Complete classification of second-order symmetric spacetimes

Oihane F Blanco$^{1*}$, Miguel Sánchez$^1$ and José M M Senovilla$^2$

$^1$ Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada
Campus Fuentenueva s/n, 18071 Granada, Spain
$^2$ Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
E-mail: oihane@ugr.es, sanchezm@ugr.es, josemm.senovilla@ehu.es

Abstract. We present the explicit local form of the metric of the second-order symmetric non-symmetric 4-dimensional Lorentzian manifolds. They turn out to be a specific subclass of plane waves.

1. Introduction
The historical roots of the subject under investigation go back to the classification of the locally symmetric spaces in the (proper) Riemannian case by Cartan [2, 3]. Locally symmetric pseudo-Riemannian manifolds are characterized by the condition $\nabla R = 0$, where $R$ denotes the curvature tensor of the manifold. The pseudo-Riemannian manifolds satisfying $\nabla^2 R = 0$ constitute a logical generalization of the locally symmetric ones. But in the Riemannian case these two conditions are equivalent. Moreover, in the Riemannian case:

$$\nabla^m R = 0, m \geq 2 \implies \nabla R = 0,$$  \hspace{1cm} (1)

a result based on the orthogonal de Rham decomposition [10, 15]. In this Riemannian case the natural generalization are the semi-symmetric spaces (introduced in [4], studied in [13, 14]) defined by the relation $\mathcal{R}_{XY} R = 0$ for all vector fields $X, Y$, where $\mathcal{R}_{XY}$ is the curvature operator $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

In the Lorentzian case (1) does not hold, because of the failure of the orthogonal de Rham decomposition [16, 17, 18]. In fact,

$$\nabla^m R = 0, m \geq 2 \implies g(\nabla^{m-1} R, \nabla^{m-1} R) = 0,$$

where the metric $g$ is extended to the inner product between tensor fields in the standard fashion. So in the Lorentzian case we can introduce the second-order symmetric (also called 2-symmetric) spaces [11], i.e., the spaces that satisfy the condition $\nabla^2 R = 0$.

The following relations hold: \{$\nabla R = 0$\} $\subset$ \{$\nabla^2 R = 0$\} $\subset$ \{$\mathcal{R}_{XY} R = 0$\}.

In this contribution we present a specific result from [1]. Namely, we find the 2-symmetric non-symmetric 4-dimensional spacetimes: the explicit local form of the metric, classification and properties. The main result is summarized in the following theorem:
Theorem 1.1 A 2-symmetric non-symmetric 4-dimensional spacetime \((M, g)\) is locally isometric to \(\mathbb{R}^4\) endowed with the metric
\[
ds^2 = -2du(dv + Hdu) + dx^2 + dy^2,
\]
where \(H(u, x, y) = (\alpha_1 u + \beta_1)x^2 + (\alpha_2 u + \beta_2)y^2 + (\alpha_3 u + \beta_3)xy\) for some constants \(\{\alpha_A, \beta_A\}_{A=1,2,3}\) with \(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0\).

2. Sketch of the proof of theorem 1.1
The starting point for this proof is theorem 4.2 in [11]. From this theorem follows that a 2-symmetric non-symmetric spacetime must have a null parallel vector field and therefore its metric is of Brinkmann type:
\[
ds^2 = -2du(dv + Hdu + W_j dx^j) + g_{ij} dx^i dx^j, \quad i, j \in \{2, 3\},
\]
where \(H, W_i\) and \(g_{ij} = g_{ji}\) are functions independent of \(v\), otherwise arbitrary.

Using the 2-symmetry condition, one can prove that \(g_{ij}\) must be locally symmetric on each slice \(u = u_0\) [1]. Since the only locally symmetric 2-dimensional Riemannian spaces are the constant curvature ones, we have that in appropriate local coordinates
\[
g_{ij} dx^i dx^j = \frac{1}{\left(1 + \frac{k(u)}{4} (x^2 + y^2)\right)^2 (dx^2 + dy^2)}.
\]

In fact, \(k(u)\) must vanish. To prove this, we use the Petrov classification [12]. By [7] we know that all the semi-symmetric spacetimes (of dimension 4) are of type \(D, N\) or \(O\). Moreover, as a consequence of this result, it was also proven that the 2-symmetric spacetimes are of Petrov type \(N\), or its degenerate case, \(O\). And this only happens when \(k(u) = 0\).

