Running coupling constant of background perturbation theory and lattice interquark interaction.

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Abstract

The conventional parametrizations of lattice static interquark force are shown to produce a mismatch of coupling constant near the matching point $R_m \approx 0.2\text{fm}$. The running coupling constant of the background perturbation theory yields instead a selfconsistent description of lattice data over all interval of distances, $0.05\text{fm} \lesssim R \lesssim 1.1\text{fm}$, and is an analytic function with the property of asymptotic freedom at small $R$ and it is freezing at large $R$. 
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1. Recently the interaction between static quark and antiquark was intensively studied in lattice version of pure gauge $SU(3)$ and $SU(2)$ theories [1-6]. These investigations seem to be very important because just in lattice approach both perturbative (P) and nonperturbative (NP) effects and their interference can be measured and thus we have an opportunity to get an information out of the framework of pure P theory. To this end static potential $V(R)$, or force $F(R) = \frac{dV}{dR}$, has to be measured over large interval of interquark separations $R$ which is possible only on large volume lattices. Such lattice data were presented in [4, 5] for $SU(3)$ gauge theory and here we shall mostly discuss results obtained in [4] (lattice $36^4$ at $\beta = 6.5$, $a^{-1} = 4.13$ GeV or lattice spacing $a \approx 0.048$ fm) where the force was measured on the large interval $a \lesssim R \lesssim 23a$ or $0.05$ fm $\lesssim R \lesssim 1.1$ fm.

At small distances, $R \lesssim 0.2$ fm, the regime of asymptotic freedom was observed and lattice data were well described by only $P$ contribution $F_P(R) = \frac{4}{3} \frac{\alpha_F(R)}{R^2}$; (1) with the effective running coupling constant (c.c.) $\alpha_F(R)$ defined by two-loop expression which was taken in [4, 5] in the form:

$$\alpha_F(R) = \frac{4\pi}{b_0 t} \left[ 1 + \frac{b_1}{b_0} \ln \frac{t}{t} \right]^{-1}, \quad t = \ln \frac{1}{(\Lambda R R)^2}. \quad (2)$$

Here $b_0 = 11$ and $b_1 = 102$ are the usual coefficients for the $\beta$-function with the number of flavours $n_f = 0$.

The "experimental" lattice points $\alpha(R)$ were defined in following way

$$\alpha \left( \frac{1}{2}(R_1 + R_2) \right) = \frac{3}{4} R_1 R_2 \frac{V_c(R_1) - V_c(R_2)}{R_1 - R_2}, \quad (3)$$

where $V_c(R)$ is the corrected static potential which takes into account the lattice artefacts responsible for the lack of rotational invariance. If we accept that NP contribution is small at small separations, $R < 4a$, then $\alpha(R)$ (3) has to coincide with $\alpha_F(R)$ in (1). Below we give values of $\alpha(R)$ at two points $R/a$ which are important for our further discussion (the numbers are taken from Table 3 in paper [4]):

$$\frac{R}{a} = 3.3839 \quad \text{or} \quad R \approx 0.16 \, fm, \quad \alpha(R) = 0.317 \quad (10) \quad (2)$$

$$\frac{R}{a} = 3.9241 \quad \text{or} \quad R \approx 0.19 \, fm, \quad \alpha(R) = 0.333 \quad (6) \quad (1)$$
From the lattice analysis [4] it is clear that these two points $R/a$ lie just near the boundary of the region (denote it as $R_m = 4a \cong 0.191$ fm) where P contribution (1) dominates. At $R \gtrsim 4a$, when both P and NP contributions are becoming important, another parametrization is widely used [4-6], it corresponds to the Cornell potential type of interaction. This force can be written (in lattice units) as

$$F(R) = a^2 \frac{\Delta V}{\Delta R} = \frac{E}{(R/a)^2} + Ka^2, \quad R \gtrsim 4a$$ (5)

and from lattice fit the following values of $K$ and $E$ were obtained

$$E = 0.278 \quad (7), \quad Ka^2 = 0.0114 \quad (2)$$ (6)

The string tension $K$ in (5) can be found from (6) and for $a^{-1} = 4.13$ GeV it gives $\sqrt{K} = 440$ MeV. Also from (6) it follows that the asymptotic value of c.c. is

$$\alpha_F(\text{sym}) = \frac{3}{4} E = 0.2085 \quad (5),$$

which we can compare with c.c. values (4) near the matching point $R_m = 4a$. From this comparison one can see that the asymptotic value (7) is about 40% less than c.c. $\alpha(R_m) \cong 0.33$ which was directly measured on the lattice. Thus if the parametrization (5) is used at $R \gtrsim 4a$ and the purely P force (1) is taken at $R < 4a$ then the resulting function $\alpha_F(R)$ is discontinuous.

