Inexact Alternating Minimization Algorithm for Distributed Optimization with an Application to Distributed MPC

Ye Pu, Colin N. Jones and Melanie N. Zeilinger

Abstract

In this paper, we propose the inexact alternating minimization algorithm (inexact AMA), which allows inexact iterations in the algorithm, and its accelerated variant, called the inexact fast alternating minimization algorithm (inexact FAMA). We show that inexact AMA and inexact FAMA are equivalent to the inexact proximal-gradient method and its accelerated variant applied to the dual problem. Based on this equivalence, we derive complexity upper-bounds on the number of iterations for the inexact algorithms. We apply inexact AMA and inexact FAMA to distributed optimization problems, with an emphasis on distributed MPC applications, and show the convergence properties for this special case. By employing the complexity upper-bounds on the number of iterations, we provide sufficient conditions on the inexact iterations for the convergence of the algorithms. We further study the special case of quadratic local objectives in the distributed optimization problems, which is a standard form in distributed MPC. For this special case, we allow local computational errors at each iteration. By exploiting a warm-starting strategy and the sufficient conditions on the errors for convergence, we propose an approach to certify the number of iterations for solving local problems, which guarantees that the local computational errors satisfy the sufficient conditions and the inexact distributed optimization algorithm converges to the optimal solution.

I. INTRODUCTION

First-order optimization methods, see e.g. [15], [3] and [12], play a central role in large-scale convex optimization, since they offer simple iteration schemes that only require information of the function value and the gradient, and have shown good performance for solving large problems with moderate accuracy requirements in many fields, e.g. optimal control [20], signal processing [7] and machine learning [6]. In this paper, we will study a sub-group of first-order methods, called splitting methods, and apply them to distributed optimization problems. Splitting methods, which are also known as alternating direction methods, are a powerful tool for general mathematical programming and optimization. A variety of different splitting methods exist, requiring different assumptions on the problem setup, while exhibiting different properties, see e.g. [12] and [7] for an overview. The main concept is to split a complex convex minimization problem into simple and small sub-problems and solve them in an alternating manner. For a problem with multiple objectives, the main strategy is not to compute the descent direction of the sum of several objectives, but to take a combination of the descent directions of each objective.

The property of minimizing the objectives in an alternating way provides an efficient technique for solving distributed optimization problems, which arise in many engineering fields [5]. By considering the local cost functions, as well as local constraints, as the multiple objectives of a distributed optimization problem, splitting methods allow us to split a global constrained optimization problem into sub-problems according to the structure of the network, and solve them in a distributed manner. The advantages of using distributed optimization algorithms include the following three points: in contrast to centralized methods, they do not require global, but only local communication, i.e., neighbour-to-neighbour communication; secondly, they parallelize the computational tasks and split the global problem into small sub-problems, which reduces the required computational power for each subsystem; thirdly, distributed optimization algorithms preserve the privacy of each subsystem in the sense that each subsystem computes an optimal solution without sharing its local cost function and local constraint with all the entities in the network.

In this paper, we consider a distributed Model Predictive Control problem as the application for the distributed optimization to demonstrate the proposed algorithms, as well as the theoretical findings. Model Predictive Control (MPC) is a control technique that optimizes the control input over a finite time-horizon in the future and allows for constraints on the states and control inputs to be integrated into the controller design. However, for networked systems, implementing an MPC controller becomes challenging, since solving an MPC problem in a centralized way requires full communication to collect information from each sub-system, and the computational power to solve the global problem in one central entity. Distributed model predictive control [21] is a promising tool to overcome the limiting computational complexity and communication requirements associated with

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centralized control of large-scale networked systems. The research on distributed MPC has mainly focused on the impact of distributed optimization on system properties such as stability and feasibility, and the development of efficient distributed optimization algorithms.

However, a key challenge in practice is that distributed optimization algorithms, see e.g. [5], [6] and [11], may suffer from inexact local solutions and unreliable communications. The resulting inexact updates in the distributed optimization algorithms affect the convergence properties, and can even cause divergence of the algorithm.

In this work, we study inexact splitting methods and aim at answering the questions of how these errors affect the algorithms and under which conditions convergence can still be guaranteed. Seminal work on inexact optimization algorithms includes [14], [9], [13] and [22]. In [14], the authors studied the convergence rates of inexact dual first-order methods. In [9], the authors propose an inexact decomposition algorithm for solving distributed optimization problems by employing smoothing techniques and an excessive gap condition. In [13], the authors proposed an inexact optimization algorithm with an accelerating strategy. The algorithm permits inexact inner-loop solutions. Sufficient conditions on the inexact inner-loop solutions for convergence are shown for different assumptions on the optimization problem.

In [22], an inexact proximal-gradient method, as well as its accelerated version, are introduced. The proximal gradient method, also known as the iterative shrinkage-thresholding algorithm (ISTA) [3], has two main steps: the first one is to compute the gradient of the smooth objective and the second one is to solve the proximal minimization. The conceptual idea of the inexact proximal-gradient method is to allow errors in these two steps, i.e. the error in the calculation of the gradient and the error in the proximal minimization. The results in [22] show convergence properties of the inexact proximal-gradient method and provide conditions on the errors, under which convergence of the algorithm can be guaranteed.

Building on the results in [22], we propose two new inexact splitting algorithms, the inexact Alternating Minimization Algorithm (inexact AMA) and its accelerated variant, inexact Fast Alternating Minimization Algorithm (inexact FAMA). The inexact FAMA has been studied in [19], and is expanded in this paper. The contributions of this work are the following:

- We propose the inexact AMA and inexact FAMA algorithms, which are inexact variants of the splitting methods, AMA and FAMA in [23] and [12]. We show that applying inexact AMA and inexact FAMA to the primal problem is equivalent to applying the inexact proximal-gradient method (inexact PGM) and the inexact accelerated proximal-gradient method (inexact APGM) in [22] to the dual problem. Based on this fact, we extend the results in [22], and show the convergence properties of inexact AMA and inexact FAMA. We derive complexity upper bounds on the number of iterations to achieve a certain accuracy for the algorithms. By exploiting these complexity upper-bounds, we present sufficient conditions on the errors for convergence of the algorithms.
- We study the convergence of the algorithms under bounded errors that do not satisfy the sufficient conditions for convergence and show the complexity upper bounds on the number of iterations for this special case.
- We apply inexact AMA and inexact FAMA for solving distributed optimization problems with local computational errors. We present the complexity upper bounds of the algorithms for this special case, and show sufficient conditions on the local computational errors for convergence.
- We study the special case of quadratic local objective functions, relating to a standard form of distributed MPC problems. We show that if the local quadratic functions are positive definite, then the algorithms converge to the optimal solution with a linear rate. We propose to use the proximal gradient method to solve the local problems. By exploiting the sufficient condition on the local computational errors for the convergence together with a warm-starting strategy, we provide an approach to certify the number of iterations for the proximal gradient method to solve the local problems to the accuracy required for convergence of the distributed algorithm. The proposed on-line certification method only requires on-line local information.
- We demonstrate the performance and the theoretical results for inexact algorithms by solving a randomly generated example of a distributed MPC problem with 40 subsystems.

II. Preliminaries

A. Notation

Let \( v \in \mathbb{R}^n \) be a vector, \( \|v\| \) denotes the \( l_2 \) norm of \( v \). Let \( \mathbb{C} \) be a subset of \( \mathbb{R}^n \). The projection of any point \( v \in \mathbb{R}^n \) onto the set \( \mathbb{C} \) is denoted by \( \text{Proj}_\mathbb{C}(v) := \arg\min_{w \in \mathbb{C}} \|w - v\| \). Let \( f : \Theta \to \Omega \) be a function. The conjugate function of \( f \) is defined as \( f^*(v) = \sup_{w \in \Theta} (v^T w - f(w)) \). For a conjugate function, it holds that \( q \in \partial f(p) \iff p \in \partial f^*(q) \), where \( \partial(\cdot) \) denotes the set of sub-gradadients of a function at a given point. Let \( f \) be a strongly convex function. \( \sigma_f \) denotes the convexity modulus \( \langle p - q, v - w \rangle \geq \sigma_f \|v - w\|^2 \), where \( p \in \partial f(v) \) and \( q \in \partial f(w) \), \( \forall v, w \in \Theta \). \( L(f) \) denotes a Lipschitz constant of the function \( f \), i.e., \( \|f(v) - f(w)\| \leq L(f)\|v - w\|, \forall v, w \in \Theta \). Let \( C \) be a matrix, \( \rho(C) \) denotes the \( l_2 \) norm of the matrix \( C \). The proximity operator is defined as

\[
\text{prox}_f(v) = \arg\min_w f(w) + \frac{1}{2}\|w - v\|^2 .
\]

We note the following equivalence:

\[
w^* = \text{prox}_f(v) \iff v - w^* \in \partial f(w^*)
\]
We refer to [4] and [2] for details on the definitions and properties above. In this paper, * is used to denote an inexact solution of an optimization problem. The proximity operator with an extra subscript * is allowed in the proximal objective function:

\[
f(\hat{w}) + \frac{1}{2}\|\hat{w} - v\|^2 \leq \epsilon + \min_w \left\{ f(w) + \frac{1}{2}\|w - v\|^2 \right\}
\] (3)

### B. Inexact Proximal-Gradient Method

In this section, we will introduce the inexact proximal-gradient method (inexact PGM) proposed in [22]. It addresses optimization problems of the form given in Problem 2.1 and requires Assumption 2.2 for convergence, and Assumption 2.3 for linear convergence. The algorithm is presented in Algorithm 1.

**Problem 2.1:**

\[
\min_{w \in \mathbb{R}^n} \quad \Phi(w) = \phi(w) + \psi(w).
\]

**Assumption 2.2:**

- \(\phi\) is a convex function with Lipschitz continuous gradient with Lipschitz constant \(L(\nabla \phi)\)
- \(\psi\) is a lower semi-continuous convex function, not necessarily smooth.

**Assumption 2.3:**

- \(\phi\) is a strongly convex function with Lipschitz continuous gradient with a convexity modulus \(\sigma_\phi\).
- \(\psi\) is a lower semi-continuous convex function, not necessarily smooth.

**Algorithm 1 Inexact Proximal-Gradient Method**

**Require:** Require \(\hat{w}^0 \in \mathbb{R}^n\) and \(\tau < \frac{L}{L(\nabla \phi)}\)

**for** \(k = 1, 2, \ldots\) **do**

1: \(\hat{w}^k = \text{prox}_{\tau \phi, \epsilon k}(\hat{w}^{k-1} - \tau(\nabla \phi(\hat{w}^{k-1}) + e^k))\)

**end for**

Inexact PGM in Algorithm 1 allows two kinds of errors: \(\{e^k\}\) represents the error in the gradient calculations of \(\phi\), and \(\{\epsilon^k\}\) represents the error in the computation of the proximal minimization in (3) at every iteration \(k\). The following propositions state the convergence property of inexact PGM with different assumptions.

