A sense-preserving Sobolev homeomorphism with negative Jacobian almost everywhere

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Abstract
For every $1 \leq p < \frac{3}{2}$ we construct a Sobolev homeomorphism $f \in W^{1,p}([-1,1]^4, [-1,1]^4)$ such that $f(x) = x$ for every $x \in \partial [-1,1]^4$ but $J_f < 0$ a.e.

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1 INTRODUCTION

The Sobolev space, $W^{1,p}$, is the completion of $C^\infty$-smooth real functions having finite Sobolev norm, but it is not obvious if given a homeomorphism $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^n)$ one can preserve the injectivity of the $C^\infty$-smooth approximation mappings. This question was formulated and promoted by Ball [2, 3] in the following form.
**Question 1.1.** If \( f \in W^{1,p}(\Omega, \mathbb{R}^n) \) is invertible, can it be approximated in \( W^{1,p} \) by piecewise affine invertible mappings (or diffeomorphisms)?

Ball attributes this question to Evans and points out its relevance to the regularity of minimizers in the Calculus of Variations.

The first positive results were achieved by Mora-Corral [20] on planar homeomorphisms smooth outside a point and by Bellido and Mora-Corral [4] on approximation in Hölder continuous maps. Let us also note that the problems of approximation by smooth or piecewise affine planar homeomorphisms are in fact equivalent, thanks to the result of Mora-Corral and Pratelli [21]. The celebrated breakthrough result in the area, which stimulated much interest in the subject, was given by Iwaniec, Kovalev and Onninen in [18, 19], where they found diffeomorphic approximations to any homeomorphism \( f \in W^{1,p}(\Omega, \mathbb{R}^2) \), for any \( 1 < p < \infty \) in the \( W^{1,p} \) norm and for \( n = 2 \). The remaining open case in the plane, \( p = 1 \), has been solved by Hencl and Pratelli in [16] using a different method. This method was extended by Campbell [5] to give a different proof of the \( W^{1,p}, p > 1 \) case and to prove the result also for Orlicz–Sobolev spaces.

An interesting open problem is the approximation of Sobolev homeomorphism in dimension greater than 2. One result in higher dimension is due to Hencl and Vejnar in [17], where they found a homeomorphism in \( W^{1,1} \), for \( n \geq 4 \), which cannot be approximated by diffeomorphisms. The following extension, which shows that the problem is not in the special choice of nonreflexive space \( W^{1,1} \) was proven in [6].

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n, n \geq 4 \) and \( 1 \leq p < \lfloor n/2 \rfloor \), then there is a homeomorphism \( f \in W^{1,p}((-1,1)^n, \mathbb{R}^n) \) such that \( J_f > 0 \) on a set of positive measure and \( J_f < 0 \) on a set of positive measure. Moreover, \( f \) satisfies the Lusin (N) condition.

Here \( \lfloor n/2 \rfloor \) denotes the integer part of \( n/2 \), that is, \( 1 \leq p < 2 \) for \( n = 4, 5 \), \( 1 \leq p < 3 \) for \( n = 6, 7 \) and so on.

This result is deeply connected with the sign of the Jacobian of a homeomorphism. Indeed, the study of the Jacobian of a homeomorphism is a natural question in the context of the study of nonlinear elasticity, quasiconformal mappings and mappings of finite distortion. In models of nonlinear elasticity (see, for instance, the pioneering work by Ball [2] or the monograph of Ciarlet [7]), one is led to study the existence and the properties of minimizers of energy functionals of the form

\[
I(f) = \int_{\Omega} W(Df) \, dx. \tag{1.1}
\]

Here \( f : \mathbb{R}^n \supseteq \Omega \to \Delta \subseteq \mathbb{R}^n \) models the deformation of a homogeneous elastic material with respect to a reference configuration \( \Omega \) and prescribed boundary values, while \( W : \mathbb{R}^{n \times n} \to \mathbb{R} \) is the stored-energy functional. In order for the model to be physically relevant, as pointed out by Ball in [2, 3], one has to require that \( f \) is a homeomorphism or at least one to one a.e., which corresponds to the nonimpenetrability of the material. Further a typical requirement is that

\[
W(A) \to +\infty \quad \text{as} \quad \det A \to 0, \quad \quad W(A) = +\infty \quad \text{if} \quad \det A \leq 0. \tag{1.2}
\]

The first condition in (1.2) prevents too high compressions of the elastic body, while the latter guarantees that the orientation is preserved (at least in the analytical sense). In particular, if \( f \) is
an admissible deformation with finite energy, then one has that

\[ J_f := \det Df > 0 \quad \text{a.e. in } \Omega \tag{1.3} \]

and, hence, we restrict our attention to mappings which do not change orientation.

A key question of interest is to prove the regularity of the solutions of this problem. A regular solution, a diffeomorphism, must of necessity have Jacobian that does not change sign. Critically, the Jacobian of any topologically sense-preserving diffeomorphism must be positive. On the other hand, working only with diffeomorphisms would be too restrictive to the tools of the Calculus of Variations. Thus, one is, naturally, led to Sobolev homeomorphisms and to questions about their Jacobian. This is the essence of the questions promoted by Hajlasz; see, for example, Goldstein and Hajlasz [10, 11].

Question 1.3. Let \( Q \subset \mathbb{R}^n \) be the open unit cube.

(a) Does there exist a homeomorphism \( f \in W^{1,p}(Q, \mathbb{R}^n) \) with \( J_f = \det Df \) positive on a set of positive measure in \( Q \) and negative on a set of positive measure in \( Q \)?

(b) Does there exist a homeomorphism \( f \in W^{1,p}(Q, \mathbb{R}^n) \) with \( J_f = \det Df \), negative almost everywhere on \( Q \) but \( f = \text{id} \) on \( \partial Q \)?

A positive answer to the questions of Hajłasz was published in [15]. Precisely

Theorem 1.4. Let \( n = 2, 3 \) and \( p \geq 1 \) or \( n \geq 4 \) and \( p > [n/2] \). Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n) \) be homeomorphism. Then either \( J_f \geq 0 \) a.e. in \( \Omega \) or \( J_f \leq 0 \) a.e. in \( \Omega \).

This result has since been pushed to the limiting case \( p = [n/2] \) under additional assumptions (including on the inverse), see [12].

The combination of Theorems 1.4 and 1.2 answers, up to the critical case \( p = [n/2] \), Question 1.3(a). Let us note that constructions of almost everywhere approximately differentiable homeomorphisms with everywhere negative (approximate) Jacobian are to be found in Goldstein and Hajłasz [10, 11]. These maps lack the Sobolev regularity but have other striking properties, for example, measure preservation or Hölder continuity of the map and its inverse.

The construction in [6] opens the question of point (b) from Question 1.3, can the construction be pushed to give negative Jacobian almost everywhere? It turns out that this question is of even greater relevance to the problem of nonlinear elasticity than the previous (point (a)), especially in connection with the Ball–Evan’s approximation problem.

Assume for contradiction that \( f \) of Theorem 1.2 can be approximated by diffeomorphisms (or piecewise affine homeomorphisms) \( \{f_k\}_{k=1}^{\infty} \) then the pointwise limit of a subsequence (which we denote in the same way) satisfies:

\[ Df_k(x) \to Df(x) \quad \text{and} \quad J_{f_k}(x) \to J_f(x) \quad \text{a.e. } x \in (-1,1)^n. \]

As \( f_k \) are locally Lipschitz we know that \( J_{f_k} \geq 0 \) a.e. in \((-1,1)^n\) or \( J_{f_k} \leq 0 \) a.e. in \((-1,1)^n\). So, the pointwise limit of nonnegative (or nonpositive) \( J_{f_k} \) cannot change sign, which gives a contradiction.

On the other hand, as noted by Buttazzo in 2016 in Naples, such a homeomorphism by (1.2) would of necessity have infinite energy and so not be of great relevance to the minimization
process. Conversely, an answer to Question 1.3(a) would supply a homeomorphism $f$, $f = \text{id}$ on $\partial Q$ with $J_f < 0$ a.e. Therefore, $\tilde{f}(x_1, x_2, ..., x_n) = f(-x_1, x_2, ..., x_n)$ would have $J_{\tilde{f}} > 0$ a.e. On the other hand, $\deg(\tilde{f}, Q) = -1$ and any approximating diffeomorphism $f_k$ with $f_k = f$ on $\partial Q$ has also $\deg(f_k, Q) = -1$ while $J_{f_k} < 0$ a.e. and $Df_k \nrightarrow D\tilde{f}$ a.e.

The main result of this paper shows exactly the existence of such a mapping for $n = 4$.

**Theorem 1.5.** There exists a Sobolev homeomorphism $f \in W^{1,p}([-1,1]^4, [-1,1]^4)$ for every $1 \leq p < \frac{3}{2}$ such that $f(x) = x$ for every $x \in \partial [-1,1]^n$ but $J_f(x) < 0$ for a.e. $x \in (-1,1)^4$. Further $f$ satisfies the Lusin $(N)$ and $(N^{-1})$ conditions.

(For the definition of the $(N)$ and $(N^{-1})$ condition see Definition 2.2). This result yields the following:

**Corollary 1.6.** Set $\tilde{f}(x_1, x_2, x_3, x_4) = f(-x_1, x_2, x_3, x_4)$ where $f$ is from Theorem 1.5. Then $J_{\tilde{f}}(x) > 0$ a.e. but there are no diffeomorphisms (or piecewise affine homeomorphisms) $f_k$ such that $f_k = f$ on $\partial [-1,1]_n$ and $f_k \rightarrow \tilde{f}$ in $W^{1,p}$ for $1 \leq p < \frac{3}{2}$.

1.1 | A brief overview of the construction and the main ideas

Just for the convenience of the reader, we recall some ingredients of the construction of [6, Theorem 1.2]. First, they fix a Cantor-type set $C_A \subset (-1,1)$ of positive measure and they set

$$
\mathcal{K}_A = (C_A \times C_A \times C_A \times [-1,1]) \cup (C_A \times C_A \times [-1,1] \times C_A) \\
(C_A \times [-1,1] \times C_A \times C_A) \cup ([[-1,1] \times C_A \times C_A \times C_A).
$$

They also fix a Cantor-type set $C_B \subset (-1,1)$ of zero measure and consider the corresponding $\mathcal{K}_B$. The first mapping $S_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ squeezes $\mathcal{K}_A$ onto $\mathcal{K}_B$ homeomorphically in a natural way. The key ingredient is the construction of a bi-Lipschitz sense-preserving homeomorphism $F$ such that

$$
F(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \text{ for every } x \in \mathcal{K}_B.
$$

At last, they find a mapping $S_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which stretches $C_B \times C_B \times C_B \times C_B$ back to $C_A \times C_A \times C_A \times C_A$ such that lines in $\mathcal{K}_B$ are not prolonged too much. Since they control the behavior of $f = S_t \circ F \circ S_q$ on lines parallel to coordinate axes, it is possible to check that $f$ satisfies the ACL property and by delicate computations that even $f \in W^{1,p}$. By (1.4) and $S_t = S_q^{-1}$ on $C_B \times C_B \times C_B \times C_B$ we obtain that $f$ behaves like $[x_1, x_2, x_3, -x_4]$ on $C_A \times C_A \times C_A \times C_A$ and, hence, $J_f < 0$ on this set of positive measure. One of the key properties of $F$ is that line segments parallel to coordinate axes and close to $\mathcal{K}_B$, but, far away from $C_B \times C_B \times C_B \times C_B$, are mapped to segments parallel to coordinate axes close to $\mathcal{K}_B$ which allows the estimates of the derivatives of $f$.

The first idea to construct our mapping is to iterate the procedure done by [6], that is, we construct $f_1$ with $J_{f_1} < 0$ on a closed set of positive measure $E_1$ and $f_1 = \text{id}$ on $\partial Q$. We cover $Q \setminus E_1$ by small cubes such that $f_1$ is close to a linear mapping on each of these cubes. We define $f_2 = f_1$ on $E_1$ and $f_2$ is a composition of $f_1$ and a scaled and translated copy of $f_1$ on cubes covering the
rest of $Q$. Then $J_{f_2} < 0$ on a new set $\mathcal{E}_2 \subset Q \setminus \mathcal{E}_1$, we cover $Q \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ with small cubes and continue. By induction we construct a sequence of maps $\{f_m\}_{m=2}^\infty$, and, we want them to converge in $W^{1,p}$ to some $f$ which satisfies our thesis. The derivative of $f$ is calculated as the derivative of an (almost everywhere finite) composition. The huge problem with this approach is that the integral of $|Df|^p$ is too big and we cannot control it.

Hence, we need to substantially modify the construction of $f_1$. Our aim is to construct a map $f_1$ such that the integral of $|D_jf_1|^p$, $j = 1, 2, 3$, is much smaller than $|D_4f_1|^p$ in average (see $v$ in Theorem 3.1 below). Then, we need a mapping $f_2$ that is a clever rotation of $f_1$ in the well-chosen small cubes covering $Q \setminus \mathcal{E}_1$. The main aim of this rotation is that the big derivative from $f_1$ (that is, $D_4f_1$) does not multiply with the big derivative of the next mapping $f_2$ in the matrix multiplication. The cubes are rotated to send $x_1$ in the direction of $D_4f_1$, and the derivative of $f_2$ in direction $x_1$ is small. Then the integral of the derivative of the composition is small and we can make everything work.

We construct our Cantor sets slightly differently than in [6]. We choose a big parameter $K > 0$ so that the Cantor set $C_{A,K} \subset [-1,1]$ has measure almost 2. We denote by $C = C_{A,K} \times C_{A,K} \times C_{A,K} \times C_{A,K}$ the corresponding Cantor set in dimension 4. For $m \in \mathbb{N}$ we refer to the set

$$C_{A,K,m} := \bigcup_{n_1,n_2,n_3 = -m}^m (C + [2n_1, 2n_2, 2n_3, 0])$$

as a four-dimensional ‘$m$-Cantor plate’ so that in each of $(2m + 1)^3$ cubes $Q$ we have a copy of $C$. This choice of $m$-Cantor plate allows us to make the integral outside of Cantor set of $|D_jf_1|^p$, $j = 1, 2, 3$, rather small. Since $f_1(x) = x$ on $\partial([-2m – 1, 2m + 1]^3 \times [-1,1])$ the derivative close to the boundary is big, but, the integral of the derivative between the copies of $C$ inside (in $x_1$, $x_2$, $x_3$ directions) is quite small, as the measure of the complement of $C$ is small. In our proof, we fix $K$ so that the measure of complement of $C_{A,K}$ is small and we choose $m$ so big that in average only the derivative between the copies of $C$ is important and, thus, the integral of $|D_jf_1|^p$ is small. Analogously we define a Cantor $m$-plate of zero measure $C_{B,m}$.

The paper is organized as follows. In Section 2, we introduce some notation and give some preliminary results to facilitate reading the paper. Sections 3–7 are dedicated to prove Theorem 3.1. More precisely in Section 3 we construct a special homeomorphism $\tilde{G}_{K,N,m,\eta}$, that squeezes the Cantor set, that is, it maps $C_{A,K,m}$ onto $C_{B,m}$. We cannot use an analogy of mapping $S_q$ from [6] because for our iteration procedure we need a mapping which is locally bi-Lipschitz outside of $C_{A,K,m}$. In Section 3 we also construct a specific homeomorphism $G_{K,N,m,\eta}$ that stretches $C_{B,m}$ onto $C_{A,K,m}$. We follow the construction in [6, Lemma 3.2] with a difference that our set $C_{A,K}$ is bigger and occupies almost all of $[-1,1]$. The key middle map $F_\beta$, that satisfies the crucial property (1.4) on $C_{A,K,m}$, is constructed in Section 4. The last Section is dedicated to proving Theorem 1.5.

2 | PRELIMINARIES

A point $x \in \mathbb{R}^n$ in coordinates is denoted as $(x_1, \ldots, x_n)$. We denote by $|x| := \sqrt{\sum_{i=1}^n x_i^2}$ the Euclidean norm of a point $x \in \mathbb{R}^n$, and $||x|| := \sup_i |x_i|$ denotes the supremum norm of $x$. For $c \in \mathbb{R}^n$ and $r > 0$ we denote the cube as $Q(c,r) = (c_1 - r, c_1 + r) \times \cdots \times (c_n - r, c_n + r)$. 
Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $f : \Omega \to \mathbb{R}^n$ belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, $1 \leq p < \infty$, if $f$ is $p$-integrable and if the coordinate functions of $f$ have $p$-integrable distributional derivatives. We say that $f$ belongs to the space $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ if $f \in W^{1,p}(\Omega', \mathbb{R}^n)$ for every subdomain $\Omega' \subset \Omega$.

By $Df$ we denote the derivative of mapping $f$ with respect to first coordinate $x_i$, that is, the matrix $Df$ consists of rows $D_1f$, $D_2f$, $D_3f$ and $D_4f$.

### 2.1 General notation

In Sections 3–8, the following symbols have the following meaning:

- $C_{A,K}, C_{A,K,m}, C_B, C_{B,m}$ (see Section 3.3);
- $\hat{F}_{\beta,m}$ — a bi-Lipschitz map sending $C_{B,m}$ onto itself defined in Section 4;
- $g$ — the Lipschitz function on the hyperplane $\mathbb{R}^3 \times \{0\}$ from Section 4;
- $G_{K,N,m,\eta}$ — a stretching mapping defined in Section 3.5;
- $\hat{G}_{K,N,m,\eta}$ — a squeezing mapping defined in Section 3.6;
- $H^n_K$ and $[H^n_K]$ are standard frame-to-frame maps defined in Section 3.3;
- $J^n_K$ and $[J^n_K]$ are standard frame-to-frame maps defined in Section 3.3;
- $K$ — a parameter such that $\mathcal{L}^4(Q(0,1) \setminus C_{A,K}) \to 0$ as $K \to \infty$;
- $\mathcal{K}_{B,m}$ — the set defined in (4.3);
- $m$ — the number of rows of cubes in the Cantor plate construction;
- $M$ — the number from Proposition 4.3;
- $M^*$ — the number from Lemma 5.11;
- $N$ — a parameter for the mappings $G_{K,N;m,\eta}$;
- $Q(0,1) = [-1,1]^4$;
- $R_{m,t} = [-2m - 5, 2m + 5]^3 \times [-1 - t, 1 + t]$;
- $R_{g,v}$ — the rubber band mapping from (4.21);
- $u = [-v_1, -v_2, -v_3, v_4]$ a vector chosen based on the vector $v$;
- $U^n_{k} = \bigcup_{v \in \mathcal{V}_k} Q_v$, see Section 3;
- $\hat{U}^n_{K,k} = \bigcup_{v \in \mathcal{V}_k} \hat{Q}_{K,v}$, see Section 3;
- $\nu = (\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 1)$ — a vector chosen based on a choice of small $t$;
- $\mathcal{V}_q$ is the set of $2^n$ vertices of the cube $[-1,1]^n$;
- $Z_m = [-2m - 1, 2m + 1]^3 \times [-1,1]$;
- $\alpha = \frac{2 + p}{3 - 2p}$ — a parameter dictated by $p$ dictating the geometry of $C_{A,K}$;
- $\beta$ — a parameter dictating the geometry of $C_{A,K}$ and $C_{A,K,m}$; and
- $\eta$ — a small parameter determining the size of the gap above and below the Cantor plates.

### 2.2 ACL condition

It is a well-known fact (see, for example, [1, Section 3.11]) that a mapping $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ is in $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ if and only if there is a representative which is an absolutely continuous function on almost all lines parallel to coordinate axes and the derivative on these lines is integrable. More precisely, let $i \in \{1, 2, \ldots, n\}$ and denote by $\pi_i$ the projection on to the hyperplane perpendicular to the $x_i$-axis. Suppose that $Q(c, r) := (c_1 - r, c_1 + r) \times \cdots \times (c_n - r, c_n + r) \subset \Omega$ for some $c \in \mathbb{R}^n$, ...)
$r > 0$ and set $Q'(c, r) = \pi_i(Q(c, r))$. Let $y \in Q'(c, r)$ and denote

$$u_{i, y}(t) = u(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n) \quad \text{for} \quad t \in (c_i - r, c_i + r).$$

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$. Then $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ if and only if the following happens. For every cube $Q(c, r) \subset \subset \Omega$ and for every $i \in \{1, \ldots, n\}$ there is a representative $u$ of $u$ such that the function $u_{i, y}(t)$ is absolutely continuous on $(c_i - r, c_i + r)$ (that is, each coordinate function is absolutely continuous) for $\mathbb{L}^{n-1}$ almost every $y \in Q_i(c, r)$ and moreover

$$\int_{Q'(c, r)} \int_{c_i - r}^{c_i + r} |\nabla u_{i, y}(t)| \, dt \, dy < \infty.$$

### 2.3 Topological degree

Given a smooth map $f$ from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ we can define the topological degree as

$$\deg(f, \Omega, y_0) = \sum_{\{x \in \Omega : f(x) = y_0\}} \text{sgn}(J_f(x))$$

if $J_f(x) \neq 0$ for each $x \in f^{-1}(y_0)$. This definition can be extended to arbitrary continuous mappings and each point, see, for example, [9].

A continuous mapping $f : \Omega \to \mathbb{R}^n$ is called sense-preserving if

$$\deg(f, \Omega', y_0) > 0$$

for all domains $\Omega'$ compactly contained in $\Omega$ and all $y_0 \in f(\Omega') \setminus f(\partial \Omega')$. Similarly, we call $f$ sense-reversing if $\deg(f, \Omega', y_0) < 0$ for all $\Omega'$ and $y_0$. Let us recall that each homeomorphism on a domain is either sense-preserving or sense-reversing; see [23, II.2.4., Theorem 3].

### 2.4 Composition and integration

**Definition 2.2.** Let $f : \mathbb{R}^n \supset G \to \mathbb{R}^n$, we say that $f$ satisfies the Lusin $(N)$ condition on $G$ if $\mathbb{L}^n(f(E)) = 0$ for every $E \subset G$ such that $\mathbb{L}^n(E) = 0$. We say that $f$ satisfies the Lusin $(N^{-1})$ condition on $G$ if $\mathbb{L}^n(f^{-1}(E) \cap G) = 0$ for every $E \subset \mathbb{R}^n$ such that $\mathbb{L}^n(E) = 0$.

Obviously a mapping locally bi-Lipschitz of $G$ satisfies both these conditions on $G$.

For the following theorem, see [1, Theorem 3.16 and Corollary 3.19]:

**Theorem 2.3.** Let $\Omega, \Delta \subset \mathbb{R}^n$ be open. Let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^d)$ and suppose that $F : \Delta \to \Omega = F(\Delta)$ is a bi-Lipschitz homeomorphism then $u \circ F \in W^{1,1}_{\text{loc}}(\Delta, \mathbb{R}^d)$ and

$$Du \circ F(x) = Du(F(x))DF(x) \quad \text{for almost all} \ x \in \Delta.$$

**Lemma 2.4.** Let $p \in [1, \infty)$ and let $a, b \geq 0$ then

$$|a^p - b^p| \leq p(a + b)^{p-1}|a - b| \leq p2^{p-1}(a^{p-1} + b^{p-1})|a - b|.$$  \hspace{1cm} (2.1)
Proof. We have

\[ |a^p - b^p| = \int_{\max\{a,b\}}^{\min\{a,b\}} pt^{p-1} \leq p(a + b)^{p-1}|a - b|. \]

In the case that \( a \geq b \) then \((a + b)^{p-1} \leq (2a)^{p-1}\), alternatively \((a + b)^{p-1} \leq (2b)^{p-1}\). In either case we have \((a + b)^{p-1} \leq 2^{p-1}(a + b)^{p-1}\), which proves our claim. \(\square\)

Let \( f : Q(0, 1) \to \mathbb{R} \). For each \( c \in \mathbb{R}^n \) and \( r > 0 \) we denote the translated and scaled function as

\[ f_{c,r}(x) = r f\left( \frac{x - c}{r} \right) + c. \]

A simple linear change of variables shows that for \( f \in W^{1,p}(Q(0, 1)) \) we have

\[ f_{c,r} \in W^{1,p}(Q(c, r)) \text{ and } \int_{Q(c,r)} |Df_{c,r}|^p \leq r^n \int_{Q(0,1)} |Df|^p. \] (2.2)

The following lemma is the key for the estimate of the derivative of the composition. At point \( a = F^{-1}(c) \) of differentiability of \( F \) we know that \( F \) is really close to \( F(a) + DF(a)(x - a) \) and we can estimate the derivative of \( f_{c,r} \circ F \) by the linearization (see (2.3)). This form is crucial for us as later we will perform a linear change of variables on the right-hand side of (2.3) since its Jacobian is a constant it can be put out of the integral.

**Lemma 2.5.** Assume that \( f \in W^{1,p}(Q(0, 1)) \). Let \( G \subset \mathbb{R}^n \) be an open set and let \( F : G \to \mathbb{R}^n \) be a mapping locally bi-Lipschitz on \( G \). Let \( Q(c, r) \subset \subset F(G) \). Then for every \( \rho > 0 \) and almost every \( c \in G \) there exists an \( r_c \) such that for all \( 0 < r < r_c \), denoting \( a = F^{-1}(c) \) we have

\[ \int_{F^{-1}(Q(c,r))} |Df_{c,r}(F(x))DF(x)|^p \, dx \]
\[ \leq \int_{[DF(a)]^{-1}(Q(0,r))} |Df_{c,r}(c + DF(a)(x - a))DF(a)|^p \, dx + \rho r^n. \] (2.3)

Similarly, for any measurable \( X \subset Q(0, 1) \) and \( X_{c,r} = rX + c \)

\[ \int_{F^{-1}(X_{c,r})} |Df_{c,r}(F(x))DF(x)|^p \, dx \]
\[ \leq \int_{[DF(a)]^{-1}(rX)} |Df_{c,r}(c + DF(a)(x - a))DF(a)|^p \, dx + \rho r^n. \] (2.4)

**Proof.** We start with rough sketch of the proof and then expound the individual steps in detail. We call

\[ A_{c,r} = F^{-1}(Q(c, r)) \cap (a + [DF(a)]^{-1}(Q(0, r))). \]
Given that $r$ is very small and $a = F^{-1}(c)$ is a differentiation point for $F$ we can restrict just to $A_{c,r}$ because the complement of $A_{c,r}$ becomes very small and the integral over it disappears by absolute continuity of the integral. We can approximate $f$ by a smooth mapping $f^n$ as well as we like. For $f^n_{c,r} \in C^2$ it becomes obvious that the difference between the linearization and the nonlinear integral is very small as soon as $r$ is very small concluding the proof.

From (2.2) we have $Df_{c,r} \in L^p$ and Theorem 2.3 shows that $f \circ F \in W^{1,p}$. For every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $L^n(E) < \delta$ we have $\int_E |Df|^p < \epsilon$. Therefore, it follows from a simple change of variables, that when

$$L^n(E) < \delta r^n \text{ then } \int_E |Df_{c,r}| < \epsilon r^n. \quad (2.5)$$

Almost every point $c$ is such that $a = F^{-1}(c)$ is a point of differentiability of $F$ and a point of approximate continuity of $D F$. Take any such $c$, fix $r_0$ such that $F$ is $L$-bi-Lipschitz on $F^{-1}(Q(c,r_0))$, fix $0 < r_1 \leq r_0$ such that when $0 < r < r_1$ the set

$$E_{c,r} = \left\{ x \in F^{-1}(Q(c,r)) : |DF(x) - DF(a)| > \frac{\rho}{C_{22} 10p(2L)^p L^n \int_{Q(0,1)} |Df|^p} \right\}$$

is so small that by (2.5)

$$\int_{F(E_{c,r})} |Df_{c,r}|^p < r^n \frac{\rho}{10p(2L)^p L^n}. \quad (2.6)$$

Calculate using Lemma 2.4, the $L$-Lipschitz quality of $F$, the definition of $E_{c,r}$, $J_{F^{-1}} < L^n$ in the change of variables formula, (2.6) and (2.2)

$$\int_{A_{c,r}} |Df_{c,r}(F(x)) DF(x)|^p - |Df_{c,r}(F(x)) DF(a)|^p \ dx$$

$$\leq p \int_{A_{c,r}} |Df_{c,r}(F(x)) DF(x)|^p - |Df_{c,r}(F(x)) DF(a)|^p \ dx$$

$$\leq p(2L)^{p-1} \int_{A_{c,r}} |Df_{c,r}(F(x))|^p |DF(x) - DF(a)| \ dx$$

$$\leq \frac{\rho}{C_{22} 10L^n \int_{Q(0,1)} |Df|^p} \int_{A_{c,r} \setminus E_{c,r}} |Df_{c,r}(F(x))|^p \ dx + p(2L)^{p-1} \int_{E_{c,r}} |Df_{c,r}(F(x))|^p 2L \ dx$$

$$\leq \frac{\rho}{C_{22} 10L^n \int_{Q(0,1)} |Df|^p} \int_{P^{-1}(Q(c,r))} |Df_{c,r}(F(x))|^p \ dx + p(2L)^p L^n \int_{F(E_{c,r})} |Df_{c,r}|^p$$

$$\leq \frac{\rho}{C_{22} 10} \int_{Q(0,1)} |Df|^p \int_{Q(c,r)} |Df_{c,r}|^p + \rho \frac{r^n}{10} \leq \rho \frac{r^n}{5}. \quad (2.7)$$

Let us fix $\eta > 0$. We can fix $f^n$ a smooth approximation of $f$ with

$$\int_{Q(0,1)} |Df - Df^n|^p < \eta \int_{Q(0,1)} |Df|^p$$
and calling $f_{c,r}^\eta(x) = rf_{\frac{x-c}{r}} + c$ we have that

$$\int_{Q(c,r)} |Df_{c,r} - Df_{c,r}^\eta|^p < \eta \int_{Q(c,r)} |Df_{c,r}|^p. \quad (2.8)$$

Clearly

$$|Df_{c,r}(F) - Df_{c,r}(c + DF)|^p \leq 3^p|Df_{c,r}(F) - Df_{c,r}^\eta(F)|^p + 3^p|Df_{c,r}^\eta(F) - Df_{c,r}^\eta(c + DF)|^p + 3^p|Df_{c,r}^\eta(c + DF) - Df_{c,r}(c + DF)|^p$$

and therefore we can estimate with the help of Lemma 2.4, $|DF(a)| \leq L$ and notation $a = F^{-1}(c)$

$$\int_{A_{c,r}} \left|Df_{c,r}(F(x))DF(a)^p - Df_{c,r}(c + DF(a)(x-a))DF(a)^p\right|$$

$$\leq 3^p pL^p \int_{A_{c,r}} 2^{p-1}\left(|Df_{c,r}(F(x))|^p - |Df_{c,r}^\eta(F(x))|^p\right)$$

$$|Df_{c,r}(F(x)) - Df_{c,r}^\eta(F(x))|$$

$$+ 3^p pL^p \int_{A_{c,r}} 2^{p-1}\left(|Df_{c,r}^\eta(F(x))|^p - |Df_{c,r}^\eta(c + DF(a)(x-a))|^p\right)$$

$$|Df_{c,r}^\eta(c + DF(a)(x-a)) - Df_{c,r}^\eta(F(x))|$$

$$+ 3^p pL^p \int_{A_{c,r}} 2^{p-1}\left(|Df_{c,r}^\eta(c + DF(a)(x-a))|^p - |Df_{c,r}(c + DF(a)(x-a))|^p\right)$$

$$|Df_{c,r}(c + DF(a)(x-a)) - Df_{c,r}(c + DF(a)(x-a))|.$$  \( (2.9) \)

Expanding the parentheses we get six terms. We estimate the first term using the Hölder inequality and (2.8):

$$\int_{A_{c,r}} |Df_{c,r}(F(x))|^p |Df_{c,r}(F(x)) - Df_{c,r}^\eta(F(x))|$$

$$\leq \left( \int_{A_{c,r}} |Df_{c,r}(F(x))|^p \right)^{\frac{p-1}{p}} \left( \int_{A_{c,r}} |Df_{c,r}(F(x)) - Df_{c,r}^\eta(F(x))|^p \right)^{\frac{1}{p}}$$

$$\leq L^n \left( \int_{Q_{c,r}} |Df_{c,r}(y)|^p \, dy \right)^{p-1} \left( \int_{Q_{c,r}} |Df_{c,r}(y) - Df_{c,r}^\eta(y)|^p \, dy \right)^{\frac{1}{p}}$$

$$\leq L^n \eta^p r^n \int_{Q_{0,1}} |Df|^p.$$  \( (2.10) \)

Since $\|Df_{c,r}^\eta\|^p \leq \|Df_{c,r}\|^p$ we have the same estimate for the second term. The fifth and sixth terms are almost identical and yield the same estimate (up to slightly changing the multiplicative
constant). From (2.10) we see that by choosing $\eta$ sufficiently small we have
\[ 3^p pL^p 2^{p-1} \int_{A_{c,r}} |Df_{c,r}(F(x))|^{p-1} |Df_{c,r}(F(x)) - Df_{c,r}^\eta(F(x))| \leq \frac{\rho}{10} r^n \] (2.11)
and the same estimate holds for the second, fifth and sixth terms.

