Quantum Modified Gravity at Low Energy in the Ricci Flow of Quantum Spacetime

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Quantum treatment of physical reference frame leads to the Ricci flow of quantum spacetime, which is a quite rigid framework to quantum and renormalization effect of gravity. The theory has a low characteristic energy scale described by a unique constant: the critical density of the universe. At low energy long distance (cosmic or galactic) scale, the theory modifies Einstein’s gravity which naturally gives rise to a cosmological constant as a counter term of the Ricci flow at leading order and an effective scale dependent Einstein-Hilbert action.

In the weak and static gravity limit, the framework gives rise to a transition trend away from Newtonian gravity and similar to the MOdified Newtonian Dynamics (MOND) around the characteristic scale. When local curvature is large, Newtonian gravity is recovered. When local curvature is low enough to be comparable with the asymptotic background curvature corresponding to the characteristic energy scale, the transition trend produces the baryonic Tully-Fisher relation. For intermediate general curvature around the background curvature, the interpolating Lagrangian function yields a similar transition trend to the observed radial acceleration relation of galaxies. When the baryonic matter density is much lower than the critical density at the outskirt of a galaxy, there may be a universal “acceleration floor” corresponding to the acceleration expansion of the universe, which differs from MOND at its deep-MOND limit.

The critical acceleration constant \( a_0 \) introduced in MOND is related to the low characteristic energy scale of the theory. The cosmological constant gives a universal leading order contribution to \( a_0 \) and the flow effect gives the next order scale dependent contribution, which equivalently induces the “cold dark matter” to the theory. \( a_0 \) is consistent with galaxian data when the “dark matter” is about 5 times the baryonic matter.

I. INTRODUCTION

A wealth of astronomical observations indicate the presence of missing masses or acceleration discrepancies in the universe based on the classical gravity theory (general relativity) although the theory is well tested within solar system very precisely. One possible approach to solve the problem is by separately introducing the missing masses components into the universe, for instance, the dark energy (DE or the cosmological constant \( \Lambda \) (CC) ) (Equation Of State \( w = -1 \)) and cold dark matter (CDM) \( (w \approx 0) \) in the so called \( \Lambda \)CDM-model. Another approach is by modifying the law of gravity, within which the problem should be more appropriately reconsidered as a gravity/acceleration discrepancy between the (cosmic or galactic) long distance scale and the (solar system or laboratory) short distance scale. There are some phenomenological supports for the latter approach, since both the acceleration expansion of universe (corresponding to the DE or \( \Lambda \)) and galactic rotation/acceleration anomalies (corresponding to the CDM) empirically manifest a particular acceleration scale \( a_0 \approx 1.2 \times 10^{-10} m/s^2 \approx \sqrt{\frac{\Lambda}{6 \cdot 8}} \), first proposed in the MOdified Newtonian Dynamics (MOND) by Milgrom \(^{[1]}\) (see reviews \(^{[2, 3]}\) and references therein, or long publication list of Milgrom’s). The baryonic Tully-Fisher law \(^{[4, 5]}\) and an amazing “mass discrepancy-acceleration relation” \(^{[6]}\) with little scatter are also observed, which do not occur naturally in the \( \Lambda \)CDM-model. Although the modified gravity approach might face its own difficulties (e.g. MOND without CDM is failed in fitting the third and subsequent acoustic peaks in the Cosmic Microwave Background (CMB)), this line of thinking might lead us to a more ambitious and unified view to our universe. The internal relation between the cosmological constant and MOND has been generally conjectured, and varieties of underpinning proposals and possible relativistic generalizations of MOND are suggested in literature, they are still more or less similar with the Kepler’s law as a phenomenological description, there is no first principle to determine the exact form of the interpolating function between the standard gravity limit and the modified one, thus lacking a fundamental underlying principle and theoretical framework remains its essential weakness.

Recent years the author based on the quantum treatment of physical reference frame, proposed a framework of quantum spacetime and gravity \(^{[7–14]}\). The basic idea of the theory is that when quantum theory is reformulated on the new foundation of relational quantum state (an entangled state) describing the “relation” between a state of a

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under-studied quantum system and a state of a quantum spacetime reference frame system, a gravitational theory is automatically contained in the quantum framework. Gravitational phenomenon is given by a relational quantum state describing a relative motion of the under-studied quantum system with respect to the material quantum reference frame system. And the 2nd order central moment of quantum fluctuations of the quantum reference system introduces the Ricci flow to the quantum spacetime,

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu},$$

where $g_{\mu\nu}$ and $R_{\mu\nu}$ are the metric and Ricci curvature of spacetime, and $t$ is the flow parameter.

The Ricci flow is historically invented independently from the physics and mathematics points of views. From the physics angle, the Ricci flow was introduced by Friedan [15,16] as a renormalization flow of a non-linear sigma model in $2+\epsilon$ dimension. From the mathematics angle, the Ricci flow was introduced first by Hamilton [17] as a useful tool to deform an initial Riemannian manifold into a more and more “simple” and “good” manifold whose topology is conserved and finally can be easily recognized in order to prove geometric theorems (like the Poincare conjecture). But certain singularity may develop during the Ricci flow and becomes the stumbling block of Hamilton’s program. Around 2003, Perelman introduced several monotonic functionals [18–20] to control the singularity during the Ricci flow. What Perelman treated is in fact a density manifold $(M^D, g, u)$ with density $u$ as a generalization of the Riemannian manifold $(M^D, g)$, in which the density $u$ describes a local density of the Riemannian manifold and physically coming from the quantum fluctuation or uncertainty at each point of the manifold. The Ricci-DeTurck flow

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2(R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\log u)$$

of the density manifold (equivalent to the Ricci flow [1] up to a diffeomorphism given by the gradient of the $u$ density) is shown to be the gradient flow of Perelman’s functionals, so that he could overcome the stumbling block by using his functionals and finally complete the Hamilton’s program.

In fact the underlying physics of Perelman’s formalism is not fully clear for physicist, the quantum spacetime reference frame picture is proposed by the author to lay the physical foundation. In the framework of the quantum spacetime reference frame, the spacetime is measured by physical rods and clocks as reference frame system and hence subject to quantum fluctuation. When the quantum fluctuation of the reference frame system is unimportant (mean fields approximation), the quantum framework recovers the standard textbook quantum theory without gravity. When the 2nd order central moment quantum fluctuation as the quantum correction of the reference frame system is important and be taken into account (Gaussian approximation), Ricci flow and gravity emerge in the quantum framework, as if one introduces gravitation into the standard textbook quantum mechanics. The physical reference frame modeled by the frame fields system is prepared and calibrated in a laboratory, which is mathematically described by a non-linear sigma model using lab’s spacetime dimension $4-\epsilon$ (a more practical example is a multi-wire chamber using the electrons as frame fields to measure the coordinates of events in a laboratory). An under-studied quantum system (e.g. the events) has physical meaning only with respect to the quantum reference frame system (the multi-wire chamber). The 2nd order central moments of quantum fluctuations of the reference frame fields blur the event and equivalently give quantum variance to the spacetime coordinates. The variance of the coordinates directly modifies the quadratic form metric of the Riemannian spacetime geometry, making the spacetime vary with the scale of the quantum fluctuation. Such scale dependent quantum correction to the metric continuously deforms the spacetime geometry driven by its Ricci curvature, which is exactly the Ricci flow: a renormalization flow of the spacetime. The $t$ parameter is related to the cutoff energy scale of the Riemannian spacetime components of the spacetime coordinates promoted to be quantum frame fields. As the Ricci flow starts from short distance scale (UV) $t \rightarrow -\infty$ and flows to long distance scale (IR) $t \rightarrow 0$, or from the astronomical viewpoint, the energies of spectral lines (as the tracers of astronomical observations) start from short distance laboratory scale and are redshifted to long distance galactic or cosmic scale. During the process, the spacetime coordinates and metric at a long distance scale $t$ are given by averaged out the shorter distance finer details which produces an effective correction to them. In a more intuitive picture, as the wave packet of the reference frame fields, such as the spectral lines, gradually Gaussian (2nd order) broaden when they travel a long distance, at long distance (e.g. cosmic or galactic) scale, the 2nd moment (i.e. the intrinsic spectral lines broadening) correction to the spacetime coordinate or metric becomes significant and hence can not be ignored. The 2nd order moment quantum fluctuation of spacetime gives rise to correction to the 2nd or quadratic order, thus quantities like curvature or acceleration as the second spacetime derivative obtain additional coarse-graining corrections in a natural and rigid way at long distance scale, which is considered as the root of the acceleration discrepancies in astronomical scale observations and the quantum modified gravity at low energy.

Further, in the framework, the 2nd order quantum corrections to gravity and acceleration are in a universal way, so that the correction is not merely the correction to specific spectral line itself but the correction to the spacetime. In
this sense the Equivalence Principle retains at the quantum level, which lays the physical foundation for the physical measurement of spacetime geometry and geometric description of gravity. The quantum description of the spacetime reference frame together with the quantum version of the Equivalence Principle leads to a completely different view on the behavior of quantum modified gravity: it is at the long distance scale where the quantum correction is significant. Beside the above intuitive “wave pocket broaden” picture of spacetime fuzziness at long distance scale, it is also reflected in the characteristic scale of the gravity theory described by the only input dimensional constant $\lambda$ of the quantum spacetime reference frame (in fact the only input constant for the $d = 4 - \epsilon$ non-linear sigma model). As we will see in the section-II-B that, to recover the standard Einstein gravity the constant must be exactly the critical density $\lambda = \frac{3H_0^2}{8\pi G} = \frac{1}{\Omega_{C,0}^{1/4}} \approx (10^{-5} eV)^4$. In contrast to the general believing of quantum gravity, the input constant is not the single Newton’s constant but the critical density $\lambda$ of the universe, as a combination of the Newton’s constant $G$ and Hubble’s constant $H_0$. As a consequence, the characteristic energy scale of the gravity theory is not the Planck scale, but the low critical density scale which is a long distance cosmic scale. As the $t$ parameter in the framework is a ratio of the cutoff energy scale $k^2$ of the frame fields over the critical density, $t = \frac{1}{\Omega_{C,0}^{1/4}} k^2$, when the energy scale of the frame fields is highly redshifted by the scale factor $a$, i.e. $k^2 \propto a^2 \rightarrow 0$, or $t \rightarrow 0^-$, it is at the low energy limit that gravity is strongly modified. In this paper, when we mention “scale $t$” (or later $\tau$) of an astronomical object, it can be understood physically as the the scale factor $t \propto -a^2$ or related redshift in the sense of the standard expanding universe picture.

