Counting Plane Rational Curves: Old and New Approaches

Aleksey Zinger

March 29, 2022

Abstract

These notes are intended as an easy-to-read supplement to part of the background material presented in my talks on enumerative geometry. In particular, the numbers $n_3$ and $n_4$ of plane rational cubics through eight points and of plane rational quartics through eleven points are determined via the classical approach of counting curves. The computation of the latter number also illustrates my topological approach to counting the zeros of a fixed vector bundle section that lie in the main stratum of a compact space. The arguments used in the computation of the number $n_4$ extend easily to counting plane curves with two or three nodes, for example. Finally, an inductive formula for the number $n_d$ of plane degree-$d$ rational curves passing through $3d-1$ points is derived via the modern approach of counting stable maps. This method is far simpler.

Contents

1 Introduction 2

2 The Low-Degree Numbers 3
   2.1 The Degree-One Number ........................................... 3
   2.2 The Degree-Two Number ............................................ 4
   2.3 The Degree-Three Number ........................................... 4

3 The Degree-Four Number 6
   3.1 Summary ................................................................. 6
   3.2 Quartics with One Singular Point .................................. 8
   3.3 Quartics with Two Singular Points ................................ 10
   3.4 Quartics with Three Simple Nodes ............................... 14
   3.5 Generalization to Arbitrary-Degree Curves ..................... 18
      3.5.1 The Numbers $\mathcal{N}_1$ and $\mathcal{N}_1'$ ........................ 18
      3.5.2 The Numbers $\mathcal{K}_1$ and $\mathcal{K}_1'$ ......................... 19
      3.5.3 The Number $\mathcal{T}_1$ ........................................... 19
      3.5.4 The Numbers $\mathcal{N}_2$ and $\mathcal{N}_2'$ ........................ 19
      3.5.5 The Number $\mathcal{K}_2$ ........................................... 20
      3.5.6 The Number $\mathcal{N}_3$ ........................................... 21

*Supported by an NSF Postdoctoral Fellowship
Enumerative geometry of algebraic varieties is a field of mathematics that dates back to the nineteenth century. The general goal of this subject is to determine the number of geometric objects that satisfy pre-specified geometric conditions. The objects are typically (complex) curves in a smooth algebraic manifold. Such curves are usually required to represent the given homology class, to have certain singularities, and to satisfy various contact conditions with respect to a collection of subvarieties. One of the most well-known examples of an enumerative problem is

**Question 1.1** If \( d \) is a positive integer, what is the number \( n_d \) of degree-\( d \) rational curves that pass through \( 3d-1 \) points in general position in the complex projective plane \( \mathbb{P}^2 \)?

Since the number of (complex) lines through any two distinct points is one, \( n_1 = 1 \). A little bit of algebraic geometry and topology gives \( n_2 = 1 \) and \( n_3 = 12 \); see Section 2. It is far harder to find that \( n_4 = 620 \), but this number was computed as early as the middle of the nineteenth century; see [Ze, p378]. We give a “classical-style” computation of this number in Section 3. Along the way, we determine the number of plane quartics that pass through 12 points and have two nodes and the number of plane quartics that pass through 11 points and have a cusp and a simple node; see Table 1. The derivations of Subsections 3.2-3.4 easily extend to counting arbitrary-degree plane curves with two nodes, a node and a cusp, and with three nodes; see Table 2 for explicit formulas. These curves are of course not rational in general. Subsections 3.3 and 3.4 also illustrate our approach to determining the number of zeros of a fixed vector bundle section that lie in the main stratum of a space. This approach is one of the two main tools that we have applied to a number of enumerative problems; see [Z1] and [Z2], for example.

The higher-degree numbers \( n_d \) remained unknown until the early 1990s, when a recursive formula for the numbers \( n_d \) was announced in [KoMa] and [RuT]:

\[
    n_d = \frac{1}{6(d-1)} \sum_{d_1 + d_2 = d} \left( d_1 d_2 - 2 \frac{(d_1 - d_2)^2}{3d - 2} \right) \frac{(3d-2)}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}.
\]
We describe the argument of the latter paper in Section 4. It can also be used to solve the natural generalization of Question 1.1 to the higher-dimensional projective spaces; see Section 10 in [RuT].

Remark: A derivation of (1.1), which is classical in spirit, appears in [Ra2] and is based on [Ra1]. The approach of Section 3 is more direct and involves no blowups.

Subsection 2.3 and Section 3, which are not used in Section 4, assume some familiarity with cohomology groups and chern classes. All other non-elementary terms, including those used in Question 1.1, are described in Appendix A. A different (and far more extensive) introduction to enumerative geometry, as well as to its relations with physics, is given in [Ka].

2 The Low-Degree Numbers

2.1 The Degree-One Number

We start by computing the number \( n_1 \) topologically. Throughout these notes, we will use the homogeneous coordinates \([X, Y, Z]\) on the complex projective plane of Question 1.1, i.e. we take

\[
P^2 = \{(X, Y, Z) \in \mathbb{C}^3: (X, Y, Z) \neq (0, 0, 0)\} / \mathbb{C}^* = \{[X, Y, Z]: (X, Y, Z) \in \mathbb{C}^3 - (0, 0, 0)\}.
\]

In this section, we use the following lemma.

**Lemma 2.1** If \( \gamma \to P^2 \) is the tautological line bundle, \( d \) is positive integer, and \( s \in \Gamma(P^2; \gamma^* \otimes d) \) is transverse to the zero set, the set \( s^{-1}(0) \) is a smooth two-dimensional submanifold of \( P^2 \) of genus

\[
g(s^{-1}(0)) = \binom{d-1}{2}.
\]

This lemma is proved in Subsection A.3. It can easily be verified directly in the \( d=1 \) and \( d=2 \) cases.

A line, or degree-one curve, in \( P^2 \) is the quotient by the \( \mathbb{C}^* \)-action of the zero set of a nonzero homogeneous polynomial

\[
s_{a_{100}, a_{010}, a_{001}} = a_{100}X + a_{010}Y + a_{001}Z
\]

of degree one on \( \mathbb{C}^3 - \{0\} \). In other words, a degree-one curve in \( P^2 \) has the form

\[
C = C_{a_{100}, a_{010}, a_{001}} = \{[X, Y, Z] \in P^2: a_{100}X + a_{010}Y + a_{001}Z = 0\}
\]

for some \((a_{100}, a_{010}, a_{001}) \in \mathbb{C}^3 - \{0\}\). Furthermore,

\[
C_{a_{100}, a_{010}, a_{001}} = C_{b_{100}, b_{010}, b_{001}} \iff (a_{100}, a_{010}, a_{001}) = \lambda(b_{100}, b_{010}, b_{001}) \text{ for some } \lambda \in \mathbb{C}^*.
\]

Thus, the space of all degree-one curves in \( P^2 \) is

\[
D_1 = \{(a_{100}, a_{010}, a_{001}): (a_{100}, a_{010}, a_{001}) \neq (0, 0, 0)\} / \mathbb{C}^* \approx P^2.
\]

A homogeneous polynomial \( s = a_{100}X + a_{010}Y + a_{001}Z \) of degree one on \( \mathbb{C}^3 \) determines a section \( s_{a_{100}, a_{010}, a_{001}} \) of the bundle \( \gamma^* \to P^2 \). If \((a_{100}, a_{010}, a_{001}) \neq (0, 0, 0)\), this section is transverse to
the zero set. Thus, by Lemma 2.1 for all \( [a_{100}, a_{010}, a_{001}] \in D_1 \) the genus of \( C_{a_{100}, a_{010}, a_{001}} \) is zero, i.e. this is a rational curve.

Finally, let \( p_1 = [X_1, Y_1, Z_1] \) and \( p_2 = [X_2, Y_2, Z_2] \) be two distinct points in \( \mathbb{P}^2 \). The curve \( C_{a_{100}, a_{010}, a_{001}} \) passes through the point \( p_i \) if and only if \( s_{a_{100}, a_{010}, a_{001}} (p_i) = 0 \). Thus, the number \( n_1 \) is the number of elements \( [a_{100}, a_{010}, a_{001}] \in D_1 \) such that

\[
\begin{aligned}
a_{100}X_1 + a_{010}Y_1 + a_{001}Z_1 &= 0; \\
a_{100}X_2 + a_{010}Y_2 + a_{001}Z_2 &= 0.
\end{aligned}
\tag{2.1}
\]

The solution of each of these equations on \( D_1 \) is a line. Since \( [X_1, Y_1, Z_1] \neq [X_2, Y_2, Z_2] \), the two lines are distinct. Since two lines in a plane, or \( \mathbb{P}^2 \), intersect in a single point, \( n_1 = 1 \). Stated differently, \( n_1 = 1 \) because the space of solutions of the system (2.1) in \( (a_{100}, a_{010}, a_{001}) \in \mathbb{C}^3 \) is a line through the origin.

### 2.2 The Degree-Two Number

The computation of the number \( n_2 \) is very similar. A degree-two curve in \( \mathbb{P}^2 \) is described by a nonzero degree-two homogeneous polynomial

\[
s_{a_{2,0,0},a_{1,1,0},a_{0,2,0},a_{0,1,1},a_{0,0,2}} = \sum_{j+k+l=2} a_{jkl}X^jY^kZ^l.
\]

Thus, the space of degree-two curves in \( \mathbb{P}^2 \) is

\[
D_2 = \{ (a_{2,0,0}, a_{1,1,0}, a_{0,2,0}, a_{0,1,1}, a_{0,0,2}) \in \mathbb{C}^6 - \{0\} \}/\mathbb{C}^* \approx \mathbb{P}^5.
\]

If \( p_i = [X_i, Y_i, Z_i] \) for \( i = 1, \ldots, 5 \) are five points in \( \mathbb{P}^2 \), the subset of conics that pass through these points is the set of elements \( ([a_{jkl}]_{j+k+l=2}) \in D_2 \) such that

\[
\sum_{j+k+l=2} a_{jkl}X^j_i Y^k_i Z^l_i = 0 \quad \text{for} \quad i = 1, \ldots, 5.
\tag{2.2}
\]

Each of these five linear equations determines a hyperplane \( H_i \) in \( D_2 \).

We assume that the five points \( p_i \) do not lie on any pair of lines in \( \mathbb{P}^2 \). Then by Lemma 2.1 every conic passing through the five points \( p_i \) is smooth and of genus zero. It follows that any two distinct conics \( C_1 \) and \( C_2 \) passing through the five points \( p_i \) must intersect at most \( 2 \cdot 2 = 4 \) points; see Lemma A.5. Thus, the system (2.2) of five equations must have at most one solution \( D_2 \), and such a solution represents a plane rational conic through the five points in \( \mathbb{P}^2 \). On the other hand, the five hyperplanes \( H_i \) in \( D_2 \) must have at least a point in common, since the poincare dual of a hyperplane generates \( H^*(\mathbb{P}^n; \mathbb{Z}) \). In simpler terms, the solution space of the system (2.2) of five linear homogeneous equations on \( \mathbb{C}^6 \) must contain a line through the origin. We conclude that \( n_2 = 1 \).

### 2.3 The Degree-Three Number

Computing the number \( n_3 \) requires a bit more care. Similarly to the previous two subsections, the space of cubics in \( \mathbb{P}^2 \) is described by

\[
D_3 = \{ (a_{jkl})_{j+k+l=3} \in \mathbb{C}^{10} - \{0\} \}/\mathbb{C}^* \approx \mathbb{P}^9.
\]
For a generic \( a \in D_3 \), the section \( s_a \) of the bundle \( \gamma^{* \otimes 3} \to \mathbb{P}^2 \) is transverse to the zero set. Thus, by Lemma 2.1, a typical cubic is smooth and of genus one, not zero.

Let \( p_i = [X_i, Y_i, Z_i] \) for \( i = 1, \ldots, 8 \) be eight points in \( \mathbb{P}^2 \) that do not lie on the union of any line and any conic in \( \mathbb{P}^2 \). It can then be shown that if the cubic \( C_a \) passes through these eight points, the section \( s_a \) has at most one singular point. In such a case, the curve \( C_a \) is a sphere with two points identified. In other words, a circle on a torus collapses to a point. This fact is immediate from the algebraic-geometry point of view, but can also be checked directly. Thus, the number \( n_3 \) is the number of plane cubics that pass through the eight points \( p_1, \ldots, p_8 \) and have a singular point. This singular point will be a simple node; see Figure 1 on page 8.

As in the previous two subsections, the space \( H_i \) of elements \( a \in D_3 \) such that \( p_i \in C_a \) is a hyperplane. With our assumption on the eight points, the eight hyperplanes intersect transversally, and thus

\[
D \equiv \bigcap_{i=1}^{i=8} H_i \approx \mathbb{P}^1.
\]

In simpler words, the eight equations analogous to (2.2) are linearly independent. Thus, the space of solution of the corresponding system of equations on \( \mathbb{C}^10 \) is a plane through the origin, which corresponds to a line \( \mathbb{P}^1 \) in \( D_3 \approx \mathbb{P}^9 \).

By the above, we need to determine the cardinality of the set

\[
Z = \{ ([a], x) \in S : ds_a|_x = 0 \}, \text{ where } S = \{ ([a], x) \in D \times \mathbb{P}^2 : s_a(x) = 0 \}.
\]

An element of the subspace \( S \) of \( D \times \mathbb{P}^2 \) is a cubic through the eight points \( p_1, \ldots, p_8 \) with a choice of a point on it. Such an element \( ([a], x) \) lies in \( Z \) if \( s_a \) is not transverse to the zero set at \( x \).

Let \( \pi_0, \pi_1 : D \times \mathbb{P}^2 \to D, \mathbb{P}^2 \) be the two projection maps. If \( \gamma_D \to D \) and \( \gamma_{\mathbb{P}^2} \to \mathbb{P}^2 \) are the tautological line bundles, we set

\[
\gamma_0 = \pi_0^* \gamma_D \to D \times \mathbb{P}^2 \quad \text{and} \quad \gamma_1 = \pi_1^* \gamma_{\mathbb{P}^2} \to D \times \mathbb{P}^2.
\]

A homogeneous polynomial in three variables of degree \( d \) induces a section of the bundle \( \gamma^{* \otimes d} \to \mathbb{P}^2 \).

For the same reason, the map

\[
\{ a \in \mathbb{C}^2 : [a] \in D \} \times \mathbb{P}^2 \to \gamma_1^{* \otimes 3}, \quad (a, x) \mapsto s_a(x),
\]

induces a section \( \psi_0 \) of the line bundle \( \gamma_0^{* \otimes 1} \to D \times \mathbb{P}^2 \). This section is transverse to the zero set. Thus, \( S = \psi_0^{-1}(0) \) is a smooth submanifold of \( D \times \mathbb{P}^2 \); see Lemma 2.2 below.

If \( ([a], x) \in S, s_a(x) = 0 \), and thus \( ds_a|_x \) is well-defined. The map

\[
\{ ([a], x) \in \mathbb{C}^2 \times \mathbb{P}^2 : s_a(x) = 0 \} \to \gamma_1^{* \otimes 3} \otimes T^* \mathbb{P}^2, \quad (a, x) \mapsto ds_a|_x,
\]

induces a section \( \psi_1 \) of the vector bundle \( \gamma_0^{* \otimes 1} \otimes \gamma_1^{* \otimes 3} \to D \times \mathbb{P}^2 \). This section is transverse to the zero set. Thus, by Lemma 2.2

\[
n_3 = |Z| = |\psi_1^{-1}(0)| = \langle e(\gamma_0^{* \otimes 1} \otimes T^* \mathbb{P}^2), [S] \rangle
\]

\[
= \langle e_2(\gamma_0^{* \otimes 1} \otimes T^* \mathbb{P}^2)^{PD_{D \times \mathbb{P}^2}}([S]), [D \times \mathbb{P}^2] \rangle
\]

\[
= \langle (3a^2 + 3a^2)(y + 3a), [D \times \mathbb{P}^2] \rangle = 12.
\]
where \( y = \pi_0^* c_1(\gamma_P^*) \) and \( a = \pi_1^* c_1(\gamma_P^*) \).

