QUASI-DOUBLING OF SELF-SIMILAR MEASURES WITH OVERLAPS

KATHRYN E. HARE, KEVIN G. HARE, AND SASCHA TROSCHEIT

Abstract. The Assouad and quasi-Assouad dimensions of a metric space provide information about the extreme local geometric nature of the set. The Assouad dimension of a set has a measure theoretic analogue, which is also known as the upper regularity dimension. One reason for the interest in this notion is that a measure has finite Assouad dimension if and only if it is doubling.

Motivated by recent progress on both the Assouad dimension of measures that satisfy a strong separation condition and the quasi-Assouad dimension of metric spaces, we introduce the notion of the quasi-Assouad dimension of a measure. As with sets, the quasi-Assouad dimension of a measure is dominated by its Assouad dimension. It dominates both the quasi-Assouad dimension of its support and the supremal local dimension of the measure, with strict inequalities possible in all cases.

Our main focus is on self-similar measures in $\mathbb{R}$ whose support is an interval and which may have ‘overlaps’. For measures that satisfy a weaker condition than the weak separation condition we prove that finite quasi-Assouad dimension is equivalent to quasi-doubling of the measure, a strictly less restrictive property than doubling. Further, we exhibit a large class of such measures for which the quasi-Assouad dimension coincides with the maximum of the local dimension at the endpoints of the support. This class includes all regular, equicontractive self-similar measures satisfying the weak separation condition, such as convolutions of uniform Cantor measures with integer ratio of dissection. Other properties of this dimension are also established and many examples are given.

1. Introduction

The Assouad dimension of a metric space is an indication of its ‘thickness’ and is of great use in solving embedding problems, see [1]. Recently, the Assouad dimension has attracted significant attention in the metric geometry community, especially when studying dynamical objects such as attractors and fractals, see for instance [4, 15, 18]. It gives quantitative information about the ‘worst’ possible scaling of a set. The quasi-Assouad dimension was introduced by Lü and Xi [16] and differs from the Assouad dimension by ignoring some subexponential effects. It is a lower bound to the Assouad dimension and an upper bound on the Hausdorff and upper box dimensions of the set. Although these dimensions will often coincide, such as for self-similar sets satisfying the open set condition, there are examples where the dimensions are different. We refer the reader to [7] and [8] for deterministic and stochastic examples.

As with the Hausdorff dimension, there is an analogue of the Assouad dimension of the measure. This dimension is also known as the upper regularity dimension and was first
studied by Käenmäki in [12, 13]. As with the Assouad dimension of a set, it captures the worst scaling behaviour of a measure. One reason the Assouad dimension of a measure is of interest is because it is finite if and only if the measure is doubling [6].

In this article we introduce and investigate the measure-theoretic analogue of the quasi-Assouad dimension. In Section 2 we define the quasi-Assouad dimension of a measure and prove basic properties such as its relation to the quasi-Assouad dimension of the support of the measure, the Assouad dimension of the measure, and the local dimensions of the measure.

We then focus our attention on self-similar measures. As remarked in [6], “self-similar measures not satisfying the strong separation condition are typically not doubling” and hence have infinite Assouad dimension. In contrast, we show that doubling is not a requirement for finite quasi-Assouad dimension of self-similar measures. Thus it becomes relevant to study the quasi-Assouad dimension of self-similar measures with ‘overlap’. Most of our results are for self-similar measures that have support equal to $[0, 1]$ and satisfy the asymptotic gap weak separation condition (AGWSC), a separation property that we introduce in this article that is strictly less restrictive than the weak separation condition. In Section 4 we characterize finite quasi-Assouad dimension for this class of measures in terms of a geometric doubling-like condition that is strictly weaker than doubling. An example of a class of AGWSC measures with finite quasi-Assouad dimension (but not necessarily finite Assouad dimension) are the equicontractive, regular, self-similar measures. Here, regular means that the probabilities associated with the left and right-most contractions from the underlying iterated function system are minimal. Examples include convolutions of uniform Cantor measures with integer ratio of dissection and Bernoulli convolutions with contraction ratios the inverse of Pisot, or even Salem, numbers. A more general class of measures with the AGWSC which need not be equicontractive and have finite quasi-Assouad dimension are what we call weakly comparable measures. For these we obtain an upper bound on the quasi-Assouad dimension that is sharper than the Assouad dimension. This is accomplished in Section 5.

In the final section, Section 6, we specialize to the case of generalized regular, self-similar measures satisfying the AGWSC. This is a subclass of the weakly comparable measures that properly contains the equicontractive, regular measures satisfying the AGWSC. The main result of this section is to show that the quasi-Assouad dimension of these measures coincides with the supremum of the local dimensions of the measure, which in this case is the local dimension at the two endpoints, 0 and 1.

We conclude by giving an example that demonstrates the necessity of the assumption that the support of the self-similar measure is $[0, 1]$.

Many of our results rely upon good estimates of the measure of net intervals, certain subintervals of the self-similar set that arise naturally through the iterative process. These technical results can be found in Section 3 and were motivated by the study of the local dimension theory of a special class of measures satisfying the AGWSC known as finite type. See [2] or [11] for more details on finite type. More background on the related asymptotic weak separation condition can also be found in Section 3.
2. Basic properties of the quasi-Assouad dimension

Given $X$, a compact subset of $\mathbb{R}^d$, we write $N_r(E)$ for the least number of sets of diameter at most $r$ that are required to cover $E$. Let

$$h(\delta) = \inf \left\{ \alpha : (\exists C_1, C_2 > 0)(\forall 0 < r < R^{1+\delta} < R < C_1) \sup_{x \in X} N_r(B(x, R) \cap E) \leq C_2 \left( \frac{R}{r} \right)^\alpha \right\}.$$ 

The Assouad dimension of $E$ is given by

$$\dim A E = h(0).$$

The quasi-Assouad dimension is characterized by an exponential gap between $r$ and $R$ and is given by

$$\dim_{qA} E = \lim_{\delta \to 0} h(\delta).$$

Note that it is not necessary to have both constants, $C_1$ and $C_2$, in the definition above. However, we introduced both constants as it is convenient to be able to change between the two definitions.

Our interest in this paper is to study a natural analogue of the Assouad and quasi-Assouad dimension for measures. By a measure we will always mean a Borel probability measure on $\mathbb{R}^d$.

Definition 2.1. Given a measure $\mu$ and $\delta \geq 0$, set

$$H(\delta) = \inf \left\{ s : (\exists C_1, C_2 > 0)(\forall 0 < r \leq R^{1+\delta} \leq R \leq C_1) \sup_{x \in \text{supp} \mu} \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_2 \left( \frac{R}{r} \right)^s \right\}.$$ 

The Assouad dimension of $\mu$ is $\dim A \mu = H(0)$ and the quasi-Assouad dimension of $\mu$ is $\dim_{qA} \mu = \lim_{\delta \to 0} H(\delta)$.

We note that the limit must exist by monotonicity, but may be infinite. The Assouad dimension of a measure has also been referred to as the upper regularity dimension, see [6] and [13].

Since $H(\delta)$ is a non-decreasing function, we clearly have $\dim_{qA} \mu \leq \dim A \mu$. As with the quasi-Assouad/Assouad dimensions of sets, the quasi-Assouad and Assouad dimensions of a measure need not coincide and it is even possible for the Assouad dimension of a measure to be infinite, while the quasi-Assouad dimension is finite. See Example 2.3.

In [6, Theorem 2.1] it is shown that $\dim A \text{supp} \mu \leq \dim A \mu$. The analogous statement holds for the quasi-Assouad dimension of measures.

Proposition 2.2. Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. Then

$$\dim_{qA} \text{supp} \mu \leq \dim A \mu \leq \dim_{qA} \mu.$$ 

Proof. Suppose that $s = \dim_{qA} \mu$ and $t = \dim_{qA} \text{supp} \mu$. Then, given $\varepsilon > 0$ and $\delta > 0$ there exists a constant $c_1 > 0$ such that for all $r \leq R^{1+\delta}$ and all $x \in \text{supp} \mu$,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq c_1 \left( \frac{R}{r} \right)^{s+\varepsilon}.$$
Similarly, there exists \( c_2 > 0 \) and points \( y \in \text{supp} \mu \) and \( r, R \) with \( 2r \leq R^{1+\delta} \), such that

\[
N_{2r}(B(y, R) \cap \text{supp} \mu) \geq c_2 \left( \frac{R}{2r} \right)^{t-\varepsilon}.
\]

Let \( B_j(x_j, r), \ j = 1, \ldots, k \) be a maximal collection of disjoint balls with centres in \( B(y, R) \cap \text{supp} \mu \). Then \( \bigcup_{j=1}^{k} B(x_j, 2r) \) covers \( B(y, R) \cap \text{supp} \mu \), and so \( k \geq N_{2r}(B(y, R) \cap \text{supp} \mu) \).

