ON FOLDED CLUSTER PATTERNS OF AFFINE TYPE

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Abstract. A cluster algebra is a commutative algebra whose structure is decided by a skew-symmetrizable matrix or a quiver. When a skew-symmetrizable matrix is invariant under an action of a finite group and this action is admissible, the folded cluster algebra is obtained from the original one. Any cluster algebra of non-simply-laced affine type can be obtained by folding a cluster algebra of simply-laced affine type with a specific G-action. In this paper, we study the combinatorial properties of quivers in the cluster algebra of affine type. We prove that for any quiver of simply-laced affine type, G-invariance and G-admissibility are equivalent. This leads us to prove that the set of G-invariant seeds forms the folded cluster pattern.

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1. Introduction

Cluster algebras are commutative algebras introduced and studied by Fomin–Zelevinsky and Berenstein–Fomin–Zelevinsky in a series of articles [13, 14, 4, 16]. They were invented in the context of total positivity and dual canonical bases in Lie theory; since then, connections and applications have been discovered in diverse areas of mathematics. A cluster algebra is a commutative algebra with certain generators, called cluster variables, defined recursively. The recursive structure of cluster algebra is encoded in the combinatorial datum of an exchange matrix, which is a skew-symmetrizable integer matrix (see Definition 2.1). More precisely, a cluster algebra is defined by a bunch of seeds and each seed consists of cluster variables and an exchange matrix. The structure of this cluster (called the seed pattern) is decided recursively via an operation (called the mutations) given by the exchange matrix in each seed (see Section 2).

A cluster algebra is said to be of finite type if the cluster pattern has only a finite number of seeds. Fomin and Zelevinsky [14] showed that the cluster algebras of finite type can be classified in terms of the Dynkin diagrams of finite-dimensional simple Lie algebras. A wider class of cluster algebras consists of cluster algebras of finite mutation type, which have finitely many exchange matrices but are allowed to have infinitely many seeds. Felikson, Shapiro, and Tumarkin proved...
in [10] that a skew-symmetric cluster algebra of rank \( n \) is finite mutation type if and only if \( n \leq 2 \); or it is of surface type; or it is one of 11 exceptional types:

\[
E_6, E_7, E_8, \quad E_6, E_7, E_8, \quad E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7.
\]

Here, we notice that cluster algebras of surface type can be of other remaining simply-laced Dynkin type: \( A_n, D_n, A_{p,q}, D_n \) as provided in [11, Table 1].

A skew-symmetric matrix can be considered as the adjacency matrix of a finite directed multi-graph that does not have directed cycles of length at most 2. We call such directed graph a quiver (see Figure 1 for examples/non-examples of quivers). To study skew-symmetric cluster algebras of finite mutation type, Felikson, Shapiro, and Tumarkin used the folding and unfolding procedures of cluster algebras in [9]. Indeed, they consider a certain symmetry on the quivers and their quotients which leads to prove that skew-symmetrizable cluster algebras correspond to the non-simply-laced Dynkin diagrams are of finite mutation type by folding simply-laced Dynkin diagrams. We present in Table 1 how simply-laced affine Dynkin diagrams and non-simply-laced affine Dynkin diagrams are related (also, see figures in Appendix A). Folding procedure produces all non-simply-laced affine Dynkin diagrams using simply-laced affine Dynkin diagrams.

| X | \( \tilde{A}_{2,2} \) | \( \tilde{A}_{n,n} \) | \( \tilde{D}_1 \) | \( \tilde{D}_n \) | \( \tilde{D}_{2n} \) | \( \tilde{E}_6 \) | \( \tilde{E}_7 \) |
|---|---|---|---|---|---|---|
| G | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| Y | \( \tilde{A}_1 \) | \( D_{n+1}^{(2)} \) | \( A_n^{(2)} \) | \( D_4^{(3)} \) | \( \tilde{C}_{n-2} \) | \( \tilde{B}_n \) | \( A_n^{(2)} \) |

| G | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| Y | \( \tilde{A}_1 \) | \( D_{n+1}^{(2)} \) | \( A_n^{(2)} \) | \( D_4^{(3)} \) | \( \tilde{C}_{n-2} \) | \( \tilde{B}_n \) | \( A_n^{(2)} \) |

Table 1. Foldings appearing in affine Dynkin diagrams. For \((X, G, Y)\) in each column, the quiver of type \( X \) is globally foldable with respect to \( G \), and the corresponding folded cluster pattern is of type \( Y \).

In this paper, we investigate combinatorial properties of quivers in the seed pattern of affine type. To state our main theorem, we prepare some terminologies. We say two quivers are mutation equivalent if one can be obtained from the other by applying finitely many mutations. For a simply-laced Dynkin type \( X \), a quiver is of type \( X \) if it is mutation equivalent to a quiver whose underlying graph is the Dynkin diagram of \( X \). For a finite group \( G \) acting on the set of vertices of a quiver \( Q \), the quiver \( Q \) is \( G \)-invariant if for any \( g \in G \), the quiver \( Q \) is isomorphic to \( g \cdot Q \) as a directed graph. A \( G \)-invariant quiver \( Q \) is \( G \)-admissible if for any two vertices \( i \) and \( i' \) in the same \( G \)-orbit, there is no arrow connecting \( i \) and \( i' \) and whenever there is an arrow \( i \rightarrow j \) (respectively, \( j \rightarrow i \)), we should have \( i' \rightarrow j \) (respectively, \( j \rightarrow i' \)). See Section 3 for more precise definitions. In general, a \( G \)-invariant quiver might not be \( G \)-admissible as explained in Example 3.6. Nevertheless, when we concentrate on quivers of affine type, the \( G \)-invariance ensures the \( G \)-admissibility, which is the main result of the paper.

**Theorem 1.1** (Theorem 3.7). Let \((X, G, Y)\) be a triple given by a column of Table 1. Let \( Q \) be a quiver of type \( X \). If \( Q \) is \( G \)-invariant, then it is \( G \)-admissible. Indeed, \( G \)-invariance and \( G \)-admissibility are equivalent.

The proof of the theorem is provided by type-by-type arguments and we make great use of the fact that the corresponding cluster algebra is of finite mutation type. Because the cluster algebras of affine type are of finite mutation type as we already mentioned, using the computer program SageMath [21], one can get the same result as Theorem 1.1 for quivers of type \( E \) or type \( A_{n,n}, D_n \) for a given \( n \), which is an experimental proof. On the other hand, we provide a combinatorial proof by observing the combinatorics of quivers.

We also study an application of Theorem 1.1 to the folded cluster pattern. Let \( F \) be the rational function field with \( n \) algebraically independent variables over \( \mathbb{C} \). Suppose that a finite group \( G \) acts on \( [n] := \{1, \ldots, n\} \). Let \( F^G \) be the field of rational functions in \( \#([n]/G) \) independent variables and \( \psi: F \rightarrow F^G \) be a surjective homomorphism. A seed \( \Sigma = (x, Q) \), which is a pair of variables
\( \mathbf{x} = (x_1, \ldots, x_n) \) in \( \mathbb{F} \) and a quiver \( Q \) on \([n]\), is called \((G, \psi)\)-invariant or \((G, \psi)\)-admissible if for any indices \( i \) and \( i' \) in the same \( G \)-orbit, we have \( \psi(x_i) = \psi(x_{i'}) \) and \( Q \) is \( G \)-invariant or \( G \)-admissible, respectively.

For a \((G, \psi)\)-admissible seed \( \Sigma \), if the admissibility is preserved under orbit mutations, then one can fold the seed \( \Sigma \), which will be denoted by \( \Sigma^G \). Here, an orbit mutation is a composition of mutations for all vertices in the same \( G \)-orbit. A cluster pattern given by the folded seed \( \Sigma^G \) can be identified with the set of \((G, \psi)\)-admissible seeds in the original cluster pattern given by \( \Sigma \). Indeed, the folded cluster pattern consists of seeds defined recursively via a sequence of orbit mutations. We prove that the set of \((G, \psi)\)-invariant seeds forms the folded cluster pattern.

**Corollary 1.2** (Corollary 5.3). Let \((X, G, Y)\) be a triple given by a column of Table 1. Let \( \Sigma_{t_0} = (x, Q) \) be a seed. Suppose that \( Q \) is of type \( X \). Define \( \psi: \mathbb{F} \to \mathbb{F}^G \) so that \( \Sigma_{t_0} \) is a \((G, \psi)\)-admissible seed. Then, any \((G, \psi)\)-invariant seed can be reached by a sequence of orbit mutations from \( \Sigma_{t_0} \). Moreover, the set of \((G, \psi)\)-invariant seeds forms the ‘folded’ cluster pattern given by \( \Sigma^G_{t_0} \) of \( Y \) via folding.

This provides an answer to the question in [8, Problem 9.5] which asks whether any \((G, \psi)\)-invariant seed can be reached by sequences of orbit mutations from the initial seed for the case of cluster algebras of affine type (see Remark 5.4). Moreover, this result is useful when studying Lagrangian fillings of Legendrians of affine type as exhibited in the forthcoming paper An–Bae–Lee [1].

The paper is organized as follows. In Section 2, we review the definition of cluster algebras and mutations. In Section 3, we consider the \( G \)-invariance and \( G \)-admissibility of quivers. In Section 4, we present the proof of the main theorem by analyzing each type of quiver. In Section 5, we provide an application of the main theorem by considering the folded version of cluster algebras and cluster patterns. We describe finite group actions on quivers of affine Dynkin type in Appendix A.

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## 2. Preliminaries: cluster algebras

Cluster algebras, introduced by Fomin and Zelevinsky [13], are commutative algebras with specific generators, called *cluster variables*, defined recursively. In this section, we recall basic notions in the theory of cluster algebras. For more details, we refer the reader to [13, 14].

Throughout this section, we fix \( n \in \mathbb{Z}_{>0} \) and we let \( \mathbb{F} \) be the rational function field with \( n \) algebraically independent variables over \( \mathbb{C} \). We also denote the set \( \{1, \ldots, n\} \) by simply \([n]\).

**Definition 2.1** (cf. [13, 14]). A seed \( \Sigma = (\mathbf{x}, B) \) is a pair of

- a tuple \( \mathbf{x} = (x_1, \ldots, x_n) \) of algebraically independent generators of \( \mathbb{F} \), that is, \( \mathbb{F} = \mathbb{C}(x_1, \ldots, x_n) \);
- an \( n \times n \) skew-symmetrizable integer matrix \( B = (b_{i,j}) \), that is, there exist positive integers \( d_1, \ldots, d_n \) such that \( \text{diag}(d_1, \ldots, d_n) \cdot B \) is a skew-symmetric matrix.

