We unify the power laws of size distributions of solar flare and nanoflare energies. We present three models that predict the power law slope $\alpha_E$ of flare energies defined in terms of the 2-D and 3-D fractal dimensions ($D_A, D_V$): (i) The spatio-temporal standard SOC model, defined by the power law slope $\alpha_{E_1} = 1 + 2/(D_V + 2) = (13/9) \approx 1.44$; (ii) the 2-D thermal energy model, $\alpha_{E_2} = 1 + 2/D_A = (7/3) \approx 2.33$, and (iii) the 3-D thermal energy model, $\alpha_{E_3} = 1 + 2/D_V = (9/5) \approx 1.80$. The theoretical predictions of energies are consistent with the observational values of these three groups, i.e., $\alpha_{E_1} = 1.47 \pm 0.07$; $\alpha_{E_2} = 2.38 \pm 0.09$, and $\alpha_{E_3} = 1.80 \pm 0.18$. These results corroborate that the energy of nanoflares does not diverge at small energies, since ($\alpha_{E_1} < 2$) and ($\alpha_{E_3} < 2$), except for the unphysical 2-D model ($\alpha_{E_2} > 2$). This conclusion adds an additional argument against the scenario of coronal heating by nanoflares.

1. Introduction

The concept of self-organized criticality (SOC) specifies nonlinear (avalanching) phenomena based on next-neighbor interactions in a lattice grid (Bak et al. 1987; 1988). The spatio-temporal evolution of avalanches in such complex environments has been numerically simulated by cellular automation methods, which exhibit fractal structures (Pruessner 2012). Alternatively, SOC-related avalanches can be considered analytically, as instabilities with a nonlinear initial growth phase and subsequent saturation (Rosner et al. 1978; Aschwanden 2011, 2012, 2014).

Let us introduce the first definition of a SOC energy, which we call the spatio-temporal energy of a SOC avalanche, labeled as $E_1$ in this Letter. The analytical SOC approach takes both the spatial as well as the temporal evolution of a SOC avalanche into account. Hence, the total energy of a SOC avalanche is determined from the spatial and temporal integration over the unstable pixel areas (or voxel volumes). Theoretical predictions of SOC parameters based on the size distribution of avalanche energies are described in Section 2 and are summarized in Table 1. The necessary parameters to characterize the spatio-temporal energy requires the measurement of the mean flux $F$ and the time duration $T$ of an event, while the spatio-temporal energy is defined by $E_1 = F \ast T$. Measurements of spatio-temporal energies were originally applied to soft X-rays in solar flares (Drake 1971), hard X-rays (Crosby et al. 1993; Lu et al. 1993), gamma rays (Perez-Enriquez and Miroshnichenko 1999), as well as to EUV small-scale brightenings or nanoflares (Brkovic et al. 2001; Uritsky et al. 2013). We compile these data sets in Table 2.

The second definition of a SOC energy was introduced by the 2-D definition of the thermal energy in a high-temperature plasma (called $E_2$ here), i.e., $E_2 = 3k_B T_e n_e V$ (where $k_B$ is the Boltzmann constant, $T_e$ the electron temperature, $n_e$ the electron density, and $V$ the fractal volume. The avalanche volumes of solar
flares and nanoflares are generally close to fractal geometries, if they are measured on a pixel-by-pixel basis (Aschwanden and Aschwanden 2008a, 2008b), in contrast to an encompassing (non-fractal) circle or square (with length scale \( L \)). An additional relationship is the definition of the emission measure, \( EM = n_e^2 V \), which can be used to substitute the electron density, i.e., \( n_e = \sqrt{EM/V} \). Furthermore, a relationship for the (fractal) event volume \( V \) needs to be specified. The projected area \( A \), which is fractal, can be measured directly in the image plane, but the line-of-sight depth is unknown, which led some pioneers to quantify it with a constant height \( h_0 \), leading to the expression \( V = A h_0 \) for the volume (Krucker and Benz 1998; Parnell and Jupp 2000; Benz and Krucker 2002, Table 3). In hindsight, this choice of a constant depth \( h_0 \) has been criticized to be unphysical, because it is very unlikely that the line-of-sight depth is equal from the smallest nanoflares (of the size of a solar granulation cell) to the largest flares (of the size of an active region). Furthermore, the assumption of a constant height \( h_0 \) introduces a crucial bias that modifies the power law slope of the energy size distribution substantially, from \( \alpha_{E3} = 1.80 \) to \( \alpha_{E2} = 2.33 \), across the critical value of \( \alpha_E = 2 \), as we will see in the remainder of this Letter.