It remains to estimate the third and fourth terms on the right-hand side of (2.9). Let us call $M_\eta$ the Lipschitz constant of $|Df_{c,r}^\eta|$. Our $\eta$ is fixed so $M_\eta$ is an absolute constant. Since $a = F^{-1}(c)$ is a point of differentiability of $F$ we choose $0 < r_2 < r_1$ so small that for all $0 < r < r_2$ we have
\[ \|F(x) - c - DF(a)(x - a)\|_{L^\infty(Q(c, r))} < \frac{\rho}{C_{2.12}(p)M_\eta \int_{Q(0,1)} |Df|^{p-1}} \] (2.12)
for a fixed constant $C_{2.12}(p)$. Now by (2.12) we have for a well-chosen value of $C_{2.12}(p)$
\[ \int_{A_{c,r}} \left( |Df_{c,r}^\eta(F(x))|^{p-1} + |Df_{c,r}^\eta(F(x)) - Df_{c,r}^\eta(a + DF(a)(x - a))|^{p-1} \right) \]
\[ \leq \int_{A_{c,r}} |Df_{c,r}^\eta(F(x)) - Df_{c,r}^\eta(a + DF(a)(x - a))|^{p-1} \]
\[ \leq \int_{A_{c,r}} \left( |Df_{c,r}^\eta(F(x))|^{p-1} + |Df_{c,r}^\eta(c + DF(c)(x - c))|^{p-1} \right) \]
\[ M_\eta \|F(x) - c - DF(a)(x - a)\|_{L^\infty(Q(c, r))} \]
\[ < \frac{\rho}{C_{2.12}(p) \int_{Q(0,1)} |Df|^{p-1}} \int_{A_{c,r}} |Df_{c,r}|^{p-1} \]
\[ \leq \frac{\rho r^n}{10^3 pL^p 2^{p-1}}. \] (2.13)

Now we apply the estimate (2.7) and in (2.9) we estimate the first, second, fifth and sixth term by (2.11), the third and fourth terms by (2.13) to get
\[ \int_{A_{c,r}} \left| |Df_{c,r}(F(x))DF(x)|^p - |Df_{c,r}(c + DF(a)(x - a))DF(a)|^p \right| \leq \frac{4}{5} \rho r^n. \] (2.14)
We consider the remaining part. We choose $\delta$ so that
\[ \int_E |Df|^p < \frac{\rho}{10L^n} \text{ as soon as } L^n(E) \leq \delta. \] (2.15)
Since $a = F^{-1}(c)$ is a point of differentiability of $F$ we can estimate that
\[ L^n \left( [F^{-1}(Q(c,r)) \cup (a + [DF(a)]^{-1}(Q(0,r))) \right] \setminus A_{c,r} \right) \leq \delta r^n L^{-n} \]
as soon as $r \leq r_3$ (without loss of generality assume that $r_3 \leq r_2$) and hence by (2.15)
\[ \int_E |Df_{c,r}|^p < \frac{\rho r^n}{10L^n} \text{ as soon as } L^n(E) \leq \delta r^n. \]
Therefore, calling

\[ S_{c,r} = F^{-1}(Q(c,r)) \setminus A_{c,r} \quad \text{and} \quad Z_{c,r} = (a + [DF(a)]^{-1}(Q(0,r))) \setminus A_{c,r}, \]

we have \( L^n(S_{c,r}) \leq \delta r^n L^{-n} \), \( L^n(Z_{c,r}) \leq \delta r^n L^{-n} \) and so

\[
\int_{S_{c,r}} |Df_{c,r}(F(x))|^p dx \leq \frac{\rho r^n}{10L^n} \quad \text{and} \quad \int_{Z_{c,r}} |Df_{c,r}(F(x))|^p dx \leq \frac{\rho r^n}{10L^n}.
\]

Adding this to (2.14) we get

\[
\left| \int_{F^{-1}(Q(c,r))} |Df_{c,r}(F(x))DF(x)|^p dx \right| \leq \frac{\rho r^n}{10L^n}.
\]

which proves the claim of (2.3). The Equation (2.4) is proved by applying the above estimates to the set \( F^{-1}(X_{c,r}) \) (respectively, \( a + [DF(a)]^{-1}(rX) \)). \( \Box \)

2.5  Covering lemma

In our main proof, we construct a mapping and some closed set where the Jacobian is negative. We do not alter our mapping on this closed set and we cover the remaining open set \( \Omega \) by (small enough) disjoint cubes where we compose our mapping with a translated and rotated copy of the previous construction.

**Lemma 2.6.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( \mathcal{N} \subset \Omega \) satisfy \( L^n(\mathcal{N}) = 0 \). Assume that for every \( c \in \Omega \setminus \mathcal{N} \) we have a number \( r_c > 0 \). Further let \( S : \Omega \setminus \mathcal{N} \rightarrow \mathbb{R}^4 \) be a mapping such that \( |S(c)| = 1 \). By \( O_c \) denote a sense-preserving unitary map such that \( O_c S(c) = e_1 \). By \( Q_{c,r} \) denote the set \( c + O_c^{-1}(Q(0,r)) \). Then, we can find a countable system of rotated cubes \( Q_{c_i,r_i} \subset \Omega \) with pairwise disjoint interiors such that \( c_i \in \Omega \setminus \mathcal{N}, r_i < r_c \) for every \( i \in \mathbb{N} \) and \( L^n(\Omega \setminus \bigcup_{i=1}^{\infty} Q_{c_i,r_i}) = 0 \).

**Proof.** To prove our claim it suffices to prove that there exists an \( \alpha > 0 \) such that for any open \( \Omega \) we can find a finite number of \( Q_{c_i,r_i} \subset \Omega, i = 1, 2, \ldots, I_\Omega \) with disjoint interiors so that

\[
L^n\left( \bigcup_{i=1}^{I_\Omega} Q_{c_i,r_i} \right) \geq \alpha L^n(\Omega).
\]

By iterating the above process (applied to the new open set \( \Omega \setminus \bigcup_{i=1}^{I_\Omega} \overline{Q_{c_i,r_i}} \)) we get precisely the claim. Therefore, we prove the above claim.

We segregate \( \Omega \) by a Whitney decomposition and select a finite number of cubes \( Q_1, Q_2, \ldots, Q_m \) of that covering so that \( L^n(\bigcup_{i=1}^{m} Q_i) \geq \frac{1}{2} L^n(\Omega) \). From here on we work inside a single cube \( Q_1 \) and therefore, without loss of generality, we may assume that \( Q_1 = Q(0,1) \).
We can find an $r_0$ small enough such that
\[ L^n \left( \{ c \in Q(0,1) : r_c > r_0 \} \right) > L^n(Q(0,1)) - \frac{1}{2} \tag{2.17} \]

Now we choose $k \in \mathbb{N}$ so that $\frac{1}{k} < r_0$. We separate $Q(0,1)$ into $k^n$ identical cubes of type $Q(x, \frac{1}{k})$. Clearly for any unitary map $O : \mathbb{R}^n \to \mathbb{R}^n$ we have that $O(Q(0, \frac{r}{\sqrt{n}})) \subset B(0,r) \subset Q(0,r)$. Therefore, given a cube of type $Q(x, \frac{1}{k})$ and any $y \in Q(0, \frac{1}{2k})$ we have that
\[ Q(x, \frac{1}{k}) \supset Q(x+y, \frac{1}{2k}) \supset B(x+y, \frac{1}{2k}) \supset x+y + O(Q(0, \frac{1}{\sqrt{n2k}})) \tag{2.18} \]

The center of each rotated cube in (2.18) is of the form $x + y$ where $x$ is the center of the large cube of radius $\frac{1}{k}$ and $y \in Q(0, \frac{1}{2k})$. The number of such cubes in $Q(0,1)$ is $k^n$. Therefore, the measure of the set of possible centers of the rotated cubes is $k^n(\frac{1}{2k})^n 2^n = 1$. Therefore by (2.17) there must be at least $\frac{k^n}{2}$ cubes of type-$Q(x, \frac{1}{2k})$ which intersect the set $\{ c \in Q(0,1) : r_c < r_0 \}$ from (2.17). That is to say we can find at least $k^n/2$ points $z = x + y$, each in a separate cube of type-$Q(x, \frac{1}{2k})$ such that $z + O_z(Q(0, \frac{1}{\sqrt{n2k}}))$ are all pairwise disjoint by (2.18) where $O_z$ is a sense-preserving unitary map with $O_z(S(z)) = e_1$. The measure of the set covered by this collection of rotated cubes is at least
\[ \frac{k^n}{2} \cdot \frac{2^n}{n^2 2^n k^n} = \frac{1}{2n^{n/2}} = 2\alpha. \]

We repeat this in all of the chosen cubes $Q_1, ... Q_m$ which together make at least half of the measure of $\Omega$ and so our rotated cubes (of which we have a finite number) cover a set of measure at least $\alpha L^n(\Omega)$. Iterating this technique we arrive at a covering of all of $\Omega$ up to a closed null set by rotated cubes with pairwise disjoint interiors. \qed

**Corollary 2.7.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathcal{N} \subset \Omega$ with $L^n(\mathcal{N}) = 0$. Assume that for every $c \in \Omega \setminus \mathcal{N}$ we have $r_c > 0$. Then, we can find a countable system of cubes $Q(c_i, r_i) \subset \Omega$ with pairwise disjoint interiors such that $c_i \in \Omega \setminus \mathcal{N}, r_i < r_c$ for every $i \in \mathbb{N}$ and $L^n(\Omega \setminus \bigcup_i Q(c_i, r_i)) = 0$.

**Proof.** It suffices to apply Lemma 2.6 with $S(c) = e_1$ for all $c \in \Omega \setminus \mathcal{N}$ and $O_c = \text{id}$. \qed

### 3 CONSTRUCTION OF CANTOR SETS AND MAPPINGS BETWEEN THEM

#### 3.1 A map that has negative Jacobian on a set of positive measure and equals to identity on the boundary

Sections 3–7 are dedicated to prove Theorem 3.1. Section 3 introduces the concept of Cantor sets and defines some mappings between them. Section 4 ensures identity on the boundary of our map. Sections 5 and 6 list the necessary estimates of the derivatives needed to prove Theorem 3.1 and we combine all the results in the proof in Section 7.
Theorem 3.1. Let \( n = 4 \) and \( 1 < p < \frac{3}{2} \). For every \( \varepsilon > 0 \) there exists a closed set \( E \subset Q(0,1) \) and a map \( f_1 \in W^{1,p}(Q(0,1), \mathbb{R}^4) \) such that

(i) \( f_1(x) = x \) for \( x \in \partial Q(0,1) \);
(ii) \( f_1 \) is locally bi-Lipschitz on \( Q(0,1) \setminus E \);
(iii) \( J_{f_1} < 0 \) on \( E \);
(iv) \( L^4(Q(0,1) \setminus E) < \varepsilon \); and
(v) for \( j = 1, 2, 3 \) we have

\[
\int_{Q(0,1) \setminus E} |D_j f_1|^p \int_{Q(0,1) \setminus E} |D_4 f_1|^p \leq \frac{1}{12}.
\]

3.2 Notation

The construction depends on a large parameter \( m \in \mathbb{N} \) whose value is chosen later. We record some notation here, which we use throughout Sections 3–7. We define

\[
R_{m,t} = [-2m - 5, 2m + 5]^3 \times [-1 - t, 1 + t]. \tag{3.1}
\]

Later we work in detail with \( R_{m,2}, R_{m,13} \) and especially \( R_{m,\eta} \) for \( \eta > 0 \) small. Eventually we choose \( \eta = K^{-\alpha} \), where \( K \) and \( \alpha \) are parameters that define our Cantor-type set of positive measure (see (3.4)).

We define

\[
Z_m = [-2m - 1, 2m + 1]^3 \times [-1, 1].
\]

3.3 Construction of Cantor-type sets

Let \( n = 3 \) or \( n = 4 \). Given a sequence of numbers \( \{s_k\}_{k=0}^\infty \), such that \( 2^k s_k \) is decreasing with \( s_0 = 1 \), we define the Cantor-type set in \( \mathbb{R}^n \) corresponding to the sequence \( \{s_k\} \) as is described next. Call \( \mathcal{V}_n \) the set of \( 2^n \) vertices of the cube \( [-1, 1]^n \). We set \( z_0 = \tilde{z}_0 = 0 \) and \( [-1, 1]^n = Q(z_0, \tilde{s}_0) \) and further we proceed by induction. For \( v = [v_1, \ldots, v_k] \in \mathcal{V}_n \) we denote \( w = [v_1, \ldots, v_{k-1}] \) and we define

\[
z_v = z_w + \frac{1}{2} s_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^{k} s_{j-1} v_j. \tag{3.2}
\]

Then we define (Figure 1)

\[
Q'_v = Q(z_v, \frac{s_{k-1}}{2}) \text{ and } Q_v = Q(z_v, s_k). \tag{3.3}
\]

We refer to the set

\[
C = \bigcap_{k=1}^\infty \bigcup_{v \in \mathcal{V}_n^k} Q_v
\]

as the Cantor set corresponding to the sequence \( s_k \).
The number of the cubes \( \{Q_v : v \in \mathcal{V}_n^k\} \) is \( 2^{nk} \) and the measure of each cube of generation \( k \) is \( 2^ns^n_k \). Therefore

\[
\mathcal{L}^n(C) = \mathcal{L}^n\left( \bigcap_{k=1}^{\infty} \bigcup_{v \in \mathcal{V}_n^k} Q_v \right) = \lim_{k \to \infty} 2^{nk}2^n s^n_k.
\]

Note that the projection of the sets \( \bigcup_{v \in \mathcal{V}_n^k} Q_v \) onto \( \mathbb{R}^d \), \( d < n \) is the same as the repeating the original construction with \( s_k \) in \( \mathbb{R}^d \). It follows that \( C(n) \) is a Cartesian product of corresponding one-dimensional Cantor sets \( C(1) \).

If \( C(4) \) is the Cantor set constructed in dimension 4 corresponding to \( s_k \), we refer to the set

\[
C_m := \bigcup_{n_1,n_2,n_3=-m}^{m} (C + [2n_1, 2n_2, 2n_3, 0])
\]

as a four-dimensional '\( m \)-Cantor plate' corresponding to the sequence \( s_k \), that is, in each of the \( (2m + 1)^3 \) cubes \( Q((2n_1, 2n_2, 2n_3, 0), 1) \) we have a copy of \( C \), for \( -m \leq n_1, n_2, n_3 \leq m \).

We use two specific sets \( C_{A,K} \), \( C_B \subset [-1,1]^n \). For \( n = 4 \) we use the notation \( C_{A,K,m} \) and \( C_{B,m} \) for the corresponding four-dimensional \( m \)-Cantor plates in \( Z_m^n \). We fix the parameters \( \alpha > 0 \) and \( K \in \mathbb{N} \) whose exact value is chosen later. In fact at the end we choose \( \alpha = (2 + p)/(3 - 2p) \) (recall that \( 1 \leq p < 3/2 \)) and \( K \) is so big that the measure of \( C_{A,K} \) is really close to \( L^4([-1,1]^n) \). We define

\[
\bar{r}_k(K) = \frac{2^{-k}}{1 + (K + 1)^{-\alpha}}(1 + (K + k + 1)^{-\alpha}).
\]

(3.4)

Note that \( 2^k \bar{r}_k(K) \) is decreasing and \( \bar{r}_0(K) = 1 \). By \( C_{A,K} \) we denote the Cantor set constructed using the sequence \( \bar{r}_k(K) \). We use the construction predominantly in the four-dimensional case but also sometimes in three dimensions. When necessary we denote the set constructed in \( \mathbb{R}^n \) by \( C_{A,K}(n) \), respectively, \( C_B(n) \). We call the points calculated by Equation (3.2) for this sequence \( \bar{z}_{K,v} \). Given a \( v \in \mathcal{V}_n^k \) we call

\[
\bar{Q}_{K,v} = Q(\bar{z}_{K,v}, \bar{r}_k(K)) \quad \text{and} \quad \bar{U}_{K,k} = \bigcup_{v \in \mathcal{V}_n^k} \bar{Q}_{K,v}.
\]
The set \( \tilde{U}_{K,k}^n \) equals to the Cartesian product \([ \tilde{U}_{K,k}^1 ]^n\). Clearly
\[
\mathcal{L}^n(C_{A,K}) = \lim_{k \to \infty} \mathcal{L}^n(\tilde{U}_{K,k}) = \lim_{k \to \infty} 2^{nk}(2\tilde{r}_k(K))^n
\]
\[
= \lim_{k \to \infty} 2^n \frac{(1 + \frac{1}{(K+k+1)^2})^n}{\left(1 + \frac{1}{(K+1)^2}\right)^n} = \frac{2^n}{\left(1 + (K + 1)^{-\alpha}\right)^n} \tag{3.5}
\]
and this tends to \(2^n = \mathcal{L}^n(Q(0,1))\) as \(K \to \infty\). More precisely
\[
\mathcal{L}^n([-1,1]^n \setminus C_{A,K}) = 2^n - \frac{2^n}{\left(1 + (K + 1)^{-\alpha}\right)^n} \leq C(n,\alpha)(K + 1)^{-\alpha}. \tag{3.6}
\]

If by \(I\) we denote one of the intervals of \(\tilde{U}_{K,k}^1 \setminus \tilde{U}_{K,k+1}^1\), then we have
\[
\text{diam } \tilde{I} = \frac{1}{2} \tilde{r}_k(K) - \tilde{r}_{k+1}(K)
= \frac{2^{-k-1}}{1 + (K + 1)^{-\alpha}}((K + k + 1)^{-\alpha} - (K + k + 2)^{-\alpha})
\approx C 2^{-k}(K + k)^{-\alpha-1} \tag{3.7}
\]
and the constant \(C\) does not depend on \(K\).

We fix a parameter \(\beta > 2\) whose exact value we specify later. In fact this is an absolute constant and we fix it big enough to that there is enough room around \(C_B\) to perform our construction. For \(k \in \mathbb{N}_0\) we write
\[
r_k = \frac{1}{2^k} 2^{-\beta k} \tag{3.8}
\]
and we call the Cantor set constructed using the sequence \(r_k\) as \(C_B\). We denote the points calculated by Equation (3.2) for this sequence \(z_v\). Given a \(v \in \mathcal{V}_n\) we call
\[
Q_v = Q(z_v, r_k) \quad \text{and we call } U^n_k = \bigcup_{v \in \mathcal{V}_n} Q_v.
\]
Again \(U^n_k = [U_k^1]^n\). If \(I\) is an interval of \(U_k^1 \setminus U_{k+1}^1\) then
\[
\text{diam } I = \frac{1}{2} r_k - r_{k+1}
= 2^{-k(\beta+1)}\left(\frac{1}{2} - 2^{-\beta}\right)
\approx C 2^{-k(\beta+1)} \tag{3.9}
\]
because we consider \(\beta > 2\).
3.4 A standard homeomorphism $J^n_K$ that maps $C_{A,K}$ onto $C_B$

The construction of the standard ‘frame-to-frame’ map can be found in [14, Chapter 4.3]. An $\infty$-norm radial map maps cubes onto cubes. It is standard that we can map $\tilde{Q}_{K,v} \setminus Q_{K,v}$ onto $Q'_v \setminus Q_v$ for each $v \in V_n$ for each $k$ by a (\infty-norm) radial map centered at $\tilde{z}_{K,v}$ and $z_v$. This radial map is injective and since the cubes $Q'_v$ are pairwise disjoint (up to their possible common boundary) the map is injective from each $U_{K,k} \setminus U_{K,k}$. We conduct this on $U_{K,k} \setminus U_{K,k}$ for each $k$. Take a $v \in V_n$, and call $v(k) \in V_k$ the element that is equal to the first $k$ components of $v$. The map we are constructing sends $\tilde{Q}_{K,v(k)}$ onto $Q_v(k)$ for each $k$. Therefore it becomes immediately obvious that each $\tilde{z}_{K,v} \in C_{A,K}$ is mapped onto $z_v$ and that the mapping is continuous on $C_{A,K}$. It is standard that this map is continuous on each $(0,1) \setminus U_{k}$ for each $k$ and extends as a homeomorphism onto $(0,1)$ sending $C_{A,K}$ onto $C_B$. We denote this map by $J^n_K$. The same construction can be conducted in the opposite direction, mapping $C_B$ onto $C_{A,K}$ and we call it $H^n_K$. Especially, we denote $J^1_K = q_K$ and $H^1_K = t_K$; to be explicit

$$q_K \text{ is the continuous extension of the map that is linear on each interval}$$

in $U_{K,k-1} \setminus U_{K,k}$ sending it onto the corresponding interval in $U_{k-1} \setminus U_k$ (for $k \in \mathbb{N}$) and the constructed function $t_K$ is its inverse.

Further, for $l \in \mathbb{N}$ we define a continuous map $[J^n_K]_l$ such that $\lim_{l \to \infty} [J^n_K]_l = J^n_K$. We define

$$[J^n_K]_l = J^n_K \text{ on } Q(0,1) \setminus U_{K,l}$$

and it maps each $\tilde{Q}_{K,v} = Q(\tilde{z}_{K,v}, \tilde{r}_l(K))$, $v \in V_l$, linearly onto $Q_v$, that is,

$$[J^n_K]_l(Q(\tilde{z}_{K,v}, \tilde{r}_l(K))) = Q(z_v, r_l) = Q_v \text{ for every } v \in V_l.$$ (3.11)

The same construction applied in reverse gives $([J^n_K]_l)^{-1} = [H^n_K]_l$. That is $[H^n_K]_l = H^n_K \text{ on } Q(0,1) \setminus U_l$ and it maps $Q_v$ linearly on $\tilde{Q}_{K,v}$ for each $v \in V_l$.

3.5 A specific homeomorphism that maps $C_B(4)$ onto $C_{A,K}(4)$ dependent on a parameter $N$

In this section, we define a map $G_{K,N,m,n}$ of course depending on $\alpha$ and $\beta$ that maps $C_B(4)$ onto $C_{A,K}(4)$. In order to do so we use mappings of type $[H^n_K]_l$ on three-dimensional hyperplanes perpendicular to $e_i$, the $i$th canonical vector. We define

$$T_l(x_1, x_2, x_3, x_4) := (x_{j_1}, x_{j_2}, x_{j_3}), \text{ where } j_1 < j_2 < j_3 \text{ and } j_l \neq i \text{ for all } l.$$ (3.12)

Further we define

$$T^i(x_{j_1}, x_{j_2}, x_{j_3}) := (x_1, \ldots, 0, \ldots, x_4), \text{ where the 0 is on the } i \text{th place.}$$ (3.13)
Then
\[ H_K^{3,i}(x) = T^i \left( H_K^3(T_i(x)) \right) \text{ and } [H_K^{3,i}](x) = T^i \left( [H_K^3](T_i(x)) \right). \]

In this case the map \([H_K^{3,i}]_k\) corresponds to the definition of \(H_K^3\) from [6, Section 3].

We divide \(Q'_v \setminus Q_v, v \in \mathbb{V}_4^k\), into parts where we are farthest from \(z_v\) in the \(i\)th direction, that is,
\[
S_{v,i} := \{ x \in Q'_v \setminus Q_v : \| x - z_v \|_\infty = |x_i - (z_v)_i| \}. \tag{3.14}
\]

It was proved in [6, Section 3] that the map defined in each frame \(Q'_v \setminus Q_v\) (of the four-dimensional cubes) for \(v \in \mathbb{V}_4^k\) and \(x \in S_{v,i}\) by the convex combination of \([H_K^{3,i}]_{3k} + t(x_i)e_i\) and \([H_K^{3,i}]_{3k+3} + t(x_i)e_i\) is a homeomorphism which maps each frame \(Q'_v \setminus Q_v\) onto \(\tilde{Q}'_0, v \setminus \tilde{Q}_0, v\). In fact the difference \(3k\) and \(3k + 3\) used there is immaterial. We define our map analogously, we find a continuous function \(\zeta_{K,k}\) so that
\[
\zeta_{K,k}(s) = \begin{cases} 
1 & s \in U_{k+1}^1 \\
0 & s \in \mathbb{R} \setminus U_k^1 \\
\text{linear on intervals in } U_k^1 \setminus U_{k+1}^1.
\end{cases}
\]

Fix \(l_1, l_2 \in \mathbb{N}\) with \(l_2 > l_1 \geq k\). We define a mapping \(G_{K,l_1,l_2,k}\) on \(U_4^k \setminus U_{k+1}^4\), which for each \(w \in \mathbb{V}_4^{k+1}\) and \(x \in S_{w,i} \subset Q'_w \setminus Q_w\) is defined as
\[
G_{K,l_1,l_2,k}(x) := \zeta_{K,k}(x_i)[H_K^{3,i}]_{l_1}(x) + [1 - \zeta_{K,k}(x_i)][H_K^{3,i}]_{l_2}(x) + t_K(x_i)e_i. \tag{3.15}
\]

We claim that for any \(l_2 > l_1 \geq k\), the mapping \(G_{K,l_1,l_2,k}\) is a homeomorphism.

The injectivity of \(G_{K,l_1,l_2,k}\) is easily proved. In fact it suffices to check injectivity on a single frame and injectivity follows on all other frames by self-similarity of the construction. The situation on a given frame is illustrated in Figure 2. We have three different considerations, which are represented in Figure 2 by the blue, red and green parts. In \(S_{v,i}\), the part of the frame farthest from \(z_v\) in the \(e_i\) direction, we send hyperplane parts perpendicular to \(e_i\) onto hyperplane parts perpendicular to \(e_i\). Distinct hyperplane parts are mapped onto distinct hyperplane parts because \(t_K\) is an increasing function. On the part of the hyperplane inside \(U_{l_1}^3\) (blue in Figure 2), injectivity on each hyperplane is easy to see from the convexity of the cube. Injectivity on the hyperplane outside \(U_{l_1}^3\) (red in Figure 2) follows immediately from the fact that in this part
\[
H_K^{3,i}(x) + t_K(x_i)e_i = [H_K^{3,i}]_{l_1}(x) + t_K(x_i)e_i = [H_K^{3,i}]_{l_2}(x) + t_K(x_i)e_i.
\]

In the case considered in [6], the fact was proved that \(S_{i,w} \subset Q'_w \setminus Q_w\) has disjoint image from \(S_{j,w}\) for \(j \neq i\). Continuity of the map at the common boundary of the two sets was also proved. The entire argument applies here with the only difference being slightly changed indices, that is, now we have general \(K \in \mathbb{N}\) and not only \(K = 0\) and we have \(l_2 > l_1 \geq k\) instead of \(3k + k > 3k \geq k\). Arguments are however the same and we refer the curious reader to [6, Lemma 3.2], especially Step 1 of the proof.

Now we fix an important parameter \(N \in \mathbb{N}\). In fact at the end we put \(N = 2K\) but we prefer to use a different notation for it so it is easy to track. Its role is that we squeeze more in some direc-
A scheme of the maps $\tilde{G}_{K, l_1, l_2, k}$ and $G_{K, l_1, l_2, k}$. Hyperplane parts get mapped to hyperplane parts as in the dark green line. Injectivity on the red parts mapped to the red parts is thanks to the injectivity of the frame-to-frame maps. Injectivity on the blue parts can be seen easily from the convexity of the cube. The fact that the green parts have image disjoint from the blue/red part was proved in $[6, \text{Section 3}]$. Continuity on the hyperplane parts is thanks to continuity of the frame-to-frame mappings. Continuity of the maps at the boundary of the green and red/blue parts was proved in $[6, \text{Section 3}]$. Red lines parallel to $e_1$ are mapped to red lines parallel to $e_1$.

We set $l_1 = 3k + N$ and $l_2 = 3k + 3 + N$. We have that $G_{K, 3k+N, 3k+3+N, k}$ is a homeomorphism on the whole frame $Q'_w \setminus Q_w$, $w \in V_{k+1}^4$. The union of these frames is the set $U_k^4 \setminus U_{k+1}^4$.

Now we define the map that this subsection is dedicated to; its scheme is in Figure 3. Let $\eta > 0$ and call $n = (n_1, n_2, n_3, n_4)$, where $n_1, n_2, n_3$ are even numbers between $-2m$ and $+2m$ and $n_4 = 0$.

We define $G_{K,N,m,\eta}$ on $Z_m$ as follows; when $y = x + n$ for $x \in Q(0,1)$,

$$G_{K,N,m,\eta}(y) = G_{K,N,m,\eta}(x + n) = \begin{cases} G_{K,3k+N,3k+3+N,k}(x) + n & x \in U_k^4 \setminus U_{k+1}^4, \\ (t_K(x_1), t_K(x_2), t_K(x_3), t_K(x_4)) + n & x \in C_B. \end{cases} \quad (3.16)$$

On $R_{m,13} \setminus Z_m = ([-2m - 5, 2m + 5] \times [-14, 14]) \setminus ([-2m - 1, 2m + 1] \times [-1, 1])$ we define our $G_{K,N,m,\eta}$ as an interpolation between values on $\partial([-2m - 1, 2m + 1] \times [-1, 1])$ (which is some form of $[H_{K,3}^1]_N$) and between identity on outer boundary. Moreover, we need to get that $G$ is stretching (and thus $DG$ is big) only on $([-2m - 2, 2m + 2] \times [-2, 2]) \setminus ([-2m - 1, 2m + 1] \times [-1, 1])$ and it is Lipschitz outside of this region (see Figure 3).
A scheme of the definition of $G_{K,N,m,\eta}$. The gray-shaded area is where we use the mapping $G_{K,3k+N,3k+3+N,k}$. The top and bottom (purple) parts are squeezed from size 13 to size $\eta$. In the preimage the blue area represents $[-2m-5,-2m-1] \times U_3^N$, and $(U_3^N+n) \times [1,14]$, respectively.

When $y = x + n$, $x_1, x_2, x_3 \in [-1,1]$, and $x_4 \in [-14,-1] \cup [1,14]$ we put

$$G_{K,N,m,\eta}(y) = G_{K,N,m,\eta}(x+n)$$

$$= \begin{cases} 
[H_K^{3,4}]_N(x) + n + \frac{n}{13}(x_4 - 1)e_4 + e_4 & x_4 \in [1, \frac{3}{2}], \\
2(2-x_4)[H_K^{3,4}]_N(x) + 2(x_4 - \frac{3}{2})(x_1, x_2, x_3, 0) + \\
+n + \frac{n}{13}(x_4 - 1)e_4 + e_4 & x_4 \in [\frac{3}{2}, 2], \\
[H_K^{3,4}]_N(x) + n + \frac{n}{13}(x_4 + 1)e_4 - e_4 & x_4 \in [-\frac{3}{2}, -1], \\
2(2+x_4)[H_K^{3,4}]_N(x) + 2(-x_4 - \frac{3}{2})(x_1, x_2, x_3, 0) + \\
+n + \frac{n}{13}(x_4 + 1)e_4 - e_4 & x_4 \in [-2, -\frac{3}{2}], \\
(x_1, x_2, x_3, 0) + n + \frac{n}{13}(x_4 - 1)e_4 + e_4 & x_4 \in [2, 14], \\
(x_1, x_2, x_3, 0) + n + \frac{n}{13}(x_4 + 1)e_4 - e_4 & x_4 \in [-14, -2], 
\end{cases}$$

(3.17)

It is easy to see from (3.15), (3.16) and (3.17) that those mapping agree on the boundary $[-2m-1, 2m+1]^3 \times \{-1,1\}$, that is, they are both equal to the shifted $[H_K^{3,4}]_N(x)$. For $x_4 \in [-2, -1] \cup$
[1, 2] we just interpolate between identity and $[H^3_K]_N(x)$ in the first three coordinates and in the last coordinate we squeeze the slab of thickness 1 to slab of thickness $\frac{n}{13}$. Finally for $x_4 \in [-14, -2] \cup [2, 14]$ we have identity in the first three coordinates and in the last coordinate we squeeze the slab of thickness 12 to slab of thickness $\frac{12n}{13}$.

