1 Implicitization

Suppose that \( \phi : \mathbb{P}^2 \to \mathbb{P}^3 \) is a map (possibly with basepoints) whose image is a surface \( S \subset \mathbb{P}^3 \). In computer graphics, such a map is given by homogeneous polynomials \((a, b, c, d)\) with real coefficients, and knowing these polynomials allows one to draw the real points of \( S \) on a computer screen. Most geometric models in CAD (Computer Aided Design) use parametric surfaces. This includes car bodies, airplanes, and animated figures in movies such as *Toy Story* or *Dinosaur*.

The *implicitization problem*, as explained in [CLO1], is to compute the implicit equation \( F = 0 \) of \( S \) using the parametrization \( \phi \). This leads to the following natural question: if our goal is to draw \( S \) on a computer screen, why would we be interested in the implicit equation of \( S \)?

One answer is that we can use implicitization to help find curve intersections. Namely, when drawing two parametrized surfaces in 3-dimensional space, one sometimes wants to highlight the curve where they intersect. Suppose that we have parametrizations \( \phi_i : \mathbb{P}^2 \to \mathbb{P}^3 \), \( i = 1, 2 \), and for simplicity assume that there are no basepoints. Then we want to find \( S_1 \cap S_2 \), where \( S_i \) is the image of \( \phi_i \).

To solve this problem, let \( F_1 = 0 \) be the implicit equation of \( S_1 \), and let \( C \subset \mathbb{P}^2 \) be defined by \( F_1 \circ \phi_2 = 0 \). This curve may be singular of high genus, but since we have an explicit equation for \( C \), there are known algorithms for drawing it (see, for example, [Hoffmann, Section 6.5]). Then \( \phi_2(C) \) is the desired intersection \( S_1 \cap S_2 \).

Another use of implicitization occurs when one creates new geometric models by applying Boolean operations to existing models. For example, if two balls intersect in \( \mathbb{R}^3 \), then removing one from the other involves knowing the curve where their boundaries intersect.
The next question concerns how to find the implicit equation, assuming we are given the parametrization $\phi$. In practice, three methods are used:

- Gröbner bases.
- Resultants.
- Syzygies.

This paper will concentrate on the third of these methods.

## 2 Syzygies and Equations of Curves

This section will report on joint work [CSC] with Tom Sederberg (Brigham Young University) and Falai Chen (University of Science and Technology of China).

We will work over $\mathbb{C}$. The idea is that we want to implicitize the map

$$\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2,$$

given by

$$(2.1) \quad \phi(s, t) = (a(s, t), b(s, t), c(s, t)), $$

where $a, b, c \in R = \mathbb{C}[s, t]$ are homogeneous polynomials of degree $n$ (here, $s, t$ are homogeneous coordinates on $\mathbb{P}^1$). We will also assume that $\gcd(a, b, c) = 1$. This ensures that $\phi$ has no basepoints (a point $(s_0, t_0) \in \mathbb{P}^1$ is a basepoint of $\phi$ if $a, b, c$ all vanish at $(s_0, t_0)$).

In [SSQR, SC, SGD], Sederberg and his co-workers introduced the idea of a moving line in $\mathbb{P}^2$. If we let $x, y, z$ be homogeneous coordinates for $\mathbb{P}^2$, then a moving line is an equation of the form

$$(2.2) \quad A(s, t)x + B(s, t)y + C(s, t)z = 0,$$

where $A, B, C \in R$ are homogeneous of the same degree. We can regard (2.2) as a family of lines parameterized by $(s, t) \in \mathbb{P}^1$.

One can easily imagine how the point of intersection of two moving lines traces out a curve in $\mathbb{P}^2$. This leads to the question of whether the map $\phi$ from (2.1) arises this way. If we dehomogenize by setting $t = 1$ (in $\mathbb{P}^1$) and $z = 1$ (in $\mathbb{P}^2$), then we get an easy answer, for in the case, (2.1) gives the curve in $\mathbb{C}^2$ parametrized by

$$x = \frac{a(s)}{c(s)}$$
$$y = \frac{b(s)}{c(s)},$$

where $a(s)$ is short for $a(s, 1)$, and similarly for $b(s)$ and $c(s)$. This can be thought of as the point of intersection of the moving vertical line $c(s)x - a(s) = 0$
and the moving horizontal line \(c(s)y - b(s) = 0\). Note that these are moving lines of degree \(n\). As we will see below, we get significantly lower degrees by allowing more general moving lines.

To formalize the above discussion, we make the following definition.

**Definition 2.1** The moving line (2.2) follows the parametrization (2.1) if

\[
A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) = 0
\]

for all \((s,t) \in \mathbb{P}^1\).

Geometrically, this means that for all \((s,t)\), the point on the parametrized curve lies on the corresponding line. More important is the algebraic interpretation of Definition 2.1, which says that \(A, B, C\) is a syzygy on \(a, b, c\). We write this as

\[
(A, B, C) \in \text{Syz}(a, b, c),
\]

where \(\text{Syz}(a, b, c) \subset R^3\) is the syzygy module of \((a, b, c)\).

Since \(\text{Syz}(a, b, c)\) is a graded module, we can speak of its graded piece in dimension \(d\), denoted \(\text{Syz}(a, b, c)_d\). We will now describe how \(\text{Syz}(a, b, c)_{n-1}\) determines the implicit equation of the image of (2.1).

To see how this works, consider the map

\[
R^3_{n-1} \xrightarrow{(a,b,c)} R^2_{2n-1},
\]

where subscripts indicated graded pieces (remember that \(a, b, c\) have degree \(n\)). The kernel of this map is \(\text{Syz}(a, b, c)_{n-1}\). Note also that \(\dim R^3_{n-1} = 3n\) and \(\dim R^2_{2n-1} = 2n\), so that

\[
\dim \text{Syz}(a, b, c)_{n-1} = n \iff (2.3) \text{ has maximal rank.}
\]

Later, we will see that (2.3) always has maximal rank by regularity. We will assume this for now. By (2.4), it follows that we can find \(n\) linearly independent moving lines which follow \(\phi\). Write these moving lines as follows:

\[
A_i x + B_i y + C_i z = \sum_{j=0}^{n-1} L_{i,j}(x,y,z) s^j t^{n-1-j}, \quad i = 0, \ldots, n-1.
\]

Note that \(L_{i,j}(x,y,z)\) is a linear form with coefficients in \(C\). Then one of the main results of [SC] is the following.

**Theorem 2.2** Let \(C\) be the image of (2.1), and let \(d\) be the generic degree of the induced map \(\mathbb{P}^1 \to C\). Then

\[
\det(L_{i,j}) = \lambda F^d,
\]

where \(\lambda \in C \setminus \{0\}\) and \(F = 0\) is the (irreducible) implicit equation of \(C \subset \mathbb{P}^2\).
Note how the numbers work in this theorem: since $a, b, c$ have degree $n$, the curve $C$ traced out by $\phi$ has degree $n/d$, where $d$ is the generic degree. Thus $F^d$ has degree $n$. On the other hand, the determinant in Theorem 2.2 also has degree $n$ since $(L_{i,j})$ is an $n \times n$ matrix of linear forms.

So far, we have used only one graded piece of the syzygy module, namely $\text{Syz}(a, b, c)_{n-1}$. We next turn our attention to the entire syzygy module. As we will see below, the structure of this module will give deeper insight into the determinant $\det(L_{i,j})$ used in Theorem 2.2.

