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NON-ABELIAN CUT CONSTRUCTIONS AND HYPERKÄHLER MODIFICATIONS

Abstract. We discuss a general framework for cutting constructions and reinterpret in this setting the work on non-Abelian symplectic cuts by Weitsman. We then introduce two analogous non-Abelian modification constructions for hyperkähler manifolds: one modifies the topology significantly, the other gives metric deformations. We highlight ways in which the geometry of moment maps for non-Abelian hyperkähler actions differs from the Abelian case and from the non-Abelian symplectic case.

1. Introduction

Cuts and modifications were introduced as constructions of symplectic or hyperkähler manifolds from examples in the same dimension with circle or torus symmetry [11, 4, 7]. These constructions start with a space $M$ that has an action of an Abelian group $G$ preserving the geometric structure and admitting a moment map. One then chooses a space $X$ with the same type of geometric structure and also with a $G$-action, such that the moment reduction of $X$ is a point. The reduction of $M \times X$ by the anti-diagonal $G$-action then gives a new space $\hat{M}$ of the same dimension as $M$, inheriting a geometrical structure of the same type as that on $M$ together with a $G$-action from the diagonal action on $M \times X$. For symplectic manifolds $\hat{M}$ is referred to as the cut space of $M$. For hyperkähler manifolds, where the construction has a somewhat different character, we introduced the term modification in [7].

In Lerman’s original construction for symplectic manifolds with circle action [11], one takes $X = \mathbb{C} = \mathbb{R}^2$; in the case $G = T^n$ of [4] the space $X$ is a toric variety of real dimension $2n$. Lerman’s construction removes part of $M$ and collapses circle orbits on the resulting boundary to give a smooth symplectic manifold $\hat{M}$. In the hyperkähler setting [7], we took $X = \mathbb{H} = \mathbb{R}^4$ when $G$ is a circle, and one may take $X$ to be a hypertoric variety [3] of real dimension $4n$ when $G = T^n$. Hyperkähler modifications by circles change the topology of $M$ whilst preserving completeness properties of the Ricci-flat metric. The whole of $M$ plays a role, no part is removed. In favourable situations the modification increases the second Betti number by one and in small dimensions the construction may be interpreted as adding a D6-brane.

It is natural to ask whether these constructions have analogues for non-Abelian groups $G$. In symplectic geometry, the case of $G = U(n)$ and $X = \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n)$ was analysed by Weitsman [15] and applied to geometric quantization. We shall provide a simple approach to his construction in terms of the polar decomposition of matrices in section 3.

The main concern of this paper is to discuss related constructions for hyperkähler manifolds with actions of non-Abelian groups, with a particular focus on the...
case $G = U(n)$. A general theme of our approach is that the geometry of the cutting or modification is controlled by the moment map geometry of $X$. We are therefore led to consider hyperkähler moment maps for non-Abelian actions. As we shall see, these exhibit rather different behaviour from moment maps in the Abelian hyperkähler and the (Abelian or non-Abelian) symplectic settings.

We describe two different constructions. In the first, section 4, we take $X = \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n) = \mathbb{R}^{2n^2}$ and only consider actions of $U(n)$. However, there are applications to $SU(n)$ manifolds. The construction has features of both the cut and of the modification: parts of $M$ are removed, the modified space contains a copy of the hyperkähler reduction of the original space, and equivariant bundle structures are changed on open sets. A new feature also occurs: some parts of $M$ are blown-up. One example of this construction is provided by the deformed instanton spaces of Nakajima [13].

The second construction, section 5, is applicable for any compact Lie group $G$ and uses the hyperkähler structure found on $X = T^*G\mathbb{C}$ [10, 5]. The latter construction preserves the topology but gives deformations of the hyperkähler geometry. The paper starts by reviewing the known cut and modification constructions.

2. The double fibration picture

**Symplectic Abelian case.** Suppose $(M, \omega)$ is a symplectic manifold, so $\omega$ is a closed non-degenerate two-form. If a group $G$ acts on $M$ preserving $\omega$, then a moment map is by definition an equivariant map $\mu: M \rightarrow \mathfrak{g}^*$ such that $d\langle \mu, A \rangle = \xi_A \lrcorner \omega$ for each $A \in \mathfrak{g}$, see e.g. [9]. If $\mu$ exists then the $G$-action is said to be Hamiltonian.

Suppose $X$ is a Hamiltonian $G$-manifold with moment map $\phi$, that $G$ acts freely on an open set of $X$ and that the reduction $\phi^{-1}(\varepsilon)/G$ is (where non-empty) a point for each regular value $\varepsilon \in \mathfrak{z}^*$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$. If $G$ is Abelian, we define the cut $\hat{M}$ to be

$$\hat{M} = (\mu - \phi)^{-1}(\varepsilon)/G$$

where $G$ acts on $M \times X$ via the anti-diagonal action $(m, x) \mapsto (g \cdot m, g^{-1} \cdot x)$ and $(\mu - \phi)(m, x) = \mu(m) - \phi(x)$. Thus $\hat{M}$ is the symplectic reduction of $M \times X$ by the anti-diagonal action of $G$. If smooth, the manifold $\hat{M}$ is now a symplectic $G$-manifold. In general, $\hat{M}$ will be a stratified symplectic space, cf. [14]. We will work in the smooth category.

