The coarse Baum-Connes conjecture for certain group extensions and relative expanders

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Abstract

Let \((1 \to N_n \to G_n \to Q_n \to 1)_{n \in \mathbb{N}}\) be a sequence of extensions of finitely generated groups with uniformly finite generating subsets. We show that if the sequence \((N_n)_{n \in \mathbb{N}}\) with the induced metric from the word metrics of \((G_n)_{n \in \mathbb{N}}\) has property A, and the sequence \((Q_n)_{n \in \mathbb{N}}\) with the quotient metrics coarsely embeds into Hilbert space, then the coarse Baum-Connes conjecture holds for the sequence \((G_n)_{n \in \mathbb{N}}\), which may not admit a coarse embedding into Hilbert space. It follows that the coarse Baum-Connes conjecture holds for the relative expanders and group extensions exhibited by G. Arzhantseva and R. Tessera, and special box spaces of free groups discovered by T. Delabie and A. Khukhro, which do not coarsely embed into Hilbert space, yet do not contain a weakly embedded expander. This in particular solves an open problem raised by G. Arzhantseva and R. Tessera [3].

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1 Introduction

The coarse Baum-Connes conjecture is a geometric analogue of the Baum-Connes conjecture which has important applications to geometry, topology and analysis. More precisely, it states that a certain assembly map

$$\mu : \lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X))$$

for a metric space $X$ is an isomorphism, where on the left hand side $\lim_{d \to \infty} K_*(P_d(X))$ is the limit of the locally finite $K$-homology group of the Rips complexes of $X$, while on the right hand side $K_*(C^*(X))$ is the $K$-theory group of the Roe algebra of $X$. The left hand side of this conjecture is local and therefore computable, while the right hand side is global and is the receptacle of higher indices of elliptic operators. A positive answer to this conjecture would provide a complete solution to the problem of computing $K$-theoretic indices for elliptic operators on non-compact spaces. In particular, it implies the Novikov conjecture on homotopy invariance of higher signatures for closed manifolds when $X$ is a finitely generated group (e.g. the fundamental group of a closed manifold) equipped with a word length metric, and the Gromov’s zero-in-the-spectrum conjecture and the positive scalar curvature conjecture when $X$ is a Riemannian manifold. See [55] for a comprehensive survey for the coarse Baum-Connes conjecture, and [5, 9, 18, 19, 20, 21, 22, 24, 41, 48, 49, 50, 51, 53] for some recent developments.

The notion of coarse embedding into Hilbert space was introduced by M. Gromov [25, p. 211] in relation to the Novikov conjecture (1965). A sequence of metric spaces $(X_n, d_n)_{n \in \mathbb{N}}$ is said to admit a coarse embedding into Hilbert space, or briefly, CE, if there exist a sequence of maps $f_n : X_n \to H$ from $X_n$ to a Hilbert space $H$, and two proper maps $\rho_1$ and $\rho_2$ from $[0, \infty)$ to $[0, \infty)$ such that

$$\rho_1(d_n(x, x')) \leq \|f_n(x) - f_n(x')\| \leq \rho_2(d_n(x, x')).$$

for all $n \in \mathbb{N}$ and all $x, x' \in X_n$. The third author of this paper G. Yu established the coarse Baum-Connes conjecture for all metric spaces with bounded geometry which admit a coarse embedding into a Hilbert space [54]. In the same paper, G. Yu introduced a geometric version of amenability, called property $A$, which implies coarse embeddability into Hilbert space. A sequence of countable discrete metric spaces $(X_n)_{n \in \mathbb{N}}$ has Property $A$ if and only if for every $R > 0$ and $\varepsilon > 0$, there exist $S > 0$, and a sequence
of functions $\xi_n : X_n \to \ell^2(X_n)$ such that (0) $\|\xi_x\| = 1$; (1) if $d(x, x') \leq R$, then $\|\xi_x - \xi_{x'}\| \leq \varepsilon$; (2) $\text{Supp}(\xi) \subset \text{Ball}_{X_n}(x, S)$, for all $n \in \mathbb{N}$ and all $x, x' \in X_n$ (see [38]).

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $(1 \to N_n \to G_n \to Q_n \to 1)_{n \in \mathbb{N}}$ be a sequence of extensions of finitely generated groups with uniformly finite generating subsets. If the sequence $(N_n)_{n \in \mathbb{N}}$ with the induced metric from the word metrics of $(G_n)_{n \in \mathbb{N}}$ has property A, and the sequence $(Q_n)_{n \in \mathbb{N}}$ with the quotient metrics coarsely embeds into Hilbert space, then the coarse Baum-Connes conjecture holds for the sequence $(G_n)_{n \in \mathbb{N}}$.

We will say that a sequence of group extensions $(1 \to N_n \to G_n \to Q_n \to 1)_{n \in \mathbb{N}}$ is an "A-by-CE" extension, or has the "A-by-CE" structure, if it satisfies the assumptions in the theorem. Similarly, we may talk about "CE-by-CE", "CE-by-A", "abelian-by-Haagerup", and so on for obvious meanings. For a long time, weakly embedded expanders (see Definition 2.4 below) were the only known obstruction for a metric space with bounded geometry to coarsely embed into Hilbert space [26, 27, 36]. In [3], G. Arzhantseva and R. Tessera introduce the notion of relative expanders (see Definition 2.4 below) to construct the first sequence of finite Cayley graphs which does not coarsely embed into any $L^p$-space for any $1 \leq p < \infty$, nor into any uniformly curved Banach space, and yet does not admit any weakly embedded expander. In [14], T. Delabie and A. Khukhro construct a certain box space of a free group to answer in the affirmative an open problem asked in [3]: does there exist a sequence of finite graphs with bounded degree and large girth which does not coarsely embed into a Hilbert space and yet does not contain a weakly embedded expander? We observe that all these examples by G. Arzhantseva and R. Tessera, and T. Delabie and A. Khukhro have the A-by-CE group extension structure. Hence, we have

**Corollary 1.2.** The coarse Baum-Connes conjecture holds for all relative expanders exhibited by G. Arzhantseva and R. Tessera, and the special box spaces of free groups discovered by T. Delabie and A. Khukhro, which do not coarsely embed into Hilbert space, yet do not contain a weakly embedded expander.

This in particular solves an open problem raised by G. Arzhantseva and R. Tessera in [3] Section 8 Open Problems. For a single extension of groups, the above theorem may be restated as follows.

**Theorem 1.3.** Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of finitely generated groups. If $N$ has Property A and $Q$ coarsely embeds into Hilbert space, then the coarse Baum-Connes conjecture holds for $G$.

In [4], G. Arzhantseva and R. Tessera answer in the negative the following well-known question [12, 28]: Does coarse embeddability into Hilbert spaces is preserved under group extensions of finitely generated groups? Their constructions also provide the first example of finitely generated group which does not coarsely embed into a Hilbert space and yet does not contain weakly embedded expander, answering in the affirmative another open problem in [28], see also [3, 38]. The first group $\mathbb{Z} \wr_{G} H$ constructed by G. Arzhantseva and R. Tessera in [4] is an extension of two groups with the Haagerup property. It satisfies the strong Baum-Connes conjecture [29] which is strictly stronger than the Baum-Connes conjecture with coefficients [7], and consequently, the coarse Baum-Connes conjecture. It has been proved in [5] Proposition 2.11] by B. M. Braga, Y. C. Chung and K. Li that the second group $\mathbb{Z} \wr_{G} (H \times F_n)$ constructed in [11] also satisfies the Baum-Connes conjecture with coefficients by applying a permanence result of H. Oyono-Oyono [10], and hence the coarse Baum-Connes conjecture. Since both groups are A-by-CE split extensions, our result provides an alternative proof to these facts. Notice that the reason why $\mathbb{Z} \wr_{G} H$ and $\mathbb{Z} \wr_{G} (H \times F_n)$ do not coarsely embed into Hilbert space is that both groups contain isometrically
in their Cayley graphs a technical relative expander $W_n = \mathbb{Z}_2 \wr G_n H_n$, where $(G_n)$ is a sequence of finite groups whose Cayley graphs are Ramanujan graphs, and $H_n$ is a certain finite-sheeted cover of $G_n$ [11 Proposition 2.16]. We observe that this relative expander $W_n = \mathbb{Z}_2 \wr G_n H_n$ also has the A-by-CE structure (see Remark 2.10 below), and hence satisfies the coarse Baum-Connes conjecture as well. So, the coarse Baum-Connes conjecture holds for all currently known examples of spaces or groups which do not coarsely embed into Hilbert space, yet do not contain a weakly embedded expander.

When the group $G$ has a classifying space of finite type, Theorem 1.3, together with the decent principle, recovers the extension results, in the case where $N$ has property A, on the Novikov conjecture in [15, Theorem 1.1] by J. Deng, and in [17, Theorem 33] (together with [16, Theorem 61]) by H. Emerson and R. Meyer.

The basic strategy of the proof of Theorem 1.3 is to apply the fundamental ideas in [54] to the case of group extensions, by using localization algebras, twisted Roe algebras, and a geometric version of infinite dimensional Bott periodicity. We use the coarse embedding of the quotients $(Q_n)_{n \in \mathbb{N}}$ into a Hilbert space to furnish the twisted Roe algebra of $(G_n)_{n \in \mathbb{N}}$ and its localization algebra with twisted coefficients, and then apply cutting-and-pasting techniques to decompose these twisted algebras, so as to reduce the coarse Baum-Connes conjecture for $(G_n)_{n \in \mathbb{N}}$ to the coarse Baum-Connes conjecture for $(N_n)_{n \in \mathbb{N}}$. At this point, it is natural to expect to complete the proof merely under the assumption that $(N_n)_{n \in \mathbb{N}}$ coarsely embeds into a Hilbert space. However, there is a subtle issue about different completions of ideals inside the maximal twisted Roe algebras, for which we cannot settle under this assumption. Fortunately, this technical difficulty disappears under the assumption that the sequence $(N_n)_{n \in \mathbb{N}}$ has Property A.

The paper is organized as follows. In Section 2, we briefly review all the examples of relative expanders and group extensions constructed by G. Arzhantseva and R. Tessera, and a special box space of free group discovered by T. Delabie and A. Khukhro, which do not coarsely embed into Hilbert spaces and yet do not contain a weakly embedded expander. We in particular indicate that all these examples has the "A-by-CE" structure. In Section 3, we recall the concept of the Roe algebras, localization algebras, and the coarse Baum-Connes conjecture. In Section 4, we introduce uniformly twisted Roe algebras and uniformly twisted localization algebras for the extension groups with coefficients coming from the coarse embedding of the quotient groups into Hilbert space, and prove that the twisted coarse Baum-Connes conjecture for the sequence of extensions. In Section 5, we introduce a version of the geometric analogue of the infinite-dimensional Bott periodicity of Higson-Kasparov-Trout to complete the proof.

As a comparison, we mention that "CE-by-A" implies "CE". This was proved by M. Dadalart and E. Guentner in [12] for a single short exact sequence of groups, and by A. Khukhro [31] for a sequence of extensions of finite groups with uniformly finite generating subsets. So the following question is natural:

**Problem.** Does "CE-by-CE" implies the coarse Baum-Connes conjecture?

## 2 A-by-CE group extension structure of relative expanders and box spaces

In this section, we briefly review the recent discoveries of relative expanders and certain groups extensions due to G. Arzhantseva and R. Tessera, and of certain box spaces of free groups due to T. Delabie and A. Khukhro, which do not coarsely embed into Hilbert space and yet contain no weakly embedded expanders.
We observe that all these spaces or groups have “A-by-CE” structure as (sequences of) group extensions.

For a finite connected graph $X$ with $|X|$ vertices and a subset $A \subset X$, denote by $\partial A$ the set of edges between $A$ and $X \setminus A$. The Cheeger constant of $X$ is defined as

$$h(X) := \min_{1 \leq |A| \leq |X|/2} \frac{|\partial A|}{|A|}.$$  

An expander is a sequence $(X_n)_{n \in \mathbb{N}}$ of finite connected graphs with uniformly bounded degree, such that $|X_n| \to \infty$, and $h(X_n) \geq c$ uniformly over $n \in \mathbb{N}$ for some constant $c > 0$.

**Definition 2.1** (weakly embedded expander). Let $(X_n)_{n \in \mathbb{N}}$ be an expander and let $Y$ be a discrete metric space with bounded geometry. A sequence of maps $f_n : X_n \to Y$ is a weak embedding of $(X_n)_{n \in \mathbb{N}}$ into $Y$ if there exists $D > 0$ such that all $f_n$ are $D$-Lipschitz and

$$\lim_{n \to \infty} \sup_{x \in X_n} \frac{f_n^{-1}(f_n(x))}{|X_n|} = 0.$$  

Let us recall the concepts of semidirect product and wreath product of groups. Let $H$ be a group, let $Q$ be a group of automorphisms of $H$, and let $K$ be a group such that there is a surjective homomorphism $\phi : K \twoheadrightarrow Q$. The restricted semi-direct product $H \rtimes_Q K$ is the product $H \times K$ with the multiplication role

$$(h_1, k_1)(h_2, k_2) = (h_1 \cdot \phi(k_1)h_2, k_1k_2).$$

Note that a semi-direct product is a split extension.

Let $H$ and $K$ be finitely generated groups, and let $\phi : K \twoheadrightarrow Q$ be a surjective homomorphism from $K$ to a countable discrete group $Q$. The restricted permutational wreath product of $H$ by $K$ through $Q$ is the semi-direct product

$$H \wr_Q K := \bigoplus_Q H \rtimes K,$$

where $\bigoplus_Q H$ is the group of finitely supported functions $\xi : Q \to H$ with the pointwise multiplication, and $K$ acts on $\bigoplus_Q H$ by permuting the indices by multiplications on the left via $\phi : K \twoheadrightarrow Q$.

Let $S$ and $T$ be finite generating sets of $H$ and $K$, respectively. Then

$$\{(\delta_s, 1_K) : s \in S\} \cup \{(1, t) : t \in T\}$$

is a finite generating subset of $H \wr_Q K$, where

$$\delta_s(q) = \begin{cases} s, & \text{if } q = 1_Q; \\ 1_H, & \text{otherwise}, \end{cases}$$

and $1$ is the constant function in $\bigoplus_Q H$ such that $1(q) \equiv 1_H$ for all $q \in Q$.

### 2.1 Relative expanders à la G. Arzhantseva and R. Tessera

A well known obstruction for a metric space to coarsely embed into a Hilbert space is to admit a weakly embedded expander [26, 27]. A long standing open problem is: does a weakly embedded expander is the only obstruction to coarse embeddability into Hilbert space? In a ground breaking article [3], G. Arzhantseva and R. Tessera gave first examples of sequences of finite Cayley graphs of uniformly bounded degree which do not coarsely embed into a Hilbert space but do not contain any weakly embedded expander.
Theorem 2.2 ([3] Theorem 1 or 7.1). There exist a finitely generated residually finite group \( G \) and a box space \((Y_n)_{n \in \mathbb{N}}\) of \( G \) which does not coarsely embed into any \( L^p \) space for \( p \in [1, \infty) \), neither into any uniformly curved Banach space, and yet does not admit any sequence of weakly embedded expanders.

The proof of this theorem relies on two major observations. The first one says that there are no expanders weakly contained in "CE-by-CE" group extensions.

Proposition 2.3 (Observation 1). (Proposition 2 in [3]) A "CE-by-CE" sequence of group extensions does not contain any weakly embedded expander. More precisely, let

\[
  \left( 1 \to N_n \to G_n \to Q_n \to 1 \right)_{n \in \mathbb{N}}
\]

be a sequence of extensions of finitely generated groups with uniformly finite generating subsets, such that

1. for each \( n \in \mathbb{N} \), the group \( G_n \) is equipped with the word length induced by a given generating set \( S_n \);
2. the sequence \((N_n)_{n \in \mathbb{N}}\) equipped with the induced metric as subgroups of \( G_n \) admit a coarse embedding into a Hilbert space;
3. the sequence \((Q_n)_{n \in \mathbb{N}}\) equipped with the quotient metric from \( G_n \) admit a coarse embedding into a Hilbert space.

Then the sequence \((G_n)_{n \in \mathbb{N}}\) does not contain any weakly embedded expander.

The second observation is a refined strengthening to an early observation of John Roe [43] that relative property (T), as opposite to the Haagerup property, leads to non-embeddability into Hilbert space. This is done by introducing a notion of relative expander in terms of Poincaré inequality [3].

Definition 2.4 (Relative expanders [3], 1.4). Let \((G_n)_{n \in \mathbb{N}}\) be a sequence of finite groups with generating subsets \( S_n \) such that \( \sup_n |S_n| < \infty \), and let \( Y_n \subseteq G_n \) be an unbounded sequence of subsets of \( G_n \), i.e. for any \( R > 0 \) there exists \( n \) such that \( Y_n \) is not contained entirely in the ball of radius \( R \) around the neutral element in \( G_n \). Then the sequence of Cayley graphs \((G_n, S_n)\) is said to be a relative expander with respect to \((Y_n)\) if it satisfies the "relative Poincaré inequality": there exists \( C > 0 \) such that for every \( n \in \mathbb{N} \), for every function \( f : G_n \to \mathcal{H} \) from \( G_n \) to a Hilbert space \( \mathcal{H} \), and for every \( y \in Y_n \), we have

\[
\sum_{g \in G_n} \|f(gy) - f(g)\|^2 \leq C \sum_{g \in G_n, s \in S_n} \|f(gs) - f(g)\|^2.
\]

Proposition 2.5 (Observation 2). ([3], Proposition 3 and Corollary 1.1) Let \( G \) be a finitely generated residually finite group, with a finite generating subset \( S \). If \( G \) has relative property (T) with respect to an infinite subset \( Y \subset G \), then for any filtration \((N_n)\) of \( G \), with the quotient maps \( \pi_n : G \to G/N_n \), the sequence of Cayley graphs \((G/N_n, \pi_n(S))\) is a relative expander with respect to the unbounded sequence \((\pi_n(Y))_n\). In particular, the box space \( \bigsqcup_n G/N_n \) does not coarsely embed into a Hilbert space.

In the paper [3], G. Arzhantseva and R. Tessera provide three examples of relative expanders which are box spaces of certain carefully defined semi-direct product groups. We observe that all these examples are "A-by-CE" sequence of extensions of finite groups.

2.1.1 Relative expander example 1: a box space of \( \mathbb{Z}^2 \rtimes_{\alpha} \mathbb{F}_3 \)

Consider the semi-direct product

\[
G := \mathbb{Z}^2 \rtimes_{\alpha} \mathbb{F}_3,
\]
where $Q$ is the kernel of the surjection $\text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}_2)$, which is generated by 3 elements:

\[
\alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.
\]

Take a surjective homomorphism $\pi : F_3 \to Q$ mapping the standard generating subset $\{a, b, c\}$ of $F_3$ onto some generating subset $\{\alpha, \beta, \gamma\}$ of $Q$, so that $F_3$ acts on $\mathbb{Z}^2$ via the standard action of $Q$ on $\mathbb{Z}^2$. Then the set $S := \{(1, 0), (0, 1), a, b, c\}$ is a finite generating subset of $G$.

Define a sequence of semi-direct products (i.e. split extensions) of finite groups as follows:

\[
G_n := \left(\mathbb{Z}_{2^n}\right)^2 \rtimes_{Q_n} \left(F_3/\Gamma_{3n-3}(F_3)\right)
\]

where, for all $n \geq 1$,

- $Q_n < \text{SL}(2, \mathbb{Z}_{2^n})$ is the image of $Q < \text{SL}(2, \mathbb{Z})$ under the quotient map $\pi_{2^n} : \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}_{2^n})$;

  $Q_n$ is a 2-group of order $2^{3n-3}$.

- For any group $G$ and every $n \geq 0$, define inductively the characteristic subgroup $\Gamma_n(G)$ of $G$ as: $\Gamma_0(G) = G$, and $\Gamma_{n+1}(G)$ is the subgroup of $\Gamma_n(G)$ generated by squares of elements of $\Gamma_n(G)$.

A central tool of Arzhantseva-Tessera [3, Lemma 2.3] is the following observation. If $G$ is a finite groups of order $2^n$, then $\Gamma_n(G) = \{e\}$. In particular, if $G$ admits a finite generating subset $U$, then there is a surjective homomorphism $\pi : F_3/\Gamma_n(F_3) \to G$ preserving the generating subsets $U$.

In the above case of interest, since $Q_n$ has order $2^{3n-3}$ and a generating subset of 3 elements, there is a surjective homomorphism $\pi^{Q_n} : F_3/\Gamma_{3n-3}(F_3) \to Q_n$ preserving the corresponding 3 generators, so that the semi-direct product of $G_n$ is defined.

**Proposition 2.6** (Section 7 in [3]). The following statements are true for the sequence of $(G_n)_{n \in \mathbb{N}}$.

1. Let $K_n$ be the kernel of the surjective homomorphism $p_n : G \to G_n$. Then $\bigcap_n K_n = \{e\}$. Therefore, $\bigcup_n (G_n, S_n)$ is a box space of $G$ with respect to the generating subset $S$ and the filtration $(K_n)_n$, where $S_n = p_n(S)$.

2. The pair $(G, \mathbb{Z}^2)$ has relative property (T), by the proof of M. Burger in [6, Proposition 1]. It follows that the sequence of Cayley graphs $(G_n, S_n)_n$ is a relative expander with respect to the unbounded sequence of subsets $(\mathbb{Z}_{2^n})^2 \subset G_n$, $n \in \mathbb{N}$. In particular, the sequence $(G_n, S_n)_n$ does not coarsely embed into a Hilbert space.

