The relative Drinfeld commutant of a fusion category and $\alpha$-induction

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Abstract

We establish a correspondence among simple objects of the relative commutant of a full fusion subcategory in a larger fusion category in the sense of Drinfeld, irreducible half-braidings of objects in the larger fusion category with respect to the fusion subcategory, and minimal central projections in the relative tube algebra. Based on this, we explicitly compute certain relative Drinfeld commutants of fusion categories arising from $\alpha$-induction for braided subfactors. We present examples arising from chiral conformal field theory.

1 Introduction

The notion of a Drinfeld center has been studied well within the Jones theory of subfactors [14]. Around 1990, Ocneanu realized that his construction of the asymptotic inclusion $M \vee (M' \cap M_{\infty}) \subset M_{\infty}$ from a hyperfinite type II$_1$ subfactor $N \subset M$ with finite index and finite depth gives an operator algebraic counterpart of the Drinfeld center construction, also called the Drinfeld double or the “quantum double”. We refer the reader to [10] for Ocneanu’s theory and related results, and to [13] for an approach based on Longo’s sector theory [18], [19].

The notion of a Drinfeld center is similar to that of a usual center of an algebra, as the name shows. Henriques [12] recently studies the Drinfeld version of a (relative/double) commutant of a fusion category. In this paper, we study the notion of the relative commutant of a full fusion subcategory in another fusion category and clarify its relations to (the relative version of) Ocneanu’s tube algebra and half-braidings along the line of [13]. We have made several computations of the Drinfeld centers for certain fusion categories arising from $\alpha$-induction in [7]. (Here $\alpha$-induction is a certain induction machinery originally introduced for an extension of a chiral conformal field theory in [20].) In this paper, we
make analogous computations for the relative commutants of these fusion categories arising from \(\alpha\)-induction. Our methods are similar to those in [7] and rely on half-braidings arising from relative braidings studied in [4].

We refer the reader to [11] for a general theory of subfactors and to [15] for a review on subfactor theory, category theory, and conformal field theory.

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2 The relative tube algebra and the relative Drinfeld commutant

Let \(D\) be a unitary fusion category and \(C\) its full subcategory. (We consider only unitary fusion categories in this paper.) We may and do assume that \(D\) is realized as a category of endomorphisms of a type III factor \(M\) with finite index. We fix a representative in each equivalence class of simple objects of \(D\) and let \(\text{Irr}(D)\) be the set of such representatives. The set \(\text{Irr}(C)\) is a subset of \(\text{Irr}(D)\) consisting of objects in \(C\). We assume that the identity morphism is in \(\text{Irr}(C)\) and denote it by \(\text{id}\). For an object \(\lambda\) in \(D\), we write \(d(\lambda)\) for its dimension, the square root of the minimal index of the subfactor \(\lambda(M) \subset M\). We set \(\dim C\) to be the square sum of the dimensions of the equivalence classes of the simple objects of \(C\). (That is, we have \(\dim C = \sum_{\lambda \in \text{Irr}(C)} d(\lambda)^2\).) We similarly define \(\dim D\). For an object \(\lambda\) in \(D\), we write \(\phi_{\lambda}\) for its standard left inverse. (See [18, page 238] for a left inverse.)

We first have a notion of a half-braiding of an object in \(D\) with respect to \(C\) as in [7, Definition 2.2] which is a slight generalization of [13, Definition 4.2].

**Definition 2.1** Let \(\sigma\) be an object of \(D\). We call a family of unitary intertwiners \(\mathcal{E}_\sigma = \{\mathcal{E}_\sigma(\beta)\}_{\beta \in \text{Irr}(C)}\) a half-braiding of \(\sigma\) with respect to \(C\) if it satisfies the following two conditions.

1. We have \(\mathcal{E}_\sigma(\beta) \in \text{Hom}(\sigma\beta, \beta\sigma)\) for all \(\beta \in \text{Irr}(C)\).
2. For \(\beta_1, \beta_2, \beta_3 \in \text{Irr}(C)\), we have

\[X\mathcal{E}_\sigma(\beta_3) = \beta_1(\mathcal{E}_\sigma(\beta_2))\mathcal{E}_\sigma(\beta_1)\sigma(X)\]

for any \(X \in \text{Hom}(\beta_3, \beta_1\beta_2)\).

Two pairs \((\sigma, \mathcal{E}_\sigma), (\sigma', \mathcal{E}'_{\sigma'})\) of objects \(\sigma, \sigma'\) in \(D\) with respective half-braidings \(\mathcal{E}_\sigma, \mathcal{E}'_{\sigma'}\) are said to be equivalent of there is a unitary intertwiner \(u \in \text{Hom}(\sigma', \sigma)\) with

\[\mathcal{E}_\sigma(\beta) = \beta(u)\mathcal{E}'_{\sigma'}(\beta)u^*\]

for all \(\beta \in \text{Irr}(C)\).
The equality in (2) is called the braiding-fusion equation.

