SOME GENERALIZATIONS OF PREPROJECTIVE ALGEBRAS
AND THEIR PROPERTIES

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Abstract. In this note we consider a notion of relative Frobenius pairs of
commutative rings $S/R$. To such a pair, we associate an $\mathbb{N}$-graded $R$-algebra
$\Pi_R(S)$ which has a simple description and coincides with the preprojective
algebra of a quiver with a single central node and several outgoing edges in
the split case. If the rank of $S$ over $R$ is 4 and $R$ is noetherian, we prove
that $\Pi_R(S)$ is itself noetherian and finite over its center and that each $\Pi_R(S)_d$
is finitely generated projective. We also prove that $\Pi_R(S)$ is of finite global
dimension if $R$ and $S$ are regular.

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1. Introduction

1.1. Relative Frobenius Pairs. For the purpose of this paper, we consider pairs
of commutative rings $R, S$ equipped with a map $R \to S$. We often refer to such
a pair as $S/R$. Moreover we will always assume $R$ is Noetherian, although some of
the results also hold in higher generality.

Definition 1.1. We say that $S/R$ is relative Frobenius of rank $n$ if:

- $S$ is a free $R$-module of rank $n$.
- $\text{Hom}_R(S, R)$ is isomorphic to $S$ as $S$-module.

The second author is an aspirant of the FWO.
Remark 1.2.  

- It is clear that if $R$ is a field, then a relative Frobenius pair coincides with a finite dimensional Frobenius algebra in the classical sense.
- Let $e_1, \ldots, e_n$ be any basis for $S$ as an $R$-module. Then the second condition is equivalent to the existence of a $\lambda \in \text{Hom}_R(S, R)$ such that the $R$-matrix $(\lambda(e_i e_j))_{i,j}$ is invertible.
- Although it also makes sense to consider relative Frobenius pairs of non-commutative rings, we won’t consider these in this paper.
- We may equally well assume that $S/R$ is projective of rank $n$. However all results we prove may be reduced to the free case by suitably localizing $R$.

In this paper we only consider the case that the rank is 4. If $R$ is an algebraically closed field, it is an easy exercise to describe all such algebras:

Lemma 1.3. Let $k$ be an algebraically closed field and $F$ a commutative Frobenius algebra of dimension 4 over $k$. Then $F$ is isomorphic to one of the following algebras:

\[
\begin{align*}
&k \oplus k \oplus k \oplus k \\
&k[t]/(t^2) \oplus k \oplus k \\
&k[s]/(s^2) \oplus k[t]/(t^2) \\
&k[t]/(t^3) \oplus k \\
&k[t]/(t^4) \\
&k[s, t]/(s^2, t^2)
\end{align*}
\]

Proof. First recall that

- a direct sum of Frobenius algebras is Frobenius.
- a finite dimensional commutative local $k$-algebra is Frobenius if and only if it has a unique minimal ideal.

It follows immediately that $k[t]/(t^n)$ is Frobenius (of dimension $n$) over $k$ as it has a unique minimal ideal $(t^{n-1})$ and $k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ is also Frobenius (of dimension $2^n$) with unique minimal ideal $(x_1 \cdots x_n)$. Thus the algebras in the above list are certainly Frobenius.

Now let $F$ be Frobenius of dimension 4. Since $F$ is Artinian, the structure theorem for Artinian rings [1] Theorem 8.7] states that $F$ must (uniquely) decompose as a direct sum of local, Artinian $k$-algebras:

\[F \cong F_1 \oplus \ldots \oplus F_n\]

We can now use the classification of local $k$-algebras of small rank in [10] Table 1]. If $n = 4$, then clearly $F = k \oplus k \oplus k \oplus k$.

If $n = 3$, then $F \cong A_1 \oplus k \oplus k$ where $\text{dim}_k(A_1) = 2$, hence $A_1 \cong k[t]/(t^2)$ which is Frobenius.

If $n = 2$, then either $F$ splits as a sum of 2-dimensional local $k$-algebras, in which case we again obtain $F \cong k[s]/(s^2) \oplus k[t]/(t^2)$ or $F = A_1 \oplus k$ where $\text{dim}_k(A_1) = 3$. This again yields 2 possibilities: either $A_1 \cong k[t]/(t^3)$, which is Frobenius, or $A_1 \cong k[s, t]/(s, t)^2$. The latter is however not Frobenius, because it is not self-injective (the morphism $A_1 t \rightarrow A_1 : t \mapsto s$ cannot be lifted to $A_1 \rightarrow A_1$).

Finally, assume $n = 1$. In this case $F$ is a local $k$-algebra of dimension 4 and by
 takes one of the five following forms:
\[
\begin{align*}
&\frac{k[t]}{(t^4)} \\
&\frac{k[s, t]}{(s^2, t^2)} \\
&\frac{k[s, t]}{(s^2, st, t^3)} \\
&\frac{k[s, t, u]}{(s, t, u)^2} \\
&\frac{k[s, t]}{(s^2 + t^2, st)} \quad \text{(if } k \text{ has characteristic } 2.)
\end{align*}
\]

The first two algebras are Frobenius whereas the other three are not as they are not self-injective by a similar argument as above.

The 6 Frobenius algebras listed in the above Lemma are related to each other by deformation. We shall use the following ad hoc notion of deformation:

**Definition 1.4.** Let \( F \) and \( G \) be Frobenius algebras over \( k \). A Frobenius deformation of \( F \) to \( G \) is a \( k[[u]] \)-algebra \( D \) such that \( D/k[[u]] \) is relatively Frobenius and

1. \( D/uD \cong F \) as a \( k \)-algebra
2. \( D(u) \cong G \otimes_k k((u)) \) as a \( k((u)) \)-algebra

We write \( F \stackrel{\text{def}}{\longrightarrow} G \).

**Remark 1.5.** Instead of requiring that \( D/k[[u]] \) is relative Frobenius we may equivalently require that \( D \) is free over \( k[[u]] \) with rank equal to the dimension of \( F \). The condition that \( \text{Hom}_{k[[u]]}(D, k[[u]]) \) should be isomorphic to \( D \) as \( D \)-modules is immediate by the corresponding condition on \( F/k \).

**Lemma 1.6.** There is a diagram of Frobenius deformations

\[
\begin{array}{c}
\frac{k[t]}{(t^2)} \oplus \frac{k[s]}{(s^2)} \\
\frac{k[s, t]}{(s^2, t^2)} \quad \frac{k[t]}{(t^4)} \\
\frac{k[t]}{(t^2)} \oplus k \oplus k \quad \frac{k[t]}{(t^3)} \oplus k
\end{array}
\]

\[
\begin{array}{ccc}
\text{def} & \text{def} & \text{def} \\
\delta \ 2 & \delta \ 4 & \delta \ 5 \\
\delta \ 3 & \delta \ 6 & \delta \ 6
\end{array}
\]

\( (1) \)

**Proof.** We first describe \( F := \frac{k[s, t]}{(s^2, t^2)} \stackrel{\text{def}}{\longrightarrow} G := \frac{k[t]}{(t^4)} \). Let \( R := k[[u]] \), \( K := k((u)) \) and define

\[
D := R[s, t]/(us - t^2, s^2, t^4)
\]

We claim that \( D \) defines a deformation from \( F \) to \( G \). It is clear that \( D/uD \cong F \) as a \( k \)-algebra and the map

\[
D \longrightarrow K[t]/(t^4) : u \mapsto u, \ s \mapsto t^2/u, \ t \mapsto t
\]

factors through an isomorphism

\[
D(u) \longrightarrow K[t]/(t^4) = G \otimes_k K
\]
Hence by the above remark it suffices to check that $D$ is a free $R$-module of rank 4. This is obviously the case with $e_1 = 1, e_2 = s, e_3 = t, e_4 = st$ providing an $R$-basis for $D$.

The other cases are similar. We first use the Chinese remainder theorem to find an alternate presentation for $F$ of the forms $k[t]/(f(t))$. Then for each deformation $F \xrightarrow{\text{def}} G$, we try to find an alternate presentation for $G \otimes_k K$ (again using the Chinese remainder theorem) of the form $K[t]/(g(t))$ in such a way that $g(t)|_{t=0} = f(t)$. We then exhibit an $R$-algebra $D := R[t]/(g(t))$. We leave the reader to check that in each of our choices, $(1, t, t^2, t^3)$ defines an $R$-basis.

| number | 2 | 3 | 4 | 5 | $6^*$ ($\text{char}(k) \neq 2$) |
|--------|---|---|---|---|-------------------------------|
| $g(t)$ | $t^2(t-u)^2$ | $t^3(t-u)$ | $(t-1)^2(t-u)$ | $t^2(t-1)(t-u)$ | $(t^2-u^2)(t^2-1)$ |

* In case $k$ has characteristic 2, one has to choose $D = R[t]/(t(t-u)) \oplus T_{R^2}$ for the 6th deformation. In this case $(1, 0, 0), (t, 0, 0), (0, 1, 0), (0, 0, 1)$ provides an $R$-basis for $D$. □

1.2. Generalized Preprojective Algebras. We shall need the following notation: for a relative Frobenius pair $S/R$, let $M := R_S$. This $R - S$-bimodule can be considered a $R \oplus S$ bimodule by letting the $R$-component act on the left and the $S$-component on the right, the other actions being trivial. Similarly, we let $N := sS_R$ and consider it an $R \oplus S$-bimodule by only letting the $S$-component act on the left and the $R$-component act on the right, the other actions again begin trivial. We now define

$$T(R, S) := T_{R \otimes S}(M \oplus N)$$

Note that by construction, we have $M \otimes_{R \otimes S} M = N \otimes_{R \otimes S} N = 0$, hence

$$T(R, S)_2 = (M \otimes_{R \otimes S} N) \oplus (N \otimes_{R \otimes S} M) = (R \otimes S \otimes R \otimes S) \oplus (S \otimes R \otimes S)$$

The algebra we are interested in, will be a quotient of $T(R, S)$ as follows: let $\lambda$ be a generator of $\text{Hom}_R(S, R)$ as an $S$-module. The $R$-bilinear form $(a, b) := \lambda(ab)$ is clearly nondegenerate and hence we can find dual $R$-bases $(e_i), (f_j)$ satisfying

$$\lambda(e_i f_j) = \delta_{ij}$$

**Definition 1.7.** For a relative Frobenius pair, the **generalized preprojective algebra** $\Pi_R(S)$ is given by

$$T(R, S)/(\text{rels})$$

where the relations are in degree 2 given by

$$1 \otimes 1 \in R \otimes S \otimes S_R$$

$$\sum_i e_i \otimes f_i \in S \otimes R \otimes S_S$$

**Remark 1.8.** Up to isomorphism, the above construction is independent of choice of generator and dual basis.

The name generalized preprojective algebra is motivated by the following:
Lemma 1.9. Let $S$ be the ring $R^\oplus n$. Then $\Pi_R(S)$ is isomorphic to the preprojective algebra over $R$ associated to the quiver with one central vertex and $n$ outgoing arrows.

