STACKY ABELIANIZATION OF ALGEBRAIC GROUPS

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ABSTRACT. We prove a conjecture of Drinfeld regarding restriction of central extensions of an algebraic group \( G \) to the commutator subgroup. As an application, we construct the “true commutator” of \( G \). The quotient of \( G \) by the action of the true commutator is the universal commutative group stack to which \( G \) maps.

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1. Introduction

One of the goals of geometric character theory is constructing sheaves on an algebraic group \( G \) over a finite field \( \mathbb{F}_q \) whose “trace of Frobenius functions” are the irreducible characters of \( G(\mathbb{F}_q) \) [Lus85], [Lus03], [BD06]. A one-dimensional character sheaf is easy to define: it is an irreducible local system \( \mathcal{K} \) equipped with an isomorphism \( m^*\mathcal{K} \cong \mathcal{K} \boxtimes \mathcal{K} \), where \( m : G \times G \to G \) is the group multiplication; see for instance, [Gai03], §4.3. One-dimensional character sheaves are closely related to central extensions. Their relationship can be summarized in

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Here $A$ is a finite abelian group. To go from left to right one takes the trace of Frobenius acting on the stalks of the sheaf (in the case of central extensions, the sheaf of local sections of $\pi$). To go from top to bottom, one needs the extra data of a homomorphism $A \to \mathbb{Q}_\ell^\times$.

Recall that every one-dimensional character of $G(\mathbb{F}_q)$ is trivial on the commutator subgroup $[G(\mathbb{F}_q), G(\mathbb{F}_q)]$. There exist, however, examples of one-dimensional character sheaves whose restriction to $[G, G]$ is nontrivial. For instance, we have the central extension

$$1 \to \mu_n \to \text{SL}_n \to \text{PGL}_n \to 1.$$ 

Our goal is to construct a commutator for $G$ which is suitable for doing geometric character theory. To accomplish this, we need to study the relationship between central extensions of $G$ and those of $[G, G]$.

Henceforth, let $G$ denote a connected algebraic group over a perfect field $k$. We will prove that there exists a pro-étale group scheme $\Pi_{\text{et}}(G)$ such that for every commutative étale group scheme $A$, $\text{Hom}(\Pi_{\text{et}}(G), A)$ equals the set of isomorphism classes of central extensions of $G$ by $A$.

**Example 1.1.** Let $G_a$ denote the additive group over $\bar{\mathbb{F}}_q$. Then

$$\Pi_{\text{et}}(G_a) = \text{Hom}(\bar{\mathbb{F}}_q, \mathbb{C}^\times).$$

Restricting central extensions of $G$ to $[G, G]$ defines a morphism of group schemes

$$\Pi_{\text{et}}(i) : \Pi_{\text{et}}([G, G]) \to \Pi_{\text{et}}(G).$$

The following theorem was conjectured by V. Drinfeld.

**Theorem 1.2.** The image of $\Pi_{\text{et}}(i)$ is finite.

Let $A^\text{et}$ be the image of $\Pi_{\text{et}}(i)$ and let

$$1 \to A^\text{et} \to [G, G]^\text{true} \to [G, G] \to 1$$

be the central extension corresponding to the canonical epimorphism $\Pi_{\text{et}}([G, G]) \to A^\text{et}$. We propose to think of $[G, G]^\text{true}$ as the “true commutator” of $G$. True commutator is characterized by the following properties:

1) $[G, G]^\text{true}$ is a connected étale central extension of $[G, G]$.

---

1This is the geometric analogue of the following fact: For a perfect group $\Gamma$, $\text{Hom}(H^2(\Gamma, \mathbb{Z}), A) = H^2(\Gamma, A)$: see for instance, [Mil72].
P2) The pullback of every étale central extension of $G$ to $[G, G]^{\text{true}}$ is trivial.

P3) The commutator map has a lift to $[G, G]^{\text{true}}$; that is to say, there exists a morphism of algebraic varieties $G \times G \to [G, G]^{\text{true}}$ such that the following diagram commutes

\[
\begin{array}{ccc}
G \times G & \xrightarrow{c} & [G, G], \\
\downarrow & & \downarrow \\
[G, G]^{\text{true}} & \cong & [G, G],
\end{array}
\]

where $c(g, h) = [g, h] := g^{-1}h^{-1}gh$ for $g, h \in G$.

Remark 1.3. Over an algebraically closed field, the fact that P1)-P3) characterize the true commutator is equivalent to the exactness of the following complex of profinite groups:

\[
\pi_1(G \times G) \to \Pi_{\text{et}}([G, G]) \to \Pi_{\text{et}}(G).
\]

Remark 1.4. If $k$ is algebraically closed of characteristic zero and $G$ is reductive, then the true commutator of $G$ is the simply connected cover of the semisimple group $[G, G]$, see Example 4.1. The fact that the commutator map lifts to $[G, G]^{\text{true}}$ was observed by P. Deligne [Del79, §2.0.2].

We now study the quotient of $G$ by the true commutator. Note that the composition $[G, G]^{\text{true}} \to [G, G] \hookrightarrow G$ defines an action of $[G, G]^{\text{true}}$ on $G$ by left translation. The quotient stack

\[
G^{\text{ab, st}} := [G/[G, G]^{\text{true}}]
\]

is called the \textit{stacky abelianization} of $G$. Observe that stabilizers of the action of $[G, G]^{\text{true}}$ on $G$ are isomorphic to $A^{\text{et}}$. In particular, stacky abelianization is a Deligne-Mumford stack [DM69]. We will show, moreover, that it is a strictly commutative Picard stack [Del73]. Roughly speaking, this means that $G^{\text{ab, st}}$ is equipped with a commutative group structure. The following theorem states that $G^{\text{ab, st}}$ is the universal commutative group stack to which $G$ maps

\textbf{Theorem 1.5.} Let $\mathcal{P}$ be a strictly commutative Deligne-Mumford Picard stack, and let $G \to \mathcal{P}$ be a 1-morphism of group stack. Then there

\footnote{For a more precise formulation, see Theorem 4.10.}
exists a 1-morphism of Picard stacks $G^{ab, st} \to \mathcal{P}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
G^{ab, st} & \xrightarrow{\text{constant}} & \\
\downarrow & & \downarrow \\
G & \xrightarrow{\phi} & \mathcal{P}.
\end{array}
\]

1.1. Organization of the text. We start §2 by proving an algebra-geometric analogue of the fact that a morphism from a connected space to a finite space is constant. We then show that every connected algebraic group has a “universal pro-cover”, and use this to define $\Pi_{et}$. We will see that over an algebraically closed field, $\Pi_{et}$ is a quotient of the Grothendieck fundamental group. We conclude the section by providing a summary of our calculations of fundamental group schemes. The details of these calculations appear in Appendix B.

In §3 we study the relationship between central extensions of $G$ and those of $[G, G]$. In particular, we provide examples of central extensions of unipotent groups whose restriction to $[G, G]$ is nontrivial. Moreover, we give a necessary and sufficient criterion for “lifting” central extension of $[G, G]$ to $G$. The proof of Theorem 1.2 is also given in this section.

