VALERY ISAEV

Abstract. In this paper, we propose an abstract definition of dependent type theories as essentially algebraic theories. One of the main advantages of this definition is its composability: simple theories can be combined into more complex ones, and different properties of the resulting theory may be deduced from properties of the basic ones. We define a category of algebraic dependent type theories which allows us not only to combine theories but also to consider equivalences between them. We also study models of such theories and show that one can think of them as contextual categories with additional structure.

1. Introduction

Type theories with dependent types originally were defined by Per Martin-Löf, who introduced several versions of the system [10, 8, 9]. There were also several theories and extensions of Martin-Löf’s theory proposed by different authors ([3, 6] to name a few). These theories may have different inference rules, different computation rules, and different constructions. Many of these theories have common parts and similar properties, but the problem is that there is no general definition of a type theory such that all of these theories would be a special case of this definition, so that their properties could be studied in general and applied to specific theory when necessary. In this paper we propose such a definition based on the notion of essentially algebraic theories.

Another problem of the usual way of defining type theories is that they are not composable. Some constructions in type theories are independent of each other (such as Π, Σ, and \(Id\) types), and others may dependent on other constructions (such as universes), so we could hope that we can study these constructions independently (at least if they are of the first kind) and deduce properties of combined theory from the properties of these basic constructions. But this is not the way it is usually done. For example, constructing models of dependent type theories is a difficult task because of the so called coherence problem. There are several proposed solutions to this problems, but the question we are interested in is how to combine them. Often only the categorical side of the question is considered, but some authors do consider specific theories [13, 12], and the problem in this case is that their work cannot be applied to other similar theories (at least formally).

When defining a type theory there are certain questions to be addressed regarding syntactic traits of the theory. One such question is how many arguments to different construction can be omitted and how to restore them when constructing a model of the theory. For example, we want to define application as a function of two arguments \(app(f, a)\), but sometimes it is convenient to have additional arguments which allows to infer a type of \(f\). It is possible to prove that additional information in the application term may be omitted (for example, see [13]), but it is a nontrivial
task. Another question of this sort is whether we should use a typed or an untyped equality. Typed equality is easier to handle when defining a model of the theory, but untyped is closer to actual implementation of the language. Algebraic approach allows us to separate these syntactic details from essential aspects of the theory.

Yet another problem is that some constructions may be defined in several different ways. For example, $\Sigma$ types can be defined using projections (Example 5.6) and using an eliminator (Example 5.7). The question then is whether these definitions are equivalent in some sense. The difficulty of this question stems from the fact that some equivalences may hold in one definition judgmentally, but in the other only propositionally; so it may be difficult (or impossible) to construct a map from the first version of the definition to the second one.

In this paper, using the formalism of essentially algebraic theories, we introduce the notion of algebraic dependent type theories which provide a possible solution the problems described above. We define a category of algebraic dependent type theories. Coproducts and more generally colimits in this category allow us to combine simple theories into more complex ones. For example, the theory with $\Sigma$, $\Pi$ and $Id$ types may be described as coproduct $T_\Sigma \coprod T_\Pi \coprod T_{Id}$ where $T_\Sigma$, $T_\Pi$ and $T_{Id}$ are theories of $\Sigma$, $\Pi$ and $Id$ types respectively.

There is a natural notion of a model of an essentially algebraic theory. Thus the algebraic approach to defining type theories automatically equips every type theory with a (locally presentable) category of its models. We will show that models of the initial theory are precisely contextual categories, and that models of an arbitrary theory are contextual categories with an additional structure (which depends on the theory). An example of a general construction that works for all theories with enough structure is the construction of a model structure on the category of models described in [4].

Since we have a category of type theories, there is a natural notion of equivalence between them, namely the isomorphism. In most cases this equivalence is too strong, so it is necessary to consider weaker notions of equivalence, but in some cases it might be useful. For example, if two theories differ only by the amount of arguments to some of the constructions, then they are isomorphic (assuming omitted arguments can be inferred from the rest). A weaker notion of equivalence of theories is Morita equivalence. Two theories are Morita equivalent if there is a Quillen equivalence between the categories of models of these theories. We will not consider this notion in this paper.

One of the key features of type theories is the use of variable bindings in terms. Such bindings are represented by de Bruijn indices in our approach. If we want to use ordinary representation of terms with named bindings, then we need to put some additional structure on out theories. We call theories with such structure stable. This additional structure allows us to use all constructions of the theory in every context and guarantees that they commute with substitutions.

The paper is organized as follows. In section 2, we define the category of partial Horn theories and discuss its properties. In section 3, we define algebraic dependent type theories in terms of partial Horn theories and prove that the category of models of the initial theory with substitution is equivalent to the category of contextual categories. In section 4, we define the concept of stable theories, which formalizes the idea that every construction in a type theory can be lifted to a larger context. Finally, we give a few standard examples of algebraic dependent type theories.
2. Partial Horn theories

There are several equivalent ways of defining essentially algebraic theories ([1], [2], [11], [5, D 1.3.4]). We use approach introduced in [11] under the name of partial Horn theories since it is the most convenient one. We define morphisms of partial Horn theories in terms of morphisms of monads and left modules over them. In this section we review necessary for our development parts of the theory of monads, left modules over them and partial Horn theories. We also define algebraic dependent type theories as certain partial Horn theories.

2.1. Monads and left modules over them. We recall definitions of monads and left modules over a monad. For our purposes the following definitions (see [7]) will be more convenient than the ordinary ones.

**Definition 2.1.** A monad \((T, \eta, (\cdot)^*)\) on a category \(C\) consists of a function \(T : Ob(C) \to Ob(C)\), a function \(\eta\) that to each \(A \in Ob(C)\) assign a morphism \(\eta_A : A \to T(A)\), and a function that to each \(A, B \in Ob(C)\) assigns a function \((\cdot)^* : \text{Hom}_C(A, T(B)) \to \text{Hom}_C(T(A), T(B))\), satisfying the following conditions:

- \(\eta_A = id_{T(A)}\).
- For every \(\rho : A \to T(B)\), \(\rho^* \circ \eta_A = \rho\).
- For every \(\rho : A \to T(B)\), \(\sigma : B \to T(C)\), \(\sigma^* \circ \rho^* = (\sigma^* \circ \rho)^*\).

A left module \((M, (\cdot)^\circ)\) over a monad \((T, \eta, (\cdot)^*)\) with values in a category \(D\) consists of a function \(M : Ob(C) \to Ob(D)\) and a function that to each \(A, B \in Ob(C)\) assigns a function \((\cdot)^\circ : \text{Hom}_C(A, T(B)) \to \text{Hom}_D(M(A), M(B))\), satisfying the following conditions:

- \(\eta_A^\circ = id_{M(A)}\).
- For every \(\rho : A \to T(B)\), \(\sigma : B \to T(C)\), \(\sigma^\circ \circ \rho^\circ = (\sigma^* \circ \rho)^\circ\).

These data and axioms imply that \(T\) and \(M\) are functorial: if \(f : A \to B\), then we can define \(T(f)\) as \((\eta_B \circ f)^*\) and \(M(f)\) as \((\eta_B \circ f)^\circ\). Moreover, \(\eta, (\cdot)^*\) and \((\cdot)^\circ\) are natural.

**Definition 2.2.** A morphism of monads \((T, \eta, (\cdot)^*)\) and \((T', \eta', (\cdot)^*)\) on \(C\) is a function \(\alpha\) that to each \(A \in Ob(C)\) assigns a morphism \(\alpha_A : T(A) \to T'(A)\), satisfying the following conditions:

- \(\alpha_A \circ \eta_A = \eta_A'\).
- For every \(\rho : A \to T(B)\), \(\alpha_B \circ \rho^* = (\alpha_B \circ \rho)^* \circ \alpha_A\).

Let \((M, (\cdot)^\circ)\) and \((M', (\cdot)^\circ)\) be left modules with values in \(D\) over monads \((T, \eta, (\cdot)^*)\) and \((T', \eta', (\cdot)^*)\) respectively. A morphism between them is a pair of functions \((\alpha, \beta)\), where \(\alpha\) is a morphism of monads \(T\) and \(T'\), and \(\beta\) assigns to each \(A \in Ob(C)\) a morphism \(\beta_A : M(A) \to M'(A)\), such that, for every \(\rho : A \to T(B)\), \(\beta_B \circ \rho^\circ = (\alpha_B \circ \rho)^\circ \circ \beta_A\).

These data and axioms imply that \(\alpha\) and \(\beta\) are natural.

Let \(S\) be a set of sorts, and let \((T, \eta, (\cdot)^*)\) be a monad on the category of \(S\)-sets. We think of elements of \(T(V)_s\) as terms of sort \(s\) with free variables in \(V\). Given \(t \in T(V)_s\) and \(\rho : V \to T(V')\), we will write \(t[\rho] \in T(V')_s\) for \(\rho^*(t)\). Let \((F, (\cdot)^\circ)\) be a left module over \(T\) with values in \(\text{Set}\). We think of elements of \(F(V)\) as formulas with free variables in \(V\). Given \(\varphi \in F(V)\) and \(\rho : V \to T(V')\), we will write \(\varphi[\rho] \in F(V')\) for \(\rho^\circ(\varphi)\).
Let $T : \text{Set}^S \rightarrow \text{Set}^S$ be a monad. Then a free variables structure on $T$ is a function $FV$ that to each $t \in T(V)$ assigns a subset of $V$, that is $FV(t) \subseteq V$, called the set of free variables of $t$. This function must satisfy the following conditions:

$$FV(\eta(x)) = x$$
$$FV(t[\rho]) = \bigcup_{x \in FV(t)} FV(\rho(x))$$

Let $F : \text{Set}^S \rightarrow \text{Set}$ be a left module over $T$. Then a free variables structure on $F$ is a function $FV$ that to each $\varphi \in F(V)$ assigns a subset of $V$, that is $FV(\varphi)$, called the set of free variables of $\varphi$. This function must satisfy the following condition:

$$FV(\varphi[\rho]) = \bigcup_{x \in FV(\varphi)} FV(\rho(x))$$

A module of formulas over $T$ is a left module $F$ over $T$ together with a function $\wedge : F(V) \times F(V) \rightarrow F(V)$ and a constant $\top \in F(V)$ for every $V \in \text{Set}^S$, satisfying the following conditions:

- For every $\rho : V \rightarrow T(V')$, $\top[\rho] = \top$.
- For every $\rho : V \rightarrow T(V')$, $(\varphi \wedge \psi)[\rho] = \varphi[\rho] \wedge \psi[\rho]$.

For every monad $T$ on $\text{Set}^S$ we define a left module $E$ with values in $\text{Set}$. For every $V \in \text{Set}^S$, let $E(V)$ be the set of triples $(s,t,t')$, where $s \in S$, and $t,t' \in T(V)$. For every $\rho : V \rightarrow T(V')$ and $(s,t,t') \in E(V)$, we let $(s,t,t')[\rho] = (s,t[\rho],t'[\rho])$. We think of $(s,t,t')$ as a formula asserting the equality of terms $t$ and $t'$. We write $t =_s t'$ (or simply $t = t'$) for $(s,t,t')$. A module of formulas with equality over $T$ is a module $F$ of formulas over $T$ together with a morphism $\epsilon : E \rightarrow F$.

**Definition 2.3.** A monadic presentation of a partial Horn theory is a triple $(T,F,\mu)$, where $T : \text{Set}^S \rightarrow \text{Set}^S$ is a finitary monad with a free variables structure, $F : \text{Set}^S \rightarrow \text{Set}$ is a finitary module of formulas with equality and a free variables structure, and $\mu_V : T(V) \times F(V) \rightarrow T(V)$ is a function such that the following conditions hold:

- For every $\rho : V \rightarrow T(V')$, $\mu_V(t,\varphi)[\rho] = \mu_V(t[\rho],\varphi[\rho])$.
- $\mu_V(t,\top) = t$.
- $\mu_V(t,\varphi \wedge \psi) = \mu_V(\mu_V(t,\varphi),\psi)$.

A morphism of triples $(T,F,\mu)$ and $(T',F',\mu')$ is a morphism $f$ of left modules $(T,F)$ and $(T',F')$ such that $f$ preserves free variables, equality, $\top$, $\wedge$ and $\mu$. The category of monadic presentations of partial Horn theories with $S$ as the sort of sorts is denoted by $\text{PMnd}_S$.

**2.2. The category of partial Horn theories.** Let $S$ be a set of sorts, $T : \text{Set}^S \rightarrow \text{Set}^S$ a monad with a free variables structure, and $P$ a set of predicate symbols together with a function that to each $R \in P$ assigns its signature $R : s_1 \times \ldots \times s_n$, where $s_1,\ldots,s_n \in S$.

Let $F$ be a set of function symbols together with a function that to each $\sigma \in F$ assigns its signature $\sigma : s_1 \times \ldots \times s_n \rightarrow s$, where $s_1,\ldots,s_n,s \in S$. Then we can define an example of a monad over $\text{Set}^S$. For each $V \in \text{Set}^S$ we can define a set $\text{Term}_F(V)_s$ of terms of sort $s$ inductively:

- If $x \in V_s$, then $x \in \text{Term}_F(V)_s$. 

If \( \sigma : s_1 \times \ldots \times s_n \to s \) and \( t_i \in \text{Term}_F(V)_s \), then \( \sigma(t_1, \ldots, t_n) \in \text{Term}_F(V)_s \).

If \( \rho : V \to \text{Term}_F(V') \), then substitution is defined as follows:

\[
x[\rho] = \rho(x) \\
\sigma(a_1, \ldots, a_k)[\rho] = \sigma(a_1[\rho], \ldots, a_k[\rho])
\]

Thus \( \text{Term}_F : \mathbf{Set}^S \to \mathbf{Set}^S \) is a monad, which we call the standard monad (over \( F \)).

An atomic formula with free variables in \( V \) is an expression either of the form \( t_1 = t_2 \) (we will usually omit \( s \) in the notation), where \( s \in \mathcal{S} \) and \( t_1, t_2 \in T(V)_s \), or of the form \( R(t_1, \ldots, t_n) \), where \( R \in \mathcal{P} \), \( R : s_1 \times \ldots \times s_n \) and \( t_i \in T(V)_{s_i} \).

A Horn formula (over \( \mathcal{P} \)) with free variables in \( V \) is an expression of the form \( \varphi_1 \land \ldots \land \varphi_n \) where \( \varphi_i \) are atomic formulas. If \( n = 0 \), then we write such a formula as \( \top \). The set of Horn formulas with free variables in \( V \) is denoted by \( \text{Form}_F(V) \).

If \( \varphi \in \text{Form}_F(V) \) and \( \rho : V \to T(V') \), then we will write \( \varphi[\rho] \) for a formula defined as follows:

\[
(t = t')[\rho] = (t[\rho] = t'[\rho]) \\
R(t_1, \ldots, t_k)[\rho] = R(t_1[\rho], \ldots, t_k[\rho]) \\
(\varphi_1 \land \ldots \land \varphi_n)[\rho] = \varphi_1[\rho] \land \ldots \land \varphi_n[\rho]
\]

Thus \( \text{Form}_F \) is a left module over \( T \). Moreover, a free variables structure on \( \text{Form}_F \) is defined as follows:

\[
\text{FV}(t = t') = \text{FV}(t) \cup \text{FV}(t') \\
\text{FV}(R(t_1, \ldots, t_k)) = \text{FV}(t_1) \cup \ldots \cup \text{FV}(t_k) \\
\text{FV}(\varphi_1 \land \ldots \land \varphi_n) = \text{FV}(\varphi_1) \cup \ldots \cup \text{FV}(\varphi_n)
\]

A Horn sequent is an expression of the form \( \varphi \vdash \psi \), where \( \varphi \) and \( \psi \) are Horn formulas with free variables in \( V \). We will often write \( \varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k \) instead of \( \varphi_1 \land \ldots \land \varphi_n \vdash \psi_1 \land \ldots \land \psi_k \). A partial Horn theory is a set of Horn sequents. The rules of partial Horn logic are listed below. If \( \mathcal{A} \) is a partial Horn theory, then a theorem of \( \mathcal{A} \) is a sequent derivable from \( \mathcal{A} \) in this logic.

\[
\varphi \vdash \varphi \quad \text{(b1)} \quad \varphi \vdash \psi \quad \psi \vdash \chi \quad \text{(b2)} \quad \varphi \vdash \top \quad \text{(b3)}
\]

\[
\varphi \land \psi \vdash \varphi \quad \text{(b4)} \quad \varphi \land \psi \vdash \psi \quad \text{(b5)} \quad \varphi \vdash \psi \quad \varphi \vdash \psi \land \chi \quad \text{(b6)}
\]

\[
\varphi \vdash x \quad \text{(a1)} \quad x = y \land \varphi \vdash [x/y] \varphi[x/y] \quad \text{(a2)}
\]

\[
\varphi \vdash \psi \quad \varphi[t/x] \vdash \psi[t/x] \quad \text{(a3)}
\]

Here, \( t/x \) denotes a function \( \rho : V \to T(V \cup V') \) such that \( \rho(x) = t \) and \( \rho(y) = y \) if \( y \neq x \).
Note that this set of rules is a generalization of the one described in [11]. If $T$ is the standard monad $\text{Term}_T$, then these rules are equivalent to the rules from [11]. In particular, the following sequents are derivable if $x \in FV(t)$:

\[ R(t_1, \ldots, t_k) \vdash t = t_i \]  \hspace{1cm} (a4)
\[ t_1 = t_2 \vdash t_i = t_i \]  \hspace{1cm} (a4')
\[ t[t'/x] = t' \]  \hspace{1cm} (a5)

We will need the following lemmas from [11]:

**Lemma 2.4.** For every $u_1, v_i \in T(V)_s$, and $t \in T(\{x_1 : s_1, \ldots, x_n : s_n\})_s$, sequents $u_1 = v_1 \land \ldots \land u_n = v_n \vdash t[x_i \mapsto u_i] \iff t[x_i \mapsto v_i]$ are theorems of any theory.

