Sheaves on non-reduced curves in a projective surface

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Abstract Sheaves on non-reduced curves can appear in moduli spaces of 1-dimensional semistable sheaves over a surface and moduli spaces of Higgs bundles as well. We estimate the dimension of the stack $M_{X}(nC, \chi)$ of pure sheaves supported at the non-reduced curve $nC$ ($n \geq 2$) with $C$ an integral curve on $X$. We prove that the Hilbert-Chow morphism $h_{L, \chi}: M_{X}^{H}(L, \chi) \to |L|$, $F \mapsto \text{supp}(F)$, sending each semistable 1-dimensional sheaf to its support has all its fibers of the same dimension for $X$ Fano or with the trivial canonical line bundle and $|L|$ contains integral curves.

Keywords 1-dimensional pure sheaves on projective surfaces, Hilbert-Chow morphism, Hitchin fibration, stack

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1 Introduction

1.1 Motivations

Sheaves on non-reduced curves can appear in two types of moduli stacks $M_{X}^{H}(L, \chi)$ and $M_{C_{0}, D}^{\text{Higgs}}(n, \chi)$ as follows:

(A) $M_{X}^{H}(L, \chi)$ parametrizes semistable sheaves $F$ with respect to the polarization $H$ on a projective surface $X$, supported at a curve in the linear system $|L|$ and of Euler characteristic $\chi$. We have a Hilbert-Chow morphism

$$h_{L, \chi}: M_{X}^{H}(L, \chi) \to |L|, \quad F \mapsto \text{supp}(F).$$

In general, $|L|$ contains singular curves and also curves with non-reduced components.

(B) $M_{C_{0}, D}^{\text{Higgs}}(n, \chi)$ parametrizes semistable Higgs bundles $(\mathcal{E}, \Theta)$ with respect to the effective divisor $D$ on a smooth curve $C_{0}$ with $\mathcal{E}$ rank $n$ and Euler characteristic $\chi$. We have the Hitchin fibration

$$h_{D, \chi}: M_{C_{0}, D}^{\text{Higgs}}(n, \chi) \to \bigoplus_{i=1}^{n} H^{0}(C_{0}, \mathcal{O}_{C_{0}}(iD)),$$

sending $(\mathcal{E}, \Theta)$ to the coefficients $(-\text{tr}(\Theta), \ldots, (-1)^{n}\text{det}(\Theta))$ of the characteristic polynomial $\text{char}(\Theta)$ of $\Theta: \mathcal{E} \to \mathcal{E}(D)$. 
Denote by \( \text{Tot}(\mathcal{O}_{C_0}(D)) \) the total space of the line bundles \( \mathcal{O}_{C_0}(D) \). Let \( p : \text{Tot}(\mathcal{O}_{C_0}(D)) \to C_0 \) be the projection. Then every Higgs bundle \((\mathcal{E}, \theta)\) on \( C_0 \) gives a 1-dimensional pure sheaf \( F_\mathcal{E} \) on \( \text{Tot}(\mathcal{O}_{C_0}(D)) \) via the following exact sequence:

\[
0 \to p^* \mathcal{E} \xrightarrow{\lambda \times p^* \text{id}_\mathcal{E} - p^* \theta} p^* \mathcal{E}(D) \to F_{\mathcal{E}} \to 0.
\]

We naturally have that \( \text{supp}(F_{\mathcal{E}}) \) is defined by the characteristic polynomial \( \chi(\theta) \), where \( \lambda \) is the variable on the fiber of \( p \). Hence the fiber of the Hitchin fibration over \( \lambda^a \), which is also called the central fiber of the Hitchin fibration, consists of sheaves (with some semistability) on the non-reduced curve in \( \text{Tot}(\mathcal{O}_{C_0}(D)) \) defined by \( \lambda^a \).

Let \( X := \mathbb{P}(\mathcal{O}_{C_0}(D) \oplus \mathcal{O}_{C_0}) \) be the ruled surface over \( C_0 \) which is a compactification of \( \text{Tot}(\mathcal{O}_{C_0}(D)) \). Then \( X \) is projective and the central fiber of \( h_{D,X} \) consists of sheaves (with some semistability) on the non-reduced curve \( nC_\epsilon \) with \( C_\epsilon \) the section satisfying \( C_\epsilon^2 = \text{deg}(D) \) and \( C_\epsilon.K_X = -\text{deg}(D) + 2g_{C_\epsilon} - 2 \).

People have to consider the dimension of each fiber of \( h_{L,X} \) in (A) or \( h_{D,X} \) in (B), if they want to study the flatness of the fibrations, or if they want to compute the cohomology of (constructible or coherent) sheaves over the moduli space \( \mathcal{M}_X^{\text{alg}}(L,\chi) \) (resp. \( \mathcal{M}_X^{\text{alg}}(n,\chi) \)) via the fibration \( h_{L,X} \) (resp. \( h_{D,X} \)).

The fibers of \( h_{L,X} \) (resp. \( h_{D,X} \)) over integral supports are their compactified Jacobians and hence are of equal dimension denoted by \( N_L \) (resp. \( N_{C_{0,D,n}} \)) which only depends on \( |L| \) (resp. \( (C_0,D,n) \)). But the fibers over non-integral supports are much more complicated, among which the worst are fibers over non-reduced curves. People expect those fibers would also have dimension \( N_L \) or \( N_{C_{0,D,n}} \), respectively, but in principle they can be of higher dimensions. So we pose the following question.

**Question 1.1.** Are all the fibers of \( h_{L,X} \) in (A) (resp. \( h_{D,X} \) in (B)) of the expected dimension \( N_L \) (resp. \( N_{C_{0,D,n}} \))?

Some results are already known for Question 1.1. For example, Ginzburg [3] showed that the central fiber of \( h_{D,X} \) is of expected dimension \( N_{C_{0,D,n}} \) for \( D \) the canonical divisor of \( C_0 \); Chaudouard and Laumon [2] showed that the central fiber of \( h_{D,X} \) is of dimension less than or equal to \( N_{C_{0,D,n}} \) for \( \text{deg}(D) > 2g - 2 \); finally, Maulik and Shen [5] showed that the fibers of \( h_{L,X} \) are of the expected dimension \( N_L \) for \( X \) a toric del Pezzo surface and \( L \) ample effective on \( X \). There seems to be no more general result. Especially the lack of the estimate of the dimensions of all the fibers of \( h_{L,X} \) prevents the main result in [5] from generalizing to all the del Pezzo surfaces not necessarily toric (see the paragraph after [5, Remark 0.2]).

In this paper, we give a complete answer to Question 1.1.

### 1.2 Notations and conventions

Throughout the paper, let \( X \) be a projective surface over an algebraically closed field \( k \). Let \( C \) be an integral curve on \( X \). Let \( \delta_C \in H^0(X,\mathcal{O}_X(C)) \) be the section defining \( C \). Denote by \( nC \) the non-reduced curve with the multiplicity \( n \) over \( C \), i.e., the 1-dimensional closed subscheme of \( X \) defined by \( \delta_C^n \).

For a pure 1-dimensional sheaf on a surface, one can define its **schematic support** (see [4, Subsection 2.2] or [7, Proposition 3.0.2]), which is a curve representing the first Chern class of the sheaf. For any 1-dimensional sheaf \( \mathcal{F} \) on a surface \( X \), we say that \( \mathcal{F} \) has the schematic support \( \overline{\mathcal{C}} \) if \( \mathcal{F} \) is an \( \mathcal{O}_{\overline{\mathcal{C}}} \)-module and also \( \mathcal{F}/\mathcal{T}_\mathcal{F} \) has \( \overline{\mathcal{C}} \) as its schematic support, where \( \mathcal{T}_\mathcal{F} \) is the maximal 0-dimensional subsheaf of \( \mathcal{F} \). People also call it the “fitting support” or just “support” for short.