On other parts of the boundary we do a similar interpolation between values on $\partial[-2m - 1, 2m + 1]^3 \times [-1, 1]$ and the identity on the outer boundary: Assume that there is exactly one index $i = 1, 2, 3$ such that $y_i \in [-2m - 5, -2m - 1] \cup [2m + 1, 2m + 5]$ and $y_j \in [-2m - 1, 2m + 1]$ for other $j \in \{1, 2, 3\} \setminus \{i\}$. Then we put $x_i = y_i$ and $n_i = 0$, for $j \in \{1, 2, 3\} \setminus \{i\}$ we find $x_j \in [-1, 1]$ so that $y_j = x_j + n_j$ and $x_4 \in [-1, 1]$. We define

$$G_{K,N,m,n}(y) = G_{K,N,m,n}(x + n) = \begin{cases} x + n & x_i \in [2m + 5, 2m + 2], \\ n + x_ie_i + (x_i - 2m - 1)(x - x_ie_i) & n + x_i e_i + (x_i - 2m - 1)(x - x_i e_i) + (2m + 2 - x_i) [H^3_K]_N(x) & x_i \in [2m + 1, 2m + 2], \\ x + n & x_i \in [-2m - 5, -2m - 2]. \end{cases}$$

(3.18)

In the remaining part around the edges of our block $[-2m - 5, 2m + 5]^3 \times [-14, 14]$ we just do a simple linear squeezing. That is for $y \in [-2m - 5, 2m + 5]^3 \times [-14, 14]$ so that $y \notin [-2m - 1, 2m + 1]^3 \times [-14, 14]$ and $y \notin [2m - 5, 2m + 5] \times [-2m - 1, 2m + 1] \times [-1, 1]$ we define

$$G_{K,N,m,n}^j(y) = y_j \text{ for } j = 1, 2, 3$$

(3.19)

and in the last coordinate we linearly stretch

$$G_{K,N,m,n}^4(y) = \begin{cases} y_4 & y_4 \in [-1, 1] \\ 1 + \frac{n}{13}(y_4 - 1) & y_4 \in [1, 14], \\ -1 + \frac{n}{13}(y_4 + 1) & y_4 \in [-14, -1]. \end{cases}$$

(3.20)

Similarly, we define it for the permutation of the first three coordinates, that is, $y \notin [-2m - 1, 2m + 1]^3 \times [-14, 14]$ and $(y \notin [2m - 1, 2m + 1] \times [2m - 5, 2m + 5] \times [-2m - 1, 2m + 1] \times [-1, 1] \text{ or } y \notin [-2m - 1, 2m + 1]^2 \times [2m - 5, 2m + 5] \times [-1, 1])$.

3.6 A specific homeomorphism that maps $C_{A,K}(4)$ onto $C_B(4)$

dependent on a parameter $N$

Now we define a map that squeezes the Cantor set; its scheme is in Figure 4. Let $\eta > 0$. The map this section is dedicated to is called $\tilde{G}_{K,N,m,n}$ (of course it depends on $\alpha$ and $\beta$) that maps $C_{A,K}$ onto $C_B$. In order to do so, we use mappings of type $[J^3_K]_m$ on three-dimensional hyperplanes per-
A scheme of the definition of $\tilde{G}_{K,N,m,\eta}$. The gray-shaded area is where we use the mapping $\tilde{G}_{K,4k+2N,4k+4+2N,k}$. In the preimage the blue area is $[-2m-5,-2m-1] \times \tilde{U}^{3}_{K,2N}$, respectively, $(\tilde{U}^{3}_{K,2N} + n) \times [1,1+\eta]$. The top and bottom (red) parts are stretched from size $\eta/2$ to size $13-\eta/2$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{A detail of the image near the boundary at the top of $R_{m,13}$.}
\end{figure}

We call
\[ J^{3,i}_{K}(x) = T_{i}(J^{3}_{K}(T_{i}(x))). \] (3.21)

Of course we also define
\[ [J^{3,i}_{K}]_{m}(x) = T_{i}([J^{3}_{K}]_{m}(T_{i}(x))). \] (3.22)

In order to define $G_{K,N,m,\eta}$ we define the continuous function
\[
\tilde{\xi}_{K,k}(s) = \begin{cases} 
1 & s \in \tilde{U}^{1}_{K,k+1} \\
0 & s \in \mathbb{R} \setminus \tilde{U}^{1}_{K,k} \\
\text{linear on intervals in} & \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1}. 
\end{cases}
\]

We define a mapping $\tilde{G}_{K,l_{1},l_{2},k}$, which for each $w \in \mathbb{R}^{k+1}$ is defined on the part of $\tilde{Q}^{1}_{K,w} \setminus \tilde{Q}_{K,w}$ farthest from $\tilde{z}_{K,w}$ in the direction $e_{i}$ by the expression
\[ \tilde{G}_{K,l_{1},l_{2},k}(x) := \tilde{\xi}_{K,k}(x_{i})[J^{3,i}_{K}]_{l_{1}}(x) + [1 - \tilde{\xi}_{K,k}(x_{i})][J^{3,i}_{K}]_{l_{2}}(x) + q_{K}(x_{i})e_{i}. \] (3.23)

We claim that for any $l_{2} > l_{1} \geq k$, the mapping $\tilde{G}_{K,l_{1},l_{2},k}$ is a homeomorphism sending $\tilde{U}^{4}_{K,k} \setminus \tilde{U}^{4}_{K,k+1}$ onto $U^{4}_{k} \setminus U^{4}_{k+1}$. This is clear from the fact that $\tilde{G}_{K,l_{1},l_{2},k} = G^{-1}_{K,l_{1},l_{2},k}$ and this is a homeomorphism by Section 3.5.
From the reasoning of the previous paragraph we have that

\[ \tilde{G}_{K,4k+2N,4k+4+2N,k} \text{ is a homeomorphism.} \]  

(3.24)

Recall that the parameter \( N \) was fixed in the previous subsection.

Call \( \eta > 0 \) and \( n = (n_1, n_2, n_3, 0) \), where \( n_1, n_2, n_3 \) are even numbers between \(-2m\) and \(2m\). We define \( \tilde{G}_{K,N,m,\eta} \) on \([-2m-1, 2m+1]^3 \times [-1,1] \) as follows; when \( y = x + n \ x \in Q(0,1) \),

\[
\tilde{G}_{K,N,m,\eta}(y) = \tilde{G}_{K,N,m,\eta}(x + n) = \begin{cases}
\tilde{G}_{K,4k+2N,4k+4+2N,k}(x) + n & x \in \tilde{U}_{K,k}^4 \setminus \tilde{U}_{K,k+1}^4, \\
(q_K(x_1), q_K(x_2), q_K(x_3), q_K(x_4)) + n & x \in C_{A,K}.
\end{cases}
\]  

(3.25)

Note that the last line together with (3.16) and \( q_K = (t_K)^{-1} \) imply that

\[ G_{K,N,m,\eta} \circ G_{K,N,m,\eta}(x) = x \text{ on } C_{A,K,m}. \]

When \( y = x + n \), \( x_1, x_2, x_3 \in [-1,1] \), and \( x_4 \in [-1 - \frac{\eta}{2}, -1] \cup [1, 1 + \frac{\eta}{2}] \) we put

\[
G_{K,N,m,\eta}(y) = G_{K,N,m,\eta}(x + n) = \begin{cases}
[J^{3,4}_K]_{2N}(x) + n + \frac{26-\eta}{\eta}(x_4 - 1)e_4 + e_4 & x_4 \in [1, 1 + \frac{\eta}{2}], \\
[J^{3,4}_K]_{2N}(x) + n + \frac{26+\eta}{\eta}(x_4 + 1)e_4 - e_4 & x_4 \in [-1 - \frac{\eta}{2}, -1].
\end{cases}
\]  

(3.26)

It is easy to see from (3.25) and (3.26) that those mapping agree on the boundary \([-2m-1, 2m+1]^3 \times \{-1,1\} \), that is, they are both equal to the shifted \([J^{3,4}_K]_{2N}(x) \). Moreover, the definition (3.26) gives us that we stretch a slab of thickness \( \frac{\eta}{2} \) onto slab of thickness \( 13 - \frac{\eta}{2} \) in the \( x_4 \)-coordinate (see Figure 4). Further when \( y = x + n \), \( x_1, x_2, x_3 \in [-1,1] \), and \( x_4 \in [-1 - \eta, -1 - \frac{\eta}{2}] \cup [1 + \frac{\eta}{2}, 1 + \eta] \) we define

\[
\tilde{G}_{K,N,m,\eta}(y) = \tilde{G}_{K,N,m,\eta}(x + n) = \begin{cases}
n + (x_4 + 13 - \eta)e_4 & x_4 \in [1 + \frac{\eta}{2}, 1 + \eta], \\
 + \frac{\eta}{2}(x_4 - 1 - \frac{\eta}{2})(x_1, x_2, x_3, 0) & x_4 \in [1 + \frac{\eta}{2}, 1 + \eta], \\
 + \frac{\eta}{2}(1 + \eta - x_4)[J^{3,4}_K]_{2N}(x) & x_4 \in [1 + \frac{\eta}{2}, 1 + \eta], \\
n + (x_4 - 13 + \eta)e_4 & x_4 \in [-1 - \eta, -1 - \frac{\eta}{2}], \\
 + \frac{\eta}{2}(-1 - \frac{\eta}{2} - x_4)(x_1, x_2, x_3, 0) & x_4 \in [-1 - \eta, -1 - \frac{\eta}{2}], \\
 + \frac{\eta}{2}(1 + \eta + x_4)[J^{3,4}_K]_{2N}(x) & x_4 \in [-1 - \eta, -1 - \frac{\eta}{2}].
\end{cases}
\]  

(3.27)

If fact the definition (3.27) is just a linear interpolation between \([J^{3,4}_K]_{2N}(x) \) on \([-2m-1, 2m+1]^3 \times \{1 + \frac{\eta}{2}\} \) and identity on \([-2m-1, 2m+1]^3 \times \{1 + \eta\} \) (see blue parts in Figure 4).

On other parts of the boundary we do a similar interpolation between values on \( \delta([-2m-1, 2m+1]^3 \times [-1,1] \) and the identity on the outer boundary: When there is exactly one index
\( i = 1, 2, 3 \) such that \( y_i \in [-2m - 5, -2m - 1] \cup [2m + 1, 2m + 5] \) (then put \( x_i = y_i \) and \( n_i = 0 \)) but for \( j \neq i \) \( y_j = x_j + n_j \) and \( x_4 \in [-1, 1] \) we put

\[
\tilde{G}_{K,N,m,\eta}(y) = \tilde{G}_{K,N,m,\eta}(x + n)
\]

\[
= \begin{cases} 
\left[J_{K}^{3,1}\right]_{2N}(x) + n + x_i e_i & x_i \in [2m + 1, 2m + 4], \\
(2m + 5 - x_i)\left[J_{K}^{3,1}\right]_{2N}(x) & x_i \in [2m + 4, 2m + 5], \\
\left[J_{K}^{3,1}\right]_{2N}(x) + n + x_i e_i & x_i \in [-2m - 4, -2m - 1], \\
(2m + 5 + x_i)\left[J_{K}^{3,1}\right]_{2N}(x) & x_i \in [-2m - 5, -2m - 4].
\end{cases}
\tag{3.28}
\]

When there are at least two indices \( i = 1, 2, 3 \) such that \( y_i \in [-2m - 1 - \eta, -2m - 1] \cup [2m + 1, 2m + 1 + \eta] \) (or at least one index \( i = 1, 2, 3 \) such that \( y_i \in [-2m - 1 - \eta, -2m - 1] \cup [2m + 1, 2m + 1 + \eta] \) and \( y_4 \in [-1 - \eta, -1] \cup [1, 1 + \eta] \)) we define the \( j \)th coordinate function of \( \tilde{G}_{N,K,m,\eta} \) as follows:

\[
\tilde{G}_{j,N,m,\eta}(\tilde{y}) = y_j \quad \text{for} \quad j = 1, 2, 3, \tag{3.29}
\]

and we just linearly stretch the corners as

\[
\tilde{G}_{4,N,m,\eta}(y) = \begin{cases} 
y_4 & y_4 \in [-1, 1], \\
1 + \frac{26 - \eta}{\eta}(y_4 - 1) & y_4 \in [1, 1 + \frac{\eta}{2}], \\
y_4 + 13 - \eta & y_4 \in [1 + \frac{\eta}{2}, 1 + \eta], \\
-1 + \frac{26 - \eta}{\eta}(y_4 - 1) & y_4 \in [-1 - \frac{\eta}{2}, -1], \\
y_4 - 13 + \eta & y_4 \in [-1 - \eta, -1 - \frac{\eta}{2}].
\end{cases} \tag{3.30}
\]

## 4 \quad A MAPPING EQUALING A REFLECTION ON \( C_B \) WITH IDENTITY ON THE BOUNDARY

Recall that \( R_{m,\eta} = [-2m - 5, 2m + 5]^3 \times [-1 - \eta, 1 + \eta] \). In Section 3, we define maps \( \tilde{G}_{K,N,m,\eta} \) and \( G_{K,N,m,\eta} \) such that

\[
G_{K,N,m,\eta} \circ \tilde{G}_{K,N,m,\eta}(x) = x \quad \text{on} \quad \partial R_{m,\eta} \quad \text{and on} \quad C_{A,K,m},
\]

\[
\tilde{G}_{K,N,m,\eta}(R_{m,\eta}) = R_{m,13} \quad \text{and} \quad G_{K,N,m,\eta}(R_{m,13}) = R_{m,\eta}.
\]

The main aim of this section is the construction of a mapping \( \tilde{F}_{\beta,m} \) from \( R_{m,13} \) onto \( R_{m,13} \) with \( \tilde{F}_{\beta,m}(x) = x \) on \( \partial R_{m,13} \) so that \( \tilde{F}_{\beta,m} \) behaves like a reflection \( x \mapsto [x_1, x_2, x_3, -x_4] \) on \( C_{B,m} = \bigcup_{n_1, n_2, n_3 = -m}^{m} C_B + (2n_1, 2n_2, 2n_3, 0) \). Then we have

\[
G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x) = x \quad \text{on} \quad \partial R_{m,\eta}
\]
and

\[ J_{G_{K,N,m,\eta}} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta} = -1 < 0 \text{ on } C_{A,K,m}. \]

The basic building block for the construction of $\hat{F}_{\beta,m}$ is the mapping $F_{\beta}$ defined in [6, Theorem 5.1], which is the subject of the following theorem. We define the set

\[ A_{i,k,l}(\beta) := \{ x \in \mathbb{R}^4 : x_i \in [-1,1] \setminus U_{k}^{1}, x_o \in U_{l}^{1} \text{ for all } o \in \{1,2,3,4\} \setminus \{i\} \}. \]

Also, by $\mathcal{K}_B$ we denote the set of lines intersecting $C_B$ parallel to coordinate axes, that is,

\[ \mathcal{K}_B = (C_B^1 \times C_B^1 \times C_B^1 \times [-1,1]) \cup (C_B^1 \times C_B^1 \times [-1,1] \times C_B^1) \cup ([-1,1] \times C_B^1 \times C_B^1 \times C_B^1). \]

**Theorem 4.1** [6, Theorem 5.1]. There is $\beta_0 > 0$ such that for all $\beta > \beta_0$ there exists a mapping $F_{\beta} : (-1,1)^4 \rightarrow (-1,1)^4$, which is a sense-preserving bi-Lipschitz extension of the map

\[ F_{\beta}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \quad x \in \mathcal{K}_B. \] (4.1)

Moreover, there exists a constant $M_0 \in \mathbb{N}$ such that for each $k, l \in \mathbb{N}$ satisfying $M_0 < k \leq l$ and for every line parallel to $e_i, L$, we have that $F(L \cap A_{i,k-M_0-1,l+M_0}(\beta))$ is a line segment parallel to $e_i$ which lies in the set $A_{i,k-1,l}(\beta)$. Moreover, the derivative along $L$ satisfies

\[ D_i F_{\beta}(x) = \begin{cases} e_i & \text{if } i = 1,2,3 \\ -e_i & \text{if } i = 4. \end{cases} \] (4.2)

for every $x \in L \cap A_{i,k-M_0-1,l+M_0}(\beta)$.

To construct $\hat{F}_{\beta,m}$ we extend $F_{\beta}$ periodically and then tweak it to get identity on the boundary $\partial R_{m,13}$. Moreover, we slightly extend its behavior from $Z_m$ to its neighborhood so that we get the key identity (4.5) not only on $\mathcal{K}_B$ but also on lines through the Cantor set in $[-2m-3,2m+3]^3 \times [-3,3]$. We call

\[ \mathcal{K}_{B,m} = \bigcup_{n_1,n_2,n_3=-m}^{m} (C_B + (2n_1, 2n_2, 2n_3, 0)) + \bigcup_{i=1}^{4} \mathbb{R} e_i. \] (4.3)

**Theorem 4.2.** Let $F_{\beta}$ be the map from Theorem 4.1. There exists a bi-Lipschitz map $\hat{F}_{\beta,m} : R_{m,13} \rightarrow R_{m,13}$, such that

\[ \hat{F}_{\beta,m}(x + n) = F_{\beta}(x) + n \quad \text{for } x \in Q(0,1) \text{ and } \]

(4.4)

for $n = (2n_1, 2n_2, 2n_3, 0), n_1, n_2, n_3 \in \{-m, ..., m\}$. Moreover, $\hat{F}_{\beta,m}(x) = x$ on $\partial R_{m,13}$. Further

\[ \hat{F}_{\beta,m}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \] (4.5)
on \( K_{B,m} \cap [-2m - 3, 2m + 3] \times [-3, 3] \). Also
\[
D_1 \hat{F}_{\beta,m}(x + (0, 2n_2, 2n_3, 0)) = e_1 \text{ for } x \in ([-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times U^3_1, \tag{4.6}
\]
\[
D_2 \hat{F}_{\beta,m}(x + (2n_1, 0, 2n_3, 0)) = e_2 \text{ for } x \in U^1_1 \times ([-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times U^2_1, \tag{4.7}
\]
\[
D_3 \hat{F}_{\beta,m}(x + (2n_1, 2n_2, 0, 0)) = e_3 \text{ for } x \in U^2_1 \times ([-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times U^1_1 \tag{4.8}
\]
and
\[
D_4 \hat{F}_{\beta,m}(x + (2n_1, 2n_2, 2n_3, 0)) = -e_4 \text{ for } x \in U^3_2 \times ([-3, 3] \setminus [-1, 1]). \tag{4.9}
\]

Let us first prove the following estimate, which is a simple result of the bi-Lipschitz nature of \( F_\beta \) and (4.2).

**Proposition 4.3.** There is \( M \in \mathbb{N}, M \geq M_0 \), such that the mapping \( F_\beta \) from Theorem 4.1 maps \([-1, 1] \times \left[ U^3_l \setminus U^3_{l+1} \right] \) into \([-1 - C(\beta) r_l, 1 + C(\beta) r_l] \times \left[ U^3_{l-M} \setminus U^3_{l+M+1} \right] \) for every \( l \in \mathbb{N} \) and similarly for all permutations of the coordinates. Likewise for every \( k, l \in \mathbb{N} \) our \( F_\beta \) maps \( (U^k_l \setminus U^k_{l+1}) \times \left[ U^3_l \setminus U^3_{l+1} \right] \) into \( (U^k_{l-M} \setminus U^k_{l+M+1}) \times \left[ U^3_{l-M} \setminus U^3_{l+M+1} \right] \) and maps \( U^4_l \setminus U^4_{l+1} \) into \( U^4_{l-M} \setminus U^4_{l+M+1} \).

**Proof.** Denote by \( C_\beta(3) \) the Cantor set constructed in \( \mathbb{R}^3 \). From (4.1) we have \( F_\beta(x) = (x_1, x_2, x_3, -x_4) \) on \([-1, 1] \times C_\beta(3) \) and \( F_\beta \) is bi-Lipschitz. Hence, the neighborhoods of \([-1, 1] \times C_\beta(3) \) are mapped onto neighborhoods of \([-1, 1] \times C_{B,K}(3) \) (and similarly for all permutations of the coordinates). Now \( U^3_l \) is a neighborhood of \( C_\beta(3) \), further there exists a \( C_1 \) such that for each \( l \) we have that
\[
[-1, 1] \times [C_\beta(3) + B_3(0, C^{-1}_1 2^{-l(\beta+1)})] \subset [-1, 1] \times [U^3_l] \text{ and }
\]
\[
[-1, 1] \times [U^3_l] \subset [-1, 1] \times [C_\beta(3) + B_3(0, C_1 2^{-l(\beta+1)})].
\]

Then call \( C \) the bi-Lipschitz constant of \( F_\beta \) and choose \( M \) so that \( CC^2_1 < 2^{M(\beta+1)} \). Further call \( \delta_l = CC_1 2^{-l(\beta+1)} \). We have
\[
[-1 + \delta_l, 1 - \delta_l] \times [U^3_{l+M}] \subset [-1 + \delta_l, 1 - \delta_l] \times [C_\beta(3) + B_3(0, C_1 2^{-(l-M)(\beta+1)})]
\]
\[
\subset [-1 + \delta_l, 1 - \delta_l] \times [C_\beta(3) + B_3(0, C^{-1}_1 2^{-l(\beta+1)})]
\]
\[
\subset F_\beta([-1, 1] \times U^3_l)
\]
\[
\subset [-1 - \delta_l, 1 + \delta_l] \times [C_\beta(3) + B_3(0, CC_1 2^{-l(\beta+1)})]
\]
\[
\subset [-1 - \delta_l, 1 + \delta_l] \times [C_\beta(3) + B_3(0, C^{-1}_1 2^{-(l-M)(\beta+1)})]
\]
\[
\subset [-1 - \delta_l, 1 + \delta_l] \times [U^3_{l-M}].
\]

This fact yields the claim immediately since clearly the choice of \( M \) does not depend on \( l \).
The argument that $F_\beta$ maps $U^4_i \setminus U^4_{i+1}$ into $U^4_{i-M} \setminus U^4_{i+M+1}$ is similar. The map $F_\beta$ sends $C_B$ onto $C_B$ and $U^4_i$ are neighborhoods of $C_B$. The rest of the argument remains analogous to the above. □

Let us recall some useful notation and results from [6, Section 4]. Let $t \in (0,1]$ and let $v = \left(\frac{t}{4}, \frac{t}{8}, \frac{t}{16}, 1\right)$ be vector. We define a projection $P_v : \mathbb{R}^4 \to \mathbb{R}^3 \times \{0\}$ in the direction of $-v$ as follows:

$$P_v(x) = x - \frac{x_4}{v_4}v.$$  

It was shown in [6, Lemma 4.2] that for $t = 1$ the mapping $P_v(x)$ is one to one on the Cantor set $C_B$ and thus the whole construction of $F_\beta$ from Theorem 4.1 is possible. It follows from the proof in [6, Theorem 5.1] that there are many admissible choices of $v$ and specifically any of the vectors $\left(\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 1\right)$ for $t \in (0,1]$ has the same properties once we choose $\beta$ sufficiently large. Thus we may assume that $\max\{|v_1|, |v_2|, |v_3|\}$ is as small as we like. During the proof of Theorem 4.2, we fix one single vector $v = v_t$, for $t \leq 1$ and work with that vector.

**Definition 4.4.** Given a Lipschitz function $g : \mathbb{R}^3 \times \{0\} \to [-3,3]$ we define the spaghetti strand map $F_{g,v} : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$F_{g,v}(x) = x + g(P_v(x))v.$$  

Spaghetti strand mappings defined above take a line through $y \in \mathbb{R}^3 \times \{0\}$ in the direction $v$ and move it in the direction of $v$ by the amount $g(y)$.

The equality $D_i F_\beta(x) = \pm e_i$ from (4.2) was attained as follows. A Lipschitz function $g$ was defined on the hyperplane $\mathbb{R}^3 \times \{0\}$. This function was constant on lines parallel to $e_i$, $i = 1, 2, 3$ and linear on lines parallel to $P_v(\mathbb{R}e_4)$ (see the proof of Lemma 5.7, specifically [6, Steps 3–5, Equations (5.27) and (5.30), pp. 789–796]). The function allows a Lipschitz periodic extension which means we are able to prove the translation Equation (4.4).

**Proof of Theorem 4.2.** Let us recall that $\beta$ and $r_k$ are from the construction of $C_B$ (3.8). We assume that $\beta > \max\{6, \beta_0\}$ where $\beta_0$ comes from Theorem 4.1.

**Step 1. Fixing a vector $v$.**

Now we prove that we can choose $t$ so that for $v = \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 1\right)$ and for each $v \in \mathbb{V}^1_4$ we have

$$P_v(Q(z_v, r_1) + [-3,3]e_4) \subset Q_3(0, \frac{3}{4}) \times \{0\}. \quad (4.11)$$

Since we have $P_v(z_v) \to (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0)$ as $t \to 0^+$, $r_1 = 2^{1-\beta} < 2^{-7}$ (because $\beta > 6$) and $\text{diam}(P_v([-3,3]e_4)) \to 0$ as $t \to 0^+$ this is clearly possible. We choose and fix this $v$ with $t \leq 1$ so that $\max\{|v_1|, |v_2|, |v_3|\} \leq \frac{1}{14}$ and (4.11) holds.

**Step 2. The periodic extension of $F_\beta$ on $[-2m - 1, 2m + 1]^3 \times [-3,3]$.**

It was shown in [6, Section 4 and Lemma 5.7] that for $\beta >> 1$ large enough we have that

$$P_v \text{ is one to one on } \left(C_B + \bigcup_{i=1}^{4} \mathbb{R}e_i\right).$$
It follows rather quickly from the fact that \( P_v(C_B) \subset [-1, 1]^3 \times \{0\} \) that \( P_v \) is one to one on \( C_{B,m} \).

By symmetry therefore \( P_v \) is one to one on \( K_{B,m} \). Thus, in the following we assume that \( \beta \) is fixed and that \( P_v \) is one to one on \( K_{B,m} \). Thus, \( v = v(t) \) and \( \beta \) are absolutely fixed geometrical constants which do not change at all during the calculations. In doing so we have fixed the bi-Lipschitz constant of \( F_\beta \) and the constant \( M \) from Proposition 4.3. None of these constants depend on \( m \).

From [6, Section 4] we know that

\[ g \text{ is Lipschitz } \Rightarrow F_{g,v} \text{ is bi-Lipschitz.} \]

Moreover, let \( u = [-v_1, -v_2, -v_3, v_4] \) and let \( g : \mathbb{R}^3 \times \{0\} \to [-3, 3] \) be a Lipschitz function satisfying

\[ g(P_v(x)) = -x_4 \text{ for every } x \in K_{B,m} \cap \left((-2m - 3, 2m + 3)^3 \times [-3, 3]\right). \tag{4.12} \]

This \( g \) was constructed in [6, Lemma 5.7] for a single \( C_B + \bigcup_{i=1}^4 \mathbb{R}e_i \). Also by a simple cut-off argument we may assume that

\[ -3 \leq \min_{x \in \mathbb{R}^3 \times \{0\}} g(x) \leq \max_{x \in \mathbb{R}^3 \times \{0\}} g(x) \leq 3. \tag{4.13} \]

In order to extend \( g \) periodically by \( g(x + n) = g(x) \) for all \( n = (2n_1, 2n_2, 2n_3) \) with \(-m \leq n_1, n_2, n_3 \leq m \) we need that

\[ g(x) = g(x \pm 2e_i) \text{ for all } x \text{ such that } x, x + 2e_i \in \partial_3 Q_3(0, 1) \times \{0\} \tag{4.14} \]

for each \( i = 1, 2, 3 \). This was not proved explicitly in the original theorem, however \( g \) was defined so that

\[ g(P_v(x)) = -x_4 \text{ on } C_B + \bigcup_{i=1}^4 \mathbb{R}e_i. \tag{4.15} \]

Therefore \( g \) is constant on lines parallel to the \( x_1, x_2 \) and \( x_3 \) axis intersecting \( C_B \) and so (4.14) holds at the intersection of these lines with \( \partial_3 Q_3(0, 1) \times \{0\} \) (in fact \( g \) was defined so that (4.14) holds not only on these lines but also on their neighborhoods). The question of the value of \( g \) on the projection of lines in direction \( e_4 \) is not an issue of the values of \( g \) on \( \partial_3 Q_3(0, 1) \) because by (4.11) we have

\[ P_v \left( (C_B + \mathbb{R}e_4) \cap (-1, 1]^3 \times [-3, 3]) \right) \subset Q_3(0, \frac{3}{4}) \]

and therefore the projection of \( (C_B + \mathbb{R}e_4 + n) \cap (n + [-1, 1]^3 \times [-3, 3]) \) is a subset of \( Q_3((2n_1, 2n_2, 2n_3), \frac{3}{4}) \) and so far away from \( \partial_3 Q_3((2n_1, 2n_2, 2n_3), 1) \). Since the exact values of \( g \) are important only close to \( K_{B,m} \) (and that only while the values are in \([-3, 3] \); see (4.13)) and away from this set we define \( g \) by an arbitrary Lipschitz extension, we may assume that \( g \) has been extended to satisfy (4.14).

Therefore, we define \( g(x + n) = g(x) \) for \( x \in [-1, 1]^3 \times \{0\} \) and \( n = (2n_1, 2n_2, 2n_3, 0) \) where \( n_1, n_2, n_3 \in \{-m, \ldots, m\} \) and we have that \( g \) is Lipschitz and defined on a subset of \([-2m - 2, 2m + 2]^3 \times \{0\} \).
This last fact together with (4.15) means we have \( g(P_v(x)) = -x_4 \) for all \( x \in [-2m - 1, 2m + 1] \times C_{B,m}(3) \), all \( x \in C_{B,m}(1) \times [-2m - 1, 2m + 1] \times C_{B,m}(2) \) and all \( x \in C_{B,m}(2) \times [-2m - 1, 2m + 1] \times C_{B,m}(1) \). By extending \( g \) constant on lines a little further we easily get

\[
g(P_v(x)) = -x_4 \quad \text{for all } x \in ([-2m - 3, 2m + 3] \times C_{B,m}(3)) \cup (C_{B,m}(1) \times [-2m - 3, 2m + 3] \times C_{B,m}(2)) \cup (C_{B,m}(2) \times [-2m - 3, 2m + 3] \times C_{B,m}(1))
\]  

(4.16)

and (because \( \max\{|v_1|, |v_2|, |v_3|\} \leq \frac{1}{14} \) ) the projection of that set is a subset of \([-2m - 3 - \frac{1}{14}, 2m + 3 + \frac{1}{14}]^3 \times \{0\} \).

We extend our \( g \) in a way that it is constant not only on lines through the Cantor set but also on their neighborhoods, that is, lines parallel to \( e_1 \) lying inside the set \((-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times U_1^3\). It follows that \( g(P_v(x)) = g(P_v(x + te_1)) \) there. Note that this leads to a correct definition since it is easy to prove (see [6, Proposition 5.3]) that for each distinct pair of cubes \( Q_v \) and \( Q_w \) in \( U_1^3 \) we have that

\[
P_v(([-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times Q_v) \cap P_v(([-2m - 3, -2m - 1] \cup [2m + 1, 2m + 3]) \times Q_w) = \emptyset.
\]

Because \( g \) is defined constant on these lines parallel to \( e_1 \) (especially \( g(P_v(x)) = g(P_v(x + te_1)) \)) we have

\[
F_{g,v}(x + te_1) = x + te_1 + v g(P_v(x + te_1)) = x + te_1 + v g(P_v(x)) = F_{g,v}(x) + te_1
\]

and further, because the similar identity holds for \( F_{g,u} \), we have

\[
\hat{F}_{\hat{g},m}(x + (0, 2n_2, 2n_3, 0) + te_1) = F_{g,u}(F_{g,v}(x + (0, 2n_2, 2n_3, 0) + te_1))
\]

\[
= F_{g,u}(F_{g,v}(x + (0, 2n_2, 2n_3, 0))) + te_1,
\]

when we put (same as in [6, proof of Theorem 5.1])

\[
\hat{F}_{\hat{g},m}(x) = F_{g,u} \circ F_{g,v} : \mathbb{R}^4 \rightarrow \mathbb{R}^4.
\]

(4.17)

Therefore, we get

\[
D_1 \hat{F}_{\hat{g},m}(x + (0, 2n_2, 2n_3, 0)) = e_1
\]

which is precisely (4.6). We get Equations (4.7) and (4.8) the same way; the only difference is a permutation of the coordinates.