We call elements \( x_1, \ldots, x_n \) *cluster variables* and call \( B \) *exchange matrix*.

In general, cluster variables consist of *unfrozen* (or *mutable*) variables and *frozen* variables but we assume the following.

**Assumption 2.2.** Throughout this paper, we assume that all cluster variables are mutable.

A finite directed multigraph \( Q \) with the set \([n]\) of vertices is called a *quiver* on \([n]\) if it does not have directed cycles of length at most 2. In Figure 1, we provide examples/non-examples of quivers. The left directed graph in Figure 1(2) is not a quiver because of the one-cycle on the vertex 2; neither is the right one because it has a directed two-cycle connecting vertices 1 and 2.
The adjacency matrix of a quiver is always a skew-symmetric matrix. To define cluster algebras, we introduce mutations on seeds, exchange matrices, and quivers as follows.

**Definition 2.3.** The mutation on seeds, exchange matrices, or quivers is defined as follows.

1. **(Mutation on seeds)** For a seed \( \Sigma = (x, B) \) and an integer \( k \in [n] \), the *mutation* \( \mu_k(\Sigma) = (x', B') \) is defined as follows:

   \[
   x_i' = \begin{cases} 
   x_i & \text{if } i \neq k; \\
   x_k^{-1} \left( \prod_{j, k > 0} x_j^{b_{j,k}} + \prod_{j, k < 0} x_j^{-b_{j,k}} \right) & \text{otherwise.}
   \end{cases}
   \]

   \[
   b_{i,j}' = \begin{cases} 
   -b_{i,j} & \text{if } i = k \text{ or } j = k; \\
   b_{i,j} + \frac{|b_{i,k}|b_{k,j} + b_{i,k}|b_{k,j}|}{2} & \text{otherwise.}
   \end{cases}
   \]

2. **(Mutation on exchange matrices)** We define \( \mu_k(B) = (b_{i,j}') \), and say that \( B' = (b_{i,j}') \) is the *mutation of \( B \) at \( k \).

3. **(Mutation on quivers)** Let \( Q \) be a quiver on \( [n] \) and \( B(Q) \) its adjacency matrix. For each \( k \in [n] \), the mutation \( \mu_k(B(Q)) \) is again the adjacency matrix of a quiver \( Q' \). We define \( \mu_k(Q) \) is a quiver satisfying

   \[
   B(\mu_k(Q)) = \mu_k(B(Q))
   \]

   and say that \( \mu_k(Q) \) is the *mutation of \( Q \) at \( k \).

An immediate check shows that \( \mu_k(\Sigma) \) is again a seed, and a mutation is an involution, that is, its square is the identity. For a skew-symmetrizable matrix \( B \) of size \( n \times n \), and for \( k \in [n] \), we have \( \mu_k(B^T) = (\mu_k(B))^T \) by the definition of mutations. Therefore, the mutation preserves the skew-symmetry. Because of (2.1), we sometimes denote a seed by

   \[
   \Sigma = (x, Q) = (x, B(Q)).
   \]

**Example 2.4.** Let \( Q \) be a quiver on the left side of Figure 1(1). The adjacency matrix \( B \) of \( Q \) and the mutation \( \mu_3(B) \) are given by

\[
B = B(Q) = \begin{bmatrix}
0 & -1 & 1 & 1 \\
1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0
\end{bmatrix}, \quad \mu_3(B) = \begin{bmatrix}
0 & 0 & -1 & 2 \\
0 & 0 & 1 & -1 \\
1 & -1 & 0 & -1 \\
-2 & 1 & 1 & 0
\end{bmatrix},
\]

which produces the quiver \( \mu_3(Q) \):

![Diagram](image)

One can easily check that \( \mu_4\mu_3(Q) \) becomes the quiver on the right side of Figure 1(1).
Remark 2.5. Let $k$ be a vertex in a quiver $Q$. The mutation $\mu_k(Q)$ can also be described via a sequence of three steps:

1. For each directed two-arrow path $i \rightarrow k \rightarrow j$, add a new arrow $i \rightarrow j$.
2. Reverse the direction of all arrows incident to the vertex $k$.
3. Repeatedly remove directed 2-cycles until unable to do so.

Let $T_n$ denote the $n$-regular tree whose edges are labeled by $1, \ldots, n$. Except for $n = 1$, there are infinitely many vertices on the tree $T_n$. A cluster pattern (or seed pattern) is an assignment

$$T_n \rightarrow \{\text{seeds in } F\}, \ t \mapsto \Sigma_t = (x_t, B_t)$$

such that if $t \xrightarrow{k} t'$ in $T_n$, then $\mu_k(\Sigma_t) = \Sigma_{t'}$.

Definition 2.6 (cf. [14]). Let $\{\Sigma_t = (x_t, B_t)\}_{t \in T_n}$ be a cluster pattern with $x_t = (x_{1:t}, \ldots, x_{n:t})$. The cluster algebra $A(\{\Sigma_t\}_{t \in T_n})$ is defined to be the $C$-subalgebra of $F$ generated by all the cluster variables $\bigcup_{t \in T_n} \{x_{1:t}, \ldots, x_{n:t}\}$.

If we fix a vertex $t_0 \in T_n$, then a cluster pattern $\{\Sigma_t\}_{t \in T_n}$ is constructed from the seed $\Sigma_{t_0}$. In this case, we call $\Sigma_{t_0}$ an initial seed. Moreover, up to isomorphism on $F$, a cluster algebra depends only on $B_{t_0}$ in the initial seed $\Sigma_{t_0}$. Because of this reason, we simply denote by $A(B_{t_0})$ the cluster algebra given by the cluster pattern constructed from the initial seed $\Sigma_{t_0} = (x_{t_0}, B_{t_0})$. Moreover, when $B_{t_0} = B(Q_{t_0})$ for a quiver $Q_{t_0}$, we denote by $A(Q_{t_0})$ the cluster algebra $A(B_{t_0})$.

We say that a skew-symmetrizable matrix $B'$ is mutation equivalent to another skew-symmetrizable matrix $B$ if they are connected by a sequence of mutations

$$B' = (\mu_{j_1} \cdots \mu_{j_{\ell}})(B),$$

and say that $B$ is acyclic if there are no sequences $j_1, j_2, \ldots, j_{\ell}$ with $\ell \geq 3$ such that

$$b_{j_1 j_2}, b_{j_2 j_3}, \ldots, b_{j_{\ell-1} j_{\ell}}, b_{j_{\ell} j_1}, > 0.$$

Similarly, we say that a quiver $Q'$ is mutation equivalent to another quiver $Q$ if $B(Q')$ is mutation equivalent to $B(Q)$, and say that $Q$ is acyclic if so is $B(Q)$, which is also equivalent to that $Q$ has no directed cycles.

The Cartan counterpart $C(B) = (c_{i,j})$ of $B$ is defined by

$$c_{i,j} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{i,j}| & \text{if } i \neq j. \end{cases}$$

Definition 2.7. For a Dynkin type $X$, we define a quiver $Q$ or a matrix $B$ of type $X$ as follows.

1. For a quiver $Q$, we say that $Q$ is of type $X$ if it is mutation equivalent to an acyclic quiver whose underlying graph is isomorphic to the Dynkin diagram of type $X$.

2. For a skew-symmetrizable matrix $B$, we say $B$ is of type $X$ if it is mutation equivalent to an acyclic skew-symmetrizable matrix whose Cartan counterpart $C(B)$ is isomorphic to the Cartan matrix of type $X$.

Here, we say that two matrices $C_1$ and $C_2$ are isomorphic if they are conjugate to each other via a permutation matrix, that is, $C_2 = P^{-1}C_1P$ for some permutation matrix $P$. It is proved in [7, Corollary 4] that if two acyclic skew-symmetrizable matrices are mutation equivalent, then there exists a sequence of mutations from one to other such that intermediate skew-symmetrizable matrices are all acyclic. Indeed, if two acyclic skew-symmetrizable matrices are mutation equivalent, then their Cartan counterparts are isomorphic. Accordingly, a quiver or a matrix of type $X$ is well-defined.

On the other hand, all Dynkin diagrams of finite or affine type but $\tilde{A}_{n-1}$ are acyclic and therefore the acyclicity in Definition 2.7(1) can be omitted for all $X$ but $\tilde{A}_{n-1}$. Here is one of the reason why the acyclicity is needed for $\tilde{A}_{n-1}$ as follows: Let $Q$ be a quiver of $n \geq 3$ vertices whose underlying graph is isomorphic to the $n$-cycle $\tilde{A}_{n-1}$ and so we have to say that $Q$ is of type $\tilde{A}_{n-1}$ unless the acyclicity. However, it is known from [22, Type IV] that the quiver $Q$ is of type $D_n$ when $Q$ is a directed $n$-cycle.
Even for acyclic quivers $Q$ of $\widetilde{\mathbb{A}}_{n-1}$, we have finer separation. Recall from [11, Lemma 6.8] that the mutation equivalence class of $Q$ does depend on the orientation of the edges in the quiver. More precisely, let $Q$ and $Q'$ be of type $\widetilde{\mathbb{A}}_{n-1}$. Suppose that in $Q$, there are $p$ edges of one direction and $q = n - p$ edges of the opposite direction. Also, in $Q'$, there are $p'$ edges of one direction and $q' = n - p'$ edges of the opposite direction. Then two quivers $Q$ and $Q'$ are mutation equivalent if and only if the unordered pairs $\{p, q\}$ and $\{p', q'\}$ coincide.

We say that a quiver $Q$ is of type $\widetilde{\mathbb{A}}_{p,q}$ if $Q$ is mutation equivalent to the quiver of type $\widetilde{\mathbb{A}}_{p+q}$ with $p$ edges of one direction and $q$ edges of the opposite direction. We depict some examples for quivers of type $\widetilde{\mathbb{A}}_{p,q}$ in Figure 2.

In Tables 2 and 3, we present lists of standard affine root systems and twisted affine root systems, respectively. They are the same as presented in Tables Aff 1, Aff2, and Aff 3 of [18, Chapter 4], and we denote by $\widetilde{X} = X^{(1)}$. We notice that the number of vertices of the Dynkin diagram of type $X_{n-1}$ is $n$ while we do not specify the vertex numbering.

