AN INFINITELY GENERATED VIRTUAL COHOMOLOGY GROUP FOR NONCOCOMPACT ARITHMETIC GROUPS OVER FUNCTION FIELDS

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Abstract. Let $G(O_S)$ be a noncocompact irreducible arithmetic group over a global function field $K$ of characteristic $p$, and let $\Gamma$ be a finite-index, residually $p$-finite subgroup of $G(O_S)$. We show that the cohomology of $\Gamma$ in the dimension of its associated Euclidean building with coefficients in the field of $p$ elements is infinite.

Let $K$ be a global function field that contains the field with $p$ elements, $\mathbb{F}_p$. We let $S$ be a finite nonempty set of inequivalent valuations of $K$. The ring $O_S \subseteq K$ will denote the corresponding ring of $S$-integers. For any $v \in S$, we let $K_v$ be the completion of $K$ with respect to $v$ so that $K_v$ is a locally compact field.

We denote by $G$ a connected noncommutative absolutely almost simple $K$-group, and we let

$$k(G, S) = \sum_{v \in S} \text{rank}_{K_v} G$$

so that $k(G, S)$ is the dimension of the Euclidean building on which the arithmetic group $G(O_S)$ acts as a lattice. Thus for example, $k(\text{SL}_n, S) = |S|(n-1)$.

If $G$ is $K$-anisotropic, then $G(O_S)$ contains a torsion-free finite-index subgroup that acts freely and cocompactly on a Euclidean building of dimension $k(G, S)$. Determining the finiteness properties of arithmetic groups $G(O_S)$ in the case that $G$ is $K$-isotropic has been more difficult. The model for the $K$-isotropic case was provided by the following theorem of Stuhler [14].

**Theorem 1.** The arithmetic group $\text{SL}_2(O_S)$ is of type $F_{k(\text{SL}_2, S)-1}$, and if $\Gamma$ is any finite-index subgroup of $\text{SL}_2(O_S)$ whose only torsion elements are $p$-elements, then $H^{k(\text{SL}_2, S)}(\Gamma; \mathbb{F}_p)$ is infinite.

Recall that a group $\pi$ is of type $F_n$ if there exists a $K(\pi, 1)$ with finite $n$-skeleton.

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It is well-known, by Selberg’s Lemma, that $\text{SL}_2(O_S)$, or that any arithmetic group over function fields $G(O_S)$ as above, contains a finite-index subgroup whose only torsion elements are $p$-elements.

Bux-Köhl-Witzel [5] completely generalized “half” of Theorem 1 with the following theorem.

**Theorem 2.** If $G$ is $K$-isotropic, then $G(O_S)$ is of type $F_{k(G,S)-1}$.

Important evidence for the theorem of Bux-Köhl-Witzel was contributed by Behr [3], Abels [1], Abramenko [2], and Bux-Wortman [7].

There are now three proofs that $G(O_S)$ as in Theorem 2 is not of type $F_{k(G,S)}$ due to Bux-Wortman [6], Bux-Köhl-Witzel [5], and Kropholler [11] as observed by Gandini [9]. However, outside of the case that $k(G, S) = 1$, the “second half” of Stuhler’s Theorem 1 had not been generalized to include any other arithmetic groups. This paper uses the results of Bux-Köhl-Witzel and Schulz [13] to further generalize the results of Stuhler by proving

**Theorem 3.** Suppose $G$ is $K$-isotropic. If $\Gamma$ is a finite-index subgroup of $G(O_S)$ that is residually $p$-finite, then $H^{k(G,S)}(\Gamma; \mathbb{F}_p)$ is infinite.

A group $\Gamma$ is residually $p$-finite if for any nontrivial $\gamma \in \Gamma$, there is a homomorphism of $\Gamma$ onto a finite $p$-group that evaluates $\gamma$ nontrivially. Such finite-index subgroups of $G(O_S)$ are well-known to exist, by Platanov’s Theorem, and we provide a proof of their existence in Section 7 for completeness.

To compare Theorems 1 and 3, notice that any torsion element of a residually $p$-finite group has order a power of $p$. The author does not know of an example of a finite-index subgroup $\Gamma \leq G(O_S)$ whose only torsion elements are $p$-elements, but such that $\Gamma$ is not residually $p$-finite.

As an example of Theorem 3, there is a finite-index subgroup of $\text{SL}_n(O_S)$ whose cohomology in dimension $|S|(n - 1)$ with coefficients in $\mathbb{F}_p$ is infinite. In particular, there is a finite-index subgroup $\Gamma$ of $\text{SL}_n(\mathbb{F}_p[t])$ such that $H^{n-1}(\Gamma; \mathbb{F}_p)$ is infinite.

0.1. **Outline of the proof.** To prove Theorem 1 Stuhler analyzed the cell stabilizers of the $\text{SL}_2(O_S)$-action on the associated Euclidean building which is a product of regular $(p+1)$-valent trees. The cell stabilizers of $\Gamma$ as in Theorem 1 are products of the group $\mathbb{F}_p$, but the cell stabilizers of a random arithmetic group acting on its associated Euclidean building are more difficult to describe and to work with, so our proof of Theorem 3 proceeds in a different direction.

The main tool in our proof of Theorem 3 is the work of Bux-Köhl-Witzel, and we spend a good portion of the beginning of our proof.
recalling their work. Let \( k = k(G, S) \) and let \( X \) be the Euclidean building that \( G(O_S) \) acts on as a lattice. Bux-Köhler-Witzel finds a \( G(O_S) \)-invariant, cocompact, \( (k-2) \)-connected complex \( X_{k-2} \subseteq X \). We attach \( k \)-cells and \( (k+1) \)-cells to \( X_{k-2} \) to produce a \( k \)-connected complex \( X_k \) endowed with a \( \Gamma \)-action and a \( \Gamma \)-equivariant map \( \psi : X_k \to X \).

We find an unbounded sequence of points \( \Gamma y_n \in \Gamma \setminus X \), and a sequence of normal subgroups \( \Gamma_n \) of \( \Gamma \) with index a power of \( p \) such that each \( y_n \in X \) is contained in a neighborhood of \( X \) that injects into \( \Gamma_n \setminus X \), and such that the \( p \)-group \( \Gamma / \Gamma_n \) acts on the homology of the image of the neighborhood in the quotient, with coefficients in \( \mathbb{F}_p \). The action of the \( p \)-group on the homology group produces a functional that nontrivially, and \( \Gamma \)-invariantly, evaluates the image under \( \psi \) of the attached \( k \)-cells in \( X_k \). Therefore, for each \( n \), we have an assignment of \( k \)-cells in \( \Gamma \setminus X_k \) to elements of \( \mathbb{F}_p \). This produces an infinite sequence in \( H^k(\Gamma \setminus X_k; \mathbb{F}_p) \). The group \( \Gamma \) may not act freely on \( X_k \), but the lack of freeness is confined to a cocompact subspace of \( X_k \), namely \( X_{k-2} \), and that implies that \( H^k(\Gamma; \mathbb{F}_p) \) is infinite.

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1. Preliminaries on \( G(O_S) \) and its action on a Euclidean building

This section establishes some conventions for notation.

1.1. Basic group structure. Let \( K, O_S, \) and \( G \) be as in Theorem 3. Because \( G \) is \( K \)-isotropic, it contains a proper minimal \( K \)-parabolic subgroup \( J \). Let \( A \) be a maximal \( K \)-split torus in \( J \), and let \( P \) be a maximal proper \( K \)-parabolic subgroup of \( G \) that contains \( J \).

Recall the Langlands decomposition that

\[
P = \mathbf{U}H \mathbf{T}
\]

where \( \mathbf{U} \) is the unipotent radical of \( P \), \( H \) is a reductive \( K \)-group with \( K \)-anisotropic center, \( T \) is a 1-dimensional connected subtorus of \( A \), and \( T \) commutes with \( H \).
In the remainder of this paper we denote the product over $S$ of local points of a $K$-group by “unbolding”, so that, for example,

$$G = \prod_{v \in S} G(K_v)$$

1.2. Euclidean building. Let $X$ be the Euclidean building for the semisimple group $G$. We let $k = k(G, S)$ so that $k = \dim(X).

For each $v \in S$ we choose a maximal $K_v$-split torus in $J$ that contains $A$, and name it $A_v$. We let $\Sigma \subseteq X$ be the apartment corresponding to the group $\prod_{v \in S} A_v(K_v)$.

2. Review of Bux-Köhl-Witzel and an unbounded sequence of points $y_n \in X$

Our proof makes use of two results from Bux-Köhl-Witzel [5]: the existence of a $G(O_S)$-invariant, $(k - 2)$-connected subcomplex $X_{k-2} \subseteq X$ that is cocompact modulo $G(O_S)$, and a lemma that will allow us to extend certain “local” $k$-disks about neighborhoods of points in $X$ to “global” $k$-disks in $X$—Lemma [9] and Corollary [10] below. Most of this section is devoted to recalling the work of Bux-Köhl-Witzel. For details omitted from the account in this paper, see [5].

