ASYMPTOTIC BEHAVIOR FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH MULTIPLE PROPAGATION SPEEDS IN THREE SPACE DIMENSIONS

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Abstract. We consider the Cauchy problem for systems of nonlinear wave equations with multiple propagation speeds in three space dimensions. Under the null condition for such systems, the global existence of small amplitude solutions is known. In this paper, we will show that the global solution is asymptotically free in the energy sense, by obtaining the asymptotic pointwise behavior of the derivatives of the solution. Nonetheless we can also show that the pointwise behavior of the solution itself may be quite different from that of the free solution. In connection with the above results, a theorem is also developed to characterize asymptotically free solutions for wave equations in arbitrary space dimensions.

1. Introduction

We consider the Cauchy problem for a system of nonlinear wave equations of the following type with small initial data:

\[ \square_{c_j} u_j(t, x) = F_j(\partial u(t, x), \partial^2 u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \]

\[ u_j(0, x) = \varepsilon f_j(x), \quad (\partial_t u_j)(0, x) = \varepsilon g_j(x), \quad x \in \mathbb{R}^3 \]

for \( j = 1, \ldots, N \), where \( \square_c = \partial_t^2 - c^2 \Delta_x = \partial_t^2 - c^2 \sum_{k=1}^3 \partial_{x_k}^2 \) for \( c > 0 \), and the propagation speeds \( c_1, \ldots, c_N \) are positive constants, while \( \partial u \) and \( \partial^2 u \) denote the first and second derivatives of \( u = (u_l)_{1 \leq l \leq N} \), respectively. More specifically, we write

\[ \partial u = (\partial_a u_l)_{1 \leq l \leq N, 0 \leq a \leq 3}, \quad \partial^2 u = (\partial_a \partial_b u_l)_{1 \leq l \leq N, 0 \leq a, b \leq 3} \]

with the notation

\[ \partial_0 := \partial_t = \frac{\partial}{\partial t}, \quad \text{and} \quad \partial_k := \partial_{x_k} = \frac{\partial}{\partial x_k} \text{ for } k = 1, 2, 3. \]

We assume that \( f = (f_j)_{1 \leq j \leq N}, g = (g_j)_{1 \leq j \leq N} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N) \). \( \varepsilon \) in (1.2) is a small and positive parameter. \( F_j \) can include nonlinear terms of higher order, but for simplicity we assume that \( F_j \) can be written as

\[ F_j = \sum_{k,l=1}^N \left( \sum_{a,b,b'=0}^3 P_{jkl}^{ab'} (\partial_a u_k)(\partial_{b'} u_l) + \sum_{a,b=0}^3 q_{jkl}^{ab} (\partial_a u_k)(\partial_b u_l) \right) \quad (1.3) \]

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for $1 \leq j \leq N$ with appropriate real constants $p_{jkl}^{ab\prime}$ and $q_{jkl}^{ab}$, where $p_{jkl}^{00} = 0$ for $1 \leq j, k, l \leq N$ and $0 \leq a \leq 3$. To assure the hyperbolicity, we assume the symmetry condition

$$p_{jkl}^{ab\prime} = p_{kjl}^{ab\prime} = p_{jkl}^{ab\prime b}$$

(1.4)

for $1 \leq j, k, l \leq N$, and $0 \leq a, b, b' \leq 3$.

For general quadratic nonlinearity, the classical solution to (1.1)–(1.2) may blow up in finite time for some $(f, g)$ no matter how small $\varepsilon$ is. Klainerman [16] introduced a sufficient condition for small data global existence for the single speed case where $c_1 = \cdots = c_N (= 1)$ (see also Christodoulou [2]). This sufficient condition, called the null condition, was extended to the multiple speed case: To simplify the description, we assume that the speeds are distinct, and that

$$0 < c_1 < c_2 < \cdots < c_N.$$  (1.5)

Let the constants $p_{jkl}^{ab\prime}$ and $q_{jkl}^{ab}$ be from (1.3). We say that the null condition (associated with the speeds $c_1, \ldots, c_N$) is satisfied if we have

$$\sum_{a, b, b' = 0}^{3} p_{jijj}^{ab\prime} X_a X_b X_{b'} = \sum_{a, b = 0}^{3} q_{jjjj}^{ab} X_a X_b = 0, \quad X \in N_j, \quad 1 \leq j \leq N,$$  (1.6)

where $N_j := \{X = (X_a)_{0 \leq a \leq 3} \in \mathbb{R}^4; \quad X_0^2 - c_j^2 \sum_{k=1}^{3} X_k^2 = 0\}$. Small data global existence under the null condition for the multiple speed case was obtained by Yokoyama [29] (see also Sideris-Tu [27], Sogge [28], and Hidano [4]; see Kubota-Yokoyama [18], the author [7], and Metcalfe-Nakamura-Sogge [22] for the case where nonlinearity of higher order depends not only on $(\partial u, \partial^2 u)$, but also on $u$). We introduce the null forms

$$Q_0(\varphi, \psi; c) = (\partial_i \varphi)(\partial_i \psi) - c^2 \sum_{k=1}^{3} (\partial_k \varphi)(\partial_k \psi),$$

(1.7)

$$Q_{ab}(\varphi, \psi) = (\partial_a \varphi)(\partial_b \psi) - (\partial_b \varphi)(\partial_a \psi), \quad 0 \leq a, b \leq 3.$$  (1.8)

Then it is shown in [29] that the quadratic nonlinearity satisfying the null condition can be written as

$$F_j(\partial u, \partial^2 u) = N_j(\partial u, \partial^2 u) + R_j^I(\partial u, \partial^2 u) + R_j^H(\partial u, \partial^2 u),$$

(1.9)

where

$$N_j = \sum_{0 \leq |a| \leq 1} A_a^0 Q_0(u_j, \partial^a u_j; c_j) + \sum_{a, b, b' = 0}^{3} B_{j}^{ab\prime} Q_{ab}(u_j, \partial_b u_{b'}),$$

(1.10)

$$R_j^I = \sum_{\{(k, l); k \neq l\}} \left( \sum_{a, b, b' = 0}^{3} p_{jkl}^{ab\prime}(\partial_a u_k)(\partial_b \partial_{b'} u_l) + \sum_{a, b = 0}^{3} q_{jkl}^{ab}(\partial_a u_k)(\partial_b u_l) \right),$$

(1.11)

$$R_j^H = \sum_{\{k; k \neq j\}} \left( \sum_{a, b, b' = 0}^{3} p_{jjkk}^{ab\prime}(\partial_a u_k)(\partial_b \partial_{b'} u_k) + \sum_{a, b = 0}^{3} q_{jkk}^{ab}(\partial_a u_k)(\partial_b u_k) \right).$$

(1.12)
with some constants $A_j^a$ and $B_j^{ab}$. $(p_{jkl}^a)$ and $q_{jkl}^{ab}$ are from (1.3)). Here $\partial = (\partial_a)_{0 \leq a \leq 3}$, and $\alpha = (\alpha_a)_{0 \leq a \leq 3}$ is a multi-index. We refer to terms involved in $R^I = (R^I_j)_{1 \leq j \leq N}$ and $R^{II} = (R^{II}_{j})_{1 \leq j \leq N}$ as the non-resonant terms of type I and type II, respectively.

Now we turn our attention to the asymptotic behavior of the global solutions. Let $c > 0$. It is known that if $G \in L^1((0, \infty); L^2(\mathbb{R}^3))$, then the solution $v$ to $\Box_c v(t, x) = G(t, x)$ is asymptotically free in the energy sense, that is to say, there exists a solution $v^+$ to the free wave equation $\Box_c v^+ = 0$ such that

$$\lim_{t \to \infty} \| (v - v^+) (t) \|_{E,c} = 0,$$

where the energy norm $\| \varphi(t) \|_{E,c}$ is defined by

$$\| \varphi(t) \|^2_{E,c} = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{1}{c^2} |\partial_t \varphi(t, x)|^2 + |\nabla_x \varphi(t, x)|^2 \right) dx$$

with $\nabla_x = (\partial_1, \partial_2, \partial_3)$. For the single speed case, investigating the proof in [16], we have

$$F_j(\partial u, \partial^2 u) \in L^1((0, \infty); L^2(\mathbb{R}^3))$$

under the null condition because of the extra decay for the null forms, and as an immediate consequence we see that the global solution $u$ is asymptotically free in the energy sense. As for the multiple speed case, by the estimates obtained in [18] (cf. (5.6) and (5.7) below), it is easy to see that

$$N_j(\partial u, \partial^2 u) + R^I_j(\partial u, \partial^2 u) \in L^1((0, \infty); L^2(\mathbb{R}^3))$$

for the global solution $u$ to (1.1)–(1.2), because we can expect some gain in the decay rate for the null forms and the non-resonant terms of type I, compared to general quadratic nonlinearity (see (5.13) and (5.14) below). Therefore, if $R^{II} \equiv 0$ in (1.9), then (1.13) implies that the global solution $u$ is asymptotically free in the energy sense; namely for $j = 1, \ldots, N$, there exists a solution $u^+_j$ of the free wave equation $\Box_c u^+_j = 0$ such that

$$\lim_{t \to \infty} \| (u_j - u^+_j)(t) \|_{E,c_j} = 0.$$ 

By contrast, there is no explicit gain in the decay rate for the non-resonant terms of type II unless they can be written in terms of the null forms, and we cannot expect $R^{II}_j(\partial u, \partial^2 u) \in L^1((0, \infty); L^2(\mathbb{R}^3))$ in general (see (5.15) below), although its influence is weak enough for the solution to exist globally. Hence it is not clear whether the global solution $u$ is asymptotically free or not when $R^{II}$ is not written in terms of the null forms. Our aim in this paper is to determine the asymptotic behavior in the presence of the non-resonant terms of type II by modifying the method developed in [10].

2. Main Results

2.1. Asymptotically free functions in the energy sense. To begin with, we will characterize the asymptotically free functions in the energy sense. We only need the three space dimensional result for an application in this paper, but we consider the general space dimensional case here for future applications.
Let $n$ be a positive integer. Let $\dot{H}^1(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\varphi\|_{\dot{H}^1(\mathbb{R}^n)} = \|\nabla_x \varphi\|_{L^2(\mathbb{R}^n)}$. We put

$$X_n := C([0, \infty); \dot{H}^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)).$$

We say that a function $v = v(t, x) \in X_n$ is asymptotically free in the energy sense associated with the speed $c$, if there is $(v_0^+, v_1^+) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that

$$\lim_{t \to \infty} \|v(t, \cdot) - v^+(t, \cdot)\|_{E,c} = 0,$$

where $v^+ \in X_n$ is a unique solution to

$$\Box_c v^+(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n$$

with initial data $(v^+(0), \partial_t v^+(0)) = (v_0^+, v_1^+)$, and the energy norm (associated with the speed $c$) is given by

$$\|\varphi(\cdot, t)\|_{E,c}^2 := \frac{1}{2} \int_{\mathbb{R}^n} \left( \frac{1}{c^2} |\partial_t \varphi(t, x)|^2 + |\nabla_x \varphi(t, x)|^2 \right) dx.$$

Here $\Box_c = \partial_t^2 - c^2 \sum_{k=1}^n \partial_k^2$ with the notation $\partial_k = \partial/\partial x_k$ for $1 \leq k \leq n$. Note that we do not suppose that $v$ is a solution to some wave equation here.

For $c > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, we define

$$\bar{\omega}_c(x) = (\omega_{c,a}(x))_{0 \leq a \leq n} := (-c, |x|^{-1}x).$$

**Theorem 2.1.** Let $n \geq 2$ and $c > 0$. A function $v \in X_n$ is asymptotically free in the energy sense associated with the speed $c$, if and only if there is a function $V = V(\sigma, \omega) \in L^2(\mathbb{R} \times S^{n-1})$ such that

$$\lim_{t \to \infty} \|\partial v(t, \cdot) - \bar{\omega}_c(\cdot)V(t, \cdot)\|_{L^2(\mathbb{R}^n)} = 0,$$

where $\partial = (\partial_0, \partial_1, \ldots, \partial_n)$, and

$$V^*(t, x) := |x|^{-(n-1)/2}V(|x| - ct, |x|^{-1}x), \quad (t, x) \in (0, \infty) \times (\mathbb{R}^n \setminus \{0\}).$$

This theorem for $n = 3$ was implicitly proved in [10]. The author believes that this result for general space dimensions is a new observation, but it possibly has already appeared in some literature. We will give the proof in Section 3. See [14] for an application of Theorem 2.1 to semilinear wave equations in two space dimensions. Another application in three space dimensions can be found in [13].

### 2.2. Asymptotic behavior of the global solutions

Next we examine the asymptotic behavior of the global solutions to (1.1)–(1.2). For $h \in C^\infty(\mathbb{R}^3)$, we define its Radon transform $\mathcal{R}[h]$ by

$$\mathcal{R}[h](\sigma, \omega) = \int_{y \cdot \omega = \sigma} h(y) dS(y),$$

where $dS(y)$ denotes the surface element on the plane $\{y \cdot \omega = \sigma\}$. Now, restricting our attention to the three space dimensional case, we introduce the Friedlander radiation field (its definition in general space dimensions will
be given in (3.25) below: For \((\varphi, \psi) \in C^\infty_0(\mathbb{R}^3) \times C^\infty_0(\mathbb{R}^3)\), we define the Friedlander radiation field
\[
\mathcal{F}_0[\varphi, \psi](\sigma, \omega) = \frac{1}{4\pi} \left( \mathcal{R}[\psi](\sigma, \omega) - (\partial_\sigma \mathcal{R}[\varphi])(\sigma, \omega) \right) \tag{2.3}
\]
for \((\sigma, \omega) \in \mathbb{R} \times S^2\). For \(z \in \mathbb{R}^d\) with a positive integer \(d\), we put \(\langle z \rangle := \sqrt{1 + |z|^2}\).