3. The sequence of subgroups $(\mathbb{Z}_{2^n})_{n \in \mathbb{N}}$ of $(G_n)_{n \in \mathbb{N}}$, equipped with the induced metric from $(G_n, S_n)$, are uniformly amenable [3, Lemma 2.5]. Consequently, it satisfies the Haagerup property uniformly, has Property A uniformly, and uniformly coarsely embed into a Hilbert space.
(4) It follows from an earlier remarkable result of G. Arzhantseva, E. Guentner, and J. Špakula \cite{[1]} that the box space \( \{ \mathbb{F}_3/\Gamma_{3n-3}(\mathbb{F}_3) \}_{n \in \mathbb{N}} \) coarsely embeds into a Hilbert space. Together with (3), we have that the sequence \((G_n)_{n \in \mathbb{N}}\) has "A-by-CE" structure. In particular, the sequence \((G_n, S_n)_{n \in \mathbb{N}}\) does not contain a weakly embedded expander.

(5) The sequence \((G_n, S_n)_{n \in \mathbb{N}}\) does not admit a fibred coarse embedding into a Hilbert space \cite{[8]}. However, the maximal coarse Baum-Connes conjecture holds for the box space \( \bigsqcup_n G_n \) \cite{[4], Corollary 4.18}.

Note that the maximal coarse Baum-Connes conjecture does not imply the (reduced) coarse Baum-Connes conjecture. Nevertheless, it follows from the main result in this paper, the coarse Baum-Connes conjecture holds for the box space \( \bigsqcup_n G_n \). This solves an open problem raised by G. Arzhantseva and R. Tessera in \cite{[3]}.

2.1.2 Relative expander example 2: a box space of \(\mathbb{Z} \wr \mathbb{Q} \mathbb{F} \mathbb{U}\)

Consider the generalized wreath product

\[
G := A \wr_{Q} \mathbb{F}_U := \left( \bigoplus_{Q} A \right) \rtimes \mathbb{F}_U,
\]

where \(A = \langle V \rangle\) is a finitely generated, residually finite, amenable group, with a finite generating subset \(V\) and a filtration \(\{ A_n \}_{n}\), and \(Q := \ker(\text{SL}(3, \mathbb{Z}) \to \text{SL}(3, \mathbb{Z}_2))\), equipped with a finite generating subset \(U\). For each \(v \in V\) and \(g \in Q\), let \(v_g : Q \to A\) denote the function such that \(v_g(g) = v \in A\) and \(v_g(h) = e_A\), the identity element of \(A\), if \(g \neq h \in Q\). Then (1) the set \(S := \{ v_v : v \in V \}\) \(\sqcup U\) is a finite generating subset of \(G\); (2) the set \(Y := \{ v_v : v \in V, g \in Q \}\) is an unbounded subset of \(\bigoplus_Q A\). It follows from a result by I. Chifan and A. Ioana \cite{[10]} that the pair \((G, Y)\) has relative property (T).

Define a sequence of generalized wreath products of finite groups as follows:

\[
G_n := (A/A_n) \wr_{Q_n} \left( \mathbb{F}_U/\Gamma_{8n-8}(\mathbb{F}_U) \right) := \left( \bigoplus_{Q_n} \left( A/A_n \right) \right) \rtimes \left( \mathbb{F}_U/\Gamma_{8n-8}(\mathbb{F}_U) \right)
\]

where, for all \(n \geq 1\),

- \(Q_n < \text{SL}(3, \mathbb{Z}_{2^n})\) is the image of \(Q < \text{SL}(3, \mathbb{Z})\) under the quotient map

\[
\pi_{2^n} : \text{SL}(3, \mathbb{Z}) \to \text{SL}(3, \mathbb{Z}_{2^n});
\]

\(Q_n\) is a 2-group of order \(2^{8n-8}\).
• Since \( Q_n \) has order \( 2^{8n-8} \) and a finite generating subset \( U \), by the central tool of Arzhantseva-Tessera [3] Lemma 2.3 again, there is a surjective homomorphism \( \varphi_{8n-8} : F_U / \Gamma_{8n-8}(F_U) \rightarrow Q_n \) preserving the corresponding generating subsets \( U \), so that the generalized wreath product of \( G_n \) is defined.

**Proposition 2.7** (Theorem 7.3 in [3]). The following statements are true for the sequence of \( (G_n)_{n \in \mathbb{N}} \).

1. Let \( K_n \) be the kernel of the surjective homomorphism \( p_n : G \twoheadrightarrow G_n \). Then \( \cap_n K_n = \{ e \} \). Therefore, \( \sqcup_n (G_n, S_n) \) is a box space of \( G \) with respect to the generating subset \( S \) and the filtration \( (K_n)_{n \in \mathbb{N}} \), where \( S_n = p_n(S) \).

2. The pair \( (G, Y) \) has relative property \( (T) \), by a result of I. Chifan and A. Ioana [11]. It follows that the sequence of Cayley graphs \( (G_n, S_n) \) is a relative expander with respect to the unbounded sequence of subsets \( Y_n := p_n(Y) \subset G_n, n \in \mathbb{N} \). In particular, the sequence \( (G_n, S_n) \) does not coarsely embed into a Hilbert space.

3. Since amenability of an amenable group \( A \) does not depend on the choice of a proper metric on \( A \), the sequence of finite quotient groups \( (A/A_n)_{n \in \mathbb{N}} \) as subgroups of \( (G_n)_{n \in \mathbb{N}} \), equipped with the induced metric from \( (G_n, S_n) \), has Yu’s property \( A \) by a result of E. Guentner as recorded in the book of J. Roe [3].

4. It follows from an earlier remarkable result of G. Arzhantseva, E. Guentner, and J. Špakula [11] that the box space \( \{ F_U / \Gamma_{8n-8}(F_U) \}_{n \in \mathbb{N}} \) coarsely embeds into a Hilbert space. Together with (3), we have that the sequence \( (G_n)_{n \in \mathbb{N}} \) has "\( A \)-by-CE" structure. In particular, the sequence \( (G_n, S_n) \) does not contain a weakly embedded expander.

5. The sequence \( (G_n, S_n) \) does not admit a fibred coarse embedding into a Hilbert space [8]. However, the maximal coarse Baum-Connes conjecture holds for the box space \( \bigsqcup_n G_n \) [4, Corollary 4.18].

Note that the maximal coarse Baum-Connes conjecture does not imply the (reduced) coarse Baum-Connes conjecture. Nevertheless, it follows from the main result in this paper, the coarse Baum-Connes conjecture holds for the box space \( \bigsqcup_n G_n \). This solves an open problem raised by G. Arzhantseva and R. Tessera in [3].

### 2.1.3 Relative expander example 3: a box space of \( \mathbb{Z} \wr_\mathbb{Q} F_3 \)

Consider the generalized wreath product

\[
G := A \wr_\mathbb{Q} F_3 := \left( \bigoplus_{\mathbb{Q}} A \right) \times F_3,
\]

where \( A = (V) \) is a finitely generated, residually finite, amenable group, with a finite generating subset \( V \) and a filtration \( \{ A_n \}_{n \in \mathbb{N}} \), and \( Q := \ker \left( \text{SL}(2, \mathbb{Z}) \twoheadrightarrow \text{SL}(2, \mathbb{Z}_2) \right) \), equipped with a generating subset \( U = \{ \alpha, \beta, \gamma \} \) as in Example No 1.

As \( Q \) is a subgroup of a group with Haagerup property, by [11] Theorem 1.5, the group \( G := A \wr_\mathbb{Q} F_3 \) has the Haagerup property.

For each \( v \in V \) and \( g \in Q \), let \( v_g : Q \rightarrow A \) denote the function such that \( v_g(g) = v \in A \) and \( v_g(h) = e_A \), the identity element of \( A \), if \( g \neq h \in Q \). Then (1) the set \( S := \{ v_e : v \in V \} \sqcup U \) is a finite generating subset of \( G \); (2) the set \( Y := \{ v_g : v \in V, g \in Q \} \) is an unbounded subset of \( \bigoplus_{\mathbb{Q}} A \).
Define a sequence of generalized wreath products of finite groups as follows:

\[ G_n := \left( A/A_n \right) \wr_{Q_n} \left( \mathbb{F}_3/\Gamma_3n-3(\mathbb{F}_3) \right) := \bigoplus_{Q_n} \left( A/A_n \right) \rtimes \left( \mathbb{F}_3/\Gamma_3n-3(\mathbb{F}_3) \right). \]

**Proposition 2.8** (Section 7.2 in [3]). The following statements are true for the sequence of \((G_n)_{n \in \mathbb{N}}\).

1. Let \(K_n\) be the kernel of the surjective homomorphism \(p_n : G \rightarrow G_n\). Then \(\cap_n K_n = \{e\}\). Therefore, \(\bigsqcup_n (G_n, S_n)\) is a box space of \(G\) with respect to the generating subset \(S\) and the filtration \((K_n)_n\), where \(S_n = p_n(S)\).

2. Since the sequence \((Q_n)_{n \in \mathbb{N}}\) has a uniform spectral gap by a famous result of A. Selberg [33], it follows from the proof of [10, Theorem 3.1] that the sequence \((G_n)_{n \in \mathbb{N}}\) has relative property (T) with respect to \(Y\), in restriction to unitary representations which are direct sums of representations which factor through some \(G_n\). This restricted version of relative property (T) implies the relative Poincaré inequality. Namely, the sequence of Cayley graphs \((G_n, S_n)_n\) is a relative expander with respect to the unbounded sequence of subsets \(Y_n := p_n(Y) \subset G_n\), \(n \in \mathbb{N}\). In particular, the sequence \((G_n, S_n)_n\) does not coarsely embed into a Hilbert space.

3. The sequence \((G_n)_{n \in \mathbb{N}}\) has "A-by-CE" structure. In particular, the sequence \((G_n, S_n)_n\) does not contain a weakly embedded expander.

4. Since \(A\) is amenable and \(Q\) is a subgroup of a Haagerup group, by [11, Theorem 1.5], the group \(G = A \wr Q \mathbb{F}_3\) has the Haagerup property. It follows that the sequence \((G_n, S_n)_n\) admits a fibred coarse embedding into a Hilbert space [9]. Hence, the maximal coarse Baum-Connes conjecture holds for the box space \(\bigsqcup_n G_n\) [9].

Note that the maximal coarse Baum-Connes conjecture does not imply the (reduced) coarse Baum-Connes conjecture. Nevertheless, it follows from the main result in this paper, the coarse Baum-Connes conjecture holds for the box space \(\bigsqcup_n G_n\). This solves an open problem raised by G. Arzhantseva and R. Tessera in [3].

### 2.2 Group extensions à la G. Arzhantseva and R. Tessera

In a very recent work [4], G. Arzhantseva and R. Tessera answer in the negative the following well-known question [12]: Does coarse embeddability into Hilbert space is preserved by group extensions of finitely generated groups? Their discoveries also answer in the affirmative another open problem [28]: Does there exist a finitely generated group which does not coarsely embed into Hilbert space and yet has no weakly embedded expander? Their examples are in fact "A-by-Haagerup" group extensions.

**Theorem 2.9** ([4]). (1) There exists a finitely generated group, e.g. \(\mathbb{Z}_2 \wr_G H\), which is a split extension of an (infinite rank) abelian group by an finitely generated group with the Haagerup property which does not coarsely embed into Hilbert space.

(2) There exists a finitely generated group, e.g. \(\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_3)\), which is a split extension of a finitely generated group with Property A by a finitely generated group with the Haagerup property that does not coarsely embed into Hilbert space.

In the first case, a certain restricted permutational wreath product

\[ \mathbb{Z}_2 \wr_G H = \bigoplus_G \mathbb{Z}_2 \rtimes H \]
meets the requirements, where \( G \) is a Gromov monster group: a finitely generated group which contains in its Cayley graph an isometrically embedded expander (cf. [39] Theorem 4 or Corollary 3.3), and \( H \) is a Haagerup monster group: a finitely generated group with the Haagerup property which does not have Property A (cf. [2] Theorems 1.2 and 5.1 and [39] Theorems 3 and 6.3), such that there is a surjective homomorphism \( H \to G \) which induces the action of \( H \) on \( G \) by left translations so as to define the wreath product. Recall that \( \bigoplus G \mathbb{Z}_2 \) is the abelian group of finitely supported functions \( \varphi : G \to \mathbb{Z}_2 \) with pointwise additions, which is not finitely generated.

In the second case, G. Arzhantseva and R. Tessera modify the first example by introducing an additional action by a finitely generated free group \( \mathbb{F}_n \). Namely, since \( G \) is finitely generated, say by \( n \) generators, there is a surjective homomorphism \( \mathbb{F}_n \to G \) so that one can form the restricted permutational wreath product

\[
\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) := \left( \bigoplus_G \mathbb{Z}_2 \right) \times (H \times \mathbb{F}_n)
\]

where \( H \) acts on \( G \) as above by left translations via its surjection onto \( G \), and \( \mathbb{F}_n \) acts on \( G \) by right translations via its surjection onto \( G \). Since these two actions commute, the group \( G \) is an \((H \times \mathbb{F}_n)\)-set so that the wreath product \( \mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) \) is defined, and it is an "A-by-Haagerup" split extension of the finitely generated group

\[
\mathbb{Z}_2 \wr_G \mathbb{F}_n = \left( \bigoplus_G \mathbb{Z}_2 \right) \times \mathbb{F}_n,
\]

which has Property A, and the Haagerup monster \( H \).

The groups \( \mathbb{Z}_2 \wr_G H \) and \( \mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) \) do not coarsely embed into Hilbert space because they contain an embedded sequence of relative expanders ([1], Theorem 4.1, Corollary 3.3 and 4.2). Both groups do not contain weakly embedded expanders because they are "CE-by-CE" group extensions ([3] Proposition 2).

The first group \( \mathbb{Z}_2 \wr_G H \) is an "abelian-by-Haagerup" extension of two groups. It satisfies the Strong Baum-Connes Conjecture ([29] Theorem 1.1], which is strictly stronger than the Baum-Connes Conjecture with coefficients ([2] Corollary 3.14]. It is proved in ([8] Proposition 2.11] that, by [10] Theorem 7.1], the second group

\[
\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) = (\mathbb{Z}_2 \wr_G H) \times \mathbb{F}_n = \left( \bigoplus_G \mathbb{Z}_2 \right) \times H \times \mathbb{F}_n
\]

also satisfies the Baum-Connes conjecture with coefficients. Consequently, the Coarse Baum-Connes Conjecture holds for both \( \mathbb{Z}_2 \wr_G H \) and \( \mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) \). Since both groups are "A-by-CE" extensions, the main result in this paper provides an alternative proof to this fact.

Remark 2.10. Recall that the groups \( \mathbb{Z}_2 \wr_G H \) and \( \mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) \) do not coarsely embed into Hilbert space because they contain an isometrically embedded sequence of relative expanders \( W_n = \mathbb{Z}_2 \wr_G H_n \), with generators \( \sigma_n = \{ \delta_{\alpha_n} \} \cup T_n, n \in \mathbb{N} \), with respective to an unbounded sequence of subsets \( X_n = \{ \delta_{g}, g \in G_n \} \subset W_n \), where \( G_n \) isometrically embeds in \( G \), and \( H_n \) isometrically embeds in \( H \), see ([1], Theorem 4.1, Corollary 3.3 and 4.2) for more details on these notations. It follows that the sequence of split extensions \( W_n = \mathbb{Z}_2 \wr_G H_n \) has the "A-by-CE" structure. Therefore, by the main theorem of this paper, the coarse Baum-Connes conjecture holds for the relative expander \( W_n = \mathbb{Z}_2 \wr_G H_n \) inside the groups \( \mathbb{Z}_2 \wr_G H \) and \( \mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) \).
2.3 Box spaces of free groups à la T. Delabie and A. Khukhuro

In [14] T. Delabie and A. Khukhro answer an open question asked by G. Arzhantseva and R. Tessera in [3] Section 8: Open Problems: Does there exist a sequence of finite graphs with bounded degree and large girth that does not coarsely embed into a Hilbert space and yet has no weakly embedded expander (in particular, a box space of the free group $F_m$)?

**Theorem 2.11** ([14]). *There exists a filtration of the free group $F_3$ such that the corresponding box space does not coarsely embed into a Hilbert space, but does not admit a weakly embedded expander.*

The overall structure of the proof is as follows. They first construct a particular sequence of nested finite index normal subgroups $\{N_i\}$ of $F_3$ with trial intersection so that the corresponding box space is an expander, and consider the sequence of $q$-homology covers, where $q$ is a certain prime number, of the quotients $\{F_3/N_i\}$: this gives rise to another sequence of subgroups $\Theta(N_i) < N_i$ of $F_3$ such that the corresponding box space $\Box_{\Theta(N_i)}F_3$ coarsely embeds into a Hilbert space, following a previous work of A. Khukhro [32]. Then they consider the quotients of $F_3$ by intersections of these two sequences of subgroups to obtain the following magic triangle, where the arrows represent quotient maps.

Finally, they find an ingenuous way to choose a subsequence of quotients

$$\left\{ \Box_{N_{n_i} \cap \Theta(N_{k_i})} F_3 \right\}$$

which lie on some path that moves sufficiently slowly away the horizontal expander sequence in this "triangle" of intersections.

They show [14] Corollary 4.4 that there exist increasing sequences $k_i$ and $n_i$ such that the box space

$$\Box_{N_{n_i} \cap \Theta(N_{k_i})} F_3$$

contains a generalized expander and therefore does not coarsely embed into Hilbert space, by the characterization of R. Tessera [47].

On the other hand, T. Delabie and A. Khukhro have essentially proved the following general fact [14] Proposition 2.4: Let $G$ be a finitely generated residually finite group with two filtrations $\{N_i\}$ and $\{M_i\}$...
such that \( M_i < N_i \) for all \( i \in \mathbb{N} \). Consider the sequence of group extensions
\[
\left( 1 \rightarrow N_i/M_i \rightarrow G/M_i \rightarrow G/N_i \rightarrow 1 \right)_{i \in \mathbb{N}},
\]
where \( G/M_i \) and \( G/N_i \) are considered with the metric induced by the restriction of the respective box space metrics, and \( N_i/M_i \) is considered with the metric induced by viewing \( N_i/M_i \) as a subspace of \( G/M_i \). Then the quotient maps
\[
G \twoheadrightarrow G/M_n; \quad G \twoheadrightarrow G/N_n
\]
are asymptotically faithful, so that the quotient maps
\[
\pi_n : G/M_n \twoheadrightarrow G/N_n
\]
is asymptotically faithful as well. It follows that there exists a sequence \( r_n \rightarrow \infty \) such that any two points of \( N_n/M_n \) are at distance at least \( r_n \) from each other. In other words, the asymptotic dimension of the coarse disjoint union \( \bigsqcup \in\mathbb{N} \) \( N_i/M_i \) is 0, which implies property A. Thus, if \( \{ G/N_i \} \) coarsely embeds into a Hilbert space, then the sequence \( \{ G/M_i \} \) has "A-by-CE" structure, so that it does not contain weakly (and hence coarsely) embedded expanders. Applied to the above magic triangle, by our main Theorem 1.1 in this paper, we actually have the following

**Corollary 2.12.** For all increasing sequences \( k_i \) and \( n_i \) of \( \mathbb{N} \), the corresponding sequence of extensions in the Delabie-Khukhro magic triangle:
\[
\left( 1 \rightarrow \Theta(N_{k_i})/(N_{n_i} \cap \Theta(N_{k_i})) \rightarrow \mathbb{F}_3/\left( N_{n_i} \cap \Theta(N_{k_i}) \right) \rightarrow \mathbb{F}_3/\Theta(N_{k_i}) \rightarrow 1 \right)_{i \in \mathbb{N}}
\]
has "A-by-CE" structure. Therefore, the coarse Baum-Connes conjecture holds for the box space
\[
\quad \bigsqcup_{i \in \mathbb{N}} \mathbb{F}_3/\left( N_{n_i} \cap \Theta(N_{k_i}) \right)
\]
for all increasing sequences \( k_i \) and \( n_i \) of \( \mathbb{N} \).

## 3 The coarse Baum–Connes conjecture

In this section, we shall recall the concepts of Roe algebras, localization algebras and the coarse Baum–Connes conjecture.

### 3.1 Roe algebras and localization algebras

Recall that a metric space is proper if every closed bounded subset is compact. Let \( X \) be a proper metric space. An ample \( X \)-module is a separable Hilbert space \( H_X \) equipped with a faithful and non-degenerate \(*\)-representation of \( C_0(X) \) whose range contains no nonzero compact operators, where \( C_0(X) \) is the algebra of all complex-valued continuous functions on \( X \) vanishing at infinity.

**Definition 3.1.** Let \( X \) be a proper metric space and \( H_X \) an ample \( X \)-module.

(1) The support of a bounded linear operator \( T : H_X \to H_X \) is defined to be the complement to the set of points \( (x, y) \in X \times X \) for which there exist \( f, g \in C_0(X) \) with \( f(x) \neq 0 \) and \( g(y) \neq 0 \) such that \( fTg \neq 0 \). The support of \( T \) is denoted by \( \text{Supp}(T) \).
For a bounded linear operator $T : H_X \to H_X$, the propagation of $T$ is defined by

$$\text{propagation}(T) := \sup \{d(x, y) : (x, y) \in \text{Supp}(T)\}.$$ 

An operator $T$ is said to have finite propagation if $\text{propagation}(T) < \infty$.

