Enhanced variety of higher level and 
Kostka functions associated to complex reflection groups

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Abstract. Let $V$ be an $n$-dimensional vector space over an algebraic closure of a finite field $\mathbb{F}_q$, and $G = GL(V)$. A variety $\mathcal{X} = G \times V^{r-1}$ is called an enhanced variety of level $r$. Let $\mathcal{X}_{uni} = G_{uni} \times V^{r-1}$ be the unipotent variety of $\mathcal{X}$. We have a partition $\mathcal{X}_{uni} = \bigsqcup \lambda X_\lambda$ indexed by $r$-partitions $\lambda$ of $n$. In the case where $r = 1$ or 2, $X_\lambda$ is a single $G$-orbit, but if $r \geq 3$, $X_\lambda$ is, in general, a union of infinitely many $G$-orbits. In this paper, we prove certain orthogonality relations for the characteristic functions (over $\mathbb{F}_q$) of the intersection cohomology $\text{IC}(X_\lambda, \overline{\mathbb{Q}}_l)$, and show some results, which suggest a close relationship between those characteristic functions and Kostka functions associated to the complex reflection group $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

Introduction

Let $V$ be an $n$-dimensional vector space over an algebraic closure of a finite field $\mathbb{F}_q$, and $G = GL(V) \simeq GL_n$. In 1981, Lusztig showed in [L1] that Kostka polynomials $K_{\lambda,\mu}(t)$ have a geometric interpretation in terms of the intersection cohomology associated to the closure of unipotent classes in $G$ in the following sense. Let $C_\lambda$ be the unipotent class corresponding to a partition $\lambda$ of $n$, and $K = \text{IC}(\overline{C}_\lambda, \overline{\mathbb{Q}}_l)$ be the intersection cohomology complex on the closure $\overline{C}_\lambda$ of $C_\lambda$. He proved that $\mathcal{H}^i K = 0$ for odd $i$, and that for partitions $\lambda, \mu$ of $n$,

\[ t^{n(\mu)} K_{\lambda,\mu}(t^{-1}) = t^{n(\lambda)} \sum_{i \geq 0} (\text{dim} \mathcal{H}^{2i} K) t^i, \]

where $x \in C_\mu \subset \overline{C}_\lambda$, and $n(\lambda)$ is the usual $n$-function.

Kostka polynomials are polynomials indexed by a pair of partitions. In [S1], [S2], as a generalization of Kostka polynomials, Kostka functions $K_{\lambda,\mu}(t)$ associated to the complex reflection group $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ were introduced, which are a-priori rational functions in $\mathbb{Q}(t)$ indexed by $r$-partitions $\lambda, \mu$ of $n$ (see 3.10 for the definition of $r$-partitions of $n$). It is known by [S2] that $K_{\lambda,\mu}(t)$ are actually polynomials if $r = 2$. Although those Kostka functions are defined in a purely combinatorial way, and have no geometric background, recently various generalizations of Lusztig’s result for those Kostka functions were found. Under the notation above, consider a variety $\mathcal{X} = G \times V$, which is called the enhanced variety, and its subvariety $\mathcal{X}_{uni} = G_{uni} \times V$ is isomorphic to the enhanced nilpotent cone introduced by Achar-Henderson [AH] (here $G_{uni}$ is the unipotent variety of $G$). The set of $G$-orbits under the diagonal action of $G$ on $\mathcal{X}_{uni}$ is parametrized by double partitions of $n$ ([AH], [T]). In [AH], they proved that Kostka polynomials indexed by double partitions have a geometric interpretation as in (*) in terms of the intersection cohomology associated to the closure of $G$-orbits in $\mathcal{X}_{uni}$.

On the other hand, let $V$ be a $2n$-dimensional symplectic vector space over an algebraic closure of $\mathbb{F}_q$ with $\text{ch} \mathbb{F}_q \neq 2$, and consider $G = GL(V) \supset H = Sp(V)$. The variety
$\mathcal{X} = G/H \times V$ is called the exotic symmetric space, and its “unipotent variety” $\mathcal{X}_{\text{uni}}$ is isomorphic to the exotic nilpotent cone introduced by Kato [K1]. $H$ acts diagonally on $\mathcal{X}_{\text{uni}}$, and the set of $H$-orbits on $\mathcal{X}_{\text{uni}}$ is parametrized by double partitions of $n$ ([K1]). As in the enhanced case, it is proved by [K2], and [SS1], [SS2], independently, that Kostka polynomials indexed by double partitions have a geometric interpretation in terms of the intersection cohomology associated to the closure of $H$-orbits in $\mathcal{X}_{\text{uni}}$.

As a generalization of the enhanced variety $G \times V$ or the exotic symmetric space $G/H \times V$, we consider $G \times V^r$ or $G/H \times V^r$ for any $r \geq 1$. $G$ acts diagonally on $G \times V^r$, and $H$ acts diagonally on $G/H \times V^r$. $\mathcal{X} = G \times V^r$ is called the enhanced variety of level $r$, and a certain $H$-stable subvariety $\mathcal{X}$ of $G/H \times V^r$ is called the exotic symmetric space of level $r$. For those varieties $\mathcal{X}$, one can consider $G$-stable subvariety $\mathcal{X}_{\text{uni}}$ (unipotent variety). The crucial difference for the general case is that the number of $G$-orbits (or $H$-orbits) on $\mathcal{X}_{\text{uni}}$ is no longer finite if $r \geq 3$. Nevertheless, it was shown in [S3], for the exotic case or the enhanced case, that one can construct subvarieties $X_\lambda$ of $\mathcal{X}_{\text{uni}}$ indexed by $r$-partitions $\lambda$ of $n$, and the intersection cohomologies $IC(\mathcal{X}_\lambda, \mathbb{Q}_l)$ enjoy similar properties as in the case $r = 1$ or $2$, more precisely, a generalization of the Springer correspondence holds for $\mathcal{X}_{\text{uni}}$.

So it is natural to expect that those intersection cohomologies will have a close relation with Kostka functions indexed by $r$-partitions of $n$. In this paper, we consider this problem in the case where $\mathcal{X}$ is the enhanced variety of level $r$. In this case, we have a partition $\mathcal{X}_{\text{uni}} = \coprod X_\lambda$ parametrized by $r$-partitions $\lambda$ of $n$. In the case where $r = 1$ or $2$, $X_\lambda$ is a single $G$-orbit, but if $r \geq 3$, $X_\lambda$ is, in general, a union of infinitely many $G$-orbits. By applying the strategy employed in the theory of character sheaves in [L2], [L3] (and in [SS2]), we show (Theorem 5.5) that the characteristic functions of the Frobenius trace over $\mathbb{F}_q$ associated to $IC(\mathcal{X}_\lambda, \mathbb{Q}_l)$ satisfy certain orthogonality relations, which are quite similar to the case of $r = 1$ or $2$. In fact, in the case where $r = 1$ or $2$, the geometric interpretation is deduced from this kind of orthogonality relations. However, in the case where $r \geq 3$, these orthogonality relations are not enough to obtain the required formula. We show some partial results which suggest an interesting relationship between those characteristic functions and Kostka functions.

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1. Complexes on the enhanced variety

1.1. Let $k$ be an algebraic closure of a finite field $\mathbb{F}_q$. Let $V$ be an $n$-dimensional vector space over $k$, and put $G = GL(V)$. For a fixed integer $r \geq 2$, we consider the variety $G \times V^{r-1}$, on which $G$ acts diagonally. We call $G \times V^{r-1}$ the enhanced variety of level $r$. Put $\mathcal{D}_{n,r} = \{m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r \mid \sum_i m_i = n\}$. For each $m \in \mathcal{D}_{n,r}$, we define integers $p_i = p_i(m)$ by $p_i = m_1 + \cdots + m_i$ for $i = 1, \ldots, r$. Let $B = TU$ be a Borel subgroup of $G$, $T$ a maximal torus of $B$, and $U$ the unipotent radical of $B$. Let $(M_i)_{1 \leq i \leq n}$ be the total flag in $V$ such that the stabilizer of $(M_i)$ in $G$ coincides with $B$. We define...
In the case where \( \mathbf{m} \) and \( \pi \) and the map \( \pi \) and the map \( X \) of \( \pi \) varieties denote \( \tilde{\pi} \) to the permutation group \( S \). Then \( \tilde{\pi} \) is spanned by \( e \) of \( Y \). It is known that \( \pi \) is smooth and irreducible, and \( \pi \) is a proper surjective map. In particular, \( \tilde{\pi} \) is a closed subvariety of \( G \times V^{r-1} \).

Let \( G_{\text{uni}} \) be the set of unipotent elements in \( G \). Similarly to the above, we define

\[
\tilde{\mathcal{X}}_{\pi,\text{uni}} = \{(x, v, gB) \in G \times V^{r-1} \times G/B \mid g^{-1}xg \in U, g^{-1}v \in \prod_{i=1}^{r-1} M_{p_i}\},
\]

\[
\mathcal{X}_{\pi,\text{uni}} = \bigcup_{g \in G} g(U \times \prod_{i=1}^{r-1} M_{p_i}),
\]

and the map \( \pi_{\pi,1} : \tilde{\mathcal{X}}_{\pi,\text{uni}} \to \mathcal{X}_{\pi,\text{uni}} \) by \( (x, v, gB) \mapsto (x, v) \). Note that \( \tilde{\mathcal{X}}_{\pi,\text{uni}} = \pi_{\pi,1}^{-1}(\mathcal{X}_{\pi,\text{uni}}) \), and \( \pi_{\pi,1} \) is the restriction of \( \pi_{\pi} \) on \( \tilde{\mathcal{X}}_{\pi,\text{uni}} \). Hence \( \pi_{\pi,1} \) is proper, and \( \tilde{\mathcal{X}}_{\pi,\text{uni}} \) is a closed subvariety of \( G_{\text{uni}} \times V^{r-1} \). In the case where \( \mathbf{m} = (n, 0, \ldots, 0) \), we denote \( \tilde{\mathcal{X}}_{\pi}, \mathcal{X}_{\pi} \) by \( \tilde{\mathcal{Y}}, \mathcal{Y} \). Hence \( \mathcal{Y} = G \times V^{r-1} \).

Let \( T_{\text{reg}} \) be the set of regular semisimple elements in \( T \), and put \( G_{\text{reg}} = \bigcup_{g \in G} g T_{\text{reg}} g^{-1} \), \( B_{\text{reg}} = G_{\text{reg}} \cap B \). We define varieties

\[
\mathcal{Y} = \{(x, v, gB) \in G_{\text{reg}} \times V^{r-1} \times G/B \mid g^{-1}xg \in B_{\text{reg}}, g^{-1}v \in \prod_{i=1}^{r} M_{p_i}\},
\]

\[
\tilde{\mathcal{Y}} = \bigcup_{g \in G} g(B_{\text{reg}} \times \prod_{i} M_{p_i}) = \bigcup_{g \in G} g(T_{\text{reg}} \times \prod_{i} M_{p_i}).
\]

Then \( \tilde{\mathcal{Y}} = \pi_{\pi,1}^{-1}(\mathcal{Y}) \) and we define the map \( \psi_{\pi} : \tilde{\mathcal{Y}} \to \mathcal{Y} \) by the restriction of \( \pi_{\pi} \) on \( \tilde{\mathcal{Y}} \). It is known that

**Lemma 1.2** ([S3, Lemma 4.2]).

(i) \( \mathcal{Y} \) is open dense in \( \mathcal{X}_{\pi} \) and \( \tilde{\mathcal{Y}} \) is open dense in \( \tilde{\mathcal{X}}_{\pi} \).

(ii) \( \dim \mathcal{X}_{\pi} = \dim \tilde{\mathcal{X}}_{\pi} = n^2 + \sum_{i=1}^{r} (r-i)m_i \).

1.3. We fix a basis \( e_1, \ldots, e_n \) of \( V \) such that \( e_i \) are weight vectors of \( T \) and that \( M_i \) is spanned by \( e_1, \ldots, e_i \). Let \( W = N_G(T)/T \) be the Weyl group of \( G \), which is isomorphic to the permutation group \( S_n \) of the basis \( \{e_1, \ldots, e_n\} \). We denote by \( W_{\mathbf{m}} \) the subgroup of \( W \) which permutes the basis \( \{e_j\} \) of \( M_{p_i} \) for each \( i \). Hence \( W_{\mathbf{m}} \) is isomorphic to the Young subgroup \( S_{m_1} \times \cdots \times S_{m_r} \) of \( S_n \). Let \( M_{p_i}^0 \) be the set of \( v = \sum a_j e_j \in M_{p_i} \) such that \( a_j \neq 0 \) for \( p_{i-1} + 1 \leq j \leq p_i \). We define a variety \( \hat{\mathcal{Y}}_{\mathbf{m}} \) by
\[ \tilde{\mathcal{Y}}_m^0 = G \times^T (T_{\text{reg}} \times \prod_i M_{p_i}^0). \]

Since \( \tilde{\mathcal{Y}}_m \simeq G \times^T (T_{\text{reg}} \times \prod_i M_{p_i}) \), \( \tilde{\mathcal{Y}}_m^0 \) is identified with the open dense subset of \( \tilde{\mathcal{Y}}_m \).

Then \( \mathcal{Y}_m^0 = \psi_m(\tilde{\mathcal{Y}}_m^0) \) is an open dense smooth subset of \( \mathcal{Y}_m \). The map \( \psi_m^0 : \tilde{\mathcal{Y}}_m^0 \rightarrow \mathcal{Y}_m^0 \) obtained from the restriction of \( \psi_m \) turns out to be a finite Galois covering with group \( W_m \) (apply the discussion in [S3, 1.3] to the enhanced case).

We consider the diagram

\[ T \leftarrow \tilde{\mathcal{X}}_m \longrightarrow \mathcal{X}_m, \]

where \( \alpha \) is the map defined by \( (x, v, gB) \mapsto (p_T(g^{-1}xg)) \) (\( p_T : B \rightarrow T \) is the natural projection). Let \( \mathcal{E} \) be a tame local system on \( T \). Let \( W_m,\mathcal{E} \) the stabilizer of \( \mathcal{E} \) in \( W_m \). Let \( \alpha_0 \) be the restriction of \( \alpha \) on \( \mathcal{Y}_m \). We also denote by \( \alpha_0 \) the restriction of \( \alpha \) on \( \mathcal{Y}_m^0 \). Since \( \psi_m^0 \) is a finite Galois covering, \( (\psi_m^0) \alpha_0^*\mathcal{E} \) is a local system on \( \mathcal{Y}_m^0 \) equipped with \( W_m,\mathcal{E} \)-action, and is decomposed as

\[ (\psi_m^0) \alpha_0^*\mathcal{E} \simeq \bigoplus_{\rho \in W^\wedge_m,\mathcal{E}} \rho \otimes \mathcal{L}_\rho, \]

where \( \mathcal{L}_\rho = \text{Hom}(\rho, (\psi_m^0) \alpha_0^*\mathcal{E}) \) is the simple local system on \( \mathcal{Y}_m^0 \). The following results were proved in Proposition 4.3 and Theorem 4.5 in [S3].

**Theorem 1.4 ([S3]).** Take \( m \in Q_{n,r} \), and put \( d_m = \dim \mathcal{Y}_m \).

(i) \( (\psi_m) \alpha_0^*\mathcal{E}[d_m] \) is a semisimple perverse sheaf on \( \mathcal{Y}_m \) equipped with \( W_m,\mathcal{E} \)-action, and is decomposed as

\[ (\psi_m) \alpha_0^*\mathcal{E}[d_m] \simeq \bigoplus_{\rho \in W^\wedge_m,\mathcal{E}} \rho \otimes \text{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)[d_m]. \]

(ii) \( (\pi_m) \alpha^*\mathcal{E}[d_m] \) is a semisimple perverse sheaf on \( \mathcal{X}_m \) equipped with \( W_m,\mathcal{E} \)-action, and is decomposed as

\[ (\pi_m) \alpha^*\mathcal{E}[d_m] \simeq \bigoplus_{\rho \in W^\wedge_m,\mathcal{E}} \rho \otimes \text{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[d_m]. \]

**1.5.** Let \( P \) be the stabilizer of the partial flag \( (M_{p_i}) \) in \( G \), which is a parabolic subgroup of \( G \) containing \( B \). Let \( L \) be the Levi subgroup of \( P \) containing \( T \), and \( U_P \) the
unipotent radical of \( P \). We consider the varieties

\[
\mathcal{F}^P_m = \bigcup_{g \in P} g(B \times \prod_{i=1}^{r-1} M_{p_i}) = P \times \prod_{i=1}^{r-1} M_{p_i},
\]

\[
\mathcal{F}^P_m = G \times P \mathcal{F}^P_m = G \times P (P \times \prod_{i=1}^{r-1} M_{p_i}),
\]

\[
\mathcal{F}^P_m = P \times B (B \times \prod_{i=1}^{r-1} M_{p_i}).
\]

We define \( \pi' : \mathcal{F}^-_m \rightarrow \mathcal{F}^P_m \) as the map induced from the inclusion map \( G \times (B \times \prod M_{p_i}) \rightarrow G \times (P \times \prod M_{p_i}) \) under the identification \( \mathcal{F}^-_m \simeq G \times B (B \times \prod M_{p_i}) \), and define \( \pi'' : \mathcal{F}^-_m \rightarrow \mathcal{F}^P_m \) by \( g \ast (x, v) \mapsto (gxg^{-1}, gv) \). (Here we denote by \( g \ast (x, v) \) the image of \( (g, (x, v)) \in G \times \mathcal{F}^-_m \) on \( \mathcal{F}^P_m \).) Thus we have \( \pi_m = \pi'' \circ \pi' \). Since \( \pi_m \) is proper, \( \pi' \) is proper. \( \pi'' \) is also proper.

Let \( B_L = B \cap L \) be the Borel subgroup of \( L \) containing \( T \), and put \( M_{p_i} = M_{p_i}/M_{p_{i-1}} \) under the convention \( M_{p_0} = 0 \). Then \( L \) acts naturally on \( M_{p_i} \), and by applying the definition of \( \pi_m : \mathcal{F}^-_m \rightarrow \mathcal{F}^L_m \) to \( L \), we can define

\[
\mathcal{F}^L_m = L \times B(L \times \prod_{i=1}^{r-1} M_{p_i}),
\]

\[
\mathcal{F}^L_m = \bigcup_{g \in L} g(B_L \times \prod_{i=1}^{r-1} M_{p_i}) = L \times \prod_{i=1}^{r-1} M_{p_i}
\]

and the map \( \pi^L_m : \mathcal{F}^-_m \rightarrow \mathcal{F}^L_m \) similarly. We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}^-_m & \xleftarrow{\bar{p}} & G \times \mathcal{F}^P_m \\
\pi' \downarrow & & \downarrow r \\
\mathcal{F}^-_m & \xrightarrow{p} & G \times \mathcal{F}^P_m \\
\pi'' \downarrow & & \downarrow \pi^L_m \\
\mathcal{F}^L_m & \xrightarrow{q} & \mathcal{F}^L_m
\end{array}
\]

(1.5.1)

where the map \( q \) is defined by \( (g, x, v) \mapsto (x, v) \), with \( x \mapsto x, v \mapsto v \) natural maps \( P \rightarrow L, \prod M_{p_i} \rightarrow \prod M_{p_i} \). \( \bar{q} \) is defined as the composite of the projection \( G \times \mathcal{F}^-_m \rightarrow \mathcal{F}^-_m \) and the map \( \mathcal{F}^-_m \rightarrow \mathcal{F}^-_m \) induced from the projection \( P \times (B \times \prod M_{p_i}) \rightarrow L \times (B_L \times \prod M_{p_i}) \).

\( \pi' \) is the natural map \( \mathcal{F}^-_m \rightarrow \mathcal{F}^P_m \), \( g \ast (x, v) \mapsto (gxg^{-1}, gv) \).

Here both squares in the diagram are cartesian. Moreover, we have

(i) \( p \) is a principal \( P \)-bundle.
(ii) \( q \) is a locally trivial fibration with fibre isomorphic to \( G \times U_P \times \prod_{i=1}^{r-2} M_{p_i} \).
Put \( a = \dim P \), \( b = \dim G + \dim U_P + \dim \prod_{i=1}^{n} M_p \). By (i) and (ii), the following property holds.

(1.5.2) For any \( L \)-equivariant simple perverse sheaf \( A_1 \) on \( \mathcal{Y}_m^L \), \( q^* A_1[b] \) is a \( G \times P \)-equivariant simple perverse sheaf on \( G \times \mathcal{X}_m^P \), and there exists a unique \( G \)-equivariant simple perverse sheaf \( A_2 \) on \( \mathcal{X}_m^P \) (up to isomorphism) such that

\[
p^* A_2[a] \simeq q^* A_1[b].
\]

We define a perverse sheaf \( K_{m,T,E} \) on \( \mathcal{X}_m \) by the right hand side of the formula (1.4.2). Thus \( K_{m,T,E} \simeq (\pi_m)_{\alpha^* E}[d_m] \). We consider the perverse sheaf \( K_{m,T,E}^L \) on \( \mathcal{X}_m^L \) defined similarly to \( K_{m,T,E} \). Put \( \mathcal{L} = \alpha^* E \). Since \( \dim \mathcal{X}_m^P = d_m \), we see by (1.5.2) that \( \pi'_m \mathcal{L}[d_m] \) is a perverse sheaf on \( \mathcal{X}_m^P \) satisfying the property

(1.5.3)

\[
p^* \pi'_m \mathcal{L}[d_m + a] \simeq q^* K_{m,T,E}^L[b].
\]

Since \( K_{m,T,E}^L \) is decomposed as

(1.5.4)

\[
K_{m,T,E}^L = \bigoplus_{\rho \in W^\wedge_m,E} \rho \otimes \IC(\mathcal{X}_m^L, \mathcal{L}_\rho^L)[d_m^L],
\]

where \( \mathcal{L}_\rho^L, d_m^L \) are defined similarly to the theorem, again by (1.5.2), \( \pi'_m \mathcal{L}[d_m] \) is a semisimple perverse sheaf, equipped with \( W_{m,E} \)-action, and is decomposed as

(1.5.5)

\[
\pi'_m \mathcal{L}[d_m] \simeq \bigoplus_{\rho \in W^\wedge_m,E} \rho \otimes A_\rho,
\]

where \( A_\rho \) is a simple perverse sheaf on \( \mathcal{X}_m^P \) such that \( p^* A_\rho[a] \simeq q^* \IC(\mathcal{X}_m^L, \mathcal{L}_\rho^L)[d_m^L + b] \).

By applying \( \pi'_m \) on both sides, we have a decomposition

(1.5.6)

\[
K_{m,T,E} \simeq \bigoplus_{\rho \in W^\wedge_m,E} \rho \otimes \pi''_m A_\rho.
\]

Comparing this with the decomposition in (1.4.2), we have

**Proposition 1.6.** \( \pi''_m A_\rho \simeq \IC(\mathcal{X}_m^L, \mathcal{L}_\rho^L)[d_m] \).

**Proof.** By replacing \( \mathcal{X}_m^P \), etc. by \( \mathcal{Y}_m^0 \), etc., we have a similar diagram as (1.5.1),
and an $F \hookrightarrow M$. Let $(m \in \pi)$ be the corresponding Frobenius maps. We fix an $F$-module structure on $w$. There exists $w_0$ such that $w_0$ is open dense in $P$, and let $\psi$ be an $F$-stable basis of $P$, and let $\psi'$ be the parabolic subgroup of $T$ and $T_0$ obtained from the Galois covering $\mathcal{L}$. Since $\psi'_m = \psi'' \circ \psi'$ and $\psi'_m$ is a finite Galois covering with group $W_m$, we see that $\psi''$ is an isomorphism. Now $\mathcal{Y}_m^{P,0}$ is open dense in $\mathcal{Y}_m^P$, and the $W_m$-module structure on $\pi'_m \mathcal{L}$ is determined from the corresponding structure on $\psi'_m \mathcal{L}$ obtained from the Galois covering $\psi'$. On the other hand, the $W_m$-module structure on $\pi'_m \mathcal{L}$ is determined from the corresponding structure on $\psi'_m \mathcal{L}$ obtained from the Galois covering $\psi'_m$. Since $\psi''$ is an isomorphism, this shows that the operation $\pi'_m$ is compatible with the $W_m$-module structures of $\pi'_m \mathcal{L}$ and of $\pi'_m \mathcal{L}$. The proposition is proved.

2. Green functions

2.1. We now assume that $G$ and $V$ are defined over $\mathbb{F}_q$, and let $F : G \to G, F : V \to V$ be the corresponding Frobenius maps. We fix an $F$-stable Borel subgroup $B_0$ and an $F$-stable maximal torus $T_0$ contained in $B_0$. We define $W_0$ as $W_0 = N_G(T_0)/T_0$. Let $(M_{0,i})$ be the total flag corresponding to $B_0$. Thus $M_{0,i}$ are $F$-stable subspaces. For $m \in \mathcal{P}_{n,r}$, let $P_m$ be the parabolic subgroup of $G$ containing $B_0$ which is the stabilizer of the partial flag $(M_{0,p_i})$. Then the Weyl subgroup of $W_0$ corresponding to $P_m$ is given by $(W_0)_m$.

