ON DIRAC’S DELTA CALCULUS

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Abstract

It is shown that theories already presented as rigorous mathematical formalizations of widespread manipulations of Dirac’s delta function are all unsatisfactory, and a new alternative is proposed.

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I. Introduction

There is a “myth” in Mathematical Physics according to which:

*The widespread manipulations of Dirac’s delta function can be rigorously formalized by the Theory of Distributions.*

It is easy to verify, however, that this statement can be taken as true only by someone who does not know the nature of such manipulations or by someone who does not know the available theories about the concept of distribution.

The most elementary manipulations of the delta function, and also the most often found in literature, involve the composition of $\delta$ and a real function. For example:

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x - a) + \delta(x + a)] \quad (a > 0),$$

or, more generally:

$$\delta[g(x)] = \sum_{i=1}^{n} \frac{1}{|g'(a_i)|}\delta(x - a_i),$$

where $a_1, a_2, \ldots, a_n$ are the simple roots of a real function $g$.

Considering the delta function as a distribution according to the best known definition of this concept—linear functionals on some test functions space—the above formulae could not be written as such, since the composite of a distribution and a generic real function simply cannot be done.

Using the sequential approach we can define the composition of a distribution and some very simple real functions, but the function which sends $x$ to $(x^2 - a^2)$ is not among them. Therefore, also in this context, not even the first formula above could have been written.

It is clear that we can use the above equality to “define” the symbol $\delta[g(x)]$, but in any way could this be considered a “formalization” of the relatively simple idea which one has in mind when writing it down. It is obvious that *the formal meaning ascribed to the basic and elementary manipulations of Delta Calculus must be related to the mental process usually understood in these manipulations.*

In this sense, the sequential approaches to the concept of distribution are much superior to that of Schwartz-Sobolev, which systematically uses the “theorems” of Delta Calculus as “definitions” of the involved concepts. For instance: the distributions are defined as linear functionals on some test functions space so we can characterize the delta
function by its sifting property:

$$\int_{-\infty}^{\infty} \delta(x) f(x) \, dx = f(0).$$

That procedure, considered “elegant” by many mathematicians, merely dismisses the fact that the sifting property itself is a basic result of the Delta Calculus to be formally proved.

Dirac has used a simple argument, based on the integration by parts formula, to get the sifting property of the derivative $\delta'$ of the delta function:

$$\int_{-\infty}^{\infty} \delta'(x) f(x) \, dx = -f'(0).$$

The theory of distributions as linear functionals, instead of defining the integral of a distribution and so proving that it satisfies some kind of integration by parts formula, just uses the formula deduced by Dirac for the delta function:

$$\int_{-\infty}^{\infty} \delta'(x) f(x) \, dx = -\int_{-\infty}^{\infty} \delta(x) f'(x) \, dx$$

as “distributions derivative definition”.

Proceeding systematically in this fashion, it is not surprising that some “unnecessary” constructions must be ignored after certain symbols and operations have been “defined”, because we know Delta Calculus has a nontrivial mathematical content.

The real surprise is the consideration of this Theory of Distributions as a satisfactory formalization of Delta Calculus, despite its ignoring the basic operational rules of this Calculus.

All proposed formalizations of Delta Calculus fail because they try to “generalize” the real functions without having generalized its variables. This will become clear if we compare those formalizations to the one proposed in this work.

In Ref. 4 we presented an extension process that can be applied to any set. This process was applied$^5$ to the ordered field of real numbers, formally defining what we call virtual numbers. Among those numbers we have infinitesimal and infinite quantities, which were so used in an reorganization of Differential and Integral Calculus. The same extension process was applied$^6$ to the set of real functions, introducing what we call virtual functions. So, the basic techniques of Infinitesimal Calculus were generalized to these mathematical objects.

The concept of “virtual function” is very close to the original idea of “improper function” created by Dirac: the limit of a sequence of real functions$^3$. The essential difference
is that Dirac did not have access to “improper numbers” which could have been used as argument of these functions, whereas the virtual functions can be evaluated at (previously defined) virtual numbers, exactly as we do with real functions and numbers.

We will here use many concepts and results presented in Refs. 4, 5 and 6, including the notational conventions adopted then (like the notation for reduced integrals stated in Ref. 6).

This paper begins by defining, in Sec. II, the class of virtual functions having the characteristics supposed by Dirac for his delta function. In Sec. III we prove the “Dirac’s functions” sifting property. Section IV is dedicated to discussing the meaning of the operational rules which make up Delta Calculus. In Sec. V we prove the basic rule for compositions involving delta function (above). Other well known results of Delta Calculus are demonstrated in the last two sections.

II. Dirac’s Functions

The aim of this section is to characterize the virtual functions which possess the basic properties assumed by Dirac for his “delta function”, and to show that there exist virtual functions in \( \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}) \) with those properties.