An operator $T$ is locally finite if the operators $fT$ and $Tf$ are compact operators on $H_X$ for all $f \in C_0(X)$.

In the case when $X$ is a countable metric space, we can choose a specific ample $X$-module as follows. Let $H_0$ be any separable infinite-dimensional Hilbert space. Consider the Hilbert space $\ell^2(X) \otimes H_0$. For each $f \in C_0(X)$, we define a bounded linear map on $\ell^2(X) \otimes H_0$ by linearly extending the map

$$f : (\xi \otimes v) = (f\xi) \otimes v$$

where $\xi \in \ell^2(X)$ and $v \in H_0$.

Since $\ell^2(X) \otimes H_0 = \bigoplus_{x \in X} C \cdot \delta_x \otimes H_0$, we can express each bounded operator $T \in B(\ell^2(X) \otimes H_0)$ as an $X$-by-$X$ matrix,

$$T = (T_{x,y})_{x,y}$$

where $T_{x,y}$ is a bounded linear operator on $H_0$. We have that

$$\text{propagation}(T) = \sup \{d(x, y) : T_{x,y} \neq 0\}.$$ 

If $T$ is locally compact, then $T_{x,y}$ is a compact operator on $H_0$ for all $x, y \in X$.

Now we are ready to recall the definition of Roe algebras due to John Roe [44].

**Definition 3.2.** Let $X$ be a proper metric space and $H_X$ an ample $X$-module.

1. The algebraic Roe algebra, denoted by $\mathbb{C}[X]$, is the $*$-algebra of all locally compact, finite propagation operators on $H_X$.
2. The Roe algebra, denoted by $C^*(X)$, is defined to be the completion of $\mathbb{C}[X]$ under the operator norm on $H_X$.

Let us now recall some basic properties of Roe algebras. Let $X$ and $Y$ be proper metric spaces. A map $f : X \to Y$ is said to be a coarse embedding if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that

1. $\lim_{t \to \infty} \rho_i(t) = \infty$ for $i = 1, 2$;
2. $\rho_1(d(x, y)) \leq d(f(x), f(y)) \leq \rho_2(d(x, y))$, for all $x, y \in X$.

A metric space $X$ is said to be coarsely equivalent to another metric space $Y$, if there exist a coarse embedding $f : X \to Y$ such that $Y$ is equal to the $C$-neighborhood of the image $f(X)$ for some $C > 0$.

Let us recall the Construction 4.2.5 in [52] about the homomorphism between Roe algebras induced by a coarse embedding between metric spaces. Let $f : X \to Y$ be a coarse embedding. Let $\{U_i\}_{i \in I}$ be a Borel cover of $Y$ with the following properties:
(1) \( \{U_i\}_{i \in I} \) is mutually disjoint;
(2) each \( U_i \) has non-empty interior;
(3) for any compact \( K \subseteq Y \), the set \( \{i \in I | U_i \cap K = \emptyset\} \) is finite;
(4) the diameter of \( U_i \) is uniformly bounded over \( i \in I \).

Note that the \( \ast \)-representation of \( C_0(Y) \) on \( H_Y \) extends to a \( \ast \)-representation of the algebra of all bounded Borel functions on \( Y \). Since the range of the representation \( C_0(Y) \) contains no non-zero compact operators, \( \chi_{U_i} H_Y \) is an infinite dimensional subspace of \( H_Y \) for each \( i \), where \( \chi_{U_i} \) is the projection of the characteristic function on \( U_i \). Hence, for each \( i \in I \), there always exists an isometry

\[
V_i : \chi_{f^{-1}(U_i)} H_X \to \chi_{U_i} H_Y.
\]

Consequently, we have an isometry

\[
V = \bigoplus_{i \in I} V_i : H_X = \bigoplus_i \chi_{f^{-1}(U_i)} H_X \to H_Y = \bigoplus_i \chi_{U_i} H_Y.
\]

For any operator \( T \in C^\ast_{alg}(X) \) with finite propagation, Condition (3) above implies that the operator \( VTV^\ast \) has finite propagation, and it follows from Condition (4) above that the operator \( VTV^\ast \) is locally compact. Therefore, the map

\[
\text{Ad}_V : C^\ast[X] \to C^\ast[Y]
\]

defined by \( T \mapsto VTV^\ast \) is well-defined.

It is obvious that when \( f : X \to X \) is the identity map, the isometry \( V \) can be chosen as a unitary. Thus we have that the definition of \( C^\ast[X] \) is independent of the choice of \( H_X \) up to \( \ast \)-isomorphisms.

Accordingly, we obtain a homomorphism

\[
\text{Ad}_V : K_\ast(C^\ast(X)) \to K_\ast(C^\ast(Y))
\]
on \( K \)-theory induced by \( \text{Ad}_V \), where \( V : H_X \to H_Y \) is induced by the coarse embedding \( f : X \to Y \). The map induced by \( \text{Ad}_V \) on \( K \)-theory does not depend on the choice of \( V \). Moreover, it is obvious that if \( f \) is a coarse equivalence, the isometry \( V \) can be chosen as a unitary.

Remark 3.3. Let \( Z_1 \) and \( Z_2 \) be countable metric spaces and \( f : Z_1 \to Z_2 \) an injective coarse map. Then we have an explicit construction of the homomorphism between the algebraic Roe algebras:

\[
C[Z_1] \to C[Z_2]
\]

by

\[
(\text{Ad}_V(T))_{y,y'} = \begin{cases} T_{x,x'} & \text{if } y = f(x), y' = f(y) \\ 0 & \text{otherwise} \end{cases}.
\]

, let us briefly recall the definition of analytic \( K \)-homology groups. More details can also be found in [51].

Let \( X \) be a proper metric space. The \( K \)-homology groups \( K_0(X) \) and \( K_1(X) \) are groups generated by certain cycles modulo a certain homotopy equivalence relation.
(1) A cycle for $K_0(X)$ is a pair $(H_X, F)$ where $H_X$ is an $X$-module and $F$ is a bounded linear operator on $H_X$ such that $f(F^* F - 1)$, $f(FF^* - 1)$ and $FF - Ff$ are compact operators for all $f \in C_0(X)$;

(2) A cycle for $K_1(X)$ is a pair $(H_X, F)$ where $H_X$ is an $X$-module and $F$ is a self-adjoint operator on $H_X$ such that $f(F^2 - 1)$ and $FF - Ff$ are compact operators for all $f \in C_0(X)$.

In the above description of of cycles, the $X$-module can be chosen to be ample.

Next, we shall recall the assembly map

$$
\mu : K_1(X) \to K_1(C^*(X)).
$$

An open cover $\{U_j\}_{j \in J}$ of $X$ is locally finite if every point $x \in X$ is contained in only finitely many elements of the cover $\{U_j\}_{j \in J}$. Let $\{U_j\}_{j \in J}$ be a locally finite and uniformly bounded open cover of $X$. Let $\{\phi_j\}_{j \in J}$ be a partition of unity subordinate to the open cover $\{U_j\}_{j \in J}$. Let $(H_X, F)$ be a cycle for $K_0(X)$ such that $H_X$ is an ample $X$-module. Define

$$
F' = \sum_j \phi_j^\frac{1}{2} F \phi_j^\frac{1}{2},
$$

where the infinite sum converges in strong operator topology. Note that $(H_X, F')$ is equivalent to $(H_X, F)$ via $(H_X, (1 - t)F + tF')$, where $t \in [0, 1]$. Note that $F'$ has finite propagation, so $F'$ is a multiplier of $C^*(X)$. It is obvious that $F'^2 - 1 \in C^*(X)$. Thus, $F'$ is invertible modulo $C^*(X)$. Hence $F'$ gives rise to an element, denoted by $\partial[F']$ in $K_0(C^*(X))$, where

$$
\partial : K_1(M(C^*(X))/C^*(X)) \to K_0(C^*(X))
$$

is the boundary map on $K$-theory, and $M(C^*(X))$ is the multiplier algebra of $C^*(X)$. We define

$$
\mu([H_X, F]) = \partial[F'].
$$

Similarly, we can define the index map from $K_1(X)$ to $K_1(C^*(X))$.

Recall that a discrete metric space is said to have bounded geometry if for any $r > 0$ there exists $N > 0$ such that any ball of radius $r$ in $X$ contains at most $N$ elements. Now we are ready to introduce the coarse Baum–Connes conjecture for a metric space with bounded geometry.

**Definition 3.4.** Let $\Gamma$ be a metric space with bounded geometry. For each $d \geq 0$, the Rips complex $P_d(\Gamma)$ is defined to be the simplicial complex where the set of vertices is $\Gamma$, and a finite subset $\{\gamma_0, \gamma_1, \cdots, \gamma_n\} \subset \Gamma$ spans a simplex if and only if $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.

The Rips complex $P_d(\Gamma)$ is equipped with the spherical metric that is the maximal metric whose restriction to each simplex $\{\sum_{i=0}^n t_i \gamma_i : t_i \geq 0, \sum_i t_i = 1\}$ is the metric obtained by identifying the simplex with $S^n_+$ via the map:

$$
\sum_{i=0}^n t_i \gamma_i \to \left( \frac{t_0}{\sqrt{\sum_{i=0}^n t_i}}, \cdots, \frac{t_n}{\sqrt{\sum_{i=0}^n t_i}} \right),
$$

where $S^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_{i=0}^n x_i^2 = 1\}$ endowed with the standard Riemannian metric. The distance between a pair of points in different connected components is defined to be infinity.

For each $s \geq r$, note that $P_r(\Gamma) \subseteq P_s(\Gamma)$. Then the canonical inclusion map $P_r(\Gamma) \xrightarrow{i_{sr}} P_s(\Gamma)$ induces a $\ast$-homomorphism from $C^*(P_r(\Gamma))$ to $C^*(P_s(\Gamma))$. Furthermore, for $s \geq r \geq d$, the inclusion map $i_{sd}$ and the composition $i_{sr} \circ i_{rd}$ induce the same $\ast$-homomorphism from $C^*(P_d(\Gamma))$ to $C^*(P_s(\Gamma))$. 

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The coarse Baum–Connes conjecture. For any discrete metric space $\Gamma$ with bounded geometry, the coarse assembly map

$$\mu: \lim_{d \to \infty} K_\ast(P_d(\Gamma)) \to \lim_{d \to \infty} K_\ast(C_\ast(P_d(\Gamma))) \cong K_\ast(C_\ast(\Gamma))$$

is an isomorphism.

Now, we shall recall the definition of localization algebras \cite{52} and the relation between its $K$-theory and the $K$-homology groups.

**Definition 3.5.** Let $X$ be a proper metric space.

1. The algebraic localization algebra, denoted by $\mathbb{C}_L[X]$, is defined to be the $\ast$-algebra of all uniformly bounded and uniformly norm-continuous functions $g : [0, \infty) \to \mathbb{C}[X]$ such that propagation $g(t) \to 0$ as $t \to \infty$.

2. The localization algebra $\mathbb{C}_L^\ast(X)$ is the norm closure of $\mathbb{C}_L[X]$ under the norm

$$\|f\| = \sup_{t \in [0, \infty)} \|f(t)\|.$$  

Naturally, the evaluation-at-zero map

$$e : C_\ast^\ast(X) \to C_\ast(X)$$

defined by

$$e(g) = g(0)$$

for $g \in C_\ast^\ast(X)$ is a $\ast$-homomorphism.

Next, we recall some basic properties of localization algebras. A Borel map $f : X \to Y$ is said to be Lipschitz, if there exists a positive constant $C$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. Following the construction in \cite{52}, a Lipschitz map $f : X \to Y$ induces a $\ast$-homomorphism $\text{Ad}(V_f) : C_\ast^\ast(X) \to C_\ast^\ast(Y)$ as follows.

Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers with $\lim_{n \to \infty} \epsilon_n = 0$. For each $k$, there exists an isometry $V_k : H_X \to H_Y$ such that $\text{Supp}(V_k) \subset \{(y, x) \in Y \times X : d(y, f(x)) \leq \epsilon_k\}$.

Define a family of isometries

$$(V_f(t))_{t \in [0, \infty)} : H_X \oplus H_X \to H_Y \oplus H_Y$$

by

$$V_f(t) = R(t - k + 1)(V_k \oplus V_{k+1})R^\ast(t - k + 1),$$

for $t \in [k - 1, k)$, where

$$R(t) = \begin{pmatrix}
\cos(\pi t/2) & \sin(\pi t/2) \\
-sin(\pi t/2) & \cos(\pi t/2)
\end{pmatrix}.$$  

Then $V_f(t)$ induces a homomorphism on unitization

$$\text{Ad}(V_f(t)) : C_\ast^\ast(X) \to C_\ast^\ast(Y) \otimes M_2(\mathbb{C})$$

by

$$\text{Ad}(V_f(t))(u(t) + cI) = V_f(t)(u(t) \oplus 0)V_f^\ast(t) + cI$$

for all $u(t) \in C_\ast^\ast(X)$ and $c \in \mathbb{C}$. 

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**Definition 3.6.** Let $X$ and $Y$ be two proper metric spaces and $f, g$ two Lipschitz maps from $X$ to $Y$. A continuous homotopy $F(t, x): [0, 1] \times X \to Y$ between $f$ and $g$ is said to be strongly Lipschitz if

1. $F(t, x)$ is coarse map from $X$ to $Y$ for each $t$;
2. there exists a positive constant $C$ such that $d(F(t, x), F(t, y)) \leq Cd(x, y)$ for all $x, y \in X$ and $t \in [0, 1]$;
3. for any $\epsilon > 0$, there exists $\delta > 0$ such that $d(F(t_1, x), F(t_2, x)) < \epsilon$ for all $x \in X$ and $|t_1 - t_2| < \delta$;
4. $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$.

**Definition 3.7.** A metric space $X$ is said to be strongly Lipschitz homotopy equivalent to $Y$, if there exist two Lipschitz maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ and $g \circ f$ are strongly Lipschitz homotopy equivalent to the identity maps $\text{id}_X$ and $\text{id}_Y$, respectively.

The $K$-theory of localization algebras is invariant under strongly Lipschitz homotopy equivalence (see [52]). The following Mayer-Vietoris sequence was introduced by the third author, and more details can be found in [52].

**Proposition 3.8.** Let $X$ be a simplicial complex endowed with the spherical metric, and $X_1, X_2 \subset X$ its simplicial subcomplexes endowed with the subspace metric. Then we have the following six-term exact sequence:

$$
\begin{align*}
K_0(C^*_L(X_1 \cap X_2)) & \longrightarrow K_0(C^*_L(X_1)) \oplus K_0(C^*_L(X_2)) \longrightarrow K_0(C^*_L(X_1 \cup X_2)) \\
K_1(C^*_L(X_1 \cap X_2)) & \longrightarrow K_1(C^*_L(X_1)) \oplus K_1(C^*_L(X_2)) \longrightarrow K_1(C^*_L(X_1 \cup X_2))
\end{align*}
$$

We then recall a local index map from the $K$-homology group $K_*(X)$ to the $K$-theory group $K_*(C^*_L(X))$. For every positive integer $n$, let $\{U_{n, i}\}_i$ be a locally finite open cover for $X$ with diameter $(U_{n, i}) \leq \frac{1}{n}$ for all $i$. Let $\{\phi_{n,i}\}_i$ be the partition of unity subordinate to the open cover $\{U_{n, i}\}_i$. For any $[H_X, F] \in K_0(X)$, we define a family of operators $(F(t))_{t \in [0, \infty)}$ by

$$
F(t) = \sum_i \left( (n-t)^{\frac{1}{2}} \phi_{n,i}^\frac{1}{2} F \phi_{n,i}^\frac{1}{2} + (t-n+1)^{\frac{1}{2}} \phi_{n+1,i}^\frac{1}{2} F \phi_{n+1,i}^\frac{1}{2} \right),
$$

for all $t \in [n-1, n]$, where the infinite sum converges under the strong operator topology. Since diameter $(U_{n, i}) \to 0$ as $n \to \infty$, we have that propagation $(F(t)) \to 0$ as $t \to 0$. Moreover, we have that the path $(F(t))_{t \in [0, \infty)}$ is a multiplier of $C^*_L(X)$ and it is a unitary modulo $C^*_L(X)$. Hence we can define a local index map

$$
\text{ind}_L : K_0(X) \to K_0(C^*_L(X))
$$

by

$$
\text{ind}_L([H_X, F]) = \partial([F(t)]),
$$

where $\partial : K_1(M(C^*_L, \text{max}(X))/C^*_L(X)) \to K_0(C^*_L(X))$ is the boundary map in $K$-theory, and $M(C^*_L, \text{max}(X))$ is the multiplier algebra of $C^*_L(X)$. Similarly we can define the local index map

$$
\text{ind}_L : K_1(X) \to K_1(C^*_L, \text{max}(X)).
$$

The following result establishes the relation between the $K$-homology groups and the $K$-theory of localization algebras.
Theorem 3.9.\footnote{52} For any finite-dimensional simplicial complex \(X\) endowed with the spherical metric, the local index map

\[ \text{ind}_L : K_\ast(X) \to K_\ast(C^\ast_L(X)) \]

is an isomorphism.

In general, Roe and Qiao\footnote{42} prove the local index map is isomorphism for any proper metric space. If \(\Gamma\) is a discrete metric space with bounded geometry, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{lim}_{d \to \infty} K_\ast(C^\ast_L(P_d(\Gamma))) & \xrightarrow{\text{ind}_L} & \text{lim}_{d \to \infty} K_\ast(P_d(\Gamma)) \\
\downarrow e_\ast & & \downarrow \mu \\
\text{lim}_{d \to \infty} K_\ast(C^\ast(P_d(\Gamma))) & \xrightarrow{\text{ind}_L} & \text{lim}_{d \to \infty} K_\ast(C^\ast(P_d(\Gamma))).
\end{array}
\]

Therefore, the coarse Baum–Connes conjecture is a consequence of the result that the map

\[ e_\ast : \lim_{d \to \infty} K_\ast(C^\ast_L(P_d(\Gamma))) \to \lim_{d \to \infty} K_\ast(C^\ast(P_d(\Gamma))) \]

induced by evaluation-at-zero map on K-theory is an isomorphism.

3.2 The coarse Baum–Connes for a sequence of metric spaces

In this paper, we denote \(I\) to be either the singleton \(\{1\}\) or \(\mathbb{N}\). Let \(\{(X_m, d_m)\}_{m \in I}\) be a sequence of metric spaces with uniform bounded geometry in the sense that for each \(R > 0\), the number of the elements in the set \(B_{X_m}(x, R) = \{y \in X_m : d(x, y) \leq R\}\) is at most \(M_R\) for some \(M_R > 0\) which does not depend on \(m\). In the case when the sequence \((X_m)_m\) is a sequence of finitely generated groups \((G_m)_m\), \((G_m)_m\) has uniform bounded geometry if the numbers of generating subsets for all groups are uniformly bounded.

For each \(d > 0\), and each \(m \in I\), we choose a countable dense subset \(X^m_d \subset P_d(X_m)\) such that \(X^m_d \subset X^m_{d'}\) for each \(m\) if \(d < d'\). Let \(K\) be the algebra of compact operators on the infinite-dimensional separable Hilbert space \(H\).

Definition 3.10. For each \(d > 0\), the algebraic Roe algebra \(C_a[(P_d(X_m))_{m \in I}]\) is the collection of tuples \(T = (T^{(m)})_{m \in I}\) satisfying

1. each \(T^{(m)}\) is a bounded function from \(X^m_d \times X^m_d\) to \(K\);
2. there exists \(r > 0\) such that for each \(m\), \(T^{(m)}_{x,y} = 0\) for all \(x, y \in X^m_d\) with \(d(x, y) \geq r\);
3. there exists \(L > 0\) such that for each \(m\) and \(x \in X^m_d\),

\[
\sharp\{y \in X^m_d : T^{(m)}_{x,y} \neq 0\} \leq L, \quad \text{and} \quad \sharp\{y \in X^m_d : T^{(m)}_{y,x} \neq 0\} \leq L;
\]
4. for each \(m\) and each bounded subset \(B \subset P_d(G_m)\), the set

\[
\{(x, y) \in (B \times B) \cap (X^m_d \times X^m_d) : T^{(m)}_{x,y} \neq 0\}
\]

is finite.
We can then view $C_u(P_d(X_m))_{m \in I}$ as a $*$-subalgebra of $\prod_m C^*_u((P_d(X_m))_{m \in I})$. Let

$$E = \bigoplus \ell^2(X^m_d) \otimes H.$$ 

The algebraic Roe algebra $C_u((P_d(X_m))_{m \in I})$ admits a $*$-representation on $E$ by the restriction of the $*$-representation of $\prod_m C^*_u(P_d(X_m))$.

The Roe algebra for the sequence $(P_d(G_m))_m$, denoted by $C^*_u((P_d(X_m))_{m \in I})$, is the completion of $C_u(P_d(X_m))_{m \in I}$ under the operator norm on $E$.

The following definition was essentially defined by the third author in \cite{52}.

