Weighted norm inequalities for commutators of Littlewood-Paley functions related to Schrödinger operators

Lin Tang

Abstract Let \( L = -\Delta + V \) be a Schrödinger operator, where \( \Delta \) is the Laplacian operator on \( \mathbb{R}^n \), while the nonnegative potential \( V \) belongs to certain reverse Hölder class. In this paper, we establish some weighted norm inequalities for commutators of Littlewood-Paley functions related to Schrödinger operators.

1. Introduction

In this paper, we consider the Schrödinger differential operator
\[
L = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 3,
\]
where \( V \) is a nonnegative potential satisfying certain reverse Hölder class.

We say a nonnegative locally \( L^q \) integral function \( V(x) \) on \( \mathbb{R}^n \) is said to belong to \( B_q(1 < q \leq \infty) \) if there exists \( C > 0 \) such that the reverse Hölder inequality
\[
\left( \frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y)dy \right)^{1/q} \leq C \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)dy \right)
\]
holds for every \( x \in \mathbb{R}^n \) and \( 0 < r < \infty \), where \( B(x,r) \) denotes the ball centered at \( x \) with radius \( r \). In particular, if \( V \) is a nonnegative polynomial, then \( V \in B_\infty \). It is worth pointing out that the \( B_q \) class is that, if \( V \in B_q \) for some \( q > 1 \), then there exists \( \epsilon > 0 \), which depends only \( n \) and the constant \( C \) in (1.1), such that \( V \in B_{q+\epsilon} \). Throughout this paper, we always assume that \( 0 \not\equiv V \in B_{n/2} \).

The study of Schrödinger operator \( L = -\Delta + V \) recently attracted much attention; see [1, 2, 3, 4, 10, 14]. In particular, it should be pointed out that Shen [10] proved the Schrödinger type operators, such as \( \nabla (-\Delta + V)^{-1} \nabla, \nabla (-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2} \nabla \) with \( V \in B_n, (-\Delta + V)^{\gamma} \) with \( \gamma \in \mathbb{R} \) and \( V \in B_{n/2} \), are standard Calderón-Zygmund operators.

Recently, Bongioanni, etc, [1] proved \( L^p(\mathbb{R}^n)(1 < p < \infty) \) boundedness for commutators of Riesz transforms associated with Schrödinger operator with \( BMO(\rho) \) functions.
Weighted norm inequalities for commutators

which include the class $BMO$ function, and in [2] established the weighted bounded-
ness for Riesz transforms, fractional integrals and Littlewood-Paley functions associated
with Schrödinger operator with weight $A^p_\rho$ class which includes the Muckenhoupt weight
class. Very recently, the author [13] established the weighted norm inequalities for some
Schrödinger type operators, which include Riesz transforms and fractional integrals and
their commutators.

In this paper, we will continue to study weighted norm inequalities for commutators
of Littlewood-Paley functions related to Schrödinger operators. More precisely, we have
the following results.

**Theorem 1.1.** Let $1 < p < \infty$. If $b \in BMO(\rho)$ (defined in Section 2), $\omega \in A^p_\rho$ (defined in Section 2), then there exists a constant $C$ such that

$$\|g_b(f)\|_{L^p(\omega)} \leq C\|b\|_{BMO(\rho)}\|f\|_{L^p(\omega)},$$

where the Littlewood-Paley $g$ function related to Schrödinger operators is defined by

$$g(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-tL} (f)(x) \right|^2 t dt\right)^{1/2},$$

and the commutator $g_b$ of $g$ with $b \in BMO(\rho)$ is defined by

$$g_b(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-tL} ((b(x) - b(\cdot))f)(x) \right|^2 t dt\right)^{1/2}.$$ (1.3)

In addition, we denote $g^*(f)(x)$ and $g^*_b(f)(x)$ in (1.2) and (1.3) if $L = \Delta$

The weighted weak-type endpoint estimate for the commutator is the following.

**Theorem 1.2.** Let $b \in BMO(\rho)$ and $\omega \in A^1_\rho$. There exists a constant $C > 0$ such that for any $\lambda > 0$

$$\omega(\{x \in \mathbb{R}^n : |g_b f(x)| > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) \omega(x) dx.$$

Throughout this paper, we let $C$ denote constants that are independent of the main
parameters involved but whose value may differ from line to line. By $A \sim B$, we mean
that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Preliminaries

We first recall some notation. Given $B = B(x, r)$ and $\lambda > 0$, we will write $\lambda B$ for
the $\lambda$-dilate ball, which is the ball with the same center $x$ and with radius $\lambda r$. Similarly,
$Q(x, r)$ denotes the cube centered at $x$ with the sidelength $r$ (here and below only cubes
with sides parallel to the coordinate axes are considered), and $\lambda Q(x, r) = Q(x, \lambda r)$. Given
a Lebesgue measurable set $E$ and a weight $\omega$, $|E|$ will denote the Lebesgue measure of $E$
and $\omega(E) = \int_E \omega dx$. $\|f\|_{L^p(\omega)}$ will denote $(\int_{\mathbb{R}^n} |f(y)|^p \omega(y) dy)^{1/p}$ for $0 < p < \infty$. 
The function $m_V(x)$ is defined by
\[
\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.
\]
Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

**Lemma 2.1** ([10]). There exists $l_0 > 0$ and $C_0 > 1$ such that
\[
\frac{1}{C_0} (1 + |x - y|m_V(x))^{-l_0} \leq \frac{m_V(x)}{m_V(y)} \leq C_0 (1 + |x - y|m_V(x))^{l_0/(l_0 + 1)}.
\]
In particular, $m_V(x) \sim m_V(y)$ if $|x - y| < C/m_V(x)$.

In this paper, we write $\Psi(B) = (1 + rm_V(B))^\theta$ where $m_V(B) = \frac{1}{|B|} \int_B m_V(x) dx$ and $\theta > 0$, and $r$ denotes the radius of $B$.

Obviously,
\[
\Psi(B) \leq \Psi(2B) \leq 2^\theta \Psi(B).
\]

(2.1)

A weight will always mean a positive function which is locally integrable. As [2], we say that a weight $\omega$ belongs to the class $A_p^\theta$ for $1 < p < \infty$, if there is a constant $C$ such that for all ball $B = B(x,r)$
\[
\left( \frac{1}{\Psi(B)|B|} \int_B \omega(y) dy \right) \left( \frac{1}{\Psi(B)|B|} \int_B \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} \leq C.
\]

We also say that a nonnegative function $\omega$ satisfies the $A_1^\theta$ condition if there exists a constant $C$ for all balls $B$
\[
M_V(\omega)(x) \leq C \omega(x), \text{ a.e. } x \in \mathbb{R}^n.
\]

where
\[
M_V f(x) = \sup_{x \in B} \frac{1}{\Psi(B)|B|} \int_B |f(y)| dy.
\]

When $V = 0$, we denote $M_0 f(x)$ by $M f(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_V f(x) \leq M f(x)$ for a.e. $x \in \mathbb{R}^n$.

