On the equations $x^2 - 2py^2 = -1, \pm 2$

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December 7, 2018

Abstract

Let $E \in \{-1, \pm 2\}$. We improve on the upper and lower densities of primes $p$ such that the equation $x^2 - 2py^2 = E$ is solvable for $x, y \in \mathbb{Z}$. We prove that the natural density of primes $p$ such that the narrow class group of the real quadratic number field $\mathbb{Q}(\sqrt{2p})$ has an element of order 16 is equal to $\frac{1}{64}$. We give an application of our results to the distribution of Hasse’s unit index for the CM-fields $\mathbb{Q}(\sqrt{2p}, \sqrt{-1})$. Our results are consequences of a twisted joint distribution result for the 16-ranks of class groups of $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{-2p})$ as $p$ varies.

2010 Mathematics Subject Classification. 11R29, 11R45, 11N45

1 Introduction

Let $p$ denote an odd prime number. Gauss’s genus theory implies that the 2-torsion subgroup of the narrow class group $\text{Cl}^+(2p)$ of the real quadratic number field $\mathbb{Q}(\sqrt{2p})$ is isomorphic to the group of two elements, and that it is generated by the classes of the ideals $t = (2, \sqrt{2p})$ and $p = (p, \sqrt{2p})$ in $\mathbb{Z}[\sqrt{2p}]$ [12] Lemma 9.8(a), p.78. Hence exactly one of the ideals $t, p,$ and $tp = (\sqrt{2p})$ is principal in the narrow sense, and the remaining two are in the class of order 2 in $\text{Cl}^+(2p)$. Let $E_p$ denote the unique integer in the set $\{-1, \pm 2\}$ such that the equation

$$x^2 - 2py^2 = E_p$$

(1.1)

has a solution with $x, y \in \mathbb{Z}$; in other words, if we denote the class of an ideal $n \subset \mathbb{Z}[\sqrt{2p}]$ in $\text{Cl}^+(2p)$ by $[n]$, then

$$E_p = \begin{cases} -1 & \text{if } [tp] = 1, \\ 2 & \text{if } [t] = 1, \\ -2 & \text{if } [p] = 1. \end{cases}$$

Stevenhagen conjectures that each of the three cases above occurs equally often [13] p.127. More precisely, let $E \in \{-1, \pm 2\}$, let $X \geq 3$ be a real number, and let $\delta(X)$ denote the proportion

$$\delta(X; E) = \frac{\{|p \leq X : E_p = E\}|}{|\{p \leq X\}|}.$$
Theorem 2. We have one prime number. Recall that the $K$ is defined to be the index $Q$ then $K = Q$ of a quadratic field and denote the subgroup of $U$ conditional on Conjecture $C_n$ with $n = 8$ from $[2]$; thanks to the recent work of Koymans $[4]$, one can remove this conditionality on Conjecture $C_n$. In this paper, we prove the same upper bounds for $E = \pm 2$ as well as improve the lower bounds in all cases. Our results are unconditional.

**Theorem 1.** Let $E \in \{-1, \pm 2\}$, let $X \geq 3$ be a real number, and define $\delta(X; E)$ as in $(1.2)$. Then

$$\frac{21}{64} \leq \liminf_{X \to \infty} \delta(X; E) \leq \limsup_{X \to \infty} \delta(X; E) \leq \frac{11}{32}.$$  

Furthermore, we will see that the “remaining” primes, which form a set of natural density $1 - 3 \cdot \frac{21}{64} = \frac{1}{64}$, are exactly those for which $\Cl^+(2p)$ has an element of order 16. This yields the first density result on the 16-rank in any family of real quadratic fields parametrized by one prime number. Recall that the 2-part of $\Cl^+(2p)$ is cyclic and set $h^+(−2p) = |\Cl^+(2p)|$.

**Theorem 2.** We have

$$\lim_{X \to \infty} \left| \frac{\{p \leq X : h^+(2p) \equiv 0 \mod 16\}}{|\{p \leq X\}|} \right| = \frac{1}{64}.$$  

Finally, we also have an application to the distribution of Hasse’s unit index for the biquadratic fields $\Q(\sqrt{2p}, -1)$. Let $U_F$ denote the unit group of the ring of integers of an algebraic number field $F$. Let $K$ be a CM-field with maximal real subfield $K^+$ and let $W_K$ denote the subgroup of $U_K$ consisting of units of finite order. Then Hasse’s unit index $Q(K)$ of $K$ is defined to be the index $[U_K : U_K \cdot W_K]$; it is always equal to 1 or 2. If $K = \Q(\sqrt{2p}, -1)$, then $Q(K) = 1$, so the simplest case from the standpoint of arithmetic statistics is when $K = \Q(\sqrt{2p}, -1)$. In that case, $Q(K) = 1$ if and only if a fundamental unit $\epsilon_{2p}$ of $\Z[\sqrt{2p}]$ and $\sqrt{-1}$ generate the full unit group $U_K$. This occurs if and only if the ideal $\frak{t}$ from above is not principal in the ordinary sense $[7]$ Theorem 1.ii.2, i.e., if and only if $E_p \neq \pm 2$. Hence we can deduce the following corollary of Theorem 1.

**Corollary 3.** Let $K = \Q(\sqrt{2p}, -1)$, let $U_K$ denote the group of its integral units, and let $\epsilon_{2p}$ denote a fundamental unit of $\Z[\sqrt{2p}]$. For a real number $X \geq 3$, let

$$\delta_H(X) = \left| \frac{\{p \leq X : U_K \text{ is generated by } \epsilon_{2p} \text{ and } -1\}}{|\{p \leq X\}|} \right|.$$  

Then

$$\frac{21}{64} \leq \liminf_{X \to \infty} \delta_H(X) \leq \limsup_{X \to \infty} \delta_H(X) \leq \frac{11}{32}.$$
2 Main strategy

Our results are possible in large part thanks to the work of Kaplan and Williams [3] and Leonard and Williams [8]. Kaplan and Williams relate the existence of an element of order 16 in the narrow class group of the real quadratic field $\mathbb{Q}(\sqrt{2p})$ to the existence of elements of order 16 in the class groups of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{-2p})$. A similar type of “reflection principle” for the 16-rank was also later proved by Stevenhagen [13], but the results from [3] appear to be more suitable for our purposes and our analytic techniques.

