Complementarity and additivity for depolarizing channels

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Abstract

In this paper we use the method of the paper [10] to compute complementary channels for certain important cases, such as depolarizing and transpose-depolarizing channels. This method allows us to easily obtain the minimal Kraus representations from non-minimal ones. We also study the properties of the output purity of the tensor product of a channel and its complement.

1 Introduction

In the recent paper [10] complementarity between output and environment of a quantum channel (or, more generally, CP map) was explored in detail. It was observed that the output purity characteristics for mutually complementary CP maps coincide, making the validity of the multiplicativity/additivity conjecture for a class of CP maps equivalent to its validity for complementary maps. A similar observation was independently made in [12] in the context of channels. In [10] a regular method for computation of complementary maps was proposed, thus providing an efficient construction of new cases for the solution of the multiplicativity/additivity problem. In this paper we use this method to compute complementary channels for certain important cases,
such as depolarizing and transpose-depolarizing channels. This method easily yields minimal Kraus representations from non–minimal ones. We also study the properties of the output purity of the tensor product of a channel and its complement.

Let us fix some notation. \( \mathcal{M}(\mathcal{H}) \) will denote the algebra of all operators, and \( \mathcal{S}(\mathcal{H}) \) – the convex set of all density operators (quantum states) in a finite-dimensional Hilbert space \( \mathcal{H} \). The output purity of a CP map \( \Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}') \), is measured by the quantity

\[
\nu_p(\Phi) := \max_{\rho \in \mathcal{S}(\mathcal{H})} \{||\Phi(\rho)||_p\}, \quad 1 \leq p < \infty, \tag{1}
\]

where \( ||\Phi(\rho)||_p = \left[\text{Tr} (\Phi(\rho))^p\right]^{1/p} \) is the \( p \)-norm of \( \Phi(\rho) \), or equivalently, by the minimal output Rényi \( p \)-entropy

\[
\tilde{R}_p(\Phi) = -\frac{p}{p-1} \log \nu_p(\Phi).
\]

Recall that the Rényi \( p \)-entropy of a density matrix \( \sigma, p > 1 \), is defined as

\[
R_p(\sigma) := -\frac{1}{p-1} \log (\text{Tr} \sigma^p) = -\frac{p}{p-1} \log ||\sigma||_p. \tag{2}
\]

The Rényi entropies have the monotonicity property \([2]\)

\[
R_q(\sigma) \leq R_p(\sigma), \quad 1 < p \leq q.
\]

In the limit \( p \to 1 \), they converge monotonically and hence uniformly to the von Neumann entropy \( H(\sigma) = -\text{Tr} \sigma \log \sigma \). Therefore we can extend the notation of the Rényi entropy by letting \( R_1(\sigma) := H(\sigma) \). The minimal output Rényi \( p \)-entropy of a channel \( \Phi \) is then

\[
\tilde{R}_p(\Phi) = \min_{\rho \in \mathcal{S}(\mathcal{H})} R_p(\Phi(\rho)),
\]

and for \( p = 1 \) it is equal to the minimum output entropy

\[
\tilde{H}(\Phi) := \min_{\rho} H(\Phi(\rho)). \tag{3}
\]
2 Representations of CP maps

Given three Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ and a linear operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, the relations
\[ \Phi(\rho) = \text{Tr}_{\mathcal{H}_C} V \rho V^*; \quad \tilde{\Phi}(\rho) = \text{Tr}_{\mathcal{H}_B} V \rho V^*; \quad \rho \in \mathcal{M}(\mathcal{H}_A), \]
(4)
define two CP maps $\Phi : \mathcal{M}(\mathcal{H}_A) \to \mathcal{M}(\mathcal{H}_B)$, $\tilde{\Phi} : \mathcal{M}(\mathcal{H}_A) \to \mathcal{M}(\mathcal{H}_C)$, which will be called mutually complementary. If $V$ is an isometry, both maps are trace preserving (TP) i.e. channels.

