Density fluctuations in $\kappa$-deformed inflationary universe

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We study the spectrum of metric fluctuation in $\kappa$-deformed inflationary universe. We write the theory of scalar metric fluctuations in the $\kappa$-deformed Robertson-Walker space, which is represented as a non-local theory in the conventional Robertson-Walker space. One important consequence of the deformation is that the mode generation time is naturally determined by the structure of the $\kappa$-deformation.

We expand the non-local action in $H^2/\kappa^2$, with $H$ being the Hubble parameter and $\kappa$ the deformation parameter, and then compute the power spectra of scalar metric fluctuations both for the cases of exponential and power law inflations up to the first order in $H^2/\kappa^2$. We show that the power spectra of the metric fluctuation have non-trivial corrections on the time dependence and on the momentum dependence compared to the commutative space results. Especially for the power law inflation case, the power spectrum for UV modes is weakly blue shifted early in the inflation and its strength decreases in time. The power spectrum of far-IR modes has cutoff proportional to $k^3$ which may explain the low CMB quadrupole moment.

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I. INTRODUCTION

The history of the studies on the Cosmic Microwave Background (CMB) anisotropies and on the cosmological fluctuations is closely linked to that of the study of the standard cosmological model [1, 2]. We now have high resolution maps of the anisotropies in the temperature of the cosmic microwave background [3], and its accuracy of the resolution is improving further. In relation to these observational data, overviews of the theory of cosmological perturbation applied to inflationary cosmology have been presented in Refs. [4, 5]. The cosmological observations reveal that the Universe has non-random fluctuations on all scales smaller than the present Hubble radius. In the most currently studied models of the very early universe it is assumed that the perturbations originate from quantum vacuum fluctuations, which was first proposed in a paper by Sakharov [6]. With this, the inflationary cosmology bears in it the “trans-Planckian problem”: Since inflation has to last for long enough time to solve several problems of big-bang model and to provide a causal generation mechanism for CMB fluctuations, the corresponding physical wavelength of these fluctuations has to be smaller than the Planck length at the beginning of the inflation [7]. Both of the theories of gravity and of matter break down at the trans-Planckian scale. Many methods have been proposed to cure the problem. The modification of the dispersion relation, which was used to study the thermal spectrum of black hole radiation [8], was applied to cosmology [9]. Modifications of the evolution of cosmological fluctuations due to the string-motivated space-time uncertainty relations, $\delta x_{\text{phys}} \delta t \geq l_s^2$, have been introduced by Brandenberger and Ho [10]. It was shown that the uncertainty relation plays a significant role in the spectrum of the metric fluctuation [11]. Greene et al. [12] proposed the initial states which give an oscillatory contribution to the primordial power spectrum of inflationary density perturbations. There have also been some attempts to explain the low CMB quadrupole moment contribution [13] by using the pre-Big Bang scenario in string theory [14]. The ambiguity of the action in the presence of a minimal length cutoff in inflation by the boundary terms are studied by Ashoorioon, Kempf, and Mann [15]. However, it is not easy to construct a consistent field theoretic model which satisfies both the stringy space-time uncertainty relation and the spatial homogeneity and isotropy of the Robertson-Walker space. A direct
non-commutative deformation of the commutation relation,
\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \]  
(1)
introduces a preferred direction in space, which breaks the isotropy of 3-dimensional space. Therefore, it would be interesting to construct a space-time non-commutative theory which keeps the spatial homogeneity and isotropy demanded by the Robertson-Walker space-time.

Much attention has been given on the possibility of explaining the observational data as a quantum gravitational effect. As a theoretical framework to study these quantum gravity effects phenomenologically “Doubly Special Relativity” (DSR, also called Deformed Special Relativity) was proposed by Amelino-Camelia [16], where there exist two relativistically invariant scales, the speed of light and the Planck scale, and extensive studies have been followed [17, 18, 20, 21, 22]. Recently, it was argued that the coordinate space of the DSR theory defined in curved momentum space is described by the \( \kappa \)-Minkowski space. Therefore, a good candidate to study the quantum gravity effect to cosmology is to extend the \( \kappa \)-Minkowski theory to \( \kappa \)-Robertson-Walker Space (\( \kappa \)-RWS) and to study the effect of the deformation in the cosmological evolution. In \( \kappa \)-Minkowski space the space-time coordinates are non-commuting generators of a quantized Minkowski space-time. The \( \kappa \)-deformed Minkowski space-time introduces a dimensionful quantum deformation parameter, \( \kappa \), which can be chosen to have the dimension of mass [23, 24, 25]. A natural choice of this deformation parameter is the Planck mass \( \kappa = M_P \). It is therefore important to construct consistent quantum field theoretic framework in \( \kappa \)-Minkowski space, and explore the physical effects in cosmological evolution. In this respect, Kowalski-Glikman [26] have studied the effects on the density fluctuations of the quantum \( \kappa \)-Poincaré algebra.

In Sec. II, we construct the theory of cosmological fluctuations in \( \kappa \)-Deformed Inflationary Universe (\( \kappa \)-DIU). Starting from the scalar-gravity theory in a flat Robertson-Walker space we briefly summarize the linearized theory of scalar metric fluctuations. After developing the \( \kappa \)-RWS, we write the theory of scalar metric fluctuations in the deformed space. We show that the scalar theory in \( \kappa \)-RWS space is described by a nonlocal field theory in the conventional Robertson-Walker space. The nonlocal action is series expanded in \( \frac{H^2}{\kappa^2} \) in Sec. III, where \( H \) is the Hubble parameter, and is quantized. In Sec. IV and V, we calculate the power spectra of scalar metric fluctuations for the cases of exponential and power law inflations, respectively. We show that the \( \kappa \)-deformation alters both the time dependence and the frequency dependence of the power spectrum nontrivially. In Sec. V, we summarize our results.

II. DENSITY FLUCTUATION IN \( \kappa \)-DEFORMED ROBERTSON-WALKER SPACE-TIME

If the Universe is quantum mechanically created with vacuum energy dominance, it will inflate from the beginning. Even though there may be other choices for the pre-inflationary universes, we assume that the inflation starts from the beginning of the universe.

The calculations in this paper are carried out in 4-dimensional spatially flat Robertson-Walker metric,
\[ ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega^2), \]  
(2)
where \( a(t) \) denotes the scale factor of expanding universe. For later use, we introduce the conformal time \( \eta \) defined by
\[ \eta = \int \frac{dt}{a(t)}. \]  
(3)

The Einstein-Hilbert action for gravity coupled to scalar matter field is
\[ S = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi G} + \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right], \]  
(4)
where \( R \) is the Ricci curvature scalar. Starting from the action [19], it was shown that the scalar and tensor parts of the linear metric fluctuation are described by the action (See Refs. [17, 18, 20]),
\[ S = \frac{1}{2}\int d\eta\left[ (\partial_\eta v_k)^2 + \left( \frac{\partial_\eta^2 z}{z} - k^2 \right) v_k^2 \right], \]  
(5)
where \[ \int_k \equiv \int \frac{d^3k}{(2\pi)^3} \] and
\[ z \equiv \frac{a(t)\dot{\varphi}_0}{H}, \]  
(6)
with $\varphi_0$ being the scalar zero mode and $H = \dot{a}/a$ the Hubble parameter. It is noted that in cases of power law inflation and of slow roll inflation, $H$ is proportional to $\dot{\varphi}_0$, hence $z \propto a$. The action (7) can be cast into the covariant form

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu
u} \partial_\mu R \partial_\nu R,$$

(7)

where $R$ is the gauge invariant metric fluctuation and plays the role of a massless scalar field in the inflating universe. The Fourier mode of $R$, $R_k$, is related to the field by $\psi_k = z R_k$. Therefore, its power spectrum is

$$P_k = \frac{|k|^3}{2\pi^2 z^2(t)} \langle 0 | \psi_k^2 | 0 \rangle.$$

(8)

We now want to write the action (7) in $\kappa$–RWS. For this purpose, we develop scalar field theory in $\kappa$–RWS in the following subsections. The 4-dimensional field theory in $\kappa$–Minkowski space has been constructed in Ref. [17, 20, 22] and references therein. We briefly summarize the results obtained in these references in the next subsection.

### A. $\kappa$-deformed Minikowski space-time

The $\kappa$-deformed Hopf algebra $H_x$ describing the $\kappa$–Minkowski space is generated by the coordinates $\hat{x}_\mu$ determined by the following relations:

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad \Delta(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu.$$

(9)

The dual Hopf algebra $H_k$ of functions on $\kappa$–deformed four-momenta is described by the Hopf subalgebra of the $\kappa$–deformed Poincaré algebra as follows:

$$[k_\mu, k_\nu] = 0, \quad \Delta(k_i) = k_i \otimes e^{-k_0/\kappa} + 1 \otimes k_i, \quad \Delta(k_0) = k_0 \otimes 1 + 1 \otimes k_0.$$

(10)

The first Casimir operator of the algebra (9) and (10) is

$$M^2 = \left(2\kappa \sinh \frac{k_0}{2\kappa} \right)^2 - k^2 e^{k_0/\kappa}.$$

(11)

It follows from this that for positive $\kappa$ the on-shell three-momentum is bounded from above by

$$k^2 \leq \kappa^2,$$

(12)

and the maximal value of momentum results in an infinite energy [27].

