Abstract

We study axiomatic foundations for different classes of constant function automated market makers (AMMs). We focus particularly on independence, homogeneity and convexity properties and give axiomatic characterizations of a natural generalization of constant product AMMs that has not been considered so far in the decentralized finance context. Our results add to a formal analysis of mechanisms that are currently used for decentralized exchanges and introduces tools of decision theory to the topic.

1 Introduction

One of the first and so far most successful applications of Decentralized Finance (DeFi) are so called Automated Market Makers (AMMs). They are used to trade a pair of cryptocurrencies algorithmically without relying on a custodian or trusted third party. The mechanics of AMMs are simple. The state of an AMM consists of the current inventories of the traded pairs of tokens. Trades are made such that some invariant of these inventory sizes is kept constant. Traders who want to exchange one token \( A \) for another token \( B \), add \( A \) tokens to the inventory and in return obtain an amount of \( B \) tokens from the inventory so that the invariant is maintained. The simplicity of these so called constant function AMMs stems from the limited storage and computation capabilities of smart contract blockchains. While maintaining

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A traditional order book is prohibitively expensive on smart contract blockchain such as Ethereum, updating a simple function and inventories isn’t. Moreover, the AMMs allow market participants to provide liquidity passively without having to trade themselves. This makes liquidity provision in AMMs a popular way of getting (negative) exposure to volatility while being compensated by a steady stream of fee income.

AMMs were introduced to DeFi by Uniswap [Adams et al., 2021] that in its original version keeps a product of inventory sizes constant. Other protocols use weighted geometric means [Martinelli and Mushegian, 2019], a combination of a mean and a geometric mean [Egorov, 2019] or a translation of the product so that it intersects with the axes [Adams et al., 2021]. While these AMMs proved to be very popular and reliable, the construction of invariants to define them seems in many ways ad-hoc and not founded in much theory.

In this paper, we want to fill this gap and provide a theoretical approach to constructing constant function AMMs that is inspired by axiomatic theories of measurement as developed for example in Krantz et al. [1971] that play a role in economics, psychology and decision theory but also connect to the natural sciences. The approach is, as in any axiomatic theory, to formalize simple principles that are implicitly or explicitly used when constructing trading functions in practice (scale invariance to make liquidity positions fungible, symmetry, convexity to counter-weigh impermanent loss, independence of the exchange rate of a traded token pair from the inventory levels of tokens not involved in the trade, etc.) and to check which classes of function satisfy these principles, beyond those functions already used in practice. The axiomatic approach leads us to considerations and classes of function familiar from other fields in economics, consumer theory and production theory in particular.

We particularly focus on three types of axioms that

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1 This contrasts with the earlier literature on AMMs [Chen and Pennock, 2007; Abernethy et al., 2011; Othman et al., 2013] for prediction markets where different theoretical contributions have been made. However, the case of prediction markets is practically and theoretically quite different. Assets in a prediction market are binary Arrow-Debreu securities and AMMs are usually constructed so that the price converges quickly to an extreme, 0 or 1. In contrast to this, AMMs used in DeFi, in particular those used for stable coins, are constructed so that it is much harder/costlier to move the price in an extreme way. For example, the rule $f(I) = \log(\sum_{j \in J} e^{I_j})$ based on Hanson’s logarithmic scoring rule [Hanson, 2007] is frequently discussed in the literature on prediction markets. The rule induces very concave liquidity curves, whereas typical AMMs for DeFi such as the constant product rule induce convex liquidity curves.

2 Other connections to classical economic theory can be drawn as well. In Angeris et al. [2021], the authors introduce a dual way of defining AMMs through “portfolio value functions”. The concept is mathematically equivalent to the concept of an expenditure function familiar from consumer theory with the completely analogous duality theory of expenditure minimization and preference maximization. Note in particular, that the authors assume homogeneity of trading functions so
each are satisfied by a large class of AMMs, but in conjunction restrict the space of trading functions immensely:

The first one is scale invariance, or, in a stronger form, homogeneity of the trading function: If liquidity is tokenized and the tokenized liquidity is fungible, then trading functions are scale invariant. Geometrically this means that liquidity curves through different liquidity levels can be obtained from each other, by projection along rays through the origin, analogous to how homothetic preferences in consumer theory can be constructed. The second is independence: The terms of trade for trading a subset of token types should not depend on the inventory level of not-traded token types. Independence leads to the possibility of an additive representation of the trading function. The combination of scale invariance and independence leads to constant inventory elasticity: the terms of trade are fully determined by the inventory ratio of the pair traded, and, at the margin, percentage changes in exchange rates are proportional to percentage changes in inventory ratio. This is analogous to how the constant elasticity of substitution production function relates to relative factor prices. The third is aversion to (im)permanent loss: the exchange rate for swapping two tokens should be increasing in the size of the trade. Geometrically this translates to convex liquidity curves which is analogous to the quasi-concavity of utility functions in consumer theory. Combining the three axioms we obtain the class of constant inventory elasticity AMMs where aversion to (im)permanent loss guarantees that the elasticity is positive. This general class contains as special cases constant product AMMs, geometric averages as well as means.

The paper is organized in the following way. In the next subsection, we briefly review the literature. In Section 2 we introduce notation and introduce various axioms. In Section 3 we provide characterization results for the case of two asset types. In Section 3 we provide results for the case of more than two asset types.

1.1 Related Work

Automated market makers have first been analyzed scientifically in the context of prediction markets (Hanson, 2007; Othman and Sandholm, 2010; Chen and Pennock, 2007; Chen et al., 2008; Abernethy et al., 2011; Othman et al., 2013). See Footnote 1 for the differences between this application and cryptocurrency trading. More recent contribution focus on the application to DeFi: In Angeris and Chitra (2020) the authors study how well different AMMs function as a price oracle. In Engel and Herlihy (2021) the authors study axiomatization of composable AMMs. AMMs allowing exchanges of more than 2 tokens arise naturally by merging different AMMs.

that the AMM is globally defined by the liquidity curve through one inventory level.
exchanging only 2 token types. That is, a trader can split his trade across different AMMs. To fix the potential drain of resources from the AMM, the paper Krishna-machari et al. (2021) proposes dynamic curves that use an additional input from a market price oracle to dynamically adjust the liquidity curve.

2 Model and Axioms

Following the notation in Abernethy et al. (2011), we can define an automated market maker (AMM) for a finite set of asset types \( \mathcal{J} = \{A, B, \ldots\} \) as follows. The AMM holds an initial inventory \( I^0 \in \mathbb{R}_{+}^{\mathcal{J}} \) of tokens of type \( A, B, \ldots \) and makes a market where for each feasible sequence of trades \( r^1, r^2, \ldots, r^t \in \mathbb{R}^{\mathcal{J}} \) so far realized, there is a cost (measured in some numéraire)

\[
Cost(r|I^0, r^1, r^2, \ldots, r^t)
\]

of a new trade \( r \in \mathbb{R}^{\mathcal{J}} \). Cost is assumed to be positive and increasing in all dimensions of \( r \). For feasibility, we require that the current inventory is non-negative after the trade for both tokens, for \( t = 0, 1, 2 \ldots \),

\[
I^0 - (r^1 + \ldots + r^t + r) \geq 0.
\]

In decentralized exchanges, the inventory itself is usually tokenized and the unit of liquidity provider (LP) tokens can be used as numéraire. Moreover, instead of allowing for arbitrary trades, the AMM only allows for

1. (bilateral) swapping tokens without exchanging into a numéraire. Without transaction fees, this corresponds to trades \( r \) with

\[
Cost(r|I^0, r^1, r^2, \ldots, r^t) = 0.
\]

The curve of all trades \( r \) satisfying the previous equation is often called a "liquidity curve". Swaps are bilateral if \( r \) has only two non-zero entries.

2. adding or removing liquidity in exchange for the numéraire (LP tokens). Without transaction fees this corresponds to trades \( r \) that are multiples of the current inventory

\[
r = \lambda(I^0 - (r^1 + \ldots + r^t)),
\]

for some \( \lambda \in \mathbb{R} \).

\[3\text{Under path independence (see below) this restriction is without loss of generality.}\]
Abstracting away from transaction fees, popular AMMs usually satisfy the following axiom:

**Path Independence**: For each sequence of trades \( r^1, r^2, \ldots, r^t \in \mathbb{R}^J \), initial inventory \( I^0 \in \mathbb{R}^J \), and trades \( r, q \in \mathbb{R}^J \) we have

\[
\text{Cost}(r + q|I^0, r^1, r^2, \ldots, r^t) = \text{Cost}(r|I^0, r^1, r^2, \ldots, r^t) + \text{Cost}(q|I^0, r^1, r^2, \ldots, r^t, r).
\]

