp. 243–244 and [12], Theorem 1.1, p. 312). This implies that

\[
E(P, 2 + \alpha) + \phi(\alpha + it)\phi(-it)E(P, 2 - \alpha) = 2a(P) - \frac{\phi'(it)}{\phi(it)} \text{res}_{s=2} E(P, s) + O(\alpha)
\]

where \( a(P) \) denotes the constant term in the Laurent expansion of \( E(P, 2 + \alpha) \) about \( \alpha = 0 \). Using the explicit form of the scattering matrix (1.16) we obtain

\[
\frac{\phi'(it)}{\phi(it)} = 2\log \left( \frac{2\pi}{\sqrt{|d_K|}} \right) - \frac{\Gamma'}{\Gamma}(1 + it) - \frac{\Gamma'}{\Gamma}(1 - it) - \frac{\zeta_K}{\zeta_K}(1 + it) - \frac{\zeta_K}{\zeta_K}(1 - it)
\]
The main term in the asymptotics comes from the terms involving the \( \Gamma \) function whereas the other terms are absorbed in the error term. Using the estimate of [33, p. 485]
\[
\frac{\zeta'(1 \pm it)}{\zeta(1 \pm it)} \ll \frac{\log t}{\log \log t},
\]
for the logarithmic derivative of the Dedekind zeta function we obtain that its contribution to the error term is
\[
O\left(\|\phi_R\|_1 \frac{\log t}{\log \log t}\right).
\]
Furthermore, Stirling’s formula implies that
\[
\Gamma(1 + it) + \Gamma(1 - it) = \log (1 + t^2) + O \left( t^{-2} \right).
\]
Thus we infer
\[
\langle \mathcal{E}, \phi_R \rangle = \frac{\pi^2 |O_K|}{\omega_K(\xi)} \log(1 + t^2) (1, \phi_R) + O \left( t^{-2} \|\phi_R\|_1 \right) + O \left( \|\phi_R\|_1 \frac{\log t}{\log \log t} \right).
\]

**9.2. The contribution of the discrete spectrum.** In this section we determine the contribution of the discrete spectrum. Using (6.5), Bessel’s inequality and (7.1), the contribution of the discrete spectrum \( u_j \in \mathcal{B} \) in the spectral expansion of (9.1) is bounded as follows:
\[
\sum_{u_j \in \mathcal{B}} \langle \langle E(\cdot, 1 + it) \rangle^2, u_j \rangle \langle u_j, \phi_R \rangle
\]
\[
\ll \|\phi_R\|_2 \left( \sum_{t_j \leq R^{-1}t^\epsilon} \left| \langle \langle E(\cdot, 1 + it) \rangle^2, u_j \rangle \right|^2 \right)^{1/2} + O_M \left( \|\phi_R\|_2 t^{-M} \right) \tag{9.3}
\]
\[
\ll \|\phi_R\|_2 \left( \sum_{t_j \leq R^{-1}t^\epsilon} \frac{|L(1/2, u_j)|^2 |L(1/2 + it, u_j)|^2}{(1 + |t_j|)(1 + |t|)^2} \right)^{1/2} + O_M \left( \|\phi_R\|_2 t^{-M} \right).
\]

Let us now consider the case that \( \Gamma = \text{PSL}_2(\mathbb{Z}[i]) \). In this case, Proposition 8.2 and summation by parts imply
\[
\sum_{t_j \leq R^{-1}t^\epsilon} \frac{|L(1/2, u_j)|^2}{(1 + |t_j|)^B} \ll R^{-3 + B} t^\epsilon.
\]

Using the point-wise estimate (8.3) we bound the first term on the right-hand side of (9.3) by
\[
\|\phi_R\|_2 (1 + |t|)^{-\frac{1-2\theta}{8} + \epsilon} \left( \sum_{t_j \leq R^{-1}t^\epsilon} \frac{|L(1/2, u_j)|^2}{(1 + |t_j|)^{\frac{11+2\theta}{4} + \epsilon}} \right)^{1/2} \ll \|\phi_R\|_2 (1 + |t|)^{-\frac{1-2\theta}{8} + \epsilon} R^{\frac{15+2\theta}{8} - \epsilon}. \tag{9.4}
\]
For \( \Gamma \) different to \( \text{PSL}_2(\mathbb{Z}[i]) \) we use the point-wise estimate (8.3) also for the central \( L \)-value \( L(1/2, u_j) \) in (9.4) and the Weyl law to get the upper bound
\[
\|\phi_R\|_2 (1 + |t|)^{-\frac{1-2\theta}{8} + \epsilon} R^{\frac{11+2\theta}{4} + \epsilon}.
\]
Remark 9.3. We mention that the large sieve of Watt (8.12) and consequently Proposition 8.2 should hold for any \( \Gamma = \text{PSL}_2(\mathcal{O}_K) \) with \( \mathcal{O}_K \) being the ring of integers of an imaginary quadratic number field of class number one. However, we treat the two cases separately to show that Proposition 8.2 is not necessary to obtain equidistribution results in some range. In particular, our exponents can be improved if one can prove better subconvexity results either for the discrete spectrum or for the continuous part.