By using $\kappa$–deformed Fourier transform, the fields on $\kappa$–Minkowski space with non-commutative space-time coordinates $\hat{x} = (\hat{x}_0, \hat{x}_i)$ is written as $(k\hat{x} \equiv k_i \hat{x}_i - k_0 \hat{x}_0)$:

$$\Phi(\hat{x}) = \int_k \tilde{\Phi}(k) : e^{ik\hat{x}} :,$$

(13)

where $\int_k \equiv \int \frac{d^4k}{(2\pi)^4}$ and $\tilde{\Phi}(k)$ is a classical function on commuting four-momentum space $k = (k_0, k_i)$ and the normal ordering is defined by

$$: e^{ik\hat{x}} : \equiv e^{-ik_0 \hat{x}_0} e^{ik\hat{x}}.$$

(14)

Multiplication of two normal ordered $\kappa$–deformed exponentials follows from Eqs. (9) and (10):

$$: e^{ik\hat{x}} : : e^{iq\hat{x}} : = e^{i(k\hat{x} - q\hat{x})} \hat{x} - i(k_0 + q_0) \hat{x}_0 :,$$

(15)

which follows from the four momentum addition rule described by the coproduct (10). From this we get the conjugate field,

$$\Phi^\dagger(\hat{x}) = \int_k \Phi^\dagger(k) : e^{ik\hat{x}} :, \quad \Phi^\dagger(k, k_0) = e^{3k_0/\kappa} \Phi^*(-e^{k_0/\kappa}k, -k_0),$$

(16)
The differential calculus and its covariance properties under the action of \( \kappa \)-deformed Poincaré group have been constructed in Ref. [23, 25]. The left or right partial derivatives \( \hat{\partial}_A \) to define \( \kappa \)-deformed vector field are given by

\[
\hat{\partial}_A \Phi(\hat{x}) =: \chi_A \left( \frac{1}{i} \partial_\mu \right) \Phi(\hat{x}),
\]

where \( \chi_A : e^{ik\hat{x}} := \chi_A(k_\mu)e^{ik\hat{x}} : \) and

\[
\chi_i(k_\mu) = e^{k_0/\kappa}k_i, \quad \chi_0 = \kappa \sinh \frac{k_0}{\kappa} + \frac{k^2}{2\kappa} e^{k_0/\kappa}.
\]

The adjoint derivative \( \hat{\partial}^\dagger \) can be defined to satisfy

\[
\int d^4\hat{x} \Phi_1(\hat{x}) \hat{\partial}_0 \Phi_2(\hat{x}) = \int d^4\hat{x} [\hat{\partial}^\dagger_0 \Phi_1(\hat{x})] \Phi_2(\hat{x}),
\]

which leads to

\[
\hat{\partial}^\dagger_\mu \Phi(x) = \chi^\dagger_\mu(\partial_\mu/i)\Phi(\hat{x}),
\]

where

\[
\chi^\dagger_\mu(k) = \chi_\mu(-e^{k_0/\kappa}k_i,-k_0).
\]

Based on these, the \( \lambda \partial^A \) field theory was constructed in Ref. [23]. In the next subsection we generalize the formulation to the case of the Robertson-Walker space-time.

**B. The scalar field theory in \( \kappa \)-deformed Robertson Walker space-time**

To generalize the field theories in the \( \kappa \)-Minkowski space to the curved space case, we must be careful in selecting the coordinates which satisfy the commutation relation [29]. Since any non-decreasing reparametrization of \( t \) is an equally good time coordinate in commutative space-time, it is important to choose the time coordinate for which the commutation relation,

\[
[s(\hat{x}_0),a(\hat{x}_0)] = a(\hat{x}_0)[s(\hat{x}_0),\hat{x}_0] + a(\hat{x}_0)s(\hat{x}_0), \hat{x}_0] = 0.
\]

is imposed. We note that a natural time coordinate consistent with the commutation relation is the cosmological time \( x_0 = t \). This choice ensures the same form of commutation relation satisfied by the locally flat coordinates \( (\hat{t}, \hat{X}_i) = (\hat{t}, i\hat{X}_i/\kappa) \). This simplicity cannot be attained for other choices of time coordinate. For example, consider a time coordinate \( x_0 \) defined by the 00-part of the metric \( g_{00} = -s^2(\hat{x}_0) \). In the locally flat coordinates, the commutator \([s(\hat{x}_0),a(\hat{x}_0)] = -s(\hat{x}_0)s(\hat{x}_0), s(\hat{x}_0) = 0 \) is not simply reduced to a well defined form of Eq. [18]. In this sense the natural choice for the time coordinate is the cosmological time \( t \) where \( s(t) = 1 \). With the commutative coordinate \( t \), the equations [19] – [25] can be used without modification.

The generalization of \( \kappa \)-deformed vector fields in \( \kappa \)-RWS can be written as Eq. [19], with the operator \( \chi_\mu \) defined by,

\[
\chi_i = e^{\partial_\mu/\kappa} \frac{\partial_\mu}{i}, \quad \chi_0 = \kappa \sinh \frac{\partial_\mu}{i\kappa} - \frac{1}{2\kappa} e^{\partial_\mu/2i\kappa} \gamma^i \gamma^j e^{\partial_\mu/2i\kappa}.
\]

where the ordering of the time-dependent metric, \( g^{ij} \), and derivatives are determined by demanding the adjoint derivatives to satisfy Eq. [20], which gives

\[
\chi^\dagger_i(k) = -k_i,
\]

\[
\hat{\partial}^\dagger_0 \Phi(\hat{x}) = -\kappa \sinh \frac{\partial_0}{i\kappa} \Phi(\hat{x}) - \frac{1}{2\kappa} e^{\partial_\mu/(2i\kappa)} \partial_i \left[ e^{\partial_\mu/(2i\kappa)} \partial_j \Phi(\hat{x}) \right] g^{ij}(\hat{t}).
\]
In addition to Eq. (20) we demand the condition
\[ \hat{\partial}_i^a \Phi^\dagger (\hat{x}) = (\hat{\partial}_0 \Phi (\hat{x}))^\dagger, \] (27)
to determine $\hat{\partial}_i^a$.

Given the covariant derivatives and its adjoint derivatives, we can write the action of a massless scalar field in $\kappa$–RWS as
\[
S = -\frac{1}{2} \int d^4 x (\hat{\partial}_i^a \Phi^\dagger (\hat{x})) \sqrt{-g} g^{\mu \nu} (\hat{x}) \hat{\partial}_\mu \Phi (\hat{x})
= \frac{1}{2} \int d^4 x \left[ (\hat{\partial}_i^a \Phi^\dagger (\hat{x})) n^3 (\hat{t}) \hat{\partial}_i \Phi (\hat{x}) - (\hat{\partial}_i^a \Phi^\dagger (\hat{x})) a (\hat{t}) \hat{\partial}_i \Phi (\hat{x}) \right],
\] (28)
where we choose the symmetric form in the action so that the metric dependent factor is placed in the middle of the operator products. It turns out that this choice gives the simplest form of the interaction between different modes. In addition we do not consider the change of measure \[28\] due to the complication of the non-commutative multiplication since what we are interested in in this paper is to understand the main feature of the $\kappa$–deformation on the metric fluctuation.

Using Eqs. (13), (15), (16), (24), and (25) we obtain for the action
\[
S = \frac{\kappa^2}{4} \int_\kappa \left\{ g (k, t) \tilde{\Phi}_{-k} (t - \frac{i}{\kappa} \tilde{\Phi}_k (t + \frac{i}{\kappa}) - a^3 (t) \left[ \rho_3 (t) + g_3 (t) \tilde{k}^2 (t) \right] \tilde{\Phi}_{-k} (t) \tilde{\Phi}_k (t) \right\},
\] (29)
where $\tilde{k}$ denotes the relative ratio between the physical momentum ($|k|/a$) and the non-commutative scale ($\tilde{k}(t) = |k|/a(t)\kappa$) and the coefficients $g(k, t)$, $g_1(t)$ and $\rho_n(t)$ are given by
\[
g(k, t) = a^3 (t) \left[ 1 + (g_0 (t) - 2) \tilde{k}^2 + \frac{g_2 (t)}{2} \tilde{k}^4 \right],
g_1(t) = \frac{1}{2} \left[ \frac{a^2 (t)}{a^2 (t + \frac{i}{2\kappa})} + \frac{a^2 (t)}{a^2 (t - \frac{i}{2\kappa})} \right],
g_2(t) = \frac{a^2 (t + \frac{i}{2\kappa}) a^2 (t - \frac{i}{2\kappa})}{a^3 (t + \frac{i}{2\kappa}) a^2 (t + \frac{i}{2\kappa})},
g_3(t) = \frac{1}{2} \left[ \frac{a^3 (t - \frac{i}{\kappa})}{a^3 (t + \frac{i}{\kappa})} + \frac{a^3 (t + \frac{i}{\kappa})}{a^3 (t + \frac{i}{\kappa})} \right],
\rho_n(t) = \frac{1}{2} \left[ \frac{a^n (t - \frac{i}{\kappa})}{a^n (t)} + \frac{a^n (t + \frac{i}{\kappa})}{a^n (t)} \right].
\] (30)

Note that the left hand sides of Eq. (30) are defined to satisfy $\lim_{\kappa t \to \infty} g_i = 1 = \lim_{\kappa t \to \infty} \rho_n$. The action (29) is highly non-local in that the fields are non-locally multiplied in the action, in addition to the nonlocal coupling between the background metric and the field modes. Each mode of the field, $\Phi_{\pm k}$, is diagonalized so that it is not coupled to other modes of different $k$.