Recall that the condition \( g(P_v(x)) = -x_4 \) from (4.16) is precisely the condition that guarantees (see [6, Lemma 4.5])

\[
\hat{F}_{\hat{g},m}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)
\]

(4.18)

on the set in (4.16).
In [6, Step 4 in the proof of Lemma 5.7] the function $g$ was defined linear on lines parallel to $P_v(e_4)$ lying inside $P_v(U^3_{k+2} \times \left([-1, 1 \ \setminus \ U_k]\right))$. Since the set $P_v(U^3_{k} \times [-3, 3]) \subset Q_5(0, \frac{3}{4})$ we can define $g$ linear on lines parallel to $P_v(e_4)$ in $P_v(U^3_{2} \times \left([-3,3] \ \setminus \ [-1,1]\right))$. This means that for all $x \in U^3_{2} \times \left([-3,3] \ \setminus \ [-1,1]\right)$ we have

$$D_4\hat{F}_{\beta,m}(x + (2n_1, 2n_2, 2n_3, 0)) = -e_4,$$

which is (4.9). Especially, since $g(P_v(x)) = -x_4$ for all $x \in C^0_B \times [-3,3]$ we can combine with (4.18) to get (4.5). Therefore, in the following we may assume that $g$ is Lipschitz and defined on $[-2m - 3 - \frac{1}{14}, 2m + 3 + \frac{1}{14}] \times \{0\}$.

Then the map $\hat{F}_{\beta,m}$ satisfies (see [6, Lemma 4.5])

$$\hat{F}_{\beta,m}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \quad \text{for} \quad x \in K_{B,m} \cap [-2m - 3, 2m + 3] \times [-3,3].$$

Thus, we see that $\hat{F}_{\beta,m}(x + n) = \hat{F}_{\beta,m}(x) + n$, which is a bi-Lipschitz map because $g$ is a Lipschitz function.

**Step 3. Obtaining identity on the boundary.**

We have defined $\hat{F}_{\beta,m}$ on $[-2m - 3, 2m + 3] \times [-3,3]$. Now we want to define $\hat{F}_{\beta,m}$ on $R_{m,13} \setminus [-2m - 3, 2m + 3] \times [-3,3]$ so that $\hat{F}_{\beta,m}(x) = x$ on $\partial R_{m,13}$. Since by Step 1 we have that $\max\{|v_1|, |v_2|, |v_3|\} \leq \frac{1}{14}$ it is easy to check that

$$P_v([-2m - 3, 2m + 3] \times [-3,3] \subset [2m - 3 - \frac{3}{14}, 2m + 3 + \frac{3}{14}] \times \{0\})$$

and hence $F_{g,u}(x) \in [-2m - 3 - \frac{3}{14}, 2m + 3 + \frac{3}{14}] \times [-6,6]$ and

$$P_u([-2m - 3 - \frac{3}{14}, 2m + 3 + \frac{3}{14}] \times [-6,6]) \subset [2m - 3 - \frac{9}{14}, 2m + 3 + \frac{9}{14}] \times \{0\}.$$ 

Thus we can define

$$g \equiv 0 \text{ on } (\mathbb{R}^3 \ \setminus \ [-2m - 3 - \frac{12}{14}, 2m + 3 + \frac{12}{14}] \times \{0\})$$

without altering the behavior of $\hat{F}_{\beta,m}$ on $[-2m - 3, 2m + 3] \times [-3,3]$. Clearly, there exists a Lipschitz extension of $g$ satisfying the above and (4.12). Now, since $g = 0$ on $([-2m - 5, 2m + 5] \setminus [-2m - 4, 2m + 4]) \times \{0\}$ and $-14 \leq x_4 \leq 14$ and $\max\{|v_1|, |v_2|, |v_3|\} \leq \frac{1}{14}$ we have that the image in $P_v$ of the hyperplane parts of $\partial R_{m,13}$ which correspond to $x_i = \pm(2m + 5)$, $i = 1, 2, 3$ belong in $(\hat{\partial}_3[-2m - 5, 2m + 5] \setminus B_3(0, 1)) \times \{0\}$ and on this set we have $g = 0$. The same holds for $P_u$. But by the definition of $\hat{F}_{\beta,m}$, the fact that $g = 0$ on $P_v(x)$ and $g = 0$ on $P_u(x)$ means that (see (4.4)) on those parts of $\partial R_{m,13}$

$$\hat{F}_{\beta,m}(x) = F_{g,u}(F_{g,u}(x)) = F_{g,u}(x - 0v) = x - 0v - 0u = x.$$

Now, to get identity on the remaining parts of $\partial R_{m,13}$ where $x_4 = \pm 14$ it suffices to alter the map $\hat{F}_{\beta,m}$ on $[-2m - 5, 2m + 5] \times ((-14, -3) \cup [3, 14])$. The spaghetti strand mappings of (4.10) moves the entire line parallel to $v$ intersecting point $(y, 0)$ by the vector $v g(y)$. Instead we define a ‘rubber band’ mapping, which moves only part of the line (see Figure 5) as

$$R_{g,u}(x) = x + r(x)g(P_v(x))v,$$

(4.21)
where $r$ is a fixed Lipschitz function with Lipschitz constant $\frac{1}{7}$ such that

$$
\begin{align*}
    r(x) &= \begin{cases} 
        0 & x_4 \geq 13, \\
        \frac{13-x_4}{7} & x_4 \in [6, 13], \\
        1 & x_4 \in [-6, 6], \\
        \frac{13+x_4}{7} & x_4 \in [-13, -6], \\
        0 & x_4 \leq -13.
    \end{cases}
\end{align*}
$$

Then it is easy to check that the fourth coordinate of $R_{g,v}(x)$, that is,

$$
    t \to t + r(t)g(y)
$$

is an increasing function for any $g(y) \in [-3, 3]$. (4.22)

Now we define $\tilde{F}_{\beta,m} := R_{g,u} \circ R_{g,v}$. Since $R_{g,v}(x) = x$ whenever $|x_4| \geq 13$ it is easy to check that $\tilde{F}_{\beta,m}(x) = x$ whenever $13 \leq |x_4| \leq 14$. Moreover, as $R_{g,u}(x) = F_{g,u}(x)$ whenever $|x_4| \leq 3$ it is not difficult to check that the new definition of $\tilde{F}_{\beta,m}$ using rubber band maps is equal to the definition of the previous definition of $\tilde{F}_{\beta,m}$ using spaghetti strand maps on $[-2m - 3, 2m + 3] \times [-3, 3]$.

Since $g$ is a Lipschitz map and since $r$ is Lipschitz with coefficient $\frac{1}{7}$ (thus giving us (4.22)) we see that $R_{g,u}$ and $R_{g,v}$ are bi-Lipschitz maps as were $F_{g,u}$ and $F_{g,v}$. Therefore $\tilde{F}_{\beta,m}$ is a bi-Lipschitz map, whose constant depends on the Lipschitz constant of the function $g$, which is fixed by the choice of $v$ and $\beta$.  

We include the following lemma about the image of certain points in $\tilde{F}_{\beta,m}$. Later, in the course of proving Theorem 3.1, it is necessary to apply derivative estimates of the composition $G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ G_{K,N,m,\eta}$. For certain points $x$ we will need to know that $|DG_{K,N,m,\eta}(\tilde{F}_{\beta,m} \circ G_{K,N,m,\eta}(x))|$ is not too large. We do this by proving that these points are not mapped by $\tilde{F}_{\beta,m} \circ G_{K,N,m,\eta}$ onto the set where the derivative of $G_{K,N,m,\eta}$ is large. This is the content of the following lemma.
Lemma 4.5. Let \( x \in G \) where

\[
G = R_{m,13} \setminus \left( \bigcup_{n_1,n_2,n_3=-m}^m \left( [-2m - 3, 2m + 3] \times U_1^3 + (0, 2n_2, 2n_3, 0) \right) \cup U_1^1 \times [-2m - 3, 2m + 3] \times U_1^2 + (2n_1, 0, 2n_3, 0) \cup U_1^2 \times [-2m - 3, 2m + 3] \times U_1^1 + (2n_1, 2n_2, 0, 0) \cup U_1^3 \times [-3, 3] + (2n_1, 2n_2, 2n_3, 0) \right),
\]

then

\[
\hat{F}_{\beta, m}(x) \in R_{m,13} \setminus T,
\]

where

\[
T = \bigcup_{n_1,n_2,n_3=-m}^m \left( [-2m - 2, 2m + 2] \times U_{M+1}^3 + (0, 2n_2, 2n_3, 0) \cup U_{M+1}^1 \times [-2m - 2, 2m + 2] \times U_{M+1}^2 + (2n_1, 0, 2n_3, 0) \cup U_{M+1}^2 \times [-2m - 2, 2m + 2] \times U_{M+1}^1 + (2n_1, 2n_2, 0, 0) \cup U_{M+1}^3 \times [-2, 2] + (2n_1, 2n_2, 2n_3, 0) \right).
\]

Proof. We may assume that \( M \) is so large that \( C(\beta)r_M = C(\beta)2^{-r(\beta+1)}k < r_1 = 2^{-\beta-1} \), where \( C(\beta) \) is the constant from Proposition 4.3. Nevertheless \( M \) is an absolute constant depending only on the construction of \( F_\beta \). Specifically it depends on \( \beta \) and \( \nu \), which are fixed and do not change at any point. Note that the Proposition 4.3 holds not only for \( F_\beta \) but also for \( F^{-1}_\beta \) since both these maps are bi-Lipschitz and send \( \mathcal{K}_B \) onto \( \mathcal{K}_B \). In that case we have

\[
F^{-1}_\beta([-1,1] \times U_{M+1}^3) \subset [-2,2] \times U_1^3
\]

and similarly for coordinates 2 and 3. Moreover, the fact that \( \hat{F}_{\beta, m} \) is constructed by translation of \( F_\beta \) on these sets we have the same for \( \hat{F}_{\beta, m} \), precisely that

\[
\hat{F}_{\beta, m}([-2m - 1, 2m + 1] \times U_{M+1}^3 + (0, 2n_2, 2n_3, 0)) \subset [-2m - 2, 2m + 2] \times U_1^3 + (0, 2n_2, 2n_3, 0)
\]

and similarly for the permutations of the coordinates. In fact, by the extension in (4.16), we have

\[
\hat{F}_{\beta, m}([-2m - 2, 2m + 2] \times U_{M+1}^3 + (0, 2n_2, 2n_3, 0)) \subset [-2m - 3, 2m + 3] \times U_1^3 + (0, 2n_2, 2n_3, 0)
\]

because \( \hat{F}_{\beta, m} \) sends \([-2m - 2, 2m + 2] \times C_{B,m}(3) \) onto \([-2m - 2, 2m + 2] \times C_{B,m}(3) \) and so the reasoning from Proposition 4.3 applies here too. Similar inclusions hold for the permutations of the coordinates 2 and 3.

The last necessary observation is that

\[
\hat{F}_{\beta, m}^{-1}(U_{M+1}^3 \times [-2,2] + (2n_1, 2n_2, 2n_3, 0)) \subset U_1^3 \times [-3,3] + (2n_1, 2n_2, 2n_3, 0)
\]
for each $n$. The reasoning here is essentially the same as before, and especially using the fact that $\hat{F}_{\beta,m}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$ on $C_{B,m}(3) \times [-3, 3]$.

This precisely means that if $x \in G$ then $x \notin R_{m,13} \setminus G \supset \hat{F}_{\beta,m}^{-1}(T)$ and therefore $\hat{F}_{\beta,m}(x) \notin T$, which was exactly our claim. □

5 ESTIMATES OF THE DERIVATIVE OF THE STRETCHING AND SQUEEZING MAPPING

Let us briefly recall the role of various constants involved in the estimates. The sequence $\tilde{r}_k(K)$ used in the definition of $C_{A,K}$ was defined in (3.4). Analogously we define $C_{B}$ with the help of sequence $r_k$ and parameter $\beta$ in (3.8). The parameter $N$ was introduced in the construction of $G_{K,N,m,\eta}$ in Section 3.5. Parameters $m$ and $\eta$ denote the size of the boxes (see (3.1) for the definition of $R_{m,\eta}$).

The following lemma on the size of the derivative of $J^n_K$, $H^n_K$, $[J^n_K]$ and $[H^n_K]$ (defined in Section 3.4) is standard and the proof can be found in [14, proof of Theorem 4.10] in combination with (3.7).

**Lemma 5.1.** Let $n \geq 2$, $k \in \mathbb{N}$, $l \geq k$, $\mathbf{v} \in \mathbb{V}^{k+1}$, let $x \in Q(\tilde{z}_{K,\mathbf{v}}, 1/2\tilde{r}_k(K)) \setminus Q(\tilde{z}_{K,\mathbf{v}}, \tilde{r}_{k+1}(K))$ be a point such that

$$|x_1 - (\tilde{z}_{K,\mathbf{v}})_1| > \max \{|x_2 - (\tilde{z}_{K,\mathbf{v}})_2|, |x_3 - (\tilde{z}_{K,\mathbf{v}})_3|, |x_4 - (\tilde{z}_{K,\mathbf{v}})_4|\}.$$  

Then it holds that

$$|D_1J^n_K(x)| = |D_1[J^n_K]_1(x)| \leq C2^{-\beta k}(k + K)^{\alpha + 1},$$  

(5.1)

further for $j \neq 1$

$$|D_jJ^n_K(x)| = |D_j[J^n_K]_1(x)| \leq C2^{-\beta k}$$  

(5.2)

and

$$|D[J^n_K]_k(y)| \leq C2^{-\beta k} \text{ for } y \in D^n_{K,k+1}.$$  

(5.3)

Now assume that $w \in Q(z_{\mathbf{v}}, 1/2r_k) \setminus Q(z_{\mathbf{v}}, r_{k+1})$ be a point† such that

$$|w_1 - (z_{\mathbf{v}})_1| > \max \{|w_2 - (z_{\mathbf{v}})_2|, |w_3 - (z_{\mathbf{v}})_3|, |w_4 - (z_{\mathbf{v}})_4|\}.$$  

Then it holds that

$$|D_1H^n_K(w)| = |D_1[H^n_K]_1(w)| \leq C2^{\beta k}(k + K)^{-\alpha - 1},$$  

(5.4)

further for $j \neq 1$

$$|D_jH^n_K(w)| = |D_j[H^n_K]_1(w)| \leq C2^{\beta k}$$  

(5.5)

† The condition on point $w$ is the same as saying that $w \in S_{\mathbf{v},1}$ from (3.14).
\[ |D_1 [H^n_K]_k(z)| \leq C 2^{\beta k} \text{ for } z \in U^n_{k+1}. \]  

(5.6)

5.1 \hspace{1cm} \textbf{Estimates on the derivative of } \tilde{G}_{K,N,m,\eta} \text{ defined in Section 3.6}

The following proposition gives an estimate on \(|D_1 \tilde{G}_{K,N,m,\eta}| \) at points which are very close to the Cantor set in coordinates \((x_2, x_3, x_4)\) compared to the \(x_1\) variable. Especially we use it to estimate the derivative of \(\tilde{G}_{K,N,m,\eta}\) along lines that go through \(C_{A,K}\).

\textbf{Proposition 5.2.} Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \(n = (n_1, n_2, n_3, 0)\) and let \(2N + 4k \leq l\) with \(k, l \in \mathbb{N}_0\). Let \(x\) be a point such that \(x_1 \in U^1_{K,k} \setminus U^1_{K,k+1}, (x_2, x_3, x_4) \in U^3_{k,l}\). Then the following estimates hold

\[ D_1 \tilde{G}^1_{K,N,m,\eta}(x + n) = \frac{1}{2^k r_k - r_{k+1}} \leq C 2^{-k \beta (K + k)^{\alpha+1}} \]  

(5.7)

and

\[ |D_1 \tilde{G}_{K,N,m,\eta}(x + n) - D_1 \tilde{G}^1_{K,N,m,\eta}(x + n)e_1| \leq C 2^{-4k \beta - 2N \beta (K + k)^{\alpha+1}}. \]  

(5.8)

Further it holds that

\[ \tilde{G}_{K,N,m,\eta}(x + n) - n \in [U^1_{k} \setminus U^1_{k+1}] \times U^3_{4k+2N}. \]  

(5.9)

The same holds for each rotation, that is, for \(D_2 \tilde{G}^2_{K,N,m,\eta}(x + n)\) when \(x_2 \in U^1_{K,k} \setminus U^1_{K,k+1}\) and \((x_1, x_3, x_4) \in U^3_{k,l}\) and similarly for coordinates 3 and 4.

\textbf{Proof.} The claim of rotational symmetry results directly from the definition of \(\tilde{G}_{K,N,m,\eta}(x + n)\) in (3.25) and (3.23). Clearly, we have \(k \leq l\) and so \(x \in U^4_{k,k} \setminus U^4_{k,k+1}\). Thus, there is exactly one \(v \in \mathbb{V}^{k+1}_4\) such that \(x \in Q(\bar{z}_{K,v}, \frac{1}{2} \bar{r}_{K}(K)) \setminus Q(\bar{z}_{K,v}, \bar{r}_{k+1}(K))\). Since \(k \leq l\) point \(x\) is farthest from \(\bar{z}_{K,v}\) in the \(x_1\) coordinate. Therefore we are on the set where (see (3.23) and (3.25))

\[ \tilde{G}_{K,N,m,\eta}(x + n) = \tilde{G}_{K,4k+2N,4k+4+2N,k}(x) + n \]

(5.10)

\[ = \tilde{\xi}_{K,k}(x_1)[J^{3,1}_K]_{4k+2N}(x) + q_K(x_1)e_1 \]

\[ + \left[1 - \tilde{\xi}_{K,k}(x_1)\right][J^{3,1}_K]_{4k+4+2N}(x) + n. \]

Both \([J^{3,1}_K]_{4k+2N}\) and \([J^{3,1}_K]_{4k+4+2N}\) are constant 0 in the \(x_1\) variable. Therefore

\[ D_1 \tilde{G}_{K,N,m,\eta}(x + n) = q'_K(x_1)e_1 + \tilde{\xi}'_{K,k}(x_1)\left([J^{3,1}_K]_{4k+2N}(x) - [J^{3,1}_K]_{4k+4+2N}(x)\right). \]  

(5.11)
On the other hand, by (3.10)

\[ q'_k(x_1) = \frac{1}{2} r_k - r_{k+1} \]

thus proving (5.7). Further, since \([J_{K}^{3,1}]\) maps each \(\tilde{Q}_{K,v}\) into corresponding \(Q_v\) (for each \(v \in \mathbb{V}_3^l\)) and \(\text{diam } Q_v < C2^{-l-l\beta}\) we have

\[
\|[J_{K}^{3,1}]_{4k+2N} - [J_{K}^{3,1}]_{4k+4+2N}\|_\infty \leq C2^{-k}2^{-4k\beta-2N\beta}. (5.12)
\]

On the other hand, we have that each interval in \(U_{K,k}^1 \setminus U_{K,k+1}^1\) has length smaller than \(C2^{-k}(K + k)^{-\alpha-1}\) by (3.7). Therefore, for \(x_1 \in U_{K,k}^1 \setminus U_{K,k+1}^1\)

\[
\tilde{\varphi}_{K,k}(x_1) \leq C2^k(K + k)^{\alpha+1}. (5.13)
\]

Now (5.8) follows immediately from (5.11), (5.12) and (5.13).

Now (5.10) implies (5.9) by noting that \([J_{K}^{3,1}]_{4k+2N}\) sends \([U_{K,k}^1 \setminus U_{K,k+1}^1] \times [U_{K,l}^3 \setminus U_{K,l+1}^3]\) onto \([0] \times U_l^3\). Of course \(q_K\) sends \([U_{K,k}^1 \setminus U_{K,k+1}^1]\) into \([U_k^1 \setminus U_{k+1}^1]\).

The following proposition gives an estimate on \(|D_1 \tilde{G}_{K,N,m,\eta}|\) at points which are closer to the Cantor set in coordinates \((x_2, x_3, x_4)\) than in the \(x_1 \in \tilde{U}_{K,k}^1 \setminus \tilde{U}_{K,k+1}^1\) variable on lines that do not intersect the Cantor set (that is, they go through \(\tilde{U}_{K,l}^3 \setminus \tilde{U}_{K,l+1}^3\)) as the distance from the Cantor set in the first variable decreases.

**Proposition 5.3.** Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \(n = (n_1, n_2, n_3, 0)\) and let \(k < l < 4k + 2N\) with \(k, l \in \mathbb{N}_0\). Let \(x\) be a point such that \(x_1 \in \tilde{U}_{K,k}^1 \setminus \tilde{U}_{K,k+1}^1\), \((x_2, x_3, x_4) \in \tilde{U}_{K,l}^3 \setminus \tilde{U}_{K,l+1}^3\). Then

\[
D_1 \tilde{G}_{K,N,m,\eta}(x + n) = \frac{1}{2} r_k - r_{k+1} \frac{1}{2} r_K(K) - r_{K+1}(K) e_1 (5.14)
\]

and

\[
G_{K,N,m,\eta}(x + n) - n \in [U_k^1 \setminus U_{k+1}^1] \times [U_l^3 \setminus U_{l+1}^3]. (5.15)
\]

The same holds for each rotation, that is, for \(D_2 \tilde{G}_{K,N,m,\eta}(x + n)\) when \(x_2 \in \tilde{U}_{K,k}^1 \setminus \tilde{U}_{K,k+1}^1\) and \((x_1, x_3, x_4) \in \tilde{U}_{K,l}^3 \setminus \tilde{U}_{K,l+1}^3\) and similarly for coordinates 3 and 4.

**Proof.** Again we have (5.10). Since \(l < 4k + 2N\) we have that (see (3.22), (3.12), (3.13) for the notation)

\[
[J_{K}^{3,1}]_{4k+2N}(x) = T_1([J_{K}^{3,1}]_{4k+2N}(T_1(x))) = [J_{K}^{3,1}]_{4k+4+2N}(x) = T_1([J_{K}^{3,1}]_{4k+4+2N}(T_1(x)))
\]
because \( [J_3^3]_o = J_3^3 \) for all \( o \in \mathbb{N} \) on \( \mathbb{R}^3 \setminus \bar{U}_3^3 \). Therefore, we have
\[
\tilde{G}_{K,N,m,\eta}(x + n) = \tilde{G}_{K,4k+N,4k+4+N,k}(x) + n = J_3^{3,1}(x) + q_K(x_1)e_1. \tag{5.16}
\]
Obviously, \( D_1J_3^{3,1}(x) = 0 \) (see the definition in Section 3.6) and as in Proposition 5.2 we have by (3.10)
\[
q'_K(x_1) = \frac{1}{2}r_K - r_{k+1}. \tag{5.18}
\]
Applying the facts of the previous sentence to (5.16) gives (5.14) immediately.

We prove (5.15) the same way we proved (5.9). The only nuance is the observation that \( [J_3^3]_{4k+2N} \) sends \( \bar{U}_{k,l}^3 \setminus \bar{U}_{k,l+1}^3 \) onto \( U_k^3 \setminus U_{l+1}^3 \) for all \( k \leq l < 4k + 2N \). □

The following proposition gives an estimate on \( |D_1\tilde{G}_{K,N,m,\eta}| \) at points which are farthest away from the Cantor set in the second coordinate (index \( l \)), with respect to their distance from the Cantor set in the first coordinate (index \( k \)).

**Proposition 5.4.** Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \( n = (n_1, n_2, n_3, 0) \) and let \( k, l \in \mathbb{N}_0 \). Let \( v \in \mathbb{V}_4^{l+1} \) and let \( x \in Q(\tilde{z}_{K,v}, \frac{1}{2} \check{r}_l(K)) \setminus Q(\tilde{z}_{K,v}, \check{r}_{l+1}(K)) \) be a point such that
\[
\max\{|x_1 - (\tilde{z}_{K,v})_1|, |x_3 - (\tilde{z}_{K,v})_3|, |x_4 - (\tilde{z}_{K,v})_4|\} < |x_2 - (\tilde{z}_{K,v})_2|.
\]
Further assume that \( l \leq k \leq 4l + 4 + 2N \), \( x_1 \in \bar{U}^1_{K,k} \setminus \bar{U}^1_{K,k+1} \) and \( x_2 \in \bar{U}^1_{K,l} \setminus \bar{U}^1_{K,l+1} \) and \( (x_3, x_4) \in \bar{U}^2_{K,k} \). Then the following estimates hold:
\[
|D_1\tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^{-k\beta}(K + k)^{\alpha + 1}, \tag{5.17}
\]
\[
|D\tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^{-l\beta}(K + l)^{\alpha + 1} \tag{5.18}
\]
and
\[
\tilde{G}_{K,N,m,\eta}(x + n) - n \in [U_k^1 \setminus U_{k+1}^1] \times [U_l^1 \setminus U_{l+1}^1] \times U_k^2 \text{ for } k \leq 4l + 2N. \tag{5.19}
\]
In case of \( 4l + 2N < k \leq 4l + 4 + 2N \) we get \( [U_k^1 \setminus U_{4l+4+2N}^1] \times [U_l^1 \setminus U_{l+1}^1] \times U_k^2 \) in the last inclusion.

Secondly, assume that \( l \in \mathbb{N}_0, x_1 \in \bar{D}^1_{K,4l+4+2N}, x_2 \in \bar{D}^1_{K,l} \setminus \bar{D}^1_{K,l+1} \) and \( (x_3, x_4) \in \bar{D}^2_{K,4l+4+2N} \). Then the following estimate holds:
\[
|D_1\tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^{-4l\beta-4\beta-2N\beta} \tag{5.20}
\]
and
\[
\tilde{G}_{K,N,m,\eta}(x + n) - n \in U_{K,4l+4+2N}^1 \times [U_l^1 \setminus U_{l+1}^1] \times U_{4l+4+2N}^2. \tag{5.21}
\]
These estimates are rotational in the sense that they also hold when we swap the roles of indices $x_2$, $x_3$ and $x_4$. Further the same estimate holds for $D_i\tilde{G}_{K,N,m,\eta}(x + n)$ for $i = 2, 3, 4$ and corresponding permutation of $x$ coordinates.

**Proof.** To prove (5.17) it suffices to note the following facts. We are farthest from the Cantor set in the direction $x_2$ and so (see (3.23) and paragraph before that)

$$\tilde{G}_{K,N,m,\eta}(x + n) = \tilde{G}_{K,4k+2N,4k+4+2N,k}(x) + n$$

$$= \tilde{\xi}_{K,l}(x_2)\left[J^{3,2}_K\right]_{4l+2N}(x) + q_K(x_2)e_2$$

$$+ \left[1 - \tilde{\xi}_{K,l}(x_2)\right]\left[J^{3,2}_K\right]_{4l+4+2N}(x) + n.$$  

But we have $x_1 \in U_{K,k}^1 \setminus U_{K,k+1}^1$ and $k \leq 4l + 2N$ and therefore we have

$$\left[J^{3,2}_K\right]_{4l+2N}(x) = \left[J^{3,2}_K\right]_{4l+4+2N}(x) = J^{3,2}_K(x)$$

which implies

$$\tilde{G}_{K,N,m,\eta}(x + n) = q_K(x_2)e_2 + J^{3,2}_K(x).$$

Now the estimates (5.1) and (5.2) estimate $D_1\tilde{G}_{K,N,m,\eta}$ as we desire in (5.17). The estimates of $D_3$ and $D_4$ are the same or even slightly better as $(x_3, x_4) \in U^2_{K,k}$ and the right-hand side of (5.17) is decreasing in $k$. Finally, we can estimate

$$|D_2\tilde{G}_{K,N,m,\eta}| \leq |q'_K(x_2)| \leq C(K + l)^{\alpha + 2l - \beta}.$$  

The inclusion (5.19) follows immediately from the fact that $\tilde{G}_{K,4k+2N,4k+4+2N,k}(x)$ maps $U^4_{K,l} \setminus U^4_{K,l+1}$ onto $U^4_{l} \setminus U^4_{l+1}$ and on parts of hyperplanes perpendicular to $x_2$ furthest from the center of the nearest cube in direction $x_2$ we apply the map $[J^{3,2}_K]_{4l+2N}$ (given that $k \leq 4l + 2N$) and $[J^{3,2}_K]_{4l+2N}$ maps each $U^3_{K,o} \setminus U^3_{K,o+1}$ onto $U^3_{o} \setminus U^3_{o+1}$ for $l \leq o \leq 4l + 2N$. Especially $[J^{3,2}_K]_{4l+2N}$ maps $U^3_{K,k} \setminus U^3_{K,k+1}$ onto $U^3_{k} \setminus U^3_{k+1}$.

The proof of (5.20) is also similar but by applying (5.3) instead of (5.1). Finally (5.21) is proved by noting that both $[J^{3,2}_K]_{4l+2N}$ and $[J^{3,2}_K]_{4l+4+2N}$ map $U^3_{K,4l+2N} \setminus U^3_{K,4l+2N}$ onto $U^3_{4l+2N} \setminus U^3_{4l+2N}$ and the rest of the argument remains the same as in the previous.  

**Proposition 5.5.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $x \in C_{A,K}$. Then the classic differential of $D\tilde{G}_{K,N,m,\eta}(x + n)$ exists and equals 0. Further, for fixed $K, \alpha, \eta$ the map $\tilde{G}_{K,N,m,\eta}$ is Lipschitz with Lipschitz constant independent of $N, m$ and $\beta$. The map $\tilde{G}_{K,N,m,\eta}$ is locally bi-Lipschitz on $R_{m,\eta} \setminus C_{A,K,m}$.

**Proof.** It is an easy observation that for each $k$ our $\tilde{G}_{K,4k+2N,4k+4+2N,k}(x)$ maps $U^4_{K,k} \setminus U^4_{K,k+1}$ onto $U^4_{k} \setminus U^4_{k+1}$ with $Q_{K,v(k)}$ being mapped onto $Q_{v(k)}$. Now

$$\frac{\text{diam}(Q_{v(k)})}{\text{diam}(Q_{K,v(k)})} = \frac{r_k}{r_{K,v(k)}} \leq C2^{-k\beta}$$
and that tends to 0 as \( k \to \infty \) and therefore \( \tilde{G}_{K,N,m,\eta}(x + n) = 0 \) for each \( x \in C_{A,K} \). The parameter \( \eta \) influences the mapping \( \tilde{G}_{K,N,m,\eta} \) only outside \( Z_m \). For fixed \( K \) and \( \alpha \) the estimates in Propositions 5.2–5.4 are decreasing in \( N \) (the value of \( \beta \) is a fixed constant not dependent on any of the other parameters) and we consider \( N \in \mathbb{N}_0 \) and \( \beta > \beta_0 \gg 1 \).

On each \( \tilde{U}^4_{K,k} \setminus \tilde{U}^4_{K,k+1} \) we have that \( \tilde{G}_{K,4k+2N,4k+4+2N,k}(x) \) is bi-Lipschitz and the derivative is smallest on sets of type \( [\tilde{U}^1_{K,k} \setminus \tilde{U}^1_{K,k+1}] \times \tilde{U}^3_{K,4k+4+2N} \). Here we send three-dimensional cubes on the hyperplane roughly of size \( 2^{-(4k+4+2N)} \) onto cubes roughly of size \( 2^{-(4k+4+2N)(\beta+1)} \) and we are linear on each cube of \( \tilde{U}^3_{K,4k+4+2N} \). Since we are bi-Lipschitz on each \( \tilde{U}^4_{K,k} \setminus \tilde{U}^4_{K,k+1} \) we are locally bi-Lipschitz on \( R_{m,\eta} \setminus C_{A,K,m} \).

**5.2 | Estimates on the derivative of \( G_{K,N,m,\eta} \)**

The following proposition gives an estimate on \( |D_1 G_{K,N,m,\eta}| \) at points which are very close to the Cantor set in coordinates \((x_2, x_3, x_4)\) compared to the \( x_1 \) variable.

**Proposition 5.6.** Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \( n = (n_1, n_2, n_3, 0) \) and let \( 3k + N \leq l \) with \( k, l \in \mathbb{N}_0 \). Let \( x \) be a point such that \( x_1 \in U^1_{k} \setminus U^1_{k+1}, (x_2, x_3, x_4) \in U^3_{l} \). Then the following estimates hold:

\[
\left| D_1 G_{K,N,m,\eta}(x + n) - \frac{1}{2} \tilde{r}_k(K) - \tilde{r}_{k+1}(K) e_1 \right| \leq C 2^{-3k-N} 2^{k(\beta+1)}, \quad D_j G^1 = 0 \text{ for } j = 2, 3, 4 \quad (5.22)
\]

and

\[
|D G_{K,N,m,\eta}(x + n)| \leq C 2^{(3k+3+N)\beta}. \quad (5.23)
\]

The same holds for each rotation, that is, for \( D_2 G^2_{K,m,\eta}(x + n) \) when \( x_2 \in U^1_{k} \setminus U^1_{k+1} \) and \((x_1, x_3, x_4) \in U^3_{l} \) and similarly for coordinates 3 and 4.