**Lemma 2.5.** Sequent $y = x \land \varphi[y/x] \vdash \varphi$ is a theorem of any theory.

Using the previous lemma we prove the following fact:

**Lemma 2.6.** For every $u_i, v_i \in T(V)_s$, and $\varphi \in \text{Form}_P(\{x_1 : s_1, \ldots, x_n : s_n\})$, sequent $u_1 = v_1 \land \ldots \land u_n = v_n \land \varphi[x_i \mapsto u_i] \vdash \varphi[x_i \mapsto v_i]$ is a theorem of any theory.

**Proof.** By the previous lemma we have $y_n = x_n \land \varphi[y_n/x_n] \vdash y_n = x_n \land \varphi[y_n/x_n]$, then we get sequent $y_{n-1} = x_{n-1} \land y_n = x_{n-1} \land \varphi[y_n/x_n, y_{n-1}/x_{n-1}] \vdash y_{n-1} = x_{n-1} \land \varphi[y_n/x_n, y_{n-1}/x_{n-1}]$. By (b2) we get sequent $y_{n-1} = x_{n-1} \land y_n = x_{n-1} \land \varphi[y_n/x_n, y_{n-1}/x_{n-1}] \vdash \varphi$. Repeating this argument we can conclude that $y_1 = x_1 \land \ldots \land y_n = x_1 \land \varphi[y_1/x_1, \ldots, y_n/x_n] \vdash \varphi$. By (a3) we conclude that the required sequent is derivable. \hfill \square

Now we define a functor $PT : \text{Set}^S \rightarrow \text{Set}^S$ of partial terms. We let $PT(V)_s$ to be the set of expressions $t|_\varphi$ where $t \in T(V)_s$ and $\varphi \in \text{Form}_P(V)$. If $\varphi = \top$, then we will write $t|_\varphi$ simply as $t$. If $p \in PT(V)_s$, $p = t|_\varphi$ and $\psi \in \text{Form}_P(V)$, then we will write $p|_\psi$ for $t|_{\varphi \land \psi}$.

We will use the following abbreviations:

- $t \downarrow$ means $t = t$
- $\varphi \downarrow t \Downarrow s$ means $\varphi \land t \Downarrow s$
- $\varphi \Downarrow t \Downarrow s$ means $\varphi \land t \Downarrow s$ and $\varphi \Downarrow s \Downarrow t \Downarrow s$
- $\varphi \Downarrow \psi$ means $\varphi \Downarrow \psi$ and $\varphi \Downarrow \psi$
- $R(t_1|_{\varphi_1}, \ldots, t_k|_{\varphi_k})$ means $R(t_1, \ldots, t_k) \land \varphi_1 \land \ldots \land \varphi_k$
- $t|_{\varphi} = s|_\psi$ means $t = s \land \varphi \land \psi$
- $t|_{\varphi} \Downarrow$ means $t \Downarrow \land \varphi$
- $\chi \Downarrow t|_{\varphi} \Downarrow \Downarrow s|_\psi$ means $\chi \land t|_{\varphi} \Downarrow \Downarrow s|_\psi \Downarrow t \Downarrow s$
- $\chi \Downarrow t|_{\varphi} \Downarrow \Downarrow s|_\psi$ means $\chi \land t|_{\varphi} \Downarrow \Downarrow t \Downarrow s$ and $\chi \land s|_\psi \Downarrow t \Downarrow s \land \varphi$


We also can define $f$, call it the monadic presentation of partial Horn theory. It is easy to see that $(\_)$ with these properties.

Then there is a morphism of left modules $f$ such that $FV(t_1, \ldots, t_k)[\varphi] = \varphi_1[\varphi_2(\ldots)]$ for every $\varphi, \varphi_2, \ldots, \varphi_k \in \text{Form}_P(V)$ and of a left module over it on $P$ and let $P, F, \mu$ over $S$. For every $x \in V_s, \eta_{V}(x)$ is the equivalence class of $x|_T$. Substitution functions respect equivalence relations, and it is easy to see that they define a structure of a monad and of a left module over it on $T$ and $F$. For every $t, t' \in PT(V)_s$, $t \sim t'$ if and only if $FV(t) = FV(t')$ and $t \equiv t'$ is a theorem of $T$. Then let $P(V)_s = PT(V)_s/\sim$ and $F(V) = \text{Form}_P(V)/\sim$. For every $x \in V_s, \eta_{V}(x)$ is the equivalence class of $x|_T$. Substitution functions respect equivalence relations, and it is easy to see that they define a structure of a monad and of a left module over it on $T$ and $F$. For every $t, t' \in PT(V)_s$, $e(s, t, t')$ is the equivalence class of $t \sim t'$. For every $t \in T(V)_s$ and $\varphi \in F(V)$, let $\mu_{V}(t, \varphi) = t|_{\varphi}$. It is easy to see that $(P, F, \mu)$ satisfies axioms of monadic presentations. We will call it the monadic presentation of partial Horn theory $T$ and denote by $P(T)$.

The category of partial Horn theories over $S$ has tuples $(T, P, A)$ as objects, where $T$ is a finitary monad with a free variables structure, $P$ is a set of predicate symbols and $A$ is a set of axioms. Morphisms of partial Horn theories $T$ and $T'$ are morphisms of their monadic presentations. The category of partial Horn theories over $S$ is denoted by $\text{Th}_S^T$.

**Proposition 2.7.** Let $T = (T, P, A)$ and $T' = (T', P', A')$ be partial Horn theories, and let $P(T) = (P, F, \mu)$ and $P(T') = (P', F', \mu')$ be their monadic presentations. To construct a morphism of these theories, it is enough to specify the following data:

- A morphism of monads $\alpha : T \to T'$ that preserves free variables.
- For every $R \in P$, $R : s_1 \times \ldots \times s_k$, a formula $\beta(R) \in F'(\{x_1 : s_1, \ldots, x_k : s_k\})$ such that $FV(\beta(R)) = \{x_1, \ldots, x_k\}$.

Then there is a morphism of left modules $f : (T, \text{Form}_P) \to (T', \text{Form}_P')$ such that $f(\sigma(x_1, \ldots, x_k)) = \alpha(\sigma)$ and $f(R(x_1, \ldots, x_k)) = \beta(R)$. If $f$ preserves axioms of $T$, then it extends to a morphism of theories. Moreover, there is at most one morphism with these properties.

**Proof.** Morphism $f$ is already defined on terms, and we can define it on formulas as follows:

$$f(a = b) = f(a) = f(b)$$
$$f(R(a_1, \ldots, a_k)) = \beta(R)[x_i \mapsto f(a_i)]$$
$$f(\varphi_1 \land \ldots \land \varphi_n) = f(\varphi_1) \land \ldots \land f(\varphi_n)$$

We also can define $f$ on partial terms:

$$f(t|_{\varphi}) = f(t)|_{f(\varphi)}$$
It is easy to see that \( f \) preserves substitution. Thus to prove that \( f \) extends to a morphism of theories, we only need to show that it preserves theorems of \( T \). By assumption, it preserves axioms, thus we only need to check that application of \( f \) preserves inference rules. This is obvious for (b1)-(b6) and (a1). For (a2) and (a3) it follows from the facts that \( f(\varphi[t/x]) = f(\varphi)[f(t)/x] \) and \( \text{FV}(f(\varphi)) = \text{FV}(\varphi) \).

Now, let us prove that \( f \) is unique. Let \( f \) and \( f' \) be morphisms of theories such that \( f(t) = f'(t) \) for every \( t \in T(V)_s \), and \( f(R(x_1, \ldots, x_k)) = f'(R(x_1, \ldots, x_k)) \) for every \( R \in \mathcal{P} \). Then we prove that \( f = f' \).

Let us prove that \( f(\varphi) = f'(\varphi) \) for every \( \varphi \in \text{Form}_P(V) \). It is enough to prove this for atomic formulas \( \varphi \). If \( \varphi \) equals to \( t = t' \), then \( f(\varphi) \) equals to \( f(t) = f'(t') \) and \( f'(\varphi) \) equals to \( f'(t) = f'(t') \). We know that \( \vdash f(t) \equiv f'(t) \) and \( \vdash f'(t') \equiv f'(t') \). Thus by transitivity and symmetry we can conclude that \( \vdash f(t) = f'(t) \). If \( \varphi = R(t_1, \ldots, t_k) \), then \( f(\varphi) = f(R(x_1, \ldots, x_k))[x_i \mapsto f(t_i)] \) and \( f'(\varphi) = f'(R(x_1, \ldots, x_k))[x_i \mapsto f'(t_i)] \). We know that \( f(R(x_1, \ldots, x_k)) \models f'(R(x_1, \ldots, x_k)) \). Since \( \text{FV}(f(R(x_1, \ldots, x_k))) = \{x_1, \ldots, x_k\} \), by (a3) we can conclude that \( \vdash f(\varphi) \models f'(R(x_1, \ldots, x_k))[x_i \mapsto f(t_i)] \). Since \( f'(R(x_1, \ldots, x_k))[x_i \mapsto f(t_i)] \models f'(\varphi) \). By (b2) we conclude that \( \vdash f(\varphi) \models f'(\varphi) \). The same argument shows that \( \vdash f'(\varphi) \models f(\varphi) \).

Finally, it is easy to see that \( f(t) = f'(t) \) for every \( t \in \text{PT}(V)_s \). Thus \( f = f' \). \( \square \)

Note that if \( T \) is the standard monad \( \text{Term}_F \), then to define a morphism of monads \( T \to T' \), it is enough to specify for every \( \sigma \in \mathcal{F} \), \( \sigma : s_1 \times \ldots \times s_k \to s' \), a partial term \( \alpha(\sigma) \in T'(\{x_1 : s_1, \ldots, x_k : s_k\}) \) such that \( \text{FV}(\alpha(\sigma)) = \{x_1, \ldots, x_k\} \). Then there is a unique morphism of monads \( f : T \to T' \) such that \( f(\sigma(x_1, \ldots, x_k)) = \alpha(\sigma) \).

Now, let us define a category \( \text{Th}_S \) of standard partial Horn theories. Its objects are tuples \( (\mathcal{S}, \mathcal{F}, \mathcal{P}, A) \), where \( \mathcal{F} \) is a set of function symbols, \( \mathcal{P} \) is a set of relation symbols, and \( A \) is a set of axioms over \( (\text{Term}_F, \text{Form}_P) \). Morphisms of standard partial Horn theories are morphisms of corresponding partial Horn theories. Thus \( \text{Th}_S \) is (equivalent to) a full subcategory of \( \text{Th}_{S}^{T} \).

2.3. Models of partial Horn theories. Given a monad \( T : \text{Set}^S \to \text{Set}^S \), we define a category of its partial algebras. A partial algebra over \( T \) is a pair \( (A, \alpha) \), where \( A \) is an \( S \)-set and \( \alpha : \text{Hom}_{\text{PSet}^{S}}(V, A) \to \text{Hom}_{\text{PSet}^{S}}(T(V), A) \), where \( \text{PSet} \) is the category of sets and partial functions between them. This pair must satisfy the following conditions:

- For every partial function \( f : V \to A \), \( \alpha_V(f) \circ \eta_V = f \).
- For every total function \( \rho : V \to T(V') \) and every partial function \( f : V' \to A \), \( \alpha_V(f) \circ \rho = \alpha_V(f) \circ \rho' \).

A morphism of partial algebras \( (A, \alpha) \) and \( (A', \alpha') \) is a total morphism \( h : A \to A' \) of \( S \)-sets such that, for every partial function \( f : V \to A \) and every \( t \in T(V)_s \), if \( \alpha_V(f)(t) \) is defined, then \( \alpha_V'(h \circ f)(t) \) is also defined and \( h(\alpha_V(f)(t)) = \alpha_V'(h \circ f)(t) \).

Lemma 2.8. If \( \text{Term}_F \) is the standard monad, then categories of partial algebras over \( \text{Term}_F \) and partial structures for signature \( (\mathcal{S}, \mathcal{F}, \emptyset) \) as defined in [11] are isomorphic.
Proof: A partial structure for signature \((\mathcal{S}, \mathcal{F}, \emptyset)\) is an \(\mathcal{S}\)-set \(A\) together with a collection of partial functions \(A(\sigma) : A_{s_1} \times \ldots \times A_{s_n} \rightarrow A\) for every \(\sigma \in \mathcal{F}\), \(\sigma : s_1 \times \ldots \times s_n \rightarrow \mathcal{S}\). Given such partial structure, we define a partial algebra \(F(A)\) over \(\text{Term}_\mathcal{F}\) as \((A, \alpha)\), where \(\alpha\) is defined as follows:

\[
\alpha_V(f)(x) = f(x) \\
\alpha_V(f)(\sigma(t_1, \ldots, t_n)) = A(\sigma)(\alpha_V(f)(t_1), \ldots, \alpha_V(f)(t_n))
\]

For every morphism \(h : A \rightarrow A'\) of partial structures, let \(F(h) = h\).

For every partial algebra \((A, \alpha)\), we define a partial structure \(G(A, \alpha)\). Let \(G(A, \alpha) = A\) and \(G(A, \alpha)(\sigma(a_1, \ldots, a_n) = \alpha_{x_1, \ldots, x_n}(x_i \mapsto a_i)(\sigma(x_1, \ldots, x_n))\). For every morphism \(h : (A, \alpha) \rightarrow (A', \alpha')\) of partial algebras, let \(G(h) = h\). It is easy to see that functors \(F\) and \(G\) determine isomorphisms of categories. □

If \(F : \text{Set}^\mathcal{S} \rightarrow \text{Set}\) is a left module of formulas over \(\mathcal{T}\), then we define a category of its partial algebras. A partial algebra over \((\mathcal{T}, F)\) is a partial algebra \((A, \alpha)\) over \(\mathcal{T}\) together with a function \(\beta_V : \text{Hom}_{\text{PSet}^\mathcal{S}}(V, A) \rightarrow \text{Hom}_{\text{Set}}(F(V), \Omega)\), where \(\Omega = \{\top, \bot\}\) is the set of truth-values. This function must satisfy the following conditions:

- For every total function \(\rho : V \rightarrow T(V')\) and every partial function \(f : V' \rightarrow A\), \(\beta_V(\alpha_V(f) \circ \rho) = \beta_V(f) \circ \rho^\circ\).
- For every partial function \(f : V \rightarrow A\), \(\beta_V(f)(\top) = \top\).
- For every partial function \(f : V \rightarrow A\), \(\beta_V(f)(\varphi \land \psi) = \beta_V(f)(\varphi \land \beta_V(f)(\psi))\), where \(P \land Q = \top\) if and only if \(P = \top\) and \(Q = \top\).

A morphism of partial algebras \((A, \alpha, \beta)\) and \((A', \alpha', \beta')\) is a morphism \(h\) of partial algebras \((A, \alpha)\) and \((A', \alpha')\) such that, for every partial function \(f : V \rightarrow A\) and every \(\varphi \in F(V)\), if \(\beta_V(f)(\varphi) = \top\), then \(\beta_V'(h \circ f)(\varphi) = \top\).

We define a function \(\epsilon_V : \text{Hom}_{\text{PSet}^\mathcal{S}}(V, A) \rightarrow \text{Hom}_{\text{Set}}(E(V), \Omega)\) for the left module \(E\) of equality. Let \(\epsilon_V(e(s, t)) = \top\) if and only if \(\alpha_V(f)(t) = \alpha_V'(f)(t')\) are defined and equal. If \(F\) is a left module of formulas with equality over \(\mathcal{T}\), then we say that a partial algebra \((A, \alpha, \beta)\) is standard if, for every partial function \(f : V \rightarrow A\), \(\epsilon_V \circ \beta_V(f) = \epsilon_V(f)\), where \(\epsilon_V : E(V) \rightarrow F(V)\).

Lemma 2.9. If \(\text{Term}_\mathcal{F}\) is the standard monad and \(\text{Form}_\mathcal{P}\) is the left module of Horn formulas, then categories of partial algebras over \((\text{Term}_\mathcal{F}, \text{Form}_\mathcal{P})\) and partial structures for signature \((\mathcal{S}, \mathcal{F}, \mathcal{P})\) are isomorphic.

Proof. A partial structure for signature \((\mathcal{S}, \mathcal{F}, \mathcal{P})\) is a partial structure \(A\) for signature \((\mathcal{S}, \mathcal{F}, \emptyset)\) together with a relation \(A(R) \subseteq A_{s_1} \times \ldots \times A_{s_n}\) for every \(R \in \mathcal{P}\), \(R : s_1 \times \ldots \times s_n\). Given such partial structure, we define a partial algebra \(F(A)\) over \((\text{Term}_\mathcal{F}, \text{Form}_\mathcal{P})\) as \((A, \alpha, \beta)\), where \((A, \alpha)\) is the partial algebra defined in Lemma 2.8, and \(\beta\) defined as follows:

\[
\beta_V(f)(t = t') = \epsilon_V(e(s, t, t')) \\
\beta_V(f)(R(t_1, \ldots, t_n)) = \top\text{ if and only if } (\alpha_V(f)(t_1), \ldots, \alpha_V(f)(t_n)) \in A(R) \\
\beta_V(f)(\varphi_1 \land \ldots \land \varphi_n) = \beta_V(f)(\varphi_1 \land \ldots \land \beta_V(f)(\varphi_n))
\]

For every morphism \(h : A \rightarrow A'\) of partial structures, let \(F(h) = h\).