If path independence is satisfied we can define the liquidity curve by a **trading function** \( f : \mathbb{R}^J_+ \to \mathbb{R} \) that maps each inventory level, \( I \in \mathbb{R}^J_+ \), to a number \( f(I) \) that we can interpret as the value of the inventory measured in the numéraire. Then, defining the inventory after trades \( r^1, r^2, \ldots, r^t \) by

\[
I^t = I^0 - (r^1 + \ldots + r^t),
\]
we have

\[
\text{Cost}(r|I^0, r^1, r^2, \ldots, r^t) = 0 \iff f(I^t - r) = f(I^t),
\]
so that

\[
\text{Cost}(r|I^0, r^1, r^2, \ldots, r^t) = f(I^t) - f(I^t - r).
\]

Note that the function \( f \) needs to be increasing in all variables to guarantee that cost is increasing in trade size.

We introduce the following equivalence notion for path-independent AMMs: two trading functions \( f, g \) are **equivalent** (for market making) if for all inventories \( I, J \) we have

\[
f(I) = f(J) \iff g(I) = g(J).
\]

Two equivalent AMMs will always make the market in the same way in the sense that they induce the same family of liquidity curves and therefore offer the same terms of trade at the same inventory levels. However, they can differ in the cardinal measure of liquidity, e.g. by using different numéraires. Since trading functions \( f, g \) are increasing in all arguments, \( f \) and \( g \) are equivalent if and only if there is a strictly increasing function \( M \) on the reals such that \( M(f(I)) = g(I) \) for each \( I \in \mathbb{R}^J_+ \).

Under this equivalence relation it is without loss of generality to focus on trading functions that are non-negative \( f(I) \geq 0 \) for each \( I \in \mathbb{R}^J_+ \) since every trading function can be brought to this form by applying a monotonic transformation. From now on we focus on these kind of trading functions.

Popular examples of path independent liquidity curve AMMs are:
• (weighted) geometric means where

\[ f_{\text{product}}(I) = \prod_{j \in J} I_j^{\alpha_j}, \]

which for the case of two assets \( A, B \) and \( \alpha_A = \alpha_B \) contain the popular constant product rule,

• (weighted) means

\[ f_{\text{mean}}(I) = \sum_{j \in J} c_j I_j, \]

• "Uniswap V3" rules defined for the case of two assets

\[ f_{\text{V3}}(I_A, I_B) = \sqrt{(I_A + \alpha)(I_B + \beta)}, \]

for \( \alpha, \beta \geq 0 \),

• "curve rules"

\[ f_{\text{curve}}(I) = \sum_{j \in J} I_j - \alpha \prod_{j \in J} I_j^{-1}, \]

for \( \alpha > 0 \) To guarantee non-negativity, we can apply a monotonic transformation, such as the exponential function, to obtain an equivalent non-negative trading function.

We also define a class of AMMs that to our knowledge have not been studied in the literature so far\(^4\) which contain the geometric means and means as special cases. They are characterized by the property that, at the margin, the same percentage change in relative prices (for the definition see below) of two token types \( A \) and \( B \) always corresponds to the same percentage change in inventory ratio \( I_A/I_B \) (See Remark 1).

• **Constant inventory elasticity** AMMs parameterized by \(-\infty < k \leq 1\),

\[ f(I) = c \left( \sum_{j \in J} \alpha_j I_j^k \right)^{1/k}, \text{ for } k \neq 0 \]

and

\[ f(I) = c \prod_{j \in J} I_j^{\alpha_j}, \text{ for } k = 0 \]

for constants \( c \) and \( \alpha_j > 0 \) with \( \sum_{j \in J} \alpha_j = 1 \).

\(^4\)However, the functional form is of course well known in economics, in particular in the theory of production to describe production processes where percentage changes in choices of production factors change proportionally to percentage changes in factor prices.
2.1 Basic properties of AMMs

Next, we introduce several axioms on AMMs. In the following, we assume path independence so that we can formulate the axioms on the trading function $f$ directly, and we assume that $f$ is increasing in all arguments so that cost is increasing in trade size. It usually will be straightforward to generalize the notions to path-dependent AMMs.