**Lemma 2.2** If \( M \) is a compact oriented manifold, \( V \rightarrow M \) is an oriented vector bundle, and \( \psi \in \Gamma(M; V) \) is transverse to the zero set, the space \( \psi^{-1}(0) \) is a smooth oriented submanifold of \( M \) and

\[
PD_M([\psi^{-1}(0)]) = e(V) \in H^*(M; \mathbb{Z}),
\]

where \( e(V) \) is the euler class of \( V \).

This lemma is a standard fact in differential topology; see Sections 9-12 of [MiSt]. It implies that if the dimension of \( M \) and the rank of \( V \) are the same, the set \( s^{-1}(0) \) is finite and its signed cardinality is given by

\[
\pm |s^{-1}(0)| = \langle e(V), [M] \rangle.
\]

In fact, this is the only case of Lemma 2.2 we would have needed if we extended the section \( \psi_1 \) over the entire space \( D \times \mathbb{P}^2 \) by using the canonical connection of the hermitian holomorphic vector bundle \( \gamma \rightarrow \mathbb{P}^2 \); see [GriH].

### 3 The Degree-Four Number

#### 3.1 Summary

In this section we use the general approach of Subsection 2.3 to compute the number \( n_4 \). Since the genus of a smooth plane quartic is three by Lemma 2.1, we will need to determine the number of quartics that pass through 11 points in \( \mathbb{P}^2 \) and have three nodes. This number is one-sixth the cardinality of the set

\[
\mathcal{N}_3 \equiv \{ ([a], x_1, x_2, x_3) \in D \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 : x_i \neq x_j \ \forall i \neq j; \ \sigma_2(x_i) = 0, \ ds_2|_{x_i} = 0 \ \forall i = 1, 2, 3 \},
\]

where \( D \approx \mathbb{P}^3 \) is the space of quartics that pass through the eleven chosen points and \( \mathbb{P}^2_i = \mathbb{P}^2 \).

Similarly to Subsection 2.3, each of the sections

\[
\varphi_i ([a], x_i) = (s_2(x_i), ds_2|_{x_i}) \in \gamma_0^* \otimes \gamma_i^* \otimes \mathbb{P}^2_i
\]

is transverse to the zero set over \( D \times \mathbb{P}^2_i \). However, the section

\[
\varphi \equiv \varphi_1 \oplus \varphi_2 \oplus \varphi_3
\]

is not transverse to the zero set over \( D \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \). For example, the zero set of \( \varphi \) contains the two-dimensional space

\[
\{ ([a], x, x, x) : s_2(x) = 0, \ ds_2|_x = 0 \}.
\]

Thus, \(|\mathcal{N}_3|\) is not the euler class of the bundle

\[
V \equiv \bigoplus_{i=1}^{3} \gamma_0^* \otimes \gamma_i^* \otimes \mathbb{P}^2_i \rightarrow M \equiv D \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3.
\]

On the other hand, \( \varphi \) is transverse to the zero set over the “main stratum” of \( M \):

\[
M^0 \equiv \{ ([a], x_1, x_2, x_3) \in M : x_i \neq x_j \ \forall i \neq j \}.
\]
Thus, $|\tilde{N}_3|$ is the euler class of the bundle $V$ minus the $\varphi$-contribution to $e(V)$ from the “boundary” of $M$:

$$|\tilde{N}_3| = \langle e(V), M \rangle - C_{\partial M}(\varphi),$$

where $\partial M = M - M^0$.

The number $C_{\partial M}(\varphi)$ is the signed number of zeros of the bundle section $\varphi + \nu$, for a small generic perturbation $\nu$, that lie near $\partial M$. If $\partial M = \bigcup_i Z_i$ is a stratification of $\partial M$,

$$C_{\partial M}(\varphi) = \sum_i C_{Z_i}(\varphi).$$

If this stratification is sufficiently fine, each of the numbers $C_{Z_i}(\varphi)$ is a certain multiple of the number of zeros of an affine bundle map between vector bundles over $\bar{Z}_i$. The latter number can be computed through a reductive procedure, described in detail in [Z1] and [Z2] and implemented in the relevant cases in Subsections 3.3 and 3.4 below.

In order to simplify the computation of $|\tilde{N}_3|$, we will essentially be adding one point at a time. This computation will require knowing the numbers of plane quartics with various one- and two-point singularities. These numbers, along with $|\tilde{N}_3|$, are given in Table 1. For example, according to this table, the cardinality of the set $\mathcal{N}_{2,1}$ of plane quartics that pass through 11 points in general position and have two nodes, one of which lies on a fixed general line, is 170. Figure 1 shows a simple node, a simple cusp, and a simple tacnode. If $s$ is a section of $\gamma^* \otimes d$ and $x \in s^{-1}(0)$ is a node of $s^{-1}(0)$, then $ds|_x = 0$. We describe the analogous cuspidal and tacnodal condition on $s$ in the next subsection. All numbers in Table 1 are computed in Subsections 3.3 and 3.4 below.

Finally, we note that a plane quartic that has 3 nodes and passes through 11 points is either irreducible, in which case it is rational, or a union of a smooth cubic, passing through 9 of the points, and a line, passing through the remaining 2 points. By the same argument as in Subsections 2.1 and 2.2, the number of plane cubics passing through 9 points in general position is 1. Thus, by the last row of Table 1, the number of rational quartics passing through 11 points in general position in $\mathbb{P}^2$ is

$$n_4 = 675 - \left(\begin{array}{c} 11 \\ 2 \end{array}\right) \cdot 1 \cdot 1 = 620.$$
(1) Let $D \approx \mathbb{P}^1 \subset D_4$ denote the subspace of plane quartics that pass through the points $p_1, \ldots, p_{13}$. With notation as in Subsection 2.3 let

$$\mathcal{N}_1 = \{([a], x) \in D \times \mathbb{P}^2 : \varphi([a], x) = 0\},$$

where

$$\varphi \in \Gamma(D \times \mathbb{P}^2; \gamma_0 \otimes \gamma_1^{*4} \oplus \gamma_0 \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2), \quad \varphi([a], x) = (s_\mathfrak{a}(x), ds_\mathfrak{a}|x).$$

Since the section $\varphi$ is transverse to the zero set, by Lemma 2.2

$$|\mathcal{N}_1| = |\varphi^{-1}(0)| = \langle e(\gamma_0 \otimes \gamma_1^{*4} \oplus \gamma_0 \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2), D \times \mathbb{P}^2 \rangle$$

$$= \langle c_1(\gamma_0^{*4} \otimes \gamma_1^{*4}), c_2(\gamma_0^{*4} \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2), D \times \mathbb{P}^2 \rangle$$

$$= \langle (y + 4a)(y^2 + 5ya + 7a^2), D \times \mathbb{P}^2 \rangle = 27.$$ (2) Let $D \approx \mathbb{P}^2 \subset D_4$ denote the subspace of plane quartics that pass through the points $p_1, \ldots, p_{12}$. Let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a general line in $\mathbb{P}^2$. We put

$$\mathcal{N}_{1,1} = \{([a], x) \in D \times \mathbb{P}^1 : \varphi([a], x) = 0\},$$

where

$$\varphi \in \Gamma(D \times \mathbb{P}^1; \gamma_0^{*4} \otimes \gamma_1^{*4} \oplus \gamma_0^{*4} \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2|_{\mathbb{P}^1}), \quad \varphi([a], x) = (s_\mathfrak{a}(x), ds_\mathfrak{a}|x).$$

Since the section $\varphi$ is transverse to the zero set, by Lemma 2.2

$$|\mathcal{N}_{1,1}| = |\varphi^{-1}(0)| = \langle e(\gamma_0^{*4} \otimes \gamma_1^{*4} \oplus \gamma_0^{*4} \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2), D \times \mathbb{P}^1 \rangle$$

$$= \langle c_1(\gamma_0^{*4} \otimes \gamma_1^{*4}), c_2(\gamma_0^{*4} \otimes \gamma_1^{*4} \otimes T^* \mathbb{P}^2), D \times \mathbb{P}^1 \rangle$$

$$= \langle (y + 4a)(y^2 + 5ya + 7a^2), D \times \mathbb{P}^1 \rangle = 9.$$
Lemma 3.2 The number $|K_1|$ of plane quartics that have a cusp and pass through 12 points in general position is 72. The number $|K_{1,1}|$ of plane quartics that have a cusp on a fixed general line and pass through 11 points in general position is 20.

Proof: (1) Let $D \cong \mathbb{P}^2$ be as in (2) of the proof of Lemma 3.1. We put

$$N'_1 = \{([a], x) \in D \times \mathbb{P}^2 : s_a(x) = 0, \ ds_a|_x = 0\}.$$ If $([a], x) \in N'_1$, we denote by $H_{a,x} \in \Gamma(N'_1; \text{Hom}(TP^2; \gamma_0^* \otimes \gamma_1^{*4} \otimes T^*\mathbb{P}^2))$ the Hessian of $s_a$ at $x$, i.e. the total second derivative of $s_a$ at $x$. Let

$$K_1 = \{([a], x) \in N'_1 : \varphi([a], x) = 0\} \quad \text{where} \quad \varphi \in \Gamma(N'_1; (\gamma_0^* \otimes \gamma_1^{*4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{*2}), \quad \varphi([a], x) = \det H_{a,x}.$$ Since the section $\varphi$ is transverse to the zero set, by Lemma 3.2.

$$|K_1| = |\varphi^{-1}(0)| = \langle e((\gamma_0^* \otimes \gamma_1^{*4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{*2}), N'_1 \rangle = 2\langle y+a, N'_1 \rangle = 2 (|N_1| + |N_{1,1}|) = 2 (27+9) = 72.$$ (2) Similarly, let $D \cong \mathbb{P}^3 \subset D_4$ denote the subspace of plane quartics that pass through the points $p_1, \ldots, p_{11}$. Let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a general line in $\mathbb{P}^2$. We put

$$N'_{1,1} = \{([a], x) \in D \times \mathbb{P}^1 : s_a(x) = 0, \ ds_a|_x = 0\} ; \quad K_{1,1} = \{([a], x) \in N'_{1,1} : \det H_{a,x} = 0\}.$$ Then, by Lemma 2.2.

$$|K_{1,1}| = \langle e((\gamma_0^* \otimes \gamma_1^{*4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{*2}), N'_{1,1} \rangle = 2\langle y+a, N'_{1,1} \rangle = 2 (|N_{1,1}| + \langle a, N'_{1,1} \rangle) = 2 (9+1) = 20.$$ Note the number $\langle a, N'_{1,1} \rangle$ of plane quartics that pass through 11 points and have a node at a fixed twelfth point is 1, since all conditions on $a \in D_4$ are linear, as in Subsections 2.1 and 2.2.

Lemma 3.3 The number $|T_1|$ of plane quartics that have a tacnode and pass through 11 points in general position is 200.

Proof: Let $D \cong \mathbb{P}^3$ be as in (2) of the proof of Lemma 3.2. We put

$$N''_1 = \{([a], x) \in D \times \mathbb{P}^2 : s_a(x) = 0, \ ds_a|_x = 0\}, \quad M = \mathbb{P}TP^2|_{N''_1}.$$ We denote by $\gamma \rightarrow M$ the tautological line bundle and by

$$\tilde{H}_{a,x} \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*4} \otimes T^*\mathbb{P}^2))$$ the bundle map induced by $H_{a,x}$. Let

$$K'_1 = \{([a], x) \in M : \tilde{H}_{a,x} = 0\} ; \quad T_1 = \{([a], x) \in K'_1 : \varphi([a], x) = 0\},$$ where $\varphi \in \Gamma(M; \text{Hom}(\gamma^{*3}, \gamma_0^* \otimes \gamma_1^{*4})), \quad \varphi([a], x) = \mathbb{P}^3|_{\tilde{H}_{a,x}}$.
and $D^3_{2,x}$ is the third derivative of $s_2$ at $x$. Let $\lambda = c_1(\gamma^*)$. Since the sections $\varphi$ and $\tilde{H}_\cdot$ are transverse to the zero set, by Lemma \ref{lem:transverse}

$$|T_1| = |\varphi^{-1}(0)| = \langle e(\gamma^* \otimes \gamma^* \otimes \gamma^* \otimes T^* \mathbb{P}^2), M \rangle$$

\begin{align*}
&= \langle e(\gamma^* \otimes \gamma^* \otimes \gamma^* \otimes T^* \mathbb{P}^2), M \rangle \\
&= \left\langle 3\lambda^3 + (7y + 19a)\lambda^2 + (5y^2 + 28ya + 41a^2)\lambda, M \right\rangle \\
&= \left\langle 5y^2 + 7ya + 2a^2, N''_1 \right\rangle \\
&= 5|N_1| + 7|N_{1,1}| + 2\langle a, N''_{1,1} \rangle = 5 \cdot 27 + 7 \cdot 9 + 2 \cdot 1 = 200.
\end{align*}

### 3.3 Quartics with Two Singular Points

In this subsection, we compute the three numbers of Table 11 that involve two-point singularities. As the relevant bundle sections are no longer transverse everywhere, each of these numbers is the euler class of the corresponding vector bundle minus the contribution from the "boundary" for the given bundle section.

Suppose $E, V \to M$ are vector bundle such that $\dim M + \text{rk} E = \text{rk} V$ and

$$\alpha \in \Gamma(M; \text{Hom}(E, V)).$$

If $\nu \in \Gamma(M; V)$ is a generic section, the affine bundle map

$$\psi_{\alpha, \nu}: E \to V, \quad \psi_{\alpha, \nu}(m; e) = \alpha(m; e) + \nu(m),$$

has a finite number of transverse zeros. By Lemma 3.14 in \cite{Z1} and Proposition 2.18A in \cite{Z2}, the signed cardinality of $\psi_{\alpha, \nu}^{-1}(0)$ is independent of the choice of $\nu$. We denote this cardinality by $N(\alpha)$.

**Lemma 3.4** The number $|N_2|$ of plane quartics that have two nodes and pass through 12 points in general position is 225. The number $|N'_{2,1}|$ of plane quartics that have two nodes, one of which lies on a fixed general line, and pass through 11 points in general position is 170.

