The Obstacle Problem for Quasilinear Stochastic PDEs with non-homogeneous operator

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Abstract: We prove the existence and uniqueness of solution of the obstacle problem for quasilinear Stochastic PDEs with non-homogeneous second order operator. Our method is based on analytical technics coming from the parabolic potential theory. The solution is expressed as a pair $(u, \nu)$ where $u$ is a predictable continuous process which takes values in a proper Sobolev space and $\nu$ is a random regular measure satisfying minimal Skohorod condition. Moreover, we establish a maximum principle for local solutions of such class of stochastic PDEs. The proofs are based on a version of Itô’s formula and estimates for the positive part of a local solution which is non-positive on the lateral boundary.

Keywords and phrases: parabolic potential, regular measure, stochastic partial differential equations, non-homogeneous second order operator, obstacle problem, penalization method, Itô’s formula, comparison theorem, space-time white noise.

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1. Introduction

In this paper we study the following SPDE with obstacle (in short OSPDE):

$$
\begin{align*}
\frac{du_t(x)}{dt} &= \partial_i (a_{i,j}(t,x)\partial_j u_t(x)) + g_i(t, x, u_t(x), \nabla u_t(x)) + f(t, x, u_t(x), \nabla u_t(x)) + \\
&\quad + \sum_{j=1}^{+\infty} h_j(t, x, u_t(x), \nabla u_t(x))dB_t^j + \nu(t, dx), \\
&\quad u_t \geq S_t, \\
&\quad u_0 = \xi.
\end{align*}
$$

(1)

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where $a$ is a time-dependant symmetric, uniformly elliptic, measurable matrix defined on some open domain $O \subset \mathbb{R}^d$, with null Dirichlet condition. The initial condition is given as $u_0 = \xi$, a $L^2(O)$-valued random variable, and $f, g = (g_1, ..., g_d)$ and $h = (h_1, ..., h_i, ...)$ are non-linear random functions. Given an obstacle $S : \Omega \times [0,T] \times O \to \mathbb{R}$, we study the obstacle problem for the SPDE (1), i.e. we want to find a solution of (1) which satisfies "$u \geq S$" where the obstacle $S$ is regular in some sense and controlled by the solution of a SPDE.

In recent work [10] we have proved in the homogeneous case, existence and uniqueness of the solution of equation (1) with Dirichlet boundary condition under standard Lipschitz hypotheses and $L^2$-type integrability conditions on the coefficients. Moreover in [11], still in the homogeneous case, we have obtained a maximum principle for local solutions. In these papers we have assumed that $a$ does not depend on time and so many proofs are based on the notion of semigroup associated to the second order operator and on the regularizing property of the semigroup. The aim of this paper is to extend all the results to the non homogeneous case.

Let us recall that the solution is a couple $(u, \nu)$, where $u$ is a process with values in the first order Sobolev space and $\nu$ is a random regular measure forcing $u$ to stay above $S$ and satisfying a minimal Skohorod condition. In order to give a rigorous meaning to the notion of solution, inspired by the works of M. Pierre in the deterministic case (see [23, 24]), we introduce the notion of parabolic capacity. We construct a solution which admits a quasi continuous version hence defined outside a polar set and use the fact that regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets.

There is a huge literature on parabolic SPDE’s without obstacle. The study of the $L^p$—norms w.r.t. the randomness of the space-time uniform norm on the trajectories of a stochastic PDE was started by N. V. Krylov in [16] (see also Kim [14]), for a more complete overview of existing works on this subject see [8, 9] and the references therein. Let us also mention that some maximum principle have been established by N. V. Krylov [17] for linear parabolic spde’s on Lipschitz domain. Concerning the obstacle problem, there are two approaches, a probabilistic one (see [20, 15]) based on the Feynmann-Kac’s formula via the backward doubly stochastic differential equations and the analytical one (see [12, 22, 27]) based on the Green function.

The main results of this paper are first an existence and uniqueness Theorem for the solution with null Dirichlet condition and a maximum principle for local solutions. This yields for example:

**Theorem 1.** Let $(M_t)_{t \geq 0}$ be an Itô process satisfying some integrability conditions, $p \geq 2$ and $u$ be a local weak solution of the obstacle problem (1). Assume that $\partial O$ is Lipschitz and $u \leq M$ on $\partial O$, then for all $t \in [0,T]$: 

$$E \left\| (u - M)^+ \right\|_{p,\infty,t}^p \leq k(p, t) C(S, f, g, h, M)$$

where $C(S, f, g, h, M)$ depends only on the barrier $S$, the initial condition $\xi$, coefficients $f, g, h$, the boundary condition $M$ and $k$ is a function which only depends on $p$ and $t$, $\| \cdot \|_{\infty,\infty,t}$ is the uniform norm on $[0,t] \times O$. 


2. Hypotheses and preliminaries

2.1. Settings

Let $\mathcal{O}$ be an open bounded domain in $\mathbb{R}^d$. The space $L^2(\mathcal{O})$ is the basic Hilbert space of our framework and we employ the usual notation for its scalar product and its norm,

$$(u, v) = \int_{\mathcal{O}} u(x)v(x) \, dx, \quad \|u\| = \left( \int_{\mathcal{O}} u^2(x) \, dx \right)^{\frac{1}{2}}.$$

In general, we shall extend the notation

$$(u, v) = \int_{\mathcal{O}} u(x)v(x) \, dx,$$

where $u, v$ are measurable functions defined on $\mathcal{O}$ such that $uv \in L^1(\mathcal{O})$.

The first order Sobolev space of functions vanishing at the boundary will be denoted as usual by $H^1_0(\mathcal{O})$. Its natural scalar product and norm are

$$(u, v)_{H^1_0(\mathcal{O})} = (u, v) + \sum_{i=1}^{d} \int_{\mathcal{O}} \partial_i u(x) \partial_i v(x) \, dx, \quad \|u\|_{H^1_0(\mathcal{O})} = \left( \|u\|^2_2 + \|\nabla u\|^2_2 \right)^{\frac{1}{2}}.$$

We shall denote by $H^1_{loc}(\mathcal{O})$ the space of functions which are locally square integrable in $\mathcal{O}$ and which admit first order derivatives that are also locally square integrable.

Another Hilbert space that we use is the second order Sobolev space $H^2_0(\mathcal{O})$ of functions vanishing at the boundary and twice differentiable in the weak sense.

We consider a sequence $((B^i(t))_{t \geq 0})_{i \in \mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.