### III. First Order Approximation and the Hamiltonian Formulation

It is not possible to solve the nonlocal equation of motion derived from the action (29) exactly. The canonical formalism for Lagrangians with non-locality of finite extent has been proposed by Woodard. However, we do not follow the formalism since our purpose is to obtain information on how the non-commutativity affects the evolution of the metric fluctuation in inflationary Universe in a simple calculable form. Instead, we use a perturbative expansion in the parameter $\tilde{H}^2 \equiv H^2/\kappa^2$ and construct the Hamiltonian for the action up to the first order in $\tilde{H}^2$.

To have an approximation of the action (29), we expand the integrand in $\tilde{H}^2$ as
\[
g(t) \Phi (t) = g (0) (t) \Phi (t) + \frac{1}{2} g (1) (t) \Phi (t)^2 + \frac{1}{8} g (2) (t) \Phi (t)^3
+ \frac{1}{16} g (3) (t) \Phi (t)^4 + \frac{1}{64} g (4) (t) \Phi (t)^5 + O (\tilde{H}^6),
\] (31)
where \(g^{(-n)}(t)\) denotes the \(n^{th}\) indefinite integrals of \(g(t)\), and
\[
g_s^{(-n)}(t) = \frac{1}{2} \left[ g^{(-n)}(t + i/\kappa) + g^{(-n)}(t - i/\kappa) \right], \quad g_A^{(-n)}(t) = \frac{\kappa}{2i} \left[ g^{(-n)}(t + i/\kappa) - g^{(-n)}(t - i/\kappa) \right].
\] (32)

From Eqs (30) and (31), we get the action up to the order \(O(\bar{H}^2)\),
\[
S = \frac{1}{2} \int dt \ a^3(t) \left\{ \mu(k, t) \dot{\Phi}_k(t) \dot{\Phi}_{-k}(t) - \frac{k^2}{a^2(t)} \left( g_4 - \frac{g_5 k^2}{4} \right) \dot{\Phi}_k \dot{\Phi}_{-k} + \frac{\gamma(k, t)}{3\kappa^2} \dot{\Phi}_k \dot{\Phi}_{-k} + \cdots \right\},
\] (33)

where the coefficients are given by
\[
\mu(k, t) = a^{-3}(t)g_A^{(-1)}(k, t),
\]
\[
g_4(t) = \rho_l + \frac{1}{4} \left\{ g_3 - \frac{1}{2} \left[ \frac{a(t)a^2(t + 3i/(2\kappa))}{a(t)a^2(t - 3i/(2\kappa))} \right] \right\},
\]
\[
g_5(t) = \frac{1}{2} \left[ \frac{a(t)a^3(t + i/\kappa)}{a^2(t + i/2\kappa)a^2(t + 3i/(2\kappa))} + \frac{a(t)a^3(t - i/\kappa)}{a^2(t - i/2\kappa)a^2(t - 3i/(2\kappa))} \right],
\]
\[
\gamma(k, t) = \frac{3}{2a^3(t)} \left[ -6\kappa^4(g^{(-4)} - g_5^{(-4)}) + \kappa^2 \left( 8g_A^{(-3)} - 4g_5^{(-2)} - g^{(-2)} \right) - \frac{g}{4} \right].
\]

The asymptotic values of these coefficient functions are
\[
g_4(\infty) = g_5(\infty) = \mu(0, \infty) = \gamma(0, \infty) = 1,
\] (35)

which make it easier to guess the asymptotic behaviors of the coefficient functions for large \(t\).

For notational simplicity, we use the change of variables \(\Phi_{k, +} = \frac{1}{2}(\Phi_k + \Phi_{-k}), \quad \Phi_{k, -} = \frac{1}{2}(\Phi_k - \Phi_{-k})\), to write the action in a diagonal form:
\[
S = \frac{1}{2} \int dt a^3(t) \left\{ \mu(k, t) \dot{\phi}_k^2 - \frac{k^2}{a^2} \left( g_4 - \frac{g_5 k^2}{4} \right) \dot{\phi}_k^2 + \frac{\gamma(k, t)}{3\kappa^2} \dot{\phi}_k^2 + \cdots \right\},
\] (36)

where \(\alpha = (k, \pm)\). Note that the coefficients \(\mu, \gamma, \) and \(g_i\) are exactly calculable once \(a(t)\) is given.

Introducing the conformal time \(\eta\) and the rescaling of the field, \(\phi_\alpha = a(t(\eta))\phi_\alpha\), we reduce the action (36) into the form, up to the order \(\bar{H}^2\),
\[
S = \frac{1}{2} \int d\eta \left\{ \bar{\mu}(\partial_\eta \phi_\alpha)^2 - \omega(k, t) \phi_\alpha^2 + \frac{\gamma(k, t)}{3\kappa^2 a^2(t)} \left( \partial_\eta^2 \phi_\alpha \right)^2 \right\},
\] (37)

where
\[
\bar{\mu} = \mu + \frac{4\gamma\bar{H}^2}{3\kappa^2} \left( 1 + \frac{5\bar{H}}{4\bar{H}^2} + \frac{3\bar{\gamma}}{4\bar{H}\gamma} \right),
\]
\[
\omega(k, t) = k^2 \left( g_4 - \frac{g_5 k^2}{4} \right) - \frac{\partial_\eta(A^2 \bar{H})}{a} + \frac{\partial_\eta^2(\gamma \partial_\eta \bar{H})}{3\kappa^2 a},
\]
\[
A^2 = a^2(\eta) \left[ \mu(k, t) + \frac{\partial_\eta(\gamma \partial_\eta a)}{3\kappa^2 a^2} \right].
\]

Note that the action (37) contains a higher derivative term, which may lead to nonunitary evolution of the system. Since this higher derivative term is a term of order \(\bar{H}^2\) and our purpose in this paper is to obtain the effect of the deformation on the cosmological evolution up to the 1st order in \(\bar{H}^2\), we require that the higher order derivative term is written as a function of the field, its first time derivative, and time:
\[
\partial_\eta^2 \phi = \Psi(\phi, \partial_\eta \phi, \eta).
\] (40)

Explicitly, we use the linearized approximation,
\[
\partial_\eta^2 \phi = \frac{a(\eta)}{\gamma^{1/2}(k, \eta)}[c(\eta)\phi + d(\eta)\partial_\eta \phi],
\] (41)
where the coefficients $c$ and $d$ are to be determined by consistency.

This requirement is equivalent to the perturbative calculation up to the 1st order in $\bar{H}^2$. This can be shown as follows: The equation of motion for $\phi = \phi_0 + \frac{1}{\kappa^2} \phi_1 + \cdots$ can be written as
\[
\partial_\eta (\bar{\mu} \partial_\eta \phi_0) + \omega \phi_0 = 0, \\
\partial_\eta (\bar{\mu} \partial_\eta \phi) + \omega \phi = \frac{1}{3 \kappa^2} \partial^2 \left( \frac{\gamma}{\alpha^2} \partial^2 \phi_0 \right),
\]
where the first equation is the 0th order equation and the second is the full equation written explicitly up to the 1st order. The first equation of Eq. (42) defines $\partial^2 \phi_0$ as a linear function of $\partial \phi_0$ and $\phi_0$. Then, the second line can be understood as a defining equation of $\partial^2 \phi$ as a linear function of $\partial \phi$ and $\phi$ up to $O(\bar{H}^2)$, which is Eq. (41).

With this reasoning and Eq. (41), the action (37) is perturbatively equivalent up to $\bar{H}^2$ to the following unitary action
\[
S = \frac{1}{2} \int d\eta \left[ m(\partial_\eta \phi_0)^2 - f(k, \eta)\phi_0^2 \right],
\]
where $k = |k|$ and we have
\[
m = \bar{\mu} + \frac{d^2}{3 \kappa^2}, \\
f = \omega(k, t) - \frac{c^2 - \partial_\eta (cd)}{3 \kappa^2}.
\]
Substituting Eq. (41) into (37) and requiring the resultant action to be the same as the action (43) with (41) as its equation of motion, we find
\[
c = -\frac{f \gamma^{1/2}}{ma}, \quad d = -\frac{m \gamma^{1/2}}{m}.
\]
Eqs. (44) and (45) imply that $m$ and $f$ satisfy the following differential equations:
\[
m = \bar{\mu} + \frac{\gamma}{3 \kappa^2} \frac{(\dot{m})^2}{m^2}, \\
f^2 = \frac{\bar{\mu}^2}{3 \kappa^2 m^2} + f = \omega(k, t) + \frac{a(t)}{3 \kappa^2} \frac{\dot{m} \gamma}{am^2}.
\]
Note that these conditions make the action to be a functional of $\partial_\eta \phi$ and $\phi$. The time evolution for the theory (43) is unitary and quantum mechanically well defined.