Denote by \( \overline{\mathcal{M}}_X(nC,\chi) \) the stack of all the 1-dimensional sheaves with the schematic support \( nC \) and the Euler characteristic \( \chi \), and let \( \mathcal{M}_X(nC,\chi) \subset \overline{\mathcal{M}}_X(nC,\chi) \) be the substack consisting of 1-dimensional pure sheaves. Then we have that \( h_{\mathcal{O}_X(nC),\chi}^{-1}(nC) \subset \mathcal{M}_X(nC,\chi) \), where \( h_{\mathcal{O}_X(nC),\chi} \) is the Hilbert-Chow morphism defined in (1.1).

Because the stack \( \mathbf{Vect}_C(n,\chi) \) of rank \( n \) Euler characteristic \( \chi \) vector bundles over \( C \) is a substack of \( \mathcal{M}_X(nC,\chi) \), we have

\[
\dim \overline{\mathcal{M}}_X(nC,\chi) \geq \dim \mathcal{M}_X(nC,\chi) \geq \dim \mathbf{Vect}_C(n,\chi) = n^2(g_C - 1).
\]
We use $K_X$ to denote both the canonical divisor of $X$ and the canonical line bundle as well. For any two (not necessarily integral) curves $C'$ and $C''$, we write $C'.K_X$ the intersection number of the divisor class of $C'$ with $K_X$, $C''.K_X$ the intersection number of the divisor classes of these two curves and $C'':=C'.C''$.

Denote by $g_{C'}$ the arithmetic genus of $C'$. We have $g_{C'}-1=C''+C'.K_X$.

For a sheaf $F$ on $X$, let $\mathcal{F}(\Sigma):=\mathcal{F} \otimes \mathcal{O}_X(\Sigma)$ for $\Sigma$ a curve or a divisor class.

Let $K(X)$ and $K(C)$ be the Grothendieck groups of coherent sheaves on $X$ and $C$, respectively. Let $*$ stand for $X$ or $C$. Denote by $\chi_*(-,-):K(*) \times K(*) \to \mathbb{Z}$ the bilinear integral form on $K(*)$ such that for every two coherent sheaves $S$ and $T$ on $*$,

$$\chi_*(-([S],[T]))=\sum_{j \geq 0}(-1)^j \dim \text{Ext}_*^j(S,T).$$

1.3 Results and applications

Our main result is the following theorem.

**Theorem 1.2** (See Theorem 4.1). *For any integral curve $C \subset X$, we have*

$$\dim M_X(nC,\chi) = \begin{cases} \frac{n^2C^2}{2} + \frac{nC.K_X}{2} = g_{nC} - 1, & \text{if } C.K_X \leq 0, \\ \frac{n^2C^2}{2} + \frac{n^2C.K_X}{2} = n^2(g_C-1), & \text{if } C.K_X > 0. \end{cases}$$

Theorem 1.2 suggests that in order to have a positive answer to Question 1.1, it is reasonable to ask the surface $X$ in (A) to be Fano or with $K_X$ trivial and to ask the divisor $D$ in (B) to satisfy $\deg(D) \geq 2g-2$.

**Corollary 1.3.** *Let $C'=\prod_{j=1}^s n_j C_j$ be a curve in the linear system $|L|$ on $X$ with $C_j$ pairwise distinct integral curves. If $C_j.K_X \leq 0$ for $j=1,\ldots,s$, then*

$$\dim h^{-1}_{L,\chi}(C') \leq g_{C'} - 1 = N_L.$$  

In particular, if $X$ is Fano or with $K_X$ trivial, and if either $|L|$ contains integral curves, or $H^1(\mathcal{O}_X) = H^1(L) = 0$ and $M_X^{H}(\mathcal{O}_X,\chi)^s$ the substack of $M_X^{H}(\mathcal{O}_X,\chi)$ consisting of stable sheaves, is not empty, then the Hilbert-Chow morphism $h^{-1}_{L,\chi}$ in (1.1) has all the fibers the expected dimension $g_{|L|} - 1$ with $g_{|L|}$ the arithmetic genus of any curve in $|L|$.

**Proof.** Every sheaf in $h^{-1}_{L,\chi}(C')$ can be realized as a successive extension of pure sheaves on $n_j C_j$ ($j=1,\ldots,s$). Since $C_j$'s are pairwise distinct integral curves, for sheaves $F_j \in M_X(n_j C_j,\chi_j)$ we have

$$\text{Hom}_{\mathcal{O}_X}(F_i,F_j) = \text{Ext}_X^2(F_i,F_j) = 0, \quad \dim \text{Ext}_X^1(F_i,F_j) = -\chi_X(F_i,F_j), \quad \forall i \neq j.$$

Because as a sheaf on $X$, $F_j$ is of rank 0 and the first Chern class $n[C_j]$, by Riemann-Roch theorem, we have

$$\chi_X(F_i,F_j) = -n_i n_j C_i.C_j.$$

Therefore, we have

$$\dim h^{-1}_{L,\chi}(C') \leq \sum_{j=1}^s \dim M_X(n_j C_j,\chi_j) - \sum_{i<j} \chi_X(F_i,F_j) = \sum_{j=1}^s \left( \frac{n_j^2 C_j^2}{2} + \frac{n_j C_j.K_X}{2} \right) + \sum_{i<j} n_i n_j C_i.C_j = (C')^2 + C'.K_X - g_{C'} - 1 = N_L. \quad (1.3)$$

Let $X$ be Fano or with $K_X$ trivial. If $|L|$ contains integral curves, then by semicontinuity every fiber of $h^{-1}_{L,\chi}$ is of dimension no less than $g_{|L|} - 1$. If $H^1(\mathcal{O}_X) = H^1(L) = 0$, then $M_X^{H}(\mathcal{O}_X,\chi)^s \neq \emptyset$ is smooth of dimension $g_{|L|} - 1 + \dim |L|$. Every fiber of $h^{-1}_{L,\chi}$ is a closed subscheme of $M_X^{H}(\mathcal{O}_X,\chi)$ defined by $\dim |L|$ equations, and hence of dimension no less than $g_{|L|} - 1$. We have proved the corollary. \qed
With Corollary 1.3, we can generalize [5, Theorem 0.1] to all the del Pezzo surfaces. For the reader’s convenience, we write the explicit statement as follows.

**Theorem 1.4** (Generalization of [5, Theorem 0.1] to all the del Pezzo surfaces). Let $X$ be a del Pezzo surface not necessarily toric with the polarization $H$, and let $L$ be an ample curve class on $X$. Let $M^H_X(L, \chi)$ be the coarse moduli space of 1-dimensional semistable sheaves with schematic supports in $|L|$ and the Euler characteristic $\chi$. Then for any $\chi, \chi' \in \mathbb{Z}$, there are isomorphisms of graded vector spaces

$$IH^p(M^H_X(L, \chi)) \cong IH^p(M^H_X(L, \chi')),$$

where $IH^p(-)$ denotes the intersection cohomology. Moreover, those isomorphisms respect perverse and Hodge filtrations carried by these vector spaces.

We refer to [5] for the details of the proof of Theorem 1.4 while [5, Proposition 2.6] can be extended to any del Pezzo surface $X$ by Corollary 1.3.

Another application of Corollary 1.3 is to generalize [8, Theorem 5.17] to all the del Pezzo surfaces. For simplicity, we only write down here the statement for $X$ a del Pezzo surface and $L$ ample on $X$.

**Theorem 1.5** (Generalization of [8, Theorem 5.17] for del Pezzo surfaces). Let $X$ be a del Pezzo surface and let $L$ be ample on $X$. Let non-integral curves in $|L|$ form a closed subset of codimension $p_L > 0$ in $|L|$. Denote by $M_\bullet(L, \chi)$ the stack of pure 1-dimensional sheaf with the first Chern class $L$ and the Euler characteristic $\chi$. Denote by $N(L, \chi)$ the substack of $M_\bullet(L, \chi)$ consisting of sheaves with integral supports. Then the complement $M_\bullet(L, \chi) \setminus N(L, \chi)$ is of codimension greater than or equal to $p_L$ and the coarse moduli space $M^H_X(L, \chi)$ of 1-dimensional semistable sheaves is irreducible.

