Eggert’s Conjecture for 2-Generated Nilpotent Algebras

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Abstract

Let \( A \) be a commutative nilpotent finitely-dimensional algebra over a field \( F \) of characteristic \( p > 0 \). A conjecture of Eggert says that \( p \cdot \dim A^{(p)} \leq \dim A \).

This conjecture gives an answer to the problem, when a finite abelian group is isomorphic to the adjoint group of some finite commutative nilpotent \( F \)-algebra. Recall that the adjoint group of \( A \) is the set with the operation \( x^0 = x + y + xy \) for every \( x, y \in A \).

Validity of this hypothesis would also have influence on an estimation of a (Prüfer) rank of a product of two (abelian) \( p \)-groups.

N. Eggert proved his conjecture only when \( \dim A^{(p)} \leq 2 \). Five years later, R. Bautista [2] proved it when \( \dim A^{(p)} = 3 \). C. Stack confirmed this results in Stack et al. [3,4], but provided shorter proofs. Finally, Amberg and Kazarin [5] proved the conjecture for the case \( \dim A^{(p)} \leq 4 \).

Another type of results presented by McLean [6,7]. He showed that this conjecture is true if the algebra \( A \) is either radical of a group algebra of a finite abelian group or \( A \) is graded and at least one of the following conditions is fulfilled:

(i) \( p = 2 \) and \( (A^{(p)})^4 = 0 \).
(ii) \( A^{(p)} \) is \( 2 \)-generated.
(iii) \( (A^{(p)})^4 = 0 \).
(iv) \( n < 3p \) and \( 3 \leq s - 1 \leq p \), where \( n \) is the number of generators of \( A^{(p)} \) and \( s \) is the index of nilpotence of \( A^{(p)} \).

We also should mention the result of Gorlov [8]. He proved the conjecture for nilpotent algebras \( A \) with a metacyclic adjoint group.

One paper concerning Eggert’s conjecture appeared in 2002 and the author L. Hammoudi [9] claimed he proved it. But, as Amberg [10] and McLean [7] have shown, his proof was incorrect.

In this short note we sketch out the main steps of the proof that Eggert’s conjecture is true if the subalgebra \( A^{(p)} \) has at most two generators. For the details, the reader is referred to Korbelar [11].

Since we will deal with nilpotency and commutativity only, we point out that the word ‘algebra’ will mean a commutative one and not necessary possessing a unit.

For an algebra \( A \) and a subset \( X \subseteq A \) we denote \( \langle X \rangle \) (\( [X] \), resp.) the algebra (vector space, resp.) generated by \( X \).

An algebra \( A \) is called nilpotent if \( A^n = 0 \) for some \( m \in N \).

Through this paper let always \( F \) be a field of characteristic \( p > 0 \) and \( R = F[x,y] \) be the ring of polynomials over the variables \( x, y \) and the field \( F \).

We start with the remark, that the number of any minimal generating set of a finite generated nilpotent \( F \)-algebra \( A \) is equal to \( \dim A/A' \). This implies the following:

**Lemma 1.1.** Suppose that Eggert’s conjecture holds for every nilpotent \( 2 \)-generated \( F \)-algebra. Then it also holds for every nilpotent \( F \)-algebra \( A \) such that \( A(p) \) is a \( 2 \)-generated \( F \)-algebra.

In the rest we deal with \( 2 \)-generated nilpotent algebras.

Bases of Nilpotent Algebras

We will use the well-known concept of monomial ordering and standard bases.

For \( \alpha = (i,j) \in N^2 \) put \( x^{\alpha} = x^i y^j \in F[x,y] \).

Denote \( [X]_0 = \{ x^{\alpha} | \alpha \in N^2 \} \cup \{ 0 \} \) the multiplicative monoid with the lexicographical ordering \( \leq \) such that \( x^{(i,j)} \leq x^{(i',j')} \iff i < i' \lor (i = i' \land j \leq j') \) and \( x^{(0,0)} = 0 \)

for every \( (i,j),(i',j') \in N^2 \).

