Statistical properties of the energy in time-dependent homogeneous power law potentials

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Abstract
We study classical 1D Hamilton systems with a homogeneous power law potential and their statistical behavior, assuming a microcanonical distribution of the initial conditions and describing its change under a monotonically increasing time-dependent function \( a(t) \) (the prefactor of the potential). Using the nonlinear Wentzel–Kramers–Brillouin-like method of Papamikos and Robnik 2012 J. Phys. A: Math. Theor. 44 315102 and following a previous work by Papamikos and Robnik 2011 J. Phys. A: Math. Theor. 45 015206, we specifically analyze the mean energy, the variance and the adiabatic invariant (action) of the systems for large time \( t \to \infty \) and we show that the mean energy and variance increase as powers of \( a(t) \), while the action oscillates and finally remains constant. By means of a number of detailed case studies, we show that the theoretical prediction is excellent, which demonstrates the usefulness of the method in such applications.

Keywords: time-dependent hamilton systems, homogeneous power law potentials, nonlinear WKB method
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(Some figures may appear in colour only in the online journal)

1. Introduction

Time-dependent Hamiltonian systems [1–4] are very interesting and important dynamical models, and many important questions about their dynamical behavior can be studied. While the energy of the system is not conserved, the Liouville theorem of course still applies and thus the phase space volume is preserved by the flow. One of the central questions is that of
the evolution of the energy of a certain ensemble of initial conditions for such systems. In particular, we can find a very rich behavior in time-periodic systems, from integrability to full chaoticity (ergodicity) and including also the scenario in between, namely the case of a mixed phase space, even in one-dimensional systems [3, 5–7]. We will consider the general family of classical 1D Hamiltonian systems with a homogeneous power law potential and quadratic kinetic energy, as follows:

\[ H(q, p, t) = \frac{p^2}{2} + a(t)V(q), \]  

where \( p \) is the momentum, \( a(t)V(q) = a(t)\frac{q^{2m}}{2m} \) is the potential as a function of the coordinate \( q \) and time \( t \), \( a(t) \) is a time-dependent function and \( m \) is an integer \( m = 1, 2, \ldots \). In a recent work, Papamikos and Robnik [8] have developed the first-order nonlinear WKB-like method for such homogeneous power law potentials as an approximation of the general solution, which can be used successfully to generalize a series of studies on the time-dependent linear oscillator by Robnik and Romanovski [9–13], where the rigorous linear WKB method (to all orders) has been employed [14]. Using these tools we shall analyze the statistical properties of the energies of systems (1).

We are particularly interested in the time evolution of a microcanonical ensemble of initial conditions. The most natural ensemble, and the most important one, is the microcanonical ensemble of the initial conditions, because if we have a large ensemble of identical systems with the same (‘prepared’) energy, and we do not have any further information about them, the uniform distribution with respect to the canonical angle (‘the phases’) is the most appropriate one. Moreover, in the context of the statistical mechanics, also for such low dimensional systems, the energy has a special status as the state variable (see the introductory discussion in reference [15]).

If the evolution is ideally adiabatic (i.e. infinitely slow), then the adiabatic invariant, which is also the action of the system, or the area inside the contour of constant energy in the phase space (divided by \( \pi \)), is conserved, and this is precisely the adiabatic theorem on one-dimensional Hamilton systems [1], provided that we do not cross a separatrix during the adiabatic process. The energy is sharply distributed (with a Dirac delta function) and its value is fully determined by the value of the adiabatic invariant. For faster changes of the system parameter the adiabatic invariant is no longer conserved and the energy becomes distributed over a certain interval. For the linear oscillator this turns out to be precisely the arcsine distribution for any frequency \( \omega(t) \) as a function of time \( t \), where in this case we have \( m = 1 \) and \( a(t) = \omega(t)^2 \) in (1). It has also been proven in [9] and further discussed in [10–13] that the value of the adiabatic invariant (action) always increases, except in an ideal adiabatic (= infinitely slow) process, where it is constant.

