ON ONE INVERSE SPECTRAL PROBLEM RELATIVELY DOMAIN

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ABSTRACT. Different practical problems, especially, problems of hydrodynamics, elasticity theory, geophysics and aerodynamics can be reduced to finding of an optimal shape. The investigation of these problems is based on the study of depending of the functionals on the domain, their first variation and gradient.

In the paper the inverse problem relatively domain is considered for two-dimensional Schrodinger operator and operator $Lu = \Delta^2 u$ and the definition of s—functions is introduced. The method is proposed for the determination of the domain by given set of functions.

Key words: Shape optimization, inverse problems, domain variation, convex domains, support functions.

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1. Introduction

One of the well studied classes of the inverse problems—is inverse spectral problems. The papers dedicated to the investigation of these problems traditionally focus on the construction of a function (potential) by given spectral data (scattering data, normalizing numbers, eigenvalues) and obtaining necessary and sufficient conditions providing unequivocal determination of the sought function. More detail review can be found in the paper [1].

There exists a wide class of practical problems requiring determination of the domain by some experimental data. For example, it is very important to find the domain of the plate under across vibrations by the quantities, which may be measured from distance [2]. There are different formulations of the inverse problems relatively domain for the different cases [3-5]. Note that in differ from traditional inverse problems, the inverse problems relatively domain have some special specifications. First, these problems require to find no function, but domain. Second, choice of data (results of measure) sufficient for determination of the domain is also enough difficult problem.

In the paper we consider formulation and investigation of one inverse problem relatively domain for the two dimensional Schrodinger operator that, in particular, describes the vibration of membrane. In the end of the work we put and solve the similar problem for the operator describing across vibrations of the plate.

2. Problem setting and preliminary results

The object under investigation is the problem

$$-\Delta u + q(x) u = \lambda u, \quad x \in D,$$

$$u(x) = 0, \quad x \in S_D,$$

where $q(x)$ is differentiable non-negative function, satisfying condition $t^2 q(xt) = q(x), \quad t \in R, 0 \notin D \subset R^2$—bounded convex domain, $S_D \in C^2$—its boundary, $\Delta$-Laplace operator.

It is known [6,p.333] that under these conditions eigenfuntions $u_j(x)$ of the problem $\square_1$, $\square_2$ belong to the class $C^2(D) \cap C^1(\bar{D})$ and eigenvalues $\lambda_j$ are positive and may be numbered as $\lambda_1 \leq \lambda_2 \leq ...$ considering their multiplicity.

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The set of all convex bounded domains $D \in \mathbb{R}^2$ we denote by $M$. Let

$$K = \left\{ D \in M : S_D \in \dot{C}^2 \right\},$$

where $\dot{C}^2$ is a class of the piece-wise twice continuous differentiable functions.

**Definition 1.** The functions

$$J_j (x, D) = \frac{\left| \nabla u_j (x) \right|^2}{\lambda_j}, \quad x \in E^2, \; j = 1, 2, \ldots$$

are said to be $s$- functions of the problem (1), (2) in the domain $D$.

The problem is: To find a domain $D \in K$, such that

$$J_j (x, D) = s_j (x), \quad x \in S_D, \; j = 1, 2, \ldots,$$

where $u_j (x)$ and $\lambda_j$ is an eigenfunction and eigenvalue of the problem (1)- (2) in the domain $D$ correspondingly, $s_j (x)$-given continuous functions defined on $\mathbb{R}^2$.

First of all we give the considerations, which led us to this formulation.

Investigation of the dependence of the eigenvalues of the operators on the domain is an important problem, as the mechanical characteristics of some systems indeed are eigenvalues of the corresponding operators, which may be expressed by the functionals depending on the domain [7, 8]. One of the important steps for the investigation of the properties of these characteristics is calculation of the variation of these functionals relatively domain.

But to do this we need to define the space of domains, give a scalar product and a definition for the domain variation in that space.