9.3. The contribution of the continuous spectrum. In this section we estimate the contribution of the continuous spectrum to the asymptotics. Using the properties of \( \phi_R \) (see (7.1)) as in [69], (4.9), p. 975 and Lemma 6.2 we obtain that

\[
\int_{-\infty}^{\infty} \langle \langle E(\cdot, 1 + it) \rangle^2, E(\cdot, 1 + it') \rangle_{\text{reg}} \langle E(\cdot, 1 + it'), \phi_R \rangle \, dt'
\]

\[
= \int_{|t'| \leq R^{-1}t} \langle \langle E(\cdot, 1 + it) \rangle^2, E(\cdot, 1 + it') \rangle_{\text{reg}} \langle E(\cdot, 1 + it'), \phi_R \rangle \, dt'
\]

\[
+ O \left( \| \phi_R \|_2 t^{-M} \right) \quad (9.5)
\]

\[
\ll \| \phi_R \|_2 \left( \int_{|t'| \leq R^{-1}t} \left| \langle \langle E(\cdot, 1 + it) \rangle^2, E(\cdot, 1 + it') \rangle_{\text{reg}} \right|^2 \, dt' \right)^{1/2} + \| \phi_R \|_2 t^{-M}
\]

where we applied the Cauchy–Schwarz inequality and [12], (3.13), p. 268. Thus it remains to estimate

\[
I(t) := \int_{|t'| \leq R^{-1}t} \left| \langle \langle E(\cdot, 1 + it) \rangle^2, E(\cdot, 1 + it') \rangle_{\text{reg}} \right|^2 \, dt'.
\]  

By Lemma 6.2 and standard bounds on the Dedekind zeta function we infer

\[
I(t) \ll \frac{R^{-\epsilon}}{(1 + |t|)^{2-\epsilon}} \int_{|t'| \leq R^{-1}t} \left| \zeta_K \left( \frac{1+it'}{2} \right) \right|^4 \left| \zeta_K \left( \frac{1+it'}{2} - it \right) \right|^2 \left| \zeta_K \left( \frac{1+it'}{2} + it \right) \right|^2 \, dt'.
\]  

(9.7)

If we were to use only Heath-Brown’s Weyl bound (see [20]) for the Dedekind zeta function

\[
\zeta_K(1/2 + it) \ll_K (1 + |t|)^{1/3 + \epsilon},
\]  

(9.8)

then we would get a worse bound for the contribution of the continuous spectrum than for the contribution of the discrete spectrum. Although the Weyl bound suffices to give a good estimate in the case of the modular surface, in our case we need something better than this. We can do slightly better using moment estimates for the Dedekind zeta function. As, so far, there is no mean Lindelöf bound available for the fourth moment of \( \zeta_K \) known for any class number one field \( K \) we proceed as follows: we use the well-known identity

\[
\zeta_Q(\sqrt{D}) = \zeta(s) L \left( s, \chi_{|D|} \right)
\]

and Hölder’s inequality to get

\[
\int_{-T}^{T} \left| \zeta_Q(\sqrt{D})(1/2 + it) \right|^4 \, dt
\]

\[
\ll \left( \int_{-T}^{T} |\zeta(1/2 + it)|^{12} \, dt \right)^{1/3} \left( \int_{-T}^{T} |L(1/2 + it, \chi_{|D|})|^6 \, dt \right)^{2/3}.
\]  

(9.9)
First we bound the second integral appearing on the right-hand side of (9.9). Unfortunately, the known results for the sixth moment of Dirichlet $L$-functions (see [8] and [51]) do not give a sufficiently good upper bound for our application. Instead we bound two of the six $L$-factors point-wise by a Weyl bound (see [51], (1.1)) and use a mean Lindelöf result for the fourth moment of our Dirichlet $L$-function (see e.g. [51], (1.8)) so that we obtain

$$\int_{-T}^T \left| L \left( \frac{1}{2} + it, \chi \right) \right|^6 dt \ll_D T^{4/3+\epsilon}.$$ 