Proof. Let $e_1, \ldots, e_n$ be the set of complete orthogonal idempotents in $S$ and write $x_1, \ldots, x_n$ (resp $y_1, \ldots, y_n$) for the corresponding elements in the bimodules $N$ (respectively $M$). We can describe the tensor algebra $T(R, S)$ as the free algebra $F:= R\langle e_1, \ldots, e_n, x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ subject to the relations

1. $e_i e_j = \delta_{ij} e_i$,
2. $e_i x_j = \delta_{ij} x_i$ and $y_i e_j = \delta_{ij} y_i$
3. $x_i e_j = e_i y_j = 0$
4. $x_i x_j = y_i y_j = 0$

The first relation defining $\Pi_R(S)$ is given by $1 \otimes 1 \in M \otimes_S N$. The first 1 is given by $1 = \sum x_i$ whereas the second 1 = $\sum y_i$, we obtain

5. $y_1 x_1 + \ldots + y_n x_n = 0$

To compute the second relation, we note that

$$\lambda : S \to R : \sum_{i=1}^n r_i e_i \mapsto \sum_i r_i$$

is a generator of $Hom_R(S, R)$ as an $S$-module and hence $(e_i)_i$ is a basis, selfdual for the associated form $\langle , \rangle$. The relation inside $sS \otimes_R S$ now becomes

6. $x_1 y_1 + \ldots + x_n y_n = 0$

It now remains to show that $F$ subject to the above 6 relations is isomorphic to the preprojective algebra of the quiver $Q$:

We let $\overline{Q}$ denote the formally doubled quiver of $Q$ and consider the map $F \to R\overline{Q}$ defined by

- sending $e_i$ to the outer node $n_i$
- sending $y_i$ to the arrow $a_i$ and $x_i$ to the formal inverse $a_i^*$

The first 4 relations now precisely describe the multiplication in the path algebra of $\overline{Q}$ and the relations 5 and 6 precisely map to the two relations defining a preprojective algebra $\sum a_i a_i^* = 0 = \sum a_i^* a_i$. □
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2. Statement of the Results

Throughout this paper we assume $S/R$ is relative Frobenius of rank 4 (with the exception of Lemmas 3.1 and 3.2 which are stated in higher generality). Moreover $R$ will always be a noetherian ring. We prove three basic properties of the algebra $\Pi R(S)$ (under the above assumptions). Section 3 is dedicated to the following result:

**Theorem** (see 3.8). $\Pi R(S) d$ is projective of rank $\left\{\begin{array}{ll}5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd}\end{array}\right.$

In section 4 we investigate the center of $\Pi R(S)$ in degree 4, which we denote by $Z_4(R, S)$, and prove

**Theorem** (see 4.1, 4.2 and 4.3). $Z_4(R, S)$ is a split submodule of $\Pi R(S) _4$, projective of rank 2. We deduce from it that $Z_4(R, S)$ is compatible with base change.

Section 5 is dedicated to constructing a map

$$\sigma_{R,S} : R[Z_4(R, S)]^\oplus n \longrightarrow \Pi R(S)$$

and we prove

**Theorem** (see 5.2 and 5.1). $\sigma_{R,S}$ is surjective, in particular $\Pi R(S)$ is Noetherian and finite over its center.

The final section covers the global dimension of $\Pi R(S)$, giving as main result:

**Theorem** (see 6.1). If $R$ and $S$ have finite global dimension, then so does $\Pi R(S)$. We have the following explicit upper bound:

$$\text{gr.gl.dim}(\Pi R(S)) \leq \max(\text{gl.dim}(R), \text{gl.dim}(S)) + 2$$

These theorems are all proven using a similar technique, namely we first prove them in case $R$ is an algebraically closed field and $S$ is extremal in the deformation graph $\mathcal{F}$. Then we extend the results step by step, increasing the generality of $R$ as follows (with references to the applied lemmas):

(2)

alg. closed field $\xrightarrow{[6, 1.4.4]}$ field $\xrightarrow{[6, 1.4.4]}$ local domain

2 specific cases

field $\xrightarrow{[1.6]}$ local domain

domain

ring $\xleftarrow{3.6}$ local ring $\xleftarrow{3.6}$ local ring with $\mathcal{T} = k$
Finally we also mention the following result which will be proven in an upcoming paper:

**Theorem 2.1.** If $R$ and $S$ have finite global dimension, then $\Pi_R(S)$ has finite global dimension as well.

3. Computing $\text{rk}(\Pi_R(S)_d)$

The construction of $\Pi_R(S)$ is compatible with base change in the following way:

**Lemma 3.1** (Base Change for $\Pi_R(S)$). Let $S/R$ be relative Frobenius of finite rank and $R \rightarrow R'$ a morphism of rings. Then

1. $(R' \otimes_R S)/R'$ is relative Frobenius of rank $n$
2. there is a canonical isomorphism

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$

**Proof.** Assume that $S/R$ is relative Frobenius with generator $\lambda$ and basis $e_1, \ldots, e_n$, then $(R' \otimes_R S)/R'$ is relative Frobenius with generator $1 \otimes \lambda$ and basis $1 \otimes e_1, \ldots, 1 \otimes e_n$. With this data we can thus construct $\Pi_{R'}(R' \otimes_R S)$. Moreover,

$$R' \otimes_R (mS \oplus S_R) \cong R'(R' \otimes_R S)_{R' \otimes_R S} \oplus R' \otimes_R S(R' \otimes_R S)_{R'}$$

as an $(R', R' \otimes_R S)$-bimodule, and we obtain a canonical isomorphism

$$R' \otimes_R T(R, S) \cong T(R', R' \otimes_R S)$$

which by our choice of basis preserves the relations, inducing an isomorphism

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$

□

To prove that the $R$-modules $\Pi_R(S)_d$ are projective and to compute their ranks, following diagram (2), we first treat the case where $R$ is an algebraically closed field. We have the following lemma relating these vector spaces under deformation:

**Lemma 3.2.** Let $F$ and $G$ be Frobenius algebras over $k$ and let $F \overset{\text{def}}{\rightarrow} G$ be a Frobenius deformation. Then for all $d$, we have

$$\text{dim}_k(\Pi_k(F)_d) \geq \text{dim}_k(\Pi_k(G)_d)$$

**Proof.** Let $R = k[[u]]$ and $K = k((u))$.

Let $m = \text{dim}_k(\Pi_k(F)_d)$. Assume that $D$ is the $R$-algebra deforming $F$ to $G$. Then

$$\Pi_k(F) = \Pi_k(k \otimes_R D) = k \otimes_R \Pi_R(D)$$

by Lemma 3.1. Nakayama’s lemma implies that a $k$-basis of length $m$ for $\Pi_k(F)_d$ lifts to a set of generators for $\Pi_R(D)_d$. Moreover, as

$$K \otimes_k \Pi_k(G) = \Pi_K(K \otimes_k G) = \Pi_K(K \otimes_R D) = K \otimes_R (\Pi_R(D))$$

this set of generators contains a $K$-basis for $K \otimes \Pi_k(G)$. It follows that

$$\text{dim}_K(K \otimes_k (\Pi_k(G)_d)) = \text{dim}_k(\Pi_k(G)_d) \leq m$$

□

From now on we will only focus on the rank 4 case for the rest of the paper. i.e. when using the notation $S/R$, we will always assume this is a relative Frobenius pair of rank 4. Similarly all upcoming Frobenius algebras $F$ or $G$ will have dimension 4 over $k$.

We will now prove that in the case of Frobenius algebras of rank 4 the above inequality is actually an equality. We first compute the ranks in two explicit cases:
Lemma 3.3. We have

\[ \dim_k \left( \Pi_k \left( \frac{k[s, t]}{(s^2, t^2)} \right)_d \right) \leq \begin{cases} 5(d + 1) & \text{if } d \text{ is even} \\ 4(d + 1) & \text{if } d \text{ is odd} \end{cases} \]

Proof. This is proven in appendix A.1. \qed

Lemma 3.4. Let \( k \) be an algebraically closed field, then

\[ \dim_k \left( \Pi_k (k^{\otimes 4})_d \right) = \begin{cases} 5(d + 1) & \text{if } d \text{ is even} \\ 4(d + 1) & \text{if } d \text{ is odd} \end{cases} \]

Proof. By Lemma 1.9, \( \Pi_k(S) \) is the preprojective algebra over \( k \) associated to the extended Dynkin quiver of \( Q = \tilde{D}_4 \).