**Proof.** Analogously to (5.10) we obtain from (3.15) that

\[
G_{K,N,m,\eta}(x + n) = \zeta_{K,k}(x_1)[H^3_{K}]_{3k+N}(x) + t_{K}(x_1)e_1 + [1 - \zeta_{K,k}(x_1)][H^3_{K}]_{3k+3+N}(x) + n.
\]

The reasoning for the first part of (5.22) is now exactly the same as in the proof of Proposition 5.2, where we use \( |\xi'_{K,k}| \leq 2^k 2^\beta k \) and

\[
\left\| [H^3_{K}]_{3k+N}(x) - [H^3_{K}]_{3k+3+N}(x) \right\|_\infty \leq 2^{-3k-N}.
\]

The other part \( D_j G^1 = 0 \) is easy to see as \( G^1_{K,m,\eta} = t_{k}(x_1)e_1 \).

The reasoning for (5.23) is exactly the same as in the proof of Proposition 5.4 especially (5.20). The only difference is that we work with \( H \) instead of \( J \) and indexes of type \( 3k + N \), and \( 3k + 3 + N \) instead of \( 4k + 2N \) and \( 4k + 4 + 2N \). Otherwise the arguments are identical. \( \square \)

The following proposition gives an estimate on \( |D_1 G_{K,N,m,\eta}| \) at points which are closer to the Cantor set in coordinates \((x_2, x_3, x_4)\) than in the \( x_1 \) variable on lines parallel to \( e_1 \) that do not
intersect the Cantor set (that is, they go through $U^3_i \setminus U^3_{i+1}$) as distance from the Cantor set in the first variable decreases.

**Proposition 5.7.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $k \leq l < 3k + N$ with $k, l \in \mathbb{N}_0$. Let $x$ be a point such that $x_1 \in U^1_k \setminus U^1_{k+1}$, $(x_2, x_3, x_4) \in U^3_i \setminus U^3_{i+1}$ and the following estimates hold:

$$D_1 G_{K,N,m,\eta}(x + n) = \frac{1}{2} \frac{r_k(K) - r_{k+1}(K)}{r_k - r_{k+1}} e_1. \quad (5.24)$$

The same holds for each rotation, that is, for $D_2 G_{K,N,m,\eta}(x + n)$ when $x_2 \in U^1_k \setminus U^1_{k+1}$ and $(x_1, x_3, x_4) \in U^3_i \setminus U^3_{i+1}$ and similarly for coordinates 3 and 4. Further, under the same assumptions,

$$|DG_{K,N,m,\eta}(x + n)| < C 2^\beta. \quad (5.25)$$

**Proof.** For (5.24), the reasoning is exactly the same as in the proof of Proposition 5.3. The only difference is that we work with $H$ instead of $J$ and indexes of type $3k + N$, and $3k + 3 + N$ instead of $4k + 2N$ and $4k + 4 + 2N$.

Otherwise the argument is identical. The estimate (5.25) is a direct application of result of (5.5) because on hyperplanes $G_{K,N,m,\eta}$ is a combination of $[H^3_{K,1}]_{3k+N}$ and $[H^3_{K,1}]_{3k+3+N}(x)$ and $(x_2, x_3, x_4) \in U^3_i \setminus U^3_{i+1}$. □

**Proposition 5.8.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $k, l \in \mathbb{N}_0$. Assume that $l \leq k < 3l + N$, $x_1 \in U^1_k \setminus U^1_{k+1}$, and $(x_2, x_3, x_4) \in U^3_i \setminus U^3_{i+1}$ and similarly for coordinates 3 and 4. Further, under the same assumptions,

$$|DG_{K,N,m,\eta}(x + n)| \leq C 2^{k\beta}. \quad (5.26)$$

Assume alternatively that $x_1 \in U^1_{3l+N}$ and $(x_2, x_3, x_4) \in U^3_i \setminus U^3_{i+1}$ then

$$|DG_{K,N,m,\eta}(x + n)| \leq C 2^{(3l+N)\beta}. \quad (5.27)$$

This estimate is rotational in the sense that it also holds when we swap the roles of indices $x_1, x_2, x_3$ and $x_4$.

**Proof.** The proof of this claim follows the proof of Proposition 5.4, especially the proof of (5.20) but in this case we use (5.6). The only difference that in the proof of Proposition 5.4 the estimates of $D_3$ and $D_4$ are the same or better than the estimate of $D_1$ as $(x_3, x_4) \in \bar{U}^2_{K,k}$ there. Here they are better than the estimate of $D_1$ as $(x_3, x_4) \in U^2_i \setminus U^2_{i+1}$, $l \leq k$ and $2^{k\beta}$ is increasing in $k$. □

**Proposition 5.9.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $l \in \mathbb{N}_0$. Assume that $l \leq N - 1$, $x_4 \in [-2, -1] \cup [1, 2]$, and $(x_1, x_2, x_3) \in U^3_i \setminus U^3_{i+1}$. Then the following estimate holds,

$$|DG_{K,N,m,\eta}(x + n)| \leq C 2^{k\beta}. \quad (5.28)$$
Assume alternatively that \((x_1, x_2, x_3) \in U^3_N\) then
\[
|DG_{K,N,m,\eta}(x + n)| \leq C 2^{N\beta}. \tag{5.29}
\]
This estimate is rotational in the sense that it also holds when we swap the roles of indices \(x_1, x_2, x_3\) and \(x_4\).

**Proof.** The proof of (5.28) is a simple application of (5.5) (see also (3.15) and (3.16)) and the proof of (5.29) is a simple application of (5.6). \(\square\)

**Proposition 5.10.** The map \(G_{K,N,m,\eta}\) is \(C \cdot 2^M\beta\)-Lipschitz on \(R_{m,13} \setminus \mathcal{T}\), where \(\mathcal{T}\) is the set from Lemma 4.5. The map \(G_{K,N,m,\eta}\) is locally bi-Lipschitz on \(R_{m,13} \setminus C_{B,m}\).

**Proof.** The proof that \(G_{K,N,m,\eta}\) is locally bi-Lipschitz on \(R_{m,13} \setminus C_{B,K,m}\) follows the same argument as given in the proof of Proposition 5.5 but using Propositions 5.6–5.8 instead of the propositions in Section 5.1.

Note that \(|DG_{K,N,m,\eta}| \leq C\) when \(|x_4| \in [2, 14]\) in (3.17) or when \(|x_i| \in [2m + 2, 2m + 5]\) in (3.18). Therefore, it is a simple application of (5.5) to prove that \(G_{K,N,m,\eta}\) is \(C \cdot 2^M\beta\) Lipschitz on \(R_{m,13} \setminus \mathcal{T}\). \(\square\)

Now we improve on Proposition 4.3 and we estimate the position of the composition with \(\tilde{G}_{K,N,m,\eta}\) from Section 3.6.

**Lemma 5.11.** Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers and \(n = (n_1, n_2, n_3, 0)\). There exists an \(M^* = M^*(\beta) \geq M\) (where \(M\) is from Proposition 4.3) such that for all \(k, l, \in \mathbb{N}_0\) such that \(l \geq k + M^*\) and all \(x_1 \in U^1_k \setminus U^1_{k+1}\) and \((x_2, x_3, x_4) \in U^3_l\), we have
\[
[\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n) - n]_1 \in U^1_{k-1} \setminus U^1_{k+2}. \tag{5.30}
\]

**Proof.** The image of \(x\) in \(\tilde{G}_{K,N,m,\eta}\) is in the set \((U^1_k \setminus U^1_{k+1}) \times U^3_l\) (see Figure 2 and (5.15)). The set \([-1, 1] \times U^3_l\) is a neighborhood of \([-1, 1] \times C_B(3)\) and the diameter of each cube in \(U^3_l\) is less than \(C 2^{-l(\beta+1)}\). Call the linear map \(L : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)\). Now on \([-1, 1] \times C_B(3)\) we have (4.19) and therefore because \(\hat{F}_{\beta,m}\) is Lipschitz we have that
\[
|[\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)]_1 - [L \circ \tilde{G}_{K,N,m,\eta}(x + n)]_1| < C 2^{-l(\beta+1)}.
\]
Now \(l \geq k + M^*\) and given that \(M^*\) is large enough we have that the right-hand side is much smaller than the size of the intervals in \(U^1_k\), which have diameter \(\approx 2^{-k(\beta+1)}\). This immediately yields (5.30). \(\square\)

## 6 ESTIMATES OF THE DERIVATIVE OF COMPOSITIONS

See the beginning of the previous section for the role of various parameters. In this section we further use fixed constants \(M\) from Proposition 4.3 (see also Theorem 4.1 as \(M \geq M_0\)) and \(M^*\) from Lemma 5.11.
6.1 | The composition of $\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}$

Consider a segment very close to being parallel to $e_i$ (as is the image of segments parallel to $e_i$ in $\tilde{G}_{K,N,m,\eta}$ by (5.7) and (5.8)). The image of this curve in $P_u$ onto $\mathbb{R}^3 \times \{0\}$ is very close to a segment parallel to $e_i$ since $g$ is constant on segments parallel to $e_i$ near the projection of $\mathcal{K}_{B,m}$ and since it is Lipschitz, the function is very close to being constant on the projection of the image of the segment. Therefore the image of segments very close to being parallel to $e_i$ in $\hat{F}_{\beta,m}$ is a curve, which has parametrization whose derivative has very small components in all directions except the $i$ component, that is, the subject of Proposition 6.1.

**Proposition 6.1.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $l, k \in \mathbb{N}$. Assume that $2N > M^*$ and $M \leq M^* \leq l$, $2N + 4k \leq l$ and let $x$ be a point such that $x_1 \in \bar{U}^1_{k,k} \setminus \hat{U}^1_{k,k+1}$, $(x_2, x_3, x_4) \in \hat{U}^3_{k,l}$. Then

$$\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n) - n \in [U^1_{k-1,k+1} \setminus U^1_{k+2,k+4}] \times [U^3_{4k,2N-M} \setminus U^3_{4k+2N+M+1}]$$

(6.1)

and the following estimate holds:

$$\left| D_1(\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)) - \frac{1}{2} r_k - r_{k+1} \frac{1}{2} \tilde{r}_k(K) - \tilde{r}_{k+1}(K) e_1 \right| \leq C 2^{-4k\beta - 2N\beta} (K + k)^{\alpha + 1}. 

(6.2)$$

All of the above holds for each rotation, that is, for $D_2 \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)$ when $x_2 \in \bar{U}^1_{k,k} \setminus \hat{U}^1_{k,k+1}$ and $(x_1, x_3, x_4) \in \hat{U}^3_{k,l}$ and similarly for coordinates 3 and 4.

**Proof.** The claim (6.1) is derived easily from (5.9), Proposition 4.3 and (5.30) of Lemma 5.11 (note that $2N > M^*$ easily implies that $k + M^* \leq 4k + M^* < 4k + 2N < l$).

Now the $i$th coordinate of the derivative equals

$$D_1(\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x))^i = \sum_{j=1}^{4} D_j(\hat{F}_{\beta,m})^j(\tilde{G}_{K,N,m,\eta}(x)) D_1(\tilde{G}_{K,N,m,\eta})^j(x).$$

By (4.2) we know that $D_1 \hat{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(x)) = e_1$ (recall that $M_0 \leq M$) and hence the first term on the right-hand side (corresponding to $j = 1$) is nonzero only for $i = 1$. By (5.7) we know that this first term for $i = 1$ equals to $\frac{1}{2} r_k - r_{k+1} \frac{1}{2} \tilde{r}_k(K) - \tilde{r}_{k+1}(K)$. Other terms for all $i \in \{1, 2, 3, 4\}$ can be estimated by (5.8) and the fact that $\hat{F}_{\beta,m}$ is Lipschitz and we get (6.2). □

The following proposition gives an estimate on $|D_1 \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|$ at points which are closer to the Cantor set in coordinates $(x_2, x_3, x_4)$ than in the $x_1 \in \hat{U}^1_{k,k} \setminus \hat{U}^1_{k,k+1}$ variable on lines that do not intersect the Cantor set (that is, they go through $\hat{U}^3_{k,l} \setminus \hat{U}^3_{k,l+1}$ as distance from the Cantor set in the first variable decreases. The key here is that $\tilde{G}_{K,N,m,\eta}$ sends segments parallel to $e_1$ onto segments parallel to $e_1$ and so does $\hat{F}_{\beta,m}$.

**Proposition 6.2.** Let $-2m \leq n_1, n_2, n_3 \leq 2m$ be even numbers, call $n = (n_1, n_2, n_3, 0)$ and let $l, k \in \mathbb{N}$. Assume that $M \leq M^* \leq l$ and $k + M + M^* + 1 < l < 2N + 4k$ and let $x$ be a point such that $x_1 \in \bar{U}^1_{k,k} \setminus \hat{U}^1_{k,k+1}$, $(x_2, x_3, x_4) \in \hat{U}^3_{k,l}$ and $(x_1, x_2, x_3) \in \hat{U}^3_{k,l+1}$.
\[ \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1}, (x_2, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1}. \]

Then
\[ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n) - n \in [U^{1}_{k-1} \setminus U^{1}_{k+2}] \times [U^{3}_{l-M} \setminus U^{3}_{l+M+1}] \]  
\hspace{1cm} (6.3)

and
\[ D_1 \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n) = \frac{1}{2} \frac{r_k - r_{k+1}}{\tilde{r}_k(K) - \tilde{r}_{k+1}(K)} e_1. \]  
\hspace{1cm} (6.4)

The same holds for each rotation, that is, for \( D_2 \tilde{G}_{K,N,m,\eta}(x + n) \) when \( x_2 \in \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1} \) and \( (x_1, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1} \) and similarly for coordinate 3. The difference in the fourth coordinate is that in (6.7), (6.8) and (6.10) the derivative a positive multiple of \(-e_4\), but the estimates are otherwise the same.

Proof. The proof is similar to Proposition 6.1 with the difference that we use Proposition 5.3 specifically (5.14) to get that segments parallel to \( e_1 \) are mapped to segments parallel to \( e_1 \) by \( \tilde{G}_{K,N,m,\eta} \) and the same is true for \( \hat{F}_{\beta,m} \) by (4.2), thus (6.4) is proved. The reasoning for (6.3) is identical to that of (6.1). \( \square \)

Proposition 6.3. Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \( n = (n_1, n_2, n_3, 0) \) and let \( k, l \in \mathbb{N} \). Assume that \( 1 \leq l - M - M^* - 1 \leq k \leq 3l - 2M - 2 + N \) and let \( x \) be a point such that \( x_1 \in \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1}, (x_2, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1} \). Then
\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C 2^{\beta + 2} + C 2^{-3k-N}(K + k)^{\alpha+1}. \]  
\hspace{1cm} (6.5)

In the case that, \( x_1 \in \tilde{U}^{1}_{K,3l-2M-2+N}, (x_2, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1} \) then
\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C 2^{\beta + 2} + C 2^{-3k-N}(K + k)^{\alpha+1}. \]  
\hspace{1cm} (6.6)

These statements are rotational in the sense that they also hold when we swap the roles of the indices.

Proof. The claim (6.5) follows from Propositions 5.4 and 4.3. The inclusion (6.6) is exactly the second claim of Proposition 4.3. \( \square \)

### 6.2 Derivative estimates of \( G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta} \)

Proposition 6.4. Let \(-2m \leq n_1, n_2, n_3 \leq 2m\) be even numbers, call \( n = (n_1, n_2, n_3, 0) \) and let \( k, l \in \mathbb{N} \). Assume that \( N \geq 3M^* + 3M + 6 \). First, let \( M^* \leq l \) and \( k \leq \frac{1}{3}(l - 2N) \) and let \( x \) be a point such that \( x_1 \in \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1}, (x_2, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1} \). Then
\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C 2^{\beta + 2} + C 2^{-3k-N}(K + k)^{\alpha+1}. \]  
\hspace{1cm} (6.7)

Second, let \( M^* \leq l \) and \( \frac{1}{3}(l - 2N) < k \leq \frac{1}{3}(l - N - M) - 1 \) and let \( x \) be a point such that \( x_1 \in \tilde{U}^{1}_{K,k} \setminus \tilde{U}^{1}_{K,k+1}, (x_2, x_3, x_4) \in \tilde{U}^{3}_{K,l} \setminus \tilde{U}^{3}_{K,l+1} \). Then
\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C 2^{\beta + 2} + C 2^{-3k-N}(K + k)^{\alpha+1}. \]  
\hspace{1cm} (6.8)
In the intermediary case assume that $M^* \leq l$ and $\frac{1}{3}(l - N - M) - 1 < k \leq \frac{1}{3}(l - N) + 1$ and let $x$ be a point such that $x_1 \in U_{K,k}^1 \setminus U_{K,k+1}^1$, $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3$. Then

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq +C2^{\beta+2} + C2^{3k-N}(K + k)^{\alpha+1}. \quad (6.9)$$

In the third case assume that $M^* \leq l$ and $\frac{1}{3}(l - N) + 1 < k \leq l - M - M^* - 1$ and let $x$ be a point such that $x_1 \in \bar{U}_{K,k}^1 \setminus \bar{U}_{K,k+1}^1$, $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3$. Then

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^\beta. \quad (6.10)$$

Further, for $M^* \leq l, k$ and $l - M - M^* - 1 < k \leq 3l - 4M + N - 1$ and let $x_1 \in \bar{U}_{K,k}^1 \setminus \bar{U}_{K,k+1}^1$, and $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3 \setminus \bar{U}_{K,l+1}^3$,

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^M\beta(K + k)^{\alpha+1}. \quad (6.11)$$

Given that $M \leq l$ and $3l - 4M + N \leq k \leq 4l + 4 + 2N$ and $x_1 \in \bar{U}_{K,k}^1 \setminus \bar{U}_{K,k+1}^1$, and $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3 \setminus \bar{U}_{K,l+1}^3$, if $x_1 \in \bar{U}_{K,3l-3M-2}^1$ and $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3 \setminus \bar{U}_{K,l+1}^3$, then

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^{3l+3M+N-k}\beta(K + k)^{\alpha+1}. \quad (6.12)$$

Finally, given that $M \leq l$ and $x_1 \in \bar{U}_{K,4l+5+2N}^1$, and $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3 \setminus \bar{U}_{K,l+1}^3$, then

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| \leq C2^{3M-l-N}\beta. \quad (6.13)$$

The same holds for each rotation, that is, for $D_2 \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)$ when $x_2 \in \bar{U}_{K,k}^1 \setminus \bar{U}_{K,k+1}^1$ and $(x_1, x_3, x_4) \in \bar{U}_{K,l}^3$ and similarly for $3$. The difference in the fourth coordinate is that in (6.7), (6.8) and (6.10) the derivative is close to $-e_4$, but the estimates are otherwise the same.

Proof. Note that the cases are all nonempty. This is obvious once one uses $N \geq 3M^* + 3M + 6$; we have

$$\frac{1}{4}(l - 2N) < \frac{1}{3}(l - N - M - 3) < \frac{1}{3}(l - N) + 1 < l - M - M^* - 1. \quad (6.14)$$

By the chain rule we have

$$D_1(G \circ \hat{F} \circ \tilde{G}) = \sum_{j=1}^{4} D_j G \cdot D_1(\hat{F} \circ \tilde{G})^j = D_1 G \cdot D_1(\hat{F} \circ \tilde{G})^1 + \sum_{j=2}^{4} D_j G \cdot D_1(\hat{F} \circ \tilde{G})^j. \quad \text{(6.14)}$$

Let us first prove (6.7). By (6.1) we know that

$$\hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x) \in [U_{k-1}^1 \setminus U_{k+1}^1] \times [U_{4k+2N-M}^3 \setminus U_{4k+2N+M+1}^3].$$
Hence we can use (5.22) at this point to obtain

\[ |D_1 G_{K,N,m,\eta}| \leq \frac{1}{2} \frac{\hat{r}_{k+1}(K) - \hat{r}_{k+2}(K)}{r_{k+1} - r_{k+2}} + C 2^{-3k-N} 2^{k(\beta+1)} \]

and from (6.2) we obtain

\[ |D_1 (\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta})| \leq \frac{1}{2} \frac{r_k - r_{k+1}}{r_k(K) - \hat{r}_{k+1}(K)} + C 2^{-4\beta k-2N\beta} (K + k)^{\alpha+1}. \]

This allows us to estimate the first term on the right-hand side of (6.14). We estimate the other terms in (6.14) using (6.2) and (5.23)

\[ |D_j G_{K,N,m,\eta}| \cdot |D_1 (\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta})| \leq C 2^{-4k\beta-2N\beta} (K + k)^{\alpha+1} \cdot 2^{(3(k+1)+N)\beta}. \]

Inequality (6.7) follows leaving out minor order terms.

We prove (6.8) much the same way as (6.7). We have that \( \frac{1}{4}(l - 2N) < k \leq \frac{1}{3}(l - M - N - 3) < l - M - M^* - 1 \) because \( N \geq 3M^* + 3M \). Therefore we apply (6.4) and (6.3) and get

\[ D_1 \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) = \frac{1}{2} \frac{r_k - r_{k+1}}{r_k(K) - \hat{r}_{k+1}(K)} e_1 \]

with \( \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) \in [U^1_{k-1} \setminus U^1_{k+2}] \times [U^3_{l-M} \setminus U^3_{l+M+1}] \). Call \( k' = k - 1, k, k+1 \) and \( l' = l - M \) and apply (5.22) for \( k' \) and \( l' \) (we need \( 3k' + N \leq l' \), the strictest condition is \( 3k + 3 + N \leq l - M \) which is \( k \leq \frac{1}{3}(l - M - N - 3) \)). We calculate \( D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) \) using the chain rule, specifically by (6.4) we have \( \sum_{j=2}^4 (6.14) \) is now zero and therefore

\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)| < 2^{\beta+2} + C 2^{-3k-N} 2^{2(k+1)\beta} \]

\[ < 2^{\beta+2} + C 2^{-3k-N} (K + k)^{\alpha+1}. \]

The calculation in (6.10) is even simpler. We apply (6.4) (recall that \( N \geq M + M^* \)) to calculate \( D_1 \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) \) and the image lies in \( \bigcup_{i=-1}^{l+1} (U_{k+i} \setminus U_{k+i+1}) \times U_{l-M} \) by (6.3) (because \( k+1 \leq l - M - M^* \)) and Proposition 4.3. Therefore we put \( l' = l - M \) and \( k' = k - 1, k, k+1 \). Then from the condition \( \frac{1}{3}(l - N) + 1 < k \leq l - M - M^* - 1 \) we get \( \frac{1}{3}(l' - N) < \frac{1}{3}(l' + M - M^*) \) \( k - 1 \leq k' \leq k+1 \leq l' - M^* < l' \) then \( k' < l' < 3k' + N \), which is the condition from Proposition 5.7. Therefore we can apply (5.24) (again after using the fact that \( \sum_{j=2}^4 = 0 \) in (6.14)) to get

\[ |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)| \]

\[ < \frac{1}{2} \frac{r_k - r_{k+1}}{r_k(K) - \hat{r}_{k+1}(K)} \cdot \frac{1}{2} \frac{\hat{r}_{k+1}(K) - \hat{r}_{k+2}(K)}{r_{k+1} - r_{k+2}} \]

\[ < C 2^{\beta}. \]
The intermediary case in (6.9) is similar. We should either apply (5.22) or (5.24) with the calculations in the first case identical to those from the proof of (6.8) and the calculations in the second case identical to those from the proof of (6.10). Since the estimate in (6.8) is larger we estimate using that one.

We prove (6.11) as follows. In the case that $k \leq l$ we estimate $|D_1 \tilde{G}_{K,N,m,\eta}(x + n)|$ by (5.14) and the right-hand side of (5.7). In the case that $k > l$ we estimate $|D_1 \tilde{G}_{K,N,m,\eta}(x + n)|$ using (5.17). The map $\tilde{F}_{\beta, m}$ is bi-Lipschitz and we know the position of the image of point $\tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}(x + n)$ by (6.5) with $k - M \leq k' \leq k + M$ and $l - M \leq l' \leq l + M$. Then $k \leq 3l - 4M + N - 1$ implies $k' \leq k + M \leq 3(l - M) + N + 1 = 3l' + N + 1$, which means we are able to apply (5.26) to estimate $|DG_{K,N,m,\eta}|$ at that point in the image. We get

$$|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < C 2^{-k\beta} (K + k)^{\alpha + 1} 2^{k\beta + M\beta} = C 2^{M\beta} (K + k)^{\alpha + 1}.$$ 

To get (6.12) we estimate $|D_1 \tilde{G}_{K,N,m,\eta}(x + n)|$ using (5.17). The map $\tilde{F}_{\beta, m}$ is bi-Lipschitz and by Proposition 4.3 we have

$$\tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}(x + n) - n \in (U_{k-M} \setminus U_{k+M}) \times (U_{l-M}^3 \setminus U_{l+M}^3).$$

Obviously, the estimate of $|DG_{K,N,m,\eta}|$ from (5.26) is greater than that in (5.27) for all $k < 3l + N$. Therefore, we use (5.26) with $l' = l + M$ to estimate $|DG_{K,N,m,\eta}| < C 2^{(3l+3M+N)\beta}$ for all $3l - 4M + N \leq k \leq 4l + 4 + 2N$. This gives

$$|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < C 2^{-k\beta} (K + k)^{\alpha + 1} 2^{(3l+3M+N)\beta} = C 2^{(3l+3M+N-k)\beta} (K + k)^{\alpha + 1}.$$ 

The estimate (6.13) is proved simply by applying (5.20), (6.6) and (5.27) to get

$$|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < C 2^{-(4l+2N)\beta} 2^{(3l+3M+N)\beta} = C 2^{(3M-l-N)\beta}.$$ 

\[\square\]

### 6.3 ACL condition and norm estimates of $G_{K,N,m,\eta} \circ \tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta}$

Recall that parameter $m$ denotes the size of the boxes (see (3.1)) and that $Z_m = [-2m - 1, 2m + 1]^3 \times [-1, 1]$.

##### Proposition 6.5

Let $1 \leq p < 2$, let $M^* \geq M$ be the numbers from Lemma 5.11 and Proposition 4.3 and let $\alpha \geq \frac{4}{2-p}$. There exists an $N_0 \geq M^*$ such that when $2K \geq N \geq 3M + K \geq K \geq N_0$ and $N \geq 3M^* + 3M + 6$ we have

(i) $G_{K,N,m,\eta} \circ \tilde{F}_{\beta, m} \circ \tilde{G}_{K,N,m,\eta} \in W^{1,p}((-2m - 1, 2m + 1)^3 \times (-1, 1), \mathbb{R}^4)$. 

(ii) For every $j \in \{1, 2, 3, 4\}$

$$\int_{Z_m} |D_j G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq C(\alpha, \beta, p) \left( m^3 + m^2 K^{(\alpha+1)(p-1)} + m^3 (1 + \eta^p K^{p(\alpha+1)}) K^{-\alpha-1} \right).$$

(iii) There exists a $K_0(\alpha, \beta, p)$ such that if $K \geq K_0$, then

$$\int_{Z_m \setminus C_{A,K,m}} |D_j G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq C(\alpha, \beta, p) \left( m^3 K^{2-(\alpha+1)(2-p)} + m^2 K^{(\alpha+1)(p-1)} + m^3 (1 + \eta^p K^{p(\alpha+1)}) K^{-\alpha-1} \right).$$

**Proof.**

**Step 1. Absolute continuity on lines.**

By Proposition 5.5 we have that $\tilde{G}_{K,N,m,\eta}$ is locally bi-Lipschitz on $Z_m \setminus C_{A,K,m}$. By Theorem 4.2 we have that $\tilde{F}_{\beta,m}$ is a bi-Lipschitz map which sends $C_{B,m}$ onto $C_{B,m}$. Proposition 5.10 implies that $G_{K,N,m,\eta}$ is locally bi-Lipschitz on $Z_m \setminus C_{A,K,m}$. Therefore, $G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}$ is absolutely continuous on lines parallel to coordinate axes that do not intersect $C_{A,K,m}$.

On those lines that do intersect $C_{A,K,m}$ we apply the following facts. We have that $\tilde{G}_{K,N,m,\eta}(x) = (q_K(x_1), q_K(x_2), q_K(x_3), q_K(x_4))$ on $C_B$, we have (4.19) and $G_{K,N,m,\eta}(x) = (t_K(x_1), t_K(x_2), t_K(x_3), t_K(x_4))$. Thus

$$G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x) = (x_1, x_2, x_3, -x_4)$$

on $C_{A,K,m}$ (6.15)

and

$$G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n) = G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x) + n.$$ 

Further by Proposition 6.4 (specifically (6.7)) we see that $G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}$ is continuous and has bounded derivative along those lines through $C_{A,K,m}$. Therefore we have that $G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}$ is absolutely continuous on all lines parallel to coordinate axes. To prove i) it now suffices to prove the integral estimate (ii).

We estimate the integral over $Z_m$ by integrating over $Q(0, 1)$ and multiplying by the number of cubes $(2m + 1)^3$. The integral over $Q(0, 1)$ is decomposed further into the integral over $\tilde{U}^4_{K,0} \setminus \tilde{U}^4_{K,M^*}$ and the integral over $\tilde{U}^4_{K,M^*}$. We now proceed to estimate the integral over $\tilde{U}^4_{K,M^*}$ in the following two steps.

**Step 2. Integral estimates on lines not intersecting $C_{A,K,m}$.**

We have to integrate the estimates from Proposition 6.4 over their corresponding lines and then multiply by $(2m + 1)^3$. We assume that we are working on a line parallel to $e_1$, although for other lines the estimates work in the same way. Call $e'$ the intersection of this line with $Q(0, 1)$ and call $e'_k$ the subset of $e'$ such that $x_1 \in U^1_{K,k} \setminus U^1_{K,k+1}$. Also we assume that the line we are working on is farther from $C_{A,K,m}(3)$ in direction $x_2$ than in directions $x_3$ and $x_4$. Thanks to the symmetry of the map we use these estimates to calculate in the other cases. Therefore, choose $l \in \mathbb{N}$, $l \geq M^*$ and assume that $(x_2, x_3, x_4) \in D^3_{K,l} \setminus D^3_{K,l+1}$. In the following sums we sum between the maximum of
the lower bound written and $M^*$ and the upper bound written (if it is larger than $M^*$, otherwise the sum is empty), but we simplify the notation by excluding the maximum. We calculate

$$\int_{\mathcal{L} \cap U^4_{K,M^*}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t)|^p d\mathcal{L}^1(t)$$

$$= \sum_{M^* \leq k \leq 4l+4+2N} \int_{\mathcal{L}_k} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t)|^p d\mathcal{L}^1(t)$$

$$+ \int_{\mathcal{L} \cap \{x_1 \in U^1_{K,4l+2N+5}\}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t)|^p d\mathcal{L}^1(t).$$

Since $N \geq K$ we easily have $2^{-3k-N}(K+k)^{\alpha+1} \leq C$. Hence we estimate by (6.7), given $N \geq 3M + K$, and $N \geq N_0$, (and by (3.7) we have that $\mathcal{L}^1(U^1_{K,K} \setminus U^1_{K,K+1}) \leq C(K+k)^{-\alpha-1}$) that

$$\sum_{M^* \leq k \leq \frac{1}{4}(l-2N)} \int_{\mathcal{L}_k} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t)|^p d\mathcal{L}^1(t)$$

$$\leq \sum_{M^* \leq k \leq \frac{1}{4}(l-2N)} C(K+k)^{-\alpha-1}(1 + 2^{-3k-N}(K+k)^{\alpha+1})^p$$

(6.16)

Moreover by (6.7), by our choice of $N$ and by (3.7) this $C$ depends only on $\alpha, \beta$ and $p$, which are fixed for our construction. In fact in the following all of the constants $C$ depend only on $\alpha, \beta$ and $p$ but not on the parameters $K, N, m$ or $\eta$. Similarly to above we can use (6.8) and (6.9) and we see that $|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| \leq C(\beta)$ and so

$$\sum_{\frac{1}{4}(l-2N)+1 \leq k \leq \frac{1}{3}(l-N)+1} \int_{\mathcal{L}_k} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t)|^p d\mathcal{L}^1(t)$$

$$\leq \sum_{\frac{1}{4}(l-2N) \leq k \leq \frac{1}{3}(l-N)+1} C(K+k)^{-\alpha-1}.$$
The combination of the preceding estimates gives

\[
\int_{l-M-M^*-1}^{l-M-M^*+1} \left| D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t) \right|^p d\mu(t) \\
\leq \sum_{k=M^*}^{l-M-M^*-1} C(K + k)^{-\alpha - 1},
\]

(6.17)

with \( C \) independent of \( K \). Using (6.11) we get

\[
\int_{3l-4M+N-1}^{3l-4M+N} \left| D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t) \right|^p d\mu(t) \\
\leq \sum_{3l-4M+N \leq k \leq 3l-4M+N-1} C^2 \beta p(K + k)^{(\alpha+1)(p-1)}.
\]

(6.18)

We apply (6.12) to get

\[
\int_{4l+4+2N}^{4l+4+2N+4} \left| D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t) \right|^p d\mu(t) \\
\leq \sum_{3l-4M+N \leq k \leq 4l+4+2N} C(K + k)^{(\alpha+1)(p-1)}.
\]

Finally, we use (6.13) to show

\[
\int_{\mathcal{E}(x_1 \in \Omega_{K,4l+2N+q})} \left| D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t) \right|^p d\mu(t) \\
\leq C^2 (3M-l-N) \beta p.
\]

(6.20)

The summary of the estimates (6.17), (6.18), (6.19), (6.20),

\[
\int_{\mathcal{E}} \left| D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(t) \right|^p d\mu(t) \\
\leq \sum_{k=1}^{4l+4+2N} C(K + k)^{(\alpha+1)(p-1)} + C^2 \beta p(l+N-3M)
\]

\[
\leq C(4l + 4 + 2N + K)^{(\alpha+1)(p-1)+1} + C^2 (3M-l-N) \beta p
\]

\[
\leq C(4l + 4 + 2N + K)^{(\alpha+1)(p-1)+1},
\]

(6.21)

where we have used the fact, that \( M \) is an absolute constant introduced in Theorem 4.1. Multiplying this by the measure of \( \Omega_{K,l} \setminus \Omega_{K,l+1} \approx (K + l)^{-\alpha - 1} \) (see (3.7)) and summing over \( l \geq M^* \) we
have

\[
\int_{U^1_{K,M^*}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \, d\mathcal{L}^4 \\
\leq \sum_{l=M^*}^{\infty} C(4l + 4 + 2N + K)^{(\alpha + 1)(p - 1) + 1} (K + l)^{-\alpha - 1} \\
\leq C \sum_{l=1}^{\infty} (K + l)^{(\alpha + 1)(p - 1) + 1 - \alpha - 1} \\
\leq CK^{(\alpha + 1)(p - 2) + 2}
\]

(6.22)

since $1 \leq p < 2$ is fixed, $\alpha \geq \frac{4}{2-p}$ and $2K \geq N \geq K + 3M \geq N_0$.

