Einstein-Scalar Field System with a cosmological constant on the type I Bianchi space-time

Alexis Nangue
Department of Mathematics
Higher Teacher’s Training College,
University of Maroua, PO.Box 55, Maroua, Cameroon
alexnanga02@yahoo.fr

Abstract

In many cases a scalar field can lead to accelerated expansion in cosmological models. This paper contains mathematical results on this subject particularly on type I Bianchi space-time. In this paper, global existence to the coupled Einstein-scalar field system which rules the dynamics of a kind of pure matter in the presence of a scalar field and cosmological constant is proven.

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1 Introduction

General Relativity is a theory of gravitation, which states that the gravitational attraction that is observed between the masses caused by deformation of space and time through these masses and not as an attractive force between the masses as in the theory of Newton gravitation law. From then space and time are no longer indissociable. General Relativity abandons the notion of force and replaces it with the concept of curvature of space-time. To be complete, this theory must also provide a means of calculating the curvature of space-time created by mass distribution. It does this through a complex system of mathematical formulas : the Einstein equations linking the geometry of space-time and properties of the matter. In global dynamics, the search for solutions to Einstein equations coupled in different material fields remains an active area of research particularly. When considering the Einstein equations on Space-time which have surface symmetry, we can always eliminate any phenomenon of wave propagation by a suitable coordinates choice.
But experiments have shown the existence of gravitational waves. In the context of General Relativity, gravitational waves are defined as disturbances of the metric that, from the point of view of the Einstein equations are decoupled from disturbances of energy-momentum tensor. One way to model the phenomenon of gravitational waves is to introduce a scalar field in the gravitation sources. This is the focus of this paper. Several studies have already been carried out on the notion of scalar field like the works of [6], [7] [2] [4] and [5].

We choose the Einstein equations with constant cosmological; this interest is a physical reason. Indeed, astrophysical observations, based on the redshift light spectrum, showed that the universe was accelerating expansion. It is the presence of the cosmological constant in Einstein equations that can mathematically model this phenomenon. An important part of General Relativity is cosmology, which is the study of the structure and evolution of the whole universe. The geometric frame selected here is the type I Bianchi space-time of Generalizing the Robertson-Walker space-time is homogeneous and isotropic: the latter being the background area of cosmology. The phenomena studied here are called homogeneous, that is, they depend only on time. Indeed, in the space-time, observers located on the same constant time hypersurface see exactly the same events so that only the evolution over time to be really significant. We study here the existence of a global solution, that is, defined on, \([0; +\infty[\), of the homogeneous Einstein-scalar field system on a type I Bianchi space-time with a perfect fluid model pure radiation type.

Unless otherwise specified, Greek indices range from 0 to 3, and Latin indices from 1 to 3. We adopt the Einstein summation convention \(a_\alpha b^\alpha = \sum_{\alpha=0}^{3} a_\alpha b^\alpha\). We consider the Bianchi type I space-time \((\mathbb{R}^4, g)\) and we denote by \(x^\alpha = (x^0, x^i) = (t, x^i)\), the usual coordinates in \(\mathbb{R}^4\); \(g\) stands for the unknown metric tensor of Lorentzian type with signature \((-\,+,\,+,\,+\)) which can be written:

\[
g = -dt^2 + a^2(t)(dx^1)^2 + b^2(t) \left[(dx^2)^2 + (dx^3)^2\right]
\]  

(1)

where \(a > 0\) and \(b > 0\) are unknown functions of the single variable \(t\).

The Einstein-Scalar Field system with cosmological constant reads as follows, according to [1]:

\[
\begin{cases}
R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta}) \\
\nabla_\alpha \phi = 0 \\
T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2}g_{\alpha\beta}\nabla^\lambda \phi \nabla_\lambda \phi \\
\tau_{\alpha\beta} = \frac{4}{3}\rho u_\alpha u_\beta + \frac{1}{3}\rho g_{\alpha\beta}
\end{cases}
\]

(2)  

(3)  

(4)  

(5)
where:

- (2) are the Einstein equations for the metric tensor $g = (g_{\alpha\beta})$ which represents the gravitational field; $R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor; $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature, contracted of the Ricci tensor.

