Integrable models of coupled Heisenberg chains

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Abstract. – We show that the solutions of the Yang-Baxter equation invariant under the action of the Yangian \( Y(sl_2) \) lead to inhomogeneous vertex models. Starting from a four-dimensional representation of \( Y(sl_2) \), we obtain an integrable family of coupled Heisenberg spin-\( \frac{1}{2} \) chains. Some thermodynamical properties of this model are studied by means of the algebraic Bethe ansatz.

Integrable models of one-dimensional quantum systems and the related two-dimensional classical statistical models have provided much of the insight into correlation effects in low-dimensional systems. In particular, Bethe ansatz methods have been used to study the thermodynamical properties of these systems as well as the excitation spectrum and asymptotic behaviour of correlation functions.

The construction of integrable models within the Quantum Inverse Scattering Method (QISM) is based on solutions of the Quantum Yang-Baxter equation [1]

\[
R_{12}(\lambda - \mu) \ R_{13}(\lambda) \ R_{23}(\mu) = R_{23}(\mu) \ R_{13}(\lambda) \ R_{12}(\lambda - \mu) .
\]  

(1)

The central objects in this approach are the so-called \( R \)-matrices: \( R_{ij}(\lambda) \) is a linear operator acting on \( V_i \otimes V_j \) depending on a complex spectral parameter \( \lambda \). Hence (1) is an equation on the product space \( V_1 \otimes V_2 \otimes V_3 \) with certain finite-dimensional vector spaces \( V_i \). Given a solution of (1) one can construct a family of commuting operators (which include the Hamiltonian) and compute the spectrum using the algebraic Bethe ansatz.

Particular solutions of eq. (1) with rational dependence on the spectral parameter are found by looking for \( R \)-matrices invariant under the action of a simple Lie algebra [2]. Identifying \( V_i \) as the representation space of irreducible finite-dimensional representation of \( sl_2 \) this con-
struction leads to the series of higher-spin SU(2) Heisenberg chains [3] starting with the $S = \frac{1}{2}$ Hamiltonian $H = \sum_n S_n \cdot S_{n+1}$. On the other hand, eq. (1) for given $R_{12}$ can be interpreted as the definition of a quadratic (Hopf) algebra of operators on $V_1$ called Yangian. In the following we shall concentrate on the Yangian associated to $sl_2$ which will eventually result in the construction of SU(2)-invariant soluble spin chains. 

For the simplest so-called evaluation representations $V_m(a)$ of the Yangian $Y(sl_2)$ the generators $x$ and $J(x)$ of the algebra can be given in terms of irreducible $(m + 1)$-dimensional representations of $sl_2$ with generators $x \in \{x^+, x^-, h\}$ and $J(x) = iax$ with an arbitrary complex number $a$. As shown by Chari and Pressley every finite-dimensional irreducible representation of $Y(sl_2)$ is isomorphic to a tensor product of these evaluation representations [4]. Associated to such a representation of the Yangian Drinfel’d has proven that there exists a unique “universal $R$-matrix” which is a rational function of the spectral parameter [5]. Later, Chari and Pressley wrote down explicitly the solution of eq. (1) associated to a finite-dimensional irreducible representation of $Y(sl_2)$, i.e. $R$-matrices intertwining between two copies of this representation.

In this letter we show that these $R$-matrices can be interpreted as plaquettes containing several vertices of certain inhomogeneous 6 vertex models. This fact allows for the diagonalization of the corresponding transfer matrices using the algebraic Bethe ansatz. Specifically we study the quantum spin chain corresponding to the simplest representation of the Yangian beyond evaluation ones (which result in the known spin-$(m/2)$ Heisenberg models) namely tensor products of two evaluation representations $V_1(a) \otimes V_1(b)$. This leads to a model of two $S = \frac{1}{2}$ Heisenberg chains coupled by two- and three-spin exchange terms with coupling constants depending on a real parameter.