An important feature of the framework is that the critical density $\lambda$ is the characteristic scale of the quantum gravity, as a consequence, the cosmological constant problem appearing in the naive quantum general relativity is more readily understood. Since the natural scale of the cosmological constant is no longer the Planck scale, which is $10^{120}$ times the observed value, but of order of the critical density $\lambda$. And the fraction in the critical density $\Omega_\Lambda = \frac{\lambda}{\Omega_{C,0}^{1/4}} \approx 0.7$ of order one is given by the counter term to the spacetime volume flow, which is related to a Ricci flow of the late epoch isotropic and homogeneous spacetime. Phenomenologically speaking, the Ricci flow and its counter term blurs the spacetime coordinate and equivalently universally broadens the spectral lines (as the universe expanding tracers). The broadening contributes a universal variance to the redshift, thus the redshift-distance relation is modified at second order in Taylor’s series expanding the distance in powers of the redshift, which gives rise to an equivalent accelerating expansion of the universe as the quantum version Equivalence Principle asserts $\Omega_\Lambda \approx 1$. Since the redshift variance (over the redshift mean squared) is independent to the specific energies of the spectral lines, so they are seen universally accelerating “free-falling” (in fact expanding), and the uniform acceleration now is not merely a specific property of the spectral lines, but measures and be interpreted as the universal property of the quantum spacetime.

When a distant earth observer measures the rotation velocities of spiral galaxies at galactic long distance scale, the mechanism works in a similar way. What the observer measures is not directly the rotation velocity of the spiral galaxy but its Doppler (red and blue) shifts induced broadening of the spectral lines (as the rotation tracers) with respect to the ones in laboratory (as the starting reference). As the galaxy is sufficiently redshifted at long distance scale, the spectral lines themselves are intrinsically quantum broadened, interpreted as the quantum variance or fluctuation of the ones in laboratory (as the starting reference). As the galaxy is sufficiently redshifted at long distance scale, the spectral lines themselves are intrinsically quantum broadened, interpreted as the quantum variance or fluctuation of the ones in laboratory (as the starting reference). 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In treating the quantum fluctuation correction effect to spacetime by the Ricci flow, it is useful to introduce an important and special solution of the Ricci flow (or more general the Ricci-DeTurck flow for a density manifold) which only shrinks the local size or volume of a manifold but its local shape unchanged, named the Ricci Soliton (or later the Ricci flow for a density manifold) [18]. Its Ricci curvature is proportional to the metric $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu}$ (or more general a gradient normalized Ricci curvature is proportional to the metric)

$$R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u = \frac{1}{2\tau} g_{\mu\nu}$$

where $\tau = t_\ast - t$ is a backwards Ricci flow parameter from a limit scale $t_\ast$ (see section-II-A for details). The Gradient Shrinking Ricci Soliton is a (temporary $t_\ast \neq 0$ or final $t_\ast = 0$) limit spacetime configuration or a (local or global) fixed point in the RG-sense, it (locally or globally) maximizes the Perelman’s monotonic functionals (at finite scale $t_\ast \neq 0$ or IR limit scale $t = 0$). In some simple cases, including for examples, the homogeneous and isotropic late epoch spacetime [10][11], the spatial inflationary early universe [13], and static thermal equilibrium black hole [13], and the topics concerned in the paper about the local galaxies, for all these limit (or nearly limit) spacetime, the Gradient Shrinking Ricci Soliton equation is more useful as simple examples.

Further more, at the fundamental level, the proposed new framework of quantum spacetime and related gravity seem to avoid several fundamental difficulties that other approaches to quantum gravity typically face. For examples, the renormalizability of the quantum gravity is the renormalizability of the $d = 4 - \epsilon$ non-linear sigma model, or at the Gaussian level correlates to the mathematical problem of the convergence of the Ricci flow. The problem is solved based on the works of Hamilton, Perelman and further developed by many other mathematicians, especially
after the discoveries of several monotonic functionals for the Ricci flow in general dimensions and the generalization of techniques to the non-compact and pseudo-Riemannian (Lorentzian) spacetime. The Hilbert space of quantum gravity correlates to the classification of spacetime geometries by using the Ricci flow approach, which is fully solved in 3-space, and can be fully understood in 4-spacetime by using the Ricci flow approach without fundamental obstacle. The unitarity of quantum gravity correlates to the problem of intrinsic diffeomorphism anomaly of the spacetime, which is given by the functional integral method for the quantum spacetime reference frame and found deeply related to the thermodynamic nature of the quantum spacetime. The local conformal stability of quantum spacetime correlates to the sign of the lowest eigenvalue related to the F-functional of Perelman, and the collapsibility of the quantum spacetime correlates to the finiteness of the W-functional of Perelman. The background independence of quantum gravity correlates to the initial (metric) condition independence of the Ricci flow, i.e. the Ricci flow and quantum fluctuations about all general initial background are on equal footings. The problem is considered fully solved by the Perelman’s formalism of the Ricci flow for general initial condition of manifold (not be restricted on some special initial condition). In the sense that the 2nd order quantum fluctuation of spacetime has been important at low energy, it also has a completely different view on the “graviton” w.r.t. the flat background. The “graviton” as the low energy excitation degrees of freedom of metric have been averaged out in the Ricci flow and effectively contribute to the general curved spacetime at certain scale, and hence it seems not to be a good signature of missing-energy in particle collision, unlike some high energy modified versions of quantum gravity.

As the quantum framework modifies the gravity at long distance scale in such a very tight and rigid way, the primary objective of this paper is to explore the weak and static gravity limit of the theory, and to determine if MOND can be derived from the theory, and if so, whether there is anything different or beyond MOND in the framework.

To avoid overlong pages, the general background of the quantum reference frame and the Ricci flow is given in the introduction section, in the next section, we skip the detail of the quantum reference frame and direct starting from the partition function derived from it, which can be found from the previous works. In the section II, we derived the low energy effective action of gravity from the partition function. And in the subsequent sections, several phenomenological consequences, e.g. the baryonic Tully-Fisher Relation in section III, the radial acceleration relation in section IV and “missing matter” in section V are discussed. Finally we discuss the relation between the theory and MOND and conclude the paper.

II. EFFECTIVE GRAVITY AT LOW ENERGY

A. Partition Function of Quantum Reference Frame and Pure Gravity

By using the quantum frame fields described by a \( d = 4 - \epsilon \) non-linear sigma model, in [11, 12] the author has derived the partition function of a pure gravity in terms of the relative Shannon entropy \( \tilde{N} \) of the spacetime 4-manifold \( M^d=4 \)

\[
Z(M^d) = e^{\lambda \tilde{N}(M^d) - \frac{D}{2} - \nu} = \frac{e^{\lambda \tilde{N}(M^d)} - \frac{D}{2} - \nu}{e^{\lambda \tilde{N}_*(M^d)}}
\]

where \( D = 4 \) and \( \lambda = \frac{3H^2}{8\pi G} \approx (10^{-3}\text{eV})^4 \) is the critical density of the universe. Up to a constant multiple, it is in fact the inverse of Perelman’s partition function [13] (the inverse is not physical important) he used to deduce his thermodynamics analogous functionals, although the underlying physical interpretation and relation to gravity is unclear in his seminal paper. The partition function is a proper starting point to pure gravity. Let us explain some quantities appearing in the partition function as follows.

1. \( N \) and \( \tilde{N}_* \) terms

The relative Shannon entropy \( \tilde{N}(M^4, t) \) is the Shannon entropy \( N(M^4, t) \) w.r.t. the extreme value \( \tilde{N}_*(M^4, t_* \) given by a Gradient Shrinking Ricci Soliton [3], as the Ricci flow limit \( t \to t_* \), i.e. \( \tilde{N} = N - N_* \). The quantum reference frame theory is described by a non-linear sigma model in \( d = 4 - \epsilon \), the trivialness of the homotopy group \( \pi_d(M^4) \) of the mapping of the non-linear sigma model makes the Ricci flow globally singularity free, simply giving \( t_* = 0 \), even though local singularities may developed at finite \( t_* \neq 0 \) during the Ricci flow for some general initial spacetime. The Shannon entropy is given by the \( u \) density

\[
N(M^4, t) = -\int_{M^4} d^4 X u \log u,
\]

in which

\[
u(X, t) = \frac{d^4 x}{d^4 X}
\]
is a volume ratio between a fiducial volume element $d^4x$ of the lab’s frame and the local spacetime (classical diffeomorphism) invariant volume element $d^4X \equiv dX^0dX^1dX^2dX^3\sqrt{|g|}$. $u$ is a dimensionless manifold density at each point $X$ of the Riemannian spacetime endowed with 2nd order moment quantum fluctuations, satisfying the density normalization condition

$$\lambda \int d^4X u = \lambda \int d^4x = 1. \quad (7)$$

from which one can see that the full volume of the spacetime or the fiducial lab’s frame is about the inverse of the critical density $O(1/\lambda)$, i.e. the volume of the universe, the characteristic scale of the theory. Without loss of generality, one can attribute the Ricci flow effects only to $\sqrt{|g|}$ and leaving $dX^0dX^1dX^2dX^3$ the fiducial volume, then $u$ is just the inverse of the Jacobian, $u = 1/\sqrt{|g|}$. By using the generalized Ricci-DeTurk flow (2), and finally we obtain the conjugate-heat equation of the $u$ density $\frac{\partial u}{\partial \tau} = (R - \Delta)u$, where $\Delta$ is the Laplace-Beltrami operator of the spacetime, $R$ the scalar curvature of the spacetime. The equation is often written in the form of a backwards flow along $d\tau = -dt$, i.e.

$$\frac{\partial u}{\partial \tau} = (\Delta - R)u, \quad (8)$$

so that the sign in front of $\Delta$ is just correct to have solution exist, as an analogous heat equation. The relative Shannon entropy $N$ part in the exponential comes from the diffeomorphism anomaly when one performs functional integration over the frame fields of the spacetime manifold [11], which also has profound thermodynamic interpretation of spacetime [13].

(2) $D/2$ term

The $D/2 = 2$ term arises from the classical action of a laboratory (walls and clock) frame given by a non-linear sigma model, which is considered classical, fiducial, rigid (volume-fixed), infinitely precise and hence the quantum fluctuations of the laboratory frame are ignored. We have

$$e^{-\frac{\bar{Q}}{2}} = \exp (-S_{cl}) = \exp \left(-\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\mu \bar{x}_\nu \partial_\mu \bar{x}_\nu \right) = \exp \left(-\frac{1}{2} \lambda \int d^4x g^{\mu\nu} g_{\mu\nu} \right) \quad (9)$$

where $x_\mu$ is the coordinates of the laboratory frame identified with the classical value of the frame fields, i.e. $x_\mu = \langle X_\mu \rangle$ ignoring the quantum fluctuation. The quantum effects are all attributed into the diffeomorphism anomaly term $\lambda N$, thus the effective action of the partition function can also be seen as an anomaly induced action of the quantum non-linear sigma model.