**Proof:** (1) Let $N'_1 \subset \mathcal{D} \times \mathbb{P}^2_1$ be defined as in (1) of the proof of Lemma 3.2. We put

$$M = N'_1 \times \mathbb{P}^2_1, \quad M^0 = \{([a], x_1, x_2) \in M : x_1 \neq x_2\}, \quad \partial M = M - M^0, \quad \tilde{N}_2 = \varphi^{-1}(0) \cap M^0,$$

where $\varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^* \otimes \gamma_0^* \otimes \gamma_2^* \otimes T^* \mathbb{P}^2_1)$, $\varphi([a], x_1, x_2) = (s_a(x_2), ds_a(x_2))$, $\gamma_2 = \pi_2^* \gamma_{\mathbb{P}^2_2}$, and $\pi_2: M \to \mathbb{P}^2_2$ is the projection onto the last component. Since $\varphi|_{M^0}$ is transverse to the zero set,

$$|\tilde{N}_2| = \frac{1}{2} |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^* \otimes \gamma_0^* \otimes \gamma_2^* \otimes T^* \mathbb{P}^2_2), M \rangle - C_{\partial M}(\varphi)$$

\begin{align*}
&= \langle (y + 4a_2)(y^2 + 5ya_2 + 7a_2^2), N'_1 \times \mathbb{P}^2_2 \rangle - C_{\partial M}(\varphi) \\
&= 27\langle y, N'_1 \rangle - C_{\partial M}(\varphi) = 27|N_1| - C_{\partial M}(\varphi) = 27 \cdot 27 - C_{\partial M}(\varphi),
\end{align*}

where $a_2 = \pi_2^* c_1(\gamma_{\mathbb{P}^2_2}^*)$. In order to determine $C_{\partial M}(\varphi)$, we split $\partial M$ into two strata:

$$Z_1 = \{([a], x, x) : ([a], x) \in N'_1 - \mathcal{K}_1\}, \quad Z_0 = \{([a], x, x) : ([a], x) \in \mathcal{K}_1\}.$$
With appropriate identifications, for some \( C \in C(\mathcal{N}_0'; \mathbb{R}^+) \),
\[
|\varphi([a], x, v) - H_{[a], x}v| \leq C([a], x) |v|^2 \quad \forall \ (([a], x), x) \in \partial M, \ v \in \text{Norm}_M \partial M \big|_{([a], x, x)} \approx T_x \mathbb{P}^2_1.
\]
By definition of the set \( K_1 \),
\[
|H_{[a], x}v| \geq C([a], x) |v|^{-1} \quad \forall \ (([a], x), x) \in \mathcal{N}_1' - K_1, \ v \in T_x \mathbb{P}^2_1.
\]
By (3.2), (3.3), and a rescaling and cobordism argument as in Subsection 3.1 of [Z1],
\[
C_{Z_1}(\varphi) = N(\alpha), \quad \text{where} \quad \alpha \in \Gamma(\mathcal{N}_1'; \text{Hom}(T_{\mathbb{P}^2}, \gamma^*_0 \otimes \gamma^*_1 \otimes \gamma^*_0 \otimes \gamma^*_1 \otimes T^* \mathbb{P}^2)), ~ \alpha([a], x; v) = (0, H_{[a], x}v).
\]
On the other hand, suppose \((([a], x), x) \in K_1 \). We denote by \( \mathcal{L}_{([a], x)} \subset T \mathbb{P}^2 \) the kernel of \( H_{[a], x} \) and by \( \mathcal{L}_{([a], x)}^+ \) its orthogonal complement. Let \( N_{([a], x)} \) be the normal bundle of \( K_1 \) in \( \mathcal{N}_1' \) at \(([a], x)\). Then, with appropriate identifications, for some \( \beta_2, \beta_3, \beta_4 \in \mathbb{C}^* \), \( \beta_4 \in \mathbb{C} \), and \( C \in \mathbb{R}^+ \),
\[
|\varphi([a], x; u, v, w) - \alpha_0(u, v, w)| \leq C(|v|^4 + |w|^2) \quad \forall \ u \in N_{([a], x)}, \ v \in \mathcal{L}_{([a], x)}, \ w \in \mathcal{L}_{([a], x)}^+, \quad \text{where} \quad \alpha_0(u, v, w) = \frac{1}{2} uv^2 + \frac{3}{2} \beta_3 v^3, uv + \beta_3 v^2 + \beta_4 v^3, \beta_2 w).
\]
Here \( \beta_2 \) is the second derivative of \( s_{[a]} \) at \( x \) along \( \mathcal{L}_{([a], x)}^+ \) and \( 2 \beta_3 \) is the third derivative of \( s_{[a]} \) at \( x \) along \( \mathcal{L}_{([a], x)} \). Since the polynomial \( \alpha_0 \) is three-to-one near the origin, it follows from (3.3) that each point of \( Z_0 \approx K_1 \) contributes 3 to \( C_{Z_0}(\varphi) \). From (3.4) and Lemmas 3.2 and 3.5 we conclude that
\[
C_{\partial M}(\varphi) = C_{Z_1}(\varphi) + C_{Z_0}(\varphi) = 63 + 3|K_1| = 63 + 3 \cdot 72 = 279.
\]
The first claim of the lemma follows from (3.4) and (3.5), since \( N_2 = N_2/S_2 \), where \( S_2 \) is the symmetric group on two elements.

(2) Similarly, let \( \mathcal{N}_{1,1} \subset \mathcal{D} \times \mathbb{P}^2_1 \) be defined as in (2) of the proof of Lemma 3.2. We put
\[
M = \mathcal{N}_{1,1} \times \mathbb{P}^2_1, \quad M^0 = \{([a], x_1, x_2) \in M: x_1 \neq x_2 \}, \quad \partial M = M - M^0, \quad N_{2,1} = \varphi^{-1}(0) \cap M^0,
\]
where \( \varphi \in \Gamma(M; \gamma^*_0 \otimes \gamma^*_2 \otimes \gamma^*_0 \otimes \gamma^*_2 \otimes T^* \mathbb{P}^2), \quad \varphi([a], x_1, x_2) = (s_{[a]}(x_2), ds_{[a]}|_{x_2}) \).
Since \( \varphi|M^0 \) is transverse to the zero set,
\[
|N_{2,1}| = \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma^*_0 \otimes \gamma^*_2 \otimes \gamma^*_0 \otimes \gamma^*_2 \otimes T^* \mathbb{P}^2), M \rangle - C_{\partial M}(\varphi) = 27|y, \mathcal{N}_{1,1}'| - C_{\partial M}(\varphi) = 27 \cdot 9 - C_{\partial M}(\varphi).
\]
We split \( \partial M \) into two strata:
\[
Z_1 = \{([a], x, x): ([a], x) \in \mathcal{N}_{1,1}' - K_{1,1} \}, \quad Z_0 = \{([a], x, x): ([a], x) \in K_{1,1} \}.
\]
By the same argument as in (1) above,
\[
C_{Z_1}(\varphi) = N(\alpha), \quad \text{where} \quad \alpha \in \Gamma(\mathcal{N}_{1,1}; \text{Hom}(T \mathbb{P}^2, \gamma^*_0 \otimes \gamma^*_1 \otimes \gamma^*_0 \otimes \gamma^*_1 \otimes T^* \mathbb{P}^2)), \quad \alpha([a], x; v) = (0, H_{[a], x}v),
\]
while \( C_{Z_0}(\varphi) = 3|K_{1,1}| \). Using Lemmas 3.2 and 3.5 we conclude that
\[
C_{\partial M}(\varphi) = C_{Z_1}(\varphi) + C_{Z_0}(\varphi) = 13 + 3 \cdot 20 = 73.
\]
The second claim of the lemma follows immediately from (3.7) and (3.8).
Lemma 3.5  If \( N'_1 \subset D \times \mathbb{P}^2 \) is as in (1) of the proof of Lemma 3.2 and
\[
\alpha \in \Gamma(N'_1; \text{Hom}(TP^2, \gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\alpha], x; v) = (0, H_{\alpha,x}v),
\]
then \( N(\alpha) = 63 \). If \( N'_{1,1} \subset D \times \mathbb{P}^2 \) is as in (2) of the proof of Lemma 3.2 and
\[
\alpha \in \Gamma(N'_{1,1}; \text{Hom}(TP^2, \gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\alpha], x; v) = (0, H_{\alpha,x}v),
\]
then \( N(\alpha) = 13 \).

Proof: (1) We put
\[
M = \mathbb{P}TP^2|_{N'_1}; \quad \partial M = \{(\alpha, x) \in M: \tilde{H}_{\alpha,x} = 0\} \approx K_1,
\]
where \( \tilde{H}_\alpha \) is as in the proof of Lemma 3.3. Let
\[
\tilde{\alpha} = (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2))
\]
be the section induced by \( \alpha \). By Lemma 3.14 in [Z1] or Proposition 2.18A in [Z2],
\[
N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2)c(TP^2)^{-1}, N'_1 \rangle - C_{\alpha^{-1}(0)}(\tilde{\alpha}^\perp) = (3y + 6a, N'_1) - C_{\partial M}(\tilde{\alpha}^\perp), \tag{3.9}
\]
where \( \tilde{\alpha}^\perp \) is the composition of the linear bundle map \( \tilde{\alpha} \) with the quotient projection map
\[
\gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2 \longrightarrow \left( \gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2 \right)/C\nu,
\]
for a generic nonvanishing section \( \nu \). The claim \( 3.9 \) can in fact be easily seen directly from the definition of \( N(\alpha) \). Since the section \( \tilde{H} \) is transverse to the zero set, so is the section \( \tilde{\alpha}^\perp \) if \( \nu \) is generic. Thus,
\[
C_{\alpha^{-1}(0)}(\tilde{\alpha}^\perp) = \pm |\tilde{\alpha}^{-1}(0)| = |K_1|. \tag{3.10}
\]
The first claim of the lemma follows from \( 3.9 \) and \( 3.10 \), along with Lemmas 3.1 and 3.2.

(2) Similarly, we put
\[
M = \mathbb{P}TP^2|_{N'_{1,1}}; \quad \partial M = \{(\alpha, x) \in M: \tilde{H}_{\alpha,x} = 0\} \approx K_{1,1},
\]
\[
\tilde{\alpha} = (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2)).
\]
By Lemma 3.14 in [Z1] or Proposition 2.18A in [Z2],
\[
N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^{* \otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{* \otimes 4} \otimes T^*\mathbb{P}^2)c(TP^2)^{-1}, N'_{1,1} \rangle - C_{\alpha^{-1}(0)}(\tilde{\alpha}^\perp) = (3y + 6a, N'_{1,1}) - C_{\partial M}(\tilde{\alpha}^\perp), \tag{3.11}
\]
As in (1), \( \tilde{\alpha}^\perp \) is transverse to the zero, and thus
\[
C_{\alpha^{-1}(0)}(\tilde{\alpha}^\perp) = \pm |\tilde{\alpha}^{-1}(0)| = |K_{1,1}|. \tag{3.12}
\]
The second claim of the lemma follows from \( 3.11 \) and \( 3.12 \), along with Lemmas 3.1 and 3.2.
Lemma 3.6  The number $|\mathcal{K}_2|$ of plane quartics that have one node and one cusp and pass through 11 points in general position is 840.

Proof: Let $\mathcal{N}_1'' \subset D \times \mathbb{P}^2$ and $\mathcal{K}' \subset \mathbb{P} \mathbb{P}^2 |\mathcal{N}_1''$ be as in the proof of Lemma 3.3. We denote by

$$\tilde{\pi}_1: \mathbb{P} \mathbb{P}^2 |\mathcal{N}_1'' \rightarrow \mathbb{P}^2$$

the composition of the bundle projection $\mathbb{P} \mathbb{P}^2 |\mathcal{N}_1'' \rightarrow \mathcal{N}_1''$ with $\pi_1$. We put

$$M = \mathcal{K}' \times \mathbb{P}^2, \quad M^0 = \{([a], x_1, x_2) \in M : \tilde{\pi}_1([a], x_1) \neq x_2\}, \quad \partial M = M - M^0, \quad \mathcal{K}_2 = \varphi^{-1}(0) \cap M^0,$$

where $\varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*04} \oplus \gamma_0^* \otimes \gamma_1^{*04} \otimes T^* \mathbb{P}^2)$, $\varphi([a], x_1, x_2) = (s_{[a]}(x_2), ds_{[a]}(x_2))$, $\gamma_2 = \pi_2^* \gamma_2^*$.

Since $\varphi |_{M^0}$ is transverse to the zero set, similarly to (3.1),

$$|\mathcal{K}_2| = \pm |\varphi^{-1}(0) \cap M^0| = \langle (y^4 + 4a_2(y^2 + 5a_2) + 7a_0^2), \mathcal{K}' \times \mathbb{P}^2 \rangle - C_{\partial M}(\varphi)$$

$$= 27 \|y, \mathcal{K}'\| - C_{\partial M}(\varphi) = 27 |\mathcal{K}_1| - C_{\partial M}(\varphi) = 27 \cdot 72 - C_{\partial M}(\varphi). \quad (3.13)$$

We split $\partial M$ into two strata:

$$\mathcal{Z}_1 = \{([a], x, \tilde{\pi}_1([a], x)) : ([a], x) \in \mathcal{K}' - \mathcal{T}_1\}, \quad \mathcal{Z}_0 = \{([a], x, \tilde{\pi}_1([a], x)) : ([a], x) \in \mathcal{T}_1\}.$$

Let $\gamma^+ \rightarrow \mathcal{K}'$ be the orthogonal complement of $\gamma$ in $\pi^* \mathbb{P}^2$. We define the bundle map

$$\alpha \in \Gamma(\mathcal{K}'_1; Hom(\gamma^{02} \oplus \gamma^+, \gamma_0^* \otimes \gamma_1^{*04} \oplus \gamma_0^* \otimes \gamma_1^{*04} \otimes T^* \mathbb{P}^2))$$

by

$$\alpha(\tilde{v}, w) = (0, \frac{1}{2} D^3_{\tilde{a}, x} \tilde{v}, \tilde{H}_{\tilde{a}, x} w) \in \gamma_0^* \otimes \gamma_1^{*04} \oplus \gamma_0^* \otimes \gamma_1^{*04} \otimes \gamma^+ \oplus \gamma_0^* \otimes \gamma_1^{*04} \otimes \gamma^+.$$  

Note that by definition of the set $\mathcal{T}_1$, for some $C \in C(\mathcal{K}'_1; \mathbb{R}^+)$,

$$|\alpha_{[a], x}(\tilde{v}, w)| \geq C(|[a], x|^{-1}(|\tilde{v}| + |w|)) \quad \forall ([a], x) \in \mathcal{K}' - \mathcal{T}_1, (\tilde{v}, w) \in (\gamma^{02} \oplus \gamma^+)|([a], x). \quad (3.14)$$

On the other hand, with appropriate identifications,

$$|\varphi([a], x, v, w) - \alpha_{[a], x}(v^{02}, w)| \leq C([a], x)(|v|^3 + |w|^2) \quad (3.15)$$

$$\forall ([a], x, \tilde{\pi}_1([a], x)) \in \partial M, v \in \gamma([a], x), w \in \gamma^+([a], x).$$

Since the bundle map

$$\mathbb{P}^2 = \gamma \oplus \gamma^+ \rightarrow \gamma^{02} \oplus \gamma^+, \quad (v, w) \rightarrow (v^{02}, w),$$

is two-to-one, outside of the proper subbundle $\gamma^+$,

$$C_{\mathcal{Z}_1}(\varphi) = 2 \cdot N(\alpha), \quad (3.16)$$

by (3.14), (3.15), and a rescaling and cobordism argument as in Subsection 3.1 of [21]. Suppose next that $([a], x) \in \mathcal{T}_1$. Let $N([a], x)$ be the normal bundle of $\mathcal{T}_1$ in $\mathcal{K}'$ at $([a], x)$. Then, with appropriate identifications, for some $\beta_2, \beta_4 \in \mathbb{C}$ and $C \in \mathbb{R}^+$,

$$|\varphi([a], x; u, v, w) - \alpha_0(u, v, w)| \leq C(|v|^5 + |w|^2) \quad \forall u \in N([a], x), v \in \gamma([a], x), w \in \gamma^+([a], x), \quad (3.17)$$

where $\alpha_0(u, v, w) = \frac{1}{6} u v^3 + \frac{1}{4} \beta_4 v^4, \frac{1}{2} u v^2 + \beta_4 v^3, \beta_2 w).$
Lemma 3.7. The lemma follows from (3.13) and (3.18). From (3.16) and Lemmas 3.3 and 3.7 we conclude that

\[ C_{\partial M}(\varphi) = C_{Z_1}(\varphi) + C_{Z_0}(\varphi) = 2 \cdot 152 + 4|T_1| = 2 \cdot 152 + 4 \cdot 200 = 1104. \]  

(3.18)

The lemma follows from (3.13) and (3.18).