In §4, we study the true commutator and the stacky abelianization. We will show that the commutator map lifts to the true commutator and use this to prove that the stacky abelianization is Picard. We then prove the characterization of the true commutator and the universal property of the stacky abelianization (Theorem 1.5).

Throughout the text we assume that the reader is familiar with group stacks and their relationship to crossed modules. For the convenience of the reader, we have gathered all the relevant results in Appendix A. The only part of this appendix which appears to be new is the stacky analogue of the First Isomorphism Theorem (Lemma A.11).

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2. Central extensions of algebraic groups

2.1. Constant morphisms. Let $\varphi : X \to Y$ be a morphism of schemes over a noetherian base-scheme $S$.

Definition 2.1. $\varphi$ is constant if for every scheme $T$ over $S$, the map of sets $\varphi_T : \text{Hom}_S(T, X) \to \text{Hom}_S(T, Y)$ is constant.
Let $S'$ be a scheme over $S$.

**Lemma 2.2.** (i) If $\varphi$ is constant, so is $\varphi \times_S S'$.
(ii) Assume $S' \to S$ is flat and $Y \to S$ is finite. Then $\varphi$ is constant if and only if $\varphi \times_S S'$ is constant.

**Proof.** Statement (i) follows from the fact that for every scheme $T'$ over $S'$,
\[
\text{Hom}_S(T', X \times_S S') = \text{Hom}_S(T', X).
\]
Under the conditions of (ii), for every scheme $T$ over $S$, the base change map
\[
\text{Hom}_S(T, Y) \to \text{Hom}_{S'}(T \times_S S', Y \times_S S)
\]
is an isomorphism (EGA 1, ch. 0, §6.2.2). This proves statement (ii). □

**Lemma 2.3.** Assume $X$ admits an $S$-section $\sigma : S \to X$. Then the following are equivalent:
(1) $\varphi$ is constant.
(2) $\varphi$ has a factorization $X \to S \to Y$.

**Proof.** If $\varphi$ is constant then $\varphi = (\varphi \circ \sigma) \circ h$, where $h : X \to S$ is the structure map. Conversely, if $\varphi$ has a factorization $X \to S \to Y$ then $\varphi_T(\text{Hom}_S(T, X))$ is the set consisting of one element; namely, the composition $T \to S \to Y$. □

**Proposition 2.4.** Let $k$ be a perfect field, $A$ a finite scheme over $k$, and $X$ a connected reduced scheme over $k$ which admits a $k$-section. Then every morphism $\varphi : X \to A$ of $k$-schemes is constant.

**Remark 2.5.** As $k$ is perfect, $X$ is reduced if and only if it is geometrically reduced (EGA IV, part 2, prop. 4.6.1). As $X$ admits a $k$-section, it is connected if and only if it is geometrically connected (loc. cit., cor. 4.5.14).

**Proof of Proposition 2.4.** In view of Lemma 2.2 and Remark 2.5, we may assume that $k$ is algebraically closed. We may suppose, moreover, that $X$ is affine.

Let $X = \text{spec}(R)$ and $A = \text{spec}(M)$, where $R$ and $M$ are $k$-algebras. Let $\Phi : M \to R$ be the $k$-algebra homomorphism corresponding to $\varphi$. Let $N \subseteq R$ be the image of $\Phi$. Since $R$ has no nontrivial idempotents, $N$ is a finite dimensional local $k$-algebra. As $R$ is reduced, $N$ is isomorphic to $k$. By Lemma 2.3, $\varphi$ is constant. □
Remark 2.6. (M. Boyarchenko) Let $X$ be as in Proposition 2.4, $S$ be a reduced scheme over $k$, and $A$ be a finite scheme over $S$. Then one can show that every $S$-morphism $X \times_{\text{spec}(k)} S \to A$ is constant. On the other hand, if $S$ is not reduced, then this results fails. For example, let $T$ be a $k$-algebra containing a nonzero element $u$ satisfying $u^d = 0$, for some positive integer $d$. Let $X = \mathbb{A}_k^1$, $S = \text{spec}(T)$, and $A = \text{spec}(T[\epsilon]/\epsilon^d)$. The map $\epsilon \mapsto u \cdot x$ extends to a $T$-algebra homomorphism $T[\epsilon]/\epsilon^d \to T[x]$. The corresponding morphism $X \times_{\text{spec}(k)} S \to A$ of schemes over $S$ is not constant.

2.2. Fundamental group schemes of algebraic groups. Let $k$ be a perfect field. Let $\mathfrak{G}$ denote the category of connected algebraic groups over $k$.

Lemma 2.7. Fiber products exists in $\mathfrak{G}$.

Proof. Given a diagram $G_1 \to G \leftarrow G_2$ in $\mathfrak{G}$, its fiber product equals $(G_1 \times_G G_2)_{\text{red}}$: the reduced neutral connected component of the scheme-theoretic fibre product $G_1 \times_G G_2$. □

Definition 2.8. A morphism $\tilde{G} \to G$ in $\mathfrak{G}$ is an isogeny if it is surjective and its kernel is finite. (Of course, the kernel is computed in the category of all group schemes.)

Definition 2.9. A group cover of $G$ is a connected algebraic group $\tilde{G}$ equipped with an isogeny $\tilde{G} \to G$. The group $\tilde{G}$ is a central cover if the kernel of $\tilde{G} \to G$ is central.³

Let $\text{Cov}(G)$ denote the category of coverings of $G$. (An arrow in $\text{Cov}(G)$ is a morphism $\tilde{G} \to \tilde{G}'$ of group schemes over $G$.) It is clear that $\text{Cov}(G)$ is essentially small.

Lemma 2.10. $\text{Cov}(G)$ is anti-equivalent to a partially ordered directed set $I = I(G)$.

Proof. Proposition 2.4 shows that there is at most one morphism between two coverings of $G$. Thus, $\text{Cov}(G)$ is a partially ordered set. The supremum of two element of $\text{Cov}(G)$ is given by their fibre product (Lemma 2.7).

³It is easy to show that an étale group cover is central.
corresponding to group covers of \( G \) with central, étale, and discrete kernel. (A discrete group scheme is a finite étale scheme on which the Galois group acts trivially.)

Every element of \( I \) dominated by an element of \( I_{\text{cent}} \) belongs to \( I_{\text{cent}} \). Furthermore, the supremum of two elements of \( I_{\text{cent}} \) belongs to \( I_{\text{cent}} \). These facts remain true if \( I_{\text{cent}} \) is replaced by \( I_{\text{et}} \) or \( I_{\text{disc}} \).

**Convention 2.11.** Set

\[
\Pi_{\text{cent}}(G) := \lim_{\longleftarrow} A_i, \\
\Pi_{\text{et}}(G) := \lim_{\longleftarrow} A_i, \\
\Pi_{\text{disc}}(G) := \lim_{\longleftarrow} A_i.
\]

We refer to these profinite group schemes as the *fundamental group schemes* of \( G \).

**Remark 2.12.** The group scheme \( \Pi_{\text{et}}(G) \) is the maximal pro-étale quotient of \( \Pi_{\text{cent}}(G) \). The group \( \Pi_{\text{disc}}(G) \) is the maximal quotient of \( \Pi_{\text{et}}(G) \) on which the absolute Galois group of \( k \) acts trivially.