We will say that a virtual function \( \delta \in \mathcal{F}(\mathbb{R}) \) is a \textit{Dirac’s function} when:

(i) \( \delta \) is defined at any \( \xi \in \mathbb{R} \) and is non-negative:

\[
\delta(\xi) \geq 0, \quad \text{for every } \xi \in \mathbb{R} ;
\]

(ii) \( \delta \) is integrable with

\[
\int_{-\infty}^{\infty} \delta(x) \, dx = 1 ; \quad \text{and}
\]

(iii) there exists a positive infinitesimal \( \varepsilon \in \mathbb{R} \) such that

\[
|\xi| \geq \varepsilon \ \Rightarrow \ \delta(\xi) = 0.
\]

Before proceeding, it is convenient to verify that there exist virtual functions under those conditions. This is not so difficult: we know there exists a real function \( f: \mathbb{R} \rightarrow \mathbb{R} \) infinitely derivable which vanishes outside the interval \((-1, 1)\), increasing between \(-1\) and 0, where it takes the value \( f(0) = 1 \), decreasing between 0 e 1, and with unitary integral:

\[
\int_{-1}^{1} f(x) \, dx = 1.
\]
From such a function $f$, we can define a virtual function \( \delta_0: \mathbb{R} \to \mathbb{R} \) by:

\[
\delta_0(\xi) = \infty f(\infty \xi),
\]
i.e., \( \delta_0 \in \mathcal{F}(\mathbb{R}) \) is the virtual function represented by the sequence \((f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]\) given by:

\[
f_n(x) = nf(nx).
\]

It is easy to see that this virtual function satisfies the three conditions above:

\[
\delta_0(\xi) \geq 0, \quad \text{for every } \xi \in \mathbb{R},
\]

\[
\int_{-\infty}^{\infty} \delta_0(x) \, dx = 1,
\]

and

\[
|\xi| \geq \varepsilon \Rightarrow \delta_0(\xi) = 0.
\]

Therefore \( \delta_0 \) is a Dirac's function.

That virtual function \( \delta_0 \) perfectly corresponds to the image given by Dirac\(^3\):

“To get a picture of \( \delta(x) \), take a function of the real variable \( x \) which vanishes everywhere except inside a small domain, of length \( \varepsilon \) say, surrounding the origin \( x = 0 \), and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations (for example provided the function is always of order \( \varepsilon^{-1} \)). Then in the limit \( \varepsilon \to 0 \) this function will go over into \( \delta(x) \).”

Moreover, the function \( \delta_0 : \mathbb{R} \to \mathbb{R} \) constructed above is infinitely derivable (so continuous), normalized in the strong sense:

\[
\int_{-\infty}^{\infty} \delta_0(\xi) \, d\xi = 1,
\]

and, evaluated at real values of its argument, provides:

\[
\delta_0(x) = \begin{cases} 
\infty, & \text{if } x = 0; \\
0, & \text{if } x \neq 0.
\end{cases}
\]

In order for a virtual function \( \delta: \mathbb{R} \to \mathbb{R} \) to be a Dirac’s function, it is not necessary that \( \delta(0) = \infty \), or even that \( \delta(0) \approx \infty \). But the condition \( \delta(x) = 0 \), for every real \( x \neq 0 \), immediately follows from the existence of an infinitesimal \( \varepsilon \) such that \( |\xi| > \varepsilon \Rightarrow \delta(\xi) = 0 \).
For instance, the virtual function \( \psi : \mathbb{R} \to \mathbb{R} \) given by:

\[
\psi(\xi) = \frac{\infty}{1 + \infty^2 \xi^2}
\]

is such that \( \psi(x) \approx 0 \) for every \( x \neq 0 \):

\[
\psi(x) = \frac{1}{\partial + \infty x^2} \approx 0.
\]

But \( \psi(x) \neq 0 \), for any \( \xi \in \mathbb{R} \), and so \( \psi \) is not a Dirac's function. (That function is the derivative of \( \phi(\xi) = \arctan(\infty \xi) \), and it was used in various examples in Ref. 5.)

It should be clear that there exist many distinct Dirac's functions in \( \mathcal{F} (\mathbb{R}) \). For example, if \( \delta_0 : \mathbb{R} \to \mathbb{R} \) is the function above constructed then the virtual functions \( \delta_+ : \mathbb{R} \to \mathbb{R} \) and \( \delta_- : \mathbb{R} \to \mathbb{R} \) given by:

\[
\delta_+ (\xi) = \delta_0 (\xi - 2\partial),
\]

and

\[
\delta_- (\xi) = \delta_0 (\xi + 2\partial)
\]

are two other infinitely differentiable Dirac's functions distinct from \( \delta_0 \). The function \( \delta_+ \) is the class of the sequence \( (g_1, g_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})] \) defined by:

\[
g_n(x) = f_n(x - \frac{2}{n}),
\]

where \( f_n \) are the above functions which represent \( \delta_0 \in \mathcal{F} (\mathbb{R}) \). Analogously, \( \delta_- \) is the class of the sequence \( (h_1, h_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})] \) defined by:

\[
h_n(x) = f_n(x + \frac{2}{n}).
\]

A Dirac's function like \( \delta_+ \) can be convenient if we intend, for instance, to apply the Laplace transformation, since it vanishes for every \( \xi < 0 \), but is normalized in the sense:

\[
\int_0^\infty \delta_+(x) \, dx = 1.
\]

Furthermore, it is easy to verify that:

If \( \delta_1 \) and \( \delta_2 \) are two Dirac's functions then the virtual function \( \delta_3 : \mathbb{R} \to \mathbb{R} \) defined by:

\[
\delta_3(\xi) = \frac{1}{2} [\delta_1(\xi) + \delta_2(\xi)]
\]
is also a Dirac’s function.

