The phenomena of collapse and dispersal for a massless scalar field has drawn considerable interest in recent years, mainly from a numerical perspective. We give here a sufficient condition for the dispersal to take place for a scalar field that initially begins with a collapse. It is shown that the change of the gradient of the scalar field from a timelike to a spacelike vector must be necessarily accompanied by the dispersal of the scalar field. This result holds independently of any symmetries of the spacetime. We demonstrate the result explicitly by means of an example, which is the scalar field solution given by Roberts. The implications of the result are discussed.

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I. INTRODUCTION

The spherically symmetric collapse of a massless scalar field has been of much interest towards understanding the dynamical evolutions in general relativity. Both analytical [1] and numerical [2] investigations have been undertaken by various authors to gain more insight into the formation of black holes. One remarkable finding of these numerical investigations is the demonstration of criticality in gravitational collapse. Specifically, it was found that for a range of values of the parameter characterizing the solution, black hole forms and there was a critical value of the parameter beyond which the solutions are such that the scalar field disperses without forming any black hole. However, this result has been obtained mainly through numerical studies and a proper theoretical understanding of this phenomenon is still lacking (see e.g. [2] and the references therein).

Therefore, it is important to understand better the phenomena of collapse and dispersal analytically. Since very few solutions of the massless scalar field coupled to gravity are known, it is not possible to reach any conclusion in this case only by studying such particular solutions. To obtain more insight into this, it is necessary then to identify some general features associated with the collapse and dispersal of the scalar fields. We study here the collapse of the scalar field in the sense that the expansion is negative to begin with along a comoving world line, and later it turns positive, thus causing the dispersal of the field. If this happens along all the world lines, the field enters a dispersal phase as a whole, and in the case otherwise the gravitational collapse continues.

Specifically, we give here a sufficient condition for the dispersal of scalar field along its world line. We show that if along the worldline, the gradient of the scalar field approaches a null value from a timelike vector, then a dispersal of the scalar field must take place somewhere along this world line, before the gradient changes. This result applies to any massless scalar field spacetime in general, and is not dependent on any symmetry of the spacetime like spherical symmetry or self-similarity etc. Here we discuss the dispersal in a local sense to begin with (along one world line). However, if this condition of dispersal is satisfied along all the comoving world lines, then the scalar field will globally disperse away without forming a black hole. We demonstrate explicitly that this is actually the case in the solution given by Roberts [3], Brady [3] and Oshiro et al [6]. We analyze this solution here in some detail along these lines, which provides some interesting new information on its structure. Therefore, in some classes of models, this condition can be used to get more insight into the phenomena of global dispersal of a scalar field. The above authors found qualitatively different behaviour of the solution for different ranges of values of a certain parameter \( p \), which characterizes the solution. Specifically for \( p > 1 \), black hole forms due to collapse, whereas for \( 0 < p < 1 \) the scalar field first collapses and then disperses away without forming any black hole. The case \( p = 1 \) is the critical case and a null singularity forms in this situation. Near the critical point, the black hole mass was found to satisfy a power law behaviour.

Here we show, in order to demonstrate our result mentioned above, that the gradient of the scalar field necessarily changes from timelike to spacelike along any world line in the case \( 0 < p < 1 \), and then it is shown that a dispersal of the scalar field occurs in this case. In the case when the field goes to a black hole, no such gradient change takes place. Another interesting point to note is that, in this case, the change of gradient is related with the global dispersal of the scalar field, in the sense that the field as a whole disperses as mentioned above.

We shall use comoving coordinates here. It is useful to note the limitation of comoving coordinates for massless scalar field spacetimes, and its domain of validity. We know that in terms of the eigenvectors of the energy-momentum tensor, matter fields can be classified in four distinct types [4]. However, all known physical matter fields in the universe fall under the first two (namely Type I. INTRODUCTION

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II. COLLAPSE CONDITION IN COMOVING FRAME

For matter fields which have a timelike four-velocity vector, we can always have a comoving reference frame where we can define a timelike matter congruence locally as,

\[ u^a = \frac{dx^a}{d\tau} : u^a u_a = -1. \]  

Here \( x^a \) denotes the position for the matter and \( \tau \) is the proper time along the congruence. Given this congruence \( u^a \), we can define a unique projection tensor as \( h^a_b = g^a_b + u^a u_b \). This determines the orthogonal metric properties of the instantaneous rest spaces of observers moving with four-velocity \( u^a \) and projects all the geometrical quantities to the three-space orthogonal to \( u^a \).