**Proposition 2.4** (Proposition 1 in [22]): Let \(\{\hat{w}^k\}\) be generated by inexact PGM defined in Algorithm 1. If Assumption 2.2 holds, then for any \(k \geq 0\) we have:

\[
\Phi\left(\frac{1}{k} \sum_{p=1}^{k} \hat{w}^p\right) - \Phi(w^*) \leq \frac{L(\nabla \phi)}{2k} \left(\|\hat{w}^0 - x^*\|^2 + 2\Gamma^k + \sqrt{2\Lambda^k}\right)
\]

where \(\Phi(\cdot)\) is defined in Problem 2.1.

\[
\Gamma^k = \sum_{p=1}^{k} \frac{\|e^p\|^2}{L(\nabla \phi)} + \sqrt{\frac{2\epsilon^p}{L(\nabla \phi)}} \quad \Lambda^k = \sum_{p=1}^{k} \epsilon^p
\]

and \(\hat{w}^0\) and \(w^*\) denote the initial sequences of Algorithm 1 and the optimal solution of Problem 2.1 respectively. As discussed in [22], the complexity upper-bound in Proposition 2.4 allows one to derive sufficient conditions on the error sequences \(\{e^k\}\) and \(\{\epsilon^k\}\) for the convergence of the algorithm to the optimal solution \(w^*\):

- The series \(\{\|e^k\|\}\) and \(\{\sqrt{\epsilon^k}\}\) are finitely summable, i.e., \(\sum_{k=1}^{\infty} \|e^k\| < \infty\) and \(\sum_{k=0}^{\infty} \sqrt{\epsilon^k} < \infty\).
- The sequences \(\{\|e^k\|\}\) and \(\{\sqrt{\epsilon^k}\}\) decrease at the rate \(O\left(\frac{1}{k^\kappa}\right)\) for any \(\kappa \geq 0\).

**Proposition 2.5** (Proposition 3 in [22]): Let \(\{x^k\}\) be generated by inexact PGM defined in Algorithm 1. If Assumption 2.3 holds, then for any \(k \geq 0\) we have:

\[
\|\hat{w}^k - w^*\| \leq (1 - \gamma)^k \cdot (\|w^0 - w^*\| + \Gamma^k), \quad (4)
\]

where \(\gamma = \frac{\sigma_\phi}{L(\nabla \phi)}\) and \(w^0\) and \(w^*\) denote the initial sequence of Algorithm 1 and the optimal solution of Problem 2.1 respectively, and

\[
\Gamma^k = \sum_{p=1}^{k} (1 - \gamma)^{-p} \cdot \left(\frac{1}{L(\nabla \phi)} \|e^p\|^2 + \sqrt{\frac{2}{L(\nabla \phi)} \epsilon^p}\right).
\]

From the discussion in [22], we can conclude that, if the series \(\{\|e^k\|\}\) and \(\{\sqrt{\epsilon^k}\}\) decrease at a linear rate, then \(\|x^k - x^*\|\) converges to the optimal solution.
C. Inexact Accelerated Proximal-Gradient Method

In this section, we introduce an accelerated variant of inexact PGM, named the inexact accelerated proximal-gradient method (inexact APGM) proposed in [22]. It addresses the same problem class in Problem 2.1 and similarly requires Assumption 2.2 for convergence.

Algorithm 2 Inexact Accelerated Proximal-Gradient Method

| Require: Initialize $v^1 = \tilde{w}^0 \in \mathbb{R}^{n_w}$ and $\tau < \frac{1}{\nabla\phi}$ |
| for $k = 1, 2, \cdots$ do |
| 1: $\tilde{w}^k = \text{prox}_{\tau \Psi, \epsilon^k}(v^{k-1} - \tau(\nabla\phi(v^{k-1}) + \epsilon^k))$ |
| 2: $v^k = \tilde{w}^k + \frac{\epsilon^k}{k+2}(\tilde{w}^k - \tilde{w}^{k-1})$ |
| end for |

Differing from inexact PGM, inexact APGM involves one extra linear update in Algorithm 2. If Assumption 2.2 holds, it improves the convergence rate of the complexity upper-bound from $O\bigl(\frac{1}{\kappa}\bigr)$ to $O\bigl(\frac{1}{\kappa^2}\bigr)$. The following proposition states the convergence property of inexact APGM.

Proposition 2.6 (Proposition 2 in [22]): Let $\{\tilde{w}_k\}$ be generated by inexact APGM defined in Algorithm 1. If Assumption 2.2 holds, then for any $k \geq 1$ we have:

$$
\Phi(\tilde{w}^k) - \Phi(w^*) \leq \frac{2L(\nabla\phi)}{(k+1)^2} \left(\|\tilde{w}^0 - x^*\| + 2\Gamma^k + \sqrt{2\Lambda^k}\right)^2
$$

where $\Phi(\cdot)$ is defined in Problem 2.1

$$
\Gamma^k = \sum_{p=1}^{k} p \left(\frac{\|e^p\|}{L(\nabla\phi)} + \sqrt{\frac{2\epsilon^p}{L(\nabla\phi)}}\right), \quad \Lambda^k = \sum_{p=1}^{k} p^2\epsilon^p,
$$

and $\tilde{w}^0$ and $w^*$ denote the starting sequence of Algorithm 2 and the optimal solution of Problem 2.1, respectively.

The complexity upper-bound in Proposition 2.6 provides similar sufficient conditions on the error sequences $\{e^k\}$ and $\{\epsilon^k\}$ for the convergence of Algorithm 2:

- The series $\{k\|e^k\|\}$ and $\{k\sqrt{\epsilon^k}\}$ are finite summable.
- The sequences $\{\|e^k\|\}$ and $\{\sqrt{\epsilon^k}\}$ decrease at the rate $O\left(\frac{1}{\kappa^2}\right)$ for $\kappa \geq 0$.

III. INEXACT ALTERNATING MINIMIZATION ALGORITHM AND ITS ACCELERATED VARIANT

The inexact proximal gradient method, as well as its accelerated version is limited to the case where both objectives are a function of the same variable. However, many optimization problems from engineering fields, e.g. optimal control and machine learning [6], are not of this problem type. In order to generalize the problem formulation, we employ the alternating minimization algorithm (AMA) and its accelerated variant in [23] and [12], which cover optimization problems of the form of Problem 3.1. In this section, we extend AMA and its accelerated variant to the inexact case and present the theoretical convergence properties.

Problem 3.1:

$$
\min_{x, z} \quad f(x) + g(z)
\text{s.t.} \quad Ax + Bz = c
$$

with variables $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$, where $A \in \mathbb{R}^{n_c \times n_x}$, $B \in \mathbb{R}^{n_c \times n_z}$ and $c \in \mathbb{R}^{n_c}$. $f : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g : \mathbb{R}^{n_z} \to \mathbb{R}$ are convex functions. The Lagrangian of Problem 3.1 is:

$$
L(x, z, \lambda) = f(x) + g(z) - \lambda^T(Ax + Bz - c), \quad (5)
$$

and the dual function is:

$$
D(\lambda) = \inf_{x, z} L(x, z, \lambda), \quad (6a)
$$

$$
= -\sup_x \{\lambda^TAx - f(x)\} - \sup_z \{\lambda^TBz - g(z)\} + \lambda^Tc
$$

$$
= -f^*(A^T\lambda) - g^*(B^T\lambda) + \lambda^Tc, \quad (6b)
$$

where $f^*$ and $g^*$ are the conjugate functions of $f$ and $g$. The dual problem of Problem 3.1 is:

Problem 3.2:

$$
\min_{\phi, \psi} -D(\lambda) = \underbrace{f^*(A^T\lambda)}_{\phi(\lambda)} + \underbrace{g^*(B^T\lambda)}_{\psi(\lambda)} - c^T\lambda.
$$
A. Inexact alternating minimization algorithm (inexact AMA)

We propose the inexact alternating minimization algorithm (inexact AMA) presented in Algorithm 3 for solving Problem 3.1. The algorithm allows errors in Step 1 and Step 2, i.e. both minimization problems are solved inexactly with errors \( \delta^k \) and \( \theta^k \), respectively.

**Algorithm 3 Inexact alternating minimization algorithm (Inexact AMA)**

**Require:** Initialize \( \lambda^0 \in \mathbb{R}^{N_k} \), and \( \tau < \sigma_f / \rho(A) \)

for \( k = 1, 2, \ldots \) do

1: \( \tilde{x}^k = \text{argmin}_x \{ f(x) + \langle \lambda^{k-1}, -Ax \rangle \} + \delta^k \).

2: \( \tilde{z}^k = \text{argmin}_z \{ g(z) + \langle \lambda^{k-1}, -Bz \rangle + \frac{\tau}{2} ||c - A\tilde{x}^k - B\tilde{z}^k||^2 \} + \theta^k \).

3: \( \lambda^k = \lambda^{k-1} + \tau (c - A\tilde{x}^k - B\tilde{z}^k) \)

end for

We study the theoretical properties of inexact AMA under Assumption 3.3. If Assumption 3.3 holds, we show that inexact AMA in Algorithm 3 is equivalent to applying inexact PGM to the dual problem defined in Problem 3.2 with the following correspondence: the gradient computation error in Algorithm 2 is equal to \( \epsilon^k = A\delta^k \) and the error of solving the proximal minimization is equal to \( \epsilon^k = \tau^2 L(\psi) \| B\theta^k \| + \frac{\tau^2}{2} \| B\theta^k \|^2 \). With this equivalence, the complexity bound in Proposition 2.4 can be extended to the inexact AMA algorithm in Theorem 3.5.

**Assumption 3.3:** We assume that

- \( f \) is a strongly convex function with convexity modulus \( \sigma_f \),
- and \( g \) is a convex function, not necessarily smooth.

**Lemma 3.4:** If Assumption 3.3 is satisfied and inexact AMA and inexact PGM are initialized with the same dual and primal starting sequence, then applying the inexact AMA in Algorithm 3 to Problem 3.1 is equivalent to applying inexact PGM in Algorithm 1 to the dual problem defined in Problem 3.2 with the errors \( \epsilon^k = A\delta^k \) and \( \epsilon^k = \tau^2 L(\psi) \| B\theta^k \| + \frac{\tau^2}{2} \| B\theta^k \|^2 \), where \( L(\psi) \) denotes the Lipschitz constant of the function \( \psi \).

The proof of Lemma 3.4 is provided in the appendix in Section VI-A. This proof is an extension of the proof of Theorem 2 in [12] and the proof in Section 3 in [23]. Based on the equivalence shown in Lemma 3.4, we can now derive an upper-bound on the difference of the dual function value of the sequence \( \{ \lambda^k \} \) from the optimal dual function value in Theorem 3.5.

**Theorem 3.5:** Let \( \{ \lambda^k \} \) be generated by the inexact AMA in Algorithm 3. If Assumption 3.3 holds, then for any \( k \geq 1 \)

\[
D(\lambda^*) - D\left( \frac{1}{k} \sum_{p=1}^{k} \lambda^p \right) \leq \frac{L(\nabla \phi)}{2k} \left( \| \lambda^0 - \lambda^* \| + 2\Gamma^k + \sqrt{2\Lambda^k} \right)^2
\]

where \( L(\nabla \phi) = \sigma_f^{-1} \cdot \rho(A) \), \( \Gamma^k = \sum_{p=1}^{k} \frac{\| A\delta^p \| + \tau \sqrt{2L(\psi) \| B\theta^p \| + \| B\theta^p \|^2}}{L(\nabla \phi)} \), and \( \Lambda^k = \sum_{p=1}^{k} \frac{\tau^2 (2L(\psi) \| B\theta^p \| + \| B\theta^p \|^2)}{2L(\nabla \phi)} \)

and \( \lambda^0 \) and \( \lambda^* \) denote the initial sequences of Algorithm 3 and the optimal solution of Problem 3.1, respectively.

**Proof:** Lemma 3.4 shows the equivalence between Algorithm 3 and Algorithm 1 with \( \epsilon^k = A\delta^k \) and \( \epsilon^k = \tau^2 L(\psi) \| B\theta^k \| + \frac{\tau^2}{2} \| B\theta^k \|^2 \). Then we need to show that the dual defined in Problem 3.2 satisfies Assumption 2.2 and that \( \phi(\lambda) \) and \( \psi(\lambda) \) are both convex, since the conjugate functions and linear functions as well as their weighted sum are always convex (the conjugate function is the point-wise supremum of a set of affine functions). Furthermore, since \( f(x) \) is strongly convex with \( \sigma_f \) by Assumption 3.3, then we know \( f^* \) has Lipschitz-continuous gradient with Lipschitz constant:

\[
L(\nabla f^*) = \sigma_f^{-1}.
\]

It follows that the function \( \phi \) has Lipschitz-continuous gradient \( \nabla \phi \) with a Lipschitz constant:

\[
L(\nabla \phi) = \sigma_f^{-1} \cdot \rho(A).
\]

Hence, the functions \( \phi \) and \( \psi \) satisfy Assumption 2.2 and Proposition 2.4 completes the proof of the upper-bound in (7).

