A note on composite operators in $N = 4$ SYM

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Abstract

We discuss composite operators in $N = 4$ super Yang-Mills theory and their realisations as superfields on different superspaces. The superfields that realise various operators on analytic superspace may be different in the free, interacting and quantum theories. In particular, in the quantum theory, there is a restricted class of operators that can be written as analytic tensor superfields. This class includes all series B and C operators in the theory as well as some series A operators which saturate the unitarity bounds. Operators of this type are expected to be protected from renormalisation.
Over the past few years the Maldacena conjecture [1] has rekindled interest in four-dimensional superconformal field theories and this has led to the discovery of many new and interesting results. Most of these results have concerned properties of short (series C) operators and their correlation functions derived both directly in field theory and from supergravity via the AdS/CFT correspondence. Some recent reviews and lists of references can be found in [2, 3, 4, 5]. A striking feature of such operators is that their shortness protects them from renormalisation - they cannot develop anomalous dimensions because the representations under which they transform determine these dimensions uniquely. More recently, however, it has been found that certain series A operators, which are not short in the above sense and which had not been anticipated to be protected from renormalisation, turn out to also have vanishing anomalous dimensions. These results have been established using the OPE and AdS/CFT [6, 7], from partial non-renormalisation of four-point functions [8, 9], in perturbation theory [10] and, most recently, using the OPE in $N=2$ harmonic superspace [11].

The representations of the superconformal group are well-known [12] and their realisations on superfields have been studied by many authors, see for example [13, 14, 15, 16, 17, 18]. In particular, shortening conditions for series A representations which saturate unitarity bounds have been discussed in [14, 17]. In this note we point out that the series A operators fall into three distinct classes when looked at as explicit functions of the underlying $N = 4$ supersymmetric Yang-Mills field strength superfield. There are 3 different types of behaviour: (i) operators which do not saturate the unitarity bounds, even in the free theory, (ii) operators which saturate the unitarity bounds in the free theory but for which the number of components changes in the interacting theory and (iii) operators which saturate unitarity bounds in the interacting theory. This classification holds in the classical theory where the dimensions are still (half) integral. In the quantum theory operators of types (i) and (ii) can develop anomalous dimensions because there are “nearby” representations with non-integral quantum numbers which have the same number of components. On the other hand, for operators of type (iii) this is not the case, and one therefore expects them to be protected in a similar fashion to the short representations of series B and C. All the operators which have been found to be non-renormalised in references [6, 7, 8, 9, 10, 11] are of type (iii) as one might expect, but this classification suggests that there are very many more of them.

Operators of type (i) take care of themselves in that there are no shortening conditions even in the free case. However, it is not so easy to distinguish between operators of types (ii) and (iii) merely by looking at the quantum numbers of the representations or at their realisation as (abstract) superfields in Minkowski superspace. It turns out that the operators of type (iii) are those that can be written as products of chiral primary operators, possibly with spacetime or spinorial derivatives. Operators of type (ii) include single trace operators (with the exception of the chiral primaries) and more complicated operators which include such single trace functions as factors. The basic reason for this is that the constraints (on Minkowski superspace) which type (iii) superfields must satisfy in order to saturate unitarity bounds follow from the constraints on the gauge-invariant factors whereas, for operators of type (ii), this is not the case, so that the corresponding interacting multiplets have more components than the free ones. One way of seeing this is to work on analytic superspace, this having the advantage that there are no further constraints to be imposed apart from analyticity. A general analytic superconformal
field will transform under the isotropy subgroup of the superconformal group which defines analytic superspace in a non-trivial manner, i.e. it will have superindices (whereas one can find analytic superspaces for the series C operators where no indices are required [17, 18]). We shall work on the analytic superspace with the smallest number of odd coordinates and the smallest number of additional even coordinates compatible with these. All representations with (half) integral dimensions can be constructed from a set of free Maxwell field strength superfields and derivatives with respect to the coordinates of this analytic superspace.\footnote{This is briefly discussed in [19]; a detailed account is in preparation.} The difference between operators of types (ii) and (iii) can be stated very simply in this context: operators of type (ii) cannot be so represented in the interacting case because this would involve applying gauge-covariant derivatives to the non-Abelian SYM field strength superfield and this is not allowed because the Yang-Mills potential is not itself a field on analytic superspace. Operators of type (iii) are therefore composite operators for which the analytic superspace derivatives only act on gauge-invariant factors. The claim, therefore, is that all such series A operators which satisfy a unitarity bound should be protected from renormalisation. In the quantum theory operators of types (i) and (ii) both cease to be realised as analytic tensor superfields. They can still be viewed as analytic fields but their transformation properties are not of the usual tensorial type. On the other hand, operators of type (iii) are analytic tensor superfields even in the quantum theory. In this sense one can view protection from renormalisation as being due to analyticity even for series A operators.