In order to distinguishing different half-braidings for the same \( \sigma \), we use the notation \( \mathcal{E}^\alpha_\sigma \) as in [13], where \( \alpha \) denotes an index.

The objects in \( \mathcal{D} \) with half-braidings with respect to \( \mathcal{C} \) make a fusion category as in [12, Definition 2.1]. We call it the relative Drinfeld commutant of \( \mathcal{C} \) in \( \mathcal{D} \) and write \( \mathcal{C}' \cap \mathcal{D} \) for it. (It is called simply the commutant of \( \mathcal{C} \) in \( \mathcal{D} \) in [12, Section 2.1], but we add the word “Drinfeld” in order to emphasize that this is different from a usual relative commutant of a subalgebra.) Note that the conjugate half-braiding \( \mathcal{E}^\alpha_\sigma \) of a half-braiding \( \{ \mathcal{E}^\alpha_\sigma(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \) is defined by \( \mathcal{E}^\alpha_\sigma(\beta) = d(\sigma) R_\sigma^\ast (\mathcal{E}^\alpha_\sigma(\beta^\ast \beta(\bar{R}_\sigma))) \) as in [13, Theorem 4.6 (iv)]. For half-braidings \( \{ \mathcal{E}_\sigma(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \) and \( \{ \mathcal{E}_{\sigma'}(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \), the fusion product is given by \( \{ \mathcal{E}_\sigma(\beta) \sigma(\mathcal{E}_{\sigma'}(\beta)) \}_{\beta \in \text{Irr}(\mathcal{C})} \). For half-braidings \( \{ \mathcal{E}_\sigma(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \) and \( \{ \mathcal{E}_{\sigma'}(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \), an intertwiner from the former to the latter is given by \( X \in \text{Hom}(\sigma, \sigma') \) satisfying \( \mathcal{E}_{\sigma'}(\beta)X = \beta(X)\mathcal{E}_\sigma(\beta) \) for all \( \beta \in \text{Irr}(\mathcal{C}) \).

Obviously, the fusion category \( \mathcal{C}' \cap \mathcal{C} \), the Drinfeld center of \( \mathcal{C} \), is a full subcategory of \( \mathcal{C}' \cap \mathcal{D} \), but note that \( \mathcal{D}' \cap \mathcal{D} \) is not a full subcategory of \( \mathcal{C}' \cap \mathcal{D} \). If \( \mathcal{C} \) is a trivial category \( \text{Vec} \) of finite dimensional Hilbert spaces, then \( \mathcal{C}' \cap \mathcal{D} \) is simply \( \mathcal{D} \).

We choose a representative \( \{ \mathcal{E}^\alpha_\sigma \} \) from each equivalence class of simple objects in \( \mathcal{C}' \cap \mathcal{D} \) and write \( \text{Irr}(\mathcal{C}' \cap \mathcal{D}) \) for the set consisting of them.

We next introduce the relative tube algebra which generalizes a notion of Ocneanu’s tube algebra studied in [10, Section 3], [13, Section 3].

**Definition 2.2** We set the relative tube algebra \( \text{Tube}(\mathcal{C}, \mathcal{D}) \) to be

\[
\bigoplus_{\lambda, \nu \in \text{Irr}(\mathcal{D}), \mu \in \text{Irr}(\mathcal{C})} \text{Hom}(\lambda \mu, \mu \nu)
\]

as a linear space. We define its algebra structure and \( * \)-structure by the same formulas as in [13, page 134].

As in [13, Section 3], we write \( (\lambda \mu | X | \mu \nu) \) for \( X \in \text{Hom}(\lambda \mu, \mu \nu) \) for indicating which space \( X \) belongs to.

For \( (\lambda \mu | X | \mu \nu) \in \text{Tube}(\mathcal{C}, \mathcal{D}) \), we set

\[
\varphi_{\mathcal{C}, \mathcal{D}}((\lambda \mu | X | \mu \nu)) = d(\lambda)^2 \delta_{\lambda, \nu} \delta_{\mu, 0} X.
\]

Note that \( X \) on the right hand side is a scalar in \( \text{Hom}(\lambda, \lambda) \) if the right hand side does not vanish. We remark that \( \text{Tube}(\mathcal{C}, \mathcal{D}) \) is a finite dimensional C*-algebra as exactly in [13, Proposition 3.2].