The matter energy-momentum tensor can be decomposed relative to \( u^a \) in the form

\[ T_{ab} = \rho u^a u^b + q^a u^b + q^b u^a + p h^{ab} + \Pi_{ab} \]  

where \( \rho = T_{ab} u^a u^b \) is the relativistic energy density relative to \( u^a \), \( q^a = -T_{bc} b^c u^b \) denotes the relativistic energy flux, \( p = \frac{1}{3} T_{ab} h^{ab} \) is the isotropic pressure, and \( \Pi_{ab} = T_{ab} h_c^c h_d^d \) is the trace-free anisotropic pressure. The physics of the situation is defined in the equation of state relating these thermodynamical quantities of the matter field. For an isotropic perfect fluid with a general equation of state \( p = \rho (\rho) \), we impose the conditions \( q^a = \Pi_{ab} = 0 \). The volume expansion for this matter congruence is defined as \( \theta = h^a_b \nabla_a u^b \), where \( \nabla \) denotes the full covariant derivative. Also the ‘time (dot) derivative’ of any function \( f(x^a) \) is defined as the derivative along \( u^a \), that is, \( f = u^a \nabla_a f \). In terms of the above quantities the energy conservation law for an isotropic perfect fluid is then given by

\[ \dot{\rho} = -\theta (\rho + p). \]  

The continual collapse condition in a comoving frame can now be defined in the following way: For the matter field which is undergoing continual collapse because of its self gravity, the volume element defined by the matter congruence always decreases with the proper time along the congruence, that is, the volume expansion is negative.

The spacetime singularity corresponds to the points where this volume element goes to zero. This implies that for a continually collapsing matter congruence we must have \( \theta < 0 \) along the world lines. In that case, for any collapsing isentropic perfect fluid obeying the weak energy condition, since we have \( \rho \geq 0 \) and \( \rho + p \geq 0 \), therefore we have \( \dot{\rho} \geq 0 \), i.e. the density is a non-decreasing function of the proper time along the matter congruence.

III. A SUFFICIENT CONDITION FOR DISPERAL

The energy-momentum tensor of a massless scalar field \( \phi(x^a) \), is given by

\[ T_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} (\phi_{,a} \phi_{,b} g^{cd}) . \]  

The massless scalar field can be thought of as a stiff fluid when the gradient of the scalar field \( \phi_{,a} \) is timelike. Now we are in a position to state and prove the sufficient condition for the dispersal of a massless scalar field: If, during the dynamical evolution of a gravitationally coupled massless scalar field, a comoving world line along which \( \phi_{,a} \) is timelike, approaches the limit where \( \phi_{,a} \) is null, then dispersal must take place somewhere along that worldline.

In order to see this, first we note that in the entire region of the spacetime where \( \phi_{,a} \) is timelike, we can always set up a comoving coordinate system, where the massless scalar field behaves like a stiff fluid with the equation of state \( p = \rho \). So we can consider the comoving world lines of this stiff fluid. Now let us consider the case that there is such a comoving world line which approaches the point where \( \phi_{,a} \) becomes null. Now from the Einstein’s equations we know that \( R = \phi_{,a} \phi_{,b} = -2p \). This implies that \( R \) and \( p \) go to zero in the limit of approaching any
point where $\phi_{\mu}$ is null. This implies that $\dot{\rho}$ must be negative at least somewhere along the comoving world line that we are considering. Since $\rho$ and $p$ are positive by construction, from (4), we conclude that $\theta$ would have to be positive whenever $\dot{\rho}$ is negative in the comoving patch. This implies that $\theta$ has to change sign at least once along this comoving world line. The changing of sign of the volume expansion from negative to positive necessarily denotes a dispersal of the scalar field.

We note that this result is independent of any symmetry assumed of the spacetime, such as spherical symmetry or self-similarity. Since so far we have been considering dispersal of the scalar field along a comoving world line, the above result does not, by itself, provide any information as to whether the scalar field will collapse into a blackhole, or disperse away without forming one. However, in the case when all the comoving world lines approach the limit where $\phi_{\mu}$ is null, there must be a dispersal along all the comoving world lines in that patch, and then there would be no blackhole formation. This in fact happens in the Roberts solution which we discuss below as an example. In the next section, we consider this solution in order to demonstrate explicitly, how the dispersal of the scalar field accompanies changing of the gradient $\phi_{\mu}$ from a timelike to a spacelike vector.