What happens for higher power law potentials when the potential, namely \( a(t) \), changes in time? For the single-parametric kick (an instantaneous discontinuous jump of the parameter \( a(t) \) it has been shown in [6], and further developed (even for a large family of potentials) in [15], that the action at the average final energy increases under very general conditions, thus exhibiting the so-called PR property. If the variation of \( a(t) \) is slow, but not ideally adiabatic, the action can decrease, due to the nonlinearity. The evolution of the energy and its distribution has been studied there only for large times up to \( T = 1/\epsilon \), where \( \epsilon \) is the adiabatic parameter, and \( T \) is the time interval of the variation of \( a(t) \).

In this paper we study the case where the time-dependent function \( a(t) \) increases monotonically and in an unbounded fashion, very much beyond \( 1/\epsilon \), for the systems (1). We will use higher order symplectic numerical techniques [16–23], namely the eighth-order
symplectic integrator, and on the theoretical side the WKB-like nonlinear method for homogeneous power law potentials [8] in order to describe the evolution of a microcanonical ensemble of initial conditions for large times. We calculate, along with the numerical energy distribution as a function of time, in particular the corresponding mean energy and the variance, and we show that these results agree with the general theory, which we will present in the next section.

In the special case of time evolution of a microcanonical ensemble of such systems with \( a(t) = 1 + \varepsilon t \), the dynamics exhibits a structure consisting of an energy layer (see figure 1). We shall calculate the width of this layer as a function of time using the nonlinear WKB method adapted for such systems and we will explore the behavior of the average energy and of the variance of the energy for an initial microcanonical ensemble for large times \( t \to \infty \). If \( a(t) \) is a monotonic and unbounded increasing function of time, then both quantities are powers of \( a(t) \). In this limit the action first oscillates and finally remains asymptotically constant, but assumes a value slightly larger than its initial value. This is an interesting result for time-dependent Hamiltonian systems in a more general context [15, 24]. The distribution function of the final energy will be calculated numerically, but cannot be derived theoretically, unlike for the case of the linear oscillator \( m = 1 \), for which Robnik, Romanovski and Stöckmann have proven the explicit exact formula for the distribution, which is the arcsine distribution [11, 13].

2. General theory of a time-dependent homogeneous power law potential

We consider the Hamilton systems with one degree of freedom in the form of a quadratic kinetic energy plus a homogeneous power law potential, as defined in equation (1). Using the nonlinear WKB-like method [8] to the first-order approximation for such nonlinear Hamiltonian systems, we have the following general solution:

\[
q(t) = a(t)^{-\frac{1}{2m}} g \left( \int_{b_0}^{t} a(x)^{\frac{1}{2m}} \, dx \right),
\]

\[
p(t) = -a'(t) \left( a(t)^{-\frac{1}{2m}} \right)^{-1} \left( \int_{b_0}^{t} a(x)^{\frac{1}{2m}} \, dx \right) + \left( a(t) \right)^{\frac{1}{2m}} g \left( \int_{b_0}^{t} a(x)^{\frac{1}{2m}} \, dx \right),
\]

where \( g \) is the solution of the corresponding time-independent Hamiltonian system with \( a(t) = 1 \) and \( g' \) is its first derivative with respect to time \( t \). The function \( g \) satisfies the following differential equation:

\[
g'' + f(g) = 0,
\]

where \( f(g) = dV(g)/dg \). The differential equation (3) admits the first integral, namely the total energy of the oscillator, which is a function of the initial energy \( E_0 \) and the initial phase (canonical angle) \( \phi_0 \) of the oscillator (1):

\[
\frac{(g'(t))^2}{2} + \frac{(g(t))^2}{2m} = c(\phi_0, E_0).
\]

The dependence on \( E_0 \) and \( \phi_0 \) enters through the lower integration limit in equation (2). The expression for the energy of (1) is
The function $g$ and its derivative $g' = \frac{dg}{dt}$ are periodic and bounded functions, being the solutions of the time-independent Hamiltonian system (3). Assuming that $a(t)$ is an unbounded monotonically increasing function of time, and also that $\frac{d\log(a(t))}{dt}$ is a decreasing function of time, we see that if $t \to \infty$, then $q(t) \to (0, \infty)$. The energy depends on the initial conditions $\phi_0$ and $E_0$, through the expression in the square brackets, and this is the term which remains non-vanishing for large time $t \to \infty$:

$$E(t) \simeq c(\phi_0, E_0) a(t)^{\frac{1}{m}}.$$  \hspace{1cm} (6)