The precise way in which $M$ and $\hat{M}$ are related can be understood in terms of the moment map geometry of the $G$-action on the particular space $X$.

For $G = T^n$, one takes $X$ to be a toric variety of dimension $2n$ [4]. The moment map $\phi$ on $X$ has image a convex polyhedron $\Delta$ in $\mathbb{R}^n$. Moreover $T^n$ acts transitively on the fibres of $\phi$. In fact this map gives a trivial $T^n$-fibration over the interior of $\Delta$. On the boundary of $\Delta$, the fibres are tori of lower dimension as the $T^n$-action is no longer free: in particular $\phi$ is injective over vertices of $\Delta$. 

The cut manifold \( \hat{M} \) fits into what we call a “double fibration picture”

\[
M \xleftarrow{\pi_1} N \xrightarrow{\pi_2} \hat{M}
\]
(though the left hand map \( \pi_1 \) can have some special fibres), with \( N = \{ (m,x) : \mu(m) - \phi(x) = \varepsilon \} \) and \( \pi_2 \) the quotient map for the anti-diagonal \( T^n \) action. We assume we have chosen \( \varepsilon \) so that the \( T^n \)-action on \( N \) is free, so \( \pi_2 \) is a \( T^n \)-fibration.

The map \( \pi_1 \) is defined by \( (m,x) \mapsto m \), so the fibre of \( \pi_1 \) over \( m \) may be identified with the fibre of \( \phi \) over \( \mu(m) - \varepsilon \). Hence the image of \( \pi_1 \) is just \( \mu^{-1}(\Delta + \varepsilon) \). Moreover on the interior of this set \( \pi_1 \) is a trivial \( T^n \)-fibration, while on the boundary the fibres are lower-dimensional tori. So \( \hat{M} \) may be obtained by removing the complement of \( \mu^{-1}(\Delta + \varepsilon) \) and performing collapsing by the appropriate tori on the boundary.

In the particular case of Lerman’s original construction we have \( G = S^1 \) and \( X = \mathbb{C} \), so \( \Delta \) is the non-negative half-line in \( \mathbb{R} \). Now \( \phi \) is injective over the origin and is a trivial circle fibration on the positive half-line. Then \( \hat{M} \) is obtained by removing \( \mu^{-1}(-\infty, \varepsilon) \) and collapsing circle fibres on \( \mu^{-1}(\varepsilon) \).

**Hyperkähler Abelian case.** The hyperkähler case studied in [7] presents some new features. The data of a hyperkähler manifold \( M \) consists of a metric \( g \) and three compatible complex structures \( I, J, K \) with \( IJ = K = -JI \) such that the two-forms \( \omega_I(\cdot, \cdot) = g(I\cdot, \cdot) \), \( \omega_J \), and \( \omega_K \) are closed. This implies that \( g \) is Ricci-flat and that \( M \) is symplectic in three ways. If the action of \( G \) is tri-Hamiltonian, that is, if \( G \) acts in a Hamiltonian way with respect to each of the symplectic forms \( \omega_I, \omega_J, \omega_K \), we consider the hyperkähler moment map \( \mu : M \to g^* \otimes \mathbb{R}^3 \) given by \( \mu = (\mu_I, \mu_J, \mu_K) \). When \( G \) acts freely \( \mu^{-1}(\varepsilon)/G \) is again a hyperkähler manifold of dimension \( 4 \dim G \) less than \( M \) for each \( \varepsilon \in g^* \otimes \mathbb{R}^3 \).

For \( G \) Abelian, the hyperkähler modification is defined in an analogous way to the symplectic cut: one takes \( X \) to be an appropriate hyperkähler manifold with tri-Hamiltonian \( G \)-action, moment map \( \phi : X \to g^* \otimes \mathbb{R}^3 \), and puts

\[
M_{\text{mod}} = (\mu - \phi)^{-1}(\varepsilon)/G
\]
for the anti-diagonal \( G \)-action on \( M \times X \). Now consider the double fibration picture

\[
M \xleftarrow{\pi_1} N \xrightarrow{\pi_2} M_{\text{mod}}.
\]

For circle actions, we take \( X = \mathbb{H} = \mathbb{R}^4 \), and now \( \phi : X \to \mathbb{R}^3 \) is \( \phi(q) = 7iq \), which is surjective, injective over the origin and a non-trivial circle fibration away from the origin. (In fact, it is the Hopf map.) This gives that the map \( \pi_1 \) is surjective, is injective over \( \mu^{-1}(\varepsilon) \), and is a non-trivial circle fibration away from \( \mu^{-1}(\varepsilon) \).

Hence we still collapse circle fibres on the set \( \mu^{-1}(\varepsilon) \) but do not now remove any part of \( M \). The complements \( M \setminus \mu^{-1}(\varepsilon) \) and \( M_{\text{mod}} \setminus (\mu^{-1}(\varepsilon)/S^1) \) are not diffeomorphic; rather they have a common space \( N \setminus (\mu^{-1}(\varepsilon) \times \{0\}) \) sitting above them as the total space of non-trivial circle fibrations.
Non-Abelian constructions. Let us now consider the case of non-Abelian $G$. As the diagonal and anti-diagonal actions no longer commute, the above method will no longer produce a space with a $G$-action. One circumvents this problem by instead requiring $X$ to have a $G \times G$-action. We denote the left and right copies of $G$ by $G_L$ and $G_R$ respectively.