We will use the notation of [5] in our Section 2 except for the following: we will refer to cells in the spherical building for $G$ by the parabolic groups they represent. For example, if $g \in G$ and we write that $g \in P$, then we are treating $P$ as a parabolic group, but if $x$ is a point in the visual boundary of $X$ and we write that $x \in P$, then we are treating $P$ as the simplex in the visual boundary of $X$ that corresponds to $P$. The correct interpretation should always be clear from context.

2.1. Busemann function for $P$. For each $v \in S$, let $X_v$ be the Euclidean building for $G(K_v)$, so that $X = \prod_{v \in S} X_v$. If $O_v \subseteq K_v$ is the ring of integers, then we let $x_v$ be the vertex in $X_v$ stabilized by $G(O_v)$.

Let $\mathbb{A}_K$ be the ring of adeles for $K$, and let $\mathbb{A}_S$ be the subring of $S$-adeles. The group $G(\mathbb{A}_S)$ has a natural left action on $X$. Given a point $y \in X$ we let $G(\mathbb{A}_S)_y$ be the stabilizer of $y$ in $G(\mathbb{A}_S)$.

Following Harder ([10]) and [5], for any $y \in \prod_{v \in S} G(K_v)x_v$, we let

$$\bar{\beta}_P(y) = \log_q \left[ \text{vol}\left[U(\mathbb{A}_K) \cap G(\mathbb{A}_S)_y\right] \right]$$

where $q$ is the cardinality of the field of constants in $K$.

We let $\chi_P$ be the canonical character of $P$. (See Section 1.3 [10] for the definition of $\chi_P$.) The essential feature of $\chi_P$ that will be used below is that the determinant of conjugation by $g \in P$ on $U$ is $\chi_P(g)$. 


If \( g \in P \), then we have the following transformation rule from Harder [10] Satz 1.3.2:

\[
\tilde{\beta}_P(gy) = \tilde{\beta}_P(y) + \log_q(||\chi_P(g)||)
\]

where \( || \cdot || \) denotes the idele norm. (There is a difference in sign in the line above with [10] and [5] that comes from our convention of using left actions in this paper rather than right actions as in [10] and [5].)

Recall that a Busemann function on the Euclidean building \( X \) is given by first choosing a unit speed geodesic \( \rho \subseteq X \) and then assigning to any point \( x \in X \) the limit as \( t \to \infty \) of the difference between the distance between \( \rho(t) \) and \( \rho(0) \) and the distance between \( \rho(t) \) and \( x \).

**Proposition 4.** There is some \( s > 0 \) and a Busemann function \( \beta_P : X \to \mathbb{R} \) such that \( \beta_P(y) = \tilde{\beta}_P(y) \) for all \( y \in \prod_{v \in S} G(K_v)x_v \), and such that \( \beta_P \) is nonconstant on factors of \( X \).

**Proof.** This is Proposition 12.2 of [5]. \( \square \)

**Lemma 5.** The Busemann function \( \beta_P \) is invariant under the actions of \( U \), \( H \), and \( T(O_S) \) on \( X \), and thus is invariant under the action of \( P(O_S) \leq UHT(O_S) \).

**Proof.** Any \( K \)-defined character on \( P \), including the canonical character \( \chi_P \), evaluates \( U \) trivially since it is unipotent and \( H \) trivially since it is reductive with \( K \)-anisotropic center. Thus the result for \( U \) and \( H \) follows from the transformation rule above.

Similarly, we need to observe that \( ||\chi_P(t)|| = 1 \) for any \( t \in T(O_S) \). This follows from the product formula (since \( \chi_P(t) \in K \)) and from the fact that \( T(K_w) \) is bounded if \( w \notin S \). \( \square \)

### 2.2. Descending chambers at a vertex.

Given a vertex \( x \in X \), we let \( \text{St}(x) \subseteq X \) denote the star of \( x \), the union of all chambers in \( X \) that contain \( x \). Thus, the boundary of the star – denoted as \( \partial\text{St}(x) \) – is the link of \( x \).

We let \( \text{St}^+(x) \) denote the union of chambers \( \mathcal{C} \subseteq X \) containing \( x \) with the property that \( \beta_P(z) < \beta_P(x) \) for all \( z \in \mathcal{C} \) with \( z \neq x \). We let \( B\text{St}^+(x) = \text{St}^+(x) \cap \partial\text{St}(x) \).

Recall that a special vertex \( x \in \Sigma \) is a vertex that is contained in a representative from each parallel family of walls in the Coxeter complex \( \Sigma \). Thus, the Coxeter complex of an apartment in the spherical building \( \partial\text{St}(x) \) is isomorphic to the Coxeter complex of an apartment in the boundary of \( X \) when \( x \) is special.

The following result is due to Schulz [13].
Lemma 6. If \( x \in X \) is a special vertex, then \( BSt^4(x) \) is homotopy equivalent to a noncontractible wedge of \((k-1)\)-spheres.

Proof. Recall that the Busemann function \( \beta_P \) is nonconstant on the factors of \( X \). Since \( x \) is a special vertex, the join factors of \( \partial St(x) \) correspond to the factors of \( X \). Therefore, \( \beta_P \) is nonconstant on the join factors of \( \partial St(x) \). That is to say, in the terminology used in \([5]\), the “vertical part” of \( \partial St(x) \) is \( \partial St(x) \) in its entirety.

Notice that \( BSt^4(x) \) is exactly the maximal subcomplex of \( \partial St(x) \) that is supported on the complement of the closed ball of radius \( \frac{\pi}{2} \) around the gradient direction of \( \beta_P \) in \( \partial St(x) \). Thus, by Theorem B of \([13]\) – restated in Theorem 4.6 of \([5]\) – \( BSt^4(x) \) is \((k-1)\)-dimensional, \((k-2)\)-connected, and noncontractible. \( \square \)

See also Theorem A.2 of Dymara-Osajda \([8]\).

2.3. Reduction datum. If \( M_a \) is a maximal proper \( K \)-parabolic subgroup of \( G \), then we can define a Busemann function \( \beta_{M_a} \) with respect to \( M_a \) similarly to how we defined \( \beta_P \) with respect to \( P \).

In \([5]\), and following \([10]\), there are real constants \( r < R \) such that the collection of Busemann functions \( \beta_{M_a} \) forms what is called a uniform \( G(O_S) \)-invariant and cocompact reduction datum. (See Theorem 1.9 of \([5]\).) The remainder of Section 2.3 is a recollection of what this sort of datum entails. In Section 2.3 we will use \( M_a \) to denote a maximal proper \( K \)-parabolic subgroup of \( G \). We will use \( M_i \) to denote a minimal \( K \)-parabolic subgroup of \( G \).

For \( x \in X \) and a \( K \)-parabolic subgroup \( Q \leq G \), we let \( \beta_Q(x) \) be the maximum of all \( \beta_{M_a}(x) \) with \( Q \leq M_a \).

Given an apartment \( \Sigma' \subseteq X \) that contains \( Q \) as a cell in its boundary, and given \( t \in \mathbb{R} \), we let

\[
Y_{\Sigma',Q}(t) = \{ x \in \Sigma' \mid \beta_Q(x) \leq t \}
\]

This set is convex in \( \Sigma' \) as it is the intersection of the convex sets \( \Sigma' \cap \beta_{M_a}^{-1}(\mathbb{R} \leq t) \) for \( M_a \) containing \( Q \). Thus, there is a closest point projection

\[
pr_{\Sigma',Q} : \Sigma' \to Y_{\Sigma',Q}(t)
\]

The group \( \sigma_t(x,Q) \) is defined to be the group of \( \prod_{v \in S} K_v \)-points of the intersection of all \( M_a \) that contain \( Q \) and such that \( \beta_{M_a}(pr_{\Sigma',Q}(x)) = t \). We have that \( \sigma_t(x,Q) \simeq Q \) (as groups, not as cells in the boundary) and we say that \( Q \) \( t \)-reduces \( x \in X \) if \( \sigma_t(x,Q) = Q \).

To say that the collection of \( \beta_{M_a} \) is an \((r,R)\) reduction datum for \( r < R \) means that if \( M_i \) is a minimal \( K \)-parabolic subgroup of \( G \) that \( r \)-reduces \( x \in X \), then \( M_i \leq \sigma_R(x,M_i) \).
To say that the reduction datum is uniform means that there exists a constant $d$ such that any point in a subset of $X$ whose diameter is less than $d$ can be $r$-reduced by a common minimal $K$-parabolic. We can assume, as in [5], by perhaps choosing a lesser $r$, that $d$ is greater than the diameter of closed stars of cells in $X$.

The reduction datum is $G(O_S)$-invariant since
\[ \beta_{M_a}(\gamma x) = \beta_{M_a}(x) \]
for all $x \in X$, $\gamma \in G(O_S)$, and maximal proper $K$-parabolic $M_a$. (Here $\gamma M_a = \gamma M_a \gamma^{-1}$.)

