Essential ideals represented by mod-annihilators of modules

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Abstract
Let \( R \) be a commutative ring with unity, \( M \) be a unitary \( R \)-module and \( G \) a finite abelian group (viewed as a \( \mathbb{Z} \)-module). The main objective of this paper is to study properties of mod-annihilators of \( M \). For \( x \in M \), we study the ideals \( [x : M] = \{ r \in R \mid rM \subseteq Rx \} \) of \( R \) corresponding to mod-annihilator of \( M \). We investigate as when \( [x : M] \) is an essential ideal of \( R \). We prove that the arbitrary intersection of essential ideals represented by mod-annihilators is an essential ideal. We observe that \( [x : M] \) is injective if and only if \( R \) is non-singular and the radical of \( R/[x : M] \) is zero. Moreover, if essential socle of \( M \) is non-zero, then we show that \( [x : M] \) is the intersection of maximal ideals and \( [x : M]^2 = [x : M] \). Finally, we discuss the correspondence of essential ideals of \( R \) and vertices of the annihilating graphs realized by \( M \) over \( R \).

Keywords Module · Ring · Essential ideal · Annihilator · Graph

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1 Introduction
A nonzero ideal in a commutative ring is called essential if it intersects with every other nonzero ideal nontrivially. The study of essential ideals in a ring \( R \) is a classical problem. For instance, Green and Van Wyk in [8] characterized essential ideals in certain classes of commutative and non-commutative rings. The authors in [5, 12] studied essential ideals in \( C(X) \), where \( C(X) \) denotes the set of continuous functions on \( X \). They topologically characterized the socle and essential ideals. Moreover, essential ideals have been investigated in rings of measurable functions [14] and \( C^* \)-algebras [11]. For more on essential ideals, see [4, 9, 10, 21].
Throughout, $R$ is a commutative ring (with $1 \neq 0$) and all modules are unitary unless otherwise stated. $[N : M] = \{ r \in R \mid rM \subseteq N \}$ denotes an ideal of $R$. The symbols $\subseteq$ and $\subset$ have usual set theoretic meaning as containment and proper containment. We will denote the ring of integers by $\mathbb{Z}$, positive integers by $\mathbb{N}$ and the ring of integers modulo $n$ by $\mathbb{Z}/n$. For basic definitions from ring and module theory we refer to [7, 25].

For a $R$-module $M$ and $x \in M$, set $[x : M] = \{ r \in R \mid rM \subseteq Rx \}$, which clearly is an ideal of $R$ and an annihilator of the factor module $M/Rx$, whereas the annihilator of $M$ denoted by $ann(M)$ is $[0 : M]$.

Recently in [17], the elements of a module $M$ have been classified into full-annihilators, semi-annihilators and star-annihilators. We recall a definition concerning full-annihilators, semi-annihilators and star-annihilators of a module $M$.

**Definition 1.1** An element $x \in M$ is a

(i) full-annihilator, if either $x = 0$ or $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $[y : M] \neq R$,

(ii) semi-annihilator, if either $x = 0$ or $[x : M] \neq 0$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $0 \neq [y : M] \neq R$,

(iii) star-annihilator, if either $x = 0$ or $ann(M) \subset [x : M]$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $ann(M) \subset [y : M] \neq R$.

We denote by $A_f(M)$, $A_s(M)$ and $A_t(M)$, respectively, the sets of full-annihilators, semi-annihilators and star-annihilators for any module $M$ over $R$ and call these annihilators as mod-annihilators. We set $\widehat{A_f(M)} = A_f(M)/[0]$, $\widehat{A_s(M)} = A_s(M)/[0]$ and $\widehat{A_t(M)} = A_t(M)/[0]$.

This paper is organized as follows. In Sect. 2, we study the correspondence of essential ideals in $R$ and submodules of $M$ represented by mod-annihilators. For some finite abelian group $G$ (viewed as a $\mathbb{Z}$-module), we determine the value of $n$ such that $[x : G] = n\mathbb{Z}$, where $x \in G$. We characterize all $\mathbb{Z}$-module $M$ such that $[x : M]$ is an essential ideal of $R$. Furthermore, we discuss when $[x : M]$ as a $R$-module is injective. Also, if essential socle of $M$ is non-zero, then we prove that $[x : M]$ is the intersection of maximal ideals and $[x : M]^2 = [x : M]$. In Sect. 3, we discuss the correspondence of essential ideals of $R$ and the vertices of the annihilating graphs realized by modules over commutative rings. We conclude this paper with a discussion on some problems in this area of research.