- (3) is the wave equation in $\phi$ which represents the scalar field. Recall that $\nabla_\alpha$ is the covariant differentiation in $g$, and are raised and lowered following the rules: $V^\alpha = g^{\alpha\beta}V_\beta; V_\alpha = g_{\alpha\beta}V^\beta$, where $(g^{\alpha\beta} = g_{\alpha\beta})^{-1}$.

- The ordinary matter is modeling by (5), which represents the relativistic perfect fluid of pure radiation type, in which $\rho \geq 0$ is an unknown function of single variable $t$, representing the matter density. For simplicity, we consider a co-moving fluid, which means that $u^i = u_i = 0$, where $u = (u^\alpha)$ is a future time-like unit vector (i.e $g_{\alpha\beta}u^\alpha u^\beta = -1$, $u^0 > 0$).

- (4) represents the stress-matter-energy tensor associated to a scalar field $\phi$, which is as $\rho$ a real-valued function of $t$.

Now, recall that, solving the Einstein equations is determining both the gravitational field and its sources: this means that we have to determine every unknown function introduced above, namely: $a$, $b$, $\rho$ and $\phi$. Notice that the spatially homogeneous coupled Einstein-Scalar Field system turns out to be a non linear second differential system. What we call global solution in this paper, is a solution defined all over the interval $[0, +\infty[$.

The paper is organized as follows:

- In section 2, we write the Einstein-Scalar field system in a explicit form.

- In section 3, we introduce the Cauchy problem and we prove the local existence of solutions.

- In section 4, we prove the global existence of existence.
2 Einstein-Scalar Field System in \( a, b, \rho, \phi \)

In this section we are going to write the equations (2) in explicit form, and afterwards we proceed to a suitable change of unknown functions. The evolution of solutions of the Einstein-Scalar Field system with a cosmological constant on the Bianchi type I space-time models, described by a perfect fluid with matter density \( \rho \), are governed following [5], by the constraint equation

\[
\left( \frac{\dot{b}}{b} \right)^2 + 2 \frac{\ddot{a}}{a} - \Lambda = 8\pi \rho + 4\pi \dot{\phi}^2, \tag{6}
\]

named Hamiltonian equation, \(^1\) the evolution equations,

\[
\left( \frac{\dot{b}}{b} \right)^2 + 2 \frac{\ddot{b}}{b} - \Lambda = -\frac{8\pi \rho}{3} - 4\pi \dot{\phi}^2, \tag{7}
\]

\[
\frac{\ddot{a}}{a} + \frac{\dot{a} \dot{b}}{ab} + \frac{\dot{b}}{b} - \Lambda = -\frac{8\pi \rho}{3} - 4\pi \dot{\phi}^2, \tag{8}
\]

and the equations in \( \phi \) and \( \rho \), resulting from (3) and conservation equation, given by :

\[
\ddot{\phi} \dot{\phi} + \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) \dot{\phi}^2 = 0 \tag{9}
\]

and

\[
\dot{\rho} + \frac{4}{3} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \rho = 0. \tag{10}
\]

In the next paragraphs, we study the local and global existence of solutions \( a, b, \rho \) and \( \phi \) to the coupled system (7), (8), (9), (10) subject to constraint (6). For this purpose, we make a change of unknown functions in order to deduce an equivalent first order differential system to which standard theory is applied. We set :

\[
u = \frac{\dot{a}}{a} ; \ v = \frac{\dot{b}}{b} ; \ \psi = \frac{1}{2} \dot{\phi}^2. \tag{11}\]

We deduce from (11) :

\[
\frac{\ddot{a}}{a} = u + u^2 ; \ \frac{\ddot{b}}{b} = v + v^2. \tag{12}
\]