We denote $A_\infty^\theta = \bigcup_{p \geq 1} A_p^\theta$. Since $\Psi(B) \geq 1$, obviously, $A_p \subset A_p^\theta$ for $1 \leq p < \infty$, where $A_p^\theta$ denote the classical Muckenhoupt weights; see [6] and [7]. We will see that $A_p \subset A_p^\theta$ for $1 \leq p < \infty$ in some cases. In fact, let $\theta > 0$ and $0 \leq \gamma \leq \theta$, it is easy to check that $\omega(x) = (1 + |x|)^{-(n+\gamma)} \not\in A_\infty^\theta$ and $\omega(x) dx$ is not a doubling measure, but $\omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1^\theta$ provided that $V = 1$ and $\Psi(B(x_0,r)) = (1 + r)^\theta$.

From the definition of $A_p^\theta$ for $1 \leq p < \infty$, it is easy to see that

**Lemma 2.2.** Let $1 \leq p < \infty$. Then

(i) If $1 \leq p_1 < p_2 < \infty$, then $A_p^{\theta_1} \subset A_p^{\theta_2}$.
(ii) \( \omega \in A_p^\theta \) if and only if \( \omega^{-1/p} \in A_{p'}^\theta \), where \( 1/p + 1/p' = 1 \).

Bongioanni, etc, [1] introduce a new space \( \text{BMO}(\rho) \) defined by

\[
\|f\|_{\text{BMO}(\rho)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\Psi(B)|B|} \int_B |f(x) - f_B| dx < \infty,
\]

where \( f_B = \frac{1}{|B|} \int_B f(y) dy \) and \( \Psi(B) = (1 + r/\rho(x_0))^{\theta'} \), \( B = B(x_0,r) \) and \( \theta > 0 \).

In particular, Bongioanni, etc, [1] proved the following result for \( \text{BMO}(\rho) \).

**Lemma 2.3.** Let \( \theta > 0 \) and \( 1 \leq s < \infty \). If \( b \in \text{BMO}(\rho) \), then

\[
\left( \frac{1}{|B|} \int_B |b - b_B|^s \right)^{1/s} \leq C_{\theta,s} \|b\|_{\text{BMO}(\rho)} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta'},
\]

for all \( B = B(x,r) \), with \( x \in \mathbb{R}^n \) and \( r > 0 \), where \( \theta' = (l_0 + 1)\theta \).

Obviously, the classical \( \text{BMO} \) is properly contained in \( \text{BMO}(\rho) \); more examples see [1].

From Lemma 2.3, the author [13] proved the John-Nireberg inequality for \( \text{BMO}(\rho) \).

**Proposition 2.1.** Suppose that \( f \) is in \( \text{BMO}(\rho) \). There exist positive constants \( \gamma \) and \( C \) such that

\[
\sup_B \frac{1}{|B|} \int_B \exp \left\{ \frac{\gamma}{\|f\|_{\text{BMO}(\rho)} \Psi_\rho(B)} |f(x) - f_B| \right\} dx \leq C,
\]

where \( f_B = \frac{1}{|B|} \int_B f(y) dy \) and \( \Psi_\rho(B) = (1 + r/\rho(x_0))^{\theta'} \), \( B = B(x_0,r) \) and \( \theta' = (l_0 + 1)\theta \).

We remark that balls can be replaced by cubes in definitions of \( A_p^\theta, \text{BMO}(\rho) \) and \( M_V \) by (2.1).

The dyadic maximal operator \( M_{V,\lambda}f(x) \) is defined by

\[
M_{V,\lambda}f(x) := \sup_{x \in Q(\text{dyadic cube})} \frac{1}{\Psi(Q)|Q|} \int_Q |f(x)| dx.
\]

The dyadic sharp maximal operator \( M_{V,\delta}f(x) \) is defined by

\[
M_{V,\delta}f(x) := \sup_{x \in Q, r < \rho(x_0)} \frac{1}{|Q|} \int_{Q_{x_0}} |f(y) - f_Q| dy + \sup_{x \in Q, r \geq \rho(x_0)} \frac{1}{\Psi(Q)|Q|} \int_{Q_{x_0}} |f| dx
\]

\[
\simeq \sup_{x \in Q, r < \rho(x_0)} \inf_C \frac{1}{|Q|} \int_{Q_{x_0}} |f(y) - C| dy + \sup_{x \in Q, r \geq \rho(x_0)} \frac{1}{\Psi(Q)|Q|} \int_{Q_{x_0}} |f| dx
\]

where \( Q_{x_0} \) denotes dyadic cubes \( Q(x_0,r) \) and \( f_Q = \frac{1}{|Q|} \int_Q f(x) dx \).

A variant of dyadic maximal operator and dyadic sharp maximal operator

\[
M_{\delta,V}^{\lambda}f(x) = M_{V,\lambda}(|\delta|^1/\delta)(x)
\]

and

\[
M_{\delta,V}^{\delta}f(x) = M_{V,\delta}(|\delta|^1/\delta)(x),
\]

which will become the main tool in our scheme.

In [13], the author proved the following Lemmas.
**Theorem 2.1.** Let $\omega \in A^p_{\infty}$. Then there exist constant $C, \delta^1$ such that for a locally integrable function $f$, and for $b$ and $\gamma$ positive $\gamma < b < b_0 = (8nC_0)^{-(\eta_0+2)\alpha}$, we have the following inequality

$$\omega(\{x \in \mathbb{R}^n : M^\Delta_V f(x) > \lambda, M^\sharp_V f(x) \leq \gamma \lambda\}) \leq Ca^\delta_1 \omega(\{x \in \mathbb{R}^n : M^\Delta_V f(x) > b \lambda\})$$  \hspace{1cm} (2.1)

for all $\lambda > 0$, where $a = 2^n \gamma/(1 - \frac{b}{b_0})$.

As a consequence of Theorem 2.1, we have the following result.

**Corollary 2.1.** Let $0 < p, \delta < \infty$ and $\omega \in A^p_{\infty}$. There exists a positive constant $C$ such that

$$\int_{\mathbb{R}^n} |M_V f(x)|^p \omega(x) dx \leq C \int_{\mathbb{R}^n} |M^\sharp_V f(x)|^p \omega(x) dx.$$  \hspace{1cm} (2.2)

Let $\varphi : (0, \infty) \to (0, \infty)$ be a doubling function. Then there exists a positive constant $C$ such that

$$\sup_{\lambda > 0} \varphi(\lambda) \omega(\{x \in \mathbb{R}^n : M^\Delta_V f(x) > \lambda\}) \leq C \sup_{\lambda > 0} \varphi(\lambda) \omega(\{x \in \mathbb{R}^n : M^\sharp_V f(x) > \lambda\})$$

for any smooth function $f$ for which the left handside is finite.