A third definition of a SOC energy \( (E_3) \) was made by abandoning the unphysical assumption of a constant height, i.e., \( V = A h_0 \), and instead replacing it with a more physical assumption of an isotropic volume, i.e., \( V = A^{3/2} \), which corresponds to a line-of-sight depth of \( h = \sqrt{A} \), while \( D_A \) is the fractal dimension of the area, \( A = L^{D_A} \). This approach, which we call the 3-D thermal energy model (Table 4), has been applied frequently (Shimizu 1995; Berghmans et al. 1998; Berghmans and Clette 1999; Parnell and Jupp 2000; Aschwanden and Parnell 2002; Uritsky et al. 2007; Joulin et al. 2016; Nhalil et al. 2020; Purkhart and Veronig 2022; Kawai and Imada 2022).

In this Letter we calculate the power law indices \( \alpha_E \) of the energy size distributions in a unified way, which provides us a diagnostic whether flare and nanoflare energies diverge at the lower or upper end of the size distribution, and this way we can assess the importance (or non-importance) of coronal heating by nanoflares. The mathematical derivation of the SOC models is given in Section 2, a discussion in Section 3, and conclusions in Section 4.

2. Theoretical Models and Observations

A SOC model should be able to predict the (occurrence frequency) size distribution functions, which can be formulated in terms of power law function slopes \( \alpha_x \) to first order. Common SOC parameters \( x = [L, A, V, T, F, P, E] \) include the length scale \( L \), the 2-D area \( A \), the 3-D volume \( V \), the time duration \( T \), the mean flux or intensity \( F \), the peak flux or peak intensity \( P \), and the fluence or energy \( E \). In this study we reconcile three different definitions of the energy \( (E) \) that have been used in the past, namely the spatio-temporal definition of the standard SOC model \( (E_1) \), the 2-D fractal model \( (E_2) \), and the 3-D fractal model \( (E_3) \).

2.1. The Standard SOC Model

The standard SOC model is derived from first principles in previous studies (Aschwanden 2012, 2014, 2022). A brief summary of the calculations that clarify the assumptions made here is given in the following, while a more detailed description is provided in Aschwanden (2022).

We start with the size distribution \( N(L) \) of length scales \( L \), also called the scale-free probability conject-
ture,\[ N(L) \, dL \propto L^{-d} \, dL , \tag{1} \]
where \( d \) is the Euclidean space dimension, which is set to \( d = 3 \) for most real-world data. The power law indices \( \alpha_x \) of the size distribution functions can then be calculated for every SOC parameter \( x \) by variable substitution \( L \mapsto x \),\[ N(x) \, dx = L(x)^{-d} \frac{dL}{dx} \, dx = x^{-\alpha_x} \, dx , \tag{2} \]
which just requires the scaling law \( x(L) \) as a function of the length scale \( L \), the inverted scaling law function \( L(x) \), and its derivative \( dL/dx \). Thus, we need to make an assumption of a scaling law \( x(L) \) of each SOC parameter of interest.

