PARITIES OF $v$-DECOMPOSITION NUMBERS AND
AN APPLICATION TO SYMMETRIC GROUP ALGEBRAS

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Abstract. We prove that the $v$-decomposition number $d_{\lambda\mu}(v)$ is an even or odd polynomial according to whether the partitions $\lambda$ and $\mu$ have the same relative sign (or parity) or not. We then use this result to verify Martin’s conjecture for weight 3 blocks of symmetric group algebras — that these blocks have the property that their projective (indecomposable) modules have a common radical length 7.

1. Introduction

Throughout this paper, let $v$ be an indeterminate, and let $e$ be an integer greater than 1. The Fock space representation $\mathcal{F}$ of $U_v(\hat{\mathfrak{sl}}_e)$, as a $C(v)$-vector space, has two distinguished bases, the standard basis $\{s(\lambda) \mid \lambda \in \mathcal{P}\}$ and the canonical basis $\{G(\lambda) \mid \lambda \in \mathcal{P}\}$, both being indexed by the set $\mathcal{P}$ of all partitions of non-negative integers. The $v$-decomposition number $d_{\lambda\mu}(v) \in C(v)$ is the coefficient of $s(\lambda)$ when the canonical basis element $G(\mu)$ is expressed in terms of the standard basis elements, i.e.

$$G(\mu) = \sum_{\lambda \in \mathcal{P}} d_{\lambda\mu}(v)s(\lambda).$$

Varagnolo and Vasserot [19] showed that the $v$-decomposition numbers are parabolic Kazhdan-Lusztig’s polynomials. As an immediate consequence, a non-zero $v$-decomposition number is a sum of monomials in $v$, either all of which are of odd degree or of even degree. In the first part of this paper, we provide a combinatorial criteria, in terms of relative signs (or parities) of the partitions $\lambda$ and $\mu$, that determines exactly which of the two possibilities occurs.

In the second part of this paper, we investigate a conjecture of Martin concerning the blocks of symmetric group algebras with Abelian defect groups. A block of a symmetric group algebra in positive characteristic $p$ is parametrised by a pair $(\kappa, w)$, where $\kappa$ is a $p$-core partition and $w$ is a non-negative integer called its $(p)$-weight of the block. The defect group of such a block is Abelian if and only if $w < p$. Martin’s conjecture [11] asserts that in such a case, the projective (indecomposable) modules of this block have a common radical length $2w + 1$. This conjecture have been verified for $w \leq 2$, and for the case of $w = 3$, Martin and the author [13, 14] showed

Date: September 2006.

2000 Mathematics Subject Classification. 17B37, 20C30.

The author, supported by Academic Research Fund R-146-000-043-112 of National University of Singapore, thanks Bernard Leclerc for his suggestion of the proof of Theorem 2.4 and Hung Yean Loke for directing him to [7] in which the formula for the length of a general element of an affine Weyl group can be found.
that certain weight 3 blocks indeed have such a property, and obtained some
sufficient conditions for the inheritance of such a property for one weight 3
block from another with which they form a [3 : 1]- or [3 : 2]-pair. One
of these sufficient conditions is the assumption that the pair of blocks con-
cerned have bipartite Ext-quivers. This condition is recently shown to hold
for all weight 3 blocks in [6], with the bipartition being given by the relative
signs (or parities) of partitions labelling the simple modules. Our result
on the parities of the v-decomposition numbers can therefore be applied to
obtain information on the module structures of some important modules.
This new ingredient proves sufficient to provide a complete verification of
Martin’s conjecture for \( w = 3 \).

We now indicate the layout of this paper. In the remainder of this section,
we introduce some non-standard notation and give a summary of the com-
binatorics of partitions which we shall require. In Section 2 we review the
theory of v-decomposition numbers, state the first main result of this paper
and prove it while assuming Proposition 2.3. Section 3 is devoted entirely
to the proof of Proposition 2.3. The remainder of the paper deals with the
representation theory of the symmetric groups. We give a brief account of
the general theory in Section 4 while in Section 5 we specialise to weight 3
blocks. We also state the second main result of this paper in Section 5 and
prove it while assuming Proposition 5.6. Section 6 is devoted entirely to the
proof of Proposition 5.6.

1.1. Notation. Given a ring \( R \), a simple left \( R \)-module \( S \) and an arbitrary
left \( R \)-module \( M \) with a composition series, we write \([M : S]\) for the multi-
plcity of \( S \) as a composition factor of \( M \).

1.2. Partitions. A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a weakly decreasing se-
quence of non-negative integers, where for sufficiently large \( i \), \( \lambda_i = 0 \). If
\( \sum_i \lambda_i = n \), we say \( \lambda \) is a partition of \( n \). The length of \( \lambda \), denoted \( l(\lambda) \),
equals \( \max(i \mid \lambda_i > 0) \). Denote the set of partitions of \( n \) by \( \mathcal{P}_n \), and let
\( \mathcal{P} = \bigcup_n \mathcal{P}_n \) be the set of all partitions.

A strictly decreasing sequence \( \beta = (\beta_1, \beta_2, \ldots, \beta_s) \) of non-negative integ-
ergies is a sequence of \( \beta \)-numbers for \( \lambda \) if \( s \geq l(\lambda) \), and \( \beta_i = \lambda_i + s - i \) for all
\( 1 \leq i \leq s \). Every strictly decreasing sequence of non-negative integers is a
sequence of \( \beta \)-numbers for a unique partition.

The James e-abacus has \( e \) vertical runners, labelled 0, 1, \ldots, \( e - 1 \). Its
positions are labelled from left to right, and top down, starting from 0. The
partition \( \lambda \) may be displayed on the abacus as follows: if \( \beta = (\beta_1, \beta_2, \ldots, \beta_s) \)
is a sequence of \( \beta \)-numbers for \( \lambda \), then we place a bead at position \( \beta_i \) for
each \( i \). This is the (e-)abacus display of \( \lambda \) with \( s \) beads.

In an abacus display of \( \lambda \), moving a bead from position \( a \) to a vacant
position \( b \), with \( a > b \), corresponds to removing a (rim) hook of length \( a - b \)
from \( \lambda \). The number of beads crossed in so doing (i.e. the number of occupied
positions between \( b \) and \( a \)) is the leg-length of the hook. The \( e \)-core of \( \lambda \) is
thus obtained when we slide the beads as far up their respective runners as
possible. The \( e \)-weight of \( \lambda \) is the total number of times we slide the beads
one position up their respective runners to obtain its \( e \)-core. The relative
(e-)sign of $\lambda$, denoted as $\sigma_e(\lambda)$, can be defined as $(-1)^t$, where $t$ is the total number of beads crossed to obtain the e-core (see $[15, \S 2]$).

The conjugate partition of $\lambda$, denoted $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, is defined by $\lambda'_j = |\{i \mid \lambda_i \geq j\}|$ for all $j \in \mathbb{Z}^+$. Given an abacus display of $\lambda$, we can obtain the abacus display of $\lambda'$ by rotating the abacus of $\lambda$ through an angle of $\pi$, and read the vacant positions as occupied and the occupied positions as vacant. Thus, $\lambda'$ has the same e-weight as $\lambda$, and its e-core is the conjugate partition of the e-core of $\lambda$.

The partition $\lambda$ is e-regular if there does not exist $i$ such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} > 0$, and is e-restricted if $\lambda'$ is e-regular. In $[16]$, Mullineux formulated an involution $\lambda \mapsto m(\lambda)$ on the set of e-regular partitions of $P_n$. This bijection plays an important role in the representation theory of Iwahori-Hecke algebras of the symmetric groups and the Fock space representation of $U_v(\hat{\mathfrak{sl}}_e)$, which we shall describe later. We refer the reader to $[16]$ for a combinatorial description of this bijection.

2. The Fock space representation of $U_v(\hat{\mathfrak{sl}}_e)$

In this section, we define the $v$-decomposition numbers arising from the Fock space representation of $U_v(\hat{\mathfrak{sl}}_e)$, and briefly discuss some of the remarkable properties they enjoy.

The quantum affine algebra $U_v(\hat{\mathfrak{sl}}_e)$ is an associative algebra over $\mathbb{C}(v)$ generated by $e_r, f_r, k_r, k_r^{-1}$ $(0 \leq r \leq e - 1)$, $d, d^{-1}$ subject to certain relations which we do not need here. The Fock space representation $\mathcal{F}$ of $U_v(\hat{\mathfrak{sl}}_e)$ is a $\mathbb{C}(v)$-vector space with basis $\{s(\lambda) \mid \lambda \in \mathcal{P}\}$. For our purposes, an explicit description of the actions of $e_r$ and $f_r$ will suffice.

Display a partition $\lambda$ on the e-abacus with $t$ beads, where $t \geq l(\lambda)$ and $e \nmid (r + t)$. Let $i$ be the residue class of $(r + t)$ modulo $e$. Suppose there is a bead on runner $i - 1$ whose succeeding position on runner $i$ is vacant; let $\mu$ be the partition obtained when this bead is moved to its succeeding position. Let $N_>(\lambda, \mu)$ (resp. $N_<(\lambda, \mu)$) be the number of beads on runner $i - 1$ below (resp. above) the bead moved to obtained $\mu$ minus the number of beads on runner $i$ below (resp. above) the vacant position that becomes occupied in obtaining $\mu$. We have

$$f_r(s(\lambda)) = \sum_{\mu} v^{N_>(\lambda, \mu)} s(\mu);$$

$$e_r(s(\mu)) = \sum_{\lambda} v^{-N_<(\lambda, \mu)} s(\lambda),$$

where $\mu$ in the first sum runs over all partitions that can be obtained from $\lambda$ by moving a bead on runner $i - 1$ to its vacant succeeding position on runner $i$, while $\lambda$ in the second sum runs over all partitions that can be obtained from $\mu$ by moving a bead on runner $i$ to its vacant preceding position on runner $i - 1$.

Let $\langle - , - \rangle$ be the inner product on $\mathcal{F}$ with respect to which $\{s(\lambda) \mid \lambda \in \mathcal{P}\}$ is orthonormal. With respect to this inner product, the operators $e_r$ and $f_r$ are adjoints to each other, i.e. $\langle f_r x, y \rangle = \langle x, e_r y \rangle$ for all $x, y \in \mathcal{F}$.

The Fock space $\mathcal{F}$ possesses another distinguished basis $\{G(\lambda) \mid \lambda \in \mathcal{P}\}$, called the canonical basis. For $\lambda, \mu \in \mathcal{P}$, define the $v$-decomposition number
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d_{\lambda\mu}(v) \in \mathbb{C}(v) to be the coefficient of s(\lambda) in the expansion of G(\mu), i.e.

d_{\lambda\mu}(v) = \langle G(\mu), s(\lambda) \rangle.

Varagnolo and Vasserot [19] showed that the v-decomposition numbers are parabolic Kazhdan-Lusztig’s polynomials. We give a brief account of this.