**Theorem 2.2.** Suppose that (1.3), (1.4), (1.5), and the null condition (1.6) are fulfilled. Let \(\sigma = (\partial_0, \partial_1, \partial_2, \partial_3)\), and the null condition (1.6) be given by (2.2).

1. We fix small \(\delta > 0\). Then for any \(f, g \in C^\infty_0(\mathbb{R}^3; \mathbb{R}^N)\) and sufficiently small \(\varepsilon > 0\), there is a function \(P = (P_j)_{1 \leq j \leq N}\) of \((\sigma, \omega) \in \mathbb{R} \times S^2\) such that
\[
|x|\partial u_j(t, x) = \varepsilon \overline{\omega}_c(x) P_j(|x| - c_j t, |x|^{-1} x) + O(\varepsilon \langle t + |x| \rangle^{-1+\delta}(c_j t - |x|)^{-\delta}) \tag{2.4}
\]
for \((t, x) \in [0, \infty) \times (\mathbb{R}^3 \setminus \{0\})\) and \(1 \leq j \leq N\), where \(u = (u_j)_{1 \leq j \leq N}\) is the global solution to the Cauchy problem (1.1)–(1.2). Moreover we have
\[
P_j(\sigma, \omega) = \partial_\sigma \mathcal{F}_0[f_j, c_j^{-1} g_j](\sigma, \omega) + O(\varepsilon \langle \sigma \rangle^{-1}), \quad (\sigma, \omega) \in \mathbb{R} \times S^2. \tag{2.5}
\]

2. We further assume that \(R^H\) \((R^H_j)_{1 \leq j \leq N}\) has the null structure, that is to say
\[
R^H_j = \sum_{\{k, k' \neq j\}} \left( \sum_{0 \leq |a| \leq 1} A^a_{jk} Q_0(u_k, \partial^a u_k; c_k) + \sum_{a,b,b'=0}^3 B^{ab}_{jk} Q_{ab}(u_k, \partial_{b'} u_k) \right)
\]
for \(1 \leq j \leq N\) with some constants \(A^a_{jk}\) and \(B^{ab}_{jk}\). Then for any \(f, g \in C^\infty_0(\mathbb{R}^3, \mathbb{R}^N)\) and sufficiently small \(\varepsilon > 0\), there is a function \(U = (U_j)_{1 \leq j \leq N}\) of \((\sigma, \omega) \in \mathbb{R} \times S^2\) such that
\[
|x|u_j(t, x) = \varepsilon U_j(|x| - c_j t, |x|^{-1} x) + O(\varepsilon \langle t + |x| \rangle^{-1} \log(2 + t)) \tag{2.6}
\]
for \((t, x) \in [0, \infty) \times (\mathbb{R}^3 \setminus \{0\})\) and \(1 \leq j \leq N\), where \(u = (u_j)_{1 \leq j \leq N}\) is the global solution to the Cauchy problem (1.1)–(1.2). Moreover we have
\[
\partial_\sigma^m U_j(\sigma, \omega) = \partial_\sigma^m \mathcal{F}_0[f_j, c_j^{-1} g_j](\sigma, \omega) + O(\varepsilon \langle \sigma \rangle^{-1-m}), \quad (\sigma, \omega) \in \mathbb{R} \times S^2. \tag{2.8}
\]
for \(m = 0, 1\) and \(1 \leq j \leq N\).

By (2.4) we see that the asymptotic pointwise behavior of \(\partial u_j\) is similar to that of the derivatives of the free solution (see Lemma 3.3 below). Combining (2.4) and (2.5) with Theorem 2.1 we see that the solution \(u\) is asymptotically free in the energy sense.

**Corollary 2.3.** Suppose that (1.3), (1.4), (1.5), and the null condition (1.6) are fulfilled. Then, for any \(f, g \in C^\infty_0(\mathbb{R}^3, \mathbb{R}^N)\) and sufficiently small \(\varepsilon > 0\),
there exist $f^+ = (f^+_j)_{1 \leq j \leq N} \in \dot{H}^1(\mathbb{R}^3; \mathbb{R}^N)$ and $g^+ = (g^+_j)_{1 \leq j \leq N} \in L^2(\mathbb{R}^3; \mathbb{R}^N)$ such that

$$\lim_{t \to \infty} \|(u_j - u_j^+)(t)\|_{E,c_j} = 0, \quad 1 \leq j \leq N,$$

where $u = (u_j)_{1 \leq j \leq N}$ is the global solution to the Cauchy problem (1.1) - (1.2), and $u_j^+$ is the solution to $\Box_j u_j^+ = 0$ with initial data $(u_j^+; \partial_t u_j^+) = (f_j^+; g_j^+)$ at $t = 0$.

Theorem 2.2 and Corollary 2.3 will be proved in Section 5 after giving some preliminaries in Section 4.

When $R^{II}$ has the null structure, we see from (2.6) that the solution $u_j$ itself also behaves similarly to the free solution. The next result shows that the lack of the estimate corresponding to (2.6) in the presence of $R^{II}$ without the null structure is inevitable.

**Theorem 2.4.** Let $0 < c_1 < c_2$, and let $A_1, A_2$ be real constants. We consider the Cauchy problem for

$$\begin{cases}
\Box c_1 u_1 = A_1(\partial_t u_2)^2, \\
\Box c_2 u_2 = A_2(\partial_t u_1)^2
\end{cases} \quad \text{in } (0, \infty) \times \mathbb{R}^3. \quad (2.9)
$$

If $A_1 \neq 0$, then there exist $(f, g) \in C_0^{\infty}(\mathbb{R}^3; \mathbb{R}^2) \times C_0^{\infty}(\mathbb{R}^3; \mathbb{R}^2)$, $M > 0$, $T_0 \geq 2$, and $C > 0$ such that

$$C^{-1} \varepsilon (1 + \varepsilon \log(2 + t)) \leq |ru_1(t, x)| \leq C \varepsilon \left(1 + \varepsilon \log(2 + t)\right) \quad (2.10)$$

for any $(t, x)$ satisfying $T_0 \leq c_1 t \leq r(=|x|) \leq c_1 t + M$, where $u = (u_1, u_2)$ is the global solution to (2.9) with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$ for sufficiently small $\varepsilon (> 0)$.

This theorem will be proved in Section 6. From this result, we see that there is some loss in the decay rate of $u_1$: (2.10) says that $u_1(t, x)$ decays like $(1 + t)^{-1} \log(2 + t)$ along the ray $r = c_1 t + \sigma$ for $0 \leq \sigma \leq M$, while (2.6) implies that $u_j$ decays like $(1 + t)^{-1}$ and has the same decay property as the free solution along the ray $r = c_j t + \sigma$ if $R^{II}$ has the null structure. Accordingly we find that estimates like (2.6) cannot hold in the presence of the non-resonant terms of type II without the null structure.

To sum up the results, an interesting character of the non-resonant terms of type II is revealed: Their effect is weak enough for the solution to exist globally (Theorem 1.1 in [29]) and to be asymptotically free in the energy sense (Corollary 2.3); however it is strong enough to affect the decay rate of the solution $u$ itself (Theorem 2.4), though that of $\partial u$ is not affected (Theorem 2.2).

Throughout this paper, various positive constants are denoted by the same letter $C$. Thus the actual value of $C$ may change line by line.

### 3. Asymptotics for Homogeneous Wave Equations

Our aim in this section is to prove Theorem 2.1. Most of the necessary materials for this purpose are rather standard, but we give the details and
proofs to make this section self-contained. In what follows, we use the formal expression of writing distributions as if they are functions.

3.1. Solutions to the free wave equation. We put \( H_0(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), and we define

\[
\|(\varphi, \psi)\|_{H_0(\mathbb{R}^n)} := \frac{1}{2} \left( \|\nabla_x \varphi\|_{L^2(\mathbb{R}^n)}^2 + \|\psi\|_{L^2(\mathbb{R}^n)}^2 \right).
\]

\( H_0(\mathbb{R}^n) \) can also be understood as the completion of \( C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \) with respect to the norm \( \| \cdot \|_{H_0(\mathbb{R}^n)} \). Let \( c > 0 \), and consider the Cauchy problem for the free wave equation

\[
\Box w(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,
\]

\[
(w(0, x), (\partial_t w)(0, x)) = (w_0(x), w_1(x), \quad x \in \mathbb{R}^n.
\]

Let \( X_n \) be defined by \( (2.1) \). It is known that for \( (w_0, w_1) \in H_0(\mathbb{R}^n) \), \((3.2)–(3.3)\) admits a unique solution \( w \in X_n \), and we have the conservation of the energy

\[
\|w(t, \cdot)\|_{E,c} = \|w(0, \cdot)\|_{E,c} = \|(w_0, c^{-1}w_1)\|_{H_0(\mathbb{R}^n)}.
\]

For \( \text{Re} a > -1 \), let \( \chi^a_+ \) be defined by

\[
\chi^a_+(s) := \begin{cases} \frac{s^a}{\Gamma(a+1)}, & s > 0, \\ 0, & s \leq 0, \end{cases}
\]

where \( \Gamma \) denotes the Gamma function. Then we have \( \chi^a_+(s) = (\chi^{a+1}_+)'(s) \) for \( \text{Re} a > -1 \). We can extend the definition of the distribution \( \chi^a_+ \) to all \( a \in \mathbb{C} \) so that we have \( \chi^a_+(s) = (\chi^{a+1}_+)'(s) \) for any \( a \in \mathbb{C} \). Note that \( \chi^a_+ \) can have its singularity only at \( s = 0 \). Especially we have

\[
\chi^{-k}_+(s) = \delta^{(k-1)}(s), \quad k \in \mathbb{N},
\]

where \( \delta \) is the Dirac function and \( \delta^{(j)} \) denotes its \( j \)-th derivative. For a positive integer \( m \), we define

\[
E_m(t, x) = \frac{1}{2\pi^{(m-1)/2}} \chi_+^{(1-m)/2}(t^2 - |x|^2).
\]

Then the solution \( w \) to \((3.2)–(3.3)\) with \( (w_0, w_1) \in (C^\infty_0(\mathbb{R}^n))^2 \) can be written as

\[
w(t, x) = c^{-1} \partial_t \left( E_n(ct, \cdot) * w_0 \right)(x) + c^{-1} \left( E_n(ct, \cdot) * w_1 \right)(x),
\]

where the convolution \( * \) is taken with respect to \( x \)-variable (see Hörmander [5, Section 6.2] for instance).

From now on, we suppose that \( (w_0, w_1) \in (C^\infty_0(\mathbb{R}^n))^2 \), and that \( w_0(x) = w_1(x) = 0 \) for \( |x| \geq M \) with some positive constant \( M \). Since \( \text{supp} \ E_n(t, \cdot) \subset \{ x; |x| \leq t \} \), it follows from \((3.3)\) that

\[
w(t, x) = 0, \quad |x| \geq ct + M.
\]

If \( n \geq 3 \) is odd, then \( \text{supp} \ E_n(t, \cdot) = \{ x; |x| = t \} \) and we also get

\[
w(t, x) = 0, \quad |x| \leq ct - M,
\]
which is called the (strong) Huygens principle. If \( n \) is even, (3.6) is not valid in general, but we have a faster decay away from the light cone \( ct = |x| \); Let \( t/2 \geq |x| \) and \( t \geq 4M \), say. Then \( t^2 - |x - y|^2 \geq 7t^2/16 \geq C \langle t + |x| \rangle^2 > 0 \) for \( |y| \leq M \). Hence, observing that \( \chi^{(1-n)/2}(s) = A_n s^{(1-n)/2} \) for \( s > 0 \) with an appropriate constant \( A_n \), we get
\[
|\partial_{t,x}^\alpha E_n(t, x - y)| \leq C_\alpha \langle t + |x| \rangle^{-|\alpha|+(1-n)}, \quad t/2 \geq \max\{|x|, 2M\}, \quad |y| \leq M
\]
with a positive constant \( C_\alpha \). Therefore (3.4) leads to
\[
|\partial^\alpha w(t, x)| \leq C_\alpha \langle t + |x| \rangle^{-|\alpha|+(1-n)}, \quad ct/2 \geq |x|, \tag{3.7}
\]
because (3.7) for \( 2M \geq ct/2 \geq |x| \) is easily shown.