First, we consider a smoothness condition that allows us, among other things, to define marginal prices:

**Existence of Marginal Prices:** The function $f$ is differentiable everywhere.

Note that all of the above examples satisfy this property. Under this assumption, the marginal price of token $A \in \mathcal{J}$ measured in the numéraire is $\frac{\partial f}{\partial I_A}$. Thus, the relative marginal price of tokens $A$ and $B$ for $A, B \in \mathcal{J}$ is

$$p_{A,B}(I) \equiv \frac{\frac{\partial f}{\partial I_A}}{\frac{\partial f}{\partial I_B}}.$$

This means that if the current inventory of the AMM is $I$, then swapping token $A$...
for token $B$ has, at the margin, an exchange rate of $p_{A,B}(I)$.

**Remark 1.** For the constant elasticity AMMs defined above, there is a proportional relation between the inventory ratio $I_B/I_A$ and changes in the relative marginal price of tokens $A$ and $B$: For the constant product AMM a one percent increase in the inventory ratio leads to a one percent increase in the relative marginal price. More generally, for AMMs in this class, a one percent increase in the inventory ratio leads to an increase of $\frac{1}{1-k}$ percent in the relative marginal price.

Sometimes we require the stronger smoothness condition that also the marginal prices themselves are differentiable:

**Existence and Smoothness of Marginal Prices:** The function $f$ is twice differentiable everywhere.

It is a desirable property of an AMM to have increasing (in the number of swapped tokens) marginal cost for swapping tokens. In other words, liquidity curves should be convex. This is related to the notion of (im)permanent loss for liquidity providers: The (im)permanent loss of a liquidity provider is given by the market exposure of the liquidity position: If the relative price of the two tokens moves a lot, holding and providing liquidity performs worse than keeping the equivalent (before the price change) amount of A and B tokens without providing liquidity. If the marginal cost for the appreciating token is increasing than inducing a price change that induces (im)permanent loss is more expensive for the market.

**Aversion to permanent loss:** For each constant $k > 0$ the curve

$$f(\cdot) = k$$

is convex, i.e., for each $0 < \lambda < 1$, $I, J \in \mathbb{R}_+^2$ with $f(I) = f(J)$ we have

$$f(\lambda I + (1 - \lambda)J) \geq f(I) = f(J).$$

If the trading function $f$ satisfies the existence of marginal prices, then aversion to permanent loss for two asset types is equivalent to assuming that the relative marginal price is decreasing as a function of $I_A$ along each liquidity curve, i.e. $p_{A,B}(I_A, I_B(I_A))$ is decreasing in $I_A$. All examples of trading functions considered above, satisfy aversion to permanent loss.

Next, we consider the property that the AMM is always able to make the market no matter how large the trade size. Geometrically this means that liquidity curves
do not intersect the axes. We write \( I > 0 \) if all dimensions of \( I \in \mathbb{R}_+^J \) are strictly positive and define:

**Sufficient Funds:** For each \( I, J \in \mathbb{R}_+^J \) with \( f(I) = f(J) \) we have

\[
I > 0 \Rightarrow J > 0.
\]

**Remark 2.** If the sufficient funds condition is violated, we have concentrated liquidity, the AMM accepts trades only up to a bounded size. This sometimes is desirable, for example for the purpose of risk management. The Uniswap V3 rules are an example of a trading function with concentrated liquidity and therefore of a violation of the sufficient funds condition. Geometric means satisfy the sufficient funds condition whereas means do not satisfy it. The constant elasticity AMMs satisfy it for elasticity smaller or equal to 1 and violate it for elasticity of more than 1.

The next requirement is that liquidity curves should be invariant under scaling of the inventory. The interpretation in the context of liquidity curve AMMs is straightforward: If liquidity is added or removed from the AMM proportionally to the current inventory levels for each token then the relative price of the token should not change.

**Scale invariance of relative prices:** For each \( I, J \in \mathbb{R}_+^J \) and \( \lambda > 0 \)

\[
f(I) = f(J) \Rightarrow f(\lambda I) = f(\lambda J).
\]

In practice liquidity is tokenized so that it becomes fungible and tradable. For this to be possible we need that the value of the inventory (that is tokenized) scales proportionally with the inventory. This leads to the stronger property of

**Homogeneity in Liquidity:** For each \( I \in \mathbb{R}_+^J \) and \( \lambda > 0 \) we have

\[
f(\lambda I) = \lambda f(I).
\]

It is straightforward to see that each scale invariant trading function is equivalent to a homogenous trading function.

