QUATERNION ANALYSIS

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(1996)

Quaternion analysis is considered in full details where a new analyticity condition in complete
analogy to complex analysis is found. The extension to octonions is also worked out.

I. INTRODUCTION

The ultimate goal of theoretical physics is to understand nature at any scale. Whatever this task
seems too ambitious, we have already made some success. Guided by the gauge symmetry principle,
we reached the standard model. Well tested but not well understood. It contains many puzzles [1].
We can go beyond the standard model by enlarging the gauge symmetry [1]. Some of the standard
model problems can be solved but many remain. Unifying gravity with the other forces of nature
is still a challenge. Having a large symmetry in our model is a good guide. Supersymmetry and
supergravity have a much better renormalization properties than the standard field theory. Simply,
symmetry kills divergences but, only, an infinite dimensional symmetric model is the best choice.
Superstring profited from this characteristic, over the string sheet, the theory posses superconfor-
mal symmetry [2]. In two dimensions (and only two), the conformal symmetry group is infinite
dimensions. Working over $\mathbb{R}^2$, we can define a single natural complex coordinate

$$x^\mu = \{x^1, x^2\} \Rightarrow z = x^1 + i x^2.$$ (1)

Any general conformal coordinate transformation in two dimensions is analytic in this complex
coordinate.

$$z' = f(z), \quad \bar{z}' = \overline{f(\bar{z})}.$$ (2)

Owing to the Cauchy-Riemann condition;

$$\frac{\partial f}{\partial x_0} = -e_1 \frac{\partial f}{\partial x_1},$$ (3)

which is, but not in a formal way [3], equivalent to

$$\frac{\partial}{\partial \bar{z}} f(z) = 0.$$ (4)

The quest of extending this notion of analyticity to four dimensions was studied extensively in
the recent years. The most promising line of attack is quaternion analysis [4–6]. In this article, we
want to investigate in details the standard generalization of complex analysis to the quaternionic
case. We will be able to show that a new generalization of complex analysis to the quaternionic
case indeed exists. The main novel result will be : An infinite set of quaternionic analytic functions
obey our new analyticity condition.

II. THE PROBLEM

In four dimensions, quaternions are the natural extension of complex numbers. Over $\mathbb{R}^4$, coor-
dinates are well known to be unified into a quaternion

$$x_\mu = \{x_0, x_1, x_2, x_3\} \Rightarrow q = x_0 + e_i x_i.$$ (5)

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where \( e_i \) are the imaginary quaternion units satisfying \( e_i e_j = -\delta_{ij} + e_{ij}e_k \) \((i, j, k = 1, 2, 3)\). The standard conjugation, involution, is defined by

\[
\bar{q} = x_0 - e_ix_i.
\]

But, unfortunately, the quaternion analysis has proved to be a very delicate subject.

In complete analogy to complex analysis, (following Ahlfors), a quaternionic function

\[
f(q) = f_0(x_\mu)e_0 + f_1(x_\mu)e_1 + f_2(x_\mu)e_2 + f_3(x_\mu)e_3,
\]

is analytic if it obeys the following Generalized-Cauchy-Riemann condition

\[
\frac{\partial f(q)}{\partial x_0} = -e_1 \frac{\partial f(q)}{\partial x_1} = -e_2 \frac{\partial f(q)}{\partial x_2} = -e_3 \frac{\partial f(q)}{\partial x_3}.
\]

The different four equalities ensures the same derivative regardless of the direction of the approaching. We have just met the first obstacle, namely the position of the \( e_i \). Why not

\[
\frac{\partial f(q)}{\partial x_0} = -\frac{\partial f(q)}{\partial x_1} e_1 = -\frac{\partial f(q)}{\partial x_2} e_2 = -\frac{\partial f(q)}{\partial x_3} e_3.
\]

There is an ambiguity due to the non-commutativity, we will call \((8, 9)\) left and right quaternion analytic condition respectively. But if we want to translate the four-dimensional physics to quaternionic language then which derivative should we use? Actually, both of \((8, 9)\) are too restricted. To show this, just plug \( f(q) \) into \((8, 9)\), we get for the different imaginary units the following equalities, for the left case