To overcome this difficulty and get selfconsisent description of static lattice potential or force we suggest to use another approach which takes into account the behaviour of running c.c. in background vacuum fields. As it was shown by Yu.Simonov in [7, 8] this running c.c., $\alpha_B(R)$, is an analytical function at all distances $R$ and its expression simplifies at small and large distances. At small distances, $R^{-2} \gg \Lambda^2_R$, $\alpha_B(R)$ coincides with standard P coupling constant, i.e. manifests the property of asymptotic freedom. At large values of $t_B$ (see formula (9)) the following expression for $\alpha_B(R)$ was obtained,

$$\tilde{\alpha}_B(R) = \frac{4\pi}{b_0 t_B} \left( 1 - \frac{b_1}{b_0^2} \frac{ln \, t_B}{t_B} \right), \quad Rm_B \gg 1$$ (8)

Here

$$t_B = ln \frac{1 + R^2 m_B^2}{\Lambda^2_R R^2}$$ (9)

and $m_B$ is a characteristic background mass which defines the asymptotic (exponential) falling off of self-energy part of two-point function $\Pi(x, y)$. This mass can depend on the process or channel considered and, in particular, in $e^+e^-$ annihilation $m_B$ coincides with the lowest hybrid mass, $M_h(0^{++})[8]$. The lattice measurements [4] has given $M_h(0^{++}) \cong 1.5$ GeV.

In this letter we analyze $\alpha_B(R)$ as obtained from static $Q\bar{Q}$ force. On the lattice the static interaction was measured for Euclidean time $T_E$ which is usually not large, e.g. in [4] lattice measurements were done at $T_E \cong 3a / 5a \sim 0.2$ fm and for such $T_E$ quark and antiquark are situated rather close to each other. In this case background mass $m_B$ can be smaller $M_h$ but larger than two constituent gluon masses $M(2g) = 2M_g \approx 1.0$ GeV [8]. In our analysis we shall use two different values of $m_B$:

$$m_B^{(1)} = M_h(0^{++}) = 1.5 GeV, \quad m_B^{(2)} = M(2g) = 1.0 GeV$$ (10)
From (8) one can see that this expression, derived at large $t_B$, has a correct $P$ behaviour at small $R$, i.e. has the property of asymptotic freedom. The important second feature of $\alpha_B(R)$ is that it has no infrared singularity and can be applied at any distances. Also the expression (8) gives the correct limit at large $R$ when $\alpha_B(R)$ tends to a constant value as it was shown in [7]. For this reasons we can use (8) as a convenient interpolation formula over all interval of interquark separations $R$.

Now to coordinate our analysis with lattice approach we introduce, instead of (8), slightly modified expression for $\alpha_B(R)$,

$$\alpha_B(R) = \frac{4\pi}{b_0 t_B} \left( 1 + \frac{b_1}{b_0^2} \ln \frac{t_B}{t_B} \right)^{-1},$$  

(11)

which is going over into the $P$ expression (2) at $m_B = 0$. In (11) $t_B$ is defined by (9). From (11) one can easily see that $\alpha_B(R)$ is freezing at large distances, $\alpha_B(R) \to const$ because $t_B \to const$ if $m_B^2 R^2 \gg 1$:

$$t_B(R \to \infty) \to t_\infty = \ln \frac{m_B^2}{\Lambda^2_R}.$$  

(12)

The asymptotic value of $\alpha_B(R)$ is given by

$$\alpha_B(\text{asym}) = \frac{4\pi}{b_0 t_\infty} \left( 1 + \frac{b_1}{b_0^2} \ln \frac{t_\infty}{t_\infty} \right)^{-1}$$  

(13)

To find $\alpha_B(\text{asym})$ we should know, besides $m_B (10)$, also $\Lambda_R$. Here we shall choose $\Lambda_R$ to be very close to those values $\Lambda_R$ which were obtained in [4], $\Lambda_{\pi MS} = 256(20)$MeV, and in [5], $\Lambda_{\pi MS} = 293 (18)$ MeV. As it is well known $\Lambda_{\pi MS} = 1.048\Lambda_{\pi MS}$ i.e. $\Lambda_R$ defining the force and $\Lambda_{\pi MS}$ practically coincide within lattice errors.

Then in the presence of background fields the force $F(R)$ can be presented as a sum

$$F(R) = F^{(PB)}(R) + F^{NP}(R),$$  

(14a)

where instead of pure $P$ term (1) we introduce the modified term,

$$F^{PB}(R) = \frac{4}{3} \frac{\alpha_B(R)}{R^2}$$  

(14b)

with $\alpha_B(R)$ taken in the form (11).