We introduce mode dependent conformal time $\eta_k$ by
\[
d\eta_k = m^{-1}(k, \eta) d\eta = \frac{dt}{a(t)m(k, \eta)}.
\]
Then the action (43) can be written in a simplified form:
\[
S = \frac{1}{2} \int d\eta_k \left[ \phi_0'^2 - \Omega_\alpha^2(\eta_k)\phi_\alpha^2 \right],
\]
where $'$ denotes the derivative with respect to $\eta_k$ and
\[
\Omega_\alpha^2(\eta_k) \equiv m(k, \eta_k)f(k, \eta_k).
\]
The Hamiltonian for mode $\alpha$ is the same as that of the time-dependent harmonic oscillator with frequency squared $\Omega_\alpha^2$:
\[
H_\alpha = \frac{\hat{\pi}_\alpha^2}{2} + \frac{1}{2} \Omega_\alpha^2 \phi_\alpha^2.
\]
The time evolution of each mode can be described by introducing invariant creation and annihilation operators [30],
\[
\hat{A}_\alpha = -\frac{i}{\hbar^{1/2}}(\varphi_\alpha^* \hat{\pi}_\alpha - \varphi_\alpha^* \dot{\phi}_\alpha), \quad \hat{A}_\alpha^\dagger = \frac{i}{\hbar^{1/2}}(\varphi_\alpha \hat{\pi}_\alpha - \varphi_\alpha \dot{\phi}_\alpha),
\]
where \( \varphi_\alpha \) is the mode solution of the differential equation (52) below and \( \hat{A}_\alpha \) and \( \hat{A}_\alpha \) satisfy the Liouville-von Neumann equation,

\[
i \hbar \hat{\partial}_\eta \hat{A}_\alpha + [\hat{A}_\alpha, H_\alpha] = 0. \tag{52}\]

One may invert Eq. (51) to construct the field operator in terms of the creation and annihilation operators as

\[
\hat{\phi}_\alpha = \hbar^{1/2} \left[ \varphi_\alpha(\eta_0) \hat{A}_\alpha + \varphi_\alpha^*(\eta_0) \hat{A}_\alpha \right],
\]
\[
\hat{\pi}_\alpha = \hbar^{1/2} \left[ \varphi_\alpha'(\eta_0) \hat{A}_\alpha + \varphi_\alpha'^*(\eta_0) \hat{A}_\alpha \right].
\]  

Note also that the Liouville-von Neumann equation is equivalent to the following differential equation for the coefficients \( \varphi_\alpha \),

\[
\varphi''_\alpha(\eta_0) + \Omega^2_k(\eta_0) \varphi_\alpha(\eta_0) = 0. \tag{54}\]

The commutation relation \([\hat{A}_\alpha, \hat{A}_\beta^\dagger]\) = \( \delta_{\alpha\beta} \) restricts the mode solution \( \varphi_\alpha \) to satisfy \( \varphi_\alpha \varphi_\alpha^* - \varphi_\alpha^* \varphi_\alpha = i \).

We present the first order approximation of \( m \) and \( f \) for later use. To first order in \( 1/\kappa^2 \), Eq. (46) gives

\[
m \simeq \bar{\mu} + \frac{\gamma}{3\kappa^2} \frac{(\bar{\mu})^2}{\bar{\mu}^2} \simeq \bar{\mu},
\]
\[
f = 2\omega(k, t) \left[ 1 + \left( 1 + \frac{4\omega}{3\kappa^2 m^2 a^2} \right)^{1/2} \right]^{-1} \simeq \omega \left( 1 - \frac{\omega}{3\kappa^2 m^2 a^2} \right).
\]

Using the explicit form for \( \omega \) and \( \mu \), we have

\[
m \simeq \bar{\mu} \simeq 1 - \frac{\bar{H}^2}{6}(1 - 7\epsilon_1) - k^2,
\]
\[
f \simeq k^2 \left[ 1 + \frac{\bar{H}^2}{3}(1 - \epsilon_1) - \frac{7}{12} k^2 \right] - 2\bar{H}^2 a^2 \left[ 1 + \frac{\epsilon_1}{2} - \frac{\bar{H}^2}{6} \left( 1 - 16\epsilon_1 - \epsilon_1^2 + \frac{\epsilon_2 + \epsilon_3}{2} \right) \right],
\]

where \( \epsilon_n = \frac{H^{(n)}}{(H^{(n)})^{n+1}} \), with \( H^{(n)} = \frac{d^n H}{dt^n} \), are constant numbers for power law inflation and vanish for exponential inflation. From these we have

\[
m^{-1}(k, \eta) \simeq 1 + \alpha_n \frac{H^2}{\kappa^2} + \frac{k^2}{a^2 \kappa^2} + \cdots, \tag{57}\]
\[
\Omega^2_k(\eta_0) \simeq k^2(1 - w_1 \bar{H}^2 - w_2 k^2) - (2 + \epsilon_1) H^2 a^2(\eta_0)(1 - w_3 \bar{H}^2) + \cdots.
\]

For the power law inflation, the values of \( \alpha_n \) and \( w_i \) are given by

\[
\alpha_n = \frac{n + 7}{6n}, \quad w_1 = \frac{1}{6}(13 - 11/n), \quad w_2 = 19/12, \quad w_3 = \frac{1}{3(1 - 1/(2n))} \left( 1 + \frac{45}{4n} - \frac{7}{4n^2} - \frac{3}{n^3} \right), \tag{58}\]

and for the exponential inflation their values are given by the limits \( n \to \infty \) of Eq. (58).

### IV. METRIC FLUCTUATIONS IN \( \kappa \)-DIU: THE EXPONENTIAL INFLATION

The simplest inflationary model is the exponential inflation, in which the scale factor \( a(t) \) increases as,

\[
a(t) = a_0 e^{Ht}, \quad -\infty < t < \infty. \tag{59}\]

Here \( a_0 \) is the scale factor at \( t = 0 \) and \( H \) is the Hubble constant. Using the conformal time \( \eta \), we get

\[
Ht = -\ln(-a_0 H \eta), \quad a(\eta) = \frac{1}{-H \eta}, \tag{60}\]
where the conformal time $\eta$ varies from $-\infty$ to $0$ as $t$ varies from $-\infty$ to $\infty$. From Eqs. (54) and (58), we have

$$
\tilde{\mu} = \frac{\sin 3\tilde{H}}{3H} + \frac{4}{3}\tilde{H}^2\xi(3\tilde{H}) + \left(-\frac{\sin \tilde{H}}{H} + \frac{2}{3}\tilde{H}^2\xi(\tilde{H})\right)\left[2 - \cos \tilde{H} - \frac{k^2}{2}\right],
$$

$$
\gamma(k, t) = \xi(3\tilde{H}) - \xi(\tilde{H})\left[2 - \cos \tilde{H} - \frac{k^2}{2}\right],
$$

(61)

where $\xi(x) = 3\left[-3\frac{1 - \cos x}{x^4} + 4\frac{\sin x}{x^3} - \frac{4\cos x + 1}{2x^2} - \frac{1}{8}\right] \approx 1 - \frac{13}{80}x^2 + \cdots$.

Since the action (48) is obtained in the $\tilde{H}$ expansion, the normalized Hubble constant, $\tilde{H} = H/\kappa$ is assumed to be smaller than one. Moreover, the Eq. (67) below restricts $k = k/(a(t)\kappa)$ to be smaller than one. Then we get $m(k, \eta)$ from the differential equation (46) and (61) by series expansion in $\tilde{H}^2$ and $\tilde{k}$,

$$
m \simeq \tilde{\mu} + \frac{\tilde{H}^2}{3\tilde{H}^2\tilde{\mu}^2} \simeq \left(1 - \frac{\tilde{H}^2}{6}\right)(1 - \tilde{H}^2k^2\eta^2) + O(\tilde{H}^4).
$$

(62)

By integrating $m^{-1}$ over $\eta$ using Eq. (47), we get the mode-dependent conformal time $\eta_k$,

$$
\eta_k \simeq \frac{1}{2Hk(1 - H^2/6)} \ln \left(\frac{1 + \tilde{H}k\eta}{1 - Hk\eta}\right) = \frac{\eta}{1 - H^2/6} \left(1 + \frac{\tilde{H}^2}{3}k^2\eta^2 + \cdots\right),
$$

(63)

which is normalized so that $\eta_k = 0$ at $\eta = 0$. A crucial point is that there is a global rescaling of the conformal time due to the non-commutative effect.