Hence, the line element of a 2-symmetric non-symmetric 4-dimensional spacetime \((M, g)\) simplifies to
\[
ds^2 = -2du(dv + Hdu + W_2 dx + W_3 dy) + dx^2 + dy^2,
\]
From now on, an overdot means derivative with respect to \(u\), \(f, x = \partial f / \partial x\) and \(f, y = \partial f / \partial y\).

Using the Cartan method (see, for example, [5]) to calculate the connection 1-forms and the curvature 2-forms for the null coframe \(\{\theta^0, \theta^1, \theta^2, \theta^3\} = \{du, dv + Hdu + W_2 dx + W_3 dy, dx, dy\}\) and solving the 2-symmetry equations in this frame, we conclude that there exist functions \(\{a_{rs}(u), b_{rs}(u)\}_{r,s=0,1,2}\), \(\omega(u, x, y), w_2(u, v), w_3(u)\) and \(r(u)\) such that
\[
W_2(u, x, y) = \omega_{,x}(u, x, y) + w_2(u)y, \quad \tag{2}
\]
\[
W_3(u, x, y) = \omega_{,y}(u, x, y) + w_3(u)x, \quad \tag{3}
\]
\[
H(u, x, y) = \frac{a_{10}}{2} x^2 + \frac{b_{11}}{2} y^2 + b_{10}xy + a_{00}x + b_{00}y + r(u) + Q(u, x, y) \quad \tag{4}
\]
where \(Q_{,y}(u, x, y) = \dot{W}_3(u, x, y)\), and the following equations must be satisfied:
\[
-\ddot{a}_{10} + 2g\ddot{y} + 2\dot{g}a_{10} - 2g^2(b_{01} - a_{10}) + 4g\dot{a}_{01} = 0, \quad \tag{5}
\]
\[
\dot{b}_{01} - 2g\ddot{y} + 2g\dot{b}_{10} + 4g\dot{b}_{10} - 2g^2(b_{01} - a_{10}) = 0, \quad \tag{6}
\]
\[
\ddot{a}_{01} + \ddot{g}g^2 + g(a_{10} - b_{01}) - 2g^2a_{01} - 2g(a_{10} - b_{01}) = 0, \quad \tag{7}
\]
\[
b_{10} - a_{01} = 2g, \quad \tag{8}
\]

2
\[ g(u) = \frac{1}{2}(W_{2y} - W_{3x}). \]  

(9)

The proof can then be completed by noticing that Eqs. (2), (3) and (4) together with (5)–(9) permit the coordinate change needed to arrive at the final form of the metric. This change of coordinates is given by

\[ u' = u + u_0, \]
\[ v' = v + F(u, x, y), \]
\[ x' = \cos(\theta(u)) x - \sin(\theta(u)) y + B_2(u), \]
\[ y' = \sin(\theta(u)) x + \cos(\theta(u)) y + B_3(u). \]

It is easily verified that the functions \( \theta, B_2, B_3 \) and \( F \) can be chosen such that the line-element becomes

\[ ds'^2 = -2du'(dv' + H'du') + (dx')^2 + (dy')^2, \]  

(10)

where

\[ H'(u', x', y') = p_1(u')(x')^2 + p_2(u')(y')^2 + p_3(u')x'y' \]  

(11)

for some new functions \( \{p_A(u')\}_{A=1,2,3} \), because the integrability conditions of the resulting differential equations are always met by virtue of equations (2)–(9).

To finalize the proof of theorem 1.1, one simply has to solve the equations (5)-(9) appropriately restricted to the new form of the metric (10) with (11). This readily implies that \( p_1(u'), p_2(u') \) and \( p_3(u') \) must be linear functions:

\[ p_A(u') = \alpha_A u' + \beta_A, \quad \alpha_A, \beta_A \in \mathbb{R}, \quad A \in \{1, 2, 3\}. \]

The condition \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0 \) must be enforced as otherwise \( \nabla R = 0 \) and the spacetime would be locally symmetric.