Let $a$ be a measurable and $d \times d$ symmetric matrix defined on $\mathbb{R}_+ \times \mathcal{O}$. We assume that there exist positive constants $\lambda, \Lambda$ and $M$ such that for all $\eta \in \mathbb{R}^d$ and almost all $(t, x) \in \mathbb{R}_+ \times \mathcal{O}$:

$$\lambda |\eta|^2 \leq \sum_{i,j} a_{i,j}(t, x) \eta^i \eta^j \leq \Lambda |\eta|^2 \text{ and } |a_{i,j}(t, x)| \leq M.$$

(2)

Let $\Delta = \{(t, x, s, y) \in \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+ \times \mathcal{O}; \ t > s\}$. We denote by $G : \Delta \to \mathbb{R}^+$ the weak fundamental solution of the problem

$$\partial_t G(t, x; s, y) - \sum_{i=1}^{d} \partial_i a_{i,j}(t, x) \partial_j G(t, x; s, y) = 0$$

with Dirichlet boundary condition $G(t, x; s, y) = 0$, for all $(t, x) \in (s, +\infty) \times \partial \mathcal{O}$.

We consider the quasilinear stochastic partial differential equation (1) with initial condition $u(0, \cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t, x) = 0$, $\forall (t, x) \in \mathbb{R}_+ \times \partial \mathcal{O}$.

We assume that we have predictable random functions

$$f : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},$$

$$g = (g_1, ..., g_d) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,$$

$$h = (h_1, ..., h_i, ...) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{N_i},$$
In the sequel, $|\cdot|$ will always denote the underlying Euclidean or $l^2$-norm. For example

$$|h(t, \omega, x, y, z)|^2 = \sum_{i=1}^{+\infty} |h_i(t, \omega, x, y, z)|^2.$$ 

**Assumption (H):** There exist non-negative constants $C$, $\alpha$, $\beta$ such that for almost all $\omega$, the following inequalities hold for all $(x, y, z, t) \in \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$:

1. $|f(t, \omega, x, y, z) - f(t, \omega, x, y', z')| \leq C(|y - y'| + |z - z'|)$,
2. $(\sum_{i=1}^{d} |g_i(t, \omega, x, y, z) - g_i(t, \omega, x, y', z')|^2)^{\frac{1}{2}} \leq C|y - y'| + \alpha|z - z'|$,
3. $(|h(t, \omega, x, y, z) - h(t, \omega, x, y', z')|^2)^{\frac{1}{2}} \leq C|y - y'| + \beta|z - z'|$,
4. the contraction property: $2\alpha + 2\beta < 2\lambda$.

**Remark 1.** This last contraction property ensures existence and uniqueness for the solution of the SPDE without obstacle (see [9]).

Moreover for simplicity, we fix a terminal time $T > 0$, we assume that:

**Assumption (I):**

$$\xi \in L^2(\Omega \times \mathcal{O}) \text{ is an } \mathcal{F}_0 - \text{measurable random variable}$$

$$f(\cdot, \cdot, \cdot, 0, 0) := f^0 \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R})$$

$$g(\cdot, \cdot, \cdot, 0, 0) := g^0 \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$$

$$h(\cdot, \cdot, \cdot, 0, 0) := h^0 \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^{N^*})$$

We denote by $\mathcal{H}_T$ the space of $H^1_T(\mathcal{O})$-valued predictable $L^2(\mathcal{O})$-continuous processes $(u_t)_{t \in [0, T]}$ which satisfy

$$\|u\|_T = E \sup_{t \in [0, T]} \|u_t\|^2 + E \int_0^T \|\nabla u_t\|^2 dt < +\infty.$$ 

It is the natural space for solutions.

The space of test functions is denote by $\mathcal{D} = C^\infty_c(\mathbb{R}^+) \otimes C^2(\mathcal{O})$, where $C^\infty_c(\mathbb{R}^+)$ is the space of all real valued infinitely differentiable functions with compact support in $\mathbb{R}^+$ and $C^2(\mathcal{O})$ the set of $C^2$-functions with compact support in $\mathcal{O}$.

**Main example of stochastic noise**

Let $W$ be a noise white in time and colored in space, defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ whose covariance function is given by:

$$\forall s, t \in \mathbb{R}_+, \forall x, y \in \mathcal{O}, \quad E[\tilde{W}(x, s)\tilde{W}(y, t)] = \delta(t - s)k(x, y),$$

where $k : \mathcal{O} \times \mathcal{O} \mapsto \mathbb{R}_+$ is a symmetric and measurable function.

Consider the following SPDE driven by $W$:

$$du_t(x) = \left( \sum_{i,j=1}^{d} \partial_i a_{i,j}(t, x) \partial_j u_t(x) + f(t, x, u_t(x), \nabla u_t(x)) + \sum_{i=1}^{d} \partial_i g_i(t, x, u_t(x), \nabla u_t(x)) \right) dt$$

$$+ \tilde{h}(t, x, u_t(x), \nabla u_t(x)) W(dt, x),$$

where $(a_{i,j}(t, x))_{i,j=1}^{d}$, $(g_i(t, x, u_t(x), \nabla u_t(x)))_{i=1}^{d}$, $(\tilde{h}(t, x, u_t(x), \nabla u_t(x)))_{t \in [0, T]}$ are measurable functions.
where \( f \) and \( g \) are as above and \( \tilde{h} \) is a random real valued function.

We assume that the covariance function \( k \) defines a trace class operator denoted by \( K \) in \( L^2(\mathcal{O}) \). It is well known that there exists an orthogonal basis \((e_i)_{i \in \mathbb{N}^*}\) of \( L^2(\mathcal{O}) \) consisting of eigenfunctions of \( K \) with corresponding eigenvalues \((\lambda_i)_{i \in \mathbb{N}^*}\) such that

\[
\sum_{i=1}^{+\infty} \lambda_i < +\infty,
\]

and

\[
k(x, y) = \sum_{i=1}^{+\infty} \lambda_i e_i(x)e_i(y).
\]

It is also well known that there exists a sequence \(((B^i(t))_{t \geq 0})_{i \in \mathbb{N}^*}\) of independent standard Brownian motions such that

\[
W(dt, \cdot) = \sum_{i=1}^{+\infty} \lambda_i^{1/2} e_i B^i(dt).
\]

So that equation (11) is equivalent to equation (1) without obstacle and with \( h = (h_i)_{i \in \mathbb{N}^*} \) where

\[
\forall i \in \mathbb{N}^*, \ h_i(s, x, y, z) = \sqrt{\lambda_i} \tilde{h}(s, x, y, z)e_i(x).
\]

Assume as in [26] that for all \( i \in \mathbb{N}^* \), \( \|e_i\|_\infty < +\infty \) and

\[
\sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 < +\infty.
\]

Since

\[
\left( |h(t, \omega, x, y, z) - h(t, \omega, x, y', z')|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 \right) \left( \|\tilde{h}(t, x, y, z) - \tilde{h}(t, x, y', z')\|_{\infty}^2 \right),
\]

\( h \) satisfies the Lipschitz hypothesis (H)-(ii) if \( \tilde{h} \) satisfies a similar Lipschitz hypothesis.

### 2.2. Parabolic potential analysis

In this section we will recall some important definitions and results concerning the obstacle problem for parabolic PDE in [23] and [24].

