A Microscopic Limit on Gravitational Waves from D-brane Inflation

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We derive a microscopic bound on the maximal field variation of the inflaton during warped D-brane inflation. By a result of Lyth, this implies an upper limit on the amount of gravitational waves produced during inflation. We show that a detection at the level \( r > 0.01 \) would falsify slow roll D-brane inflation. In DBI inflation, detectable tensors may be possible in special compactifications, provided that \( r \) decreases rapidly during inflation. We also show that for the special case of DBI inflation with a quadratic potential, current observational constraints imply strong upper bounds on the five-form flux.

1. INTRODUCTION

In the foreseeable future it may be possible to detect primordial gravitational waves produced during inflation. This would be a spectacular opportunity to reveal physics at energy scales that are unattainable in terrestrial experiments. In light of this possibility, it is essential to understand the predictions made by various inflationary models for gravitational wave production. As we shall review, a result of Lyth connects detectably large gravitational wave signals to motion of the inflaton over Planckian distances in field space. It is interesting to know when suitably flat potentials over such large distances are attainable in string compactifications, allowing a potentially observable tensor signal in the associated string inflation models. In this paper we analyze this issue for the case of warped D-brane inflation models, and use compactification constraints to derive a firm upper bound on the inflaton field range in Planck units.

For slow roll warped brane inflation, our result implies that the gravitational wave signal is undetectably small. This constraint is model-independent and holds for any slow roll potential. For DBI inflation, the limit on the field range forces the tensor signal to be much smaller than the current observational bound. Detection in a future experiment may be possible only if \( r \) decreases rapidly soon after scales observable in the cosmic microwave background (CMB) exit the horizon. This does occur in some models, but it has a striking correlate: the scalar spectrum will typically have a strong blue tilt and/or be highly non-Gaussian during the same epoch.

We also consider compactification constraints on the special case of DBI inflation in which the potential is quadratic. We find that observational constraints, together with our bound on the field range, exclude scenarios with a large amount of five-form flux. For a DBI model realized in a warped cone over an Einstein manifold \( X_5 \), this translates into a very strong requirement on the volume of \( X_5 \) at unit radius. However, we show that manifolds obeying this constraint do exist, at least in noncompact models. This translates the usual problem of accommodating a large flux into the problem of arranging that \( X_5 \) has small volume.

2. THE LYTH BOUND

In slow roll inflation the tensor fluctuation two-point function is proportional to

\[
\frac{H^2}{M_P^2},
\]

where \( H \) is the Hubble expansion rate. The scalar fluctuation two-point function is proportional to

\[
H^2 \left( \frac{H}{\dot{\phi}} \right)^2.
\]

The first factor in (2) represents the two-point function of the scalar field, while the second factor comes from the conversion of fluctuations of the scalar field into fluctuations of the scale factor in the metric (or scalar curvature fluctuations). This implies that the ratio between the tensor and scalar two-point functions is proportional to

\[
r \equiv 8 \left( \frac{\dot{\phi}}{H M_P} \right)^2 = 8 \left( \frac{d\phi}{dN} \frac{1}{M_P} \right)^2,
\]

where \( dN = H dt \) represents the differential of the number of e-folds. We have fixed the numerical prefactor in (3) so that \( r \) is defined as in the WMAP conventions.

This implies that the total field variation during inflation is

\[
\frac{\Delta \phi}{M_P} = \frac{1}{8^{1/2}} \int_0^{N_{\text{end}}} dN r^{1/2},
\]

where \( N_{\text{end}} \sim 60 \) is the total number of e-folds from the time the CMB quadrupole exits the horizon to the end of inflation.

In any given model of inflation, \( r \) is determined as a function of \( N \). We therefore define

\[
N_{\text{eff}} \equiv \int_0^{N_{\text{end}}} dN \left( \frac{r}{r_{\text{CMB}}} \right)^{1/2},
\]
so that
\[ \frac{\Delta \varphi}{M_P} = \left( \frac{r_{\text{CMB}}}{8} \right)^{1/2} N_{\text{eff}}. \]  
(6)
Here \( r_{\text{CMB}} \) denotes the tensor-to-scalar ratio \( r \) evaluated on CMB scales (2 \( \leq \ell \leq 100 \)), 0 < \( N \) < \( N_{\text{CMB}} \) \( \approx 4 \). We use \( N_{\text{eff}} \) to parameterize how far beyond \( N_{\text{CMB}} \) the support of the integral in (4) extends. If \( r \) is precisely constant then \( N_{\text{eff}} = N_{\text{end}} \), if \( r \) is monotonically increasing then \( N_{\text{eff}} > N_{\text{end}} \), and if \( r \) decreases then \( N_{\text{eff}} < N_{\text{end}} \). For a detailed discussion of related issues, see \( [9] \).

