A NOTE ON PRIME NUMBER RACES AND ZERO FREE REGIONS FOR L FUNCTIONS

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Abstract. Let \( \chi \) be a real and non-principal Dirichlet character, \( L(s, \chi) \) its Dirichlet \( L \)-function and let \( p \) be a generic prime number. We prove the following result: If for some \( 0 \leq \sigma < 1 \) the partial sums \( \sum_{p \leq x} \chi(p)p^{-\sigma} \) change sign only for a finite number of \( x \), then there exists \( \epsilon > 0 \) such that \( L(s, \chi) \) has no zeros in the half plane \( \text{Re}(s) > 1 - \epsilon \). Moreover, if \( f : \mathbb{N} \to [-1, 1] \) is a completely multiplicative function that is small on average, i.e., \( \sum_{n \leq x} f(n) = o(x^{1-\delta}) \) for some \( \delta > 0 \), and if the Dirichlet series of \( f \), say \( F(s) \), is such that \( F(1) \neq 0 \), then there exists \( \epsilon > 0 \) such that \( F(s) \neq 0 \) for all \( \text{Re}(s) > 1 - \epsilon \) if and only if there exists \( 0 \leq \sigma < 1 \) such that the partial sums \( \sum_{p \leq x} f(p)p^{-\sigma} \) change sign only for a finite number of \( x \).

1. Introduction.

Let \( \chi \) be a non-principal Dirichlet character and \( L(s, \chi) \) its Dirichlet \( L \)-function. Many central problems in analytic number theory such as questions about the distribution of primes in arithmetic progressions can be phrased in terms of zero-free regions for \( L(s, \chi) \). The typical zero free-region for \( L(s, \chi) \) known up to date is: If \( q \) is the modulus of \( \chi \), then there exists a constant \( c > 0 \) such \( L(\sigma + it) \neq 0 \) for

\[
\sigma > 1 - \frac{c}{\log q(2 + |t|)},
\]

with at most one possible exception – A real zero \( \beta < 1 \) – in the case that \( \chi \) is real (see [6] pg. 360).

Let \( p \) be a generic prime number and \( \mathcal{P} \) be the set of primes. Let \( \chi_4 \) be the real and non-principal Dirichlet character \( \mod 4 \), i.e., \( \chi_4(n) = 1 \) if \( n \equiv 1 \mod 4 \), \( \chi_4(n) = -1 \) if \( n \equiv 3 \mod 4 \) and \( \chi_4(n) = 0 \) if \( n \) is even. Then the sum \( \sum_{p \leq x} \chi_4(p) \) is the prime number race \( \mod 4 \): the number of primes up to \( x \) of the form \( 4n + 1 \) minus the number of primes up to \( x \) of the form \( 4n + 3 \).

In 1853, in a letter to Fuss, it has been observed by Tchébyhev that seems to be more primes of the form \( 4n + 3 \) than primes of the form \( 4n + 1 \). In other words, it seems that \( \sum_{p \leq x} \chi_4(p) \leq 0 \) for most values of \( x \). This observation led to many investigations on prime number races for a generic modulus \( q \). For an historical background on prime number races we refer reader to the expository paper...
of Granville and Martin [3], and for recent results in this topic we refer to the paper of Harper and Lamzouri [4] and the references therein.

In the prime number race mod 4, the partial sums \( \sum_{p \leq x} \chi_4(p) \) change sign for an infinite number of \( x \). However, for \( 0 < \sigma < 1 \), it is possible that the weighted prime number race \( \sum_{p \leq x} \frac{\chi_4(p)}{p^\sigma} \) change sign only for a finite number of \( x \). If this is the case, then we have:

**Theorem 1.1.** Let \( \chi \) be a real and non-principal Dirichlet character. If for some \( 0 \leq \sigma < 1 \) the partial sums \( \sum_{p \leq x} \frac{\chi(p)}{p^\sigma} \) change sign only for a finite number of \( x \), then there exists \( \epsilon > 0 \) such that \( L(s, \chi) \neq 0 \) for all \( s \) in the half plane \( \text{Re}(s) > 1 - \epsilon \).

Let \( \mathcal{P} \) be the set of prime numbers.

**Corollary 1.1.** Under the hypothesis of Theorem 1.1, we have that for some \( \epsilon > 0 \), \( \sum_{p \in \mathcal{P}} \frac{\chi(p)p^{-(1-\epsilon)}}{p^\sigma} \) converges, and hence, the Euler product formula

\[
L(s, \chi) = \prod_{p \in \mathcal{P}} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}
\]

holds for all \( \text{Re}(s) > 1 - \epsilon \).