For a Dynkin type $X$, we say that $X$ is simply-laced if its Dynkin diagram has only single edges, otherwise, $X$ is non-simply-laced. Recall that the Cartan matrix associated to a Dynkin diagram $X$ can be read directly from the diagram $X$ as follows:

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix}
\]  \hspace{1cm} \text{(2.2)}

Therefore, for each non-simply-laced Dynkin diagram $X$, any exchange matrix $B$ of type $X$ is not skew-symmetric but skew-symmetrizable. Hence it never come from any quiver.

One of the beauties of a skew-symmetrizable matrix of Dynkin type is that they are used to classify cluster algebras of finite type or finite mutation type. A cluster algebra is said to be of finite type if the cluster pattern has only a finite number of seeds. A wider class of cluster algebras consists of cluster algebras of finite mutation type, which have finitely many exchange matrices but are allowed to have infinitely many seeds.

**Theorem 2.8** ([14] for finite Dynkin type; [11, 10] for affine Dynkin type). Let $\Sigma_{t_0} = (x_{t_0}, B_{t_0})$ be an initial seed.

1. The exchange matrix $B_{t_0}$ is of finite Dynkin type if and only if there are only finitely many seeds in the cluster pattern $\{\Sigma_t\}_{t \in \mathbb{T}_n}$. 

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix}
\]
Table 2. Dynkin diagrams of standard affine root systems

Table 3. Dynkin diagrams of twisted affine root systems

(2) If the exchange matrix $\mathcal{B}_{t_0}$ is of affine Dynkin type, then there are only finitely many exchange matrices in the cluster pattern $\{\Sigma_t\}_{t \in T_n}$ while there might be infinitely many seeds.

Assumption 2.9. Throughout this paper, we consider the case where $\mathcal{B}$ is an acyclic matrix of affine type.
Remark 2.10. Let $B$ be a skew-symmetrizable matrix of size $n \times n$. We have already seen that $\mu_k(B^T) = \mu_k(B)^T$ for $k \in [n]$. Accordingly, for a skew-symmetrizable matrix $B$ of affine type, there is a bijective correspondence between the set of exchange matrices in the cluster pattern $\{\Sigma_t\}_{t \in \tau_n}$ given by $\Sigma_{t_0} = (x_{t_0}, B)$ and that in the cluster pattern $\{\Sigma'_t\}_{t \in \tau_n}$ given by $\Sigma'_{t_0} = (x'_{t_0}, B^T)$. We present pairs $(X, X')$ of affine Dynkin type whose Cartan matrices are transposed to each other.

$$(\tilde{A}_{n-1}, \tilde{A}_{n-1}), \quad (\tilde{B}_{n-1}, A_2^{(2)}(2n_{n-1}), 1), \quad (\tilde{C}_{n-1}, D_n^{(2)}), \quad (\tilde{D}_{n-1}, \tilde{D}_{n-1}),$$

$$(\tilde{E}_n, \tilde{E}_n) \text{ for } n = 6, 7, 8, \quad (\tilde{F}_4, E_6^{(2)}), \quad (G_2, D_4^{(3)}), \quad (A_2^{(2)}, A_2^{(2)}).$$

Remark 2.11. The number of isomorphism classes of exchange matrices in the cluster pattern $\{\Sigma_t\}_{t \in \tau_n}$ of type $\tilde{A}_{p,q}$ is provided in [3]:

$$\begin{align*}
\begin{cases}
\frac{1}{2} \sum_{k|p,k\neq q} \phi(k) \left( \frac{2p/k}{p/k} \right) \left( \frac{2q/k}{q/k} \right) & \text{if } p \neq q; \\
\frac{1}{2} \left( \frac{2p}{p} \right) + \sum_{k|p} \phi(k) \left( \frac{2p/k}{p/k} \right)^2 & \text{if } p = q,
\end{cases}
\end{align*}$$

where $\phi(k)$ is the number of $1 \leq d \leq k$ coprime to $k$, called Euler’s totient function. Moreover, the following table is obtained from [19, Theorem 4.15].

| Dynkin type $X$ | $\tilde{E}_6$ | $\tilde{E}_7$ | $\tilde{E}_8$ | $\tilde{F}_4$ or $E_6^{(2)}$ | $G_2$ or $D_4^{(3)}$ |
|-----------------|---------------|---------------|---------------|------------------|------------------|
| $\# \text{ of exchange matrices}$ | 130 | 1080 | 7660 | 60 | 6 |

The number of exchange matrices in the cluster pattern of other affine Dynkin type is conjectured in [19, Conjecture 4.14] which is still an open problem to the authors’ knowledge.

For the sake of convenience, we define the restriction of a seed $\Sigma = (x, Q)$ as follows. Suppose that $Q$ is a quiver on $[n]$. For each subset $I = \{i_1, \ldots, i_k\} \subset [n]$ with $i_1 < i_2 < \cdots < i_k$, the restriction $\Sigma|_I$ is defined as the pair $\Sigma|_I = (x|_I, Q|_I)$, where $x|_I = (x_{i_1}, \ldots, x_{i_k})$ and $Q|_I$ is the induced subquiver with the set $I$ of vertices. Also, we denote by $B|_I$ the submatrix of $B$ obtained by considering the columns and rows in $I$ simultaneously, that is, $B|_I = (b_{i,j})_{i,j \in I}$. The following is an example of restriction of $Q$ of type $\tilde{D}_{2n}$,

$$Q|_{\{4, \ldots, 2n-2\}} =$$

which is of type $A_{2n-5}$.

We enclose this section by recalling the following result for later use.

Lemma 2.12 ([20] and also see [1, Theorem 2.12]). Let $\Sigma_{t_0} = (x_{t_0}, Q_{t_0})$ be an initial seed with a quiver $Q_{t_0}$ on $[n]$. Suppose that $Q_{t_0}$ is an acyclic quiver of affine type. Then for any seed $\Sigma = (x, Q)$ in the cluster pattern given by the initial seed $\Sigma_{t_0}$, there exists an index $i \in [n]$ such that the quiver $Q$ is obtained from $Q_{t_0}$ by applying mutations on vertices $[n] \setminus \{i\}$.

3. Invariance and admissibility of quivers

Under certain condition, one can fold quivers to produce new ones. This procedure is used to study quivers of non-simply-laced affine type from those of simply-laced affine type. In this section, we recall from [12] the invariance and admissibility of a finite group action on the quiver. We also refer the reader to [8].
Let $Q$ be a quiver on $[n]$ and let $G$ be a finite group acting on the set $[n]$. For $i, i' \in [n]$, the notation $i \sim i'$ will mean that $i$ and $i'$ lie in the same $G$-orbit. To study folding of exchange matrices or quivers, we prepare some terminologies.

For each $g \in G$, let $Q' = g \cdot Q$ be the quiver whose adjacency matrix $B(Q')$ is given by

$$B(Q') = (b'_{ij}), \quad b'_{ij} = b_{g(i), g(j)}.$$

**Definition 3.1** (cf. [12, §4.4] and [8, §3]). Let $Q$ be a quiver on $[n]$ and let $G$ be a finite group acting on the set $[n]$.

1. A quiver $Q$ is $G$-invariant if $g \cdot Q = Q$ for each $g \in G$.
2. A $G$-invariant quiver $Q$ is $G$-admissible if for any $i \sim i'$,
   a. $b_{i,i'} = 0$;
   b. $b_{i,j}b_{i',j} \geq 0$ for any $j$, where $B(Q) = (b_{i,j})$.

**Remark 3.2.** The $G$-admissibility can also be defined for an exchange matrix and a seed, and furthermore those with frozen vertices. Note that this definition is simplified due to Assumption 2.2.

For a $G$-admissible quiver $Q$, we define the matrix $B^G = B(Q)^G = (b^G_{I,J})$ whose rows and columns are labeled by the $G$-orbits by

$$b^G_{I,J} = \sum_{i \in I} b_{i,j},$$

where $I$ and $J$ are $G$-orbits and $j$ is an arbitrary index in $J$. We then say $B^G$ is obtained from $B$ (or from the quiver $Q$) by folding with respect to the given $G$-action.

**Example 3.3.** Let $Q$ be a quiver of type $\tilde{A}_{2,2}$ given as follows:

$$\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \xrightarrow{\mu_1} 
\begin{array}{c}
1 \\
2 \\
3 =: Q \\
4
\end{array}$$

The adjacency matrix $B(Q)$ of $Q$ is

$$B(Q) = \begin{bmatrix}
0 & -1 & -1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & -1 & -1 & 0
\end{bmatrix}.$$

Suppose that the finite group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the set $[4] = \{1, \ldots, 4\}$ such that the generator sends $1 \mapsto 4 \mapsto 1$ and $2 \mapsto 3 \mapsto 2$. Then, the quiver $Q$ is $G$-admissible, and by setting $I_1 = \{1, 4\}$ and $I_2 = \{2, 3\}$, we obtain

$$b^G_{I_1,I_2} = \sum_{i \in I_1} b_{i,2} = b_{1,2} + b_{4,2} = -2,$$

$$b^G_{I_2,I_1} = \sum_{i \in I_2} b_{i,1} = b_{2,1} + b_{3,1} = 2.$$

This provides

$$B^G = \begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix},$$

whose Cartan counterpart is the Cartan matrix of type $\tilde{A}_1$, and moreover, it is the adjacency matrix of the quiver $\begin{array}{c}
1 \\
\downarrow
\end{array} \longleftrightarrow \begin{array}{c}
2
\end{array}$.

In Example 3.3, the folded matrix $B^G$ is again skew-symmetric. However, as we will see in the example below, the folded matrix is not skew-symmetric but skew-symmetrizable in general.
Example 3.4. Let $Q$ be a quiver of type $\widetilde{E}_6$ whose adjacency matrix $B(Q)$ is given as follows.

$$B(Q) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$  

Suppose that the finite group $G = \mathbb{Z}/3\mathbb{Z}$ acts on the set $[7] = \{1, \ldots, 7\}$ as depicted in (9) of Appendix A. One may check that the quiver $Q$ is $G$-admissible. By setting $I_1 = \{1\}$, $I_2 = \{2, 4, 6\}$, and $I_3 = \{3, 5, 7\}$, we obtain

$$b_{1,2}^G = \sum_{i \in I_1} b_{i,2} = b_{1,2} = 1,$$
$$b_{1,3}^G = \sum_{i \in I_1} b_{i,3} = b_{1,3} = 0,$$
$$b_{2,3}^G = \sum_{i \in I_2} b_{i,3} = b_{2,3} + b_{4,3} + b_{6,3} = -1,$$
$$b_{2,1}^G = \sum_{i \in I_2} b_{i,1} = b_{2,1} + b_{4,1} + b_{6,1} = -3,$$
$$b_{3,1}^G = \sum_{i \in I_3} b_{i,1} = b_{3,1} + b_{5,1} + b_{7,1} = 0,$$
$$b_{3,2}^G = \sum_{i \in I_3} b_{i,2} = b_{3,2} + b_{5,2} + b_{7,2} = 1.$$ 

Accordingly, we obtain the matrix

$$B^G = \begin{bmatrix}
0 & 1 & 0 \\
-3 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}$$

whose Cartan counterpart is isomorphic to the Cartan matrix of type $\widetilde{G}_2$ (cf. (2.2)).

