ON THE NONUNIQUENESS AND INSTABILITY OF SOLUTIONS OF TRACKING-TYPE OPTIMAL CONTROL PROBLEMS

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Abstract. We study tracking-type optimal control problems that involve a non-affine, weak-to-weak continuous control-to-state mapping, a desired state $y_d$, and a desired control $u_d$. It is proved that such problems are always nonuniquely solvable for certain choices of the tuple $(y_d, u_d)$ and instable in the sense that the set of solutions (interpreted as a multivalued function of $(y_d, u_d)$) does not admit a continuous selection.

1. Introduction. This paper is concerned with the uniqueness and the stability of solutions of tracking-type optimal control problems of the form

$$\begin{align*}
\min_{(y,u)\in Y\times U} & \quad \|y - y_d\|_Y^p + \|u - u_d\|_U^p \\
\text{s.t.} & \quad y = S(u).
\end{align*}$$

(P)

Our standing assumptions on the quantities in (P) are as follows:

Assumption 1.1.
(i) $(Y, \|\cdot\|_Y)$ and $(U, \|\cdot\|_U)$ are uniformly convex, uniformly smooth Banach spaces,
(ii) $p \in (1, \infty)$ is arbitrary but fixed,
(iii) $y_d \in Y$ and $u_d \in U$ are given,
(iv) $S: U \to Y$ is a function that is not affine-linear and satisfies

$u_n \xrightarrow{n\to\infty} u \text{ in } U \implies S(u_n) \xrightarrow{n\to\infty} S(u) \text{ in } Y.$

Here, the symbol "$\xrightarrow{}$" denotes weak convergence.

Due to their simple structure and since they allow to easily construct situations with known analytic solutions (just choose $u_d := \bar{u}$ and $y_d := S(\bar{u})$ for some given $\bar{u} \in U$), tracking-type optimal control problems of the form (P) are considered very frequently in the literature - in particular in the case where the exponent $p$ is equal to two and the spaces $Y$ and $U$ are Hilbert. Compare, for instance, with [1, 4, 8, 11, 14, 15, 16] and the tangible examples in Section 2 in this context. Very recently, it

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was demonstrated in [24] by means of an explicit construction for a boundary control problem with \(u_d = 0\) governed by a semilinear elliptic partial differential equation that problems of the type \(\text{P}\) can possess multiple global solutions. The aim of this brief note is to point out that tracking-type optimal control problems which involve a desired state \(y_d\), a desired control \(u_d\), and a non-affine, weak-to-weak continuous control-to-state map \(S: u \mapsto y\) are indeed always nonuniquely solvable for certain choices of the tuple \((y_d, u_d)\) - regardless of whether the control-to-state operator arises from a partial differential equation, a variational inequality, a differential inclusion or something else. We further demonstrate that the same effects that are responsible for this nonuniqueness of solutions also cause the problem \(\text{P}\) to be instable in the sense that the set of solutions of \(\text{P}\) (interpreted as a multivalued map of \((y_d, u_d))\) does not admit a continuous selection. For the main results of this note, we refer the reader to Theorems 2.2 and 2.3.

Although, at the end of the day, just consequences of classical results from nonlinear approximation theory and a simple identification with a metric projection, we believe that the observations made in this paper are of sufficient interest to justify pointing them out and making them available in a tangible format - in particular due to their very general nature and their potential consequences for, e.g., the study of turnpike properties, cf. the discussion in [24] and the references therein. We remark that, for the special case of trajectory control problems, arguments analogous to those in this note have already been used in [10, 21, 27].

The organization of the remainder of this paper (i.e., Section 2) is as follows: In Proposition 2.1, we briefly check that the problem \(\text{P}\) possesses at least one global solution for every choice of the tuple \((y_d, u_d)\). The subsequent Theorem 2.2 then establishes our first main result - the nonuniqueness of solutions of \(\text{P}\) for certain choices of the desired state and the desired control. Afterwards, in Theorem 2.3, we demonstrate that the solution set of \(\text{P}\) indeed does not admit a selection which depends continuously on the problem data \((y_d, u_d)\). The paper concludes with some additional comments and tangible examples in Remark 2.4 and Examples 2.5 to 2.8.

2. Nonuniqueness and Instability of Solutions. Before we turn our attention to our main observations, we note the following:

**Proposition 2.1 (Existence of Global Minimizers).** In the situation of Assumption 1.1, the minimization problem \(\text{P}\) admits at least one global solution \((\bar{y}, \bar{u}) \in Y \times U\) for every choice of the tuple \((y_d, u_d) \in Y \times U\).