Before discussing this in more detail we shall briefly discuss an example of each type of operator in $N = 4$ super Minkowski space. The field strength superfield $W_I$ transforms under the 6 of $SO(6)$, and is subject to the constraint

$$\nabla_{\alpha i} W_I = (\sigma_I)_{ij} \Lambda^j_\alpha$$

(1)

where $\alpha$ is a 2-component spinor index, $i$ is an $SU(4)$ index and $\sigma_I$ is an $SO(6)$ $\sigma$-matrix. The spinorial derivative includes a gauge field in the non-Abelian case. The leading component of $W_I$ is the set of six scalar fields of $N = 4$ SYM while the leading component of $\Lambda^i_\alpha$ is the quartet of spin one-half fields. The supercurrent is $T^I_J = \text{tr}(W_I W_J) - 1/6 \text{tr}(W_K W_K)$. From (1) it obeys the constraint that when $D_{\alpha i}$ is applied to it only the 20-dimensional representation of $SU(4)$ survives. The quantum numbers specifying a representation of the $N = 4$ superconformal group are $(L, J_1, J_2, a_1, a_2, a_3)$ where $L$ is the dilation weight, $J_1$ and $J_2$ are spin labels and $(a_1, a_2, a_3)$ are $SU(4)$ Dynkin labels. We thus see that $T^I_J$ has quantum numbers $(2, 0, 0, 2, 0)$. The unitarity bounds are:

Series A : \[ L \geq 2 + 2J_1 + 2m_1 - \frac{m}{2} \quad \text{or} \quad L \geq 2 + 2J_2 + \frac{m}{2} \]

Series B : \[ L = \frac{m}{2}; \quad L \geq 1 + m_1 + J_1, \quad J_2 = 0 \]

Series C : \[ L = m_1 = \frac{m}{2}; \quad J_1 = J_2 = 0 \]
where \( m \) is the total number of boxes in the Young tableau of the \( SU(4) \) representation and \( m_1 \) the number of boxes in the first row.

An operator of type (i) is given by \( T_{IJ}T_{IJ} \). This has quantum numbers \( (4, 0, 0, 0, 0, 0) \). It is a series A operator which does not saturate either unitarity bound and is simply an unconstrained scalar superfield on Minkowski superspace. In the quantum theory there is nothing to prevent this operator developing an anomalous dimension.

An example of a type (ii) operator is the \( N = 4 \) Konishi multiplet, \( K = \text{tr}(W_I W_J) \) \([20, 21]\). In the free theory this operator obeys the constraint

\[
D_{ij}K := D_{\alpha i}D^\alpha_j K = 0 \tag{3}
\]

However, in the interacting theory one finds

\[
D_{ij}K \sim \text{tr}([W_{ik}, W_{jl}] W^{kl}) := S_{ij} \tag{4}
\]

so that \( K \) is now an unconstrained superfield \( (W_{ij} := (\sigma_I)_{ij} W_I) \). This is similar in some respects to the behaviour of the Yang-Mills supercurrent in ten dimensions. In the free theory this consists of a quasi-superconformal multiplet \( (128 + 128) \) together with a constrained scalar superfield \[22\] whereas in the interacting theory the scalar superfield is unconstrained \[23\]. As in the type (i) case, in the quantum theory, there is nothing to stop \( K \) developing an anomalous dimension and it is well-known that this indeed happens \[24, 25\].