Let \( \overline{Q} \) be the formally doubled quiver. Let 0 denote the central vertex and 1, 2, 3, 4 the outer vertices. Then for each \( d \in \mathbb{N} \) we consider the matrix \( W_d \in \mathbb{N}^{5 \times 5} \) where \( (W_d)_{ij} \) gives the number of paths of length \( d \) in \( \overline{Q} \) starting at vertex \( i \) and ending at vertex \( j \), modulo relations. Finally write \( W(t) = \sum_{d=0}^{\infty} W_d t^d \in \mathbb{N}^{5 \times 5}[t] \). Then by [5, Proposition 3.2.1] we have

\[ W(t) = \frac{1}{1 - t \cdot C + t^2} \]

Where \( C \) is the adjacency matrix of \( \overline{Q} \), i.e.

\[
W(t) = \left( 1 - t \cdot \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + t^2 \right)^{-1}
\]

\[
= \frac{1}{(1 - t^2)^2(1 + t^2)} \cdot \begin{pmatrix}
(t + t^2)^2 & t(1 + t^2) & t(1 + t^2) & t(1 + t^2) & t(1 + t^2) \\
(t + t^2) & 1 - t^2 + t^4 & t^2 & t^2 & t^2 \\
(t + t^2) & t^2 & 1 - t^2 + t^4 & t^2 & t^2 \\
(t + t^2) & t^2 & t^2 & 1 - t^2 + t^4 & t^2 \\
(t + t^2) & t^2 & t^2 & t^2 & 1 - t^2 + t^4
\end{pmatrix}
\]
This gives the desired result as the Hilbert series of $\Pi_k(S)$ now becomes

\[
h_{\Pi_k(S)}(t) = \sum_{d=0}^{\infty} \left( \sum_{i,j=0}^{4} (W_d)_{i,j} \right) t^d
\]

\[
= \sum_{i,j=0}^{4} \sum_{d=0}^{\infty} (W_d)_{i,j} t^d
\]

\[
= \frac{(1 + t^2)^2 + 8t(1 + t^2) + 4(1 - t^2 + t^4) + 12t^2}{(1 - t^2)^2(1 + t^2)}
\]

\[
= \frac{5 + 8t + 5t^2}{(1 - t^2)^2}
\]

\[
= (5 + 8t + 5t^2) \sum_{l=0}^{\infty} (l + 1)t^{2l}
\]

\[
= \sum_{l=0}^{\infty} (5l + 5(l + 1))t^{2l} + 8(l + 1)t^{2l+1}
\]

\[
= \sum_{l=0}^{\infty} (5(2l + 1))t^{2l} + 4((2l + 1) + 1)t^{2l+1}
\]

\[\square\]

**Corollary 3.5.** Let $k$ be a field and $F$ a Frobenius algebra (of rank 4) over $k$ then:

\[\dim_k (\Pi_k(F)) = \begin{cases} 
5(d + 1) & \text{if } d \text{ is even} \\
4(d + 1) & \text{if } d \text{ is odd}
\end{cases}\]

**Proof.** By Lemma 3.1 we can reduce to the case where $k$ is algebraically closed. The statement then follows as a combination of Lemmas 1.3, 1.6, 3.2, 3.3 and 3.4. \[\square\]

To extend the result from fields to general rings we will need the following two lemmas. They essentially show that locally every relative Frobenius pair is a base change of a relative Frobenius pair where the ground ring is a polynomial ring over the integers.

**Lemma 3.6.** Let $R$ be a local ring with residue field $k$. Then there is a faithfully flat morphism $R \to \overline{R}$ where $\overline{R}$ is a local ring with residue field $\overline{k}$.

**Proof.** This is an immediate application of \[7\] 10.3.1 \[\square\]

**Lemma 3.7.** Let $R$ be a local ring with an algebraically closed residue field $k$. Let $S/R$ be relative Frobenius of rank 4. Then there exists a domain $\tilde{R}$, together with a morphism $\tilde{R} \to R$ and a ring $\tilde{S}$ with $\tilde{S}/\tilde{R}$ relative Frobenius of rank 4 such that $\tilde{S} \otimes_{\tilde{R}} R \cong S$.

Moreover $\tilde{R}$ can be chosen to be chosen of the form $\mathbb{Z}[x_1, \ldots, x_m]_f$, the localization of a polynomial ring over $\mathbb{Z}$ at some non-zero element $f$.

**Proof.** We prove the theorem in a specific case and quickly sketch the other cases, leaving some details to the reader. By Lemmas 3.1 and 1.4, $S \otimes k$ is one of 6 Frobenius algebras. Assume $S \otimes k = k[s, t]/(s^2, t^2)$ and let $s, t \in S$ be lifts of $s$ and
t. Since \((1, s, t, st)\) is a basis for \(S_k\). By Nakayama’s lemma \((1, s, t, st)\) forms a set of \(R\)-generators for \(S\). In particular we can write:

\[
\begin{align*}
\tilde{s}^2 &= a_1 + b_1 s + c_1 t + d_1 st \\
\tilde{t}^2 &= a_2 + b_2 s + c_2 t + d_2 st
\end{align*}
\]

where \(a_1, \ldots, d_2\) all lie in the maximal ideal of \(R\) (because \(s^2 = t^2 = 0\) in \(S \otimes k\)). We thus have a canonical morphism

\[
\pi : R[\tilde{s}, \tilde{t}]/(a_1 + b_1 s + c_1 t + d_1 st - \tilde{s}^2, a_2 + b_2 s + c_2 t + d_2 st - \tilde{t}^2) \to S
\]

such that \(\pi \otimes_R k\) is the identity morphism. It follows that \(\pi\) is surjective, moreover since \(S\) is free over \(R\), we have \(0 = \text{Ker}(\pi \otimes_R k) = \text{Ker}(\pi) \otimes_R k\) and \(\text{Ker}(\pi) = 0\) by Nakayama’s lemma. \(\pi\) is thus an isomorphism. There is a canonical morphism

\[
A := \mathbb{Z}[a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2] \to R
\]

Let \(f = 1 - d_1 d_2\) and denote \(\tilde{R} = A_f\), then as the image of \(f\) in \(R\) is invertible (because \(d_1, d_2\) lie in the maximal ideal of \(R\)), the above morphism factors through a morphism \(\tilde{R} \to R\). Finally set \(\tilde{S} = \tilde{R}[\tilde{s}, \tilde{t}]/(a_1 + b_1 \tilde{s} + c_1 \tilde{t} + d_1 \tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2 \tilde{s} + c_2 \tilde{t} + d_2 \tilde{s}\tilde{t} - \tilde{t}^2)\). By construction we have

\[
\tilde{S} \otimes_{\tilde{R}} R \cong R[\tilde{s}, \tilde{t}]/(a_1 + b_1 \tilde{s} + c_1 \tilde{t} + d_1 \tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2 \tilde{s} + c_2 \tilde{t} + d_2 \tilde{s}\tilde{t} - \tilde{t}^2) \cong S
\]

It hence suffice to prove \(\tilde{S} / \tilde{R}\) is relative Frobenius of rank \(4\). For this note that \((e_i)_1 := (1, s, t, st)\) is an \(\tilde{R}\)-base for \(\tilde{S}\) and if we let \(\lambda \in \text{Hom}_R(\tilde{S}, \tilde{R})\) denote the projection onto the component \(\tilde{R} s\tilde{t}\), the matrix of \(\lambda(e_i, e_j)\) is of the form

\[
\Theta = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & d_1 & 1 & * \\
0 & 1 & d_2 & * \\
1 & * & * & *
\end{bmatrix}
\]

Hence \(\Theta\) has determinant \(1 - d_1 d_2\), which by construction is invertible in \(\tilde{R}\), proving that \(\tilde{S}\) is indeed Frobenius of rank \(4\) over \(\tilde{R}\) by Remark 1.2

In the 5 other cases from Lemma 1.3 we have \(S \otimes k = k[t]/(t^4 + at^3 + bt^2 + ct + d)\) for some \(a, b, c, d \in k\) and we can choose \(\tilde{R}, \tilde{S}\) of the form \(\tilde{R} := \mathbb{Z}[\alpha, \beta, \gamma, \delta]\) and \(\tilde{S} := \tilde{R}[t]/(t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta)\). For each choice of \(\alpha, \beta, \gamma, \delta\) we have that \(\tilde{S} / \tilde{R}\) is relative Frobenius of rank \(4\), because the corresponding matrix \(\Theta\) will have determinant exactly \(1\). We leave the details to the reader.

We can now prove the main theorem of this section:

**Theorem 3.8.** \(\Pi_R(S)_d\) is projective of rank \(\begin{cases}
5(d + 1) & \text{if } d \text{ is even} \\
4(d + 1) & \text{if } d \text{ is odd}
\end{cases}\)

**Proof.** First let \(R\) be a local domain with residue field \(k\) and field of fractions \(K\). By Corollary 3.3 and Lemma 3.1 we have for each degree \(d\):

\[
\dim_K(K \otimes_R \Pi_R(S)_d) = \dim_K(\Pi_K(K \otimes R S)_d) = \dim_k(\Pi_k(k \otimes R S)_d) = \dim_k(k \otimes_R \Pi_R(S)_d)
\]

[6] 1.4.4 implies that \(\Pi_R(S)_d\) is free of the stated ranks.

Next, let \(R\) be any domain. Then for each \(p \in \text{Spec}(R)\), \(R_p \otimes_R \Pi_R(S) \cong \Pi_{R_p} (R_p \otimes S)\) is a generalized preprojective algebra over the local domain \(R_p\) and hence in each degree is a free module of the stated rank. As these ranks do not depend on the
choice of \( p \), Serre’s theorem (see for example [12]) now implies that \( \Pi_R(S)_d \) is projective of the stated rank.

Now let \( R \) be a local ring with algebraically closed residue field. Then by Lemma 3.7 there is a domain \( \tilde{R} \), a morphism \( \tilde{R} \to R \) and a ring \( \tilde{S} \) such that \( \tilde{S}/\tilde{R} \) is relative Frobenius of rank 4 and \( S \cong \tilde{S} \otimes \tilde{R} R \). By the above \( \Pi_{\tilde{R}}(\tilde{S})_d \) is a projective \( \tilde{R} \)-module of the given ranks and hence \( \Pi_R(S)_d = \Pi_{\tilde{R}}(\tilde{S})_d \otimes R \) is a projective \( R \)-module of the above rank.