After calculating the mean energy and the mean square of the energy, and dropping the decaying terms, we arrive at the following expressions for the leading terms for the average energy and the variance, $\text{Var}(E(t)) = \langle (E(t))^2 \rangle - \langle E(t) \rangle^2$, in the limit $t \to \infty$:

$$\langle E(t) \rangle \simeq \frac{a(t)^{\frac{1}{m}}}{2\pi} \int_0^{2\pi} c(\phi_0, E_0) \, d\phi_0,$$  \hspace{1cm} (7)

$$\text{Var}(E(t)) \simeq \frac{a(t)^{\frac{1}{m}}}{2\pi} \left[ \int_0^{2\pi} \left( c(\phi_0, E_0) \right)^2 \, d\phi_0 - \frac{1}{2\pi} \left( \int_0^{2\pi} c(\phi_0, E_0) \, d\phi_0 \right)^2 \right].$$  \hspace{1cm} (8)

Using the previous expression for the average energy, we calculate also the action ratio [15] of the final and initial action. The general formula for the action at the mean energy is
I(⟨E(t)⟩) = \frac{1}{2\pi} \int\int H(q,p) = \langle E(t) \rangle \ p \ dq = \frac{2^{m+1}m^{m}n\Gamma\left(\frac{1}{2m}\right)}{2m\sqrt{\pi} \ (a(t))^{\frac{1}{2m}}} \ ⟨E(t)⟩^{\frac{m+1}{2m}}. \quad (9)

Hence the action ratio for large times \( t \to \infty \) is

\[
\frac{I(⟨E_1⟩)}{I(⟨E_0⟩)} \approx \left( a(t_0) \right)^{\frac{1}{2m}} \ \left[ \int_0^{2\alpha c} c\left(\phi_0, E_0\right) \ d\phi_0 \right]^{\frac{m+1}{2m}} = \text{const.} \quad (10)
\]

This is an important result, showing that the action ratio remains constant for large times. The numerical empirics shows that it is always slightly larger than its initial value.

3. Examples

In this section we show two examples: the linear oscillator and the quartic oscillator. For the linear oscillator we have exact analytical results for the energy distribution function, as well as for the average energy, the variance and the action ratio [9–13], but we reproduce them here using the more general, although only leading order, nonlinear WKB-like method [8], now for arbitrarily large times. For the quartic oscillator it is not easy to have explicit theoretical results due the complexity of the Jacobi elliptic functions, but using the numerical technique, for the case with \( \epsilon = a \), we calculate and analyze the solutions and show that the general theory of section 2 is correct. For the higher order potentials, \( m = 3, 4, \ldots \), we calculate numerically the energy distribution as a function of time (originating from a microcanonical ensemble of initial conditions) and its mean value and the variance, and establish whether they obey the theoretical power laws from section 2.

3.1. The linear oscillator

We consider the Hamiltonian of the linear oscillator in the following form:

\[
H(q, p, t) = \frac{p^2}{2} + \omega^2(t) \frac{q^2}{2}, \quad (11)
\]

where \( \omega(t) \) is an unbounded monotonically increasing function of time. The general solution obtained using the general nonlinear WKB-like method (which of course is identical to the linear WKB method in the leading order), to the first-order approximation, is

\[
q(t) = \frac{c_1}{\sqrt{\omega(t)}} \sin \left( c_2 + \int_{t_0}^{t} \omega(x) \ dx \right),
\]

\[
p(t) = c_1 \sqrt{\omega(t)} \cos \left( c_2 + \int_{t_0}^{t} \omega(x) \ dx \right) - \frac{c_1 \omega'(t)}{2(\omega(t))^3} \ \sin \left( c_2 + \int_{t_0}^{t} \omega(x) \ dx \right), \quad (12)
\]

