

Enclosure method for the biharmonic equation

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Abstract. Herein, we study an inverse problem for detecting unknown obstacles by the
enclosure method using the Dirichlet–to–Neumann map for measurements. We justify the
method for an impenetrable obstacle case involving a biharmonic equation. We use complex
geometrical optics solutions with a logarithmic phase to reconstruct some non–convex parts
of the obstacle.

Keywords: Enclosure method, Biharmonic operator, inverse problems

1. Introduction

The inverse problem in this study involves determining an unknown obstacle or jump in the
inclusions embedded in a known background medium via near-field measurements. Several
methods have been proposed to detect this jump based on two main types of special solutions.
One is Green’s type or singular solutions of elliptic equations, which were introduced by
Isakov [14]. Subsequently, other methods have been developed, such as the linear sampling
method in [2] and, [3], and the factorization method in [8] and, [19]. Another set of solutions
is the complex geometrical optics (CGO) solutions, which were considered by Ikehata, [9],
to develop the enclosure method to reconstruct the unknown inclusions. The main objective
of this study is to discuss the enclosure method for the zeroth-order perturbation of the
biharmonic–type operator.

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded, smooth domain. We assume \( D(\subset \subset \Omega) \) to be an unknown
obstacle with a \( C^1 \)-regular boundary \( \partial D \), such that \( \Omega \setminus \overline{D} \) is connected. As a model
problem, we consider the zeroth–order perturbation of the biharmonic equation with the Navier boundary condition:

\[
\begin{cases}
\Delta^2 u + \tilde{n} u = 0 & \text{in } \Omega \\
u = f_1 & \text{on } \partial \Omega \\
\Delta u = f_2 & \text{on } \partial \Omega.
\end{cases}
\] (1.1)

We assume that \( \tilde{n}(x) = 1 + n_D(x)\chi_D(x) \), for all \( x \in \Omega \), such that \( \tilde{n} \in L^\infty(\Omega, \mathbb{R}) \). Here, \( \chi_D \) is the characteristic function of \( D \). Let us also assume that \( n_D \in L^\infty_+(D) \), where \( L^\infty_+(D) := \{ f \in L^\infty(D); f \geq C > 0 \text{ for some } C \in \mathbb{R} \} \).

In practice, the biharmonic equation appears in many areas of physics, such as in the elasticity theory, plate plasma, and Stokes flow \[5\]. In \[15\], they posed inverse boundary value problems for thin elastic plates in the planner domain for the system \( \nabla \nabla (C \nabla^2 u) = 0 \), where \( C \) is the fourth-order tensor. Recently, an anisotropic nonlocal fractional \( p \)-biharmonic system was considered \[16\]. They studied the existence and uniqueness of weak solutions to the associated interior source and exterior value problems, unique continuation properties, monotonicity relations, and inverse problems for exterior Dirichlet–to–Neumann maps.

By the standard well–posedness of the boundary value problem for the fourth–order elliptic equation, problem (1.1) has a unique solution \( u \in H^4(\Omega) \) for any \( f_1 \in H^{7/2}(\partial \Omega) \) and \( f_2 \in H^{3/2}(\partial \Omega) \); see Lemma 8 for more details. We define the Dirichlet–to–Neumann map corresponding to the biharmonic problem above as follows:

\[
\mathcal{N}_D : H^{7/2}(\partial \Omega) \times H^{3/2}(\partial \Omega) \rightarrow H^{5/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)
\]

by

\[
\mathcal{N}_D(f_1, f_2) = \left( \frac{\partial u}{\partial \nu}|_{\partial \Omega}, \frac{\partial}{\partial \nu}(\Delta u)|_{\partial \Omega} \right),
\]

where \( u \) is the solution to (1.1), and \( \nu \) denotes the outward unit normal vector to \( \partial \Omega \). The inverse problem in this study is to determine the shape and location of \( D \) from the knowledge of the Dirichlet–to–Neumann map \( \mathcal{N}_D \) measured at boundary \( \partial \Omega \).

We mainly applied the enclosure method proposed by Ikehata \[12\] (see also \[9\], \[10\], and \[11\]) to solve the inverse problem. In \[12\], the author considered the inverse problem for a conductivity equation as a model problem and used complex geometrical optics solutions with a linear phase to detect the convex hull of an obstacle. Many studies have been conducted on detecting unknown obstacles using this enclosure method for various other types of partial differential equations. See, for example the Maxwell system \[18\], \[24\], \[27\], elasticity equation \[17\], \[23\], and Helmholtz equation \[26\]. In \[26\], the authors considered the problem of determining the unknown obstacle for divergence form elliptic equations with lower–order terms from the knowledge of the Dirichlet–to–Neumann map. The result was proved using
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Meyers $L^p$–estimates \cite{25} to eliminate some geometrical assumptions on the obstacle surface. The ideas of using $L^p$–estimates to prove the enclosure method for the reconstruction, have also been implemented in \cite{17} and \cite{18}. The Calderón problem corresponding to the biharmonic operator was first studied by Krupchyk–Lassas–Uhlmann \cite{20} to prove the unique
determination of the first–order perturbation $-i^{-1}A(x) \cdot \nabla + q$ of the biharmonic equation from the Dirichlet–to–Neumann map measured on a part of the boundary. Further results on the inverse problems for the biharmonic equation can be found in various studies; see, for example, \cite{1}, \cite{21}, \cite{22}, and the references therein.