Now we follow the arguments in [13, page 146]. Let \( \{ \mathcal{E}^\alpha_\sigma(\beta) \}_{\beta \in \text{Irr}(\mathcal{C})} \) be a half-braiding of an object \( \sigma \) in \( \mathcal{D} \) with respect to \( \mathcal{C} \) where we have \( [\sigma] = \bigoplus_{\lambda \in \text{Irr}(\mathcal{D})} n_{\lambda} [\lambda] \). We fix an orthonormal basis \( \{ W_{\sigma}(\lambda)_i \}_{i=1}^{n_{\lambda}} \subset \text{Hom}(\lambda, \sigma) \) and put

\[
\mathcal{E}^\alpha_\sigma(\beta)(\lambda, i), (\mu, j) = \beta(W_{\sigma}(\mu)_{j})\mathcal{E}^\alpha_\sigma(\beta)W_{\sigma}(\lambda)_i \in \text{Hom}(\lambda \beta, \beta \mu) \subset \text{Tube}(\mathcal{C}, \mathcal{D}),
\]

where \( \beta \) is in \( \text{Irr}(\mathcal{C}) \). We then have

\[
\mathcal{E}^\alpha_\sigma(\beta) = \sum_{\lambda, \mu \in \text{Irr}(\mathcal{D})} \sum_{i=1}^{n_{\lambda}} \sum_{j=1}^{n_{\mu}} \beta(W_{\sigma}(\mu)_{j})\mathcal{E}^\alpha_\sigma(\beta)(\lambda, i), (\mu, j) W_{\sigma}(\lambda)_i^*.
\]
We next put
\[ e(E^\alpha_{\sigma}(\lambda,i),(\mu,j)) = \frac{d(\sigma)}{(\dim C)d(\lambda)^{1/2}d(\mu)^{1/2}} \sum_{\beta \in \Irr(C)} d(\beta)\lambda(\beta|E^\alpha_{\sigma}(\beta)(\lambda,i),(\mu,j)|\beta\mu) \in \text{Tube}(C, D). \]

The following is a slight generalization of [13, Lemma 4.7] with essentially the same proof.

**Lemma 2.3** For \( e(E^\alpha_{\sigma})(\lambda,i),(\mu,j) \) as above and \( X \in \text{Hom}(\nu\sigma, \tau\sigma) \subset \text{Tube}(C, D) \) where \( \sigma \in \Irr(C) \) and \( \nu, \tau \in \Irr(D) \), we have the following.

1. We have \( e(E^\alpha_{\sigma}(\lambda,i),(\mu,j)) = e(E^\alpha_{\sigma}(\mu,j),(\lambda,i)) \).
2. We have
   \[
   e(E^\alpha_{\sigma}(\lambda,i),(\mu,j))(\nu\sigma|X|\tau\sigma) = \delta_{\mu,\nu} \frac{d(\sigma)d(\tau)^{1/2}}{d(\mu)^{1/2}} \sum_k \phi_\nu(XE^\alpha_{\sigma}(\sigma)^*(\mu,j),(\tau,k))e(E^\alpha_{\sigma}(\lambda,i),(\tau,k)).
   \]
3. We have
   \[
   (\nu\sigma|X|\tau\sigma)e(E^\alpha_{\sigma}(\lambda,i),(\mu,j)) = \delta_{\tau,\lambda} \frac{d(\sigma)d(\nu)^{1/2}}{d(\lambda)^{1/2}} \sum_k \phi_\nu(E^\alpha_{\sigma}(\sigma)^*(\nu,k),(\lambda,i)X)e(E^\alpha_{\sigma}(\nu,k),(\mu,j)).
   \]

The following is also a slight generalization of [13, Corollary 4.8] with essentially the same proof.

**Corollary 2.4** Let \( E^\alpha_{\sigma} \in \Irr(C' \cap D) \). Then \( z(E^\alpha_{\sigma}) = \sum_{\lambda,i} e(E^\alpha_{\sigma}(\lambda,i),(\lambda,i)) \) is in the center of \( \text{Tube}(C, D) \).

The following is also a slight generalization of [13, Lemma 4.9] with essentially the same proof.

**Lemma 2.5** In the above setting, we have the following.

1. We have \( \sum_{E^\alpha_{\sigma} \in \Irr(C,D)} d(E^\alpha_{\sigma})^2 = \dim C \dim D \).
2. We have \( \varphi_{C,D}(z(E^\alpha_{\sigma})) = d(\sigma)^2/\dim C \).
3. We have \( \varphi_{C,D}(1) = \dim D \).

The following is again a slight generalization of [13, Theorem 4.10] with essentially the same proof.

