LEGENDRIAN AMBIENT SURGERY AND LEGENDRIAN CONTACT HOMOLOGY

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Abstract. Let \( L \subset Y \) be a Legendrian submanifold of a contact manifold, \( S \subset L \) be a framed embedded sphere bounding an isotropic disc \( D_S \subset Y \setminus L \), and use \( L_S \) to denote the manifold obtained from \( L \) by surgery on \( S \). Under some additional assumptions, we describe a construction denoted by Legendrian ambient surgery, which produces a Legendrian embedding of \( L_S \) into an arbitrarily small neighbourhood of \( L \cup D_S \subset Y \). In the case when the disc is subcritical, we produce an isomorphism of the Chekanov-Eliashberg algebra of \( L_S \) with the Chekanov-Eliashberg algebra of \( L \) twisted by a count of holomorphic discs with boundary-point constraints on \( S \). This isomorphism induces a one-to-one correspondence between the augmentations of the Chekanov-Eliashberg algebras of \( L \) and \( L_S \).

1. Introduction

1.1. Basic definitions. A contact manifold is a \((2n+1)\)-dimensional manifold \( Y \) together with a maximally non-integrable hyper-plane field \( \xi \subset TY \). For us, this hyper-plane field will be of the form \( \xi = \ker(\lambda) \), where \( \lambda \) is a fixed global one-form called the contact form. Consequently, \( \lambda \) satisfies the property that \( \lambda \wedge (d\lambda)^n \neq 0 \) is everywhere non-zero.

The Reeb vector field on \((Y,\lambda)\) is the vector field \( R \) defined by the requirements

\[
\iota_R d\lambda = 0, \quad \lambda(R) = 1.
\]

The definition of \( R \) clearly depends on the choice of \( \lambda \).

An embedded submanifold \( L \subset Y \) is called isotropic if \( TL \subset \xi \), which implies that \( \dim L \leq n \). In the case \( \dim L = n \) we say that \( L \) is Legendrian, and in the case \( \dim L < n \) we say that \( L \) is subcritical isotropic. A Reeb chord on \( L \) is an integral curve of \( R \) starting and ending on two different sheets of \( L \). Unless stated otherwise, we will make the assumption that a Legendrian submanifold \( L \subset Y \) is closed.

By a Legendrian isotopy we mean a smooth one-parameter family of Legendrian embeddings. Determining whether two Legendrian embeddings are Legendrian isotopic is a very subtle problem in general, but powerful invariants have been developed in the last decades. The results of this paper

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mainly concerns Legendrian contact homology, which is such an isotopy invariant.

Understanding Legendrian isotopy classes are important for understanding contact and symplectic manifolds \cite{BEE}, but has also lead to interesting applications in other fields. For instance, knot contact homology \cite{EENS} is a powerful invariant of smooth knots which is based on Legendrian contact homology.

Let \((Y, \lambda)\) be a contact manifold. The exact symplectic manifold \((\mathbb{R} \times Y, d(e^t \lambda))\),

where \(t\) is a coordinate on the \(\mathbb{R}\)-factor, is called the symplectisation of \((Y, \lambda)\).

By an exact Lagrangian cobordism from the Legendrian manifold \(L^- \subset Y\) to \(L^+ \subset Y\) we mean a properly embedded submanifold \(V \subset \mathbb{R} \times Y\) of the form

\[ V = ((-\infty, A) \times L^-) \cup \overline{V} \cup ((B, +\infty) \times L^+) , \]

where \(\overline{V} \subset [A, B] \times Y\) is compact, and such that the pull-back of \(e^t \lambda\) to \(V\) has a primitive which is globally constant when restricted to either \(V \cap \{t \leq A\}\) or \(V \cap \{t \geq B\}\).

For a Legendrian submanifold \(L \subset Y\), the trivial cylinder \(\mathbb{R} \times L \subset \mathbb{R} \times Y\) is obviously an exact Lagrangian cobordism from \(L\) to \(L\). Observe that the relation of being exact Lagrangian cobordant is in general not a symmetric relation.

Recall that a symplectic manifold \((X, \omega)\) has many compatible almost complex structures, that is, \(J \in \text{End}(TX)\) for which \(J^2 = -\text{id}\) and \(\omega(\cdot, J\cdot)\) is a Riemannian metric. An almost complex structure \(J\) on \(\mathbb{R} \times Y\) is called cylindrical if it is compatible with \(d(e^t \lambda)\), invariant under the natural \(\mathbb{R}\)-action on \(\mathbb{R} \times Y\), satisfies \(J\partial_t = R\) for the Reeb vector field \(R \subset TY\), and preserves the contact planes \(J\xi = \xi\).

Symplectic field theory, introduced in \cite{EGH} by Eliashberg, Givental and Hofer, provides a framework for invariants of, among others, contact manifolds and Legendrian submanifolds by algebraically encoding information about related spaces of pseudo-holomorphic curves. In particular, Legendrian contact homology is a Legendrian isotopy invariant obtained from symplectic field theory, which was defined in \cite{EGH} by the previous authors, and in \cite{Che} by Chekanov.

We will make use of the version of Legendrian contact homology defined in \cite{Ekh} by Ekholm. To that end, we have to make the following additional assumptions on \(Y\) (see Remark 3.1 below and \cite{Ekh}, Appendix B.2)). We will assume that, either

1. \((Y, \lambda)\) is closed, every periodic Reeb orbit is non-degenerate, and each finite-energy \(J_{cyl}\)-holomorphic plane in \(\mathbb{R} \times Y\) has expected dimension at least two for a cylindrical almost complex structure \(J_{cyl}\).
2. \((Y, \lambda) = (P \times \mathbb{R}, dz + \theta)\), where \(z\) denotes the coordinate on the \(\mathbb{R}\)-factor, and \((P, d\theta)\) is an exact symplectic manifold having finite
geometry at infinity for some compatible almost complex structure (see [EES3, Definition 2.1]).

The contact manifold of the latter kind is the so-called contactisation of the exact symplectic manifold \((P, d\theta)\). Observe that the Reeb vector-field is given by \(R = \partial_z\) for this choice of contact form and that, in particular, there are no periodic Reeb orbits.

**Remark 1.1.** The condition on the closed contact manifold \((Y, \lambda)\) given in (1) above can be expressed in terms of the Conley-Zehnder indices of the Reeb orbits of \((Y, \lambda)\), see e.g. [EGH, Proposition 1.7.1].

Moreover, in addition to the above conditions, for simplicity we will only consider the case when \((Y, \xi)\) has vanishing first Chern class.

**Example 1.2.** We now give important examples of contact manifolds satisfying the above conditions.

1. The standard contact \((2n+1)\)-sphere \((S^{2n+1}, \lambda)\), where \(S^{2n+1} \subset \mathbb{C}^{2n}\) is the unit-sphere, is a contact manifold of the first kind given that

\[
\lambda := \frac{1}{2} \sum_{i=1}^{n} a_i (x_i dy_i - y_i dx_i)
\]

for rationally independent \(a_i > 0, i = 1, \ldots, n\).

2. The jet-space \(J^1(M)\) of a smooth (possibly non-closed) manifold \(M\) can be endowed with the contact structure

\[
(J^1(M) \simeq T^*M \times \mathbb{R}, \lambda_0)\quad \lambda_0 := dz + \theta_M,
\]

where \(z\) is the coordinate on the \(\mathbb{R}\)-factor and where \(-\theta_M\) is the Liouville form on \(T^*M\). This is the contactisation of the exact symplectic manifold \((T^*M, d\theta_M)\), which furthermore can be seen to have finite geometry at infinity.

Specialising to the case \(M = \mathbb{R}^n\) and \(\theta_{\mathbb{R}^n} = -\sum_{i=1}^{n} y_i dx_i\), where \(x_i\) are coordinates on \(\mathbb{R}^n\) and \(y_i = df(\partial_{x_i})\) are the induced coordinates on \(T^*\mathbb{R}^n\), we obtain the so-called standard contact \((2n + 1)\)-space.

In Section 3, we give an outline of the definition of Legendrian contact homology. Roughly speaking, it is constructed as follows. For a compact Legendrian submanifold \(L \subset Y\) and a cylindrical almost complex structure \(J_{cyl}\) on \(\mathbb{R} \times Y\), one associates its Chekanov-Eliashberg algebra \((\mathcal{A}(L), \partial)\) which is a differential graded algebra (DGA for short). Using \(Q(L)\) to denote the set of Reeb chords on \(L\), the algebra \(\mathcal{A}(L)\) is unital, non-commutative, graded, and freely generated by \(Q(L)\) over \(\mathbb{Z}_2\). The generators \(Q(L)\) are graded using the Conley-Zehnder index (see Section 3.1.1). The differential counts rigid (up to translation) \(J_{cyl}\)-holomorphic discs in \(\mathbb{R} \times Y\) with boundary on \(\mathbb{R} \times L\), and boundary-punctures asymptotic to Reeb chords on \(L\), of which exactly one is positive. By the results in [EKh] the homotopy type of the
Chekanov-Eliashberg algebra is invariant under Legendrian isotopy and independent of the choice of a generic cylindrical almost complex structure (see Theorem 3.5 below).

Following the principles of symplectic field theory, an exact Lagrangian cobordism $V$ from $L_-$ to $L_+$ induces a unital DGA morphism

$$
\Phi_V : (A(L_+), \partial_+) \to (A(L_-), \partial_-)
$$

which is defined by counting rigid $J_V$-holomorphic discs in $\mathbb{R} \times Y$ having boundary on $V$, a single positive boundary-puncture asymptotic to a Reeb chord on $L_+$, and several negative boundary-punctures asymptotic to Reeb chords on $L_-$. Here we assume that $J_V$ coincides with the cylindrical almost complex structure used to define $\partial_+$ and $\partial_-$ in sets of the form $\{t \geq B\}$ and $\{t \leq A\}$, respectively. It follows by the results in [Ekh] that the homotopy class of $\Phi_V$ is invariant under compactly supported deformations of the almost complex structure and compactly supported Hamiltonian isotopies of $V$ (see Theorem 3.3 below).

1.2. Results. Let $(Y, \lambda)$ be a contact manifold of dimension $2n+1$ and $L \subset Y$ a Legendrian submanifold, which hence is of dimension $n$. In the following we will fix a number $E > 0$ and only consider the Reeb chords on $L$ having action at most $E$. Observe that in the case when $(Y, \lambda) = (P \times \mathbb{R}, dz + \theta)$ is a contactisation, there are only finitely many Reeb chords on a generic closed Legendrian submanifold, so this condition imposes no constraint.

For a general contact manifold, there might be Reeb chords of arbitrarily large action. In this case a direct limit must be taken in order to deduce a result for the full DGA.

1.2.1. Legendrian ambient surgery. Let $k < n$ and consider a framed embedded $k$-sphere $S \subset L$, i.e. an embedded sphere together with a choice of frame of its normal-bundle $NS \subset TL$. Suppose that $D_S \subset Y$ is an isotropic surgery disc compatible with this framed sphere (see Definition 4.2). In particular, we require $D_S$ to be an embedded closed isotropic disc satisfying $\partial D_S = S$ and having interior disjoint from $L$. The isotropic surgery disc determines an ambient Legendrian surgery on $S$ (see Definition 4.5), which produces a Legendrian embedding $L_S \subset Y$ of the manifold obtained from $L$ by surgery on the framed sphere $S$. We may assume that $L_S$ is contained in an arbitrarily small neighbourhood of $L \cup D_S \subset Y$.

Furthermore, we will make the assumption that

$$
Q(L_S) = Q(L) \cup \{c_S\}.
$$

In the case when $k < n - 1$ this may be achieved by a general position argument, while we must make some additional assumptions on $D_S$ in the case $k = n - 1$ (see Lemma 4.8 for more details). The new Reeb chord $c_S$ is graded by $|c_S| = n - k - 1$ and may be assumed to have arbitrarily small action.
The construction also provides an exact Lagrangian cobordism $V_S \subset \mathbb{R} \times Y$ from $L$ to $L_S$ which is diffeomorphic to the elementary cobordism of index $k+1$ induced by the surgery. We will say that $V_S$ is an \textit{elementary Lagrangian cobordism of index} $k+1$.

Let $\Phi_{V_S} : (\mathcal{A}(L), \partial) \rightarrow (\mathcal{A}(L_S), \partial)$ denote the DGA morphism induced by an elementary Lagrangian cobordism $V_S$. In Section 5.4 we prove the following result.

\textbf{Theorem 1.3.} For an appropriate regular compatible almost complex structure on $\mathbb{R} \times Y$, the DGA morphism $\Phi_{V_S}$ is a surjection which, moreover, satisfies $\ker \Phi_{V_S} = \{\langle c_{\text{new}} \rangle, \ k < n - 1, \ \langle c_{\text{new}} - 1 \rangle, \ k = n - 1\}$.

In other words, for the above choice of almost complex structure, $(\mathcal{A}(L), \partial)$ is the quotient of $(\mathcal{A}(L_S), \partial_{L_S})$ by a two-sided algebra ideal generated by one element.

By an \textit{augmentation} of a DGA $(\mathcal{A}, \partial)$ we mean a DGA morphism $\epsilon : (\mathcal{A}, \partial) \rightarrow (\mathbb{Z}_2, 0)$ to the trivial DGA. In case when the grading of $c_S$ is non-zero, it follows by definition that any augmentation satisfies $\epsilon(c_S) = 0$. A purely algebraic consequence of the above theorem is thus the following result. Recall that augmentations can be pulled back by the pre-composition with a unital DGA morphism. Let $\mu \in H^1(V_S, \mathbb{Z})$ denote the Maslov class of $V_S$.

\textbf{Corollary 1.4.} If $k < n - 1$ and, moreover

$$ (n - k - 1) \notin \mu(H_1(V_S)), $$

then $\Phi_{V_S}$ induces a bijection from the set of 0-graded augmentations of $(\mathcal{A}(L), \partial)$ to the set of 0-graded augmentations of $(\mathcal{A}(L_S), \partial_{L_S})$ via pull-back.

1.2.2. The Chekanov-Eliashberg algebra of $L$ twisted by $S \subset L$. Assume that we are given an embedded $k$-dimensional submanifold $S \subset L$ with $k < n-1$, together with the choice of a non-vanishing normal vector-field $\mathbf{v}$ in $NS \subset TL$. In Section 6 we define a version $(\mathcal{A}(L; S), \partial_{S, \mathbf{v}})$ of the Chekanov-Eliashberg algebra whose differential is twisted by data determined by $S$ and $\mathbf{v}$.

The algebra $\mathcal{A}(L; S)$ is unital, non-commutative, graded and freely generated over $\mathbb{Z}_2$ by $\mathcal{Q}(L) \cup \{s\}$. Here we grade the Reeb chord generators as before, while $s$ is a generator in degree $n - k - 1$.

To define the twisted differential we will first need to describe the appropriate moduli spaces. We fix a cylindrical almost complex structure $J_{cyl}$ on the symplectisation and lift $\mathbf{v}$ to a vector-field normal to $\mathbb{R} \times S \subset \mathbb{R} \times L$ which is invariant under translation of the $t$-coordinate. Let $\mathbf{w} = (w_1, \ldots, w_{m+1}) \in$
Consider the moduli space \( M_{a;b;A}(L; J_{cyl}) \) defined in Section 3.2 consisting of \( J_{cyl} \)-holomorphic discs
\[
u: (D^2, \partial D^2) \to (\mathbb{R} \times Y, \mathbb{R} \times L)
\]
having one positive boundary-puncture \( p_0 \) asymptotic to \( a \), and \( m \) negative boundary-punctures \( p_i \) asymptotic to \( b_i \), where \( i = 1, \ldots, m \). Here we require that the boundary-punctures satisfy \( p_1 < \ldots < p_m \) with respect to the total order on \( \partial D^2 \setminus \{p_0\} \) induced by the orientation.

Let \( \delta > 0 \) and let \( g \) be a Riemannian metric on \( L \). Similarly to the constructions in [BEE, Section 8.2.D], in Section 8 we define the moduli spaces
\[
M_{a;b;w;A}^{\delta}(L; S, v; J_{cyl}) \subset M_{a;b;A}(L; J_{cyl})
\]
consisting of the solutions \( u \) satisfying the following boundary-point constraints at parallel copies of \( \mathbb{R} \times S \subset \mathbb{R} \times L \). There are \( w := w_1 + \ldots + w_{m+1} \) distinct boundary points \( \{q_i\} \subset \partial D^2 \), satisfying \( q_1 < \ldots < q_w \) with respect to the order on \( \partial D^2 \setminus \{p_0\} \), for which
\[
u(q_i) \in \mathbb{R} \times \exp_S((i-1)\delta v) \subset \mathbb{R} \times L, \quad i = 1, \ldots, w.
\]
Here \( \exp_p \) denotes the exponential map at \( p \in L \) induced by \( g \). Moreover, we require that \( w_{i+1} \) of the points \( \{q_i\} \) are situated on the boundary arc between the punctures \( p_i \) and \( p_{i+1} \) (here we set \( p_{m+1} := p_0 \)).

Since the data used to define these moduli spaces is invariant under translations of the \( t \)-coordinate, they carry a natural \( \mathbb{R} \)-action induced by such translations.

Remark 1.5. If one takes the limit \( \delta \to 0 \), the solutions in a moduli space as above converge to \( J_{cyl} \)-holomorphic discs having several boundary points that are mapped to \( \mathbb{R} \times S \). If, for such a convergent sequence of solutions, a cluster of \( m \) of boundary points mapping to \( \mathbb{R} \times \exp_S(i\delta v) \) (for different \( i \in \mathbb{Z}_{\geq 0} \)) collide, the solution in the limit can moreover be seen to have a tangency to \( \mathbb{R} \times \exp_S(\mathbb{R}v) \) of order \( m \). We refer to [DR2, Part II, Sections 8 and 9] for more details concerning the moduli spaces of discs with jet-constraints at boundary points. Following the ideas in [ES], transversality for these moduli spaces is there shown by using an almost complex structure \( J_{cyl} \) that is integrable in a neighbourhood of \( \mathbb{R} \times S \). It is also shown that there is an isomorphism between the moduli space with appropriate jet-constraints at boundary points and the above moduli space with boundary-point constraints at parallel copies, given that \( \delta > 0 \) is sufficiently small (see [BEE, Lemma 8.3] for a similar statement). Moduli spaces of pseudo-holomorphic curves with jet-constraints at boundary points are also treated in [Zeh] using a different approach.
The differential is now defined as follows. For the formal generator \( s \) we define \( \partial_{S,v}(s) := 0 \), while for a Reeb chord generator \( a \in Q(L) \) we define
\[
\partial_{S,v}(a) := \sum_{|a|-|s|-|b|+\mu(A)=1} M^{g,\delta}_{a,b,w;c}(L;S,v)/R|s^{w_1}b_1s^{w_2}\cdot\cdot\cdot s^{w_m}b_1s^{w_{m+1}},
\]
where \( s := s^{w_1+\cdots+w_{m+1}} \) and \( J_{cyl} \) is a cylindrical almost complex structure. Finally, the differential is extended to the whole algebra using the Leibniz rule.

Observe that the two-sided algebra ideal \( \langle s \rangle \) generated by \( s \) is preserved by \( \partial_{S,v} \). Furthermore, it follows by construction that the equality
\[
(A(L;S),\partial_{S,v})/\langle s \rangle = (A(L),\partial)
\]
holds under the canonical identification of generators.

Remark 1.6. The formal generator \( s \) can be taken to correspond to the critical manifold \( \mathbb{R} \times S \subset \mathbb{R} \times L \) of a Morse-Bott function on \( \mathbb{R} \times L \) having its minimum along \( \mathbb{R} \times S \), and which moreover is invariant under translation of the \( t \)-coordinate. The differential may be seen as counting rigid configurations consisting of pseudo-holomorphic discs, together with several negative gradient flow-lines in \( \mathbb{R} \times L \) (which must be trivial by the rigidity) starting at the boundary of the disc and ending at this critical manifold.

The following theorem follows from Lemma 6.1 together with Proposition 6.3.

**Theorem 1.7.** Let \( S \subset L \) be a \( k \)-dimensional submanifold with a non-vanishing normal vector-field \( v \), where \( k < n - 1 \). For a generic cylindrical almost complex structure on \( \mathbb{R} \times Y \), the DGA \((A(L;S),\partial_{S,v})\) is well-defined. Furthermore, its tame-isomorphism class is independent of the choice of a generic pair \((g,\delta)\), and invariant under isotopy of the pair \((S,v)\).

Remark 1.8. (1) The invariance proof of the Chekanov-Eliashberg algebra should be possible to extend to show that the homotopy-type of the twisted Chekanov-Eliashberg algebra is invariant under Legendrian isotopy and independent of the choice of a cylindrical almost complex structure. The proof depends on an abstract perturbation argument, and is omitted.

(2) In the case when there is an embedded null-cobordism of \( S \) inside \([0,1] \times L\), to which \( v \) extends as a non-vanishing normal vector-field, Corollary 6.4 implies that \((A(L;S),\partial_{S,v})\) is tame-isomorphic to the free product of the Chekanov-Eliashberg algebra with the trivial DGA generated by \( s \).

(3) The homotopy type of the Chekanov-Eliashberg algebra of \( L \) twisted by \( S \) will in general depend on the homotopy class of the non-vanishing normal vector-field \( v \), as follows by the computation in Section 2.2 (see the case \( k = (n - 1)/2 \)).
1.2.3. The Chekanov-Eliashberg algebra of $L_S$ in terms of the Chekanov-Eliashberg algebra of $L$ twisted by $S$. We now assume that $S \subset L$ is a framed $k$-sphere with $k < n - 1$, and that one can perform a Legendrian ambient surgery on $S$, thus producing the Legendrian submanifold $L_S \subset Y$.

In Section 6.3, we show the following connection between the Chekanov-Eliashberg algebra of $L$ twisted by $S$, and the Chekanov-Eliashberg algebra of $L_S$.

**Theorem 1.9.** Suppose that $k < n - 1$ and that $L_S$ is obtained from $L$ by a Legendrian ambient surgery on a framed embedded sphere $S \subset L$. Let $v$ be a non-vanishing normal vector-field to $S$ that is constant with respect to this frame. There is a tame isomorphism of DGAs

$$\Psi : (A(L_S), \partial_{L_S}) \to (A(L; S), \partial_{S, v}),$$

where $\Psi(c_S) = s$, and for which the DGA morphism

$$\Phi_{V_S} \circ \Psi^{-1} : (A(L; S), \partial_{S, v}) \to (A(L), \partial) = (A(L; S), \partial_{S, v})/\langle s \rangle$$

is the natural quotient-projection.

The map $\Psi$ is defined by counting rigid pseudo-holomorphic discs in $\mathbb{R} \times Y$ having boundary on $V_S$ and boundary-point constraints at parallel copies of the core disc $C_S \subset V_S$ of the elementary Lagrangian cobordism $V_S$ (see Section 4.3 for its definition). Note that $C_S$ coincides with $(-\infty, -1) \times S$ outside of a compact set.

1.2.4. Relations to other results. In the case when $L$ is a Legendrian knot inside standard contact three-space $J^1(\mathbb{R})$, the map $\Phi_{V_S}$ as well as the Chekanov-Eliashberg algebra $(A(L), \partial)$ was computed in terms of data on $L_S$ in [EHK]. This is the case of a Legendrian ambient surgery on a 0-sphere in a Legendrian knot. Theorem 1.3 provides a computation of the Chekanov-Eliashberg algebra of $L$ in terms of the Chekanov-Eliashberg algebra of $L_S$ for a general Legendrian ambient surgery. However, since we do not compute $\Phi_{V_S}$ explicitly, we do not provide an explicit identification.

As a particular case, Theorem 1.9 allows us to compute the Legendrian contact homology of the cusp connected sum $L_1 \# L_2 \subset J^1(M^n)$ of two Legendrian submanifolds $L_1, L_2 \subset J^1(M^n)$ when $n \geq 2$. This operation was described in [DES2, Section 4.2] and is the special case of a Legendrian ambient surgery on a 0-sphere in a Legendrian surface $L_1 \cup L_2$. The Chekanov-Eliashberg algebra of $L_1 \# L_2$ can in this setting also be computed from data on $L_1, L_2 \subset J^1(M)$ by the result in [HS].

In [EH], Lemma 3.2] it was shown that the cusp connected sum on one-dimensional knots is a well-defined operation. In Proposition 4.9 we generalise this result by showing that Legendrian ambient 0-surgery is well-defined in higher dimensions as well.
Generating family homology is a Legendrian isotopy invariant introduced in \cite{PC} and \cite{FR} for Legendrian submanifolds of $J^1(M)$ admitting a generating family. When the Chekanov-Eliashberg algebra of $L$ has an augmentation, it can be linearised to a complex which is a vector-space spanned by the Reeb chords. In the case when there exists a generating family for $L \subset J^1(M)$, it is expected that there is an augmentation that induces a complex which is equivalent to that induced by the generating family. This is shown in \cite{FR} in case when the contact manifold is the standard contact three-space $J^1(\mathbb{R})$. Forthcoming work by Bourgeois, Sabloff, and Traynor will provide formulas for computing the effect of a Legendrian ambient surgery on the generating family homology for $L \subset J^1(M)$ in a jet-space, under the assumption that the generating family extends over the corresponding elementary Lagrangian cobordism.