Lemma 3.8. In this subsection we compute the last number of Table 1. We start with the following structural lemma.

3.4 Quartics with Three Simple Nodes

The lemma follows from (3.19)-(3.21) along with Lemma 3.3.

Similarly to the proof of the Lemma 3.5,

\[ N(\alpha) = N(\bar{\alpha}), \quad \text{where} \]

\[ \bar{\alpha} \in \Gamma(K_1, H_{\mathbb{P}^2}^{\gamma^* \otimes \gamma^* \otimes \gamma^* \otimes \gamma^*}), \quad \alpha([a], x, v, w) = (0, 1, 2D^3_{\mathbb{P}^2} v, \bar{H}_{\mathbb{P}^2} w). \]

(3.19)

Since the section \( D^3 \) is transverse to the zero set, so is the section \( \bar{\alpha}^\perp \) if \( \nu \) is generic. Thus,

\[ C_{\bar{\alpha}^-1(0)}(\bar{\alpha}^\perp) = \pm |\bar{\alpha}^-1(0)| = |T_1|. \]  

(3.21)

The lemma follows from (3.19)-(3.21) along with Lemma 3.3.

3.4 Quartics with Three Simple Nodes

In this subsection we compute the last number of Table 1. We start with the following structural lemma.

Lemma 3.8. Let \( N''_1 \subset D \times \mathbb{P}^2 \) be as in the proof of Lemma 3.5, and let

\[ \bar{N}''_{2,0} = \{(a, x, x), x \in N''_1 \times \mathbb{P}^2 : x_1 \neq x_2, \varphi_2([a], x, x) = 0\}, \quad \text{where} \]

\[ \varphi_2 \in \Gamma(N''_1 \times \mathbb{P}^2, \gamma^* \otimes \gamma^* \otimes \gamma^* \otimes \gamma^* \otimes \mathbb{P}^2_1), \quad \varphi_2([a], x, x) = (s_a(x_1), ds_a(x_2)). \]

If \( \bar{N}''_2 \) is the closure of \( \bar{N}''_{2,0} \) in \( N''_1 \times \mathbb{P}^2 \), then

\[ \partial \bar{N}''_2 \equiv \bar{N}''_2 - \bar{N}''_{2,0} = \{(a, x, x) \in N''_1 \times \mathbb{P}^2 : (a, x) \in T_1\}. \]
Lemma 3.9

With notation as in the statement of Lemma 3.8, let $\varphi_2$ is continuous, $x_2 = x_1$ by definition of $\partial N''_2$. If $([a], x_1) \in N''_2 - K'_1$, by (3.2) and (3.3) and with appropriate identifications,

$$|\varphi_2([a], x_1, v)| \geq C([a], x_1)^{-1}|v|$$

for all $v \in T_{x_1}\mathbb{P}^2_1$ sufficiently small. Thus, $([a], x_1, x_1)$ is not in the closure of $\tilde{N}_{2,0}$. Suppose next that $([a], x_1) \in K'_1 - \mathcal{T}_1$. Then, by (3.5),

$$|\varphi_2([a], x_1, u, v, w)| \geq C([a], x_1)^{-1}(|u|^3 + |w|)$$

(3.22)

for all $u \in N_{([a], x_1)}$, $v \in \mathcal{L}_{([a], x_1)}$, and $v \in \mathcal{L}'_{([a], x_1)}$ sufficiently small. In this case, $N$ is the normal bundle of $K'_1$, viewed as a submanifold of $N''_1$, in $N''_1$, while the line bundles $\mathcal{L}$ and $\mathcal{L}'$ over $K'_1$ are defined as in (1) of the proof of Lemma 3.4. From (3.22), we conclude that $([a], x_1, x_1)$ is not in the closure of $\tilde{N}_{2,0}$.

Lemma 3.9 The number $|N_3|$ of plane quartics that have three nodes and pass through 11 points in general position is 675.

Proof: With notation as in the statement of Lemma 3.8, let

$$M = \tilde{N}_3 \times \mathbb{P}^2_3$$

$$M^0 = \{([a], x_1, x_2, x_3) \in M : x_3 \neq x_1, x_2\}, \quad \partial M = M - M^0, \quad \tilde{N}_3 = \varphi_3^{-1}(0) \cap M^0,$$

where $\varphi_3 \in \Gamma(M; \gamma_3 \otimes \gamma_3^* \otimes T_0 \mathbb{P}^2_3, \gamma_3), \quad \varphi_3([a], x_1, x_2, x_3) = (s_2(x_3), ds_2|_{x_3}, \gamma_3 = |\pi_3|^2)\mathbb{P}^2_3,$

and $\pi_3 : M \to \mathbb{P}^2_3$ is the projection onto the last component. Since $\varphi_3|_{M^0}$ is transverse to the zero set,

$$|\tilde{N}_3| = |\varphi_3^{-1}(0) \cap M^0| = \langle e(\gamma_3 \otimes \gamma_3^* \otimes T_0 \mathbb{P}^2_3, M) - C_{\partial M}(\varphi_3)$$

$$= \langle y + 4a_3(y^2 + 5ya_3 + 7a_3^2), N_2' \times \mathbb{P}^2_3 - C_{\partial M}(\varphi_3)$$

$$= 27(y, N_2') - C_{\partial M}(\varphi_3) = 27 \cdot 2|N_2| - C_{\partial M}(\varphi_3) = 27 \cdot 450 - C_{\partial M}(\varphi_3),$$

(3.23)

where $a_3 = \pi_3^*c_1(\gamma_3^*).$ In order to determine $C_{\partial M}(\varphi_3)$, we split $\partial M$ into five strata:

$$Z_{1,i} = \{([a], x_1, x_2, x_3) : x_3 = x_1, x_3 \neq x_2, ([a], x_1) \in N''_2 - K'_1\}, \quad \{i, j\} = \{1, 2\};$$

$$Z_{0,i} = \{([a], x_1, x_2, x_3) : x_3 = x_1, x_3 \neq x_2, ([a], x_1) \in K'_1 - \mathcal{T}_1\} \approx K_2, \quad \{i, j\} = \{1, 2\};$$

$$Z_{0,12} = \{([a], x, x, x) : ([a], x) \in \mathcal{T}_1\}.$$

Note that Lemma 3.8 implies that the union of these five spaces is indeed $\partial M$. Similarly to the proof of Lemma 3.4, we have

$$C_{Z_{1,1}}(\varphi_3) = C_{Z_{1,2}}(\varphi_3) = N(\alpha), \quad \alpha \in \Gamma(\tilde{N}_2; \text{Hom}(T_0 \mathbb{P}^2_1, \gamma_0 \otimes \gamma_1^* \otimes T_0 \mathbb{P}^2_1), \quad \alpha([a], x_1, x_2, v) = (0, H_{[a], x_1}, v),$$

while

$$C_{Z_{0,1}}(\varphi_3) = C_{Z_{0,2}}(\varphi_3) = 3|K_2|.$$
Finally, suppose that \((\alpha, x) \in \mathcal{T}_1\). Let \(N_{1(x)}^1\) and \(N_{2(x)}^2\) be the normal bundles of \(\mathcal{T}_1\) in \(\mathcal{K}_1\) and of \(\mathcal{K}_1\) in \(\mathcal{N}_1\), respectively, at \((\alpha, x)\). Let \(\mathcal{L}_{1(x)}\) and \(\mathcal{L}_{2(x)}^2\) be as in the proof of Lemma 3.8. Then, with appropriate identifications, for some \(\beta_2, \beta_4 \in \mathbb{C}^*\), \(C \in \mathbb{R}^+\), and \(i = 2, 3\),
\[
|\varphi_1(\alpha, x; u_1, u_2, v, w)| \leq C (|v_1|^5 + |w_1|^2) \tag{3.26}
\]
\[
\forall u_1 \in N_{1(x)}^1, u_2 \in N_{2(x)}^2, v_1 \in \mathcal{L}_{1(x)}, w_1 \in \mathcal{L}_{2(x)}^1,
\]
where \(\alpha_0(u_1, u_2, v, w) = \left(\frac{1}{6} u_1 v^3 + \frac{1}{2} u_2 v^2 + \frac{1}{12} \beta_4 v^4, \frac{1}{2} u_1 v^2 + u_2 v + \frac{1}{3} \beta_4 v^3, \beta_2 w\right)\).

Since \(\hat{N}_{2,0} = \varphi_2^{-1}(0)\), the \(\varphi_3\)-contribution of \((\alpha, x, x, x)\) is the number of small solutions of the system
\[
\begin{align*}
\varphi_2(u_1, u_2, v_2, w_2) &= 0, \\
\varphi_3(u_1, u_2, v_3, w_3) &= t \nu(u_1, u_2, v_2, w_2, v_3, w_3), \quad (u_1, u_2, v_2, w_2, v_3, w_3) \in \mathbb{C}^6 \tag{3.27}
\end{align*}
\]
for a generic \(\nu \in \mathbb{C}^3\) and \(t \in \mathbb{R}^+\) sufficiently small. By \(3.26\) and a rescaling and cobordism argument as in Subsection 3.1 of \(Z\), the number of small solutions of \((3.27)\) is the same as the number of solutions of the system
\[
\begin{align*}
\begin{cases}
\frac{1}{6} u_1 v^3 + \frac{1}{2} u_2 v^2 + \frac{1}{12} \beta_4 v^4 = 0 \\
\frac{1}{2} u_1 v^3 + u_2 v^2 + \frac{1}{3} \beta_4 v^4 = 0 \\
\frac{1}{2} u_1 v^3 + \frac{1}{2} u_2 v^2 + \frac{1}{3} \beta_4 v^4 = \nu \\
\frac{1}{2} u_1 v^3 + u_2 v^2 + \frac{1}{3} \beta_4 v^4 = 0
\end{cases}
\quad (u_1, u_2, v_2, v_3) \in \mathbb{C}^4\tag{3.28}
\end{align*}
\]
for a generic \(\nu \in \mathbb{C}\). Dividing the first two equations by \(v_2^2\) and the last equation by \(v_3^2\) and then solving for \(u_2\) and \(u_1\) in terms of \(v_1\) and \(v_2\), we find that the system \((3.28)\) is equivalent to
\[
\begin{align*}
\begin{cases}
u &= v_3 \quad \text{or} \quad v_2 = 2 v_3 \\
-\frac{1}{2} v_2 v_3^3 + \frac{1}{12} v_2^2 v_3^2 + \frac{1}{12} v_4^4 = \nu
\end{cases}
\quad (u_1, u_2, v_2, v_3) \in \mathbb{C}^4.
\end{align*}
\tag{3.29}
\]
If \(v_2 = v_3\), the last equation has no solutions for \(\nu \neq 0\). On the other hand, if \(v_2 = 2v_3\), the last equation in \((3.29)\) has four solutions. We conclude that
\[
C_{Z_{1,12}}(\varphi_3) = 4|\mathcal{T}_1|.
\tag{3.30}
\]
From \((3.21)\), \((3.25)\), and \((3.30)\), along with Lemmas 3.8 and 3.6, we conclude that
\[
C_{JM}(\varphi_3) = 2 C_{Z_{1,1}}(\varphi_3) + 2 C_{Z_{0,1}}(\varphi_3) + C_{Z_{0,12}}(\varphi_3) = 2 \cdot 1130 + 6 \cdot 840 + 4 \cdot 200 = 8100.
\tag{3.31}
\]
The lemma follows from \((3.28)\) and \((3.31)\), since \(N_3 = \hat{N}_3/S_3\).

**Lemma 3.10** If \(\mathcal{N}_2 \subset \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P}_2^2\) is as in Lemma 3.8 and
\[
\alpha \in \Gamma(\mathcal{N}_2; \text{Hom}(\mathcal{T}^2, \gamma_0^* \otimes_1 \gamma^* \otimes T^* \mathbb{P}_2^1, \mathcal{N}_2; x_1, x_2; v) = (0, H_{\alpha, x_1}, v),
\]
then \(N(\alpha) = 1130\).
Proof: We put
\[
M = \mathbb{P}T\mathbb{P}^2|_{\mathbb{A}'_2}, \quad \partial M = \{([a], x_1, x_2) \in M : \tilde{H}_{a,x} = 0\},
\]
where $\tilde{H}_{a,x}$ is as in the proof of Lemma 3.3. Using Lemma 3.8, we split $\partial M$ into two subsets:
\[
\mathcal{Z}_{0,1} = \{([a], x_1, x_2) \in M : \pi_1([a], x_1) \neq x_2, ([a], x_1) \in \mathcal{K}'_1 - \mathcal{T}_1\},
\]
\[
\mathcal{Z}_{0,2} = \{([a], x_1, x_2) \in M : \pi_1([a], x_1) = x_2, ([a], x_1) \in \mathcal{T}_1\},
\]
where $\pi_1$ is as in the proof of Lemma 3.6. Here $\mathcal{K}'_1$ and $\mathcal{T}_1$ are viewed as subspaces of $\mathbb{P}T\mathbb{P}^2|_{\mathbb{A}'_2}$, as defined in the proof of Lemma 3.3. Let
\[
\tilde{a} = (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^* \otimes \gamma_0^* \otimes \gamma_1^* \otimes \mathbb{T}_\mathbb{P}^2))
\]
be the section induced by $\alpha$. Similarly to the proof of Lemma 3.8,
\[
N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^* \otimes \gamma_0^* \otimes \gamma_1^* \otimes \mathbb{T}_\mathbb{P}^2) \rangle - c(\mathbb{T}_\mathbb{P}^2)^{-1}(\mathbb{A}'_2' - \mathcal{K}'_1) - c_{\partial M}(\tilde{a}^\perp)
\]
\[
= \langle 3y + 6a_1, \mathbb{A}'_2' - \mathcal{K}'_1 - \mathcal{T}_1 \rangle - c_{\partial M}(\tilde{a}^\perp).
\]
(3.32)

As in the proof of Lemma 3.8, we have
\[
C_{\mathcal{Z}_{0,1}}(\tilde{a}^\perp) = \pm |\mathcal{Z}_{0,1}| = |\mathcal{K}'_1|.
\]
(3.33)

On the other hand, suppose $([a], x_1, x_2) \in \mathcal{Z}_{0,2}$ and thus $([a], x_1) \in \mathcal{T}_1$, while $x_2 = \pi_1([a], x_1)$. Then, with identifications similar to the ones used at the end of the proof of Lemma 3.9, the $a^\perp$-contribution of $([a], x_1, x_2)$ is the number of small solutions of the system
\[
\begin{cases}
\varphi_2(u_1, u_2, v_2, w_2) = 0 & (u_1, u_2, v_2, w_2, w_3) \in \mathbb{C}^5 \\ \\
\tilde{a}^\perp(u_1, u_2, w_3) = t \nu(u_1, u_2, v_2, w_2, w_3) & (v_2, w_2) \neq (0, 0),
\end{cases}
\]
(3.34)

for a generic $\nu \in \mathbb{C}^2$ and $t \in \mathbb{R}^+$ sufficiently small. In this case, $w_3 \in \gamma_0^* \otimes \gamma_1^*$ for a good choice of identifications
\[
\tilde{a}^\perp(u_1, u_2, v_2, w_2, w_3) = (u_2, w_3).
\]
(3.35)