Let $h(-p)$ and $h(-2p)$ denote the class numbers of $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{-2p})$, respectively. Note that Gauss’s genus theory implies that the 2-parts of the narrow class groups of $\mathbb{Q}(\sqrt{-p})$, $\mathbb{Q}(\sqrt{-2p})$, and $\mathbb{Q}(\sqrt{2p})$ are cyclic and hence determined by the highest power of 2 dividing $h(-p)$, $h(-2p)$, and $h^+(2p)$, respectively. The following facts can be found in or readily deduced from [3][13]. We have

$$h^+(2p) \equiv 0 \mod 8 \iff h(-p) \equiv h(-2p) \equiv 0 \mod 8$$
$$\iff p \text{ splits completely in } \mathbb{Q}(\zeta_{16}, \sqrt{2}).$$

Here $\zeta_n$ denotes a primitive $n$th root of unity. Let $F$ be any one of the three quadratic fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{2})$. It is not hard to check that in each case

$$\text{Gal}(\mathbb{Q}(\zeta_{16}, \sqrt{2})/F) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

As this group is abelian, class field theory implies that if a prime $p$ that splits in $F/\mathbb{Q}$, then the splitting type of $p$ in $\mathbb{Q}(\zeta_{16}, \sqrt{2})/\mathbb{Q}$ can be detected via congruence conditions on a prime $\pi$ in $F$ lying above $p$. Concretely, $p$ splits completely in $\mathbb{Q}(\zeta_{16}, \sqrt{2})/\mathbb{Q}$ if and only if there exist integers $a$, $b$, $c$, $d$, $u$, and $v$ such that

$$p = a^2 + b^2 = c^2 + 2d^2 = u^2 - 2v^2$$

and

$$a \equiv 1 \mod 8, \ b \equiv 0 \mod 8, \ c \equiv 1 \mod 8, \ d \equiv 0 \mod 4, \ u \equiv 1 \mod 8, \ v \equiv 0 \mod 4. \quad (2.2)$$

For such a prime $p$, define

$$\alpha_p = (-1)^{h(-2p)/8} \quad \text{and} \quad \beta_p = (-1)^{(a-1+b+2d+h(-p)+h(-2p))/8}. \quad (2.3)$$

By studying the number of quadratic residues modulo $p$ that are less than $p/8$, Kaplan and Williams [3] Theorem, p.26] proved that

$$\alpha_p = \beta_p = 1 \implies h^+(2p) \equiv 0 \mod 16, \quad (2.4)$$
$$\alpha_p = 1, \beta_p = -1 \implies h^+(2p) \equiv 8 \mod 16, \ E_p = -2, \quad (2.5)$$
$$\alpha_p = -1, \beta_p = 1 \implies h^+(2p) \equiv 8 \mod 16, \ E_p = +2, \quad (2.6)$$
$$\alpha_p = \beta_p = -1 \implies h^+(2p) \equiv 8 \mod 16, \ E_p = -1. \quad (2.7)$$

We will prove that each of the four possibilities $\{2.4\} - \{2.7\}$ occurs equally often, and this will imply both Theorem [1] and Theorem [2]. Indeed, the natural density of primes $p$ that split completely in $\mathbb{Q}(\zeta_{16}, \sqrt{2})/\mathbb{Q}$ is equal to $\frac{1}{16}$, by the Chebotarev Density Theorem; hence we
will prove that the natural density of primes satisfying each of the four possibilities above is equal to \( \frac{1}{16} \). This immediately implies Theorem 2. We note that the classical bounds (1.3) derive from primes \( p \) that do not split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) / \mathbb{Q} \) (this is a set of primes of natural density \( \frac{1}{15} \)), and the improvements in Theorem 1 say for \( E = -2 \), come from adding to the lower bound in (1.3) the fraction of primes satisfying (2.5) and subtracting from the upper bound in (1.3) the fraction of primes satisfying (2.6) or (2.7).

To prove that each of the four possibilities (2.4)-(2.7) occurs equally often, we restrict to the set of primes \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) / \mathbb{Q} \) and consider indicator functions of the type

\[
\frac{1}{4} (1 + \alpha_p + \beta_p + \alpha_p \beta_p) = \begin{cases} 
1 & \text{if } \alpha_p = \beta_p = 1, \\
0 & \text{otherwise},
\end{cases}
\]

(2.8)

It then suffices to prove that each of the three sums

\[
\sum_{p \leq X}^* \alpha_p, \quad \sum_{p \leq X}^* \beta_p, \quad \sum_{p \leq X}^* \alpha_p \beta_p
\]

is \( o(X / \log X) \) as \( X \to \infty \); here * denotes the restriction to primes that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) / \mathbb{Q} \).

The sum \( \sum_{p \leq X}^* \alpha_p \) encodes the behavior of the 16-rank in the family \( \{ \mathbb{Q}(\sqrt{-2p}) : p \equiv 1 \text{ mod } 4 \} \); it was the subject of a paper of Koymans and the author [5], where we proved that

\[
\sum_{p \leq X} \alpha_p \ll X^{1/3200},
\]

Here ' restricts the sum to primes \( p \equiv 1 \text{ mod } 4 \) such that \( h(-2p) \equiv 0 \text{ mod } 8 \); these are exactly the primes that split completely in \( \mathbb{Q}(\zeta_8, \sqrt{2}) \). We will show that the same result holds when we further restrict the sum to \( p \) such that \( h^+(2p) \equiv 0 \text{ mod } 8 \), i.e., \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \).

**Proposition 2.1.** We have

\[
\sum_{p \leq X}^* \alpha_p \ll X^{1 - \frac{1}{3200}},
\]

where * restricts the sum to primes \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \), and where \( \alpha_p \) is defined in (2.3).