For the spatial parameters we define the fractal dimensions for 2-D areas \( A \), which is identical to the fractal Hausdorff dimension \( D_A \approx 3/2, \)
\[ A = L^{D_A} , \tag{3} \]
and like-wise for the 3-D volume \( V \), which is identical to the fractal Hausdorff dimension \( D_V \approx 5/2, \)
\[ V = L^{D_V} . \tag{4} \]
We can estimate the numerical values of the fractal dimensions \( D_A \) and \( D_V \) from the mean of the minimum and maximum value in each Euclidean domain,
\[ D_A = \frac{(D_{A,\text{min}} + D_{A,\text{max}})}{2} = \frac{3 + 2}{2} = 1.50 , \tag{5} \]
and correspondingly,
\[ D_V = \frac{(D_{V,\text{min}} + D_{V,\text{max}})}{2} = \frac{5 + 2}{2} = 2.50 . \tag{6} \]
In the standard SOC model we need four more scaling laws. The time duration \( T \) of a SOC avalanche can be linked to spatial (fractal) structures by the diffusive behaviour,
\[ T \propto L^{2/\beta} , \tag{7} \]
where the coefficient is \( \beta = 1 \) for classical diffusion, \( \beta < 1 \) for sub-diffusive transport, and \( \beta > 1 \) for hyper-diffusive transport (also called Levy flight). Furthermore we need a relationship between the mean flux \( F \) and the emitting volume \( V \),
\[ F \propto V^{\gamma} = L^{D_V \gamma} , \tag{8} \]
which is generally found to be near to proportional, hence we set \( \gamma = 1 \). A relationship between the peak flux \( P \) and the length scale \( L \) is,
\[ P \propto V^{\gamma} = L^{d \gamma} , \tag{9} \]
where the flux \( F \) (Eq. 8) is maximized to the peak flux \( P \), i.e., \( P(t_{\text{peak}}) = \max[F(t)] \), by replacing the dimension \( D_V \) in Eq. (8) with the Euclidean dimension \( d \), i.e., \( D_V \mapsto d \).

Finally, the fluence or energy \( E_1 \), which is expressed by the product of the mean flux \( F \) and the event duration \( T \) (for a spatio-temporal SOC event) yields (Crosby et al. 1993),
\[ E_1 = (F \ast T) \propto L^{(D_V \gamma + 2/\beta)} . \tag{10} \]
If we assume classical diffusion ($\beta = 1$) and flux-volume proportionality ($\gamma = 1$), the four basic scaling laws are reduced further to $T \propto L^2$ (Eq. 7), $F \propto L^{2.5}$ (Eq. 8), $P \propto L^3$ (Eq. 9), and $E_1 \propto L^{4.5}$ (Eq. 10). In this framework, there are no free parameters, and the power law slopes $\alpha_x$ of the size distributions,

$$N(x) \, dx = x^{-\alpha_x} \, dx,$$

of all SOC parameters $x = [A, V, T, F, P, E]$ can be predicted by variable substitution (Eq. 2), yielding the values $D_A = 3/2$, $D_V = 5/2$, $\alpha_A = 7/3 \approx 2.33$, $\alpha_V = 9/5 \approx 1.80$, $\alpha_T = 2$, $\alpha_F = 9/5 \approx 1.80$, $\alpha_P = 5/3 \approx 1.67$, and $\alpha_{E_1} = 13/9 \approx 1.44$, as listed in Table 1.

Comparison of these theoretical predictions of power law slopes $\alpha_x^{\text{theo}}$ with observed size distributions $\alpha_x^{\text{obs}}$ have been presented in Aschwanden (2022). Among the solar flare data sets that apply the spatio-temporal energy model (Eq. 10; Table 2), we identify hard X-ray data (Crosby et al. 1993; Lu et al. 1993), gamma ray data (Perez-Enriquez and Miroshnichenko 1999), soft X-ray data (Drake 1971), and EUV data (Brkovic et al. 2001; Uritsky et al. 2013), which exhibit a mean power slope of $\alpha_{E_1}^{\text{obs}} = 1.47 \pm 0.07$, agreeing well with the theoretical prediction $\alpha_{E_1}^{\text{theo}} = (13/9) \approx 1.44$.

### 2.2. The 2-D Thermal Energy Model

The energy of a spatio-temporal SOC event is defined in the standard SOC model by the product of the count rate ($F$) and the event duration ($T$) (Eq. 10), which is appropriate for nonthermal energies that are quantified by hard X-ray counts (or intensity) in solar and stellar flares. In both solar or stellar flares, down to nanoflares, one can estimate thermal (radiative) energies at the peak time of an event, defined by

$$E_2 = (3 k_B n_e T_e) \, V,$$

where $k_B$ is the Boltzmann constant, $n_e$ the electron density, $T_e$ the electron temperature, and $V$ the 3-D volume, all measured at the peak time of an event. The 3-D volume $V$ cannot be measured directly, which led some authors to approximate the volume with a constant height $h_0$ in the line-of-sight,