The key tool for understanding $\text{Syz}(a, b, c)$ is the Hilbert Syzygy Theorem, which implies that syzygy modules of homogeneous polynomials in $R = \mathbb{C}[s, t]$ are always free. More precisely, if we set $I = \langle a, b, c \rangle \subset R = \mathbb{C}[s, t]$, then the syzygy Theorem and an easy argument using the Hilbert Polynomial imply that $R/I$ has a free resolution

$$0 \to R(-n - \mu_1) \oplus R(-n - \mu_2) \to R(-n)^3 \to R \to R/I \to 0,$$

where $\mu_1 + \mu_2 = n$ and the map $R^3 \to R$ is given by $a, b, c$. (See [CLO2, CSC] for the details.) We can assume $\mu_1 \leq \mu_2$, and following [CSC], we let $\mu = \mu_1$, so that $\mu \leq n - \mu = \mu_2$. In down to earth terms, the above resolution means that $\text{Syz}(a, b, c)$ is a free $R$-module of rank two with generators, say $p$ and $q$, of respective degrees $\mu$ and $n - \mu$. We call $p, q$ a $\mu$-basis of the syzygy module.

The existence of a $\mu$-basis has some nice consequences. First, note that syzygies of degree $n - 1$ can be uniquely written in the form

$$h_1 p + h_2 q,$$

where

$$\deg h_1 = n - \mu - 1 \quad \text{and} \quad \deg h_2 = \mu - 1.$$

Computing dimensions, we conclude that

$$\dim \text{Syz}(a, b, c)_{n-1} = n,$$

so that by (2.4), we see that (2.3) always has rank $n$, as claimed earlier.

Another consequence (2.6) is that a basis of $\text{Syz}(a, b, c)_{n-1}$ is given by

$$s^i t^{n-\mu-1-i} p, \quad 0 \leq i \leq n - \mu - 1, \quad s^i t^{\mu-1-i} q, \quad 0 \leq i \leq \mu - 1.$$

If we use this basis to form the linear forms $L_{i,j}$ as in (2.7), then we easily obtain

$$\det(L_{i,j}) = \text{Res}(p, q).$$

Thus the syzygies of degree $n - 1$ compute the resultant of the $\mu$-basis. If we combine this with Theorem 2.2, then we conclude that

$$\text{Res}(p, q) = \lambda F^d, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$ 

This shows that the $\mu$-basis computes the implicit equation of the curve. A careful proof can be found in [CSC].
We can also make some comments from a computational point of view. In order to use Theorem 2.2, one needs a basis of \( \text{Syz}(a, b, c)_{n-1} \). Since the matrix of (2.3) is \( 2n \times 3n \), the complexity of computing a basis is \( O(n^3) \) by standard methods in numerical linear algebra. However, [SZ] gives an \( O(n^2) \) method, based on variant of the Buchberger algorithm, for finding a \( \mu \)-basis of \( \text{Syz}(a, b, c) \).

Here are some other results from [CSC] concerning \( \mu \)-bases:

- The regularity of the ideal \( I = \langle a, b, c \rangle \subset \mathbb{C}[s, t] \) is \( n - \mu - 1 \). Thus knowing \( \mu \) is equivalent to knowing the regularity. Note also that \( n - \mu - 1 \leq n - 1 \). This explains why syzygies of degree \( n - 1 \) work so well—regularity always holds for this degree, no matter what \( \mu \) is.
- We defined \( \mu \) so that \( \mu \leq n - \mu \), which implies \( 0 \leq \mu \leq \lfloor n/2 \rfloor \). When one considers all triples \( a, b, c \) in \( R \) of degree \( n \) with \( \gcd(a, b, c) = 1 \), one can show that \( \mu = \lfloor n/2 \rfloor \) is generic.
- The usual Sylvester form of the resultant expresses \( \text{Res}(p, q) \) as the \( n \times n \) determinant described above. One can also express \( \text{Res}(p, q) \) as a \( (n - \mu) \times (n - \mu) \) determinant where \( n - 2\mu \) rows are linear in the entries of \( p \) and \( \mu \) rows are quadratic and are built from the Bézoutian of \( p \) and \( q \).

Finally, I should comment that “moving lines” represent an independent discovery of the concept of syzygy by the computer science community. Furthermore, in their definition of “\( \mu \)-basis”, Sederberg and Chen essentially conjectured a special case of the Hilbert Syzygy Theorem.

It is interesting to note that this special case, which asserts that \( \text{Syz}(a, b, c) \) is a graded free \( R \)-module, was actually proved by Franz Meyer in 1887 in [Meyer]. Meyer also conjectured that a similar result should hold for \( \text{Syz}(a_1, \ldots, a_m) \), where the \( a_i \in R = \mathbb{C}[s, t] \) are homogeneous, but he was unable to prove this in general. Hilbert, in his great paper [Hilbert] of 1890, proves the general form of the Hilbert Syzygy Theorem and explains how to use the Hilbert polynomial. His very first application is to Meyer’s conjecture from 1887.

### 3 Syzygies and Tensor Product Surfaces

We now turn our attention to surface parametrizations. This section will report on the paper [CGZ] written with Ron Goldman and Ming Zhang, both of Rice University. Again, most proofs will be omitted.

In this section, we will consider a tensor product parametrization, which is a map

\[
\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3
\]

defined by

\[
\phi(s, u; t, v) = (a(s, u; t, v), b(s, u; t, v), c(s, u; t, v), d(s, u; t, v)),
\]

where \( a, b, c, d \) are homogeneous polynomials of degree \( n \) in \( s, t, u, v \).
where $a, b, c, d \in R = \mathbb{C}[s, u; t, v]$ are bihomogeneous polynomials of bidegree $(m, n)$. Here, we think of $s, u$ as homogeneous coordinates on the first factor of $\mathbb{P}^1$ and $t, v$ as homogeneous coordinates on the second. We will also assume that $a, b, c, d$ have no common factors, which implies that $\phi$ has at most finitely many basepoints (a point $(s_0, u_0; t_0, v_0) \in \mathbb{P}^1 \times \mathbb{P}^1$ is a basepoint if $a, b, c, d$ all vanish at $(s_0, u_0; t_0, v_0)$).

If $\phi$ has no basepoints (which we will assume throughout this section) and is generically one-to-one, then it is well-known that the image of $\phi$ is a surface $S \subset \mathbb{P}^3$ of degree $2mn$. The goal of this section is to find the defining equation of $S$ using syzygies.

The analog of a moving line in $\mathbb{P}^2$ is clearly a moving plane in $\mathbb{P}^3$. This is an equation of the form

$$A(s, u; t, v)x + B(s, u; t, v)y + C(s, u; t, v)z + D(s, u; t, v)w = 0,$$

where $x, y, z, w$ are homogeneous coordinates on $\mathbb{P}^3$ and $A, B, C, D \in R$ are bihomogeneous of the same bidegree. Moving planes were first considered in [SC]. Then we say that the above moving plane follows the parametrization (3.1) if

$$A(s, u; t, v)a(s, u; t, v) + B(s, u; t, v)b(s, u; t, v) + C(s, u; t, v)c(s, u; t, v) + D(s, u; t, v)d(s, u; t, v) = 0$$

for all $(s, u; t, v) \in \mathbb{P}^1 \times \mathbb{P}^1$. Thus the moving plane follows the parametrization if and only if

$$(A, B, C, D) \in \text{Syz}(a, b, c, d).$$

As we will soon see, moving planes are not sufficient—to get the implicit equation of the image of $\phi$, we will need to use moving surfaces of higher degree. This idea appears in [SC], and even for the curve case, one can use moving conics to get some interesting results concerning implicitization (see [SC, ZCC]).