From Eq. (57), the effective frequency squared $\Omega_k^2$ is written as

$$
\Omega_k^2(\eta_k) = k^2(1 - w_1\tilde{H}^2 - w_3\tilde{H}^2) - 2\tilde{H}^2a^2(\eta)(1 - w_3\tilde{H}^2) + \cdots
$$

(64)

$$
\simeq \tilde{k}^2 - \frac{\nu^2 - 1/4}{\eta_k^2} - w_2\tilde{H}^2k^4\eta_k^2,
$$

where $w_1 = 13/6$, $w_2 = 19/12$, $w_3 = 1/3$, and

$$
\tilde{k}^2 \equiv k^2\left[1 - (w_1 + 2/3)\tilde{H}^2\right], \quad \nu^2 - 1/4 \equiv \frac{2(1 - w_3\tilde{H}^2)}{(1 - H^2/6)^2} \simeq 2.
$$

(65)

The explicit value of $\nu$ is $\nu = 3/2$ to this order, which is the same as the commutative space result. We note that both the frequency and the mode solution have corrections from non-commutativity for large $|\eta_k|$. The last term of the second line in Eq. (63) becomes negligible for $\eta_k \sim 0$.

A. Mode generation and the initial condition

In the $\kappa$–RWS, spatial momentum is also restricted similarly as in (12). In terms of comoving momentum $k = |k|$, we have

$$
\frac{k}{a(t)} \leq \kappa.
$$

(66)

This gives the upper bound of $k$

$$
k \leq k_{\text{max}}(t) \equiv \kappa a(t).
$$

(67)

The maximal value of the wave-number (67) is very similar to that used by Brandenberger and Ho [11], except for the fact that the maximal value used in Ref. [10] is determined by an effective scale factor modified by the Moyal star product in the action.

Since $m$ is positive definite, the mode-dependent conformal time $\eta_k$, Eq. (64), is well defined and is an increasing function of $t$. The relation (67) implies that for a given $k$, there exists a conformal time $\eta_k^0$, the time saturating the relation (67):

$$
a(t(\eta_k^0)) \equiv \frac{k}{\kappa}.
$$

(68)
Then, the mode $\phi_k$ cannot exist before $\eta_0^k$. In other words, $\eta_0^k$ is the generating time of the mode $\phi_k$. This provides a hint to one of the major issues in which state the fluctuations are generated. To satisfy the continuity of the number of quanta of the $k$ mode when the mode becomes physical at $\eta_0^k$, it must be in the adiabatic vacuum state. This vacuum state can be chosen to be the WKB mode solution,

$$\lim_{\eta_k \to \eta_0^k} \varphi_k(\eta_k) = \frac{1}{\sqrt{2\Omega_k(\eta_k)}} \exp i \left( \int_{\eta_0^k}^{\eta_k} \Omega_k d\eta_k + \psi_k \right),$$

where the constant phase $\psi_k$ can be chosen conveniently.

### B. inflationary evolution

The inflationary evolution of the mode solution is determined by identifying $\Omega_k^2$ of Eq. (64). The corresponding commutative space values can be obtained by setting $\ddot{H} = 0$ in Eq. (64). With the $\Omega_k^2$, we have two different time scales $\eta_c^k$ and $\eta_i^k$ defined by $\Omega'(\eta_c^k) = 0$ and $\Omega^2_k(\eta_i^k) = 0$, respectively. These time scales are given by

$$\eta_c^k = -\left(\nu^2 - 1/4\right)^{1/4} \frac{k w_2}{H^{1/2}}, \quad \eta_i^k \simeq -\left(\nu^2 - 1/4\right)^{1/2} \frac{1}{k} \left( 1 + \frac{(\nu^2 - 1/4) w_2 H^2}{2} + \cdots \right),$$

where $\Omega_k^2(\eta_k)$ increases while $\eta_k < \eta_c^k$ and decreases later as shown in Fig. 1. It is positive definite when $\eta_k < \eta_i^k$ and negative later. For $\eta_k > \eta_i^k$, the modulus of $\phi_k$ increases in time. By Eq. (68), the mode $\phi_k$ is generated at $\eta_0^k \simeq -(k\dot{H})^{-1}$. Note that these three time scales satisfy

$$\eta_0^k \ll \eta_c^k \ll \eta_i^k,$$

if $\dot{H} \ll 1$. Note also that during $\eta_0^k < \eta_k < \eta_c^k$, the condition for the WKB approximation,

$$\frac{\partial^2 \Omega_k^2}{\partial \eta_k^2} \sim 2\omega_2 |\eta_k| \dot{H}^2 \ll 1,$$

is valid. In Fig. 1 we present schematic plot of $\Omega_k^2(\eta_k)$ for a given mode $\phi_k$. Therefore, during this period, the WKB mode solution,

$$\phi_k(\eta_k) = \frac{1}{\sqrt{2\Omega_k(\eta_k)}} \exp i \left( \int_{\eta_0^k}^{\eta_k} \Omega_k d\eta_k + \psi_k \right),$$

FIG. 1: Schematic plot of $\Omega_k^2$ for the exponential inflation and for the UV modes in the power law inflation. There is no mode $\phi_k$ for $\eta_k < \eta_0^k$. The mode solution for $\eta_0^k < \eta_k < \eta_c^k$ is given by the WKB solution and the mode solution for $\eta_k > \eta_c^k$ is given by the Bessel functions. The two solutions are matched at $\eta_c^k$. Therefore, during this period, the WKB mode solution,
can be used to describe the time evolution. We use this solution to determine the matching condition at \( \eta_k = \eta^c_k \),

\[
\varphi_k(\eta^c_k) = \frac{1}{\sqrt{2\Omega_c}}, \quad \varphi'_k(\eta^c_k) = i\sqrt{\frac{\Omega_c}{2}},
\]

(74)

where \( \Omega^2_c \equiv \Omega^2_k(\eta^c_k) \simeq \tilde{k}^2[1 - 2(\nu^2 - 1/4)^{1/2}\tilde{H}], \) \( \tilde{k} \) given in (68), and \( \psi_k \) is chosen to give this matching condition (74).

For \( \eta_k > \eta^c_k \), the last term in Eq. (72) becomes much smaller than other terms. Thus we ignore this term and use the Bessel function as the solution for \( \eta_k > \eta^c_k \),

\[
\varphi_k(\eta_k) = A_k \sqrt{-\eta_k} J\nu(\tilde{k} \eta_k) + B_k \sqrt{-\eta_k} Y\nu(\tilde{k} \eta_k).
\]

(75)

Matching the two solutions at \( \eta_k = \eta^c_k \), we get

\[
A_k = \frac{\pi}{2} \left[ i\sqrt{-\eta^c_k} Y\nu(\tilde{k} \eta^c_k) \varphi'_k(\eta^c_k) + \frac{Y\nu(\tilde{k} \eta^c_k) - 2\tilde{k} \eta^c_k Y\nu'(\tilde{k} \eta^c_k)}{2\sqrt{-\eta^c_k}} \varphi_k(\eta^c_k) \right],
\]

\[
B_k = -\frac{\pi}{2} \left[ i\sqrt{-\eta^c_k} J\nu(\tilde{k} \eta^c_k) \varphi'_k(\eta^c_k) + \frac{J\nu(\tilde{k} \eta^c_k) - 2\tilde{k} \eta^c_k J\nu'(\tilde{k} \eta^c_k)}{2\sqrt{-\eta^c_k}} \varphi_k(\eta^c_k) \right].
\]

(76)

Note that \( A_k \) and \( B_k \) are independent of \( k \) since \( \tilde{k} \eta^c_k \) is independent of \( k \) due to Eq. (70) and \( \Omega_c \propto k \).

As \( k \eta_k \to 0 \), the Bessel functions become

\[
J\nu \to \frac{(\tilde{k} \eta_k)\nu}{2^\nu \nu!}, \quad Y\nu \to \frac{2^\nu(\nu - 1)!}{\pi(\tilde{k} \eta_k)^\nu},
\]

(77)

and the second term in Eq. (75) dominates in the later time (\( -k \eta_k \to 0 \)). Therefore, the power spectrum of the scalar metric perturbation has the form

\[
P_k(t) = \frac{k^3}{2\pi^2} \frac{|\varphi_k(t)|^2}{z^2(t)} \simeq \frac{2^\nu(\nu - 1)!!|B|^2\tilde{H}^2}{2\pi^4(\tilde{z}/a)^2} \frac{(\tilde{H}/6)^2}{(1 - 17\tilde{H}/6)^{3/2}} \left( \tilde{k} \eta_k \right)^{3-2\nu},
\]

(78)

where \( z/a, \tilde{H}, \nu, \) and \( B \) are constant numbers. Note that \( 3 - 2\nu \simeq \frac{2}{3}(w^3 - 2\alpha)\tilde{H}^2 = 0 \) in the present case since \( w_3 = 1/3 \). Therefore, the spectrum of the metric fluctuation for exponential inflation in \( \kappa \)-RWS is time independent and is scale invariant up to the first order in \( \tilde{H}^2 \). The only effect of the non-commutativity to the power spectrum is the global rescaling of the power spectrum, which is of the order \( \tilde{H}^2 \). It is an interesting fact that \( w_3 \) and \( 2\alpha \) are the same. Note that \( \alpha \) originates from the scale factor of the mode-dependent conformal time \( \eta_k \) with respect to the conformal time \( \eta \), and \( w_3 \) comes from the \( \tilde{H}^2 \) order correction term of the frequency squared. Since there is no physical reason for the coincidence, it is possible that the next order correction may give a result of \( w_3 > 1/3 \).