We choose to look for a \( C^2 \)-non-decreasing scalar field (i.e. \( \dot{\phi} \geq 0 \)), then (11) gives :

\[
\dot{\phi} = \sqrt{2} \psi^{\frac{1}{2}}. \tag{13}
\]

\(^1\) Overdot denotes differentiation with respect to time \( t \).
According to (2) and (3) we deduce from (7), (8), (9), (10) the equivalent first order differential system:

\[
\begin{align*}
\frac{du}{dt} &= \frac{2}{3}\Lambda - u^2 + \frac{1}{3}v^2 - \frac{4}{3}uv - \frac{8}{3}\pi\psi \\
\frac{dv}{dt} &= \frac{2}{3}\Lambda - \frac{5}{3}v^2 - \frac{1}{3}uv - \frac{8}{3}\pi\psi \\
\frac{d\phi}{dt} &= \sqrt{2}\psi^\frac{1}{2} \\
\frac{d\rho}{dt} &= -\frac{4}{3}(u + 2v)\rho \\
\frac{d\psi}{dt} &= -2(u + 2v)\psi
\end{align*}
\]

subject to the constraint:

\[v^2 + 2uv - \Lambda = 8\pi\rho + 8\pi\psi,\]

which we are going to study.

### 3 Cauchy problem and constraint

Let \(a_0 > 0, \dot{a}_0, b_0 > 0, \dot{b}_0, \phi_0, \dot{\phi}_0 > 0, \rho_0\) be given real numbers. We look for solutions \(a, b, \rho\) and \(\phi\) of the Einstein-Scalar field system over \([0, T], T \leq +\infty\) satisfying:

\[
a(0) = a_0; \dot{a}(0) = \dot{a}_0; b(0) = b_0; \dot{b}(0) = \dot{b}_0; \phi(0) = \phi_0; \dot{\phi}(0) = \dot{\phi}_0; \rho(0) = \rho_0.
\]

Our objective now is to prove the local existence of solution satisfying (20), called initial conditions, with the given numbers \(a_0, \dot{a}_0, b_0, \dot{b}_0, \phi_0, \dot{\phi}_0, \rho_0\) being the initial data.

It is well known that equation (6) called Hamiltonian constraint is satisfied all over the domain of the solutions of evolution equations, if and only if equation (6) is satisfied at \(t = 0\) i.e given (20) if the initial data satisfy:

\[
\left(\frac{\dot{b}_0}{b_0}\right)^2 + 2\frac{\dot{a}_0}{a_0}\frac{b_0}{b_0} - \Lambda = 8\pi\rho_0 + 4\pi\dot{\phi}_0^2,
\]

which is calling the initial constraint. Now we are going to study the equivalent first order differential system (14) to (18), subject to constraint (18) and with the initial conditions at \(t = 0\), provided by (20):

\[
u(0) := u_0 = \frac{\dot{a}_0}{a_0}; \quad v(0) := v_0 = \frac{\dot{b}_0}{b_0}; \quad \rho(0) = \rho_0; \quad \psi(0) := \psi_0 = \dot{\phi}_0; \quad \phi(0) = \phi_0.
\]
4 Local existence of solutions

We use an iterative scheme.

4.1 Construction of the iterated sequence

We construct the sequence $S_n = (u_n, v_n, \rho_n, \psi_n, \phi_n), n \in \mathbb{N}$, as follows:

- Set $u_0 = u(0); v_0 = v(0); \rho_0 = \rho(0); \psi_0 = \psi(0); \phi_0 = \phi(0)$ where $u_0$, $v_0$, $\rho_0$, $\psi_0$, $\phi_0$ are initial data which satisfy constraint equation.

- Define $S_{n+1} = (u_{n+1}, v_{n+1}, \rho_{n+1}, \psi_{n+1}, \phi_{n+1})$ as solution of the ordinary differential equations obtained by substituting $u, v, \rho, \psi, \phi$ in the right hand side of the evolution system (14) to (18).