**2 Essential ideals represented by mod-annihilators**

In this section, we discuss the correspondence of essential ideals in $R$ represented by elements of $\widehat{A_f(M)}$, and submodules of $M$ generated by elements of $\widehat{A_f(M)}$. We characterize essential ideals corresponding to $\mathbb{Z}$-modules. We discuss the cases of finite abelian groups where essential ideals which are represented by elements of $\widehat{A_f(M)}$ corresponding to submodules of $M$ are isomorphic. If $M$ is a non-simple $R$-module, then for $x \in \widehat{A_f(M)}$, we show that an ideal $[x : M]$ considered as an $R$-module is injective. We also study essential ideals represented by mod-annihilators over hereditary and regular rings.

By Definition 1.1, we see that there is a correspondence of ideals in $R$ represented by elements of $\widehat{A_f(M)}$, $\widehat{A_s(M)}$, and $\widehat{A_t(M)}$ and cyclic submodules of $M$ generated by elements of sets $\widehat{A_f(M)}$, $\widehat{A_s(M)}$, and $\widehat{A_t(M)}$. Furthermore, the containment $A_t(M) \subseteq A_s(M) \subseteq
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of $n$ denoted by $\lambda \vdash n$. For any $\mu \vdash n$, we have an abelian group of order $p^n$ and conversely every abelian group corresponds to some partition of $n$. In fact, if $H_{\mu,p} = \mathbb{Z}/p^{\mu_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\mu_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\mu_r}\mathbb{Z}$ is a subgroup of $G_{\lambda,p} = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\lambda_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\lambda_r}\mathbb{Z}$, then $\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2, \ldots, \mu_r \leq \lambda_r$. If these inequalities hold, we write $\mu \subset \lambda$, that is a “containment order” on partitions. For example, a $p$-group $\mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is of type $\lambda = (5, 1, 1)$. The possible types for its subgroup are $(5, 1, 1), (4, 1, 1), (3, 1, 1), (2, 1, 1), (1, 1, 1), 2(5, 1), 2(4, 1), 2(3, 1), 2(2, 1), 2(1, 1), (5), (4), (3), (2), 2(1)$. Note that the types $(5, 1), (4, 1), (3, 1), (2, 1), (1, 1)$ are appearing twice in the sequence of partitions for a subgroup.

Let $\lambda = (1, 1, \ldots, 1) = (1^n)$. A group of type $\lambda$ is nothing but the $\mathbb{Z}/p\mathbb{Z}$-vector space $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$. Its subgroups are of type $(1^r)$, where $0 \leq r \leq n$. The essential ideals corresponding to subspaces of vector space $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$ (represented by elements of the set $A_f(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z})$) are same. In fact, $[x : \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}] = \text{ann}(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}) = p\mathbb{Z}$.

More generally, for a finite abelian $p$-group of the type $\mathbb{Z}/p^\alpha\mathbb{Z} \oplus \mathbb{Z}/p^\alpha\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^\alpha\mathbb{Z}$, where $\alpha \geq 2$, the essential ideals represented by elements of the set $A_f(\mathbb{Z}/p^\alpha\mathbb{Z} \oplus \mathbb{Z}/p^\alpha\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^\alpha\mathbb{Z}) = p^{\alpha}\mathbb{Z}$.

A finite abelian $p$-group is isomorphic to the group of the form $\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z}$ whereas a finitely generated abelian $p$-group with Betti number $n$ is of the from $\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z}$. It is very difficult to determine the exact ideals represented by mod-annihilators of sets $A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z})$ and $A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z})$. However, it is clear from the definition of mod-annihilators that for some $x \in A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z})$, $[x : \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) = \text{some ideal in } \mathbb{Z}.$

Using the description given above, we now characterize all essential ideals represented by elements of $A_f(M)$ and corresponding to $\mathbb{Z}$-modules.

**Lemma 2.1** If $M$ is any $\mathbb{Z}$-module, then $[x : M]$ is an essential ideal if and only if $[x : M]$ is non-zero for all $x \in A_f(M)$.