For us, the crucial ingredient will be moving quadrics, which are equations of the form

$$A(s, u; t, v)x^2 + B(s, u; t, v)xy + \cdots + I(s, u; t, v)zw + J(s, u; t, v)w^2 = 0,$$

where $A, B, \ldots, I, J \in R$ are bihomogeneous of the same bidegree. It should be clear what it means for a moving quadric to follow the parametrization (3.1), and one easily sees that this is equivalent to

$$(A, B, \ldots, I, J) \in \text{Syz}(a^2, ab, \ldots, cd, d^2) \subset R^{10}.$$

For curves, moving lines of degree $n - 1$ played a crucial role. For a tensor product surface, it thus makes sense to consider moving planes and quadrics of bidegree $(m - 1, n - 1)$ which follow the parametrization. The moving planes of this degree are the kernel of the map

$$MP : R_{m-1,n-1}^4 \xrightarrow{(a,b,c,d)} R_{2m-2,2n-2}.$$
(remember that $a, b, c, d$ have bidegree $(m, n)$). Both of these vector spaces have dimension $4mn$, so that generically, we expect $MP$ to be an isomorphism. In other words, there should usually be no moving planes of bidegree $(m-1, n-1)$.

Thus we turn our attention to moving quadrics which follow the parametrization. In bidegree $(m-1, n-1)$, these are given by the kernel of the map

$$MQ : R_{m-1,n-1}^{10} \to R_{3m-1,3n-1}^{31}$$

In this case, one easily sees that

$$\dim \text{Syz}(a, b, c, d)_{m-1,n-1} = mn \iff (3.3) \text{ has maximal rank.}$$

For now, we will assume that we have precisely $mn$ linearly independent moving quadrics of bidegree $(m-1, n-1)$ which follow the parametrization. Label these as $Q_i$ for $1 \leq i \leq mn$. The idea is to construct a square matrix by writing out the $Q_i$ as we did in (2.5). Here, we first dehomogenize by setting $u = v = 1$ to simplify the resulting formulas. Thus $Q_i$ can be written

$$Q_i = A_i x^2 + \cdots + J_i w^2$$

$$= \left( \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} A_{i,jk} s^j t^k \right) x^2 + \cdots + \left( \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} J_{i,jk} s^j t^k \right) w^2$$

$$= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} Q_{i,jk}(x, y, z, w) s^j t^k,$$

where $Q_{i,jk}$ is a quadric in $x, y, z, w$ with coefficients in $\mathbb{C}$. Furthermore, note that $i$ ranges over the $mn$ numbers 1 to $mn$ while $(j, k)$ ranges over the $mn$ pairs $(0, 0)$ to $(m-1, n-1)$. It follows that we can arrange the $Q_{i,jk}$ into a square matrix of size $mn \times mn$, where each entry is a quadric in $x, y, z, w$. We write this as

$$M = (Q_{i,jk}).$$

Notice that $\det M$ has degree $2mn$ in $x, y, z, w$.

We can now state one of the main results of [CGZ].

**Theorem 3.1** Suppose that $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ has no basepoints and is generically one-to-one. If $MP$ from (3.2) has maximal rank, then so does $MQ$ from (3.3) and furthermore, the image of $\phi$ is defined by the equation $\det M = 0$.

**Proof.** I will sketch some parts of the proof. One begins by changing coordinates on $\mathbb{P}^3$ so that $a, b, c$ have no basepoints. Then consider the map

$$MQ' : R_{m-1,n-1}^9 \to R_{3m-1,3n-1}^{31}$$
obtained from (3.3) by omitting $d^2$. Here, $\dim R^0_{m-1,n-1} = \dim R_{3m-1,3n-1} = 9mn$. Thus $\det MQ' \neq 0$ implies that $MQ$ has maximal rank. This in turn will give the $mn$ linearly independent moving quadrics of bidegree $(m-1, n-1)$ needed to construct the matrix $M$.

I will discuss two proofs that $\det MQ' \neq 0$. For the first, suppose that $\det MQ' = 0$. Then there is a nontrivial syzygy

$$Aa^2 + Bab + \cdots + Icd = 0,$$

where $A, B, \ldots, I$ are bihomogeneous of bidegree $(m-1, n-1)$. Since every term contains $a, b$ or $c$ (we got rid of $d^2$), we obtain

\begin{equation}
(Aa + Bb + Cc + Dd)a + (Eb + Fc + Gd)b + (Hc + Id)c = 0.
\end{equation}

This is a syzygy on $a, b, c$ of bidegree $(2m-1, 2n-1)$. I remember when Ron Goldman showed me this equation and asked me if it implied that

\begin{equation}
Hc + Id = -h_1a - h_3b
\end{equation}

for bihomogeneous polynomials $h_1, h_3$ of bidegree $(m-1, n-1)$. If (3.6) is true, then we get a nontrivial syzygy on $a, b, c, d$, which contradicts our assumption that $MP$ has maximal rank.

Hence I needed to show that (3.5) implies (3.6). The idea, of course, is that there is a Koszul complex lurking in the background. In general, if

$$\mathcal{A}a + \mathcal{B}b + \mathcal{C}c = 0,$$

then we say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a Koszul syzygy if there are $h_1, h_2, h_3$ such that

\begin{align*}
\mathcal{A} &= h_1c + h_2b \\
\mathcal{B} &= -h_2a + h_3c \\
\mathcal{C} &= -h_1a - h_3b.
\end{align*}

If we were working in $\mathbb{P}^2$, then $a, b, c$ having no basepoints would imply that they were a regular sequence, and it would follow immediately that their Koszul complex is exact. In particular, every syzygy on $a, b, c$ would be Koszul, so that (3.5) $\Rightarrow$ (3.6) is automatic in $\mathbb{P}^2$.

But we are in $\mathbb{P}^1 \times \mathbb{P}^1$, where $a, b, c$ are bihomogeneous. In this bigraded situation, $a, b, c$ almost never form a regular sequence, and their Koszul complex need not be exact (it is easy to give counterexamples). Instead, I had to use the vanishing of a certain sheaf cohomology group to show that every syzygy of bidegree $(2m-1, 2n-1)$ is Koszul (see [CGZ] for details). Hence (3.6) is true, and as explained above, this completes the first proof that $\det MQ' \neq 0$.

The second proof that $\det MQ' \neq 0$ is much quicker. In [ZCG], it was conjectured that

\begin{equation}
\det MQ' = \text{Res}(a, b, c)(\det MP)^3.
\end{equation}

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Since \(a, b, c\) have no basepoints, their resultant is nonvanishing, and \(\det MP \neq 0\) by assumption. Then \(\det MQ' \neq 0\) would follow immediately from the above identity. In a recent paper [D’Andrea], Carlos D’Andrea not only proved (3.8) but also generalized it moving surfaces of degree \(>2\) as well.

Once we know that \(\det MQ' \neq 0\), we can construct the desired matrix \(M\). It is easy to see that \(\det M\) vanishes on the image of \(\phi\) since the moving quadrics used in \(M\) all follow the parametrization. Furthermore, standard techniques from resultant theory show that \(\det M\) is not identically zero (one shows that the coefficient of \(w^{2mn}\) is nonzero). It follows that \(\det M\) is a nonzero polynomial of degree \(2mn\) which vanishes on the image of \(\phi\). But since \(\phi\) is generically one-to-one, its image has degree \(2mn\). Hence \(\det M = 0\) must be the irreducible equation of the image.

Theorem 3.1 assumes that \(\phi\) has no basepoints, that \(MP\) has maximal rank, and that \(\phi\) is generically one-to-one. Recently, D’Andrea has been able to weaken some of these hypotheses:

- If \(MP\) has maximal rank and \(d\) is the generic degree of \(\phi\), then \(\det M = \lambda F^d\), where \(\lambda \in \mathbb{C} \setminus \{0\}\) and \(F = 0\) is the irreducible equation of the image. This is proved in [D’Andrea].

- If \(\phi\) is generically one-to-one, the \(MP\) has maximal rank. This is unpublished.

It follows that when \(\phi\) has no basepoints, we can modify Theorem 3.1 to assert that \(\det M = \lambda F^d\) if either \(MP\) has maximal rank or \(\phi\) is generically one-to-one.