We have not required a Dirac’s function to be always continuous or derivable, so we can take “square pulses” like the function \( \delta_n \in F(\mathbb{R}) \) represented by the sequence:

\[
g_n(x) = \begin{cases} 
n/2, & \text{if } |x| < 1/n; \\
0, & \text{if } |x| \geq 1/n. 
\end{cases}
\]

This Dirac’s function, which is discontinuous at \( \xi = -\partial \) and \( \xi = \partial \), can also be useful in specific situations. It is such that:

\[
\delta_\Gamma(\xi) = \begin{cases} 
\infty/2, & \text{if } |\xi| < \partial; \\
0, & \text{if } |\xi| \geq \partial.
\end{cases}
\]

III. Sifting Property

Our aim in this section is to demonstrate the sifting property of Dirac’s functions. We will say that a real function is continuous around \( x \in \mathbb{R} \) when there exists a real open interval containing \( x \) in which it is defined and continuous.

If \( \delta: \mathbb{R} \to \mathbb{R} \) is a Dirac’s functions, and \( f: \mathbb{R} \to \mathbb{R} \) is a real function continuous around the origin, then:

\[
\int_{-\infty}^{\infty} \delta(x) f(x) \, dx = f(0).
\]

Proof: First, we note that there exists a positive infinitesimal \( \varepsilon \) such that:

\[
|\xi| \geq \varepsilon \Rightarrow \delta(\xi) = 0,
\]

for \( \delta \) is a Dirac’s function. By the additivity with respect to the virtual integration interval \( \delta(\xi) f(\xi) \, d\xi \), we have:

\[
\int_{-\infty}^{\infty} \delta(\xi) f(\xi) \, d\xi = \int_{-\varepsilon}^{\varepsilon} \delta(\xi) f(\xi) \, d\xi + \int_{-\varepsilon}^{\varepsilon} \delta(\xi) f(\xi) \, d\xi + \int_{\varepsilon}^{\infty} \delta(\xi) f(\xi) \, d\xi.
\]

On the right-hand side of the above equation, the integrands of the first and third integrals vanish, for any function \( f \). So those integrals exist and are equal to zero. In the second integral (right-hand side) we have the product of two integrable functions, so the left-hand side integral exists and:

\[
\int_{-\infty}^{\infty} \delta(\xi) f(\xi) \, d\xi = \int_{-\varepsilon}^{\varepsilon} \delta(\xi) f(\xi) \, d\xi.
\]

It is clear that, in \( \mathbb{R} \), the following statement holds: if \( g \) and \( h \) are two real functions defined between \(-a\) and \( a\), where \( a \) is a positive real number, with \( g \) integrable and
non-negative and $h$ continuous, then there exist two real numbers $c_1$ and $c_2$ between $-a$ and $a$ such that:

$$h(c_1) \int_{-a}^{a} g(x) \, dx \leq \int_{-a}^{a} g(x)h(x) \, dx \leq h(c_2) \int_{-a}^{a} g(x) \, dx.$$  

So we have, by the Virtual Extension Theorem (VET, Ref. 4), that if $\phi$ and $\psi$ are two virtual functions defined between $-\alpha$ and $\alpha$, where $\alpha$ is a positive virtual number, with $\phi$ integrable and non-negative and $\psi$ continuous, then there exist two virtual numbers $\gamma_1$ and $\gamma_2$ between $-\alpha$ and $\alpha$ such that:

$$\psi(\gamma_1) \int_{-\alpha}^{\alpha} \phi(\xi) \, d\xi \leq \int_{-\alpha}^{\alpha} \phi(\xi)\psi(\xi) \, d\xi \leq \psi(\gamma_2) \int_{-\alpha}^{\alpha} \phi(\xi) \, d\xi.$$  

Now making $\alpha = \varepsilon$, $\phi = \delta$ and $\psi = f$, we conclude that there exist two virtual numbers $\gamma_1$ and $\gamma_2$ between $-\varepsilon$ and $\varepsilon$ such that:

$$f(\gamma_1) \int_{-\varepsilon}^{\varepsilon} \delta(\xi) \, d\xi \leq \int_{-\varepsilon}^{\varepsilon} \delta(\xi)f(\xi) \, d\xi \leq f(\gamma_2) \int_{-\varepsilon}^{\varepsilon} \delta(\xi) \, d\xi.$$  

Since $\gamma_1$ and $\gamma_2$ are between the infinitesimals $-\varepsilon$ and $\varepsilon$, they are also infinitesimals themselves. Then it follows from the continuity of $f$ at the origin that:

$$f(\gamma_1) \approx f(0) \approx f(\gamma_2),$$  

and therefore:

$$\int_{-\infty}^{\infty} \delta(\xi) f(\xi) \, d\xi \approx f(0).$$

It is important to note that the sifting property demonstrated above holds for all Dirac’s functions. If we are working with a particular Dirac’s function in a specific context then we can deduce variants of this property by analogous methods. For instance, we can use the function $\delta_+$ from the previous section to “unilaterally” sift a real function $f$ defined only for positive values of $x$.