The sum \( \sum_{p \leq X}^* \alpha_p \beta_p \) concerns a twisted version of the 16-rank in the family \( \{ \mathbb{Q}(\sqrt{-p}) \} \); the twist here is the factor \( (-1)^{(a-1+b+2d)/8} \). This sum too was the subject of another paper of Koymans and the author [6], where we proved that

\[
\sum_{p \leq X}^* \alpha_p \beta_p \ll X^{1 - \frac{1}{800}}.
\]

Here \( \delta > 0 \) is a conjectural constant that appears in Conjecture \( C_n \) for \( n = 8 \) in [2]; see [6] Theorem 3, p.102 and its proof in [6] Section 7. Koymans [4] recently gave an unconditional proof of [3] Theorem 2, p.102, i.e., a similar power-saving estimate for the sum \( \sum_{p \leq X}^* (-1)^{h(-p)/8} \). We will show that the proof in [4] can be modified slightly to also give an unconditional estimate for \( \sum_{p \leq X}^* \alpha_p \beta_p \).
Proposition 2.2. We have
\[ \sum_{p \leq X} \alpha_p \beta_p \ll X^{1 - \frac{1}{25000}}, \]

where * restricts the sum to primes \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \), and where \( \alpha_p \) and \( \beta_p \) are defined in (2.3).

It remains to show a similar estimate for the sum \( \sum^* \beta_p \), which concerns a twisted version of the joint distribution of the 16-ranks in the families \( \{ \mathbb{Q}(\sqrt{-p}) \} \) and \( \{ \mathbb{Q}(\sqrt{-2p}) \} \). This forms the main subject of the present paper. We will prove

Proposition 2.3. We have
\[ \sum_{p \leq X} \beta_p \ll X^{1 - \frac{1}{200}}, \]

where * restricts the sum to primes \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \), and where \( \beta_p \) is defined in (2.3).

Propositions 2.1, 2.2, and 2.3, in conjunction with (2.8) and similar identities, imply Theorems 1 and 2. We start by laying the groundwork for the proof of Proposition 2.3 with the appropriate set-up. Propositions 2.1 and 2.2 will follow readily from [5] and [4], respectively. Ultimately we will see that Proposition 2.3 is connected to the equidistribution results from [9], although this connection is far less obvious – in fact, [9] features a result for a family parametrized by primes \( p \equiv -1 \) mod 4, while the sum in Proposition 2.3 is supported on primes \( p \equiv 1 \) mod 4.

3  Algebraic Criteria

Leonard and Williams [8] use the theory of binary quadratic forms, along with composition laws of Gauss and Dirichlet, to derive formulas for \( (-1)^{h(-p)/8} \) and \( (-1)^{h(-2p)/8} \) for \( p \equiv 1 \) mod 8 in terms of prime ideals lying above \( p \) in \( \mathbb{Z}[\sqrt{2}] \). The proofs of their results rely in part on very clever manipulations of Legendre and Jacobi symbols, and they produce formulas which vaguely resemble two different types of spin symbols, one appearing in the work of Friedlander and Iwaniec [11 (20.1), p.1021] and the other being the main object of study by Friedlander, Iwaniec, Mazur, and Rubin in [2]. In the case of a similar criterion for \( (-1)^{h(-2p)/8} \) for \( p \equiv -1 \) mod 8 [8 Theorem 3, p.205], we translated their proof to the language of rings and ideals, and this translation revealed additional structure that allowed us to embed a Jacobi symbol appearing in this criterion into a sequence conducive to certain sieving methods; see [9]. As a result, it may be interesting to translate also the proofs of the following criteria to the language of rings and ideals, but we avoid doing so since the results of [9] already suffice for our applications.

3.1 Preliminaries

Given an integer \( n \geq 1 \), let \( \zeta_n \) denote a primitive \( n \)-th root of unity. Let \( F \) be a finite Galois extension of \( \mathbb{Q} \) containing \( \zeta_n \), and let \( \mathcal{O}_F \) denote the ring of integers of \( F \). Let \( \mathfrak{N}_{F/\mathbb{Q}} \) denote the norm map from \( F \) to \( \mathbb{Q} \). Given \( \alpha \in \mathcal{O}_F \) and a prime ideal \( \mathfrak{p} \) in \( \mathcal{O}_F \) coprime to \( n \), the \( n \)-th
power residue symbol \((\alpha/p)_{F,n}\) is defined to be the unique element of \(\{0, 1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}\) such that
\[
\left(\frac{\alpha}{p}\right)_{F,n} \equiv \alpha^{2^f} \mod p.
\]

It is evident from this definition that \((\alpha/p)_{F,n}\) depends only on the congruence class of \(\alpha\) modulo \(p\); that \((\alpha\alpha_2/p)_{F,n} = (\alpha/p)_{F,n}(\alpha_2/p)_{F,n}\) for all \(\alpha, \alpha_2 \in \mathcal{O}_F\); and that \(\alpha\) is an \(n\)-th power modulo \(p\) if and only if \((\alpha/p)_{F,n} = 1\). Moreover, if \(n\) is even, \(\alpha\) is an \((n/2)\)-th power but not an \(n\)-th power modulo \(p\) if and only if \((\alpha/p)_{F,n} = -1\). For an ideal \(\mathfrak{b}\) in \(\mathcal{O}_F\) coprime to \(n\), we set \((\alpha/\mathfrak{b})_{F,n} = \prod_{p\mathfrak{b} | \mathfrak{b}} (\alpha/p)_{F,n}^{\mathfrak{p}}\), where \(\mathfrak{p}\) is the exact power of \(p\) dividing \(\mathfrak{b}\). For an odd element \(\beta \in \mathcal{O}_F\), we set \((\alpha/\beta)_{F,n} = (\alpha/\beta_{\mathcal{O}_F})_{F,n}\). If \(F = \mathbb{Q}\), \(n = 2\), and \(\beta \in \mathbb{Z}\) is positive, the symbol \((\cdot/\beta)_{\mathbb{Q},2}\) coincides with the usual Jacobi symbol, and so we suppress the subscripts \(\mathbb{Q},2\) and simply write \((\cdot/\beta)\).