**Proposition 2.2([13]).** Let $1 < p < \infty$ and suppose that $\omega \in A^p_{\infty}$. If $p < p_1 < \infty$, then the equality

$$\int_{\mathbb{R}^n} |M_V f(x)|^{p_1} \omega(x) dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^{p_1} \omega(x) dx.$$  \hspace{1cm} (2.3)

Further, let $1 \leq p < \infty$, $\omega \in A^p_{\infty}$ if and only if

$$\omega(\{x \in \mathbb{R}^n : M^\Delta_V f(x) > \lambda\}) \leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$  \hspace{1cm} (2.4)

From proposition 4.1, we know that $M_V$ may be not bounded on $L^p(\omega)$ for all $\omega \in A^p_{\infty}$ and $1 < p < \infty$. We now need to define a variant maximal operator $M_{V,\eta}$ for $0 < \eta < \infty$ as follows

$$M_{V,\eta} f(x) = \sup_{x \in B} \frac{1}{(\Psi(B))^{\eta}|B|} \int_{B} |f(y)| dy.$$  \hspace{1cm} (2.5)

**Theorem 2.2([13]).** Let $1 < p < \infty$, $p' = p/(p - 1)$ and suppose that $\omega \in A^p_{\infty}$. There exists a constant $C > 0$ such that

$$\|M_{V,p'} f\|_{L^p(\omega)} \leq C \|f\|_{L^{p'}(\omega)}.$$  \hspace{1cm} (2.6)

We next recall some basic definitions and facts about Orlicz spaces, referring to [9] for a complete account.

A function $B(t) : [0, \infty) \to [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $B \to \infty$ as $t \to \infty$. If $B$ is a Young function,
we define the \( B \)-average of a function \( f \) over a cube \( Q \) by means of the following Luxemberg norm:

\[
\| f \|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.
\]

If \( A, B \) and \( C \) are Young functions such that

\[
A^{-1}(t)B^{-1}(t) \leq C^{-1}(t),
\]

where \( A^{-1} \) is the complementary Young function associated to \( A \), then

\[
\| fg \|_{C,R} \leq 2 \| f \|_{A,R} \| g \|_{B,R}.
\]

The examples to be considered in our study will be \( A^{-1}(t) = \log(1+t) \), \( B^{-1}(t) = \frac{t}{\log(e+t)} \) and \( C^{-1}(t) = t \). Then \( A(t) \sim e^t \) and \( B(t) \sim t \log(e+t) \), which gives the generalized Hölder’s inequality

\[
\frac{1}{|Q|} \int_Q |fg| dy \leq \| f \|_{A,Q} \| g \|_{B,Q}
\]

holds. For these example and using Theorem 2.1, if \( b \in BMO(\rho) \) and \( b_Q \) denotes its average on the cube \( Q \), then

\[
\| (b - b_Q)/\Psi_{\theta'}(Q) \|_{\exp L,Q} \leq C \| b \|_{BMO(\rho)}.
\]

where \( \theta' = (1 + l_0)\theta \).

And we define the corresponding maximal function

\[
M_B f(x) = \sup_{Q: x \in Q} \| f \|_{B,Q}
\]

and

\[
M_{V,B} f(x) = \sup_{Q: x \in Q} \Psi(Q)^{-1}\| f \|_{B,Q}.
\]

3. Some Lemmas

Bongioanni, etc, [2] proved the following result.

\textbf{Lemma 3.1.} Let \( g_{loc}^*(f)(x) = g^*(f \chi_{B(x,\rho(x))})(x) \). Let \( 1 < p < \infty \) and suppose that \( \omega \in A_p^\rho \). Then

\[
\int_{\mathbb{R}^n} |g_{loc}^*(f)(x)|^p \omega(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.
\]

Furthermore, suppose that \( \omega \in A_1^\rho \). Then, there exists a constant \( C \) such that for all \( \lambda > 0 \)

\[
\omega(\{ x \in \mathbb{R}^n : g_{loc}^*(f)(x) > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x)dx.
\]
Lemma 3.2. Let $b \in BMO(\rho)$, and $(l_0 + 1) \leq \eta < \infty$. Set $g_{lo,b}^*(f)(x) = g^*(b(x) - b(\cdot))f_{B(x, \rho(x))}(x)$. Let $0 < 2\delta < \epsilon < 1$, then

$$M_{\delta,\eta}^2(g_{lo,b}^*(f))(x) \leq C\|b\|_{BMO(\rho)}(M_{\epsilon,\eta}^2(g_{lo}^*(f)))(x) + M_{L\log L,V,\eta}(f)(x), \quad \text{a.e } x \in \mathbb{R}^n, \quad (3.1)$$

holds for any $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. We fix $x \in \mathbb{R}^n$ and let $x \in Q = Q(x_0, r)$ (dyadic cube). To prove (3.1), we consider two cases about $r$, that is, $r < \rho(x_0)$ and $r \geq \rho(x_0)$.

Case 1. when $r < \rho(x_0)$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\bar{Q}}$, where $Q = Q(x_0, 4\sqrt{n}r)$. Let $\lambda$ be a constant and $C_Q$ a constant to be fixed along the proof. Since $0 < \delta < 1$, we then have

$$\left(\frac{1}{|Q|} \int_Q |g_{lo,b}^*(f)(y)|^\delta - |C_Q|^\delta \right) dy^{1/\delta}$$

$$\leq \left(\frac{1}{|Q|} \int_Q |g_{lo,b}^*(f)(y) - C_Q|^\delta dy \right)^{1/\delta}$$

$$\leq C \left(\frac{1}{|Q|} \int_Q |(b(y) - \lambda)g_{lo}^*(f)(y)|^\delta dy \right)^{1/\delta}$$

$$+ C \left(\frac{1}{|Q|} \int_Q |g_{lo}^*((b - \lambda)f_1)(y)|^\delta dy \right)^{1/\delta}$$

$$+ C \left(\frac{1}{|Q|} \int_Q |g_{lo}^*((b - \lambda)f_2)(y) - C_Q|^\delta dy \right)^{1/\delta}$$

$$:= I + II + III.$$  

To deal with $I$, we first fix $\lambda = b_{\bar{Q}}$, the average of $b$ on $\bar{Q}$. Then for any $1 < \gamma < \epsilon/\delta$, note that $m_V(x) \sim m_V(x_0)$ for any $x \in \bar{Q}$ and $\Psi(\bar{Q}) \sim 1$, by Lemma 2.3, we then obtain

$$I \leq C \left(\frac{1}{|Q|} \int_Q |b(y) - b_{\bar{Q}}|^\delta \gamma dy \right) \gamma'/\delta \left(\frac{1}{|Q|} \int_Q |g_{lo}^*(f)(y)|^\delta \gamma dy \right)^{\delta\gamma}$$

$$\leq C\|b\|_{BMO(\rho)}M_{\epsilon,\eta}^2(g_{lo}^*(f))(x), \quad (3.2)$$

where $1/\gamma' + 1/\gamma = 1$.

