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Macroscopic excitations in confined Bose-Einstein condensates, searching for quantum turbulence

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Abstract We present a survey of macroscopic excitations of harmonically confined Bose-Einstein condensates (BEC), described by Gross-Pitaevskii (GP) equation, in search of routes to develop quantum turbulence. These excitations can all be created, in principle, by phase imprinting techniques on an otherwise equilibrium Bose-Einstein condensate. We analyze two crossed vortices, two parallel anti-vortices, a vortex ring, a vortex with topological charge $Q = 2$, and a tangle of 4 vortices. Since GP equation is time-reversal invariant, we are careful to distinguish time intervals in which this symmetry is preserved and those in which rounding errors play a role. With those considerations, we find that the system tends to reach stationary states that may be widely classified as having array of vortices or an agitated state with excitations at different length scales. We study the evolution by observing velocity and density plots and by analyzing the incompressible kinetic energy spectrum.

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1 Introduction

Turbulence, be it classical or quantum, remains as a fundamental problem in the dynamics of fluids. Its high complexity makes it a very difficult, yet a fascinating, state of matter to be described in simple terms. While there does not exist a clear definition or specification of what turbulence should be, there are several aspects or signatures that indicate whether or not one is facing a turbulent flow. Most notorious is the cascade of energy through vortices or eddies, in the classical case,
or by a tangle of vortices in the quantum version. Among the relevant questions in the problem, specially in the quantum case, is the initiation or generation of such a complex state and its characterization. That is the motivation of the study presented in this article. We point out that there are already experimental realizations of quantum turbulence as well of studies of the route to turbulence.

In this work, through the Gross-Pitaevskii (GP) model of an ultracold quantum gas, and assuming that arbitrary phase-imprinted states can be generated, we present here a survey of macroscopic excitations that lead a Bose-Einstein condensate (BEC), confined in an external harmonic potential, to stationary agitated or chaotic states that, under appropriate conditions, may be considered to pass through turbulent transients. Although the details of the model and of the different excitations that we explore are presented below, we want to advance here a general result that we find may be of relevance in a more complete study. The first one is the fact that the GP model is capable of showing a clear cascade of energy that, we shall argue, starts with the large scale excitation and ends up in a “stationary” state, sometimes with an array of stable of vortices or with nothing, but always accompanied by an agitated background composed mainly of Bogoliubov phonons. This apparent “irreversible” behavior of GP is actually a complicated non-linear dephasing effect since GP obeys time-reversal invariance. Thus, it is relevant to take it into account since it will compete with true irreversible development induced by actual dissipative effects. In this work, we limit ourselves to the numerical solutions of the time-reversal invariant GP equation, valid for temperatures as close as possible to Absolute Zero, but we recognize that in order to make a direct comparison with actual experiments, being these at finite temperature, one must take into account dissipative effects. There are already efforts along these lines including the effects of the thermal cloud that surrounds a confined BEC. True stationary states, of course, are found in those cases.

2 Gross-Pitaevskii quantum fluid

Our model is summarized in the Gross-Pitaevskii equation for a three dimensional, one-component BEC confined by an external potential,

\[ \imath \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_{\text{ext}} \psi + g |\psi|^2 \psi. \]  

(1)

The mean-field interaction coupling term \( g = 4\pi \hbar^2 Na/m \), where \( N \) is the number of bosonic atoms of mass \( m \), and \( a \) is the s-wave scattering length, assumed positive throughout. The GP wave function \( \psi \) is normalized to one. Although our numerical analysis will be limited to an isotropic harmonic potential \( V_{\text{ext}} = 1/2m\omega r^2 \), there are several comments and aspects that should be generic for any potential. In particular, due to the problem at hand, we are interested in also posing relevant questions in terms of a hydrodynamic formulation. This is done as follows.

