LQGNET: HYBRID MODEL-BASED AND DATA-DRIVEN LINEAR QUADRATIC STOCHASTIC CONTROL

Solomon Goldgraber Casspi, Oliver Hüsser, Guy Revach, and Nir Shlezinger

ABSTRACT

Stochastic control deals with finding an optimal control signal for a dynamical system in a setting with uncertainty, playing a key role in numerous applications. The linear quadratic Gaussian (LQG) is a widely-used setting, where the system dynamics is represented as a linear Gaussian state-space (SS) model, and the objective function is quadratic. For this setting, the optimal controller is obtained in closed form by the separation principle. However, in practice, the underlying system dynamics often cannot be faithfully captured by a fully known linear Gaussian SS model, limiting its performance. Here, we present LQGNet, a stochastic controller that leverages data to operate under partially known dynamics. LQGNet augments the state tracking module of separation-based control with a dedicated trainable algorithm. The resulting system preserves the operation of classic LQG control while learning to cope with partially known SS models without having to fully identify the dynamics. We empirically show that LQGNet outperforms classic stochastic control by overcoming mismatched SS models.

Index Terms— Stochastic control, LQG, deep learning.

1. INTRODUCTION

Stochastic optimal control is a sub-field of mathematical optimization with applications spanning from operations research to physical sciences and engineering, including aerospace, vehicular systems, and robotics [1]. Stochastic control considers dynamical systems under the existence of uncertainty, either in its evolution or in its observations. The aim is to find an optimal control signal for a given objective function. In the fundamental linear quadratic Gaussian (LQG) setting [2], the system dynamics obey a linear Gaussian state-space (SS) model, and the controller should minimize a quadratic objective. The optimal LQG controller follows the separation principle [3], where state estimation is decoupled from control, and it comprises a Kalman filter (KF) followed by a conventional linear quadratic regulator (LQR) [4].

While LQG control is simple and tractable, it relies on the ability to faithfully describe the dynamics as a fully known linear Gaussian SS model. In practice, SS models are often approximations of the system’s true dynamics, while its stochasticity can be non-Gaussian. The presence of such mismatched domain knowledge notably affects the performance of classical model-based policies.

To overcome the drawbacks of oversimplified modeling, one can resort to learning. The main learning-based approach in sequential decision making and control is reinforcement learning (RL) [5, 6] where an agent is trained via experience-driven autonomous learning to maximize a reward [7]. The growing popularity of model-agnostic deep neural networks (DNNs) and their empirical success in various tasks involving complex data, such as visual and language data, has led to a growing interest in deep RL [8]. Deep RL systems based on black-box DNNs were proposed for implementing controllers for various tasks, including robotics and vehicular systems [9, 10]. Despite their success, these architectures cannot naturally incorporate the domain knowledge available in partially known SS models, are complex and difficult to train, and lack the interpretability of model-based methods [11]. An alternative approach uses DNNs to extract features processed with model-based methods [12–14]. This approach still requires one to impose a fully known SS model of the features, motivating the incorporation of deep learning into classic controllers to bypass the need to fully characterize the dynamics.

In this work, we propose LQGNet, a hybrid stochastic controller designed via model-based deep learning [11, 15]. LQGNet preserves the structure of the optimal LQG policy, while operating in partially known settings. We adopt the recent KalmanNet architecture [16], which implements a trainable KF, in a separation-based controller. The resulting LQGNet architecture utilizes the available domain knowledge by exploiting the system’s description as a SS model, thus preserving the simplicity and interpretability of the model-based policy while leveraging data to overcome partial information and model mismatches. By converting the optimal model-based LQG controller into a trainable discriminative algorithm [17] our LQGNet learns to control in an end-to-end manner. We empirically demonstrate that LQGNet approaches the performance of the optimal LQG policy with fully known SS models, and notably outperforms it in the presence of mismatches.

The rest of this paper is organized as follows: Section 2 reviews the SS model and details the LQG control task. Section 3 presents LQGNet, which is evaluated in Section 4.

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2. SYSTEM MODEL AND PRELIMINARIES

As a preliminary step to deriving LQGNet (LNet), we describe the system model in Subsection 2.1. Subsection 2.2 formulates the data-driven LQG control task in partially known SS models, and Subsection 2.3 reviews basics in optimal model-based LQG control.