The Lyth bound \( [8] \) relates the maximal amount of gravitational waves to the field variation \( \Delta \varphi \) during inflation
\[ r_{\text{CMB}} = \frac{8}{(N_{\text{eff}})^{3/2}} \left( \frac{\Delta \varphi}{M_P} \right)^2. \]  
(7)
The bound implies that a model producing a detectably large quantity of gravitational waves necessarily involves field variations of order the Planck mass. In the remainder of the paper we will determine whether such large field variations are possible in a class of string inflation models.

In slow roll inflation \( r \) is proportional to the slow roll parameter \( \epsilon \). We can define a second slow roll parameter \( \eta \) as the fractional variation of \( \epsilon \) during one e-fold. Then we have
\[ \frac{d \ln \epsilon}{d N} = \frac{d \ln \epsilon}{d N} = \frac{r}{8}, \]  
(8)
where the last equality is just the definition of \( \eta \). We can also write \( [9] \) in terms of the spectral indices of the scalar and tensor power spectra
\[ \frac{d \ln r}{d N} = n_T - (n_S - 1) = - \left( n_S - 1 + \frac{r}{8} \right), \]  
(9)
where we have used the usual single-field consistency condition \( n_T = -r/8 \).

To determine \( N_{\text{eff}} \), we notice that \( N_{\text{eff}} < N_{\text{end}} \) only if \( \eta \) is negative. Present observations \( [10, 11] \) indicate that \( |\tilde{\eta}_{\text{CMB}}| \) is very small on scales probed by CMB (\( N \lesssim 4 \)) and large-scale structure observations (\( N \lesssim 10 \)). In particular, \( \tilde{\eta}_{\text{CMB}} \gtrsim -0.03 \). Since the variation of \( \eta \) is second order in slow roll we may assume that \( \eta \) remains small throughout inflation. Integrating \( [8] \), we find a range \( N_{\text{eff}} \sim 30 - 60 \) in \( [10, 11] \). Nearly all of the range for \( \tilde{\eta}_{\text{CMB}} \) allowed by WMAP3+SDSS \( [10, 11] \) actually corresponds to \( N_{\text{eff}} \gtrsim 50 \). To get a conservative bound, we have considered the most negative allowed values of \( \tilde{\eta}_{\text{CMB}} \), corresponding to the largest allowed values of \( n_S - 1 \) and \( r \), and this gives \( N_{\text{eff}} \sim 30 \). Direct observation of gravitational waves by some futuristic gravitational wave detector such as the Big Bang Observer (BBO) would put a similar lower bound on \( N_{\text{eff}} \) (see e.g. \( [12] \)).

With this input, the Lyth bound for slow roll models of inflation becomes
\[ r_{\text{CMB}} \lesssim \frac{8}{30^2} \left( \frac{\Delta \varphi}{M_P} \right)^2. \]  
(10)

3. CONSTRAINT ON FIELD VARIATION IN COMPACT SPACES

In this section we determine the maximum field range of the inflaton in warped D-brane inflation.\(^1\) By \( [11] \) or \( [10] \), this will imply a model-independent upper limit on gravitational wave production in this scenario.

3.1. Warped Throat Compactifications

Consider a warped flux compactification of type IIB string theory to four dimensions \( [14] \), with the line element
\[ ds^2 = h^{-1/2}(y)g_{\mu \nu}dx^\mu dx^\nu + h^{1/2}(y)g_{ij}dy^i dy^j. \]  
(11)
We will be interested in the case that the internal space has a conical throat, i.e. a region in which the metric is locally of the form\(^2\),
\[ g_{ij}dy^i dy^j = d\rho^2 + \rho^2 ds^2_{X_5}, \]  
(12)
for some five-manifold \( X_5 \). The metric on this cone is Calabi-Yau provided that \( X_5 \) is a Sasaki-Einstein space. If the background contains suitable fluxes, the metric in the throat region can be highly warped.

Many such warped throats can be approximated locally, i.e. for a small range of \( \rho \), by the geometry \( AdS_5 \times X_5 \), with the warp factor
\[ h(\rho) = \left( \frac{R}{\rho} \right)^4, \]  
(13)
where \( R \) is the radius of curvature of the \( AdS \) space. In the case that the background flux is generated entirely by \( N \) dissolved D3-branes placed at the tip of the cone, we have the relation \( [15] \)
\[ \frac{R^4}{(\alpha')^2} = 4\pi g_s N \frac{\pi^3}{\text{Vol}(X_5)}. \]  
(14)
Here \( \text{Vol}(X_5) \) denotes the dimensionless volume of the space \( X_5 \) with unit radius.\(^3\) Generically, we expect this

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\(^1\) The implications of field range limits for eternal D-brane inflation have been discussed in \( [13] \).

\(^2\) We use \( \rho \) to denote the radial direction, because the conventional symbol \( r \) is already in use.