**Remark 3.** From the above examples, the Uniswap V3 and the curve rule violate scale invariance whereas the other rules satisfy it. This means that liquidity can be made fungible for all of these rules except for the Uniswap V3 and curve rule.
Finally, we consider a symmetry axiom that requires that in the absence of additional information on the nature of the assets, the market should be made in the same way if the names of the assets are flipped:

**Symmetry** For each permutation $\pi : \mathcal{J} \rightarrow \mathcal{J}$ and $I \in \mathbb{R}_+^\mathcal{J}$ we have

$$f(I) = f((I_{\pi(j)})_{j \in \mathcal{J}}).$$

All above examples (for the constant elasticity coefficients need be chosen the same for all assets $c_A = c_B$ for $A, B \in \mathcal{J}$) satisfy symmetry.

### 3 Characterizations for two assets

In this section, we provide various axiomatizations of different classes of constant function AMMs for the case of two tokens. In Section 4 we consider extensions to the case of more than two token types.

#### 3.1 Concave AMMs under Symmetry

We start with a simple characterization result for the case of symmetric AMMs. The characterization result states that the space of all symmetric, scale invariant AMMs can be described by concave increasing bijections on the unit interval. The bijection encodes how the AMM should react to imbalances in the liquidity ratio $I_B/I_A$.

**Theorem 1.** A trading function $f$ satisfies homogeneity in liquidity, symmetry, sufficient funds and aversion to permanent loss if and only if it is of the form

$$f(I) = \begin{cases} 
  cI_A g \left( \frac{I_B}{I_A} \right), & \text{for } I_A \geq I_B \\
  cI_B g \left( \frac{I_A}{I_B} \right), & \text{for } I_B \geq I_A 
\end{cases}$$

for a constant $c > 0$ and a concave bijection $g : [0, 1] \rightarrow [0, 1]$.

**Proof.** It is straightforward to see that an AMM that is represented in this way satisfies the axioms. For the opposite direction, note that aversion to permanent loss means that $f$ is quasi-concave. By a theorem of Friedman (1973), monotonicity, homogeneity and quasi-concavity of $f$ implies that $f$ is a concave function. Next define $g(z) := f(1, z)/f(1, 1)$ and $c := f(1, 1)$. By homogeneity we have

$$cI_A g \left( \frac{I_B}{I_A} \right) = I_A f(1, \frac{I_B}{I_A}) = f(I).$$
Concavity implies coordinate-wise concavity, and therefore concavity of \( f \) implies that \( g \) is concave. By symmetry, the representation also works for \( I_B \geq I_A \). By concavity the function \( g \) is continuous. By sufficient funds, for each \( k > 0 \) the curve \( I_B(I_A) \) implicitly defined by \( f(I) = k \) satisfies \( \lim_{I_A \to 0} I_B(I_A) = \infty \) and therefore we have \( \lim_{z \to 0} g(z) = \lim_{I_A \to 0} g(I_A/I_B(I_A))) = \lim_{I_A \to 0} k \frac{I_A}{I_B(I_A)} = 0 \). By continuity we have \( g(0) = 0 \). By construction we have \( g(1) = 1 \). By monotonicity, this implies that \( g \) is a bijection of \([0, 1]\).

With the representation above the product AMM can be obtained through the function \( g(z) = \sqrt{z} \). Immediately from this representation it follows that relative price is proportional to the inventory ratio for these AMMs (as we already discussed previously), as through differentiation we obtain

\[
p_{A,B}(z) = \frac{g(z)}{g'(z)} - z
\]

where \( z \equiv I_B/I_A \) is the inventory ratio. The expression is linear in \( z \) for the square root function.
3.2 Additivity Axioms

Many popular AMMs used in practice, satisfy an additivity property in the sense that they can be represented by a liquidity curve of the form:

$$\phi_A(I_A) + \phi_B(I_B) = \text{const.}$$

Also the constant product AMM (or more generally weighted geometric means) belongs to this class because the curve can be brought to the above form by using a logarithmic transformation.