By the $i = 2$ case of (3.25) and (3.35), the number of small solutions of the system (3.34) is the same as the number of solutions of the system
\[
\begin{cases}
\frac{1}{6}u_1v_2^3 + \frac{1}{2}u_2v_2^3 + \frac{1}{12}\beta_1v_2^6 = 0 \\
\frac{1}{4}u_1v_2^3 + u_2v_2^2 + \frac{1}{3}\beta_4v_2^4 = 0 & (u_1, u_2, v_2) \in \mathbb{C}^3, \\
w_2 = \nu
\end{cases}
\]
for a generic $\nu \in \mathbb{C}$. Thus, each point of $\mathcal{Z}_{0,2}$ contributes two, and
\[
C_{\mathcal{Z}_{0,2}}(\tilde{a}^\perp) = 2|\mathcal{T}_1|.
\]
(3.36)

The lemma follows from (3.32), (3.33), and (3.36), along with Lemmas 3.3, 3.4, and 3.6.
| set | singularities | $d >$ | co-# pts | cardinality |
|------|--------------|------|----------|-------------|
| $\mathcal{N}_1$ | 1 node | 1 | 1 | $3(d-1)^2$ |
| $\mathcal{N}_{1,1}$ | 1 node on a fixed line | 1 | 2 | $3(d-1)$ |
| $\mathcal{K}_1$ | 1 cusp | 1 | 2 | $12(d-1)(d-2)$ |
| $\mathcal{K}_{1,1}$ | 1 cusp on a fixed line | 3 | 3 | $4(2d-3)$ |
| $\mathcal{T}_1$ | 1 tacnode | 3 | 3 | $2(25d^2 - 96d + 84)$ |
| $\mathcal{N}_2$ | 2 nodes | 1 | 2 | $3(d-1)(d-2)(3d^2 - 3d - 11)/2$ |
| $\mathcal{N}_{2,1}$ | 2 nodes, one on a fixed line | 3 | 3 | $9d^4 - 27d^2 - d + 30$ |
| $\mathcal{K}_2$ | 1 node and 1 cusp | 3 | 3 | $12(d-3)(3d^3 - 6d^2 - 11d + 18)$ |
| $\mathcal{N}_3$ | 3 nodes | 3 | 3 | $(9d^6 - 54d^5 + 9d^4 + 423d^3 - 458d^2 - 829d + 1050)/2$ |

Table 2: Some Characteristic Numbers of Degree-$d$ Plane Curves

3.5 Generalization to Arbitrary-Degree Curves

The computations in the previous subsections generalize to higher-degree curves, as well as to other types of singularities. We list the results of the generalization to arbitrary-degree curves in Table 2. The number in the third column is the lowest value of the degree $d$ for which the formula given in the last column is applicable. Note that in the cases when this number is higher than one, the constraints are $−1$ points for $d=1$ and two points for $d=2$. So, the corresponding count of curves makes no sense for $d=1$, while for $d=2$ this is a count of structures on the double line through two distinct points in $\mathbb{P}^2$. The number in the fourth column is the difference between

$$\dim(d) \equiv \dim \{\deg. - d \text{ curves} \} = \frac{d(d+3)}{2}$$

and the number of points in general position. Below we state the changes that are needed to be made in the above lemmas to obtain these results.

3.5.1 The Numbers $\mathcal{N}_1$ and $\mathcal{N}_{1,1}$

In order to compute the number $\mathcal{N}_1$, we take $\mathcal{D} \approx \mathbb{P}^1$ to be the subspace of degree-$d$ plane curves that pass through a set of $\dim(d)$ points in general position. We define $\mathcal{N}_1$ as in (1) of the proof of Lemma 3.1 except now

$$\varphi \in \Gamma(\mathcal{D} \times \mathbb{P}^2; \gamma_0^* \otimes \gamma_1^* \otimes d \otimes T^* \mathbb{P}^2).$$

Since $\varphi$ is transverse to the zero set, we obtain

$$|\mathcal{N}_1| = |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^* \otimes d \otimes T^* \mathbb{P}^2), \mathcal{D} \times \mathbb{P}^2 \rangle = \langle (y+da)(y^2+(2d-3)ya+(d^2-3d+3)a^2), \mathcal{D} \times \mathbb{P}^2 \rangle = 3(d-1)^2.$$ 

With the analogous changes in (2) of the proof of Lemma 3.1 we find that

$$|\mathcal{N}_{1,1}| = |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^* \otimes d \otimes d \otimes T^* \mathbb{P}^2), \mathcal{D} \times \mathbb{P}^1 \rangle = \langle (y+da)(y^2+(2d-3)ya+(d^2-3d+3)a^2), \mathcal{D} \times \mathbb{P}^1 \rangle = 3(d-1).$$
3.5.2 The Numbers $K_1$ and $K_{1,1}$

We take $D \approx \mathbb{P}^2$ to be the subspace of degree-$d$ plane curves that pass through a set of dim$(d) - 2$ points in general position. We define $N'_1$ and $K_1$ as in (1) of the proof of Lemma 3.2, except now

$$H_{\mathbb{P}^2} \in \Gamma(\mathcal{N}'_1; \text{Hom}(TP^2, \gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2)) \quad \text{and} \quad \varphi \in \Gamma(\mathcal{N}'_1; \gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2) \otimes 2).$$

Since $\varphi$ is transverse to the zero set,

$$|K_1| = |\varphi^{-1}(0)| = \langle \epsilon((\gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2) \otimes 2), \mathcal{N}'_1 \rangle$$

$$= 2\langle y + (d-3)a, N'_{1,1} \rangle = 2(|N_{1,1}| + (d-3)|N|_{1,1})$$

$$= 2(3(d-1)^2 + (d-3) \cdot 3(d-1)) = 12(d-1)(d-2).$$

With the analogous changes in (2) of the proof of Lemma 3.2, we find that

$$|K_{1,1}| = \langle \epsilon((\gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2) \otimes 2), \mathcal{N}'_{1,1} \rangle$$

$$= 2\langle y + (d-3)a, N'_{1,1} \rangle = 2(|N_{1,1}| + (d-3)|a, N'_{1,1})$$

$$= 2(3(d-1) + (d-3)) = 4(2d-3).$$

3.5.3 The Number $T_1$

In this case, we take $D \approx \mathbb{P}^3$ to be the subspace of degree-$d$ plane curves that pass through a set of dim$(d) - 3$ points in general position. We define $N''_1$, $M$, $K'_1$, and $T_1$ as in the proof of Lemma 3.3, except now

$$\tilde{H}_{\mathbb{P}^3} \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2)) \quad \text{and} \quad \varphi \in \Gamma(M; \text{Hom}(\gamma_3, \gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2)).$$

Since the sections $\varphi$ and $\tilde{H}_{\mathbb{P}^3}$ are transverse to the zero set, we obtain

$$|T_1| = |\varphi^{-1}(0)| = \langle \epsilon(\gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2), M \rangle$$

$$= \langle (3\lambda^2 + (7y + (7d-9)a) \lambda^2 + (5y^2 + (10d-12)ya + (5d^2 - 12d + 9)a^2), M \rangle$$

$$= \langle 5N_1 + (10d-33)ya + (5d^2 - 33d + 54)a^2, N''_1 \rangle$$

$$= 5|N_1| + (10d-33)N_{1,1} + (5d^2 - 33d + 54)(a, N'_{1,1})$$

$$= 5 \cdot 3(d-1)^2 + (10d-33) \cdot 3(d-1) + (5d^2 - 33d + 54) = 2(25d^2 - 96d + 84).$$

3.5.4 The Numbers $N_2$ and $N_{2,1}$

In order to compute the number $N_2$, we take $D \approx \mathbb{P}^2$ to be the subspace of degree-$d$ plane curves that pass through a set of dim$(d) - 2$ points in general position. We define $N'_2$, $M$, $\partial M$, $N_2$, $Z_1$, $Z_0$, and $\alpha$ as in (1) of the proof of Lemma 3.4, except now

$$\varphi \in \Gamma(M; \gamma_0^* \otimes_{\gamma_1} \Lambda^2 T^* \mathbb{P}^2), M \rangle \quad \text{and} \quad \alpha \in \Gamma(N'_2; \text{Hom}(TP^2, \gamma_0^* \otimes_{\gamma_2} \Lambda^2 T^* \mathbb{P}^2)).$$

Since $\varphi|_{M^0}$ is transverse to the zero set,

$$|N_2| = \pm |\varphi^{-1}(0) \cap M^0| = \langle \epsilon(\gamma_0^* \otimes_{\gamma_2} \Lambda^2 T^* \mathbb{P}^2), M \rangle - C_{\partial M}(\varphi)$$

$$= \langle (y + da)(y^2 + (2d-3)ya + (d^2 - 3d + 3)a^2), N'_1 \times \mathbb{P}^2 \rangle - C_{\partial M}(\varphi)$$

$$= 3(d-1)^2(y, N'_1) - C_{\partial M}(\varphi) = 3(d-1)^2|N_1| - (C_{Z_0}(\varphi) + C_{Z_1}(\varphi)). \quad (3.37)$$
As in (1) of the proof of Lemma 3.4 we have

\[ C_{Z_1}(\varphi) = N(\alpha) \quad \text{and} \quad C_{Z_0}(\varphi) = 3|K_1|. \]  

(3.38)

Similarly to (1) of the proof of Lemma 3.5

\[ N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^* \ast \otimes T^\ast \mathbb{P}_2) \rangle - C_{\partial M}(\alpha^\perp) \]

\[ = \langle 3y + 3(d-2)a, N_1' \rangle - C_{\partial M}(\alpha^\perp) = (3|N_1| + 3(d-2)|N_{1,1}|) - C_{\partial M}(\alpha^\perp), \]

where \( C_{\partial M}(\alpha^\perp) = |K_1|. \)

Combining these observations with (3.37) and (3.38), we obtain

\[ |\tilde{N}_2| = 3d(d-2)|N_1| - 3(d-2)|N_{1,1}| - 2|K_1| = 3(d-1)(d-2)(3d^2 - 3d - 11). \]

With the analogous modifications in (2) of the proof of Lemma 3.4 we obtain

\[ |N_{2,1}| = \frac{1}{2} |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^* \ast \otimes T^\ast \mathbb{P}_2), N_{1,1}' \times \mathbb{P}_2^2 \rangle - C_{\partial M}(\varphi) \]

\[ = 3(d-1)^2(y, N_{1,1}') - C_{\partial M}(\varphi) = 3(d-1)^2|N_{1,1}| - (C_{Z_0}(\varphi) + C_{Z_1}(\varphi)), \]  

(3.39)

where \( C_{Z_1}(\varphi) = N(\alpha) \quad \text{and} \quad C_{Z_0}(\varphi) = 3|K_{1,1}|. \)  

(3.40)

By the argument in (2) of the proof of Lemma 3.5

\[ N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^* \ast \otimes T^\ast \mathbb{P}_2) \rangle - C_{\partial M}(\alpha^\perp) \]

\[ = \langle 3y + 3(d-2)a, N_{1,1}' \rangle - C_{\partial M}(\alpha^\perp) = (3|N_{1,1}| + 3(d-2)) - C_{\partial M}(\alpha^\perp), \]

where \( C_{\partial M}(\alpha^\perp) = |K_{1,1}|. \)

Combining these identities with (3.39) and (3.40), we obtain

\[ |N_{2,1}| = 3d(d-2)|N_{1,1}| - 3(d-2) - 2|K_{1,1}| = 9d^3 - 27d^2 - d + 30. \]

### 3.5.5 The Number \( K_2 \)

We take \( D \cong \mathbb{P}^3 \) to be the subspace of degree-\( d \) plane curves that pass through a set of \( \dim(d) - 3 \) points in general position. We define \( N_{1,1}', K_1', M, \partial M, K_2, Z_1, Z_0, \) and \( \alpha \) as in the proof of Lemma 3.6 except now

\[ \varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^* \ast \otimes T^\ast \mathbb{P}_2), \quad \alpha \in \Gamma(K_1'; H^m(\gamma^\otimes \otimes \gamma_1^* \ast \otimes \gamma_0^* \otimes \gamma_2^* \ast \otimes T^\ast \mathbb{P}_2)). \]

Since \( \varphi|_{M^0} \) is transverse to the zero set,

\[ |K_2| = \frac{1}{2} |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^* \ast \otimes T^\ast \mathbb{P}_2), M \rangle - C_{\partial M}(\varphi) \]

\[ = \langle (y + da_2)(y^2 + (2d-3)ya_2 + (d^2 - 3d + 3)a_2^2), K_1' \times \mathbb{P}^2, C_{\partial M}(\varphi) \]  

(3.41)

\[ = 3(d-1)^2(y, K_1') - C_{\partial M}(\varphi) = 3(d-1)^2|K_1| - (C_{Z_0}(\varphi) + C_{Z_1}(\varphi)). \]

As in (1) of the proof of Lemma 3.6 we have

\[ C_{Z_1}(\varphi) = 2N(\alpha) \quad \text{and} \quad C_{Z_0}(\varphi) = 4|T_1|. \]  

(3.42)
Similarly to (1) of the proof of Lemma 3.7

\[ N(\tilde{\alpha}) = \langle c(\gamma_0^* \otimes \gamma_1^* \otimes d \oplus \gamma_0^* \otimes \gamma_1^* \otimes d \otimes \gamma^*)c(\gamma^* \otimes 2)^{-1}, K'_1 \rangle - C_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \]

\[ = (3\lambda + 2y + 2da)(\lambda^2 + 2y + (2d - 3)a)\lambda + (2d - 3)ya + (d^2 - 3d + 3)a^2 \}

\[ = (7y^2 + (14d - 39)ya + (7d^2 - 39d + 54)a^2), N''_2) - C_{\tilde{T}_1}(\tilde{\alpha}^\perp) \]

\[ = (7|N_1| + (14d - 39)|N_{1,1}| + (7d^2 - 39d + 54)) - C_{\tilde{T}_1}(\tilde{\alpha}^\perp), \]

where \( C_{\partial M}(\tilde{\alpha}^\perp) = |T_1| \).

