Multi-Objective Robust Control for Vehicle Active Suspension Systems via Parameterized Controller

ZHONG CAO1, WENJING ZHAO2, XIAORONG HOU3, AND ZHAOHUI CHEN4

1School of Physics and Electronic Engineering, Guangzhou University, Guangzhou 510006, China
2Lab Center, Guangzhou University, Guangzhou 510006, China
3School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China
4School of Mathematics, Physics and Data Science, Chongqing University of Science and Technology, Chongqing 401331, China

Corresponding author: Zhaohui Chen (z.chen@cqust.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 61374001, in part by the National High Technology Research and Development Program of China under Grant 2015AA015408, in part by the Science and Technology Research Program of Chongqing Municipal Education Commission of China under Grant KJQN201901506, in part by the Chongqing Science and Technology Commission under Grant CSCT2019 jcyj-msxmX0171, and in part by the China Scholarship Council under Grant 201609945011.

ABSTRACT  A parameterized controller design approach is proposed to solve the problem of multi-objective control for vehicle active suspension systems by using symbolic computation. The considered model is a quarter-vehicle model of the active suspension system. The multi-objective robust control performances include the sprung mass acceleration, suspension deflection, and tire deflection. Based on dissipative Hamiltonian systems and Lyapunov function, a multi-objective $H_{\infty}$ controller design approach is developed, which can avoid solving Hamilton-Jacobi-Issacs equations. Then, an algorithm of solving semi-positive definite polynomial with tuning parameters is proposed by using symbolic computation. Furthermore, a method of parameter optimization is proposed. Simulations and comparison show that the control performance is significantly improved comparing with passive controlled systems and existing other control systems for active suspension systems.

INDEX TERMS  Multi-objective control, active suspension system, parameterized controller, symbolic computation, dissipative Hamiltonian systems.

I. INTRODUCTION

Vehicle suspension system plays a vital role in both the driving safety and comfort, where the ride comfort and the road-handling capacity are important in modern vehicles. In order to meet these vehicle performance requirements, three types of vehicle suspension systems, including passive [1], [2], semi-active [3], [4], and active suspensions [5], [6], are currently being studied and applied in practice in the past decades. For active suspension control design, more energy should be added and dissipated by actuators between the wheel-axle and sprung mass than passive and semi-active suspension control. However, the control performance in active suspension system is better than passive system and semi-active suspension system. The active suspension controller for vehicle system obtains more and more attention [7]–[10].

The aim of this paper is to improve the ride performance for the vehicle active suspension system. Generally, acceleration of sprung mass, suspension stroke, and tire deflection are the multi-objective performances, which should maintain an acceptable level as handling measures. There are various approaches can improve the performance of active suspension. The $H_{\infty}$ control of active suspension has obtained much attention in recent years. In particular, [11], [12] pointed out that $H_{\infty}$ control method for vehicle active suspension can be a particularly effective way to manage the tradeoff between conflicting performances. An approach of $H_{\infty}$ robust control and method of linear matrix inequality (LMI) optimization for non-stationary running condition and parameter uncertainty has been proposed in [13]. For the same problems, [13], [14] proposed an input delay method with sampling measurements. References [15], [16] studied the $H_{\infty}$ control for half-vehicle active suspensions with time delay. These controllers obtained in [13]–[16] were based on LMI. However, this inequality method has high complexity. A saturated
adaptive robust control strategy has been proposed in [17] for system uncertainties and actuator saturation. Based on accurate models, an adaptive sliding mode fault-tolerant controller was proposed for the active suspension systems in [18]. Reference [19] proposed a sliding-mode control method for vehicle active suspension system by using Lyapunov stability theory. However, the multi-objective control problem was not considered in [17]-[19]. Vehicle performance control is complex and multi-objective. An adaptive law for multi-objective suspension control based on the frequency domain was established from the road profile [20]. Reference [21] established suspension control based on the frequency domain was estimated in [22]. These approaches in [20]–[22] resolved multi-objective control problems for active suspension systems. However, they are the control strategy, which proposed only one fixed controller for active suspension systems. A parameterization design method for all stabilizing controllers was derived for a given plant [23]. However, too many constraints were required to obtain the controller [23].

For a system, designing a controller to satisfy some desired performance objectives is a critical thing. To this end, it is a sophisticated and efficient method to find a parameterized controller to solve multi-objective control problems. Therefore, the parameterized controller can provide a wonderful solution in control theory [24]–[27]. Youla firstly proposed the idea of parameterizing of stabilizing controllers in a linear feedback system in [24]. In the system, the state and the external disturbance are measurable, [25] proposed a family of nonlinear controllers via state-feedback. Yung extended the formulas of state-space and a family of $H_{\infty}$ controllers for the $n$-dimensional system in [26]. Reference [27] proposed a family of reliable controllers via solving the Hamilton-Jacobi-Issacs (HJI) inequality (or equations). The process of designing these controllers can be finished by solving a class of HJI inequalities (or equations), which is very difficult [24]–[27]. In order to obtain the parameterized controller, [28] studied the general Hamiltonian system and proposed a set of parameterized $H_{\infty}$ controllers, which avoided solving HJI inequalities (or equations) by applying good structure and clear physical expression of a general Hamiltonian system. However, so many hard constraints in controller design could not even be achieved. A family of robust simultaneous controllers with tuning parameters for a set of dissipative Hamiltonian systems has been proposed [29], [30].

In this paper, we propose a multi-objective $H_{\infty}$ parameterized controller design approach by Lyapunov function and symbolic computation for vehicle active suspension systems. The main features of the proposed parameterized controller are as follows:

1) The controller can satisfy the multi-objective control performance for active suspension systems.

2) The controller can optimize the robustness by adjusting parameters’ values through tuning parameters.

3) The controller design method can realize easily to avoid solving HJI inequalities (or equations).

The remainder of this paper is organized as follows. In section 2, the problem to be addressed is presented. Section 3 presents some preliminaries and results of controller parameterization, a controller with tuning parameters, an algorithm for solving parameters and a method for optimization parameters, respectively. Then, a family of parameterized controllers is proposed for a quarter-vehicle model with the active suspension system in Section 4. Some examples and simulations for illustrating the effectiveness and feasibility of the multi-objective robust controller are given in Section 5. Finally, Section 6 concludes this paper.