**Definition 3.11.** The algebraic localization algebra, $C_{u,L}((P_d(X_m))_{m \in I})$, is the $*$-algebra of all uniformly bounded and uniformly continuous functions

$$f : [0, \infty) \to C_u((P_d(X_m))_{m \in I})$$

such that $f(t)$ is of the form $f(t) = (f^m(t))_{m \in I}$ for all $t \in [0, \infty)$, where the path of the tuples $(f^m(t))_{m \in I}$ satisfy the conditions in Definition 3.10 with uniform constants and there is a bounded function $r : [0, \infty) \to [0, \infty)$ with $\lim_{t \to \infty} r(t) = 0$, such that

$$(f^m(t))_{x,y} = 0 \text{ whenever } d(x,y) > r(t)$$

for all $m \in I$, $x, y \in X^m_d$ and $t \in [0, \infty)$.

The uniform localization algebra $C^*_u((P_d(X_m))_{m \in I})$ is defined to be the completion of

$$C_{u,L}((P_d(X_m))_{m \in I})$$

under the norm

$$\|f\| = \|f(t)\|_{C^*((P_d(X_m))_{m \in I})}$$

for all $f \in C_{u,L}(P_d(X_m))_{m \in I}$.

Naturally, we have a $*$-homomorphism

$$e : C^*_u((P_d(X_m))_{m \in I}) \to C^*_u((P_d(X_m))_{m \in I})$$

defined by the evaluation-at-zero map. Moreover, this homomorphism induces a map

$$e_* : \lim_{d \to \infty} K_*(C^*_u((P_d(X_m))_{m \in I})) \to \lim_{d \to \infty} K_*(C^*_u((P_d(X_m))_{m \in I}))$$

on the $K$-theory level.

**The coarse Baum–Connes conjecture for a sequence of metric spaces.** For any sequence of discrete metric spaces $(X_m, d_m)_m$ with uniform bounded geometry, the map

$$e_* : \lim_{d \to \infty} K_*(C^*_u((P_d(X_m))_{m \in I})) \to \lim_{d \to \infty} K_*(C^*_u((P_d(X_m))_{m \in I}))$$

induced by the evaluation-at-zero map on $K$-theory is an isomorphism.

Let $(X_m, d_m)_{m \in \mathbb{N}}$ be a sequence of metric spaces with uniform bounded geometry. The sequence of metric spaces $(X_m, d_m)_m$ is said to be coarsely embeddable into Hilbert space if for each $m \in I$ there exists a map $f_m$ from $X_m$ to a Hilbert space $H_m$, and two non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that
that the sequences \( (X_m)_{m \in I} \) is coarsely embeddable into Hilbert space if and only if the metric space of separated disjoint union of the sequence \( (X_m)_{m \in I} \) is coarsely embeddable into Hilbert space. By the third author's result in \([54]\), the coarse Baum–Connes conjecture for the sequence \( (X_m)_{m \in I} \) is true when the sequence \( (X_m)_{m} \) can be coarsely embedded into Hilbert space.

The concept of Property A was introduced by the third author in \([54]\). As a generalization of amenability in the context of metric spaces, Property A has many equivalent characterizations, see \([38]\). In this paper, we will consider the concept of Property A for a sequence of metric spaces. A sequence of metric space \( \{(X_m, d_m)_{m \in I}\} \) is said to have Property A, if for each \( \epsilon > 0 \) and each \( R > 0 \), there exists a sequence of functions \( \{k_m : X_m \times X_m \to \mathbb{R}\} \) and a constant \( S > 0 \), such that

1. for each \( m \), the function \( k_m \) is of positive definite type, i.e., \( \sum_{i,j}^N k_m(x_i, x_j)t_it_j \geq 0 \) for each finite subset \( \{x_i\}_{i=1}^N \subset X_m \) and \( \{t_i\}_{i=1}^N \subset \mathbb{R} \);
2. for each \( m \in I \), \( |1 - k_m(x, y)| \leq \epsilon \) for all \( x, y \in X_m \) with \( d(x, y) < R \);
3. for each \( m \in I \), \( k_m(x, y) = 0 \) for all \( x, y \in X_m \) with \( d(x, y) \geq S \).

In the rest of this paper, we shall prove the following result. Theorem \( \text{[3.3]} \) and \( \text{[3.4]} \) follows from the following result.

**Theorem 3.12.** Let \((1 \to N_m \to G_m \to Q_m \to 1)_{m \in I}\) be a sequence of exact sequences of groups such that the sequences \((N_m)_m\), \((G_m)_m\) and \((Q_m)\) have uniform bounded geometry. If \((N_m)_m\) has Property A and \((Q_m)_m\) is coarsely embeddable into Hilbert space, then the coarse Baum–Connes conjecture holds for the sequence \((G_m)_{m \in I}\), that is, the map

\[
eq \lim_{d \to \infty} K_s(C^*_u(L((P_d(G_m))_{m \in I}))) \to \lim_{d \to \infty} K_s(C^*_u((P_d(G_m))_{m \in I}))
\]

induced by the evaluation-at-zero map on \( K\)-theory is an isomorphism.

When \( I = \{1\} \), Theorem \( \text{[3.3]} \) follows from the Theorem 3.12.

Let us consider \( I = \mathbb{N} \) and \((1 \to N_m \to G_m \to Q_m \to 1)_{m \in I}\) be a sequence of extensions of finite groups. A coarse disjoint union \((X, d)\) is the disjoint union \( \bigsqcup_m G_m \) as a set endowed with a metric \( d \) satisfying the following conditions:

- for each \( m \), \( d(x, y) = d_m(x, y) \) for all \( x, y \in G_m \);
- \( \text{dist}(G_m, G_{m'}) \to \infty \), as \( m + m' \to \infty \) for all \( m \neq m' \).

The separated disjoint union is the metric space \((Y, d')\) be the disjoint union equipped with the metric \( d' \) satisfying the following conditions:

- for each \( m \), \( d'(x, y) = d_m(x, y) \) for all \( x, y \in G_m \);
- \( \text{dist}(G_m, G_{m'}) = \infty \) for all \( m \neq m' \).
Recall that the index set $I$ is neither the singleton $\{1\}$ or the natural numbers $\mathbb{N}$. Note that when $I = \{1\}$, then the coarse Baum–Connes conjecture for the sequence $(X_m)_{m \in I}$ is the coarse Baum–Connes conjecture for a metric space $X_1$ with bounded geometry, while the coarse Baum–Connes conjecture for the sequence $(X_m)_{m \in I}$ is the coarse Baum–Connes conjecture for the separated disjoint union of $X_m$'s when $I = \mathbb{N}$.

**Theorem 3.13.** Let $(G_m)_{m \in I}$ be a sequence of finite groups endowed with word length metrics such that the metric spaces $X$ and $Y$ defined above have bounded geometry. If the coarse Baum–Connes conjecture holds for the separated disjoint union $(Y, d')$, then the coarse Baum–Connes conjecture holds for the coarse disjoint union $(X, d)$.

**Proof.** We have the following commutative diagram:

$$
\begin{array}{c}
0 \\
\oplus_m K_*(C_\ell^+(P_d(G_m))) \\
\searrow \\
K_*(C_\ell^+(P_d(Y))) \\
\oplus_m K_*(C_\ell^+(P_d(G_m))) \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\oplus_m K_*(C_u((P_d(G_m))_{m \in I})) \\
\searrow \\
K_*(C^+(P_d(Y))) \\
\oplus_m K_*(C_\ell^+(P_d(G_m))) \\
0
\end{array}
$$

where the horizontal maps are induced by evaluation-at-zero maps and the vertical maps are induced by inclusions. Note that the diagram is compatible with increasing of the Rips complex scale, so we have the commutative diagram:

$$
\begin{array}{c}
\lim_{d \to \infty} \oplus_m K_*(C_\ell^+(P_d(G_m))) \\
\searrow \\
\lim_{d \to \infty} K_*(C_\ell^+(P_d(Y))) \\
\oplus_m K_*(C_\ell^+(P_d(G_m))) \\
0
\end{array}
\quad
\begin{array}{c}
\lim_{d \to \infty} \oplus_m K_*(C^+(P_d(G_m))) \\
\searrow \\
\lim_{d \to \infty} K_*(C^+(P_d(Y))) \\
\oplus_m K_*(C^+(P_d(G_m))) \\
0
\end{array}
$$
Since each group $G_m$ is finite, so the Rips complex $P_d(G_m)$ is strongly Lipschitz homotopy equivalent to a point when $d$ is larger than the diameter of $G_m$. As a result, the map

$$e_* : \lim_{d \to \infty} \bigoplus_m K_* (C^*_L (P_d (G_m))) \to \lim_{d \to \infty} \bigoplus_m K_* (C^* (P_d (G_m)))$$

induced by the evaluation-at-zero map is an isomorphism. In addition, we assume that the middle horizontal map in the above diagram is an isomorphism. As a consequence of diagram chasing, we have that the map

$$e_* : \lim_{d \to \infty} \bigoplus_m K_* (C^*_L (P_d (Y))) \to \lim_{d \to \infty} \bigoplus_m K_* (C^* (P_d (G_m)))$$

is an isomorphism.

Let $H_d$ be an ample $(\bigcup_m P_d (G_m))$-module. We have a commutative diagram:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
K_* (C^*_L (\Delta_d)) \bigoplus_{m > N_d} K_* (C^*_L (P_d (G_m))) & \to & K_* (K (H_d)) \\
\downarrow & & \downarrow \\
K_* (C^*_L (P_d (X))) & \to & K_* (C^* (P_d (X))) \\
\downarrow & & \downarrow \\
K_* (C^*_L (P_d (X))) & \to & K_* (C^* (P_d (X))) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

The diagram is also compatible with increasing of the Rips complex scale $d$, so we may take the limit as $d$ goes to infinity. Note that $\Delta_d$ is strongly homotopy to a point, we have that the restriction of the top horizontal map

$$e_* : K_* (C^*_L (\Delta_d)) \to K_* (K (H_d))$$

is an isomorphism. As $d$ tends to infinity, we have that the top horizontal map is an isomorphism.

For each $d > 0$, we have that

$$C^* (P_d (X)) = K (H_d) + C^* (P_d (Y))$$

and

$$C^* (P_d (Y)) \cap K (H_d) = \bigoplus_m C^* (P_d (G_m)).$$

Therefore, we have

$$\frac{C^* (P_d (X))}{K (H_d)} \cong \frac{C^* (P_d (Y))}{\bigoplus_m C^* (P_d (G_m))}$$

In addition,

$$\lim_{d \to \infty} \frac{K_* (C^*_L (P_d (Y)))}{\bigoplus_m K_* (C^*_L (P_d (G_m)))} \cong \lim_{d \to \infty} \frac{K_* (C^* (P_d (Y)))}{\bigoplus_m K_* (C^* (P_d (G_m)))}$$

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As a result, the bottom horizontal map is an isomorphism when taking the limit on $d$ as $d$ tends to infinity. By Five Lemma, the middle horizontal map is an isomorphism as $d$ tends to infinity, i.e., the map

$$e_* : \lim_{d \to \infty} K_*(C^*_L(P_d(X))) \to \lim_{d \to \infty} K_*(C^*(P_d(X)))$$

induced by the evaluation-at-zero map is an isomorphism.

\[\Box\]

For a sequence of finite groups $(N_m)_{m \in \mathbb{N}}$, we have that the coarse disjoint union has Property A if and only if the sequence $(N_m)_{m \in \mathbb{N}}$ has Property A.

In the constructions of relative expanders by Arzhantseva–Tessera (in \[3\]) and Delabie–Khukhro (in \[14\]), the normal subgroups are a sequence of quotient groups of an abelian group. As a result of Proposition 11.39 in \[43\], the coarse disjoint union $\bigsqcup N_m$ has property A, which implies that the sequence of finite metric spaces $(N_m)_{m \in \mathbb{N}}$ has Property A. In addition, the coarse disjoint union of the quotient groups $(Q_m)_{m \in \mathbb{N}}$ in the relative expanders constructed by Arzhantseva–Tessera and Delabie–Khukhro are all coarsely embeddable into Hilbert space which implies the sequence of the quotient groups $(Q_m)_{m \in \mathbb{N}}$ are uniformly coarsely embeddable into Hilbert space. Therefore, Theorem 3.12 holds. By Theorem 3.13, we have that the coarse Baum–Connes conjecture holds for the coarse disjoint union $\bigsqcup G_m$, that is, the map

$$e_* : \lim_{d \to \infty} K_*(C^*_L(P_d(\bigsqcup_m G_m))) \to \lim_{d \to \infty} K_*(C^*(P_d(\bigsqcup_m G_m)))$$

is an isomorphism.

4 Twisted Roe algebras and twisted localization algebras

In this section, we will define twisted Roe algebras and twisted localization algebras for a sequence of metric spaces. The constructions of these algebras are similar to those defined by the third author in \[54\] and by the second and third author with X. Chen in \[9\].

4.1 Preliminary

In this subsection, we shall recall a $C^*$-algebra associated with an infinite-dimensional Euclidean space introduced by Higson, Kasparov and Trout \[30\].

Let $H$ be a countably infinite-dimensional Euclidean space. Denote by $V_a, V_b$ the finite-dimensional affine subspaces of $H$. Let $V^0_a$ be the finite dimensional linear subspace of $H$ consisting of differences of elements in $V_a$. Let Cliff$(V^0_a)$ be the complexified Clifford algebra on $V^0_a$ and $C(V_a)$ be the graded $C^*$-algebra of continuous functions from $V_a$ to Cliff$(V^0_a)$ vanishing at infinity. Let $S$ be the $C^*$-algebra of all continuous functions on $\mathbb{R}$ vanishing at infinity. Then $S$ is graded according to the odd and even functions. Define the graded tensor product

$$A(V_a) = S \hat{\otimes} C(V_a).$$

If $V_a \subseteq V_b$, we have a decomposition $V_b = V^0_{ba} + V_a$, where $V^0_{ba}$ is the orthogonal complement of $V^0_a$ in $V^0_b$. For each $v_b \in V_b$, we have a unique decomposition $v_b = v_{ba} + v_a$, where $v_{ba} \in V^0_{ba}$ and $v_a \in V_a$. Every function $h$ on $V_a$ can be extended to a function $\tilde{h}$ on $V_b$ by the formula $\tilde{h}(v_{ba} + v_a) = h(v_a)$.
Definition 4.1.  (1) If \( V_a \subseteq V_b \), we define \( C_{ba} \) to be the Clifford algebra-valued function \( V_b \to \text{Cliff}(V_b^0) \), \( v_b \mapsto v_{ba} \) \( \in V_b^0 \subseteq \text{Cliff}(V_b^0) \). Let \( X \) be the function multiplication by \( x \) on \( \mathbb{R} \), considered as a degree one and unbounded multiplier of \( S \). Define a homomorphism \( \beta_{ba} : A(V_a) \to A(V_b) \) by
\[
\beta_{ba}(g \circ h) = g(X \otimes 1 + 1 \otimes C_{ba})(1 \otimes h)
\]
for all \( g \in S \) and \( h \in A(V_a) \), and \( g(X \otimes 1 + 1 \otimes C_{ba}) \) is the functional calculus of \( g \) on the unbounded, essentially self-adjoint operator \( X \otimes 1 + 1 \otimes C_{ba} \).

(2) We define a \( C^* \)-algebra \( A(H) \) by:
\[
A(H) = \lim_{\to} A(V_a),
\]
where the direct limit is taken over the directed set of all finite-dimensional affine subspaces \( V_a \subseteq H \).

Remark 4.2. If \( V_a \subseteq V_b \subseteq V_c \), then we have \( \beta_{fb} \circ \beta_{ba} = \beta_{ca} \). Therefore, the above homomorphisms give a directed system \( (A(V_a))_{V_a} \) as \( V_a \) ranges over finite dimensional affine subspaces of \( V \).

The set \( \mathbb{R}_+ \times H \) is equipped with a topology as follows. Let \( \{(t_i, v_i)\} \) be a net in \( \mathbb{R}_+ \times H \), it converges to a point \( (t, v) \in \mathbb{R}_+ \times H \) if
\[
\begin{align*}
(1) & \quad t_i^2 + \|v_i\|^2 \to t^2 + \|v\|^2, \text{ as } i \to \infty; \\
(2) & \quad (v_i, u) \to (v, u) \text{ for any } u \in H, \text{ as } i \to \infty.
\end{align*}
\]

It is obvious that \( \mathbb{R}_+ \times H \) is a locally compact Hausdorff space. Note that for each \( v \in H \) and each \( r > 0 \), \( B(v, r) = \{(t, w) \in \mathbb{R}_+ \times H : t^2 + \|v - w\|^2 < r^2\} \) is an open subset of \( \mathbb{R}_+ \times H \). For finite dimensional subspaces \( V_a \subseteq V_b \subseteq V_c \), since \( \beta_{ba} \) takes \( C_0(\mathbb{R}_+ \times V_a) \) into \( C_0(\mathbb{R}_+ \times V_b) \), then the \( C^* \)-algebra \( \lim_{\to} C_0(\mathbb{R}_+ \times V_a) \) is \( * \)-isomorphic to \( C_0(\mathbb{R}_+ \times H) \).

Definition 4.3. The support of an element \( a \in A(H) \) is the complement to the set of all points \( (t, v) \in \mathbb{R}_+ \times H \) such that there exists \( g \in C_0(\mathbb{R}_+ \times H) \) with \( g(t, v) \neq 0 \) and \( g \cdot a = 0 \).

Let \( G \) be a finitely generated group with a finite symmetric generating subset \( S \subseteq G \). Recall that a set is symmetric in the sense that \( s^{-1} \in S \) for all \( s \in S \). We then can define the word length \( |g|_S \) of \( g \in G \) as follows:
\[
|g|_S = \min \{ n \mid g = g_1 \cdots g_n, \text{ where } g_1, \cdots, g_n \in S \}.
\]

Definition 4.4. Let \( G \) be a finitely generated discrete group with a finite symmetric generating subset \( S \). The word length metric \( d \) associated with \( S \), is defined by
\[
d(g, h) = |gh^{-1}|_S
\]
for all \( g, h \in G \).

Note that a finitely generated group \( G \) equipped with any word length metric is a metric space with bounded geometry. In addition, the metric spaces of \( G \) associated with different word length metrics are coarsely equivalent. In general, any countable discrete group admits a proper metric and different proper metrics on such a group are coarsely equivalent.

Let \( 1 \to N \to G \to Q \to 1 \) be a short exact sequence of finitely generated discrete groups. Let \( \pi : G \to Q \) be the quotient map, and \( S \subseteq G \) a finite symmetric generating subset of \( G \). It is easy to verify
that the image \(\pi(S)\) is a generating subset of the quotient group \(Q\). As a result, we obtain a right invariant word length metric on \(Q\). In addition, the normal subgroup \(N\) admits a metric which is a restriction of the metric on \(G\).

Let \(f : Q \to H\) be a coarse embedding, and \(G = \bigcup_{g \in \Lambda} gN\) a coset decomposition of \(G\) where \(\Lambda \subset G\) is a set of representatives of cosets in \(Q\). For each \(d > 0\), we shall extend the map \(f\) to a map from the Rips complex \(P_d(Q)\) to the Hilbert space \(H\) as follows. For any point \(z = \sum_{g \in \Lambda} c_g gN \in P_d(Q)\), define

\[
f(z) = \sum_{g \in \Lambda} c_g f(gN).
\]

For each \(z = \sum_{g \in \Lambda} c_g gN \in P_d(Q)\) and each \(n \in \mathbb{N}\), define a subspace \(W_n(z)\) of \(H\) as

\[
W_n(g) = \text{span}\{f(g'N) : g'N \in Q, d_Q(g'N, gN) \leq n^2\}
\]

for all \(g\) such that \(c_g > 0\). The bounded geometry property of \(Q\) implies that \(W_n(g)\) is finite dimensional.

The quotient map \(\pi : G \to Q\) induces a map \(\pi : P_d(G) \to P_d(Q)\) by \(\pi(\sum_i c_i g_i) = \sum_i c_i \pi(g_i)\) for all \(\sum_i c_i g_i \in P_d(G)\). For any element \(x \in P_d(G)\), for any \(n \in \mathbb{N}\), we define \(W_n(x) = W_n(\pi(x))\) to be the finite-dimensional Euclidean space of Hilbert space \(H\).

**Remark 4.5.** (1) Since \(f : Q \to H\) is a coarse embedding and \(G\) has bounded geometry, then for any \(n \in \mathbb{N}\), there exists \(R_{n,d} > 0\) such that \(\dim(W_n(x)) \leq R_{n,d}\) for all \(x \in P_d(G)\).

(2) For each \(r > 0\), there exists \(N > 0\) such that \(W_n(x) \subset W_{n+1}(y)\) for all \(n \geq N\) and \(x, y \in P_d(G)\) satisfying \(d(x, y) \leq r\).

### 4.2 Twisted Roe algebras and twisted localization algebras

In this subsection, we shall define the twisted Roe algebras and twisted localization algebras for the sequence \((G_m)_m\) using the coarse embeddability of the sequence \((Q_m)_m\).

Let \((1 \to N_m \to G_m \to Q_m \to 1)_{m \in I}\) be a sequence of extensions of discrete groups. For each \(m \in I\), we have word length metrics on \(N_m, G_m\) and \(Q_m\). Here, we do not assume the groups \(N_m, G_m\), and \(Q_m\) are finite. Assume that the sequences of metric spaces \((N_m)_{m \in I}\), \((G_m)_{m \in I}\) and \((Q_m)_{m \in I}\) have uniform bounded geometry.