The enclosure method is based on the asymptotic behavior of $I_{x_0}(h,t)$. The main idea behind this method is as follows. First, we define an indicator function, $I_{x_0}(h,t)$, as in (2.7). The indicator function represents the energy difference between when an obstacle is in $\Omega$ and when no obstacle is in $\Omega$. Subsequently, an asymptotic estimate of the indicator function for a small parameter, $h > 0$, is studied (see Theorem 2). This indicates whether the level set of the phase function touches the obstacle surface. Finally, the intersection of all level sets touching the interface determines convex hull of the obstacle and its non–convex part.

Most of our efforts are devoted to the proof of Theorem 2. Because of Lemma 3 providing the lower and upper bounds of the indicator function when $t = h_D(x_0)$ is sufficient. Because of Lemma 4 an appropriate estimate of the corresponding reflected solution $w := u - v$ is required, where $u$ satisfies (1.1), and $v$ is the CGO solution of the Equation (2.8). In this study, we verified that the reflected solution $w$ of Equation (3.3) satisfies

$$\|w\|_{L^2(\Omega)} \leq C\|v\|_{L^1(\partial D)}, \quad (1.2)$$

see Proposition 4. To justify the enclosure method, we must construct the CGO solutions with an appropriate decay estimate in the correction term. We used the CGO solutions proposed in \cite{20}, and constructed CGO solutions of the form

$$v(x;h) = e^{\frac{\phi+i\psi}{h}}(a_0(x) + ha_1(x) + r(x,h)) \quad (1.3)$$

where $\phi \in C^\infty(\tilde{\Omega}, \mathbb{R})$ is a limiting Carleman weight for the semiclassical Laplacian on $\tilde{\Omega}$, where $\Omega \subset \subset \tilde{\Omega}$. Functions $a_0$ and $a_1$ are smooth, and the correction term $r$ satisfies $\|r\|_{H^4_{sc}(\Omega)} = O(h^2)$.

The remainder of this paper is organized as follows. In Section 2, we state our main results and discuss the CGO solutions that are useful in our proof. Section 3 provides proof of the main results. Finally, Section 4 presents the existence and uniqueness of the boundary value problem for the fourth-order elliptic equation.
2. Main result

In this section, we present our main results. We first discuss the CGO solutions for the following fourth–order elliptic equation:

$$\mathcal{L}_{A,q}v := (\Delta^2 - i^{-1}A \cdot \nabla + q(x))v = 0 \text{ in } \Omega,$$

where $A = (A_j)_{1 \leq j \leq n} \in C^4(\overline{\Omega}, \mathbb{C}^n)$ and $q \in L^\infty_+(\Omega, \mathbb{C})$. The CGO solutions of the form

$$v(x; h) = e^{\frac{\phi + i\psi}{h}} (a_0(x) + ha_1(x) + r(x; h)),$$

were derived in [20] to determine the first–order perturbation of the biharmonic operator. Here, function $a_0$ solves the first transport equation

$$T^2a_0 = 0 \text{ in } \Omega,$$

where $T = (\nabla \phi + i\nabla \psi) \cdot \nabla + \frac{1}{2}(\Delta \phi + i\Delta \psi)$. In addition, function $a_1$ solves the second transport equation in $\Omega$:

$$T^2a_1 = -\frac{1}{2}(\Delta \circ T + T \circ \Delta)a_0 + \frac{1}{4}A \cdot (i^{-1}\nabla \phi + \nabla \psi)a_0 \text{ in } \Omega.$$

The Carleman weight is of the form

$$\phi(x) = \frac{1}{2} \log |x - x_0|^2$$

and

$$\psi(x) = \frac{\pi}{2} - \tan^{-1} \frac{\omega \cdot (x - x_0)}{\sqrt{(x - x_0)^2 - (\omega \cdot (x - x_0))^2}} = \text{dist}_{S^{n-1}} \left( \frac{x - x_0}{|x - x_0|}, \omega \right),$$

where $\omega \in S^{n-1}$ is chosen such that $\psi$ is smooth near $\overline{\Omega}$, and $x_0$ is a fixed point outside the convex hull of $\Omega$. In particular, in [20], they proved the following proposition.

**Proposition 1.** [Proposition 2.4 in [20]] Let $A \in C^4(\overline{\Omega}, \mathbb{C}^n)$, and $q \in L^\infty_+(\Omega, \mathbb{C})$. Then, for
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If $h > 0$ sufficiently small, there exist solutions $v(x; h) \in H^4(\Omega)$ to the equation:

$$\Delta^2 v - i^{-1} A \cdot \nabla v + qv = 0 \text{ in } \Omega$$  \hspace{1cm} (2.5)

of the form $v(x; h) = e^{\frac{\phi}{i\pi}}(a_0(x) + ha_1(x) + r(x; h))$, where $\phi$ is a limiting Carleman weight for the semi–classical Laplacian, and $\psi$ is defined as \((2.4)\). The amplitudes $a_0 \in C^\infty(\overline{\Omega})$ and $a_1 \in C^4(\Omega)$ satisfy Equations \((2.2)\) and \((2.3)\), respectively, and the correction term $r$ satisfies $\|r\|_{H^4_{\text{cl}}} = O(h^2)$.