II. PROBLEM FORMULATION

The quarter-vehicle model used in this paper is as shown in Fig. 1. Sprung mass $m_s$ is the car body and may vary for loading different passengers and cargo; unsprung mass $m_u$ is the wheel assembly mass; $k_s$ and $c_s$ represent the coefficient of stiffness and damping, respectively; $k_t$ and $c_t$ stand for the tires’ elasticity coefficient and damping coefficient, respectively; $z_s$ and $\dot{z}_s$ represent the displacements of $m_s$ and $m_u$, respectively; $z_r$ denotes the road displacement of external input; $u(t)$ stands for the input control power of suspension system. Then, the dynamic equations for the model as follows:

$$
\begin{align*}
\dot{z}_s(t) &= -k_s [z_s(t) - z_a(t)] - c_s [\dot{z}_s(t) - \dot{z}_d(t)] + u(t) \\
\dot{z}_d(t) &= k_s [z_s(t) - z_a(t)] c_s [\dot{z}_s(t) - \dot{z}_d(t)] \\
&\quad - k_t [z_u(t) - z_r(t)] - c_t [\dot{z}_u(t) - \dot{z}_r(t)] - u(t)
\end{align*}
$$

We define the following state variable for establishing the state-space form:

- $x_1(t) = z_s(t) - z_a(t)$, suspension deflection,
- $x_2(t) = z_u(t) - z_r(t)$, tire deflection,
\[ x_1 (t) = \dot{z}_1 (t), \text{ sprung mass velocity}, \]
\[ x_4 (t) = \dot{z}_2 (t), \text{ unsprung mass velocity.} \]

After choosing the disturbance input \( \omega (t) = \dot{z}_r (t) \) and the variables as:

\[ x (t) = [x_1 \ x_2 \ x_3 \ x_4]^T \]

The dynamic equations (1) can be expressed as the following state-space form:

\[ \dot{x} (t) = Ax (t) + B_1 u (t) + B_2 \omega (t) \]

where 
\[
A = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
-\frac{1}{m_s} & 0 & -\frac{1}{m_s} & \frac{\xi_s}{m_s} \\
-\frac{1}{m_u} & -\frac{1}{m_u} & 0 & -\frac{\xi_u}{m_u}
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{1}{m_s}
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{1}{m_u}
\end{bmatrix}.
\]

The performances of ride comfort, road holding ability, and suspension deflection are three main performance criteria in any vehicle suspension design, which are exactly the multi-objective of robust control in this paper.

1) As known, the ride comfort and the vehicle body’s vertical acceleration have a direct relationship. Consequently, the sprung mass acceleration (SMA), \( \text{SMA} = |\dot{x}_1 (t)| \), should be as small as possible.

2) The structural properties of the vehicle active suspension system impose restrictions on the suspension deflection within the rattle space as the limit \( x_{\text{rmax}}, x_1 (t) < x_{\text{rmax}} \). The definition of relative suspension deflection \( \text{RSD} = \frac{|x_1|}{x_{\text{rmax}}} \). Therefore, the performance requirement is \( \text{RSD} < 1 \).

3) External disturbances caused by road bumps are harmful to safe driving. How to guarantee the uninterrupted contact between wheels and road for vehicle handling is an extremely important issue. The relative tire force (RTF) is \( \text{RTF} = \frac{\dot{z}_1 + \dot{z}_2 + \dot{z}_3}{m_s + m_u} \). Therefore, another performance requirement is that \( \text{RTF} < 1 \).

III. PARAMETERIZED CONTROLLER DESIGN

A. PRELIMINARIES

Consider the dissipative Hamiltonian system [31]

\[
\begin{align*}
\dot{x} &= [J (x) - R (x)] \nabla H (x) + g_1 (x) u (t) + g_2 (x) \omega (t) \\
y &= g_2^T (x) \nabla H (x) \\
z &= h (x) g_1^T (x) \nabla H (x)
\end{align*}
\]

(4)

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \) and \( \omega \in \mathbb{R}^q \) are the state vector, the controller with parameters and the disturbances, respectively. \( J (x) \) and \( R (x) \) are the skew-symmetric matrix and the positive semi-definite matrix. \( g_1 (x) \) and \( g_2 (x) \) are sufficiently smooth functions with proper order. \( y \in \mathbb{R}^p \) and \( z \in \mathbb{R}^q \) are the output and the penalty, respectively. \( h (x) \) is the weight matrix for a specific penalty. The Hamiltonian function \( H (x) \) must be a positive semi-definite polynomial and has a local minimum at the equilibrium \( x_0 \) of system, and \( \nabla H (x) = (\partial H / \partial x) (x) \).

Assumption 1: The Hessian matrix \( \text{Hess} (H (x_0)) > 0 \) and \( H (x) \in C^2 \).

Remark 1: From Assumption 1 for the system (4), we can find \( \text{Hess} (H (x_0)) > 0 \) guarantees that \( H (x) \) is strict convex on some neighborhood of the equilibrium \( x_0 \) and \( H (x) \in C^2 \) guarantees the existence of \( \text{Hess} (H (x)) \).

In this paper, the aim of proposing a parameterized controller approach for systems (4) can be described as: for a given \( H_\infty \) disturbance attenuation level \( \gamma > 0 \), we can obtain a parameterized controller of the form \( u = \alpha (x) (\alpha (x_0) = 0) \) such that the \( L_2 \) gain is not more than \( \gamma \) for the closed-loop system (from \( \omega \) to \( z \)), and the closed-loop system is locally asymptotically stable when disturbance \( \omega = 0 \).

Definition 1: [32] If \( y (t) = 0 \) and \( \omega (t) = 0, \forall t \geq 0, \) then \( \nabla H (x (t)) = 0, \forall t \geq 0, \) system (4) is called zero-energy-gradient (ZEG) observable to \( \gamma \); If \( y (t) = 0 \) and \( \omega (t) = 0, \forall t \geq 0, \) then \( \lim_{t \to \infty} \nabla H (x (t)) = 0, \) system (4) is called ZEG detectable with respect to \( \gamma \).

