Tensor monopoles and negative magnetoresistance effect in four-dimensional optical lattices

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We propose that a kind of four-dimensional (4D) Hamiltonians, which host tensor monopoles related to quantum metric tensor in even dimensions, can be simulated by ultracold atoms in the optical lattices. The topological properties and bulk-boundary correspondence of tensor monopoles are investigated in detail. By fixing the momentum along one of the dimensions, it can be reduced to an effective three-dimensional model manifesting with a nontrivial chiral insulator phase. Using the semiclassical Boltzmann equation, we calculate the longitudinal resistance against the magnetic field \(B\) and find a negative relative magnetoresistance effect of approximately \(-B^2\) dependence when a hyperplane is cut through the tensor monopoles in the parameter space. We also propose an experimental scheme to realize this 4D Hamiltonian by extending an artificial dimension in 3D optical lattices. Moreover, we show that the quantum metric tensor can be detected by applying an external drive in the optical lattices.

I. INTRODUCTION

In 1931, Dirac introduced the concept of monopoles to explain the quantization of electron charge [1]. Since then, the development of gauge theory has shown that monopoles emerge in a natural way in all theories of grand unification. However, the existence of the monopole as an element parcticles has not been confirmed by any experiments till today. Monopoles in momentum space have attracted extensive studies in condensed matter physics and artificial quantum systems, for example, Dirac monopoles in Weyl semimetals. The celebrated Nielsen-Ninomiya theorem states that Weyl points in the first Brillouin zone must emerge and annihilate in pairs with opposite chirality, which provides a mechanism of anomaly cancellation in the field theorem framework [2]. Moreover, negative magnetoresistance (MR) effect, for which the longitudinal conductivity increases along with the increasing magnetic field, has been reported in several experiments and can be interpreted as a result of the suppression of backscattering due to the opposite chirality of the monopoles in Weyl semimetals [3–9]. Besides those monopoles in odd dimensions, recent research shows that another kind of monopoles can emerge in even dimensions, named “tensor monopoles”, which are Abelian monopoles associated with the tensor (Kalb-Ramond) gauge field [10]. The topological charge of the tensor monopole is related to the so-called quantum metric which measures the distance of two nearby states in the parameter space. Recently, by using controllable quantum systems, several experiments have been reported to directly measure the quantum metric tensor, which characterizes the geometry and topology of underlying quantum states in parameter space [11–14].

The technology of ultracold atom provides an excellent platform to study different topological systems of condensed matter and high-energy physics, because of its perfect cleanliness and high controllability [15]. Recently, 4D quantum Hall effect has also been experimentally simulated by ultracold atom, which opens up the research of high-dimensional physics in realistic systems [16, 17]. The extra dimension can be introduced to the 3D optical lattice with a cyclical parameter varying from \(-\pi\) to \(\pi\), which plays the role of the pseudo-momentum of the fourth dimension [16–23]. Besides, it shows that synthetic dimension can also be engineered by a set of internal atomic levels as an artificial lattice dimension [24, 25]. In order to measure the Berry phase of topological systems in cold atoms, many experimental approaches have been proposed and conducted, including state tomography [26, 27], interferometry [28], and atomic transport [29]. Recent development on how to measure the quantum metric tensor by shaking the optical lattice has promoted the research of tensor monopoles with cold atoms [12, 30].

In this paper, we propose two minimal Hamiltonians in 4D which host tensor monopoles [10, 31–34], and then study their topological properties. Tensor monopoles in these two systems can be considered as one conductance band, one valence band and one flat band touching at the common points and the topological properties of the tensor monopoles can be controlled by a tunable parameter. After fixing the momentum of the fourth dimension in the parameter space, we obtain a 3D model. Rich phase diagrams can be derived from this 3D model, including trivial phase and chiral insulator. In addition, we calculate the MR with the semiclassical Boltzmann equation. When a hyperplane cuts through the tensor monopoles, the relative MR approximately proportional to \(-B^2\) sig-
tured by tensor Berry connection. The associated gauge connections, i.e., vector gauge field. But for tensor Yang monopoles defined in 3D and 5D parameter spaces, touch commonly.

\[ u^k = \text{components of the lowest band} |u_- \rangle \]  

This tensor Berry connection satisfies the following gauge transformation \[ B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \]  

where \( \xi_\mu \) is a vector that contains the redundant gauge degree of freedom of the field. Related 3-form curvature is \( H = dB \), whose components are given by

\[ H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}. \]  