Following the arguments in Hörmander [6, Section 6.2], we will obtain a useful expression of \( E_m(t) \ast \varphi \) for \( \varphi \in C_0^\infty(\mathbb{R}^n) \) with \( \varphi(x) = 0 \) for \( |x| \geq M \). Note that we have
\[
(E_m(t) \ast \varphi)(x) = 0, \quad |x| \geq t + M, \quad \tag{3.8}
\]
and
\[
(E_m(t) \ast \varphi)(x) = 0, \quad |x| \leq t - M \quad \text{when} \quad m(\geq 3) \quad \text{is odd} \quad \tag{3.9}
\]
as in (3.5) and (3.6). Let \( x = r\omega \) with \( r = |x| \) and \( \omega \in S^{n-1} \). We assume
\[
2M \leq \frac{t}{2} \leq r \leq t + M.
\]
We put \( \sigma = r - t \). Then we get \( -r \leq \sigma \leq M \). Since we have
\[
t^2 - |x - y|^2 = 2r(\omega \cdot y - \sigma) + \sigma^2 - |y|^2,
\]
the homogeneity of \( \chi_+^{(1-m)/2} \) implies that
\[
(2\pi t)^{(m-1)/2} (E_m(t) \ast \varphi)(x) = \frac{1}{2} \int_{\mathbb{R}^n} \chi_+^{(1-m)/2} \left( \omega \cdot y - \sigma + \frac{\sigma^2 - |y|^2}{2r} \right) \varphi(y) dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^n} \chi_+^{(1-m)/2} \left( s - \sigma + \frac{\sigma^2}{2r} \right) \mathcal{G}[\varphi](s, \omega, r^{-1}) ds,
\]
where \( \mathcal{G}[\varphi](s, \omega, z) \) is given by
\[
\mathcal{G}[\varphi](s, \omega, z) = \int_{\mathbb{R}^n} \delta \left( s - \omega \cdot y + \frac{|y|^2}{2} \right) \varphi(y) dy
\]
for \( (s, \omega, z) \in \mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}] \). If we put \( \rho = s - \omega \cdot y + |y|^2z/2 \), then \( \nabla_y \rho = -\omega + zy \). Since \( |\nabla_y \rho| \geq 1 - |z| |y| \geq 1/2 \) for \( z \in [0, (2M)^{-1}] \) and \( |y| \leq M \), \( \mathcal{G}[\varphi] \) can be written as an integral of a compactly supported function \( \varphi \) over a hyper-surface \( \{ y \in \mathbb{R}^n; s - \omega \cdot y + |y|^2z/2 = 0 \} \) which smoothly depends on \( (s, \omega, z) \in \mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}] \), and we see that
\[
\mathcal{G}[\varphi] \in C^\infty(\mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}]).
\]
We also see that if \( \mathcal{G}[\varphi](s, \omega, z) \neq 0 \) for some \( (\omega, z) \in S^{n-1} \times [0, (2M)^{-1}] \), then we have \(-5M/4 \leq s \leq M \). Indeed, since the assumption implies \( s = \omega \cdot y - \)
(\|y\|^2/2)z for some $y$ and $(\omega, z)$ with $\|y\| \leq M$ and $(\omega, z) \in S^{n-1} \times [0, (2M)^{-1}]$, we get

$$-\frac{5M}{4} \leq -\frac{\|y\|^2}{2} \frac{1}{2M} \leq \omega \cdot y - \frac{\|y\|^2}{2}z (= s) \leq \|y\| \leq M.$$  

For a multi-index $\alpha$ with $|\alpha| = k \leq 1$, we have

$$G[\partial^\alpha x \varphi](s, \omega, z) = \partial^k s \int_{\mathbb{R}^n} \delta \left( s - \omega \cdot y + \frac{\|y\|^2}{2}z \right) (\omega - z y)^\alpha \varphi(y) dy. \quad (3.10)$$

We define

$$H_m[\varphi](\sigma, \omega, z) := \frac{1}{2(2\pi)^{(m-1)/2}} \int_{\mathbb{R}} \chi_{+}^{(1-m)/2} \left( s - \sigma + \frac{\sigma^2}{2}z \right) (\partial^k s G[\varphi]) (s, \omega, z) ds.$$ 

Then we obtain

$$r^{(m-1)/2} (E_m(t) * \varphi)(x) = H_m[\varphi](\sigma, \omega, r^{-1}). \quad (3.11)$$

Since we have

$$H_m[\varphi](\sigma, \omega, z) = \frac{(-1)^k}{2(2\pi)^{(m-1)/2}} \int_{\mathbb{R}} \chi_{+}^{k+(1-m)/2} \left( s - \sigma + \frac{\sigma^2}{2}z \right) (\partial^k s G[\varphi]) (s, \omega, z) ds$$

for any nonnegative integer $k$, and since we have $\chi_a^+ \in C^1(\mathbb{R})$ for $a > 1$, we can easily see that $H_m[\varphi] \in C^\infty(\mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}])$. Moreover we have

$$\sigma \leq M \text{ in supp } H_m[\varphi] \text{ when } m \text{ is even,} \quad (3.12)$$

and

$$|\sigma| \leq M \text{ in supp } H_m[\varphi] \text{ when } m(\geq 3) \text{ is odd.} \quad (3.13)$$

Indeed (3.12) and (3.13) for $z \neq 0$ follow from (3.8), (3.9), and (3.11), while they follow immediately from the definition of $H_m$ when $z = 0$.

### 3.2. The Radon transform and the Friedlander radiation field.

Let $S(\mathbb{R}^n)$ denote the set of rapidly decreasing functions on $\mathbb{R}^n$. For $\varphi \in S(\mathbb{R}^n)$ we define the Radon transform $R[\varphi]$ of $\varphi$ by

$$R[\varphi](\sigma, \omega) := \int_{y \cdot \omega = \sigma} \varphi(y) dS(y), \quad (\sigma, \omega) \in \mathbb{R} \times S^{n-1}, \quad (3.14)$$

where $dS(y)$ denotes the surface element on the hyperplane $\{y \in \mathbb{R}^n; y \cdot \omega = \sigma\}$. It is easy to see that $R[\varphi] \in S(\mathbb{R} \times S^{n-1})$. Since we have $R[\varphi](\sigma, \omega) = G[\varphi](\sigma, \omega, 0)$, it follows from (3.10) that

$$R[\partial^\alpha x \varphi](\sigma, \omega) = \omega^\alpha \partial^k s R[\varphi](\sigma, \omega) \quad (3.15)$$

for any multi-index $\alpha$ with $|\alpha| = k \leq 1$. 

For a positive integer $m$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we put
\[
\mathcal{R}_m[\varphi](\sigma, \omega) := \frac{1}{2(2\pi)^{(m-1)/2}} \int_{\mathbb{R}} \chi_{+}^{(1-m)/2}(s - \sigma)\mathcal{R}[\varphi](s, \omega)ds
\]
\[
= \frac{1}{2(2\pi)^{(m-1)/2}} \left( \chi_{+}^{(1-m)/2} * \mathcal{R}[\varphi](\cdot, \omega) \right)(\sigma), \quad (\sigma, \omega) \in \mathbb{R} \times S^{n-1},
\]
where $\chi_+^a(\sigma) := \chi_+^a(-\sigma)$ for $a \in \mathbb{C}$, and $*$ is the convolution with respect to $\sigma$-variable. Note that we have $\mathcal{R}_m[\varphi] = H_m \ast 0 \in C^\infty(\mathbb{R} \times S^{n-1})$. For any multi-index $\alpha$ with $|\alpha| = k \leq 1$, we obtain from (3.15) that
\[
\mathcal{R}_m[\partial_x^\alpha \varphi](\sigma, \omega) = \frac{1}{2(2\pi)^{(m-1)/2}} \int_{\mathbb{R}} \chi_{+}^{(1-m)/2}(s - \sigma)\omega^\alpha \partial_x^\alpha \mathcal{R}[\varphi](s, \omega)ds
\]
\[
= \frac{(-1)^k \omega^\alpha}{2(2\pi)^{(m-1)/2}} \int_{\mathbb{R}} \chi_{+}^{(1-m)/2-k}(s - \sigma)\mathcal{R}[\varphi](s, \omega)ds
\]
\[
= \omega^\alpha \partial_\sigma^k \mathcal{R}_m[\varphi](\sigma, \omega). \tag{3.16}
\]
If $\varphi(x) = 0$ for $|x| \geq M$, then we immediately see by (3.14) that
\[
\mathcal{R}[\varphi](\sigma, \omega) = 0, \quad |\sigma| \geq M, \quad \omega \in S^{n-1}. \tag{3.17}
\]
Consequently we get
\[
\mathcal{R}_m[\varphi](\sigma, \omega) = 0, \quad \sigma \geq M, \quad \omega \in S^{n-1}. \tag{3.18}
\]
When $m(\geq 3)$ is odd, since $\text{supp} \chi_{+}^{(1-m)/2} = \{0\}$, we obtain
\[
\mathcal{R}_m[\varphi](\sigma, \omega) = 0, \quad |\sigma| \geq M, \quad \omega \in S^{n-1}. \tag{3.19}
\]

**Lemma 3.1.** Let $m$ be a positive integer with $m \geq 2$. For $\varphi \in C^\infty_0(\mathbb{R}^n)$, a nonnegative integer $j$, and a multi-index $\alpha$ with $|\alpha| = k \leq 1$, there is a positive constant $C$ such that
\[
\left| \partial_\sigma^j \mathcal{R}_m[\partial_x^\alpha \varphi](\sigma, \omega) \right| \leq C \langle \sigma \rangle^{-j-k+(1-m)/2}, \quad (\sigma, \omega) \in \mathbb{R} \times S^{n-1}.
\]

**Proof.** Suppose that $\varphi(x) = 0$ for $|x| \geq M$ with a positive constant $M$. Since $\partial_\sigma^j \mathcal{R}_m[\partial_x^\alpha \varphi] \in C^\infty(\mathbb{R} \times S^{n-1})$, we get
\[
\left| \partial_\sigma^j \mathcal{R}_m[\partial_x^\alpha \varphi](\sigma, \omega) \right| \leq C \leq C \langle 2M \rangle^{j+k+(m-1)/2} \langle \sigma \rangle^{-j-k+(1-m)/2} \tag{3.20}
\]
for $(\sigma, \omega) \in [-2M, M] \times S^{n-1}$, which implies the desired result for odd $m$ because of (3.19).

Let $m$ be even. In view of (3.18) and (3.20), it suffices to consider the case where $\sigma \leq -2M$. Then we have $s - \sigma \geq |\sigma|/2 \geq C \langle \sigma \rangle > 0$ for $|s| \leq M$. Hence we obtain from (3.16) that
\[
\left| \partial_\sigma^j \mathcal{R}_m[\partial_x^\alpha \varphi](\sigma, \omega) \right| = \left| \frac{(-1)^{j+k} \omega^\alpha}{2(2\pi)^{(m-1)/2}} \int_{-M}^{M} \chi_{+}^{j-k+(1-m)/2}(s - \sigma)\mathcal{R}[\varphi](s, \omega)ds \right|
\]
\[
\leq C \int_{-M}^{M} (s - \sigma)^{-j-k+(1-m)/2}ds \leq C \langle \sigma \rangle^{-j-k+(1-m)/2},
\]
because of (3.17). This completes the proof. \qed
Lemma 3.2. Let $n \geq 2$, and $m$ be a positive integer with $m \geq 2$. Suppose that $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi(x) = 0$ for $|x| \geq M$ with a positive constant $M$. Then, for any integer $j$ with $0 \leq j \leq 2$, and a multi-index $\alpha$ with $|\alpha| = k \leq 1$, there is a positive constant $C$ such that

\[
| r^{(m-1)/2} \partial_t \left( E_m(t) \ast \partial_x^\alpha \varphi \right)(x) - \left( (-\partial_\sigma)^j \mathcal{R}_m[\partial_x^\alpha \varphi] \right)(r-t, \omega) | 
\leq C r^{-1} (t-r)^{-j-k+(3-m)/2}, \quad r \geq \frac{t}{2} \geq 2M, \quad (3.21)
\]

where $r = |x|$ and $\omega = |x|^{-1}x$.

Proof. Recall the definitions of $G$ and $H_m$ in the previous subsection, and that we have

\[
G[\partial_x^\alpha \varphi](s, \omega, 0) = \mathcal{R}[\partial_x^\alpha \varphi](s, \omega), \quad H_m[\partial_x^\alpha \varphi](\sigma, \omega, 0) = \mathcal{R}_m[\partial_x^\alpha \varphi](\sigma, \omega).
\]

We suppose that $r \geq t/2 \geq 2M$. By (3.8) and (3.18), we may also assume $r \leq t + M$.

First we assume that $j = 0$. We put $\sigma = r-t$ as before. By (3.11) we obtain

\[
| r^{(m-1)/2} \left( E_m(t) \ast \partial_x^\alpha \varphi \right)(x) - \left( \mathcal{R}_m[\partial_x^\alpha \varphi] \right)(r-t, \omega) | 
\leq r^{-1} \int_0^1 \left| (\partial_z H_m[\partial_x^\alpha \varphi])(\sigma, \omega, \theta r^{-1}) \right| d\theta,
\]

which leads to (3.21) with $j = 0$ if we can show

\[
| \partial_z H_m[\partial_x^\alpha \varphi](\sigma, \omega, z) | \leq C \langle \sigma \rangle^{-k+(3-m)/2} \quad (3.22)
\]

for $(\sigma, \omega, z) \in \mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}]$. Since $H_m[\partial_x^\alpha \varphi] \in C^\infty(\mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}])$, we have

\[
| \partial_z H_m[\partial_x^\alpha \varphi](\sigma, \omega, z) | \leq C \leq C \langle \sigma \rangle^{-k+(3-m)/2} \quad (3.23)
\]

for $(\sigma, \omega, z) \in [-2M, M] \times S^{n-1} \times [0, (2M)^{-1}]$, which leads to (3.22) for odd $m$ because of (3.13). Let $m$ be even. In view of (3.12) and (3.23), it suffices to show (3.22) for $(\sigma, \omega, z) \in (-\infty, -2M] \times S^{n-1} \times [0, (2M)^{-1}]$. Suppose $\sigma \leq -2M$, $\omega \in S^{n-1}$, and $0 \leq z \leq (2M)^{-1}$. We compute

\[
2(2\pi)^{(m-1)/2} \partial_z H_m[\partial_x^\alpha \varphi](\sigma, \omega, z) 
= \int_{\mathbb{R}} \frac{\sigma^2}{2} \chi_{+}^{-(1+m)/2} \left( s - \sigma + \frac{\sigma^2}{2} z \right) G[\partial_x^\alpha \varphi](s, \omega, z) ds 
+ \int_{\mathbb{R}} \chi_{+}^{(1-m)/2} \left( s - \sigma + \frac{\sigma^2}{2} z \right) (\partial_z G[\partial_x^\alpha \varphi])(s, \omega, z) ds =: I_1 + I_2.
\]
Since \( s - \sigma + \sigma^2 z/2 \geq -5M/4 + |\sigma| \geq 3|\sigma|/8 \geq C \langle \sigma \rangle > 0 \) for \( \sigma \leq -2M \), \( z \geq 0 \), and \( s \geq -5M/4 \), recalling (3.10) we obtain

\[
|I_1| \leq \left| \int_{\mathbb{R}} \frac{\sigma^2}{2} \chi_+^{-k-(1+m)/2} \left( s - \sigma + \frac{\sigma^2}{2} \right) \Phi_\alpha(s, \omega, z) ds \right|
\]

\[
\leq C |\sigma|^2 \langle \sigma \rangle^{-k-(1+m)/2} \leq C \langle \sigma \rangle^{-k+(3-m)/2},
\]

where

\[
\Phi_\alpha(s, \omega, z) = \int_{\mathbb{R}^n} \delta \left( s - \omega \cdot y + \frac{|y|^2}{2} \right) (\omega - zy)^\alpha \varphi(y) dy
\]

which is a \( C^\infty \) function in \( \mathbb{R} \times S^{n-1} \times [0, (2M)^{-1}] \) with \(-5M/4 \leq s \leq M \) in \( \text{supp} \Phi_\alpha \). Similarly we get \( |I_2| \leq C \langle \sigma \rangle^{-k+(3-m)/2} \). This completes the proof of (3.22) for even \( m \), and (3.21) with \( j = 0 \) is established.