These comments imply

\[
\mu(B(y, 2R)) \geq k \min_j \mu(B(x_j, r)) = k \mu(B(x_{j_0}, r))
\]

for a suitable choice of index \( j_0 \). Moreover, \( B(y, 2R) \subseteq B(x_{j_0}, 4R) \), thus

\[
c_2 \left( \frac{R}{2r} \right)^{t-\varepsilon} \leq N_{2r}(B(y, R) \cap \text{supp} \mu) \leq k \leq \frac{\mu(B(y, 2R))}{\mu(B(x_{j_0}, r))} \leq \frac{\mu(B(x_{j_0}, 4R))}{\mu(B(x_{j_0}, r))} \leq c_1 \left( \frac{2R}{r} \right)^{s+\varepsilon}.
\]

Since we can find arbitrarily small \( R \) satisfying this inequality, we must have \( s \geq t \). \( \square \)

**Example 2.3.** (i) A measure \( \mu \) satisfying \( \dim_{qA} \text{supp} \mu < \dim_A \mu < \dim_A \mu^* \): Let \( C \) be the classic middle third Cantor set. We will label the Cantor intervals at step \( n \) of the standard construction as \( I^\omega \), where \( \omega \in \{0, 1\}^n \), with the meaning that if \( I^\nu \) is a Cantor interval of step \( n - 1 \), then its left descendent is the interval labelled \( I_0^\nu \) and the right descendent is labelled \( I_1^\nu \). Choose a sparse sequence \( (n_k) \). Put \( p_0^{(n_k)} = 1/3, \) \( p_1^{(n_k)} = 2/3 \) if \( n \neq n_k \) and \( p_0^{(n_k)} = 1/4, \) \( p_1^{(n_k)} = 3/4 \). We define the measure \( \mu \) by the rule that \( \mu(I_0^\nu) = p_0^{(1)} p_2^{(2)} \cdots p_\nu^{(n_k)} \) for \( \omega = (\omega_1, \ldots, \omega_n) \). The support of \( \mu \) is \( C \). Provided \( (n_k) \) is sufficiently sparse, it can be verified that

\[
\dim_A \text{supp} \mu = \dim_{qA} \text{supp} \mu = \dim_H C = \log 2 / \log 3,
\]

\[
1 = -\frac{\log p_0^{(n_k)}}{\log 3} = \dim_{qA} \mu,
\]

and

\[
\log 4 / \log 3 = -\log(\lim \inf p_0^{(n_k)}) / \log 3 = \dim_A \mu.
\]

If instead \( p_0^{(n_k)} = 1/k \), then \( \dim_A \mu = \infty \). The details are left to the reader.

(ii) A measure \( \mu \) satisfying \( \dim_A \text{supp} \mu > \dim_{qA} \mu^* \): By taking as \( \mu \) the uniform Cantor measure on a Cantor set \( C \) with suitably varying ratios of dissection, we can arrange for

\[
\dim_{qA} \text{supp} \mu = \log 2 / \log 3 = \dim_{qA} \mu < \dim_A \text{supp} \mu = \dim_A \mu.
\]

(iii) By varying both the probabilities and ratios we can construct measures \( \mu \) satisfying

\[
\dim_{qA} \text{supp} \mu < \dim_A \text{supp} \mu < \dim_{qA} \mu < \dim_A \mu \text{ or }
\]

\[
\dim_{qA} \text{supp} \mu < \dim_{qA} \mu < \dim_A \text{supp} \mu < \dim_A \mu.
\]

In [6, Proposition 3.1] it was observed that the Assouad dimension of a measure is finite if and only if the measure is doubling. In Example [6,1] we show that a measure can fail to be doubling, but have finite quasi-Assouad dimension. In Section 4 we characterize finite quasi-Assouad dimension in terms of a weaker doubling-type condition.

Recall that the **upper local dimension** of a Borel probability measure \( \mu \) at \( x \in \text{supp} \mu \) is

\[
\dimloc \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]
The lower local dimension, denoted $\dim_{\text{loc}} \mu(x)$, is defined analogously, replacing $\lim \sup$ by $\lim \inf$. If the upper and lower local dimensions coincide, their common value is known as the local dimension of $\mu$ at $x$ and we write $\dim_{\text{loc}} \mu(x)$.

The Assouad dimension is bounded below by the upper local dimension for any point in the support of $\mu$, see [6]. The same holds for the quasi-Assouad dimension.

**Proposition 2.4.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$. Then

$$\dim_{qA} \mu \geq \sup \{ \dim_{\text{loc}} \mu(x) : x \in \text{supp} \mu \}. \quad (2.1)$$

**Proof.** Let $s = \sup \{ \dim_{\text{loc}} \mu(x) : x \in \text{supp} \mu \}$. Temporarily fix $\tau > 0$. We will show the quasi-Assouad dimension of $\mu$ is at least $s - \tau$.

To begin the proof, fix $\delta > 0$, choose $0 < \varepsilon < \tau/2$ so small that $\delta(\tau - 2\varepsilon) > 3\varepsilon$ and select $x \in \text{supp} \mu$ such that $\dim_{\text{loc}} \mu(x) \geq s - \varepsilon$. Choose a decreasing sequence $R_n \to 0$ such that $R_{n+1} \leq R_n^{\delta+\varepsilon}$ and satisfying the property

$$s - 2\varepsilon \leq \frac{\log \mu(B(x, R_n))}{\log R_n} \leq s + \varepsilon$$

for all $n$. Then

$$\frac{\mu(B(x, R_n))}{\mu(B(x, R_{n+1}))} \geq \frac{R_n^{s+\varepsilon}}{R_n^{s-2\varepsilon}}.$$

The choice of $\varepsilon$ ensures that $R_{n+1}^{\tau-2\varepsilon} \leq R_n^{(1+\delta)(\tau-2\varepsilon)} \leq R_n^{s+\varepsilon}$ and therefore

$$\frac{R_n^{s+\varepsilon}}{R_n^{s-2\varepsilon}} \geq \left( \frac{R_n}{R_{n+1}} \right)^{s-\tau}.$$

That implies $\dim_{qA} \mu \geq s - \tau$, as claimed, and as $\tau$ was arbitrary the desired conclusion holds.

Note that the measure of Example 2.3(i) has its supremal local dimension occurring at 0 and this value coincides with the quasi-Assouad (but not the Assouad) dimension of the measure. In Theorem 6.6 we exhibit a class of measures for which this continues to be true. However, Example 5.1 shows that it is also possible for the quasi-Assouad dimension to be strictly larger.

3. Self-similar Iterated Function Systems

3.1. Iterated function systems and separation conditions. Let $\{S_j\}$ be a finite family of contractions on $\mathbb{R}$ such that $S_j(x) = r_jx + d_j$, $j = 0, ..., m-1$, where $r_j > 0$. Let $(p_0, ..., p_{m-1})$ be a non-degenerate probability vector, i.e. $p_j > 0$ and $\sum_{j=0}^{m-1} p_j = 1$. We refer to the collection $\{S_j\}$ as an iterated function system (IFS) and the collection of tuples $\{S_j, p_j\}$ as a weighted iterated function system.

There exists a unique non-empty compact set satisfying $F = \bigcup_{j=0}^{m-1} S_j(F)$, called the attractor, associated with the collection $S_j$. As all maps are similarities the attractor is known as a self-similar set. Throughout this article we will assume that the self-similar set is the unit line $[0, 1]$. There is no loss of generality in assuming $S_0(0) = 0$ and $S_{m-1}(1) = 1$ as $r_j > 0$.