To extend the result to general local rings, we invoke Lemma 3.6 to find a faithfully flat morphism \( R \to \tilde{R} \). By the above \( \Pi_R(R \otimes S)_d \cong \tilde{R} \otimes \Pi_R(S)_d \) is a free \( \tilde{R} \)-module of the desired rank. By the faithfully flatness of \( R \to \tilde{R} \), \( \Pi_R(S)_d \) is itself a free \( R \)-module of the desired rank.

Finally we extend the statement from local rings to general commutative rings by again applying Serre’s theorem. \( \square \)

The following Lemma is a slight improvement of Theorem 3.8 which we will need in the final section of this paper.

**Lemma 3.9.** \( (1_R \cdot \Pi_R(S))_d \) and \( (1_S \cdot \Pi_R(S))_d \) are projective \( R \)-modules of ranks respectively

\[
\begin{cases}
  d + 1 & \text{if } d \text{ is even} \\
  2(d + 1) & \text{if } d \text{ is odd}
\end{cases}
\]

and

\[
\begin{cases}
  4(d + 1) & \text{if } d \text{ is even} \\
  2(d + 1) & \text{if } d \text{ is odd}
\end{cases}
\]

**Proof.** Note that we can write \( \Pi_R(S) = 1_R \cdot \Pi_R(S) \oplus 1_S \cdot \Pi_R(S) \) and that this decomposition is compatible with base change and Frobenius deformations in the obvious sense. An argument similar to the proof of Theorem 3.8 shows that it suffices to check the cases \( S = k^{\oplus 4} \) and \( S = k[s, t]/(s^2, t^2) \).

For the first case we notice that \( h_{1_R \cdot \Pi_R(S)}(t) \) can be deduced from the proof of Lemma 3.4 by adding the entries in the first column of \( W(t) \), giving

\[
\begin{align*}
h_{1_R \cdot \Pi_R(S)}(t) & = \frac{(1 + t^2)^2 + 4 \cdot t(1 + t^2)}{(1 - t^2)^2(1 + t^2)} \\
& = \frac{1 + 6t + t^2}{(1 - t^2)^2} \\
& = (1 + 6t + t^2) \sum_{l=0}^{\infty} (l + 1) t^{2l} \\
& = \sum_{l=0}^{\infty} (2l + 1) t^{2l} + \sum_{l=0}^{\infty} 2((2l + 1) + 1) t^{2l+1}
\end{align*}
\]

Similarly we find

\[
\begin{align*}
h_{1_S \cdot \Pi_R(S)}(t) & = \sum_{l=0}^{\infty} 4(2l + 1) t^{2l} + \sum_{l=0}^{\infty} 2((2l + 1) + 1) t^{2l+1}
\end{align*}
\]

For the case \( S = k[s, t]/(s^2, t^2) \) this is an immediate corollary of the “Type I”-“Type II”-classification of the generators of \( \Pi_k(S) \) found in appendix A.1. \( \square \)
4. Base Change for $Z_4(R, S)$ and $\text{rk}(Z_4(R, S))$

Throughout this section we prove the following results for the center of $\Pi_R(S)$:

**Theorem 4.1.** $Z_4(R, S)$ is a split $R$-submodule of $\Pi_R(S)_4$.

**Theorem 4.2.** Let $S/R$ be relative Frobenius of rank 4 and $R \to R'$ a morphism of rings. Then the base change map

$$Z_4(R, S) \otimes_R R' \to Z_4(R', S \otimes_R R')$$

is an isomorphism.

**Theorem 4.3.** $Z_4(R, S)$ is a projective $R$-module of rank 2.

The proofs of these theorems are heavily intertwined, we shall prove them according to the following diagram of implications:

```
Theorem 4.3 when $R$ is a field
  ↓
Theorems 4.3 and 4.1 when $R$ is a local domains
  ↓
Theorem 4.1 for general $R$
  ↓
Theorem 4.2 for general $R$
  ↓
Theorem 4.3 for general $R$
```

In several of these steps we use the fact that in each degree the center $Z_d(R, S)$ can be obtained as kernel of a morphism between (projective) $R$-modules. For this recall that (with the notations of Section 2) there exists $R$-bases $a_0, \ldots, a_5$ for $\Pi_R(S)_0$ and $a_1 = e_1, \ldots, a_8 = f_4$ for $\Pi_R(S)_1$. Moreover, since $\Pi_R(S)$ is generated in degrees 0 and 1, for each $d$ there is a map

$$\phi_{R,S} : \Pi_R(S)_d \to \Pi_R(S)_d \oplus \Pi_R(S)_{d+1} : x \mapsto \left( ([x, a_i^0]), ([x, a_j^1]) \right)$$

whose kernel is precisely $Z_d(R, S)$. I.e. there is a left-exact sequence

$$0 \to Z_d(R, S) \to \Pi_R(S)_d \xrightarrow{\phi_{R,S}} \Pi_R(S)_{d+5} \oplus \Pi_R(S)_{d+8}$$

In particular we have the following special case of Theorem 4.2

**Lemma 4.4 (flat base change).** Let $R \to R'$ be a flat morphism of rings. Then the canonical map

$$R' \otimes_R Z_d(R, S) \to Z_d(R' \otimes_R S)$$

is an isomorphism.

**Proof.** The construction of $\phi_{R,S}$ is compatible with base change and tensoring with flat modules preserves left exact sequences. Hence

$$R' \otimes Z_d(R, S) = R' \otimes \ker(\phi_{R,S}) = \ker(R' \otimes \phi_{R,S}) = \ker(\phi_{R', R \otimes S}) = Z_d(R', R' \otimes S)$$

$\square$
As stated in Theorem 4.2 we will show that in case \( d = 4 \) we have base change for arbitrary morphisms. As in diagram (2) we shall first compute the dimension of \( Z_4(k, S) \) in two specific cases.

**Lemma 4.5.** Let \( k \) be an algebraically closed field of characteristic different from 2 and \( F = k^{2^4} \), then \( \Pi_k(F) \) is Morita equivalent to \( k[x, y] \# BD_8 \) where

\[
BD_8 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, ab = ba^3 \rangle
\]

is the binary dihedral group of order 8 acting on \( k[x, y] \) via

\[
a \cdot x = ix, \quad a \cdot y = -iy, \quad b \cdot x = y, \quad b \cdot y = x
\]

**Proof.** Let \( Q \) be \( \tilde{D}_4 \) and \( \overline{Q} \) the formally doubled quiver. Then \( \overline{Q} \) is the McKay-quiver of \( BD_8 \) and by [3, Corollary 4.2] (which was already announced in [11]) the preprojective algebra on \( Q \) is Morita equivalent to \( k[x, y] \# BD_8 \), the result now follows from Lemma 1.9.\( \Box \)

**Lemma 4.6.** Let \( R = k \) be an algebraically closed field of characteristic different from 2 then \( \dim_k(Z_4(k, k^{2^4})) = 2 \).

**Proof.** By Lemma 4.5 and the fact that the center of a ring is invariant under Morita equivalence, we only need to show that the degree 4 polynomials in \( k[x, y] \) invariant under the action of \( BD_8 \) span a 2-dimensional vectorspace. One easily checks that these invariants are given by \( kx^2y^2 \oplus k(x^4 + y^4) \). \( \Box \)

In order to include characteristic 2 as well, we need some direct computations which are done in Appendix B. In particular we prove:

**Lemma 4.7.** Let \( k \) be an algebraically closed field of characteristic 2, then \( \dim_k(Z_4(k, k^{2^4})) \geq 2 \).

For the second specific case we have:

**Lemma 4.8.** Let \( k \) be a field, then \( \dim_k(Z_4(k, k[s, t]/(s^2, t^2))) = 2 \).

**Proof.** This is proven in section A.2. \( \Box \)

We now use the following Lemma to compute the \( \dim_k(Z_4(k, F)) \) for all fields \( k \) and Frobenius algebras \( F \).

**Lemma 4.9.** Let \( F \) and \( G \) be two Frobenius algebras over a field \( k \) such that \( F \rightarrow G \). Then for each \( d \in \mathbb{N} \),

\[
\dim_k(Z_d(k, F)) \geq \dim_k(Z_d(k, G))
\]

**Proof.** Let \( D \) be the algebra deforming \( F \) to \( G \) and denote \( R = k[[u]], K = k((u)) \). As in [3], we write \( Z_d(R, D) = Ker(\phi) \) and let \( \Phi \) be the matrix corresponding to \( \phi \).

Let \( \Phi_K \) denote the same matrix with coefficients viewed in the fraction field \( K \) and \( \Phi_k \) denote the matrix with coefficients viewed in the residue field \( k \). Then by construction,

\[
Ker(\Phi_K) = Ker(K \otimes_R \phi) = Z_d(K, K \otimes_R D) \quad \text{and} \quad Ker(\Phi_k) = Ker(k \otimes_R \phi) = Z_d(k, k \otimes_R D)
\]
Now, 
\[
\dim_k(Z_d(k, G)) = \dim_K(K \otimes_k (Z_d(k, G)) \\
= \dim_K(Z_d(K, K \otimes_k G)) \\
= \dim_K(Z_d(K, K \otimes_R D)) \\
= \dim_K Ker(\Phi_K)
\]
Since clearly \(\dim_k(Ker(\Phi_k)) \geq \dim_K(Ker(\Phi_K))\), the claim follows.