Let $T$ be an $F$-stable maximal torus of $G$, and $B \supset T$ a not necessarily $F$-stable Borel subgroup of $G$. Let $(M_i)$ be the total flag of $G$ whose stabilizer is $B$. We assume that $M_{p_i}$ is $F$-stable for each $i$. Let us construct $\mathcal{Y}_m, \mathcal{Y}_m^{P,0}$, etc. as in Section 1 by using these $T$ and $B$. There exists $h \in G$ such that $B = hB_0h^{-1}, T = hT_0h^{-1}$, and that $h^{-1}F(h) = \hat{w},$ where $\hat{w}$ is a representative of $w \in (W_0)_m$ in $N_G(T_0)$. We fix an $F$-stable basis $e_1, \ldots, e_n$ of $V$ which are weight vectors for $T_0$. Then $h e_1, \ldots, h e_{p_i}$ are basis of $M_{p_i}$ consisting of weight
vectors for $T$. If we define $M_0^\circ_\pi$, as in 1.3 by using this basis, then $M_0^\circ_\pi$ is $F$-stable for each $i$. Since $\mathcal{Y}_m^0 \cong G \times T(T_{\text{reg}} \times \prod_i M_0^\circ_\pi)$, $\mathcal{Y}_m^0$ has a natural $\mathbb{F}_q$-structure. $\mathcal{Y}_m^0$ is $F$-stable, and the maps $\psi^0_m : \mathcal{Y}_m^0 \to \mathcal{Y}_m^0$ and $\alpha_0 : \mathcal{Y}_m^0 \to T$ are $F$-equivariant. Let $\mathcal{E}$ be a tame local system on $T$ such that $F^* \mathcal{E} \cong \mathcal{E}$. We fix an isomorphism $\varphi_0 : F^* \mathcal{E} \cong \mathcal{E}$. Then $\varphi_0$ induces an isomorphism $\tilde{\varphi}_0 : F^* \mathcal{L} \cong \mathcal{L}$, where $\mathcal{L}$ is the local system $(\psi^0_m)\alpha^0_0 \mathcal{E}$ on $\mathcal{Y}_m^0$. By (1.3.1), we have $\mathcal{L} \cong \bigoplus_{\rho \in W_m} \rho \otimes \mathcal{L}_\rho$. As in 1.5, we define a complex $K_{m,T,\mathcal{E}}$ on $\mathcal{Y}_m$ by

$$
K_{m,T,\mathcal{E}} = \text{IC}(\mathcal{Y}_m, \mathcal{L}^\bullet)[d_m] \cong \bigoplus_{\rho \in W_m} \rho \otimes \text{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)[d_m].
$$

$\tilde{\varphi}_0$ can be extended to a unique isomorphism $\varphi : F^* K_{m,T,\mathcal{E}} \cong K_{m,T,\mathcal{E}}$. Note that by Theorem 1.4 (ii), $(\pi_m)\alpha^0_0 \mathcal{E}[d_m]$ is isomorphic to $K_{m,T,\mathcal{E}}$. But the $\mathbb{F}_q$-structure of $(\pi_m)\alpha^0_0 \mathcal{E}[d_m]$ is not defined directly from the construction.

2.2. Let $\mathcal{D}_X = \mathcal{D}_c^s(X)$ be the bounded derived category of $\mathbb{Q}_l$-constructible sheaves on a variety $X$ over $\mathbb{k}$. Assume that $X$ is defined over $\mathbb{F}_q$, and let $F : X \to X$ be the corresponding Frobenius map. Recall that for a given $K \in \mathcal{D}_X$ with an isomorphism $\phi : F^* K \cong K$, the characteristic function $\chi_{K,\phi} : X^F \to \mathbb{Q}_l$ is defined by

$$
\chi_{K,\phi}(x) = \sum_i (-1)^i \text{Tr} (\phi^*, \mathcal{H}_x \tilde{K}), \quad (x \in X^F),
$$

where $\phi^*$ is the induced isomorphism on $\mathcal{H}_x \tilde{K}$.

Returning to the original setting, we consider a tame local system $\mathcal{E}$ on $T$ such that $F^* \mathcal{E} \cong \mathcal{E}$. Since the isomorphism $F^* \mathcal{E} \cong \mathcal{E}$ is unique up to scalar, we fix $\varphi_0 : F^* \mathcal{E} \cong \mathcal{E}$ so that it induces the identity map on the stalk $\mathcal{E}_e$ at the identity element $e \in T$. We consider the characteristic function of $K_{m,T,\mathcal{E}}$ with respect to the map $\varphi$ induced from this $\varphi_0$, and denote it by $\chi_{m,T,\mathcal{E}}$. Since $K_{m,T,\mathcal{E}}$ is a $G$-equivariant perverse sheaf, $\chi_{m,T,\mathcal{E}}$ is a $G^F$-invariant function on $\mathcal{D}_m$.

The following result is an analogue of Lustig’s result ([L2, (8.3.2)]) for character sheaves. The gap of the proof in [L2] was corrected in [L4] in a more general setting of character sheaves on disconnected reductive groups. The analogous statement in the case of exotic symmetric space (of level 2) was proved in [SS2, Prop. 1.6] based on the argument in [L4]. The proof for the present case is quite similar to that of [SS2], so we omit the proof here.

**Proposition 2.3.** The restriction of $\chi_{m,T,\mathcal{E}}$ on $\mathcal{X}_m$ is independent of the choice of $\mathcal{E}$.

We define a function $Q_{m,T} = Q^G_{m,T}$ as the restriction of $\chi_{m,T,\mathcal{E}}$ on $\mathcal{X}_m$, and call it the Green function on $\mathcal{X}_m$.

2.4. Let $T = T_w$ be an $F$-stable maximal torus in $G$ as in 2.1, namely $T = hT_0h^{-1}$ with $h \in G$, such that $h^{-1}F(h) = w \in (W_0)_m$. We consider the isomorphism $\varphi = \varphi_T : F^* K_{m,T,\mathcal{E}} \cong K_{m,T,\mathcal{E}}$ as in 2.2, defined from the specific choice of $\varphi_0$. Let $\mathcal{E}_0$ be the tame local system on $T_0$ defined by $\mathcal{E}_0 = (ad h)^* \mathcal{E}$. Then we have an isomorphism $\varphi_{T_0} : F^* K_{m,T_0,\mathcal{E}_0} \cong K_{m,T_0,\mathcal{E}_0}$. For later use, we shall describe the relationship between $\varphi_T$ and $\varphi_{T_0}$. We write the varieties and maps $\mathcal{Y}_m, \mathcal{Y}_m^0, \alpha_0$, etc. as $\mathcal{Y}_{m,T}, \mathcal{Y}_{m,T_0,\mathcal{E}_0}$, etc. to
indicate the dependence on $T$. $\tilde{\mathcal{Y}}_{m,T_0}$ has a natural Frobenius action $F : (x,v,gT_0) \mapsto (F(x), F(v), F(gT_0))$, and similarly for $\mathcal{Y}_{m,T}$. The map $(x,v,gT) \mapsto (x,v,ghT_0)$ gives a morphism $\delta : \mathcal{Y}_{m,T} \to \mathcal{Y}_{m,T_0}$ commuting with the projection to $\mathcal{Y}_{m}$ (note that $\mathcal{Y}_{m}$ is independent of the choice of $T$). We define a map $a_w : \mathcal{Y}_{m,T_0}^0 \to \mathcal{Y}_{m,T_0}^0$ by $(x,v,ghT_0) \mapsto (x,v,gw^{-1}T_0)$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}_{m,T}^0 & \xrightarrow{\delta} & \mathcal{Y}_{m,T_0}^0 \\
F \downarrow & & \downarrow a_w F \\
\mathcal{Y}_{m,T}^0 & \xrightarrow{\delta} & \mathcal{Y}_{m,T_0}^0 \\
\end{array}
$$

(2.4.1)

Let $\mathcal{L}_0^* = \left(\psi_{m,T_0}^0\right)^*((\alpha_0,T_0)^*E_0)$ be the local system on $\mathcal{Y}_{m}^0$. We know $\End \mathcal{L}_0^* \simeq Q_{t\left([W_0]_{m,0}\right)}$. This isomorphism is given as follows; the map $w \mapsto a_w$ gives a homomorphism $(W_0)_m^0 \to \text{Aut}(\mathcal{Y}_{m,T_0}^0)$. If $w \in (W_0)_m^0$, $a_w$ induces an isomorphism $\tilde{a}_w$ on $\mathcal{L}_0^*$, and the map $w \mapsto \tilde{a}_w$ gives the isomorphism $Q_{t\left([W_0]_{m,0}\right)} \to \End \mathcal{L}_0^*$. By the property of the intermediate extensions, we have an isomorphism on $K_{m,T_0,0}$ induced from $\tilde{a}_w$, which we denote by $\theta_w$. Also $\delta$ induces an isomorphism $\mathcal{L}_0^* \to (\psi_{m,T}^0)^*E_0 \to \mathcal{L}_0^*$, and so induces an isomorphism $K_{m,T,0} \to K_{m,T_0,0}$, which we denote by $\tilde{\delta}$. Then the diagram (2.4.1) implies the following commutative diagram.

$$
\begin{array}{ccc}
F^*K_{m,T,E} & \xrightarrow{F^*}(\tilde{\delta}) & F^*K_{m,T_0,0,E} \\
\varphi_T \downarrow & & \downarrow \varphi_{T_0}F^*(\theta_w) \\
K_{m,T,0,E} & \xrightarrow{\tilde{\delta}} & K_{m,T_0,0,0,E} \\
\end{array}
$$

(2.4.2)

Note that $F^*(\theta_w) = \theta_{F(w)}$. Since $F$ acts trivially on $W_0$, we have $F^*(\theta_w) = \theta_w$.

2.5. For a semisimple element $s \in G^F$, we consider $Z_G(s) \times V^{r-1}$. Then $V$ is decomposed as $V = V_1 \oplus \cdots \oplus V_t$, where $V_i$ is an eigenspace of $s$ with $\dim V_i = n_i$, and $F$ permutes the eigenspaces. $Z_G(s) \simeq G_1 \times \cdots \times G_t$ with $G_i = GL(V_i)$. Hence

$$
Z_G(s) \times V^{r-1} \simeq \prod_{i=1}^t (G_i \times V_i^{r-1}),
$$

(2.5.1)

and the diagonal action of $Z_G(s)$ on the left hand side is compatible with the diagonal action of $G_i$ on $G_i \times V_i^{r-1}$ under the isomorphism $Z_G(s) \simeq G_1 \times \cdots \times G_t$. The definition of $\mathcal{F}_m, \mathcal{F}_m, \mathcal{F}_m, \mathcal{F}_m$, etc. make sense if we replace $G$ by $Z_G(s)$, hence one can define the complex $K_{m,T,0}$ associated to $Z_G(s)$. By (2.5.1), $\mathcal{F}_m, \mathcal{F}_m$ with respect to $Z_G(s)$ are a direct product of similar varieties appeared in 1.1 with respect to $G_i$, hence Theorem 1.4 holds also for $Z_G(s)$ under a suitable adjustment. Concerning the $\mathcal{F}_m$-structure, Proposition 2.3 holds also for $Z_G(s)$. In fact, this is easily reduced to the case where $F$ acts transitively on $Z_G(s) \simeq G_1 \times \cdots \times G_t$. In that case, $G^F \simeq G_1^{F_1}$, and we may assume that
s ∈ T₀. Thus Proposition 2.3 holds in this case, hence holds for the general case. In particular, one can define a Green function \( Q_{m,T}^{Z_G(s)} \) on \( \mathfrak{X}^{m,uni}_T \) for \( T \subset Z_G(s) \).

For an \( F \)-stable torus \( S \), put \((S^F)\wedge = \text{Hom}(S^F, \bar{Q}_\epsilon^I)\). As in [SS2, 2.3], the set of \( F \)-stable tame local systems on \( S \) is in bijection with the set \((S^F)\wedge\) in such a way that the characteristic function \( \chi_{\epsilon,\varphi_0} \) on \( S^F \) gives an element of \((S^F)\wedge\) (under the specific choice of \( \varphi_0 : F^* \epsilon \cong \epsilon \) as in 2.2). We denote by \( \sigma_\theta \) the \( F \)-stable tame local system on \( S \) corresponding to \( \theta \in (S^F)\wedge\).

The following theorem is an analogue of Lusztig’s character formula for character sheaves [L2, Theorem 8.5]. A similar formula for the case of exotic symmetric space (of level 2) was proved in [SS2, Theorem 2.4]. The proof of the theorem is quite similar to the proof given in [SS2, Theorem 2.4], so we omit the proof here.

**Theorem 2.6 (Character formula).** Let \( s,u \in G^F \) be such that \( su = us \), where \( s \) is semisimple and \( u \) is unipotent. Then

\[
\chi_{m,T,\epsilon}(su,v) = |Z_G(s)|^{F^{-1}} \sum_{x \in G^F} Q_{m,xT^{-1}}^{Z_G(s)}(u,v)(x^{-1}sx),
\]

where \( \theta \in (T^F)\wedge \) is such that \( \epsilon = \epsilon_\theta \).

**2.7.** For later use, we shall introduce another type of Green functions. We follow the notation in 1.5. By using the basis \( \{e_1, \ldots, e_n\} \) of \( V \), we identify \( M_p^i \) with the subspace \( M^+_{p_i} \) of \( V \) so that \( V = M^+_{p_1} \oplus \cdots \oplus M^+_{p_r} \).

Hence \( B_L \) stabilizes each \( M^+_{p_i} \). As an analogue of the construction of \( \tilde{\mathfrak{X}}_{m} \), we consider the diagram

\[
T \leftarrow \alpha^+ : \tilde{\mathfrak{X}}^+_{m} \to \mathfrak{X}^+_{m}.
\]

where

\[
\tilde{\mathfrak{X}}^+_{m} = \{(x,v,gB_L) \in G \times V^{r-1} \times G/B_L \mid g^{-1}xg \in B, g^{-1}v \in \prod_{i=1}^{r-1} M^+_{p_i}\} \cong G \times B_L (B \times \prod_{i} M^+_{p_i}) \]

and \( \alpha^+(x,v,gB_L) \mapsto p_T(g^{-1}xg), \pi^+_m : (x,v,gB_L) \mapsto (x,v) \). Put

\[
(2.7.1) \quad \mathfrak{X}^+_{m} = \bigcup_{g \in G} g(B \times \prod_i M^+_{p_i}).
\]

Then \( \tilde{\mathfrak{X}}^+_{m} \) is smooth, irreducible, and \( \text{Im} \pi^+_m = \mathfrak{X}^+_{m} \), but \( \pi^+_m \) is not proper, and \( \mathfrak{X}^+_{m} \) is not necessarily a locally closed subset of \( \mathfrak{X}_{m} \). We consider a variety

\[
(2.7.2) \quad \tilde{\mathfrak{X}}^+_{mL} = G \times L (P \times \prod_i M^+_{p_i})
\]
and put $d^+_m = \dim \hat{\mathcal{F}}^L_m$. For a tame local system $\mathcal{E}$ on $T$, we consider the complex $(\pi^+_m)_!(\alpha^+)^* \mathcal{E}[d^+_m]$ on $\mathcal{F}^L_m$, which we denote by $K^L_{m,T,\mathcal{E}}$.

2.8. Let $\hat{\mathcal{F}}^L_m$ be as in 2.7. We further define

\[
\mathcal{F}^P_m = \bigcup_{g \in L} g(B \times \prod_i M^+_{p_i}) = P \times \prod_i M^+_{p_i},
\]

\[
\hat{\mathcal{F}}^L_m = L \times B_L (B \times \prod_i M^+_{p_i}).
\]

We define maps $\pi'_m : \hat{\mathcal{F}}^L_m \to \hat{\mathcal{F}}^L_m$, $\pi''_m : \hat{\mathcal{F}}^L_m \to \mathcal{F}^L_m$ in a similar way as in 1.5. Thus we have $\pi^+_m = \pi''_m \circ \pi'_m$. We consider a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{F}}^L_m & \xrightarrow{\tilde{\phi}} & G \times \hat{\mathcal{F}}^L_m, \\
\pi'_m & \downarrow & \pi'_m \\
\hat{\mathcal{F}}^L_m & \xrightarrow{\hat{q}'} & \mathcal{F}^L_m,
\end{array}
\]

(2.8.1)

\[
\begin{array}{ccc}
\mathcal{F}^P_m & \xrightarrow{\tilde{q}'} & G \times \mathcal{F}^P_m \\
\pi''_m & \downarrow & \pi''_m \\
\mathcal{F}^P_m & \xrightarrow{q'_m} & \mathcal{F}^L_m,
\end{array}
\]

where $q'_m$ is the composite of the projection $G \times \mathcal{F}^P_m \to \mathcal{F}^P_m$ and the map $\mathcal{F}^P_m \to \mathcal{F}^L_m, (x, v) \mapsto (x, v)$ under the identification $M^+_p = \overline{M}_{p_i}$ ($\overline{\pi}$ is as in 1.5). The map $\tilde{q}'_m$ is the composite of the projection $G \times \hat{\mathcal{F}}^L_m \to \hat{\mathcal{F}}^L_m$ and the natural map $\hat{\mathcal{F}}^L_m \to \hat{\mathcal{F}}^L_m$ induced from the map $L \times (B \times \prod_i M^+_{p_i}) \to L \times (B_L \times \prod_i M_{p_i})$. $r'_m = \text{id} \times r'_m$, where $r'_m$ is the natural map $\hat{\mathcal{F}}^L_m \to \mathcal{F}^P_m$. Both squares are cartesian, and we have

(i) $p'_m$ is a principal $L$-bundle.

(ii) $q'_m$ is a locally trivial fibration with fibre isomorphic to $G \times U_P$.

Then (i) and (ii) imply a similar statement as (1.5.2). We consider the perverse sheaf $K^L_{m,T,\mathcal{E}}$ on $\mathcal{F}^L_m$ as in 1.5. Put $\mathcal{L}^+ = (\alpha^+)^* \mathcal{E}$. By a similar discussion as in 1.5, $(\pi'_m)_! \mathcal{L}^+[d^+_m]$ is a perverse sheaf on $\hat{\mathcal{F}}^L_m$ satisfying the property

\[
(2.8.2) \quad p'_m(\pi'_m)_! \mathcal{L}^+[d^+_m + a'] \simeq q'_m K^L_{m,T,\mathcal{E}}[b'],
\]

where $a' = \dim L$ and $b' = \dim G + \dim U_P$. Since $K^L_{m,T,\mathcal{E}}$ is decomposed as in (1.5.4), it follows that $(\pi'_m)_! \mathcal{L}^+[d^+_m]$ is a semisimple perverse sheaf on $\hat{\mathcal{F}}^L_m$, equipped with $W_{m,\mathcal{E}}$-action, and is decomposed as

\[
(2.8.3) \quad (\pi'_m)_! \mathcal{L}^+[d^+_m] \simeq \bigoplus_{\rho \in W_{m,\mathcal{E}}} \rho \otimes B_\rho,
\]

where $B_\rho$ is a simple perverse sheaf on $\hat{\mathcal{F}}^L_m$. 

\[\]
By applying \((\pi''_+)^\rho\); on both sides of (2.8.3), we have a similar formula as (1.5.6),

\[
(2.8.4) \quad K_{m_T,\mathscr{E}}^+ \simeq \bigoplus_{\rho \in W^\wedge_{m,\mathscr{E}}} \rho \otimes (\pi''_+)^\rho B_\rho.
\]

But note that \((\pi''_+)B_\rho\) is not necessarily a perverse sheaf.

2.9. We now consider the \(F_q\)-structure, and assume that \(T, B\) are both \(F\)-stable, hence \(P\) and \(L\) are also \(F\)-stable. Then all the varieties involved in the diagram (2.8.1) have natural \(F_q\)-structures, and all the maps are \(F\)-equivariant. This is also true for the diagram (1.5.1). Recall that \(K^{L}_{m_T,\mathcal{E}}\) has the natural \(F_q\)-structure \(\varphi^L : F^*K^{L}_{m_T,\mathcal{E}} \simeq K^{L}_{m_T,\mathcal{E}}\). Then the decomposition (1.5.4) determines the isomorphism \(\varphi^L : F^* \text{IC}(\mathcal{F}_m^L, \mathcal{L}_\rho^L) \simeq \text{IC}(\mathcal{F}_m^L, \mathcal{L}_\rho^L)\) for each \(\rho\) such that \(\varphi^L = \sum_\sigma \sigma \otimes \varphi^L_\sigma\), where \(\sigma_\rho\) is the identity map on the representation space \(\rho\). By (2.8.1), the map \(\varphi^L_\rho\) induces an isomorphism \(h_\rho : F^*B_\rho \simeq B_\rho\). We also obtain an isomorphism \(f^*_\rho : F^*A_\rho \simeq A_\rho\) by a similar discussion applied to the diagram (1.5.1).

We have the following lemma.

**Lemma 2.10.** For any \(z \in (\mathcal{F}_m^*)^L\) and \(\rho \in W_{m,\mathcal{E}}^\wedge\), we have

\[
\text{Tr}(h_\rho^*, H^i_c((\pi''_+)^{-1}(z), B_\rho)) = \text{Tr}(f_\rho^*, H^i_c(\dim U_\rho((\pi''_+)^{-1}(z), A_\rho))) \cdot q^{\dim U_\rho},
\]

where \(f_\rho^*, h_\rho^*\) are the linear maps on the cohomologies induced from \(f_\rho, h_\rho\).

**Proof.** We consider the following commutative diagram

\[
\begin{array}{c}
\mathcal{F}_m^L \\
\eta \downarrow \\
\begin{array}{c}
G \times^L (P \times \prod M_{p_i}) \\
\xi \downarrow \\
\widetilde{\mathcal{F}}_m^P
\end{array}
\end{array}
\begin{array}{ccc}
\leftarrow^p & 
\leftarrow^p & \\
\uparrow^{q_+} & 
\uparrow^{q} & \\
\begin{array}{c}
G \times (P \times \prod M_{p_i})^+ \\
\eta_\mathcal{E} \downarrow \\
\widetilde{\mathcal{E}}_m^L
\end{array} & 
\begin{array}{c}
G \times (P \times \prod M_{p_i})^+ \\
\eta \downarrow \\
\mathcal{F}_m^L
\end{array} & 
\begin{array}{c}
G \times (P \times \prod M_{p_i})^+ \\
\xi \downarrow \\
\mathcal{F}_m^P
\end{array}
\end{array}
\]

(2.10.1)

where \(\eta\) is the inclusion map, and \(\eta_\mathcal{E}\) is the map induced from \(\eta\) by taking the quotients by \(L\). \(\xi\) is the natural map from the quotient by \(L\) to the quotient by \(P\), and \(p_*\) is the quotient by \(L\). Note that \(\eta\) is injective, and \(\xi\) is a locally trivial fibration with fibre isomorphic to \(U_\rho\).

By a similar construction of \(A_\rho\) and \(B_\rho\), one can define a simple perverse sheaf \(\widetilde{B}_\rho\) on \(G \times^L (P \times \prod M_{p_i})\). By (2.10.1), we have

\[
(2.10.2) \quad \xi^* A_\rho[d] \simeq \widetilde{B}_\rho, \quad \eta^* \widetilde{B}_\rho \simeq B_\rho,
\]

where \(d = \dim U_\rho\). (The latter formula follows from the fact that the upper left square in (2.10.1) is cartesian.) We now define \(\pi''_* : G \times^L (P \times \prod M_{p_i}) \to \mathcal{F}_m^L\) by \((g * (x, v)) \mapsto (gxg^{-1}, gv)\). Then \((\pi''_*)^{-1}(\mathcal{F}_m^+) = \mathcal{F}_m^L\) and the restriction of \(\pi''_*\) on \(\mathcal{F}_m^L\) coincides with \(\pi''_+\). It follows, for \(z \in (\mathcal{F}_m^+)\), that
since \((\pi''_\omega)^{-1}(z)\rightarrow (\pi''_\omega)^{-1}(z)\) is a locally trivial fibration with fibre isomorphic to \(U_P\), and so \(\xi^*_\omega A_\rho \cong A_\rho[-2d](d)\), where \((d)\) denotes the Tate twist. Note that the Frobenius actions \(h_\rho^*, f_\rho^*\) on these cohomologies come from the Frobenius action \(\varphi_\rho^L\) on \(\mathcal{I}C(\mathcal{E}_{\mathbf{m}_1}, \mathcal{L}_\rho^L)\). Hence the isomorphism in (2.10.3) is compatible with the maps \(h_\rho^*, f_\rho^*\). This proves the lemma. 

\[\Box\]

2.11. We now consider the general case. We assume that \(T, L\) and \(P\) are \(F\)-stable, but \(B\) is not necessarily \(F\)-stable. Since \(P\) is \(F\)-stable, in particular all the subspaces \(M_\rho\) are \(F\)-stable. We consider the diagram (2.8.1). Except the top row, all the objects involved in (2.8.1) are \(F\)-equivariant. We consider the isomorphism \(\varphi^L : F^*K_{m,T,\epsilon}^L \cong K_{m,T,\epsilon}^L\). Since \(p_+, q_+\) are \(F\)-equivariant, \(\varphi^L\) induces an isomorphism \(\varphi^L : F^*((\pi''_\omega)^L), \mathcal{L}^+ \cong (\pi''_\omega^L), \mathcal{L}^+.\) Since \(\pi''_\omega\) is \(F\)-equivariant, \(\varphi^L\) induces an isomorphism \(\varphi^L : F^*K_{m,T,\epsilon}^+ \cong K_{m,T,\epsilon}^+\). We define a function \(\chi_{m,T,\epsilon}^+\) by

\[\chi_{m,T,\epsilon}^+ = (-q)^{-\dim U_P}\chi_{K_{m,T,\epsilon}^+}^+ ; \varphi^L.\]

2.12. Let \(T_0 \subset B_0\) be as in 2.1. We also consider the \(F\)-stable parabolic subgroup \(P_0\) containing \(B_0\) which is conjugate to \(P\) under \(G^F\). Let \(L_0\) be the Levi subgroup of \(P_0\) containing \(T_0\). Since \(T\) is \(G^F\)-conjugate to an \(F\)-stable maximal torus in \(L_0\), in considering the \(\mathbf{F}_q\)-structure of \(K_{m,T,\epsilon}\), we may assume that \(T = T_w \subset L_0\) for \(w \in (W_0)_{\mathbf{m}}\). We shall consider two complexes \(K_{m,T,\epsilon}\) and \(K_{m,T_0,\epsilon_0}\) as discussed in 2.4. We follow the notation in 2.4. In particular, let \(\varphi_T\) (resp. \(\varphi_{T_0}\)) be the isomorphism \(\varphi\) with respect to \(K_{m,T,\epsilon}\) (resp. \(K_{m,T_0,\epsilon_0}\)). By the decomposition of \(K_{m,T_0,\epsilon_0}\) in (1.4.2), \(\varphi_{T_0}\) determines an isomorphism \(\varphi_\rho : F^*\mathcal{I}C(\mathcal{E}_{\mathbf{m}}, \mathcal{L}_\rho)|[d_{\mathbf{m}}] \cong \mathcal{I}C(\mathcal{E}_{\mathbf{m}}, \mathcal{L}_\rho)|[d_{\mathbf{m}}]\) such that \(\varphi_{T_0} = \sum_{\rho \in (W_0)_{\mathbf{m}, \epsilon_0}} \sigma_\rho \otimes \varphi_\rho\), where \(\sigma_\rho\) is the identity map on the representation space \(\rho\). By (2.4.2), under the isomorphism \(\tilde{\delta} : K_{m,T,\epsilon} \cong K_{m,T_0,\epsilon_0}\), the map \(\varphi_T\) can be described by the map \(\varphi_{T_0}\) and \(\theta_w\), where \(\theta_w\) corresponds to the action of \(w\) on each \((W_0)_{\mathbf{m}, \epsilon_0}, \rho\)-module \(\rho\). It follows that

\[\varphi_T \simeq \sum_{\rho \in (W_0)_{\mathbf{m}, \epsilon_0}} w_\rho| \otimes \varphi_\rho,\]

where \(w_\rho|\) denotes the action of \(w\) on \(\rho\).