\(^3\) An equivalent definition of \( \text{Vol}(X_5) \), which may be more clear when it is difficult to define a radius, is as the angular factor in the integral defining the volume of a cone over \( X_5 \).
volume to obey \( \text{Vol}(X_5) = O(\pi^3) \), e.g. \( \text{Vol}(S^5) = \pi^3 \), \( \text{Vol}(T^{1,1}) = \frac{6}{5} \pi^3 \). However, very small volumes are possible, for example by performing orbifolds.

Warped throats have complicated behavior both in the infrared and the ultraviolet. For almost all \( X_5 \), no smooth tip geometry, analogous to that of the Klebanov-Strassler throat \( [16] \), is known. Furthermore, the ultraviolet end of the throat, where the conical metric is supposed to be glued into a compact bulk, is poorly understood. These regions are geometric realizations of what are called the ‘IR brane’ and ‘Planck brane’ in Randall-Sundrum models. In this note we study constraints that are largely independent of the properties of these boundaries. We take the throat to extend from the tip at \( \rho = 0 \) up to a radial coordinate \( \rho_{\text{UV}} \), where the ultraviolet end of the throat is glued into the bulk of the compactification. Background data, in particular three-form fluxes, determine \( \rho_{\text{UV}} \), but we will find that \( \rho_{\text{UV}} \) cancels from the quantities of interest.

To summarize our assumptions: we consider a throat that is a warped cone over some Einstein space \( X_5 \), but may have complicated modifications in the infrared and ultraviolet. This very large class of geometries includes the backgrounds most often studied for warped brane inflation, but it would be interesting to understand even more general warped throats.

3.2. A Lower Bound on the Compactification Volume

Standard dimensional reduction gives the following relation between the four-dimensional Planck mass \( M_P \), the warped volume of the compact space \( V_6^w \), the inverse string tension \( \alpha' \), and the string coupling \( g_s \):

\[
M_P^2 \equiv \frac{V_6^w}{\kappa_{10}^2},
\]

where \( \kappa_{10}^2 \equiv \frac{1}{2} (2\pi)^7 \alpha'(\alpha')^4 \). The warped volume of the internal space is

\[
V_6^w = \int d^5y \sqrt{g} h. \tag{16}
\]

Formally this may be split into separate contributions from the bulk and the throat region

\[
V_6^w \equiv (V_6^w)_{\text{bulk}} + (V_6^w)_{\text{throat}}. \tag{17}
\]

The throat contribution is

\[
(V_6^w)_{\text{throat}} \equiv \text{Vol}(X_5) \int_0^{\rho_{\text{UV}}} d\rho' \rho'^2 h(\rho) = \frac{1}{2} \text{Vol}(X_5) R^4 \rho_{\text{UV}}^2 = 2\pi^4 g_s N (\alpha')^2 \rho_{\text{UV}}^2. \tag{18}
\]

A key point is that the \textit{warped} throat volume is independent of \( \text{Vol}(X_5) \).

The result \( [18] \) is rather robust. To confirm that \( [13] \) is a suitable approximation for the warp factor, we note that in the Klebanov-Tseytlin regime \( [17] \) of a Klebanov-Strassler throat, the warp factor may be written as

\[
h(\rho) = \left( \frac{L}{\rho} \right)^4 \ln \frac{\rho}{\rho_s}, \tag{19}
\]

where \( L^4 \equiv \frac{3}{2\pi} \frac{g_s M^2}{3} R^4 \), \( \ln \frac{\rho_{\text{UV}}}{\rho_s} \approx \frac{2\pi}{3} \frac{K}{g_s M} \) and \( N \equiv MK \). Integrating \( [19] \) one finds

\[
(V_6^w)_{\text{throat}} = \frac{1}{2} \text{Vol}(T^{1,1}) R^4 \rho_{\text{UV}}^2, \tag{20}
\]

in agreement with equation \( [18] \).

The bulk volume is model-dependent, but we can impose a very conservative lower bound on the total warped volume by omitting the bulk volume,

\[
V_6^w > (V_6^w)_{\text{throat}}. \tag{21}
\]

This implies a lower limit on the four-dimensional Planck mass in string units

\[
M_P^2 > \left( \frac{V_6^w}{\kappa_{10}^2} \right)_{\text{throat}}. \tag{22}
\]

3.3. An Upper Bound on the Field Range

Let us now consider inflation driven by the motion of a D3-brane in the background \( [11] \). The canonically-normalized inflaton field is

\[
\varphi^2 = T_3 \rho^2, \quad T_3 \equiv \frac{1}{(2\pi)^3} \frac{1}{g_s (\alpha')^2}. \tag{23}
\]

The maximal radial displacement of the brane in the throat is the length of the throat, from the tip \( \rho_{IR} \approx 0 \) to the ultraviolet end, \( \rho_{UV} \), so that \( \Delta \rho \lesssim \rho_{UV} \). Naively one could think that the range of the inflaton could be made arbitrarily large by increasing the length of the throat. However, what is relevant is the field range in four-dimensional Planck units, which is

\[
\left( \frac{\Delta \varphi}{M_P} \right)^2 < \frac{T_3 \rho_{\text{UV}}^2}{M_P^2} < \frac{T_3 \rho_{\text{UV}}^2}{(V_6^w)_{\text{throat}}}. \tag{24}
\]