Lemma 1: [31] Consider a nonlinear system

\[
\begin{align*}
\dot{x} &= f (x) + g (x) \omega, f (x_0) = 0 \\
z &= h (x)
\end{align*}
\]

where \( x \in \mathbb{R}^n, \omega \in \mathbb{R}^q \) and \( z \in \mathbb{R}^q \) are the state vector, the disturbances, and the penalty, respectively. If there exists a function \( V (x) \geq 0 (V (x_0) = 0) \) such that HJI inequality

\[
\left( \frac{\partial V}{\partial x} \right)^T f (x) + \frac{1}{2 \gamma^2} \left( \frac{\partial V}{\partial x} \right)^T g (x) g^T (x) \frac{\partial V}{\partial x} + \frac{1}{2} h^T (x) h (x) \leq 0
\]

holds, it’s indicated that the \( L_2 \) gain is not more than \( \gamma (\gamma > 0) \) for the closed-loop system (5) (from \( \omega \) to \( z \)), i.e.,

\[
\int_0^\infty \| z \|^2 dt \leq \int_0^\infty \gamma^2 \| \omega \|^2 dt
\]

(7)

B. \( H_\infty \) CONTROLLER DESIGN

An \( H_\infty \) parameterized controller for the system (4) will be designed in this section. For simplification, we denote \( J = J (x), R = R (x), \nabla H = \nabla H (x), u = u (t), g_1 = g_1 (x), g_2 = g_2 (x), h = h (x) \) in this section.

Theorem 1: Assumption 1 holds and system (4) is generalized ZEG detectable (when \( \omega = 0 \)) and

\[
R + \frac{1}{2 \gamma^2} (g_1 g_1^T - g_2 g_2^T) \geq 0
\]

(8)

\[
\nabla H^T g_1 \alpha (x, a) \leq 0
\]

(9)

hold simultaneously (the matrix \( \nabla H^T g_1 \alpha (x, a) \) is negative semi-definite and \( R + \frac{1}{2 \gamma^2} (g_1 g_1^T - g_2 g_2^T) \) is positive semi-definite). \( \alpha (x, a) \) is the item with tuning parameters and \( a \) are the tuning parameters. Then, \( H_\infty \) control of the system (4) can be realized by the following controller

\[
u = \alpha (x, a) + \beta (x)
\]

(10)
where $\beta(x) = -\frac{1}{2} \left[ h^T h + \frac{1}{\gamma^2} I_m \right] g_1^T \nabla H$, $I_m$ is an $m \times m$ identity matrix.

**Proof:** Consider the candidate Lyapunov function $V(x) = H(x) - c \geq 0$ ($c = H(x_0)$). From Lemma 1 and controller (10), we have

\[
\left( \frac{\partial V}{\partial x} \right)^T f(x) + \frac{1}{2\gamma^2} \left( \frac{\partial V}{\partial x} \right)^T g_1 g_1^T \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} x^T x \\
= -\nabla H^T R \nabla H + \nabla g_1^T g_1 u + \frac{1}{2\gamma^2} \nabla H^T g_2 g_2^T \nabla H + \frac{1}{2} x^T x
\]

\[
= -\nabla H^T \left( R + \frac{1}{2\gamma^2} g_1 h_1^T h_1 g_1^T + \frac{1}{2\gamma^2} g_1 g_1^T H g_1 + \frac{1}{2\gamma^2} g_2 g_2^T \nabla H \\
+ \frac{1}{2\gamma^2} \nabla H^T g_2 g_2^T \nabla H \right) \nabla H \\
+ \nabla H^T g_1 a(x, a)
\]

\[
\leq 0 \tag{11}
\]

which implies the $L_2$ gain of the closed-loop system (4) controlled by the controller (10) (from $\omega$ to $z$) is bounded by given prescribed disturbance attenuation value $\gamma$. Next, we prove that the closed-loop system is asymptotically stable at $x_0$, when $\omega = 0$. From system (4), controller (10) and $\omega = 0$, it follows that

\[
\dot{V}(x) = \left( \frac{\partial V}{\partial x} \right)^T \left[ J - R \right] \left( \frac{\partial V}{\partial x} \right) + \left( \frac{\partial V}{\partial x} \right)^T g_1 u \\
= -\nabla H^T R \nabla H + \nabla H^T g_1 \\
\times \left[ -\frac{1}{2} \left( h^T h + \frac{1}{\gamma^2} I_m \right) g_1^T \nabla H + \alpha(x, a) \right] \\
= -\nabla H^T \left( R + \frac{1}{2\gamma^2} \left( g_1 g_1^T T - g_2 g_2^T \right) \right) \nabla H \\
- \frac{1}{2} \nabla H^T g_1 h_1^T h_1 g_1 \nabla H \\
- \frac{1}{2\gamma^2} \nabla H^T g_2 g_2^T \nabla H + \nabla H^T g_1 a(x, a) \\
= -\nabla H^T \left( R + \frac{1}{2\gamma^2} \left( g_1 g_1^T T - g_2 g_2^T \right) \right) \nabla H \\
- \frac{1}{2} \left\| h_1^T \nabla H \right\|^2 \\
- \frac{1}{2\gamma^2} \left\| g_2^T \nabla H \right\|^2 + \nabla H^T g_1 a(x, a) \leq 0 \tag{12}
\]

Hence, the closed-loop system converges to the largest invariant set, which is contained in

\[
S := \{x: \dot{V}(x) = 0\} \subset \{x: y = g_2^T \nabla H = 0, \frac{z}{h_1^T \nabla H} = 0, \forall t \geq 0\} \tag{13}
\]

From Assumption 1 and the generalized ZEG detectable system (4), we obtain

\[
\lim_{t \to \infty} \nabla H(x(t)) = \nabla H \left( \lim_{t \to \infty} x(t) \right) = 0 \tag{14}
\]

On the other hand, $\nabla H(x_0) = 0$ holds because of the equilibrium $x_0$ is a local minimum of $H(x)$. So, there exists a neighborhood of $x_0$, in which $\nabla H \left( \lim_{t \to \infty} x(t) \right) = 0$ implies $\lim_{t \to \infty} x(t) = x_0$. Therefore, it holds that the trajectory of $\dot{V}(x) = 0$ is strict convex at the equilibrium point $x_0$. The closed-loop system (4) is locally asymptotically stable at $x_0$ under controller (10) from the LaSalle invariant principle. This completes the proof.

