Volatility of Linear and Nonlinear Time Series

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Previous studies indicate that nonlinear properties of Gaussian time series with long-range correlations, \( u_i \), can be detected and quantified by studying the correlations in the magnitude series \( |u_i| \), i.e., the “volatility”. However, the origin for this empirical observation still remains unclear, and the exact relation between the correlations in \( u_i \) and the correlations in \( |u_i| \) is still unknown. Here we find analytical relations between the scaling exponent of linear series \( u_i \) and its magnitude series \( |u_i| \). Moreover, we find that nonlinear time series exhibit stronger (or the same) correlations in the magnitude time series compared to linear time series with the same two-point correlations. Based on these results we propose a simple model that generates multifractal time series by explicitly inserting long range correlations in the magnitude series; the nonlinear multifractal time series is generated by multiplying a long-range correlated time series (that represents the magnitude series) with uncorrelated time series (that represents the sign series \( \text{sgn}(u_i) \)). Our results of magnitude series correlations may help to identify linear and nonlinear processes in experimental records.

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I. INTRODUCTION

Natural systems often exhibit irregular and complex behavior that at first look erratic but in fact posses scale invariant structure [e.g., 1, 2]. In many cases this nontrivial structure points to long-range temporal correlations, which means that very far events are actually (statistically) correlated with each other. Long-range correlations are usually characterized by scaling laws where the scaling exponents quantify the strength of these correlations. However, it is clear that the two-point long-range correlations reveal just one aspect of the complexity of the system under consideration and that higher order statistics is needed to fully characterized the statistical properties of the system.

The two-point correlation function is usually used to quantify the scale invariant structure of time series (long-range correlations), while the \( q \)-point correlation function quantifies the higher order correlations. In some cases the \( q \)-point correlation function is trivially related to the two-point correlation function, and the scaling exponents of different moments are linearly dependent on the second moment scaling exponent. These kind of processes are termed “linear” and “monofractal”, since just a single exponent that determines the two-point correlations (and thus the linear correlations) quantifies the entire spectrum of \( q \) order scaling exponents. In other cases, the relation between the \( q \)-point correlation function has nontrivial relation to the two-point correlation function, and a (nontrivial) spectrum of scaling exponents is needed to quantify the statistical properties of the system; these processes are called “nonlinear” and “multifractal”. The classification into linear and nonlinear processes is important for the understandings of the underlying dynamics of natural time series and for models development. Moreover, the nonlinear properties of natural time series may have practical diagnosis use [e.g., 3].

Recently a simple measure for nonlinearity of time series was suggested 3. Given a time series \( u_i \), the correlations in the magnitude series (volatility) \( |u_i| \) may be related (in some cases) to the nonlinear properties of the time series; basically, when the magnitude series is correlated the time series \( u_i \) is nonlinear. It was also shown that the scaling exponent of the magnitude series may be related, in some cases, to the multifractal spectrum width. However, these observations are empirical and the reasons underly these observations still remain unclear.

Here we develop an analytical relation between the scaling exponent of the original time series \( u_i \) and the scaling exponent of the magnitude time series \( |u_i| \) for linear series. We show that when the original time series is nonlinear, the corresponding scaling exponent of the magnitude series is larger (or in some cases equal) compared to that of linear series and that the correlations in the magnitude series increase as the nonlinearity of the original series increases. These relations may help to identify nonlinear processes and quantify the nonlinearity strength. Based on these results we suggest a generic model for multifractality by multiplying random signs with long-range correlated noise, and show that the multifractal spectrum width and the volatility exponent increase as these correlations become stronger.

The paper is organized as follows: in Section II we present some background regarding non-linear processes and magnitude (volatility) series correlations. In Section III we develop an analytical relation between the original time series scaling exponent \( \alpha \) and the magnitude series exponent \( \alpha_v \); we confirm the analytical relation using numerical simulation. We then study in Section IV the relation between volatility correlations and the multifractal spectrum width of several multifractal models, and introduce a simple model that generates multifractal series by explicitly inserting long range correlations in...
the magnitude series. A summary of the results is given in Section [7].