Substituting equation \( [18] \) gives the following important constraint on the maximal field variation in four-dimensional Planck units

\[
\left( \frac{\Delta \varphi}{M_P} \right)^2 < \frac{4}{N}. \tag{25}
\]

Two comments on this result are in order. First, the field range in Planck units only depends on the background charge \( N \) and is manifestly independent of the choice of \( X_5 \), so our result is the same for any throat that is a warped cone over some \( X_5 \). Second, the size of the
that D-brane inflation models are usually considered of the level can be falsified by a detection of gravitational waves at controlled. We therefore find that warped D-brane inflation waves in such an experiment would therefore imply that Near-future CMB polarization experiments \[1, 18\] will least respond to an unwarped throat, we require at the very mass to be large in string units. Because \( N = 0 \) corresponds to an unwarped throat, we require at the very least \( N \geq 1 \); in practice, \( N \gg 1 \) is required for a controllable supergravity description.

The bound \( \ref{eq:bound} \) is extremely conservative, because we have neglected the bulk volume, which in many cases will actually be larger than the throat volume. Modifications of the geometry at the tip of the throat, where \( \rho \ll R \), provide negligible additional field range. One might also try to evade this bound by considering a stack of \( n \) D3-branes moving down the throat, which increases the effective tension. However, the backreaction from such a stack is important unless \( n \ll N \), so this will not produce a bound weaker than \( \ref{eq:bound} \) with \( N = 1 \).

4. IMPLICATIONS FOR SLOW ROLL BRANE INFLATION

Via the Lyth relation \( \ref{eq:lyth} \), the bound \( \ref{eq:bound} \) translates into a microscopic constraint on the maximal amount of gravitational waves produced during warped brane inflation

\[
\frac{r_{\text{CMB}}}{0.009} \lesssim \left( \frac{1}{N \sqrt{N_{\text{eff}}}} \right)^2. \tag{26}
\]

As explained in \[2, 3\] for slow roll inflation, recent observations \[10, 11\] imply

\[
30 \lesssim N_{\text{eff}} \lesssim N_{\text{end}} \approx 60 \, . \tag{27}
\]

Let us stress that the lower part of this range is occupied only by models with a large, positive scalar running or a blue scalar spectrum and a large tensor fraction.

This implies

\[
\frac{r_{\text{CMB}}}{0.009} \lesssim \frac{4}{N} \, . \tag{28}
\]

Near-future CMB polarization experiments \[1, 18\] will probe\(^4\) \( r_{\text{CMB}} \gtrsim \mathcal{O}(10^{-2}) \). Detection of gravitational waves in such an experiment would therefore imply that \( N < 4 \). This implies that the space is effectively unwarped, and that the supergravity description is uncontrolled. We therefore find that warped D-brane inflation can be falsified by a detection of gravitational waves at the level \( r_{\text{CMB}} \gtrsim 0.01 \).

One might have anticipated this result on the grounds that D-brane inflation models are usually considered of the ‘small-field’ type, and are typically thought to predict an unobservably small tensor fraction. Let us stress, however, that extracting precise predictions from D-brane inflation scenarios is rather involved, and requires careful consideration, and fine-tuning, of the potential introduced by moduli stabilization \[20\]. It is quite unlikely that the fully corrected potential will enjoy the same exceptional flatness as the uncorrected potential given in \[3\]. As moduli stabilization effects increase \( \epsilon \), they increase \( r \), and \emph{a priori} this may be expected to lead to observable gravitational waves. Indeed, it has been argued in the context of more general single-field inflation that minimally tuned models correlate with maximal gravitational wave signals \[21\]. Nevertheless, our result implies that even the \emph{maximal} signal in warped D-brane inflation is undetectably small. We have thus excluded the possibility of detectable tensors on purely kinematic grounds, \emph{i.e.} by using only the size of the field space.

5. IMPLICATIONS FOR DBI INFLATION

A very interesting alternative to slow roll inflation arises when nontrivial kinetic terms drive inflationary expansion. The DBI model \[7, 8, 22, 23\] is a string theory realization of this possibility in which a D3-brane moves rapidly in a warped background. In this section we combine the Lyth bound with our field range bound \( \ref{eq:bound} \) to constrain the tensor signal in DBI inflation.