**Theorem 2.6** Let \( e(E^\alpha_{\sigma})(\lambda,i),(\mu,j) \) be as above. Then we have the following.

1. The system \( \{ e(E^\alpha_{\sigma})(\lambda,i),(\mu,j) \}_{(\lambda,i),(\mu,j)} \) is a system of matrix units of a simple component of \( \text{Tube}(C, D) \).
2. The operators \( \{ z(E^\alpha_{\sigma}) \}_{E^\alpha_{\sigma} \in \Irr(C' \cap D)} \) are mutually orthogonal minimal central projections of \( \text{Tube}(C, D) \) with \( \sum_{E^\alpha_{\sigma} \in \Irr(C' \cap D)} z(E^\alpha_{\sigma}) = 1 \).
3 A half-braiding and $\eta$-extension

We keep the notation of Section 2. Let $M \otimes M^{\text{opp}} \subset R$ be the Longo-Rehren subfactor [20] corresponding to $\text{Irr}(\mathcal{C})$ with the dual canonical endomorphism $\Theta = \bigoplus_{\lambda \in \text{Irr}(\mathcal{C})} \lambda \otimes \lambda^{\text{opp}}$ and the inclusion map $\iota_{LR} : M \otimes M^{\text{opp}} \hookrightarrow R$. Here we use the anti-isomorphism $j : M \to M^{\text{opp}}$ and $\sigma^{\text{opp}} = j \cdot \sigma \cdot j^{-1}$ for an endomorphism $\sigma$ of $M$ which is an endomorphism of $M^{\text{opp}}$. We have an isometry $V \in R$ with $(M \otimes M^{\text{opp}})V = R$ and $Vx = \iota_{LR} \cdot \iota_{LR}(x)V$ for $x \in R$ since the Longo-Rehren subfactor has a finite index.

The following is a direct analogue of [13, Theorem 4.1] and can be proved in the same way. (The Longo-Rehren subfactor studied in [13] is dual to the one studied in [7] and here, but this difference is only superficial.)

Proposition 3.1 (1) The set $\{\lambda \otimes \mu^{\text{opp}}\}_{\lambda \in \text{Irr}(\mathcal{D}), \mu \in \text{Irr}(\mathcal{C})}$ gives mutually inequivalent $M \otimes M^{\text{opp}}-M \otimes M^{\text{opp}}$ sectors and the sectors associated with the Longo-Rehren subfactor $M \otimes M^{\text{opp}} \subset R$ give its subset.

(2) The set $\{\iota_{LR} \cdot (\lambda \otimes \text{id}^{\text{opp}})\}_{\lambda \in \text{Irr}(\mathcal{D})}$ gives mutually inequivalent $R-M \otimes M^{\text{opp}}$ sectors. We have $[\iota_{LR} \cdot (\lambda \otimes \text{id}^{\text{opp}})] = [\iota_{LR} \cdot (\text{id} \otimes \bar{\lambda}^{\text{opp}})]$ for $\lambda \in \text{Irr}(\mathcal{C})$, and

$$[\iota_{LR} \cdot (\lambda \otimes \mu^{\text{opp}})] = \bigoplus_{\nu \in \text{Irr}(\mathcal{D})} N_{\mu, \lambda}[\iota_{LR} \cdot (\nu \otimes \text{id}^{\text{opp}})].$$

for $\lambda \in \text{Irr}(\mathcal{D})$ and $\mu \in \text{Irr}(\mathcal{C})$.

Statement (1) above holds also true for $\mu \in \text{Irr}(\mathcal{D})$, but it is important to consider only $\mu \in \text{Irr}(\mathcal{C})$ for (2).

For a half-braiding $\mathcal{E}_\sigma$ of an object $\sigma$ in $\mathcal{D}$, we set as follows as in [13, page 139].

$$U(\sigma, \mathcal{E}_\sigma) = \sum_{\lambda \in \text{Irr}(\mathcal{C})} W_\lambda(\mathcal{E}_\sigma(\lambda) \otimes 1)(\sigma \otimes \text{id}^{\text{opp}})(W^*_\lambda),$$

$$U(\sigma^{\text{opp}}, \mathcal{E}_\sigma) = \sum_{\lambda \in \text{Irr}(\mathcal{C})} W_\lambda(1 \otimes j((\mathcal{E}_\sigma(\lambda))))(\text{id} \otimes \sigma^{\text{opp}})(W^*_\lambda),$$

where $\{W_\lambda\}_{\lambda \in \text{Irr}(\mathcal{C})}$ is a set of isometries in $M \otimes M^{\text{opp}}$ with $\sum_{\lambda \in \text{Irr}(\mathcal{C})} W_\lambda W^*_\lambda = 1$ and $W^*_\lambda V \in (\iota_{LR}, \iota_{LR} \cdot (\lambda \otimes \lambda^{\text{opp}})).$