This is an interesting possibility since this positive power of \( k \eta \) makes the power spectrum decrease in time. If the present analysis is applied to the tensor mode fluctuation, the time dependence can be used to solve the gravitational hierarchy problem \( (\tilde{H}/M_P \sim 10^{-5}) \) [31]. This is what happens in the power law inflation considered in the next section. The spectrum (78) is scale invariant in contrast to that of Ref. [26] with the same initial vacuum state. The difference may be attributed to the different choice of the initial conditions. At the present case, the initial time is dependent on the mode through Eq. (68), which uniquely fixes the initial state (69) for the mode solutions.

V. METRIC FLUCTUATIONS IN \( \kappa \)-DIU: POWER LAW INFLATION

In this section we calculate the metric fluctuation in the power law inflationary model, in which the scale factor increases as,

\[
a(t) = a_0(\kappa t)^n, \quad 0 < t < t_f,
\]

(79)

where \( n \neq 1 \), \( a_0 \) is the scale factor at the Planck time \( t = 1/\kappa \), and \( t_f \) is the instance when the inflation ends. This model, the variable \( z(t) \) in Eq. (8) is given by \( z(t) = \sqrt{\frac{2}{n}} M_P a(t) \). For \( n \neq 1 \), we have

\[
\kappa t = \left( \frac{\eta}{\eta_0} \right)^{\frac{1}{1-n}}, \quad a(\eta) = a_0 \left( \frac{\eta}{\eta_0} \right)^{\frac{1}{1-n}}.
\]

(80)
where $\mu = \frac{3n-1}{2(n-1)}$ and $\eta_0$ is the conformal time corresponding to $\kappa t = 1$, given by

$$\eta_0 = -\frac{\mu - 3/2}{a_0\kappa}.$$  \hfill (81)

$\eta_f = \eta_0(\kappa t_f)^{-1/(2\mu-3)}$ is the time when the inflation ends.

When $\kappa t \gg 1$ is large and $k^2/(\kappa^2) \ll 1$ is small, we have

$$m^{-1}(k, \eta) \simeq 1 + \alpha_n H^2(\eta) + \frac{k^2}{a^2(\eta)\kappa^2} + \cdots.$$  \hfill (82)

The mode-dependent conformal time $\eta_k$ can be approximated as

$$\eta_k \simeq \eta \left[ 1 + \frac{\alpha_n n^2}{2(\mu - 1)} \eta^{2\mu-3} + \frac{k^2}{2\mu\kappa^2 a_0^2} \eta^{2\mu-1} + \cdots \right],$$  \hfill (83)

where we use the notation

$$\bar{\eta} = \frac{\eta}{\eta_0} = -\frac{a_0\kappa\eta}{\mu - 3/2}.$$  \hfill (84)

and we normalize the time so that $\eta_k = 0$ when $\eta = 0$. The function $\Omega_k^2$ in Eq. (57) for $H^2 \ll 1$ is approximated to be

$$\Omega_k^2(\eta_k) \simeq k^2(1 - w_1 H^2_k - w_2 k^2) - \frac{2n-1}{n} H^2 a^2(\eta)(1 - w_3 h^2) + \delta,$$  \hfill (85)

where $\delta$ represents the smaller terms proportional to the differences of the Hubble parameter from its time-averaged values,

$$\delta = -w_1 k^2(\bar{H}^2 - H^2_k) + 2w_3 H^2 a^2(\bar{H}^2 - h^2).$$  \hfill (86)

$w_i$'s are given in Eq. (57) and the time-averaged values of the Hubble parameters, $H_k$ and $h$ are defined by

$$H^2_k = \frac{\int \frac{d\eta_k H^2(\eta)}{\kappa^2} d\eta_k}{\int d\eta_k}, \quad h^2 = \frac{\int \frac{d\eta_k a^2(\eta)H^4(\eta)}{\kappa^2} d\eta_k}{\int d\eta_k a^2(\eta)H^2(\eta)},$$

with the $\eta_k$ integrations performed over the validity range of the differential equation (57) for a given mode solution, which will be clarified in the next subsections. We do not put the index $k$ to $h$ since $h$ depends on $k$ very weakly.

**A. Mode generation and the initial condition**

As in the case of the $\kappa$–deformed exponentially inflating universe, due to the condition (67), the mode $\phi_k$ is generated at the conformal time $\eta(k)$,

$$\eta(k) = \eta_0 \left( \frac{k}{a_0\kappa} \right)^{-\frac{1}{2\mu - 3}},$$  \hfill (88)

where we assume the Universe is inflating with $n \gg 1$.

A serious obstacle in finding physics of low comoving momentum modes is that the action (48) is not well defined for large $\bar{H}^2$ since the action is approximated by expansion in $\bar{H}^2$. The condition $\bar{H}(\eta_m) \sim 1$ is attained at

$$\eta_m \sim \eta_0 n^{-\frac{1}{2\mu - 3}}.$$  \hfill (89)

Before $\eta_m$, our approximation for the action (48) is not valid. This condition restricts the validity range of the present approximation to the modes $\phi_k$ with comoving momentum

$$k > k_m \equiv n^{\frac{\mu+1}{2\mu - 3}} a_0\kappa.$$  \hfill (90)

In this subsection, we restrict ourselves to the ultra-violet modes satisfying the condition (90). To know the behavior of smaller frequency modes, one should use better approximation of the action (29) instead of (48). We present some reasonable arguments for the evolution of those low frequency modes in Sec. V.C.
For modes \( k > k_m \), all the arguments for the initial state in Sec. IV.A hold true. Therefore, the initial state is given by the WKB ground state,

\[
\varphi_k(\eta_k) = \frac{1}{\sqrt{2\Omega_k(\eta_k)}} \exp i \left( \int_{\eta_k^0}^{\eta_k} \Omega_k(\eta_k) d\eta_k + \psi_k \right), \quad k \geq k_m,
\]

where \( \eta_k \) is close to the mode generation time \( \eta_k^0 \) given by Eq. (68).

\[\bar{k}(t)\]

\[\eta_k^0 \quad \eta_k^1 \quad \eta_m \quad \eta_k^2 \]

\[\text{perturbatively unreliable} \quad \text{perturbatively reliable}\]

**FIG. 2:** Schematic plot for the mode generation time in power law inflation. Each curve describes the value of the normalized physical wave-number \( \bar{k}(t) = \frac{k}{a \kappa} \) for a given conformal time \( \eta \). Since \( a(\eta) \) increases for expanding universe, the value of \( \bar{k} \) always decreases. Due to Eq. (88) each mode becomes physical when it crosses the horizontal line \( \bar{k} = 1 \). We denote the unphysical part of each modes using the dashed curve. Since \( \bar{H} = 1 \) at \( \eta = \eta_m \), the region \( \eta > \eta_m \) is perturbatively reliable and the region \( \eta < \eta_m \) is perturbatively unreliable, which are divided by a shaded vertical line. The UV modes resides in the perturbatively reliable region. The IR and the Far Infra-Red (\( \frac{1}{a_0 \kappa} < 1 \)) modes pass through the perturbatively unreliable region.

**B. inflationary evolution**

The inflationary evolution of the mode solution is governed by \( \Omega_k^2 \). When \( k > k_m \), we approximate \( \Omega_k^2 \) in Eq. (89) using Eqs. (83) and (86), and dropping the term \( \delta \) in Eq. (85), as

\[
\Omega_k^2(\eta_k) \simeq \tilde{k}^2 - \frac{\nu^2 - 1/4}{\eta_k^2} - \frac{w_2 k^4}{k^2 a_0^2} \eta_k^{2\mu - 1} + \cdots.
\]

where

\[
\tilde{k}^2 \equiv k^2 \left[ 1 - (w_1 + \frac{2n - 1}{\mu n})H_k^2 \right], \quad \nu^2 - \frac{1}{4} \equiv \left( \mu^2 - \frac{1}{4} \right) \left[ 1 + \left( \frac{\alpha_n}{\mu - 1} - w_3 \right) H^2 \right].
\]

Eq. (92) looks similar to Eq. (64) except for the final term. With the \( \Omega_k^2(\eta_k) \) in Eq. (92), we have two different time scales defined by \( \Omega_k^2(\eta_k^0) = 0 \) and \( \Omega_k^2(\eta_k^1) = 0 \):

\[
\eta_k^0 = \eta_0 \left[ \frac{\nu^2 - 1/4}{w_2 (\mu - 1/2)(\mu - 3/2)^2 k^2} \right]^{1/(2\mu + 1)},
\]

\[
\eta_k^2 = \frac{\eta_0^2}{k^2} \left[ 1 + w_2 (\mu - 3/2) \left( \frac{\nu^2 - 1/4}{\mu - 3/2} \right)^{2\mu} \right]^{3-2\mu}.
\]
where \( \bar{k}_0 = k/(\kappa a_0) \). Since \( \bar{k}_0 \gg 1 \) for modes under consideration, we have
\[
\eta_k^0 \ll \eta_k^c \ll \eta_k^i.
\]
(95)

Since the condition for the WKB approximation, \( \Omega_k^2/\Omega_1^1 \ll 1 \), holds during \( \eta_k < \eta_k^c \), we may use the WKB solution (91). Therefore, we get the mode-dependent conformal time \( \eta_k^c \) given by Eq. (74) with
\[
\Omega_k^2 \equiv \Omega_k^2(\eta_k^c) \simeq \bar{k}^2 \left[ 1 - \frac{2\mu + 1}{2\mu - 1} \frac{\nu^2 - 1/4}{(k\eta_k^c)^2} \right],
\]
(96)

where the second term in the parenthesis is smaller than 1 for \( \bar{k}_0 \gg 1 \). For \( \eta_k > \eta_k^c \), we ignore the last term in Eq. (12) since \( \eta_k^c \) is very small there. The solution for the differential equation (44) is given by the Bessel function in Eq. (65) with parameters given in Eq. (80). With the initial condition (74) and \( \Omega_c \) in Eq. (80), \( A_k \) and \( B_k \) are given by Eq. (74).