Combining these observations with (3.41) and (3.42), we obtain

\[ |K_2| = 3(d - 1)^2|K_1| - 2(7|N_1| + (14d - 39)|N_{1,1}| + (7d^2 - 39d + 54)) - 2|T_1| \]

\[ = 12(d - 3)(3d^3 - 6d^2 - 11d + 18). \]

3.5.6 The Number \( N_3 \)

We take \( D \approx \mathbb{P}^3 \) as above and define \( N''_1, \tilde{N''}_2, N_0, \tilde{N}_3, M, M^0, \tilde{N}_3, Z_{k,i} \) for \( k = 0, 1 \) and \( i = 1, 2, Z_{0,12} \), and \( \alpha \) as in Lemmas 3.8 and Lemma 3.9 except now

\[ \varphi_2 \in \Gamma(N''_1 \otimes \tilde{N''}_2; \gamma_0^* \otimes \gamma_1^* \otimes d \oplus \gamma_0^* \otimes \gamma_1^* \otimes d \otimes T^*\mathbb{P}^2_2), \quad \varphi_3 \in \Gamma(M; \gamma_0^* \otimes \gamma_3^* \otimes d \oplus \gamma_0^* \otimes \gamma_3^* \otimes d \otimes T^*\mathbb{P}^2_3), \]

and \( \alpha \in \Gamma(\tilde{N}_3; \text{Hom}(T^*\mathbb{P}^2_1, \gamma_0^* \otimes \gamma_1^* \otimes d \oplus \gamma_0^* \otimes \gamma_1^* \otimes d \otimes T^*\mathbb{P}^2_1)). \)

Since \( \varphi_3|_{M^0} \) is transverse to the zero set,

\[ |\tilde{N}_3| = \pm |\varphi_3^{-1}(0) \cap M^0| = \langle c(\gamma_0^* \otimes \gamma_3^* \otimes d \oplus \gamma_0^* \otimes \gamma_3^* \otimes d \otimes T^*\mathbb{P}^2_3), M \rangle - C_{\partial M}(\varphi_3) \]

\[ = \langle (y + da_3)(y^2 + (2d - 3)ya_3 + (d^2 - 3d + 3)a^2_3), \tilde{N}_2 \otimes \mathbb{P}^2_3 \rangle - C_{\partial M}(\varphi_3) \]

\[ = 3(d - 1)^2|\varphi, \tilde{N}_2| - C_{\partial M}(\varphi_3) = 6(d - 1)^2|\tilde{N}_2| - 2C_{Z_{1,1}}(\varphi_3) - 2C_{Z_{1,0}}(\varphi_3) - C_{Z_{0,12}}(\varphi_3). \]

Similarly to the proof of Lemma 3.9 we have

\[ C_{Z_{1,1}}(\varphi) = N(\alpha), \quad C_{Z_{1,0}}(\varphi) = 3|K_2|, \quad \text{and} \quad C_{Z_{0,12}}(\varphi) = 4|T_1|. \]

In order to compute \( N(\alpha) \), we define \( M, Z_{0,1} \), and \( Z_{0,2} \) as in the proof of Lemma 3.10. By the same argument as before, we find that

\[ N(\alpha) = \langle c(\gamma_0^* \otimes \gamma_1^* \otimes d \oplus \gamma_0^* \otimes \gamma_1^* \otimes d \otimes T^*\mathbb{P}^2_1)c(T^*\mathbb{P}^2_1)^{-1}, \tilde{N}_2 \rangle - C_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \]

\[ = (3y + (3d - 2)a_1, \tilde{N}_2) - C_{\partial M}(\tilde{\alpha}^\perp) = (6|N_2| + 3(d - 2)|N_{2,1}|) - C_{Z_{0,1}}(\tilde{\alpha}^\perp) - C_{Z_{0,2}}(\tilde{\alpha}^\perp), \]

where \( C_{Z_{0,1}}(\tilde{\alpha}^\perp) = |K_2| \) and \( C_{Z_{0,2}}(\tilde{\alpha}^\perp) = 2|T_1| \).

Combining this result with (3.33) and (3.44), we conclude that

\[ |\tilde{N}_3| = 6((d - 1)^2 - 2)|N_2| - 6(d - 2)|N_{2,1}| - 4|K_2| \]

\[ = 3(9d^6 - 54d^5 + 9d^4 + 423d^3 - 458d^2 - 829d + 1050). \]

Remark: For \( d \geq 5 \), the middle component of the polynomial \( \alpha_0 \) in the proof of Lemma 3.9 should be increased by \( \frac{1}{3} \beta_3 \nu^5 \). However, this term vanishes as we proceed from (3.27) to (3.28).
4 Stable Maps and Recursive Formula

4.1 The Moduli Space of Four Marked Points on a Sphere

In this section, we derive recursion \([1,1]\), following the argument in \([\text{RuT}]\). We start by defining an invariant that counts holomorphic maps into \(\mathbb{P}^{n}\). A priori, the number we describe depends on the cross ratio of the chosen four points on a sphere. However, it turns out that this number is well-defined. We use its independence to express this invariant in terms of the numbers \(n_{d}\) in two different ways. By comparing the two expressions, we obtain \([1,1]\).

Let \(x_{0}, x_{1}, x_{2}\) and \(x_{3}\) be the four points in \(\mathbb{P}^{2}\) given by
\[
x_{0} = [1, 0, 0], \quad x_{1} = [0, 1, 0], \quad x_{2} = [0, 0, 1], \quad x_{3} = [1, 1, 1].
\]
We denote by \(H^{0}(\mathbb{P}^{2}; \gamma^{\otimes 2})\) the space of holomorphic sections of the holomorphic line bundle \(\gamma^{\otimes 2} \to \mathbb{P}^{2}\), or equivalently of the degree-two homogeneous polynomials in three variables; see Lemma \([A,3]\). Let
\[
U = \left\{ ([s], x) \in \mathbb{P}^{1} \times \mathbb{P}^{2}: s(p_{i}) = 0 \quad \forall i = 0, 1, 2, 3; \quad s(x) = 0 \right\}
\]
\[
\approx \left\{ ([A, B], [z_{0}, z_{1}, z_{2}]) \in \mathbb{P}^{1} \times \mathbb{P}^{2}: A z_{0} z_{1} - (A - B) z_{0} z_{2} - B z_{1} z_{2} = 0 \right\}.
\]
The space \(U\) is a compact complex two-manifold. Let \(\pi: U \to \overline{\mathcal{M}}_{0,4} \equiv \mathbb{P}^{1}\) denote the projection onto the first component. If \([A, B] \in \overline{\mathcal{M}}_{0,4}\), the fiber \(\pi^{-1}([A, B])\) is the conic
\[
\mathcal{C}_{A, B} = \left\{ [z_{0}, z_{1}, z_{2}] \in \mathbb{P}^{2}: A z_{0} z_{1} - (A - B) z_{0} z_{2} - B z_{1} z_{2} = 0 \right\}.
\]
If \([A, B] \neq [1, 0], [0, 1], [1, 1]\), \(\mathcal{C}_{A, B}\) is a smooth complex curve of genus zero. In other words, \(\mathcal{C}_{A, B}\) is a union of two lines. One of the lines contains two of the four points \(x_{0}, \ldots, x_{3}\), and the other line passes through the remaining two points. The two lines intersect in a single point. Figure 2 shows the three singular fibers of the projection map \(\pi: U \to \overline{\mathcal{M}}_{0,4}\). The other fibers are smooth conics. The fibers should be viewed as lying in planes orthogonal to the horizontal line in the figure.

We conclude this subsection with a few remarks concerning the family \(U \to \overline{\mathcal{M}}_{0,4}\). These remarks are irrelevant for the purposes of the next subsection and can be omitted.

If \([A, B] \in \overline{\mathcal{M}}_{0,4} - \{ [1, 0], [0, 1], [1, 1] \}\), \(\mathcal{C}_{A, B}\) is a smooth complex curve of genus zero, i.e. it is a sphere holomorphically embedded in \(\mathbb{P}^{2}\). Thus, there exists a one-to-one holomorphic map \(f: \mathbb{P}^{1} \to \mathcal{C}_{A, B}\). Using Lemma \([A,1]\) it can be shown directly that if \([u_{i}, v_{i}] = f^{-1}(x_{i})\),
\[
\begin{align*}
\frac{u_{0}/v_{0} - u_{2}/v_{2}}{u_{0}/v_{0} - u_{3}/v_{3}} : \frac{u_{1}/v_{1} - u_{2}/v_{2}}{u_{1}/v_{1} - u_{3}/v_{3}} &= \frac{B}{A}.
\end{align*}
\]
The cross-ratio is the only invariant of four distinct points on \(\mathbb{P}^{1}\); see \([A]\), for example. Thus,
\[
\mathbb{P}^{1} - \{ [1, 0], [0, 1], [1, 1] \} = \overline{\mathcal{M}}_{0,4} \equiv \left\{ (x_{0}, x_{1}, x_{2}, x_{3}) \in (\mathbb{P}^{1})^{4}: x_{i} \neq x_{j} \text{ if } i \neq j \right\} / \sim,
\]
where \((x_{0}, x_{1}, x_{2}, x_{3}) \sim (\tau(x_{0}), \tau(x_{1}), \tau(x_{2}), \tau(x_{3}))\) if \(\tau \in \text{PSL}_{2} \equiv \text{Aut}(\mathbb{P}^{1})\).

Furthermore, the restriction of the projection map \(\pi: U|_{\overline{\mathcal{M}}_{0,4}} \to \overline{\mathcal{M}}_{0,4}\) to each fiber \(\mathcal{C}_{[A, B]}\) is the cross ratio of the points \(x_{0}, \ldots, x_{3}\) on \(\mathcal{C}_{[A, B]}\), viewed as an element of \(\mathbb{P}^{1} \subset \mathbb{C}\).
Figure 2: The Family $\mathcal{U} \longrightarrow \overline{\mathfrak{M}}_{0,4}$

### 4.2 Counts of Holomorphic Maps

If $d$ is an integer and $C$ is a complex curve, which may be a wedge of spheres, let

$$\mathcal{H}_d(C) = \{ f \in C^\infty(C;\mathbb{P}^2) : f \text{ is holomorphic}, \ f_*[C] = d[L] \},$$

where $[L] \in H_2(\mathbb{P}^2;\mathbb{Z})$ is the homology class of a line in $\mathbb{P}^2$. We give a more explicit description of the space $\mathcal{H}_d(C)$ in the relevant cases below.

Suppose $\ell_0, \ell_1$ and $p_2, \ldots, p_{3d-1}$ are two lines and $3d-2$ points in general position in $\mathbb{P}^2$. If $\sigma \in \overline{\mathfrak{M}}_{0,4}$, let $N^0_d(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$ denote the cardinality of the set

$$\{ f \in \mathcal{H}_d(C) : f(x_0) \in \ell_0, \ f(x_1) \in \ell_1, \ f(x_2) = p_2, \ f(x_3) = p_3, \ p_i \in \text{Im} f \ \forall i \}. \quad (4.2)$$

Here $C_\sigma$ denotes the rational curve with four marked points, $x_0$, $x_1$, $x_2$, and $x_3$, whose cross ratio is $\sigma$; see Subsection 4.1. If $\sigma \neq [1,0], [0,1], [1,1]$, $C_\sigma$ is a sphere with four, distinct, marked points. In this case, the condition $f \in \mathcal{H}_d(C_\sigma)$ means that $f$ has the form

$$f([u,v]) = [P_0(u,v), P_1(u,v), P_2(u,v)] \quad \forall [u,v] \in \mathbb{P}^1,$$

for some degree-$d$ homogeneous polynomials $P_0, P_1, P_2$ that have no common factor; see Lemma A.1. If $\sigma = [1,0], [0,1], [1,1]$, $C_\sigma$ is a wedge of two spheres, $C_{\sigma,1}$ and $C_{\sigma,2}$, with two marked points each. In this case, the first condition in (4.1) means that $f$ is continuous and $f|_{C_{\sigma,1}}$ and $f|_{C_{\sigma,2}}$ are holomorphic. The second condition in (4.1) means that $d = d_1 + d_2$ if $f_*[C_{\sigma,1}] = d_1[L]$ and $f_*[C_{\sigma,2}] = d_2[L]$.

The requirement that the two lines, $\ell_0$ and $\ell_1$, and the $3d-2$ points, $p_2, \ldots, p_{3d-1}$, are in general position means that they lie in a dense open subset $\mathcal{U}_d$ of the space of all possible tuples $(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$:

$$\mathcal{X} \equiv \text{Gr}_2 \mathbb{C}^3 \times \text{Gr}_2 \mathbb{C}^3 \times (\mathbb{P}^2)^{3d-2}.$$

Here $\text{Gr}_2 \mathbb{C}^3$ denotes the Grassmanian manifold of two-planes through the origin in $\mathbb{C}^3$, or equivalently of lines in $\mathbb{P}^2$. The dense open subset $\mathcal{U}_d$ of $\mathcal{X}$ consists of tuples $(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$ that satisfy a number of geometric conditions. In particular, $\ell_0 \neq \ell_1$, none of the points $p_2, \ldots, p_{3d-1}$ lies on either $\ell_0$ or $\ell_1$, the $3d-1$ points $\ell_0 \cap \ell_1, p_2, \ldots, p_{3d-1}$ are distinct, no three of them lie on the same line, and so on. In addition, we need to impose certain cross-ratio conditions on the
rational curves that pass through \( \ell_0, \ell_1, p_2, p_3 \), and a subset of the remaining \( 3d-4 \) points. These conditions can be stated more formally. Define

\[
\text{ev}_\sigma: \mathcal{H}_d(C_\sigma) \times (C_\sigma)^{3d-4} \rightarrow (\mathbb{P}^2)^d \quad \text{by} \quad \text{ev}_\sigma(f; x_4, \ldots, x_{3d-1}) = (f(x_1), \ldots, f(x_{3d-1})).
\]

Lemma A.1 implies that \( \mathcal{H}_d(C_\sigma) \) is a dense open subset of \( \mathbb{P}^{3d+2} \) and the evaluation map \( \text{ev}_\sigma \) is holomorphic. The space \( \mathcal{H}_d(C_\sigma) \) has a natural compactification \( \overline{\mathcal{M}}_d(\mathbb{P}^2, d) \), which is the union of spaces of holomorphic maps from various wedges of spheres into \( \mathbb{P}^2 \). The complex dimension of each such boundary stratum is less than that of \( \mathcal{H}_d(C_\sigma) \). The evaluation map \( \text{ev}_\sigma \) admits a continuous extension over \( \partial \overline{\mathcal{M}}_d(\mathbb{P}^2, d) \), whose restriction to each stratum is holomorphic. The elements \( (\ell_0, \ell_1, p_2, \ldots, p_{3d-1}) \) of the subspace \( \mathcal{U}_\sigma \) of \( X \) are characterized by the condition that the restriction of the evaluation map to each stratum of \( \overline{\mathcal{M}}_d(\mathbb{P}^2, d) \) is transversal to the submanifold

\[
\ell_0 \times \ell_1 \times p_2 \times \ldots \times p_{3d-1} \subset (\mathbb{P}^2)^d.
\]

This condition implies that

\[
\text{ev}_\sigma^{-1}(\ell_0 \times \ell_1 \times p_2 \times \ldots \times p_{3d-1}) \cap \partial \overline{\mathcal{M}}_d(\mathbb{P}^2, d) = \emptyset
\]

and the set in (4.2) is a finite subset of \( \mathcal{H}_d(C_\sigma) \).

The set \( \mathcal{U}_\sigma \) of "general" tuples \( (\ell_0, \ell_1, p_2, \ldots, p_{3d-1}) \) is path-connected. Indeed, it is the complement of a finite number of proper complex submanifolds in \( X \). It follows that the number in (4.2) is independent of the choice of two lines and \( 3d-2 \) points in general position in \( \mathbb{P}^2 \). We thus may simply denote it by \( N^\sigma_d \). If \( \sigma \neq [1,0], [0,1], [1,1], \), \( C_\sigma \) is a sphere with four distinct points. In such a case, it is fairly easy to show that the number \( N^\sigma_d \) does not change with small variations \( \sigma \), or equivalently of the four points on the sphere. Thus, \( N^\sigma_d \) is independent of

\[
\sigma \in \mathcal{M}_{0,4} = \mathbb{P}^1 - \{ [1,0], [0,1], [1,1] \} = \overline{\mathcal{M}}_{0,4} - \{ [1,0], [0,1], [1,1] \}.
\]

It is far harder to prove

**Proposition 4.1** The function \( \sigma \rightarrow N^\sigma_d \) is constant on \( \overline{\mathcal{M}}_{0,4} \).

This proposition is a special case of the gluing theorems first proved in [McSa] and [RuT]. A more straightforward proof can be obtained via the approach of [LT].