**IV. Dirac’s Equivalence**

Delta Calculus is based on a set of operational rules, whose meaning has been established clearly by Dirac himself:

“There are a number of elementary equations which one can write down about $\delta$ functions. These equations are essentially rules of manipulation for algebraic work involving
\( \delta \) functions. The meaning of any of these equations is that its two sides give equivalent results as factors in an integrand.”

We will say that two integrable virtual functions \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) and \( \psi: \mathbb{R} \rightarrow \mathbb{R} \) are equivalent when, for every continuous real function \( f: \mathbb{R} \rightarrow \mathbb{R} \), we have:

\[
\int_{-\infty}^{\infty} \phi(x) f(x) \, dx = \int_{-\infty}^{\infty} \psi(x) f(x) \, dx.
\]

It is important to note that, according to this definition, both integrals must be reducible in order for the two functions to be equivalent. That means, if \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) is such that there exists a continuous real function \( f: \mathbb{R} \rightarrow \mathbb{R} \) which makes the virtual integral

\[
\int_{-\infty}^{\infty} \phi(\xi) f(\xi) \, d\xi
\]

irreducible, then \( \phi \) is equivalent only to itself. For example, if \( \delta \) is a Dirac’s function and \( k \in \mathbb{R} \) a non-zero real constant, then the sum function \( (\delta + k) \) is equivalent only to itself, since the integral

\[
\int_{-\infty}^{\infty} [\delta(\xi) + k] \, d\xi = \int_{-\infty}^{\infty} \delta(\xi) \, d\xi + \int_{-\infty}^{\infty} k \, d\xi \approx 1 + 2k\infty
\]

is not reducible.

We will indicate that two virtual functions \( \phi \) and \( \psi \) are equivalent by writing:

\[
\phi(x) \equiv \psi(x).
\]

For instance:

\[
\delta(2x) \equiv \frac{1}{2} \delta(x),
\]

for every Dirac’s function \( \delta \) (this is a simple consequence from the “composition rule” which will be stated and proved in the next section). We use a real variable in this notation (Latin, not Greek letter) to reinforce that the two expressions are interchangeable only in reduced integrals.

It is not difficult to verify that “\( \equiv \)” is an equivalence relation on the set of integrable virtual functions defined at any \( \xi \in \mathbb{R} \). This relation will be called Dirac’s equivalence.

The above example can be used to gain a flash of intuition about the nature of this equivalence: the graph of function \( \delta(2x) \) is “the same height as one of the function \( \delta \), but half of its width”; whereas the graph of function \( (1/2)\delta(x) \) is “the same width as the
one of function $\delta$, but half of its height”. Thus, although fairly different, those functions are equivalent as factors in an integrand:
\[
\int_{-\infty}^{\infty} \delta(2x) f(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{2} \delta(x) f(x) \, dx = \frac{1}{2} f(0),
\]
since both delimit “the same area infinitely concentrated around the origin”, whereas any continuous real function $f$ “varies slowly”.

To discuss the equivalence of Dirac’s functions, let us first consider the following definition: an integrable virtual function $\phi: \mathbb{R} \to \mathbb{R}$ is sifting when:
\[
\int_{-\infty}^{\infty} \phi(x) f(x) \, dx = f(0),
\]
for any continuous real function $f: \mathbb{R} \to \mathbb{R}$.

It is not difficult to see that the set of all sifting functions is an equivalence class according to the Dirac’s relation.

Every Dirac’s function is sifting (as we saw in the previous section), but not every sifting virtual function is a Dirac’s function. As a counterexample, we can take the sequence $(f_1, f_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$ used in Sec. I to define $\delta_0$, and alter all $f_n$ at one and only one point:
\[
g_n(x) = \begin{cases} 
  f_n(x), & \text{if } x \neq 7; \\
  3, & \text{if } x = 7.
\end{cases}
\]
This sequence $(g_1, g_2, \ldots) \in \Sigma[\mathcal{F}(\mathbb{R})]$ represents an integrable virtual function $\phi: \mathbb{R} \to \mathbb{R}$ which is clearly sifting, but not a Dirac’s function, for $\psi(7) = 3$.

Even so, we can affirm:

*Any two Dirac’s functions are always equivalent to each other.*

This fact makes two conventions which are part of the traditional language of Delta Calculus compatible:

(i) All Dirac’s functions are represented by the same symbol: “$\delta$”.

(ii) The Dirac’s equivalence is indicated simply by the equality symbol: “$=$”.

Those conventions drastically simplify notation, and certainly do not jeopardize the rigour of a scientific work if used properly. Nevertheless, since they might generate some confusion, we will continue to explicitly distinguish equality ($=$) from equivalence ($\equiv$), and to represent distinct Dirac’s functions by distinct symbols.

The traditional language of Delta Calculus also does not establish a clear distinction between sifting functions and Dirac’s functions, since the latter are generally handled using the sifting property. That is another possible source of confusion, which we should keep in mind, mainly when the above conventions are used.
V. Composition Rule

Dirac’s functions, as defined in Sec. II, are virtual functions $\delta \in \mathcal{F}(\mathbb{R})$ with certain specific characteristics. Therefore, it is clear that they can be composed with any other virtual function $\phi$. Besides, the composite virtual function $(\delta \circ \phi)$ is defined at every point in the domain of $\phi$, since Dirac’s functions are defined at any virtual number $\xi \in \mathbb{R}$.

