BOSONIC FORMULA FOR LEVEL-RESTRICTED PATHS

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Abstract. We prove a bosonic formula for the generating function of level-restricted paths for the infinite families of affine Kac-Moody algebras. In affine type $A$ this yields an expression for the level-restricted generalized Kostka polynomials.

1. Introduction

Let $g$ be an affine Kac-Moody algebra, $V$ a $U_q(g)^+$-submodule of a finite direct sum $V'$ of irreducible integrable highest weight $U_q(g)$-modules, and $I$ the limit of the Demazure operator for an element $w$ of the Weyl group as $\ell(w) \to \infty$. The main theorem of this paper gives sufficient conditions on $V$ so that the formula

$$I \text{ch}(V) = \text{ch}(V')$$

(1.1)

holds, where $\text{ch}(V)$ is the character of $V$. When $V$ is the one-dimensional $U_q(g)^+$-module generated by the dominant integral weight $\Lambda$ then (1.1) is the Weyl-Kac character formula. The above result is well-known when $V$ is a union of Demazure modules for any Kac-Moody algebra $g$.

Let $g'$ be the derived subalgebra of $g$. Consider the $g'$-module $V$ given by a tensor product of finite-dimensional $U_q(g')$-modules that admit a crystal of level at most $\ell$, with the one-dimensional subspace generated by a highest weight vector of an irreducible integrable highest weight $U_q(g')$-module of level $\ell$. Such modules $V$ can be given the structure of a $U_q(g)^+$-module and as such, satisfy the above conditions. Then a special case of (1.1) is a bosonic formula for the $q$-enumeration of level-restricted inhomogeneous paths by the energy function. In type $A_{n-1}^{(1)}$ this formula was conjectured in [3], stated there as a $q$-analogue of the Goodman-Wenzl straightening algorithm for outer tensor products of irreducible modules over the type $A$ Hecke algebra at a root of unity [4]. In the isotypic component of the vacuum, the bosonic formula coincides with half of the bose-fermi conjecture in $[20, (9.2)]$.

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2. Notation

Most of the following notation is taken from ref. [7]. Let $X$ be a Dynkin diagram of affine type with vertices indexed by the set $I = \{0, 1, 2, \ldots, n\}$ as in [7], Cartan matrix $A = (a_{ij})_{i,j\in I}$, $g = g(A)$ the affine Kac-Moody algebra, and $h$ the Cartan subalgebra. Let $\{\alpha_i^\vee : i \in I\} \subset h$ and $\{\alpha_j : j \in I\} \subset h^*$ be the simple coroots and roots, which are linearly independent subsets that satisfy $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for $i,j \in I$ where $\langle \cdot, \cdot \rangle : h \otimes h^* \to \mathbb{C}$ is the natural pairing. Let $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the root lattice. Let the null root $\delta = \sum_{i \in I} a_i \alpha_i$ be the unique element of the positive cone $\bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ in $Q$, that generates the one-dimensional lattice $\{\beta \in Q | \langle \alpha_i^\vee, \beta \rangle = 0 \text{ for all } i \in I\}$. Let $K = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in h$ be the canonical central element, where the integers $a_i^\vee$ are the analogues of the integers $a_i$ for the dual algebra $g^\vee$ defined by the transpose $^tA$ of the Cartan matrix $A$. Let $d \in h$ (the degree derivation) be defined by the conditions $\langle d, \alpha_i \rangle = \delta_0$ where $\delta_i$ is the Kronecker delta; $d$ is well-defined up to a summand proportional to $K$. Then $\{\alpha_0^\vee, \ldots, \alpha_n^\vee, d\}$ is a basis of $h$. Let $\{\Lambda_0, \ldots, \Lambda_n, \delta\}$ be the dual basis of $h^*$; the elements $\{\Lambda_0, \ldots, \Lambda_n\}$ are called the fundamental weights. The weight lattice is defined by $P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \bigoplus \mathbb{Z} a_0^\vee$; in the usual definition the scalar $a_0^\vee$ is absent. The weight lattice contains the root lattice since $\alpha_j = \sum_{i \in I} a_{ij} \alpha_i$ for $j \in I$. Define $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i \bigoplus \mathbb{Z} a_0^\vee \delta$. Say that a weight $\Lambda \in P^+$ has level $\ell$ if $\ell = \langle K, \Lambda \rangle$.

Consider the standard symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $h^*$. Since $\{\alpha_0^\vee, \ldots, \alpha_n^\vee, \delta\}$ is a basis of $h^*$, this form is uniquely defined by $\langle \alpha_i^\vee, \alpha_j \rangle = \delta_0 a_{ij}^{-1}$ for $i,j \in I$, $(\alpha_i | \Lambda_0) = \delta_0 a_0^{-1}$ for $i \in I$ and $(\alpha_i | \Lambda_0) = 0$. This form induces an isomorphism $\nu : h \to h^*$ defined by $\nu(\alpha_i^\vee) = a_i a_i^\vee$ for $i \in I$ and $\nu(d) = a_0 \Lambda_0$. Also $\nu(K) = \delta$.

The Weyl group $W$ is the subgroup of $GL(h^*)$ generated by the simple reflections $r_i (i \in I)$ defined by

$$r_i(\beta) = \beta - \langle \alpha_i^\vee, \beta \rangle \alpha_i.$$  

The form $\langle \cdot, \cdot \rangle$ is $W$-invariant. Suppose $\alpha \in Q$ is a real root, that is, the $\alpha$-weight space of $g$ is nonzero and there is a simple root $a_i$ and a Weyl group element $w \in W$ such that $\alpha = w(a_i)$. Define $\alpha^\vee \in h$ by $w(\alpha_i^\vee)$. This is independent of the expression $\alpha = w(a_i)$. Define $r_\alpha \in W$ by

$$r_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha \quad \text{for } \beta \in h^*.$$  

Let $g'$ be the derived algebra of $g$, obtained by “omitting” the degree derivation $d$. Its weight lattice is $P_d = P/\mathbb{Z} \delta$. Denote the canonical projection $P \to P_d$ by $cl$. Write $\alpha_i^{cl} = cl(\alpha_i)$ and $\Lambda_i^{cl} = cl(\Lambda_i)$ for $i \in I$. The elements $\{\alpha_i^{cl} \mid i \in I\}$ are linearly dependent. Write $af : P_d \to P$ for the section of $cl$ given by $af(\Lambda_i^{cl}) = \Lambda_i$ for all $i \in I$. Write $P_d^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i^{cl}$. Define the level of $\mu \in P_d^+$ to be $\langle K, af(\mu) \rangle$.