It is gauge invariant and antisymmetric. For Hamiltonian in Eq. (1), the corresponding 3-form curvature is

\[ H_{\mu\nu\lambda} = \text{sgn}(\alpha_x \alpha_y \alpha_z \alpha_w) \epsilon_{\mu\nu\lambda\gamma} \frac{nk_\gamma}{(k_1^2 + k_2^2 + k_3^2 + k_4^2)^2}. \]  

A topological charge associated with this curvature \( H_{\mu\nu\lambda} \) can be defined by surrounding the tensor monopole with a sphere \( S^3 \),

\[ Q_n = \frac{1}{2\pi^2} \int_{S^3} dk^\mu \wedge dk^\nu \wedge dk^\lambda H_{\mu\nu\lambda}. \]  

This is a topological invariant known as the Dixmier-Douady (DD) invariant, which is related to the (first) Dixmier-Douady class of U(1) "bundle gerbes" [39–42].

Inspired by Ref. [10], we can also find a direct relation between the components of 3-form curvature and the quantum metric (or Fubini-Study metric),

\[ H_{\mu\nu\lambda} = \text{sgn}(\alpha_x \alpha_y \alpha_z \alpha_w) \epsilon_{\mu\nu\lambda\gamma}(4\sqrt{\det g_{\mu\nu}}), \]  

where \( g_{\mu\nu} \) is the 3 \times 3 quantum-metric tensor defined in the proper 3D subspace. This equation provides a feasible method to detect the 3-form curvature, which will be discussed in section V. Quantum metric tensor is the real part of the quantum geometric tensor, whose imaginary part is just Berry curvature [43] and has been directly measured in some engineered systems [26–29, 44, 45].

Physically, if the Hamiltonian of a system is parametrized as \( H \equiv H(\vec{\lambda}) \), quantum metric tensor measures the (infinitesimal) distance between two nearby quantum states, \( ds^2 = 1 - |\langle \psi_\lambda | \psi_{\lambda + \delta\lambda} \rangle|^2 \), in \( \vec{\lambda} \) space [46] as

\[ ds^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu, \]  

in which the metric tensor can be explicitly written as

\[ g_{\mu\nu} = \text{Re} \left( \langle \frac{\partial \psi}{\partial \lambda_\mu} | \frac{\partial \psi}{\partial \lambda_\nu} \rangle - \langle \frac{\partial \psi}{\partial \lambda_\nu} | \frac{\partial \psi}{\partial \lambda_\mu} \rangle \langle \psi | \frac{\partial \psi}{\partial \lambda_\mu} \rangle \langle \psi | \frac{\partial \psi}{\partial \lambda_\nu} \rangle \right). \]  

Obviously, the metric is positive and satisfies \( g_{\mu\nu} = g_{\nu\mu} \). For our model, we parametrize the momentum space \( \vec{k} = (k_x, k_y, k_z, k_w) \) in Eq. (1) with the hyperspherical coordinates \((k, \theta_1, \theta_2, \varphi)\) as

\[ k_x = k \sin \theta_1 \sin \theta_2 \cos \varphi, \]

\[ k_y = k \sin \theta_1 \sin \theta_2 \sin \varphi, \]

\[ k_z = k \sin \theta_1 \cos \theta_2, \]

\[ k_w = k \cos \theta_1, \]
where \( k = \sqrt{(k_x^2 + k_y^2)^n + k_z^2 + k_w^2} \) is the radius of the 3-hypersphere encircling the monopole in momentum space. If the lowest energy band \( \varepsilon_k = -k \) is filled, the topological charge of the tensor monopole can be defined as

\[
Q_n = \frac{1}{2\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{2\pi} d\varphi \mathcal{H}_{\theta_1 \theta_2 \varphi},
\]

where \( \mathcal{H}_{\theta_1 \theta_2 \varphi} = 4\text{sgn}(\alpha_x \alpha_y \alpha_z \alpha_w)\epsilon_{\theta_1 \theta_2 \varphi} \sqrt{\det g} \) with \( \epsilon \) the Levi-Civita antisymmetric tensor. By considering the hypersphere surrounding the monopole, we derive \( \det g = \frac{1}{16} n^2 \sin^4 \theta_1 \sin^2 \theta_2 \) and the topological charge is obtained as

\[
Q_n = n\text{sgn}(\alpha_x \alpha_y \alpha_z \alpha_w).
\]