II. NONLINEARITY AND VOLATILITY CORRELATIONS

A. Two-point correlations

The long range correlations of a time series \( \{u_t\} \) \( (i = 0, 1, 2, \ldots, N) \) can be evaluated using the two-point correlation function \( \langle u_i u_j \rangle \) (\( \langle \cdot \rangle \) stands for expectation value); when \( u_t \) is long-range correlated and stationary the two-point correlation function is \( \langle u_i u_j \rangle \sim |i - j|^{-\gamma} \) \( (0 < \gamma < 1) \) [4, 5]. It is possible to get good estimation of the scaling exponent using various methods, such as the power spectrum, Fluctuation Analysis (FA) [6], Detrended Fluctuation Analysis (DFA) [6, 7, 8], wavelet transform [8], and others; see [3] for more details. The different techniques characterize the linear two point correlations in a time series with a scaling exponent which is related to the scaling exponent \( \gamma \).

In this study we use the FA method for the analytical derivations since this method is relatively simple. In the FA method the sequence is treated as steps of a random walk; then the variance of its displacement, \( \sum_{i=0}^{t} u_i \), is found by averaging over different time windows of length \( t \). The scaling exponent \( \alpha \) of the series (also referred to as the Hurst exponent \( H \)) can be measured using the relation \( \text{Var}(X_t) = \langle X_t^2 \rangle \sim t^{2\alpha} \) where \( \text{Var}(\cdot) \) is the variance; the scaling exponent \( \alpha \) is related to the correlation exponent \( \gamma \) by \( 2 - \gamma = 2\alpha \).

B. High order correlations

A more complete description of stochastic process \( \{u_t\} \) with a zero mean is given by its multivariate distribution: \( P(u_0, u_1, u_2, \ldots) \). It is equivalent to the knowledge stored in the correlation functions of different orders [4, 10]: \( \langle u_i \rangle, \langle u_i u_j \rangle, \langle u_i u_j u_k \rangle, \langle u_i u_j u_k u_l \rangle \), etc. In many cases it is useful to use the cumulants of different orders \( C_q \) which are related to the \( q \) order correlation function by [10]:

\[
\begin{align*}
C_1 &= \langle u_i \rangle \\
C_2 &= \langle u_i u_j \rangle \\
C_3 &= \langle u_i u_j u_k \rangle \\
C_4 &= \langle u_i u_j u_k u_l \rangle - \langle u_i u_j \rangle \langle u_k u_l \rangle - \langle u_i u_k \rangle \langle u_j u_l \rangle - \langle u_i u_l \rangle \langle u_j u_k \rangle,
\end{align*}
\]

and so on.

For a linear process (sometimes referred to as “Gaussian” process), all cumulants above the second are equal to zero (Wicks theorem) [10]. Thus, in this case, the two-point correlation fully describes the process [11, 12], since all correlation functions (of positive and even order) may be expressed as products of the two-point correlation function \( \langle u_i u_j \rangle \).

Processes that are nonlinear (or “multifractal”) have nonzero high order cumulants. The nonlinearity of these processes may be detected by measuring the multifractal spectrum [11, 12] using advanced techniques, such as the Wavelet Transform Modulus Maxima [8] or the Multifractal DFA (MF-DFA) [13]. In MF-DFA we calculate the \( q \) order correlation function of the profile \( X_t = \sum_{i=0}^{t} u_i \) and the partition function is \( Z_q(t) \equiv \langle |X_t|^q \rangle \). For time series that obey scaling laws the partition function is \( Z_q(t) \sim t^{q\alpha(q)} \). Thus, the “spectrum” of scaling exponents \( \alpha(q) \) characterizes the correlation functions of different orders. For a linear series, the exponents \( \alpha(q) \) will all give a single value \( \alpha \) for all \( q \) [14].

C. Volatility correlations

A known example for the use of volatility correlations (defined below) is econometric time series [12]. Econometric time series exhibit irregular behavior such that the changes (logarithmic increments) in the time series have a white noise spectrum (uncorrelated). Nonetheless, the magnitudes of the changes exhibit long-range correlations that reflect the fact that economic markets experience quiet periods with clusters of less pronounced price fluctuations (up and down), followed by more volatile periods with pronounced fluctuations (up and down). This type of correlation is referred to as “volatility correlations”.

Given a time series \( u_t \), the magnitude (volatility) series may be defined as \( |\Delta u_t| = |u_{t+1} - u_t| \). The scaling exponent of the magnitude series is the volatility scaling exponent \( \alpha_v \). Correlations in the magnitude series are observed to be closely related to nonlinearity and multifractality [3, 10, 14].