It is worth mentioning that a converse result holds for Theorem 1.1:

**Theorem 1.2.** Let \( \chi \) be a real and non-principal Dirichlet character. If for some \( \epsilon > 0 \) we have that \( L(s, \chi) \neq 0 \) for all \( \text{Re}(s) > 1 - \epsilon \), then there exists \( 1 - \epsilon < \sigma < 1 \) such that \( \sum_{p \leq x} \chi(p)p^{-\sigma} \) change sign only for a finite number of \( x \).

The proof of Theorem 1.1 is an application of an integral version of Landau’s oscillation Theorem: If \( A : [0, \infty) \to \mathbb{R} \) is a bounded Riemann-integrable function in any finite interval \([1, x]\), and such that \( A(x) \geq 0 \) for all \( x \geq x_0 > 1 \), then the function

\[
F(s) = \int_1^\infty \frac{A(x)}{x^s} \, dx
\]

has a singularity in its abcissa of convergence.

In fact, the proof of Theorem 1.1 is done by the following steps: For \( \text{Re}(s) > 1 \) we can write

\[
\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s} = (s - \sigma) \int_1^\infty \frac{\sum_{p \leq x} \chi(p)}{x^{s+1-\sigma}} \, dx.
\]

For \( \chi \) non-principal, \( L(1, \chi) \neq 0 \) and since \( L(s, \chi) \) is analytic in \( \text{Re}(s) > 0 \), there exists an open ball \( B \) of center 1 and radius \( \delta > 0 \) in which \( L(s, \chi) \neq 0 \). The union
of the half plane $Re(s) > 1$ with this open ball is a simply connected domain, and since $L(s, \chi) \neq 0$ in this domain, there exists a branch of the Logarithm for $L(s, \chi)$. The existence of this branch implies that $\int_1^\infty \sum_{n \leq x} \frac{\chi(n)}{n^s} dx$ is analytic at $s = 1$, and hence, by the Landau’s oscillation Theorem, this integral converges for $s = 1 - \epsilon$, for some $\epsilon > 0$.

If $f : \mathbb{N} \to [-1, 1]$ is a completely multiplicative function that is small on average, i.e., $\sum_{n \leq x} f(n) = o(x^{1-\delta})$ for some $\delta > 0$, then the Dirichlet series $F(s) := \sum_{n=1}^\infty f(n)n^{-s}$ is analytic in $Re(s) > 1 - \delta$. In [2], Koukoulopoulos proved that if $f$ is small on average and if $F(1) \neq 0$, then $\sum_{p \leq x} f(p) \log p \ll x \exp(-c\sqrt{\log x})$, for some constant $c > 0$. Let $\mathcal{P}$ be the set of primes. In [11] it has been proved that under biased assumptions, i.e., if at primes $(f(p))_{p \in \mathcal{P}}$ is a sequence of independent random variables such that $\mathbb{E}f(p) < 0$ for all primes $p$, then the assumptions that $f$ is small on average almost surely (a.s.) and $F(1) \neq 0$ a.s. imply that $\sum_{p \in \mathcal{P}} f(p)p^{-(1-\epsilon)}$ converges for some $\epsilon > 0$ a.s., and hence that $F(s) \neq 0$ for all $Re(s) > 1 - \epsilon$, a.s.

The same lines of the proof of Theorems 1.1 and 1.2 allow us to show that:

**Theorem 1.3.** If $f : \mathbb{N} \to [-1, 1]$ is a completely multiplicative function that is small on average, and if the Dirichlet series of $f$, say $F(s)$, is such that $F(1) \neq 0$, then there exists $\epsilon > 0$ such that $F(s) \neq 0$ for all $Re(s) > 1 - \epsilon$ if and only if there exists $0 \leq \sigma < 1$ such that the partial sums $\sum_{p \leq x} f(p)p^{-\sigma}$ change sign only for a finite number of $x$.

2. Proof of the main results

**Notation.** Here $\chi$ is a Dirichlet character and $L(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}$. We use both $f(x) \ll g(x)$ and $f(x) = O(g(x))$ whenever there exists a constant $C > 0$ such that for all large $x > 0$ we have that $|f(x)| \leq C|g(x)|$. Further, $\ll_{\delta}$ means that the implicit constant may depend on $\delta$. We let $\mathcal{P}$ for the set of primes and $p$ for a generic element of $\mathcal{P}$. For a real number $a$, we denote the half plane $\{s \in \mathbb{C} : Re(s) > a\}$ by $H_a$.