It was shown [9] the pairs $(A, B) \in M \times M$ form a linear space with operations

$$(A, B) + (C, D) = (A + C, B + D),$$

$$(A, B) = (C, D), \quad \text{if} \quad A + B = C + D,$$

$$\lambda (A, B) = (|\lambda| A, |\lambda| B).$$

Here $A + B$ is taken in the sense of Minkowsky, i.e.

$$A + B = \{ a + b : \; a \in A, \; b \in B \}.$$

The scalar product is introduced by formula

$$(a, b) = \int_{S_B} P_1 (\xi) P_2 (\xi) d\xi,$$

where

$$a = (A_1, A_2), \; b = (B_1, B_2),$$

$$P_1 (x) = P_{A_1} (x) - P_{A_2} (x),$$

$$P_2 (x) = P_{B_1} (x) - P_{B_2} (x),$$

$S_B$- unit sphere, $P_D (x) = \max \left\{ (x, l), x \in E^2 \right\}$- support function of the domain $D$.

The obtained space we define by $ML_2$.

For any fixed $D \in M$ eigenvalue $\lambda_j$ of the problem (1), (2) is defined as ([10], p.182)

$$\lambda_j = \inf I (u, D), \; (u, u_p) = 0, \; p = 1, j - 1,$$
where

$$I (u, D) = \frac{\int_D \left[ |\nabla u (x)|^2 + q (x) u^2 (x) \right] dx}{\int_D u^2 (x) dx}.$$ 

Thus we can consider $\lambda_j$ as a functional of $D \in K$ and define by $\lambda_j (D)$. The following formula is obtained (see [9], p.98) for the first variation of the functional $\lambda_j (D)$ in the space $ML_2$

$$\delta \lambda_j (D) = - \max_{u_j} \int_{S_D} |\nabla u_j (x)|^2 \delta P_D (n (x)) ds,$$

where $|\nabla u_j (x)|^2 = \sum_{i=1}^2 \left( \frac{\partial u_j (x)}{\partial x_i} \right)^2$, $n (\xi)$- external normal to $S_D$ in the point $\xi$, $\max$ is taken over all eigenfunctions corresponding to the eigenvalue $\lambda_j$ in the case of its multiplicity.

Using (5) the following formula may be obtained for the eigenvalues of the problem (1), (2) in the domain $D$

$$\lambda_j (D) = \frac{1}{2} \max_{u_j} \int_{S_D} |\nabla u_j (x)|^2 P_D (n (x)) ds.$$

Really, let's take $D_0 \in K$, $D (t) = t \cdot D_0$, $t > 0$.

By $u_j$ we define $j$-th eigenfunction of the problem (1), (2) corresponding to the domain $D_0$. Then

$$-\Delta u_j (x) + q (x) u_j (x) = \lambda_j u_j (x), \; x \in D_0.$$

This relation may be written in the following equivalent from

$$-\frac{1}{t^2} \Delta t u_j \left( \frac{x}{t} \right) + \frac{1}{t^2} q \left( \frac{x}{t} \right) u_j \left( \frac{x}{t} \right) = \lambda_j \frac{(D_0)}{t^2} u_j \left( \frac{x}{t} \right), \; x \in D (t).$$

Since the function

$$\tilde{u}_j (x) = u_j \left( \frac{x}{t} \right), \; x \in D (t)$$

satisfies the relation

$$\Delta \tilde{u}_j (x) = \frac{1}{t^2} \Delta u_j \left( \frac{x}{t} \right)$$

from the condition $t^2 q (tx) = q (x)$and (8) one may get

$$-\Delta \tilde{u}_j (x) + q (x) \tilde{u}_j (x) = \lambda_j \frac{(D_0)}{t^2} \tilde{u}_j (x), \; x \in D (t).$$

It shows that $\Delta \tilde{u}_j (x)$ is an eigenfunction, and $\lambda (t) = \frac{\lambda_j (D_0)}{t^2}$ eigenvalue for the problem (1), (2) in the domain $D(t)$. Then using (5) we can write

$$\lambda_j (t + \Delta t) - \lambda_j (t) = \lambda_j \left( D (t + \Delta t) \right) - \lambda_j \left( D (t) \right) =$$

$$= \int_{S_{D(t)}} |\nabla u (x)|^2 \left[ P_{D(t+\Delta t)} (n (x)) - P_{D(t)} (n (x)) \right] ds + o (\Delta t), \; \xi \in S_{D(t)}.$$

If support function $P_{D(t)} (x)$ of the domain $D (t)$ is differentable relatively $t$, then dividing both sides of (9) by $t$ we obtain

$$\lambda_j (t) = - \max_{u_j} \int_{S_{D(t)}} |\nabla u_j (x)|^2 P'_{D(t)} (n (x)) ds,$$
where \( P'_{D(t)}(x) = \frac{\partial}{\partial t} P_{D(t)}(x) \).

