ON HOMOLOGY PLANES AND MAZUR MANIFOLDS

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Abstract. We call a non-trivial homology sphere a Kirby-Ramanujam sphere if it bounds both a homology plane and a Mazur type manifold. In 1980, Kirby found the first example by proving that the boundary of the Ramanujam surface bounds a Mazur manifold and it has remained a single example since then. By tracing their initial step, we provide the first additional examples and we present three infinite families of Kirby-Ramanujam spheres. Also, we show that one of our families of Kirby-Ramanujam spheres is diffeomorphic to the splice of two certain families of Brieskorn spheres. Since this family of Kirby-Ramanujam spheres bound contractible 4-manifolds, they lie in the class of the trivial element in the homology cobordism group; however, both splice components are separately linearly independent in that group.

1. Introduction

In algebraic geometry, the homology planes are defined to be algebraic complex smooth quasi-projective surfaces with the same homology groups of the complex plane $\mathbb{C}^2$ in integer coefficients. Ramanujam provided the first example of a homology plane [Ram71], today this object is known as Ramanujam surface $W(1)$ with the dual graph shown in Figure 1. Further, he proved that $W(1)$ is a contractible 4-manifold with a non-trivial homology sphere boundary and it is different than $\mathbb{C}^2$ up to algebraic isomorphism. Since $W(1) \times \mathbb{C}$ is diffeomorphic but not algebraically isomorphic to the complex space $\mathbb{C}^3$, the Ramanujam surface provided the first exotic algebraic structure on $\mathbb{C}^3$. It was also used to produce the first examples of exotic symplectic structures on Euclidean space which are convex at infinity by Seidel and Smith [SS05].

![Figure 1. A generalization of Ramanujam surface $W(n)$.](image)

The Mazur manifolds are compact contractible smooth 4-manifolds built with a single 0-, 1-, and 2-handle, so they are obtained by adding a 2-handle to $S^1 \times B^3$, the unknotted disk exterior. It first appeared in the eminent article of Mazur [Maz61] and then systematically explored in the celebrated work of Akbulut and Kirby [AK79]. Since they are cores of Akbulut corks [Akb91], they have been extensively studied in low-dimensional topology, see the recent papers [DHM20], [HP20], [Akb21], and [Lad22]. The second author introduced the concept of generalized Mazur manifolds as vast generalizations of Mazur manifolds in [Şav20a]; they are again compact contractible smooth 4-manifolds constructed by attaching an appropriate 2-handle to any ribbon disk exterior of the 4-ball $B^4$, see Section 3 for details.

A classical problem in low-dimensional topology asks which homology spheres bound contractible 4-manifolds, see [Kir78b, Problem 4.2]. Around the 1980s, Kirby was able to find a Mazur manifold that has the same boundary as the Ramanujam surface up to diffeomorphism, see [Man80, Pg. 56]. This valuable observation provides the initial motivation behind our definition.
We aim to enrich the problem of 3- and 4-manifolds above by addressing the algebro-geometric objects as well.

**Definition.** A non-trivial homology sphere is said to be a **Kirby-Ramanujam sphere** if it bounds both a homology plane and a Mazur type manifold.

After the ground-breaking work of Ramanujam, several novel techniques for the constructions of homology planes appeared in the works of Gurjar and Miyanishi [GM88], tom Dieck and Petrie [tDP89, tDP93], and Zaidenberg [Zai93]. We use their constructions for the existence of our homology spheres and we present the first new examples after Kirby and Ramanujam.

**Theorem A.** All dual graphs are depicted in Figure 2.
- Let $X(n)$ be the dual graphs of tom Dieck-Petrie homology planes of log-Kodaira dimension 2. Then Kirby-Ramanujam spheres $\partial X(n)$ bound Mazur manifolds with one 0-handle, one 1-handle and one 2-handle.
- Let $Y(n)$ be the dual graphs of Zaidenberg homology planes of log-Kodaira dimension 2. Then Kirby-Ramanujam spheres $\partial Y(n)$ bound generalized Mazur manifolds with one 0-handle, two 1-handles and two 2-handles.
- Let $Z(n)$ be the dual graphs of Gurjar-Miyanishi homology planes of log-Kodaira dimension 1. Then Kirby-Ramanujam spheres $\partial Z(n)$ bound generalized Mazur manifolds with one 0-handle, $n + 1$ 1-handles and $n + 1$ 2-handles.