5.1. A Generalized Lyth Bound

We present the Lyth bound in a theory with a general kinetic term, then specialize to the DBI case. Consider the action \[24\]

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_P^2 R + 2P(X, \varphi) \right], \tag{29}
\]

where \( P(X, \varphi) \) is a general function of the inflaton \( \varphi \) and of \( X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \). For slow roll inflation

\[
P(X, \varphi) \equiv X - V(\varphi), \tag{30}
\]

while DBI inflation may be parameterized by \[8\]

\[
P(X, \varphi) \equiv -f^{-1}(\varphi) \sqrt{1 - 2f(\varphi)X + f^{-1}(\varphi) - V(\varphi)}, \tag{31}
\]

where \( f^{-1}(\varphi) = T_3 h^{-1}(\varphi) \) is the rescaled warp factor. From \[24\] we find the energy density in the field to be \( \rho = 2XP_X - P \). We also define the speed of sound as

\[
c_s^2 \equiv \frac{dP}{d\rho} = \frac{P_X}{P} = \frac{P_X}{P_X + 2XP_{XX}}. \tag{32}
\]

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\(^4\) The ultimate detection limit is probably around \( r \sim 10^{-3} - 10^{-4} \). Measuring even lower \( r \) is prohibited by the expected magnitude of polarized dust foregrounds and by the lensing conversion of primordial E-modes to B-modes. \[18\]
We define slow variation parameters in analogy with the standard slow roll parameters
\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{XP_X}{M_p^2 H^2}, \]
\[ \bar{\eta} \equiv \frac{\dot{c_s}}{c_s H}, \]
\[ s \equiv \frac{\dot{c_s}}{c_s H}. \]

To first order in these parameters the basic cosmological observables are
\[ P_S = \frac{1}{8\pi^2 M_p^2 c_s^4 \epsilon}, \]
\[ P_T = \frac{2}{\pi^2} \frac{H^2}{M_p^2}, \]
\[ n_S - 1 = -2\epsilon - \bar{\eta} - s, \]
\[ n_T = -2\epsilon. \]

The tensor-to-scalar ratio in these generalized inflation models is
\[ r = 16 c_s \epsilon. \]

This nontrivial dependence on the speed of sound implies a modified consistency relation
\[ r = -8 c_s n_T. \]

As discussed recently by Lidsey and Seery [25] for the case of DBI inflation, equation [46] provides an interesting possibility of testing fast roll inflation. The standard slow roll predictions are recovered in the limit \( c_s = 1 \).

Restricting to the homogeneous mode \( \varphi(t) \) we find from [43] that
\[ \frac{d\varphi}{M_P} = \sqrt{\frac{2\epsilon}{P_X}} dN \]
and hence
\[ \frac{\Delta \varphi}{M_P} = \int_{N_{\text{end}}}^{N} \sqrt{\frac{r - 1}{8 c_s P_X}} dN. \]

Notice the nontrivial generalization of the slow roll result [41] through the factor \( c_s P_X \). For DBI inflation this factor happens to be
\[ c_s P_X = 1, \]
where
\[ c_s^2 = 1 - 2f(\varphi)X \equiv \frac{1}{\gamma^2(\varphi)}, \]
so that the Lyth bound remains the same as for slow roll inflation.\(^5\)

The variation of \( r \) during inflation follows from [40]
\[ \frac{d\ln r}{dN} = \frac{d\ln \epsilon}{dN} + \frac{d\ln c_s}{dN} = \bar{\eta} + s. \]

While the observed near scale-invariance of the density perturbations restricts the magnitude of \( s = d\ln c_s/dN \) in the range \( 0 \leq N \leq 10 \), outside that window \( s \) can in principle become large and negative. By [40] this would source a rapid decrease in \( r \). Note, however, that the results given in this section are first order in \( s \) and so receive important corrections when \( s \) is large. Furthermore, omission of terms in the DBI action involving two or more derivatives of \( \varphi \) may not be consistent when \( s \) is sufficiently large.

Constraints on the evolution of \( r \) may also be understood by rewriting equation [40] as
\[ \frac{d\ln r}{dN} = n_T - (n_S - 1) \]
\[ = - \left[ (n_S - 1) + \frac{r}{8c_s} \right]. \]

This implies that \( r \) can decrease significantly only if the scalar spectrum becomes very blue \((n_S - 1 > 0)\) and/or the speed of sound becomes very small, so that \( r/c_s \) is large. During the time when observable scales exit the horizon this possibility is significantly constrained, but outside that window \( r \) may decrease rapidly in some models.

### 5.2. Constraints on Tensors

Just as in slow roll inflation we can write
\[ r_{CMB} < \frac{1}{N} \left( \frac{60}{N_{\text{eff}}} \right)^2. \]

However, in DBI inflation we have to allow for the possibility that a nontrivial evolution of the speed of sound allows \( N_{\text{eff}} \) to be considerably smaller than \( N_{\text{end}} \), which weakens the Lyth bound. The precise value of \( N_{\text{eff}} \) will be highly model-dependent.

In light of the constraint [48], constructing a successful DBI model with detectable tensors is highly nontrivial. First of all, such a model must produce a spectrum of scalar perturbations consistent with observations, i.e., with the appropriate amplitude and with a suitably small level of non-Gaussianity. Then, the model should include each of the following additional elements related to the large tensor signal:

1. A consistent compactification in which
\[ (V_6^u)_{\text{bulk}} \ll (V_6^u)_{\text{throat}}, \]
so that the inequality in [25] may be nearly saturated.