Note that by (3.21) with \( j = 0 \) and Lemma 3.1 we get

\[
|r^{(m-1)/2}(E'_m(t) * \partial_x^\alpha \varphi)(x)| \leq C \langle \sigma \rangle^{-k+(1-m)/2}.
\]  \hspace{1cm} (3.24)

Direct calculations lead to

\[
\partial_t E'_m(t, x) = 2\pi t E_{m+2}(t, x), \quad -\partial_\omega R_m[\partial_x^\alpha \varphi](\sigma, \omega) = 2\pi R_{m+2}[\partial_x^\alpha \varphi](\sigma, \omega).
\]

Hence it follows from (3.24) and (3.21) with \( j = 0 \) that

\[
|r^{(m-1)/2} \partial_t(E'_m(t) * \partial_x^\alpha \varphi)(x) - (-\partial_\omega)R_m[\partial_x^\alpha \varphi](\sigma, \omega)| \leq 2\pi|r^{(m+1)/2}(E_{m+2}(t) * \partial_x^\alpha \varphi)(x) - R_{m+2}[\partial_x^\alpha \varphi](\sigma, \omega)| + 2\pi|r^{(m-1)/2}(t-r)(E_{m+2}(t) * \partial_x^\alpha \varphi)(x)| \leq Cr^{-1} \langle \sigma \rangle^{-k+(1-m)/2},
\]

which is (3.21) for \( j = 1 \). Observing that

\[
\partial_t^2 E'_m(t, x) = 2\pi E_{m+2}(t, x) + (2\pi t)^2 E_{m+4}(t)
\]

and

\[
(-\partial_\omega)^2 R_m[\partial_x^\alpha \varphi](\sigma, \omega) = (2\pi)^2 R_{m+4}[\partial_x^\alpha \varphi](\sigma, \omega),
\]

we can show (3.21) for \( j = 2 \) in a similar way. \( \square \)

For \((\varphi, \psi) \in (\mathcal{S}(\mathbb{R}^n))^2\), we define the Friedlander radiation field \( \mathcal{F}_0[\varphi, \psi] \) by

\[
\mathcal{F}_0[\varphi, \psi](\sigma, \omega) = -\partial_\omega \mathcal{R}_n[\varphi](\sigma, \omega) + \mathcal{R}_n[\psi](\sigma, \omega), \quad (\sigma, \omega) \in \mathbb{R} \times S^{n-1}. \]  \hspace{1cm} (3.25)

Observe that this definition is a generalization of the previous definition (2.3) that was given only for \( n = 3 \) and \((\varphi, \psi) \in (C^\infty_0(\mathbb{R}^3))^2\), because \( \mathcal{R}_n[h] = (4\pi)^{-1} \mathcal{R}[h] \). The next lemma is a slight refinement of [6, Theorem 6.2.1] (see also Friedlander [3] and Katayama-Kubo [12]).

**Lemma 3.3.** Let \( n \geq 2 \) and \( c > 0 \). Let \( w \) be the solution to the Cauchy problem (3.2)-(3.3). If \((w_0, w_1) \in (C^\infty_0(\mathbb{R}^n))^2\), then there is a positive constant \( C \) such that

\[
|r^{(n-1)/2}w(t, x) - W(r-ct, \omega)| \leq C \langle t + r \rangle^{-1} \langle ct - r \rangle^{(3-n)/2}
\]  \hspace{1cm} (3.26)

and

\[
|r^{(n-1)/2} \partial w(t, x) - \overline{w_c}(x)(\partial_\sigma W)(r-ct, \omega)| \leq C \langle t + r \rangle^{-1} \langle ct - r \rangle^{(1-n)/2}
\]  \hspace{1cm} (3.27)
for all \((t, x) \in [0, \infty) \times (\mathbb{R}^n \setminus \{0\})\), where \(W(\sigma, \omega) = \mathcal{F}_0[w_0, c^{-1} w_1]((\sigma, \omega), \omega_\tau(x))\) is given by (2.2), \(r = |x|\), \(\omega = |x|^{-1} x\), and \(\partial = (\partial_0, \partial_1, \ldots, \partial_n)\).

**Proof.** We may assume that \(c = 1\), because the general result is easily obtained by a change of variables. We assume that \(w_0(x) = w_1(x) = 0\) for \(|x| \geq M\).

Firstly we suppose that \(r \geq t/2 \geq 2M\). Then we have \(r^{-1} \leq C \langle t + r \rangle^{-1}\). Let \(|\alpha| = k\), and suppose that \(j\) and \(k\) are nonnegative integers with \(0 \leq j + k \leq 1\). From (3.4) we get

\[
\partial^j_t \partial^k_x w(t, x) = \partial^{j+1}_t (E_n(t) * \partial^k_x w_0)(x) + \partial^j_t (E_n(t) * \partial^k_x w_1)(x).
\]

Then Lemma 3.2 (with \(m = n\)) and (3.16) lead to

\[
r^{(n-1)/2} \partial^j_t \partial^k_x w(t, x) = (-1)^j \omega^n (\partial^{j+k}_\sigma W)(r - t, \omega) + O \left( (t + r)^{-1} \langle t - r \rangle^{-j-k+(3-n)/2} \right),
\]

which implies (3.26) when \(j + k = 0\), and (3.27) when \(j + k = 1\).

Secondly we suppose that either \(r \leq t/2\) or \(t \leq 4M\) holds. Then we have \(\langle t - r \rangle^{-1} \leq C \langle t + r \rangle^{-1}\). Hence by Lemma 3.1 (with \(m = n\)) we get

\[
|\langle \partial^j_\sigma W \rangle (r - t, \omega)| \leq C \langle t - r \rangle^{-j+(1-n)/2} \leq C \langle t + r \rangle^{-j+(1-n)/2}.
\]

Now we are going to prove

\[
|r^{(n-1)/2} \partial^\alpha w(t, x)| \leq C \langle t + r \rangle^{-|\alpha|+(1-n)/2}
\]

for \((t, x)\) with either \(r \leq t/2\) or \(t \leq 4M\), which implies (3.26) and (3.27) with the help of (3.28). If we assume \(-2M \leq r - t \leq M\) in addition, then we get \(t \leq 4M\) and \(r \leq 5M\). Now, from (3.4) we can easily obtain

\[
|r^{(n-1)/2} \partial^\alpha w(t, x)| \leq C \langle t + r \rangle^{-|\alpha|+(1-n)/2}, \quad t \leq 4M, \ r \leq 5M.
\]

When \(n\) is odd, this shows (3.29) because of (3.5) and (3.6). Thus we assume \(n\) is even and \(r - t \leq -2M\). Accordingly we have \(t/2 \geq r\), and (3.7) immediately implies the desired result. This completes the proof. \(\square\)

3.3. **The translation representation.** For \(\varphi \in \mathcal{S}(\mathbb{R}^n)\), we write \(\hat{\varphi}(= \mathcal{F}[\varphi])\) for its Fourier transform. To be more precise, we put

\[
\hat{\varphi}(\xi) = \mathcal{F}[\varphi](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n,
\]

where \(i = \sqrt{-1}\). For a function \(\psi = \psi(\sigma, \omega) \in \mathcal{S}(\mathbb{R} \times S^{n-1})\), we define

\[
\tilde{\psi}(\rho, \omega) = \tilde{\mathcal{F}}[\psi](\rho, \omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i \sigma \cdot \omega} \psi(\sigma, \omega) d\sigma, \quad (\rho, \omega) \in \mathbb{R} \times S^{n-1},
\]
which is the one-dimensional Fourier transform of $\psi(\cdot, \omega)$ with a parameter $\omega \in \mathbb{S}^{n-1}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. For $(\rho, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}$, we have

$$\widehat{\varphi}(\rho \omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\rho(y \cdot \omega)} \varphi(y) dy = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\rho \sigma} \mathcal{R}[\varphi](\sigma, \omega) d\sigma$$

$$= \frac{1}{(2\pi)^{(n-1)/2}} \widehat{\mathcal{R}[\varphi]}(\rho, \omega).$$

Equation (3.30) implies

$$\widehat{\mathcal{R}[\varphi]}(-\rho, -\omega) = (2\pi)^{(n-1)/2} \widehat{\varphi}(\rho \omega) = \widehat{\mathcal{R}[\varphi]}(\rho, \omega), \quad (\rho, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}. \quad (3.31)$$

In other words $\widehat{\mathcal{R}[\varphi]}$ is an even function in $(\rho, \omega)$.

**Lemma 3.4.** For $(\varphi, \psi) \in (\mathcal{S}(\mathbb{R}^n))^2$ we have

$$\|\partial_\sigma \mathcal{F}_0[\varphi, \psi]\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = \|\langle \varphi, \psi \rangle\|_{H_0(\mathbb{R}^n)}.$$  

**Proof.** Since we have $\mathcal{R}_n[h](\sigma, \omega) = B_n(\chi_{(1-n)/2}^* \mathcal{R}[h](\cdot, \omega))(\sigma)$ for $h \in \mathcal{S}(\mathbb{R}^n)$ with $B_n = 1/(2(2\pi)^{(n-1)/2})$, we get

$$\widehat{\mathcal{R}_n[h]}(\rho, \omega) = \mathcal{F}_1[\mathcal{R}_n[h](\cdot, \omega)](\rho) = \sqrt{2\pi} B_n \mathcal{F}_1[\chi_{(1-n)/2}^*](\rho) \widehat{\mathcal{R}[h]}(\rho, \omega), \quad (3.32)$$

where $\mathcal{F}_1$ denotes the one-dimensional Fourier transformation (of a tempered distribution). It is known that

$$\mathcal{F}_1[\chi_{a}^*](\rho) = \frac{1}{\sqrt{2\pi}} e^{i\pi(a+1)/2}(\rho + i0)^{-a-1}$$

for every $a \in \mathbb{C}$, where $z^b$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $b \in \mathbb{C}$ is given by $z^b = \exp(b \log z)$ if we write $\log z = \log |z| + i \arg z$ with $-\pi < \arg z < \pi$, and $(\rho + i0)^b$ is defined by $(\rho + i0)^b = \lim_{\epsilon \rightarrow +0}(\rho + i\epsilon)^b$ (see Hörmander [5] Example 7.1.17) for instance. Especially we have

$$|\mathcal{F}_1[\chi_{(1-n)/2}^*](\rho)|^2 = \frac{1}{2\pi} |\rho|^{n-3}, \quad \rho \in \mathbb{R}.$$ 

Therefore, for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and nonnegative integers $j, k$, it follows from the Plancherel formula for the one-dimensional Fourier transform and (3.32) that

$$\left\langle \partial_\sigma^j \mathcal{R}_n[\varphi], \partial_\sigma^k \mathcal{R}_n[\psi] \right\rangle_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$= \langle (-i)^k B_n^2 \int_{\mathbb{S}^{n-1}} \left( \int_{-\infty}^{\infty} \mathcal{R}[\varphi](\rho, \omega) \mathcal{R}[\psi](\rho, \omega) |\rho|^{n-3} \rho^{j+k} d\rho \right) dS_\omega, \quad (3.33)$$

where $dS_\omega$ denotes the surface element on $\mathbb{S}^{n-1}$. By (3.31), we see that the integrand on the right-hand side of (3.33) is an odd (resp. even) function in $(\rho, \omega)$ if $j + k$ is odd (resp. even). Hence we have $\langle \partial_\sigma^2 \mathcal{R}_n[\varphi], \partial_\sigma \mathcal{R}_n[\psi] \rangle_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = 0$.