We can similarly define a unique Borel probability measure by assigning weights to the maps. The resulting measure has support $F$. In fact, this measure is simply the projection of a Bernoulli measure from the underlying symbolic dynamics onto $\mathbb{R}$ and is referred to
as the self-similar measure associated with the weighted iterated function system \( \{S_j, p_j\} \). More precisely, the self-similar measure is the unique probability measure \( \mu \) satisfying 
\[
\mu(E) = \sum_{j=0}^{m-1} p_j \mu(S_j^{-1}(E)) \quad \text{for all Borel sets } E.
\]
We let 
\[
\lambda = \min r_j
\]
and write \( \Omega \) for the set of finite words on the alphabet \( \{0, 1, \ldots, m-1\} \). If all \( r_j \) are equal, we say that the iterated function system is \textit{equiconttractive}. Given a (finite) word \( w = (w_1, \ldots, w_n) \), we let \( w^- = (w_1, \ldots, w_{n-1}) \) and denote the length of \( w \) by \( |w| \). We usually write words by concatenation, that is \( w = w_1 w_2 w_3 \cdots \), and we define 
\[
r_w = r_{w_1} r_{w_2} \cdots r_{w_n} \quad \text{and} \quad p_w = p_{w_1} p_{w_2} \cdots p_{w_n}.
\]

Commonly, separation conditions are employed to give precise results about these attractors and measures. If \( F \) is the attractor of the IFS \( \{S_j\} \) and \( S_j(F) \cap S_i(F) = \emptyset \) for all \( i \neq j \), we say that the IFS satisfies the \textit{strong separation condition (SSC)}. If there is an open set \( U \) such that \( S_i(U) \subseteq U \) for all \( i \in \Lambda \) and \( S_i(U) \cap S_j(U) = \emptyset \) for all \( i \neq j \), we say that IFS \( \{S_i\} \) satisfies the \textit{open set condition (OSC)}. The OSC is a less restrictive condition than the SSC, but in both cases the Hausdorff and Assouad dimensions coincide for the attractor and their common value is given by the unique \( s \) that satisfies \( \sum r_i^s = 1 \), see [4, Cor. 2.11].

There are, however, some important distinctions for self-similar measures. Fraser and Howroyd proved that in the case when self-similar measures \( \mu \) satisfy the strong separation condition, the Assouad and quasi-Assouad dimensions coincide with the supremal local dimension of \( \mu \). [6, Theorem 2.4].

\section*{Example 3.1.}
A self-similar measure satisfying the OSC, with \( \text{dim}_{qA} \mu > \sup_{x} \{\text{dim}_{loc} \mu(x)\} \).

Consider the IFS \( S_0(x) = x/2 \) and \( S_1(x) = x/2 + 1/2 \), with probabilities \( p_0 > p_1 \). Although the self-similar set of the IFS is \( [0, 1] \), the open set condition is satisfied with the open set \( U = (0, 1) \). Temporarily fix \( \delta > 0 \). Choose \( N \) large and let \( k = \lfloor \delta N \rfloor \). Take \( x \) to be the midpoint of the interval \( S_{01^{N+k}}[0,1] \), where \( 1^{N+k} \) is the word consisting of \( N + k \) many letters \( 1 \). This interval has left endpoint \( 1/2 \) and length \( 2^{-(N+k)} \). Choose \( R = 2^{-N} \) and \( r = 2^{-(N+k+2)} \), so \( r \leq R^{1+\delta} \). Then \( B(x, R) \) contains \( S_{10^{N}}[0,1] \), so \( \mu(B(x, R)) \geq p_0^{-N} \), while \( B(x, r) \subseteq S_{01^{N+k}}[0,1] \) and hence has \( \mu \)-measure at most \( p_0^{-N} \).

Thus if we are to have 
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_1 \left( \frac{R}{r} \right)^s
\]
for all large \( N \), it must be true that \( 2^{\delta s} \geq p_0^{-1-\delta} \), in other words,
\[
s \geq \log 2 \left( \frac{1}{\delta} + 1 \right) \left| \log p_1 \right| - \left| \log p_0 \right| = \left| \log p_1 \right| + \left| \log p_1 \right| - \left| \log p_0 \right|.
\]
This inequality shows that \( H(\delta) \to \infty \) as \( \delta \to 0 \), hence \( \text{dim}_{qA} \mu \) is infinite.

As this IFS satisfies the open set condition, it is known that 
\[
\{\text{dim}_{loc} \mu(x) : x \in \text{supp } \mu\} = \left[ \frac{\left| \log p_0 \right|}{\log 2}, \frac{\left| \log p_1 \right|}{\log 2} \right],
\]
thus the inequality of Proposition 2.4 can be strict.
In this article we will focus on the quasi-Assouad dimension and will show that various desirable properties hold under even weaker conditions that we will now state.

Recall that $\lambda = \min r_j$. Set

$$\Lambda_n = \{ u \in \Omega : r_u \leq \lambda^n \text{ and } r_u^{-} > \lambda^n \}$$

for the set of words that are comparable to $\lambda^n$. In the equicontractive setting, $\Lambda_n$ are simply the words of length $n$.

Lau and Ngai [14] studied self-similar IFS under a weaker separation condition that limits the number of overlapping distinct images. This so-called weak separation condition subsequently turned out to be the ‘right’ separation condition to consider when dealing with the Assouad dimension of sets. Indeed, the Assouad dimension coincides with the Hausdorff dimension for self-similar sets satisfying the weak separation condition and is maximal otherwise, see [5].

We now recall this definition. Let

$$A(x, r) = \{ v \in \Omega : |S_v([0,1])| \leq r, \ |S_v^{-}([0,1])| > r \text{ and } x \in S_v([0,1]) \}$$

and

$$M(x, r) = \{ S_v : v \in A(x, r) \}.$$  

The self-similar IFS $\{S_i\}$ is said to satisfy the weak separation condition if

$$\sup_{r \in (0,1)} \sup_{x \in [0,1]} \#M(x, r) < \infty.$$

Zerner [19] showed that this is equivalent to the identity not being an accumulation point of

$$\mathcal{E} = \{ S_v^{-1} \circ S_w : v, w \in \Omega \}$$

with respect to the pointwise topology.

Iterated function systems generating Bernoulli convolutions where the contraction ratio is the reciprocal of a Pisot number, as well as iterated function systems of the form $(S_j)$ where $S_j(x) = x/d + d_j$ where $d \in \mathbb{N}$ and $d_j \in \mathbb{Q}$ are examples that satisfy the WSC.

Using Zerner’s definition, it is straightforward to show that Definition 3.2 below is another equivalent way of stating the weak separation condition. We have opted to state it in this version as this is the form that we will use in this article.

**Definition 3.2.** An iterated function system $\{S_j\}$ satisfies the weak separation condition (WSC) if there exists $a > 0$ such that if $u, w \in \Lambda_n$ and $S_u(0) \neq S_w(0)$, then

$$|S_u(0) - S_w(0)| \geq a \lambda^n \text{ and } |S_u(1) - S_w(1)| \geq a \lambda^n.$$

Motivated by Ngai and Lau’s original definition of the weak separation condition, Feng [3] introduced a separation condition, known as the asymptotic weak separation condition, also in terms of overlapping images.

**Definition 3.3.** Let $\{S_i\}$ be a self-similar IFS. We say that $\{S_i\}$ satisfies the asymptotic weak separation condition (AWSC) if there exists non-decreasing function $g(r)$ such that

$$\log g(r) / \log r \to 0 \text{ and } \sup_{x \in [0,1]} \#M(x, r) \leq g(r).$$

It is easily observed that the AWSC is weaker than the WSC.
In a similar fashion, we define a useful separation condition on the asymptotic separation of images of 0 and 1.

**Definition 3.4.** An iterated function system \( \{S_j\} \) satisfies the **asymptotic gap weak separation condition (AGWSC)** if there exists some non-increasing function \( f(n) > 0 \) such that \( (\log f(n))/n \to 0 \) as \( n \to \infty \) and
\[
|S_u(0) - S_w(0)| \geq f(n)\lambda^n \quad \text{and} \quad |S_u(1) - S_w(1)| \geq f(n)\lambda^n
\]
whenever \( u, w \in \Lambda_n \) and \( S_u(0) \neq S_w(0) \).

Note that the AGWSC is similar in spirit to the AWSC by allowing the defining feature to vary on a subexponential scale rather than be finite. We will show below that the AGWSC implies the AWSC. While closely related, we are not able to show that these two conditions are equivalent. Note, however, that the two notions coincide in the only known family to satisfy the AWSC (or AGWSC), but not the WSC. This family are the IFSs generating Bernoulli convolutions with contraction ratio the inverse of Salem numbers, see [3].

