Characterization of entanglement using sum uncertainty relations for N-level systems

Holger F. Hofmann and Shigeki Takeuchi
PRESTO, Japan Science and Technology Corporation (JST),
Research Institute for Electronic Science,
Hokkaido University, Sapporo 060-0812
Tel/Fax: 011-706-2648
e-mail: h.hofmann@osa.org

Abstract

The efficient experimental verification of entanglement requires an identification of the essential physical properties that distinguish entangled states from non-entangled states. Since the most characteristic feature of entanglement is the extreme precision of correlations between spatially separated systems, we propose a quantitative criterion based on local uncertainty relations (quant-ph/0212090). Some basic sum uncertainty relations for N-level systems are introduced and the amount of entanglement that can be verified by violations of the corresponding local uncertainty limit is discussed.

Keywords: entanglement detection, N-level entanglement, N-level uncertainties

1 Introduction

As the word itself suggests, entanglement is one of the most mysterious aspects of quantum physics. Although it is by now possible to verify the predictions of entanglement theory in a variety of experiments, there remains a considerable gap between the formal definition of entanglement and the observable effects that are associated with this property. The formal definition of entanglement is that any quantum state of two systems that cannot be decomposed into a mixture of product states of the two systems is an entangled state. However, the decomposition of a mixed state does not represent any real physical process, it is only one of many possible interpretations of the origin of the statistical mixture. Consequently, the physical meaning of this definition is unclear. Indeed, it remains a highly non-trivial task to identify physical properties that can be used to distinguish mixed entangled states from separable ones.

In order to identify some characteristic physical properties of entanglement, it may be useful to reconsider the original discussions of entanglement. These discussions focused on the fact that maximally entangled states have perfect correlations between every single observable property, that is, the outcome of any measurement of a property A in system one can be predicted by measuring a corresponding property B in system two. In a classical system, this would not be surprising at all. System one is simply an exact copy of system two. However, such a high level of precision appears to contradict the uncertainty principle. Indeed, complete knowledge of the local quantum states of two systems requires that there are no correlations between the two systems, so that the uncertainties of the two pure states represent statistically independent fluctuations of the respective measurement outcomes. Therefore, the correlations that exist in mixtures of local states can never overcome the limits set by local uncertainties. Only the strong correlations between measurement fluctuations associated with entangled states can violate such local uncertainty limits. The definition of local uncertainty relations can thus provide a directly observable measure of entanglement. This measure of entanglement naturally provides an evaluation of the usefulness of an entangled state for applications such as quantum teleportation and dense coding, where the main purpose of entanglement is the suppression of uncertainty related noise in the transfer of information. It may indeed be an interesting question whether the useful properties of entanglement really arise from the theoretical lack of separability, or if this formal requirement of entanglement is only a necessary condition for the improvement of precision described by the violation of local uncertainties.

There remains one fundamental problem that has to be solved in order to derive measures of entangle-
moment from local uncertainty relations. Previously, uncertainty arguments were mostly restricted to continuous variables, where a non-vanishing uncertainty limit for the product of position and momentum uncertainties provides a useful definition of local uncertainties. Unfortunately, such product uncertainties do not provide any real uncertainty limits for N-level systems because the product uncertainties will always be zero for eigenstates of one of the two properties. It is therefore necessary to find a different formulation of uncertainty limits for the properties of N-level systems. As we have pointed out recently, such uncertainty limits can be obtained by using the sum of uncertainties instead of their product \( 3 \). It is then possible to obtain sum uncertainty relations for any set of operators, although in general, the determination of the correct uncertainty limit may be quite difficult. In the following, we present some typical examples of sum uncertainty relations and discuss their applications to the experimental characterization and quantification of entanglement.

2 Uncertainty relations for N-level systems

The uncertainty \( \delta A_i^2 \) of a hermitian operator property \( \hat{A}_i \) is defined as the variance of the measurement statistics of a precise measurement of \( \hat{A}_i \). For a given density matrix \( \rho \), this uncertainty is given by

\[
\delta A_i^2 = \text{Tr}(\rho \hat{A}_i^2) - \left( \text{Tr}(\rho \hat{A}_i) \right)^2.
\]

(1)

The uncertainty of \( \hat{A}_i \) can only be zero if \( \rho \) represents an eigenstate of \( \hat{A}_i \). Therefore, the sum of all uncertainties for a given set of operators \( \{\hat{A}_i\} \) can only be zero if all the operators in the set have at least one common eigenstate. In all other cases, there exists a non-trivial uncertainty limit given by

\[
\sum_i \delta A_i^2 \geq U.
\]