The approach used in ref. [4] to compute the solutions of (1) depends on the concept of the intertwiner $I(\alpha - \beta) = R(\alpha - \beta) \sigma$ (with the “switch map” $\sigma$). Given a tensor product of two evaluation representations $V_m(\alpha)$ and $V_n(\beta)$ the intertwiner is uniquely defined through the following properties: It interchanges the order of the factors, i.e.

$$V_m(\beta) \otimes V_m(\alpha) \longrightarrow V_m(\alpha) \otimes V_m(\beta),$$

the Yangian highest weight state $\Omega_m \otimes \Omega_m$ is mapped onto $\Omega_m \otimes \Omega_n$ and $I(\alpha - \beta)$ commutes with all elements of the Yangian, in particular with the action of $sl_2$. This has first been used by Kulish et al. [2] who determined the $R$-matrix associated to the product $V_m(\alpha) \otimes V_n(\beta)$. As shown in ref. [4] $R$-matrices associated to any tensor product of evaluation representations and hence any irreducible representation of $Y(sl_2)$ can be given as a product of such $R$-matrices involving only two factors of the product space.

Hence the simplest $R$-matrices beyond the ones of [2] are obtained by considering the tensor product of four-dimensional representations $V_A(\lambda)$ and $V_B(\mu)$ of $Y(sl_2)$

$$V_A(\lambda) \otimes V_B(\mu) = \frac{A}{1} \otimes \frac{B}{4} = V_1(\lambda) \otimes V_1(\lambda + \kappa) \otimes V_1(\mu) \otimes V_1(\mu + \epsilon).$$

(2)

Following Chari and Pressley the $R$-matrix on this space is given by $R_{AB}(\lambda) \equiv R_{23}(\lambda + \kappa) R_{24}(\lambda + \kappa - \epsilon) R_{13}(\lambda) R_{14}(\lambda - \epsilon)$, where $R_{kl}$ intertwines between the $k$-th and $l$-th factor in $V_A \otimes V_B$. For the case of two-dimensional (spin $\frac{1}{2}$) evaluation representations $V_1$ this is Yang’s $R$-matrix [6] ($\sigma_{kl}$ interchanges space $k$ and $l$)

$$R_{kl}(\lambda) = \frac{\lambda - i \sigma_{kl}}{\lambda - i}.$$  

(3)
In the language of vertex models $R_{\mathcal{M}}(\lambda - \mu)$ can be represented graphically by

$$
\begin{array}{c}
\lambda - \mu \\
\mu \\
\lambda - \mu \\
R(\lambda - \mu)
\end{array}
\quad
\begin{array}{c}
\lambda \\
\lambda \\
\lambda \\
R(0) = \sigma
\end{array}
$$

where one identifies horizontal lines with the matrix space $k$ and vertical ones with the quantum space $l$. In this framework the $R$-matrix on (2) is just a group of vertices in an inhomogeneous vertex model (fig. 1).

Multiplying in matrix space $N$ of these objects acting on different copies of the quantum space gives the monodromy matrix $T_{AB}(\lambda - \mu)$ satisfying a Yang-Baxter equation similar to (1) whose trace in matrix space, the so-called transfer matrix $T_{AB}(\lambda)$, commutes for different values of the spectral parameter $\lambda$. Hence $T_{AB}(\lambda)$ is a generating functional of commuting operators in quantum space. Obviously, $T_{AB}$ can be written as $T_{AB}(\lambda) = \tau(\lambda + \kappa) \otimes \tau(\lambda)$, where $\tau(\lambda)$ is the monodromy matrix of the inhomogeneous vertex model constructed from $R_{1B}(\lambda) \equiv R_{13}(\lambda) R_{14}(\lambda - \epsilon)$ (see (2) for the numbering of the different spaces). Hence we obtain the following relation between $T_{AB}$ and the transfer matrix $t(\lambda) = \text{tr} \, \tau(\lambda)$