### 3.2 Symbols over \(\mathbb{Z}\)

Let \(\varepsilon = 1 + \sqrt{2}\), and note that \(\varepsilon\) is a unit of infinite order in \(\mathbb{Z}[\sqrt{2}]\). For the remainder of Section 3, let \(p \equiv 1 \mod 8\) be a prime number. Since \(p\) splits in the unique factorization domain \(\mathbb{Z}[\sqrt{2}]\), there exist rational integers \(u\) and \(v\) such that
\[
p = (u + v\sqrt{2})(u - v\sqrt{2}) = u^2 - 2v^2, \quad u, v > 0. \tag{3.1}
\]

Note that \(u\) and \(v\) must be odd and even, respectively. Moreover, after multiplying \(u + v\sqrt{2}\) by \(\varepsilon^2\) if necessary, we can choose \(u\) in \((3.1)\) so that
\[
u \equiv 1 \mod 4. \tag{3.2}
\]

Let \(g\) and \(h\) be positive rational integers such that
\[
h + g\sqrt{2} = (u + v\sqrt{2}) \cdot \varepsilon,
\]

so that \(p = 2g^2 - h^2\). Now assume also that \(h(-p) \equiv h(-2p) \equiv 0 \mod 8\) so that \(p\) splits completely in \(\mathbb{Q}(\zeta_{16}, \sqrt{2})\) and \(h^+(2p) \equiv 0 \mod 8\). Then Leonard and Williams show that
\[
\left(\frac{u}{p}\right) = \left(\frac{-2}{u}\right) = \left(\frac{g}{p}\right) = \left(\frac{-1}{g}\right) = 1; \tag{3.3}
\]
in particular, \(u \equiv 1 \mod 8\) and \(g \equiv 1 \mod 4\). As \(p \equiv 1 \mod 16\), we see that \(v \equiv 0 \mod 4\).

Recall also \((2.1)\) and \((2.2)\) above.

For an integer \(n\) satisfying \((n/p) = 1\), set
\[
\left[\frac{n}{p}\right]_4 = \begin{cases} 
1 & \text{if } n \text{ is a fourth power residue modulo } p \\
-1 & \text{otherwise}.
\end{cases}
\]

Two of the main results in [8] are then as follows. Suppose \(u\) satisfies \((3.1)-(3.2)\). First, [8, Theorem 2, p.204] implies that
\[
(-1)^{h(-2p)/8} = \left[\frac{u}{p}\right]_4.
\]
Second, [8, Theorem 1, p.201] implies that
\[
(-1)^{h(-p)/8} = \left[\frac{g}{p}\right]_4 \left(\frac{2h}{g}\right).
\]
Hence,
\[ (-1)^{(h(-p)+h(-2p))/8} = \left[ \frac{u}{p} \right]_4 \left[ \frac{g}{p} \right]_4 \left( \frac{2h}{g} \right). \] (3.4)

### 3.3 From \( \mathbb{Z} \) to \( \mathbb{Z}[\zeta_8] \)

To fully exploit the multiplicative properties underlying the symbol \([\cdot]/p)_4\), we will rewrite the above criterion in a field containing \( \zeta_4 = \sqrt{-1} \), a primitive fourth root of unity. Moreover, as the criterion naturally depends on the splitting of \( p \) in \( \mathbb{Q}(\sqrt{2}) \), we will work over the field
\[ K = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) = \mathbb{Q}(\zeta_8). \]

Let \( \varepsilon = 1 + \sqrt{2} \). Note that the ring of integers of \( K \) is \( \mathbb{Z}[\zeta_8] \), that \( \mathbb{Z}[\zeta_8] \) is a principal ideal domain, and that its group of units is generated by \( \zeta_8 \) and \( \varepsilon \).

Since the prime \( p \equiv 1 \mod 8 \) splits completely in \( K \), we can choose a prime element \( \varpi \in \mathbb{Z}[\zeta_8] \) such that \( \mathfrak{p} = \mathbb{Z}[\zeta_8] \) is a principal ideal domain, and that its group of units is generated by \( \zeta_8 \) and \( \varepsilon \).

Hence, as \( \varpi = \sqrt{2} \) is a fourth power in \( \mathbb{Z}[\zeta_8] \), we have
\[ (\varpi^4)_2 = (\varpi^4)_2 = (\varpi_4)_2 \]
and
\[ (\varpi^2)_2 = (\varpi^2)_2 = (\varpi_2)_2. \]

Combining the above formulas, and noting that
\[ (\varpi)^2 (\varpi^2) = (\varpi^2) (\varpi^2) = (\varpi^2), \]
one can rewrite (3.4) as
\[ (-1)^{(h(-p)+h(-2p))/8} = \left( \frac{2\sqrt{2} \varepsilon^{-1} \tau(\varpi) \sigma(\varpi)}{\varpi} \right)_4 \left( \frac{\tau(\varpi) \sigma(\varpi)}{\varpi} \right)_2 \left( \frac{2h}{g} \right). \] (3.5)
### 3.4 The first factor

We have

$$
\left( \frac{2\sqrt{2}z^{-1}}{\varpi} \right)_{K,4} = \left( \frac{2(2 - \sqrt{2})}{\varpi} \right)_{K,4} = \left( \frac{2}{\varpi} \right)_{K,4} \left( \frac{e_2}{\varpi} \right)_{K,4},
$$

where $e_2 = 2 - \sqrt{2}$ as in [3] (5), p.24. As $\varpi$ lies above a prime $p$ that splits completely in $\mathbb{Q}(\zeta_{16}, \sqrt{2})$, we see that $2$ is a fourth power modulo $p$. Thus

$$
\left( \frac{2}{\varpi} \right)_{K,4} = \left[ \frac{2}{p} \right]_{4} = 1.
$$

Moreover, Kaplan and Williams already computed in [3] p.25] that

$$
\left( \frac{e_2}{\varpi} \right)_{K,4} = \left[ \frac{e_2}{p} \right]_{4} = (-1)^{(b+2d)/8}.
$$

Hence

$$
\left( \frac{2\sqrt{2}z^{-1}}{\varpi} \right)_{K,4} = (-1)^{(b+2d)/8}. \hspace{1cm} (3.6)
$$