Notice that Theorem 1.5 also generalizes [5, Theorem 2.3].

**Remark 1.6.** To have the irreducibility of $M^H_X(L, \chi)$, it is not necessary to ask that the Hilbert-Chow morphism has all the fibers of the same dimension. It might suffice to have sheaves supported at non-integral curves form a subset of positive codimension.

One can also generalize [8, Theorem 6.11] by improving the estimate of $p_L$. In order to avoid introducing many complicated notations in [8], we only write down as follows: the statement for $X = \mathbb{P}^2$, which in other words is the generalization of [8, Theorem 1.1]. Notice that as $\mathbb{P}^2$ is toric, Maulik-Shen’s result (see [5, Proposition 2.6]) is enough to obtain Theorem 1.7.

**Theorem 1.7.** Let $X = \mathbb{P}^2$ and $H$ be the hyperplane class on $X$. Let $M_{\mathbb{P}^2}(d, \chi)$ be the coarse moduli space of 1-dimensional semistable sheaves with schematic supports in $|dH|$ and the Euler characteristic $\chi$. Then for $0 \leq i \leq 2d - 3$, we have

$$
\begin{align*}
&b_i^v(M_{\mathbb{P}^2}(d, \chi)) = 0 \\
&b_i^v(M_{\mathbb{P}^2}(d, \chi)) = b_i^v(X^{(\frac{2d-3}{2})} - \chi_0) \\
&h^{i-p}(M_{\mathbb{P}^2}(d, \chi)) = 0 \\
&h^{i-p}(M_{\mathbb{P}^2}(d, \chi)) = h^{i-p}(X^{(\frac{2d-3}{2})} - \chi_0)
\end{align*}
$$

for $i$ odd, for $i$ even, for $p \neq i - p$, for $i = 2p$,

where $b_i^v$ (resp. $h_i^q$) denotes the $i$-th virtual Betti number (resp. $(p,q)$-th virtual Hodge number), $X^{[n]}$ is the Hilbert scheme of $n$-points on $X$ and finally $\chi_0 \equiv \chi(d)$ with $-2d - 1 \leq \chi_0 \leq -d + 1$.

Notice that by [1], $h_i^q(M_{\mathbb{P}^2}(d, \chi)) = 0$ for any $p \neq q$.

By Theorem 1.7, both the virtual Betti number $b_i^v(M_{\mathbb{P}^2}(d, \chi))$ and the virtual Hodge number $h_i^q(M_{\mathbb{P}^2}(d, \chi))$ stabilize as $d \to \infty$. Write down the generating function

$$Z(t, q) := \sum_{d \geq 0} q^d \left( \sum_{i=0}^{2 \dim M_{\mathbb{P}^2}(d, \chi)} b_i^v(M_{\mathbb{P}^2}(d, \chi)) t^i \right).$$

Then the coefficient of $t^i$ in $(1 - q)Z(t, q)$ is a polynomial in $q$. However, whether $Z(t, q)$ is a rational function is still a wide open question.
2 Filtrations for sheaves on non-reduced curves

For any \( F \in \overline{\text{M}}_X(nC, \chi) \), we can describe \( F \) via two filtrations in the following proposition.

**Proposition 2.1.** Let \( F \in \overline{\text{M}}_X(nC, \chi) \). Then there are two filtrations of \( F \):

1. **The lower filtration of \( F \)**:
   \[
   0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = F,
   \]
   such that \( Q_i := F_i/F_{i-1} \) are coherent sheaves on \( C \) with the rank \( t_i \). \( \sum t_i = n \), and moreover, there are injections \( f^i_j : Q_i(-C) \hookrightarrow Q_{i-1} \) induced by \( F \) for all \( 2 \leq i \leq l \).

2. **The upper filtration of \( F \)**:
   \[
   0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^m = F,
   \]
   such that \( R_i := F^i/F^{i-1} \) are coherent sheaves on \( C \) with the rank \( r_i \). \( \sum r_i = n \), and moreover, there are surjections \( g^i_j : R_i(-C) \twoheadrightarrow R_{i-1} \) induced by \( F \) for all \( 2 \leq i \leq m \).

Moreover, we have

(i) \( l = m \);

(ii) \( \forall 1 \leq i \leq m, \ t_i = r_{m-i+1} \).

Proposition 2.1 is not difficult to prove. Since \( F \) is an \( O_{nC} \)-module, the map \( F \xrightarrow{\delta^m_n} F(nC) \) is zero.

We define \( F_i := \ker(F \xrightarrow{\delta^m_n} F(iC)) \), and then the length of the lower filtration is

\[
\ell = \min\{k \geq 0 \mid F \xrightarrow{\delta^m_n} F(kC) \text{ is zero}\}.
\]

We can define the upper filtration \( F^i := \text{im}(F((i-m)C) \xrightarrow{\delta^{m-i}_{i-1}} F) \) with \( m = l \). Therefore, we have natural maps \( F_i(-C) \xrightarrow{\delta^m_n} F_{i-1} \) and \( F^i(-C) \xrightarrow{\delta^{m-i}_{i-1}} F^{i-1} \) inducing \( f^i_j : Q_i(-C) \hookrightarrow Q_{i-1} \) and \( g^i_j : R_i(-C) \twoheadrightarrow R_{i-1} \), respectively.

In particular, we have \( Q_i \cong F_i/\text{Tor}_{\mathcal{O}_C}^1(F, O_{(i-1)C}) \) and \( R_i \cong F^i \otimes O_C \). Since \( Q_i \)'s are factors of a filtration of \( F \), we have \( \sum c_1(Q_i) = c_1(F) = n[C] \). On the other hand, \( c_1(Q_i) = t_i[C] \) and hence \( \sum r_i = n \). For the same reason, we have \( \sum t_i = n \).

The reader can figure out the proof of Proposition 2.1 by themselves or look at [8, Propositions 5.7 and 5.10 and Lemma 5.11] for more details.

**Remark 2.2.** Using the same notations as in Proposition 2.1, we can see that \( m \leq n \) and

\[
m = \max\{k \mid F \xrightarrow{\delta^m_n} F((k-1)C) \text{ is not zero}\}.
\]

Recall that \( K(C) \) is the Grothendieck group of coherent sheaves on \( C \). For any class \( \beta \in K(C) \), denote by \( r(\beta) \) (resp. \( \chi(\beta) \)) the rank (resp. Euler characteristic) of \( \beta \).

Obviously, both upper and lower filtrations are uniquely determined by \( F \). Hence we can stratify \( \overline{\text{M}}_X(nC, \chi) \) by the filtration type. Let \( \overline{\text{M}}_X(nC, \chi)_{\beta_1, \ldots, \beta_m} \) (resp. \( \overline{\text{M}}_X(nC, \chi)_{\beta_1, \ldots, \beta_m} \)) with \( \beta_1, \ldots, \beta_m \in K(C) \) be the substack of \( \overline{\text{M}}_X(nC, \chi) \) consisting of sheaves with upper (resp. lower) filtrations satisfying \( [R_i] = \beta_i \) (resp. \( [Q_i] = \beta_i \)). Notice that \( \overline{\text{M}}_X(nC, \chi)_{\beta_1, \ldots, \beta_m} \) (resp. \( \overline{\text{M}}_X(nC, \chi)_{\beta_1, \ldots, \beta_m} \)) is not empty only if \( r(\beta_i) \geq r(\beta_{i-1}) \) (resp. \( r(\beta_i) \leq r(\beta_{i-1}) \)) for all \( 2 \leq i \leq m \).