Using the transformation \( \psi = \sqrt{\rho} e^{i\phi} \) and the identification \( v = (\hbar/m) \nabla \phi \), where \( \rho \) is the particle density and \( v \) may be interpreted as the fluid velocity field,
one finds the following hydrodynamic equations,

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2} \]

and

\[ \rho \partial_t \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \cdot \mathbf{\sigma} - \frac{\rho}{m} \nabla V_{\text{ext}}. \tag{3} \]

where the stress tensor is,

\[ \sigma_{ij} = \left( \frac{\hbar^2}{4m^2} \partial_{kk} \rho - \frac{g}{2m} \rho^2 \right) \delta_{ij} - \frac{\hbar^2}{m^2} \partial_i \sqrt{\rho} \partial_j \sqrt{\rho} \tag{4} \]

and with the additional condition that the fluid is irrotational \( \nabla \times \mathbf{v} = 0. \) It is of interest to note that the stress tensor has a hydrostatic pressure \( p_0 = gp^2/2m, \) due to the interatomic interactions, and other volumetric and shear stresses of pure quantum origin, namely, all those proportional to \( \hbar^2. \)

As it is known, and we discuss and review it here, there is a ground stationary state, \( \psi(\mathbf{r}, t) = e^{-i\mu t/\hbar} \phi_0(\mathbf{r}), \) with \( \mu \) the chemical potential and \( \phi_0(\mathbf{r}) \) the ground state macroscopic wave function, solution to,

\[ \mu \phi_0 = -\frac{\hbar^2}{2m} \nabla^2 \phi_0 + V_{\text{ext}} \phi_0 + g |\phi_0|^2 \phi_0. \tag{5} \]

For \( V_{\text{ext}}, \ g \) and \( m \) given, the chemical potential \( \mu \) and \( \phi_0 \) are uniquely given. For an isotropic harmonic potential, and with dimensionless units \( \hbar = m = \omega = 1, \) for \( g = 8000, \) the chemical potential is \( \mu = 19.63. \) The wave function \( \phi_0(\mathbf{r}) \) can be accurately found numerically\(^{31,32} \). This is our starting point for all the forthcoming calculations. From the hydrodynamic point of view, this fluid has an inhomogenous density \( \rho_0(\mathbf{r}) = |\phi_0(\mathbf{r})|^2 \) and zero velocity field everywhere, \( \mathbf{v} = 0. \)

Any macroscopic excitation on top of the state \( \phi_0 \) will necessarily evolve in time (except perhaps a line vortex at an axis, but this is unstable\(^{33} \)).

While there may be many more excitations, we consider for the moment three of them, vortices, collective modes (such as breathing or scissors modes\(^{34,35} \)) and sound waves (or Bogoliubov phonons\(^{20} \)), and their interactions. The line vortex, for \( V_{\text{ext}} = 0, \) is the actual solution given by Gross\(^{14} \) and Pitaevskii\(^{15} \), with zero density at the vortex and the velocity field yielding a non-zero circulation,

\[ \oint \mathbf{v} \cdot d\mathbf{l} = \frac{2\pi \hbar}{m} Q, \tag{6} \]

with \( Q = 1, 2, \ldots \) the topological charge. On the other extreme, we have that any small perturbation is composed of Bogoliubov sound waves, as we know briefly review it.

Let us assume an stationary solution to the equations, \( \rho_0, \) which is a uniform one \( \rho_0 = \mu/g \) if \( V_{\text{ext}} = 0, \) and the non-uniform solution \( \rho_0(\mathbf{r}) = |\phi_0(\mathbf{r})|^2 \) of the harmonic trap, discussed above. In both cases, the velocity field is zero everywhere, \( \mathbf{v} = 0. \) Let us now consider a perturbation in the free-field case, \( V_{\text{ext}} = 0, \)