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\(^5\) This result has been obtained independently by S. Kachru and by H. Tye.
2. A *small* five-form flux $N$, together with a demonstration that the supergravity corrections and brane backreaction are under control in this difficult limit.

3. A decrease in $r$ that is rapid enough to ensure that $N_{\text{eff}} \ll 30$. In this situation the slow variation parameters $\tilde{\eta}$, $s$ cannot both be small, substantially complicating the analysis.

It would be extremely interesting to find a system that satisfies all these constraints, especially because this would be a rare example of a complete string inflation model with detectable tensors.

### 5.3. Constraints on Quadratic DBI Inflation

In this section we illustrate our considerations for one important class of DBI models, those with a quadratic potential.\(^6\)

We consider an action of the form \(\mathcal{S}_1\), with

\[ V(\varphi) = \frac{1}{2} m^2 \varphi^2. \]  

(50)

At sufficiently late times, the Hubble parameter is \(\tilde{H}(\varphi) = c \varphi\),

(51)

for some constant $c$. Using this in \(\mathcal{S}_3\), one finds

\[ \epsilon \gamma(\varphi) = 2M_P^2 \left(\frac{H}{\tilde{H}}\right)^2 = 2 \left(\frac{M_P}{\varphi}\right)^2. \]  

(52)

This relates the DBI Lorentz factor $\gamma$ to the slow roll parameter $\epsilon < 1$ and to the inflaton field value.

#### 5.3.1. Microscopic Constraint from Limits on Non-Gaussianity

Observational tests of the non-Gaussianity of the primordial density perturbations are most sensitive to the three-point function of the comoving curvature perturbations. It is usually assumed that the three-point function has a form that would follow from the field redefinition

\[ \zeta = \zeta_0 - \frac{3}{5} f_{NL} \zeta_0^2, \]  

(53)

where $\zeta_0$ is Gaussian. The scalar parameter $f_{NL}$ quantifies the amount of non-Gaussianity. It is a function of three momenta which form a triangle in Fourier space.

Here we cite results for the limit of an equilateral triangle. Slow roll models predict $f_{NL} \ll 1$ \cite{Garriga:1999vw}, which is far below the detection limit of present and future observations. For generalized inflation models represented by the action \(\mathcal{S}_1\) one finds

\[ f_{NL} = \frac{35}{108} \left(\frac{1}{c_s^2} - 1\right) - \frac{5}{81} \left(\frac{1}{c_s^2} - 1 - 2\Lambda\right), \]  

(54)

where

\[ \Lambda \equiv \frac{X^2 P_{XX} + \frac{4}{3}X^3 P_{XXX}}{XP_{X} + 2X^2 P_{XX}}. \]  

(55)

For the case of DBI inflation \(\mathcal{S}_1\), the second term in \(\mathcal{S}_1\) is identically zero \cite{Garriga:1999vw}, so that the prediction for the level of non-Gaussianity \(\mathcal{S}_1\) is\(^7\)

\[ f_{NL} = \frac{35}{108} \left(\frac{1}{c_s^2} - 1\right) \approx \frac{35}{108} \gamma^2, \]  

(56)

where the second relation holds when $c_s \ll 1$. This result leads to an upper bound on $\gamma$ from the observed limit on the non-Gaussianity of the primordial perturbations. The recent analysis of the WMAP3 \cite{Komatsu:2008hk} data in \cite{Garriga:2008sh} gives $-256 < f_{NL} < 332$ (95% confidence level), which implies

\[ \gamma \lesssim 32. \]  

(57)

Using the expressions \(\mathcal{S}_1\) and \(\mathcal{S}_2\), we have

\[ N < 4 \left(\frac{M_P}{\varphi}\right)^2 = \frac{r \gamma^2}{8} = \frac{27}{70} r f_{NL}. \]  

(58)

Combining the observational bound on gravitational waves \cite{Ade:2013lnn}, $r < 0.3$, with the bound on non-Gaussianity, we find

\[ N \lesssim 38. \]  

(59)

Quadratic DBI inflation with a larger amount of five-form flux is hence excluded by current observations. The Planck satellite may be sensitive enough to give the limits $|f_{NL}| < 50$ \cite{Garriga:2008sh} and $r < 0.05$. Non-observation at these levels would give the bound $N < 1$, excluding quadratic DBI inflation.

#### 5.3.2. Microscopic Constraint from the Amplitude of Primordial Perturbations

Garriga and Mukhanov \cite{Garriga:2000yf} have derived the perturbation spectrum for theories with non-canonical kinetic terms

\[ P_S = \frac{1}{8\pi^2 M_P^2 c_s^2} \left| \frac{H^2}{c_k a H} \right|_{c_k = a H}. \]  

(60)

\(\text{---}\)

\(^6\) We consider the so-called ‘UV model’, i.e. with a D3-brane moving toward the tip of the throat; cf. \cite{Garriga:1999vw} for an interesting alternative.