Note that, for a given $h > 0$ and $k \in \mathbb{N}$, the semi–classical norm of $r$ is defined as follows:

$$\|r\|_{H^k_{\text{cl}}(\Omega)} := \left[ \sum_{|\alpha| \leq k} \int_{\Omega} |(hD)\alpha u|^2 dx \right]^{1/2}.$$  \hspace{1cm} (2.7)

See [28], Chapter 7 for an extensive study on these spaces and their properties. Let $t$ be a constant and $h > 0$ a small parameter. Maintaining the same notation as in Proposition \(\text{II}\) we define

$$v(x, h, t) = e^{\frac{\psi}{i\pi}}(t^{-\frac{1}{2}}\log|x-x_0|^2 - \frac{\psi(x)}{i\pi})(a_0(x) + ha_1(x) + r(x; h))$$  \hspace{1cm} (2.6)

to be a complex geometric optics solution with spherical phases for Equation \((2.5)\). In our model problem, $A$ is taken as zero, and the potential $q$ is a real–valued function.

Using the CGO solutions with spherical phases, we define an indicator function as follows:

$$I_{x_0}(h, t) = \langle (N_D - N_\emptyset)f, f \rangle = \int_{\partial\Omega} (N_D - N_\emptyset)f \cdot f dS,$$  \hspace{1cm} (2.7)

where $dS$ denotes the surface measure of $\partial\Omega$. Here, $N_D$ is the Dirichlet–to–Neumann map corresponding to the solution $u$ of problem \((1.1)\), and $N_\emptyset$ denotes the Dirichlet–to–Neumann map corresponding to a CGO solution with a spherical phase function, as described in Equation \((2.6)\), which satisfies

$$\begin{cases}
\Delta^2 v + v = 0 & \text{in } \Omega \\
v = f_1 & \text{on } \partial\Omega \\
\Delta v = f_2 & \text{on } \partial\Omega.
\end{cases}$$  \hspace{1cm} (2.8)

Note that $f$ is a vector–valued function defined as $f = (f_1, f_2) = (u|_{\partial\Omega}, (\Delta u)|_{\partial\Omega}) = (v|_{\partial\Omega}, (\Delta v)|_{\partial\Omega})$, such that the boundary values corresponding to \((1.1)\) and \((2.8)\) are the
same. \( \langle \mathcal{N}_D(f), f \rangle \) can be defined as follows:

\[
\langle \mathcal{N}_D(f), f \rangle = \int_{\partial \Omega} \langle \frac{\partial u}{\partial \nu}|_{\partial \Omega}, \frac{\partial}{\partial \nu} (\Delta u)|_{\partial \Omega}, (\bar{f}_1, \bar{f}_2) \rangle ) dS
\]

\[
= \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \bar{f}_2 + \frac{\partial}{\partial \nu} (\Delta u) \bar{f}_1 \right) dS.
\]

We introduce the distance function as

\[
h_D(x_0) := \inf_{x \in D} \frac{1}{2} \log |x - x_0|^2,
\]

where \( x_0 \in \mathbb{R}^3 \setminus \text{conv}(\Omega) \), and \( \text{conv}(\Omega) \) denotes the convex hull of the domain \( \Omega \). Note that, \( e^{h_D(x_0)} \) measures the distance from \( x_0 \) to \( D \).

We are now ready to formulate the main theorem of this study.

**Theorem 2.** Let \( x_0 \in \mathbb{R}^3 \setminus \text{conv}(\Omega) \). Then, we have the following asymptotic behavior for \( I_{x_0}(h, t) \). There exist constants \( c, C > 0 \) independent of \( h \) such that

\[
c \leq |h^{-3} \text{Re} I_{x_0}(h, h_D(x_0))| \leq Ch^{-2}, \ h \ll 1.
\]

Moreover, it holds that

\[
t - h_D(x_0) = \lim_{h \to 0} \frac{1}{2} h \log |\text{Re} I_{x_0}(h, t)|.
\]

Using (2.9) and (2.10), we can easily prove the following:

(i) When \( t < h_D(x_0) \), we have

\[
|h^{-3} \text{Re} I_{x_0}(h, t)| \leq C e^{-\frac{t}{h}}, \ h \ll 1.
\]

In particular, \( \lim_{h \to 0} |\text{Re} I_{x_0}(h, t)| = 0 \).

(ii) When \( t > h_D(x_0) \), we have

\[
|h^{-3} \text{Re} I_{x_0}(h, t)| \geq C e^{\frac{t}{h}}, \ h \ll 1.
\]

In particular, \( \lim_{h \to 0} |\text{Re} I_{x_0}(h, t)| = \infty \).

From this theorem, we obtain the certain asymptotic behavior of the indicator function required to reconstruct the unknown obstacle from the boundary data. Specifically, let us fix a point \( x_0 \in \mathbb{R}^3 \setminus \text{conv}(\Omega) \). Then, we observe that the complex geometric optics solution (see Proposition [1]) exhibits an asymptotic behavior, that is, it grows exponentially inside the sphere \( S = \{ x \in \mathbb{R}^2; |x-x_0| = e^t \} \) for a sufficiently small \( h > 0 \) and decays exponentially fast outside the sphere. Using this feature of the CGO solution, we can observe that, when \( t < h_D(x_0) \), the indicator function \( I_{x_0}(h, t) \) vanishes exponentially for a sufficiently small \( h > 0 \). Now, we can expand the sphere such that when time \( t \geq h_D(x_0) \), the obstacle
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Intersects the sphere, and by Theorem 2, the indicator function becomes large for a small \( h \). Finally, moving point \( x_0 \) around \( \partial h(\Omega) \), we can enclose the unknown obstacle using the spheres. In this manner, we can recover the convex hull of the obstacle and some of its non-convex part.