### 3.5 The middle factor

We now deal with the middle factor on the right-hand-side of (3.5). As $\varpi$ is a prime of degree one in $\mathbb{Z}[\zeta_8]$, setting $\pi = \varpi \sigma(\varpi) = \mathfrak{m}_{Q(\zeta_8)}/Q(\sqrt{2})$, the inclusion of rings $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{Z}[\zeta_8]$ induces an isomorphism of fields $\mathbb{Z}[\sqrt{2}]/\pi \mathbb{Z}[\sqrt{2}] \cong \mathbb{Z}[\zeta_8]/\varpi \mathbb{Z}[\zeta_8]$. Letting $\tau$ denote the restriction of $\tau$ to $Q(\sqrt{2})$, we note that $\tau(\varpi) \sigma(\varpi) = \tau(\varpi \sigma(\varpi)) = \tau(\sigma(\varpi)) = \tau(\pi) \in \mathbb{Z}[\sqrt{2}]$, and so

$$
\left( \frac{\tau(\varpi) \sigma(\varpi)}{\varpi} \right)_{K,2} = \left( \frac{\overline{\tau(\pi)}}{\varpi} \right)_{K,2} = \left( \frac{\overline{\tau(\pi)}}{\pi} \right)_{Q(\sqrt{2}),2}
$$

Writing $\pi = u + v\sqrt{2}$ as above, we see that

$$
\left( \frac{\overline{\tau(\pi)}}{\pi} \right)_{Q(\sqrt{2}),2} = \left( \frac{\overline{\tau(\pi)} + \pi}{\pi} \right)_{Q(\sqrt{2}),2} = \left( \frac{2u}{\pi} \right)_{Q(\sqrt{2}),2}.
$$

Again, as $\pi$ is a prime of degree one and $2u \in \mathbb{Z}$, we can use the canonical isomorphism $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}[\sqrt{2}]/\pi \mathbb{Z}[\sqrt{2}]$ to write

$$
\left( \frac{2u}{\pi} \right)_{Q(\sqrt{2}),2} = \left( \frac{2u}{p} \right).
$$

By (3.5), we see that the Legendre symbol above is equal to 1, and so the middle factor in (3.5) is trivial, i.e.,

$$
\left( \frac{\tau(\varpi) \sigma(\varpi)}{\varpi} \right)_{K,2} = 1. \hspace{1cm} (3.7)
$$
3.6 The last factor: essential spin

Next, we deal with the last factor on the right-hand-side of (3.5). This factor is essential in ensuring that $\beta_p$ can be written as a genuine spin symbol. We now relate it to the spin symbol $[u + v\sqrt{2}] = \left(\frac{u}{u}\right)$ appearing in [9]. Recall that $u \equiv 1 \mod 8$, that $v \equiv 0 \mod 4$, and that $u$ and $v$ are positive. Writing $e_v$ for the highest power of 2 dividing $v$, so that $v' = v2^{-e_v}$ is odd, we have

$$\left(\frac{2h}{g}\right) = \left(\frac{2u + 4v}{u + v}\right) = \left(\frac{2v}{u + v}\right) = \left(\frac{2}{u + v}\right)^{e_v+1}\left(\frac{v'}{u + v}\right).$$

Using that $u \equiv 1 \mod 8$ and $v \equiv 0 \mod 4$, so that $\left(\frac{2}{u+v}\right) = \left(\frac{2}{1+v}\right)$ and $\left(\frac{v'}{u}\right) = \left(\frac{v'}{u}\right)$, we arrive at

$$\left(\frac{2h}{g}\right) = \left(\frac{2}{1+v}\right)^{e_v+1}\left(\frac{v'}{u}\right).$$

As

$$\left(\frac{2}{1+v}\right)^{e_v+1} = \begin{cases} 1 & \text{if } v \equiv 0 \mod 8 \\ -1 & \text{if } v \equiv 4 \mod 8, \end{cases}$$

we obtain the formula

$$\left(\frac{2h}{g}\right) = (-1)^{v/4}\left(\frac{v}{u}\right). \quad (3.8)$$

3.7 The formula for $\beta_p$

Recall from (2.3) that $\beta_p$ is defined to be

$$\beta_p = (-1)^{(a-1+b+2d+h(-p)+h(-2p))/8}.$$

Combining (3.5), (3.6), (3.7), and (3.8), we arrive at the formula

$$\beta_p = (-1)^{(a-1)/8}(-1)^{v/4}\left(\frac{v}{u}\right).$$

Using (2.1) and (2.2), namely that $p = a^2 + b^2 \equiv 1 \mod 16$ with $a - 1 \equiv b \equiv 0 \mod 8$, one can check that

$$\frac{a - 1}{8} \equiv \frac{p - 1}{16} \mod 2.$$

We thus arrive at the final form of the formula for $\beta_p$ that we will subsequently extend into an oscillating sequence to be sieved for primes:

$$\beta_p = (-1)^{(p-1)/16}(-1)^{v/4}\left(\frac{v}{u}\right). \quad (3.9)$$
3.8 A comment on the formula for $\beta_p$

Due to the presence of fourth power residue symbols in the algebraic criteria of Leonard and Williams, it seems most natural to define $\beta_p$ over $\mathbb{Q}(\zeta_8)$ as in (3.5). However, the final formula (3.9) suggests that $\beta_p$ also has a natural definition over the smaller field $\mathbb{Q}(\sqrt{2})$, which is very advantageous. Roughly speaking, if we were only able to define $\beta_p$ over $\mathbb{Q}(\zeta_8)$, our analytic arguments would require us to prove significant cancellation (namely power-saving in $X$) in sums resembling

$$\sum_{\alpha=a_0+a_1\zeta_8+a_2\zeta_8^2+a_3\zeta_8^3 \atop a_i \in \mathbb{Z}, |a_i| \leq X^{1/4}} \left( \frac{\nu}{u} \right).$$

This appears to be a very difficult problem that we see how to solve only by appealing to a standard conjecture on short character sums that is just out of reach of the deep Burgess’s inequality (see Conjecture $C_n$ for $n = 4$ from [2]). Instead, working over $\mathbb{Q}(\sqrt{2})$ leads to sums of the form

$$\sum_{u, v \in \mathbb{Z}, u \text{ odd}, >0 \atop 2|v| < u \leq X^{1/2}} \left( \frac{\nu}{u} \right),$$

which can be handled even with just the classical Pólya-Vinogradov inequality.