\( \mathcal{K} \) denotes \( L^\infty([0, T]; L^2(\mathcal{O})) \cap L^2([0, T]; H^1_0(\mathcal{O})) \) equipped with the norm:

\[
\| v \|_\mathcal{K} = \| v \|_{L^\infty([0, T]; L^2(\mathcal{O}))} + \| v \|_{L^2([0, T]; H^1_0(\mathcal{O}))} = \sup_{t \in [0, T]} \| v_t \|_2^2 + \int_0^T \left( \| v_t \|_2^2 + \| \nabla v_t \|_2^2 \right) dt.
\]

\( \mathcal{C} \) denotes the space of continuous functions on compact support in \([0, T] \times \mathcal{O}\) and finally:

\[
\mathcal{W} = \{ \varphi \in L^2([0, T]; H^1_0(\mathcal{O})); \ \frac{\partial \varphi}{\partial t} \in L^2([0, T]; H^{-1}(\mathcal{O})) \}.
\]
endowed with the norm $$\| \varphi \|_W = \| \varphi \|_{L^2([0,T]; H^1_0(\Omega))} + \| \frac{\partial \varphi}{\partial t} \|_{L^2([0,T]; H^{-1}(\Omega))}.$$  It is known (see [18]) that $$W$$ is continuously embedded in $$C([0,T]; L^2(\Omega))$$, the set of $$L^2(\Omega)$$-valued continuous functions on $$[0,T]$$. So without ambiguity, we will also consider $$W_T = \{ \varphi \in W; \varphi(T) = 0 \}$$, $$W^+ = \{ \varphi \in W; \varphi \geq 0 \}$$, $$W_0^+ = W_T \cap W^+$$.

We now introduce the notion of parabolic potentials and regular measures which permit to define the parabolic capacity.

**Definition 1.** An element $$v \in K$$ is said to be a parabolic potential if it satisfies:

$$\forall \varphi \in W_T^+, \int_0^T -\left(\frac{\partial \varphi}{\partial t}, v_t\right) dt + \int_0^T \mathcal{E}(\varphi_t, v_t) dt \geq 0.$$  

We denote by $$\mathcal{P}$$ the set of all parabolic potentials.

The next representation property is crucial:

**Proposition 1.** (Proposition 1.1 in [24]) Let $$v \in \mathcal{P}$$, then there exists a unique positive Radon measure on $$[0,T] \times \Omega$$, denoted by $$\nu^v$$, such that:

$$\forall \varphi \in W_T \cap C, \int_0^T \left( -\frac{\partial \varphi}{\partial t}, v_t \right) dt + \int_0^T \mathcal{E}(\varphi_t, v_t) dt = \int_0^T \int_\Omega \varphi(t,x) d\nu^v.$$  

Moreover, $$v$$ admits a right-continuous (resp. left-continuous) version $$\hat{v}$$ (resp. $$\bar{v}$$) : $$[0,T] \mapsto L^2(\Omega)$$.

Such a Radon measure, $$\nu^v$$ is called a regular measure and we write:

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

**Remark 2.** As a consequence, we can also define for all $$v \in \mathcal{P}$$:

$$v_T = \lim_{t \uparrow T} \bar{v}_t \in L^2(\Omega).$$

**Definition 2.** Let $$K \subset [0,T] \times \Omega$$ be compact, $$v \in \mathcal{P}$$ is said to be $$\nu$$—superior than 1 on $$K$$, if there exists a sequence $$v_n \in \mathcal{P}$$ with $$v_n \geq 1$$ a.e. on a neighborhood of $$K$$ converging to $$v$$ in $$L^2([0,T]; H^1_0(\Omega))$$.

We denote:

$$K = \{ v \in \mathcal{P}; \nu \text{ is } \nu - \text{superior to 1 on } K \}.$$  

**Proposition 2.** (Proposition 2.1 in [24]) Let $$K \subset [0,T] \times \Omega$$ compact, then $$K$$ admits a smallest $$v_K \in \mathcal{P}$$ and the measure $$\nu^v_K$$ whose support is in $$K$$ satisfies

$$\int_0^T \int_\Omega d\nu^v_K = \inf_{v \in \mathcal{P}} \{ \int_0^T \int_\Omega d\nu^v; v \in K \}.$$  

**Definition 3.** (Parabolic Capacity)

- Let $$K \subset [0,T] \times \Omega$$ be compact, we define $$\text{cap}(K) = \int_0^T \int_\Omega d\nu^v_K$$;
- let $$O \subset [0,T] \times \Omega$$ be open, we define $$\text{cap}(O) = \sup \{ \text{cap}(K); K \subset O \text{ compact} \}$$;
- for any borelian $$E \subset [0,T] \times \Omega$$, we define $$\text{cap}(E) = \inf \{ \text{cap}(O); O \supset E \text{ open} \}.$$
Definition 4. A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.

Definition 5. (Quasi-continuous)
A function \( u : [0, T] \times \mathcal{O} \to \mathbb{R} \) is called quasi-continuous, if there exists a decreasing sequence of open subsets \( O_n \) of \( [0, T] \times \mathcal{O} \) with:
1. for all \( n \), the restriction of \( u_n \) to the complement of \( O_n \) is continuous;
2. \( \lim_{n \to +\infty} \text{cap} (O_n) = 0. \)

We say that \( u \) admits a quasi-continuous version, if there exists \( \tilde{u} \) quasi-continuous such that \( \tilde{u} = u \) a.e.

The next proposition, whose proof may be found in [23] or [24] shall play an important role in the sequel:

Proposition 3. Let \( K \subset \mathcal{O} \) a compact set, then \( \forall t \in [0, T[ \)
\[
\text{cap}(\{t\} \times K) = \lambda_d(K),
\]
where \( \lambda_d \) is the Lebesgue measure on \( \mathcal{O} \).

As a consequence, if \( u : [0, T] \times \mathcal{O} \to \mathbb{R} \) is a map defined quasi-everywhere then it defines uniquely a map from \( [0, T] \) into \( L^2(\mathcal{O}) \). In other words, for any \( t \in [0, T[ \), \( u_t \) is defined without any ambiguity as an element in \( L^2(\mathcal{O}) \). Moreover, if \( u \in \mathcal{P} \), it admits version \( \tilde{u} \) which is left continuous on \( [0, T] \) with values in \( L^2(\mathcal{O}) \) so that \( u_T = \tilde{u}_T \) is also defined without ambiguity.

Remark 3. The previous proposition applies if for example \( u \) is quasi-continuous.

Proposition 4. (Theorem III.1 in [24]) If \( \varphi \in \mathcal{W} \), then it admits a unique quasi-continuous version that we denote by \( \tilde{\varphi} \). Moreover, for all \( v \in \mathcal{P} \), the following relation holds:
\[
\int_{[0,T[\times\mathcal{O}} \tilde{\varphi} dv^v = \int_0^T (-\partial_t \varphi, v) + \mathcal{E}(\varphi, v) \, dt + (\varphi_T, v_T).
\]

We end this section by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [24]):

Lemma 1. If \( v^n \in \mathcal{P} \) is a bounded sequence in \( \mathcal{K} \) and converges weakly to \( v \) in \( L^2([0,T]; H^1_0(\mathcal{O})) \); if \( u \) is a quasi-continuous function and \( |u| \) is bounded by an element in \( \mathcal{P} \). Then
\[
\lim_{n \to +\infty} \int_0^T \int_{\mathcal{O}} u dv^{v^n} = \int_0^T \int_{\mathcal{O}} u dv^v.
\]

Remark 4. For the more general case one can see [24] Lemma 3.8.