For \( f \neq 0 \) \( \sum_{\alpha} a_{\alpha} x^{\alpha} \in F[x,y] \) put \( m(f) = \min \{ x^{\alpha} | a_{\alpha} \neq 0 \} \).

Finally, \( f \) will be called normal iff \( \lambda_{\alpha} = 1 \), where \( m(f) = x^\alpha \), and \( m(f) < \pi x^\alpha \) implies \( \lambda_{\alpha} = 0 \) for every

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\( \alpha \in \mathbb{N}_0 \)

This function \( m: F[x, y] \to [X]_n \) has common properties of a valuation:

(i) \( m(fg) = m(f)m(g) \).

(ii) \( m(f + g) \geq \min\{m(f); m(g)\} \). Moreover, \( m(f + g) = m(f) \) if \( m(f) < m(g) \).

(iii) \( m(f(x', y')) = m(f) \).

for every \( f, g \in F[x, y] \).

Finally, a set \( \mathcal{X} \subseteq \{x^a \mid \alpha \in \mathbb{N}_0 \} \) will be called upper (lower, resp.) if \( x^a \in \mathcal{X} \) and \( x^b(x'^a \mid x^a \in \mathcal{X}) \), resp. \( x^b \in \mathcal{X} \) for every \( x^a, x \in [X]_n \).

Definition 2.1. Let \( A \) be a nilpotent \( F \)-algebra generated by \( \{a_i, a_j\} \). Put

\[ C_A(a_i, a_j) = \{u \in [X]_0 \mid f \in Rx + Ry, m(f) = u \wedge f(a_i, a_j) = 0\} \]

and

\[ B_A(a_i, a_j) = [X]_0 \setminus C_A(a_i, a_j). \]

Proposition 2.2. Let \( A \) be a nilpotent \( F \)-algebra generated by \( \{a_i, a_j\} \).

(i) \( C_A(a_i, a_j) \) is an upper set and \( 0 \in C_A(a_i, a_j) \).

(ii) \( B_A(a_i, a_j) \) is a lower set and \( 1 \in B_A(a_i, a_j) \).

(iii) The set \( \{x^a \mid \alpha \in \mathbb{N}_0 \} \setminus (3f \in Rx + Ry, m(f) = u \wedge f(a_i, a_j) = 0 \wedge f \) normal \( ) \{0\} \) is a basis of \( A \). In particular, \( B_A(a_i, a_j) \) is finite.

(iv) \( C_A(a_i, a_j) = \{u \mid [X]_0 \mid f \in Rx + Ry, m(f) = u \wedge f(a_i, a_j) = 0 \wedge f \) is normal \} \{0\} \) is a basis of \( A \).

Definition 2.3. Let \( A \) be a nilpotent \( F \)-algebra generated by \( \{a_i, a_j\} \). Denote

\[ n_0 = \#\{x^a \in B_A(a_i, a_j) \mid x^a \in \mathbb{N}_0 \times \{0\}, \}
\]

\[ d_i = \#\{x^a \in B_A(a_i, a_j) \alpha \in [i] \times \{0\}, \}
\]

\[ \overline{n_0} = \#\{x^a \in B_A(a_i, a_j), a_i^2, a_j^2 \alpha \in \mathbb{N}_0 \times \{0\}, \}
\]

\[ \overline{d_i} = \#\{x^a \in B_A(a_i, a_j), a_i^2, a_j^2 \alpha \in [i] \times \{0\}, \}
\]

and

\[ D_i = \sum_{k=p_i}^{p_i+p-1} d_k \]

for \( i \in \mathbb{N}_0 \).

Lemma 2.4. Let \( A \) be a nilpotent \( F \)-algebra generated by \( \{a_i, a_j\} \).