3. Proof of the Theorem 2

In this section, we provide a proof of Theorem 2. We begin with the following lemma.

**Lemma 3.** It holds that

\[
I_{x_0}(h, t) = e^{2(t-hD(x_0))} I_{x_0}(h, hD(x_0)).
\]

The lemma follows directly from the definition of the indicator function and complex geometric optics solutions.

Because of Lemma 3, proving Equation (2.9) in Theorem 2 is sufficient. Let us recall the integration by parts formula, which is often useful in estimates. For any \( \phi \in H^4(\Omega) \) and \( \psi \in H^2(\Omega) \), Green’s theorem holds as follows:

\[
\int_{\Omega} \nabla \cdot \nabla (\Delta \phi) \psi dx = \int_{\Omega} \nabla (\Delta \phi) \cdot \nabla \psi dx + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta \phi) \psi dS
\]

\[
= \int_{\Omega} (\Delta \phi) \Delta \psi dx - \int_{\partial \Omega} \Delta \phi \frac{\partial \psi}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta \phi) \psi dS. \tag{3.1}
\]

3.1. Lower and upper bound of \( I_{x_0}(h, hD(x_0)) \)

Let \( v \) be a CGO solution of the biharmonic equation

\[
\begin{align*}
\Delta^2 v + v &= 0 \quad \text{in } \Omega \\
v &= f_1 \quad \text{on } \partial \Omega \\
\Delta v &= f_2 \quad \text{on } \partial \Omega.
\end{align*} \tag{3.2}
\]

Let \( w := u - v \) be the reflected solution, where \( u \) is the solution to problem (1.1). Then, \( w \) satisfies the following boundary value problem:

\[
\begin{align*}
\Delta^2 w + \tilde{n}(x) w &= -(\tilde{n} - 1) v \quad \text{in } \Omega \\
w &= 0 \quad \text{on } \partial \Omega \\
\Delta w &= 0 \quad \text{on } \partial \Omega. \tag{3.3}
\end{align*}
\]

The main step in proving the lower and upper bounds of \( I_{x_0}(h, hD(x_0)) \) is to prove the following proposition.
**Proposition 4.** Let $\Omega$ be a smooth domain in $\mathbb{R}^3$ and the inclusion $D$ to be strictly embedded inside $\Omega$. Then, there exists $C > 0$ such that

$$\|w\|_{L^2(\Omega)} \leq C \|v\|_{L^1(D)}.$$

**Proof.** Let us define a function space

$$X := \{ \phi \in H^4(\Omega); \phi = \Delta \phi = 0 \text{ on } \partial \Omega \}.$$

Suppose $\Phi \in X$ is a weak solution of the equation

$$\begin{cases}
    \Delta^2 \Phi + \tilde{n} \Phi = w & \text{in } \Omega \\
    \Phi = 0 & \text{on } \partial \Omega \\
    \Delta \Phi = 0 & \text{on } \partial \Omega,
\end{cases} \quad (3.4)$$

where $w$ satisfies Equation (3.3). By multiplying Equation (3.4) by $\overline{w}$ and integrating by parts, we obtain,

$$\int_{\Omega} |w(x)|^2 dx = \int_{\Omega} \nabla \cdot \nabla (\Delta \Phi) \overline{w} dx + \int_{\Omega} \tilde{n}(x) \Phi(x) \overline{w(x)} dx
= \int_{\Omega} \Delta \Phi \Delta w dx - \int_{\partial \Omega} \Delta \Phi \frac{\partial w}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta \Phi) \overline{w(x)} dS
+ \int_{\Omega} \tilde{n}(x) \Phi(x) \overline{w(x)} dx.$$

Because $w = 0$, $\Delta \Phi = 0$ on $\partial \Omega$, the above identity becomes

$$\int_{\Omega} |w(x)|^2 dx = \int_{\Omega} \Delta \Phi \Delta w dx + \int_{\Omega} \tilde{n}(x) \Phi(x) \overline{w(x)} dx. \quad (3.5)$$

By multiplying Equation (3.3) by $\Phi$ and integrating by parts, we obtain

$$\int_{\Omega} \Delta \overline{w} \Delta \Phi dx + \int_{\Omega} \tilde{n}(x) \overline{w(x)} \Phi(x) dx = - \int_{\Omega} (\overline{n} - 1) \overline{v} \Phi dx. \quad (3.6)$$

Then, by combining the real parts of Equation (3.5) and (3.6), we obtain

$$\|w\|^2_{L^2(\Omega)} = - \text{Re} \int_{D} n_D \overline{v} \Phi dx.$$ 

Using the Cauchy-Schwarz inequality, we obtain

$$\|w\|^2_{L^2(\Omega)} \leq C \|v\|_{L^1(D)} \|\Phi\|_{L^\infty(D)}. \quad (3.7)$$
Now, we apply the Sobolev embedding and elliptic estimate (see Lemma 8) to obtain
\[
\|\Phi\|_{L^\infty(D)} \leq C\|\Phi\|_{L^\infty(\Omega)} \leq C\|\Phi\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}.
\] (3.8)
Finally, the conclusion follows from Equations (3.7) and (3.8).