(3) $\nu$ term

When the Wick rotated spacetime $M^4$ is topologically equivalent to a 4-sphere $S^4$ which puts the space and time on an equal footing, the Ricci flow will gradually deform the initial $M^4$ into an isotropic and homogeneous $S^4$ at IR. In some cases, the flow process is singularity-free, for example $M^4$ has already been an isotropic and homogeneous spacetime with positive curvature (e.g. late epoch universe), the Ricci flow just shrinks its volume but deforms its shape. In other initial cases, the local irregularities of the spacetime are not large and hence irrelevant, the flow then smooths out them and flows the spacetime to the final IR isotropic and homogeneous $S^4$ spacetime. In some general cases, singularities might developed in some local places during the Ricci flow, then some local surgeries are needed to remove the singularities and then continues the flow, and finally it flows to several disconnected isotropic and homogeneous $S^4$ as well, and an observer living in one of the disconnected part of the universe, sees an isotropic and homogeneous $S^4$ universe. Anyway, if one starts from the final $S^4$ with a proper choice of a final density $u(t_\tau) = u_\tau$ (up to a gauge), and traces it backwardly by $\tau = t_\tau - t$, where $t_\tau$ is certain IR singular scale of the flow (singularity-free case simply gives $t_\tau = 0$) and $t$ is the flow parameter of the forwards Ricci flow, from IR $\tau = 0$ backwardly to UV $\tau \rightarrow \infty$ ($S^4$ allows the existence of the limit, i.e. $t \rightarrow -\infty$ called ancient solution), we get

$$\nu(S^4) \equiv \lambda \left[\tilde{N}(S^4, \tau \rightarrow \infty) - \tilde{N}(S^4, \tau = 0)\right] = \log \tilde{V}(S^4) \approx -0.8 \quad (10)$$

which is calculated from the log of the Reduced Volume or Relative Volume $\tilde{V}$ of Perelman [18–20], close to the observed $-\Omega_4$, $\nu$ is the finite difference of $\lambda \tilde{N}(M^4 \cong S^4)$ between UV ($\tau \rightarrow \infty$) and IR ($\tau = 0$) following the analogous H-theorem of the Ricci flow [18]. And because of the analogous irreversible H-theorem (parallel to the a-theorem [21] in 4d and c-theorem [22] in 2d in essential), the Ricci flow monotonically maximized the Shannon entropy $N$ to $N_\tau$ at IR, so $\tilde{N}_{\tau=0} = 0$ trivially. $\nu$ playing the role as a counter term and the fraction of the cosmological constant, ensures the diffeomorphism anomaly given by $\lambda \tilde{N}(M^4)$ can be completely canceled in the laboratory frame up to UV scale, leaving only the classical action $D/2$ of the laboratory frame, if the frame has been pre-assuming classical, fiducial and rigid.
B. Low Energy Expansion

At low energy IR scale $t \to 0^-$, or equivalently small $\tau \to 0^+$, the relative Shannon entropy can be expanded in powers of $\tau$

$$\tilde{N}(M^4, \tau) = \frac{d\tilde{N}}{d\tau} \bigg|_{\tau \to 0} \tau + \mathcal{O}(\tau^2) = \tau \tilde{F} \bigg|_{\tau \to 0} + \mathcal{O}(\tau^2)$$

(11)

in which $\tilde{N} = N - N_*$ and $\tilde{N}_{\tau=0} = 0$. By using the conjugate heat equation (8) and Ricci flow of the volume element $\frac{d}{d\tau} (d^4X) = R (d^4X)$, we have

$$\frac{dN}{d\tau} = -\frac{d}{d\tau} \int_{M^4} d^4X u \log u = \int_{M^4} d^4X u \left(R + |\nabla \log u|^2\right)$$

(12)

and the extreme value $N_*$ is given by a fundamental solution of the Maxwell-Boltzmann type

$$\lim_{\tau \to 0} u = u_* = \frac{1}{\lambda(4\pi\tau)^2} \exp\left(-\frac{1}{4\tau} |X - x|^2\right)$$

(13)

and hence

$$\frac{dN_*}{d\tau} = -\frac{d}{d\tau} \int_{M^4} d^4X u_* \log u_* = \frac{2}{\lambda\tau}$$

(14)

so finally

$$\tilde{F} = \int_{M^4} d^4X u \left(R + |\nabla \log u|^2 - \frac{2}{\tau}\right)$$

(15)

is the normalized F-functional [18] of Perelman $\tilde{F} = F - F_*$ w.r.t. the maximized value $F_* = \frac{2}{\pi\tau}$ in 4-spacetime.

At IR limit since $\lambda \int d^4X u_* |\nabla \log u_*|^2 = \frac{2}{\tau}$, so the low energy effective action of the partition function [4] can be written as

$$S_{eff} = -\log Z(M^4) = \lambda \int_{M^4} d^4X u_* \left[2 - \tau R_0 + \nu + \mathcal{O}(R^2\tau^2)\right], \quad (\tau \to 0^+).$$

(16)

From the Ricci flow equation (1) of the metric, it is straightforward to give the flow equation of the scalar curvature $\frac{\partial R}{\partial \tau} = -\Delta R - 2R_{\mu\nu}R^{\mu\nu}$. Since in the IR flow limit $\tau \to 0$, the Ricci curvature is homogeneous and isotropic $\Delta R_0 = 0$, and $R_{\mu\nu}(\tau \to 0) = R_0 g_{\mu\nu}$, so the equation becomes $\frac{\partial R}{\partial \tau} = -\frac{2}{\tau} R^2 = -\frac{1}{2} R^2$ in 4-spacetime, thus at small $\tau$ the solution is

$$R_\tau = \frac{R_0}{1 + \frac{2}{3} R_0 \tau},$$

(17)

where $R_0 = D(D-1)H_0^2 = 12H_0^2$. As a consequence the first two terms in (16) can be interpreted as a scale($t$ or $\tau$)-dependent Einstein-Hilbert-like action

$$2\lambda - \lambda R_\tau = \frac{R_\tau}{16\pi G},$$

(18)

in which $\lambda = \frac{3H_0^2}{8\pi G} = \frac{R_0}{2\pi^2 G}$ has been used. Therefore the 3rd term $\lambda \nu$ as the counter term of $\lambda \tilde{N}$ is naturally the cosmological constant in the unit of the critical density $\lambda$,

$$\lambda \nu = -\Omega_\Lambda \lambda = \frac{-2A}{16\pi G}.$$

(19)

The 4th and subsequent terms $\mathcal{O}(R^n \tau^n)$ give high energy corrections. In the IR limit, the fundamental solution of $u_*$ degenerates to a delta function, but considering that the spacetime at IR limit is homogeneous and isotropic, and the general solution of $u_*$ recovers a homogeneous density at IR limit, thus the integral measure $d^4X u_*$ recovers the classical invariant measure $d^4X \equiv dX^0 dX^1 dX^2 dX^3 \sqrt{|g|}$, the density manifold $(M^4, g, u)$ recovers the Riemannian manifold $(M^4, g)$. The introducing of matter to the quantum reference frame and pure gravity can be found in [11] [12].
Finally, including the Lagrangian density of matters $\mathcal{L}_M$, the effective action $[16]$ can be rewritten as the standard Einstein-Hilbert+Cosmological Constant (EH + CC) action,

$$S_{\text{eff}} = \int_{M^4} d^4x \left[ \frac{R_\tau - 2\Lambda}{16\pi G} + O(R^2\tau^2) + \mathcal{L}_M \right], \quad (\tau \to 0^+ ) \tag{20}$$

simply the scalar curvature and the matter Lagrangian becomes scale($t$ or $\tau$)-dependent.

Note that the standard EH + CC action has two input constants, the Newton’s constant $G$ and the cosmological constant (CC) $\Lambda$, while strictly speaking, the action $[16]$ has only one input constant, the critical density $\lambda = \frac{3H_0^2}{8\pi G}$ as a combination of the two, which is a low energy density compared with the Planck scale. The Planck scale instead is considered play no fundamental role in the theory. It is clearly that in an ancient solution configuration (limit $\tau \to \infty$ exists without local singularities during the Ricci flow) the energy scale could safely go beyond the Planck scale, the situation of the high energy modification is beyond the scope of the paper when the whole partition function $[14]$ is more appropriate to consider. At the low energy limit $[20]$, there are some crucial observations worth stressing.

(i) The characteristic energy scale of the theory is as low as the critical density and the cosmological constant, which gives a characteristic scale or size to the universe in contrast to the naive predictions of a quantum general relativity plus the cosmological constant (e.g. leading to the cosmological constant problem).

(ii) At low energy, $\tau = t_s - t \to 0$, ($t_s = 0$ is the cosmic far infrared limit), there is a non-flat IR asymptotic spacetime background $R_0 = 32\pi G \lambda$, even in the pure gravity theory where matter is not included. The cosmological constant, as the counter term of the Ricci flow, or equivalently, the constant asymptotic background curvature, is the leading quantum correction at low energy. This not only modifies the behavior of the asymptotic spacetime (i.e. the acceleration of its expansion) but also affects the behavior of some long-distance scale objects (i.e. the acceleration discrepancies in galaxies).

(iii) The next leading order quantum correction for the standard gravity at low energy is that the observed scalar curvature $R_\tau$ and metric in $[20]$ are scale($t$ or $\tau$)-dependent. Based on the Ricci flow of the scalar curvature $R$ and the metric $g$, we have

$$R_g = \frac{R_b}{1 - \frac{1}{2}R_b \Delta t} \quad \text{and} \quad g_{\mu\nu}(t_g) = g_{\mu\nu}(t_b) \left( 1 - \frac{1}{2}R_b \Delta t \right), \quad (\Delta t = t_g - t_b > 0) \tag{21}$$

where $-\infty < t_b < t_g \leq 0$. As a result, the gravity generated by curvature $R_b$ and metric $g_{\mu\nu}(t_b)$ is expected to be distinct from the gravity generated by $R_g$ and $g_{\mu\nu}(t_g)$. The former is predicted from the baryonic matter at the local scale $t_b$ (b for baryon), which is a short-distance fiducial lab scale used as a starting point for calibrating and extrapolating with the optical law used for optical measurement of baryon. The latter is at the galactic scale $t_g$ (g for galaxy), which is related to a relatively redshifted and quantum broadened long-distance scale and observed by a distant observer.