The following table summarizes our computations of fundamental group schemes of certain connected algebraic groups. See Appendix B for the details.

**Convention 2.13.** For a semisimple group \( G \), \( \pi_{1}^{\text{ss}}(G) \) denotes the weight lattice modulo the root lattice \([\text{Bor69}], \S 24.1\). For an algebraic group \( G \), \( \pi_{1}(G) \) denotes the algebraic fundamental group of \( G \) \([\text{GR72}]\). In characteristic zero, \( \pi_{1}^{\text{ss}}(G) = \pi_{1}(G) \).

| \( G \) defined over \( \mathbb{C} \) | \( \Pi_{\text{et}}(G) = \pi_{1}(G) \) |
|----------------|------------------|
| \( G \) semisimple | \( \Pi_{\text{et}}(G) = \pi_{1}^{\text{ss}}(G) \) |
| \( G \) commutative defined over \( \mathbb{F}_q \) | \( \Pi_{\text{disc}}(G) = G(\mathbb{F}_q) \) |
| \( G = \mathbb{G}_a \) additive group over \( \mathbb{F}_q \) | \( \Pi_{\text{et}}(G) = \text{Hom}(k, \mathbb{Q}/\mathbb{Z}) \) |

**2.3. Classifying central extensions.** Let \( A \) be a finite commutative group scheme over \( k \). Let

\[
1 \to A \to \tilde{G} \to G \to 1
\]

be a central extension. Then \( (\tilde{G})_{\text{red}}^0 \to G \) is a central cover. Let \( f : \Pi_{\text{cent}}(G) \to A \) denote the composition

\[
\Pi_{\text{cent}}(G) \to \ker((\tilde{G})_{\text{red}}^0 \to G) \hookrightarrow A.
\]
Proposition 2.14. The map $\tilde{G} \mapsto f_{\tilde{G}}$ defines a canonical isomorphism
\begin{equation}
H^2(G,A) \xrightarrow{\sim} \text{Hom}(\Pi_{\text{cent}}(G), A).
\end{equation}
If $A$ is étale (resp. discrete), then we can replace $\Pi_{\text{cent}}(G)$ by $\Pi_{\text{et}}(G)$ (resp. $\Pi_{\text{disc}}(G)$).

Proof. Let $\varphi \in \text{Hom}(\Pi_{\text{et}}(G), A)$. Let $B := \text{im}(A)$. The group $B$ is a finite quotient of $\Pi_{\text{cent}}(G)$; thus, $B \cong A_i$ for some $i \in I_{\text{cent}}$. Let $G^\varphi$ be the pushforward of the central extension
\begin{equation}
1 \to A_i \to G_i \to G \to 1
\end{equation}
along the morphism $A_i \xrightarrow{\sim} B \hookrightarrow A$. One checks that $\varphi \mapsto G^\varphi$ is the inverse of $\tilde{G} \mapsto f_{\tilde{G}}$. \hfill \Box

2.4. $\Pi_{\text{et}}$ is a quotient of $\pi_1$. In this section, we assume $k$ is algebraically closed. Let $\pi_1(G) := \pi_1(G,e)$ denote the fundamental group of $G$ in the sense of [GR72]. For every abelian group $A$ one has a homomorphism
\begin{equation}
\text{Hom}(\Pi_{\text{et}}(G), A) \xrightarrow{f_A} \{\text{central extensions of } G \text{ by } A\}
\end{equation}
\begin{equation}
\text{Hom}(\pi_1(G), A) \xrightarrow{\text{central extensions of } G \text{ by } A} \{\text{A-torsors on } G \text{ trivialized over } e\}.
\end{equation}
Since $f_A$ is functorial in $A$, it comes from a homomorphism $f : \pi_1(G) \to \Pi_{\text{et}}(G)$.

Lemma 2.15. (i) $f_A$ is injective.

(ii) $\varphi : \pi_1(G) \to A$ belongs to the image of $f_A$ if and only if the diagram
\begin{equation}
\pi_1(G \times G) \xrightarrow{m^*} \pi_1(G) \to A
\end{equation}
\begin{equation}
\pi_1(G) \times \pi_1(G) \quad A \times A
\end{equation}
commutes.

Proof. A based $A$-torsor $\tilde{G} \to G$ is a central extension if and only if $m^*\tilde{G}$ is isomorphic to $\tilde{G} \boxtimes \tilde{G}$ (as $A$-torsors on $G \times G$). This proves (i).

To prove (ii), let $\tilde{G} \to G$ denote the based $A$-torsor corresponding to $\varphi$. One checks that the commutativity of the diagram is equivalent
to the existence of an \((A \times A)\)-equivariant morphism of based schemes \(\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}\) such that the diagram

\[
\begin{array}{ccc}
\tilde{G} \times \tilde{G} & \xrightarrow{\tilde{m}} & \tilde{G} \\
\downarrow & & \downarrow \\
G \times G & \to & G
\end{array}
\]

commutes. Proposition 2.4 then implies that \(\tilde{m}\) satisfies the axioms defining a group.

The following is a reformulation Lemma 2.15.

**Corollary 2.16.** \(\Pi_{\text{et}}(G)\) equals the coequalizer of the following homomorphisms:

\[
\begin{align*}
\pi_1(G \times G)^{\text{ab}} & \xrightarrow{m_*} \pi_1(G)^{\text{ab}} \\
\pi_1(G \times G)^{\text{ab}} & \xrightarrow{\pi_1(G)^{\text{ab}} \times \pi_1(G)^{\text{ab}}} \pi_1(G)^{\text{ab}}
\end{align*}
\]

**Remark 2.17.** Corollary 2.16 remains valid if one replaces \(\Pi_{\text{et}}(G)\) with \(\Pi_{\text{cent}}(G)\) and \(\pi_1(G)\) with the fundamental group scheme of \(G\) defined \[Nors82\].

3. **Relationship between central covers of \(G\) and \([G,G]\)**

3.1. **Restricting central covers of \(G\) to \([G,G]\)**. Let \(k\) be a perfect field, \(G\) a connected algebraic group over \(k\), and \(\tilde{G} \to G\) a central cover (Definition 2.9). It is clear that \((\tilde{G} \times G[G,G])^{\text{red}}\) is a central cover of \([G,G]\).

**Convention 3.1.** We call \((\tilde{G} \times G[G,G])^{\text{red}}\) the *restriction* of \(\tilde{G} \to G\) to \([G,G]\). Restricting central covers defines a homomorphism

\[\Pi_{\text{cent}}([G,G]) \to \Pi_{\text{cent}}(G)\]

In the introduction, we mentioned that if \(\tilde{G} \to G\) is a nontrivial central cover of a *semisimple* non-simply-connected group \(G\), then the restriction of \(\tilde{G}\) to \([G,G]\) is nontrivial. We now give examples of covers of *unipotent* groups whose restrictions to \([G,G]\) is nontrivial.