Finally, the results in this paper can be seen as the first steps for obtaining a relative version of the results in \cite{BEE}, which explains the linearised contact homology of the boundary of a Weinstein manifold $X$ in terms of pseudo-holomorphic curves related to a decomposition of $X$ into elementary cobordisms. Linearised contact homology is an invariant of a contact manifold obtained from symplectic field theory, which would correspond to the linearisation of the Chekanov-Eliashberg algebra in the relative case under consideration here. An important difference, however, is that it is not clear if a general exact Lagrangian cobordism can be decomposed into a sequence of elementary Lagrangian cobordisms (that is, handles corresponding to Legendrian ambient surgeries). Even the case when the symplectisation coordinate restricted to the exact Lagrangian cobordism is without critical points is not well understood. The known examples of such cobordisms seem to suggest that (up to Hamiltonian isotopy) they all are obtained from traces of Legendrian isotopies, but the proof of such a result is currently beyond reach.

1.2.5. The case $k = n - 1$. In this case the sphere $S \subset L$ is of codimension one. Since the boundary-point constraint on any number of parallel copies of $S$ is an open condition for the moduli-space $\mathcal{M}_{\alpha;\beta;A}(L; J_{cyl})$ of pseudo-holomorphic discs, we cannot expect the above theory to fit into the same algebraic framework. We refer to Proposition 7.2 for a partial result along the lines of Theorem 1.9

2. Examples and computations

2.1. Twisting by an even number of points. Part (2) of Remark 1.8 shows that

Proposition 2.1. Let $L$ be connected and of dimension at least two, and let $S \subset L$ be an even number of points. It follows that, for any choice of $\mathbf{v}$, the DGA $(A(L; S), \partial_S, \mathbf{v})$ is isomorphic to the free product of the Chekanov-Eliashberg algebra with the trivial DGA having one generator in degree $n - 1$. 

In the case when the number of points $S \subset L$ is odd, the situation gets more complicated. See the example below for the case of the Whitney sphere.

2.2. Computations for the Whitney sphere. The Whitney immersion is the exact Lagrangian immersion

$$S^n \to \tilde{L}_{Wh} \subset \mathbb{C}^n,$$

$$(x, y) \mapsto \frac{1}{1 + y^2}(x + iy, (x, y)) \in S^n \subset \mathbb{R}^{n+1}.$$

There exists a lift of this immersion to a Legendrian embedding $L_{Wh} \subset (\mathbb{C}^n \times \mathbb{R}, dz - \sum y_i dx_i)$. This Legendrian submanifold will be referred to as the Whitney $n$-sphere. We use $c$ to denote its unique Reeb chord, which is of degree $|c| = n$, and corresponds to the unique transverse double-point of $\tilde{L}_{Wh}$ situated at the origin.

In the following we suppose that $S \subset L_{Wh}$ is an embedded $k$-dimensional submanifold, where $k < n - 1$.

**Case** $n - k - 1 \nmid n - 1$: By degree reasons it follows that, for any choice of non-vanishing normal vector-field $v$ to $S$, the twisted Chekanov-Eliashberg algebra

$$(A(L_{Wh}; S), \partial_{S,v})$$

is the free product of the Chekanov-Eliashberg algebra with the trivial DGA having one generator in degree $|s| = n - k - 1$.

**Case** $k = 0$: In the case when $|S|$ is even, Proposition 2.1 implies that

$$(A(L_{Wh}; S), \partial_{S,v})$$

is the free product of the Chekanov-Eliashberg algebra with the trivial DGA having one generator in degree $|s| = n - 1$.

In the general case, using an almost complex structure as described in Section 2.3 below, Lemma 2.2 shows that there is a unique holomorphic disc passing through a given point for the standard almost complex structure. In particular, the differential is given by

$$\partial_{S,v}(c) = |S|s.$$

In the following we let

$$C := S^{n-1} \subset \mathbb{R}^n$$

denote the unit-sphere inside the real-part of $\mathbb{C}^n$, and observe that $C \subset L_{Wh}$ is an embedded sphere of codimension one. We now consider the special case when $S \subset C$ is the embedded $k$-dimensional sphere being the boundary of the isotropic disc

$$D_S := \left\{ \begin{array}{l} x_1^2 + \ldots + x_{k+1}^2 \leq 1, \\ x_{k+2} = \ldots = x_n = 0, \\ y = 0 \end{array} \right\} \subset \mathbb{R}^n.$$

This isotropic disc lifts to an isotropic disc embedded in $(\mathbb{C}^n \times \mathbb{R}, \lambda_0)$ having boundary on $L_{Wh}$. Furthermore, one can find a Lagrangian frame of the
symplectic normal bundle of $D_S$ that makes it into an isotropic surgery disc compatible with some choice of frame of the normal bundle of $S \subset L_{Wh}$. In other words, we can perform a Legendrian ambient surgery on $S$, producing the Legendrian submanifold $(L_{Wh})_S$. Theorem 1.9 implies that

$$(A((L_{Wh})_S), \partial_{(L_{Wh})_S}) \simeq (A(L_{Wh}; S), \partial_S, v),$$

where $v$ may be taken to be (the restriction of) the vector-field $\sum_{i=1}^n x_i \partial_{y_i}$ in $TL_{Wh}$ along $C$ (see condition (c) in Definition 4.2).

**Case $0 \leq k < (n-1)/2$:** Since $\pi_k(S^n) = 0$, and since $S$ is of codimension at least $k+2$, it follows that $S$ is null-cobordant in $[0,1] \times L_{Wh}$ by a null-cobordism to which $v$ extends as a non-vanishing normal vector-field. Part (2) of Remark 1.8 shows that $(A(L_{Wh}; S), \partial_S, v)$ is the free product of the Chekanov-Eliashberg algebra with the trivial DGA having one generator in degree $|s| = n-k-1$.

**Case $k = n-2$:** It can be explicitly checked that $S$ is null-cobordant in $[0,1] \times C \subset [0,1] \times L_{Wh}$ by a null-cobordism to which $v$ extends as a non-vanishing normal vector-field. Part (2) of Remark 1.8 again shows that $(A(L_{Wh}; S), \partial_S, v)$ is the free product of the Chekanov-Eliashberg algebra with the trivial DGA having one generator in degree $|s| = 1$.

**Case $k = 1$:** Since $k < n-1$ by assumption, it follows that $n \geq 3$. This case is thus covered by the case $n-k-1 \downarrow n-1$ above. Note that, when $n > 3$, the Lagrangian frame of the isotropic surgery disc $D_S$ can be chosen to be compatible with any given frame of the normal bundle of $S$, as follows from part (3) of Remark 4.3. In particular, a Legendrian ambient surgery can be performed for which the corresponding elementary Lagrangian cobordism $V_S$ is not spin.

**Case $k = (n-1)/2$:** In this case $|s| = (n-k-1) = (n-1)/2$. Since the normal bundle of $S$ is trivial, the choice of a non-vanishing normal vector-field (up to homotopy) lives in

$$\pi_{(n-1)/2}(S^{n-k-1}) = \pi_{(n-1)/2}(S^{(n-1)/2}) \simeq \mathbb{Z}.$$  

One obtains an explicit identification of groups by the algebraic count of zeros of the orthogonal projection of this vector-field to the normal bundle of $S \subset C$ (recall that $C \subset L_{Wh}$ is of codimension one). We use $v_m$ to denote the non-vanishing normal vector-field whose projection has an algebraic number $m \in \mathbb{Z}$ of zeros (for a fixed choice of orientation). Note that the non-vanishing vector-field $v$ above is homotopic to $v_0$.

For the almost complex structure in Section 2.3 below, the descriptions of the holomorphic discs in $\mathbb{C}^n$ with boundary on $L_{Wh}$ given by Lemma 2.2 can be used for explicitly computing the differential. For a generic perturbation of the vector-fields $v_m$, one readily checks that

$$\partial_S v_m (c) = ms^2.$$
In particular, it follows that the homotopy type of the Chekanov-Eliashberg algebra twisted by $S$ depends on the homotopy class of the non-vanishing normal vector-field.

2.3. Holomorphic discs on the Whitney sphere. As in Section 5.1.2 we will consider the unique cylindrical almost complex structure $J_0$ on the symplectisation $\mathbb{R} \times (\mathbb{C}^n \times \mathbb{R})$ for which the canonical projection $\mathbb{R} \times (\mathbb{C}^n \times \mathbb{R}) \to \mathbb{C}^n$ is holomorphic.

By [DR1, Theorem 2.1] this projection induces a bijection between the moduli space $\mathcal{M}_{c,0}(\mathbb{R} \times L_{Wh}; J_0)$ and the moduli space of holomorphic polygons in $\mathbb{C}^n$ having boundary on $\tilde{L}_{Wh}$. The following lemma can thus be used to compute the twisted Chekanov-Eliashberg algebra of the Whitney sphere.

For any point $x \in C \subset \Re \mathbb{C}^n$ in the real unit-sphere we consider the complex half-plane $H_x := \{x + iy; x \geq 0\} x \subset \mathbb{C}x$.

Observe that $\mathbb{C}x$ intersects $\tilde{L}_{Wh}$ in a figure-eight curve, and that each half-plane $H_x$ intersects $\tilde{L}_{Wh}$ in a closed curve.

![Figure 1. The image of $\tilde{L}_{Wh} = \tilde{L}_+ \cup \tilde{L}_-$ under the orthogonal projection $\pi_x: \mathbb{C}^n \to \mathbb{C}x \simeq \mathbb{C}$, where $x \in C$. The arrow denotes the behaviour of the boundary of the hypothetical disc in the proof of Lemma 2.2](image)

Lemma 2.2. The moduli space of (non-trivial) holomorphic discs in $(\mathbb{C}^n, i)$ having boundary on $\tilde{L}_{Wh}$ and one puncture mapping to the double-point $c$ is $(n-1)$-dimensional and transversely cut out. Moreover, every solution is contained in a unique one-dimensional complex half-plane $H_x \subset \mathbb{C}x$ as above.

Proof. It is easily seen that each such one-dimensional complex half-plane contains a unique such disc. Conversely, we now show that every non-trivial holomorphic disc

$$u: (D^2, \partial D^2) \to (\mathbb{C}^n, \tilde{L}_{Wh})$$
having one boundary-puncture is of this form.

In the following we let \( \pi_x : \mathbb{C}^n \to \mathbb{C} \) denote the orthogonal projection onto the one-dimensional complex vector-space spanned by \( x \in \mathbb{R} \mathbb{C}^n \subset \mathbb{C}^n \).

The image of the Whitney sphere \( \pi_x(\tilde{L}_{Wh}) \subset \mathbb{C} \) under these projections can be seen to be a filled figure-eight curve as shown in Figure 1. The composition \( \pi_x \circ u : D^2 \to \mathbb{C} \) is holomorphic and maps the boundary into \( \pi_x(\tilde{L}_{Wh}) \). Furthermore, the boundary-puncture is mapped to the origin. By the open mapping theorem, it thus follows that

\[
\pi_x \circ u(D^2 \setminus \partial D^2) \subset \operatorname{int}\pi_x(\tilde{L}_{Wh}),
\]

unless \( \pi_x \circ u \equiv 0 \) vanishes identically (i.e. maps constantly to the double-point of the figure-eight curve).

The boundary of the disc \( u \) has the following behaviour at the boundary-puncture \( p_0 \in \partial D^2 \) mapping to the double-point \( 0 \in \tilde{L}_{Wh} \). Let \( \tilde{L}_{Wh}^+ \subset \tilde{L}_{Wh} \) denote the image of the hemisphere \( \{\pm y \geq 0\} \cap S^n \) under the Whitney immersion. For \( \epsilon > 0 \) sufficiently small, we have that

\[
\begin{align*}
\left\{ e^{i\theta} \right\} &\in \left\{ \tilde{L}_{Wh}^- \subset \mathbb{C}^n, \quad 0 > \theta > -\epsilon, \right\}, \\
\left\{ x_0 \right\} &\in \left\{ \tilde{L}_{Wh}^+ \subset \mathbb{C}^n, \quad \theta = 0, \right\}, \\
\left\{ \tilde{L}_{Wh}^- \subset \mathbb{C}^n, \quad \epsilon > \theta > 0. \right\}
\end{align*}
\]

From this, it follows that the boundary of \( u \) must intersect the sphere

\[
C \subset \tilde{L}_{Wh}^- \cap \tilde{L}_{Wh}^+ \subset \tilde{L}_{Wh}
\]

of codimension one in at least one point, say \( x_0 \in \mathbb{R} \mathbb{C}^n \).

Since \( u \) is holomorphic, we may furthermore assume that the following property holds for at least one point \( p \in \partial D^2 \) that maps to \( x_0 \). There exists some \( \epsilon > 0 \) for which

\[
(2.1) \quad u(e^{i\theta} p) \in \left\{ \tilde{L}_{Wh}^- \subset \mathbb{C}^n, \quad 0 > \theta > -\epsilon, \right\}, \\
\left\{ x_0 \right\} \in \left\{ \tilde{L}_{Wh}^+ \subset \mathbb{C}^n, \quad \theta = 0, \right\}, \\
\left\{ \tilde{L}_{Wh}^- \subset \mathbb{C}^n, \quad \epsilon > \theta > 0. \right\}
\]

We will now show that the projection \( \pi_{x_1} \circ u \) must vanish whenever \( \mathbb{C}x_1 \) is in the orthogonal complement of \( \mathbb{C}x_0 \). From this it follows that \( u \) is a map of the required form.

To see this claim, observe that the holomorphic map \( \pi_{x_1} \circ u \) vanishes at the above boundary-point \( p \in \partial D^2 \) that maps to \( x_0 \). Moreover, the behaviour (2.1) of \( u \) near \( p \) implies that, unless \( \pi_{x_1} \circ u \) vanishes constantly, one of \( \pm \pi_{x_1} \circ u \) maps the oriented boundary near \( p \) as depicted by the arrow in Figure 1. In the latter case, the open mapping theorem can be used to get a contradiction with the fact that \( \pi_{x_1} \circ u(D^2 \setminus \partial D^2) \subset \operatorname{int}\pi_{x_1}(\tilde{L}_{Wh}). \) In conclusion, \( \pi_{x_1} \circ u \equiv 0 \).

Finally, transversality follows by an argument similar to the proof of Lemma 5.3. \( \square \)
3. Background

Legendrian contact homology was developed in [EGH] by Eliashberg, Givental and Hofer, and in [Che] by Chekanov. It is a theory that associates a differential graded algebra, or DGA for short, to a Legendrian submanifold $L$ of a contact manifold. This DGA will be called the Chekanov-Eliashberg algebra of $L$, and its homotopy type has been shown to be a Legendrian isotopy invariant of $L \subset (Y,\lambda)$, given that the contact manifold $(Y,\lambda)$ satisfies the assumptions in Section 1.1.

Both the construction and the invariance have been worked out in the case $(\mathcal{J}^1(\mathbb{R}^1),\lambda_0)$ by Chekanov [Che], and in arbitrary contactisations $(P \times \mathbb{R}, dz + \theta)$ by Ekholm, Etnyre and Sullivan [EES3], where $(P, d\theta)$ is an exact symplectic manifold having finite geometry at infinity.

We will be using the version of Legendrian contact homology obtained as a special case of a (more general) invariant called relative symplectic field theory, which was constructed by Ekholm [Ekh]. In addition to the contactisations as above, this version of Legendrian contact homology is also well-defined for closed contact manifolds $(Y,\lambda)$ satisfying the conditions described in the introduction.

Finally, one should note that there are transversality issues that need to be resolved in order to define Legendrian contact homology for an arbitrary contact manifold.

3.1. Definition of the Chekanov-Eliashberg algebra. For a closed Legendrian submanifold $L \subset (Y,\lambda)$ we will use $\mathcal{Q}(L)$ to denote the set of Reeb-chords on $L$. A Reeb chord $c$ has an associated action given by

$$\ell(c) := \int_c \lambda.$$ 

Throughout this paper we will assume that $L$ is chord generic, that is, the Reeb flow takes the tangent plane of $L$ at the starting-point of a Reeb chord to a plane transverse to the tangent plane of $L$ at the end-point of the Reeb chord. This assumption implies that the set of Reeb chords on $L$ below any given action is a discrete and compact, and hence finite, set.

3.1.1. The algebra and the grading. The underlying graded algebra is defined as follows. We let

$$\mathcal{A}(L) = \mathbb{Z}_2(\mathcal{Q}(L))$$

be the unital non-commutative algebra over $\mathbb{Z}_2$ which is freely generated by the Reeb chords $\mathcal{Q}(L)$ on $L$.

We say that a Reeb chord $c$ on $L$ is pure if it has both its endpoints on the same component. Otherwise, we say that it is mixed.

To grade a generator $c$ corresponding to a pure Reeb chord we fix a capping path $\gamma_c$ for $c$ on $L$, by which we mean a path on $L$ which starts at the end-point of $c$, and ends at the starting-point of $c$. We grade each
Reeb-chord generator $c$ by the formula

$$|c| := \nu(\Gamma_c) - 1 \in \mathbb{Z}/\mu(H_1(L)),$$

where $\nu(\Gamma_c)$ denotes the Conley-Zehnder index of the Lagrangian tangent-planes of $L \subset \xi$ along $\gamma_c$, and where $\mu \in H^1(L; \mathbb{Z})$ is the Maslov class. We refer to [EES1, Section 2.2] for the definitions.

In the case when $c$ is mixed, there is no canonical grading. However, one can still make a choice of grading, by making the following modification of the notion of a capping path. Fix two components $L_1, L_2 \subset L$ and a path $\gamma \subset Y$ connecting a point $p_1 \in L_1$ with a point $p_2 \in L_2$. We also choose a path $\Gamma$ of Lagrangian tangent-planes in $\xi$ along $\gamma$ starting at $T_{p_2}L_2 \subset \xi$ and ending at $T_{p_1}L_1 \subset \xi$. We define a capping-path of a mixed Reeb chord $c$ starting at $L_1$ and ending at $L_2$ to be a path $\gamma_2$ from the end-point of $c$ to $p_2$, followed by $\gamma$, and ultimately followed by a path $\gamma_2$ from $p_1$ to the starting-point of $c$.

Finally, we define the grading by applying the above formula to the path $\Gamma_c$ of Lagrangian tangent-planes obtained by taking the path of tangent-planes $T L_2 \subset \xi$ along $\gamma_2$, followed by the path $\Gamma$ chosen above, and ultimately followed by the path of tangent planes $T L_1 \subset \xi_1$ along $\gamma_1$.

In the case $(J^1(M), \lambda_0)$, the following description of the Conley-Zehnder index and Maslov class is given in [EES2, Lemma 3.4]. The canonical projection

$$\Pi_F : J^1(M) \to M \times \mathbb{R}$$

is called the front projection. We may moreover assume that $L$ is generic under the front projection, which implies that the singularities of codimension one of $\Pi_F(L)$ consist of self-intersections and cusp-edges.

Suppose that $c$ is a Reeb chord on $L$ contained above a point $p \in M$. Let $f_s$ and $f_e$ be the smooth real-valued functions corresponding to the $z$-coordinates of the sheets of $L$ at the starting-point and end-point of the Reeb chord $c$, respectively, which are well-defined in a neighbourhood of $p \in M$. Phrased differently, in a neighbourhood of $p \in M$ the front projection of the two sheets of $L$ containing the endpoints of $c$ are given as the graphs of $f_s$ and $f_e$.

The Conley-Zehnder index of the path $\Gamma_c$ of Lagrangian tangent-planes along $\gamma_c$ can now be expressed as

$$\nu(\Gamma_c) = D(\gamma_c) - U(\gamma_c) + \text{index}_p(f_e - f_s) - 1,$$

where $D(\gamma_c)$ and $U(\gamma_c)$ denote the number of cusp-edges in the front projection traversed by (a generic perturbation of) $\gamma_c$ in downward and upward direction, respectively, and where $\text{index}_p(f_e - f_s)$ denotes the Morse index of the function $f_e - f_s$ at $p$.

Similarly, the Maslov class evaluated on $[\gamma] \in H_1(L)$ can be computed by the formula

$$\mu([\gamma]) = D(\gamma) - U(\gamma),$$
where $\gamma$ is a generic smooth closed curve on $L$, and where $D$ and $U$ are as above.

### 3.1.2. The differential

Fix a cylindrical almost complex structure $J_{\text{cyl}}$ on the symplectisation $\mathbb{R} \times Y$. For a Reeb-chord generator $a \in \mathcal{Q}(L)$, we define

$$\partial(a) := \sum_{|a| - |b| + \mu(A) = 1} |\mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}})| / |\mathbb{R}|b,$$

where $\mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}})$ denotes the moduli space of $J_{\text{cyl}}$-holomorphic discs defined in Section 3.2 below, and where the sum is taken over all possible words $b = b_1 \cdot \ldots \cdot b_m$ of Reeb chords (including the empty word) and all homology classes $A \in H_1(L)$.

Observe that, since $J_{\text{cyl}}$ is cylindrical, these spaces have a natural $\mathbb{R}$-action induced by translation of the $t$-coordinate.

Here we assume that $J_{\text{cyl}}$ is chosen so that the above moduli spaces all are transversely cut-out, and hence of their expected dimensions (see Section 3.2.3 below). In general, we will call such an almost complex structure regular, where it should be understood from the context to which moduli spaces this refers.

That this count is well-defined follows from the Gromov-Hofer compactness theorem for these moduli spaces, which is outlined in Section 3.2.2 below. Note that the compactness theorem applies for the moduli spaces appearing in the definition of $\partial(a)$ since, for a fixed $a \in \mathcal{Q}(L)$, the total energy of a solution in a moduli space $\mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}})$ is bounded from above by $2\ell(a)$ according to Proposition 3.12.

Finally, we extend the differential to all of $\mathcal{A}(L)$ via the Leibniz rule

$$\partial(ab) = \partial(a)b + a\partial(b).$$

The dimension formula below shows that

$$\dim \mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}}) = |a| - |b| + \mu(A)$$

and that $\partial$ hence is of degree $-1$.

The following standard argument shows that $\partial^2 = 0$. The compactness result together with pseudo-holomorphic gluing shows that the coefficient in front of the word $b$ in the expression $\partial^2(a)$ is given by the number of boundary-points of the compact one-dimensional moduli space

$$\bigcup_{\dim \mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}}) = 2} \mathcal{M}_{a;b;A}(\mathbb{R} \times L; J_{\text{cyl}})/\mathbb{R}.$$

In particular this coefficient is even, and hence vanishing.

**Remark 3.1.** Recall that we assume that our contact manifolds satisfy the property that every $J_{\text{cyl}}$-holomorphic plane in $\mathbb{R} \times Y$ is of expected dimension at least two. The reason is to prevent the following potential scenario. A priori, a one-dimensional moduli space as above could have a boundary point corresponding to a broken configuration consisting of a $J_{\text{cyl}}$-holomorphic disc together with a $J_{\text{cyl}}$-holomorphic plane (i.e. a one-
punctured $J_{cyl}$-holomorphic sphere). By the additivity of the formula for expected dimension, the $J_{cyl}$-holomorphic disc would necessarily have negative expected dimension. However, the existence of such a disc would contradict the above transversality assumption, and hence this kind of breaking cannot occur. This is important, since our boundary operator does not take any $J_{cyl}$-holomorphic planes into account.

One finally defines the Legendrian contact homology of $L$ to be the homology

$$HC_\bullet(L) := H_\bullet(A(L), \partial)$$

of the Chekanov-Eliashberg algebra.

3.1.3. The DGA morphism induced by an exact Lagrangian cobordism and invariance. Let $V \subset \mathbb{R} \times Y$ be an exact Lagrangian cobordism from $L^-$ to $L^+$ which is cylindrical outside of the set $[A, B] \times Y$. We will describe the associated unital DGA morphism

$$\Phi_V : (A(L^+), \partial_+) \to (A(L^-), \partial_+).$$

Suppose that $\partial_\pm$ is defined using a cylindrical almost complex structure $J_\pm$. Choose a compatible almost complex structure $J$ on $V$ which coincides with the cylindrical almost complex structure $J_+$ and $J_-$ in the sets $[B, +\infty) \times Y$ and $(-\infty, A] \times Y$, respectively.

The map $\Phi$ is defined on the Reeb-chord generator $a \in Q(L^+)$ by the formula

$$\Phi_V(a) = \sum_{|a| - |b| + \mu(A) = 0} |M_{\alpha;b:A}(V; J)| b,$$

where the sum is taken over all words $b = b_1 \cdot \ldots \cdot b_m$ of Reeb chords in $Q(L^-)$ (including the empty word) and homology classes $A \in H_1(V)$. Here we assume that $J$ is chosen so that the above moduli spaces are transversely cut out (see Section 3.2.3 below), i.e. that $J$ is regular.

The Gromov-Hofer compactness implies that the above sum is well-defined. Note that the compactness result applies for the moduli spaces appearing in the definition of $\Phi_V(a)$ since the total energy of a solution in $M_{\alpha:b:A}(V; J)$ is bounded from above by $2e^{B - A} \ell(a)$ according to Proposition 3.12.

We extend $\Phi_V$ to a unital algebra map. The dimension formula

$$\dim M_{\alpha;b:A}(V; J) = |a| - |b| + \mu(A)$$

shows that $\Phi$ is of degree zero. Observe that we are required to use a grading in the group

$$\mathbb{Z}/\mu(H_1(V))$$

for the Chekanov-Eliashberg algebras of both $L^+$ and $L^-$ in order for $\Phi_V$ to respect the grading. Moreover, the gradings of the mixed chords have to be chosen with some care.

The fact that $\Phi_V$ is a chain-map now follows similarly to the proof that $\partial^2 = 0$. Namely, the coefficient in front of the word $b \in A(L^-)$ in the
expression \((\partial_+ \circ \Phi_V - \Phi_V \circ \partial_+)(a)\) is given by the number of boundary-points of the compact one-dimensional moduli space

\[ \bigcup_{\dim \mathcal{M}_{a;b;A}(V;J)=1} \mathcal{M}_{a;b;A}(V;J). \]

In particular these coefficients are even, and thus vanishing.