where \( c_1 \) and \( c_2 \) are integration constants. The action–angle variables, for \( t = t_0 \), are

\[
q_0 = \frac{\sqrt{2E_0}}{\omega_0} \sin \phi_0, \quad p_0 = \sqrt{2E_0} \cos \phi_0, \quad (13)
\]
where $\omega_0 = \omega(t_0)$, $E_0$ is the initial energy and $\phi_0$ is the initial angle. Using this we get

$$q(t) = \frac{b_1}{\sqrt{\omega(t)}} \left[ (b_2 + \cot \phi_0) \sin \phi_0 \sin \left( \int_{t_0}^{t} \omega(x) \, dx \right) + \sin \phi_0 \cos \left( \int_{t_0}^{t} \omega(x) \, dx \right) \right],$$

$$p(t) = b_1 \sqrt{\omega(t)} \left[ (b_2 + \cot \phi_0) \sin \phi_0 \cos \left( \int_{t_0}^{t} \omega(x) \, dx \right) - \sin \phi_0 \sin \left( \int_{t_0}^{t} \omega(x) \, dx \right) \right]$$

$$- \frac{b_1 \omega'(t)}{2(\omega(t))^2} \left[ (b_2 + \cot \phi_0) \sin \phi_0 \sin \left( \int_{t_0}^{t} \omega(x) \, dx \right) \right]$$

$$+ \sin \phi_0 \cos \left( \int_{t_0}^{t} \omega(x) \, dx \right). \quad (14)$$

where $b_1 = \frac{2E_0}{\omega_0}$ and $b_2 = \frac{\omega_0^2}{2m_0}$. The average energy after the integration over $\phi_0$ in the interval $[0, 2\pi]$ is

$$\langle E(t) \rangle = \left( \frac{b_1^2}{2} + \frac{1}{4} b_1^2 b_2^2 \right) \omega(t) - \frac{b_1^2 b_2^2 (\omega'(t))^2}{32 \omega^3(t)} \cos \left( 2 \int_{t_0}^{t} \omega(x) \, dx \right) + \frac{b_1^2 b_2^2 (\omega'(t))^2}{32 \omega^3(t)}$$

$$- \frac{b_1^2 b_2^2 \omega'(t) \sin \left( 2 \int_{t_0}^{t} \omega(x) \, dx \right)}{8 \omega(t)} + \frac{b_1^2 b_2 (\omega'(t))^2}{16 \omega^3(t)} \sin \left( 2 \int_{t_0}^{t} \omega(x) \, dx \right)$$

$$- \frac{b_1^2 b_2 \omega'(t) \cos \left( 2 \int_{t_0}^{t} \omega(x) \, dx \right)}{4 \omega(t)} + \frac{b_1^2 (\omega'(t))^2}{16 \omega^3(t)}. \quad (15)$$

For large time and after a long calculation of the mean squared energy, and dropping the constant and decaying terms, we arrive at the following expressions:

$$\langle E(t) \rangle = \left( \frac{b_1^2}{2} + \frac{1}{4} b_1^2 b_2^2 \right) \omega(t),$$

$$\text{Var}(E(t)) = \frac{1}{32} b_1^4 b_2^2 \left( 4 + b_2^2 \right) \omega^2(t). \quad (16)$$

While the higher order terms of $\langle E(t) \rangle$ are clearly visible in equation (15), for the variance they are too complex to be presented here, but we only mention that they, as a function of time through $\omega(t)$, are constant in the next order, and decay as $1/\omega^2(t)$ in the next to next order. In manipulations it is necessary to use computer symbolic calculations.

### 3.2. The quartic oscillator

We consider the Hamiltonian of the quartic oscillator in the following form:

$$H(q, p, t) = \frac{p^2}{2} + a(t) \frac{q^4}{4}, \quad (17)$$

where $a(t)$ is an unbounded monotonically increasing function of time. The general solution obtained using the nonlinear WKB-like method [8], to the first-order approximation, is
where \( c_1 \) and \( c_2 \) are integration constants, and sn is the snoidal Jacobi elliptic function [25].