In this paper we refer to “volatility” with two small differences. First, we consider the square of the series elements rather than their absolute values. According to our observations, this transformation has negligible effect on the scaling exponent \( \alpha_v \), but it substantially simplifies the analytical treatment. Second, for simplicity, we also consider the series itself rather than the increment series. That is: the volatility series is defined as \( u_t^2 \) rather than \( |\Delta u_t| \). Note that in most applications the absolute values of the increment series are considered instead of the absolute values of the series itself, since the original series is mostly nonstationary (defined below); here we overcome this problem by first considering stationary series.

D. Stationary and nonstationary time series

Series with \( 0 < \alpha < 1 \) are stationary, that is, their correlation function depends only on the difference between points \( i \) and \( j \), i.e., \( \langle u_i u_j \rangle = f(|i - j|) \). Their variance is a finite constant, that does not increase with the sequence length. Sequences with \( \alpha > 1 \) are non stationary and
have a different form of correlation function where the correlation function depends also on the absolute indices $i$ and $j$, $(u_i,u_j) = i^{2\alpha-2}j^{2\alpha-2} - |i-j|^{2\alpha-2}$; see Fig. 3. Scaling exponents of nonstationary series (or series with polynomial trends) may be calculated using methods that can eliminate constant or polynomial trends from the data \cite{2,5,8}.

III. VOLATILITY CORRELATIONS OF LINEAR TIME SERIES

We proceed to study the relation between the volatility correlation exponent $\alpha_v$, and the original scaling exponent $\alpha$ for linear processes, both numerically and analytically.

A. Simulations

We generate artificial long-range correlated linear sequences $u_t$ with different values of $\alpha$ in the range $\alpha \in (0,1.5]$ as follows \cite{13}:

(i) generate Gaussian white noise series, (ii) apply Fourier transform on that series, (iii) multiply the power spectrum $S(f)$ by $1/f^\beta$ where $\beta = 2\alpha - 1$ and $f > 0$, and (iv) apply inverse Fourier transform. The resultant series is long-range correlated with a scaling exponent $\alpha$. We measure the volatility scaling exponent $\alpha_v$, i.e. the scaling exponent of $|u_t|^2$ and $|u_t|$, versus the original scaling exponent. The results are plotted in Fig. 4.

The simulations indicate that the dependence of $\alpha_v$ on $\alpha$ for linear series may be divided into three regions: for $\alpha < 3/4$ we obtain $\alpha_v \approx 1/2$, for $\alpha > 1.25$ we obtain $\alpha_v \approx \alpha$, while for $0.75 < \alpha < 1.25$ there is a transition region between those two behaviors. These results were obtained using the DFA method which can handle nonstationary time series.

It is important to note that in Fig. 4 we used the absolute value of the series $u_i$, to calculate the volatility scaling exponent. Nevertheless, the usual method for calculating the volatility scaling exponent is performed by taking the absolute value of the differences series, $\Delta u_i$, rather than $u_i$ itself \cite{3,17}. The reason for this is that in many cases the given time series has an exponent in the range $\frac{1}{2} < \alpha < \frac{1}{2}$, By differentiating the sequence we get a new sequence $\Delta u_i$ with an exponent $\hat{\alpha} = \alpha - 1 < \frac{1}{2}$. According to our analysis, if the sequence is linear, the volatility exponent for this series will be $\hat{\alpha} = \frac{1}{2}$ (whereas for the original series $\alpha_v$ may be higher than $\frac{1}{2}$ even for linear data). Thus, if the scaling exponent of the magnitude of the differences series, $|\Delta u_i|$, differs from $1/2$ this is an indication for nonlinearity.

We note that for $\alpha \gg 1$ the series is highly nonstationary, i.e., it is most of the time either above or below 0, apart from a discrete set of crossing points. The behavior of the series $u_i$ is not very different than the behavior of its absolute value $|u_i|$, and therefore it is not surprising that $\alpha_v = \alpha$.

