The colored-noise driven Kuramoto model in the unified colored noise approximation

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Many natural and social phenomena are characterized by synchronization. The Kuramoto model, taking into account the basic ingredients for observing synchronized states, allows to study mathematically synchronization in a simplified but nontrivial picture. Here we study how a noise that is correlated on a finite time-scale $\tau$ impacts on the ability of the Kuramoto model to achieve synchronization. We develop an approximated theory that allows to compute the critical coupling constant $k_c$ as a function of the correlation time $\tau$. We obtain that that $k_c(\tau)$ decreases as $\tau$ increases indicating that colored noise promotes synchronization. Moreover we show that theory describes qualitatively well the degree of synchronization near $k_c$ obtained numerically. Finally we show that, independently on the value of $\tau$, the curves of the order parameter versus $k$ scale on the same master curve even at values of $k$ very far from $k_c$.

I. INTRODUCTION

The firing of neurons in the visual cortex [1, 2], the frequency locking in Josephson arrays [3], and the flashing in large groups of fireflies [4] are just three examples that range from biology, to physics and animal behavior where synchronization plays a fundamental role [5]. Winfree realized that nonlinear interactions are a key ingredient for synchronization phenomena that happens above a threshold value of the coupling constant [6, 7]. The Kuramoto model (KM), originally introduced by Kuramoto in 1975 contains all the basic ingredients needed to observe synchronization [8]. In the KM a population of oscillators, each of them characterized by its own natural frequency, are coupled globally through a non-linear interaction. The model is thus general enough to describe diverse situations, contains a small number of control parameters, and, because of the global coupling, is analytically tractable.

The model undergoes a continuous phase transition towards a partially synchronous state [9]. In absence of noise, the control parameter of the transition is the coupling constant $k$ between the oscillators. When white noise is taken into account, this acts as a random perturbation that prevents the system from reaching a perfectly synchronized state and makes the critical value of the coupling constant noise-dependent, i. e., $k_c = k_c(T)$, with the “temperature” $T$ representing the strength of the noise. However, in many biological systems [10], the external noise has no flat spectrum. A colored noise, that is exponentially correlated in time on a finite time scale $\tau$, is a step towards more realistic description of the system.

In this paper, we study the KM in presence of Gaussian exponentially correlated noise [11-13]. We show that, within the Unified Colored Noise Approximation (UCNA) [14-16], it is possible to solve the colored-noise driven KM for the critical coupling constant $k_c$ which becomes decreasing function of $\tau$. This is in agreement with early numerical and theoretical works [12, 13] and we further test it by computer simulations which, however, reveal significant quantitative deviations from the UCNA predictions at large values of $\tau$. Moreover, we compute analytically the critical behavior of the order parameter that is found to grow as $[k - k_c(\tau)]^{1/2}$, i.e. with the same mean-field exponent of the KM in presence of thermal noise. Finally we show that, intriguingly, the synchronization order parameter of the KM displays a universal behavior which holds well even when $k$ is very far from $k_c$, at all values of $\tau$ simulated.

II. THEORY

We consider the KM defined by the set of equations [9]

$$\dot{\phi}_i = \omega_i - \frac{k}{N} \sum_{j=1}^{N} \sin(\phi_i - \phi_j) + \eta_i$$

(1)

where $\phi_i \in [0, 2\pi]$ is the phase of the $i-$th oscillator $(i = 1, \ldots, N)$, $k$ is the coupling strength and the $\omega_i$ is the (random) natural frequency distributed among the oscillators according to the probability function $g(\omega)$ which is assumed to be symmetric and with zero mean. The $\eta_i$ are a set colored noise sources evolving according to the stochastic equations:

$$\dot{\eta}_i = -\tau^{-1} \eta_i + D^{1/2} \tau^{-1} \zeta_i$$

(2)

where $\tau$ is the correlation time of the noise, $D$ is the noise amplitude and the $\zeta_i$ are a set of uncorrelated standard white noise sources: $\langle \zeta_i(0) \zeta_j(t) \rangle = 2\delta_{ij} \delta(t)$. 

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Following Ref. [17] we define the (complex) order parameter $\sigma \exp(i \phi_0)$ as

$$\sigma \exp(i \phi_0) = \frac{1}{N} \sum_{j=1}^{N} \exp(i \phi_j) \quad (3)$$

By using Eq. (3) we can rewrite Eq. (1) as

$$\dot{\psi} = \omega - k \sigma \sin \psi + \eta \quad (4)$$

where $\psi = \phi_i - \phi_0$ and we have dropped the index $i$ for lightening the notation. By using the UCNA [15] we can write an approximated evolution equation for the probability distribution $n = n(t, \psi, \omega)$

$$\dot{n} = -\partial_\psi \left[ f \gamma n - D \gamma \partial_\psi (\gamma n) \right] \quad (5)$$

where $f = \omega - k \sigma \sin \psi$ is the external field appearing in Eq. (1) and $\gamma = [1 - \tau \partial_\psi f]^{-1}$ is the non-homogeneous friction of the UCNA. The stationary probability distribution $n(\psi, \omega)$ is found from Eq. (5) by setting the probability current to a constant:

$$f \gamma n - D \gamma \partial_\psi (\gamma n) = \text{const} \quad (6)$$

The solution of Eq. (6) can be obtained by imposing periodic the boundary condition $n(0, \omega) = n(2\pi, \omega)$. Following the method of Ref. [18] we thus have

$$n(\psi, \omega) = \frac{N \exp(F(\psi, \omega))}{1 - e^{2\pi \omega / D}} \left( \int_0^\psi d\psi' e^{-F(\psi', \omega)} G(\psi') + e^{2\pi \omega / D} \int_2^\psi d\psi' e^{-F(\psi', \omega)} G(\psi') \right) \quad (7)$$

where

$$F(\psi, \omega) \equiv k \sigma \left[ k \sigma \tau \cos(2\psi) + 4 \tau \omega \sin \psi + 4 \cos \psi \right] + 4 \psi \omega$$

and $G(\psi) \equiv |1 + k \sigma \tau \cos \psi|$

and the constant $N$ is set by the normalization condition $\int d\psi n(\psi, \omega) = 1$. The self-consistent equation for the order parameter $\sigma$ is obtained by integrating over the “disorder” $\omega$, i.e.

$$\sigma = \int_{-\infty}^{\infty} d\omega g(\omega) \int_0^{2\pi} d\psi n(\psi, \omega) \exp(i \psi) \quad (8)$$

To find the critical coupling strength $k_c$ and the behavior of $\sigma$ near $k_c$ we expand the r.h.s. of Eq. (8) up to $\sigma^3$ so that this equation becomes

$$\sigma - k \sigma \int_{-\infty}^{\infty} d\omega Q(\omega) + k^3 \sigma^3 \int_{-\infty}^{\infty} d\omega P(\omega) = 0 \quad (9)$$

where

$$Q(\omega) = \frac{1}{2} g(\omega) \left( \frac{D}{D^2 + \omega^2} + \tau \right)$$

$$P(\omega) = \frac{g(\omega) \left[ 2D^3 - 4D \omega^2 + \tau^2 (6D^5 + 6D^3 \omega^2) \right]}{8 (D^2 + \omega^2)^2 (4D^2 + \omega^2)}$$

Obviously $\sigma = 0$ is always a solution of Eq. (9), however an additional real positive root of the form

$$\sigma = \left[ k \int_{-\infty}^{\infty} d\omega Q(\omega) - 1 \right]^{1/2} \quad (10)$$

appears when $k \int_{-\infty}^{\infty} d\omega Q(\omega) > 1$ yielding the equation

$$k_c = 1/ \int d\omega Q(\omega), \text{ i.e.}$$

$$k_c(\tau) = \frac{2}{\tau + D \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{(\tau + D)^3}} \quad (11)$$

Note that Eq. (11) reduces to the well known formula for the critical coupling of the KM in presence of white noise.
in the limit $\tau \to 0$ (see Ref.s [17] and [12]). Moreover from Eq. (10) we see that, for $k$ close to $k_c$ and independently on the value of $\tau$, the order parameter grows as

$$\sigma \sim |k - k_c(\tau)|^{1/2} \quad (12)$$

Summarizing our results by using the UCNA we have found that, while the critical coupling strength decreases with $\tau$ (Eq. (11)), the critical behavior of the order parameter near $k_c$ is still determined by the mean-field exponent $1/2$ (Eq. (12)).

### III. SIMULATIONS

To test the results of the previous section we integrate numerically the equations of motion (1) and (2) by using an Euler algorithm that, taking advantage of peculiar form of the interaction, results in a computational cost $O(N)$ [19]. Following previous works [11, 12] we simulate $N = 5000$ oscillators fixing $D = 1$ scanning several values of $k$ and $\tau$. We make the usual choice of distributing the $\omega_i$ according to a Cauchy-Lorentz probability function

$$g(\omega) = \frac{1}{\pi} \frac{\lambda}{\omega^2 + \lambda^2} \quad (13)$$

setting $\lambda = 1/2$. In all simulations the time-step is set to $10^{-3}$, each simulation runs at fixed $k, \tau$ and $D$ for $2 \times 10^6$ steps and the order parameter $\sigma$ is computed as in Eq. (5) over the last $10^6$ steps averaging over all spins.

Simulation results are shown in Fig. 1 where its clear that the order parameter goes to zero at $k$ values that become progressively lower as $\tau$ increases. In Fig. 1 we also compared the simulation data with the numerical solution of the theoretical Eq. (3) for $\tau = 0$. To obtain this solution we evaluate the integrals in Eq. (3) by the trapezoidal rule and solve at each value of $k$ by using the Newton method. The close agreement between the simulations and Eq. (3) confirms that a system with $N = 5000$ behaves essentially as in the thermodynamic limit (as noticed e.g. in Ref. [12]) with some deviation observed for $k < k_c$.