We consider two special cases of the Hamiltonians in Eq. (1) as

\[
\mathcal{H}_1 = k_x \lambda_1 + k_y \lambda_2 + k_z \lambda_6 + k_w \lambda_7,
\]

for \( n = 1 \), and

\[
\mathcal{H}_2 = (k_x^2 - k_y^2) \lambda_1 + 2k_x k_y \lambda_2 + k_z \lambda_6 + k_w \lambda_7,
\]

for \( n = 2 \). In both cases, the three energy bands all crossing at the \( k = (0,0,0,0) \), which hosts tensor monopoles with the topological charges being \( Q_1 = 1 \) and \( Q_2 = 2 \), respectively. Also, the tensor monopoles are stable under smooth deformations of the Hamiltonians as a result of its topological nature [10]. Actually this kind of monopoles can be viewed as magnetic monopoles in parameter space and the generalized curvature can be regarded as the magnetic field of the monopole [47].

III. THE MINIMAL MODELS IN MOMENTUM SPACE

Now we construct the Hamiltonians in Eqs. (14) and (15) with the tight-binding models in the momentum space as

\[
\mathcal{H}_n = d_{n,x} \lambda_1 + d_{n,y} \lambda_2 + d_{n,z} \lambda_6 + d_{n,w} \lambda_7,
\]

with \( n = 1, 2 \).

For \( n = 1 \), the explicit form of \( d \)’s are

\[
\begin{align*}
d_{1,x} &= 2t \sin k_x, \\
d_{1,y} &= 2t \sin k_y, \\
d_{1,z} &= 2t \sin k_z, \\
d_{1,w} &= 2t(h - \cos k_x - \cos k_y - \cos k_z - \cos k_w),
\end{align*}
\]

where we have set the lattice constant \( a = 1 \). Here, \( t \) is hopping energy and \( h \) is a tunable parameter. The corresponding spectrum is given by \( \{0, \pm \sqrt{d_{1,x}^2 + d_{1,y}^2 + d_{1,z}^2 + d_{1,w}^2}\} \). When \( h = 3 \), there exist a pair of triple-degenerate Dirac-like points at \( K_\pm = (0,0,0,\pm \pi/2) \), which are tensor monopoles with topological charges \( \pm 1 \), and the \( k \cdot p \) Hamiltonian near the two nodes with \( q = k - K_\pm \) yields the low-energy effective Hamiltonian as Eq. (14).

Similarly, for \( n = 2 \), the \( d \)’s can be written as

\[
\begin{align*}
d_{2,x} &= 2t(\sin^2 k_x - \sin^2 k_y), \\
d_{2,y} &= 4t \sin k_x \sin k_y, \\
d_{2,z} &= 2t \sin k_z, \\
d_{2,w} &= 2t(h - \cos k_x - \cos k_y - \cos k_z - \cos k_w),
\end{align*}
\]

by which the energy dispersion is obtained as \( \{0, \pm \sqrt{d_{2,x}^2 + d_{2,y}^2 + d_{2,z}^2 + d_{2,w}^2}\} \). For \( h = 3 \), there are also a pair of tensor monopoles at \( K_\pm = (0,0,0,\pm \pi/2) \) with topological charges \( \pm 2 \) as shown in Fig. 1(b) with \( k_x = 0 \). The low-energy effective Hamiltonian near the two nodes is obtained as Eq. (15).

For both cases of \( n = 1 \) and \( n = 2 \), the combination and division of tensor monopoles inside the first Brillouin zone (FBZ) are controlled by the parameter \( h \). For \( h = 0 \), there are six monopoles at \( (\pi,0,0,0) \), \( (\pi,0,0,\pi) \), \( (0,\pi,0,\pi) \), \( (0,\pi,\pi,0) \), \( (0,0,\pi,\pi) \), and \( (\pi,0,\pi,\pi) \), including three of them with positive topological charge and three others with negative topological charge as a result of the generalized Nielsen-Ninomiya theorem [2]. Increasing \( h \), the six monopoles begin to move in FBZ. When \( h = 1 \), the six degenerate points move to \( (0,0,\pi,\pm \pi/2), (0,\pi,0,\pm \pi/2), (\pi,0,0,\pm \pi/2) \) with the same topology. When continuously increasing \( h \) to \( h = 2 \), there are four monopoles left at \( (\pi,0,0,0), (\pi,0,0,\pi), (0,\pi,0,\pi), (0,0,\pi,\pi) \), with two others of opposite topological charges annihilated each together to open a gap. For \( h = 3 \), only two monopoles are left at \( (0,0,0,\pm \pi/2) \). For \( h = 4 \), the two monopoles move toward \( (0,0,0,0) \) and combine to open a gap. Finally, it becomes a topologically-trivial insulator for \( h > 4 \).