For a $G$-admissible quiver $Q$ and a $G$-orbit $I$, we consider a composition of mutations given by

$$\mu_I = \prod_{i \in I} \mu_i$$

which is well-defined because of the definition of admissible quivers. We call $\mu_I$ an orbit mutation. If $\mu_I(Q)$ is again $G$-admissible, then we have that

$$(\mu_I(B))^G = \mu_I(B^G).$$

We notice that the quiver $\mu_I(Q)$ may not be $G$-admissible in general (cf. Example 3.6). Therefore, we present the following definition.

Definition 3.5. Let $G$ be a group acting on the vertex set of a quiver $Q$. We say that $Q$ is globally foldable with respect to $G$ if $Q$ is $G$-admissible, and moreover, for any sequence of $G$-orbits $I_1, \ldots, I_t$, the quiver $(\mu_{I_t} \cdots \mu_{I_1})(Q)$ is $G$-admissible.

Example 3.6. Let $Q$ be a quiver with 6 vertices given as follows.

![Quiver Diagram]
Consider an action of $G = \mathbb{Z}/2\mathbb{Z}$ such that $1 \sim 4$, $2 \sim 5$, and $3 \sim 6$. One can easily see that the quiver $Q$ is $G$-invariant, and moreover, is $G$-admissible. However, by considering mutations on vertices 1 and 4, we obtain the quiver $\mu_1 \mu_4(Q)$ which is $G$-invariant but not $G$-admissible. This is because for indices $2 \sim 5$ and $3$, we have $b_{2,3} b_{5,3} = 1 \cdot (-1) = -1 \not\geq 0$ which violates the condition (b) in Definition 3.1(2). Accordingly, the quiver $Q$ is $G$-admissible but not globally foldable with respect to $G$.

As we saw in Example 3.6, a $G$-invariant quiver may not be $G$-admissible. The following theorem says that the converse holds when we consider the folding $s$ presented in Table 1. In Appendix A, we describe the finite group action explicitly for each triple $(X, G, Y)$. The proof of the following theorem will be given in Section 4.

**Theorem 3.7.** Let $(X, G, Y)$ be a triple given by a column of Table 1. Let $Q$ be a quiver of type $X$. If $Q$ is $G$-invariant, then it is $G$-admissible. Indeed, $G$-invariance and $G$-admissibility are equivalent.

As a direct consequence of the above theorem, we obtain the following.

**Corollary 3.8.** Let $(X, G, Y)$ be a triple given by a column of Table 1. Then any quiver of type $X$ is globally foldable with respect to $G$.

**Remark 3.9.** Because the cluster algebras of affine type is finite mutation type (see Theorem 2.8(2)), using the computer program SageMath [21], one can get the same result as Theorem 3.7 for quivers of type $\tilde{E}$ or type $\tilde{A}_{n,n}, \tilde{D}_n$ for a given $n$. More precisely, the command `mutation_class` produces all quivers which are mutation equivalent to a given one. For more details, we refer the reader to [19, §4.4]. Using this command, one may provide an experimental proof while we provide a combinatorial proof by observing the combinatorics of quivers.

### 4. Type-by-type arguments for admissibility

In this section, we will prove Theorem 3.7. We say that a quiver $Q$ is of finite mutation type (or is mutation-finite) if there is only finitely many quivers mutation equivalent to $Q$. Otherwise, we say that $Q$ is of infinite mutation type (or is mutation-infinite). As we have already seen in Theorem 2.8, a quiver of affine or finite Dynkin type is mutation-finite.

Before providing a proof, we study some mutation-infinite quivers. We first recall the following lemma:

**Lemma 4.1** ([10, Lemma 6.4]). Let $Q$ be a quiver on $[n]$ and let $Q_0 = Q|_I$ be the restriction onto a subset $I \subset [n]$. Then, for any quiver $Q_1$ mutation equivalent to $Q_0$, there exists $Q'$ mutation equivalent to $Q$ such that $Q'|_I = Q_1$.

We say that a quiver $Q$ is reduced to $Q'$ if $Q'$ is obtained by applying a sequence of mutations on $Q$ and restrictions, denoted by $Q \succ Q'$.

Then the lemma below is the direct consequence of the definition and Lemma 4.1.

**Lemma 4.2.** Let $Q$ and $Q'$ be quivers with $Q \succ Q'$. Then, we obtain the following.

1. If $Q$ is mutation-finite, then so is $Q'$.
2. If $Q'$ is mutation-infinite, then so is $Q$.

Obviously, any quiver with two vertices is mutation-finite and here are known mutation-infinite quivers of three vertices (cf. [2, Section 2.3]):

- $\bullet \quad a \quad b \quad \bullet \quad ab \geq 2,$
- $\bullet \quad a \quad b \quad \bullet \quad ab > 2c \geq 2,$
- $\bullet \quad a \quad b \quad \bullet \quad abc \geq 2.$
where $a, b, c$ represent the number of arrows in the shown direction. We call these quivers the \textit{linear quiver} of type $(a, b)$, the \textit{cyclic triangle} of type $(a, b, c)$ and the \textit{acyclic triangle} of type $(a, b, c)$, respectively.

**Lemma 4.3.** Let $Q$ be a quiver on $|4| = \{1, 2, 3, 4\}$ such that every pair of distinct vertices is connected. Then $Q$ is mutation-infinite unless $Q$ is the following quiver:

$$
Q = \\
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
$$

\text{Proof.} \text{ According to the number of sources—vertices only with outward edges— and sinks—vertices only with inward edges— in $Q$, there are only four quivers up to isomorphisms as depicted in Figure 3. Then it is easy to check that a quiver with one sink is mutation equivalent to a quiver with one source.}

\text{Figure 3. Quivers of the complete 4-graph}

Suppose that $Q$ is mutation-finite. Then it can not be reduced to the acyclic triangle of type $(a, b, c)$ with $abc \geq 2$ and therefore any acyclic triangle is of type $(1, 1, 1)$.

Since every edge of the quiver $Q$ with a sink or a source is a part of an acyclic triangle, all edges of $Q$ is \textit{simple}, that is, it consists of a single arrow. For a quiver without sink and source, the only edge from 2 to 1 may have multiple edges.

Since a cyclic triangle of type $(1, 1, b)$ with $b \geq 3$ is mutation-infinite, $|b_{1,2}| \leq 2$. Hence either,

$$
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
$$

or

$$
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
$$

but we exclude the latter by assumption.
Then the following can be checked directly:

\[ \begin{array}{c}
2 & 1 & 3 & 4 \\
3 & 1 & 4 & 2 \\
2 & 1 & 4 & 2 \\
3 & 1 & 4 & 2 \\
\end{array} \xrightarrow{\mu_4 \mu_3} \begin{array}{c}
2 & 1 & 3 & 4 \\
3 & 1 & 4 & 2 \\
2 & 1 & 4 & 2 \\
3 & 1 & 4 & 2 \\
\end{array}
\]

**Remark 4.4.** Indeed, the quiver that we exclude in the assumption of the lemma above is *block-decomposable* in the sense of [11, 10] and so mutation-finite.

**Corollary 4.5.** Let \( Q \) be a quiver on \([4]\) such that every pair of distinct vertices is connected. If \( Q \) contains a cyclic triangle of type different from \((1,1,2)\), then it is mutation-infinite.

**Corollary 4.6.** Let \( Q \) be a quiver on \([n]\) of standard affine type. Then \( Q \) can not be reduced to any of the following:

1. mutation-infinite quivers;
2. \( Q(2\tilde{A}_1) \),

where

\[ Q(2\tilde{A}_1) = Q(\tilde{A}_1) \prod Q(\tilde{A}_1) = \begin{array}{c}
\ast \\
\end{array} \]

**Proof.** (1) This is obvious since any standard affine type quiver is mutation-finite by Theorem 2.8(2) and by Lemma 4.2.

(2) Assume on the contrary that \( Q \succ Q(2\tilde{A}_1) \). Since \( 2\tilde{A}_1 \) is not of standard affine type, it must be a proper restriction of a quiver \( Q' \), which is mutation equivalent to \( Q \). We denote by \( I \subseteq [n] \) the subset satisfying \( Q'|_I = Q(2\tilde{A}_1) \). Consider a cluster pattern \( \{\Sigma_i\}_{i \in I} \) with the initial seed \( \Sigma' = (X', Q') \). Since the quiver \( Q' \) is of standard affine type, by Lemma 2.12, the number of seeds in the cluster pattern \( \{\Sigma_i\}_{i \in I} \), obtained from the initial seed \( \Sigma' \) by applying mutations on vertices \([n]\) \{i\} is finite for any \( i \in [n] \). This implies that the number of seeds obtained from the restriction \( \Sigma'|_I \) by mutations on vertices \( I \) \{i\} is finite as well. This is impossible because of Theorem 2.8(1) and therefore \( Q \) can not be reduced to \( Q(2\tilde{A}_1) \) as claimed.