\[ \rho \approx \rho_0 + \rho_1(\mathbf{r}, t) \quad \text{and} \quad \mathbf{v} \approx \mathbf{v}_1(\mathbf{r}, t). \tag{7} \]
Then, a simple linearization of the hydrodynamic equations, eqs. (2) and (3), yields,
\[ \partial_{tt} \rho_1 - \frac{\hbar^2}{4m^2} \nabla^2 (\nabla^2 \rho_1) + \frac{g\rho_0}{m} \nabla^2 \rho_1 = 0. \] (8)
This is a fourth-order wave equation whose solution is \( \rho_1 = R_k e^{i(k \cdot r - \omega_k t)} \), with \( R_k \) an arbitrary complex amplitude, yielding,
\[ \omega_k^2 = \frac{\hbar^2 k^4}{4m^2} + \frac{g\rho_0 k^2}{m} \] (9)
which, as expected, is Bogoliubov dispersion relation for the elementary excitations in a weakly-interacting Bose gas\(^{20}\). The velocity field is, in turn, found as \( v_1 = \rho_0 k (R_k \partial_\theta / k^2) e^{i(k \cdot r - \omega_k t)} \), which shows that the waves (phonons) are longitudinal, also as expected. The purpose of this derivation is to argue that any macroscopic excitation that one may consider will always create phonons because of the non-linearities of GP. These are the weakest excitations, similarly to any classical compressible fluid. In our calculations we always observe this phenomenon, as we see below.

A further general but very important observation is that while there is a lack of true dissipation in the GP equation, in general, the fluid appears to “relax” to “stationary” states. These may sometimes be arrays of parallel vortices and sometimes agitated or chaotic states, as exemplified in the following section. These states, still time-dependent, cannot decay to true equilibrium states because there is not viscosity in the fluid, otherwise, being the system closed, energy would be dissipated into heat yielding an increase in temperature. Nevertheless, an apparent “relaxation” to stationary states are observed, and one should be quite precise and avoid considering it as true decoherence or relaxation. It is rather a dephasing effect, common in the evolution of true Schrödinger equations with arbitrary anharmonic potentials in which the wave function spreads out covering all the available space with almost no probability of rephasing for very long times. This is a dephasing effect since GP, as actual Schrödinger equations, are time-reversal invariant under the transformation \( t \to -t \) and \( \psi \to \psi^* \). Thus, if at any time \( t \) we change \( \psi \) to \( \psi^* \), the system returns to its initial state at time \( 2t \). We shall check this point further below. It is important to understand this effective “irreversible” behavior since it will contribute to the true irreversible one, that ensues in real situations at finite temperature due to the fact that Bose-Einstein condensates are surrounded by thermal clouds that bear true viscosity. We mention here that, for instance, Kobayashi and Tsubota\(^{22}\) and Reeves et al\(^{26}\) do take into account true dissipative mechanisms into GP with a damping coefficient into the equation, and also that Jackson et al\(^{25}\) and Allen et al\(^{28}\) consider dissipative effects by coupling GP to a thermal component using the ZNG proposal\(^{24}\).

In the following section we shall analyze several macroscopic excitations that are imposed on the ground state. We will see that, since by themselves most are unstable\(^{36}\), they decay typically to two different stationary states (in the sense described above), either an agitated state or one with stable orbiting vortices. In the process, there is a clear “emission” of sound waves or phonons, and in intermediate times, there appears to be a quite complex flow that perhaps is related to some kind of turbulence. Although not necessarily an unambiguous check, we present,
together with images of the evolution of the velocity field of the fluid, incompressible kinetic energy spectrum $K_i$ as a function of wavenumber $k$, in order to compare with Kolmogorov law $k^{-5/3}$. Such a spectrum is calculated as

$$K_i = \int \frac{1}{2} \rho v^2 d^3r = \int \epsilon_i(k) dk.$$  

(10)

Certainly, as mentioned, this comparison may not be a definitive test of turbulence, yet, as we shall see, it does serve as a direct characterization of the evolution of the fluid. Details of the calculations are given below.