Now we state another interesting consequence of the above result, which is as follows: If the dynamical evolution of a massless scalar field, starting from a timelike hypersurface on which the gradient of the scalar field is timelike, and the volume expansion for the comoving congruence is negative to begin with, is such that the gradient changes from being timelike to null in the future, then there must be a bounce or dispersal of the scalar field in the part of the spacetime where the gradient is timelike.

As should be clear, we have here a situation in mind where the scalar field is gravitationally collapsing initially, when it begins the dynamical evolution. To see why the above result is true, let us consider the case of the massless scalar field, where the dynamical evolution takes place from an initial spacelike hypersurface $t = t_i$, and the gradient $\phi_{\mu}$ is timelike on that surface. This implies that we can construct a comoving congruence evolving from the initial hypersurface.

Next, if there is a hypersurface $S$ in the future of the initial surface $t = t_i$ where $\phi_{\mu}$ is null, this would then imply that any comoving congruence defined in the past of $S$ must break down at $S$. In other words, the comoving coordinate system describing the scalar field breaks down at $S$. Since $S$ is the boundary of the comoving patch, there are necessarily comoving lines which approach arbitrarily close to $S$. Then using the earlier result, it is seen that there must be dispersal somewhere along any such comoving worldline. Therefore there must be a dispersal of the scalar field in the part of the spacetime where the gradient is timelike.

In the argument above, we have implicitly assumed that the density does not go to a vanishing value asymptotically on the initial spacelike hypersurface. If the density does go to zero asymptotically, then it would not be possible to argue in the above manner, that the density has to decrease from the initial value if the gradient of the field becomes null in the future. In fact, for the Roberts solution discussed below, this turns out to be the case. However, even in that case, the result we mentioned above holds true as we shall show. In general also, even when the density asymptotically falls off on a given spacelike surface, the result above holds true with some minor changes. The proof of this would be given elsewhere.

IV. THE ROBERTS SOLUTION AS AN EXAMPLE

Now we shall consider a particular analytic solution for massless scalar field to demonstrate how dispersal of the scalar field arises in this model. The metric for this solution to the Einstein’s field equations was given by Roberts (3). For a massless scalar field, one can construct a comoving coordinate system when the gradient of the scalar field is timelike. In the Roberts solution, the gradient changes sign, therefore a single comoving coordinate system cannot be used to cover the whole spacetime manifold. However, when the continual gravitational collapse starts from an initial spacelike surface, the gradient of the scalar field is necessarily timelike. In that case the equation of state of the massless scalar field in the comoving frame is that of a stiff fluid, with $\rho = p$.

To demonstrate the dispersal of the scalar field, we need only consider the comoving patch of the whole manifold where the gradient is timelike. We then show explicitly that the continual collapse turns into a dispersal of the field when the gradient $\phi_{\mu}$ changes from timelike to a spacelike vector in the future of the initial surface.

Using notations in (3) the Roberts solution takes the following form in double null coordinates

$$ds^2 = -dudv + R(u, v)^2 d\Omega^2$$

$$R(u, v) = \frac{1}{2} \sqrt{[(1 - p^2)v^2 - 2vu + u^2]} , \tag{5}$$

and

$$\phi = \pm \frac{1}{2} \ln \frac{(1 - p)v - u}{(1 + p)v - u} . \tag{6}$$

Here the constant $p$ can be chosen to be positive without any loss of generality. The units used are such that $8\pi G = c = 1$. The Ricci scalar for this solution has the form

$$R = \frac{p^2 uv}{2R(u, v)^4} \tag{7}$$

The spacetime structure depends crucially on the parameter $p$. For $p > 1$, the singularity becomes timelike in the region $v < 1$, the singularity becomes timelike in the region $v > 1$. For
\[ p = 1 \text{ the singularity becomes null. For } 0 < p < 1, \text{ the solution has only timelike singularities.} \]

Now the singularity in the past can be removed by matching the solution with a Minkowski patch, as shown in the figures. Then there is an evolution from regular initial data to a future singularity, just like a collapse scenario.

**A. Transformation into comoving coordinates**

For massless scalar field, the Einstein equations can be written in the form \( R_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} \). From this it follows that the norm of \( \phi_{,\mu} \) is equal to the Ricci scalar. Since we would like to construct a comoving coordinate system in the part of the manifold where \( \phi_{,\mu} \) is timelike, we have to ensure that the Ricci scalar is negative. From (7), it is then seen that we must have \( uv < 0 \) in this comoving patch.