We then define an $\eta$-extension of $\sigma \otimes \text{id}$, an endomorphism of $M \otimes M^{\text{opp}}$, to $R$ as follows as in [7, Definition 2.3].

$$\eta(\sigma, \mathcal{E}_\sigma)(a) = (\sigma \otimes \text{id})(a), \quad a \in M \otimes M^{\text{opp}},$$

$$\eta(\sigma, \mathcal{E}_\sigma)(V) = U(\sigma, \mathcal{E}_\sigma)^*V.$$ 

This is indeed an endomorphism of $R$, which can be shown as in [13, Definition 4.4 (i)]. We also define $\eta^{\text{opp}}(\sigma, \mathcal{E}_\sigma)$, an extension of $\text{id} \otimes \sigma^{\text{opp}}$ to $R$ in a similar way.

Theorem 3.2 The category $\mathcal{C}' \cap \mathcal{D}$ is equivalent to the category of $R-R$ morphisms arising from decompositions of $[\iota_{LR}][\lambda \otimes \text{id}^{\text{opp}}][\iota_{LR}]$ where $\lambda \in \text{Irr}(\mathcal{D})$. We have $\dim(\mathcal{C}' \cap \mathcal{D}) = \dim \mathcal{C} \cdot \dim \mathcal{D}$. 

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Proof. If we have an irreducible half-braiding $E_\sigma$ of an object $\sigma$ in $D$ with respect to $C$, we have an extension $\eta(\sigma, E_\sigma)$. By definition, we have $[\eta(\sigma, E_\sigma) \cdot \iota_{LR}] = [\iota_{LR}(\sigma \otimes \text{id}^{\text{opp}})]$, so the extension $\eta(\sigma, E_\sigma)$ appears in the decomposition of $[\iota_{LR}(\sigma \otimes \text{id}^{\text{opp}})][\iota_{LR}]$.

Suppose an irreducible endomorphism $\rho$ of $R$ appears in the decomposition of $[\iota_{LR}(\xi \otimes \text{id}^{\text{opp}})][\iota_{LR}]$ for some irreducible object $\xi \in D$. Then $\rho \cdot \iota_{LR}$ is contained in

$[\iota_{LR}(\xi \otimes \text{id}^{\text{opp}})][\iota_{LR}] = \bigoplus_{\lambda \in \text{Irr}(C)} [\iota_{LR}(\xi \cdot \lambda \otimes \lambda^{\text{opp}})],$

and Proposition 3.1 (2) shows that there exists an object $\sigma \in D$ satisfying $[\rho \cdot \iota_{LR}] = [\iota_{LR}(\sigma \otimes \text{id}^{\text{opp}})]$, which means $\rho$ is an extension of $\sigma \otimes \text{id}^{\text{opp}}$ as an endomorphism. We then have the following for some $U \in M \otimes M^{\text{opp}}$.

$\rho(a) = (\sigma \otimes \text{id})(a)$, $a \in M \otimes M^{\text{opp}}$,

$\rho(V) = U^* V$.

We then have $U \in \text{Hom}((\sigma \otimes \text{id}^{\text{opp}}) \cdot \Theta, \Theta \cdot (\sigma \otimes \text{id}^{\text{opp}}))$ by a similar argument to the one in the middle of [13, Page 141]. A further argument similar to the one in [13, Pages 141–143] shows that $U$ is of the form $U(\sigma, E_\sigma)$ for some half-braiding $E_\sigma$ of $\sigma$ with respect to $\sigma$.

For the conjugate half-braiding, we have $[\eta(\sigma, E_\sigma^\text{opp})] = [\eta(\bar{\sigma}, E_{\bar{\sigma}})]$ as in [13, Theorem 4.6 (iv)].

It is easy to see that the above correspondence indeed gives equivalence of the two categories.

By what we have proved so far, $\dim(C \cap D)$ is equal to the square sum of the dimensions of the irreducible $R$-$R$ sectors arising from decompositions of $[\iota_{LR}(\lambda \otimes \text{id}^{\text{opp}})][\iota_{LR}]$ where $\lambda \in \text{Irr}(D)$. The latter is then equal to the square sum of the dimensions of the irreducible $M \otimes M^{\text{opp}}$-$M \otimes M^{\text{opp}}$ sectors $\lambda \otimes \mu^{\text{opp}}$ where $\lambda \in \text{Irr}(D)$ and $\mu \in \text{Irr}(C)$, so we have the conclusion. \qed