3 Macroscopic excitations and stationary states

In this section we show a survey of the time evolution of several initial states, that we describe in detail below. The different initial states are prepared, first, by letting the system achieve its ground state $\phi_0(\mathbf{r})$ (for an isotropic harmonic potential of frequency $\omega$) and then by imposing on it different types of vortex perturbations. In such a case, the initial state is

$$\psi(\mathbf{r}, 0^+) = \phi_0(\mathbf{r}) e^{iQ\theta(\mathbf{r})}.$$  

(11)

In the examples described below, we describe in detail the functions $\theta(\mathbf{r})$, and give the corresponding topological charges $Q$.

Our calculations are performed using parallel computing with Graphic Processors Units (GPU), which allows us to develop large and fast calculations; details of the numerical methods and programming will be reported elsewhere. While we monitor wave function normalization, energy and angular momentum conservation, in order to partially ensure numerical convergence of our calculations, we have also used a further dynamical criterion since we are interested in time evolution behavior rather than in static properties. As mentioned above, GP is time-reversal invariant, however, its numerical implementation, due to rounding errors, eventually looses its reversibility. Thus, we recognize two time regimes, one in which time-reversal invariance is maintained and another that goes beyond this time interval. For this, we define a dephasing time $\tau_d$ such that, if we let the system evolve from a given time $t_0$ up to a time $t_0 + \tau_d$, at which we change $\psi$ to $\psi^*$, then, at the time $t_0 + 2\tau_d$, the system returns to the state the system had at $t_0$. If we let the system run for time intervals longer than $\tau_d$, due to rounding errors, the system does not return to the state at $t_0$, if one makes the mentioned transformation. As we specify below, the dephasing time may depend on the initial state at $t_0$. Our calculations are performed in a grid of size $256^3$, in double-precision and with time steps of $\Delta t = 0.0005$, in dimensionless units. Our calculations correspond fairly well to a gas of $N = 1.4 \times 10^5$, atoms of $^{87}$Rb in an isotropic trap of frequency $\omega = 2 \pi (100)$ Hz with a scattering length $a = 100$ Å, yielding a dimensionless equilibrium chemical potential of $\mu = 19.63$ and a unit of time corresponding to $0.00159$ seconds.

Using the above criterion, we note that for time intervals of the order of $\tau_d$ the dynamics is “trustable”, and thus, the evolution to the obtained state can be considered as a true consequence of the initial state. Evolutions of time intervals longer than $\tau_d$ also yield important indications of the overall dynamics, specially
regarding the existence of stationary states. To clarify this latter statement, we point out that GP being a non-linear equation has what one may call "basins of attraction" of stationary or stable states, that may be reached either by slightly changing initial conditions in an otherwise time-reversal invariant evolution, or by letting rounding errors to effectively lead the system to such states. The relevance of this behavior is that if the system enters one of those stable states, it does not leave them, neither for time intervals smaller nor for longer intervals than $\tau_d$. In this sense, one can consider those as true stationary states. For the cases that we have analyzed, we have found two different types of stationary states, either an array of orbiting vortices or an agitated seemingly chaotic state, and we highlight here, both with well defined energy spectra. We shall specify this in more detail with the examples below.

3.1 Crossing of two orthogonal vortices.

In this case, the initial state is given by two line vortices at orthogonal directions, see Fig. 1, one with $Q = +1$ parallel to the $x-$axis at $y = -2.0$ and $z = 0$, the other with $Q = -1$ parallel to the $z-$axis at $x = 0$ and $y = 2.0$,

$$\theta(r) = \arctan\left(\frac{y + 2.0}{z}\right) - \arctan\left(\frac{x}{y - 2.0}\right)$$

As one may expect, see the set of snapshots for the magnitude of the velocity field in Fig. 1, the vortices join and reconnect few times, in our case we have observed three of these reconnections, and this occur within a time interval of 50 units of time. Then, the vortices stop crossing and become almost but not quite parallel and keep orbiting around each other, in a apparent stationary state. This is the most stable case we present since the dephasing time is very large, $\tau_d > 200$. We observe that for times longer than 100 units the state appears as truly stationary, that is, the system remains all the time in the mentioned configuration. On top of the orbiting vortices, one observes a mildly agitated fluid with acoustic waves.