![Fig. 1](image1.png)

**Fig. 1**: Numerical values of the order parameter for a system of $N = 5000$ oscillators as a function of $k$. Each colored curve corresponds to a different value of $\tau$ (see legend). The dashed line is the numerical solution of the self-consistent equation for $\tau = 0$.

![Fig. 2](image2.png)

**Fig. 2**: Symbols are the values of $k_c$ obtained numerically. The full line is the theoretical prediction of the UCNA. The dashed dotted line is the expected asymptote of $k_c(\tau)$ in the limit $\tau \to \infty$. The dashed line is a fit with an empirical formula obtained by slightly modifying the theory.

From Fig. 1 $k_c$ is evidently a decreasing function of $\tau$ as predicted by theory (see Eq. (11)), however a more accurate comparison shows a clear quantitative deviation. To see this we calculate $k_c(\tau)$ from Eq. (11) with the choice $D = 0$

$$k_c(\tau) = \frac{2(D + \lambda)}{1 + \tau(D + \lambda)} \quad (14)$$

and compare this formula with the values of $k_c$ obtained numerically in Fig. 2. We see that Eq. (14) is close to the numerical $k_c$ for small $\tau$ yielding the exact result $k_c(0) = 2(D + \lambda)$ at $\tau = 0$, however it quickly deviates from the numerical data as $\tau$ increases. In particular, while Eq. (14) predicts that $\lim_{\tau \to \infty} k_c(\tau) = 0$, the numerical $k_c$ seems to approach an asymptotic value $k_c(\infty) > 0$. This behavior is easily understood if we consider that the colored noise “intensity” $\langle n^2 \rangle = D/\tau$ (from Eq. (2)) goes to zero in the $\tau \to \infty$ limit at fixed $D$. Therefore we expect that, in this limit, the critical coupling coincides with the $k_c = 2\lambda$ of the KM in absence of noise [12]. Based on these considerations it is possible to derive a simple empirical formula for $k_c(\tau)$. Assuming that the actual $k_c(\tau)$ has the form of a hyperbola as in Eq. (11)
and requiring that this coincides with the known limiting value \( k_c(0) \) and \( k_c(\infty) \) we have

\[
k_c(\tau) = k_c(\infty) + \frac{k_c(0) - k_c(\infty)}{1 + \alpha \tau}
\]

(15)

where \( \alpha \) is an adjustable parameter. In Fig. 2 we show that such a simple formula interpolates well the data (with \( \alpha \approx 0.5 \)) signaling that, while the UCNA leads to a quantitatively wrong result, the behavior of \( k_c \) is still well captured by a hyperbola as predicted by theory.

Finally we discuss the scaling of the order parameter \( \sigma \) near the transition. As shown Eq. (12) when \( k - k_c(\tau) \) is small the critical behavior of \( \sigma \) should be invariant with respect to the correlation time of the noise \( \tau \). Therefore all data-points, collected varying \( \tau \), should collapse in the same power-law when properly scaled. To test this in Fig. 3 we plot the scaled \( \sigma \) as a function of the scaled coupling \( (k - k_c(\tau))/k_c(\tau) \) (using the \( k_c \) obtained numerically). To scale the \( y \)-value in this plot we minimize numerically the difference between the simulation values of \( \sigma \) and the theoretical curved \( \sigma(k - k_c) \) obtained by solving numerically Eq. (8) for \( \tau = 0 \) (also shown in Fig. 1) in the near-critical region \( (k - k_c)/k_c < 0.5 \). Quite surprisingly we find that a very good data collapse is observed even far from the critical region (i.e. up to values of \( (k - k_c) \approx 2k_c \)).

In conclusions simulations show that the prediction of the UCNA is qualitatively correct but quantitatively inaccurate for locating the critical coupling strength in the presence of colored noise. The critical scaling of the order parameter predicted by the theory is confirmed numerically and it is found to extend well beyond the critical region suggesting that the functional form of the order parameter \( \sigma(k) \) is invariant with respect to the correlation time of the noise \( \tau \).

IV. CONCLUSIONS

In this paper we have studied the KM driven by exponentially correlated noise. We have shown that the UCNA allows to obtain analytical predictions that are in qualitatively agreement with numerical simulations. In particular, we computed the critical coupling \( k_c(\tau) \) that results to be a decreasing function of the correlation time of the noise \( \tau \). This finding is confirmed by numerical data that clearly show the increasing of the magnitude of the order parameter \( \sigma \) as \( \tau \) increases. The observed phenomenology is in agreement with early works on similar models \[11\],[13]. The fact that a persistent noise helps the system to reach more ordered phases has been also observed in different systems driven out-of-equilibrium by an exponentially correlated noise \[20\],[21].

Although the approximated theory does not reproduce quantitatively numerical simulations, the analytical expression for \( k_c(\tau) \) can be employed as a fitting function for the numerical data. In particular, its functional form reproduces quite well numerical simulations in a wide range of \( \tau \) values. Moreover, the theory predicts that colored noise does not change the critical behavior of the order parameter \( \sigma \), as confirmed by numerical simulations that provide evidence for the universal scaling of \( \sigma \) as a function of the relative deviation from the critical coupling \( (k - k_c)/k_c \).

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