By taking a slice of these two 4D models, i.e., fixing \( k_w = 0 \), 3D models can be derived from the 4D systems. The topological nature of the 3D system is captured by DD invariant. This invariant is equivalent to the winding number, which characterizes 3D topological insulators in
class $AIII$ [38, 48, 49]

$$\Gamma_n = \frac{1}{12\pi^2} \int_{BZ} d^3 k e^{\alpha\beta\gamma\rho} \epsilon^{\mu\nu\tau} \frac{1}{E^n_+} \partial_\alpha d_\beta \partial_\gamma d_\rho$$

which the indexes of the Levi-Civita symbol with $\alpha, \beta, \gamma, \rho$ and $\mu, \nu, \tau$ represent $\{x, y, z, w\}$ and $\{k_x, k_y, k_z\}$, respectively.

Another equivalent way to characterize the topology of the 3D models is the Chern-Simons invariant (CSI), which takes the form as

$$CS = \frac{1}{4\pi} \int_{BZ} dk e^{\mu\nu\tau} A_\mu(k) \partial_\nu A_\tau(k),$$

where $A_\mu(k) = \langle u(k) | i\partial_\mu | u(k) \rangle$ ($\mu = x, y, z$) [48, 50]. We plot CSI against $h$ in Fig. 2(a) with $n = 1$ and 2(b) with $n = 2$ for the three energy bands at $k_w = 0$. The relation of the value of it between different bands is

$$CS_n(+) = CS_n(-) = \frac{1}{4} CS_n(0).$$

As indicated in Fig. 2, the topological phase transitions occur at $h = 0, \pm 2, 4$, when the three bands touch at the hyperplane of $k_w = 0$. For $h \in (-2, 4)$, the $CS_n$ is nonzero, it is topologically nontrivial phases here. A detailed calculation shows that the relation between winding number and the Chern-Simons term is

$$\frac{\pi}{4} \Gamma_n = CS_n(-).$$

When $h = 5$, according to Fig. 2, the 3D systems is trivial and no surface state exists, which is confirmed by the numerical calculation shown in Fig. 3 (c) and (d).

The energy spectrum and surface states with the open boundary along $\hat{z}$ direction are shown in Fig. 3. As discussed before, for $h = 3$, two tensor monopoles are located at $(0, 0, 0, \pm \pi/2)$ and the spectrum is gapped and topologically nontrivial for $k_w \in (-\pi/2, \pi/2)$. They are chiral insulator phases. Therefore, there are surface states of Dirac cones for those sliced 3D systems until the slicing hyperplane hits the tensor monopoles. Namely, surface Dirac cone survives when $-\pi/2 < k_w < \pi/2$.

The spectra with $k_w = 0$ are shown in Fig. 3 (a) and (b) for $n = 1$ and $n = 2$, respectively. When we take the slicing hyperplane perpendicular to $\hat{z}$ axis, the sets of those Dirac points constitute the Fermi arcs connecting two tensor monopoles for $n = 1$ and $n = 2$, which are shown in Fig. 4 (a) and (c), respectively. Fig. 4 (b) and (d) shows the density distribution of surface states. By using the method in Ref [48], $k_w \in (-\pi/2, \pi/2)$, the low-energy spectra of surface states around $(k_x, k_y, k_z) = (0, 0, 0)$ are $\pm v \sqrt{k_x^2 + k_y^2}$ and $\pm v(k_x^2 + k_z^2)$ for $n = 1$ and $n = 2$, respectively. Here $v = 2t$ is the effective Fermi velocity. Detailed derivation of these spectra can be found in the Appendix.
IV. TRANSPORT PROPERTY

Negative magnetoresistance effect has already been extensively discussed in topological semimetals. This fantastic transport phenomena is widely believed to be caused by chiral anomaly, which is the violation of the conservation of chiral current [5]. In some topological insulators, there also emerges the same effect, although the chiral anomaly is not well defined in those systems [51, 52]. By using semiclassical equation, we can calculate MR.