Next we consider two complexes \(K_{m,T,\epsilon}^+\) and \(K_{m,T_0,\epsilon_0}^+\). Let \(\varphi_T^+\) (resp. \(\varphi_{T_0}^+\)) be the isomorphism with respect to \(K_{m,T,\epsilon}^+(\text{resp. } K_{m,T_0,\epsilon_0}^+)\). The isomorphism \(\varphi_T^+\) and \(\varphi_{T_0}^+\) are defined similarly with respect to \(K_{m,T,\epsilon}^L\) and \(K_{m,T_0,\epsilon_0}^L\). Then a similar formula as (2.12.1) holds for \(\varphi_T^+\) and \(\varphi_{T_0}^+\). We denote by \(K_{m,T,\epsilon}^L\) (resp. \(K_{m,T_0,\epsilon_0}^L\)) the complex \((\pi''_\omega^L), \mathcal{L}^+|[d_{\mathbf{m}}]\) defined with respect to \(T\) (resp. \(T_0\)). By the diagram (2.8.1), the isomorphism \(\varphi_T^L\) induces an
isomorphism \( h_T : F^* K_T^+ \simeq K_T^+ \), and \( h_{T_0} : F^* K_{T_0}^+ \simeq K_{T_0}^+ \). Then again by (2.8.1), we have

\[(2.12.2) \quad h_T \simeq \sum_{\rho \in (W_0)_{\rho,0}} w|_{\rho} \otimes h_{\rho}, \]

where \( h_{\rho} \) is the isomorphism in 2.9 defined with respect to \( T_0 \). By applying \((\pi_{++})_!\), we see that

\[(2.12.3) \quad \varphi_T^+ \simeq \sum_{\rho \in (W_0)_{\rho,0}} w|_{\rho} \otimes \varphi_{\rho}^+, \]

where \( \varphi_{\rho}^+ : F^*(\pi_{++})_!B_{\rho} \simeq (\pi_{++})_!B_{\rho} \) is an isomorphism induced from \( h_{\rho} \).

**Proposition 2.13.** Under the setting in 2.11, we have

\[ \chi_{m,T,\mathcal{E}}^+(z) = \begin{cases} \chi_{m,T,\mathcal{E}}(z) & \text{if } z \in (\mathcal{X}_m^+)^F, \\ 0 & \text{otherwise.} \end{cases} \]

*Proof.* Since \( \text{Im} \pi_{++}^m = \mathcal{X}_m^+ \), it is clear that \( \chi_{m,T,\mathcal{E}}^+(z) = 0 \) unless \( z \in (\mathcal{X}_m^+)^F \). Take \( z \in (\mathcal{X}_m^+)^F \). By (2.12.1), we have

\[ \chi_{m,T,\mathcal{E}}(z) = \sum_{\rho \in (W_0)_{\rho,0}} \text{Tr}(w, \rho) \chi_{I,\rho,\varphi,\rho} \]

where \( I_{\rho} = \text{IC}(\mathcal{X}_m, \mathcal{L})[d_{\rho}] \). Here \( \mathcal{H}^i(I_{\rho}) \simeq \mathcal{H}^i_c((\pi_{++})^{-1}(z), A_{\rho}) \) by Proposition 1.6, and the isomorphism on \( \mathcal{H}^i(I_{\rho}) \) induced from \( \varphi_{\rho} \) coincides with the isomorphism \( f_{\rho}^* \) on \( \mathcal{H}^i_c((\pi_{++})^{-1}(z), A_{\rho}) \). On the other hand, by (2.12.3) we have

\[ \chi_{K_{m,T,\mathcal{E}},\mathcal{F}}^+(z) = \sum_{\rho \in (W_0)_{\rho,0}} \text{Tr}(w, \rho) \chi_{J_{\rho},\varphi_{\rho}^+} \]

where \( J_{\rho} = (\pi_{++})_!B_{\rho} \). Here \( \mathcal{H}^i(J_{\rho}) \simeq \mathcal{H}^i_c((\pi_{++})^{-1}(z), B_{\rho}) \), and the isomorphism on \( \mathcal{H}^i(J_{\rho}) \) induced from \( \varphi_{\rho}^+ \) coincides with \( h_{\rho}^* \) on \( \mathcal{H}^i_c((\pi_{++})^{-1}(z), B_{\rho}) \). The proposition now follows from Lemma 2.10 and (2.11.1). \( \square \)

2.14. As in the case of Green functions \( Q_{m,T} \), we define Green functions \( Q_{m,T}^+ \) as the restriction of \( \chi_{m,T,\mathcal{E}}^+ \) on \( \mathcal{X}_{m,\text{uni}} \). In view of Proposition 2.13 and Proposition 2.3, \( Q_{m,T}^+ \) does not depend on the choice of \( \mathcal{E} \). As in the case of Green functions, the definition of \( Q_{m,T}^+ \) can be generalized to the case where \( G \) is replaced by \( Z_G(s) \), in which case we denote it as \( Q_{m,T}^{+,Z_G(s)} \). Now take \((su, v) \in (\mathcal{X}_m^+)^F \) under the setting in Theorem 2.6. Then by Proposition 2.13 and Theorem 2.6, the value \( \chi_{m,T,\mathcal{E}}^+(su, v) = \chi_{m,T,\mathcal{E}}(su, v) \) can be described as a linear combination of various Green functions \( Q_{m,Tx^{-1}}^{Z_G(s)}(u, v) \) such that \( s \in xTx^{-1} \). One can check that if \((su, v) \in \mathcal{X}_m^+ \), then \((u, v) \in \mathcal{X}_m^+ \text{uni}^+(s) \). It follows, by
Proposition 2.13, that $Q^{Z_0(s)}_{m, xT_x^{-1}}(u, v) = Q^{+, Z_0(s)}_{m, xT_x^{-1}}(u, v)$. Thus as a corollary to Theorem 2.6, we have

**Corollary 2.15** (Character formula for $\chi^{+}_{m,T,\varepsilon}$). Under the assumption in Theorem 2.6, we have

$$\chi^{+}_{m,T,\varepsilon}(su, v) = |Z_G(s)^F|^{-1} \sum_{x \in G^F} Q^{+, Z_0(s)}_{m, xT_x^{-1}}(u, v) \theta(x^{-1} sx).$$

### 3. Orthogonality relations

**3.1.** For a fixed $m \in \mathcal{Z}_{n,r}$, we have defined in the previous sections, $K_{m,T,\varepsilon}$, $\chi_{m,T,\varepsilon}$, $Q_{m,T}$, etc. and $K^{+}_{m,T,\varepsilon}$, $\chi^{+}_{m,T,\varepsilon}$, $Q^{+}_{m,T}$, etc. From this section by changing the notation, we denote $K_{m,T,\varepsilon}$, $\chi_{m,T,\varepsilon}$, $Q_{m,T}$, etc. by $K^{-}_{m,T,\varepsilon}$, $\chi^{-}_{m,T,\varepsilon}$, $Q^{-}_{m,T}$, etc. by attaching the sign “−”. In this section, we shall prove the orthogonality relations for the functions $\chi^{\pm}_{m,T,\varepsilon}$ and $Q^{\pm}_{m,T}$. Before stating the results, we prepare the following.

**Proposition 3.2.** Let $K = K^{+}_{m,T,\varepsilon}$ (resp. $K' = K^{+}_{m', T', \varepsilon'}$) be a complex on $\mathcal{X}$ associated to $m$ and $(M_i)$ ( resp. $m'$ and $(M_i')$), where $\varepsilon, \varepsilon' \in \{+, -\}$. Assume that $\varepsilon'$ is the constant sheaf on $T'$, and $\varepsilon$ is a non-constant sheaf on $T$. Then we have

$$H^i_c(\mathcal{X}, K \otimes K') = 0 \quad \text{for all } i.$$  

**Proof.** We prove the proposition by a similar argument as in the proof of Proposition 7.2 in [L2]. In the discussion below, we consider the case where $\varepsilon = -, \varepsilon' = +$. The other cases are dealt similarly. Let $B'$ be the Borel subgroup of $G$ containing $T'$, which is the stabilizer of the total flag $(M_i')$, and $P'$ the parabolic subgroup of $G$ containing $B'$ which is the stabilizer of the partial flag $(M_i'')$. Let $L'$ be the Levi subgroup of $P'$ containing $T'$, and $U_{P'}$ the unipotent radical of $P'$. Put $B'_L = B' \cap L'$. We consider the fibre product $Z = \mathcal{F}_m \times \mathcal{X} \mathcal{F}_m^+$, where $Z$ can be written as

$$Z = \{(gB, g'B'_L, x, v) \in G/B \times G/B'_L \times G \times V^{-1} \mid g^{-1}xg \in B, g^{-1}xg' \in B', g^{-1}v \in \prod_i M_{p_i}, g^{-1}v' \in \prod_i M_{p_i}' \}.$$  

Let $\mathcal{L} = \alpha^* \varepsilon$ and $\mathcal{L}' = (\alpha^+)^* \varepsilon'$. Since $K = (\pi_m)_* \mathcal{L}$ and $K' = (\pi_{m}')_* \mathcal{L}'$, up to shift, by the Künneth formula, we have

$$H^i_c(\mathcal{X}, K \otimes K') \simeq H^i_c(Z, \mathcal{L} \otimes \mathcal{L}')$$

up to the degree shift. Hence in order to prove the proposition, it is enough to show that the right hand side of (3.2.1) is equal to zero for each $i$. For each $G$-orbit $\mathcal{O}$ of $G/B \times G/B'$, put

$$Z_\mathcal{O} = \{(gB, g'B'_L, (x, v)) \in Z \mid (gB, g'B') \in \mathcal{O}\}.$$
Then \( Z = \bigsqcup_{\partial} Z_{\partial} \) is a finite partition, and \( Z_{\partial} \) is a locally closed subvariety of \( Z \). Hence it is enough to show that \( H^i_c(Z_{\partial}, \mathcal{L} \boxtimes \mathcal{L}') = 0 \) for any \( i \). We consider the morphism \( \varphi_{\partial} : Z_{\partial} \to \mathcal{O}, (gB, g'B_L, (x, v)) \mapsto (gB, g'B') \). Then by the Leray spectral sequence, we have

\[
H^i_c(\mathcal{O}, R^j(\varphi_{\partial})_!(\mathcal{L} \boxtimes \mathcal{L}')) \Rightarrow H^{i+j}_c(Z_{\partial}, \mathcal{L} \boxtimes \mathcal{L}').
\]

Thus it is enough to show that \( R^j(\varphi_{\partial})_!(\mathcal{L} \boxtimes \mathcal{L}') = 0 \) for any \( j \), which is equivalent to the statement that \( H^i_c(\varphi^{-1}_{\partial}(\xi), \mathcal{L} \boxtimes \mathcal{L}') = 0 \) for any \( j \) and any \( \xi \in \mathcal{O} \). Since \( \mathcal{L} \boxtimes \mathcal{L}' \) is a \( G \)-equivariant local system, it is enough to show this for a single element \( \xi \in \mathcal{O} \). Thus, we may choose \( \xi = (B, nB') \in \mathcal{O} \), where \( n \in G \) is such that \( nT'n^{-1} = T \). Then \( \varphi^{-1}_{\partial}(\xi) \) is given as

\[
\varphi^{-1}_{\partial}(\xi) = \{(B, nuB'_L, x, v) \mid x \in B, n^{-1}xn \in B', \varepsilon \in \prod_i M_{p_i}, u^{-1}n^{-1}\varepsilon \in \prod_i M_{p_i}^{r+}, u \in U \}.
\]

Thus \( Y = \varphi^{-1}_{\partial}(\xi) \) is isomorphic to \( (B \cap nB'n^{-1}) \times Y_1 \), where

\[
Y_1 = \{(u, \varepsilon) \in U_{p'} \times V^{r-1} \mid \varepsilon \in \prod_i M_{p_i} \cap \prod_i nu(M_{p_i}^{r+})\}.
\]

Let \( h : Y \to U_{p'} \) be the map obtained from the projection \( h_1 : Y_1 \to U_{p'} \). Again by using the Leray spectral sequence associated to the map \( h \), in order to show \( H^i_c(Y, \mathcal{L} \boxtimes \mathcal{L}') = 0 \), it is enough to see that \( H^i_c(h^{-1}(u), \mathcal{L} \boxtimes \mathcal{L}') = 0 \) for any \( u \in U_{p'} \). For each \( u \in U_{p'} \), the fibre \( h^{-1}(u) \) has a structure of an affine space. Hence \( h^{-1}(u) \) is isomorphic to the direct product of \( (B \cap nB'n^{-1}) \) with an affine space. Since \( B \cap nB'n^{-1} \simeq T \times (U \cap nU'n^{-1}) \), where \( U' \) is the unipotent radical of \( B' \), \( h^{-1}(u) \) can be written as \( h^{-1}(u) \simeq T \times Y_2 \) with an affine space \( Y_2 \). If we denote by \( p : h^{-1}(u) \to T \) the projection on \( T \), the restriction of \( \mathcal{L} \boxtimes \mathcal{L}' \) on \( h^{-1}(u) \) coincides with \( p^*(\mathcal{O} \boxtimes f^*\mathcal{O}') = p^*\mathcal{O} \simeq \mathcal{O} \boxtimes Q_i \) since \( \mathcal{O}' \) is the constant sheaf, where \( f : T \to T' = n^{-1}Tn \). Hence we have only to show that \( H^i_c(T, \mathcal{O}) = 0 \) for any \( i \). But since \( \mathcal{O} \) is a non-constant tame local system on \( T \), we have \( H^i_c(T, \mathcal{O}) = 0 \) for any \( i \). This proves the proposition. \( \square \)

### 3.3.

We consider the complexes \( K_{m,T,\mathcal{O}}, K_{m',T,\mathcal{O}'} \), and their characteristic functions \( \chi_{m,T,\mathcal{O}}, \chi_{m',T,\mathcal{O}'} \). We put \( N(T, T') = \{ n \in G \mid n^{-1}Tn = T' \} \). For \( \varepsilon, \varepsilon' \in \{+, -\} \) and \( n \in N(T, T') \), we define \( a_{\varepsilon, \varepsilon'}(m, m'; n) \) by

\[
a_{\varepsilon, \varepsilon'}(m, m'; n) = \sum_{i=1}^{r-1} \dim(M_{p_i}^\varepsilon \cap n(M_{p_i}^\varepsilon')) = \sum_{i=1}^{r-1} \dim(M_{p_i}^\varepsilon \cap n(M_{p_i}^\varepsilon')),
\]

where \( M_{p_i}^+ \) is as before, and we put \( M_{p_i} = M_{p_i}^- \). We define \( M_{p_i}^{\varepsilon'} \) similarly. Also put

\[
p_{-}(m) = \dim \prod_{i=1}^{r-1} M_{p_i} = \sum_{i=1}^{r-1} p_i, \quad p_{+}(m) = \dim \prod_{i=1}^{r-1} M_{p_i}^+ = p_{r-1}.
\]
We have the following orthogonality relations, which is an analogue of Theorems 9.2 and 9.3 in [L2]. Also see Theorems 3.4 and 3.5 in [SS2] for the case of exotic symmetric space of level 2.

**Theorem 3.4** (Orthogonality relations for $\chi_{m,T,\epsilon}^{\pm}$). Let $\mathcal{E} = \mathcal{E}_g, \mathcal{E}' = \mathcal{E}_{\theta'}$ with $\theta \in (T^F)^\wedge, \theta' \in (T^F)^\wedge$. Then we have

$$(-1)^{p_{\epsilon}(m)+p_{\epsilon'}(m')}|G^F|^{-1} \sum_{z \in \mathcal{F}} \chi_{m,T,\epsilon}^{\epsilon}(z)\chi_{m',T',\epsilon'}^{\epsilon'}(z)\tag{3.4.1}$$

$$= |T^F|^{-1}|T'^F|^{-1} \sum_{n \in N(T,T')^F} \theta(t)\theta'(n^{-1}tn)q^{a_{\epsilon,\epsilon'}(m,m';n)}.$$  

**Theorem 3.5** (Orthogonality relations for Green functions).

$$(-1)^{p_{\epsilon}(m)+p_{\epsilon'}(m')}|G^F|^{-1} \sum_{z \in \mathcal{F}_{uni}} Q_{m,T}^{\epsilon}(z)Q_{m',T'}^{\epsilon'}(z)\tag{3.5.1}$$

$$= |T^F|^{-1}|T'^F|^{-1} \sum_{n \in N(T,T')^F} q^{a_{\epsilon,\epsilon'}(m,m';n)}.$$  

### 3.6.

As was discussed in Section 2, the functions $\chi_{m,T,\epsilon}^{\pm}, Q_{m,T}^{\pm}$ make sense if we replace $G$ by its subgroup $Z_G(s)$, and $G \times V^{r-1}$ by $Z_G(s) \times V^{r-1}$ for a semisimple element $s \in G^F$. Theorem 3.4 and 3.5 are then formulated for this general setting. In what follows, we shall prove Theorem 3.4 and 3.5 simultaneously under this setting.

First we note

(3.6.1) **Theorem 3.4** holds if $\theta'$ is the trivial character and $\theta$ is a non-trivial character.

In fact, by the trace formula for the Frobenius maps, the left hand side of (3.4.1) coincides, up to scalar, with

$$\sum_i (-1)^i \text{Tr} (F^*H^i(\mathcal{F}, K_{m,T,\epsilon}^{\epsilon} \otimes K_{m',T',\epsilon'}^{\epsilon'}),$$

where $F^*$ is the isomorphism induced from $\varphi : F^*K_{m,T,\epsilon}^{\epsilon} \cong K_{m,T,\epsilon}^{\epsilon}$ and $\varphi' : F^*K_{m',T',\epsilon'}^{\epsilon'} \cong K_{m',T',\epsilon'}^{\epsilon'}$. Since $\mathcal{E}'$ is a constant sheaf and $\mathcal{E}$ is a non-constant sheaf, by Proposition 3.2, the left hand side of (3.4.1) is equal to zero. On the other hand, the right hand side of (3.4.1) is equal to zero by the orthogonality relations for irreducible characters of $T^F$. Hence (3.6.1) holds.

Next we show

(3.6.2) **Theorem 3.4** holds if there exist $F$-stable Borel subgroups $B, B'$ such that $B \supset T, B' \supset T'$.

Since $B, B'$ are $F$-stable, we can compute $\chi_{m,T,\epsilon}^{\pm}$ by using the complexes $(\pi_m)_{\alpha^* \mathcal{E}}$ and $(\pi_m^+)_{(\alpha'^-) \mathcal{E}'}$ defined in Section 1 and 2. Then by the trace formula for the Frobenius map, we have
\[ (-1)^{d_m} \chi_{m,T',\varphi}(x,v) = |B^F|^{-1} \sum_{g \in G^F \atop g^{-1}xg \in B^F} \chi_{\varphi,\varphi_0}(p_T(g^{-1}xg)), \]

\[ (-1)^{d'_m} \chi_{m',T',\varphi'}(x,v) = (-q)^{-\dim U_{P'}} |B'_{L'}^F|^{-1} \sum_{g \in G^F \atop g^{-1}xg \in B'^{F'} \atop g^{-1}v \in \prod_i M_{p_i}^{(q)}} \chi_{\varphi',\varphi'_0}(p_{T'}(g^{-1}xg)). \]

Since \( d_m = n^2 + p_-(m) \) by Lemma 1.2 and \( d'_m = n^2 + \dim U_{P'} + p_+(m') \) by (2.7.2), we have

\[ (-1)^{p_\varepsilon(m)+p_{e'}(m')} |G^F|^{-1} \sum_{(x,v) \in \mathcal{X}^F} \chi_{\varepsilon,m,T,\varphi}(x,v) \chi_{\varepsilon',m',T',\varphi'}(x,v) \]

\[ = |G^F|^{-1} |B^F|^{-1} |B'_{L'}^F|^{-1} \sum_{(x,v)} \theta(p_T(g^{-1}xg)) \theta'(p_{T'}(g^{-1}xg')), \]

where the sum \((*)\) is taken over all \( x \in G^F, v \in (V^{r-1})^F, g, g' \in G^F \) such that \( g^{-1}xg \in B^F, g^{-1}xg' \in B'^{F'}, g^{-1}v \in \prod_i (M_{p_i}^{(q)})^F, g^{-1}v' \in \prod_i (M_{p_i}^{(q)})^F \). We change the variables; put \( y = g^{-1}xg, h = g^{-1}g', v' = g^{-1}v \). Then the condition \((*)\) is equivalent to the condition \((**): g \in G^F, h \in G^F, y \in B^F, v' \in \prod_i (M_{p_i}^{(q)})^F \) such that \( h^{-1}yh \in B^F, h^{-1}v' \in \prod_i (M_{p_i}^{(q)})^F \). We consider the partition of \( G \) into double cosets \( B\backslash G \). For each \( F \)-stable coset \( BnB' \) of \( G \), we may assume that \( n^{-1}Tn = T' \) since \( T \) and \( T' \) are \( G^F \)-conjugate. Then the sum \( \sum_{(x,v)} \) in (3.6.3) can be written as

\[ |G^F||U^F||U'^{F'}| \sum_{n \in (B \backslash G \backslash B')^F} \sum_{h \in (TnT')^F} \sum_{t \in T'^{F'}} \theta(t) \theta'(n^{-1}tn) q_0(m,m';n). \]

Thus Theorem 3.4 holds. (3.6.2) is proved.

**3.7.** In this subsection, we show that Theorem 3.4 holds for \( G \) under the assumption that Theorem 3.5 holds for subgroups \( Z_G(s) \) of \( G \). By making use of the character formulas in Theorem 2.6 and Corollary 2.15, we have

\[ (-1)^{p_\varepsilon(m)+p_{e'}(m')} |G^F|^{-1} \sum_{z \in \mathcal{X}^F} \chi_{\varepsilon,m,T,\varphi}(z) \chi_{\varepsilon',m',T',\varphi'}(z) \]

\[ = |G^F|^{-1}(-1)^{p_\varepsilon(m)+p_{e'}(m')} \sum_{s \in G^F \atop x,x' \in G^F \atop x^{-1}sx \in T^F \atop x^{-1}sx' \in T'^{F'}} f(s,x,x') |Z_G(s)^F|^{-2} \theta(x^{-1}sx) \theta'(x'^{-1}sx'), \]
By applying Theorem 3.5 for $Z_G(s)$, we see that

$$f(s, x', x) = (-1)^{p_e(m)+p_e'(m')} |Z_G(s)|^F |T^F|^{-1} |T'^F|^{-1} \sum_{n \in Z_G(s)} q_{\alpha_e, \alpha_e'}(m, m'; n).$$

It follows that the previous sum is equal to

$$|G^F|^{-1} |T^F|^{-1} |T'^F|^{-1} \sum_{s \in G^F} |Z_G(s)|^{-1} \sum_{n \in Z_G(s)} \theta(x^{-1}sx) \theta'(x'^{-1}sx') q_{\alpha_e, \alpha_e'}(m, m'; n).$$

Now put $t = x^{-1}sx \in T^F$ and $y = x^{-1}nx'$. Then $y \in G^F$ such that $y^{-1}Ty = T'$. Under this change of variables, the above sum can be rewritten as

$$|G^F|^{-1} |T^F|^{-1} |T'^F|^{-1} \sum_{x \in G^F} |Z_G(t)|^{-1} \sum_{y \in N(T, T')^F} \theta(t) \theta'(y^{-1}ty) q_{\alpha_e, \alpha_e'}(m, m'; y),$$

which is equal to

$$|T^F|^{-1} |T'^F|^{-1} \sum_{t \in T^F} \sum_{y \in N(T, T')^F} \theta(t) \theta'(y^{-1}ty) q_{\alpha_e, \alpha_e'}(m, m'; y).$$

Thus our assertion holds.