Several of the operational rules of Delta Calculus involve the composition of a Dirac’s function $\delta$ and a real function $g$. We looked at the meaning of those rules in the previous section: they establish Dirac’s equivalences for the composite virtual function $(\delta \circ g)$. To discuss these equivalences, we will suppose that the real function $g$ is defined on the whole virtual extension $\mathbb{R}$ of the real line.

Dirac’s functions do not vanish only around the origin, so:

If $\delta: \mathbb{R} \to \mathbb{R}$ is a Dirac’s function and $g: \mathbb{R} \to \mathbb{R}$ a real function for which there exists a positive $r \in \mathbb{R}$ with $|g(x)| > r$ for every $x \in \mathbb{R}$, then the composite function $(\delta \circ g): \mathbb{R} \to \mathbb{R}$ is identically null:

$$\delta[g(\xi)] = 0, \quad \text{for every } \xi \in \mathbb{R}.$$  

In this case, it is clear that $\delta[g(\xi)] \equiv 0$. For instance:

$$\delta(x^2 + 1) \equiv 0 \quad \text{and} \quad \delta(\sin x + 2) \equiv 0.$$  

Let us consider now a real function whose image approximates the origin: $g(x) = x^2$, for example. If we take the Dirac’s function $\delta_-$ defined in Sec. II:

$$\delta_-(\xi) = \delta_0(\xi + 2\partial),$$

then:

$$\delta_-(\xi^2) = 0, \quad \text{for every } \xi \in \mathbb{R},$$

since $\delta_-(\xi) = 0$ for any $\xi \geq 0$. So it is clear that:

$$\delta_-(x^2) \equiv 0.$$  

On the other hand, composing the “square” Dirac’s function $\delta_\square$ (also defined in Sec. II) and that same function $g(x) = x^2$, we get a non-vanishing composite:

$$|\xi| < \sqrt{\partial} \Rightarrow \delta_\square(\xi^2) = \frac{\infty}{2}.$$
Furthermore, it easy to see that:

$$\delta_{\gamma}(\xi^2) \neq 0,$$

for:

$$\int_{-\infty}^{\infty} \delta_{\gamma}(\xi^2) \, d\xi = \int_{-\sqrt{\gamma}}^{\sqrt{\gamma}} \frac{\infty}{2} \, d\xi = \frac{\infty}{2} 2\sqrt{\gamma} = \sqrt{\infty}.$$

So, we have:

$$\delta_{-}(x^2) \neq \delta_{\gamma}(x^2),$$

which shows that the equivalence class of this composite depends on the particular Dirac’s function chosen. That means, there is not a virtual function $\phi$ such that

$$\delta(x^2) \equiv \phi(x)$$

for any Dirac’s function $\delta$. Therefore, there is not an operational rule for the composite $\delta(x^2)$ in the traditional language of Delta Calculus, which deals only with generic equivalences that hold for all Dirac’s functions.

This dependence of the class of the composite with respect to the selection of the Dirac’s function might also occur if the graph of the real function $g: \mathbb{R} \to \mathbb{R}$ is asymptotic of the $x$-axis. As an example, for $g(x) = e^x$ we have:

$$\delta_{-}(e^x) = 0 \quad \text{for every } \xi \in \mathbb{R},$$

whereas:

$$\delta_{+}(e^x) \neq 0.$$

However, the class of the composite function will not depend on the chosen Dirac’s function if the graph of the real function $g: \mathbb{R} \to \mathbb{R}$ only “crosses” the $x$-axis in a finite number of roots, not approximating it in any other region:

Let $\delta: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be any Dirac’s function and $g: \mathbb{R} \to \mathbb{R}$ a real function with a finite number of roots $a_1, a_2, \ldots, a_n$; and such that there exist $n + 1$ positive real numbers $r, r_1, r_2, \ldots, r_n$ having the following properties:

(i) the $n$ intervals $[a_i - r_i, a_i + r_i]$ are disjoint in pairs and

$$|g(x)| > r,$$

for any $x \in \mathbb{R}$ outside those intervals; and
(ii) inside the \( n \) intervals \([a_i - r_i, a_i + r_i]\), the function \( g \) is differentiable and its derivative does not vanish.

Then, for any real function \( f: \mathbb{R} \to \mathbb{R} \) continuous on the \( n \) intervals \([a_i - r_i, a_i + r_i]\), we have:

\[
\int_{-\infty}^{\infty} \delta[g(x)] f(x) \, dx = \sum_{i=1}^{n} \frac{f(a_i)}{|g'(a_i)|},
\]

Proof: Since the composition \( \delta[g(\xi)] \) vanishes outside the \( n \) intervals \([a_i - r, a_i + r]\), we have:

\[
\int_{-\infty}^{\infty} \delta[g(\xi)] f(\xi) \, d\xi = \sum_{i=1}^{n} \int_{a_i - r_i}^{a_i + r_i} \delta[g(\xi)] f(\xi) \, d\xi.
\]

We will calculate the integrals in the above sum separately. First, we note that, for each \( i = 1, 2, \ldots, n \), the restriction of the function \( g \) to the interval \([a_i - r_i, a_i + r_i]\) admits an inverse \( h_i \):

\[
h_i[g(x)] = x, \quad \text{for every } x \in [a_i - r_i, a_i + r_i],
\]

and this inverse \( h_i \) is monotonic, differentiable and its derivative does not vanish between \( g(a_i + r_i) \) and \( g(a_i - r_i) \).