For any linear map $\Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}')$ the dual map $\Phi^* : \mathcal{M}(\mathcal{H}') \to \mathcal{M}(\mathcal{H})$ is defined by the formula
\[ \text{Tr} \Phi(\rho) X = \text{Tr} \rho \Phi^*(X); \quad \rho \in \mathcal{M}(\mathcal{H}), X \in \mathcal{M}(\mathcal{H}'). \]
If $\Phi$ is CP, then $\Phi^*$ is also CP. Relations (4) are equivalent to
\[ \Phi^*(X) = V^*(X \otimes I_C)V; \quad X \in \mathcal{M}(\mathcal{H}_B), \]
(5)
\[ \tilde{\Phi}^*(X) = V^*(I_B \otimes X)V; \quad X \in \mathcal{M}(\mathcal{H}_C), \]
(6)where $I$ is the identity operator in the corresponding Hilbert space. Considering $\tilde{\Phi}$ as dual to CP map $\Phi^*$, we conclude that there should also be a representation of the form
\[ \tilde{\Phi}(\rho) = S_C(\rho \otimes I_B)S_C^*, \]
(7)where $S_C : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ (in the case of channel $\text{Tr}_{\mathcal{H}_B} S_C^* S_C = I_A$). There is a simple general relation between this representation and the second formula in (4) for an arbitrary CP map; namely, given $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$ choose an orthonormal basis $\{e_j^B\}$ in $\mathcal{H}_B$ and define $S_C : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ by the relation $\langle e_j^B | V = S_C | e_j^B \rangle$, or, more precisely,
\[ \langle \tilde{\psi}_B \otimes \psi_C | V | \psi_A \rangle = \langle \psi_C | S_C | \psi_A \otimes \psi_B \rangle, \]
where $\tilde{\psi}_B$ denotes the complex conjugate of $\psi_B$ in the basis $\{e_j^B\}$. Alternatively, introducing the maximally entangled vector
\[ |\Omega^{BB}\rangle = \frac{1}{\sqrt{d_B}} \sum_{j=1}^{d_B} |e_j^B\rangle \otimes |e_j^B\rangle, \]
3
in $\mathcal{H}_B \otimes \mathcal{H}_B$, we have the reciprocity relations

$$S_C = \sqrt{d_B} \langle \Omega^{BB} \vert (V \otimes I_B); \quad V = \sqrt{d_B} (I_B \otimes S_C) \vert \Omega^{BB} \rangle.$$ 

The representation (7) is in fact nothing but the dual form (6) of the Stinespring representation for the map $\tilde{\Phi}$, if it is considered (somewhat “unnaturally”) as a map in the Heisenberg–rather than in the Schrödinger picture. To give a kind of physical interpretation to the representation (7), consider the polar decomposition

$$S_C = \vert S_C^* \vert W,$$

where $\vert S_C^* \vert = \sqrt{S_C S_C^*}$ is a Hermitian operator in $\mathcal{H}_C$ and $W : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ is a partial isometry. Denote $D_C = \sqrt{d_B} \vert S_C^* \vert$ and choose the basis in which this operator is diagonal. Then (7) takes the form

$$\tilde{\Phi}(\rho) = D_C W \left( \rho \otimes \frac{I_B}{d_B} \right) W^* D_C,$$

and (with some strain) can be interpreted as an interaction of the system $A$ with environment $B$ in the chaotic state followed by partial dephasing, cf. [6]. Note however, that the “interaction” is only partially unitary and the dephasing CP map is in general not TP (i.e. channel).

By interchanging the roles of $\mathcal{H}_B, \mathcal{H}_C$ we of course obtain a similar representation for the initial map $\Phi$

$$\Phi(\rho) = S_B (\rho \otimes I_C) S_B^*,$$

where $S_B : \mathcal{H}_A \otimes \mathcal{H}_C \to \mathcal{H}_B$ (in the case of channel $\text{Tr}_{\mathcal{H}_C} S_B^* S_B = I_A$). This representation is especially nice in the case where $A = B$ and $\Phi$ is unital: then $S_B$ is co-isometry, $S_B S_B^* = I_A$.

The Kraus representation for the map $\Phi$

$$\Phi(\rho) = \sum_{j=1}^{d_C} V_j \rho V_j^*; \quad \rho \in \mathcal{M}(\mathcal{H}_A),$$

follows from (4) by letting $V_j = \langle e_j^C \vert V$ where $\{e_j^C\}$ is an orthonormal basis in $\mathcal{H}_C$. Conversely,

$$V = \sum_{j=1}^{d_C} V_j \otimes \vert e_j^C \rangle,$$
whence, applying the second relation in (4), we have explicit formula for the complementary map

\[
\tilde{\Phi}(\rho) = \sum_{j,k=1}^{dc} |e_j^C\rangle\langle e_k^C| \text{Tr} V_j \rho V_k^*; \quad \rho \in \mathcal{M}(\mathcal{H}_A).
\]  

(9)

It follows that the Kraus representation for $\tilde{\Phi}$ is

\[
\tilde{\Phi}(\rho) = \sum_{k=1}^{dB} \tilde{V}_k \rho \tilde{V}_k^*;
\]

where $\tilde{V}_k : \mathcal{H}_A \rightarrow \mathcal{H}_C$ are given by

\[
\tilde{V}_k = \sum_{j=1}^{dc} \langle e_j^B| V_j \otimes |e_j^C\rangle,
\]

and hence satisfy

\[
\langle e_j^C| \tilde{V}_k = \langle e_k^B| V_j.
\]  