Now we find

$$\|\partial_\sigma \mathcal{F}_0[\varphi, \psi]\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 = \|\partial_\sigma^2 \mathcal{R}_n[\varphi]\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 + \|\partial_\sigma \mathcal{R}_n[\psi]\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2.$$
We obtain from (3.33), (3.31), and (3.30) that
\[ \| \partial_\sigma R_n[\psi] \|_{L^2(\mathbb{R} \times S^{n-1})} = 2B_n^2 \int_{S^{n-1}} \left( \int_0^\infty \left| \mathcal{R}[\psi](\rho, \omega) \right|^2 \rho^{n-1} d\rho \right) dS_\omega \]
\[ = \frac{1}{2} \| \hat{\psi} \|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2} \| \psi \|_{L^2(\mathbb{R}^n)}^2. \]

Just in the same manner, we obtain
\[ \| \partial_\sigma^2 R_n[\psi] \|_{L^2(\mathbb{R} \times S^{n-1})} = \frac{1}{2} \| \cdot \| \hat{\psi} \|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2} \| \nabla_x \psi \|_{L^2(\mathbb{R}^n)}. \]

This completes the proof. \( \square \)

By Lemma 3.4, we can uniquely extend the linear mapping
\( (S(\mathbb{R}^n))^2 \ni (\varphi, \psi) \mapsto \partial_\sigma \mathcal{F}_0[\varphi, \psi] \in L^2(\mathbb{R} \times S^{n-1}) \)

\( \) to the linear mapping \( T \) from \( H_0(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) with
\[ \| T[\varphi, \psi] \|_{L^2(\mathbb{R} \times S^{n-1})} = \| (\varphi, \psi) \|_{H_0(\mathbb{R}^n)}, \quad (\varphi, \psi) \in H_0(\mathbb{R}^n). \]

(3.34)

This mapping \( T \) is called the translation representation in Lax-Phillips [20].

The following lemma was essentially proved in [20] for odd \( n \), and in [19] for even \( n \) (see also Melrose [21]).

**Lemma 3.5.** The mapping \( T \) defined above is an isometric isomorphism from \( H_0(\mathbb{R}^n) \) to \( L^2(\mathbb{R} \times S^{n-1}) \).

**Proof.** What is left to show is that \( T \) is surjective. Because of (3.34), we only have to prove the following: For any function \( v \) in some dense subset of \( L^2(\mathbb{R} \times S^{n-1}) \), there is \( (\varphi, \psi) \in H_0(\mathbb{R}^n) \) such that \( T[\varphi, \psi] = v \).

We write \( h \in S_0(\mathbb{R} \times S^{n-1}) \) if \( h \in S(\mathbb{R} \times S^{n-1}) \) and there is a positive constant \( \delta \) such that \( h(\rho, \omega) = 0 \) for all \( (\rho, \omega) \in (-\delta, \delta) \times S^{n-1} \). We put
\[ S_1(\mathbb{R} \times S^{n-1}) = \left\{ \tilde{\mathcal{F}}^{-1}[h] ; h \in S_0(\mathbb{R} \times S^{n-1}) \right\}, \]

where
\[ \tilde{\mathcal{F}}^{-1}[h](\sigma, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\rho\sigma} h(\rho, \omega) d\rho, \]

which is the one-dimensional inverse Fourier transform of \( h(\cdot, \omega) \) with a parameter \( \omega \in S^{n-1} \). It is easy to see that \( S_0(\mathbb{R} \times S^{n-1}) \) is dense in \( L^2(\mathbb{R} \times S^{n-1}) \), and hence \( S_1(\mathbb{R} \times S^{n-1}) \) is also dense in \( L^2(\mathbb{R} \times S^{n-1}) \).

Let \( v \in S_1(\mathbb{R} \times S^{n-1}) \). We want to find \( (\varphi, \psi) \in H_0(\mathbb{R}^n) \) such that \( T[\varphi, \psi] = v \). We put
\[ v_0(\rho, \omega) = \frac{1}{\sqrt{2\pi B_n \mathcal{F}_1[\chi^{(1-n)/2}]}(\rho)} \tilde{v}(\rho, \omega), \quad (\rho, \omega) \in \mathbb{R} \times S^{n-1}. \]
Note that we have \( v_0 \in \mathcal{S}_0(\mathbb{R} \times \mathbb{S}^{n-1}) \), because \( \tilde{v} \in \mathcal{S}_0(\mathbb{R} \times \mathbb{S}^{n-1}) \), and the singularity of \( 1/\mathcal{F}_1[\chi_{-1}^{(1-n)/2}] \) lies only at \( \rho = 0 \). For \( \xi \in \mathbb{R}^n \setminus \{0\} \) we put

\[
\begin{align*}
v_1(\xi) &= v_0(|\xi|, |\xi|^{-1}\xi) + v_0(-|\xi|, -|\xi|^{-1}\xi) \left( \frac{2(2\pi)^{(n-1)/2}|\xi|^2}{2i(2\pi)^{(n-1)/2}|\xi|} \right), \\
v_2(\xi) &= v_0(|\xi|, |\xi|^{-1}\xi) - v_0(-|\xi|, -|\xi|^{-1}\xi) \left( \frac{2(2\pi)^{(n-1)/2}|\xi|^2}{2i(2\pi)^{(n-1)/2}|\xi|} \right).
\end{align*}
\]

We also set \( v_1(0) = v_2(0) = 0 \). Then, since \( v_0 \in \mathcal{S}_0(\mathbb{R} \times \mathbb{S}^{n-1}) \), we find that \( v_1, v_2 \in \mathcal{S}(\mathbb{R}^n) \). Hence if we set \( \varphi = \mathcal{F}^{-1}[v_1] \) and \( \psi = \mathcal{F}^{-1}[v_2] \), then we get \( (\varphi, \psi) \in (\mathcal{S}(\mathbb{R}^n))^2 \subset H_0(\mathbb{R}^n) \). Using (3.30) and (3.32), we get

\[
\tilde{\mathcal{F}} [\partial_\sigma \mathcal{F}_0[\varphi, \psi]](\rho, \omega) = \rho^2 \tilde{\mathcal{R}}_n[\varphi](\rho, \omega) + i\rho \tilde{\mathcal{R}}_n[\psi](\rho, \omega) = \tilde{v}(\rho, \omega)
\]

for \((\rho, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}\), which shows \( \mathcal{T}[\varphi, \psi] = \partial_\sigma \mathcal{F}_0[\varphi, \psi] = v \). This completes the proof. \( \square \)

Theorem 2.4 is an immediate consequence of the following lemma.

**Lemma 3.6.** Let \( c > 0 \), and \((w_0, w_1) \in H_0(\mathbb{R}^n)\). Let \( w \) be a solution to the Cauchy problem (3.2)-(3.3), and

\[
\mathcal{W}(t, x) = |x|^{-(n-1)/2}\mathcal{T}[w_0, c^{-1}w_1](|x| - ct, |x|^{-1}x)
\]

for \((t, x) \in [0, \infty) \times (\mathbb{R}^n \setminus \{0\})\). Then we have

\[
\lim_{t \to \infty} \|\partial^{(c)}w(t, \cdot) - \tilde{\omega}(\cdot)\mathcal{W}(t, \cdot)\|_{L^2(\mathbb{R}^n)} = 0,
\]

where \( \partial^{(c)} := (c^{-1}\partial_t, \partial_1, \ldots, \partial_n) \) and \( \tilde{\omega}(x) := \tilde{\omega}_1(x) = (-1, |x|^{-1}x) \).

**Proof.** Let \( \varepsilon > 0 \). There is \((w_0^*, w_1^*) \in \left( C_0^\infty(\mathbb{R}^n) \right)^2\) such that

\[
\left\| (w_0, c^{-1}w_1) - (w_0^*, c^{-1}w_1^*) \right\|_{H_0(\mathbb{R}^n)}^2 < \left( \frac{\varepsilon}{3} \right)^2.
\]

Let \( w^* \) and \( \mathcal{W}^* \) be defined similarly to \( w \) and \( \mathcal{W} \), respectively, by replacing \((w_0, w_1)\) with \((w_0^*, w_1^*)\). Recalling (3.1), we obtain from the energy identity that

\[
\frac{1}{2} \left\| \partial^{(c)}w(t, \cdot) - \partial^{(c)}w^*(t, \cdot) \right\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2} \left( \|c^{-1}(w_1 - w_1^*)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla_x(w_0 - w_0^*)\|_{L^2(\mathbb{R}^n)}^2 \right) = \left\| (w_0, c^{-1}w_1) - (w_0^*, c^{-1}w_1^*) \right\|_{H_0(\mathbb{R}^n)}^2 < \left( \frac{\varepsilon}{3} \right)^2.
\]
Since $|\tilde{\gamma}(x)| = 2$, using (3.34) we get
\[
\frac{1}{2} \left\| \tilde{\omega}(\cdot)(W(t, \cdot) - W^*(t, \cdot)) \right\|_{L^2(\mathbb{R}^n)}^2 \\
= \int_0^\infty \left( \int_{S^{n-1}} |T[w_0 - w_0^*, c^{-1}(w_1 - w_1^*)](r - ct, \omega)|^2 dS_\omega \right) dr \\
\leq \left\| T[w_0 - w_0^*, c^{-1}(w_1 - w_1^*)] \right\|_{L^2(\mathbb{R} \times S^{n-1})}^2 \\
= \left\| (w_0, c^{-1}(w_1) - (w_0^*, c^{-1}w_1^*) \right\|_{H_0(\mathbb{R}^n)}^2 < \left( \frac{\varepsilon}{3} \right)^2.
\]
By Lemma 3.3 we obtain
\[
\frac{1}{2} \left\| \partial^{(c)} w^*(t, \cdot) - \tilde{\omega}(\cdot)W^*(t, \cdot) \right\|_{L^2(\mathbb{R}^n)}^2 \\
\leq \frac{1}{2} \int_0^\infty \left( \int_{S^{n-1}} |r^{(n-1)/2}\partial^{(c)} w^*(t, r\omega) - \tilde{\omega}(r\omega)(\partial_\sigma \mathcal{F}_0^*)(r - ct, \omega)|^2 dS_\omega \right) dr \\
\leq C \int_0^\infty (1 + t + r)^{-2}(1 + |t - r|)^{-(n-1)} dr \leq C(1 + t)^{-1},
\]
where $\mathcal{F}_0^*(\sigma, \omega) = \mathcal{F}_0[w_0^*, c^{-1}w_1^*](\sigma, \omega)$. Hence there is a positive constant $t_0 > 0$ such that $t \geq t_0$ implies
\[
\frac{1}{2} \left\| \partial^{(c)} w^*(t, \cdot) - \tilde{\omega}(\cdot)W^*(t, \cdot) \right\|_{L^2(\mathbb{R}^n)}^2 < \left( \frac{\varepsilon}{3} \right)^2.
\]
To sum up, for any $\varepsilon > 0$ there is a positive constant $t_0$ such that we have
\[
\frac{1}{\sqrt{2}} \left\| \partial^{(c)} w(t, \cdot) - \tilde{\omega}(\cdot)W(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} < \varepsilon, \quad t \geq t_0.
\]
This completes the proof. $\square$

Now we are in a position to prove Theorem 2.1.

**Proof of Theorem 2.1.** Suppose that there is $V \in L^2(\mathbb{R} \times S^{n-1})$ such that
\[
\lim_{t \to \infty} \left\| \partial_v(t, \cdot) - \tilde{\omega}_c(\cdot)V^2(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} = 0, \quad (3.35)
\]
where
\[
V^2(t, x) = |x|^{-(n-1)/2} V(|x| - ct, |x|^{-1}x).
\]
Then we get
\[
\lim_{t \to \infty} \left\| \partial^{(c)} v(t, \cdot) - \tilde{\omega}(\cdot)V^2(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} = 0, \quad (3.36)
\]
where $\partial^{(c)}$ and $\tilde{\omega}(x)$ are defined as in Lemma 3.6. We define $(w_0, c^{-1}w_1) = T^{-1}[V](\in H_0(\mathbb{R}^n))$, and let $w$ be the solution to the Cauchy problem (3.2)–(3.3) for this $(w_0, w_1)$. Then it follows from Lemma 3.6 that
\[
\lim_{t \to \infty} \left\| \partial^{(c)} w(t, \cdot) - \tilde{\omega}(\cdot)V^2(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} = 0, \quad (3.37)
\]
which, together with (3.37) implies
\[
\lim_{t \to \infty} \left\| v(t, \cdot) - w(t, \cdot) \right\|_{E,c} = \lim_{t \to \infty} \frac{1}{\sqrt{2}} \left\| \partial^{(c)} v(t, \cdot) - \partial^{(c)} w(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} = 0. \quad (3.39)
\]
Conversely, suppose that there is \((w_0, w_1) \in H_0(\mathbb{R}^n)\) such that \((3.39)\) holds for the solution \(w\) to the Cauchy problem \((3.2)–(3.3)\). If we put
\[ V(\sigma, \omega) = \mathcal{T}[w_0, c^{-1}w_1](\sigma, \omega) (\in L^2(\mathbb{R} \times S^{n-1})) \]
and define \(V^\sharp\) by \((3.36)\), then Lemma \(3.6\) implies \((3.38)\). Now \((3.39)\) and \((3.38)\) yield \((3.37)\), which is equivalent to \((3.35)\). This completes the proof. \(\square\)