In order to bound the first term on the right-hand side of (9.9) we note that Heath-Brown’s bound [19] for the twelfth moment of the Riemann zeta function

$$\int_{-T}^T |\zeta(1/2 + it)|^{12} dt \ll T^{2+\epsilon}$$

implies $\zeta(1/2 + it) \ll t^{1/12+\epsilon}$ on average. Assuming the improved bound

$$\int_{-T}^T |\zeta(1/2 + it)|^{12} dt \ll T^{1+a+\epsilon},$$

$a \in [0,1]$, we finally obtain the bound

$$\int_{-T}^T |\zeta(\sqrt{D})(1/2 + it)|^4 dt \ll T^{13/8+\epsilon}.$$ 

If we replace in the integral (9.7) the last two Dedekind zeta functions by the Weyl bound (9.8), this bound implies

$$I(t) \ll \frac{R^{-\epsilon}}{(1 + |t|)^{2/3-\epsilon}} \int_{|t| \leq R^{-1+\epsilon}} \left| \zeta_K \left( \frac{1 + it'}{2} \right) \right|^4 \left( 1 + |t'| \right)^{1/3+\epsilon} dt'$$

$$\ll R^{-14/9+\epsilon} (1 + |t|)^{-2/3+\epsilon}. $$

Thus the contribution of the continuous spectrum in (9.1) is bounded by

$$\|\phi\|_2 R^{-14/9+\epsilon} (1 + |t|)^{-1/3+\epsilon} + \|\phi_R\|_2 t^{-M}$$

This concludes the proof of Proposition 9.1.

**Remark 9.4.** The fourth moment for the Dedekind zeta function of $K = \mathbb{Q}(i)$ was extensively studied in the deep work of Bruggeman and Motohashi [5]. Nevertheless, even in this case the mean Lindelöf bound for the fourth moment of $\zeta_K$ remains out of reach.

9.4. **Proof of Theorem 1.7 (b) and (c).** We now combine our previous results to prove parts (b) and (c) of Theorem 1.7. Let $\phi$ approximate the characteristic function of a ball of radius $R$ and normalize it such that $\langle \phi, 1 \rangle \asymp R^{3}$. For $K \neq \mathbb{Q}(i)$ it follows from $\|\phi\|_2 \asymp R^{3/2}$ and Proposition 9.1 that

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |E(Q, 1 + it)|^2 d\nu(Q) - \frac{|O_K^*| \sqrt{|d_K|} \log(1 + t^2)}{4 \text{vol}(\Gamma \setminus \mathbb{H}^3)}$$

$$= O_{\epsilon} \left( (1 + |t|)^{-1/8+\epsilon} R^{-\frac{17}{4}+2\epsilon} \right) + O_{\epsilon} \left( (1 + |t|)^{-1/3+\epsilon} R^{-\frac{41}{18}+\epsilon} \right)$$

$$+ O \left( \frac{\log t}{\log \log t} \right) \ (9.15)$$


Thus we have equidistribution up to
\[ R \gg \max \left\{ t^{-\frac{1-2\theta}{27+2\theta} + \epsilon}, t^{-\frac{6}{51+3\theta} + \epsilon} \right\} = t^{-\frac{1-2\theta}{27+2\theta} + \epsilon}. \]

For \( \theta = 7/64 \) this exponent equals 25/1102.

If \( K = \mathbb{Q}(i) \) then the error term coming from the discrete spectrum can be improved. This corresponds to the first error term on the right-hand side of (9.15) which can be replaced by
\[ O\left( (1 + |t|)^{-\frac{1-2\theta}{8} + \epsilon} R^{\frac{-27+2\theta}{8} - \epsilon} \right). \]

Consequently, we see that equidistribution holds up to
\[ R \gg \max \left\{ t^{-\frac{1-2\theta}{27+2\theta} + \epsilon}, t^{-\frac{6}{51+3\theta} + \epsilon} \right\} = t^{-\frac{1-2\theta}{27+2\theta} + \epsilon}. \]

Using the best known result for the Ramanujan–Peterson conjectures gives us the exponent 25/871.

If we assume the Generalized Lindelöf Hypothesis for the \( L \)-functions appearing in (9.3) and (9.7) we get for the contribution of the discrete spectrum the improved bound
\[ \|\phi\|_2 \left( \sum_{t_j \leq R^{-1/\epsilon}} \frac{|t + t_j|^\epsilon}{(1 + |t_j|)(1 + |t|)^2} \right)^{1/2} \ll \|\phi\|_2 (1 + |t|)^{-1+\epsilon} R^{-1-\epsilon} \]
and for the contribution of the continuous spectrum the improved bound
\[ \|\phi\|_2^2 R^{-\epsilon} \left( \int_{|t'| \leq R^{-1/\epsilon}} \frac{|t'|^\epsilon}{1 + |t'|} dt' \right)^{1/2} \ll \|\phi\|_2^2 R^{-\epsilon} \frac{1}{(1 + |t|)^{1-\epsilon}}. \]

Thus the discrete contribution is \( o(1) \) if \( R \gg t^{-2/5+\epsilon} \) whereas the continuous contribution is \( o(1) \) if \( R \gg t^{-2/3+\epsilon} \).