**Corollary 3.6:** Let \( \{ \lambda^k \} \) be generated by the inexact AMA in Algorithm 3. If Assumption 3.3 holds, and the constant \( L(\psi) < \infty \), the following sufficient conditions on the error sequences \( \{ \delta^k \} \) and \( \{ \theta^k \} \) guarantee the convergence of Algorithm 3.
• The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) are finitely summable, i.e., \(\sum_{k=1}^{\infty} \|\delta^k\| < \infty\) and \(\sum_{k=0}^{\infty} \|\theta^k\| < \infty\).
• The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) decrease at the rate \(O(1/k^{\kappa})\) for any \(\kappa > 0\).

**Proof:** By Assumption 3.3, the dual Problem 3.2 satisfies Assumption 1 and the complexity upper-bound in Proposition 2.4 holds. By extending the sufficient conditions on the error sequences for the convergence of inexact proximal-gradient method discussed after Proposition 2.4, we can derive sufficient conditions on the error sequences for inexact AMA with the errors defined in Lemma 3.4 \(e^k = A\delta^k\) and \(e^k = \tau^2 L(\psi)\|B\theta^k\| + \frac{\tau}{2}\|B\theta^k\|^2\). Since \(L(\psi) < \infty\), we have that if the error sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) satisfy the conditions in Corollary 3.6, the complexity upper-bound in Theorem 3.5 converges to zero, as the number of iterations \(k\) goes to infinity, which further implies that the inexact AMA algorithm converges to the optimal solution.

**Remark 3.7:** If the function \(\psi\) is an indicator function on a convex set, then the constant \(L(\psi)\) is equal to infinity, if for any iteration the inexact solution is infeasible with respect to the convex set. However, if we can guarantee that for every iteration the solutions are feasible with respect to the convex set, then the constant \(L(\psi)\) is equal to zero.

1) **Linear convergence of inexact AMA for a quadratic cost:** In this section, we study the convergence properties of inexact AMA with a stronger assumption, i.e. the first objective \(f\) is a quadratic function and coupling matrix \(A\) has full-row rank.

We show that with this stronger assumption, the convergence rate of inexact AMA is improved to be linear. The applications guarantee the convergence of the inexact AMA algorithm.

**Assumption 3.8:** We assume that
• \(f\) is a quadratic function \(f = x^THx + h^Tx\) with \(H \succeq 0\),
• \(A\) has full-row rank.

**Remark 3.9:** If Assumption 3.8 holds, we know that the first objective \(\phi(\lambda)\) in the dual problem in Problem 3.2 is equal to \(\phi(\lambda) = \frac{1}{2}(A^T\lambda - h)^TH^{-1}(A^T\lambda - h)\). Then, a Lipschitz constant \(L(\nabla \phi)\) is given by the largest eigenvalue of the matrix \(\frac{1}{4}AH^{-1}A^T\), i.e., \(L(\nabla \phi) = \lambda_{\max}(\frac{1}{4}AH^{-1}A^T)\). In addition, the convexity modulus of \(\phi(\lambda)\) is equal to the smallest eigenvalue, i.e., \(\sigma_\phi = \lambda_{\min}(\frac{1}{4}AH^{-1}A^T)\).

**Theorem 3.10:** Let \(\{\lambda^k\}\) be generated by inexact AMA in Algorithm 3. If Assumption 3.3 and 3.8 hold, then for any \(k \geq 1\)
\[
\|\lambda^k - \lambda^*\| \leq (1 - \gamma)^k \cdot (\|\lambda^0 - \lambda^*\| + \Gamma^k),
\]
with
\[
\gamma = \frac{\lambda_{\min}(AH^{-1}A^T)}{\lambda_{\max}(AH^{-1}A^T)},
\]
\[
\Gamma^k = \sum_{p=1}^{k} (1 - \gamma)^{-p} \cdot \left(\frac{\|A\delta^p\|}{L(\nabla \phi)} + \tau \sqrt{\frac{L(\psi)\|B\theta^p\| + \|B\theta^p\|^2}{L(\nabla \phi)}}\right),
\]
and \(\lambda^0\) and \(\lambda^*\) denote the initial sequences of Algorithm 3 and the optimal solution of Problem 3.1 respectively.

**Proof:** By Assumption 3.3 and 3.8, the dual problem in Problem 3.2 satisfies Assumption 2.3 and the complexity upper-bound in Proposition 2.5 holds for the dual problem. The proof of Theorem 3.10 follows directly from this fact.

By using the complexity upper-bounds in Theorem 3.10, we derive sufficient conditions on the error sequences, which guarantee the convergence of the inexact AMA algorithm.

**Corollary 3.11:** Let \(\{\lambda^k\}\) be generated by the inexact AMA in Algorithm 3. If Assumption 3.3 and 3.8 hold, and the constant \(L(\psi) < \infty\), the following sufficient conditions on the error sequences \(\{\delta^k\}\) and \(\{\theta^k\}\) guarantee the convergence of Algorithm 3
• The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) are finitely summable, i.e., \(\sum_{k=1}^{\infty} \|\delta^k\| < \infty\) and \(\sum_{k=0}^{\infty} \|\theta^k\| < \infty\).
• The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) decrease at \(O(\frac{1}{k^{\kappa}})\) for any \(\kappa \in \mathbb{Z}_+\). For this case the complexity upper-bound in (10) reduces to the same rate as the error sequences.
• The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) decrease at a linear rate.

**Proof:** By using the complexity upper-bound in Theorem 3.10, we can derive sufficient conditions on the error sequences for inexact AMA. Since \(L(\psi) < \infty\), we have that if the error sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) satisfy the first and third conditions in Corollary 3.11, the complexity upper-bound in Theorem 3.10 converges to zero, as the number of iterations \(k\) goes to infinity, which further implies that the inexact AMA algorithm converges to the optimal solution. For the second sufficient condition in Corollary 3.11 we provide Lemma 3.12 to prove that the second sufficient condition guarantees the convergence of the algorithm.

**Lemma 3.12:** Let \(\alpha\) be a positive number \(0 < \alpha < 1\). The following series \(S^k\) converges to zero, as the index \(k\) goes to infinity
\[
\lim_{k \to \infty} S^k := \lim_{k \to \infty} \alpha^k \cdot \sum_{p=1}^{k} \frac{\alpha^{-p}}{p} = 0.
\]
Furthermore, the series \(S^k\) converges at the rate \(O(\frac{1}{k})\).
The proof of Lemma 3.12 is provided in the appendix in Section VI-B. Lemma 3.12 provides that if the sequences \{\|\delta^k\|\} and \{\|\theta^k\|\} decrease at \(O(1/k)\), the complexity upper-bound in (10) converges at the rate \(O(1/k^2)\). Note that this result can be extended to the case that \{\|\delta^k\|\} and \{\|\theta^k\|\} decrease at \(O(1/k^\kappa)\) for any \(\kappa \in \mathbb{Z}_+\), by following a similar proof as for Lemma 3.12.

B. Inexact fast alternating minimization algorithm (inexact FAMA)

In this section, we present an accelerated variant of inexact AMA, named the inexact fast alternating minimization Algorithm (inexact FAMA), which is presented in Algorithm 4. It addresses the same problem class as Problem 3.1 and requires the same assumption as Assumption 3.3 for convergence. Similar to inexact AMA, inexact FAMA allows computation errors in the two minimization steps in the algorithm. Differing from inexact AMA, inexact FAMA involves one extra linear update in Algorithm 4. It addresses the same problem class as Problem 3.1 and requires the

**Algorithm 4** Inexact Fast alternating minimization algorithm (Inexact FAMA)

**Require:** Initialize \(\hat{\lambda}^0 = \lambda^0 \in \mathbb{R}^N\), and \(\tau < \sigma_f / \rho(A)\)

for \(k = 1, 2, \cdots\) do

1: \(\hat{x}^k = \text{argmin}_x \{f(x) + \langle \hat{\lambda}^{k-1}, -Ax \rangle\} + \delta^k\).

2: \(\hat{z}^k = \text{argmin}_z \{g(z) + \langle \hat{\lambda}^{k-1}, -Bz \rangle + \frac{\tau}{2} \|c - A\hat{x}^k - Bz\|^2\} + \theta^k\).

3: \(\lambda^k = \hat{\lambda}^{k-1} + \tau (c - A\hat{x}^k - B\hat{z}^k)\).

4: \(\hat{\lambda}^k = \lambda^k + \frac{1}{k+2}(\lambda^{k-1} - \lambda^k)\).

end for

**Lemma 3.13:** If Assumption 3.3 is satisfied and inexact FAMA and inexact APGM are initialized with the same dual and primal starting sequence, respectively, then applying the inexact FAMA in Algorithm 4 to Problem 3.1 is equivalent to applying inexact APGM to the dual problem. With this equivalence, we further show a complexity upper bound for inexact FAMA by using the result in Proposition 2.6 for inexact APGM. The main results for inexact FAMA have been presented in [19], and are restated in this section.

**Algorithm 4** Inexact Fast alternating minimization algorithm (Inexact FAMA)

**Require:** Initialize \(\hat{\lambda}^0 = \lambda^0 \in \mathbb{R}^N\), and \(\tau < \sigma_f / \rho(A)\)

for \(k = 1, 2, \cdots\) do

1: \(\hat{x}^k = \text{argmin}_x \{f(x) + \langle \hat{\lambda}^{k-1}, -Ax \rangle\} + \delta^k\).

2: \(\hat{z}^k = \text{argmin}_z \{g(z) + \langle \hat{\lambda}^{k-1}, -Bz \rangle + \frac{\tau}{2} \|c - A\hat{x}^k - Bz\|^2\} + \theta^k\).

3: \(\lambda^k = \hat{\lambda}^{k-1} + \tau (c - A\hat{x}^k - B\hat{z}^k)\).

4: \(\hat{\lambda}^k = \lambda^k + \frac{1}{k+2}(\lambda^{k-1} - \lambda^k)\).

end for

**Lemma 3.13:** If Assumption 3.3 is satisfied and inexact FAMA and inexact APGM are initialized with the same dual and primal starting sequence, respectively, then applying the inexact FAMA in Algorithm 4 to Problem 3.1 is equivalent to applying inexact APGM in Algorithm 2 to the dual problem defined in Problem 3.2 with the errors \(\epsilon^k = A\delta^k\) and \(e^k = \tau^2 L(\psi)\|B\theta^k\| + \frac{\tau^2}{2} \|B\theta^k\|^2\), where \(L(\psi)\) denotes the Lipschitz constant of the function \(\psi\).

**Proof:** The proof follows the same flow of the proof of Lemma 3.4 by replacing \(\lambda^{k-1}\) by \(\hat{\lambda}^{k-1}\) computed in Step 4 in Algorithm 4 and showing the following equality

\[
\lambda^k = \text{prox}_{\tau \psi, \epsilon_k} \left(\hat{\lambda}^{k-1} - \tau (\nabla \phi(\hat{\lambda}^{k-1}) + e^k)\right)
\]

(11)

Based on the equivalence shown in Lemma 3.13, we can now derive an upper-bound on the difference of the dual function value of the sequence \(\{\lambda^k\}\) for inexact FAMA in Theorem 3.14.