**Example 3.2.** A direct consequence of Example 3.14 below is that \(\Phi_{\mathbb{R}\times L} = \text{id}_{A(L)}\) in the case when a cylindrical almost complex structure has been used to define the DGA morphism.

**Theorem 3.3 (Ekh).** Let \(V \subset Y\) be an exact Lagrangian cobordism from the Legendrian submanifold \(L_-\) to \(L_+\). For a regular almost complex structure as above, the induced map

\[ \Phi_V : (A(L_+), \partial_+) \rightarrow (A(L_-), \partial_-) \]

is a DGA morphism whose homotopy-class is invariant under compactly supported deformations of the almost complex structure and compactly supported Hamiltonian isotopies of \(V\).

This result follows from the more general invariance for relative symplectic field theory proven in [Ekh, Section 4]. Also, see [EHK, Lemma 3.13] for a description that carries over to our setting.

**Remark 3.4.** The proof of this invariance theorem contains one substantial difficulty. One must take into account pseudo-holomorphic discs of formal dimension \(-1\), which arise in generic one-parameter families of moduli spaces. The abstract perturbation scheme outlined in [Ekh, Section 4] is crucial for getting control of the situation.

The above invariance theorem is also the main ingredient in the proof of the following invariance result for Legendrian contact homology.

**Theorem 3.5 (Ekh).** Let \(L \subset Y\) be a Legendrian submanifold. The homotopy type of its Chekanov-Eliashberg algebra \((\mathcal{A}(L), \partial)\) is independent of the choice of a regular cylindrical almost complex structure, and invariant under Legendrian isotopy. In particular, \(HC^\bullet(L)\) is a Legendrian isotopy invariant.

**Sketch of proof.** Let \(V_1\) be an exact Lagrangian cobordism from \(L_-\) to \(L_+\), and \(V_2\) be an exact Lagrangian cobordism from \(L\) to \(L_+\). After a suitable translation, one can form their concatenation

\[ V_1 \circ V_2 := (V_1 \cap \{t \leq 0\}) \cup (V_2 \cap \{t \geq 0\}) \]

which is an exact Lagrangian cobordism from \(L_-\) to \(L_+\). Observe that \(V_1 \circ V_2\) is diffeomorphic to the cobordism obtained by gluing \(V_1\) to \(V_2\) along their common end.

Suppose that \(L_1\) is Legendrian isotopic to \(L_2\) and fix such an isotopy. Arguing as in [Cha, Theorem 1.1] or [Ekh Lemma A.2], the isotopy induces
Lagrangian cobordisms $U, V, W \subset \mathbb{R} \times Y$, each diffeomorphic to cylinders, satisfying

- $V$ is an exact Lagrangian cobordism from $L_1$ to $L_2$ which, moreover, is smoothly isotopic to the trace of the isotopy from $L_1$ to $L_2$.
- $U, W$ are exact Lagrangian cobordisms from $L_2$ to $L_1$ which, moreover, are smoothly isotopic to the trace of the above isotopy in inverse time.
- The concatenations $U \circ V$ and $V \circ W$ are exact Lagrangian cobordisms isotopic to $\mathbb{R} \times L_2$ and $\mathbb{R} \times L_1$, respectively, by compactly supported Hamiltonian isotopies.

Using the splitting construction \[BEH^+\] Section 3.4] together with the compactness result one can show the following, given that we use suitable almost complex structures. First, we have the identities

$$
\Phi_{U \circ V} = \Phi_U \circ \Phi_V,
\Phi_{V \circ W} = \Phi_V \circ \Phi_W,
$$

while, using Theorem 3.3 together with Example 3.2, we get that

$$
\Phi_{U \circ V} \sim \Phi_{\mathbb{R} \times L_2} \sim \text{id}_{\mathcal{A}(L_2)},
\Phi_{V \circ W} \sim \Phi_{\mathbb{R} \times L_1} \sim \text{id}_{\mathcal{A}(L_1)}.
$$

In other words, $\Phi_U$ and $\Phi_W$ are left and right homotopy inverses, respectively, of

$$
\Phi_V : (\mathcal{A}(L_1), \partial_1) \to (\mathcal{A}(L_2), \partial_2).
$$

3.1.4. **Augmentations.** An augmentation is a unital DGA morphism

$$
\varepsilon : (\mathcal{A}(L), \partial) \to (\mathbb{Z}_2, 0),
$$

where $\mathbb{Z}_2$ is considered as the unital DGA having trivial differential and being freely generated by the empty set. The existence of an augmentation implies that one can define a linearised version of Legendrian contact homology, originally due to Chekanov \[Che\] Section 5].

For a unital DGA morphism $\Phi$, it obviously follows that the pull-back $\varepsilon \circ \Phi$ of an augmentation is again an augmentation. In particular, if $L_-$ is exact Lagrangian cobordant to $L_+$, the existence of an augmentation of $(\mathcal{A}(L_-), \partial_-)$ implies the existence of an augmentation of $(\mathcal{A}(L_+), \partial_+)$, since there is a unital DGA morphism induced by the exact Lagrangian cobordism. The converse is however not true in general.

3.1.5. **Tame isomorphisms.** Let $\mathcal{A}$ and $\mathcal{A}'$ be unital algebras over $\mathbb{Z}_2$ which are freely generated by the sets of generators $\{a_i\}_{i \in I}$ and $\{a'_i\}_{i \in I}$, respectively. A unital isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ of DGAs is called tame if, after some identification of the generators of $\mathcal{A}$ and $\mathcal{A}'$, it can be written as a
composition of \( \text{elementary automorphisms} \), i.e. automorphisms which are of the form
\[
\Psi(a_i) = \begin{cases} 
  a_i, & i \neq j, \\
  a_j + B, & i = j,
\end{cases}
\]
for some fixed \( j \), where \( B \) is an element of the unital subalgebra generated by \( \{a_i; \ i \neq j\} \).

### 3.2. Definitions of the moduli spaces.

In this section we give an overview of the moduli spaces appearing in the above constructions. We also recall some important properties.

In the following we let \( V \subset \mathbb{R} \times Y \) be an exact Lagrangian cobordism from the Legendrian submanifold \( L_- \) to \( L_+ \) which is cylindrical outside of the set \( I \times Y \), where \( I = [A, B] \). We allow the case when \( L_- = \emptyset \), as well as the case when \( I = \emptyset \) and \( V = \mathbb{R} \times L \). For each Reeb chord in \( Q(L_\pm) \) we will fix a capping path on \( L_\pm \) as in Section 3.1.2 above. In case when \( V = \mathbb{R} \times L \) we will moreover use the same capping paths on both ends of \( V \).

Observe that a Reeb chord \( c \) has a natural parametrisation \( \eta_c(t): [0, \ell(c)] \to c \subset Y \), \( \dot{\eta}_c(t) = R_{\eta_c(t)} \) by the Reeb flow.

Let \( \hat{D}^2 = D^2 \setminus P \), be the closed unit disc with \( m+1 \) fixed boundary points \( P \subset \partial D^2 \) removed and let
\[
u = (a,v): \ (\hat{D}^2, \partial \hat{D}^2) \to (\mathbb{R} \times Y, V)
\]
be a continuous map. For a conformal structure on \( \hat{D}^2 \), there is an induced conformal identification of
\[
[0, +\infty) \times [0, 1] \subset \mathbb{C} = \{s + it\}
\]
with a set of the form \( \overline{U} \setminus \{p\} \subset \hat{D}^2 \), where \( \overline{U} \subset D^2 \) is a compact neighbourhood of \( p \in P \).

**Definition 3.6.** We say that \( p \in P \) is a

1. **positive puncture asymptotic to the Reeb chord** \( c \in Q(L_+) \) if there exists a \( S_0 \in \mathbb{R} \) such that
\[
\lim_{s \to +\infty} (a(s,t) - s/\ell(c), v(s,t)) = (S_0, \eta_c(t/\ell(c)))
\]
uniformly in the above coordinates, given some choice of metric.

2. **negative puncture asymptotic to the Reeb chord** \( c \in Q(L_-) \) if there exists a \( S_0 \in \mathbb{R} \) such that
\[
\lim_{s \to +\infty} (a(s,t) + s/\ell(c), v(s,t)) = (S_0, \eta_c((1 - t)/\ell(c)))
\]
uniformly in the above coordinates, given some choice of metric.

Consider the compactification \( \overline{V} \) of \( V \) obtained by first making the topological identification \( V \simeq V \cap ((A - 1, B + 1) \times Y) \) and then taking
\[
\overline{V} := \text{cl}(V \cap ((A - 1, B + 1) \times Y)) \subset [A - 1, B + 1] \times Y.
\]
Let $u$ be a map with the asymptotic properties as above. By abuse of notation, we say that the boundary of $u$ is in homology class $A \in H_1(V)$ if closing up the boundary in $\nabla$ using the capping paths produces a cycle in class $A' \in H_1(\nabla)$, such that $A$ is identified with $A'$ under the isomorphism $H_1(V) \to H_1(\nabla)$ induced by the inclusion.

Consider a compatible almost complex structure $J$ on $(\mathbb{R} \times Y, d(e^t \lambda))$ which is cylindrical outside of $I \times Y$. Let $a$ be a Reeb chord on $L_+$, $b = b_1 \cdots b_m$ a word of Reeb chords on $L_-$ (we allow the empty word), $A \in H_1(V)$ a homology class, and $\mathcal{D}^2 = D^2 \setminus \{p_0, \ldots, p_m\}$ the closed unit disc with $m+1$ boundary points removed. We require the punctures to be ordered by $p_1 < \ldots < p_m$ relative the order on $\partial D^2 \setminus \{0\}$ induced by the orientation.

**Definition 3.7.** We use $\mathcal{M}_{a;b;A}(V;J)$ to denote the moduli space of $J$-holomorphic maps $u = (a,v) : (\mathcal{D}^2, \partial \mathcal{D}^2) \to (\mathbb{R} \times Y,V)$, i.e. solutions to the non-linear Cauchy-Riemann equation $du + J du \circ i = 0$ for some conformal structure $(\mathcal{D}^2, i)$, for which

- $p_0$ is a positive boundary puncture asymptotic to $a \in \mathcal{Q}(L_+)$,
- $p_i$ is a negative boundary puncture asymptotic to $b_i \in \mathcal{Q}(L_-)$ for $i = 1, \ldots, m$, and
- the boundary of $u$ is in homology class $A \in H_1(V)$.

Moreover, we identify two such solutions which are related by a conformal reparametrisation of the domain.

**Example 3.8.** For any cylindrical almost complex structure $J_{cyl}$ and $V = \mathbb{R} \times L$, each cylinder $\mathbb{R} \times c$ of a Reeb chord $c \in \mathcal{Q}$ is an element of $\mathcal{M}_{c;c;0}(\mathbb{R} \times L; J_{cyl})$.

**Remark 3.9.** In the case when $V = \mathbb{R} \times L$ and when the almost complex structure is cylindrical, translations of the $t$-coordinate induces an action of $\mathbb{R}$ on the above moduli spaces.

**Remark 3.10.** An elementary application of Stoke’s theorem shows the following. Since $V \subset (\mathbb{R} \times Y, d(e^t \lambda))$ is exact there are no non-constant $J$-holomorphic curves in $\mathbb{R} \times Y$, either closed or with boundary on $V$, having compact image. Moreover, any $J$-holomorphic disc as above must have at least one positive puncture. The latter fact can be seen to follow from either part (2) or (3) of Proposition 3.12 below.

3.2.1. **Energies of pseudo-holomorphic curves.** Let $\varphi_I(t) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be the continuous and piecewise smooth function satisfying $\varphi_I(t) = e^t$ for $t \in I$ and $\varphi(t) = 0$ for $t \notin I$. We let $\mathcal{C}^+_I$ consist of the smooth functions $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ supported in $\{t \leq A\}$ and satisfying $\int_{\mathbb{R}} \rho(t) = 1$. Similarly, we let $\mathcal{C}^-_I$ consist of the smooth functions $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ supported in $\{t \geq B\}$ and satisfying $\int_{\mathbb{R}} \rho(t) = e^{B-A} \geq 1$. Finally, we define $\mathcal{C}_I := \mathcal{C}^+_I \cup \mathcal{C}^-_I$. 
Definition 3.11. For a smooth map 
\[ u : (\dot{D}^2, \partial \dot{D}^2) \to (\mathbb{R} \times Y, V), \]
satisfying the above asymptotic properties near its punctures, we define the quantities
\[
E_{d\lambda}(u) := \int_u d\lambda, \\
E_{d(\varphi I \lambda)}(u) := e^{-A} \int_u d(\varphi(t)\lambda), \\
E_{\lambda,I}(u) := \sup_{\rho \in \mathcal{C}_I} \int_u \rho(t) dt \land \lambda, \\
E_I(u) := E_{d(\varphi I \lambda)}(u) + E_{\lambda,I}(u),
\]
which will be called the $d\lambda$-energy, $d(\varphi I \lambda)$-energy, $\lambda$-energy, and total energy, respectively.

Observe that in the case when $V = \mathbb{R} \times L$ is cylindrical, we have $\varphi_I(t) \equiv 1$, and the $d\lambda$-energy coincides with the $d(\varphi I \lambda)$-energy. The following properties of the above energies are standard, see e.g. [Ekh, Lemma B.3].

Proposition 3.12. Let \( u : (\dot{D}^2, \partial \dot{D}^2) \to (\mathbb{R} \times Y, V) \) be a smooth map as above with positive punctures asymptotic to
\[ a_1, \ldots, a_{m_+} \in \mathcal{Q}(L_+) \]
and negative punctures asymptotic to
\[ b_1, \ldots, b_{m_-} \in \mathcal{Q}(L_-). \]
Here we allow either of $m_+$ and $m_-$ to vanish. We assume that $V$ is an exact Lagrangian cobordism which is cylindrical outside of the set $I \times Y$, where $I = [A, B]$.

1. In the case when $V = \mathbb{R} \times L$ we have
\[
E_{d\lambda}(u) = \sum_{i=1}^{m_+} \ell(a_i) - \sum_{i=1}^{m_-} \ell(b_i).
\]
If $u$ is $J$-holomorphic for a cylindrical $J$, it moreover follows that $E_{d\lambda}(u) \geq 0$ where equality holds if and only if the image of $u$ is contained in $\mathbb{R} \times c$ for a Reeb chord $c$.

2. We have
\[
E_{d(\varphi I \lambda)}(u) = e^{B-A} \sum_{i=1}^{m_+} \ell(a_i) - \sum_{i=1}^{m_-} \ell(b_i).
\]
Given that $u$ is $J$-holomorphic, where $J$ is cylindrical outside of $I \times Y$, it moreover follows that
\[
E_{d(\varphi I \lambda)}(u) \geq 0.
\]
Again, equality holds if and only if the image of \( u \) is contained in \( \mathbb{R} \times c \) for a Reeb chord \( c \).

(3) If \( u \) is \( J \)-holomorphic for a compatible almost complex structure \( J \) which is cylindrical outside of \( I \times Y \), it follows that

\[
0 < E_{\lambda,I}(u) = e^{B-A} \sum_{i=1}^{m_+} \ell(a_i).
\]

Proof. (1): This follows by an elementary application of Stoke’s theorem. Observe that \( \lambda \) vanishes on \( V = \mathbb{R} \times L \).

(2): By the assumptions on \( V \) it follows that that \( \varphi_I(t) \lambda \) pulls-back to an exact one-form on \( V \) which moreover has a primitive that is constant when restricted to either \( V \cap \{ t \in (-\infty,A] \} \) or \( V \cap \{ t \in [B,\infty) \} \). The equality then follows by Stoke’s theorem.

(3): Since \( V \) is exact \( u \) must have at least one boundary puncture. Using its asymptotic behaviour, one easily computes \( E_{\lambda}(u) > 0 \). We proceed to calculate the expression of \( E_{\lambda}(u) \) in terms of the action of the asymptotic Reeb chords.

Consider a compactly supported bump-function \( \rho(t) \in C^+_I \). Using the fact that \( \int_{\mathbb{R}} \rho(t) dt = e^{B-A} \), together with the asymptotic properties of \( u \), one computes

\[
\lim_{N \to +\infty} \int_{-\infty}^t \rho(t-N) dt \wedge \lambda = e^{B-A} \sum_{i=1}^{m_+} \ell(a_i).
\]

In other words, we have the inequality

\[
E_{\lambda}(u) \geq e^{B-A} \sum_{i=1}^{m_+} \ell(a_i).
\]

We start by taking \( \rho(t) \in C^-_I \), which we use to define the continuous function

\[
P(t) := \int_{-\infty}^t (\rho(s) + \chi(s)s^{-A}e^s) ds,
\]

\[
\chi_I(s) := \begin{cases} 1, & s \in I = [A,B], \\ 0, & s \notin I = [A,B]. \end{cases}
\]

Observe that \( P(t) = e^{-A}t^t \) for \( t \in [A,B] \), while \( P(t) = e^{B-A} \) for \( t \geq B \).

It follows that

\[
\int_u \rho(t) dt \wedge \lambda = \int_u d(P(t)\lambda) - \int_u (\chi_I(t)e^t dt \wedge \lambda + P(t) d\lambda) \leq \int_u d(P(t) \lambda),
\]

where the latter inequality holds since the two-form

\[
\chi_I(t)e^{-A} e^t dt \wedge \lambda + P(t)d\lambda
\]
is positive on $J$-complex lines (here we use the assumption that $J$ is cylindrical outside of $I \times Y$). We thus get

$$\int_{u} \rho(t) dt \wedge \lambda \leq \int_{u} d(P(t) \lambda) = \int_{\partial u} P(t) \lambda + e^{B-A} \sum_{i=1}^{m_A} \ell(a_i),$$

where the last equality follows from the fact that $\lim_{t \to -\infty} P(t) \lambda = 0$, while $\lim_{t \to +\infty} P(t) \lambda = e^{B-A} \lambda$.

The exactness of $V$ implies that $P(t) \lambda$ pulls back to an exact one-form on $V$ which has a primitive that is constant when restricted to either $V \cap \{ t \leq A \}$ or $V \cap \{ t \geq B \}$. From this it follows that

$$\int_{u} \rho(t) dt \wedge \lambda \leq \int_{\partial u} P(t) \lambda + e^{B-A} \sum_{i=1}^{m_A} \ell(a_i) = e^{B-A} \sum_{i=1}^{m_A} \ell(a_i).$$

For $\rho(t) \in C^{+}_t$, if we instead take $P(t) := \int_{-\infty}^{t} \rho(s) ds$, the above calculation gives the same upper bound. In conclusion, we have the inequality

$$E_{\lambda}(u) \leq e^{B-A} \sum_{i=1}^{m_A} \ell(a_i)$$

which, together with the opposite inequality proven above, finally gives

$$E_{\lambda}(u) = e^{B-A} \sum_{i=1}^{m_A} \ell(a_i).$$

3.2.2. Gromov-Hofer compactness. Gromov’s compactness theorem was the starting-point for pseudo-holomorphic curve techniques in symplectic geometry. In its original form [Gro, Section 1.5.B] it establishes compactness for the moduli spaces of closed pseudo-holomorphic curves inside a compact symplectic manifold. This was later generalised to the setting in which symplectic field theory is formulated [BEH+], where the symplectic manifold is non-compact and the closed pseudo-holomorphic curves are allowed to have interior punctures asymptotic to periodic Reeb orbits. This version of the compactness theorem is sometimes referred to as Gromov-Hofer compactness.

We will need a version of the latter compactness theorem for pseudo-holomorphic curves with boundary and boundary-punctures asymptotic to Reeb chords. An outline of this result is given in [BEH+ Section 11.3], and it will be treated thoroughly in [Abb].

In order to formulate the compactness theorem we need the concept of a holomorphic building, which was introduced in [BEH+ Section 7]. In our situation, the following less general definition will suit our needs. Again, assume that $J$ is a compatible almost complex structure on $\mathbb{R} \times Y$ which coincides with the cylindrical almost complex structures $J_-$ and $J_+$ in the sets $\{ t \leq A \}$ and $\{ t \geq B \}$, respectively.
Definition 3.13. A holomorphic building consists of a finite collection of pseudo-holomorphic curves in $\mathbb{R} \times Y$, where each curve has an associated level $i \in \{N, N+1, \ldots, N+L\}$, such that each disc in level $i$ is an element of a moduli space

1. $M_{a:b;A}(V; J)$ if $i = 0$,
2. $M_{a:b;A}(\mathbb{R} \times L_+; J_+)$ if $i > 0$, and
3. $M_{a:b;A}(\mathbb{R} \times L_-; J_-)$ if $i < 0$.

A choice of a bijection between the positive punctures of the discs in the $i$:th level and the negative punctures of the discs in the $(i+1)$:th level for each $i < N+L$ is also part of the data. Moreover, if two punctures correspond under this bijection, we require them to be asymptotic to the same Reeb chord. We call a level trivial if it consists only of trivial cylinders, and we assume that there is at most one trivial level.

By [BEH], [Abb], any sequence $u_n$ of $J$-holomorphic curves in moduli spaces $M_{a:b;A}(V; J)$ having a uniform bound on the total energy $E(u_n) < E$, has a subsequence converging (in the appropriate sense) to a pseudo-holomorphic building as above.

3.2.3. Transversality. One says that a compatible almost complex structure $J$ on $\mathbb{R} \times Y$ is regular if the moduli spaces $M_{a:b;A}(V; J)$ all are transversely cut out smooth manifolds. Their dimension is then given by

$$\dim M_{a:b;A}(V; J) = |a| - |b_1| - \ldots - |b_m| + \mu(A),$$

as follows from the computation of the Fredholm index in [EES1, Section 6] (also see [EES3, Proposition 2.3]). We say that a solution is rigid if it lives inside a zero-dimensional transversely cut-out moduli space.

Example 3.14. For a regular cylindrical almost complex structure $J_{cyl}$, it follows that a rigid solution in $M_{a:b;A}(\mathbb{R} \times L; J_{cyl})$ must be translation-invariant, and hence equal to a trivial strip $\mathbb{R} \times c$.

Since the above moduli spaces consist of discs with exactly one positive puncture, each solution can be seen to be embedded in some set of the form \(\{t \geq N\}\). Hence the standard technique applies to show that the set of regular $J$ form a Baire subset in the set of all compatible almost complex structures, see for example [MS, Theorem 3.1.6].

However, if one wishes to find a regular cylindrical almost complex structure, the argument needs to be refined. This was done in [Dra] for curves without boundary. This result can be generalised to the moduli spaces under consideration here.

Proposition 3.15. (1) There is a Baire set of cylindrical almost complex structures which are regular for the moduli spaces $M_{a:b;A}(\mathbb{R} \times L; J_{cyl})$ as above.

(2) The moduli spaces of the form $M_{a:b;A}(\mathbb{R} \times L; J_{cyl})$ may be supposed to be transversely cut out after a cylindrical perturbation of $J_{cyl}$ supported in an arbitrarily small neighbourhood of $\mathbb{R} \times a$. 
Proof. First, observe that the moduli spaces $M_{a;2A}(\mathbb{R} \times L; J_{cyl})$ always are transversely out, as follows by an explicit calculation. We thus need to apply a transversality argument for the moduli spaces consisting of non-trivial discs, i.e. discs that are not contained in a trivial strip.

The proof of [Dra, Theorem 4.1], which shows the first statement in the case when the domain is a Riemann surface without boundary, can be generalised to the current setting. To that end, it is crucial that a solution $u = (\alpha, v)$ satisfies the property that $v$ is somewhere injective in the following sense: there should exist a point $z_0 \in \mathring{D}^2 \setminus \partial \mathring{D}^2$ for which

$$\begin{cases}
\pi_\xi \circ D_{z_0} v \neq 0, \\
v^{-1}(v(z_0)) = \{z_0\},
\end{cases}$$

is satisfied, where $\pi_\xi : TY \rightarrow \xi \subset TY$ denotes the linear projection to the contact-planes along the Reeb vector-field. To show the second statement of the proposition, it will be enough to infer that $v(z_0)$ may be taken to be arbitrarily close to the Reeb chord $a$.

To find a point $z_0$ satisfying the above, we argue as follows. Let $a \subset Y$ denote the asymptotic Reeb chord of the positive puncture of $u$. Observe that this is the only puncture asymptotic to $a$, as follows by the formula for the $d\lambda$-area given in Proposition 3.12 together with the assumption that $u$ is not a trivial strip.

Let $Q \subset \mathring{D}^2$ denote the set of points for which $\pi_\xi \circ Dv = 0$. Observe that $u$ has positive $d\lambda$-energy by assumption, from which it follows that $Q \subsetneq \mathring{D}$. The generalised similarity principle [HIZ, Theorem 12 in A.6] can be applied to any locally defined section $\pi_\xi \circ Dv(X)$, where $X \in TD$ is a locally defined holomorphic vector-field (see the proof of the second statement of [HIWZ, Theorem 5.2]). It follows that the set of limit-points $Q'$ of $Q$ satisfies the property that $Q' \cap (\mathring{D}^2 \setminus \partial \mathring{D}^2) \subset (\mathring{D}^2 \setminus \partial \mathring{D}^2)$ is open. Since this set is closed as well, the non-triviality of $u$ implies that there are no limit-points of $Q$ inside $\mathring{D}^2 \setminus \partial \mathring{D}^2$.

Take a neighbourhood $U \subset \mathring{D}^2$ of the positive puncture, where $U$ can be conformally identified with

$$\{s + it; s \geq 0, 0 \leq t \leq 1\} \subset \mathbb{C},$$

and where $s + it$ denotes the standard holomorphic coordinate.