3. Quasi-continuity of the solution of SPDE without obstacle

We consider the SPDE without obstacle:
\[
d u_t(x) = \partial_i (a_{i,j}(t,x) \partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))) \, dt + f(t,x,u_t(x),\nabla u_t(x)) \, dt + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x)) dB_t^j,
\]
As a consequence of well-known results (see for example [9], Theorem 11), we know that under assumptions (H) and (I), SPDE (5) with zero Dirichlet boundary condition, admits a unique solution in $H_T$, we denote it by $U(\xi, f, g, h)$, moreover it satisfies the following estimate:

$$E[\|u\|_{H_T}^2] \leq cE \left[ \|\xi\|^2 + \int_0^T \left( \|f_t^0\|^2 + \|g_t^0\|^2 + \|h_t^0\|^2 \right) dt \right]$$  \quad (6)

The main theorem of this section is the following:

**Theorem 2.** Under assumptions (H) and (I), $u = U(\xi, f, g, h)$ the solution of SPDE (5) admits a quasi-continuous version denoted by $\tilde{u}$ i.e. $u = \tilde{u} dP \otimes dt \otimes dx$ a.e. and for almost all $w \in \Omega$, $(t, x) \rightarrow \tilde{u}(w, x)$ is quasi-continuous.

Before giving the proof of this theorem, we need the following lemmas. The first one is proved in [24], Lemma 3.3:

**Lemma 2.** There exists $C > 0$ such that, for all open set $\vartheta \subset [0, T] \times \Omega$ and $v \in P$ with $v \geq 1$ a.e. on $\vartheta$:

$$\text{cap} \vartheta \leq C \|v\|_{K}^2.$$

Let $\kappa = \kappa(u, u^+(0))$ be defined as following

$$\kappa = \text{ess inf}\{v \in P; \ v \geq u \ a.e. \ and \ v(0) \geq u^+(0)\}.$$

One has to note that $\kappa$ is a random function. From now on, we always take for $\kappa$ the following measurable version

$$\kappa = \sup_n v^n,$$

where $(v^n)$ is the non-decreasing sequence of random functions given by

$$\begin{cases}
\frac{\partial v^n}{\partial t} = L v^n + n(v^n - u) \\
v^n_0 = u^+(0).
\end{cases}$$  \quad (7)

From F.Mignot and J.P.Puel [21], we know that for almost all $w \in \Omega$, $v^n(w)$ converges weakly to $v(w) = \kappa(u(w), u^+(0))(w)$ in $L^2([0, T]; H^1_0(\Omega))$ and that $v \geq u$.

**Lemma 3.** We have the following estimate:

$$E \|\kappa\|_{K}^2 \leq C \left( E \| u^+_0 \|^2 + E \| u_0 \|^2 + E \int_0^T \| f_t^0 \|^2 + \| g_t^0 \|^2 \| h_t^0 \|^2 \ dt \right),$$

where $C$ is a constant depending only on the structure constants of the equation.

Thanks to [2], the proof of Lemma 3 in [10] can be easily extended to the case of non-homogeneous operator.

**Proof of Theorem 2** First of all, we remark that we only need to prove this result in the linear case, namely we consider that $f$, $g$ and $h$ only depend on $t$, $x$ and $\omega$. Then, we approximate the coefficients, the domain and the second order operator in the following way:
1. We mollify coefficients $a_{i,j}$ and so consider sequences $(a^{n}_{i,j})_{n}$ of $C^\infty$ functions such that for all $n \in \mathbb{N}^*$, the matrix $a^n$ satisfies the same ellipticity and boundedness assumptions as $a$ and

$$\forall 1 \leq i, j \leq d, \quad \lim_{n \to +\infty} a^{n}_{i,j} = a_{i,j} \text{ a.e.}$$

2. We approximate $\mathcal{O}$ by an increasing sequence of smooth domains $(\mathcal{O}^n)_{n \geq 1}$.

3. We consider a sequence $(\xi^n)$ in $C_c^\infty(\mathcal{O})$ which converges to $\xi$ in $L^2(\mathcal{O})$ and such that for all $n$, $\text{supp} \xi^n \subset \mathcal{O}^n$.

4. For each $i \in \mathbb{N}^*$, we construct a sequence of predictable functions $(h^n_i)$ in $(L^2(\Omega) \otimes C_c([0, +\infty)) \otimes C_c^\infty(\mathcal{O}))$ which converges in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O}))$ to $h_i$ such that for all $n$, $\text{supp} \ h^n_i \subset \mathcal{O}^n$ and

$$\forall t \geq 0, \quad E[\int_0^t \|h^n_i\|^2 ds] \leq E[\int_0^t \|h_i\|^2 ds],$$

so that

$$E[\int_0^t \|h^n\|^2 ds] \leq E[\int_0^t \|h\|^2 ds] < +\infty.$$  

5. We consider a sequence of predictable functions $(f^n)$ in $(L^2(\Omega) \otimes C_c([0, +\infty)) \otimes C_c^\infty(\mathcal{O}))$ which converges in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O}))$ to $f$ and such that for all $n$, $\text{supp} f^n \subset \mathcal{O}^n$.

6. Finally, let $(g^n)$ be a sequence in $(L^2(\Omega) \otimes C_c([0, +\infty)) \otimes C_c^\infty(\mathcal{O})$ which converges in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O})^d)$ to $g$ and such that for all $n$, $\text{supp} g^n \subset \mathcal{O}^n$.

For all $n \in \mathbb{N}^*$, we put $\Delta^n = \{(t, x, s, y) \in \mathbb{R}_+ \times \mathcal{O}^n \times \mathbb{R}_+ \times \mathcal{O}^n; t > s\}$. We denote by $G^n : \Delta^n \mapsto \mathbb{R}_+$ the weak fundamental solution of the problem (8) associated to $a^n$ and $\mathcal{O}^n$:

$$\partial_t G^n(t, x; s, y) - \sum_{i,j=1}^d \partial_i a^n_{i,j}(t, x) \partial_j G^n(t, x; s, y) = 0$$

with Dirichlet boundary condition $G^n(t, x, s, y) = 0$, for all $(t, x) \in (s, +\infty) \times \partial \mathcal{O}^n$.

In a natural way we extend $G^n$ on $\Delta$ by setting: $G^n \equiv 0$ on $\Delta \setminus \Delta^n$.