**Lemma 2.1.** Let $\chi$ be a real and non-principal Dirichlet character. Then there exists an analytic function $B : H_{1/2} \to \mathbb{C}$ such that for $a > 1/2$, $B(s) \ll_a 1$ in the half plane $H_a$, and for $s \in H_1$:

$$\log L(s, \chi) = \sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s} + B(s).$$
Proof. This follows from the Euler product formula valid for \( s \in H_1 \):

\[
L(s, \chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.
\]

Thus

\[
\log L(s, \chi) = \sum_{p \in \mathbb{P}} \log \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{p \in \mathbb{P}} \sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}}
\]

where in the last equality above we used the Taylor expansion for each term \( \log \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \). Now, we split this double infinite sum into two infinite sums:

\[
\log L(s, \chi) = \sum_{p \in \mathbb{P}} \frac{\chi(p)}{p^s} + \sum_{p \in \mathbb{P}} \sum_{m=2}^{\infty} \frac{\chi(p)^m}{mp^{ms}}.
\]

Let \( B(s) := \sum_{p \in \mathbb{P}} \sum_{m=2}^{\infty} \frac{\chi(p)^m}{mp^{ms}} \). Then the inner sum \( \sum_m \) is \( \ll \frac{1}{p^s \Re(s)} \). Thus \( B(s) \) converges absolutely for each \( s \in H_{1/2} \), and hence, it defines an analytic function in this half plane. Moreover, for each fixed \( a > 1/2 \), \( B(s) \ll \sum_{p \in \mathbb{P}} \frac{1}{p^{2a}} \). \( \Box \)

Lemma 2.2 (Lemma 15.1 of [8], Landau’s oscillation Theorem). Let \( A : [0, \infty) \to \mathbb{R} \) be a bounded Riemann-integrable function in any finite interval \([1, x]\), and assume that for some large \( x_0 > 0 \) we have that \( A(x) \geq 0 \) for all \( x \geq x_0 > 0 \). Let \( \sigma_c \) be the infimum of those \( \sigma \) for which \( \int_1^x \frac{|A(x)|}{x^\sigma} dx < \infty \). Then the function

\[
F(s) = \int_1^x \frac{A(x)}{x^s} dx
\]

is analytic in \( H_{\sigma_c} \) and has a singularity at \( \sigma_c \).

Lemma 2.3 (Corollary 6.17 of [2]). Let \( G \) be a simply connected domain and \( f : G \to \mathbb{C} \) an analytic function such that \( f(s) \neq 0 \) for all \( s \in G \). Then there exists an analytic function \( g : G \to \mathbb{C} \) such that \( f(z) = \exp(g(z)) \). If \( w : G \to \mathbb{C} \) is another analytic function such that \( f(s) = \exp(w(s)) \) for all \( s \in G \), then there exists \( c \in \mathbb{C} \) such that \( g(s) - w(s) = c \), for all \( s \in G \).

Proof of Theorem 1.1. Let \( A(x) = \sum_{p \leq x} \frac{\chi(p)}{p^s} \), \( 0 \leq \sigma < 1 \). Assume that for some \( x_0 > 0 \), \( A(x) \) is either \( A(x) \geq 0 \) for all \( x \geq x_0 \) or \( A(x) \leq 0 \) for all \( x \geq x_0 \). Clearly \( A(x) \) is a bounded Riemann-integrable function in any finite interval \([1, x]\). Let \( s \in H_1 \). Then

\[
\sum_{p \in \mathbb{P}} \frac{\chi(p)}{p^s} = \sum_{p \in \mathbb{P}} \frac{\chi(p)}{p^s} \frac{1}{p^{s-\sigma}} = \int_1^\infty \frac{1}{u^{s-\sigma}} dA(u) = (s-\sigma) \int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du.
\]

Since the partial sums \( \sum_{n \leq x} \chi(n) \ll 1 \), we have that \( L(s, \chi) \) converges for all \( s \in H_0 \), and hence, it is analytic in this half plane. Further, by Lemma 2.1, we have that \( L(s, \chi) \neq 0 \) for \( s \in H_1 \). Moreover, if \( \chi \) is non-principal, \( L(1, \chi) \neq 0 \). Thus
there exists an open ball $B$ with positive radius and centered at $s = 1$ such that $L(s, \chi) \neq 0$ for all $s \in B$. It follows that $L(s, \chi) \neq 0$ for all $s \in H_1 \cup B$. The set $H_1 \cup B$ is simply connected. Thus, by Lemma 2.3, there exists an analytic function $\log^* L(\cdot, \chi) : H_1 \cup B \to \mathbb{C}$ such that $L(s, \chi) = \exp(\log^* L(s, \chi))$ for all $s \in H_1 \cup B$. By Lemma 2.1, we have for $s \in H_1$

$$L(s, \chi) = \exp \left( \sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s} + B(s) \right).$$