By the asymptotic properties of $u$, after possibly shrinking the neighbourhood $U$, we may moreover assume that $u|_U$ is an embedding. In particular, each pseudo-holomorphic disc $u|_{\{0 \leq s \leq A\}}$ satisfies the assumptions of [HIWZ, Theorem 1.14]. Applying this result we conclude that, for any $A > 0$, the subset

$$\left\{0 \leq s \leq A, \pi_\xi \circ D_{s+it} v \neq 0, (v|_{\{0 \leq s \leq A\}})^{-1}(v(s + it)) = \{s + it\}\right\} \subset U$$
is open and dense. From this it now follows that
\[
\left\{ \pi_\xi \circ D_{s+it} v \neq 0, (v|_U)^{-1}(v(s+it)) = \{s+it\} \right\} \subset U
\]
is dense as well.

We will now show the existence of the required point \(z_0 \in \dot{D}^2 \setminus \partial \dot{D}^2\) assuming the existence of a point \(y_0 \in a \setminus v(\dot{D}^2 \setminus U)\), which is a fact that we will establish below. The asymptotic properties of \(v\) together with the above property of \(v|_U\) shows that there exists some \(z_0 \in U \setminus \partial D^2\) satisfying \(\pi_\xi \circ D_{z_0} v \neq 0\) and \((v|_U)^{-1}(v(z_0)) = \{z_0\}\), and for which \(v(z_0)\) is arbitrarily close to \(y_0 \in a \setminus v(\dot{D}^2 \setminus U)\). It now follows that this choice of \(z_0\) may be supposed to satisfy \(v^{-1}(v(z_0)) = \{z_0\}\) as well, and hence \(z_0 \in \dot{D}^2 \setminus \partial \dot{D}^2\) is the sought point.

It remains show the existence of a point \(y_0 \in a \setminus v(\dot{D}^2 \setminus U)\). To that end, let \(z' \in \dot{D}^2\) be a limit point of \(v^{-1}(a) \subset \dot{D}^2\). Since \(\mathbb{R} \times a\) is pseudo-holomorphic for any cylindrical almost complex structure, an application of the similarity principle in [Laz, Lemma 4.2] shows the following. The limit point must satisfy \(z' \in Q \cup \partial \dot{D}^2\). If not, \(u\) would map a non-empty open set into \(\mathbb{R} \times a\), thus contradicting the fact that \(Q \setminus \partial \dot{D}^2 \subset \dot{D}^2 \setminus \partial \dot{D}^2\) is a discrete subset.

Consequently, the points in \(\dot{D}^2 \setminus \partial \dot{D}^2\) that are mapped to \(a\) by \(v\) form a discrete subset of \(\dot{D}^2 \setminus \partial \dot{D}^2\). Use \(\tilde{a}\) to denote the Reeb chord \(a\) with both end-points removed. Since the boundary condition of \(u\) implies that
\[
v(\partial \dot{D}^2) \cap a \subset a \setminus \tilde{a},
\]
the existence of a point \(y_0 \in \tilde{a} \setminus v(\dot{D}^2 \setminus U)\) now follows. \(\square\)

Another approach to attain a regular cylindrical almost complex structure is to generalise the method used in [DES3, Lemma 4.5] to the symplectisation. To that end, one must work with almost complex structure that are integrable in some neighbourhood of the Reeb chords, in which \(L\) moreover satisfies a real-analyticity condition.

Pseudo-holomorphic buildings consisting of transversely cut-out levels can be glued to form transversely cut-out solution [DES3, Proposition 4.6]. To that end, the following properties of a glued pseudo-holomorphic buildings is elementary, but important.

**Lemma 3.16.** Let \(u\) be a pseudo-holomorphic map obtained by gluing the levels in a pseudo-holomorphic building. It follows that

1. The expected dimension of \(u\) is equal to the sum of the expected dimensions of the discs in each level.
2. \(E_{d(\varphi,\lambda)}(u)\) is the sum of the \(E_{d(\varphi,\lambda)}\)-energy of the discs in level 0 together with the sum of the \(E_{d\lambda}\)-energies of the discs in the other levels.
Together with Gromov-Hofer compactness it thus follows that, for a regular almost complex structure, the moduli spaces in Section 3.2 can be compactified to smooth manifolds with boundary (with corners), such that the boundary-points are in bijection with pseudo-holomorphic buildings of the appropriate type.

4. **Legendrian ambient surgery**

Let $(Y, \lambda)$ be a contact manifold of dimension $2n + 1$ with contact distribution $\xi := \ker \lambda$, and let $L \subset Y$ be a Legendrian submanifold, which thus satisfies $\dim L = n$.

Here we describe the following construction. Suppose that we are given a Legendrian embedding $L \subset Y$ containing a framed embedded sphere $S \subset L$, together with a so-called isotropic surgery disc $D_S \subset Y$ compatible with the frame of the normal bundle of $S$ (see Definition 4.2). Using $L_S$ to denote the manifold obtained from $L$ by surgery on the framed sphere $S$, the above data will be used to construct a Legendrian embedding $L_S \subset Y$ contained in an arbitrarily small neighbourhood of $L \cup D_S$. We say that $L_S$ is obtained from $L$ by a Legendrian ambient surgery on $S$ (see Definition 4.5).

Finally, the construction also produces an exact Lagrangian cobordism $V_S \subset \mathbb{R} \times Y$ from $L$ to $L_S$ which is diffeomorphic to the elementary cobordism induced by the surgery.

4.1. **The isotropic surgery disc.** For a linear subspace $A \subset (W, \omega)$ of a symplectic vector-space, recall the definition of the symplectic complement of $A$, which is the subspace

$$A^\omega := \{ w \in W; \ \omega(A, w) = \{0\} \}.$$  

An isotropic submanifold $D \subset (Y, d\lambda)$ has an associated symplectic normal bundle defined by $(TD)^{d\lambda} \subset \xi$. In case when $\dim D = k + 1$, this is naturally a symplectic $2(n - k - 1)$-dimensional bundle over $D$ whose symplectic form is given by the restriction of $d\lambda$. We will say that a $(n - k - 1)$-dimensional subframe of this vector-bundle is Lagrangian if it spans Lagrangian subbundle. The following result is standard.

**Lemma 4.1.** A symplectic trivialisation of an $2l$-dimensional symplectic bundle, i.e. a trivialisation which identifies each fibre with the standard symplectic vector-space

$$\left( \mathbb{R}^l \oplus i\mathbb{R}^l = \mathbb{C}^l, \omega_0 = \sum_{i=1}^{l} dx_i \wedge dy_i \right),$$

induces a Lagrangian frame by taking the first $l$-vectors of the trivialising frame (i.e. the subframe that spans the real-part of $\mathbb{C}^l$). Conversely, any Lagrangian frame can be extended to such a symplectic trivialisation.

The Legendrian ambient surgery will depend on the following data.
Definition 4.2. Let $S \subset L$ be an embedded $k$-sphere with a choice of frame $F$ of its normal-bundle $NS \subset TL|_S$. An isotropic surgery disc compatible with the framed sphere $S \subset L$ is an embedded isotropic closed $(k + 1)$-disc $D_S \subset Y$, together with the choice of a Lagrangian frame of its symplectic normal bundle, satisfying

(a) $\partial D_S = S$ and $\text{int}D_S \subset Y \setminus L$.
(b) Any outward normal vector-field to $D_S$ is nowhere tangent to $L$, or equivalently, $(TD_S)^{d\lambda}|_S \cap NS$ is $(n - k - 1)$-dimensional.
(c) Let $H$ denote a vector-field in $NS$ satisfying $d\lambda(G,H) > 0$ for any outward normal $G$ to $D_S$ (these vector fields form a convex and non-empty set). We require that the frame obtained by adjoining $H$ to the Lagrangian frame of $(TD_S)^{d\lambda}|_S$ is a frame of $NS$ which is homotopic to $F$.

In particular, the last condition implies that the Lagrangian frame of the symplectic normal bundle $(TD_S)^{d\lambda}$ of $D_S$ is required to be tangent to $L$ along $\partial D_S = S$. Moreover, assuming that $D_S$ satisfies (a) and (b), the following can be said about condition (c).

Remark 4.3. (1) In the case when $k = n - 1$ the disc $D_S$ is Lagrangian and hence its symplectic normal bundle is zero-dimensional. The last condition is thus equivalent to the requirement that $H = e^f n$, where $\{n\}$ denotes the chosen frame of the one-dimensional normal bundle $NS$ and where $f: S \to \mathbb{R}$.

(2) In the case when $k = 0$ and $n > 1$, there is always a Lagrangian frame that makes $D_S$ into an isotropic surgery disc compatible with the framed sphere $S$. To that end, observe that $U(n-1)$ is connected. Furthermore, up to homotopy relative the boundary, the choice of such a Lagrangian frame lives in $\pi_1 U(n-1) \simeq \mathbb{Z}$.

(3) In the case when $k = 1$ and $n > 3$, the fact that $\pi_1 SO(n-1) \simeq \mathbb{Z}_2$ implies that there are exactly two oriented frames of $NS$ up to homotopy, say $F_0$ and $F_1$. Suppose there exists an isotropic surgery disc $D_S$ compatible with the frame $F_0$ of $NS$. In this case, there exists a different Lagrangian frame of the symplectic normal bundle of $D_S$ which is compatible with the frame $F_1$ of $NS$. To see this, observe that the inclusion $SO(n-2) \subset U(n-2)$ induces the trivial map on $\pi_i$, $i \leq 1$, and that the inclusion $SO(n-2) \subset SO(n-1)$ is 1-connected.

The case when $k < n - 1$ will be called the subcritical case. Observe that in this case the isotropic surgery $(k + 1)$-disc $D_S$ is subcritical. As follows by work of Gromov and Lees, see for example [EM, Section 14.1], there is an h-principle for subcritical isotropic embeddings. In particular, the existence and deformations of such submanifolds can be formulated in purely homotopy-theoretic terms.
Remark 4.4. It follows that the existence of a subcritical surgery disc, and its deformations up to isotopies fixing $L$, are governed by an $h$-principle.

4.2. The standard model of a Lagrangian handle. Recall that the jet-space $(J^1(M), \lambda_0)$, $\lambda_0 := dz + \theta_M$, has the natural projections

$$\Pi_F : J^1(M) \rightarrow M \times \mathbb{R},$$

$$\Pi_{\text{Lag}} : J^1(M) \rightarrow T^*M,$$

called the front projection and the Lagrangian projection, respectively. A Legendrian submanifold of $J^1(M)$ is determined by its image under $\Pi_F$ and, up to a translation of the $z$-coordinate, it is determined by its image under $\Pi_{\text{Lag}}$ as well.

The image $\Pi_{\text{Lag}}(L) \subset (T^*M, d\theta)$ of a Legendrian submanifold $L \subset J^1(M)$ is an exact immersed Lagrangian submanifold, and the Reeb chords on $L$ are in bijective correspondence with the self-intersections of $\Pi_{\text{Lag}}(L)$. In the front projection, Reeb chords correspond to two sheets of $\Pi_F(L)$ having a common tangency above a given point in the base $M$.

We will use the above projections to construct a Legendrian standard model $L_{0,k} \subset J^1(\mathbb{R}^n)$ of a neighbourhood of a framed $k$-sphere $S_{0,k} \subset L_{0,k}$, together with a Legendrian submanifold $L_{\epsilon,k} \subset J^1(\mathbb{R}^n)$ that is obtained from $L_{0,k}$ by surgery on $S_{0,k}$. This will be the standard model of a Legendrian ambient surgery.

The construction will also provide an exact Lagrangian cobordism $W_{\epsilon,k} \subset \mathbb{R} \times J^1(\mathbb{R}^n)$ from $L_{0,k}$ to $L_{\epsilon,k}$ which is diffeomorphic to the manifold obtained by a $(k + 1)$-handle attachment on $(-\infty, -1] \times L_{0,k}$ along

$$S_{0,k} \subset L_{0,k} = \partial((-\infty, -1] \times L_{0,k}).$$

4.2.1. Identifying the symplectisation with a cotangent bundle. Let $(q, p, z)$ be canonical coordinates on $(J^1(\mathbb{R}^n), dz - \sum_{i=1}^n p_i dq_i)$, and recall that $t$ is the coordinate of the $\mathbb{R}$-factor of the symplectisation $\mathbb{R} \times J^1(\mathbb{R}^n)$. Also, we endow $T^*(\mathbb{R}^n \times \mathbb{R}_{>0})$ with the canonical coordinates

$$((x, x_n+1), (y, y_n+1)) = (((x_1, \ldots, x_n), x_{n+1}), ((y_1, \ldots, y_n), y_{n+1})), $$

where $(x, x_{n+1})$ form standard coordinates on the base $\mathbb{R}^n \times \mathbb{R}_{>0}$ and $(y, y_{n+1})$ form coordinates induced by the coframe $\{dx_1\}$. The symplectomorphism

$$(T^*(\mathbb{R}^n \times \mathbb{R}_{>0}), d\theta_{\mathbb{R}^n \times \mathbb{R}_{>0}}) \rightarrow (\mathbb{R} \times J^1(\mathbb{R}^n), d(e^t\lambda_0)), $$

$$((x, x_{n+1}), (y, y_{n+1})) \mapsto (\log x_{n+1}, (x, y/y_{x+n+1}, y_{n+1})), $$

is exact, given that we take the primitive of the symplectic form $d\theta_{\mathbb{R}^n \times \mathbb{R}_{>0}}$ on $T^*(\mathbb{R}^n \times \mathbb{R}_{>0})$ to be

$$-y_1 dx_1 - \ldots - y_n dx_n + x_{n+1} dy_{n+1} = \theta_{\mathbb{R}^n \times \mathbb{R}_{>0}} + d(x_{n+1}y_{n+1}).$$

Since $-y_1 dx_1 - \ldots - y_n dx_n + x_{n+1} dy_{n+1}$ and $\theta_{\mathbb{R}^n \times \mathbb{R}_{>0}}$ differ by an exact one-form, it follows that the above map also identifies Lagrangian submanifolds of $T^*(\mathbb{R}^n \times \mathbb{R}_{>0})$ which are exact with respect to the primitive $\theta_{\mathbb{R}^n \times \mathbb{R}_{>0}}$ with Lagrangian submanifolds of the symplectisation $\mathbb{R} \times J^1(\mathbb{R}^n)$ which are exact.
with respect to the primitive $e^t \lambda_0$. In other words, an exact Lagrangian submanifold of the symplectisation $\mathbb{R} \times J^1(\mathbb{R}^n)$ has a Legendrian lift to $J^1(\mathbb{R}^n \times \mathbb{R}_{>0})$, and can thus be represented by the corresponding image in $(\mathbb{R}^n \times \mathbb{R}_{>0}) \times \mathbb{R}$ under the front projection.

Assume that a part of the front projection is given as the graph of a smooth function $f(x, x_{n+1})$ above some region in $\mathbb{R}^n \times \mathbb{R}_{>0}$. It thus follows that the corresponding sheet of the Lagrangian submanifold is determined by

$$\{y_i = \partial_{x_i} f(x, x_{n+1})\} \subset T^* (\mathbb{R}^n \times \mathbb{R}_{>0}),$$

on which the pull-back of $e^t \lambda_0$ has a primitive given by

$$- f(x, x_{n+1}) + x_{n+1} y_{n+1} + C, C \in \mathbb{R}.$$

Finally, under above the identification of $T^*(\mathbb{R}^n \times \mathbb{R}_{>0})$ with $\mathbb{R} \times J^1(\mathbb{R}^n)$, this sheet corresponds to a subset of a cylinder over a Legendrian submanifold in $J^1(\mathbb{R}^n)$ if and only if the above function is of the form

$$f(x, x_{n+1}) = g(x) x_{n+1} + C, C \in \mathbb{R}.$$

**4.2.2. The model for an exact Lagrangian handle $W_{\epsilon,k}$.** Here we construct the standard model $W_{\epsilon,k}$ of an elementary Lagrangian cobordism, where $k \in \{0, \ldots, n-1\}$.

---

**Figure 2.** A part of the front projection of the standard model $W_{\epsilon,k}$ of an elementary Lagrangian cobordism of index $k + 1$ in the case when $k = 0$ and $n = 1$. The cusp-edge corresponds to the core disc $C_S$.

For each $0 < \epsilon < 1$, we let

$$\rho_\epsilon : \mathbb{R} \to [0, 1]$$

be a smooth symmetric function satisfying

- $\rho_\epsilon(0) = 1$,
- $\rho_\epsilon'(x) \geq 0$ for $x \geq 0$,
- $\rho_\epsilon(x^2) = 0$ for $x^2 \geq 1 + (2/3)\epsilon$, and
\[ \frac{d^2}{dx^2} \left( x^2 + \rho(x^2) \left( 1 + \epsilon/2 \right) \right) > 0 \text{ for all } x \in \mathbb{R}. \]

In particular, for each \( 0 \leq C \leq 1 + \epsilon/2 \), the function \( x^2 + C \rho(x^2) \) is strictly convex, coincides with \( x^2 \) for \( x^2 \geq 1 + (2/3)\epsilon \), and takes the value \( C \) at \( x = 0 \).

For each \( 0 < \epsilon < 1 \) we also consider a smooth non-decreasing function

\[ \sigma_\epsilon : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \]

satisfying

- \( 0 \leq \sigma_\epsilon'(x) \leq (1 + \epsilon)\epsilon^{-1/3} \),
- \( \sigma_\epsilon(x) = 0 \) for \( x \leq 1 - \epsilon^{1/3} \),
- \( \sigma_\epsilon(x) = 1 + \epsilon/2 \) for \( x \geq 1 + \epsilon^{1/3} \), and
- \( \sigma_\epsilon(1) = 1, \sigma_\epsilon'(1) > 0, \sigma_\epsilon''(1) \geq 0 \).

Finally, we define

\[ \varphi_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \]

\[ \varphi_\epsilon(x, x_{n+1}) = \rho(x_1^2 + \ldots + x_{k+1}^2) \sigma_\epsilon(x_{n+1}). \]

We now consider the front in \((\mathbb{R}^n \times \mathbb{R}_{>0}) \times \mathbb{R}\) that above the domain

\[ \{(x_1^2 + \ldots + x_{k+1}^2) - (x_{k+2}^2 + \ldots + x_n^2) + \varphi_\epsilon(x, x_{n+1}) \geq 1\} \subset \mathbb{R}^n \times \mathbb{R}_{>0}, \]

is given by the graphs of the two functions \( \pm F_\epsilon(x)x_{n+1} \), where

\[ F_\epsilon(x, x_{n+1}) := \left( \frac{x_1^2 + \ldots + x_{k+1}^2 - (x_{k+2}^2 + \ldots + x_n^2) + \varphi_\epsilon(x, x_{n+1}) - 1}{1} \right)^{3/2}, \]

and which has a cusp-edge along the boundary of the same domain. We use \( W_{\epsilon,k} \subset T^*(\mathbb{R}^n \times \mathbb{R}_{>0}) \) to denote the corresponding (non-compact!) exact Lagrangian submanifold.

Observe that \( W_{\epsilon,k} \) can be identified with an exact Lagrangian submanifold of \( \mathbb{R} \times J^1(\mathbb{R}^n) \) which is cylindrical over a Legendrian submanifold above the complement of

\[ \left\{ x_1^2 + \ldots + x_{k+1}^2 \leq 1 + \epsilon, \\
1 - \epsilon^{1/3} \leq x_{n+1} \leq 1 + \epsilon^{1/3} \right\} \subset \mathbb{R}^n \times \mathbb{R}_{>0}. \]

Consequently, the primitive of the pull-back of the one-form

\[ -y_1 dx_1 - \ldots - y_n dx_n + x_{n+1} dy_{n+1} \]

may be taken to vanish in the same set.

We also consider the exact Lagrangian submanifold \( W_{0,k} \) corresponding to the following front in \((\mathbb{R}^n \times \mathbb{R}_{>0}) \times \mathbb{R}\). Above the domain

\[ \{(x_1^2 + \ldots + x_{k+1}^2) - (x_{k+2}^2 + \ldots + x_n^2) \geq 1\} \subset \mathbb{R}^n \times \mathbb{R}_{>0}, \]

we require it to correspond to the graphs of the two functions

\[ \pm \left( (x_1^2 + \ldots + x_{k+1}^2) - (x_{k+2}^2 + \ldots + x_n^2) - 1 \right)^{3/2} x_{n+1}, \]
while it has a cusp-edge along the boundary of the same domain. Obviously, $W_{0,k}$ is cylindrical over a Legendrian submanifold, and the primitive of the pull-back of the one-form

$$-y_1dx_1 - \ldots - y_n dx_n + x_{n+1}dy_{n+1}$$

may be taken to vanish. We use $L_{0,k}$ to denote this Legendrian submanifold. It follows that $W_{\epsilon,k}$ is an exact Lagrangian cobordism from $L_{0,k} \subset J^1(\mathbb{R}^n)$ to a Legendrian submanifold that will be denoted by $L_{\epsilon,k} \subset J^1(\mathbb{R}^n)$.

Observe that $L_{\epsilon,k}$ is diffeomorphic to the manifold obtained from $L_{0,k}$ by a surgery on the $k$-sphere

$$S_{0,k} := \begin{cases} \frac{q_1^2 + \ldots + q_{k+1}^2}{z} = 1, \\ \frac{q_{k+2}}{z} = \ldots = \frac{q_n}{z} = 0, \\ p = 0, \\ z = 0 \end{cases} \subset L_{0,k} \subset J^1(\mathbb{R}^n)$$

with the choice of frame

$$N_{0,k} := (q_1 \partial_{p_1} + \ldots + q_{k+1} \partial_{p_{k+1}}, \partial_{q_{k+2}}, \ldots, \partial_{q_n})$$

of its normal bundle. The cobordism $W_{\epsilon,k}$ is diffeomorphic to the manifold obtained by a handle-attachment on $(-\infty, -1] \times L_{0,k}$ along

$$S_{0,k} \subset L_{0,k} = \partial ((-\infty, -1] \times L_{0,k}).$$

By applying Formula 3.2, one sees that the Maslov class vanishes for $W_{\epsilon,k}$, and hence the same is true for both $L_{0,k}$ and $L_{\epsilon,k}$.

**Figure 3.** The front projection of the standard model $L_{0,k}$ before the surgery in the case when $k = 0$ and $n = 1$.

**Figure 4.** The front projection of the standard model $L_{\epsilon,k}$ after the surgery in the case when $k = 0$ and $n = 1$. 
4.2.3. The Reeb chords on $L_{0,k}$ and $L_{\varepsilon,k}$. Note that $L_{0,k}$ has no Reeb-chords, and that $L_{\varepsilon,k}$ has exactly one Reeb chord $c_0$ above $q = 0$. One computes

$$\ell(c_0) = \varepsilon^{3/2}/\sqrt{2}.$$ 

Furthermore, this Reeb chord satisfies

$$|c_0| = n - k - 1,$$

which can be seen by applying Formula 3.1. To that end, we observe that the front projection $\Pi_F(L_{\varepsilon,k})$ consists of two sheets being the graphs of the functions $f_\varepsilon < 0 < f_\varepsilon$, and that the critical point of $f_\varepsilon - f_\varepsilon$ has Morse index $n - (k + 1)$ (see Figure 4). Furthermore, in the case $n > 1$, the obvious capping-path of $c_0$ traverses exactly one cusp-edge in downward direction.

4.2.4. The standard model of the isotropic surgery disc. Consider the embedded isotropic disc

$$D_{0,k} := \begin{cases} \sum_{i=1}^{k+1} q_i^2 \leq 1, \\ q_{k+2} = \ldots = q_n = 0, \\ p = 0, \\ z = 0 \end{cases} \subset J^1(\mathbb{R}^n)$$

with boundary $\partial D_{0,k} = S_{0,k}$. If we endow the symplectic normal bundle $(TD_{0,k})^{d\lambda_0}$ with the Lagrangian frame

$$\{\partial_{q_{k+2}}, \ldots, \partial_{q_n}\},$$

it is easily seen that $D_{0,k}$ becomes an isotropic surgery disc compatible with the above frame of the normal bundle of $S_{0,k} \subset L_{0,k}$.

4.3. The definition of a Legendrian ambient surgery. Suppose that we are given a Legendrian submanifold $L \subset Y$ containing a framed embedded $k$-sphere $S \subset L$, together with a compatible isotropic surgery $(k+1)$-disc $D_S \subset Y$. We use $NS \subset TL|_S$ to denote the normal bundle of $S$.

We will construct an exact Lagrangian cobordism $V_{S,\varepsilon} \subset \mathbb{R} \times Y$ from $L$ to a Legendrian submanifold which will be denoted by $L_S$, and which is diffeomorphic to a manifold obtained from $L$ by $k$-surgery on $S$ with the above frame of $NS$. Furthermore, the cobordism $V_{S,\varepsilon}$ will be diffeomorphic to the elementary cobordism of index $k + 1$ induced by the surgery. We will say that $L_S$ is obtained from $L$ by a Legendrian ambient surgery on $S$.

4.3.1. Identifying the isotropic surgery disc with the standard model. There is a neighbourhood $U \subset J^1(\mathbb{R}^n)$ of $D_{0,k}$ for which there is a contact-form preserving diffeomorphism

$$\phi: (U, \lambda_0) \rightarrow (\phi(U), \lambda)$$

identifying $U$ with a neighbourhood $\phi(U) \subset Y$ of $D_S$. We can moreover require that

1. $\phi(D_{0,k}) = D_S$, and hence $\phi(S_{0,k}) = S$, 

where $D_S$ is the isotropic surgery disc compatible with $S_{0,k}$.
(2) $D\phi$ maps the Lagrangian frame of $(TD_{0,k})^{d\lambda_0}$ in the standard model (as given in Section 4.2.4) to the choice of Lagrangian frame of $(TD_S)^d\lambda$ for the isotropic surgery disc, and

(3) $\phi(L_{0,k} \cap U) = L \cap \phi(U)$.