We define the process $u^n$ by setting for all $(t, x) \in \mathbb{R}_+ \times \mathcal{O}$:

$$u^n(t, x) = \int_\mathcal{O} G(t, \cdot, 0, y) \xi^n(y) dy + \int_0^t \int_\mathcal{O} G(t, \cdot, s, y) f^n_s(y) dyds + \sum_{i=1}^d \int_0^t \int_\mathcal{O} G(t, \cdot, s, y) \partial_i g^n_s(y) dyds + \sum_{j=1}^{+\infty} \int_0^t \int_\mathcal{O} G(t, \cdot, s, y) h^n_{j,s}(y) dB^j_s.$$

The main point is that there exists a subsequence of $(G^n)_{n \geq 1}$ which converges everywhere to $G$ on $\Delta$, where $G$ still denotes the fundamental solution of $\square$, see Lemma 7 in [4]. From Proposition 6 in [4], we know that $u^n \in \mathcal{H}_T$ is the unique weak solution of $\square$. $G$ is uniformly continuous in space-time variables on any compact away from the diagonal in
Lemma 3 to we get the following relation
\[ \kappa \]
Let \( P \), \( \epsilon \) Then we take \( \kappa \), \( \hat{u}^n \) such that a sequence of convex combinations \( (\hat{u}^n)_k \) which converges weakly in \( L^2([0,T] \times \Omega; H^1_0(O)) \) and such that a sequence of convex combinations \( (\tilde{u}^n) \) of the form
\[ \tilde{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k} \]
converges strongly to \( u \) in \( L^2(\Omega \times [0,T]; H^1_0(O)) \). It is clear that for all \( n \in \mathbb{N}^* \), \( (\tilde{u}^n)_n \) is \( P \)-almost surely continuous in \( (t,x) \).

We consider a sequence of random open sets
\[ \psi_n = \{ |\hat{u}^n - \tilde{u}^n| > \epsilon_n \}, \quad \Theta_p = \bigcup_{n=p}^{+\infty} \psi_n. \]
Let \( \kappa_n = \kappa(\frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n), \frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n)(0)) + \kappa(-\frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n), \frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n)(0)) \), from the definition of \( \kappa \) and the relation (see [2])
\[ \kappa(|v|) \leq \kappa(v, v^+(0)) + \kappa(-v, v^-(0)), \]
we know that \( \kappa_n \) satisfy the conditions of Lemma [2] i.e. \( \kappa_n \in \mathcal{P} \) et \( \kappa_n \geq 1 \) a.e. on \( \psi_n \), thus we get the following relation
\[ \text{cap}(\Theta_p) \leq \sum_{n=p}^{+\infty} \text{cap}(\psi_n) \leq \sum_{n=p}^{+\infty} \kappa_n \frac{\|\|}{K}. \]
Thus, remarking that \( \hat{u}^{n+1} - \tilde{u}^n = \mathcal{U}((\xi^{n+1} - \xi^n), \xi^n, \xi^{n+1} - \xi^n, \xi^n, \delta^{n+1} - \delta^n, \delta^n, \delta^{n+1} - \delta^n) \), we apply Lemma [3] to \( \kappa(\frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n), \frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n)(0)) \) and \( \kappa(-\frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n), \frac{1}{\epsilon_n} (\hat{u}^{n+1} - \tilde{u}^n)(0)) \) and obtain:
\[ E[\text{cap}(\Theta_p)] \leq \sum_{n=p}^{+\infty} E \|\kappa_n\|_K^2 \leq 2C \sum_{n=p}^{+\infty} \frac{1}{\epsilon_n^2} (E \|\xi^{n+1} - \xi^n\|^2 + E \int_0^T \|f_t^{n+1} - f_t^n\|^2 dt)
+ \|g_t^{n+1} - g_t^n\|^2 + \|h_t^{n+1} - h_t^n\|^2 dt). \]
Then, by extracting a subsequence, we can consider that
\[ E \|\xi^{n+1} - \xi^n\|^2 + E \int_0^T \|f_t^{n+1} - f_t^n\|^2 + \|g_t^{n+1} - g_t^n\|^2 + \|h_t^{n+1} - h_t^n\|^2 dt \leq \frac{1}{2n}. \]
Then we take \( \epsilon_n = \frac{1}{n^2} \) to get
\[ E[\text{cap}(\Theta_p)] \leq \sum_{n=p}^{+\infty} \frac{2Cn^4}{2^n}. \]
Therefore
\[
\lim_{p \to +\infty} E[\text{cap}(\Theta_p)] = 0.
\]

For almost all \( \omega \in \Omega \), \( \hat{u}^n(\omega) \) is continuous in \((t, x)\) on \((\Theta_p(w))^c\) and \((\hat{u}^n(\omega))_n\) converges uniformly to \( u \) on \((\Theta_p(w))^c\) for all \( p \), hence, \( u(\omega) \) is continuous in \((t, x)\) on \((\Theta_p(w))^c\), then from the definition of quasi-continuous, we know that \( u(\omega) \) admits a quasi-continuous version since \( \text{cap}(\Theta_p) \) tends to 0 almost surely as \( p \) tends to \(+\infty\). □

4. Existence and uniqueness result

From now on, similarly to the homogeneous case studied in [10], we make the following assumptions on the obstacle:

**Assumption (O):** The obstacle \( S \) is assumed to be an adapted process, quasi-continuous, such that \( S_0 \leq \xi P\)-almost surely and controlled by the solution of a SPDE, i.e. \( \forall t \in [0, T] \),
\[
S_t \leq S'_t
\]  
where \( S' \) is the solution of a linear SPDE
\[
\begin{aligned}
dS'_t(x) &= \partial_i(a_{i,j}(t, x)\partial_jS'_t(x) + g'_{i,t}(x))dt + f'_t(x)dt + \sum_{j=1}^{+\infty} h'_{j,t}(x)dB^j_t \\
S'_0 &= S'_0
\end{aligned}
\]
with \( S'_0 \in L^2(\Omega \times \mathcal{O}) \), \( F_0 \)-measurable, \( f' \), \( g' \) and \( h' \) adapted processes respectively in \( L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}) \), \( L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^d) \) and \( L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^{d^2}) \).

**Remark 5.** Here again, we know that \( S' \) uniquely exists and satisfies the following estimate:
\[
E \sup_{t \in [0, T]} \| S'_t \|^2 + E \int_0^T \| \nabla S'_t \|^2 dt \leq CE \left[ \| S'_0 \|^2 + \int_0^T (\| f'_t \|^2 + \| g'_t \|^2 + \| h'_t \|^2) dt \right].
\]

Moreover, from Theorem 2, \( S' \) admits a quasi-continuous version.

Let us also remark that even if this assumption seems restrictive since \( S' \) is driven by the same operator and Brownian motions as \( u \), it encompasses a large class of examples.