(2)

In principle, there are infinitely many sum uncertainty relations, since any set of operators may be used. However, in most cases, the physical situation considered will determine which choices are useful. In the following, we will focus on the angular momentum operators \( \hat{L}_x, \hat{L}_y, \) and \( \hat{L}_z \) of a spin \( l \) system. In general, this spin algebra can be defined for any N-level system, where \( N = 2l + 1 \). In many cases, the components of the spin algebra will even have a clear physical meaning. For example, these operators can also be used to describe the Stokes parameters \( \hat{S}_i \) of an \( n \) photon state, where \( n = 2l \) and \( \hat{S}_1 = 2\hat{L}_x, \hat{S}_2 = 2\hat{L}_y, \) and \( \hat{S}_3 = 2\hat{L}_z \).

The uncertainty relation for all three spin components of an N-level system can be derived directly from the properties of the operators,

\[
\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = l(l+1)
\]

and \( (\hat{L}_i)^2 \leq l^2 \).

(3)

Since the spin system is isotropic, the expectation value limit of one component is enough to define the limit for the total averaged spin vector. The uncertainty relation then reads

\[
\delta L_x^2 + \delta L_y^2 + \delta L_z^2 \geq l.
\]

(4)

Likewise, the uncertainty relation for the Stokes parameters of an \( n \) photon system reads

\[
\delta S_1^2 + \delta S_2^2 + \delta S_3^2 \geq 2n.
\]

(5)

In the case of a single photon (\( n = 1 \)), the Stokes parameters are equivalent to the Pauli matrices. This means that the single photon polarization state can be interpreted more generally as a physical implementation of a single qubit. Relation \( 4 \) is therefore particularly useful for the description of quantum bits. In particular, the uncertainty limit of two at \( n = 1 \) expresses the fact that information can only be encoded in a single component of the spin vector.

It is also of interest to know the uncertainty relations for only two spin components. However, it is a non-trivial task to derive the proper uncertainty limit. Here, we would only like to note that the best strategy to minimize the uncertainty of \( \hat{L}_x \) and \( \hat{L}_y \) seems to be the application of spin squeezing to a coherent spin state, e.g. the eigenstate of \( \hat{L}_x \) with \( L_x = +l \). Spin squeezing can then redistribute the quantum fluctuations of the state from \( \hat{L}_y \) to \( \hat{L}_z \) until the gradual increase in \( \delta L_x^2 \) prevents any further reduction in the sum uncertainty. However, care must be taken to ensure that the minimum obtained is actually the global minimum for all possible quantum states.

For two level systems, it is still very easy to obtain the uncertainty limit for \( \hat{L}_x \) and \( \hat{L}_y \), since every eigenstate of a spin component has the same maximal amount of uncertainty in the other two components and a redistribution of uncertainty by spin squeezing is not possible. The uncertainty relations therefore read

\[
N = 2 \text{ uncertainties} : \quad \delta L_x^2 + \delta L_y^2 \geq \frac{1}{4}
\]

or

\[
\delta S_1^2 + \delta S_2^2 \geq 1.
\]

As mentioned above, the relation for the Stokes parameters also represents the uncertainty for two of the
three Pauli matrices of a general quantum bit. Therefore, relation (11) is a very compact formulation of the limitations of encoding and readout for any quantum communication scheme that encodes information in the eigenstates of both $\hat{S}_1$ and $\hat{S}_2$.

For three level systems, the uncertainty distribution is variable, and the minimum uncertainty state must be determined by optimizing this distribution. However, the three dimensional Hilbert space is still small enough to allow an analytical determination of the global maximum. The result we have obtained was first reported in [3] and reads

$$N = 3 \text{ uncertainties} :$$

$$\delta L_x^2 + \delta L_y^2 \geq \frac{7}{16} \quad (8)$$

or $$\delta S_i^1 + \delta S_i^2 \geq \frac{7}{4} \quad (9)$$

As suggested above, the minimum uncertainty states for this relation are spin squeezed states with average spins in the xy-plane. In the $\hat{L}_z$ basis, these minimal uncertainty states read

$$|\phi\rangle = \frac{\sqrt{3}}{4} e^{-i\phi} | -1 \rangle + \frac{\sqrt{6}}{4} | 0 \rangle + \frac{\sqrt{3}}{4} e^{+i\phi} | +1 \rangle. \quad (10)$$

It may be interesting to note that these states are quite obviously not minimal uncertainty states of relation (4). The notion of minimal uncertainty states is therefore strongly dependent on the uncertainty relation selected for their definition. In general, all pure states are minimal uncertainty states for some set of uncertainty relations.