Additivity is more natural as an assumption for more than two asset types, since it is related to independence in that case (see Section 4): the terms of trade for exchanging a subsets of types of tokens should not depend on the inventory of other types of tokens not exchanged. For two assets, independence has no bite. However, additivity is still with loss of generality.\footnote{Examples of non-additive AMMs are the curve family defined above. There are also scale invariant AMMs that do not have an additive representation. For example polynomials such as $f(I) = I_A I_B^2 + I_A^2 I_B$.} To guarantee additivity, instead we need that liquidity provision is additive in the following sense: if starting from an inventory level $I$ adding $x$ units of type $A$ tokens to the inventory leads to the same issuance of LP tokens as adding $y$ units of type $B$ tokens, and if starting from an inventory level $(I_A + x, I_B + y)$ adding $\bar{x}$ units of type $A$ tokens to the inventory leads to the same issuance of LP tokens as adding $\bar{y}$ units of type $B$, then starting from $I$ adding $x + \bar{x}$ units of type $A$ tokens to the inventory leads to the same issuance of LP tokens as adding $y + \bar{y}$ units of type $B$ tokens. This leads to the following condition which is often called the Thomsen condition (Krantz et al. [1971]) in decision theory:

**Liquidity Provision (LP) Additivity:** For each $I$ and quantities $x, y, \bar{x}, \bar{y}$ such that

$$f(I_A, I_B + y) = f(I_A + x, I_B), \quad f(I_A + x + \bar{x}, I_B + y) = f(I_A + x, I_B + y + \bar{y})$$

we have

$$f(I_A + x + \bar{x}, I_B) = f(I_A, I_B + y + \bar{y}).$$

Combining additivity with scale invariance imposes additional structure on the functions $\phi_A$ and $\phi_B$. With scale invariance we can show that the functions $\phi_A$ and $\phi_B$ have to be of the power form ($\phi_A = c_A I_A^k$ resp. $\phi_B = c_B I_B^k$) or logarithmic form so that we obtain the following theorem:
Theorem 2. A trading function satisfies existence and smoothness of marginal prices, \textit{LP} additivity, scale invariance and aversion to permanent loss if and only if it is equivalent to an AMM with constant inventory elasticity.

Proof. One readily checks that constant inventory elasticity AMMs satisfy all of the axioms.

Monotonicity and additivity imply by a classical result (see Krantz et al. [1971] Section 6.2) that \( f \) is equivalent to a trading function \( \tilde{f} \) for which there are one-dimensional, increasing functions \( \phi_A, \phi_B \) such that

\[
\tilde{f}(I) = \phi_A(I_A) + \phi_B(I_B).
\]

The construction of the additive representation proceeds by constructing a self mapping of the plane that maps liquidity curves into lines with negative slope, and lines parallel to the axes into lines parallel to the axes. It is straightforward to see that if liquidity curves are smooth, then the mapping can be constructed to be smooth as well, i.e. the existence and smoothness of marginal prices allows to obtain a twice differentiable \( \tilde{f} \), as well as twice differentiable \( \phi_A \) and \( \phi_B \). Next, we show that the functions \( \phi_A, \phi_B \) have a derivative with constant elasticity.
Let $I, J$ be inventory levels with $f(I) = f(J)$. By scale invariance for each $\lambda > 0$,

$$\phi_A(\lambda I_A) + \phi_B(\lambda I_B) = \phi_A(\lambda J_A) + \phi_B(\lambda J_B).$$

Differentiating the above with respect to $\lambda$ and evaluating it at $\lambda = 1$ we have

$$I_A \phi'_A(IA) + I_B \phi'_B(IB) = J_A \phi'_A(IA) + J_B \phi'_B(IB).$$

This identity is often called the Euler theorem for homogeneous functions.

Next, we fix a liquidity curve through an inventory level $J$ and consider the curve $I_B(IA)$ implicitly defined by $f(I) = f(J)$. Differentiating with respect to $IA$, we obtain:

$$I_A \phi''_A(IA) + \phi'_A(IA) + (I_B \phi''_B(IB) + \phi'_B(IB)) \frac{dIA}{dIB} = I_A \phi''_A(IA) + \phi'_A(IA) -(I_B \phi''_B(IB) + \phi'_B(IB)) \frac{\phi'_A(IA)}{\phi'_B(IB)} = 0$$

Rearranging the above we obtain

$$I_A \frac{\phi''_A(IA)}{\phi'_A(IA)} = I_B \frac{\phi''_B(IB)}{\phi'_B(IB)}.$$

Since the inventory $J$ was chosen arbitrarily, the above equation holds for any inventory levels. In particular, for $I_A \neq IA$ and $I_B$ we have

$$I_A \frac{\phi''_A(IA)}{\phi'_A(IA)} = I_B \frac{\phi''_B(IB)}{\phi'_B(IB)} = I_A \frac{\phi''_A(IA)}{\phi'_A(IA)} \equiv k - 1.$$