(iv) If we assume the validity of the classical Einstein’s equation, then the flow change of the scalar curvature $\Delta R = R_g - R_b$ at different scales would give rise to the missing matter density $\Delta \rho = \rho_g - \rho_b$ between these scales. In this sense, the “missing matter” is seen as unavoidable in the theory when one predicts the effective masses from the gravity coupled to them in different scales. However, since the classical Einstein’s equation is modified at long distances and is no longer exact, the “missing matter” does not actually exist; they are merely an illusion created by the Ricci flow of the curvature.

C. Weak and Static Gravity Approximation

In this subsection, first we derive a relativistic but weak fields approximate action in the sense that the scalar curvature $R_\tau$ is low enough and comparable with the asymptotic background curvature $R_0$ corresponding to the characteristic scale $\lambda$, which describes how the galactic scale curvature $R_g$ is correlated to the local scale baryonic matter Lagrangian density $\mathcal{L}_M$. And then we study the static or non-relativistic approximation and compare it with MOND.

Dropping the higher order term and considering $\lambda = \frac{R_0}{32\pi G}$, the effective action $[20]$ can be rewritten in terms of the dimensionless ratio $R_\tau / R_0 = R_g / R_0$, in the unit of $2\lambda$,

$$S_{\text{eff}} = \int_{M^4} d^4x \left[ 2\lambda \left( \frac{R_g}{R_0} - \frac{\Omega_\Lambda}{2} \right) + \mathcal{L}_M \right] = \int_{M^4} d^4x \left[ 2\lambda \frac{R_g}{R_0} \left( 1 - \frac{\Omega_\Lambda}{2} \frac{R_0}{R_g} \right) + \mathcal{L}_M \right]. \tag{22}$$

In the situation that gravity is strong, $R_g \gg R_0$, for example, gravity well within the luminary region of galaxies or in the solar system, the action can ignore the cosmological constant term $\frac{\Omega_\Lambda}{2}$ and recovers the well-tested standard Einstein’s gravity, and hence the static Newtonian gravity.
However, at long distance scales, when the asymptotic gravity is weak, i.e. the flow-averaged curvature \( R_g \) is low and comparable to the asymptotic background curvature \( R_0 \) at the coarse-graining level, ratio \( \frac{R_0}{R_g} \approx O(0.35) \) is still a small number less than 1, we could go from (22) to

\[
S_{\text{eff}} \approx \int_{M^4} d^4X \left[ \frac{1}{2} R_g + \frac{1}{1 + \Omega R_0/R_g} L_M \right] = \int_{M^4} d^4X \left[ \frac{1}{16\pi G} \frac{R_g}{\sqrt{1 + \Omega R_0/R_g}} + L_M \right], \quad (R_g \approx O(R_0))
\]  

under the approximation condition

\[
R_\tau = R_g \approx O(R_0).
\]  

The purpose of the approximation is to incorporate the global effects of the cosmological constant or background curvature into a local gravitational system, which is more suitable as a starting action to study the modified law of local gravity. In the action, the matter term is still the dominant factor in the effective curvature, as if the cosmological constant were absent in the usual consideration of local gravitational binding systems. However, the effect of the cosmological constant has also become increasingly important at the outskirts of the binding systems, where the curvature becomes low, as indicated by its absorption into the effective scalar curvature under (24). In this approximation, the contribution of the cosmological constant just effectively gives an extra factor \((1 + \Omega R_0/R_g)^{-1/2}\) to the scalar curvature.

Indeed, there are many other functions that can be used to approximate the action (22). The form of the approximation (23) we choose is to take MOND as a phenomenological reference. In fact, there are also other allowed forms of interpolation functions in MOND that work about equally well phenomenologically. However, the accuracy of current cosmic measurements is not good enough to uniquely fit or to distinguish them very well phenomenologically. Nevertheless, if the theory could (at leading order) give one of the allowed interpolating functions of MOND, then we could say that the theory could qualitatively give a similar transition trend of MOND at least at the leading order within the allowed range of current observational accuracy. The form of the approximation (23) is the most direct and closest form in all possible and allowed interpolating functions of MOND appearing in literature that can be derived from the action (22) at the leading order. If one chooses other forms and plots them by curves "at leading order", they will work about equally well to fit data.

At this point, we further consider the weak \( \Phi \ll 1 \) and static \( \dot{\Phi} \approx 0 \) gravity approximation

\[
g^{00} \approx -(1 - 2\dot{\Phi}), \quad g^{ij} \approx (1 + 2\dot{\Phi})\delta_{ij}.
\]  

where \( \Phi \) without subscript is the Newtonian potential in weak and static approximation from the metric in general scale. The potential taking a subscript e.g. \( \Phi_g \) is the observed potential at the specific galactic scale, \( \Phi_b \) at the baryonic scale. Roman letters \( i, j \) are used for spatial indices. In the limit, scalar curvature \( R \) includes \( \Delta \Phi \), and second order ones such as \((\nabla \Phi)^2\) and \(\Phi \Delta \Phi\). To derive the fields equation, at the linear level, the terms \(\Delta \Phi\) becomes immaterial in the action, as a complete derivative, and hence we are left with terms \((\nabla \Phi)^2\) (note that \(\Phi \Delta \Phi\) terms is also \((\nabla \Phi)^2\) up to a derivative). At linear level \(R_g\) can be replaced by \(2(\nabla \Phi)^2\) without significantly change the classical fields equation of Newtonian potential we concern in the weak and static gravity limit.

In contrast to the fact that the gravity part of the Lagrangian as geometric quantity varies with the Ricci flow, the matter part \(L_M\) is assumed not, so we write down local \(L_M\) at the conventional fixed baryonic scale. The proper stress tensor, which does not contain metric and using the covariant index, is given by

\[
T_{00} = \rho_M, \quad T_{ij} \approx 0,
\]  

in which we have considered the proper velocity and pressure of the baryonic matters in galaxy are low. The proper stress tensor is coupled to the local metric \(g^{00}(t_b)\) or gravity potential \(\Phi_b\) at the baryonic scale. Thus it is the conventional baryonic scale \(L_M\) inserting into the total effective action

\[
L_M \approx \frac{1}{2} g^{00}_{\text{eq}} T_{00} = -\frac{1}{2} (1 - 2\Phi_b) \rho_M = -\frac{1}{2} (1 - 2\Phi_g) \left(1 - \frac{1}{2} R_b \Delta \tau\right) \rho_M
\]  

in which the local comoving metric \(g^{00}_b\) has been transformed to the galactic scale metric \(g^{00}_g\) observed by a distant observer via the Ricci flow

\[
g^{00}(t_g) = \frac{g^{00}(t_b)}{1 - \frac{1}{2} R_b \Delta \tau}.
\]
Finally we have the action for the potential observed at the galaxy scale coming from a local baryonic density

\[ S_{\text{eff}} = \int d^4X \left[ \frac{|\nabla \Phi_{\text{g}}|^2}{8\pi G} \frac{1}{1 + \Omega_\Lambda \frac{\rho M}{|\nabla \Phi_{\text{g}}|^2}} - \frac{1}{2} (1 - 2\Phi_{\text{g}}) \left( 1 - \frac{1}{2} R_0 \Delta t \right) \rho_M \right]. \]  

(29)

Compared with the action of the standard Newtonian gravity (Poisson action), there are two extra factors appear. First is the factor \( \left( 1 + \Omega_\Lambda \frac{\rho M}{|\nabla \Phi_{\text{g}}|^2} \right)^{-1/2} \) modifying the quadratic term \( |\nabla \Phi_{\text{g}}|^2 \), which will give a transition trend similar with MOND. The second is the factor \( 1 - \frac{1}{2} R_0 \Delta t \) modifying the Newtonian potential coupled to the baryonic density source, which will play the role of a scale dependent “missing matter”. In the Newtonian limit when these two extra factors are trivially unity, the effective action can recover back to obtain the Poisson equation of Newtonian potential \( \Phi_N \approx \Phi_{\text{g}} \approx \Phi_b \)

\[ \Delta \Phi_N = 4\pi G \rho_M \]  

(30)

when the scale difference is small \( \Delta t \to 0 \) and the gravity is much stronger than the background one \( |\nabla \Phi_N|^2 \gg H_0^2 \).

Further, there is a transition trend away from Newtonian gravity in the low curvature region \( R_g \approx O(R_0) \), where the gravity appears to be strongly modified as the deep-MOND limit. More precisely, the limit requires \( R_g \ll R_0 \) which is beyond the approximation \( R_g \approx 2|\nabla \Phi_{\text{g}}|^2 + \Delta \Phi_{\text{g}} \).

\[ \Delta \Phi_{\text{g}} \ll |\nabla \Phi_{\text{g}}|^2 \ll 6\Omega_\Lambda H_0^2 = 2\Lambda \]  

(31)

but we have an extrapolating deep-MOND action from \( S_{\text{eff}} \)

\[ S_{\text{eff}} \to \int d^4X \left[ \frac{1}{8\pi G \sqrt{2\Lambda}} |\nabla \Phi_{\text{g}}|^3 - \frac{1}{2} (1 - 2\Phi_{\text{g}}) \left( 1 - \frac{1}{2} R_0 \Delta t \right) \rho_M \right]. \]  

(32)

Since little matter locate far from the galaxy center, so in \( |\nabla \Phi_{\text{g}}|^2 \gg \Delta \Phi_{\text{g}} \approx 0 \) is used and \( |\nabla \Phi_{\text{g}}|^2 \) dominating the curvature \( R_g \approx 2|\nabla \Phi_{\text{g}}|^2 + \Delta \Phi_{\text{g}} \).

However, it is worth stressing that although \( a_N \sim |\nabla \Phi_N| \) could be very small, the condition of the deep-MOND action \( \frac{|\nabla \Phi_{\text{g}}|^2}{a^2} \ll 2\Lambda \) is unreachable at the coarse-graining level, since in the limit action, the approximation condition \( \frac{|\nabla \Phi_{\text{g}}|^2}{a^2} \ll 2\Lambda \) and action \( \frac{|\nabla \Phi_{\text{g}}|^2}{a^2} \ll 2\Lambda \) fail globally. The action \( S_{\text{eff}} \) shows a transition trend at low curvature region \( R_g \approx O(R_0) \) towards the extrapolating limit \( S_{\text{eff}} \).