Now we consider the solution for which \( \phi \) has a negative sign in (6). To make the transformation to the comoving co-ordinates \((t, r, \theta, \phi)\), we choose \( t(u, v) \) and \( r(u, v) \) in such a manner that \( \phi(u, v) = t \) and \( g_{rr}^c = 0 \), i.e. the comoving metric is diagonal and is given in the form

\[ ds^2 = -g_{tt}^c(t, r)dt^2 + g_{rr}^c(t, r)dr^2 + R^2(t, r)d\Omega^2 \] \quad (8)

Putting \( \phi = t \), in the expression for \( \phi \), we have \( v = -x(t)u \), where \( x(t) = [e^{-2t} - 1]/(1 + p)(1 - e^{-2t}) - 2p \).

Using the constraint \( g_{rr}^c = 0 \) and the relation between \( v \) and \( u \), we can solve explicitly for \( u(t, r) \) and \( v(t, r) \) as

\[ u(t, r) = \frac{f(r)}{\sqrt{x}}, \quad v(t, r) = -f(r)\sqrt{x} \] \quad (9)

where \( f(r) \) is the arbitrary function of integration. The metric components in the comoving frame are

\[ g_{tt}^c = -\frac{1}{4} f^2(r) \left( \frac{dx}{dt} \right)^2, \quad g_{rr}^c = f'^2(r) \] \quad (10)

and

\[ R(t, r)^2 = \frac{1}{4} f'^2(r) \left[ (1 - p^2)x + \frac{1}{x} + 2 \right] \] \quad (11)

Also the Ricci scalar can be calculated as

\[ R = -\frac{8p^2}{f'^2(r) \left[ (1 - p^2)x^2 + 2x + 1 \right]^2} \] \quad (12)

The volume expansion for the timelike congruence of comoving shells ‘r’ is given by \( \theta = \frac{2pR_t}{|R|} \).

As discussed earlier, the comoving coordinate system breaks down when \( \phi_{,\mu} \) becomes null. This corresponds to \( R = 0 \). This happens at \( x = 0 \) and as \( x \to \infty \). From the expression for \( x \), it is seen that \( x = 0 \) implies \( t = 0 \) and \( x \to \infty \) implies \( t \to -\frac{1}{2} \ln \frac{1 + x}{1 - p} \). Therefore for \( p > 1 \), the allowed range for \( t \) in the comoving patch is \( 0 < t < \infty \) and for \( 0 < p < 1 \), it is \( 0 < t < -\frac{1}{2} \ln \frac{1 + x}{1 - p} \). For all these cases, we can take a \( t = t_1 \) spacelike surface inside the comoving patch and consider the dynamic evolution from that surface. From our earlier discussions, we know that \( \phi_{,\mu} \) is timelike in the initial spacelike surface. We note here that for all the three cases, \( u = 0 \) is another null hypersurface, where \( \phi_{,\mu} \) becomes null. But this surface is in the past of \( t = t_1 \) (\( t_1 \) can be chosen in such a way), hence not relevant for our purpose.
Without any loss of generality, we can take the initial time \( t_i \) to be zero. We note that if \( R \to \infty \), then there is a spacetime singularity. This occurs at \((1-p^2)x^2+2x+1=0\). The solution of this equation is \( x = -\frac{1}{1+p} \). But only \( x = \frac{1}{1+p} \) can be realized for the time to be real. This implies that there would be a singularity when \( e^{-2t} = 0 \) or \( t \to \infty \). Let us now discuss the three cases depending on the value of \( p \).

**B. The case of \( 0 < p < 1 \)**

This is an interesting case for the Roberts solution in which a future timelike singularity develops. In this case, however, \( \phi_{\mu} \) becomes a spacelike vector from a timelike one. The surface \( v = 0 \) is the hypersurface where \( \phi_{\mu} \) is null. Therefore in this case, the continual collapse condition should be violated within the comoving patch. In what follows, we shall show that this is indeed the case.

In this case, \( \theta = \frac{2}{1-t} \frac{f(r)}{R(t)}(1-p^2) - \frac{1}{x^2} \frac{dx}{dt} \). From this relation, it is seen that \( \theta = 0 \) at \( x = \pm \frac{1}{\sqrt{1-p^2}} \). Only the positive sign gives a real value of \( t \). If we denote the instant when \( \theta = 0 \) as \( t = t_b \), then we have \( t_b = -\frac{1}{2} \ln \left( \frac{1}{1+p} \right) \).