For an example of type (iii) we consider the operator \( O_{IJ} := T_{IK}T_{JK} - \frac{1}{6} \delta_{IJ}T_{KL}T_{KL} \), which transforms under the \( 20' \) representation of \( SU(4) \). This has quantum numbers \( (4, 0, 0, 0, 2, 0, 0) \); it is a series A operator which saturates both unitarity bounds. This operator obeys the same constraints in the interacting theory as it does in the free theory because they can be derived from the gauge-invariant constraints that \( T_{IJ} \) satisfies. There is a representation related to this one by changing \( L = 4 \) to \( L = 4 + 2\gamma \) where \( \gamma \) is a real number, but it has many more components and so one expects \( O_{IJ} \) to be protected from renormalisation. Indeed, this operator is one of those found to have vanishing anomalous dimensions in references \[6, 7, 8, 9, 10, 11\].

To discuss these operators more generally we shall use super Dynkin diagrams. For the \([4]\) superconformal group \( SL(4|N) \) acting on \( \mathbb{C}^4|N \), the Dynkin diagram depends on the choice of basis. If the basis is ordered in the standard fashion, 4 even - \( N \) odd, we have the distinguished basis with one odd root, but we shall use a different basis, which we shall refer to as physical, in which the basis has the ordering, 2 even - \( N \) odd - 2 even. The physical basis has two odd roots so that the Dynkin diagram is

\[
\begin{array}{c}
\circ \quad \cdots \quad \cdots \quad \circ \\
N-1
\end{array}
\]

Any representation can be specified by giving labels associated to each node of the Dynkin diagram. The labels associated with the two external even (black) nodes are determined by the spin quantum numbers \( (J_1, J_2) \) and the \((N - 1)\) internal even labels are fixed by the Dynkin
labels of $SL(N)$. The two odd (white) labels are then determined by the dilation ($L$) and the R-symmetry ($R$) quantum numbers. All the Dynkin labels should be non-negative integers except for the odd ones which can be positive real numbers. These continuous labels are directly related to anomalous dimensions of operators.

The super Dynkin diagram can also be used to represent coset spaces determined by parabolic subgroups. With respect to a given basis the Borel subalgebra consists of lower triangular matrices, and a parabolic subalgebra (which by definition is one which contains the Borel subalgebra) consists of lower block triangular matrices. The size of these blocks is determined by a set of at most $N + 3$ positive integers $k_1 < k_2 \ldots$ and can be represented on the Dynkin diagram by placing crosses through the $k_i$th nodes (starting from the left). For example, super Minkowski space is represented by

$$\bullet \otimes \bullet \bullet \bullet \ldots \bullet \otimes \bullet$$

(6)

Chiral superspaces have a single cross through one of the odd nodes, harmonic superspaces have crosses through both odd nodes and some internal nodes, and analytic superspaces have crosses only through internal nodes. Superspaces with crosses through the external nodes include projective super twistor space, but such spaces are inconvenient for representation theory and so will not be considered further here.

The crosses on a super Dynkin diagram factorise the diagram into sub-(super)-Dynkin diagrams corresponding to the semi-simple subalgebra of the Levi subalgebra (the diagonal blocks in the parabolic), while the Dynkin labels above the crosses correspond to charges under internal $U(1)$’s or dilation and $R$ weights. In general the Levi subalgebra will be a superalgebra and so the fields can carry superindices. Only in cases where both odd nodes have crosses through (such as for super Minkowski space and harmonic superspaces) does the Levi subalgebra contain no superalgebra.