3 Formulation of local uncertainty limits

It is now possible to construct local uncertainty relations from any selection of basic uncertainties. As mentioned in the introduction, the uncertainties of local states cannot be correlated. Therefore, the uncertainty sums of any set of joint properties $\{\hat{A}_i + \hat{B}_i\}$ cannot be lower than the sum of the local uncertainty limits for $\{\hat{A}_i\}$ and $\{\hat{B}_i\}$.

- Limit for system A: $$\sum_i \delta A_i^2 \geq U_A$$
- Limit for system B: $$\sum_i \delta B_i^2 \geq U_B$$
- Local limit for A+B: $$\sum_i \delta (A_i + B_i)^2 \geq U_A + U_B \quad (11)$$

Any violation of this local uncertainty limit indicates that systems A and system B are entangled. (For a simple mathematical proof of this property, see [2].)

In general, it is not even necessary to choose the same type of uncertainty relation in each system. For example, it is possible to determine local uncertainty limits for $N \times M$ systems simply by selecting one N-level uncertainty and one M-level uncertainty. However, the basic principle of local uncertainty violations is best illustrated by symmetric relations, where the operators $\hat{A}_i$ and $\hat{B}_i$ describe the same physical properties in system A and system B, respectively. For general $N \times N$ entanglement, such a local uncertainty limit can be obtained from relation (11) or (12), respectively. According to these uncertainty limits, separable states must fulfill the conditions

$$\sum_{i=x,y,z} \delta (L_i(A) + L_i(B))^2 \geq 2l, \quad (12)$$

$$\sum_{i=1}^3 \delta (S_i(A) + S_i(B))^2 \geq 4n. \quad (13)$$

The maximal violation of these local uncertainty relations is obtained for the singlet state, which has a total uncertainty of zero. Consequently, these uncertainty relations provide a measure of how close any given state is to the maximal entanglement described by the singlet state of the two spin l systems. This closeness can be quantified using the relative violation of local uncertainty [3].

$$C_{L3} = 1 - \frac{\sum_{i=x,y,z} \delta (L_i(A) + L_i(B))^2}{2l} \quad (14)$$

$$C_{S3} = 1 - \frac{\sum_{i=1}^3 \delta (S_i(A) + S_i(B))^2}{4n}. \quad (15)$$

These measures of entanglement allow a direct experimental evaluation of the errors in entanglement generation, especially if the experiment is intended to generate pure singlet state entanglement. No special assumptions about any detailed properties of the actual density matrix are necessary, although it is of course interesting to study the effects of some typical errors such as decoherence effects on the relative violation of local uncertainties. For example, the addition of white noise represents a kind of worst case scenario,

$$\hat{\rho}(p_W) = (1-p_W)|\text{sing.}\rangle \langle \text{sing.}| + p_W \frac{\hat{1} \otimes \hat{1}}{N^2} \quad (16)$$

The relation between the noise level $p_W$ and the relative violation of local uncertainty then reads

$$C_{L3/S3} = 1 - p_W \frac{N + 1}{2}. \quad (17)$$
It may be interesting to note that the effect of white noise errors on the relative violation of local uncertainty increases with the number of level $N$. However, it should be remembered that white noise errors become less and less likely in larger systems, as the nearly macroscopic properties of many level systems become easier to control. This example therefore also illustrates the problems associated with the choice of a noise model in larger Hilbert spaces.

4 Application to $2 \times 2$ entanglement

The most simple case of entanglement is that between a pair of two level systems. Since this kind of entanglement has been studied extensively in the context of application in quantum information technologies, much is known about the fundamental properties of such entanglement. Nevertheless, the application of local uncertainty relations can be very useful in the study of $2 \times 2$ entanglement, since it provides a greatly simplified access to some of the most characteristic features of this kind of entanglement.