Recall that a sequence of maps \((f_m : Q_m \to H_m)_{m \in I}\) is a coarse embedding for the sequence \((Q_m)_m\) if there exist two non-decreasing maps, \(\rho_i : [0, \infty) \to [0, \infty)\), satisfying

- \(\lim_{t \to \infty} \rho_1(t) = \infty\) as \(t \to \infty\);
- for each \(m \in I\), \(\rho_1(d(x, y)) \leq \|f_m(x) - f_m(y)\| \leq \rho_2(d(x, y))\) for all \(x, y \in Q_m\).

For each \(m \in I\) and \(d > 0\), let \(f_m : P_d(Q_m) \to H_m\) be the linear extension of \(f_m : Q_m \to H_m\), and \(\pi : P - d(G_m) \to P_d(Q_m)\) the linear extension of the quotient map \(\pi : G_m \to Q_m\). For each \(m \in I\) and \(x \in P_d(G_m)\), let \(W_n(x)\) is the linear subspace of \(H_m\) defined by

\[
W_n(x) = \text{span}\{f_m(\pi(y)) : d(\pi(x), \pi(y)) \leq n^2\}.
\]
For each \( d > 0 \) and each \( m \in I \), we firstly defined a spherical metric on each \( P_d(G_m) \), then choose a countable dense subset \( X^m_d \) of \( P_d(G_m) \) such that \( X^m_d \subseteq X^m_{d_2} \) if \( d_1 \leq d_2 \) for each \( m \in I \).

**Definition 4.6.** For each \( d > 0 \), the algebraic twisted Roe algebra \( C^{\ast}_{alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \) is the set of all tuples \( T = (T^{(m)})_{m \in I} \) satisfying

1. for each \( m \), \( T^{(m)} \) is a bounded function from \( X^m_d \times X^m_d \) to \( \mathcal{A}(H_m) \tilde{\otimes} K \) where \( K \) is the algebra of all compact operators on an infinite-dimensional separable Hilbert space;

2. there exists an integer \( N \) such that
   \[
   T^{(m)}(x, y) \to (\beta_N(x))(\mathcal{A}(W_N(x)) \tilde{\otimes} K) \subseteq \mathcal{A}(H_m) \tilde{\otimes} K
   \]
   for each \( m \) and \( x, y \in X^m_d \), where \( \beta_N(x) : \mathcal{A}(W_N(x)) \to \mathcal{A}(H_m) \) is the \( * \)-homomorphism associated with the inclusion of \( W_N(x) \) into \( H_m \);

3. there exists \( L > 0 \) such that, for each \( m \) and each \( y \in X^m_d \),
   \[
   \sharp\{ x : T^{(m)}(x, y) \neq 0 \} \leq L, \quad \sharp\{ x : T^{(m)}(y, x) \neq 0 \} \leq L;
   \]

4. for each \( m \) and each bounded subset \( B \subset P_d(G_m) \), the set
   \[
   \sharp\{(x, y) \in (B \times B) \cap (X^m_d \times X^m_d) : T^{(m)}(x, y) \neq 0 \}
   \]
   is finite;

5. there exists \( r_1 > 0 \) such that \( T^{(m)}(x, y) = 0 \) for all \( m \in I \) and \( x, y \in X^m_d \) with \( d(x, y) > r_1 \) (The least such \( r_1 \) is the propagation of the tuple \( T = (T^{(m)})_{m \in I} \));

6. there exists \( r_2 > 0 \) such that \( \text{Supp}(T^{(m)}(x, y)) \subseteq B(f_m(\pi(x)), r_2) \) for all \( x, y \in X_d \) where
   \[
   B(f_m(\pi(x)), r_2) = \{(s, h) \in \mathbb{R}_+ \times H : s^2 + \| h - f_m(\pi(x)) \|^2 < r_2^2 \};
   \]

7. there exists \( c > 0 \) such that if \( T^{(m)}(x, y) = \beta_N(x)(S^{(m)}(x, y)) \) for some \( S^{(m)}(x, y) \in \mathcal{A}(W_N(x)) \tilde{\otimes} K \), then
   \[
   \sup_m \| D_Y(S^{(m)}(x, y)) \| \leq c
   \]
   for each \( m \in I \), \( x, y \in X^m_d \) and all \( Y = (s, h) \in \mathbb{R}_+ \times W_N(x) \) satisfying \( \| Y \| = \sqrt{s^2 + \| h \|^2} \leq 1 \), where \( D_Y(S^{(m)}(x, y)) \) is the derivative of the function \( S^{(m)}(x, y) : \mathbb{R}_+ \times W_N(x) \to \text{Cliff}(W_N(x)) \tilde{\otimes} K \), in the direction of \( Y \).

**Definition 4.7.** For each tuple \( T = (T^{(m)})_{m \in I} \), the propagation of \( T \) is defined to be

\[
\text{propagation}(T) = \sup_m \text{propagation}(T^{(m)}) = \sup_m \sup_{x, y \in X^m_d} \{ d(x, y) : T^{(m)}(x, y) \neq 0 \}.
\]

We define the multiplication and adjoint operations on \( C^{\ast}_{alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \) by:

\[
(TS)^{(m)}(x, y) = \sum_{z \in X_d} T^{(m)}(x, z)S^{(m)}(z, y),
\]

and

\[
(T^*)^{(m)}(x, y) = \left( T^{(m)}(y, x) \right)^*.
\]
for all $T = (T^{(m)})_{m \in I} = (T^{(1)}, T^{(2)}, \ldots)$ and $S = (S^{(m)})_{m \in I} = (S^{(1)}, S^{(2)}, \ldots)$ in $C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$. It is obvious that $C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ is a $*$-algebra.

For each $m \in I$, define

$$E^m = \left\{ \sum_{x \in X_d^m} a_x [x] : \sum_x a_x^* a_x \text{ converges in strong operator topology, where } a_x \in \mathcal{A}(H_m) \hat{\otimes} K \right\}.$$ 

The space $E^m$ can be equipped with a Hilbert $\mathcal{A}(H_m) \otimes K$-module by

$$\langle \sum_{x \in X_d^m} a_x [x], \sum_{x \in X_d^m} b_x [x] \rangle = \sum_{x \in X_d^m} a_x^* b_x,$$

for all $\sum_{x \in X_d^m} a_x [x], \sum_{x \in X_d^m} b_x [x] \in E^m$. Let $B(E^m)$ be the algebra of all adjointable $\mathcal{A}(H_m) \otimes K$-linear homomorphisms. We then define a $*$-representation

$$\phi^m : C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to B(E^m)$$

by

$$T \cdot \left( \sum_{x \in X_d^m} a_x [x] \right) = \sum_{x \in X_d^m} \left( \sum_{y \in X_d^m} T_{x,y}^{(m)} a_y \right) [x],$$

for all $T = (T^{(m)})_{m \in I} \in C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ and $\sum_{x \in X_d^m} a_x [x] \in E^m$. Note that this is a $*$-representation. Taking direct sum of these $*$-representations, we obtain a faithful $*$-representation for the $*$-algebra $C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ by

$$\bigoplus_m \phi^m : C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to B(\bigoplus_m E^m).$$

The uniform twisted Roe algebra $C_u^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ is defined to be the completion of

$$C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$$

under the operator norm in $B(\bigoplus_m E^m)$.

Let $C_{u,L,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ be the $*$-algebra of all uniformly bounded and uniformly norm-continuous functions

$$g : \mathbb{R}_+ \to C_{u,alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$$

such that:

1. there exists $N$ such that $(g(t))^{(m)}(x,y) \in (\beta_N(x)) (\mathcal{A}(W_N(x)) \hat{\otimes} K) \subseteq \mathcal{A}(H_m) \hat{\otimes} K$ for all $t \in \mathbb{R}_+$, $m \in I$ and $x, y \in X_d^m$;

2. there exists a bounded function $r(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t \to \infty} r(t) = 0$ and if $d(x,y) > r(t)$, then $(g(t))^{(m)}(x,y) = 0$;

3. there exists $R > 0$ such that $\text{Supp}\{(g(t))^{(m)}(x,y)\} \subseteq B(f_m(\pi(x)), R)$ for all $t \in \mathbb{R}_+$, $m \in I$ and $x, y \in X_d^m$;

4. there exists $c > 0$ such that $\|D_T(h^{(m)}(t)(x,y))\| \leq c$ for all $t \in \mathbb{R}_+$, $m \in I$, $x, y \in X_d^m$ and $Y \in \mathbb{R} \times W_N(x)$ satisfying $\|Y\| \leq 1$, where $h^{(m)}(t)(x,y) \in \mathcal{A}(W_N(x))$ satisfying $(\beta_N(x))(h^{(m)}(t)(x,y)) = (g(t))^{(m)}(x,y)$. 

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Definition 4.8. The twisted localization algebra $C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ is defined to be the norm completion of $C^*_{u,L,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ with respect to the norm

$$\|g\| = \sup_{t \in \mathbb{R}_+} \|g(t)\|.$$ 

There is a natural evaluation-at-zero map

$$e : C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$$

by

$$e((g^{(1)}(t), g^{(2)}(t), \cdots)) = (g^{(1)}(0), g^{(2)}(0), \cdots)$$

for all $(g^{(1)}(t), g^{(2)}(t), \cdots) \in C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$. The map $e$ is a *-homomorphism, then it induces a homomorphism

$$e_* : \lim_{d \to \infty} K_*(C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})) \to \lim_{d \to \infty} K_*(C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}))$$
on $K$-theory.

4.3 The twisted Baum–Connes conjecture

In this section, we will prove the twisted coarse Baum–Connes conjecture.

Theorem 4.9. Let $(1 \to N_m \to G_m \to G_m/N_m \to 1)_{m \in I}$ be a sequence of extensions of discrete groups. Assume that the sequences of metric spaces $(N_m)_{m \in I}$, $(G_m)_{m \in I}$ and $(Q_m)_{m \in I}$ have uniformly bounded geometry. If the sequence of metric spaces $(N_m)_{m \in I}$ has Property A and the sequence $(Q_m)_{m \in I}$ is coarsely embeddable into Hilbert space. Then the homomorphism

$$e_* : \lim_{d \to \infty} K_*(C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})) \to \lim_{d \to \infty} K_*(C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I}))$$
on $K$-theory is an isomorphism, where $C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ and $C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ are the $C^*$-algebras defined in Section 4.2.

To prove this result, we need to discuss ideals of the twisted algebras associated with open subsets of $\mathbb{R}_+ \times H_m$.

Definition 4.10. Let $O = (O_m)_{m \in I}$ be a sequence of subsets such that each $O_m$ is an open subset of $\mathbb{R}_+ \times H_m$.

1. Denoted by $C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$ the $C^*$-subalgebra of $C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ consisting of all tuples $T = (T^{(m)})_{m \in I}$ such that

$$\text{Supp}(T^{(m)}(x, y)) \subset O_m$$

for all $m \in I$ and $x, y \in X^m_d$.

2. Denoted by $C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$ the $C^*$-subalgebra of $C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ consisting of all tuples $g = (g^{(m)}(t))_{m \in I}$ such that

$$\text{Supp}(g^{(m)}(t)(x, y)) \subset O_m$$

for all $t \in [0, \infty)$, $m \in I$ and $x, y \in X^m_d$.
For any two sequence of open sets \( O = (O_m)_m \) and \( O' = (O'_m)_m \), we say \( O \subset O' \), if \( O_m \subset O'_m \) for all \( m \), and denote \( O \cup O' = (O_m \cup O'_m)_m \) and \( O \cap O' = (O_m \cap O'_m)_m \). Note that the \( C^* \)-subalgebras \( C^*((P_d(G_m), A(H_m))_{m \in I})_O \) and \( C^*_L((P_d(G_m), A(H_m))_{m \in I})_{O'} \) are closed two-sided ideals of \( C^*((P_d(G_m), A(H_m))_{m \in I})_{O'} \) and \( C^*_u((P_d(G_m), A(H_m))_{m \in I})_{O'} \), respectively.

Given a sequence of open sets \( O = (O_m)_m \), we have a homomorphism
\[
\epsilon_* : \kappa((C^*_u(P_d(G_m), A(H_m))_{m \in I})_O) \rightarrow K_*(((C^*_u(P_d(G_m), A(H_m))_{m \in I})_O)
\]
induced by the evaluation-at-zero map on \( K \)-theory.

**Lemma 4.11.** Let \( O^{(1)} = (O^{(1)}_m) \) and \( O^{(2)} = (O^{(2)}_m) \) be two sequences of sets such that each \( O^{(i)}_m \) is an open subset of \( \mathbb{R}_+ \times H_m \) for \( i = 1, 2 \) and for all \( m \in I \). Then we have

\begin{enumerate}
\item \( C^*_u((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cup O^{(2)}} = C^*((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cup O^{(2)}} \),
\item \( C^*_u,(P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} = C^*_u,(P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} \),
\item \( C^*_u((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} = C^*_u,(P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} \),
\item \( C^*_u,(P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} = C^*_u,(P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cap O^{(2)}} \).
\end{enumerate}

**Proof.** We shall prove the first equality and others can be dealt with similarly. To prove the first equality, it suffices to show that
\[
C^*_u,alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cup O^{(2)}} = C^*_u,alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)}} + C^*_u,alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(2)}},
\]
Given \( T \in C^*_alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)} \cup O^{(2)}} \), without loss of generality, assume that
\[
\text{Supp}(T^{(m)}(x,y)) \subset (Z_d^m \times Z_d^m) \times (C_m^m \cup C_2^m) \subset (Z_d^m \times Z_d^m) \times (O^{(1)} \cup O^{(2)}),
\]
where \( C_1^m \subset O^{(1)}_m \) and \( C_2^m \subset O^{(2)}_m \) are closed subsets. Taking smooth functions \( \phi_1 \in C_0(O^{(1)}_m) \) and \( \phi_2 \in C_0(O^{(2)}_m) \) such that \( \phi_1^m + \phi_2^m = 1 \) on \( C_1^m \cup C_2^m \). Since the space \( \mathbb{R}_+ \times H_m \) is locally compact, the existence of the functions \( \phi_1^m \) and \( \phi_2^m \) is guaranteed for each \( m \).

Define
\[
(T_1^{(m)})(x,y) = \phi_1^m T^{(m)}(x,y)
\]
\[
(T_2^{(m)})(x,y) = \phi_2^m T^{(m)}(x,y)
\]
for \( x,y \in X_d^m \). Then we have
\[
T_1 = \left( T_2^{(m)} \right)_{m \in I} \in C^*_alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(1)}},
\]
and
\[
T_2 = \left( T_2^{(m)} \right)_{m \in I} \in C^*_alg((P_d(G_m), A(H_m))_{m \in I})_{O^{(2)}},
\]
In addition, we have \( T = T_1 + T_2 \). \( \square \)
Proposition 4.12. Let $r > 0$. If $O = (O_m)_{m \in I}$ is a sequence of sets such that each $O_m$ is the union of a family of open subsets $\{O_{m,j}\}_{j \in J}$ of $\mathbb{R}_+ \times H_m$ satisfying:

1. for each $j \in J$, $O_{m,j} \subseteq B(f_m(\pi(x_j^m)), r)$ for some $x_j^m \in G_m$, where
   
   $$B(f_m(\pi(x_j^m)), r) = \{(t, h) \in \mathbb{R}_+ \times H_m : t^2 + \|h - f_m(\pi(x_j^m))\|^2 < r^2\},$$

   and $f_m$ is the coarse embedding $f_m : Q_m \to H_m$;

2. for each $m$, $O_{m,j} \cap O_{m,j'} = \emptyset$ if $j \neq j'$.

then the map

$$e_* : \lim_{d \to \infty} K_*(C_u L((P_d(G_m), A(H_m))_{m \in I})_O) \to \lim_{d \to \infty} K_*(C_u L((P_d(G_m), A(H_m))_{m \in I})_O)$$

induced by the evaluation-at-zero map on $K$-theory is an isomorphism.

In order to prove this result, we need to analyze the algebraic structure of the twisted localization algebras and twisted Roe algebras.

Let $T = (T^{(m)})_{m \in I} \in C^*((P_d(G_m), A(H_m))_{m \in I})_O$. Since each $O_m$ is the disjoint union of a family of open subsets $\{O_{m,j}\}_{j \in J}$, then for each $m$ the operator $T^{(m)}$ can be expressed as

$$T^{(m)} = \left(T_j^{(m)}\right)_{m,j},$$

where $T_j^{(m)}$ is an $X_d^m \times X_d^m$-matrix with $\text{Supp}(T_j^{(m)}) \subset X_d^m \times X_d^m \cap O_{m,j}$. Furthermore, by Condition (6) in Definition 4.6, we know that each $T_j^{(m)}$ has support contained in

$$P_d(B_{G_m}(x_j^m N_m, R)) \times P_d(B_{G_m}(x_j^m N_m, R)) \times O_{m,j}$$

for some $R > 0$ which is independent on $m$, $j$.

For a fixed $R > 0$, the collection of metric spaces $(B_{G_m}(x_j^m N_m, R))_{m,j}$ satisfies the following properties:

1. the sequence $(B_{G_m}(x_j^m N_m, R))_{m,j}$ has uniform bounded geometry;
2. the sequence $(B_{G_m}(x_j^m N_m, R))_{m,j}$ has Property A.

Let $N_{m,j} = N_m$ for each $m$ and each $j$. The sequence of metric spaces $(B_{G_m}(x_j^m N_m, R))_{m,j}$ is uniformly coarsely equivalent to the metric spaces $(N_{m,j})_{m,j}$ in the sense that there exists a constant $R > 0$ such that $N_{m,j}$ is an $R$-net of $B_{G_m}(x_j^m N_m, R)$ for each $m$ and each $j$.

Let $A(H_m)_{O_{m,j}}$ be the $C^*$-subalgebra of $A(H_m)$ with support contained in $O_{m,j}$ for each $m$ and each $j$. For the sequence of metric spaces $(B_{G_m}(x_j^m N_m, R))_{m,j}$ and $C^*$-algebras $(A(H_m)_{O_{m,j}})_{m,j}$, we can define a $*$-subalgebra of Roe algebra $C^*((P_d(G_m), A(H_m))_{m \in I})_O$.

For brevity, set up $Z^m_{d,j,R} = P_d(B_{G_m}(x_j^m N_m, R))$. Given $R > 0$, for each $d > 0$, let $X_d^m \subseteq X_d^m \cap P_d(B_{G_m}(x_j^m N_m, R))$ be the countable dense subset of $P_d(B_{G_m}(x_j^m N_m, R))$, where $X_d^m \subset P_d(G_m)$ is the countable sense subset in Definition 4.6.
Definition 4.13. Let $C^*_{u,alg}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ be the set of tuples $T = \left(T^{(m)}_j\right)_{m,j}$ where each $T^{(m)}_j$ is a bounded function on $X^m_{d,j,R} \times X^m_{d,j,R}$ such that

1. there exists an integer $N$ such that
   $$T^{(m)}_j(x,y) \in (\beta_N(x))(\mathcal{A}(W_N(x)) \hat{\otimes} K) \subseteq \mathcal{A}(H_m) \hat{\otimes} K$$
   for all $x, y \in X^m_{d,j,R}$, where $\beta_N(x) : \mathcal{A}(W_N(x)) \to \mathcal{A}(H_m)$ is the $*$-homomorphism associated to the inclusion of $W_N(x)$ into $H_m$, and $K$ is the algebra of compact operators on an infinite-dimensional separable Hilbert space;

2. there exists $L > 0$ such that, for each $m$ and each $y \in X^m_{d,j,R}$,
   $$\sharp\{x : T^{(m)}_j(x,y) \neq 0\} \leq L, \quad \sharp\{x : T^{(m)}_j(x,y) \neq 0\} \leq L;$$

3. for each $m, j$ and each bounded subset $B \subset Z^m_{d,j,R}$, the set
   $$\sharp\{(x,y) \in (B \times B) \cap (X^m_{d,j,R} \times X^m_{d,j,R}) : T^{(m)}_j(x,y) \neq 0\}$$
   is finite;

4. there exists $r_1 > 0$, $T^{(m)}_j(x,y) = 0$ for all $m, j$ and $x, y \in X^m_{d,j,R}$ with $d(x, y) > r_1$;

5. there exists $r_2 > 0$ such that $\text{Supp}(T^{(m)}_j(x,y)) \subseteq B(f_m(\pi(x^m)), r_2)$ for all $x, y \in X^m_{d,j,R}$ where
   $$B(f_m(\pi(x)), r_2) = \{(s, h) \in \mathbb{R}_+ \times H : s^2 + \|h - f(\pi(x))\|^2 < r_2^2\};$$

6. there exists $c > 0$ such that if $\beta_N(x)(S^{(m)}_j(x,y)) = T^{(m)}_j(x,y)$ for some $S^{(m)}_j(x,y) \in \mathcal{A}(W_N(x))$, then
   $$D_Y(S^{(m)}_j(x,y)) \in \mathcal{A}(W_N(x)) \hat{\otimes} K$$
   exists and
   $$\sup_m \|D_Y(S^{(m)}_j(x,y))\| \leq c$$
   for all $x, y \in X^m_{d,j,R}$ and $Y = (s, h) \in \mathbb{R}_+ \times W_N(x)$ satisfying $\|Y\| = \sqrt{s^2 + \|h\|^2} \leq 1$.

The algebra $C^*_{u,alg}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ admits a $*$-algebra structure defined by coordinatewise multiplication and adjoint.