By using these lemmas, we will prove Theorem 3.7. We note that for any triple \((X,G,Y)\) in Table 1, we have

\[ X = \tilde{A}_{n,n}, \quad \tilde{D}_n, \quad \text{or} \quad \tilde{E}. \]

We will prove the theorem using type-by-type arguments.
Before presenting the proof, we explain our strategy. In order to show the $G$-admissibility of a $G$-invariant quiver $Q$, it is enough to show the conditions (a) and (b) in Definition 3.1(2) since $Q$ is $G$-invariant by the assumption. Moreover, if any element $g \in G$ is of order 2, the condition (a) holds:

**Lemma 4.7.** Let $(X, G, Y)$ be a triple given by a column of Table 1. Let $Q$ be a quiver on $[n]$ of type $X$. Suppose that every element $g \in G$ is of order 2. Then, for each $i \in [n]$ and $g \in G$, we obtain

$$b_{i,g(i)} = 0.$$

**Proof.** Since every $g \in G$ is of order 2, we have

$$b_{i,g(i)} = b_{g(i),g^2(i)} = b_{g(i),i} = -b_{i,g(i)}.$$

Therefore $b_{i,g(i)} = 0$. □

### 4.1. Admissibility of quivers of type $\tilde{A}_{n,n}$

#### 4.1.1. $(X, G, Y) = (\tilde{A}_{2,2}, \mathbb{Z}/2\mathbb{Z}, \tilde{A}_1)$. Let $Q$ be the quiver of type $\tilde{A}_{2,2}$:

There are four quivers mutation equivalent to $Q$ up to isomorphisms. We present all of them:

For the $\mathbb{Z}/2\mathbb{Z}$-action defined by

$$\tau(1) = 4, \quad \tau(2) = 3, \quad \tau(3) = 2, \quad \tau(4) = 1,$$

the second quiver is the only $\mathbb{Z}/2\mathbb{Z}$-invariant quiver, which is $\mathbb{Z}/2\mathbb{Z}$-admissible. See (1) in Appendix A. This proves the following lemma:

**Lemma 4.8.** Let $Q$ be the $\mathbb{Z}/2\mathbb{Z}$-invariant quiver of type $\tilde{A}_{2,2}$. Then, $Q$ is $\mathbb{Z}/2\mathbb{Z}$-admissible.

#### 4.1.2. $(X, G, Y) = (\tilde{A}_{n,n}, \mathbb{Z}/2\mathbb{Z}, D_n^{(2)})$. Let $Q$ be a $\mathbb{Z}/2\mathbb{Z}$-invariant quiver on $[2n]$ of type $\tilde{A}_{n,n}$ and $B = B(Q)$. See (2) in Appendix A for the $\mathbb{Z}/2\mathbb{Z}$-action. To show the admissibility, it is enough to check the condition (b) because of Lemma 4.7.

**Lemma 4.9.** For any $i, j \in [2n]$, we obtain

$$b_{i,j}b_{i,\tau(j)} \geq 0.$$

**Proof.** If one of $i$ or $j$ is 1 or $2n$, then we are done since $\tau(i) = i$ or $\tau(j) = j$. Assume on the contrary that $b_{i,j}b_{i,\tau(j)} < 0$ for some $i, j \neq 1, 2n$. We may assume that $1 < i, j \leq n$ and

$$b_{\tau(i),\tau(j)} = b_{i,j} < 0 < b_{i,\tau(j)} = b_{\tau(i),j}.$$
Then we have a directed cycle $Q|_{\{i,j,\tau(i),\tau(j)\}}$

\[
Q|_{\{i,j,\tau(i),\tau(j)\}} = \\
\begin{array}{c}
  i \\
  \tau(i) \\
  \tau(j) \\
  j
\end{array}
\]

On the other hand, $Q$ is obtained from the initial quiver $Q(\tilde{A}_{n,n})$ via a sequence of mutations

\[
Q = (\mu_{j_L}^{\tilde{A}_{n,n}} \cdots \mu_{j_1}^{\tilde{A}_{n,n}})(Q(\tilde{A}_{n,n})),
\]

where the sequence $j_1, \ldots, j_L$ misses at least index $\ell \in [2n]$ by Lemma 2.12.

If $\ell \notin \{i, j, \tau(i), \tau(j)\}$, then the restriction

\[
Q|_{[2n] \setminus \{\ell\}} = (\mu_{j_L}^{\tilde{A}_{2n-1}} \cdots \mu_{j_1}^{\tilde{A}_{2n-1}})(Q(\tilde{A}_{2n-1}))
\]

is of type $A_{2n-1}$. However, this yields a contradiction as any quiver mutation equivalent to $A_{2n-1}$ never have a directed cycle of length 4, as shown in [6, Proposition 2.4].

If $\ell \in \{i, j, \tau(i), \tau(j)\}$, then the sequence $j_1, \ldots, j_L$ misses $\tau(\ell)$ as well by Lemma 2.12. Hence the directed cycle $Q|_{\{i,j,\tau(i),\tau(j)\}}$ should be contained in one of two restrictions $Q|_R$ and $Q|_S$ of type $A_{2\ell-1}$ and $A_{2(n-\ell)+3}$, where

\[
R = \{1, \ldots, \ell, n + 1, \ldots, n + \ell - 1\}, \quad S = \{\ell, \ldots, n, n + \ell - 1, \ldots, 2n\}.
\]

On the other hand, two restrictions of $Q(\tilde{A}_{n,n})$ are of type $A_{2\ell-1}$ and $A_{2(n-\ell)+3}$, respectively.

\[
\begin{array}{c}
Q(\tilde{A}_{n,n})|_R = 1 \bullet \quad \text{is of type } A_{2\ell-1}, \\
2 \quad \ell \quad \bullet \quad \bullet \quad 2n \\
n + 1 \quad n + \ell - 1 \quad 2n - 1
\end{array}
\]

\[
\begin{array}{c}
Q(\tilde{A}_{n,n})|_S = 1 \bullet \quad \text{is of type } A_{2(n-\ell)+3}, \\
2 \quad \ell \quad \bullet \quad \bullet \quad 2n \\
n + 1 \quad n + \ell - 1 \quad 2n - 1
\end{array}
\]

Hence, there is a sequence of mutations either from $Q(\tilde{A}_{n,n})|_R$ to $Q|_R$ or $Q(\tilde{A}_{n,n})|_S$ to $Q|_S$, which yields a contradiction again. \hfill \Box

**Proof of Theorem 3.7** for $X = \tilde{A}_{n,n}$. For a $\mathbb{Z}/2\mathbb{Z}$-invariant quiver $Q$ of type $\tilde{A}_{n,n}$, the conditions (a) and (b) in Definition 3.1(2) follows from Lemmas 4.7 and 4.9. Combining this with Lemma 4.8, the quiver $Q$ is $\mathbb{Z}/2\mathbb{Z}$-admissible as claimed. \hfill \Box

### 4.2. Admissibility of quivers of type $\tilde{E}$.

#### 4.2.1. $(X,G,Y) = (\overline{E}_6,\mathbb{Z}/3\mathbb{Z},\overline{G}_2)$. Let $Q$ be a $\mathbb{Z}/3\mathbb{Z}$-invariant quiver on [7] of type $\overline{E}_6$ and $\mathcal{B} = \mathcal{B}(Q)$. See (9) in Appendix A.

**Lemma 4.10.** We have

\[
b_{2,4} = b_{4,6} = b_{6,2} = 0 \quad \text{and} \quad b_{3,5} = b_{5,7} = b_{7,3} = 0.
\]
Proof. Suppose that the assertion does not hold. Then by relabelling vertices if necessary, we may
assume that $b_{2,4} = b_{4,6} = b_{6,2} = n \geq 1$ and so $Q_{\{2,4,6\}}$ is a cyclic triangle of type $(n,n,n)$.

If $b_{1,2} = b_{1,4} = b_{1,6} \neq 0$, then every pair of distinct vertices in $Q_{\{1,2,4,6\}}$ is connected and it
contains a cyclic triangle of type $(n,n,n)$. This restriction is mutation-infinite by Corollary 4.5
and so is $Q$. Since $\mathcal{E}_6$ is mutation-finite, this is a contradiction. Hence, we obtain

$$b_{1,2} = b_{1,4} = b_{1,6} = 0.$$ 

Similarly, we have

$$b_{3,5} = b_{5,7} = b_{7,3} = 0,$$

otherwise, the restriction $Q_{\{1,3,5,7\}}$ is mutation-infinite. However, since $Q$ is connected, we have

$$b_{1,3} = b_{1,5} = b_{1,7} = m \neq 0$$

and at least one of

$$x = b_{3,2} = b_{5,4} = b_{7,6}, \quad y = b_{3,4} = b_{5,6} = b_{7,2}, \quad z = b_{3,6} = b_{5,2} = b_{7,4}$$

is non-zero.

If none of $x, y$ and $z$ is zero, then the restriction $Q_{\{2,3,4,6\}}$ is again mutation-infinite by Corol-

lary 4.5 since every pair of distinct vertices in $Q_{\{2,3,4,6\}}$ is connected and it contains a cyclic
triangle of type $(n,n,n)$. Hence, we may assume that either

(i) $x \neq 0, y = z = 0$ or (ii) $x \neq 0, y \neq 0, z = 0$

Case (i) Suppose that $x \neq 0$ but $y = z = 0$. By taking a mutation $\mu_1$ if necessary, and mutations
$\mu_3, \mu_5, \mu_7$, we obtain a new quiver $Q'$ with $mx$ edges from 1 to 2. That is, for $B' = B(Q') = (b'_{i,j})$

$$b'_{1,2} = b'_{1,4} = b'_{1,6} = mx \neq 0.$$ 

Hence $Q > Q'_{\{1,2,4,6\}}$, which is mutation-infinite and therefore this is a contradiction.

Case (ii) Suppose that $x \neq 0, y \neq 0, z = 0$. Similarly, if $xy > 0$ (equivalently, the signs of $x$ and
$y$ are same), then by taking a mutation $\mu_1$ if necessary, and mutations $\mu_3, \mu_5, \mu_7$, we obtain $Q'$ so that

$$b'_{1,2} = b'_{1,4} = b'_{1,6} = m(x + y) \neq 0.$$ 

Even if $xy < 0$, unless $Q_{\{2,3,4\}}$ is cyclic and $n + xy = 0$, the same argument still holds.
Suppose that $y < 0 < x$, $n + xy = 0$, $m > 0$, and quivers $Q|_{\{2,3,4\}}$, $Q|_{\{4,5,6\}}$, and $Q|_{\{2,6,7\}}$ are cyclic. Then the quiver $Q'$ obtained by applying the mutations $\mu_3, \mu_5$, and $\mu_7$ on $Q$ looks like

$$Q' = \begin{array}{c}
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both are reduced to mutation-infinite quivers:

![Quiver Diagrams](image-url)

This yields a contradiction and we are done. □

4.2.2. \((X, G, Y) = (\tilde{E}_6, \mathbb{Z}/2\mathbb{Z}, E_6^{(2)})\). Let \(Q\) be a \(\mathbb{Z}/2\mathbb{Z}\)-invariant quiver on \([7]\) of type \(\tilde{E}_6\) and \(B = B(Q)\). See (10) in Appendix A. To show the admissibility, it is enough to check the condition (b) because of Lemma 4.7.

**Lemma 4.12.** For any \(i, j \in [7]\),

\[ b_{i,j}b_{i,\tau(j)} \geq 0. \]

**Proof.** If \(i \leq 3\) or \(j \leq 3\), then \(b_{i,j}b_{i,\tau(j)} = b_{i,j}^2 \geq 0\) since \(\tau(i) = i\) or \(\tau(j) = j\).