The neighbourhood theorem [Gei, Theorem 6.2.2] for isotropic submanifolds gives neighbourhoods and a map $\phi$ as above satisfying (1) and (2). We may furthermore assume that (3) holds infinitesimally, i.e. that

$$D\phi(TL_{0,k}|_{S_{0,k}}) = TL_S.$$  

After a perturbation of $L$ by a Legendrian isotopy, and after possibly shrinking the above neighbourhoods, we may hence assume that (3) is satisfied as well.

4.3.2. The constructions. The identification of a neighbourhood of $D_S$ with a neighbourhood of $D_{0,k}$ via the contact-form preserving map $\phi$ constructed above induces an exact symplectomorphism of the form

$$(\text{id}_\mathbb{R}, \phi) : (\mathbb{R} \times U, d(e^t \lambda_0)) \rightarrow (\mathbb{R} \times \phi(U), d(e^t \lambda))$$

from the neighbourhood

$$\mathbb{R} \times U \supset \mathbb{R} \times D_{0,k}$$

in $\mathbb{R} \times J^1(\mathbb{R}^n)$ to the neighbourhood

$$\mathbb{R} \times \phi(U) \supset \mathbb{R} \times D_S$$

in $\mathbb{R} \times Y$. Observe that, by construction, this symplectomorphism identifies $W_{0,k}$ with $\mathbb{R} \times L$.

Recall the identification of $\mathbb{R} \times J^1(\mathbb{R}^n)$ with $T^*(\mathbb{R}^n \times \mathbb{R}_{>0})$ described in Section 4.2. After choosing a small enough $\epsilon > 0$, we may assume that the cylindrical neighbourhood

$$\bar{U}_\epsilon := \left\{ \begin{array}{lcl} x_1^2 + \ldots + x_{k+1}^2 < 1 + \epsilon, \\
 x_{k+2}^2 + \ldots + x_n^2 < 2\epsilon, \\
 |y_i| < 3\sqrt{\epsilon(1+\epsilon)}x_{n+1}, \ i \leq n, \\
 |y_{n+1}| < 10\epsilon^{1/6} \end{array} \right\} \subset T^*(\mathbb{R}^n \times \mathbb{R}_{>0}).$$

of $\mathbb{R} \times D_{0,k}$ is identified with a neighbourhood $\mathbb{R} \times U_\epsilon \subset \mathbb{R} \times U$ under this identification.

The two exact Lagrangian cobordisms $W_{0,k}, W_{\epsilon,k} \subset \mathbb{R} \times J^1(\mathbb{R}^n)$ constructed in Section 4.2.2 coincide outside of the set $\bar{U}_\epsilon$ as follows by Lemma 4.7 below. Replacing the set $(\mathbb{R} \times L) \cap (\mathbb{R} \times \phi(U_\epsilon))$ with $(\text{id}_\mathbb{R}, \phi)(W_{\epsilon,k} \cap (\mathbb{R} \times U_\epsilon))$ thus produces an exact Lagrangian cobordism that will be denoted by

$$V_{S,\epsilon} \subset \mathbb{R} \times Y.$$  

This is an exact Lagrangian cobordism from $L \subset Y$ to a Legendrian submanifold that we will denote by

$$L_{S,\epsilon} \subset Y.$$
Observe that $L_{S,\epsilon}$ is diffeomorphic to the manifold obtained by performing a $k$-surgery on $S$ with the specified frame of its normal bundle, and that $V_{S,\epsilon}$ is diffeomorphic to the corresponding elementary cobordism of index $k + 1$.

Moreover, we will fix a choice of an embedding of the core disc

$$C_{S,\epsilon} := (\text{id}_\mathbb{R}, \phi)(W_{\epsilon,k} \cap \{ p = 0, t \leq 0 \}) \subset V_{S,\epsilon}$$

of the handle-attachment. It is easily checked that $C_{S,\epsilon} \subset \mathbb{R} \times V_{S,\epsilon}$ is a smoothly embedded $(k + 1)$-disc that coincides with $(-\infty, -1) \times S$ outside of a compact set (also, see Figure 2).

**Definition 4.5.** Given an isotropic surgery $(k + 1)$-disc $D_S \subset Y$ compatible with a framed embedded $k$-sphere $S \subset L \subset Y$ inside a Legendrian manifold, we will say that the Legendrian submanifold $L_{S,\epsilon} \subset Y$ constructed above is obtained from $L$ by a Legendrian ambient surgery on $S$. Furthermore, the construction provides the exact Lagrangian cobordism $V_{S,\epsilon} \subset \mathbb{R} \times Y$ from $L$ to $L_{S,\epsilon}$, which will be called an elementary Lagrangian cobordism of index $k + 1$.

The above construction can be seen to satisfy the following properties.

**Remark 4.6.**

1. The new Reeb chord on $L_S$ that corresponds to the unique Reeb chord on the local model $L_{0,k}$ will be denoted by $c_S$ and satisfies

$$|\ell(c_S)| = n - k - 1,$$

$$\ell(c_S) = \epsilon^{3/2}/\sqrt{2},$$

as follows from the description in Section 4.2.3.

2. Changing the parameter $\epsilon > 0$ above does not change the Hamiltonian isotopy class of $V_{S,\epsilon}$, nor the Legendrian isotopy class of $L_{S,\epsilon}$. We will sometimes omit $\epsilon$ from the notation.

3. Assume that we are given two embedded spheres $S_i \subset L_i \subset Y$, $i = 1, 2$, with compatible isotropic surgery discs $D_{S_i}$. If there exists a contact isotopy taking $L_1 \cup D_{S_1}$ to $L_2 \cup D_{S_2}$, whose tangent-map moreover takes the Lagrangian frame of the symplectic normal bundle of $D_{S_1}$ to that of $D_{S_2}$, it follows that $(L_1)_{S_1}$ is Legendrian isotopic to $(L_2)_{S_2}$.

4. In the case $k < n - 1$, the isotropic surgery disc $D_S$ is subcritical. This means that, for generic data, there are no Reeb chords on $L \cup D_S$ starting or ending on $D_S$. Given any $E > 0$, after shrinking $\epsilon > 0$, we may thus assume that there is a natural identification

$$\{ c \in \mathcal{Q}(L_S); \ell(c) \leq E \} = \{ c \in \mathcal{Q}(L); \ell(c) \leq E \} \cup \{ c_S \}$$

of Reeb chords. In Section 4.5 we show that this also can be achieved in the case when $k = n - 1$, given that we first deform $L$ by a Legendrian isotopy.
(5) It can be seen that the symplectisation-coordinate $t$ restricted to $V_{S,\epsilon}$ is a Morse function with a unique critical point of index $k + 1$. Our choice of core disc $C_S$ moreover passes through this critical point.

**Lemma 4.7.** For each $0 < \epsilon < 1$, $W_{\epsilon,k}$ and $W_{0,k}$ coincide outside of the open set $\tilde{U}_\epsilon$ given that $\epsilon' \geq (2/3)\epsilon$.

**Proof.** We show that $W_{\epsilon,k}$ and $W_{0,k}$ coincide outside of the set $\tilde{U}_\epsilon(2/3)\epsilon$, from which the general statement obviously follows. We thus fix $0 < \epsilon < 1$ and set $\epsilon' := (2/3)\epsilon$.

Consider the function
\[ f_y(x) := x^2 + y\rho_\epsilon(x^2), \]
where $\rho_\epsilon$ is as defined in Section 4.2. Recall that $f_y(x)$ is symmetric in $x$, satisfies $f_y(x) = x^2$ for $x^2 \geq 1 + \epsilon'$, and that $f''_y(x) > 0$ holds for each $0 \leq y \leq 1 + \epsilon/2$. From this we conclude that the inequality
\[ y = f_y(0) \leq f_y(x) < 1 + \epsilon' \]
holds for all $x^2 < 1 + \epsilon'$, $0 \leq y \leq \eta + \epsilon/2$ or, in other words, that
\[ 0 \leq y = f_y(0) \leq x^2 + y\rho_\epsilon(x^2) - 1 < \epsilon' \]
holds in the same set.

This inequality implies that we have the inclusion
\[ W_{\epsilon,k} \cap \{ x_1^2 + \ldots + x_{k+1}^2 < 1 + \epsilon' \} \subset \{ x_{k+2}^2 + \ldots + x_n^2 < \epsilon' \} \]
and hence that the inequality
\[ F_\epsilon(x, x_{n+1}) < (\epsilon')^{3/2} \]
holds on the same set.

Furthermore, since $\rho_\epsilon(x^2)$ vanishes for $x^2 \geq 1 + \epsilon'$, it follows that $W_{\epsilon,k}$ and $W_{0,k}$ coincide outside of the set
\[ \{ x_1^2 + \ldots + x_{k+1}^2 < 1 + \epsilon' \}, \]
and by (4.1), outside of $O := \{ x_1^2 + \ldots + x_{k+1}^2 < 1 + \epsilon', \quad x_{k+2}^2 + \ldots + x_n^2 < \epsilon' \}$.

Since $f_y(x)$ is symmetric in $x$ and has increasing derivative by assumption, and since $f''_y(\pm \sqrt{1 + \epsilon'}) = \pm 2\sqrt{1 + \epsilon'}$, the inequality
\[ |f'_y(x)| < 2\sqrt{1 + \epsilon'} \]
holds for all $x^2 < 1 + \epsilon'$.

Using (4.2) and (4.3) one computes that
\[ \left| \frac{\partial}{\partial x_i} F_\epsilon(x, x_{n+1}) \right| \leq (3/2) F_\epsilon(x, x_{n+1})^{1-2/3} 2\sqrt{1 + \epsilon'} < 3\sqrt{\epsilon'} \sqrt{(1 + \epsilon')}, \]
holds inside $O$ for each $i = 1, \ldots, k + 1$, and that

$$
(4.5) \quad \left| \frac{\partial}{\partial x_i} F_\epsilon(x, x_{n+1}) \right| \leq (3/2) F_\epsilon(x, x_{n+1})^{1-2/3} 2\sqrt{\epsilon'} < 3\sqrt{\epsilon'} \sqrt{(1 + \epsilon')}
$$

holds inside $O$ for each $i = k + 2, \ldots, n$. In conclusion, $W_{\epsilon,k} \cap O$ is contained in the set

$$
\left\{ |y_i| < 3\sqrt{\epsilon'}(1 + \epsilon')x_{n+1}, \ 1 \leq i \leq n \right\}.
$$

Finally, from the inequality

$$
0 \leq \sigma'_\epsilon(x_{n+1}) \leq (1 + \epsilon)\epsilon^{-1/3},
$$

together with the fact that $\sigma'_\epsilon(x_{n+1}) = 0$ holds for $x_{n+1} \geq 1 + \epsilon^{1/3}$, we get the bound

$$
\left| \frac{\partial}{\partial x_{n+1}} F_\epsilon(x, x_{n+1})x_{n+1} \right| \leq
$$

$$
\leq F_\epsilon(x, x_{n+1}) + \frac{3}{2} F_\epsilon(x, x_{n+1})^{1-2/3} \sigma'_\epsilon(x_{n+1})x_{n+1}
$$

$$
< (\epsilon')^{3/2} + \frac{3}{2}(\epsilon')^{1/2}((1 + \epsilon)\epsilon^{-1/3})(1 + \epsilon^{1/3})
$$

$$
< (\epsilon')^{1/6} + 6(\epsilon')^{1/2} \epsilon^{-1/3}
$$

$$
= (\epsilon')^{1/6} + 6(\epsilon')^{1/2} (3/2)\epsilon^{1/3}
$$

$$
< (\epsilon')^{1/6} + 6(\epsilon')^{1/6}
$$
on $O$, where we again have used the inequality (4.2). In other words, $W_{\epsilon,k} \cap O$ is contained inside

$$
\left\{ |y_{n+1}| < 7(\epsilon')^{1/6} \right\}.
$$

In conclusion, we have shown that $W_{\epsilon,k}$ and $W_{0,k}$ coincide outside of the set $\tilde{U}'$. □

4.4. The effect of a Legendrian ambient surgery on the Maslov class. The Maslov class of a Lagrangian cobordism pulls back to the Maslov classes of its respective Legendrian ends under the inclusion maps. It follows that

$$
\mu(H_1(L)), \mu(H_1(L_S)) \subset \mu(H_1(V_S)) \subset \mathbb{Z}.
$$

4.4.1. The case of a 0-surgery. Suppose we are given an embedded isotropic 1-disk $D_S \subset Y$ that bounds $S \subset L$ and satisfies (a) and (b) in Definition 4.2. We fix an arbitrary frame of the normal bundle of $S \subset L$. When $n > 1$ we can always find a Lagrangian frame of the symplectic normal bundle of $D_S$ which makes it into an isotropic surgery disc compatible with $S$, as follows by part (2) of Remark 4.3. Furthermore, the choice of such a Lagrangian frame lives in

$$
\pi_1(U(n - 1)) \simeq \mathbb{Z}.
$$

Under the additional assumption that the 1-handle attachment adds a generator $\gamma \in H_1(V_S)$, the following can be said. Consider two choices
m_1, m_2 ∈ \mathbb{Z} \cong \pi_1(U(n-1)) of Lagrangian frames of D_S. The two corresponding Legendrian ambient surgeries give rise to two diffeomorphic cobordisms, and the evaluation of the Maslov class on γ for these two choices can be seen to differ by 2(m_1 - m_2).

4.4.2. The case of a k-surgery with 0 < k < n - 1. Let X ∪ (D^{k+1} × D^{n-k}) be a (k + 1)-handle attachment on the manifold X with non-empty boundary. From the associated long exact sequence in singular homology

\[ \ldots \rightarrow H_2(D^{k+1} × D^{n-k}, S^k × D^{n-k}) \rightarrow H_1(X) \rightarrow H_1(X ∪ (D^{k+1} × D^{n-k})) \rightarrow H_1(D^{k+1} × D^{n-k}, S^k × D^{n-k}) \rightarrow H_0(X) \rightarrow \ldots, \]

together with the fact that

\[ H_1(D^{k+1} × D^{n-k}, S^k × D^{n-k}) \cong H_1(S^{k+1}) \]

we conclude that the above map

\[ H_1(X) \rightarrow H_1(X ∪ (D^{k+1} × D^{n-k})) \]

is surjective, since k > 0.

Setting X = (−∞, −1] × L and identifying X ∪ (D^{k+1} × D^{n-k}) with V_S, we conclude that

\[ \mu(H_1(L)) = \mu(H_1(V_S)) \subset \mathbb{Z} \]

whenever L_S is obtained from L by an ambient k-surgery with k > 0.

Since V_S equivalently can be viewed as a smooth manifold obtained by a (n − k)-handle attachment on (−∞, −1] × L_S, the same argument shows that

\[ \mu(H_1(L_S)) = \mu(H_1(L)) = \mu(H_1(V_S)) \subset \mathbb{Z}, \]

whenever L_S is obtained from L by a Legendrian ambient k-surgery with 0 < k < n - 1.

4.5. Contracting the isotropic surgery disc. Assume that we are given an isotropic surgery disc D_S of dimension k + 1 which is compatible with a framed embedded sphere S ⊂ L inside a Legendrian submanifold. Let D_{S'} ⊂ D_S be a smooth embedding of a closed (k + 1)-disc. Assume that there is an isotopy of D_S taking S to the boundary ∂D_{S'}.

This isotopy can be extended to a contact isotopy of the ambient contact manifold Y (see e.g. Lemma 8.3) having support in some arbitrarily small neighbourhood of D_S. We use (L', S') to denote the image of (L, S) induced by this isotopy, where L' thus is Legendrian isotopic to L and S' = ∂D_{S'}.

Moreover, choosing this extension with some care, we may assume that L_S may be obtained from L' by a Legendrian ambient surgery on S' using the isotropic surgery disc D_{S'}.

Take any E > 0. After choosing D_{S'} ⊂ D_S to be sufficiently small, we may assume that all Reeb chords on L' ∪ D_{S'} that start or end on D_{S'} have action at least E. We conclude that
Lemma 4.8. For $0 \leq k \leq n - 1$ we may assume that $L_S = (L')_{S'}$ is obtained from $L'$ by a Legendrian ambient $k$-surgery on $S'$, where $(L', S')$ is isotopic to $(L, S)$ by a contact isotopy, and for which

$$\{ c \in Q(L_S); \ell(c) \leq E \} = \{ c \in Q(L'); \ell(c) \leq E \} \cup \{ c_S \}.$$

Observe that when $k < n - 1$ this result also follows by a general position argument (see part (4) of Remark 4.6).

4.6. Well-definedness of the Legendrian ambient 0-surgery. It was proven in [EH, Lemma 3.2] that cusp connected sum is a well-defined operation on Legendrian knots. Since Legendrian ambient 0-surgery is a generalisation of cusp connected sum, the below proposition provides a positive answer to a question posed in [EES2].

For simplicity we here only consider the case when $(\mathbb{C}^n \times \mathbb{R}, dz - \sum_i y_i dx_i)$ is the standard contact $(2n + 1)$-space. We also restrict ourselves to the case $n > 1$. Recall that, given an framed embedded 0-sphere $S \subset L$, we can always find a compatible isotropic surgery disc $D_S$, as follows by part (2) of Remark 4.3.

Proposition 4.9. Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a Legendrian submanifold, where $n > 1$, and let $S \subset L$ a framed embedded 0-sphere.

1. Suppose that $L \cap \{ x_1 = 0 \} = \emptyset$ and that the two points $S \subset L$ are separated by the half-space $\{ x_1 = 0 \} \subset \mathbb{C}^n \times \mathbb{R}$. It follows that the Legendrian isotopy class of $L_S$ is invariant under isotopy of the framed 0-sphere $S$ and independent of the choice of a compatible isotropic surgery disc $D_S$.

2. If $L$ is connected, then the Legendrian isotopy class of $L_S$ is invariant under isotopy of the framed 0-sphere $S$ and independent of the choice of an isotropic surgery disc $D_S$. However, the parametrised Legendrian isotopy class does depend on the choice of a Lagrangian frame of the symplectic normal bundle of $D_S$.

Proof. Consider the following set-up. Let $S, S' \subset L$ be isotopic framed embeddings of a 0-sphere, and let $D_S$ and $D_{S'}$ be isotropic surgery 1-discs compatible with $S$ and $S'$, respectively.

Since contact isotopies can be seen to act transitively on points in a Legendrian submanifold (see Lemma 8.3), after a contact isotopy we may assume that $S = S'$ and that the frames of $NS = NS'$ agree. This also shows that the outward-normal vector fields of the two isotropic surgery discs $D_S$ and $D_{S'}$ may be assumed to coincide. Moreover, again after a contact isotopy, it is readily checked that we may assume that the isotropic surgery discs themselves agree in some neighbourhood of $S \subset Y$.

Since the ambient contact manifold is of dimension $2n + 1 \geq 5$, it now follows by the h-principle for isotropic embeddings that the isotropic surgery discs are isotopic by a contact isotopy supported outside of some small neighbourhood of $L$ (also, see Remark 4.3).
In conclusion, the above reasoning shows that it suffices to consider the case $S = S'$ and $D_S = D_S'$ when proving the two results.

If the Lagrangian frames are homotopic relative the boundary, it follows that the two Legendrian embeddings produced by an ambient Legendrian surgery are Legendrian isotopic (see part (3) of Remark 4.6). It thus remains to consider the case when the Lagrangian frames are not homotopic relative the boundary.

(1): By assumption we have $L \cap \{ -\epsilon < x_1 < \epsilon \} = \emptyset$ for some $\epsilon > 0$, where the two components of $L$ containing the respective component of $S$ are separated by the hypersurface $\{ x_1 = 0 \}$. Without loss of generality, we may thus assume that

$$D_S \cap \{ -\epsilon < x_1 < \epsilon \} = \left\{ \begin{array}{l}
|x_1| < \epsilon,
\{ x_2 = \ldots = x_n = 0, \\
y_1 = \ldots = y_n = 0, \\
z = 0
\end{array} \right\}.$$  

Consider the autonomous Hamiltonian

$$H : \mathbb{C}^n \to \mathbb{R},$$

$$H(x, y) := \varphi(x_1)(x_2^2 + y_2^2)/2,$$

on $(\mathbb{C}^n, \sum_i dx_i \wedge dy_i)$ where $\varphi(x_1) \geq 0$ is a smooth function satisfying $\varphi(x_1) = 1$ for $x_1 < 0$ and $\varphi(x_1) = 0$ for $x_1 \geq \epsilon$. The corresponding Hamiltonian isotopy of $(\mathbb{C}^n, \sum_i dx_i \wedge dy_i)$ has a unique lift to a contact-form preserving isotopy $\phi^t$ of $(\mathbb{C}^n \times \mathbb{R}, \lambda_0)$ that satisfies

$$\phi^{2\pi l}_{\{ x_1 \leq 0 \}} = \text{id}_{\{ x_1 \leq 0 \}}, \quad l \in \mathbb{Z},$$

$$\phi^t_{\{ x_1 \geq \epsilon \}} = \text{id}_{\{ x_1 \geq \epsilon \}}, \quad t \in \mathbb{R}.$$  

Moreover, $\phi^t$ can be seen to preserve $D_S \cap \{ -\epsilon < x_1 < \epsilon \}$. It is readily seen that $\{ \phi^{2\pi l} \}_{l \in \mathbb{Z}} \simeq \mathbb{Z}$ acts transitively on the homotopy classes of Lagrangian frames of the symplectic normal bundle of $D_S$ relative its boundary.

(2): The discussion in Section 4.4 shows that different choices of homotopy classes of Lagrangian frames induce Legendrian embeddings of $L_S \subset Y$ that have different Maslov classes with respect to a fixed parametrisation of $L_S$. It follows that the parametrised Legendrian isotopy class of $L_S \subset Y$ indeed depends on the homotopy class of this Lagrangian frame. □

5. PSEUDO-HOLOMORPHIC DISCS WITH BOUNDARY ON THE ELEMENTARY COBORDISM

In this section we will investigate the pseudo-holomorphic discs in $\mathbb{R} \times Y$ having boundary on an elementary Lagrangian cobordism $V_S$ from $L$ to $L_S$. In this section we also prove Theorem 1.3 which depends on this analysis.

In the following we start by fixing an $(n + 1)$-dimensional elementary Lagrangian cobordism $V_{S, \epsilon_S} \subset \mathbb{R} \times Y$ of index $k + 1$ constructed using the parameter $\epsilon_S > 0$ (see Definition 4.5). Later it might become necessary to
"shrink" this handle by replacing $V_{S,\epsilon}$ with $V_{\epsilon,\epsilon}$, where $0 < \epsilon \leq \epsilon_S$. To that end, we will use a fixed identification $(\text{id}_R, \phi)$ of a neighbourhood of the standard model $D_{0,k}$ with a neighbourhood of the isotropic surgery disc $D_S$, as in Section 4.3. For each $0 < \epsilon \leq \epsilon_S$, we now use this choice of identification $(\text{id}_R, \phi)$ to construct $V_{S,\epsilon}$ from the standard model $W_{\epsilon,k}$.

We will in the following only consider Reeb chords on $L$ below some fixed action $E > 0$. We furthermore assume that all Reeb chords on $L \cup D_S$ that start or end on $D_S$ have action at least $2E$. When $k < n - 1$ this can be arranged by a general position argument, while for $k = n - 1$ one may achieve this after a Legendrian isotopy as described in Section 4.8. We will moreover choose $\epsilon_S > 0$ sufficiently small, so that

$$\{c \in Q(L_{S,\epsilon}); \ell(c) \leq E\} = \{c \in Q(L); \ell(c) \leq E\} \cup \{c_S\}$$

holds for each $0 < \epsilon \leq \epsilon_S$.

5.1. Preliminaries. We fix an exact symplectomorphism $(\text{id}_R, \phi) : \mathbb{R} \times U_{\epsilon_S} \rightarrow \mathbb{R} \times \phi(U_{\epsilon_S})$ as used in the construction of $V_{S,\epsilon}$ (see Section 4.3.2). Here, $\phi(U_{\epsilon_S})$ is a neighbourhood of $D_S$, and

$$\mathbb{R} \times U_{\epsilon_S} := \left\{ \begin{array}{l}
q_1^2 + \ldots + q_{k-1}^2 < 1 + \epsilon_S, \\
q_k^2 + q_{k+1}^2 + \ldots + q_n^2 < 2\epsilon_S, \\
|p_i| < 3\sqrt{\epsilon_S(1 + \epsilon_S)}, \ i \leq n, \\
|z| < 10\epsilon_S^{1/6}(1 + \epsilon_S) \end{array} \right\} \subset \mathbb{R} \times J^1(\mathbb{R}^n).$$

By construction, $(\text{id}_R, \phi)$ maps $W_{\epsilon,k} \cap (\mathbb{R} \times U_{\epsilon_S})$ to $V_{S,\epsilon} \cap (\mathbb{R} \times \phi(U_{\epsilon_S}))$ for any $0 < \epsilon \leq \epsilon_S$. We recall the definition of the core disc

$$C_{S,\epsilon} := (\text{id}_R, \phi)(W_{\epsilon,k} \cap \{p = 0, t \leq 0\}).$$

We also fix an open embedding

$$\Sigma \subset L \setminus \phi(U_{\epsilon_S}) \subset Y, \quad \Sigma \simeq S \times S^{n-k-1} \times (-1, 1),$$

for which the induced embedding of $\Sigma_0 := S \times S^{n-k-1} \times \{0\}$ can be identified with the unit normal-bundle of $S \subset L$.

We let $\phi^t_R$ denote the Reeb flow on $(Y, \lambda)$. After choosing both $\epsilon_S > 0$ and the subset $\Sigma$ sufficiently small, by the assumptions made on the Reeb flow restricted to $L \cup D_S$, we may require that

$$\phi^\pm(0, 2E)(\Sigma) \cap (L \cup \phi(U_{\epsilon_S})) = \emptyset$$

holds for the above constant $E > 0$. 