4 The relative Drinfeld commutants arising from $\alpha$-induction

Now we change the notations and let $C$ be a modular tensor category realized as a full subcategory of $\text{End}(N)$ for a type III factor $N$. Let $(\theta, \psi, x)$ be a $Q$-system with $\theta$ being an object in $C$, $\mathcal{M} \supset N$ the corresponding subfactor and $\iota$ the inclusion map $N \hookrightarrow M$. We have $\alpha$-induction $\alpha^\pm_\lambda$ for an object $\lambda$ in $C$ as in [20], [23], [2], [3], [1], [5], [6], [7]. Set $\mathcal{D}^\pm$ to be the fusion category generated by $\alpha^\pm_\lambda$ where $\lambda$ is an object of $C$. Set $\mathcal{D}$ to be the fusion category generated by $\mathcal{D}^+$ and $\mathcal{D}^-$, and set $\mathcal{D}^0$ to be the fusion category whose set of objects consists of those which are objects of both $\mathcal{D}^+$ and $\mathcal{D}^-$. Note that $\mathcal{D}$ is equal to the category generated by $\iota \circ \lambda \circ \iota$ for $\lambda \in \text{C}$ by [5, Theorem 5.10]. (The objects of $\mathcal{D}^0$ have been called ambichiral in [6], and they correspond to dyslectic/local modules in the terminology of [8], [9].)

The following result has been shown in [7, Corollary 4.8]. (Also see [8, Corollary 3.30], where unitarity is not assumed.) See [1, Lemma 3.20, Remark 4.17] for an opposite braided category. Here $(\mathcal{D}^0)^{\text{opp}}$ means a modular tensor category with its braiding reversed.
Theorem 4.1 The Drinfeld center $(D^+)' \cap D^+$ of $D^+$ is equivalent to $C \boxtimes (D^0)^{opp}$ as modular tensor categories.

Now $D^0$ is a full fusion subcategory of $D^\pm$ and $D^\pm$ is a full fusion subcategory of $D$. We first compute $(D^+)' \cap D$ explicitly. We use a relative braiding introduced in [4 Proposition 3.12]. We remark that the arguments there use only the braided structure of $C$ and do not depend on a net structure or locality (as noted in [6 page 739].)

For an object $\beta$ in $D$, we choose an isometry $T \in \text{Hom}(\beta, \alpha^+_\lambda \alpha^-_\lambda)$ with some objects $\nu, \nu'$ in $C$. For any object $\lambda \in C$, we set

$$\mathcal{E}^+_{\alpha^+_\lambda}(\beta) = T^* \varepsilon^+(\lambda, \nu') \alpha^+_\lambda(T),$$

as in [7 (10)]. (We have changed the notations slightly here from those in [7].) By [7 Lemma 3.1], $\{\mathcal{E}^+_{\alpha^+_\lambda}(\beta)\}$ gives a half-braiding of $\alpha^+_\lambda$ with respect to $D$ and $\mathcal{E}^+_{\alpha^+_\lambda}(\beta)$ does not depend on the choices of $T$ and $\nu, \nu'$. In particular, $\{\mathcal{E}^+_{\alpha^+_\lambda}(\beta)\}_{\beta \in \text{Irr}(D^+)}$ gives a half-braiding of $\alpha^+_\lambda$ with respect to $D^+$. By [7 Corollary 3.8], we have the following. (Note that [7 Proposition 2.6] works here since $\alpha^+_\lambda$ is an object of $D^+$ rather than $D$.)

Proposition 4.2 In the above setting, we have $[\eta^{opp}(\alpha^+_\lambda, \mathcal{E}^+_{\alpha^+_\lambda})] = [\eta(\alpha^+_\lambda, \mathcal{E}^+_{\alpha^+_\lambda})]$.

Also, [7 Lemma 3.7] gives the following about the conjugate half-braiding $\bar{\mathcal{E}}^+_{\alpha^+_\lambda}(\beta)$.

Proposition 4.3 We have $\bar{\mathcal{E}}^+_{\alpha^+_\lambda}(\beta) = \mathcal{E}^+_{\alpha^+_\lambda}(\beta)$.

We next follow the first paragraph of [7 Section 4]. Recall from [4 Subsection 3.3] that for an object $\beta_\pm$ in $D^\pm$, the operators

$$\mathcal{E}_\tau(\beta_+, \beta_-) = S^* \alpha^-_\mu(T)^* \varepsilon^+(\lambda, \mu) \alpha^+_\lambda(S)T$$

are unitaries in $\text{Hom}(\beta_+, \beta_-, \beta_-, \beta_+)$ for objects $\lambda, \mu$ in $C$ and isometries $T \in \text{Hom}(\beta_+, \alpha^+_\lambda)$ and $S \in \text{Hom}(\beta_-, \alpha^-_\mu)$. They do not depend on the choices of $\lambda, \mu, S, T$. They give a “relative braiding” between $D^+$ and $D^-$. For an object $\tau$ in $D^-$ and an object $\beta$ in $D^+$, we put $\mathcal{E}_\tau^-(\beta) = \mathcal{E}_\tau(\beta, \tau)^*$, and this gives a half-braiding $\{\mathcal{E}_\tau^-(\beta)\}_{\beta \in \text{Irr}(D^+)}$ of $\tau$ with respect to $D^+$ by [7 Lemma 4.1].