In order to have unitary representations (of the real superconformal group $SU(2,2|N)$) the Dynkin labels on the odd nodes must exceed those of the adjacent external nodes by at least one unless one or both pairs of these adjacent nodes are zero. This gives three series of unitarity bounds. We label the nodes from the left $n_1 \ldots n_{N+3}$ so that the two odd nodes are $n_2$ and $n_{N+2}$ and the adjacent external nodes are $n_1$ and $n_{N+3}$ respectively. For series A we have $n_2 \geq n_1 + 1$ and $n_{N+3} \geq n_{N+2} + 1$. For series B we have either $n_1 = n_2 = 0$ and $n_{N+3} \geq n_{N+2} + 1$ or we have $n_2 \geq n_1 + 1$ and $n_{N+3} = n_{N+2} = 0$. Finally series C requires that $n_1 = n_2 = n_{N+3} = n_{N+2} = 0$. For general $N$ we have

$$n_2 = \frac{1}{2}(L - R) + J_1 + \frac{m}{N} - m_1$$
$$n_{N+2} = \frac{1}{2}(L + R) + J_2 - \frac{m}{N}$$

(7)

where $m$ is the total number of boxes in the internal Young tableau determined by the $SU(N)$ Dynkin labels $(a_1, \ldots a_{N-1}) = (n_3, \ldots n_{N+1})$ and $m_1$ is the number of boxes in the first row. The external black labels are $(n_1, n_{N+3}) = (2J_1, 2J_2)$. For $N = 4$ we need to impose $R = 0$ in order to have representations of $PSU(2,2|4)$. 

4
The above discussion implies that all of the unitary representations can be represented in various ways on superfields defined on superspaces, and that these fields will transform linearly under representations of the Levi subalgebra. In particular, in \( N = 4 \), all of the representations can be realised as (analytic) superfields on \((N, p, q) = (4, 2, 2)\) analytic superspace:

\[
\begin{array}{c}
\bullet & \bigcirc & \times & \bullet & \bigcirc
\end{array}
\]  

This space is a super Grassmannian with local coordinates

\[
X^{AA'} = \begin{pmatrix}
x^{\alpha\dot{\alpha}} \\
\pi^{\alpha\dot{\alpha}} \\
y^{\alpha\dot{\alpha}}
\end{pmatrix}.
\]

where \( x \) are spacetime coordinates, \( \lambda, \pi \) are odd coordinates and \( y \) are coordinates for the internal manifold. The indices \((\alpha, \dot{\alpha})\) are 2-component spacetime spinor indices while \((a, a')\) are 2-component spinor indices for the internal space which is (locally) the same as spacetime in the complexified case. The capital indices span both spacetime and internal indices, \( A = (\alpha, a), \ A' = (\dot{\alpha}, a') \), and we use the convention that \((\alpha, \dot{\alpha})\) are even indices while \((a, a')\) are odd. As we remarked previously an important feature of analytic superspace is that superfields carrying irreducible representations are completely specified by the super Dynkin labels and analyticity; no further constraints need to be imposed.

In the free theory the Maxwell field strength superfield, corresponding to the representation with \( n_4 = 1 \) and all other Dynkin labels zero, is a single component analytic superfield \( W \). In the interacting case \( W \) is covariantly analytic and so is not a superfield on analytic superspace. However, gauge-invariant products of \( W \) are. The operators \( A_p := \text{tr}(W^p) \) \( p = 2, 3, \ldots \) which transform under the representations which have only the central Dynkin label non-zero are in one-to-one correspondence with the Kaluza-Klein supermultiplets of IIB supergravity on \( AdS_5 \times S^5 \) [26, 27, 28]. The operator \( A_2 := T \) is special; it is the supercurrent multiplet. The diagram for \( A_p \) is \( \begin{array}{c}
\bullet & \bigcirc & \times & \bullet & \bigcirc
\end{array} \). This means that \( A_p \) is a scalar under \( \mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2) \) and has charge \( p \) under the \( U(1) \) corresponding to the central node of the super Dynkin diagram. All other representations transform non-trivially under the sub-algebra \( \mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2) \). The series B superfields must transform under the totally (generalised) antisymmetric tensor representation (or the trivial representation) of one of the \( \mathfrak{sl}(2|2) \) subgroups and the series C superfields must transform under the totally antisymmetric representation of both \( \mathfrak{sl}(2|2) \) subgroups (trivially in the KK case). For a general representation the highest weight state is obtained from the tensor component which has the most number of internal \((a \text{ or } a')\) indices.