B. Analytical Treatment

Let us consider a Gaussian distributed linear sequence $u_t$ of length $t$ with scaling exponent $\alpha$. For simplicity, we assume that the sequence is stationary. Consider the magnitude series: $v_t = u_t^2$. In order to calculate the magnitude series scaling exponent $\alpha_v$, we will calculate the variance of the displacement $V_t = \sum_{i=0}^{t} v_i$:

\[
Var(V_t) = \langle V_t^2 \rangle - \langle V_t \rangle^2 = \sum_{i=0}^{t} \sum_{j=0}^{t} \langle v_i v_j - \langle v_i \rangle \langle v_j \rangle \rangle = \sum_{i=0}^{t} \sum_{j=0}^{t} \langle u_i^2 u_j^2 \rangle - \langle u_i^2 \rangle \langle u_j^2 \rangle.
\]

Because the series $u_i$ is linear, the fourth cumulant is $C_4 = 0$ (Wicks theorem), and by using Eq. 4 we get,

\[
\langle u_i^2 u_j^2 \rangle = \langle u_i^2 \rangle \langle u_j^2 \rangle + 2 \langle u_i u_j \rangle^2,
\]

and thus,

\[
Var(V_t) = 2 \sum_{i=0}^{t} \sum_{j=0}^{t} \langle u_i u_j \rangle^2.
\]

Substituting the two-point correlation function for long-range correlated data:

\[
\rho(i-j) = \langle u_i u_j \rangle \sim \begin{cases} |i-j|^{-\gamma} & i \neq j \\ 1 & i = j \end{cases}
\]

FIG. 1: Magnitude series scaling exponent $\alpha_v$ vs. the two-point correlation exponent $\alpha$ for linear sequences $u_i$. The solid line represents results for synthesized sequences of length $2^{15}$, averaged over 15 configurations, for $u_i^2$ and $|u_i|$ (these two coincide). The circles represent the analytical reconstruction taking into account corrections due to finite size effects and non-stationarity. Analytical results for $N \to \infty$ are given by the dashed line.
we obtain
\[
\text{Var}(V_t) \sim t + \sum_{i \neq j} |i - j|^{-2\gamma} \sim t + t^{-2\gamma+2}. \tag{7}
\]

Since \(2 - \gamma = 2\alpha\) the above expression becomes
\[
\langle V_t^2 \rangle \sim t + t^{2\alpha-2} = t^{2\alpha}. \tag{8}
\]

For \(\alpha < \frac{1}{2}\) the first term, \(t\), is dominant and for \(t \to \infty\) we obtain \(\alpha = \frac{1}{2}\). Otherwise the second term, \(t^{2\alpha-2}\), is dominating and thus \(\alpha = 2 \alpha - 1\). However, the simulation results (Fig. 11) deviate from \(\alpha_v \approx 2 \alpha - 1\) as \(\alpha \to 1\). This is because as \(\alpha \to 1\), logarithmic and polynomial corrections due to strong finite size effects and non-stationarity must be taken into account in our calculations (i.e., the variance of the sequence depends on its length; see Appendix A). This is done by dividing \(\langle V_t^2 \rangle \) [Eq. 8] by the variance of the sequence:
\[
\text{Var}(u_i) = \frac{1}{1 - \alpha} 2^{2\alpha - 2} (1 - t^{2\alpha - 2}) = \begin{cases} \text{const} & \alpha < 1 \\ \ln t & \alpha = 1 \\ t^{2\alpha - 2} & \alpha > 1 \end{cases} \tag{9}
\]

This modification yields an \(\alpha_v\) that is very close to the one obtained from the numerical simulation (in the transition region \(0.75 < \alpha < 1.25\), and also for \(\alpha > 1.25\) with \(\alpha_v = \alpha\); see Fig. 11). The relation \(\alpha_v = \alpha\) for \(\alpha > 1.25\) can now be proved analytically: It is noticeable that the dominant scaling term of \(\langle V_t^2 \rangle\) for the nonstationary case is proportional to \(t^{2\alpha - 2}\) [Eq. 8]. Dividing by the variance term, \(t^{2\alpha - 2}\) [Eq. 8], yields \(t^{2\alpha} \sim t^{2\alpha}\) and hence \(\alpha = \alpha_v\).

IV. VOLATILITY CORRELATIONS AND THE MULTIFRACTAL SPECTRUM WIDTH

A. Random multifractal cascades

Following [3] 17, we study the relation between the volatility scaling exponent \(\alpha_v\) and the multifractal spectrum width of nonlinear multifractal time series. We generate artificial noise with multifractal properties according to the algorithm proposed in 10. The algorithm is based on random cascades on wavelet dyadic trees. The multifractal series is constructed by building its wavelet coefficients at different scales recursively, where at each stage the coefficients of the coarser scale are multiplied by a random variable \(W\) in order to build the coefficients of the finer scale. Note that we now consider the increments series, hence the generated time series is stationary. The multifractal spectrum \(f(\alpha)\) depends on the statistical properties of the random variable \(W\).