Lemma 5. Assume that functions \(v\) and \(w\) are the solutions to Equations (3.2) and (3.3), respectively. Then, we obtain:

1. the lower bound of the indicator function
\[
|\text{Re} I_{x_0}(h, t)| \geq C \int_D |v(x)|^2 dx - c \int_D |w(x)|^2 dx,
\]
2. and the upper bound of the form
\[
|\text{Re} I_{x_0}(h, t)| \leq C \int_D |v(x)|^2 dx + c \int_D |w(x)|^2 dx,
\]
where \(C\) and \(c > 0\) are constants.

Proof. Let us recall that \(\mathcal{N}_D\) denotes the Dirichlet–to–Neumann map that encodes the current measurement on the boundary \(\partial \Omega\) corresponding to the boundary voltage \(u = f\) prescribed on \(\partial \Omega\), when an obstacle \(D\) is embedded in the domain \(\Omega\). We write the weak form of \(\mathcal{N}_D\) as:
\[
\langle \mathcal{N}_D f, f \rangle = \int_{\partial \Omega} \langle \frac{\partial u}{\partial \nu}, \frac{\partial}{\partial \nu} (\Delta u), (f_1, f_2) \rangle dS
= \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} f_2 + \frac{\partial}{\partial \nu} (\Delta u) f_1 \right) dS,
\] (3.9)
where \(u\) satisfies Equation (1.1). Moreover, we denote \(\mathcal{N}_\emptyset\) as the Dirichlet–to–Neumann map when no obstacle is in \(\Omega\). It has the following weak form:
\[
\langle \mathcal{N}_\emptyset f, f \rangle = \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} f_2 + \frac{\partial}{\partial \nu} (\Delta v) f_1 \right),
\]
where \(v\) satisfies Equation (3.2). By multiplying problem (1.1) by \(v\) and integrating by parts, we obtain
\[
0 = \int_\Omega \nabla \cdot \nabla (\Delta u) v dx + \int_\Omega \tilde{\nu} u v dx
= \int_\Omega \Delta u \Delta v dx - \int_{\partial \Omega} f_2 \frac{\partial v}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta u) f_1 dS + \int_\Omega \tilde{\nu} u v dx.
\] (3.10)
Using Equation (3.10), the Dirichlet–to–Neumann map can be written as
\[ \langle N_D f, f \rangle = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} f_2 + \frac{\partial \overline{u}}{\partial \nu} f_2 \right) dS - \int_{\Omega} \Delta u \Delta \overline{v} dx - \int_{\Omega} \overline{n} u \overline{v} dx. \] (3.11)

Moreover, by multiplying Equation (3.2) by \( u \) and integrating by parts, we obtain
\[ 0 = \int_{\Omega} (\Delta^2 v) \overline{u} dx + \int_{\Omega} \overline{uv} dx = \int_{\Omega} \Delta \overline{u} \Delta v dx - \int_{\partial \Omega} f_2 \frac{\partial v}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta v) \overline{f_1} dS + \int_{\Omega} \overline{uv} dx. \] (3.12)

By taking the real part of Equation (3.12), we compute the following Dirichlet–to–Neumann map:
\[ \text{Re} \langle N_\emptyset f, f \rangle = \text{Re} \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} f_2 dS + \int_{\partial \Omega} \frac{\partial (\Delta \overline{v})}{\partial \nu} f_1 dS ight) \]
\[ = \text{Re} \int_{\partial \Omega} \frac{\partial v}{\partial \nu} f_2 dS - \text{Re} \int_{\Omega} \Delta u \Delta \overline{v} dx + \text{Re} \int_{\partial \Omega} f_2 \frac{\partial u}{\partial \nu} dS - \text{Re} \int_{\Omega} u \overline{v} dx. \] (3.13)

Then, Equations (3.11) and (3.13) provide that
\[ - \text{Re} I_{x_0}(h, t) = - \text{Re} \int_{\partial \Omega} \langle (N_D - N_\emptyset)f, f \rangle dS \]
\[ = \text{Re} \int_{\Omega} (\overline{n} - 1) u \overline{v} dx. \] (3.14)

By applying the Cauchy-Schwartz inequality to Equation (3.14), we obtain the upper bound of \(- I_{x_0}(h, t)\).

To estimate the lower bound of \(- \text{Re} I_{x_0}(h, t)\), we use Cauchy’s \( \epsilon \) inequality (see [4, Appendix]), as follows:
\[ - \text{Re} I_{x_0}(h, t) = \text{Re} \int_{\Omega} (\overline{n} - 1) u \overline{v} dx \geq C \int_D |v|^2 dx - c \int_D |w|^2 dx, \]
where \( C \) and \( c > 0 \) are positive constants. Finally, conclusions can be derived.

\[ \square \]

3.2. End of the proof of Theorem 2

We provide a detailed estimate for the complex geometrical optics solutions related to the bi-Laplace equation. Let \( B(\alpha, \delta) \) denote a ball of radius \( \delta \) centered at \( \alpha \). We define,
We first compute the following integral using the Hölder inequality.

Simplifying this, we obtain

\[
\int_{D \setminus D_\delta} e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx = O(\frac{1}{h}) \text{ as } h \to 0.
\]

We introduce a change of co-ordinates as in [26], \( y' = x' \), \( y_3 = \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \), where \( x' = (x_1, x_2) \), \( y' = (y_1, y_2) \), \( x = (x', x_3) \), and \( y = (y', y_3) \). We also denote the parameterization \( \partial D \) near \( \alpha_j \) by the \( C^1 \) function \( l_j(y') \). Then, we have the following lemma.