That the reduction datum is cocompact means that for any real number $t \geq R$, the set of $x \in X$ for which $\beta_{M_i}(x) \leq t$ for all minimal $K$-parabolics $M_i$ that $r$-reduce $x$ is cocompact with respect to the action of $G(O_S)$.

### 2.4. Definition of height.

In [5], the reduction datum is used to define a height function $h : X \to \mathbb{R}_{\geq 0}$. In Section 2.4, we recall this definition.

Choose a special vertex $z \in \Sigma$, and let $W_z$ be the spherical Coxeter group that fixes $z$ in $\Sigma$.

The affine space $\Sigma$ may be realized as a vector space with origin $z$. Let $V_z$ be the set of all differences of vertices in $\Sigma$ whose closed stars intersect, where we regard vertices in this context as vectors in $\Sigma$. Notice that $V_z$ is finite.

We let $D = W_z V_z$. Again, realizing points of $D$ as vectors of the vector space $\Sigma$ with origin $z$, we let
\[ Z(D) = \left\{ \sum_{d \in D} a_d d \mid 0 \leq a_d \leq 1 \text{ for all } d \in D \right\} \]

The set $Z(D) \subseteq \Sigma$ depended on the choice of vertex $z$, but modulo isometric translations of $\Sigma$, $Z(D)$ is defined intrinsically in terms of the geometry of $\Sigma$. Furthermore, if $\Sigma' \subseteq X$ is any apartment in $X$, then $\Sigma'$ is isometric to $\Sigma$ as Coxeter complexes, and thus $x + Z(D)$ is a well-defined subset of $\Sigma'$ for any $x \in \Sigma'$.

To define a height function, a suitably large $R^* > R$ is chosen. For any apartment $\Sigma' \subseteq X$, any $x \in \Sigma'$, and any minimal $K$-parabolic $M_i$ such that $M_i$ represents a cell in the boundary of $\Sigma'$ that $r$-reduces $x$; the point $x_{\Sigma',M_i}^*$ is defined to be the closest point to $x$ in $Y_{\Sigma',M_i}(R^*) - Z(D)$. Then $h(x)$ is defined as the distance between $x$ and $x_{\Sigma',M_i}^*$, and it is shown in Proposition 5.2 of [5] to be independent of $\Sigma'$ or $M_i$.

If $h(x) > 0$, then $e(x)$ is defined as the point in the visual boundary of $\Sigma'$ that is determined as the limit point of the geodesic ray in $\Sigma'$.
from $x_{\Sigma,M_i}$ through $x$. The point $e(x)$ is also shown to be independent of $\Sigma'$ or $M_i$ in Proposition 5.2 of [5]. If we let $\sigma(x)$ denote the group of $\prod_{v \in s} K_v$-points of the $K$-parabolic subgroup of $G$ that is minimal with respect to the property that $\sigma(x)$ contains every $\sigma_R(x,M_i)$ for which $M_i$ $r$-reduces $x$, then $e(x) \in \sigma(x)$.

As the reduction datum used in this section is $G(O_S)$-invariant, we have that $h(\gamma x) = h(x)$ for any $\gamma \in G(O_S)$. And if $h(x) > 0$, then $e(\gamma x) = \gamma e(x)$ and $\sigma(\gamma x) = \gamma \sigma(x)$.

The subsets of $X$ whose values under $h$ are bounded from above are shown to have bounded quotient on $G(O_S) \setminus X$ (See Proposition 2.4 and Observation 5.5 of [5]).

2.5. Choice of $y_n$. We still have more to discuss about the results of [5], but we take a short break from our account of [5] to establish a Lemma 7.

Lemma 7. Let $N^* > 0$ be twice the maximum diameter of stars in $X$. We can choose $R^* \gg 0$ as above to satisfy the following: There is a constant $C^* \in \mathbb{R}$, and a geodesic ray $\ell_Y \subseteq \Sigma$ that limits to a point $\ell_Y(\infty)$ in the simplex $P$ and is orthogonal to level sets of $\beta_P$ in $\Sigma$, such that every point $z$ in the $N^*$-neighborhood of $U \ell_Y$ in $X$ is $r$-reduced by $J$, has $h(z) = \beta_P(z) + C^* > 0$, and has $e(z) = \ell_Y(\infty) \in P$.

Furthermore, there is a sequence of special vertices $y_n \in \Sigma$ that are contained in chambers of $\Sigma$ that intersect $\ell_Y$, such that $\beta_P(y_n)$ is a strictly increasing sequence of numbers, and such that the set of all $(y_n)_{\Sigma,P}$ is a bounded set.

Proof. There are rank$_K G \leq \dim(\Sigma)$ maximal proper $K$-parabolic subgroups that contain $J$. The space $Y_{\Sigma,J}(R^*) \subseteq \Sigma$ is the intersection of one half-apartment of $\Sigma$ for every maximal proper $K$-parabolic subgroup that contains $J$, and the set $\beta_P^{-1}(R^*) \cap Y_{\Sigma,J}(R^*)$ is an unbounded face of the boundary of $Y_{\Sigma,J}(R^*)$. We call this face $F_{P,R^*}$. It has dimension equal to $\dim(\Sigma) - 1$.

We let

$$\Omega(r,R^*,J,P) = \{x \in \Sigma \mid \sigma_r(x,J) = J \text{ and } \sigma_{R^*}(x,J) = P\}$$

For $x \in \Sigma$, we let $B_{\Sigma}(x;N^*) \subseteq \Sigma$ be the ball in $\Sigma$ centered at $x$ with radius $N^*$. Notice that by replacing $R^*$ with a greater constant, we may assume that there is some $x \in F_{P,R^*} \cap \Omega(r,R^*,J,P)$ such that

$$F_{P,R^*} \cap [B_{\Sigma}(x;N^*) + Z(D)] \subseteq F_{P,R^*} \cap \Omega(r,R^*,J,P)$$

Furthermore, if $y$ is contained in the geodesic ray $\ell_Y \subseteq \Sigma$ that begins at $x$, is orthogonal to $F_{P,R^*}$, and is contained in $\Omega(r,R^*,J,P)$, then
\( B_\Sigma(y; N^*) + Z(D) \subseteq \Omega(r, R^*, J, P) \) as long as the distance between \( y \) and \( x \) is sufficiently large. We replace \( \ell_Y \) with a subray so that \( B_\Sigma(y; N^*) + Z(D) \subseteq \Omega(r, R^*, J, P) \) for any \( y \in \ell_Y \).

If \( z \) is contained in the interior of \( \Omega(r, R^*, J, P) \), then \( e(z) \) is given by the direction of the gradient of \( \beta \) restricted to \( \Sigma \) — which is the direction of \( \ell_Y(\infty) \). Thus by Lemma 5, \( UHe(z) = UH\ell_Y(\infty) = \ell_Y(\infty) = e(z) \). And \( T \leq A \) acts trivially on the boundary of \( \Sigma \), so we have \( P\ell(z) = UHTe(z) = e(z) \) which implies that \( e(z) \in P \).

We let \( d_0 \) be the constant difference of the distance between \( y \in \ell_Y \) and \( \Sigma \cap \beta^{-1}_P(R^*) \) and the distance between \( y + Z(D) \) and \( \Sigma \cap \beta^{-1}_P(R^*) \). (Note that the latter of the two distances is \( h(y) \).) Then for \( z \in B_\Sigma(y; N^*) \), \( h(z) = \beta_P(z) - R^* - d_0 \). Thus we let \( C^* = -R^* - d_0 \).

Again let \( y \in \ell_Y \) and now let \( z \in B_X(y; N^*) \), where \( B_X(y; N^*) \) is the ball in \( X \) of radius \( N \) that is centered at \( y \). We will show that \( z \) is \( r \)-reduced by \( J \), has \( h(z) = \beta_P(z) + C^* > 0 \), and has \( e(z) = \ell_Y(\infty) \in P \).

For every \( v \in S \), let \( J_v \leq G \) be a minimal \( K_v \)-parabolic subgroup of \( G \) such that \( A_v \leq J_v \leq J \). We let \( U_v \) be the unipotent radical of \( J_v \), so that \( U_v \leq J \leq P \) and \( U_v \leq UH \).

If \( X_v \) is the Euclidean building for \( G(K_v) \), and \( \Sigma_v \) is the apartment that \( A_v(K_v) \) acts on, then because any point in \( X_v \) is contained in a \( J_v(K_v) \) translate of \( \Sigma_v \)

\[
X_v = J_v(K_v)\Sigma_v = U_v(K_v)Z_G(A_v)(K_v)\Sigma_v = U_v(K_v)\Sigma_v
\]

where \( Z_G(A_v) \) is the centralizer of \( A_v \) in \( G \), and thus is a Levi subgroup of \( J_v \). Therefore,

\[
X = \prod_{v \in S} U_v(K_v)\Sigma
\]

and there is a distance nonincreasing retraction

\[
\varrho : X \to \Sigma
\]

defined on each \( u\Sigma \) for \( u \in \prod_{v \in S} U_v(K_v) \) as the map \( u^{-1} : u\Sigma \to \Sigma \).