Consider the Dynkin diagram $\overline{X}$ obtained by removing the vertex 0 from the diagram $X$, with corresponding Cartan matrix $\overline{A}$ indexed by the set $J = I - \{0\}$, and let $\overline{g} = g(\overline{A})$ be the simple Lie algebra. One has the inclusions $\overline{g} \subset g' \subset g$. Let $\{\overline{\alpha}_i : i \in J\}$ be the simple roots, $\{\overline{\Lambda}_i : i \in J\}$ the fundamental weights, and $\overline{P} = \bigoplus_{i \in J} \mathbb{Z} \overline{\alpha}_i$ the root lattice for $\overline{g}$. The weight lattice of $\overline{g}$ is $\overline{P} = \bigoplus_{i \in J} \mathbb{Z} \overline{\alpha}_i$ and $\overline{P} \cong P_d/\mathbb{Z} \Lambda_0$. The image of $\Lambda \in P$ into $\overline{P}$ is denoted by $\overline{\Lambda}$. We shall use the signed of the natural projection $P_d \to \overline{P}$ given by the map $\overline{P} \to P_d$ that sends $\overline{\alpha}_i \mapsto \Lambda_i^{cl} - \Lambda_0^{cl}$ for $i \in J$. By abuse of notation, for $\Lambda \in P$, $\overline{\Lambda}$ shall also denote the image of the element $\overline{\Lambda}$ under the lifting map $\overline{P} \to P$ specified above.


Let $\mathcal{P}^+ = \bigoplus_{i \in J} \mathbb{Z}_{\geq 0} \Lambda_i$. For $\lambda \in \mathcal{P}^+$, denote by $V(\lambda)$ the irreducible integrable highest weight $U_q(\mathfrak{g})\text{-module of highest weight } \lambda$.

Let $\theta = \delta - a_0a_0 = \sum_{i \in J} a_i \alpha_i \in \mathcal{Q}$. One has the formulas $(\theta/\theta) = 2a_0$, $\theta = a_0\nu(\theta^\vee)$, and $\alpha_0^\vee = K - a_0\theta^\vee$. Observe that

$$\text{cl}(\alpha_0) = -a_0^{-1} \sum_{i \in J} a_i \alpha_i^c = -\text{cl}(\nu(\theta^\vee)).$$

For $\Lambda \in P^+$ let $\mathbb{V}(\Lambda)$ be the irreducible integral highest weight module of highest weight $\Lambda$ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$, $\mathbb{B}(\Lambda)$ the crystal base of $\mathbb{V}(\Lambda)$, and $u_\Lambda \in \mathbb{B}(\Lambda)$ the highest weight vector.

By restriction from $U_q(\mathfrak{g})$ to $U_q(\mathfrak{g}')$, the module $\mathbb{V}(\Lambda)$ is an irreducible integral highest weight module for $U_q(\mathfrak{g}')$ of highest weight $\text{cl}(\Lambda)$, with crystal $\mathbb{B}(\Lambda)$ that is $P_{ij}$-weighted by composing the weight function $\mathbb{B}(\Lambda) \rightarrow P$ with the projection $\text{cl}$. Conversely, any integrable irreducible highest weight $U_q(\mathfrak{g}')\text{-module can be obtained this way.}$

3. Short review of affine crystal theory

3.1. Crystals. A $P$-weighted $I$-crystal $B$ is a colored graph with vertices indexed by $b \in B$, directed edges colored by $i \in I$, and a weight function $\text{wt} : B \rightarrow P$, satisfying the axioms below. First some notation is required. Denote an edge from $b$ to $b'$ colored $i$, by $b' = f_i(b)$ or equivalently $b = e_i(b')$. Write $\phi_i(b)$ (resp. $\epsilon_i(b)$) for the maximum index $m$ such that $f_i^m(b)$ (resp. $e_i^m(b)$) is defined.

1. If $b' = f_i(b)$ then $\text{wt}(b') = \text{wt}(b) - \alpha_i$.
2. $\phi_i(b) - \epsilon_i(b) = \langle \alpha_i^\vee, \text{wt}(b) \rangle$.

An element $u \in B$ is a highest weight vector if $\epsilon_i(u)$ is undefined for all $i \in I$. The $i$-string of $b \in B$ consists of all elements $e_i^m(b)$ ($0 \leq m \leq \epsilon_i(b)$) and $f_i^m(b)$ ($0 \leq m \leq \phi_i(b)$). The nondominant part of the $i$-string is comprised of all elements which emit $e_i$.

We also define the crystal reflection operator $s_i : B \rightarrow B$ by

$$s_i(b) = \begin{cases} f_i^{\phi_i(b) - \epsilon_i(b)}(b) & \text{if } \phi_i(b) > \epsilon_i(b) \\ b & \text{if } \phi_i(b) = \epsilon_i(b) \\ e_i^{\epsilon_i(b) - \phi_i(b)}(b) & \text{if } \phi_i(b) < \epsilon_i(b). \end{cases}$$

It is obvious that $s_i$ is an involution. Observe that

$$\text{wt}(s_i(b)) = r_i \text{wt}(b) = \text{wt}(b) - \langle \alpha_i^\vee, \text{wt}(b) \rangle \alpha_i.$$

Define the notation $\phi(b) = \sum_{i \in I} \phi_i(b)\Lambda_i$ and $\epsilon(b) = \sum_{i \in I} \epsilon_i(b)\Lambda_i$.

If a $U_q(\mathfrak{g})\text{-module (resp. } U_q(\mathfrak{g}')\text{-module, resp. } U_q(\mathfrak{g})\text{-module) has a crystal base then the latter is naturally a } P\text{-weighted (resp. } P_{ij}\text{-weighted, resp. } \mathcal{P}\text{-weighted) } I\text{-crystal (resp. } I\text{-crystal, resp. } J\text{-crystal).}$

3.2. Tensor products. Given crystals $B_1$ and $B_2$, contrary to the literature (but consistent with the Robinson-Schensted-Knuth correspondence in type A), define the following crystal structure on the tensor product $B_2 \otimes B_1$. The elements are denoted $b_2 \otimes b_1$ for $b_i \in B_i$ ($i \in \{1, 2\}$) and one defines

$$\phi_i(b_2 \otimes b_1) = \phi_i(b_2) + \max(0, \phi_i(b_1) - \epsilon_i(b_2))$$

$$\epsilon_i(b_2 \otimes b_1) = \epsilon_i(b_1) + \max(0, -\phi_i(b_1) + \epsilon_i(b_2)).$$
When \( \phi_i(b_2 \otimes b_1) > 0 \) (resp. \( \epsilon_i(b_2 \otimes b_1) > 0 \)) one defines
\[
f_i(b_2 \otimes b_1) = \begin{cases} 
  b_2 \otimes f_i(b_1) & \text{if } \phi_i(b_1) > \epsilon_i(b_2) \\
  f_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) \leq \epsilon_i(b_2)
\end{cases}
\]
and respectively
\[
e_i(b_2 \otimes b_1) = \begin{cases} 
  b_2 \otimes e_i(b_1) & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2) \\
  e_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) < \epsilon_i(b_2).
\end{cases}
\]
An element of a tensor product of crystals is called a path.

3.3. **Energy function.** The definitions here follow [10]. Suppose that \( B_1 \) and \( B_2 \) are crystals of finite-dimensional \( U_q(\mathfrak{g}') \)-modules such that \( B_2 \otimes B_1 \) is connected. Then there is an isomorphism of \( P_{cl} \)-weighted \( \mathcal{I} \)-crystals \( B_2 \otimes B_1 \cong B_1 \otimes B_2 \). This is called the local isomorphism. Let the image of \( b_2 \otimes b_1 \in B_2 \otimes B_1 \) under this isomorphism be denoted \( b'_1 \otimes b'_2 \). Then there is a unique (up to a global additive constant) map \( H : B_2 \otimes B_1 \to \mathbb{Z} \) such that
\[
H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} 
-1 & \text{if } i = 0, \ e_0(b_2 \otimes b_1) = e_0(b_2) \otimes b_1 \\
& \text{and } e_0(b'_1 \otimes b'_2) = e_0(b'_1) \otimes b'_2, \\
1 & \text{if } i = 0, \ e_0(b_2 \otimes b_1) = b_2 \otimes e_0(b_1) \\
& \text{and } e_0(b'_1 \otimes b'_2) = b'_1 \otimes e_0(b'_2), \\
0 & \text{otherwise.}
\end{cases}
\]
This map is called the local energy function.