**Lemma 6.** The following upper and lower estimates hold for \( h \ll 1 \):

(i) For \( 1 \leq q < 2 \), we have

\[
\int_D |v(x)|^q dx \leq C h \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q|l_j(y')|}{h}} dy' + \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-2\frac{q|l_j(y')|}{h}} dy' \right)^{\frac{2}{q+1}} \right) + \text{exponentially decaying terms.}
\]

(ii) When \( q = 2 \), it follows that

\[
\int_D |v(x)|^2 dx \leq C h \sum_{j=1}^N \int_{|y'|<\delta} e^{-2\frac{|l_j(y')|}{h}} dy' + \text{exponentially decaying terms}
\]

and

\[
\int_D |v(x)|^2 dx \geq C h \sum_{j=1}^N \int_{|y'|<\delta} e^{-2\frac{|l_j(y')|}{h}} dy' + \text{exponentially decaying terms.}
\]

**Proof.** Recall that, when \( t = h_D(x_0) \), the complex geometrical optics solution \( v \) is of the form \( v(x, h) = e^{\frac{\phi}{h}} (a_0(x) + ha_1(x) + r(x, h)) \) where \( \phi = h_D(x_0) - \frac{1}{2} \log |x-x_0|^2 \). In addition, the correction term \( r \) satisfies

\[
\|r\|_{H^{2}_{\text{reg}}(\Omega)} \leq O(h^2).
\]

Simplifying this, we obtain

\[
\|r\|_{L^2(\Omega)} \leq h^2. \tag{3.15}
\]

(1) We first compute the following integral using the Hölder inequality.

\[
\int_D |v(x)|^q dx \leq C \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))}(a_0^q + h^q a_1^q + r^q) dx
\]
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\[
= \left( \int_{D_\delta} + \int_{D \setminus D_\delta} \right) e^{-\frac{q}{\kappa} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} (a_0^q + h^q a_1^q + r^q) dx
\]

\[
\leq C(1 + h^q) \sum_{j=1}^N \int_{|y'|<\delta} dy' \int_{l_j(y')} \frac{e^{-q y_j(y')}{\kappa}}{h} dy_n
\]

\[
+ C \left( \int_{D_\delta} e^{-\frac{q}{\kappa} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx \right)^{\frac{1}{p}} \left( \int_{D_\delta} r^2 \right)^{\frac{q}{2}} + C e^{-\frac{c}{h}},
\]

where \( p = \frac{2}{2-q} \). Then, the estimate \( \|r\|_{L^2(\Omega)} \leq h^2 \) yields

\[
\int_D |v(x)|^q dx
\]

\[
\leq C(h + h^{q+1}) \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q y_j(y')}{h}} dy' + h^{2q+\frac{2-q}{2}} \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q y_j(y')}{h}} dy' \right)^{\frac{1}{p}}
\]

\[- C \frac{(1 + h^q)h}{q} e^{-\frac{q h^q}{h}} + Ch^{2q+\frac{2-q}{2}} e^{-\frac{q h}{h}} + C e^{-\frac{c}{h}}.
\]

Here, we observe that

\[
h^{q+1} \leq o(h) \quad \text{and} \quad h^{2q+\frac{2-q}{2}} \leq o(h)
\]

for a sufficiently small \( h \), and the last three terms are exponentially decaying. Therefore, it follows that

\[
\int_D |v(x)|^q dx \leq C h \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q y_j(y')}{h}} dy' + \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q y_j(y')}{h}} dy' \right)^{\frac{1}{p}} \right)
\]

+ exponentially decaying terms.

(2) We then compute the upper bound estimate of the \( L^2 \)-norm of \( v \).

\[
\int_D |v(x)|^2 dx
\]

\[
\leq C \int_D e^{-\frac{q}{\kappa} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx + Ch^2 \int_D e^{-\frac{q}{\kappa} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx
\]

\[
+ C \int_D e^{-\frac{q}{\kappa} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} r^2 dx
\]

\[
:= I_1 + I_2 + I_3.
\]

\( \dagger \) Here, \( o(\cdot) \) denotes small \( o \) notation.
For $I_1$ and $I_2$, we have

\[ I_1 := \int_D e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx \]

\[ \leq \left( \int_{D_\delta} + \int_{D \setminus D_\delta} \right) e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx \]

\[ \leq \int_{D_\delta} e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx + \text{exponentially decaying terms} \]

\[ \leq C \sum_{j=1}^N \int_{|y'|<\delta} dy' \int_{I_j(y')} e^{-\frac{2|y'|}{\delta}} dy_n + \text{exponentially decaying terms} \]

\[ \leq C h \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2|y'|}{\delta}} dy' + \text{exponentially decaying terms}, \]

and

\[ I_2 := h^2 \int_D e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx \]

\[ = h^2 \left( \int_{D_\delta} + \int_{D \setminus D_\delta} \right) e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} dx \]

\[ \leq c h^3 \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2|y'|}{h}} dy' + \text{exponentially decaying terms}. \]

Now, we estimate the remainder. The Hölder inequality yields

\[ I_3 := \int_D e^{-\frac{2}{\pi} \left( \frac{1}{2} \log |x-x_0|^2 - h_D(x_0) \right)} r^2 dx \]

\[ \leq C \|r\|_{L^2(\Omega)}^2 \leq C h^4. \]