**Lemma 3.5.** Let \( \{S_j\} \) be a self-similar IFS of the unit line that satisfies the AGWSC. Then \( \{S_j\} \) satisfies the AWS.

**Proof.** Let \( x_0 \in [0, 1] \) and let \( v \in \Omega \) be such that \( x_0 \in S_v([0, 1]) \). Set \( r = |S_v([0, 1])| \), then any element \( S \in M(x_0, r) \) satisfies \( r < |S([0, 1])| \leq r \) for some uniform \( \gamma > 0 \). Thus
\[
S(0), S(1) \in [S_v(0) - r, S_v(0) + 2r]
\]
Now, since the IFS satisfies the AGWSC, no two distinct maps \( S_1, S_2 \in M(x_0, r) \) may have both \( |S_1(0) - S_2(0)| < \lambda^n \cdot f(n) \) and \( |S_1(1) - S_2(1)| < \lambda^n \cdot f(n) \), where \( n \) is such that \( r \leq \lambda^{n+1} \) and \( r > \lambda^{n+2} \). Thus, there are at most
\[
\frac{3r}{\lambda^n f(n)} \leq \frac{3}{\lambda f(n)}
\]
choices for \( S(0) \) and \( S(1) \), giving
\[
\#M(x_0, r) \leq \left(\frac{3}{\lambda f(n)}\right)^2.
\]
Now \( v \), and thus \( r \), was arbitrary and so the above inequality holds for all \( n \). We obtain
\[
\lim_{r \to 0} \frac{\#M(x_0, r)}{|\log r|} \leq \lim_{n \to \infty} \frac{2\log 3 - 2\log(\lambda f(n))}{|\log \lambda^{n+2}|} = 0,
\]
showing that the AWSC is satisfied. \( \square \)

Since we will fix a self-similar measure by fixing a weighted iterated function system, we will also refer to a measure \( \mu \) as satisfying the AGWSC or WSC, where we should say “the weighted iterated function system associated with \( \mu \)”.

In the weak separation case we may assume without loss of generality that the constant \( c \) arising in the definition of the weak separation condition satisfies \( a < \lambda \). Similarly, in the asymptotic gap weak separation case we can assume that \( f(n) < \lambda \) for all \( n \).

**Definition 3.6.** Let \( \{S_j\} \) be an equicontractive iterated function system with contraction ratio \( \lambda \). Further, let \( c > 0 \) be such that \( c(1 - \lambda) \) is the diameter of the associated attractor. The IFS is said to be of **finite type** if there is a finite set \( F \subseteq \mathbb{R} \) such that if \( u, w \in \Lambda_n \) then either
\[
|S_u(0) - S_w(0)| > c\lambda^n \quad \text{or} \quad \lambda^{-n}(S_u(0) - S_w(0)) \in F.
\]
The notion of finite type was introduced by Ngai and Wang [17]. It can also be defined for IFS that are not equicontractive, but as this is more technical and not needed in this article, we omit its definition. Examples of iterated function systems of finite type include Bernoulli convolutions with contraction ratio the inverse of Pisot numbers. For further information, the interested reader may peruse [11] and the references cited therein.

We have the following inclusions among these classes, all of which are known to be proper except for the last:

\[ \text{OSC} \subset \text{Finite Type} \subset \text{WSC} \subset \text{AGWSC} \subset \text{AWSC}. \]

3.2. **Net intervals**. Recall that the attractor of the IFS is assumed to be \([0,1]\). For each \(n \in \mathbb{N}\), let \(h_1,\ldots,h_{s_n}\) denote the elements of the set \(\{S_u(0), S_u(1) : u \in \Lambda_n\}\), listed in increasing order. Put

\(\mathcal{F}_n = \{[h_j, h_{j+1}] : 1 \leq j \leq s_n - 1\}\).

The elements of \(\mathcal{F}_n\) are called the net intervals of level \(n\). Of course, the net intervals of a given level \(n \geq 1\) cover \([0,1]\). The interval \([0,1]\) will be the (unique) net interval of level 0. Since \(|S_u(0) - S_u(1)| \leq r_u \leq \lambda^n\) when \(u \in \Lambda_n\), any net interval of level \(n\) has length at most \(\lambda^n\). We denote the length of the net interval \(\Delta\) by \(l(\Delta)\).

Each \(x \in [0,1]\) belongs to either one or two net intervals of level \(n\). The point \(x\) will belong to two net intervals if and only if \(x\) is an endpoint of a net interval of level \(n\). In both cases we refer to the net interval of level \(n\) containing \(x\) by \(\Delta_n(x)\), choosing arbitrarily when it is not unique. Each net interval \(\Delta\) of level \(n\) is contained in a unique net interval of level \(n - 1\) which we refer to as the parent of \(\Delta\).

Given a net interval \(\Delta\) of level \(n\), let

\[ P_n(\Delta) = \sum_{u \in \Lambda_n \cap [0,1]} p_u. \]

Note that \(P_n(\Delta) \geq \min p_u^n\). Since \(P_n(\Delta)\) is the sum of all weights of words whose images cover the net interval \(\Delta\), we must have \(P_n(\Delta_n(x)) \geq \mu(\Delta_n(x))\). As \(l(\Delta_n(x)) \leq \lambda^n\), the ball \(B(x, \lambda^n)\) contains \(\Delta_n(x)\). In particular, if \(u \in \Lambda_n\) and \(x \in S_u([0,1])\), then \(S_u([0,1]) \subseteq B(x, \lambda^n)\). Thus,

\[ \mu(B(x, \lambda^n)) \geq P_n(\Delta_n(x)) \geq \mu(\Delta_n(x)). \]

To compare the minimal and maximal contraction rate we define \(\Theta \in \mathbb{N}\) implicitly as the least integer satisfying

\[ (\max_{j \in \Lambda} r_j)^{\Theta+1} < \lambda^2 = \left(\min_{j \in \Lambda} r_j\right)^2. \]

We collect some further properties of \(P_n(\Delta)\) below.

**Lemma 3.7.** Let \(\Theta\) be as above. Then

(a) \(P_n(\Delta_n(x)) \geq \min p_u^{\Theta} P_{n-1}(\Delta_{n-1}(x))\) and
(b) \(P_n(\Delta_n(x)) \leq P_{n-1}(\Delta_{n-1}(x))\).

**Proof.** (a) Let \(u \in \Lambda_{n-1}\) and suppose \(S_u[0,1] \supseteq \Delta_{n-1}(x)\). Then there exists some word \(w\) such that \(uw \in \Lambda_n\) and \(S_uw[0,1] \supseteq \Delta_n(x)\). Note that \(r_{uw} > \lambda\) and so \(r_w \geq \lambda^2\). Using the definition of \(\Theta\) we find that \(|w| \leq \Theta\). As \(p_u = p_{uw} p_w^{-1}\), we have

\[ P_{n-1}(\Delta_{n-1}(x)) \leq P_n(\Delta_n(x)) (\min p_w)^{-1} \leq P_n(\Delta_n(x)) (\min p_u^{\Theta})^{-1}. \]
(b) Suppose \( v \in \Lambda_n \) and \( S_v[0,1] \supseteq \Delta_n(x) \). Then \( v = uw \) where \( u \in \Lambda_{n-1} \) and \( S_u[0,1] \supseteq \Delta_{n-1}(x) \). Furthermore, the sum of \( p_w \) taken over such \( w \) is at most one. Thus
\[
P_n(\Delta_n(x)) = \sum_{uw \in \Lambda_n, w \in \Lambda_{n-1}} p_u p_w \leq P_{n-1}(\Delta_{n-1}(x)).
\]

Now suppose the iterated function system satisfies the asymptotic gap weak separation condition with function \( f \). For each \( n \), choose the minimal integer \( \kappa_n \) such that \( \lambda^{\kappa_n} \leq f(n) \). By definition,
\[
\kappa_n \leq \frac{\log f(n)}{\log \lambda} + 1
\]
and therefore \( \kappa_n/n \to 0 \). As \( \lambda^{n+\kappa_n} \leq f(n)\lambda^n \),
\[
B(x, \lambda^{n+\kappa_n}) \subseteq B(x, f(n)\lambda^n).
\]
If \( u \in \Lambda_{n+\kappa_n} \), then \( r_u \leq \lambda^{n+\kappa_n} \). If also \( S_u[0,1] \supseteq \Delta_{n+\kappa_n}(x) \), then \( S_u[0,1] \subseteq B(x, f(n)\lambda^n) \), and so by (3.4),
\[
\mu(B(x, f(n)\lambda^n)) \geq P_{n+\kappa_n}(\Delta_{n+\kappa_n}(x)).
\]
If \( u \in \Lambda_n \) and \( S_u(0) = 0 \), then \( S_u(1) = r_u \geq \lambda^{n+1} \geq f(n)\lambda_n \), hence \( l(\Delta_n(0)) \geq f(n)\lambda^n \). Consequently,
\[
B(0, f(n)\lambda^n) \cap [0,1] \subseteq \Delta_n(0) \subseteq B(0, \lambda^n).
\]

Combining these observations with the previous lemma gives the following bounds.