For $II$, note that $m_V(x) \sim m_V(x_0)$ for any $x \in \bar{Q}$ and $\Psi(\bar{Q}) \sim 1$, by Kolmogorov’s inequality and and Proposition 2.1 and Lemma 3.1, we then have

$$II \leq C \frac{1}{|Q|} \|g((b - b_{\bar{Q}})f_1)\|_{L^{1,\infty}}$$

$$\leq C \frac{1}{|Q|} \int_Q |(b - b_{\bar{Q}})f(y)| dy$$

$$\leq CM_{L\log L,V,\eta}f(x). \quad (3.3)$$
For III, we first fix the value of \(C_Q\) by taking \(C_Q = g_{lo}^*((b - b_Q)f_2)(y_0)\) with \(y_0 \in Q\). Let \(b_{Q_k} = b_{Q(x_0,2^{k+1}r)}\). By Proposition 2.1, we then obtain

\[
II \leq \frac{C}{|Q|} \int_Q |g_{lo}^*((b - b_Q)f_2)(y) - g_{lo}^*((b - b_Q)f_2)(y_0)| \, dy
\]

\[
\leq \frac{C}{|Q|} \int_Q \int_0^\infty \left( \int_{2r < |z - x_0| \leq c\rho(x_0)} |f(z)||b(z) - b_Q| \frac{(t-n/2)|y - y_0|/\sqrt{t}}{(1 + |z - y_0|/\sqrt{t})^{n+2}} \, dz \right)^2 \, dtdy
\]

\[
\leq \frac{C}{|Q|} \int_Q \int_0^\infty \left( \int_{2r < |z - x_0| \leq c\rho(x_0)} |f(z)||b(z) - b_Q| \frac{rt}{(t + |z - x_0|)^{2(n+2)}} \, dz \right)^2 \, dtdy
\]

\[
\leq \frac{C}{|Q|} \int_Q \int_0^\infty \left( \int_{z - x_0 \leq c\rho(x_0)} r|f(z)||b(z) - b_Q| \frac{1}{(t + |z - x_0|)^{2(n+2)}} \, dt \right)^2 \, dy
\]

\[
\leq \frac{C}{|Q|} \sum_{k=1}^{k_0} \frac{2^{-k}}{(2^{k}r)^n} \int_{z - x_0 \leq 2^{k+1}} |f(z)||b(z) - b_Q| \, dz
\]

\[
\leq C\|b\|_{BMO(\rho)} M_{L\log L, \eta}(f)(x),
\]

where the integer \(k_0\) satisfies \(2^{k_0}r \leq c\rho(x_0) \leq 2^{k_0+1}\) and \(c = C_0n2^{lo+4}\).

Case 2. When \(r \geq \rho(x_0)\). Decompose \(f = f_1 + f_2\), where \(f_1 = f\chi_Q\), where \(Q = Q(x_0,C_02^{lo+4}\sqrt{n}r)\). Since \(0 < 2\delta < \epsilon < 1\), so \(a = \eta/\delta\) and \(\epsilon/\delta > 2\), then

\[
\frac{1}{\Psi(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| g_{lo,b}^*(f) \right| |y\|^\delta \, dy \right)^{1/\delta}
\]

\[
\leq \frac{1}{\Psi(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| (b(y) - \lambda)g_{lo}^*(f)(y) + g_{lo}^*((b - \lambda)f)(y) \right| |y\|^\delta \, dy \right)^{1/\delta}
\]

\[
\leq C \frac{1}{\Psi(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| (b(y) - \lambda)g_{lo}^*(f)(y) \right| |y\|^\delta \, dy \right)^{1/\delta}
\]

\[
+ C \frac{1}{\Psi(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| g_{lo}^*((b - \lambda)f_1)(y) \right| |y\|^\delta \, dy \right)^{1/\delta}
\]

\[
+ C \frac{1}{\Psi(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| g_{lo}^*((b - \lambda)f_2)(y) \right| |y\|^\delta \, dy \right)^{1/\delta}
\]

\[
:= I + II + III.
\]

To deal with I, we first fix \(\lambda = b_Q\), the average of \(b\) on \(Q\). Then for any \(2 \leq \gamma < \epsilon/\delta\), note that \(l_0 + 1 \leq \eta\), by Lemma 2.3, we then have

\[
I \leq C \frac{1}{\Psi^{(r)}(Q)^a} \left( \frac{1}{|Q|} \int_Q \left| b(y) - b_Q \right| |y\|^\delta \, dy \right)^{1/(r\delta)}
\]

\[
\times \frac{\Psi^{(r)}(Q) \Psi^{(\eta)}(Q) \Psi^{(\eta/2\ gamma)}(Q)}{(\Psi(Q)^a \eta/(2\delta) \int_Q \left| g_{lo}^*(f) \right| |y\|^\gamma \, dy)}^{1/(\delta\gamma)}
\]
where $1/\gamma' + 1/\gamma = 1$.

For II, we recall that $g^*_{loc}$ is weak type $(1,1)$ by Lemma 3.1. By Kolmogorov’s inequality and Proposition 2.1, we then have

\begin{equation}
II \leq \frac{C}{\Psi(Q)^a |Q|^\gamma} \|g^*_{loc}((b - b_Q)f_1)\|_{L^1(Q)} \leq \frac{C}{\Psi(Q)^a |Q|} \int_Q |(b - b_Q)f(y)| dy \leq CM_{L,\log L,V,\eta} f(x).
\end{equation}

Finally, for III, notice that $B(y, \rho(y)) \subset Q(x_0, C_0 \rho^{\eta_0 + \frac{4}{3} \sqrt{nr}})$ for any $y \in Q$, then $III = 0$.

From (3.2)–(3.6), we get (3.1). Hence the proof is finished. \hfill $\Box$

We next consider several maximal operators, which play an important role in this paper.

\begin{equation}
M_{V,\eta}f(x) = \sup_{x \in B} \frac{1}{\Psi(B)^\eta |B|} \int_B |f(y)| dy,
\end{equation}

\begin{equation}
\tilde{M}_{V,\eta}^b f(x) = \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon\psi(B(x,\epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-\eta} \varphi\left(\frac{x - y}{\epsilon}\right) |f(y)| dy,
\end{equation}

and their commutators

\begin{equation}
M_{V,\eta}^b f(x) = \sup_{x \in B} \frac{1}{\Psi(B)^\eta |B|} \int_B |b(x) - b(y)||f(y)| dy,
\end{equation}

\begin{equation}
\tilde{M}_{V,\eta}^b f(x) = \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon\psi(B(x,\epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-\eta} \varphi\left(\frac{x - y}{\epsilon}\right) |b(x) - b(y)||f(y)| dy,
\end{equation}

where $\psi(B(x,\epsilon)) = \frac{1}{\rho(B(x,\epsilon))} \int_{B(x,\epsilon)} \rho(y)^{-1} dy$.