2.1. Dynamical System Model

We consider dynamic systems characterized by a SS model in discrete-time $t \in \mathbb{Z}$. This SS model describes the relationship between a state vector $x_t \in \mathbb{R}^m$, an input control signal $u_t \in \mathbb{R}^q$, and the noisy observations $y_t \in \mathbb{R}^n$, at each time instance $t$. We focus on linear Gaussian models, given by

$$
\begin{align*}
\dot{x}_t &= F \cdot x_{t-1} + G \cdot u_{t-1} + w_t, \quad w_t \sim \mathcal{N}(0, W), \quad (1a) \\
\dot{y}_t &= H \cdot x_t + v_t, \quad v_t \sim \mathcal{N}(0, V). \quad (1b)
\end{align*}
$$

Here, $F \in \mathbb{R}^{m \times m}$, $G \in \mathbb{R}^{m \times q}$, and $H \in \mathbb{R}^{n \times m}$ are the evolution, control, and observation (emission) matrices, respectively, and $w_t$, $v_t$ are additive white Gaussian noise signals with covariances $W$, $V$, respectively.

2.2. Data Driven Stochastic Control Task

The stochastic control task considers the optimization of a policy $\psi: \mathbb{R}^n \mapsto \mathbb{R}^q$, which maps a noisy observation signal $y_t$ into a control signal $u_t$, under quadratic control loss. Over a finite-horizon of $T$ time steps, the loss is given by

$$
J(\psi) = \bar{x}_T^T Q_T \bar{x}_T + \sum_{t=0}^{T-1} (\bar{x}_t^T Q_x \bar{x}_t + \bar{u}_t^T R_u \bar{u}_t). \quad (2)
$$

The loss (2) balances the stability of the closed-loop system and the aggressiveness of control. Here, $\bar{u}_t$ and $\bar{x}_t$ are the deviations of $u_t$ and $x_t$, respectively, from predefined target values. The former is typically set to zero, while the latter is a desired state that depends on the regulation problem. The matrices $Q_x$, $Q_T \succeq 0$, and $R_u \succ 0$ are predefined weighting costs for state, final state, and input control, respectively. While we assume to have access to a (possibly mismatched) estimate of the SS design matrices $F$, $G$, $H$, we do not assume prior knowledge of the distribution of the noises.

To overcome this missing domain knowledge, we are given access to a simulator $\Gamma: \mathbb{R}^q \mapsto \mathbb{R}^m \times \mathbb{R}^n$ that emulates the underlying dynamics. The simulator allows generating random trajectories of inputs while measuring their observations and state values, i.e., $(y_t, x_t) = \Gamma(u_{t-1})$.

2.3. Optimal LQG Control

LQG is one of the most fundamental optimal control frameworks [2]. It considers dynamics with a linear Gaussian SS model as in (1) with a quadratic objective as in (2). For such setting, (1), the separation principle [3, 4] applies, and the optimal policy $\psi^*$ which minimizes (2) can be decomposed into two separate modules: a mean-squared error (MSE) optimal state estimator, namely the KF, followed by full-state optimal LQR. Both modules, detailed next, require full knowledge of the SS model parameters.

2.3.1. Kalman filter - MSE optimal state estimator

The KF is an efficient linear recursive state estimator, which for every time step $t$ produces an estimate $\hat{x}_t$ for $x_t$, based on all previous and current observations $\{y_{\tau}\}_{\tau \leq t}$. It can be described as a two-step procedure; prediction and update, that uses only the new observation and the previous estimate as sufficient statistics to compute first and second order moments. The prediction is given by

$$
\begin{align*}
\hat{x}_{t|t-1} &= F \hat{x}_{t-1} + G \hat{u}_{t-1}, \\
\Sigma_{t|t-1} &= F \Sigma_{t-1} F^\top + W, \quad (3a) \\
\hat{y}_{t|t-1} &= H \hat{x}_{t|t-1}, \\
S_t &= H \Sigma_{t|t-1} H^\top + V, \quad (3b)
\end{align*}
$$

and the update is given by

$$
\begin{align*}
\hat{x}_t &= \hat{x}_{t|t-1} + K_t \cdot \Delta y_t, \\
\Delta y_t &= y_t - \hat{y}_{t|t-1}, \\
K_t &= \Sigma_{t|t-1} H^\top S_t^{-1}, \\
\Sigma_t &= \Sigma_{t|t-1} - K_t S_t K_t^\top. \quad (4b)
\end{align*}
$$

Here, $K_t$ is the Kalman gain (KG), computed recursively based on tracking the second-order moments of the signals.