\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_2}{\partial x_2} &= \frac{\partial f_3}{\partial x_3}, \\
\frac{\partial f_1}{\partial x_0} &= -\frac{\partial f_0}{\partial x_1}, & \frac{\partial f_2}{\partial x_2} &= -\frac{\partial f_3}{\partial x_3}, \\
\frac{\partial f_2}{\partial x_0} &= -\frac{\partial f_1}{\partial x_1}, & \frac{\partial f_3}{\partial x_3} &= -\frac{\partial f_0}{\partial x_0}, \\
\frac{\partial f_3}{\partial x_0} &= -\frac{\partial f_2}{\partial x_2}.
\end{align*}
\]

and for the right case

\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_2}{\partial x_2} &= \frac{\partial f_3}{\partial x_3}, \\
\frac{\partial f_1}{\partial x_0} &= -\frac{\partial f_0}{\partial x_1}, & \frac{\partial f_2}{\partial x_2} &= -\frac{\partial f_3}{\partial x_3}, \\
\frac{\partial f_2}{\partial x_0} &= -\frac{\partial f_1}{\partial x_1}, & \frac{\partial f_3}{\partial x_3} &= -\frac{\partial f_0}{\partial x_0}, \\
\frac{\partial f_3}{\partial x_0} &= -\frac{\partial f_2}{\partial x_2}.
\end{align*}
\]

We can put the common part of \((10, 11)\) into the following form

\[
\begin{align*}
\frac{\partial^2 f_0}{\partial x_0^2} + \frac{\partial^2 f_0}{\partial x_1^2} &= 0, & \frac{\partial^2 f_0}{\partial x_0^2} + \frac{\partial^2 f_0}{\partial x_2^2} &= 0, \\
\frac{\partial^2 f_0}{\partial x_0^2} + \frac{\partial^2 f_0}{\partial x_3^2} &= 0, & \frac{\partial^2 f_0}{\partial x_1^2} + \frac{\partial^2 f_0}{\partial x_2^2} &= 0, \\
\frac{\partial^2 f_0}{\partial x_1^2} + \frac{\partial^2 f_0}{\partial x_3^2} &= 0, & \frac{\partial^2 f_0}{\partial x_2^2} + \frac{\partial^2 f_0}{\partial x_3^2} &= 0.
\end{align*}
\]

which leads eventually to

\[
\begin{align*}
\frac{\partial^2 f_0}{\partial x_0^2} &= -\frac{\partial^2 f_0}{\partial x_1^2}, & \frac{\partial^2 f_0}{\partial x_0^2} &= -\frac{\partial^2 f_0}{\partial x_2^2}, & \frac{\partial^2 f_0}{\partial x_0^2} &= -\frac{\partial^2 f_0}{\partial x_3^2} = 0,
\end{align*}
\]

and similar forms can be obtained for \( f_1, f_2 \) and \( f_3 \). After considering the non-common terms, it is easy to see that \((8, 9)\) admit only a special form of the linear function as a solution. In complete disagreement with the complex case where analytic functions are wide class.
The only successful attempt to relax this constraint is due to Feuter \cite{4}, in analogy with \cite{4}. He defined the analyticity condition to be
\[
\frac{\partial}{\partial q} f(q) = 0, \quad (14)
\]
where
\[
\frac{\partial}{\partial q} \equiv \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}. \quad (15)
\]
But, again, we have a problem due to the non-commutativity of quaternions. Feuter defined left quaternionic analytic condition by
\[
\overrightarrow{\frac{\partial}{\partial q}} f(q) = \frac{\partial}{\partial x_0} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) \\
+ e_1 \frac{\partial}{\partial x_1} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) \\
+ e_2 \frac{\partial}{\partial x_2} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) \\
+ e_3 \frac{\partial}{\partial x_3} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) = 0, \quad (16)
\]
and right quaternionic analytic condition by
\[
\overleftarrow{\frac{\partial}{\partial q}} f(q) = \frac{\partial}{\partial x_0} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) \\
+ \frac{\partial}{\partial x_1} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) e_1 \\
+ \frac{\partial}{\partial x_2} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) e_2 \\
+ \frac{\partial}{\partial x_3} (f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3) e_3 = 0. \quad (17)
\]
After substituting by \( f(q) \), we find, for the left analyticity condition
\[
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0; \\
\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = 0; \\
\frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} + \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} = 0; \\
\frac{\partial f_3}{\partial x_0} + \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_0}{\partial x_3} = 0. \quad (18)
\]
whereas for the right case, we get
\[
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0; \\
\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1} - \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0; \\
\frac{\partial f_2}{\partial x_0} + \frac{\partial f_3}{\partial x_1} + \frac{\partial f_0}{\partial x_2} - \frac{\partial f_1}{\partial x_3} = 0; \\
\frac{\partial f_3}{\partial x_0} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_0}{\partial x_3} = 0. \quad (19)
\]
It is clear that these conditions are completely different from the standard left/right analytic conditions defined in \cite{4, 11}. Such Feuter quaternionic analytic function admits right/left Weierstrass-like series for left/right Feuter conditions respectively \cite{4}. But, it is not a straightforward generalization of complex analysis as it is not based on \cite{3}. One can try simply
\[
f(q) = q^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2(e_1 x_1 + e_2 x_2 + e_3 x_3) x_0. \quad (20)
\]
It does not satisfy neither the left nor the right Feuter analytic condition.