**Proposition 3.8.** Suppose \( \mu \) is a self-similar measure that satisfies the asymptotic gap weak separation condition with function \( f(n) \). There is a constant \( 0 < A < 1 \) such that for any \( x, n \)
\[
A^{\kappa_n} P_n(\Delta_n(x)) \leq \mu(B(x, f(n)\lambda^n)).
\]

In particular,
\[
A^{\kappa_n} P_n(\Delta_n(0)) \leq \mu(B(0, f(n)\lambda^n)) \leq \mu(\Delta_n(0)) \leq P_n(\Delta_n(0)).
\]

**Proof.** Lemma 3.7(a) implies \( P_{n+\kappa_n}(\Delta_{n+\kappa_n}(x)) \geq A^{\kappa_n} P_n(\Delta_n(x)) \) for \( A = \min \{ p_j^\Theta \} \). This fact, coupled with (3.4), gives the first statement. The second statement follows similarly from (3.2) and (3.5). \( \square \)

Let \( s, t \) be
\[
s = \liminf \left( P_n(\Delta_n(0)) \right)^{1/n} \quad \text{and} \quad t = \liminf \left( P_n(\Delta_n(1)) \right)^{1/n}.
\]

Note that \( s, t > 0 \) since \( P_n(\Delta_n(x)) \geq (\min_j p_j)^n \) for all \( x, n \).

**Proposition 3.9.** Suppose \( \mu \) is a self-similar measure that satisfies the asymptotic gap weak separation condition. Then
\[
\dim_{\text{loc}} \mu(0) = \frac{\log s}{\log \lambda} \quad \text{and} \quad \dim_{\text{loc}} \mu(1) = \frac{\log t}{\log \lambda}.
\]

Furthermore, \( s, t < 1 \).
Proof. Our earlier observations show that
\[ B(0, \lambda_n^{n+\kappa}) \cap [0, 1] \subseteq B(0, f(n)\lambda^n) \cap [0, 1] \subseteq \Delta_n(0) \subseteq B(0, \lambda^n) \]
and by Proposition 3.8,
\[ A_{\kappa_n} P_n(\Delta_n(0)) \leq \mu(B(0, \lambda^n)) \leq \mu(\Delta_{n-\kappa_n}(0)) \leq P_{n-\kappa_n}(\Delta_{n-\kappa_n}(0)) \leq A_{-\kappa_n} P_n(\Delta_n(0)). \]

Since \( \kappa_n/n \to 0 \) as \( n \to \infty \),
\[
\limsup_{n \to \infty} \frac{\log(P_n(\Delta_n(0)))}{n \log \lambda} \leq \limsup_{n \to \infty} \frac{-\kappa_n \log A + \log(P_n(\Delta_n(0)))}{n \log \lambda} \leq \limsup_{n \to \infty} \frac{\log(\mu(B(0, \lambda^n)))}{n \log \lambda} = \lim_{n \to \infty} \frac{\log(P_n(\Delta_n(0)))}{n \log \lambda}.
\]
This proves the first equality. The second equality follows similarly, and is omitted for brevity.

To prove that \( s < 1 \), note that there must be at least one index \( j \) such that \( S_j(0) \neq 0 \). Without loss of generality assume the index is \( j_0 \). Thus, if \( \tau \in \Lambda_n \) with \( S_\tau[0,1] \supseteq \Delta_n(0) \), and \( \tau = uw \) where \( u \in \Lambda_{n-1} \) and \( S_u[0,1] \supseteq \Delta_{n-1}(0) \), then \( w \) does not contain the letter \( j_0 \). It follows that
\[ P_n(\Delta_n(0)) \leq (1 - p_{j_0}) \sum_{u \in \Lambda_{n-1}, S_u[0,1] \supseteq \Delta_{n-1}} p_u = (1 - p_{j_0}) P_{n-1}(\Delta_{n-1}(0)) \]
and hence \( P_n(\Delta_n(0)) \leq (1 - p_{j_0})^n \). Thus \( s \leq 1 - p_{j_0} < 1 \). The case for \( t < 1 \) follows along the same lines and is omitted. \( \square \)

4. Measures with finite quasi-Assouad dimension

In [6, Proposition 3.1] it was proven that the Assouad dimension of a measure is finite if and only if the measure is doubling. This is not required for finite quasi-Assouad dimension, as is shown in Example 6.1.

In this section we characterize the measures with finite quasi-Assouad dimension that satisfy the asymptotic gap weak separation condition in terms of a weak doubling-like condition. The main result of this section is the characterization in Corollary 4.5.

Definition 4.1. A Borel probability measure \( \mu \) is **quasi-doubling** if for every \( q > 1 \), constant \( b > 0 \) and positive, non-increasing sequence \((g(n))_n\) satisfying \( \log g(n)/n \to 0 \), there exists \( c > 0 \) such that
\[
\mu(B(x, g(n)\lambda^n)) \geq c q^{-n} \mu(B(x, b\lambda^n)),
\]
for all \( x \in [0, 1] \) and \( n \in \mathbb{N} \).

To state our results we continue developing some notation for self-similar measures \( \mu \) with support \([0,1]\) and minimal contraction factor \( \lambda \).

Given a level \( n \) net interval, \( \Delta_n \), other than \( \Delta_n(1) \) or \( \Delta_n(0) \), we let \( \Delta_n^R \) be the union of the two net intervals of level \( n \) immediately to the right of \( \Delta_n \), and let \( \Delta_n^L \) be the union of the two net intervals immediately to the left, with the understanding that \( \Delta_n^R = \Delta_n(1) \) if
\( \Delta_n \) is immediately adjacent to \( \Delta_n(1) \) and similarly for \( \Delta_n' \). If \( \Delta_n = \Delta_n(1) \) we only define \( \Delta_n^L \) and if \( \Delta_n = \Delta_n(0) \) we only define \( \Delta_n^R \).

We remark that if the IFS associated with \( \mu \) satisfies the asymptotic gap weak separation condition with function \( f(n) \), any net interval whose endpoints are \( S_u(0) \) and \( S_w(0) \) for some \( u, w \in \Lambda_n \) has length at least \( f(n)\lambda^n \) and likewise if the endpoints are both iterates of 1. Consequently, the length of the union of any two adjacent net intervals has length at least \( f(n)\lambda^n \). In particular, this is true for \( \Delta_n^L \) and \( \Delta_n^R \) since \( \Delta_n(0) \) and \( \Delta_n(1) \) also have length at least \( f(n)\lambda^n \).

**Definition 4.2.** A self-similar measure \( \mu \) that satisfies the asymptotic gap weak separation condition with function \( f(n) \) is quasi-net doubling if for every \( q > 1 \) there exist \( c_1, c_2 > 0 \) such that if \( l(\Delta_n) \geq f(n+1)\lambda^{n+1} \), then

\[
(4.2) \quad c_1 q^{-n}\mu(\Delta_n^*) \leq \mu(\Delta_n) \leq c_2 q^n \mu(\Delta_n^*)
\]

for \( \Delta_n^* = \Delta_n^L \) and \( \Delta_n^* = \Delta_n^R \), where defined.