**Proof.** The claim follows straightforwardly from the direct method of calculus of variations. Indeed, if we consider a minimizing sequence \(\{(y_n, u_n)\}_{n \in \mathbb{N}} \subset Y \times U\) of \(\text{P}\), then the sequences \(\{y_n\}_{n \in \mathbb{N}} \subset Y\) and \(\{u_n\}_{n \in \mathbb{N}} \subset U\) are trivially bounded by the structure of the objective function of \(\text{P}\), and it follows from our assumption of uniform convexity and the theorems of Milman-Pettis and Banach-Alaoglu, see [23] and [26, Section V-2], that the spaces \(Y\) and \(U\) are reflexive and that we may extract a subsequence of \(\{(y_n, u_n)\}_{n \in \mathbb{N}}\) (for simplicity denoted by the same symbol) such that \(\{y_n\}_{n \in \mathbb{N}}\) converges weakly in \(Y\) to some \(\bar{y} \in Y\) and \(\{u_n\}_{n \in \mathbb{N}}\) converges weakly in \(U\) to some \(\bar{u} \in U\). Note that the weak-to-weak continuity of \(S\) implies that \(\bar{y} = S(\bar{u})\) has to hold. In combination with the weak lower semicontinuity of continuous and
convex functions, see [5, Corollary 4.1.14], it now follows immediately that
\[
\inf_{(y,u) \in Y \times U, y = S(u)} \left\| y - y_d \right\|_Y^p + \left\| u - u_d \right\|_U^p = \lim_{n \to \infty} \left\| y_n - y_d \right\|_Y^p + \left\| u_n - u_d \right\|_U^p \\
\geq \left\| \bar{y} - y_d \right\|_Y^p + \left\| \bar{u} - u_d \right\|_U^p.
\]
This shows that \((\bar{y}, \bar{u})\) is a global solution of \((P)\) and completes the proof. \(\square\)

We are now in the position to prove our first main result:

**Theorem 2.2 (Nonuniqueness of Global Minimizers).** In the situation of Assumption 1.1, there always exists a tuple \((y_d, u_d) \in Y \times U\) such that the problem \((P)\) possesses more than one global solution.

**Proof.** The main idea of the proof is to identify \((P)\) with a metric projection problem onto the graph of the control-to-state mapping \(S\), i.e., the set
\[
M := \{(S(u), u) \mid u \in U\} \subset Y \times U
\]
and to subsequently invoke classical results on the convexity of Chebyshev sets. To pursue this approach, we argue by contradiction.

Assume that the minimization problem \((P)\) possesses for each tuple \((y_d, u_d) \in Y \times U\) precisely one global solution \((\bar{y}, \bar{u}) \in Y \times U\). Then, the monotonicity of the function \(0, \infty) \ni x \mapsto x^{1/p} \in [0, \infty)\) implies that, for every \((y_d, u_d)\), the unique global minimizer of \((P)\) is also the sole solution of the problem
\[
\min_{(y, u) \in M} \left\| (y, u) - (y_d, u_d) \right\|_{Y \times U},
\]
where \(M\) is the set in (1) and where \(\cdot\) is the norm on \(Y \times U\) defined by
\[
\|(y, u)\|_{Y \times U} := \left\| y \right\|_Y^p + \left\| u \right\|_U^p \quad \forall (y, u) \in Y \times U.
\]
Note that the space \(Y \times U\) endowed with the norm \(\cdot\) is trivially Banach, and that [7, Theorem 1] and our assumptions on \(U\) and \(Y\) imply that \((Y \times U, \cdot)\) is uniformly convex. Further, the space \((Y \times U, \cdot)\) is also uniformly smooth. Indeed, from [20, Theorem 5.5.12], we obtain that the uniform smoothness of the spaces \((Y, \cdot)\) and \((U, \cdot)\) is equivalent to the uniform convexity of the duals \((Y^*, \cdot)\) and \((U^*, \cdot)\), and, using a standard calculation, it is easy to check that the dual of \((Y \times U, \cdot)\) is isometrically isomorphic to the product space \(Y^* \times U^*\) endowed with the norm
\[
\|(y^*, u^*)\|_{Y^* \times U^*} := \left(\left\| y^* \right\|_{Y^*}^{p/(p-1)} + \left\| u^* \right\|_{U^*}^{p/(p-1)} \right)^{(p-1)/p} \quad \forall (y^*, u^*) \in Y^* \times U^*.
\]
In combination with [7, Theorem 1], the above implies that \((Y^* \times U^*, \cdot)\) is uniformly convex, and, by [20, Proposition 5.2.7 and Theorem 5.5.12], that the space \((Y \times U, \cdot)\) is uniformly smooth as claimed.