(10)

The representation (7) takes place with

\[
S_C = \sum_{k=1}^{dB} \tilde{V}_k \otimes \langle e_k^B|.
\]

Finally, consider the case where $A = B$ and $\Phi$ is unital, which is equivalent to

\[
\sum_{j=1}^{dc} V_j V_j^* = I_A.
\]

By using (10) we obtain that this is the same as

\[
\text{Tr} \tilde{V}_j^* \tilde{V}_k = \delta_{jk}.
\]

Since $S_C^*S_C = \sum_{j,k=1}^{dB} \tilde{V}_j^* \tilde{V}_k \otimes |e_j^B\rangle\langle e_k^B|$, this is equivalent to $\text{Tr}_{\mathcal{H}_B} S_C^*S_C = I_A$. 

5
3 Depolarizing channel

Consider the depolarizing channel

$$\Phi(\rho) = (1 - p)\rho + \frac{p}{d} I \text{Tr}\rho, \quad 0 \leq p \leq \frac{d^2}{d^2 - 1},$$

(11)

where $\rho \in \mathcal{M}(\mathcal{H})$, with $\mathcal{H} \simeq \mathbb{C}^d$. If $\{|j\rangle : j = 1, \ldots, d\}$ is a complete set of orthonormal basis vectors in $\mathcal{H}$, then writing the channel as

$$\Phi(\rho) = (1 - p)\rho + \frac{p}{d} \sum_{i,j=1}^d |i\rangle\langle j| \rho |j\rangle\langle i|,$$

yields a Kraus representation with the operators

$$V_0 = \sqrt{1-p}I, \quad V_{ij} = \sqrt{\frac{p}{d}} |i\rangle\langle j|.$$

Let us relabel these Kraus operators as follows. Define a variable

$$c(i,j) := i + (j - 1)d, \quad 1 \leq i, j \leq d$$

which takes integer values from 1 to $d^2$. Then the Kraus operators can be denoted as $A_k$, $0 \leq k \leq d^2$, where

$$A_0 := V_0 \quad \text{and} \quad A_{c(i,j)} := V_{ji} \quad 1 \leq i, j \leq d.$$

(12)

Note that $A^*_{c(i,j)} = V_{ji}^* = V_{ij}$. The channel complementary to the depolarizing channel is given by [10]

$$\tilde{\Phi}(\rho) = \left[ \text{Tr} A_\alpha \rho A_\beta^* \right]_{\alpha,\beta = 0,1,...,d^2}.$$

It is easy to see that

$$\text{Tr} A_0 \rho A_0^* = (1 - p)\text{Tr} (I \rho I) = (1 - p)\text{Tr}\rho;$$

$$\text{Tr} A_0 \rho A_{c(i,j)}^* = \sqrt{(1-p)} \text{Tr}(\rho V_{ij}) = \sqrt{\frac{p(1-p)}{d}} (j|\rho|i);$$

$$\text{Tr} A_{c(i,j)} \rho A_0^* = \sqrt{\frac{p(1-p)}{d}} (i|\rho|j);$$

$$\text{Tr} A_{c(i,j)} \rho A_{c(i',j')}^* = \text{Tr} V_{i'j'} \rho V_{i'j'} = \frac{p}{d} (i|\rho|i') \delta_{jj'}$$

(13)
To express the complementary channel in a compact form, let us define a $d^2$-dimensional row vector\(^1\)

$$\vec{\rho} := \sum_{i,j=1}^{d^2} \rho_{ji} \langle ij |,$$

where $\rho_{ji} = \langle j | \rho | i \rangle$. (14)

In terms of this vector and its transpose $\vec{\rho}^T$, the complementary channel $\tilde{\Phi}(\rho)$ can be represented by a $(d^2 + 1) \times (d^2 + 1)$ matrix

$$\tilde{\Phi}(\rho) = \begin{bmatrix} \sqrt{p(1-p)} d \rho \sqrt{p(1-p)} \\ \sqrt{p(1-p)} \rho^T \end{bmatrix}. \quad (15)$$