We now describe how the three operators discussed earlier can be written as fields on analytic superspace. The first one, \( T_{IJ}T_{IJ} \) in super Minkowski space, has super Dynkin labels \((0200020)\). On analytic superspace its behaviour with respect to both of the \( \mathfrak{sl}(2|2) \) subalgebras is given by the super Dynkin labels \((020)\). It can be constructed from two \( T \)'s and four derivatives with both sets of indices, primed and unprimed, in the representation corresponding to the super Young tableau with two boxes in the first and second rows.

The free Konishi multiplet on \((4, 2, 2)\) analytic superspace is \( \begin{array}{c}
\begin{array}{c}
\bigcirc & \times & \bigcirc
\end{array}
\end{array} \). This saturates the bounds of series A and as a tensor superfield has indices \( K_{AB,A'B'} \) with generalised symmetry on both pairs. \( (A \text{ corresponds to the left } \mathfrak{sl}(2|2) \text{ and } A' \text{ to the right one.}) \) In the interacting
theory, the diagram is the same with the 1 replaced by $1 + \gamma$, $\gamma > 0$. For $\gamma$ non-integral the representation $\textbullet \otimes \textbullet^{1+\gamma}$ of $\mathfrak{sl}(2|2)$ is non-tensorial; it can be explicitly described by putting a cross through the odd node which gives rise to a purely fermionic coset space of $SL(2|2)$ with four odd coordinates. The representation then has 16 components whose transformation properties can be read off. (If we cross both odd nodes in the full $N = 4$ diagram we get a field on harmonic superspace which, being analytic with respect to the internal compact manifold, is equivalent to an unconstrained superfield on super Minkowski space.) The representations $\textbullet \otimes \textbullet^{n}$ of $\mathfrak{sl}(2|2)$ for $n$ integral, $n \geq 2$, all have the same dimension as $\textbullet \otimes \textbullet^{1+\gamma}$, $\gamma > 0$, so that the non-tensorial representation is closely related to these tensorial representations. In terms of the underlying Maxwell supermultiplet, the free Konishi superfield can be written [5]

$$K_{AB,A'B'} = \partial(A'W \partial B)W - \frac{1}{6} \partial(A' \partial B')W^2$$

(10)

However, this expression cannot be generalised to the interacting case since there is no gauge covariant derivative $\nabla_{\mathcal{A}}$ on analytic superspace. Moreover (10) is misleading in the quantum theory. The quantum Konishi multiplet resembles more closely the operator $\textbullet \otimes \textbullet^{2} \otimes \textbullet \otimes \textbullet^{2}$ which has four primed and unprimed indices (both in the 2 times 2 box tableau) and which, by the above discussion, has the same number of components as the interacting quantum $K$.

We now consider the multiplet of type (iii) discussed above which is protected. As a field on analytic superspace it is determined by the diagram $\textbullet \otimes \textbullet \otimes \textbullet \otimes \textbullet^{1+\gamma}$. Again this representation has an associated anomalous representation $\textbullet \otimes \textbullet^{1+\gamma} \otimes \textbullet^{2}$ and there is also a tower of representations with $1 + \gamma$ replaced by $n \geq 2$. These all have the same dimension (as representations of the analytic isotropy group) as opposed to the original representation (with 1’s over both the white nodes) which is smaller. However, as a superfield $\textbullet^{1} \otimes \textbullet^{2} \otimes \textbullet^{2} \otimes \textbullet^{1}$ can be expressed in terms of derivatives of the supercurrent $T = \textbullet \otimes \textbullet^{2} \otimes \textbullet \otimes \textbullet$, and so in this case there is no difficulty in generalising the representation to the interacting case. Explicitly, the superfield for this representation is

$$T_{AB,A'B'} = \partial(A'T \partial B')T - \frac{1}{5} \partial(A' \partial B')T^2.$$  