### 4.3 Holomorphic Maps vs. Complex Curves

In this subsection, we express the numbers \( N_d^{[1,0]} \) and \( N_d^{[0,1]} \) of Subsections 4.2 in terms of the numbers \( n_{d,d'} \), with \( d' \leq d \), of Question 1.1. By Proposition 4.1, \( N_d^{[1,0]} = N_d^{[0,1]} \). We obtain a recursion for the numbers of Question 1.1 by comparing the expressions for \( N_d^{[1,0]} \) and \( N_d^{[0,1]} \).

Let \( C_1 \) denote the component of \( C_{[1,0]} \) containing the marked points \( x_0 \) and \( x_3 \); see Figure 2. We denote by \( C_2 \) the other component of \( C_{[1,0]} \). By definition,

\[
N_d^{[1,0]} = \sum_{d_1 + d_2 = d} N_{d_1,d_2}^{[1,0]} \quad \text{where}
\]

\[
N_{d_1,d_2}^{[1,0]} = \left| \{ f \in \mathcal{H}_d(C_{[0,1]}; \mathbb{P}^2) : f_*[C_1] = d_1[L], f_*[C_2] = d_2[L]; p_i \in \text{Im} f \forall i; f(x_0) \in \ell_0, f(x_1) \in \ell_1, f(x_2) = p_2, f(x_3) = p_3 \} \right|.
\]
Since the group $PSL_2$ of holomorphic automorphisms acts transitively on triples of distinct points on the sphere,

$$N_{d_1,d_2}^{[1,0]} = |\{(f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : f_1(\infty) = f_2(\infty), \ p_i \in f_1(S^2) \cup f_2(S^2) \ \forall i; \\
f_1(0) \in \ell_0, \ f_1(1) = p_3, \ f_2(0) \in \ell_1, \ f_2(1) = p_2\}|.$$ 

Since the maps $f_1$ and $f_2$ above are holomorphic, $d_1, d_2 \geq 0$ if $N_{d_1,d_2}^{[1,0]} \neq 0$. Since every degree-zero holomorphic map is constant and $p_3 \not\in \ell_0$, $N_{0,d}^{[1,0]} = 0$. Similarly, $N_{d,0}^{[1,0]} = 0$. Thus, we assume that $d_1, d_2 > 0$. Since the points $p_3, \ldots, p_{3d-1}$ are in general position, $f_1(S^2)$ contains at most $3d_1-2$ of the points $p_4, \ldots, p_{3d-1}$. Similarly, the curve $f_2(S^2)$ passes through at most $3d_2-2$ of the points $p_4, \ldots, p_{3d-1}$. Thus, if $I = \{4, \ldots, 3d-1\}$,

$$N_{d_1,d_2}^{[1,0]} = \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1,d_2}^{[1,0]}(I_1, I_2),$$

where $N_{d_1,d_2}^{[1,0]}(I_1, I_2)$ is the cardinality of the set

$$S_{d_1,d_2}^{[1,0]}(I_1, I_2) = \{(f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : p_i \in f_1(S^2) \ \forall i \in I_1, \ p_i \in f_2(S^2) \ \forall i \in I_2; \\
f_1(\infty) = f_2(\infty), \ f_1(0) \in \ell_0, \ f_1(1) = p_3, \ f_2(0) \in \ell_1, \ f_2(1) = p_2\}.$$

If $(f_1, f_2) \in S_{d_1,d_2}^{[1,0]}(I_1, I_2)$, $f_1(S^2)$ is one of the $n_{d_1}$ curves passing through the points $\{p_i : i \in \{3\} \setminus I_1\}$. Similarly, $f_2(S^2)$ is one of the $n_{d_2}$ curves passing through the points $\{p_i : i \in \{2\} \setminus I_2\}$. The point $f_1(\infty) = f_2(\infty)$ must be one of the $d_1 d_2$ points of $f_1(S^2) \cap f_2(S^2)$; see Lemma A.5. Finally, $f_1(0)$ must be one of the $d_1$ points of $f_1(S^2) \cap \ell_0$, while $f_2(0)$ must be one of the $d_2$ points of $f_2(S^2) \cap \ell_1$. Thus, we conclude that

$$N_d^{[1,0]} = \sum_{d_1+d_2=d} N_{d_1,d_2}^{[1,0]} = \sum_{d_1+d_2=d} \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1,d_2}^{[1,0]}(I_1, I_2)$$

$$= \sum_{d_1+d_2=d} \sum_{I_1 \sqcup I_2, |I_1|=3d_1-2} (d_1 d_2)(d_1 n_{d_1})(d_2 n_{d_2})$$

$$= \sum_{d_1+d_2=d} \left(\frac{3d-4}{3d_1-2}\right) d_1^2 d_2^2 n_{d_1} n_{d_2}; \tag{4.3}$$

where $I = \{4, \ldots, 3d-1\}$.

We compute the number $N_d^{[0,1]}$ similarly. We denote by $C_1$ the component of $C_{[0,1]}$ containing the points $x_0$ and $x_1$ and by $C_2$ the other component of $C_{[0,1]}$. By definition,

$$N_d^{[0,1]} = \sum_{d_1+d_2=d} N_{d_1,d_2}^{[0,1]}, \quad \text{where}$$

$$N_{d_1,d_2}^{[0,1]} = \left|\{(f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : f_1(\infty) = f_2(\infty), \ p_i \in f_1(S^2) \cup f_2(S^2) \ \forall i; \\
f_1(0) \in \ell_0, \ f_1(1) \in \ell_1, \ f_2(0) = p_2, \ f_2(1) = p_3\}\right|.$$ 

Since every degree-zero holomorphic map is constant, $N_{d,0}^{[0,1]} = 0$ as before. However,

$$N_{0,d}^{[0,1]} = \left|\{f_2 \in \mathcal{H}_{d}(S^2) : f_2(\infty) \in \ell_0 \cap \ell_1, \ p_i \in f_2(S^2) \ \forall i = 2, \ldots, 3d-1\}\right| = n_d.$$
If \( d_1, d_2 > 0 \),
\[
N_{d_1, d_2}^{[0,1]} = \sum_{I: |I| = 3d_1 - 1} N_{d_1, d_2}^{[0,1]}(I, I_2),
\]
where \( N_{d_1, d_2}^{[0,1]}(I, I_2) \) is the cardinality of the set
\[
S_{d_1, d_2}^{[0,1]}(I, I_2) = \{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : p_i \in f_1(S^2) \forall i \in I_1, p_i \in f_2(S^2) \forall i \in I_2; f_1(\infty) = f_2(\infty), f_1(0) = \ell_0, f_1(1) = \ell_1, f_2(0) = p_2, f_2(1) = p_3 \}.
\]

Proceeding as in the previous paragraph, we conclude that
\[
N_d^{[0,1]} = \sum_{d_1 + d_2 = d} N_{d_1, d_2}^{[0,1]} = n_d + \sum_{d_1 + d_2 = d} \sum_{I: |I| = 3d_1 - 1} N_{d_1, d_2}^{[0,1]}(I, I_2)
\]
\[
= n_d + \sum_{d_1 + d_2 = d} \sum_{I: |I| = 3d_1 - 1} (d_1 d_2)(d_1^2 n_d_1)(n_{d_2})
\]
\[
= n_d + \sum_{d_1 + d_2 = d} \left( 3d_1 - 4 \right) d_1^2 d_2 n_d_1 n_{d_2};
\]  
(4.4)

Comparing equations (4.3) and (4.4), we obtain
\[
n_d = \sum_{d_1 + d_2 = d} \left( \frac{3d_1 - 4}{3d_1 - 2} d_1 d_2 - \frac{3d_1 - 4}{3d_1 - 1} d_1^2 \right) d_1 d_2 n_{d_1} n_{d_2}.
\]  
(4.5)

The recursive formula (1.1) is the symmetrized version of (4.5).

A The Basics

A.1 Complex Projective Spaces

The complex projective space \( \mathbb{P}^n \) is the space of (complex) lines through the origin in \( \mathbb{C}^{n+1} \). Equivalently,
\[
\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*, \quad \text{where} \quad (z_0, \ldots, z_n) \sim (t z_1, \ldots, t z_n) \quad \text{if} \ t \in \mathbb{C}^*.
\]

This space is a smooth \( 2n \)-manifold. For \( i = 0, \ldots, n \), let
\[
U_i = \{ [z_0, \ldots, z_n] \in \mathbb{P}^n : z_i \neq 0 \},
\]
\[
\phi_i : \mathbb{C}^n \rightarrow U_i, \quad \phi_i(z_1, \ldots, z_n) = [w_1, \ldots, w_i, 1, w_{i+1}, \ldots, w_n].
\]

The set \( \{(U_i, \phi_i, \mathbb{C}^n)\} \) is the standard atlas for \( \mathbb{P}^n \). If \( i < j \), the corresponding overlap map is given by
\[
\phi_{ij} = \phi_i^{-1} \circ \phi_j |_{\phi_j^{-1}(U_i)} : \{ (w_1, \ldots, w_n) \in \mathbb{C}^n : w_{i+1} \neq 0 \} \rightarrow \{ (w_1, \ldots, w_n) \in \mathbb{C}^n : w_{j+1} \neq 0 \}
\]
\[
(w_1, \ldots, w_n) \rightarrow \left( \frac{w_1}{w_{i+1}}, \ldots, \frac{w_i}{w_{i+1}}, \frac{w_{i+1}}{w_{i+1}}, \ldots, \frac{w_j}{w_{i+1}}, \frac{w_{j+1}}{w_{i+1}}, \ldots, \frac{w_n}{w_{i+1}} \right).
\]

Each map \( \phi_{ij} \) is a diffeomorphism. In fact, this map is holomorphic, and so is its inverse \( \phi_{ij}^{-1} \). In other words, \( \mathbb{P}^n \) is naturally a complex \( n \)-manifold.
Suppose $X$ and $Y$ are complex manifolds, of complex dimensions $m$ and $n$, and with (holomorphic) atlases $\{(U_i, \phi_i, U'_i)\}_{i \in I}$ and $\{(V_j, \varphi_j, V'_j)\}_{j \in J}$ respectively. Smooth map $f: X \to Y$ is called \textit{holomorphic} if for all $i \in I$ and $j \in J$, the map

$$\varphi_j^{-1} \circ f \circ \phi_i: \varphi_i^{-1}(f^{-1}(V_j)) \to \mathbb{C}^n$$

is holomorphic as a $\mathbb{C}^n$-valued function on an open subset of $\mathbb{C}^m$. In the case of interest to us, i.e. $X = \mathbb{P}^1$ and $Y = \mathbb{P}^n$, the holomorphic maps have a much simpler description, see Lemma A.1 below. This lemma can be checked directly. The simpler characterization of Lemma A.1 can be taken as the definition of what it means to be a holomorphic map between $\mathbb{P}^1$ and $\mathbb{P}^n$.

\textbf{Lemma A.1} If $f: \mathbb{P}^1 \to \mathbb{P}^n$ is a holomorphic map, there exist homogeneous polynomials $p_0, \ldots, p_n$ in two variables such that $p_0, \ldots, p_n$ are of the same degree, have no common factor, and

$$f([z_0, z_1]) = [p_0(z_0, z_1), \ldots, p_n(z_0, z_1)] \quad \forall [z_0, z_1] \in \mathbb{P}^1.$$  \hspace{1cm} (A.1)

Conversely, if $p_0, \ldots, p_n$ are homogeneous polynomials in two variables that are of the same degree and have no common factor, the map $f: \mathbb{P}^1 \to \mathbb{P}^n$ given by (A.1) is well-defined and holomorphic.

\section*{A.2 Almost Complex and Symplectic Structures}

This subsection is not relevant for understanding Sections 2-4. However, it puts the last section in perspective.

Let $X$ be a smooth manifold. An \textit{almost complex structure} on $X$ is a smooth section $J$ of the bundle $\text{End}(TX) \to X$ such that $J^2 = -I$. In other words, an almost complex structure is a smooth family of linear maps $J_p: T_pX \to T_pX$ such that $J_p J_p v = -v$ for all $v \in T_pX$ and $p \in X$. For example, if $X = \mathbb{C}^n$, $T_p \mathbb{C}^n = \mathbb{C}^n$ and the desired endomorphism on $T_p \mathbb{C}^n$ is simply the multiplication by $i$.

Every complex $n$-manifold $X$ carries a natural almost complex structure $J$, defined as follows. Let $\{(U_i, \phi_i, U'_i)\}_{i \in I}$ be the (holomorphic) atlas for $X$. If $p \in U_i$, we set

$$J_p = d\phi_i|_{\phi_i^{-1}(p)} \circ i \circ d\phi_i^{-1}|_p.$$

Since all overlap maps $\phi_i^{-1} \circ \phi_j$ are holomorphic, the endomorphism $J_p$ is independent of the choice of $i \in I$ such that $p \in U_i$. An almost complex structure arising in such a way is called \textit{complex} or \textit{integrable}.

A typical almost complex structure is not integrable, unless the real dimension of the manifold is two. In fact, there is a criterion that characterizes integrable almost complex structures. If $(X, J)$ is an almost complex manifold, $p \in X$, and $V$ and $W$ are vector fields on $X$, let

$$N^J_p(V_p, W_p) = \frac{1}{4} ([V, W]_p + J_p [JV, W]_p + J_p[V, JW]_p - [JV, JW]_p).$$

The vector $N^J_p(V_p, W_p) \in T_pX$ depends only on the values $V_p$ and $W_p$ of the vector fields $V$ and $W$ at the point $p$. In addition, $N^J_p$ is linear in each of the two inputs. Thus,

$$N^J \in \Gamma(X; \text{Hom}(TX \otimes TX, TX)).$$
i.e. $N^J$ is a $(2,1)$-tensor field on $X$. This tensor field is called the Nijenhuis torsion of $J$. It is easy to see that $N^J = 0$ if $J$ is an integrable almost complex structure. The converse is proved in [NeNi].

Since $N^J = 0$ if $(X, J)$ is an almost complex manifold of real dimension two, it follows every almost complex structure on a smooth two-manifold is integrable. Such a manifold is called a Riemann surface.

Suppose $(X, j)$ and $(Y, J)$ are almost complex manifolds and $f : X \to Y$ is a smooth map. If $z \in X$, we set
\[ \bar{\partial}_{I,J}f|_z = df|_z + J_{f(z)} \circ df|_z \circ j_z \in \text{Hom}(T_zX, T_{f(z)}Y). \]
Note that $\bar{\partial}_{I,J}f|_z = -J_{f(z)} \circ \bar{\partial}_{I,J}f|_z$, i.e. the linear map $\bar{\partial}_{I,J}f|_z$ is $(J, j)$-antilinear. Thus,
\[ \bar{\partial}_{I,J}f \in \Gamma(X, \Lambda^0_1 T^*X \otimes f^*TY), \]
where $\Lambda^0_1 T^*X \otimes f^*TY \to X$ is the bundle of $(f^*J, j)$-antilinear homomorphisms from $(TX, j)$ to $f^*(TY, J)$. The smooth map $f : X \to Y$ is called $(J, j)$-holomorphic, or pseudoholomorphic, if $\bar{\partial}_{I,J}f = 0$. If $(X, j)$ and $(Y, J)$ are complex manifolds, this definition agrees with the one given in the previous subsection. More generally, if $(X, j)$ is a wedge of finitely many almost complex manifolds $(X_l, j_l)$, we will call a continuous map $f : X \to Y (J, j)$-holomorphic if $f|_{X_l}$ is $(J, j_l)$-holomorphic for all $l$.