One can then define $\hat{M}$ (or $M_{\text{mod}}$) to be the moment reduction of $M \times X$ by $G$ at level $\epsilon \in z^*$ (respectively $z^* \otimes \mathbb{R}^3$), where $G$ acts in the given way on $M$ and by $G_R$ on $X$. The action of $G_L$ now induces a $G$-action on $\hat{M}$. Again, we have a double fibration picture

\[
\begin{array}{ccc}
M & \xrightarrow{\pi_1} & N \\
\downarrow & & \downarrow \\
\hat{M}, M_{\text{mod}} & \xleftarrow{\pi_2} & 
\end{array}
\]

where

\[N = \{(m,x) : \mu(m) + \phi(x) = \epsilon\}\]

and $\phi$ is the moment map for the $G_R$-action on $X$. As in the Abelian case, to analyse this picture we must understand the fibres of $\phi$.

3. Symplectic cuts by unitary groups

Let us now study from the above point of view Weitsman’s construction of cuts for symplectic manifolds with $U(n)$ action [15]. We shall find that it may be understood in terms of the geometry of the polar decomposition.

Weitsman takes $X$ to be $\text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n)$ with a $U(n)_L \times U(n)_R$-action

\[(U,V) : A \mapsto UAV^{-1}.
\]

The moment map $\phi : X \to u(n)$ for the $U(n)_R$-action is

\[A \mapsto iA^*A
\]

which of course is equivariant for the $U(n)_R$-action and is invariant for the $U(n)_L$-action. The image of $\phi$ is now just $i\Delta(n)$, where $\Delta(n)$ is the set of non-negative Hermitian $n \times n$ matrices.

We can study $\phi$ in terms of the polar decomposition of $A$. Write $A = UP$, where $P$ is non-negative and $U$ is unitary. Now $P$ is uniquely determined by $A$, while $U$ is uniquely determined by $A$ if $A$ is invertible. Explicitly, $P$ is the unique non-negative square root $H^{1/2}$ of $H = A^*A$. So knowing $iH = \phi(A)$ determines $P$ but does not impose any conditions on $U$. Moreover, the map $H \mapsto H^{1/2}$ is a section for $\phi$.

Thus $\phi$ maps $X$ onto $i\Delta(n)$ and is a trivial $U(n)$-fibration over its interior $i\Delta(n)^0$, which is $i$ times the set of positive Hermitian matrices. Over the boundary of $i\Delta(n)$, the fibre is $U(n)/U(n-k)$, where $k$ is the number of positive eigenvalues of $H$. In particular, $\phi$ is injective over the zero matrix.

Let us see what this means in the double fibration picture (1). Now the image of $\pi_1$ is $\mu^{-1}(-i\Delta(n) + \epsilon)$, and over $\mu^{-1}(-i\Delta(n)^0 + \epsilon)$ the map $\pi_1$ is a trivial $U(n)$-fibration.
So we can view $\hat{M}$ as being obtained by removing the complement of
\[ \mu^{-1}(-i\Delta(n) + \epsilon) \]
and performing collapsing on the boundary. More precisely, if $H$ has $k$ positive eigenvalues we replace $\mu^{-1}(\epsilon - iH)$ by $\mu^{-1}(\epsilon - iH)/U(n-k)$. In particular, $\mu^{-1}(\epsilon)$ is replaced by $\mu^{-1}(\epsilon)/U(n)$. We thus recover the results of Weitsman.

4. A recipe for hyperkähler unitary modifications

Let us consider modifications of hyperkähler manifolds with $U(n)$ action, for $n > 1$. The most obvious choice of $X$ in the light of Weitsman’s symplectic construction is the flat hyperkähler vector space
\[ X = \mathbb{H}^2 = \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n). \]
This has a hyperkähler action of $U(n) \times U(n)$ (which we shall denote, as above, by $U(n)_L \times U(n)_R$) given as follows:
\[ (U,V) : (A,B) \mapsto (UAV^{-1},VBU^{-1}). \]
Note that the locus where $U(n)_R$ (or $U(n)_L$) acts freely includes the open set where $A$ or $B$ is invertible.

**Definition 1.** Let $M$ be a hyperkähler manifold with tri-Hamiltonian $U(n)$-action and moment map $\mu$. The hyperkähler modification $M_{\text{mod}}$ of $M$ at level $\epsilon \in \mathfrak{z} \otimes \mathbb{R}^3$ with respect to $X = \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n)$ is the hyperkähler quotient
\[ (\mu + \phi)^{-1}(\epsilon)/U(n) \]
of $M \times X$ by $U(n)$, where $U(n)$ acts on $M$ by the given action and on $X$ by $U(n)_R$ and $\phi$ is the hyperkähler moment map for the $U(n)_R$-action on $X$:
\[ \phi(A,B) = \left( \frac{1}{2}(AA^* - BB^*), BA \right) \in u(n) \oplus \mathfrak{gl}(n, \mathbb{C}). \]

In the formula for $\phi : X \to u(n) \otimes \mathbb{R}^3$, we have chosen a splitting $\mathbb{R}^3 = \mathbb{R} + \mathbb{C}$ and used $u(n) \otimes \mathbb{C} = \mathfrak{gl}(n, \mathbb{C})$, so $\phi = (\phi^R, \phi^C)$. Note that $\mathfrak{z} = i\mathbb{R}\text{Id}$, so $\epsilon$ is just a point of $\mathbb{R}^3$.