It is very important to notice that, for every $n$ the initial data for the ordinary differential equations are the same initial data $u_0, v_0, \rho_0, \psi_0$ and $\phi_0$. We obtain through this way a sequence $S_n = (u_n, v_n, \rho_n, \psi_n, \phi_n), n \in \mathbb{N}$ defined in a maximal interval $[0, T_n], T_n$.

4.2 Boundedness of the iterated sequence

**Proposition 4.1** There exits $T > 0$, $T$ independent on $n$, such that the iterated sequence $S_n = (u_n, v_n, \rho_n, \psi_n, \phi_n)$ is defined and uniformly bounded over $[0, T_n], T_n$.

**Proof 4.1** Let $N \in \mathbb{N}, N > 1$, be an integer. Suppose that we have, for $n \leq N - 1$, the inequalities

$$|u_n - u_0| \leq C_1, \quad |v_n - v_0| \leq C_2, \quad |\rho_n - \rho_0| \leq C_3, \quad |\psi_n - \psi_0| \leq C_4, \quad |\phi_n - \phi_0| \leq C_5$$

(22)

where $C_i > 0, i = 1, \ldots, 5$ are given constants. We are going to prove that one can choose the constants $C_i$ such that (22) still holds for $n = N$ on $[0, T], T > 0$ sufficiently small. Integrating over $[0, t], 0 \leq t \leq T$, the ordinary differential equations satisfied by $u_N, v_N, \rho_N, \psi_N, \phi_N$ yields:

$$|u_N - u_0| \leq B_1 t, \quad |v_N - v_0| \leq B_2 t, \quad |\rho_N - \rho_0| \leq B_3 t, \quad |\psi_N - \psi_0| \leq B_4 t \leq, \quad |\phi_N - \phi_0| \leq B_5 t$$

(23)

where $B_i > 0, i = 1, \ldots, 5$ are constants depending only on the constant $C_i$. If we choose $T > 0$ such that $B_i T < C_i, i = 1, \ldots, 5$. Hence for $n = N$, the iterated sequence $(S_n)$ is defined and uniformly bounded over $[0, T]$. 
4.3 Local existence and uniqueness of solution

**Theorem 4.2** The initial value problem for the Einstein-Scalar Field system on Bianchi type I space-time has a unique local solution.

**Proof 4.2** We are going to prove that the iterated sequence \((S_n)\) converges uniformly on each bounded interval \([0, \zeta] \subset [0, T]\), \(\zeta\), towards a solution \(S = (u, v, \rho, \psi, \phi)\) of the evolution system. For this purpose, we study the difference \(S_{n+1} - S_n\). But given the evolution equation (14) to (18) in \(\phi\) and \(\psi\), we will deal with the difference :

\[
\sqrt{2}\psi_n - \sqrt{2}\psi_{n+1} = \frac{2(\psi_n - \psi_{n+1})}{\sqrt{2}\psi_n - \sqrt{2}\psi_{n+1}}.
\]

We then need to show first of all that the sequence \(\frac{1}{\sqrt{2}\psi_n}\) is uniformly bounded.

- By (18), the iterated equation providing \(\psi_{n+1}\) writes :

\[
\dot{\psi}_{n+1} = -2(u_n + 2v_n)\rho_n
\] **(24)**

but by proposition 4.1, there exists a constant \(C > 0\) such that we have over \([0, T]\) :

\[
| -2(u_n + 2v_n)\psi_n| \leq C;
\]

**(24)** then gives :

\[
\frac{d\psi_{n+1}}{dt} \geq -C.
\]

and integrating over \([0, t]\), \(0 \leq t \leq T\) yields:

\[
\psi_{n+1} \geq \psi_0 - Ct.
\]

Recall that \(\psi_0 > 0\); then taking \(t\) sufficiently small such that \(Ct \leq \frac{\psi_0}{2}\), we have \(\psi_{n+1} \geq \frac{\psi_0}{2}\). Then

\[
\frac{1}{\sqrt{2}\psi_{n+1}} \leq \frac{1}{\sqrt{\psi_0}}
\]

which shows that \(\frac{1}{\sqrt{2}\psi_n}\) is uniformly bounded over \([0, T]\), \(T > 0\) small enough.