The generalization of the Ramanujam surface shown in Figure 1 appeared in [Şav20a], and the second author proved that $\partial W(n)$ bound generalized Mazur manifolds with one 0-handle, two 1-handles, and two 2-handles. During the course of this paper, we also investigate that $W(n)$ originate from the homology planes for every $n \geq 1$, see Proposition 2.5. Therefore, they also provide examples of Kirby-Ramanujam spheres. Since our all Kirby-Ramanujam spheres bound contractible 4-manifolds, they are homology cobordant to the 3-sphere $S^3$; and therefore, they represent the trivial element in the homology cobordism group $\Theta^3_Z$, see [Man18] and [Şav22].

The algebraic complexity and structure of $\Theta^3_Z$ have been always studied by using Brieskorn spheres $\Sigma(p, q, r)$, which are the links of the complex surface singularities $x^p + y^q + z^r = 0$ in the complex space $\mathbb{C}^3$ where $p, q,$ and $r$ are pairwise relatively prime integers. Their links are Seifert fibered spheres over the 2-sphere with three singular fibers having the multiplicities $p, q,$
and $r$. In [DHST18], Dai, Hom, Stoffregen, and Truong provided the first family of Brieskorn spheres generating an infinite rank summand $\mathbb{Z}\infty$ in $\Theta^3_\mathbb{Z}$. Recently, even more families were found by Karakurt and the second author [KŚ22].

Next, we show that one of our families of Kirby-Ramanujam spheres comes from the splice of two resolutions of Brieskorn singularities. Note that the $n = 1$ case corresponds to the splice of the Poincaré homology sphere $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 7)$ along their singular fibers of degree 5 and 7.

**Theorem B.** Let $K_1(n) = K(n^2 + 3n + 1)$ and $K_2(n) = K(n^2 + 3n + 3)$ denote the singular fibers of Brieskorn spheres $\Sigma_1(n) = \Sigma(n + 1, n + 2, n^2 + 3n + 1)$ and $\Sigma_2(n) = \Sigma(n + 1, n + 2, n^2 + 3n + 3)$ respectively. Then there is a diffeomorphism between Kirby-Ramanujam spheres $\partial Z(n)$ and the splice of $\Sigma_2(n)$ along $K_1(n)$ and $K_2(n)$:

$$
\partial Z(n) \approx \Sigma_1(n)^{\infty}_{K_1(n)} \Sigma_2(n)
$$

for each $n \geq 1$.

From perspectives of both algebraic geometry and low-dimensional topology, the result presented above is to be considered an unexpected novelty due to the following reasons:

- Since their intersection matrices are not negative-definite, the dual graphs in Figure 2 do not originate from the resolution of singularities in normal surfaces, see Artin’s article [Art66, Proposition 2].
- Brieskorn spheres may bound Mazur manifolds, see [Sav20b] and references therein. However, one cannot realize any Brieskorn sphere as a boundary of a homology plane due to Orevkov [Ore97].
- The Brieskorn spheres $\Sigma_1(n)$ and $\Sigma_2(n)$ uniquely bound negative-definite resolution dual graphs shown in Figure 3. Using the algorithm of Neumann and Raymond [NR78, Section 7], one can easily compute the Neumann-Siebenmann invariants of $\Sigma_2(n)$ as follows:

$$
\bar{\mu}(\Sigma_2(n)) = \begin{cases} 
\frac{n^2 + 4n + 3}{8}, & \text{if } n \geq 1 \text{ odd,} \\
\frac{n^2 + 2n}{8}, & \text{if } n \geq 2 \text{ even.}
\end{cases}
$$

Since $\bar{\mu}$ is splice additive due to Saveliev [Sav95, Theorem 1], we conclude that $\bar{\mu}(\Sigma_1(n)) = -\bar{\mu}(\Sigma_2(n))$ by using Theorem A and Theorem B. Thus, $\Sigma_1(n)$ and $\Sigma_2(n)$ are homology cobordant to neither $S^3$ nor each other for each $n \geq 1$ because $\bar{\mu}$ is a homology cobordism invariant for plumblings, see Saveliev’s article [Sav02a].

- Furuta’s gauge theoretic argument [Fur90] guarantees that $\{\{\Sigma_1(n)\}\}_n^\infty$ are linearly independent in $\Theta^3_\mathbb{Z}$. Relying on the computations [Ném07] arising from Némethi’s lattice homology [Ném05], we know that $\Sigma_2(n)$ have vanishing Ozsváth-Szabó $d$-invariants [OS03a] and they are of projective type. Since the difference between $\bar{\mu}$- and $d$-invariants is arbitrarily large, we can apply the involutive Floer theoretic linear independence argument of Dai and Manolescu [DM19] to see that $\{\{\Sigma_2(n)\}\}_n^\infty$ also generate a $\mathbb{Z}\infty$ subgroup in $\Theta^3_\mathbb{Z}$.