**Lemma 4.10.** For any field \(k\) and Frobenius algebra \(F\) of dimension 4, we have \(\dim_k(Z_4(k, F)) = 2\)

*Proof.* If \(k\) is algebraically closed, this follows from Lemmas 3.3, 3.6, 3.9, 4.6, 4.7 and 4.8. For the general case we use Lemma 4.4. □

**Lemma 4.11.** Theorems 4.3 and 4.7 hold in the case where \(R\) is a local domain.

*Proof.* Let \(\phi_{R, S}\) be as in [3], then \(\phi_{R, S}\) is a morphism between free \(R\)-modules of finite rank and hence can be represented by a matrix \(\Phi\) with respect to some chosen basis for \(V := \Pi_R(S)_4\) and \(W := \Pi_R(S)_4^5 \oplus \Pi_R(S)_5^8\). Let \(\Phi_k\) be the matrix obtained by replacing each entry of \(\Phi\) by its corresponding class in \(k = R/m\), then \(\Phi_k\) is a matrix representation for \(k \otimes \phi\) using the induced \(k\)-basis for \(k \otimes_R V\) and \(k \otimes_R W\). Let \(a = rk(\Phi_k)\), then there is an invertible \(a \times a\) submatrix \(\Psi_k\) in \(\Phi_k\). The corresponding submatrix \(\Psi\) of \(\Phi\) has a determinant which does not lie in \(m\) and is thus itself invertible. By a suitable base change on \(V\) and \(W\) we can now rewrite \(\Phi\) in the form:
\[
\Phi = \begin{bmatrix}
Id_{a \times a} & 0 \\
0 & \Psi'
\end{bmatrix}
\]
all entries of \(\Psi'\) lie in \(m\) (any entry not in \(m\) would give rise to a an invertible submatrix of rank \(a + 1\) by elementary row and column operations). Hence we can decompose \(V\) and \(W\) as a direct sum of free submodules \(V = V_1 \oplus V_2\) and \(W = W_1 \oplus W_2\) such that \(\phi = \phi_1 \oplus \phi_2\) where \(\phi_1 : V_1 \rightarrow W_1\) and \(\phi_2 : V_2 \rightarrow W_2\) satisfies \(k \otimes_R \phi_2 = 0\). This implies that \(Z_4(k, k \otimes_R S) = ker(k \otimes \phi) = k \otimes V_2\) and hence \(V_2\) is free of rank 2 by Lemma 4.10.

Now, by construction \(Ker(\phi) \subset V_2\) and hence \(K \otimes Ker(\phi) \subset K \otimes V_2\). But then, since \(K\) is flat over \(R\), Lemma 3.1 gives:
\[
\dim_K(K \otimes_R Ker(\phi)) = \dim_K(K \otimes Z_4(R, S)) = \dim(\Pi_R(S)_4) = 2 = \dim(K \otimes V_2)
\]
It follows that \(Ker(\phi) = V_2\) from which \(\phi_2 = 0\) and hence \(Z_4(R, S) \rightarrow \Pi_R(S)_4\) splits. It follows that \(Z_4(R, S)\) is projective of finite rank and this rank equals 2 by Lemma 4.10. □

We can now finish the proofs of the main results of this section. This is done in a way similar to the proof of Theorem 3.8.

*Proof of Theorem 4.7.* By Lemma 4.11 we already know that the result holds if \(R\) is a local domain and by the local nature of splitting (see for example [9], Exercise 4.13, p.105) hence also if \(R\) is any domain.

Now let \(R\) be a local ring with algebraically closed residue field. Then by Lemma 3.7, \(S/R\) is a base change of \(\tilde{S}/\tilde{R}\) by a morphism \(\tilde{R} \rightarrow R\) for some domain \(\tilde{R}\) and
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the result follows in this case as the base change of a split embedding is a split embedding.

If $R$ is any local ring, we can consider the faithfully flat morphism $R \rightarrow \overline{R}$ provided by Lemma 3.6. As the residue field of $\overline{R}$ is locally closed the monomorphism $\iota_{\overline{R},S} : \iota_{R,S} \otimes R \rightarrow \overline{R}$ is split. This implies that $\iota_{R,S} : A \rightarrow B$ must be split itself by Lemma 4.12. Finally, again using the local nature of splitting, we have the result for any ring $R$.

□

Proof of Theorem 4.2. This is an immediate consequence of Theorem 4.1 and the fact that the construction of $\phi_{R,S}$ in (4) is compatible with base change.

□

Proof of Theorem 4.3. First let $R$ be a local domain with residue field $k$ and field of fractions $K$. Then by Lemma 4.10,

$$\dim_K(K \otimes_K(Z_4(R,S))) = \dim_k(Z_4(k,S)) = 2 = \dim_k(k \otimes_R Z_4(R,S)).$$

Hence by [6, Chapitre 1, Corollaire 4.4], $Z_4(R,S)$ is free of rank 2.

If $R$ is a domain, then for any $p \in \text{Spec}(R)$, $R_p$ is a local domain such that $R_p \otimes_R Z_4(R,S) = Z_4(R_p, R_p \otimes_R S)$ is a free module of rank 2. Serre’s theorem then proves that $Z_4(R,S)$ is projective of rank 2.

Now let $R$ be a local ring with algebraically closed residue field and let $\overline{S}/\overline{R}$ be as in Lemma 3.7. Then we know that $Z_4(\overline{R}, \overline{S})$ is projective over $\overline{R}$ of rank 2. Hence $Z_4(R,S) = Z_4(\overline{R}, \overline{S}) \otimes R$ is free of rank 2 over $R$.

To extend the statement to general local rings we just use Lemma 3.6. Finally Serre’s theorem extends the statement to non-local rings as well.

□

We now prove the technical lemma used in the proof of Theorem 4.1.

Lemma 4.12. Let $R$ be a local ring and let $R \rightarrow \overline{R}$ be as in Lemma 3.6. Let $\iota : A \rightarrow B$ be an embedding of finitely generated $R$-modules in which $B$ is projective. Moreover assume $\iota \otimes \overline{R} : A \otimes \overline{R} \rightarrow B \otimes \overline{R}$ is split. Then $\iota$ is a split monomorphism.

Proof. Let $k$ be the residue field of $R$ and $\overline{k}$ its algebraic closure, then there is a commutative diagram

$$
\begin{array}{ccc}
R & \rightarrow & k \\
\downarrow & & \downarrow \\
\overline{R} & \rightarrow & \overline{k}
\end{array}
$$

As $\iota \otimes \overline{R}$ is split, $\iota \otimes \overline{k}$ is a monomorphism. The above commutative diagram (and the faithfully flatness of $k \rightarrow \overline{k}$) implies $\iota \otimes k$ is a monomorphism. Let $C = \text{coker}(\iota)$, then we have a long exact sequence

$$\ldots \rightarrow \text{Tor}_1^R(B,k) \rightarrow \text{Tor}_1^R(C,k) \rightarrow A \otimes k \rightarrow B \otimes k \rightarrow C \otimes k \rightarrow 0$$

As $B$ is a projective $R$-module and $R$ is local, $B$ is also flat, implying $\text{Tor}_1^R(B,k) = 0$. From this it follows that $\text{Tor}_1^R(C,k) = 0$ and, again because $R$ is a local noetherian ring, this implies $C$ is a projective $R$-module such that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is split.

□
5. $\Pi_R(S)$ is noetherian and finite over its center

With the assumptions as before ($S/R$ relative Frobenius of rank 4 and $R$ noetherian), our main result of the section is the following

**Theorem 5.1.** $\Pi_R(S)$ is noetherian.

For this we define a map $\sigma_{R,S} : R[Z_4(R,S)] \otimes^N R \to \Pi_R(S)$ as follows: first choose an $R$-basis $(x, y, z, w)$ for $S$ and let $e$ be the element corresponding to $1_S \in N$ and $f$ be the element corresponding to $1_S \in M$. There is a map

$$\pi : R < x, y, z, w, e, f > \to T_{R\otimes S}(M \oplus N)$$

Where $x, y, z, w$ have degree 0 and $e, f$ have degree 1 in $R < x, y, z, w, e, f >$. The $R$-module $T(R, S)_0$ is generated by $(1_R, x, y, z, w)$ and these 5 elements are the images under $\pi$ of the corresponding elements in $R < x, y, z, w, e, f >$, hence $\pi$ is surjective in degree 0. Moreover, $T(R, S)_1 = R S \oplus S R$ is generated by $(xe, ye, ze, we, fx, fy, fz, fw)$ as an $R$-R-bimodule and hence $\pi$ is also surjective in degree 1. Finally since $T(R, S)$ is a tensor algebra, it is generated in degree 0 and 1 and $\pi$ is surjective. Composing with the canonical quotient map $T(R, S) \twoheadrightarrow \Pi_R(S)$ yields a surjection

$$\chi : R < x, y, z, w, e, f > \to \Pi_R(S)$$

Now, the $R$-module $\Pi_R(S)_{\leq 6}$ is generated the image of the words of length at most 6 in $\{e, f\}$. We can reduce this set by making the following remarks

1. since $\{1_R, x, y, z, w\}$ forms an $R$-basis for $\Pi_R(S)_0$, we can assume that any subword of degree zero is precisely a letter in this set
2. by the definition of the multiplication of $\Pi_R(S)$, we have $e^2 = f^2 = 0$

Hence if we let $H$ be the finite set set of words in $\{x, y, z, w, e, f\}$ of length at most 6 in $\{e, f\}$ and such any two instances of $x, y, z, w$ are separated by at least one $e$ or $f$. We obtain $\chi(R \cdot H) = \Pi_R(S)_{\leq 6}$. If we list this set as

$$H = \{a_1, \ldots, a_n\}$$

we can define $\sigma_{R,S}$ as

$$\sigma_{R,S} : R[Z_4(R,S)] \otimes^N R \to \Pi_R(S) : (z_i)_{i=1}^n \mapsto \sum_{i=1}^n z_i \chi(a_i)$$

We shall prove the following theorem

**Theorem 5.2.** $\sigma_{R,S}$ is a surjective map. In particular $\Pi_R(S)$ is finite over its center.

From this Theorem 5.1 will readily follow as $Z_4(R, S)$ is clearly finitely generated over $R$. In turn we prove Theorem 5.2 by first considering fields and then lifting the theorem via Nakayama.