## 4 Syzygies and Triangular Surfaces

Here, we will indicate how the results of the previous section can be modified in the case of a triangular parametrization

\[
\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^3,
\]

which is given by homogeneous polynomials \(a, b, c, d \in R = \mathbb{C}[s, t, u]\) of degree \(n\). As above, we assume that \(a, b, c, d\) have no basepoints. Then the analogs of \(MP\) and \(MQ\) are

\[
\begin{align*}
MP : R_{n-1}^4 & \xrightarrow{(a,b,c,d)} R_{2n-1} \\
MQ : R_{n-1}^{10} & \xrightarrow{(a^2, ab, \ldots, cd, d^2)} R_{3n-1}.
\end{align*}
\]

Assuming that \(MP\) and \(MQ\) have maximal rank, one easily computes that

\[
\dim \text{Syz}(a, b, c, d)_{n-1} = n \\
\dim \text{Syz}(a^2, ab, \ldots, cd, d^2)_{n-1} = (n^2 + 7n)/2.
\]
However, each moving plane of degree $n - 1$ which follows $\phi$ can be multiplied by $x, y, z, w$ to get a moving quadric. This gives a subspace of $\text{Syz}(a^2, \ldots, d^2)_{n-1}$ of dimension $4n$, and if we pick a complementary subspace, then we obtain

- $n$ linearly independent moving planes which follow $\phi$
- $(n^2 - n)/2$ linearly independent moving quadrics not coming from moving planes which follow $\phi$.

Note also that there are $(n^2 + n)/2$ monomials in $s, t, u$ of degree $n - 1$. It follows that if we expand the above moving planes and quadrics as in (3.4), then we get a matrix $M$ of size $(n^2 + n)/2 \times (n^2 + n)/2$, where the first $n$ rows (coming from the moving planes) are linear in $x, y, z, w$ and the remaining $(n^2 - n)/2$ rows (coming from moving quadrics) are quadratic in $x, y, z, w$. It follows that

$$\text{deg} (\det M) = 1 \cdot n + 2 \cdot (n^2 - n)/2 = n^2.$$ 

Then the following theorem is proved in [CGZ].

**Theorem 4.1** Suppose that $\phi : \mathbb{P}^2 \to \mathbb{P}^3$ has no basepoints and is generically one-to-one. If $M \mathbb{P}$ from (4.1) has maximal rank, then so does $M Q$ from (4.1) and furthermore, the image of $\phi$ is defined by the equation $\det M = 0$.

As in the tensor product case, the exactness of a certain Koszul complex plays a key role in the proof. We also note that the improvements that D’Andrea made to Theorem 3.1 apply to Theorem 4.1 as well.

### 5 Syzygies and Basepoints

This section discusses new results which (we hope!) will shed light on how syzygies can be used to compute implicit equations of parametrized surfaces in the presence of basepoints. For simplicity, we will concentrate on the triangular case, where

$$\phi : \mathbb{P}^2 \to \mathbb{P}^3$$

is the rational map given by homogeneous polynomials $a, b, c, d \in R = \mathbb{C}[s, t, u]$ of degree $n$. Note that $\phi$ is a morphism outside the set of basepoints.

#### 5.1 Strong $\mu$-Bases for Surfaces

We first ask if it ever happens that the syzygy module $\text{Syz}(a, b, c, d)$ is free. While this always happens in the curve case, it is quite rare for surfaces. In the discussion which follows, we will make frequent use of standard results in commutative algebra. A good reference is [Eisenbud], especially Chapters 18–20.

We first consider the case when $\phi$ has no basepoints.
Proposition 5.1 Let \(a, b, c, d \in R = \mathbb{C}[s, t, u]\) be homogeneous polynomials of degree \(n\), and assume that \(a, b, c, d\) have no common zeros on \(\mathbb{P}^2\). Then \(\text{Syz}(a, b, c, d)\) is not a free \(R\)-module.

Proof. Let \(I = \langle a, b, c, d \rangle \subset R\), and let \(\mathfrak{m}\) denote the maximal ideal of \(R/I\). This ring has Krull dimension 0 since there are no basepoints, and thus \(\mathfrak{m}\) has codimension 0. The usual inequality between depth and codimension implies that \(\mathfrak{m}\) has depth 0 as well. Then the Auslander-Buchsbaum Theorem easily implies that the projective dimension of \(R/I\) is 3.

However, if \(\text{Syz}(a, b, c, d)\) were free, then we would get the free resolution
\[
0 \to \text{Syz}(a, b, c, d) \to R(-n)^4 \to R \to R/I \to 0,
\]
which would imply that the projective dimension is \(\leq 2\).

We next consider the case when there are basepoints.

Proposition 5.2 Let \(a, b, c, d \in R = \mathbb{C}[s, t, u]\) be homogeneous polynomials of degree \(n\), and assume that \(\gcd(a, b, c, d) = 1\). Set \(I = \langle a, b, c, d \rangle \subset R\). If \(a, b, c, d\) have at least one common zero in \(\mathbb{P}^2\), then the following are equivalent:

1. \(\text{Syz}(a, b, c, d)\) is a free graded \(R\)-module.
2. \(R/I\) has projective dimension 2.
3. \(R/I\) is Cohen-Macaulay.
4. \(\langle s, t, u \rangle \not\in \text{Ass}(R/I)\).
5. \(I\) is saturated.

Proof. The equivalence of (1) and (2) follows from the proof of Proposition 5.1, and the equivalence of (2) and (3) follows easily from the Auslander-Buchsbaum Theorem and the definition of Cohen-Macaulay. The equivalence of (2) and (4) follows from Corollary 19.10 of [Eisenbud] since \(I\) is an ideal of codimension 2. (This corollary is for local rings, but as is typical, the proof also applies to the graded situation considered here.) Finally, \(\langle s, t, u \rangle \in \text{Ass}(R/I)\) if and only if \(\langle s, t, u \rangle = \text{Ann}(a)\) for some \(a \in R/I\). The latter is equivalent to \(I: \langle s, t, u \rangle \neq I\), and the equivalence of (4) and (5) follows.

The original version of this proposition gave only the equivalence of (1), (2) and (3). I am grateful to Hal Schenk for pointing out the relevance of (4) and (5).

Now suppose that \(\text{Syz}(a, b, c, d)\) is a free graded \(R\)-module. Then an easy Hilbert polynomial argument shows that we have an exact sequence of the form
\[
0 \to R(-n - \mu_1) \oplus R(-n - \mu_2) \oplus R(-n - \mu_3) \xrightarrow{A} R(-n)^4 \xrightarrow{B} R \to R/I \to 0,
\]
where \(\mu_1 + \mu_2 + \mu_3 = n\). Here \(B\) is the map given by \((a, b, c, d)\), and the columns of \(A\) give three syzygies \(p_1, p_2, p_3\) of respective degrees \(\mu_1, \mu_2, \mu_3\) which are free.
generators of \text{Syz}(a, b, c, d). Furthermore, as is well-known, the Hilbert-Burch Theorem implies that \(a, b, c, d\) are (up to sign) the maximal minors of the matrix \(A\).

In this situation, we say that \(p_1, p_2, p_3\) form a \textit{strong} \(\mu\)-basis, where \(\mu = (\mu_1, \mu_2, \mu_3)\) for \(\mu_1 \leq \mu_2 \leq \mu_3\). We next describe how \(\mu\) influences the map \(\phi\).