Suppose that \(b_{i,j}b_{i,\tau(j)} < 0\) for some \(i, j \geq 4\). Then the only possibility is up to relabelling,

\[ b_{7,6} = b_{5,4} < 0 < b_{5,6} = b_{7,4}. \]

As seen in the previous lemma, \(b_{4,6} = 0\) and therefore by considering the restriction \(Q|_{\{4,5,6\}}\), we have \(b_{4,5} = b_{5,6} = b_{6,7} = b_{7,4} = 1\). Therefore \(Q|_{\{4,5,6,7\}}\) is a cyclic graph as follows:

![Cyclic Graph](image-url)

Since \(Q\) is connected, \(b_{i,j} \neq 0\) for some \(i \leq 3 < j\). Let us assume that \(b_{1,j} \neq 0\). Then by \(\mathbb{Z}/2\mathbb{Z}\)-invariance, the restriction \(Q|_{\{1,4,5,6,7\}}\) is one of the following: up to relabelling and mutation \(\mu_1\),

![Restriction Examples](image-url)
Then indeed, first two quivers are mutation equivalent via \( \mu_5\mu_7\mu_4\mu_6 \).

\[
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\xrightarrow{\mu_5\mu_7\mu_4\mu_6}
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\]

Finally, the last two out of the above three can be reduced as follows:

\[
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\xrightarrow{\mu_1}
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\xrightarrow{\mu_5\mu_7}
\begin{array}{c}
\begin{array}{c}
5 \quad \bullet \\
6 \\
7 \\
1 \quad \bullet
\end{array}
\end{array}
\]

This yields a contradiction so we are done. \( \square \)

4.2.3. \((X,G,Y) = (\tilde{E}_7,\mathbb{Z}/2\mathbb{Z},\tilde{F}_4)\). Let \( Q \) be a \( \mathbb{Z}/2\mathbb{Z} \)-invariant quiver on \([8]\) of type \( \tilde{E}_7 \) and \( B = B(Q) \). See (11) in Appendix A. To show the admissibility, it is enough to check the condition (b) because of Lemma 4.7.

**Lemma 4.13.** For any \( i, j \in [8] \),

\[ b_{i,j} b_{i,\tau(j)} \geq 0. \]

**Proof.** If \( i \leq 2 \) or \( j \leq 2 \), then \( b_{i,j} b_{i,\tau(j)} = b_{i,j}^2 \geq 0 \) since \( \tau(i) = i \) or \( \tau(j) = j \). Suppose that \( b_{i,j} b_{i,\tau(j)} < 0 \) for some \( i, j \geq 3 \). Then up to relabelling, we may assume that

\[ b_{8,7} = b_{5,4} < 0 < b_{5,7} = b_{8,4}. \]

As before, \( Q|_{\{4,5,7,8\}} \) is a cyclic graph and so we must have \( b_{4,5} = b_{5,7} = b_{7,8} = b_{8,4} = 1. \)

\[
Q|_{\{4,5,7,8\}} = \begin{array}{c}
\begin{array}{c}
4 \quad \bullet \\
3 \\
7 \\
8
\end{array}
\end{array}
\]

Suppose that \( b_{i,j} \neq 0 \) for \( i \leq 2 \) and \( j \leq 7 \). Then the restriction \( Q|_{\{1,4,5,7,8\}} \) will be reduced to a mutation-infinite quiver by Lemma 4.12. Hence we may assume that for \( i \leq 2 \),

\[ b_{i,4} = b_{i,5} = b_{i,7} = b_{i,8} = 0, \]

and since \( Q \) is connected, \( b_{1,3} = b_{1,6} \neq 0 \) for some \( i \leq 2 \), say \( i = 1 \). Then the mutation-finiteness of the restriction \( Q|_{\{1,3,6\}} \) implies that \( b_{1,3} = b_{1,6} = \pm 1. \)

If \( b_{3,j} b_{3,\tau(j)} < 0 \) for some \( j = 4, 5, 7, 8 \), then the restriction \( Q|_{\{1,3,6,j,\tau(j)\}} \) is mutation-infinite as before. Therefore, \( b_{3,j} b_{3,\tau(j)} \geq 0 \) for all \( j \in [8] \) since \( b_{3,j} b_{3,\tau(j)} \geq 0 \) for any \( j = 1, 2, 3, 6. \)
Again, since \( Q \) is connected, one of \( b_{3,4}, b_{3,5}, b_{3,7} \) and \( b_{3,8} \) is nonzero. Suppose that none of \( b_{3,j} \) for \( j = 4, 5, 7, 8 \) is zero. Then two restrictions \( Q|_{\{3,4,7\}} \) and \( Q|_{\{3,5,8\}} \) force us to have 
\[
b_{3,4} = b_{3,7} = \pm 1 \quad \text{and} \quad b_{3,5} = b_{3,7} = \pm 1.
\]
Then the restriction \( Q|_{\{3,4,5,7,8\}} \) is up to mutation \( \mu_3 \) one of the following:

However, as seen in (4.3) and (4.4), these three are reduced to either a mutation-infinite quiver or \( Q(2\tilde{A}_1) \), which are impossible. Therefore, at least one of \( b_{3,4}, b_{3,5}, b_{3,7} \) and \( b_{3,8} \) is zero.

**Case (i)** Suppose that \( b_{3,8} = 0 \) but \( b_{3,4}, b_{3,7}, b_{3,5} \neq 0 \). Then the restriction \( Q|_{\{3,4,5,7,8\}} \) is reduced to a mutation-infinite quiver as follows:

This yields a contradiction so this case cannot occur.

**Case (ii)** Suppose that \( b_{3,8} = b_{3,5} = 0 \) but \( b_{3,4}, b_{3,7} \neq 0 \). Then \( b_{3,4} = b_{3,7} = \pm 1 \) by the mutation-finiteness of \( Q|_{\{3,4,7\}} \) and so the restriction \( Q|_{\{3,4,5,7,8\}} \) is reduced to \( Q(2\tilde{A}_1) \) as seen in (4.2) and (4.3), which yields a contradiction so this case cannot happen.

**Case (iii)** Suppose that \( b_{3,7} = b_{3,8} = 0 \). Considering \( Q|_{\{8\}\setminus\{2\}} \), there are four cases as follows:

One can check easily that the first two are reduced to \( Q(2\tilde{A}_1) \) as follows:
For the third case, we further reduce it to the quiver $Q_{\{3,4,5,7\}}$, which will be reduced to a linear quiver of type $(2,1)$.

![Diagram of quiver](image)

The fourth quiver is mutation equivalent to the quiver $Q'$

![Diagram of quiver](image)

and it reduces to the next case.

**Case (iv)** Suppose that $b_{3,4} \neq 0$ but $b_{3,5} = b_{3,7} = b_{3,8} = 0$. Then it looks like the last quiver in the previous case. Then up to mutation $\mu_1$, we may assume that $b_{1,3}b_{3,4} > 0$. Namely,

![Diagram of quiver](image)

Since the last quiver is reduced to $Q(2\tilde{A}_1)$ by (4.2) and (4.3), this is a contradiction, which completes the proof.

**Proof of Theorem 3.7 for $X = \tilde{E}$.** For a $G$-invariant quiver $Q$ of type $\tilde{E}_6$ or $\tilde{E}_7$, with $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, the condition (a) in Definition 3.1(2) follows from Lemmas 4.7 and 4.10. The condition (b) follows from Lemmas 4.11, 4.12 and 4.13. Therefore $Q$ is $G$-admissible as claimed.

**4.3. Admissibility of quivers of type $\tilde{D}$.** Throughout this section, we denote $G = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$ and $Y = \tilde{D}_n^G$. Let $Q$ be a $G$-invariant quiver on $[n+1]$ of type $\tilde{D}_n$ and $B = B(Q)$.

**Lemma 4.14.** For each $i \in [n+1]$ and $g \in G$, we obtain

$$b_{i,g(i)} = 0.$$

**Proof.** Suppose that $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we prove the claim by Lemma 4.7. Let $G = \mathbb{Z}/3\mathbb{Z}$ which acts on $\tilde{D}_1$ and identifies $\{3,4,5\}$ (see (4) in Appendix A). Since $Q$ is connected, $b_{1,3} = b_{1,4} = b_{1,5} = b \neq 0$ for some $i \leq 2$. Then every pair of distinct vertices in $Q_{\{1,3,4,5\}}$ is connected and it contains a cyclic triangle of type $(b,b,b)$, which is mutation-infinite by Corollary 4.5. This yields a contradiction and we are done.

Hence, to prove the admissibility, it is sufficient to show that the condition (b) holds, that is, $b_{i,j}b_{i,\tau(j)} \geq 0$ for all $i, j \in [n+1]$. 

4.3.1. \((X,G,Y) = (\tilde{D}_4, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, A^{(2)}_2)\) or \((\tilde{D}_4, \mathbb{Z}/3\mathbb{Z}, D^{(3)}_4)\).

**Lemma 4.15.** For each \(i, j \in [5]\), we have 
\[
   b_{i,j}b_{i,\tau(j)} \geq 0.
\]

**Proof.** Suppose that \(G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) and \(\tilde{D}^{(2)}_4 = A^{(2)}_2\). Then we only need to prove that 
\[
   b_{1,2}b_{1,\tau(2)} \geq 0
\]
for \(2 \leq j\), which is obvious since \(b_{1,j} = b_{\tau(1),\tau(j)} = b_{1,\tau(j)}\).

For \(G = \mathbb{Z}/3\mathbb{Z}\), we have \(\tilde{D}^{(3)}_4 = D^{(3)}_4\). If \(i \leq 2\) or \(j \leq 2\), then there is nothing to prove. Otherwise, \(b_{i,j} = 0\) by Lemma 4.14 and we are done. \(\square\)

4.3.2. \((X,G,Y) = (\tilde{D}_n, \mathbb{Z}/2\mathbb{Z}, \tilde{C}_{n-2})\). The action of the generator \(\tau \in \mathbb{Z}/2\mathbb{Z}\) is as follows:
\[
   \tau(i) = \begin{cases} 
   i & \text{if } i = 1, 4, \ldots, n-1; \\
   3 & \text{if } i = 2; \\
   2 & \text{if } i = 3; \\
   n+1 & \text{if } i = n; \\
   n & \text{if } i = n+1.
   \end{cases}
\]

See (5) in Appendix A.