This representation is not minimal since the number of Kraus operators $A_k$ (defined by (12)) is $d^2 + 1$. However, a minimal representation for $\tilde{\Phi}$ can be obtained from (15) as follows. Note that (15) can be equivalently written as

$$\tilde{\Phi}(\rho) = T \rho T^*,$$

where

$$T^* = \begin{bmatrix} \sqrt{d(1-p)} |\Omega_{12}\rangle & \sqrt{\frac{d}{d^2 - 1}} I_{12} \end{bmatrix}.$$

with $|\Omega_{12}\rangle = d^{-1/2} \sum_{j=1}^{d} |jj\rangle$ the maximally entangled vector in $\mathcal{H} \otimes \mathcal{H}$ and $I_{12}$ is the identity operator in $\mathcal{H} \otimes \mathcal{H}$. Let $T = US$ be its polar decomposition, where $S = |T|$ is a positive Hermitian operator in $\mathcal{H} \otimes \mathcal{H} \simeq \mathcal{H}_{d^2}$ and $U$ is an isometry from $\mathcal{H}_{d^2}$ to $\mathcal{H}_{d^2+1}$, which is irrelevant for the minimal representation we are looking for. Since

$$T^* T = \frac{d}{d^2} I_{12} + d(1-p) |\Omega_{12}\rangle \langle \Omega_{12}|$$

is easily diagonalizable, we find

$$S = \sqrt{T^* T} = \sqrt{\frac{d}{d^2}} I_{12} + \sqrt{d} \left( -\frac{\sqrt{d}}{d^2} + \sqrt{1 - p \left( \frac{d^2 - 1}{d^2} \right)} \right) |\Omega_{12}\rangle \langle \Omega_{12}|,$$

and the minimal representation of the complementary channel is

$$\tilde{\Phi}(\rho) = S (\rho \otimes I) S^* \quad (16)$$

\(^1\)Here and henceforth, we use the notation $|ij\rangle$ to denote the vector $|i\rangle \otimes |j\rangle$. Consequently, $\langle ij | = \langle i \otimes |j\rangle$. 

7
While the depolarizing channel is globally unitarily covariant, the complementary channel has the covariance property
\[
\tilde{\Phi}[U \rho U^*] = (U \otimes \tilde{U})\tilde{\Phi}[\rho](U \otimes \tilde{U})^*
\]
for arbitrary unitary operator \(U\) in \(\mathcal{H}\).

By the results in [10], [12], the complementary channel (16) has the same multiplicativity/additivity properties as the depolarizing channel established in [11].

### 4 Transpose-depolarizing channel

Consider the one-parameter family of channels in \(\mathcal{H} \simeq \mathbb{C}^d\)

\[
\Phi(\rho) = t\rho^T + (1-t)\text{Tr}\rho\frac{I}{d},
\]

where

\[
-\frac{1}{d-1} \leq t \leq \frac{1}{d+1}.
\]

Here \(\rho^T\) denotes transpose of the matrix \(\rho\) in a fixed basis. The channel \(\Phi\) is irreducibly covariant since for any arbitrary unitary transformation \(U\)

\[
\Phi(U \rho U^*) = \tilde{U}\Phi(\rho)\tilde{U}^*,
\]

where \(\tilde{U}\) is the complex conjugate of \(U\) in the fixed basis. For this class of channels, additivity of the minimum output entropy and the multiplicativity of its maximal \(p\)-norm for \(1 \leq p \leq 2\), has been proved in [7, 5, 4]. As it was shown in [5], this channel can also be written as

\[
\Phi(\rho) = c^+\Phi^+(\rho) + c^-\Phi^-(\rho),
\]

where

\[
c^\pm = \left(\frac{d^2-1}{2d}\right)\left(\frac{1}{d\mp 1} \pm t\right),
\]

and

\[
\Phi^\pm(\rho) := \frac{1}{d \pm 1} (I\text{Tr}\rho \pm \rho^T).
\]
Note that the extreme channel $\Phi^{-}(\rho)$ is the well known Werner-Holevo (WH) channel [15]. The channels $\Phi^{\pm}(\rho)$ have Kraus operators

$$V_{ij}^{\pm} := \frac{1}{\sqrt{2(d \pm 1)}}(|i\rangle\langle j| \pm |j\rangle\langle i|),$$

where $|i\rangle, |j\rangle$ denote orthonormal basis vectors in $\mathcal{H}$. Let us relabel these operators using the variable

$$c(i, j) = i + (j - 1)d, \quad 1 \leq i \leq d, \quad 1 \leq j \leq 2d.$$

and the relations

$$A^{+}_{c(i, j)} = \sqrt{c^{+}}V^{+}_{ji} \quad \text{for} \quad 1 \leq i, j \leq d;$$

$$A^{-}_{c(i, j)} = \sqrt{c^{-}}V^{-}_{(d-j)i} \quad \text{for} \quad 1 \leq i \leq d, \quad (d+1) \leq j \leq 2d.$$