**Proposition 5.3** Suppose that \(\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^3\) is given by \(a, b, c, d\) as above and that \text{Syz}(a, b, c, d) has a strong \(\mu\)-basis for \(\mu = (\mu_1, \mu_2, \mu_3)\), \(\mu_1 + \mu_2 + \mu_3 = n\). Assume in addition that \(\phi\) is generically one-to-one and that \(V(a, b, c, d) \subset \mathbb{P}^2\) is a local complete intersection. Then:

1. The degree of the image of \(\phi\) in \(\mathbb{P}^3\) is
   \[
   \frac{1}{2}(n^2 - (\mu_1^2 + \mu_2^2 + \mu_3^2)) = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3.
   \]

2. The sum of the multiplicities of the basepoints of \(\phi\) is
   \[
   \frac{1}{2}(n^2 + (\mu_1^2 + \mu_2^2 + \mu_3^2)) = n^2 - (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3).
   \]

**Proof.** Let \(I = \langle a, b, c, d \rangle \subset R\) and \(Z = V(I) \subset \mathbb{P}^2\). Since \(\phi\) is generically one-to-one, we know that the image of \(\phi\) is a surface in \(\mathbb{P}^3\) of degree

\[
(5.1) \quad n^2 - \sum_{p \in Z} e(I_{Z,p}, \mathcal{O}_{\mathbb{P}^2,p}),
\]

where \(I_{Z,p} \subset \mathcal{O}_{\mathbb{P}^2}\) is the ideal sheaf of \(Z \subset \mathbb{P}^2\) and \(e(I_{Z,p}, \mathcal{O}_{\mathbb{P}^2,p})\) is the multiplicity, as defined in \([BH, 4.5]\). (A proof of \((5.1)\) is sketched in the Appendix.) It follows that part 1 of the proposition is an immediate consequence of part 2.

Since \(Z\) is a local complete intersection, we have \(e(I_{Z,p}, \mathcal{O}_{\mathbb{P}^2,p}) = \dim_{\mathcal{C}} \mathcal{O}_{Z,p}\) for all \(p \in Z\). In other words,

\[
\sum_{p \in Z} e(I_{Z,p}, \mathcal{O}_{\mathbb{P}^2,p}) = \dim_{\mathcal{C}} \mathcal{H}^0(Z, \mathcal{O}_Z).
\]

Using the usual vanishing theorems for the sheaf cohomology of \(\mathbb{P}^2\), one obtains

\[
\dim_{\mathcal{C}} \mathcal{H}^0(Z, \mathcal{O}_Z) = \dim_{\mathcal{C}} (\mathcal{R}/I)_d
\]

for \(d \gg 0\). Since \(\dim_{\mathcal{C}} \mathcal{R}_d = \binom{d+2}{2}\), the above resolution for \(\mathcal{R}/I\) shows that

\[
(5.2) \quad \dim_{\mathcal{C}} (\mathcal{R}/I)_d = \frac{1}{2} (n^2 + (\mu_1^2 + \mu_2^2 + \mu_3^2))
\]

for \(d \gg 0\). Combining the above equalities completes the proof.

In using Proposition 5.3 note that

\[
\mu_1^2 + \mu_2^2 + \mu_3^2 = (\mu_1 - n/3)^2 + (\mu_2 - n/3)^2 + (\mu_3 - n/3)^2 + n^2/3
\]

since \(\mu_1 + \mu_2 + \mu_3 = n\). It follows that

\[
\mu_1^2 + \mu_2^2 + \mu_3^2 \geq n^2/3.
\]
Let \( N = \sum_{p \in Z} e(I_{Z,p}, \mathcal{O}_{\mathbb{P}^2, p}) \) be the number of basepoints, counted with multiplicity. Combining the above inequality with Proposition 5.3 we see that if \( \phi \) has a strong \( \mu \)-basis, then the number of basepoints of \( \phi \) is bounded below by

\[ N \geq \frac{2n^2}{3}. \]

Thus surface parametrizations with strong \( \mu \)-bases have lots of basepoints.

The classic example of Proposition 5.3 is when \( n = 3 \). Then \( \mu_1 + \mu_2 + \mu_3 = 3 \) implies \( \mu_1 = \mu_2 = \mu_3 = 1 \) (since \( \mu_i = 0 \) would imply that the image of \( \phi \) lies in a plane). It follows that the surface has degree

\[ \frac{1}{2}(3^2 - (1^2 + 1^2 + 1^2)) = 3, \]

and the number of basepoints is

\[ \frac{1}{2}(3^2 + (1^2 + 1^2 + 1^2)) = 6. \]

This, of course, is the usual representation of a cubic surface in \( \mathbb{P}^3 \) as \( \mathbb{P}^2 \) blown up at 6 points. In the 19th century, algebraic geometers were aware that cubic surfaces have strong \( \mu \)-bases. For example, the 1915 edition of [Salmon, Vol. II] states on p. 264 that “Clebsch has used the theorem that any cubic may be generated as the locus of the intersection of three corresponding planes, each of which passes through a fixed point.” This refers to three moving planes, which Salmon called “sheaves of planes” (in a footnote on p. 25 of Volume I, Salmon notes that the term “sheaf” comes from the German “Bündel”). We should also mention that [Sommerville, p. 389] calls a moving plane a “bundle of planes”. I am grateful to Tom Sederberg for supplying these references.

One feature of Proposition 5.3 is the requirement that the basepoint locus \( Z \) be a local complete intersection. This is because the degree of the surface naturally involves the multiplicity of the basepoints, while the Hilbert polynomial computation given in (5.2) computes the degree of the basepoints. As is well-known, these agree only for a local complete intersection. It would be interesting to study what happens to Proposition 5.3 when the basepoints are not a local complete intersection. Since having a strong \( \mu \)-basis is such a restrictive condition, it is possible that the basepoints are very special.

### 5.2 Syzygies Which Vanish at Basepoints

At one point in the proof of Theorem 3.1, we needed to know that a certain syzygy on \( a, b, c \) was a Koszul syzygy (this was (3.5) \( \Rightarrow \) (3.6)). We now study what happens when basepoints are present.

We will consider homogeneous polynomials \( a, b, c \in R = \mathbb{C}[s, t, u] \) of degree \( n \), where \( \gcd(a, b, c) = 1 \). If \( a, b, c \) have no basepoints, then they are a regular sequence, so that every syzygy

\[ Aa + Bb + Cc = 0 \]
must be a Koszul syzygy

\[ A = h_1c + h_2b \]
\[ B = -h_2a + h_3c \]
\[ C = -h_1a - h_3b, \]

as in (3.7).

Now suppose that \(a, b, c\) have some basepoints. Then it is easy to make examples of syzygies which are not Koszul. One easy observation is that Koszul syzygies also vanish at the basepoints. This leads to the following question:

\begin{equation}
\text{If } Aa + Bb + Cc = 0 \text{ and } A, B, C \text{ vanish at the basepoints, then is } A, B, C \text{ a Koszul syzygy?} \tag{5.3}
\end{equation}

To make this precise, we need the following definition.

**Definition 5.4** Let \(a, b, c \in R = \mathbb{C}[s, t, u]\) be homogeneous with no common factors, and let \(Z = V(a, b, c)\) be their basepoint locus, regarded as a 0-dimensional subscheme of \(\mathbb{P}^2\). Then a homogeneous polynomial \(A \in R\) of degree \(d\) **vanishes at the basepoints** if \(A\) is in the kernel of the map

\[ H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow H^0(Z, \mathcal{O}_Z(d)). \]

Equivalently, \(A\) vanishes at the basepoints if and only if \(A\) is in the saturation of \(I = \langle a, b, c \rangle \subset R\).

It would probably be better to say “vanishes scheme-theoretically at the basepoints” or “vanishes with multiplicity at the basepoints” in Definition 5.4. We hope that the simpler phrase “vanishes at the basepoints” will not cause confusion.

We say that a syzygy \(Aa + Bb + Cc = 0\) **vanishes at the basepoints** if \(A, B, C\) vanish at the basepoints of \(a, b, c\) in the sense of Definition 5.4. Thus (5.3) now has a precise meaning.