The extended affine Weyl group \( W = \mathfrak{S}_n \rtimes \mathbb{Z}^n \) acts on \( \mathbb{Z}^n \) via

\[ \sigma(t_1, \ldots, t_n) \cdot (a_1, \ldots, a_n) = (t_{\sigma^{-1}(1)} + a_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)} + a_{\sigma^{-1}(n)}) \]

(Here, and hereafter, \( \mathfrak{S}_n \) denotes the symmetric group on \( n \) letters.) The set \( A = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid -e < x_1 \leq \cdots \leq x_n \leq 0\} \) is a fundamental domain of this action, and \( \{\varepsilon_i - \varepsilon_j \mid i \neq j\} \) is a root system for \( W \), where \( \varepsilon_i \) is the \( i \)-th standard basis element of \( \mathbb{Z}^n \). We take the positive roots to be \( \{\varepsilon_i - \varepsilon_j \mid i < j\} \). For \( a \in \mathbb{Z}^n \), write \( w_a \) for the (unique) element in \( \mathfrak{S}_n \rtimes \mathbb{Z}^n \) having the minimal length (with respect to this positive root system) such that \( w_a^{-1} \cdot a \in A \).

Given a partition \( \lambda \) with \( l(\lambda) \leq n \), write \( \widehat{\lambda} \) for the strictly increasing sequence of non-negative integers \( (a_1, \ldots, a_n) \) such that \( (a_n, \ldots, a_1) \) is a sequence of \( \beta \)-numbers of \( \lambda \). Let \( \mu \) be another partition with \( l(\mu) \leq n \), and assume \( \lambda \) and \( \mu \) have the same \( e \)-weight and \( e \)-core. Then \( \widehat{\lambda} \) and \( \widehat{\mu} \) lie in the same \( W \)-orbit, which intersects \( A \) at \( \alpha \), say. Let \( w^\alpha \) be the longest element in the stabilizer of \( \alpha \) (under the action of \( W \)). We have

**Theorem 2.1** ([19]; see also [10] Theorem 13).

\[
d_{\lambda\mu}(v) = \sum_{y \in \mathfrak{S}_n} (-v)^{\ell(y)} P_{w^\alpha w^\mu, w^\alpha w^\mu}(v),
\]

where \( \ell(y) \) is the length of \( y \) as an element of \( W \) (with respect to the chosen positive root system), and \( P_{x, w} \) is the coefficient of \( T_x \) in the expansion of the Kazhdan-Lusztig base element \( C'_x \) of the Hecke algebra associated with \( W \) (following Soergel’s convention on the normalisation of the generators \( T_i \) [18]).

The upshot of this is:

**Corollary 2.2.** Keeping the above notations, we have

\[
d_{\lambda\mu}(v) \in \begin{cases} \mathbb{N}_0[v^2], & \text{if } (-1)^{\ell(w_{\lambda})} = (-1)^{\ell(w_{\mu})}; \\ v\mathbb{N}_0[v^2], & \text{if } (-1)^{\ell(w_{\lambda})} \neq (-1)^{\ell(w_{\mu})}. \end{cases}
\]

**Proof.** This follows from the fact that

\[
P_{x, w}(v) \in \begin{cases} \mathbb{N}_0[v^2], & \text{if } (-1)^{\ell(w)} = (-1)^{\ell(x)}; \\ v\mathbb{N}_0[v^2], & \text{if } (-1)^{\ell(w)} \neq (-1)^{\ell(x)}. \end{cases}
\]

\[ \square \]

**Proposition 2.3.** Let \( \lambda \) be a partition with \( e \)-weight \( W \) and \( e \)-core \( \kappa \), and assume that \( l(\lambda) \leq n \). Then

\[
(-1)^{\ell(w_{\lambda})} = (-1)^{W(n-1)+\ell(w_{\lambda})} \sigma_e(\lambda).
\]

We devote the next section to the proof of Proposition 2.3. The following theorem immediately follows from Corollary 2.2 and Proposition 2.3.
Theorem 2.4. If \(d_{\lambda\mu}(v) \neq 0\), then
\[
d_{\lambda\mu}(v) \in \begin{cases} \mathbb{N}_0[v^2], & \text{if } \sigma_e(\lambda) = \sigma_e(\mu); \\ v\mathbb{N}_0[v^2], & \text{if } \sigma_e(\lambda) \neq \sigma_e(\mu). \end{cases}
\]

It is proved in [6, Proposition 2.19] that when \(p\) is an odd prime and \(\lambda\) is a \(p\)-regular partition with \(p\)-weight 3, \(\sigma_p(m(\lambda)) \neq \sigma_p(\lambda)\). Furthermore, it is remarked that \(\sigma_p(m(\lambda)) = (-1)^w\sigma_p(\lambda)\) holds in general when \(\lambda\) has \(p\)-weight \(w\) (and is \(p\)-regular), by using \(v\)-decomposition numbers, without giving details. Here we use Theorem 2.4 to provide a formal proof of a more general version of this statement.

Proposition 2.5. Suppose \(\lambda\) is an e-regular partition having e-weight \(w\). Then \(\sigma_e(m(\lambda)) = (-1)^w\sigma_e(\lambda)\).

Proof. Since \(d_{m(\lambda)\lambda}(v) = v^w\) [9, Corollary 7.7], we have \(\sigma_e(m(\lambda)')\sigma_e(\lambda) = (-1)^w\) by Theorem 2.4. Note that \(\sigma_e(\mu') = (-1)^{\epsilon-1}w\sigma_e(\mu)\) for a e-regular partition \(\mu\) with e-weight \(w\). Thus, \(\sigma_e(m(\lambda)) = (-1)^w\sigma_e(\lambda)\).

\(\square\)

3. The extended affine Weyl group \(W = \mathfrak{S}_n \ltimes \mathbb{Z}^n\) action on \(\mathbb{Z}^n\)

We provide a proof of Proposition 2.3 in this section. Our goal is to relate the (parity of the) length of \(w_\lambda\) to that of \(w_\kappa\), where \(\kappa\) is the e-core of \(\lambda\).

Recall the set of positive roots of \(W\), and the fundamental domain \(\mathcal{A}\) of the action of \(W\) on \(\mathbb{Z}^n\), as described in the previous section.

For \(T = \sum_{i=1}^n T_i \varepsilon_i \in \mathbb{Z}^n\), and \(\sigma \in \mathfrak{S}_n\), the length of \(T\sigma\) as an element of \(W\) may be calculated based on the following formula, which is a specialization of that for general extended affine Weyl groups found in Proposition 1.23 of [7]:

Theorem 3.1.
\[\ell(T\sigma) = \sum_{\sigma^{-1}(i) < \sigma^{-1}(j)} |T_i - T_j| + \sum_{\sigma^{-1}(i) > \sigma^{-1}(j)} |T_i - T_j - 1|.
\]

Observe that the formula agrees with that for the ordinary Weyl group \(\mathfrak{S}_n\) upon restriction.

Fix an element \(a = \sum_{i=1}^n a_i \varepsilon_i \in \mathbb{Z}^n\), and for each \(i = 1, \ldots, n\), let \(t_i = [a_i/e]\) so that \(-e < a_i - et_i \leq 0\). Write \(t = \sum_{i=1}^n t_i \varepsilon_i\) and \(c = \sum_{i=1}^n c_i \varepsilon_i = -t \cdot a\). Note that \(w^{-1} \cdot a \in \mathcal{A}\) for \(w \in \mathfrak{S}_n \ltimes \mathbb{Z}^n\) if and only if \(w = t\sigma\) for some \(\sigma \in \mathfrak{S}_n\) with \(\sigma^{-1} \cdot c \in \mathcal{A}\).

A description of \(w_a\) is particularly easy when \(a_1 \leq a_2 \leq \cdots \leq a_n\).

Proposition 3.2. If \(T = \sum_{i=1}^n T_i \varepsilon_i \in \mathbb{Z}^n\) with \(T_1 \leq T_2 \leq \cdots \leq T_n\), and \(\sigma \in \mathfrak{S}_n\), then
\[\ell(T\sigma) = \sum_{i<j} (T_j - T_i) + \sum_{\sigma^{-1}(i) > \sigma^{-1}(j)} 1 = \sum_{i<j} (T_j - T_i) + \ell(\sigma).
\]
In particular, if \( a, t \) and \( c \) are as defined above, with \( a_1 \leq a_2 \leq \cdots \leq a_n \), then \( w_\alpha = t \sigma_c \), where \( \sigma_c \) is defined as the unique element of \( \mathcal{S}_n \) satisfying \( \sigma_c^{-1}(i) > \sigma_c^{-1}(j) \) if and only if \( c_i > c_j \), and

\[
\ell(w_\alpha) = \sum_{i<j} (t_j - t_i) + l(\sigma_c) = \sum_{i<j} (t_j - t_i) + |\{(i,j) \mid 1 \leq i < j \leq n, \ c_i > c_j\}|.
\]

**Proof.** The formula for \( \ell(T \sigma) \) follows immediately from Theorem \[3.1\]. If \( a_1 \leq a_2 \leq \cdots \leq a_n \), then necessarily \( t_1 \leq t_2 \leq \cdots \leq t_n \), so that \( w_\alpha = t \sigma_c \), where \( \sigma_c \) is the shortest element of \( \mathcal{S}_n \) satisfying \( \sigma_c^{-1} c \in \mathcal{A} \). The description of \( \sigma_c \) and \( \ell(w_\alpha) \) then follows. \( \square \)

Now, assume \( a_1 < a_2 < \cdots < a_n \), and suppose \( a_{s-1} < a_r - e < a_s \) for some \( 1 \leq s \leq r \leq n \). Let \( u \) be the least index such that \( t_u = t_r \). Then \( s \leq u \leq r \). Furthermore,

**Lemma 3.3.**

1. \( t_u = t_{u+1} = \cdots = t_r \), and if \( s < u \), then \( t_s = t_{s+1} = \cdots = t_{u-1} = t_r - 1 \).
2. \( c_u < c_{u+1} < \cdots < c_r \), and if \( s < u \), then \( c_r < c_s < c_{s+1} < \cdots < c_{u-1} \).

**Proof.** Since \( a_r - e < a_i < a_r \) for all \( s \leq i \leq r - 1 \), we have \( t_r - 1 \leq t_i \leq t_r \). This yields part (1). Part (2) then follows since \( c_i = a_i - et_i \) for all \( i \) (note that if \( s < u \), then \( c_r = a_i - et_r < a_s + e - e(t_s + 1) = c_s \)). \( \square \)

Write \( [s,r] \) for \( (s,s+1,\ldots,r) \in \mathcal{S}_n \), and let

\[
a' = (a'_1, \ldots, a'_n) = [s,r](-e\varepsilon_r \cdot a) = (a_1, \ldots, a_{s-1}, a_r - e, a_s, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)
\]

For each \( i \), let \( t'_i = [a'_i/e] \) and \( c'_i = a'_i - et'_i \), and write \( t' = (t'_1, \ldots, t'_n) \) and \( c' = (c'_1, \ldots, c'_n) \). Then \( c' = [s,r] \cdot c \), and

**Lemma 3.4.** \( t' = t - \varepsilon_u \).

**Proof.** This follows from part (1) of Lemma \[3.3\]. \( \square \)

We wish to compare \( \ell(w_\alpha) \) and \( \ell(w_{\alpha'}) \). First, we compare \( \ell(\sigma_c) \) and \( \ell(\sigma_{c'}) \).