$$V = A \, h_0 = L^{D_A} \, h_0,$$

while the fractal area is defined as $A = L^{D_A}$. The fractality is not explicitly mentioned in some of these studies, but every pattern recognition code that measures an area on a pixel-by-pixel basis (at different spatial resolutions) yields approximately the fractal area $A \propto L^{D_A}$ with $D_A < 2$, rather than the encompassing Euclidean area $A = L^2$. Inserting the area fractal dimension $D_A = 3/2$ (Eq. 5) into the expression for the thermal energy $E_2 \propto L^{D_A}$ (Eq. 12), we obtain

$$E_2 = (3 k_B n_e T_e h_0) L^{D_A}.$$

The same way as we substituted the variable $L$ in the size distribution with the energy $x = E$ (Eqs. 1 and 2),

$$N(E_2) \, dE_2 = L(E_2)^{-d} \frac{dL}{dE_2} \, dE_2 = E_2^{-\alpha_{E_2}} \, dE_2,$$

yielding the power law slope $\alpha_{E_2}$, for $d = 3$ and $D_A = 3/2$,

$$\alpha_{E_2} = 1 + \frac{(d-1)}{D_A} = \frac{7}{3} \approx 2.33.$$


Note that we treat the variables $n_e$, $T_e$, $h_0$ as constants here, while the scaling law hinges entirely on the correlation between the thermal energy $E_2$ and the length scale $L$, rendering a first-order approximation to the power law slope $\alpha_{E_2}$. Since the thermal energy $E_2 \propto V \propto L^{D_A}$ (Eq. 14) and the fractal area $A \propto L^{D_A}$ (Eq. 13) have the same scaling law, the power law index for the size distribution of areas $\alpha_A$ has the same power law index $\alpha_{E_2}$ too,

$$\alpha_A = \alpha_{E_2} = \frac{7}{3} \approx 2.33 \ .$$  \hspace{1cm} (17)

The definition of the energy made here (Eq. 12) invokes an isothermal plasma. Nevertheless, the definition of the thermal energy can accommodate a multi-thermal formalism, which involves a differential emission measure distribution function $dEM(T_e)/dT_e$, characterized by the increase in the emission measure $EM$, the (mean) electron density $n_e$, and the volume $V$,

$$EM = n_e^2 V\ ,$$  \hspace{1cm} (18)

which inserted into Eq. (12) yields,

$$E_2 = (3k_B T_e) \sqrt{EM * V} = (3k_B T_e) \sqrt{EM * A h_0} \ .$$  \hspace{1cm} (19)

Size distribution of thermal energies, based on emission measure changes $EM$, yield power law slopes of $\alpha_{E_2} = 2.38 \pm 0.09$ (Krucker and Benz 1998; Parnell and Jupp 2000; Benz and Krucker 2002), which match closely the theoretically expected value of $\alpha_E = 2.33$ (Eq. 16).

### 2.3. The 3-D Thermal Energy Model

In the 3-D version of the thermal model, the SOC avalanche volume $V \propto L^{D_V}$ (Eq. 4) is defined by the (mean) Hausdorff dimension $D_V = (5/2)$ (Eq. 6), which inserted into the expression for the thermal energy is,

$$E_3 = (3k_B n_e T_e) L^{D_V} \ .$$  \hspace{1cm} (20)

We substitute the variable $L$ in the size distribution of the thermal energy $E_3$ (Eq. 14),

$$N(E_3) dE_3 = L(E_3)^{-d} \frac{dL}{dE_3} dE_3 = E_3^{\alpha_{E_3}} dE_3 \ ,$$  \hspace{1cm} (21)

yielding the power law slope $\alpha_{E_3}$, for $d = 3$ and $D_V = 5/2$,

$$\alpha_{E_3} = 1 + \frac{(d-1)}{D_V} = \frac{9}{5} = 1.80 \ .$$  \hspace{1cm} (22)