\(^7\) Notice that this result is generic to DBI inflation and is independent of the choice of the potential and the warp factor. This is in contrast to other observables like $n_S$, $P_S$, etc.
Using \( c_s^{-2} = \gamma(\varphi) = \sqrt{1 + 4M_P^2f(\varphi)(H_\varphi)^2} \) for DBI inflation, this becomes
\[
P_S = \frac{16 \gamma^2(\gamma^2 - 1)}{\pi^2 (r\gamma^2)^2} \frac{1}{M_P^2 f}.
\]
(61)

For the AdS warp factor \( M_P f(\varphi) = \lambda \left( \frac{M_P}{\varphi} \right)^4 = \lambda \left( \frac{r \gamma^2}{32} \right)^2, \)

where
\[
\lambda = T_3 R^4 = \frac{\pi}{2 \text{Vol}(X_5)},
\]
we find
\[
P_S = \left( \frac{32}{\pi} \right)^3 \times \frac{\gamma^2(\gamma^2 - 1)}{(r\gamma^2)^4} \frac{\text{Vol}(X_5)}{N}.
\]
(64)

In the relativistic limit we have \( \gamma^2(\gamma^2 - 1) \sim \gamma^4 = 9f_{NL}^2 \)
and \( (64) \) becomes
\[
P_S = \left( \frac{32}{3\pi} \right)^3 \times \frac{3}{r^4 f_{NL}^2} \frac{\text{Vol}(X_5)}{N} \gtrsim 0.1 \frac{\text{Vol}(X_5)}{N},
\]
(65)

where the last relation comes from the current observational bounds on \( r \) and \( f_{NL} \) on CMB scales. COBE or WMAP give the normalization \( P_S \approx 10^{-9} \), so that we arrive at the condition
\[
N \gtrsim 10^8 \text{Vol}(X_5).
\]
(66)

The requirements \( (60) \) and \( (66) \) are clearly inconsistent for the generic case, \( \text{Vol}(X_5) \sim \mathcal{O}(\pi^3). \) We conclude that quadratic DBI inflation in warped throats\(^8\) cannot simultaneously satisfy the observational constraints on the amplitude and Gaussianity of the primordial perturbations unless \( \text{Vol}(X_5) \lesssim 10^{-7}. \) In particular, this excludes realization of this scenario in a cut-off AdS model or in a Klebanov-Strassler \( 10 \) throat. Cones with very small values of \( \text{Vol}(X_5) \) can be constructed by taking orbifolds or considering \( Y^{p,q} \) spaces in the limit that \( q \) is fixed and \( p \to \infty. \) However, it seems rather unlikely that one could embed these spaces into a string compactification.

6. CONCLUSIONS

We have established a firm upper bound on the canonical field range in Planck units for a D3-brane in a warped throat. This range can never be large, and can be of order one only in the limit of an unwarped throat attached to a bulk of negligible volume. Combined with the Lyth relationship \( 3 \) between the variation of the inflaton field during inflation and the gravitational wave signal, this implies a constraint on the tensor fraction in warped D-brane inflation. The tensor signal is undetectably small in slow roll warped D-brane inflation, regardless of the form of the potential. In DBI inflation, detectable tensors may be possible only in a poorly-controlled limit of small warping, moderately low velocity, rapidly-changing speed of sound, and substantial backreaction. In this case, the scalar spectrum will typically have a strong blue tilt and/or become highly non-Gaussian shortly after observable scales exit the horizon.

We have also presented stronger constraints for the case of DBI inflation with a quadratic potential, finding that combined observational constraints on tensors and non-Gaussianity imply an upper bound \( N \lesssim 38 \) on the amount of five-form flux. Near-future improvements in the experimental limits could imply \( N < 1 \) and thus exclude the model. For models realized in a warped cone over a five-manifold \( X_5 \), current limits imply that the dimensionless volume of \( X_5 \), at unit radius, is smaller than \( 10^{-7}. \) Manifolds of this sort do exist; extremely high-rank orbifolds and cones over special \( Y^{p,q} \) manifolds are examples, but it is not clear that these can be embedded in a string compactification.

Although our result resonates with some well-known effective field theory objections (see \( \textit{e.g.} 31 \)) to controllably flat inflaton potentials involving large field ranges, we stress that our analysis was entirely explicit and did not rely on notions of naturalness or of fine-tuning.