Taking into account all of the above and the structure of the problem (2), we may conclude that, in the considered situation and under the assumption that the problem \((P)\) is uniquely solvable for all \((y_d, u_d) \in Y \times U\), the metric projection in the uniformly convex and uniformly smooth Banach space \((Y \times U, \cdot)\) onto the set \(M\) defined in (1) is well-defined and single-valued everywhere. In other words, \(M\) is a Chebyshev subset of \((Y \times U, \cdot)\) in the sense of [18, Section 0]. From the weak-to-weak continuity of the control-to-state mapping \(S\), we further obtain that every sequence \(\{(y_n, u_n)\}_{n \in \mathbb{N}} \subset M\) that converges weakly in \(Y \times U\) to some \((\tilde{y}, \tilde{u})\) has to satisfy
\[
\tilde{y} \mathop{\lim\limits_{n \to \infty}} y_n = S(u_n) \mathop{\lim\limits_{n \to \infty}} S(\tilde{u}).
\]
The set $M$ is thus not only Chebyshev but also weakly closed and we may invoke [18, Corollary 4.2] to deduce that $M$ has to be convex, i.e., we have
\[
\lambda(y_1, u_1) + (1 - \lambda)(y_2, u_2) = (\lambda S(u_1) + (1 - \lambda)S(u_2), \lambda u_1 + (1 - \lambda)u_2) \in M
\] (4)
for all $\lambda \in [0, 1]$ and all $(y_1, u_1), (y_2, u_2) \in M$. Due to the definition of $M$, (4) can only be true if
\[
S(\lambda u_1 + (1 - \lambda)u_2) = \lambda S(u_1) + (1 - \lambda)S(u_2)
\] (5)
holds for all $\lambda \in [0, 1]$ and all $u_1, u_2 \in U$. This property, however, implies in combination with our assumptions on $S$ that the map $L(\cdot) := S(\cdot) - S(0)$ is linear and continuous as a function from $U$ to $Y$. Indeed, for every arbitrary but fixed $u \in U$, (5) yields
\[
L(\alpha u) = S(\alpha u + (1 - \alpha)0) - S(0) = \alpha S(u) - \alpha S(0) = \alpha L(u) \quad \forall \alpha \in [0, 1]
\] and
\[
\alpha L(u) = \alpha L \left( \frac{1}{\alpha} u \right) = L(\alpha u) \quad \forall \alpha \in (1, \infty).
\]
From these equations, it readily follows that
\[
L(u_1 + u_2) = S \left( \frac{1}{2}(2u_1) + \frac{1}{2}(2u_2) \right) - S(0) = \frac{1}{2}S(2u_1) + \frac{1}{2}S(2u_2) - S(0)
\]
\[
= \frac{1}{2}L(2u_1) + \frac{1}{2}L(2u_2) = L(u_1) + L(u_2) \quad \forall u_1, u_2 \in U.
\]
In particular, $L(-u) = -L(u)$ for all $u \in U$, and we may conclude that
\[
L(\alpha u_1 + u_2) = L(\alpha u_1) + L(u_2) = \alpha L(u_1) + L(u_2) \quad \forall u_1, u_2 \in U \quad \forall \alpha \in \mathbb{R}.
\]
The function $L: U \to Y$ is thus linear as claimed and, since the weak closedness of the set $M$ immediately yields the closedness of the graph of $L$ in $Y \times U$, also continuous by the closed graph theorem, see, e.g., [26, Section II-6].

In summary, we now arrive at the conclusion that the map $S$ has to be an affine-linear function. This contradicts our standing assumptions and establishes that (P) cannot possess precisely one solution for all $(y_d, u_d) \in Y \times U$. As we already know that (P) possesses at least one solution for each $(y_d, u_d)$ by Proposition 2.1, the assertion of the theorem now follows immediately.

Next, we address the issue of instability:

**Theorem 2.3 (Nonexistence of a Continuous Selection of Minimizers).**

In the situation of Assumption 1.1, there always exist a tuple $(y_d, u_d) \in Y \times U$, sequences $\{(y_{d,n}, u_{d,n})\}_{n \in \mathbb{N}} \subset Y \times U$ and $\{(y'_{d,n}, u'_{d,n})\}_{n \in \mathbb{N}} \subset Y \times U$, and elements $(\bar{y}, \bar{u}) \in Y \times U$ and $(\bar{y}', \bar{u}') \in Y \times U$ such that the following is true:

(i) $\{(y_{d,n}, u_{d,n})\}_{n \in \mathbb{N}}$ and $\{(y'_{d,n}, u'_{d,n})\}_{n \in \mathbb{N}}$ converge strongly in $Y \times U$ to $(y_d, u_d)$,