4. Vector Fields Associated with the Wave Equations

We restrict our attention to the three space dimensional case from now on. We introduce vector fields
\[ S := t\partial_t + x \cdot \nabla_x, \]
\[ \Omega = (\Omega_1, \Omega_2, \Omega_3) := x \times \nabla_x = (x_2\partial_3 - x_3\partial_2, x_3\partial_1 - x_1\partial_3, x_1\partial_2 - x_2\partial_1), \]
where the symbols \(\cdot\) and \(\times\) denote the inner and exterior products in \(\mathbb{R}^3\), respectively. We also use \(\partial = (\partial_t, \nabla_x) = (\partial_0)_{0 \leq a \leq 3}\). We define
\[ Z = (Z_0, Z_1, \ldots, Z_7) = (S, \Omega, \partial) = (S, \Omega_1, \Omega_2, \Omega_3, \partial_0, \partial_1, \partial_2, \partial_3). \]
These vector fields are compatible with the system of wave equations having multiple propagation speeds, since we have \([\square_c, \Omega_j] = [\square_c, \partial_a] = 0\) for \(1 \leq j \leq 3\) and \(0 \leq a \leq 3\), and \([\square_c, S] = 2\square_c\), where \(c > 0\), and \([P, Q] = PQ - QP\) for operators \(P\) and \(Q\). Using a multi-index \(\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_7)\), we write \(Z^\alpha = Z_0^\alpha Z_1^{\alpha_1} \cdots Z_7^{\alpha_7}\). For a nonnegative integer \(s\) and a (scalar- or vector-valued) smooth function \(\varphi = \varphi(t, x)\), we define
\[ |\varphi(t, x)|_s = \sum_{|\alpha| \leq s} |Z^\alpha \varphi(t, x)|. \]
We can check that we have \([Z_0, Z_0] = \sum_{d=0}^7 C_d^a Z_d\) and \([Z_0, \partial_0] = \sum_{d=0}^3 D_d^a \partial_d\) with appropriate constants \(C_d^a\) and \(D_d^a\). Hence, for any multi-indices \(\alpha\) and \(\beta\), and any nonnegative integer \(s\), there exist some positive constants \(C_{\alpha, \beta}\) and \(C_s\) such that we have
\[ |Z^\alpha Z^\beta \varphi(t, x)| \leq C_{\alpha, \beta} |\varphi(t, x)|_{|\alpha| + |\beta|}, \]
\[ C_s^{-1} |\partial \varphi(t, x)|_s \leq \sum_{|\alpha| \leq s} \sum_{a=0}^3 |\partial_a Z^\alpha \varphi(t, x)| \leq C_s |\partial \varphi(t, x)|_s \]
for any smooth function \(\varphi\).

We write \(r = |x|, \omega = (\omega_1, \omega_2, \omega_3) = |x|^{-1} x, \) and \(\partial_r = \sum_{j=1}^3 \omega_j \partial_j\). Then we can write \(S = t\partial_t + r\partial_r\). For \(c > 0\), we define
\[ \partial^{(c)}_\pm = \partial_t \pm c\partial_r, \quad \text{and} \quad D^{(c)}_\pm = \pm \frac{1}{2c} \partial^{(c)}_\pm = \frac{1}{2}(\partial_t \pm c^{-1}\partial_r). \]
Since we have \(\partial_t = -c (D^{(c)}_- - D^{(c)}_+), \partial_r = D^{(c)}_- + D^{(c)}_+\), and
\[(1 + r) \partial^{(c)}_+ = cS - (ct - r) \partial_t + (\partial_t + c\partial_r), \]

\[ \square = \partial_t^2 - \Delta_x, \quad \square^{(c)} = \partial^{(c)}_+ \partial^{(c)}_- - \Delta^{(c)}_x. \]
there exists a positive constant $C$ such that
\[
\left| (\partial_t - (-c)D^{(e)})(\varphi(t, x)) \right| + \left| (\partial_r - D^{(e)}_{-})(\varphi(t, x)) \right| \\
\leq \frac{C}{1 + r} \left( |Z\varphi(t, x)| + |ct - r| |\varphi(t, x)| \right) \tag{4.1}
\]
for any smooth function $\varphi$. Since we have $\nabla_x = \omega \partial_r - r^{-1} \omega \times \Omega$, we get
\[
| (\partial_k - \omega_k \partial_r )\varphi(t, x) | \leq C(1 + r)^{-1}|Z\varphi(t, x)|, \quad k = 1, 2, 3. \tag{4.2}
\]
From (4.1) and (4.2) we obtain
\[
\left| \partial_x \varphi(t, x) - \tilde{\omega}_c(x)D^{(e)}_{-}(r\varphi(t, x)) \right| \leq \frac{C}{1 + r} \left( |Z\varphi(t, x)| + |ct - r| |\varphi(t, x)| \right) \tag{4.3}
\]
for any smooth function $\varphi$, where $\tilde{\omega}_c(x)$ is given by (2.2). As an immediate consequence we also get
\[
\left| r\partial \varphi(t, x) - \tilde{\omega}_c(x)D^{(e)}_{-}(r\varphi(t, x)) \right| \leq C \left( |\varphi(t, x)|_1 + |ct - r| |\varphi(t, x)| \right). \tag{4.4}
\]
We remark that the term $|ct - r| |\varphi|$ was not needed in the estimates used in [10] instead of (4.3) and (4.4); however the vector field $L = t\nabla_x + x\partial_t$, which is not compatible with the multiple speed case, was involved. We need the term $|ct - r| |\varphi|$ here to compensate the lack of the vector field $L$. These kinds of identities and estimates without the vector field $L$ were developed and used in [17], [1], [23], [27], [29], and so on (see also [9], [11], [24], [25], and [26] for the related topics).

Using (4.3), we can easily show the following estimate for the null forms $Q_0$ and $Q_{ab}$ given by (1.7) and (1.8) (see [23], [27] and [29] for the details of the proof):

**Lemma 4.1.** Let $c > 0$. Then we have
\[
|Q_0(\varphi, \psi; c)| + \sum_{0 \leq a < b \leq 3} |Q_{ab}(\varphi, \psi)| \leq C \langle r \rangle^{-1} (|Z\varphi| |\varphi| + |\varphi| |Z\psi|) \]
\[
+ C \langle r \rangle^{-1} |ct - r| |\varphi| |\partial \varphi|
\]
at $(t, x) \in (0, \infty) \times \mathbb{R}^3$ with $r = |x|$.

**Outline of proof.** Fix $c > 0$, and we define $R = (R_a)_{0 \leq a \leq 3} = \partial - \tilde{\omega}_c(x)D^{(e)}_{-}$. Substituting $\partial = \tilde{\omega}_c(x)D^{(e)}_{-} + R$, we get
\[
|Q_0(\varphi, \psi; c)| + \sum_{a,b} |Q_{ab}(\varphi, \psi)| \leq C \left( |D^{(e)}_{-}\varphi| |R\psi| + |R\varphi| |D^{(e)}_{-}\psi| + |R\varphi| |R\psi| \right) \]
\[
\leq C \left( |\partial \varphi| |R\psi| + |R\varphi| |\partial \varphi| \right),
\]
where we have used $|R\psi| \leq C |\partial \varphi|$ to obtain the last line. The point here is that terms including $(D^{(e)}_{-}\varphi)(D^{(e)}_{-}\psi)$ are canceled out because of the structure of the null forms. Now, using (4.3) to evaluate $|R\varphi|$ and $|R\psi|$, we obtain the desired result. \[\square\]
Let $c_1, \ldots, c_N$ satisfy (1.5). We define
\[
\Lambda_0 = \left\{ (t, r) \in \mathbb{R}_+ \times \mathbb{R}_+ ; r \geq \frac{c_j t}{2} \geq 1 \right\},
\] (4.5)
where $\mathbb{R}_+ = [0, \infty)$. For $1 \leq j \leq N$, we define
\[
t_{0,j}(\sigma) := \max\{-2(2c_j - c_1)^{-1}\sigma, 2c_1^{-1}\}
\] (4.6)
so that we have $\bigcup_{\sigma \in \mathbb{R}} \{(t, c_j t + \sigma); t \geq t_{0,j}(\sigma)\} = \Lambda_0$. We also define $r_{0,j}(\sigma) := c_j t_{0,j}(\sigma) + \sigma$ for $\sigma \in \mathbb{R}$. The following lemma is a modification of a key lemma in [10] to obtain the asymptotic behavior.

**Lemma 4.2.** Assume (1.5), and let $j \in \{1, \ldots, N\}$. Suppose the following:
- $\mu_j > 1$ and $\kappa_j \geq 0$.
- $\mu_k \geq 0$ and $\kappa_k > 1$ for any $k = 1, \ldots, N$ with $k \neq j$.

If $v = v(t, r, \omega) \in C^1(\Lambda_0 \times \mathbb{S}^2)$ satisfies
\[
\partial_+^{(c_j)} v(t, r, \omega) = G(t, r, \omega), \quad (t, r) \in \Lambda_0, \, \omega \in \mathbb{S}^2,
\] (4.7)
and $G$ satisfies
\[
|G(t, r, \omega)| \leq \sum_{k=1}^N B_k \langle t + r \rangle^{-\mu_k} \langle c_j t - r \rangle^{-\kappa_k}, \quad (t, r) \in \Lambda_0, \, \omega \in \mathbb{S}^2
\] (4.8)
with some positive constants $B_1, \ldots, B_N$, then there exists a positive constant $C$ such that
\[
|v(t, r, \omega) - V(r - c_j t, \omega)| \leq C B_j \langle t + r \rangle^{-\mu_j + 1} \langle c_j t - r \rangle^{-\kappa_j}
\] \[\quad + C \sum_{1 \leq k \leq N} B_k \langle t + r \rangle^{-\mu_k}
\] (4.9)
for any $(t, r) \in \Lambda_0$ and $\omega \in \mathbb{S}^2$, where $V$ is defined by
\[
V(\sigma, \omega) = v(t_{0,j}(\sigma), r_{0,j}(\sigma), \omega) + \int_{t_{0,j}(\sigma)}^{\infty} G(s, c_j s + \sigma, \omega) ds.
\] (4.10)
The constant $C$ above is determined only by $c_k, \mu_k$, and $\kappa_k$ with $1 \leq k \leq N$.

**Proof.** First we note that $V$ is well-defined because of (4.8). By (4.7) we find
\[
v(t, c_j t + \sigma, \omega) = v(t_{0,j}(\sigma), r_{0,j}(\sigma), \omega) + \int_{t_{0,j}(\sigma)}^{t} G(s, c_j s + \sigma, \omega) ds, \quad t \geq t_{0,j}(\sigma)
\]
for $(\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^2$. From (4.10) and (4.8) we get
\[
|v(t, c_j t + \sigma, \omega) - V(\sigma, \omega)| \leq \int_{t}^{\infty} |G(s, c_j s + \sigma, \omega)| ds \leq C \sum_{k=1}^{N} B_k I_k(t, \sigma)
\] (4.11)
where
\[
I_k(t, \sigma) = \int_{t}^{\infty} (1 + \tau + \rho)^{-\mu_k} (1 + |c_k \tau - \rho|)^{-\kappa_k} \bigg|_{(\tau, \rho) = (s, c_j s + \sigma)} ds
\]
for $1 \leq k \leq N$. By direct calculations, we obtain
\[
I_j(t, \sigma) = \frac{1}{(c_j + 1)(\mu_j - 1)}(1 + (c_j + 1)t + \sigma)^{-\mu_j+1}(1 + |\sigma|)^{-\kappa_j},
\] (4.12)
since $\mu_j > 1$. If $k \neq j$, then we have
\[
I_k(t, \sigma) \leq (1 + (c_j + 1)t + \sigma)^{-\mu_k} \int_t^\infty \frac{1}{|c_k - c_j|} (1 + |\tau|)^{-\kappa_k} d\tau
\]
\[
\leq \frac{2}{(\kappa_k - 1)|c_k - c_j|} (1 + (c_j + 1)t + \sigma)^{-\mu_k}
\] (4.13)
for $t \geq t_{0,j}(\sigma)$ and $\sigma \in \mathbb{R}$, because we have $\mu_k \geq 0$, $\kappa_k > 1$, and $c_k \neq c_j$. Now we obtain (4.9) immediately by putting $\sigma = r - c_j t$ in (4.11), (4.12), and (4.13).

5. Proof of Theorem 2.2 and Corollary 2.3

In the following, we write $r = |x|$ and $\omega = |x|^{-1} x$. Let the assumptions in Theorem 2.2 be fulfilled. Then we have the global solution $u$ to the Cauchy problem (1.1)–(1.2) for sufficiently small $\varepsilon$ by the global existence theorems in [27], [28], or [29]. For $j = 1, \ldots, N$, we put
\[
u_j^j(t, x) = u_j(t, x) - \varepsilon u_j^0(t, x),
\]
where $u_j^0$ is the solution to $\Box_v u_j^0 = 0$ with initial data $u_j^0 = f_j$ and $\partial_t u_j^0 = g_j$ at $t = 0$. From (1.1) and (1.2), we have
\[
\Box_v u_j^1 = F_j(\partial u, \partial^2 u) \quad \text{in } (0, \infty) \times \mathbb{R}^3,
\]
\[
u_j^1(0, x) = (\partial_t u_j^1)(0, x) = 0, \quad x \in \mathbb{R}^3.
\] (5.2)

We put
\[
U_j^0(\sigma, \omega) = F_j[f_j, c_j^{-1} g_j](\sigma, \omega).
\]
Suppose that we have $f(x) = g(x) = 0$ for $|x| \geq M$. Then, as in (3.5), (3.6), and (3.19), we get
\[
u_j^0(t, x) = 0, \quad |r - c_j t| \geq M,
\] (5.3)
\[
U_j^0(\sigma, \omega) = 0, \quad |\sigma| \geq M
\] (5.4)
for $1 \leq j \leq N$. Hence Lemma 3.3 implies
\[
|ru_j^0(t, x) - U_j^0(r - c_j t, x)| + |r \partial u_j^0(t, x) - \omega c_j(x) (\partial_x U_j^0)(r - c_j t, x)|
\leq C (t + r)^{-1} (c_j t - r)^{-1}, \quad 1 \leq j \leq N
\] (5.5)
for $(t, x) \in [0, \infty) \times (\mathbb{R}^3 \setminus \{0\})$, because (5.3) and (5.4) imply $1 \leq \langle c_j t - r \rangle \leq \langle M \rangle$ on the support of the functions on the left-hand side of (5.5).
Let \( \Lambda \) be the set of \((t, x) \in [0, \infty) \times \mathbb{R}^3 \) with \((t, r) \in \Lambda_0 \), where \( \Lambda_0 \) is given by \((1.5) \); namely we put
\[
\Lambda = \left\{ (t, x) \in [0, \infty) \times (\mathbb{R}^3 \setminus \{0\}); |x| \geq \frac{c_k t}{2} \geq 1 \right\}.
\]
We set \( \Lambda^c = \left( [0, \infty) \times (\mathbb{R}^3 \setminus \{0\}) \right) \setminus \Lambda \). Recall the definitions of \( \partial_{(c)}^\pm \) and \( D_{(c)}^\pm \) in the previous section.