**3.8.** We shall show that Theorem 3.5 holds for $G$ under the assumption that it holds for $Z_G(s)$ if $s$ is not central. Hence Theorem 3.4 holds for such groups by 3.7. Put

$$A = (-1)^{p_e(m)+p_e'(m')} |G^F|^{-1} \sum_{z \in Z_{G(s)}^F} Q_{m, T}^e(z) Q_{m', T'}^e(z),$$

$$B = |T^F|^{-1} |T'^F|^{-1} \sum_{n \in N(T, T')^F} q_{\alpha_e, \alpha_e'}(m, m'; n).$$

We want to show that $A = B$. By making use of a part of the arguments in 3.7 (which can be applied to the case where $s \notin Z(G)^F$), we see that

$$(-1)^{p_e(m)+p_e'(m')} |G^F|^{-1} \sum_{z \in Z^F} \lambda_{m, T, \delta}(z) \lambda_{m', T', \delta'}(z) - A \sum_{s \in Z(G)^F} \theta(s) \theta'(s)$$

$$= |T^F|^{-1} |T'^F|^{-1} \sum_{t \in T^F} \sum_{y \in N(T, T')^F} \theta(t) \theta'(y^{-1}ty) q_{\alpha_e, \alpha_e'}(m, m'; y) - B |Z(G)^F|. \quad (3.8.1)$$
This formula holds for any \( \theta \in (T^F)^\wedge \) and \( \theta' \in (T'^F)^\wedge \). The case where \( G = T \) is included in the case discussed in (3.6.2). So we may assume that \( G \neq T \), hence \( Z(G) \neq G \). Now assume that \( q > 2 \). Then one can find a linear character \( \theta \) on \( T^F \) such that \( \theta|_{Z(G)^F} = \text{id} \) and that \( \theta \neq \text{id} \). We choose \( \theta' \) the identity character on \( T'^F \). Then by (3.6.1), the first term of the left hand side of (3.8.1) coincides with the first term of the right hand side. This implies that \( A = B \) as asserted. The remaining case is the case where \( q = 2 \). If \( T \) is not a split torus, still one can find a linear character \( \theta \) satisfying the above property, and the above discussion can be applied. So we may assume that both of \( T, T' \) are \( F_q \)-split. But in this case Theorem 3.4 holds by (3.6.2). Hence the first term of the left hand side of (3.8.1) coincides with the first term of the right hand side. By choosing the identity characters \( \theta, \theta' \), again we have \( A = B \). Thus our assertion holds.

3.9. We are now ready to prove Theorems 3.4 and 3.5. First note that Theorem 3.4 and 3.5 hold for \( Z_G(s) \) in the case where \( Z_G(s) = T \), i.e., in the case where \( s \) is regular semisimple. In fact, in this case, we have \( T = T' \) and \( N(T,T') = T \). We have \( \chi_{\lambda,T,\varepsilon}(t,v) = (-1)^{d_m(t)} \) if \( v \in \prod_i (M_i^T)^F \) and is equal to zero otherwise. A similar formula holds for \( \chi_{\lambda,T,\varepsilon}^+ \) by replacing \( d_m \) by \( d_m^+ \). Theorem 3.4 and 3.5 follows from this. Now by induction of the semisimple rank of \( Z_G(s) \), we may assume that Theorem 3.5 holds for \( Z_G(s) \) if \( s \) is non central. Then by 3.8, Theorem 3.5 holds for \( G \), and by 3.7, Theorem 3.4 holds for \( G \). This completes the proof of Theorems 3.4 and 3.5.

3.10. An \( r \)-tuple of partitions \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is called an \( r \)-partition of \( n \) if \( n = \sum_{i=1}^r |\lambda^{(i)}| \). We denote by \( \mathcal{P}_{n,r} \) the set of all \( r \)-partitions of \( n \). In the case where \( r = 1 \), we denote \( \mathcal{P}_{n,1} \) simply by \( \mathcal{P}_n \). For each \( \mathbf{m} \in \mathcal{D}_{n,r} \), we denote by \( \mathcal{P}(\mathbf{m}) \) the subset of \( \mathcal{P}_{n,r} \) consisting of \( \lambda \) such that \( |\lambda^{(i)}| = m_i \). Let \( S_{\mathbf{m}} = S_{m_1} \times \cdots \times S_{m_r} \) be the Young subgroup of \( S_n \) corresponding to \( \mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{D}_{n,r} \). Recall that irreducible characters of \( S_n \) are parametrized by partitions of \( n \). We denote by \( \chi^\lambda \) the irreducible character of \( S_n \) corresponding to \( \lambda \in \mathcal{P}_n \) (here \( \chi^{(n)} \) corresponds to the trivial character, and \( \chi^{(1^n)} \) corresponds to the sign character). Thus the irreducible characters of \( S_{\mathbf{m}} \) are parametrized by \( \mathcal{P}(\mathbf{m}) \). We denote by \( \chi^\lambda \) the irreducible character of \( S_{\mathbf{m}} \) corresponding to \( \lambda \in \mathcal{P}(\mathbf{m}) \).

We identify \( (W_0)_{\mathbf{m}} \) with \( S_{\mathbf{m}} \). For each \( \mathbf{w} \in S_{\mathbf{m}} \), let \( T_\mathbf{w} \) be an \( F \)-stable maximal torus in \( G \) as given in 2.1. For each \( \lambda \in \mathcal{P}(\mathbf{m}) \), we define functions \( Q^\pm_\lambda \) on \( \mathcal{Y}_\text{uni}^F \) by

\[
Q^\pm_\lambda = |S_{\mathbf{m}}|^{-1} \sum_{w \in S_{\mathbf{m}}} \chi^\lambda(w) Q^\pm_{\mathbf{m},T_\mathbf{w}}.
\]

Let \( \mathcal{C}_q = \mathcal{C}_q(\mathcal{Y}_\text{uni}) \) be the \( \mathbb{Q}_l \)-space of all \( G^F \)-invariant \( \mathbb{Q}_l \)-functions on \( \mathcal{Y}_\text{uni}^F \). We define a bilinear form \( \langle f, h \rangle \) on \( \mathcal{C}_q \) by

\[
\langle f, h \rangle = \sum_{z \in \mathcal{Y}_\text{uni}^F} f(z) h(z).
\]

Concerning the functions \( Q^\pm_\lambda \), the following formula holds.
Proposition 3.11. For any $\lambda \in \mathcal{P}(m)$, $\mu \in \mathcal{P}(m')$ and $\varepsilon, \varepsilon' \in \{+,-\}$, we have

$$
\langle Q_{\lambda}^{\varepsilon}, Q_{\mu}^{\varepsilon'} \rangle = (-1)^{p_{\varepsilon}(m)+p_{\varepsilon'}(m')}|S_m|^{-1}|S_{m'}|^{-1}|G^F| \times \sum_{\theta \in S_m/S_n/S_{m'}} q^{a_{\varepsilon,\varepsilon'}(m,m';n_\theta)} \sum_{w \in S_m \cap \omega S_{m'} x^{-1}} |T_w^{-1}|^{-1} \chi^{\lambda}(w) \chi^{\mu}(x^{-1}w),
$$

where $n_\theta$ is an element in $N(T_w,T_{x^{-1}w})^F$ attached to $\theta$.

Proof. By making use of the orthogonality relations for Green functions (Theorem 3.5), we have

$$
(-1)^{p_{\varepsilon}(m)+p_{\varepsilon'}(m')}|G^F|^{-1}\langle Q_{\lambda}^{\varepsilon}, Q_{\mu}^{\varepsilon'} \rangle = |S_m|^{-1}|S_{m'}|^{-1}(-1)^{p_{\varepsilon}(m)+p_{\varepsilon'}(m')}|G^F|^{-1} \sum_{w \in S_m} \chi^{\lambda}(w) \chi^{\mu}(w') |Q_{\lambda, T_w}^{\varepsilon}, Q_{\mu, T_{w'}}^{\varepsilon'}\rangle
$$

$$
= |S_m|^{-1}|S_{m'}|^{-1} \sum_{w \in S_m} \sum_{w' \in S_{m'}} \chi^{\lambda}(w) \chi^{\mu}(w') |T_w^{-1}T_{w'}^{-1}|^{-1} \sum_{n \in N(T_w,T_{w'})^F} q^{a_{\varepsilon,\varepsilon'}(m,m';n)}.\n$$

Let $P_m$ be the parabolic subgroup of $G$ containing $B_0$ which is the stabilizer of the partial flag $(M_{0,p_i})$ with respect to $m$, and define $P_{m'}$ similarly. We put $P_m = L_m U_m$, where $L_m$ is the Levi subgroup of $P_m$ containing $T_0$ and $U_m$ is the unipotent radical of $P_m$. Define $P_{m'} = L_m U_{m'}$ similarly. We may assume that $T_w \subset P_m$ and $T_{w'} \subset P_{m'}$. Thus $a_{\varepsilon,\varepsilon'}(m,m';n)$ is defined with respect to $(M_{0,p_i}^0),(M_{0,p_i}^{0'})$. It follows that $a_{\varepsilon,\varepsilon'}(m,m';n)$ is independent of the choice of $n \in \tilde{G}^F$ for each orbit $\tilde{G} \in L_m \setminus G/L_{m'}$. Let $\tilde{G}$ be an element in $P_m \setminus G/P_{m'}$ containing $\tilde{G}$. Then

$$
\tilde{G} \cap N(T_w,T_{w'})^F = \tilde{G} \cap N(T_w,T_{w'})^F
$$

since $\tilde{G} = \bigcup_{u \in U_m, w' \in U_{m'}} u\tilde{G}u'$. It follows that $a_{\varepsilon,\varepsilon'}(m,m';n)$ is independent of the choice for any $n \in \tilde{G} \cap N(T_w,T_{w'})^F$. Note that $P_m \setminus G/P_{m'} \simeq S_m \setminus S_n/S_{m'}$. We denote by $\tilde{G} \in S_m \setminus S_n/S_{m'}$ the orbit in $S_n$ corresponding to $\tilde{G}$, and choose $n_\theta \in \tilde{G} \cap N(T_w,T_{w'})^F$ for each $\tilde{G}$. Then the last sum is equal to

$$
|S_m|^{-1}|S_{m'}|^{-1} \sum_{w \in S_m} \sum_{w' \in S_{m'}} \chi^{\lambda}(w) \chi^{\mu}(w') |T_w^{-1}T_{w'}^{-1}|^{-1} |N_{L_m}(T_w)^F| |N_{L_{m'}}(T_{w'})^F|
$$

$$
\times \sum_{\theta \in S_m \setminus S_n/S_{m'} \cap N(T_w, T_{w'})^F \cap n_\theta} q^{a_{\varepsilon,\varepsilon'}(m,m';n_\theta)} |T_w n_\theta = T_{w'}|^{-1} \sum_{n_\theta^{-1} T_w n_\theta = T_{w'}} |N_{L_m}(T_w)^F| |T_w^{-1}|^{-1} N_{L_{m'}}(T_{w'})^F |n_\theta|^{-1} |N_{L_{m'}}(T_{w'})^F|^{-1} q^{a_{\varepsilon,\varepsilon'}(m,m';n_\theta)}.\n$$

Here we have

$$
|N_{L_m}(T_w)^F| |T_w^{-1}|^{-1} N_{L_{m'}}(T_{w'})^F |T_{w'}^{-1}|^{-1} = |Z_{S_m}(w)|, \quad |N_{L_{m'}}(T_{w'})^F| |T_{w'}^{-1}|^{-1} = |Z_{S_{m'}}(w')|.\n$$
Moreover since \( n_\theta^{-1}T_w n_\theta = T_w' \) for \( n_\theta \in G^F \), there exists \( x_\theta \in S_n \) such that \( x_\theta^{-1}w x_\theta = w' \). Now \( \text{ad } n_\theta^{-1} \) gives an isomorphism \( N_{G}(T_w) \rightarrow N_{G}(T_w) \), which induces an isomorphism \( N_{G}(T_w)/T_w \rightarrow G(T_w')/T_w \), hence by taking the \( F \)-fixed point subgroups on both side, an isomorphism \( Z_{S_n}(w) \rightarrow Z_{S_n}(w') \). We may assume that this isomorphism is induced by \( \text{ad } x_\theta^{-1} \). It follows that

\[
(N_{L_m}(T_w)^F \cap n_\theta N_{L_m}(T_w')^F n_\theta^{-1})/T_w^F \simeq Z_{S_m}(w) \cap x_\theta Z_{S_m'}(w') x_\theta^{-1},
\]

and so

\[
(N_{L_m}(T_w)^F \cap n_\theta N_{L_m}(T_w')^F n_\theta^{-1})/T_w^F \simeq |Z_{S_m}(w) \cap x_\theta Z_{S_m'}(w') x_\theta^{-1}|.
\]

Substituting (3.11.2) and (3.11.3) into (3.11.1), the formula (3.11.1) turns out to be

\[
|S_m|^{-1}|S_m'|^{-1} \sum_{w \in S_m} \chi^\lambda(w) \chi^\mu(w') |T_w^F|^{-1} |Z_{S_m}(w)| |Z_{S_m'}(w')| \sum_{\theta \in S_n \setminus S_n/S_m'} |Z_{S_m}(w) \cap x_\theta Z_{S_m'}(w') x_\theta^{-1}|^{-1} q_{a',e'}(m, m'; n_\theta) \times |Z_{S_m}(w)| |Z_{S_m'}(w')| |Z_{S_m}(w) \cap x_\theta Z_{S_m'}(w') x_\theta^{-1}|^{-1} = |Z_{S_m}(w) x_\theta Z_{S_m'}(w')|,
\]

but we have

\[
|Z_{S_m}(w)| |Z_{S_m'}(w')| |Z_{S_m}(w) \cap x_\theta Z_{S_m'}(w') x_\theta^{-1}|^{-1} = |Z_{S_m}(w) x_\theta Z_{S_m'}(w')|,
\]

and for given \( w \in S_m, w' \in S_m' \), the choice of \( x \in S_n \) such that \( w' = x^{-1}wx \) is given by \( x \in Z_{S_m}(w) x_\theta Z_{S_m'}(w') \). Hence the last formula is equal to

\[
|S_m|^{-1}|S_m'|^{-1} \sum_{w \in S_m} \sum_{\theta \in S_n \setminus S_n/S_m'} \chi^\lambda(w) \chi^\mu(w') |T_w^F|^{-1} q_{a',e'}(m, m'; n_\theta).
\]

This proves the proposition. \( \square \)

4. Unipotent Variety

4.1. We express an \( r \)-partition \( \lambda \) by \( \lambda = (\lambda^{(i)}_j) \), where \( \lambda^{(i)} = (\lambda^{(i)}_1, \lambda^{(i)}_2, \ldots, \lambda^{(i)}_m) \) is a partition with \( \lambda^{(i)}_m \geq 0 \) for some fixed number \( m \). For an \( r \)-partition \( \lambda \), we define a sequence of non-negative integers \( c(\lambda) \) associated to \( \lambda \) by

\[
c(\lambda) = (\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(r)}_1, \lambda^{(2)}_2, \lambda^{(r)}_2, \ldots, \lambda^{(1)}_m, \lambda^{(2)}_m, \ldots, \lambda^{(r)}_m).
\]

For a sequence of non-negative integers, \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m), \mu = (\mu_1, \mu_2, \ldots, \mu_m) \), we denote by \( \lambda \leq \mu \) if

\[
\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k
\]

for \( k = 1, 2, \cdots, m \). We define a dominance order \( \leq \) on \( \mathcal{P}_{n,r} \) by the condition that \( \lambda \leq \mu \) if \( c(\lambda) \leq c(\mu) \). In the case where \( r = 1 \), this is the standard dominance order on the set
of partitions. In the case of \( r = 2 \), this coincides with the partial order used in [AH] and [SS2].

For a partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_n \), we define \( n(\lambda) \in \mathbb{Z} \) by \( n(\lambda) = \sum_i (i - 1) \lambda_i \). For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \), we define \( n(\lambda) \in \mathbb{Z} \) by \( n(\lambda) = \sum_{i=1}^{r} n(\lambda^{(i)}) \). Hence if we put \( \nu = \lambda^{(1)} + \cdots + \lambda^{(r)} \), we have \( n(\lambda) = n(\nu) \).

4.2. Let \( \mathcal{X}_\text{uni} = G_\text{uni} \times V^{r-1} \). In the case where \( r = 2 \), it is known by [AH], [T] that \( G \)-orbits of \( \mathcal{X}_\text{uni} \) are parametrized by \( \mathcal{P}_{n,2} \). The parametrization is given as follows; take \((x, v) \in G_\text{uni} \times V \). Put \( E_x = \{ y \in \text{End}(V) \mid xy = yx \} \). \( E_x \) is a subalgebra of \( \text{End}(V) \) containing \( x \). Put \( W = E_x v \). Then \( W \) is an \( x \)-stable subspace of \( V \). We denote by \( \lambda^{(1)} \) the Jordan type of \( x|_W \), and by \( \lambda^{(2)} \) the Jordan type of \( x|_{V/W} \). Thus one can define \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_{n,2} \). We denote by \( \mathcal{O}_\lambda \) the \( G \)-orbit containing \((x, v) \). This gives the required parametrization. Note that if \((x, v) \in \mathcal{O}_\lambda \), the Jordan type of \( x \) is given by \( \lambda^{(1)} + \lambda^{(2)} \). For \((x, v) \in G_\text{uni} \times V \), we say that \((x, v) \) has type \( \lambda \) if \((x, v) \in \mathcal{O}_\lambda \).

If \( r \geq 3 \), the number of \( G \)-orbits in \( \mathcal{X}_\text{uni} \) is infinite. In [S3, 5.3], the partition of \( X \) into \( G \)-stable pieces \( X_\lambda \) (possibly a union of infinitely many \( G \)-orbits) labelled by \( \lambda \in \mathcal{P}_{n,r} \) is given;

\[
\mathcal{X}_\text{uni} = \prod_{\lambda \in \mathcal{P}_{n,r}} X_\lambda.
\]

Following [S3], we define \( X_\lambda \) by induction on \( r \) as follows. Take \((x, v) \in \mathcal{X}_\text{uni} \) with \( v = (v_1, \ldots, v_{r-1}) \). Put \( W = E_x v_1 \), \( W' = V/W \) and \( G = GL(W') \). We consider the variety \( \mathcal{X}'_{\text{uni}, r} = \mathcal{X}'_{\text{uni}} \times V^{r-2} \). Assume that \((x, v_1) \in G_\text{uni} \times V \) is of type \((\lambda^{(1)}, \nu') \), where \( \nu = \lambda^{(1)} + \nu' \) is the Jordan type of \( x \). Let \( \nu \) be the restriction of \( x \) on \( V' \). Then the Jordan type of \( \nu \in GL(V') \) is \( \nu' \). Put \( \nu = (\nu_2, \ldots, \nu_{r-1}) \), where \( \nu_i \) is the image of \( v_i \) on \( V \). Thus \((x, \nu) \in \mathcal{X}'_{\text{uni}, r-1} \). By induction, we have a partition \( \mathcal{X}'_{\text{uni}} = \prod_{\nu \in \mathcal{P}_{n', r-1}} X'_\nu \), where \( \dim W' = n' \). Thus there exists a unique \( X'_\nu \) containing \((\nu, \nu)\). If we write \( \nu' = (\lambda^{(2)}, \ldots, \lambda^{(r)}) \), we have \( \lambda^{(2)} + \cdots + \lambda^{(r)} = \nu' \). It follows that \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \). We define the type of \((x, v) \) by \( \lambda \), and define a subset \( X_\lambda \) of \( \mathcal{X}_\text{uni} \) as the set of all \((x, v)\) with type \( \lambda \). Then \( X_\lambda \) is a \( G \)-stable subset of \( \mathcal{X}_\text{uni} \), and we obtain the required partition (4.2.1).

\( X_\lambda \) has an alternate description. Assume that \( \lambda \in \mathcal{P}(m) \) for \( m \in \mathcal{D}_{n,r} \), and let \( P = P_m = LU_P \) be the parabolic subgroup of \( G \) attached to \( m \) as in 1.5. Put \( M_{p_i} = M_{p_i} / M_{p_{i-1}} \). Then \( L \simeq G_1 \times \cdots \times G_r \) with \( G_i = GL(M_{p_i}) \). Let \( \mathcal{M}_\lambda \) be the subset of \( U \times \prod_{i=1}^{r-1} M_{p_i} \) defined by the following properties; take \((x, v) \) with \( x \in U \) and \( v_i \in M_{p_i} \). Put \( x_i = x|M_{p_i} \), and let \( \nu_i \in M_{p_i} \) be the image of \( v_i \). Then \((x, v) \in \mathcal{M}_\lambda \) if the Jordan type of \( x \) is \( \lambda^{(1)} + \cdots + \lambda^{(r)} \), the \( G_i \)-orbit of \((x_i, \nu_i) \in (G_i)_{\text{uni}} \times M_{p_i} \) has type \( (\lambda^{(i)}, \emptyset) \) for \( i = 1, \ldots, r-1 \), and the Jordan type of \( x_r \) is \( \lambda^{(r)} \). Let \( \overline{X}_\lambda \) be the closure of \( X_\lambda \) in \( \mathcal{X}_\text{uni} \). Then by a similar argument as in the proof of [S3, Lemma 6.17], we have
Proposition 4.4 ([S3, Prop. 5.11]). Let \( \lambda \) be the closure of \( X_\lambda \) in \( \mathcal{X}_{\text{uni}} \). Then

\[
\overline{X}_\lambda = \bigcup_{g \in G} g \cdot \mathcal{M}_\lambda.
\]

By Propositions 5.4 and 5.14 in [S3], we have

**Proposition 4.3.** \( X_\lambda \) is a \( G \)-stable, smooth, irreducible, locally closed subvariety of \( \mathcal{X}_{\text{uni}} \) with

\[
\dim X_\lambda = (n^2 - n - 2n(\lambda)) + \sum_{i=1}^{r-1} (r - i)|\lambda^{(i)}|.
\]

Let \( \leq \) be the dominance order on \( \mathcal{P}_{n, r} \) defined in 4.1. Concerning the closure relations of \( X_\lambda \), we have

**Proposition 4.4 ([S3, Prop. 5.11]).** Let \( \overline{X}_\lambda \) be the closure of \( X_\lambda \) in \( \mathcal{X}_{\text{uni}} \). Then

\[\overline{X}_\lambda \subset \bigcup_{\mu \leq \lambda} X_\mu.\]

For each \( \mathbf{m} \in \mathcal{D}_{n, r} \), recall the map \( \pi_{\mathbf{m}, 1} : \mathcal{X}_{\text{uni}} \to \mathcal{X}_{\text{uni}} \) in 1.1. Define \( \lambda(\mathbf{m}) = ((m_1), (m_2), \ldots, (m_r)) \). Then we have

**Proposition 4.5 ([S3, Prop. 5.9]).** For \( \mathbf{m} \in \mathcal{D}_{n, r} \), we have

(i) \( \mathcal{X}_{\text{uni}} = \overline{X}_{\lambda(\mathbf{m})} \).

(ii) \( \dim \mathcal{X}_{\text{uni}} = n^2 - n + \sum_{i=1}^{r-1} (r - i) m_i. \)

(iii) For \( \mu \in \mathcal{P}(\mathbf{m}) \), \( X_\mu \subset \mathcal{X}_{\text{uni}} \).

### 4.6.

As in [S3, 5.10], we define a distinguished element \( z_\lambda \in X_\lambda \) as follows. Put \( \nu = (\nu_1, \ldots, \nu_\ell) \in \mathcal{P}_n \) for \( \nu = \lambda^{(1)} + \cdots + \lambda^{(\ell)} \). Take \( x \in G_{\text{uni}} \) of Jordan type \( \nu \), and let \( \{u_{j,k} \mid 1 \leq j \leq \ell, 1 \leq k \leq \nu_j\} \) be a Jordan basis of \( x \) in \( V \) having the property \((x - 1)u_{j,k} = y_{j,k-1} \) with the convention \( u_{j,0} = 0 \). We define \( v_i \in V \) for \( i = 1, \ldots, r - 1 \) by the condition that

\[
(4.6.1) \quad v_i = \sum_{1 \leq j \leq \ell} u_{j, \lambda_j^{(1)} + \cdots + \lambda_j^{(i)}}.
\]

Put \( \mathbf{v} = (v_1, \ldots, v_{r-1}) \) and \( z_\lambda = (x, \mathbf{v}) \). Then \( z_\lambda \in X_\lambda \). We denote by \( \mathcal{O}_\lambda^- \subset X_\lambda \) the \( G \)-orbit containing \( z_\lambda \). We also define \( z_\lambda^+ \in X_\lambda \) as follows; put

\[
(4.6.2) \quad v'_i = \sum_{1 \leq j \leq \ell} \overline{\nu}_{j, \lambda_j^{(1)} + \cdots + \lambda_j^{(i)}},
\]

where \( \overline{\nu}_{j, \lambda_j^{(1)} + \cdots + \lambda_j^{(i)}} = u_{j, \lambda_j^{(1)} + \cdots + \lambda_j^{(i)}} \) if \( \lambda_j^{(i)} \neq 0 \) and is equal to zero if \( \lambda_j^{(i)} = 0 \). Put \( \mathbf{v}' = (v'_1, \ldots, v'_{r-1}) \), and \( \mathbf{z}_\lambda^+ = (x, \mathbf{v}') \). Then \( \mathbf{z}_\lambda^+ \in X_\lambda \), and we denote by \( \mathcal{O}_\lambda^+ \) the \( G \)-orbit in \( X_\lambda \) containing \( \mathbf{z}_\lambda^+ \). The \( G \)-orbits \( \mathcal{O}_\lambda^-, \mathcal{O}_\lambda^+ \subset X_\lambda \) are determined, independently of the choice of the Jordan basis \( \{u_{j,k}\} \).
Now take \( m \in \mathcal{D}_{n,r} \), and let \( M^+_p \) be the subspace of \( M_p \) isomorphic to \( \overline{M}_p \) defined in 2.7. Recall the set \( \mathcal{X}^+_m \) in (2.7.1). We define a set \( \mathcal{X}^+_{m,\text{uni}} \) by

\[
\mathcal{X}^+_{m,\text{uni}} = \bigcup_{g \in G} g(U \times \prod_i M^+_p),
\]

which coincides with \( \mathcal{X}_{\text{uni}} \cap \mathcal{X}^+_m \). Note that for each \( \lambda \in \mathcal{P}(m) \), \( \mathcal{O}^+_{\lambda} \subset X_\lambda \cap \mathcal{X}^+_{m,\text{uni}} \).