(ii) $(\bar{y}, \bar{u})$ is the unique solution of (P) with data $(y_d, u_d)$, i.e.,

\[
\{ (\bar{y}, \bar{u}) \} = \arg \min_{(y,u) \in Y \times U, \ y = S(u)} \|y - y_{d,n}\|_Y^p + \|u - u_{d,n}\|_U^p \quad \forall n \in \mathbb{N},
\]

(iii) $(\bar{y}', \bar{u}')$ is the unique solution of (P) with data $(y'_{d,n}, u'_{d,n})$ for all $n \in \mathbb{N}$, i.e.,

\[
\{ (\bar{y}', \bar{u}') \} = \arg \min_{(y,u) \in Y \times U, \ y = S(u)} \|y - y'_{d,n}\|_Y^p + \|u - u'_{d,n}\|_U^p \quad \forall n \in \mathbb{N},
\]

(iv) $(\bar{y}, \bar{u}) \neq (\bar{y}', \bar{u}')$. 

\[\square\]
**Proof.** In the considered situation, we obtain from exactly the same arguments as in the proof of Theorem 2.2 that (P) is equivalent to the projection problem (2) and from Theorem 2.2 itself that there exists a tuple \((y_{d}, u_{d}) \in Y \times U\) such that (P) (and thus also (2)) possesses two nonidentical global solutions \((\bar{y}, \bar{u}) \in Y \times U\) and \((\tilde{y}', \tilde{u}') \in Y \times U\). Define

\[
(y_{d,t}, u_{d,t}) := t(\bar{y}, \bar{u}) + (1 - t)(y_{d}, u_{d}) \quad \forall t \in (0,1)
\]

and

\[
(y_{d,t}', u_{d,t}') := t(\tilde{y}', \tilde{u}') + (1 - t)(y_{d}, u_{d}) \quad \forall t \in (0,1).
\]

Then, the uniform convexity of the space \((Y \times U, \|\cdot\|_{Y \times U})\) (with \(\|\cdot\|_{Y \times U}\) defined as in (3), see again [7, Theorem 1]) and exactly the same calculations as in the proof of [17, Theorem 2.1] yield that

\[
\{(\bar{y}, \bar{u})\} = \arg\min_{(y,u) \in M} \|(y, u) - (y_{d,t}, u_{d,t})\|_{Y \times U} \quad \forall t \in (0,1)
\]

and

\[
\{(\tilde{y}', \tilde{u}')\} = \arg\min_{(y,u) \in M} \|(y, u) - (y_{d,t}', u_{d,t}')\|_{Y \times U} \quad \forall t \in (0,1)
\]

holds, where \(M\) is the set in (1). To establish the assertion of the theorem, it now suffices to choose an arbitrary sequence \(\{t_{n}\}_{n \in \mathbb{N}} \subset (0,1)\) with \(t_{n} \to 0\), to define \((y_{d,n}, u_{d,n}) := (y_{d,t_{n}}, u_{d,t_{n}})\) and \((y_{d,n}', u_{d,n}') := (y_{d,t_{n}}', u_{d,t_{n}}')\) for all \(n \in \mathbb{N}\), and to again exploit the equivalence between the problems (P) and (2). \(\square\)

Some remarks regarding the last two results are in order:

**Remark 2.4.**

(i) The assumption that both the desired state \(y_{d}\) and the desired control \(u_{d}\) can be chosen at will in Theorem 2.2 cannot be dropped. If, e.g., \(u_{d}\) is fixed to be zero, then it is perfectly possible that a problem of the type (P) is uniquely solvable for all \(y_{d} \in Y\) even if the control-to-state mapping \(S\) is non-affine. An example of such a configuration can be found in [9, Corollary 5.3].

(ii) The nonuniqueness of global minimizers in Theorem 2.2 implies that numerical solution algorithms for problems of the type (P) may produce sequences of iterates with several accumulation points and that termination criteria which consider the distance between successive iterates cannot be expected to reliably detect stationarity. The instability of the solutions in Theorem 2.3 further shows that numerical errors and small inaccuracies in the problem data may prevent a proper identification of a global optimum.

(iii) Theorem 2.3 shows that, in the situation of Assumption 1.1, every function \(F: Y \times U \to U\) with the property

\[
F(y_{d}, u_{d}) \in \arg\min_{u \in U} \|S(u) - y_{d}\|_{Y}^{p} + \|u - u_{d}\|_{U}^{p} \quad \forall (y_{d}, u_{d}) \in Y \times U
\]

is discontinuous. There thus does not exist a continuous selection from the set of optimal controls of (P) (in the sense of set-valued analysis). Theorem 2.3 further illustrates that, in the presence of nonlinearity, adding a Tikhonov-type regularization term to an objective function may fail to properly regularize an inverse problem.