3. Concluding remarks

We would like to stress that, even though the family of metrics obtained in Theorem 1.1 depend on six parameters, a further analysis shows that only four of them are essential parameters, and the other two can be removed by the remaining freedom of coordinates.

Similarly, we want to remark that our result is local. A global classification is also feasible, though some global hypotheses on the spacetime seem unavoidable to that end. Such kind of hypotheses may involve the topology of the manifold, geodesics or causality (see [6, 8, 9] for a study of the last two issues in some wave-type spacetimes). In particular, it seems that the local solutions obtained in Theorem 1.1 are also the unique global 4-dimensional solutions under the requirements of simple connectedness and geodesic completeness.

An open question is whether or not the above pattern is maintained for higher-order symmetric 4-dimensional spacetimes. Specifically, we wonder if the \( m \)-symmetric spacetimes (those with \( \nabla^m R = 0 \)) are given, for general \( m \), precisely by the line-element (10) with (11) where the functions \( p_1(u'), p_2(u') \) and \( p_3(u') \) are polynomials of degree \( m \).

A more important open question is the extension of the result to higher dimensions. Of course, many more possibilities appear in this general case, which are being analyzed in [1].

Acknowledgments

JMMS is supported by grants FIS2004-01626 (MICINN) and GIU06/37 (UPV/EHU). OFB and MS are partially supported by grants P06-FQM-01951 (J. Andalucía) and MTM2007-60731 (Spanish MEC with FEDER funds).
References

[1] Blanco O F, Sánchez M and Senovilla J M M 2010 2nd-order symmetric Lorentzian manifolds II: classification and explicit local form in preparation
[2] Cartan É 1926 Sur une classe remarquable d’espaces de Riemann Bull. Soc. Math. France 54 214-64
[3] Cartan É 1927 Sur une classe remarquable d’espaces de Riemann Bull. Soc. Math. France 55 114-34
[4] Cartan É Leçons sur la Géométrie des Espaces de Riemann 2nd edn (Paris: Gauthier-Villars)
[5] Chandrasekhar S The Mathematical Theory of Black Holes (Oxford University Press, Oxford, 1983)
[6] Candela A M, Flores J L and Sánchez M 2003 On general plane fronted waves. Geodesics Gen. Relat. Gravit. 35 631-649.
[7] Eriksson I and Senovilla J M M 2010 Note on (conformally) semi-symmetric spacetimes Class. Quantum Grav. 27 027001 (Preprint arXiv:0908.3246)
[8] Flores J L and Sánchez M 2003 Causality and conjugate points in general plane waves Class. Quantum Gravit. 20 2275-2291
[9] Flores J L and Sánchez M 2008 The causal boundary of wave-type spacetimes J. High Energy Phys. no. 3 036 43
[10] Nomizu K and Ozeki H 1962 A theorem on curvature tensor fields Proc. Nat. Acad. Sci. USA 48 206-7
[11] Senovilla J M M 2008 2nd-order symmetric Lorentzian manifolds I: characterization and general results, Class. Quantum Grav. 25 245011
[12] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E, Exact Solutions of Einstein’s Field Equations, 2nd ed. (Cambridge University Press., Cambridge, 2003)
[13] Szabó Z I 1982 Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$: I. The local version J. Differ. Geom. 17 531-82
[14] Szabó Z I 1985 Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$: I. The global versions Geom. Dedicata 19 65-108
[15] Tanno S 1972 Curvature tensors and covariant derivatives Ann. Mat. Pura Appl. 96 233-41
[16] Wu H 1964 On the de Rham decomposition theorem Illinois J. Math. 8 291-311
[17] Wu H 1964 Decomposition of Riemannian manifolds Bull. Am. Math. Soc. 70 610-7
[18] Wu H 1967 Holonomy groups of indefinite metrics Pacific J. Math. 20 351-92