(11)

We next consider operators of the form $\partial^pT \partial^qT$ on analytic superspace. Since all such operators are compatible with non-Abelian gauge invariance they are all either type (i) or type (iii). We shall consider these operators first in the classical theory where the Dynkin labels are all integers. Those that are type (i) can then develop anomalous dimensions in the quantum theory whereas the others will be protected. The result is simple: those operators which have vanishing internal Dynkin labels are type (i) and all the others are type (iii). This is in agreement with the results derived in [11] using the OPE in $N = 2$ harmonic superspace.

To study these operators we first define $Q = L - (J_1 + J_2)$. Since $Q(\bar{\partial}) = 0$ and $Q(T) = 2Q(W) = 2$, it follows that $Q = 4$ for any of these operators. In terms of the Dynkin labels $Q = \sum_{i=2}^{6} n_i - (n_1 + n_7)$, so that we have

$$n'_2 + n'_6 + m_1 = 4$$

(12)
where \( n'_2 := n_2 - n_1 \geq 1; \) \( n'_6 := n_6 - n_7 \geq 1, \) the inequalities following from the unitarity bounds. The requirement that the R-charge be zero gives

\[
n_3 + 2n_2 - n_1 = n_5 + 2n_6 - n_7
\]

Due to the bounds we need only consider the cases \( m_1 = 0, 1, 2. \) For \( m_1 = 2 \) we have \( n'_2 = n'_6 = 1. \) The possible internal Dynkin labels are \([020], [110], [011], [101], [200] \) and \([002].\)

\( m_1 = 2; [020] \)

For \([020]\) we find the super Dynkin labels are \([k(k + 1)020(k + 1)k].\) These operators can be written in the form \( T \partial^{k+2}T \) with the \( k + 2 \) \( A \) and \( A' \) indices completely symmetrised\(^2\). Clearly such operators saturate the bounds and so are type (iii).\(^2\)

\( m_1 = 2; [110] \)

For this case the super Dynkin labels are \([k(k + 1)110(k + 1)k].\) These operators can be written as \( T \partial^{k+3}T \) where the \((k + 3)\) unprimed and primed indices are both in the representation with symmetrisation over \((k + 2)\) indices but not over all of them. Again these operators saturate the bounds.

\( m_1 = 2; [010] \)

The super Dynkin labels are \([k(k + 1)010(k + 2)(k + 1)].\) In this case the left-hand \( \mathfrak{sl}(2|2) \) representation corresponding to the unprimed indices is the same as the previous case whereas the right-hand one is totally symmetric in \( k + 3 \) indices. These operators cannot be written with all the derivatives hitting one of the \( T \)'s, but can be written in the form \( \partial T \partial^{k+2}T. \) Again these are saturated. The case \([011]\) is conjugate to this one. Note that the leading component of this supermultiplet is fermionic. In super Minkowski space it will involve an odd derivative acting on one of the \( T \)'s.

\( m_1 = 2; [200] \)

Here the super Dynkin labels are \([k(k + 1)200(k + 3)(k + 2)].\) The left-hand \( \mathfrak{sl}(2|2) \) representation has Young tableau \(<k + 2, 1, 1>\) while the right one is \(<k + 4>\) where the notation denotes the number of boxes in the first, second, third row, and so on. It is not possible to construct this representation from derivatives acting on two \( T \)'s by symmetry. The case \([002]\) is conjugate to this one and also cannot be constructed.