First we give some base change arguments: Let $R \to R'$ be any morphism of rings, then by Theorem 4.2 we have a diagram

$$\begin{array}{ccc}
Z_4(R, S) & \to & Z_4(R', S) \\
\downarrow & \searrow & \downarrow \\
\Pi_R(S) & \to & \Pi_{R'}(S')
\end{array}$$
Proof of Theorem 5.2. If follows by the above and Nakayama.

Lemma 5.3. Let \( \Pi_R(R') \). Proof.

Corollary 5.6. This is proven in A.3.

Lemma 5.5. For any morphism \( \varphi : R \rightarrow R' \), the diagram in (3) is commutative.

\[
\begin{array}{c}
R'[Z_4(R', R' \otimes_R S)]^\otimes_n & \xrightarrow{\sigma_{R', R' \otimes_R S}} & \Pi_R(R' \otimes_R S) \\
\downarrow \eta & & \downarrow \eta \\
R' \otimes_R R[Z_4(R, S)]^\otimes_n & \xrightarrow{\sigma_{R, R} \otimes_R S} & R' \otimes_R \Pi_R(S)
\end{array}
\]

Let \( z_i \) be an element in \( R[Z_4(R, S)] \) considered as the \( i \)th component of \( R[Z_4(R, S)]^\otimes_n \), then

\[
\eta \circ (1_{R'} \otimes_R (\sigma_{R, S}))(r' \otimes z_i) = \eta (r' \otimes z_i(\chi_R(a_i))) = r'(1 \otimes z_i)\chi_R(a_i) = r'(1 \otimes z_i)\chi_R(1 \otimes a_i) = \sigma_{R', R' \otimes_R S}(r'(1 \otimes z_i)) = \sigma_{R', R' \otimes_R S} \circ \eta (r' \otimes z_i)
\]

Lemma 5.4. Let \( F \) and \( G \) be Frobenius algebras over \( k \) such that \( F \rightarrow G \). If \( \sigma_{k, F} \) is surjective, then so is \( \sigma_{k, G} \).

Proof. Let \( D \) be the algebra deforming \( F \) to \( G \) and write \( R := k[[u]] \) and \( K := k((u)) \).

Then Lemma 4.11 and Lemma 3.1 imply the vertical maps in (5) isomorphism, hence \( k \otimes_R \sigma_{R, D} = \sigma_{k, F} \). Thus Nakayama’s lemma implies that \( \sigma_{R, D} \) is surjective whenever \( \sigma_{k, F} \) is. A second application of (5) together with Lemma 3.1 and Theorem 1.2 shows that \( K \otimes_k \sigma_{k, G} = K \otimes_R \sigma_{R, D} \), showing that \( K \otimes_k \sigma_{k, G} \) is surjective in this case and hence also \( \sigma_{k, G} \) because \( K \) is faithfully flat over \( k \).

Lemma 5.5. Let \( F := k[s, t]/(s^2, t^2) \). Then the map \( \sigma_{k, F} \) is surjective.

Proof. This is proven in A.3.

Corollary 5.6. Let \( F \) be Frobenius over a field \( k \). Then \( \sigma_{k, F} \) is surjective.

Proof. If \( k \) is algebraically closed, then any Frobenius algebra \( F \) over \( k \) can be obtained from \( k[s, t]/(s^2, t^2) \) by a finite number of Frobenius deformations. Hence it follows immediately from Lemma 5.3 and Lemma 5.4.

For a general field we use that \( F \) is faithfully flat over \( k \).

Proof of Theorem 5.8. If \( R \) is a local ring, then \( k \otimes_R \sigma_{R, S} \cong \sigma_{k, k \otimes_R S} \) and the result follows by the above and Nakayama.
If $R$ is any ring, for any $p \in \text{Spec}(R)$, we have $R_p \otimes_R \sigma_{R,S} = \sigma_{R_p,R_p \otimes_R S}$, which is a surjective morphism. As this holds for all $p \in \text{Spec}(R)$, $\sigma_{R,S}$ is itself surjective. □

6. THE GLOBAL DIMENSION OF $\Pi_R(S)$

In this section we prove the following:

**Theorem 6.1.** The global dimension of $\Pi_R(S)$ is bounded by the number

$$\max(\text{gl. dim}(R), \text{gl. dim}(S)) + 2$$

We first bound the projective dimension of $R$ and $S$ as $\Pi_R(S)$-modules.

**Lemma 6.2.** There is a projective resolution of $R \oplus S$ of the following form:

$$0 \to \Pi_R(S)(-2) \xrightarrow{\alpha_2} (S \otimes_R S_R) \otimes \Pi_R(S)(-1) \xrightarrow{\alpha_1} \Pi_R(S) \xrightarrow{\alpha_0} R \oplus S \to 0$$

**Proof.** $\alpha_0$ is the canonical projection with kernel $\Pi_R(S)_{\geq 1}$. This module is generated by $\Pi_R(S)_1 = S \otimes_R \oplus R S$, hence $\text{im}(\alpha_1) = \ker(\alpha_0)$. Since the relations of $\Pi_R(S)$ are generated in degree 2, we also have $\text{im}(\alpha_2) = \ker(\alpha_1)$. Only the injectivity of $\alpha_2$ remains to be checked.

The sequence splits into the following two subsequences:

$$0 \to 1_R \cdot \Pi_R(S)(-2) \to 1_S \cdot \Pi_R(S)(-1) \to 1_R \cdot \Pi_R(S) \to R \to 0$$

$$0 \to 1_S \cdot \Pi_R(S)(-2) \to (1_R \cdot \Pi_R(S)(-1))^{\oplus 4} \to 1_S \cdot \Pi_R(S) \to S \to 0$$

By Lemma 3.1 exactness can be checked after localization at each prime ideal of $R$, hence we may assume all terms in (7) and (8) are free $R$-modules of finite rank in each degree by Lemma 3.9. The claim reduces to the following relation on the Hilbert series: for each $d \in \mathbb{N}$ we must have

$$h_{d-2}(1_R \cdot \Pi_R(S)(-2)) - h_{d-1}(1_S \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - \delta_{d,0} = 0$$

$$h_{d-2}(1_S \cdot \Pi_R(S)(-2)) - 4h_{d-1}(1_R \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - 4\delta_{d,0} = 0$$

(where $h_d(-)$ denotes the rank of the degree $d$-part as an $R$-module)

Using Lemma 3.9 we see that this is indeed the case. □

**Lemma 6.3.** Each simple $\Pi_R(S)$-module is either a simple $R$-module or a simple $S$-module.

**Proof.** Each simple $R$ or $S$-module is clearly simple when considered as a $\Pi_R(S)$-module. Conversely if $M$ is a simple $\Pi_R(S)$-module, then $M = 1_R M$ or $M = 1_S M$ since $M = 1_R M \oplus 1_S M$. Moreover we claim that $\Pi_R(S)_{\geq 1} M = 0$ or equivalently $\Pi_R(S)_1 M = 0$. For this assume for example that $M = 1_R M$. If $x \in S \otimes_R S$ then

$$xM = (1_S x) M = 1_S (x M) = 0$$

and if $x \in R \otimes_R S$ then

$$xM = (x 1_S) M = x (1_S M) = 0$$

Hence only the $R$-component in degree 0 acts non-trivially on $M$, it follows in particular that $M$ is also a simple $R$-module. The case $M = 1_S M$ is completely similar. □
Proof of Theorem 6.1. By [2, Proposition III.6.7(a)] it suffices to check that if $M$ is a simple $\Pi_R(S)$-module then:

$$pd_{\Pi_R(S)}(M) \leq \max(gl.\dim(R), gl.\dim(S)) + 2$$

By Lemma 6.3 $M$ is a simple $R$-module or a simple $S$-module. We assume the former, the other case being completely similar. Let $P_\bullet \rightarrow M$ be a resolution of $M$ by projective $R$-modules of length $pd_R(M) \leq gl\dim(R)$. Then for each $i$, by Lemma 6.2 we have

$$pd_{\Pi_R(S)}(P_i) \leq pd_{\Pi_R(S)}(R) \leq pd_{\Pi_R(S)}(R \oplus S) \leq 2$$

A standard long exact sequence-argument now gives the desired result. \qed
APPENDIX A. EXPLICIT COMPUTATIONS FOR $S = \frac{k[s, t]}{(s^2, t^2)}$

We describe $\Pi_k(S)$ through generators and relations:

- $\Pi_k(S)_0 = k \oplus S$. Let $a$ denote $(1_k, 0)$ and $b = (0, 1_S)$ then since $a + b = 1$, $a, 1, s, t, st$ is a $k$-basis for $\Pi_k(S)_0$. It is clear that this set satisfies the relations
  \[ a^2 = a, as = sa = at = ta = 0 \]

- $\Pi_k(S)_1 = kS \oplus S_k$. Let $f$ be $(1_S, 0)$ and $e = (0, 1_S)$, then we can write $\Pi_k(S)_1 = fS \oplus Se$. Hence $f, fs, ft, f(st, e, se, te, ste$ is a $k$-basis for $\Pi_k(S)_1$. By construction, each generator $\neq 1$ of $\Pi_k(S)_0$ acts nontrivially on exactly one side of each component. Hence we have the relations
  \[ ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0 \]

Note that this implies $e^2 = f^2 = 0$ since for example
  \[ e^2 = (ea)e = c(ea) = 0 \]

- It is clear that the relation $1 \otimes 1 \in kS \otimes sS_k$ takes the form $fe = 0$. To compute the second relation, note that projection onto $kst$ provides the duality isomorphism $\text{Hom}_R(S, R) \cong S$ (see Lemma 1.6). It immediately follows that $(e, se, te, ste)$ is dual to $(fst, ft, fs, f)$ in the sense of Definition 1.7. The relation now takes the form
  \[ (9) \]
  \[ efst + seft + tefs + stef = 0 \]

To summarize $\Pi_k(S)$ is a quotient of the free algebra $k < a, s, t, e, f >$ by the relations

\[
\begin{cases}
  s^2 = t^2 = st - ts = 0 \\
  a^2 = a, as = sa = at = ta = 0 \\
  ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0 \\
  fe = efst + seft + tefs + stef = 0
\end{cases}
\]

Note that $\Pi_k(S)$ is a graded algebra via $\text{deg}(a) = \text{deg}(s) = \text{deg}(t) = 0$ and $\text{deg}(e) = \text{deg}(f) = 1$.