In general, \( X_\lambda \cap \mathcal{X}^+_m \) consists of infinitely many \( G \)-orbits even if \( \lambda \in \mathcal{P}(m) \). The following special case would be worth mentioning.

**Lemma 4.7.** Assume that \( m \in \mathcal{D}_{n,r} \) is of the form \( m_j = 0 \) for \( j \neq i_0, r \) for some \( i_0 \) (possibly \( i_0 = r \)). Then \( X_\lambda \cap \mathcal{X}^+_{m,\text{uni}} = \mathcal{O}^+_{\lambda} \) for any \( \lambda \in \mathcal{P}_{n,r} \) if it is non-empty.

**Proof.** Assume that \( 1 \leq i_0 \leq r - 1 \) and that \( X_\lambda \cap \mathcal{X}^+_m \neq \emptyset \). Take \( (x, v) \in X_\lambda \cap \mathcal{X}^+_m \). Since \( (x, v) \in \mathcal{X}^+_m \), we must have \( v_j = 0 \) for \( j \neq i_0 \). Then the description of \( X_\lambda \) in 4.1 implies that \( \lambda^{(i)} = \emptyset \) for \( i \neq i_0 \). It follows that \( \lambda \in \mathcal{P}(m') \) with \( m' = (m'_1, \ldots, m'_{i_0}) \) such that \( m'_{i_0} \leq m_{i_0}, m'_{j} = 0 \) for \( j \neq i_0, r \). But this is essentially the same as the case of \( r = 2 \), hence \( X_\lambda \cap \mathcal{X}^+_m \) coincides with \( \mathcal{O}^+_{\lambda} \). The case where \( i_0 = r \) is dealt with similarly. \( \square \)

Note that irreducible representations of \( S_m \) are, up to isomorphism, parametrized by \( \mathcal{P}(m) \) (see 3.10). We denote by \( V_{\lambda} \) the irreducible representation of \( S_m \) corresponding to \( \lambda \in \mathcal{P}(m) \). The following result gives the Springer correspondence between \( \mathcal{X}_{m,\text{uni}} \) and \( S_m \).

**Theorem 4.8** ([S3, Theorem 8.13]). For any \( m \in \mathcal{D}_{n,r} \), put \( d'_m = \dim \mathcal{X}_{m,\text{uni}} \). Then \( (\pi_{m,1})_! \mathbb{Q}_t[d'_m] \) is a semisimple perverse sheaf on \( \mathcal{X}_{m,\text{uni}} \), equipped with the action of \( S_m \), and is decomposed as

\[
(\pi_{m,1})_! \mathbb{Q}_t[d'_m] \cong \bigoplus_{\lambda \in \mathcal{P}(m)} V_{\lambda} \otimes \text{IC}(X_\lambda, \mathbb{Q}_t)[\dim X_\lambda].
\]

**4.9.** For each \( z = (x, v) \in \mathcal{X}_m \), we consider the Springer fibre \( (\pi_{m})^{-1}(z) \cong \mathcal{B}^{(m)}_z \), where \( \mathcal{B}^{(m)}_z \) is a closed subvariety of \( \mathcal{B} = G/B \) defined as

\[
\mathcal{B}^{(m)}_z = \{ gB \in \mathcal{B} \mid g^{-1}xg \in B, g^{-1}v \in \prod_i M^+_p \}. \]

For each \( \lambda \in \mathcal{P}(m) \), put \( d_{\lambda} = (\dim \mathcal{X}_{m,\text{uni}} - \dim X_\lambda)/2 \). One can check that \( d_{\lambda} = n(\lambda) \) (see 4.1 for the definition of \( n(\lambda) \)). We have

**Lemma 4.10.** Assume that \( \lambda \in \mathcal{P}(m) \). Then for any \( z \in X_\lambda \), we have \( \dim \mathcal{B}^{(m)}_z = d_{\lambda} \).

**Proof.** By [S3, Lemma 8.5], we have \( \dim \mathcal{B}^{(m)}_z \geq d_{\lambda} \). On the other hand, clearly we have \( \mathcal{B}^{(m)}_z \subset \mathcal{B}_x \), where \( \mathcal{B}_x = \{ gB \in \mathcal{B} \mid g^{-1}xg \in B \} \). Here the Jordan type of \( x \) is given by \( \nu = \lambda^{(1)} + \cdots + \lambda^{(r)} \). By the classical result for the case of \( GL_n \), we know that \( \dim \mathcal{B}_x = n(\nu) \). Since \( n(\nu) = n(\lambda) \), we have \( \dim \mathcal{B}^{(m)}_z \leq d_{\lambda} \). The lemma is proved. \( \square \)
5. Kostka functions

5.1. Kostka functions associated to complex reflection groups were introduced in [S1], [S2] as a generalization of Kostka polynomials. In this section, we discuss the relationship between our Green functions and those Kostka functions. In [S1], [S2], Kostka functions $\tilde{K}_{\lambda,\mu}^{\pm}(t)$ indexed by $\lambda, \mu \in \mathcal{P}_{n,r}$ (and depending on the sign $+$, $-$) were introduced as coefficients of the transition matrix between the basis of Schur functions and those of Hall-Littlewood functions, as in the case of original Kostka polynomials. A-priori they are rational functions in $Q(t)$. We define a modified Kostka function $\tilde{K}_{\lambda,\mu}^{\pm}(t)$ by

\[ \tilde{K}_{\lambda,\mu}^{\pm}(t) = t^{a(\mu)}K_{\lambda,\mu}^{\pm}(t^{-1}), \]

where the $a$-function $a(\lambda)$ is defined by

\[ a(\lambda) = r \cdot n(\lambda) + |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r-1)|\lambda^{(r)}| \]

for $\lambda \in \mathcal{P}_{n,r}$. Note that in the case where $r = 1$, $a(\lambda)$ coincides with $n(\lambda^{(1)})$, and in the case where $r = 2$, $a(\lambda)$ coincides with the $a$-function on $\mathcal{P}_{n,2}$ used in [AH] and [SS2] (in [AH], the notation $b(\mu; \nu)$ is used instead of $a(\lambda)$ for $\lambda = (\mu; \nu)$).

Following [S1], we give a combinatorial characterization of modified Kostka functions $\tilde{K}_{\lambda,\mu}^{\pm}(t)$. Let $W_{n,r}$ be the complex reflection group $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$. For a (not necessarily irreducible) character $\chi$ of $W_{n,r}$, we define the fake degree $R(\chi)$ by

\[ R(\chi) = \prod_{i=1}^{n}(t^{ir} - 1) - \sum_{w \in W_{n,r}} \frac{\det_{V}(w)\chi(w)}{\det_{V}(t - w)}, \]

where $V$ is a representation space (over $Q(t)$) of the reflection representation of $W_{n,r}$, and $\det_{V}$ means the determinant on $V$. Here we have $|W_{n,r}| = n!r^n$. Note that $R(\chi) \in \mathbb{Z}_{\geq 0}[t]$; if $\chi$ is irreducible, $R(\chi)$ is given as the graded multiplicity of $\chi$ in the coinvariant algebra $R(W_{n,r})$ of $W_{n,r}$. Let $N^*$ be the number of reflections of $W_{n,r}$, which is given as the maximum degree of $R(W_{n,r})$, and is explicitly given as

\[ N^* = \frac{rn(n+1)}{2} - n = \binom{n}{2}r + (r-1)n. \]

It is known that irreducible characters of $W_{n,r}$ are parametrized by $\mathcal{P}_{n,r}$. We denote by $\rho^\lambda$ the irreducible character of $W_{n,r}$ corresponding to $\lambda \in \mathcal{P}_{n,r}$. (For example, $\rho^\lambda$ is the trivial character for $\lambda = (n; -; \cdots ; -)$. See 5.6 for details). For $\lambda, \mu \in \mathcal{P}_{n,r}$, we define a square matrix $\Omega = (\omega_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_{n,r}}$ by

\[ \omega_{\lambda,\mu} = t^{N^*}R(\rho^\lambda \otimes \rho^\mu \otimes \overline{\det}_V), \]

where $\overline{\chi}$ denotes the complex conjugate of the character $\chi$ (in fact, the function $\overline{\chi}$ is defined by $\overline{\chi}(w) = \chi(w^{-1})$ for $w \in W_{n,r}$). Here we fix a total order $\lambda \preceq \mu$ on $\mathcal{P}_{n,r}$ compatible with the partial order $\lambda \preceq \mu$ defined in 4.1, and consider the square matrix with respect to this total order. Note that $\omega_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}[t]$. We have the following result.
Theorem 5.2 ([S1, Theorem 5.4]). There exist unique matrices \( P^\pm = (p_{\lambda,\mu}^\pm) \), \( \Lambda = (\xi_{\lambda,\mu}) \) over \( \mathbb{Q}(t) \) satisfying the equation

\[
(5.2.1) \quad P^{-1} \Lambda^{t} P^{+} = \Omega
\]

subject to the conditions that \( \Lambda \) is a diagonal matrix and that

\[
P_{\lambda,\mu}^\pm = \begin{cases} 
0 & \text{unless } \mu \leq \lambda, \\
\rho(\lambda) & \text{if } \lambda = \mu.
\end{cases}
\]

Then the entry \( p_{\lambda,\mu}^\pm \) of the matrix \( P^\pm \) coincides with \( K_{\lambda,\mu}(t) \).

Remarks 5.3. (i) Since \( \Omega \) is non-symmetric unless \( r = 1 \) or \( 2 \), \( P^+ \) does not coincide with \( P^- \) if \( r \geq 3 \).

(ii) Our construction of Kostka functions \( K_{\lambda,\mu}(t) \) depends on the choice of the total order \( \preceq \) on \( \mathcal{P}_{n,r} \). In the case where \( r = 1 \) or \( 2 \), it is known that Kostka functions are independent of the choice of the total order whenever it is compatible with the partial order \( \preceq \) (see [M] for \( r = 1 \), and [S2], [AH], [SS2] for \( r = 2 \)).

5.4. We define an involution \( \tau \) on \( \mathcal{P}_{n,r} \) by

\[
(5.4.1) \quad \tau : (\lambda^{(1)}, \ldots, \lambda^{(r)}) \mapsto (\lambda^{(r-1)}, \lambda^{(r-2)}, \ldots, \lambda^{(1)}, \lambda^{(r)})
\]

for \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \). We shall prove the following result.

Theorem 5.5. For any \( \lambda \in \mathcal{P}(m), \mu \in \mathcal{P}(m') \), we have

\[
q^{a(\lambda) - a(\tau(\mu))} \omega_{\lambda,\mu}(q) = (-1)^{p_{-}(m) + p_{+}(m')} q^{-r(\mu) + n(\mu)} \sum_{z \in \mathbb{F}_{uni}^{r}} Q_{\mu}(z) Q_{\mu}(z).
\]

5.6. The right hand side of (5.5.1) can be computed by applying Proposition 3.11 to the case where \( \varepsilon = -, \varepsilon' = + \). Hence we will compute the left hand side of (5.5.1), i.e., \( \omega_{\lambda,\mu}(t) \) for the indeterminate \( t \). In order to compute them, we recall some known facts on the characters of \( W_{n,r} \). Let \( \delta : W_{n,r} \to \mathbb{Q}^*_{l} \) be the linear character of \( W_{n,r} \) defined by \( \delta|_{S_n} = 1_{S_n} \) and \( \delta(s_0) = \zeta \), where \( s_0 \) is a (simple) reflection of order \( r \), and \( \zeta \) is a primitive \( r \)-th root of unity in \( \mathbb{Q}_l \). Assume given \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}(m) \). Then the irreducible character \( \rho^\lambda \) is constructed by

\[
(5.6.1) \quad \rho^\lambda = \text{Ind}_{W_{m,r}}^{W_{n,r}}(\chi^{(1)} \boxtimes \delta \chi^{(2)} \boxtimes \cdots \boxtimes \delta^{r-1} \chi^{(r)}),
\]

where \( W_{m,r} = W_{m_1,r} \times \cdots \times W_{m_r,r} \), and \( \chi^{(i)} \) is the irreducible character of \( S_{m_i} \) corresponding to the partition \( \lambda^{(i)} \), which is regarded as a character of \( W_{m_i,r} \) by the natural surjection \( W_{m_i,r} \to S_{m_i} \), and \( \delta^{i-1} \chi^{(i)} \) is the irreducible character of \( W_{m_i,r} \) obtained by multiplying \( \delta^{i-1} \) on it. (Here we use the same symbol \( \delta \) to denote the corresponding linear character of \( W_{m_i,r} \) for each \( i \).) Let \( \varepsilon \) be the irreducible character of \( W_{n,r} \) obtained by the
pull-back of the sign character of $S_n$ under the map $W_{n,r} \to S_n$. (Do not confuse the character $\varepsilon$ of $W_{n,r}$ with the signatures $\{\varepsilon, \varepsilon'\} = \{-, +\}$. In the present discussion, the signatures are fixed.) Then we have $\det V = \varepsilon \delta$. For $i = 1, \ldots, r$, put $\lambda_i = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ with $\lambda^{(j)} = 0$ for $j \neq i$ and $\lambda^{(i)} = (n)$, and put $\mu_i = (\mu^{(1)}, \ldots, \mu^{(r)})$ with $\mu^{(j)} = 0$ for $j \neq i$ and $\mu^{(i)} = (1^n)$. Then the following fact is easily verified from (5.6.1).

$$\rho^\lambda = \delta^{i-1}, \quad \rho^{\mu_i} = \varepsilon \delta^{i-1}.$$  

In particular, we have

$$\rho^\lambda = \delta^{i-1}, \quad \delta = \rho^\lambda, \quad \det V = \rho^\mu, \quad \overline{\det V} = \rho^\mu = \varepsilon \delta^{-1}.$$  

Although the following facts (5.6.4) ~ (5.6.6) are not used in the discussion below, we write them for the reference.

$$\rho^\lambda = \rho^\lambda' \quad \text{for } \lambda' = (\lambda^{(1)}, \lambda^{(r)}, \lambda^{(r-1)}, \ldots, \lambda^{(2)}),$$  

$$\rho^\lambda = \varepsilon \rho^\lambda,$$

where $^t \lambda = (^t \lambda^{(1)}, \ldots, ^t \lambda^{(r)})$. It follows from (5.6.5) that

$$\omega_{\lambda, \mu} = \omega_{^t \lambda, ^t \mu}$$

for any $\lambda, \mu \in \mathcal{P}_{n,r}$.

For a given $m \in \mathcal{Q}_r$, we consider $W_{m,r} \simeq S_m \times (\mathbb{Z}/r\mathbb{Z})^n$. As in 3.10, we denote by $\chi^\lambda$ the irreducible character of $S_m$ corresponding to $\lambda \in \mathcal{P}(m)$. We define a linear character $\delta_m$ of $W_{m,r} = W_{m_1,r} \times \cdots \times W_{m_r,r}$ by

$$\delta_m = \delta^0 \boxtimes \delta^1 \boxtimes \cdots \boxtimes \delta^{r-1}.$$  

Let $\tilde{\chi}^\lambda$ be the irreducible character $\delta_m \chi^\lambda$ on $W_{m,r}$. Then (5.6.1) can be rewritten as

$$\rho^\lambda = \text{Ind}_{W_{n,r}}^{W_{m,r}} \tilde{\chi}^\lambda.$$  

5.7. We now compute $R(\rho^\lambda \otimes \rho^\mu \otimes \overline{\det V})$ for $\lambda \in \mathcal{P}(m), \mu \in \mathcal{P}(m')$. In the computation below, we write $W_{n,r}, W_{m,r}$, etc. simply as $W_n, W_m$, etc. by omitting $r$.

$$R(\rho^\lambda \otimes \rho^\mu \otimes \overline{\det V}) = \frac{\prod_{k=1}^{n} (tk^r - 1)}{|W_n|} \sum_{w \in W_n} \frac{(\text{Ind}_{W_m}^{W_n} \tilde{\chi}^\lambda)(w) (\text{Ind}_{W_m}^{W_n} \tilde{\chi}^\mu)(w)}{\det V(t - w)}$$

$$= \frac{\prod_{k=1}^{n} (tk^r - 1)}{|W_n||W_m||W_m'|} \sum_{w_1, w_2 \in W_n, \ w_1 w_2 \in W_m} \frac{\tilde{\chi}^\lambda(w_1^{-1}w_2) \tilde{\chi}^\mu(w_2^{-1}w_1)}{\det V(t - w)}$$
where the last formula follows from the change of variables $x = w_1^{-1}w_2, y = w_1^{-1}ww_1$.

We consider the set of double cosets $W_m \backslash W_n / W_{m'}$. This set is described by a certain set of matrices as given in Section 5 in [AH]. We define $\mathcal{M}_{m,m'}$ as the set of degree $r$ matrices $(h_{ij})$ with entries in $\mathbb{Z}_{\geq 0}$ satisfying the following properties:

\begin{equation}
(5.7.2) \sum_{i=1}^r h_{ij} = m_j \text{ for all } j, \quad \sum_{j=1}^r h_{ij} = m'_i \text{ for all } i.
\end{equation}

Then there exists a bijective correspondence

$$
\mathcal{M}_{m,m'} \simeq S_m \backslash S_n / S_{m'} \simeq W_m \backslash W_n / W_{m'}
$$

satisfying the properties

$$
S_m \cap xS_{m'}x^{-1} \simeq \prod_{i,j} S_{h_{ij}} \quad \text{and} \quad W_m \cap xW_{m'}x^{-1} \simeq \prod_{i,j} W_{h_{ij}}
$$

if $x$ is contained in the orbit $\mathcal{O} \in S_m \backslash S_n / S_{m'}$ corresponding to $(h_{ij}) \in \mathcal{M}_{m,m'}$. Note that if $x \in S_n$, we have $W_m \cap xW_{m'}x^{-1} \simeq (S_m \cap xS_{m'}x^{-1}) \rtimes (\mathbb{Z}/r\mathbb{Z})^n$.

By applying the above expression, we see that

$$
\frac{1}{|W_m \cap xW_{m'}x^{-1}|} \sum_{y \in W_m \cap xW_{m'}x^{-1}} \frac{\tilde{\chi}_0^\lambda(y)\tilde{\chi}_0^\mu(x^{-1}yx)}{\det_V(t-y)} = \frac{R(\tilde{\chi}_0^\lambda \otimes \tilde{\chi}_0^\mu \otimes \det_V)}{\prod_{i,j} \prod_{k=1}^{h_{ij}} (t^{kr} - 1)}
$$

for $x \in \mathcal{O}$ corresponding to $(h_{ij}) \in \mathcal{M}_{m,m'}$, where $\tilde{\chi}_0^\lambda$ (resp. $\tilde{\chi}_0^\mu$) is the restriction of $\tilde{\chi}^\lambda$ (resp. $\tilde{\chi}^\mu$) on $W_m \cap xW_{m'}x^{-1}$. Substituting this into the last formula of (5.7.1), we have

\begin{equation}
(5.7.3) \quad R(\rho^\lambda \otimes \rho^\mu \otimes \det_V) = \sum_{\mathcal{O} \in W_m \backslash W_n / W_{m'}} \frac{\prod_{k=1}^n (t^{kr} - 1)}{\prod_{i,j} \prod_{k=1}^{h_{ij}} (t^{kr} - 1)} R(\tilde{\chi}_0^\lambda \otimes \tilde{\chi}_0^\mu \otimes \det_V)
\end{equation}

since $|W_m||W_{m'}|/|W_m \cap xW_{m'}x^{-1}| = |\mathcal{O}|$ for $\mathcal{O} = W_m xW_{m'}$.

5.8. Our next aim is to compute $R(\tilde{\chi}_0^\lambda \otimes \tilde{\chi}_0^\mu \otimes \det_V)$. For each $\mathcal{O} \subset W_n$, we choose a representative $x \in \mathcal{O}$ such that $x \in S_n$. Then under the isomorphism $W_m \cap xW_{m'}x^{-1} \simeq (S_m \cap xS_{m'}x^{-1}) \rtimes (\mathbb{Z}/r\mathbb{Z})^n$, we have

$$
\tilde{\chi}_0^\lambda = \chi^\lambda|_{W_m \cap xW_{m'}x^{-1}} = \chi^\lambda|_{S_m \cap xS_{m'}x^{-1}} \otimes \delta_m,
\tilde{\chi}_0^\mu = x(\tilde{\chi}_0^\lambda)|_{W_m \cap xW_{m'}x^{-1}} = x(\chi^\lambda)|_{S_m \cap xS_{m'}x^{-1}} \otimes x\delta_m',
$$
where $\delta_m, t_{m'}$ are the restriction of the corresponding linear characters of $W_n$ to $W_m \cap xW_{m'}x^{-1}$. It follows from this that

$$\bar{\chi}^t_0 \otimes \chi_0^{m'} \otimes \det V = (\chi^t \otimes \chi^m \otimes \varepsilon)|_{S_m \cap xS_{m'}x^{-1}} \otimes \delta_1 \varepsilon,$$

where $\delta_1 = \delta_m \otimes \bar{\chi}^{m'} \otimes \det V$, and $\varepsilon$ is as in 5.6. Note that $\det V = \varepsilon \delta_{r}^{-1}$ by (5.6.3). Hence we have $\delta_1 \varepsilon = \delta_m \otimes \bar{\chi}^{m'} \otimes \delta_{r}^{-1}$, which is a linear character of $(\mathbb{Z}/r\mathbb{Z})^n$ extended to that of $W_m \cap xW_{m'}x^{-1}$ by the trivial action of $xS_{m'}x^{-1}$. We claim that

$$(5.8.1) \quad R(\chi^t \otimes \chi^m \otimes \varepsilon)|_{t \rightarrow t'} R(\delta_1 \varepsilon),$$

where we denote by $f|_{t \rightarrow t'}$ the polynomial $f(t')$ obtained from $f(t) \in \mathbb{Z}[t]$.

We show (5.8.1). Let $R = R(W_n)$ be the coinvariant algebra of $W_n$. Then we have

$$R \simeq \mathcal{Q}_t[x_1, \ldots, x_n]/I_+(W_n),$$

where $I_+(W_n)$ is the ideal of $\mathcal{Q}_t[x_1, \ldots, x_n]$ generated by $e_1(x^r), \ldots, e_n(x^r)$. Here $e_i(x) = e_i(x_1, \ldots, x_n)$ is the $i$th elementary symmetric polynomial with variables $x_1, \ldots, x_n$, and $e_i(x^r)$ denotes such a polynomial with variables $x_1^r, \ldots, x_n^r$. Note that $(\mathbb{Z}/r\mathbb{Z})^n$ acts on $R$ via $t_i : x = \zeta x_i$ and $t_i : x = x_j$ if $j \neq i$ for a generator $t_i$ of the $i$th factor cyclic group $\mathbb{Z}/r\mathbb{Z}$. Let $\psi_a$ be a linear character of $(\mathbb{Z}/r\mathbb{Z})^n$ defined by $\psi_a(t_i) = \zeta^{a_i}$ for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then the $\psi_a$-isotypic subspace of $R$ is given by

$$x_1^{a_1} \cdots x_n^{a_n} \mathcal{Q}_t[x_1^r, \ldots, x_n^r]/(I_+(W_n) \cap \mathcal{Q}_t[x_1^r, \ldots, x_n^r])$$

with $0 \leq a_i < r$. If $\psi_a$ can be lifted to the linear character of $W_n$, then $R(\psi_a) = \prod I_{a_i}$. We have

$$\mathcal{Q}_t[x_1^r, \ldots, x_n^r]/(I_+(W_n) \cap \mathcal{Q}_t[x_1^r, \ldots, x_n^r]) \simeq \mathcal{Q}_t[x_1^r, \ldots, x_n^r]/I_+,$$

where $I_+$ is the ideal of $\mathcal{Q}_t[x_1^r, \ldots, x_n^r]$ generated by $e_1(x^r), \ldots, e_n(x^r)$, hence it is isomorphic to the coinvariant algebra of $S_n$ with respect to the variables $x_1^r, \ldots, x_n^r$. Now suppose that $S_n$ stabilizes $\psi_a$. Then for an irreducible character $\chi$ of $S_n$, $\rho = \chi \otimes \psi_a$ gives rise to an irreducible character of $W_n$. Moreover any irreducible submodule in $R$ affording $\rho$ is contained in the $\psi_a$-isotypic subspace $R^{\psi_a}$ of $R$, and the graded multiplicity of $\rho$ is determined by the graded multiplicity of $\chi$ in the graded $S_n$-module $R^{\psi_a}$. Hence in this case we have $R(\rho) = R(\chi)|_{t \rightarrow t'} R(\psi_a)$. A similar argument works also for the group $W_m \cap xW_{m'}x^{-1} \simeq \prod_{i,j} W_{h_{ij}}$. Thus (5.8.1) holds.

The definition of the fake degree in (5.1.3) makes sense for the case of symmetric groups, and we have

$$R(\chi^t \otimes \chi^m \otimes \varepsilon)|_{t \rightarrow t'} = \frac{\prod_{i,j} h_{ij}}{[S_m \cap xS_{m'}x^{-1}]} \sum_{y \in S_m \cap xS_{m'}x^{-1}} \frac{\chi^y(y)\chi^m(x^{-1}yx)}{\det V(t' - y)}.$$
formula (5.7.3) turns out to be

\[
(5.8.2) \quad R(\rho^\lambda \otimes \rho^\mu \otimes \det \chi) = \prod_{k=1}^n \frac{(tkr - 1)}{[S_m || S_{m'}]_H} \sum_{\theta \in S_m \backslash S_n / S_{m'}} t^{A_\theta} \sum_{x \in \theta} \sum_{y \in S_m \cap x S_{m'} x^{-1}} \frac{\chi^\lambda(y) \chi^\mu(x^{-1}yx)}{\det \chi(t^r - y)} ,
\]

Next we compute the value $A_\theta$ for each $\theta$. For a matrix $(h_{ij}) \in \mathcal{M}_{m,m'}$, we introduce the notation $h_{i \leq i,j} = \sum_{k \leq i} \sum_{l \leq j} h_{k,l}$, $h_{i \leq j} = \sum_{k \leq j} h_{i,k}$, etc.. Assume that $\theta$ corresponds to $(h_{ij}) \in \mathcal{M}_{m,m'}$. We define an integer $B_\theta(\lambda, \mu)$ by

\[
B_\theta(\lambda, \mu) = \binom{n}{2} - n(\lambda) - n(\mu) + \sum_{i=1}^{r-1} h_{i \leq i}.
\]

We have the following lemma.