**Proof of Theorem 2.2** (1). Following the proof of the global existence theorem in [18] (see also [7, Proposition 4.2] and its proof) we obtain
\[
|u_k(t, x)|_2 + \varepsilon^{-1}|u_k^1(t, x)|_2 \leq C\varepsilon (t + r)^{-1} \log \left( 1 + \frac{1 + c_k t + r}{1 + |c_k t - r|} \right), \quad (5.6)
\]
\[
|\partial u_k(t, x)|_1 + \varepsilon^{-1}|\partial u_k^1(t, x)|_1 \leq C\varepsilon (r)^{-1} (c_k t - r)^{-1} \quad (5.7)
\]
for \( 1 \leq k \leq N \). We choose small \( \delta > 0 \). Then, by \((5.6)\) we get
\[
|u_k(t, x)|_2 + \varepsilon^{-1}|u_k^1(t, x)|_2 \leq C\varepsilon (t + r)^{-1+\delta} (c_k t - r)^{-\delta} \quad (5.8)
\]
for \( 1 \leq k \leq N \), because there is a positive constant \( C_{\delta} \) such that we have \( \log(1+z) \leq C_{\delta} z^\delta \) for \( z \geq 1 \).

Fix \( j = 1, 2, \ldots, N \). Switching to the polar coordinates, we get
\[
\frac{d}{dt}(\Box_{c_j} u_j)(t, r \omega) = \partial_{(c)}^+(\partial_{(c)}^j \partial_{(c)}^j(r u_j^1(t, r \omega) - c_j^2 r^{-1} \Delta_{\omega} u_j^1(t, r \omega)) \quad (5.9)
\]
for \((t, r, \omega) \in (0, \infty) \times (0, \infty) \times \mathbb{S}^2 \), where \( \Delta_{\omega} = \sum_{k=1}^3 \Omega_k^2 \) is the Laplace-Beltrami operator on \( \mathbb{S}^2 \). We put
\[
v_j(t, r, \omega) := D_{(c)}^j(r u_j^1(t, r \omega)). \quad (5.10)
\]
Then \((5.1)\) and \((5.9)\) lead to
\[
\partial_{(c)}^+(v_j)(t, r, \omega) = G_j(t, r, \omega), \quad (5.11)
\]
where
\[
G_j(t, r, \omega) := \frac{1}{2c_j} \left\{ c_j^2 r^{-1} \Delta_{\omega} u_j^1(t, r \omega) + r F_j(\partial u(t, r \omega), \partial^2 u(t, r \omega)) \right\}.
\]

We suppose \((t, r) \in \Lambda_0 \) and \( \omega \in \mathbb{S}^2 \) for a while. Note that we have
\[
\langle 1 + 2c_j^{-1} \rangle^{-1} (t + r) \leq r \leq \langle t + r \rangle, \quad (t, r) \in \Lambda_0.
\]
From \((5.8)\) we obtain
\[
|r^{-1} \Delta_{\omega} u_j^1(t, r \omega)| \leq C\varepsilon^2 \langle t + r \rangle^{-2+\delta} \langle c_j t - r \rangle^{-\delta}. \quad (5.12)
\]
Recall the definitions of \( N_j, R_j^1, \) and \( R_j^{11} \) given in \((1.10), (1.11), \) and \((1.12)\), respectively. By Lemma \((4.1)\) \((5.7), \) and \((5.8)\), we get
\[
|r N_j(\partial u, \partial^2 u)| \leq C\varepsilon^2 \langle t + r \rangle^{-2+\delta} \langle c_j t - r \rangle^{-1-\delta}. \quad (5.13)
\]
Noting that we have
\[
\langle c_k t - r \rangle \langle c_k t - r \rangle \geq C \langle t + r \rangle \min\{\langle c_k t - r \rangle, \langle c_k t - r \rangle\}
\]
for $c_k \neq c_l$, we obtain from (5.7) that
\[
|r| R_j^1(\partial u, \partial^2 u)| \leq C\varepsilon^2 \left( t + r \right)^{-1} \sum_{k \neq l} c_k t - r^{-1} c_l t - r^{-1} \\
\leq C\varepsilon^2 \left( t + r \right)^{-2} \sum_{k=1}^N \langle c_k t - r \rangle^{-1}.
\] (5.14)

By (5.7) we also have
\[
|r| R_j^1(\partial u, \partial^2 u)| \leq C\varepsilon^2 \sum_{k \neq j} (t + r)^{-1} \langle c_k t - r \rangle^{-2}.
\] (5.15)

Now (5.12), (5.13), (5.14), and (5.15) lead to
\[
|G_j(t, r, \omega)| \leq C\varepsilon^2 \left( t + r \right)^{-2+\delta} \langle c_j t - r \rangle^{-\delta} \\
+ C\varepsilon^2 \sum_{k \neq j} (t + r)^{-1} \langle c_k t - r \rangle^{-2}.
\] (5.16)

We define
\[
V_j(\sigma, \omega) = v_j(t_0,j(\sigma), r_0,j(\sigma), \omega) + \int_{t_0,j(\sigma)}^{\infty} G_j(s, c_j s + \sigma, \omega) ds,
\] (5.17)
where $t_0,j(\sigma)$ is defined by (4.6), and $r_0,j(\sigma) = c_j t_0,j(\sigma) + \sigma$. Since we have (5.11) and (5.16), Lemma 4.2 implies
\[
|v_j(t, r, \omega) - V_j(r - c_j t, \omega)| \leq C\varepsilon^2 \left( \left( t + r \right)^{-1+\delta} \langle c_j t - r \rangle^{-\delta} + (t + r)^{-1}\right)
\leq C\varepsilon^2 \left( \langle c_j t - r \rangle^{-1} + \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta} \right)
\leq C\varepsilon^2 \langle c_j t - r \rangle^{-1}.
\] (5.18)

for $(t, r) \in \Lambda_0$ and $\omega \in \mathbb{S}^2$. By (5.10), (5.7), and (5.8), we get
\[
|v_j(t, r, \omega)| \leq C (|r| \partial u_j^1(t, r\omega)| + |u_j^1(t, r\omega)|)
\leq C\varepsilon^2 \left( \langle c_j t - r \rangle^{-1} + \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta} \right)
\leq C\varepsilon^2 \langle c_j t - r \rangle^{-1}.
\] (5.19)

Hence, putting $r = c_j t + \sigma$ in (5.18), we get
\[
|V_j(\sigma, \omega)| \leq |v_j(t, c_j t + \sigma, \omega)| + C\varepsilon^2 \langle (c_j + 1)t + \sigma \rangle^{-1+\delta} \langle \sigma \rangle^{-\delta}
\leq C\varepsilon^2 \langle \langle \sigma \rangle^{-1} + \langle (c_j + 1)t + \sigma \rangle^{-1+\delta} \langle \sigma \rangle^{-\delta} \rangle
\] (5.20)
for all $(\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}$ and $t \geq t_0,j(\sigma)$. Taking the limit in (5.20) as $t \to \infty$, we obtain
\[
|V_j(\sigma, \omega)| \leq C\varepsilon^2 \langle \sigma \rangle^{-1}, \quad (\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^2.
\] (5.21)

Now we define $P^1_j(\sigma, \omega) = \varepsilon^{-1}V_j(\sigma, \omega)$. Then, recalling the definition of $v_j$, we obtain from (4.4), (5.7), and (5.8) that
\[
|r \partial u_j^1(t, x) - \tilde{\omega}_j(x)v_j(t, r, \omega)| \leq C\varepsilon^2 \left( t + r \right)^{-1+\delta} \langle c_j t - r \rangle^{-\delta}, \quad (t, x) \in \Lambda.
\]
Indeed, similarly to (5.13) and (5.14), we obtain from (5.27) and (5.28) that
\[ |r \partial u_j^1(t, x) - \varepsilon \widetilde{w}_j(x) P_j^1(r - c_j t, \omega)| \leq C \varepsilon^2 \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta} \] (5.22)
for \((t, x) \in \Lambda\). Finally we define
\[ P_j^1(\sigma, \omega) = \partial_s U_j^0(\sigma, \omega) + P_j^1(\sigma, \omega). \] (5.23)
Then (5.5) and (5.22) yield (2.4) for \((t, x) \in \Lambda\). Noting that (5.21) and the definition of \(P_j^1\) lead to
\[ |P_j^1(\sigma, \omega)| \leq C \varepsilon \langle \sigma \rangle^{-1}, \quad (\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^2, \] (5.24)
and recalling (5.4), we immediately obtain (2.5) by (5.23).

Now what is left to prove is (2.4) for \((t, x) \in \Lambda^c\). Since we have either \(t \leq 2c_1^{-1}\) or \(r \leq c_1 t/2 (\leq c_2 t/2)\) in \(\Lambda^c\), we get \(\langle c_j t - r \rangle^{-1} \leq C \langle t + r \rangle^{-1}\). Therefore, by (5.1) we get
\[ |r \partial u_j(t, x)| \leq C \varepsilon \langle c_j t - r \rangle^{-1} \leq C \varepsilon \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta} \] (5.25)
for \((t, x) \in \Lambda^c\). It follows from (5.4), (5.24), and (5.23) that
\[ |P_j(\sigma, \omega)| \leq C \langle \sigma \rangle^{-1}, \quad (\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^2, \]
which immediately yields
\[ |P_j(r - c_j t, \omega)| \leq C \langle c_j t - r \rangle^{-1} \leq C \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta} \] (5.26)
for \((t, x) \in \Lambda^c\). From (5.25) and (5.26) we obtain
\[ |r \partial u_j(t, x) - \varepsilon \widetilde{w}_j(x) P_j(r - c_j t, \omega)| \leq C \varepsilon \langle t + r \rangle^{-1+\delta} \langle c_j t - r \rangle^{-\delta}, \quad (t, x) \in \Lambda^c, \]
which is (2.4) for \((t, x) \in \Lambda^c\). This completes the proof. □

**Proof of Theorem 2.2** (2). Suppose that \(R^I\) has the null structure, and fix \(\rho \in (1/2, 1)\). Then following the proof of the global existence theorem in [8] (see Proposition 5.2 and its proof in [8] specifically), we have
\[ |u_k(t, x)|_2 + \varepsilon^{-1}|u_k^1(t, x)|_2 \leq C \varepsilon \langle t + r \rangle^{-1} \langle c_k t - r \rangle^{-1}, \] (5.27)
\[ |\partial u_k(t, x)|_2 + \varepsilon^{-1}|\partial u_k^1(t, x)|_2 \leq C \varepsilon \langle r \rangle^{-1} \langle c_k t - r \rangle^{-1-\rho}, \] (5.28)
which are better than (5.6) and (5.7). Using decay estimates in [15] (see Lemmas 3.2 and 6.1 in [15]), we can further improve (5.28) in the decay rate and we get
\[ |\partial u_k(t, x)|_1 + \varepsilon^{-1}|\partial u_k^1(t, x)|_1 \leq C \varepsilon \langle r \rangle^{-1} \langle c_k t - r \rangle^{-2}. \] (5.29)

Indeed, similarly to (5.13) and (5.14), we obtain from (5.27) and (5.28) that
\[ r |N_j|_1 \leq C \varepsilon^2 \langle t + r \rangle^{-1} \langle c_j t - r \rangle^{-2-\rho} + \langle r \rangle^{-2} \langle c_j t - r \rangle^{-1-2\rho}, \]
\[ \leq C \varepsilon^2 \langle t + r \rangle^{-2} w_-(t, r)^{-1-2\rho}, \]
\[ r |R_j^1|_1 \leq C \varepsilon^2 \langle r \rangle^{-1} \sum_{k \neq l} \langle c_k t - r \rangle^{-1-\rho} \langle c_l t - r \rangle^{-1-\rho} \]
\[ \leq C \varepsilon^2 \langle t + r \rangle^{-2} w_-(t, r)^{-1-2\rho} \]
at \((t, x) \in (0, \infty) \times \mathbb{R}^3\), where \(w_-(t, r) = \min_{0 \leq k \leq N} \langle c_k t - r \rangle\) with \(c_0 = 0\). Here we have used the estimates 

\[ \langle r \rangle \langle c_j t - r \rangle \geq C \langle t + r \rangle w_-(t, r) \] 

and we obtain (5.29) from the following estimate, which comes from Lemma in [15] (especially see the estimates (3.7), (6.1) and (6.2) in [15]): For \(c, \kappa, \mu > 0\), we have used the estimates

\[ |r| |F(\partial u, \partial^2 u)|_1 \leq C\varepsilon^2 \langle t + r \rangle^{-2} w_-(t, r)^{-1-2\rho}, \quad (5.30) \]

and we obtain (5.29) from the following estimate, which comes from Lemmas 3.2 and 6.1 in [15] (especially see the estimates (3.7), (6.1) and (6.2) in [15]): For \(c, \kappa, \mu > 0\), it holds that

\[ \langle r \rangle \langle ct - r \rangle^{1+\kappa} |\partial \phi(t, x)| \leq C \sup_{y \in \mathbb{R}^3} \langle y \rangle^{2+\kappa} |\partial \phi(\tau, y)|_1 \bigg|_{\tau = 0} + C \sup_{(\tau, y) \in [0, t] \times \mathbb{R}^3} |y| \langle r + |y| \rangle^{1+\kappa} w_-(\tau, |y|)^{1+\mu} |\Box \phi(\tau, y)|_1. \]