- One can extract monotone graded subroots [DM19] from the graded roots [Ném05] by using the recipe in [DM19]. Némethi’s computations [Ném07] yield that $\Sigma_2(n)$ have complicated monotone graded subroots. Computing the invariants of Dai, Hom, Stoffregen, and Truong, one can conclude that $\{\{\Sigma_2(n)\}\}_n^\infty$ generate a $\mathbb{Z}\infty$ summand in $\Theta^3_\mathbb{Z}$. However, the monotone graded subroots of $\Sigma_1(n)$ are trivial due to Tweedy’s calculations [Twe13]. Therefore, we cannot address their powerful invariants.

In [Sav98, Section 6], Saveliev computed instanton Floer homology of surgeries along the generalized square knots. His result includes our family $\Sigma_1(n)^{\infty}_{K_1(n)} \Sigma_2(n)$ as well. He also described the representation spaces of irreducible representations of the fundamental groups of splices in SU(2) in [Sav98, Section 5].

Our other Kirby-Ramanujam spheres a priori have the splice decompositions of some Brieskorn spheres, but we do not know currently what they are. After finding these components, one can study these objects by following Saveliev’s strategies.
Organization. In Section 2, we review the construction of certain homology planes. In Section 3, we provide a background for the Mazur type manifolds and the splicing of homology spheres. We present the proofs of Theorem A and Theorem B together in Section 4. Finally, we discuss some further research topics and list several open problems in Section 5.

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2. Preliminaries in Algebraic Geometry

Consider a curve $D = D_1 \cup \ldots \cup D_i$ in a projective smooth surface $\bar{Y}$ with the irreducible components $D_j$ of $D$. We say that the curve $D$ is simple normal crossing or SNC for short if each $D_j$ is smooth and around every point $p \in D$ there exists local complex coordinates $(z_1, z_2)$ such that $D = \{z_1 z_2 = 0\}$ or $D = \{z_1 = 0\}$. For any SNC curve $D$, we can associate a dual graph $\Delta_D$ as follows:

- For every irreducible component $D_j$, we have a vertex $v_j$.
- For every point in $D_k \cap D_l$, we have an edge connecting $v_k$ and $v_l$.
- Every curve $D_j$ has a self-intersection number $D_j^2$ corresponding to weight of the vertex $v_j$.

The weights constitute the diagonal entries of the intersection matrix of $\Delta_D$. For the rest, we write 1 if there is an edge connecting vertices and we assign 0 otherwise.

An algebraic complex smooth quasi-projective surface $X$ is called a homology plane if it has the same of homology groups of $\mathbb{C}^2$ in integer coefficients, i.e., $H_*(X; \mathbb{Z}) = H_*(\mathbb{C}^2; \mathbb{Z})$. It can be compactified to a projective variety $\bar{X}$ and by blowing up points, this compactification $\bar{X}$ can be chosen to be a smooth projective surface such that $D = \bar{X} \setminus X$ is an SNC curve.

Moreover, for a homology plane $X$, there always exists a compactification $\bar{X}$ such that the dual graph $\Delta_D$ for $D = \bar{X} \setminus X$ is absolutely minimal, in the sense that any weight of a at most linear vertex (linear or an end) of $\Delta_D$ does not exceed $-2$. See [Zai93, Appendix A.2] for details. Such a dual graph determines uniquely the homology plane $X$. Therefore, we will use $X$ to denote both the homology plane and its absolutely minimal dual graph.

The SNC curve $D$ is of a very special type due to the following folklore result, for a proof see [tDP93, Proposition 2.1].

Proposition 2.1. Let $X$ be a homology plane and let $D = D_1 \cup \ldots \cup D_i = \bar{X} \setminus X$ be an SNC curve. Then the irreducible components $D_j$ of $D$ are isomorphic to $\mathbb{CP}^1$. Moreover, the dual graph $\Delta_D$ is a tree, i.e., it is connected and has no cycles.
Therefore, there exists a compact regular tubular neighborhood $U$ of $D$ in $\bar{X}$ and its boundary $\partial U$ is a \textit{plumbed homology sphere} with the plumbing graph $\Delta_D$. By abusing of language, we also call $\partial U$ the boundary of the homology plane $X$.

2.1. The Classification of Homology Planes. The partial classification of homology planes follows the lines of classification of open smooth algebraic surfaces. We refer to the book of Miyanishi \cite{Miy01} for an overview of this classification. A main ingredient is the invariant called the logarithmic Kodaira dimension.