**Theorem 3.14 (Theorem III.5 in [19]):** Let \(\{\lambda^k\}\) be generated by the inexact FAMA in Algorithm 4. If Assumption 3.3 holds, then for any \(k \geq 1\)

\[
D(\lambda^*) - D(\lambda^k) \leq \frac{2L(\nabla \phi)}{(k+1)^2} \left(\|\lambda^0 - \lambda^*\| + 2\Gamma_k + \sqrt{2\Lambda_k}\right)^2
\]

(12)

where

\[
\Gamma_k = \sum_{p=1}^{k} \frac{p^2 \tau^2 L(\psi)\|B\theta^p\|^2 + 2L(\psi)\|B\theta^p\|}{2L(\nabla \phi)}
\]

(13)

\[
\Lambda_k = \sum_{p=1}^{k} \frac{p^2 \tau^2 L(\psi)\|B\theta^p\|^2 + \|B\theta^p\|^2}{2L(\nabla \phi)}
\]

(14)

and \(L(\nabla \phi) = \sigma_f^{-1} \cdot \rho(A)\).

**Proof:** The proof is a similar to the proof of Theorem 3.5. Lemma 3.13 shows the equivalence between Algorithm 4 and Algorithm 2. Proposition 2.6 completes the proof of the upper-bound in inequality (12).

With the results in Theorem 3.14, the sufficient conditions on the errors for the convergence of inexact APGM presented in Section II.C can be extended to inexact FAMA with the errors defined in Lemma 3.13.

**Corollary 3.15:** Let \(\{\lambda^k\}\) be generated by the inexact AMA in Algorithm 4. If Assumption 3.3 holds, and the constant \(L(\psi) < \infty\), the following sufficient conditions on the error sequences \(\{\delta^k\}\) and \(\{\theta^k\}\) guarantee the convergence of Algorithm 3.

- The series \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) are finitely summable, i.e., \(\sum_{k=1}^{\infty} \|\delta^k\| < \infty\) and \(\sum_{k=0}^{\infty} \|\theta^k\| < \infty\).
- The sequences \(\{\|\delta^k\|\}\) and \(\{\|\theta^k\|\}\) decrease at the rate \(O(1/k^{\kappa+\varepsilon})\) for any \(\kappa > 0\).
Proof: By Assumption 3.3, the dual Problem 3.2 satisfies Assumption 1 and the complexity upper-bound in Proposition 2.6 holds. By extending the sufficient conditions on the error sequences discussed after Proposition 2.6, we obtain sufficient conditions on the error sequences for inexact FAMA with the errors defined in Lemma 3.13. Since \( L(\psi) < \infty \), we have that if the error sequences \( \{\|\delta^k\|\} \) and \( \{\|\theta^k\|\} \) satisfy the conditions in Corollary 3.15, the complexity upper-bound in Theorem 3.14 converges to zero, as the number of iterations \( k \) goes to infinity, which further implies that the inexact FAMA algorithm converges to the optimal solution.

\[ \text{C. Discussion: inexact AMA and inexact FAMA with bounded errors} \]

In this section, we study the special case that the error sequences \( \delta^k \) and \( \theta^k \) are bounded by constants. This special case is of particular interest, as it appears in many engineering problems in practice, e.g. quantized distributed computation and distributed optimization with constant local computation errors. Previous work includes [14], where the authors studied the complexity upper-bounds for a distributed optimization algorithm with bounded noise on the solutions of local problems. In this section, we will study errors satisfying Assumption 3.16 and derive the corresponding complexity upper-bounds for inexact AMA, as well as for inexact FAMA, with different assumptions. We show that if the problem satisfies the stronger assumption in Assumption 3.3, i.e. the cost function \( f \) is a quadratic function, then the complexity bounds for the inexact algorithms with bounded errors converge to a finite positive value, as \( k \) increases. It is important to point out that if only the conditions in Assumption 3.3 are satisfied, convergence of the complexity upper bound to a small constant cannot be shown. We present the complexity upper-bound of inexact FAMA in details for this case, and the result can be easily extended to inexact AMA.

Assumption 3.16: We assume that the error sequences \( \delta^k \) and \( \theta^k \) are bounded by \( \|\delta^k\| \leq \bar{\delta} \) and \( \|\theta^k\| \leq \bar{\theta} \) for all \( k \geq 0 \), where \( \bar{\delta} \) and \( \bar{\theta} \) are positive constants.

Corollary 3.17: For the inexact FAMA in Algorithm 4, if Assumption 3.3 and Assumption 3.16 hold, i.e. the cost is not necessarily quadratic, we can also derive the following complexity upper bound

\[
D(\lambda^*) - D(\lambda^k) \leq \left( \frac{2L(\nabla\phi)\|\lambda^0 - \lambda^*\|}{(k + 1)} + k \cdot \Delta \right)^2
\]

with

\[
\Delta = \frac{\|A\| \cdot \bar{\delta}}{L(\nabla\phi)} + \frac{3\tau}{2} \cdot \sqrt{\frac{L(\psi)\|B\| \cdot \bar{\theta} + \|B\| \cdot \bar{\theta}^2}{L(\nabla\phi)}}
\]
and $L(\nabla g) = \sigma^{-1} \cdot \rho(A)$. The proof follows the same flow of the proof for Corollary 3.17 by replacing Theorem 3.10 with Theorem 3.14. Compared to the FAMA algorithm without errors, we see that the inexact FAMA with bounded errors has one extra term $k \cdot \Delta$ in the complexity upper-bound in [16]. Unfortunately, the term $k \cdot \Delta$ increases as $k$ increases. Hence, the complexity bound for the inexact FAMA with bounded errors does not converge, as $k$ goes to infinity.

IV. INEXACT AMA FOR DISTRIBUTED OPTIMIZATION WITH AN APPLICATION TO DISTRIBUTED MPC

A. Distributed optimization problem

In this section, we consider a distributed optimization problem on a network of $M$ sub-systems (nodes). The sub-systems communicate according to a fixed undirected graph $G = (V, E)$. The vertex set $V = \{1, 2, \cdots, M\}$ represents the sub-systems and the edge set $E \subseteq V \times V$ specifies pairs of sub-systems that can communicate. If $(i, j) \in E$, we say that sub-systems $i$ and $j$ are neighbours, and we denote by $\mathcal{N}_i = \{j | (i, j) \in E\}$ the set of the neighbours of sub-system $i$. Note that $\mathcal{N}_i$ includes $i$. The cardinality of $\mathcal{N}_i$ is denoted by $|\mathcal{N}_i|$. The global optimization variable is denoted by $z$. The local variable of sub-system $i$, namely the $i$th element of $z$ and $z = [z_1^T, \cdots, z_M^T]^T$, is denoted by $[z]_i$. The concatenation of the variable of sub-system $i$ and the variables of its neighbours is denoted by $z_i$. With the selection matrices $E_i$ and $F_{ji}$, the variables have the following relationship: $z_i = E_i z$ and $[z]_i = F_{ji} z_j$, $j \in \mathcal{N}_i$, which implies the relation between the local variable $[z]_i$ and the global variable $z$, i.e. $[z]_i = F_{ji} E_j z$, $j \in \mathcal{N}_i$. We consider the following distributed optimization problem:

Problem 4.1:

$$\min_{z, u} \sum_{i=1}^M f_i(z_i)$$

$$s.t. \quad z_i \in C_i, \quad z_i = E_i v, \quad i = 1, 2, \cdots, M.$$ 

where $f_i$ is the local cost function for sub-system $i$, and the constraint $C_i$ represents a convex local constraint on the concatenation of the variable of sub-system $i$ and the variables of its neighbours $z_i$.

Assumption 4.2: Each local cost function $f_i$ in Problem 4.1 is a strongly convex function with a convexity modulus $\sigma_f$, and has a Lipschitz continuous gradient with Lipschitz constant $L(\nabla f_i)$. The set $C_i$ is a convex set, for all $i = 1, \cdots, M$.

Remark 4.3: Recall the problem formulation of inexact AMA and FAMA defined in Problem 3.1. The two objectives are defined as $f(z) = \sum_{i=1}^M f_i(z_i)$ subject to $z_i \in C_i$ for all $i = 1, \cdots, M$ and $g = 0$. The matrices are $A = I$, $B = -[E_1^T, E_2^T, \cdots, E_M^T]$ and $c = 0$. The first objective $f(z)$ consists of a strongly convex function on $z$ and convex constraints. The convex constraints can be considered as indicator functions, which are convex functions. Due to the fact that the sum of a strongly convex and a convex function is strongly convex, the objective $f(z)$ is strongly convex with the modulus $\sigma_f$ and Problem 4.1 satisfies Assumption 3.3.

B. Application: distributed model predictive control

In this section, we consider a distributed linear MPC problem with $M$ sub-systems, and show that it can be written in the form of Problem 4.1. The dynamics of the $i$th agent are given by the discrete-time linear dynamics:

$$x_{i}(t+1) = \sum_{j \in \mathcal{N}_i} A_{ij} x_j(t) + B_{ij} u_j(t) \quad i = 1, 2, \cdots, M. \tag{17}$$

where $A_{ij}$ and $B_{ij}$ are the dynamical matrices. The states and inputs of agent $i$ are subject to local convex constraints:

$$x_{i}(t) \in X_i, \quad u_{i}(t) \in U_i \quad i = 1, 2, \cdots, M. \tag{18}$$

The distributed MPC problem, as e.g. considered in [8], is given in Problem 4.4.

Problem 4.4:

$$\min_{x, u} \sum_{i=1}^M \sum_{t=0}^{N-1} l_i(x_i(t), u_i(t)) + \sum_{i=1}^M l_i^f(x_i(N))$$

$$s.t. \quad x_{i}(t+1) = \sum_{j \in \mathcal{N}_i} A_{ij} x_j(t) + B_{ij} u_j(t)$$

$$x_{i}(t) \in X_i, \quad u_{i}(t) \in U_i$$

$$x_{i}(N) \in X_i^f, \quad x_{i}(0) = \bar{x}_i, \quad i = 1, 2, \cdots, M.$$ 

where $l_i(\cdot, \cdot)$ and $l_i^f(\cdot)$ are strictly convex stage cost functions and $N$ is the horizon for the MPC problem. The state and input sequences along the horizon of agent $i$ are denoted by $x_i = [x_i^T(0), x_i^T(1), \cdots, x_i^T(N)]^T$ and $u_i = [u_i^T(0), u_i^T(1), \cdots, u_i^T(N-1)]^T$. We denote the concatenations of the state and input sequences of agent $i$ and its neighbours by $x_{\mathcal{N}_i}^i$ and $u_{\mathcal{N}_i}^i$. The corresponding constraints are $x_{\mathcal{N}_i}^i \in X_{\mathcal{N}_i}$ and $u_{\mathcal{N}_i}^i \in U_{\mathcal{N}_i}$. We define $v = [x_1^T, x_2^T, \cdots, x_M^T, u_1^T, u_2^T, \cdots, u_M^T]^T$ to be the global variable and $z_i = [x_{\mathcal{N}_i}^i, u_{\mathcal{N}_i}^i]$ to be the local variables. $Z_{\mathcal{N}_i} = X_{\mathcal{N}_i} \times U_{\mathcal{N}_i}$ denotes the local constraints on $z_i$ and
functions with respect to the input variables and the local linear dynamical system $A_i x(t)$ are linear dynamics. The distributed MPC problem further satisfies Assumption 3.8.

Remark 4.5: If the stage cost functions $l_i(\cdot, \cdot)$ and $l^T_i(\cdot)$ are strictly convex functions, and the state and input constraints $X_i$ and $U_i$ are convex sets, then the conditions in Assumption 4.2 are all satisfied. Furthermore, if the state cost functions $l_i(\cdot, \cdot)$ and $l^T_i(\cdot)$ are set to be positive definite quadratic functions, then the distributed optimization problem originating from the distributed MPC problem further satisfies Assumption 3.8.