Consider \( B = B_L \otimes \cdots \otimes B_1 \) with \( B_j \) the crystal of a finite-dimensional \( U_q(\mathfrak{g}') \)-module for \( 1 \leq j \leq L \). Assume that for all \( 1 \leq i < j \leq L \), \( B_j \otimes B_i \) is a connected \( P_{cl} \)-weighted \( \mathcal{I} \)-crystal. Given \( b = b_L \otimes \cdots \otimes b_1 \in B \), denote by \( b^{(i+1)} \) the \((i+1)\)-th tensor factor in the image of \( b \) under the composition of local isomorphisms that switch \( B_j \) with \( B_k \) as \( k \) goes from \( j - 1 \) down to \( i + 1 \). Then define the energy function
\[
E_B(b) = \sum_{1 \leq i < j \leq L} H_{j,i}(b^{(i+1)} \otimes b_i)
\]
where \( H_{j,i} : B_j \otimes B_i \to \mathbb{Z} \) is the local energy function. It satisfies the following property.

**Lemma 1.** [10, Prop. 1.1] Suppose \( i \in I, b \in B \) and \( e_i(b) \) is defined. If \( i \neq 0 \) then \( E_B(e_i(b)) = E_B(b) \). If \( i = 0 \) and \( b \) has the property that for any of its images \( b' = b'_L \otimes \cdots \otimes b'_1 \) under a composition of local isomorphisms, \( e_0(b') = b'_L \otimes \cdots \otimes e_0(b'_k) \otimes \cdots \otimes b'_1 \) with \( k \neq 1 \), then \( E_B(e_0(b)) = E_B(b) - 1 \).

3.4. **Classically restricted paths.** Say that \( b \in B := B_L \otimes \cdots \otimes B_1 \) is classically restricted if \( b \) is a \( \mathfrak{g}' \)-highest weight vector, that is, \( e_i(b) \) is undefined for all \( i \in J \).

For \( \lambda \in \overline{\mathcal{P}}^+ \) denote by \( \mathcal{P}(B, \lambda) \) the set of classically restricted \( b \in B \) of weight \( \lambda \). Define the polynomial
\[
K(B, \lambda)(q) = \sum_{b \in \mathcal{P}(B, \lambda)} q^{E_B(b)}
\]
where \( E_B \) is the energy function on \( B \). For \( \mathfrak{g} \) of type \( A_{n-1}^{(1)} \) \( K(B, \lambda)(q) \) is the generalized Kostka polynomial [10, 13, 20].
3.5. Almost perfect crystals. Let $B$ be the crystal of a finite-dimensional $U_q(\mathfrak{g})$-module. Say that $B$ is almost perfect of level $\ell$ if it satisfies the following weakening of the definition of a perfect crystal [7, Def. 4.6.1]:

1. $B \otimes B$ is connected.
2. There is a $\Lambda' \in P_{cl}$ such that there is a unique $b' \in B$ such that $\text{wt}(b') = \Lambda'$ and for every $b \in B$, $\text{wt}(b) \in \Lambda' + \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i$.
3. For every $b \in B$, $(K, \epsilon(b)) \geq \ell$.
4. For every $\Lambda \in P_{cl}^+$ of level $\ell$, there is a $b, b' \in B$ such that $\epsilon(b) = \phi(b') = \Lambda$.

$B$ is said to be perfect if the elements $b$ and $b'$ in item 4 are unique.

3.6. Level restricted paths. From now on, fix a positive integer $\ell$ (the level).

For $1 \leq j \leq L$ let $B_j$ be the crystal of a finite-dimensional $U_q(\mathfrak{g})$-module, that is almost perfect of level at most $\ell$.

Let $B = B_L \otimes \cdots \otimes B_1$, $\Lambda, \Lambda' \in P_{cl}^+$ weights of level $\ell$, and $\mathcal{P}(B, \Lambda, \Lambda')$ the set of paths $b = b_L \otimes \cdots \otimes b_1 \in B$ such that $b \otimes u_\Lambda \in B \otimes \mathbb{B}(\Lambda)$ is a highest weight vector of weight $\Lambda'$.

In the special case that $\Lambda = \ell \Lambda_0$, the elements of $\mathcal{P}(B, \Lambda, \Lambda')$ are called the level-$\ell$ restricted paths of weight $\Lambda'$.

**Theorem 2.** [7] [3] Appendix A. Let $\mathfrak{g}$ be an affine Kac-Moody algebra in one of the infinite families. Let $B$ be the tensor product of crystals of finite-dimensional $U_q(\mathfrak{g})$-modules, that are almost perfect of level at most $\ell$, and $\Lambda \in P_{cl}^+$ a weight of level $\ell$. Then there is an isomorphism of $P_{cl}$-weighted $I$-crystals

$$B \otimes \mathbb{B}(\Lambda) \cong \bigoplus_{\Lambda' \in P_{cl}^+} \bigoplus_{b \in \mathcal{P}(B, \Lambda, \Lambda')} \mathbb{B}(\Lambda')$$

where $\Lambda'$ is of level $\ell$.

This isomorphism of $P_{cl}$-weighted crystals can be lifted to one of $P$-weighted crystals by specifying an integer multiple of $a_0^{-1} \delta$ for each highest weight vector in $B \otimes \mathbb{B}(\Lambda)$. However for our purposes this should be done in a way that extends the definition of the energy function for $B$. To this end, choose a perfect crystal $B_0$ of level $\ell$, and assume that for all $0 \leq i < j \leq L$, $B_j \otimes B_i$ is connected. Let $b_0 \in B_0$ be the unique element such that $\phi(b_0) = \Lambda$. Define the energy function $E : B \rightarrow \mathbb{Z}$ by $E(b) = E_{B, B_0}(b \otimes b_0)$ where $E_{B, B_0} : B \otimes B_0 \rightarrow \mathbb{Z}$ is the energy function defined in (3.3). For $b \in \mathcal{P}(B, \Lambda, \Lambda')$, define an affine weight function $\text{wt}(b \otimes u_\Lambda) = a(\Lambda') - E(b)a_0^{-1} \delta$. This defines the $P$-weight of every highest weight vector in $B \otimes \mathbb{B}(\Lambda)$ and hence a $P$-weight function for all of $B \otimes \mathbb{B}(\Lambda)$.