Considering this we have
\[
-2\frac{\lambda_j(D_0)}{t^3} = -\frac{1}{t^2} \max_{u_j(x)} \int_{S_D} \left| \nabla u_j \left( \frac{\xi}{t} \right) \right|^2 P_{D_0}(n(\xi)) \, dS, \quad \xi \in S_D.
\]

Taking \( t = 1 \) from this we get (9).

As we see from (9) the boundary values of the function \( |\nabla u_j(x)|^2 \) unequivocally define eigenvalue \( \lambda_j \).

From (9) taking into account (4) we obtain
\[
\int_{S_D} s_j(\xi) P_{D}(n(\xi)) \, ds = 2, \quad j = 1, 2, \ldots.
\] *(11)*

This is the basic relation for the solving of the considered problem.

**Note.** As we take \( s \)-functions as a given data, let’s consider them for some concrete cases. For one dimensional case
\[
u'' + q(x) y = \lambda u,
\] *(12)*
\[u(a) = u(b) = 0,
\] *(13)*

where \( q(x) = \frac{c}{x^2}, \ c \geq 0, \ 0 \notin (a, b) \subset \mathbb{R}, \) \( s \)-functions defined by (3) indeed are
\[
\frac{u_j^2(a)}{J_j(a)} = J_j(a), \quad \frac{u_j^2(b)}{J_j(b)} = J_j(b).
\]

Thus the expression (8) takes a form
\[
J_j(b) \cdot b - J_j(a) \cdot a = 2, \quad j = 1, 2, \ldots
\] *(14)*

Let’s take \( a = 0 \), i.e. consider the problem (12), (13) in the interval \((0, b)\). For this case from (9) we get the following

**Consequence.** All \( s \)-functions of the problem (12), (13) satisfy to the condition
\[
J_j(b) = \frac{2}{b}, \quad j = 1, 2, \ldots
\] *(15)*

This formula allows to solve the inverse problem: Let the set of functions \( s_j(x), \ j = 1, 2, \ldots \) is given. In this case the problem of finding of the domain satisfying (4) is reduced to determination of the point \( b \), which may be done using (15).

As noted in consequence all \( s \)-functions satisfy to the condition (15), which is equivalent to the one condition. This condition is sufficient for finding of the point \( b \). Really as one may get from (15)
\[
\begin{align*}
    b &= \frac{2}{J_j(b)}.
\end{align*}
\]

Similarly, if \( b = 0 \), then we have
\[
\begin{align*}
    a &= -\frac{2}{J_j(a)}.
\end{align*}
\]

Note that if \( J_j(x) \equiv c_j, \ x \in S_D, \ c_j = const, \ j = 1, 2, \ldots \) then as follows from (15) they all are equal to each other for all \( j = 1, 2, \ldots \).

In two dimensional case from (11) is obtained that if the functions \( J_j(x, D) \) are constant, then
\[
J_j(x, D) = \frac{1}{\text{mes}D}, \quad j = 1, 2, \ldots \quad (\text{see} \ [9]).
\]
Now we prove the lemma that will be used later on.

**Lemma 1.** Let \( f(x) \) be a continuous function defined on the unit sphere \( S_B \). Then for any \( D_1, D_2 \in K \)

\[
\int_{S_{D_1+D_2}} f(n(\xi)) ds = \int_{S_{D_1}} f(n(\xi)) ds + \int_{S_{D_2}} f(n(\xi)) ds,
\]

where \( D_1 + D_2 \) is taken in the sense of Minkowsky i.e.

\[
D_1 + D_2 = \{ x : x = x_1 + x_2, x_1 \in D_1, x_2 \in D_2 \}.
\]

**Proof.** It is known [11], that \( f(x) \) may be continuously, positive-homogeneously extended over all the space and presented as a limit of the difference of two convex functions

\[
f(x) = \lim_{n \to \infty} [g_n(x) - h_n(x)].
\]

First consider

\[
f(x) = g(x) - h(x),
\]

where \( g(x), h(x) \) are convex, positively-homogeneous functions.