**Example 3.2.** Let \(\tilde{G}\) be a connected noncommutative unipotent algebraic group over a field of positive characteristic. Let \(A\) be an arbitrary finite subgroup of the center \(Z(\tilde{G})\) such that \(A \cap [\tilde{G}, \tilde{G}] \neq \{1\}\).\footnote{For example, \(A\) can be any nontrivial finite subgroup of \(C(\tilde{G}) := Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]\). Note that \(\dim(C(\tilde{G})) > 0\) since the last nonzero term of the lower central series of \(\tilde{G}\) is a subgroup of \(C(\tilde{G})\). In particular, \(C(\tilde{G})\) has many finite subgroups.} Let
$G := \tilde{G}/A$. Then $\tilde{G} \overset{\pi}{\longrightarrow} G$ is a central cover of $G$ whose restriction to $[G, G]$ is nontrivial.

**Convention 3.3.** For a central cover $\tilde{G} \rightarrow G$, let $d(\tilde{G})$ denote the degree of the central cover $\left(\tilde{G} \times_G [G, G]\right)_\text{red}^0 \rightarrow [G, G]$.

**Theorem 3.4.** There exists a constant $C$, depending only on $G$, such that $d(\tilde{G}) < C$ for all central covers $\tilde{G} \rightarrow G$.

In view of Remark 2.12, the corresponding results for $\Pi_{\text{et}}$ and $\Pi_{\text{disc}}$ follow immediately. In particular, the image of the map

$$\Pi_{\text{et}}([G, G]) \rightarrow \Pi_{\text{et}}(G)$$

is finite (Theorem 1.2).

To prove Theorem 3.4, we need a lemma. For every positive integer $n$, let $c^n : G^{2n} \rightarrow [G, G]$ denote the map

$$(3.1) \quad (g_1, g_2, g_3, g_4, \ldots, g_{2n-1}, g_{2n}) \mapsto [g_1, g_2][g_3, g_4]\cdots[g_{2n-1}, g_{2n}]$$

**Lemma 3.5.** There exists a positive integer $n$ such that $c^n$ is surjective.

**Proof.** See, for instance, [Mil06], cor. 11.13. □

The following remark is a key observation used in the proof of Theorem 3.4.

**Remark 3.6.** Let $\tilde{G}$ be a central cover of $G$ and let $H$ be its restriction to $[G, G]$. The commutator map $\tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ descends to a morphism of algebraic varieties $G \times G \rightarrow H$. In this case, we say that the commutator map of $G$ lifts to $H$.

**Proof of Theorem 3.4.** Choose $n$ large enough so that $c^n$ is surjective. Remark 3.6 implies that $c^n$ has a lift to $H$; that is to say, there exists a morphism of algebraic varieties $G^{2n} \rightarrow H$ such that the following diagram commutes:

$$\begin{CD}
H @>>> \Pi_{\text{et}}(G) \\
@VVV @VVV \\
G^{2n} @>>{c^n}> [G, G]
\end{CD}$$

(3.2)

In fact, one can take $n = 2 \dim(G)$. 

As \( c^n \) is surjective, there exists a closed subvariety \( X \subseteq G^{2n} \) such that the generic fiber of the morphism \( c^n|_X : X \to [G,G] \) is finite. The number \( d(G) \) divides the degree of \( c^n|_X \); hence, it is bounded. \( \square \)

### 3.2. Lifting central covers from \([G,G]\) to \(G\)

We keep the notation and conventions of the previous section. Let \( H \to [G,G] \) be a central cover. The following result is a converse of Remark 3.6.

**Proposition 3.7.** Suppose the commutator map of \( G \) lifts to \( H \). Then there exists a central cover \( \tilde{G} \to G \) whose restriction to \([G,G]\) is isomorphic to \( H \).

The proof of this proposition will take up the rest of this section. Let \( H \) be a central cover of \([G,G]\) which has a lift of the commutator map. Let \( \delta : H \to G \) denote the composition \( H \to \delta : [G,G] \to G \).

**Lemma 3.8.** One can endow \( H \xrightarrow{\delta} G \) with a structure of is a strictly stable crossed module (Definition A.7).

**Proof.** Let \( \{-,-\} : G \times G \to H \) denote the lift of the commutator. We may assume that \( \{1,1\} = 1 \). Define a morphism of varieties \( G \times H \to H \) by
\[
(g,h) \mapsto h^g := h \{ \delta(h), g \}.
\]
Using Proposition 2.4, it is easy to show that this morphism defines an action of \( G \) on \( H \) making \( H \xrightarrow{\delta} G \) into a strictly stable crossed module. \( \square \)

In the proof of Proposition 3.7, we employ a theorem about extension of sheaves. To state this result, we need some notation. Let \( k_{\text{fppf}} \) denote category of schemes over \( k \) equipped with the topology generated by faithfully flat morphisms of finite type. In what follows, \( \text{Ext} \) will denote the extension of abelian sheaves in \( k_{\text{fppf}} \). Let \( \text{Ab} \) (resp. \( \text{Fin}_{\text{k}} \)) denote the category of finite abelian group (resp. the category of finite commutative group schemes over \( k \)). Let \( J \) be a connected commutative algebraic group over \( k \).

**Theorem 3.9.** \( \text{Ext}^2(J,-) : \text{Fin}_k \to \text{Ab} \) is an effaceable functor; that is to say, given \( A \in \text{Fin}_k \) and \( \alpha \in \text{Ext}(J,A) \), there exists a monomorphism \( A \hookrightarrow B \) in \( \text{Fin}_k \), such that the image of \( \alpha \) under the map \( \text{Ext}^2(J,A) \to \text{Ext}^2(J,B) \) is zero.

**Remark 3.10.** This theorem was formulated and proved by V. Drinfeld (unpublished). It is closely related to the fact that \( \text{Ext}^2(G,\mathbb{Q}/\mathbb{Z}) \) vanishes. For a proof of this vanishing result see [Boy07], lem. 3.2.2.

\( ^6 \)Note that we do not use the fact that \( H \) is connected.
Convention 3.11. All Picard stacks considered are assumed to be strictly commutative, see Definition A.2.

Proof of Proposition 3.7. Step 1: Let $H$ be a central cover of $[G, G]$ equipped with a lift of the commutator map. By Lemma 3.8, $\delta : H \to G$ is a strictly stable crossed module. As explained in §§A.3-A.4, this implies that the quotient stack $\mathcal{P}$ of $G$ by the action of $H$ is a Deligne-Mumford Picard stack. Set $A := \ker(\delta)$. Note that

\[ \pi_0(\mathcal{P}) = \coker(\delta) = G^{ab}, \quad \text{and} \quad \pi_1(\mathcal{P}) = \ker(\delta) = A. \]  

Step 2: It follows from [Del73], prop. 1.4.15, that the set of isomorphism classes of Picard stacks with $\pi_0 = G^{ab}$ and $\pi_1 = A$ equals $\text{Ext}^2(G^{ab}, A)$. Let $\alpha \in \text{Ext}^2(G^{ab}, A)$ denote a representative for the isomorphism class of $\mathcal{P}$. By Theorem 3.9, there exists a monomorphism $\kappa : A \to B$ of finite commutative group schemes, such that the image of $\alpha$ under the induced morphism $\text{Ext}^2(G^{ab}, A) \to \text{Ext}^2(G^{ab}, B)$ is zero. Let $H'$ be the unique central extension of $[G, G]$ for which the following diagram is commutative\footnote{$H'$ is the pushforward of $H$ with respect to $\kappa$.}

\[ \begin{array}{cccccc}
1 & \to & A & \to & H & \to & [G, G] & \to & 1 \\
\kappa \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & B & \to & H' & \to & [G, G] & \to & 1.
\end{array} \]

Step 3: The composition $G \times G \xrightarrow{(-,-)} H \to H'$ endows $H'$ with a lift of the commutator map. Applying Lemma 3.8, we conclude that $H' \to G$ is a strictly stable crossed module. Let $\mathcal{P}'$ denote the corresponding quotient stack. The class of $\mathcal{P}'$ in $\text{Ext}^2(G^{ab}, B)$ is trivial; therefore,

$\mathcal{P}' \cong G/H \times B - \text{Tors}$.