Note that $c(i, j)$ takes integer values from 1 to $2d^2$. In terms of the above operators, the Kraus operators of the transpose depolarizing channel $\Phi$, (20), can be expressed as

$$A_{c(i, j)} := A^{+}_{c(i, j)}\mathcal{I}(1 \leq j \leq d) + A^{-}_{c(i, j)}\mathcal{I}(d + 1 \leq j \leq 2d),$$

where $\mathcal{I}(\cdot)$ denotes an indicator function. Its complementary channel is given by

$$\tilde{\Phi}(\rho) := \left[\text{Tr}A_{\alpha}^{+}\rho A_{\beta}^{+}\right]_{\alpha, \beta = 1, \ldots, 2d^2}.$$

Let us first consider the case $1 \leq \alpha, \beta \leq d^2$, for which $\alpha = c(i, j)$ and $\beta = c(i', j')$ for some $1 \leq i, i' \leq d$ and $1 \leq j, j' \leq d$. From (23) it follows that

$$\text{Tr}A_{c(i, j)}^{+}\rho A_{c(i', j')}^{+} = \text{Tr}c^{+}V^{+}_{ji}\rho V^{+}_{i'j'}$$

$$= \frac{c^{+}}{2(d + 1)}\left[\delta_{jj'}\rho_{ii'} + \delta_{jj'}\rho_{ij'} + \delta_{ij'}\rho_{i'j'} + \delta_{i'j}\rho_{ij'}\right].$$

$$= \frac{c^{+}}{2(d + 1)}\left[\rho \otimes I + (\rho \otimes I)F + F(\rho \otimes I) + F(\rho \otimes I)F\right]_{ij, i'j'}$$

$$= \frac{c^{+}}{2(d + 1)}\left[(I_{12} + F)(\rho \otimes I)(I_{12} + F)\right]_{ij, i'j'}$$

Here the flip operator $F$ is defined by its action

$$F|i\rangle = |ji\rangle.$$
on basis vectors $|ij⟩$ in $\mathcal{H} \otimes \mathcal{H}$ and $I_{12}$ is the identity operator in $\mathcal{H} \otimes \mathcal{H}$. Moreover, $\rho_{ij} := \langle i | \rho | j \rangle$.

Similarly, for $(d^2 + 1) \leq \alpha, \beta \leq 2d^2$ we have $\alpha = c(i, j)$ and $\beta = c(i', j')$ for some $1 \leq i, i' \leq d$ and $d + 1 \leq j, j' \leq 2d$. Defining $j = j - d$ and $j' = j' - d$, we get

$$\text{Tr} A_{c(i,j)} \rho A_{c(i',j')}^* = \text{Tr} e^{-V_{ji}} \rho V_{i'j'}^- = \frac{c^-}{2(d-1)} [\delta_{j'j} \rho_{ii'} - \delta_{ji'} \rho_{ij'} + \delta_{ij'} \rho_{j'j} - \delta_{ij} \rho_{j'i'} - \delta_{ji'} \rho_{ij'} + \delta_{ij'} \rho_{j'i'}]$$

$$= \frac{c^-}{2(d-1)} [(I_{12} - F)(\rho \otimes I)(I_{12} - F)]_{ij,i'j'}.$$

For $\alpha = c(i, j), \beta = c(i', j')$ for some $1 \leq i, i', j, j' \leq d$ and $d + 1 \leq j, j' \leq 2d$, we have

$$\text{Tr} A_{c(i,j)} \rho A_{c(i',j')}^* = \text{Tr} \sqrt{c^+ c^-} V_{ji}^+ \rho V_{i'j'}^- = \frac{1}{2} \sqrt{\frac{c^+ c^-}{(d+1)(d-1)}} [(I_{12} + F)(\rho \otimes I)(I_{12} - F)]_{ij,i'j'}.$$

By symmetry, for $\alpha = c(i, j), \beta = c(i', j')$ for some $1 \leq i, i', j, j' \leq d$ and $d + 1 \leq j, j' \leq 2d$, we have

$$\text{Tr} A_{c(i,j)} \rho A_{c(i',j')}^* = \frac{1}{2} \sqrt{\frac{c^+ c^-}{(d+1)(d-1)}} [(I_{12} - F)(\rho \otimes I)(I_{12} + F)]_{ij,i'j'}.$$