Let $X$ be a homology plane and $\bar{X}$ a smooth projective compactification with an SNC $D = \bar{X} \setminus X$. Define its \textit{logarithmic Kodaira dimension} $\bar{k}(X) \in \{-\infty, 0, 1, 2\}$ as the unique value that satisfies the following inequalities: for some positive constants $\alpha$ and $\beta$ we have

$$\alpha m \bar{k}(X) \leq \dim H^0(\bar{X}, m(K_{\bar{X}} + D)) \leq \beta m \bar{k}(X)$$

where $m$ is sufficiently large and divisible positive integer. Here, $H^0$ stands for the sheaf cohomology and $K_{\bar{X}}$ denotes the canonical divisor of $\bar{X}$. The invariant $\bar{k}(X)$ does not depend on the chosen compactification $\bar{X}$, see \cite[Chapter 11.1]{Lit77}.

We know that $\bar{k}(X) = -\infty$ if and only if $X \cong \mathbb{C}^2$ as algebraic varieties, see the articles of Fujita \cite{Fuj79} and Miyanishi and Sugie \cite{MS80}. Fujita also prove that if $X$ is not isomorphic to $\mathbb{C}^2$ then $\bar{k}(X) \geq 1$ \cite{Fuj82}. Furthermore, the homology planes with $\bar{k}(X) = 1$ are completely classified by Gurjar and Miyanishi in \cite{GM88}. Also, there are explicit constructions of homology planes with $\bar{k}(X) = 2$ due to Tom Dieck and Petrie \cite{tDP93} and Zaidenberg \cite{Zai93, Zai94}.

In order to construct infinite families of homology planes, we first define an operation on a dual graph $\Delta_D$ of an SNC curve $D$ in a projective surface $Y$ called \textit{expanding an edge}.

Let $v_k$ and $v_l$ be two vertices of $\Delta_D$ and let $e$ be an edge joining them. Fix two coprime positive integers $a$ and $b$. We use the following short-hand notation for the continued fraction expansion:

$$[d_s, d_{s-1}, \ldots, d_1] = d_s - \frac{1}{d_{s-1} - \frac{1}{\ddots - \frac{1}{d_1}}}$$

Now set

$$\frac{a+b}{b} = [c_{-r}, \ldots, c_{-1}] \quad \text{and} \quad \frac{a+b}{a} = [c_s, \ldots, c_1].$$

Replace the edge $e$ by a linear graph divided in three linear subgraphs with $r - 1, 1$ and $s - 1$ vertices respectively:

$$\Delta_{a,b} = \{E_{-r+1}, \ldots, E_{-1}\}, \quad E_0, \quad \Delta'_{a,b} = \{E_1, \ldots, E_{s-1}\}$$

with weights:

$$E_h^2 = \begin{cases} -c_h, & \text{if } h \neq 0, \\
-1, & \text{if } h = 0, \end{cases}$$

and we modify the weights of $v_k$ and $v_l$ to $D_k^2 - c_{-r} + 1$ and $D_l^2 - c_s + 1$ respectively, see Figure 4.

![Figure 4. Expanding an edge.](image-url)
Lemma 2.2. Consider \( a = n \) and \( b = 1 \), then we have
\[
\frac{a + b}{b} = n + 1 \quad \frac{a + b}{a} = \frac{[2, \ldots, 2]}{n-times}.
\]

2.2. Gurjar-Miyanishi Homology Planes. Now we review the following constructions of homology planes appeared in [GM88]. We call the resulting surfaces Gurjar-Miyanishi homology planes. They can be obtained in the following fashion.

Suppose that \( \Delta_1 \subset \Delta_D \) is a cycle, that the vertices \( v_k, v_l \) and the edge \( e \) are in \( \Delta_1 \). We call the following procedure cutting a cycle:

- Fix two coprime positive integers \( a \) and \( b \) and expand the edge \( e \) as above,
- Remove the vertex \( E_0 \) and its two adjacent edges.

Consider the union of four lines in the complex projective plane
\[
L(1, 4) = \bigcup_{k=1}^{4} l_k \subset \mathbb{CP}^2
\]
where the first three lines intersect in a point \( P \) and the fourth line is in general position. Blow-up the point \( P \) to obtain an SNC curve \( D \). Denote its dual graph by \( \Delta_D \). We will cut the cycles in \( \Delta_D \) at the edges corresponding to the points \( l_1 \cap l_4 \) and \( l_2 \cap l_4 \) and two pairs of coprime positive integers \( a, b \) and \( c, d \) such that \( ac - ad - bc = \pm 1 \). This step is expressed by dashed lines in Figure 5.

![Figure 5. The procedure for the construction of Gurjar-Miyanishi homology planes.](image)

We also need to blow-up a smooth point in the strict transform of \( l_4 \), which can be followed by a sequence of \( k \) many blow up operations, and each time a smooth point of the exceptional divisor is created by the precedent blow up. This yields a chain of \(-2\)-curves and one \(-1\)-curve. We do not consider this \(-1\)-curve in the divisor \( D' \), whose dual graph is shown in Figure 6. The divisor \( D' \) lies in a projective smooth surface \( \bar{X} \) obtained by sequences of blow ups in \( \mathbb{CP}^2 \).