Remark 4.6: For the case that the distributed MPC problem has only input constraints and the state coupling matrices in the linear dynamics are $A_{ij} = 0$ for any $i \neq j$, we can eliminate all state variables in the distributed MPC problem and only have the input variables as the optimization variables. For this case, if the stage cost functions $l_i(\cdot, \cdot)$ and $l^T_i(\cdot)$ are strictly convex functions with respect to the input variables and the local linear dynamical system $x_i(t+1) = A_{ii} x_i(t) + \sum_{j \in N_i} B_{ij} u_j(t)$ is controllable, then the resulting distributed optimization problem satisfies Assumption 3.8. The details of this formulation can be found in [18].

C. Inexact AMA and Inexact FAMA for distributed optimization

In this section, we apply inexact AMA and inexact FAMA to the distributed optimization problem in Problem 4.1 originating from the distributed MPC problem in Problem 4.4. The concept is to split the distributed optimization into small and local problems according to the physical couplings of the sub-systems. Algorithm 5 and Algorithm 6 represent the algorithms. Note that Step 2 in inexact AMA and inexact FAMA, i.e., Algorithm 5 and Algorithm 6, are simplified to be a consensus step in Step 3 in Algorithm 5 and Algorithm 6 which requires only local communication. In the algorithms, $\delta^k_i$ represents the computational error of the local problems.

Algorithm 5 Inexact Alternating Minimization Algorithm for Distributed Optimization

Require: Initialize $\lambda^0_i = 0 \in \mathbb{R}^{z_i}$, and $\tau < \min_{1 \leq i \leq M} \{\sigma_{f_i}\}$

for $k = 1, 2, \cdots$ do

1: $\tilde{z}^k_i = \arg\min_{z_i \in C_i} \{f_i(z_i) + \langle \lambda^k_{i-1}, -z_i \rangle \} + \delta^k_i$
2: Send $\tilde{z}^k_i$ to all the neighbours of agent $i$.
3: $[\hat{v}^k]_i = \sum_{j \in N_i} [\tilde{z}^k_j]_i$.
4: Send $[\hat{v}^k]_i$ to all the neighbours of agent $i$.
5: $\lambda^k_i = \lambda^k_{i-1} + \tau (E_i \hat{v}^k - \tilde{z}^k_i)$

end for

Algorithm 6 Inexact fast alternating minimization algorithm for Distributed Optimization

Require: Initialize $\lambda^0_i = \tilde{\lambda}^0_i \in \mathbb{R}^{z_i}$, and $\tau < \min_{1 \leq i \leq M} \{\sigma_{f_i}\}$

for $k = 1, 2, \cdots$ do

1: $\tilde{z}^k_i = \arg\min_{z_i \in C_i} \{f_i(z_i) + \langle \tilde{\lambda}^k_{i-1}, -z_i \rangle \} + \delta^k_i$
2: Send $\tilde{z}^k_i$ to all the neighbours of agent $i$.
3: $[\hat{v}^k]_i = \sum_{j \in N_i} [\tilde{z}^k_j]_i$.
4: Send $[\hat{v}^k]_i$ to all the neighbours of agent $i$.
5: $\lambda^k_i = \lambda^k_{i-1} + \tau (E_i \hat{v}^k - \tilde{z}^k_i)$
6: $\tilde{\lambda}^k_i = \lambda^k_i + \frac{k-1}{k+2} (\lambda^k_i - \lambda^{k-1}_i)$

end for

Remark 4.7: Note that for every iteration $k$, Algorithm 5 and 6 only need local communication and the computations can be performed in parallel for every subsystem.

We provide a lemma showing that considering Algorithm 5 there exists a Lipschitz constant $L(\psi)$ equal to zero. The results can be easily extended to Algorithm 6. This result is required by the proofs of the complexity upper-bounds in Corollary 4.9, 4.10, and 4.11.

Lemma 4.8: Let the sequence $\lambda^k$ be generated by Algorithm 5. For all $k \geq 0$ it holds that $E^T \lambda^k = 0$ and the Lipschitz constant of the second objective in the dual problem of Problem 4.1 $L(\psi)$ is equal to zero.

Proof: We first prove that for all $k \geq 0$, the sequence $\lambda^k$ satisfies $E^T \lambda^k = 0$. We know that Step 3 in Algorithm 5 is equivalent to the following update

$$\hat{v}^k = M \sum_{i=1}^M E^T_i \cdot \hat{z}_i^k = M \cdot E^T \cdot \hat{z}^k,$$
with $M = \text{blkdiag}(\frac{1}{|N_i|} \cdot I_1, \cdots, \frac{1}{|N_i|} \cdot I_i, \cdots, \frac{1}{|N_i|} \cdot I_M) = (E^T E)^{-1}$, where $|N_i|$ denotes the number of the elements in the set $N_i$, and $I_i$ denotes an identity matrix with the dimension of the $i$th component of $v$, denoted as $[v]_i$. From Step 5 in Algorithm 5 for all $k \geq 1$ we have that

$$\lambda^k = \lambda^{k-1} + \tau(E \tilde{v}^k - \tilde{z}^k).$$

By multiplying the matrix $E^T$ to both sides, we have

$$E^T \lambda^k = E^T \lambda^{k-1} + \tau(E^T E \tilde{v}^k - E^T \tilde{z}^k) = E^T \lambda^{k-1} + \tau(E^T E \tilde{z}^k - E^T \tilde{z}^k).$$

Since $M = (E^T E)^{-1}$, the above equality becomes

$$E^T \lambda^k = E^T \lambda^{k-1} + \tau(E^T \tilde{z}^k - E^T \tilde{z}^k) = E^T \lambda^{k-1}.$$

From the initialization in Algorithm 5, we know $E^T \lambda^0 = E^T \cdot 0 = 0$. Then by induction, we can immediately prove that for all $k \geq 0$ it holds that $E^T \lambda^k = 0$. We can now show that for all $E^T \lambda = 0$, a Lipschitz constant of the second objective in the dual problem in Problem 3.2 $L(\psi)$ is equal to zero. Since $g = 0$, $B = -E$ and $c = 0$, then the second objective in the dual problem is equal to

$$\psi(\lambda) = g^*(B^T \lambda) - c^T \lambda = g^*(E^T \lambda) = \sup_w (\psi^T E^T \lambda - 0) = \begin{cases} 0 & \text{if } E^T \lambda = 0 \\ \infty & \text{if } E^T \lambda \neq 0. \end{cases}$$

The function $\psi(\lambda)$ is an indicator function on the nullspace of matrix $E^T$. For all $\lambda$ satisfying $E^T \lambda = 0$, the function $\psi(\lambda)$ is equal to zero. Hence, zero is a Lipschitz constant of the function $\psi(\lambda)$ for all $E^T \lambda = 0$. ■

After proving Lemma 4.8 we are ready to show the main theoretical properties of Algorithm 5 and 6.

**Corollary 4.9:** Let $\{\lambda^k = [\lambda^0_k, \cdots, \lambda^M_k]^T\}$ be generated by Algorithm 5. If Assumption 4.2 is satisfied and the inexact solutions $\tilde{z}^k_i$ for all $k \geq 1$ are feasible, i.e. $\tilde{z}^k_i \in C_i$, then for any $k \geq 1$

$$D(\lambda^*) - D\left(\frac{1}{k} \sum_{p=1}^{k} \lambda^p\right) \leq \frac{L(\nabla \phi)}{2k} \left(\|\lambda^0 - \lambda^*\| + 2 \sum_{p=1}^{k} \frac{\|\delta^p\|}{L(\nabla \phi)}\right)^2,$$

where $D(\cdot)$ is the dual function of Problem 4.1, $\lambda^0 = [\lambda^0_1, \cdots, \lambda^0_M]^T$ and $\lambda^*$ are the starting sequence and the optimal solution of the Lagrangian multiplier, respectively, and $\delta^p = [\delta^0_1, \cdots, \delta^M_1]^T$ denotes the global error sequence. The Lipschitz constant $L(\nabla \phi)$ is equal to $\sigma_f^{-1}$, with $\sigma_f = \min\{\sigma_{f_1}, \cdots, \sigma_{f_M}\}$.

**Proof:** As stated in Remark 4.3 Problem 4.1 is split as follows: $f = \sum_{i=1}^{M} f_i(z_i)$ with the constraints $z_i \in C_i$ for all $i = 1, \cdots, M$ and $g = 0$. The matrices are $A = I$, $B = -E$ and $c = 0$. If Assumption 4.2 holds, then this splitting problem satisfies Assumption 3.3 with the convexity modulus $\sigma_f$. From Theorem 3.5 we know that the sequence $\{\lambda^k\}$ generated by inexact AMA in Algorithm 5 satisfies the complexity upper bound in (7) with $T^k$ and $A^k$ in (8) and (9) with $\delta^k = [\delta^0_1, \cdots, \delta^M_1]^T$ and $\delta^0 = 0$. By Lemma 4.8 it follows that the constant $L(\psi)$ in $\Lambda^k$ is equal to zero. The Lipschitz constant of the gradient of the dual objective is equal to $L(\nabla \phi) = \sigma_f^{-1} \cdot \rho(A) = \sigma_f^{-1}$ with $\sigma_f = \min\{\sigma_{f_1}, \cdots, \sigma_{f_M}\}$. Hence, we can simplify the complexity upper bound in (7) for Algorithm 5 to be inequality (21).

As we discussed in Remark 4.5 if the state cost functions $l_i(\cdot)$ and $l_i^T(\cdot)$ in the distributed MPC problem are strictly positive quadratic functions, then the distributed optimization problem originating from the distributed MPC problem satisfies Assumption 3.8 which according to Theorem 3.5 implies a linearly decreasing upper-bound given in Corollary 4.10.

**Corollary 4.10:** Let $\{\lambda^k = [\lambda^0_k, \cdots, \lambda^M_k]^T\}$ be generated by Algorithm 5. If Assumption 4.2 is satisfied, the local cost function $f_i$ is a strictly positive quadratic function, and the inexact solutions $\tilde{z}^k_i$ for all $k \geq 1$ are feasible, i.e. $\tilde{z}^k_i \in C_i$, then for any $k \geq 1$

$$\|\lambda^k - \lambda^*\| \leq (1 - \gamma)(\|\lambda^0 - \lambda^*\| + \|A^k\|) \left(\sum_{p=0}^{k} (1 - \gamma)^{p-1} \cdot \frac{L(\nabla \phi)}{L(\nabla \phi)}\right),$$

where $\gamma = \frac{\lambda_{\min}(M)}{\lambda_{\max}(M)}$, and $\lambda^0$ and $\lambda^*$ are the starting sequence and the optimal sequence of the Lagrangian multiplier, respectively. The Lipschitz constant $L(\nabla \phi)$ is equal to $\sigma_f^{-1}$, where $\sigma_f = \min\{\sigma_{f_1}, \cdots, \sigma_{f_M}\}$.

**Proof:** In Algorithm 6 the variable $\lambda^k_i$ is a linear function of $\lambda^k_i$ and $\lambda^{k-1}_i$. This preserves all properties shown in Lemma 4.8 for Algorithm 6. Then, Corollary 4.10 can be easily proven by following the same steps as in the proof of Corollary 4.9 by replacing Theorem 3.5 by Theorem 3.10. ■

**Corollary 4.11:** Let $\{\lambda^k = [\lambda^0_k, \cdots, \lambda^M_k]^T\}$ be generated by Algorithm 6. If Assumption 4.2 is satisfied and the inexact solutions $\tilde{z}^k_i$ for all $k \geq 1$ are feasible, i.e. $\tilde{z}^k_i \in C_i$, then for any $k \geq 1$

$$D(\lambda^*) - D(\lambda^k) \leq \frac{2L(\nabla \phi)}{(k+1)^2} \left(\|\lambda^0 - \lambda^*\| + 2M \sum_{p=1}^{k} \frac{\|\delta^p\|}{L(\nabla \phi)}\right)^2.$$
where $D(\cdot)$ is the dual function of Problem 4.1, $\lambda^0$ and $\lambda^*$ are the starting sequence and the optimal sequence of the Lagrangian multiplier, respectively. The Lipschitz constant $L(\nabla \phi)$ is equal to $\sigma_f^{-1}$, where $\sigma_f = \min\{\sigma_{f_1}, \ldots, \sigma_{f_M}\}$.