3
III. THE SOLUTION

The aim of this work is to find a ring of analytic differentiable quaternionic function that generalizes the notion of complex analysis. The only way to relax the condition (10, 11) is to use both of left and right derivative together to define a new analytic condition. The simplest case is

\[ f'(q) = \frac{\partial f(q)}{\partial x_0} = \frac{-1}{2} (e_1 \frac{\partial f(q)}{\partial x_1} + \frac{\partial f(q)}{\partial x_1} e_1) = \frac{-1}{2} (e_2 \frac{\partial f(q)}{\partial x_2} + \frac{\partial f(q)}{\partial x_2} e_2) = \frac{-1}{2} (e_3 \frac{\partial f(q)}{\partial x_3} + \frac{\partial f(q)}{\partial x_3} e_3). \] (21)

But unfortunately this simple generalization is also limited. After substituting by \( f(q) \), we get

\[ f'(q) = \frac{\partial f_0}{\partial x_0} + \frac{\partial f_1}{\partial x_0} e_1 + \frac{\partial f_2}{\partial x_0} e_2 + \frac{\partial f_3}{\partial x_0} e_3 = \frac{\partial f_1}{\partial x_1} - \frac{\partial f_0}{\partial x_1} e_1 = \frac{\partial f_2}{\partial x_2} - \frac{\partial f_0}{\partial x_2} e_2 = \frac{\partial f_3}{\partial x_3} - \frac{\partial f_0}{\partial x_3} e_3. \] (22)

leading to

\[ \frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3}; \]
\[ \frac{\partial f_1}{\partial x_0} = \frac{\partial f_0}{\partial x_1} = 0; \]
\[ \frac{\partial f_2}{\partial x_0} = \frac{\partial f_0}{\partial x_2} = 0; \]
\[ \frac{\partial f_3}{\partial x_0} = \frac{\partial f_0}{\partial x_3} = 0. \] (23)

Which can work for some special forms but if one tries a simple function like \( q^2 \), it will fail in contrast to the complex analysis. So we should modify again this condition. After some trials, one can find the possible definition mimicking complex analysis is the following

\[ f'(q) = \frac{\partial f(q)}{\partial x_0} = \frac{-1}{2} (e_1 \frac{\partial f(q)}{\partial x_1} + \frac{\partial f(q)}{\partial x_1} e_1) - \frac{\partial f_0}{\partial x_2} e_2 - \frac{\partial f_0}{\partial x_3} e_3 = \frac{-1}{2} (e_2 \frac{\partial f(q)}{\partial x_2} + \frac{\partial f(q)}{\partial x_2} e_2) - \frac{\partial f_0}{\partial x_1} e_1 - \frac{\partial f_0}{\partial x_3} e_3 = \frac{-1}{2} (e_3 \frac{\partial f(q)}{\partial x_3} + \frac{\partial f(q)}{\partial x_3} e_3) - \frac{\partial f_0}{\partial x_1} e_1 - \frac{\partial f_0}{\partial x_2} e_2. \] (24)