The arguments similar to those below [7 (18)] give the following.

Proposition 4.4 For $\tau, \tau' \in \text{Irr}(D^-)$, we have $\text{Hom}(\eta(\tau, \mathcal{E}_{\tau'}^-), \eta(\tau', \mathcal{E}_{\tau'}^-)) = \delta_{\tau, \tau'} C$.

Since we now assume $C$ is a modular tensor category, [7 Lemma 4.2] produces the following.

Proposition 4.5 For $\lambda, \mu \in \text{Irr}(C)$, we have $\text{Hom}(\eta(\alpha^+_\lambda, \mathcal{E}_{\alpha^+_\lambda}^+), \eta(\alpha^+_\mu, \mathcal{E}_{\alpha^+_\mu}^+)) = \delta_{\lambda, \mu} C$.

In the same way as in [7 Lemma 4.3], we have the following. (Note that we assumed $\tau \in M\mathcal{X}_M^0$ in [7 Lemma 4.3] while we have $\tau \in \text{Irr}(D^-)$ here, but in the proof of [7 Lemma 4.3], we used only the condition $\tau \in M\mathcal{X}_M^-$ in the fourth line of the proof.)
Proposition 4.6 For $\lambda \in \text{Irr}(C)$ and $\tau \in \text{Irr}(D^-)$, we have $\text{Hom}(\eta(\alpha^+_\lambda, E^+_{\alpha^+_\lambda}), \eta(\tau, E^-_{\tau})) = \delta_{\lambda, \text{id}} \delta_{\tau, \text{id}} C$.

Now [7] Lemma 4.4 gives the following about the conjugate half-braiding.

Proposition 4.7 We have $E^-_{\tau}(\beta) = E^+_{\tau}(\beta)$ for objects $\beta$ in $D^+$ and $\tau$ in $D^-$. We then have the following as in [7] Theorem 4.6.

Proposition 4.8 For $\lambda, \lambda' \in \text{Irr}(C)$ and $\tau, \tau' \in \text{Irr}(D^-)$, we have

$$\langle \eta(\alpha^+_{\lambda'}, E^+_{\alpha^+_{\lambda'}}) \eta(\tau, E^-_{\tau}), \eta(\alpha^+_{\lambda}, E^+_{\alpha^+_{\lambda}}) \eta(\tau', E^-_{\tau'}) \rangle = \delta_{\lambda, \lambda'} \delta_{\tau, \tau'} C.$$

Using Theorem 3.2, we know that the set $\{\eta^{\text{opp}}(\alpha^+_{\lambda'}, E^+_{\alpha^+_{\lambda'}}) \eta(\tau, E^-_{\tau})\}$ with $\lambda \in \text{Irr}(C)$ and $\tau \in \text{Irr}(D^-)$ gives all representatives of the equivalence classes of simple objects of $(D^+)' \cap D$. We then have the following as a direct analogue of [7] Corollary 4.8 (along similar arguments to those in [7] page 18), which are based on [13] Corollary 7.2).

Theorem 4.9 The fusion category $(D^+)' \cap D$ is equivalent to $C \boxtimes D^-.$

Remark 4.10 Note that the above result has some formal similarity to [12] Theorem A in the appearance of $D^-.$

We next consider $(D^0)' \cap D^+$. In a way similar to the above, we have a half-braiding $\{E^+_{\tau}(\beta)\}_{\beta \in \text{Irr}(D^0)}$ with respect to $D^0$ for $\tau \in \text{Irr}(D^+).$ We also have a half-braiding $\{E^-_{\tau}(\beta)\}_{\beta \in \text{Irr}(D^0)}$ with respect to $D^0$ for $\tau \in \text{Irr}(D^0).$ We similarly have the following.

Proposition 4.11 The set $\{\eta(\tau, E^+_{\tau})\eta^{\text{opp}}(\tau', E^-_{\tau})\}$ with $\tau \in \text{Irr}(D^+)$ and $\tau' \in \text{Irr}(D^0)$ gives all representatives of the equivalence classes of simple objects of $(D^0)' \cap D^+.$

We then have the following again along similar arguments to those in [7] page 18.