We conclude this paper with some tangible examples of problems that are covered by Theorems 2.2 and 2.3. (Note that the following list is far from exhaustive.)
Example 2.5 (Finite-Dimensional Tracking-Type Problems). Consider a finite-dimensional optimization problem of the form

\[
\min_{y \in \mathbb{R}^l, u \in \mathbb{R}^m} \frac{1}{2} (y - y_d)^T A(y - y_d) + \frac{\nu}{2} (u - u_d)^T B(u - u_d) \quad \text{s.t.} \quad y = S(u)
\]

(6)

with some \( l, m \in \mathbb{N} \), an arbitrary but fixed Tikhonov parameter \( \nu > 0 \), symmetric positive definite matrices \( A \in \mathbb{R}^{l \times l} \) and \( B \in \mathbb{R}^{m \times m} \), vectors \( y_d \in \mathbb{R}^l \) and \( u_d \in \mathbb{R}^m \), and a non-affine, continuous mapping \( S : \mathbb{R}^m \to \mathbb{R}^l \). Then, by defining

\[
Y := \mathbb{R}^l, \quad \|y\|_Y := \left( \frac{1}{2} y^T A y \right)^{1/2}, \quad U := \mathbb{R}^m, \quad \|u\|_U := \left( \frac{\nu}{2} u^T B u \right)^{1/2}, \quad p := 2,
\]

we can recast (6) as a problem of the form (P) that satisfies all of the conditions in Assumption 1.1 (as one may easily check). Theorems 2.2 and 2.3 are thus applicable to (6), and we may deduce that there exist choices of the tuple \((y_d, u_d)\) for which (6) possesses more than one global solution and that the solution set of (6) does not admit a continuous selection. Note that problems of the type (6) arise very frequently in optimal control when a continuous tracking-type problem is discretized, e.g., by means of the finite element method, cf. [8, Section 5.1] and [12, Sections 4.3, 5.3].

Example 2.6 (Optimal Control of a Nonsmooth Semilinear Elliptic PDE). Consider an optimal control problem of the form

\[
\min_{y \in H_0^1(\Omega), \quad u \in L^2(\Omega)} \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u - u_d\|_{L^2(\Omega)}^2
\]

(7)

w.r.t. \( y \in H_0^1(\Omega), \quad u \in L^2(\Omega) \),

s.t. \( -\Delta y + \max(0, y) = u \in \Omega \),

where \( \Omega \subset \mathbb{R}^m \), \( m \in \mathbb{N} \), is a bounded domain, \( y_d \in L^2(\Omega) \) and \( u_d \in L^2(\Omega) \) are given, \( \nu > 0 \) is an arbitrary but fixed Tikhonov parameter, \( L^2(\Omega) \) and \( H_0^1(\Omega) \) are defined as in [2], \( \Delta \) is the distributional Laplacian, and the function \( \max(0, \cdot) : \mathbb{R} \to \mathbb{R} \) acts as a Nemytskii operator. Then, it follows from [8, Proposition 2.1, Corollary 3.8] that (7) possesses a well-defined and weak-to-weak continuous control-to-state mapping \( S : L^2(\Omega) \to L^2(\Omega), \quad u \mapsto y \). Further, the map \( S \) is also non-affine. Indeed, if we choose an arbitrary but fixed \( z \in H_0^1(\Omega) \cap H^2(\Omega) \) that is positive almost everywhere in \( \Omega \) (such a \( z \) exists by [8, Lemma A.1]) and if we define

\[
\begin{align*}
  u_1 &:= 2(-\Delta z + z) \in L^2(\Omega) \quad \text{and} \quad u_2 := 2\Delta z \in L^2(\Omega), \\
  \frac{1}{2} S(u_1) + \frac{1}{2} S(u_2) &\neq S(z) = S\left( \frac{1}{2} u_1 + \frac{1}{2} u_2 \right),
\end{align*}
\]