**Step 3. Integral estimates on lines intersecting $C_{A,K,m}$**

Assuming that we have $(x_2, x_3, x_4) \in C_{A,K}(3)$, then the segment $\ell = (t, x_2, x_3, x_4)$, $t \in U^1_{M^*}$ intersects $C_{A,K}(4)$. We use (6.7) to estimate the derivative on $\{y \in \ell'; y_1 \in U^1_{K,k} \setminus U^1_{K,k+1}\}$ analogously to (6.16). By (6.15) we have $D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta} = e_1$ on $C_{A,K}(4)$, that is, for $t \in C_{A,K}(1)$, and thus

\[
\int_{\ell} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \, d\mathcal{L}^1 \leq \sum_{k=M^*}^{\infty} C(K + k)^{-(\alpha - 1)} + \mathcal{L}^1(C_{A,K}(1)) \\
\leq CK^{-\alpha} + \mathcal{L}^1(C_{A,K}(1)).
\]

This holds for every $(x_2, x_3, x_4) \in C_{A,K}(3)$ and $\mathcal{L}^3(C_{A,K}(3)) \leq 2^3$ and hence

\[
\int_{U^1_{M^*} \times C_{A,K}(3)} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \, d\mathcal{L}^4 \leq CK^{-\alpha} + \mathcal{L}^4(C_{A,K}).
\]

(6.23)

Now this together with (6.22) and Step 1 proves (i).

**Step 4. Integral estimates on $U^4_{K,0} \setminus U^4_{K,M^*}$ on lines close to $C_{A,K}(3)$**

In steps 4 and 5 we deal with the messy part of $Z_m = [-2m - 1, 2m + 1]^3 \times [-1, 1]$ close to its boundary. The main aim is to prove the following estimate

\[
\int_{n+U^4_{K,0} \setminus U^4_{K,M^*}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \, d\mathcal{L}^4 \leq C(\alpha, \beta, p)K^{1-(2-p)(\alpha + 1)}
\]

(6.24)

where $n = (2n_1, 2n_2, 2n_3, 0)$ for $|n_1| \leq m - 1$ and $|n_2|, |n_3| \leq m$.

We want to estimate $|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)|$ for $x$ in

\[
(U^1_{K,0} \setminus U^1_{K,M^*}) \times U^3_{K,M^*} = (U^1_{K,0} \setminus U^1_{K,M^*}) \times (\bigcup_{l=M^*}^{2N-1} (U^3_{K,l} \setminus U^3_{K,l+1}) \cup \bigcup_{l=M^*}^{2N-1} (U^1_{K,0} \setminus U^1_{K,M^*}) \times U^3_{K,2N})
\]
and integrate it over lines parallel to $e_1$ in that set. From Lemma 5.11 and Proposition 4.3 we get for each $x \in [U^1_k \setminus U^1_{k+1}] \times [U^3_l \setminus U^3_{l+1}]$, $1 \leq k \leq M^* - 1, l \geq k + M^*$ that

$$\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x) \in [U^1_{k-1} \setminus U^1_{k+2}] \times [U^3_{l-M} \setminus U^3_{l+M+1}] \quad (6.25)$$

and for $x \in [U^1_0 \setminus U^1_1] \times [U^3_l \setminus U^3_{l+1}]$

$$\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x) \in [(−2,2) \setminus U^1_2] \times [U^3_{l-M} \setminus U^3_{l+M+1}]. \quad (6.26)$$

Given $k \geq 1$ and $l \geq 4k + 2N$, then surely $l \geq k + M^*$ (because $N \geq 3M + 3M^*$) and so (6.25) holds. Then we estimate in the same way as in (6.7), that is, we use Proposition 6.1 and Proposition 5.6 to obtain

$$|D_1 \hat{G}_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)| \leq C2^\beta + 2 + C2^\beta 2^{-3k-N}(K + k)^{2+1}. \quad (6.27)$$

Given $k \geq 1$ and $3k + N + M \leq l \leq 4k + 2N − 1$, then surely $l \geq k + M^*$ (because $N \geq 3M + 3M^*$) and so (6.25) holds. Then we estimate in the same way as in (6.8), that is, we use Proposition 6.2 and Proposition 5.6 to obtain

$$|D_1 \hat{G}_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)| \leq C2^\beta + 2 + C2^\beta 2^{-3k-N}(K + k)^{2+1}. \quad (6.27)$$

The calculations for the case when $k \geq 1$ and $k + M + M^* \leq l \leq 3k + N + M$ (again $l \geq k + M^*$) are the same as (6.9) and (6.10) and again

$$|D_1 \hat{G}_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)| \leq C2^\beta + 2 + C2^\beta 2^{-3k-N}(K + k)^{2+1}. \quad (6.27)$$

Now consider the case when $k = 0$. In all the above cases (that is, for $l = M + M^* + 1, \ldots, 2N$) (6.26) holds and the estimate of $|D_1 \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)|$ is the same as before. The main difference is the fact that $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)$ might not lie in $Q(n,1)$ but instead it might lie in the neighboring cube $Q((2n_1 \pm 2, 2n_2, 2n_3, 0),1)$. In the case when $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) \in Q(n,1)$ we obtain the same estimate (6.27) as before. If this is not the case then

$$\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) \in (U^1_0 \setminus U^1_1) \times (U^3_{l-M} \setminus U^3_{l+M+1}) + (2n_1 \pm 2, 2n_2, 2n_3, 0)$$

(the argument from Proposition 5.11 that a point cannot skip more than one frame still holds also for points originating outside the cube). For $n_1 \in \{-m + 1, \ldots, m - 1\}$ we are in the neighboring cube where $G_{K,N,m,\eta}$ is given by similar formula and we can estimate its derivatives as before and we have again (6.27). The problem is for $n_1 = m$ (or similarly for $n_1 = -m$) if the first coordinate of $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)$ is bigger than $2m + 1$. Then $G_{K,N,m,\eta}$ is defined by a convex combination of the frame to frame on hyperplane maps $[H^3_{K}]_N$ and the identity (see (3.18)). We expound the estimates in detail next.

We use the same estimate of $D_1 G_{K,N,m,\eta}(x + n)$ as before, that is, from (5.14) for any $1 \leq l < 2N$ and (almost) every $x \in (U^1_{K,0} \setminus U^1_{K,1}) \times (U^3_{K,l} \setminus U^3_{K,l+1})$ we have that

$$D_1 G_{K,N,m,\eta}(x + n) = \frac{1}{2}r_0 - r_1 e_1 - f_{K,0} + f_{K,1} e_1$$

(6.28)
and
\[
\frac{1}{2}r_0 - r_1 \leq CK^{\alpha+1}.
\] (6.29)

For \( x \in (\bar{U}_{K,0}^1 \setminus \bar{U}_{K,1}^1) \times (\bar{U}_{K,2N}^3) \) we get from (5.8) that
\[
|D_1 \bar{G}_{K,N,m,\eta}(x + n) - \frac{1}{2}r_0 - r_1 e_1| \leq C 2^{-2N\beta} K^{\alpha+1} \leq C 2^{-N\beta}
\] (6.30)
because \( N \geq K \).

In the case of \( 1 \leq l < 2N \) we get by (6.28) that
\[
D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) = D_1 \hat{F}_{\beta,m} |D_1 \bar{G}_{K,N,m,\eta}(x + n)|.
\]
Applying (4.6) (using the fact that \( l \geq 1 \) and \( x_1 \in [-2m - 2, -2m - 1] \cup [2m + 1, 2m + 2] \)) we get that
\[
D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) = \frac{1}{2}r_0 - r_1 e_1.
\] (6.31)

In the case \( x \in (\bar{U}_{K,0}^1 \setminus \bar{U}_{K,1}^1) \times (\bar{U}_{K,2N}^3) \) we argue by mimicking the proof of (6.2) that thanks to (6.30) and (4.6) we have that \( D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) \) has a very small component perpendicular to \( e_1 \), specifically
\[
|D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) - [D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n)]^1 e_1| \leq C 2^{-N\beta},
\] (6.32)
while
\[
[D_1 \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n)]^1 \leq CK^{\alpha+1}
\] (6.33)
by (6.30) and (6.29).

As mentioned earlier we are considering points which are mapped by \( \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta} \) onto \([-2m - 2, -2m - 1] \cup [2m + 1, 2m + 2] \times (U_{l-M}^3 \setminus U_{l+M+1}^3) \). On this set \( \bar{G}_{K,N,m,\eta} \) is a convex combination of \([H^3_K]_N\) and id (on hyperplanes perpendicular to \( e_1 \)) and the convex combination goes over a segment of length 1, that is, \( \pm [2m + 1, 2m + 2] \) (see (3.18)). Both \([H^3_K]_N\) and id send \( Q_3(0,1) \) onto itself. Therefore, the derivative \( |D_1 \bar{G}_{K,N,m,\eta}| \) is bounded by
\[
C(1 + \|[H^3_K]_N - \text{id}\|_{\infty}) \leq C(1 + \text{diam}(Q_3(0,1))) \leq C.
\]

On the other hand, for \( i = 2, 3, 4 \) we have \( |D_i \bar{G}_{K,N,m,\eta}| \leq C(|D_i[H^3_K]_N| + 1) \) which is calculated in Proposition 5.1 (see (5.5) and (5.6)). In summary at point \( \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) \) we have
\[
|D_1 \bar{G}_{K,N,m,\eta}| \leq C
\] (6.34)
and

$$|DG_{K,N,m,n}| \leq C(\min\{2^{(l+M)\beta}, 2^{N\beta}\} + 1)$$

(6.35)

because $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n) = (\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n})(x + n) \in [(-2m - 2, -2m - 1] \cup [2m + 1, 2m + 2]) \times (U_{1-M}^3 \setminus U_{l+M+1}^3)$.

We calculate $|D_1G_{K,N,m,n} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)|$ as follows. In the case that $1 \leq l < 2N$ we multiply (6.31) with (6.34) using (6.29) to estimate

$$|D_1G_{K,N,m,n} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)| \leq CK^{\alpha+1}.$$  (6.36)

In the case that $x \in (\hat{U}_{1,k,0}^1 \setminus \hat{U}_{1,k,1}^1) \times (\hat{U}_{2,2N}^3)$ we use the chain rule with the same estimates as used in the proof of (6.7). We estimate the first component using (6.34) and (6.33) by

$$|D_1G_{K,N,m,n}(\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)) \cdot (D_1\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n))| \leq CK^{\alpha+1}$$

and the other components are estimated by

$$|DG_{K,N,m,n}(\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)) \cdot |D_1\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n) - \frac{1}{2}r_0 - \frac{1}{2}r_1 e_1|.$$  (6.37)

We estimate (6.37) using (6.32), which we multiply with (6.35) to estimate (6.37) by $C2^{N\beta} \cdot 2^{-2N\beta}K^{\alpha+1} \leq C$ because $N \geq K$. Altogether we have

$$|D_1G_{K,N,m,n} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)| \leq CK^{\alpha+1}$$

for $x \in (\hat{U}_{1,k,0}^1 \setminus \hat{U}_{1,k,1}^1) \times (\hat{U}_{2,2N}^3) + n$ mapped by $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}$ outside $Z_m$. Now combining the two cases (that is, the last estimate and (6.36)) we obtain

$$|D_1G_{K,N,m,n} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)| \leq CK^{\alpha+1}$$  (6.38)

for all $x \in (\hat{U}_{1,k,0}^1 \setminus \hat{U}_{1,k,1}^1) \times (\hat{U}_{2,2N}^3) + n$ (in this step we consider only those $x$ with $(x_2, x_3, x_4) \in (\hat{D}_{1,k,M}^3) + (2n_2, 2n_3, 0)$). In summary we obtain

$$|D_1G_{K,N,m,n} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)| \leq C + C2^{-N}(K + k)^{\alpha+1} \leq C$$  (6.39)

for $0 \leq k \leq M^*$, $k \leq l - M^* - M - 1$ and $K \leq N \leq 2K$ for $|n_1| \leq m - 1$, but only (6.38) holds for $n_1 \in \{-m, m\}$.

The case of $0 \leq k \leq M^*$, $l \geq M^*$ and $l - M - M^* \leq k \leq M^*$ for $|n_1| \leq m - 1$ is much the same as the proof of (6.38). We have $l \leq 2M^* + M$. The calculation is the same for all values of $n$. By (5.14) we have

$$|\hat{D}_1\hat{G}_{K,N,m,n}(x)| \leq C2^{-\beta k}(K + k)^{\alpha+1}.$$  

Further by Proposition 4.3 we obtain that $\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n)$ is far away from $\mathcal{K}_{B,m}$, that is,

$$\hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,n}(x + n) \notin U_{k+M}^1 \times U_{l+M}^3 + n.$$
Analogously to reasoning in Proposition 5.10 we obtain that on this set we have
\[ |D_G K, N, m, \eta| \leq C 2^{(2M^* + M)\beta} \leq C \]
since \( k \leq M^* \) and \( l \leq 2M^* + M \). It follows that in this case we obtain
\[ |D_1 G K, N, m, \eta \circ \hat{F}_{\beta, m} \circ \hat{G}_{K, N, m, \eta}(x)| \leq C 2^{-\beta k} (K + k)^{\alpha + 1} \leq CK^{\alpha + 1} \] (6.40)
as \( K \geq \frac{N}{2} \geq M^* \geq k \).

From (3.7) we obtain for all \( K \)
\[ \mathcal{L}^1(\tilde{U}_1^{1} K, 0 \setminus \tilde{U}_1^{1} K, M^*) \leq C \sum_{k=1}^{M^*} 2^{k} \frac{1}{2^{k}} \frac{1}{(k + K)^{\alpha + 1}} \leq CM^* K^{-\alpha - 1} \leq CK^{-\alpha - 1}. \] (6.41)

Analogously we can conclude that
\[ \mathcal{L}^1(\tilde{U}_1^{1} K, M^* \setminus \tilde{U}_1^{1} K, 2M^* + M) \leq CK^{-\alpha - 1}. \]

Integrating (6.39) over \((\tilde{U}_1^{1} K, 0 \setminus \tilde{U}_1^{1} K, M^*) \times \tilde{U}_3^{3} K, M^* \) and (6.40) (over the set where \( 0 \leq k \leq M^* \) and \( M^* \leq l \leq 2M^* + M \)) we obtain that for \( |n_1| \leq m - 1 \) we have
\[ \int_{n+(\tilde{U}_1^{1} K, 0 \setminus \tilde{U}_1^{1} K, M^*) \times \tilde{U}_3^{3} K, M^*} |D_1 G K, N, m, \eta \circ \hat{F}_{\beta, m} \circ \hat{G}_{K, N, m, \eta}(x)|^p \, dx \]
\[ \leq CK^{-\alpha - 1} C^p + CK^{-\alpha - 1} K^{-\alpha - 1} K^{(\alpha + 1)p} \]
\[ \leq CK^{-(\alpha + 1)(2-p)} . \] (6.42)

Similarly we obtain using (6.38) for \( n_1 \in \{-m, m\} \)
\[ \int_{n+(\tilde{U}_1^{1} K, 0 \setminus \tilde{U}_1^{1} K, M^*) \times \tilde{U}_3^{3} K, M^*} |D_1 G K, N, m, \eta \circ \hat{F}_{\beta, m} \circ \hat{G}_{K, N, m, \eta}(x)|^p \, dx \]
\[ \leq CK^{-\alpha - 1} K^{(\alpha + 1)p} + CK^{-\alpha - 1} K^{-\alpha - 1} K^{(\alpha + 1)p} \]
\[ \leq CK^{(\alpha + 1)(p-1)} . \] (6.43)

**Step 5. Integral estimates on \( \tilde{U}_4^{4} K, 0 \setminus \tilde{U}_4^{4} K, M^* \) on lines far away from \( C_{A,K}(3) \).**

The remaining lines are those lines parallel to \( e_1 \) that lie in the set \( \tilde{U}_4^{4} K, 0 \setminus ([{-1,1}] \times \tilde{U}_3^{3} K, M^* \) . When \( 0 \leq l \leq M^* \) and \( 0 \leq k \leq M^* \) we obtain (6.40) with the same reasoning as before and thus
\[ \int_{(\tilde{U}_1^{1} K, 0 \setminus \tilde{U}_1^{1} K, M^*)^2 \times \tilde{U}_2^{2} K, M^*} |D_1 G K, N, m, \eta \circ \hat{F}_{\beta, m} \circ \hat{G}_{K, N, m, \eta}(x)|^p \leq CK^{-(\alpha + 1)(2-p)} \] (6.44)
and the same holds for other permutations of \((x_2, x_3, x_4)\). It remains to consider \( 0 \leq l \leq M^* \) and \( M^* \leq k \).
Let us first consider $0 \leq l \leq M^*$ and $M^* \leq k \leq 4l + 4 + 2N$, $x_1 \in \bar{U}_{K,k}^1 \setminus \bar{U}_{K,k+1}^1$ and $(x_2, x_3, x_4) \in \bar{U}_{K,l}^3 \setminus \bar{U}_{K,l+1}^3$. From Proposition 5.4 we obtain

$$|D_1 \bar{G}_{K,N,m,\eta}(x)| \leq C 2^{-k} \beta (K + k)^{\alpha + 1}.$$ 

Our $\hat{F}_{\beta,m}$ is bi-Lipschitz and we claim that we can estimate

$$|D G_{K,N,m,\eta}(\hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n))| \leq C (k + M) \beta. \quad (6.45)$$

Indeed, our $\bar{G}_{K,N,m,\eta}$ maps

$$(U_{K,k}^1 \setminus U_{K,k+1}^1) \times (U_{K,l}^3 \setminus U_{K,l+1}^3) \text{ into } (U_{k}^1 \setminus U_{k+1}^1) \times (U_{l}^3 \setminus U_{l+1}^3).$$

By Proposition 4.3 the index $k$ is shifted to index at most $k + M$ by the mapping $\hat{F}_{\beta,m}$. If $\hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) \in Q(n,1)$ then we can estimate $|DG_{K,N,m,\eta}|$ by the minimum of (5.26) and (5.27). If the point is outside $Q(n,1)$ then $G_{K,N,m,\eta}$ is defined by a convex combination of $[H_{K}^{3,j}]_N$ and identity and so it has derivative majorized by the estimate the minimum of (5.26) and (5.27), and therefore in both cases we have (6.45). It follows that in this case we have

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n)| \leq C (K + k)^{\alpha + 1}. \quad (6.46)$$

It remains to consider $0 \leq l \leq M^*$ and $x \in U_{K,4l+4+2N}^1 \times (U_{K,l}^3 \setminus U_{K,l+1}^3)$. By (5.20) we estimate

$$|D_1 \bar{G}_{K,N,m,\eta}(x + n)| \leq 2^{-2N \beta - 4l \beta}. \quad (6.47)$$

By Proposition 4.3 the index $l$ shifts to at most $l + M \leq M^* + M$ in the mapping $\hat{F}_{\beta,m}$. If $\hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n) \in Q(n,1)$ then we can estimate $|DG_{K,N,m,\eta}|$ by (5.27). If the point is outside $Q(n,1)$ then $G_{K,N,m,\eta}$ is defined by a convex combination of $[H_{K}^{3,j}]_N$ and identity and so it has derivative majorized by the estimate from (5.27), that is, we have

$$|DG_{K,N,m,\eta}| \leq C 2^{3M^* + 3M + N} \beta.$$

Therefore we obtain

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x + n)| \leq C 2^{-2N \beta} 2^{(3M^* + 3M + N) \beta}$$

$$\leq C 2^{-N \beta} \leq C. \quad (6.47)$$

Using the estimates of the derivatives are in (6.46) and (6.47) we estimate on each such line $\ell$

$$\int_{\ell} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \bar{G}_{K,N,m,\eta}(x)|$$

$$\leq \sum_{k=0}^{4l+2N} C (K + k)^{(\alpha+1)p} (K + k)^{-(\alpha+1)} + C.$$
\[
\leq C \sum_{k=0}^{4M^*+2N} (K + k)^{(\alpha+1)(p-1)} + C
\]
\[
\leq C(K + N)^{(\alpha+1)(p-1)+1}
\]
\[
\leq CK^{(\alpha+1)(p-1)+1}
\]
(6.48)
because \(N_0 \leq K \leq N \leq 2K\).

By (3.7) we have analogously to (6.41) that
\[
\mathcal{L}^3\left(\overline{U}^3_{K,0} \setminus \overline{U}^3_{K,M^*}\right) \leq CK^{-\alpha-1}.
\]

Multiplying (6.48) by this measure estimate and adding to (6.44) we get
\[
\int_{\overline{U}^4_{K,0} \setminus ([1,1] \times \overline{U}^2_{K,M^*})} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| \leq CK^{-(\alpha+1)(p-2)+1}
\]

Combining the previous estimate with Step 4 we have
\[
\int_{n+\overline{U}^4_{K,0} \setminus \overline{U}^4_{K,M^*}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq CK^{-(\alpha+1)(p-2)} + CK^{-(\alpha+1)(p-2)+1}
\]
\[
\leq CK^{1-(\alpha+1)(2-p)}
\]
for \(|n_1| \leq m-1\) which gives us (6.24). In case \(n_1 \in \{-m, m\}\) we obtain \(CK^{(\alpha+1)(p-1)}\) on the right-hand side by (6.43).

**Step 6. Proving (ii) and (iii) for \(j = 1, 2, 3\).**

Finally adding (6.24), (6.23) and (6.22) (and excluding the \(\mathcal{L}^4(C_{A,K})\) term in (6.23)) we get for \(|n_1| \leq m-1\)
\[
\int_{n+Q(0,1) \setminus C_{A,K}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p d\mathcal{L}^4
\]
\[
\leq CK^{1-(2-p)(\alpha+1)} + CK^{-\alpha} + CK^{(\alpha+1)(p-2)+2}
\]
(6.49)
since \(1 \leq p < 2\) is fixed and \(\alpha \geq \frac{4}{2-p}\). For \(n_1 \in \{-m, m\}\) we obtain \(CK^{(\alpha+1)(p-1)}\) on the right-hand side by (6.43). We sum (6.49) over the \((2m-1)(2m+1)^2\) cubes in the Cantor plate construction together with the corresponding case for \(n_1 \in \{-m, m\}\) and get,
\[
\int_{Z_m \setminus C_{A,K,m}} |D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq CK^{-(\alpha+1)(2-p)+2}m^3 + CK^{(\alpha+1)(p-1)}m^2.
\]
The inequality in (iii) for \( j = 2, 3 \) is proved by rotational symmetry and exchanging the role of \( |n_1| \leq m - 1 \) for \( n_j \). From here (ii) is proved by adding the \( m^3 \mathcal{L}^4(C_{A,K}) \leq 16m^3 \) term in (6.23) and noting that \(- (\alpha + 1)(2 - p) + 2 < 0 \). In fact, in the cases \( j = 1, 2, 3 \) we do not need the \( m^3(1 + \eta^p K^{p(\alpha + 1)})K^{-\alpha - 1} \) term from (ii) and (iii) as the integral is already estimated by the first two terms.

**Step 7. Proving (ii) and (iii) for \( j = 4 \).**

The majority of the proof for \( j = 4 \) is the same as in the previous cases. The difference is as follows, in direction \( e_4 \) the Cantor plate construction \( Z_m \) is only one cube thick, that is, all cubes fall into the equivalent category of the previous step for \( |n_1| = m \). We continue to estimate \( |D_4 \hat{\Phi}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n) - (\hat{\Phi}_{\beta,m}(y))^4| \leq C2^{-N\beta} \).

We have

\[
\hat{G}_{K,N,m,\eta}(U^{3}_{K,2N} \times (U^{1}_{K,0} \setminus U^{1}_{K,1})) = U^{3}_{2N} \times (U^{1}_{0} \setminus U^{1}_{1}).
\]

Each cube in \( U^{3}_{2N} \) has diameter \( C2^{-2N-2N\beta} \). The map \( \hat{\Phi}_{\beta,m} \) is Lipschitz and (4.9) holds and therefore

\[
|\hat{\Phi}_{\beta,m}(y)|^4 - (-y_4) < C2^{-2N-2N\beta}.
\]

Thus, as soon as \( N \) is large enough (that is, we assume that \( N \geq K \geq N_0 \)) we have \( |(\hat{\Phi}_{\beta,m}(y))^4 - (-y_4)| < \frac{1}{2} \). That is to say any point in \( U^{3}_{K,2N} \times (U^{1}_{K,0} \setminus U^{1}_{K,1}) \) mapped outside \( Z_m \) by \( \hat{\Phi}_{\beta,m} \circ \hat{G}_{K,N,m,\eta} \) lands in the set \( U^{3}_{2N-M} \times ([-\frac{3}{2}, -1] \cup [1, \frac{3}{2}]) \). By (3.17) we have that at these points

\[
D_4 \hat{G}_{K,N,m,\eta}(\hat{\Phi}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}(x + n)) = \frac{\eta}{13} e_4.
\]

The application of the chain rule (analogous to the proof of (6.7)) leads us to sum of

\[
|D_4 G_{K,N,m,\eta}(\hat{\Phi}_{\beta,m} \circ G_{K,N,m,\eta}(x + n))[D_4 \hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)]^4| \leq C\eta K^{\alpha+1}
\]

with

\[
|D_j G_{K,N,m,\eta}(\hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n))[D_4 \hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)]^j| \leq C2^{-N\beta}2^{N\beta} = C
\]

(recall by (3.17) and (5.6) that \( |D_j G_{K,N,m,\eta}| \leq 2^{N\beta} \) and the second factor is bounded by (6.50)) giving

\[
|D_4 G_{K,N,m,\eta} \circ \hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < C(\eta K^{\alpha+1} + 1)
\]

for (almost) all \( x \in (U^{3}_{K,2N} \times (U^{1}_{K,0} \setminus U^{1}_{K,1}) \) with \( x + n \) mapped outside \( Z_m \) by \( \hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta} \). On the other hand, if \( x \) is mapped inside \( Q(n, 1) \) we have

\[
|D_4 G_{K,N,m,\eta} \circ \hat{\Phi}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < C
\]

for (almost) all \( x \in (U^{3}_{K,2N} \times (U^{1}_{K,0} \setminus U^{1}_{K,1}) \) (calculations the same as in Step 4).
If \( x \in (U_{K,1}^3 \setminus U_{K,2N}^3) \times (U_{K,0}^1 \setminus U_{K,1}^1) \) we use a simple estimate, that is, for (almost) all \( x \) we have \( D_4G_{K,N,m,\eta}(x + n) \) is parallel with \( e_4 \) and has size \( C_{K,\alpha+1} \). By (4.9) we are interested only in \( D_4G_{K,N,m,\eta}(\tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)) \) which is either bounded by \( C \) in the case that \( |\tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)|^4 > \frac{3}{2} \) and there all we have is \( \tilde{G}_{K,N,m,\eta} \) Lipschitz or \( CK^{-\alpha-1} \) in the case the point stays inside the cube. The larger of these is obviously \( C \) and (similar to (6.38)) we have

\[
|D_4G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < CK^{\alpha+1}
\]

for (almost) all \( x \in (U_{K,1}^3 \setminus U_{K,2N}^3) \times (U_{K,0}^1 \setminus U_{K,1}^1) \). For \( x \in (U_{K,0}^3 \setminus U_{K,2N}^3) \times (U_{K,0}^1 \setminus U_{K,1}^1) \) we simply apply (5.17), Lemma 4.5 and Proposition 5.10 to get the same estimate of \( CK^{\alpha+1} \). Therefore,

\[
|D_4G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)| < CK^{\alpha+1}
\]

(6.53)

for (almost) all \( x \in (U_{K,0}^3 \setminus U_{K,2N}^3) \times (U_{K,0}^1 \setminus U_{K,1}^1) \).

Clearly we have (independently of choice of \( N \))

\[
\mathcal{L}^3(U_{K,0}^3 \setminus U_{K,2N}^3) \leq \mathcal{L}^3(U_{K,0}^3 \setminus C_{A,K}(3)) \leq CK^{-\alpha} \text{ and } \mathcal{L}^3(U_{K,2N}^3) < 8.
\]

Again we have the estimate \( \mathcal{L}^3(U_{K,0}^3 \setminus U_{K,1}^1) < CK^{-\alpha-1} \). Now from (6.51) and (6.52) we get

\[
\int_{U_{K,2N}^3 \times (U_{K,0}^3 \setminus U_{K,1}^1)} |D_4G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)|^p < C(1 + \eta^p K^{p(\alpha+1)})K^{-\alpha-1}.
\]

(6.54)

and from (6.53) we obtain

\[
\int_{(U_{K,0}^3 \setminus U_{K,1}^1) \times (U_{K,0}^3 \setminus U_{K,1}^1)} |D_4G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x + n)|^p < CK^{-(2-p)(\alpha+1)+1}.
\]

(6.55)

The sum of (6.54) over the \((2m + 1)^3\) cubes in \( Z_m \) is estimated by the \( Cm^3 (1 + \eta^p K^{p\alpha+p})K^{-\alpha-1} \) term of (ii) and (iii). The sum of (6.55) over the \((2m + 1)^3\) cubes in \( Z_m \) is estimated by the \( Cm^3 K^{2-(2-p)(\alpha+1)} \)–term of (ii) and (iii). The other cases, that is, when \( k \geq 1 \), are identical by rotational symmetry to those dealt with in Steps 4–6 and are estimated already by the \( Cm^3 K^{2-(2-p)(\alpha+1)} + Cm^2 K^{(p-1)(\alpha+1)} \)–terms of (ii) and (iii). \( \square \)

7 PROOF OF THEOREM 3.1

Recall that parameter \( m \) denotes the size of the boxes (see (3.1)) and that \( Z_m = [-2m - 1, 2m + 1]^3 \times [-1,1] \) and \( R_{m,\eta} = [-2m - 5, 2m + 5]^3 \times [-1 - \eta, 1 + \eta] \).