As is known [12] for any continuous, convex, positively-homogeneous function \( P(x) \) there exists the only convex bounded set \( D \) such, that \( P(x) \) is a support function of \( D \), i.e. \( P(x) = P_D(x) \). The opposite statement also is true.

It is also known that \( D \) is found as subdifferential of its support function in the point \( x = 0 \)

\[
D = \partial P(0) = \{ l \in E^n : P(x) \geq (l, x), \forall x \in R^n \}.
\]

So, there exist the domains \( G \) and \( H \), such that

\[
g(x) = P_G(x), \quad h(x) = P_H(x).
\]

Considering (18), (19) we get

\[
\int_{S_{D_1+D_2}} f(n(\xi)) ds = \int_{S_{D_1+D_2}} [g(n(\xi)) - h(n(\xi)) ds] =
\]

\[
= \int_{S_{D_1+D_2}} P_{G}(n(\xi)) ds - \int_{S_{D_1+D_2}} P_{H}(n(\xi)) ds.
\]

As for any \( D_1, D_2 \in K \) the following relation is valid [9]

\[
\int_{S_{D_1}} P_{D_2}(n(\xi)) ds = \int_{S_{D_2}} P_{D_1}(n(\xi)) ds,
\]

from (20) one may obtain

\[
\int_{S_{D_1+D_2}} f(n(\xi)) ds = \int_{S_{G}} P_{D_1+D_2}(n(\xi)) ds - \int_{S_{H}} P_{D_1+D_2}(n(\xi)) ds.
\]

As \( P_{D_1+D_2}(x) = P_{D_1}(x) + P_{D_2}(x) \) [12], applying (21) again we get (16). Lemma is proved.

3. Main results

Now we investigate the main problem of the work-construction of \( D \) by given set of functions \( s_j(x), \quad j = 1, 2, \ldots \).

Let \( B \subset E^2 \) be unit ball with the center at the origin and \( S_B \) its boundary. By \( \varphi_k(x), \quad k = 1, 2, \ldots \) we denote some basis in \( C(S_B) \)-space of continuous in \( S_B \) functions. These functions may be continuously, positive-homogeneously extended to \( B \). It may be done as:
\[ \tilde{\varphi}_k(x) = \begin{cases} \varphi_k \left( \frac{x}{\|x\|} \right) \cdot \|x\|, & x \in B, x \neq 0, \\ 0, & x = 0. \end{cases} \]

One may check, that these functions are continuous and satisfy to the positive homogeneity condition

\[ \tilde{\varphi}_k(\alpha x) = \alpha \tilde{\varphi}_k(x), \quad \alpha > 0. \]

Without loss of generality we can denote \( \tilde{\varphi}_k(x) \) by \( \varphi_k(x) \).

Thus we obtain the set of continuous, positive-homogeneous functions defined on \( B \).

As we noted above each positive-homogeneous, continuous function \( \varphi_j(x) \) may be presented in the form

\[ \varphi_k(x) = \lim_{n \to \infty} \left[ g_n^k(x) - h_n^k(x) \right] \]  

(22)

and there exist satisfying above mentioned properties domains \( G_n^k \) and \( H_n^k \) such, that

\[ g_n^k(x) = P_{G_n^k}(x), \quad h_n^k(x) = P_{H_n^k}(x). \]

These domains we call basic domains. Substituting these into (22) we get

\[ \varphi_k(x) = \lim_{n \to \infty} \left[ P_{G_n^k}(x) - P_{H_n^k}(x) \right]. \]  

(23)

First we consider

\[ \varphi(x) = P_{G_n^k}(x) - P_{H_n^k}(x), \]  

(24)

where \( G^k \) and \( H^k \) are closed, bounded convex domains.