Hence, we see that in the co-moving patch, for \( t_b < t < t_b \), we have \( \theta < 0 \), and hence the massless scalar field is collapsing. However, for \( t_b < t < -\frac{1}{2} \ln \left( \frac{1}{1+p} \right) \), we have \( \theta > 0 \) which imply that the scalar field has bounced back from the collapsing state at \( t = t_b \), and is expanding in this range. This shows clearly that the continual collapse condition is violated in the comoving patch and also \( \phi_{\mu} \) becomes null during evolution. This agrees with the statement made in the section III.

**C. The case of \( p > 1 \)**

In this case we see that the singularity, which is spacelike, lies in the comoving patch. Also all the shells collapse to a spacelike singularity in a finite proper time and the final state is necessarily a black hole. The volume expansion for the comoving shells remains negative throughout and at the singularity, \( \theta(t, r) \to -\infty \). Therefore in this case, \( \phi_{\mu} \) does not become null in the dynamic evolution from the surface \( t = t_i \) and there is continual collapse starting from \( t = t_i \).

**D. The case of \( p = 1 \)**

In the marginal case of \( p = 1 \), as seen from the Fig 2, the singularity is null and \( x = \frac{1-e^{-2t}}{2e^{-2t}} \). Putting \( p = 1 \) in \( \beta(\tau) \), we get \( R = -\frac{8}{f(x)(2x^2+1)} \). This shows that \( R \) can diverge only at \( x = -\frac{1}{2} \). From the expression for \( x \), we find however, that this cannot be realised for any \( t > 0 \). So the comoving shells \( r > 0 \) never reach the null singularity. Moreover, since the singularity is null, no timelike or null geodesic can come out of the singularity and reach the comoving observer. In this case also, \( \phi_{\mu} \) does not become null in the dynamic evolution from the surface \( t = t_i \). In this case, \( \theta \) is always negative but goes to zero asymptotically as \( t \to \infty \).

**V. DISCUSSION**

In the above, we have identified the change of the gradient of the scalar field as a sufficient condition for the dispersal of the scalar field. We can consider such a dispersal of the scalar field either along a single world line, or for a local congruence of the same, and this does not by itself determine whether the scalar field would collapse into a black hole or disperse away as a whole. To determine that, we need to impose additional condition, such as the said behaviour should take place along all the world lines. We indicated such a scenario in Sec III.

Here we emphasise that this dispersal occurs in that region of the spacetime where the gradient of the scalar field is timelike and hence no black hole can form in that part of the spacetime. So the question that whether a black hole can form in that part of the spacetime where \( \phi_{\mu} \) is spacelike is still open. However, there are indications that at least for spherically symmetric spacetimes, if there is dispersal along all the comoving world lines in the part where \( \phi_{\mu} \) is timelike, then no black hole can form even in the part of the spacetime where \( \phi_{\mu} \) is spacelike. We are currently investigating this issue.

The physical significance of the above is the following: Consider a massless scalar field that begins collapse from an initial spacelike surface, and its gradient is then timelike at all points. The condition we gave then ensures dispersal of this initially collapsing field later, as it evolves. It is possible that the field may refocus again later even after the dispersal, but that is an issue to be considered and examined separately.

We pointed out that the massless scalar field solution given by Roberts obeys this result, and further, in the case where no dispersal takes place, a black hole forms. On the other hand, if there is a dispersal of the scalar field along all the world lines, then a black hole does not form. It would be interesting to investigate whether the Roberts solution is the only solution which has such a behaviour. It would be interesting to explore and examine if there are other such solutions which behave in a similar manner. It is also possible that even if the condition for the local dispersal of the scalar field is satisfied not along all the world lines, but along a certain fraction of them, even in that case the scalar field could disperse away as a whole without forming a black hole.

Our study indicates that to gain insight into the phenomena of both dispersal of scalar fields and collapse, it may be useful to consider more closely the gradient change of the scalar fields during their dynamical evolution. This is important because, if there is indeed such a
connection, then we can expect to see critical behaviour in such cases. Since our result here does not depend on any symmetry of the spacetime, this suggests that the dispersal phenomena might be a possible feature of massless scalar field solutions in general.

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