**Lemma 3.5.** \( \ell(\sigma_c) - \ell(\sigma_{c'}) = 2u - (r + s) \).

**Proof.** We define a partial correspondence between the sets \( X = \{(i,j) \mid i < j, \ c_i > c_j\} \) and \( Y = \{(i,j) \mid i < j, \ c'_i > c'_j\} \) as follows:

\[
(i,j) \leftrightarrow (i^+, j^+) \quad (j \neq r) \\
(i,r) \leftrightarrow (i,s) \quad (i < s),
\]

where

\[
i^+ = \begin{cases} 
    i + 1, & \text{if } s \leq i < r; \\
    i, & \text{otherwise}.
\end{cases}
\]

This correspondence yields part (1) of Lemma \[3.3\]. Part (2) then follows since \( c_i = a_i - et_i \) for all \( i \) (note that if \( s < u \), then \( c_r = a_i - et_r < a_s + e - e(t_s + 1) = c_s \)). \( \square \)
By part (2) of Lemma 3.3, this is actually a one-to-one correspondence between the set \( X \setminus X' \) and \( Y \setminus Y' \), where
\[
X' = \{(i, r) \mid s \leq i < u\},
\]
\[
Y' = \{(s, j) \mid u < j \leq r\}.
\]
Thus, \( \ell(\sigma_c) - \ell(\sigma'_c) = |X| - |Y| = |X'| - |Y'| = (u - s) - (r - u) = 2u - (r + s) \). □

**Corollary 3.6.** \( \ell(w_a) = \ell(w_{a'}) + 4u - n - 1 - r - s \).

**Proof.** By Proposition 3.2, Lemmas 3.4 and 3.5, we have
\[
\ell(w_a) = \sum_{i<j}(t_j - t_i) + \ell(\sigma_c)
\]
\[
= \sum_{i<j}(t'_j - t'_i) + 2u - n - 1 + \ell(\sigma_{c'}) + 2u - r - s
\]
\[
= \ell(w_{a'}) + 4u - n - 1 - r - s.
\]
□

We are now ready to prove Proposition 2.3.

**Proof of Proposition 2.3.** We prove by induction on the \( e \)-weight \( W \) of \( \lambda \), with \( W = 0 \) being trivial. Let \( \lambda \) have positive \( e \)-weight \( W \), and let \( \mu \) be the partition obtained when one particular \( e \)-hook is removed from \( \lambda \). Let \( \tilde{\lambda} = (a_1, \ldots, a_n) \). Then there exist some integers \( r, s \) with \( 1 \leq s \leq r \leq n \) such that \( \tilde{\mu} = [s, r]|(-e\varepsilon_r) \cdot \tilde{\lambda} \). Thus,
\[
(-1)^{\ell(w_{\tilde{\lambda}})} = (-1)^{\ell(w_{\tilde{\mu}})}(-1)^{(n-1)+(r-s)}
\]
\[
= (-1)^{\ell(w_{\tilde{\mu}})+W-1(n-1)}\sigma_e(\mu)(-1)^{(n-1)+(r-s)}
\]
\[
= (-1)^{\ell(w_{\tilde{\lambda}})+W(n-1)}\sigma_e(\lambda)
\]
by Corollary 3.6 and induction hypothesis (note that the \( e \)-hook removed from \( \lambda \) to obtain \( \mu \) has leg-length \( r - s \)). □

4. The symmetric group algebras

For the remainder of this paper, we look at the weight 3 blocks of symmetric group algebras with Abelian defect group. Our goal is to prove Martin’s conjecture that the projective (indecomposable) modules in these blocks have a common radical length 7. We shall see that Theorem 2.4 will come in useful in our proof.

In this section, we give an account of the background of the representation theory of the symmetric group where the underlying field \( \mathbb{F} \) has positive characteristic \( p \). Throughout, \( \mathfrak{S}_n \) denotes the symmetric group on \( n \) letters.

For each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}_n \), one defines the Specht module \( S^\lambda \) of \( \mathbb{F}\mathfrak{S}_n \). When \( \lambda \) is \( p \)-regular, the Specht module \( S^\lambda \) has a simple, self-dual head \( D^\lambda \), and the modules \( D^\lambda \), as \( \lambda \) runs over all the \( p \)-regular partitions of \( n \), give a complete list of simple modules of \( \mathbb{F}\mathfrak{S}_n \).
The projective cover $P(D^\mu)$ of $D^\mu$, where $\mu \in \mathcal{P}_n$ and is $p$-regular, has a (distinguished) filtration in which all factors are Specht modules. The multiplicity of $S^\lambda$ as a factor in this filtration equals the composition multiplicity $[S^\lambda : D^\mu]$ of $D^\mu$ in $S^\lambda$.

Two Specht modules $S^\lambda$ and $S^\mu$ of $\mathbb{F}S_n$ lie in the same block if and only if $\lambda$ and $\mu$ have the same $p$-core. As such each block $B$ of a symmetric group algebra is parametrised by a $p$-core partition $\kappa$ and a non-negative integer $w$. A Specht module $S^p$ lies in $B$ if and only if $p$ has $p$-core $\kappa$ and $p$-weight $w$. We call $\kappa$ the $p$-core of $B$, and $w$ the ($p$-)weight of $B$. We also say $\rho$ is a partition in $B$ if $p$ has $p$-core $\kappa$ and $p$-weight $w$. The defect group of $B$ is isomorphic to a Sylow $p$-subgroup of $\mathbb{G}_w$. As such, $B$ has Abelian defect group if and only if $w < p$.

4.1. Restriction and induction. Let $B$ be a weight $w$ block of $\mathbb{F}S_n$, with $p$-core $\kappa_B$ (so $\kappa_B \in \mathcal{P}_{n-wp}$). Fix an abacus display of $\kappa_B$, and consider the $p$-core partition $\kappa_C$ having an abacus display in which all the runners have the same number of beads as that in $\kappa_B$ except for runners $i-1$ and $i$, where respectively there are $k$ beads more and $k$ beads less than those in $\kappa_B$. We assume that $\kappa_C \in \mathcal{P}_m$ with $m \leq n-k$, and let $C$ be the block of $\mathbb{F}S_{n-k}$ with $p$-core $\kappa_C$.

For every module $M$ of $B$, there is a module $M'$ of $C$ such that $M_{\downarrow C} \cong (M')^\oplus k!$, and for every module $N$ of $C$, there is a module $N'$ of $B$ such that $N'_{\uparrow B} \cong (N')^\oplus k!$. If $M$ (resp. $N$) is simple, then $M'$ (resp. $N'$) is either zero, or has a simple socle (and hence is indecomposable). This induces, via Frobenius reciprocity, a bijection between a subset of $p$-regular partitions in $B$ with another subset of $p$-regular partitions in $C$, as follows:

**Theorem 4.1** (See Section 11.2 of [8]). Let $\mathcal{P}_B$ (resp. $\mathcal{P}_C$) be the set of $p$-regular partitions indexing the simple modules of $B$ (resp. $C$) which do not vanish upon restriction to $C$ (resp. induction to $B$). Then there is a bijection $\Phi = \Phi_{B,C} : \mathcal{P}_B \to \mathcal{P}_C$ such that

$$\text{soc}(D^\lambda_{\downarrow C}) \cong (D^{\Phi(\lambda)}_{\downarrow B})^\oplus k! \quad \text{and} \quad \text{soc}(D^{\Phi(\lambda)}_{\uparrow B}) \cong (D^\lambda)_{\uparrow B}^\oplus k!.$$ 

Furthermore, $\mathcal{P}_B$, $\mathcal{P}_C$ and $\Phi$ can be combinatorially described.

We have a nice description of $\text{End}_B(D^{\Phi(\lambda)})_{\uparrow B}$ when $k = 1$.

**Theorem 4.2** ([8] Theorem 11.2.8(i,ii)). Assume $k = 1$, and suppose that $\lambda \in \mathcal{P}_B$ with $[D^{\Phi(\lambda)}_{\uparrow B} : D^\lambda] = r$. Then $\text{End}_B(D^{\Phi(\lambda)})_{\uparrow B} \cong \mathbb{F}[x]/(x^r)$, where $(x^r)$ is the ideal of the polynomial ring $\mathbb{F}[x]$ generated by $x^r$.

Now suppose further that runner $i$ of the abacus display of $\kappa_B$ has exactly $k$ beads more than runner $i-1$. Then $C$ also has weight $w$, and the abacus display of $\kappa_C$ can be obtained from that of $\kappa_B$ by interchanging the runners $i$ and $i-1$. We say that $(B,C)$ is a $[w:k]$-pair. In this case, $\Phi$ is a bijection between the entire sets of $p$-regular partitions in $B$ and $C$. Furthermore, $D^\lambda_{\downarrow C}$ is semi-simple for many simple modules of $B$ (so $D^\lambda_{\downarrow C} \cong (D^{\Phi(\lambda)})^\oplus k!$ for these partitions) — we call these simple modules of $B$ non-exceptional (with respect to the $[w:k]$-pair $(B,C)$), and the others exceptional. Analogously, we define the exceptional and non-exceptional simple modules of
C (with respect to \((B, C)\)). It turns out that the bijection \(\Phi\) sends partitions indexing exceptional (resp. non-exceptional) simple modules of \(B\) to partitions indexing exceptional (resp. non-exceptional) simple modules of \(C\).

A necessary and sufficient condition for the absence of exceptional simple modules of \(B\) and \(C\) is \(w \leq k\). In this case, \(B\) and \(C\) are Morita equivalent — we shall say that \(B\) and \(C\) are Scopes equivalent — and the effect of \(\Phi\) on a \(p\)-regular partition in \(B\) is merely to interchange runners \(i\) and \(i - 1\) of its abacus display. It is clear that Scopes equivalence can be extended to an equivalence relation on the set of all blocks of symmetric group algebras.

4.2. Rouquier blocks. Let \(B\) be a weight \(w\) block of \(F\mathfrak{S}_n\). We say that \(B\) is Rouquier if its abacus display of its \(p\)-core has the following properties: whenever runner \(i\) is on the left of runner \(j\), either runner \(j\) has at least \(w - 1\) beads more than runner \(i\), or runner \(i\) has at least \(w\) beads more than runner \(j\). It is easy to check that such a property is independent of the choice of abacus display of the \(p\)-core of \(B\), and that the Rouquier blocks of a fixed weight form a single Scopes equivalence class. These blocks are well understood in the Abelian defect case, by the results of [2]. In particular, we have the following Theorem.