2.3.2. Optimal LQR policy

For the system in (1) with known $(F, G)$, the optimal control input is given by

$$
u_t = -L_t \hat{x}_t, \quad L_t = [R_u + G^\top P_{t+1} G]^{-1} G^\top P_{t+1} F. \quad (5)
$$

$L_t$ is control gain and $P_t$ is the solution to the discrete-time algebraic Riccati equation [18], computed as

$$
P_{t+1} = Q_x + F^\top P_t F - F^\top P_t G \cdot L_t. \quad (6)
$$

The above controller coincides with the LQR policy, which minimizes the quadratic objective for linear Gaussian SS models where one has a noise-free observation of the state.

3. LQGNET

Next, we present LQGNet, which learns to implement LQG control under the considered partially known SS model. We begin by detailing the architecture of LQGNet in Subsection 3.1, after which we describe the training procedure and provide a discussion in Subsections 3.2-3.3, respectively.
3.1. LQGNet Architecture

The optimal LQG controller detailed in Subsection 2.3 requires full access to the SS model (1). While the design matrices $F, G, H$ are assumed to be partially known, the distributions of the noise signals are not known at all. Consequently, to cope with this missing knowledge without imposing a model and estimating the statistics of the noise signals, LQGNet augments the state estimator with deep learning.

Since the LQG optimal controller employs the KF for state estimation, we replace it with our recently proposed KalmanNet architecture [16]. KalmanNet is particularly suitable here, as it preserves the flow and interpretable nature of the KF [19]. While the model-based KF requires the noise statistics to formulate the KG $K_t$ (4b), KalmanNet uses a trainable recurrent neural network (RNN) to compute it, bypassing the need to impose a model on the noises. More specifically, KalmanNet predicts the first order moments using the design matrices, as in (3), and then updates using its learned (surrogate) KG, as in (4a). The state estimate produced by KalmanNet is then processed by the model-based LQR (5) and the gain $L_t$ using the available (though possibly approximated) design matrices, where the noise covariance matrices are not required. The resulting architecture is illustrated in Fig. 1.

3.2. Training Algorithm

The trainable parameters of LQGNet are the weights of the internal RNN used to compute the KG, denoted $\Theta$. In principle, one can use the data simulator $\Gamma$ to train the state estimator separately from the control task, i.e., to minimize the state estimate MSE in a supervised manner [16], or to optimize its internal predictions in an unsupervised manner [20]. However, inspired by RL [7], since the LQR policy (5) uses the possibly mismatched design matrices, we aim to train the overall system in an end-to-end manner, based on the quadratic control objective (2). By doing so, we leverage data to jointly cope with mismatches in both the state estimate and the regulator.

To formulate the training loss, we use $J_{\Gamma}(\Theta)$ to denote the objective (2) evaluated when applying LQGNet with parameters $\Theta$ to control the dynamics of the simulator $\Gamma$. The random nature of the simulator indicates that restarting it multiple times yields different trajectories. We can thus formulate the regularized LQG loss as

$$L_{\Gamma}(\Theta) = J_{\Gamma}(\Theta) + \gamma \cdot \|\Theta\|_2^2,$$

where $\gamma > 0$ is a regularization coefficient. By (5), the input $u_t$ is differentiable with respect to the state estimate $\hat{x}_t$, which is in turn differentiable with respect to $\Theta$ [16]. Consequently, the loss in (7) is differentiable, allowing to train LQGNet end-to-end as a discriminative model [17] via gradient-based learning. The resulting training procedure is summarized in Algorithm 1.

3.3. Discussion

LQGNet is a hybrid model-based/data-driven implementation of the optimal LQG controller. Comprising a trainable KF and the LQR policy, it preserves the interpretable and low-complexity operation of the model-based controller. By augmenting the KG computation, which encapsulates the noise statistics, with an RNN, LQGNet operates without imposing a model on the noise signals. The training procedure adapts LQGNet based on the overall control objective, facilitating coping with mismatches in the SS model parameters. In particular, LQGNet can overcome uncertainty by learning an alternative state estimate that yields high performance control, as demonstrated in Section 4.