![Figure 6. The dual graph of Gurjar-Miyanishi homology planes.](image)

The Gurjar-Miyanishi homology plane corresponding to the numerical data \((k, (a, b), (c, d))\) is given by \( X = \bar{X} \setminus D' \). If \((a, b) \neq (1, 1) \neq (c, d)\), then \( X \) has logarithmic Kodaira dimension 1 and it is contractible.

Lemma 2.3. Let \((a, b) = (n + 1, n)\) then
\[
\frac{a + b}{a} = [2, n + 1], \quad \frac{a + b}{b} = \frac{[3, 2, \ldots, 2]}{n-times}.
\]
2.3. **tom Dieck-Petri Homology Planes.** Next, we consider homology planes introduced by tom Dieck and Petri. In Figure 7, we have the line arrangement \( L(3) \) together with its blow ups at two points of multiplicity three.

In this way, we can obtain an SNC curve and a dual graph \( \Delta_D \) where we cut the cycles corresponding to intersections of lines \( L_2 \cap L_3, L_2 \cap L_7, L_5 \cap L_6, \) and \( L_1 \cap L_4 \). Here, the cutting data \((1,1)\) stands for the first three intersections and \((a, b)\) for the last one. The resulting dual graphs depicted in Figure 8 are called tom Dieck-Petri homology planes.

\[
\begin{align*}
L_1 & L_2 & L_3 & L_4 & L_5 \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
L_6 & L_7 & L_1 & L_3 & L_6 \\
\end{align*}
\]

**Figure 7.** The \( L(3) \) arrangement.

**Figure 8.** The dual graph of tom Dieck-Petri homology planes.

**Remark 1.** In order to obtain a homology plane from the conditions \((a, b)\) above, they must satisfy: 
\[4b - 3a = \pm 1,\] see [AA21, Section 5.2.3].

**Lemma 2.4.** For \( a = 4n + 1 \) and \( b = 3n + 1 \), we have
\[
\frac{a + b}{b} = [3, 2, 2, n + 1], \quad \frac{a + b}{a} = [2, 5, 2, 2, \ldots, 2]_{n-1\text{-times}}.
\]

2.4. **Zaidenberg Homology Planes.** We finally consider the homology planes constructed by Zaidenberg [Zai93]. His approach provides the generalization of the original Ramanujam surface [Ram71].

Consider \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) with coordinates \((x : y, u : v)\) and the curves
\[
c_s = \{u^2y^s = x^sy^2\}, \quad e_0 = \{v = 0\}, \quad e_1 = \{y = 0\}, \quad l_1 = \{x = y\}.
\]
Denote by \( d_s = c_s \cup e_0 \cup e_1 \cup l_1 \), let \( \pi : \tilde{X}_s \to \mathbb{CP}^1 \times \mathbb{CP}^1 \) be the minimal resolution of singularities of \( d_s \) and let \( B_s = \pi^{-1}(d_s) \). In Figure 9, there is a representation in the affine plane \( \mathbb{C}^2 \) with coordinates \((x, u)\) of the curve \( d_s \).

Consider the integer valued \( 2 \times 2 \) matrix
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(M) = 1
\]
and apply the cutting cycles procedure to \((\tilde{X}_s, B_s)\) in two points \( z_1 \) and \( z_2 \) corresponding to \( l_1 \cap c_s \) according to the numerical data in \( M \) and denote by \((\tilde{X}_s, W_s)\) the resulting pair. Zaidenberg proved that the surface \( X_s = \tilde{X}_s \setminus W_s \) is a contractible homology plane [Zai93].

Here we only use the case of \( s = 3 \). A partial resolution of singularities of the curve \( d_s \) is shown in Figure 10. This can be used to obtain the dual graph of \( W_s \), depicted in Figure 11. They are said to be Zaidenberg homology planes.
Example 2.1. The original Ramanujam surface corresponds to the numerical data: \( s = 3, M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \).

Analysing the Zaidenberg’s construction, we see that Ramanujam manifolds studied by the second author in [Şav20a] are also homology planes.

**Proposition 2.5.** Let \( W(n) \) be Ramanujam manifolds depicted in Figure 1. Then the boundaries \( \partial W(n) \) are Kirby-Ramanujam homology spheres.

**Proof.** By [Şav20a], we already know that the homology spheres \( \partial W(n) \) bound generalized Mazur manifolds. Also, their dual graph corresponds to the numerical data \( s = 3 \) and \( M = \begin{pmatrix} 1 & 1 \\ n & n+1 \end{pmatrix} \).