Define

\[
\overline{\text{M}}_X(nC, \chi)^{r_1, \ldots, r_m} := \coprod_{1 \leq i \leq m} \overline{\text{M}}_X(nC, \chi)^{\beta_1, \ldots, \beta_m},
\]

\[
\overline{\text{M}}_X(nC, \chi)^{r'_1, \ldots, r'_m} := \coprod_{1 \leq i \leq m} \overline{\text{M}}_X(nC, \chi)^{\beta'_1, \ldots, \beta'_m},
\]

\[
\text{M}_X(nC, \chi)^{r_1, \ldots, r_m} := \text{M}_X(nC, \chi) \cap \overline{\text{M}}_X(nC, \chi)^{r_1, \ldots, r_m},
\]
\begin{align*}
M_X(nC,\chi)_{r_1,\ldots,r_m} &: = M_X(nC,\chi) \cap M_X(nC,\chi)_{r_1,\ldots,r_m}.
\end{align*}

By Proposition 2.1, we have
\begin{align*}
M_X(nC,\chi)^{r_1,\ldots,r_m} &= M_X(nC,\chi)_{r_m,\ldots,r_1}, \\
M_X(nC,\chi)^{r_1,\ldots,r_m} &= M_X(nC,\chi)_{r_m,\ldots,r_1}.
\end{align*}

The following lemma is straightforward.

**Lemma 2.3.** Let \( F \in M_X(nC,\chi)^{r_1,\ldots,r_m} \), and let \( F' \) and \( F'' \), respectively, lie in the following two exact sequences over \( X \):
\begin{align*}
0 \to F \to F' \to T' \to 0, \quad 0 \to T'' \to F'' \to F \to 0,
\end{align*}
where \( T' \) and \( T'' \) are 0-dimensional sheaves of \( O_{nC} \)-module. Then we have
\begin{align*}
F' \ (\text{resp. } F'') \in M_X(nC,\chi(\text{resp. } \chi''))^{r_1,\ldots,r_m},
\end{align*}
where \( \chi' = \chi + \text{length}(T') \) and \( \chi'' = \chi + \text{length}(T'') \).

**Remark 2.4.** Use the same notations as in Proposition 2.1. If \( F \in M_X(nC,\chi) \), i.e., \( F \) is pure, then the factors \( Q_i \) of the lower filtration are torsion-free over \( C \) while the factors \( R_i \) of the upper filtration may still contain torsion. We can take another upper filtration
\begin{align*}
0 = \tilde{F}^0 \subseteq \tilde{F}^1 \subseteq \cdots \subseteq \tilde{F}^m = F,
\end{align*}
such that \( \tilde{R}_i := \tilde{F}^i/\tilde{F}^{i-1} \) are torsion-free sheaves on \( C \) and every \( \tilde{F}^i \) is an extension of some 0-dimensional sheaf by \( F^i \) in the upper filtration. Actually, \( \tilde{R}_i \) is the maximal torsion-free quotient of \( \tilde{F}^i \otimes O_C \) and \( r(\tilde{R}_i) = r(R_i) \) for all \( 1 \leq i \leq m \) by Lemma 2.3.

There are also morphisms \( \tilde{g}_j^i : \tilde{R}_i(C) \to \tilde{R}_{i-1} \) induced by \( F \) for all \( i \) \((2 \leq i \leq m)\). But \( \tilde{g}_j^i \) is not necessary surjective. We call this filtration the torsion-free upper filtration of \( F \).

### 3 The case with reduced curve smooth

In this section, we let \( \beta_i \in K(C)_{\num} := K(C)/\sim \), where \( \sim \) is the numerical equivalence. We have the following proposition which implies the main theorem for \( C \) smooth.

**Proposition 3.1.** Let \( C \) be a smooth curve. Then we have
\begin{align*}
\dim M_X(nC,\chi)^{r_1,\ldots,r_m} &\leq \frac{n^2C^2}{2} + \frac{C.K_X}{2} \left( \sum_{j=1}^m r_j \right).
\end{align*}

In particular,
\begin{align*}
\dim M_X(nC,\chi) &\begin{cases}
\leq \frac{n^2C^2}{2} + \frac{n.C.K_X}{2} = g_{nC} - 1, & \text{if } C.K_X \leq 0, \\
= \frac{n^2C^2}{2} + \frac{n^2.C.K_X}{2} = n^2(g_{nC} - 1), & \text{if } C.K_X > 0.
\end{cases}
\end{align*}

**Proof.** Since the possible choices of \((\beta_1,\ldots,\beta_m)\in K(C)_{\num}^m\) such that \( r(\beta_j) = r_j \ (j = 1,\ldots,m)\) form a discrete set, Proposition 3.1 follows straightforwardly from the following lemma. \( \square \)

**Lemma 3.2.** For every \((\beta_1,\ldots,\beta_m)\in K(C)_{\num}^m\), we have
\begin{align*}
\dim M_X(nC,\chi)^{\beta_1,\ldots,\beta_m} &\leq \frac{n^2C^2}{2} + \frac{C.K_X}{2} \left( \sum_{j=1}^m r(\beta_j)^2 \right).
\end{align*}
Before proving Lemma 3.2, we need to define some stacks. For any $\eta \in K(C)_{num}$, denote by $\text{Coh}_{\eta}$ the stack of coherent sheaves on $C$ of the class $\eta$. Let $\underline{\eta} = (\eta_1, \ldots, \eta_m) \in K(C)_{m \text{ num}}^m$. Denote by $\overline{\text{Coh}}_{\underline{\eta}}$ the stack of chains $C_\bullet$, i.e.,

$$C_\bullet := [S_m \xrightarrow{d_m} S_{m-1}(C) \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} S_1((m-1)C)],$$

where $S_i$ are coherent sheaves on $C$ of the class $\eta_i$. Two chains $C_\bullet$ and $C'_\bullet$ are isomorphic if we have the following commutative diagram with vertical arrows all isomorphisms:

$$
\begin{array}{ccc}
S_m & \xrightarrow{d_m} & S_{m-1}(C) \\
\downarrow & & \downarrow \\
S'_m & \xrightarrow{d'_m} & S'_{m-1}(C)
\end{array}
\quad
\begin{array}{ccc}
\cdots & \xrightarrow{d_2} & S_1((m-1)C) \\
\downarrow & & \downarrow \\
\cdots & \xrightarrow{d'_2} & S'_1((m-1)C).
\end{array}
$$

Let $\overline{\text{Coh}}_{\underline{\eta}}$ be the stack of the pairs $(H, H_\bullet)$, where $H$ is a coherent sheaf on $C$ of the class $\sum_{j=1}^m \eta_j$ and $H_\bullet$ is a filtration

$$H_1 \subset H_2 \subset \cdots \subset H_m = H$$

satisfying $\lbrack H_k \rbrack = \sum_{j=1}^k \eta_j$ for $k = 1, \ldots, m$.

Define $\gamma_i := \beta_m - \beta_{m-i} \in [\mathcal{O}_X(iC)]$, $\alpha_i := \gamma_i - \gamma_{i-1}$ (with $\gamma_0 = \gamma_\alpha = 0$). We have several natural maps as follows:

$$\begin{array}{ccc}
\text{Coh}_{\beta_m} \times \cdots \times \text{Coh}_{\beta_1} & \xrightarrow{\pi_m} & \overline{\text{Coh}}_{\underline{\eta}} \\
\downarrow \pi_s & & \downarrow \pi_s \\
\overline{M}(nC, \chi)^{\beta_1 \cdots \beta_m} & \xrightarrow{\Phi} & \text{Coh}_{\alpha_1} \times \cdots \times \text{Coh}_{\alpha_m},
\end{array}$$

(3.1)

where $\Phi$ (resp. $\pi_m$) is defined by sending

$$[S_m \xrightarrow{d_m} S_{m-1}(C) \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} S_1((m-1)C)]$$

to $(H = S_m, H_i = \text{Ker } d_{m-i+1} \circ \cdots \circ d_m)$ (resp. $(S_m, \ldots, S_1)$), $\pi_q$ is defined by sending $(H, H_\bullet)$ to $(H_1, H_2/H_1, \ldots, H/H_{m-1})$, and finally $\pi_s$ is defined by sending $F$ to its factors $(R_m, \ldots, R_1)$ of the upper filtration.