**Remark 2:**
1) Compared with the controller proposed in [28], the controller (10) has a much simpler form and is easier to realize.
2) For the proposed controller, the parameters’ range of polynomial vector $a(x, a)$ can be obtained by solving the condition (9).
3) The proposed controller can be applied to general nonlinear systems with the dissipative Hamiltonian realization method [33], [34].

**C. SOLVING TUNING PARAMETERS**

From condition (8), we can obtain a special given disturbance attenuation value $\gamma^*$. Let $\gamma \geq \gamma^*$ such that condition (8) holds. Then we propose an algorithm to find the ranges of parameters in the controller (10) via solving the parameters of $a(x, a)$ in condition (9) by using cylindrical algebraic decompositions (CAD), which is one method of symbolic computation [35]. The solving turning parameters (STP) algorithm is proceeded as follows [29].

**STP Algorithm**

**INPUT:** System’s parameters $S_{ys} = \{x_i, n, r, \nabla H, a(x, a)\}$. $/x_i$ is the state variable, $n$ is the number of the state variable, $a(x, a)$ is the constructed polynomial with parameters and $r$ is the such polynomial’s highest order, $a$ is the tuning parameters.

**OUTPUT:** Tuning parameters’ range $U$. $/U$ is an empty set, initially.

The flow diagram of the proposed STP algorithm for condition (9) is outlined in Fig. 2. The obtained solution set $U$ is the partial solution, not a complete solution by using CAD.

**D. PARAMETER OPTIMIZATION**

The parameters optimization will be discussed in the tuning parameters ranges as follows. For system (5), add the controller and rewrite it,

\[
\dot{X} = F(X, \mu) + a(X, \mu) \tag{15}
\]

According to Lyapunov Theorem, there must have a positive definite symmetric matrix $P = P^T$ in a stabilized system by using the controller (10). The fastest stabilization parameter search problem of the controlled system (15) is
Problem. The design steps for the controller parameter optimization method are given below.

Step1. According to the Jacobi matrix at the equilibrium point of the system, solve the positive definite symmetric matrix P. Suppose \( J_E \) is the equilibrium of the system, which can be written at the equilibrium as follows:

\[
\dot{X} = J_E X
\]  
(16)

Then it is certainly that existing a positive definite symmetric matrix \( P \) and a unit matrix \( Q \), which makes the follows holds,

\[
V(X) = X^T PX, \quad \dot{V}(X) = -X^T QX
\]  
(17)

where \( V(X) \) is the Lyapunov function of the controlled system, \( \dot{V}(X) = \frac{dV(X)}{dt} \). Then according to the linearization of the system, obtain the following equation,

\[
\dot{V}(X) = \frac{dV(X)}{dt} = \frac{d(X^T PX)}{dt} = \dot{X}^T PX + X^T P \dot{X} = X^T (J_E^T P + PJ_E) \dot{X}
\]  
(18)

That is

\[
J_E^T P + PJ_E = -Q
\]  
(19)

where \( J_E \) and \( Q \) can be obtained from the system (15). The positive definite symmetric matrices \( P \) can be obtained from formula (19).

Step2. Solve the characteristic equation \( H(\lambda) = \det (\lambda I - P) \). If the time of adjustment for system states is minimized, it is equivalent to the fastest decay rate of the system. According to Lyapunov theorem, the decay rate of the system can be expressed as follows,

\[
\eta = \lambda \left( QP^{-1} \right)
\]  
(20)

Because \( Q \) is a unit matrix, considering \( \lambda \left( P^{-1} \right) = \frac{1}{\lambda(P)} \), the decay rate of the controlled system to equilibrium point depends on the characteristic root of \( P^{-1} \). So, the problem can be solved by solving the characteristic root extremum to matrix \( P \).

Step3. Solve the extremum problem of \( H(\lambda) = 0 \). Because \( J_E \) and \( P \) include the controller parameters \( k \), let \( H(\lambda) = 0 \) be the implicit function of \( \lambda = \bar{H}(k) \). Solve the following equations for obtaining the fastest convergent control parameters,

\[
\begin{align*}
\frac{dH(\lambda)}{dk} &= 0 \\
\frac{\partial H(\lambda)}{\partial k} &= 0
\end{align*}
\]  
(21)

So, we can obtain the optimized tuning parameters for the proposed controller by solving equations (21).

IV. DESIGN EXAMPLE

In this section, the above formulated problem will be solved by the Hamiltonian function method. We rewrite the system (3) as a dissipative Hamiltonian system, firstly.

\[
\begin{align*}
\dot{x} &= (J(x) - R(x)) \nabla H(x) + g_1(x) u(t) + g_2(x) \omega(t) \\
\dot{z} &= h(x) g_2^T(x) \nabla H(x)
\end{align*}
\]  
(22)

where

\[
J(x) = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & \frac{k_t}{m_u} & \frac{c_s}{m_s}
\end{bmatrix}
\]  

\[
R(x) = \begin{bmatrix}
0 & 0 & 0 & \frac{k_s}{m_u} & 0 \\
0 & 0 & 0 & \frac{c_s}{m_s} & \frac{c_s + c_t}{m_u}
\end{bmatrix}
\]  

\[
H(x) = \frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right),
\]

\[
g_1(x) = \begin{bmatrix} 0 & 0 & \frac{1}{m_s} \frac{1}{m_u} \end{bmatrix}^T, g_2(x) = \begin{bmatrix} 0 & -1 \frac{c_t}{m_u} \end{bmatrix}^T, h(x)
\]  

is the weight matrix for a specific penalty.
From system (22), it is easy to get \( \nabla H(x)|_{x=0} = 0 \), \( \text{Hess}(H(x))|_{x=0} = \text{Diag}\{1, 1, 1, 1\} > 0 \). So, assumption 1 holds.