In semiclassical limit, the electronic transport can be described by the equations of motion

$$\dot{r} = \frac{1}{\hbar} \nabla_k \tilde{\varepsilon}_k - \tilde{k} \times \Omega_k,$$
$$\tilde{k} = -\frac{e}{\hbar} (E + \dot{r} \times B),$$
$$\tilde{\varepsilon}_k = \varepsilon_k - M \cdot \tilde{B},$$
$$M = -\frac{e}{2\hbar} \text{Im} \left( \frac{\partial u}{\partial k} \right) \left( \tilde{\varepsilon}_0 - \tilde{H}_0(k) \right) \frac{\partial u}{\partial k},$$

(23)

which describe the dynamics of the wave packet [53–55]. Here, \(r\) is the position of the wave packet in real space, and \(k\) corresponds to the wave vector. \(\tilde{\varepsilon}_k\) is the energy dispersion of the valence band, and \(M\) is orbital magnetic moment of the wave packet which is analogous to the magnetic moment of a electron motions around the nucleus [56].

Using the semiclassical Boltzmann equation, the longitudinal conductivity can be calculated by

$$\sigma^{\mu \nu} = \int \frac{d^3k}{(2\pi)^3} \frac{e^2 \tau}{D_k} \left( \tilde{\varepsilon}_0 + \frac{e}{\hbar} B^a \tilde{\varepsilon}_k \Omega_k^a \right)^2 \left( -\frac{\partial \tilde{f}_0}{\partial \tilde{\varepsilon}} \right),$$

(24)

$$D_k = 1 + \frac{e}{\hbar} B \cdot \Omega_k,$$

where \(\tilde{f}_0\) is the equilibrium Fermi distribution, and \(\tau\) is the life time of the quasiparticle in the semiclassical limit [52, 57]. In Fig. 5, for \(n = 1, k_x = 0, \) and \(h = 3, \) we plot the relative MR of the longitudinal resistance against the magnetic field \(B_z\), which is defined as [52].

$$MR_z(B_z) = \frac{1/\sigma_{zz}(B_z) - 1/\sigma_{zz}(0)}{1/\sigma_{zz}(0)},$$

(25)

and the results are plotted in Fig. 5, there is a typical \(-B^2\) -dependence of the MR, which signifies the negative MR effect along \(\tilde{z}\). It is because the hyperplane for \(k_x = 0\) cuts through the two monopoles and a 3D Weyl semimetal is realized consequently. The typical experimental parameters of ultracold atoms in the optical lattice have been used in our calculation, while the magnetic field can be realized by artificial gauge field for the experimental setup. Since the flat zero-energy band doesn’t contribute to the conductance because of the vanishing velocity of the wave packet, we don’t take the flat band into consideration.

V. IMPLEMENTATION SCHEME

A. Realization with optical lattices

In this subsection, we propose a scheme to realize the 4D Hamiltonian in Eq. (16) for \(n = 1\) using ultracold atoms [7, 58–60]. The simulation of 4D system is achieved by parameterizing the momentum along \(\tilde{w}\) on the 3D optical lattice. For \(n = 1\), we can use noninteracting fermionic atoms in a cubic optical lattice and choose three atomic internal states in the ground state manifold into consideration.

FIG. 5. The relative MR with \(n = 1\) and \(h = 3,\) \(k_x = 0.\) The magnetic field applied along \(\tilde{z}\) direction, the lattice spacing \(a = 382\) nm, \(t/M = 2\pi \times 40\) Hz, and \(E_F = -0.1 t.\)

In this tight-binding model, the spin-dependent hopping is written as

$$\tilde{H} = t \sum_{\tau} \left[ \tilde{H}_{\tilde{r}y} + \tilde{H}_{\tilde{r}y} + \tilde{H}_{\tilde{r}x} + \tilde{H}' \right]$$
$$\tilde{H}_{\tilde{r}x} = -ia_{\tilde{r},\tilde{x},0} (a_{\tau,\tau} + a_{\tau,\tau}) + ia_{\tilde{r},\tilde{y},0} (a_{\tau,\tau} - a_{\tau,\tau}) + H.c.$$  
$$\tilde{H}_{\tilde{r}y} = a_{\tilde{r},\tilde{y},0} (a_{\tau,\tau} - ia_{\tau,\tau}) - a_{\tilde{r},\tilde{y},0} (a_{\tau,\tau} + ia_{\tau,\tau}) + H.c.$$  
$$\tilde{H}_{\tilde{r}z} = -2ia_{\tilde{r},\tilde{z},0}a_{\tau,\tau} + H.c.$$  
$$\tilde{H}' = 2i \gamma a_{\tau,\tau} + H.c.$$