**Lemma 3.3.** Let $\Phi, \pi_q, \pi_m, \pi_s, \beta$ and $\alpha$ be as in (3.1). Then we have

(i) $\Phi$ is an isomorphism;

(ii) $\dim \pi_q^{-1}((H_1, H_2/H_1, \ldots, H/H_{m-1})) = - \sum_{i<j} \chi_C(\alpha_j, \alpha_i);

(iii) for every $(S_m, \ldots, S_1) \in \text{Im}(\pi_m)$,

$$\dim \pi_m^{-1}((S_m, \ldots, S_1)) = \sum_{i=2}^m \dim \text{Hom}_{\mathcal{O}_C}(S_i, S_{i-1}(C));$$

(iv) $\text{Im}(\pi_s) \subset \text{Im}(\pi_m);

(v) for every $(R_m, \ldots, R_1) \in \text{Im}(\pi_s)$,

$$\dim \pi_s^{-1}((R_m, \ldots, R_1)) \leq \sum_{i<j} r(\beta_i)r(\beta_j)C^2 + \sum_{i=1}^{m-1} \dim \text{Hom}_{\mathcal{O}_C}(R_i, R_{i+1}(K_X)).$$

**Proof.** (i) is obvious.

(ii) is analogous to [6, Proposition 3.1(ii)], and hence we omit the proof here and refer to [6].

(iii) It is easy to see that

$$\dim \pi^{-1}_m((S_m, \ldots, S_1)) = \sum_{i=2}^m \dim \text{Hom}_{\mathcal{O}_C}(S_i, S_{i-1}(C))^\text{num},$$

where $\text{num}$ denotes the number of terms.
where $\text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))^{\text{sur}}$ is the subset of $\text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))$ consisting of surjective maps. The semicontinuity $\text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))^{\text{sur}}$ is always open in $\text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))$. Therefore, we have

$$\dim \text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))^{\text{sur}} = \dim \text{Hom}_{\mathcal{O}_C}(S_1, S_0(C)).$$

if $\text{Hom}_{\mathcal{O}_C}(S_1, S_0(C))^{\text{sur}}$ is not empty which is true for any $(S_0, \ldots, S_1) \in \text{Im}(\pi_m)$.

(iv) is also obvious.

We prove (v) by induction on $m$. For $m = 1$, there is nothing to prove.

For $m \geq 2$, we have the commutative diagram

$$\begin{CD}
\mathcal{M}_X(nC, \chi)^{\beta_1, \ldots, \beta_m} @> \pi''_s \times \text{Coh}_{\beta_m} \times \mathcal{M}_X((n-r(\beta_m))C, \chi - \chi(\beta_m))^{\beta_1, \ldots, \beta_{m-1}}
\end{CD}$$

where $\pi''_s$ is defined by sending $F$ to $(R_m, F^{m-1})$ with $0 = F^0 \subset F^1 \subset \cdots \subset F^m = F$ the upper filtration.

We can obtain $F$ as an extension of $R_m$ by $F^{m-1}$:

$$0 \to F^{m-1} \to F \to R_m \to 0.$$

Notice that every element $\sigma \in \text{Aut}_{\mathcal{O}_X}(F)$ induces an element in $\text{Aut}_{\mathcal{O}_X}(R_m) \times \text{Aut}_{\mathcal{O}_X}(F^{m-1})$ because $R_m \cong F \otimes \mathcal{O}_C$. Moreover, $\text{Aut}_{\mathcal{O}_X}(R_m) \cong \text{Aut}_{\mathcal{O}_C}(R_m)$. Hence we have

$$\text{Aut}_{\mathcal{O}_X}(F) \xrightarrow{f_1} \text{Aut}_{\mathcal{O}_C}(R_m) \times \text{Aut}_{\mathcal{O}_X}(F^{m-1})$$

and $\text{Hom}_{\mathcal{O}_X}(R_m, F^{m-1}) \subset \text{Ker}(f_1)$.

Denote by $\text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1})$ the stack of extensions of $F^{m-1}$ by $R_m$. For any extension $0 \to F^{m-1} \xrightarrow{h} F \xrightarrow{p} R_m \to 0$, an automorphism of $\sigma$ is the following commutative diagram:

$$\begin{CD}
0 @>>> F^{m-1} @> h > F @> p > R_m @>>> 0
\end{CD}$$

Denote by $\text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1})'$ the substack of $\text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1})$ consisting of the extensions $[0 \to F^{m-1} \to F \to R_m \to 0]$ such that $R_m \cong F \otimes \mathcal{O}_C$. Then for every $\sigma \in \text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1})'$, we have $\text{Ker}(f_1) = \text{Aut} (\sigma) \subset \text{Aut}_{\mathcal{O}_X}(F)$, and hence we have a surjective morphism of stacks

$$\text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1}') \xrightarrow{(\pi'')^{-1}} \text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1})).$$

On the other hand, $\text{Hom}_{\mathcal{O}_X}(R_m, F^{m-1}) \subset \text{Ker}(f_1) = \text{Aut} (\sigma)$ for every $\sigma \in \text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1}')$. Therefore,

$$\dim (\pi'')^{-1}((R_m, F^{m-1})) \leq \dim \text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1}') \leq \dim \text{Ext}^1_{\mathcal{O}_X}(R_m, F^{m-1}) - \dim \text{Hom}_{\mathcal{O}_X}(R_m, F^{m-1})$$

$$= -\chi_X(R_m, F^{m-1}) + \dim \text{Ext}^2_{\mathcal{O}_X}(R_m, F^{m-1}).$$

By the Serre duality, we have

$$\dim \text{Ext}^2_{\mathcal{O}_X}(R_m, F^{m-1}) = \dim \text{Hom}_{\mathcal{O}_X}(F^{m-1}, R_m(K_X)).$$

Since $R_m$ is a sheaf of $\mathcal{O}_C$-module, $F^{m-2} \subset \text{Ker}(g), \forall g \in \text{Hom}_{\mathcal{O}_X}(F^{m-1}, R_m(K_X))$. Hence,

$$\dim \text{Hom}_{\mathcal{O}_X}(F^{m-1}, R_m(K_X)) = \dim \text{Hom}_{\mathcal{O}_X}(R_m, R_m(K_X))$$
By Riemann-Roch theorem, we have
\[\chi_X(R_m, \mathcal{F}^{m-1}) = -\sum_{j=1}^{m-1} r(\beta_j)r(\beta_m)C^2.\]

Therefore,
\[\dim(\pi^n_u)^{-1}((R_m, \mathcal{F}^{m-1})) \leq \sum_{j=1}^{m-1} r(\beta_j)r(\beta_m)C^2 + \dim \text{Hom}_{\mathcal{O}_C}(R_{m-1}, R_m(K_X)).\]

On the other hand,
\[\dim \pi_{\alpha}^{-1}((R_m, \ldots, R_1)) \leq \dim(\pi_u^1)^{-1}((R_{m-1}, \ldots, R_1)) + \dim(\pi^n_u)^{-1}((R_m, \mathcal{F}^{m-1})).\]

We obtain (v) by applying the induction assumption to \(\mathcal{F}^{m-1}\).