In particular, we can consider the case of a pair of entangled photons with no local polarizations and different degrees of correlation between identical components of the Stokes vector. Such a state can be expressed as a mixture of four Bell states,

$$\hat{\rho}(p_S; p_i) = p_S |S\rangle\langle S| + p_1 |T1\rangle\langle T1| + p_2 |T2\rangle\langle T2| + p_3 |T3\rangle\langle T3|, \quad (18)$$

where $|S\rangle$ is the singlet state and the three states $|T_i\rangle$ are the triplet states defined by

$$|T_i\rangle = \lambda_i |A\rangle + |B\rangle, \quad (19)$$

The quantum state $\hat{\rho}(p_S; p_i)$ is entangled if the probability of one of the maximally entangled states exceeds one half. Specifically, if $p_S > 1/2$, the concurrence of the quantum state is given by

$$C = 2p_S - 1. \quad (20)$$

The concurrence is a precise measure of the total entanglement in the $2 \times 2$ system and is directly related to the entanglement of formation. It is therefore interesting to compare this basis independent measure of entanglement with the relative violation of local uncertainties.

The properties of the quantum state $\hat{\rho}(p_S; p_i)$ can be determined completely by measuring the three uncertainties $\delta(S_i(A) + S_i(B))^2$. For the quantum state $\hat{\rho}(p_S; p_i)$, these uncertainties are given by

$$\delta(S_i(A) + S_i(B))^2 = 4 - 4(p_S + p_i). \quad (21)$$

The corresponding violation of local uncertainty given by (6) reads

$$\sum_{i=1}^{3} \delta(S_i(A) + S_i(B))^2 = 8(1 - p_s) < 4,$$

$$C_{S3} = 2p_S - 1 = C. \quad (22)$$

In this case, the relative violation of local uncertainties is indeed equal to the concurrence. The local uncertainty relation (9) therefore identifies the physical properties that "concur". Local uncertainty relations can thus specify the actual physical situation represented by a given entangled state.

In order to reduce the experimental effort, it may also be desirable to apply the local uncertainty limit for two components,

$$\delta(S_1(A) + S_1(B))^2 + \delta(S_2(A) + S_2(B))^2 \geq 2. \quad (23)$$

In this case, there is a slight difference between the relative violation of local uncertainty and the concurrence,

$$\sum_{i=1}^{2} \delta(S_i(A) + S_i(B))^2 = 8(1 - p_s + p_3) < 2,$$

$$C_{S2} = 2p_S - 1 - 2p_3 \leq C. \quad (24)$$

The relative violation $C_{S2}$ of the local two component uncertainty may therefore serve as a lowest limit of the actual concurrence. This lowest estimate corresponds to the "worst case scenario" that $p_3$ is zero and that the uncertainty in $\hat{S}_3(A) + \hat{S}_3(B)$ has the maximal value consistent with the other two uncertainties. It is thus possible to identify the quantum state associated with this lowest estimate from only two measurement settings.

In the context of photon pair entanglement, it may also be useful to identify the precise relationship between coincidence counts and the joint uncertainties $\delta(S_i(A) + S_i(B))^2$. Experimentally, the correlation between Stokes parameter components is usually evaluated by comparing the maximal rate of coincidence counts observed at anti-correlated polarizer settings with the minimal rate of coincidence counts at correlated settings. The normalized difference between the two defines the visibility $V_i$ for the polarization component $i$. In the absence of local polarizations, this measurement result is related to the uncertainty by

$$\delta(S_i(A) + S_i(B))^2 = 2(1 - V_i). \quad (25)$$

It is therefore possible to derive an estimate of the concurrence directly from a measurement of the visibilities $V_1$ and $V_2$,

$$C \geq V_1 + V_2 - 1. \quad (26)$$
The visibilities thus allow an even greater simplification of the experimental evaluation of $2 \times 2$ entanglement. However, it should be pointed out that local uncertainties generally do include the effect of non-vanishing local polarizations and are therefore a more precise measure of the actual entanglement correlations.

5 Application to $3 \times 3$ entanglement

Recently, the experimental generation of entanglement in parametric downconversion has been extended to pairs of three level systems [5]. However, the great variety of possible error sources in such systems has made a precise analysis of the experimental results very difficult. We therefore believe that the local uncertainty relations can provide a powerful tool for a quantitative estimate of the amount of entanglement created in such experiments.