Obviously, we have

\begin{equation}
M_{V,\eta}^b f(x) \leq C \tilde{M}_{V,\eta}^b f(x),
\end{equation}

where $\eta' = (l_0 + 1)\eta$ and $\eta > 0$.

**Lemma 3.3.** Let $b \in BMO(\rho)$, and $(l_0 + 1)(1 + 1/\theta) \leq \eta < \infty$, $\eta_1 = (l_0 + 1)\eta$ and $\eta_2 = (l_0 + 1)\eta_1(1 + 1/\theta)$. Let $0 < 2\delta < \epsilon < 1$, then

\begin{equation}
M_{\delta,\eta}(\tilde{M}_{V,\eta^2}^b f)(x) \leq C\|b\|_{BMO(\rho)} (M_{\epsilon,\eta}^\Delta (\tilde{M}_{V,\eta^2}^b f))(x) + M_{L,\log L,V,\eta} f(x), \quad \text{a.e. } x \in \mathbb{R}^n,
\end{equation}

holds for any $f \in C_0^\infty(\mathbb{R}^n)$.

**Proof.** We fix $x \in \mathbb{R}^n$ and let $x \in Q = Q(x_0, r)(\text{dyadic cube})$. To prove (3.9), we consider two cases about $r$, that is, $r < \rho(x_0)$ and $r \geq \rho(x_0)$.

Case 1. when $r < \rho(x_0)$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\bar{Q}}$, where $\bar{Q} = Q(x_0, 4\sqrt{nr})$. Let $\lambda$ be a constant and $C_Q$ a constant to be fixed along the proof. Since $0 < \delta < 1$, we then have

\begin{equation}
\end{equation}
Weighted norm inequalities for commutators

\[
\left( \frac{1}{|Q|} \int_Q |\tilde{M}_{V,\eta_2}^b(f)(y)|^\delta - |C_Q|^\delta \right)^{1/\delta} \\
\leq \left( \frac{1}{|Q|} \int_Q |\tilde{M}_{V,\eta_2}^b(f)(y) - C_Q|^\delta \right)^{1/\delta} \\
\leq C \left( \frac{1}{|Q|} \int_Q |(b(y) - \lambda)\tilde{M}_{V,\eta_2}^b(f)(y)|^\delta \right)^{1/\delta} \\
+ C \left( \frac{1}{|Q|} \int_Q |\tilde{M}_{V,\eta_2}^b((b - \lambda)f_1)(y)|^\delta \right)^{1/\delta} \\
+ C \left( \frac{1}{|Q|} \int_Q |\tilde{M}_{V,\eta_2}^b((b - \lambda)f_2)(y) - C_Q|^\delta \right)^{1/\delta} \\
:= I + II + III.
\]

To deal with \( I \), we first fix \( \lambda = b_Q \), the average of \( b \) on \( Q \). Then for any \( 1 < \gamma < \epsilon/\delta \), note that \( m_V(x) \sim m_V(x_0) \) for any \( x \in Q \) and \( \Psi(Q) \sim 1 \), by Proposition 2.1, we then obtain

\[
I \leq C \left( \frac{1}{|Q|} \int_Q |b(y) - b_Q|^{\delta'} \right)^{\gamma'/\delta} \left( \frac{1}{|Q|} \int_Q |\tilde{M}_{V,\eta_2}^b(f)(y)|^{\delta'} \right)^{\delta'} \tag{3.10}
\]

where \( \frac{1}{\gamma'} + \frac{1}{\delta} = 1 \).

For II, note that \( m_V(x) \sim m_V(x_0) \) for any \( x \in Q \) and \( \Psi(Q) \sim 1 \), by Kolmogorov’s inequality and Theorem 2.1, by the weak \((1,1)\) of \( \tilde{M}_{\eta_2} \), we then have

\[
II \leq C \frac{1}{|Q|} \|\tilde{M}_{V,\eta_2}^b((b - b_Q)f_1)\|_{L^{1,\infty}} \\
\leq C \frac{1}{|Q|} \int_Q |(b - b_Q)f(y)| \, dy \tag{3.11}
\]

where \( \frac{1}{\gamma'} + \frac{1}{\delta} = 1 \).

For III, we fix the value of \( C_Q \) by taking \( C_Q = \tilde{M}_{V,\eta_2}((b - b_Q)f_2)(y_0) \) for some \( y_0 \in Q \). We now estimate \( E := |\tilde{M}_{V,\eta}((b - b_Q)f_2)(y) - C_Q| \) for any \( y \in Q \).

\[
E = \left| \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon \Psi(B(y, \epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-n} \varphi\left( \frac{y - z}{\epsilon} \right) |b(z) - b_Q||f_2(z)| \, dz \right| \\
- \left| \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon \Psi(B(y, \epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-n} \varphi\left( \frac{y_0 - z}{\epsilon} \right) |b(z) - b_Q||f_2(z)| \, dz \right| \\
\leq \left| \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon \Psi(B(y, \epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-n} \varphi\left( \frac{y - z}{\epsilon} \right) - \varphi\left( \frac{y_0 - z}{\epsilon} \right) |b(z) - b_Q||f_2(z)| \, dz \right| \\
+ \left| \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon \Psi(B(y, \epsilon)))^\eta} \int_{\mathbb{R}^n} e^{-n} \varphi\left( \frac{y_0 - z}{\epsilon} \right) |b(z) - b_Q||f_2(z)| \, dz \right|
\]