10. Quantum variance of Eisenstein series for shrinking balls on \( \Gamma \setminus \mathbb{H}^3 \)

In this section we prove Theorem 1.9. Note that for \( \Gamma = \text{PSL}_2(\mathbb{Z}[i]) \) we have \( |O^*_K| = 4 \) and \( d_K = 4 \). To bound the second moment of
\[ \frac{1}{\log(1 + t^2) \text{vol}(B_R(P))} \int_{B_R(P)} |E(Q, 1 + it)|^2 dv(Q) - \frac{2}{\text{vol}(\Gamma \setminus \mathbb{H}^3)} \]
we use Theorem 4.1 in the version for the hyperbolic 3-space as in the proof of Proposition 9.1. For this we chose \( \phi_R \) as in (7.1).

10.1. The contribution of \( \lambda_0 \) and \( \langle \mathcal{E}, \phi_R \rangle \). It follows from Section 9.1 that the contribution of the eigenvalue \( \lambda_0 = 0 \) and of \( \langle \mathcal{E}, \phi_R \rangle \) in 10.1 is
\[ O\left( \frac{1}{\log \log t} \right). \]
10.2. The contribution of the discrete spectrum. Using (6.5) as in Section 9.2, (9.3), we infer that the contribution of the discrete spectrum to (10.1) is majorized by

\[
\frac{1}{R^{3/2}} \left( \sum_{u_j \in B, \ t_j \leq R^{-1}t_{\epsilon}} \frac{|L(1/2, u_j)|^2 |L(1/2 + it, u_j)|^2}{(1 + |t_j|)(1 + |t|)^2} \right)^{1/2} + O_M \left( R^{-3/2t_{\epsilon}} \right).
\]

Keeping in mind that Proposition 8.4 implies

\[
\int_T^{2T} \frac{|L(1/2 + it, u_j)|^2}{(1 + |t|)^2} \, dt \ll \frac{1}{T^2} \int_T^{2T} |L(1/2 + it, u_j)|^2 \, dt \ll (1 + T)^{-1+\epsilon}(1 + |t_j|)^{\epsilon}
\]

we therefore infer that the contribution of the discrete spectrum to the left-hand side of (1.23) is bounded by

\[
M_D(R, T) := \int_T^{2T} \frac{1}{R^3} \sum_{u_j \in B, \ t_j \leq R^{-1}t_{\epsilon}} \frac{|L(1/2, u_j)|^2 |L(1/2 + it, u_j)|^2}{(1 + |t_j|)(1 + |t|)^2} \, dt
\]

\[
\ll \frac{1}{R^3} \sum_{u_j \in B, \ t_j \leq R^{-1}(2T)^{\epsilon}} \frac{|L(1/2, u_j)|^2 |L(1/2 + it, u_j)|^2}{(1 + |t_j|)} \int_T^{2T} \frac{|L(1/2 + it, u_j)|^2}{(1 + |t|)^2} \, dt
\]

\[
\ll \frac{T^{-1+\epsilon}}{R^3} \sum_{u_j \in B, \ t_j \leq R^{-1}(2T)^{\epsilon}} \frac{|L(1/2, u_j)|^2}{(1 + |t_j|)^{1-\epsilon}}.
\]

The last sum on the right-hand side of (10.2) is estimated using partial summation and the mean-subconvexity estimate (8.5) so that

\[
M_D(R, T) \ll R^{-5-\epsilon T^{-1+\epsilon}}.
\]

10.3. The contribution of the continuous spectrum. In order to estimate the contribution of the continuous spectrum to the error term in (1.23) we adapt the approach of Section 4 to the situation of the hyperbolic 3-space. By the same arguments as in the proof of Proposition 9.1 (see Section 9.3) we infer that

\[
M_C(R, T) := \int_T^{2T} \frac{1}{R^3} \int_\infty^{-\infty} \langle |E(\cdot, 1 + it)|^2, E(\cdot, 1 + it') \rangle_{\text{reg}} \langle E(\cdot, 1 + it'), \phi_R \rangle \, dt' \, dt
\]

\[
\ll \frac{\|\phi_R\|_2^2}{R^6} \int_T^{2T} I(t) \, dt
\]

where \( I(t) \) is defined in (9.6). By (9.7) and the Cauchy–Schwarz inequality we therefore obtain
\[ M_C(R, T) \]
\[
\ll \frac{1}{R^{3+\epsilon}} \int_T^{2T} \int_{|t'| \leq R^{-1}t} \left| \zeta_K \left( \frac{1+it'}{2} \right) \right|^4 \left| \zeta_K \left( \frac{1+it'}{2} - it \right) \right|^2 \left| \zeta_K \left( \frac{1+it'}{2} + it \right) \right|^2 \frac{dt'}{(1 + |t'|)(1 + |t|)^{2-\epsilon}} dt \]
\[
\ll \frac{T^{-2+\epsilon}}{R^{3+\epsilon}} \int_{|t'| \leq R^{-1}T} \left| \zeta_K \left( \frac{1+it'}{2} \right) \right|^4 \left| \zeta_K \left( \frac{1+it'}{2} - it \right) \right|^2 \left| \zeta_K \left( \frac{1+it'}{2} + it \right) \right|^2 \frac{dt'}{(1 + |t'|)^\epsilon}.
\]