As \( n(x) \in S_B \), for any \( x \in S_D \), we can decompose \( P_D(x), x \in S_B \) by basic functions \( \varphi_k(x) \)

\[ P_D(x) = \sum_{k=1}^{\infty} \alpha_k \varphi_k(x), \quad x \in S_B, \quad \alpha_k \in R, \quad k = 1, 2, \ldots. \]  

(25)

Thus, to determine \( P_D(x) \) we have to find the coefficients \( \alpha_k, \quad k = 1, 2, \ldots. \)

**Theorem 1.** Let the set of functions \( s_j(x), j = 1, 2, \ldots \) is given. Then the coefficients \( \alpha_k, \quad k = 1, 2, \ldots \) of the support function of sought domain \( D \) for which (4) is true, satisfy the equation

\[ \sum_{k,m=1}^{\infty} A_{k,m}(j) \alpha_k \alpha_m = 2, \quad j = 1, 2, \ldots. \]  

(26)

with coefficients

\[ A_{k,m}(j) = \int_{S_{G_n^k}} s_j(x) \left[ P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds - \int_{S_{H_n^k}} s_j(x) \left[ P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds. \]  

(27)

**Proof.** Considering (24) from (25) one may get

\[ P_D(x) = \sum_{k=1}^{\infty} \alpha_k (P_{G_n^k}(x) - P_{H_n^k}(x)), x \in S_B. \]  

(28)

The set of all indexes for which \( \alpha_k \geq 0 (\alpha_k < 0) \) denote by \( I^+ (I^-) \).
Then the relation (28) may be written as

\[ P_D(x) - \sum_{k \in I^-} \alpha_k P_{G^k}(x) + \sum_{k \in I^+} \alpha_k P_{H^k}(x) = \]
\[ = \sum_{k \in I^+} \alpha_k P_{G^k}(x) - \sum_{k \in I^-} \alpha_k P_{H^k}(x), \quad x \in S_B. \]

(29)

From last taking into account the properties of support functions [12] we obtain

\[ D - \sum_{k \in I^-} \alpha_k G^k + \sum_{k \in I^+} \alpha_k H^k = \sum_{k \in I^+} \alpha_k G^k - \sum_{k \in I^-} \alpha_k H^k. \]

The use of (29) and the lemma gives

\[ \int_{S_D} s_j(\xi) P_D(n(\xi)) d\xi + \sum_{k \in I^-} (-\alpha_k) s_{G^k} + \]
\[ + \sum_{k \in I^+} \alpha_k s_{H^k} = \int_{S_D} s_j(\xi) P_D(n(\xi)) d\xi + \]
\[ = \int_{S_D} s_j(\xi) P_D(n(\xi)) d\xi + \]
\[ + \sum_{k \in I^-} (-\alpha_k) s_{H^k}. \]

From this considering (11) we have

\[ \int_{S_D} s_j(\xi) P_D(n(\xi)) d\xi = \sum_{k=1}^{\infty} \alpha_k \left[ \int_{S_{G^k}} s_j(\xi) P_D(n(\xi)) d\xi - \right. \]
\[ \left. - \int_{S_{H^k}} s_j(\xi) P_D(n(\xi)) d\xi \right] = 2. \]

Substituting here (25) one may get (26) with coefficients (27). Theorem is proved.

We assumed that the considered problem has a solution in general case. For interesting cases as the functions \( s_j(x), \ j = 1, 2, \ldots \) are defined as experimental data, this problem always has a solution. The function \( P_D(x) \) is constructed by the help of the solution of (26), using (25).

As we noted above domain \( D \) is unequivocally defined by its support function \( P_D(x) \). Suppose that (26) has the only solution providing convexity of the support function \( P(x) \).

Let’s show that the expressions \( \frac{\| \nabla u_j(\xi) \|^2}{\lambda_j} \), \( j = 1, 2, \ldots \) for the problem (2), (3) in the constructed by the help of this solution, using (25) domain \( D \) indeed are \( s \)-functions. Really, if \( D \) is a domain in which the problem (2), (3) has given by formula (25) \( s \)-functions then decomposition \( D \) by formulæ (25) and making above done transformations we get the equation (26) with the same coefficients. From the assumption that this equation has the only solution, it follows \( D = D \).