where \( \frac{1}{\gamma'} + \frac{1}{\delta} = 1 \).
\[
= \sup_{\varepsilon > r} \left( \frac{1}{(1 + \varepsilon \psi(B(y, \varepsilon)))^{\theta_n}} \right) \int_{\mathbb{R}^n} e^{-n} |\varphi(y^{2} - z^{2}) - \varphi(y^{0} - z^{0})||b(z) - b_Q||f_2(z)|dz \\
+ \sup_{\varepsilon > r} \left( \frac{1}{(1 + \varepsilon \psi(B(y, \varepsilon)))^{\theta_n}} \right) \frac{1}{(1 + \varepsilon \psi(B(y, \varepsilon)))^{\theta_n}} \int_{\mathbb{R}^n} e^{-n} \varphi(y^{0} - y^{0})|b(z) - b_Q||f_2(z)|dz \\
\leq \sup_{\varepsilon > r} \left( \frac{C}{(1 + \varepsilon \rho(y))^{\theta_n}} \right) \int_{|y - y| \leq 8\varepsilon} e^{-n} \frac{1}{\varepsilon} |b(y) - b_Q||f(y)|dy \\
+ C \sup_{\varepsilon > r} \frac{\varepsilon}{\rho(y)} \left( \frac{1}{1 + \varepsilon \rho(y)} \right)^{\theta_n} \int_{|y - y| \leq 8\varepsilon} e^{-n} \frac{1}{\varepsilon} |b(y) - b_Q||f(y)|dy \\
\leq \sup_{\varepsilon > r} \frac{C}{(1 + \varepsilon \rho(y))^{\theta_n}} \int_{|y - y| \leq 8\varepsilon} e^{-n} \frac{1}{\varepsilon} |b(y) - b_Q||f(y)|dy \\
\leq \sup_{\varepsilon > r} \sum_{k=1}^{[\ln(\frac{\varepsilon}{8\rho(y)})]+1} \frac{C}{(1 + \varepsilon \rho(y))^{\theta_n}} \int_{|y - y| \leq 2^k \varepsilon} \frac{1}{|y|} |b(y) - b_Q||f(y)|dy \\
\leq \sup_{\varepsilon > r} \sum_{k=1}^{[\ln(\frac{\varepsilon}{8\rho(y)})]+1} \frac{r}{\varepsilon} k \|b\|_{BMO(\rho)}M_{L\log L,V,\eta}f(x) \\
\leq C \|b\|_{BMO(\rho)}M_{L\log L,V,\eta}f(x). \\
\]

Hence,

\[
III \leq C \|b\|_{BMO(\rho)}M_{L\log L,V,\eta}f(x). \quad (3.12)
\]

Case 2. when \(r > \rho(x_0)\). Let \(f_1, f_2\) be above. We then have

\[
\left( \frac{1}{|Q|} \int_{Q} |\tilde{M}_{V,\eta_2}(f)(y)|^\delta dy \right)^{1/\delta} \leq C \left( \frac{1}{|Q|} \int_{Q} |(b(y) - \lambda)\tilde{M}_{V,\eta_2}(f)(y)|^\delta dy \right)^{1/\delta} \\
+ C \left( \frac{1}{|Q|} \int_{Q} |\tilde{M}_{V,\eta_2}((b - \lambda)f_1)(y)|^\delta dy \right)^{1/\delta} \\
+ C \left( \frac{1}{|Q|} \int_{Q} |\tilde{M}_{V,\eta_2}((b - \lambda)f_2)(y)|^\delta dy \right)^{1/\delta} \\
:= I_1 + II_1 + III_1.
\]

To deal with \(I_1\), we first fix \(\lambda = b_Q\), the average of \(b\) on \(Q\). Then for any \(2 \leq \gamma < \varepsilon/\delta\), by
Lemma 2.3, we then obtain that
\[
I \leq \frac{1}{\Psi'(Q)} \left( \frac{1}{|Q|} \int_{Q} |b(y) - b_Q|^{\delta \gamma'} dy \right)^{1/(r' \delta)} 
\times \frac{\Psi'(Q)}{\Psi(Q)^{a - \eta/(2s)}} \left( \frac{1}{\Psi(Q)^{\eta(1)}} \int_{Q} |g^*_{loc}(f)(y)|^{\delta \gamma'} dy \right)^{1/(\delta \gamma)}
\]
(3.13)
\[
\leq C\|b\|_{BMO(\rho)} M_{\epsilon,\eta}(\widetilde{M}_{V,\eta})(f)(x),
\]
where $1/\gamma' + 1/\gamma = 1$.

For $II_1$, by Kolmogorov’s inequality and Proposition 2.1, by the weak (1,1) of $\widetilde{M}_{V,\eta}$, we then have
\[
II_1 \leq \frac{C}{|Q|} \|\widetilde{M}_{V,\eta}((b - b_Q)f_1)\|_{L^{1,\infty}} 
\leq \frac{C}{|Q|} \int_{Q} |(b - b_Q)f(y)| dy 
\leq C\|b\|_{BMO(\rho)} M_{L \log L,V,\eta}f(x).
\]
(3.14)

For $III_1$, we have for any $y \in Q$,
\[
\widetilde{M}_{V,\eta}((b - b_Q)f_2)(y) = \sup_{\epsilon > 0} \frac{1}{(1 + \epsilon \psi(B(y,\epsilon)))^{\eta/2}} \int_{\mathbb{R}^n} \epsilon^{-n}\nu((y - z)/(\epsilon\rho))|b(z) - b_Q||f_2(z)|dz 
\leq \sup_{\epsilon > r} \frac{C}{\rho(y)^{\eta/2}} \int_{|z - y| \leq 8\epsilon} \epsilon^{-n}|b(y) - b_Q||f(y)|dy 
\leq \sup_{\epsilon > r} \frac{C}{\rho(y)^{\eta/2}} \int_{|z - y| \leq 8\epsilon} \epsilon^{-n}|b(y) - b_Q||f(y)|dy 
\leq \sup_{\epsilon > r} \frac{C}{\rho(y)^{\eta/2}} \int_{|z - y| \leq 8\epsilon} \epsilon^{-n}|b(y) - b_Q||f(y)|dy 
\leq C\|b\|_{BMO(\rho)} M_{L \log L,V,\eta}f(x).
\]
(3.15)

From (3.10)–(3.15), we get (3.9). Hence the proof is finished. \qed

**Lemma 3.4.** Let $2 \leq \eta < \infty$, $\omega \in A_1^\infty$ and $B(t) = t \log(e + t)$. Then there exists a constant $C > 0$ such that for all $t > 0$
\[
\omega(\{x \in \mathbb{R}^n : M_{B,V,\eta}f(x) > t\}) \leq C \int_{\mathbb{R}^n} B\left(\frac{|f(x)|}{t}\right) \omega(x)dx.
\]
(3.16)

**Proof.** Let $K$ be any compact subset in $\{x \in \mathbb{R}^n : M_{L \log L,V,\eta}(f)(x) > \lambda\}$. For any $x \in K$, by a standard covering lemma, it is possible to choose cubes $Q_1, \ldots, Q_m$ with pairwise...
disjoint interiors such that $K \subset \bigcup_{j=1}^{m} 3Q_j$ and with $\|f\|_{L^{\log} L, \varphi, Q_j} > \lambda$, $j = 1, \cdots, m$. This implies
\[
\Psi(Q_j)^2 |Q_j| \leq \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy.
\]
From this, by (vi) in Lemma 2.1 with $p = 1$ and $E = Q$, we obtain that
\[
\omega(3Q_j) \leq C \Psi(Q_j) \omega(Q_j)
\]
\[
= C \frac{\omega(Q_j)^2 |Q_j|}{\Psi(Q_j)|Q_j|} \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy
\]
\[
\leq C \inf_{Q_j} \omega(x) \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy
\]
\[
\leq C \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) \omega(y) dy.
\]
Thus, (3.16) holds, hence, the proof is complete. □

Finally, the author [13] proved the following result.