We choose \(W\) to follow log-normal distribution, such that the \(\ln|W|\) is normally distributed with \(\mu\) and \(\sigma^2\) being the mean and variance. For this case the multifractal spectrum \(f(\alpha)\) is known analytically 10 and by assigning \(f(\alpha) = 0\) it is possible to obtain \(\alpha_{\text{min,max}}\):
\[
\alpha_{\text{min}} = -\frac{\sqrt{2} \sigma}{\ln 2} \frac{\mu}{\ln 2} \tag{10}
\]
\[
\alpha_{\text{max}} = \frac{\sqrt{2} \sigma}{\ln 2} \frac{\mu}{\ln 2}. \tag{11}
\]

Thus, the multifractal width, \(\Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}} = 2 \frac{\sigma^2}{\ln 2}\), depends just on \(\sigma\) while the scaling exponent \(\alpha(0)\) depends on \(\mu\), i.e., \(\alpha(0) = -\frac{\mu}{\ln 2}\).

Using the above algorithm, we generate multifractal time series with a fixed multifractal width \(\Delta \alpha\) (by fixing \(\sigma\)) and different scaling exponents \(\alpha(q = 2)\) (by changing \(\mu\)), and calculate their volatility exponents \(\alpha_v\) (see Fig. 2 10). We perform the same analysis for their respective surrogate time series, which are linearized series (after phase randomization) that have the same two-point correlations with exponent \(\alpha(q = 2)\) as the original series 21; note that \(\alpha(q)\) is the \(q\)-point correlation exponent. We find that the volatility exponent \(\alpha_v\), calculated in section III for the linear case, is the lower bound for all multifractal sequences (studied here) with same \(\alpha(q = 2)\). Nonetheless, for \(\alpha(q = 2) > 1\), i.e., for nonstationary series, \(\alpha_v = \alpha(q = 2)\) as in linear series. It is clearly seen in Fig. 2 that for stationary time series \((\alpha(2) < 1)\), the volatility correlations increase as the multifractal spectrum width becomes wider, or alternatively, as the nonlinearity of the original series strengthens.

B. A simple model for multifractality

We now propose a simple model for generating multifractal records, based on the property that multifractal series exhibit long range correlations in the volatility series. Following 21, we multiply a long range correlated series \(\eta_i\) (with a scaling exponent \(\alpha_\eta > 0.75\)) with a series of uncorrelated random signs \(\epsilon_i = \pm 1\). The resultant series, \(u_i = \epsilon_i \eta_i\), has a two-point correlation exponent \(\alpha = 1/2\) because of the random signs \(\epsilon_i\). The magnitude exponent \(\alpha_v\) is the same as the magnitude exponent \(\alpha_{v,\eta}\) for \(\eta_i\), because \(|u_i| = |\eta_i|\). Thus, using our results from Section III if we take \(\alpha_v > 0.75\) we get a sequence with \(\alpha = 1/2\) and \(\alpha_v \approx 2 \alpha_\eta - 1 > 1/2\) (see Fig. 4) full diamond symbol for \(\alpha_\eta = 0.95\). Note that in Fig. 2 the theoretical value of \(\alpha_v = 2 \times 0.95 - 1 = 0.9\) is higher than that of the numerical estimation \(\alpha_v = 0.8\), most probably due to finite size effects.

According to our derivation in Section III this sequence is nonlinear/ multifractal (because linear series with a two-point correlation exponent \(\alpha = 1/2\) should have \(\alpha_v = 1/2\)). Indeed, one can see from Fig. 3 that the multifractal width for this model increases as \(\alpha_\eta\) increases beyond 0.75.