The inner integral can be evaluated using the Cauchy–Schwarz inequality and the estimate for the fourth moment of the Dedekind zeta function (9.12). We conclude
\[
M_C(R, T) \ll \frac{T^{-2+\epsilon}}{R^{3+\epsilon}} \int_{|t'| \leq R^{-1}T} \left| \zeta_K \left( \frac{1+it'}{2} \right) \right|^4 \left( T + t' \right) \frac{29+3\epsilon+\epsilon}{3+\epsilon} dt'.
\]

Using that \( t' \leq R^{-1}T^\epsilon \ll T \) and (9.12) again we get
\[
M_C(R, T) \ll \frac{T^{-\frac{7}{3}+\frac{7}{3}+\epsilon}}{R^{\frac{29}{3}+\frac{29}{3}+\epsilon}}.
\]

10.4. **Proof of Theorem 1.9.** To prove Theorem 1.9 we combine our previous results for the contribution of the discrete and the continuous spectrum and infer
\[
\int_T^{2T} \left| \int_{B_R(P)} E(Q, 1 + it) |dv(Q)\right|^2 dt \ll \frac{2}{\log(1 + t^2) \text{vol}(B_R(P))} \left( \frac{T^{-1+\epsilon}}{R^5+\epsilon} + \frac{T^{-7/9+a/3+\epsilon}}{R^{29/9+a/3+\epsilon}} + \frac{T}{(\log \log T)^2} \right).
\]

This is \( o(T) \) if and only if
\[
R \gg \max \left\{ T^{-2/5+\epsilon}, T^{-16/29+3a+\epsilon} \right\}.
\]

As \( f(a) = -\frac{16-3a}{29+3a} \) is increasing, \( f(a) \leq f(1) = -13/32 < -2/5 \) and \( T^{-16/29+3a+\epsilon} \leq T^{-2/5+\epsilon} \) for \( a \in [0, 1] \). Thus Heath-Brown’s result for the twelfth moment of the Riemann zeta function suffices to prove that quantum ergodicity holds for \( R \gg T^{-2/5+\epsilon} \sim t^{-2/5+\epsilon} \). 

11. **Massive irregularities in shrinking balls of large dimension**

In this final section we generalize Theorem 1.6 to compact arithmetic quotients of the \( n \)-dimensional hyperbolic space for \( n \geq 4 \). Our result follows from an explicit estimate of the Selberg transform for the characteristic kernel and a lower bound of Donnelly [11] for the sup-norm of Laplace eigenfunctions on \( n \)-dimensional compact arithmetic manifolds. Let us write \( \lambda_j = (n-1)^2/4 + t_j^2 \) for the Laplace eigenvalue of the cusp form \( \phi_j \) and we will always assume \( t_j \geq 0 \). The trivial bound for the sup-norm of the Laplace eigenfunction \( \phi_j \) on \( \Gamma \setminus \mathbb{H}^n \) is
\[
\| \phi_j \|_\infty \ll t_j^{\frac{n-1}{2}}.
\]

For \( n \geq 5 \) Donnelly proved:
Theorem 11.1 (Donelly [11]). For every \( n \geq 5 \) there exist compact arithmetic manifolds \( \Gamma \setminus \mathbb{H}^n \) admitting sequences of eigenfunctions satisfying

\[
\|\phi_j\|_\infty = \Omega \left( t_j^{\frac{n-4}{2}} \right) \quad j \to \infty. \tag{11.2}
\]

The sequences \((\phi_j)_j\) that satisfy (11.2) are called exceptional sequences. Furthermore, it follows from the work of Brumley and Marshall [6] that there also exist 4-manifolds which admit eigenfunctions with large sup-norm: they obtain the bound

\[
\|\phi_j\|_\infty = \Omega \left( t_j^{b_4} \right) \tag{11.3}
\]

for some \( b_4 > 0 \). In fact, similarly to [47] and [55], in [6] and [11] the following stronger statements are proved: we can find thin subsequences \((\lambda_{j_k})_{j_k}\) and fixed arithmetic points \( P \in \Gamma \setminus \mathbb{H}^n \) such that, as \( j_k \to \infty \), we have

\[
|\phi_{j_k}(P)| \gtrsim \begin{cases} 
 t_j^{b_4} & \text{if } n = 4, \\
 t_j^{(n-4)/2} & \text{if } n \geq 5.
\end{cases} \tag{11.4}
\]

Thus we can prove our next result.