We now are able to define rigorously the notion of solution to the problem with obstacle:

**Definition 6.** A pair \((u, \nu)\) is said to be a solution of the obstacle problem for (5) if

1. \( u \in \mathcal{H}_T \) and \( u(t, x) \geq S(t, x) \), \( dP \otimes dt \otimes dx \) - a.e. and \( u_0(x) = \xi \), \( dP \otimes dx \) - a.e.;
2. \( \nu \) is a random regular measure defined on \([0, T] \times \mathcal{O}\);
3. the following relation holds almost surely, for all \( t \in [0, T] \) and \( \forall \varphi \in \mathcal{D} \),
\[
(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)ds - \sum_{i,j=1}^d \int_0^t \int_{\mathcal{O}} a_{i,j}(s, x)\partial_i u_s(x)\partial_j \varphi_s(x)dx ds
\]
\[
= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s)ds + \sum_{i=1}^d \int_0^t (g^i_s(u_s, \nabla u_s), \partial_i \varphi_s)ds
\]
\[
+ \sum_{j=1}^{+\infty} \int_0^t (h^j_s(u_s, \nabla u_s), \varphi_s)dB^j_s + \int_0^t \int_{\mathcal{O}} \varphi_s(x)\nu(dx, ds).
\] (13)
4. $u$ admits a quasi-continuous version, $\tilde{u}$, and we have
\[ \int_0^T \int_{\Omega} (\tilde{u}(s,x) - S(s,x))\nu(dx, ds) = 0 \text{ a.s.} \]

The first important result of this paper is:

**Theorem 3.** Under assumptions (H), (I) and (O), there exists a unique weak solution of the obstacle problem for the SPDE (5) associated to $(\xi, f, g, h, S)$. We denote by $\mathcal{R}(\xi, f, g, h, S)$ the solution of SPDE (5) with obstacle when it exists and is unique.

*Proof.* As we have Itô's formula and comparison theorem for the solution of non homogeneous SPDE (5), see Proposition 9 and Theorem 16 in [10], we can make the same proof as in the homogeneous case (see [11]). More precisely, we first establish the result in the linear case by following Section 5.2 in [10]. Then, we prove an Itô formula for the difference of two (linear) solutions of SPDE's with obstacle similarly to Section 5.4 in [10] and finally conclude thanks to a Picard iteration procedure as in Section 5.5 in [10].

We can also establish the following Itô formula and comparison theorem for the solution of SPDE (11) with obstacle. Here again, the proofs are the same as in [10].

**Theorem 4.** Let $u$ be the solution of OSPDE (11) and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^{1,2}$. We denote by $\Phi'$ and $\Phi''$ the derivatives of $\Phi$ with respect to the space variables and by $\frac{\partial \Phi}{\partial x}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\Phi'(t,0) = 0$ for all $t \geq 0$. Then for all $t \in [0, T]$,

\[
\int_{\Omega} \Phi(t, u_t(x))dx + \int_0^t \int_{\Omega} a_{i,j}(s,x)\Phi''(s, u_s(x))\partial_i u_s(x)\partial_j u_s(x)dxds = \int_{\Omega} \Phi(0, \xi(x))dx \\
+ \int_0^t \int_{\Omega} \frac{\partial \Phi}{\partial s}(s, u_s(x))dxds + \int_0^t (\Phi'(s, u_s), f_s)ds - \sum_{i=1}^d \int_0^t \int_{\Omega} \Phi''(s, u_s(x))\partial_i u_s(x)g_i(x)dxds \\
+ \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s), h_j)dB_j^s + \frac{1}{2} \sum_{j=1}^{+\infty} \int_0^t \int_{\Omega} \Phi''(s, u_s(x))(h_{j,s}(x))^2dxds \\
+ \int_0^t \int_{\Omega} \Phi'(s, \tilde{u}_s(x))\nu(dxds), \quad P-a.s.
\]

This Itô formula naturally leads to a comparison theorem, the proof being the same as in the homogeneous case (see Theorem 8 in [11]). More precisely, consider $(u^1, \nu^1) = \mathcal{R}(\xi^1, f^1, g, h, S^1)$ the solution of the SPDE with obstacle

\[
\begin{align*}
\left\{ \begin{array}{l}
du^1_t(x) = \partial_i(a_{i,j}(t,x)\partial_j u^1_t(x) + g_i(t, x, u^1_t(x), \nabla u^1_t(x))dt + f^1(t, x, u^1_t(x), \nabla u^1_t(x))dt \\
+ \sum_{j=1}^{+\infty} h_j(t, x, u^1_t(x), \nabla u^1_t(x))dB^j_t + \nu^1(x, dt)
\end{array} \right. \\
u^1 \geq S^1, \quad u^1_0 = \xi^1,
\end{align*}
\]

where we assume $(\xi^1, f^1, g, h)$ satisfy hypotheses (H), (I) and (O).

We consider another coefficients $f^2$ which satisfies the same assumptions as $f^1$, another
obstacle \( S^2 \) which satisfies \((O)\) and another initial condition \( \xi^2 \) belonging to \( L^2(\Omega \times \mathcal{O}) \) and \( \mathcal{F}_0 \) adapted such that \( \xi^2 \geq S^2_0 \). We denote by \( (u^2, \nu^2) = \mathcal{R}(\xi^2, f^2, g, h, S^2) \).

**Theorem 5.** Assume that the following conditions hold

1. \( \xi^1 \leq \xi^2, \text{ } dx \otimes dP - \text{a.e.} \)
2. \( f^1(u^1, \nabla u^1) \leq f^2(u^1, \nabla u^1), \text{ } dt \otimes dx \otimes dP - \text{a.e.} \)
3. \( S^1 \leq S^2, \text{ } dt \otimes dx \otimes dP - \text{a.e.} \)

Then for almost all \( \omega \in \Omega \), \( u^1(t, x) \leq u^2(t, x) \) q.e.

5. Maximum principle for local solutions of the OSPDE

5.1. \( L^{p,q} \)-spaces

For each \( t > 0 \) and for all real numbers \( p, q \geq 1 \), we denote by \( L^{p,q}([0, t] \times \mathcal{O}) \) the space of (classes of) measurable functions \( u : [0, t] \times \mathcal{O} \rightarrow \mathbb{R} \) such that

\[
\|u\|_{p,q; t} := \left( \int_0^t \left( \int_\mathcal{O} |u(s, x)|^p \, dx \right)^{q/p} \, ds \right)^{1/q}
\]

is finite. The limiting cases with \( p \) or \( q \) taking the value \( \infty \) are also considered with the use of the essential sup norm.

The space of measurable functions \( u : \mathbb{R}_+ \rightarrow L^2(\mathcal{O}) \) such that \( \|u\|_{2,2; t} < \infty \), for each \( t \geq 0 \), is denoted by \( L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathcal{O})) \), where \( \mathbb{R}_+ \) denotes the set of non-negative real numbers.

Similarly, the space \( L^2_{\text{loc}}(\mathbb{R}_+; H^1_0(\mathcal{O})) \) consists of all measurable functions \( u : \mathbb{R}_+ \rightarrow H^1_0(\mathcal{O}) \) such that

\[
\|u\|_{2,2; t} + \|\nabla u\|_{2,2; t} < \infty,
\]

for any \( t \geq 0 \).