If $(X, J)$ is an almost complex manifold, $A \in H_2(X; \mathbb{Z})$, and $g$ and $n$ are nonnegative integers, let
\[ \mathcal{M}_{g,n}(X, A; J) = \{(\Sigma, j, x_1, \ldots, x_n; f) : (\Sigma, j) = \text{Riemann surface of genus } g; \]
\[ x_i \in \Sigma, x_i \neq x_j \text{ if } i \neq j; f \in C^\infty(\Sigma; X), f_*[\Sigma] = A, \bar{\partial}_{I,J}f = 0\}/, \]
where $(\Sigma, j, z_1, \ldots, z_n; f) \sim (\Sigma', j', \tau(z_1), \ldots, \tau(z_n), f \circ \tau^{-1})$ if $\tau \in C^\infty(\Sigma; \Sigma'), \bar{\partial}_{j,j'}\tau = 0$.

This moduli space has a natural topology, as well as $n$ evaluation maps
\[ \text{ev}_i : \mathcal{M}_{g,n}(X, A; J) \to X, \quad [\Sigma, j, z_1, \ldots, z_n; f] \to f(z_i). \]
In general, $\mathcal{M}_{g,n}(X, A; J)$ is not a compact topological space. However, under certain conditions on $(X, J)$, $\mathcal{M}_{g,n}(X, A; J)$ admits a natural compactification and in fact carries a (virtual) fundamental class.

Let $X$ be a smooth manifold. A symplectic form on $X$ is a closed two-form $\omega$ on $X$ which is nondegenerate at every point of $X$. In other words, $d\omega = 0$, and for every point $p$ in $X$ and nonzero tangent vector $v \in T_pX$, there exists $w \in T_pX$ such that $\omega_p(v, w) \neq 0$. For example, if $(x_1, y_1, \ldots, x_n, y_n)$ are the standard coordinates on $\mathbb{C}^n$,
\[ \omega \equiv dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n \]
is a symplectic form on $\mathbb{C}^n$. More generally, if $X$ admits a symplectic form, the (real) dimension of $X$ is even.

If $(X, \omega)$ is a symplectic manifold, the almost complex structure $J$ on $X$ is $\omega$-tame if for every point $p$ in $X$ and nonzero tangent vector $v \in T_pX$, $\omega_p(v, J_p v) > 0$. The $\omega$-tame almost complex structure $J$ is $\omega$-compatible if
\[ \omega_p(Jv, Jw) = \omega_p(v, w) \quad \forall p \in X, v, w \in T_pX. \]
For example, if $\omega$ is the standard symplectic form on $\mathbb{C}^n$, defined in the previous paragraph, the standard complex structure $i$, defined in the second paragraph of this subsection, is $\omega$-compatible. For a general symplectic manifold $(X, \omega)$, the spaces of $\omega$-tame and $\omega$-compatible almost complex structures on $X$ are non-empty and contractible. The most fundamental result in the theory of pseudoholomorphic curves is Gromov’s Compactness Theorem, stated roughly below.

**Theorem A.2** \cite{Gro} Suppose $(X, \omega)$ is a compact symplectic manifold and $J$ is an almost complex $\omega$-tame structure on $X$. If $A \in H_2(X; \mathbb{Z})$ and $g$ and $n$ are nonnegative integers, the moduli space $\mathcal{M}_{g,n}(X, A; J)$ admits a natural compactification $\overline{\mathcal{M}}_{g,n}(X, A; J)$. In particular, the evaluation maps $ev_i$ extend continuously over $\overline{\mathcal{M}}_{g,n}(X, A; J)$.

The compactification $\overline{\mathcal{M}}_{g,n}(X, A; J)$ consists of equivalence classes of tuples $(\Sigma, j, x_1, \ldots, x_n, f)$, where $(\Sigma, j)$ is a possibly singular genus-$g$ Riemann surface, i.e. a wedge of smooth Riemann surfaces, $x_1, \ldots, x_n$ are distinct points on $\Sigma$, and $f: \Sigma \to X$ is a $(J, j)$-holomorphic map such that $f_*[\Sigma] = A$. Notice that the space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ is described by the almost complex structure $J$, and not the symplectic form $\omega$. However, this space may not be compact if $J$ is not $\omega$-tame for some symplectic form $\omega$ on $X$.

Since the space of $\omega$-tame almost complex structures on $X$ is contractible, up to an appropriate equivalence, the space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ is independent of the choice of $J$. In particular, the "equivalence class" of $\overline{\mathcal{M}}_{g,n}(X, A; J)$ is determined by $(X, \omega)$ and thus is a symplectic invariant. This is essentially the Gromov-Witten invariant of $(X, \omega)$.

### A.3 Tautological Line Bundle

We continue with the notation of Subsection A.1. Let

$$\gamma = \{ (\ell; z_0, \ldots, z_n) \in \mathbb{P}^n \times \mathbb{C}^{n+1}; (z_0, \ldots, z_n) \in \ell \}.$$  

We denote by $\pi: \gamma \to \mathbb{P}^n$ the projection map. For each $\ell \in \mathbb{P}^n$, the fiber $\gamma_\ell \equiv \pi^{-1}(\ell)$ over a point $\ell \in \mathbb{P}^n$ is the line $\ell$ through the origin in $\mathbb{C}^n$. For each $i = 0, \ldots, n$, let

$$\tilde{U}_i = \pi^{-1}(U_i) = \{ (\ell; z_0, \ldots, z_n) \in \gamma; z_i \neq 0 \},$$  

$$\tilde{\phi}_i: \mathbb{C}^n \times \mathbb{C} \to \tilde{U}_i, \quad \tilde{\phi}_i(w_1, \ldots, w_n; \lambda) = (\phi_i(w_1, \ldots, w_n); \lambda w_1, \ldots, \lambda w_i, \lambda w_{i+1}, \ldots, \lambda w_n).$$

The set $\{(\tilde{U}_i, \tilde{\phi}_i, \mathbb{C}^n) \times \mathbb{C}\}$ is the standard atlas for $\gamma$. If $i < j$, the corresponding overlap map is given by

$$\tilde{\phi}_{ij} \equiv \tilde{\phi}_j^{-1} \circ \tilde{\phi}_i|_{\tilde{U}_i} \circ (\tilde{U}_i) \times \mathbb{C} \to \tilde{U}_j \times \mathbb{C}, \quad (w_1, \ldots, w_n; \lambda) \to (\phi_{ij}(w_1, \ldots, w_n); w_{i+1} \lambda).$$

Each map $\tilde{\phi}_{ij}$ is holomorphic, and so is its inverse $\tilde{\phi}_{ij}^{-1}$. Thus, $\gamma$ is a complex $(n+1)$-manifold. Furthermore, if $p: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is the projection map,

$$\pi \circ \tilde{\phi}_i = \phi_i \circ p \quad \forall \ i = 0, \ldots, n,$$

and $\tilde{\phi}_i: p^{-1}(w) \to \pi^{-1}(\phi_i(w))$ is a $\mathbb{C}$-linear map for all $w \in \mathbb{C}^n$. Thus, $\gamma \to \mathbb{P}^n$ is a holomorphic rank-one vector bundle, i.e. a holomorphic line bundle.
Each homogeneous polynomial,

\[ p = \sum_{i_0 + \ldots + i_n = d} a_{i_0 \ldots i_n} z_0^{i_0} \ldots z_n^{i_n}, \]

of degree \( d \) in \( n+1 \) variables determines a section \( s_p \) of the bundle \( \gamma \otimes \rightarrow \mathbb{P}^n \), described as follows. At each point \( \ell \in \mathbb{P}^n \), \( s_p(\ell) \) is to be a map from \( \gamma_p \) to \( \mathbb{C} \) such that

\[ \{ s_p(\ell) \}(t\zeta) = t^d \{ s_p(\ell) \}(\zeta) \quad \forall \zeta \in \gamma_p = \ell. \]

Thus, we define \( s_p \) by

\[ \{ s_p(\ell) \}(\ell; z_0, \ldots, z_n) = p(z_0, \ldots, z_n). \]

Lemma A.3 below can be checked directly from the relevant definitions.

**Lemma A.3** If \( p \) is a homogeneous polynomial of degree \( d \) in \( n+1 \) variables, \( s_p \) is a holomorphic section of the holomorphic line bundle \( \gamma \otimes \rightarrow \mathbb{P}^n \). Conversely, if \( s \) is a holomorphic section of \( \gamma \otimes \rightarrow \mathbb{P}^n \), \( s = s_p \) for some homogeneous polynomial \( p \) of degree \( d \) in \( n+1 \) variables.

If \( s \) is a section of a vector bundle \( V \) over a smooth manifold \( X \) and \( x \in s^{-1}(0) \), the differential of \( s \) at \( x \) is a well-defined linear map:

\[ ds|_x : T_x X \rightarrow V_x. \]

It can be constructed using either a chart for \( V \) or a connection in \( V \). If \( ds|_x \) is surjective, \( s \) is said to be transversal to the zero set at \( x \). If \( ds|_x \) is surjective for all \( x \in s^{-1}(0) \), \( s \) is to be transverse to the zero set. If \( V \) is a complex vector bundle of rank \( n \), \( X \) is a complex \( n \)-manifold, and \( s \) is transversal to the zero set at \( x \in s^{-1}(0) \), \( x \) is an isolated point of \( s^{-1}(0) \) and \( ds|_x : T_x X \rightarrow V_x \) is an \( \mathbb{R} \)-linear map between complex (and thus, oriented) vector spaces. The point \( x \) is assigned the plus sign if this map is orientation-preserving and the minus sign otherwise. Note that if \( s \) is a holomorphic section, \( ds|_x \) is \( \mathbb{C} \)-linear and thus orientation-preserving.

We conclude this subsection by proving Lemma 2.1. With notation as before,

\[ g(s^{-1}(0)) = \frac{2 - \chi(s^{-1}(0))}{2}, \tag{A.2} \]

where \( \chi(s^{-1}(0)) \) is the euler characteristic of the surface \( s^{-1}(0) \). On the other hand, by Corollary 11.12 in [MSt] and by Lemma 2.2

\[ \chi(s^{-1}(0)) = \langle e(Ts^{-1}(0)), s^{-1}(0) \rangle = \langle c_1(T\mathbb{P}^2) - c_1(\gamma \otimes \rightarrow \mathbb{P}^n), s^{-1}(0) \rangle = \langle (3a - da) \cdot da, \mathbb{P}^2 \rangle = 3d - d^2. \tag{A.3} \]

Lemma 2.1 follows immediately from (A.2) and (A.3).

**A.4 Plane Curves**

A (reduced, complex) curve \( C \) in \( \mathbb{P}^2 \) is a subset of \( \mathbb{P}^2 \) of the form

\[ C = C_a = \{ [X, Y, Z] \in \mathbb{P}^2 : \sum_{j+k+l=d} a_{jkl} X^j Y^k Z^l = 0 \}, \]
for some positive integer $d$ and some tuple $a = (a_{jkl})_{j+k+l=d}$ of complex numbers, not all zero. In
other words, a curve in $\mathbb{P}^2 \equiv (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$ is the quotient of the zero set of a nonzero homogeneous
polynomial on $\mathbb{C}^3 - \{0\}$ by the $\mathbb{C}^*$-action. The degree $d(C)$ of the curve $C$ in $\mathbb{P}^2$ is the minimal
degree of a homogeneous polynomial giving rise to $C$. Alternatively, $d(C)$ is the positive number
such that
\[ |C| = d(C) \cdot \ell \in H_2(\mathbb{P}^2; \mathbb{Z}), \]
where $\ell$ is the homology class of a line in $\mathbb{P}^2$.

If $C \subset \mathbb{P}^2$ is a curve, there exists a smooth Riemann surface $\Sigma$, possibly not connected, and a
holomorphic map $f: \Sigma \rightarrow \mathbb{P}^2$ such that $C = f(\Sigma)$. The degree of such a map $f$ is the number $d(f)$
such that
\[ f_*[\Sigma] = d(f) \cdot \ell \in H_2(\mathbb{P}^2; \mathbb{Z}). \]
If $C = f(\Sigma)$, $d(C) \leq d(f)$. If $d(C) = d(f)$, $f: \Sigma \rightarrow C$ is a normalization of $C$. If $f: \Sigma \rightarrow C$ is a
normalization of $C$, the (geometric) genus, $g(C)$, of the curve $C$ is the genus of Riemann surface $\Sigma$.

The following two lemmas can be proved using basic facts from complex analysis and algebraic
topology.

**Lemma A.4** Every complex curve $C \subset \mathbb{P}^2$ admits a normalization $f: \Sigma \rightarrow C$. If $f_1: \Sigma_1 \rightarrow C$
and $f_2: \Sigma_2 \rightarrow C$ are normalizations of $C$, there exists a biholomorphism $\tau: \Sigma_1 \rightarrow \Sigma_2$ such that
$f_1 = f_2 \circ \tau$.

**Lemma A.5** If $C_1$ and $C_2$ are complex plane curves that intersect at a finite number points, then
the number of intersection points counted with appropriate positive multiplicities is $d(C_1) \cdot d(C_2)$.

**References**

[A] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.

[GriH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Willey & Sons, 1994.

[Gro] M. Gromov, *Pseudoholomorphic Curves in Symplectic Manifolds*, Invent. Math. 82 (1985),
no. 2, 307–347.

[Ka] S. Katz, *Introduction to Enumerative Geometry and its Interaction with Theoretical Physics*,
PCMI Lecture Notes, 2001.

[KoMa] M. Kontsevich and Yu. Manin, *Gromov-Witten Classes, Quantum Cohomology, and Enum-
erative Geometry*, Comm. Math. Phys. 164 (1994), no. 3, 525–562.

[LT] J. Li and G. Tian, *Virtual Moduli Cycles and Gromov-Witten Invariants of General Symplec-
tic Manifolds*, Topics in Symplectic 4-Manifolds, 47-83, First Int. Press Lect. Ser., I, Intern-
at. Press, 1998.

[McSa] D. McDuff and D. Salamon, *Introduction to J-Holomorphic Curves*, American Mathematical
Society, 1994.

[MiSt] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.
[NeNi] A. Newlander and L. Nirenberg, Complex Analytic Coordinates in Almost-Complex Manifolds, Ann. of Math. 65 (1957), 391-404.

[Ra1] Z. Ran, Enumerative Geometry of Singular Plane Curves, Invent. Math. 97 (1989), no. 3, 447–465.

[Ra2] Z. Ran, On the Quantum Cohomology of the Plane, Old and New, and a $K3$ Analogue, Collect. Math. 49 (1998), no. 2-3, 519–526.

[RuT] Y. Ruan and G. Tian, A Mathematical Theory of Quantum Cohomology, J. Diff. Geom. 42 (1995), no. 2, 259-367.

[Ze] H. Zeuthen, Almindelige Egenskaber ved Systemer af Plane Kurver, Kongelige Danske Videnskabernes Selskabs Skrifter, 10 (1873), 285-393. Danish.

[Z1] A. Zinger, Enumeration of Genus-Two Curves with a Fixed Complex Structure in $\mathbb{P}^2$ and $\mathbb{P}^3$, J. Diff. Geom. 65 (2003), no. 3, 341-467.

[Z2] A. Zinger, Counting Rational Curves of Arbitrary Shape in Projective Spaces, Geom. Top. 9 (2005), 571-697.