![Fig. 1](Color online) Snapshots of the magnitude of the velocity field for the collision of two orthogonal vortices.

Fig. 2 shows the time evolution of the incompressible kinetic energy spectrum of the excitation that initially is two orthogonal vortices. The spectrum does not show complicated features in its evolution, varying very little in time, showing that although the initial state is somewhat unstable, it does not differ much from the stationary one. We show the evolution up to 200 units of time, which is within...
the dephasing time $\tau_d > 200$ in this case. One can also see that although appears to be parallel to the $k^{-5/3}$ line in a segment of the spectrum, it clearly deviates for “long” values of $k$. In the figure, we also show the location of the Thomas-Fermi radius $R_{TF} = (2\mu / m\omega^2)^{1/2} \approx 6.7$, which is comparable with the actual size of the cloud, and the healing length $\xi$ which is of the order of the vortices core. Clearly, the behavior of the kinetic energy changes at these two length scales. As we will see below, the energy spectrum can be used as a signature of the stationary state.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(Color online) Time evolution of the incompressible kinetic energy spectrum of two orthogonal vortices. For reference, we show straight lines at slopes $k^{-5/3}$ and $k^2$, as well as the Thomas-Fermi radius $R_{TF}$ and the healing length $\xi$, the former indicating the size the of cloud, the other the size of the vortices cores.}
\end{figure}

3.2 Collision of two anti-vortices

Fig. 3 shows the evolution of the collision of two parallel vortices, along the $z-$axis, one with topological charge $Q = 1$, at $y = 2.0$, and the other with $Q = -1$ at $y = -2.0$, namely,

$$\theta(r) = \arctan \left( \frac{x}{y - 2.0} \right) - \arctan \left( \frac{x}{y + 2.0} \right)$$ (13)

The behavior of this case is quite the opposite of the two orthogonal vortices. Referring to Fig. 3 first, the vortices approach each other and reconnect ($t = 10.4$) forming two vortices at directions orthogonal to the original ones ($t = 12.0$), but because they have opposite circulations, eventually join at their extrema forming a twisted vortex ring (19.2), that further folds into itself. Then, it appears to break into smaller rings that eventually become what appears to be a very agitated state.
with excitations and phonons at all length scales and in all directions. The behavior is so complex that its dephasing time is short, \( \tau_d < 50 \). Therefore, while we cannot definitely conclude about the full evolution of this state, we can certainly say that the stationary state reached by rounding errors, as mentioned above, is a stable state; that is, once the system enters such a state, it effectively remains there for any interval of time. As we now discuss, this stationary state is not only different in its observed macroscopic features but it also has a very different and distinctive energy spectrum.

Fig. 3 (Color online) Snapshots of the magnitude of the velocity field for the collision of two anti-vortices.

Fig. 4 shows the time evolution of the energy spectrum of the collision of two anti-vortices. The figure shows that the system is evolving in towards a stationary state. Two features we point out, the first one, without claiming it as evidence of turbulence, we observe an intermediate transient state, from \( t \approx 20 \) to \( t \approx 50 \), in which there is cascade of energy with a slope of \(-5/3\). Then the state evolves towards a very characteristic state with oscillating features in the spectrum. While it is difficult to identify the origin of those features, we find that their wavelengths correspond to the typical sizes of the excitations that one observes in the stationary state at \( t = 200 \).