Theorem 11.2. Let \( \Gamma \setminus \mathbb{H}^n \) be a hyperbolic \( n \)-manifold satisfying (11.2) or (11.3), respectively and \( P \) a fixed arithmetic point as in (11.4). Furthermore, choose \( \delta_n \) such that

\[
\delta_n = \begin{cases} 
 1 - \frac{2b_4}{5} & \text{if } n = 4, \\
 \frac{n+1}{n+1} & \text{if } n \geq 5.
\end{cases} \tag{11.5}
\]

If \( R \ll t_j^{-\delta_n+\epsilon} \), then

\[
\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |\phi_j(Q)|^2 dv(Q) = \Omega \left( t_j^\epsilon \right).
\]

In particular, QUE on shrinking balls fails for radii away from the Planck-scale.

Remark 11.3. We mention that in [6] no explicit value for \( b_4 \) is given. However, by (11.1) we get \( b_4 \leq 3/2 \) and this allows to bound the exponent \( \delta_4 \) appearing in Theorem 11.2. Namely, we have \( 2/5 \leq \delta_4 < 1 \).

The proof of Theorem 11.2 relies on the following lemma, which generalizes the third case of [27, Lemma 4.2] and provides also a modified generalization of the third case in [7, Lemma 2.4].

Lemma 11.4. For \( n \geq 3 \) there exists a constant \( c_n \) such that, as \( R \to 0 \) and \( Rt \to \infty \), the Selberg transform \( h_{R,n}(t) \) associated to the characteristic function via (5.1) satisfies

\[
h_{R,n}(t) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma \left( \frac{n}{2} + 1 \right) (Rt)^{-\frac{n+1}{4}} \cos \left( \frac{Rt - (n+1)\pi}{4} \right) + O_{n,M} \left( (tR)^{-\frac{n+3}{2}} + R^2(tR)^{-\frac{n+1}{2}} + R^{2M+2} \right)
\]

for any fixed integer \( M \geq 1 \).
Proof. By [52, Eq. (4.2)] and [38] the Selberg transform of the kernel \( k(u) \) defined by (5.1) is given by

\[
h_{R,n}(t) = \frac{a_n}{\text{vol}(B_R)} \int_0^R (\cosh R - \cosh u)^{\frac{n-1}{2}} \cos(tu) \, du
\]

\[
= \frac{a_n R}{\text{vol}(B_R)} \int_0^1 (\cosh R - \cosh Ru)^{\frac{n-1}{2}} \cos(tRu) \, du
\]

where the constant

\[
a_n = \frac{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}}}{(n-1) \Gamma \left( \frac{n-1}{2} \right)}
\]

depends only on the dimension \( n \). In order to determine the asymptotic behaviour of \( h_{R,n} \), we distinguish between the case \( n \) even and \( n \) odd. If \( n \) is even, we get using the Taylor expansion of the hyperbolic cosine and of \((1 + x)^{\frac{n+1}{2}}\)

\[
(\cosh R - \cosh Ru)^{\frac{n+1}{2}} = \left( \frac{R^2 (1 - u^2)}{2} \right)^{\frac{n+1}{2}} \left( 1 + 2 \sum_{k=1}^{M} \frac{R^{2k}}{(2k)!} \sum_{l=0}^{k} u^{2l} + O \left( R^{2M+2} \right) \right)^{\frac{n+1}{2}}
\]

with certain positive coefficients \( c_{k,l,r} \) and \( N \leq M \). If \( n \) is odd, we can avoid the Taylor expansion of \((1 + x)^{\frac{n+1}{2}}\) and obtain immediately

\[
(\cosh R - \cosh Ru)^{\frac{n+1}{2}} = \left( \frac{R^2 (1 - u^2)}{2} \right)^{\frac{n+1}{2}} \left( 1 + \sum_{r=1}^{N} \sum_{k=r}^{M} \sum_{l=0}^{k} c_{k,l,r} R^{2k} u^{2l} + O_M \left( R^{2M+2} \right) + O_N \left( R^{2N+2} \right) \right)
\]

By (11.6) this implies that

\[
h_{R,n}(t) = MT_1 + MT_2 + O_{n,M} \left( R^{2M+2} \right)
\]

where the main term is given by

\[
MT_1 = \frac{a_n R^n}{2^{\frac{n-1}{2}} \text{vol}(B_R)} \int_0^1 (1 - u^2)^{\frac{n-1}{2}} \cos(tRu) \, du
\]

and

\[
MT_2 = \frac{a_n R^n}{2^{\frac{n-1}{2}} \text{vol}(B_R)} \int_0^1 \sum_{r=1}^{N} \sum_{k=r}^{M} \sum_{l=0}^{k} c_{k,l,r} R^{2k} u^{2l} (1 - u^2)^{\frac{n-1}{2}} \cos(tRu) \, du.
\]