where the inequality \( 0 \neq S(z) \) follows immediately from the PDE in (7) and our assumption \( z > 0 \) a.e. in \( \Omega \). Since (7) can be recast as a problem of the form (P) (with \( Y := L^2(\Omega), \quad U := L^2(\Omega), \quad p := 2 \), and appropriately rescaled norms) and since Hilbert spaces are trivially uniformly convex and uniformly smooth, we may now conclude that the optimal control problem (7) satisfies all of the conditions in Assumption 1.1. Theorems 2.2 and 2.3 are thus applicable and it follows that (7) is not uniquely solvable for certain choices of the tuple \((y_d, u_d)\) \( \in L^2(\Omega) \times L^2(\Omega) \) and that the solution set of (7) does not admit a continuous selection. Note that the above setting is precisely the one considered in [8].
Example 2.7 (\(L^p\)-Boundary Control for a Signorini-Type VI). Consider an optimal control problem of the form
\[
\min \frac{1}{p} \|y - y_d\|_{L^p(\Omega)}^p + \frac{\nu}{p} \|u - u_d\|_{L^p(\partial \Omega)}^p
\]
\[\text{w.r.t. } y \in H^1(\Omega), \quad u \in L^p(\partial \Omega), \quad \text{and} \quad u_d \in L^p(\partial \Omega) \]
\[\text{s.t. } \int_{\partial \Omega} \nabla y \cdot \nabla (v - y) + y(v - y) \, dx \geq \int_{\partial \Omega} u(v - y) \, ds \quad \forall v \in K,
\]
where \(\Omega \subset \mathbb{R}^m, m \in \mathbb{N}\), is a bounded Lipschitz domain with boundary \(\partial \Omega\), \(y_d \in L^p(\Omega)\) and \(u_d \in L^p(\partial \Omega)\) are given, \(\nu > 0\) is an arbitrary but fixed Tikhonov parameter, \(p\) is an exponent that satisfies \(p \in [2, \infty)\) for \(m \leq 2\) and \(p \in [2, m/(m - 2)]\) for \(m \geq 3\), \(L^p(\partial \Omega)\), \(L^p(\Omega)\), and \(H^1(\Omega)\) are defined as in [2], \(\nabla\) is the weak gradient, and \(K\) is the set of all elements of \(H^1(\Omega)\) whose trace is nonnegative a.e. on \(\partial \Omega\). Then, using [19, Theorem II-2.1], the Sobolev embeddings, see [22, Theorem 2-3.4], and the compactness of the trace operator, see [22, Theorem 2-6.2], it is easy to check that the elliptic variational inequality in (8) possesses a well-defined and weak-to-weak continuous solution operator \(S : L^p(\partial \Omega) \rightarrow H^1(\Omega) \rightarrow L^p(\Omega), \quad u \mapsto y\). To see that this \(S\) is non-affine, we note that, for every a.e.-positive control \(u \in L^p(\partial \Omega)\), the trace of \(S(u)\) has to be positive a.e. on a set of positive surface measure. Indeed, if the latter was not the case for an a.e.-positive control \(u\), then the variational inequality in (8) and the inclusion \(H^1_0(\Omega) \subset K\) would imply that \(y = S(u) \in H^1(\Omega)\) is also the solution of
\[-\Delta y + y = 0 \text{ in } \Omega, \quad y = 0 \text{ on } \partial \Omega.
\]
This, however, would yield \(y = 0\) and, as a consequence,
\[0 \geq \int_{\partial \Omega} uv \, ds = \int_{\partial \Omega} |uv| \, ds \quad \forall v \in K,
\]
which is a contradiction. The trace of \(S(u)\) thus has to be positive on a non-negligible subset of \(\partial \Omega\) for all a.e.-positive \(u \in L^p(\partial \Omega)\) as claimed. Since we trivially have \(S(0) = 0\) and since \(S(u)\) has to be an element of \(K\) for all \(u\) by the definition of \(S\), it now follows immediately that \(S(u) + S(-u) \neq S(0)\) holds for all \(u \in L^p(\partial \Omega)\) that are positive a.e. on \(\partial \Omega\). In combination with our previous observations on \(S\) and the fact that \(L^q\)-spaces are uniformly convex and uniformly smooth for \(1 < q < \infty\) (see [20, Theorems 5.2.11, 5.5.12]), this shows that (8) satisfies the conditions in Assumption 1.1 (with \(Y := L^p(\Omega), U := L^p(\partial \Omega),\) and again appropriately rescaled norms). We may thus again invoke Theorems 2.2 and 2.3 to deduce that (8) is not uniquely solvable for certain tuples \((y_d, u_d) \in L^p(\Omega) \times L^p(\partial \Omega)\) and that the solution set of (8) does not admit a continuous selection.