Note that the power law slope is substantially steeper in the 2-D model ($\alpha_{E_2} = 2.33$) than in the 3-D version ($\alpha_{E_3} = 1.80$). Moreover, the two models predict power law slopes below ($\alpha_{E_2} < 2$), as well as above ($\alpha_{E_3} > 2$) the critical value of $\alpha_E = 2$, which decides whether the nanoflare population diverges at the low end or upper end of the size distribution. Calculations of the multi-thermal energy using a 3-D model have been performed using Yohkoh, TRACE, SOHO, AIA, and IRIS data (Shimizu 1995; Berghmans et al. 1998; Berghmans and Clette 1999; Parnell and Jupp 2000; Aschwanden and Parnell 2002; Uritsky et al. 2007; Joulin et al. 2016; Nhalil et al. 2020; Purkhart and Veronig 2022; Kawai and Imada 2022), as listed in Table 4.
3. Discussion

3.1. Scaling Laws

Scaling laws, typically expressed by variables \((x, y, ...)\) with power law dependencies, \(x^\alpha y^\beta \ldots = \text{const}\), are powerful tools to test parameter correlations and size distribution functions. If a scaling law function \(y(x)\) and a single size distribution \(N(x)\) is known, we can derive the size distribution \(N(y)\) of a correlated parameter by variable substitution, \(N(y)dy = N(x[y])(dx/dy)dy\). This way we can predict theoretical size distributions \(N(x)\) based on observed size distributions \(N(x)\), as well as significant correlations between variables. Here we explore the size distributions of 9 variables \(x = [L, A, V, T, F, P, E_1, E_2, E_3]\) in a unified scheme (Table 1). We focus mainly on the three energy parameters \(x = [E_1, E_2, E_3]\), which represent the spatio-temporal energy \((E_1)\), and the 2-D \((E_2)\) and 3-D fractal thermal energies \((E_3)\). Additional forms of energy definitions in solar and stellar flares, such as magnetic energies, radiative energies, conductive energies, coronal mass ejection kinetic or potential energies, etc.) are studied elsewhere (e.g., Aschwanden et al. 2017). The fact that we can predict energy size distribution functions, \([N_{E_1}, N_{E_2}, N_{E_3}]\), within the statistical uncertainties, corroborates the validity of the unified scaling laws derived here. Specifically, the scaling laws used here involve fractality, diffusive transport, flux-volume proportionality, spatio-temporal energy, and thermal energies in a fractal volume. The unified formalism to calculate size distributions based on the scale-free probability conjecture (Eq. 1) appears to be a sound method to obtain (macroscopic) physical scaling laws in (microscopic) SOC systems. We mention as a caveat however, that careful treatment has to be applied to small number statistics, truncation biases, data undersampling, background subtraction, inadequate fitting ranges, and deviations from ideal power law functions.

3.2. Power Law Slopes

Our unified method of implementing physical scaling laws in the calculation of size (or occurrence rate) distribution functions yields a power law slope \(\alpha_x\) for every SOC parameter \(x\). Thus we have a unique correspondence of a scaling law with the power law slope \(\alpha\). Our results yield a power law slope of \(\alpha_{E_1} = (13/9) = 1.44\) for a SOC system with spatio-temporal avalanche energies, a slope of \(\alpha_{E_2} = (7/3) = 2.33\) for the thermal energy in a SOC system with 2-D geometry, and \(\alpha_{E_3} = (9/5) = 1.80\) for the thermal energy in a SOC system with 3-D geometry. We can discard the model with the unphysical fractal 2-D geometry, but it explains why some researchers found relatively high values of \(\alpha_E > 2\). So we are left with relatively low values of \(\alpha_E < 2\) for realistic energy models, such as \(\alpha_{E_1} \approx 1.44\) for spatio-temporal avalanches, or \(\alpha_{E_3} \approx 1.80\) for 3-D fractal avalanches. Although we obtain a well-defined value for the power law slope \(\alpha_E\) for each size distribution, we should keep in mind that the estimation of fractal dimensions has some uncertainties within the fractal domains, such as in the range of \(1 \leq D_A \leq 2, \text{ and } 2 \leq D_V \leq 3\), respectively (Aschwanden and Aschwanden 2008a, 2008b). In principle, one can measure the values of the fractal dimensions \(D_A\) and \(D_V\) from the observed (fitted) power law slopes \(\alpha_A\) and \(\alpha_V\), i.e., \(D_A = 2/(\alpha_A - 1)\), and \(D_V = 2/(\alpha_V - 1)\) (Table 1).