From the above relations one concludes that the complementary channel of the transpose-depolarizing channel has the (non–minimal) representation

$$\tilde{\Phi}(\rho) = T(\rho \otimes I)T^*,$$

where

$$T^* = [a^+(I_{12} + F) \quad a^-(I_{12} - F)].$$

with

$$a^\pm := \sqrt{\frac{c^\pm}{2(d \pm 1)}}.$$

Let $T = US$ denote the polar decomposition of the matrix $T$, where $S = |T| = \sqrt{T^*T}$ is a positive Hermitian operator in $\mathcal{H} \otimes \mathcal{H} \simeq \mathcal{H}_{d^2}$ and $U$
is an isometry from $\mathcal{H}_d$ to $\mathcal{H}_{2d}$. By using the fact that $(I_{12} \pm F)/2$ are projection operators we obtain the minimal representation

$$\tilde{\Phi}(\rho) = S(\rho \otimes I)S^*, \quad (24)$$

where

$$S = \sqrt{T^*T} = (a^+ + a^-)I_{12} + (a^+ - a^-)F.$$

The covariance property of the channel (24) is

$$\tilde{\Phi}(U\rho U^*) = (U \otimes U)\tilde{\Phi}(\rho)(U^* \otimes U^*),$$

as follows from the fact that $F(U \otimes U) = (U \otimes U)F$.

### 5 Coupling channel with its complementary

Let us now study the properties of a channel which is a tensor product of the WH channel

$$\Phi(\rho) := \frac{1}{d-1} (ITr\rho - \rho T), \quad (25)$$

and the complementary channel

$$\tilde{\Phi}(\rho) = \frac{1}{2(d-1)}(I_{12} - F)(\rho \otimes I)(I_{12} - F). \quad (26)$$

The particular significance of the WH channel lies in the fact that it provides a counterexample for the multiplicativity of the maximal output $p$-norm for $p > 4.79$ and $d = 3$ [15]. It is interesting to investigate whether a similar violation of multiplicativity is exhibited for the product channel $\Phi \otimes \tilde{\Phi}$. The multiplicativity of the maximal output $p$-norm and hence, additivity of the minimum output Rényi $p-$ entropies of the WH channel for $p \in [1, 2]$ was established in [13, 1, 3]. It is also interesting to study whether these additivity properties hold for the channel $\Phi \otimes \tilde{\Phi}$.

For the WH channel, $\tilde{R}_p(\Phi) = \tilde{R}_p(\tilde{\Phi}) = \log(d - 1)$ for all $p \geq 1$, since $\nu_p(\Phi) = (d - 1)^{(1-p)/p}$ as shown in [15]. Further, it was observed in [14] that if for some channel $\Phi$

$$\tilde{R}_p(\Phi) = \tilde{R}_q(\Phi) \quad \text{for} \ 1 \leq q \leq p,$$
then the additivity of the minimal output Rényi $p$-entropy implies the additivity of the minimal output Rényi $q$-entropy. By using these facts, the proof of the additivity relation

$$\tilde{R}_p(\Phi \otimes \tilde{\Phi}) = \tilde{R}_p(\Phi) + \tilde{R}_p(\tilde{\Phi})$$

(27)

reduces to proving

$$\tilde{R}_2(\Phi \otimes \tilde{\Phi}) = 2\tilde{R}_2(\Phi).$$

(28)

We can restate the additivity conjecture (28) as a multiplicativity of maximal $2$–norms

$$\nu_2(\Phi \otimes \tilde{\Phi}) = \nu_2(\Phi) \nu_2(\tilde{\Phi}) = \nu_2(\Phi)^2,$$

(29)

where

$$\nu_2(\Phi \otimes \tilde{\Phi}) := \max_{|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \ |\psi_{12}\rangle = 1} \left\{ \||\Phi \otimes \tilde{\Phi})(|\psi_{12}\rangle\langle\psi_{12}|)\||_p \right\},$$

(30)

and we have made use of the relation $\nu_2(\tilde{\Phi}) = \nu_2(\Phi)$ [10]. To prove (29), it is sufficient to show that the maximum on the right hand side of (30) is achieved for unentangled vectors $|\psi_{12}\rangle$, which in turn corresponds to the reduced states $\rho_1 := \text{Tr}_{\mathcal{H}_2} |\psi_{12}\rangle\langle\psi_{12}|$ and $\rho_2 := \text{Tr}_{\mathcal{H}_1} |\psi_{12}\rangle\langle\psi_{12}|$ being pure.