**Lemma 4.3.** Let \( \mu \) be a self-similar measure with support \([0,1]\).

1. If \( \dim_{\mu^A} \mu < \beta \), then \( \mu \) is quasi-doubling.
2. Suppose \( \mu \) satisfies the asymptotic gap weak separation condition with function \( f(n) \). Further assume that \( \mu \) satisfies (4.1) for all \( q > 1 \) with \( g(n) = f(n+1)\lambda/2 \) and \( b = 3 \). Then \( \mu \) is quasi-net doubling.

**Proof.** (1) Suppose \( \dim_{\mu^A} \mu < \beta < \infty \) and let \( q > 1 \). Select \( \delta > 0 \) so that \( \lambda^{-\delta \beta} = q \). The definition of the quasi-Assouad dimension ensures that, for some \( C > 0 \),

\[
(4.3) \quad \frac{\mu(B(x, b\lambda^n))}{\mu(B(x, (b\lambda^n)^{1+\delta}))} \leq C \lambda^{-n \delta \beta} = Cq^n.
\]

As \( \log g(n)/n \to 0 \), \( g(n) \geq b^{1+\delta} \lambda^n \) for \( n \) sufficiently large and thus \( g(n)\lambda^n \geq (b\lambda^n)^{1+\delta} \). Hence \( B(x, (b\lambda^n)^{1+\delta}) \subseteq B(x, g(n)\lambda^n) \) and

\[
C^{-1} q^{-n} \mu(B(x, b\lambda^n)) \leq \mu(B(x, (b\lambda^n)^{1+\delta})) \leq \mu(B(x, g(n)\lambda^n))
\]

for sufficiently large \( n \). Thus, redefining \( C > 0 \) if necessary, (4.1) holds for all \( n \) as required.

(2) Let \( \Delta_n \) be such that \( l(\Delta_n) \geq f(n+1)\lambda^{n+1} \) as there is nothing to prove otherwise. Let \( z \) be the midpoint of \( \Delta_n \) and let \( \Delta_n^* \) refer to either \( \Delta_n^L \) or \( \Delta_n^R \). Then \( B(z, f(n+1)\lambda^{n+1}/2) \subseteq \Delta_n \) and \( B(z, 3\lambda^n) \supseteq \Delta_n^* \). Thus, using (4.1),

\[
\mu(\Delta_n) \geq \mu(B(z, f(n+1)\lambda^{n+1}/2)) \geq c q^{-n} \mu(B(z, 3\lambda^n)) \geq c q^{-n} \mu(\Delta_n^*).
\]

This proves the left hand inequality in (4.2).

Similarly, we can prove the other inequality. Recall that \( l(\Delta_n^*) \geq f(n)\lambda^n \) and so, letting \( z \) be the midpoint of \( \Delta_n^* \), and using (4.1),

\[
\mu(\Delta_n^*) \geq \mu(B(z, f(n)\lambda^n)) \geq c q^{-n} \mu(B(z, 3\lambda^n)) \geq c q^{-n} \mu(\Delta_n).
\]

This proves the right hand inequality. \( \square \)

**Theorem 4.4.** Suppose the self-similar measure \( \mu \) has support \([0,1]\) and satisfies the asymptotic gap weak separation condition. Then \( \dim_{\mu^A} \mu < \infty \) if and only if \( \mu \) is quasi-net doubling.
Proof. Note that by Lemma 4.3 any self-similar measure satisfying the AGWSC with finite quasi-Assouad dimension is quasi-net doubling. It remains to prove the opposite direction.

Without loss of generality we can assume \( \lambda < 1/2 \), for if not, we can replace the IFS \( \{ S_j \} \) with suitable \( k \)-fold compositions of the maps \( S_j \).

Fix \( \delta > 0 \) and let \( N_0 \) be large enough such that
\[
(4.4) \quad f(N + 1) \geq \lambda^{N\delta/2}
\]
for all \( N \geq N_0 \). Such an \( N_0 \) exists as the asymptotic gap weak separation condition guarantees \( \log(f(n))/n \to 0 \) as \( n \to 0 \). We will be using the bounds of (3.2) with \( q = 2^{\delta/(1+\delta)} > 1 \) and consider \( \Delta_N(x) \).

**Case 1:** Assume that \( l(\Delta_N(x)) \geq f(N + 1)\lambda^{N+1} \). As \( \Delta_N^R \) and \( \Delta_N^L \) have length at least \( f(N)\lambda^N \),
\[
B(x, f(N)(1-\lambda)\lambda^N) \cap [0, 1] \subseteq \Delta_N(x) \cup \Delta_N^R \cup \Delta_N^L
\]
and thus by the quasi-net doubling condition,
\[
\mu(B(x, f(N)(1-\lambda)\lambda^N)) \leq \mu(\Delta_N(x)) + \mu(\Delta_N^R) + \mu(\Delta_N^L) \\
\leq c q^N \mu(\Delta_N(x))
\]
for some \( c > 0 \). From (3.2), we have \( \mu(\Delta_N(x)) \leq P_N(\Delta_N(x)) \) and \( \mu(B(x, \lambda^n)) \geq P_n(\Delta_n(x)) \) for any \( n \). Let \( t = (\min p_j)^{-\Theta} \), where \( \Theta \) is given in (3.3). It now follows from Lemma 3.7 that
\[
\frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq c q^N \frac{P_N(\Delta_N(x))}{P_n(\Delta_n(x))} \leq c q^N t^{n-N}.
\]

The “gap” between \( r \) and \( R \) in the definition of the quasi-Assouad dimension means that we can restrict our attention to the case where
\[
\lambda^n \leq (f(N)(1-\lambda)\lambda^N)^{1+\delta}.
\]
We can therefore assume without loss of generality that \( n \geq N(1+\delta') \) for all \( \delta' < \delta \). In particular, this holds for \( \delta' = \delta/(1+\delta) \). Rearranging gives \( N\delta/(1+\delta) = N\delta' \leq (n - N) \) and hence \( q^n = 2^{N\delta'} \leq 2^{n-N} \). Taking \( \beta = \log 2t/|\log \lambda| \) we have
\[
\frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq c \lambda^{-\delta(n-N)}.
\]

Using (1.4), we have
\[
\frac{f(N + 1)(1-\lambda)\lambda^N}{3\lambda^n} \geq c_1 \lambda^{N-n} \lambda^{N\delta/2} \geq c_1 \lambda^{-(n-N)/2},
\]
for all \( n \geq (1+\delta')N > N \geq N_0 \) and some \( c_1 > 0 \). Redefining \( c \), if necessary, we obtain
\[
(4.5) \quad \frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq c \left( \frac{f(N + 1)(1-\lambda)\lambda^N}{3\lambda^n} \right)^{2\beta}.
\]

We next show that in the second case we obtain the same bound, before establishing that this is sufficient to guarantee finite quasi-Assouad dimension.

**Case 2:** Assume \( l(\Delta_N(x)) < f(N + 1)\lambda^{N+1} \). In this case, \( \Delta_N(x) \) cannot contain two net subintervals of level \( N + 1 \) as their union would have length at least \( f(N + 1)\lambda^{N+1} \). Thus
\( \Delta_{N+1}(x) = \Delta_N(x) \). Fix \( n \) such that \( 3 \lambda^n \leq (f(N)(1 - \lambda) \lambda^N)^{i+\delta} \) and choose the maximal integer \( j \) such that \( N < j \leq n \) and \( \Delta_N(x) = \cdots = \Delta_j(x) \).

Since the union of two adjacent level \( N \) net intervals has length at least \( f(N) \lambda^N \), it follows that the level \( N \) net intervals immediately adjacent to \( \Delta_N(x) \) have length at least \( f(N)(1 - \lambda) \lambda^N \). Denote the left and right net intervals of \( \Delta_N(x) \) by \( \Delta_N^l \) and \( \Delta_N^r \) respectively. Thus

\[
B(x, f(N)(1 - \lambda) \lambda^N) \cap [0, 1] \subseteq \Delta_N(x) \cup \Delta_N^l \cup \Delta_N^r.
\]

Let \( x_1, x_2 \) be the midpoints of \( \Delta_N^r \) and \( \Delta_N^l \) respectively. As each level \( j \) net interval has length at most \( \lambda^j \) we have \( B(x_i, \lambda^j) \subseteq B(x, 3 \lambda^j) \) for \( i = 1, 2 \).