- Taking the difference between two consecutive iterated equations we deduce from the evolution equations, using \(S_n(0) = S_0, \forall n\), that there exists a constant \(C_2 > 0\) such that :

\[
|u_{n+1}(t) - u_n(t)| + |v_{n+1}(t) - v_n(t)| + |\rho_{n+1}(t) - \rho_n(t)| + |\psi_{n+1}(t) - \psi_n(t)|
\]

\[
+ |\phi_{n+1}(t) - \phi_n(t)| \leq C_2 \int_0^t (|u_n(s) - u_{n-1}(s)| + |v_n(s) - v_{n-1}(s)| + |\rho_n(s) - \rho_{n-1}(s)|
\]

\[
+ |\psi_n(s) - \psi_{n-1}(s)| + |\phi_n(s) - \phi_{n-1}(s)|)ds
\] **(25)**
For the same reasons we have:

\[
\left| \frac{du_{n+1}}{dt}(t) - \frac{du_n}{dt}(t) \right| + \left| \frac{dv_{n+1}}{dt}(t) - \frac{dv_n}{dt}(t) \right| + \left| \frac{d\rho_{n+1}}{dt}(t) - \frac{d\rho_n}{dt}(t) \right| + \left| \frac{d\psi_{n+1}}{dt}(t) - \frac{d\psi_n}{dt}(t) \right|
\]

\[
+ \left| \frac{d\phi_{n+1}}{dt}(t) - \frac{d\phi_n}{dt}(t) \right| \leq C_3 \left| u_n(t) - u_{n-1}(t) \right| + \left| v_n(t) - v_{n-1}(t) \right| + \left| \rho_n(t) - \rho_{n-1}(t) \right|
\]

\[
+ \left| \psi_n(t) - \psi_{n-1}(t) \right| + \left| \phi_n(t) - \phi_{n-1}(t) \right| ds.
\]  

(26)

For \( n \in \mathbb{N} \), we set:

\[
\beta_n(t) = \left| u_{n+1}(t) - u_n(t) \right| + \left| v_{n+1}(t) - v_n(t) \right| + \left| \rho_{n+1}(t) - \rho_n(t) \right| + \left| \psi_{n+1}(t) - \psi_n(t) \right| + \left| \phi_{n+1}(t) - \phi_n(t) \right|
\]  

(27)

(25) and (27) give:

\[
\beta_n(t) \leq C_2 \int_0^t \beta_{n-1}(s) ds.
\]  

(28)

By induction on \( n \geq 2 \), we obtain, from (28):

\[
\left| \beta_n(t) \right| \leq \left\| \beta_2 \right\| \frac{(C_2 t)^{n-2}}{(n-2)!} \leq \left\| \beta_2 \right\| \frac{(C_2 \zeta)^{n-2}}{(n-2)!}
\]  

(29)

for \( 0 \leq t \leq \zeta \) and \( 0 < \zeta < T \). But the series \( \sum_{n=0}^{+\infty} \frac{C^n}{n!} \) converges. Hence we obtain from (29) that:

\[
\lim_{t \to +\infty} \sup_{0 \leq t \leq \zeta} \beta_n(t) = 0.
\]

According to definition (28) of \( \beta_n \), we conclude that every sequence \( u_n, v_n, \rho_n, \psi_n \) and \( \phi_n \) converges uniformly on every interval \( [0, \zeta] \), \( 0 < \zeta < T \) and we denote the different limits by \( u, v, \rho, \psi \) and \( \phi \) are continuous functions of \( t \).