**Lemma 5.9.** Under the above notation, we have

\[
N^* - a(\lambda) - a(\tau(\mu)) + A_\theta = r \cdot B_\theta(\lambda, \mu).
\]

**5.10.** Assuming the lemma, we continue the proof. By (5.8.2) together with Lemma 5.9, we have

\[
(5.10.1) \quad t^{-a(\lambda) - a(\tau(\mu))} \omega_{\lambda, \mu}(t) = \frac{\prod_{k=1}^n (tkr - 1)}{[S_m || S_{m'}]} \sum_{\theta \in S_m \backslash S_n / S_{m'}} t^{A_\theta} \sum_{x \in \theta} \sum_{y \in S_m \cap x S_{m'} x^{-1}} \frac{\chi^\lambda(y) \chi^\mu(x^{-1}yx)}{\det \chi(t^r - y)} .
\]

Here we note that the set of double cosets $P_m \cap G/P_{m'}$ is also in bijective correspondence with $\mathcal{M}_{m,m'}$ in such a way that if $\hat{\theta} \in P_m \cap G/P_{m'}$ corresponds to $(h_{ij}) \in \mathcal{M}_{m,m'}$, then we have

\[
\dim(M_{p_i} \cap g(M'_{p_i})) = h_{i \leq i,j} \text{ for } g \in \hat{\theta} \text{ and for any } i, j.
\]

We have a decomposition $M'_{p_i} = M'_{p_i}^+ \oplus \cdots \oplus M'_{p_i}^+$, and the maximal torus $T = T_0$ is contained in $L_m$ and $L_{m'}$. Recall that $\{e_1, \ldots, e_n\}$ gives a basis of $V$ consisting of weight vectors of $T$, and bases of $M_{p_i}, M'_{p_i}$ are given by subsets of this basis. For each $\hat{\theta} \in P_m \cap G/P_{m'}$, one can choose a representative $n_\theta \in \hat{\theta}$ such that $n_\theta \in N_G(T)$, hence $n_\theta$ permutes the basis $\{e_1, \ldots, e_n\}$ up to scalar. It follows that

\[
\dim(M_{p_i} \cap n_\theta(M'_{p_i})) = h_{i \leq i,j} - h_{i-1 \leq i} = h_{i \leq i} ,
\]

[Note: The rest of the text is not provided due to the limitations of the preview tool.]
and we see that

\begin{equation}
(5.10.2) \quad a_{-+}(m, m'; n_0) = \sum_{i=1}^{r-1} h_i \leq i.
\end{equation}

We now compare Proposition 3.11 with (5.10.1). Since \(q^r \prod_{k=1}^{n} (q^{kr} - 1) = |G^F|\), and \(|\det V(q^r - y)| = |T_y|\), we have, by (5.10.2), that

\begin{equation}
(-1)^{p_- + p_+} q^{-r(n(\lambda) + n(\mu))} \sum_{z \in \mathcal{P}_{m,uni}} Q_\lambda^-(z) Q_\mu^+(z) = q^{-a(\lambda) - a(\tau(\mu))} \omega_{\lambda, \mu}(q).
\end{equation}

This proves the theorem modulo Lemma 5.9.

5.11. We shall prove Lemma 5.9. Note that the isomorphism \(W_m \cap xW_{m'} x^{-1} \cong \prod_{i,j} W_{h_{ij}}\) is chosen so that it satisfies the following properties; the factor of \(\delta_m\) corresponding to \(W_{h_{ij}}\) is equal to \(\delta_0^{-1}\), and the factor of \(x \delta_m\) corresponding to \(W_{h_{ij}}\) is equal to \(\delta_0^{-1}\), where we denote by \(\delta_0\) the linear character of \(W_{h_{ij}}\) corresponding to \(\delta\) given in 5.6. It follows that the factor of \(\delta_1 \varepsilon = \delta_m \otimes x \delta_m' \otimes \delta^{-1}\) corresponding to \(W_{h_{ij}}\) is given by

\begin{equation}
(5.11.1) \quad \delta_0^{(j-1)+(r-i+1)+(r-1)} = \delta_0^{(j-1)+(r-i)}.
\end{equation}

Since \(\delta_0 = \psi_a\) with \(a = (1, \ldots, 1)\) in the notation of 5.8, we see that

\begin{equation}
(5.11.2) \quad R(\delta_0^k) = q^{kh_{ij}}
\end{equation}

for \(k = 0, \ldots, r - 1\). It follows that

\[A_{\rho} = \sum_{1 \leq i,j \leq r} [j - i - 1] h_{ij},\]

where \([j - i - 1]\) is an integer between 0 and \(r - 1\) which is congruent to \((j - 1) + (r - i)\) modulo \(r\). On the other hand, by (5.1.2), (5.1.4) and (5.4.1), we have

\[N^* - a(\lambda) - a(\tau(\mu)) = r \left\{ \binom{n}{2} - n(\lambda) - n(\mu) \right\} + C,
\]

where

\[C = (r - 1)n - \left\{ |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r - 1)|\lambda^{(r)}| \right\}
- \left\{ |\mu^{(r-2)}| + 2|\mu^{(r-3)}| + \cdots + (r - 2)|\mu^{(1)}| + (r - 1)|\mu^{(r)}| \right\}.
\]

Hence in order to prove the lemma, it is enough to show that

\begin{equation}
(5.11.3) \quad C + \sum_{1 \leq i,j \leq r} [j - i - 1] h_{ij} = r \sum_{i=1}^{r-1} h_i \leq i.
\end{equation}
We show (5.11.3). Since \( n = \sum_{1 \leq i, j \leq r} h_{ij} \), we have

\[
(r - 1)n + \sum_{1 \leq i, j \leq r} [j - i - 1]h_{ij} = \sum_{1 \leq i, j \leq r} \{[j - i - 1] + (r - 1)\}h_{ij}.
\]

(5.11.4)

On the other hand, since \( |\lambda(j)| = m_j \) and \( |\mu(i)| = m_i' \), by (5.7.2), we have

\[
(j - 1)|\lambda(j)| = \sum_{1 \leq j \leq r} (j - 1)h_{ij} \quad \text{for } j = 1, 2, \ldots, r,
\]

\[
(r - 1 - i)|\mu(i)| = \sum_{1 \leq j \leq r} (r - 1 - i)h_{ij} \quad \text{for } i = 0, 1, \ldots, r - 1,
\]

where we understand that \( \mu(0) = \mu(r) \) and \( h_{0j} = h_{rj} \). It follows that

\[
\begin{align*}
\{ |\lambda^{(2)}| + 2|\lambda^{(3)}| + \ldots + (r - 1)|\lambda^{(r)}| \} \\
+ \{ |\mu^{(r-2)}| + 2|\mu^{(r-3)}| + \ldots + (r - 2)|\mu^{(1)}| + (r - 1)|\mu^{(r)}| \} \\
= \sum_{1 \leq j \leq r} (j - 1)h_{ij} + \sum_{1 \leq j \leq r} (r - 1 - i)h_{ij} \\
= \sum_{1 \leq j \leq r} (j - 1 + r - 2)h_{ij} + \sum_{1 \leq j \leq r} (j + r - 2)h_{rj}.
\end{align*}
\]

(5.11.5)

Subtracting (5.11.5) from (5.11.4), we see that the left hand side of (5.11.3) is equal to

\[
\begin{align*}
&\sum_{1 \leq i < j \leq r} ((j - i - 1) - (j - i - 1))h_{ij} \\
&+ \sum_{1 \leq j \leq r} ((j - r - 1) - (j - 1))h_{rj} \\
&= r \sum_{i=1}^{r-1} h_{i, i}. 
\end{align*}
\]

Hence (5.11.3) holds, and the lemma is proved. The proof of Theorem 5.5 is now complete.

6. Geometric properties of Kostka functions

6.1. We consider the complex \( K_1 = (\pi_{m,1})_! \overline{Q}_l \) on \( \mathscr{F}_{m, \text{uni}} \). By Theorem 4.8, \( K_1[d'_{m}] \) is a semisimple perverse sheaf equipped with \( W_m \)-action, and is decomposed as

\[
K_1[d'_{m}] \simeq \bigoplus_{\lambda \in \mathcal{P}(m)} V_\lambda \otimes A_\lambda,
\]
where $A_\lambda = \text{IC}(\mathfrak{X}_\lambda, \mathbb{Q}_l)[\dim X_\lambda]$ is a simple perverse sheaf on $\mathfrak{X}_{\text{uni}}$. Assume that the map $\pi_{\text{uni}} : \mathfrak{X}_{\text{uni}} \to \mathfrak{X}_{\text{uni}}$ is defined with respect to an $F$-stable Borel subgroup $B$. Then $\mathfrak{X}_{\text{uni}}$ has a natural $F_q$-structure, and $\pi_{\text{uni}}$ is $F$-equivariant. Thus one can define a canonical isomorphism $\varphi_1 : F^*K_1 \cong K_1$. Since each $X_\lambda$ is $F$-stable, we have $F^*A_\lambda \cong A_\lambda$, and $\varphi_1$ induces an isomorphism $F^*(V_\lambda \otimes A_\lambda) \cong (V_\lambda \otimes A_\lambda)$. It follows that there exists a unique isomorphism $\varphi_{\lambda,1} : F^*A_\lambda \cong A_\lambda$ such that $\varphi_1 = \sum \sigma_\lambda \otimes \varphi_{\lambda,1}$, where $\sigma_\lambda$ is the identity map on the representation space $V_\lambda$. Let $\phi_\lambda : F^*A_\lambda \cong A_\lambda$ be the natural isomorphism induced from the $F_q$-structure of $X_\lambda$. Since $A_\lambda$ is a simple perverse sheaf, $\phi_\lambda$ coincides with $\varphi_{\lambda,1}$ up to scalar. Let $d_\lambda$ be as in 4.9. The following result can be verified in a similar way as in [SS2, (4.1.1)].

(6.1.1) $\varphi_{\lambda,1} = q^{d_\lambda} \phi_\lambda$. In particular, the map $\varphi_{\lambda,1}$ coincides with the scalar multiplication $q^{d_\lambda}$ on $A_\lambda|_{X_\lambda}$.

In fact, for $z \in \mathfrak{X}_{\text{uni}}$, we have $\mathcal{H}_z^i K_1 \cong H^i(\mathcal{B}_z^{(m)}, \mathbb{Q}_l)$. For $z \in X_\lambda^F$, we have, by Theorem 4.8 (and by [S3, Prop. 8.16]),

$$H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \mathbb{Q}_l) \cong V_\lambda \otimes \mathcal{H}_z^0 \text{IC}(\mathfrak{X}_\lambda, \mathbb{Q}_l) \cong V_\lambda.$$  

Here $d_\lambda = \dim \mathcal{B}_z^{(m)}$ by Lemma 4.10, and $H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \mathbb{Q}_l)$ is an irreducible $W_\lambda$-module. Since the Frobenius action on $H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \mathbb{Q}_l)$ commutes with the $W_\lambda$-action, the Frobenius map acts on $H^{2d_\lambda}(\mathcal{B}_z^{(m)}, \mathbb{Q}_l)$ as a scalar multiplication by Schur’s lemma. In particular, all the irreducible components of $\mathcal{B}_z^{(m)}$ are $F$-stable, and this scalar is given by $q^{d_\lambda}$. It follows that $\varphi_{\lambda,1}$ acts as a scalar multiplication by $q^{d_\lambda}$ on $\mathcal{H}_z^0 \text{IC}(\mathfrak{X}_\lambda, \mathbb{Q}_l) \cong V_\lambda$. Since $\phi_\lambda$ acts as the identity map on this space, we obtain (6.1.1).

6.2. In general, let $K$ be a complex on a variety $X$ defined over $\mathbb{F}_q$ such that $F^*K \cong K$. An isomorphism $\phi : F^*K \cong K$ is said to be pure at $x \in X^F$ if the eigenvalues of $\phi$ on $\mathcal{H}_x^i K$ are algebraic numbers all of whose complex conjugates have absolute value $q^{i/2}$.

We prove the following.

**Proposition 6.3.** Let $\phi_\lambda : F^*A_\lambda \cong A_\lambda$ be as in 6.1. Assume that $z \in \mathcal{O}_{\mu}^+ \subset X_\mu^F$. Then $\phi_\lambda$ is pure at $z$.

**Proof.** Note that if $z \in \mathcal{O}_{\mu}^-$ (resp. $z \in \mathcal{O}_{\mu}^+$), then $z = (x, \nu)$ with $\nu = (v_1, \ldots, v_{r-1})$ satisfies the condition (a) (resp. (b)), where

(a) $v_i = v_{i-1}$ if $\mu^{(i)} = 0$,  
(b) $v_i = 0$ if $\mu^{(i)} = 0$.

First we show

(6.3.1) $\varphi_1 : F^*K_1 \cong K_1$ is pure at $z \in X_\mu$ if $z$ satisfies the condition (a) or (b).

Assume that $\mu \in \mathcal{P}(\mathfrak{m}')$ with $\mathfrak{m}' = (m'_1, \ldots, m'_n) \in \mathcal{D}_{n,r}$, and let $0 \leq p'_1 \leq \cdots \leq p'_n = n$ be integers associated to $\mathfrak{m}'$. For given $\mathfrak{m}, \mathfrak{m}'$, we define a sequence of integers $\mathfrak{m}'' = (m''_1, \ldots, m''_r)$, where $m''_i \in \mathcal{D}_{m'_i, r_i}$ with $\sum_{i=1}^r (r_i - 1) = r - 1$ as follows; write the sequences $\{p'_i\}, \{p'_j\}$ in the increasing order $0 \leq p''_1 \leq \cdots \leq p''_r = n$. Then the sequence $\{p''_i\}$ determines a composition of $m'_i$ which we denote by $\mathfrak{m}''_i$. Here $r_i$ is given by $r_i = \# \{ j \mid p'_{i-1} < p_j \leq p'_i \} + 1$. We consider a parabolic subgroup $P = P_{\mathfrak{m}'}$, which is the
stabilizer of a partial flag \((M_p')\) in \(G\). Let \(L = L_{m'}\) be the Levi subgroup of \(P\) containing \(T\), and \(U_P\) the unipotent radical of \(P\). Note that \(L \simeq G_1 \times \cdots \times G_r\) with \(G_i = GL_{m'_i}\). We define varieties

\[
\widetilde{\mathcal{F}}^L_{m', \text{uni}} = \overline{\mathcal{F}}^{G_1}_{m^{(1)}, \text{uni}} \times \cdots \times \overline{\mathcal{F}}^{G_r}_{m^{(r)}, \text{uni}},
\]

\[
\mathcal{F}^L_{m', \text{uni}} = \mathcal{F}^{G_1}_{m^{(1)}, \text{uni}} \times \cdots \times \mathcal{F}^{G_r}_{m^{(r)}, \text{uni}},
\]

where \(\overline{\mathcal{F}}^{G_i}_{m^{(i)}, \text{uni}}\) and \(\mathcal{F}^{G_i}_{m^{(i)}, \text{uni}}\) are varieties with respect to \(G_i\) defined similarly to \(\overline{\mathcal{F}}_{\text{uni}}\) and \(\mathcal{F}_{\text{uni}}\). Thus we can define a map \(\pi^{L,1}_{m',} : \overline{\mathcal{F}}^L_{m', \text{uni}} \to \mathcal{F}^L_{m', \text{uni}}\) as the product of \(\pi^{G_i}_{m^{(i)},1} : \overline{\mathcal{F}}^{G_i}_{m^{(i)}, \text{uni}} \to \mathcal{F}^{G_i}_{m^{(i)}, \text{uni}}\).

By generalizing the diagram in (1.5.1), we obtain a diagram

\[
\begin{array}{ccc}
\widetilde{\mathcal{F}}_{\text{uni}} & \xymatrix{\leftarrow & \overline{\mathcal{F}}^P_{m, \text{uni}} & \ar[r]^-{\overline{\pi}_1} & \overline{\mathcal{F}}^L_{m, \text{uni}} & \ar[d]^-{\overline{\pi}_1} } \\
\overline{\mathcal{F}}^P_{m, \text{uni}} & \xymatrix{\leftarrow & G \times \overline{\mathcal{F}}^P_{m, \text{uni}} & \ar[r]^-{\pi_1} & \mathcal{F}^L_{m, \text{uni}} & \ar[d]^-{\pi_1} } \\
\overline{\mathcal{F}}^L_{m, \text{uni}} & \xymatrix{\leftarrow & G \times \overline{\mathcal{F}}^L_{m, \text{uni}} & \ar[r]^-{\pi_1} & \mathcal{F}^L_{m, \text{uni}} & \ar[d]^-{\pi_1} } \\
\mathcal{F}_{\text{uni}} & \xymatrix{\leftarrow & G \times \mathcal{F}^L_{m, \text{uni}} & \ar[r]^-{\pi_1} & \mathcal{F}^L_{m, \text{uni}} & \ar[d]^-{\pi_1} }
\end{array}
\]

(6.3.2)

where, by putting \(P_{\text{uni}} = L_{\text{uni}} U_P\) (the set of unipotent elements in \(P\)),

\[
\mathcal{F}^P_{m, \text{uni}} = \bigcup_{g \in P} g(U \times \prod_i M_{p_i}),
\]

\[
\overline{\mathcal{F}}^P_{m, \text{uni}} = G \times \mathcal{F}^P_{m, \text{uni}},
\]

\[
\overline{\mathcal{F}}^L_{m, \text{uni}} = P \times B (U \times \prod_i M_{p_i}).
\]

The maps are defined similarly to (1.5.1). In particular, \(\overline{\pi}_1 \circ \pi'_1 = \pi_{m,1}\). As in (1.5.1), both squares are cartesian. The map \(p_1\) is a principal \(P\)-bundle, and the map \(q_1\) is a locally trivial fibration with fibre isomorphic to \(G \times U_P \times \prod_{i=1}^{r-1} M_{p_i}\). We assume that \(B\) and \(L\) are \(F\)-stable. Then all the objects in (6.3.2) are defined over \(F\). Put \(K'_1 = (\pi'_1)^* Q_l\) and \(K^L_1 = (\pi^L_{m',1})^* Q_l\). As in the discussion in 1.5, we see by (6.3.2)

\[
(6.3.3) \quad p_1^* K'_1 \simeq q_1^* K^L_1.
\]

Take \(z \in (\mathcal{F}_{\text{uni}} \cap X_{\mu})^F\) satisfying the condition (a) or (b). We have

\[
\overline{\mathcal{F}}^P_{m, \text{uni}} \simeq \{(y, gP) \in (G_{\text{uni}} \times V^{r-1}) \times G/P \mid g^{-1}y \in \mathcal{F}^P_{m, \text{uni}}\}.
\]

Up to \(G\)-conjugate, one can choose \(\xi = (z, P) \in (\overline{\mathcal{F}}^P_{m, \text{uni}})^F\) such that \(\overline{\pi}_1^u(\xi) = z\). Choose \(\eta \in (G \times \mathcal{F}^P_{m, \text{uni}})^F\) such that \(p_1(\eta) = \xi\). Put \(z' = q_1(\eta) \in (\overline{\mathcal{F}}^L_{m', \text{uni}})^F\). By (6.3.3), we have
By Theorem 4.8, we have

\[ \mathcal{H}_\xi^i(K_1^f) \simeq \mathcal{H}_\eta^i(p^*K_1^f) \simeq \mathcal{H}_\eta^i(q_1^*K_1^L) \simeq \mathcal{H}_\xi^i(K_1^L). \]

Put \( z' = (z'_1, \ldots, z'_r) \in \mathcal{X}_{m_0, \text{uni}}^{r} \) with \( z'_i \in \mathcal{X}_{m_0, \text{uni}}^{G_i} \). If we denote \((\pi_{G_i}^{G_{i(0)}}, Q_1)\) by \( K_1^{G_i} \), \( K_1^L \) can be written as

\[ K_1^L \simeq K_1^{G_1} \otimes \cdots \otimes K_1^{G_r} \]

and so

\[ \mathcal{H}_\xi^i K_1^L \simeq \bigoplus_{i_1 + \cdots + i_r = i} \mathcal{H}_{z'_1}^{i_1} K_1^{G_1} \otimes \cdots \otimes \mathcal{H}_{z'_r}^{i_r} K_1^{G_r}. \]

The isomorphism \( \varphi_1^L : F^*K_1^L \simeq K_1^L \) is defined similar to \( \varphi_1 \), and under the isomorphism (6.3.5), \( \varphi_1^L \) can be written as \( \varphi_1^L = \varphi_1^{G_1} \otimes \cdots \otimes \varphi_1^{G_r} \), where \( \varphi_1^{G_i} : F^*K_1^{G_i} \simeq K_1^{G_i} \). Note that \( z'_i \in \mathcal{X}_{m_0, \text{uni}}^{G_i} \) also satisfies the condition (a) or (b). (6.3.1) holds in the case where \( r = 2 \) by [AH, Corrigendum, Prop. 3]. So by double induction on \( n \) and \( r \), we may assume that \( \varphi_1^{G_i} \) is pure at \( z_i \) for each \( i \) unless \( m' = (n; \ldots; \ldots; -) \). Now assume that \( m' \neq (n; \ldots; -; -) \). Then \( \varphi_1^L \) is pure at \( z' \) by (6.3.6). By (6.3.4), \( K_1^L \) is pure at \( \xi \) with respect to \( \varphi_1^L : F^*K_1^L \simeq K_1^L \). Note that \( P = P_{m'} \) is the unique parabolic subgroup of \( G \), conjugate to \( P \), containing \( x \) such that the image of \( x \) on \( L \simeq G_1 \times \cdots \times G_r \) has Jordan type \((\mu^{(1)}, \ldots, \mu^{(r)}) \). Thus \((\pi''_m)^{-1}(z) = \{(z, P)\} \), and so \( \mathcal{H}_1^i K_1 \simeq \mathcal{H}_1^i K_1^L \). This proves (6.3.1) in the case where \( m' \neq (n; \ldots; -; -) \).

It remains to consider the case \( m' = (n; \ldots; -; -) \). In this case, by our assumption we have \( v_1 = v_2 = \cdots = v_{r-1} \) or \( v_2 = \cdots = v_{r-1} = 0 \). Hence the complex \( K_1^L \) is isomorphic to a similar complex in the case where \( r = 2 \). So in this case, (6.3.1) holds by [AH].

(6.3.1) implies that the eigenvalues of \( F^* \) on \( H^i(\mathcal{X}^{(m)}; Q_1) \) have absolute value \( q^{i/2} \). By Theorem 4.8, we have

\[ \mathcal{H}_\xi^i K_1 \simeq \bigoplus_{\lambda \in \mathcal{P}(m)} V_\lambda \otimes \mathcal{H}_{z}^{i-d_m^i + \dim X_\lambda} IC(X_\lambda, Q_1) \]

Since \( d_\lambda = (d_m^i - \dim X_\lambda)/2, i - d_m^i + \dim X_\lambda = i - 2d_\lambda \). Since the eigenvalues of \( \varphi_1 \) on \( \mathcal{H}_\xi^i K_1 \) have absolute value \( q^{i/2} \) by (6.3.1), the eigenvalues of \( \varphi_{\lambda, 1} \) on \( \mathcal{H}_{z}^{i-2d_\lambda} IC(X_\lambda, Q_1) \) also have absolute value \( q^{i/2} \). By (6.1.1), this implies that the eigenvalues of \( \phi_\lambda \) have absolute value \( q^{i/2} \). The proposition is proved. \( \square \)

**Remark 6.4.** In the case where \( r = 2 \), \( X_\lambda \) is a single \( G \)-orbit. In this case the corresponding fact was proved in [AH, Corrigendum, Prop. 3], by making use of the contraction of a suitable transversal slice, and by the result of [MS]. The purity result for the exotic case with \( r = 2 \) was also proved in [SS2, Prop. 4.4] based on the discussion in [L3], again by making use of the transversal slice. However for \( r \geq 3 \), the discussion by using the transversal slice will not work since it often happens that \( Z_G(z) \) turns out to be a unipotent group for \( z \in \mathcal{X}_{m_0, \text{uni}}^{r} \), and one cannot construct a one-parameter subgroup. It is not clear whether \( \phi_\lambda \) is pure for all \( z \in X_\mu^F \).
6.5. For $\lambda, \mu \in \mathcal{P}_{n,r}$, we define a polynomial $IC_{\lambda,\mu}^{-}(t) \in \mathbb{Z}[t]$ by
\[
IC_{\lambda,\mu}^{-}(t) = \sum_{i \geq 0} \dim \mathcal{H}_{z}^{2i} IC(\mathcal{X}_{\lambda}, Q_{t}) t^{i}
\]
for $z \in \mathcal{O}_{\mu}^{-}$. Also we define a polynomial $IC_{\lambda,\mu}^{+}(t) \in \mathbb{Z}[t]$ by the same formula as above if $z \in \mathcal{O}_{\mu}^{+} \cap \mathcal{Z}_{m,uni}^{+}$ and by 0 if $\mathcal{O}_{\mu}^{+} \cap \mathcal{Z}_{m,uni}^{+} = \emptyset$, where we assume that $\lambda \in \mathcal{P}(m)$.  