Thus, changing variables:

\[
\mu = g(\xi), \quad \xi = h_i(\mu) \quad \text{and} \quad d\mu = g'(\xi) \, d\xi,
\]

we get:

\[
\int_{a_i - r_i}^{a_i + r_i} \delta[g(\xi)] f(\xi) \, d\xi = \int_{a_i - r_i}^{a_i + r_i} \delta[g(\xi)] \frac{f(\xi)}{g'(\xi)} g'(h_i(\mu)) \, d\xi = \int_{g(a_i - r_i)}^{g(a_i + r_i)} \delta(\mu) \frac{f(h_i(\mu))}{g'[h_i(\mu)]} \, d\mu.
\]

If \( g'(a_i) > 0 \) then \( g(a_i - r_i) < 0 < g(a_i + r_i) \), so the sifting property guarantees that:

\[
\int_{g(a_i - r_i)}^{g(a_i + r_i)} \delta(\mu) \frac{f[h_i(\mu)]}{g'[h_i(\mu)]} \, d\mu \approx \frac{f[h_i(0)]}{g'[h_i(0)]} = \frac{f(a_i)}{g'(a_i)}.
\]

On the other hand, if \( g'(a_i) < 0 \) then \( g(a_i + r_i) < 0 < g(a_i - r_i) \), so the sifting property provides:

\[
\int_{g(a_i - r_i)}^{g(a_i + r_i)} \delta(\mu) \frac{f[h_i(\mu)]}{g'[h_i(\mu)]} \, d\mu \approx -\frac{f[h_i(0)]}{g'[h_i(0)]} = -\frac{f(a_i)}{g'(a_i)}.
\]

Thus, in any of both cases we have:

\[
\int_{a_i - r_i}^{a_i + r_i} \delta[g(\xi)] f(\xi) \, d\xi \approx \frac{f(a_i)}{|g'(a_i)|}.
\]
For any number $a \in \mathbb{R}$, the function $g(x) = x - a$ satisfies the hypothesis of the above result, so we have the well known sifting property of translated Dirac’s functions:

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx = f(a),$$

for any $f: \mathbb{R} \to \mathbb{R}$ continuous around $a \in \mathbb{R}$.

So, we conclude that, for any real function $g$ under the above conditions, and for any Dirac’s function $\delta$:

$$\delta[g(x)] \equiv \sum_{i=1}^{n} \frac{1}{|g'(a_i)|} \delta(x - a_i).$$

This formula will be called *composition rule*.

The sifting property of translated Dirac’s functions implies:

$$f(x) \delta(x - a) \equiv f(a) \delta(x - a),$$

for any function $f: \mathbb{R} \to \mathbb{R}$ continuous around $a \in \mathbb{R}$.

The composition rule, on its turn, shows that:

$$\delta(ax) \equiv \frac{1}{|a|} \delta(x) \quad (a \neq 0),$$

and

$$\delta(x^2 - a^2) \equiv \frac{1}{2a} [\delta(x - a) + \delta(x + a)] \quad (a > 0),$$

for any Dirac’s function $\delta$.

**VI. Contraction of Dirac’s Functions**

Dirac’s functions, as defined in Sec. II, are integrable virtual functions $\delta: \mathbb{R} \to \mathbb{R}$. Thus, it is clear that, for any $\beta \in \mathbb{R}$, the virtual function which assigns $\delta(\xi - \beta)$ to $\xi$ is also integrable. The result below will be used as a lemma in the following demonstration:

*If $\delta: \mathbb{R} \to \mathbb{R}$ is a Dirac’s function, and $\beta \in \mathbb{R}$ a finite virtual number, then:*

$$\int_{-\infty}^{\infty} \delta(\xi - \beta) \, d\xi = \int_{-\infty}^{\infty} \delta(\xi) \, d\xi,$$

*Proof:* Changing the variables $\mu = \xi - \beta$ we get:

$$\int_{-\infty}^{\infty} \delta(\xi - \beta) \, d\xi = \int_{-\infty}^{\infty} \delta(\mu) \, d\mu$$

$$= \int_{-\infty}^{-\beta} \delta(\mu) \, d\mu + \int_{-\beta}^{\infty} \delta(\mu) \, d\mu + \int_{-\infty}^{-\beta} \delta(\mu) \, d\mu,$$
but the integrands of the above first and third integrals vanish, for \( \beta \) is finite and \( \delta \) is a Dirac’s function. ■

If \( \delta_1 \) and \( \delta_2 \) are two continuous Dirac’s functions then the virtual function \( \delta_3: \mathbb{R} \rightarrow \mathbb{R} \) defined by:

\[
\delta_3(\xi) = \int_{-\infty}^{\infty} \delta_1(\xi - \beta)\delta_2(\beta) \, d\beta
\]

is also a Dirac’s function. Besides, we have:

\[
\int_{-\infty}^{\infty} \delta_1(\xi - \beta)\delta_2(\beta - \alpha) \, d\beta = \delta_3(\xi - \alpha),
\]

for any finite \( \alpha \in \mathbb{R} \).