Since $H_1$ also is simply connected, by Lemma 2.3 it follows that there exists a constant $c \in \mathbb{C}$ such that for all $s \in H_1$

$$\log^* L(s, \chi) = (s - \sigma) \int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du + B(s) + c.$$ 

Thus:

$$\int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du = \frac{\log^* L(s, \chi) - B(s) - c}{s - \sigma}.$$ 

It follows that $\int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du$ has an analytic continuation to $H_1 \cup B'$, where $B'$ is an open ball of positive radius and centered at 1. In particular, this integral is analytic at $s = 1$. Hence, by Landau’s oscillation Theorem (Lemma 2.2), we have that $\int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du$ converges for $s = 1 - \epsilon$, for some $\epsilon > 0$. Since $A(x)$ changes sign only for a finite number of $x$, this convergence is absolute, and hence $\int_1^\infty \frac{A(u)}{u^{s+1-\sigma}} du$ is an analytic function in $H_{1-\epsilon}$. It follows that $\log L(s, \chi)$ has an analytic continuation to $H_{1-\epsilon}$, and hence, $L(s, \chi) \neq 0$ for $s \in H_{1-\epsilon}$. \hfill \Box

**Proof of Corollary 1.1.** In view of Lemma 2.1, we only need to show that the series $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s}$ converges for all $s \in H_{1-\epsilon}$.

A classical result for Dirichlet series (see Theorem 15, pg. 119, [1]) states that:

Let $F(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$ be a Dirichlet series with finite abscissa of convergence $\sigma_c$. Let $\sigma_0 > \sigma_c$. Then uniformly for $\sigma_0 \leq \sigma \leq \sigma_c + 1$ we have that $F(\sigma + it) \ll |t|^{1-(\sigma-\sigma_c)+\delta}$. Since $L(s, \chi)$ is convergent for $s \in H_0$, we have for some constant $A > 0$, $L(\sigma + it) \ll |t|^A$.

On the hypothesis of Theorem 1.1, we have that $L(s, \chi) \neq 0$ for $s \in H_{1-\epsilon}$. Hence, for $\sigma > 1 - \epsilon$, $\log |L(\sigma + it, \chi)| \ll A \log(|t| + 2)$. By applying the Borel-Caratheodory theorem, we can conclude, in the same line of reasoning of Theorem 14.2 of [8] that $\log L(\sigma + it, \chi) \ll \log(|t| + 2)$. Thus, by Lemma 2.1 and Lemma 2.3 we have that $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s}$ has an analytic continuation to $H_{1-\epsilon}$ given by $F(s) = \log L(s, \chi) - B(s) + c$, for some constant $c$. This analytic continuation is, for $\sigma > 1 - \epsilon$, $F(\sigma + it) \ll \log(|t| + 2) \ll t^\delta$, for all $\delta > 0$. 


Another classical result for Dirichlet series states (Theorem 4, pg. 134, [7]): If $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has a finite abscissa of convergence and if $\sigma_0$ is some real number for which $F(s)$ has an analytic continuation to $\mathbb{H}_{\sigma_0}$ satisfying, for each $\sigma > \sigma_0$, $F(\sigma + it) \ll t^\delta$, for all $\delta > 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for all $s \in \mathbb{H}_{\sigma_0}$. Thus, $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s}$ converges for all $s \in \mathbb{H}_{1-\epsilon}$.

Proof of Theorem 1.2. In the proof of Corollary 1.1, we showed that the hypothesis $L(s, \chi) \neq 0$ for all $s \in \mathbb{H}_{1-\epsilon}$ implies that the series $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s}$ converges for all $s \in \mathbb{H}_{1-\epsilon}$. We claim that there exists $\sigma \in (1-\epsilon, 1)$ for which $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^\sigma} \neq 0$. By contradiction, if no such $\sigma$ exists, then $\sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s} = 0$ for all $s \in (1-\epsilon, 1)$, and since this Dirichlet series is an analytic function, it follows that this analytic function is equal to zero everywhere. Hence, by Theorem 1.6 of [6], we have that $\chi(p) = 0$ for all $p \in \mathcal{P}$, which is a contradiction. Hence, there exists $\sigma \in (1-\epsilon, 1)$ such that the partial sums $\sum_{p \leq x} \frac{\chi(p)}{p^\sigma}$ converges, as $x \to \infty$, to a non-zero value. Hence, this partial sums can change sign only for a finite number of $x$. □

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