Since we are considering modes with \( \bar{k} \gg 1 \), we always have \( |\bar{k}\eta_k^c| \gg 1 \). Using the asymptotic expansion of the Bessel functions,
\[
J_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \cos[x - (\nu + 1/2)\frac{\pi}{2}], \quad Y_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \sin[x - (\nu + 1/2)\frac{\pi}{2}],
\]
(97)

we get
\[
A_k = \frac{\sqrt{\pi}}{2} \frac{\Omega_c}{\bar{k}} e^{-i(\bar{k}\eta_k^c + (\nu+1/2)\pi/2)} = iB_k.
\]
(98)

Note that the absolute values of \( A_k \) and \( B_k \) are very weakly dependent on \( k \) since \( \Omega_c \sim \bar{k} \).

From this, we have the power spectrum,
\[
P_{UV,k}(t) = \frac{k^3}{2\pi^2} \frac{|\varphi_k(t)|^2}{z^2(t)} \simeq \frac{k^2}{M_P^2} \frac{n[2\nu(\nu-1)]^2(\mu - 3/2)^{2\nu - 1}}{16\pi^3} \left( \frac{k}{a(t)} \right)^{3-2\nu} \left( \frac{\eta_k}{\eta_0} \right)^{2(\mu - \nu)} + \cdots.
\]
(99)

Since we are interested in the time evolution for \( \eta_k^c < \eta_k < \eta_f \), we have the time-averaged Hubble parameters,
\[
H_k^2 \simeq \frac{n^2}{2(2-\mu)} \frac{(\bar{\eta}_f)^{3-2\mu} - (\bar{\eta}_k^c)^{3-2\mu}}{\bar{\eta}_f - \bar{\eta}_k^c} \simeq \frac{n^2}{2(2-\mu)} \frac{1}{(\bar{\eta}_k^c)^{2\mu-3}},
\]
(100)

\[
k^2 \simeq \frac{n^2}{2(2-\mu)} \frac{\bar{\eta}_f^{2\mu-4} - (\bar{\eta}_k^c)^{2\mu-4}}{\bar{\eta}_f^{-1} - (\bar{\eta}_k^c)^{-1}} \simeq \frac{n^2}{2(2-\mu)} (\bar{\eta}_f)^{2\mu-3}.
\]
(101)

The mode-dependent conformal time \( \eta_f \) at the end of the inflation is almost independent of the comoving momentum \( k \). Note that from Eq. (93),
\[
\nu \simeq \mu - c h^2, \quad c \equiv \frac{\mu^2 - 1/4}{2\mu} \left( w_3 - \frac{\alpha_n}{\mu - 1} \right),
\]
(102)

\[
3 - 2\nu \simeq 3 - 2\mu + 2 c h^2 = - \frac{2}{n - 1} + 2 c h^2,
\]

where \( c \) is a positive number for \( n > 1.53 \). \( 3 - 2\nu \) changes sign from positive to negative at \( \eta_k = \eta_{f,\text{lat}} \), where
\[
\eta_{f,\text{lat}} = \left[ \frac{(2-\mu)(2\mu - 3)}{cn^2} \right]^{\frac{1}{2\mu-3}} \eta_0.
\]
(103)

Since \( |3 - 2\nu| \ll 1 \) always, the slightly blue-shifted spectrum changes into the slightly red-shifted spectrum at \( \eta = \eta_{f,\text{lat}} \).

The power spectrum decreases in time since \( \mu - \nu \) is positive even though it is very small. For example, for \( n = 14 \), we have \( a(\eta_{f,\text{lat}})/a_0 \sim e^{41} \) and \( P_k \sim 0.075 \). Since the currently discussed minimal duration of inflation is roughly \( a/a_0 \sim e^{60} \), \( \eta_f \) must be later than \( \eta_{f,\text{lat}} \) and the size of the spectrum decreases in time. We will observe the decreasing red spectrum of the scalar density fluctuation for high comoving momentum \( k > k_m \).
If $n = 19$, on the other hand, we have $a(\eta_{\text{flat}})/a_0 \sim e^{\delta_0}$ and we will observe fully scale invariant power spectrum for the high comoving momentum modes for minimal inflation.

There are a couple of features in our result (59), which can be used to resolve the gravitational hierarchy problem, the extreme weakness of the gravitational wave (the tensor mode fluctuation) relative to the large scale density fluctuations. The first is the fact that the power spectrum decreases as $k$ increases for later time, which is also the case in the commutative space results. The second is the time-dependence of the power spectrum, which is a new feature of the present paper, that may describe the difference of the power spectra between the structure formation case in the commutative space results. The features may solve the gravitational hierarchy problem without imposing any fine tuning on $H$ or on the non-commutativity scale $\kappa$.

C. Speculation for IR modes

In the previous subsection, we deferred the discussion on the evolution of the modes of smaller frequencies than $k_m$. Due to the definition of $k_m$, (41), these modes always satisfy in its physical region,

$$\frac{k}{a(t)\kappa} \ll 1.$$  \hspace{1cm} (104)

The Far Infra-Red (FIR) modes are created at $\eta = \eta_k^0 < \eta_0$. For $\eta_k^0 < \eta < \eta_m$, the condition (104) can be interpreted as

$$k^2 \ll a^2 H^2, \quad \text{for } \eta < \eta_m,$$

(105)

since $\eta_m$ is the time for $H/\kappa \sim 1$.

To figure out the dynamics of these modes, we must have reasonable approximation on the action around $H^2/\kappa^2 \sim 1$. We assume that the action (45) is still valid even though the explicit functional forms of $m$ and $f$ are not known. We keep Eq. (10) since the equation comes from the requirements of quantization and unitarity of the time evolution. To show that the mode solution at time $\eta_m$ is almost independent of $k$, we use several steps of reasoning. First, we show that $m(k, \eta_k) < \kappa$ is very small until $\eta_k < \eta_m$. Next, we argue that $\Omega^2_k$ is also very small. Finally, we argue that $\Omega^2_k$ is almost independent of $k$ for this region of time. With these, we have the result that the matching condition of $\varphi_k(\eta_k)$ at $\eta_k \sim \eta_m$ is almost independent of $k$.

$m(k, \eta)$ is defined by the first order differential equation (46) with $m = 1$ at $k \tau \to \infty$. The value of $m$ increases in time asymptotically approaching to, $m \sim \bar{\mu}$ at $\eta = \infty$. Therefore, to get the behavior of $m$, we need to analyze its behavior around $m \sim 0$. In this case, the differential equation (46) becomes

$$\frac{\dot{m}}{m} \approx \sqrt{-\frac{3\kappa^2 \bar{\mu}}{\gamma}},$$

(106)

Thus, to have a consistent, real-numbered value of $m$, we must have $\bar{\mu}/\gamma < 0$ in this region and

$$m(k, \eta_k) \approx \left[ \Lambda - \int_{\eta_k^0}^{\eta_k} \sqrt{-\frac{3\kappa^2 \bar{\mu}^2}{\gamma}} \, d\eta_k \right]^{-1},$$

(107)

where the integration constant $\Lambda$ is a large number which ensures the value of $m$ to be small. The value of $m$ increases to $O(1)$ around $\bar{\mu}/\gamma \sim 0$ since for positive $\bar{\mu}$ we have $m \sim \bar{\mu}$ as a $0^{\text{th}}$ order approximation. Let $\eta_k^c$ be the value of $\eta_k$ satisfying $\bar{\mu}(\eta_k^c)/\gamma(\eta_k^c) = 0$. Then, we have $m \sim 1/\Lambda \ll 1$ for $\eta_k \ll \eta_k^c$ and $m \sim \bar{\mu}$ for $\eta_k > \eta_k^c$. Rough estimation on the value of $\eta_k^c$ is possible by using the asymptotic form of $m(k, \eta_k) \sim \bar{\mu}$. Since

$$\frac{k^2}{\kappa^2 a^2(t_k^c)} \approx 1 - \alpha_n \frac{H^2(t_k^c)}{\kappa^2} < 1,$$

(108)

where $t_k^c$ is the time corresponding to the mode-dependent conformal time $\eta_k^c$, and the mode $\phi_k$ is generated before $\eta_k^c$. Since $H(\eta_m) \sim \kappa$, we must have $\eta_k^c \sim \eta_m + \cdots$. Therefore, the zeroth order of $\eta_k^c$ is independent of $k$.