Using Lemma 2.3 and the dual graph in Figure 10, we are done. \( \square \)

### 3. Preliminaries in Low-Dimensional Topology

#### 3.1. Mazur and Generalized Mazur Manifolds

A knot \( K \) in \( S^3 \) is said to be a **ribbon knot** if it can be built by attaching bands between components of an unlink. The minimal number of bands is recorded by the quantity called **fusion number**, see [Hos67]. Since the Euler characteristic of a disk is one, the fusion number also determines the number of components of the corresponding unlink. The basic examples of ribbon knots are the unknot \( U \), the square knot \( T(2, 3) \# T(2, 3) \), and the generalized square knot \( T(n+1, n+2) \# T(n+1, n+2) \) for \( n \geq 1 \). They have fusion numbers 0, 1, and \( n \), respectively. Here, the notation \( T(p, q) \) stands for the left-handed \( (p, q) \)-torus knot given coprime positive integers \( p \) and \( q \) and \( \overline{T(p, q)} \) denotes its mirror image.
The generalized Mazur manifolds can be defined by attaching a 2-handle to the ribbon disk exteriors of $B^4$ in order to kill its fundamental group. In the following lemma, changing the role of the ribbon knot with the unknot recovers the dual version of the original argument of Mazur.

**Lemma 3.1** (Lemma 2.1, [Şav20a]). Let $Y$ be the 3-manifold obtained by 0-surgery on a ribbon knot in $S^3$ with the fusion number $n$. Then any homology sphere obtained by an integral surgery on a knot in $Y$ bounds a generalized Mazur manifold with one 0-handle, $n+1$ 1-handles, and $n+1$ 2-handles.

In order to shorten the proof of Theorem A, we need the following Kirby calculus trick of Akbulut and Larson. It was found in [AL18] and played a key role in the proof of their main theorem. Later, it was also effectively used by the second author [Şav20a, Şav20b].

**Definition 3.2.** The Akbulut-Larson trick is an observation about describing the iterative procedure for passing from the surgery diagram of a homology sphere to a consecutive one by using a single blow-up with an isotopy, see Figure 12.

3.2. **Splicing Homology Spheres.** The splice operation was defined by Siebenmann [Sie80] and later it was elaborated systematically in the novel book of Eisenbud and Neumann [EN85].

Let $(Y_1, K_1)$ and $(Y_2, K_2)$ be two pairs of homology spheres and knots together with meridians and longitudes $(\mu_1, \lambda_1)$ and $(\mu_2, \lambda_2)$, and tubular neighborhoods $\nu(K_1)$ and $\nu(K_2)$ inside $Y_1$ and $Y_2$. The splice operation between $(Y_1, K_1)$ and $(Y_2, K_2)$ produces a homology sphere given by

$$Y_1 \bowtie_{K_1} Y_2 = \left( Y_1 \setminus \nu(K_1) \right) \cup \left( Y_2 \setminus \nu(K_2) \right)$$

where the pasting homeomorphism along tori boundaries sends $\mu_1$ onto $\lambda_2$ and $\lambda_1$ onto $\mu_2$.

Since Brieskorn spheres bound unique negative definite plumbing graphs [Sav02b, Example 1.17], we can describe their splice on a joint plumbing graph by using the recipe in [EN85, Chapter V.20]. These plumblings graphs are the same as their resolution graphs of singularities.

Let $(\Sigma(a_1, a_2, a_3), K(a_n))$ and $(\Sigma(b_1, b_2, b_3), K(b_m))$ be a pairs of Brieskorn spheres and singular fibers with plumbing graphs $G$ and $G'$ respectively. In Figure 13, the arrows indicate the singular fibers. The weights $e_n$ and $e_m$ label the endmost vertices in the branches of $K(a_n)$ and $K(a_m)$ respectively. The additional weights $x$ and $y$ appeared after the splice operation are given by the formula $x = \det(G_0)$ and $y = \det(G'_0)$ where $G_0$ and $G'_0$ are the portions of $G$ and $G'$ obtained by removing $e_n$ and $e_m$ respectively. These type of determinants can be easily found, consult [EN85, Chapter V.21].

4. **Proofs of Theorem A and Theorem B**

Since we need a portion of the proof Theorem A in the proof argument of Theorem B, we first show Theorem A.
Proof of Theorem A. We present our proof case by case.