To investigate the region when the baryonic matter density \( \rho_M \) is much lower than the critical density \( \lambda \), or equivalently the limit \( a_N \to 0 \). Comparing the Poisson action and the standard EH + CC action \( S_{\text{eff}} \), if we effectively consider \( \frac{1}{2} (R_g - 2\Lambda) \sim |\nabla \Phi_N|^2 \sim a_N^2 \), then as \( a_N \to 0 \) it leads to a universal acceleration lower bound \( R_g \sim 2\Lambda \sim O(R_0) \). If the observed acceleration is related to the curvature \( a^2 \sim |\nabla \Phi_{\text{g}}|^2 = \frac{1}{2} R_g \), it leads to a minimal observed acceleration \( a \sim a_{\text{min}} \sim \sqrt{\Lambda} \) corresponding to the universal cosmic expanding acceleration. Here “\( \sim \)” means up to certain correction factor. The existence of the minimal acceleration might occur when crossing over from a local galaxy binding gravitational system to the cosmological expanding system. It is natural to imagine, since the rotational acceleration of a galaxy is measured by the Doppler broadening of the (e.g. 21cm) spectral lines at the outskirts of the galaxy, so when the acceleration is low enough, the line broadening will be finally dominated by the universal quantum broadening due to the Ricci flow of spacetime so that the acceleration seems to achieve a universal value related to the acceleration expansion of the spacetime. In other equivalent words, when the binding gravity is so low at the outskirts of the galaxy that the satellites of the galaxy will escape from its binding and experience the universal cosmic expanding acceleration given by the cosmological constant.

The possible existence of a minimal acceleration differs from MOND at the deep-MOND limit when \( \rho_M \ll \lambda \) or \( a_N \ll \sqrt{\Lambda} \), which predicts that the rotation acceleration of galaxy can really reach a low value in the deep-MOND limit since the cosmological constant and the background acceleration expansion of the spacetime are in fact not taken into account in MOND. There are certain observational hints \( S_{\text{eff}} \) for such a coarse-grained “acceleration floor” (see Figure 1 citing from \( S_{\text{eff}} \)), especially for those ultrafaint dSphs having some tension with the prediction of MOND. The acceleration floor is fitted to be about \( (9.2 \pm 0.2) \times 10^{-12} \text{m/s}^2 \) which is lower than the order of the cosmological constant \( O(\Lambda) \). There are also possible explanations to weaken such tension, for instance, the transforming from the cosmological constant \( \sqrt{\Lambda} \) to an acceleration \( a \) is not simply equal. Some nontrivial extra factors due to the Ricci flow might contribute, a reasonable renormalization factor is \( 1 - \frac{1}{2} R_0 \Delta t \), in which \( 1 - \frac{1}{2} R_0 \Delta t \approx (0.13 \sim 0.18) \) (see later) is for the renormalization of the coupling between matter and potential/acceleration, so the minimal acceleration might be suppressed to \( a_{\text{min}} \sim (1 - \frac{1}{2} R_0 \Delta t) \sqrt{\Lambda} \sim a_0 \sim O(10^{-10} \text{m/s}^2) \). To fit the observed order \( O(10^{-11} \text{m/s}^2) \), further correction is needed, for example, the renormalization of curvature. Actually if the acceleration is able to reach the
which has a correct order but several times larger than the observed value. The interpolating Lagrangian can be

defined as

\[ L_{gra} = \frac{2\Lambda}{8\pi G} \frac{\nabla \Phi_g^2}{2\Lambda} \frac{1}{\sqrt{1 + \frac{2\Lambda}{|\nabla \Phi_g|^2}}} = \frac{\Lambda}{4\pi G} F(x), \quad \left( x = \frac{|\nabla \Phi_g|^2}{2\Lambda} \right), \tag{33} \]

where \( \Lambda = \Omega \frac{3H_0^2}{8\pi G} \) is the cosmological constant and

\[ F(x) = \frac{x}{\sqrt{1 + x^{-1}}} = \begin{cases} x & x \gg 1, \quad \text{(Newtonian Limit)} \\ x^{3/2} & x \ll 1, \quad \text{(Unreachable Asymptotic deep-MOND Limit)} \end{cases} \tag{34} \]

is an interpolating Lagrangian function.

To see its relation to the MOND, the curvature ratio \( R/R_0 \) here is replaced by the acceleration ratio \( a/a_0 \) in MOND, if we take \( a = |\nabla \Phi_g| \) and

\[ a_0 = \frac{2}{3} \sqrt{2\Lambda}, \tag{35} \]

which has a correct order but several times larger than the observed value. The interpolating Lagrangian can be

rewritten as an interpolating Lagrangian similar with a non-relativistic generalization of MOND [23]

\[ L_{gra} = \frac{a_0^2}{4\pi G} \frac{\nabla \Phi_g^2}{a_0^2} \frac{1}{\sqrt{1 + \frac{2}{3} \frac{a_0^2}{|\nabla \Phi_g|^2}}} = \frac{a_0^2}{4\pi G} f \left( \frac{|\nabla \Phi_g|^2}{a_0^2} \right) \tag{36} \]

where

\[ f(y) = \frac{y}{\sqrt{1 + \frac{9}{4} y^{-1}}} = \begin{cases} y & y \gg 1, \quad \text{(Newtonian Limit)} \\ \frac{y^{3/2}}{2} & y \ll 1, \quad \text{(Unreachable Asymptotic deep MOND Limit)} \end{cases} \quad \left( y = \frac{a^2}{a_0^2} \right). \tag{37} \]

So at leading order, the theory predicts a critical acceleration \( a_0 \) having a correct order \( O(\sqrt{\Lambda}) \). The next leading order corrections to \( a_0 \) is considered in the baryonic Tully-Fisher relation in the next section.
The baryonic Tully-Fisher relation \( v^4 = GMa_0 \) is an empirical tight relationship between the baryonic mass \( M \) (including the visible and invisible baryonic mass in gas) and its asymptotic flat rotation velocity \( v \) of disk galaxies, in which \( G \) is the Newton’s constant and \( a_0 \approx 1.2 \times 10^{-10} \text{m/s}^2 \approx \sqrt{\frac{\Lambda}{(10^{37})}} \) is a nearly universal constant interpreted as a critical acceleration in MOND. The baryonic Tully-Fisher relation, especially the 4th power of the rotation velocity can not be naturally explained in a cold dark matter model, in which a naive expectation is about 3rd power.

Similar with MOND, as long as the deep-MOND limit can be asymptotic approached (not necessarily be reached), the Tully-Fisher relation can be produced from the MOND-like transition trend \((29)\) and its asymptotically approached limit \((32)\), the derivation of the relation is basically similar with MOND with a little correction from the Ricci flow.

A simple way to derive the baryonic Tully-Fisher relation is to take the analogy of deriving a non-relativistic dispersion relation from a relativistic one. Comparing the Poisson action and the Einstein-Hilbert + CC action, we can see that the effective baryonic (Newtonian) acceleration is \( a_{N} \sim \frac{a^2}{\sqrt{2\Lambda}} \), and if the observed acceleration is simply taken as \( a^2 \sim |\nabla \Phi_g|^2 = \frac{1}{2} R_g^2 \), then

\[
a_N \sim \sqrt{\frac{1}{2} R_g - \Lambda} \sim \sqrt{a^2 - \Lambda} \quad (38)
\]

The relation is in analogy with the relativistic dispersion relation \( E = \sqrt{p^2 + m^2} \), where we have the analogies \( \Lambda \sim -m^2 \sim a_0^2 \), the observed acceleration \( a^2 \sim p^2 \), and baryonic Newtonian acceleration \( a_N \sim E \). In a non-relativistic, \( p^2 \lesssim m^2 \), we obtain a non-relativistic trend \( E \sim \frac{p^2}{2m} \) up to a rest mass constant. The low matter density limit \( \rho_M \ll \lambda \) is in analogy with the non-relativistic limit, (but note the minus sign in front of \( \Lambda \), so \( a^2 \) could not actually lower than \( \Lambda \), otherwise \( a^2 - \Lambda \) in the squared root would become negative), but when \( a^2 \rightarrow O(\Lambda) \) or \( R_g \approx O(R_0) \), the transition trend of the magnitude of the acceleration is given by the MOND dynamics

\[
|a_N| \sim \frac{a^2}{2\sqrt{\Lambda}}. \quad (a^2 \approx O(\Lambda))
\]

up to a background acceleration constant. If one considers the limit rotational velocity \( v \) of a galaxy at radius \( r \), i.e. \( a = \frac{v^2}{r} \), and Newtonian acceleration at the same radius, \( a_N = \frac{GM}{r^2} \), then the transition trend \( |a_N| \sim \frac{a^2}{2\sqrt{\Lambda}} \) gives rise to the baryonic Tully-Fisher relation \( v^4 \sim 2GM\sqrt{\Lambda} \sim GMa_0 \) although the deep-MOND limit \( a \ll O(\Lambda) \) is unreachable.

If the deep-MOND region is really unreachable as \( \rho_M \rightarrow 0 \) or \( a_N \rightarrow 0 \), there is a maximal radius \( r_{max} \) corresponding to the minimal acceleration floor \( a_{min} \sim O(a_0) \sim O(\sqrt{\Lambda}) \), i.e. \( \frac{GM}{r_{max}^2} \sim a_{min} \), when \( r \gtrsim r_{max} \) the outskirt satellites finally escape from the binding of the local galaxy and experience the background acceleration expansion. Given the rotating platform velocity \( v(r \lesssim r_{max}) \) of the satellites, relation \( v^2 = a_{min} \cdot r_{max} \) also gives the baryonic Tully-Fisher relation \( v^2 \sim \sqrt{GMa_{min}} \sim \sqrt{GMa_0} \).

The above hand waving derivations suggest a simple picture that around (rather than deep below) \( O(\sqrt{\Lambda}) \), the asymptotic behavior of acceleration discrepancy in MOND and the Tully-Fisher relation might qualitatively come from the local effect of the cosmological constant.