Then, we check that condition (8) holds for all \( x \) and given \( \gamma \). From system (22), we have

\[
R(x) + \frac{1}{2\gamma^2} \left( g_1(x)g_1^T(x) - g_2(x)g_2^T(x) \right) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 - \frac{1}{2\gamma^2} & 0 & -\frac{k_i}{m_u} & 1 \\
k_s & m_s - 1 & 0 & -\frac{c_s}{m_s} - \frac{1}{2\gamma^2 m_s m_a} \\
1 - k_s & 0 & \frac{c_s}{m_s} & -\frac{1}{2\gamma^2 m_s m_a} \\
0 & 0 & -\frac{c_s}{m_s} & -\frac{1}{2\gamma^2 m_s m_a} \\
-\frac{c_s}{m_s} & -\frac{1}{2\gamma^2 m_s m_a} & 0 & \frac{1}{m_u} \\
\end{bmatrix}
\]

We assume the specific given disturbance attenuation value \( \gamma^* = 0.01 \).

\[
\gamma \geq 0.01 \quad (23)
\]

ensure condition (8) holds.

Then, if system (22) satisfies robustness performance in \( H_\infty \) control, condition (9) must hold. The Hamiltonian function \( H(x) \) can be used as candidate Lyapunov function \( V(x) \), i.e., \( V(x) = H(x) \). It follows from controller (10) that

\[
u(x) = a(x, a) + \beta(x)
= a(x, a) - \frac{1}{2} \left( h(x)^T h(x) + \frac{1}{2} I_m \right) g_1^T(x) \nabla H(x)
\quad (24)
\]

From system (22), \( n = 4 \). Let \( r = 1 \), we have

\[a(x, a) = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4\]

where \( a_i \) \((i = 1, 2, 3, 4)\) are the tuning parameters. From system (22), we obtain that \( \nabla H(x) = [x_1 \ x_2 \ x_3 \ x_4]^T \).

Let \( \frac{1}{m_1} = b_1, \frac{1}{m_2} = b_2 \) and \( S = -\nabla H(x)^T g_1 a(x, a) \), we have

\[
S = -a_1 b_1 x_1^2 + a_2 b_1 x_2 x_4 - a_3 b_1 x_2^2 - a_4 b_1 x_3 x_4 + a_1 b_2 x_1 x_4 - a_2 b_2 x_2 x_3 + a_3 b_2 x_3^2 - a_4 b_2 x_4^2
\quad (26)
\]

Rewrite coefficient of \( S \) as a quadratic form \( M \):

\[
M = \begin{bmatrix}
0 & 0 & -a_1 b_1 & -a_1 b_2 \\
0 & 0 & -a_2 b_1 & -a_2 b_2 \\
-a_1 b_1 & -a_2 b_1 & -2a_3 b_1 & -a_4 b_1 + a_3 b_2 \\
-a_1 b_2 & -a_2 b_2 & -a_4 b_2 + a_3 b_2 & 2a_4 b_2
\end{bmatrix}
\quad (27)
\]

As we all know, all principal diagonals of \( M \) must be positive semi-definite. So, we can obtain inequalities \( B \) from \( M \).

From \( B \), we can easily to obtain that \( a_1 = 0, a_2 = 0, a_3 \leq 0 \) and \( a_4 \geq 0 \). Substitute \( a_1 \) and \( a_2 \) into inequalities \( B \) for simplifying computation, we obtain simplified inequality \( -(2a_3 a_4 b_1 b_2 + a_3^2 b_1^2 + a_4^2 b_2^2) \geq 0 \). Solving the inequality by using CAD, we obtain \( a_3 = -\frac{b_2}{b_1} a_4 \). Let \( a_4 = a_4 \). So, we have

\[
U = \left\{ a_1 = 0, a_2 = 0, a_3 = -\frac{b_2}{b_1} k, a_4 = k, k \leq 0 \right\}
\quad (28)
\]

| Parameter | Value | Units |
|-----------|-------|-------|
| \( m_r \) | 320   | kg    |
| \( m_u \) | 40    | kg    |
| \( k_s \) | 18000 | N/m   |
| \( k_e \) | 20000 | N/m   |
| \( c_s \) | 1000  | N/m   |
| \( c_t \) | 0     | N/m   |
| \( z_{\text{max}} \) | 0.08  | m     |

Substitute \( U \) into the controller (10), we obtain a family of controllers with parameter

\[
u = -\frac{1}{2} \left( h(x)^T h(x) + \frac{1}{\gamma^2} I_m \right) g_1^T(x) \nabla H(x) + \frac{1}{\gamma^2} \left( x_3 - \frac{a_3}{m_s} \right)^T \left( \frac{a_3}{m_s} \right) - \frac{b_2}{b_1} k x_3 + k x_4
\quad (29)
\]

According to the parameter optimization steps above, we can get the tuning parameter optimization \( k = 1525 \) for the system (22).

V. SIMULATIONS TO VEHICLE ACTIVE SUSPENSION CONTROL

We will verify our method by some examples and simulations. First, the peak value of the state is the direct reflection of the state change of the system under external disturbance and action of the controller. That is the direct embodiment of the controller’s effect. Then, the root-mean-square (RMS) values are used to describe the vibrations of the vehicle body in the vehicle suspension system, especially as the ride comfort of passengers. Hence, we take the peak value and RMS to describe the quantitative indexes.

The quarter-vehicle model parameters have the following values [36] and list in Table 1.

The quarter-vehicle model parameters have the following values [36] and list in Table 1.

\[z_r(t) = \begin{cases}
H \left(1 - \cos \left(\frac{2\pi V_0 t}{L}ight)\right), & \text{if } 0 \leq t \leq \frac{L}{V_0} \\
0, & \text{if } t > \frac{L}{V_0}
\end{cases}
\quad (30)
\]

where \( H \) and \( L \) are the height and the length of the bump, respectively. Assume \( H = 0.1m, L = 10m \), and the vehicle forward velocity is \( V_0 = 45km/h \).