Let $E^m_j = \ell^2(X^m_{d,j,R}) \otimes K \otimes \mathcal{A}(H_m)_{O_{m,j}}$. $X^m_{d,j,R} \subset X^m_d$, $O_{m,j} \cap O_{m,j'} = \emptyset$ for all $j \neq j'$, and $\mathcal{A}(H_m)_{O_{m,j}} \subset \mathcal{A}(H_m)$ is an ideal, we have that $E^m_j$ is sub-module of $E^m$. Moreover, we have that the direct sum $\bigoplus_j E^m_j$ is a sub-module of $E^m$. Naturally, the $*$-algebra $C^*_{u,alg}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ admits by faithful $*$-representation by matrix operation on the Hilbert module $\bigoplus_{m,j} E^m_j$.

Let $C^*_{u}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ be the completion of $C^*_{u,alg}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ under the norm on the Hilbert module $\bigoplus_{m,j} E^m_j$. Note that $C^*_{u}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ is a $C^*$-subalgebra of $C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O}$. Similarly, we have the localization algebras $C^*_{u,L}((Z^m_{d,j,R}, \mathcal{A}(H_m)_{O_{m,j}})_{m \in I,j \in J})$ and it is a $C^*$-subalgebra of $C^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O}$.
By the Condition (6) in Definition 4.6 we have that

\[ C_u^*(\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I})_O = \lim_{R \to \infty} C_u^*\langle P_d(B_{G_m}(x_{jN_m}, R)), \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J} \]

and

\[ C_{u,L}^*\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I} = \lim_{R \to \infty} C_{u,L}^*\langle P_d(B_{G_m}(x_{jN_m}, R)), \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}. \]

For each \( R > 0 \), we have the map

\[ e_* : \lim_{d \to \infty} K_*(C_{u,L}^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}) \to \lim_{d \to \infty} K_*(C_u^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}) \]

induced by the evaluation-at-zero map on \( K \)-theory, where \( Z_{d,j,R}^m = P_d(B_{G_m}(x_{jN_m}^m, R)) \) is the Rips complex of \( B_{G_m}(x_{jN_m}^m, R) \) for each \( m, j \).

Now, let us recall the result of the third author about the coarse Baum–Connes conjecture in [54]. For every discrete metric space \( Z \) with bounded geometry, the third author showed that if \( Z \) admits a coarse embedding into a Hilbert space, then the coarse Baum–Connes conjecture holds for \( Z \), i.e., the map

\[ e_* : \lim_{d \to \infty} C_u^*(P_d(Z)) \to \lim_{d \to \infty} C_u^*(P_d(Z)) \]

induced by evaluation-at-zero map on \( K \)-theory is an isomorphism.

In this paper, we need the following generalized version of the third author’s result in [54].

**Theorem 4.14 (Yu, [54]).** Let \( R > 0 \) be any positive number. Then the map

\[ e_* : \lim_{d \to \infty} K_*(C_{u,L}^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}) \to \lim_{d \to \infty} K_*(C_u^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}) \]

induced by the evaluation-at-zero map on \( K \)-theory is an isomorphism.

Although the algebras in this theorem are different from the ones in [54], it can be proved by the same ideas. Using Dirac-dual-Dirac method reduces the above version of coarse Baum–Connes conjecture to a twisted version of coarse Baum–Connes conjecture which can be proved using the same cutting-and-pasting techniques in [54].

**Proof of Proposition 4.12** For each \( R > 0 \), we have the commutative diagram:

\[
\begin{array}{ccc}
K_*(C_{u,L}^*\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I})_O & \xrightarrow{e_*} & K_*(C_u^*\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I})_O \\
\cong & & \cong \\
\lim_{R \to \infty} K_*(C_{u,L}^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}) & \xrightarrow{e_*} & \lim_{R \to \infty} K_*(C_u^*\langle Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}} \rangle_{m \in I, j \in J}).
\end{array}
\]

By Theorem 4.14 the bottom horizontal map is an isomorphism, as a result, the map

\[ e_* : \lim_{d \to \infty} K_*(C_{u,L}^*\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I})_O \to \lim_{d \to \infty} K_*(C_u^*\langle P_d(G_m), \mathcal{A}(H_m) \rangle_{m \in I})_O \]

is an isomorphism.
Proof of Theorem 4.9: For any \( r > 0 \) and for each \( m \), let \( O_{m,r} = \bigcup_{x \in G_m/N_m} B(f_m(x),r) \subset \mathbb{R}_+ \times H_m \), where \( f_m : Q_m \to H_m \) is the coarse embedding and
\[
B(f_m(x),r) = \{(t,h) \in \mathbb{R}_+ \times H_m : t^2 + ||h - f(x)||^2 < r\}.
\]
Let \( O_r = (O_{m,r})_{m \in I} \) be the sequence of open subsets. By definition, we have
\[
C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) = \lim_{r \to \infty} C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r},
\]
and
\[
C^*_u,(((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) = \lim_{r \to \infty} C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r}.
\]
So it suffices to show that, for any \( r > 0 \), the map
\[
e_* : \lim_{d \to \infty} C^*_u,(((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r} = \lim_{d \to \infty} C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r},
\]
induced by the evaluation-at-zero map on \( K \)-theory is an isomorphism. Since the coarse disjoint union \( \sqcup_m Q_m \) has bounded geometry property and \( f_m : Q_m \to H_m \) are coarse embedding uniformly, then there exist finitely many, say \( k_r \), independent on \( m \), mutually disjoint subsets \( \{ \Lambda_{m,i} \}_{i=1}^{k_r} \) of \( Q_m \) such that \( G_m/N_m = \sqcup_{i=1}^{k_r} \Lambda_{m,i} \), where for each \( i \) and each \( m \), \( d(f_m(g), f_m(g')) > 3r \) for different elements \( g, g' \) in \( \Lambda_{m,i} \).

Set \( O_{r,i} = \bigcup_{g \in \Lambda_{m,i}} B(f_m(g),r) \), then we have a sequence of open sets \( O_{r,i} = (O_{m,r,i})_m \) for each \( r \) and each \( i \).

By Proposition 4.12 we have
\[
e_* : \lim_{d \to \infty} K_*(C^*_u,(((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}}) \to \lim_{d \to \infty} K_*(C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}})
\]
is an isomorphism.

By the Mayer–Vietoris sequence and Five Lemma, we have that the map
\[
e_* : \lim_{d \to \infty} C^*_u,(((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r} = \lim_{d \to \infty} C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r},
\]
induced by the evaluation-at-zero map on \( K \)-theory is an isomorphism.

Passing to infinity, we have that the map
\[
e_* : \lim_{d \to \infty} K_*(C^*_u,(((P_d(G_m), \mathcal{A}(H_m))_{m \in I})) \to \lim_{d \to \infty} K_*(C^*_u((P_d(G_m), \mathcal{A}(H_m))_{m \in I}))
\]
induced by the evaluation-at-zero map on \( K \)-theory is an isomorphism. \( \square \)

4.4 The maximal twisted Roe algebras and twisted Roe algebra

In this subsection, we shall show that the \( K \)-theory of the maximal uniform twisted Roe algebra is isomorphic to the \( K \)-theory of uniform twisted Roe algebra when the sequence of metric spaces \( (N_m)_{m \in I} \) has Property A and the sequence of metric spaces \( (Q_m)_{m \in I} \) is uniformly coarsely embeddable into Hilbert spaces. This is important in the definition of the Dirac map in the last section.
Lemma 4.15. For each $T \in C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$, there exists a non-negative number $N_T$ such that

$$\|\phi(a)\| \leq N_T \cdot \sup_{m,x,y} \|T^m_{x,y}\|_{\mathcal{A}(H_m) \otimes K}$$

for any $*$-representation $\phi : C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to B(H_\phi)$.

Let $\phi : C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to B(H_\phi)$ be any faithful $*$-representation. By Lemma 4.15, we can define the maximal norm on the $*$-algebra $C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ by

$$\|T\|_{\text{max}} = \sup \{ \|\phi(T)\| : \phi : C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to B(H_\phi) \text{ is a } *\text{-representation} \} .$$

The maximal twisted Roe algebra $C^*_{u,max}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ is the norm completion of

$$C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$$

under the maximal norm.

By the universal property of the maximal norm, the identity map on $C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ continuously extends to the canonical quotient

$$\lambda : C^*_{u,max}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}).$$

In the rest of this subsection, we shall follow the arguments in Section 4.3 to analyze the ideal structure of the algebras $C^*_{u,max}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ and $C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$. Using a similar cutting-and-pasting machinery, we will show that this canonical quotient map is an isomorphism when the sequence of metric spaces $(N_m)_m$ has Property A and the sequence $(Q_m)_m$ is uniformly coarsely embeddable into Hilbert space.

Let $r > 0$. If $O = (O_m)_{m \in I}$ is a sequence of sets such that each $O_m$ is a union of a family of open subsets $\{O_{m,j}\}_{j \in J}$ of $\mathbb{R}_+ \times H_m$ satisfying:

(1) for each $j \in J$, $O_{m,j} \subseteq B(f_m(\pi(x^m_j)), r)$ for some $x^m_j \in G_m$, where

$$B(f_m(\pi(x^m_j)), r) = \left\{ (t, h) \in \mathbb{R}_+ \times H_m : t^2 + \| h - f_m(\pi(x^m_j)) \|^2 < r^2 \right\},$$

and $f_m$ is the coarse embedding $f_m : G/N \to H_m$;

(2) for each $m$, $O_{m,j} \cap O_{m,j'} = \emptyset$ if $j \neq j'$.

Recall in Definition 4.12 of Section 4.2 for the sequence of open sets $O = (O_m)_{m \in I}$, we define a $*$-algebra $C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$. Then $C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$ is a $*$-subalgebra of the maximal Roe algebra $C^*_{u,max}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$, and the completion of $C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$ under the norm in maximal twisted Roe algebra is denoted by $C^*_{u,\phi}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$. We can also view

$$C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$$

as a $*$-subalgebra of $C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})$ and let $C^*_{u}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O$ be the completion of

$$C^*_{u,alg}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_O.$$
Then the canonical quotient map $\lambda$ restrict to a $\ast$-homomorphism

$$\lambda : C^\ast_u((P_d(G_m), A(H_m))_{m \in I})_O \to C^\ast_u((P_d(G_m), A(H_m))_{m \in I})_O$$

Recall that $Z^m_{d,j,R} = P_d(B_{G_m}(x_j N_m, R))$ for each $m$, each $j$ and each $R$. For each $d > 0$, let

$$X^m_{d,j,R} = X^m_d \cap P_d(B_{G_m}(N_m, R))$$

be the countable dense subset of $P_d(B_{G_m}(N_m, R))$, where $X^m_d \subset P_d(G_m)$ be the countable dense subset in Definition 4.6.

Let $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$ be the $\ast$-algebra defined in Definition 4.13. We have that

$$C^\ast_{u,alg}((P_d(G_m), A(H_m))_{m \in I})_O = \lim_{R \to \infty} C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J}).$$

The inductive limit can be viewed as the union of the nested $\ast$-algebras. For each $R > 0$, the $\ast$-algebra $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$ is a subalgebra of $C^\ast_{u,alg}((P_d(G_m), A(H_m))_{m \in I})_O$, and the completion of $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$ is denoted by $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$. Note that

$$C^\ast_{u,alg}((P_d(G_m), A(H_m))_{m \in I})_O = \lim_{R \to \infty} C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J}).$$

When viewing $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$ as a $\ast$-subalgebra of $C^\ast_{u,alg}((P_d(G_m), A(H_m))_{m \in I})_O$, we get a completion of $C^\ast_{u,alg}((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$, denoted by $C^\ast_u((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$. We have that

$$C^\ast_u((P_d(G_m), A(H_m))_{m \in I})_O = \lim_{R \to \infty} C^\ast_u((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J}).$$

To prove that the restriction of quotient map

$$\lambda : C^\ast_u((P_d(G_m), A(H_m))_{m \in I})_O \to C^\ast_u((P_d(G_m), A(H_m))_{m \in I})_O$$

is an isomerism, it suffices to show that the restriction of the quotient map

$$\lambda : C^\ast_u((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J}) \to C^\ast_u((Z^m_{d,j,R}, A(H_m)_{O_{m,j}})_{m \in I, j \in J})$$

is an isomorphism.

Let us now recall the concept of Property A. A metric space with this property was introduced by the third author in [61] as an example of metric spaces which are coarsely embeddable into Hilbert space.

**Definition 4.16** [ES]. A metric space $X$ with bounded geometry is said to have Property A if for any $R > 0$, $\epsilon > 0$, there exists a map $\xi : X \to \ell^2(X)$ and $S > 0$ such that

- for all $x, y \in X$, $0 \geq \xi_x(y) \leq 1$;
- for all $x \in X$, $\|\xi_x\| = 1$;
• for all $x \in X$, $\text{Supp} (\xi_x) \subset B_X(x, S)$;
• if $d(x, y) \leq R$, then $|1 - \langle \xi_x, \xi_y \rangle| < \epsilon$.

Then we can define a kernel $k : X \times X \rightarrow \mathbb{R}$ by

$$k(x, y) = \langle \xi_x, \xi_y \rangle$$

for all $x, y \in X$. The kernel $k$ is support on a stripe with length $S$ and it satisfies that $|1 - k(x, y)| < \epsilon$ for all $x, y \in X$ with $d(x, y) \leq R$. For each $d$, we can extend the kernel to Rips complex

$$k : P_d(X) \times P_d(X) \rightarrow [0, 1]$$

by

$$k(\sum a_i x_i, \sum b_i x_i) = \sum a_i b_i k(x_i, x_i)$$

for all $\sum a_i x_i, \sum b_i x_i \in P_d(X)$. So we do not distinguish the domain of the kernel in the Definition of Property A.

Assume that the sequence of metric space $(N_m)_{m \in I}$ has Property A, since for each fixed $R$, the sequence of metric spaces $(Z_{d,j,R}^m = B_G(x_j^m, N_m, R))_{m,j}$ is uniformly coarse equivalent to $(N_m)_{m \in I}$, we have that $(Z_{d,j,R}^m = B_G(x_j^m, N_m, R))_{m,j}$ also has Property A.

Then for each $m$, each $\epsilon > 0$ and $R > 0$ we can find a kernel $k_j^m : X_{d,j,R}^m \times X_{d,j,R}^m \rightarrow [0, 1]$ and $S > 0$ such that $k_j^m$ is supported on a stripe of length $S$ uniformly.

Now we can define a map

$$(M_{k^m})_{m,j} : C^*_{u,alg}((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}}))_{m,j} \rightarrow C^*_{u,alg}((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}}))_{m,j}$$

by

$$(M_{k_j^m})_{m,j} : (T^{(m)}_j)_{m,j} \mapsto (M_{k_j^m} \cdot T^{(m)}_j)_{m,j}$$

for all $(T^{(m)})_{m,j} \in C^*_{u,alg}((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}}))_{m,j}$, where

$$(M_{k_j^m} \cdot T^{(m)}_j)(x, y) = k_j^m(x, y)T^{(m)}(x, y)$$

for all $m$, $j$ and all $x, y \in X_{d,j,R}^m$.

We then have the following general result which can be proved by the same arguments as Corollary 2.3 in [40].

**Lemma 4.17.** Let $(k_j^m)_{m,j}$ be the sequence of kernels defined from Property A of the sequence of metric spaces $(X_{d,j,R}^m)_{m,j}$. Then the map

$$(M_{k^m})_{m,j} : C^*_{u,alg}((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}}))_{m,j} \rightarrow C^*_{u,alg}((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}}))_{m,j}$$

extends continuously to a contractive completely positive map on any $C^*$-algebraic completion of

$$(Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,j}})_{m,j}.$$
Consider the commutative diagram:

\[
\begin{array}{ccc}
C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) & \xrightarrow{\left( M_{k_{m}} \right)_{m \in I,j \in J}} & C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \\
\downarrow \lambda & & \downarrow \lambda \\
C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) & \xrightarrow{\left( M_{k_{m}} \right)_{m \in I,j \in J}} & C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J})
\end{array}
\]

where \( \left( M_{k_{m}} \right)_{m \in I,j \in J} \) and \( \left( M_{k_{m}} \right)_{m \in I,j \in J} \) are the contractive completely positive maps given by Lemma 4.17. For each \( s > 0 \), let \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \) be the subspace of all tuples \( T = (T_{j, m}^{(m)})_{m \in I,j \in J} \) with propagation of \( T_{j, m}^{(m)} \) at most \( s \) for all \( m \) and all \( j \). Then \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \) is closed in \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \) under the norm topology. Consider a sequence of tuples \( \{ T_n = (T_{j, m}^{(m)}) \}_{n=1}^{\infty} \) in \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \) which converges in norm to an operator \( T = (T_{j, m}^{(m)})_{m \in I,j \in J} \). Let

\[
\lambda : C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \to C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J})
\]

be the restriction of the canonical quotient map from the maximal twisted uniform Roe algebra to the twisted uniform Roe algebra. It follows that \( \lambda(T_n) \to \lambda(T) \) as \( n \to \infty \). Note that \( \lambda(T_n) = T_n \) since

\[
\lambda : C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \to C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J})
\]

is the identity map restricted to \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \). By Lemma 4.13, we have that each entry \( T_{j, m}^{(m)}(x, y) \) of \( T_{j, m}^{(m)} \) converges to the entry \( \lambda(T_{j, m}^{(m)})(x, y) \) uniformly in the norm topology of \( A(H_m) \otimes K \). As a result of Lemma 4.15, \( T_{j, m}^{(m)}(x, y) = 0 \) for all \( x, y \in X_{d, j, R}^m \) with \( d(x, y) \geq s \). In addition, we have that

\[
\left( M_{k_{m}} \right)_{m \in I,j \in J} \left( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \right) \subset C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}).
\]

**Proposition 4.18.** Let \( R > 0 \). Assume that the sequence of metric spaces \( \{ B_{s_m}(x_{m, n}, N_m, R) \}_{m \in I,j \in J} \) has Property A. Then the canonical quotient map

\[
\lambda : C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \to C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J})
\]

is an isomorphism.

**Proof.** Using property A, we obtain a kernels \( (k_{j, n}^m)_{m \in I,j \in J} \) associated to \( R = n \) and \( \epsilon = 1/n \) for each \( n \in \mathbb{N} \) in Definition 4.16. By Lemma 4.13, we have that for each \( S \in C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}), \)

\[
\left( M_{k_{j, n}^m} \right)_{m \in I,j \in J}(S) \text{ converges in the } C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \text{ norm to } S \text{ as } n \to \infty.
\]

In addition, for each \( T \in C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}), \) the image \( \left( M_{k_{j, n}^m} \right)_{m \in I,j \in J}(T) \) is an element in \( C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \).

It suffices to show that \( \lambda \) is injective. Let \( T \in C_{\ast, \phi}((Z_{d,j,R}^m, A(H_m)_{O_{m,j}})_{m \in I,j \in J}) \) in the kernel of \( \lambda \). Then we have that

\[
T = \lim_n \left( M_{k_{j, n}^m} \right)_{m \in I,j \in J}(T) = \lambda \left( \left( M_{k_{j, n}^m} \right)_{m \in I,j \in J}(T) \right) = \lim_n \left( M_{k_{j, n}^m} \right)_{m \in I,j \in J}(\lambda(T)) = 0.
\]

Therefore, \( \lambda \) is injective. 

\( \square \)
Theorem 4.19. Let \( (1 \to N_m \to G_m \to Q_m \to 1)_{m \in I} \) be a sequence of extensions of discrete groups. Assume that the sequence of metric spaces \((N_m)_m, (G_m)_m \) and \((Q_m)_m \) have uniform bounded geometry. If the sequence of metric spaces \((N_m)_m \) has Property A and the sequence \((Q_m)_m \) is coarsely embeddable into Hilbert space, then the canonical quotient map

\[
\lambda : C_{u,max}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) \to C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})
\]

is an isomorphism.

Proof. By the definition the maximal twisted Roe algebra and twisted Roe algebra, and the proof of Proposition 4.12 we have that

\[
C_{u,max}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) = \lim_{r} C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r},
\]

and

\[
C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I}) = \lim_{r} C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r},
\]

where \( O_r = (O_m, r)_m \) is the sequence of open sets as in Theorem 4.9 and \( C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r} \)

is the completion of \( C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r} \)

under the norm obtained by the inclusion

\[
C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r} \hookrightarrow C_{u,max}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_r}.
\]

Moreover, for each \( r > 0 \), as in the proof of Theorem 4.9 there are finitely many sequences of open sets

\[
O_{r,i} = (O_{m,r,i,j})_{m,j},
\]

for \( 1 \leq i \leq k_r \) for some \( k_r \in \mathbb{N} \), where each \( O_{m,r,i,j} \) is an open subsets of \( \mathbb{R}_+ \times H_m \) such that

- \( O_{m,r,i,j} \subset B(f_m(\pi(x^m_j))) \) for some \( x^m_j \in G_m \);
- \( O_{m,r,i,j} \cap O_{m,r,i,j'} = \emptyset \) for all \( j \neq j' \).