A.1. Proof of Lemma 3.3 In this subsection we give sets of generators in each degree, hence giving an upper bound for $\text{dim}_k(\Pi_k(S)_d)$. More explicitly we prove that

\[
\text{dim}_k \left( \Pi_k \left( \frac{k[s, t]}{(s^2, t^2)} \right)_d \right) \leq \begin{cases}
  5(d+1) & \text{if } d \text{ is even} \\
  4(d+1) & \text{if } d \text{ is odd}
\end{cases}
\]

For this we make the following remarks:

- In each degree there are generators of two types:
  Type I) Elements of the form $f * ef * e f * \ldots * ef * c(f(\ast))$ where each $\ast$ is either $s, t$ or $st$
  Type II) Elements of the form $(\ast) ef * e f * \ldots * ef * c(f(\ast))$ where each $\ast$ is either $s, t$ or $st$
Let $\mathcal{R}$ denote the relation (10), then $f\mathcal{R}e, t\mathcal{R}, s\mathcal{R}, st\mathcal{R}$ take the form

\begin{align}
(10) & \quad fsefte = -ftefse \\
(11) & \quad steft = -tefst \\
(12) & \quad stefs = -sefst \\
(13) & \quad stefst = 0
\end{align}

As a consequence of the above equalities, we know that for any non-zero element there is at most one appearance of $st$. For example:

$$fste fsefst = fste f(sefst) = -fste f(stefs) = -f(stefst)efs = 0$$

We say any of the above elements is of bidegree $(m, n)$ if there are $m$ appearances of $s$ and $n$ appearances of $t$. It is easy to see that the above relations do not violate this bidegree and that it turns $\Pi_k(S)$ into a $\mathbb{Z} \times \mathbb{Z}$-graded ring. Using the above remarks we create (minimal) sets of generators by a case-by-case study:

- **Case 1: $d$ even and Type I**
  All words in this case take the form $(f*e)\ldots(f*e)$. We can use relations (10), (11), (12) to write the element in the form $\pm (fse)^i (fte)^j$ where $\varepsilon = 0, 1$. For $\varepsilon = 0$ we have $\frac{d}{2} + 1$ choices for $i$ and $j$ and for $\varepsilon = 1$ we have $\frac{d}{2}$ choices, giving a total $d + 1$ generators.

- **Case 2: $d$ even and Type II**
  These are elements of the form $(*) (ef*) \ldots (ef*) ef(*)$ and since there is at most one occurrence of $st$ the bidegree satisfies $\frac{d}{2} - 1 \leq m + n \leq \frac{d}{2} + 2$.
  If $m + n = \frac{d}{2} - 1$ the element can be written in the form $\pm (efs)^m (ef)n ef$, giving $\frac{d}{4}$ choices. Similarly if $m + n = \frac{d}{2} + 2$ the element can be written in the form $\pm (sef)^m st(eft)^n$. Giving $\frac{d}{2} + 1$ choices.
  Assume $m + n = \frac{d}{2}$. If $(m, n) = \left(\frac{d}{2}, 0\right)$ (or $(n, m) = \left(0, \frac{d}{2}\right)$) we have 2 generators: $(sef)^{\frac{d}{2}}$ and $(ef)^{\frac{d}{2}}$ (or $(tef)^{\frac{d}{2}}$ and $(eft)^{\frac{d}{2}}$).
  In all other cases we need 3 generators: $(sef)^{m} (tef)^{n}, (efs)^{m} (ef)^{n}$ and $(efs)^{m-1} efseft(ef)^n$. This gives a total of $\frac{3d}{2} + 1$ generators for this subcase.
  Finally assume $m + n = \frac{d}{2} + 1$. If $(m, n) = \left(\frac{d}{2} + 1, 0\right)$ (or $(m, n) = \left(0, \frac{d}{2} + 1\right)$) we have 1 generator: $(sef)^{\frac{d}{2} s}$ (or $(tef)^{\frac{d}{2} t}$).
  In all other cases we need 3 generators: $(sef)^{m} (tef)^{n-1} t, (efs)^{m-1} efst(ef)^{n-1}$ and $(sef)^{m-1} stef(tef)^n$. This gives a total of $\frac{3d}{2} + 1$ generators for this subcase.
  For case 2 this results in $\frac{d}{2} + \left(\frac{3d}{2} + 1\right) + \left(\frac{3d}{2} + 2\right) + \left(\frac{d}{2} + 1\right) = 4(d + 1)$ generators.
  Finally adding up the number of generators from Case 1 and Case 2 yields $5(d + 1)$ generators.

- **Case 3: $d$ odd and Type I**
  All elements in this case take the form $(f*e)(f*e)\ldots(f*e)f(*)$. By a completely similar argument as above, we conclude that generators can be chosen of the following forms:
  
  $(fse)^{m}(fte)^{n}f$, $(fse)^{m}(fte)^{n-1}ft$, $(fse)^{\frac{d+1}{2}}fs$, $(fse)^{n-1}(fte)^{m-1}fst$ and
\[ fste(fse)^{n-1}(fte)^{m-1}f. \] This gives a total of
\[
\frac{d+1}{2} + \frac{d+1}{2} + 1 + \frac{d+1}{2} + \frac{d-1}{2} = 2(d+1)
\]
generators
- **Case 4:** \( d \) odd and Type II
  Elements in this case are of the form \((*)e(f*e)(f*e)\ldots(f*e)\). Note that any such word can be obtained by taking a word from Case 3, reading it from right to left and interchanging \( e \) and \( f \). Applying this “procedure” to the generators of Case 3 yeilds a set of generators for the current case by symmetry. Hence in the current case we have \( 2(d+1) \) generators, adding up to \( 4(d+1) \) generators in case \( d \) is odd.

**A.2. Proof of Lemma 4.8** Consider the elements
\[
u := sef + efs + fse\quad\text{and}\quad v := tef + eft + fte\]
It is easy to see that \( u \) is normalizing with respect to the automorphism \( \sigma \) on \( \Pi_k(S) \) which sends \( t \) to \( -t \) and is the identity on the other generators. As \( \sigma^2 = Id \) we have as an immediate consequence that \( u^2 \) is central. A completely similar discussion yields that \( v^2 \) is central. Using the relations in \( \Pi_k(S) \) we can write \( u^2 \) and \( v^2 \) as
\[
sefsef + efsefs + fsefse \quad\text{and}\quad teftef + eftef + ftefte\]
In what follows we explain why there are no other central elements. This is done by constructing a basis for \( \Pi_k(S)_4 \) such that finding the central elements boils down to some linear algebra. By the arguments in A.1 we see that the following 25 elements generate \( \Pi_k(S)_4 \):

Type I)
- 1 element of bidegree \((1,1)\): \( fsefte \)
- 1 element of bidegree \((2,0)\): \( fsefse \)
- 1 element of bidegree \((0,2)\): \( teftef \)
- 1 element of bidegree \((2,1)\): \( fsefst \)
- 1 element of bidegree \((1,2)\): \( fstefte \)

Type II)
- 1 element of bidegree \((1,0)\): \( efsef \)
- 1 element of bidegree \((0,1)\): \( eftef \)
- 2 elements of bidegree \((2,0)\): \( sefsef, efseds \)
- 2 elements of bidegree \((0,2)\): \( teftef, eftef \)
- 1 element of bidegree \((3,0)\): \( sefsefs \)
- 1 element of bidegree \((0,3)\): \( teftef \)
- 3 elements of bidegree \((1,1)\): \( seftef, eftefse, efsteft \)
- 3 elements of bidegree \((2,1)\): \( sefseft, efsedsf, fsteft \)
- 3 elements of bidegree \((1,2)\): \( sefteft, eftseft, steftef \)
- 1 element of bidegree \((2,2)\): \( fsefteft \)
- 1 element of bidegree \((3,1)\): \( sefsefst \)
- 1 element of bidegree \((1,3)\): \( steftef \)

Corollary 3.5 implies that they form a \( k \)-basis for \( \Pi_k(S)_4 \).

Since the center of a graded ring is a homogeneous subring, we can write \( Z_d(k,S) \) as
\[
Z_d(k,S) = \bigoplus_{(m,n)} Z_d(k,S)_{m,n}\]
Where $Z_d(k, S)_{m,n}$ consists of the central elements in $\Pi_k(S)$ of degree $d$ and bidgree $(m,n)$. It follows that generators for $Z_4(k, S)$ can be chosen as linear combinations of elements of fixed bidgree. This reduces the computations to 12 linear combinations of at most 4 elements. A brute force computation shows that $sefsef + efsefs + fsefse$ and $teftef + efteft + ftefte$ are the only linear combinations that are central.

A.3. **Proof of Lemma 5.5.** Let $u$ and $v$ be the normalizing elements as above. And let $V \subset \Pi_k(S)_2$ be the $k$-vector space spanned by $u$ and $v$. Let $\mu_3$ be the multiplication morphism given by the composition

$$\mu_3 : V \otimes \Pi_k(S)_1 \rightarrow \Pi_k(S)_2 \otimes \Pi_k(S)_1 \rightarrow \Pi_k(S)_3$$

Then we use a brute force computation to show that $\mu_3$ must be surjective. I.e. we show that any element of $\Pi_k(S)_3$ can be written as a linear combination of elements of the form $u \cdot x$ or $v \cdot x$ with $x \in \Pi_k(S)_1$. It suffices to check this for the generators of $\Pi_k(S)_3$:

**Type I**: elements of the form $f \ast ef(*)$.
These can all be put into the form $fsef(*)$ or $ftef(*)$ where $\ast$ is either $s$, $t$ or $st$, Now use $fsef(*) = u \cdot f(*)$ and similarly $ftef(*) = v \cdot f(*)$.