5.1.1. The energies of pseudo-holomorphic discs with boundary on $V_S$. By construction, $V_{S,\varepsilon}$ is cylindrical outside of the set $I_\varepsilon \times Y$, where

$$I_\varepsilon = [\log (1 - \varepsilon^{1/3}), \log (1 + \varepsilon^{1/3})].$$

Furthermore, the primitive of $e^t\lambda$ pulled back to $V_{S,\varepsilon}$ may be supposed to vanish outside of a compact set. Let $J_{cyl}$ denote a cylindrical almost complex structure on $\mathbb{R} \times Y$. Proposition 3.12 can be applied, giving us

$$0 \leq E_{d(\varphi_{I,\lambda})}(u) = \frac{1 + \varepsilon^{1/3}}{1 - \varepsilon^{1/3}}(\ell(a) - (\ell(b_1) + \ldots + \ell(b_m))),$$

$$0 < E_{\lambda,I}(u) = \frac{1 + \varepsilon^{1/3}}{1 - \varepsilon^{1/3}}(\ell(a),$$

$$0 < E_{I}(u) = \frac{2 + \varepsilon^{1/3}}{1 - \varepsilon^{1/3}}(\ell(a) - (\ell(b_1) + \ldots + \ell(b_m))),$$

for any $J_{cyl}$-holomorphic disc $u \in M_{a;I;A}(V_{S,\varepsilon}; J_{cyl})$ with $b = b_1 \cdot \ldots \cdot b_m$.

5.1.2. A cylindrical almost complex structure that is integrable near the core disc. There is a unique cylindrical almost complex structure $J_0$ on $\mathbb{R} \times J^1(\mathbb{R}^n)$ having the property that the canonical projection

$$\pi : \mathbb{R} \times J^1(\mathbb{R}^n) \to T^*\mathbb{R}^n = \mathbb{C}^n$$

is $(J_0,i)$-holomorphic. Furthermore, it can be checked that the identification

$$(\mathbb{R} \times J^1(\mathbb{R}^n), J_0) \to (\mathbb{C}^n \times \mathbb{C}, i),$$

$$(t, ((q,p),z)) \mapsto (q + ip, t - \|p\|^2/2 + iz),$$

is a biholomorphism. In particular, the function

$$\pi_C : \mathbb{R} \times (J^1(\mathbb{R}^n), J_0) \to (\mathbb{C}, i),$$

$$(t, ((q,p),z)) \mapsto (t - \|p\|^2/2 + iz),$$

is holomorphic.

We use $(id_{\mathbb{R}}, \phi)$ to push forward $J_0$ to an almost complex structure on $\mathbb{R} \times \phi(U_{\varepsilon_S}) \subset \mathbb{R} \times Y$. We then choose an extension to a cylindrical almost complex structure on $\mathbb{R} \times Y$ that we denote by $J_S$.

5.1.3. An anti-holomorphic involution in a neighbourhood of the unit normal-bundle of $S$. We will need to make some further assumptions on $J_S$ in a neighbourhood of $\mathbb{R} \times \Sigma_0 \subset \mathbb{R} \times Y$, where we recall that $\Sigma_0 \subset L$ can be identified with the unit normal-bundle of $S \subset L$.

A standard result implies that there exists an integrable almost complex structure $J_\Sigma$ on $T^*\Sigma$ for which the zero-section $0_\Sigma \subset T^*\Sigma$ is real-analytic. After shrinking the neighbourhood of the zero-section further, we may moreover assume that there exists an anti-holomorphic involution $\iota$ defined on this neighbourhood that fixes the zero-section pointwise.

It follows by Proposition [CE, Proposition 2.15] that, on some sufficiently small neighbourhood $D^*\Sigma \ni 0_\Sigma$, the squared distance $\rho : D^*\Sigma \to \mathbb{R}_{\geq 0}$ to
the zero-section induced by some choice of Hermitian metric on \( T^*\Sigma \) is a smooth plurisubharmonic function. In other words, writing

\[
\theta_\rho := -(d\rho) \circ J_\Sigma,
\]

it follows that \( d\theta_\rho \) is an exact symplectic form defined on \( D^*\Sigma \) for which \( J_\Sigma \) is a compatible almost complex structure. The restriction of \( \theta_\rho \) to \( 0_\Sigma \) vanishes, which implies that the zero-section is an exact Lagrangian submanifold on which the primitive of \( \theta_\rho \), moreover, can be taken to vanish.

The neighbourhood theorem \([\text{Gei}, \text{Theorem 6.2.2}]\) for isotropic submanifolds can now be used to identify a neighbourhood of \( \Sigma \subset L \subset Y \) with the contact manifold

\[
(D^*\Sigma \times [-2E, 2E], dz + \theta_\rho)
\]

by a contact-form preserving diffeomorphism that moreover identifies

\[
0_\Sigma \times \{0\} \subset D^*\Sigma \times [-2E, 2E]
\]

with \( \Sigma \subset L \). Here have used the assumption in Section 5.1 regarding the behaviour of the Reeb flow of \((Y, \lambda)\) restricted to \( \Sigma \subset Y \) to conclude that the neighbourhood may be taken on this very form, where \( E > 0 \) is as chosen above.

Again, there is a unique cylindrical almost complex structure \( J_1 \) on the symplectisation \( \mathbb{R} \times (D^*\Sigma \times [-2E, 2E]) \) defined by the requirement that the canonical projection

\[
(\mathbb{R} \times (D^*\Sigma \times [-2E, 2E]), J_1) \to (D^*\Sigma, J_\Sigma)
\]

is holomorphic. We will require the cylindrical almost complex structure \( J_S \) to coincide with the push-forward of \( J_1 \) under the above identification. It thus follows that there is a neighbourhood \( \mathbb{R} \times O_\Sigma \subset \mathbb{R} \times Y \) for which

- \( \phi_{[-2E, 2E]}[\Sigma] \subset O_\Sigma \),
- \( O_\Sigma \cap L = \Sigma \), and such that
- there exists an anti-holomorphic involution of \( (\mathbb{R} \times O_\Sigma, J_S) \) that fixes \( \mathbb{R} \times \Sigma \) pointwise.

To see the last property, we argue as follows. It is readily checked that there is a holomorphic open embedding

\[
(\mathbb{R} \times (D^*\Sigma \times [-2E, 2E]), J_1) \to (D^*\Sigma \oplus \mathbb{C}, J_\Sigma \oplus i),
\]

\[(t, ((q, p), z)) \mapsto ((q, p), t - \rho(q, p) + iz).\]

Using this embedding, the anti-holomorphic involution \( \iota \) of \( (D^*\Sigma, J_\Sigma) \) can be seen to lift to an anti-holomorphic involution of \( (\mathbb{R} \times (D^*\Sigma \times [-2E, 2E]), J_1) \).

Finally, by using the above identification, one obtains the required anti-holomorphic involution.
5.2. Properties of pseudo-holomorphic discs with boundary on $V_S$.

**Lemma 5.1.** There is a constant $E_0 > 0$ that only depends on the cylindrical almost complex structure $J_{cyl}$ such that, for each $0 < \epsilon \leq \epsilon_S$, any solution $u \in \mathcal{M}_{a;b,A}(V_{S,\epsilon}; J_{cyl})$ passing through $\mathbb{R} \times \partial \phi(U_{\epsilon_S})$ satisfies

$$E_u(u) \geq E_0 > 0.$$

**Proof.** For each $t_0 \in \mathbb{R}$ we consider the symplectic form

$$\omega_{t_0} = e^{-(t_0 + 1/2)}d(e^t \lambda) = e^{l-(t_0 + 1/2)} dt \wedge \lambda + e^{-l(t_0+1/2)} d\lambda$$

on $\mathbb{R} \times Y$ and observe that $J_{cyl}$ is compatible with $\omega_{t_0}$.

The subset $\partial \phi(U_{(3/4)\epsilon_S}) \subset Y$ is compact and, moreover, has a compact neighbourhood $O \subset Y$ satisfying

$$V_{S,\epsilon} \cap (\mathbb{R} \times O) = V_{S,\epsilon_S} \cap (\mathbb{R} \times O) = \mathbb{R} \times (L \cap O)$$

for each $0 < \epsilon \leq \epsilon_S$.

The monotonicity property for the $\omega_{t_0}$-area of $J_{cyl}$-holomorphic curves with and without boundary [Sik Propositions 4.3.1 and 4.7.2] applies to pseudo-holomorphic curves $u$ as above passing through a point in $\{t_0\} \times \partial \phi(U_{(3/4)\epsilon_S})$.

More precisely, there is a constant $C > 0$ only depending on $J_{cyl}$ for which the following holds. Any non-trivial $J_{cyl}$-holomorphic curve $u$ inside the compact set $[t_0 - 1/2, t_0 + 1/2] \times O$ that passes through $\{t_0\} \times \partial \phi(U_{(3/4)\epsilon_S})$ and whose boundary is contained inside the compact set

$$([t_0 - 1/2, t_0 + 1/2] \times (L \cap O)) \cup \partial ([t_0 - 1/2, t_0 + 1/2] \times O)$$

satisfies

$$E_u := \int_{\omega_{t_0}} \omega_{t_0} \geq C > 0.$$

Now, let $u$ be a $J_{cyl}$-holomorphic curve as in the assumption of the lemma. It follows that the above inequality holds for its $\omega_{t_0}$-area contained in the set $[t_0 - 1/2, t_0 + 1/2] \times O$.

By the construction of the total energy, it follows that we have the inequality

$$2E_{I_e}(u) \geq \int_{[t_0 - 1/2, t_0 + 1/2]} dt \wedge \lambda + \int_{[t_0 - 1/2, t_0 + 1/2]} \frac{2}{1 - \epsilon^{1/3}} d(\varphi_{I_e}(t) \lambda)$$

for any choice of functions $\rho^-(t)$ and $\rho^+(t)$ having support in the sets $\{t \leq \log (1 + \epsilon^{1/3})\}$ and $\{t \geq \log (1 - \epsilon^{1/3})\}$, respectively, and satisfying

$$\int_{\mathbb{R}} \rho^+(t) dt = 1.$$
can now readily be seen to follow. The statement thus holds for the choice of constant $E_0 := C/2 > 0$. □

Using (5.3) and Remark 4.6 we conclude that $u \in M_{cS; b; A}(V_{S; \epsilon}; J_{cyl})$ satisfies

$$0 < E_{I_i}(u) = 2 \frac{1 + \epsilon_i^{1/3}}{1 - \epsilon_i^{1/3}} \ell(c_S) - (\ell(b_1) + \ldots + \ell(b_m)) \leq 2 \frac{1 + \epsilon_i^{1/3}}{1 - \epsilon_i^{1/3}} \epsilon^{3/2}/\sqrt{2}$$

for any $u \in M_{cS; b; A}(V_{S; \epsilon}; J_{cyl})$. Together with Lemma 5.1 it immediately follows that such a disc must be disjoint from $\mathbb{R} \times \partial \phi(U_{(3/4)\epsilon_S})$ given that $0 < \epsilon \leq \epsilon_S$ is sufficiently small. In other words

**Corollary 5.2.** For sufficiently small $0 < \epsilon \leq \epsilon_S$, every $u \in M_{cS; b; A}(V_{S; \epsilon}; J_{cyl})$ is contained inside $\mathbb{R} \times \phi(U_{(3/4)\epsilon_S})$. In particular, it must satisfy $b = \emptyset$.

**Lemma 5.3.** Let $J_S$ be a cylindrical almost complex structure as constructed in Section 7.1.2. For $0 < \epsilon \leq \epsilon_S$ sufficiently small and $a \in Q(L)$, the only solution in $\bigcup_{A \in H^1(V_{S; \epsilon})} M_{a; A}(V_{S; \epsilon}; J_S)$ is the trivial strip over $a$.

**Proof.** We will show that any sequence $u_i \in M_{a; A}(V_{S; \epsilon}; J_S)$ of strips, where $\lim_{i \to +\infty} \epsilon_i = 0$, consists of trivial strips $\mathbb{R} \times a$ for each $i > 0$ sufficiently large.

First, the above formulas for the energy yield

$$E_{d(\phi_{I_i})}(u_i) = 1 + \frac{\epsilon_i^{1/3}}{1 - \epsilon_i^{1/3}} \ell(a) - \ell(a) = 2 \frac{\epsilon_i^{1/3}}{1 - \epsilon_i^{1/3}} \ell(a),$$

$$E_{\lambda, I_i}(u_i) = 1 + \frac{\epsilon_i^{1/3}}{1 - \epsilon_i^{1/3}} \ell(a),$$

and, thus, in particular

$$\lim_{i \to +\infty} E_{d(\phi_{I_i})}(u_i) = 0,$$

$$\lim_{i \to +\infty} E_{\lambda, I_i}(u_i) = \ell(a).$$

We argue by contradiction. Assume that there is a subsequence of $\{u_i\}$ satisfying the property that the boundary of the strip $u_i$ passes through

$$\{t_i\} \times \Sigma_0 \subset \{t_i\} \times \Sigma \subset \{t_i\} \times L.$$

Recall that $\Sigma_0 \subset \Sigma \subset L$ is a smooth embedding of $S \times S^{n-k-1}$ that can be identified with the unit normal-bundle of $S$.

The target-local version of Gromov’s compactness theorem [Fis Theorem A] can be applied to the induced subsequence

$$u_i(\mathbb{R} \times [0, 1]) \cap ([t_i - 1, t_i + 1] \times O_\Sigma) \subset [t_i - 1, t_i + 1] \times O_\Sigma$$

of $J_S$-holomorphic curves to extract a convergent subsequence (we give a justification of this at the end of this proof). After a translation of
the $t$-coordinate, the limit $\tilde{u}_\infty$ may be considered to be a non-trivial $J_S$-holomorphic curve in $[-2, 2] \times O_\Sigma$ having boundary on

\[
([-2, 2] \times \Sigma) \cup \partial([-2, 2] \times O_\Sigma)
\]

passing through $\mathbb{R} \times \Sigma_0 \subset \mathbb{R} \times \Sigma$.

By Formula (5.4), the limit $\tilde{u}_\infty$ must be contained inside a trivial strip $\mathbb{R} \times c$, where $c$ is some (possibly disconnected) integral curve of the Reeb vector-field. From the assumption in Section 5.1.3 that $\phi_{[-2E, 2E]}(\Sigma) \subset O_\Sigma$, we can now compute

\[
\int_{\tilde{u}_\infty \cap \{t \in I\}} dt \wedge \lambda \geq 2E > 2\ell(a)
\]

for any interval $I \subset [-2, 2]$ of length one. Observe that the latter inequality holds by definition, since we only consider Reeb chords of action less than $E$. This shows that $\{u_i\}$ has a subsequence for which

\[
\int_{u_i \cap \{t \in I_i\}} dt \wedge \lambda > 2\ell(a)
\]

is satisfied as well, where $I_i \subset [t_i - 2, t_i + 2]$ again denotes any interval of length one. However, the latter inequality clearly contradicts the limit (5.5) of the $\lambda$-energy of the solutions $u_i$.

This contradiction shows that the strips $u_i$ have boundary disjoint from $\mathbb{R} \times \Sigma_0$ for each sufficiently big $i > 0$. Since each such solution $u_i$ thus has boundary on

\[
V_{S, \epsilon} \setminus (\mathbb{R} \times \phi(U(3/4)_{3, S})) \subset \mathbb{R} \times L,
\]

as follows from topological considerations, such a solution satisfies $E_{d\lambda}(u_i) = 0$. In conclusion, all $u_i$ coincide with $\mathbb{R} \times a$ for sufficiently big $i > 0$.

We end by arguing that the target-local version of Gromov’s compactness theorem indeed can be applied in this situation. First, by the assumptions on $J_S$ in Section 5.1.3 there exists an anti-holomorphic involution of $(\mathbb{R} \times O_\Sigma, J_S)$ that fixes $(\mathbb{R} \times O_\Sigma) \cap V_{S, \epsilon}$ pointwise. This involution can be used to perform a Schwarz-reflection of each curve $u_i(\mathbb{R} \times [0, 1]) \cap (\mathbb{R} \times O_\Sigma)$, thus producing a curve $C_i \subset [t_i - 2, t_i + 2] \times O_\Sigma$ which is $J_S$-holomorphic and whose boundary is contained in $\partial([t_i - 2, t_i + 2] \times O_\Sigma)$. The sequence $C_i$ can moreover be seen to satisfy the following properties.

- After translating the $t$-coordinate, each $C_i$ may be identified with a $J_S$-holomorphic curve inside $[-2, 2] \times O_\Sigma$ having boundary on $\partial([-2, 2] \times O_\Sigma)$.
- Each $C_i$ is of genus zero.
- Since there is a uniform bound on the total energy $E_{d\lambda}(u_i)$ of the solutions $u_i$, there is a uniform bound on the $d(c\lambda)$-area of the curves $C_i$. 

$\Box$
5.3. Pseudo-holomorphic discs intersecting the core disc.

Lemma 5.4. For sufficiently small $0 < \epsilon \leq \epsilon_S$, every $u \in \mathcal{M}_{cS;\emptyset,A}(V_{S,\epsilon}; J_S)$ is contained in the set

$$\mathbb{R} \times \phi \left( U_{\epsilon} \cap \left\{ \begin{array}{l} q_1 = \ldots = q_{k+1} = 0, \\ p_1 = \ldots = p_{k+1} = 0, \\ t \geq 0 \end{array} \right\} \right).$$

Proof. By Corollary 5.2 we may assume that the image of $u$ is contained in $\text{id}_{\mathbb{R}} \times \phi(U_{(3/4)\epsilon_S})$. We can thus use $(\text{id}_{\mathbb{R}}, \phi)^{-1}$ to identify $u$ with a $J_0$-holomorphic disc in $\mathbb{R} \times U_{\epsilon_S} \subset \mathbb{R} \times J^1(\mathbb{R}^n)$ having boundary on the exact Lagrangian cobordism $W_{\epsilon,k}$ as defined in Section 4.2.2. We will again use $u$ to denote this $J_0$-holomorphic disc.

Consider the holomorphic projections

$$\pi_i := (q_i, p_i): \mathbb{R} \times U_{\epsilon_S} \rightarrow \mathbb{C}$$

for $i = 1, \ldots, n$. Observe that, by the construction of $J_S$ in the neighbourhood $\mathbb{R} \times U_{\epsilon_S}$, the composition $\pi_i \circ u$ is a holomorphic map from the closed unit disc $\hat{D}^2$ with one boundary-point removed. The image

$$\pi_i(W_{\epsilon,k}) \subset \mathbb{C}, \ i = 1, \ldots, k + 1,$$

is shown in Figure 5. Since $\pi_i \circ u$ maps the boundary to a compact subset of $\pi_i(W_{\epsilon,k})$, the open mapping theorem implies that

$$\pi_i \circ u(\hat{D}^2) \subset \pi_i(W_{\epsilon,k}), \ i = 1, \ldots, k + 1.$$  \hfill (5.6)

By contradiction, we assume that $\pi_i \circ u$ does not vanishing identically for some $i = 1, \ldots, k + 1$. Using the asymptotic properties of $u$, and the fact that the boundary-condition has a discontinuity near the puncture, it can be seen that $\pi_i \circ u$ fills some part of the corner either above or below $c_S$ shown in Figure 5. This however contradicts the above inclusion (5.6). In other words, $u$ is contained in the set

$$\left\{ \begin{array}{l} q_1 = \ldots = q_{k+1} = 0, \\ p_1 = \ldots = p_{k+1} = 0 \end{array} \right\}$$

and, by examining the construction of $W_{\epsilon,k}$ in Section 4.2.2, its boundary is thus contained in

$$W_{\epsilon,k} \cap \left\{ \begin{array}{l} q_1 = \ldots = q_{k+1} = 0, \\ p_1 = \ldots = p_{k+1} = 0 \end{array} \right\} \subset \{ t \geq 0 \}.$$  

Recall that there is a holomorphic projection

$$\pi_C: (J^1(\mathbb{R}^n), J_0) \rightarrow (\mathbb{C}, i),$$

$$((t, ((p, q), z)) \mapsto t - \|p\|^2/2 + iz,$$

and that the real-part $\Re(\pi_C \circ u)$ thus is harmonic. Using the fact that

$$\Re(\pi_C \circ u) \geq 0$$ \hfill (5.7)
holds along the boundary, as well as in some neighbourhood of the boundary puncture, we conclude that (5.7) holds on all of the domain $\hat{D}^2$. In particular, $t \circ u \geq 0$ holds everywhere, from which the statement of the lemma follows.

It remains to establish the inequality (5.7) along the boundary $\partial \hat{D}^2$. We will do this via certain estimates in terms of the functions used in the construction of $W_{\epsilon,k}$, and we refer to Section 4.2.2 for their definitions.

The inequality in Formula (4.5) implies that

$$|p_i| = \left| \frac{\partial}{\partial x_i} F_\epsilon(x, x_{n+1}) \right| \leq (3/2) F_\epsilon(x, x_{n+1})^{1-2/3} 2\sqrt{(2/3)\epsilon}$$

holds along the boundary of $u$ for each $i = k + 2, \ldots, n$, which can be translated into the inequality

$$|p_i|^2 \leq 9 \left( (x_1^2 + \ldots + x_{k+1}^2) - (x_{k+2}^2 + \ldots + x_n^2) + \varphi_\epsilon(x, x_{n+1}) - 1 \right) (2/3)\epsilon$$

along the boundary. Since the boundary of $u$ is contained in the set

$$\left\{ \begin{array}{l} q_1 = \ldots = q_{k+1} = 0, \\ t \geq 0 \end{array} \right\} = \left\{ \begin{array}{l} x_1 = \ldots = x_{k+1} = 0, \\ x_{n+1} \geq 1 \end{array} \right\}.$$

as was shown above, this estimate becomes

$$|p_i|^2 \leq 6\epsilon \left( - (x_{k+2}^2 + \ldots + x_n^2) + \sigma_\epsilon(x_{n+1}) - 1 \right) \leq 6\epsilon (\sigma_\epsilon(x_{n+1}) - 1).$$

Observe that the right-hand side vanishes for $x_{n+1} = e^t$ when $t = 0$, and that we have the inequality $0 \leq \sigma_\epsilon(x_{n+1}) \leq (1 + \epsilon)\epsilon^{-1/3}$. Using the identity $x_{n+1} = e^t$ it follows that, for any $A > 0$,

$$(t/n - |p_i|^2/2) \circ u \geq 0$$

holds along $\partial \hat{D}^2 \cap u^{-1}\{t \in [0, A]\}$ for each sufficiently small $0 < \epsilon \leq \epsilon_S$. This can finally be used to show that $\Re(\pi_C \circ u) \geq 0$ holds along all of the boundary $\partial \hat{D}^2$. \[\square\]

**Figure 5.** The image of $W_{\epsilon,k}$ under the holomorphic projection $\pi_i$ for $i = 1, \ldots, k + 1.$
Lemma 5.5. For sufficiently small \( 0 < \epsilon \leq \epsilon_S \), each \( u \in \mathcal{M}_{c_S;\emptyset;A}(V_{S,\epsilon}; J_S) \) whose boundary intersects \( C_{S,\epsilon} \) is contained in the holomorphic strip \( \mathbb{R} \times c_S \subset \mathbb{R} \times Y \). In particular, there is a unique such disc \( u_0 \).

Proof. Let \( u \) be such a disc. Using Corollary 5.2 we may identify \( u \) with a \( J_0 \)-holomorphic disc in \( \mathbb{R} \times U_{\epsilon_S} \) having boundary on \( W_{\epsilon,k} \). Furthermore, Lemma 5.4 implies that \( u \) necessarily intersects \( C_{S,\epsilon} \) in the set

\[
C_{S,\epsilon} \cap \{q_1 = \ldots = q_{k+1} = 0\} = \{q = p = z = t = 0\}.
\]

Recall that, by the construction of \( J_S \), there are holomorphic projections

\[
\pi_i := (q_i, p_i) : \mathbb{R} \times U_{\epsilon_S} \to \mathbb{C}.
\]

We define

\[
W := W^+ \cup W^-,
\]

\[
W^\pm := W_{\epsilon,k} \cap \begin{cases} q_1 = \ldots = q_{k+1} = 0, \\ \pm z \geq 0, \\ t \geq 0 \end{cases}
\]

The image

\[
\pi_i(W) \subset \mathbb{C}, \ i = k + 2, \ldots, n,
\]

is a filled figure-eight curves as shown in Figure 6. Since \( \pi_i \circ u \) is holomorphic and maps the boundary into \( \pi_i(W) \), the open mapping theorem implies that

\[
\pi_i \circ u(\hat{D}^2) \subset \pi_i(W), \ i = k + 2, \ldots, n.
\]

We have seen that a boundary point \( p \) mapped into \( C_{S,\epsilon} \) by \( u \) necessarily is mapped to the origin by \( \pi_i \circ u \), and that the same is true for the holomorphic map

\[
\pi_C \circ u = (t - \|p\|^2/2 + iz) \circ \hat{D}^2 \to \mathbb{C}.
\]
Also, recall that \( \Re(\pi_C \circ u) \geq 0 \) holds by (5.7) established in the proof of Lemma 5.4. From this it follows that, for small values of \( \theta \in \mathbb{R} \), \( \Im(\pi_C \circ u) \) has the following behaviour at the boundary points \( e^{i\theta}p \in \partial D^2 \). There is some sufficiently small \( \delta > 0 \) for which

\[
\{ 0 \}, \quad \theta = 0,
\]

\[
\mathbb{R}_{>0}, \quad 0 > \theta > -\delta,
\]

\[
\mathbb{R}_{<0}, \quad \delta > \theta > 0.
\]

By contradiction, we assume that \( \pi_i \circ u \) does not vanish identically for \( i = k + 1, \ldots, n \). It readily follows that one of \( \pm \pi_i \circ u \) must map the oriented boundary near \( p \) as depicted by the arrow in Figure 6 (also, see the proof of Lemma 5.4). However, in this case, we can use the open mapping theorem to get a contradiction with the inclusion (5.8) established above. This contradiction thus shows that \( u \) is contained in the trivial strip \( \mathbb{R}_{\geq 0} \times c_S \) as claimed.