If (19) has more than one solution then the sought domain is among the ones, constructed by (18) using these solutions, providing convexity of \( P(x) \).

This algorithm is constructed considering (18). In general case when \( \varphi(x) \) has form (22) \( A_{k,m}(j) \) turns to
$$A_{k,m}(j) = \lim_{n \to \infty} \left[ \int_{S_{c_k}^n} s_j(x) \left[ P_{G_n}(n(x)) - P_{H_n}(n(x)) \right] ds - \int_{S_{h_k}^n} s_j(x) \left[ P_{G_n}(n(x)) - P_{H_n}(n(x)) \right] ds \right].$$

Now let’s consider the across vibrations of the plate.

Let $D \in \mathbb{R}^2$ be a domain of the plate with boundary $S_D \in C^2$.

It is known [2] that the function $\omega(x_1, x_2, t)$ describing across vibrations of the plate satisfies equation

$$\omega_{x_1 x_1} + 2\omega_{x_1 x_2} + \omega_{x_2 x_2} + \omega_{tt} = 0. \quad (30)$$

Assuming the process stabilized the solution - eigen-vibration is sought as

$$\omega(x, t) = u(x_1, x_2) \cos \lambda t,$$ 

where $\lambda$ is an eigen-frequency.

Substituting this to (30) we get

$$\Delta^2 u = \lambda u, \quad x \in D, \quad (31)$$

where $\Delta^2 = \Delta \Delta$.

For different cases different boundary conditions may be considered. The object under investigation is the freezed plate with boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad x \in S_D. \quad (32)$$

Let

$$K = \left\{ D \in M : S_D \in \mathcal{C}^2 \right\},$$

where $\mathcal{C}^2$ is a class of the piece-wise twice continuous differentiable functions.

**Definition2.** The functions $J_j(x, D) = \frac{\left| \Delta u_j(x) \right|^2}{\lambda_j}, \quad x \in E^2, \quad j = 1, 2, \ldots$ are called $s$- functions of the problem (31) - (32) in the domain $D$.

The problem is: To find a domain $D \in K$, such that

$$J_j(x, D) = s_j(x), \quad x \in S_D, \quad j = 1, 2, \ldots, \quad (33)$$

where $u_j(x)$ and $\lambda_j$ are eigen-vibration and eigen-frequency of the problem (31) - (32) in the domain $D$ correspondingly, $s_j(x), \quad j = 1, 2, \ldots$- given continuous functions defined on $R^2$.

In [9] the following formula is obtained for the eigen-frequency of the freezed plate under across vibrations

$$\lambda_j = \frac{1}{4} \max_{u_j} \int_{S_D} \left| \Delta u_j(\xi) \right|^2 P_D(n(\xi)) ds, \quad (34)$$

where $P_D(x) = \max_{l \in D} (l, x), \quad x \in E^n$ is a support function of $D$, and $\max$ is taken over all eigen-vibrations $u_j$ corresponding to eigen-frequency $\lambda_j$ in the case of its multiplicity. (As we see from (34), the boundary values of the function $\left| \Delta u_j(x) \right|^2$ unequivocally define $\lambda_j$). From (34) considering (33) we get
\[ \int_{S_D} s_j(\xi) P_D(n(x)) \, ds = 4, \quad j = 1, 2, \ldots. \]

Caring out above done considerations the following theorem is proved for considered problem. **Theorem 2.** Let the set of functions \( s_j(x), \ j = 1, 2, \ldots \) is given. Then the coefficients of the support function of sought domain \( D \) of the plate for which (33) is true, satisfy the equation

\[ \sum_{k,m=1}^{\infty} A_{k,m}(j) \alpha_k \alpha_m = 4, \quad j = 1, 2, \ldots, \]

with coefficients

\[
A_{k,m}(j) = \lim_{n \to \infty} \left[ \int_{S_{Gk}^n} s_j(x) \left[ P_{G_m}^n(n(x)) - P_{H_m}^n(n(x)) \right] \, ds - \int_{S_{Hk}^n} s_j(x) \left[ P_{G_m}^n(n(x)) - P_{H_m}^n(n(x)) \right] \, ds \right].
\]

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