Proof: It is quite easy to see that \( \delta_3(\xi) \geq 0 \), for every \( \xi \in \mathbb{R} \), since the above integrand is non-negative. To show that \( \delta_3(\xi) \) vanishes far from the origin, we first note that there exist positive infinitesimals \( \varepsilon_1 \) and \( \varepsilon_2 \) such that:

\[
|\xi| \geq \varepsilon_1 \Rightarrow \delta_1(\xi) = 0 \quad \text{and} \quad |\xi| \geq \varepsilon_2 \Rightarrow \delta_2(\xi) = 0.
\]

Taking \( \varepsilon_3 = \varepsilon_1 + \varepsilon_2 \) we get:

\[
|\xi| \geq \varepsilon_3 \Rightarrow \delta_3(\xi) = 0.
\]

To demonstrate that \( \delta_3 \) is integrable and normalized:

\[
\int_{-\infty}^{\infty} \delta_3(\xi) \, d\xi = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \delta_1(\xi - \beta)\delta_2(\beta) \, d\beta \right] \, d\xi = 1,
\]

let us consider the following affirmation in \( \mathbb{R} \): if \( a_1 \) and \( a_2 \) are two positive real numbers; \( g_1 \) is a real function defined and continuous between \(-a_1\) and \( a_1 \), and such that for every \( b \) between \(-a_2\) and \( a_2 \):

\[
\int_{-a_1}^{a_1} g_1(x - b) \, dx = \int_{-a_1}^{a_1} g_1(x) \, dx;
\]

and \( g_2 \) is a real function defined and continuous between \(-a_2\) and \( a_2 \); then:

\[
\int_{-a_1}^{a_1} \left[ \int_{-a_2}^{a_2} g_1(x - b)g_2(b) \, db \right] \, dx = \left[ \int_{-a_1}^{a_1} g_1(x) \, dx \right] \left[ \int_{-a_2}^{a_2} g_2(b) \, db \right].
\]

This affirmation can easily be proved in \( \mathbb{R} \) by Fubini’s Theorem, and so extended to \( \mathbb{R} \) by the VET:
If \( \alpha_1 \) and \( \alpha_2 \) are two positive virtual numbers; \( \phi_1 \) is a virtual function defined and continuous between \( -\alpha_1 \) and \( \alpha_1 \), and such that for every \( \beta \) between \( -\alpha_2 \) and \( \alpha_2 \):

\[
\int_{-\alpha_1}^{\alpha_1} \phi_1(\xi - \beta) \, d\xi = \int_{-\alpha_1}^{\alpha_1} \phi_1(\xi) \, d\xi;
\]

and \( \phi_2 \) is a virtual function defined and continuous between \( -\alpha_2 \) and \( \alpha_2 \); then:

\[
\int_{-\alpha_1}^{\alpha_1} \left[ \int_{-\alpha_2}^{\alpha_2} \phi_1(\xi - \beta) \phi_2(\beta) \, d\beta \right] \, d\xi = \left[ \int_{-\alpha_1}^{\alpha_1} \phi_1(\xi) \, d\xi \right] \left[ \int_{-\alpha_2}^{\alpha_2} \phi_2(\beta) \, d\beta \right].
\]

Now, making

\[
\int_{-\infty}^{\infty} \delta_3(\xi) \, d\xi = \int_{-\infty}^{\infty} \left[ \int_{-\varepsilon_2}^{\varepsilon_2} \delta_1(\xi - \beta) \delta_2(\beta) \, d\beta \right] \, d\xi,
\]

the above lemma guarantees that taking \( \alpha_1 = \infty \), \( \alpha_2 = \varepsilon_2 \), \( \phi_1 = \delta_1 \), and \( \phi_2 = \delta_2 \), we get:

\[
\int_{-\infty}^{\infty} \delta_3(\xi) \, d\xi = \left[ \int_{-\infty}^{\infty} \delta_1(\xi) \, d\xi \right] \left[ \int_{-\varepsilon_2}^{\varepsilon_2} \delta_2(\beta) \, d\beta \right] \approx 1.
\]

To calculate the integral

\[
\int_{-\infty}^{\infty} \delta_1(\xi - \beta) \delta_2(\beta - \alpha) \, d\beta
\]

we change the variables \( \mu = \beta - \alpha \):

\[
\int_{-\infty}^{\infty} \delta_1(\xi - \beta) \delta_2(\beta - \alpha) \, d\beta = \int_{-\infty}^{\infty} \delta_1(\xi - \alpha - \mu) \delta_2(\mu) \, d\mu
\]

\[
= \int_{-\infty}^{\infty} \delta_1(\xi - \alpha - \mu) \delta_2(\mu) \, d\mu
\]

\[
= \delta_3(\xi - \alpha).
\]

This result shows that the contraction of two continuous Dirac’s functions is equivalent to a third Dirac’s function:

\[
\int_{-\infty}^{\infty} \delta_1(x - \beta) \delta_2(\beta - a) \, d\beta \equiv \delta_3(x - a).
\]

VII. Differentiable Dirac’s Functions

We will now consider a differentiable Dirac’s function (derivable with continuous derivative) and demonstrate the sifting property associated to its derivative. We will
say that a real function is differentiable around \( x \in \mathbb{R} \) when there exists an open real interval containing \( x \) in which the function is defined and differentiable.