Observe that the above \( J_S \)-holomorphic disc \( u_0 \) contained in the trivial strip \( \mathbb{R}_{\geq 0} \times c_S \) and having boundary on \( V_{S,\epsilon} \) is an embedding onto its image, as follows by its asymptotic properties.

**Lemma 5.6.** For sufficiently small \( 0 < \epsilon \leq \epsilon_S \), the disc \( u_0 \in \mathcal{M}_{(S;\emptyset;A)}(V_{S,\epsilon}; J_S) \) contained in \( \mathbb{R} \times c_S \) is transversely cut out. Furthermore, the evaluation-map

\[
ev: \mathcal{M}_{(S;\emptyset;A)}(V_{S,\epsilon}; J_S) \times \partial D^2 \to V_{S,\epsilon}
\]

from the boundary is transverse to \( C_{S,\epsilon} \) in a neighbourhood of \( \{ u_0 \} \times \partial D^2 \).

**Proof.** Again, we will identify \( u_0 \) with the corresponding \( J_0 \)-holomorphic disc in \( \mathbb{R} \times U_{\epsilon_S} \subset \mathbb{R} \times J^1(\mathbb{R}^n) \). Since \( u_0 \) is an embedding, we will identify the domain of \( u_0 \) with its image.

We write

\[
\begin{align*}
q &= (q_1, q_2) = ((q_1, \ldots, q_{k+1}), (q_{k+2}, \ldots, q_n)), \\
p &= (p_1, p_2) = ((p_1, \ldots, p_{k+1}), (p_{k+2}, \ldots, p_n)), \\
z &= \psi = (\psi_1, \psi_2, \psi_3): (\mathbb{R} \times J^1(\mathbb{R}^n), J_0) \to (\mathbb{C}^{k+1} \times \mathbb{C}^{n-(k-1)} \times \mathbb{C}, i), \\
(t, ((q, p), z)) &\mapsto (q_1 + iq_1, q_2 + ip_2, t - \|p\|^2/2 + iz).
\end{align*}
\]

Since the almost complex structure is integrable in a neighbourhood of \( u_0 \), it follows that the linearisation of the Cauchy-Riemann operator at \( u_0 \) is the standard Cauchy-Riemann operator \( \partial \) acting on the trivial holomorphic vector-bundle \( (u_0)^*(T\mathbb{C}^{n+1}) \).

We parametrise the punctured boundary of the domain of \( u_0 \) by

\[
(0, 2\pi) \to \partial D^2, \\
\theta \mapsto e^{i\theta},
\]

\[
\text{for the coordinates on } J^1(\mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}).
\]

Similarly to the map in Section 5.1.2 there is a biholomorphism
for some choice of holomorphic coordinate that moreover satisfies the property that $e^{i\pi} \in \partial \hat{D}^2$ is the unique point evaluating to $u_0(e^{i\pi}) \in C_{S, \epsilon}$. Using the above biholomorphism, the linearised boundary condition
\[ A(\theta) := T_{u_0}(\theta)W_{e,k} \subset T_{u_0}(\theta)(C^{k+1} \times C^{n-k-1} \times \mathbb{C}) = C^{k+1} \oplus C^{n-k-1} \oplus \mathbb{C} \]
can be seen to split into the direct sum
\[ A(\theta) = A_1(\theta) \oplus A_2(\theta) \oplus A_3(\theta) \subset C^{k+1} \oplus C^{n-k-1} \oplus \mathbb{C} \]
respecting the above decomposition of $C^{n+1}$. Furthermore, $A_i$ are families of Lagrangian subspaces of the form
\[
\begin{align*}
A_1(\theta) &= \mathbb{R}\langle e^{i\varphi_1(\theta)}e_j; \ 1 \leq j \leq k+1 \rangle \subset C^{k+1}, \\
A_2(\theta) &= \mathbb{R}\langle e^{i\varphi_2(\theta)}e_j; \ k+2 \leq j \leq n \rangle \subset C^{n-k-1}, \\
A_3(\theta) &= \mathbb{R}e^{i\varphi_3(\theta)}e_{n+1} \subset \mathbb{C},
\end{align*}
\]
where $\{e_i\}_{i=1, \ldots, n+1}$ denotes the standard basis of $\mathbb{C}^{n+1}$. It can be checked that that $\varphi_1$ is non-increasing, while $\varphi_2$ is non-decreasing. These facts will be important in the argument below.

Since the above linearised boundary condition splits, the kernel of the linearised problem has an induced splitting $K = K_1 \oplus K_2 \oplus K_3$. Moreover, by elliptic regularity, this is a finite-dimensional space consisting of holomorphic functions
\[
\zeta: D^2 \to \mathbb{C}^{n+1},
\]
that moreover are continuous up to the boundary. Let
\[
\pi_i: \mathbb{C}^{k+1} \oplus \mathbb{C}^{n-k-1} \oplus \mathbb{C} \to \mathbb{C}
\]
be the orthogonal projection onto the $i$:th component. By the argument principle, together with the fact that $\varphi_1$ above is non-increasing, we immediately see that $\pi_i \circ \zeta \equiv 0$ vanishes identically for each $i = 1, \ldots, k+1$. In other words, $K_1 = 0$. Furthermore, the solutions $\pi_{n+1} \circ \zeta \in K_3$ correspond to infinitesimal reparametrisations of the domain, and hence $\dim K_3 = 2$.

Investigating the Fredholm index of the linearised boundary-value problem we get
\[
\dim K_2 + 2 = \dim K \geq (n - k - 1) + 2,
\]
where equality holds if and only if the cokernel vanishes, i.e. if the solution $u_0$ is transversely cut out. To show transversality it therefore suffices to show that $\dim K_2 = n - k - 1$. We will simultaneously show that the evaluation map is transverse to $C_{S, \epsilon}$ in a neighbourhood of $\{u_0\} \times \partial \hat{D}^2 \subset \mathcal{M}_{cS; \theta; A}(V_{S, \epsilon}; J_S) \times \partial \hat{D}^2$.

Recall that
\[
(u_0, e^{i\pi}) \in \mathcal{M}_{cS; \theta; A}(V_{S, \epsilon}; J_S) \times \partial \hat{D}^2
\]
is the unique point evaluating to $C_{S, \epsilon}$. Consider the linear map
\[
Dev = (Dev)_1 + (Dev)_2: K \times \mathbb{R} \to A(\pi) \subset \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \times \mathbb{C},
\]
where
\[ \text{Dev}_1(\zeta, \theta) = \zeta(\pi), \]
\[ \text{Dev}_2(\zeta, \theta) = \theta \partial_s u(e^{i\theta})|_{s=\pi}. \]
Note that the map \( \text{Dev} \) in fact the differential of the evaluation-map \( ev \) at the point \( (u_0, e^{i\pi}) \in M_{cS;A(V_s, J_s)} \times \partial \hat{D}^2 \), given that the latter space is smooth.

Under the above biholomorphism, the tangent-plane
\[ T_{ev(u_0, e^{i\pi})} C_{S, \epsilon} \subset T_{ev(u_0, e^{i\pi})} V_{S, \epsilon} \]
is identified with
\[ \mathbb{R} \mathbb{C}^{k+1} \oplus 0 \oplus 0 \subset \mathbb{C}^{k+1} \oplus \mathbb{C}^{n-k-1} \oplus \mathbb{C}, \]
while the tangent plane \( T_{ev(u_0, e^{i\pi})} V_{S, \epsilon} \) is identified with
\[ A(\pi) = \mathbb{R} \mathbb{C}^{k+1} \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C} \subset \mathbb{C}^{k+1} \oplus \mathbb{C}^{n-k-1} \oplus \mathbb{C}. \]

Consider the linear subspace
\[ V := \text{im}(\text{Dev}) \subset A(\pi) = \mathbb{R} \mathbb{C}^{k+1} \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C}. \]
First, observe that \( 0 \oplus 0 \oplus \mathbb{R} \mathbb{C} \subset V \) since the boundary of \( u_0 \) is embedded and tangent to this subspace at the boundary-point \( e^{i\pi} \in \partial \hat{D}^2 \). Since \( K_1 = 0 \) we moreover conclude that
\[ V \subset 0 \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C}. \]
Both the property that \( \dim K_2 = n - k - 1 \) and the transversality of the evaluation map will thus follow if we manage to show that the linear map
\[ (\text{Dev})|_{K_2} : K_2 \rightarrow 0 \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus 0, \]
\[ \zeta \mapsto \zeta(\pi), \]
is injective. To see this, recall that we have the inequality \( \dim K_2 = \dim K \geq n - k - 1 \) by the above considerations of the Fredholm index and, therefore, if \( (\text{Dev})|_{K_2} \) is injective it is necessarily also an isomorphism. The transversality of the evaluation map would thus also follow from this, since
\[ V = 0 \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C} \subset \mathbb{R} \mathbb{C}^{k+1} \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C} \]
is transverse to
\[ \mathbb{R} \mathbb{C}^{k+1} \oplus 0 \oplus 0 \subset \mathbb{R} \mathbb{C}^{k+1} \oplus \mathbb{R} \mathbb{C}^{n-k-1} \oplus \mathbb{R} \mathbb{C} \]
in this case.

To establish the injectivity of \( (\text{Dev})|_{K_2} \) we proceed as follows. Consider the holomorphic projections
\[ \pi_i := (x_i, y_i) : \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \times \mathbb{C} \rightarrow \mathbb{C}, \quad i = k + 2, \ldots, n. \]
Take any solution \( \zeta \in K_2 \) to the linearised problem. Because of the boundary condition, the holomorphic map \( \pi_i \circ \zeta \) has boundary values inside the cone
By the open-mapping theorem, it thus follows that
\[ \pi_i \circ \zeta(D^2 \setminus \partial D^2) \subset \text{int} \pi_i(A_2(0, 2\pi)). \]

Observe that \( \pi_i \circ \zeta(e^{i\theta}) \in \pi_i A_2(\theta) \), and that \( \arg(\pi_i \circ \zeta(e^{i\theta})) = \varphi_2(\theta) \) thus is non-decreasing. Since
\[ \pi_i \circ \zeta(e^{i\pi}) \in \pi_i A_2(\pi) = \Re \mathbb{C}, \]
the open-mapping theorem can again be used to show that \( \pi_i \circ \zeta(e^{i\pi}) = 0 \) if and only if \( \pi_i \circ \zeta \equiv 0 \) vanishes identically. In other words, the above map \( (Dev)_1|_{K_2} \) into \( \Re \mathbb{C}^{n-k-1} \) has no kernel. \( \square \)

![Figure 7](image-url)

**Figure 7.** The \( i: \text{th} \) component of the linearised boundary condition \( \pi_i(A_2(0, 2\pi)) \) for \( i = k + 2, \ldots, n \).

**Lemma 5.7.** The moduli space \( \mathcal{M}_{c_S; \emptyset; A}(\mathbb{R} \times L_{S,\varepsilon}; J_S)/\mathbb{R} \) satisfies the following properties when \( 0 < \varepsilon \leq \varepsilon_S \) is sufficiently small.

- **Case 1:** \( k < n - 2 \): There is a unique \( A \in H_1(L_{S,\varepsilon}) \) for which this moduli space is non-empty. In this case, it has expected dimension at least one.
- **Case 2:** \( k = n - 2 \): There is a unique \( A \in H_1(L_{S,\varepsilon}) \) for which this moduli space is non-empty. In this case, it consists of precisely two transversely cut out solutions.
- **Case 3:** \( k = n - 1 \): The moduli space is empty.

**Proof.** After a translation of the \( t \)-coordinate, we may assume that \( u \in \mathcal{M}_{c_S; \emptyset; A}(\mathbb{R} \times L_{S,\varepsilon}; J_S) \) also is contained in \( \mathcal{M}_{c_S; \emptyset; A}(\mathbb{R} \times V_{S,\varepsilon}; J_S) \). Since Lemma 5.4 thus applies, it follows that \( u \) has image contained in the subset
\[ \mathbb{R} \times \phi \left( U_{c_S} \cap \left\{ q_1 = \ldots = q_{k+1} = 0, \quad p_1 = \ldots = p_{k+1} = 0 \right\} \right) \subset \mathbb{R} \times Y. \]

Again, we will identify \( u \) with the corresponding \( J_0 \)-holomorphic disc in \( \mathbb{R} \times U_{c_S} \subset \mathbb{R} \times J^1(\mathbb{R}^n) \) having boundary on \( \mathbb{R} \times L_{c,k} \). From this fact we obtain the result in the case \( k \neq n - 2 \), and it thus remains to consider the case \( k = n - 2 \).
Since the projection
\[ \pi_n := (q_n, p_n) : \mathbb{R} \times U_{\epsilon_S} \rightarrow \mathbb{C} \]
is holomorphic, and since \( \pi_n(\mathbb{R} \times (L_{\epsilon,k} \cap U_{\epsilon_S})) \) is a figure-eight curve (see Figure 6), it follows that \( \pi_n \circ u \) is a holomorphic disc having boundary on this figure-eight curve and precisely one corner at the double-point.

Up to parametrisation, there are exactly two such holomorphic polygons in \( \mathbb{C} \), which moreover are embedded, as follows by the assumption that there is a unique positive puncture. As in the proof of Lemma 5.6 it can be checked that the two corresponding holomorphic polygons in
\[ \left\{ \begin{array}{l}
q_1 = \ldots = q_{k+1} = 0, \\
p_1 = \ldots = p_{k+1} = 0
\end{array} \right\} \subset T^* \mathbb{R}^n = \mathbb{C}^n \]
having boundary on \( \Pi_{\text{Log}}(L_{\epsilon,k}) \subset T^* \mathbb{R}^n = \mathbb{C}^n \) are transversely cut out.

Using [DR1, Theorem 1.2] it follows the above holomorphic polygons inside \( (T^* \mathbb{R}^n = \mathbb{C}^n, \Pi_{\text{Log}}(L_{\epsilon,k})) \) can be lifted to the symplectisation. In other words, they are in bijective correspondence with \( J_0 \)-holomorphic discs in \( \mathbb{R} \times U_{\epsilon_S} \subset \mathbb{R} \times J^1(\mathbb{R}^n) \) having boundary on \( \mathbb{R} \times L_{\epsilon,k} \). Moreover, the latter discs are transversely cut out as well. Finally, under the map \( (\text{id}_\mathbb{R}, \phi) \), these two \( J_0 \)-holomorphic discs are in bijective correspondence with the solutions in \( \mathcal{M}_{c_S; \emptyset; A}(\mathbb{R} \times L_{S,\epsilon}; JS)/\mathbb{R} \). \( \square \)

5.4. **Proof of Theorem 1.3.** Let \( JS \) be a cylindrical almost complex structure on \( \mathbb{R} \times Y \) as constructed in Section 5.1.2. Part (2) of Proposition 3.15 implies that we may assume \( JS \) to be regular for the moduli spaces \( \mathcal{M}_{a;b;A}(\mathbb{R} \times L; JS) \) and \( \mathcal{M}_{a;b;A}(\mathbb{R} \times L_{S,\epsilon}; JS) \) for all \( a \neq c_S \) simultaneously. The transversality for the moduli spaces \( \mathcal{M}_{c_S; \emptyset; A}(\mathbb{R} \times L_{S,\epsilon}; JS) \) of expected dimension one follows from Lemma 5.7. In conclusion, \( JS \) may be assumed to be regular for the moduli spaces in the definition of the Chekanov-Eliashberg algebras of \( L \) and \( L_{S,\epsilon} \) simultaneously.

We begin with the proof of surjectivity. Consider \( a \in \mathcal{Q}(L_{S,\epsilon}) \). The inequality (5.1) implies that, for \( \epsilon > 0 \) small enough, the moduli space \( \mathcal{M}_{a;b;A}(V_{S,\epsilon}; JS) \) is empty whenever \( b \) is a word containing a generator \( b \in \mathcal{Q}(L) \) satisfying \( \ell(b) > \ell(a) \).

For sufficiently small \( \epsilon > 0 \) and \( a \neq c_S \), Lemma 5.3 implies that the trivial strip is the only solution in \( \mathcal{M}_{a;a;A}(V_{S,\epsilon}; JS) \). By an explicit calculation of the cokernel of the linearised \( \partial_{JS} \)-operator along a trivial strip \( \mathbb{R} \times a \), the fact that \( JS \) is cylindrical implies that such a solution is transversely cut out.

In order to achieve transversality for the moduli spaces \( \mathcal{M}_{a;b;A}(V_{S,\epsilon}; JS) \) in the case when \( b \neq a \), we might have to perturb \( JS \) by a compactly supported (and thus non-cylindrical) perturbation. However, after a sufficiently small such perturbation, the above counts \( |\mathcal{M}_{a;a;A}(V_{S,\epsilon}; JS)| = 1 \) still remain true.
For such a choice of $J_S$, we have thus concluded that

$$
\Phi_{V_{S,\epsilon}}(a) = a + B(a), \quad a \neq c_S,
$$

$$
\Phi_{V_{S,\epsilon}}(c_S) \in \mathbb{Z}_2,
$$

where $B(a)$ is spanned by words of generators having action strictly less than $\ell(a)$. In particular the surjectivity of $\Phi_{V_{S,\epsilon}}$ follows.

It remains to investigate $\ker \Phi_{V_{S,\epsilon}}$. Since the induced map $\Phi_{V_{S,\epsilon}}: (A(L_S), \partial_S) / \langle c_S - \Phi_{V_{S,\epsilon}}(c_S) \rangle \rightarrow (A(L), \partial)$ can be seen to be an isomorphism of DGAs, we get that

$$
\ker \Phi_{V_{S,\epsilon}} = \langle c_S - \Phi_{V_{S,\epsilon}}(c_S) \rangle,
$$

where the latter denotes the two-sided ideal generated by $c_S - \Phi_{V_{S,\epsilon}}(c_S)$.

Let $u \in \mathcal{M}_{c_S;\emptyset;A}(V_{S,\epsilon}, J_S)$ be a disc contributing to $\Phi_{V_{S,\epsilon}}(c_S)$. Lemma \ref{lem:disc-contributing} implies that the image of $u$ is contained in the set $\mathbb{R} \times \phi(U_{c_S}) \cap \{q = p = 0\}$.

**Case** $k < n - 1$: By this property of $u$, it follows that the moduli space $\mathcal{M}_{c_S;\emptyset;A}(V_{S,\epsilon}, J_S)$ is of expected dimension at least one whenever it is non-empty. In particular, we have

$$
\Phi_{V_{S,\epsilon}}(c_S) = 0.
$$

**Case** $k = n - 1$: It can be seen that there is a unique such solution $u$ which moreover is contained inside $\mathbb{R} \times c_S$. Furthermore, Lemma \ref{lem:transverse-cut-out} may be applied to show that $\mathcal{M}_{c_S;\emptyset;A}(V_{S,\epsilon}, J_S) = \{u\}$ is transversely cut out. In other words,

$$
\Phi_{V_{S,\epsilon}}(c_S) = 1.
$$

6. **The Chekanov-Eliashberg algebra twisted by a submanifold**

Let $L \subset (Y, \lambda)$ be a chord-generic Legendrian submanifold of a contact manifold of dimension $2n + 1$. In the following we assume that $S \subset L$ is an embedded submanifold of dimension $k$ that admits a non-vanishing normal vector-field $v \subset NS \subset TL$. We here require that $k < n - 1$, so that the codimension of $S \subset L$ is at least two. Furthermore, we assume that there are no Reeb chords on $L$ having endpoints on $S$, which can be achieved after a generic perturbation of $S \subset L$.

6.1. **Definitions.** In this section we construct the *Chekanov-Eliashberg algebra of $L$ twisted by $S$*, which is a differential graded algebra that will be denoted by $(A(L; S), \partial_{S,v})$.

6.1.1. **The graded algebra.** Consider the unital non-commutative algebra

$$
A(L; S) := \mathbb{Z}_2 \langle Q(L) \cup \{s\} \rangle,
$$

freely generated over $\mathbb{Z}_2$ by the Reeb chords on $L$ together with a formal generator $s$. We give the Reeb-chord generators the usual grading induced
by the Conley-Zehnder index (see Section 3.1.1), while we grade the formal variable by

$$|s| = n - k - 1.$$  

6.1.2. The boundary map. Let $J_{cyl}$ be a cylindrical almost complex structure. For a number $\delta > 0$ and a Riemannian metric $g$ on $L$, consider the moduli spaces $M_{a:b,w,A}^{g,\delta}(L;S,v;J_{cyl})$ as defined in Section 8. We define the boundary operator by

$$\partial_{S,v}(a) := \sum_{|a|-|b|-w|+\mu(A) = 1} |M_{a:b,w,A}^{g,\delta}(L;S,v)|/\mathbb{R} |s^{w_1}b_1s^{w_2}\cdots s^{w_m}b_ms^{w_{m+1}},$$

$$\partial_{S,v}(s) := 0,$$

where $a \in Q(L)_E$, $b = b_1 \cdots b_m$ is a (possibly empty) word of Reeb chords on $L$, $A \in H_1(L)$, $w \in \mathbb{Z}_{\geq 0}^{m+1}$, and $w := w_1 + \ldots + w_{m+1}$. We extend $\partial_{S,v}$ to $\mathcal{A}(L;S)$ using the Leibniz rule. It immediately follows that $\partial_{S,v}$ has degree $-1$.

**Lemma 6.1.** For a generic $J_{cyl}$ the boundary map $\partial_{S,v}$ is a well-defined differential. Finally, there is a canonical identification $$(\mathcal{A}(L;S),\partial_{S,v})/\langle s \rangle = (\mathcal{A}(L),\partial)$$

of DGAs.

**Proof.** For a generic $J_{cyl}$ the union of moduli spaces in the definition of $\partial_{S,v}(a)$ is a compact zero-dimensional manifold, as follows by Proposition 8.2 together with Theorem 8.5. It follows that the above count makes sense.

By Theorem 8.5, the coefficient in front of the word

$$s^{w_1}b_1s^{w_2}\cdots s^{w_m}b_ms^{w_{m+1}}$$

in the expression $\partial_{S,v}^2(a)$ is given by the count of boundary-points of the one-dimensional compact moduli space $M_{a:b,w,A}^{g,\delta}(L;S,v;J_{cyl})$, from which $\partial_{S,v}^2(a) = 0$ follows.

The last statement immediately follows from the definition of the Chekanov-Eliashberg algebra of $L$. \qed

6.2. Maps induced by cobordisms. Suppose that $V$ is an exact Lagrangian cobordism from $L_-$ to $L_+$ and that $M \subset V$ is a $(k+1)$-dimensional submanifold coinciding with

$$((-\infty,-A) \times S_-) \cup ((B,\infty) \times S_+)$$

outside of a compact set. We assume that $v$ is a non-vanishing normal vector-field to $M$ in $TV$ that moreover coincides with translation invariant vector-fields $v_- \subset TL_-$ and $v_+ \subset TL_+$ in the sets $\{t \leq -A\}$ and $\{t \geq B\}$, respectively.
We use $s_{\pm}$ to denote the formal generators of $A(L_{\pm}; S_{\pm})$. For a choice of compatible almost complex structure $J$ which, moreover, is cylindrical on the sets $(-\infty, A) \times Y$ and $[B, +\infty) \times Y$, we define

\[ \Phi_{V; M, v}(a) := \sum_{|a| - |\mu(A)| = 0} |M_{a;b,w}^{\delta_{\epsilon}}(V; M, v; J)| s_{-}^{w_1} b_1 s_{-}^{w_2} \cdots s_{-}^{w_m} b_m s_{-}^{w_{m+1}}, \]

\[ \Phi_{V; M, v}(s_{+}) := s_{-}, \]

where $a \in Q(L_{+})$, $b = b_1 \cdots b_m$ is a (possibly empty) word of Reeb chords on $L_{-}$, $A \in H_1(V)$, $w \in (Z_{\geq 0})^{m+1}$, and $w := w_1 + \cdots + w_{m+1}$. We extend $\Phi_{V; M, v}$ to a unital algebra map

\[ \Phi_{V; M, v} : A(L_{+}; S_{+}) \to A(L_{-}; S_{-}). \]

**Proposition 6.2.** For a generic compatible almost complex structure $J$ the above map

\[ \Phi_{V; M, v} : (A(L_{+}; S_{+}), \partial_{S_{+}}, v_{+}) \to (A(L_{-}; S_{-}), \partial_{S_{-}}, v_{-}) \]

is well defined unital DGA morphism. Furthermore, it descends to the DGA morphism

\[ \Phi_{V} : (A(L_{+}), \partial_{+}) \to (A(L_{-}), \partial_{-}) \]

under the natural projections to the respective Chekanov-Eliashberg algebras.

**Proof.** As in the proof of Lemma 6.1, the fact that this map is well-defined again follows from Proposition 8.2 together with Theorem 8.5.

Furthermore, by Theorem 8.5 the coefficient in front of the word $s_{-}^{w_1} b_1 s_{-}^{w_2} \cdots s_{-}^{w_m} b_m s_{-}^{w_{m+1}}$ in the expression

\[(\Phi_{V; M, v} \circ \partial_{S_{+}, v_{+}} - \partial_{S_{-}, v_{-}} \circ \Phi_{V; M, v})(a)\]

is given by the count of boundary-points of the one-dimensional compact moduli space $M_{a;b,w}^{\delta_{\epsilon}}(V; M, v; J)$. The chain-map property now follows.