3.3. Nanoflares and Coronal Heating

It was pointed out early on that powerlaw distributions \(N(E) \propto E^{-\alpha}\) of energies, with a slope flater than the critical value of \(\alpha_E = 2\) imply that the energy integral diverges at the upper end \(E_{max}\), and thus
the total energy of the distribution is dominated by the largest events (Hudson 1991),

\[ E_{\text{tot}} = \int_{E_{\text{min}}}^{E_{\text{max}}} E \cdot N(E) \, dE = \int_{E_{\text{min}}}^{E_{\text{max}}} (\alpha - 1)E^{1-\alpha} \, dE = \left( \frac{\alpha - 1}{2 - \alpha} \right) \left[ E_{\text{max}}^{2-\alpha} - E_{\text{min}}^{2-\alpha} \right]. \]  

(23)

In the opposite case, however, when the powerlaw distribution is steeper than the critical value, it will diverge at the lower end \( E_{\text{min}} \), and thus the total energy budget will be dominated by the smallest detected events, an argument that was used for dominant nanoflare heating (Krucker and Benz 1998). However, subsequent simulations demonstrated that there exists a strong bias towards a steeper slope (\( \alpha_{E_2} \approx 2.3 - 2.6 \)) if the assumption of a constant line-of-sight depth is assumed (\( h_0 = \text{const} \)), while the application of an isotropic geometry (\( h = A^{1/2} \)) lowers the power law slope to \( \alpha \approx 2.0 \) (Parnell and Jupp 2000; Benz and Krucker 2002). In our analytical 2-D fractal model we predict a power law slope of \( \alpha_{E_2} = (7/3) \approx 2.33 \) (Table 3), which agrees well with the spread of observed values, \( \alpha_{E_2} = 2.38 \pm 0.09 \) (Table 3). This result shows clearly that the size distribution of nanoflares has a power law slope of \( \alpha < 2 \), for both the spatio-temporal model (\( \alpha_{E_1} = (13/9) \approx 1.44 \)), as well as for the 3-D fractal thermal energy model (\( \alpha_{E_3} = (9/5) = 1.80 \)), which implies that the energy in nanoflares does not diverge at the lower end, \( E_{\text{min}} \leq 10^{24} \) erg, and that nanoflares are not the dominant contributor to the heating of the solar corona.

4. Conclusions

In this study we test whether the standard self-organized criticality model can predict the size (or occurrence frequency) distribution functions \( N(x) dx \propto x^{-\alpha_x} \) of physical parameters \( x \) in solar flares, down to the nanoflare regime with energies of \( E \gtrsim 10^{24} \) erg. We focus mostly on energy parameters, such as the spatio-temporal avalanche energy (\( E_1 \)), the 2-D fractal energy model (\( E_2 \)), and the more realistic 3-D fractal energy model (\( E_3 \)). For this three energy models, power law slopes of \( \alpha_{E_1} = 1.44 \), \( \alpha_{E_2} = 2.33 \), and \( \alpha_{E_3} = 1.80 \) are predicted. We test these predictions from literature values and find mean slopes of \( \alpha_{E_1} = 1.47 \pm 0.07 \) from 9 data sets (Table 2), \( \alpha_{E_2} = 2.38 \pm 0.09 \) from 4 data sets (Table 3), and \( \alpha_{E_3} = 1.80 \pm 0.18 \) from 17 data sets (Table 4), which all are fully self-consistent with the predicted values.

The related observations include solar flares observed in hard X-rays, soft X-rays, and EUV wavelengths, from large flares with energies of \( E \lesssim 10^{33} \) erg down to nanoflares (specified as EUV transients, coronal brightenings, or blinkers). We consider both the spatio-temporal (or standard SOC) model as well as the 3-D fractal energy model, based on emission measure analysis, as realistic tools to quantify the energy of flares and nanoflares, while the 2-D version of the fractal energy model (\( E_2 \)) significantly over-estimates the power law slope of the energy size distributions. The analytical approach clearly demonstrates that the size distribution of nanoflares has a power law slope of \( \alpha < 2 \), and thus the energy in nanoflares does not diverge at the lower end of the size distributions, so that nanoflares do not qualify to be dominant contributors to the heating of the solar corona.