To find the precise value of \( a_0 \), we could derive the baryonic Tully-Fisher relation in another more precise approach, in which the normalization of the coupling between matter and the potential can be taken into account. We consider the Euler-Lagrangian equation \( \nabla_i \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_g} - \frac{\partial \mathcal{L}}{\partial \Phi_g} = 0 \) of the limit Lagrangian \((32)\)

\[
\frac{3}{8\pi G\sqrt{2\Lambda}} \nabla \cdot \left( \nabla \Phi_g | \nabla \Phi_g \right) - \left( 1 - \frac{1}{2} R_b \Delta t \right) \rho_M = 0. \quad (40)
\]

By using the Poisson equation \( \Delta \Phi_N = 4\pi G \rho_M \), the baryonic matter density can be replaced by the Newtonian potential

\[
\frac{3}{2\sqrt{2\Lambda} \left( 1 - \frac{1}{2} R_b \Delta t \right)} \nabla \cdot \left( \nabla \Phi_g | \nabla \Phi_g \right) = \nabla \cdot \nabla \Phi_N. \quad (41)
\]

Naturally to assume that the gravity potential \( \Phi_g \) and \( \Phi_N \) are spherical symmetric and curl-less, so

\[
\frac{3}{2\sqrt{2\Lambda} \left( 1 - \frac{1}{2} R_b \Delta t \right)} |\nabla \Phi_g| |\nabla \Phi_g| = \nabla \Phi_N = |\nabla \Phi_N| \epsilon_{r}, \quad (42)
\]
where $\vec{e}_r$ is a unit vector in the radial direction. Therefore

$$\frac{3}{2\sqrt{2\Lambda}} |\nabla_r \Phi_g|^2 = |\nabla_r \Phi_N| = \frac{GM}{r^2}, \quad (43)$$

where $M$ is the total mass of the baryonic matters, simply assuming that baryonic matters are centrosymmetric distributed (it is true in most galaxies with a little distribution correction). Considering e.g. spiral galaxies, a star at the outskirt of the galaxy, rotating with an observed rotating platform velocity $v$, radius $r$ and rotation radial acceleration $a_r = \frac{v^2}{r} = -\nabla_r \Phi_g$, so we have

$$a_r^2 = \left(\frac{v^2}{r}\right)^2 = \frac{2}{3} \sqrt{2\Lambda} \left(1 - \frac{1}{2} R_b \Delta t\right) \frac{GM}{r^2} \quad (44)$$

in which the correction factor $1 - \frac{1}{2} R_b \Delta t$ can be considered as a renormalization to the coupling between matter and potential or acceleration $a_N = \frac{GM}{r^2}$. So

$$v^4 = \frac{2}{3} \sqrt{2\Lambda} \left(1 - \frac{1}{2} R_b \Delta t\right) GM, \quad (45)$$

which is just the baryonic Tully-Fisher relation with a critical acceleration

$$a_0 = \frac{2}{3} \sqrt{2\Lambda} \left(1 - \frac{1}{2} R_b \Delta t\right), \quad (46)$$

The predicted critical acceleration is the universal part $\left(\frac{2}{3}\right)$ plus a scale dependent correction term. Although there are many possible corrections to the universal part $\left(\frac{2}{3}\right)$, such as the mass distribution (not point mass) correction in galaxies, the scale dependent part $R_b \Delta t$ is also a possible correction rigidly predicted in the Ricci flow of spacetime which may be important amount other contributions. To fit the observed value $a_0 \approx \frac{\sqrt{2\Lambda}}{10^6 \cdot 0.87}$ for spiral galaxies at the galactic scale $t_g$, w.r.t. $t_b$, if we attempt to attribute all the discrepancy to the scale correction, it equivalent to require a rough value

$$\frac{1}{2} R_b \Delta t = \frac{1}{2} R_b (t_g - t_b) \approx (0.82 \sim 0.87). \quad (47)$$

As is shown in eq. (21), the correction factor $1 - \frac{1}{2} R_b \Delta t \approx (0.13 \sim 0.18)$ also renormalizes the scalar curvature $R_g$, in this case, the action (22) differs from the standard EH action in $R_b$ by about $\frac{R_g - R_b}{R_b} \approx (4.6 \sim 6.7)$ times. Note that the approximation in the action (23) deviates from (22) by about 10%. In this sense, the approximated action (23) is closer to MOND than the standard Einstein-Hilbert theory.

The correction term is seen scalar curvature $R_b$ and scale difference $\Delta t$ dependent, but here the combination of the two roughly takes a constant value and varies little with $\Delta t$. There indeed exists a special reason for the case. Since the Ricci flow is highly non-linear, the spacetime under the Ricci flow not always be smoothed out as the linear heat equation. During the flow, some spacetime regions with high curvature might develop local singularities at certain finite singular scale $t_s \neq 0$ (although the global singularity could avoid by the topological consideration in the dimension of the base space of the quantum reference frame). Galaxies as relative high curvature local regions in the universe might play such role, meaning that $t_g$ might be close to a finite singular scale $t_g \approx t_s$. Near the singular scale, the galaxy region roughly resembles a local Shrinking Ricci soliton-like spacetime configuration (3), for which the combination of $R$ and $\tau = t_s - t_b$ is just about a constant of order one $\left(18\right)$, and so $R_b \Delta t \sim O(1)$. As the flow limit configuration, the Shrinking Ricci soliton flow equation (6) only deforms its local volume but local shape so it is a self-similar configuration, and it is for this reason the combination $R_b \Delta t$ could weakly depend on its scale. According to the explanation, the nearly constant value of the correction term $R_b \Delta t$ might imply that galaxy (and surrounding spacetime configuration) is close to and resembles a Shrinking Ricci Soliton configuration, and hence $a_0$ is roughly universal for different galaxies. In fact whether $a_0$ is universal for all galaxies is still in controversy $\left(25\right)$, $\left(26\right)$. To our knowledge, some obscure evidences show $a_0$ might correlate with the central surface brightness $\left(27\right)$, if so, it might be roughly understood as a possible correlation to the curvature, while there is still little evidence $\left(28\right)$ for a redshift evolution of $a_0$.

**IV. THE RADIAL ACCELERATION DISCREPANCY OF GALAXIES**

The baryonic Tully-Fisher relation is a simple scaling relation with no apparent dependence on other properties like size or surface brightness of galaxies, and it mainly tests the asymptotic low curvature or low acceleration limit
at limit galaxian radius. To detail test the interpolating Lagrangian function for other intermediate accelerations at different radii in different galaxies, a sharp empirical relation, the Radial Acceleration Relation of galaxies, by McGaugh, Lelli and Schombert is worth considering. The relation subsumes and generalizes the Tully-Fisher relation. The radial acceleration $a$ are measured followed by 2693 points in 153 galaxies with very different morphologies, mass, sizes and gas fractions, distributed up to the range $a_{\text{obs}} \gtrsim O(10^{-11} m/s^2)$ shown in the Figure 1. It is surprisingly found their strong correlations with that expected from the only baryonic acceleration $a_{\text{bar}}$ without any particular halo distribution model of the dark matter. The blue data points (ignoring the “acceleration floor”) of Figure 1 is fitted [9] [23] as

$$a = \frac{a_N}{1 - e^{-\sqrt{\frac{\Delta}{a_0}}} \sqrt{\frac{\Delta}{a_0}}} \tag{48}$$

where $a_0 \approx 1.2 \times 10^{-10} m/s^2 \approx \sqrt{\frac{\psi}{[\text{obs} - 8]}}$ is again empirically the critical acceleration. The circles and diamonds in the Figure 1 given by the dwarf spheroidals (dSphs) (distinguish between MW and M31 satellites) data points shows a possible “acceleration floor” at about $(9.2 \pm 0.2) \times 10^{-12} m/s^2$.

We plot the radial acceleration discrepancy by the black curve in the Figure 2. Note in the eq.(29) that the coupling between matter and potential (and acceleration) is normalized by a factor $1 - \frac{1}{2} R_b \Delta t \approx (0.13 \sim 0.18)$. Let both $a$ and $a_N$ coming from potential $\Phi_g$ and $\Phi_N$ are normalized by such a factor, then the observed acceleration is

$$a = \left(1 - \frac{1}{2} R_b \Delta t\right) |\nabla \Phi_g| = \left(1 - \frac{1}{2} R_b \Delta t\right) \sqrt{\frac{1}{2} R_g}$$

and the effective Newtonian acceleration

$$a_N = \left(1 - \frac{1}{2} R_b \Delta t\right) |\nabla \Phi_N| = \left(1 - \frac{1}{2} R_b \Delta t\right) \sqrt{\frac{1}{2} (R_g - 2 \Lambda)} = \sqrt{a^2 - \left(1 - \frac{1}{2} R_b \Delta t\right)^2} \times \sqrt{\frac{a^2 - \frac{9}{8} a_0^2}{8}} \tag{49}$$

in which by $a_0 = 1.2 \times 10^{-10} m/s^2$ is used. So the observed acceleration $a = \sqrt{a^2 + \frac{9}{8} a_0^2} \approx \sqrt{a^2 + a_0^2}$ is plotted by the black curve in the Figure 2, in which the curve is away from the Newtonian prediction (orange dotted line) at about $O(\sqrt{R_0})$, and has an asymptotic acceleration floor at about $a_0 \sim 10^{-11} m/s^2$ (dash-dot line). $a_0$ can be seen as the normalization version corresponding to the background curvature $\sqrt{R_0}$ (dot-dash line), so naively there should be no accelerations below the acceleration floor, i.e. no curvatures below the asymptotic background curvature $R_0$ by the coarse-graining effect of the Ricci flow to the spacetime. However, there are indeed accelerations data points below $a_0$, and the acceleration floor $a_{\text{min}}$ in Figure 1 is seen lower than $a_0$. A possible reason might be that further normalization corrections are also needed, for instance, the renormalization to the curvature square-root $a \sim \sqrt{R}$ itself also by the factor $\left(1 - \frac{1}{2} R_b \Delta t\right)^{1/2}$, (see [21]), to about $a_{\text{min}}^e \sim (1 - \frac{1}{2} R_b \Delta t)^{1/2} a_0 \sim 10^{-11} m/s^2$, which is about the order of the observed acceleration floor in Figure 1. But unfortunately if we further renormalize and lower the minimal acceleration, the transition trend would further deviate from the fitting (48), so there would be a global fit in both the transition trend and the acceleration floor which predict the range of the minimal acceleration $a_{\text{min}}^e \sim (10^{-11} \sim 10^{-10} m/s^2)$.

The dashed black curve is given by the transition trend [36] around $O(a_0)$, i.e.

$$a_N^2 = \frac{a^2}{\sqrt{1 + \frac{9}{4} \frac{a_0^2}{a^2}}} \tag{50}$$

in which by $a_0 = 1.2 \times 10^{-10} m/s^2$ is used. The approximation of the formula is hold around $O(a_0)$ (dash-dot line) which is a normalized version of $O(\sqrt{R_0}) \sim O(\sqrt{\Lambda})$. But we have continuously extended it to the unreachable deep-MOND limit, which agrees to the fitting (48) (red curve) within uncertainties. There is also problem in the predicted dash curve in the Figure 2, although an acceleration floor is qualitatively predicted in the deep-MOND limit, the acceleration floor predicted at $a_{\text{min}} \sim 10^{-10} m/s^2$ appears higher than the observational hint if the transition trend is fit. As a result, the validity range of the transition trend is smaller than the range of the observed blue data points.