Furthermore, we observe that for $h \ll 1$,

\[ \int_{|y'|<\delta} e^{-\frac{2|y'|}{h}} dy' \geq C \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2|y'|}{h}} dy' \]

\[ \geq C h^2 \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2|y'|}{\pi}} dy' \]

\[ \geq C h^2. \]
Here, we use $l_j(y') \leq C|y'|$ if $\partial D$ is $C^1$. Hence, we obtain
\[
I_3 \leq Ch^2 \int_{|y'|<\delta} e^{-\frac{2l_j(y')}{h}dy'}.
\]
Because the first term $I_1$ dominates the remaining terms, it follows that
\[
\int_D |v(x)|^2 dx \leq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2l_j(y')}{h}dy'} + \text{exponentially decaying terms}.
\]

For the lower bound estimate of the $L^2$-norm of $v$, we observe that
\[
\int_D |v(x)|^2 dx \geq C \int_D e^{-\frac{2}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx - Ch^2 \sum_{j=1}^N \int_D e^{-\frac{2}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx \]
\[
- C \int_D e^{-\frac{2}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} r^2 dx
\]
\[
= I_1 - I_2 - I_3.
\]
For $I_1$, we have
\[
I_1 = \int_D e^{-\frac{2}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx
\]
\[
\geq \int_{D_{\delta}} e^{-\frac{2}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx
\]
\[
\geq C \sum_{j=1}^N \int_{|y'|<\delta} dy' \int_{l_j(y')}^\delta e^{-\frac{2l_j(y')}{h}dy'} dy_n
\]
\[
\geq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2l_j(y')}{h}dy'} - \frac{C}{2} he^{-\frac{2\delta}{h}}.
\]
Therefore, with the previous estimates for $I_2$ and $I_3$, we conclude that
\[
\int_D |v(x)|^2 dx \geq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2l_j(y')}{h}dy'} + \text{exponentially decaying terms}.
\]
Proof of Theorem 2. We first prove

\[ \frac{\|w\|^2_{L^2(D)}}{\|v\|^2_{L^2(D)}} \leq Ch, \ h \ll 1. \]

Proposition 4 gives that

\[ \frac{\|w\|^2_{L^2(D)}}{\|v\|^2_{L^2(D)}} \leq C \frac{\|v\|^2_{L^1(D)}}{\|v\|^2_{L^2(D)}}. \]

Using Lemma 6 with an elementary inequality

\[ \sum_{j=1}^{N} \int_{|y'|<\delta} \left| y' \right| e^{- \frac{lj(y')}{h}} dy' \leq C \left( \sum_{j=1}^{N} \int_{|y'|<\delta} e^{- \frac{2lj(y')}{h}} dy' \right)^{1/2}, \]

we get

\[ \frac{\left( \int_{D} |v| dx \right)^2}{\int_{D} |v|^2 dx} \leq C h^2 \left( \sum_{j=1}^{N} \int_{|y'|<\delta} e^{- \frac{lj(y')}{h}} dy' \right)^2 + \text{exponentially decaying terms} \]

\[ \leq C h \sum_{j=1}^{N} \int_{|y'|<\delta} e^{- \frac{2lj(y')}{h}} dy' + \text{exponentially decaying terms} \]

\[ \leq Ch \sum_{j=1}^{N} \int_{|y'|<\delta} e^{- \frac{2lj(y')}{h}} dy' + \text{exponentially decaying terms} \]

\[ = \mathcal{O}(h) \ (h \to 0). \]

Therefore, from Lemma 5 (1), we obtain

\[ \frac{\left| \text{Re} I_{x_0}(h, hD(x_0)) \right|}{\int_{D} |v|^2 dx} \geq C - c \int_{D} |w|^2 dx \geq C - ch \geq C, \ (h \to 0). \]

Using Lemma 6 (2), and

\[ \sum_{j=1}^{N} \int_{|y'|<\delta} e^{- \frac{2lj(y')}{h}} dy' \geq Ch^2, \]

we have

\[ \int_{D} |v(x)|^2 dx \geq Ch^3, \ h \ll 1. \]
Therefore, it follows that

\[ |\text{Re} I_{x_0}(h, t)| \geq C \int_D |v(x)|^2 dx \geq Ch^3, \]

which yields

\[ |h^{-3} \text{Re} I_{x_0}(h, t)| \geq C > 0 \quad \text{for } h \ll 1. \]

For the upper bound, we use Lemma 6 (2), Proposition 4, and Lemma 6 (1) as follows:

\[
|\text{Re} I_{x_0}(h, t)| \leq C \int_D |v(x)|^2 dx + C \int_D |w(x)|^2 dx
\leq C \int_D |v(x)|^2 dx + \left( \int_D |v(x)| dx \right)^2
\leq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2j(y')}{h}} dy' 
+ h^2 \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2j(y')}{h}} dy' \right)^{\frac{1}{2}} + \text{exponentially decaying terms}
\leq Ch.
\]

\[ \square \]

Remark 7. Theorem 2 likely holds for domains with Lipschitz regularity, excluding the cusp-type domains. In [13], Ikehata argued to prove a lemma similar to Lemma 6 for the Helmholtz equation. See [13, Section 4] for the counterexample and [13, Lemma 3.2] for the proof of the case of a linear phase. In our case, because domain \( D \) is bounded \( C^1 \) regular, the technique in the proof of Lemma 6 holds.