Proposition 7.1. Let \( 1 \leq p \leq 2 \) and \( \alpha \geq \frac{4}{2-p} \). For each \( K \) put \( \eta = \eta(K) = K^{-\alpha-1} \). Call \( f_{K,N,m,\eta} = G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta} : R_{m,\eta} \to \mathbb{R}^4 \) with \( K \geq K_0 \) and \( 2K \geq N \geq 3M + K \) and \( N \geq 3M^* + 3M + 6 \). Then,

(i) \( f_{K,N,m,\eta} \in W^{1,p}(R_{m,\eta}, \mathbb{R}^4) \),

(ii) \( f_{K,N,m,\eta} \in \mathcal{C}^{1,p}(R_{m,\eta}, \mathbb{R}^4) \),

(iii) \( f_{K,N,m,\eta} \in \mathcal{L}^p(R_{m,\eta}, \mathbb{R}^4) \),

(iv) \( f_{K,N,m,\eta} \in \mathcal{L}^{2,p}(R_{m,\eta}, \mathbb{R}^4) \),

(v) \( f_{K,N,m,\eta} \in \mathcal{L}^{q,p}(R_{m,\eta}, \mathbb{R}^4) \),

where \( q = \frac{2p}{2-2p} \).
\[ f_{K,N,m,\eta}(x) = x \text{ for } x \in \partial R_{m,\eta}. \]

(iii) \textit{the map } \( f_{K,N,m,\eta} \) \textit{is locally bi-Lipschitz on } \( R_{m,\eta} \setminus C_{A,K,m} \),

(iv) \( J_{f_{K,N,m,\eta}} < 0 \) on \( C_{A,K,m} \).

(v) \[ \int_{R_{m,\eta} \setminus C_{A,K,m}} |D_j f_{K,N,m,\eta}|^p \leq C(\alpha, \beta, K)(\alpha + 1)(p - 1) + 1((m + 1)^2 + (m + 1)^3 \eta) \]

\[ + C(\alpha, \beta)K^{-(\alpha + 1)(2 - p) + 2} (m + 1)^3 \text{ for every } j \in \{1, 2, 3\}, \]

(vi) \[ \int_{R_{m,\eta} \setminus C_{A,K,m}} |D_4 f_{K,N,m,\eta}|^p \leq C(\alpha, \beta, K)(\alpha + 1)(p - 1) + 1((m + 1)^2 \eta)^{1 - p} + C(\alpha, \beta)K^{-(\alpha + 1)(2 - p) + 2} (m + 1)^3. \]

\[ \text{Proof. Step 1. Proof of (iii).} \]

In Step 1 of the proof Proposition 6.5 we showed that \( f_{K,N,m,\eta} \) is locally bi-Lipschitz on \( Z_m \setminus C_{A,K,m} \). The fact that \( f_{K,N,m,\eta} \) is bi-Lipschitz on \( R_{m,\eta} \setminus Z_m \) is obvious from Propositions 5.5, 5.10 and Theorem 4.2 (the fact that \( \hat{F}_{\beta,m} \) is the identity on the boundary and that \( \hat{F}_{\beta,m}(C_{B,m}) = C_{B,m} \)).

\[ \text{Step 2. Proof of (i).} \]

The fact that \( f_{K,N,m,\eta} \in W^{1,p}((-2m - 5, 2m + 5)^3 \times (-1 - \eta, 1 + \eta), \mathbb{R}^4) \) comes from Step 1 (on the part outside \( Z_m \)) and Proposition 6.5 point (i) on the union of the cubes.

\[ \text{Step 3. Proof of (ii).} \]

This holds from the fact that \( \hat{F}_{\beta,m} \) is identity on \( \partial R_{m,\eta} \) and that \( \hat{G}_{K,N,m,\eta} \) is the inverse to \( \hat{G}_{K,N,m,\eta} \) on the boundary as can easily be observed from (3.29), (3.30), (3.27), (3.28), (3.17) and (3.19).

\[ \text{Step 4. Proof of (iv).} \]

The fact that \( J_{f_{K,N,m,\eta}} < 0 \) on \( C_{A,K,m} \) follows from two facts. The first one is that \( \hat{G}_{K,N,m,\eta}(x) = (q_K(x_1), q_K(x_2), q_K(x_3), q_K(x_4)) \) on \( C_{A,K} \) and \( G_{K,N,m,\eta}(x) = (t_K(x_1), t_K(x_2), t_K(x_3), t_K(x_4)) \) on \( C_B = \hat{G}_{K,N,m,\eta}(C_{A,K}) \) and these two maps are mutually inverse on these sets. The combination of the above fact and (4.19) precisely prove that the approximative derivative satisfies

\[ D_j f_{K,N,m,\eta}(x) = e_j \text{ for } j = 1, 2, 3 \text{ and } D_4 f_{K,N,m,\eta}(x) = -e_4 \text{ for a.e. } x \in C_{A,K}. \]

It is well known that the weak (Sobolev) derivative equals to approximative derivative a.e. and so \( J_{f_{K,N,m,\eta}}(x) = -1 \) for a.e. \( x \in C_{A,K} \).

\[ \text{Step 5. Proof of (v).} \]

By symmetry it is enough to show this for \( j = 1 \). We calculate using (iii) of Proposition 6.5. Note that since we have already chosen \( \eta = K^{-\alpha - 1} \) the right-hand side of (iii) of Proposition 6.5 simplifies because the term

\[ C^3(1 + \eta^p K^{p\alpha + p})K^{-\alpha - 1} = C^3 K^{-\alpha - 1} \]

but because \( p \geq 1 \)

\[ C^3 K^{\alpha - 1} \leq C^3 K^{2-(2-p)(\alpha + 1)} \]

which is the first term of (iii) of Proposition 6.5, therefore we use only the first two terms, we point out that the constant \( C \) depends on \( p, \alpha, \beta \). Then we continue to apply the simplified estimate from (iii) of Proposition 6.5 and we get

\[ \int_{R_{m,\eta} \setminus C_{A,K,m}} |D_1 \hat{G}_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}|^p \leq C(\alpha, \beta, p)K^{-(\alpha + 1)(2 - p) + 2} m^3 + \]

\[ + C(\alpha, \beta, p)K^{(\alpha + 1)(p - 1)} m^2 + \int_{R_{m,\eta} \setminus Z_m} |D_1 \hat{G}_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \hat{G}_{K,N,m,\eta}|^p. \]
By the definition of $\overline{G}_{K,N,m,\eta}$ especially (3.26) we can easily calculate that

$$G_{K,N,m,\eta}^{-1}(R_{m,2}) = [-2m - 5, 2m + 5]^3 \times \left([-1 - \frac{2\eta}{26 - \eta}, 1 + \frac{2\eta}{26 - \eta}\right]$$

(7.2)

and

$$G_{K,N,m,\eta}^{-1}(R_{m,1} \setminus R_{m,2}) = [-2m - 5, 2m + 5]^3 \times \left([-1 - \eta, 1 + \eta] \setminus \left([-1 - \frac{2\eta}{26 - \eta}, 1 + \frac{2\eta}{26 - \eta}\right]\right).$$

(7.3)

With respect to Lemma 4.5 and the definition of $G_{K,N,m,\eta}$ it is necessary to calculate separately on the sets in (7.2) and (7.3). We decompose the integral over $R_{m,\eta} \setminus Z_m$ into several sets. The sets around the sides of $Z_m$ are

$$E_1 = \left([-2m - 5, 2m + 5] \setminus [-2m - 1, 2m + 1]\right) \times [-2m - 1, 2m + 1] \times [-1, 1],$$

$$E_2 = [-2m - 5, 2m + 5] \setminus [-2m - 1, 2m + 1] \times \left([-2m - 1, 2m + 1]^2 \setminus [-2m - 1, 2m + 1]^2\right) \times [-1, 1]$$

$$E_3 = [-2m - 1, 2m + 1] \times \left([-2m - 5, 2m + 5]^2 \setminus [-2m - 3, 2m + 3]^2\right) \times [-1, 1]$$

$$\cap \left((C_{A,K,m} + \mathbb{R}e_2) \cup (C_{A,K,m} + \mathbb{R}e_3)\right)$$

$$E_4 = [-2m - 1, 2m + 1] \times \left([-2m - 5, 2m + 5]^2 \setminus [-2m - 3, 2m + 3]^2\right) \times [-1, 1]$$

$$\setminus \left((C_{A,K,m} + \mathbb{R}e_2) \cup (C_{A,K,m} + \mathbb{R}e_3)\right)$$

$$E_5 = [-2m - 1, 2m + 1] \times \left([-2m - 3, 2m + 3]^2 \setminus [-2m - 1, 2m + 1]^2\right) \times [-1, 1]$$

$$\cap \left((C_{A,K,m} + \mathbb{R}e_2) \cup (C_{A,K,m} + \mathbb{R}e_3)\right)$$

$$E_6 = [-2m - 1, 2m + 1] \times \left([-2m - 3, 2m + 3]^2 \setminus [-2m - 1, 2m + 1]^2\right) \times [-1, 1]$$

$$\setminus \left((C_{A,K,m} + \mathbb{R}e_2) \cup (C_{A,K,m} + \mathbb{R}e_3)\right).$$

The sets above and below $Z_m$ are

$$E_7 = [-2m - 5, 2m + 5]^3 \times \left([-1 - \frac{2\eta}{26 - \eta}, -1] \cup [1, 1 + \frac{2\eta}{26 - \eta}]\right) \setminus \left(C_{A,K,m} + \mathbb{R}e_4\right)$$

$$E_8 = [-2m - 5, 2m + 5]^3 \times \left([-1 - \frac{2\eta}{26 - \eta}, -1] \cup [1, 1 + \frac{2\eta}{26 - \eta}]\right) \cap \left(C_{A,K,m} + \mathbb{R}e_4\right)$$

$$E_9 = [-2m - 5, 2m + 5]^3 \times \left([-1 - \eta, -1 - \frac{2\eta}{26 - \eta}] \cup [1 + \frac{2\eta}{26 - \eta}, 1 + \eta]\right) \setminus \left(C_{A,K,m} + \mathbb{R}e_4\right)$$

$$E_{10} = [-2m - 5, 2m + 5]^3 \times \left([-1 - \eta, -1 - \frac{2\eta}{26 - \eta}] \cup [1 + \frac{2\eta}{26 - \eta}, 1 + \eta]\right) \cap \left(C_{A,K,m} + \mathbb{R}e_4\right).$$

It holds that $R_{m,\eta} \setminus Z_m = \bigcup_{i=1}^{10} E_i$ and (see (3.6))

$$\mathcal{L}^4(E_1), \mathcal{L}^4(E_3), \mathcal{L}^4(E_5) \leq C(m + 1)^2,$$

$$\mathcal{L}^4(E_2) \leq C(m + 1).$$
\[ \mathcal{L}^4(E_4), \mathcal{L}^4(E_6) \leq CK^{-\alpha}(m + 1)^2, \]
\[ \mathcal{L}^4(E_7), \mathcal{L}^4(E_9) \leq CK^{-\alpha}(m + 1)^3, \]
\[ \mathcal{L}^4(E_8), \mathcal{L}^4(E_{10}) \leq C\eta(m + 1)^3. \] (7.4)

We estimate \( D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta} \) on \( E_1 \) as follows. On
\[ E_{1,a} := ([2m - 5, 2m + 5] \setminus [2m - 4, 2m + 4]) \times [2m - 1, 2m + 1]^2 \times [-1, 1] \]
we have \( |D_1G_{K,N,m,\eta}| \leq C \) by (3.28). By the definition of \( \tilde{G}_{K,N,m,\eta} \) we have
\[ \tilde{G}_{K,N,m,\eta}(E_{1,a}) = E_{1,a} \]
Further
\[ P_v(E_{1,a}) \subset \mathbb{R}^3 \setminus [-2m - 4 + \frac{1}{14}, 2m + 4 - \frac{1}{14}]^3 \]
(because \( \max\{|v_1|, |v_2|, |v_3|\} \leq \frac{1}{14} \)) and on this set we have \( g = 0 \) by (4.20). Therefore (see (4.17) and (4.10)) \( \hat{F}_{\beta,m} = \text{id} \) on \( E_{1,a} \). Further, on \( E_{1,a} \) we have \( G_{K,N,m,\eta} = \text{id} \) by (3.18). Therefore on \( E_{1,a} \) we have
\[ |D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| = |D_1\text{id} \circ \text{id} \circ \tilde{G}_{K,N,m,\eta}(x)| = |D_1\tilde{G}_{K,N,m,\eta}(x)| \leq C. \]
For (almost) all \( x \) in
\[ E_{1,b} := ([2m - 4, 2m + 4] \setminus [2m - 3, 2m + 3]) \times [2m - 1, 2m + 1]^2 \times [-1, 1] \]
we have \( D_1\tilde{G}_{K,N,m,\eta}(x) = e_1 \) by (3.28) and
\[ \tilde{G}_{K,N,m,\eta}(E_{1,b}) = E_{1,b}. \]
On this set we use the fact that \( \hat{F}_{\beta,m} \) is Lipschitz to estimate \( |D_1\hat{F}_{\beta,m}| < C \). From Lemma 4.5 we obtain \( \hat{F}_{\beta,m}(x) \notin \mathcal{T} \) since \( E_{1,b} \subset \mathcal{G} \). Applying Proposition 5.10 for these \( \hat{F}_{\beta,m}(x) \) we have
\[ |D_1G_{K,N,m,\eta}| \leq C(M, \beta). \]
Therefore
\[ |D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| \leq |DG_{K,N,m,\eta}| \cdot |D\hat{F}_{\beta,m}| \cdot |D_1\tilde{G}_{K,N,m,\eta}(x)| \leq C. \]
For (almost) all \( x \) in
\[ E_{1,c} := ([2m - 3, 2m + 3] \setminus [2m - 1, 2m + 1]) \times [2m - 1, 2m + 1]^2 \times [-1, 1] \]
we have \( D_1\tilde{G}_{K,N,m,\eta}(x) = e_1 \) by (3.28). Let us consider two options
\( x \in ([2m - 3, 2m + 3] \setminus [2m - 1, 2m + 1]) \times \bar{U}_{K,M+1}^2 \times [-1, 1] \) or
\( y \in ([2m - 3, 2m + 3] \setminus [2m - 1, 2m + 1]) \times ([2m - 1, 2m + 1]^2 \setminus \bar{U}_{K,M+1}^2) \times [-1, 1] \). (7.5)
Of course we assume that $N \geq M + 1$ and therefore by (3.28) and (3.11)

$$\tilde{G}_{K,N,m,\eta} \left( \left[ -2m - 3, 2m + 3 \right] \setminus \left[ 2m - 1, 2m + 1 \right] \right) \times U_{K,M+1}^2 \times [-1, 1]$$

$$= \left( \left[ -2m - 3, 2m + 3 \right] \setminus \left[ -2m - 1, 2m + 1 \right] \right) \times U_{M+1}^2 \times [-1, 1]$$

$$\tilde{G}_{K,N,m,\eta} \left( \left[ -2m - 3, 2m + 3 \right] \setminus \left[ 2m - 1, 2m + 1 \right] \right) \times \left( \left[ -2m - 1, 2m + 1 \right] \setminus U_{K,M+1}^2 \right) \times [-1, 1]$$

$$= \left( \left[ -2m - 3, 2m + 3 \right] \setminus \left[ 2m - 1, 2m + 1 \right] \right) \times \left( \left[ -2m - 1, 2m + 1 \right] \setminus U_{M+1}^2 \right) \times [-1, 1].$$

From Lemma 4.5 we obtain $\tilde{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(y)) \not\in \mathcal{T}$ since $\tilde{G}_{K,N,m,\eta}(y) \in \mathcal{G}$. Applying Proposition 5.10 for these $\tilde{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(y))$ we have $|D\tilde{G}_{K,N,m,\eta}| \leq C(M, \beta)$. We use the fact that $\tilde{F}_{\beta,m}$ is Lipschitz and combine the above estimates to get

$$|D^1 \tilde{G}_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(y)| \leq |D\tilde{G}_{K,N,m,\eta}| \cdot |D\tilde{F}_{\beta,m}| \cdot |D^1 \tilde{G}_{K,N,m,\eta}(x)| \leq C.$$

Recall that for $x$ as in (7.5) we still have $D^1 \tilde{G}_{K,N,m,\eta}(x) = e_1$. By (4.6) we have $D^1 \tilde{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(x)) = e_1$. The map $\tilde{F}_{\beta,m}$ is bi-Lipschitz and satisfies (4.5) on $K_{B,m}$ part. Therefore calling $z = \tilde{G}_{K,N,m,\eta}(x)$ we have by Theorem 4.1 (recall that $M \geq M_0$)

$$|\tilde{F}_{\beta,m}(z) - (z_1, z_2, z_3, -z_4)| < C 2^{-M\beta} < \frac{1}{4},$$

where $C$ is the bi-Lipschitz constant of $\tilde{F}_{\beta,m}$ and up to increase of $M$ (still a fixed finite constant) the above holds. Then $\tilde{F}_{\beta,m}(z)$ is furthest from $C_{B,m}$ in the $e_1$ coordinate and $(\tilde{F}_{\beta,m}(z))^1 \in [-2m - 3 - \frac{1}{4}, -2m - 1] \cup [2m + 1, 2m + 3 + \frac{1}{4}] \cup U_1$. On the set where $(\tilde{F}_{\beta,m}(z))^1 \in [-2m - 3 - \frac{1}{4}, -2m - 1] \cup [2m + 1, 2m + 3 + \frac{1}{4}]$ we have $|D^1 \tilde{G}_{K,N,m,\eta}| < C$ by (3.18). On the set where $(\tilde{F}_{\beta,m}(z))^1 \in U_1$ we use (3.16) and (3.15) for $k = 1$ together with $|D^1 \xi_{K,1}(x_1)| \leq C$, $D^1 [H^{1,1}_{K}] = 0$ and $|D^1 t_K| \leq C$ to obtain $|D^1 \tilde{G}_{K,N,m,\eta}| < C$. Therefore we calculate

$$|D^1 \tilde{G}_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| = |D^1 \tilde{G}_{K,N,m,\eta}(\tilde{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(x)))|$$

$$= |D^1 \tilde{G}_{K,N,m,\eta}(\tilde{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(x)))|$$

$$\leq C.$$
On the set $E_4$ we estimate as follows. It is easy to check from the definition of $\tilde{G}_{K,N,m,\eta}$ (3.28) using Proposition 5.1 that because $2K \geq N$ we have

$$|D_1 \tilde{G}_{K,N,m,\eta}| \leq |D_1 [J_{K}^{3,1}]_{2N}| \leq CK^{\alpha+1} \text{ on } E_4.$$  

On $E_4$ we use Lemma 4.5 and Proposition 5.10 (again $\hat{F}_{\beta,m}(\tilde{G}_{K,N,m,\eta}(x)) \in \mathcal{G}$) to get $|DG_{K,N,m,\eta}| \leq C$ and thus

$$|D_1 G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| < CK^{\alpha+1} \text{ on } E_4.$$  

Let us consider $x \in E_6$ now and let us assume that for example (other permutations can be treated analogously)

$$x_3 \in [-2m - 3, 2m + 3] \setminus [-2m - 1, 2m + 1]$$  

and

$$x_1, x_2 \in [-2m - 1, 2m + 1] \text{ and } x_4 \in [-1, 1].$$

For almost all $x$ there exists exactly one cube $Q(n,1)$ closest to $x$ with $-m \leq n_1, n_2 \leq m$ and $n = (2n_1, 2n_2, 2m, 0)$. On the set $[2n_1 - 1, 2n_1 + 1] \times [2n_2 - 1, 2n_2 + 2] \times ([-2m - 3, 2m + 3] \setminus [-2m - 1, 2m + 1]) \times [-1, 1]$ the map $\tilde{G}_{K,N,m,\eta}$ is defined as $[J_{K}^{3,3}]_{2N}(x) + n + x_3 e_3$ (see (3.28)). Either there exists some $0 \leq k < 2N$ such that $(x_1, x_2, x_4) \in U_{3,k,k+1}^3 \setminus U_{3,k,k+1}^3$ or $(x_1, x_2, x_4) \in U_{3,k,2N}^3$. In the case that $(x_1, x_2, x_4) \in U_{3,k,k+1}^3 \setminus U_{3,k,k+1}^3$ we use a rotated version of (5.1) to estimate

$$|D_1 \tilde{G}_{K,N,m,\eta}(x)| \leq |D_1 [J_{K}^{n}]_{2N}(x)| \leq C2^{-k\beta}K^{\alpha+1}$$  

(7.6)

and in the second case, where $(x_1, x_2, x_4) \in U_{3,k,2N}^3$ we use (5.3) to estimate

$$|D_1 \tilde{G}_{K,N,m,\eta}(x)| \leq |D_1 [J_{K}^{n}]_{2N}(x)| \leq C2^{-2N\beta}.$$  

(7.7)

It holds, by (4.5), that $\hat{F}_{\beta,m}$ maps $\mathcal{K}_{B,m} \cap [-2m - 3, 2m + 3] \times [-3, 3]$ onto itself. The map $\hat{F}_{\beta,m}$ is bi-Lipschitz and so $\text{dist}(\mathcal{K}_{B,m}, \hat{F}_{\beta,m}(x)) \approx \text{dist}(\mathcal{K}_{B,m}, x)$. Especially, when we call

$$O_k = \{(x_1, x_2, x_3, x_4) : (x_1, x_2, x_4) \in U_{k,3}, x_3 \in [-2m - 3, 2m + 3] \setminus [-2m - 1, 2m + 1]\}$$

(and note that each $O_k$ is roughly speaking $\mathcal{K}_{B,m} + Q(0, 2^{-k(\beta+1)})$) we use the same reasoning as used in Proposition 4.3 to get a constant $M$ such that for each $x \in O_k \setminus O_{k+1}$

$$([\hat{F}_{\beta,m}(x)]^1, [\hat{F}_{\beta,m}(x)]^2, [\hat{F}_{\beta,m}(x)]^4) \in U_{k-M}^3 \setminus U_{k+M+1}^3.$$  

(7.8)

Similarly in the $e_2$ direction

$$[\hat{F}_{\beta,m}(G_{K,N,m,\eta}(x))]^3 \in ([-2m - 3, 2m + 3] \setminus [-2m - 1, 2m + 1]) \cup (U_0 \setminus U_{M+1} + n)$$
Therefore, using (3.18) if we are outside $Q(n,1)$ and (3.15) if we are in $Q(n,1)$ for the definition of $G_{K,N,m,\eta}$ and (5.5) and (5.6) to calculate, we get

$$|DG_{K,N,m,\eta}(y)| \leq C \min\{2^{(k+M)\beta}, 2^{(N+3M)\beta}\}$$  \hspace{1cm} (7.9)

for $(y_1,y_2,y_4) \in U_{k-M}^3 \setminus U_{k+M+1}^3, \quad y_2 \in \left([-2m-3,2m+3] \setminus [-2m-1,2m+1]\right) \cup (U_0 \setminus U_{M+1+n})$.

We calculate $|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)|$ for $x \in O_k \setminus O_{k+1}$ and $0 \leq k < 2N$ by multiplying (7.6) by (7.9) (using (7.8)), which gives

$$|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| \leq C^{2-2(m+1)}(K+k)^{\alpha+1}2^{(k+M)\beta}$$

$$\leq C(K+k)^{\alpha+1}$$

$$\leq C(K + 2N)^{\alpha+1}$$

$$\leq CK^{\alpha+1}$$

because $2K \geq N$. For $x \in O_{2N}$ we multiply (7.7) by (7.9) (using (7.8)) to get

$$|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| \leq C^{2-N\beta}2^{(N+3M)\beta}$$

$$\leq C^{2-N\beta}.$$ 

The combination of these two estimates gives that

$$|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| \leq CK^{\alpha+1}$$

on $E_6$ by considering the various permutations of the coordinates.

For a point in $E_5$ such that its nearest cube is $Q(n,1)$ we use exactly the same estimates as before. Namely, because $C_{A,K} \subset D_{K,2N}^3$, we use precisely the estimates from above, that is, we multiply (7.7) by (7.9) (using (7.8)) to get

$$|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}(x)| \leq C^{2-N\beta}2^{(N+3M)\beta}$$

$$\leq C^{2-N\beta}.$$ 

Therefore

$$|D_1G_{K,N,m,\eta} \circ \hat{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| \leq C$$

on $E_5$ by considering the various permutations of the coordinates.

Estimates on $E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12}$ are similar to estimates above. The only difference is that we map slab of thickness $\eta$ on slab of thickness 13 by $\tilde{G}_{K,N,m,\eta}$ (see (3.26), (3.27)) and then slab of thickness 13 back to slab of thickness $\eta$ by $G_{K,N,m,\eta}$ (see (3.17)). The derivative $D_1\tilde{G}_{K,N,m,\eta}$ can be estimated by the same expression as before since the additional factor $\frac{1}{\eta}$ is influencing only
\( D_4 \tilde{G}_{K,N,m,\eta} \) (note that, for example, in (3.27) we have

\[
|D_1 \frac{2}{\eta} (x_4 - 1 - \frac{\eta}{2})(x_1, x_2, x_3, 0)| \leq C \text{ as } x_4 \in \left[1 + \frac{\eta}{2}, 1 + \eta\right]
\]

and so there is no additional \( \frac{1}{\eta} \) there). The derivative \( DG_{K,N,m,\eta} \) can be estimated by the same expression as the additional factor \( \eta \) in some of the terms can only help us.

The estimate of \( |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| \) on \( E_7 \) is calculated the same way as on \( E_6 \) and

\[
|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| < CK^{\alpha+1} \text{ on } E_7.
\]

Similarly \( |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| \) is estimated on \( E_8 \) is calculated the same way as on \( E_5 \) and

\[
|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| < C \text{ on } E_8.
\]

The estimate on \( E_9 \) is the same as on \( E_4 \)

\[
|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| < CK^{\alpha+1} \text{ on } E_9.
\]

The estimate on \( E_{10} \) is the same as on \( E_5 \)

\[
|D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}| < C \text{ on } E_{10}.
\]

Combining the above with (7.4) we easily calculate

\[
\int_{E_1 \cup E_2 \cup E_3 \cup E_5} |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq C(m + 1)^2,
\]

\[
\int_{E_4 \cup E_6} |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq CK^{(\alpha+1)(p-1)+1}(m + 1)^2,
\]

\[
\int_{E_7 \cup E_9} |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq CK^{(\alpha+1)(p-1)+1}\eta(m + 1)^3,
\]

\[
\int_{E_8 \cup E_{10}} |D_1 G_{K,N,m,\eta} \circ \tilde{F}_{\beta,m} \circ \tilde{G}_{K,N,m,\eta}|^p \leq C\eta(m + 1)^3.
\]

Adding the estimates in (7.10) to point (iii) of Proposition 6.5 we get precisely \( v \). By rotating the sets we get the same estimates for the \( D_j \)-derivative, \( j = 2, 3 \).

**Step 6. Proof of (vi).**

Now it is useful to slightly change the decomposition of \( R_{m,\eta} \setminus Z_m \) from Step 5. We will not use set \( E_1, E_2, ..., E_6 \) instead we define

\[
\tilde{E}_3 = [-2m - 5, 2m + 5]^3 \setminus [-2m - 3, 2m + 3]^3 \times [-1, 1] \cap \left( \bigcup_{j=1}^{3} (C_{A,K,m} + \mathbb{R}e_j) \right)
\]
\[ E_4 = [-2m - 5, 2m + 5]^3 \setminus [-2m - 3, 2m + 3]^3 \times [-1, 1] \setminus \left( \bigcup_{j=1}^{3} (C_{A,K,m} + \mathbb{R}e_j) \right) \]

\[ E_5 = [-2m - 3, 2m + 3]^3 \setminus [-2m - 1, 2m + 1]^3 \times [-1, 1] \cap \left( \bigcup_{j=1}^{3} (\mathbb{D}(A,K,m + \mathbb{R}e_j)) \right) \]

\[ E_6 = [-2m - 3, 2m + 3]^3 \setminus [-2m - 1, 2m + 1]^3 \setminus [-1, 1] \setminus \left( \bigcup_{j=1}^{3} (C_{A,K,m} + \mathbb{R}e_j) \right). \]

We need to estimate \( D_4 \hat{G}_{K,N,m,\eta} \). The estimates of measure from (7.4) still hold even when we replace \( E_i \) with \( \hat{E}_i \) for \( i = 3, 4, 5, 6 \). By the rotational symmetry of the map, the calculations of \( |D_4 G_{K,N,m,\eta} \circ \hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}| \) on \( \hat{E}_3 \cup \hat{E}_4 \cup \hat{E}_5 \cup \hat{E}_6 \) are exactly the same as estimates of \( |D_4 G_{K,N,m,\eta} \circ \hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}| \) in Step 5 for \( E_3 \cup E_4 \cup E_5 \cup E_6 \). Precisely we have

\[ |D_4 G_{K,N,m,\eta} \circ \hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}| < CK^{\alpha+1} \text{ on } \hat{E}_3 \cup \hat{E}_4 \cup \hat{E}_5 \cup \hat{E}_6. \quad (7.11) \]

It is easy, by (3.26), (3.27) and (3.30), to estimate that

\[ |D_4 \hat{G}_{K,N,m,\eta}| \leq \frac{C}{\eta} \text{ on } E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12}. \]

Especially on \( E_8 \) we have \( D_4 \hat{G}_{K,N,m,\eta} = \frac{C}{\eta} e_4 \). Again

\[ \hat{F}_\beta,m(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \text{ for } x \in \hat{G}_{K,N,m,\eta}(E_8). \]

Easily from (3.17) we see that \( |D_4 G_{K,N,m,\eta} (\hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}(x))| < C \) on \( x \in E_8 \). Therefore

\[ |D_4 G_{K,N,m,\eta} \circ \hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}| < \frac{C}{\eta} \text{ on } E_8. \quad (7.12) \]

We calculate on \( E_7 \) in two parts. The first part is when

\[ x \in \bigcup_{n_1,n_2,n_3 = -m}^{m} \left( U_{K,2}^3 \times \left([-1 - \frac{2n}{26-\eta}, -1] \cup [1, 1 + \frac{2n}{26-\eta}] \right) \right) \cup (2n_1, 2n_2, 2n_3, 0). \]

From (3.26) and (7.2) we obtain that \( \hat{G}_{K,N,m,\eta}(x) \) maps this set into

\[ \bigcup_{n_1,n_2,n_3 = -m}^{m} (U_{2}^3 \times ([3, -1] \cup [1, 3]) \cup (2n_1, 2n_2, 2n_3, 0) \]

and that \( D_4 \hat{G}_{K,N,m,\eta} = \frac{C}{\eta} e_4 \) there. By (4.9) we have \( D_4 \hat{F}_\beta,m (\hat{G}_{K,N,m,\eta}(x)) = -e_4 \) and by (3.17) we have \( |D_4 G_{K,N,m,\eta}| < C \). Therefore

\[ |D_4 G_{K,N,m,\eta} \circ \hat{F}_\beta,m \circ \hat{G}_{K,N,m,\eta}| < \frac{C}{\eta}. \]
on the first part. The second part is
\[
\left( [-2m - 5, 2m + 5]^3 \setminus \bigcup_{n_1, n_2, n_3 = -m}^m U_{K, 2}^3 + (2n_1, 2n_2, 2n_3, 0) \right) \times \left( [-1 - \frac{2\eta}{26 - \eta}, -1] \cup \left[ 1, 1 + \frac{2\eta}{26 - \eta} \right] \right).
\]

From (3.26) and (7.2) we obtain that \( \tilde{G}_{K, N, m, \eta}(x) \) maps this set into
\[
\left( [-2m - 5, 2m + 5]^3 \setminus \bigcup_{n_1, n_2, n_3 = -m}^m U_{2}^3 + (2n_1, 2n_2, 2n_3, 0) \right) \times \left( [-3, -1] \cup [1, 3] \right).
\]

On this set we have that \( |DF_{\beta, m}| < C \) and that \( \hat{F}_{\beta, m}(\tilde{G}_{K, N, m, \eta}(x)) \notin \mathcal{T} \) by Lemma 4.5. Therefore \( |DG_{K, N, m, \eta}(\hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}(x))| < C \) on this second part as well and
\[
|D_4 G_{K, N, m, \eta} \circ \hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}| < \frac{C}{\eta} \text{ on } E_7. \tag{7.13}
\]

It remains to consider \( E_9 \cup E_{10} \). From (3.26) and (7.3) we see that \( \tilde{G}_{K, N, m, \eta}(x) \) maps this set into
\[
[-2m - 5, 2m + 5]^3 \times ([-14, -3] \cup [3, 14]).
\]

From Lemma 4.5 we see that \( \hat{F}_{\beta, m}(\tilde{G}_{K, N, m, \eta}(x)) \notin \mathcal{T} \) and so \( |DG_{K, N, m, \eta}(\hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}(x))| < C \). Therefore
\[
|D_4 G_{K, N, m, \eta} \circ \hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}| < \frac{C}{\eta} \text{ on } E_9 \cup E_{10}. \tag{7.14}
\]

Then
\[
\int_{E_9 \cup E_5} |D_4 G_{K, N, m, \eta} \circ \hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}|^p \leq C(m + 1)^2,
\]
\[
\int_{E_4 \cup E_6} |D_4 G_{K, N, m, \eta} \circ \hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}|^p \leq CK^{(\alpha+1)(p-1)+1}(m + 1)^2, \tag{7.15}
\]
\[
\int_{E_9 \cup E_8 \cup E_{10}} |D_4 G_{K, N, m, \eta} \circ \hat{F}_{\beta, m} \circ \tilde{G}_{K, N, m, \eta}|^p \leq C\eta^{1-p}(m + 1)^3.
\]

Adding the estimates in (7.15) to point (iii) of Proposition 6.5 we get precisely vi). \( \Box \)

**Proof of Theorem 3.1.** We choose \( \alpha = \frac{2+p}{3-2p} \) and note that for \( p \in [1, \frac{3}{2}] \) we easily have \( \alpha > \frac{4}{2-p} \).