For every partial algebra \((A, \alpha, \beta)\), we define a partial structure \(G(A, \alpha, \beta)\). We already defined interpretation of function symbols in Lemma 2.8. For every \(R \in \mathcal{P}\), let \(G(A, \alpha, \beta)(R) = \{(a_1, \ldots, a_n) \mid \beta_{x_1, \ldots, x_n}(x_i \mapsto a_i)(R(x_1, \ldots, x_n)) = \top\}\). For every
morphism \( h : (A, \alpha, \beta) \to (A', \alpha', \beta') \) of partial algebras, let \( G(h) = h \). It is easy to see that functors \( F \) and \( G \) determine isomorphisms of categories. \( \square \)

If \((T, F, \mu)\) is a monadic presentation, then we define a category of its partial algebras as a full subcategory of partial algebras over \((T, F)\). A partial algebra \((A, \alpha, \beta)\) over \((T, F)\) is a partial algebra over \((T, F, \mu)\) if, for every partial function \( f : V \to A \), every \( t \in T(V)_s \) and every \( \varphi \in F(V) \), \( \alpha_V(f)(\mu_V(t, \varphi)) \) is defined if and only if \( \alpha_V(f)(t) \) is defined and \( \beta_V(f)(\varphi) = \top \), and \( \alpha_V(f)(\mu_V(t, \varphi)) \) equals to \( \alpha_V(f)(t) \) when it is defined. The category of partial algebras over \((T, F, \mu)\) will be denoted by \((T, F, \mu)\text{-}\text{PAlg}\).

**Lemma 2.10.** If \( \text{Term}_F \) is the standard monad and \( \mathbb{T} = (\text{Term}_F, \mathcal{P}, \mathcal{A}) \) is a partial Horn theory, then categories of partial algebras over \( \mathcal{P}(\mathbb{T}) \) and models of \( \mathbb{T} \) as defined in [11] are isomorphic.

**Proof.** Using Lemma 2.9, models of \( \mathcal{T} \) can be described as partial algebras \((A, \alpha', \beta')\) over \((\text{Term}_F, \text{Form}_F)\) such that, for every derivable sequent \( \varphi \vdash \psi \) of \( \mathbb{T} \) and every partial function \( f : V \to A \), \( \beta'_V(f)(\varphi) = \beta'_V(f)(\psi) \).

Let \((A, \alpha, \beta)\) be a partial algebra over \( \mathcal{P}(\mathbb{T}) \). Then we define a partial algebra \( F(A, \alpha, \beta) \) over \((\text{Term}_F, \text{Form}_F)\). Let \( F(A, \alpha, \beta) = (A, \alpha', \beta') \), where \( \alpha'(f)(t) = \alpha_V(f)(|t|_{\varphi}) \) and \( \beta'(f)(\varphi) = \alpha_V(f)(|\varphi|_{\varphi}) \), where \(|t|_{\varphi} \) and \(|\varphi|_{\varphi} \) are equivalence classes of \( t \gamma \) and \( \varphi \) in \( \mathcal{P}(V) \) and \( F(V) \) respectively. Then \( F(A, \alpha, \beta) \) is a model of \( \mathbb{T} \). Indeed, if \( \varphi \vdash \psi \) is a theorem of \( \mathbb{T} \), then \( \varphi \vdash \psi' \) is also a theorem of \( \mathbb{T} \), where \( \varphi' = \varphi \wedge x_1 \wedge \ldots \wedge x_n, \psi' = \psi \wedge y_1 \wedge \ldots \wedge y_k, x_1, \ldots, x_n \) is the set of free variables of \( \psi \), and \( y_1, \ldots, y_k \) is the set of free variables of \( \varphi \). It follows that \(|\varphi'|_{\varphi} = |\psi'|_{\varphi} \); hence \( \beta_V(f)(\varphi') = \beta_V(f)(\psi') \). But \( \beta_V(f)(\varphi) = \beta_V(f)(\varphi') \) and \( \beta_V(f)(\psi) = \beta_V(f)(\psi') \); hence \( F(A, \alpha, \beta) \) is a model of \( \mathbb{T} \). If \( h \) is a morphism of partial algebras over \( \mathcal{P}(\mathbb{T}) \), then let \( F(h) = h \).

Let \((A, \alpha', \beta')\) be a model of \( \mathbb{T} \). Then we define a partial algebra \( G(A, \alpha', \beta') \) over \( \mathcal{P}(\mathbb{T}) \).

Let \( G(A, \alpha', \beta') = (A, \alpha, \beta) \), where \( \beta_V(f)(|\varphi|_{\varphi}) = \beta'_V(f)(\varphi) \), and \( \alpha_V(f)(|t|_{\varphi}) = \alpha'_V(f)(\varphi) \). These definitions do not depend on the choice of a representative of the equivalence classes. Indeed, if \( \varphi \sim \psi \), then \( \varphi \vdash \psi \) is a theorem of \( \mathbb{T} \), and in this case \( \alpha_V(f)(|t|_{\varphi}) = \alpha'_V(f)(\varphi) \).

Finally, we prove a proposition which shows that if \( \mathbb{T}' \) is a partial Horn theory under \( \mathbb{T} \), then we can think of models of \( \mathbb{T}' \) as models of \( \mathbb{T} \) with additional structure.

**Proposition 2.11.** For every morphism of monadic presentations \( f : (P, F, \mu) \to (P', F', \mu') \), there is a faithful functor \( f^* : (P', F', \mu')\text{-}\text{PAlg} \to (P, F, \mu)\text{-}\text{PAlg} \) such that \( id^*_{(P, F, \mu)} \) is the identity functor and \( (g \circ f)^* = f^* \circ g^* \).

**Proof.** If \((A, \alpha, \beta)\) is a partial algebra over \((P', F', \mu')\), then let \( f^*(A, \alpha, \beta) = (A, e \mapsto \alpha_V(e) \circ f_V, e \mapsto \beta_V(e) \circ f_V) \). If \( h : (A, \alpha, \beta) \to (A', \alpha', \beta') \) is a morphism of partial algebras, then let \( f^*(h) = h \). It is easy to see that these definitions satisfy all required conditions. \( \square \)
2.4. Properties of the category of theories. Now we prove a few properties of the category of theories. We begin with a proof of the existence of colimits.

**Proposition 2.12.** Category $\mathbf{Th}_S$ is cocomplete.

*Proof.* First, let $\{T_i\}_{i \in S} = \{((S_i, F_i, P_i), A_i)\}_{i \in S}$ be a set of theories. Then we can define its coproduct $\coprod_{i \in S} T_i$ as the theory with $\coprod_{i \in S} F_i$ as the set of function symbols and $\coprod_{i \in S} A_i$ as the set of axioms. Morphisms $f_i : T_i \to \coprod_{i \in S} T_i$ are defined in the obvious way. If $g_i : T_i \to X$ is a collection of morphisms, then Proposition 2.7 implies that there is a unique morphism $g : \coprod_{i \in S} T_i \to X$ satisfying $g(\sigma(x_1, \ldots, x_n)) = g_i(\sigma(x_1, \ldots, x_n))$ and $f_i(R(x_1, \ldots, x_n)) = f_i(R(x_1, \ldots, x_n))$ for every $\sigma \in F_i$ and $R \in P_i$.

Now, let $f, g : T_1 \to T_2$ be a pair of morphisms of theories. Then we can define their coequalizer $\mathcal{T}$ as the theory with the same set of function and predicate symbols as $T_2$ and the set of axioms which consists of the axioms of $T_2$ together with $f(x_1, \ldots, x_n) \equiv g(x_1, \ldots, x_n)$ for each function symbols $\sigma$ of $T_1$ and $f(R(x_1, \ldots, x_n)) \equiv g(R(x_1, \ldots, x_n))$ for each predicate symbols $R$ of $T_1$. Then we can define $e : T_2 \to T_1$ as identity function on terms and formulas. By Proposition 2.7, $e \circ f = e \circ g$. If $h : T_2 \to X$ is such that $h \circ f = h \circ g$, then it extends to a morphism $\mathcal{T} \to X$ since additional axioms are preserved by the assumption on $h$. This extension is unique since $e$ is an epimorphism. \[\quad\square\]

Now we give a characterization of monomorphisms.

**Proposition 2.13.** A morphism of theories $f : T_1 \to T_2$ is a monomorphism if and only if, for every sequent $\varphi \vdash^V \psi$ of $T_1$, if $f(\varphi) \vdash^V f(\psi)$ is a theorem of $T_2$, then $\varphi \vdash^V \psi$ is a theorem of $T_1$.

*Proof.* First, let us prove the “if” part. Let $g, h : T \to T_1$ be a pair of morphisms such that $f \circ g = f \circ h$. If $t \in \text{PTerm}_{T_2}(V)_s$, then $\vdash^V f(g(t)) \equiv f(h(t))$; hence $\vdash^V g(t) \equiv h(t)$. If $\varphi \in \text{Form}_{P}(V)$, then $\vdash^V f(g(\varphi)) \equiv f(h(\varphi))$; hence $\vdash^V g(\varphi) \equiv h(\varphi)$. Thus $g = h$.

Now, let us prove the “only if” part. Suppose that $f$ is a monomorphism. Let $\varphi \vdash^V \psi$ be a sequent of $T_1$ such that $f(\varphi) \vdash^V f(\psi)$ is a theorem of $T_2$. Let $T$ be a theory which consists of a single predicate symbol $R : s_1 \times \cdots \times s_n \times s_1' \times \cdots \times s_k'$ where $s_1, \ldots, s_n$ are sorts of variables in $FV(\varphi)$ and $s_1', \ldots, s_k'$ are sorts of variables in $FV(\psi)$. Let $g : T \to T_1$ be a morphism defined by $g(R(x_1, \ldots, x_n, y_1, \ldots, y_k)) = \varphi \land y_1 \downarrow \land \ldots \land y_k \downarrow$ and let $h : T \to T_1$ be a morphism defined by $h(R(x_1, \ldots, x_n, y_1, \ldots, y_k)) = \varphi \land \psi$. By Proposition 2.7, $f \circ g = f \circ h$, hence $g = h$ which implies that $\varphi \vdash^V \psi$. \[\quad\square\]

Let $\mathcal{T} = ((S, F, P), A)$ and $\mathcal{T}' = ((S', F', P'), A')$ be a pair of theories. Then we say that $\mathcal{T}'$ is a subtheory of $\mathcal{T}$ if $S' \subseteq S$, $F' \subseteq F$, $P' \subseteq P$ and $A' \subseteq A$. If $\mathcal{T}'$ is a subtheory of a theory $\mathcal{T}$, then we often need to know when a theorem of $\mathcal{T}$ is a theorem of $\mathcal{T}'$. The lemma below gives us a simple criterion for this. First, we need to introduce a bit of notation. Let $t$ be a term over the signature of $\mathcal{T}$ such that there is no subterm of a sort that does not belong to $S'$. Then we define a term
Ret(t) over the signature of $\mathbb{T}'$ as follows:

\[
\begin{align*}
\text{Ret}(x) & = x \\
\text{Ret}(\sigma(t_1, \ldots, t_n)) & = \sigma(\text{Ret}(t_1), \ldots, \text{Ret}(t_n)), \text{ if } \sigma \in \mathcal{F}' \\
\text{Ret}(\sigma(t_1, \ldots, t_n)) & = x_s, \text{ if } \sigma \notin \mathcal{F}' \text{ and } \sigma : s_1 \times \ldots \times s_n \rightarrow s
\end{align*}
\]

where $x_s$ is a variable of sort $s$ that is not a free variable of $t$.

If $\varphi$ is an atomic formula over the signature of $\mathbb{T}$, then we define a formula $\text{Ret}(\varphi)$ over the signature of $\mathbb{T}'$ as follows:

\[
\begin{align*}
\text{Ret}(t = t') & = (\text{Ret}(t) = \text{Ret}(t')), \text{ if } \text{Ret}(t) \text{ and } \text{Ret}(t') \text{ are defined} \\
\text{Ret}(R(t_1, \ldots, t_n)) & = R(\text{Ret}(t_1), \ldots, \text{Ret}(t_n)), \text{ if } \text{Ret}(t_i) \text{ is defined for every } i
\end{align*}
\]

For an arbitrary Horn formula $\varphi$ we define $\text{Ret}(\varphi)$ as follows:

\[
\text{Ret}(\varphi_1 \land \ldots \land \varphi_n) = \text{Ret}(\varphi_1) \land \ldots \land \text{Ret}(\varphi_n)
\]

For every partial term $t|_\varphi$, let $\text{Ret}(t|_\varphi) = \text{Ret}(t)|_{\text{Ret}(\varphi)}$. If $S$ is sequent $\varphi \vdash \psi$ in the signature of $\mathbb{T}$, then we define sequent $\text{Ret}(S)$ in the signature of $\mathbb{T}'$ as $\text{Ret}(\varphi) \vdash_{\cup \text{FV}(\text{Ret}(\varphi))} \text{Ret}(\psi)$.

**Lemma 2.14.** Let $\mathbb{T}'$ be a subtheory of $\mathbb{T}$. Suppose that, for every axiom $S$ of $\mathbb{T}$, $\text{Ret}(S)$ is a theorem of $\mathbb{T}'$. Then if a sequent in the signature of $\mathbb{T}'$ is provable in $\mathbb{T}$, then it is also provable in $\mathbb{T}'$.

**Proof.** If $S$ is a sequent in the signature of $\mathbb{T}'$, then $\text{Ret}(S) = S$. Thus we only need to prove that if $S$ is a theorem of $\mathbb{T}$, then $\text{Ret}(S)$ is a theorem of $\mathbb{T}'$. For axioms this is true by assumption. We need to check that $\text{Ret}(-)$ preserves inference rules. This is clearly true for rules (b1)-(b6) and (a1).

Let us consider rule (a2). Let $S$ equals $x = y \land \varphi \vdash_{x:s,y:s,V} \varphi[y/x]$. Note that $\text{Ret}(\varphi[y/x])$ is defined and only if $\text{Ret}(\varphi)$ is defined, and in this case $\text{Ret}(\varphi[y/x]) = \text{Ret}(\varphi)[y/x]$. Thus $\text{Ret}(S)$ is either of the form $x = y \land \text{Ret}(\varphi) \vdash_{x:s,y:s,V,\text{FV}(\text{Ret}(\varphi))} \text{Ret}(\varphi)[y/x]$, or of the form $x \vdash_{x:s,y:s,V} \top$, or of the form $\top \vdash_{x:s,y:s,V} \top$. In all of these cases $\text{Ret}(S)$ is a theorem of $\mathbb{T}'$.

Finally, let us consider rule (a3). To prove that it preserves the required property, it is enough to show that $\varphi$ is a formula of $(S', \mathcal{F}', \mathcal{P}')$ if and only if $\varphi[t/x]$ is. If $x \notin \text{FV}(\varphi)$, then $\varphi = \varphi[t/x]$. Suppose that $x \in \text{FV}(\varphi)$ and $\varphi$ is a formula of $(S', \mathcal{F}', \mathcal{P}')$. If $x$ has sort $s$, then $s \in S'$. We need to show that a term of sort $s$ is a term of $(S', \mathcal{F}', \mathcal{P}')$. But this follows from the assumption on the set of function symbols. \[\square\]

Sometimes it is convenient to have a sort which consists of a single element. Let $S$ be a set of sorts and let $s_0$ be a sort in $S$. Then we define a theory $\mathbb{T}_{s_0}$ which consists of a single function symbol $*: s_0$ and two axioms: $\downarrow(*)$ and $\uparrow(*) = x = *$. Then, for every theory $\mathbb{T} \in \text{Th}_S$, there is at most one morphism from $\mathbb{T}_{s_0}$ to $\mathbb{T}$. If such morphism exists, we will say that $s_0$ is trivial in $\mathbb{T}$. Thus $\mathbb{T}_{s_0}/\text{Th}_S$ is (equivalent to) a full subcategory of $\text{Th}_S$.

As an application of the previous results we will prove that adding a trivial sort does not change the category of theories. Every theory $\mathbb{T} \in \text{Th}_S$ is naturally a
theory in \(\text{Th}_{\text{SH}(s_0)}\). Thus we have a functor \(i : \text{Th}_S \to T_{s_0}/\text{Th}_{\text{SH}(s_0)}\) such that \(i(T) = T \amalg T_{s_0}\).

**Proposition 2.15.** Functor \(i : \text{Th}_S \to T_{s_0}/\text{Th}_{\text{SH}(s_0)}\) is an equivalence of categories.

**Proof.** Let \(T_1, T_2 \in \text{Th}_S\) be theories with \(P(T_i) = (T_i, F_i, \mu_i)\), \(i = 1, 2\). Let \(\alpha, \beta : T_1 \to T_2\) be morphisms such that \(i(\alpha) = i(\beta)\). Then, for every \(t \in T_1\), sequent \([V] i(\alpha)(t) \cong i(\beta)(t)\) is a theorem of \(i(T_2)\). Since \(T_2\) is (isomorphic to) a subtheory of \(i(T_2)\), by Lemma 2.14, sequent \([V] \alpha(t) \cong \beta(t)\) is a theorem of \(T_2\). Analogously, we can show that \(\alpha(\varphi) \vdash V \beta(\varphi)\) is a theorem of \(T_2\) for every \(\varphi \in F_1\). Thus \(i\) is faithful.