6.6. As in 3.10, let $\mathcal{C}_{q} = \mathcal{C}_{q}(\mathcal{X}_{uni})$ be the $\mathbb{Q}_{l}$-space of all $G^{F}$-invariant $\mathbb{Q}_{l}$-valued functions on $\mathcal{Z}_{uni}^{F}$. The bilinear form $\langle \ , \ \rangle$ on $\mathcal{C}_{q}$ is defined as in (3.10.2). Let $\mathcal{C}_{q}^{\pm}$ be the $\mathbb{Q}_{l}$-subspace of $\mathcal{C}_{q}$ generated by $Q_{\lambda}^{\pm}$ for various $\lambda \in \mathcal{P}_{n,r}$. Then $\{Q_{\lambda}^{\pm} \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a basis of $\mathcal{C}_{q}^{\pm}$. For an $F$-stable $G$-orbit $C$ in $G_{uni} \times V^{r-1}$, we denote by $y_{C}$ the characteristic function of $C^{F}$. In the case where $C = \mathcal{O}_{\lambda}^{\pm}$ we denote it by $y_{\lambda}^{\pm}$. Then $Q_{\lambda}^{\pm}$ can be written as a sum of various $y_{C}$. For each $\lambda \in \mathcal{P}_{n,r}$, we define a function $Y_{\lambda}^{\pm} \in \mathcal{C}_{q}^{\pm}$ by the condition that $Y_{\lambda}^{\pm}$ is a linear combination of $Q_{\nu}^{\pm}$ with $\nu \leq \lambda$ and that
\[
Y_{\lambda}^{\pm} = y_{\lambda}^{\pm} + \sum_{C} b_{\lambda,C}^{\pm} y_{C},
\]
where $b_{\lambda,C}^{\pm} = 0$ for $y_{C} = y_{\nu}^{\pm}$ with $\nu < \lambda$. Note that $Y_{\lambda}^{\pm}$ is determined uniquely by this condition. Although $Y_{\lambda}^{\pm}$ are not characteristic functions on $(\mathcal{O}_{\lambda}^{\pm})^{F}$, they enjoy similar properties,

\[
\begin{cases}
1 & \text{if } z \in \mathcal{O}_{\lambda}^{\pm}, \\
0 & \text{if } z \in \mathcal{O}_{\nu}^{\pm} \text{ with } \nu < \lambda.
\end{cases}
\]

$\{Y_{\lambda}^{\pm} \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a basis of $\mathcal{C}_{q}^{\pm}$. We define a matrix $\tilde{\Lambda}_{q} = (\tilde{\xi}_{\lambda,\mu}(q))$ by $\tilde{\xi}_{\lambda,\mu}(q) = \langle Y_{\lambda}^{\pm}, Y_{\mu}^{\pm} \rangle$. In the case where $r = 2$, $Y_{\lambda}^{\pm} = Y_{\lambda}^{\pm}$ is the characteristic function of a single $G$-orbit, hence $\tilde{\Lambda}_{q}$ is a diagonal matrix. For $r \geq 3$, $Y_{\lambda}^{\pm}$ and $Y_{\mu}^{\pm}$ are not necessarily orthogonal for $\lambda \neq \mu$. We consider the following condition for $\tilde{\Lambda}_{q}$.

(A) The matrix $\tilde{\Lambda}_{q}$ is upper triangular with respect to the total order $\prec$.

We show the following theorem.

**Theorem 6.7.** Suppose that the condition (A) holds for $\tilde{\Lambda}_{q}$.

(i) $\tilde{K}_{\lambda,\mu}^{-}(t) = t^{d(\lambda)} IC_{\lambda,\mu}^{-}(t)$.

(ii) Assume that $z \in (\mathcal{O}_{\mu}^{\pm})^{F}$ for $\mu \leq \lambda$. Then $\mathcal{H}_{z}^{i} IC(\mathcal{X}_{\lambda}, Q_{t}) = 0$ if $i$ is odd, and the eigenvalues of $\phi_{\lambda}$ on $\mathcal{H}_{z}^{2i} IC(\mathcal{X}_{\lambda}, Q_{t})$ are $q^{i}$.

**Proof.** In view of (2.4.2), $Q_{m,T_{w}}^{-}(z)$ can be written as
\[
Q_{m,T_{w}}^{-}(z) = (-1)^{d_{m}} \sum_{i \geq 0} (-1)^{i} \text{Tr} (\varphi_{1} w, \mathcal{H}_{z}^{i} K_{1}^{i}).
\]
Thus by Theorem 4.8, noticing that $d_m - \dim X_\lambda = 2d_\lambda$ is an even integer, we have

$$Q^\pm_\lambda(z) = (-1)^{dm} \sum_{i \geq 0} (-1)^i \Tr \langle \varphi_{\lambda,1}, H^i \mathcal{Z}(\lambda, \tilde{Q}_l) \rangle. \tag{6.7.1}$$

$Q^\pm_\lambda(z)$ is obtained by restricting $Q^\pm_\lambda$ on $(\mathcal{Z}^+_{m,\text{uni}})^F$ by Proposition 2.13.

We express $Q^\pm_\lambda$ as

$$Q^\pm_\lambda = \sum_{\nu \leq \lambda} \tilde{p}^\pm_{\lambda,\nu}(q) \xi_{\nu,\nu'}(q) \tilde{p}^\pm_{\mu,\nu'}(q). \tag{6.7.2}$$

Then we have

$$\langle Q^\pm_\lambda, Q^\pm_\mu \rangle = \sum_{\nu \leq \lambda, \nu' \leq \mu} \tilde{p}^\pm_{\lambda,\nu}(q) \tilde{\xi}_{\nu,\nu'}(q) \tilde{p}^\pm_{\mu,\nu'}(q). \tag{6.7.3}$$

We also note, by (6.1.1), that $\tilde{p}^\pm_{\lambda}(q) = (-1)^{dm} q^{d_\lambda}$. Let $\tilde{Q}^\pm_\lambda \in \mathcal{Z}^\pm_{\text{uni}}$ be the functions defined by

$$\tilde{Q}^\pm_\lambda(z) = (-1)^{p_-(m)} q^{\lambda(\nu')} (q^r)^{-d_\lambda} Q^\pm_\lambda(z),$$

for $z \in \mathcal{Z}^F_{m,\text{uni}}$. Put, for $\lambda \in \mathcal{P}(m), \mu \in \mathcal{P}(m')$,

$$\tilde{p}^\pm_{\lambda,\nu}(q) = (-1)^{dm} q^{\lambda(\nu')} (q^r)^{-d_\lambda} \tilde{p}^\pm_{\lambda,\nu}(q^r),$$

$$\tilde{p}^\pm_{\mu,\nu'}(q) = (-1)^{d_{\nu'} + p_-(m')} q^{\lambda(\nu') + a(\nu')} (q^r)^{-d_\mu} \tilde{p}^\pm_{\mu,\nu'}(q^r),$$

$$\xi_{\nu,\nu'}(q) = q^{a(\nu')} (q^r)^{-d_\mu} \tilde{\xi}_{\nu,\nu'}(q^r).$$

Since $d_m = \dim G + p_-(m)$ by Lemma 1.2, we have

$$d_m + d_{m'} + p-(m') + p_+(m') \equiv p_-(m) + p_+(m') \pmod{2}.$$ 

Hence by Theorem 5.5 together with (6.7.3), we have

$$\sum_{z \in \mathcal{Z}^F_{m,\text{uni}}} \tilde{Q}^\pm_\lambda(z) \tilde{Q}^\pm_\mu(z) = \sum_{\nu \leq \lambda, \nu' \leq \mu} \tilde{p}^\pm_{\lambda,\nu}(q) \xi_{\nu,\nu'}(q) \tilde{p}^\pm_{\mu,\nu'}(q) = \omega_{\lambda,\mu}(q), \tag{6.7.4}$$

with $p^\pm_{\lambda}(q) = q^{\lambda(\nu)}$, $p^\pm_{\mu,\nu'} = (-1)^{p_-(m') + p_-(m')} q^\mu(q)$. Moreover, $p^\pm_{\lambda,\nu} = 0$ unless $\nu \leq \lambda$.

We consider the matrix $P_1^\pm = (p^\pm_{\lambda,\mu}), A_1 = (\xi_{\nu,\nu'})$ and $\Omega = (\omega_{\lambda,\mu})$. We have a matrix equation

$$P_1^- (A_1^t P_1^+) = \Omega. \tag{6.7.5}$$
Since $P_1^-$ is a lower triangular matrix with diagonal entries $q^{a(\lambda)}$, and $A_1^+P_1^+$ is an upper triangular matrix with diagonal entries $\pm q^{a(\mu)}\epsilon_{\mu,\mu}(q)$ by the condition (A), $P_1^-$ and $A_1^+P_1^+$ are determined uniquely from $\Omega$. In particular, the diagonal part of $A_1$ is determined uniquely from $\Omega$. Let $A$ be a diagonal matrix and consider the matrix equation $P^-A^+P^+ = \Omega$, where $P^\pm$ satisfy similar conditions as in Theorem 5.2. Then $P^\pm, A$ are determined uniquely from the equation, and by the uniqueness of the solution for (6.7.5), we have $P^- = P_1^-$ and $A^+P^+ = A_1^+P_1^+$. In particular, $A$ coincides with the diagonal entries of $A_1$. (But $A_1$ and $P_1^+$ are not determined from the equation (6.7.5).) Thus by Theorem 5.2, we have $p_{\lambda,\mu}(q) = K_{\lambda,\mu}(q)$.

By (6.6.1) and (6.7.2), we have $Q_\lambda^\pm(z) = \hat{p}_{\lambda,\nu}(q)$ for $z \in (\Theta_\nu^\pm)^F$. We consider the equation (6.7.1). We replace $\varphi_\lambda$ by $\phi_\lambda$, and use (6.1.1). By replacing $q$ by $q^m$ for a positive integer $m$, we have

$$
(6.7.6) \quad K_{\lambda,\nu}(q^m) = (q^m)^{a(\lambda)} \sum_{i \geq 0} (-1)^i \text{Tr}(\phi_{r^m, \mathcal{H}_2^i \text{IC}(X_\lambda, Q_t))}.
$$

By Proposition 6.3, $\phi_\lambda$ is pure at $z \in (\Theta_\nu^\pm)^F$. Thus, if we put $\mathcal{H}_2^i = \mathcal{H}_2^i \text{IC}(X_\lambda, Q_t)$, one can write as

$$
(6.7.7) \quad \text{Tr}(\phi_{r^m, \mathcal{H}_2^i}) = \sum_{j=1}^{k_i} (\alpha_{ij} q^{i/2})^{r^m},
$$

where $k_i = \dim \mathcal{H}_2^i$ and $\{\alpha_{ij} q^{i/2} | 1 \leq j \leq k_i\}$ are eigenvalues of $\phi_\lambda$ on $\mathcal{H}_2^i$ such that $\alpha_{ij}$ are algebraic numbers all of whose complex conjugates have absolute value 1. By Theorem 5.2, $K_{\lambda,\nu}(t)$ is a rational function on $t$. Here we note that

$$
(6.7.8) \quad K_{\lambda,\nu}(t) \text{ is a polynomial in } t.
$$

In fact, we can write $K_{\lambda,\nu}(t)$ as $K_{\lambda,\nu}(t) = P(t) + R(t)/Q(t)$, where $P, Q, R$ are polynomials with $\deg R < \deg Q$ or $R = 0$. By (6.7.6) and (6.7.7), the absolute value of the right hand side of (6.7.6) goes to $\infty$ when $m \to \infty$. It follows that the absolute value of $K_{\lambda,\nu}(q^m) - P(q^m)$ goes to $\infty$ when $m \to \infty$, if it is non-zero. This implies that $R = 0$, and (6.7.8) holds.

Now by applying Dedekind’s theorem, we see that $\alpha_{ij} = 0$ for odd $i$, and $\alpha_{ij} = 1$ for even $i$ such that $\mathcal{H}_2^i \neq 0$. It follows that $\mathcal{H}_2^i = 0$ for odd $i$, and $\sum_{j=1}^{k_i} \alpha_{ij} q^{i/2} = (\dim \mathcal{H}_2^i) q^{i/2}$. Thus by (6.7.6), we have $K_{\lambda,\nu}(q) = q^{a(\lambda)} IC_{\lambda,\nu}(q')$, which holds for any prime power $q$. The assertion (i) follows from this in view of (6.7.8). The assertion (ii) is already shown in the proof of (i). The theorem is proved.

By using similar arguments, we can prove the following result. Note that in this case, we do not need to appeal the condition (A).

**Proposition 6.8.** Let $\nu \in \mathcal{P}(m'')$ with $m'' = (m''_1, \ldots, m''_r)$. Assume that $m''_i = 0$ for $i = 1, \ldots, r - 2$. Then for any $\lambda \in \mathcal{P}(m)$ and $\mu \in \mathcal{P}(m')$, the following holds.
(i) We have
\[ K_{\lambda, \nu}^-(t) = t^{a(\lambda)} IC_{\lambda, \nu}(t^r), \]
\[ K_{\mu, \nu}^+(t) = t^{a(\tau(\mu)) + a(\nu) - a(\tau(\nu))} IC_{\mu, \nu}(t^r). \]

(ii) Assume that \( z \in X^P_{\nu} \) for \( \nu \leq \lambda \). Then \( H^i IC(X_{\lambda}, \tilde{Q}_i) = 0 \) if \( i \) is odd, and the eigenvalues of \( \phi \) on \( H^i IC(X_{\lambda}, \tilde{Q}_i) \) are \( q^i \).

**Proof.** Put \( \lambda_0 = (-; \cdots; -; n; -) \). Then for any \( \nu \leq \lambda_0 \), \( X_{\nu} = \mathcal{O}_\nu^\pm \) is a single \( G \)-orbit. It follows that \( Y^\pm_\lambda \) coincides with the characteristic function \( y_\lambda = y^\pm_\lambda \) of \( X_{\lambda}^P \) for \( \lambda \leq \lambda_0 \).

Assume that \( \lambda \leq \lambda_0 \) or \( \mu \leq \lambda_0 \). Then (6.7.4) can be rewritten as
\[
\sum_{z \in \mathcal{F}^F_{\text{uni}}} \tilde{Q}_\lambda(z) \tilde{Q}_\mu^+(z) = \sum_{\nu \leq \lambda} p_{\lambda, \nu}(q) \xi_{\nu, \nu}(q) p_{\mu, \nu}^+(q) = \omega_{\lambda, \mu}(q),
\]
We consider the matrix \( P^0_{\lambda, \nu} = (p_{\lambda, \nu}^\pm) \) \( A_0 = (\xi_{\lambda, \mu}) \) and \( \Omega_0 = (\omega_{\lambda, \mu}) \) indexed by \( \lambda, \mu \leq \lambda_0 \). By (6.8.1), the matrices \( P^0_{\lambda, \nu} \) \( A_0 \) and \( \Omega_0 \) satisfy a similar condition as in Theorem 5.2.

(Note that \( p_+ (m') = p_- (m') \) if \( \mu \leq \lambda_0 \), hence \( p_{\lambda, \nu}^+ (q) = q^a(\mu) \).) Thus the equation (6.8.1) determines uniquely the matrices \( P^0_{\lambda, \nu} \) and \( A_0 \). Hence by Theorem 5.2, \( p_{\lambda, \nu}^+ (q) = K_{\lambda, \nu}^+(q) \) for any \( \lambda, \mu \leq \lambda_0 \). In particular, \( p_{\lambda, \nu}^+ (q) \) and \( \xi_{\nu, \nu} \) are determined for \( \nu \leq \mu \leq \lambda_0 \). We now consider arbitrary \( \lambda \in \mathcal{P}_{n, r} \). Since \( p_{\lambda, \nu}^+ (q) = q^a(\nu) \), the equation (6.8.1) determines uniquely \( p_{\lambda, \nu}^- (q) \) for \( \nu \leq \lambda_0 \), by induction on the total order \( \prec \) on \( \mathcal{P}_{n, r} \). Again by Theorem 5.2, we see that \( p_{\lambda, \nu}^-(q) = K_{\lambda, \nu}^-(q) \). Similar argument also holds for \( p_{\lambda, \nu}^+(q) \). Thus we have proved
\[
p_{\lambda, \nu}^- (q) = K_{\lambda, \nu}^-(q), \quad p_{\lambda, \nu}^+ (q) = K_{\lambda, \nu}^+(q) \quad \text{for any } \nu \leq \lambda_0.
\]

By using a similar argument as in the proof of (6.7.6), we have
\[
K_{\lambda, \nu}^-(q^m) = (q^m)^{a(\lambda)} \sum_{i \geq 0} (-1)^i \text{Tr} (\phi^{r,m}_\lambda, H^i IC(X_{\lambda}, \tilde{Q}_i)),
\]
\[
K_{\lambda, \nu}^+(q^m) = (q^m)^{a(\tau(\mu)) + a(\nu) - a(\tau(\nu))} \sum_{i \geq 0} (-1)^i \text{Tr} (\phi^{r,m}_\mu, H^i IC(X_{\mu}, \tilde{Q}_i)).
\]

(Here in the latter formula, we assume that \( z \in \mathcal{O}_\nu^+ \cap \mathcal{F}^+_m \) for \( \mu \in \mathcal{P}(m') \). In the case where \( \mathcal{O}_\nu^+ \cap \mathcal{F}^+_m = \emptyset \), we have \( K_{\mu, \nu}^+(q^m) = 0 \).) Now the proposition follows by a similar argument as in the proof of Theorem 6.7.

The condition(A) can be verified in small rank cases. We have the following result.

**Proposition 6.9.** Assume that \( n = 1, 2 \), and \( r \) is arbitrary. Then the condition (A) holds for \( \hat{A}_q \).
Proof. First consider the case where \( n = 1 \). In this case, \( X_\nu \cap \mathcal{X}_m^+ \) coincides with \( \Theta_\nu^+ \) for any \( \nu \) and \( m \), if it is non-empty. Hence \( Y_\mu^+ = y_\mu^+ \) for any \( \mu \). One can check that \( Y_\mu^- \) coincides with the characteristic function of \( X_\mu^- \). Thus the condition (A) holds. In fact, in this case \( \Delta_\ell \) is a diagonal matrix, and by applying the arguments in the proof of Proposition 6.8, \( \tilde{K}_{\lambda,\nu}^-(t), \tilde{K}_{\mu,\nu}^+(t) \) are given as in the formulas in Proposition 6.8 (i).

Next consider the case where \( n = 2 \). We determine the pair \((m', \nu)\) satisfying the condition (*) \( X_\nu \cap \mathcal{X}_m^+ \) splits into more than two orbits. This occurs only when \( m' \) is of the form \( m' = (\ldots, 1, \ldots, 1, \ldots) \), where non-zero factors occur on \( 1 \leq k < \ell < r \), and \( \nu \) is such that \( \nu^{(k)} = (1^2) \), or \( \nu = \lambda(m') = (\ldots; 1; \ldots; 1; \ldots) \). In that case, we have \( X_\nu \cap \mathcal{X}_m^+ = \Theta_\nu^+ \mathcal{X}_{m,m'}^+ \), where \( \mathcal{X}_{m,m'}^+ \) consists of infinitely many \( G \)-orbits. Thus we have, for any \( 1 \leq k < \ell < r \),

\[(*) \quad m' = (\ldots, 1, \ldots, 1, \ldots), \quad \nu = (\ldots; 1^2; \ldots; -\ldots) \quad \text{or} \quad \nu = \lambda(m'), \]

where \( 1 \) appear in the \( k, \ell \)-th factors for \( m' \), and \( 1^2 \) appears in the \( k \)-th factor for \( \nu \). We show

\[(6.9.1) \quad \text{The function } Y_\mu^+ \text{ coincides with } y_\mu^+ \text{ unless } \mu \in \mathcal{P}(m') \text{ for } m' \text{ in } (*), \text{ in which case, } Y_\mu^+ = y_\mu^+ + \sum_{C} a_C y^C \text{ with } C \subset X_{m,m'}^+ \text{ for } \nu \text{ in } (*). \]

If \( \mu = (\ldots; 1^2; \ldots) \), any \( \nu \leq \mu \) has a similar form as \( \mu \). Hence by (*), \( Y_\mu^+ = y_\mu^+ \). If \( \mu = (\ldots; 1; \ldots; 1) \), any \( \nu \leq \mu \) has a similar form as a similar form as in the previous case. Hence again by (*), \( Y_\mu^+ = y_\mu^+ \). Assume that \( \mu = (\ldots; 1; \ldots; 1; \ldots) \) with \( 1 \leq k < \ell < r \). Then \( X_\nu \cap \mathcal{X}_m^+ \neq \emptyset \) only when \( \nu = \mu, (\ldots; 1^2; \ldots), (\ldots; 1; \ldots; 1) \). In the second and the third case, by the previous results, \( Y_\nu^+ = y_\nu^+ \). Except the case where \( \nu \) is as in (*), \( X_\nu \cap \mathcal{X}_m^+ = \Theta_\nu^+ \). Hence by subtracting those functions \( y_\mu^+ \) from \( Q_\mu^+ \), we obtain the function which has supports only on \( \Theta_\nu^+ \) and \( X_{m,m'}^+ \). Finally assume that \( \mu = (\ldots; 2; \ldots; 1) \). Then \( X_\nu \cap \mathcal{X}_m^+ \neq \emptyset \) only when \( \nu = \mu, \text{ or } \nu = (\ldots; 1^2; \ldots), \ldots; 1, \ldots; 1 \). Moreover, by (*), in each case, \( X_\nu \cap \mathcal{X}_m^+ = \Theta_\nu^+ \). Hence by subtracting those \( y_\nu^+ \) such that \( \nu \neq \mu \) from \( Q_\mu^+ \), we obtain \( Y_\nu^+ = y_\nu^+ \). This proves (6.9.1).

Next we determine the pair \((m, \nu)\) satisfying the condition (**), \( X_\nu \cap \mathcal{X}_m \neq \emptyset \) and \( X_\nu \nsubseteq \mathcal{X}_m \). If \( m = (\ldots, 2, \ldots) \), clearly \( X_\nu \subset \mathcal{X}_m \). So assume that \( m \) is of the form \( (\ldots, 1, \ldots, 1, \ldots) \) for some \( 1 \leq a < b \leq r \). If \( \nu \) is of the form \( (\ldots; 2; \ldots; 1) \) for \( k \)-factor, then \( k \geq b \) since \( \lambda(m) \geq \nu \). But in this case, \( X_\nu \subset \mathcal{X}_m \). If \( \nu = (\ldots; 1; \ldots; 1; \ldots) \) for some \( k, \ell \)-factors, then \( a \leq k, b \leq \ell \), and \( X_\nu \subset \mathcal{X}_m \). Hence we assume that \( \nu = (\ldots; 1^2; \ldots; 1) \) for some \( k \)-factor. If \( k \geq b \), then \( X_\nu \subset \mathcal{X}_m \). So, we have \( a \leq k < b \). Then \( X_\nu \nsubseteq \mathcal{X}_m \). Hence we have

\[(**) \quad m = (\ldots, 1, \ldots, 1, \ldots), \quad \nu = (\ldots; 1^2; \ldots), \]

where \( k \) factor appears for \( \nu \), and \( a, b \) factors appear for \( m \) with \( a \leq k < b \). For each \( \lambda \), let \( \bar{y}_\lambda \) be the characteristic function of \( X_\lambda^E \). We show

\[(6.9.2) \quad Y_\lambda^- \text{ coincides with } \bar{y}_\lambda \text{ unless } \lambda = (\ldots; 1; \ldots; 1; \ldots) \text{ with } \lambda^{(a)} = \lambda^{(b)} = 1 \text{ for } a < b, \text{ in which case, } Y_\lambda^- = \bar{y}_\lambda + \sum_C a_C y^C \text{ with } C \subset X_\nu \backslash \Theta_\nu^- \text{ for } \nu = (\ldots; 1^2; \ldots) \text{ with } \nu^{(a)} = (1^2). \]
If $\lambda = (\cdots ; 1^2; \cdots )$, any $\nu \leq \lambda$ has a similar type as $\lambda$. We have $X_\nu \subset \mathcal{B}_m$ and $\mathcal{B}_z^{(m)}$ has a common structure for any $z \in X_\lambda$. Thus by backwards induction on $k$ (1^2 appears in the k-th factor), we see that $Y_\lambda = \tilde{y}_\lambda$. Assume that $\lambda = (\cdots ; 1^2; \cdots ; 1^2; \cdots )$, where $\lambda^{(i)} = (1)$ for $i = a, b$ with $a < b - 1$. Put $\lambda' = (\cdots ; 1^2; \cdots ; 1^2; \cdots )$, where $\lambda'^{(a+1)} = \lambda'^{(b)} = (1)$. If $\nu \leq \lambda$, then $\nu$ has the type $\nu = (\cdots ; 1^2; \cdots )$ or $(\cdots ; 1^2; \cdots ; 1^2; \cdots )$. Assume that $\nu$ is such that $\nu^{(k)} = (1^2)$ for some $k \geq a$. We have

\[X_\nu = \{(v, \nu) \in G_{uni} \times V^{r-1} | x = 1, v_k = 0 \text{ for } i \leq k - 1, v_k \neq 0\}, \]

\[X_\nu \cap \mathcal{B}_m = \{(v, \nu) \in X_\lambda | \langle v_i \rangle = \langle 0 \rangle \text{ for } k + 1 \leq i < b\}.\]

Take $z \in X_\nu \cap \mathcal{B}_m$. Then $\mathcal{B}_z^{(m)}$ is equal to a one point if $k < b$, and is equal to $\mathcal{B}$ if $k \geq b$. If $\nu = (\cdots ; 1^2; \cdots ; 1^2; \cdots )$, then $\mathcal{B}_z^{(m)}$ is a one point for any $z \in X_\mu$. A similar property also holds for the pair $(\mathcal{B}_z^{(m)}$, $\nu$) if $\lambda' \in \mathcal{B}(\mathcal{B}_z^{(m)})$ and $\nu' \leq \lambda'$. It follows that the function $Q_{\lambda} - Q_{\lambda}'$ has supports only on $X_\lambda$ and $X_\nu \cap \mathcal{B}_m$ for $\nu$ such that $\nu^{(a)} = (1^2)$. By subtracting $\tilde{y}_\nu$ from this, we see that $Y_\lambda = \tilde{y}_\lambda + \sum C a \mu yC$, where $C \subset X_\nu \cap \mathcal{B}_m$. Assume that $\lambda = (\lambda', (1; 1; \cdots ))$ with $\nu^{(a)} = (1^2)$ in $\nu$. Next assume that $\nu = (\lambda', (1; 1; \cdots ))$ with $\nu^{(a)} = (2)$. If $a + 1 = r$, it is easy to see that $Y_\lambda = \tilde{y}_\lambda$. So assume that $a + 1 < r$. We consider $\lambda' = (\cdots ; 1^2; \cdots )$ with $\nu^{(a)} = (\lambda'^{(a+2)} = 1$. Then by a similar consideration as above, $Q_{\lambda} - Q_{\lambda}'$ has supports only on $X_\lambda$ and $X_\nu$ with $\nu$ such that $\nu^{(a)} = (1^2)$. Hence $Y_\lambda$ can be written as in the previous case. Finally assume that $\lambda = (\cdots ; 2; \cdots )$ with $\nu^{(a)} = (2)$. If $a = r$, it is easy to see that $Y_\lambda = \tilde{y}_\lambda$. So assume that $a = r$ and put $\lambda' = (\lambda', (1; 1; \cdots ))$ with $\nu^{(a)} = (1^2) + 1$. As in the previous discussion, $Q_{\lambda} - Q_{\lambda}'$ has supports only on $\lambda$, $\nu = (\lambda', (1; 2; \cdots ))$ with $\nu^{(k)} = (2)$. If $a < k$, or $\nu = (\lambda', (1; 1^2; \cdots ))$ with $\nu^{(a)} = 1$. If $\nu$ is in the former case, by induction we may assume that $Y_\lambda = \tilde{y}_\nu$. By the previous results, we also have $Y_{\tilde{y}_\lambda} = \tilde{y}_\nu$ for the latter $\nu$. Hence we have $Y_{\tilde{y}_\lambda} = \tilde{y}_\nu$. This proves (6.9.2).