The modification $M_{\text{mod}}$ is hyperkähler with a $U(n)$-action induced by the action of $U(n)_L$ on $X$. We will work in the smooth category, but in general $M_{\text{mod}}$ will decompose into a union of hyperkähler manifolds, cf. [6].

Starting with a $U(n)$-orbit $U(n) \cdot m$ in $M$, we wish to understand the corresponding subset in the modification $M_{\text{mod}}$. Let $H$ be the $U(n)$ stabiliser of $m$. We use the notation of the double fibration picture (1). The set $(U(n) \cdot m)_{\text{mod}} = \pi_2 \pi_1^{-1}(U(n) \cdot m)$
can be identified with the quotient of \( \phi^{-1}(\varepsilon - \mu(m)) \) by \( H_{R} \), where \( H_{R} \) is \( H \) acting as a subgroup of \( U(n)_{R} \). The new \( U(n) \)-orbits are the \( U(n)_{L} \)-orbits in this quotient.

Therefore we need to analyse the fibres \( \phi^{-1}(R, S) \) of the map \( \phi \). Note that \( \phi \) is equivariant under the \( U(n)_{R} \) action and invariant under the \( U(n)_{L} \) action, as expected.

First, we study the linearisation. This is given by

\[
d\Phi_{(A,B)}: (a,b) \mapsto \left( \frac{1}{2}(a^*A + A^*a - Bb^* - bB^*), Ba + bA \right).
\]

Let us now take \( S \) (and hence \( A, B \)) to be invertible. The \( U(n)_{R} \) and \( U(n)_{L} \) actions are therefore free at \( (A,B) \).

A vector \((a,b)\) in the kernel of the complex part \( d\Phi^C \) of \( d\Phi \) may now be written as \((hA, - Bh)\) for a unique \( h \in \mathfrak{gl}(n, \mathbb{C}) \). The real part of \( d\Phi \) acts on this vector by

\[
d\Phi^R_{(A,B)}: (hA, - Bh) \mapsto A^*(h + h^*)A + B(h + h^*)B^*.
\]

We obtain

**Lemma 1.** Suppose \( A \) and \( B \) are invertible. The kernel of \( d\Phi_{(A,B)} \) is the set of vectors \((hA, - Bh)\) satisfying

\[
L(h + h^*)L^* + (h + h^*) = 0
\]

where \( L = B^{-1}A^* \). In particular, the Hermitian part of \( h \) has signature 0. \( \square \)

Note that if \( h \) is skew-Hermitian then \((hA, - Bh)\) is always in \( \ker d\Phi \). This of course is just the infinitesimal version of the statement that \( \phi \) is invariant under the action of \( U(n)_{L} \). Such \( h \) give an \( n^2 \)-dimensional subspace in \( \ker d\Phi \).

Now \((A,B)\) is a critical point of \( \phi \) if and only if \( \dim \ker d\Phi_{(A,B)} > n^2 \). From Lemma 1 above, this holds (with \( A,B \) invertible) if and only if the operator \( L = L \otimes T \) acting on \( \mathbb{C}^n \otimes (\mathbb{C}^n)^* = \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) \) has \(-1\) as an eigenvalue with a Hermitian eigenvector.

Now the eigenvalues of \( L \) are just the products of eigenvalues of \( L \) with those of \( T \) (see [1]), so \(-1\) is an eigenvalue of \( L \) if and only if there is a complex number \( \lambda \) such that \( \lambda \) and \(-1/\lambda \) are both eigenvalues of \( L \). The eigenvalues of \( T \) then include \( \lambda \) and \(-1/\lambda \), so \(-1\) actually has multiplicity at least two as an eigenvalue of \( L \) on \( \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) \). If \( v, w \) are eigenvectors of \( L \) with eigenvalues \( \lambda, -1/\lambda \) respectively, we see that \( v \otimes w^* + v^* \otimes w \) is then a Hermitian eigenvector for \( L \) with eigenvalue \(-1\).

**Theorem 1.** A point \((A,B)\) with \( A,B \) invertible is a critical point for \( \phi \) if and only if \( L = B^{-1}A^* \) has a pair of eigenvalues of the form \( \lambda, -1/\lambda \).

The set of critical points for \( \phi \) therefore has real codimension two in \( X = \mathbb{H}^{n^2} \).
The regular points form an open dense set in \( X \). \( \square \)

Note that if \((R,S)\) is a regular value then \( \phi^{-1}(R,S) \) is an \( n^2 \)-dimensional manifold with an action of \( U(n)_{L} \).
THEOREM 2. Let \((R,S)\) be a regular value of \(\phi\), where \(S\) is invertible. Then \(U(n)_L\) acts freely and locally transitively on the fibre \(\phi^{-1}(R,S)\).

In fact the quotient of the fibre by \(U(n)_L\) is finite, since \(\phi\) is a polynomial map in the real and imaginary parts of \(A_{ij}, B_{ij}\).