Therefore, $\phi'_A$ has constant elasticity $k - 1$ and as a consequence so does $\phi'_B$. The derivatives are of the form

$$\phi'_A(IA) = c_A I_A^{k - 1}, \quad \phi'_B(IB) = c_B I_B^{k - 1}$$

for constants $c_A, c_B > 0$. The constants are positive since the functions $\phi_A$ and $\phi_B$ are strictly increasing. By integration, we obtain

$$\phi_A(IA) = \begin{cases} \tilde{c}_A I_A^k + d_A, & k \neq 0, \\ \tilde{c}_A \log(I_A), & k = 0. \end{cases}, \quad \phi_B(IB) = \begin{cases} \tilde{c}_B I_B^k + d_B, & k \neq 0, \\ \tilde{c}_B \log(I_B), & k = 0. \end{cases}$$

for constants $\tilde{c}_A, \tilde{c}_B > 0$ and $d_A, d_B$. In the case $k \neq 0$, (by subtracting $d_A + d_B$ which is a monotone transformation $f$ (and therefore) $f$ is equivalent to the trading function

$$\tilde{c}_A I_A^k + \tilde{c}_B I_B^k.$$

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In the case $k = 0$ (by applying the transformation $M(z) = e^{\frac{z}{c_1 + c_2}}$), $\tilde{f}$ (and therefore $f$) is equivalent to the trading function

$$\int_{I_A^{c_1+c_2}} \int_{I_B^{c_1+c_2}}.$$

Finally note that the liquidity curves defined in the above way is convex if and only if $k \leq 1$. Thus, by the aversion to permanent loss we have $k \leq 1$.

It is straightforward to see that if scale invariance is replaced by homogeneity in the above characterization the AMM is not only equivalent to a constant inventory elasticity AMM, but it actually is such an AMM.

**Corollary 1.** A trading function satisfies existence and smoothness of marginal prices, LP additivity, homogeneity in liquidity and aversion to permanent loss if and only if it is an AMM with constant inventory elasticity.

**Proof.** We have a homogeneous liquidity function $f$ and a strictly increasing function $M$ such that

$$f(I) = M(c_A I_A^k + c_B I_B^k)$$

respectively

$$f(I) = M(I_A^\alpha I_B^{1-\alpha}).$$

In the first case, by homogeneity in liquidity, the monotone transformation $M$ is of the form

$$M(z) = cz^{1/k}$$

for a constant $c > 0$. For $k = 0$, by homogeneity in liquidity, the monotone transformation is of the form $M(z) = cz$. Therefore the AMM defined by $f$ is of the constant elasticity form. \(\square\)

Constant elasticity liquidity curves intersect with the axes only if the elasticity is greater than 1. Therefore we have the following corollary:

**Corollary 2.** A trading function satisfies existence and smoothness of marginal prices, LP additivity, scale invariance and sufficient funds if and only if it is equivalent to an AMM with constant inventory elasticity $\frac{1}{1-k} \leq 1$.

**Proof.** This follows from the fact that a curve $I_B(I_A)$ implicitly defined by $c_A I_A^k + c_B I_B^k = \text{const.}$ intersects with the axes for $k > 0$ but not for $k < 0$. For $k = 0$ one immediately sees that geometric averages satisfy the sufficient funds condition. \(\square\)
4 Results for more than two asset types

In this section, we consider generalization of the previous results to the case of more than two assets. For more than two assets the key axioms is independence: Independence requires that the terms of trade for trading a subset of tokens $K \subseteq J$ does not depend on the inventory levels for the other tokens $J \setminus K$. The independence axiom only has bite for more than two token types.

**Independence:** For each subset of token types $K \subseteq J$ and inventories $I, J$ we have

$$f(I_K, J_{-K}) = f(J) \iff f(I) = f(J_K, I_{-K})$$

For more than two assets independence and monotonicity allow us to obtain an equivalent additive representation (see Krantz et al. (1971), Section 6.11) of the trading function

$$\sum_{j \in J} \phi_j(I_j) = \text{const.}$$

The possibility of such representation crucially depends on the assumption of more than two assets since we already have seen that for two assets we would need an additional axiom namely the LP additivity axiom which we do not need for the case of more than two assets. For more than two assets we obtain versions of Theorem 2 and Corollaries 1 and 2 where the independence axiom replaces the LP additivity axiom.