Now assume that $\lambda$ is of the form $\lambda = (\cdots ; 1^2; \cdots )$ with $\nu^{(a)} = (\lambda'^{(a+1)} = 1$ if $a < b$. By (6.9.2), $Y_\lambda$ has an additional support only for $\nu = (\cdots ; 1^2; \cdots )$ with $\nu^{(a)} = (1^2)$. Let $\lambda = (\lambda; 1; \cdots ; 1; \cdots ) \in \mathcal{B}(\mathcal{B}_m)$ be as in (*) with $a = k$. Then $X_\nu \cap \mathcal{B}_m = \mathcal{B}_\nu \cup X_{\nu, m'}$. We show that

\[Y_{\tilde{y}_\lambda} = Y_{\tilde{y}_\nu} = Y_{\tilde{y}_\mu}.\]

If $b < \ell$, then $Y_{\tilde{y}_\lambda} = Y_{\tilde{y}_\nu} = 0$. In particular, if $Y_{\tilde{y}_\lambda} \neq Y_{\tilde{y}_\nu}$, then $\lambda \neq \mu$. We now define a total order $\prec$ on $\mathcal{P}_{n,r}$ compatible with the partial order $\leq$ so that $\tilde{A}_q$ is upper triangular. $\mathcal{P}_{n,r}$ is decomposed as $\mathcal{P}_{n,r} = \mathcal{P}_{\nu \in \mathcal{P}_{n,r}} \mathcal{P}[\nu]$, where $\mathcal{P}[\nu] = \{\lambda \in \mathcal{P}_{n,r} | \sum \lambda(i) = \nu\}$. We arrange $\mathcal{P}[\nu]$ according to the total order compatible with the dominance order on $\mathcal{P}_{n,r}$. In our case, $\mathcal{P}_{2,r} = \mathcal{P}[2] \mathcal{P}[1^2]$. We put the lexicographic order on $\mathcal{P}[2]$ and on $\mathcal{P}[1^2]$. We define $\mathcal{P}[2] \prec \mathcal{P}[1^2]$. By (6.9.1), (6.9.2) and (6.9.4), we see that if $Y_{\tilde{y}_\lambda} \neq Y_{\tilde{y}_\nu}$, then $\lambda, \mu \in \mathcal{P}[1^2]$, and $\lambda \neq \mu$. The proposition is proved. \qed
Note that the matrix \( (7.1.1) \)

Thus the condition (A) is equivalent to the formula (6.10.1).

(ii) The matrix \( \tilde{A}_t \) is not diagonal even in the case where \( n = 2, r = 3 \). In fact, assume that \( \lambda = (1; -1), \mu = (1; 1) \) and \( \nu = (1^2; -1) \). Since \( Y^\lambda_{\lambda, r} = 0 \), we can show that \( \tilde{A}_t \) is upper triangular, and so the condition (A) holds. Since \( Q_\lambda(z) = \tilde{p}_\lambda(q) \), the condition \( P_1 = P \) is equivalent to the formula (cf. (6.7.6))

\[
\tilde{K}_{\lambda, \mu}(q) = q^a(\lambda) \sum_{i \geq 0} (-1)^i \text{Tr} (\phi_i^2 Z \mathcal{H}_z \text{IC}(X_\lambda, Q_t)).
\]

Thus the condition (A) is equivalent to the formula (6.10.1).

7. Some examples

7.1. We consider the matrix equation \( P - A^t P^+ = \Omega \) with \( P^\pm = (\tilde{K}_{\lambda, \mu}^\pm(t)) \) as in (5.2.1). In the case where \( n = 1 \), Kostka functions have an interpretation in terms of the polynomials \( \text{IC}_{\lambda, \mu}^\pm(t) \) by Proposition 6.9 (see also Proposition 6.8), namely, the following formula holds.

\[
\tilde{K}_{\lambda, \mu}(t) = t^a(\lambda) \text{IC}_{\lambda, \mu}^\pm(t^r),
\]

\[
\tilde{K}_{\mu, \nu}(t) = t^a(\mu) \text{IC}_{\mu, \nu}^\pm(t^r).
\]

In this case, \( W = G(r, 1, 1) \simeq \mathbb{Z}/r\mathbb{Z} \). Put \( \mathcal{P}_{1,r} = \{\lambda_1, \ldots, \lambda_r\} \), arranged with respect to the dominance order, \( \lambda_1 < \lambda_2 < \cdots < \lambda_r \), where \( \lambda_i = (\lambda_i^{(1)}, \ldots, \lambda_i^{(r)}) \) with \( \lambda^{(r+1-i)} = (1), \lambda^{(j)} = \emptyset \) for \( j \neq r + 1 - i \).

We first consider the simplest case, i.e., the case where \( n = 1 \) and \( r = 3 \). Hence \( \mathcal{P}_{1,3} = \{\lambda_1, \lambda_2, \lambda_3\} \). In this case the equation \( P - A^t P^+ = \Omega \) is given by

\[
\begin{pmatrix}
  t^2 & t & 1 \\
  t & t^2 - t^{-1} & t^2 - t \\
  1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 1 & t \\
  t^2 & t^3 & t^4 \\
  t^3 & t^4 & t^4
\end{pmatrix} = \begin{pmatrix}
  t^4 & t^2 & t^3 \\
  t^3 & t^4 & t^2 \\
  t^2 & t^3 & t^4
\end{pmatrix}.
\]

Note that the matrix \( A \) has non-polynomial entries. However, the left hand side of this equation is changed to the form

\[
P^t A^t P'' = \begin{pmatrix}
  t^2 & t & 1 \\
  t & t^3 & t^3 - 1 \\
  1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & t^2 - 1 & t \\
  t^3 & t^3 - 1 & t^3 - 1 \\
  t^4 & t^4 & t^4
\end{pmatrix} = \begin{pmatrix}
  t^2 & 1 & t \\
  t & t^3 & 1 \\
  1 & 0 & t
\end{pmatrix}.
\]
where \( P' = P^-, P'' = P^+ \Theta^{-1}, A' = \Lambda \Theta \) with a diagonal matrix \( \Theta = \text{Diag}(1, t, t^{-1}) \). In this case, we have
\[
(\text{IC}_{-\lambda, \mu}^-(t^3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (\text{IC}_{-\lambda, \mu}^+(t^3)) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]

We define polynomials \( \xi_{\lambda_1}(t) \) by \( \xi_{\lambda_1}(t) = 1, \xi_{\lambda_2}(t) = t - 1, \xi_{\lambda_3}(t) = t - 1 \). Then we have,
\[
A' = \begin{pmatrix} \xi_{\lambda_1}(t^3) \\ \xi_{\lambda_2}(t^3) \\ \xi_{\lambda_3}(t^3) \end{pmatrix}, \quad \text{and} \quad A^{-1}A' = \begin{pmatrix} 1 & t & t^{-1} \end{pmatrix}.
\]

We have \( (a(\lambda_1), a(\lambda_2), a(\lambda_3)) = (2, 1, 0) \). In our case \( \tau \) is a permutation \( \lambda_1 \leftrightarrow \lambda_1, \lambda_2 \leftrightarrow \lambda_3 \) and so \( (a(\tau(\lambda_1)), a(\tau(\lambda_2)), a(\tau(\lambda_3))) = (2, 0, 1) \). Then \( P^- = P' \) is obtained from \( (\text{IC}_{-\lambda, \mu}^-(t^3)) \) by multiplying \( t^2, t, 1 \) for corresponding rows. In turn, \( P'' \) is obtained from \( (\text{IC}_{-\lambda, \mu}^+(t^3)) \) by multiplying \( t^2, 1, t \) for corresponding rows, and \( P^+ \) is obtained from \( P'' \) by multiplying \( 1, t, t^{-1} \) for corresponding columns. Moreover, we have
\[
\Theta = (t^a(\lambda_1)-a(\tau(\lambda_1)), t^a(\lambda_2)-a(\tau(\lambda_2)), t^a(\lambda_3)-a(\tau(\lambda_3))).
\]

### 7.2.
We consider the general \( r \) with \( n = 1 \). As in 7.1, we consider the relation \( P'A^rP'' = \Omega \) where \( P' = P^-, P'' = P^+ \Theta^{-1}, A' = \Lambda \Theta \) with a diagonal matrix \( \Theta = \text{Diag}(\ldots, t^a(\lambda)-a(\tau(\lambda)), \ldots) \). We give the matrices \( P^\pm, (\text{IC}_{-\lambda, \mu}^+(t^r)), A, A' \) and \( \Theta \). The matrices \( P^\pm \) are given as follows.
\[
P^- = \begin{pmatrix} t^{r-1} \\ t^{r-2} t^{r-2} \\ t^{r-3} t^{r-3} t^{r-3} \\ \vdots \vdots \vdots \vdots \\ t t t t t \\ 1 1 1 1 1 1 \\ \end{pmatrix}, \quad P^+ = \begin{pmatrix} t^{r-1} \\ 1 t^{r-2} \\ t 0 t^{r-3} \\ \vdots \vdots \vdots \vdots \\ t^{r-3} 0 \ldots 0 t \\ t^{r-2} 0 \ldots 0 0 1 \\ \end{pmatrix}.
\]

The diagonal matrix \( A \) is given as
\[
A = \text{Diag}(1, t^2 - t^{-r+2}, t^4 - t^{-r+4}, \ldots, t^{2r-4} - t^{-r-4}, t^{2r-2} - t^{-r-2}).
\]

The permutation of \( S_{1,r} \) is given by \( \lambda_1 \leftrightarrow \lambda_1 \) and \( \lambda_i \leftrightarrow \lambda_{r-i+2} \) for \( i \neq 1 \). Then \( \Theta \) is given by
\[
\Theta = \text{Diag}(1, t^{r-2}, t^{r-4}, \ldots, t^{-r+4}, t^{-r+2}),
\]
and the matrix \( A' = \Lambda \Theta \) is given by
\[
A' = (1, t^r - 1, t^r - 1, \ldots, t^r - 1).
\]

Finally the matrices \( (\text{IC}_{-\lambda, \mu}^+(t^3)) \) are given by
\[(IC^-_{\lambda, \mu}(t^3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ \end{pmatrix}, \quad (IC^+_{\lambda, \mu}(t^3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \end{pmatrix} \]

7.3. Assume that \( W = G(3, 1, 2) \cong S_2 \times (\mathbb{Z}/3\mathbb{Z})^2 \). We arrange the elements in \( P_{3,2} \) in the total order \( \lambda_1 < \lambda_2 < \cdots < \lambda_9 \) compatible with the dominance order, where

\[
\begin{align*}
\lambda_1 &= (-; -; 1^2), & \lambda_2 &= (-; 1^2; -), & \lambda_3 &= (-; -; 2), \\
\lambda_4 &= (1^2; -; -), & \lambda_5 &= (-; 1; 1), & \lambda_6 &= (1; -; 1), \\
\lambda_7 &= (-; 2; -), & \lambda_8 &= (1; 1; -), & \lambda_9 &= (2; -; -).
\end{align*}
\]

Then the matrices \( P^\pm \) are given as follows.

\[
P^- = \begin{pmatrix} t^7 & t^5 & \cdots \\ t^5 & t^3 & \cdots \\ t^4 & 0 & \cdots \\
0 & t^3 & \cdots \\
t^6 + t^3 & t^3 & \cdots \\
t^5 + t^2 & t^2 & \cdots \\
t^2 & t^2 & \cdots \\
t^4 + t & t^4 + t & \cdots \\
1 & 1 & \cdots 
\end{pmatrix},
\]

\[
P^+ = \begin{pmatrix} t^7 & t^5 & \cdots \\ t^3 & t^5 & \cdots \\ t^4 & 0 & \cdots \\
0 & t^4 & \cdots \\
0 & 0 & \cdots \\
t^5 + t^2 & t^4 & \cdots \\
t^6 + t^3 & 0 & \cdots \\
1 & t^2 & \cdots \\
t^4 + t & t^3 & \cdots \\
t^2 & 0 & \cdots 
\end{pmatrix}.
\]

The diagonal matrix \( \Lambda \) is given as

\[
\Lambda = \text{Diag} \begin{pmatrix} 1, & t^{-2}(t^6 - 1), & (t^6 - 1), \\
t^2(t^6 - 1), & t^{-1}(t^3 - 1)(t^6 - 1), & t(t^3 - 1)(t^6 - 1), \\
t(t^3 - 1)(t^6 - 1), & t^3(t^3 - 1)(t^6 - 1), & t^5(t^3 - 1)(t^6 - 1) \end{pmatrix}.
\]
In this case the permutation $\tau$ on $P_{2,3}$ is given as

$$\lambda_2 \leftrightarrow \lambda_4, \quad \lambda_5 \leftrightarrow \lambda_6, \quad \lambda_7 \leftrightarrow \lambda_9, \quad \text{(other } \lambda_j \text{ are fixed)}.$$  

Then $\Theta$ is given by $\Theta = \text{Diag}(1, t^2, 1, t^{-2}, t, t^{-1}, t^2, 1, t^{-2})$, and the matrix $\Lambda' = \Lambda \Theta$ is given by

$$\Lambda' = \text{Diag}(1, (t^6 - 1), (t^6 - 1), (t^6 - 1), (t^3 - 1)(t^6 - 1), (t^3 - 1)(t^6 - 1), (t^3 - 1)(t^6 - 1)).$$

Now $P^\pm = (p_{\lambda, \mu}^\pm(t))$ can be modified to the matrices

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$
In contrast to the cases in 7.1 and 7.2, \((t^{-a(\tau(\lambda))}-a(\mu)+a(\tau(\mu))) p_{\lambda,\mu}^+(t))\) does not coincide with \((IC_{\lambda,\mu}^+(t^3))\).

7.4. We give the table of \(P^\pm\) and \(\Lambda\) for the case where \(n = 3\) and \(r = 3\). We fix the total order on \(\mathcal{P}_{3,3}\) as in the first column of the following table. The diagonal matrix \(\Lambda = (\xi_{\lambda,\lambda})\) and the values of \(a\)-function are given as follows.

| \(\lambda\) | \(a(\lambda)\) | \(\xi_{\lambda,\lambda}\) |
|-------------|----------------|---------------------|
| \((-,-,1^4)\) | 15             | 1                   |
| \((-,-,1^3,-)\) | 12             | \(t^{-3}(t^9 - 1)\) |
| \((-,-,21)\) | 9              | \((t^3 + 1)(t^9 - 1)\) |
| \((1^3,-,-)\) | 9              | \(t^3(t^9 - 1)\)    |
| \((-1,1,1^2)\) | 8              | \(t^{-1}(t^6 - 1)(t^9 - 1)\) |
| \((-,-,-,3)\) | 6              | \(t^3(t^6 - 1)(t^9 - 1)\) |
| \((1,-,1^2)\) | 7              | \(t(t^6 - 1)(t^9 - 1)\) |
| \((-1,1^2,1)\) | 7              | \(t(t^6 - 1)(t^9 - 1)\) |
| \((1^2,-,1)\) | 5              | \(t^5(t^6 - 1)(t^9 - 1)\) |
| \((-21,-)\) | 6              | \(t^3(t^6 - 1)(t^9 - 1)\) |
| \((-1,2)\) | 5              | \(t^2(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((1,1^2,-)\) | 5              | \(t^5(t^6 - 1)(t^9 - 1)\) |
| \((1,-,2)\) | 4              | \(t^4(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((-,2,1)\) | 4              | \(t^4(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((1^2,1,-)\) | 4              | \(t^4(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((-3,-)\) | 3              | \(t^6(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((1,1,1)\) | 2              | \(t^8(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((21,-,-)\) | 3              | \(t^9(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((1,2,-)\) | 2              | \(t^8(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((2,-,1)\) | 2              | \(t^8(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((2,1,-)\) | 1              | \(t^{10}(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
| \((3,-,-)\) | 0              | \(t^{12}(t^3 - 1)(t^6 - 1)(t^9 - 1)\) |
|       | $(-, -, 1^3)$ | $(-, 1^2, -)$ | $(-, -1, 2^1)$ | $(1^2, -, -)$ | $(-, 1, 1^2)$ | $(-, -, 3)$ | $(1, -, 1^2)$ | $(-, 1^2, 1)$ | $(1^2, -, 1)$ | $(-, 21, -)$ |
|-------|---------------|---------------|----------------|--------------|---------------|-------------|---------------|-------------|---------------|-------------|
| $(-, -, 1^3)$ | $t^{15}$ | $t^{12}$ | $t^{12}$ | $t^9$ | $0$ | $t^9$ | $0$ | $0$ | $t^9$ | $t^9$ |
| $(-, 1^3, -)$ | $t^{12}$ | $t^{12}$ | $t^{12}$ | $t^9$ | $0$ | $t^9$ | $0$ | $0$ | $t^9$ | $t^9$ |
| $(-, -, 21)$ | $t^{14} + t^{11} + t^8$ | $t^8$ | $t^8$ | $0$ | $t^8$ | $0$ | $0$ | $t^8$ | $t^8$ | $t^8$ |
| $(-, 1, 1^2)$ | $t^6$ | $0$ | $t^6$ | $0$ | $0$ | $t^6$ | $0$ | $0$ | $t^6$ | $t^6$ |
| $(1^3, -, -)$ | $t^{13} + t^{10} + t^7$ | $t^7$ | $t^7$ | $t^7$ | $0$ | $t^7$ | $0$ | $0$ | $t^7$ | $t^7$ |
| $(-, 1^2, 1)$ | $t^{11} + t^8 + t^5$ | $t^8 + t^5$ | $t^5$ | $t^5$ | $t^5$ | $0$ | $t^5$ | $t^5$ | $t^5$ | $t^5$ |
| $(1^2, -1)$ | $t^9 + t^6$ | $t^9 + t^6$ | $t^6$ | $0$ | $t^6$ | $0$ | $0$ | $t^6$ | $t^6$ | $t^6$ |
| $(-, 21, -)$ | $t^{11} + t^8 + t^5$ | $t^8 + t^5$ | $0$ | $t^5$ | $0$ | $t^5$ | $0$ | $0$ | $t^5$ | $t^5$ |
| $(1, 1^2)$ | $t^{10} + t^7 + t^4$ | $t^7 + t^4$ | $0$ | $t^4$ | $0$ | $t^4$ | $0$ | $0$ | $t^4$ | $t^4$ |
| $(-, 2, 1)$ | $t^{10} + t^7 + t^4$ | $t^{10} + t^7 + t^4$ | $t^7 + t^4$ | $t^4$ | $t^4$ | $t^4$ | $t^4$ | $t^4$ | $t^4$ | $t^4$ |
| $(1^2, 1)$ | $t^{10} + t^7 + t^4$ | $t^3$ | $t^3$ | $0$ | $t^3$ | $0$ | $0$ | $t^3$ | $t^3$ | $t^3$ |
| $(-, 3, -)$ | $t^{12} + 2t^9 + 2t^6 + t^3$ | $t^9 + 2t^6 + t^3$ | $2t^6 + t^3$ | $t^6 + t^3$ | $t^6 + t^3$ | $2t^6 + t^3$ | $t^6 + t^3$ | $t^6 + t^3$ | $t^6 + t^3$ | $t^6 + t^3$ |
| $(21, -1)$ | $t^2 + t^5 + t^2$ | $t^8 + t^5 + t^2$ | $t^5 + t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ |
| $(1, 2, -)$ | $t^8 + t^5 + t^2$ | $t^5 + t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ | $t^2$ |
| $(2, -1)$ | $t^2 + t^4 + t$ | $t^7 + t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ |
| $(2, 1, -)$ | $t^7 + t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ | $t^4 + t$ |
| $(3, -, -)$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
\[
\begin{array}{cccccccccccc}
& (-, 1, 2) & (1, 1^2, -) & (1, -, 2) & (-, 2, 1) & (1^2, 1, -) & (-, 3, -) & (1, 1, 1) & (21, -, -) & (1, 2, -) & (2, -, 1) & (2, 1, -) & (3, -, -) \\
(-, -, 1^3) & & & & & & & & & & & & \\
(-, 1^3, -) & & & & & & & & & & & & \\
(-, -, 21) & & & & & & & & & & & & \\
(1^3, -, -) & & & & & & & & & & & & \\
(-, 1, 1^2) & & & & & & & & & & & & \\
(-, -, 3) & & & & & & & & & & & & \\
(1, -, 1^2) & & & & & & & & & & & & \\
(-, 1^2, 1) & & & & & & & & & & & & \\
(1^2, -, 1) & & & & & & & & & & & & \\
(-, 21, -) & & & & & & & & & & & & \\
(-, 1, 2) & & & & & & & & & & & & \\
(1, 1^2, -) & & & & & & & & & & & & \\
(1, -, 2) & & & & & & & & & & & & \\
(-, 2, 1) & & & & & & & & & & & & \\
(1^2, 1, -) & & & & & & & & & & & & \\
(-, 3, -) & & & & & & & & & & & & \\
(1, 1, 1) & & & & & & & & & & & & \\
(21, -, -) & & & & & & & & & & & & \\
(1, 2, -) & & & & & & & & & & & & \\
(2, -, 1) & & & & & & & & & & & & \\
(2, 1, -) & & & & & & & & & & & & \\
(3, -, -) & & & & & & & & & & & & \\
\hline
& t^5 & t^5 & 0 & t^5 & 0 & t^4 & 0 & t^4 & 0 & t^4 & & \\
\end{array}
\]
| $(-, -, 1^2)$ | $(-, 1^2, -)$ | $(-, -, 21)$ | $(1^2, -, -)$ | $(-, 1, 1^2)$ | $(-, -, 3)$ | $(1, -, 1^2)$ | $(-, 1^2, 1)$ | $(1^2, -, 1)$ | $(-21, -)$ | $(-1, 2)$ | $(1, 1^2, -)$ | $(-, 2, -1)$ | $(1^2, 1, -)$ | $(-3, -)$ | $(1, 1, 1)$ | $(21, -, -)$ | $(1, 2, -)$ | $(2, -1)$ | $(2, 1, -)$ | $(3, -)$ |
|-------------|-------------|-------------|--------------|--------------|-------------|--------------|-------------|-------------|-------------|-------------|--------------|--------------|-------------|-------------|--------------|-------------|--------------|-------------|
| $t^{15}$   | $t^9$       | $t^{12}$   | $t^{12} + t^9$ | $t^6$       | $t^{12}$   | $t^{10} + t^7$ | $t^{10} + t^7$ | $t^6$       | $t^6$       | $t^6$       | $t^{10} + t^7$ | $t^6$       | $t^{10} + t^7$ | $t^{10} + t^7$ | $t^3$       | $t^3$       | $t^3$       | $t^3$ |
| $t^9$      | $t^9$       | $t^9$      | $t^9$         | $t^9$       | $t^9$       | $t^9$         | $t^9$         | $t^9$       | $t^9$       | $t^9$       | $t^9$         | $t^9$       | $t^9$         | $t^9$         | $t^9$       | $t^9$       | $t^9$       | $t^9$ |
| $(-, 1^2)$ | $(-, 21)$   | $(1^2, -)$ | $(1^2, -)$    | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ | $(-, 21, -)$ |

$P_n$ for $n = 3, r \parallel 3$
\[
\begin{array}{ccc}
\text{KOSTKA POLYNOMIALS} & 51 & \\
\hline
P & + & \\
\hline
n = 3 & r = 3 & (continued)
\end{array}
\]

\[
\begin{pmatrix}
-1 & 2 & -1 \\
1 & -1 & 1 \\
-1 & -2 & 1 \\
1 & 2 & 1 \\
-1 & -3 & -1 \\
1 & -1 & 2 \\
-1 & 2 & -1 \\
1 & 2 & 1 \\
-3 & -1 & 2 \\
1 & -1 & 2 \\
-1 & 2 & -1 \\
3 & -1 & 2 \\
1 & -1 & 2 \\
-1 & 2 & -1 \\
0 & 0 & 1
\end{pmatrix}
\]

(continued) \( z = \lambda x = u \) for \( z = t \)
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