$$
T_{AB}(\lambda) = t(\lambda + \kappa) \, t(\lambda) .
$$

Since $R_{1B}$ itself is a solution of a Yang-Baxter equation (1), we have $[t(\lambda), t(\mu)] = 0$. Consequently the diagonalization of $T_{AB}$ is equivalent to the solution of the eigenvalue problem for $t(\lambda)$ which is done in the framework of the algebraic Bethe ansatz (see, e.g., [7]). Starting from the ferromagnetic vacuum $|\Omega\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$ one constructs eigenstates of $T_{AB}(\lambda)$ with $M$ overturned spins parametrized by $\lambda_j$ with eigenvalues

$$
A(\lambda|\{\lambda_j\}) = \tilde{A}(\lambda|\{\lambda_j\}) \tilde{A}(\lambda + \kappa|\{\lambda_j\}) .
$$

Fig. 1. – Plaquette of elementary vertices in the $R$-matrix $R_{AB}(\lambda - \mu)$ on the tensor product (2).
Fig. 2. – Lattice on which the spin Hamiltonian (8) is defined: The two-spin exchange coupling is $2J_1$ and $J_3$ on full and dashed lines, respectively. The three-spin exchange $\propto J_2$ couples the spins on the corners of each triangle.

Here $\tilde{\Lambda}$ are the eigenvalues of $t(\lambda)$

$$\tilde{\Lambda}(\lambda|\{\lambda_j\}) = \prod_{j=1}^l \frac{\lambda_j - \lambda + \frac{i}{2}}{\lambda_j - \lambda - \frac{i}{2}} + \left(\frac{\lambda}{\lambda + i}\right)^N \left(\frac{\lambda - \epsilon}{\lambda - \epsilon + i}\right)^N \prod_{j=1}^l \frac{\lambda - \lambda_j + \frac{3}{2}i}{\lambda - \lambda_j + \frac{i}{2}}$$

and the $\lambda_j$ are solutions of the Bethe ansatz equations

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^N \left(\frac{\lambda_j - \epsilon + \frac{i}{2}}{\lambda_j - \epsilon - \frac{i}{2}}\right)^N = \prod_{j \neq k} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}.$$  

For the construction of a Hamiltonian with local interactions it is necessary that $T_{AB}(\lambda)$ degenerates to the shift operator for some value of $\lambda$. From $R(\lambda = 0) = \sigma$ for (3) and the unitarity condition $R(\lambda)\sigma R(-\lambda)\sigma = 1$ one obtains for $\epsilon = \kappa$ a one-parameter family of monodromy matrices allowing for the construction of Hamiltonians with local interactions. These have first been studied in [8].

The corresponding spin chain Hamiltonian is obtained by taking the logarithmic derivative of the transfer matrix $T_{AB}(\lambda)$ at the shift point $\lambda = 0$ leading to $\mathcal{H}(\kappa) = \sum_{n=1}^{N} \mathcal{H}_n(\kappa)$ with

$$\mathcal{H}_n(\kappa) = J_1 (S_{2n-1}S_{2n} + 2S_{2n}S_{2n+1} + S_{2n+1}S_{2n+2} - 1) +$$
$$+ J_2 (S_{2n}(S_{2n+2} \times S_{2n+1}) + S_{2n-1}(S_{2n} \times S_{2n+1})) +$$
$$+ J_3 (S_{2n+2}S_{2n+3} + S_{2n-1}S_{2n+1} - \frac{1}{2})$$

$$J_1 = \frac{1}{1 + \kappa^2}, \quad J_2 = \frac{2\kappa}{1 + \kappa^2}, \quad J_3 = \frac{\kappa^2}{1 + \kappa^2}.$$  

For $\kappa = 0$ the Hamiltonian reduces to that of a single spin-$\frac{1}{2}$ Heisenberg chain of length $L = 2N$, for $\kappa \to \infty$ one obtains two decoupled systems of length $L = N$ each. For intermediate values of the coupling $\kappa$ the Hamiltonian (8) describes spins on the lattice shown in fig. 2.