V. THE “MISSING MATTER” BETWEEN SCALES

In order to have a natural interpretation of the required value $\frac{1}{2} R_b \Delta t \approx (0.82 \sim 0.87)$, we consider the “missing matter” interpretation of the curvature flow as the next order correction to gravity. The interpretation is more or less...
similar with the renormalization effect of electric fields, in which the “polarization charge” appears as the “missing charge” w.r.t. the “free charge” (visible charge) obeying the Maxwell’s equation. Now since the scalar curvature $R_g$ at the galactic scale is larger than the $R_b$ at the baryonic or fiducial lab scale, leading to the matter density at the galactic scale seems larger than the visible baryonic matter, if one assumes the validity of the classical Einstein’s equation. Then some matters are seen “missing” in the galactic scale beside the visible baryonic matter and the “dark energy” (the cosmological constant as the leading order correction). When the baryonic matter is slowly moving and pressure-less, using the classical Einstein equation $\Delta R = 8\pi G \Delta \rho$, the flow change of the scalar curvature \(17\) can be translated to the equivalent “missing matter” density $\Delta \rho = \rho_g - \rho_b$,

$$\Delta R = R_g - R_b = R_b \left( \frac{1}{1 - \frac{1}{2} R_b \Delta t} - 1 \right) = 8\pi G \Delta \rho. \tag{52}$$

Since the baryonic matter density gives $R_b = 8\pi G \rho_b$, by using the value of \(47\), we have

$$\frac{\Delta \rho}{\rho_b} = \frac{1}{2} \frac{R_b \Delta t}{1 - \frac{1}{2} R_b \Delta t} \approx (4.6 \sim 6.7), \tag{53}$$

which generally means the “missing matter” in the invisible halo of galaxy at the galactic scale seems about $4.6 \sim 6.7$ times larger than the baryonic matter in the luminary region of galaxy, roughly consistent with the cosmological observations that the “dark matter” density is about $5$ times the visible baryonic matter. And according to the monotonicity of the Ricci flow, $\Delta R$ is non-negative, so the “missing matter” $\Delta \rho$ is always non-negative as well, which is also consistent with the observational fact that no dark matter with negative mass. From this point of view, it is very possible that the so called “dark matter” is just a mirage of the Ricci flow of curvature. As long as the Ricci flow of spacetime is real in physics, this kind of “dark matter” is inevitable, at least contribute to a portion of the dark matter in the conventional sense, even if not the whole.

We shall not detailed compare this theory with the dark matter theory in the paper, since in fact we know little about the dark matter. Here we could do some simple qualitative calculations for the “missing stress tensor $\Delta T_{\mu\nu}$” and its Equations of State $w$ (EoS). The spatial component of the stress tensor is

$$\Delta T_{ij} = \frac{1}{8\pi G} \frac{\delta (\Delta R)}{\delta g^{ij}} = \frac{1}{8\pi G} \Delta R_{ij}. \tag{54}$$
Obviously, in the non-relativistic galaxies, since matter at the initial baryonic scale is non-relativistic, i.e. $R_{ij}(t_b) \approx 0$ as the initial condition of the Ricci flow, then we must have $\Delta R_{ij} \approx 0$ as well during the Ricci flow, consequently the “missing stress” related to it gives $\Delta T_{ij} \approx 0$ and EoS $w \approx 0$.

In the background of Hubble expansion, we have $R_{\mu\nu} = \frac{\dot{R}}{4}g_{\mu\nu} \approx 3H_0^2g_{\mu\nu}$, so

$$\Delta T_{ij} \approx \frac{3H_0^2}{8\pi G}\Delta g_{ij} \approx O(\lambda)\delta_{ij}$$

For its pressure is given by $\Delta T_{ij} = \Delta p g_{ij} = \Delta pa^2\delta_{ij}$, so we obtain a pressure proportional to the the scale factor $a^2$ of the Hubble expansion. In a sufficiently redshifted galactic scale, its pressure is sufficiently suppressed by the scale factor $g_{ij} = a^2\delta_{ij} \ll \delta_{ij}$ at the high redshift and hence seems “cold” taking the equation of state (EoS) $w \approx 0$.

Therefore, the “missing matter” and the conventional dust-like baryonic matter are diluted in the same way with the redshift, and hence the ratio $\Delta a_{\rho_b} \approx (4.6 \sim 6.7)$ is almost unchanged in the expanding history of the universe. While at low redshift, the “missing matter” halo surrounding a galaxy becomes warm but the pressure is still as low as $O(\lambda) \approx (10^{-3}eV)^4$.
Certainly, the classical Einstein’s equation is not exactly true in the framework, so in fact the “missing matter” is not real matter, it is just the mirage of the flow of curvature or gravity, in other words, the “missing matter” is part of the gravity itself arisen from the Ricci flow of initial spacetime from the baryonic matter. Thus although the “missing matter” seems not possible pressure support or rotation support, it still does not expected collapse gravitationally, if the pure gravity itself is intrinsically non-collapsing [12], so the induced “missing matter” halo surrounding a galaxy is considered stable.

In the framework, there are several observations in the internal relation between the “missing matter” and the cosmological constant. First, roughly speaking, the “missing matter” term \( R \Delta t \) and \( \nu \approx -\Omega_\Lambda \) could be of the same order shown in [16], as a consequence that the “dark matter” and the “dark energy” (cosmological constant) most naturally are both of the same order of the critical density \( O(\lambda) \) in the theory. Second, from [10], we can see in [2] \( \nu \) as a counter term measures the whole difference between IR \((t = 0)\) and UV \(t \approx -\infty\), so that \( \nu \) could completely cancel the scale change of the Shannon entropy \( \Delta N \) (higher order correction \( O(R^n) \) might appear in the Shannon entropy \( N \) when beyond the low energy expansion), leaving a short distance theory of the fiducial lab with no “missing matter” and “dark energy”. However, in galactic observations, the “missing matter” term \( R \Delta t \) only measures the finite scale difference, i.e. the difference between finite galactic scale \( t_g < 0 \) and baryonic scale \( -\infty < t_b, (-\infty < t_b < t_g < 0) \), so \( -\nu \) do not completely cancel the “missing matter” \( \Delta R \). At galactic scale, it leaves a theory with mixture of \( -\nu \) (dark energy) and \( R \Delta t \) (dark matter). Third, difference from \( -\nu \) as a constant counter term, which leads to a “4-spatetime volume energy” of EoS \( w = -1 \), the “missing matter” term \( \Delta R \) is dynamic, its pressure can be suppressed by the expanding of the 3-space, so it is a “3-spatial volume matter” of EoS \( w \approx 0 \), thus they behave like different missing components of the universe.

Further more, since \( R \Delta t \) is roughly a constant of order one shown previously, the fact might also imply that the distribution of the supposed “missing matter” and corresponding curvature \( R_{ij}(t_g) \) (in the invisible halo surrounding a galaxy) can be modeled by the local Shrinking Ricci Soliton equation [3], \( \frac{1}{\sqrt{\lambda}} g_{ij} \) (if simply setting a rough constant \( u \) density in the galaxy). In other words, the supposed “dark halo” may be nothing but a Shrinking Ricci Soliton Ricci flow limit configuration seen by a long distant observer. Since the Ricci flow tends to gradually smooth out the initial inhomogeneous and anisotropic baryonic matter distribution, for instance, an anisotropic disk galaxy. During the process, such local initial configuration becomes more and more homogeneous and isotropic, thus the “dark halo” of the disk galaxy will be rounder and rounder. More precisely, the static Shrinking Ricci Soliton equation in 3-space \( R_{ij} = \frac{1}{\sqrt{\lambda}} g_{ij} \) at linear approximation gives a Helmholtz equation for the metric \( (\Delta + \frac{1}{\sqrt{\lambda}}) g_{ij} \approx 0 \) (\( \Delta \) is the Laplacian operator of the space, \( \Delta t \) the finite scale difference). The equation indicates a Yukawa type profile with a characteristic radius \( \sqrt{\Delta t} \approx \sqrt{t_g - t_b} \) of the round “dark halo”, which is much larger than the radius of the visible galaxy. It is analogous to the phenomenon that the screening effect of a medium tends to gradually smooth out the inhomogeneous and anisotropic free charges distribution by dipolarizing the dielectric medium surrounding the free charges. During the process, the observed total charges (free charge + polarization charge) distribution becomes more and more homogeneous and isotropic. If the Shrinking Ricci Soliton configuration really resembles the “dark halo” of a galaxy, it also supports the possible explanation that the correction \( \frac{\nu}{\Omega_\Lambda} \) of \( a_0 \) should be almost a constant.

To sum up, the cosmological constant as the counter term of the Ricci flow modifies the gravity at the leading order, manifesting transition trend similar with MOND, but the deep-MOND limit is unreachable in the theory. The theory not only eliminates the leading missing mass components in the universe but also gives a universal and leading part of the critical acceleration \( a_0 = \frac{3}{2} \sqrt{2 \Lambda} \), while it does not remove the missing components completely and \( a_0 \) is still several times larger than the best fit, the next order contribution is required. The Ricci flow induces the “cold” missing matter and a correction term \( \frac{1}{2} R_0 \Delta t \) to \( a_0 \). In fact, a galaxy spacetime configuration just resembles but exactly be a Shrinking Ricci soliton, so the correction term \( \frac{1}{2} R_0 \Delta t \) is not precisely a constant. Then the next order correction depends on specific scales of different cosmic objects, and in essential non-universal. In this sense, the theory seems like a mixture of MOND and the “cold missing matter”. Considering the “missing matter” is about 5 times the baryonic matter, then we obtain a roughly universal baryonic Tully-Fisher relation of spiral galaxies with the best fit \( a_0 = \frac{3}{2} \sqrt{2 \Lambda} (1 - \frac{1}{2} R_0 \Delta t) \approx \frac{\sqrt{2 \Lambda}}{6 - 8} \). The correction term \( R_0 \Delta t \) here not only gives a correction to \( a_0 \) but also the density of “missing matter”. At this point, it is treated as an approximate constant correction, if we consider the galactic scale \( t_g \) close to the finite singular scale \( t_s \approx t_\ast \). But in fact \( t_g \) is not precisely \( t_\ast \), it is yet unclear whether it is feasible to have a more detailed testing of the scale \( t_g \) or curvature dependence for different galaxies.