While numerical Monte Carlo-type simulations leave the option of a super-critical value of \( \alpha_E \gtrsim 2 \) open (Krucker and Benz 1998; Parnell and Jupp 2000), we demonstrate in this Letter that this conclusion is true only for the unrealistic 2-D fractal energy model \( E_2 \), observationally (\( \alpha_{E_2} = 2.38 \pm 0.09 \)), as well as theoretically (\( \alpha_{E_2} = (7/3) \approx 2.33 \)). Consequently, the power law slope is flatter (\( \alpha_E < 2 \)) for at least two energy models (the spatio-temporal standard SOC model \( \alpha_{E_1} = 1.44 \), and the 3-D fractal energy model \( \alpha_{E_3} = 1.80 \)), which implies that heating of the corona in active regions is dominated by large (M- and X-class) flares. The same argument holds for Quiet Sun regions, where the largest events in each size distribution (of nanoflares, microflares, EUV transients, coronal brightenings, blinkers, etc.) dominate the energy budget.
(see power law slopes $\alpha_E$ of energies in Table 4), rather than the smallest events.

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Table 1: Parameters of the standard SOC Model, with fractal dimensions $D_x$ and power law slopes $\alpha_x$ of size distributions.

| Parameter                          | Power law slope | Power law slope |
|------------------------------------|-----------------|-----------------|
|                                    | analytical      | numerical       |
| Euclidean Dimension                | $d = 3.00$      |                 |
| Diffusion type                     | $\beta = 1.00$  |                 |
| Area fractal dimension             | $D_A = d - (3/2) = 1.50 = (3/2)$ |
| Volume fractal dimension           | $D_V = d - (1/2) = 1.20 = (5/2)$ |
| Length                             | $\alpha_L = d = 3.00$ |
| Area                               | $\alpha_A = 1 + (d - 1)/D_A = 2.33 = (7/3)$ |
| Volume                             | $\alpha_V = 1 + (d - 1)/D_V = 1.80 = (9/5)$ |
| Duration                           | $\alpha_T = 1 + (d - 1)\beta/2 = 2.00$ |
| Mean flux                          | $\alpha_F = 1 + (d - 1)/(\gamma D_V) = 1.80 = (9/5)$ |
| Peak flux                          | $\alpha_P = 1 + (d - 1)/(\gamma d) = 1.67 = (5/3)$ |
| Spatio-temporal energy             | $\alpha_{E_1} = 1 + (d - 1)/(\gamma D_V + 2/\beta) = 1.44 = (13/9)$ |
| Thermal energy (h=const)           | $\alpha_{E_2} = 1 + 2/D_A = 2.33 = (7/3)$ |
| Thermal energy (h=A^{1/2})         | $\alpha_{E_3} = 1 + 2/D_V = 1.80 = (9/5)$ |

Table 2: Observed frequency distributions of spatio-temporal energies $E = F \ast T$, by integrating the flux rate $F$ in space and time $T$.

| Powerlaw slope of energy $\alpha_E$ | Instrument | Observed | Reference |
|-------------------------------------|------------|----------|-----------|
| 1.53±0.02                           | HXRBS (>25 keV) | solar flares | Crosby et al. (1993) |
| 1.51±0.04                           | HXRBS (>25 keV) | solar flares | Crosby et al. (1993) |
| 1.48±0.02                           | HXRBS (>25 keV) | solar flares | Crosby et al. (1993) |
| 1.53±0.02                           | ISEE3 (>25 keV) | solar flares | Crosby et al. (1993) |
| 1.51                               | Explorer SXR 2-12 A | solar flares | Lu et al. (1993) |
| 1.39±0.01                           | PHEBUS (>100 keV) | solar flares | Perez-Enriquez and Miroshnichenko (1999) |
| 1.44                               | SMM/FCS, OV | blinkers | Brkovic et al. (2001) |
| 1.50±0.04                           | SOHO/EIT 195,HMI | EUVE events | Uritsky et al. (2013) |
| 1.47±0.07                           | Mean of 9 observations |
| 1.44                               | Theoretical prediction |
Table 3: Observed frequency distributions of thermal energies $E_2$ calculated from peak emission measures and temperatures with 2-D fractal model and constant line-of-sight depth ($h_0=\text{const}$).