THEOREM 3. There exist critical values \((0,S)\) with \(S\) invertible for which the fibre \(\phi^{-1}(0,S)\) has dimension strictly greater than \(n^2\).

Proof. Start by considering \(n=2\) and take \(S=L=\text{diag}(1,-1)\); a possible choice of point in \(X\) giving such an \(L\); \((A,B) = (L,I)\). Now \(L = L \otimes \mathbb{C}\) has a non-trivial space of Hermitian eigenvectors for eigenvalue \(-1\); in fact such eigenvectors are the real anti-diagonal matrices.

Let us consider the fibre of \(\phi\) over \(\phi(L,I) = (0,L)\). It is convenient to write \((A,B) \in \phi^{-1}(0,L)\) as \(A = UT\) and \(B = TV\), where \(U, V\) are unitary and \(T, \bar{T}\) are lower and upper triangular respectively with real positive entries on the diagonal.

The real equation \(\phi^R(A,B) = 0\) is \(A^*A = BB\) which becomes \(T^*T = \bar{T}\bar{T}\), and it is easy to check this implies \(\bar{T} = T^*\).

The complex equation \(\phi^C(A,B) = L\), is now the condition \(T^*VUT = L\), i.e., the requirement that \((T^*)^{-1}LT^{-1}\) be unitary. Each such \(T\) gives a unique \((A,B)\) up to the action of \(U(2)_L\). Writing

\[
T = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},
\]

where \(a, d \in (0, \infty)\) and \(b \in \mathbb{C}\), this is equivalent to the equations

\[
\begin{align*}
d^2(d^2 + |b|^2) &= a^2d^4, \\
b(2d^2 - |b|^2 - d^2) &= 0, \\
d^2|b|^2 + (a^2 - |b|^2)^2 &= a^4d^4.
\end{align*}
\]

If \(b \neq 0\) the second equation gives \(d^2 = a^2 - |b|^2\) and the remaining equations are now both equivalent to \(a^2d^2 = 1\). So we have

\[
a^2 = \frac{1}{d^2}, \quad |b|^2 = \frac{1}{d^2} - d^2, \quad d \in (0, 1).
\]

We have a set of solutions parametrised by the punctured disc. If \(b = 0\) then \(a = d = 1\), corresponding to the centre of the disc.

Thus the quotient by the free \(U(2)_L\)-action on the fibre \(\phi^{-1}(0,L)\) is diffeomorphic to the open disc in \(\mathbb{C}\).

For general \(n\), consider \(S = L \oplus L'\) with \(L'\) an invertible \((n-2) \times (n-2)\) matrix. The analysis above shows that the fibre is determined by the upper triangular matrices \(Y = T^{-1}\) with positive entries on the diagonal such that \(Y'SY\) is unitary and each such matrix determines a unique \(U(n)_L\)-orbit in the fibre. The result follows by taking direct sums of solutions for the \(n=2\) case with solutions for \(L'\). \(\square\)
We may also get fibres with positive-dimensional quotient by $U(n)_L$ if $S$ is non-invertible.

**THEOREM 4.** The quotient $\phi^{-1}(0,0)/U(n)_L$ may be identified with the set of non-negative Hermitian matrices of rank at most $n/2$.

**Proof.** Write $(A,B) \in \phi^{-1}(0,0)$ as $A = UP, B = QV$ where $U,V$ are unitary and $P,Q$ are non-negative Hermitian matrices (that is, we are taking left and right polar decompositions of $A$ and $B$). The real equation $A^*A = BB^*$ now says $P = Q$, and the complex equation becomes

$$PVUP = 0.$$ 

This holds precisely when $\text{Im}P$ may be mapped by a unitary $W = VU$ transformation into $\ker P$, which is true if and only if $\text{rank}(P) \leq n/2$.

Given $P$ non-negative Hermitian of rank at most $n/2$, it remains to show that there is a unique $U(n)_L$-orbit in $\phi^{-1}(0,0)$ determined by $P$. For $(A_1,B_1), (A_2,B_2) \in \phi^{-1}(0,0)$ with $A_1 = U_1P, B_1 = PV_1$ and $A_2 = U_2P, B_2 = PV_2$ for unitary matrices $U_1, V_1, U_2, V_2$ we have that $W_1 = V_1U_1$ and $W_2 = V_2U_2$ both represent transformations that map $\text{Im}P$ into $\ker P = (\text{Im}P)^\perp$.

Let $X = \text{Im}P, Y = W_1^{-1}\text{Im}P$ and $Z = W_2^{-1}\text{Im}P$. Then $Y, Z \leq X^{\perp}$ and $W_2^{-1}W_1$ maps $Y$ isometrically on to $Z$, so there is a unitary transformation $w \in U(n)$ with $w|_X = \text{Id}_X$ and $w|_Y = W_2^{-1}W_1|Y$. Put $u = U_1w^{-1}U_2^{-1}$ which is unitary. Then $B_1 = PV_1 = PW_1U_1^{-1} = PW_2wU_1^{-1} = B_2w^{-1}$ and $uA_2 = U_1w^{-1}P = U_1P = A_1$, giving $(A_1,B_1) = (u,1) \cdot (A_2,B_2)$, as required. □

The existence of fibres of $\phi$ of dimension greater than $n^2$, and more generally fibres whose quotient by $U(n)_L$ is of positive dimension, is a phenomenon that does not occur in the cutting/modification constructions for symplectic or hyperkähler manifolds with torus actions, or even in the case of symplectic manifolds with $U(n)$ actions. Indeed, as noted earlier, in these cases the action of the group on the fibres is transitive.