Moreover, we remark that although the essential spin factor in $\beta_p$ comes from the formula for $(-1)^{h(-p)+h(-2p)/8}$ (see the last factor in (3.5)), the twist by $(-1)^{(a-1+b+2d)/8}$ in the definition of $\beta_p$ (see (2.3)) is exactly what allows us to cancel the seemingly innocuous but fatal first factor appearing in (3.5). In fact, determining the symbol

$$\left( \frac{2\sqrt{2} \varepsilon^{-1}}{\omega} \right)_{K,4} = \left( \frac{2(2 - \sqrt{2})}{\omega} \right)_{K,4}$$

is tantamount to determining the splitting of $p$ in the extension $L/\mathbb{Q}$, with

$$L = \mathbb{Q} \left( \zeta_8, \frac{4}{\sqrt{2} - \sqrt{2}} \right).$$

The key observation is that the Galois group of $L/\mathbb{Q}(\sqrt{2})$ is isomorphic to the quaternion group $Q_8$, which is not abelian; hence, by class field theory, the first factor in (3.3) cannot be determined by congruence conditions on a prime lying above $p$ in $\mathbb{Z}[\sqrt{2}]$. This would then force us to work over the bigger field $\mathbb{Q}(\zeta_4)$. We challenge the reader to give an unconditional proof that there exists a $\delta > 0$ such that for all $X \geq 3$ we have

$$\sum_{p \leq X}^* (-1)^{(h(-p)+h(-2p))/8} \ll X^{1-\delta},$$

where, as before, $*$ restricts the summation to primes $p$ that split completely in $\mathbb{Q}(\zeta_{16}, \sqrt{2})$.

4 Construction of the spin sequence

Following the method in [9], we wish to construct a bounded sequence of complex numbers $\{b_n\}_n$ indexed by non-zero ideals $n$ of $\mathbb{Z}[\sqrt{2}]$ such that $b_p = \beta_p$ whenever $p$ is a prime ideal
lying above a prime $p$ that splits completely in $\mathbb{Q}(\zeta_{16}, \sqrt{2})/\mathbb{Q}$ and such that we can prove power-saving estimates for sums of the form

$$\sum_{\mathfrak{a} \in \mathcal{A}} b_{\mathfrak{a}} \quad \text{and} \quad \sum_{\mathfrak{a} \in \mathcal{A}} \sum_{\mathfrak{b} \in \mathcal{B}} v_{\mathfrak{a}}w_{\mathfrak{b}}b_{\mathfrak{a} \mathfrak{b}}.$$ 

To this end, we will now state the key result from [9] that lets us do so. Let $\varepsilon = 1 + \sqrt{2}$, as before. For an odd, totally positive (and not necessarily prime) element $\alpha = u + v\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, define

$$[\alpha] = \left(\frac{v}{u}\right).$$

Then [9] Proposition 2, p. 979 implies that

$$[\alpha] = [\varepsilon^8 \alpha]. \quad (4.1)$$

Now, since $\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain and $\varepsilon$ is a unit of norm $-1$, every non-zero ideal of $\mathbb{Z}[\sqrt{2}]$ can be generated by a totally positive element. Let $\alpha = u + v\sqrt{2}$ be a totally positive generator of a non-zero ideal $\mathfrak{n}$; all the other totally positive generators of $\mathfrak{n}$ are of the form $\varepsilon^{2k}\alpha$ for some integer $k$. Suppose now that $\mathfrak{n}$ is odd, i.e., that $\mathfrak{n}\mathfrak{m}$ is odd. Then (4.1) implies that the quantity

$$[\alpha] + [\varepsilon^2 \alpha] + [\varepsilon^4 \alpha] + [\varepsilon^6 \alpha]$$

depends only on the ideal $\mathfrak{n} = \alpha\mathbb{Z}[\sqrt{2}]$ and not on the choice of the totally positive generator $\alpha$ of $\mathfrak{n}$. Eventually we will need to be able to detect $\alpha$ with $u$ and $v$ satisfying $u - 1 \equiv 2v \equiv 0 \mod 8$ (see (2.2)) as well as detect when $v \equiv 0 \mod 8$ (to control the factor $(−1)^v/4$ in (3.9)). The idea is to first detect $\alpha$ with $\mathfrak{m}(\alpha) = u^2 - 2v^2 \equiv 1 \mod 16$ via multiplicative Dirichlet characters modulo 16. Then we will detect when $u \equiv 1 \mod 8$; this already ensures that $v \equiv 0 \mod 4$ provided that $u^2 - 2v^2 \equiv 1 \mod 16$. To detect when $u \equiv 1 \mod 8$, we should study how $(u \mod 8, v \mod 4)$ changes as we multiply $\alpha$ by successive powers of $\varepsilon^2$. We compute that

$$\varepsilon^2 \alpha = (3 + 2\sqrt{2})(u + v\sqrt{2}) = (3u + 4v) + (2u + 3v)\sqrt{2}.$$ 

The orbits of the map $(u \mod 8, v \mod 4) \mapsto (3u + 4v \mod 8, 2u + 3v \mod 4)$ for $u$ odd and $v$ even can be listed as follows (note that $u$ odd implies that $2u \equiv 2 \mod 4$):

$$
\begin{align*}
(u, 0) &\mapsto (3u, 2) \mapsto (u, 0) \\
(u, 2) &\mapsto (3u, 0) \mapsto (u, 2).
\end{align*}
$$

Hence if $u' + v'\sqrt{2} = \varepsilon^2(u + v\sqrt{2})$ with $v, v'$ even, then either $\{u \mod 8, u' \mod 8\} = \{1 \mod 8, 3 \mod 8\}$ or $\{u \mod 8, u' \mod 8\} = \{5 \mod 8, 7 \mod 8\}$.