**Proof** Let $M$ be a $\mathbb{Z}$-module. Clearly, $M$ is an abelian group in a unique way. For all $x \in A_f(M)$, we have $[x : M] = n\mathbb{Z}$, $n \in \mathbb{N}$. The ideal $n\mathbb{Z}$ intersects non-trivially with any ideal $m\mathbb{Z}$, $m \in \mathbb{N}$ in $\mathbb{Z}$. So, if $M$ is a non-simple $\mathbb{Z}$-module, then for every $x \in M$, it follows that $[x : M]$ is an essential ideal. Note that $M$ is simple if and only if $A_f(G) = \emptyset$.

If possible, suppose $[x : M] = \{0\}$, then $[x : M]$ does not intersect non-trivially with non-trivial ideals of $\mathbb{Z}$, a contradiction. \hfill $\Box$

Since it is possible to have some finitely generated $\mathbb{Z}$-modules such that the set of mod-annihilators is equal to zero only which of course by definition is not an essential ideal. Consider a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, which is a direct sum of $n$ copies of $\mathbb{Z}$. It is easy to verify that $A_f(M) = \hat{M}$ with $[x : M][y : M]M = 0$ for all $x, y \in M$. The cyclic submodules generated by elements of $A_f(M)$ are simply lines with integral coordinates passing through the origin in the hyperplane $\mathbb{R} \ominus \mathbb{R} \ominus \cdots \ominus \mathbb{R}$ and these lines intersect at the origin only. Thus, for each $x \in M$, it follows that $[x : M]$ is not an essential ideal in $\mathbb{Z}$. In fact $[x : M]$ is a zero-ideal in $\mathbb{Z}$.

For any $R$-module and $x \in A_f(M)$, it would be interesting to characterize essential ideals $[x : M]$ represented by elements of $A_f(M)$ such that the intersection of all essential
ideals is again an essential ideal. It is easy to see that a finite intersection of essential ideals in any commutative ring is an essential ideal. But an infinite intersection of essential ideals need not to be an essential ideal, even a countable intersection of essential ideals in general is not an essential ideal, as can be seen in [5]. If the cardinality of \( M \) is finite over \( R \), then the submodules determined by elements of \( \hat{A_f(M)} \) are finite and therefore the ideals corresponding to submodules are finite in number. Thus, we conclude that for every \( x \in \hat{A_f(M)} \), the intersection of essential ideals \([x : M] \) in \( R \) is an essential ideal. For the other case, that is, if the cardinality of \( M \) is infinite over \( R \), we have the following result. Note that, a nonzero submodule of a module \( M \) is said to be an essential submodule of \( M \) if it intersects non-trivially with other nonzero submodules of \( M \).

**Theorem 2.2** Let \( M \) be an \( R \)-module such that every proper submodule of \( M \) is cyclic over \( R \). For \( x \in \hat{A_f(M)} \), if the submodule generated by \( x \) intersects non-trivially with every other nonzero submodule of \( M \), then \([x : M] \) is an essential ideal in \( R \).

**Proof** Assume that \( \bigcap_{0 \neq x \in M} Rx \neq 0 \). If \( \hat{A_f(M)} = \emptyset \), then \( M \) is simple, a contradiction. Let \( x \in \hat{A_f(M)} \) and let \( Rx \) be the submodule generated by \( x \). Since \( Rx \) intersects non-trivially with every other submodule, so there exists \( y \in \hat{A_f(M)} \) such that \( Rx \cap Ry \neq 0 \). It suffices to prove the result for \( Rx \cap Ry \). Let \( z \in Rx \cap Ry \) and let \([x : M], [y : M], [z : M] \) be ideals of \( R \) corresponding to submodules \( Rx, Ry \) and \( Rz \). Then \([z : M] \subseteq [x : M] \cap [y : M] \neq 0 \), which implies \([x : M] \) intersects non-trivially with every nonzero ideal corresponding to the submodule generated by an element of \( \hat{A_f(M)} \). For any other ideal \( I \) of \( R \), it is clear that \( IM = \{ \sum_{f in \text{finite}} am : a \in I, m \in M \} = Ra \) for some \( a \in M \). Thus \( I \) corresponds to the cyclic submodule generated by \( a \in M \). It follows that \([x : M] \cap I \neq 0 \), for every nonzero ideal of \( R \) and we conclude that \([x : M] \) is an essential ideal for each \( x \in \hat{A_f(M)} \). \( \Box \)

The converse of Theorem 2.2 is not true in general. We can easily construct examples from \( \mathbb{Z} \)-modules such that an ideal corresponding to the submodule generated by some element of \( \hat{A_f(M)} \) is an essential ideal, but the intersection of all submodules determined by elements of \( \hat{A_f(M)} \) is empty. However, if every ideal \([x : M] \), where \( x \in \hat{A_f(M)} \), corresponds to an essential submodule of \( M \), then we have a non-zero intersection.