Theorem 4.12 The fusion category $(D^0)' \cap D^+$ is equivalent to $D^+ \boxtimes D^0.$

We next consider $(D^0)' \cap D$. In a way similar to the above, we have a half-braiding $\{E^+_{\tau}(\beta)\}_{\beta \in \text{Irr}(D^0)}$ with respect to $D^0$ for $\tau \in \text{Irr}(D^+).$ We also have a half-braiding $\{E^-_{\tau}(\beta)\}_{\beta \in \text{Irr}(D^0)}$ with respect to $D^0$ for $\tau \in \text{Irr}(D^-).$ Using Theorem 3.2 and 6 Theorem 4.2, we have the following.

Proposition 4.13 The set $\{\eta(\tau, E^+_{\tau})\eta(\tau', E^-_{\tau})\}$ with $\tau \in \text{Irr}(D^+)$ and $\tau' \in \text{Irr}(D^-)$ gives all representatives of the equivalence classes of simple objects of $(D^0)' \cap D.$

However, we do not know whether $\eta(\tau', E^-_{\tau})$ is equivalent to $\eta^{\text{opp}}(\tau', E^-_{\tau})$ since $\tau'$ is not in $D^0$ in general. So we now need an extra argument.

Proposition 4.14 For $\tau \in \text{Irr}(D^+)$ and $\tau' \in \text{Irr}(D^-)$, the relative braiding $E_r(\tau, \tau')$ gives an element in $\text{Hom}(\eta(\tau, E^+_{\tau}), \eta(\tau', E^-_{\tau}), \eta(\tau', E^-_{\tau}) \eta(\tau, E^+_{\tau}).)$
Proof. We show that $E_r(\tau, \tau')$ gives an intertwiner on the level of half-braiding. Then the arguments in the proof of [4 Proposition 3.12] based on [2 Lemma 3.25] give the desired conclusion. \qed

We also have the following.

**Proposition 4.15** For (possibly reducible) objects $\tau, \tau'$ of $D^+$, we have $\text{Hom}(\eta(\tau, E^+_{\tau}), \eta(\tau', E^+_{\tau'})) = \text{Hom}(\tau, \tau')$.

**Proof.** It is easy to see that the right hand side is contained in the left hand side. The dimensions of the both hand sides are equal, so we have the equality. \qed

We also have a similar equality for $\eta(\tau, E^-_{\tau})$ for an object of $D^-$. Now consider the fusion category whose irreducible objects are given by $\{\eta(\tau, E^+_{\tau})\eta(\tau', E^-_{\tau'})\}$ with $\tau \in \text{Irr}(D^+)$ and $\tau' \in \text{Irr}(D^-)$. We consider the $6j$-symbols of this fusion category. Then it splits as a product of two $6j$-symbols as in Fig. [1]. In this diagram, we follow the graphical convention of [5 Section 3]. In particular, we compose morphisms from the top to the bottom. The thick wire represents an irreducible object of the fusion category $\text{Irr}(D^+)$ and the thin wires represents an irreducible object of the fusion category $\text{Irr}(D^-)$. The inner products in Fig. [1] represents those between two intertwiners and the compositions give complex numbers. When we compose two irreducible morphisms, we switch two components using Proposition 4.14 and this give a relative braiding $E_r$ at the upper left corner of Fig. [1]. All the crossings in Fig. [1] represent the relative braiding $E_r$ or its conjugate.

We thus obtain the following theorem.

**Theorem 4.16** The fusion category $(D^0)' \cap D$ is equivalent to $D^+ \boxtimes D^-$. 

We show some example now. A typical appearance of $\alpha$-induction is an extension of a completely rational local conformal net in the sense of [17 page 498], [16 Definition 8], [20 Definition 3.1]. Note that strong additivity and split property in the definition of complete rationality [16 Definition 8] are unnecessary due to [21] and [22], respectively. Let $\{A(I) \subset B(I)\}$ be such an extension, where $I$ is an interval contained in $S^1$. Let $\mathcal{C}$ be the representation category of the local conformal net $\{A(I)\}$ and consider the $\alpha$-induction for a subfactor $A(I) \subset B(I)$ for some interval $I$ as in [2 Definition 3.3]. Then we have $D^0, D^\pm, D$ from this $\alpha$-induction as in [5], so the above results apply. Note that $D^0$ is the representation category of $B(I)$ and $D^\pm$ are the categories of soliton sectors.

Consider an extension of a completely rational local conformal net arising from a conformal embedding $SU(2)_{10} \subset SO(5)_1$ as in [6 Example 2.2]. In this case, the category $\mathcal{C}$ has 11 simple objects, and $D^0, D^+, D^-$, $D$ have 3, 6, 6, and 12 simple objects, respectively, as in [4 Fig. 2]. This setting gives concrete examples to which the above results apply.
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Figure 1: 6j-symbols