Our microscopic limit on the evolution of the inflaton implies that a detection of primordial gravitational waves would rule out most models of warped D-brane inflation, and place severe pressure on the remainder. We expect that compactification constraints on canonical field ranges imply similar bounds in many other string inflation models \( 32 \). In this sense, current models of string inflation do not readily provide detectable gravitational waves \( 34. \) However, this is not yet by any means a firm prediction of string theory, and it is more important than ever to search for a compelling model of large-field string inflation that overcomes this obstacle. Given the apparent difficulty of achieving super-Planckian field variations with controllably flat potentials for scalar fields in string theory, a detection of primordial gravitational waves would provide a powerful selection principle for string inflation models and give significant clues about the fundamental physics underlying inflation.

\(^8\) Throats that are not cones over Einstein manifolds could evade this constraint, and may be a more natural setting for realizing the DBI mechanism. We thank E. Silverstein for explaining this to us.

\(^9\) We should mention one promising string inflation scenario, N-inflation \( 33 \), that does predict observable tensors. It would be very interesting to understand whether this model can indeed be realized in a string compactification.
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[1] J. Bock et al., “Task Force on Cosmic Microwave Background Research,” arXiv:astro-ph/0604101.
[2] A. H. Guth, Phys. Rev. D 23, 347 (1981); A. D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[3] D. H. Lyth, Phys. Rev. Lett. 78, 1861 (1997) arXiv:hep-ph/9606387.
[4] G. R. Dvali and S. H. H. Tye, Phys. Lett. B 450, 72 (1999) arXiv:hep-ph/9812483.
[5] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister and S. P. Trivedi, JCAP 0310, 013 (2003) arXiv:hep-th/0308055.
[6] S. H. Tye, arXiv:hep-th/0610221.
[7] E. Silverstein and D. Tong, Phys. Rev. D 70, 103505 (2004) arXiv:hep-th/0310221.
[8] M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D 70, 123505 (2004) arXiv:hep-th/0404084.
[9] R. Easther, W. H. Kinney and B. A. Powell, JCAP 0608, 004 (2006) arXiv:astro-ph/0601270.
[10] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003) arXiv:astro-ph/0302209.
[11] M. Tegmark et al., arXiv:astro-ph/0608632.
[12] L. A. Boyle and P. J. Steinhardt, arXiv:astro-ph/0512014.
[13] X. Chen, S. Sarangi, S. H. Henry Tye and J. Xu, arXiv:hep-th/0608082.
[14] M. R. Douglas and S. Kachru, arXiv:hep-th/0610102.
[15] S. S. Gubser, Phys. Rev. D 59, 025006 (1999) arXiv:hep-th/9807164.
[16] I. R. Klebanov and M. J. Strassler, JHEP 0008, 052 (2000) arXiv:hep-th/0007191.
[17] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B 578, 123 (2000) arXiv:hep-th/0002159.
[18] A. C. Taylor et al., arXiv:astro-ph/0407148.
[19] D. Spergel, private communication.
[20] D. Baumann, A. Dymarsky, I. R. Klebanov, J. Maldacena, L. McAllister and A. Murugan, arXiv:hep-th/0607050; D. Baumann et al., in preparation.
[21] L. A. Boyle, P. J. Steinhardt and N. Turok, Phys. Rev. Lett. 96, 111301 (2006) arXiv:astro-ph/0507455.
[22] S. E. Shandera and S. H. Tye, JCAP 0605, 007 (2006) arXiv:hep-th/0601099; S. Kecskemeti, J. Maiden, G. Shiu and B. Underwood, arXiv:hep-th/0605189; G. Shiu and B. Underwood, arXiv:hep-th/0610151.
[23] X. Chen, JHEP 0508, 045 (2005) arXiv:hep-th/0501184; X. Chen, Phys. Rev. D 72, 123518 (2005) arXiv:astro-ph/0507053.
[24] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999) arXiv:hep-th/9904176.
[25] J. E. Lidsey and D. Seery, arXiv:astro-ph/0610398.
[26] J. M. Maldacena, JHEP 0305, 013 (2003) arXiv:astro-ph/0210603.
[27] X. Chen, M. X. Huang, S. Kachru and G. Shiu, arXiv:hep-th/0605045.
[28] P. Creminelli, L. Senatore, M. Zaldarriaga and M. Tegmark, arXiv:astro-ph/0610600.
[29] E. Komatsu and D. N. Spergel, Phys. Rev. D 63, 063002 (2001) arXiv:astro-ph/0005036.
[30] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Adv. Theor. Math. Phys. 8, 711 (2004) arXiv:hep-th/0403002.
[31] D. H. Lyth and A. Riotto, Phys. Rept. 314, 1 (1999) arXiv:hep-ph/9807278.
[32] J. P. Conlon and F. Quevedo, JHEP 0601, 146 (2006) arXiv:hep-th/0509012; J. J. Blanco-Pillado et al., JHEP 0609, 002 (2006) arXiv:hep-th/0603129.
[33] S. Dimopoulos, S. Kachru, J. McGreevy and J. G. Wacker, arXiv:hep-th/0507205.
[34] S. Kachru, talks at Solvay Conference, November 2005; KITP String Phenomenology workshop, September 2006; and COSMO 06, September 2006.