**Lemma 3.5.** Let $0 < \eta < \infty$ and $M_{V, \eta/2} f$ be locally integral. Then there exist positive constants $C_1$ and $C_2$ independent of $f$ and $x$ such that
\[
C_1 M_{V, \eta} M_{V, \eta+1} f(x) \leq M_{L^{\log} L, \eta+1} f(x) \leq C_2 M_{V, \eta/2} M_{V, \eta/2} f(x).
\]

4. **Proof of some theorems**

**Proof of Theorem 1.1.** We adapt a similar argument of Theorem 5 in [2]. As before, we define
\[
g_{\text{loc}, b}(f)(x) = g((b(x) - b(\cdot)) f_{XB(x, \rho(x))})(x), \quad g_{\text{glob}, b}(f)(x) = g((b(x) - b(\cdot)) f_{XB^{*}(x, \rho(x))})(x).
\]
Thus
\[
\|g_{\text{b}}(f)\|_{L^p(\omega)} \leq \|g_{\text{loc}, b}(f)\|_{L^p(\omega)} + \|g_{\text{glob}, b}(f)\|_{L^p(\omega)}.
\]
We start with $g_{\text{glob}, b}$. Denoting by $q_t$ the kernel of $\frac{d}{dt} e^{-tL}$, from (2.7) of [4], for any $N > 0$, we have
\[
|q_t(x, y)| \leq \frac{C_N}{t^{n/2+1}} \left(1 + \frac{t}{\rho(x)^2} + \frac{t}{\rho(y)^2}\right)^{-N} e^{-\frac{|x-y|^2}{ct}}.
\]
(4.1)

Hence,
\[
\left|\int_{|x-y| > \rho(x)} q_t(x, y)(b(x) - b(y)) f(y) dy\right|
\]
\[
\leq Ct^{-n/2-1} \left(1 + \frac{t}{\rho(x)^2}\right)^{-N} \int_{|x-y| > \rho(x)} e^{-\frac{|x-y|^2}{ct}} |b(x) - b(y)||f(y)| dy
\]
\[
\leq Ct^{n/4-1} \left(1 + \frac{t}{\rho(x)^2}\right)^{-N} \int_{|x-y| > \rho(x)} \frac{|b(x) - b(y)||f(y)|}{|x-y|^M} dy
\]
\[
\leq C \frac{M-n-1}{\rho(x)^{M-n}} \left(1 + \frac{t}{\rho(x)^2}\right)^{-N} \sum_{k=1}^{\infty} \frac{2^{-k(M-n-\theta)}}{2^{k\rho(x)}^{n}} \int_{|x-y| < 2^k \rho(x)} |b(x) - b(y)||f(y)| \, dy \\
\leq C \frac{M-n-1}{\rho(x)^{M-n}} M^b_{V,\eta} f(x).
\]

Then,
\[
g_{\text{glob},b}(f)(x) \leq CM^b_{V,\eta} f(x) \left( \int_0^\infty \left( \frac{t}{\rho(x)^2} \right)^{M-n} \left(1 + \frac{t}{\rho(x)^2}\right)^{-2N} \, dt \right)^{1/2} \leq CM^b_{V,\eta} f(x).
\]

Choose \(M\) and \(N\) such that \(M - n > \theta\eta\) and \(2N > M - n\). Therefore, the estimates for \(g_{\text{glob},b}\) follow from those for \(M^b_{V,\eta} f(x)\) by Lemmas 3.3 and 3.5.

To deal with \(g_{\text{loc},b}\), we write
\[
g_{\text{loc},b}(f)(x) \leq I(x) + g^*_{\text{loc},b}(f)(x) + II(x),
\]
where \(g^*_{\text{loc},b}(f)(x)\) is defined in Lemma 3.2,
\[
I(x) = \left( \int_0^{\rho(x)^2} \left| \int_{|x-y| < \rho(x)} [\tilde{q}_t(x,y) - \tilde{q}_t(x,y)](b(x) - b(y)) f(y) \, dy \right|^2 \, dt \right)^{1/2},
\]
where \(\tilde{q}_t\) is the kernel of \(d/dt e^{t\Delta}\), and
\[
II(x) = \left( \int_0^{\rho(x)^2} \left| \int_{|x-y| < \rho(x)} q_t(x,y)(b(x) - b(y)) f(y) \, dy \right|^2 \, dt \right)^{1/2}.
\]

For \(II(x)\), by (4.1) with \(N = 1/2\),
\[
II(x) \leq C \left( \int_0^{\rho(x)^2} \left( \frac{\rho(x)^2}{t} \right)^2 \left| \int_{|x-y| < \rho(x)} t^{-n/2} e^{-\frac{|x-y|^2}{ct}} |b(x) - b(y)||f(y)| \, dy \right|^2 \, t \, dt \right)^{1/2}
\]
\[
\leq C \left( \int_0^{\rho(x)^2} \left( \frac{\rho(x)^2}{t} \right)^2 \rho(x)^{-n} \left| \int_{|x-y| < \rho(x)} |b(x) - b(y)||f(y)| \, dy \right|^2 \, t \, dt \right)^{1/2}
\]
\[
\leq CM^b_{V,\eta} f(x) \left( \int_0^{\rho(x)^2} \left( \frac{\rho(x)^2}{t} \right)^2 \, dt \right)^{1/2}
\]
\[
\leq CM^b_{V,\eta} f(x).
\]

For \(I(x)\), adapting the same argument of pages 578-579 in[2], we obtain for some \(\delta > 0\) and \(\epsilon > 0\)
\[
I(x) \leq C \left( \int_0^{\rho(x)^2} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \left| \int_{|x-y| < \rho(x)} t^{-n/2} e^{-\epsilon \frac{|x-y|^2}{t}} |b(x) - b(y)||f(y)| \, dy \right|^2 \, dt \right)^{1/2}
\]
\[
\leq C \left( \int_0^{\rho(x)^2} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \left| \int_{|x-y| < \rho(x)} t^{-n/2} e^{-\epsilon \frac{|x-y|^2}{t}} |b(x) - b(y)||f(y)| \, dy \right|^2 \, dt \right)^{1/2}
\]
\[
+ C \left( \int_0^{\rho(x)^2} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \left| \int_{|x-y| < \sqrt{t}} t^{-n/2} e^{-\epsilon \frac{|x-y|^2}{t}} |b(x) - b(y)||f(y)| \, dy \right|^2 \, dt \right)^{1/2}
\]
From (4.2), (4.3) and (4.4), we can obtain the desired result by Lemmas 3.2, 3.3, 3.5 and Theorem 2.2.