One way to think about (5.3) is that vanishing at the basepoints is a local condition, while being a Koszul syzygy is a global condition. Because of this, it turns out that the answer to (5.3) is sometimes “no”. For an example, suppose that

\[ a = s^2u + st^2 \]
\[ b = stu + 2t^3 \]
\[ c = t^2u + s^3. \]

One can show without difficulty that the \(p = (0, 0, 1) \in \mathbb{P}^2\) is the unique basepoint. It has multiplicity 4 and degree 3, and locally looks like \(\mathbb{C}[s, t]/(s^2, st, t^2)\). Using Macaulay, one finds the syzygy of degree 5 given by

\[ A = t^2u^3 - 2s^2t^2u \]
\[ B = -stu^3 + s^3tu \]
\[ C = st^2u^2. \tag{5.4} \]
It is obvious that this syzygy vanishes at $p$ in the sense of Definition 5.4. This is because every term contains either $s^2$, $st$ or $t^2$. With a little more work, one can show that (5.4) is not a Koszul syzygy. So (5.3) is not always true.

So the next question is whether there exist special classes of base points for which the answer to (5.3) is “yes”. We do not yet have a complete answer to this question, but we do know one class of basepoints for which this works.

Given $a, b, c \in R$ and $Z = V(a, b, c) \subset \mathbb{P}^2$ as usual, we say that $p \in Z$ has embedding dimension at most one if the Zariski tangent space of $Z$ at $p$ has dimension $\leq 1$. One easily sees that this is equivalent to either of the following conditions:

- One of $a, b, c$ has a nonvanishing partial derivative at $p$.
- There are local analytic coordinates $u, v$ at $p \in \mathbb{P}^2$ such that near $p$, $Z$ is defined as a formal scheme by $u = v^k = 0$.

The second characterization shows that $Z$ is a local complete intersection at $p$. It follows easily that $k$ is the multiplicity of the basepoint $p$. These basepoints were introduced under the name aligned by Iarrobino in 1981 [Iarrobino1], though these days, the name curvilinear is more common—see, for example, [Iarrobino2, Le Barz].

Then we have the following result.

**Theorem 5.5** Suppose that $a, b, c \in R$ are as usual, and let $Z = V(a, b, c)$ be the basepoint locus. If all basepoints are curvilinear (i.e., have embedding dimension at most one), then every syzygy on $a, b, c$ which vanishes on $Z$ is a Koszul syzygy.

Before we can give the proof, we need some preliminary results, the first of which is the following vanishing lemma.

**Lemma 5.6** Let $X$ be a smooth complete surface with $q := \dim \mathbb{C} H^1(X, \mathcal{O}_X) = 0$, and let $L \subset X$ be a smooth rational curve with self-intersection $L^2 = 1$. Then $H^1(X, \mathcal{O}_X(mL)) = 0$ for all $m \in \mathbb{Z}$.

**Proof.** Tensoring the exact sequence $0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_L \to 0$ with $\mathcal{O}_X(mL)$ gives

$$0 \to \mathcal{O}_X((m - 1)L) \to \mathcal{O}_X(mL) \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0$$

since $L \cong \mathbb{P}^1$ and $L^2 = 1$. Then the long exact sequence in cohomology yields

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) \to H^1(X, \mathcal{O}_X((m - 1)L)) \to H^1(X, \mathcal{O}_X(mL)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)).$$

Using $q = 0$ and the vanishing of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ for $m \geq 0$, the lemma follows for $m \geq 0$ by induction.
Observe that $H^0(X, \mathcal{O}_X(mL)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ is an isomorphism for $m \leq 0$, since it is $\mathbb{C} \simeq \mathbb{C}$ for $m = 0$ and $H^0(X, \mathcal{O}_X(mL)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = \{0\}$ for $m < 0$. It follows that for $m \leq 0$, (5.5) gives the exact sequence

$$0 \to H^1(X, \mathcal{O}_X((m-1)L)) \to H^1(X, \mathcal{O}_X(mL)).$$

Using $q = 0$ and induction, the lemma follows for $m \leq 0$.

We next study curvilinear basepoints using toric geometry, as in [Fult on2, Chapter 2]. Our goal is to construct a toric blow-up of $0 \in \mathbb{C}^2 = \text{Spec}(\mathbb{C}[s, t])$ such that $I = \langle s, t^k \rangle \subset \mathbb{C}[s, t]$ becomes principal on $X$. For this purpose, let $N = \mathbb{Z}^2$, with basis $e_1, e_2$. Then let

$$v_i = \begin{cases} ie_1 + e_2 & 0 \leq i \leq k \\ e_1 & i = k + 1 \end{cases}$$

and define the cones $\sigma_0, \ldots, \sigma_k$ by

$$\sigma_i = \text{Cone}(v_i, v_{i+1}), \quad 0 \leq i \leq k.$$ 

Finally, let $\Delta_k$ be the fan consisting of the $\sigma_i$ and all of their faces. When $k = 3$, here is a picture of the fan $\Delta_3$ in $N_\mathbb{R} \simeq \mathbb{R}^2$:

The first quadrant $\sigma = \text{Cone}(e_2, e_1) = \text{Cone}(v_0, v_{k+1})$ is the union of the $\sigma_i$, and one sees that $\Delta_k$ is obtained from $\sigma$ by a sequence of $k$ stellar subdivisions. Turning to the corresponding toric varieties, $\sigma$ gives $\mathbb{C}^2$, and $\Delta_k$ gives the smooth toric surface $X_k = X(\Delta_k)$. We also have a natural map $\pi : X_k \to \mathbb{C}^2$ which is the successive blow-up of smooth points.

Let $Z \subset \mathbb{C}^2$ be the subscheme defined by $I = \langle s, t^k \rangle$, with ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{C}^2}$. Under $\pi : X_k \to \mathbb{C}^2$, we get the inverse image ideal sheaf

$$\mathcal{I}_Z' = \pi^{-1}\mathcal{I}_Z \cdot \mathcal{O}_{X_k} \subset \mathcal{O}_{X_k},$$

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as defined in [Hartshorne, II.7]. We can now explain how \( \pi : X_k \to \mathbb{C}^2 \) relates to the ideal \( I \). Recall that \( K_X \) denotes the canonical divisor of a smooth surface \( X \).

**Proposition 5.7** Let \( \pi : X_k \to \mathbb{C}^2 \) be as above. Then \( \mathcal{I}' = \mathcal{O}_{X_k}(-E) \), where \( K_{X_k} = \pi^* K_{\mathbb{C}^2} + E \).

**Proof.** As usual, each interior ray \( \rho_i = \text{Cone}(v_i) \), \( 1 \leq i \leq k \), corresponds to an orbit closure \( E_i \simeq \mathbb{P}^1 \) in \( X \). We will show that \( E = E_1 + 2E_2 + 3E_3 + \cdots + kE_k \) satisfies the two conditions of the proposition.