So for \( z \in B_X(y; N^*) \) we choose \( u \in \prod_{v \in S} U_v(K_v) \) such that \( u^{-1}z \in \Sigma \). Because \( \varrho \) is distance nonincreasing and \( \varrho(y) = y \), we have that \( u^{-1}z \in B_X(y; N^*) \). By Lemma 5

\[
\beta_P(z) + C^* = \beta_P(u^{-1}z) + C^* = h(u^{-1}z) > 0
\]

If \( Q \) is a proper \( K \)-parabolic subgroup of \( G \) containing \( J \), then \( Q \) contains \( U_v \) and thus \( u^{-1}Qu = Q \), so applying the clear analogue of Lemma 5 to each maximal proper \( K \)-parabolic group containing \( J \) yields \( uY_{\Sigma,J}(R^*) = Y_{u\Sigma,J}(R^*) \) and that \( z \in u\Omega(r, R^*, J, P) \) since \( u^{-1}z \in B_\Sigma(y; N^*) \subseteq \Omega(r, R^*, J, P) \). Thus, \( z \) is \( r \)-reduced by \( uJu^{-1} = J \).
and \( u^{-1}(z^*_{n,\Sigma, J}) = (u^{-1}z)^*_{\Sigma, J} \) and
\[
h(z) = h(u^{-1}z) = \beta_p(z) + C^*
\]

Furthermore, as the set \( \Omega(r, R^*, J, P) \) limits to the cell \( P \) and \( u \in P \), the set \( u\Omega(r, R^*, J, P) \) also limits to \( P \) and thus
\[
e(z) = e(u^{-1}z) = \ell_Y(\infty) \in P
\]

To review, we have shown that for any \( z \) in the \( N^* \)-neighborhood of \( \ell_Y \) in \( X \) that \( z \) is \( r \)-reduced by \( J \), has \( h(z) = \beta_p(z) + C^* > 0 \), and has \( e(z) = \ell_Y(\infty) \in P \). We still need to show the same results apply to the weaker condition that \( z \) is contained in the \( N^* \)-neighborhood of \( U\ell_Y \) in \( X \). For that, recall that \( U \) is unipotent, so \( U(\mathcal{O}_S) \) is a cocompact lattice in \( U \). That is, there is a compact set \( B \subset U \) such that \( U(\mathcal{O}_S)B = U \). Since \( \ell_Y \) limits to \( P \) and \( U \) is the unipotent radical of \( \mathcal{P} \), any element of \( U \) fixes pointwise a subray of \( \ell_Y \). Therefore, there is a common subray of \( \ell_Y \) that is fixed pointwise by every element of \( B \). Thus, by replacing \( \ell_Y \) with a subray we may assume that \( B \) fixes \( \ell_Y \) and thus that
\[
U\ell_Y = U(\mathcal{O}_S)B\ell_Y = U(\mathcal{O}_S)\ell_Y
\]

Hence, if \( z \in UB_X(\ell_Y; N^*) = U(\mathcal{O}_S)B_X(\ell_Y; N^*) \) then \( uz \in B_X(\ell_Y; N^*) \) for some \( u \in U(\mathcal{O}_S) \), and since \( h \) is \( G(\mathcal{O}_S) \)-invariant and \( \beta_p \) is \( U \)-invariant,
\[
h(z) = h(uz) = \beta_p(uz) + C^* = \beta_p(z) + C^*
\]

Since the reduction datum is \( G(\mathcal{O}_S) \)-invariant and \( uz \) is \( r \)-reduced by \( J \), we see that \( z \) is \( r \)-reduced by \( u^{-1}Ju = J \). Last, since \( u \in U(\mathcal{O}_S) \leq P \) and \( e(uz) \in P \) we have \( e(z) = u^{-1}e(uz) = e(uz) = \ell_Y(\infty) \).

To find the sequence of \( y_n \), just choose an unbounded sequence of chambers in \( \Sigma \) that intersect \( \ell_Y \). Any chamber in \( X \) contains a special vertex, and this produces the sequence of \( y_n \). Because each of the \( y_n \in \Sigma \) are uniformly bounded distances from \( \ell_Y \), each \( (y_n)^*_{\Sigma, P} \in F_{P, R^*} \) is a uniformly bounded distance from the point \( x \in F_{P, R^*} \).

In the remainder of this paper, we shall abbreviate \( \text{St}(y_n) \) as \( S_n \). Similarly, we shall abbreviate \( \text{St}^+(y_n) \) and \( B\text{St}^+(y_n) \) as \( S_n^+ \) and \( BS_n^+ \) respectively.

2.6. Morse function. Section 2.6 is the final section in which we recount the work of Bux-Köhler-Witzel. In this section we recall the definition of a combinatorial Morse function from [3] that is defined on the vertices of the barycentric subdivision of \( X \) and used to deduce connectivity properties of subsets of \( X \).

For any cell \( \tau \in X \) we let \( \text{dim}(\tau) \) be its dimension. There is also a number defined in [5] as \( \text{dp}(\tau) \) which refers to the “depth” of a cell.
We refer the reader to Section 8 of [5] for the definition of the depth of a cell.

We let \( \hat{X} \) be the barycentric subdivision of the Euclidean building \( X \). For any cell \( \tau \subseteq X \), we let \( \hat{\tau} \) be its barycenter. Bux-Köhler-Witzel assigned to \( \hat{\tau} \) the triple of real numbers

\[
f_{\text{BKW}}(\hat{\tau}) = \left( \max_{x \in \tau} h(x), \text{dp}(\tau), \dim(\tau) \right)
\]

The function \( f_{\text{BKW}} \) is a combinatorial Morse function when triples of real numbers are ordered lexicographically.

For any triple of real numbers \( s \) that is greater than or equal to the triple \( s_0 = (1, 0, 0) \), we let \( \hat{X}(s) \) be the subcomplex of \( \hat{X} \) spanned by the \( \hat{\tau} \) for which \( f_{\text{BKW}}(\hat{\tau}) \leq s \). Since \( f_{\text{BKW}} \) is \( G(O_S) \)-invariant, so too is \( \hat{X}(s) \). Since \( \hat{X}(s) \) is a closed subset of \( \hat{X} \) whose height is bounded, it is cocompact modulo \( G(O_S) \). The values of \( f_{\text{BKW}} \) are finite below any given bound, and we let \( s + 1 \) denote the least value of \( f_{\text{BKW}} \) that is greater than \( s \).

We let \( \text{Lk}(\hat{\tau}) \) be the link of \( \hat{\tau} \) in \( \hat{X} \), and we define the \textit{Morse descending link} of \( \hat{\tau} \) with respect to the Morse function \( f_{\text{BKW}} \) to be the complex of simplices \( \sigma \subseteq \text{Lk}(\hat{\tau}) \) such that \( f_{\text{BKW}}(v) < f_{\text{BKW}}(\hat{\tau}) \) for every vertex \( v \in \sigma \). To obtain \( \hat{X}(s+1) \) we attach to \( \hat{X}(s) \) the descending links of cells \( \hat{\tau} \subseteq \hat{X} \) with \( f_{\text{BKW}}(\hat{\tau}) = s + 1 \). The work of Bux-Köhler-Witzel is to have defined \( f_{\text{BKW}} \) in such a way as to utilize the work of Schulz [13] in showing that the Morse descending links of vertices in \( \hat{X} \) are either contractible or spherical of dimension \( (k-1) \). Thus, up to homotopy equivalence, \( \hat{X}(s+1) \) is obtained by attaching \( k \)-cells to \( \hat{X}(s) \). This process induces an isomorphism of homotopy groups \( \pi_i(\hat{X}(s)) \cong \pi_i(\hat{X}(s+1)) \) for \( i \leq k-2 \). Since \( X \) is contractible and the union of the \( \hat{X}(s) \), we have that \( \hat{X}(s) \) is \( (k-2) \)-connected for any \( s \geq s_0 \). It is the existence of a \( G(O_S) \)-cocompact \( (k-2) \)-connected space that can be viewed as the main result of [5] as it immediately implies that \( G(O_S) \) is of type \( F_{k-1} \).

In what remains, we will let \( X_{k-2} = X(s_0) \). In particular, \( X_{k-2} \) is a \( (k-2) \)-connected subcomplex of \( X \) that is invariant and cocompact under the action of \( G(O_S) \). We will also pass to a subsequence of the \( y_n \) to assume that \( S_n \cap X_{k-2} = \emptyset \) for all \( n \).

The following lemma demonstrates the compatibility of \( \beta_P \) and \( f_{\text{BKW}} \) on \( S_n \).

**Lemma 8.** The Morse descending link of \( y_n \) with respect to \( f_{\text{BKW}} \) equals \( BS_n^4 \).
Proof. As in Section 6 of [5], the height function $h$ forces a decomposition of the link of $y_n \in X$ into a join of a “horizontal link” of $y_n$ and a “vertical link” of $y_n$ where the horizontal link of $y_n$ is the join of all factors of the link of $y_n$ whose points are evaluated by $h$ as $h(y_n)$.