VI. DISCUSSIONS AND CONCLUSIONS

In the paper, starting from the Ricci flow of quantum spacetime reference frame as the first principle, a quantum modified gravity is obtained at low energy and long distance scale via the small \( \tau \) scale parameter expansion. As the gravity theory has a low characteristic energy scale \( \lambda \), the gravity suffers from a major correction at low energy. The
Ricci flow effects to the modification of gravity is two folds: the effect of the counter term of the Ricci flow and the Ricci flow of the specific spacetime metric and curvature. The low energy effective action is rather conservative, it can be simply considered as a Ricci flowing Einstein-Hilbert action plus a cosmological constant.

The leading correction to the standard Einstein’s gravity is a cosmological constant as a counter term of the Ricci flow, which makes the spacetime asymptotic a Shrinking Ricci soliton or deSitter. It not only modifies the behavior of gravity at the cosmic scale (i.e. acceleration expansion of the universe), but also contributes to the acceleration discrepancies at the galactic scale (i.e. the baryonic Tully-Fisher relation, and the radial acceleration discrepancies of galaxies) when the local curvature is low enough to be comparable with the characteristic background curvature $R_0$ corresponding to the low energy scale $\lambda$. The leading low energy correction of the gravity coming from the cosmological constant gives a similar transition trend and radial acceleration discrepancy like MOND. This suggests a simple picture that the asymptotic behavior of acceleration discrepancy around $a \sim (10^{-11} \sim 10^{-8}) m/s^2$ or $O(a_0)$ can qualitatively come from the local effect of the cosmological constant. The internal relation between the cosmological constant and the transition trend of MOND is manifested in such a way that the global effects of the cosmological constant or asymptotic background curvature affects a local gravitational system, such as galaxy, around the curvature $O(R_0)$ and around the related acceleration $O(a_0)$.

However, the cosmological constant seems prevent the observed acceleration being much lower than a coarse-grained lower bound, so it gives a flattening “acceleration floor” when $a_N \ll a_0$ or $\rho_M \ll \lambda$ at the outskirt of galaxies, which is qualitatively consistent with some hints of the observations. When some correction factor is introduced, the predicted bound could roughly close to the observed bound. The existence of the “acceleration floor” differs from MOND at the deep-MOND limit $a_N \ll a_0$. If the acceleration is able to reach the deep-MOND limit, then as $a \rightarrow 0$, the final constant rotation curve $v^2 = a \cdot r$ of a galaxy would be extended to infinity $r \rightarrow \infty$, which is impossible from the cosmological viewpoint. It is more natural to exist an escaping maximal radius of rotating satellites beyond which the curvature and related acceleration achieve minimum, rather than reaches the deep-MOND limit $a \rightarrow 0$. In fact the deep-MOND limit might not be applied the principle of general covariance [29]. In general, since in the framework, the acceleration (defined by the second spacetime derivative of the coordinate) of a test satellite (spectral line) at the outskirts of a galaxy is affected by the variance or second moment fluctuation of the spacetime coordinate, if the Ricci flow and the second moment quantum fluctuation of spacetime coordinate at long distance scale are intrinsically unavoidable, and the rotation velocity and related acceleration of the satellites are essentially measured by the broadening (variance) of the spectral lines, there would very likely be a final spectral line broadening due to the Ricci flow and a related fundamental lower bound for the acceleration of the test satellites. In the framework, the acceleration discrepancy observed at the outskirts of galaxies arises from a process that is initially dominated by Doppler broadening caused by local proper motion (baryonic matter giving), but gradually becomes dominated by quantum broadening caused by background spacetime expansion (Ricci flow giving). From this point of view, if we desire a unified view of gravity that not only modifies its behavior at the outskirts of a galaxy but also accurately accounts for the cosmological scale, a deviation from the deep-MOND behavior at the cosmic scale seems inevitable.

The Ricci flow of metric or curvature plays a role in providing next order corrections to gravity at low energy. The starting scale used to calibrate measurements by optical law and to construct the distance ladder is different from the observed scale, such as the cosmic or galactic scale, which is relatively redshifted and quantum broadened. The scale change due to the Ricci flow as a next order correction is non-negligible. The first consequence of the flow effect is that it gives a scale dependent correction to the universal part of the critical acceleration $a_0$, which can be interpreted as the normalization of the coupling between the matter and the potential or acceleration. The corrected $a_0$ can be consistent with the observed fitting value for galaxies, if the equivalent “missing matter” in the invisible halo of galaxy is about five times the baryonic matter in the luminary region of galaxy. The second consequence of the flow effect is that it is equivalent to the “missing mass” mimicking the cold dark matter with the equation of state $w \approx 0$ at high redshift, while at low redshift the “missing matter” becomes warm with low enough pressure.

In the framework, if we consider the leading order correction to gravity from the cosmological constant is intrinsically universal, the next order are corrections depending on specific scales $t_g$ of various cosmic objects, only approximately universal if galaxy spacetime configuration resembles a Shrinking Ricci soliton limit. In this picture, a rough distribution of the “missing matter” invisible halo surrounding a visible galaxy is approximately determined by a Shrinking Ricci soliton equation, $R_{ij} = \frac{1}{\lambda^2} g_{ij}$. The Ricci flow smooths out the initial inhomogeneous and anisotropic baryonic matter distribution of a local visible galaxy, making the local spacetime more and more like a local homogeneous and isotropic Shrinking Ricci soliton configuration as the flow limit, and making the “missing matter” halo surrounding the galaxy rounder and rounder as the coarse-graining process of the Ricci flow tends to do. This picture might provide us a possible way to model the profile of an invisible halo of galaxy by the Ricci flow approach.

The leading order and next order corrections to gravity at low energy produce a theory like a mixture of MOND and “cold missing matter”, similar to the $\Lambda$CDM model, which is mainly a mixture of the cosmological constant and cold dark matter. The difference is that the transition trend and “cold missing matter” here are correlated to each other and have a common origin: the Ricci flow of spacetime. To understand the astronomical phenomenon,
both of them are necessary. The framework is not motivated by empirical fitting of astronomical data, but purely a framework of gravity coming from alternative theoretical considerations, which happened to contain these speculative “dark” components. The Ricci flow might provide us with a different paradigm for cosmological and astronomical observations.

The acceleration expansion of universe and acceleration discrepancy of galaxies given by the low energy quantum modified gravity has a unified understanding by the fact that the quantum spacetime with 2nd order moment fluctuation modifies the physical quantities at 2nd order in spacetime coordinates, like the curvature and acceleration, which are second spacetime derivative induced. In other words, curvature and acceleration can be roughly seen as the same thing in the sense that they both suffer from the quantum fluctuation corrections at the same (2nd) order. In the framework, the curvature ratio \( R/R_0 \) plays a similar role of the acceleration ratio \( a/a_0 \) in MOND, but essentially, the curvature ratio is more fundamental and has many advantages than the acceleration ratio. Curvature is a more fundamental and general covariant concept, and hence the framework is a relativistic theory. But the concept of acceleration is not generally covariant, which might lead to confusions in MOND, e.g. the acceleration is absolute or relative to any specific frames, or direction dependent, internal or external, etc.? Some relativistic generalizations of MOND could be seen as attempts to solve this kind of conceptual difficulties making it more well-defined especially in cosmology. The framework of the paper is based on the first principle for alternative motivations, and hence it differs from most of the relativistic generalization of MOND. As one of the implications of MOND and its relativistic generalization, the Equivalence Principle is considered explicitly violated. In fact any modifications of gravity has to face the issue that the effective gravitational mass differs from the inertial mass of the unmodified gravity, or effective metric differs from the one of the unmodified gravity, so it seems that one has to deal with at least two spacetime, but only one of which we can sense directly, e.g. the bi-metric theory \[30\]. In some theories, the difference of masses or metric of spacetime is interpreted as the violation of the Equivalence Principle \[31\]. Or in some other theories, the difference is alternatively interpreted as missing mass or other kind of new matter e.g. cold dark matter and dipole or metric of spacetime is interpreted as the violation of the Equivalence Principle \[31\]. Or in some other theories, the difference of masses or metric of spacetime is interpreted as the violation of the Equivalence Principle \[31\]. However, it is worth stressing that the Equivalence Principle retains in the framework even at the quantum level, the price to pay is that the metric or curvature of the spacetime must flow or renormalize with scale at the quantum level. Different metrics of spacetime (e.g. \( g(t_y) \) and \( g(t_{b_i}) \)) in the modified gravity at different scales have equal realities. Since the Equivalence Principle plays a fundamental role in the quantum reference frame theory. The Equivalence Principle is the physical foundation of measuring the spacetime by physical material reference frame even at the quantum level, and it is the physical foundation of the geometric interpretation of gravity through curved spacetime, so that the gravity is simply a relational phenomenon that the motion of a test particle in gravity is manifested as a relative motion w.r.t. the (quantum) material reference frame. Without the Equivalence Principle, we would lost the physical foundation of all these concepts, such as the metric and acceleration that all the theories are based on.

There are limitations in the studies. The transition trend away from Newtonian gravity of the theory is studied and tested with reference to the universal and successful part of MOND. However, the theory and MOND have completely different behaviors at the deep-MOND limit \( a_N < a_0 \). In this picture, although an acceleration floor is qualitatively predicted in the deep-MOND limit, the acceleration floor predicted at about \( a_{min} \sim 10^{-10} m/s^2 \) appears higher than the observational hint if the transition trend is fit. As a result, the validity range of the transition trend is smaller than the range of the observed blue data points. Thus, if the idea that the MOND-like transition trend and the acceleration floor are both due to the cosmological constant (i.e. its broadening effect to the spectral lines) is on the right track, further effects must be taken into account. The question of the acceleration floor is still open in the paper. It is also unclear whether the full relativistic theory can solve some of the challenges of MOND, such as the bullet cluster and/or missing baryons in groups or clusters of galaxies. Qualitatively, the framework may provide a promising picture, as clusters colliding or for other possible reasons, it is possible to solve a separation of the mass-centers of the baryonic matter and the effective “missing matter” in a full relativistic theory. This is analogous to the separation of the charge-centers of the free charge and the polarization charge when the free charge is sharply accelerated, and the response of the polarization charge is delayed due to its own inertia. Additionally, it is unclear whether the full scale-dependent theory can satisfactorily explain some of the sharp challenges in cosmology that are considered to be the big successes of CDM, such as a consistent history of structure formation and fitting the third and subsequent peaks of the acoustic spectrum in the Cosmic Microwave Background. Furthermore, there are several unexplained discrepancies to the global fit \( a_0 \) reported in e.g. \[25, 27\], but whether these discrepancies are truly correlated to the curvature or scale-dependent correction \[46\] is yet unclear. These issues require further study.
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