**Theorem 4.3** ([2, Theorem 6.4]). Suppose \(B\) is a Rouquier block of weight \(w\), with an Abelian defect group. Then the projective indecomposable modules of \(B\) have a common radical length \(2w + 1\).

An arbitrary weight \(w\) block can always be induced to a Rouquier block through a sequence of \([w : k]\)-pairs.

**Lemma 4.4.** Suppose \(A\) is a weight \(w\) block of \(F\mathfrak{S}_n\). Then there exists a sequence \(B_0, B_1, \ldots, B_s\) of weight \(w\) blocks of symmetric group algebras such that \(B_0 = A\), \(B_s\) is Rouquier, and for each \(1 \leq i \leq s\), \((B_i, B_{i-1})\) is a \([w : k_i]\)-pair for some \(k_i \in \mathbb{Z}^+\).

**Proof.** This is Lemma 3.1 of [4] in the context of the Iwahori-Hecke algebras of the symmetric groups, which is a deformation of the symmetric group algebras, and hence includes the Lemma as a special case.

4.3. Conjugate block. Let \(B\) be a block of \(F\mathfrak{S}_n\), with \(p\)-core \(\kappa\). The block of \(F\mathfrak{S}_n\) conjugate to \(B\) is the one with \(p\)-core \(\kappa'\). This conjugate block, which we denote as \(B'\), is Morita equivalent to \(B\) via the functor \(- \otimes \text{sgn}\), where \(\text{sgn}\) is the one-dimensional sign representation of \(\mathfrak{S}_n\).

If \(\lambda\) is a partition in \(B\), we have \(S^\lambda \otimes \text{sgn} \cong \text{dual of } S^{\lambda'}\), and if \(\lambda\) is \(p\)-regular, \(D^\lambda \otimes \text{sgn} = D^{m(\lambda)}\), where \(m\) is the Mullineux map discussed earlier. In particular, \(S^\lambda\) has a simple socle \(D^{m(\lambda')}\) when \(\lambda\) is \(p\)-restricted.

4.4. Connection with the Fock space. The connection between the representation theory of the symmetric groups and the Fock space representation of \(U_q(\mathfrak{sl}_n)\) is through the Iwahori-Hecke algebra \(\mathcal{H}_n = \mathcal{H}_{F,q}(\mathfrak{S}_n)\) of the symmetric group. As an algebra, \(\mathcal{H}_n\) is generated by \(T_1, T_2, \ldots, T_{n-1}\)
subject to the following relations:

\[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n - 2); \]
\[ T_i T_j = T_j T_i \quad (|i - j| \geq 2); \]
\[ (T_i - q)(T_i + 1) = 0 \quad (1 \leq i \leq n - 1). \]

Here, \( q \) is an invertible element of the ground field \( \mathbb{F} \), where in this subsection we allow its characteristic to be zero too. Clearly, \( \mathcal{H}_n \) is isomorphic to \( \mathbb{F} \S_n \) if \( q = 1 \), so that \( \mathcal{H}_n \) is a deformation of \( \mathbb{F} \S_n \) in general.

Much of the representation theory of \( \S_n \) carries over to that of \( \mathcal{H}_n \): for \( \lambda \in \mathcal{P}_n \), we also have the Specht module \( S^\lambda \), and this has a simple head \( D^\lambda \) if \( q \) is a root of unity and \( \lambda \) is \( e \)-regular, where \( e \) is the least positive integer such that \( 1 + q + \cdots + q^{e-1} = 0 \). As \( \lambda \) varies over the set of \( e \)-regular partitions in \( \mathcal{P}_n \), the \( D^\lambda \)'s give a complete list of non-isomorphic simple module of \( \mathcal{H}_n \). The projective cover \( P(D^\lambda) \) of \( D^\lambda \) (when \( \lambda \) is \( e \)-regular) has an analogous Specht filtration, and the blocks of \( \mathcal{H}_n \) are also similarly parametrised.

When \( \mathbb{F} \) has characteristic \( p \), the composition multiplicity \([S^\lambda : D^\mu]\) of \( D^\mu \) in \( S^\lambda \) of \( \mathbb{F} \S_n \) is bounded above by the corresponding composition multiplicity \([S^\lambda : D^\mu]\) of \( D^\mu \) in \( S^\lambda \) of \( \mathcal{H}_{e,q}(\S_n) \), where \( q \) is a primitive \( p \)-th root of unity.

Ariki \( \Pi \) established the connection between the Fock space representation of \( U_v(s\mathfrak{l}_e) \) and representation theory of \( \mathcal{H}_n \) in characteristic zero where \( q \) is a primitive \( e \)-th root of unity.

**Theorem 4.5** \( \Pi \). Evaluating the \( v \)-decomposition number \( d_{\lambda\mu}(v) \) at \( v = 1 \) gives the composition multiplicity \([S^\lambda : D^\mu]\) of \( D^\mu \) in \( S^\lambda \) of \( \mathcal{H}_n \) in characteristic zero.

This has the following consequence:

**Proposition 4.6.** Let \( B \) and \( C \) be blocks of \( \mathcal{H}_n \) and \( \mathcal{H}_{n-1} \) in characteristic zero. Suppose that their \( e \)-cores, denoted \( \kappa_B \) and \( \kappa_C \) respectively, are of the form described in subsection 4.1. Let \( r \) be the residue class of \( i - t \) modulo \( e \), where \( t \) is the number of beads used in the abacus display of \( \kappa_C \) (or \( \kappa_B \)), and let \( \lambda \) be an \( e \)-regular partition in \( C \). If

\[ f_r^{(k)} G(\lambda) = \sum_{\rho \in \mathcal{P}_B} a_\rho(v) G(\rho), \]

then

\[ P(D^\lambda) \uparrow^B = \bigoplus_{\rho \in \mathcal{P}_B} P(D^\rho)^{\otimes k \alpha_\rho(1)}. \]

**Note.** It is sometimes possible to recover \( a_\rho(v) \) from \( a_\rho(1) \). This is because \( a_\rho(v) = a_\rho(v^{-1}) \) and \( a_\rho(v) \in \mathbb{N}_0[v, v^{-1}] \) (see \( \mathbb{K} \) Proposition 2.4). Thus, \( a_\rho(v) = 0 \) if and only if \( a_\rho(1) = 0 \), while \( a_\rho(v) = 1 \) if and only if \( a_\rho(1) = 1 \). We shall use these facts later.

5. **Weight 3 Blocks of \( \mathbb{F} \S_n \)**

In this section, we focus our attention on weight 3 blocks of \( \mathbb{F} \S_n \) with Abelian defect groups. Thus the characteristic \( p \) of \( \mathbb{F} \) is assumed to be at
least 5. These blocks enjoy many nice properties. From now on, the \(v\)-decomposition numbers we are concerned with come from the Fock space representation of \(U_v(\mathfrak{sl}_n)\) (i.e. \(v = p\)).

**Theorem 5.1.** Let \(B\) be a weight 3 block of \(\mathbb{F}S_n\), and let \(\lambda\) and \(\mu\) be partitions in \(B\).

1. If \(\mu\) is \(p\)-regular, then \([S^\lambda : D^\mu] = 0\) or 1. Furthermore, \([S^\lambda : D^\mu] = d_{\lambda \mu}(1)\).
2. If \(\lambda\) and \(\mu\) are \(p\)-regular, then \(\text{Ext}^1(D^\lambda, D^\mu) = 0\) unless \(\sigma_p(\lambda) \neq \sigma_p(\mu)\). Furthermore, if \(\text{Ext}^1(D^\lambda, D^\mu) \neq 0\), then \(\dim_{\mathbb{F}} \text{Ext}^1(D^\lambda, D^\mu) = 1\).

**Proof.** Part (1) is proved by Fayers in \([5]\). For part (2), the first statement is the main result of \([6]\), while the second statement is proved in \([12]\). □

**Corollary 5.2.** Let \(\lambda\) and \(\mu\) be partitions having \(p\)-weight 3. If \(\mu\) is \(p\)-regular and \(d_{\lambda \mu}(v) \neq 0\), then

\[
d_{\lambda \mu}(v) = \begin{cases} 
1, & \text{if } \lambda = \mu; \\
v, & \text{if } \lambda \notin \{\mu, m(\mu)'\} \text{ and } \sigma_p(\lambda) \neq \sigma_p(\mu); \\
v^2, & \text{if } \lambda \notin \{\mu, m(\mu)'\} \text{ and } \sigma_p(\lambda) = \sigma_p(\mu); \\
v^3, & \text{if } \lambda = m(\mu)'.
\end{cases}
\]

**Proof.** Since \(d_{\lambda \mu}(v)\) is a parabolic Kazhdan-Lusztig polynomial, it follows that \(d_{\lambda \mu}(v) \in \mathbb{N}_0[v]\), so that \(d_{\lambda \mu}(v)\) is a monic monomial by Theorem 5.1(1). The Corollary thus follows from Theorem 6.8 and Corollary 7.7 of \([3]\), and Theorem 2.4. □

The following result is proved in \([4]\).

**Lemma 5.3 (\([6]\) Proposition 2.18).** Let \((B, C)\) be a [3 : \(k\)]-pair, and let \(\lambda\) be a \(p\)-regular partition in \(B\). Then \(\sigma_p(\lambda) \neq \sigma_p(\Phi(\lambda))\) if and only if \(k = 1\) and \(D^\lambda\) is an exceptional simple module (with respect to \((B, C)\)).

We now state the second main result of this paper.

**Theorem 5.4.** Let \(B\) be a weight 3 block of \(\mathbb{F}S_n\). Then the projective (indecomposable) modules of \(B\) have a common radical length 7.

The proof of Theorem 5.4 relies on the following two propositions.

**Proposition 5.5.** If Theorem 5.4 holds for one of the blocks in a [3 : 1]-pair, then it holds for the other.

**Proof.** Let \((B, \widetilde{B})\) be a [3 : 1]-pair. From the main result of \([13]\) (see from page 109 onwards), we only need to show the following:

- whenever an exceptional simple module \(D^\mu_{B}\) of \(B\) extends a non-exceptional simple module \(D^\lambda\), then \(D^{\Phi(\alpha)}\) does not extend \(D^{\Phi(\lambda)}\);
- whenever an exceptional simple module \(D^{\Phi(\alpha)}\) of \(B\) extends a non-exceptional simple module \(D^{\Phi(\mu)}\), then \(D^{\alpha_k}\) does not extend \(D^\mu\).