Here $G/H$ is the discrete Picard stack defined by the commutative algebraic group $G/H$ (Example A.3), and $B - \text{Tors}$ denotes the (Deligne-Mumford Picard) stack of $B$-torsors on $k_{\text{fppf}}$. The composition

$G \to \mathcal{P}' := [G/H'] \to G \times B - \text{Tors} \to B - \text{Tors},$

defines a 1-morphism of gr-stacks $G \to B - \text{Tors}$.

Step 4: Let $\text{Hom}(G, B - \text{Tors})$ (resp. $\text{Cent}(G, B)$) denote the Picard stack of 1-morphisms of gr-stacks $G \to B - \text{Tors}$ (resp. central extensions of $G$ by $B$). According to [Gro72], §1.1, these two Picard stacks are naturally equivalent. Therefore, the 1-morphism of gr-stacks
$G \to B - \text{Tors}$ defined in the previous step, gives rise to a central extension

$$1 \to B \to G' \to G \to 1.$$ 

This central extension is a lift of the central extension corresponding to the 1-morphism of gr-stacks

$$[G, G] \to A - \text{Tors} \xrightarrow{\kappa} B - \text{Tors}.$$ 

It follows that the restriction of $G'$ to $[G, G]$ equals $H'$. Moreover, $(H')^0_{\text{red}} \cong H$, as required. □

4. True commutator and stacky abelianization

4.1. Stacky abelianization is Picard. Let $k$ be an algebraically closed field and $G$ be a connected algebraic group over $k$. Let $A^\text{et}$ be the image of $\Pi_{\text{et}}([G, G]) \to \Pi_{\text{et}}(G)$. By Theorem 3.4, $A^\text{et}$ is finite. By Proposition 2.14, we obtain a central extension

$$1 \to A^\text{et} \to [G, G]^\text{true} \to [G, G] \to 1.$$ 

We call $[G, G]^\text{true}$ the true commutator of $G$. We refer to the quotient stack of $G$ by the action of $[G, G]^\text{true}$ as the stacky abelianization of $G$ and denote it by $G_{\text{ab, st}}$.

Example 4.1. Let $G$ be a connected reductive group over an algebraically closed field of characteristic zero. Let $G^\text{ad} := G/Z(G)$ be the associated adjoint semisimple group. The group cover $[G, G] \to G^\text{ad}$ defines an injection $\Pi_{\text{et}}([G, G]) \to \Pi_{\text{et}}(G^\text{ad})$. It follows that the natural morphism $\Pi_{\text{et}}([G, G]) \to \Pi_{\text{et}}(G)$ is also injective. As we will see in §B.2, $\Pi_{\text{et}}([G, G]) = \pi_1^\text{ss}([G, G])$. Therefore $[G, G]^\text{true}$ identifies with the simply connected cover of $[G, G]$.

Lemma 4.2. The true commutator is the restriction of a central cover of $G$ (Conventions 3.1).

To prove this lemma, we need an easy result from the theory of profinite groups whose proof we omit.

Lemma 4.3. Let $A \hookrightarrow C$ be an inclusion of a finite group into a profinite abelian group. Then there exists an epimorphism $C \twoheadrightarrow B$, where $B$ is a finite group, such that the composition $A \hookrightarrow C \twoheadrightarrow B$ is an injection.

Proof of Lemma 4.2. By Lemma 4.3, there exists a finite quotient $B$ of $\Pi_{\text{et}}(G)$ such that the composition $A^\text{et} \hookrightarrow \Pi_{\text{et}}(G) \twoheadrightarrow B$ is an injective morphism. Let $\tilde{G}$ be the central cover of $G$ corresponding to $\Pi_{\text{et}}(G) \twoheadrightarrow B$. The restriction of $\tilde{G}$ to $[G, G]$ is isomorphic to $[G, G]^\text{true}$. □
Corollary 4.4. The commutator map lifts to $[G, G]^{\text{true}}$.

Proof. This follows from Lemma 4.2 and Remark 3.6.

Corollary 4.5. $G^{\text{ab, st}}$ is a (strictly commutative) Picard stack.

Proof. By Corollary 4.4 and Lemma 3.8, $[G, G]^{\text{true}} \to G$ is a strictly stable crossed module. Therefore, the corresponding quotient is a Picard stack.

4.2. Characterization of the true commutator. We keep the notation and conventions of the previous section. Let $H$ be an étale central cover of $[G, G]$.

Lemma 4.6. The following are equivalent:

(i) The pullback of every central extension of $G$ - by an étale group scheme - to $H$ is trivial.

(ii) For every étale central cover $\tilde{G} \xrightarrow{\pi} G$, we have a morphism $H \to \pi^{-1}([G, G])^0_{\text{red}}$ of algebraic groups over $[G, G]$.

Corollary 4.7. Assume that the commutator map lifts to $H$. Then we have a morphism $[G, G]^{\text{true}} \to H$ of algebraic groups over $[G, G]$.

Proof. By Proposition 3.7, $H$ is a restriction of an étale central cover of $G$. By the previous lemma, we obtain a morphism $[G, G]^{\text{true}} \to H$.

Proposition 4.8. The true commutator is the unique, up to isomorphism, étale central cover of $[G, G]$ satisfying the following properties:

P1) The pullback of every central extension of $G$ - by an étale group scheme - to $[G, G]^{\text{true}}$ is trivial.

P2) The commutator map lifts to $[G, G]^{\text{true}}$.

Proof of Proposition 4.8. Let $H$ be an étale central cover of $[G, G]$. Suppose the true commutator lifts to $H$. Then by Corollary 4.7, we have a morphism $[G, G]^{\text{true}} \to H$. Assume, furthermore, that the pullback of every central extension of $G$ - by an étale group scheme - to $H$ is trivial. In view of the fact that $[G, G]^{\text{true}}$ is a restriction of an étale cover of $G$, Lemma 4.6 provides us with a morphism $H \to [G, G]^{\text{true}}$.

4.3. Universal property of stacky abelianization.

Convention 4.9. All stacks discussed below are Deligne-Mumford stacks. The 2-category of Deligne-Mumford gr-stacks is denoted by $\text{DMgrst}$. All Picard stacks are assumed to be strictly commutative.

Let $\Phi : G \to G^{\text{ab, st}}$ denote the canonical 1-morphism of gr-stacks.
Theorem 4.10. Let $\mathcal{P}$ be a Picard stack over $k$. Then composition with $\Phi$ defines an equivalence of Picard groupoids

$$\text{Hom}_{\text{DMgrst}}(G^{\text{ab, st}}, \mathcal{P}) \sim \rightarrow \text{Hom}_{\text{DMgrst}}(G, \mathcal{P}).$$

Lemma 4.11. Let $C$ and $C'$ be gr-stacks over $k$. Assume that $\pi_0(C)$ is representable by a connected algebraic group over $k$. Then, the groupoid $\text{Hom}_{\text{DMgrst}}(C, C')$ is discrete (i.e., the objects have no nontrivial automorphisms).