Note that for small radii \( R \) the hyperbolic volume of \( B_R \) is asymptotic to the Euclidean volume, i.e. \( \text{vol}(B_R) = \pi^\frac{n}{2} R^n / \Gamma \left( \frac{n}{2} + 1 \right) + O \left( R^{n+2} \right) \) as \( R \to 0 \). It remains to determine
the main term $MT_1$ and the contribution of the second error term $MT_2$. Using \cite[Eq. (17.34.10)]{14} we get for the main term

$$\frac{a_n R^n}{2 \frac{n+1}{2}} \int_0^1 (1 - u^2)^{\frac{n+1}{2}} \cos(t Ru) \, du = \frac{a_n \sqrt{\pi} R^n}{\sqrt{2} \text{vol}(B_t)} \Gamma\left(\frac{n+1}{2}\right) \frac{J_{\frac{n}{2}}(Rt)}{(Rt)^{\frac{n}{2}}},$$

(11.9)

where $J_n(x)$ denotes the $J$-Bessel function. As the asymptotic behaviour of the Bessel function is given by

$$J_{\frac{n}{2}}(Rt) = \sqrt{\frac{2}{\pi Rt}} \cos \left( Rt - \frac{(n+1)\pi}{4} \right) + O_n \left( (Rt)^{-3/2} \right)$$

(11.10)

(see e.g. \cite[(B.35)]{30}), we infer that

$$MT_1 = \frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \frac{J_{\frac{n}{2}}(Rt)}{(Rt)^{\frac{n}{2}}}$$

$$+ O_n \left( (Rt)^{-\frac{n+3}{2}} \right).$$

In order to treat $MT_2$ we consider the integral

$$I_l := \int_0^1 (1 - u^2)^{\frac{n+1}{2}} u^{2l} \cos(t Ru) \, du,$$

$l \in \mathbb{N}$, that appears in $MT_2$. By \cite[Eq. (17.34.10)]{14} and \cite[Eq. (8.471.2)]{14} we obtain that

$$I_l = \frac{(-1)^l}{R^{2l}} \frac{\partial^{2l}}{\partial t^{2l}} \int_0^1 (1 - u^2)^{\frac{n+1}{2}} \cos(t Ru) \, du$$

$$= \frac{(-1)^l 2^{\frac{n}{2} - 1}}{R^{2l}} \sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right) \frac{J_{\frac{n}{2}}(Rt)}{(Rt)^{\frac{n}{2}}}$$

$$= \frac{(-1)^l 2^{\frac{n}{2} - 1}}{R^{2l}} \sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right) \sum_{k=0}^{2l} \alpha_k \left( (Rt)^{-\frac{n}{2}} \right)^k \left( J_{\frac{n}{2}}(Rt) \right)^{(2l-k)}$$

$$= \frac{(-1)^l 2^{\frac{n}{2} - 1}}{R^{2l}} \sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right) \sum_{k=0}^{2l} \alpha_k \sum_{m=-\frac{2l-k}{2l-k}} \beta_m J_{\frac{n}{2}+m}(Rt)$$

holds with certain coefficients $\alpha_k, \alpha_{k,n}, \beta_m \in \mathbb{R}$. Thus (11.8) and (11.10) imply

$$MT_2 = O \left( R^2 (Rt)^{-\frac{n+1}{2}} \right).$$

This proves the lemma. \hfill \Box

What is crucial in the proof of Lemma (11.4) is that we have freedom in the choice of $M$. If $M$ is chosen such that $R \ll t^{-\frac{n+1}{4(n+1)+\eta+1}}$, then

$$R^{2M+2} \ll (Rt)^{-\frac{n+1}{2}}$$

Taking $M$ sufficiently large we infer that our main term is a good approximation of the Selberg transform $h_{R,n}$ in the range $R \sim t^{-\delta}$ with $\delta \in (0,1)$. Hence

$$h_{R,n}(t) = \Omega \left( (Rt)^{-\frac{n+1}{2}} \right).$$
We now prove Theorem 11.2.