**Type II**: elements of the form $(*)(ef \ast e$)
- $efse = u \cdot e$ and $efte = v \cdot e$
- $sefse = u \cdot se$ and $tefte = v \cdot te$
- $sefste = u \cdot ste$ and $tefste = v \cdot ste$
- $sefte = -tefse - efsef = v \cdot (-se)$
- $tefse = -sefste - fste = u \cdot (-te)$

Which shows that $\mu_3$ is indeed surjective.

Now for each degree $d$ we have a commutative diagram

$$
\begin{array}{ccc}
V \otimes \Pi_k(S)_{d+1} & \rightarrow & \Pi_k(S)_{d+3} \\
\downarrow & & \downarrow \\
V \otimes \Pi_k(S)_1 \otimes \Pi_k(S)_d & \rightarrow & \Pi_k(S)_3 \otimes \Pi_k(S)_d \\
\mu_3 \otimes \Pi_k(S)_d & & \\
\end{array}
$$

where the top horizontal arrow must be a surjection as the other three are surjective. Hence by induction (and the fact that $V \otimes -$ is right exact) we have for each $n \in \mathbb{N}$ a surjection

$$\mu_{2n+\varepsilon} : V^{\otimes n} \otimes \Pi_k(S)_\varepsilon \rightarrow V^{\otimes n-1} \otimes \Pi_k(S)_{2+\varepsilon} \rightarrow \ldots \rightarrow \Pi_k(S)_{2n+\varepsilon}$$

Next let $W$ be the vector space spanned by $u^2$ and $v^2$, then for each $n$ and $\omega = 1, 2$ there is a surjection

$$W^{\otimes n} \otimes V^{\otimes \omega} \rightarrow V^{\otimes 2n+\omega}$$

and we have a commutative diagram

$$
\begin{array}{ccc}
W^{\otimes n} \otimes V^{\otimes \omega} \otimes \Pi_k(S)_\varepsilon & \rightarrow & W^{\otimes n} \otimes \Pi_k(S)_{2\omega+\varepsilon} \\
\downarrow & & \downarrow \\
V^{\otimes 2n+\omega} \otimes \Pi_k(S)_\varepsilon \otimes \Pi_k(S)_d & \rightarrow & \Pi_k(S)_{4n+2\omega+\varepsilon} \\
\mu_{4n+2\omega+\varepsilon} & & \\
\end{array}
$$
where \( \rho_{4n+2\omega+\epsilon} \) must be surjective because the other three morphisms are. Then using the commutative triangle

\[
\begin{array}{ccc}
W^\otimes n \otimes \Pi_k(S)^{2\omega+\epsilon} & \xrightarrow{\rho_{4n+2\omega+\epsilon}} & \Pi_k(S)^{4n+2\omega+\epsilon} \\
 k[Z_4(k,S)]^n \otimes \Pi_k(S)^{2\omega+\epsilon} & \xrightarrow{\rho_{4n+2\omega+\epsilon}} & \\
\end{array}
\]

we must have that \( \rho_{4n+2\omega+\epsilon} : k[Z_4(k,S)]^n \otimes \Pi_k(S)^{2\omega+\epsilon} \longrightarrow \Pi_k(S)^{4n+2\omega+\epsilon} \) must be surjective. As \( 2\omega + \epsilon \) takes the values \( 3, 4, 5, 6 \) we have an induced surjection:

\( \overline{\rho} : k[Z_4(k,S)] \otimes \Pi_k(S)_{\leq 6} \longrightarrow \Pi_k(S) \)

(where we included \( \Pi_k(S)_d \) for \( d = 0, 1, 2 \) on the left hand side to guarantee surjectivity in these three lowest degrees).

Now \( \sigma_{k,S} \) factors as \( \overline{\rho} \circ \zeta \) where \( \zeta \) is the morphism:

\[
\zeta : k[Z_4(k,S)]^\otimes \otimes \Pi_k(S)_{\leq 6} : (z_i)_i \mapsto \sum_{i=1}^{\infty} z_i \otimes \chi(a_i)
\]

By the choice of the \( a_i \) in \( H \), \( \zeta \) is surjective and hence also \( \sigma_{k,S} \) proving the Lemma.

**Appendix B. Proof of Lemma 4.7**

In this section we do some computations in characteristic 2 in order to prove Lemma 4.7. For this let \( k \) be an algebraically closed field of characteristic 2, take \( F = k^{\otimes 4} \) and let \( a, b, c, d \) be the 4 idempotent elements. Our goal is to find two linearly independent central elements of degree 4 in \( \Pi_k(F) \). Remark that the elements of \( \Pi_k(F) \) are generated by elements of two types:

Type I) Elements of the form \( f * e f * e f \ldots * e f * e (f*) \) where each \( * \) is either \( a, b, c \) or \( d \)

Type II) Elements of the form \( * e f * e f \ldots * e f * e (f*) \) where each \( * \) is either \( a, b, c \) or \( d \)

The relations on \( \Pi_k(F) \) imply the following relations for these generators:

- \( a e f a = b e f b = c e f c = d e f d = 0 \)
- \( f a e + f b e + f c e + f d e = 0 \)

Our strategy in constructing central elements in degree 4 is the same as before: we first find elements in degree 2 which are normalizing with respect to some automorphism \( \sigma \) satisfying \( \sigma^2 = Id \). One such element is the following:

\[
u_{ab/cd} := a e f b + b e f a + c e f d + d e f c + f a e + f b e = a e f b + b e f a + c e f d + d e f c + f e c + f d e
\]

(where the equality follows from the above relations and the fact that we work in characteristic 2)

We use the subscript to denote that this element only depends on the partition of \( \{a, b, c, d\} \) in the subsets \( \{a, b\} \) and \( \{c, d\} \). Hence there are 3 such elements, the other two being \( u_{ac/bd} \) and \( u_{ad/bc} \).

This element is normalizing with respect to

\[
\sigma_{ab/cd} : a \mapsto b, \ b \mapsto a, \ c \mapsto d, \ d \mapsto c
\]

because for example:

\[
(ce)u_{ab/cd} = c e f d e = u_{ab/cd}(de) = u_{ab/cd} \cdot \sigma_{ab/cd}(ce)
\]
In particular the following 3 elements are central:

\[ x_{ab/cd} := u^2_{ab/cd} = aefbea + befaefb + cefdefc + defcefd + faefbe + fbefae \]
\[ x_{ac/bd} := u^2_{ac/bd} = aefcefa + befdefb + cefaefc + defbefd + faefce + fcefae \]
\[ x_{ad/bc} := u^2_{ad/bc} = aefdefa + befcefb + cefbefc + defaefd + faefde + fdefae \]

These 3 elements are pairwise linearly independent. Because suppose for example that \( x_{ab/cd} \) and \( x_{ac/bd} \) were linearly dependent. Then \( faefbe + fbefae \) and \( faefce + fcefae \) should be linearly dependent. By the nature of the relations and the fact that we are working in characteristic 2 there are only 3 possibilities:

- \( faefbe + fbefae = 0 \)
- \( faefce + fcefae = 0 \)
- \( faefbe + fbefae + faefce + fcefae = 0 \)

The first two options are obviously impossible and third option gives \( faefde + fdefae = 0 \), leading to a contradiction. Hence \( x_{ab/cd} \) and \( x_{ac/bd} \) must be linearly independent. On the other hand these elements satisfy the relation \( x_{ab/cd} + x_{ac/bd} + x_{ad/bc} = 0 \). This shows that there are 3 central elements satisfying one linear relation, hence \( \dim_k(Z_4(k,k^{2|4})) \geq 2 \).

References

[1] M.F. Atiyah and I.G. Macdonald. Introduction to commutative algebra. Addison-Wesley publishing company, 1969.
[2] H. Bass. Algebraic K-Theory. W.A.Benjamin,Inc., 1968.
[3] R. Bocklandt, T. Schedler, and M. Wemyss. Superpotentials and higher order derivations. Journal of Pure and Applied Algebra, 214(9):1501 – 1522, 2010.
[4] D. Eisenbud. Commutative Algebra: With a View Toward Algebraic Geometry. Graduate Texts in Mathematics. Springer, 1995.
[5] P. Etingof and C. Eu. Koszulity and the hilbert series of preprojective algebras. Math. Research Letters, 14(4):589–596, 2007.
[6] A. Grothendieck. revetement etals et groupe fondamental (SGA 1), volume 224 of Lecture Notes in Mathematics. Springer, 1971.
[7] A. Grothendieck and J. Dieudonné. éléments de géométrie algébrique III: étude cohomologique des faisceaux cohérents, premiere partie (EGA 3a), volume 11 of publications mathematique de l’IHÉS, IHES, 1971.
[8] R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics. Springer, 8 edition, 1997.
[9] T.Y. Lam. Exercises in modules and rings. Problem books in mathematics. Springer, 2007.
[10] B. Poonen. Isomorphism types of commutative algebras of finite rank over an algebraically closed field. In Computational arithmetic geometry, volume 463 of Contemp. Math., pages 111–120. Amer. Math. Soc., Providence, RI, 2008.
[11] I. Reiten and M. Van den Bergh. Two-dimensional tame and maximal orders of finite representation type. Memoirs of the American Mathematical Society, 80(408), 1989.
[12] J-P. Serre. Faisceaux algebriques coherents. Annals of Mathematics, 61(2):197–278, 1955.