- **The Family X(n)**: First, we verify that X(n) for every n ≥ 1 originate from tom Dieck-Petri homology planes by addressing Subsection 2.3. Note that we can contract −1’s in the dual graph shown in Figure 8. Then we set a = 4n + 1 and b = 3n + 1 as in Lemma 2.4. Writing $\Delta_{a,b}$ and $\Delta'_{a,b}$ explicitly, we obtain the dual graph X(n) displayed in Figure 2. Since X(n) does not come from a Gurjar-Mayanishi type dual graph and it is absolutely minimal, the logarithmic Kodaira dimension of X(n) is 2, see [Zai93, Appendix A.5]

In Figure 14, we first draw the surgery diagram of $\partial X(1)$. The additional 2-handle is shown in dark black. Applying the several Kirby moves indicated in the picture, we end up with the surgery diagram of $S^3_0(U)$. Since these moves do not change the diffeomorphism type of a 3-manifold [Kir78a], we can use Lemma 3.1 to conclude that $\partial X(1)$ bounds a Mazur manifold with one 0-handle, one 1-handle, and one 2-handle. We can guarantee that all $\partial X(n)$ for n ≥ 2 bound such Mazur manifolds as well by using the Akbulut-Larson trick consecutively.

- **The Family Y(n)**: By following Subsection 2.4, we initially verify that Y(n) for n ≥ 1 are indeed Zaidenberg homology planes. We use Lemma 2.3 and the dual graph in Figure 10 to guarantee
that the resolution graph of $Y(n)$ is encoded by the numerical data $s = 3$ and $M = \left( \begin{array}{cc} 1 & 1 \\ n+1 & n \end{array} \right)$. Since Zaidenberg classified his homology planes in terms of their logarithmic Kodaire dimensions [Zai93], we know that $\bar{\kappa}(Y(n)) = 2$.

Then we address our previous strategy for the family $Y(n)$. We first start with the surgery diagram of $\partial Y(1)$ and then attach a $(-1)$-framed dark black 2-handle. Now we perform various blow down operations and we eventually find the surgery diagram of $S_0^3(T(2,3)\# \overline{T(2,3)})$. Recall that the square knot has fusion number one, by using Lemma 3.1, we can say that $\partial Y(1)$ bounds a generalized Mazur manifold with one 0-handle, two 1-handles, and two 2-handles.

Applying the Akbulut-Larson trick again, we can pass the surgery diagram of $\partial Y(2)$, and so on. Since our moves all respect the diffeomorphism type of a 3-manifold, we are done.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{diagram.png}
\caption{(-1)-surgery from $\partial Y(1)$ to $S_0^3(T(2,3)\# \overline{T(2,3)})$.}
\end{figure}

- **The Family $Z(n)$**: We first see that $Z(n)$ for $n \geq 1$ are Gurjar-Miyanishi homology planes by following Subsection 2.2. Let $k = 1, (a, b) = (n+2, 1)$ and $(c, d) = (n+1, n)$. Note that the vertex $v_3$ in Figure 6 has weight $-1$, so we can contract it. Using Lemma 2.2 and Lemma 2.3, we can write the linear graphs obtained by cutting the cycles explicitly. Thus, the dual graph of $Z(n)$ corresponds to one shown in Figure 2. Moreover, $\bar{\kappa}(Z(n)) = 1$.

Now, we begin with the surgery diagram of $\partial Z(n)$ for every $n \geq 1$, see Figure 4. Next, we apply a blow up and find the second picture. Performing many blow down operations along both right- and left-hand sides of our figure, we finally reach the surgery diagram of the manifold $S_{-1}^3(T(n+1, n+2)\# \overline{T(n+1, n+2)})$. To address Lemma 3.1, we can add a $(-1)$-framed 2-handle to pass to the surgery diagram $S_0^3(T(n+1, n+2)\# \overline{T(n+1, n+2)})$. Since the generalized square knot has fusion number $n$, we can conclude that for each $n \geq 1$ $\partial Z(n)$ bounds a generalized Mazur manifold built with one 0-handle, $n+1$ 1-handles, and $n+2$ 2-handles.

\begin{proof}[Proof of Theorem B] We already know from the previous proof that homology spheres $\partial Z(n)$ and $S_{-1}^3(T(n+1, n+2)\# \overline{T(n+1, n+2)})$ are diffeomorphic for every $n \geq 1$. We claim that there is also a diffeomorphism between the latter family of 3-manifolds and the splice of Brieskorn spheres $\Sigma(n+1, n+2, n^2 + 3n + 1)$ $\times K(n^2+3n+1)$ $\times K(n^2+3n+3)$ $\times \Sigma(n+1, n+2, n^2 + 3n + 3)$.
\end{proof}
We first consider the base case \( n = 1 \). Using the recipe in Subsection 3.2, we can easily find the plumbing graph of \( \Sigma(2, 3, 5) \cong K(K(5) \bowtie K(7)) \Sigma(2, 3, 7) \), shown in Figure 17. The additional weights \(-2\) and \(-1\) correspond to the determinants of the \( E_7 \) graph and the linear graph with vertices \(-2\), \(-1\), and \(-7\), respectively. Next, we apply blow down operations six many times and we obtain the last picture in Figure 17.