Then one has the following $P$-weighted analogue of (3.4):

$$B \otimes \mathbb{B}(a(\Lambda)) \cong \bigoplus_{\Lambda' \in P_{cl}^+} \bigoplus_{b \in \mathcal{P}(B, \Lambda, \Lambda')} \mathbb{B}(\text{wt}(b \otimes u_\Lambda))$$

where $\Lambda'$ is of level $\ell$. This decomposition can be described by the polynomial

$$K(B, \Lambda, \Lambda', B_0)(q) = \sum_{b \in \mathcal{P}(B, \Lambda, \Lambda')} q^{E(b)}.$$

Our goal is to prove a formula for the polynomial $K(B, \Lambda, \Lambda', B_0)(q)$. 
Let $J$ be the antisymmetrizer

$$J = \sum_{w \in W} (-1)^w w.$$ 

Write

$$R = \prod_{\alpha \in \Delta_+} (1 - \exp(-\alpha))^{\text{mult}(\alpha)}$$

where $\Delta_+$ is the set of positive roots of $\mathfrak{g}$ and $\text{mult}(\alpha)$ is the dimension of the $\alpha$-weight space in $\mathfrak{g}$.

Let $\rho \in P^+$ be the unique weight defined by $\langle \alpha_i^\vee, \rho \rangle = 1$ for all $i \in I$ and $\langle d, \rho \rangle = 0$. It satisfies $\langle \theta^\vee, \rho \rangle = a - 1$ and $\langle K - \alpha_i^\vee, \rho \rangle = a - 1$ where $h^\vee = \sum_{i \in I} a_i^\vee$ is the dual Coxeter number. Define the operator

$$\Pi(p) = R^{-1} e^{-\rho} J(e^\rho p).$$

where $R^{-1}$ makes sense by expanding the reciprocals of the factors of $R$ in geometric series. The computation is defined in a suitable completion of $\mathbb{Z}[P]$. One has $\Pi(e^\Lambda) = \text{ch} V(\Lambda)$ for all $\Lambda \in P^+$, which is the Weyl-Kac character formula [7, Theorem 10.4].

**Theorem 3.** Let $\mathfrak{g}$ be an affine Kac-Moody algebra in one of the infinite families, $B'$ the crystal of a finite direct sum of irreducible integrable highest weight $U_q(\mathfrak{g})$-modules and $B \subset B'$ a subset such that:

1. $B$ is closed under $e_i$ for all $i \in I$.
2. $B'$ is generated by $B$.
3. For all $b \in B$ and $i \in I$, if $\epsilon_i(b) > 0$ then the $i$-string of $b$ in $B'$ is contained in $B$.

Then

$$\Pi \text{ ch}(B) = \text{ch}(B').$$

**Proof.** Without loss of generality it may be assumed that $B' = \mathcal{B}(\Lambda)$ for some $\Lambda \in P^+$. Multiplying both sides of (4.1) by $Re^\rho$, one obtains

$$\sum_{(w, b) \in W \times B} (-1)^w w(e^{\text{wt}(b) + \rho}) = \sum_{w \in W} (-1)^w w(e^{\Lambda + \rho}).$$

Observe that both sides are $W$-alternating. The $W$-alternants have a basis given by $J(\Lambda + \rho)$ where $\Lambda \in P^+$. Taking the coefficient of $e^{\Lambda + \rho}$ on both sides,

$$\sum_{(w, b) \in S} (-1)^w = 1$$

where $S$ is the set of pairs $(w, b) \in W \times B$ such that

$$\text{wt}(b) = w^{-1}(\Lambda + \rho) - \rho.$$ 

Observe that if $(w, b) \in S$ is such that $b$ is a highest weight vector, then $w = 1$ and $b = u_\Lambda$, for both of the regular dominant weights $\text{wt}(b) + \rho$ and $\Lambda + \rho$ are in the same $W$-orbit and hence must be equal. Conditions 1 and 2 ensure that $u_\Lambda \in B$. Let $S' = S - \{(1, u_\Lambda)\}$. It is enough to show that there is an involution $\Phi : S' \to S'$ with no fixed points, such that if $\Phi(w, b) = (w', b')$ then $w$ and $w'$ have opposite sign. In this case $\Phi$ is said to be sign-reversing. Let $S_i$ be the set of pairs $(w, b) \in S'$
such that $\epsilon_i(b) > 0$. Define the map $\Phi_i : S_i \to S_i$ by $\Phi_i(w, b) = (wr_i, s_ie_i(b))$. Note that $s_ie_i(b) \in B$ by condition 3. The condition (4.3) for $\Phi_i(w, b)$ is

$$(wr_i)^{-1}(\Lambda + \rho) - \rho = r_iw^{-1}(\Lambda + \rho) - r_i\rho + r_i\rho - \rho$$

$$= r_i(w^{-1}(\Lambda + \rho) - \rho) - (\alpha_i^\vee, \rho)\alpha_i$$

$$= r_i(\text{wt}(b)) - \alpha_i = \text{wt}(f_is_i(b)) = \text{wt}(s_ie_i(b)).$$

Since $s_ie_i(b) = f_is_i(b)$, $\epsilon_i(s_ie_i(b)) > 0$, so that $(wr_i, s_ie_i(b)) \in S_i$. This shows that $\Phi_i$ is well-defined. It follows directly from the definitions that $\Phi_i$ is a sign-reversing involution.

Since $S' = \cup_i S_i$ it suffices to define a global involutive choice of the canceling root direction for each pair $(w, b) \in S'$, that is, a function $v : S' \to I$ such that if $v(w, b) = i$ then

(V1) $(w, b) \in S_i$.

(V2) $v(wr_i, s_ie_i(b)) = i$.

Let $\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_j}$ be an expression of $\Lambda$ as a sum of fundamental weights. By [6, Lemma 8.3.1], $B(\Lambda)$ is isomorphic to the full subcrystal of $B(\Lambda_{i_1}) \otimes \cdots \otimes B(\Lambda_{i_j})$ generated by $u_{\Lambda_{i_j}} \otimes \cdots \otimes u_{\Lambda_{i_1}}$.

Given $(w, b) \in S'$, let $b_l \otimes \cdots \otimes b_1$ be the image of $b$ in the above tensor product of crystals of modules of fundamental highest weight. Let $r$ be minimal such that $b_r \otimes b_{r-1} \otimes \cdots \otimes b_1$ is not a highest weight vector. Then $b_{r-1} \otimes \cdots \otimes b_1$ is a highest weight vector, say of weight $\Lambda'$. Let $B$ be a perfect crystal of the same level as $\Lambda_{i_r}$. Given any $L > 0$, the theory of perfect crystals [6, Section 4.5] gives an isomorphism of $P$-weighted crystals

$$B(\Lambda_{i_r}) \cong B^{\otimes L} \otimes B(\Lambda_j)$$

where $j$ is determined by $i_r$ and $L$ and $B^{\otimes L}$ is $P$-weighted using the energy function.

Let $b_r \in B(\Lambda_{i_r})$ have image $p_{-1} \otimes \cdots \otimes p_{-L} \otimes u'$ where $u' \in B(\Lambda_j)$. Assume that $L$ is large enough so that $u' = u_{\Lambda_j}$. If one takes the image of $b_r$ in such a tensor product for $L' > L$ the tensor factors $p_{-1}$ through $p_{-L}$ do not change.

Let $k$ be minimal such that $p_k \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is not a highest weight vector. Observe that $k$ is independent of $L$ as long as $L$ is big enough. Then $p_{k-1} \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is a highest weight vector, say of weight $\Lambda''$.