In the thermodynamic limit the solution of the Bethe ansatz equations (7) describing the ground state in the absence of a magnetic field is made up of $N$ real $\lambda$’s (as is the case in the Heisenberg model). Standard Bethe ansatz methods can be used to compute the ground-state energy:

$$\frac{E_0}{2N} = -\ln 2 - \frac{1}{4} \left(\psi\left(1 - \frac{i}{2}\kappa\right) - \psi\left(1 + \frac{i}{2}\kappa\right) + \psi\left(1 + \frac{i}{2}\kappa\right) - \psi\left(1 - \frac{i}{2}\kappa\right)\right).$$

The term containing digamma-functions $\psi(x)$ increases from $-\ln 2$ to $0$ as $\kappa$ varies between $0$ and $\infty$. 
As in the Heisenberg model the low-lying excitations above the ground state are made up of an even number of spin-1/2 objects [9] called spinons parametrized by real rapidities $\lambda$ with dispersion

$$
\epsilon(\lambda) = \tilde{\epsilon}(\lambda) + \tilde{\epsilon}(\lambda - \kappa), \quad k(\lambda) = \tilde{k}(\lambda) + \tilde{k}(\lambda - \kappa),
$$

(10)

where $\tilde{\epsilon}(\lambda) = \pi/(2 \cosh(\lambda))$ and $\tilde{k}(\lambda) = \arctan(\lambda) - \frac{\pi}{2}$. As a direct consequence of the derivation of the Hamiltonian from (4) both the quasiparticle energies and momenta written as a function of the rapidity are a sum of two contributions corresponding to the two commuting operators $\partial_\lambda \ln t(\lambda)|_{\lambda=0}$ and $\partial_\lambda \ln t(\lambda)|_{\lambda=\kappa}$ into which $\mathcal{H}(\kappa)$ can be split (a similar situation exists in the models considered in [10]). In fig. 3 spinon dispersion $\epsilon(k)$ is shown for several values of $\kappa$.

The simplest physical excitations built from pairs of spinons form a continuum of degenerate singlet and triplet states. Hence the spectrum is gapless, the conformal properties can be found by standard methods: the model has central charge $c = 1$ and critical exponents independent of $|\kappa| < \infty$. This situation is different from the so-called Heisenberg ladder which has been introduced recently to study the effect of coupling between parallel chains present in any real quasi-one-dimensional spin system [11]: perturbation theory and numerical investigations indicate the existence of a gap separating the triplet states from the ground state for any nonzero coupling.

The structure of excitations and the thermodynamics of this model clearly deserves more detailed studies. In addition, several extensions to the system discussed in this paper are possible: First, the above methods are easily extended to representations $V_m(a) \otimes V_n(b)$ of $Y(sl_2)$. This leads to one-parametric families of integrable quantum spin chains with alternating spin $S = m/2$ and $S = n/2$. In fact, the case of $m = 1$, $n = 2$ is the isotropic limit of the alternating spin model constructed by de Vega and Woynarovich [10] at its conformal point(1) (see also [12]). This work also suggests an extension of the $sl_2$ invariant systems studied here to quantum group invariant ones (with hyperbolic rather than rational $R$-matrices). Second, irreducible representations of $Y(sl_2)$ of the type $V_{m_1}(a_1) \otimes \ldots \otimes V_{m_k}(a_k)$ built from more than two evaluation representations can be considered to construct integrable models of $k$ coupled spin chains. In addition to the chiral three spin coupling in (8) these obtain multi-spin couplings. Some of these can be removed by adjusting the $k - 1$ free parameters (similar to $\kappa$), but the resulting systems still exhibit the one-dimensional character found in the present two-chain model. The excitations of these models are again built from spinons leading

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(1) In ref. [10] two commuting Hamiltonian operators $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are constructed for each value of $\kappa$. The method used here produces the rotational invariant vertex model with $\mathcal{H} = \mathcal{H} + \tilde{\mathcal{H}}$. 

to a gapless spectrum. This is not what would be expected for a generic two-dimensional system. To gain a better understanding of the spectral properties of the models considered in this letter (e.g., the vanishing of the gap) the effect of the multispin couplings which are crucial for integrability needs to be studied in more detail (see also [8, 13]).

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