Let \(\alpha : i(T_1) \to i(T_2)\) be a morphism. For every \(t \in T_1(V)_s\), let \(\beta(t) = \text{Ret}(\alpha(t))\) and, for every \(\varphi \in F_1(V)\), let \(\beta(\varphi) = \text{Ret}(\alpha(\varphi))\). Since \(\text{Ret}\) preserves substitution, \(\land\) and, \(\top\), this defines a morphism \(\beta : T_1 \to T_2\). Since \(s_0\) is trivial in \(i(T_2)\), \(\text{Ret}(t) = t\) and \(\text{Ret}(\varphi) = \varphi\) for every partial term \(t\) and every formula \(\varphi\). Thus \(i(\beta) = \alpha\); hence \(i\) is full.

Let \(T \in \text{Th}_{\text{SH}(s_0)}\) be a theory with trivial \(s_0\). Then we define a theory \(T' \in \text{Th}_S\). It has a predicate symbol \(R : s'_1 \times \ldots \times s'_n\) for every predicate symbol \(R : s_1 \times \ldots \times s_n\) of \(T\), where \(s'_1, \ldots, s'_n\) is the subsequence of \(s_1, \ldots, s_n\) consisting of sorts from \(S\). It has a function symbol \(\sigma : s'_1 \times \ldots \times s'_n \to s\) for every function symbol \(\sigma : s_1 \times \ldots \times s_n \to s\) of \(T\) such that \(s \in S\). Also, for every function symbol \(\sigma : s_1 \times \ldots \times s_n \to s_0\) of \(T\), there is a predicate symbol \(R_\sigma : s'_1 \times \ldots \times s'_n\) in \(T'\).

For every term \(t\) of \(T\) of a sort from \(S\), we can define a term \(r(t)\) of \(T'\). Term \(r(t)\) is obtained from \(t\) by omitting subterms of sort \(s_0\). For every formula \(\varphi\) of \(T\), we can define a formula \(r(\varphi)\) of \(T'\):

\[
(r(t = s_0 t')) = \top
\]
\[
n(t = s t') = (r(t) =_s r(t'))
\]
\[
r(R(t_1, \ldots, t_n)) = R(r(t'_1), \ldots, r(t'_n))
\]
\[
r(\varphi_1 \land \ldots \land \varphi_n) = r(\varphi_1) \land \ldots \land r(\varphi_n)
\]

where \(t'_1, \ldots, t'_n\) is the subsequence of \(t_1, \ldots, t_n\) consisting of the terms of sorts from \(S\). Axioms of \(T'\) are sequents of the form \(r(\varphi) \vdash_{FV(r(\varphi)) \cup FV(r(\psi))} r(\psi)\) for every axiom \(\varphi \vdash V \psi\) of \(T\). It is easy to see that \(i(T')\) is isomorphic to \(T\). Thus \(i\) is essentially surjective on objects. \(\square\)

### 3. Algebraic dependent type theories

In this section we define algebraic dependent type theories and theories with substitution, describe two partial Horn theories \(T_0, T_1\) in terms of which algebraic dependent type theories are defined, and prove that the category of models of \(T_1\) is equivalent to the category of contextual categories.

**3.1. Algebraic dependent type theories.** Let \(C = \{\text{ctx}, tm\} \times \mathbb{N}\) be the set of sorts. We will write \((ty, n)\) for \((\text{ctx}, n+1)\). Sort \((tm, n)\) represents terms in contexts of length \(n\), sort \((ctx, n)\) represents contexts of length \(n\), and sort \((ty, n)\) represents types in contexts of length \(n\).

We define \(T_0 \in \text{Th}_C\) as the theory with the set of function symbols \(ft_n : (ty, n) \to (ctx, n), ty_n : (tm, n) \to (ty, n)\) and axioms asserting that \((ctx, 0)\) is trivial. Let
\( ft^n_i : (ctx, n + i) \rightarrow (ctx, n) \) be the following derived operation:

\[
\begin{align*}
ft^0_n(A) &= A \\
ft^{i+1}_n(A) &= ft^n_i(ft^{n+i}_n(A))
\end{align*}
\]

Let \( ctx_{p,n} : (p, n) \rightarrow (ctx, n) \) be defined as follows: \( ctx_{ty,n}(t) = ft^n_n(t) \) and \( ctx_{tm,n}(t) = ft^n_n(ty_n(t)) \).

Now, we can define the notion of algebraic dependent type theories.

**Definition 3.1.** An algebraic dependent type theory is a theory \( T \in \text{Th} \) together with a morphism \( T_0 \rightarrow T \). The category \( \text{TT}^0 \) of algebraic dependent type theories is the under category \( T_0/\text{Th} \).

Often (algebraic) type theories involve substitution operations. So we describe a theory of substitution, which we call \( T_1 \). There are two ways to define substitution: either to substitute the whole context (full substitution) or only a part of it (partial substitution). Using ordinary type theoretic syntax the full substitution can be described by the following inference rule:

\[
\begin{align*}
A_1, \ldots, A_n \vdash A \text{ type} & & \Gamma \vdash a_1 : A_1[\_] & & \ldots & & \Gamma \vdash a_n : A_n[a_1, \ldots, a_{n-1}] \\
\Gamma \vdash A[a_1, \ldots, a_n] \text{ type}
\end{align*}
\]

The partial substitution is described by the following inference rule:

\[
\begin{align*}
\Gamma, A_1, \ldots, A_n \vdash A \text{ type} & & \Gamma \vdash a_1 : A_1 & & \ldots & & \Gamma \vdash a_n : A_n[a_1, \ldots, a_{n-1}] \\
\Gamma \vdash A[a_1, \ldots, a_n] \text{ type}
\end{align*}
\]

The partial substitution was used in [14], but we will use the full version since it is stronger. To make these operations equivalent, we need to add another operation to the partial substitution, and even more axioms. Thus our approach seems to be slightly simpler.

The set of function symbols of \( T_1 \) consists of the symbols of \( T_0 \) and the following symbols:

\[
v_{n,i} : (ctx, n) \rightarrow (tm, n), 0 \leq i < n
\]

\[
\text{subst}_{p,n,k} : (ctx, n) \times (p, k) \times (tm, n)^k \rightarrow (p, n), p \in \{tm, ty\}
\]

Auxiliary predicates \( Hom_{n,k} : (ctx, n) \times (ctx, k) \times (tm, n)^k \rightarrow (p, n) \) are defined as follows: \( Hom_{n,k}(B, A, a_1, \ldots, a_k) \) holds if and only if

\[
ty_n(a_i) = \text{subst}_{ty,n,i-1}(B, ft_i^{k-i}(A), a_1, \ldots, a_{i-1}) \text{ for each } 1 \leq i \leq k
\]

The idea is that a tuple of terms should represent a morphism in a contextual category. So \( Hom_{n,k}(B, A, a_1, \ldots, a_k) \) holds if and only if \( (a_1, \ldots, a_k) \) is a morphism with domain \( A \) and codomain \( B \). Note that if \( Hom_{n,k}(B, A, a_1, \ldots, a_k) \), then \( ft_n(ty_n(a_i)) = B \).

The set of axioms of \( T_1 \) consists of the axioms of \( T_0 \) and the axioms we list below. The following axioms describe when functions are defined:

\[
\Gamma \vdash ft^n_i(A) \downarrow \quad (1)
\]

\[
Hom_{n,k}(B, ctx_p,k(a), a_1, \ldots, a_k) \overset{B:a: a_i}{\underset{\text{subst}_{p,n,k}}{\downarrow}} Hom_{n,k}(B, a, a_1, \ldots, a_k) \downarrow \quad (2)
\]
The following axioms describe the “typing” of the constructions we have:

\[ t_y n (v_n, i(A)) = subst_{ty, n, n-1}(A, ft_{n-1}^i(A), v_{n, n-1}(A), \ldots v_{n, i+1}(A)) \] (3)

\[ ft_n (subst_{ty, n, k}(B, A, a_1, \ldots a_k)) = B \] (4)

\[ ty_n (subst_{tm, n, k}(B, a, a_1, \ldots a_k)) = subst_{ty, n, k}(B, ty_k(a), a_1, \ldots a_k) \] (5)

The following axioms prescribe how \( subst_{p, n, k} \) must be defined on indices \( (v_n, i) \):

\[ subst_{p, n, n}(ctx_{p, n}(a), a, v_{n, n-1}(ctx_{p, n}(a)), \ldots v_{n, 0}(ctx_{p, n}(a))) = a \] (6)

\[ Hom_{n, k}(B, A, a_1, \ldots a_k) \xrightarrow{B, a_i, A} subst_{tm, n, k}(B, v_{k, i}(A), a_1, \ldots a_k) = a_{k-i} \] (7)

The last axiom say that substitution must be “associative”:

\[ Hom_{n, k}(C, B, b_1, \ldots b_k) \wedge Hom_{k, m}(B, ctx_{p, m}(a), a_1, \ldots a_m) \xrightarrow{C, b_i, B, a_i, a} \]

\[ subst_{p, n, m}(C, subst_{p, k, m}(B, b_1, \ldots b_k), b_1, \ldots b_k) = subst_{p, n, m}(C, a, subst_{tm, n, k}(C, a_1, b_1, \ldots b_k), \ldots subst_{tm, n, k}(C, a_m, b_1, \ldots b_k)) \] (8)

**Definition 3.2.** An algebraic dependent type theory with substitution is a theory \( T \in ThC \) together with a morphism \( T_1 \to T \). The category \( TT^1 \) of algebraic dependent type theories with substitution is the under category \( T_1/ThC \).

### 3.2. Models of \( T_1 \)

Here we will show that the category of models of \( T_1 \) is equivalent to the category of contextual categories. First, we construct a functor \( F : T_1{-}\text{Mod} \to CCat \). Let \( M \) be a model of \( T_1 \). Then the set of objects of level \( n \) of \( F(M) \) is \( M(ctx, n) \). For each \( A \in M(ctx, n) \), \( B \in M(ctx, k) \) morphisms from \( A \) to \( B \) are tuples \( (a_1, \ldots a_k) \) such that \( a_i \in M(tm, n) \) and \( Hom_{n, k}(A, B, a_1, \ldots a_k) \).

For each \( 0 \leq i \leq n \) axiom (3) implies

\[ \xrightarrow{A} Hom_{n, n-1}(A, ft_{n-1}^i(A), v_{n, n-1}(A), \ldots v_{n, 1}(A)). \]

For each \( A \in M(ctx, n) \) we define \( id_A : A \to A \) as tuple

\[(v_{n, n-1}(A), \ldots v_{n, 0}(A))\]

and \( p_A : A \to ft(A) \) as tuple

\[(v_{n, n-1}(A), \ldots v_{n, 1}(A))\].

Now, we introduce some notation. If \( B \in M(ctx, n) \), \( a \in M(p, k) \), and \( f = (a_1, \ldots a_k) : B \to ctx_{p, k}(a) \) is a morphism, then we define \( a[f] \in M(p, n) \) as \( subst_{p, n, k}(B, a, a_1, \ldots a_k) \). By axiom (2) this construction is total.

If \( A \in M(ctx, n) \), \( B \in M(ctx, k) \), \( C \in M(ctx, m) \), \( f : A \to B \), and \( (c_1, \ldots c_m) : B \to C \), then we define composition \( (c_1, \ldots c_m) \circ f \) as \( (c_1[f], \ldots c_m[f]) \). The following sequence of equations shows that \( (c_1, \ldots c_m) \circ f : A \to C \).

\[ ty_n(c_1[f]) = (\text{by axiom (5)}) \]

\[ ty_k(c_i)[f] = (\text{since } Hom_{k, m}(c_1, \ldots c_m)) \]

\[ ft_{n-1}^m(C)[c_1, \ldots c_{i-1}][f] = (\text{by axiom (8)}) \]

\[ ft_{n-1}^m(C)[c_1[f], \ldots c_{i-1}[f]] \]
With these notations we can rewrite axioms (5), (6) and (8) as follows:

\[ t y_n (a[f]) = A[f] \]
for each \( f : B \to ft_k(A) \), where \( A = t y_k(a) \)

\[ a[i_1 id_{ctx,n}(a)] = a \]
\[ a[g][f] = a[g \circ f] \]
for each \( f : C \to B \) and \( g : B \to ctx_p,m(a) \)

Associativity of the composition follows from axiom (8), and the fact that \( id \) is identity for it follows from axioms (6) and (7).

For every \( A \in M(ty,k) \) there is a bijection \( \varphi \) between the set of \( a \in M(tm,k) \) such that \( t y_k(a) = A \) and the set of morphisms \( f : ft_k(A) \to A \) such that \( p_A \circ f = id_{ft_k(A)} \). For every such \( a \in M(tm,k) \) we define \( \varphi(a) \) as

\[ \langle v_{k-1}(ft_k(A)), \ldots, v_{k,0}(ft_k(A)), a \rangle. \]

Note that if \( (a_1, \ldots, a_{k+1}) : B \to A \) is a morphism, then axiom (7) implies that \( p_A \circ (a_1, \ldots, a_{k+1}) \) equals to \( (a_1, \ldots, a_k) \). Thus \( \varphi(a) \) is a section of \( p_A \). Clearly, \( \varphi \) is injective. Let \( f : ft_k(A) \to A \) be a section of \( p_A \); then first \( k \) components of \( f \) must be identity on \( ft_k(A) \). So if \( a \) is the last component of \( f \), then \( \varphi(a) \) equals to \( f \). Hence \( \varphi \) is bijective.

If \( A \in M(ty,k), B \in M(ctx,n), \) and \( f = (a_1, \ldots, a_k) : B \to ft_k(A) \), then we define \( f'(A) \) as \( A[f] = \text{subst}_{ty,n,k}(B,A,a_1,\ldots,a_k) \). Map \( q(f,B) \) defined as the tuple with \( i \)-th component equals to

\[
\begin{cases}
  a_i[v_{n+1,n}(A[f]), \ldots, v_{n+1,1}(A[f])] & \text{if } 1 \leq i \leq k \\
  v_{n+1,0}(A[f]) & \text{if } i = k + 1
\end{cases}
\]

Now we have the following commutative square:

\[
\begin{array}{ccc}
A[f] & \xrightarrow{q(f,A)} & A \\
\downarrow{p_{A[f]}} & & \downarrow{p_A} \\
B & \xrightarrow{f} & ft_k(A)
\end{array}
\]

We need to prove that this square is cartesian. By proposition 2.3 of [15] it is enough to construct a section \( s_{f'} : B \to A[f] \) of \( p_{A[f]} \) for each \( f' = (a_1, \ldots, a_k, a_{k+1}) : B \to A \) and prove a few properties of \( s_{f'} \). We define \( s_{f'} \) to be equal to \( \varphi(a_{k+1}) \). Axioms (7) and (8) implies that \( q(f,B) \circ s_{f'} = f \). To complete the proof that the square above is cartesian we need, for every \( g : ft_k(A) \to ft_m(C) \) and \( A = C[g] \), prove that \( s_{f'} = s_{q(g,C) \circ f'} \). The last component of \( q(g,C) \circ f' \) equals to \( v_{n+1,0}(C[g])[f'] = a_{k+1} \). Thus the last components of \( q(g,C) \circ f' \) and \( f' \) coincide, hence \( s_{f'} = s_{q(g,C) \circ f'} \).

We are left to prove that operations \( A[f] \) and \( q(f,A) \) are functorial. Equations \( A[id_{ft_k(A)}] = A \) and \( A[f \circ g] = A[f][g] \) are precisely axioms (6) and (8). The fact that \( q(id_{ft_k(A)}, A) = id_A \) follows from axiom (7). Now let \( g : C \to B \) and \( f : B \to ft_k(A) \) be morphisms; we need to show that \( q(f \circ g, A) = q(f, A) \circ q(g, A[f]) \). The last component of \( q(f, A) \circ q(g, A[f]) \) equals to \( v_{n+1,0}(A[f])[q(g, A[f])] = v_{m+1,0}(A[f][g]) \), which equals to the last component of \( q(f \circ g, A) \), namely \( v_{m+1,0}(A[f \circ g]) \). If \( 1 \leq i \leq k \), then \( i \)-th component of \( q(f, A) \circ q(g, A[f]) \) equals to

\[
a_i[v_{n+1,n}(A[f]), \ldots, v_{n+1,1}(A[f])][q(g, A[f])] = a_i[b_1', \ldots, b_n']
\]
where $a_i$ is $i$-th component of $f$, $b_i$ is $i$-th component of $g$, and $b'_i$ equals to $b_i[v_{m+1,n}(A[f][g]), \ldots v_{m+1,1}(A[f][g])]$. $i$-th component of $q(f \circ g, A)$ equals to

$$a_i[v_{m+1,n}(A[f \circ g]), \ldots v_{m+1,1}(A[f \circ g])] = a_i[b'_1, \ldots b'_n],$$

where $b'_i = b_i[v_{m+1,n}(A[f \circ g]), \ldots v_{m+1,1}(A[f \circ g])]$. Thus $q(f \circ g, A) = (f, A) \circ q(g, A[f])$. This completes the construction of contextual category $F(M)$.

**Proposition 3.3.** $F$ is functorial, and functor $F : T_1\text{-Mod} \to CCat$ is an equivalence of categories.

**Proof.** Given a map of $T_1$ models $\alpha : M \to N$, we define a map of contextual categories $F(\alpha) : F(M) \to F(N)$. $F(\alpha)$ is already defined on objects. Let

$$f = (a_1, \ldots a_k) \in Hom_{n,k}(B, A).$$

We define $F(\alpha)(f)$ as $(\alpha(a_1), \ldots \alpha(a_k)) \in Hom_{n,k}(\alpha(B), \alpha(A))$. $F(\alpha)$ preserves identity morphisms, compositions, $f^*(A)$, and $q(f, A)$ since all of these operations are defined in terms of $T_1$ operations. Clearly, $F$ preserves identity maps and compositions of maps of $T_1$ models. Thus $F$ is a functor.