Proof. Let $F$ be a 1-morphism of stacks $C \rightarrow C'$. By Remark A.6, a 2-morphism $F \Rightarrow F$ defines a morphism of schemes $\epsilon : \pi_0(C) \rightarrow \pi_1(C')$. As $C'$ is a Deligne-Mumford stack, $\pi_1(C')$ is finite. By Proposition 2.4, $\epsilon$ is constant.

Proof of Theorem 4.10. In view of the above lemma, it is enough to show that every 1-morphism of gr-stacks $F : G \rightarrow \mathcal{P}$ has a canonical factorization

$$G \rightarrow G^{\text{ab, st}} \xrightarrow{\Phi} \mathcal{P}.$$  

By the First Isomorphism Theorem for gr-stacks (Lemma A.14) and the remarks following it, there exists a strictly stable crossed module of algebraic groups $\delta : \tilde{H} \rightarrow G$ such that $F$ has a factorization $G \rightarrow [G/\tilde{H}] \rightarrow \mathcal{P}$. Let $H := \delta(\tilde{H})$. Then $H$ is a subgroup of $[G, G]$ containing $[G, G]$, and $\tilde{H}$ is a central extension of $H$ by an étale group scheme. Without loss of generality, we may assume $H = [G, G]$.

By Remark 4.7, we have a morphism of algebraic groups $[G, G]^{\text{true}} \rightarrow H$, which in turn, defines a morphism of strictly stable crossed modules $([G, G]^{\text{true}} \rightarrow G) \rightarrow (H \rightarrow G)$. Hence, we obtain a 1-morphism of Picard stacks $G^{\text{ab, st}} = [G/[G, G]^{\text{true}}] \rightarrow [G/\tilde{H}] \xrightarrow{\sim} \mathcal{P}$, providing the required factorization. □

Appendix A. Crossed modules and gr-stacks

A.1. Gr-categories.

Convention A.1. All categories we consider are essentially small. A monoidal category is denoted by $(\mathcal{M}, \otimes, 1)$. In other words, we suppress the associativity and unit constraints [Mac98]. Occasionally, we suppress $\otimes$ and $1$ as well. A monoidal category is strict if the associativity and unit constraints are trivial. With the usual abuse of notation, $x \in \mathcal{M}$ means $x$ is an object of $\mathcal{M}$. We denote the set of isomorphism classes of objects of $\mathcal{M}$ with $\pi_0(\mathcal{M})$. For every $x \in \mathcal{M}$, $\pi_1(\mathcal{M}, x)$ denotes the abelian group $\text{Aut}_\mathcal{M}(x)$. We set $\pi_1(\mathcal{M}) := \pi_1(\mathcal{M}, 1)$. 

Definition A.2. A $gr$-category is a monoidal groupoid all of whose objects have a weak inverse; that is to say, for every $x \in \mathcal{M}$ there exists $y \in \mathcal{M}$ such that $x \otimes y \cong y \otimes x \cong 1$. $\mathcal{M}$ is a strict $gr$-category if it is a strict monoidal groupoid such that for every $x \in \mathcal{M}$ there exists $y \in \mathcal{M}$ satisfying $x \otimes y = y \otimes x = 1$. A (strictly commutative) Picard groupoid is a (strictly) symmetric $gr$-category $[Del73]$, §1.4.

Example A.3. Let $G$ be a group. The discrete groupoid whose set of objects equals $G$ is a $gr$-category which, by an abuse of notation, is also denoted by $G$. If $G$ is commutative, the corresponding $gr$-category is a strictly commutative Picard groupoid.

Remark A.4. Let $\mathcal{M}$ be a $gr$-category. For every $x \in \mathcal{M}$, the map $u \mapsto u \otimes \text{id}_x$ defines an isomorphism $\pi_1(\mathcal{M},1) \cong \pi_1(\mathcal{M},x)$.

Definition A.5. A $1$-morphism of $gr$-categories (resp. Picard groupoids) is a monoidal functor (resp. symmetric monoidal functor) $F : \mathcal{M} \to \mathcal{M}'$. $F$ is a monomorphism (resp. epimorphism, resp. isomorphism) if it is fully faithful (resp. essentially surjective, resp. equivalence). Gr-categories and Picard groupoids form a 2-category where 2-morphism are monoidal natural transformations.

Remark A.6. Let $\mathcal{M}$ and $\mathcal{M}'$ be $gr$-categories. Let $F$ be a (not necessarily monoidal) functor $\mathcal{M} \to \mathcal{M}'$, and let $\eta$ be a natural transformation $F \Rightarrow F$. For every $m \in \mathcal{M}$, we obtain an element of $\text{Aut}_{\mathcal{M}'}(F(m))$ which, by the identification of Remark A.4, gives us an element of $\pi_1(\mathcal{M}')$. One checks that this element depends only on the isomorphism class of $m$ in $\mathcal{M}$. Therefore, we obtain a well-defined map $\eta_0 : \pi_0(\mathcal{M}) \to \pi_1(\mathcal{M}')$. Note that if $\eta_0$ is trivial, then so is $\eta$.

A.2. Crossed modules.

Definition A.7. A (right) crossed module $\mathfrak{G}$ is the data consisting of a group homomorphism $\delta : H \to G$ and a right action of $G$ on $H$, denoted by $h \mapsto h^g$, such that for every $g \in G$ and $h_1, h_2 \in H$,

$$(h_2)^{\delta(h_1)} = h_1^{-1}h_2h_1, \quad \delta(h^g) = g^{-1}\delta(h)g.$$ $\mathfrak{G}$ is strictly stable if there exists a map $\{ -, - \} : G \times G \to H$ such that for every $g, g_0, g_1, g_2 \in G$ and $h, h_0, h_1 \in H$, we have:

1. $\delta\{g_1, g_2\} = [g_1, g_2]$
2. $\{\delta h_1, \delta h_2\} = [h_1, h_2]$
3. $\delta h, g = h^{-1}(h^g)$
4. $\{g, \delta h\} = (h^g)^{-1}h$
5. $\{g_0, g_1g_2\} = \{g_0, g_2\}\{g_0, g_1\}^{g_2}$
6. $\{g_0g_1, g_2\} = \{g_0, g_2\}^{g_1}\{g_1, g_2\}$
(7) \(\{g_1, g_2\}\{g_2, g_1\} = 1\)
(8) \(\{g, g\} = 1\).

In view of axiom (1), \(\{-, -\}\) is called a lift of the commutator map to \(H\). By an abuse of notation, we denote a crossed module (or a strictly stable crossed module) by \(H \rightarrow G\).

**Definition A.8.** Let \(\mathfrak{G} := (H \rightarrow G)\) and \(\mathfrak{G}' := (H' \rightarrow G')\) be crossed modules. A morphism of crossed modules is a pair of homomorphisms \(a : H \rightarrow H'\) and \(b : G \rightarrow G'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
H & \rightarrow & G \\
\downarrow^a & & \downarrow^b \\
H' & \rightarrow & G',
\end{array}
\]

and for every \(h \in H\) and \(g \in G\), \(a(h^g) = a(h) b(g)\).