For each \( r \) and each \( 1 \leq i \leq k_r \), the completion of the \( * \)-algebra \( C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}} \)

is denoted by \( C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}} \)

under the \( * \)-representation induced by the inclusion

\[
C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}} \hookrightarrow C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}}.
\]

Note that

\[
C_{u,\phi}^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}} = \lim_{R \to \infty} C_{u,\phi}^\ast((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,r,i,j,m}})_{m \in I,j \in J}),
\]

and

\[
C_u^\ast((P_d(G_m), \mathcal{A}(H_m))_{m \in I})_{O_{r,i}} = \lim_{R \to \infty} C_u^\ast((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,r,i,j,m}})_{m \in I,j \in J}).
\]

By Proposition 4.18 we have that the restriction of the canonical quotient map

\[
\lambda : C_{u,\phi}^\ast((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,r,i,j,m}})_{m \in I,j \in J}) \to C_u^\ast((Z_{d,j,R}^m, \mathcal{A}(H_m)_{O_{m,r,i,j,m}})_{m \in I,j \in J})
\]

is an isomorphism.
is an isomorphism. It follows that the map

\[ \lambda : C^*_u((P_d(G_m), A(H_m))_{m \in I}) \rightarrow C^*_u((P_d(G_m), A(H_m))_{m \in I}) \]

is an isomorphism for each \( r \) and each \( i \). By the Mayer–Vietoris sequence and Five Lemma, we have that the restriction of the canonical map

\[ \lambda : C^*_u((P_d(G_m), A(H_m))_{m \in I}) \rightarrow C^*_u((P_d(G_m), A(H_m))_{m \in I}) \]

is an isomorphism. By taking limit on \( r \), we have that the canonical quotient map

\[ \lambda : C^*_u,max((P_d(G_m), A(H_m))_{m \in I}) \rightarrow C^*_u((P_d(G_m), A(H_m))_{m \in I}) \]

is an isomorphism.

\[ \square \]

5 Geometric analogue of Bott periodicity and proof of the main theorem

In this section, we shall prove Theorem 3.12 using a geometric analogue of the infinite-dimensional Bott periodicity introduced by the third author in [54]. The geometric analogue of infinite-dimensional Bott periodicity is used to reduce the coarse Baum–Connes conjecture to the twisted coarse Baum–Connes conjecture.

5.1 Bott maps \( \beta \) and \( \beta_L \)

In this subsection, we shall define the Bott map

\[ \beta : S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}) \rightarrow C^*_u((P_d(G_m), A(H_m))_{m \in I}) \]

and the localized Bott map

\[ \beta_L : S \hat{\otimes} C^*_{u,L}((P_d(G_m))_{m \in I}) \rightarrow C^*_{u,L}((P_d(G_m), A(H_m))_{m \in I}). \]

For \( x \in X_d^m \). Let \( E \) be a finite-dimensional Euclidean space of \( H_m \). Define a \( * \)-homomorphism

\[ \beta_E(x) : S \rightarrow A(E) \]

by

\[ \beta_E(x)(g) = g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{E,x}) \]

for all \( g \in S \), where \( X \) is the operator of multiplication by \( x \) on \( \mathbb{R} \) viewed as a degree one and unbounded multiplier of \( S \), \( C_{E,x} \) is the Clifford algebra-valued function on \( E \) denoted by \( C_{E,x}(v) = v - f_m(\pi(x)) \in E \subset \text{Cliff}(E) \) for all \( v \in E \) and \( g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{E,x}) \) is defined by functional calculus. By definition of the algebra \( A(H_m) \), we have a \( * \)-homomorphism

\[ \beta(x) : S \rightarrow A(H_m). \]

For each \( t \in [1, \infty) \), we define a map

\[ \beta : S \hat{\otimes} C^*_u((P_d(G))_{m \in I}) \rightarrow C^*_u,alg((P_d(G), A(H_m))_{m \in I}) \]

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by
\[(\beta_t(g \hat{\otimes} T))^{(m)}(x, y) = \beta(x)(g_t \hat{\otimes} T^{(m)})(x, y)\]
for all \(g \in S, T = (T^{(m)})_m \in C_u([P_d(G)]_{m \in I})\), where \(g_t(s) = g(t^{-1}s)\) for all \(t \geq 1, s \in \mathbb{R}\), and \(\beta(x): S \rightarrow \mathcal{A}(H_m)\) is the \(*\)-homomorphism associated to the inclusion of the zero-dimensional affine space \(0\) into \(H_m\) by mapping \(0\) to \(f_m(\pi(x))\), where \(\mathcal{A}(H_m)\) is defined in Section 4.2.

**Lemma 5.1.** The Bott map \(\beta\) extends to an asymptotic morphism
\[
\beta : S \hat{\otimes} C_u^*((P_d(G))_{m \in I}) \rightarrow C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I}).
\]

**Proof.** Let \(E = \bigoplus_m E^m\) as in Section 4.2, where \(E^m = \ell^2(X_d^m) \otimes \mathcal{A}(H_m) \otimes K\). For each \(m\) and each \(g \in S\), we define a module homomorphism
\[
V^m_g : E^m \rightarrow E^m
\]
by
\[
V^m_g \left( \sum_{x \in X_d^m} a_x[x] \right) = \sum_{x \in X_d^m} (\beta(x)(g) \otimes 1)a_x[x]
\]
for all \(g \in S\) and all \(\sum_{x \in X_d^m} a_x[x] \in E^m\). Then taking direct sum gives us a module homomorphism
\[
V_g = \bigoplus_m V^m_g : \bigoplus_m E^m \rightarrow \bigoplus_m E^m.
\]

Note that
\[
\beta_t(g \hat{\otimes} (T^{(m)})) = V^m_g \circ (1 \hat{\otimes} (T^{(m)}))_m
\]
for all \(g \in S\) and \((T^{(m)})_m \in C^*((P_d(G_m))_{m \in I})\), where \(1 \hat{\otimes} (T^{(m)})_m\) is the adjointable homomorphism on \(E = \bigoplus_m E^m\) defined by
\[
1 \hat{\otimes} (T^{(m)})_m \colon \bigoplus_m \sum_{x \in X_d^m} a_x[x] \mapsto \bigoplus_m \sum_{y \in X_d^m} \left( \sum_{x \in X_d^m} (1 \hat{\otimes} T^{(m)}(y, x))a_x \right)[y]
\]
for all \(\sum_{x \in X_d^m} a_x[x] \in \bigoplus_m E^m\).

Since \(\|(T^{(m)})_m\| = \sup_m \|T^{(m)}\|_{E^m}\), we have that
\[
\|\beta_t(g \hat{\otimes} (T^{(m)})) - (1 \hat{\otimes} T^{(m)})\| \leq \|g\| \cdot \|(T^{(m)})_m\|. \tag{1}
\]
As a result, for each \(t \geq 1\), the linear map \(\beta_t : S \hat{\otimes} C_u([P_d(G_m)]_{m \in I}) \rightarrow C_{u, alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})\) extends to a bounded linear map
\[
\beta_t : S \hat{\otimes} C_u^*(P_d(G_m))_{m \in I}) \rightarrow C_{u, alg}^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})
\]
where \(S \hat{\otimes} algC_u^*((P_d(G))_{m \in I})\) is the algebraic tensor product. Since
\[
\|f_m(\pi(x)) - f_m(\pi(y))\| \leq \rho_2(d(\pi(x), \pi(y)))
\]
for all \(x, y \in E^m\) by the uniformly coarse embeddability of the sequence of quotient groups \((Q_m)_m\), in addition with Inequality (1) above, we have that
\[
\|\beta_t((g_1 \hat{\otimes} T_1) \cdot (g_2 \hat{\otimes} T_2)) - \beta_t((g_1 \hat{\otimes} T_1) \cdot (g_2 \hat{\otimes} T_2))\| \rightarrow 0 \text{ as } t \rightarrow \infty.
\]

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for all \( g_i \in \mathcal{S} \) and \( T_i \in C_u^*((P_d(G))_{m \in I}), \ i = 1, 2 \).

So we have a \(*\)-homomorphism

\[
\beta : \mathcal{S} \widehat{\otimes}_{alg} C_u^*((P_d(G))_{m \in I}) \rightarrow \frac{C_b([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})]}{C_0([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})]}
\]

by

\[
g \widehat{\otimes} T \mapsto [\beta_i(g \widehat{\otimes} T)]
\]

for all \( g \in \mathcal{S} \) and all \( T \in C^*((P_d(G), \mathcal{A}(H_m))_{m \in I}) \), where \( C_b([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})] \) is the \( C^*\)-algebra all norm-continuous and bounded functions from \([1, \infty)\) to \( C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I}) \) and \( C_0([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})] \) is the \( C^*\)-subalgebra of \( C_b([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})] \) consisting of functions vanishing at infinity.

Accordingly, we obtain a \(*\)-homomorphism between \( C^*\)-algebras from the maximal tensor product \( \mathcal{S} \widehat{\otimes}_{max} C^*((P_d(G))_{m \in I}) \) to the \( C^*\)-algebra

\[
\frac{C_b([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})]}{C_0([1, \infty), C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I})]}
\]

By nuclearity of \( \mathcal{S} \), we have that

\[
\mathcal{S} \widehat{\otimes} C_u^*((P_d(G))_{m \in I}) \cong \mathcal{S} \widehat{\otimes}_{max} C_u^*((P_d(G))_{m \in I}).
\]

Then the \(*\)-homomorphism above gives rise to an asymptotic morphism

\[
\beta_t : \mathcal{S} \widehat{\otimes} C_u^*((P_d(G))_{m \in I}) \rightarrow C_u^*((P_d(G), \mathcal{A}(H_m))_{m \in I}),
\]

for \( t \in [0, \infty) \).

Similarly, we can define asymptotic morphism between the localization algebras

\[
\beta_{L,t} : \mathcal{S} \widehat{\otimes} C_u^*_{u,L}((P_d(G_m))_{m \in I}) \rightarrow C_u^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I}).
\]

In addition, the asymptotic morphisms \((\beta_t)_{t \in [0, \infty)}\) and \((\beta_{L,t})_{t \in [1, \infty)}\) induce homomorphisms on \( K\)-theory

\[
\beta_* : K_*\left(\mathcal{S} \widehat{\otimes} C_u^*((P_d(G_m))_{m \in I})\right) \rightarrow K_*\left(C_u^*((P_d(G_m), \mathcal{A}(H_m))_{m \in I})\right)
\]

and

\[
\beta_{L,*} : K_*\left(\mathcal{S} \widehat{\otimes} C_u^*_{u,L}((P_d(G_m))_{m \in I})\right) \rightarrow K_*\left(C_u^*_{u,L}((P_d(G_m), \mathcal{A}(H_m))_{m \in I})\right).
\]

### 5.2 Dirac maps \( \alpha \) and \( \alpha_L \)

In this subsection, we shall define the Dirac maps \( \alpha \) and \( \alpha_L \). We will firstly recall the definition of the Bott–Dirac operator on an infinite dimensional Euclidean space, then we extend the construction of Dirac maps to the geometric analogues on Roe algebras and localization algebras.

Let \( H \) be an infinite-dimensional separable Hilbert space. Let \( V \) be the countable infinite-dimensional Euclidean dense subspace of \( H \). If \( V_a \subset V \) be a finite-dimensional affine subspace, we use \( L^2(V_a, \text{Cliff}(V_a^0)) \) to denote the Hilbert space of square integrable functions from \( V_a \) to \( \text{Cliff}(V_a^0) \). If \( V_a \) and \( V_b \) are finite dimensional affine subspaces with \( V_a \subset V_b \), then there exists a decomposition \( V_b = V_{ba} \oplus V_a \). Then

\[
L^2(V_b, \text{Cliff}(V_b^0)) \cong L^2(V_{ba}, \text{Cliff}(V_{ba})) \widehat{\otimes} L^2(V_a, \text{Cliff}(V_a^0)),
\]

for all \( g_i \in \mathcal{S} \) and \( T_i \in C_u^*((P_d(G))_{m \in I}), \ i = 1, 2 \).
where $L^2(V_{ba}, \text{Cliff}(V_{ba}))$ is the the Hilbert space associated with $V_{ba}$. We define a unit vector

$$\xi_{ba} \in L^2(V_{ba}, \text{Cliff}(V_{ba}))$$

by

$$\xi_{ba}(v_{ba}) = \pi^{-\frac{\dim(V_{ba})}{4}} \exp(-\frac{1}{2} \|v_{ba}\|^2)$$

for all $v_{ba} \in V_{ba}^0$. Regarding $L^2(V_a, \text{Cliff}(V_a^0))$ as a subspace of $L^2(V_b, \text{Cliff}(V_b))$ via the isometry $\xi \mapsto \xi_0 \otimes \xi$ for $V_a \subset V_b$, we define

$$\mathcal{H} = \lim_{\to} L^2(V_a, \text{Cliff}(V_a^0)).$$

For each finite dimensional affine subspace $V_a \subset V$, define $s_a$ to be the space of Schwartz functions from $V_a$ to $\text{Cliff}(V_a^0)$. Let $s = \lim s_a$ be the algebraic direct limit of the Schwartz subspaces $s_a \subset V_a$. If $V_a \subset V \subset \mathcal{H}$ is a finite-dimensional affine subspace, then the Dirac operator $D_{V_a}$ is an unbounded operator on $L^2(V_a, \text{Cliff}(V_a^0))$ with domain $s_a$, defined by

$$D_a \xi = \sum_{i=1}^n (-1)^{deg \xi} \frac{\partial \xi}{\partial x_i} v_i$$

where $\{v_1, \cdots, v_n\}$ is an orthonormal basis for $V_a^0$ and $\{x_1, \cdots, x_n\}$ are the dual coordinates to $\{v_1, \cdots, v_n\}$.

The Clifford operator $C_{V_a, v_0}$ of $V_a$ at $v \in V_a$ is an unbounded operator on $L^2(V_a, \text{Cliff}(V_a^0))$ defined by

$$(C_{V_a, v_0} \xi)(v) = (v - v_0) \cdot \xi(v)$$

for any $\xi \in L^2(V_a, \text{Cliff}(V_a^0))$ and $v \in V_a$, where the multiplication is the Clifford multiplication of

$$v - v_0 \in V_a^0 \in \text{Cliff}(V_a^0)$$

and $\xi(v) \in \text{Cliff}(V_a^0)$. The domain of the Clifford operator is the space of Schwartz functions $s_a$. When $V_a$ is a linear subspace and $v_0 = 0 \in V_a$, we denote $C_{V_a} = C_{V_a, 0}$.

Given an algebraic decomposition

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$$

where each $V_i$ is a finite-dimensional linear subspace of $V$. For each $n \in \mathbb{N}$ and each $t \in [1, \infty)$, we define an unbounded, selfadjoint operator $B_{n,t}$ on $\mathcal{H}$ by

$$B_{n,t} = t_0D_0 + t_1D_1 + \cdots + t_{n-1}D_{n-1} + t_n(D_n + C_n) + t_{n+1}(D_{n+1} + C_{n+1}) + \cdots$$

where $t_i = 1 + i/t$, $C_i$ and $D_i$ are the Clifford operator and the Dirac operator on $L^2(V_i, \text{Cliff}(V_i))$, respectively.

Let $\mathcal{K}$ be the algebra of compact operators on the Hilbert space $\mathcal{H}$ and $\mathcal{S} \otimes \mathcal{K}$ the graded tensor product where $\mathcal{K}$ is endowed with the trivial grading. Let $W_n = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$, then we have an asymptotic morphism

$$(\alpha_{n,t})_{t \in [1, \infty]} : \mathcal{A}(W_n) \to \mathcal{S} \otimes \mathcal{K}$$

by

$$\alpha_{n,t} : f \otimes h \mapsto f(X \otimes 1 + 1 \otimes B_{n,t}) \cdot (1 \otimes M_{h})$$

where $M_{h}$ is the multiplication operator on $\mathcal{S} \otimes \mathcal{K}$ with symbol $h$.
for all \(f \in S\) and \(h \in C(W_n)\) where \(h_t(v) = h(v/t)\) and \(M_{h_t}\) is the operator on \(H\) of left multiplication by \(h_t\). By Definition 2.8, Lemma 2.9 and Proposition 4.2 in [30], this map is well-defined. In addition, we have the following asymptotically commutative diagram:

\[
\begin{array}{ccc}
\mathcal{A}(W_n) & \overset{\alpha_n,t}{\longrightarrow} & S \otimes K \\
\downarrow \beta_{W_n,W_{n+1}} & & \downarrow \\
\mathcal{A}(W_{n+1}) & \overset{\alpha_{n+1,t}}{\longrightarrow} & S \otimes K 
\end{array}
\]

where \(\beta_{W_n,W_{n+1}} : \mathcal{A}(W_n) \to \mathcal{A}(W_{n+1})\) is the homomorphism induced by the inclusion from \(W_n\) to \(W_{n+1}\). Accordingly, we have an asymptotic morphism

\[\alpha_t : \mathcal{A}(H) \to S \otimes K,\]

for all \(t \in [1, \infty)\). Moreover, it has be proved in [30] that the composition

\[S \overset{\beta_t}{\longrightarrow} \mathcal{A}(H) \overset{\alpha_t}{\longrightarrow} S \otimes K\]

is asymptotically equivalent to the homomorphism

\[S \to S \otimes K\]

defined by

\[f \mapsto f \otimes p,\]

for all \(f \in S\), where \(p\) is a rank-one projection. As a result, the asymptotic morphism of the composition \(\alpha_t \circ \beta_t\) induces the identity map the the \(K\)-theory of \(S\).

For each \(m\), recall \(f_m : Q_m \to H_m\) is the coarse embedding, and choose \(V_m \subset H_m\) to be the infinite-dimensional dense Euclidean subspace of \(H_m\) defined by

\[V_m = \text{span}\{f_m(x) : x \in Q_m\}\]

as in Section 4. Recall that for each \(m \in I\) and \(x \in P_d(G_m)\), \(W_n(x) \subset H_m\) is the subspace

\[W_n(x) = \text{span}\{f_m(\pi(y)) : y \in G_m, d(\pi(x), \pi(y)) \leq n^2\}.\]

Let \(V_n(x) = W_{n+1}(x) \oplus W_n(x)\) if \(n \geq 1\), \(V_0(x) = W_1(x)\), where \(x \in P_d(G_m)\) and \(W_n(x)\) is as in Section 4. For each \(x \in P_d(G_m)\), we have the algebraic decomposition:

\[V_m = V_0(x) \oplus V_1(x) \oplus \cdots \oplus V_n(x) \oplus \cdots.\]

For each \(m \in I\), we define a Hilbert space \(\mathcal{H}_m\) by choosing a linear dense subspace \(V_m\) of \(H_m\). For each \(n \in \mathbb{N}\), we define an unbounded operator \(B_{n,t}^m\) on \(\mathcal{H}_m\) as follows:

\[B_{n,t}^m(x) = t_0 D_0^m + t_1 D_1 + \cdots + t_{n-1} D_{n-1}^m + t_n(D_n^m + C_n^m) + t_{n+1}(D_{n+1}^m + C_{n+1}^m) + \cdots\]

where \(t_j = 1 + t^{-1} j\), \(D_j^m\) and \(C_j^m\) are respectively the Dirac operator and Clifford operator associated to \(V_n(x)\). The operator \(B_{n,t}^m(x)\) plays the role of the Dirac operator in the infinite-dimensional case. Note that \(B_{n,t}^m(x)\) is an unbounded and essentially selfadjoint operator.
For each non-negative integer \( n, m \in I \) and \( x \in X_d^m \), we have an asymptotic morphism 
\[
\alpha_{n,t}(x) : \mathcal{A}(W_n(x)) \hat{\otimes} K \to \mathcal{S} \hat{\otimes} \mathcal{K}_m \hat{\otimes} K.
\]
by
\[
\alpha_{n,t}(x) ((g \hat{\otimes} h) \hat{\otimes} k) = g_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B^n_{n,t}(x)) (1 \hat{\otimes} M_{h_t}) \hat{\otimes} k
\]
for every \( g \in \mathcal{S}, h \in C(W_n(x)), k \in K \) and \( t \geq 1 \), where \( g_t(s) = g(t^{-1}s) \) for all \( t \geq 1 \) and \( s \in \mathbb{R} \), 
\( h_t(v) = h(t^{-1}v) \) for all \( t \geq 1 \) and \( v \in W_n(x) \) and \( M_{h_t} \) acts on \( \mathcal{H}_m \) by point-wise multiplication.

In order to define the geometric Dirac maps, we need to introduce new \( C^* \)-algebras. Note that each element in \( \mathcal{S} \hat{\otimes} \mathcal{K}_m \hat{\otimes} K \) can be viewed as a \( \mathcal{K}_m \hat{\otimes} K \)-valued function on \( \mathbb{R} \). We define \( C^*_{u,alg}(P_d(G_m), \mathcal{S} \hat{\otimes} \mathcal{K}_m)_{m \in I} \) to be the algebra of all tuples \( T = (T^{(m)})_{m \in I} \) such that

1. for each \( m \), \( T^{(m)} \) is a bounded function from \( X_d^m \times X_d^m \) to \( \mathcal{S} \hat{\otimes} \mathcal{K}_m \hat{\otimes} K \), where \( \mathcal{K}_m \) is the algebra of all compact operators acting on \( \mathcal{H}_m \) and \( K \) is the algebra of compact operators on the infinite-dimensional separable Hilbert space;

2. there exists \( L > 0 \) such that for each \( x \in X_d^m \) and each \( m \),
   \[
   \sharp \{ x : T^{(m)}(x,y) \neq 0 \} \leq L, \text{ and } \sharp \{ y : T^{(m)}(x,y) \neq 0 \} \leq L;
   \]

3. there exists \( r > 0 \), \( T^{(m)}(x,y) = 0 \) for all \( m \) and \( x, y \in X_d^m \) with \( d(x,y) > r \);

4. for each \( m \in I \) and each bounded subset \( B \subset P_d(G_m) \), the set
   \[
   \left\{(x,y) \in (B \times B) \cap (X_d^m \times X_d^m) : T^{(m)}(x,y) \neq 0 \right\}
   \]
   is finite;

5. there exists \( r_1 > 0 \) such that \( \text{Supp}(T^{(m)}(x,y)) \subset [-r_1, r_1] \) for all \( m \) and \( x, y \in X_d^m \);

6. there exists \( c > 0 \),
   \[
   \left\| \frac{d}{dt} T^{(m)}(x,y) \right\| \leq c
   \]
   where \( T^{(m)}(x,y) \in \mathcal{S} \hat{\otimes} \mathcal{K}_m \hat{\otimes} K \) can be viewed a \( \mathcal{K}_m \hat{\otimes} K \)-valued function on \( \mathbb{R} \) for all \( m \) and all \( x, y \in X_d^m \).