**Proof of Lemma 3.2.** As \(C\) is smooth, \(\dim \text{Coh}(\alpha) = -\chi_C(\alpha, \alpha) = (g_C - 1)r(\alpha)^2\) for any \(\alpha \in K(C)\) such that \(\text{Coh}(\alpha)\) is not empty. Combining (i)--(v) of Lemma 3.3, we have
\[\dim \text{Ext}_{\mathcal{O}_C}^1(R_{i+1}, R_i(C)) = \dim \text{Hom}_{\mathcal{O}_C}(R_i, R_{i+1}(K_X)) = \dim \text{Hom}_{\mathcal{O}_C}(R_i, R_{i+1}(K_X)).\]

By the Serre duality on \(C\), we have
\[\dim \text{Hom}_{\mathcal{O}_C}(R_i, R_{i+1}(K_X)) = \dim \text{Ext}_{\mathcal{O}_C}^1(R_{i+1}, R_i(C)),\]

since \(\mathcal{O}_C(C + K_X)\) is the canonical line bundle on \(C\). Therefore, we have
\[\dim \text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m) \leq \sum_{i<j} r(\beta_i)r(\beta_j)C^2 - \sum_{i<j} \chi_C(\alpha_j, \alpha_i) - \sum_{i=2}^{m} \chi_C(R_i, R_{i-1}(C)).\]

It is easy to see that \(\alpha_i = [R_m - iC] - [R_{m-i}(iC)]\). Therefore, from (3.2) we have
\[\dim \text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m) \leq \sum_{i<j} r(\beta_i)r(\beta_j)C^2 - \sum_{i=1}^{m} \chi_C(\alpha_i, \alpha_i) - \sum_{i=1}^{m} \chi_C(R_i, R_i) - \sum_{i=2}^{m} \chi_C(R_i, R_{i-1}(C)) + \chi_C(R_{i-1}(C), R_i)).\]

Because we have
\[\chi_C(\alpha, \beta) + \chi_C(\beta, \alpha) = 2(1 - g_C)r(\alpha)r(\beta), \quad \forall \alpha, \beta \in K(C),\]

we obtain Lemma 3.2 from (3.3) by a direct computation.

**Remark 3.4.** Although \(\text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m)\) has finite dimension, it is not of finite type and actually contains infinite many connected components. In general even \(\text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m)\) is only locally of finite type. However, it will be an interesting question to ask whether the p-reduction of \(\text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m)\) to a finite field \(\mathbb{F}_p\) is of finite volume.

**Remark 3.5.** If \(F\) is a locally free sheaf of \(\mathcal{O}_{nC}\)-module, then \(m = n\) and \(R_n\) is torsion-free of rank 1 and hence \(R_{n-i} \cong R_n(-iC)\) for all \(i = 1, \ldots, n - 1\). Thus we have
\[\chi(F) = \sum_{j=1}^{n} \chi(R_j) = n\chi(R_n) - \frac{n(n - 1)}{2}C^2.\]

Therefore, if \(n \not\mid 2\chi\), there is no locally free sheaf of \(\mathcal{O}_{nC}\)-module in \(\text{Ext}_{\mathcal{O}_C}^1(nC, \chi^\beta_1, \ldots, \chi^\beta_m)\).
At the end of the section, we would like to state a result for extensions on $C \cong \mathbb{P}^1$ which generalizes [8, Lemma 5.2]. We will not need Lemma 3.6 in the rest of this paper, and the reader who only concerns the main theorem can also skip it.

**Lemma 3.6.** Let $C \cong \mathbb{P}^1$. Take an exact sequence on $X$:

$$0 \to O_C(s_1) \to E \to O_C(s_2) \to 0.$$  \hfill (3.4)

If $s_1 < s_2 - C^2$, then $E$ is a locally free sheaf of rank 2 on $C$ and hence splits into the direct sum of two line bundles.

**Proof.** We only need to show the following equality for all $s_1 < s_2 - C^2$:

$$\dim \text{Ext}^1_{O_C}(O_C(s_2), O_C(s_1)) = \dim \text{Ext}^1_{O_X}(O_C(s_2), O_C(s_1)).$$  \hfill (3.5)

Let LHS and RHS stand for the left-hand side of (3.5) and the right-hand side of (3.5), respectively. By the Serre duality, we have $LHS = \dim H^0(O_{C^*}(s_2 - s_1 - 2)) = \max\{0, s_2 - s_1 - 1\}$ and

$$\text{RHS} = -\chi(O_C(s_2), O_C(s_1)) + \dim \text{Hom}_{O_C}(O_C(s_2), O_C(s_1)) + \dim \text{Ext}^2_{O_C}(O_C(s_2), O_C(s_1))$$

$$= C^2 + \max\{0, s_1 - s_2 + 1\} + \dim \text{Hom}_{O_C}(O_C(s_1), O_C(s_2 + K_C)).$$

$$= C^2 + \max\{0, s_1 - s_2 + 1\} + \max\{0, s_2 - s_1 + 1 + C.K_C\}.$$  

If $C.K_C \geq -1$, then $C^2 \leq -1$ and we have

$$\begin{aligned}
\text{LHS} = \text{RHS} = s_2 - s_1 - 1, & \quad \text{if } s_1 \leq s_2 - 1, \\
\text{LHS} = \text{RHS} = 0, & \quad \text{if } s_2 - 1 < s_1 \leq s_2 - 1 - C^2.
\end{aligned}$$

If $C.K_C \leq -2$, then $C^2 \geq 0$ and we have LHS = RHS = $s_2 - s_1 - 1$, if $s_1 \leq s_2 - 1 - C^2$. Hence, (3.5) holds if $s_1 < s_2 - C^2$. \hfill \square

## 4 Proof of the main theorem

In this section, let $C$ be any integral curve on $X$, not necessarily smooth. We want to prove the following theorem.

**Theorem 4.1.** For any integral curve $C \subset X$, we have

$$\dim M_X(nC, \chi) \leq \frac{n^2.C^2}{2} + \frac{C.K_C}{2} \left( \sum_{j=1}^{n} r^2_j \right).$$

In particular,

$$\dim M_X(nC, \chi) \begin{cases} 
\leq \frac{n^2.C^2}{2} + \frac{n.C.K_C}{2} = g_{nC} - 1, & \text{if } C.K_C \leq 0, \\
= \frac{n^2.C^2}{2} + \frac{n^2.C.K_C}{2} = n^2(g_{C} - 1), & \text{if } C.K_C > 0.
\end{cases}$$

We proceed the proof of Theorem 4.1 by induction on the arithmetic genus $g_C$ of $C$. If $g_C = 0$, then $C \cong \mathbb{P}^1$ and Theorem 4.1 follows immediately from Proposition 3.1. Let us assume that $g_C > 0$ and $C$ is not smooth. Denote by $P$ a singular point of $C$ and $\theta \geq 2$ the multiplicity of $C$ at $P$. Let $\bar{X} \xrightarrow{\bar{f}} X$ be the blow-up at $P$ and $\bar{C} := f^{-1}(C)$. Then $\bar{C} = C_0 + \theta E$ with $C_0$ an integral curve with $g_{C_0} < g_C$ and $E \cong \mathbb{P}^1$ the exceptional divisors. We have analogous definitions for stacks $\overline{M}_X(n\bar{C}, \chi)$ and $M_{\bar{X}}(n\bar{C}, \chi)$ although $\bar{C}$ is not integral.

**Proposition 4.2.** Let $C' \subset X$ be any curve not necessarily integral. Let $\bar{C}' = f^{-1}(C')$. For every $F \in M_X(C', \chi)$, the pull-back $f^*F \in M_{\bar{X}}(\bar{C}', \chi)$. Moreover, $f_* f^* F \cong F$ and $f^* : M_X(C', \chi) \to M_{\bar{X}}(\bar{C}', \chi)$ is injective.
Proof. Since $\mathcal{F}$ is pure, we can take a locally free resolution of it, i.e.,

$$0 \to A \overset{f}{\to} B \to \mathcal{F} \to 0,$$

(4.1)

where $A$ and $B$ are locally free. Pull back (4.1) to $\tilde{X}$ and we obtain

$$f^*A \overset{f^*f}{\longrightarrow} f^*B \to f^*\mathcal{F} \to 0,$$

where $f^*A$ has to be injective because it is generically injective and $f^*A$ is a locally free sheaf over $\tilde{X}$. So we have a locally free resolution of length 1 for $f^*\mathcal{F}$, i.e.,

$$0 \to f^*A \overset{f^*f}{\longrightarrow} f^*B \to f^*\mathcal{F} \to 0,$$

(4.2)

Hence, $f^*\mathcal{F}$ has to be pure of dimension 1 and obviously is supported at $f^{-1}(C') = \tilde{C}'$.