(26)

where \(\tilde{H}_{\tilde{r}x}, \tilde{H}_{\tilde{r}y} \) and \(\tilde{H}_{\tilde{r}z}\) represent the hoppings along the \(x, y \) and \(z\) axis, respectively, with the tunneling amplitude \(t, \gamma = h - \cos \theta, \) \(h\) is a constant and \(\theta\) is a cyclic parameter that vary from \(\theta = -\pi\) to \(\theta = \pi\).

The generalized 3D tight-binding model on a simple cubic lattice Hamiltonian

$$H = \sum_{k, s, s'} \tilde{a}_{k,s}^+ \left[ \tilde{H}_1(k) \right]_{s,s'} \tilde{a}_{k,s},$$

(27)
where $\mathcal{H}_1(k)$ is Bloch Hamiltonian as in Eq. (16). Taking the parameter $\theta$ as the pseudo-momentum $k_w$, we can study the 3D system in a 4D parameter space $k = (k_x, k_y, k_z, \theta)$.

B. Detecting the quantum metric tensor

We now turn to address an experimental method to detect the tensor monopole in our system. The basic procedure is preparing the system in a given Bloch state and introducing an external drive by shaking the lattice [30, 62–70]. Then the quantum metric can be measured by establishing its relationship with integrated excitation rate, which is a measurable quantity in experiments [12, 62, 71–75].

In order to measure the quantum metric tensor related to Eq. (16) with $n = 1$, the system is first prepared in the state of $|u_-\rangle$. Shaking the lattice along the $x$ direction results in a circular time-periodic perturbation given by

$$\hat{H}_x(t) = \hat{H}_{\text{lattice}} + 2E x \cos(\omega t),$$

where $E$ is drive amplitude, $\omega$ is the frequency of shaking driving interband transitions [12, 30, 62]. By introducing this external drive, according to Ref. [11, 12], the relation between the total integrated excitation rate $\Gamma^\text{int}_x = \int d\omega \Gamma_x(\omega)$ and the diagonal quantum metric tensor is given as

$$\Gamma^\text{int}_x = \frac{2\pi E^2}{\hbar^2} g_{xx},$$

where the diagonal component of quantum metric tensor $g_{xx} = \frac{\hbar^2}{2\pi E^2} \sum_{n \neq -} |\langle u_n | \hat{d}_{k_z} | u_- \rangle |^2$. The relation in Eq. (29) provides an experimentally feasible approach to measure the quantum metric tensor. In practice, one can change the frequency to get the integrated excitation rate as [62, 67, 76]

$$\Gamma^\text{int}_x = \sum_i \Gamma_x (\omega_i) \Delta \omega.$$  

(30)

To obtain the off-diagonal components of quantum metric tensor, we can measure the excitation rate by applying the shaking along the different directions. Taking $g_{yz}$ as an example, the shaking can be applied along the directions $\hat{y} \pm \hat{z}$ [12, 30] and then the total Hamiltonian can be written as

$$\hat{H}_{y\pm z}(t) = \hat{H}_{\text{lattice}} + 2E (y \pm z) \cos(\omega t),$$

from which we can obtain the excitation rates $\Gamma^\text{int}_{y\pm z}$ and the difference of those two excitation rates is related to the off-diagonal quantum metric tensor as

$$\Gamma^\text{int}_{y+z} - \Gamma^\text{int}_{y-z} = \frac{8\pi E^2}{\hbar^2} g_{yz}.$$  

(32)

By shaking the optical lattice, the topological properties of tensor monopoles can be derived through the measurement of quantum metric tensor. For our 4D system, the generalized Berry curvature $\mathcal{H}_{\mu\nu\lambda} = 4\epsilon_{\mu\nu\lambda}\sqrt{\det g_{\mu\nu}}$ can be obtained after extracting the quantum metric tensor, with which the topological charge may be obtained consequently.