Example 2.8 (Distributed Control of the Parabolic Obstacle Problem). Consider an optimal control problem of the form
\[
\min \frac{1}{2} \|y(T) - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u - u_d\|_{L^2(0,T;L^2(D))}^2
\]
\[\text{w.r.t. } u \in L^2(0,T;L^2(D)), \quad y \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \text{and} \quad u_d \in L^2(0,T;L^2(D))
\]
\[\text{s.t. } y(t) \geq \psi \text{ a.e. in } \Omega \text{ for a.a. } t \in (0,T), \quad y(0) = 0 \text{ a.e. in } \Omega,
\]
\[
\int_0^T (\partial_t y - \Delta y - Bu, v - y) \, dt \geq 0
\]
\[\forall v \in L^2(0,T;H^1_0(\Omega)), \quad v(t) \geq \psi \text{ a.e. in } \Omega \text{ for a.a. } t \in (0,T),
\]
where $\Omega \subset \mathbb{R}^n$, $m \in \mathbb{N}$, is a bounded domain, $D$ is an open, non-empty subset of $\Omega$, $T > 0$ is a given final time, $\nu > 0$ is an arbitrary but fixed Tikhonov parameter, the appearing Lebesgue-, Sobolev-, and Bochner spaces are defined as in [2] and [13], $y_d \in L^2(\Omega)$ and $u_d \in L^2(0, T; L^2(D))$ are given, $\psi \in L^2(\Omega)$ is a given function that satisfies $\psi \leq 0$ a.e. in $\Omega$, $\partial_t h$ is the time derivative in the Sobolev-Bochner sense, $\Delta$ is the distributional Laplacian, $B$ denotes the canonical embedding of $L^2(0, T; L^2(D))$ into $L^2(0, T; L^2(\Omega))$, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing in $H^1_0(\Omega)$. Then, using [3, Theorem 1.13, Equation (4.9)], [6, Theorem 2.3], and the lemma of Aubin-Lions, see [25, Theorem 10.12], it is easy to check that the evolution variational inequality in (9) possesses a well-defined weak-to-weak continuous solution map $G: L^2(0, T; L^2(D)) \rightarrow H^1(0, T; L^2(\Omega))$, $u \mapsto y$. (Note that, in order to apply [3, Theorem 1.13], one has to define the function $\Phi$ appearing in this theorem as in [3, Equation (4.9)].) As $H^1(0, T; L^2(\Omega))$ embeds continuously into $C([0, T]; L^2(\Omega))$ by [25, Theorem 10.9], the above implies in particular that (9) possesses a well-defined weak-to-weak continuous control-to-state (or, in this context, more precisely control-to-observation) operator $S: L^2(0, T; L^2(D)) \rightarrow L^2(\Omega)$, $u \mapsto G(u)(T)$, where $G(u)(T)$ denotes the value of the $C([0, T]; L^2(\Omega))$-representative of $G(u)$ at the final time $T$. To see that the map $S$ is non-affine, we proceed similarly to Examples 2.6 and 2.7. Suppose that $E$ is an open, non-empty set whose closure is contained in $D$, and that $\varepsilon \in (0, T)$ is fixed. Then, it follows from [8, Lemma A.1] that there exists a function $z \in C^\infty_c((0, T) \times \Omega)$ that is positive in $(\varepsilon, T) \times E$ and zero everywhere in $(0, T) \times \Omega \setminus (\varepsilon, T) \times E$. If, for such a $z$, we define $\tilde{u} := (\partial_z z - \Delta z)\big|_{(0, T) \times \Omega}$, where the vertical bar denotes a restriction, then it clearly holds $S(\tilde{u}) = z(T) > 0$ a.e. in $E$. From the $C([0, T]; L^2(\Omega))$-regularity and the properties of the solutions of the evolution variational inequality in (9) and the closedness of the set $\{v \in L^2(\Omega) \mid v \geq \psi \text{ a.e. in } \Omega\}$ in $L^2(\Omega)$, we further obtain that $S(\alpha \tilde{u}) \geq \psi$ has to hold a.e. in $\Omega$ for all $\alpha \in \mathbb{R}$. In combination with the trivial identity $S(0) = 0$ and $z(T) > 0$ a.e. in $E$, it now follows immediately that
\[
\psi \leq S(\alpha \tilde{u}) = S(\alpha \tilde{u}) - S(0) = \alpha(S(\tilde{u}) - S(0)) = \alpha S(\tilde{u}) = \alpha z(T)
\]
cannot be true a.e. in $\Omega$ for all $\alpha \in \mathbb{R}$. This shows that the map $S$ is indeed non-affine in the situation of (9). In summary, we may now again conclude that (9) satisfies all of the conditions in Assumption 1.1 (with $p := 2$, $Y := L^2(\Omega)$, $U := L^2(0, T; L^2(D))$, and appropriately rescaled norms). Theorems 2.2 and 2.3 thus apply to (9), and we obtain that this optimal control problem is not uniquely solvable for certain choices of the tuple $(y_d, u_d) \in L^2(\Omega) \times L^2(0, T; L^2(D))$ and that the solution set of (9) does not admit a continuous selection.