In Figure 18, we first draw the surgery diagram of the last plumbing graph appeared in Figure 17. Applying blow down operations seven many times more, we get the surgery diagram of \( S^3_{-1}(T(2, 3) \# T(2, 3)) \), as expected. Our argument and procedure can be simply generalized to all values of \( n \), so they are left to readers as exercises.

Remark 2. A complete and alternative proof argument for Theorem B can be given as follows.
One can prove our initial claim in full generality by using 3-dimensional techniques of Fukuhara and Maruyama [FM88, Pg. 285-286]. They provided a surgery formula for splicing operation of homology spheres. Their observation is a generalization of Gordon’s previous arguments in [Gor75] and [Gor83, Pg. 700-701]. Their useful observation was recently reproved by Karakurt, Lidman, and Tweedy by using Kirby calculus [KLT21, Lemma 2.1]. Thus we have

1. $\Sigma_1 \bowtie_{K_1} \Sigma_2 \approx S^3_{-1}(T(n+1, n+2) \# T(n+1, n+2))$.

A famous result of Gordon indicates that a homology sphere obtained by $(\pm 1)$-surgery along a slice knot in $S^3$ bounds a contractible 4-manifold [Gor75, Theorem 3].

5. FURTHER DIRECTIONS

We state some problems and questions for possible further research directions engaging with Kirby-Ramanujam spheres and their modifications.

**Question A.** Let $X$ be a homology plane and $\partial X$ be a Kirby-Ramanujam sphere bounding a generalized Mazur manifold $W$. Does $W$ always admit a complex structure? If yes, is there a complex structure such that $W$ is biholomorphic to $X$?
In a similar vein to the concept of Kirby-Ramanujam spheres, we introduce the objects called Ramanujam spheres. They are non-trivial homology spheres bounding homology planes. In contrast to Theorem B, we have a certain constraint due to the article of Orevkov [Ore97] in which he proved that Brieskorn spheres as well as Seifert fibered spheres cannot be Ramanujam spheres.

**Problem A.** Which type of homology spheres arise as Kirby-Ramanujam spheres or Ramanujam spheres?

In order to compare our objects, we may raise the following naive question:

**Problem B.** Are Ramanujam spheres always Kirby-Ramanujam spheres?

A homology plane $X$ is known to be affine [Fuj82], see also [Zai98, Lemma 2.1]. Hence $X$ is Stein and therefore $\partial X$ admits a contact structure. However, there are examples of Mazur manifolds with Brieskorn sphere boundaries such that they cannot admit Stein structure due to Mark and Tosun [MT18].

**Question B.** Is there any example of a Kirby-Ramanujam sphere that bounds a Stein Mazur manifold or an Akbulut cork?

Since there are several examples of homology planes that are known to be non-contractible [Miy01],[AA21], we curiously ask the following question. Compare with [Şav22, Problem G].

**Question C.** Does a Kirby-Ramanujam sphere bound a non-contractible homology plane?

It would be interesting to study Ramanujam spheres by using the certain invariants of 3-manifolds [OS03b], [Ném08], [GM21], [AM22]. Hence we inquire:

**Question D.** Is it possible to extend invariants of 3-manifolds to Ramanujam spheres?

If the answer is positive, then one can pursue to extend invariants to $\mathbb{Q}$-homology planes. Similarly, they are affine complex smooth surfaces having the same $\mathbb{Q}$-homology as $\mathbb{C}^2$. For the basic properties and the partial classification of $\mathbb{Q}$-homology planes, see [Miy01].

Using the notion of bad vertices in [Ném05], one can show that the graphs in Figure 2 have two vertices and that by reducing the weight of these vertices one obtains rational singularities.

**Question E.** Is it possible to characterize Ramanujam spheres by using the lattice cohomology?

In [Sav99, Chapter 14], the representation varieties of Seifert fibered spheres were discussed. In contrast to the Orevkov’s theorem [Ore97], there exist homology planes of log-Kodaira dimension 1 with the fundamental group isomorphic to that of a Seifert manifold. Also, there is a surjective homorphism $\pi_1(\partial X) \rightarrow \pi_1(X)$ for a homology plane.

**Problem C.** For a non-contractible homology plane $X$, study $\text{Hom}(\pi_1(X), \text{SU}(2))$ the representation variety of the Ramanujam sphere $\partial X$ as a subvariety of the representation variety $\text{Hom}(\pi_1(\partial X), \text{SU}(2))$ of $X$.

This leads the final question of our paper:

**Question F.** Which invariants or representations of a Ramanujam sphere $\partial X$ can be extended to a homology plane $X$?

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