Proof of Theorem 11.2. First of all, we note that the $n$-dimensional analogue of Proposition 5.2 is a standard property of the Selberg transform [60, eq. (1.8)]. Then, as in the proof of Theorem 1.6 the Cauchy–Schwarz inequality implies

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |\phi_j(Q)|^2 dv(Q) \geq |h_{R,n}(t_j)|^2 |\phi_j(P)|^2.$$  

Now let us consider the exceptional subsequence $(\lambda_{j_k})_k$. For $n = 4$ Lemma 11.4 and (11.3) imply that

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |\phi_{j_k}(Q)|^2 dv(Q) \gtrsim t_{j_k}^{2\delta_4 - 5(1 - \delta_4)}$$

if $R \ll t_{j_k}^{-\delta_4}$ and $t_{j_k}^{2\delta_4 - 5(1 - \delta_4)} \to \infty$ if $\delta_4 > 1 - \frac{2\delta_4}{5}$. For $n \geq 5$ we replace (11.3) by Theorem 11.1 and obtain

$$\frac{1}{\text{vol}(B_R(P))} \int_{B_R(P)} |\phi_{j_k}(Q)|^2 dv(Q) \gtrsim \left(t_{j_k}^{-1 + \delta_n}\right)^{2\frac{\delta_n}{2(1 - \delta_n)}} \left(t_{j_k}^{\frac{\delta_n}{2}}\right)^2 \left(t_{j_k}^{\frac{\delta_n(n+1)-5}{2}}\right)$$

if $R \ll t_{j_k}^{-\delta_n}$. This proves the theorem as $t_{j_k}^{\delta_n(n+1)-5} \to \infty$ if $\delta_n > \frac{5}{n+1}$. \hfill $\Box$

Remark 11.5. It is natural to expect that for $n \geq 5$ the exponent $\delta_n = \frac{5}{n+1}$ is optimal, in the same sense that we expect $\delta = 3/4$ to be optimal for arithmetic 3-manifolds.

Remark 11.6. One can also consider the more general case where the centre $w_j$ of the shrinking balls $B_{R_j}(w_j)$ varies. For the situation of the flat torus this more general case has been treated by Lester–Rudnick [39]. However, in the hyperbolic case, even for in dimension 2, the assumption that the centre $w$ is fixed or belongs to a compact set is necessary. For the hyperbolic plane and the modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$ Iwaniec and Sarnak proved that

$$t_j^{\frac{\delta}{2} - \epsilon} \ll u_j(w_j) \ll t_j^{\frac{\delta}{2} + \epsilon}$$

if $w_j = \frac{1}{4} + i \frac{t_j}{2\pi} + o(1)$ (see [56]). This purely analytic phenomenon implies that

$$\frac{1}{\text{vol}(B_{R}(w_j))} \int_{B_{R}(w_j)} |u_j(z)|^2 d\mu(z) \gg |h(t_j)|^2 |u_j(w_j)|^2 \gg t_j^{-\frac{8}{3} - \epsilon} R^{-3}$$

which is unbounded for any $R \ll t_j^{-\delta}$ with $\delta > 8/9$ fixed.

It follows from the work of Phillips and Sarnak [53] that a generic subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$ has few cusp forms, conjecturally there are only finitely many. This is a consequence of the contribution of the continuous spectrum to the Weyl law. In general, if a co-finite group $\Gamma \subset \text{SO}(n,1)$ has sufficiently large Eisenstein series in the sense that

$$\sum_a \int_{-T}^T \left| E_a \left( P, \frac{n-1}{2} + it \right) \right|^2 dt \gg_P T^n \quad \text{(11.11)}$$

for some point $P \in \Gamma \setminus \mathbb{H}^n$, then

$$E_a \left( P, \frac{n-1}{2} + it \right) = \Omega \left( t^{\frac{n-1}{2}} \right).$$
Here $E_a(P,s)$ denotes the Eisenstein series associated to the cusp $a$. Note that the modular group $\text{PSL}_2(\mathbb{Z})$ does not satisfy condition (11.11). Namely, in this case the following bound holds for $z$ fixed:

$$E(z, 1/2 + it) \ll t^{1/3+\epsilon}$$

(see [4], [49]). For groups that satisfy (11.11) we obtain the following proposition working as in Theorem 11.2:

**Proposition 11.7.** If $\Gamma$ has sufficiently large Eisenstein series in the sense of (11.11), then QUE of Eisenstein series fails in shrinking balls of radius $R \ll t^{-2/(n+1)+\epsilon}$.

Comparing Theorem 11.2 and Proposition 11.7 to the bound of Han and Hezari–Rivi`ere (1.11) it is natural to ask whether for arithmetic hyperbolic manifolds $\mathcal{M} = \Gamma \setminus \mathbb{H}^n$ their bound can be improved. If $\mathcal{M} = \Gamma \setminus \mathbb{H}^n$ is an $n$-dimensional hyperbolic manifold of finite volume, we expect quantum unique ergodicity for shrinking balls of radii

$$R \gg t^{-\frac{2}{n+1}+\epsilon}$$

for some constant $c = c_\mathcal{M} > 0$ that depends on the manifold $\mathcal{M}$ and which might be different for the discrete and the continuous spectrum.

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