Push forward (4.2) to $X$ and we obtain

$$0 \to f_*f^*A \overset{f_*f^*f}{\longrightarrow} f_*f^*B \to f_*f^*\mathcal{F} \to R^1f_*f^*A,$$

where $R^1f_*f^*A \cong A \otimes R^1f_*\mathcal{O}_X = 0$. As $R^i f_*\mathcal{O}_X = 0$, $\forall i > 0$ and $f_*\mathcal{O}_X \cong \mathcal{O}_X$, it is easy to see $f_*f^*A \cong A$, $f_*f^*B \cong B$, $f_*f^*\mathcal{F} \cong \mathcal{F}$. Also $\chi(f^*\mathcal{F}) = \chi(R^1f_*f^*\mathcal{F}) = \chi(f_*f^*\mathcal{F}) = \chi(\mathcal{F})$.

On the other hand, for any $F_1, F_2 \in \mathcal{M}_X(C', \chi)$, we have

$$\text{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}_1, f^*\mathcal{F}_2) \cong \text{Hom}_{\mathcal{O}_X}(F_1, f_*f^*\mathcal{F}_2) \cong \text{Hom}_{\mathcal{O}_X}(F_1, F_2).$$

The proposition is proved. \qed

We also have the following lemma.

Lemma 4.3. For every pure 1-dimensional sheaf $\mathcal{F}$ over $X$, $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, f_*\mathcal{O}_X) = 0$. In particular, any injective map $F_1 \overset{i_1}{\to} F_2$ with $F_2/F_1$ purely of dimension 1 remains injective after pulled back to $\tilde{X}$.

Proof. Pull back (4.1) to $\tilde{X}$ and we obtain

$$\text{Tor}_1^{\mathcal{O}_X}(A, f_*\mathcal{O}_X) \to \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, f_*\mathcal{O}_X) \to \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, f_*\mathcal{O}_X) \to f^*A \overset{f^*f}{\longrightarrow} f^*\mathcal{F} \to 0.$$

As we have already seen in the proof of Proposition 4.2, $f^*A$ is injective. $\text{Tor}_1^{\mathcal{O}_X}(A, f_*\mathcal{O}_X) = 0$ by local freeness of $A$. Hence, $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, f_*\mathcal{O}_X) = 0$. \qed

Lemma 4.4. Let $\mathcal{F} \in \mathcal{M}_X(nC, \chi)^{r_1, \ldots, r_m}$. Then

$$f^*\mathcal{F} \otimes \mathcal{O}_{nC_0} \in \mathcal{M}_{\tilde{X}}(nC_0, \chi_0)^{r_1, \ldots, r_m}$$

for some suitable $\chi_0$.

Proof. For $\mathcal{F} \in \mathcal{M}_X(nC, \chi)^{r_1, \ldots, r_m}$, we can take its torsion-free upper filtration $\tilde{\mathcal{F}}^\bullet$ as in Remark 2.4, and hence by Lemma 4.3, the pull-back of the filtration $f^*\tilde{\mathcal{F}}^\bullet$ is still a filtration of $f^*\mathcal{F}$ which generically coincides with the upper filtration of $f^*\mathcal{F} \otimes \mathcal{O}_{nC_0}$. Hence, we have $f^*\mathcal{F} \otimes \mathcal{O}_{nC_0} \in \mathcal{M}_{\tilde{X}}(nC_0, \chi_0)^{r_1, \ldots, r_m}$. \qed

Proof of Theorem 4.1. By Proposition 4.2, it is enough to estimate the dimension of $f^*(\mathcal{M}_X(nC, \chi)) \subset \mathcal{M}_{\tilde{X}}(nC, \chi)$. For any $\mathcal{F} \in \mathcal{M}_X(nC, \chi)^{r_1, \ldots, r_m}$, $f^*\mathcal{F}$ lies in the following canonical sequence:

$$0 \to \mathcal{F}_0 \to f^*\mathcal{F} \to \mathcal{F}_E \to 0,$$

(4.3)

where $\mathcal{F}_E$ is the torsion-free quotient of $f^*\mathcal{F} \otimes \mathcal{O}_{n\theta E}$ and $\mathcal{F}_0$ is the extension of the torsion-free quotient of $f^*\mathcal{F} \otimes \mathcal{O}_{nC_0}$ by the torsion of $f^*\mathcal{F} \otimes \mathcal{O}_{n\theta E}$. As $\mathcal{F}$ is pure of dimension 1 and so is $f^*\mathcal{F}$ by Proposition 4.2, the torsion of $f^*\mathcal{F} \otimes \mathcal{O}_{n\theta E}$ can only be supported at the intersection of $nC_0$ and $n\theta E$. Hence the schematic supports of $\mathcal{F}_E$ and $\mathcal{F}_0$ are $n\theta E$ and $nC_0$, respectively. Moreover, by Lemmas 2.3 and 4.4, we have $\mathcal{F}_0 \in \mathcal{M}_{\tilde{X}}(nC_0, \chi_0)^{r_1, \ldots, r_m}$ with some suitable $\chi_0$. 
Take the upper filtration $0 = \mathcal{F}^0 \subseteq \mathcal{F}^1 \subseteq \cdots \subseteq \mathcal{F}^m = \mathcal{F}$. Then we have
\[
f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \cong f^*(\mathcal{F}/\mathcal{F}^{m-i}) \otimes \mathcal{O}_{\theta E}, \quad \forall i = 1, \ldots, m-1,
\]
which is because
\[
f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \cong f^* \mathcal{F} \otimes \mathcal{O}_\mathcal{C} \otimes \mathcal{O}_{\theta E} \cong f^*(\mathcal{F} \otimes \mathcal{O}_\mathcal{C}) \otimes \mathcal{O}_{\theta E}.
\]
On the other hand, $\mathcal{F}/\mathcal{F}^{m-i}$ is not necessarily torsion-free. Let $\mathcal{T}$ be any 0-dimensional subsheaf supported at $P$ with length $t$. Then $f^* \mathcal{T}$ is a 1-dimensional sheaf with the schematic support $tE$. Hence, the schematic support of $f^*(\mathcal{F}/\mathcal{F}^{m-i})$ is
\[
\left( \sum_{j=0}^{i+1} r_{m-j} \right) \mathcal{C} + \eta_i E
\]
with $\eta_i \in \mathbb{N}$ the length of the 0-dimensional subsheaf of $\mathcal{F}/\mathcal{F}^{m-i}$ supported at $P$. Notice that $\eta_i = 0$ if and only if $\mathcal{F}/\mathcal{F}^{m-i}$ contains no torsion supported at $P$. Then the schematic support of $f^* \mathcal{F} \otimes \mathcal{O}_{\theta E}$ is
\[
\left( \sum_{j=0}^{i+1} r_{m-j} \right) \theta + \eta_i E.
\]
Define $\ell_i := (\sum_{j=0}^{i+1} r_{m-j}) \theta + \eta_i$. We can write
\[
f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \cong \mathbb{M}_{\mathcal{X}}(\ell_1 \mathcal{E}, \chi_1)_{r_1^{\infty} \cdots r_m^{\infty}},
\]
where $\theta^m \in \mathbb{N}_{>0}$, $\sum_{j=1}^m r_j^1 = \ell_1$ and $r_1^1 \leq \cdots \leq r_m^1 \in \mathbb{N}_{>0}$. We can also see that $\theta^m \leq \theta$ by Remark 2.2 and the fact that $f^* \mathcal{F} \otimes \mathcal{O}_{\theta E}$ is an $\mathcal{O}_{\theta E}$-module.

Analogously for $i = 1, \ldots, m$, we can write
\[
f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \cong \mathbb{M}_{\mathcal{X}}(\ell_i \mathcal{E}, \chi_i)_{r_i^{m-i+1} \cdots r_m^{m-i+1} \cdots r_1^{m-i+1} \cdots r_m^{\infty}},
\]
where $\theta^i \in \mathbb{N}_{>0}$, $\sum_{j=m-i+1}^m (\sum_{j=1}^i r_j^i) = \ell_i$ and $r_{m-i+1}^i \leq \cdots \leq r_m^i \in \mathbb{N}$.