Now from the inequality (26), we conclude similarly that the sequences of derivatives \( \left( \frac{du_n}{dt} \right), \left( \frac{dv_n}{dt} \right), \left( \frac{d\rho_n}{dt} \right), \left( \frac{d\psi_n}{dt} \right) \), \( \left( \frac{d\phi_n}{dt} \right) \) converge uniformly on \( [0, \zeta] \), \( 0 < \zeta < T \). In this condition, the functions \( u, v, \rho, \psi \) and \( \phi \) are of class \( C^1 \) on \( [0, T] \). Hence \( S = (u, v, \rho, \psi, \phi) \) is a local solution of the system (14) to (18).

We now prove that the solution is unique. Consider two solutions \( S_1 \) and \( S_2 \) of the same initial values problem. Define \( \beta(t) = |S_1 - S_2| \) with \( \beta(0) = 0 \). Since the functions \( u, v, \rho, \psi \) and \( \phi \) are bounded on \( [0, \zeta] \), \( 0 < \zeta < T \), there exists a constant \( C > 0 \) such that:

\[
\beta(t) \leq C \int_0^t \beta(s) ds.
\]

By Gronwall lemma, we obtain \( \beta(t) = 0 \) since \( \beta(0) = 0 \), \( S_1 = S_2 \) and the local solution is unique. This completes the proof of proposition 4.1.
5 Global existence of solutions

What we want to know now is whether, the solution found previously is global. So, by always following the standard theory on the first order differential systems, to show that the solution is global, it will be enough if we prove that \( u, v, \rho, \psi \) and \( \phi \) remain uniformly bounded.

\textbf{Remark 5.1} If equations (7), (8) admits a global solution \((a, b)\) defined on \([0, +\infty[\), then \( a \) and \( b \) will be of class \( \mathcal{C}^2 \) on \([0, +\infty[\) and hence \( u, v, \rho, \psi \) and \( \phi \) are of class \( \mathcal{C}^1 \) on \([0, +\infty[\). Inversely, if system (14), (15), (16), (17), (18) admits a global solution \((u, v, \rho, \psi, \phi)\) on \([0, +\infty[\), then in particular \( u \) and \( v \) will be of class \( \mathcal{C}^1 \) on \([0, +\infty[\) and accordingly the system (7), (8) will admit a global solution of class \( \mathcal{C}^2 \) on \([0, +\infty[\).

\textbf{Theorem 5.2} If \( \Lambda \geq 0 \) and \( \dot{b}_0 > 0 \), then the Einstein-Scalar Field system on Bianchi type I space-time has a global solution.

\textbf{Proof 5.1} Following the standard theory of the first order differential systems, it will be enough if we prove that every solution of the Cauchy problem is uniformly bounded. Suppose that \( \Lambda \geq 0 \) and \( \dot{b}_0 \).

- Firstly, the constraint (19) implies \( v(v + 2u) = \Lambda + 8\pi\rho + 8\pi\psi > 0 \) since \( 8\pi\psi > 0 \). This show that \( v \) never vanishes and has the same sign as \( v + 2u \). Since, \( v \) is continuous and \( v(0) = \frac{\dot{b}_0}{b_0} > 0 \), these imply that \( v > 0 \); the we also have :

\[ v + 2u > 0. \]

According to (17)

\[ \dot{\rho} = -\frac{4}{3}(u + 2v)\rho = -\frac{4}{3}\left(\frac{1}{2}(v + 2u) + \frac{3}{2}v\right) \rho < 0 \]

since \( v + 2u > 0, v > 0 \) and \( \rho > 0 \), therefore \( \rho \) is a decreasing function on \([0, +\infty[\); it follows that :

\[ 0 < \rho \leq \rho_0. \]

- Next according to (18),

\[ \dot{\psi} = -2(u + 2v)\psi = -2\left(\frac{1}{2}(v + 2u)\psi + \frac{3}{2}u\psi\right) = -(v + 2u)\psi - 3v\psi < 0, \]
since \( v + 2u > 0, v > 0 \) and \( \psi > 0 \); it follows that \( \psi \) is a decreasing function on \([0, +\infty]\), hence
\[
0 < \psi \leq \psi_0.
\]

• Finally, let us show that \( u \) and \( v \) are uniformly bounded. Setting \( H = u + 2v \), it will be enough to show that \( H \) is uniformly bounded on \([0, +\infty]\).