**Corollary 2.3** Let \( M \) be an \( R \)-module.

(i) For \( x \in \hat{A_f(M)} \), if the cyclic submodule \( Rx \) intersects with every other cyclic nonzero submodule of \( M \) non-trivially, then \([x : M] \) is an essential ideal in \( R \).

(ii) The intersection \( \bigcap_{x \in \hat{A_f(M)}} [x : M] \) is an essential ideal in \( R \) if and only if every submodule of \( M \) is essentially cyclic over \( R \).

In the preceding results, we proved that “arbitrary intersection of essentials ideals is an essential ideal”. We formulated this theory of essential ideals using the concept of mod-annihilators and mainly the theory involves study of cyclic submodules of \( M \). It is interesting to develop a similar theory that would employ the other finitely generated submodules of \( M \). So, motivated by [5], we have the following question regarding essential ideals represented by elements of \( \hat{A_f(M_N)} \), where \( \hat{A_f(M_N)} = \{ r \in R : rM \subseteq N \} \), \( N \) is a finitely generated submodule of \( M \).

**Problem 2.4** Let \( M \) be an \( R \)-module. For \( x \in \hat{A_f(M_N)} \), characterize essential ideals \([x : M] \) in \( R \) such that their intersection is an essential ideal.
For an $R$-module $M$, let $Z(M)$ denote the following.

$$Z(M) = \{m \in M : \text{ann}(m) \text{ is an essential ideal in } R\}.$$ 

If $Z(M) = M$, then $M$ is said to be singular and if $Z(M) = 0$, then $M$ is said to be non-singular. By $\text{rad}(M)$, we denote the intersection of all maximal submodules of $M$. So, $\text{rad}(R)$ is the Jacobson radical $J(R)$ of a ring $R$. The socle of an $R$-module $M$ denoted by $\text{Soc}(M)$ is the sum of simple submodules or equivalently the intersection of all essential submodules. To say that $\text{Soc}(M)$ is an essential socle is equivalent to saying that every cyclic submodule of $M$ contains a simple submodule of $M$. An essential socle of $M$ is denoted by $\text{esssoc}(M)$.

**Lemma 2.5** Let $M$ be an $R$-module with $\text{esssoc}(M) \neq 0$, $\cap_{0 \neq x \in M} Rx \neq 0$. Then for $x \in \overline{Af}(M)$, $R/[x : M]$ is a singular module.

**Proof** Since $\cap_{0 \neq x \in M} Rx \neq 0$ and $\text{esssoc}(M) \neq 0$, therefore, $\overline{Af}(M) \neq \emptyset$. Thus, $[x : M]$ is an essential ideal. Moreover, $Z(R/[x : M]) = R/[x : M]$. Therefore, $R/[x : M]$ is a singular module. \hfill \Box

A ring $R$ is said to be a regular ring if for all $a \in R$, $a^2x = a$ for some $x \in R$.

**Lemma 2.6** [24] A commutative ring $R$ with unity is regular if and only if every simple $R$-module is injective.

Now, we consider singular simple $R$-modules (ideals) which are injective, and obtain some properties of essential ideals corresponding to submodules generated by elements of $\overline{Af}(M)$.

**Theorem 2.7** Let $M$ be an $R$-module with $\text{esssoc}(M) \neq 0$ and $\cap_{0 \neq x \in M} Rx \neq 0$. Then every singular simple $R$-module $[x : M], x \in \overline{Af}(M)$ is injective if and only if $Z(R) = 0$ and $\text{rad}(R/[x : M]) = 0$.