Some remarks are needed about our choice of NP part of interaction. We shall consider two possibilities: case A, when NP potential is defined by linear potential at all distances or $F^{NP}(R) = K a^2 = const$ as in (5), and case B, when NP interaction is defined through bilocal correlators $D(x)$ with a finite correlation length $T_g [11]$. The first measurements of correlators and $T_g$ were done in [12] where the value $T_g \approx 0.2 \text{fm}$ (or $\delta = T_g^{-1} \approx 1 \text{Gev}$) was found. Note that the case A corresponds to vanishing correlation length $T_g$, or $\delta \gg 1 \text{Gev}$. The potential and force, corresponding to a finite $T_g$ and the exponential form of the correlator $D(x)$ were calculated in [13],

$$F^{NP}(R) = a^2 \frac{d\xi}{dR} = 2a^2 \int_0^\infty dv \int_0^R d\lambda D(\sqrt{\lambda^2 + v^2}) = \frac{2}{\pi} Ka^2 \int_0^{R_\delta} tK(t)dt$$  

(15)
where $K_1(t)$ is the modified Bessel function. From (15) one can see that (i) if $T_g$ is finite then
the force is diminishing ($F(R) \sim R$) at $R \to 0$; (ii) $F(R)$ is approaching the constant value, $Ka^2$, 
only at distances $R_{form} \gtrsim 3T_g \sim 0.6$fm. This second statement is in agreement with Bali, 
Schilling result [14] that the formation of string takes place at distances $R_{form} \sim 0.6$fm.

Now we give two sets of parameters used for our calculations,
case A:

$$\Lambda_A = 240 MeV, \quad m_B = 1.5 GeV,$$
$$\alpha_B(asym) = 0.240; \quad e_A = \frac{4}{3}\alpha_B(asym) = 0.320; \quad F^{NP}(R) = Ka^2 = 0.0114 \quad (16)$$

case B:

$$\Lambda_B = 280 MeV, \quad m_B = 1.0 GeV$$
$$\alpha_B(asym) = 0.343; \quad e_B = \frac{4}{3}\alpha_B(asym) = 0.457; \quad F^{NP}(R) \text{ is given by } (15) \quad (17)$$

with $\delta = 1$GeV or $T_g = 0.2$fm

Our calculations of the force (the cases A and B) at $R < 4a$ are presented in table 1 together 
with lattice data from [4] and the P calculations of $F^P(R)$ with $\Lambda_R = 289$ MeV, which slightly 
differs from the value $\Lambda_{MPS} = 256(20)$MeV or $\Lambda_R = 268$MeV which was predicted in [4]. We 
have used $\Lambda_R = 289$ MeV because for this value we get very good $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{52}{8})$ whereas for

$\Lambda_R = 268$MeV $\frac{\chi^2}{N_{d.o.f.}} = \frac{200}{8}$ is too large. At $R < 4a$ our calculations of $F(R)$ give also a good description of lattice data: in case A we get $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{70}{8})$ and in case B $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{45}{8}$ is better (see table 1).

At medium and large separations, $R \gtrsim 4a$, the values of $F(R)$ (the cases A and B) are 
presented in table 2 together with 15 values found on the lattice. In both cases we have 
have obtained a good agreement with lattice data. In case A we get $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{116}{15}$ and in case B $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{160}{15} which should be compared with $\chi^2(\frac{\chi^2}{N_{d.o.f.}} = \frac{125}{15}$ obtained in [4] for the parametrization 
(5) corresponding Cornell potential.

Thus we can conclude that with running c.c. of background perturbation theory it is possible to 
provide a good description of lattice static interaction over all interval of separations $R$, 
$a \leq R \leq 23a$ or 0.05fm$\lesssim R \lesssim 1.1$fm. It holds in both cases, A and B. The c.c. $\alpha_B(R)$ 
strongly depends on the parameters $m_B$ and $\Lambda_R$ (mostly on $(\frac{m_B}{\Lambda_R})^2$) but still we are not able 
to distinguish between two sets of these parameters. There are two reasons for it. First, the 
lattice errors are still too large in [4] and, secondly, the fitted values of $m_B$ and $\Lambda_R$ depend indirectly on NP contribution to the force. At present this $F^{NP}(R)$ is not known unambiguously.

The constant force, $F^{NP}(R) = Ka^2$, which is widely used in lattice analysis, corresponds to 
vanishing $T_g \ (T_g = 0)$ whereas the finite value of $T_g \sim 0.2fm$ was obtained in [12]. So it would 
be important to determine $F^{NP}(R)$ from the lattice measurements in unambiguous way. Then, 
having more precise lattice data on the force and knowing $T_g$ with good accuracy, one has an 
opportunity to determine unambiguously $m_B$ and $\Lambda_R$.