First, note that if $a \in M(tm, k)$ and $\alpha : M \to N$, then $F(\alpha)(\varphi(a)) = \varphi(\alpha(a))$. Indeed, consider the following sequence of equations:

$$F(\alpha)(\varphi(a)) =$$

$$F(\alpha)(\varphi(a)) =$$

$$F(\alpha)(v_{k,k-1}(ctx_{tm,k}(a)), \ldots v_{k,0}(ctx_{tm,k}(a)), a) =$$

$$\varphi(\alpha(a)).$$

Now, we prove that $F$ is faithful. Let $\alpha, \beta : M \to N$ be a pair of maps of $T_1$ models such that $F(\alpha) = F(\beta)$. Then $\alpha$ and $\beta$ coincide on contexts. Given $a \in M(tm, n)$ we have the following equation: $\alpha(a) = \varphi^{-1}(F(\alpha)(\varphi(a))) = \varphi^{-1}(F(\beta)(\varphi(a))) = \beta(a)$.

Now, we prove that $F$ is full. Let $\alpha : F(M) \to F(N)$ be a map of contextual categories. Then we need to define $\beta : M \to N$ such that $F(\beta) = \alpha$. If $A \subseteq M(ctx, n)$, then we let $\beta(A) = \alpha(A)$. Note that if $f : ft_n(A) \to A$ is a section of $p_A$, then $\alpha(f)$ is a section of $\alpha(A)$. If $a \in M(tm, n)$, then we let $\beta(a) = \varphi^{-1}(\alpha(\varphi(a)))$.

Maps $F(\beta)$ and $\alpha$ agree on contexts. We prove by induction on $k$ that they coincide on morphisms $f = (a_1, \ldots a_k) \in M(Hom_{n,k})(B, A)$. If $k = 0$, then $F(A)$ is terminal objects, hence $F(\beta) = \alpha$. Suppose $k > 0$ and consider the following equation: $f = \varphi((a_1, \ldots a_{k-1}), A) \circ \varphi(a_k)$. By induction hypothesis we know that $F(\beta)(\varphi((a_1, \ldots a_{k-1}), A)) = \alpha(\varphi((a_1, \ldots a_{k-1}), A))$. Thus we only need to prove that $F(\beta)(\varphi(a_k)) = \alpha(\varphi(a_k))$. But $F(\beta)(\varphi(a_k)) = \varphi(\varphi^{-1}(\alpha(\varphi(a_k)))) = \alpha(\varphi(a_k))$.

Finally, we prove that $F$ is essentially surjective on objects. Given contextual category $C$ we define $T_1$ model $M$. Let $M(ctx, n)$ be equal to $Obh_n(C)$ and $M(tm, n)$ be the set of pairs of objects $A \subseteq Obh_n(C)$ and sections of $p_A : A \to ft_n(A)$. Let $ty_n$ be the obvious projection. We will usually identify $a \in M(tm, n)$ with the section $ctx_{tm,n}(a) \to ty_n(a)$.

For each $n, k \in \mathbb{N}$ we define partial function

$$subst_{ty,n,k} : M(ctx, n) \times M(ty, k) \times M(tm, n)^k \to M(ty, n)$$

such that $ft_n(subst_{ty,n,k}(B, A, a_1, \ldots, a_k)) = B$. We also define morphism

$$q_{n,k} \in Hom_{n+1,k}(subst_{ty,n,k}(B, A, a_1, \ldots, a_k), A)$$
whenever \( \text{subst}_{y,n,k}(B, A, a_1, \ldots a_k) \) is defined. We define \( \text{subst}_{y,n,k} \) and \( q_{n,k} \) by induction on \( k \). Let \( \text{subst}_{y,n,0}(B, A) = !_B(A) \) and \( q_{n,0} = q(!_B, A) \) where \( !_B : B \to Ob_0(C) \) is the unique morphism.

\[
\begin{array}{c}
\text{subst}_{y,n,0}(B, A) \\
\downarrow^{q_{n,0}} \\
A \\
\downarrow^{p_A} \\
B \\
\downarrow^{!_B} \\
1
\end{array}
\]

Let \( \text{subst}_{y,n,k+1}(B, A, a_1, \ldots a_{k+1}) \) be defined whenever \( \text{subst}_{y,n,k}(B, ft_k(A), a_1, \ldots a_k) \) is defined and \( ty_n(a_{k+1}) = \text{subst}_{y,n,k}(B, ft_k(A), a_1, \ldots a_k) \). In this case we let \( \text{subst}_{y,n,k+1}(B, A, a_1, \ldots a_{k+1}) = f^*(A) \) and \( q_{n,k+1} = q(f, A) \) where \( f \) is the composition of \( a_{k+1} \) and \( q_{n,k} \).

\[
\begin{array}{c}
\text{subst}_{y,n,k+1}(B, A, a_1, \ldots a_{k+1}) \\
\downarrow^{q_{n,k+1}} \\
A \\
\downarrow^{p_A} \\
B \\
\downarrow^{a_{k+1}} \\
ty_n(a_{k+1}) \\
\downarrow^{q_{n,k}} \\
ft_k(A)
\end{array}
\]

It is easy to see by induction on \( k \) that axiom (2) holds. Axiom (4) holds by definition of \( \text{subst}_{y,n,k} \).

The definition of predicates \( \text{Hom}_{n,k} \) makes sense in \( M \) now. Thus we can define as before the set \( \text{Hom}^M_{n,k}(B, A) \) of morphisms in \( M \) as the set of tuples \( (a_1, \ldots a_k) \) such that \( \text{Hom}_{n,k}(B, A, a_1, \ldots a_k) \). There is a bijection \( \alpha : \text{Hom}^M_{n,k}(B, A) \to \text{Hom}_{n,k}(B, A) \) such that \( \text{subst}_{y,n,k}(B, A, a_1, \ldots a_k) = \alpha(a_1, \ldots a_k)(A) \) and \( q_{n,k} = q(\alpha(a_1, \ldots a_k), A) \). We define \( \alpha \) by induction on \( k \). Both \( \text{Hom}^M_{n,0}(B, A) \) and \( \text{Hom}_{n,0}(B, A) \) are singletons, so there is a unique bijection between them. If \( (a_1, \ldots a_k) \in \text{Hom}^M_{n,k}(B, ft_k(A)) \), then there is a bijection between morphisms \( f \in \text{Hom}_{n,k+1}(B, A) \) satisfying \( p_A \circ f = \alpha(a_1, \ldots a_k) \) and sections of \( p_\alpha(a_1, \ldots a_k)^{s}(A) \).

By induction hypothesis these sections are just sections of \( p_{\text{subst}_{y,n,k}}(B, A, a_1, \ldots a_k) \).

This gives us a bijection between \( \text{Hom}^M_{n,k+1}(B, A) \) and \( \text{Hom}_{n,k+1}(B, A) \), namely \( \alpha(a_1, \ldots a_{k+1}) = q(\alpha(a_1, \ldots a_k), A) \circ a_{k+1} \). Then the required equations hold by definition.

Now we define total functions \( v_{n,i} : M(ctx,n) \to M(tm,n) \). Let \( v_{n,i}(A) = (p^{i+1}(A)^* (ft^i_{n-i}(A)), s_{p_A}) \).

\[
\begin{array}{c}
p^{i+1}(A)^* (ft^i_{n-i}(A)) \\
\downarrow{s_{p_{\alpha}}} \\
A \\
\downarrow{p_A} \\
ft^i_{n-i}(A)
\end{array} \quad \begin{array}{c}
f^i_{n-i}(A) \\
\downarrow{p_{\alpha}} \\
ft^{i+1}_{n-i-1}(A)
\end{array}
\]

Axiom (1) holds by definition. By induction on \( n-i \) it is easy to see that \( \alpha(v_{n,n-1}(A), \ldots v_{n,i}(A)) \) equals to \( p_A^i : A \to ft^i_{n-i}(A) \). Axiom (3) follows from
the following sequence of equations:

\[
\text{subst}_{y,n,n−1}(A, f t_{n−1}^i(A), v_{n,n−1}(A), \ldots v_{n,i+1}(A)) = \\
\alpha(v_{n,n−1}(A), \ldots v_{n,i+1}(A))(f t_{n−1}^i(A)) = \\
p^{i+1}(A)^*(f t_{n−1}^i(A)) = \\
t y_n(v_{n,i}(A)).
\]

Axiom (6) follows from the facts that \(\alpha(v_{n,n−1}(f t_n(A)), \ldots v_{n,0}(f t_n(A))) = id_{f t_n(A)}\) and \(id_{f t_n(A)}(A) = A\).

Now we define partial functions \(\text{subst}_{tm,n,k} : M(\text{ctx}, n) \times M(tm, k) \times M(tm, n)^k \rightarrow M(tm, n)\). Function \(\text{subst}_{tm,n,k}(B, a, a_1, \ldots a_k)\) is defined whenever \(Hom_{n,k}(B, ctx_{tm,k}(a), a_1, \ldots a_k)\) holds. In this case we let \(\text{subst}_{tm,n,k}(B, a, a_1, \ldots a_k) = a[\alpha(a_1, \ldots a_k)]\) where \(a[f] = sa_0f\). Axioms (2) and (5) hold by definition. Axiom (6) follows from the fact that \(id_{ctx_{tm,n}(a)}(a) = a\).

To prove axiom (7) note that \(p_A \circ \alpha(a_1, \ldots a_{k+1}) = \alpha(a_1, \ldots a_k)\) by definition of \(\alpha\). Hence \(p^i(A) \circ \alpha(a_1, \ldots a_k) = \alpha(a_1, \ldots a_{k−i})\). Also note that \(s_{\alpha(a_2, \ldots a_k)} = a_k\).

Now the axiom follows from the following equations:

\[
\text{subst}_{tm,n,k}(B, v, k_i(A), a_1, \ldots a_k) = \\
v_{k_i(A)}(A) \circ a(a_1, \ldots a_k) = \\
q_{p^{i+1}(A), f t_{n−1}^i(A)}v_{k_i(A)}(A) \circ a(a_1, \ldots a_k) = \\
s_{p^i}(A) \circ a(a_1, \ldots a_k) = \\
s_{\alpha(a_1, \ldots a_{k−1})} = \\
a_{k−i}.
\]

Now we prove that \(\alpha\) preserves compositions. To do this we need to show that \(\alpha(a_1, \ldots a_k) \circ f = \alpha(a_1[f], \ldots a_k[f])\). We do this by induction on \(k\). For \(k = 0\) it is trivial and for \(k > 0\) we have the following sequence of equations:

\[
\alpha(a_1, \ldots a_k) \circ f = \\
q(\alpha(a_1, \ldots a_{k−1}), A) \circ a_k \circ f = \\
q(\alpha(a_1, \ldots a_{k−1}), A) \circ q(f, B[\alpha(a_1, \ldots a_{k−1})]) \circ a_k[f] = \\
q(\alpha(a_1, \ldots a_{k−1}) \circ f, A) \circ a_k[f] = \\
q(\alpha(a_1, \ldots a_{k−1}−i), A) \circ a_k[f] = \\
\alpha(a_1[f], \ldots a_{k−1}[f]).
\]

Now axiom (8) follows from the facts that \(\alpha\) preserves compositions and \((f \circ g)^*(A) = f^*(g^*(A))\). This completes the construction of \(\mathbb{T}_1\) model \(M\) from a contextual category \(C\). To finish the proof we need to show that \(F(M)\) is isomorphic to \(C\). The isomorphism is given by bijection \(\alpha\). We already saw that \(\alpha\) preserves the structure of contextual categories. Thus \(\alpha\) is a morphism of contextual categories, and it is easy to see that \(\alpha^{-1}\) also preserves the structure. Hence \(\alpha\) is isomorphism and \(F\) is an equivalence. \(\square\)

Let \(u : \mathbb{T}_1 \rightarrow \mathbb{T}\) be an algebraic dependent type theory with substitution. Then it follows from Proposition 2.11 and Proposition 3.3 that models of \(\mathbb{T}\) are contextual
categories with additional structure, where \( u^* : \mathbb{TMod} \to \mathbb{TMod}_1 \) is the forgetful functor.

4. Stable theories

In this section we consider algebraic dependent type theories with additional structure which we call stable. We also define category \( \mathbb{TT}^{1}_{st} \) of stable algebraic dependent type theories with substitutions and its full subcategory \( \mathbb{TT}^{reg} \) of regular theories.

4.1. Stable theories. Usually in type theories all of the function symbols are available in every context. We call such theories stable. Let \( S_0 \) be a set of (non-dependent) sorts. Then we define the corresponding set \( S \) of dependent sorts as \( S_0 \times \mathbb{N} \). Suppose that \( S_0 \) contains a distinguished sort \( \text{ctx} \). Let \( \mathbb{T}_{S_0} \) be a theory with the following function symbols:

\[
* : (\text{ctx}, 0) \\
f_{t_k} : (\text{ctx}, n + 1) \to (\text{ctx}, n) \\
c_{\text{ctx},p,n} : (p, n) \to (\text{ctx}, n) \text{ for every } p \in S_0
\]

and axioms of \( \mathbb{T}_{(\text{ctx},0)} \) as defined in section 2.4 together with the following axiom:

\[
\text{\text{ctx}_{ctx,n}}(x) = x
\]

To define stable theories, we need to introduce a few auxiliary constructions. First, we define a function \( L : \mathbb{C} \to \mathbb{C} \) as follows:

\[
L(\text{ctx}, n) = L(\text{ctx}, n + 1) \\
L(\text{tm}, n) = L(\text{tm}, n + 1)
\]

For every set \( \mathcal{F} \) of function symbols, we define another set \( L(\mathcal{F}) \) which consists of symbols \( L(\sigma) \) for every \( \sigma \in \mathcal{F} \). If \( \sigma : s_1 \times \ldots \times s_k \to s \), then \( L(\sigma) : (\text{ctx}, 1) \times L(s_1) \times \ldots \times L(s_k) \to L(s) \). For every set of variables \( V \) we define a set \( L(V) \) which contains a variable \( x \) of sort \( L(s) \) for every variable \( x \) of sort \( s \) in \( V \). For every terms \( \Gamma \in \text{Term}_{\mathcal{L}(\mathcal{F})}(L(V))(\text{ctx},1) \) and \( t \in \text{Term}_{\mathcal{F}}(V)_{(p,n)} \), we define a partial term \( L(\Gamma,t) \in P\text{Term}_{\mathcal{L}(\mathcal{F})}(L(V))_{(p,n+1)} \) as follows:

\[
L(\Gamma,x) = x|_{L(\text{ctx}_{p,n},(\mathcal{F},x))} \\
L(\Gamma,\sigma(t_1, \ldots t_k)) = L(\sigma)(\Gamma, L(\Gamma,t_1), \ldots L(\Gamma, t_k))
\]

For every set \( \mathcal{P} \) of relation symbols, we define set \( L(\mathcal{P}) \) which consists of symbols \( L(R) : (\text{ctx}, 1) \times L(s_1) \times \ldots \times L(s_k) \) for every \( R \in \mathcal{P} \). If \( R : s_1 \times \ldots \times s_k \). For every formula \( \varphi \in \text{Form}_{\mathcal{P}}(V) \) and term \( \Gamma \in \text{Term}_{\mathcal{L}(\mathcal{F})}(L(V))(\text{ctx},1) \), we define a formula \( L(\Gamma, \varphi) \in \text{Form}_{\mathcal{L}(\mathcal{P})}(L(V)) \) as follows:

\[
L(\Gamma, t_1 = t_2) = (L(\Gamma, t_1) = L(\Gamma, t_2)) \\
L(\Gamma, R(t_1, \ldots t_k)) = L(R)(\Gamma, L(\Gamma, t_1), \ldots L(\Gamma, t_k))
\]

Now, let us define a functor \( L : \mathbb{T}_{S_0}/\mathbb{Th}_{S} \to \mathbb{T}_{S_0}/\mathbb{Th}_{S} \). Let \( L((S, \mathcal{F}, \mathcal{P}), A) = ((S, L(\mathcal{F}) \cup \mathcal{F}_{S_0}, L(\mathcal{P})), A' \cup A_{S_0}) \), where \( \mathcal{F}_{S_0} \) and \( A_{S_0} \) are the sets of function symbols and axioms of \( \mathbb{T}_{S_0} \), and \( A' \) consists of the following axioms:

\[
f^{*}(c_{\text{ctx},p,n+1}(x)) = L(\text{ctx}_{p,n+1})(\Gamma, x)\]
for every $p \in S_0$,
\[
L(\sigma)(\Gamma, x_1, \ldots, x_k) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(ctx_{p,n_i}(L(\sigma)(\Gamma, x_1, \ldots, x_k))) = \Gamma \quad L(\sigma)(\Gamma, x_1, \ldots, x_k) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(ctx_{p,n_i}(x_i)) = \Gamma
\]
for every $\sigma \in \mathcal{F}$, $\sigma : (p_1, n_1) \times \ldots \times (p_k, n_k) \rightarrow (p, n)$ and every $1 \leq i \leq k$,
\[
L(R)(\Gamma, x_1, \ldots, x_k) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(ctx_{p,n_i}(x_i)) = \Gamma
\]
for every $R \in \mathcal{P}$, $R : (p_1, n_1) \times \ldots \times (p_k, n_k)$ and every $1 \leq i \leq k$, and
\[
L(\Gamma, \varphi) \wedge \bigwedge_{1 \leq i \leq k} f^{n_i}(ctx_{p,n_i}(x_i)) = \Gamma \downarrow_{\Gamma, x_1, \ldots, x_k} L(\Gamma, \psi)
\]
for every axiom $\varphi \in \mathcal{F}$, $\varphi : (p_1, n_1) \times \ldots \times (p_k, n_k)$ in $A$.