**Proof** We have $\text{esssoc}(M) \neq 0$ and $\cap_{0 \neq x \in M} Rx \neq 0$, so that $\overline{Af}(M) \neq 0$. Therefore corresponding to every cyclic submodule generated by elements of $\overline{Af}(M)$, we have an ideal in $R$. For $x \in \overline{Af}(M)$, suppose all singular simple $R$-modules $[x : M]$ are injective. If for some $z \in \overline{Af}(M)$, $I = [z : M] \subseteq Z(R)$ is a simple $R$-module, then $Z(I) = I$. This implies that $I$ is injective and thus a direct summand of $R$. However, the set $Z(R)$ is free from nonzero idempotent elements. Therefore, $I = 0$ and so $Z(R) = 0$. For $x \in \overline{Af}(M)$, clearly $A = [x : M]$ is an essential ideal of $R$. Thus, by Lemma 2.5, $R/A$ is a singular module and so is every submodule of $R/A$. Therefore every simple submodule of $R/A$ is injective, which implies that every simple submodule is excluded by some maximal submodule. Thus we conclude that $\text{rad}(R/A) = 0$.

For the converse, we again consider the correspondence of cyclic submodules of $M$ and ideals of $R$. Let $\tilde{I}$ be a singular simple $R$-module corresponding to the submodule of $M$. In order to show that $\tilde{I}$ is injective, we must show that for every essential ideal $A$ in $R$ corresponding to the submodule determined by an element $x \in \overline{Af}(M)$, every $\varphi \in \text{Hom}_R(A, \tilde{I})$ has a lift $\psi \in \text{Hom}_R(R, \tilde{I})$ such that the following diagram commutes.
Let $K = \ker(\varphi)$. We claim that $K$ is an essential ideal of $R$. For, if $K \cap J = \{0\}$, for some nonzero ideal $J$ of $R$, then $I^* = J \cap A \neq 0$ and $I^* \cap K = \{0\}$. This implies that $I^* \subseteq \varphi(I^*) \subseteq \tilde{I}$, a contradiction, since $\tilde{I}$ is a singular simple submodule and $Z(R) = 0$. For $\mu \neq 0$, it is clear that $\varphi$ induces an isomorphism $\mu : A/K \rightarrow \tilde{I}$. So, $A/K$ is a simple $R$-submodule of $R/K$. By our assumption, $\text{rad}(R/K) = 0$, so there is a maximal submodule $M/K$ such that $R/K = A/K \oplus M/K$. Let $g : R \rightarrow R/K$ be a canonical map and let $p : R/K \rightarrow A/K$ be a projection map. Then, we have $pg : R \rightarrow A/K$. Therefore the composition $\psi = \mu pg : R \rightarrow \tilde{I}$ is the required lift such that the above diagram commutes. \hfill \ensuremath{\square}

Next, we discuss some interesting consequences of the preceding theorem.

**Theorem 2.8** Let $M$ be an $R$-module with $\text{essoc}(M) \neq 0$, $\bigcap_{0 \neq x \in M} Rx \neq 0$ and for $x \in \hat{A}(M)$, let every singular simple $R$-module $[x : M]$ be injective. Then every ideal $[x : M]$ is an intersection of maximal ideals, $J(R)^2 = 0$ and $[x : M]^2 = [x : M]$.

**Proof** For any $x \in \hat{A}(M)$, clearly $[x : M]$ is an essential ideal in $R$. Therefore, $J(R) \subseteq [x : M]$, since $J(R)$ is contained in every essential ideal of $R$. On the other hand, intersection of all essential ideals in $R$ is Socle of $R$, therefore $J(R) \subseteq \text{Soc}(R)$. This implies that $J(R)^2 = 0$ and $[x : M]$ is the intersection of maximal ideals in $R$. Suppose that $[x : M]^2 \neq [x : M]$, for an essential ideal $[x : M]$ of $R$. By Theorem 2.7, $Z(R) = 0$ and therefore for every essential ideal $I$, we have $I \subseteq I^2$. In particular, $[x : M] \subseteq [x : M]^2$ for each $x \in \hat{A}(M)$. It follows that $[x : M]^2$ is an essential ideal and is the intersection of maximal ideals in $R$. Finally, if $y \in [x : M]^2$, $y \notin [x : M]$, there is some maximal ideal $P$ of $R$ such that $[x : M] \subseteq P$, $y \notin P$. Then $R = Ry + P$, that is, $1 = ry + m$. This implies that $y = yry + ym \in P$, a contradiction. Hence we conclude that $[x : M]^2 = [x : M]$. \hfill \ensuremath{\square}

**Corollary 2.9** Let $M$ be an $R$-module, where $R$ is hereditary. For $x \in \hat{A}(M)$, if $[x : M]$ is an essential ideal of $R$ and $J(R)^2 = 0$, then every singular simple $R$-module $[x : M]$ is injective.