This behaviour means that when we perform the $U(n)$ hyperkähler modification, certain loci in $M$ are being blown up. In particular the set $\mu^{-1}(\varepsilon)$, where $\varepsilon$ is the level at which we perform the modification, is not simply collapsed by $U(n)$ in the modification, as $\phi$ is not injective over the origin. As noted later in the section, the corresponding region in $M_{\text{mod}}$ includes $\mu^{-1}(\varepsilon)/U(n)$ but also includes other strata corresponding to nonzero elements of $\phi^{-1}(0,0)$. This contrasts with the Abelian case or the symplectic $U(n)$ case, where $\phi$ is injective over the origin.

**REMARK 1.** Using the techniques of the proof of Theorem 4, one may check that $\phi^{-1}(0,\lambda\text{Id})$ for each $\lambda \neq 0$ consists exactly of one $U(n)_L$-orbit.

**REMARK 2.** We saw above that the moment map $\phi$ for the $U(n)_R$ action on $X$ can have critical points, and fibres of larger than expected dimension, even on the locus where the action is free. Again, this is a new feature that appears for non-Abelian hyperkähler actions.
In the symplectic case, the kernel of the differential of a moment map is just the symplectic orthogonal of the space $\mathcal{G}$ of Killing fields for the action. In the Kähler setting, this is of course the metric orthogonal $(I\mathcal{G})^\perp$ where $I$ denotes the complex structure. Hence on the locus where the action is free the rank is maximal because $\mathcal{G}$ has the maximal dimension $\dim G$.

For hyperkähler moment maps, the kernel is now the intersection of the symplectic orthogonals to $\mathcal{G}$ relative to the Kähler forms $\omega_I, \omega_J, \omega_K$; equivalently it is the metric orthogonal $(I\mathcal{G} + J\mathcal{G} + K\mathcal{G})^\perp$. If the action is Abelian then the moment map is invariant under the group action and hence $\mathcal{G}$ is contained in the above orthogonal complement. It follows that $I\mathcal{G}, J\mathcal{G}$ and $K\mathcal{G}$ are mutually orthogonal and the above sum is direct. Again, we see that on the free locus the moment map has maximal rank. In the non-Abelian case this can fail, leading to critical points on the free locus. Note that the value of the moment map at such critical points is non-central.

In contrast to the Abelian case, the moment map $\phi$ is not surjective. For later use, we will prove a little more.

**Theorem 5.** For each $\varepsilon \in \mathbb{Z} \otimes \mathbb{R}^3$ there exist values $\varepsilon + (R,S)$ with $(R,S) \in su(n) + sl(n, \mathbb{C})$ that are not in the image of $\phi$.

**Proof.** Rotating the choice of complex structures on $X$, we may assume that $\varepsilon = (0, \lambda I)$ for some complex number $\lambda$.

Consider $(0,M) \in u(n) \oplus gl(n, \mathbb{C})$, where $M = (m_{ij})$ is any matrix that has:

(i) first column zero: $m_{i1} = 0$ for all $i = 1, \ldots, n$,
(ii) non-zero entries immediately above the diagonal: $m_{i,i+1} \neq 0, i = 1, \ldots, n-1$,
(iii) all other entries above the diagonal zero: $m_{ij} = 0$ for $j > i + 1$, and
(iv) the trace $m_{22} + \cdots + m_{nn}$ equals $n\lambda$.

Note that $M$ has rank $n - 1$.

If $\phi(A,B) = (0,M)$, the equations are

$$A^*A - BB^* = 0, \quad BA = M.$$  

The first of these implies, using $\ker A = \ker A^* A$ etc., that $A$ and $B$ have equal rank. By the second equation, this rank must be $n - 1$. Write

$$B = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}, \quad A = (w_1 \ldots w_n)$$

for column vectors $v_i$ and $w_j$. Now we have

$$BA = (v_i^T w_j) = M.$$
Condition (ii) gives that $v_1, \ldots, v_{n-1}$ and $w_2, \ldots, w_n$ are non-zero. We claim that each of these collections of $n-1$ vectors is linearly independent. To see this, suppose $0 = a_2w_2 + \cdots + a_nw_n$. Then multiplying by $v_1^T$ and using (iii), we get $0 = a_2v_1^Tw_2$ and so $a_2 = 0$. Multiplying successively with $v_i^T$, $i = 2, \ldots, n-1$ then shows $a_j = 0$ for each $j$, as required. A similar argument gives the linear independence of $v_1, \ldots, v_{n-1}$.

Since $\text{rank} A = n-1$, we have that $w_1$ is a linear combination of the vectors $w_2, \ldots, w_n$. Multiplying this combination successively by $v_i^T$, $i = 1, \ldots, n-1$, and using (i), we find this combination is zero. Thus $w_1 = 0$ and $A^T A$ has leading entry zero. The corresponding entry in $BB^*$ is $\|v_1\|^2$, which is non-zero. Thus there is no $(A, B)$ mapping to $(0, M)$ under $\phi$. Taking $R = 0$ and $S = M - \lambda \text{Id}$, we have by (iv) $S \in \mathfrak{s}(n, \mathbb{C})$ and the result follows.