We recall that the Sobolev inequality states that

\[
\|u\|_{2^*} \leq c_S \|\nabla u\|_2,
\]

for each \( u \in H^1_0(\mathcal{O}) \), where \( c_S > 0 \) is a constant that depends on the dimension and \( 2^* = \frac{2d}{d-2} \) if \( d > 2 \), while \( 2^* \) may be any number in \([2, \infty[\) if \( d = 2 \) and \( 2^* = \infty \) if \( d = 1 \).

Finally, we introduce the following norm which is obtained by interpolation in \( L^{p,q} \)-spaces:

\[
\|u\|_{\#; t} = \|u\|_{2,\infty; t} \vee \|u\|_{2^*,2; t},
\]

and we denote by \( L^*_{\#; t} \) the set of functions \( u \) such that \( \|u\|_{\#; t} \) is finite. Its dual space is a functional space: \( L^*_{\#; t} \) equipped with the norm \( \|\cdot\|_{\#; t}^* \) and we have

\[
\int_0^t \int_\mathcal{O} u(s, x) v(s, x) \, dx \, ds \leq \|u\|_{\#; t} \|v\|_{\#; t}^*, \quad (14)
\]

for any \( u \in L^*_{\#; t} \) and \( v \in L^*_{\#; t} \).
5.2. Local solutions

We define $\mathcal{H}_{loc} = \mathcal{H}_{loc}(\mathcal{O})$ to be the set of $H^1_{loc}(\mathcal{O})$-valued predictable processes defined on $[0, T]$ such that for any compact subset $K$ in $\mathcal{O}$:

$$E\left( \sup_{0 \leq s \leq T} \int_K u_s(x)^2 \, dx + E \int_0^T \int_K |\nabla u_s(x)|^2 \, dx \, ds \right)^{1/2} < \infty.$$ 

Definition 7. We say that a Radon measure $\nu$ on $[0, T] \times \mathcal{O}$ is a local regular measure if for any non-negative $\phi$ in $C^\infty_c(\mathcal{O})$, $\phi \nu$ is a regular measure.

In [11] (see Proposition 2.10), we have proved:

**Proposition 5.** Local regular measures do not charge polar sets (i.e. sets of capacity 0).

We can now define the notion of local solution:

**Definition 8.** A pair $(u, \nu)$ is said to be a local solution of the problem (1) if

1. $u \in \mathcal{H}_{loc}$, $u(t, x) \geq S(t, x)$, $dP \otimes dt \otimes dx$ and $u_0(x) = \xi$, $dP \otimes dx$-a.e.;
2. $\nu$ is a local random regular measure defined on $[0, T] \times \mathcal{O}$;
3. the following relation holds almost surely, for all $t \in [0, T]$ and all $\varphi \in \mathcal{D}$,

$$\begin{align*}
(u_t, \varphi_t) = & (\xi, \varphi_0) + \int_0^t (u_s, \partial_s \varphi_s) \, ds - \sum_{i,j} \int_0^t \int_\mathcal{O} a_{i,j}(s, x) \partial_i u_s(x) \partial_j \varphi_s(x) \, dx \, ds \\
& - \sum_{i=1}^d \int_0^t (g^i_s(u_s, \nabla u_s), \partial_i \varphi_s) \, ds + \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) \, ds \\
& + \sum_{j=1}^{+\infty} \int_0^t (h^j_s(u_s, \nabla u_s), \varphi_s) \, dB^j_s + \int_0^t \varphi_s(x) \nu(dx, ds).
\end{align*}$$

(15)

4. $u$ admits a quasi-continuous version, $\tilde{u}$, and we have

$$\int_0^T \int_\mathcal{O} (\tilde{u}(s, x) - S(s, x)) \nu(dx, ds) = 0, \quad P - a.s.$$

We denote by $\mathcal{R}_{loc}(\xi, f, g, h, S)$ the set of all the local solutions $(u, \nu)$.

5.3. Hypotheses

In order to get some $L^p$-estimates for the uniform norm of the positive part of the solution of (1), we need stronger integrability conditions on the coefficients and the initial condition. To this end, we consider the following assumptions: for $p \geq 2$:

**Assumption (HI2p)**

$$E\left( \|\xi\|_{p, \mathcal{O}}^p + \|f^0\|_{2,2;T}^p + \|g^0\|_{2,2;T}^p + \|h^0\|_{2,2;T}^p \right) < \infty.$$

**Assumption (OL):** The obstacle $S : [0, T] \times \Omega \times \mathcal{O} \to \mathbb{R}$ is an adapted random field, almost surely quasi-continuous, such that $S_0 \leq \xi$ $P$-almost surely and controlled by a local solution of an SPDE, i.e. $\forall t \in [0, T]$,

$$S_t \leq S_t^e, \quad dP \otimes dt \otimes dx - a.e.$$
where \( S' \) is a local solution (for the definition of local solution see for example Definition 1 in [8]) of the linear SPDE

\[
\begin{cases}
    dS'_t = LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_t dt + \sum_{j=1}^{+\infty} h'_j dB^j_t \\
    S'(0) = S'_0.
\end{cases}
\]

Assumption (HIL)

\[
E \int_K |\xi(x)|^2 dx + E \int_0^T \int_K (|f'_s(x)|^2 + |g'_s(x)|^2 + |h'_s(x)|^2) dx ds < \infty,
\]
for any compact set \( K \subset \mathcal{O} \).

Assumption (HOL)

\[
E \int_K |S'_0|^2 dx + E \int_0^T \int_K (|f'_t(x)|^2 + |g'_t(x)|^2 + |h'_t(x)|^2) dx dt < \infty
\]
for any compact set \( K \subset \mathcal{O} \).

Assumption (HO\(\infty\)p)

\[
S'_0 \in L^\infty(\Omega \times \mathcal{O}) \quad \text{and} \quad E \left( (||f'||_{\infty,\infty;T})^p + (||g'||_{\infty,\infty;T})^{p/2} + (||h'||_{\infty,\infty;T})^{p/2} \right) < \infty.
\]

As our approach is based on some estimates of \( u - S' \) that we obtain thanks to the Itô formula, we need to introduce the following functions:

\[
\begin{align*}
\bar{f}(t, \omega, x, y, z) &= f(t, \omega, x, y + S'_t, z + \nabla S'_t) - f'(t, \omega, x) \\
\bar{g}(t, \omega, x, y, z) &= g(t, \omega, x, y + S'_t, z + \nabla S'_t) - g'(t, \omega, x) \\
\bar{h}(t, \omega, x, y, z) &= h(t, \omega, x, y + S'_t, z + \nabla S'_t) - h'(t, \omega, x).
\end{align*}
\]

And we consider:

Assumption (HD\(\theta\)p)

\[
E((||\bar{f}'||^*_{\theta;T})^p + (||\bar{g}'||^*_{\theta;T})^{p/2} + (||\bar{h}'||^*_{\theta;T})^{p/2}) < \infty.
\]

This assumption is fulfilled in the following case:

Example 1. If \( ||\nabla S'||_{\theta;T}^* \), \( ||f'||_{\theta;T}^* \), \( ||g'||_{\theta;T}^* \) and \( ||h'||_{\theta;T}^* \) belong to \( L^p(\Omega, P) \), and assumptions (H) and (HO\(\infty\)p) hold, then:

\( \bar{f} \) satisfies the Lipschitz condition with the same Lipschitz coefficients:

\[
|\bar{f}(t, \omega, x, y, z) - \bar{f}(t, \omega, x, y', z')| = |f(t, \omega, x, y + S'_t(x), z + \nabla S'_t(x)) + f'(t, \omega, x) - f(t, \omega, x, y' + S'_t(x), z' + \nabla S'_t(x)) - f'(t, \omega, x)| \leq C |y - y'| + C |z - z'|.
\]

\( \bar{f} \) satisfies the integrability condition:

\[
||\bar{f}'||^*_{\theta;T} = ||f(S', \nabla S') - f'||^*_{\theta;T} \leq ||f(S', \nabla S')||^*_{\theta;T} + ||f'||^*_{\theta;T} \leq ||f'||^*_{\theta;T} + C ||S'||^*_{\theta;T} + C ||\nabla S'||^*_{\theta;T} + ||f'||_{\infty,\infty;T}.
\]

And the same for \( \bar{g} \) and \( \bar{h} \), which proves that (HD\(\theta\)p) holds.
5.4. The main results

We now introduce the lateral boundary condition that we consider:

**Definition 9.** If \( u \) belongs to \( \mathcal{H}_{loc} \), we say that \( u \) is non-negative on the boundary of \( \mathcal{O} \) if \( u^+ \) belongs to \( \mathcal{H}_T \) and we denote it simply: \( u \leq 0 \) on \( \partial \mathcal{O} \). More generally, if \( M \) is a random field defined on \([0,T] \times \mathcal{O} \), we note \( u \leq M \) on \( \partial \mathcal{O} \) if \( u - M \leq 0 \) on \( \partial \mathcal{O} \).

From now on, we can follow step by step the proof of the maximum principle for OSPDE in the homogeneous case in [11]; the first step consists in establishing an estimate for the positive part of the solution with null Dirichlet condition. To get this estimate, we can adapt to our case the arguments of proof of Proposition 5.2 in [11], then Itô formula for the difference of 2 elements in \( \mathcal{R}_{loc}(\xi, f, g, h, S) \) (Proposition 5.3 in [11]). This yields the comparison theorem (see Theorem 5.4 in [11]):

**Theorem 6.** Assume that \( \partial \mathcal{O} \) is Lipschitz. Let \((\xi_i, f^i, g, h, S^i), i = 1,2, \) satisfy assumptions (H), (HIL), (OL) and (HOL). Consider \((u^i, \nu^i) \in \mathcal{R}_{loc}(\xi_i, f^i, g, h, S^i), i = 1,2\) and suppose that the process \((u^1 - u^2)^+ \) belongs to \( \mathcal{H}_T \) and that one has

\[
E \left( \| f^1 (., \nu, \nabla u^2) - f^2 (., \nu, \nabla u^2) \|^\star_{\#;t} \right)^2 < \infty, \quad \text{for all} \quad t \in [0,T].
\]

If \( \xi^1 \leq \xi^2 \) a.s., \( f^1 (t, \omega, u^2, \nabla u^2) \leq f^2 (t, \omega, u^2, \nabla u^2) \), \( dt \otimes dx \otimes dP \)-a.e. and \( S^1 \leq S^2 \), \( dt \otimes dx \otimes dP \)-a.s., then one has \( u^1(t,x) \leq u^2(t,x) \), \( dt \otimes dx \otimes dP \)-a.e.

By adapting the proof of Theorem 5.5 in [11], we get first the maximum principle in the case \( u \leq 0 \) on \( \partial \mathcal{O} \):

**Theorem 7.** Assume that \( \partial \mathcal{O} \) is Lipschitz and suppose that Assumptions (H), (OL), (HOL), (H\(\text{I}2\text{p}\)), (HO\(\infty\text{p}\)) and (HD\(\theta\text{p}\)) hold for some \( \theta \in [0,1[ \), \( p \geq 2 \) and that the constants of the Lipschitz conditions satisfy

\[
\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.
\]

Let \((u, \nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)\) be such that \( u^+ \in \mathcal{H} \). Then one has

\[
E \| u^+ \|^p_{\infty;\infty; \infty} \leq k(t)c(p)E \left( \| \xi^+ - S^0_0 \|^p + \left( \| g^{0+} \|_{\theta;\infty;\infty} \right)^p + \left( \| h^{0+} \|_{\theta;\infty;\infty} \right)^p \right) + \left( \| h^0 \|_{\theta;\infty;\infty} \right)^p
\]

where \( k(t) \) is constant that depends on the structure constants and \( t \in [0,T] \).

As in the homogeneous case (see Theorem 5.6 in [11]), we can generalize the previous result by considering a real Itô process of the form

\[
M_t = m + \int_0^t b_sds + \sum_{j=1}^{\infty} \int_0^t \sigma_{j,s}dB^j_s
\]

where \( m \) is a random variable and \( b = (b_t)_{t \geq 0}, \sigma = (\sigma_{1,t}, ..., \sigma_{n,t}, ...)_{t \geq 0} \) are adapted processes.
Theorem 8. Assume that $\partial\mathcal{O}$ is Lipschitz and suppose that Assumptions (H), (OL), (HOL), (HI2p), (HO∞p) and (HDθp) hold for some $\theta \in [0, 1]$, $p \geq 2$ and that the constants of the Lipschitz conditions satisfy

$$\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.$$ 

Assume also that $m$ and the processes $b$ and $\sigma$ satisfy the following integrability conditions

$$E|m|^p < \infty, \quad E\left(\int_0^t |b_s|^{1+p} \, ds\right)^{p(1-\theta)/2} < \infty, \quad E\left(\int_0^t |\sigma_s|^{1+p} \, ds\right)^{p(1-\theta)/2} < \infty,$$

for each $t \in [0, T]$. Let $(u, \nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)$ be such that $(u - M)^+$ belongs to $\mathcal{H}_T$. Then one has

$$E \|(u - M)^+\|_{p, \infty; \theta; t}^p \leq c(p)k(t)E\left[ \|\xi - m\|^p + (\|f^0, +\|_{\theta; t}^*)^p \right.$$

$$+ \left(\|g^0\|_{\theta; t}^* \right)^{\frac{q}{2}} + \left(\|\tilde{h}\|_{\theta; t}^* \right)^{\frac{q}{2}} + \left(\|S_0^f - m\|^p \right) \left(\|f' - b\|^p_{\theta; t} \right)^{\frac{q}{2}} + \left(\|g'\|^p_{\theta; t} \right)^{\frac{q}{2}} + \left(\|h' - \sigma\|^p_{\theta; t} \right)^{\frac{q}{2}} \left(\int_0^t |b_s|^{1+p} \, ds\right)^{p(1-\theta)/2}$$

$$+ \left(\int_0^t |\sigma_s|^{1+p} \, ds\right)^{p(1-\theta)/2} \right].$$

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