By Lemma 7, the restriction of $\beta_P$ to the horizontal link of $y_n$ is constant. But $y_n$ is a special vertex, so Proposition 4 implies that the horizontal link of $y_n$ is trivial, and therefore, that the vertical link of $y_n$ equals the link of $y_n$.

Now by Proposition 9.6 of [5], the Morse descending link of $y_n$ is the subcomplex of the link of $y_n$ in $X$ that is spanned by all vertices $v$ in the link of $y_n$ such that $h(v) < h(y_n)$. (Keep in mind that any vertex of $X$ is “significant”.) Again, by Lemma 7, this complex is equal to $\overline{BS_n^k}$.

2.7. Extending local disks near $y_n$. In addition to the existence of $X_{k-2}$, we shall utilize the results of [5] to extend “local” disks near $y_n$ to “global” disks in $X$. More precisely, we have

Lemma 9. Let $\sigma : S^{k-1} \to X$ be a continuous map of a $(k - 1)$-sphere into $X$. Suppose there is some triple $s > s_0$ such that $\sigma(S^{k-1}) \subseteq \tilde{X}(s)$. Then there is a homotopy $F : S^{k-1} \times [0, 1] \to X$ such that for all $x \in S^{k-1}$ we have $F(x, t) \in \tilde{X}(s)$, $F(x, 0) = \sigma(x)$, and $F(x, 1) \subseteq \tilde{X}(s_0) = X_{k-2}$.

Proof. Let $c^0_0, \ldots, c^0_m \subseteq X$ be the image under $\sigma$ of the 0-cells of $S^{k-1}$. Let $c^0_{i,F} \subseteq \tilde{X}(s)$ be paths from $c^0_i$ to $X_{k-2}$. The boundary of each $c^0_{i,F}$ is $c^0_i$ and $b^0_i$ for some $b^0_i \in X_{k-2}$.

If $k = 1$, then $m = 2$, and $c^0_{1,F} \cup c^0_{2,F}$ is the image of the homotopy $F$.

If $k \geq 2$, then let $c^1_i \subseteq \sigma(S^{k-1})$ be the image of the 1-cell with boundary $c^0_i$ and $c^0_j$. Since $\tilde{X}(s)$ is obtained from $X_{k-2}$ by attaching $k$-cells, there is a homotopy relative $b^0_i$ and $b^0_j$ between $c^1_i \cup c^0_{i,F} \cup c^0_{j,F}$ and a 1-cell $b^1_i \subseteq X_{k-2}$. We name the image of this homotopy $c^1_{i,F}$.

If $k = 2$, then the union of the $c^1_{i,F}$ defines the homotopy $F$.

If $k \geq 3$, then we proceed as above by induction on the skeleta of $S^{k-1}$.

We let $I_n = S_n - \partial S_n$ be the interior of $S_n$. As a consequence of the above lemma, we have

Corollary 10. For $n \gg 0$, there is a $k$-disk $D^k_n \subseteq S^k_n \cup (X - G(O_S)I_n)$ with $\partial D^k_n \subseteq X_{k-2}$ and such that $D^k_n \cap S^k_n$ is a $k$-disk that represents a noncontractible $k$-sphere in the quotient space $S^k_n/BS^k_n$. 

Proof. Let \( s_n \) be the triple such that \( f_{
abla} (y_n) = s_n \). By Lemma \([7] \) and the definition of the Morse function \( f_{
abla} \), we have for any cell \( \tau \subseteq S_n \) that is not contained in \( \partial S_n \) that \( f_{
abla} (G(O) \tau) = f_{
abla} (\tau) \geq s_n \) since \( y_n \in \tau \). That is, \( G(O) \tau \cap \hat{X}(s_n - 1) = \emptyset \).

By Lemmas \([6] \) and \([8] \) there is a noncontractible \((k-1)\)-sphere \( \sigma_n^{k-1} \subseteq BS_n^k \).

We let \( d_n^k \subseteq S_n^k \) be the cone at \( y_n \in S_n^k \) on

\[
\sigma_n^{k-1} \subseteq BS_n^k \subseteq \hat{X}(s_n - 1)
\]

By Lemma \([9] \) there is a homotopy \( F \) between \( \partial d_n^k \) and a \((k-1)\)-sphere in \( X_{k-2} \) whose image is contained in \( \hat{X}(s_n - 1) \). We let \( D_n^k \) be the union of \( d_n^k \) and \( F \). Then

\[
D_n^k \subseteq S_n^k \cup \hat{X}(s_n - 1) \subseteq S_n^k \cup (X - G(O) \tau)
\]

That \( D_n^k \cap S_n^k = d_n^k \) represents a noncontractible \( k \)-sphere in \( S_n^k / BS_n^k \) follows from the natural identification of \( d_n^k / \partial d_n^k \) and \( S_n^k / BS_n^k \) with the suspensions of \( \sigma_n^{k-1} \) and \( BS_n^k \) respectively. \( \square \)

**Lemma 11.** Suppose that \( \mathfrak{C}_a, \mathfrak{C}_b \subseteq S_n^k \) are chambers in \( X \), and that there is some \( \gamma \in G(O) \) such that \( \gamma \mathfrak{C}_a = \mathfrak{C}_b \). Then \( \gamma y_n = y_n \).

*Proof.* The vertex \( y_n \) is the only vertex of any chamber in \( S_n^k \) with \( f_{
abla} (v) = f_{
abla} (y_n) \). Since \( f_{
abla} \) is \( G(O) \) invariant, we have for \( \gamma y_n \in \mathfrak{C}_b \) that \( f_{
abla} (\gamma y_n) = f_{
abla} (y_n) \) so that \( \gamma y_n = y_n \). \( \square \)

### 3. Construction of a \( k \)-connected \( G(O) \)-complex

Bux-Köhl-Witzel gives us a \((k-2)\)-connected complex that \( G(O) \) acts on properly and cocompactly, namely \( X_{k-2} \). In order to determine the cohomology of finite-index subgroups of \( G(O) \) in dimension \( k \), we will create a \( k \)-connected space that \( G(O) \) acts on. In this section we will construct such a space by attaching \( k \)-cells to \( X_{k-2} \) and then attaching \((k+1)\)-cells after that.

#### 3.1. Construction of \( X_k \)

We let \( \psi : X_{k-2} \to X \) be the inclusion. In the process of our construction of a \( k \)-connected space that contains \( X_{k-2} \), we will be extending \( \psi \) to a map from that \( k \)-connected space into \( X \).

Let \( \sigma : S^{k-1} \to X_{k-2} \) be a continuous map of a \((k-1)\)-sphere into the \((k-1)\)-skeleton of \( X_{k-2} \). We regard \( \sigma \) as an attaching map for a \( k \)-cell that we name \( D_{\sigma}^k \).

For each nontrivial \( \gamma \in G(O) \), we attach another \( k \)-cell \( D_{\gamma \sigma}^k \) to \( X_{k-2} \) using the attaching map \( \gamma \circ \sigma \). We assign a homeomorphism
γ : D^k_{1,σ} → D^k_{γ,σ} that restricts to the γ-action on ∂D^k_{1,σ}, ∂D^k_{γ,σ} ⊆ X_{k−2}. Then for any λ ∈ G(OS), we let

λ : D^k_{γ,σ} → D^k_{λγ,σ}

be the homeomorphism defined by λ = (λγ)γ−1. In this way, we have defined a G(OS)-action on the complex

X_{k−2} ∪ \bigcup_{γ ∈ G(OS)} D^k_{γ,σ}

We repeat the process above for every continuous σ : S^{k−1} → X_{k−2} with image in the (k−1)-skeleton of X_{k−2}. The resulting union of X_{k−2} with the union of every D^k_{γ,σ} for every pair of γ and σ is a k-complex that we will denote by X_{k−1}. Notice that X_{k−1} is a (k−1)-connected, G(OS)-complex. The group G(OS) will not in general act freely on X_{k−1}, but any nontrivial point stabilizers correspond to points in X_{k−2} since the interiors of each of the D^k_{γ,σ} are disjoint.

We extend ψ to each D^k_{γ,σ} — and thus to all of X_{k−1} — by assigning arbitrary continuous maps ψ : D^k_{1,σ} → X that agree with ψ on ∂D^k_{1,σ} ⊆ X_{k−2} and then by defining ψ : D^k_{γ,σ} → X as γ ◦ ψ ◦ γ−1. Notice that γ ◦ ψ = ψ ◦ γ so that ψ is G(OS)-equivariant.

Now repeat the above process, this time attaching (k+1)-cells D^k_{γ,σ} to X_{k−1} with attaching maps σ : S^{k} → X_{k−1} to obtain a k-connected complex X_k that G(OS) acts on with a G(OS)-equivariant map ψ : X_k → X that restricts to X_{k−2} ⊆ X as the inclusion map. The action of G(OS) on X_k − X_{k−2} is free.

4. Assigning attaching disks to cycles in a finite complex

In this section we will begin to focus some attention on a given finite-index subgroup Γ of G(OS) from the statement of our main result, Theorem 3. That is, we let Γ be any finite-index subgroup of G(OS) that is residually p-finite.