| Powerlaw slope of fluence or energy $\alpha_{E_2}$ | Instrument | Observed Phenomenon | Reference                           |
|--------------------------------------------------|------------|---------------------|-------------------------------------|
| 2.45±0.15                                        | EIT 171,195 | EUV transient       | Krucker & Benz (1998)               |
| 2.30±0.30                                        | TRACE 171,195 | Nanoflares       | Parnell & Jupp (2000)               |
| 2.48±0.11                                        | TRACE 171,195 | Nanoflares       | Parnell & Jupp (2000)               |
| 2.31                                             | EIT 171,195 | EUV transient       | Benz & Krucker (2002)               |
| 2.38±0.09                                        |             |                     | Mean of 4 observations             |
| 2.33                                             |             |                     | Theoretical prediction              |
Table 4: Observed frequency distributions of thermal energies $E_3$ based on 3-D fractal model with isotropic line-of-sight depth $h = \sqrt{A}$.

| Powerlaw slope of $\alpha_{E_3}$ | Instrument | Observed Phenomenon | References |
|----------------------------------|------------|----------------------|------------|
| 1.55±0.05                        | Yohkoh     | Solar flares          | Shimizu (1995) |
| 1.90                             | SOHO/EIT 195 | EUV transient         | Berghmans et al. (1998) |
| 1.73±0.28                        | SOHO/EIT 195 | EUV transient         | Berghmans and Clette (1999) |
| 2.05±0.05                        | TRACE 171,195 | nanoflares            | Parnell and Jupp (2000) |
| 1.57±0.05                        | Yohkoh SXT/AlMg | nanoflares            | Aschwanden and Parnell (2002) |
| 1.41±0.09                        | Yohkoh SXT/AlMg | nanoflares            | Aschwanden and Parnell (2002) |
| 1.81±0.10                        | TRACE 195   | nanoflares            | Aschwanden and Parnell (2002) |
| 1.70±0.17                        | TRACE 195   | nanoflares            | Aschwanden and Parnell (2002) |
| 1.86±0.07                        | TRACE 171   | nanoflares            | Aschwanden and Parnell (2002) |
| 2.06±0.10                        | TRACE 171   | nanoflares            | Aschwanden and Parnell (2002) |
| 1.66                             | SOHO/EIT 195 | nanoflares            | Uritsky et al. (2007) |
| 1.79±0.01                        | AIA/SDO 171 A | coronal brightenings | Joulin et al. (2016) |
| 1.83±0.01                        | AIA/SDO 193 A | coronal brightenings | Joulin et al. (2016) |
| 1.88±0.01                        | AIA/SDO 211 A | coronal brightenings | Joulin et al. (2016) |
| 1.80±0.01                        | IRIS        | nanoflares            | Nhalil et al. (2020) |
| 2.07±0.02                        | IRIS        | nanoflares            | Nhalil et al. (2020) |
| 2.00±0.20                        | AIA/SDO     | flares                | Kawai and Imada (2022) |
| Outliers:                        |            |                      |            |
| (2.15±0.01)$^a$                  | AIA/SDO 131 A | coronal brightenings | Joulin et al. (2016) |
| (2.53±0.01)$^a$                  | AIA/SDO 335 A | coronal brightenings | Joulin et al. (2016) |
| (2.28±0.03)$^b$                  | AIA/SDO     | nanoflares            | Purkhart and Veronig (2022) |

1.80±0.18

1.80

Mean of 17 observations

Theoretical prediction

($^a$ No large events are detected in the 131 and 335 Å high-temperature bands during the time of observations, which causes a steeper power law slope (Joulin et al. 2016). ($^b$ High-energy events could have significant uncertainties since they may heavily depend on accurate event combinations between many pixels, one of the most challenging steps in the event detection algorithm (Purkhart and Veronig 2022).