So $p_k \in B$ is such that $\epsilon_i(p_k) > \langle \alpha_i^\vee, \Lambda'' \rangle$ for some $i \in I$; let $I'$ be the set of such $i \in I$.

Fix an $i \in I'$. Consider the same constructions for $b' = s_i e_i(b)$. Let $b'_r \otimes \cdots \otimes b'_1$ be the image of $b'$ in the above tensor product of irreducible crystals of fundamental highest weights. Then $b'_{r-1} \otimes \cdots \otimes b'_1 = b_{r-1} \otimes \cdots \otimes b_1$ and $b'_r \otimes \cdots \otimes b'_1$ is not a highest weight vector; in particular it admits $\epsilon_i$. Take $L$ large enough so that the image of $b'_r$ in $B^{\otimes L} \otimes B(\Lambda_j)$ also has the form $p'_{-1} \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j}$. Then $p_{k-1} \otimes \cdots \otimes p_{-L} = p'_{k-1} \otimes \cdots \otimes p'_{-L}$ and $p'_k \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ admits $\epsilon_i$.

The level of the fundamental weight $\Lambda_i$ is $a_i^\vee$. For the affine algebras $A_n^{(1)}$ and $C_n^{(1)}$, $a_i^\vee = 1$ for all $i \in I$. For all others $1 \leq a_i^\vee \leq 2$. The theorem now follows from Lemma 4 below, applied with the dominant integral weight $\Lambda''$. \[ \square \]

**Lemma 4.** For the affine Kac-Moody algebra $\mathfrak{g}$ in one of the infinite families, there exist perfect crystals $B$ of level one and two (the latter case only for $\mathfrak{g} \neq A_n^{(1)}, C_n^{(1)}$) having the following property. Let $\Lambda$ be a dominant integral weight of positive level,
the set of elements $b_1 \in B$ such that $b_1 \otimes u_\Lambda$ is not a highest weight vector. Then there is a map $v : S \to \bar{I}$ such that if $v(b_1) = i$ then

1. $e_i(b_1 \otimes u_\Lambda) > 0$.
2. For any $b_L, \ldots, b_2 \in B$, writing $b'_L \otimes \cdots \otimes b'_2 \otimes b'_1 \otimes u_\Lambda = s_i e_i (b_L \otimes \cdots \otimes b_2 \otimes b_1 \otimes u_\Lambda)$, one has $b'_1 \in S$ and $v(b'_1) = i$.

**Proof.** For the involutive property 2, it is sufficient that $v$ is constant on the non-dominant part of every string. Hence one only needs to consider

(4.4) $b_1$ that are on the nondominant part of at least two strings of length $\geq 2$.

Perfect crystals of level one for $A_n^{(1)}$ ($n \geq 1$), $B_n^{(1)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$) and $D_{n+1}^{(2)}$ ($n \geq 2$) are listed in Table 1 (see [9, Section 6]).

Note that there are no elements satisfying (4.4). This guarantees the existence of the map $v$ with the desired properties.

The crystal $B(2\Lambda_1) \oplus B(0)$ is a level one perfect crystal for $C_n^{(1)}$ ($n \geq 2$) [8].

The crystal graph corresponding to the integrable highest weight module $V(\Lambda_1)$ of $U_q(C_n)$ is given by [14, (4.2.4)]

There are the following strings of length greater than one

(4.5a)

(4.5b)

Note that none of the elements satisfies (4.4).

For type $A_{2n-1}^{(2)}$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.6 and 6.7]. The elements are given by $x \, y$ with $x \leq y$ and $x, y \in \{1 < 2 < \cdots < n < \bar{n} < \cdots < 2 < 1\}$. The action of $f_i$ for $i = 1, 2, \ldots, n$ is the same as for the above $C_n^{(1)}$ crystal of level one, and $f_0 = \sigma \circ f_1 \circ \sigma$ where $\sigma$ is the involution that exchanges 1 and $\bar{1}$ (with appropriate reorderings).
The strings of length greater than one are the same as in (4.5a) and (4.5b). In addition there are the following 0-strings of length 2

\begin{align*}
&\begin{array}{c}
\begin{array}{c}
1 & 2 & \cdots & n-1 & n & n-1 & 2 & 1
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
1 & 2 & \cdots & n-1 & n & n-1 & 2 & 1
\end{array}
\end{array}
\end{align*}

The only elements fulfilling (4.4) are \(1 \, 1\), \(1 \, 2\), \(2 \, 1\), and \(2 \, 2\) which belong to a 0-string and a 1-string of length two. It can be checked that setting \(v(b) = 0\) for \(b\) one of these four elements guarantees the involutive condition of \(v\).
For type $B^{(1)}_n$ the crystal $B(2\Lambda_1)$ is perfect of level 2 \cite[Sec. 1.7 and 6.8]{10}. It consists of the elements $x, y$ with $x \leq y$ and $x, y \in \{1 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}\}$; $x = y = 0$ is excluded. The action of $f_i$ for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $B^{(1)}_n$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where $\sigma$ is the involution that exchanges 1 and $\overline{1}$ (with appropriate reorderings).

The strings of length greater than one are those of equations (4.5a) and (4.6) and in addition the following $n$-string of length four

\begin{equation}
\begin{array}{llllll}
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\end{array}
\end{equation}

(4.7)

The same four elements as for $A^{(2)}_{2n-1}$ satisfy (4.4) and again setting $v(b) = 0$ for these ensures the involutive property of $v$.

For type $D^{(1)}_n$ the crystal $B(2\Lambda_1)$ is perfect of level 2 \cite[Sec. 1.8 and 6.9]{10}. It consists of the elements $x, y$ with $x \leq y$ and $x, y \in \{1 < \cdots < 2 < \cdots < n < 0 < \cdots < 1\}$, the cases $x = n, y = 0$ and $x = 0, y = n$ being excluded. The action of $f_i$ for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D^{(1)}_n$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where $\sigma$ is the involution that exchanges 1 and $\overline{1}$ (with appropriate reorderings).

Again the strings of length greater than one are the same as in equations (4.5a) and (4.6) plus the following $n$-strings

\begin{equation}
\begin{array}{llllll}
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\text{n} & \text{n} & n & n & \text{n} & \text{n} \\
\end{array}
\end{equation}

In addition to the four elements $1, 1, 2, 2$ also the elements $\overline{n}, n, \overline{n}, n$, and $\overline{n}, n, \overline{n}, n$ satisfy (4.4). The latter ones are contained in an $(n-1)$-string and an $n$-string. Setting $v(b) = 0$ for the first four elements and $v(b) = n$ for the last four elements ensures the involutive property of $v$.