A. COMPARISON WITH DIFFERENT PARAMETERS OF CONTROLLERS

The initial parameters of the system (22) are \( \gamma = 1, h(x) = 0.5 \) and the parameters of the controller are \( k = 100, k = 1525 \) (optimization) and \( k = 3000 \), respectively. We obtain the corresponding controllers (31)-(33),

\[
u = -800.00195 x_3 + 100.0156 x_4
\quad (31)
\]

\[
u = -12200.00195 x_3 + 1525.0156 x_4
\quad (32)
\]

\[
u = -24000.00195 x_3 + 3000.0156 x_4
\quad (33)
\]
In this case, the sprung mass is assumed to be constant with $m_s = 320\text{kg}$ and the vehicle speed is $V_0 = 45\text{ km/h}$. Fig. 3 shows the road disturbance velocity. Fig. 4 and Fig. 5 show the bump response of the SMA and sprung mass deflection (SMD) for passive control (dot line blue) and the controller of parameter values $k = 100$ (dot-dash line purple), $k = 1525$ (solid line red) and $k = 3000$ (dash line green) in the active suspension system. The bump response of the RSD and RTF are shown in Fig. 6 and Fig. 7. Then, Fig. 8 shows the active control force for the controller of different controller parameter values, respectively.

From Fig. 4-Fig. 8, we can clearly see that good bump response quantities for SMA, SMD, RSD, and RTF can be guaranteed by using $H_\infty$ parameterized control. The time back to the equilibrium position of the vehicle system with active control is less than passive control and the robustness performance of the vehicle system can be further optimized through adjusting the values of controller parameter from $k = 100$ to $k = 3000$, where $k = 1525$ is the better parameter.

B. COMPARISON ANALYSIS WITH DIFFERENT PARAMETERS OF VEHICLE

The sprung mass may vary due to changes in passenger or cargo load and the vehicle speed is the most variable factor. The different values of the sprung mass and vehicle speed
TABLE 2. Comparison of peak and rms value for bump response.

| Measure | Peak | RMS | Peak | RMS |
|---------|------|-----|------|-----|
|     | SMA | SMA | RSD | RTF |
| m₀ = 320 kg | Passive | 2.2864 | 1.0082 | 0.2977 | 0.0955 |
| V₀ = 30 km/h | Proposed | 0.4341 | 0.1700 | 0.4095 | 0.0208 |
| m₀ = 480 kg | Passive | 2.2010 | 1.0003 | 0.4354 | 0.0960 |
| V₀ = 30 km/h | Proposed | 0.3009 | 0.1199 | 0.4776 | 0.0149 |
| m₀ = 320 kg | Passive | 6.1758 | 1.7593 | 0.5627 | 0.1588 |
| V₀ = 90 km/h | Proposed | 3.1882 | 0.8499 | 0.5312 | 0.1199 |
| m₀ = 480 kg | Passive | 3.8721 | 1.2167 | 0.5184 | 0.1109 |
| V₀ = 90 km/h | Proposed | 2.1824 | 0.5816 | 0.5850 | 0.0844 |

FIGURE 9. Sprung mass acceleration.

are considered as m₀ = 320kg, m₀ = 480kg and V₀ = 30 km/h, V₀ = 90 km/h. The controller (32) is considered in simulations.

To show clearly the comparison results between the different sprung masses and different speeds of the vehicle, the maximum peak values of the bump responses for SMA, RSD, and RTF are listed in Table 2.

From Table 2, it can be seen clearly that the effect of the controller is related to vehicle speed and mass. In the same circumstances, the proposed control method has better suspension’s adaptability to the road than the passive suspension control method. With the increase of weight and constant low speed, the passive control has little change, while the proposed controller obtains better control effect. The suspension’s adaptability to road will be reduced with the increase of speed. However, the proposed controller can still effectively control the vehicle at high speed V₀ = 90 km/h to meet the three control performances.

C. COMPARISON ANALYSIS WITH EXISTING METHODS

In order to further verify the performance of the proposed control method, two existing results in [37] and [38] are presented for the objective of the comparison. The controller (32) is considered in simulations.

In this case, the sprung mass is assumed to be constant with m₀ = 400 kg and the vehicle speed is V₀ = 45 km/h. Simulation results are shown in Fig. 9- Fig. 13. For different control strategies, the SMA, SMD, RSD and RTF and active control force are shown in Fig. 9- Fig. 13, respectively.

As shown in Fig. 9- Fig. 13, the proposed parameterized control can obtain better regulation performance of body acceleration, and simultaneously satisfy the constraint requirements than FHC in [37] and LFK in [38].

Table 3 shows the performance results for the proposed method and existing methods.

FIGURE 10. Suspension deflection.

FIGURE 11. Relative suspension deflection.

FIGURE 12. Relative tire force.
From Table 3, it can be observed that the parameterized controller is better than the passive method and the other two existing methods in multi-objective robust control.

**FIGURE 13.** Active control force.

**TABLE 3.** Performance of peak and rms values.

|            | Passive | FHC in [37] | LKF in [38] | Proposed |
|------------|---------|-------------|-------------|----------|
| Peak SMA   | 5.0928  | 2.7814      | 2.2114      | 1.2191   |
| RMS SMA    | 1.8224  | 0.8450      | 0.6361      | 0.3606   |
| Peak RSD   | 0.6760  | 0.3667      | 0.4412      | 0.5133   |
| RMS RTF    | 0.1676  | 0.0944      | 0.0625      | 0.0498   |

**FIGURE 14.** Road disturbance velocity.

D. DIFFERENT FLUCTUATION OF THE ROAD PROFILE

In order to further verify the robustness of the proposed controller for different fluctuations of the road profile. We take the road profile from [39], which is described by the following equation,

\[ z_r(t) = \begin{cases} 
-0.0592t^3 + 0.1332t^2 + b(t), & 3.5 \leq t < 5 \\
0.0592t^3 + 0.1332t^2 + b(t), & 5 \leq t < 6.5 \\
0.0592t^3 - 0.1332t^2 + b(t), & 8.5 \leq t < 10 \\
-0.0592t^3 - 0.1332t^2 + b(t), & 10 \leq t < 11.5 \\
b(t), & \text{else}
\end{cases} \]

where \( b(t) = 0.002\sin(2\pi t) + 0.002\sin(7.5\pi t) \) as the disturbance. Assume the vehicle forward velocity is \( V_0 = 45\text{km/h} \) and the quarter-vehicle model parameters are listed in Table 1. The controller (32) is considered in simulations for system (22).