But these are immediate from Theorem 5.1(2) and Lemma 5.3. □

**Proposition 5.6.** Let \((B, \widetilde{B})\) be a [3 : 2]-pair. If Theorem 5.4 holds for \(B\), then it holds for \(\widetilde{B}\).
In [14], three sufficient conditions (Y1–Y3) for which one block in a \([3 : 2]\)-pair may inherit Theorem 5.3 from the other were obtained. However, unlike the proof of Proposition 5.5, we are unable to prove that the first of these conditions (Y1) holds in general (although we believe this to be true). As such, we have to study \([3 : 2]\)-pairs more carefully, which we do in the next section, to get around this and prove Proposition 5.6.

**Proof of Theorem 5.4.** If \(B\) is Rouquier, then the Theorem holds by Theorem 4.3. If \(B\) is not Rouquier, then by Lemma 4.4, there exists a sequence of \(\Phi = \Phi_0, \ldots, \Phi_{s-1}\) such that \(\Phi_{s-1}\) is Rouquier, and for each \(1 \leq i \leq s\), \((B_i, B_{i-1})\) is a \([w : k_i]\)-pair for some \(k_i \in \mathbb{Z}^+\). By induction, we may assume that the Theorem holds for \(B_1\). If \(k_1 \geq 3\), then \(B_1\) and \(B_0\) are Scopes, and hence Morita, equivalent, so that the Theorem holds for \(B_0\). If \(k_1 = 1\) or 2, then the Theorem holds for \(B_0\) by Propositions 5.5 and 5.6 respectively.

\[\square\]

### 6. \([3 : 2]\)-pairs

In this section, \(B\) is a weight 3 block of \(\mathcal{F} \mathcal{S}_n\), forming a \([3 : 2]\)-pair with a block \(\tilde{B}\) of \(\mathcal{F} \mathcal{S}_{n-2}\). We fix an abacus display of the \(p\)-core of \(B\), such that by interchanging the runners \(i\) and \(i - 1\), we obtain the abacus display of the \(p\)-core of \(\tilde{B}\). We begin by recalling the background theory on such pairs.

Every partition in \(B\), with the exception of four, has exactly two beads on runner \(i\) of its abacus display which can be moved one position to the left. The four exceptional partitions in \(B\) are denoted as \(\alpha, \beta, \gamma\) and \(\delta\), and the runners \(i - 1\) and \(i\) of their respective abacus displays are as follows:

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha & \beta & \gamma & \delta
\end{array}
\]

Similarly, every partition in \(\tilde{B}\), with the exception of four, has exactly two beads on runner \(i - 1\) of its abacus display which can be moved one position to the right. The four exceptional partitions in \(\tilde{B}\) are denoted as \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\) and \(\tilde{\delta}\), and the runners \(i - 1\) and \(i\) of their respective abacus displays are as follows:

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & \tilde{\delta}
\end{array}
\]

We note that the partitions \(\alpha\) and \(\tilde{\alpha}\) are always \(p\)-regular, while \(\delta\) and \(\tilde{\delta}\) are always \(p\)-restricted. In fact, \(\alpha = m(\delta')\) and \(\tilde{\alpha} = m(\tilde{\delta}')\).

The map \(\Phi = \Phi_{B, \tilde{B}}\) has the following effect on the exceptional partitions (when they are \(p\)-regular):

\[
\Phi(\alpha) = \tilde{\alpha}; \ \Phi(\beta) = \tilde{\delta}; \ \Phi(\gamma) = \tilde{\gamma}; \ \Phi(\delta) = \tilde{\beta}.
\]
Furthermore, if $\beta$ is $p$-regular, then $[S^\gamma : D^\beta] = [S^\delta : D^\beta] = 1$; if $\gamma$ is $p$-regular, then $[S^\delta : D^\gamma] = 1 = [S^\delta : D^\gamma]$; if $\delta$ is $p$-regular, then $[S^\delta : D^\beta] = [S^\delta : D^\beta] = 1$.

We note the following, which can be easily verified:

$$\sigma_p(\alpha) = \sigma_p(\tilde{\alpha}) = \sigma_p(\gamma) = \sigma_p(\tilde{\gamma}) \neq \sigma_p(\beta) = \sigma_p(\tilde{\beta}) = \sigma_p(\delta) = \sigma_p(\tilde{\delta}).$$

The four exceptional partitions in $\tilde{B}$ have another characterisation: they are the partitions where there is a unique bead on runner $i$ of their respective abacus displays which can be moved one position to its left. The partition so obtained is always the same, and we denote it as $\tilde{\alpha}$. This partition has weight 0 and is the unique partition in the block $\tilde{B}$ of $\mathcal{F}S_{n-3}$. Thus $S^\tilde{\alpha} = D^\tilde{\alpha}$ is simple and projective. By the ordinary branching rule, we have

$$S^\tilde{\alpha} - \downarrow_B \cong S^\tilde{\beta} - \downarrow_B \cong S^\tilde{\gamma} - \downarrow_B \cong S^\tilde{\delta} - \downarrow_B \cong S^\alpha,$$

and $S^\tilde{\lambda} - \downarrow_B = 0$ for all $\tilde{\lambda} \notin \{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\}$, and hence $D^\tilde{\alpha} - \downarrow_B = D^\tilde{\alpha}$ while $D^\tilde{\lambda} - \downarrow_B = 0$ for all $\tilde{\lambda} \neq \tilde{\alpha}$. This gives us $[S^\tilde{\beta} : D^\tilde{\tilde{\alpha}}] = [S^\tilde{\gamma} : D^\tilde{\tilde{\alpha}}] = [S^\tilde{\delta} : D^\tilde{\tilde{\alpha}}] = 1$.

Furthermore, the module $D^\tilde{\alpha} - \downarrow_B$ is projective and has a simple head $D^\tilde{\alpha}$ by Frobenius reciprocity, and a Specht filtration filtered by $S^\tilde{\alpha}, S^\tilde{\beta}, S^\tilde{\gamma}$ and $S^\tilde{\delta}$ by the ordinary branching rule. Thus $D^\tilde{\alpha} - \downarrow_B \cong P(D^\tilde{\alpha})$ and $[P(D^\tilde{\alpha}) : D^\alpha] = 4$.

Entirely analogous results also hold for the exceptional partitions in $B$; i.e. we also have $[S^\beta : D^\alpha] = [S^\gamma : D^\alpha] = [S^\delta : D^\alpha] = 1$ and $[P(D^\alpha) : D^\alpha] = 4$.

There is exactly one exceptional simple module of $\tilde{B}$ with respect to $(B, \tilde{B})$, namely $D^\tilde{\alpha}$ (thus $D^\alpha$ is the unique exceptional simple module of $B$).

We can obtain further information on $B$ and $\tilde{B}$ by considering their conjugate blocks $B'$ and $\tilde{B}'$. The latter also form a $[3 : 2]$-pair, and denoting the exceptional partitions in $B'$ and $\tilde{B}'$ as $\alpha^c, \beta^c, \gamma^c, \delta^c$ and $\tilde{\alpha}^c, \tilde{\beta}^c, \tilde{\gamma}^c, \tilde{\delta}^c$ respectively, we have

$$\begin{align*}
\alpha^c &= \delta^c, \quad \beta^c = \gamma^c, \quad \gamma^c = \beta^c, \quad \delta^c = \alpha^c; \\
\tilde{\alpha}^c &= \tilde{\delta}^c, \quad \tilde{\beta}^c = \tilde{\gamma}^c, \quad \tilde{\gamma}^c = \tilde{\beta}^c, \quad \tilde{\delta}^c = \tilde{\alpha}^c.
\end{align*}$$

Thus, $m(\alpha) = \alpha^c$ and $m(\tilde{\alpha}) = \tilde{\alpha}^c$. As such we deduce the following:

**Lemma 6.1.** We have

1. $\alpha$ is $p$-restricted if and only if $\tilde{\gamma}^c$ is in which case $[S^\tilde{\alpha} : D^{m(\tilde{\gamma}^c)}] = [S^\tilde{\beta} : D^{m(\tilde{\gamma}^c)}] = 1$;
2. $\beta$ is $p$-restricted if and only if $\tilde{\beta}^c$ is in which case $[S^\alpha : D^{m(\beta^c)}] = 1 = [S^\alpha : D^{m(\beta^c)}]$;
3. $\gamma$ is $p$-restricted if and only if $\tilde{\alpha}^c$ is, in which case $[S^\alpha : D^{m(\gamma^c)}] = [S^\beta : D^{m(\gamma^c)}] = 1$.

**Proof.** We show part (2); parts (1) and (3) are similar. If $\beta$ is $p$-restricted, then $\gamma^c = \beta^c$ is $p$-regular, and this is equivalent to $\tilde{\gamma}^c$ being $p$-regular, or $\beta = (\tilde{\gamma}^c)'$ being $p$-restricted. Furthermore, we have $[S^\delta^c : D^{\gamma^c}] = 1 = [S^\delta^c : D^{\gamma^c}]$, so that part (2) follows by tensoring with the sign representation. □
Another block of interest in the study of the [3 : 2]-pair \((B, \tilde{B})\) is the ‘intermediate’ block of \(\mathcal{B}\) of \(\mathcal{F}\mathcal{S}_{n-1}\). The \(p\) core of \(\mathcal{B}\) can be obtained from that of \(B\) by moving one bead from runner \(i\) to runner \(i - 1\). This block is ‘intermediate’ in the following sense: if \(M\) is a \(B\)-module, then \(M \downarrow_{\mathcal{B}} = (M \downarrow_{\mathcal{B}}) \downarrow_{\mathcal{B}}\), and if \(N\) is a \(\tilde{B}\)-module, then \(N \uparrow_{\mathcal{B}} = (N \uparrow_{\mathcal{B}}) \uparrow_{\mathcal{B}}\).

**Theorem 6.2** ([14] Proposition 4.3]). Let \(D^\lambda\) be a non-exceptional simple modules of \(B\), and let \(\lambda = \Phi_{B, \tilde{B}}(\lambda)\) and \(\bar{\lambda} = \Phi_{B, \bar{B}}(\lambda)\). Then

1. \(D^\bar{\lambda} \uparrow_{B} = D^\lambda\), \(D^\bar{\lambda} \downarrow_{B} = D^\lambda\);
2. \(D^\bar{\lambda} \uparrow_{B} \cong D^{\lambda}_{\uparrow_{B}},\) and \(D^\bar{\lambda} \downarrow_{B} \) is non-simple, has a simple head and a simple socle both isomorphic to \(D^\lambda\) and the composition factors \(D^\bar{\mu}\) of its heart satisfy \(D^\bar{\mu} \uparrow_{B} = 0 = D^\bar{\mu} \downarrow_{B}\).