\( m_1 = 1; [010] \)

If we choose \( n'_2 = 2, n'_6 = 1 \) we find the super Dynkin labels are \([k(k + 2)010(k + 3)(k + 2)].\) The corresponding tensor has left Young tableau \(<k + 4>\) and right Young tableau \(<k + 2, 2>\). These operators can be written in the form \( \partial^2 T \partial^{k+2}T \) and they saturate only one of the unitarity bounds. Nevertheless, this is sufficient for them to be of type (iii).

\( m_1 = 1; [100] \)

If we choose \( n'_2 = 2, n'_6 = 1, \) the super Dynkin labels are \([k(k + 2)100(k + 4)(k + 3)].\) The left

\(^2\)Here and below we shall not write explicitly the extra terms which are required to ensure that a given operator is indeed primary.
Young tableau is $< k + 2, 2, 1 >$ while the right one is $< k + 5 >$. Such operators cannot be constructed from derivatives acting on two $T$'s by symmetry.

$m_1 = 1; \ [001]$

For $n'_2 = 2, n'_6 = 1$ the Dynkin labels are $[k(k + 2)001(k + 2)(k + 1)]$. The left Young tableau is $< k + 2, 2 >$ while the right one is $< k + 3, 1 >$. These operators can be written in the form $\partial^2 T \partial^{k+2} T$ and satisfy one unitarity bound.

$m_1 = 0$

In this case we could in principle have $n'_2 = 3, n'_6 = 1$ but these cannot be written in terms of derivatives acting on two $T$'s. So take $n'_2 = n'_6 = 2$. The super Dynkin labels are $[k(k + 2)000(k + 2)k]$, and the Young tableaux are $< k + 2, 2 >$ for both the primed and unprimed indices. So these operators can be written in the form $T \partial^{k+4} T$ and are unsaturated. Therefore these operators can acquire anomalous dimensions in the quantum theory.

Operators of the above form contain, as spacetime components, the operators constructed from spacetime derivatives acting on two factors of the leading scalars in $T$ discussed in [6]. The authors of [6] were not always able to specify which supermultiplet was involved when the component field under discussion was not the highest weight state. Here we briefly indicate how these supermultiplets can be identified using analytic superspace. Let $T_o$ be the leading component of $T$; it is a scalar field in the 20' representation of $SU(4)$. The operators of [6] are schematically of the form,

$$O^{[abc]}_{rL} \sim (\partial_{\alpha\alpha})^r \Box^r (T_o T_o)_{[abc]}$$  \hspace{1cm} (14)

where $r' = 1/2(L - (r + 4))$, $L$ being the naïve dimension. The indices on the spacetime derivatives are totally symmetrised and $[abc]$ denotes the $SU(4)$ representation. Since $T_o$ is in the 20' representation, the possible representations that can arise are 1, 20, 84, 105, 15, 175.

To illustrate the procedure let us consider operators in the 105 = [040] representation. There were two series of non-renormalised 105 operators mentioned in [6], $r = 2k, L = 4 + 2k$ and $r = 2k, L = 6 + 2k$ (where $k$ is a positive integer), the first non-renormalised operator being $r = 0, L = 8$. Now, as an operator on analytic superspace, the leading component of $T^2$ is a scalar in the 105 representation. To obtain the desired component we therefore need only include the right spacetime derivatives. To find the full multiplet we then replace the spacetime derivatives by analytic superspace derivatives.