or explicitly

\[ f'(q) = \frac{\partial f_0}{\partial x_0} + \frac{\partial f_1}{\partial x_0} e_1 + \frac{\partial f_2}{\partial x_0} e_2 + \frac{\partial f_3}{\partial x_0} e_3 = \frac{\partial f_1}{\partial x_1} - \frac{\partial f_0}{\partial x_1} e_1 = \frac{\partial f_2}{\partial x_2} - \frac{\partial f_0}{\partial x_2} e_2 = \frac{\partial f_3}{\partial x_3} - \frac{\partial f_0}{\partial x_3} e_3. \] (25)
our analytic conditions are
\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3}; \\
\frac{\partial f_1}{\partial x_0} &= -\frac{\partial f_0}{\partial x_1}; \\
\frac{\partial f_2}{\partial x_0} &= -\frac{\partial f_0}{\partial x_2}; \\
\frac{\partial f_3}{\partial x_0} &= -\frac{\partial f_0}{\partial x_3}.
\end{align*}
\]

We will call these conditions the Quaternionic Analyticity Condition (QAC). Moreover, we will call any quaternion function that satisfies the QAC simply Quaternionic Analytic Function (QAF).

It is easy to show that (26) can be put it in the following form
\[
\begin{align*}
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_1^2} &= 0, \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_2^2} &= 0, \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_3^2} &= 0,
\end{align*}
\]

Consider, again, \( f(q) = q^2 \). It satisfies the QAC without any problems leading to
\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = 2x_0; \\
\frac{\partial f_1}{\partial x_0} &= -\frac{\partial f_0}{\partial x_1} = 2x_1; \\
\frac{\partial f_2}{\partial x_0} &= -\frac{\partial f_0}{\partial x_2} = 2x_2; \\
\frac{\partial f_3}{\partial x_0} &= -\frac{\partial f_0}{\partial x_3} = 2x_3,
\end{align*}
\]

implying
\[ f'(q) = 2q. \]

And
\[
\begin{align*}
f_0'(q) &= 2x_0; \\
f_1'(q) &= 2x_1; \\
f_2'(q) &= 2x_2; \\
f_3'(q) &= 2x_3.
\end{align*}
\]

Also, we notice that
\[
\begin{align*}
\frac{\partial f_1}{\partial x_2} &= \frac{\partial f_1}{\partial x_3} = 0; \\
\frac{\partial f_2}{\partial x_2} &= \frac{\partial f_2}{\partial x_3} = 0; \\
\frac{\partial f_3}{\partial x_1} &= \frac{\partial f_3}{\partial x_2} = 0.
\end{align*}
\]

One can prove without any problems that the sum of any two QAF is again analytic. Also
\[
\begin{align*}
f(q) &= f_0(x_\mu) e_0 + f_1(x_\mu) e_1 + f_2(x_\mu) e_2 + f_3(x_\mu) e_3, \\
g(q) &= g_0(x_\mu) e_0 + g_1(x_\mu) e_1 + g_2(x_\mu) e_2 + g_3(x_\mu) e_3, \\
(fg)(q) &= (f_0g_0 - f_1g_1 - f_2g_2 - f_3g_3)(x_\mu) e_0 + (f_0g_1 + f_1g_0 + f_2g_3 - f_3g_2)(x_\mu) e_1 \\
&\quad + (f_0g_2 + f_2g_0 + f_3g_1 - f_1g_3)(x_\mu) e_2 + (f_0g_3 + f_3g_0 + f_1g_2 - f_2g_1)(x_\mu) e_3.
\end{align*}
\]
taking into account \( f_{0...3}, g_{0...3} \in R \) then after simple, but lengthy, algebraic calculations (Actually, the next relation holds trivially since \( f'(q) = \partial f \) by \( \frac{24}{24} \)) we can prove that

\[
(fg)' = f'g + fg',
\]