Let \(\bar{\alpha} = \Phi_{B, \bar{B}}(\alpha)\). Then we have the following multiplicities:

**Lemma 6.3** ([14] Lemmas 4.5 and 4.7]). \([D^\alpha]_{\uparrow_{B}} : D^\bar{\alpha}\) = 3 = \([D^\alpha]_{\downarrow_{B}} : D^\bar{\alpha}\), and \([D^\bar{\alpha}]_{\downarrow_{B}} : \bar{\alpha}\) = 2 = \([D^\bar{\alpha}]_{\uparrow_{B}} : \bar{\alpha}\). In fact, \(D^\bar{\alpha} \uparrow_{B} \) and \(D^\bar{\alpha} \downarrow_{B}\) have radical length 3.

**Remark.** It is worth mentioning that \(D^\bar{\alpha} \uparrow_{B} \not\cong D^\alpha \downarrow_{B}\) (compare this with Theorem 6.2(2)). This follows from the fact that the dimension of \(\text{Hom}_{\mathcal{B}}(D^\alpha \downarrow_{B}, D^\bar{\alpha} \uparrow_{B})\) \(\cong \text{Hom}_{\mathcal{B}}(D^\alpha \downarrow_{B}, D^\bar{\alpha})\) is two, while \(\text{End}_{\mathcal{B}}(D^\bar{\alpha} \uparrow_{B})\) is isomorphic to \(\mathbb{F}[x]/(x^3)\) by Theorem 4.2 and Lemma 6.3 and hence has dimension three.

From now on, we write \(\Phi\) for \(\Phi_{B, \bar{B}}\). Also, following [14], write \(L_5\) (resp. \(\tilde{L}_5\)) for the indecomposable direct summand of \(D^\bar{\alpha} \uparrow_{B}\) (resp. \(D^\alpha \downarrow_{B}\)). Thus, \(D^\bar{\alpha} \uparrow_{B} \cong \tilde{L}_5 \oplus \bar{\alpha}\) and \(D^\alpha \downarrow_{B} \cong L_5 \oplus \bar{\alpha}\).

**Proposition 6.4.** There exists \(\phi \in \text{End}_{B}(P(D^\bar{\alpha}))\) such that \(\phi(P(D^\bar{\alpha})) = \tilde{L}_5, \phi(\tilde{L}_5) = D^\bar{\alpha} \downarrow_{B}, \phi(D^\bar{\alpha} \downarrow_{B}) = D^\bar{\alpha}\).

**Proof.** Since \(P(D^\bar{\alpha}) \cong D^\bar{\alpha} \uparrow_{B}\) and \([P(D^\bar{\alpha}) : D^\bar{\alpha}] = 4\), we see that \(\text{End}_{\mathcal{B}}(P(D^\bar{\alpha})) \cong \mathbb{F}[x]/(x^4)\) by Theorem 4.2. Let \(\phi \in \text{End}_{\mathcal{B}}(P(D^\bar{\alpha}))\) such that \(\{1, \phi, \phi^2, \phi^3\}\) is a basis for \(\text{End}_{\mathcal{B}}(P(D^\bar{\alpha}))\). Then \(P(D^\bar{\alpha}), P\phi(D^\bar{\alpha}), \phi^2(D^\bar{\alpha})\) and \(\phi^3(P(D^\bar{\alpha}))\) are submodules of \(P(D^\bar{\alpha})\) with simple head \(D^\bar{\alpha}\), with decreasing multiplicity of \(D^\bar{\alpha}\) as a composition factor. The Proposition thus follows from (the analogue of) [14] Proposition 5.2.

**Lemma 6.5** ([14] Proposition 4.6)). Let \(D^\lambda\) be a non-exceptional simple module of \(B\), and let \(\lambda = \Phi(\lambda)\). The following table provides all the possible composition multiplicities of \(D^\lambda\) (resp. \(\bar{\lambda}\)) in \(P(D^\alpha), L_5\) and \(D^\bar{\alpha} \uparrow_{B}\) (resp. \(P(D^\bar{\alpha}), \tilde{L}_5\) and \(D^\bar{\alpha} \downarrow_{B}\)).

| \([P(D^\bar{\alpha}) : D^\bar{\alpha}]\) | \([L_5 : D^\lambda]\) | \([D^\bar{\alpha} \downarrow_{B} : D^\lambda]\) | \([D^\bar{\alpha} \uparrow_{B} : D^\lambda]\) | \([L_5 : D^{\bar{\lambda}}]\) | \([P(D^\alpha) : D^\alpha]\) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| I | 1 | 0 | 0 | 1 | 2 | 3 |
| II | 2 | 1 | 0 | 0 | 1 | 2 |
| III | 3 | 2 | 1 | 0 | 0 | 1 |
| IV(A) | 4 | 2 | 1 | 1 | 2 | 4 |
| IV(B) | 4 | 2 | 0 | 0 | 2 | 4 |
As a corollary, we have the following:

**Corollary 6.6.** In cases I, III and IV(A) of Lemma 6.5, we have \( \sigma_p(\alpha) \neq \sigma_p(\lambda) \).

**Proof.** In these cases, we have either \( D^\lambda \) occurring as a composition factor of the heart of \( D^{\mathfrak{r} B} \), or \( D^{\Phi(\lambda)} \) occurring as a composition factor of the heart of \( D^{\mathfrak{r} B} \). By Lemma 6.3 the hearts of \( D^{\mathfrak{r} B} \) and \( D^{\mathfrak{r} B} \) are semi-simple. Thus, \( \sigma_p(\lambda) = \sigma_p(\lambda) \neq \sigma_p(\alpha) = \sigma_p(\rho) \) in both instances by Theorem 5.1(2) and Lemma 5.8.

**Proposition 6.7.** In cases II and IV(B) of Lemma 6.5, we have \( \sigma_p(\alpha) = \sigma_p(\lambda) \).

To prove Proposition 6.7, we use the \( v \)-decomposition numbers \( d_{\lambda \mu}(v) \) arising from the canonical basis of the Fock space representation \( F \) of \( U_v(\mathfrak{sl}_p) \).

Suppose that the fixed abacus display of the \( p \)-core of \( B \) has \( t \) beads. Let \( r \) be the residue class of \( (i - t) \) modulo \( p \). In this section, we write \( e \) and \( f \) for the elements \( e_r \) and \( f_r \) of \( U_v(\mathfrak{sl}_p) \) respectively. Furthermore, \( e(2) = e^2/(v + v^{-1}) \) and \( f(2) = f^2/(v + v^{-1}) \).

**Lemma 6.8.** The table below records the effects of \( e(2) \) and \( f(2) \) on the standard basis elements of \( F \) labelled by exceptional partitions in \( B \) and \( \overline{B} \) respectively. The entry on a row labelled by \( \rho \) and a column labelled by \( \sigma \) is \( \langle e(2) s(\rho), s(\sigma) \rangle \) (\( = (s(\rho), f(2) s(\sigma)) \)).

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) |
|---|---|---|---|
| \( v^{-2} \) | \( v^{-1} \) | \( 1 \) | \( 0 \) |
| \( v^{-1} \) | \( 1 \) | \( 0 \) | \( 1 \) |
| \( 1 \) | \( 0 \) | \( 1 \) | \( v \) |
| \( 0 \) | \( 1 \) | \( v \) | \( v^2 \) |

**Lemma 6.9.** \( G(\alpha) = s(\alpha) + vs(\beta) + v^2 s(\gamma) + v^3 s(\delta) \) and \( G(\overline{\alpha}) = s(\overline{\alpha}) + vs(\overline{\beta}) + v^2 s(\overline{\gamma}) + v^3 s(\overline{\delta}) \).

**Proof.** Since \( \overline{\alpha} \) has weight 0, we see that \( G(\overline{\alpha}) = s(\overline{\alpha}) \) (\( \overline{\alpha} \) is a composition of \( \alpha \)). This gives \( fG(\overline{\alpha}) = f(s(\overline{\alpha}) + vs(\overline{\beta}) + v^2 s(\overline{\gamma}) + v^3 s(\overline{\delta})) \). But this implies that \( fG(\overline{\alpha}) = G(\overline{\alpha}) \) so that the second assertion follows. Similar arguments apply to the first assertion.

**Proposition 6.10.** Suppose \( \lambda \) is in case II of Lemma 6.5 and let \( \overline{\lambda} = \Phi(\lambda) \). The following is a complete list of possible \( v \)-decomposition numbers \( d_{\overline{\mu} \lambda}(v) \), \( d_{\overline{\mu} \overline{\lambda}}(v) \) where \( \mu \) are \( \overline{\mu} \) are exceptional partitions in \( B \) and \( \overline{B} \) respectively.

| \( d_{\alpha \lambda}(v) \) | \( d_{\beta \lambda}(v) \) | \( d_{\gamma \lambda}(v) \) | \( d_{\delta \lambda}(v) \) | \( d_{\alpha \overline{\lambda}}(v) \) | \( d_{\beta \overline{\lambda}}(v) \) | \( d_{\gamma \overline{\lambda}}(v) \) | \( d_{\delta \overline{\lambda}}(v) \) |
|---|---|---|---|---|---|---|---|
| \( v^2 \) | \( v^3 \) | \( . \) | \( . \) | \( v^2 \) | \( v^3 \) | \( . \) | \( . \) |
| \( v^2 \) | \( . \) | \( v^2 \) | \( . \) | \( v^2 \) | \( v^3 \) | \( . \) | \( . \) |
| \( v^2 \) | \( . \) | \( . \) | \( v \) | \( . \) | \( v^2 \) | \( v^3 \) | \( . \) |
| \( . \) | \( v \) | \( . \) | \( v^2 \) | \( . \) | \( v \) | \( . \) | \( v \) | \( v \) | \( v \) | \( v \) | \( v \) | \( v \) | \( v \) |
Proof. By Lemma 5.14, we have $[L_5 : D^\lambda] = 1$. Thus, $P(D^\lambda)^B \cong P(D^\lambda)^{\oplus 2} \oplus P(D^\alpha)^{\oplus 2}$ by Frobenius reciprocity. These arguments carry over to the Iwahori-Hecke algebras in characteristic zero, so that $P(D^\lambda)^B \cong P(D^\lambda)^{\oplus 2} \oplus P(D^\alpha)^{\oplus 2}$, and hence $f(\alpha)G(\overline{\lambda}) = G(\alpha) + G(\overline{\lambda})$ by Proposition 4.4 (and the Note immediately after that). Thus,

$$d_{\mu}(v) + d_{\mu}(v) = \langle G(\alpha) + G(\overline{\lambda}), s(\mu) \rangle = \langle f(\alpha)G(\overline{\lambda}), s(\mu) \rangle = \langle G(\overline{\lambda}), e(\alpha) \rangle$$

for all partitions $\mu$. Varying $\mu$ over the exceptional partitions in $B$ and using Lemmas 6.8 and 6.9, we obtain the following four equations:

$$d_{\alpha}(v) + 1 = v^{-2}d_{\alpha}(v) + v^{-1}d_{\beta}(v) + d_{\gamma}(v)$$
$$d_{\beta}(v) + v = v^{-1}d_{\alpha}(v) + d_{\beta}(v) + d_{\delta}(v)$$
$$d_{\gamma}(v) + v^2 = d_{\alpha}(v) + d_{\gamma}(v) + vd_{\delta}(v)$$
$$d_{\delta}(v) + v^3 = d_{\beta}(v) + vd_{\gamma}(v) + v^2d_{\delta}(v)$$

As exactly two among $d_{\alpha}(v)$, $d_{\beta}(v)$, $d_{\gamma}(v)$ and $d_{\delta}(v)$ are zero, while the other two are monic monomials (since $[P(D^\alpha) : D^\lambda] = 2$), the above equations give all the possibilities as listed in the Proposition. We illustrate this with an example. Suppose $d_{\alpha}(v) = 0$. Then the last two equations give $d_{\alpha}(v) = v^2$, $d_{\beta}(v) = v^3$, $d_{\delta}(v) = d_{\gamma}(v) = 0$. Substituting these into the first two equations, we get $d_{\alpha}(v) = v^2$, $d_{\beta}(v) = v^3$. This is precisely the first row in the Proposition.