If $f : \mathbb{T} \rightarrow \mathbb{T}'$, then let $L(f) : L(\mathbb{T}) \rightarrow L(\mathbb{T}')$ be defined as follows:
\[
L(f)(L(\sigma)(\Gamma, x_1, \ldots, x_k)) = L(\Gamma, f(\sigma(x_1, \ldots, x_k)))
\]
\[
L(f)(L(R)(\Gamma, x_1, \ldots, x_k)) = L(\Gamma, f(R(x_1, \ldots, x_k)))
\]
It is easy to see that this defines a morphism of theories and that $L$ preserves identity morphisms and compositions.

**Definition 4.1.** A stable (essentially) algebraic theory is an algebra for functor $L$, that is a pair $(\mathbb{T}, \alpha)$, where $\mathbb{T}$ is a theory under $\mathbb{T}_{S_0}$ and $\alpha : L(\mathbb{T}) \rightarrow \mathbb{T}$. The category $\mathbf{St}_{S_0}$ of stable theories is the category of algebras for $L$.

Theories $\mathbb{T}_0$ and $\mathbb{T}_1$ are stable. Indeed, we can define maps $\alpha_0 : L(\mathbb{T}_0) \rightarrow \mathbb{T}_0$ and $\alpha_1 : L(\mathbb{T}_1) \rightarrow \mathbb{T}_1$ as follows:
\[
\alpha_0(L(ty_n)(\Gamma, a)) = ty_{n+1}(a) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(ctx_{m,n_i+1}(a)) = \Gamma
\]
\[
\alpha_1(L(v_{n,i})(\Gamma, \Delta)) = v_{n+1,i}(\Delta) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(\Delta) = \Gamma
\]
and $\alpha_1(L(subst_{p,n,k})(\Gamma, \Delta, B, a_1, \ldots, a_k))$ is defined as
\[
\text{subst}_{p,n+1,k+1}(\Delta, B, v_{n+1,i}(\Delta), a_1, \ldots, a_k) \downarrow_{\Gamma, x_1, \ldots, x_k} f^{n_i}(\Delta) = \Gamma
\]

**Definition 4.2.** A stable algebraic dependent type theory is a stable theory under $(\mathbb{T}_0, \alpha_0)$. The category $(\mathbb{T}_0, \alpha_0)/\mathbf{St}_{ctx,tm}$ will be denoted by $\mathbf{TT}^{\alpha_0}_{ctx,tm}$. A stable algebraic dependent type theory with substitution is a stable algebraic dependent type theory under $(\mathbb{T}_1, \alpha_1)$. The category $(\mathbb{T}_1, \alpha_1)/\mathbf{St}_{ctx,tm}$ will be denoted by $\mathbf{TT}^{\alpha_1}_{ctx,tm}$.

The construction of colimits in Proposition 2.12 implies that $L$ preserves colimits. It follows that $\mathbf{St}_{S_0}$ is cocomplete. Since $L$ preserves colimits, the forgetful functor $\mathbf{St}_{S_0} \rightarrow \mathbb{T}_{S_0}/\mathbf{Th}_{S}$ has a left adjoint $st : \mathbb{T}_{S_0}/\mathbf{Th}_{S} \rightarrow \mathbf{St}_{S_0}$, which we call the stabilization functor. More generally, for every $(\mathbb{T}_a, \alpha) \in \mathbf{St}_{S_0}$, we define a left adjoint $st_{(\mathbb{T}_a, \alpha)} : \mathbb{T}_a/\mathbf{Th}_{S} \rightarrow (\mathbb{T}_a, \alpha)/\mathbf{St}_{S_0}$ to the forgetful functor $U_{(\mathbb{T}_a, \alpha)} : (\mathbb{T}_a, \alpha)/\mathbf{St}_{S_0} \rightarrow \mathbb{T}_a/\mathbf{Th}_{S}$. Let $\alpha : \mathbb{T}_a \rightarrow \mathbb{T}$ be an object of $\mathbb{T}_a/\mathbf{Th}_{S}$. Let $e : L^\infty(\mathbb{T}) \rightarrow E$ be the coequalizer of the following maps:
\[
\prod_{n \in \mathbb{N}} L^{n+1}(\mathbb{T}_a) \xrightarrow{\prod_{n \in \mathbb{N}} L^n(f)} \prod_{n \in \mathbb{N}} L^n(\mathbb{T}) \xrightarrow{\prod_{n \in \mathbb{N}} L^n(g)} \prod_{n \in \mathbb{N}} L^n(\mathbb{T}_a) \xrightarrow{\prod_{n \in \mathbb{N}} L^n(e)} L^\infty(\mathbb{T})
\]
where $L^\infty(X)$ is the following colimit:

$$X \rightarrow X \amalg L(X) \rightarrow X \amalg L(X \amalg L(X)) \rightarrow \ldots$$

and $f, g : L(T_a) \rightarrow L^\infty(T)$ are defined as follows: $f$ is the composite $L(T_a) \xrightarrow{\alpha} T_a \xrightarrow{\alpha} \nabla \rightarrow L^\infty(T)$, and $g$ is the composite $L(T_a) \xrightarrow{\beta} L(T) \rightarrow L^\infty(T)$. Since $L$ preserves colimits, $L(E)$ is a coequalizer of $i^{n+1} \circ \bigsqcup_{n \in \mathbb{N}} L^{n+1}(f)$ and $i^{n+1} \circ \bigsqcup_{n \in \mathbb{N}} L^{n+1}(g)$. By the universal property of coequalizers we have a map $\beta : L(E) \rightarrow E$. We define $st_{(T_a, \alpha)}(a)$ as $(E, \beta)$, and morphism $(T_a, \alpha) \rightarrow (E, \beta)$ as the composite $T_a \xrightarrow{\alpha} \nabla \rightarrow L^\infty(T) \xrightarrow{\beta} E$. This map is a morphism of algebras for $L$ since $e$ coequalizes $f$ and $g$. Moreover, if $(D, \delta)$ is an object of $(T_a, \alpha)/St_{S_i}$, then a map $L^\infty(T) \rightarrow D$ is a morphism of algebras if and only if it factors through $E$. It follows that $st_{(T_a, \alpha)}$ is left adjoint to $U_{(T_a, \alpha)}$. In particular, there are stabilization functors $st^0 : TT^0 \rightarrow TT^0_{st}$ and $st^1 : TT^1 \rightarrow TT^1_{st}$.

### 4.2. Regular theories

Regular theories are stable theories in which substitution commutes with all function symbols. To define such theories, we need to define weakening first. For every $p \in \{ty, tm\}$, the operations of weakening $wk_{p,n}^{m,l} : (ctx, n + m) \times (p, n + l) \rightarrow (p, n + m + l)$ are defined as follows:

$$wk_{p,n}^{m,0}(\Gamma, a) = subst_{p,n+m,n}(\Gamma, a, v_{n+1}, \ldots, v_{m})$$

$$wk_{p,n}^{m,l+1}(\Gamma, a) = subst_{p,n+m+l,n+l+1}(\Gamma, a, v_{n+m+l+1}, \ldots, v_{m+l+1}, v_1, \ldots, v_0),$$

where $\Gamma' = wk_{p,ty}^{m,l}(\Gamma, ctx(a))$. We also define $wk_{ctx,n}^{m,l} : (ctx, n + m) \times (ctx, n + l) \rightarrow (ctx, n + m + l)$ as follows:

$$wk_{ctx,n}^{m,0}(\Gamma, a) = \Gamma$$

$$wk_{ctx,n}^{m,l+1}(\Gamma, a) = wk_{ty,n}^{m,l}(\Gamma, a)$$

To define regular theories, we need to introduce a new derived operation. For every $m, n, k \in \mathbb{N}$ and $p \in \{ctx, ty, tm\}$, we define $subst_{p,n,k}^{m,l} : (ctx, n) \times (p, k + m) \times (tm, n) \rightarrow (p, n + m)$. First, let $\text{subst}_{ctx,n,k}^{m,l}(B, A, a_1, \ldots, a_k) = B$ and $\text{subst}_{ty,n,k}^{m+1} = \text{subst}_{ty,n,k}^{m+1}$. If $p \in \{ty, tm\}$, then let $\text{subst}_{p,n,k}(B, a, a_1, \ldots, a_k)$ be equal to

$$\text{subst}_{p,n+m,k+m}(B', a, wk_{tm,n}^{m,l}(a_1), \ldots, wk_{tm,n}^{m,l}(a_k), v_{m-1}, \ldots, v_0)$$

where $B' = \text{subst}_{ctx,n,k}(B, ctxk+m(a_1), a_1, \ldots, a_k)$.

For every stable theory $T$, every $m \in \mathbb{N}$, and every function symbol $\sigma : (p_1, n_1) \times \ldots \times (p_k, n_k) \rightarrow (p, n + m)$, we define a new derived symbol $\sigma_m : (ctx, m) \times (p_1, n_1 + m) \times \ldots \times (p_k, n_k + m) \rightarrow (p, n + m)$ as follows:

$$\sigma_0(\Gamma, x_1, \ldots, x_k) = \sigma(x_1, \ldots, x_k)$$

$$\sigma_{m+1}(\Gamma, x_1, \ldots, x_k) = \alpha(L(f^m(\Gamma), \sigma_m(\Gamma, x_1, \ldots, x_k)))$$

Analogously, we can define derived symbols $R_m : (ctx, m) \times (p_1, n_1 + m) \times \ldots \times (p_k, n_k + m)$ for every predicate symbol $R : (p_1, n_1) \times \ldots \times (p_k, n_k)$:

$$R_0(\Gamma, x_1, \ldots, x_k) = R(x_1, \ldots, x_k)$$

$$R_{m+1}(\Gamma, x_1, \ldots, x_k) = \alpha(L(f^m(\Gamma), R_m(\Gamma, x_1, \ldots, x_k)))$$
Definition 4.3. A stable algebraic dependent type theory with substitution is regular if, for every \( \sigma \in \mathcal{F} \), \( \sigma : (p_1, q_1) \times \ldots \times (p_k, q_k) \to (p, q) \) and every \( R \in \mathcal{P} \), \( R : (p_1, q_1) \times \ldots \times (p_k, q_k) \), the following sequents are derivable in it:

\[
\begin{align*}
\sigma_m(\Delta, b_1, \ldots, b_k) \bsq \varphi \quad &\vdash \quad \text{subst}_{p,n,m}(\Gamma, \sigma_m(b_1, \ldots, b_k), a_1, \ldots, a_m) = \sigma_n(\Gamma, b_1', \ldots, b_k') \\
R_m(\Delta, b_1, \ldots, b_k) \bsq \varphi \quad &\vdash \quad R_n(\Gamma, b_1', \ldots, b_k')
\end{align*}
\]

where \( \varphi = \bigwedge_{1 \leq i \leq m} ty(a_i) = \text{subst}(\Gamma, ft^{m-i}(\Delta), a_1, \ldots, a_{i-1}) \), \( V = \Gamma, \Delta, b_1, \ldots, b_k, a_1, \ldots, a_m \), and \( b_i' = \text{subst}_{p,n,m}(\Gamma, b_i, a_1, \ldots, a_m) \).

We call these sequents the regularity axioms. The full subcategory of \( \mathbf{TT}^1_{st} \) on regular theories will be denoted by \( \mathbf{TT}^1_{reg} \).

Remark 4.4. The construction of colimits in Proposition 2.12 implies that \( \mathbf{TT}^1_{reg} \) is closed under colimits in \( \mathbf{TT}^1_{st} \).

Let \( (T, \alpha) \in \mathbf{TT}^0_{st} \) be a stable theory, and let \( \mathcal{A} \) be a set of sequents. Then we will say that \( \mathcal{A} \) is stable in \( T \) if \( \alpha(L(\mathcal{A})) \) is derivable in \( T \cup \mathcal{A} \). For every set \( \mathcal{A} \), there is the minimal stable set \( st(\mathcal{A}) \) which contains \( \mathcal{A} \). We call this set the stabilization of \( \mathcal{A} \), and it is defined as \( \bigcup_{n \in \mathbb{N}} \alpha^n(\mathcal{A}) \), where \( \alpha^0(\mathcal{A}) = \mathcal{A} \), and \( \alpha^{n+1}(\mathcal{A}) = \alpha^n(\alpha(L(\mathcal{A}))) \).

If \( \mathcal{A} \) is a stable set, then \( T \cup \mathcal{A} \) is a stable theory. Note that the regularity axioms are stable in any theory. Thus we can define regularization functor \( \text{reg} : \mathbf{TT}^1_{st} \to \mathbf{TT}^1_{reg} \) as follows: \( \text{reg}(T) = T \cup \mathcal{R} \), where \( \mathcal{R} \) is the set of regularity axioms. It is easy to see that \( \text{reg} \) is a left adjoint to the inclusion \( \mathbf{TT}^1_{reg} \hookrightarrow \mathbf{TT}^1_{st} \).

5. Contextual representation of theories

The definition of algebraic dependent type theories that we used before has a disadvantage that terms contain a lot of redundant information. For example, when we describe a term we need to repeat the context in which it is defined several times. Also, we need explicitly specify when we use weakening. In this subsection, we describe a representation of stable theories that is often equivalent to our previous definition and that does not have these disadvantages. We call such theories contextual.

5.1. Contextual theories. First, let us describe the syntax of contextual theories. We will use the following set of sorts: \( T = \{ ty, tm \} \times \mathbb{N} \). A contextual theory \( T = (\mathcal{F}_0, \mathcal{P}_0, \mathcal{A}) \) consists of sets \( \mathcal{F}_0 \) and \( \mathcal{P}_0 \) of function and predicate symbols respectively and a set of axioms \( \mathcal{A} \). Each function symbol \( \sigma \in \mathcal{F}_0 \) is equipped with a signature:

\[
\sigma : s_1 \times \ldots \times s_k \to p,
\]

where \( s_1, \ldots, s_k \in T \) and \( p \in \{ ty, tm \} \). Similarly, each predicate symbol \( R \in \mathcal{P}_0 \) is equipped with a signature:

\[
\quad R : s_1 \times \ldots \times s_k,
\]

where \( s_1, \ldots, s_k \in T \).

For every \( T \)-set \( V \) of variables, we define the \( T \)-set \( CTerm(V) \) of contextual terms inductively:

1. \( v_i \in CTerm(V)_{(im,n)} \) for every \( i < n \).
2. \( x \in CTerm(V)_{(p,m)} \) for every \( x \in V_{(p,m)}, m \leq n \).
(3) \( \sigma((A_1^1, \ldots, A_{n_i}^1), t_1, \ldots, (A_i^k, A_{n_k}^k), t_k) \in CTerm(V)_{(p, n)} \) for every \( \sigma \in F_0 \), \( \sigma : (p_1, n_1) \times \ldots \times (p_k, n_k) \rightarrow p \), every \( t_i \in CTerm(V)_{(p, n + n_k)} \), and every \( \psi \in CTerm(V)_{(ty, n + j)} \).

A context with free variables in \( V \) is a finite sequence \( A_1, \ldots, A_n \), where \( A_i \in CTerm(V)_{(ty, i-1)} \). The set of contexts will be denoted by \( Ctx(V) \). The length of a context \( \Gamma \) will be denoted by \( |\Gamma| \). Let us denote by \( CForm(V)_{n} \) the set of expressions of the following forms:

1. \( a \equiv a' \), where \( a, a' \in CTerm(V)_{(p, n)} \).

2. \( R((A_1^1, \ldots, A_{n_i}^1), t_1, \ldots, (A_i^k, A_{n_k}^k), t_k), \) where \( R \in \mathcal{P}_0 \), \( R : (p_1, n_1) \times \ldots \times (p_k, n_k) \rightarrow \), \( t_i \in CTerm(V)_{(p, n + n_k)} \), and \( A_i^j \in CTerm(V)_{(ty, n + j)} \).

An atomic contextual formula with free variables in \( V \) is an expression of the form \( \Gamma \vdash \varphi \), where \( \Gamma \in Ctx(V) \) and \( \varphi \in CForm(V)_{|\Gamma|} \). We write \( \Gamma \vdash a \downarrow \) for \( \Gamma \vdash a = a \). We write \( \Gamma \vdash ctx \) for \( \top \) if \( \Gamma \) is empty, and if \( \Gamma = A_1, \ldots, A_n \), \( n > 0 \), then \( \Gamma \vdash ctx \) means \( A_1, \ldots, A_{n-1} \vdash A_n \downarrow \).

A contextual formula is a finite conjunction of atomic contextual formulas. A sequent is an expression of the form \( \varphi \vdash \psi \), where \( \varphi \) and \( \psi \) are contextual formulas such that \( FV(\varphi) \subseteq FV(\psi) \). We do not have the condition on free variables in the usual definition of sequents since we have axiom (a1). We do not have an analog of this axiom in the contextual setting, so we need to assume that the premise of every sequent explicitly mentions every variable. This also means that we do not need to explicitly list all variables since they all must appear in the left hand side of the sequent.

For simplicity, we left weakening operations implicit in this notation. This means that we need to define substitution functions slightly differently. For every term or formula \( e \), every variable \( x \), and every term \( t \), we define \( e[t/x] \) recursively as usual with the only difference: if \( x \in V_{(p, m)} \) is a term of sort \( (p, n) \), then we define \( x[t/x] \) as \( t \uparrow^{m-n} \), where \( \uparrow \) is the usual operation of lifting de Bruijn indices.