Suppose that \(\mathfrak{G}\) and \(\mathfrak{G}'\) are strictly stable crossed modules, and let \(\{-, -\}\) and \(\{-, -\}'\) be their respective lift of the commutator map. Then \((a, b)\) is a morphism of strictly stable crossed modules if the following diagram commutes:

\[
\begin{array}{ccc}
G \times G & \rightarrow & H \\
\downarrow^a \downarrow^b & & \\
G' \times G' & \rightarrow & H'.
\end{array}
\]

**A.3. Relationship between gr-categories and crossed modules.**

**Crossed modules to gr-categories:** The passage from crossed modules to gr-categories uses the notion of quotient groupoid.

**Definition A.9.** Let \(\Gamma\) be a group acting on a set \(X\). The **quotient groupoid** \([X/\Gamma]\) is the groupoid whose objects are the elements of \(X\). An arrow \(x \rightarrow x'\) in \([X/\Gamma]\) is an element \(\gamma \in \Gamma\) such that \(\gamma.x = x'\).

Let \(H \rightarrow G\) be a crossed module. Then \(H\) acts on \(G\) by right translation. One can show that \([G/H]\) is a gr-category. Furthermore, a morphism of crossed modules defines a **strict** 1-morphism between the corresponding gr-categories. These facts remain true if we replace crossed modules by strictly stable crossed modules and gr-categories by strictly commutative Picard groupoids; see, for instance, [Con84], [Bre92], [Bre94], and [Noo05].

**Gr-categories to crossed modules:** Let \(\mathcal{M}\) be a gr-category. Let \(\mathcal{M}'\) be a strict model of \(\mathcal{M}\); that is to say, \(\mathcal{M}'\) is a strict gr-category which is equivalent to \(\mathcal{M}\). Let \(G\) be the group of objects of \(\mathcal{M}'\) and let \(H\) be the group of arrows whose source is \(1\). There is a natural morphism
$H \to G$ given by taking an arrow to its target. Furthermore, $G$ acts on $H$ by conjugation. One can show that $H \to G$ is a crossed module and $[G/H]$ is equivalent to $\mathcal{M}$; see, for instance, [Noo05].

Our goal is to prove a first isomorphism theorem for gr-categories. Recall that the first isomorphism from group theory states that every homomorphism $f : \Gamma \to M$ has a canonical factorization $\Gamma \to \Gamma/(f^{-1}(1)) \hookrightarrow M$.

**Convention A.10.** Let $F : C \to D$ be a functor between groupoids. Let $d \in D$. Let $F^{-1}(d)$ denote the following groupoid:

- objects = objects of $C$ which map to $d$
- morphisms = morphisms of $C$ which map to $\text{id}_d$.

Let $G$ be a group and let $\mathcal{M} = (\mathcal{M}, \otimes, 1)$ be a gr-category. Let $F : G \to \mathcal{M}$ be a monoidal functor (Example A.3). Let $H := F^{-1}(1)$.

**Lemma A.11.** (First Isomorphism Theorem)

1. $H \to G$ is a crossed module.
2. $F$ has a factorization $G \to [G/H] \hookrightarrow \mathcal{M}$.

**Proof.** By [Noo06], thm. 7.10, $F$ has a factorization $G \xrightarrow{F'} \mathcal{M}' \xrightarrow{e} \mathcal{M}$ where $\mathcal{M}'$ is a strict gr-category, $F'$ is a strict monoidal functor, and $e$ is an isomorphism of gr-category. Let $H' := F'^{-1}(d)$. Then, $e$ defines an isomorphism of gr-categories $H' \xrightarrow{\simeq} H$. Therefore, we may assume that $\mathcal{M}$ and $F$ are strict.

By definition, the objects of $H$ are pairs $(x, \varphi)$ where $x \in G$ and $\varphi$ is an isomorphism $1 \to F(x)$ in $\mathcal{M}$. Define a multiplication on $H$ by

$$(x, \varphi).(y, \psi) := (xy, \varphi \otimes \psi).$$

Note that $\varphi \otimes \psi$ is an isomorphism $1 \to F(x) \otimes F(y) = F(xy)$; hence, this product is well-defined. As $\mathcal{M}$ is strictly associative, this product is associative. It is easy to check that the above multiplication makes $H$ into a group, where the unit is $(1, \text{id}_1)$. Furthermore, the map $H \to G$, defined by $(x, \varphi) \mapsto x$, is a group morphism.

Next, define an action of $G$ on $H$ by

$$(x, \varphi)g := (g^{-1}xg, \text{id}_{F(g)}^{-1} \otimes \varphi \otimes \text{id}_{F(g)}).$$

One checks that this map is indeed an action, making $H \to G$ into a crossed module. Finally, define a functor $K : [G/H] \to \mathcal{M}$ as follows:

$K(g) := F(g)$ for $g \in G$

$K(g \xrightarrow{h} g\delta(h)) := \text{id}_g \otimes (1 \xrightarrow{\delta(h)} \delta(h))$.

$K$ is a monomorphism of gr-categories, giving rise to the required factorization. \qed
Remark A.12. In the previous lemma, if $\mathcal{M}$ were a strictly commutative Picard groupoid, then $H \to G$ would be a strictly stable crossed module.

A.4. Gr-stacks and Picard stacks. Let $k$ be a field. Let $k_{\text{et}}$ denote the category of sheaves on the category of schemes over $k$, equipped with the étale topology. The 2-category of stacks (in groupoids) on $k_{\text{et}}$ is defined in [Gir71]. Roughly speaking, a stack $S$ is a sheaf of groupoids: for every scheme $U$ over $k$, the “sections” of $S$ above $U$ form a groupoid, which is denoted by $S_U$. The gluing conditions for these groupoids is best expressed using fibred categories and descent [GR72, Gir71]. In this text, we will only be concerned with Deligne-Mumford stacks [DM69].

Example A.13. Let $G$ be an algebraic group acting on an algebraic variety $X$. Suppose the stabilizer of every point of $X$ is an étale subgroup of $G$. Then the quotient stack $[X/G]$ is a Deligne-Mumford stack.

Gr-stacks (resp. Picard stack) are studied in [Bre92, §3 (resp. Del73)]. Roughly speaking, a gr-stack (resp. Picard stack) is a sheaf of group-categories (resp. Picard groupoids). It is easy to see that gr-stacks and Picard stacks form a 2-category. We omit the obvious analogue of the relationship between crossed modules and gr-stacks; see, for instance, [Bre92] and [Bre94]. The only result we use in the main body of the text is the First Isomorphism Theorem for Deligne-Mumford Picard stacks.

Let $G$ be a connected algebraic group over $k$. (Note that $G$ can also be considered as a Deligne-Mumford gr-stack on $k$.) Let $\mathcal{P}$ a Deligne-Mumford strictly commutative Picard stack and let $F : G \to \mathcal{M}$ be a 1-morphism of gr-stacks. Let $H = F^{-1}(1)$.

Lemma A.14. 