**Proposition 6.11.** Suppose $[P(D^\alpha) : D^\lambda] = 4$ with $\lambda \neq \alpha$, and let $\overline{\lambda} = \Phi(\lambda)$. The following is a complete list of possible $v$-decomposition numbers $d_{\mu}(v)$, $d_{\mu}(v)$ where $\mu$ and $\bar{\mu}$ are exceptional partitions in $B$ and $\bar{B}$ respectively.

$$d_{\alpha}(v) \quad d_{\beta}(v) \quad d_{\gamma}(v) \quad d_{\delta}(v) \quad d_{\alpha}(v) \quad d_{\beta}(v) \quad d_{\gamma}(v) \quad d_{\delta}(v)$$

$$\begin{array}{cccccccc}
v & v^2 & v & v^2 & v & v^2 & v & v^2 \\
v & v & v^2 & v & v^2 & v & v^2 & v
\end{array}$$

**Proof.** Note that $\lambda \notin \{\alpha, \beta, \gamma, \delta, m(\alpha)' \dot{\epsilon}, m(\beta)' \dot{\epsilon}, m(\gamma)' \dot{\epsilon}, m(\delta)' \dot{\epsilon}\}$. Thus if $\mu$ is an exceptional partition in $B$, then

$$d_{\mu}(v) = \begin{cases} v, & \text{if } \sigma_p(\lambda) \neq \sigma_p(\mu); \\
v^2, & \text{if } \sigma_p(\lambda) = \sigma_p(\mu) \end{cases}$$

by Corollary 5.2. As analogous statements hold for $d_{\mu}(v)$, the Proposition follows immediately from Lemma 5.3.

**Proof of Proposition 6.7.** By Proposition 6.10 and Corollary 5.2, we see that $\sigma_p(\lambda) = \sigma_p(\mu)$ if $\lambda$ is in case II of Lemma 6.5. By Proposition 6.11 and Corollary 5.2, it suffices to show that $d_{\overline{\alpha}}(v) \neq v$, where $\overline{\lambda} = \Phi(\lambda)$, when $\lambda$ is in case IV(B) of Lemma 6.5. Indeed, in this case, we have $[D^\alpha \mid_{\bar{B}} : D^\lambda] = 0$ by Lemma 6.5 so that $P(D^\lambda)^{\bar{B}} \cong P(D^{\overline{\lambda}})$ by Frobenius reciprocity and Theorem 6.2(1), where $\overline{\lambda} = \Phi_{B,\bar{B}}(\lambda)$. These arguments carry over to the Iwahori-Hecke algebra in characteristic zero, so that $P(D^\lambda)^{\bar{B}} \cong P(D^{\overline{\lambda}})$, and hence $fG(\overline{\lambda}) = G(\overline{\lambda})$ by Proposition 4.6 (and the Note immediately after
that. Thus \( \langle fG(\lambda), s(\sigma) \rangle \in v\mathbb{N}_0[v] \). This implies that \( d_{\alpha\lambda}(v) = \frac{v^{-1}d_{\lambda\lambda}(v)}{v} \in v\mathbb{N}_0[v] \), so that \( d_{\alpha\lambda}(v) \neq v \).

Remark. The sufficient condition (Y3) in [14] holds trivially by Proposition 6.6 and Theorem 5.1(2).

With Proposition 6.7 we conclude that \( \text{Ext}^1(D^\lambda, D^\alpha) = 0 = \text{Ext}^1(D^\tilde{\lambda}, D^\tilde{\alpha}) \) when \( \lambda \) is in Cases II or IV(B). We are also able to conclude that \( \text{Ext}^1(D^\lambda, D^\alpha) \) is non-zero when \( \lambda \) is in Cases III or IV(A) since in these cases, \( D^\lambda \) lies in the semi-simple heart of \( D^{\mathfrak{p}1+}\). However, it is as yet unclear if \( \text{Ext}^1(D^\lambda, D^\alpha) \) is non-zero when \( \lambda \) is in Case I, even though we know that \( \sigma_p(\lambda) \neq \sigma_p(\alpha) \) by Corollary 5.6. We now address this issue.

Lemma 6.12. Suppose \( [P(D^\tilde{\alpha}) : D^\tilde{\lambda}] = 1 = [S^\tilde{\alpha} : D^\tilde{\lambda}] \). Then \( \tilde{\alpha} \) is \( p \)-restricted and \( \tilde{\alpha} = m(\lambda)' \).

Proof. By Lemma 6.5 we have \( [\tilde{L}_5 : D^\lambda] = 0 \). Thus, \( P(D^\lambda)^B \cong P(D^\lambda)^{\oplus 2} \) by Frobenius reciprocity. These arguments carry over to the Iwahori-Hecke algebra in characteristic zero, so that \( P(D^\lambda)^B \cong P(D^\lambda)^\oplus 2 \), and hence \( f^{(2)}G(\tilde{\lambda}) = G(\lambda) \) by Proposition 4.6 (and the Note immediately after that). Thus,

\[
v^{-2}d_{\alpha\lambda}(v) = \langle f^{(2)}G(\tilde{\lambda}), s(\alpha) \rangle = \langle G(\lambda), s(\alpha) \rangle \in v\mathbb{N}_0[v].
\]

This implies that \( d_{\alpha\lambda}(v) = v^3 \) and \( \tilde{\alpha} = m(\lambda)' \) by Corollary 5.2.

Proposition 6.13. Suppose \( [P(D^\tilde{\alpha}) : D^\tilde{\lambda}] = 1 \). Then \( \text{Ext}^1(D^\tilde{\lambda}, D^\tilde{\alpha}) = 0 \).

Proof. Suppose for the sake of contradiction that \( \text{Ext}^1(D^\tilde{\lambda}, D^\tilde{\alpha}) \neq 0 \). If \( \tilde{\beta} \) is \( p \)-regular, then \( D^\tilde{\lambda} \) lies in the second radical layer of \( S^\tilde{\alpha} \) by Corollary 5.6 of [14]. However, by Lemma 6.12 we also have \( \tilde{\alpha} \) is \( p \)-restricted and \( \tilde{\alpha} = m(\lambda)' \), so that \( D^\tilde{\lambda} \) occurs as the socle of \( S^\tilde{\alpha} \). This is impossible, since \( S^\tilde{\alpha} \) has a nonzero heart, with composition factor such as \( m(\beta') \) by Lemma 6.11 (note that \( \tilde{\beta} \) is \( p \)-restricted since \( \tilde{\alpha} \) is).

If \( \tilde{\gamma} \) is \( p \)-restricted, then applying the argument in the last paragraph to the conjugate blocks of \( (B, \tilde{B}) \), we obtain \( \text{Ext}^1(D^{m(\tilde{\lambda})}, D^{m(\tilde{\alpha})}) \neq 0 \), so that \( \text{Ext}^1(D^\tilde{\lambda}, D^\tilde{\alpha}) \neq 0 \).

Since we cannot have \( \tilde{\beta} \) being \( p \)-singular and \( \tilde{\gamma} \) being non-\( p \)-restricted at the same time, we are done.

Remark.

1. The analogue of Proposition 6.13 also holds — if \( [P(D^\alpha) : D^\lambda] = 1 \), then \( \text{Ext}^1(D^\lambda, D^\alpha) = 0 \). Its proof however is more complicated, as \( \alpha \) being \( p \)-restricted no longer implies \( \beta \) (or \( \gamma \)) being \( p \)-restricted; also it is possible for \( \beta \) to be \( p \)-singular and \( \gamma \) to be non-\( p \)-restricted at the same time.

2. By Proposition 6.13 and its analogue, it is straightforward to see that the sufficient condition (Y2) in [14] holds.

Proposition 6.14. The radical length of \( \tilde{L}_5 \) is 5.
Proof. Let $\widetilde{M}$ be the submodule of $\widetilde{L}_5$ such that $\widetilde{L}_5/\widetilde{M} \cong D^\alpha_{\tilde{B}}$. Since $\tilde{M}$ is the largest submodule of $\tilde{M}$ with $[\tilde{M} : D^\alpha] = 1$, and $[\text{rad}(D^\alpha_{\tilde{B}}) : D^\alpha] = 1$, we have $\text{rad}(D^\alpha_{\tilde{B}}) \subseteq \tilde{M}$. By Lemma 6.16 the composition factors of $\tilde{M}/\text{rad}(D^\alpha_{\tilde{B}})$ are in cases II or IV(B), and hence are indexed by partitions having the same relative sign as $\tilde{M}$ by Proposition 6.7 and Lemma 5.3. Thus, these composition factors lie in an odd radical layer of $\tilde{L}_5$, and hence in the third radical layer, since $\widetilde{L}_5/\tilde{M} \cong D^\alpha_{\tilde{B}}$ has radical length 3 by Lemma 6.3. This implies that $\tilde{L}_5$ has radical length 5.

Theorem 6.15. The radical length of $P(D^\tilde{\alpha})$ is 7.