Now, for the rest of the paper, fix a square root of $−1$ and denote it by $i$. We define, for each Dirichlet character $\chi$ modulo 8 and each odd totally positive element $u + v\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ with $v$ even,

$$[u + v\sqrt{2}]_{\chi} = i^{v/2}\chi(u) \left(\frac{v}{u}\right).$$

Next, for each pair of Dirichlet characters $\chi$ modulo 8 and $\psi$ modulo 16 and each ideal $\mathfrak{n}$ satisfying $\mathfrak{n}\mathfrak{m} \equiv 1 \mod 8$, we define

$$b_{\mathfrak{n}}(\chi, \psi) = \frac{1}{2} \psi(\mathfrak{m}(\mathfrak{n}))i^{(\mathfrak{m}(\mathfrak{n})-1)/8} \left([\alpha]_{\chi} + [\varepsilon^2 \alpha]_{\chi} + [\varepsilon^4 \alpha]_{\chi} + [\varepsilon^6 \alpha]_{\chi}\right), \quad (4.2)$$
where $\alpha$ is any totally positive generator of $n$. To prove that the right-hand side above is a well-defined function of $n$, it suffices to show that $[\varepsilon^8 \alpha]_\chi = [\alpha]_\chi$. Indeed, $\varepsilon^8 \alpha = (577 + 408\sqrt{2})(u + v\sqrt{2}) = (577u + 816v) + (408u + 577v)\sqrt{2}$, so

$$i^{(408u+577v)/2} = i^{u/2} \cdot i^{(51u+72v)} = i^{u/2}, \quad \chi(577u + 816v) = \chi(u + 8(72u + 102v)) = \chi(u),$$

and hence

$$[\varepsilon^8 \alpha]_\chi = i^{(408u+577v)/2} \chi(577u + 408v)[\varepsilon^8 \alpha] = i^{u/2} \chi(u)[\alpha] = [\alpha]_\chi.$$

We extend $b_n(\chi, \psi)$ to the remaining non-zero ideals by 0, i.e., we set $b_n(\chi, \psi) = 0$ whenever $\mathfrak{n} \not\equiv 1 \mod 8$.

Finally, when $n = p$ is a prime ideal in $\mathbb{Z}[\sqrt{2}]$ lying above a prime $p \equiv \pm 1 \mod 8$, i.e., above a prime that splits completely in $\mathbb{Z}[\sqrt{2}]$, then

$$\sum_{\chi \mod 8} \sum_{\psi \mod 16} b_p(\chi, \psi) = \begin{cases} \beta_p & \text{if } p \text{ splits completely in } \mathbb{Q}(\zeta_{16}, \sqrt{2}) \\ 0 & \text{otherwise.} \end{cases}$$

(4.3)

5 Proof of Proposition 2.3

To prove Proposition 2.3, it suffices to prove that

$$\sum_{\mathfrak{n} \equiv 0 \mod \mathfrak{d}} b_n(\chi, \psi) \ll X^{1 - \frac{1}{200}}$$

(5.1)

for all pairs of Dirichlet characters $\chi$ modulo 8 and $\psi$ modulo 16. Indeed, if this is the case, since the contribution from $\sqrt{2}\mathbb{Z}[\sqrt{2}]$ and the inert primes is $\ll X^{1/2}$ when we order the prime ideals by norm, we see that the same estimate holds when the sum is restricted to split primes $p$. Then adding together finitely many such sums, one for each pair of Dirichlet characters $\chi$ modulo 8 and $\psi$ modulo 16, we obtain the sum in Proposition 2.3 via formula (4.3). The proof of (5.1) is essentially no different from the proof of [9, Theorem 3, p.994]. In short, one uses Vinogradov’s method of sums of type I and type II, in the form presented in [2, Section 5]. One first selects a suitable fundamental domain in $\mathbb{Z}^2$ for the action of $\varepsilon^2$ on non-zero elements of $\mathbb{Z}[\sqrt{2}]$, so that each point in the domain corresponds to exactly one non-zero ideal in $\mathbb{Z}[\sqrt{2}]$. This allows us to pass to sums over elements $\alpha$ in the domain (or a translate thereof) instead of ideals; the summand then becomes $(-1)^{\mathfrak{n}(\alpha)-1}/8 \psi(\mathfrak{N}(\alpha))[\alpha]_\chi$. To control the factors appearing in the summand other than $[\alpha]$, one breaks up the relevant sum according to the congruence class of $\alpha$ modulo 16, shows the desired estimate for each such sum, and then adds together the contributions from the finitely many congruence classes. In this way, one proves that

$$\sum_{\mathfrak{n} \equiv 0 \mod \mathfrak{d}} b_n(\chi, \psi) \ll X^{\frac{5}{12} + \varepsilon}$$

uniformly in $\mathfrak{d}$. The key idea is to breaking up the domain into horizontal segments and apply the Pólya-Vinogradov inequality to the sum over each segment. See [9, Section 5] for details. Next, one proves that

$$\sum_m \sum_{\mathfrak{n} \leq M} \sum_{\mathfrak{m} \leq N} v_{\mathfrak{m}} w_{\mathfrak{n}} b_m(\chi, \psi) \ll (MN)^{\frac{5}{12} + \varepsilon} \left( M^\frac{1}{6} + N^\frac{1}{6} \right)$$
uniformly for all bounded sequences of complex numbers \( \{v_m\} \) and \( \{w_n\} \) indexed by non-zero ideals of \( \mathbb{Z}[\sqrt{2}] \). One again isolates the key spin \([\alpha \beta]\) from the other factors in the definition of \([\alpha \beta]\), by restricting the congruence classes of \( \alpha \) and \( \beta \) modulo 16, and then uses the very key \([9]\) Proposition 8, p.1010] (modeled after [4] Lemma 20.1, p.1021) to factor \([\alpha \beta]\) into essentially \([\alpha]\), \([\beta]\), which are absorbed into the sequences \(v_m\) and \(w_n\), and a quadratic residue symbol \((\alpha/\beta)_{Q(\sqrt{2})^2}\). One finishes by applying the double oscillation result [9, Lemma 22, p.1009]. See [9, Section 6] for details. The result then follows by applying [2, Proposition 5.2, p.722].

6 Proof of Propositions 2.1 and 2.2

Finally, we explain how to obtain Propositions 2.1 and 2.2 from [5] and [4], respectively.