**Proof** Let $R$ be hereditary. From [7], the exact sequence

$$0 \rightarrow \text{ann}(x) \rightarrow R \rightarrow Rx \rightarrow 0$$

splits for any $x \in R$. Since $J(R)^2 = 0$ and $R/J(R)$ is an artinian ring, therefore $J(R) \subseteq \text{Soc}(R)$. But any essential ideal of $R$ contains $\text{Soc}(R)$. So, $J(R) \subseteq [x : M]$. This implies that $R/[x : M]$ is a completely reducible $R$-module and therefore $\text{rad}(R/[x : M]) = 0$. Thus, by Theorem 2.7, every singular simple $R$-module $[x : M]$ is injective. \hfill \ensuremath{\square}

Next, we consider the modules over regular rings.

**Theorem 2.10** Let $M$ be an $R$-module such that every submodule of $M$ is cyclic over $R$ and $\bigcap_{0 \neq x \in M} Rx \neq 0$. The following are equivalent.

(i) $R$ is regular.

(ii) $A^2 = A$ for each ideal $A$ of $R$. 

}$${}$
(iii) \([x : M]^2 = [x : M] \) for each \(x \in \hat{A_f}(M)\).

**Proof** The equivalence of (i) and (ii) is clear and certainly (ii) implies (iii). Thus, we just need to show that (iii) implies (ii). By Theorem 2.7, \([x : M]\) is an essential ideal for each \(x \in \hat{A_f}(M)\). Suppose that \([x : M]^2 = [x : M]\). Choose \(J\) to be maximal ideal of \(R\) such that \(A \cap J = 0\), where \(A\) is some non essential ideal of \(R\). Then \(A + J\) is an essential ideal of \(R\). Therefore, again by Theorem 2.7, \(A + J\) corresponds to some submodule of \(M\) and we have \(A + J = [z : M]\) for some \(z \in M\). So, \((A + J)^2 = A^2 + J^2 = A + J\). If \(x \in A\), then
\[
x - \sum_{finite} ab = \sum_{finite} mn \in A \cap J = 0.
\]
This implies that \(x \in A^2\) and we conclude that \(A = A^2\).

**Corollary 2.11** Let \(M\) be an \(R\)-module with \(\text{essoc}(M) \neq 0\) and \(\bigcap_{x \in M} Rx \neq 0\). Then every singular simple \(R\)-module \([x : M]\), where \(x \in \hat{A_f}(M)\), is injective if and only if \(R\) is regular.

**Proof** By Theorem 2.8, if every singular simple \(R\)-module \([x : M]\) is injective, then for \(x \in \hat{A_f}(M)\), we have \([x : M]^2 = [x : M]\). Therefore, by Theorem 2.10, \(R\) is regular. If \(R\) is regular, then by Lemma 2.6 every simple \(R\)-module is injective.

### 3 Representation of essential ideals by vertices of annihilating graphs

In this section, we give a brief discussion on representation of essential ideals by vertices of graphs realized by modules over commutative rings.

A simple graph \(\Gamma\) consists of a vertex set \(V(\Gamma)\) and an edge set \(E(\Gamma)\), where an edge is an unordered pair of distinct vertices of \(\Gamma\). One of the areas in algebraic combinatorics introduced by Beck [6] is to study the interplay between graph theoretical and algebraic properties of an algebraic structure. Continuing the concept of associating a graph to an algebraic structure, another combinatorial approach of studying commutative rings was given by Anderson and Livingston in [2]. They associated a simple graph to a commutative ring \(R\) with unity called the zero-divisor graph denoted by \(\Gamma(R)\) with vertex set \(Z^+(R) = Z(R)/\{0\}\), where two distinct vertices \(x, y \in Z^+(R)\) are adjacent in \(\Gamma(R)\) if and only if \(xy = 0\). The study of graph theoretical parameters and spectral properties in zero-divisor graphs of commutative rings are explored in [1–3, 15, 16, 18, 20, 22]. In [2, 18], authors have discussed chromatic number, clique number and metric dimensions of zero-divisor graphs associated with finite commutative rings whereas [16, 22] are related to eigenvalues and Laplacian eigenvalues of zero-divisor graphs associated to finite commutative rings of type \(\mathbb{Z}_n\) for \(n = p^{N_1} q^{N_2}\), where \(p < q\) are primes and \(N_1, N_2\) are positive integers. The extension of zero-divisor graphs to non-commutative rings and semigroups can be found in [13, 23].