Natural processes are often characterized by complex nonlinear and multifractal properties. However, the un-
FIG. 2: The magnitude series exponent \( \alpha_v \) vs. the two-point correlation exponent \( \alpha \) for multifractal series. The full triangles and squares represent sequences generated by the log-normal random cascade algorithm with \( \sigma = 0.1 \) (triangles) and \( \sigma = 0.05 \) (squares). The respective linear (phase randomized) surrogate data sequences are represented by empty symbols. The solid line indicates simulation results for linear sequences as derived in [12], explained in section III and shown in Fig. 1. The full diamond represents the scaling exponent of our multifractal model’s sequences shown in Fig. 1. The full diamond represents the scaling exponent of the surrogate data. All sequences are of length \( 2^{14} \) elements, and results were averaged over 15 configurations. Error bars are smaller than symbol size.

FIG. 3: (a) Multifractal spectrum width and (b) volatility exponent \( \alpha_v \) for sequences of the form \( u_i = \epsilon_i \eta_i \) of length \( 2^{19} \), averaged over 15 configurations. The error bars indicate the mean \( \pm 1 \) std. For \( \alpha_{\eta} > 0.75 \) both volatility correlation exponent and the multifractal spectrum width of the series are increasing with \( \alpha_{\eta} \).

The multifractal model described in this section is a simple model with known properties that may help to gain better understanding of multifractal processes. The model consists of two components as follows, (i) a random series (which can be also any other long-range correlated series) that may represent fast processes of a natural system, which as a first approximation may be regarded as a white noise, interacting with (ii) a long-range correlated process that may represents a slow modulation of the natural system. This interaction results in episodes with less volatile fluctuations followed by episodes with more volatile fluctuations. In the context of heart-rate variability, the fast component may represents the parasympathetic branch of the autonomic nervous system while the slow process may represents the sympathetic and the hormonal activities. In the context of geophysical phenomena, the fast component may represents the fast atmospheric processes while the slow process may represent the relatively slow oceanic processes. Our model can also be used to model other complex systems like economy and heartbeat dynamics.

V. SUMMARY

We study the behavior of the magnitude series scaling exponent \( \alpha_v \) versus the original two-point scaling exponent \( \alpha \) for linear and nonlinear (multifractal) series. We find analytically and by simulations that for linear series the dependence of \( \alpha_v \) on \( \alpha \) may be divided into three regions: for \( \alpha < 3/4 \) the volatility exponent is \( \alpha_v = 1/2 \), for \( \alpha > 1.25 \) the volatility exponent is \( \alpha_v = \alpha \), while for \( 0.75 < \alpha < 1.25 \) there is a transition region in which logarithmic corrections due to finite size effects and non-stationarity are dominant.

The results presented here provide the theory for the relation found previously [3, 17] between multifractality and the scaling exponent of the magnitude of the differences series (volatility). This relation provides a simple method for preliminary detection and quantification of nonlinear time series, a procedure which usually requires relatively complex techniques.

We also study the volatility of some known models of multifractal time series, and find that their magnitude scaling exponent is bounded from below by \( \alpha_v \) of the corresponding phase randomized linear surrogate series.

Based on the above findings, we propose a simple model that generates multifractality by explicitly inserting long range correlations (\( \alpha_{\eta} > 0.75 \)) into the magnitude series. This model may serve as a generic model for...
multifractality and may help to gain preliminary understandings of natural complex phenomena. The model, which involves interaction between fast and slow components, may represents natural fast processes that interact with slower processes. In addition, the simplicity of the model may help to identify these processes more easily in experimental records.

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APPENDIX A: FINITE SIZE EFFECTS AND NON-STATIONARITY NEAR $\alpha = 1$

A linear time sequence with scaling exponent $\alpha$ can be generated by filtering Gaussian white noise such that the power spectrum will be [18]:

$$S(f) \sim \begin{cases} 
0 & f = 0 \\
\frac{1}{|f|^{\alpha}} & f \neq 0 
\end{cases} \quad (A1)$$

where $\beta = 2\alpha - 1$. Assume a signal $u_i$ of $N$ discrete points sampled at time intervals $\Delta t$. The power spectrum consists of $N$ points in the frequency range $(-\frac{1}{2\Delta t}, \frac{1}{2\Delta t}]$ with intervals of $\Delta f = \frac{1}{N\Delta t}$. Thus, looking only at the positive frequencies, the minimal frequency (without loss of generality) is $\frac{\Delta f}{2} = \frac{1}{2N\Delta t}$. The variance of the signal is the total area under the power spectrum:

$$Var(u_i) = 2 \int_{\frac{1}{2N\Delta t}}^{\frac{1}{N\Delta t}} S(f) df = 2 \int_{\frac{1}{2N\Delta t}}^{\frac{1}{N\Delta t}} \frac{1}{f^{2\alpha-1}} df \quad (A2)$$

Assuming $\Delta t = 1$, for $\alpha = 1$ the variance is,

$$Var(u_i) = 2 \ln N. \quad (A3)$$

Thus, the variance diverges logarithmically for $\alpha = 1$.