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $D^{(2)}_{n+1}$ \cite[Sections 1.9 and 6.10]{10}. The elements of this crystal are $\emptyset$, $\overline{1}$, and $x, y$ with $x, y \in \{1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}\}$ and $x \leq y$; $x = y = 0$ is excluded. The action of $f_i$ for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D^{(2)}_{n+1}$ as given in Table 1, and the action of $f_0$ is given by

\begin{align*}
\emptyset &\rightarrow 0 \\
\overline{1} &\rightarrow 1 \\
\emptyset &\rightarrow \emptyset \\
\overline{1} &\rightarrow \emptyset \\
\overline{1} &\rightarrow \overline{1} \\
\overline{1} &\rightarrow \emptyset \\
\end{align*}

for $x \neq \overline{1}$

\begin{align*}
\emptyset &\rightarrow 1 \\
\overline{1} &\rightarrow 1 \\
\emptyset &\rightarrow \emptyset \\
\overline{1} &\rightarrow \overline{1} \\
\overline{1} &\rightarrow \emptyset \\
\end{align*}

and undefined otherwise.
Remark 7. Observe that even without the extra hypothesis on the action of $A^{(2)}_{\infty}$, there are no elements with property (4.4).

(4.9) \[
\begin{array}{ccccccc}
\tau & \uparrow & 0 & \tau & 0 & \varnothing & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]

(4.10) There are no elements with property (4.4).

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $A^{(2)}_{\infty}$ [11, Sec. 1.10 and 6.11]. The elements of this crystal are $\emptyset, \uparrow$, and $\tau \uparrow$ with $x, y \in \{1 < 2 < \cdots < n < \pi < \cdots < \bar{1}\}$ and $x \leq y$. The action of $f_i$ for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $A^{(2)}_{\infty}$ as given in Table 4, and the action of $f_0$ is the same as in (1.8).

The strings of length greater than one are as in (4.5a) for $n \geq 2$, (4.5b) and (4.9). Again there are no elements with property (4.4). \qed

Remark 5. Suppose $g$ is of type $A^{(1)}_{\infty-1}$ in Lemma 4. The function $v$ amounts to a canonical choice of a simple root $i$ among those such that the given element admits $e_i$. Consider $b \in B(\Lambda_r)$ such that $b \neq u_{\Lambda_r}$. In addition to the realization of the crystal $B(\Lambda_r)$ by the space of homogeneous paths using the crystal given in the proof of Lemma 4, one may also consider the realization in [2] by $n$-regular partitions. Suppose $\lambda$ is the partition corresponding to $b$. Then up to the Dynkin diagram automorphism that sends $r + i$ to $r - i$ modulo $n$, the choice of violation $v$ corresponds to the corner cell of $\lambda$ that is in the rightmost column of $\lambda$. This choice of corner cell is used in [18] to define the smallest Demazure crystal of $B(\Lambda_r)$ containing $b$.

5. INHOMOGENEOUS PATHS

Theorem 6. Let $g$ be as in Theorem 4, and $B$, $\Lambda$, and $B_0$ be as in (3.5). Suppose in addition that for all $1 \leq j \leq L$ and $b \in B_j$, if $b \otimes b_0 \mapsto b'_0 \otimes b'$ under the local isomorphism $B_j \otimes B_0 \to B_0 \otimes B_j$ and $e_0(b \otimes b_0) = e_0(b) \otimes b_0$ then $e_0(b'_0 \otimes b') = e_0(b'_0) \otimes b'$. Then

\[
\Pi (\text{ch}(B \otimes u_{\Lambda})) = \text{ch}(B \otimes B(\Lambda)).
\]

Proof. It is enough to verify the hypotheses of Theorem 4, applied to $B \otimes u_{\Lambda} \subset B \otimes B(\Lambda)$. $B \otimes B(\Lambda)$ is isomorphic to a direct sum of irreducible integrable highest weight modules by Theorem 2. $B \otimes u_{\Lambda}$ is obviously closed under the $e_i$. It follows from [15, Lemma 1] that $B \otimes u_{\Lambda}$ generates $B \otimes B(\Lambda)$. To check the third condition of Theorem 2, let $b \in B$ and $i \in I$ be such that $e_i(b \otimes u_{\Lambda}) \neq 0$. Then $e_i(b) \otimes \phi_i(u_{\Lambda}) = (\alpha_i^\vee, \Lambda)$. This implies that the $i$-string of $b \otimes u_{\Lambda}$ inside $B \otimes B(\Lambda)$, consists of vectors of the form $b' \otimes u_{\Lambda}$ where $b' \in B$.

Finally, Lemma 4 with $B$ replaced by $B \otimes B_0$ guarantees that the affine weight function on $B \otimes B(\Lambda)$ determined by its value on highest weight vectors, agrees on the subset $B \otimes u_{\Lambda}$ with the function $\text{wt}(b) = \lambda(b) - E_{B_0, B_0}(b \otimes b_0)\alpha_0^{-1}\delta$ where $\lambda' : B \to P_d$ is the original weight function. \qed

Remark 7. Observe that even without the extra hypothesis on the action of $e_0$ in Theorem 4 one obtains a bosonic formula. The extra condition is only needed to show that the energy function $b \mapsto E_{B_0, B_0}(b \otimes b_0)\alpha_0^{-1}\delta$ gives rise to the correct affine
weight for all elements of the form $b \otimes u_\Lambda$ and not just on the highest weight vectors. Perhaps this extra condition is always a consequence of the other hypotheses.

Now the formula (5.1) is written more explicitly. Let $m \in \mathbb{Z}$ and $\Lambda, \Lambda' \in af(P_{cl}^+)$ be of level $\ell$. A formula equivalent to (5.1) is obtained by taking the coefficient of $\text{ch} \check{V}(\Lambda' - ma_0^{-1}\delta)$ on both sides:

$$[q^m]K(B, \Lambda, \Lambda', B_0)(q) = \sum_{(w, b) \in S} (-1)^w$$

where $S$ is the set of pairs $(w, b) \in W \times B$ such that

$$w^{-1}(\Lambda' + \rho) - ma_0^{-1}\delta - \rho = \text{wt}(b \otimes u_\Lambda).$$

Let $M$ be the sublattice of $\overline{\mathfrak{P}}$ given by the image under $\nu$ of the $\mathbb{Z}$-span of the orbit $\overline{W}\mathfrak{h}^\vee$. Let $T \subset GL(\mathfrak{h}^\vee)$ be the group of translations by the elements of $M$, where $t_\alpha \in T$ is translation by $\alpha \in M$. Then $W \cong T \times \overline{W}$ and $r_0 = t_{\nu(\delta^\vee)}r_\theta$. For $\alpha \in M$ and $\Lambda \in P$ of level $\ell$, one has $[3, (6.5.2)]$

$$t_\alpha(\Lambda) = \Lambda + \ell\alpha - ((\Lambda|\alpha) + \frac{1}{2}|\alpha|^2\ell)\delta.$$  

The action of $\tau \in \overline{W}$ on the level $\ell$ weight $\Lambda$ is given by

$$\tau(\Lambda) = \tau(\overline{\Lambda} + \ell\Lambda_0) = \tau(\overline{\Lambda}) + \ell\Lambda_0.$$  

Now $\rho = h^\vee\Lambda_0 + \overline{\rho}$ where $h^\vee$ is the dual Coxeter number and $\overline{\rho}$ is the half-sum of the positive roots in $\overline{\mathfrak{g}}$.