Our goal in proving our main result is to show that H^k(Γ\backslash X_k; \mathbb{F}_p) is infinite. In the penultimate section of this paper we explain why this implies that H^k(Γ; \mathbb{F}_p) is infinite.

4.1. Definition of Γ_n. Our proof of our main result relies on forming a sequence of finite quotients of the group Γ. These quotients are described in the following

Lemma 12. For any n ≥ 0, there is a normal subgroup Γ_n ≤ Γ such that Γ/Γ_n is a finite p-group and Γ_n acts cocompactly and freely on ΓS_n.
Proof. The group \( \Gamma \) acts cocompactly on \( \Gamma S_n \).

For any cell \( \tau \subseteq S_n \), let \( \Gamma_\tau \) be the finite stabilizer of \( \tau \) in \( \Gamma \), and let \( Z_n \subseteq \Gamma \) be the finite set of the union of \( \Gamma_\tau \) over the finite set of cells \( \tau \subseteq S_n \).

Since \( \Gamma \) is residually \( p \)-finite, there is for each nontrivial \( \gamma \in Z_n \) a finite \( p \)-group, \( G_\gamma \), and a homomorphism \( \phi_\gamma : \Gamma \to G_\gamma \) such that \( \phi_\gamma(\gamma) \neq 1 \). Now let \( \phi : \Gamma \to \prod \gamma G_\gamma \) be the product of the \( \phi_\gamma \), and let \( \Gamma_n \) be the kernel of \( \phi \). Then \( \Gamma_n \leq \Gamma, \Gamma/\Gamma_n \) is a finite \( p \)-group, and \( Z_n \cap \Gamma_n = \{1\} \).

Since \( \Gamma_n \) is finite-index in \( \Gamma \), it acts cocompactly on \( \Gamma S_n \). Furthermore, if \( \gamma \in \Gamma_n \) and \( \gamma g \tau = g \tau \) for some \( g \in \Gamma \) and some cell \( \tau \subseteq S_n \), then \( g^{-1} \gamma g \in \Gamma_n \) is contained in \( \Gamma_\tau \subseteq Z_n \), and thus \( g^{-1} \gamma g \), and hence \( \gamma \), is trivial. \( \square \)

4.2. Definition of \( \theta_n \). We define

\[ \theta_n : X \to \Gamma_n \backslash X \]

to be the quotient map. Notice that \( \Gamma \) acts on \( \Gamma_n \backslash X \) since \( \Gamma_n \) is normal in \( \Gamma \). Furthermore, \( \theta_n \) is \( \Gamma \)-equivariant.

Also note that \( \Gamma \) acts on the pair \( (X, X - \Gamma I_n) \) and thus on the pair \( (\theta_n(X), \theta_n(X - \Gamma I_n)) \), and therefore on the homologies of these pairs.

(All homologies of complexes in this paper are cellular.)

4.3. Definition of \( \Theta_n(D^k_{\gamma,\sigma}) \). Given a \( k \)-cell \( D^k_{\gamma,\sigma} \) attached to \( X_{k-2} \) in the construction of \( X_k \), we have that \( \psi(\partial D^k_{\gamma,\sigma}) \subseteq X_{k-2} \).

By Lemma \( \ref{lemma} \) the sequence of \( h(y_n) \), and hence of \( f_{BKW}(\Gamma y_n) \) is unbounded. Thus we may assume that \( X_{k-2} \) intersects each \( \Gamma S_n \) trivially, which implies \( \partial \psi(D^k_{\gamma,\sigma}) \subseteq X - \Gamma I_n \) and thus that \( \psi'(D^k_{\gamma,\sigma}) \) represents a class in the homology group \( H_k(X, X - \Gamma I_n; \mathbb{F}_p) \), and further, that \( \theta_n \circ \psi(D^k_{\gamma,\sigma}) \) represents a class in the homology group \( H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p) \).

In the remainder we shall let

\[ \Theta_n(D^k_{\gamma,\sigma}) = [\theta_n \circ \psi(D^k_{\gamma,\sigma})] \in H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p) \]

Recall that \( \psi \) is \( \Gamma \)-equivariant, and that \( \theta_n \) is \( \Gamma \)-equivariant. Therefore, the group \( \Gamma \) acts on the set of all \( \Theta_n(D^k_{\gamma,\sigma}) \) by the rule that if \( g \in \Gamma \), then

\[
g \Theta_n(D^k_{\gamma,\sigma}) = g[\theta_n \circ \psi(D^k_{\gamma,\sigma})] = [\theta_n \circ \psi(g D^k_{\gamma,\sigma})] = [\theta_n \circ \psi(D^k_{g\gamma,\sigma})] = \Theta_n(D^k_{g\gamma,\sigma})
\]
4.4. Definition of $W_n$. We let $W_n$ be the vector subspace of 
$H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ generated by the classes $\Theta_n(D^{k}_{\gamma, \sigma})$ for every 
pair $\gamma$ and $\sigma$.

By the above, the $\Gamma$-action on $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ restricts 
to a $\Gamma$-action on $W_n$. Since $\Gamma_n$ acts trivially on $\theta_n(X)$, the action of $\Gamma$ 
on $W_n$ factors through the finite $p$-group $\Gamma/\Gamma_n$.

Lemma 13. The vector space $W_n$ is finite-dimensional and nonzero.

Proof. The space $X$ is the union of $\Gamma S_n$ and $X - \Gamma I_n$, so $\Gamma S_n$ surjects 
via $\theta_n$ onto the quotient $\theta_n(X)/\theta_n(X - \Gamma I_n)$. Lemma 12 gives us that 
$\theta_n(\Gamma S_n)$ is a finite complex, and thus, $\theta_n(X)/\theta_n(X - \Gamma I_n)$ is finite. The 
finiteness of $W_n$ now follows from the finite dimensionality of $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$.

Let $D^{k}_n \subseteq X$ be as in Corollary 10. We claim that $\theta_n(D^{k}_n)$ represents 
a nonzero class in $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. Indeed, $BS^1_n \subseteq X - \Gamma I_n$ 
and it suffices to prove that 
$$(\theta_n)_* : H_k(S^1_n; BS^1_n; \mathbb{F}_p) \longrightarrow H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$$
is injective. As $\theta_n(X)$ is a $k$-dimensional complex, this reduces to showing that 
$\theta_n(\mathcal{C}_a) \neq \theta_n(\mathcal{C}_b)$ for distinct chambers $\mathcal{C}_a, \mathcal{C}_b \subseteq S^1_n$. In 
other words, we want to show that $\gamma \mathcal{C}_a = \mathcal{C}_b$ for any $\gamma \in \Gamma_n$ and any 
pair of chambers $\mathcal{C}_a, \mathcal{C}_b \subseteq S^1_n$ implies that $\mathcal{C}_a = \mathcal{C}_b$. By Lemma 11 any 
such $\gamma \in \Gamma_n$ fixes $y_n \in \Gamma S_n$, and by Lemma 12 $\gamma$ is trivial so that 
$\mathcal{C}_a = \mathcal{C}_b$.

Now let $\sigma_n : S^{k-1} \to X_{k-2}$ represent $\partial D^{k}_n$, and let $D^{k}_{1, \sigma_n}$ be the 
k-disk attached to $X_{k-2}$ by $\sigma_n$ in the construction of $X_k$. Since $X$ 
is contractible and $k$-dimensional, and since $D^{k}_n$ and $\psi(D^{k}_{1, \sigma_n})$ share a 
common boundary, they represent the same $k$-chain in the homology of $X$. Therefore, by the above paragraph,
$$\Theta_n(D^{k}_{1, \sigma_n}) = [\theta_n \circ \psi(D^{k}_{1, \sigma_n})] = [\theta_n(D^{k}_n)]$$
is a nonzero class in $W_n \leq H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. \hfill $\square$

5. A Sequence of Cycles and Cocycles for $\Gamma \backslash X_k$

The action of $\Gamma$ on $W_n$ induces an action of $\Gamma$ on the dual vector 
-space $W_n^*$ by $\gamma \phi(x) = \phi(\gamma^{-1}x)$ for $\gamma \in \Gamma$, $\phi \in W_n^*$, and $x \in W_n$.

Lemma 14. For each $n$, there is a $\Gamma$-invariant $\varphi_n \in W_n^*$ and some $\lambda_n \in \mathcal{G}(O_\mathcal{S})$ and 
$\tau_n : S^{k-1} \to X_{k-2}$ such that $\varphi_n(\Theta_n(D^{k}_{\lambda_n, \tau_n})) \neq 0$. Furthermore, after passing to a subsequence, if $m > n$ then $\varphi_m(\Theta_m(D^{k}_{\lambda_n, \tau_n})) = 0$. 