We have $O^{[040]}_{2k+2} \sim (\partial_{\alpha\alpha})^{2k} (T_o^2)_{[040]}$, so the desired supermultiplet is (schematically) $(\partial_{A'A'})^{2k} T^2$ with the primed and unprimed indices symmetrised. The super Dynkin labels are $[(2k - 2)(2k - 1)020(2k - 1)(2k - 2)]$, so this operator is protected. The second operator is $O^{[040]}_{2k6+2k} \sim \Box (\partial_{\alpha\alpha})^{2k} (T_o^2)_{[040]}$. For this case we have $2k + 2$ derivatives and the primed (unprimed) indices are symmetrised with respect to $2k + 1$ of them. In other words the associated super Young tableaux are $< 2k + 1, 1 >$ for both sets of indices. The super Dynkin labels are $[(2k - 1)(2k)101(2k)(2k - 1)]$, and the operator is protected. In the third case $O^{[040]}_{08} \sim \Box^2 (T_o^2)_{[040]}$. In this case the four derivatives fall into the representation $< 2, 2 >$ for both primed and unprimed indices so the super Dynkin labels are [0200020]. This operator is unprotected.
For $k = 0$, one can have no d’Alembertians, in which case the operator is simply $T^2$, which is series C, or one can have one d’Alembertian in which case the operator has super Dynkin labels [0020200] and is again series C.

A slightly more complicated situation arises when one needs to add further internal derivatives in order to obtain the right $SU(4)$ representation. For example, consider the operator $O_{2k+1}^{[101]} \sim (\partial_{\alpha \dot{\alpha}})^{2k+1}(T^2)^{[101]}$. Here it is necessary to add three further derivatives. There are three possibilities corresponding to the super Dynkin labels $[2k(2k+2)000(2k+2)2k]$ (renormalised) and $[(2k+1)(2k+2)101(2k+2)(2k+1)]$ or $[2k(2k+2)001(2k+2)(2k+1)]$ (protected). Presumably the precise spacetime components of the three cases will not be identical because there will be different contributions from the other fields in the SYM multiplet (and from terms required to make the operators primary).

As well as the operators discussed above one can construct many more which should be protected by the same argument. To build any such operator one begins (schematically) with a product of $A_p$’s and analytic superspace derivatives, with the indices on the latter projected onto irreducible representations of the two $\mathfrak{sl}(2|2)$ superalgebras. One then requires that the operator really is primary, i.e. terms can be added in such a way to achieve this, and finally that at least one of the unitarity bounds is satisfied.

For example, representations with Dynkin labels $[k(k+1)lm(k+1)k]$ can be obtained by applying derivatives to gauge invariant operators for all positive integers $k$ and $l$ and for all positive integers $m$ such that $m \geq 4 - 2l$ or $m = 2 - 2l$. These have the form $T\partial^{k+l+2}TA_{2l+m-2}$, and since they saturate both unitarity bounds they should be protected. Another example is the representation $[(k+1)(k+2)lm(l+1)(k+1)k]$ for positive integers $k, l, m$ and $m \geq 3 - 2l$ or $m = 1 - 2l$. These are of the form $\partial^{k+l+2}T\partial TA_{2l+m-1}$, saturate both unitarity bounds and are therefore protected. There are also many more examples of protected operators that saturate just one unitarity bound.

Note that, as shown above, the only unprotected operator constructed from two $T$’s is in the singlet representation of the internal $SU(4)$ in agreement with [11]. Furthermore we cannot construct any protected operators that are singlets by using more $T$’s or $A_p$’s. There are, however, plenty of examples of unprotected operators that are not singlets. A simple example can be obtained by multiplying the $m_1 = 0$ example above by $T$. This has the form $T^2\partial^{k+4}T$ and has Dynkin labels $[k(k+2)020(k+2)k]$ and is thus in the 20$'$ representation of $SU(4)$.

To summarise, we have seen that there are many series A composite operators in $N = 4$ SYM which should be protected from renormalisation by virtue of the fact that they are short and remain short in the interacting theory, whereas the corresponding representations with anomalous dimensions are not shortened. These protected multiplets are all multi-trace operators, since the single-trace series A operators which are short in the free theory do not remain short in the presence of interactions. If we write the composite operators as fields on $(4, 2, 2)$ analytic superspace, the protected operators (from any series) are analytic tensor fields. The non-protected operators can still be interpreted as fields on analytic superspace but they are not tensor fields of the standard type.

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