6.1 Proof of Proposition 2.1

In [5], it is proved that
\[
\sum'_{p \leq X} \alpha_p \ll X^{1-\frac{1}{3200}},
\]
where ' restricts the summation to primes \( p \) that split completely in \( \mathbb{Q}(\zeta_8, \sqrt{2}) \). The proof has a similar structure to the proof in [9], but one works over \( \mathbb{Z}[\zeta_8] \) instead of \( \mathbb{Z}[\sqrt{2}] \) (although we said in Section 3.8 that working over \( \mathbb{Z}[\zeta_8] \) would only give a conditional result as it pertains to Proposition 2.3, there are other circumstances we were able to exploit in [5] to obtain an unconditional result). So if we choose an element \( \varpi \in \mathbb{Z}[\zeta_8] \) whose norm is a prime \( p \) that splits completely in \( \mathbb{Q}(\zeta_8, \sqrt{2}) \), then \( p \) further splits in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \) if and only if the quadratic residue symbol \((\zeta_8/\varpi)_{Q(\zeta_8)^2}\) is equal to 1 (since \( \zeta_{16} \) is a square root of \( \zeta_8 \)). Hence we simply multiply the symbol \( a(\chi)_n \) in [5] (2.3), p.6] by the indicator function
\[
\frac{1}{2} \left( 1 + \left( \frac{\zeta_8}{n} \right)_{Q(\zeta_8)^2} \right).
\]
After expanding, we see that one also has to prove [5, Proposition 3.7, p. 13] and [4, Proposition 3.8, p. 14] with \( a(\chi)_n \) replaced by \( a(\chi)_n(\zeta_8/n)_{Q(\zeta_8)^2} \). By quadratic reciprocity, the quadratic residue symbol \((\zeta_8/n)_{Q(\zeta_8)^2}\) is controlled by the congruence class modulo \( 8\mathbb{Z}[\zeta_8] \) of a generator \( \alpha \) of \( n \), and so the same proofs apply (since in the proofs in [5] one immediately reduces to sums over a fixed congruence class modulo \( F = 16 \)).

6.2 Proof of Proposition 2.2

In [4], it is proved that
\[
\sum^*_{p \leq X} \alpha_p \beta_p \ll X^{1-\delta},
\]
where * restricts the summation to primes \( p \) that split completely in \( \mathbb{Q}(\zeta_{16}, \sqrt{2}) \), but the result is conditional on Conjecture \( C_n \) for \( n = 8 \) from [2]. The reason we needed this conjecture is that this time we carried out the analytic estimates over \( \mathbb{Q}(\zeta_8, \sqrt{1+\zeta_4}) \), a number field of degree 8 over \( \mathbb{Q} \). In proving a result on \((-1)^{(h(-p))/8}\), Koymans [4] managed to work over
and hence controlled by the congruence class of \( \overline{\omega} \) modulo \( 32\mathbb{Z}[\zeta_8] \). As shown above in Section 3, the twist is actually equal to

\[
(-1)^{p-1} \left( \frac{2 - \sqrt{2}}{\overline{\omega}} \right)^{Q(\zeta_8),4}
\]

and hence controlled by the congruence class of \( \overline{\omega} \) modulo \( 32\mathbb{Z}[\zeta_8] \). For \( (p-1)/16 \) is determined by \( p \mod 32 \), which is certainly determined by \( \overline{\omega} \mod 32\mathbb{Z}[\zeta_8] \); the factor \( (2 - \sqrt{2}/\overline{\omega})^{Q(\zeta_8),4} \) is determined by \( \overline{\omega} \mod 8\mathbb{Z}[\zeta_8] \), by quartic reciprocity). As Koymans already restricts the sums appearing in [5] to fixed congruence classes modulo a much higher power of 2, his proof safely carries through also for \( (-1)^{(h(p)-1)/8} \) twisted by \( (-1)^{(a-1+b+2d)/8} \).

References

[1] J. B. Friedlander and H. Iwaniec. The polynomial \( X^2 + Y^4 \) captures its primes. *Ann. of Math.* (2), 148(3):945–1040, 1998.

[2] J. B. Friedlander, H. Iwaniec, B. Mazur, and K. Rubin. The spin of prime ideals. *Invent. Math.*, 193(3):697–749, 2013.

[3] P. Kaplan and K. S. Williams. On the strict class number of \( \mathbb{Q}(\sqrt{2}p) \) modulo 16, \( p \equiv 1 \mod 8 \) prime. *Osaka J. Math.*, 21(1):23–29, 1984.

[4] P. Koymans. The 16-rank of \( \mathbb{Q}(\sqrt{-p}) \). *ArXiv e-prints*, page arXiv:1809.07167, Sept. 2018.

[5] P. Koymans and D. Z. Milovic. On the 16-rank of class groups of \( \mathbb{Q}(\sqrt{-2p}) \) for primes \( p \equiv 1 \mod 4 \). *International Mathematics Research Notices*, page rny010, 2018.

[6] P. Koymans and D. Z. Milovic. Spins of prime ideals and the negative pell equation \( x^2 - 2py^2 = -1 \). *Compositio Mathematica*, 155(1):100–125, 2019.

[7] F. Lemmermeyer. Ideal class groups of cyclotomic number fields. I. *Acta Arith.*, 72(4):347–359, 1995.

[8] P. A. Leonard and K. S. Williams. On the divisibility of the class numbers of \( \mathbb{Q}(\sqrt{-p}) \) and \( \mathbb{Q}(\sqrt{-2p}) \) by 16. *Canad. Math. Bull.*, 25(2):200–206, 1982.

[9] D. Z. Milovic. On the 16-rank of class groups of \( \mathbb{Q}(\sqrt{-8p}) \) for \( p \equiv -1 \mod 4 \). *Geom. Funct. Anal.*, 27(4):973–1016, 2017.

[10] L. Rédei. Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper. I. *J. Reine Angew. Math.*, 180:1–43, 1939.

[11] A. Scholz. Über die Lösbarkeit der Gleichung \( t^2 - Du^2 = -4 \). *Math. Z.*, 39(1):95–111, 1935.

[12] P. Stevenhagen. Ray class groups and governing fields. In *Théorie des nombres, Année 1988/89, Fasc. 1*, Publ. Math. Fac. Sci. Besançon, page 93. Univ. Franche-Comté, Besançon, 1989.

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[13] P. Stevenhagen. Divisibility by 2-powers of certain quadratic class numbers. *J. Number Theory*, 43(1):1–19, 1993.

[14] P. Stevenhagen. The number of real quadratic fields having units of negative norm. *Experiment. Math.*, 2(2):121–136, 1993.