**Proof:** It follows from the same proof as Corollary 4.9 by replacing Theorem 3.5 by Theorem 3.14.

**Remark 4.12:** For the case that all the local problems are solved exactly, i.e. $\delta^k = 0$, Algorithm 5 and Algorithm 6 reduce to standard AMA and FAMA, and converge to the optimal solution at the rate of the complexity upper-bounds.

**Remark 4.13:** The sufficient conditions on the errors for convergence given in Corollary 3.6, 3.11 and 3.15 can be directly extended to the error sequence $\{\delta^k\}$.

### D. Certification of the number of local iterations for convergence

We have shown that the inexact distributed optimization algorithms in Algorithm 5 and 6 allow one to solve the local problems, i.e. Step 1 in Algorithm 5 and 6, inexactly. In this section, we will address two questions: which algorithms are suitable for solving the local problems; and what termination conditions for the local algorithms guarantee that the computational cost functions and convex constraints. From Corollary 4.9, 4.10 and 4.11, we know that the inexact solution conditions, for example $\alpha_k$ and $\beta_k$ decrease with a given rate, that satisfies the sufficient conditions derived from Corollary 4.9, 4.10 and 4.11 ensuring convergence of the inexact distributed optimization algorithm to the optimal solution. We define a decrease function $\alpha_k$ satisfying the sufficient conditions, for example $\alpha_k = \alpha_0 \cdot \frac{1}{k^\tau}$, where $\alpha_0$ is a positive number.

1) **Gradient method:** The local problems in Step 1 in Algorithm 5 and 6 are optimization problems with strongly convex cost functions and convex constraints. From Corollary 4.9, 4.10 and 4.11 we know that the inexact solution $\tilde{z}^k_i$ needs to be a feasible solution subject to the local constraint $C_i$, i.e., $\tilde{z}^k_i \in C_i$ for all $k > 0$. Therefore, a good candidate algorithm for solving the local problems should have the following three properties: the algorithm can solve convex optimization problems efficiently; if the algorithm is stopped early, i.e., only a few number of iterations are implemented, the sub-optimal solution is feasible with respect to the local constraint $C_i$; and there exists a certificate on the number of iterations to achieve a given accuracy of the sub-optimal solution. Gradient methods satisfy these requirements, have simple and efficient implementations, and offer complexity upper-bounds on the number of iterations. These methods have been studied in the context of MPC in [20], [11] and [12].

We apply the proximal gradient method in Algorithm 7 for solving the local problems in Step 1 in Algorithm 5 and 6. The local optimization problems at iteration $k$ are parametric optimization problems with the parameter $\lambda^k_i$. We denote the optimal function as

$$z_i^*(\lambda_i) := \arg\min_{z_i \in C_i} \{f_i(z_i) + \langle \lambda_i, -z_i \rangle\} .$$

(22)

The solution of the optimal function at $\lambda_i^{k-1}$ is denoted as $z_i^{k,*} := z_i^*(\lambda_i^{k-1})$. The function $z_i^*(\cdot)$ has a Lipschitz constant $L(z_i^*)$ satisfying $L(z_i^*) = \|L_i(z_i^{k-1})\|$, where $L_i(z_i^{k-1})$ is Lipschitz constant for any $\lambda_i$ and $\lambda_{ij}$. Motivated by the fact that the difference between the parameters $\lambda_i^{k-1}$ and $\lambda_i^{k-1}$ is limited and measurable for each $k$, i.e., $\beta_k^i := \|\lambda_i^{k-1} - \lambda_i^{k-1}\| = \|E_i \hat{\nu}^{k-1} - \hat{\nu}^{k-1}\|$, we use a warm-starting strategy to initialize the local problems, i.e. we use the solution $\hat{z}^{k-1}_i$ from the previous step $k-1$ as the initial solution for Algorithm 7 at step $k$.

**Algorithm 7** Gradient method for solving Step 1 in Algorithm 5 at iteration $k$

**Require:** Initialize $\alpha^k = \alpha^0 \cdot \frac{1}{k^\tau}$, $\beta_k = \|\tau(E_i \hat{\nu}^{k-1} - \hat{\nu}^{k-1})\|$, $\lambda_i^{k,i} = \tilde{z}_i^{k-1}$ and $\tau_i < \frac{1}{L_i(\nabla f_i)}$

Compute $J_k$ satisfying (23)

for $j = 1, 2, \ldots, J_k$ do

$z_i^{k,j} = \text{Proj}_{C_i}(z_i^{k,j-1} - \tau(\nabla f_i(z_i^{k,j-1}) - \lambda_i^{k-1}))$

end for

$z_i^k \leftarrow z_i^{k,J_k}$

Note that we initialize the vectors $\hat{\nu}^{k-1}$, $\hat{\nu}^{k-1}$ and $\hat{\nu}^{k-1}$ for $j = 1$ in Algorithm 5 to be zero vectors. Let $\hat{z}_i^{k,j}$ be generated by Algorithm 7. If Assumption 4.2 holds, then for any $j \geq 0$ we have:

$$\|z_i^{k,j} - \hat{z}_i^{k,j}\| \leq \|z_i^{k,0} - \hat{z}_i^{k,0}\| \cdot (1 - \gamma)^j ,$$

where $\gamma = \frac{\sigma_f}{L_i(\nabla f_i)}$ and $z_i^{k,0}$ and $\hat{z}_i^{k,0}$ denote the initial sequence of Algorithm 1 and the optimal solution of the problem in Step 6 in Algorithm 5 at iteration $k$, respectively.
2) Termination condition on the number of iterations for solving local problems: Methods for bounding the number of iterations to reach a given accuracy have been studied e.g. in [10], [16] and [9]. In [10] and [16], the authors proposed dual decomposition based optimization methods for solving quadratic programming problems and presented termination conditions to guarantee a prespecified accuracy. However, these methods do not directly guarantee feasibility of the sub-optimal solution. One approach is to tighten constraints to ensure feasibility, which can be conservative in practice. In [9], the authors propose an inexact decomposition algorithm for solving distributed optimization problems by employing smoothing techniques and an excessive gap condition as the termination condition on the number of iterations to achieve a given accuracy. To certify the termination condition, this method requires to measure the values of the global primal and dual functions on-line, which requires full communication on the network and is not satisfied in our distributed framework. In addition, this method does not provide any algorithms for solving the local problems.

By employing the complexity upper-bounds in Proposition 4.14 for Algorithm 7 we propose a termination condition in (24) to find the number of iterations $J_k$, which guarantees that the local computational error is upper-bounded by the predefined decrease function $\alpha^k$, i.e. $\|\delta_i^k\| \leq \alpha^k$.

**Lemma 4.15:** If the number of iterations $J_k$ in Algorithm 7 satisfies

$$J_k \geq \lceil \log(1-\gamma) \alpha^{k-1} + L(z_i^*) \beta^k \rceil$$

for all $k \geq 1$, then the computational error for solving the local problem in Step 6 in Algorithm 5 $\delta_i^k$ satisfies $\|\delta_i^k\| \leq \alpha^k$.

**Proof:** We will prove Lemma 4.15 by induction.

- Base case: For $k = 1$, the vectors $\delta_1^{k-1}$, $z_i^{k-1}$ and $z_i^{k-1}$ are initialized as zero vectors. By Proposition 2.4 and the fact $\|\delta_1^{k-1}\| = 0$, we know

$$\|z_i^{1,J_1} - z_i^{1,*}\| = \|z_i^{1,0} - z_i^{1,*}\| \cdot (1 - \gamma)^{J_1} = \|0 - z_i^{1,*}\| \cdot (1 - \gamma)^{J_1}.$$  

Due to the definition of the function $\alpha^k$, it follows that the term above is upper-bounded by $\alpha^0 \cdot (1 - \gamma)^{J_1}$. Using the fact that $\beta^0 = \|\tau((I_i v^0 - z_i^0))\| = 0$ and $J_1$ satisfies (24), it is further upper-bounded by $\alpha^1$:

$$\|\delta_1^1\| = \|z_i^1 - z_i^{1,*}\| = \|z_i^{1,J_1} - z_i^{1,*}\| \leq \alpha^1.$$  

- Induction step: Let $l \geq 1$ be given and suppose that $\|\delta_i^l\| \leq \alpha^l$. We will prove that $\|\delta_i^{l+1}\| \leq \alpha^{l+1}$ By Proposition 2.4 and the warm-starting strategy, i.e. $z_i^{l,0} = z_i^{l-1} - z_i^{l-1,J_{l-1}}$, we know

$$\|\delta_i^{l+1}\| = \|z_i^{l+1,J_{l+1}} - z_i^{l+1,*}\|$$

$$\leq \|z_i^{l+1,0} - z_i^{l+1,*}\| \cdot (1 - \gamma)^{J_{l+1}}$$

$$= \|z_i^{l,J_l} - z_i^{l+1,*}\| \cdot (1 - \gamma)^{J_{l+1}}$$

$$\leq (\|z_i^{l,J_l} - z_i^{l,*}\| + \|z_i^{l,L} - z_i^{l+1,*}\|) \cdot (1 - \gamma)^{J_{l+1}}$$

$$\leq (\delta^l + L(z_i^*) \beta^{l+1}) \cdot (1 - \gamma)^{J_{l+1}}.$$  

Due to the induction assumption and the fact that $J_l$ satisfies (24), it follows that $\|\delta_i^{l+1}\| \leq \alpha^{l+1}$.

We conclude that by the principle of induction, it holds that $\|\delta_i^k\| \leq \alpha^k$ for all $k \geq 1$.

**Corollary 4.16:** If Assumption 4.2 holds and the decrease rate of the function $\alpha^k$ satisfies the corresponding sufficient conditions presented in Corollary 3.6 and 3.15, then Algorithm 5 and 6 converge to the optimal solution, with Algorithm 7 solving the local problem in Step 1. Furthermore, if the local cost function $\bar{f}_i$ is a strictly positive quadratic function, and the decrease rate of the function $\alpha^k$ satisfies the sufficient conditions presented in Corollary 3.11, then Algorithm 5 converges to the optimal solution, with Algorithm 7 solving the local problem in Step 1.

**Remark 4.17:** All the information required by the proposed on-line certification method, i.e., by Algorithm 7 as well as the condition for $J_k$ in (24), can be obtained on-line and locally.

3) Computation of the Lipschitz constant $L(z_i^*)$: In the above proposed on-line certification method, the Lipschitz constant of the optimal solution function $z_i^*(\lambda_i^{k-1})$, $L(z_i^*)$, plays an important role. While it is generally difficult to compute this Lipschitz constant, it can be computed for special cases, such as positive quadratic functions.

**Lemma 4.18:** Let the local cost function be a quadratic function, i.e. $f_i(z_i) = \frac{1}{2} z_i^T H_i z_i + h_i^T z_i$ with $H_i > 0$. A Lipschitz constant of the function $z_i^*(\lambda_i)$ defined in (22) is given by $\frac{1}{\rho_{\min}(H_i)}$, i.e.

$$\|z_i^*(\lambda_i) - z_i^*(\lambda_{i2})\| \leq \frac{1}{\rho_{\min}(H_i)} \cdot \|\lambda_i - \lambda_{i2}\|.$$  

(25)
Algorithm 7 is applied to solve the local problems in Step 2 in Algorithm 5. In this experiment Algorithm 7 is stopped after findings in Corollary 3.11. The input-coupled dynamics allow us to eliminate the states of the distributed MPC problem, such that the optimization variable \( \mathbf{x}_i(t+1) = A_{ii} \mathbf{x}_i(t) + \sum_{j \in N_i} B_{ij} \mathbf{u}_j(t) \) for sub-system \( i \).