Proof. A linear transformation of a finite-dimensional nonzero vector space of characteristic $p$ is unipotent if and only if it has order $p^k$ for some $k$. Since the action of $\Gamma$ on $W_n^*$ factors through the $p$-group $\Gamma/\Gamma_n$, the elements of $\Gamma$ act on $W_n^*$ as unipotent transformations. By Kolchin’s Theorem, any group of unipotent transformations on a finite-dimensional nonzero vector space fixes a nonzero vector. That is, there is some $\Gamma$-invariant $\varphi_n \in W_n^*$ and some $k$-disk $D_{\lambda_n, \tau_n}^k$ from the construction of $X_k$ such that $\varphi_n(\Theta_n(D_{\lambda_n, \tau_n}^k)) \neq 0$.

Given the disk $D_{\lambda_n, \tau_n}^k$ above, we may assume that the $f_{\text{BKw}}$-values of the cells in $S_{n+1}$, and hence of those in $\Gamma S_{n+1}$ exceed the $f_{\text{BKw}}$-values of the finitely many cells in $\psi(D_{\lambda_n, \tau_n}^k)$. Thus, if $m > n$ we have that $\psi(D_{\lambda_n, \tau_n}^k) \subseteq X - \Gamma I_m$ and thus $\Theta_m(D_{\lambda_n, \tau_n}^k) = 0$ in $W_m$. □

5.1. Cocycles. Let $D_{\gamma_\sigma}^k$ be a $k$-cell that was attached to $X_{k-2}$ in the construction of $X_k$. Recall that $\Theta_n(D_{\gamma_\sigma}^k)$ represents a class in $W_n$ and that $\varphi_n$ is a $\Gamma$-invariant functional on $W_n$.

Lemma 15. For any $n \geq 0$, $\gamma \in G(O_S)$, $g \in \Gamma$, and $D_{\gamma_\sigma}^k$, we have $\varphi_n(\Theta_n(D_{\gamma_\sigma}^k)) = \varphi_n(\Theta_n(gD_{\gamma_\sigma}^k))$.

Proof. This is immediate since $\psi$ is $\Gamma$-equivariant, $\theta_n$ is $\Gamma$-equivariant, and $\varphi_n$ is $\Gamma$-invariant. □

Let $q : X_k \to \Gamma \backslash X_k$ be the quotient map. Note that any $k$-cell in $\Gamma \backslash X_k$ is contained in $\Gamma \backslash X_{k-2}$ or else is of the form $q(D_{\gamma_\sigma}^k)$ for some $D_{\gamma_\sigma}^k \subseteq X_k$. We define the $k$-cochain $\Phi_n$ on $k$-chains in $\Gamma \backslash X_k$ with values in $\mathbb{F}_p$ as 0 on $\Gamma \backslash X_{k-2}$ and $\Phi_n(q(D_{\gamma_\sigma}^k)) = \varphi_n(\Theta_n(D_{\gamma_\sigma}^k))$ for any $q(D_{\gamma_\sigma}^k)$, and then we extend linearly. The previous lemma tells us that $\Phi_n$ is well-defined.

Lemma 16. $\Phi_n$ is a cocycle.

Proof. The $(k+1)$ cells of $\Gamma \backslash X_k$ are of the form $q(D_{\gamma_{\sigma}}^{k+1})$, so we must check that $\Phi_n$ evaluates the boundary of any $q(D_{\gamma_{\sigma}}^{k+1})$ trivially.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be a collection of $k$-cells in $X_{k-2}$ such that the chain $\partial D_{\gamma_\sigma}^{k+1}$ equals $\sum_j \mathcal{C}_j + \sum_i D_{\gamma_{i}}^{k}$, for some $D_{\gamma_{i}}^{k}$ where we suppress in this notation the orientation of $k$-cells. Then $\partial q(D_{\gamma_{\sigma}}^{k+1}) = \sum_j q(\mathcal{C}_j) + \sum_i q(D_{\gamma_{i}}^{k})$.

Note that $\psi(\partial D_{\gamma_{\sigma}}^{k+1})$ is a $k$-sphere in the $k$-dimensional and contractible $X$, and hence it represents the 0-chain. That is, the chain
ψ(∑j C_j + ∑i D^k_{γi,σi}) ∩ ΓS_n, and hence ψ(∑i D^k_{γi,σi}) ∩ ΓS_n, is the 0-chain. Therefore, Θ_n(∑i D^k_{γi,σi}) is the 0-chain, which implies

\[ \Phi_n(∂q(D^{k+1})) = \Phi_n\left(\sum_j q(C_j) + \sum_i q(D^k_{γi,σi})\right) \]
\[ = \Phi_n\left(\sum_i q(D^k_{γi,σi})\right) \]
\[ = \varphi_n \circ Θ_n\left(\sum_i D^k_{γi,f_i}\right) \]
\[ = \varphi_n(0) \]
\[ = 0. \]

□

5.2. Cycles. Given \( D^k_{λ_n,τ_n} \) as in Lemma 14, the \( k \)-chain \( D^k_{λ_n,τ_n} - D^k_{λ_0,τ_0} \) is the difference of two \( k \)-disks in \( X_k \). We let

\[ C_n = q(D^k_{λ_n,τ_n}) - q(D^k_{λ_0,τ_0}) \]

which is a \( k \)-chain in \( Γ \backslash X_k \).

**Lemma 17.** After passing to a subsequence in \( n \), each \( C_n \) is a \( k \)-cycle over \( \mathbb{F}_p \) in \( Γ \backslash X_k \).

**Proof.** Notice that \( q(∂D^k_{γ_n,σ_n}) \) is a \((k - 1)\)-cycle in \( Γ \backslash X_{k−2} \). Since \( Γ \backslash X_{k−2} \) is compact, there are only finitely many cellular \((k - 1)\)-chains in \( Γ \backslash X_{k−2} \) with coefficients in \( \mathbb{F}_p \). Therefore, we may pass to a subsequence and assume that \( q(∂D^k_{λ_n,τ_n}) \) is a constant \( \mathbb{F}_p \)-cycle for \( n \geq 0 \).

□

We can now prove

**Proposition 18.** \( H^k(Γ \backslash X_k; \mathbb{F}_p) \) and \( H_k(Γ \backslash X_k; \mathbb{F}_p) \) are infinite.

**Proof.** Let \( m \geq n > 0 \). By the definitions of \( Φ_n \) and \( C_n \), and by Lemma 14

\[ Φ_m(C_n) = Φ_m(q(D^k_{λ_n,τ_n})) - Φ_m(q(D^k_{λ_0,τ_0})) \]
\[ = \varphi_m(Θ_m(D^k_{λ_n,τ_n})) - \varphi_m(Θ_m(D^k_{λ_0,τ_0})) \]
\[ = \varphi_m(Θ_m(D^k_{λ_n,τ_n})) \]

does not equal 0 if \( m = n \), but does equal 0 if \( m > n \). Thus, each of the terms in the sequences \([Φ_n] ∈ H^k(Γ \backslash X_k; \mathbb{F}_p)\) and \([C_n] ∈ H_k(Γ \backslash X_k; \mathbb{F}_p)\) are distinct. □
6. Proof of Theorem 3

If \( \Gamma \) acts freely on \( X_k \), then Theorem 3 is immediate from Proposition 18. And one can always choose a finite-index, residually \( p \)-finite subgroup of \( G(O_S) \) that acts freely on \( X_k \) (see the following section). However, to show Theorem 3 holds for any, and not just some, finite-index, residually \( p \)-finite subgroup of \( G(O_S) \), we need to apply one more technique. That is the goal of this section.

By our construction of \( X_k \), the group \( \Gamma \) acts freely on \( X_k \), and while it may not be true that \( \Gamma \) acts freely on \( X_{k-2} \), it does act cocompactly on \( X_{k-2} \). That is, there are only finitely many \( k \)-cells in the quotient \( \Gamma \backslash X_{k-2} \). This will imply Theorem 3 after the application of a spectral sequence.

The material from this section is taken from Chapter VII of Brown’s text on Cohomology of Groups [4].

We begin by subdividing \( X_k \) such that individual cells in \( X_k \) inject into \( \Gamma \backslash X_k \).