Next, we consider the function $f$ of Eq. (46) for $\eta_k \ll \eta_k^c$. From Eqs. (10), (109), and the change of variable $\tilde{f} = \frac{\sqrt{-\gamma \bar{\mu}}}{\kappa^2 am} f$, we get

$$\frac{3\omega}{\kappa^2} + \partial_\eta \tilde{f} \approx -\frac{\tilde{f}^2}{\gamma \bar{\mu}} > 0,$$

(109)
where we have ignored the term proportional to \( m \) since \( m \) is very small. For this differential equation to be well defined, \( \bar{f} \) must be a function of \( O(1) \). Thus, \( f = \frac{\kappa^2 am}{\sqrt{-\gamma \mu}} \bar{f} \) is of the same order as \( m \), and is very small. Therefore, the potential (\( \propto m f \propto 1/\Lambda^2 \)) is very small.

Finally, note that \( \Omega_k^2/(a^2 \kappa^2) \) is a function of \( k^2/(a^2 \kappa^2) \) and \( H^2/\kappa^2 \). Because of Eq. (103) for FIR modes, we guess that \( \Omega_k \) may be almost independent of \( k \) for \( \eta_k < \eta_k^c \). This implies that the initial mode solution created at \( \eta_k^c \) of the FIR modes are almost independent of \( k \).

Since \( m \) is small, small variation \( \delta \eta \) corresponds to a large variation in the mode dependent conformal time \( \delta \eta_k \sim 1/\Lambda \). Because of Eq. (105) for FIR modes, we have

\[
\frac{\delta \phi_k'}{\phi_k} \sim \Omega_k^2 \delta \eta_k \sim \frac{1}{\Lambda},
\]

which is small, the mode solution does not change much until \( \eta_k^c \) after the mode generation at \( \eta_k^c \). Then, the state at \( \eta_k = \eta_k^c \) is not much different from its ground state at \( \eta_k^c \).

Collecting all of the above arguments, we propose that the initial mode solutions, which are almost independent of \( k \), vary only by a small amount during \( \eta_0^k < \eta < \eta_k^c \sim \eta_m \). The evolution for \( \eta_k > \eta_k^c \) will be described by the same method as that of the UV modes with the matching condition at \( \eta_k^c \). The mode solution is given by Eq. (75) and its coefficients are given by Eq. (76). For these modes we have \(-k \eta_k^c \propto n = -k \eta_m \). In the case of a FIR modes, we have \(|k \eta_k^c| \ll 1 \). Then, we can use the form (77) of the Bessel function to determine \( B_k \):

\[
B_k \sim -\frac{\pi (\nu + 1/2)}{2^{\nu+3/2} \nu!} \sqrt{\frac{k}{\Omega_k(\eta_k^c)}} (-k \eta_k^c)^{\nu-1/2},
\]

where we have assumed \( \eta_k^c \Omega_k(\eta_k^c) \ll 1 \).

Thus the power spectrum for the FIR modes is given by

\[
P_{FIR}(k) \simeq \frac{n(\nu + 1/2)^2}{32 \pi^2 \nu^2} \frac{k^3}{a^2(\eta_m)M_p^2 \Omega_k(\eta_m)} \left( \frac{\eta_k}{\eta_k^c} \right)^{2(\nu-\nu)},
\]

since \( \eta_k^c \sim \eta_m \), we have strong (proportional to \( k^3 \)) blue spectrum for this FIR modes. Since \( \mu - \nu \) is a small positive number, the size of the spectrum slowly decreases in \( \eta_k \). It was argued that, in relation to the pre-big-bang scenario, this kinds of cutoff of power spectrum in the low frequency region can explain the low CMB quadrupole moments [13].

For the spectrum to be continuous, the red spectrum in the UV region must be continuously deformed to the blue spectrum in the FIR region. Therefore, the spectrum for \( a_0 \kappa < k < k_m \) must be continuously deformed from the weak red spectrum to flat and then to weak blue spectrum.
We have studied the effects of the $\kappa$-deformation of Robertson-Walker space on the evolution of metric fluctuations in expanding cosmological background. For a given noncommutative $\kappa$-deformed inflationary universe the cosmological background is still described by the Einstein equation since the background fields only depend on one variable $t$ so that the homogeneity and isotropy of the Robertson-Walker space are kept. The equation for linear fluctuations, however, are modified. We have shown that the modification takes the form of nonlocal interaction of the fluctuating field with itself and with the background. We have analyzed the system by perturbatively expanding the action up to the first order $H^2/\kappa^2$.

An important consequence of the space-time non-commutativity is that for each wave number $k$, there exists an earliest time $\eta^0_k$ at which the fluctuating mode is created. The origin of the mode generation phenomena is a direct consequence of the $\kappa$-deformation which introduces an upper bound of the comoving wave-number by Eqs. (12) and (27). We assume that the fluctuation starts out with its vacuum amplitude at $\eta^0_k$ since the number of excitations should be conserved during the creation process. Moreover, this condition restricts the physical frequency to be smaller than $\kappa$. This condition determines the initial condition of given modes for a given initial time. The deformation also generates the correction terms proportional to $H^2/\kappa^2$ and $k^2/(\kappa^2 a^2)$ to the frequency squared, which determines the time evolution of mode $k$.

There are two main results for the corrections to the power spectrum of the metric fluctuation due to the deformation. The first is that the deformation alters both the time dependence and the momentum dependence of the power spectrum. Especially, in the case of a power law inflation, we have shown that the power spectrum slowly decreases in time as $(\eta_k/\eta_0)^{2(\mu-\nu)}$, where $\mu - \nu \approx c\hbar^2$ is a very small but positive. This time dependence of the power spectrum is a new feature of the present approach in contrast to the result of the commutative space case and to the result of Ref. 10. The fact that the power spectrum decreases in time in addition to the existence of the red shifted spectrum, can be used to resolve the gravitational hierarchy problem for the tensor mode fluctuation. Another consequence of the deformation is that the momentum dependence of the power spectrum for UV modes is also dependent on time $k^{3-2\nu}$, where $\nu$ is given by Eq. (92). Note that $3 - 2\nu$ is a small positive number for $\eta_f < \eta_{\text{flat}}$ and is a small negative number for $\eta_f > \eta_{\text{flat}}$. Therefore, there is a period ($\eta < \eta_{\text{flat}}$) of blue spectrum in the earlier time of the inflation and the spectrum becomes red after the time $\eta_{\text{flat}}$, which is determined by $\nu$. Maximal red shift occurs to the power spectrum as $\eta_f \rightarrow \infty$ in which case $\mu = \nu$. Another interesting effect appears in the power spectrum of infra-red modes. Although we cannot obtain the dynamics of infra-red modes explicitly, we have suggested a form of the power spectrum from the consistency requirements. The power spectrum of the far infra-red modes ($k < a_0\kappa$), with $a_0$ the scale factor at $t = 1/\kappa$, have a cutoff proportional to $k^3$ even though the explicit procedure needs much refining since the dynamics at early times are not known. One can use the existence of this cutoff to explain the low CMB quadrupole moment. For the spectrum to be continuous, this $k^3$ type power may change as $k$ increases. We know that for ultra-violet mode ($k > k_m$, Eq. (89)) the spectrum becomes slightly red shifted for $\eta > \eta_{\text{flat}}$. Therefore, the spectrum may change from weak blue to flat spectrum for $a_0\kappa < k < k_m$.

Brandenberger and Ho 10 computed the effect of the stringy space-time uncertainty relation to the power spectrum of metric fluctuations for power law inflation in the Robertson-Walker space. It is interesting to compare their results with that of the present paper since we have started from a different commutation relation 24. We start from the basic commutation relation 24 and construct the theory from the first principle, although we have to use the perturbation to compute the physical effects of the deformation. Let us consider the power spectra for time $\eta_f > \eta_{\text{flat}}$. The spectra of Ref. 10 changes from weak red for ultra-violet modes to weak blue for infra-red modes. In our case, the far infra-red modes behaves as $k^3$. This is due to the fact that the far infra-red modes becomes almost independent of $k$ for the very early times $\eta < \eta_m$. Since the generation time for the ultra-violet modes is of similar form as in Ref. 10, the behavior of the spectrum in our case is similar to theirs for UV modes. The spectrum for $k > a_0\kappa$ changes from weak blue to weak red as $k$ increases. A totally new phenomena due to the $\kappa$-deformation, which is absent in the cases of Ref. 10 and the metric fluctuations in the commutative Robertson-Walker model, is the time-dependence of the spectrum. We have shown that both the spectra of the ultra-violet and infra-red modes decrease slowly in time.

We have used the perturbative approximation to obtain the generic feature of the physical effect of the $\kappa$-deformation on the cosmological evolution. It would be interesting if one could develop some nonperturbative approximation methods to extract better information from the nonlocal theory described by the action 24.
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