In particular, $f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \cong \mathbb{M}_{\mathcal{X}}(n \theta \mathcal{E}, \chi_0)_{r_1^m \cdots r_1^1 \cdots r_m^1 \cdots r_m^{\infty}}$.

Now we want to show the following inequality:
\[
\theta^j \leq \theta, \quad \forall j = 1, \ldots, m. \tag{4.4}
\]
To show (4.4), it is enough to show that the kernel of the map $f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \to f^* \mathcal{F} \otimes \mathcal{O}_{(i-1)\theta E}$ is an $\mathcal{O}_{\theta E}$-module for $i = 1, \ldots, m$. But this can easily be seen from the following exact sequence:
\[
f^* \mathcal{F} \otimes \mathcal{O}_{\theta E}(-i \theta E) \to f^* \mathcal{F} \otimes \mathcal{O}_{\theta E} \to f^* \mathcal{F} \otimes \mathcal{O}_{(i-1)\theta E} \to 0.
\]
Moreover, we have
\[
\sum_{j=k}^m \left( \sum_{i=1}^\theta r_j^i \right) = \ell_{m-k+1} \geq \theta \left( \sum_{j=k}^m r_j \right), \quad k = 2, \ldots, m, \quad \sum_{j=1}^m \left( \sum_{i=1}^\theta r_j^i \right) = \theta \left( \sum_{j=1}^m r_j \right). \tag{4.5}
\]
Notice that we also have $\mathcal{F}_E \in \mathbb{M}_{\mathcal{X}}(n \theta \mathcal{E}, \chi_0)_{r_1^m \cdots r_1^1 \cdots r_m^1 \cdots r_m^{\infty}}$ by Lemma 2.3.

Denote by $\mathbb{A}$ the set of all $f := (r_1^1, \ldots, r_1^\theta, \ldots, r_m^1, \ldots, r_m^\theta)$ satisfying both (4.4) and (4.5). We have the map between stacks induced by (4.3), i.e.,
\[
f^* (\mathbb{M}_{\mathcal{X}}(n \mathcal{C}, \chi)^{r_1^m \cdots r_m^{\infty}}) \to \prod_{\mathcal{X}} \mathbb{M}_{\mathcal{X}} \times \prod_{\mathcal{X} \in \mathbb{A}} \mathbb{M}_{\mathcal{X}}(n \mathcal{E}, \chi^\mathbb{A}).
\]
It is easy to see that the fiber of $\pi_E$ is of dimension no higher than
\[-\chi_X(\mathcal{F}_E, \mathcal{F}_0) = n \mathcal{C}_0 \cdot n(\theta E) = n^2 \mathcal{C}_0 \cdot \theta E,
\]
since $\text{Ext}^2(F_E,F_0) = 0$. By applying the induction assumption to $C_0$ and $E$, we obtain
\[
\dim M_X(nC, \chi)^{r_1, \ldots, r_m} \leq n^2 C_0. \theta E + \frac{n^2 C_0^2}{2} + \frac{C_0 K_X}{2} \left( \sum_{j=1}^{m} r_j^2 \right)
\]
\[
+ \max_{\xi \in \Delta} \left\{ \frac{n^2 \theta^2 E^2}{2} + \frac{E \cdot K_X}{2} \left( \sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} r_j^i \right) \right) \right\}.
\]
(4.6)

Since $E K_X = -1 < 0$, together with the following Lemma 4.5 we obtain
\[
\dim M_X(nC, \chi)^{r_1, \ldots, r_m} \leq n^2 C_0. \theta E + \frac{n^2 C_0^2}{2} + \frac{C_0 K_X}{2} \left( \sum_{j=1}^{m} r_j^2 \right)
\]
\[
+ \frac{n^2 \theta^2 E^2}{2} + \frac{\theta E \cdot K_X}{2} \left( \sum_{j=1}^{m} r_j^2 \right)
\]
\[
= \frac{n^2 (C_0 + \theta E)^2}{2} + \frac{(C_0 + \theta E) K_X}{2} \left( \sum_{j=1}^{m} r_j^2 \right)
\]
\[
= \frac{n^2 (C)^2}{2} + \frac{(\tilde{C}) K_X}{2} \left( \sum_{j=1}^{m} r_j^2 \right).
\]
(4.7)

By $\tilde{C}^2 = C^2$ and $\tilde{C} K_X = \tilde{C}. (f^* K_X + E) = C K_X$, we obtain this theorem. \hfill \Box

**Lemma 4.5.** Let $r_m \geq r_{m-1} \geq \ldots \geq r_1$ and let $\theta \in \mathbb{Z}_{>0}$. If we have real numbers
\[
r_{m}^{1}, r_{m}^{2}, \ldots, r_{m}^{\theta}, r_{m-1}^{1}, \ldots, r_{m-1}^{\theta}, r_{m-2}^{1}, \ldots, r_{1}^{\theta},
\]
such that
\[
\sum_{j=k}^{m} \left( \sum_{t=1}^{\theta^i} r_j^i \right) \geq \theta \left( \sum_{j=1}^{m} r_j \right), \quad k = 2, \ldots, m, \quad \sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} r_j^i \right) = \theta \left( \sum_{j=1}^{m} r_j \right),
\]
then we have
\[
\sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} (r_j^i)^2 \right) \geq \theta \left( \sum_{j=1}^{m} (r_j)^2 \right)
\]
and the equality holds if and only if $r_j^i = r_j$ for all $t = 1, \ldots, \theta$ and $j = 1, \ldots, m$.

In particular, we can remove zeros in $\{r_j^i\}$ and ask the rest of them to be positive, i.e.,
\[
r_{m}^{1}, r_{m}^{2}, \ldots, r_{m}^{\theta}, r_{m-1}^{1}, \ldots, r_{m-1}^{\theta}, r_{m-2}^{1}, \ldots, r_{1}^{\theta},
\]
with $\theta^i \leq \theta$ for $i = 1, \ldots, m$. Then
\[
\sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} (r_j^i)^2 \right) \geq \theta \left( \sum_{j=1}^{m} (r_j)^2 \right)
\]
and the equality holds if and only if $\theta^i = \theta$ and $r_j^i = r_j$ for all $t = 1, \ldots, \theta$ and $j = 1, \ldots, m$.

**Proof.** Let $\epsilon_j := r_j^i - r_j$. Then we have
\[
\sum_{j=k}^{m} \left( \sum_{t=1}^{\theta^i} \epsilon_j^i \right) \geq 0, \quad k = 2, \ldots, m, \quad \sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} \epsilon_j^i \right) = 0.
\]
Hence,
\[
\sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} (r_j^i)^2 \right) = \sum_{j=1}^{m} \left( \sum_{t=1}^{\theta^i} (r_j + \epsilon_j^i)^2 \right)
\]
\[
\theta \left( \sum_{j=1}^{m} (r_j)^2 \right) + \sum_{j=1}^{m} \left( \sum_{t=1}^{\theta} (\epsilon_{tj})^2 \right) + 2 \sum_{j=1}^{m} \left( \theta \sum_{t=1}^{\theta} (\epsilon_{tj}) \right) \\
\geq \theta \left( \sum_{j=1}^{m} (r_j)^2 \right),
\]

where the last inequality is because \( r_k \geq r_{k-1} \) and \( \sum_{j=k}^{m} (\sum_{t=1}^{\theta} \epsilon_{tj}) \geq 0 \). It is easy to see that the equality holds if and only if \( \epsilon_{tj}^i = 0 \) for all \( t = 1, \ldots, \theta \) and \( j = 1, \ldots, m \).

Remark 4.6. As Theorem 4.1 only concerns the dimension, the assumption that the base field \( k \) is algebraically closed can be removed.

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