We let \( H^\Gamma_k(X_k; \mathbb{F}_p) \) be the \( k \)-th equivariant homology group of \( \Gamma \) and \( X_k \) with coefficients in \( \mathbb{F}_p \). That is, if \( C_*^{X_k; \mathbb{F}_p} \) is the chain complex for the homology of \( X_k \) with coefficients in \( \mathbb{F}_p \), and if \( F_* \) is a projective resolution of \( \mathbb{Z} \) over \( \mathbb{Z} \Gamma \), then

\[
H^\Gamma_k(X_k; \mathbb{F}_p) = H_k(F_* \otimes \Gamma C_*^{X_k; \mathbb{F}_p})
\]

Lemma 19. \( H^\Gamma_k(X_k; \mathbb{F}_p) = H_k(\Gamma; \mathbb{F}_p) \)

Proof. The complex \( F_* \otimes \Gamma C_*^{X_k; \mathbb{F}_p} \) is a double complex with an associated spectral sequence

\[
E^1_{\ell,q} = H_q(F_\ell \otimes \Gamma C_*^{X_k; \mathbb{F}_p}) = F_\ell \otimes \Gamma H_q(X_k; \mathbb{F}_p)
\]

and

\[
E^2_{\ell,q} = H_\ell(\Gamma; H_q(X_k; \mathbb{F}_p))
\]

Notice that if \( 0 < q \leq k \) then \( E^2_{\ell,q} = H_\ell(\Gamma; 0) = 0 \) since \( X_k \) is \( k \)-connected. It follows that \( E^r_{\ell,q} = 0 \) when \( r \geq 2 \) and \( 0 < q \leq k \). Hence,

\[
H_k(\Gamma; \mathbb{F}_p) = E^2_{k,0} = E^\infty_{k,0} = \bigoplus_{\ell+q=k} E^\infty_{\ell,q}
\]

The lemma follows since the spectral sequence converges to \( H^\Gamma_k(X_k; \mathbb{F}_p) \). \( \square \)

The complex \( F_* \otimes \Gamma C_*^{X_k; \mathbb{F}_p} \) is also a double complex with an associated spectral sequence where \( E^1_{\ell,q} = H_q(F_* \otimes \Gamma C_\ell(X_k; \mathbb{F}_p)) \). The
spectral sequence converges to \( H^r_\Gamma(X_k; F_p) \), and in particular,

\[
H_k(\Gamma; F_p) = H_k^\Gamma(X_k; F_p) = \bigoplus_{\ell+q=k} E^{\infty}_{\ell,q}
\]

As in VII.7.7 of [4],

\[
E^1_{\ell,q} = \bigoplus_{c \in Y_\ell} H_q(\Gamma_c; F_p)
\]

where \( Y_\ell \) is a set of representatives of \( \ell \)-cells in \( X_k \) modulo \( \Gamma \), and \( \Gamma_c \) is the stabilizer in \( \Gamma \) of \( c \).

**Lemma 20.** If \( r, q \geq 1 \), then \( E^r_{\ell,q} \) is finite.

**Proof.** Since \( \Gamma \) acts cocompactly on \( X_{k-2} \) and freely on \( X_k - X_{k-2} \), there are only finitely many \( c \in Y_\ell \) such that \( \Gamma_c \neq 1 \). Thus, \( E^1_{\ell,q} \) is finite as it is a finite sum of homology groups of finite groups with coefficients in a finite field. The lemma follows since the dimension of \( E^r_{\ell,q} \) is bounded by that of \( E^1_{\ell,q} \). \( \square \)

**Lemma 21.** \( E^2_{\ell,0} = H_\ell(\Gamma \setminus X_k; F_p) \). In particular, by Proposition 18, \( E^2_{k,0} \) is infinite.

**Proof.** Let \( \partial' \) be the boundary operator for \( C_*(X_k; F_p) \), and for any \((\ell-1)\)-cell \( d \subseteq X_k \), let \( \pi_d \) be the projection of \( C_{\ell-1}(X_k; F_p) \) onto the coordinate represented by \( d \).

We let \( \partial \) be the boundary operator for the chain complex of \( \Gamma \setminus X_k \), denoted as \( C_*(\Gamma \setminus X_k; F_p) \).

Notice that \( E^2_{\ell,0} \) is the homology of the complex \((E^1_{k,0}, d^1)\) where \( d^1 : E^1_{\ell,0} \to E^1_{\ell-1,0} \). There is a natural identification of

\[
E^1_{\ell,0} = \bigoplus_{c \in Y_\ell} H_0(\Gamma_c; F_p) = \bigoplus_{c \in Y_\ell} F_p
\]

with

\[
C_\ell(\Gamma \setminus X_k; F_p)
\]

given by

\[
(a_c)_{c \in Y_\ell} \mapsto \sum_{\Gamma_c \subseteq \Gamma \setminus X_k} a_c(\Gamma_c)
\]

where \( a_c \in F_p \). Below we apply this identification liberally.

Our goal is to show that \( d^1 \) can be identified with \( \partial \). For this, if \( c \in Y_\ell \) then we let \( D_c \) be the set of \((\ell-1)\)-cells in \( X_k \) contained in \( c \). Then VII.8.1 of [4] tells us that if \( a_c \in F_p = H_0(\Gamma_c; F_p) \) then, up to sign,

\[
d^1(a_c) = \sum_{d \in D_c} v_d \circ u_{cd} \circ t_c(a_c)
\]
where \( t_c : H_0(\Gamma_c; \mathbb{F}_p) \to H_0(\Gamma_c; \mathbb{F}_p) \) is transfer — and thus is the identity — and where \( v_d : H_0(\Gamma_d; \mathbb{F}_p) \to H_0(\Gamma_d; \mathbb{F}_p) \) for \( d_0 \in Y_{l-1} \) is such that \( \Gamma d = \Gamma d_0 \) and \( v_d \) is induced by conjugation in \( \Gamma \) — and thus is the identity — and where \( u_{cd} : H_0(\Gamma_c; \mathbb{F}_p) \to H_0(\Gamma_d; \mathbb{F}_p) \) is induced by \( \Gamma_c \twoheadrightarrow \Gamma_d \) and \( \pi_d \circ \partial'|_c \) — and thus is identified with
\[
\pi_d \circ \partial'|_c : \{ a_c \mid a_c \in \mathbb{F}_p \} \to \{ a_d \mid a_d \in \mathbb{F}_p \}
\]

Therefore,
\[
d^1(a_c) = \sum_{d \in D_c} u_{cd}(a_c) = \sum_{d \in D_c} \pi_d \circ \partial'(a_c) = \partial(a_c(\Gamma_c))
\]

6.1. **Proof of Theorem** By the two preceding lemmas, we have for each \( r \geq 2 \) that the kernel of \( d^r : E_{k,0}^r \to E_{k-r,r-1}^r \) is infinite, which implies the infiniteness of
\[
E_{k,0}^\infty \leq \bigoplus_{\ell+q=k} E_{\ell,q}^\infty = H_k(\Gamma; \mathbb{F}_p) \cong H_k(\Gamma; \mathbb{F}_p)
\]

7. **Existence of finite-index, residually \( p \)-finite subgroups of \( \mathbf{G}(\mathcal{O}_S) \)**

In this section we give a sketch of the well-known existence statement from the title of this section. The existence essentially follows from Platonov’s Theorem on finitely-generated matrix groups. We took our account below from Nica [12].

Let \( w \) be a valuation of \( K \) that is not contained in \( S \), and let \( \mathfrak{m} \subseteq \mathcal{O}_S \) be the ideal \( \{ x \in \mathcal{O}_S \mid |x|_w < 1 \} \). Note that \( \cap_k \mathfrak{m}^k = 0 \). Furthermore, \( \mathcal{O}_S/\mathfrak{m} \) is identified with the values of elements of \( \mathcal{O}_S \) at \( w \), and hence is finite. Similarly, \( \mathfrak{m}^k/\mathfrak{m}^{k+1} \) is finite for any \( k \geq 1 \), so that \( \mathcal{O}_S/\mathfrak{m}^k \) is a finite ring.

For \( k \geq 1 \), let \( \Lambda_k \) be the kernel of
\[
\alpha_k : \mathbf{G}\mathbf{L}_n(\mathcal{O}_S) \to \mathbf{G}\mathbf{L}_n(\mathcal{O}_S/\mathfrak{m}^k)
\]
Since \( \mathcal{O}_S/\mathfrak{m}^k \) is a finite ring, \( \Lambda_k \) is a finite-index normal subgroup of \( \mathbf{G}\mathbf{L}_n(\mathcal{O}_S) \). Also note that if \( m > k \) then \( \Lambda_m \) is a normal subgroup of \( \Lambda_k \) since \( \Lambda_m \) is the kernel of \( \alpha_m \) restricted to \( \Lambda_k \).

We claim that \( \Lambda_k/\Lambda_{k+1} \) is a \( p \)-group. Indeed, if \( g \in \Lambda_k \) then the matrix entries of \( g - 1 \) are contained in \( \mathfrak{m}^k \). Thus, the matrix entries
of \((g - 1)^p\) are contained in \(m^{k+1}\). Since \(O_S \subseteq K\) has characteristic \(p\), \(g^p - 1 = (g - 1)^p\) so that \(g^p \in \Lambda_{k+1}\), establishing our claim.

Note that \(\cap_k m^k = 0\) implies \(\cap_k \Lambda_k = 1\). Thus, if \(Z \subseteq \Lambda_1\) is finite we can choose \(k \gg 0\) such that \(Z \cap \Lambda_k \subseteq \{1\}\), and

\[
[\Lambda_1 : \Lambda_k] = \prod_{i=1}^{k-1} [\Lambda_i : \Lambda_{i+1}]
\]

is a power of \(p\). Therefore, \(\Lambda_1\) is a finite-index, residually \(p\)-finite subgroup of \(\text{GL}_n(O_S)\).

For general \(G(O_S)\) we have an embedding of \(K\)-groups \(G \leq \text{GL}_n\) and we replace \(\Lambda_k\) in the above with \(\Lambda_k \cap G(O_S)\).

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