But, in contrast to complex analysis, this does not imply that every polynomial is QAF \( \text{e.g.,} \)

\[
q^3 = (q_0^3 - 3q_0q_1^2 - 3q_0q_2^2 - 3q_0q_3^2) + (3q_0^2q_1 - q_1^3 - q_1q_2^2 - q_1q_3^2)e_1
+ (3q_0^2q_2 - q_2^3 - q_2q_1^2 - q_2q_3^2)e_2 + (3q_0^2q_3 - q_3^3 - q_3q_1^2 - q_3q_2^2)e_3,
\]

it is clear that the first condition of \( \text{[26]} \) is no more valid. the possible solution is the following modification \( \text{[no summation over i=1,2,3]} \)

\[
\frac{\partial f_0}{\partial x_0} = \frac{1}{x_1} \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) f_i;
\]

\[
\frac{\partial f_1}{\partial x_0} = -\frac{\partial f_0}{\partial x_1}; \\
\frac{\partial f_2}{\partial x_0} = -\frac{\partial f_0}{\partial x_2}; \\
\frac{\partial f_3}{\partial x_0} = -\frac{\partial f_0}{\partial x_3}.
\]

I will call this condition the Modified QAF (MQAF), which is valid for any polynomial, (for integer \( n \) and \( a_{0...n} \in R \))

\[
P(q) = a_0 + a_1q + \ldots + a_nq^n,
\]

is a MQAF and

\[
P'(q) = a_1 + \ldots + na_nq^{n-1}.
\]

In complete parallel agreement with complex analysis. In summary \( \text{[26]} \) admits a chain rule, it has a simple geometric meaning something like the overlapping of three planar complex structure, in the spirit of \( \text{[7]} \), a tri-holomorphicity that appears in the context of Self-Dual Yang-Mills fields. But being non-valid for generic polynomials, we can not develop something as powerful as Laurent series whereas \( \text{[38]} \) is more attractive and may have a richer geometric structure. I have given in the appendix a Mathematica \( \text{[8]} \) program to check (38) to any desired order.

**IV. FURTHER EXTENSION**

The next step should be octonions. Every thing (about the QAC) is the same just we let the \( i \) index takes values from 1 to 7. Our Octonionic Analytic Condition OAC reads (where summation over repeated indices is understood)

\[
f'(q) = \frac{\partial f_0}{\partial x_0} + \frac{\partial f_i}{\partial x_i}e_i = \frac{\partial f_1}{\partial x_1} - \frac{\partial f_0}{\partial x_1}e_1 \\
= \frac{\partial f_2}{\partial x_2} - \frac{\partial f_0}{\partial x_2}e_2 = \frac{\partial f_3}{\partial x_3} - \frac{\partial f_0}{\partial x_3}e_3 \\
= \frac{\partial f_4}{\partial x_4} - \frac{\partial f_0}{\partial x_4}e_4 = \frac{\partial f_5}{\partial x_5} - \frac{\partial f_0}{\partial x_5}e_4 \\
= \frac{\partial f_6}{\partial x_6} - \frac{\partial f_0}{\partial x_6}e_6 = \frac{\partial f_7}{\partial x_7} - \frac{\partial f_0}{\partial x_7}e_7
\]

which implies

\[1\] That \( q^3 \) is not analytic had been noticed by S. Abdel-Rahman and G. Auberson.

\[2\] This suggestion is entirely due to S. Abdel-Rahman.
\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = \frac{\partial f_4}{\partial x_4} = \frac{\partial f_5}{\partial x_5} = \frac{\partial f_6}{\partial x_6} = \frac{\partial f_7}{\partial x_7}; \\
\frac{\partial f_1}{\partial x_0} &= -\frac{\partial f_0}{\partial x_1} \\
\frac{\partial f_2}{\partial x_0} &= -\frac{\partial f_0}{\partial x_2} \\
\frac{\partial f_3}{\partial x_0} &= -\frac{\partial f_0}{\partial x_3} \\
\frac{\partial f_4}{\partial x_0} &= -\frac{\partial f_0}{\partial x_4} \\
\frac{\partial f_5}{\partial x_0} &= -\frac{\partial f_0}{\partial x_5} \\
\frac{\partial f_6}{\partial x_0} &= -\frac{\partial f_0}{\partial x_6} \\
\frac{\partial f_7}{\partial x_0} &= -\frac{\partial f_0}{\partial x_7} \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_1^2} &= 0, \quad \frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_2^2} = 0, \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_3^2} &= 0, \quad \frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_4^2} = 0, \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_5^2} &= 0, \quad \frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_6^2} = 0, \\
\frac{\partial^2 f_0(q)}{\partial x_0^2} + \frac{\partial^2 f_0(q)}{\partial x_7^2} &= 0, \quad \frac{\partial^2 f_1(q)}{\partial x_0^2} + \frac{\partial^2 f_1(q)}{\partial x_1^2} = 0, \\
\frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_2^2} &= 0, \quad \frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_3^2} = 0, \\
\frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_4^2} &= 0, \quad \frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_5^2} = 0, \\
\frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_6^2} &= 0, \quad \frac{\partial^2 f_2(q)}{\partial x_0^2} + \frac{\partial^2 f_2(q)}{\partial x_7^2} = 0.
\end{align*}
\]