Note that diagonal matrices of the form $D = \text{diag}(0, d_2, \ldots, d_n)$ may be obtained as the limit of the matrices $M$ in the above proof. However the point $(0, D)$ is in the image of $\phi$, putting $d_1 = 0$ and writing $d_j = e^{i\theta_j} r_j^2$ with $r_j \geq 0$, we have $(0, D) = \phi(U P, P)$ where $U = \text{diag}(e^{i\theta_j})$ and $P = \text{diag}(r_j)$. Thus we have:

**Corollary 1.** The image of $\phi$ is not open.

The above results are easiest to summarise when the action of $U(n)$ on $M$ is free. An example of such a hyperkähler manifold is the space $T^*GL(n, \mathbb{C})$ discussed in the final section of the paper.

**Theorem 6.** Suppose $M$ is a hyperkähler manifold with free tri-Hamiltonian $U(n)$-action, $n > 1$, with moment map $\mu$. Let $M_{\text{mod}}$ be the hyperkähler modification at level $\varepsilon$ defined in Definition 1. Then $M_{\text{mod}}$ is a smooth hyperkähler manifold with tri-Hamiltonian $U(n)$-action. Moreover, in $M_{\text{mod}}$

(i) the set $\mu^{-1}([u(n) \otimes \mathbb{R}^3] \setminus [\varepsilon - \text{Im} \phi])$ is removed from $M$,

(ii) there is an open set $\mathcal{U} \subset u(n) \otimes \mathbb{R}^3$ and a manifold of dimension $\dim M + n^2$ that fibres over both $\mu^{-1}(\mathcal{U})$ and $(\mu^{-1}(\mathcal{U}))_{\text{mod}}$ up to finite covers both of these fibrations are principal $U(n)$-bundles,

(iii) $M_{\text{mod}}$ contains a copy of the hyperkähler quotient of $M$ at level $\varepsilon$,

(iv) there is a set $Z \subset u(n) \otimes \mathbb{R}^3$ such that $(\mu^{-1}(Z))_{\text{mod}}$ has larger dimension than $\mu^{-1}(Z)$ and is “blown-up”.

**Proof.** The diagonal action of $U(n)$ on $M \times X$ is free, so reduction at a central value $\varepsilon$ gives a smooth hyperkähler quotient. The moment map $\phi_\varepsilon$ for $U(n)_\varepsilon$ on $X$ descends to the quotient giving a moment map for the induced $U(n)$-action; explicitly

$$\phi_\varepsilon(A, B) = \left( \frac{1}{2} B^* B - AA^* , AB \right).$$

Part (i) is immediate: by Theorem 5 the set being removed can be nonempty.
For (ii), the open set $\mathcal{U}$ is given by Theorem 2 and the comment immediately following it.

The hyperkähler quotient of $M$ at level $\epsilon$ is $\mu^{-1}(\epsilon)/U(n)$ may be identified with the quotient of points $(m, p) \in M \times X$, with $\mu(m) = \epsilon$ and $p = 0$ by the diagonal $G$-action, so is a subset of $M_{\text{mod}}$. This gives part (iii).

The set $\mathcal{Z}$ in (iv) consists of those points where the fibre of $\phi$ has dimension greater than $n^2$.

We note that the intertwining of the actions on $M$ and $X$ implies that the $U(n)$-action on $M_{\text{mod}}$ can be effective. This is despite the fact that the diagonal $U(1)$-subgroup of $U(n)_L \times U(n)_R$ acts trivially on $X$.

**Example 1.** Let us consider an example of our construction when $M$ is flat. We take $M = \text{Hom}_C(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}_C(\mathbb{C}^n, \mathbb{C}^n)$, but this time with $U(n)$ acting by conjugation. The moment map $\mu = (\mu^R, \mu^I)$ is given by

$$\mu: (B_1, B_2) \mapsto ([B_1, B_1^*] + [B_2, B_2^*], [B_1, B_2]).$$

The modification $M_{\text{mod}}$ at level $\epsilon$ will now be the quotient of the space

$$\{ (B_1, B_2, A, B) : [B_1, B_1^*] + [B_2, B_2^*] + A^*A - BB^* = \epsilon^R, [B_1, B_2] + BA = \epsilon^C \}$$

by the action

$$(B_1, B_2, A, B) \sim (gB_1g^{-1}, gB_2g^{-1}, Ag^{-1}, gB), \quad g \in U(n).$$

This is just the deformed instanton space $M_{\epsilon}(n, n)$ of Nakajima (see for example [13]).

**Remark 3.** Consider taking the hyperkähler quotient of $X$ by the central $U(1)$, which is the same for both $U(n)_L$ and $U(n)_R$. If we choose a non-zero central level then we obtain the hyperkähler structure found by Calabi on $Y = T^*CP(m)$ in the special case $m = n^2 - 1$. This space inherits tri-holomorphic actions of $SU(n)_L$ and $SU(n)_R$, so may be used to construct modifications of hyperkähler manifolds with $SU(n)$ symmetry. We see all the features described above. In particular, Theorem 5 shows that parts of $M$ are cut away in this process. The $SU(n)$-modification is less flexible than the $U(n)$ case, since one can only reduce at the level $\epsilon = 0$.