4. Appendix

We provide a detailed proof of the \( L^2 \) regularity estimate of the solutions for the bi-Laplace equation with non-smooth coefficients in the \( n \)-dimensional domain.

Lemma 8. We assume \( \Omega \subset \mathbb{R}^n, n \geq 3 \) to be an open bounded set with a sufficiently smooth
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regular boundary. Let \( u \) be a solution of the following fourth–order elliptic equation:

\[
\begin{aligned}
\Delta^2 u + \tilde{n}(x)u &= f \quad \text{in } \Omega \\
u &= h_1 \quad \text{on } \partial \Omega \\
\Delta u &= h_2 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(4.1)

where coefficient \( \tilde{n}(x) \in L^\infty(\Omega) \). Then, for any \( f \in L^2(\Omega) \), \( h_1 \in H^{7/2}(\partial \Omega) \) and \( h_2 \in H^{3/2}(\partial \Omega) \), there exists a unique solution \( u \in H^4(\Omega) \) to Equation (4.1) such that

\[
\|u\|_{H^4(\Omega)} \leq C \left[ \|f\|_{L^2(\Omega)} + \|h_1\|_{H^{7/2}(\partial \Omega)} + \|h_2\|_{H^{3/2}(\partial \Omega)} \right],
\]

where \( C > 0 \) is a constant independent of the data.

Proof. Define the Sobolev space

\[
H^2(\Omega) := \{ u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), \text{ for } |\alpha| \leq 2 \}
\]

with the usual Sobolev norm

\[
\|u\|_{H^2(\Omega)} = \sum_{|\alpha| = 2} \|D^\alpha u\|_{L^2(\Omega)}.
\]

We now define another norm

\[
\|\|u\|\| := \|\Delta u\|_{L^2(\Omega)}.
\]

Notice that these two norms \( \| \cdot \| \) and \( \|\| \cdot \|\| \) are equivalent for all \( u \in H^2(\Omega) \cap H^1_0(\Omega) \). It follows by combining interpolation inequalities (see \([\text{[4]}, \text{Section 5.10, Problem 9}]\))

\[
\|\nabla u\|_{L^2(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \|D^2 u\|_{L^2(\Omega)},
\]

the Poincaré inequality \( \|u\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \), and the identity

\[
\|D^2 u\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)},
\]

(see \([\text{[6]}, \text{Corollary 9.10}]\)). Therefore, the space \( H^2_0(\Omega) \) can be defined as the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \|\| \cdot \|\| \). See \([\text{[5]}, \text{Chapter 2}]\) for more details. We now consider the following homogeneous boundary value problem

\[
\begin{aligned}
\Delta^2 u + \tilde{n}(x)u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
\Delta u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(4.2)
We then define the bilinear form
\[ \mathcal{L} : H^2(\Omega) \cap H^1_0(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) \to \mathbb{R} \]
by
\[ \mathcal{L}(u, v) := \langle u, v \rangle = \int_\Omega \Delta u \Delta v \, dx + \int_\Omega \tilde{n}(x) uv \, dx. \]
It is easy to verify that \( \mathcal{L} \) is bounded
\[ |\mathcal{L}(u, v)| \leq C \|\Delta u\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \]
as well as coercive:
\[ |\mathcal{L}(u, u)| \geq C \|\Delta u\|_{L^2(\Omega)}^2 \]
for all \( u, v \in H^2(\Omega) \cap H^1_0(\Omega) \). By the Lax-Milgram lemma, for any \( f \in L^2(\Omega) \), there is a unique weak solution \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) to Equation (4.2) such that
\[ \|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \]
where \( C > 0 \) is a constant. To obtain a strong solution to Equation (4.2), the interior and boundary regularity results can be used [7]. In this case, we obtain an \( H^4(\Omega) \) solution to Equation (4.2) and estimate of the form
\[ \|u\|_{H^4(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \]
To prove the well-posedness of Equation (4.1), we first reduce the problem to a homogeneous form. Let us define \( v := u - \tilde{h}_1 \) such that \( \tilde{h}_1|_{\partial \Omega} = h_1 \) and \( \tilde{h}_2|_{\partial \Omega} = h_2 \). Then, Equation (4.1) can be reduced to
\[
\begin{cases}
\Delta^2 v + \tilde{n}(x)v = f - \tilde{n}\tilde{h}_1 - \Delta \tilde{h}_2 & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega \\
\Delta v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Because the trace maps \( T_1 : H^4(\Omega) \to H^{7/2}(\Omega) \) by \( T_1(u) = u|_{\partial \Omega} \) and \( T_2 : H^2(\Omega) \to H^{3/2}(\Omega) \) by \( T_2(\Delta u) = (\Delta u)|_{\partial \Omega} \) are surjective and have bounded inverses, the functions \( \tilde{h}_1 \in H^4(\Omega) \) and \( \Delta \tilde{h}_2 \in H^2(\Omega) \), respectively. Therefore, the \( H^4(\Omega) \) estimates for the homogeneous Equation (4.2) imply that
\[ \|u\|_{H^4(\Omega)} \leq C \left[ \|f\|_{L^2(\Omega)} + \|h_1\|_{H^{7/2}(\partial \Omega)} + \|h_2\|_{H^{3/2}(\partial \Omega)} \right]. \]
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