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References

[1] A. Ahmad Ali, K. Deckelnick, and M. Hinze. Global minima for optimal control of the obstacle problem. ESAIM Control Optim. Calc. Var., 2020. to appear.
[2] H. Attouch, G. Buttazzo, and G. Michaille. Variational Analysis in Sobolev and BV Spaces. SIAM, Philadelphia, 2006.
[3] V. Barbu. Optimal Control of Variational Inequalities. Research Notes in Mathematics. Pitman, 1984.
[4] T. Betz, C. Meyer, A. Rademacher, and K. Rosin. Adaptive optimal control of elastoplastic contact problems. Ergebnisseberichte des Instituts für Angewandte Mathematik, TU Dortmund, Nr. 496, 2014.
[5] J. M. Borwein and J. D. Vanderwerff. *Convex Functions: Constructions, Characterizations and Counterexamples*. Cambridge University Press, Cambridge, 2010.

[6] C. Christof. Sensitivity analysis and optimal control of obstacle-type evolution variational inequalities. *SIAM J. Control Optim.*, 57(1):192–218, 2019.

[7] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.

[8] C. Christof, C. Meyer, S. Walther, and C. Clason. Optimal control of a non-smooth semilinear elliptic equation. *Math. Control Relat. Fields*, 8(1):247–276, 2018.

[9] C. Christof and G. Wachsmuth. On second-order optimality conditions for optimal control problems governed by the obstacle problem. *Optimization*, 2020. to appear.

[10] A. L. Dontchev and T. Zolezzi. *Well-Posed Optimization Problems*. Number 1543 in Lecture Notes in Mathematics. Springer, 1993.

[11] M. Gugat, G. Leugering, and G. Sklyar. Lp-optimal boundary control for the wave equation. *SIAM J. Control Optim.*, 44(1):49–74, 2005.

[12] D. Hafemeyer. *Optimal Control of the Parabolic Obstacle Problem*. PhD thesis, Technische Universität München, 2020.

[13] J. Heinonen, P. Koselka, N. Shanmugalingam, and J. T. Tyson. *Sobolev Spaces on Metric Measure Spaces*. Number 27 in New Mathematical Monographs. Cambridge University Press, 2015.

[14] M. Herty, R. Pinnau, and M. Seaid. Optimal control in radiative transfer. *Optim. Methods Softw.*, 22(6):917–936, 2007.

[15] R. Herzog, A. Rösch, S. Ulbrich, and W. Wollner. OPTPDE - A collection of problems in PDE-constrained optimization. [http://www.optpde.net](http://www.optpde.net).

[16] R. Herzog, A. Rösch, S. Ulbrich, and W. Wollner. OPTPDE: A collection of problems in PDE-constrained optimization. In G. Leugering, P. Benner, S. Engell, A. Griewank, H. Harbrecht, M. Hinze, R. Rannacher, and S. Ulbrich, editors, *Trends in PDE Constrained Optimization*, volume 165 of *International Series of Numerical Mathematics*, pages 539–543. Springer, 2014.

[17] P. C. Kainen, V. Kurková, and A. Vogt. Geometry and topology of continuous best and near best approximations. *J. Approx. Theory*, 105(2):252 – 262, 2000.

[18] V. Klee. Convexity of Chebyshev sets. *Math. Ann.*, 142:292–304, 1961.

[19] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*, volume 31 of *Classics in Applied Mathematics*. SIAM, 2000.

[20] R. E. Megginson. *An Introduction to Banach Space Theory*. Number 183 in Graduate Texts in Mathematics. Springer, 1998.

[21] E. Muselli. Affinity and well-posedness for optimal control problems in Hilbert spaces. *J. Convex Anal.*, 14(4):767 – 784, 2007.

[22] J. Nečas. *Direct Methods in the Theory of Elliptic Equations*. Springer, Berlin, 2012.

[23] B. J. Pettis. A proof that every uniformly convex space is reflexive. *Duke Math. J.*, 5(2):249–253, 1939.

[24] D. Pighin. Nonuniqueness of minimizers for semilinear optimal control problems. [arXiv:2002.04485](https://arxiv.org/abs/2002.04485), 2020.

[25] B. Schweizer. *Partielle Differentialgleichungen*. Springer, Berlin/Heidelberg, 2013.

[26] K. Yosida. *Functional Analysis*. Springer, 1980.

[27] T. Zolezzi. A characterization of well-posed optimal control systems. *SIAM J. Control Optim.*, 19(3):604–616, 1981.

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