Now, let us describe the inference rules of the logic of contextual theories. The first chunk of inference rules is given by rules (b1)-(b6) as defined in subsection (2.2). The rest of the inference rules is given below. Since we do not have an analog of (a1) here, we need to add analogs of (a4), (a4'), and (a5).

\[
(\Gamma \vdash x \equiv y) \land \varphi \longrightarrow \varphi[y/x] \quad (c2) \quad \varphi \longrightarrow \psi \quad \varphi[t/x] \longrightarrow \psi \quad (c3)
\]

\[
\Gamma \vdash R(\ldots, (A_1^1, \ldots, A_{n_i}^1), x_i, \ldots) \quad \Gamma, A_1^i, \ldots, A_{n_i}^i \vdash x_i \downarrow \quad (c4)
\]

\[
\Gamma \vdash x \equiv y \quad \Gamma \downarrow \quad (c4')
\]

\[
\Gamma \vdash \sigma(\ldots, (A_1^1, \ldots, A_{n_i}^1), x_i, \ldots) \downarrow \quad \Gamma, A_1^i, \ldots, A_{n_i}^i \vdash x_i \downarrow \quad (c5)
\]

\[
\Gamma \vdash x \downarrow \land \Delta \vdash x \downarrow \quad \Gamma = \Delta \quad (c6)
\]

Now, we define partial contextual terms as pairs of contextual terms and contextual formulas. As before, we define equivalence relations on partial contextual terms and contextual formulas. Partial contextual terms \( t \) and \( t' \) of sort \( (p, 0) \) are equivalent if and only if \( FV(t) = FV(t') \) and sequents \( \Gamma \vdash t \equiv t' \) are derivable. Contextual formulas \( \varphi \) and \( \psi \) are equivalent if and only if \( FV(\varphi) = FV(\psi) \) and sequents \( \varphi \leftrightarrow \psi \) are derivable.
Let $f$ be a function such that, for every function symbol $\sigma : (p_1, n_1) \times \ldots \times (p_k, n_k) \to p$ of $T$, $f(\sigma)$ is a partial contextual term of $T'$ of sort $(p, 0)$ with free variables in $\{x_1^0, x_1^{n_1-1}, x_2^0, x_2^{n_2-1}, \ldots, x_k^0, x_k^{n_k-1}\}$, where $x_i^j : (ty, j)$ and $x_i : (p, n_i)$ and, for every predicate symbol $R : (p_1, n_1) \times \ldots \times (p_k, n_k)$, $f(R)$ is a contextual formula of $T'$ with the same free variables as above. Then $f$ can be extended in the obvious way to a function on terms, formulas, and contexts. Morphisms of contextual theories $T$ and $T'$ are equivalence classes of functions $f$ as above which preserve axioms and such that the following sequents are derivable:

\[
\begin{align*}
f(\sigma) & \downarrow \quad x_i^0, \ldots x_i^{n_i-1} \vdash x_i^1 \\
f(R) & \downarrow \quad x_i^0, \ldots x_i^{n_i-1} \vdash x_i^1
\end{align*}
\]

It follows that such functions preserve rules (c4) and (c5), and other inference rules are always preserved. Compositions of morphisms are defined in the obvious way.

We can use the traditional notation and write $\Gamma \vdash a : A$ and $\Gamma \vdash \alpha \equiv \alpha'$ : $A$ for $\Gamma \vdash ty(a) \equiv A$ and $(\Gamma \vdash \alpha \equiv \alpha') \land (\Gamma \vdash a : A)$ respectively. We also write

\[
\begin{array}{c}
\varphi_1, \ldots, \varphi_n \\
\psi
\end{array}
\quad \begin{array}{c}
\varphi_1, \ldots, \varphi_n \\
\psi
\end{array}
\]

for $\varphi_1 \land \ldots \land \varphi_n \vdash \psi$ and $\varphi_1 \land \ldots \land \varphi_n \vdash \psi$ respectively.

A $T_0$-theory is a contextual theory with a distinguished function symbol $ty : (tm, 0) \to ty$ such that the following sequents are derivable:

\[
\begin{align*}
\Gamma \vdash a & \downarrow & A_1, \ldots, A_n \vdash ctx \\
\Gamma \vdash ty(a) & \downarrow & A_1, \ldots, A_n \vdash v_i : A_{n-i}
\end{align*}
\]

A $T_1$-theory is a $T_0$-theory with two distinguished function symbols $\nabla : (p, 1) \times (tm, 0) \to p$ such that the following sequent is derivable:

\[
\begin{align*}
\Gamma, ty(a) & \vdash b & \Gamma, ty(a) & \vdash b & \Gamma \vdash a_j, 0 \leq j \leq i \\
\Gamma \vdash b[a] & \downarrow & \Gamma \vdash ty(b[a]) \equiv ty(b)[a] & \Gamma \vdash v_i[a_0] \ldots [a_i] \equiv a_i
\end{align*}
\]

A regular contextual theory is a $T_1$-theory $T$ such that, for every function symbol $\sigma$ and every predicate symbol $R$, the following sequents are derivable:

\[
\begin{align*}
\Gamma, A \vdash \sigma(\ldots, (B_1^i, \ldots B_{n_i}^i).x_i, \ldots) & \downarrow & \Gamma \vdash a : A \\
\Gamma \vdash \sigma(\ldots, (B_1^i, \ldots B_{n_i}^i).x_i, \ldots)[a] \equiv \sigma(\ldots, x_i', \ldots)
\end{align*}
\]

where $x_i' = (B_1^i[a], \ldots B_{n_i}^i[v_0] \ldots [v_{n_i-2}[a]).x_i[v_0] \ldots [v_{n_i-1}[a])$.

**Remark 5.1.** This is only one half of the regularity axioms. The other half states that the weakening operations commute with function and predicate symbols. This property is already incorporated in the definition of contextual theories. Thus we cannot work with non-regular stable theories in this syntax. If we want to do so, we just need to make weakening operations explicit.

**Remark 5.2.** It is possible to use ordinary named representation for contextual theories. That is we can use binding instead of de Bruijn indices. The translation between two representations is straightforward. We give several examples of theories described in terms of the named syntax in the next subsection.
For every $T_1$-theory $T$, we define ordinary theory $C(T)$ under $T_1$. The set of function symbols of $C(T)$ consists of symbols 

$$\sigma_n : (ctx, n) \times (p_1, n + 1) \times \cdots \times (p_k, n + k) \to (p, n)$$

for every $n \in \mathbb{N}$ and every $\sigma \in \mathcal{F}_0$, $\sigma : (p_1, n_1) \times \cdots \times (p_k, n_k) \to p$ together with function symbols of $T_1$. Similarly, the set of predicate symbols of $C(T)$ consists of symbols 

$$R_n : (ctx, n) \times (p_1, n + 1) \times \cdots \times (p_k, n + k)$$

for every $n \in \mathbb{N}$ and every $R \in \mathcal{P}_0$, $R : (p_1, n_1) \times \cdots \times (p_k, n_k)$.

Now, let us define a function $C : Term_{C(T)}(V)(ctx, n) \times CTerm_T(V)(p, n) \to Term_{C(T)}(V)(p, n)$ as follows:

$$C(\Gamma, v_1) = v_{n,1}(\Gamma)$$

$$C(\Gamma, x) = x|_{ctx(x) = \Gamma}$$

$$C(\Gamma, \sigma((A_1^1, \ldots, A_{n_1}^1), t_1, \ldots)) = \sigma_n(\Gamma, \ldots, C(\Gamma, \ldots, C(\Gamma, A_1^1), \ldots, A_{n_1}^1)(t_1), \ldots)$$

Function $C : Term_{C(T)}(V)(ctx, n) \times CForm_T(V)_n \to Form_{C(T)}(V)$ is defined similarly:

$$C(\Gamma, a \equiv a') = (C(\Gamma, a) = C(\Gamma, a'))$$

$$C(\Gamma, R((A_1^1, \ldots, A_{n_1}^1), t_1, \ldots)) = R_n(\Gamma, \ldots, C(\Gamma, \ldots, C(\Gamma, A_1^1), \ldots, A_{n_1}^1)(t_1), \ldots)$$

For every context $\Gamma$, we define a term $C(\Gamma)$ of $C(T)$ of sort $(ctx, |\Gamma|)$. If $\Gamma$ is empty, then let $C(\Gamma)$ be equal to $\ast$. If $\Gamma = \Gamma', A$, then let $C(\Gamma)$ be equal to $C(C(\Gamma'), A)$. For every atomic formula $\Gamma \vdash \varphi$ of $T$, we define an atomic formula $C(\Gamma \vdash \varphi)$ of $C(T)$ as $C(C(\Gamma), \varphi)$. Now, it is easy to define $C$ on formulas and sequents.

The set of axioms of $C(T)$ consists of axioms of $T_1$, regularity axioms, axiom $C(S)$ for every axiom $S$ of $T$, and the following axioms:

$$\sigma_n(\Gamma, x_1, \ldots, x_k) \downarrow \frac{\Gamma, x_1, \ldots, x_k}{\Gamma, x_1, \ldots, x_k} ctx(\sigma_n(\Gamma, x_1, \ldots, x_k)) = \Gamma \land \bigwedge_{1 \leq i \leq k} f^{v_i}(ctx(x_i)) = \Gamma$$

$$\frac{\Gamma, B, a}{\Gamma, B, a} subst_{p, n, 1}(\Gamma, B, a) \equiv ([\ldots])_n(\Gamma, B, a)$$

where the first axiom is defined for every function symbol $\sigma$.

For every contextual theory $T$, we can define a function $\alpha : L(C(T)) \to C(T)$ as follows:

$$\alpha(L(\sigma_n)(\Gamma, \Delta, x_1, \ldots, x_k)) = \sigma_{n+1}(\Delta, x_1, \ldots, x_k)|_{f^{v_1}(\Delta) = \Gamma}$$

$$\alpha(L(R_n)(\Gamma, \Delta, x_1, \ldots, x_k)) = R_{n+1}(\Delta, x_1, \ldots, x_k)|_{f^{v_1}(\Delta) = \Gamma}$$

This function might not preserve axioms of $L(C(T))$. We say that a theory $T$ is stable if, for every axiom

$$\bigwedge_{1 \leq i \leq n} \Gamma_i \vdash \varphi_i \bigwedge_{1 \leq i \leq k} \Delta_i \vdash \psi_i,$$

the following sequent is derivable in $T$:

$$\vdash A \downarrow \bigwedge_{1 \leq i \leq n} A, \Gamma_i \vdash \varphi_i \bigwedge_{1 \leq i \leq k} A, \Delta_i \vdash \psi_i,$$
Note that if \( n < m \), then \( CTerm(V)_{(p,n)} \) is a subset of \( CTerm(V)_{(p,m)} \). Hence the sequent above actually makes sense. If \( T \) is stable, then \( \alpha \) preserves axioms, and so \( C(T) \) is also stable.

For every morphism \( f : T \to T' \) of stable contextual theories, we can define a stable morphism \( C(f) : C(T) \to C(T') \). For every function or predicate symbol \( \sigma \), we define \( C(f)\{\sigma_\eta(\Gamma, x_1, \ldots, x_k)\} \) as follows:

\[
C(\Gamma, f\{\sigma(\ldots, (x_0^i, \ldots x_{n_i}^{n_i-1}), x_i, \ldots)\})[f^{l_{n_i}-1}(ctx(x_i))/x_i^j].
\]

**Proposition 5.3.** Function \( C \) determines a bijection between the set of morphism between stable theories \( T \) and \( T' \) and the set of stable morphisms between theories \( C(T) \) and \( C(T') \). Moreover, \( C \) preserves compositions and identity morphisms. Thus \( C \) is a fully faithful functor from the category of stable contextual \( T_\perp \)-theories to the category of regular algebraic dependent type theories.

**Proof.** First, let us prove that \( C \) preserves substitution in a weak sense. Let \( t \) be a term, and let \( x \) be a variable. Then, for every occurrence of \( x \) in \( t \), we can define a context \( \Delta \) such that \( \Gamma \vdash t \line{tx(x)}=\Delta \). This context is defined recursively in the obvious way. It is easy to see that if \( x \in \text{FV}(t) \) and \( \Delta \) is a context corresponding to some occurrence of \( x \) in \( t \), then the following sequent is derivable:

\[
\vdash^V C(\Gamma, t[s/x]) \equiv C(\Gamma, t)[C(\Delta, s)/x]
\]

This implies that, for every term contextual \( t \) and every morphism \( f \), sequent \( \vdash^V C(f)(C(\Gamma, t)) \equiv C(\Gamma, f(t)) \) is derivable. It follows that \( C \) preserves compositions. The fact that \( C \) preserves identity morphisms is obvious.

Now, let us prove that \( C \) preserves theorems. Rules (\( b_1 \)) – (\( b_2 \)) are mapped to rules of the same form. Rule (\( c_2 \)) is mapped to the following rule:

\[
x = y \land ctx(x) = C(\Gamma) \land ctx(y) = C(\Gamma) \land C(\varphi) \vdash^x,y C(\varphi)[y/x] \land ctx(y) = \Delta,
\]

where \( \Delta \) is a context corresponding to an occurrence of \( x \) in \( \varphi \). Since \( C(\varphi) \) implies \( ctx(x) = \Delta \) and \( ctx(x) = ctx(y) \), this sequent follows from (\( a_2 \)).

Rule (\( c_3 \)) is mapped to the following rule:

\[
\begin{array}{c}
C(\varphi) \vdash^x C(\psi) \\
\hline
C(\varphi)[C(\Delta, t)/x] \vdash^V \psi \vdash^V C(\psi)[C(\Delta', t)/x]
\end{array}
\]

Note that \( C(\varphi) \vdash^x C(\psi) \) implies that \( C(\Delta) = C(\Delta') \). Hence this rule is admissible. Finally, it is easy to see that rules (\( c_4 \)) – (\( c_6 \)) are also preserved.

It follows that \( C \) is well-defined. Let us prove that \( C \) is surjective. First, we need to define functions \( D_p : \text{Term}_{C(T)}(V)_{(p,n)} \to CTerm_V(D(V))_{(p,n)} \) and \( E_p : \text{Term}_{C(T)}(V)_{(p,n)} \to Ctx(D(V))_n \), where \( D(V) \) is the union of \( V \) and the set of variables \( x^i : (ty, i) \) for every \( x : (p, n) \) in \( V \), \( p \in \{ty, tm\} \) and \( 0 \leq i < n \). They are
then we define a contextual morphism $g$

where $\Gamma \vdash \tau$ and predicate symbol $\tau$ are derivable:

$$D_p(x) = x$$

$$D_p(\sigma_n(\Gamma, t_1, \ldots, t_k)) = \sigma(\ldots, E_p(t_i) \geq n, D_p(t_i), \ldots) | \wedge_{1 \leq i \leq k} E_{ctx}(\Gamma) = E_p(t_i) < n$$

$$D_p(v_n, \Gamma)) = v_i$$

$$D_p(\text{sub}(\Gamma, b, t_1, \ldots, t_k)) = D_p(b)[D_{tm}(t_1) \ldots [D_{tm}(t_k)] | \wedge_{1 \leq i \leq k} E_{ctx}(\Gamma) = D_p(t_i)$$

$$D_{ty}(ty_n(t)) = ty(D_{tm}(t))$$

$$D_p(ft^k(ctx(x))) = x^{n-1-k}$$

$$D_{ty}(ctx) = E_{ctx}(ft(t))$$

$$D_{ti}(t) = E_{ctx}( ft(t))$$

$$D_{tm}(t) = E_{ty}( ty(t))$$

where $\Gamma \vdash \tau$ is the first $n$ types in context $\Gamma$ and $\Gamma \vdash \tau$ is the rest of the context.

Let us prove that $C$ is surjective. If $f : C(T) \to C(T')$ is a stable morphism, then we define a contextual morphism $g : T \to T'$ as follows:

$$g(\sigma(\ldots, (x_i^0, \ldots, x_i^{n-1}, x_1, \ldots)), \ldots) D_p(f(\sigma(\ldots, x_1, \ldots, x_k))) | \wedge_{1 \leq i \leq k} x_i^0 \ldots x_i^{n-1} = x_i$$

To prove that $C(g) = f$, it is enough to show that $C(g) \sigma_0 = f(\sigma_0)$ for every function and predicate symbol $\sigma$ since $f$ and $C(g)$ are stable. Unfolding the definitions, we can see that we just need to show that, for every term $t$, the following sequents are derivable:

$$\vdash_{x_1^0 \ldots x_k} C(C(E_p(t)), D_p(t)) \wedge t^{n-1-j}(ctx(x_i)) \models t$$

$$\vdash_{x_1^0 \ldots x_k} C(E_{ctx}(t)) \wedge t^{n-1-j}(ctx(x_i)) \models t$$

This is easy to prove by induction.

Finally, let us prove that $C$ is injective. We will show that if $C(f)$ and $C(g)$ agree on function symbols, then so do $f$ and $g$. Analogous statement for relation symbols is proved similarly.