LQGNet gives rise to multiple possible extensions. For instance, while we focus on linear SS models, the proposed design can potentially be enhanced to account for non-linear dynamics [21]. Furthermore, it can be enhanced to account for alternative control objectives such as model predictive control, and utilize corresponding regulators such as emerging convex optimization control policies [22]. We leave these extensions for future investigation.
Mismatched observations

Trajectory along a single state entry

4. EMPIRICAL EVALUATIONS

Next, we empirically study the performance of LQGNet\(^1\). We evaluate both the control objective (LQG loss (2)) and the state estimation MSE for different levels of observation noise. Since it was derived from the optimal LQG controller, we compare it with the model-based controller, i.e., KF-LQR, when operating with and without mismatches. Here, the noise signals obey the SS model in (1), with diagonal covariance matrices with an identical variance, i.e., \( \mathbf{V} = \sigma_w^2 \mathbf{I}, \mathbf{W} = \sigma_w^2 \mathbf{I}, \mathbf{R} = \sigma_v^2 \mathbf{I} \), and \( \sigma_w^2 = \sigma_v^2 \). In the non-mismatched case, the dynamics also obey the SS model, with the design matrices

\[
\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}
\]

In the mismatched case the design matrices stay the same, but in the ground truth dynamics, from which the data was generated, either \( \mathbf{F} \) or \( \mathbf{H} \) are replaced by

\[
\mathbf{R}_\alpha \mathbf{F}, \quad \mathbf{R}_\alpha \mathbf{H}, \quad \mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, \tag{9}
\]

namely, rotated with \( \alpha = 20^\circ \).

The loss measures achieved by LQGNet compared with model-based KF-LQR are depicted in Fig. 2 for both full and partial model information. The mismatch in Fig. 2a is in the state-evolution \( \mathbf{F} \), and the mismatch in Fig. 2b corresponds to the observation mapping \( \mathbf{H} \). As expected, in the absence of mismatches, the optimal LQG controller achieves the lowest MSE and LQG values, as it follows the separation principle for optimality. For the same setting, LQGNet approaches the optimal MSE and LQG performance. Even though LQGNet is trained to minimize the LQG loss, it is not surprising that it achieves optimal performance in both LQG and MSE metrics. Optimal control is approached because LQR and KF (i.e., linear quadratic estimation) are dual problems \([23]\), which also indicates that LQGNet learns to obey the separation principle when it is optimal.

Under the setting with partial (mismatched) model information, the LQG loss associated with the LQGNet is notably better than its model-based counterpart. However, LQGNet does not accurately estimate the state here, achieving MSE that is notably higher than the lower bound. The learned KG of LQGNet induced by training on the LQG loss produces different state estimates than the model-based controllers. However, these estimates, despite being inaccurate, enable the mismatched LQR controller to produce reliable inputs.

The result for the partial knowledge in the state-evolution in Fig. 2a is also reflected in Fig. 2c, where a single trajectory of the first entry of \( x_1 \) is presented. Here, the initial state is set to \( x_0 = [10, 0]^\top \) and \( \sigma_{w}^2 = \sigma_v^2 = 0 [\text{dB}] \). We observe that despite the fact that LQGNet operates with a mismatched state-evolution matrix \( \mathbf{F} \), it controls the state to yield a trajectory that is closely similar to that achieved for the same setting using the optimal LQG controller. These results demonstrate the ability of LQGNet to reliably learn from data to control partially known SS models.

5. CONCLUSIONS

In this work we presented LQGNet, a hybrid model-based and data-driven stochastic controller. Our design augments the optimal LQG controller with a deep learning component, using the recently proposed KalmanNet to overcome unknown noise distributions. We train LQGNet end-to-end, such that it learns from data to overcome model mismatches. Our numerical study shows that LQGNet approaches optimal control with both a full and partial SS model, and that it retains the simplicity and interpretability of its classic model-based counterpart while implicitly learning to cope with mismatched dynamics.

\(^1\)The source code and hyperparameters used are at [https://github.com/KalmanNet/LQGNet_ICASSP23](https://github.com/KalmanNet/LQGNet_ICASSP23).
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