These observations yield the bounds

\[
\mu(B(x, 3 \lambda^j)) \geq \max(\mu(B(x, \lambda^j)), \mu(B(x_1, \lambda^j)), \mu(B(x_2, \lambda^j)))
\]

and

\[
\mu(B(x, f(N)(1 - \lambda) \lambda^N)) \leq 3 \max(\mu(\Delta_N(x)), \mu(\Delta_N^l), \mu(\Delta_N^r)) \leq 3 \max(P_N(\Delta_N(x)), P_N(\Delta_N^l), P_N(\Delta_N^r)).
\]

Since \( \Delta_N(x) = \Delta_j(x) \), it follows that \( \Delta_j^r \subseteq \Delta_N^r \) and \( \Delta_j^l \subseteq \Delta_N^l \), so

\[
\frac{\mu(B(x, f(N)(1 - \lambda) \lambda^N))}{\mu(B(x, 3 \lambda^j))} \leq 3 \max(\frac{P_N(\Delta_N(x))}{P_j(\Delta_j^r)}). \frac{P_N(\Delta_N^l)}{P_j(\Delta_j^l)} \leq 3 \mu^{-N} t^j,
\]

where \( t = (\min p_j)^{-\theta} \). If \( j = n \), then as in Case 1, we have

\[
(4.6) \quad \frac{\mu(B(x, f(N)(1 - \lambda) \lambda^N))}{\mu(B(x, 3 \lambda^n))} \leq c \left( \frac{f(N+1)(1 - \lambda) \lambda^N}{3 \lambda^n} \right)^{2\beta}.
\]

Otherwise, \( j < n \) so that \( \Delta_{j+1}(x) \neq \Delta_j(x) \). That ensures \( \Delta_j(x) \) contains at least two \( (j+1) \)-level net intervals and so its length is at least \( f(j + 1) \lambda^{j+1} \). Thus the quasi-net doubling condition implies

\[
\mu(\Delta_j(x)) + \mu(\Delta_j^R) + \mu(\Delta_j^L) \leq c q^j \mu(\Delta_j(x)) \leq c q^j P_j(\Delta_j(x))
\]

and hence

\[
\frac{\mu(\Delta_j(x) \cup \Delta_j^R \cup \Delta_j^L)}{\mu(B(x, \lambda^n))} \leq c q^j \frac{P_j(\Delta_j(x))}{P_n(\Delta_n(x))} \leq c q^j t^{n-j}.
\]

We will deal with the case where the maximum of \( P_N(\Delta_N(x)), P_N(\Delta_N^r), \) and \( P_N(\Delta_N^l) \) is \( P_N(\Delta_N^r) \). The other two cases are analogous and left to the reader.

Let \( y_1 \) be the right endpoint of \( \Delta_j(x) \). Then

\[
B(y_1, f(j+1) \lambda^{j+1}) \cap [0, 1] \subseteq \Delta_j(x) \cup \Delta_j^R \cup \Delta_j^L,
\]

so applying \( \text{(4.3)} \) we have

\[
\mu(\Delta_j(x)) + \mu(\Delta_j^R) + \mu(\Delta_j^L) \geq \mu(B(y_1, f(j+1) \lambda^{j+1})) \geq P_{j+1+\kappa_{j+1}}(\Delta_{j+1+\kappa_{j+1}}(y_1)).
\]
where we are free to take $\Delta_{j+1+\kappa_{j+1}}(y_1)$ to be the net interval having $y_1$ as the left endpoint. But as $\Delta_j(x) = \Delta_N(x)$, $y_1$ is also the left endpoint of $\Delta_N$. So we can choose $\Delta_N(y_1) = \Delta_N$.

Combining these observations gives
\[
\frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(\Delta_j(x) \cup \Delta_j^R \cup \Delta_j)} \leq \frac{3P_N(\Delta_N)}{P_{j+1+\kappa_j}(\Delta_{j+1+\kappa_{j+1}}(y_1))} \leq \frac{3P_N(\Delta_N)}{P_{j+1+\kappa_j}(\Delta_{j+1+\kappa_{j+1}}(y_1))} \leq c t^{j+\kappa_{j+1}-N}.
\]

Consequently,
\[
\frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq \frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(\Delta_j(x) \cup \Delta_j^R \cup \Delta_j)} \frac{\mu(\Delta_j(x) \cup \Delta_j^R \cup \Delta_j)}{\mu(B(x, 3\lambda^n))} \leq c t^{j+\kappa_{j+1}+1-N} q^j t^{n-j} \leq c q^n t^{n-N+\kappa_n} \leq c 2^n - N t^{n-N+\kappa_n}.
\]

Since $(\log f(N+1))/N$ and $\kappa_n/n$ tend to zero for increasing $n, N$, there exists $N_1$ such that
\[
\left(1 - \frac{N}{n}\right) \log(2t\lambda^2) \leq 2\beta \frac{\log f(N+1)}{n} - \frac{\kappa_n}{n} \log t
\]
for all $n \geq (1 + \delta)N \geq N_1$. Thus
\[
(2t)^{n-N} t^{\kappa_n} \leq \left( f(N+1)\lambda^{-(n-N)} \right)^{2\beta}
\]
and that ensures
\[
(4.7) \quad \frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq c \left( \frac{f(N+1)(1-\lambda)\lambda^N}{3\lambda^n} \right)^{2\beta}
\]
for all $N \geq N_1$.

Having established the same upper bound in (4.5), (4.6), and (4.7) it remains to show that this is sufficient for the quasi-Assouad dimension to be finite. Let $N_2 = \max\{N_0, N_1\}$. For fixed $\delta' > \delta$, let $r \leq R^{1+\delta'} \leq R < f(N_2)(1-\lambda)\lambda^{N_2}$ and choose $N \geq N_2$ and $n$ such that
\[
f(N+1)(1-\lambda)\lambda^{N+1} \leq R \leq f(N)(1-\lambda)\lambda^N \quad \text{and} \quad 3\lambda^n \leq r \leq 3\lambda^{n-1}.
\]

We note that this is well-defined as $f$ is non-increasing and $r \leq R^{1+\delta'}$ gives $n \geq (1 + \delta)N$. Now, by appealing to (4.5), (4.6), and (4.7), we have
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{\mu(B(x, f(N)(1-\lambda)\lambda^N))}{\mu(B(x, 3\lambda^n))} \leq c \left( \frac{f(N+1)(1-\lambda)\lambda^N}{3\lambda^n} \right)^{2\beta} \leq c \left( \frac{R}{r} \right)^{2\beta}.
\]
This proves that $\dim_{qA} \mu \leq 2\beta < \infty$. \hfill \square

We finish this section by providing a classification of the quasi-Assouad dimension in terms of quasi-doubling.
Corollary 4.5. Let $\mu$ be a self-similar measure with $\text{supp}\, \mu = [0,1]$.

1. Suppose $\{S_j\}$ satisfies the asymptotic gap weak separation condition. Then $\dim_{qA} \mu < \infty$ if and only if $\mu$ is quasi-doubling.

2. Suppose $\mu$ satisfies the weak separation condition. Then $\dim_{qA} \mu < \infty$ if and only if for every $q > 1$ and $0 < a < b$ there exist constants $c = c(q,a,b)$ such that

$$\mu(B(x,a\lambda^n)) \geq c q^{-n} \mu(B(x,b\lambda^n)).$$

Proof. These follow immediately from Lemma 4.3 and Theorem 4.4 upon noting that in the weak separation case, the constant sequence $\alpha$ from the definition of the WSC could be chosen as the sequence $(g(n))$ in the quasi-doubling definition. \qed

5. Dimensions of weakly comparable measures

In this section we consider weakly comparable measures that were originally introduced in [11] where they were called comparable measures. In particular, we show that weakly comparable self-similar measures $\mu$ that satisfy the asymptotic gap weak separation condition are quasi-doubling and we provide an upper bound to their quasi-Assouad dimension.

Definition 5.1. A self-similar measure $\mu$ with support $[0,1]$ is weakly comparable if for each $q > 1$ there is a constant $c$, depending on $q$, such that for all $n$ and adjacent net intervals $\Delta_1, \Delta_2$ of level $n$, we have

$$\frac{1}{c} q^{-n} P_n(\Delta_2) \leq P_n(\Delta_1) \leq c q^n P_n(\Delta_2).$$

In the equicontractive case, it can be shown that if the probabilities are regular (meaning, $p_0 = p_{m-1} = \min p_j$), then the self-similar measure is weakly comparable; see Corollary 6.3.