3.3 An off-center vortex ring

This case corresponds to the evolution of a single vortex ring, shown in Fig. 5, initially placed at \((x, y, z) = (1.8, 0.51, -1.57)\) and \( Q = -1 \), with

\[
\theta(r) = \arctan \left( \frac{z - z_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 - 1}} \right)
\]

We find that, due to its circulation, the ring starts to move to one of the poles of the condensate, see Fig. 5, expanding at the surface of the cloud \((t = 10.0)\), returning to the inside but then breaking into an internal excitation and what appears to be a vortex at the edge of the condensate \((t = 20.0 \text{ and } t = 30.0)\). If the ring center had been along any axis (not shown here), then the vortex keeps going to the surface, around the cloud, and then returning to its initial place, in a periodic fashion. Since the dephasing is \( \tau_d \approx 60 \), we cannot conclude about the precise evolution, yet, it appears to enter an effective stationary state with large excitations and the remnant of a small vortex at the edge, as seen in the picture at \( t = 200 \).
Fig. 4 (Color online) Time evolution of the incompressible kinetic energy spectrum of two anti-vortices. See Fig. 2 for explanation of the straight lines.

Fig. 5 (Color online) Snapshots of the magnitude of the velocity field for the evolution of a ring vortex.

Fig. 6 shows the time evolution of the incompressible kinetic energy spectrum of the vortex ring excitation. We show two lines as guide to the eye, one with slope $k^2$ and the other at $k^{-5/3}$. The figure shows clearly that the system is evolving in time towards a stationary state but with very different features at the start and at later stages. While now quite the same, the stationary state ($t > 150$), the spectrum is similar to the one reached by the case of two colliding anti-vortices, see Fig. 4.

3.4 Vortex of charge $Q = 2$

We now analyze the case in which the initial excitation is one that corresponds a single line vortex with topological charge $Q = 2$, parallel to $z$, at $x = 2.0$ and $y = 0$, see Fig. 7. As it is known, vortices of charge $Q > 1$ are unstable and decay to two vortices of the same charge $Q = 1$. This situation is exemplified in the snapshots of times $t = 5.0, 25.0$ and $40.0$ of Fig. 7. It is very interesting to see that right after its creation, the vortex separates into two braid-entangled vortices, then these unwind until two stable parallel vortices are formed. The dephasing time is $\tau_d \approx 60$, and
Fig. 6 (Color online) Time evolution of the kinetic incompressible energy spectrum of an initial vortex ring. See Fig. 2 for explanation of the straight lines.

Within this time the system reaches a stationary state ($t = 200$) of two parallel vortices orbiting around the cloud. This late state is similar to the stationary state of the two orthogonal ones of Fig. 1, also with a mildly agitated surrounding fluid.

Fig. 7 (Color online) Snapshots of the magnitude of the velocity field for the evolution of an initial $Q = 2$ perturbation.

Fig. 8 shows the incompressible kinetic energy spectrum of an initial perturbation corresponding to a vortex with $Q = 2$ charge. One can see several stages, first, an initial featureless spectrum that about $t \sim 100$, turns into one with a complicated change of curvature, similar to Fig. 4, and finally approaches a typical one of orbiting vortices, as in the case of the stationary state of two parallel vortices, as in Fig. 1.
3.5 A tangle of 4 vortices

Here we study a “tangle” of 4 vortices, two anti vortices parallel to the $z$–axis at random positions $(x_1, y_1) = (1.20, -2.99)$ and $(x_2, y_2) = (-0.64, 0.05)$, with $Q_1 = +1$ and $Q_2 = -1$; and two cross vortices, one parallel to the $x$–axis at $(y_3, z_3) = (-1.86, 3.60)$ with $Q_3 = -1$, and the other parallel to the $y$–axis at $(x_4, z_4) = (2.57, 1.05)$ with $Q_4 = +1$, see Fig. 9. This would the simplest tangle of vortices that may lead to turbulence, as originally suggested by Tsubota et al.\textsuperscript{5} and also already treated by White et al.\textsuperscript{7}. The evolution of this case is very harsh, similarly to the case of the colliding anti-vortices; here we see that by $t = 10$ there have been several reconnections and there remain two vortices only. Then the system enters a very agitated flow, perhaps turbulent for times $t \approx 30$ to $t \approx 50$. Unfortunately, the dephasing time is also $\tau_d \approx 50$, and therefore the true evolution is lost for longer times. Nevertheless, after this agitated state, one of the vortices disappears and there remains only one with a very agitated background flow.
The evolution of the incompressible kinetic energy of the 4-vortex tangle, Fig. 10, is a complicated as the observed evolution of the velocity field of Fig. 9. The spectrum appears, for intermediate times, first close to perhaps to a turbulent state, up to times $t \sim 100$, then it moves to a stationary state similar to the case of the colliding anti-vortices, with characteristic features of excitations of very different sizes.