The local cost functions are set to be quadratic functions, i.e.

\[
\mathbf{z}_i^*(\lambda_i) = \arg \min_{\mathbf{z}_i \in \mathbb{C}_i} \frac{1}{2} z_i^T H_i z_i + (h_i - \lambda_i)^T z_i \\
= \arg \min_{z_i \in \mathbb{C}_i} \frac{1}{2} ||D^T z_i + D^{-1}(h_i - \lambda_i)||^2
\]

Let \( v = D^T z_i \). The optimization problem above becomes

\[
v^*(\lambda_i) = \arg \min_{D^{-1}v \in \mathbb{C}_i} \frac{1}{2} ||v + D^{-1}(h_i - \lambda_i)||^2 ,
\]

which can be seen as the projection of the point \( D^{-1}(h_i - \lambda_i) \) onto the set \( \mathbb{C}_i := \{ v \mid D^{-1}v \in \mathbb{C}_i \} \).

Since \( \mathbb{C}_i \) is convex, then \( \mathbb{C}_i \) is convex as well. It follows directly from Proposition 2.2.1 in [4] that

\[
\|v^*(\lambda_{i1}) - v^*(\lambda_{i2})\| \leq \|D^{-1} \cdot (\lambda_{i1} - \lambda_{i2})\|.
\]

By \( z_i = D^{-1}v \), we get

\[
\|\mathbf{z}_i^*(\lambda_{i1}) - \mathbf{z}_i^*(\lambda_{i2})\| \leq \|D^{-1}\| \cdot \|D^{-1} \cdot (\lambda_{i1} - \lambda_{i2})\| \\
\leq \|D^{-1}\|^2 \cdot \|\lambda_{i1} - \lambda_{i2}\| \\
\leq \frac{1}{\rho_{min}(H_i)} \cdot \|\lambda_{i1} - \lambda_{i2}\|.
\]

\[\square\]

V. Numerical Example

This section illustrates the theoretical findings of the paper and demonstrates the performance of inexact AMA by solving a randomly generated distributed MPC problem with 40 sub-systems. For this example, we assume that the sub-systems are coupled only in the control input:

\[
\mathbf{x}_i(t+1) = \mathbf{A}_{ii} \mathbf{x}_i(t) + \sum_{j \in N_i} \mathbf{B}_{ij} \mathbf{u}_j(t), \quad i = 1, 2, \cdots, M,
\]

The input-coupled dynamics allow us to eliminate the states of the distributed MPC problem, such that the optimization variable in the distributed optimization problems is the control sequence \( \mathbf{u} = [u_1^T, \cdots, u_M^T]^T \), with \( u_i = [u_i^T(0), u_i^T(1), \cdots, u_i^T(N)]^T \).

Examples with this structure include systems sharing one resource, e.g. a water-tank system or an energy storage system.

We randomly generate a connected network with 40 agents. Each sub-system has three states and two inputs. The dynamical matrices \( \mathbf{A}_{ii} \) and \( \mathbf{B}_{ij} \) are randomly generated, i.e. generally dense, and the local systems are controllable. The input constraint \( \mathbb{U}_i \) for sub-system \( i \) is set to be \( \mathbb{U}_i = \{ u_i(t) \mid 0.4 \leq u_i(t) \leq 0.3 \} \). The horizon of the MPC problem is set to be \( N = 11 \).

The local cost functions are set to be quadratic functions, i.e. \( l_i(\mathbf{x}_i(t), u_i(t)) = x_i^T(t) \mathbf{Q} x_i(t) + u_i^T(t) \mathbf{R} u_i(t) \) and \( l_i^f(\mathbf{x}_i(N)) = x_i^T(N) \mathbf{P} x_i(N) \), where \( Q, R \) and \( P \) are identity matrices. Therefore, the distributed optimization problem resulting from the distributed MPC satisfies Assumption 4.2: the local cost functions \( f_i \) are strictly positive quadratic functions, and the results in Corollary 4.10 hold. The initial states \( \mathbf{x}_i \) are chosen, such that more than 70% of the elements of the vector \( \mathbf{u} \) are at the constraints.

In Fig. 1 we demonstrate the convergence performance of inexact AMA for solving the distributed optimization problem in Problem 4.1 originating from the randomly generated distributed MPC problem, applying Algorithm 5. In this simulation, we compare the performance of inexact AMA with three different kinds of errors for \( \delta^k \) with exact AMA, for which the errors are equal to zero. Note that these errors are synthetically constructed to specify different error properties. We solve the local problems to optimality and then add errors with predefined decreasing rates to the local optimal solution, ensuring that the solution remains primal feasible. The black line shows the performance of exact AMA. The blue, red and green lines show the performance of inexact AMA, where the errors \( \delta^k \) are set to be decreasing at the rates of \( O(\frac{1}{k}) \), \( O(\frac{1}{k^2}) \) and \( O(\frac{1}{k^3}) \), respectively. Note that all three errors satisfy the sufficient condition for convergence in Corollary 5.11. We can observe that as the number of iterations \( k \) increases, the differences \( \|u^k - u^*\| \) decrease for all the cases, however, the convergence speed is quite different. For the exact AMA algorithm (black line), it decreases linearly, which supports the results in Corollary 4.10. For the three cases for inexact AMA (blue, red and green lines), we can see that the differences \( \|u^k - u^*\| \) decrease more slowly than for exact AMA, and the decrease rates correspond to the decrease rate of the errors, which supports the theoretical findings in Corollary 5.11.

The second simulation illustrates the convergence properties of inexact AMA, where the proximal gradient method in Algorithm 7 is applied to solve the local problems in Step 2 in Algorithm 5. In this experiment Algorithm 7 is stopped after the number of iterations providing that the local computation error \( \delta^k \) decreases at a certain rate. The error decrease rate is...
selected to be $O(\frac{1}{k})$, i.e., the decrease function $\alpha^k$ is set to be $\alpha^k = O(\frac{1}{k})$ and thereby satisfies the second sufficient condition in Corollary 3.11. In order to ensure $\|\delta^k\| \leq \alpha^k$, the number of iterations for the proximal gradient method $J_K$ in Algorithm 7 is chosen according to the certification method presented in Section IV-D such that condition (24) is satisfied. Note that we use a warm-starting strategy for the initialization of Algorithm 7.

Fig. 2 shows the comparison of the performance of exact AMA and inexact AMA. We can observe that the black (exact AMA) and red lines basically overlap (inexact AMA with Algorithm 7 solving local problems with the numbers of iterations $J_k$ satisfying (24)). Inexact AMA converges to the optimal solution as the iterations increase, and shows almost the same performance as exact AMA.

Fig. 3 shows the corresponding local error sequence $\delta^k$, where the number of iterations $J_k$ for Algorithm 7 satisfies the condition in (24). We can observe that the global error sequence $\delta^k = [\delta^k_1, \cdots, \delta^k_M]$ is upper-bounded by the decrease function $\alpha^k$. As $k$ is small, the upper-bound $\alpha^k$ is tight to the error sequence. As $k$ increases, the error decreases faster and the bound becomes loose.

Fig. 4 shows the comparison of the numbers of iterations for Algorithm 7 computed using two different approaches. Approach 1 uses the termination condition proposed in Section IV-D. In Approach 2, we first compute the optimal solution of the local problem $z^k_*$ and then run the proximal gradient method to find the smallest number of iterations providing that the difference of the local sub-optimal solution satisfies the decrease function $\alpha^k$, i.e. $\|z^k_j - z^k_*\| \leq \alpha^k$. Approach 2 is therefore the exact minimal number, whereas Approach 1 uses a bound on the minimal number. Note that the second approach guarantees $\|\delta^k\| \leq \alpha^k$ for all $k$, however, this method is not practically applicable, since the optimal solution $z^k_*$ is unknown. Its purpose is merely to compare with the proposed certification method and to show how tight the theoretical bound in (24) is. For both techniques, we use a warm-starting strategy for initialization of the proximal gradient method to solve the local problems for each $k$ in Algorithm 7. In Fig. 4, the green line and region result from the termination condition proposed in Section IV-D, and the pink line and region result from the second approach. The solid green and red lines show the average value of the numbers of iterations for the proximal gradient method for solving the local problems over the 40 sub-systems. The upper and lower boundaries of the regions show the maximal and minimal number of iterations, respectively. The maximal number of iterations for the proposed certification method (green region) is equal to 7, while for the second method (the red region) it is equal to 4. Fig. 4 shows that the certification approach in (24), which can be performed locally, is reasonably tight and the provided number of iterations is close to the minimal number of iterations required to satisfy the desired error.

![Figure 1: Comparison of the performance of AMA and inexact AMA (IAMA) with the errors decreasing at pre-defined rates.](image)

**Figure 1:** Comparison of the performance of AMA and inexact AMA (IAMA) with the errors decreasing at pre-defined rates.

**VI. APPENDIX**

**A. Proof of Lemma 3.4**

**Proof:** In order to show the equivalence, we prove that Step 1, 2 and 3 in Algorithm 3 are equivalent to Step 1 in Algorithm 1 i.e. the following equality holds:

$$\lambda^k = \text{prox}_{\psi, \tau} (\lambda^{k-1} - \tau (\nabla \phi(\lambda^{k-1}) + e^k))$$

with $e^k = A\delta^k$ and $e^k = \tau^2 L(\psi)\|B\theta_k\| + \tau^2 \|B\theta_k\|^2$. Step 2 in Algorithm 3 implies:

$$B^T\lambda^{k-1} + \tau B^T (c - A\tilde{z}^k - Bz^k) \in \partial g(z^k),$$

where $z^k = \text{argmin}_{p \in \partial f(q)} \{g(z) + \langle \lambda^{k-1}, -Bz \rangle + \frac{\tau}{2} \|c - A\tilde{z}^{k+1} - Bz\|_2^2 \} = \tilde{z}^k - \theta^k$. From the property of the conjugate function $p \in \partial f(q) \Leftrightarrow q \in \partial f^*(p)$, it follows:

$$z^k \in \partial g^*(B^T\lambda^{k-1} + \tau B^T (c - A\tilde{z}^k - Bz^k)).$$
By multiplying with $B$ and subtracting $c$ on both sides, we obtain:

$$Bz^k - c \in B\partial g^*(B^T\lambda^{k-1} + \tau B^T(c - A\tilde{z}^k - Bz^k)) - c.$$ 

By multiplying with $\tau$ and adding $\lambda^{k-1} + \tau(c - A\tilde{z}^k - Bz^k)$ on both sides, we get:

$$\lambda^{k-1} - \tau A\tilde{z}^k \in \tau B\partial g^*(B^T\lambda^{k-1} + \tau B^T(c - A\tilde{z}^k - Bz^k)) - \tau c + \lambda^{k-1} + \tau(c - A\tilde{z}^k - Bz^k).$$

Since $\psi(\lambda) = g^*(B^T\lambda) - c^T\lambda$, we have $\partial \psi(\lambda) = B\partial g^*(B^T\lambda) - c$, which implies:

$$\lambda^{k-1} - \tau A\tilde{z}^k \in \tau \partial \psi(\lambda^{k-1} + \tau(c - A\tilde{z}^k - Bz^k)) + \lambda^{k-1} + \tau(c - A\tilde{z}^k - Bz^k).$$

Since $z^k = \tilde{z}^k - \theta^k$, it follows that:

$$\lambda^{k-1} - \tau A\tilde{z}^k \in \tau \partial \psi(\lambda^{k-1} + \tau(c - A\tilde{z}^k - B\tilde{z}^k + B\theta^k)) + \lambda^{k-1} + \tau(c - A\tilde{z}^k - B\tilde{z}^k + B\theta^k).$$

By Step 3 in Algorithm 3, the above equation results in:

$$\lambda^{k-1} - \tau A\tilde{z}^k \in \tau \partial \psi(\lambda^{k} + \tau B\theta^k) + \lambda^{k} + \tau B\theta^k.$$ 

From Step 1 in Algorithm 3 and the property of the conjugate function $p \in \partial f(q) \iff q \in \partial f^*(p)$, we obtain:

$$\lambda^{k-1} - \tau A(\nabla f^*(A^T\lambda^k) + \delta^k) \in \tau \partial \psi(\lambda^{k} + \tau B\theta^k) + \lambda^{k} + \tau B\theta^k.$$
Figure 4: Comparison of the numbers of iterations for Algorithm 7, using two approaches: **Approach 1** uses a bound on the minimal number, i.e. the termination condition proposed in (24); and **Approach 2** computes the exact minimal number, which requires the optimal solution of the local problem $z_i^{k,*}$ at each iteration.