**Lemma 4.16.** For each \(i, j \in [n+1]\), we have 
\[
   b_{i,j}b_{i,\tau(j)} \geq 0.
\]

**Proof.** If \(i \not\in \{2, 3, n, n+1\}\) or \(j \not\in \{2, 3, n, n+1\}\), then there is nothing to prove since \(\tau(i) = i\) or \(\tau(j) = j\) and
\[
   b_{i,j} = b_{\tau(i),\tau(j)} = \begin{cases} 
   b_{i,\tau(j)} & \text{if } i \not\in \{2, 3, n, n+1\}; \\
   b_{\tau(i),\tau(j)} = b_{i,\tau(j)} & \text{if } j \not\in \{2, 3, n, n+1\}.
   \end{cases}
\]

If \(i, j\) are in the same \(\mathbb{Z}/2\mathbb{Z}\)-orbit, namely, either \(\{i, j\} = \{2, 3\}\) or \(\{i, j\} = \{n, n+1\}\), then we are done since \(b_{i,j} = 0\) by Lemma 4.14.

Finally, suppose that \(i \in \{2, 3\}\), \(j \in \{n, n+1\}\) and \(b_{i,j}b_{i,\tau(j)} < 0\). Then we may assume that \(i = 2\), \(j = n\) and
\[
b_{2,n+1} = b_{2,n} < b_{2,n+1} = b_{3,n}
\]
and therefore the restriction \(Q|_{\{2,3,n,n+1,\ell\}}\) is a directed cycle of length 4. On the other hand, since \(Q\) is connected, there exists \(\ell \in [n+1] \setminus \{2,3,n,n+1\}\) such that
\[
b_{2,\ell} = b_{\ell,3} \neq 0 \quad \text{or} \quad b_{\ell,n} = b_{\ell,n+1} \neq 0.
\]
Then up to \(\mu_{\ell}\), the restriction \(Q|_{\{2,3,n,n+1,\ell\}}\) looks like one of three quivers depicted in (4.1). Indeed, as seen in (4.2), (4.3) and (4.4), the quiver \(Q|_{\{2,3,n,n+1,\ell\}}\) eventually reduces to a mutation-infinite quiver or the quiver \(2\overline{A}_1\). This contradicts to Corollary 4.6. \(\square\)

4.3.3. \((X,G,Y) = (\tilde{D}_n, \mathbb{Z}/2\mathbb{Z}, A^{(2)}_r)_{2(n-1)-1})\). The action of the generator \(\tau \in \mathbb{Z}/2\mathbb{Z}\) is as follows:
\[
   \tau(i) = \begin{cases} 
   i & \text{if } i \leq n-1; \\
   n+1 & \text{if } i = n; \\
   n & \text{if } i = n+1.
   \end{cases}
\]

See (6) in Appendix A.

**Lemma 4.17.** For each \(i, j \in [n+1]\), we have 
\[
   b_{i,j}b_{i,\tau(j)} \geq 0.
\]

**Proof.** If \(i \leq n-1\) or \(j \leq n-1\), then there is nothing to prove by the same argument as Lemma 4.16. Otherwise, \(b_{i,j} = 0\) by Lemma 4.14 and we are done. \(\square\)
4.3.4. \( (X,G,Y) = (\tilde{D}_{2n}, \mathbb{Z}/2\mathbb{Z}, \mathbb{B}_n) \). The action of the generator \( \tau \in \mathbb{Z}/2\mathbb{Z} \) is as follows:

\[
\tau(i) = \begin{cases} 
2n - 1 & \text{if } i = 1; \\
2n + 1 & \text{if } i = 2; \\
2n & \text{if } i = 3; \\
2n + 2 - i & \text{if } 4 \leq i \leq 2n - 2; \\
1 & \text{if } i = 2n - 1; \\
3 & \text{if } i = 2n; \\
2 & \text{if } i = 2n + 1.
\end{cases}
\]

See (7) in Appendix A.

**Lemma 4.18.** For each \( i, j \in [2n + 1] \), we have

\[
b_{i,j}b_{i,\tau(j)} \geq 0.
\]

**Proof.** If \( i = n + 1 \) or \( j = n + 1 \), then there is nothing to prove since

\[
b_{n+1,j} = b_{\tau(n+1),\tau(j)} = b_{n+1,\tau(j)} \quad \text{and} \quad b_{i,n+1} = b_{i,\tau(n+1)}.
\]

Suppose that \( b_{i,j}b_{i,\tau(j)} < 0 \) for \( i, j \in [2n + 1] \setminus \{n + 1\} \). Then we may assume that \( i, j < n + 1 \) and

\[
b_{\tau(i,j)} = b_{i,\tau(j)} < 0 < b_{i,j} = b_{\tau(i),\tau(j)}.
\]

Therefore the restriction \( Q_{\{i,j,\tau(i),\tau(j)\}} \) is a directed cycle of length 4. We furthermore assume that this cycle is the closest one to the vertex \( n + 1 \) with respect to the length of undirected edge-path. In other words, for any vertex \( p \) closer than \( i \) and \( j \) from \( n + 1 \), we have \( b_{p,q}b_{p,\tau(q)} \geq 0 \) for all \( q \in [2n + 1] \).

On the other hand, there is a sequence of mutations

\[
Q = (\mu_{j_1}^{\tilde{D}_{2n}} \cdots \mu_{j_L}^{\tilde{D}_{2n}})(Q(\tilde{D}_{2n})),
\]

where the sequence \( j_1, \ldots, j_L \) misses at least one vertex \( \ell \in [2n + 1] \) and so \( \tau(\ell) \) as well by Lemma 2.12.

We consider subquivers separated by \( \ell, \tau(\ell) \) and observe that in \( Q \), there are no edges between pieces separated by \( \{\ell, \tau(\ell)\} \). Suppose that \( \ell = n + 1 \). Then either \( i, j \) or \( i, \tau(j) \) are contained in different subquivers separated by \( n + 1 \). However since we are assuming \( b_{i,j}b_{i,\tau(j)} < 0 \), this is a contradiction and so we may assume that \( \ell \neq n + 1 \). Then among separated quivers, there exists a central piece \( Q' \) containing \( n + 1 \), which is \( \mathbb{Z}/2\mathbb{Z} \)-invariant. Moreover, \( Q' \) is of type \( A_m \) for some \( m < 2n \).

**Claim (i)** We claim that \( \ell \in \{i,j,\tau(i),\tau(j)\} \). Assume on the contrary that \( \ell \notin \{i,j,\tau(i),\tau(j)\} \). Then, by the observation above, four vertices \( i, j, \tau(i), \tau(j) \) are contained in the central piece \( Q' \). This yields a contradiction since any \( \mathbb{Z}/2\mathbb{Z} \)-invariant quiver of type \( A_m \) is \( \mathbb{Z}/2\mathbb{Z} \)-admissible. Therefore, we get \( \ell \in \{i,j,\tau(i),\tau(j)\} \).

**Claim (ii)** We claim that \( \ell \in \{2, 3, 2n, 2n+1\} \). Assume on the contrary that \( \ell \notin \{2, 3, 2n, 2n+1\} \). Then the restriction \( Q'' \) containing vertices of \( Q' \) and \( \{\ell, \tau(\ell)\} \) is \( \mathbb{Z}/2\mathbb{Z} \)-admissible of type \( A_{2m+2} \). Then by the observation above, \( i, j, \tau(i), \tau(j) \) are contained in \( Q'' \). This yields a contradiction again and therefore we obtain \( \ell \in \{2, 3, 2n, 2n+1\} \).

Because of the above two claims (i) and (ii), we may assume that

\[
\ell \in \{i,j,\tau(i),\tau(j)\} \cap \{2, 3, 2n, 2n+1\}.
\]

Let \( i_0 = n + 1, i_1, \ldots, i_{M+1} \) be a sequence of vertices which gives us a shortest (undirected) path from \( n + 1 \) to the cycle \( Q_{\{i,j,\tau(i),\tau(j)\}} \). That is, \( i_{M+1} \in \{i,j,\tau(i),\tau(j)\} \). Then we also have another shortest (undirected) path given by a sequence of vertices \( \tau(i_{M+1}), \ldots, \tau(i_1), n + 1 \) from the cycle \( Q_{\{i,j,\tau(i),\tau(j)\}} \) to the vertex \( n + 1 \).

Notice that the set

\[
R = \{\tau(i_M), \ldots, \tau(i_1), n + 1, i_1, \ldots, i_M\} \subset [2n + 1]
\]

misses \( \{i,j,\tau(i),\tau(j)\} \).
We claim that the restriction \( Q|_R \) is a \( \mathbb{Z}/2\mathbb{Z} \)-admissible quiver of type \( A_{2M+1} \). Indeed, \( Q|_R \) is a restriction of \( Q|_{[2n+1] \setminus \{\ell, \tau(\ell)\}} \), which is of type \( A_{2n-1} \). Since any connected subquiver of a quiver mutation equivalent to \( A \) is again of type \( A \), we proved the claim.

Because we take the shortest undirected path connecting \( n+1 \) and \( i \), the restriction \( Q|_R \) is an undirected path having \( 2M+1 \) vertices, that is, the underlying graph of \( Q|_R \) is isomorphic to the Dynkin diagram of type \( A_{2M+1} \).

Now consider the quiver \( Q|_{R,i,j,\tau(i),\tau(j)} \) which is \( \mathbb{Z}/2\mathbb{Z} \)-invariant and looks like the left picture below. Finally, by a sequence of orbit mutations, the vertex \( n+1 \) can be directly connected with both \( i \) and \( \tau(i) \), so \( Q|_{R,i,j,\tau(i),\tau(j)} \) can be reduced to one of the quivers in (4.1) as displayed in the right picture below. This contradiction completes the proof.

\[
\begin{align*}
\tau_1(i) &= \begin{cases} 
  i & \text{if } i \leq n-1; \\
  n+1 & \text{if } i = n; \\
  n & \text{if } i = n+1,
\end{cases} \\
\tau_2(i) &= \begin{cases} 
  i & \text{if } i = 1, 4, \ldots, 2n-1; \\
  3 & \text{if } i = 2; \\
  2 & \text{if } i = 3; \\
  2n+1 & \text{if } i = 2n; \\
  2n & \text{if } i = 2n+1.
\end{cases}
\end{align*}
\]

See (8) in Appendix A.