The results obtained for quaternionic QAC can be generalized directly to octonions without any problems since the non-associativity of octonions is not relevant here. But generalizing the MQAF to octonions is not at all easy since, starting from \( f^3 \), any octonionic polynomial is ill defined

\[ f^3(x) \text{ is it } (f.f).f \text{ or } f.(f.f), \]

and the number of alternatives increases with the power of \( f \). It would be interesting to find out if a possible combination admits something like the MQAC.

V. CONCLUSION

To conclude, we have found a natural extension of Cauchy–Reimann condition to 4 which we hope to play the role of complex analysis in 2 dimensions. It will be interesting to look for a generalization of conformal symmetry and Virasoro algebra which will lead us eventually to find integrable models in higher dimensions. The case of octonions is still mysterious.

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APPENDIX A:

I have ran this program to $f^{10}$. Since the presence of a chain rule is not enough to guarantee the analyticity of generic polynomials, this seems to be the only way - Explicit Calculation. Below, I give the program and an example the test of analyticity of $f^{10}$

```math
\[
\begin{align*}
  e_0 &= \{(1,0), (0,1)\}; \\
  e_1 &= \{(0,1), (1,0)\}; \\
  e_2 &= \{(0,-1), (1,0)\}; \\
  e_3 &= \{(-1,0), (0,1)\}; \\
  Q &= q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3; \\
  f_0[x_] &= \text{Expand}[ \text{ComplexExpand}[ \text{Conjugate}[\text{Part}[x,1,1]] + \text{Part}[x,1,1]/2 ] ]; \\
  f_2[x_] &= \text{Expand}[ \text{ComplexExpand}[ \text{Conjugate}[\text{Part}[x,2,1]] + \text{Part}[x,2,1]/2 ] ]; \\
  f_1[x_] &= \text{Expand}[ \text{ComplexExpand}[ \text{Conjugate}[\text{Part}[x,2,1]] - \text{Part}[x,2,1]/(2 I) ] ]; \\
  f_3[x_] &= \text{Expand}[ \text{ComplexExpand}[ \text{Conjugate}[\text{Part}[x,1,1]] - \text{Part}[x,1,1]/(2 I) ] ]; \\
  d_0[x_,y_] &= \text{Expand}[ \text{ComplexExpand}[ (q_1 D[x,q_1] + q_2 D[x,q_2] + q_3 D[x,q_3]) / y ] ]; \\
  Q^{10} &= \text{Expand}[Q.Q.Q.Q.Q.Q.Q.Q.Q.Q]; \\
  \text{Print}[^{10}\ldots\ldots^]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_0]] == d_0[f_1][Q^{10}, q_1]]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_0]] == d_0[f_2][Q^{10}, q_2]]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_0]] == d_0[f_3][Q^{10}, q_3]]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_1]] == - \text{Expand}[D[f_1][Q^{10}, q_0]]]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_2]] == - \text{Expand}[D[f_2][Q^{10}, q_0]]]; \\
  \text{Print}[\text{Expand}[D[f_0][Q^{10}, q_3]] == - \text{Expand}[D[f_3][Q^{10}, q_0]]]; \\
  \text{Print}[^{\text{Ready}^}];
\end{align*}
\]
```

The output is

\begin{align*}
  10\ldots\ldots \\
  \text{True} \\
  \text{True} \\
  \text{True} \\
  \text{True} \\
  \text{True} \\
  \text{Ready}
\end{align*}