We notice that:
\[
H = u + 2v = \frac{1}{2}(v + 2u) + \frac{3}{2}v \geq \frac{3}{2}v > 0,
\]
so (14) and (15) yield:
\[
\dot{H} = \dot{u} + 2\dot{v} = 2\Lambda - u^2 - 3v^2 - 2uv - 8\pi\psi. \tag{30}
\]

We then have:
\[
-u^2 - 3v^2 - 2uv = -(u^2 + v^2 + 4uv) + v^2 + 2uv
\]
\[
= -H^2 + v(v + 2u).
\]

So (30) yields:
\[
\dot{H} = 2\Lambda - H^2 + v(v + 2u) - 8\pi\psi. \tag{31}
\]

Let us show that \( v(v + 2u) \) is bounded above since we have \( v(v + 2u) \geq 0 \). Hamiltonian constraint can be written as:
\[
v(v + 2u) - 8\pi\psi = \Lambda + 8\pi\rho. \tag{32}
\]

Setting \( \Lambda_0 = v(v + 2u) - 8\pi\psi \) hence \( \Lambda_0 = \Lambda + 8\pi\rho \), but we have:
\[
0 < \rho \leq \rho_0 \iff 0 < 0\pi\rho \leq 8\pi\rho_0
\]
\[
\iff \Lambda \leq \Lambda_0 \leq \Lambda + 8\pi\rho_0
\]

which means that \( \Lambda_0 \) is bounded. Otherwise we have \( 0 < \psi \leq \psi_0 \) hence \( \Lambda < v(v + 2u) \leq \Lambda + 8\pi\rho_0 + 8\pi\psi_0 \) which shows that \( v(v + 2u) \) is bounded. (31) implies, since \( v(v + 2u) - 8\pi\psi = \Lambda + 8\pi\rho \), that
\[
\dot{H} \leq 3\Lambda + 8\pi\rho_0 - H^2.
\]

But it is well known that with \( C^2_0 = 3\Lambda + 8\pi\rho_0 \), that:
\[
H(t) \leq W(t),
\]
where $W$ satisfies:

$$
\begin{align*}
\dot{W} &= C_0^2 - W^2 ; \\
W(0) &= H(0).
\end{align*}
$$

(33)

We give a general result that is useful here and in what follows. Consider the Cauchy problem, in which $t_0 \in \mathbb{R}$ is given:

$$
\begin{align*}
\dot{y} &= K - \alpha^2 y^2 ; \quad (a) \\
y(t_0) &= \text{given} \quad (b)
\end{align*}
$$

(34)

where $K > 0$ and $\alpha > 0$ are constants; (34)(a) is a first order differential equation of Riccati type, which admits $y_0 = \frac{K}{\alpha}$ as an evident solution. It is also well known that, setting $y = Z + \frac{K}{\alpha}$ leads to a Bernouilli equation in $Z$, which turns out to be a first order linear differential equation in $\frac{1}{Z}$. Hence, by direct calculation we obtain:

$$
y(t) = \frac{K}{\alpha} \left[ 1 + \frac{2h_1(t_0)}{h_2(t_0) \exp(2\alpha Kt) - h_1(t_0)} \right]
$$

(35)

where $h_1(t_0) = \alpha y(t_0) - K$; $h_2(t_0) = \alpha y(t_0) + K$. (35) shows that $y(t) \to \frac{K}{\alpha}$ as $t \to +\infty$ thus $y$ is bounded. Applying this result to (33) by setting $K = C_0$, $\alpha = 1$ and $t_0 = 0$, it appears that $W(t) \to C_0$ as $t \to +\infty$, then $W$ is bounded. The corresponding reduced expression (35) for $W$ ensures that $W \geq 0$. Now, since $H(t) \leq W(t)$, $t \geq 0$, and since $W$ is bounded, $H$ is bounded from above. This completes the proof of Theorem 5.2

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