![Fig. 10](Color online) Time evolution of the incompressible kinetic energy spectrum of a tangle of 4 line vortices.

4 Final Remarks

In this article we have analyzed the evolution of several representative vortex states in an otherwise equilibrium Bose-Einstein condensate, using Gross-Pitaevskii equation as a model for a gas at very low temperatures. Our goal is to add to the understanding of evolution of complicated states in the search for different way to approach a quantum turbulent state. Since the dynamics observed shows different evolutions in wide time scales, and these types of studies can only be realized numerically, one must elucidate the role of rounding errors in determining the fate of the evolution. In order to quantify the deviations of true from effective, yet informative, evolution, we defined a dephasing time $\tau_d$, as that up to which the time reversal symmetry of GP is preserved. With this criterion we were able to identify transient states, that appear to have characteristics proper of a turbulent state, in the sense that their energy spectra gets close to the Kolmogorov 5/3 law, and also, a decaying to stable stationary states. It is interesting to note that the stationary states may be loosely classified as either being arrays of orbiting line vortices with topological charge $Q = 1$, or agitated states with excitation with very different sizes. Their difference is best identified in the features of their energy spectra,
see for instance Figs. 2 and 4. It is also of relevance to mention that while all the perturbations studied are always accompanied by emission of acoustic waves or phonons, we have not yet found a simple way to quantify this process in a given evolution of an arbitrary initial state. This remains a topic to be further analyzed.

To conclude, we want to emphasize that our conclusions have been possible in great part due not only to the possibility of performing long time calculations in relative brief times (for instance, 200 time units or $4 \times 10^5$ iterations takes about 18 hours of GPU time in a NVIDIA Tesla C2075), but also due to the visualizations capabilities that GPU processors allow. In particular, the determination of dephasing times can be readily done by simply observing if the system returns to its initial state. The details of the numerical and visualizations methods will be reported elsewhere.