The initial conditions in terms of the action–angle variables, for \( t = t_0 \), are

\[
q_0 = \sqrt{4E_0 \over a_0} \left( \sqrt{R^2 \over 2} \phi_0, i \right), \quad p_0 = \sqrt{2E_0 \over 1 - \sn^2 \left( \sqrt{R^2 \over 2} \phi_0, i \right), (19)}
\]

where \( a_0 = a(t_0) \), \( E_0 \) is the initial energy and \( \phi_0 \) is the initial angle. We calculate the constants \( c_1, c_2 \) as functions of the initial conditions as follows:

\[
c_2 = \sn^{-1} \left( \frac{q_0 a_0^2}{\sqrt{2} c_1}, i \right),
\]

\[
c_1 = \sqrt{E_0 \over a_0^2} + \frac{\sqrt{2} q_0 a_0^2}{6 a_0^2} + \frac{q_0^2 (a_0^2)^2}{72 a_0^2}, (20)
\]

and obtain the following expression for the energy:

\[
E(t) = c_1^2 (a(t))^2 + c_1^2 (a'(t))^2 A^2(t) \over 36 (a(t))^2 = c_1 A(t) A(t) \sqrt{1 - A^2(t) \over 3 a(t)}, (21)
\]

where the function \( A(t) \) is the following periodic function:

\[
A(t) = \sn \left( c_2 + c_1 \int_{t_0}^t (a(x))^2 \, dx, i \right). (22)
\]

We cannot analytically calculate the average energy from this expression, unlike in the linear oscillator case. But we see that the leading order term, for \( t \gg 1 \), is proportional to the power \((a(t))^2\). The term \( E^2(t) \) has leading order, for \( t \gg 1 \), proportional to \((a(t))^2\). For the average energy and the variance, dropping the decaying terms in the limit \( t \to \infty \), we find

\[
\langle E(t) \rangle = \frac{a(t)^2}{2\pi} \int_0^{2\pi} c_1^4 \, d\phi_0,
\]

\[
\text{Var}(E(t)) = \frac{a(t)^2}{2\pi} \left[ \int_0^{2\pi} c_1^4 \, d\phi_0 - {1 \over 2\pi} \left( \int_0^{2\pi} c_1^4 \, d\phi_0 \right)^2 \right]. (23)
\]

The variance is nonzero due to the fact that \( \left( \int_0^{2\pi} c_1^4 \, d\phi_0 \right)^2 < 2\pi \int_0^{2\pi} c_1^8 \, d\phi_0 \). The higher order constant and decaying terms in equations (23) can be easily calculated, but are too complex to be displayed here. In manipulations it is necessary to use computer symbolic calculations.
The energy distribution for the general family of systems (arbitrary $m$) is difficult to find analytically in a closed formula, except for the linear oscillator with $m = 1$ [11]. For the model $a(t) = 1 + \epsilon t$, for $\epsilon = 1$, $t = 2000$ and initial energy $E_0 = 0.5$.

For the model $\epsilon = +$, we have numerically calculated the energy distribution functions and found that for all of them (any $m$) it reminds us of the arcsine distribution, as it has two integrable singularity spikes at the minimal and maximal energy, but is distorted in an asymmetric manner. For the quartic oscillator we can find an empirical approximation. Namely, we propose the following rough approximation for the probability density, which is a linearly distorted arcsine distribution:

$$h(x) = \left(1 - \frac{\alpha}{2} + \alpha x\right)x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}, \quad x \in [0, 1].$$ (24)

In the specific numerical example shown in figure 2, we empirically (by best fitting) find $\alpha = 0.40175$. We remark that $\alpha = 0$ yields the arcsine distribution, which is exact for the linear oscillator for any time dependence $\omega(t)$.