Let \((t, r) \in A_0\) and \(\omega \in \mathbb{S}^2\) for a while. Recall the proof of (1). We use (5.27) and (5.29), instead of (5.6) (or (5.8)) and (5.7). Then (5.12), (5.13), and (5.14) are replaced by

\[ |r^{-1} \Delta_\omega u_j^1(t, r\omega)| \leq C\varepsilon^2 \langle t + r \rangle^{-2} \langle c_j t - r \rangle^{-1}, \]

\[ |r| N_j(\partial u, \partial^2 u)|_1 \leq C\varepsilon^2 \langle t + r \rangle^{-2} \langle c_j t - r \rangle^{-3}, \]

\[ |r| R_j^1(\partial u, \partial^2 u)|_1 \leq C\varepsilon^2 \langle t + r \rangle^{-3} \sum_{k=1}^N \langle c_k t - r \rangle^{-2}, \]

respectively. Since \(R_j^1\) has the null structure, we can use Lemma 4.1 to get

\[ |r| R_j^1(\partial u, \partial^2 u)|_1 \leq C\varepsilon^2 \sum_{k \neq j} \langle t + r \rangle^{-2} \langle c_k t - r \rangle^{-3}, \]

instead of (5.15). These estimates lead to

\[ |G_j(t, r, \omega)| \leq C\varepsilon^2 \langle t + r \rangle^{-2} \left( \langle c_j t - r \rangle^{-1} + \sum_{k \neq j} \langle c_k t - r \rangle^{-3} \right) \quad (5.31) \]

for \((t, r) \in A_0\) and \(\omega \in \mathbb{S}^2\). Now going similar lines to (5.17) through (5.24), we can construct \(P_j^1\) satisfying

\[ |r \partial u_j^1(t, x) - \varepsilon \omega \sigma_j^1(x) P_j^1(r - c_j t, \omega)| \leq C\varepsilon^2 \langle t + r \rangle^{-1} \langle c_j t - r \rangle^{-1} \quad (5.32) \]

for \((t, x) \in \Lambda, \) and

\[ |P_j^1(\sigma, \omega)| \leq C\varepsilon \langle \sigma \rangle^{-2}, \quad (\sigma, \omega) \in \mathbb{R} \times \mathbb{S}^2. \quad (5.33) \]

We define

\[ U_j^1(\sigma, \omega) := - \int_0^\infty P_j^1(\lambda, \omega) d\lambda \quad (5.34) \]
so that we have \( \partial_\sigma U_j^1(\sigma, \omega) = P_j^1(\sigma, \omega) \). Note that the right-hand side of (5.34) is finite because of (5.33), and that \( U_j^1(\sigma, \omega) \to 0 \) as \( \sigma \to \infty \). It follows from (5.32) and (5.27) that
\[
|\partial_r (ru_j^1(t, r\omega)) - \varepsilon (\partial_\sigma U_j^1)(r - cj_t, \omega)| \leq C\varepsilon^2 (t + r)^{-1} (cj_t - r)^{-1} \tag{5.35}
\]
for \((t, x) \in \Lambda\). Since (5.27) implies \( ru_j^1(t, r\omega) \to 0 \) as \( r \to \infty \), from (5.35) we get
\[
|ru_j^1(t, r\omega) - \varepsilon U_j^1(r - cj_t, \omega)| \leq \left| -\int_r^\infty \partial_\lambda (\lambda ru_j^1(t, \lambda \omega) - \varepsilon U_j^1(\lambda - cj_t, \omega)) d\lambda \right| \\
\leq C\varepsilon^2 \int_r^\infty (1 + t + \lambda)^{-1} (1 + |cj_t - \lambda|)^{-1} d\lambda. \tag{5.36}
\]
If \( r \geq 3cj_t/2 \), then we have
\[
\int_r^\infty (1 + t + \lambda)^{-1} (1 + |cj_t - \lambda|)^{-1} d\lambda \leq C \int_r^\infty (1 + t + \lambda)^{-2} d\lambda \leq C(1 + t + r)^{-1}.
\]
For \( r \) satisfying \( 1 \leq c_1 t/2 \leq r \leq 3cj_t/2 \), we get
\[
\int_r^\infty (1 + t + \lambda)^{-1} (1 + |cj_t - \lambda|)^{-1} d\lambda \leq C(1 + t)^{-1} \int_{c_1 t/2}^{3cj_t/2} (1 + |cj_t - \lambda|)^{-1} d\lambda \\
+ C \int_{3cj_t/2}^\infty (1 + t + \lambda)^{-2} d\lambda \\
\leq C(1 + t + r)^{-1} \log(2 + t).
\]
Now it follows from (5.36) that
\[
|ru_j^1(t, x) - \varepsilon U_j^1(r - cj_t, \omega)| \leq C\varepsilon^2 (t + r)^{-1} \log(2 + t), \quad (t, x) \in \Lambda. \tag{5.37}
\]
For \((\sigma, \omega) \in \mathbb{R} \times S^2\), by (5.37) and (5.27) we get
\[
|U_j^1(\sigma, \omega)| \leq C\varepsilon \langle (cj + 1)t + \sigma \rangle^{-1} \log(2 + t) + C\varepsilon \langle \sigma \rangle^{-1}
\]
for sufficiently large \( t \). Now taking the limit as \( t \to \infty \), we obtain
\[
|U_j^1(\sigma, \omega)| \leq C\varepsilon \langle \sigma \rangle^{-1}. \tag{5.38}
\]
Finally, putting
\[
U_j(\sigma, \omega) = U_j^0(\sigma, \omega) + U_j^1(\sigma, \omega),
\]
we obtain (2.6) and (2.7) for \((t, x) \in \Lambda\) from (5.35), (5.32), and (5.37). (2.8) is an immediate consequence of (5.33) and (5.38).

As in the proof of (1), using (5.27), (5.29), (5.33), and (5.38), as well as (5.4), we can show that (2.6) and (2.7) hold also for \((t, x) \in \Lambda^c\). This completes the proof. \( \square \)

Now we are in a position to prove Corollary 2.3.
Proof of Corollary 2.3. Let the assumptions of Corollary 2.3 be fulfilled. Let $P_j$ for $1 \leq j \leq N$ be from Theorem 2.2 (1). Recalling (5.4), we see from (2.5) that $|P_j(\sigma, \omega)| \leq C \langle \sigma \rangle^{-1}$, which implies $P_j \in L^2(\mathbb{R} \times S^2)$. If we put $P_j^\ast(t, x) = x^{-1}P_j(|x| - c_j t, |x|^{-1} x)$, it follows from (2.4) that

$$
\left\| \partial u_j(t, \cdot) - \varepsilon \tilde{\omega}_c(t, \cdot) P_j^\ast(t, \cdot) \right\|^2_{L^2(\mathbb{R}^3)}
= \int_0^\infty \int_{S^2} |r \partial u_j(t, r \omega) - \varepsilon \tilde{\omega}_c(r \omega) P_j(r - c_j t, \omega)|^2 dS_\omega dr
\leq C \varepsilon^2 \int_0^\infty \langle t + r \rangle^{-2+2\delta} \langle c_j t - r \rangle^{-2\delta} dr
= C \varepsilon^2 (1 + t)^{-1} \rightarrow 0, \quad t \rightarrow \infty.
$$

Now Theorem 2.1 (with $n = 3$, $c = c_j$, $v = u_j$, and $V = \varepsilon P_j$) implies that each $u_j$ is asymptotically free in the energy sense. This completes the proof. \(\square\)

6. Proof of Theorem 2.4

Suppose that $\varepsilon$ is sufficiently small, and let $u$ be the global solution to (2.9) with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$. Since the last half of (2.10) follows from (5.6) for any $f, g \in C^\infty_0(\mathbb{R}^3)$, we will prove the existence of $(f, g)$ for which the first half holds. Without loss of generality, we may assume $A_1 > 0$ and $A_2 \geq 0$. We may also assume $c_1 = 1 < c = c_2$.

We suppose that $f \equiv 0$, and that $g = (g_1, g_2)$ is radially symmetric, i.e., $g_j(x) = g_j^\ast(|x|)$ with some function $g_j^\ast$ for $j = 1, 2$. Then $u = (u_1, u_2)$ is also radially symmetric in $x$-variable, i.e., $u_j(t, x) = u_j^\ast(t, |x|)$ with some $u_j^\ast(t, r)$ for $j = 1, 2$. For $r \in \mathbb{R}$, we put

$$
h_j(r) := \frac{r}{2} g_j^\ast(|r|), \quad v_j(t, r) := ru_j^\ast(t, |r|), \quad j = 1, 2.
$$

From (2.9) we obtain

$$
v_j(t, r) = \frac{\varepsilon}{c_j} \int_{r-c_j t}^{r+c_j t} h_j(\lambda) d\lambda + \frac{1}{c_j} \int_0^t \left( \int_{r-c_j(t-\tau)}^{r+c_j(t-\tau)} G_j(\tau, \lambda) d\lambda \right) d\tau
$$

(6.1)

for $j = 1, 2$, where

$$
G_1(t, r) = \frac{A_1 r}{2} \left( \partial_t u_2^\ast(t, |r|) \right)^2 = \frac{A_1}{2r} \left( \partial_t v_2(t, r) \right)^2,
$$

$$
G_2(t, r) = \frac{A_2 r}{2} \left( \partial_t u_1^\ast(t, |r|) \right)^2 = \frac{A_2}{2r} \left( \partial_t v_1(t, r) \right)^2.
$$

Assume that $g_j^\ast$ is a nonnegative function. Since we have

$$
\partial_t v_2(t, r) = \varepsilon \{ h_2(r + ct) + h_2(r - ct) \}
+ \int_0^t \{ G_2(\tau, r + c(t - \tau)) + G_2(\tau, r - c(t - \tau)) \} d\tau,
$$

we get

$$
\partial_t v_2(t, r) \geq \varepsilon h_2(r - ct) (\geq 0),
$$

(6.2)
provided that $r - ct \geq 0$. We fix $M > 0$, and assume that

$$C_0 := \int_0^M |h_2(\lambda)|^2 d\lambda > 0.$$ 

We suppose that $0 \leq \sigma = r - t \leq M$ in what follows. We put

$$\tau_0 = \frac{r - t}{c - 1} = \frac{\sigma}{c - 1}, \quad \tau_1 = \frac{r + t - M}{c + 1} = \frac{2t + \sigma - M}{c + 1},$$

so that we have $c\tau_0 = r - t + \tau_0$ and $c\tau_1 + M = r + t - \tau_1$. Let $2t \geq (c - 1)^{-1}(c + 1)M$ hold, so that we have $\tau_1 \geq \tau_0$. Then it is easy to see that $E \subset D$, where

$$D = \{(\tau, \lambda); 0 \leq \tau \leq t, r - t + \tau \leq \lambda \leq r + t - \tau\},$$

$$E = \{(\tau, \lambda); \tau_0 \leq \tau \leq \tau_1, c\tau \leq \lambda \leq c\tau + M\}.$$ 

Therefore, noting that we have $G_1(t, r) \geq 0$ for $r > 0$, we get

$$\int \int_D G_1(\tau, \lambda) d\tau d\lambda \geq \int \int_E G_1(\tau, \lambda) d\tau d\lambda.$$ 

From the definition of $G_1$ and (6.2), we obtain

$$\int \int_E G_1(\tau, \lambda) d\tau d\lambda \geq \frac{A_1 \varepsilon^2}{2} \int_{\tau_0}^{\tau_1} \left( \int_{c\tau}^{c\tau + M} \frac{|h_2(\lambda - c\tau)|^2}{\lambda} d\lambda \right) d\tau$$

$$\geq \frac{A_1 \varepsilon^2}{2} \int_{\tau_0}^{\tau_1} \frac{1}{c\tau + M} \left( \int_0^M |h_2(\lambda)|^2 d\lambda \right) d\tau$$

$$= \frac{C_0 A_1 \varepsilon^2}{2c} \log \frac{c\tau_1 + M}{c\tau_0 + M} \geq \frac{C_0 A_1 \varepsilon^2}{2c} \log \frac{(c - 1)(2ct + M)}{(c + 1)(2c - 1)M}.$$ 

(6.3)

Now we choose a nonnegative function $g_1^*$ satisfying $\int_M^{M+1} h_1(\lambda) d\lambda \geq 1$. Then we get

$$\int_{r-t}^{r+t} h_1(\lambda) d\lambda \geq 1$$ 

(6.4)

for $(t, r)$ with $0 \leq r - t \leq M$ and $t \geq M + 1$.

From (6.1) with $j = 1$, (6.3), and (6.4), we obtain

$$v_1(t, r) \geq C(\varepsilon + \varepsilon^2 \log(2 + t))$$

for $0 \leq r - t \leq M$ and $t \gg 1$. This completes the proof. \(\square\)

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