We know that \( X_k \) has the affine open cover given by

\[
U_i = \text{Spec}(\mathbb{C}[\sigma_i^\vee \cap M]), \quad 0 \leq i \leq k;
\]

where \( M \) is the dual of \( N \). In order to get coordinates for these affine pieces, let \( e_1, e_2 \) in \( M \) be the dual basis of \( e_1, e_2 \) in \( N \). Then \( s = \chi e_1, t = \chi e_2 \) are coordinates on

\[
\mathbb{C}^2 = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \text{Spec}(\mathbb{C}[s,t]),
\]

and one can show without difficulty that

\[
(5.6) \quad U_i = \begin{cases} 
\text{Spec}(\mathbb{C}[t^{i+1}/s, s/t^i]) & 0 \leq i \leq k-1 \\
\text{Spec}(\mathbb{C}[t, s/t^k]) & i = k.
\end{cases}
\]

As above, the orbit closures \( E_i \simeq \mathbb{P}^1 \) correspond to the interior rays of \( \Delta_k \). Of these, \( U_0 \) meets only \( E_1 \), \( U_1 \) meets only \( E_1 \) and \( E_2 \), and so on until \( U_k \) meets only \( E_k \). Furthermore, in terms of the local coordinates \((5.6)\), we have equations

\[
\text{On } U_0, \ E_1 \text{ is defined by } s = 0
\]

\[
\text{On } U_1, \ \begin{cases} 
E_1 \text{ is defined by } t^2/s = 0 \\
E_2 \text{ is defined by } s/t = 0 \\
\vdots
\end{cases}
\]

\[
\text{On } U_{k-1}, \ \begin{cases} 
E_{k-1} \text{ is defined by } t^k/s = 0 \\
E_k \text{ is defined by } s/t^{k-1} = 0
\end{cases}
\]

\[
\text{On } U_k, \ E_k \text{ is defined by } s = 0.
\]

Let \( E = E_1 + 2E_2 + 3E_3 + \cdots + kE_k \) be as above. Using the local equations for the \( E_i \), one can verify that

\[
(5.7) \quad \text{On } U_i, \ E \text{ is defined by } \begin{cases} 
s = 0 & 0 \leq i \leq k-1 \\
t^k = 0 & i = k
\end{cases}
\]

It remains to show that this divisor has the desired properties.
First, the inverse image ideal sheaf $I_Z$, when restricted to $U_i$, corresponds to the ideal generated by $s, t^k$ in the coordinate ring of $U_i$. Using (5.6), one easily computes that in the coordinate ring of $U_i$, one has

$$\langle s, t^k \rangle = \begin{cases} \langle s \rangle & 0 \leq i \leq k - 1 \\ \langle t^k \rangle & i = k. \end{cases}$$

By (5.7), this is the ideal defining $E$ on $U_i$, which implies that $I_Z = O_{X_k}(-E)$.

Second, we need to show that $K_X = \pi^* K_{\mathbb{P}^2} + E$. This follows by representing $X_k \to \mathbb{P}^2$ as a composition of successive blow-ups of smooth points

$$X_k \xrightarrow{\pi_{k-1}} X_{k-1} \xrightarrow{\pi_{k-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = \mathbb{C}^2$$

and using the well-known fact that $K_X = \pi^* K_{X_{k-1}} + E_i$. The key point that needs to be checked is that $\pi_{i+1}$ blows up a point on $E_i \setminus (E_1 \cup \cdots \cup E_{i-1})$. We omit the straightforward details.

We can now prove Theorem 5.5.

**Proof of Theorem 5.5.** Since the basepoint locus $Z = V(a, b, c)$ is curvilinear, each $p \in Z \subset \mathbb{P}^2$ is analytically equivalent to the germ of the analytic space $0 \in V(s, t^k) \subset \mathbb{C}^2$, where $k$ depends on $p$. By Proposition 5.7, there is a smooth surface $\pi : X \to \mathbb{P}^2$ such that $I_Z = O_X(-E)$ and $K_X = \pi^* K_{\mathbb{P}^2} + E$.

By the definition of $Z$, $a, b, c$ give global sections of $I_Z(n)$. In the blow-up, these become sections $\tilde{a}, \tilde{b}, \tilde{c}$ (the proper transforms of $a, b, c$) of $I_Z \otimes O_X \pi^* O_{\mathbb{P}^2}(n)$. If we let $L \subset X$ be the inverse image of a line in $\mathbb{P}^2$ missing $Z$, then this implies that $\tilde{a}, \tilde{b}, \tilde{c}$ are global sections of $O_X(nL - E)$. The key point is that $\tilde{a}, \tilde{b}, \tilde{c}$ have no basepoints on $X$. This, of course, is why we blew-up $\mathbb{P}^2$. Recall why this is true: $a, b, c$ give an exact sequence

$$O_{\mathbb{P}^2}(-n)^3 \to I_Z \to 0,$$

so that on the blow-up,

$$O_X(-nL)^3 \to I_Z \to 0$$

is still exact. Since $I_Z = O_X(-E)$ is invertible, $\tilde{a}, \tilde{b}, \tilde{c}$ give the exact sequence

$$(5.8) \quad O_X(E - nL)^3 \to O_X \to 0.$$  

Hence these sections can’t vanish simultaneously on $X$. Once we know that $\tilde{a}, \tilde{b}, \tilde{c}$ have no basepoints, they are locally a regular sequence, so that we can extend (5.8) to the Koszul complex

$$(5.9) \quad 0 \to O_X(3E - 3nL) \to O_X(2E - 2nL)^3 \to O_X(E - nL)^3 \to O_X \to 0,$$

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which is exact on \(X\).

Now suppose that \(Aa + Bb + Cc = 0\) is a syzygy, where \(A, B, C\) have degree \(m\) and vanish at \(Z\). This means that \(A, B, C\) are global sections of \(I_Z(m)\), and taking proper transforms as above, we get global sections \(\bar{A}, \bar{B}, \bar{C}\) of \(\mathcal{O}_X(mL - E)\). Now tensor (5.9) with \(\mathcal{O}_X((m + n)L - 2E)\) to obtain

\[
0 \to \mathcal{O}_X((m - 2n)L + E) \to \mathcal{O}_X((m - n)L)^3 \to \mathcal{O}_X(mL - E)^3 \to \mathcal{O}_X((m + n)L - 2E) \to 0.
\]

Splitting this into two short exact sequences in the usual way, one sees that if

\[
(5.10) \quad H^1(X, \mathcal{O}_X((m - 2n)L + E)) = \{0\},
\]

then

\[
(5.11) \quad H^0(X, \mathcal{O}_X((m - n)L))^3 \to H^0(X, \mathcal{O}_X((m - n)L)) \to H^0(X, \mathcal{O}_X((m + n)L - 2E))
\]

is exact at the middle term. For more details, see Proposition 2.2 of [CGZ].

We claim that (5.11) holds for all \(m\). To see why, note that \(K_X = \pi^*K_{\mathbb{P}^2} + E = -3L + E\). Thus Serre duality implies that

\[
H^1(X, \mathcal{O}_X((m - 2n)L + E)) \simeq H^1(X, \mathcal{O}_X(-[(m - 2n)L + E] + K_X))^* \simeq H^1(X, \mathcal{O}_X((2n - m - 3)L))^*.
\]

However, Lemma 5.6 implies \(H^1(X, \mathcal{O}_X((2n - m - 3)L)) = \{0\}\), and (5.10) follows.

This shows that (5.11) is exact at the middle term. However, the syzygy \(Aa + Bb + Cc = 0\) implies that \((\bar{A}, \bar{B}, \bar{C}) \in H^0(X, \mathcal{O}_X(mL - E))^3\) maps to 0 in \(H^0(X, \mathcal{O}_X((m + n)L - 2E))\). By exactness, this means that there is \((h_1, h_2, h_3) \in H^0(X, \mathcal{O}_X((m - n)L))^3\) which maps to \((\bar{A}, \bar{B}, \bar{C})\) in (5.11). However,

\[
H^0(X, \mathcal{O}_X((m - n)L)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-n))
\]

since \(\pi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2}\). It follows that \((h_1, h_2, h_3)\) are homogeneous polynomials of degree \(m - n\) which make \(A, B, C\) into a Koszul syzygy, as claimed. ♣

The proof of this theorem shows that a curvilinear basepoint locus \(Z \subset \mathbb{P}^2\) has the property that there is a blow-up \(\pi: X \to \mathbb{P}^2\) such that \(I_Z = \mathcal{O}_X(-E)\) and \(K_X = \pi^*K_{\mathbb{P}^2} + E\). Obviously, we can replace \(\mathbb{P}^2\) with any smooth complete surface and the same result holds. Furthermore, we suspect that the converse is true, though we have not proved this.