Recall that $\overline{W}$ leaves $\delta$ invariant. In (5.2) write $w = t_\alpha \tau$ where $\tau \in \overline{W}$ and $\alpha \in M$, obtaining

$$\text{wt}(b \otimes u_\Lambda) = \tau^{-1}t_{-\alpha}(\Lambda' + \rho) - ma_0^{-1}\delta - \rho$$

$$= - ma_0^{-1}\delta - \rho + \tau^{-1}\{\Lambda' + \rho - (\ell + h^\vee)\alpha - \{((\Lambda' + \rho)|\alpha) + \frac{1}{2}|\alpha|^2(\ell + h^\vee)\}\}$$

$$= \ell\Lambda_0 - \overline{\rho} + \tau^{-1}(\overline{\Lambda} + \overline{\rho} - (\ell + h^\vee)\alpha)$$

$$+ \{-ma_0^{-1} + (\overline{\Lambda} + \overline{\rho}|\alpha) - \frac{1}{2}|\alpha|^2(\ell + h^\vee)\}\delta$$

Since both sides are weights of level $\ell$, by equating coefficients of $\delta$ and projections into $\overline{\mathfrak{P}}$, one obtains the equivalent conditions

$$\text{wt}(b) = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda} - (\ell + h^\vee)\alpha + \overline{\rho})$$

and

$$a_0^{-1}E(b) = a_0^{-1}m - (\overline{\Lambda} + \overline{\rho}|\alpha) + \frac{1}{2}|\alpha|^2(\ell + h^\vee).$$

Therefore one has the equality

$$K(B, \Lambda, \Lambda', B_0)(q) = \sum_{\tau \in \overline{W}} \sum_{\alpha \in M} \sum_{b \in B} (-1)^\tau q^{E(b) + a_0(\overline{\Lambda} + \overline{\rho}|\alpha) - \frac{1}{2}a_0|\alpha|^2(\ell + h^\vee)}$$

where $b \in B$ satisfies

$$\text{wt}(b) = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda} - (\ell + h^\vee)\alpha + \overline{\rho}).$$
6. Type A

6.1. Conjecture of [3]. For simplicity let us assume that $\mathfrak{g}$ is of untwisted affine type, where $a_0 = 1$ and $(\rho|\theta) = h^\vee - 1$ [3, Ex. 6.2].

Let $\Lambda \in \mathcal{P}$ be a weight of level $\ell$ but not necessarily dominant. Consider the weight $\Lambda + \rho$. If it is regular (not fixed by any $w \in W$) then there is a unique $w \in W$ such that $w(\Lambda + \rho) \in \mathcal{P}^+$. It follows from the definition of $\Pi$ that

$$\Pi e^\Lambda = \begin{cases} (-1)^w \text{ch} V(w(\Lambda + \rho) - \rho) & \text{if } \Lambda + \rho \text{ is } W\text{-regular and } w(\Lambda + \rho) \in \mathcal{P}^+ \\ 0 & \text{if } \Lambda + \rho \text{ is not } W\text{-regular}. \end{cases} \quad (6.1)$$

Then for all $i \in I$,

$$\Pi e^\Lambda = \Pi e^{r_i(\Lambda + \rho) - \rho}.$$ \quad (6.2)

Suppose $i \neq 0$. Then

$$r_i(\Lambda + \rho) - \rho = (\ell + h^\vee)\Lambda_0 + r_i(\Lambda) - (h^\vee\Lambda_0 + \rho) \quad \ell \Lambda_0 - \alpha_i + r_i(\Lambda).$$

For $i = 0$, recall that

$$r_0 = t_{\nu(\theta\cdot)}r_\theta = t_\theta r_\theta = r_\theta t_{-\theta}.$$ \quad (6.2)

Then

$$t_{-\theta}(\Lambda + \rho) = \Lambda + \rho - (\ell + h^\vee)\theta + \{(\Lambda + \rho|\theta) - \frac{1}{2}(\theta|\theta)(\ell + h^\vee)\}$$

$$= (\ell + h^\vee)\Lambda_0 + \rho + \theta - (\ell + h^\vee)\theta + \{(\rho|\theta) - (1 + \ell)\} \delta.$$

and

$$r_0(\Lambda + \rho) - \rho = r_\theta\{(\ell + h^\vee)\Lambda_0 + \rho + \theta - (\ell + h^\vee)\theta$$

$$+ \{(\rho|\theta) - (1 + \ell)\} \delta\} - \rho$$

$$= (\ell + h^\vee)\Lambda_0 + \rho + \theta - (\theta^\vee, \rho|\theta) + r_\theta(\Lambda)$$

$$+ (\ell + h^\vee)\theta + \{(\rho|\theta) - (1 + \ell)\} \delta - (h^\vee\Lambda_0 + \rho)$$

$$= \ell \Lambda_0 + r_\theta(\Lambda) + (\ell + 1)\theta + \{(\rho|\theta) - (1 + \ell)\} \delta.$$ \quad (6.2)

Now let $\mathfrak{g}$ be of type $A^{(1)}_{n-1}$. Let $\mathcal{P}$ be identified with the subspace of $\mathbb{Z}^n$ given by vectors with sum zero.

For $\alpha \in \mathcal{P}$ define the Demazure operator $\tilde{\Pi}$ to be the linear operator on $\mathbb{Z}[\mathcal{P}]$ such that

$$s_\alpha := \tilde{\Pi}(e^\alpha) = \mathcal{J}^{-1}(e^\alpha) \mathcal{J}(e^{\mathcal{P}+\alpha})$$

where $\mathcal{J} = \sum_{T \in \mathcal{W}(-1)^T \tau}$. Let $q = e^{-\delta}$. Then for $\alpha \in \mathcal{P}$,

$$\Pi e^{\ell \Lambda_0} e^\alpha = \begin{cases} \Pi e^{\ell \Lambda_0} e^{r_i(\Lambda) - \alpha_i} & \text{for } i \neq 0 \\ \Pi e^{\ell \Lambda_0} e^{r_\theta(\Lambda_0) + (\ell + 1)\theta} q^{\ell + 1 - (\cdot|\theta)} & \text{for } i = 0. \end{cases} \quad (6.3)$$

These equations express the $q$-equivalence in [3]. Let $\mathbb{Z}^n$ have standard basis $\{e_i \mid 1 \leq i \leq n\}$ and $\mathcal{P}$ be the subspace of $\mathbb{Z}^n$ orthogonal to the vector $\sum_{i=1}^n e_i$. Then $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n - 1$, $\theta = e_1 - e_n$, $(\cdot|\cdot)$ is the ordinary dot product in $\mathbb{Z}^n$, and $\mathcal{W}$ is the symmetric group on $n$ letters acting on the coordinates of $\mathbb{Z}^n$. Since $\Pi \circ \tilde{\Pi} = \Pi$ and $\Pi$ is $\mathbb{Z}\Lambda_0$-linear, one may replace every term $e^\alpha$ by $s_\alpha := \tilde{\Pi} e^\alpha$ in (6.3). Define the map $\mathbb{Z}[\mathcal{P}] \mathcal{W}[q] \rightarrow \mathbb{Z}[\mathcal{P}] \mathcal{W}[q]$ given by $s_\alpha \mapsto \Pi(e^{\ell \Lambda_0 + \alpha}) e^{-\ell \Lambda_0}$. 