**Theorem 3.** A trading function for $|J| > 2$ assets satisfies existence and smoothness of marginal prices, independence, scale invariance in liquidity and aversion to permanent loss if and only if it is equivalent to an AMM with constant inventory elasticity.

**Proof.** One readily checks that constant inventory elasticity AMMs satisfy all of the axioms.

Monotonicity and additivity imply by a classical result (see Krantz et al. (1971) Section 6.11) that $f$ is equivalent to a trading function $\tilde{f}$ for which there are one-dimensional, increasing functions $\{\phi_j\}_{j \in J}$ such that

$$\tilde{f}(I) = \sum_{j \in J} \phi_j(I_j).$$

As in the two-dimensional case, the existences and smoothness of marginal prices allows to choose $\tilde{f}$ and $\{\phi_j\}_{j \in J}$ twice differentiable.
Next we show that the functions \( \{ \phi_j \}_{j \in J} \) have a derivative with constant elasticity. Let \( I, J \) be inventory levels with \( f(I) = f(J) \). By scale invariance, and an analogous argument as in the two-dimensional case:

\[
\sum_{j \in J} I_j \phi'_j(I_j) = \sum_{j \in J} J_j \phi'_j(J_j).
\]

Next, let \( A, B \in J \) be asset types. We fix a liquidity curve through an inventory level \( J \) and consider the curve \( I_B(I_A) \) implicitly defined by \( f(I_{\{A,B\}}, J-\{A,B\}) = f(J) \). Differentiating with respect to \( I_A \), we obtain:

\[
I_A \phi''_A(I_A) + \phi'_A(I_A) + (I_B \phi''_B(I_B) + \phi'_B(I_B)) \frac{d \ln}{d I_A} = I_A \phi''_A(I_A) + \phi'_A(I_A) - (I_B \phi''_B(I_B) + \phi'_B(I_B)) \frac{\phi'_A(I_A)}{\phi'_B(I_B)} = 0.
\]

Rearranging the above we obtain

\[
I_A \phi''_A(I_A) = I_B \phi''_B(I_B).
\]

Since the inventory \( J \) and the asset types \( A, B \) were chosen arbitrarily, the above equation holds for any inventory levels and pair of asset types. Therefore, all functions \( \phi'_j \) have the same constant elasticity \( k - 1 \). The derivatives are of the form

\[
\phi'_j(I_j) = c_j I_j^{k-1}, \quad \text{for } j \in J
\]

for constants \( c_j > 0 \). The constants are positive since the functions \( \{ \phi_j \}_{j \in J} \) are strictly increasing. By integration, we obtain

\[
\phi_j(I_j) = \begin{cases} 
\tilde{c}_j I_j^k + d_j, & k \neq 0, \\
\tilde{c}_j \log(I_j), & k = 0,
\end{cases} \quad \text{for } j \in J,
\]

for constants \( \tilde{c}_j > 0 \) and \( d_j \). In the case \( k \neq 0 \), (by subtracting \( \sum_{j \in J} d_j \) which is a monotone transformation) \( \hat{f} \) and therefore \( f \) is equivalent to the trading function

\[
\sum_{j \in J} \tilde{c}_j I_j^k.
\]

In the case \( k = 0 \) (by applying the transformation \( M(z) = e^{\sum_{j \in J} \tilde{c}_j z} \)), \( \hat{f} \) and therefore \( f \) is equivalent to the trading function

\[
\prod_{j \in J} I_j^{\sum_{j \in J} \tilde{c}_j}.
\]

Finally note that the liquidity curves defined in the above way is convex if and only if \( k \leq 1 \). Thus, by the aversion to permanent loss we have \( k \leq 1 \). \( \square \)
We also obtain corresponding corollaries to Corollaries 1 and 2. The proofs are completely analogous to the two-dimensional case.

**Corollary 3.** A trading function for $|\mathcal{J}| > 2$ assets satisfies existence and smoothness of marginal prices, independence, homogeneity in liquidity and aversion to permanent loss if and only if it is an AMM with constant inventory elasticity.

**Corollary 4.** A trading function for $|\mathcal{J}| > 2$ assets satisfies existence and smoothness of marginal prices, independence, scale invariance in liquidity and sufficient funds if and only if it is an AMM with constant inventory elasticity $\frac{1}{1-k} \leq 1$.

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