**Remark 4.** As in the symplectic case [15], we may generalise the modification construction by taking $X$ to be hyperkähler with a $H \times G$ action. Now, performing our construction will give a hyperkähler space with an action of $H$ rather than $G$. For example, we may take $H = U(r)$ and $X = \text{Hom}_C(\mathbb{C}^n, \mathbb{C}^r) \oplus \text{Hom}_C(\mathbb{C}^r, \mathbb{C}^n)$. The moment map $\phi$ is as in equation (2), but $A, B$ are now $r \times n$ and $n \times r$ matrices respectively.

Applying this to $M$ as in Example 1 gives the general Nakajima space $M_{\epsilon}(r, n)$.

**5. A gauge theory modification**

In this section we describe another possible choice of $X$. This is less involved topologically, but it does give a metric deformation and it applies to all compact symmetry
groups $G$.

Recall from [10] that if $G$ is compact then the cotangent bundle $T^*G_C$ of the complexification of $G$ carries a complete hyperkähler structure preserved by an action of $G \times G$. Kronheimer proved this by identifying the cotangent bundle with the moduli space of solutions to Nahm’s equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k],$$

$(ijk)$ a cyclic permutation of $(123)$, where the $T_a$ are smooth maps from $[0, 1]$ to $g$. The moduli space is formed by identifying any two such solutions which are equivalent under the action of the restricted gauge group $G_L$. Kronheimer proved this by identifying the cotangent bundle with the moduli space of solutions to Nahm’s equations smooth on $X_1$. Note that $G_0$ is a proper subset of $G_L$. Moreover $G_0$ is star-shaped with respect to the origin. This is because the moment map $\psi$ is a $G_L$-equivariant and $G_L$-invariant. The image of $\psi$ is an open set $U_0$ in $g^* \otimes \mathbb{R}^3$, consisting of the triples $(X_1, X_2, X_3)$ such that there exists a solution to the Nahm equations smooth on $[0, 1]$ with $T_i(1) = X_i$, $(i = 1, 2, 3)$. The results of [5] show that $U_0$ is star-shaped with respect to the origin. This is because the Nahm equations are preserved under the affine reparametrisation $T_i(t) \mapsto aT_i(at + a - a)$, $0 < a < 1$, which scales $T_i(1)$ by $a$ while preserving the condition of smoothness on $[0, 1]$. Moreover $U_0$ is a proper subset of $g^* \otimes \mathbb{R}^3$ if $G$ is non-Abelian. This is because we may find solutions to the Nahm equations that are smooth at $t = 1$ but blow up at some point in the interior of $[0, 1]$.

From above, given $\psi(T_0, T_1, T_2, T_3)$ we know the solutions to the reduced Nahm equations are gauge equivalent to $(T_0, T_1, T_2, T_3)$, hence we know $(T_i)$ up to the (free) $G_L$ action. So $\psi: X \to U_0 \subset g^* \otimes \mathbb{R}^3$ is a $G$-fibration, and this fibration is trivial (e.g. because $U_0$ is star-shaped).
We see that the modification of $M$ using $X = T^*G_C$ may be viewed topologically as removing the complement of $\mu^{-1}(-U_g + \varepsilon)$ from $M$. We do not perform any collapsing because there are no special fibres of $\psi$. Note that the metric on $M_{\text{mod}}$ will be complete as long as $M$ is complete.

Note that if $G = S^1$ then $X$ is just $T^*C^*$, which can be identified with $\mathbb{R}^3 \times S^1$ with the flat hyperkahler metric. As $G$ is now Abelian, constant gauge transformations act trivially, so the $G \times G$-action on $X$ just collapses to a $G$-action which is rotation of the $S^1$ factor. We may form the modification $M_{\text{mod}}$ in the same way as in [7]. Now $\psi$ is just projection of this trivial $S^1$-bundle onto $\mathbb{R}^3$, so $M_{\text{mod}}$ is diffeomorphic to $M$. However the metric on $M_{\text{mod}}$ may be different from that on $M$. For example, if $M$ is flat $\mathbb{R}^4$ then $M_{\text{mod}}$ is $\mathbb{R}^4$ with the Taub-NUT metric. So we may view $M_{\text{mod}}$ as a metric deformation of $M$ (this is what Bielawski [2] calls a Taub-NUT deformation of $M$). As any compact $G$ contains such circle subgroups we find

**Theorem 7.** Let $M$ be a hyperkahler manifold with tri-Hamiltonian $G$-action and moment map $\mu$, for some compact Lie group $G$. Then the modification of $M$ via $X = T^*G_C$ at a central level $\varepsilon$ is diffeomorphic to $\mu^{-1}(-U_g + \varepsilon)$ but need not be isometric to this subset of $M$. If $M$ is complete, then this modification is also complete.

**Remark 5.** Some analogous results may be obtained in hypersymplectic geometry. Hypersymplectic cuts for circle actions were introduced in [8] and some results in the hypersymplectic non-Abelian case have been found by T. Matsoukas in his 2010 Oxford D.Phil. thesis [12].

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