Define \( f \equiv g \) in \( \mathbb{Z}[\mathbb{P}]^w[q] \) by the condition that the above linear map sends \( f \) and \( g \) to the same element. With this definition, we have

\[
-s_{\alpha} = \begin{cases} 
  s(\alpha_1, \ldots, \alpha_{i+1}-1, \alpha_i+1, \ldots, \alpha_n) & \text{for } i \neq 0 \\
  s(\ell+1+\alpha_n, \alpha_2, \ldots, \alpha_{n-1}, -1-\ell+1)q^\ell+1-\alpha_1+\alpha_n & \text{for } i = 0.
\end{cases}
\]

It is not hard to see that this recovers the \( q \)-equivalence of Schur functions given in [3].

### 6.2. Bosonic conjecture of [24, (9.2)]

In this section it is assumed that \( g \) is of type \( A^{(1)}_{n-1} \), \( \Lambda = \ell \Lambda_0 \), and the tensor factors \( B_j \) are perfect crystals of the form \( B^{k_j,\ell} \) in the notation of [10] with \( j \leq \ell \) for all \( j \). By restriction to \( U_q(\mathfrak{g}) \), \( B_j \) is the crystal of the irreducible integrable \( U_q(\mathfrak{g}) \)-module of highest weight \( \ell_j \Lambda_k \). In this case \( B_0 \) is not needed. To see this, recall that \( B_j \) can be realized as the set of column-strict Young tableaux of the rectangular shape having \( k_j \) rows and \( \ell_j \) columns with entries in the set \( \{1, 2, \ldots, n\} \). In [10] the \( P_{\ell}\)-weight I-crystal structure on the perfect crystals \( B^{k,\ell} \) is computed explicitly. In particular, if \( b \in B_j \) is a tableau then \( e_0(b) \) is at most the number of ones in the tableau \( b \), which is at most \( \ell_j \) by column-strictness. Therefore \( b \otimes u_{\ell \Lambda_0} \) never admits \( e_0 \). Thus the energy function \( E_B \) of \((3.2)\) has the property that for any \( b \in B = B_{k,\ell} \otimes \cdots \otimes B_{1} \), such that \( e_0(b \otimes u_{\ell \Lambda_0}) = e_0(b) \otimes u_{\ell \Lambda_0} \), one has \( E_B(e_0(b)) = E_B(b) - 1 \). Thus one obtains the bosonic formula in this case.

Since \( g \) is of type \( A^{(1)}_{n-1} \), \( a_0 = 1 \) and \( h^\vee = n \). Take \( \Lambda = \Lambda' = \ell \Lambda_0 \) in \((3.3)\). The lattice \( M \) is given by the root lattice \( \mathfrak{g} \) of \( \mathfrak{g} \), which may be realized by \( \{ \beta \in \mathbb{Z}^n \mid \sum_{i=1}^n \beta_i = 0 \} \). Let \( B_{r,\beta} \) be the set of paths \( b \in B \) of weight \( -n+1(\ell+n)\beta + \beta' \). Then

\[
K(B, \ell \Lambda_0, \ell \Lambda_0)(q) = \sum_{\tau \in \mathcal{W}} \sum_{\beta \in \mathcal{M}} \sum_{b \in B_{\tau,\beta}} (-1)^\tau q^{E_B(b) + |(\mathfrak{g},\beta)| - \frac{1}{2} |\beta|^2 - \ell+n} = \sum_{\tau \in \mathcal{W}} \sum_{\beta \in \mathcal{M}} \sum_{b \in B_{\tau,\beta}} (-1)^\tau q^{E_B(b) - 1/2} \sum_{i \geq 1} (\frac{1}{2} (\ell+n) \beta_i + i \beta_i).}
\]

Notice that \( \sum_{b \in B_{r,\beta}} q^{E_B(b)} \) is (up to an overall factor) the \( q \rightarrow 1/q \) form of the superpolynomial \( S \) of ref. [24] so that \( K(B, \ell \Lambda_0, \ell \Lambda_0)(q) \) equals the left-hand side of \([24, (9.2)]\) up to an overall power of \( q \). This shows that the left-hand side of \([24, (9.2)]\) is indeed the generating function of level-\( \ell \) restricted paths. To establish the equality \([24, (9.2)]\) it remains to prove that also the right-hand side equals the generating function of level-restricted paths.

### 6.3. Identities for level one and level zero

As in the previous section let \( g \) be of type \( A^{(1)}_{n-1} \) and assume that \( B = B^{k,1} \otimes \cdots \otimes B^{k,1} \). Fix \( \ell = 1 \) and \( \Lambda, \Lambda' \in \mathbb{P}_{\ell}\) weights of level 1. It is easy to verify that \( \mathcal{P}(B, \Lambda, \Lambda') \) consists of at most one element \( p \). Choose \( B, \Lambda, \Lambda' \) such that \( p \in \mathcal{P}(B, \Lambda, \Lambda') \) exists. Then by \((3.4)\) and \((6.6)\) we find that

\[
\sum_{\tau \in \mathcal{W}} \sum_{\beta \in \mathcal{M}} \sum_{b \in B_{r,\beta,\Lambda,\Lambda'}} (-1)^\tau q^{E(b) - \sum_{i=1} \left( \frac{n+1}{2} \beta_i^2 + i \beta_i \right)} = q^{E(p)}
\]

where \( B_{r,\beta,\Lambda,\Lambda'} \) is the set of paths \( b \in B \) of weight \(-\Lambda - \beta + \tau^{-1}(\Lambda' - (n+1)\beta + \beta')\).
A similar formula exists for $\ell = 0$:

$$
\sum_{\tau \in \mathcal{W}} \sum_{\beta \in M} \sum_{b \in B_{\tau, \beta}} (-1)^{\tau} q^{E(b) - \sum_{i=1}^{n} \frac{\beta_i^2 + i\beta_i}{2}} = \delta_{B, \emptyset}
$$

(6.6)

where $B_{\tau, \beta}$ is the set of paths $b \in B$ of weight $-\mathbf{p} + \tau^{-1}(-n\beta + \mathbf{p})$. The right-hand side is the generating function of paths in $B$ of level zero since there are no level zero restricted paths unless $B$ is empty. However, the arguments of Sections 4 and 5 do not imply that also the left-hand side is the generating function of level zero paths since it was assumed in the proof of Theorem 3 that the level of the crystals $B_j$ does not exceed $\ell$. We have assumed that $B_j = B^{k_j, 1}$ which are crystals of level one. However, it is possible to define a sign-reversing involution directly on $B = B^{k_1, 1} \otimes \cdots \otimes B^{k_j, 1}$ without using the crystal isomorphisms that are used in the proof of Theorem 3. Let $b \in B$. There exists at least one $0 \leq i \leq n$ such that $c_i(b_1)$ is defined. Define $v(b) = \min \{i | c_i(b_1) \text{ is defined} \}$ which has the property that $v(b) = v(\Phi(b))$ where as before $\Phi_i = s_i e_i$. Hence define the involution $\Phi(b) = \Phi_i(b)$. It is again sign-reversing and has no fixed points when $B \neq \emptyset$. This proves that the left-hand side of (6.6) is the generating function of level 0 restricted paths.

Equation (6.5) was conjectured in [20, 21]. For $n = 2$ identity (6.6) follows from the $q$-binomial theorem, for $n = 3$ it was proven in [3], Proposition 5.1 and for general $n$ it was conjectured in [22].

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