Finally, the statement concerning the induced maps on the (non-twisted) Chekanov-Eliashberg algebras follows immediately from the definition of $\Phi_{V}$ together with Lemma 6.1. \[QED\]

**Proposition 6.3.** The map

\[ \Phi_{\mathbb{R} \times L; M, v} : (A(L; S_{+}), \partial_{S_{+}, v_{+}}) \to (A(L; S_{-}), \partial_{S_{-}, v_{-}}) \]

is a tame isomorphism of DGAs for any regular cylindrical almost complex structure.

**Proof.** Recall that trivial cylinders $\mathbb{R} \times a$ are rigid pseudo-holomorphic discs in the case when the almost complex structure is cylindrical. By the formula for the $d\lambda$-area in Proposition 8.12 it thus follows that

\[ \Phi_{\mathbb{R} \times L; M, v}(a) = a + B(a), \]
where \( B(a) \) is a linear combination of words of generators having action strictly less than \( \ell(a) \) (here we have prescribed \( \ell(s_\pm) := 0 \)). Finally, recall that
\[
\Phi_{R \times L; M, v}(s_+) = s_-
\]
holds by definition.

**Corollary 6.4.** Let \( S \subset L \) be a submanifold with non-vanishing normal vector-field \( v \). If \( S \) admits an embedded null-cobordism \((M, \partial M) \subset ([0, 1] \times L, \{1\} \times S)\) to which \( v \) moreover extends as a non-vanishing normal vector-field, then there is tame isomorphism
\[
\Phi_{V; M, v}: (A(L; S), \partial S, v) \rightarrow (A(L; \emptyset), \partial \emptyset, \emptyset)
\]
of DGAs.

### 6.3. Proof of Theorem 1.9

We assume that \( L_{S, \epsilon} \subset Y \) is obtained from \( L \subset Y \) by a Legendrian ambient surgery on the framed sphere \( S \subset L \). We let \( V_{S, \epsilon} \) be the induced exact Lagrangian cobordism from \( L \) to \( L_{S, \epsilon} \). Recall the definition of the core disc \( C_{S, \epsilon} \subset V_{S, \epsilon} \) in Section 4.3.2 which coincides with
\[
(-\infty, -1) \times S \subset V_{S, \epsilon}
\]
outside of a compact set. We also fix a non-vanishing normal vector-field \( v \) to \( C_{S, \epsilon} \subset V_{S, \epsilon} \) which, outside of a compact set, has the property that it is
\begin{itemize}
  \item invariant under translations of the \( t \)-coordinate, and
  \item coincides with a non-vanishing normal vector-field of \( S \subset L_{S, \epsilon} \).
\end{itemize}
In particular, observe that \( v \) is homotopic to a constant vector-field with respect to the frame of the normal bundle of \( S \) used for the surgery.

By construction, we have an inclusion
\[
(A(L_{S, \epsilon}), \partial L_{S, \epsilon}) \subset (A(L_{S, \epsilon}; \emptyset, \emptyset), \partial \emptyset, \emptyset)
\]
of DGAs. We show that the DGA morphism
\[
\Psi := \Phi_{V_{S, \epsilon}; C_{S, \epsilon}, v}: (A(L_{S, \epsilon}), \partial L_{S, \epsilon}) \rightarrow (A(L; S), \partial S, v),
\]
obtained as a restriction of the DGA morphism constructed above satisfies the required properties, given that we choose a particular almost complex structure. To that end, we will use the cylindrical almost complex structure \( J_S \) used in the proof of Theorem 1.3 which was constructed in Section 5.1.2.

First, for \( 0 < \epsilon \leq \epsilon_S \) sufficiently small, the proof of Theorem 1.9 immediately generalises to give the following. For each \( a \neq c_S \) we have
\[
\Psi(a) = a + B(a),
\]
where \( B(a) \) is a linear-combination of words consisting of generators of action strictly less than \( \ell(a) \) (here we have defined \( \ell(s_\pm) := 0 \)).

Second, the disc count in Lemma 5.5 together with the transversality result in 5.4 shows that
\[
\Psi(c_S) = s,
\]
given that \( 0 < \epsilon \leq \epsilon_S \) is sufficiently small.
In particular, it follows that $\Psi$ is a tame isomorphism of DGAs for such an almost complex structure $J_S$.

7. The Chekanov-Eliashberg algebra twisted by a hypersurface

We now consider the case when $S \subset L$ is a closed co-oriented hypersurface, where $L \subset (Y, \lambda)$ is a Legendrian submanifold. The requirement that the boundary of a pseudo-holomorphic disc $u: (\mathbb{D}^2, \partial \mathbb{D}^2) \to (\mathbb{R} \times Y, \mathbb{R} \times L)$ intersects $\mathbb{R} \times S$ transversely is in this case an open condition in the space of maps. Hence, introducing such a boundary-point constraint does not cut down the dimension of a moduli space and, consequently, the algebraic formalism in Section 6 cannot be expected to work. Here we present an alternative construction that, unfortunately, in general is weaker than Theorem 1.9.

There is a version of the Chekanov-Eliashberg algebra of $L$ whose coefficients are taken in the group-ring $\mathbb{Z}_2[H_1(L)]$ (see e.g. [EES1]). In [EENS, Section 2.3.3] this construction was refined to a version where the underlying algebra is the free product $A(L) \ast \mathbb{Z}_2[H_1(L)]$, i.e. where the group-ring elements do not commute with the Reeb chord generators. The Chekanov-Eliashberg algebra twisted by a hypersurface, as defined below, can in fact be obtained from this latter version of the Chekanov-Eliashberg algebra with Novikov coefficients.

For simplicity we will here only consider the case when there are finitely many Reeb chords, and we assume that $S$ is disjoint from the Reeb chords on $L$. Consider the algebra $A(L; S)$ as defined in Section 6. Recall that the generator $s$ is graded by $|s| = n - k - 1 = 0$ in this case. We extend this algebra to $\tilde{A}(L; S) := A(L; S) \ast \langle s^{-1} \rangle \supset A(L; S)$, by adding multiplicative inverses $s^{-l}$ to $s^l$ (so, in particular, this algebra is no longer free).

Consider the decomposition

$$\mathcal{M}_{a;b:A}(L; J_{cyl}) = \bigsqcup_{w \in \mathbb{Z}^{m+1}} \mathcal{M}_{a;b,w:A}(L; S; J_{cyl})$$

into different components, where $\mathcal{M}_{a;b,w:A}(L; S; J_{cyl})$ consists of those $J_{cyl}$-holomorphic discs whose boundary arc between the $i$:th and $(i+1)$:th puncture, starting the count at the positive puncture, has algebraic intersection-number $w_i \in \mathbb{Z}$ with $\mathbb{R} \times S$. One can define a differential on $\tilde{A}(L; S)$ by the formulas

$$\partial_S(a) := \sum_{|a| = |b| + \mu(A) = 1} \left[ \mathcal{M}_{a;b,w:A}(L; S; J_{cyl}) / \mathbb{R} |s|^{w_1} b_1 s^{w_2} \ldots s^{w_m} b_m s^{w_{m+1}} \right],$$

$$\partial_S(s^{\pm 1}) := 0,$$
and extend the definition to all of $\tilde{A}(L; S)$ using the Leibniz rule. The below proposition can be seen to follow from the invariance result for the Chekanov-Eliashberg algebra with Novikov coefficients.

**Proposition 7.1.** The homotopy type of the DGA $(\tilde{A}(L; S), \partial_S)$ is invariant under Legendrian isotopy and the choice of cylindrical almost complex structure $J_{cyl}$. In fact, this DGA can be obtained as a quotient of the DGA as defined in [EENS, Section 2.3.3]. Furthermore

\[(\tilde{A}(L; S), \partial_S)/\langle s - 1 \rangle \simeq (A(L), \partial).\]

We now assume that $S \subset L$ is a co-oriented sphere for which there is an isotropic surgery disc $D_S \subset Y$ compatible with $S \subset L$. We moreover assume that there are no Reeb chords on $L \cup D_S$ starting or ending on $D_S$. According to Lemma 4.8, this can be achieved after a Legendrian isotopy of $L$.

Consider the decomposition

\[\mathcal{M}_{a;b;A}(V_S; J) = \bigsqcup_{w \in \mathbb{Z}^{m+1}} \mathcal{M}_{a;b,w;A}(V_S; C_S; J)\]

into different components, where $\mathcal{M}_{a;b,w;A}(L; C_S; J)$ consists of those $J_{cyl}$-holomorphic discs whose boundary arc between the $i$:th and $(i+1)$:th puncture, starting the count at the positive puncture, has algebraic intersection-number $w_i \in \mathbb{Z}$ with the core disc $C_S \subset V_S$.

Fix a generic almost complex structure $J_S$ as in Section 5.1.2, which is used in the proof of Theorem 1.9. We use this almost complex structure together with the above decomposition of the moduli space to define a DGA morphism

\[\Psi : (A(L_S), \partial_{L_S}) \to (\tilde{A}(L; S), \partial_S)\]

by prescribing it to take the value

\[\Psi(a) = \sum_{|a| - |b| + \mu(A) = 0} \mathcal{M}_{a;b,w;A}(V_S; C_S; J)s^{w_1}b_1s^{w_2} \cdots s^{w_m}b_ms^{w_{m+1}}\]

on generators, and then extending it to a unital algebra map. The proof of Theorem 1.9 given in Section 6.3 generalises to show that

**Proposition 7.2.** The map

\[\Psi : (A(L_S), \partial_{L_S}) \to (\tilde{A}(L; S), \partial_S)\]

is a well-defined and unital DGA morphism that, moreover, is injective and satisfies

\[\Psi(c_S) = s.\]

**Remark 7.3.** (1) After adding formal inverses $c_S^{-l}$ to $c_S$ in the Chekanov-Eliashberg algebra $A(L_S)$ of $L_S$, the above map $\Psi$ becomes an isomorphism.
(2) There are circumstances when the above map $\Psi$ has its image in the free sub-algebra

$$A(L; S) \subset \tilde{A}(L; S).$$

In this case, it follows that

$$\Psi : (A(L_S), \partial) \to (A(L; S), \partial_S)$$

is an isomorphism and, in particular, $A(L; S) \subset \tilde{A}(L; S)$ is a sub-DGA.

(3) There are certain geometric constraints on the isotropic surgery disc $D_S$ for which the assumptions in (2) above can be seen to hold. In particular, this is true in situations when the boundary of a pseudo-holomorphic disc in the above moduli spaces always has positive local intersection number with $C_S$ (appropriately oriented) at each intersection point.

8. The moduli spaces with boundary-point constraints

We here define the moduli spaces of pseudo-holomorphic discs in a symplectisation $\left(\mathbb{R} \times Y, d(e^t \lambda)\right)$ having boundary in an exact Lagrangian cobordism $V \subset \mathbb{R} \times Y$, and boundary-point constraints on parallel copies of a submanifold $M \subset V$ of codimension at least two. We also establish a transversality result for these moduli spaces.

8.1. The definitions of the moduli space. Similarly to the moduli spaces in [BEE, Section 8.2.D], we make the following definition. Again, we assume that $(Y, \lambda)$ is a $(2n + 1)$-dimensional contact manifold. Let $V \subset (\mathbb{R} \times Y, d(e^t \lambda))$ be an exact Lagrangian cobordism from $L_-$ to $L_+$ and let $M \subset V$ a $(k + 1)$-dimensional submanifold of codimension at least two. We fix a non-vanishing normal vector-field $v \subset TV|_M$ to $M$, a Riemannian metric $g$ on $V$, and a compatible almost complex structure $J$ on $\mathbb{R} \times Y$. We moreover require there to exist numbers $A < B$ such that

- The compatible almost complex structure $J$ is cylindrical outside of the set $[A, B] \times Y$.
- $V \cap \{t \notin [A, B]\}$ coincides with the cylinders $((-\infty, A) \times L_-) \cup ((B, +\infty) \times L_+)$.
- $M \cap \{t \notin [A, B]\} \subset V$ coincides with $((-\infty, A) \times S_-) \cup ((B, +\infty) \times S_+)$, where $S_\pm \subset L_\pm$. We moreover require that $S_\pm$ is disjoint from the end-points of the Reeb chords on $L_\pm$.
- $v$ restricted to the sets $\{t \leq A\}$ and $\{t \geq B\}$ is given by $v_- \in TL_-$ and $v_+ \in TL_+$, respectively, where $v_\pm$ moreover is invariant under translations of the $t$-coordinate.
- $g$ is the product metric $dt^2 + g_\pm$ outside outside of a compact set, where $g_\pm$ is a Riemannian metric on $L_\pm$. 

Let \( a \) be a Reeb chord on \( L_+ \), \( b = b_1 \cdots b_m \) a word of Reeb chords on \( L_- \), and \( A \in H_1(V) \). We also fix some
\[
w = (w_1, \ldots, w_{m+1}) \in (\mathbb{Z}_{\geq 0})^{m+1}
\]
and set
\[
w = w_1 + \ldots + w_{m+1}.
\]

**Definition 8.1.** The moduli space
\[
\mathcal{M}_{a,b,w;A}^\delta(V;M,v;J) \subset \mathcal{M}_{a,b;A}(V;J)
\]
consists of the solutions
\[
u: (\hat{D}^2, \partial \hat{D}^2) \to (\mathbb{R} \times Y, V)
\]
satisfying the following boundary-point constraints. Recall that \( \hat{D} = D \setminus \{p_0, \ldots, p_m\} \) denotes the unit disk with \( m+1 \) fixed boundary-points removed.

We require there to be \( w \) boundary points on \( \partial \hat{D}^2 \) that are mapped to the submanifolds
\[
M_{i,\delta} := \exp_M((i-1)\delta v), \ i = 1, \ldots, w,
\]
where the \( i \)-th point with respect to the order on \( \partial D^2 \setminus \{p_0\} \) induced by the orientation is mapped to \( M_{i,\delta} \). Moreover, we require there to be \( w_{i+1} \) such points on the boundary arc in \( \partial \hat{D}^2 \) starting at \( p_i \in \partial D^2 \) and ending at \( p_{i+1} \in \partial D^2 \) (here we set \( p_{m+1} := p_0 \)).

The case when \( V = \mathbb{R} \times L \), \( v \), and
\[
M_{j,\delta} = \mathbb{R} \times \exp_S((j-1)\delta v) \subset \mathbb{R} \times L, \ j = 1, \ldots, w,
\]
are invariant under translations of the \( t \)-coordinate will be referred to as the **cylindrical setting**. When the almost complex structure \( J = J_{cyl} \) also is cylindrical, it follows that there is a natural action by \( \mathbb{R} \) on the above moduli spaces induced by translation of the \( t \)-coordinate. In this case we will write
\[
\mathcal{M}_{a,b,w;A}^\delta(L;S,v;J_{cyl}) := \mathcal{M}_{a,b,w;A}^\delta(\mathbb{R} \times L; \mathbb{R} \times S, v; J_{cyl}).
\]

**8.1.1. Definition in terms of the evaluation map.** In the case when \( m+1 \geq 3 \) there are no conformal orientation-preserving reparametrisations of \( \hat{D}^2 \) that fix the puncture \( p_0 \). In other words, there is a unique map \( u: \hat{D}^2 \to \mathbb{R} \times Y \) representing a solution \( u \in \mathcal{M}_{a,b;A}(V;J) \).

In the case when \( m+1 < 3 \), there is a \((2-m)\)-dimensional family of conformal automorphisms of \( \hat{D}^2 \). In order to construct an evaluation-map on these moduli-spaces, we can either choose a representative \( u: \hat{D}^2 \to \mathbb{R} \times Y \) for each \( u \in \mathcal{M}_{a,b;A}(V;J) \) (this has to be done in a way that continuously depends on the point in the moduli space), or one can choose to work on the level of moduli spaces of **parametrised maps**. We will use the latter approach, and we will use \( \tilde{\mathcal{M}}_{a,b;A}(V;J) \) to denote the moduli spaces of parametrised solutions.
There is a well-defined smooth evaluation-map
\[ ev_w : \tilde{M}_{a:b:A}(V; J) \times \partial \dot{D}^w \to V^w, \]
\[ (u, (e^{i\theta_1}, \ldots, e^{i\theta_w})) \mapsto (u(e^{i\theta_1}), \ldots, u(e^{i\theta_w})). \]

In other words, the above moduli spaces \( M_{g,\delta}^{\alpha:b,w:A}(V; M, \nu; J) \) can be identified with an appropriate connected component of a quotient of
\[ ev_w^{-1}(M_{1,\delta} \times \ldots \times M_{w,\delta}) \]
where the quotient identifies two maps \( u \) and \( u' \) that differ by a holomorphic reparametrisation of the domain.

An important feature of \( M_{1,\delta} \times \ldots \times M_{w,\delta} \subset V^w \) is that this submanifold is disjoint from the generalised diagonal
\[ \{(v_1, \ldots, v_w) \in V^w; \; v_i = v_j, \; i \neq j\} \subset V^w. \]
This property simplifies the transversality argument considerably.

### 8.2. Transversality results

The property of being transversely cut out for the moduli-space
\[ M_{\alpha:b,w:A}(V; M, \nu; J) \subset M_{a:b:A}(V; J) \]
can be reformulated into the requirements that
- \( M_{a:b:A}(V; J) \) is transversely cut out, and
- the evaluation map
\[ ev_w : \tilde{M}_{a:b:A}(V; J) \times \partial \dot{D}^w \to V^w \]
is transverse to \( M_{1,\delta} \times \ldots \times M_{w,\delta} \subset V^w \).

Recall that the first property holds for a generic choice of \( J \) by the results in Section 3.2.3.

**Proposition 8.2.** In the case when
\[ ev_w : \tilde{M}_{a:b:A}(V; J) \times \partial \dot{D}^w \to V^w \]
is transverse to \( M_{1,\delta} \times \ldots \times M_{w,\delta} \subset V^w \), it follows that
\[ \dim M_{\alpha:b,w:A}(V; M, \nu; J) = |a| - |b| + \mu(A) - (n - k - 1)w. \]

**Proof.** Recall that, for regular \( J \), we have
\[ \dim M_{a:b:A}(V; J) = |a| - |b| + \mu(A). \]
The domain of \( ev_w \) thus has dimension
\[ \dim(\tilde{M}_{a:b:A}(V; J) \times \partial \dot{D}^w) = |a| - |b| + \mu(A) + w + \eta_m(2 - m), \]
\[ \eta_m := \begin{cases} 0, & m \geq 2, \\ 1, & m < 2, \end{cases} \]
while the codimension of \( M_{1,\delta} \times \ldots \times M_{w,\delta} \subset V^w \) is given by
\[ w(n + 1) - w(k + 1) = w(n - k). \]
The statement now follows. □

We will need the following standard result.

**Lemma 8.3.** Any compactly supported smooth isotopy $\phi^s: V \to V$, where $V \subset (X,\omega)$ is a Lagrangian submanifold, extends to a Hamiltonian isotopy $\phi^s_H$ of $X$. Similarly, any compactly supported smooth isotopy $\phi^s: L \to L$, where $L \subset (Y,\xi)$ is a Legendrian submanifold, extends to a contact isotopy of $Y$.

**Proof.** We begin with the first case. Take a Weinstein neighbourhood of $V$ symplectomorphic to a neighbourhood of the zero-section of the cotagent bundle $(T^*V, d\theta_V)$, under which $V$ moreover is identified with the zero-section. Let $\Gamma_s \subset TV$ be the one-parameter family of vector-fields generating the isotopy $\phi^s$. Consider the time-dependent Hamiltonian

$$H_s: T^*V \to \mathbb{R},$$

$$\eta \mapsto \eta(\Gamma_s).$$

It is easily checked, e.g. in a choice of Darboux coordinates, that the Hamiltonian flow $\phi^s_H$ of $H_s$ is given by $\phi^s$ along the zero-section. This Hamiltonian can then be suitably cut off to generate a Hamiltonian flow on $X$ with the required properties.

The statement about the contact isotopy follow similarly. Since $L$ has a neighbourhood that is contactomorphic to a neighbourhood of the zero-section of the cotangent bundle $(J^1(L), dz + \theta_L)$, one can lift the above Hamiltonian isotopy on $T^*L$ to a (suitably cut off) contact isotopy of $J^1(L)$. It can moreover be checked that this contact isotopy preserves the zero-section of $J^1(L)$. We again obtain the sought contact isotopy by cutting off the induced contact Hamiltonian. □

Let $J$ denote a Banach manifold of compatible almost complex structures on $(\mathbb{R} \times Y, d(e^t\lambda))$ that are cylindrical outside of some compact set. The transversality results in Section 3.2.3 are established via the intermediate result that the so-called universal moduli space

$$\tilde{\mathcal{M}}_{a;b;A}(V;J) \times_J \mathcal{J} \to \mathcal{J}$$

is a Banach manifold, where the projection onto $\mathcal{J}$ is a Fredholm map. Here we have to use a suitable functional analytic set-up, which we omit from the discussion. It then follows from the Sard-Smale theorem that there is a Baire subset of regular almost complex structures in $\mathcal{J}$.

This result also holds in the case when $V = \mathbb{R} \times L$ and $\mathcal{J} = \mathcal{J}_{cyl}$ consists of cylindrical almost complex structures, as follows from the proof of Proposition 3.15. This result is a generalisation of [Dra, Theorem 1.8], which considers the case of moduli-spaces of pseudo-holomorphic maps from a closed Riemann surface into $\mathbb{R} \times Y$.

**Proposition 8.4.** The evaluation map

$$ev_w: \tilde{\mathcal{M}}_{a;b;A}(V;J) \times_J \mathcal{J} \times \partial\tilde{D}^w \to V^w$$

...
from the universal moduli space is transverse to the submanifold $M_{1,\delta} \times \ldots \times M_{w,\delta} \subset V^w$. In the cylindrical setting, the same is true when $J = J_{\text{cyl}}$ is a suitable space of cylindrical almost complex structures.

Proof. We write $u = (a, v)$. Assume that 

$$ev_w(u, J, (e^{i\theta_1}, \ldots, e^{i\theta_w})) \in M_{1,\delta} \times \ldots \times M_{w,\delta}.$$ 

We will show that the differential $D_{\left(u, (e^{i\theta_1}, \ldots, e^{i\theta_w})\right)}$ is a surjection onto the space 

$$W_1 \times \ldots \times W_w \subset T_{ev_w(u, J, (e^{i\theta_1}, \ldots, e^{i\theta_w}))} V^w,$$

defined by

$$T_{u(e^{i\theta_j})} V \supset W_j = \begin{cases} T_{u(e^{i\theta_j})} L_-, & a(e^{i\theta_j}) < A, \\ T_{u(e^{i\theta_j})} V, & A \leq a(e^{i\theta_j}) \leq B, \\ T_{u(e^{i\theta_j})} L_+, & a(e^{i\theta_j}) > B. \end{cases}$$

Since each $M_{j,\delta} \subset \mathbb{R} \times Y$ is invariant under translations of the $t$-coordinate outside of the set $[A, B] \times Y$, this will imply the claim.

Recall that $ev_w(u, J, (e^{i\theta_1}, \ldots, e^{i\theta_w}))$ is disjoint from the generalised diagonal 

$$\{(v_1, \ldots, v_w) \in V^w; v_i = v_j, i \neq j\} \subset V^w$$

by the construction of $M_{j,\delta}$. The result can now be established by the following standard argument (see e.g. [MS, Section 3.4]).

Take any vector 

$$(\zeta_1, \ldots, \zeta_w) \in W_1 \times \ldots \times W_w,$$

and write $(x_1, \ldots, x_w) := ev_w(u, J, (e^{i\theta_1}, \ldots, e^{i\theta_w}))$. Using Lemma 8.3 it follows that there is a Hamiltonian isotopy $\phi^s$ of $\mathbb{R} \times Y$ satisfying $\phi^s(V) = V$ as well as 

$$\frac{d}{ds} (\phi^s(x_1), \ldots, \phi^s(x_w)) = (\zeta_1, \ldots, \zeta_w).$$

Furthermore, outside of the set $[A, B] \times Y$, this Hamiltonian may be taken to be of the form $\phi^s = (\text{id}_{\mathbb{R}}, \tilde{\phi}^s)$, where $\tilde{\phi}^s$ is a contact isotopy.

Observe that $\phi^s$ induces a one-parameter family 

$$s \mapsto (\phi^s \circ u, (D\phi^s) \circ J \circ (D\phi^s)^{-1}) \in \widetilde{M}_{a:b;A}(V; J) \times J$$

of solutions in the universal moduli space. Finally, differentiating $ev_w$ along this curve can be seen to give the vector $(\zeta_1, \ldots, \zeta_w)$ as required. □

Using the Sard-Smale theorem for Banach manifolds, it follows that there is a Baire subset of compatible almost complex structures $J \in J$ for which the moduli space $M_{a:b;A}(V; M, v; J)$ is transversely cut-out. Together with the Gromov-Hofer compactness in [BEH + ] (also see Section 3.2.2) for the moduli spaces $M_{a:b;A}(V; J)$, we conclude that
Theorem 8.5. There is a Baire subset of $J \in \mathcal{J}$ for which the moduli space $\mathcal{M}_{g,\delta}^{a:b:w;A}(V;M,v;J)$ is transversely cut-out, and for which the compactification of

$$\mathcal{M}_{g,\delta}^{a:b:w;A}(V;M,v;J) \subset \overline{\mathcal{M}}_{g,\delta}^{a:b;A}(V;J)$$

is transverse to the boundary. In the cylindrical setting, the same is true when $\mathcal{J} = \mathcal{J}_{\text{cyl}}$ consists of cylindrical almost complex structures.

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