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References

1. A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR.* **31**, 538, (1941); *Proc. R. Soc. London,* Ser. A 434, **15**, (1991).
2. R.P. Feynman, *Progress in Low Temperature Physics,* Vol. I, 17, (1955).
3. N.G. Parker and C.S. Adams, *Phys. Rev. Lett.* **95**, 145301 (2005).
4. M. Kobayashi and M. Tsubota, *J. Low Temp. Phys.*, **145**, 209, (2006).
5. M. Kobayashi and M. Tsubota, *Phys. Rev. A,* **76**, 045603, (2007).
6. B. Villaseñor, R. Zamora-Zamora, D. Bernal and V. Romero-Rochín, *Phys. Rev. A,* **89**, 033611, (2014).
7. A.C. White, N.P. Proukakis, A.J. Youd, D.H. Wacks, A.W. Baggaley and C.F. Barenghi, *Journal of Physics: Conference Series,* **318**, 062003, (2008).
8. A.W. Baggaley, J. Laurie and C.F. Barenghi, *Phys. Rev. Lett.*, **109**, 205304 (2012).
9. E. A. Henn, J. A. Seman, G. Roati, K. M. F. Magalhaes and V. S. Bagnato. *Phys. Rev. Lett.* **103**, 045301 (2009).
10. E. A. Henn, J. A. Seman, G. Roati, K. M. F. Magalhaes and V. S. Bagnato. *J. Low Temp. Phys.* **158**, 435 (2010).
11. T.W. Neely, A.S. Bradley, E.C. Samson, S.J. Rooney, E.M. Wright, K.J.H. Law, R. Carretero-Gonzalez, P.G. Kevrekidis, M.J. Davis, and B.P. Anderson, *Phys. Rev. Lett.* **111**, 235301 (2013).
12. M. Tsubota and M. Kobayashi, *J Low Temp. Phys-** **150**, 402 (2008).
13. J. A. Seman, E. A. Henn, R. F. Shiozaki, G. Roati, F. J. Poveda-Cuevas, K. M. F. Magalhaes, V. I. Yakalov, M. Tsubota, M. Kobayashi, K. Kasamatsu and V. S. Bagnato, *Laser Phys. Lett.* **8**, 691 (2011).
14. E.P. Gross, *Il Nuovo Cimento,* **20**, 454 (1961).
15. L.P. Pitaevskii, *J. Exptl. Theoret. Physic.* **40**, 646, (1961).
16. J.E. Williams and M.J. Holland, *Nature* **401**, 568 (1999).
17. M.R. Matthews, B.P. Anderson, P.C. Haljan, D.S. Hall, C.E. Wieman and E.A. Cornell, *Phys. Rev. Lett.* **83**, 2498 (1999).
18. A.E. Leanhardt, A. Görlitz, A.P. Chikkatur, D. Kielpinski, Y. Shin, D.E. Pritchard and W. Ketterle, *Phys. Rev. Lett.* **89**, 190403-1 (2002).
19. H. Shibayama, Y. Yasaku and T. Kuwamoto, *J. Phys. B: At. Mol. Opt. Phys.* **44**, 075302 (2011).
20. N.N. Bogoliubov, *J. Phys. (USSR)* **11**, 23 (1947).
21. D. Promont, S. Nazarenko and Miguel Onorato, *Phys. Rev. A,* **80**, 051603 (2009).
22. M. Kobayashi and M. Tsubota, *Phys. Rev. Lett.,* **94**, 065302, (2005).
23. M. Tsubota, S. Ogawa and Y. Hattori, *J. Low Temp. Phys.,** **121**, 435 (2000).
24. E. Zaremba, T. Nikuni, and A. Griffin, *J. Low Temp. Phys.* **116**, 277 (1999).
25. B. Jackson, N.P. Proukakis, C.F. Barenghi and E. Zaremba, *Phys. Rev. A* **79**, 053615 (2009).
26. M.T. Reeves, B.P. Anderson and S. Bradley, *Phys. Rev. A,* **86**, 053621 (2012).
27. A.J. Allen, E. Zaremba, C.F. Barenghi and N.P. Proukakis, *Phys. Rev. A,* **87**, 013630 (2013).
28. A.J. Allen, S. Zuccher, M. Caliari, N.P. Proukakis, N.G. Parker, and C.F. Barenghi, *Phys. Rev. A,* **90**, 013601 (2014).
29. C. Nore, M. Abid and M.E. Brachet, *Phys. Fluids* **9**, 2644, (1997).
30. A.L. Fetter, *J. Low Temp. Phys.*, **161**, 445 (2010).
31. R. Zeng and Y. Zhang, *Comp. Phys. Comm.* **180**, 854-860 (2009).
32. R. Zamora-Zamora, M. Lozada-Hidalgo, S.F. Caballero-Benitez, and V. Romero-Rochin, *Phys. Rev. A* **86**, 053624 (2012).
33. Y. Castin and R. Dum, *Eur. Phys. J. D.*, **7**, 399, (1999).
34. S. Stringari, *Phys. Rev. Lett.* **77**, 2360, (1996).
35. D. Guéry-Odelin and S. Stringari, *Phys. Rev. Lett.* **83**, 4452, (1999).
36. A.L. Fetter and A.A. Svidzinsky, C.F. Barenghi, R.J. Donnelly and W.F. Vinen (Eds.): LNP **571**, 320 (2001).