Fig. 14 shows the road disturbance velocity. Fig. 15 and Fig. 16 show the bump response of the SMA and sprung mass deflection (SMD) for passive control (dot line blue) and the controller of parameter values \( k = 1525 \) (solid line in red) in the active suspension system. The bump response of the RSD and RTF are shown in Fig. 17 and Fig. 18.
From Fig. 15–Fig. 18, we can clearly see that good bump response quantities for SMA, SMD, RSD, and RTF can be guaranteed by using \( H_\infty \) parameterized control under road disturbance in Fig. 14.

VI. CONCLUSION

The sprung mass may vary due to changes of passenger load, cargo, and vehicle speed. In which, the vehicle speed is the most variable factor. A novel controller with tuning parameters is proposed to achieve the multi-objective control performances of providing the ride safe and ride comfort. The proposed parameterization method uses the Hamiltonian function method and avoids solving HJI inequalities, thus the obtained controllers with parameters are easier as compared to some existing ones. The parameters’ range can be obtained by the STP Algorithm and optimal parameters for the controller can be obtained by the parameter optimization method, respectively. Simulations results with two different road profiles validate the multi-objective robust control performances for active suspension vehicles with different sprung masses and different speeds of the vehicle. A comparison with passive system and existing methods shows that the proposed parameterized controller scheme performs better than them in terms of peak and RMS values.

ACKNOWLEDGMENT

The reviewers’ instructive and valuable suggestions for improving the quality of the paper are extremely appreciated.

REFERENCES

[1] R. S. Sharp and S. A. Hassan, “An evaluation of passive automotive suspension systems with variable stiffness and damping parameters,” Vehicle Syst. Dyn., vol. 15, no. 6, pp. 335–350, Jan. 1986.

[2] J. Tamboli and S. Joshi, “Optimum design of a passive suspension system of a vehicle subjected to actual random road excitations,” J. Sound Vib., vol. 219, no. 2, pp. 193–205, Jan. 1999.

[3] D. Hrovat, D. L. Margolis, and M. Hubbard, “An approach toward the optimal semi–active suspension,” J. Dyn. Syst., Meas., Control, vol. 110, no. 3, pp. 288–296, Sep. 1988.

[4] B. Xu, C. Xiang, Y. Qin, P. Ding, and M. Dong, “Semi–active vibration control for in-wheel switched reluctance motor driven electric vehicle with dynamic vibration absorbing structures: Concept and validation,” IEEE Access, vol. 6, pp. 60274–60285, 2018.

[5] X. Su, “Master–slave control for active suspension systems with hydraulic actuator dynamics,” IEEE Access, vol. 5, pp. 3612–3621, 2017.

[6] O. Tutsoy, D. E. Barkana, and H. Tugal, “Design of a completely model free adaptive control in the presence of parametric, non-parametric uncertainties and random control signal delay,” ISA Trans., vol. 76, pp. 67–77, May 2018.

[7] P. Brezas and M. C. Smith, “Linear quadratic optimal and risk–sensitive control for vehicle active suspensions,” IEEE Trans. Control Syst. Technol., vol. 22, no. 2, pp. 543–556, Mar. 2014.

[8] H. Pan, W. Sun, H. Gao, and J. Yu, “Finite–time stabilization for vehicle active suspension systems with hard constraints,” IEEE Trans. Intell. Transp. Syst., vol. 16, no. 5, pp. 2663–2672, Oct. 2015.

[9] W. Sun, H. Pan, and H. Gao, “Filter–based adaptive vibration control for active vehicle suspensions with electrohydraulic actuators,” IEEE Trans. Veh. Technol., vol. 65, no. 6, pp. 4619–4626, Jun. 2016.

[10] H. Pan, X. Jing, W. Sun, and H. Gao, “A bioinspired dynamics–based adaptive tracking control for nonlinear suspension systems,” IEEE Trans. Control Syst. Technol., vol. 26, no. 3, pp. 903–914, May 2018.

[11] H. Chen and K.-H. Guo, “Constrained \( H_\infty \) control of active suspensions: An LMI approach,” IEEE Trans. Control Syst. Technol., vol. 13, no. 3, pp. 412–421, May 2005.

[12] H. Du and N. Zhang, “\( H_\infty \) control of active vehicle suspensions with actuator time delay,” J. Sound Vib., vol. 301, nos. 1–2, pp. 236–252, 2007.

[13] L.-X. Guo and L.-P. Zhang, “Robust \( H_\infty \) control of active vehicle suspension under non-stationary running,” J. Sound Vib., vol. 331, no. 26, pp. 5824–5837, Dec. 2012.

[14] H. Gao, W. Sun, and P. Shi, “Robust sampled–data \( H_\infty \) control for vehicle active suspension systems,” IEEE Trans. Control Syst. Technol., vol. 18, no. 1, pp. 238–245, Jan. 2010.

[15] H. Li, H. Liu, S. Hand, and C. Hilton, “Design of robust \( H_\infty \) controller for a half-vehicle active suspension system with input delay,” Int. J. Syst. Sci., vol. 44, no. 4, pp. 625–640, 2013.

[16] H. D. Choi, C. K. Ahn, M. T. Lim, and M. K. Song, “Dynamic output-feedback \( H_\infty \) control for active half-vehicle suspension systems with time-varying input delay,” Int. J. Control Autom. Syst., vol. 14, no. 1, pp. 59–68, Feb. 2016.

[17] W. Sun, Z. Zhao, and H. Gao, “Saturated adaptive robust control for active suspension systems,” IEEE Trans. Ind. Electron., vol. 60, no. 9, pp. 3889–3896, Sep. 2013.

[18] S. Liu, H. Zhou, X. Luo, and J. Xiao, “Adaptive sliding fault tolerant control for nonlinear uncertain active suspension systems,” J. Franklin Inst., vol. 353, no. 1, pp. 180–199, Jan. 2016.

[19] R. Bai and D. Guo, “Sliding–mode control of the active suspension system with the dynamics of a hydraulic actuator,” Complexity, vol. 2018, pp. 1–6, Aug. 2018.

[20] J. Lu, “A frequency–adaptive multi–objective suspension control strategy,” J. Dyn. Syst., Meas., Control, vol. 126, no. 3, pp. 700–707, Sep. 2004.