Let the output of the product channel for an arbitrary pure input state $|\psi_{12}\rangle\langle\psi_{12}| \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, be denoted by

$$\Omega := (\Phi \otimes \tilde{\Phi})(|\psi_{12}\rangle\langle\psi_{12}|) = (\text{Id} \otimes \tilde{\Phi})(\Phi \otimes \text{Id})(|\psi_{12}\rangle\langle\psi_{12}|),$$

(31)

where $\text{Id}$ is the identity channel. Due to the unitary covariance of the channel $\Phi \otimes \tilde{\Phi}$, the state vector $|\psi_{12}\rangle$ can be chosen as

$$|\psi_{12}\rangle = \sum_{j=1}^d \sqrt{\lambda_j} |j\rangle \otimes |j\rangle,$$

(32)

where $\{|j\rangle\}$ is the fixed orthonormal basis in $\mathbb{C}^d$ (one which defines the transposition), $\lambda_j \geq 0$ and $\sum_{j=1}^d \lambda_j = 1$. The reduced density matrices $\rho_i$, $i = 1, 2$ are therefore given by

$$\rho := \rho_1 = \sum_{j=1}^d \lambda_j |j\rangle \langle j| = \rho_2.$$

(33)
Using the decomposition (32) we find that
\[
(\Phi \otimes \text{Id})(|\psi_{12}\rangle\langle\psi_{12}|) = \sum_{j,k} \sqrt{\lambda_j \lambda_k} \Phi(|j\rangle\langle k|) \otimes |j\rangle\langle k|.
\]

From the definition (21) of the WH channel it follows that
\[
\Phi(|j\rangle\langle k|) = \frac{1}{d-1} (I \delta_{jk} - |k\rangle\langle j|),
\]
which in turn implies that
\[
(\Phi \otimes \text{Id})(|\psi_{12}\rangle\langle\psi_{12}|) = \frac{1}{d-1} \left[ \sum_j \lambda_j I \otimes |j\rangle\langle j| - \sum_{j,k} \sqrt{\lambda_j \lambda_k} |kj\rangle\langle jk| \right]
\]
\[
= \frac{1}{d-1} \left[ I_{12} \otimes \rho - F(\sqrt{\rho} \otimes \sqrt{\rho}) \right],
\]
where \(\rho\) is given by (33) and hence \(\sqrt{\rho} = \sum_j \sqrt{\lambda_j} |j\rangle\langle j|\).

Due to the relation \(F(I \otimes \rho) = (\rho \otimes I) F\), the complementary channel (26) can be alternatively expressed in the following forms:
\[
\tilde{\Phi}(\rho) = \frac{1}{d-1} \left( \frac{I_{12} - F}{2} \right) (\rho \otimes I + I \otimes \rho)
\]
\[
= \frac{1}{d-1} \left( \frac{I_{12} - F}{2} \right) (\rho \otimes I + F(\rho \otimes I) F)
\]

Using the above relations we get
\[
\Omega = (\text{Id} \otimes \tilde{\Phi}) \left[ \frac{1}{d-1} \left( I_{12} \otimes \rho - F_{12}(\sqrt{\rho} \otimes \sqrt{\rho}) \right) \right]
\]
\[
= \frac{1}{(d-1)^2} \left( \frac{I_{123} - F_{23}}{2} \right) \left[ I \otimes \rho \otimes I + I \otimes I \otimes \rho \right.
\]
\[\left. - F_{12}(\sqrt{\rho} \otimes \sqrt{\rho} \otimes I) - F_{23}(F_{12}(\sqrt{\rho} \otimes \sqrt{\rho} \otimes I)) F_{23} \right],
\]
where we have defined
\[
I_{123} = I_{12} \otimes I, \quad F_{23} := I \otimes F, \quad F_{12} := F \otimes I.
\]
we can now evaluate \(\text{Tr}\Omega^2\) by employing the spectral decompositions of \(\rho\) (and hence of \(\sqrt{\rho}\)), the resolution of the identity \(I = \sum_k |k\rangle\langle k|\), and the explicit actions of the operators \(F_{12}\) and \(F_{23}\) on basis vectors, namely,
\[
F_{12}|ijk\rangle = |jik\rangle; \quad F_{23}|ijk\rangle = |ikj\rangle,
\]
where \(|ijk\rangle := |i\rangle \otimes |j\rangle \otimes |k\rangle\). This calculation yields

\[ \text{Tr}\Omega^2 = \frac{1}{(d-1)^4} [(d^2 - 4d + 5)\text{Tr}\rho^2 + 2(d - 2)], \]

which is indeed maximised when \(\text{Tr}\rho^2 = 1\), i.e., when \(\rho\) is a pure state. Thus we see that, for the product channel \(\Phi \otimes \tilde{\Phi}\), the multiplicativity (29) of the 2–norms and hence the additivity (27) of the minimum output entropy holds.