(i) $H$ is an algebraic group; that is to say, it is a reduced scheme of finite type over $k$.

(ii) $H \to G$ is a strictly stable crossed module.

(iii) $F$ has a factorization $G \to [G/H] \hookrightarrow \mathcal{P}$.

Proof. For every scheme $U$ over $\text{spec}(k)$, and every object $x \in H_U$, we have $\text{Aut}_{H_U}(x) = \{\text{id}\}$. By [LMB06, cor. 8.1.1 (iii), $H$ is representable (by an algebraic space). As $H$ has a group structure, we conclude that it is represented by a group scheme over $k$. Moreover, the kernel of the canonical morphism $H \to G$ equals $\pi_1(\mathcal{P})$ and is, therefore, an étale

\footnote{For every scheme $U$ over $k$, $[X/G]_U$ equals the quotient groupoid $[\text{Hom}(U, X)/\text{Hom}(U, G)]$, see Definition A.9}
group scheme over $k$. It follows that $H$ is an algebraic group over $k$; i.e., it is reduced of finite-type.

Statements (ii) and (iii) follow from Lemma A.11 and Remark A.12.

□

APPENDIX B. COMPUTING FUNDAMENTAL GROUP SCHEMES

B.1. **Characteristic zero.** Let $G$ be a connected algebraic over a field of characteristic zero. The Künneth formula implies that the two homomorphisms in (2.5) are equal. It follows that $\pi_1(G) = \Pi_{\text{et}}(G)$.

B.2. **Semisimple groups.** Let $G$ be a connected semisimple group over an algebraically closed field. Let $\pi^\text{ss}_1(G)$ denote the weight lattice modulo the root lattice. We claim that $\Pi_{\text{et}}(G) = \pi^\text{ss}_1(G)$.

It is enough to show that a connected simply connected semisimple algebraic group $G$ does not admit a nontrivial central extension by finite groups. Suppose $1 \to A \to \tilde{G} \xrightarrow{p} G \to 1$ is a central extension of $G$ where $A$ is a finite abelian group, and $\tilde{G}$ is connected. It is clear that $\tilde{G}$ is semisimple. Let $T$ be a connected maximal torus of $\tilde{G}$, and let $T := p(T)$ denote the corresponding maximal torus in $G$. Since the center of $\tilde{G}$ is a subgroup of $T$, we have a central extension $1 \to A \to \tilde{T} \to T \to 1$. This, in turn, defines an inclusion

$$X(T) \hookrightarrow X(\tilde{T}) \hookrightarrow \Lambda(T) = \Lambda(\tilde{T}),$$

where $X$ (resp. $\Lambda$) denotes the root (resp. weight) lattice. As $G$ is simply connected, $X(T) = \Lambda(T)$; thus, $X(T) = X(\tilde{T})$, implying that $A$ is trivial.

B.3. **Finite fields.** Let $\mathbb{F}_q$ denote a finite field with $q$ elements. Let $G$ be a connected algebraic group over $\mathbb{F}_q$. Let $\text{Fr}$ denote the Frobenius automorphism $x \mapsto x^q$. Let $A$ be a finite abelian group (considered as a discrete group scheme over $\mathbb{F}_q$). A central extension

$$1 \to A \to \tilde{G} \xrightarrow{\pi} G \to 1$$

defines a homomorphism\(^9\) $f_\pi : G(\mathbb{F}_q) \to A$ as follows. Given $g \in G(\mathbb{F}_q)$, pick $g' \in \pi^{-1}(g)$. Note that $\text{Fr}(g') = ag'$ for some $a \in A$.

Lemma B.1. (Boy07, §1.4.) The map $H^2(G, A) \to \text{Hom}(G(\mathbb{F}_q), A)$ given by the above construction is an injection. If $G$ is commutative, then it is an isomorphism.

\(^9\)The choice of the maximal torus does not matter.
\(^{10}\)This is an instance of Grothendieck’s sheaf-function correspondence. The sheaf in question, is the sheaf of local sections of $\pi$.  

\[ \text{Hom}(G(\mathbb{F}_q), A) \]
Corollary B.2. Let $G$ be a connected commutative algebraic group over $\mathbb{F}_q$. Then $\Pi_{\text{disc}}(G) = G(\mathbb{F}_q)$.

B.4. Additive group. Let $k$ be an algebraically closed field of positive characteristic. Let $G$ denote the additive group over $k$. Our aim is to prove

$$\Pi_{\text{et}}(G) = \text{Hom}(k, \mathbb{Q}/\mathbb{Z})$$

Remark B.3 (Pontryagin Duality). The category of profinite abelian groups is anti-equivalent to that of discrete torsion abelian groups: to a profinite abelian group $\Pi$, one associates $\Gamma := \text{Hom}_{\text{continuous}}(\Pi, \mathbb{Q}/\mathbb{Z})$. The inverse functor is $\Gamma \rightarrow \text{Hom}(\Gamma, \mathbb{Q}/\mathbb{Z})$.

By Pontryagin duality, it is enough to show that $H^2(G, \mathbb{Q}/\mathbb{Z}) = k$. As $G$ is connected and commutative, the natural injection $\text{Ext}(G, \mathbb{Q}/\mathbb{Z}) \hookrightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. Thus, it is enough to show that $\text{Ext}(G, \mathbb{Q}/\mathbb{Z}) = k$. On the other hand, $G$ is killed by multiplication by $p$; hence, the long exact sequence corresponding to

$$0 \to p\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \overset{p}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \to 0$$

shows that $\text{Ext}(G, \mathbb{Q}/\mathbb{Z}) = \text{Ext}(G, \mathbb{F}_p)$. Thus, we are reduced to proving $\text{Ext}(G, \mathbb{F}_p) = k$.

Observe that

$$\text{Ext}(G, \mathbb{F}_p) \subseteq H^1_{\text{et}}(G, \mathbb{F}_p) = H^1_{\text{et}}(\mathbb{A}^1_k, \mathbb{F}_p) = k[x]/A(k[x]),$$

where $A(u) = u^p - u$. Furthermore, every element of $H^1_{\text{et}}(G, \mathbb{F}_p)$ can be written as

$$h = \sum_{p^i} c_i \overline{x^i},$$

where $\overline{x^i}$ is the image of $x^i$ in the quotient $k[x]/A(k[x])$. By Lemma 2.15 $h \in \text{Ext}(G, \mathbb{F}_p)$ if and only if the polynomials

$$\sum_{p^i} c_i (x^i + y^i), \quad \text{and} \quad \sum_{p^i} c_i (x + y)^i$$

represent the same element of $H^1_{\text{et}}(\mathbb{A}^2, \mathbb{F}_p) = k[x, y]/A(k[x, y])$. This happens if and only if the polynomial $t(x, y) = \sum_{p^i} c_i [(x + y)^i - x^i - y^i]$ is of the form $u^p - u$. This is possible if and only if $c_i = 0$ for all $i \neq 1$ (otherwise, the degree of $t$ is not divisible by $p$). Therefore, $\text{Ext}(G, \mathbb{F}_p)$ identifies with

$$\{c\overline{x} | c \in k\} \subseteq k[x]/A(k[x]) = H^1_{\text{et}}(\mathbb{A}^1, \mathbb{F}_p).$$
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