Further we set \( N = 2K \) and \( \eta = K^{-\alpha-1} \). We assume that \( K \geq K_1 \) so that \( N \geq 3M^* + 3M + 6 \) and \( N \geq 3M + K \). We calculate \( J = J(K, m) \approx m \) the largest natural number strictly smaller than \( \frac{2m+5}{2+2K^{-\alpha}} \) and call \( d_J = \frac{2+2K^{-\alpha}}{2m+5} \). We divide the cube \( Q(0, 1) \) into \( J \) disjoint (up to common boundaries) ‘plates’ call them
\[
P_i = [-1, 1]^3 \times [-1 + (i-1)d_J, -1 + i \cdot d_J]
\]
for $i = 1, \ldots, J$. We define $g_{K,m}$ on each $P_i$ we apply a rescaled and translated version of the mapping from Proposition 7.1, that is, for each $x \in P_i$ we put

$$ g_{K,m}(x) = \frac{1}{2m + 5} f_{K,3M+K,m,K^{-\alpha}} (2m + 5 \left( x - (1 + (i - \frac{1}{2}) \frac{d_j}{2}) e_4 \right)) + (1 + (i - \frac{1}{2}) \frac{d_j}{2}) e_4 $$

and finally, for $x \in Q(0,1) \setminus \left( J \bigcup_{i=1}^J P_i \right)$ (a set of measure less than $3/m$) we define $g_{K,m}(x) = x$.

Points (i)–(iii) follow from Proposition 7.1 (points (i)–(iv)) given we call the set

$$ E_{K,m} = \bigcup_{i=1}^J \left( \frac{1}{2m + 5} C_{A,K,m} - e_4 + i \cdot d_j e_4 - \frac{d_j}{2} e_4 \right). $$

Now note by (3.6) we have

$$ \frac{\mathcal{L}^4(C_{A,K})}{\mathcal{L}^4(Q(0,1))} \geq 1 - CK^{-\alpha-1} \quad \text{and therefore} \quad \frac{\mathcal{L}^4(C_{A,K,m})}{\mathcal{L}^4(Z_m)} \geq 1 - CK^{-\alpha-1}. $$

Further, it is an obvious geometrical fact that $\mathcal{L}^4(Z_m)/\mathcal{L}^4(R_{m,K^{-\alpha}}) \to \frac{1}{1+K^{-\alpha}}$ as $m \to \infty$. Hence it is easy to see that given $\epsilon > 0$ we can choose $K$ big enough and $m$ big enough so that

$$ \mathcal{L}^4 \left( \bigcup_{i=1}^J P_i \setminus E_{K,m} \right) < \frac{\epsilon}{2} \quad \text{and} \quad \mathcal{L}^4 \left( Q(0,1) \setminus \bigcup_{i=1}^J P_i \right) < \frac{\epsilon}{2} $$

and so

$$ \mathcal{L}^4 \left( Q(0,1) \setminus E_{K,m} \right) < \epsilon $$

for all $m \geq m_0$ and $K > K_0$. Therefore the set $E_{K,m}$ satisfy the condition iv).

Clearly $D g_{K,m}(x) = D f_{K,K+3M+K,m,K^{-\alpha}}(\Phi_{i,m}(x))$ where $\Phi_{i,m}$ is the affine map of $P_1$ onto $R_{m,\eta}$ and hence by a simple linear change of variables we have

$$ \int_{P_i} |D_j g_{K,m}(y)|^P (2m + 5)^4 \, dy = \int_{R_{m,K^{-\alpha}}} |D_j f_{K,3M+K,m,m^{-1}}(x)|^P \, dx $$

and also

$$ \int_{P_i \setminus (E_{K,m} \cap P_i)} |D_j g_{K,m}(y)|^P (2m + 5)^4 \, dy = \int_{R_{m,K^{-\alpha}} \setminus C_{A,K,m}} |D_j f_{K,3M+K,m,m^{-1}}(x)|^P \, dx $$

for all $j = 1, 2, 3, 4$. We sum this over all $1 \leq i \leq J \approx m$, and use Proposition 7.1 points v) and vi)

$$ \int_{Q(0,1) \setminus E_{K,m}} |D_j g_{K,m}|^P \leq C(p, \alpha, \beta) K^{(\alpha+1)(p-1)+1}(m + 1)^2 \frac{J}{(2m + 5)^4} $$

$$ + C(p, \alpha, \beta) K^{(\alpha+1)(p-1)+1}(m + 1)^3 K^{-\alpha} \frac{J}{(2m + 5)^4} $$

$$ + C(p, \alpha, \beta) K^{-(\alpha+1)(2-p)+2}(m + 1)^3 \frac{J}{(2m + 5)^4} $$

$$ \leq C(p, \alpha, \beta) \left[ K^{(\alpha+1)(p-1)+1} m^{-1} + K^{-(\alpha+1)(2-p)+2} \right] $$
for \( j = 1, 2, 3 \). Similarly
\[
\int_{Q(0,1)\setminus E_{K,m}} |D_4 g_{K,m}|^p \leq C(p, \alpha, \beta) K^{(\alpha+1)(p-1)+1} (m + 1)^2 \frac{J}{(2m + 5)^4} \\
+ C(p, \alpha, \beta) (m + 1)^3 K^{-(1-p)\alpha} \frac{J}{(2m + 5)^4} \\
+ C(p, \alpha, \beta) K^{-(-\alpha+1)(2-p)+2} (m + 1)^3 \frac{J}{(2m + 5)^4} \\
\leq C(p, \alpha, \beta) \left[ K^{(\alpha+1)(p-1)+1} m^{-1} + K^{\alpha(p-1)} + K^{-(-\alpha+1)(2-p)+2} \right] \\
\leq C(p, \alpha, \beta) \left[ K^{(\alpha+1)(p-1)+1} m^{-1} + K^{\alpha(p-1)} \right]
\]
where we have used \( K^{-(-\alpha+1)(2-p)+2} \leq 1 \leq K^{\alpha(p-1)} \) in the last step (recall that \( \alpha > \frac{4}{2-p} \)). Therefore for \( j = 1, 2, 3 \) we have
\[
\int_{Q(0,1)\setminus E_{K,m}} |D_j g_{K,m}|^p \cdot \int_{Q(0,1)\setminus E_{K,m}} |D_4 g_{K,m}|^p \\
< C K^{2(\alpha+1)(p-1)+2} m^{-2} + C K^{(\alpha+1)(2p-2)-p+2} m^{-1} \\
+ C K^{-(\alpha+1)(3-2p)+3} m^{-1} + C K^{-\alpha(3-2p)+p}.
\] (7.16)

First we show that the last term is bounded. By the choice of \( \alpha = \frac{2+p}{3-2p} \) we have \( K^{-\alpha(3-2p)+p} = K^{-2} \), therefore when \( K \geq \sqrt[5]{C} \) we have \( C K^{-\alpha(3-2p)+p} \leq \frac{1}{25} \). Now we fix \( K > \max\{K_0, K_1, \sqrt[5]{C}\} \). Clearly
\[
m^{-1} \left( C K^{2(\alpha+1)(p-1)+2} m^{-1} + C K^{(\alpha+1)(2p-2)-p+2} + C K^{-(\alpha+1)(3-2p)+3} \right) \to 0 \text{ as } m \to \infty
\]
and therefore we can choose \( m \) large enough so that the other terms are smaller than \( \frac{1}{24} \). Thus we have
\[
\int_{Q(0,1)\setminus E_{K,m}} |D_j g_{K,m}|^p \cdot \int_{Q(0,1)\setminus E_{K,m}} |D_4 g_{K,m}|^p \leq \frac{1}{12}
\] (7.17)
which is (v). Having chosen \( K \) and \( m \) we put \( f_1 = g_{K,m} \) and so Theorem 3.1 is proven. \( \square \)

8 \ PROOF OF THEOREM 1.5

8.1 \ Construction of \( f_2 \) by composing \( f_1 \) with itself

Theorem 8.1. Let \( Q(0,1) \) be the cube in dimension 4, let \( \varepsilon > 0 \) and \( 1 \leq p < \frac{3}{2} \). There exists a closed set \( \mathcal{E} \subset Q(0,1) \) and a map \( f_2 \in W^{1,p}(Q(0,1), \mathbb{R}^4) \) such that,

(i) \( f_2(x) = x \) for \( x \in \partial Q(0,1) \),
(ii) \( f_2 \) is locally bi-Lipschitz on \( Q(0,1) \setminus \mathcal{E} \).
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(iii) \( Jf_2 < 0 \) almost everywhere on \( \mathcal{E} \).

(iv) \[
\int_{Q(0,1) \setminus \mathcal{E}} |Df_2|^p < \frac{1}{3} \mathcal{L}^d(Q(0,1)).
\] (8.1)

(v) \( \mathcal{L}(Q(0,1) \setminus \mathcal{E}) < \varepsilon \).

**Proof.** The starting point for our construction is the mapping \( f_1 \) from Theorem 3.1, where \( p \) is the \( p \) of our claim, \( \varepsilon \) is some small fixed positive number. By point (ii) of Theorem 3.1 we have that \( f_1 \) is locally bi-Lipschitz on \( Q(0,1) \setminus E \). In the following we have the pair \( a \) in the preimage and \( c \) in the image, so that \( c = f_1(a) \). Assume that \( a = f_1^{-1}(c) \in Q(0,1) \setminus E \) is a point of differentiability of \( f_1 \) and a point of approximate continuity of \( Df_1 \). Then we define

\[
S(c) = \frac{DF_1(a)(e_4)}{|DF_1(a)(e_4)|},
\]

that is, \( S(c) \) is the unit vector in the direction of the image of \( e_4 \) in the differential of \( f_1 \) at \( a \) (note that \( DF_1(a)(e_4) \neq 0 \) by the local bi-Lipschitz quality of \( f_1 \)). Call \( O_c \) a sense-preserving unitary map so that \( O_c(S(c)) = e_1 \). For each such \( c \) we define the cube \( Q_{c,r} = O_c^{-1}(Q(0,r)) + c \). Then we define

\[
g_{c,r}(x) = O^{-1}_c(r f_1(O_c(\frac{x-c}{r}))) + c
\]

(8.3)

for \( x \in Q_{c,r} \) for each of the \( c \) where we have defined \( S(c) \). Note especially that \( g_{c,r}(x) = x \) on \( \partial Q_{c,r} \). We intend to apply Lemma 2.5, which is formulated for \( f_{c,r} = r f_1(a - x) \) but our \( g_{c,r} \) is composed of a unitary map from inside and outside. This does not make any difference to the estimates however.

For each \( c \in Q(0,1) \) call \( a = f_1^{-1}(c) \). We denote \( A_{c,r}(x) = c + Df_1(a)(x - a) \). Now \( f_1 \) is locally Lipschitz on \( Q(0,1) \setminus E \) and so for any \( c \in Q(0,1) \setminus E \) there is \( L_c > 0 \) so that for \( 0 < r < \frac{1}{2} \) \( \text{dist}(c, E) \) we have

\[
|Df_1(x)|^p \leq L_c^p \text{ for almost all } x \in [f_1^{-1}(Q_{c,r}) \cup A_{c,r}^{-1}(Q_{c,r})].
\]

This, together with the approximate continuity of \( Df_1 \), the absolute continuity of its integral and the fact that \( a \) is a point of differentiability of \( f_1 \) means that if \( c \) is fixed then there exists an \( r_c \) such that when \( 0 < r < r_c \) it holds that

\[
|Df_1(a)|^p \mathcal{L}^d(A_{c,r}^{-1}(Q_{c,r})) \leq \frac{3}{2} \int_{f_1^{-1}(Q_{c,r})} |Df_1|^p \text{ for } j = 1, 2, 3, 4.
\]

(8.4)

We may moreover assume that \( 0 < r_c \) is small enough so that the conclusion of Lemma 2.5 holds for \( g_{c,r} \) on \( Q_{c,r} \), with \( F = f_1 \). Therefore by Lemma 2.6 we have a covering of \( Q(0,1) \setminus (f_1(E) \cup \mathcal{N}) \) (where \( E \) is the closed set from Theorem 3.1 and \( \mathcal{N} \) is the closed null set of Lemma 2.6) by rotated cubes \( \{Q_{c_i,r_i}\} \) such that \( r_i < r_{c_i} \), for \( i \in \mathbb{N} \). We can infer from our covering and from (8.4) that

\[
\sum_{i=1}^{\infty} |Df_1(a_i)|^p \mathcal{L}^d(A_{c_i,r_i}^{-1}(Q_{c_i,r_i})) \leq \frac{3}{2} \int_{Q(0,1)} |Df_1|^p \text{ for } j = 1, 2, 3, 4.
\]

(8.5)
Then we define \( f_2 \) as \( f_1 \) composed of a scaled and rotated copy of \( f_1 \) on \( f_1^{-1}(Q_{c_i,r_i}) \), that is,

\[
f_2(x) = \begin{cases} 
  f_1(x) & x \in Q(0,1) \setminus f_1^{-1}(\bigcup_{i=1}^{\infty} Q_{c_i,r_i}) \\
  g_{c_i,r_i} \circ f_1(x) & x \in f_1^{-1}(Q_{c_i,r_i}).
\end{cases}
\] (8.6)

Let us note that \( Q(0,1) \setminus f_1^{-1}(\bigcup_{i=1}^{\infty} Q_{c_i,r_i}) = E \cup f_1^{-1}(\mathcal{N}) \) and hence it is \( E \) up to a zero measure set. This immediately implies that \( f_2(x) = x \) on \( \partial Q(0,1) \) (point (i)). The fact that \( f_1 \) is a homeomorphism implies that \( g_{c,r} \) is a homeomorphism on \( Q_{c_i,r_i} \) and thus we see that \( f_2 \) is a homeomorphism. We call

\[
E_i = c_i + O_c^{-1} (r_i E)
\]

where \( N = Q(0,1) \setminus \bigcup_i Q_{c_i,r_i} \) is closed and has zero measure. Since \( Q_{c_i,r_i} \) have pairwise disjoint interiors and each \( E_i \) is closed we have that the complement of \( E \) is the union of open sets and so \( E \) is closed in \( Q(0,1) \) (point (ii) first claim). Since \( f_1 \) is locally bi-Lipschitz on \( Q(0,1) \setminus E \), \( g_{c_i,r_i} \) is locally bi-Lipschitz on \( Q(0,1) \setminus (c_i + O_c^{-1}(r_i E)) \) and so \( f_2 \) is locally bi-Lipschitz on \( Q(0,1) \setminus E \) (point (ii) second claim). Further, the fact that \( J f_1 < 0 \) on \( E \) and \( J f_1(a_i) > 0 \) easily gives that \( J g_{c_i,r_i} \circ f_1 < 0 \) on \( f_1^{-1}(c_i + O_c^{-1}(r_i E)) \) and so we get \( J f_2 < 0 \) almost everywhere on \( E \) and \( J f_2 > 0 \) almost everywhere on \( Q(0,1) \setminus E \) (point (iii)). It remains to prove point (iv).

**The main idea** is that the derivative of the composition is \( |D g_{c_i,r_i}(f_1(x)) D f_1(x)| \) and the derivative of \( f_1 \) is big only in the \( x_4 \) direction (see Theorem 3.1) and small in \( x_1, x_2, x_3 \) directions. We have rotated our \( g_{c_i,r_i} \) so that big derivative \( D_4 f_1 \) is multiplied by derivatives of \( g_{c_i,r_i} \) only in the directions where it is small (that is, \( D_1 \) of the rotated and scaled \( f_1 \)). Thus the derivative of the composition is small in average. More precisely, we use derivative of the composition, (8.3), \( O_c S(c) = e_1 \) and the fact that \( O^{-1} \) is unitary to obtain

\[
|D g_{c,r}(y) S(c)| = \left| O_c^{-1} r_4 f_1 \left( \frac{x - c}{r} \right) \right| O_c \frac{1}{r} S(c) = \left| O_c^{-1} f_1 \left( \frac{x - c}{r} \right) \right| e_1 = \left| D_1 f_1 \left( \frac{x - c}{r} \right) \right|.
\] (8.7)

By \( Z_i \) we denote the linearized preimage of \( Q_{c_i,r_i} \setminus E_i \), that is,

\[
Z_i = a_i + [D f_1(a_i)]^{-1}[(Q_{c_i,r_i} \setminus E_i) - c_i].
\]

We calculate using (8.6), (2.4) of Lemma 2.5 with \( \rho = \frac{1}{64} \), a linear change of variables, (8.2) and (8.7)

\[
\int_{f_1^{-1}(Q_{c_i,r_i} \setminus E_i)} |D f_2|^p \leq \int_{Z_i} |D g_{c_i,r_i}(c_i + D f_1(a_i)(x - a_i)) D_4 f_1(a_i)|^p \, dx + \frac{1}{64} r_i^4
\]

\[
\leq |D_4 f_1(a_i)|^p \int_{Q_{c_i,r_i} \setminus E_i} \frac{|D g_{c_i,r_i}(y) S(c_i)|^p}{\det D f_1(a_i)} \, dy + \frac{1}{64} r_i^4
\]
Using (8.5) we may sum over $Q_{c_i,r_i}$; we recall that $f$ is locally Lipschitz and that we use Theorem 3.1:

$$
\int_{Q(0,1)\setminus E} |D_4f_1(x)|^p \, dx \leq \sum_{i=1}^\infty \left[ |D_4f_1(a_i)|^p \frac{L^4\left([Df_1(a_i)]^{-1}Q_{c_i,r_i}\right)}{L^4(Q(0,1))} \int_{Q(0,1)\setminus E} |Df_1(x)|^p + \frac{1}{64}r_i^4 \right]
$$

$$
\leq \frac{3}{2} \int_{Q(0,1)\setminus E} |D_4f_1|^p \int_{Q(0,1)\setminus E} |Df_1(x)|^p + \frac{1}{64}L^4(Q(0,1))
$$

$$
\leq \frac{1}{8} + \frac{1}{4}.
$$

We estimate the other derivatives $j = 1, 2, 3$ in similar fashion to the above (recall that $f_1^{-1}(\bigcup_i Q_{c_i,r_i} \cup N) = Q(0,1) \setminus E$ and that $f_1$ is bi-Lipschitz on the set $f_1^{-1}(N)$)

$$
\int_{Q(0,1)\setminus E} |D_jf_2|^p \leq \sum_{i=1}^\infty \int_{f_1^{-1}(Q_{c_i,r_i}) \setminus E} |Dg_{c_i,r_i}(f_1(x))D_jf_1(x)|^p \, dx
$$

$$
\leq \sum_{i=1}^\infty \int_{Z_i} |Dg_{c_i,r_i}(c_i + Df_1(a_i)(x - a_i))D_jf_1(a_i)|^p \, dx + \frac{1}{64}r_i^4
$$

$$
\leq \sum_{i=1}^\infty |D_jf_1(a_i)|^p \int_{Q_{c_i,r_i} \setminus E} \frac{|Dg_{c_i,r_i}(y)|^p}{\det Df_1(a_i)} \, dy + \frac{1}{64}r_i^4
$$

$$
\leq \sum_{i=1}^\infty |D_jf_1(a_i)|^p \frac{L^4\left([Df_1(a_i)]^{-1}Q_{c_i,r_i}\right)}{L^4(Q(0,1))} \int_{Q(0,1)\setminus E} |Df_1|^p + \frac{1}{64}r_i^4
$$

$$
\leq \frac{3}{2} \int_{Q(0,1)\setminus E} |D_jf_1|^p \int_{Q(0,1)\setminus E} |Df_1|^p + \frac{1}{64}L^4(Q(0,1))
$$

$$
\leq \frac{1}{8} + \frac{1}{4}.
$$

Now

$$
\int_{Q(0,1)\setminus E} |Df_2|^p \leq \sum_{j=1}^4 \int_{Q(0,1)\setminus E} |D_jf_2|^p
$$

$$
\leq \sum_{j=1}^4 \left( \frac{1}{8} + \frac{1}{4} \right)
$$

$$
\leq \frac{1}{3}L^4(Q(0,1)),
$$
thus proving point (iv). It is obvious from Theorem 3.1 that
\[ \mathcal{L}^d(Q(0,1) \setminus \mathcal{E}) < \varepsilon, \] (8.8)
which gives (v).

\[ \square \]

\section{8.2 Proof of Theorem 1.5}

By induction we construct a sequence of maps \( \{f_m\}_{m=2}^\infty \), that converge in \( W^{1,p} \) to some \( f \), which satisfies our claim. We refer to the following properties as the induction hypothesis: for each \( f_m \) we have a closed set \( \mathcal{E}_m \) such that \( J f_m < 0 \) almost everywhere on \( \mathcal{E}_m \) and \( f_m \) is locally bi-Lipschitz on the complement of \( \mathcal{E}_m \). The start point \( f_2 \) is the map from Theorem 8.1 and we denote \( \mathcal{E}_2 = \mathcal{E} \) the set from Theorem 8.1. The fact that \( f_2 \) satisfies the induction hypothesis is included in the claim of Theorem 8.1.

Assume then that we have constructed \( f_{m'} \), \( 2 \leq m' \leq m-1 \) so that they satisfy the induction hypothesis and we now want to construct \( f_m \). We continue to construct a system of cubes \( \{Q_i = Q(c_i, r_i); i \in \mathbb{N}\} \) with pairwise disjoint interiors, which covers \( Q(0,1) \setminus \mathcal{E}_{m-1} \) up to a set \( N_m \) of measure 0. We use the notation that for a given \( c \in Q(0,1) \) we have \( a = f^{-1}(c) \), the set \( P_{c,r} = \{a + [D f_{m-1}(a)]^{-1}Q(0, r)\} \), the linearization
\[ A_{m,c,r}(x) = c + D f_{m-1}(a)(x - a) \]
for \( x \in Q(c, r) \). By the induction hypothesis we have that \( f_{m-1} \) is locally bi-Lipschitz and so for every \( c = f_2(a) \) there exists an \( R_c > 0 \) such that \( f_{m-1} \) is \( L_c \)-Lipschitz on \( B(a, R_c) \). Thus by the approximative continuity of the function \( |D f_{m-1}(a)|^p \) at almost every \( a \), we have, for almost every \( c \) with \( a = f^{-1}(c) \in Q(0,1) \setminus \mathcal{E}_{m-1} \) there is \( r_c \), so that for given \( \rho > 0 \) and every \( 0 < r < r_c \)
\[ |D f_{m-1}(a)|^p \leq 6 \int_{f_{m-1}^{-1}(Q_{c,r})} |D f_{m-1}|^p. \]

Moreover, we can assume that this \( r_c \) is so small that \( f_{m-1}^{-1}(Q(c, r)) \subset Q(0,1) \setminus \mathcal{E}_{m-1} \) and that we can apply Lemma 2.5 on \( f_{m-1} \) we see that
\[ \int_{f_{m-1}^{-1}(Q(c,r))} |D f_{c,r}(f_{m-1}(x))D f_{m-1}(x)|^p dx \]
\[ \leq \int_{A_{m,c,r}^{-1}(Q,c,r)} |D f_{c,r}(A_{m,c,r}(x))D f_{m-1}(a)|^p dx + \rho r^4, \]
(8.9)
where we fix and choose \( \rho > 0 \). Therefore we can apply Corollary 2.7 and get a system \( \{Q_i(m) = Q(c_i, r_i)\} \) covering \( Q(0,1) \setminus \mathcal{E}_{m-1} \) up to a closed null set \( N_m \) such that on each cube \( Q_i(m) \) we have (8.9) with fixed \( \rho = \frac{1}{100} \int_{Q(0,1)\setminus \mathcal{E}_{m-1}} |D f_{m-1}|^p \) and
\[ \sum_{i=1}^{\infty} \mathcal{L}^d(P_{c_i,r_i})|D f_{m-1}(a_i)|^p \leq \frac{6}{5} \int_{Q(0,1)\setminus \mathcal{E}_{m-1}} |D f_{m-1}|^p. \] (8.10)
We define \( f_{c_i, r_i} = r_i f_2(x - c_i/r_i) + c_i \) and

\[
  f_m(x) = \begin{cases} 
  f_{c_i, r_i}(f_{m-1}(x)) & x \in f_{m-1}^{-1}(Q_i(m)) \\
  f_{m-1}(x) & x \in \mathcal{E}_{m-1}.
  \end{cases}
\]

(8.11)

Thus, we see that \( f_m(x) = f_{m-1}(x) \) for \( x \in \partial Q_i(m) \) and that \( f_m \) is a homeomorphism. Call \( E_i = r_i \mathcal{E}_2 + c_i \) and call \( \mathcal{E}_m = \mathcal{E}_{m-1} \cup \bigcup_i f_{m-1}^{-1}(E_i) \cup N_m \). The set \( \mathcal{E}_m \) is closed because \( Q_i(m) \) have pairwise disjoint interiors and each \( f_{m-1}^{-1}(E_i) \) is closed in \( Q_i(m) \) and so \( Q(0,1) \setminus \mathcal{E}_m \) is the union of open sets. The fact that \( f_{m-1} \) is locally bi-Lipschitz outside \( \mathcal{E}_{m-1} \) and \( f_2 \) is locally bi-Lipschitz outside \( \mathcal{E}_2 \) shows that \( f_m \) is locally bi-Lipschitz outside \( \mathcal{E}_m \). The fact that \( J_{f_{m-1}} < 0 \) a.e. on \( \mathcal{E}_{m-1} \) and \( J_{f_2} > 0 \) a.e. outside \( \mathcal{E}_m \) together with the fact that \( J_{f_2} < 0 \) on \( \mathcal{E}_2 \), shows that \( J_{f_m} < 0 \) a.e. on \( \mathcal{E}_m \) and \( J_{f_m} > 0 \) a.e. outside \( \mathcal{E}_m \). Thus we immediately see our maps satisfy the induction hypothesis.

By (8.8), the fact that \( f_m \) is bi-Lipschitz on \( Q_i \) and \( J_{f_{m-1}} \) is close to being constant on \( Q_i \) we have that \( \mathcal{L}^4(Q(0,1) \setminus \mathcal{E}_m) < 2m \mathcal{L}^4(Q(0,1)). \)

(8.12)

The estimate (8.12) holds for \( m = 2 \) by (8.1). Call

\[
  Z_i(m) = \left\{ a_i + [Df_{m-1}(a_i)]^{-1}(Q(0,r_i) \setminus r_i \mathcal{E}_2) \right\}.
\]

By using (8.11), (8.9) with \( \rho = \frac{1}{100} \int_{Q(0,1) \setminus \mathcal{E}_m} |Df_{m-1}|^p \), the definition of \( f_{c_i, r_i} \), a linear change of variables, (8.10) and (8.1) we get for \( m \geq 3 \)

\[
  \int_{Q(0,1) \setminus \mathcal{E}_m} |Df_m|^p \leq \sum_{i=1}^\infty \int_{Z_i} |Df_{c_i, r_i}(c_i + Df_{m-1}(a_i)(x - a_i))Df_{m-1}(a_i)|^p \, dx + \rho r_i^4 \\
  \leq \sum_{i=1}^\infty |Df_{m-1}(a_i)|^p \int_{Q_i \setminus E_i} |Df_{c_i, r_i}(y)|^p \, dy + \rho r_i^4 \\
  \leq \sum_{i=1}^\infty |Df_{m-1}(a_i)|^p \mathcal{L}^4([Df_{m-1}(a_i)]^{-1}Q(0,r_i)) \int_{Q_i \setminus E_i} |Df_{c_i, r_i}(y)|^p \, dy + \rho r_i^4 \\
  \leq \frac{6}{5} \int_{Q(0,1) \setminus \mathcal{E}_m} |Df_{m-1}(y)|^p \mathcal{L}^4([Df_2(y)]^{-1}Q(0,1)) \, dy + \rho \mathcal{L}^4(Q(0,1)) \\
  \leq \frac{1}{2} \int_{Q(0,1) \setminus \mathcal{E}_m} |Df_{m-1}|^p 
\]
and so our claim by induction. Using the fact that $Df_{m-1} = Df_m$ on $\mathcal{E}_{m-1}$ and $|Df_m| = |Df_{m-1}|$ on $E_{m-1} \setminus E_m$

$$\int_{Q(0,1)} |Df_{m-1}(x) - Df_m(x)|^p \, dx \leq 2^p \int_{Q(0,1) \setminus E_{m-1}} |Df_{m-1}|^p + 2^p \int_{Q(0,1) \setminus E_{m-1}} |Df_m|^p$$

$$\leq 2^p \int_{Q(0,1) \setminus E_{m-1}} |Df_{m-1}|^p + 2^p \int_{Q(0,1) \setminus E_m} |Df_m|^p + 2^p \int_{E_{m-1} \setminus E_m} |Df_m|^p$$

$$\leq 2^p \frac{1}{2^{m-2}} L^4(Q(0,1)) + 2^p \frac{1}{2^{m-1}} L^4(Q(0,1)) + 2^p \int_{E_{m-1} \setminus E_m} |Df_m|^p$$

$$\leq 2^p \frac{1}{2^{m-2}} L^4(Q(0,1)) + 2^p \frac{1}{2^{m-1}} L^4(Q(0,1)) + 2^p \int_{Q(0,1) \setminus E_m} |Df_m|^p$$

which shows that $f_m$ is a Cauchy sequence in $W^{1,p}$. Calling $f$ the limit of the sequence we prove our claim. Concerning the Lusin properties of the function we have the following; the map $f_1$ is linear on each scaled copy of $C_{A,K,m}$ and that $f_1$ is locally bi-Lipschitz outside of the union of these sets. This means that $f_2$ is locally bi-Lipschitz on the set $E_2$. By the same argument $f_m$ is locally bi-Lipschitz on $E_m$. Because $f = f_m$ on $E_m$ we easily see that $f$ satisfies the $(N)$ and the $(N^{-1})$ conditions on $E_m$. On the other hand, the union of these sets has full measure in $Q$ and so $f$ satisfies the $(N)$ and the $(N^{-1})$ conditions on $Q$.

Remark 8.2. We would like to know if there is an example like in Theorem for every $1 \leq p < 2$.

We can slightly improve on our construction. So far we have iterated $f_1$ twice to obtain $f_2$ so that big derivative of first iteration in $x_4$ direction meets the derivative of the next iteration in the $x_1$ direction, where it is small. It would be possible to iterate $f_1$ four times and each time rotate the next iteration cleverly not to meet the big derivatives (in the $x_4$ direction) of previous iterations. This should give us the result for any $1 \leq p < \frac{7}{4}$. We have not pursued this direction as our computations are already quite technical.

To briefly hint that we note that with our choice $\eta = K^{-\alpha}$ after four iterations we get in the key estimate analogous to (7.16)

$$\left( \int_{Q(0,1) \setminus E} |DgK,m|^p \right)^3 \cdot \int_{Q(0,1) \setminus E} |DgK,m|^p \leq \frac{C(K)}{m} + CK^{3[-(\alpha+1)(2-p)+2]} K^{\alpha(p-1)}.$$

The last key term is $K^{\alpha(4p-7)+p}$ and for any $p < \frac{7}{4}$ we can choose $\alpha$ big enough so that the exponent is negative. Finally for $m$ large enough the first term on the right-hand side is as small as we wish.

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