Then:

(i) \( \overline{d} + \overline{d} \overline{d} = (a_i^2, a_j^2) = 1 + \dim A^{(0)} \).

(ii) \( D_0 + \overline{D} = (a_i, a_j) = 1 + \dim A \).

(iii) The set \( \{x^a \mid a_i^2, a_j^2 \mid 1 \neq x^a \in (a_i^2, a_j^2)\} \) is a basis of \( A^{(0)} \).

Eggert’s Conjecture for 2-generated Algebras

Let \( I \leq Rx + Ry \) be an ideal in \( R \) such that \( A = Rx + Ry/I \) is a non-zero nilpotent \( F \)-algebra.

We have \( A = \{x + I, y + I\} \) and \( A^{(0)} = \{x^* + I, y^* + I\} \).

By definition of \( C(x + I, y + I) \) there are \( f \in Rx + Ry, 0 \leq i \leq n \), such that \( m(f) = x_i^{(i)} f, e \), and \( f \) are normal.

The main idea of the proof lies in the fact that taking a normal polynomial from \( I \), dividing it by \( x \) and then multiplying by some suitable \( y^* \), we get again a member of \( I \) (3.3). Then, using binomial formula in a suitable way, we obtain a polynomial that will estimate the number \( d_0 \) (3.4 and the definition of \( B^{(0)}(a_i^2, a_j^2) \)).

Lemma 3.1. (i) \( f_0 = x^{(0)} - xh_0, \) where \( h_0 \in R \), and \( f_{n+1} = x^{(m)} \).

(ii) \( f_0 + \cdots + f_{n+1} \) for \( i = 0, \ldots, n_0 \).

Definition 3.2. Denote \( w \in \max B_A(x + I, y + I) \).

For \( 0 \leq i \leq n_0 \) denote

\[ m_i \in \mathbb{N}_0 \]

the least integer such that \( p_i \leq m_i \leq p_i + p - 1 \) and \( d_{m} = \cdots = d_{m+1} \). Put

\[ l_i = (m_i - 1) \sum_{k=p_i}^{m_i} (d_k - 1) = (p_i - 1) d_{m_i}. \]

Following lemma is obtained using induction.

Lemma 3.3. Let \( 1 \leq i \leq n_0 + 1 \) and \( 0 \neq f \in I \) be such that \( m(f) \) is. Then \( y^{(0)} \leq (f/y) \leq 1 + w + I \).

The proof of the next proposition uses only the binomial formula. It finds the particular polynomial the we need to make an estimation of the numbers \( D_i \) and Thus of the dimension of \( A^{(0)} \).

Proposition 3.4.

(i) \( 0 \leq i \leq n_0 \) and \( l_i > 0 \), then \( x^{(0)} \leq (f/y) \leq 1 + w + I \).

(ii) \( 0 \leq i < n_0 \) and \( l_i > 0 \), then \( x^{(0)} \leq (f/y) \leq 1 + w + I \).

(iii) \( 0 \leq i = n_0 \), then \( y^{(0)} \leq (f/y) \leq 1 + w + I \).

Now, only exploring carefully the previous cases for \( i \) and \( l_i \), we get the following interesting claim. It says that the inequality \( \overline{p} D_i \leq D_i \) holds for almost every \( i \).

Theorem 3.5. One of the following cases takes place:

(i) \( pD_n \leq D_{n+1} \) for \( 1 \leq i \leq \overline{n}_0 \).

(ii) \( pD_n \leq D_{n+1} \) for at most one \( 0 \leq i < \overline{n}_0 \).

(iii) \( \overline{pD_n} \leq D_{n+1} \) for \( 0 \leq i \leq \overline{n}_0 \).

And our main result is just an easy corollary of this and 1.1.

Theorem 3.6. Let \( A \) be a nilpotent \( F \)-algebra, \( \text{char}F = p > 0 \), such that \( A^{(0)} \) is 2-generated. Then \( p \dim A^{(0)} \) dim A.

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