The main unresolved question is whether curvilinear basepoints are the only basepoints which have the property of Theorem 5.3. For example, does the theorem hold when \(Z\) is a local complete intersection? When blowing up more complicated basepoints, Enriques introduced Enriques diagrams to keep track of the combinatorics of the blow-ups. This has been greatly generalized in
Our treatment of curvilinear basepoints uses toric geometry and is a special case of the toric clusters and toric constellations discussed in [G-SP]. It would also be interesting to explore higher-dimensional versions of Theorem 5.5. It is possible that the papers just mentioned might provide useful tools for attacking this problem.

6 Final Comments

The results in the paper raise many questions to pursue. For instance, the results of Sections 3 and 4 should extend to the case when there are basepoints. To give an example of how this might work, suppose that

\[ \phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \]

has bidegree \((m, n)\) with a single base point \(p\) of multiplicity one. Then the map \(MP\) of (3.2) cannot be onto, since its image lies in the subspace of polynomials of degree \(2n - 1\) which vanish at \(p\). Here is what we would expect to happen:

- The basepoint should cause the rank of \(MP\) should drop by one, so that there should be exactly one linearly independent moving plane of degree \((m - 1, n - 1)\) which follows \(\phi\).

- The basepoint should also cause the rank of \(MQ\) from (3.3) to drop by three. This makes sense since \(a^2, ab, \ldots, cd, d^2\) all vanish to second order at \(p\). This means there should be \(mn + 3\) linearly independent moving quadrics of degree \((m - 1, n - 1)\) which follow \(\phi\).

- Multiplying the moving plane of the first bullet by \(x, y, z, w\) as we did in Section 4 gives four moving quadrics. Picking a complementary subspace, we get \(mn - 1\) linearly independent moving quadrics which don’t come from moving planes.

- If we use the one moving plane and \(mn - 1\) moving quadrics to create a matrix \(M\), we get an \(mn \times mn\) matrix with one linear row and \(mn - 1\) quadratic rows. It follows that \(\det M\) has degree \(1 + 2 \cdot (mn - 1) = 2mn - 1\).

- Since the image of \(\phi\) has degree \(2mn - 1\) (assuming \(\phi\) is generically one-to-one), the equation of the image should be \(\det M = 0\).

As the number and complexity of the basepoints increases, it becomes less clear how to modify the matrix \(M\) in order to get the equation of the image.

At the end of Section 5, I mentioned some open questions concerning syzygies and basepoints. There are also numerous questions from Sections 3 and 4 that I would love to be able to answer, including the following:

- For the tensor product case, we used syzygies of bidegree \((m - 1, n - 1)\), and for the triangular case, we used degree \(n - 1\). What is the systematic reason for choosing these degrees?
• Why do we need $a^2, ab, \ldots, cd, d^2$ in order to compute the implicit equation? In a sense, we can think of $a^2, ab, \ldots, cd, d^2$ as the “second symmetric power” of $a, b, c, d$.

• When $I = \langle a, b, c, d \rangle \subset R = \mathbb{C}[s, t, u]$ is the ideal coming from a triangular surface parametrization, what is the free resolution of $R/I$? How do the Betti numbers depend on $n$? As far as I know, this is an open problem.

• Resolutions of length 3 are studied in [Weyman]. Do the results of this paper shed any light on the resolution of $R/I$? One intriguing observation is that symmetric powers (such as those mentioned in the second bullet) appear in Weyman’s description.

• In the curve case, the determinant giving the implicit equation is the resultant of a basis of the syzygy module. In the surface case, $\text{Syz}(a, b, c, d)$ is usually not free. But is it still possible to interpret the determinants of Theorems [1] and [2] as some sort of resultant? Klaus Altmann suggested that this may involve the determinant of a complex built from a free resolution of $\text{Syz}(a, b, c, d)$ (which is related to the free resolution of $R/I$ mentioned in the previous two bullets).

• In the curve case, we saw that the structure of the syzygy module was closely related to the regularity of $R/I$. Does something similar happen in the surface case? Also, how does one define the regularity of a tensor product surface? This is not obvious since everything is now bigraded.

• Beauville studies hypersurfaces in $\mathbb{P}^n$ whose defining equation is a determinant in [Beauville]. How do his results relate to the theorems of Sections 3 and 4?

• When the base points are a complete intersection, the recent preprint [Busé] shows how the implicitization problem can be solved using the residual resultants defined in [BEM]. In [Busé], these resultants are computed as the gcd of three determinants or the product of two determinants divided by a third. One question is whether one can combine residual resultants with the syzygy methods introduced here to give a determinantal formula for the implicit equation, assuming that the base points are a local complete intersection. For example, in the tensor product case, we explained at the beginning of this section how to modify the matrix $M$ in the presence of a single basepoint. Can the determinant of $M$ be interpreted as a residual resultant?

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Appendix

In the proof of Proposition 5.3, we gave a formula (5.1) for the degree of the image of a generically one-to-one map \( \phi : \mathbb{P}^2 \to \mathbb{P}^3 \). As I learned by reading [Busé], this formula follows easily from the results in [Fulton].

Let \( \phi : \mathbb{P}^2 \to \mathbb{P}^3 \) be defined by homogeneous polynomials \( a, b, c, d \in \mathbb{C}[s, t, u] \) of degree \( n \). As in Section 5, the set of basepoints is \( Z = V(a, b, c, d) \subset \mathbb{P}^2 \). We will assume that:

- \( \gcd(a, b, c, d) = 1 \), so that \( Z \) is finite. The ideal sheaf of \( Z \) is \( \mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2} \) and, for each \( p \in Z \), we get the multiplicity \( e(\mathcal{I}_Z, p, \mathcal{O}_{\mathbb{P}^2}, p) \).

- \( S = \overline{\phi(\mathbb{P}^2 \setminus Z)} \) is a surface in \( \mathbb{P}^3 \). This allows us to define the degree \( \deg(S) \) of \( S \subset \mathbb{P}^3 \) and the generic degree \( \deg(\phi) \) of \( \phi \).

These numbers are related by the degree formula, which goes as follows:

\[
\deg(\phi) \deg(S) = n^2 - \sum_{p \in Z} e(\mathcal{I}_Z, p, \mathcal{O}_{\mathbb{P}^2}, p).
\]

To prove this, set \( L = \mathcal{O}_{\mathbb{P}^2}(n) \) and let \( \pi : X \to \mathbb{P}^2 \) be the blow-up of \( \mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2} \). Since the cycle \( \pi_* [X] \) has degree \( \deg(\phi) \deg(S) \), [Fulton], Proposition 4.4 implies that

\[
\deg(\phi) \deg(S) = \int_{\mathbb{P}^2} c_1(L)^2 - \int_Z (1 + c_1(L))^2 \cap s(Z, \mathbb{P}^2),
\]

where \( s(Z, \mathbb{P}^2) \) is the Segre class. It is standard that \( \int_{\mathbb{P}^2} c_1(L)^2 = n^2 \). Furthermore, \( c_1(L) = 0 \) on \( Z \) since \( Z \) is zero-dimensional. Hence the above formula reduces to

\[
\deg(\phi) \deg(S) = n^2 - \int_Z 1 \cap s(Z, \mathbb{P}^2).
\]

However, as explained on [Fulton], p. 79], the Segre class \( s(Z, \mathbb{P}^2) \) is given by

\[
s(Z, \mathbb{P}^2) = \sum_{p \in Z} e(\mathcal{I}_Z, p, \mathcal{O}_{\mathbb{P}^2}, p) [p].
\]

The degree formula now follows immediately.

When \( \phi \) is generically one-to-one, the degree formula reduces to the formula (5.1) used in the proof of Theorem 5.3.
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