VI. CONCLUSION

In summary, we have proposed two minimal Hamiltonians, which host tensor monopoles with topological charges equal to $n$, and discuss the topological properties of them. The topological properties and the phase transitions of the tensor monopoles with $n = 1, 2$ have been considered. By increasing $h$ from zero, the tensor monopoles can be annihilated in pairs of opposite topological charges to open a gap. As $h > 4$, all tensor monopoles disappear and the system becomes a trivial insulator. The semiclassical Boltzmann equation has been used to calculate the longitudinal conductivity with the magnetic field, a $-B^2$-dependence of MR is obtained as a result of the Weyl semimetal with a hyperplane cutting through the two tensor monopoles. An experimental scheme of the topological charge 1 has been proposed. We suggest to simulate the 4D Hamiltonian of tensor monopole by the 3D optical lattice with a parametrized pseudo-momentum along the fourth dimension. The relation between the total excitation rate and the quantum metric tensor facilitates us to measure the quantum metric tensor by shaking the optical lattice.

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Appendix A: Calculation of the surface state spectrum

Expand the Hamiltonian $H_2$ around $(k_x, k_y, k_z) = (0, 0, 0)$, and consider the open boundary condition along $z$ direction, the Hamiltonian can be rewritten as

$$\mathcal{H} = \begin{pmatrix}
0 & \tilde{d}_1 - i\tilde{d}_2 & 0 \\
\tilde{d}_1 + i\tilde{d}_2 & 0 & \tilde{d}_3 + iA_{000} \\
0 & \tilde{d}_3 - iA_{000} & 0
\end{pmatrix},$$  

(\text{A1})
where
\[
\begin{align*}
\hat{d}_1 &= 2t(k_x^2 - k_y^2), \\
\hat{d}_2 &= 4tk_xk_y, \\
\hat{d}_3 &= 2tk_z = -2it\partial_z, \\
A_{000} &= 2t(h - 3 - \cos k_w).
\end{align*}
\]

Define \( A_{000} \equiv \hat{d}_4 \), and we regard \( A_{000} \) as a domain wall configuration along the \( z \)-direction, which we choose to parametrize as
\[
A_{000}(z) = A_{000} \left[ \Theta(z) - \Theta(-z) \right],
\]
(A3)

Here \( A_{000} = -A_{000} \), and \( \Theta \) is the Heaviside function with
\[
\Theta(z) = \begin{cases} 
1, & z > 0 \\
\frac{1}{2}, & z = 0 \\
0, & z < 0.
\end{cases}
\]
(A4)

Since \( k_x \) and \( k_y \) are good quantum numbers, we can use their eigenvalues to replace the momentum operators, and solve eigen-equation
\[
\mathcal{H}\psi = \varepsilon_k\psi,
\]
(A5)

with \( \psi = e^{ik_xx + ik_yy}\phi(z) \). The components of the spinor wavefunction \( \phi(z) = (f(z), g(z), h(z))^\top \). Combining above equations derive
\[
\begin{align*}
f(z) &= \frac{1}{\varepsilon_k}(d_1 - id_2)g(z), \\
h(z) &= \frac{1}{\varepsilon_k}(-2i\partial_z - iA_{000})g(z),
\end{align*}
\]
(A6)

and
\[
[-4\partial_z^2 + A_{000}^2]g(z) = \left[\varepsilon^2_k - (d_1^2 + d_2^2)\right]g(z),
\]
(A7)
at \( z \neq 0 \). The solution of Eq. A(7) is
\[
h(z) = h_0e^{-|z|/\lambda},
\]
(A8)

where \( h_0 \) is a normalization constant and \( \lambda^{-1} := \sqrt{A_{000}^2 + (d_1^2 + d_2^2) - \varepsilon^2_k} > 0 \), the discontinuity of delta function at \( z = 0 \) imposes the condition \( \lambda^{-1} = A_{000} \). Therefore, the surface states dispersions are given by
\[
\varepsilon_{\pm,k} = \pm \sqrt{(d_1^2 + d_2^2)} = \pm v(h_x^2 + k_y^2),
\]
(A9)

where \( v = 2t \) is the effective Fermi velocity. For \( A_{000} > 0 \), we only consider \( h = 3 \) in the main text, so we derive \( k_w \in (-\pi/2, \pi/2) \). The surface state spectrum of \( H_1 \) can be derived in the same method.

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