[21] H. Li, H. Liu, S. Hand, and C. Hilton, “Multi–objective \( H_\infty \) control for vehicle active suspension systems with random actuator delay,” Int. J. Syst. Sci., vol. 43, no. 12, pp. 2214–2227, Dec. 2012.

[22] V. S. Deshpande, P. D. Shendge, and S. B. Phadke, “Nonlinear control for dual objective active suspension systems,” IEEE Trans. Intell. Transp. Syst., vol. 18, no. 3, pp. 656–665, Mar. 2017.

[23] M. Smith and F.-C. Wang, “Controller parameterization for disturbance response decoupling: Application to vehicle active suspension control,” IEEE Trans. Control Syst. Technol., vol. 10, no. 3, pp. 393–407, May 2002.

[24] D. Youla, H. Jabr, and J. Bongiorno, “Modern Wiener–Hopf design of optimal controllers—Part II: The multivariable case,” IEEE Trans. Autom. Control, vol. AC-21, no. 3, pp. 319–338, Jun. 1976.

[25] W.-M. Lu and J. Doyle, “\( H_\infty \) control of nonlinear systems via output feedback: Controller parameterization,” IEEE Trans. Autom. Control, vol. 39, no. 12, pp. 2517–2521, Dec. 1994.

[26] C.-F. Yung, J.-L. Wu, and T.-T. Lee, “Parameterization of nonlinear \( H_\infty \) state-feedback controllers,” Automatica, vol. 33, no. 8, pp. 1587–1590, Aug. 1997.

[27] Y. Fu, Z. Tian, and S. Shi, “A family of reliable nonlinear \( H_\infty \) state-feedback controllers,” IET Control Theory Appl., vol. 18, no. 3, pp. 447–452, 2001.

[28] S. Xu and X. Hou, “A family of \( H_\infty \) controllers for dissipative Hamiltonian systems,” Int. J. Robust Nonlinear Control, vol. 22, no. 11, pp. 1258–1269, 2012.

[29] Z. Cao, X. Hou, and W. Zhao, “A family of robust simultaneous controllers with tuning parameters design for a set of port–controlled Hamiltonian systems,” Asian J. Control, vol. 19, no. 1, pp. 151–163, Jan. 2017.
[30] Z. Cao, X. Hou, and W. Zhao, “Adaptive robust simultaneous stabilization controller with tuning parameters design for two dissipative Hamiltonian systems,” Arch. Control Sci., vol. 27, no. 4, pp. 505–525, Dec. 2017.

[31] T. Shen, $H_\infty$ Control Theory and Its Applications, Beijing, China: Tsinghua Univ. Press, 1996.

[32] A. van der Schaft, $L_2$-Gain and Passivity Techniques in Nonlinear Control, Berlin, Germany: Springer, 1999.

[33] Y. Wang, D. Cheng, C. Li, and Y. Ge, “Dissipative hamiltonian realization and energy-based $L_2$-disturbance attenuation control of multimechanical power systems,” IEEE Trans. Autom. Control, vol. 48, no. 8, pp. 1428–1433, Aug. 2003.

[34] Y. Wang, D. Cheng, and S. Ge, “Approximate dissipative Hamiltonian realization and construction of local Lyapunov functions,” Syst. Control Lett., vol. 56, no. 2, pp. 141–149, Feb. 2007.

[35] B. F. Caviness and J. R. Johnson, Quantifier Elimination and Cylindrical Algebraic Decomposition. New York, NY, USA: Springer-Verlag, 1998.

[36] H. Li, N. Zhang, and J. Lam, “Parameter-dependent input-delayed control of uncertain vehicle suspensions,” J. Sound Vib., vol. 317, nos. 3–5, pp. 537–556, Nov. 2008.

[37] Y. Bai, Z. Li, and C. Huang, “New $H_\infty$ control approaches for interval time-delay systems with disturbances and their applications,” ISA Trans., vol. 65, pp. 174–185, Nov. 2016.

[38] P.-C. Chen and A.-C. Huang, “Adaptive sliding control of non-autonomous active suspension systems with time-varying loadings,” J. Sound Vib., vol. 282, nos. 3–5, pp. 1119–1135, Apr. 2005.

ZHONG CAO was born in Anhui, China, in 1977. He received the M.S. degree from the School of Computer Science and Educational Software, Guangzhou University, in 2005, and the Ph.D. degree from the School of Energy Science and Engineering, University of Electronic Science and Technology of China, in 2015.

From December 2016 to December 2017, he was a Visiting Scholar at the Department of Civil and Environmental Engineering, University of North Carolina at Charlotte, Charlotte, NC, USA. He is currently a Teacher of the School of Physics and Electronic Engineering, Guangzhou University. His research interests include nonlinear control systems and controlled polynomial Hamiltonian systems.

WENJING ZHAO was born in Gansu, China, in 1981. She received the M.S. degree from the School of Computer Science and Educational Software, Guangzhou University, in 2006. From December 2016 to December 2017, she was a Visiting Scholar with the Department of Civil and Environmental Engineering, University of North Carolina at Charlotte, Charlotte, NC, USA. She is currently an Experimentalist with Lab Center, Guangzhou University. Her current research interests include nonlinear systems, and robust control with the application in vehicle suspension systems.

ZHAOHUI CHEN was born in Henan, China. He received the B.S. degree from the Department of Mathematics, Zhengzhou University, in 2003, the M.S. degree from the School of Science, Liaoning University of Science and Technology, in 2006, and the Ph.D. degree from the School of Energy Science and Engineering, University of Electronic Science and Technology of China, in 2015.

From March 2017 to February 2018, he was a Visiting Scholar at the Department of Mathematics, North Carolina State University, Raleigh, NC, USA. He is currently an Associate Professor with the School of Mathematics, Physics and Data Science, Chongqing University of Science and Technology. His research interests include stability analysis and synthesis of time delay dynamical systems, fault-tolerant control, tracking control, and interconnected large-scale systems.

XIAORONG HOU was born in Shanxi, China, in 1966. He is currently a Professor with the School of Automation Engineering, University of Electronic Science and Technology of China. He has published over ninety research articles and two monographs. His research interests include symbolic computation, real algebraic geometry, control theory, and intelligent systems.