Proof. Let $\phi \in \text{End}_{\tilde{B}}(P(D^\tilde{\alpha}))$ be as described in Proposition 6.4. Then

$$\text{rad}(D^\alpha_{\tilde{B}}) = \ker(\phi) \cap D^\alpha_{\tilde{B}} = \ker(\phi|_{\tilde{L}_5}) \cap D^\alpha_{\tilde{B}}.$$ 

Let $M = D^\alpha_{\tilde{B}} + \ker(\phi)$, and $N = D^\alpha_{\tilde{B}} + (\ker(\phi) \cap \tilde{L}_5) = D^\alpha_{\tilde{B}} + \ker(\phi|_{\tilde{L}_5})$, and consider the filtration

$$0 \subseteq D^\alpha_{\tilde{B}} \subseteq N \subseteq M \subseteq P(D^\tilde{\alpha}).$$ 

We have $P(D^\tilde{\alpha})/M \cong (P(D^\tilde{\alpha})/\ker(\phi))/(M/\ker(\phi))$, and $P(D^\tilde{\alpha})/\ker(\phi) \cong \phi(P(D^\tilde{\alpha})) = \tilde{L}_5$ while

$$M/\ker(\phi) \cong D^\alpha_{\tilde{B}}/(D^\alpha_{\tilde{B}} \cap \ker(\phi)) = D^\alpha_{\tilde{B}}/\text{rad}(D^\alpha_{\tilde{B}}) = D^\tilde{\alpha},$$

so that $P(D^\tilde{\alpha})/M \cong \tilde{L}_5/D^\tilde{\alpha}$. Thus $P(D^\tilde{\alpha})/M$ has radical length 4 by Proposition 6.14. Next, $M/N = (N + \ker(\phi))/N \cong \ker(\phi)/(N \cap \ker(\phi)) = \ker(\phi)/(\ker(\phi|_{\tilde{L}_5}) + (D^\alpha_{\tilde{B}} \cap \ker(\phi))) = \ker(\phi)/\ker(\phi|_{\tilde{L}_5})$. By Lemma 6.5 we see that the composition factors of $M/N$ are in cases I or IV(A), and are thus labelled by partitions having the same relative sign, which is different from that of $\tilde{M}$ by Corollary 6.6 and Lemma 5.3. Thus they lie in even radical layers of $P(D^\tilde{\alpha})$ by Theorem 5.1(2). Since $P(D^\tilde{\alpha})/M$ has radical length 4, this implies that $P(D^\tilde{\alpha})/N$ also has radical length 4. Now, as $N \subseteq \text{rad}(\tilde{L}_5)$, and hence has radical length at most 4 by Proposition 6.14, we see that $P(D^\tilde{\alpha})$ has radical length at most 8. But at the same time, the radical length of $P(D^\tilde{\alpha})$ is greater than 5 (since $L_5$ is a proper submodule and has radical length 5 by Proposition 6.14), and must be odd (since $D^\alpha$ and hence $P(D^\alpha)$ are self-dual and the Ext-quiver of $\tilde{B}$ is bipartite). The Theorem thus follows.

Remark. In view of the symmetry between $B$ and $\tilde{B}$, statements and proofs entirely analogous to Proposition 6.14 and Theorem 6.15 also hold for the block $B$.

Corollary 6.16 (of proof). Let $\phi$ be as described in Proposition 6.4. The radical length of $\ker(\phi)$ is at most 4.

Proof. Keeping the notations used in the proof of Theorem 6.15 we have proved the following:

- The radical length of $N$ is at most 4;
- The composition factors of $M/N$ are labelled by partitions all having the same relative sign, which is different from that of $\tilde{M}$. 

This shows that the radical length of $M$ is at most 4 too, since $M$ (as well as $N$) has a simple socle $D^\alpha$. Thus, the same holds for $\ker(\phi)$ since $\ker(\phi)$ is a submodule of $M$.

\textbf{Proposition 6.17.} Let $D^\lambda$ be a non-exceptional simple module of $\bar{B}$ and suppose $[P(D^\alpha):D^\lambda] > 0$. Then the radical length of $P(D^\lambda)$ is at least 7.

\textit{Proof.} We show $\text{rad}^6(P(D^\lambda)) \neq 0$. Note that if the radical length of $P(D^\lambda)$ is $\ell$, then $\ell$ is odd by Theorem 5.12, and $\text{rad}^{\ell-1}(P(D^\lambda)) = D^\lambda$. We consider the various cases of Lemma 6.5 separately.

\textbf{Cases III and IV(A):} For these cases, $D^\lambda$ is a composition factor of $\text{rad}(D^{\pi}_{\downarrow B})$ by Lemma 6.5. Since $D^{\pi}_{\downarrow B} \subseteq \text{rad}(L_5)$ and $L_5 \subseteq \text{rad}^2(P(D^\alpha))$, we see that $D^\lambda$ is a composition factor of $\text{rad}^5(P(D^\alpha))$. As the simple modules of symmetric groups are self-dual, this implies that $D^\alpha$ is a composition factor of $\text{rad}^5(P(D^\lambda))$. Thus $\text{rad}^6(P(D^\lambda)) \neq 0$.

\textbf{Cases II and IV(B):} For these cases, $D^\lambda$ is a composition factor of $L_5$ by Lemma 6.5. Since $\sigma(\bar{\lambda}) = \sigma(\bar{\alpha})$ for these cases by Proposition 6.7, we see that in fact $D^\lambda$ is a composition factor of $\text{rad}^2(L_5)$, and hence of $\text{rad}^4(P(D^\alpha))$. By self-duality of simple modules, this implies that $D^\alpha$ is a composition factor of $\text{rad}^4(P(D^\lambda))$. Thus $\text{rad}^6(P(D^\lambda)) \neq 0$.

\textbf{Case I:} Since $\sigma(\bar{\alpha}) \neq \sigma(\bar{\lambda})$ by Corollary 6.6 and $\text{Ext}^1(D^\lambda, D^\alpha) = 0$ by Proposition 6.13, $D^\lambda$ is a composition factor of $\text{rad}^3(P(D^\lambda))$. By self-duality of simple modules, this implies that $D^\alpha$ is a composition factor of $\text{rad}^3(P(D^\lambda))$. Now if $\text{rad}^6(P(D^\lambda)) = 0$, then $\text{rad}^4(P(D^\lambda)) = D^\lambda$, so that $D^\alpha$ extends $D^\lambda$, a contradiction.

The next result is on general representation theory of finite groups and will be needed in the proof of Proposition 5.6.

\textbf{Proposition 6.18.} Let $H$ be a subgroup of a finite group $G$, and let $B$ be a block of $\mathbb{F}G$ and $C$ be a block of $\mathbb{F}H$. Suppose $N$ is a submodule of a $C$-module $M$ such that for any simple $B$-module $S$, and any $\phi \in \text{Hom}_C(M, S_{\downarrow C})$, we have $N \subseteq \ker(\phi)$. Then $N^{\uparrow B} \subseteq \text{rad}(M^{\uparrow B})$.

\textit{Proof.} This follows from Frobenius reciprocity. Suppose $N^{\uparrow B} \notin \text{rad}(M^{\uparrow B})$. Then there exists a simple $B$-module $S$ and a map $\psi \in \text{Hom}_{\mathbb{F}G}(M^{\uparrow G}, S)$ such that $\psi(N^{\uparrow G}) \neq 0$. Thus $\psi(1 \otimes n) \neq 0$ for some $1 \otimes n \in \mathbb{F}G \otimes_{\mathbb{F}H} N = N^{\uparrow G}$. Define $\phi : M \rightarrow S_{\downarrow C}$ by $\phi(m) = e_C \psi(1 \otimes m)$, where $e_C$ is the block idempotent of $C$. Then $\phi \in \text{Hom}_C(M, S_{\downarrow C})$, and

$$\phi(n) = e_C \psi(1 \otimes n) = \psi(1 \otimes e_C n) = \psi(1 \otimes n) \neq 0,$$

a contradiction.

We conclude this paper with a proof of Proposition 5.6.
Proof of Proposition 5.6. Let $D^\lambda$ be a simple module of $\widetilde{B}$, and let $l$ be the radical length of $P(D^\lambda)$. If $[P(D^\lambda) : D^\alpha] = 0$, then $P(D^\lambda)$ has the same radical length as $P(D^\lambda) \uparrow^B$, so that $l = 7$. We thus assume that $[P(D^\lambda) : D^\alpha] > 0$, and we consider the cases of $\widetilde{\lambda}$ having the same and different relative sign as $\widetilde{\alpha}$ separately.

Case 1. $\sigma_p(\widetilde{\lambda}) = \sigma_p(\widetilde{\alpha})$:

Subcase 1a. $\widetilde{\lambda} = \widetilde{\alpha}$: This is dealt with in Theorem 6.15.

Subcase 1b. $\widetilde{\lambda} \neq \widetilde{\alpha}$: By Proposition 6.17 $l \geq 7$. Suppose that $l > 7$; so we have $l \geq 7$. Let $M$ be a submodule of $\text{rad}^2(P(D^\lambda))$ having a simple head, say $D^\mu$, and radical length $l - 2$. Note that $\mu \neq \alpha$, since $P(D^\alpha)$ has radical length 7. Let $N$ be a submodule of $\text{rad}^2(M)$ having a simple head, say $D^\nu$, and radical length $l - 4$. Clearly, $[N : D^\nu] \leq 1$ with equality if and only if $\nu = \alpha$. Suppose for a contradiction that $\nu = \alpha$. Embed $N^*$ (the dual of $N$) into $P(D^\alpha)$. Let $\phi \in \text{End}_B(P(D^\alpha))$ be as described in Proposition 6.14. Then $N^* \subseteq \text{ker}(\phi)$ since $\text{ker}(\phi)$ is the unique maximal submodule of $P(D^\alpha)$ such that $[\text{ker}(\phi) : D^\alpha] \leq 1$. But $\text{ker}(\phi)$ has radical length at most 4 by Corollary 6.16 contradicting $N^*$ having radical length $l - 4$ ($\geq 5$). Thus, $\nu \neq \alpha$ and hence $[N : D^\alpha] = 0$. This implies that $N \uparrow^B$ has radical length $l - 4$. Note that $M$ and $N$ satisfy the hypothesis of Proposition 6.18 thus $N \uparrow^B \subseteq \text{rad}(M \uparrow^B)$. Furthermore, by Frobenius reciprocity, only $D^\mu$, where $\mu = \Phi^{-1}_{B,B}(\mu)$, and possibly $D^\alpha$ occur in the head of $M \uparrow^B$, while the head of $N \uparrow^B$ is isomorphic to $D^\nu \uparrow^B$ ($\cong (D^\nu)^{\otimes 2}$ say). Since $\sigma_p(\mu) = \sigma_p(\nu) = \sigma_p(\alpha)$, we see that in fact $N \uparrow^B \subseteq \text{rad}^2(M \uparrow^B)$, so that $M \uparrow^B$ has radical length at least $l - 2$. Since $P(D^\lambda)$ and $M$ also satisfy the hypothesis of Proposition 6.18 we have $M \uparrow^B \subseteq \text{rad}(P(D^\lambda) \uparrow^B)$, so that $M \uparrow^B$ has radical length at most 6. This gives $l - 2 \leq 6$, a contradiction.

Case 2. $\sigma_p(\widetilde{\lambda}) \neq \sigma_p(\widetilde{\alpha})$: Let $P$ be the projective cover of $\text{rad}(P(D^\lambda))$. Then $P$ is a direct sum of projective indecomposable modules which are all indexed by partitions having the same relative sign as $\alpha$. From Case (1), we conclude that $P$ has radical length 7, so that $\text{rad}(P(D^\lambda))$ has radical length at most 6. Thus $l \leq 7$, and hence $l = 7$ by Proposition 6.17.

$\Box$

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