As mentioned in section 4, one possible type of error in the creation of $3 \times 3$ entanglement could be white noise of the form given by $\hat{\rho}(pW)$ in equation (16). For three level systems, the corresponding relative violation of the three component local uncertainty relation (4) reads

$$\sum_{i=x,y,z} \delta(\hat{L}_i(A) + \hat{L}_i(B))^2 = 4pW < 2,$$

$$C_L3 = 1 - 2pW. \quad (27)$$

It is also possible to apply the two component uncertainty using the uncertainty limit given by relation (5),

$$\sum_{i=x,y} \delta(\hat{L}_i(A) + \hat{L}_i(B))^2 = \frac{8}{3}pW < \frac{7}{8},$$

$$C_{L2} = 1 - \frac{64}{21}pW. \quad (28)$$

White noise therefore reduces the relative violation of the two component local uncertainty about 1.5 times faster than it reduces the relative violation of the three component uncertainty. However, this sensitivity ratio between the three component and the two component uncertainties can be quite different for less symmetric noise sources.

An alternative source of error could be decoherence in the $L_x$ basis, given by

$$\hat{\rho}(pD) = (1 - pD) |\text{sing.}\rangle \langle \text{sing.}| + \frac{pD}{3} \left( |-1; +1\rangle \langle -1; +1| + |0; 0\rangle \langle 0; 0| + |+1; -1\rangle \langle +1; -1| \right). \quad (29)$$

In this case, the uncertainty in $\hat{L}_x(A) + \hat{L}_x(B)$ always remains zero, while the other two uncertainties increase as they did for white noise. The three component local uncertainty then reads

$$\sum_{i=x,y,z} \delta(\hat{L}_i(A) + \hat{L}_i(B))^2 = \frac{8}{3}pD < 2,$$

$$C_L3 = 1 - \frac{4}{3}pD. \quad (30)$$

As expected, the reduction of the local uncertainty violation by decoherence is only $2/3$ of that caused by the same amount of white noise. Likewise, the effect of decoherence on the two component local uncertainty violation is reduced to one half,

$$\sum_{i=x,y} \delta(\hat{L}_i(A) + \hat{L}_i(B))^2 = \frac{4}{3}pD < \frac{7}{8},$$

$$C_{L2} = 1 - \frac{32}{21}pD. \quad (31)$$

As a result, the two component local uncertainty is nearly as good at quantifying the remaining entanglement in the presence of decoherence as the corresponding three component uncertainty. If decoherence is assumed to be the main source of error, it is therefore sufficient to characterize experimentally generated three level entanglement using only two measurement settings.

6 A short note on the interpretation of quantum statistics

As shown in this paper, entanglement is a very useful property because it represents strong non-local correlations that can overcome local uncertainty limits. However, the non-locality of these correlations appears to be no different from the non-locality of classical correlations. Since there is no direct evidence for action at a distance, it may be more prudent to interpret the collapse of the wavefunction as the reduction of a probability distribution based on new information. More mysterious effects such as the violation of Bell’s inequalities could then be interpreted in terms of local non-classical correlations [6].

The decomposition of density matrices into mixtures of pure states unfortunately tends to suggest that there might be a fundamental difference between quantum noise and classical noise, and that the collapse of the wavefunction should be somehow different from the selection of a classical subensemble. However, the uncertainty relations indicates that it may be more realistic to define only a quantitative limit - in the same way
that the speed of light defines the relativistic limit, without suggesting a separation of velocities into a relativistic and a non-relativistic component. As mentioned in the introduction, the ambiguity in the decomposition of density matrices into pure state mixtures indicates that there is no physical meaning in an arbitrary decomposition. The uncertainty principle can explain this observation by suggesting that even pure states are very noisy. For all practical purposes, pure states also represent probability distributions over many possible measurement outcomes, and there is no observable difference between classical noise and quantum noise. A separation of mixed states into pure states therefore divides the total ensemble into a rather arbitrary choice of subensembles. However, a pure state is not sufficiently "pure" to define the physical properties of an individual representative of the total probability distribution. The interpretational problems of quantum mechanics arise from the lack of any fundamental set of noise free subensembles that could identify the physical properties of individual systems as measurement independent "elements of reality".

7 Conclusions

The definition of uncertainty relations for arbitrary sets of operators describing the physical properties of N-level systems allows a generalized interpretation of entanglement as a violation of local uncertainty relations. The quantitative measure given by the relative violation of local uncertainties seems to correspond well with more theoretical measures such as the concurrence. Since the local uncertainties are defined as variances of actual measurement results they greatly simplify the theoretical background needed to device tests of experimental realizations of entanglement sources. Moreover, the violation of local uncertainties directly identifies the improvement of precision achieved by entanglement in applications such as dense coding or quantum teleportation.

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