Proof Theorem 1.2. By (4.1)-(4.4) and using Lemmas 3.2, 3.3, 3.4, 3.5 and Proposition 2.2, by adapting an argument in [8], we can obtain the desired result.

Finally, we consider the maximal operator of the diffusion semi-group

\[ T^*_b f(x) = \sup_{t > 0} e^{-tL} f(x) = \sup_{t > 0} \int_{\mathbb{R}^n} k_t(x, y) f(y) dy, \]

and it’s commutator

\[ T^*_b f(x) = \sup_{t > 0} e^{-tL} f(x) = \sup_{t > 0} \int_{\mathbb{R}^n} k_t(x, y)(b(x) - b(y)) f(y) dy, \]

where \( k_t \) is the kernel of the operator \( e^{-tL}, t > 0 \).

Theorem 4.1. Let \( b \in BMO(\rho) \) and \( T^*_b f \) be as above.

(i) If \( 1 < p < \infty, \omega \in A^p_\rho \), then there exists a constant \( C \) such that

\[ \|T^*_b f\|_{L^p(\omega)} \leq C\|b\|_{BMO(\rho)}\|f\|_{L^p(\omega)}. \]

(ii) If \( \omega \in A^1_\rho \), then there exists a constant \( C > 0 \) such that for any \( \lambda > 0 \)

\[ \omega(\{x \in \mathbb{R}^n : |T^*_b f(x)| > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) \omega(x) dx. \]

Proof. We first recall the kernel \( k_t \) has the following property (see [4])

\[ 0 \leq k_t(x, y) \leq C_N t^{-n/2} e^{-\frac{|x-y|^2}{2t}} \left(1 + \frac{t}{\rho(x)^2} + \frac{t}{\rho(y)^2}\right)^{-N}. \]
Then
\[
|T_0^* f(x)| \leq \sup_{t > 0} \int_{\mathbb{R}^n} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
\[
\leq \sup_{t > 0} \int_{|x-y| < \rho(x)} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
\[
+ \sup_{t > 0} \int_{|x-y| \geq \rho(x)} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
\[
:= I(x) + II(x).
\]

For $I(x)$, by (4.5), we then have
\[
I(x) \leq \sup_{0 < \sqrt{t} \rho(x)} \int_{|x-y| < \sqrt{t}} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
\[
+ \sup_{0 < \sqrt{t} \rho(x)} \int_{\sqrt{t} \leq |x-y| < \rho(x)} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
\[
+ \sup_{\sqrt{t} \rho(x) \leq |x-y| \leq \rho(x)} k_t(x, y) |(b(x) - b(y)) f(y)| dy
\]
[4.6]
\[
\leq C \sup_{0 < \sqrt{t} \rho(x)} \int_{|x-y| < \sqrt{t}} t^{-n/2} |(b(x) - b(y)) f(y)| dy
\]
\[
+ C \sup_{0 < \sqrt{t} \rho(x)} \int_{\sqrt{t} \leq |x-y| < \rho(x)} \sqrt{t} |x-y|^{-(n+1)} |(b(x) - b(y)) f(y)| dy
\]
\[
+ \sup_{\sqrt{t} \rho(x) \leq |x-y| \leq \rho(x)} \rho(x)^{-n} \int_{|x-y| < \rho(x)} |(b(x) - b(y)) f(y)| dy
\]
\[
\leq CM_{V,q}^{p,\theta} f(x).
\]

For $II(x)$, by (4.5) again, we then obtain that
\[
II(x) \leq \sup_{0 < t} t^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{|x-y| \geq \rho(x)} e^{-\frac{|x-y|^2}{2t}} |(b(x) - b(y)) f(y)| dy
\]
[4.7]
\[
\leq \sup_{0 < t} \left(\frac{\sqrt{t}}{\rho(x)}\right)^N \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}
\]
\[
\times \sum_{k=1}^{\infty} \frac{2^{-k(M-n-\theta_\eta)}}{2^k \rho(x)^n} \int_{|x-y| < 2^k \rho(x)} |b(x) - b(y)||f(y)| dy
\]
\[
\leq CM_{V,q}^{p,\theta} f(x),
\]
if $N > M > n + \theta_\eta$.

Thus, by (4.6) and (4.7), and using Lemmas 3.3, 3.4, 3.5, Theorem 2.2 and Proposition 2.2, we can obtain the desired result. \qed

We remark that in fact all results in this section also hold for $BMO_{\theta_1}(\rho)$ and $A_{p,\theta_2}^p$ if $\theta_1 \neq \theta_2$. 

References

[1] B. Bongioanni, E. Harboure and O. Salinas, Commutators of Riesz transforms related to Schrödinger operators, J. Fourier Ana Appl. 17(2011), 115-134.
[2] B. Bongioanni, E. Harboure and O. Salinas, Class of weights related to Schrödinger operators, J. Math. Anal. Appl. 373(2011), 563-579.
[3] J. Dziubański and J. Zienkiewicz, Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Math. Iber. 15 (1999), 279-296.
[4] J. Dziubański, G. Garrigós, J. Torrea and J. Zienkiewicz, $BMO$ spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality, Math. Z. 249(2005), 249 - 356.
[5] Z. Guo, P. Li and L. Peng, $L^p$ boundedness of commutators of Riesz transforms associated to Schrödinger operator, J. Math. Anal and Appl. 341(2008), 421-432.
[6] J. García-Cuerva and J. Rubio de Francia, Weighted norm inequalities and related topics, Amsterdam- New York, North-Holland, 1985.
[7] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal functions, Trans. Amer. Math. Soc. 165(1972), 207-226.
[8] C. Pérez, Endpoint estimates for commutators of singular integral operators, J. Funct. Anal. 128(1995), 163-185.
[9] M. M. Rao and Z. D. Ren, Theory of Orlicz spaces, Monogr. Textbooks Pure Appl. Math. 146, Marcel Dekker, Inc., New York, 1991.
[10] Z. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier. Grenoble, 45(1995), 513-546.
[11] S. Spanne, Some function spaces defined using the mean oscillation over cubes, Ann. Scuola. Norm. Sup. Pisa, 19(1965), 593-608.
[12] E. M. Stein, Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory integrals. Princeton Univ Press. Princeton, N. J. 1993.
[13] L. Tang, Weighted norm inequalities for Schrödinger type operators, preprint.
[14] J. Zhong, Harmonic analysis for some Schrödinger type operators, Ph.D. Thesis. Princeton University, 1993.