The combinatorial properties of zero-divisors discovered in [6] have also been investigated in module theory. In [17], the authors introduced annihilating graphs realized by modules over commutative rings known as full-annihilating, semi-annihilating and star-annihilating graphs, denoted by \(\text{ann}_f(\Gamma(M))\), \(\text{ann}_s(\Gamma(M))\) and \(\text{ann}_t(\Gamma(M))\). The vertices of annihilating graphs are elements of sets \(\hat{A_f}(M)\), \(\hat{A_s}(M)\) and \(\hat{A_t}(M)\) respectively, where two vertices \(x\) and \(y\) are adjacent if and only if \([x : M][y : M]M = 0\). The three simple graphs: full-annihilating, semi-annihilating and star-annihilating with vertex sets: \(\hat{A_f}(M)\), \(\hat{A_s}(M)\), \(\hat{A_t}(M)\) are natural
generalizations of the zero-divisor graph introduced in [2]. This concept was further studied in [19].

We call a vertex $x$, an essential vertex in $ann_f(\Gamma(M))$ if the ideal represented by $x$ is essential in $R$. Recall that a graph $\Gamma$ is said to be a complete if there is an edge between every pair of distinct vertices.

By Definition 1.1, we see the containment $ann_t(\Gamma(M)) \subseteq ann_s(\Gamma(M)) \subseteq ann_f(\Gamma(M))$ as induced subgraphs of the graph $ann_f(\Gamma(M))$, since $A_t(M) \subseteq A_s(M) \subseteq A_f(M)$. If $ann_f(\Gamma(M))$ is a finite graph, then by [17,Theorem 3.3 and Example 2.2], $|A_f(M)| = |A_s(M)|$ and annihilating graphs $ann_f(\Gamma(M))$, $ann_s(\Gamma(M))$ coincide, whereas the graph $ann_t(\Gamma(M))$ with vertex set $A_t(M)$ may be different. For a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, we have by Definition 1.1, $\{x : M\}|y : M|M = 0$ for all $x, y \in A_f(M)$. Therefore $ann_f(\Gamma(M))$ is a complete graph whereas the graph $ann_s(\Gamma(M))$ is an empty graph. Thus for finitely generated infinite modules, graphs $ann_f(\Gamma(M))$ and $ann_s(\Gamma(M))$ are different.

As discussed in Sect. 2, for a module $M = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, the ideal $\{x : M\}$ represented by a vertex $x \in A_f(M)$ of the graph $ann_f(\Gamma(M))$ is not an essential ideal. So, $x$ is not an essential vertex of the graph $ann_f(\Gamma(M))$. On the other hand, every vertex of a $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is an essential vertex of the graph $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_q))$, where $p$ and $q$ are any two primes.

Finally, Problem 2.4 can be restated in the graph theoretical version as follows.

Problem 3.1 Characterize all annihilating graphs realized by a module $M$ such that every vertex $x \in A_f(M_N)$ of an annihilating graph is an essential vertex.

4 Conclusion

In this paper, we formulated a new approach of recognition of essential ideals in a commutative ring $R$. This formulation of essential ideals corresponds to mod-annihilators of a $R$-module $M$. It is interesting to characterize essential ideals such that their arbitrary intersection is an essential ideal, since it is specified in [5] that an arbitrary intersection of essential ideals may not be an essential ideal. Furthermore, we obtained the results related to ideals $\{x : M\}$ of $R$, where $x$ is a mod-annihilator of $M$ and discussed the representation of vertices of annihilating graphs by essential ideals of $R$. Apart from the research problems which we mentioned in Sects. 2 and 3, the following problems could be investigated for the future work.

1. Let $G$ be a finite abelian $p$-group (viewed as a finite $\mathbb{Z}$-module) of rank at least 3. Determine the value of $n$ for the essential ideal $\{x : G\} = n\mathbb{Z}$, where $x \in G$.
2. Let $G$ be any finite abelian group (viewed as a finite $\mathbb{Z}$-module). Determine the value of $n$ for the essential ideal $\{x : G\} = n\mathbb{Z}$, where $x \in G$.

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