This $\alpha_B(R)$ is an analytical function of $R$ at any distances and freezing at large separations. 
The main difference between cases A and B considered here is in the asymptotic value of 
$\alpha_B(R)$ or in constant $e$: $e_A = 0.32$ and $e_B \cong 0.46$, i.e. $e_B$ is about 40% larger than $e_A$. 

It is interesting to note that our $e_A$ is very close to the fitted values of $e = 0.31 \div 0.315$ obtained in [3] for static potential with linear confining term which corresponds to our case A. On the other hand our value of $e_B \approx 0.46$ is close to Coulomb parameter $e_{\text{phen}}$, usually used in phenomenological potential models, e.g. to describe heavy quarkonia spectra the values of $e_{\text{phen}}(n_f = 3) \sim 0.45 \div 0.55$ are needed. Notice also that the fitted values of $e_A$ and $e_B$ found here are about $(30 \div 60)\%$ larger than the value of $e(\text{string}) = \frac{\pi}{12} \approx 0.262$ predicted in bosonic string theory at large distances [3].

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Table 1. Lattice data and theoretical calculations of the force $F(R)$ at small distances, $R < 4a$.

| $R/a$ | PT $\Lambda_R = 289\text{MeV}^{a)}$ | Case A $\Lambda_R = 240\text{MeV}^{b)}$ | Case B $\Lambda_R = 280\text{MeV}^{c)}$ | Lattice data from $[4]^{d)}$ |
|-------|---------------------------------|---------------------------------|---------------------------------|------------------------|
| 1.2071 | 0.1662                          | 0.1623                          | 0.1639                          | 0.1607 (55)            |
| 1.7071 | 0.0956                          | 0.0944                          | 0.0940                          | 0.0930(23)             |
| 2.1180 | 0.0686                          | 0.0685                          | 0.0674                          | 0.0664(48)             |
| 2.5322 | 0.0527                          | 0.0530                          | 0.0517                          | 0.0523(6)              |
| 2.9142 | 0.0432                          | 0.0438                          | 0.0424                          | 0.0424(57)             |
| 3.0811 | 0.0400                          | 0.0404                          | 0.0393                          | 0.0368(49)             |
| 3.3839 | 0.0353                          | 0.0361                          | 0.0348                          | 0.0371(14)             |
| 3.9241 | 0.0292                          | 0.0302                          | 0.0287                          | 0.0290(6)              |

a) calculations in perturbative theory with $\Lambda_R = 289\text{MeV}$;
b) $m_B = 1.5\text{GeV}$, $Ka^2 = 0.0114$;
c) $m_B = 1.0\text{GeV}$ and $F^{NP}(R)$ is given by (15) with $Ka^2 = 0.0114$, $\delta = 1\text{GeV}$;
d) lattice errors are found from the paper $[4]$, Table 3 ($\beta = 6.5$; $a^{-1} = 4,13\text{GeV}$).
Table 2. Lattice data and theoretical calculations of force $F(R)$ at large distances, $R \geq 4a$.

| $R/a$ | Case A$^a$ $\Lambda_R = 240\text{MeV}$ | Case B$^b$ $\Lambda_R = 280\text{MeV}$ | Lattice data$^c$ from [4] |
|-------|--------------------------------|--------------------------------|---------------------------|
| 4.2361 | 0.0278                        | 0.0268                        | 0.0286(8)                 |
| 5.0645 | 0.0231                        | 0.0218                        | 0.0226(5)                 |
| 5.8284 | 0.0204                        | 0.0191                        | 0.0196(19)                |
| 6.1623 | 0.0194                        | 0.0183                        | 0.0190(16)                |
| 6.7678 | 0.0181                        | 0.0171                        | 0.0174(9)                 |
| 7.6056 | 0.0167                        | 0.0159                        | 0.0161(10)                |
| 8.2426 | 0.0160                        | 0.0152                        | 0.0161(11)                |
| 9.00   | 0.0152                        | 0.0146                        | 0.0149(3)                 |
| 11.00  | 0.0139                        | 0.0136                        | 0.0130(4)                 |
| 13.00  | 0.0133                        | 0.0131                        | 0.0134(5)                 |
| 15.00  | 0.0128                        | 0.0127                        | 0.0121(6)                 |
| 17.00  | 0.0125                        | 0.0126                        | 0.0132(6)                 |
| 19.00  | 0.0123                        | 0.0124                        | 0.0117(6)                 |
| 21.00  | 0.0121                        | 0.123                         | 0.0125(8)                 |
| 23.00  | 0.0120                        | 0.0122                        | 0.0126(7)                 |

a) see footnote (b) in Table 1;
b) see footnote (c) in Table 1;
c) lattice data for $a^2\frac{\Delta V}{\Delta R}$ are taken from [4], Table 1.