For $\alpha \neq 1$ the variance is

$$Var(u_i) = \frac{1}{1-\alpha} 2^{2\alpha-2}(1 - N^{2\alpha-2}). \quad (A4)$$

From (A4) follows: For $\alpha < 1$ the variance converges, and for $\alpha > 1$ it diverges.

**Non-stationarity**: For $\alpha \geq 1$ the variance diverges with the sequence length $N$, because of the singularity in the power spectrum, and the sequence is non-stationary. For $\alpha > 1$ the divergence is power-law, i.e. $Var(u_i) \sim N^{2\alpha-2}$, while at $\alpha = 1$ the divergence is logarithmic.

**Finite size effects**: For $\alpha < 1$ the variance converges to a finite constant so the sequence is stationary, but as $\alpha \to 1$ this convergence becomes slower. This means that as $\alpha \to 1$, larger and larger sequence lengths $N$ are required so that the variance will indeed converge to a constant value (see Fig. 4). This argument also holds for other values of the correlation functions $\rho(n)$, $n = 0, 1, \ldots, \infty$, although in a more moderate way.

The strong finite size effects around $0.75 < \alpha < 1.25$ and the non-stationarity at $\alpha \geq 1$ have to be taken into account when calculating the magnitude series scaling exponent $\alpha_v$. This is done by dividing the volatility fluctuation function $\langle V^2 \rangle$ by the variance of the sequence given in Eq. (A4) [20].

For $N \to \infty$ the finite size effects disappear and $\alpha_v$ converges to its theoretical value (see Fig. 4). This convergence is extremely slow and becomes weaker as we approach $\alpha \approx 1$.

For completeness, we show in Fig. 6 the correlation coefficients $\rho(n = i - j)$ for $\alpha < 1/2$. In this regime the sequences exhibits short range anti-correlations as can be seen in Fig. 6. The expression of the correlation function for $\alpha < 1/2$ is approximately [23]

$$\rho(i - j) = \langle u_iu_j \rangle \sim \begin{cases} 
\alpha(2\alpha - 1)|i - j|^{2\alpha-2} & i \neq j \\
2\alpha & i = j. \quad (A5)
\end{cases}$$

\[\text{FIG. 4: Correlation coefficient } \rho(0) (\text{i.e. the variance } \langle u_i^2 \rangle) \text{ for linear sequences of } N = 50,000 \text{ points. } \text{Circles indicate the simulation results. Dots represent analytical results for the variance calculated according to Eq. (A4), which takes into account the finite size effects. The solid line is the variance for } N \to \infty. \text{ It can be seen that as } \alpha \to 1 \text{ the convergence becomes slower and finite size effects become more dominant [i.e., the convergence is non-uniform in the range } \alpha \in (0, 1)].}\]
FIG. 5: Correlation coefficients $\rho(0)$ (circles) and $\rho(1)$ (squares) for linear sequences of 50,000 points, in range $\alpha < 1/2$. The dashed line indicates the analytical results for $\rho(0)$, taking into account the finite series size effects, which approximately follows results for $N \to \infty$ (solid line). $\rho(1)$ is negative for $\alpha < 1/2$ indicating anti-correlations. The solid lines are the analytical expressions of Eq. (A5).

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[26] GA and DFA actually measure the scaling of the fluctuations for window sizes ranging from 1 to t. Fluctuations for window sizes of size t are given by $\langle X_t^2 \rangle$, while fluctuations for windows of size 1 are actually the variance of the sequence. Thus, the scaling exponent is approximated by $\ln(\langle X_t^2 \rangle^{1/2} - \langle X_t^2 \rangle^{1/2})/\ln t = \ln(\langle X_t^2 \rangle^{1/2} - \ln \text{var}(u_t))/\ln t = \ln[(\langle X_t \rangle)/\text{var}(u_t)]/\ln t$. Therefore, in cases where the variance is not constant, the fluctuation function $\langle V_t^2 \rangle$ should be normalized by the variance.