It is easy to see that $D_p$ preserves inference rules. It follows that $D_p$ preserves theorems in the sense that if $\varphi \vdash_{x_1^0 \ldots x_k} \psi$ is a theorem, then $\wedge_{1 \leq i \leq k} x_i^0 \ldots x_i^{n-1} \vdash_{x_i} \wedge D(\varphi) \vdash D(\psi)$ is also a theorem, where $D$ is defined as follows:

$$D(t = t') = E_p(t) \vdash_{D_p(t)} D_p(t') \wedge E_p(t) = E_p(t')$$

$$D(R_n(\Gamma, t_1, \ldots, t_k)) = \Gamma' \vdash_{D_p(t_i) \geq n, D_p(t_i), \ldots} | \wedge_{1 \leq i \leq k} R(\ldots, E_p(t_i) = E_p(t_i) < n),$$

where $\Gamma' = D_{ty}(\Gamma)$. If $f, g : T \to T'$ are morphisms of contextual theories such that $\vdash_{x_1^0 \ldots x_k} \{ f(\sigma(\ldots, x_1, \ldots, x_k)) \} \models C(g)(\sigma(\ldots, x_1, \ldots, x_k))$ is derivable, then

$$\bigwedge_{1 \leq i \leq k} \varphi_i \vdash_{D_p(\sigma(\ldots, x_1, \ldots, x_k))} \models D_p(C(g)(\sigma(\ldots, x_1, \ldots, x_k)))$$

is also derivable, where $\varphi_i = x_i^0 \ldots x_i^{n-1} \vdash_{x_i} \wedge$. Thus we only need to prove that, for every context $\Gamma$ and every contextual term $t$ with free variables in $\{x_1, \ldots, x_k\}$, the following sequent is derivable:

$$\bigwedge_{1 \leq i \leq k} \varphi_i \vdash_{\Gamma} t \models D_p(C(\Gamma), t)$$
If $t$ is a variable, then this follows from (c6). If $t$ is of the form $\sigma(t_1, \ldots t_k)$, then this follows from the inductive hypothesis.

5.2. Examples. Now, let us describe a few examples of algebraic dependent type theories with substitution. We use the named representation, which can be translated to the contextual representation as noted in Remark 5.2. We implicitly assume that all of these theories contain function symbols and axioms of $T_1$. If we take their stabilization and regularization, then we get theories corresponding to usual constructions of the type theory. Also, to simplify the notation, we use the following agreement. For every sequent of the form $\phi \vdash \Gamma \vdash A$, there is also sequent $\Gamma \vdash A$ and, for every sequent of the form $\phi \Gamma \vdash a : A$, there is also sequent $\Gamma \vdash a$. 

Example 5.4. The theory of unit types with eta rules has function symbols $\top : (ty, 0)$ and unit : $(tm, 0)$ and the following axioms:

\[
\begin{align*}
\Gamma &\vdash \top \quad \Gamma &\vdash \text{unit} \quad \Gamma &\vdash t : \top \\
\rightarrow &\vdash \text{unit} \quad \rightarrow &\vdash t \equiv \text{unit} \\
\end{align*}
\]

Example 5.5. The theory of unit types without eta rules has function symbols $\top : (ty, 0)$, unit : $(tm, 0)$ and $\top$-elim : $(ty, 1) \times (tm, 0) \times (tm, 0) \rightarrow (tm, 0)$. The axioms for $\top$ and unit are the same, and the axioms for $\top$-elim are

\[
\begin{align*}
x : \top &\vdash D \quad \Gamma &\vdash d : D[x \mapsto \text{unit}] \\
\rightarrow &\vdash \top \vdash t \equiv \text{unit} \\
\end{align*}
\]

Example 5.6. The theory of $\Sigma$ types with eta rules has function symbols $\Sigma : (ty, 1) \rightarrow (ty, 0)$, pair : $(ty, 1) \times (tm, 0) \times (tm, 0) \rightarrow (tm, 0)$, $\text{proj}_1 : (ty, 1) \times (tm, 0) \rightarrow (tm, 0)$, $\text{proj}_2 : (ty, 1) \times (tm, 0) \rightarrow (tm, 0)$ and the following axioms:

\[
\begin{align*}
x : A &\vdash B \quad \Gamma &\vdash \Sigma(x : A. B) \\
\rightarrow &\vdash \text{pair}(x : A. B, a, b) : \Sigma(x : A. B) \\
\end{align*}
\]

\[
\begin{align*}
x : A &\vdash B \quad \Gamma &\vdash \text{proj}_1(x : A. B, p) : A \\
\rightarrow &\vdash \text{proj}_2(x : A. B, p) : B[x \mapsto \text{proj}_1(x : A. B, p)] \\
\end{align*}
\]

\[
\begin{align*}
x : A &\vdash B \quad \Gamma &\vdash \text{proj}_1(x : A. B, \text{pair}(x : A. B, a, b)) \equiv a \\
\rightarrow &\vdash \text{proj}_2(x : A. B, \text{pair}(x : A. B, a, b)) \equiv b \\
\rightarrow &\vdash \text{pair}(x : A. B, \text{proj}_1(x : A. B, p), \text{proj}_2(x : A. B, p)) \equiv p \\
\end{align*}
\]
Example 5.7. The theory of \( \Sigma \) types without eta rules has the following function symbols:

\[
\Sigma : (ty, 1) \to (ty, 0) \\
pair : (ty, 1) \times (tm, 0) \times (tm, 0) \to (tm, 0) \\
\Sigma\text{-elim} : (ty, 1) \times (ty, 1) \times (tm, 2) \times (tm, 0) \to (tm, 0)
\]

The axioms for \( \Sigma \) and \( \text{pair} \) are the same, and the axioms for \( \Sigma\text{-elim} \) are:

\[
az \colon \Sigma(x : A.B) \vdash D \text{ type} \\
x : A, y : B \vdash d : D[z \mapsto \text{pair}(x : A.B, x, y)] \\
\vdash p : \Sigma(x : A.B) \\
\vdash \Sigma\text{-elim}(x : A.B, z.D, xy.d, p) : D[z \mapsto p]
\]

Example 5.8. The theory of \( \Pi \) types with eta rules has function symbols

\[
\Pi : (ty, 1) \to (ty, 0) \\
\lambda : (tm, 1) \to (tm, 0) \\
\text{app} : (ty, 1) \times (tm, 0) \times (tm, 0) \to (tm, 0)
\]

and the following axioms:

\[
x : A \vdash b : B \\
\vdash \Pi(x : A.B) \text{ type} \\
\vdash \lambda(x : A.b) : \Pi(x : A.B) \\
\vdash a : A \\
\vdash f : \Pi(x : A.B) \\
\vdash \text{app}(x : A.B, f, a) : B[x \mapsto a] \\
\vdash b : \Pi(x : A.B) \\
\vdash \lambda(x : A. \text{app}(x : A.B, b, x)) \equiv b
\]

Example 5.9. The theory of identity types has function symbols

\[
\text{Id} : (ty, 0) \times (tm, 0) \times (tm, 0) \to (ty, 0) \\
\text{refl} : (ty, 0) \times (tm, 0) \to (tm, 0) \\
J : (ty, 0) \times (ty, 3) \times (tm, 1) \times (tm, 0) \times (tm, 0) \times (tm, 0) \to (tm, 0)
\]

and the following inference rules:

\[
\vdash a : A \\
\vdash a' : A \\
\vdash a : A \\
\vdash \text{refl}(A,a) : \text{Id}(A,a,a) \\
\vdash \text{Id}(A,a,x,y) \vdash D \text{ type} \\
\vdash J(A, xy.z, d, x, d, a, a', p) : D[x \mapsto a, y \mapsto a', z \mapsto p] \\
\vdash J(A, xy.z, d, x, d, a, a, \text{refl}(A,a)) \equiv d[x \mapsto a]
\]

where \( D' = D[y \mapsto x, z \mapsto \text{refl}(A,x)] \).
Example 5.10. We can define a theory which is isomorphic to the previous one, but terms of this theory contain less redundant information.

\[ Id : (tm, 0) \times (tm, 0) \rightarrow (ty, 0) \]
\[ refl : (tm, 0) \rightarrow (tm, 0) \]
\[ J : (ty, 3) \times (tm, 1) \times (tm, 0) \times (tm, 0) \rightarrow (tm, 0) \]

Then the axioms should look like this:

\[
\frac{\vdash ty(a) \equiv ty(a')}{\vdash Id(a, a') : type} \quad \frac{\vdash \text{refl}(a) : Id(ty(a), a, a)}{x : A, y : A, z : Id(x, y) \vdash D \text{ type}}
\]

\[
\vdash J(xyz, D, x.d, a, a', p) : D[x \mapsto a, y \mapsto a', z \mapsto p]
\]

where \( A = ty(a) \) and \( D' = D[y \mapsto x, z \mapsto \text{refl}(x)] \).

Example 5.11. We define an endofunctor \( U \) on the category of regular contextual theories. For every such theory \( \mathcal{T} \), theory \( U(\mathcal{T}) \) has the same symbols as \( \mathcal{T} \), but we also add a universe which is closed under all function symbols of \( \mathcal{T} \).

Let \( \mathcal{T} \) be a regular contextual theory. Then \( U(\mathcal{T}) \) has the same predicate symbols as \( \mathcal{T} \) and the following function symbols:

\[
U : ty
\]
\[
El : (tm, 0) \rightarrow ty
\]
\[
\sigma : s_1 \times \ldots \times s_k \rightarrow p
\]
\[
\sigma_U : U(s_1) \times \ldots \times U(s_k) \rightarrow tm
\]

for every function symbol \( \sigma : s_1 \times \ldots \times s_k \rightarrow p \) of \( \mathcal{T} \), where \( U(ty, n) = (tm, 0) \times \ldots \times (tm, n) \) and \( U(tm, n) = U(ty, n) \times (tm, n) \).

Theory \( U(\mathcal{T}) \) has the following axioms:

\[
\Gamma \vdash a : U
\]

For every function symbol \( \sigma : (p_1, n_1) \times \ldots \times (p_k, n_k) \rightarrow p \) of \( \mathcal{T} \) and every \( 1 \leq i \leq k \), if \( p_i = ty \), then we add the following axioms to \( U(\mathcal{T}) \):

\[
\Gamma \vdash \sigma_U(\ldots, t_i, \ldots) \downarrow \quad \Gamma, A_1^j, \ldots, A_{j-1} \vdash El(a_j) = A_j^{i+1},
\]

where \( t_i = a_1, \ldots, (A_1^{n_j+1}, \ldots, A_k^{n_j+1}), a_{n_j+1} \). If \( p_i = tm \), then we add the following axioms:

\[
\Gamma \vdash \sigma_U(\ldots, t_i, \ldots) \downarrow \quad \Gamma, A_1^j, \ldots, A_{j-1} \vdash El(a_j) = A_j^{i+1}
\]

\[
\Gamma \vdash \sigma_U(\ldots, t_i, \ldots) \downarrow \quad \Gamma, A_1^{n_j+1}, \ldots, A_k^{n_j+1} \vdash El(a_{n_j+1}) = ty(b)
\]

where \( t_i = a_1, \ldots, (A_1^{n_j+1}, \ldots, A_k^{n_j+1}), a_{n_j+1}, (B_1, \ldots, B_{n_i}) \). b.

Also, for every function symbol \( \sigma \) of \( \mathcal{T} \) we add the following axiom:

\[
\Gamma \vdash El(\sigma_U(\ldots, t_i, \ldots)) \equiv e_p(\sigma(\ldots, t_i', \ldots)),
\]
where
\[
t_i = \begin{cases} 
a_1, El(a_1). a_2, \ldots (El(a_1), \ldots El(a_n)). a_{n+1} & \text{if } p_i = ty \\
a_1, \ldots (El(a_1), \ldots El(a_n)). b & \text{if } p_i = tm 
\end{cases}
\]
and
\[
t_i' = \begin{cases} 
(El(a_1), \ldots El(a_n)). El(a_{n+1}) & \text{if } p_i = ty \\
(El(a_1), \ldots El(a_n)). b |_{ty(b)=El(a_{n+1})} & \text{if } p_i = tm.
\end{cases}
\]

To define the rest of the axioms of \(U(T)\), we need to introduce a few auxiliary functions. For every set of variables \(V\), we define a set \(U(V)\) as follows:
\[
V \uplus \{ x_U : (tm, n) \mid x : (p, n) \in V \}.
\]

Now, we define a function \(U : \text{Term}_T(V)_{(ty,n)} \rightarrow \text{Term}_{U(T)}(U(V))_{(tm,n)}\) as follows:
\[
U(e_p(x)) = x_U
\]
where \(e_{ty} = id, e_{tm} = ty, t'_i = U(A_1), \ldots A_{n_i}). U(t_i)\) if \(t_i\) is a term of sort \((ty, n_i)\), and \(t'_i = U(A_1), \ldots (A_1, \ldots A_{n_i}). U(ty(t_i)), (A_1, \ldots A_{n_i}). t_i\) if \(t_i\) is a term of sort \((tm, n_i)\).

We add all axioms of \(T\) to \(U(T)\) and, for every axiom \(\varphi \longrightarrow \psi\) of \(T\), we add the following axiom:
\[
\chi \land U(\varphi) \longrightarrow U(\psi),
\]
where \(U(\Gamma \vdash R(t_1, \ldots t_k))\) equals to \(\Gamma \vdash R(t_1, \ldots t_k)\), \(U(\Gamma \vdash t_1 \equiv t_2)\) equals to \(\Gamma \vdash U(e_p(t_1)) \equiv U(e_p(t_2)) \land \Gamma \vdash t_1 \equiv t_2\), and \(\chi\) equals to \(\bigwedge_{x \in FV(\varphi)} \Gamma_x \vdash e_p(x) = El(x_U)\), where \(\Gamma_x\) is a context such that \(\varphi \longrightarrow \Gamma_x \vdash x \downarrow\) is derivable in \(T\). Note that such context always exists and (e6) implies that it does not matter which one we choose.

Finally, let us show that \(U\) is a functor. Let \(f : T \rightarrow T'\) be a morphism of contextual theories. Then \(U(f)\) is defined in the obvious way on \(U, El\) and function symbols from \(T\). We only need to define \(U(f)(\sigma_U(\ldots, t_i, \ldots))\), where
\[
t_i = \begin{cases} 
a_1, \ldots (A^i_{n_1} \ldots A^i_{n_i+1}). a_{n_i+1} & \text{if } p_i = ty \\
a_1, \ldots (A^i_{n_1} \ldots A^i_{n_i+1}). a_{n_i+1}, (B_1, \ldots B_{n_i}). b & \text{if } p_i = tm
\end{cases}
\]
Let \(U(f)(\sigma_U(\ldots, t_i, \ldots)) = U(f(\sigma(\ldots, (C^1, \ldots C^n), \ldots)))[\rho]|_{\varphi}\), where
\[
\rho(C^j_U) = a_j
\]
\[
\rho(C^j) = El(a_j)
\]
\[
\rho(c^j_U) = a_{n_i+1}
\]
\[
\rho(c^j) = \begin{cases} 
El(a_{n_i+1}) & \text{if } c^j : ty \\
b & \text{if } c^j : tm
\end{cases}
\]
and \(\varphi\) is the conjunction of formulas \(A^j_1, \ldots A^i_{j-1} \vdash El(a_j) = A^j_{i+1}\) and \(A^i_1, \ldots A^i_{j-1} \vdash a_j \downarrow\) for all \(1 \leq j \leq n_i\) and also \(B_1, \ldots B_{n_i} \vdash b \downarrow\) if \(p_i = tm\).

It is easy to see that \(U(f)\) is a morphism of contextual theories and that \(U\) preserves identity morphisms and compositions. Thus, \(U\) is a functor.
Example 5.12. There is a natural map $T \to U(T)$. We define $U^\omega(T)$ as the colimit of the following sequence:

$$T \to U(T) \to U^2(T) \to \ldots$$

Then $U^\omega(T)$ is the theory with a hierarchy of universes closed under constructions of $T$.

References

1. J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge University Press, 1994.
2. John Cartmell, *Generalised algebraic theories and contextual categories*, Annals of Pure and Applied Logic 32 (1986), 209 – 243.
3. Thierry Coquand and Gérard Huet, *The calculus of constructions*, Information and Computation 76 (1988), no. 2, 95 – 120.
4. V. Isaev, *Model structures on categories of models of type theories*, (2016), arXiv:1607.07407.
5. Peter T. Johnstone, *Sketches of an elephant : a topos theory compendium*, Oxford Logic Guides, Clarendon Press, Oxford, 2002, Autre tirage : 2008.
6. Zhaohui Luo, *Computation and Reasoning: A Type Theory for Computer Science*, Oxford University Press, 1994.
7. Ernest G. Manes, *Algebraic theories*, Graduate texts in mathematics, Springer, New York, Heidelberg, 1976.
8. Per Martin-Löf, *An intuitionistic theory of types: predicative part*, Logic Colloquium ’73 (Bristol, 1973), North-Holland, Amsterdam, 1975, pp. 73–118. Studies in Logic and the Foundations of Mathematics, Vol. 80. MR 0387009 (52 #7856)
9. __________, *Constructive mathematics and computer programming*, Logic, methodology and philosophy of science, VI (Hannover, 1979), Stud. Logic Found. Math., vol. 104, North-Holland, Amsterdam, 1982, pp. 153–175. MR 682410 (85d:03112)
10. __________, *An intuitionistic theory of types*, Twenty-five years of constructive type theory (Venice, 1995), Oxford Logic Guides, vol. 36, Oxford Univ. Press, New York, 1998, pp. 127–172. MR 1686864
11. E. Palmgren and S.J. Vickers, *Partial horn logic and cartesian categories*, Annals of Pure and Applied Logic 145 (2007), no. 3, 314 – 353.
12. A. M. Pitts, *Categorical logic*, Handbook of Logic in Computer Science, Volume 5. Algebraic and Logical Structures (S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, eds.), Oxford University Press, 2000, pp. 39–128.
13. Thomas Streicher, *Semantics of type theory: Correctness, completeness, and independence results*, Birkhäuser Boston Inc., Cambridge, MA, USA, 1991.
14. V. Voevodsky, *B-systems*, (2014), arXiv:1410.5389.
15. __________, *Subsystems and regular quotients of C-systems*, (2014), arXiv:1406.7413.