3.3. Higher power law homogeneous potentials

In general it is not easy to calculate the exact expressions for the mean energy and the variance for higher order homogeneous power law potentials, because of the complicated dependence on the initial conditions, due to the complexity of the fundamental solution $g(t)$, entering into the WKB formulas, as described in section 2. Nevertheless, the power laws for the mean energy (7) and the variance (8) are predicted explicitly by the theory, and the
proportionality constants can be found empirically (numerically), if needed. We again use the specific model \( a(t) = 1 + et \), with \( e = 1 \), for \( m = 1, \ldots, 6 \), and perform the numerical integrations for very large time \( t = 2000 \), using the eighth-order symplectic integrator. In the table 1 we present the comparison of the theoretical exponents \( 1/(m + 1) \) for the mean energy, and in the table 2 the exponents \( 2/(m + 1) \) for the variance, with their corresponding numerical values. In addition, in table 3 we show the calculated action ratio, obtained using equations (7) and (10).

| \( m \) | Numerical value | Theoretical value | Error   |
|-------|----------------|------------------|---------|
| 1     | 0.9992962      | 1                | 7 \times 10^{-4} |
| 2     | 0.6661974      | 0.6666666       | 4 \times 10^{-4} |
| 3     | 0.499648       | 0.5              | 3 \times 10^{-4} |
| 4     | 0.3997184      | 0.4              | 2 \times 10^{-4} |
| 5     | 0.3330986      | 0.3333332       | 2 \times 10^{-4} |
| 6     | 0.2855071      | 0.2857142       | 2 \times 10^{-4} |

| \( m \) | Numerical value | Theoretical value | Error   |
|-------|----------------|------------------|---------|
| 1     | 1.0218739      | 1.0190229       | 2 \times 10^{-3} |
| 2     | 1.0110728      | 1.009601        | 10^{-3} |
| 3     | 1.0061271      | 1.0051507       | 9 \times 10^{-4} |
| 4     | 1.003837       | 1.0031063       | 7 \times 10^{-4} |
| 5     | 1.0026292      | 1.002045        | 6 \times 10^{-4} |
| 6     | 1.0019214      | 1.0014258       | 5 \times 10^{-4} |

Table 2. We show the power law exponents for \( \text{Var}(E(t)) \), whose theoretical value is \( 2/(m + 1) \), as derived in (8), compared to the numerical results; initial energy \( E_0 = 0.5 \). The number of points in the initial ensemble is 5000. Time is \( t = 2000 \).

Table 3. We show the exact values of the action ratio compared to the theoretical values, as derived in (10) for the initial energy \( E_0 = 0.5 \). The number of points in the initial ensemble is 5000. Time is \( t = 2000 \).

| \( m \) | Numerical value | Theoretical value | Error   |
|-------|----------------|------------------|---------|
| 1     | 10.218739      | 10.190229       | 2 \times 10^{-3} |
| 2     | 10.110728      | 10.09601        | 10^{-3} |
| 3     | 10.061271      | 10.051507       | 9 \times 10^{-4} |
| 4     | 10.03837       | 10.031063       | 7 \times 10^{-4} |
| 5     | 10.026292      | 10.02045        | 6 \times 10^{-4} |
| 6     | 10.019214      | 10.014258       | 5 \times 10^{-4} |

4. Discussion and conclusion

In this work we have analyzed the statistical properties of the one-degree-of-freedom time-dependent Hamilton systems with a homogeneous power law potential. We have calculated numerically the energy distribution as a function of time, for large times. Our main interest is in the value of the final average energy, the variance and the action at the final average energy. These are the main parameters describing the energy layer which evolves from the initial microcanonical ensemble. For the case where the dependence on time is monotonic and unbounded, we have calculated the average energy and the variance of the energy using the nonlinear WKB method [8], and derived the power laws observed in the asymptotic limit of large times. In particular, for the adiabatic invariant (action) we have proven that the value becomes constant for very large time but is (empirically) larger than the initial value. The agreement between the theory and the numerics is excellent. Unlike this limit, for small or
intermediate times, the analytic expressions become too complicated to be expressed in a closed formula.

To the best of our knowledge this is the first explicit application of the nonlinear WKB-like theory developed in [8]. There are of course many important open questions, including that of how to describe other 1D nonlinear Hamilton oscillators, like those studied e.g. in [15], possibly for arbitrary drivings \( a(t) \), for which the nonlinear WKB-like method should be generalized, and possibly improved beyond the leading order.

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