**Lemma 4.19.** For each \( i, j \in [2n+1] \) and \( g \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), we have
\[ b_{i,j}b_{i,g(j)} \geq 0. \]

**Proof.** Since \( Q \) is already \( \mathbb{Z}/2\mathbb{Z} \)-invariant and \( \tau_1 \) and \( \tau_1 \tau_2 \) generate isomorphic actions on \( \mathbb{D}_{2n} \), the only thing to check is for \( g = \tau_2 \) by Lemma 4.18.

If \( i \notin \{2, 3, 2n, 2n+1\} \) or \( j \notin \{2, 3, 2n, 2n+1\} \), then there is nothing to prove since
\[ b_{i,j} = b_{\tau_1(i),\tau_1(j)} = \begin{cases} 
  b_{i,\tau_1(j)} & \text{if } i \notin \{2, 3, 2n, 2n+1\}; \\
  b_{\tau_1(i),j} = b_{\tau_2(i),\tau_2(j)} & \text{if } j \notin \{2, 3, 2n, 2n+1\}.
\end{cases} \]

Otherwise, if \( i, j \in \{2, 3, 2n, 2n+1\} \), then \( b_{i,j} = 0 \) by Lemma 4.14 and we are done. \( \square \)

**Proof of Theorem 3.7 for \( X = \mathbb{D} \).** For a \( G \)-invariant quiver \( Q \) of type \( \mathbb{D}_n \) with \( G = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \) or \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), the condition (a) in Definition 3.1(2) follows from Lemma 4.14 and the condition (b) follows from Lemmas 4.15, 4.16, 4.17, 4.18, and 4.19. Therefore \( Q \) is \( G \)-admissible as claimed. \( \square \)

5. **Connections with cluster algebras: folded cluster patterns**

Under certain conditions, one can fold cluster patterns to produce new ones. This procedure is used to study cluster algebras of non-simply-laced affine type from those of simply-laced affine type (see Table 1). In this section, we observe the properties of folded cluster patterns of non-simply-laced affine type in Corollary 5.3.
For a globally foldable quiver \( \mathcal{Q} \) on \([n]\) with respect to \(G\)-action, we can fold all the seeds in the corresponding cluster pattern. Let \(F^G\) be the field of rational functions in \(#(\mathcal{Q})/G\) independent variables and \(\psi: F \to F^G\) be a surjective homomorphism. A seed \(\Sigma = (x, \mathcal{Q})\) is called \((G, \psi)\)-invariant (respectively, \((G, \psi)\)-admissible) if

- for any \(i \sim i'\), we have \(\psi(x_i) = \psi(x_{i'})\);
- \(\mathcal{Q}\) is \(G\)-invariant (respectively, \(G\)-admissible).

In this situation, we define a new “folded” seed \(\Sigma^G = (x^G, (B(\mathcal{Q}))^G)\) in \(F^G\) whose exchange matrix is given as before and cluster variables \(x^G = (x_1)\) are indexed by the \(G\)-orbits and given by \(x_1 = \psi(x_i)\).

**Proposition 5.1** ([12, Corollary 4.4.11]). Let \(\mathcal{Q}\) be a quiver on \([n]\) which is globally foldable with respect to a group \(G\) acting on \([n]\). Let \(\Sigma_{t_0} = (x, \mathcal{Q})\) be a seed in the field \(F\) of rational functions freely generated by a cluster \(x = (x_1, \ldots, x_n)\). Define \(\psi: F \to F^G\) so that \(\Sigma_{t_0}\) is a \((G, \psi)\)-admissible seed. Then, for any \(G\)-orbits \(I_1, \ldots, I_t\), the seed \((\mu_{I_1} \ldots \mu_{I_t})(\Sigma_{t_0})\) is \((G, \psi)\)-admissible, and moreover, the folded seeds \(((\mu_{I_1} \ldots \mu_{I_1})(\Sigma_{t_0}))^G\) form a cluster pattern in \(F^G\) with the initial seed \(\Sigma^G = (x^G, (B(\mathcal{Q}))^G)\).

**Proposition 5.2.** Let \(\mathcal{Q}\) be an acyclic quiver on \([n]\) which is globally foldable with respect to a group \(G\) acting on \([n]\). Let \(\Sigma_{t_0} = (x, \mathcal{Q})\) be a seed in the field \(F\) of rational functions freely generated by a cluster \(x = (x_1, \ldots, x_n)\). Define \(\psi: F \to F^G\) so that \(\Sigma_{t_0}\) is a \((G, \psi)\)-admissible seed. Then, the set \((G, \psi)\)-admissible seeds are connected via orbit mutations. Indeed, the set \((G, \psi)\)-admissible seeds forms a cluster pattern with the initial seed \(\Sigma^G\).

**Proof.** We denote by \(S\) the set of \((G, \psi)\)-admissible seeds in the cluster pattern obtained by \(\Sigma_{t_0}\). Consider a subset \(S'\) of \(S\) such that each of which element is connected to the initial seed \(\Sigma_{t_0}\) via a sequence of orbit mutations.

To show \(S' = S\), it is enough to prove that for each seed \(\Sigma'\) in the cluster pattern of \(\Sigma^G_{t_0}\), there exists only one \((G, \psi)\)-admissible seed \(\Sigma_i\) in \(S\) such that \(\Sigma^G_i = \Sigma'\). Take a seed \(\Sigma_{t_1}\) in \(S'\) such that \(\Sigma^G_{t_1} = \Sigma'\). We may assume that \(\Sigma_{t_1}\) is the initial seed in the cluster pattern, that is, \(\Sigma_{t_1}\) has cluster variables \(\{x_i \mid i \in [n]\}\). Assume on the contrary that there exists another \((G, \psi)\)-admissible seed \(\Sigma_{s_1} = (x_{s_1}, B_{s_1})\) satisfies \(\Sigma^G_{s_1} = \Sigma'\), then

\[
\{\psi(x_i) \mid i \in [n]\} = \{\psi(x_{s_1}) \mid i \in [n]\}.
\]

The positivity of Laurent phenomenon, which was conjectured in Fomin–Zelevinsky [13] and proved in Gross–Hacking–Keel–Kontsevich [17, Corollary 4.4.11], states that every non-zero cluster variable can be uniquely written by a rational polynomial whose numerator is a polynomial with non-negative integer coefficients in the initial cluster variables \(x_1, \ldots, x_n\). Accordingly, to get (5.1), the cluster variables \(x_{i_1}, \ldots, x_{i_2}\) should be the initial cluster variables \(x_1, \ldots, x_n\) because the non-negativity of coefficients means no cancellation exists. Since we are considering a cluster pattern whose initial seed has an acyclic quiver, the cluster variables determine a seed by [5, Theorem 4.1] (also, see [15, Conjecture 4.14]), so we have \(\Sigma_s = \Sigma_{t_1}\) and this proves the claim. \(\square\)

If a seed \(\Sigma = (x, \mathcal{Q})\) is \((G, \psi)\)-admissible, then \(\Sigma\) is \((G, \psi)\)-invariant by Definition 3.1. As a direct corollary of Theorem 3.7 and Proposition 5.2, we obtain that the converse holds when we consider the foldings presented in Table 1.

**Corollary 5.3.** Let \((X, G, Y)\) be a triple given by a column of Table 1. Let \(\Sigma_{t_0} = (X, \mathcal{Q})\) be a seed. Suppose that \(\mathcal{Q}\) is of type \(X\). Define \(\psi: F \to F^G\) so that \(\Sigma_{t_0}\) is a \((G, \psi)\)-admissible seed. Then, any \((G, \psi)\)-invariant seed can be reached by a sequence of orbit mutations from \(\Sigma_{t_0}\). Moreover, the set \((G, \psi)\)-invariant seeds forms the ‘folded’ cluster pattern given by \(\Sigma^G_{t_0}\) of \(Y\) via folding.

**Remark 5.4.** Let \(\mathcal{Q}\) be an acyclic quiver which is globally foldable with respect to a finite group \(G\). Define \(\psi: F \to F^G\) so that \(\Sigma_{t_0} = (x_{t_0}, \mathcal{Q})\) is \((G, \psi)\)-admissible. Let \(\Sigma_t = (x_t, Q_t)\) be a \((G, \psi)\)-invariant seed. Dupont asked in [8, Problem 9.5] that can \(\Sigma_t\) be reached by sequences of orbit mutations from the initial seed \(\Sigma_{t_0}\)? Corollary 5.3 implies that when the quiver \(\mathcal{Q}\) is of type \(A_{n}, B_{n}, \overline{E}_6\) or \(\overline{E}_7\) and with the specific choice of \(G\) (as in Table 1), we get an affirmative answer to the question proposed by Dupont.
Appendix A. Group actions on Dynkin diagrams of affine type

In the appendix, we provide group actions on Dynkin diagrams of affine type. More precisely, for each triple $(X, G, Y)$ given by a column of Table 1, we describe the $G$-action of the Dynkin diagram of $X$. Throughout this section, we denote by $\tau$ the generator of each finite group $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. For $i \in [n]$, we denote by $I_i$ the orbit $G \cdot i$. We decorate vertices of Dynkin diagram of type $Y$ with orbits $I_i$.

1. $(X, G, Y) = (\tilde{A}_{2,2}, \mathbb{Z}/2\mathbb{Z}, \tilde{A}_1)$

2. $(X, G, Y) = (\tilde{A}_{n,n}, \mathbb{Z}/2\mathbb{Z}, D_{n+1}^{(2)})$

3. $(X, G, Y) = (\tilde{D}_4, (\mathbb{Z}/2\mathbb{Z})^2, A_2^{(2)})$

4. $(X, G, Y) = (\tilde{D}_4, \mathbb{Z}/3\mathbb{Z}, D_4^{(3)})$

5. $(X, G, Y) = (\tilde{D}_n, \mathbb{Z}/2\mathbb{Z}, \tilde{C}_{n-2})$
(6) \((X, G, \mathcal{Y}) = (\tilde{D}_n, \mathbb{Z}/2\mathbb{Z}, A_{2n-1}^{(2)})\)

(7) \((X, G, \mathcal{Y}) = (\tilde{D}_{2n}, \mathbb{Z}/2\mathbb{Z}, \tilde{B}_n)\)

(8) \((X, G, \mathcal{Y}) = (\tilde{D}_{2n}, (\mathbb{Z}/2\mathbb{Z})^2, A_{2n-2}^{(2)})\)

(9) \((X, G, \mathcal{Y}) = (\tilde{E}_6, \mathbb{Z}/3\mathbb{Z}, \tilde{G}_2)\)

(10) \((X, G, \mathcal{Y}) = (\tilde{E}_6, \mathbb{Z}/2\mathbb{Z}, E_6^{(2)})\)
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