By definition of the function $\phi$, we get:

$$\lambda^{k-1} - \tau(\nabla \phi(\lambda^{k-1}) + A\delta^k) \in \tau \partial \psi(\lambda^k + \tau B\theta^k) + \lambda^k + \tau B\theta^k,$$

which is equivalent to:

$$\lambda^k = \text{prox}_{\tau\psi}(\lambda^{k-1} - \tau(\nabla \phi(\lambda^{k-1}) + e^k)) - \tau B\theta^k,$$

with $e^k = A\delta^k$. In order to complete the proof of equation (26), we need to show that $\lambda^k$ is an inexact solution of the proximal operator as defined in equation (3) with the error $\epsilon^k = \tau^2 \|B\theta^k\| + \frac{\tau^2}{2} \|B\theta^k\|^2$, i.e. to prove:

$$\tau \psi(\lambda^k) + \frac{1}{2} \|\lambda^k - v\|^2 \leq \epsilon^k + \min_{\lambda} \left\{ \tau \psi(\lambda) + \frac{1}{2} \|\lambda - v\|^2 \right\},$$

where $v = \lambda^{k-1} - \tau(\nabla \phi(\lambda^{k-1}) + A\delta^k)$. Finally, using

$$
\begin{align*}
\tau \psi(\lambda^k + \tau B\theta^k) + \frac{1}{2} \|\lambda^k + \tau B\theta^k - v\|^2 - \tau \psi(\lambda^k) - \frac{1}{2} \|\lambda^k - v\|^2 &\leq \tau (\psi(\lambda^k + \tau B\theta^k) - \psi(\lambda^k)) + \frac{1}{2} \|\tau B\theta^k\|^2 \\
&\leq \tau^2 L(\psi) \|B\theta^k\| + \frac{\tau^2}{2} \|B\theta^k\|^2 = \epsilon^k,
\end{align*}
$$
equation (26) is proved.

B. **Proof of Lemma 3.12**

Proof: We first prove that there exists an upper bound on the series $b^k = \sum_{p=1}^k \frac{\alpha^{-p}}{p}$. Since $0 < \alpha < 1$, there always exists a positive integer $k'$ such that $0 < \frac{\alpha^{-k'}}{k' + 1} < \frac{\alpha^{-1}}{k' + 1}$. We can write the series $b^k$ as

$$b^k = \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \sum_{p=k'+1}^k \frac{\alpha^{-p}}{p}.$$  

Since $k'$ satisfies $0 < \frac{\alpha^{-k'}}{k' + 1} < \frac{\alpha^{-1}}{k' + 1}$ and $0 < \alpha < 1$, then we know that for any $t \geq k'$ the function $\frac{\alpha^{-t}}{t}$ is a non-decreasing function with respect to $t$. Due to the fact that for any non-decreasing function $f(t)$, the following inequality holds,

$$\sum_{p \in \mathbb{Z}; y \leq p \leq x} f(p) = \int_y^x f([t]) dt + f(x) \leq \int_y^x f(t) dt + f(x)$$
where \( \lfloor \cdot \rfloor \) denotes the floor operator, the series \( b^k \) can be upper-bounded by

\[
b^k \leq \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \int_{k'}^{k} \frac{\alpha^{-t}}{t} dt + \frac{\alpha^{-k}}{k}.
\]

We know that the integral of the function \( \frac{\alpha^{-t}}{t} \) is equal to \( \text{E}_1(-x \log(\alpha)) \), where \( \text{E}_1(\cdot) \) denotes the Exponential Integral Function, defined as \( \text{E}_1(x) := \int_{x}^{\infty} \frac{e^{-t}}{t} dt \). By using the fact that \( -\text{E}_1(-x) = \text{E}_1(x) \), where \( \text{E}_1(x) := \int_{x}^{\infty} \frac{e^{-t}}{t} dt \), and inequality (5.1.20) in [1], it follows that the Exponential Integral Function \( \text{E}_1(x) \) satisfies

\[
-\log(1 + \frac{1}{x}) < e^x \cdot \text{E}_1(-x) < -\frac{1}{2} \cdot \log(1 + \frac{2}{x}).
\]

Since \( e^x > 0 \), we can rewrite the inequality as

\[
-e^{-x} \log(1 + \frac{1}{x}) < \text{E}_1(-x) < -\frac{1}{2} e^{-x} \log(1 + \frac{2}{x}).
\]

Hence, the series \( b^k \) can be further upper-bounded by

\[
b^k \leq \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \frac{\alpha^{-k}}{k} + \text{E}_1(-k \log(\alpha)) - \text{E}_1(-k' \log(\alpha))
\]

\[
< \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \frac{\alpha^{-k}}{k} - \frac{1}{2} e^{-k \log(\alpha)} \log(1 + \frac{2}{k \log(\alpha)}) + e^{-k' \log(\alpha)} \log(1 + \frac{1}{k' \log(\alpha)})
\]

\[
= \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \frac{\alpha^{-k}}{k} - \frac{1}{2} \alpha^{-k} \log(1 + \frac{2}{k \log(\alpha)}) + \alpha^{-k'} \log(1 + \frac{1}{k' \log(\alpha)})
\]

We can now find the upper-bound for the series \( s^k \) as

\[
s^k = \alpha^k \cdot b^k < \alpha^k \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} + \frac{1}{k} - \frac{1}{2} \log(1 + \frac{2}{k \log(\alpha)}) + \alpha^{k-k'} \log(1 + \frac{1}{k' \log(\alpha)})
\]

Since \( 0 < \alpha < 1 \) and the integer \( k' \) is a constant for a given \( \alpha \), the upper bound above converges to zero, as \( k \) goes to infinity. In addition, we know that the two terms \( \alpha^k \sum_{p=1}^{k'} \frac{\alpha^{-p}}{p} \) and \( \alpha^{k-k'} \log(1 + \frac{1}{k' \log(\alpha)}) \) converge to zero linearly with the constant \( \alpha \). From Taylor series expansion, we know that the term \( \frac{1}{2} \log(1 + \frac{2}{k \log(\alpha)}) \) converges to zeros at the rate \( O(\frac{1}{k}) \). Note that since \( 0 < \alpha < 1 \), the term \( \frac{1}{2} \log(1 + \frac{2}{k \log(\alpha)}) \) is always negative for all \( k > 0 \). To summarize, we know that the upper bound above converges to zero with the rate \( O(\frac{1}{k}) \). Therefore, we conclude that the series \( s^k \) converges to zero, as \( k \) goes to infinity. In addition, the convergence rate is \( O(\frac{1}{k}) \).

**REFERENCES**

[1] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Dover Publications, Incorporated, 1974.

[2] H. H. Bauschke and P. L. Combettes. Convex analysis and monotone operator theory in Hilbert spaces. Springer, 2011.

[3] A. Beck and M. Teboulle. A fast iterative shrinkage thresholding algorithm for linear inverse problems. SIAM Journal on Imaging Sciences, pages 183–202, 2009.

[4] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar. Convex analysis and optimization. Athena Scientific Belmont, 2003.

[5] D. P. Bertsekas and J. N. Tsitsiklis. Parallel and Distributed Computation: Numerical Methods. Athena Scientific, Belmont, Massachusetts, 1997.

[6] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends in Machine Learning, 3:1–122, 2011.

[7] P. L. Combettes and J-C. Pesquet. Proximal splitting methods in signal processing. In Fixed-Point Algorithms for Inverse Problems in Science and Engineering. Springer Optimization and Its Applications, volume 53, pages 185–212. Springer, New York, 2011.

[8] C. Comte, N. R. Voellmy, M. N. Zeilinger, M. Morari, and C. N. Jones. Distributed synthesis and control of constrained linear systems. In American Control Conference, 2012, pages 6017–6022, 2012.

[9] Q. T. Dinh, I. Necoara, and M. Diehl. Fast inexact decomposition algorithms for large-scale separable convex optimization. arXiv preprint arXiv:1212.4275, 2012.

[10] P. Giselsson. Execution time certification for gradient-based optimization in model predictive control. In 51th IEEE Conference on Decision and Control, pages 3165–3170, December 2012.

[11] P. Giselsson, M. D. Doan, T. Keiczky, B. D. Schutter, and A. Rantzer. Accelerated gradient methods and dual decomposition in distributed model predictive control. Automatica, 49:829–833, 2013.

[12] T. Goldstein, B. O’Donoghue, and S. Setzer. Fast alternating direction optimization methods. CAM report, pages 12–35, 2012.

[13] H. Lin, J. Mairal, and Z. Harchaoui. A universal catalyst for first-order optimization. In Advances in Neural Information Processing Systems, pages 3366–3374, 2015.

[14] I. Necoara and V. Nedelcu. Rate analysis of inexact dual first order methods: Application to distributed MPC for network systems. arXiv:1302.3129 [math], February 2013. arXiv:1302.3129.

[15] Y. Nesterov. A method of solving a convex programming problem with convergence rate \( O(1/k^2) \). In Soviet Mathematics Doklady, volume 27, pages 372–376, 1983.
[16] P. Patrinos and A. Bemporad. An accelerated dual gradient-projection algorithm for embedded linear model predictive control. *IEEE Transactions on Automatic Control*, 59:18–33, 2014.

[17] Y. Pu, M. N. Zeilinger, and C. N. Jones. Fast alternating minimization algorithm for model predictive control. In *19th World Congress of the International Federation of Automatic Control*, 2014.

[18] Y. Pu, M. N. Zeilinger, and C. N. Jones. Quantization design for distributed optimization with time-varying parameters. In *54th IEEE Conference on Decision and Control*, pages 2037–2042, 2015.

[19] Y. Pu, M. N. Zeilinger, and C. N. Jones. Inexact fast alternating minimization algorithm for distributed model predictive control. In *53rd IEEE Conference on Decision and Control*, pages 5915–5921, 2014.

[20] R. Richter, C. N. Jones, and M. Morari. Computational complexity certification for real-time MPC with input constraints based on the fast gradient method. *IEEE Transactions on Automatic Control*, 57(6):1391–1403, 2012.

[21] R. Scattolin. Architectures for distributed and hierarchical model predictive control – a review. *Journal of Process Control*, 19(5):723–731, 2009.

[22] M. Schmidt, N. L. Roux, and F. Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *25th Annual Conference on Neural Information Processing Systems*, pages 6819–6824, 2011.

[23] P. Tseng. Applications of a splitting algorithm to decomposition in convex programming and variational inequalities. *SIAM Journal on Control and Optimization*, 29:119–138, 1991.

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