To investigate violation of multiplicativity for the product channel \(\Phi \otimes \tilde{\Phi}\), let us consider the output \(\Omega^{me}\) of this channel when the input is the maximally entangled state \(|\psi_{me}\rangle\langle\psi_{me}|\).

\[ |\psi_{me}\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |jj\rangle. \]

In this case the reduced density matrix \(\rho\), defined by (33), is the completely mixed state: \(\rho = I/d\). Hence, \(\Omega^{me}\) is simply obtained from (34) by replacing \(\rho\) by \(I/d\) on its right hand side. This yields the relation

\[ \Omega^{me} = \frac{1}{d(d-1)^2} \left[ P_1 + \frac{1}{2} (I_{123} - F_{23} - F_{12} - F_{23}F_{12} + F_{23}F_{12} + F_{12}F_{23}) \right]. \]

where \(P_1 := (I_{123} - F_{23})/2\) is a projection operator.

Let us express \(\Omega^{me}\) in a more transparent form, in order to evaluate its eigenvalues. For this purpose, define a vector

\[ |\phi_{\{ijk\}}\rangle := \frac{1}{\sqrt{6}} [ |ijk\rangle + |jki\rangle + |kij\rangle - |jik\rangle - |kj\rangle - |ikj\rangle]. \]

It is of unit norm and satisfies the relations

\[ F_{12}|\phi_{\{ijk\}}\rangle = -|\phi_{\{ijk\}}\rangle \]
\[ F_{23}|\phi_{\{ijk\}}\rangle = -|\phi_{\{ijk\}}\rangle \]

Moreover,

\[ \langle \phi_{\{ijk\}} | \phi_{\{i'j'k'\}} \rangle = 0 \text{ unless } \{ijk\} = \{i'j'k'\}, \]

and hence the set of vectors

\[ \{ |\phi_{\{ijk\}}\rangle : i, j, k \in \{1, 2, \ldots, d\}, i, j, k \text{ all different} \} \]
form an orthonormal set. Therefore

\[ P_2 := \sum_{\{ijk\}, i,j,k \in \{1,2,\ldots,d\} \text{ all different}} |\phi_{\{ijk\}}\rangle\langle \phi_{\{ijk\}}|, \]

is a projection operator. Moreover

\[ \text{ran } P_2 \subset \text{ran } P_1. \]

It is easy to see that

\[ I_{123} - F_{23} - F_{12} - F_{23}F_{12}F_{23} + F_{23}F_{12} + F_{12}F_{23} = 6P_2. \]

Hence,

\[ \Omega^{me} = \frac{1}{d(d-1)^2} [P_1 + 3P_2], \]

and its eigenvalues are

1. \(4/d(d-1)^2\) with multiplicity

\[
\binom{d}{3} \equiv \text{number of distinct subsets } \{ijk\} \text{ of the set } \{1,2,\ldots,d\};
\]

2. \(1/d(d-1)^2\) with multiplicity

\[ \dim \text{ (range of } P_1 \setminus P_2) = \frac{d^2(d-1)}{2} - \binom{d}{3} = \frac{d(d^2-1)}{3} \]

3. \(0\) with multiplicity \(d(d+1)/2\)

For \(d = 3\), therefore, there is a non-degenerate eigenvalue of \(1/3\), the eigenvalue \(1/12\) with multiplicity \(8\), and the eigenvalue \(0\) with multiplicity \(6\). The non–zero eigenvalues are found to be exactly identical those of the channel \(\Phi \otimes \Phi\) for \(d = 3\), (see [15]), for which a violation of the multiplicativity of the maximal output \(p\)–norm was obtained for \(p > 4.79\). Hence, we deduce that a similar violation of multiplicativity is exhibited for the channel \(\Phi \otimes \tilde{\Phi}\) for \(p > 4.79\) and \(d = 3\).
For $d \geq 4$ we get

$$\nu_p(\Phi \otimes \tilde{\Phi})^p/\nu_p(\Phi)^p \nu_p(\tilde{\Phi})^p \geq \frac{1}{(d-1)^2} \left[ \left( \frac{d}{3} \right)^p (4/d)^p + \frac{d(d^2 - 1)}{3} (1/d)^p \right],$$

but the right hand side is always less or equal than 1 for $p \geq 1$, so contrary to the case of $\Phi \otimes \Phi$, considering the output for the maximally entangled state does not allow us to conclude violation of multiplicativity. However, this might be due to the fact that the channel $\Phi \otimes \tilde{\Phi}$ does not have the flip symmetry of the channel $\Phi \otimes \Phi$, and the maximizing input state could be different from the maximally entangled one.

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