If \( \delta : \mathbb{R} \to \mathbb{R} \) is a differentiable Dirac’s function and \( f : \mathbb{R} \to \mathbb{R} \) a real function differentiable around \( a \in \mathbb{R} \), then:

\[
\int_{-\infty}^{\infty} \delta'(x-a)f(x)\,dx = -f'(a).
\]

**Proof:** Since \( \delta \) is a Dirac’s function, there exists a positive infinitesimal \( \varepsilon \) such that:

\[
|\xi| \geq \varepsilon \Rightarrow \delta(\xi) = \delta'(\xi) = 0.
\]

By the additivity with respect to the virtual integration interval, we have:

\[
\int_{-\infty}^{\infty} \delta'(\xi-a)f(\xi)\,d\xi = \int_{-\infty}^{a-\varepsilon} \delta'(\xi-a)f(\xi)\,d\xi + \int_{a-\varepsilon}^{a+\varepsilon} \delta'(\xi-a)f(\xi)\,d\xi + \int_{a+\varepsilon}^{\infty} \delta'(\xi-a)f(\xi)\,d\xi.
\]

On the right-hand side of this equation, the integrands of the first and third integrals vanish, for any function \( f \), and therefore those integrals exist and are equal to zero. In the second integral (right-hand side) we have the product of two continuous functions, so the integral on the left-hand side exists and:

\[
\int_{-\infty}^{\infty} \delta'(\xi-a)f(\xi)\,d\xi = \int_{a-\varepsilon}^{a+\varepsilon} \delta'(\xi-a)f(\xi)\,d\xi.
\]

The VET shows that the *integration by parts formula* holds for virtual integration. Since the function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable between \( a - \varepsilon \) e \( a + \varepsilon \), we have:

\[
\int_{a-\varepsilon}^{a+\varepsilon} \delta'(\xi-a)f(\xi)\,d\xi = \left[ \delta(\xi-a)f(\xi) \right]_{a-\varepsilon}^{a+\varepsilon} - \int_{a-\varepsilon}^{a+\varepsilon} \delta'(\xi-a)f'(\xi)\,d\xi
\]

\[
= \int_{a-\varepsilon}^{a+\varepsilon} \delta(\xi-a)[-f'(\xi)]\,d\xi
\]

\[
= \int_{-\infty}^{\infty} \delta(\xi-a)[-f'(\xi)]\,d\xi
\]

\[
\approx -f'(a).
\]

Taking \( a = 0 \) and \( f(x) \) constantly equal to 1, we get:

\[
\int_{-\infty}^{\infty} \delta'(x)\,dx = 0,
\]

therefore:
If $\delta: \mathbb{R} \to \mathbb{R}$ is a differentiable Dirac’s function then its derivative $\delta': \mathbb{R} \to \mathbb{R}$ cannot be a Dirac’s function.

Proceeding on the same line of argument, it is not difficult to obtain the sifting property associated to the higher order derivatives of a sufficiently differentiable Dirac’s function:

If $\delta: \mathbb{R} \to \mathbb{R}$ is a Dirac’s function $n$ times differentiable, and $f: \mathbb{R} \to \mathbb{R}$ is a real function $n$ times differentiable around $a \in \mathbb{R}$, then:

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - a)f(x) \, dx = (-1)^{n} f^{(n)}(a).$$

These sifting properties of the higher order derivatives of a Dirac’s function only hold for sufficiently differentiable real functions, whereas the Dirac’s equivalence, as defined in Sec. IV, requires that

$$\int_{-\infty}^{\infty} \phi(x)f(x) \, dx = \int_{-\infty}^{\infty} \psi(x)f(x) \, dx$$

for every continuous real function $f: \mathbb{R} \to \mathbb{R}$. Since there are continuous non-derivable real functions, we cannot deduce equivalence formulae in this “strong” sense using those properties, even for a Dirac’s function sufficiently differentiable.

Nevertheless, if we are only dealing with differentiable real functions, we can consider “weak” Dirac’s equivalences, which require integrals equal for this kind of real functions only:

We will say that two integrable virtual functions $\phi: \mathbb{R} \to \mathbb{R}$ and $\psi: \mathbb{R} \to \mathbb{R}$ are equivalent in order $n$ when, for every real function $n$ times differentiable $f: \mathbb{R} \to \mathbb{R}$, we have:

$$\int_{-\infty}^{\infty} \phi(x)f(x) \, dx = \int_{-\infty}^{\infty} \psi(x)f(x) \, dx.$$

We will indicate that two virtual functions are equivalent in order $n$ by writing:

$$\phi(x) \equiv \psi(x) \quad \text{(order } n \text{)}.$$

According to this definition, it is not difficult to deduce, from the sifting properties above, that:

If $\delta: \mathbb{R} \to \mathbb{R}$ is a Dirac’s function $n$ times differentiable, and $g: \mathbb{R} \to \mathbb{R}$ is a real function $n$ times differentiable around $a \in \mathbb{R}$, then:

$$g(x)\delta^{(n)}(x - a) \equiv (-1)^{n} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} g^{(n-i)}(a)\delta^{(i)}(x - a) \quad \text{(order } n \text{)}.$$
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