The algebraic structure on \( C^*_{u,alg}(P_d(G_m), \mathcal{S} \hat{\otimes} K \otimes K)_{m \in I} \) is defined by:
\[
(T_1 T_2)^{(m)}(x,y) = \sum_{z \in X_d} T_1^{(m)}(x,z) T_2^{(m)}(z,y),
\]
and
\[
(T_1^*)^{(m)}(x,y) = \left(T_1^{(m)}(y,x)\right)^*
\]
for all \( x, y \in X_d^m \) and \( T_1, T_2 \in C^*_{u,alg}(P_d(G_m), \mathcal{S} \hat{\otimes} \mathcal{K}_m)_{m \in I} \).

Let \( E_m = \ell^2(X_d^m) \hat{\otimes} \mathcal{H}_m \hat{\otimes} H_0 \), where \( K \) is the algebra of all compact operators on \( H_0 \). Then \( C^*_{u,alg}(P_d(G_m), \mathcal{K}_m)_{m \in I} \) acts on \( E = \oplus_m E_m \) by
\[
T(\delta_x \hat{\otimes} h \hat{\otimes} h_0) = \sum_{y \in X_d} \delta_y \hat{\otimes} T^{(m)}(y,x)(h \hat{\otimes} h_0)
\]
for all \( m \in I, x \in X_d^m, h \in H_m, h_0 \in H_0, \) where \( \delta_x \) and \( \delta_y \) are the Dirac functions at \( x \) and \( y \), respectively.

The completion of \( C_{u,alg}^*(P_d(G_m), S \hat{\otimes} K_m)_{m \in I} \) under the operator norm on \( E \) is denoted by
\[
C^u_\ast((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}).
\]
Since \( K_m \hat{\otimes} K \) is isomorphic to \( K \) for all \( m \). For each \( T \in C_{u,alg}^*((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) \), we can view it as a function from \( \mathbb{R} \) to \( C_u((P_d(G_m))_{m \in I}) \) by
\[
(T(t))^{(m)}(x, y) = T^{(m)}(x, y)(t).
\]
Since the support of each \( T^{(m)}(x, y)(t) \) is contained in some interval \([-r_1, r_1]\) where \( r_1 \) does not depend on \( x, y, m \), so this function is of compactly supported. Moreover, the sequence of functions \( \{T^{(m)}(x, y)(t)\} \) has uniformly bounded and uniformly continuous derivative, so the functions \( t \mapsto T^{(m)}(x, y)(t) \) are equicontinuous on \( t \). It follows that the function \( t \mapsto T(t) \) is continuous on \( t \). As a result, we can viewed \( T \) as an element in \( S \hat{\otimes} C^u_\ast((P_d(G_m))_{m \in I}) \).

Denoted by \( C^u_{u,L,alg}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) \) the algebraic uniform localization algebra consisting of all uniformly bounded and uniformly continuous functions
\[
g = (g^{(m)}_{s})_{m \in I} : [0, \infty) \to C^u_{u,alg}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I})
\]
such that

1. \( \sup_m \) propagation\( g^{(m)}_{s} \) \( \to 0 \), as \( s \to \infty; \)
2. there exists \( r_1 > 0 \) such that \( \text{Supp}(g^{(m)}_{s}(x, y)) \subset [-r_1, r_1] \) for all \( m \in I, x, y \in X_d^m \) and \( s \in [0, \infty); \)
3. there exists \( c_1 > 0, \)
\[
\left\| \frac{d}{dt}g^{(m)}_{s}(x, y) \right\| \leq c_0
\]

where each \( g^{(m)}_{s} \in S \otimes K_m \hat{\otimes} K \) is viewed a \( K_m \hat{\otimes} K \)-valued function on \( \mathbb{R} \) for all \( m \), all \( x, y \in X_d^m \) and all \( s \in [0, \infty). \)

The uniform localization algebra, denoted by \( C^u_{u,L}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) \), is the completion of the \( * \)-algebra \( C^u_{u,L,alg}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) \) under the norm
\[
\|g\| = \sup_s \|g_s\|
\]
for all \( g \in C^u_{u,L,alg}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}). \) Note that
\[
C^u_{u,L}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) \cong S \hat{\otimes} C^u_{u,L}((P_d(G_m))_{m \in I}).
\]

**Definition 5.2.** For each \( d > 0 \) and each \( t \in [1, \infty) \), we define a map
\[
\alpha_t : C^u_{u,alg}((P_d(G_m), A(H_m))_{m \in I}) \to C^u_{u}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I})
\]
by
\[
(\alpha_t(T))^{(m)}(x, y) = (\alpha_{N,t}(x))(T^{(m)}_1(x, y))
\]
for all \( T = (T^{(m)}_1) \in C^u_{u,alg}((P_d(G_m), A(H_m))_{m \in I}) \) and \( t \geq 1, \) where \( N \) is a positive integer such that for every pair \( (x, y) \in X_d^m \times X_d^m \), there exists \( T^{(m)}_1(x, y) \in A(W_N(x)) \hat{\otimes} K \) satisfying \( (\beta_N(x))(T^{(m)}_1(x, y)) = T^{(m)}(x, y). \)
By Lemma 7.2, Lemma 7.3, Lemma 7.4 and Lemma 7.4 in \[54\], we know that the asymptotic morphism \((\alpha_t)_{t \in [0, \infty)}\) is well-defined. Similarly, we can define the maps between the localization algebras.

**Definition 5.3.** For each \(d > 0\) and each \(t \in [1, \infty)\), we define a map

\[
\alpha_{L,t} : C^*_u(L_{alg})((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u((P_d(G_m), S \hat{\otimes} K_m)_{m \in I})
\]

by

\[
(\alpha_{L,t}(T))^{(m)}(s) = (\alpha_t(x))(T^{m}(s))
\]

for all \((T(s))_{s \in [0,\infty)} = (T^{(m)}(s))_{m \in C^*_u(L_{alg})((P_d(G_m), A(H_m))_{m \in I})}\) and \(t \geq 1\).

**Lemma 5.4.** The maps \((\alpha_t)_{t \in [1, \infty)}\) and \((\alpha_{L,t})_{t \in [1, \infty)}\) extends to an asymptotic morphism

\[
(\alpha_t)_{t \in [1, \infty)} : C^*_u,\text{max}((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}),
\]

and

\[
(\alpha_{L,t})_{t \in [1, \infty)} : C^*_u,\text{max}((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}).
\]

**Proof.** Given any \(T, S \in C^*_u(L_{alg})((P_d(G_m), A(H_m))_{m \in I})\). By the definition of \(W_n(x)\), there exists \(N > 0\) independent of \(m\) such that \(W_n(x) \subset W_{n+1}(y)\) for all \(n \geq N\), \(m \in I\) and \(x, y \in X^m_d\) satisfying \(d(x, y) \leq \text{propagation}(T)\). By the definition of twisted Roe algebra (Definition \[4.1\]), there exists \(N_0\) such that for each \(n \geq N_0\), and for every pair \((x, y) \in X^m_d\) there exists \(T^{(m)}_1(x, y) \in A(W_n(x)) \hat{\otimes} K\) satisfying \((\beta_n(x))(T^{(m)}_1(x, y)) = T^{(m)}(x, y)\) and \((\beta_n(x))(S^{(m)}_1(x, y)) = S^{(m)}(x, y)\). It follows from Lemma 7.3, Lemma 7.4 and 7.5 in \[51\] that

\[
\|\alpha_{n,t}(x)(T^{(m)}_1(x, y)S^{(m)}_1(x, y)) - \alpha_{n,y}(x)(T^{(m)}_1(x, y))\alpha_{n,t}(x)(T^{(m)}_1(x, y))\| \to 0,
\]

and

\[
(\alpha_{n,t}(T^{(m)}_1(x, y)))^* - \alpha_{n,t}((T^{(m)}_1(x, y))^*) \to 0
\]

uniformly as \(t \to \infty\). By Lemma \[4.15\] we have that

\[
\|\alpha_t(ST) - \alpha_t(S)\alpha_t(T)\| \to 0, \text{ and } \|\alpha_t(a) - \alpha_t(a^*)\| \to 0
\]

as \(t \to \infty\). As a result, we have an asymptotic morphism

\[
(\alpha_t)_{t \in [1, \infty)} : C^*_u,\text{max}((P_d(G_m), A(H_m))_{m \in I}) \to S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}).
\]

Similarly, we have an asymptotic morphism between localization algebras

\[
(\alpha_{L,t})_{t \in [1, \infty)} : C^*_u,\text{max}((P_d(G_m), A(H_m))_{m \in I}) \to S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}).
\]

\[\square\]

By Theorem \[1.24\] we have the isomorphisms

\[
\lambda : C^*_u,\text{max}((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u((P_d(G_m), A(H_m))_{m \in I})
\]

and

\[
\lambda : C^*_u(L_{alg},\text{max})((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u,L((P_d(G_m), A(H_m))_{m \in I}).
\]
Therefore, we have the following Dirac maps on twisted Roe algebras and twisted localization algebras by composing the Dirac maps with the quotient map \( \lambda \), still denoted by

\[
(\alpha_t)_{t \in [0, \infty)} : C^*_u((P_d(G_m), A(H_m))_{m \in I}) \to C^*_u((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) = S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I})
\]

and

\[
(\alpha_{L,t})_{t \in [0, \infty)} : C^*_{u,L}((P_d(G_m), A(H_m))_{m \in I}) \to C^*_{u,L}((P_d(G_m), S \hat{\otimes} K_m)_{m \in I}) = S \hat{\otimes} C^*_{u,L}((P_d(G_m))_{m \in I}).
\]

Accordingly, we have the homomorphisms

\[
\alpha_* : K_*(S \hat{\otimes} C^*_u((P_d(G_m), A(H_m))_{m \in I})) \to K_*(S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}),
\]

and

\[
\alpha_{L,*} : K_*(C^*_{u,L}((P_d(G_m), A(H_m))_{m \in I})) \to K_*(S \hat{\otimes} C^*_{u,L}((P_d(G_m))_{m \in I}),
\]

induced by the asymptotic morphisms \((\alpha_t)_{[1, \infty)}\) and \((\alpha_{L,t})_{[1, \infty)}\) on \(K\)-theory.

### 5.3 The geometric analogue of Bott periodicity

Note that the asymptotic morphisms \(\alpha, \alpha_L, \beta\) and \(\beta_L\) induce the following commutative diagram on \(K\)-theory:

\[
\begin{array}{ccc}
K_*(S \hat{\otimes} C^*_u, L((P_d(G_m), A(H_m))_{m \in I})) & \xrightarrow{\epsilon_*} & K_*(S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}), \\
(\beta_L)_* & \downarrow{\beta_*} & \\
K_*(C^*_u, L((P_d(G_m), A(H_m))_{m \in I})) & \xrightarrow{\epsilon_*} & K_*(C^*_u((P_d(G_m), A(H_m))_{m \in I})).
\end{array}
\]

In this subsection, we shall prove a geometric analogue of the infinite-dimensional Bott periodicity introduced by Higson, Kasparov and Trout [30]. This geometric analogue was essentially proved by the third author [34].

**Proposition 5.5.** For each \(d > 0\), the compositions \(\alpha_* \circ \beta_*\) and \((\alpha_L)_* \circ (\beta_L)_*\) are the identity maps.

**Proof.** Following the arguments in Lemma 5.1 we can define the asymptotic morphism

\[
\gamma : S \hat{\otimes} C^*_u((P_d(G_m))_{m \in I}) \to S \hat{\otimes} C^*_u((P_d(G_m), K_m)_{m \in I}),
\]

by

\[
\gamma(g \hat{\otimes} T) = g \alpha (X \hat{\otimes} 1 + 1 \hat{\otimes} B^{m}_{o,t}(x)) \hat{\otimes} T^{(m)}(x, y)
\]

for each \(m, (x, y) \in X^m_d \times X^m_d\), where \(g \in C_0(\mathbb{R})\) is a continuously differentiable function with compact support and \(T = (T^{(m)})_{m} \in C_u((P_d(G_m))_{m \in I})\). Let \(g \in \mathcal{S} = C_0(\mathbb{R})\) be a continuously differentiable function with compact support, and \(T = (T^{(m)})_{m} \in C_u((P_d(G_m))_{m \in I})\). It is obvious that \(\beta_t(g \hat{\otimes} T) \in C^*_{u,alg}((P_d(G_m), A(H_m))_{m \in I})\) for all \(t \in [1, \infty)\). Accordingly, the asymptotic morphism \((\gamma_t)_{t \in [1, \infty)}\) is well defined.
For each \( m \) and \( x \in X_d^m \ t \geq 1 \), we define
\[
\eta(x) : A(W_1(x)) \hat{\otimes} K \to H_m \hat{\otimes} K
\]
by
\[
(\eta_t(x))(g \hat{\otimes} h) \hat{\otimes} k = g_t(x)X \hat{\otimes} 1 + 1 \hat{\otimes} B_{0,d}(x)M_{ht} \hat{\otimes} k
\]
for all \( g \in S, h \in C(W_1), k \in K \). Let \( \gamma' \) be the asymptotic morphism
\[
\gamma' : S \hat{\otimes} C^*_U((P_d(G_m))_{m \in I}) \to S \hat{\otimes} C^*_U((P_d(G_m), K_m)_{m \in I})
\]
by
\[
\gamma'_t(g \hat{\otimes} T)^{(m)}(x, y) = (\eta_t(x))((\beta_{W_1}(x)(g)) \hat{\otimes} T^{(m)}(x, y))
\]
for each \( g \in C_0(\mathbb{R}) \) a continuously differentiable function with compact support, \( T = (T^{(m)})_m \in C_u[(P_d(G_m))_{m \in I}], t \geq 1 \), \( m \in I \), \( (x, y) \in X_d^m \times X_d^m \), where \( \beta_{W_1}(x) : S \to A(W_1(x)) \) is the \( * \)-homomorphism induced by the map \( g \mapsto g(X \otimes 1 + 1 \otimes C_{W_1(x)}x) \). It follows from Proposition 4.2 in [30] that \( \gamma \) is asymptotic equivalent to \( \gamma' \). Therefore, \( \gamma_* = \gamma'_* \).

For every \( m \in I, t \geq 1 \) and \( x \in X_d^m \), let \( U^m_x : H_m \to H_m \) be the unitary operator acting on the Hilbert space \( H_m = \lim_{V_a \subset H_m} L^2(V_a, Cllif_c(V_a)) \) defined by
\[
(U_x \xi)(v) = \xi(v - f_m(\pi(x))),
\]
for all \( \xi \in H_m \) and all \( v \in V_a \). Then we have
\[
U^{-1}_x g_t(x)X \hat{\otimes} 1 + 1 \hat{\otimes} B_{0,d}(x))U_x = (\eta_t(x))((\beta_{W_1}(x)(g))
\]
for all \( m \in I \) and \( x \in X_d^m \), \( g \in S \).

For each \( s \in [0, 1] \), let
\[
R(s) = 
\begin{pmatrix}
\cos(\frac{\pi}{2}s) & \sin(\frac{\pi}{2}s) \\
-\sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s)
\end{pmatrix},
\]
For each \( m \in I, x \in X_d^m, s \in [0, 1] \) and \( t \geq 1 \), we define a unitary operator \( U_{x,s} \) acting on \( \ell^2(X_d^m) \hat{\otimes} H_m \hat{\otimes} H_0 \oplus \ell^2(X_d^m) \hat{\otimes} H_m \hat{\otimes} H_0 \) by:
\[
U_{x,s} = R(s) \begin{pmatrix} 1 \hat{\otimes} U_x \hat{\otimes} 1 & 0 \\ 0 & 1 \end{pmatrix} R(s)^{-1},
\]
For each \( s \in [0, 1] \), we define an asymptotic morphism
\[
\gamma(s) : S \hat{\otimes} C^*_U((P_d(G_m))_{m \in I}) \to S \hat{\otimes} C^*_U((P_d(G_m))_{m \in I}) \hat{\otimes} M_2(\mathbb{C})
\]
by
\[
(\gamma_t(s)(g \hat{\otimes} T))^{(m)}(x, y) = U_{x,s}^{-1} \begin{pmatrix}
\gamma'_t(g \hat{\otimes} T^{(m)}(x, y) & 0 \\
0 & 0
\end{pmatrix} U_{x,s}^{-1},
\]
for each \( g \in S, T = (T^{(m)})_m \in C_u[(P_d(G_m))_{m \in I}], t \geq 1, s \in [0, 1] \) and \( (x, y) \in X_d^m \times X_d^m \), where the algebra of all complex 2-by-2 matrices \( M_2(\mathbb{C}) \) is endowed with the trivial grading.
Since $f_m(\pi(x)) \in W_1(x)$, for each $m \in I$ and each $x \in X^m_d$, we can verify that

$$\gamma_t(0) - \left( \alpha_t(\beta_t(g \otimes T)) \right) \to 0$$

in norm as $t \to \infty$ for every compactly supported function $g \in S$ with continuous derivative and $T \in \mathbb{C}_u[(P_d(G_m))_{m \in I}]$. It follows that

$$\gamma(0)_* = \alpha_* \circ \beta_*.$$

On the other hand,

$$\gamma(1) = \left( \gamma'_t \ 0 \ 0 \right).$$

We have that $\gamma(1)_* = \gamma'_*$. As a result, we have that $\alpha_* \circ \beta_* = \gamma_*$. Replacing $B_{0,t}^m(x)$ with $s^{-1}B_{0,t}^m(x)$ in the definition of $\gamma$, we obtain a homotopy between $\gamma$ and the $*$-homomorphism $g \otimes (T^{(m)})_m \to g \otimes (P^{(m)} \otimes T^{(m)})$, where $P^{(m)}$ is the rank-one projection onto the one-dimensional kernel of $B_{0,t}^m(e)$, where $e \in G_m$ is the identity element of $G_m$. Since $f_m(\pi(x)) = 0 \in H_m$, $P^m$ does not depend on $x$. It follows that $\gamma_*$ is the identity homomorphism. Therefore, $\alpha_* \circ \beta_*$ is the identity.

\[\square\]

\textbf{Proof of Theorem 5.3} Consider the following commutative diagram:

\[
\begin{array}{ccc}
\lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u,L}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right) & \xrightarrow{e_*} & \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right) \\
(\beta_L)_* & & (\beta)_* \\
\lim_{d \to \infty} K_*\left(C^*_{u,L}\left(\left(P_d(G_m), A(H_m)\right)_{m \in I}\right)\right) & \xrightarrow{e_*} & \lim_{d \to \infty} K_*\left(C^*_{u}\left(\left(P_d(G_m), A(H_m)\right)_{m \in I}\right)\right) \\
(\alpha_L)_* & & (\alpha)_* \\
\lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u,L}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right) & \xrightarrow{e_*} & \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right)
\end{array}
\]

For any element $x \in \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u,L}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right)$ with $e_*(x) = 0$, we have that

$$e_* \circ (\beta_L)_*(x) = \alpha_* \circ e_*(x) = 0.$$

Since the map $e_*$ is an isomorphism by Theorem 4.9 we have that $(\beta_L)_*(x) = 0$. In addition, $id = (\alpha_L)_* \circ (\beta_L)_*$ by Theorem 5.5 we have that

$$x = (\alpha_L)_* \circ (\beta_L)_*(x) = 0.$$

Thus, the map

$$e_* : \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u,L}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right) \to \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right)$$

is injective.

For any element $y \in \lim_{d \to \infty} K_*\left(S \widehat{\otimes} C^*_{u}\left(\left(P_d(G_m)\right)_{m \in I}\right)\right)$, since the middle horizontal map is an isomorphism, we can find an element $z \in \lim_{d \to \infty} K_*\left(C^*_{u,L}\left(P_d(G_m), A(H_m)\right)_{m \in I}\right)$, such that

$$e_*(z) = \beta_*(y).$$
By Theorem 4.9 and commutativity of the above diagram, we have that

\[ y = \alpha_\ast \circ \beta_\ast (y) = \alpha_\ast \circ e_\ast (z) = e_\ast \circ (\beta_L)_\ast (z). \]

Thus, the map

\[ e_\ast : \lim_{d \to \infty} K_\ast (S \hat{\otimes} C^*_{u,L}((P_d(G_m))_{m \in I})) \to \lim_{d \to \infty} K_\ast (S \hat{\otimes} C^*_{u}((P_d(G_m))_{m \in I})) \]

is surjective.

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