If an IFS is of finite type, then the net intervals of level $n$ are comparable in size. Thus if a finite type measure is doubling, then it is weakly comparable. On the other hand, Example 6.1 gives an equicontractive, finite type IFS that is regular, hence weakly comparable, but not doubling.

Proposition 5.2. If $\mu$ is a weakly comparable, self-similar measure that satisfies the asymptotic gap weak separation condition, then $\mu$ is quasi-net doubling and hence has finite quasi-Assouad dimension.

Proof. Fix $q_0 > 1$ and let $q = (q_0)^{1/3} > 1$. By Lemma 3.6 there is a constant $A = \min p_j^g$ such that $P_n(\Delta_n(x)) \geq A P_{n-1}(\Delta_{n-1}(x))$ for all $n$ and $x \in [0,1]$. Choose $\epsilon > 0$ so that $A^\epsilon \geq q^{-1}$. Given any net interval $\Delta_n(x)$ we have

$$\mu(\Delta_n^*) \leq P_n(\Delta_n^L) + P_n(\Delta_n^R) \leq c q^{2n} P_n(\Delta_n(x)),$$

where $\Delta_n^*, \Delta_n^L, \Delta_n^R$ are as in Definition 4.2. Note that by the definition of quasi-net doubling we only need to check the case when $l(\Delta_n(x)) \geq f(n+1) \lambda^{n+1}$. Taking $z$ to be the midpoint of $\Delta_n(x)$ we have

$$\Delta_n(x) \supseteq B(z, f(n+1) \lambda^{n+1}/2) \supseteq B(z, f(n+2) \lambda^{n+2}),$$

hence Lemma 3.4 and (3.3) yield

$$\mu(\Delta_n(x)) \geq \mu(B(z, f(n+2) \lambda^{n+2})) \geq P_{n+2+\kappa_n+2}(\Delta_{n+2+\kappa_n+2}(x)) \geq A^{2+\kappa_n+2} P_n(\Delta_n(x)).$$
For large enough $n$, the weakly comparable assumption thus implies
\[
\mu(\Delta_n(x)) \geq A^2A^{n_k}P_n(\Delta_n(x)) \geq A^2q^{-n}P_n(\Delta_n(x)) \\
\geq A^2q^{-3n}c^{-1}\mu(\Delta_n^\ast) \geq A^2q_0^{-n}c^{-1}\mu(\Delta_n^\ast).
\]

The inequality $\mu(\Delta_n^\ast) \geq c_1q_0^{-n}\mu(\Delta_n(x))$ follows analogously. This shows that $\mu$ is quasi-net doubling. \hfill \Box

The next result is similar in spirit, but more technical, and will be used later to find upper bounds on the quasi-Assouad dimension.

**Lemma 5.3.** Suppose $\mu$ is a weakly comparable, self-similar measure that satisfies the asymptotic gap weak separation condition with function $f(n)$. Then for any $q > 1$ there are constants $c_1, c_2 > 0$ depending on $q$ such that, for all $x \in [0, 1]$ and $n \in \mathbb{N}$,
\[
(5.1) \quad c_1q^{-n}P_n(\Delta_n(x)) \leq P_{n+k_n}(\Delta_{n+k_n}(x)) \leq \mu(B(x, f(n)\lambda^n)) \leq c_2q^nP_n(\Delta_n(x)).
\]

**Proof.** Since $l(\Delta_n^\ast) \geq f(n)\lambda^n$, we have
\[
B(x, f(n)\lambda^n) \cap [0, 1] \subseteq \Delta_n(x) \cup \Delta_R^N \cup \Delta_L^N,
\]
so that $\mu(B(x, f(n)\lambda^n)) \leq c_q^nP_n(\Delta_n(x))$ for some $c > 0$. Similar reasoning to the above shows that $P_{n+k_n}(\Delta_{n+k_n}(x)) \geq q^{-n}P_n(\Delta_n(x))$ for $n$ sufficiently large and as we always have $P_{n+k_n}(\Delta_{n+k_n}(x)) \leq \mu(B(x, f(n)\lambda^n))$, the inequalities of (5.1) are complete. \hfill \Box

For each positive integer $n$ and $x \in [0, 1]$, let
\[
Q_n(x) = \sup_{N \in \mathbb{N}} \frac{P_N(\Delta_N(x))}{P_{N+n}(\Delta_{N+n}(x))}
\]
where $\Delta_{N+n}(x)$ is a child of $\Delta_N(x)$ containing $x$ (with $P_{N+n}(\Delta_{N+n}(x))$ minimal if there are two choices). Set
\[
Q_n = \sup_{x \in \text{supp } \mu} Q_n(x).
\]

Of course, $Q_n \leq Q_1^n$ and by Lemma 3.7, $Q_1 \leq (\min_j p_j)^{-\Theta}$ where $\Theta$ is given by (3.3).

**Theorem 5.4.** Suppose $\mu$ is weakly comparable and satisfies the asymptotic gap weak separation property. Then the quasi-Assouad dimension of $\mu$ is bounded by
\[
(5.2) \quad \dim_q \mu \leq \limsup_{n \to \infty} \frac{-\log Q_n^n}{n \log \lambda} < \infty,
\]
and for all $x \in [0, 1]$,
\[
\overline{\dim}_{\text{loc}} \mu(x) = \limsup_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda} \quad \quad \underline{\dim}_{\text{loc}} \mu(x) = \liminf_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda}.
\]

**Proof.** We first prove the upper bound. Let
\[
d = \limsup_n \frac{-\log Q_n}{n \log \lambda}
\]
It will be enough to prove that $\dim_q \mu \leq d + \varepsilon$ for all $\varepsilon > 0$. Choose $M$ such that
\[
(5.3) \quad (1/m) \log Q_m \leq -(d + \varepsilon/2) \log \lambda,
\]
for all $m \geq M$.

Recall that $f(n)$ is decreasing in $n$. Thus, given $r \leq R^{1+\delta} \leq R \leq C_0$ we can choose $N$ and $n$ such that

$$f(N+1)\lambda^{N+1} < R \leq f(N)\lambda^{N}$$

and

$$f(n)\lambda^{n} \leq r < f(n-1)\lambda^{n-1},$$

where $C_0$ is chosen sufficiently small to ensure that $N+1 \leq n-1$ and $n-N \geq M$.

Take $q > 1$. Then (5.1) yields

$$\mu(B(x,R)) \leq \mu(B(x,f(N)\lambda^{N})) \leq Cq^{N}P_{N}(\Delta_{N}(x))$$

and

$$\mu(B(x,r)) \geq \mu(B(x,f(n)\lambda^{n})) \geq P_{n+\kappa_{n}}(\Delta_{n+\kappa_{n}}(x)),$$

where $C > 0$ depends on $q$. Thus

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq \frac{Cq^{N}P_{N}(\Delta_{N}(x))}{P_{n+\kappa_{n}}(\Delta_{n+\kappa_{n}}(x))} \leq C q^{N}Q_{n+\kappa_{n}-N}(x).$$

Using (5.3) and the fact that $\lambda^{\kappa_{n}} \geq \lambda f(n)$, we have

$$Cq^{N}Q_{n+\kappa_{n}-N}(x) \leq Cq^{N}\lambda^{-(n+\kappa_{n}-N)(d+\varepsilon/2)} \leq C q^{N} \lambda^{-(n-N)(d+\varepsilon/2)} f(n)^{-(d+\varepsilon/2)},$$

redefining $C > 0$ as appropriate.

Further,

$$\left(\frac{R}{r}\right)^{d+\varepsilon} \geq \left(\frac{f(N+1)}{f(n-1)}\right)^{d+\varepsilon} \lambda^{(N-n+1)(d+\varepsilon)} \geq c\lambda^{(N-n)(d+\varepsilon)}.$$ 

Thus, in order to satisfy

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq c \left(\frac{R}{r}\right)^{d+\varepsilon}$$

for some constant $c$, it will be enough to satisfy the inequality

$q^{N} \lambda^{-(n-N)(d+\varepsilon/2)} f(n)^{-(d+\varepsilon/2)} \leq \lambda^{(N-n)(d+\varepsilon)}$

for all $n \geq (1+\delta)N$, $N$ sufficiently large and for some suitable $q > 1$. Equivalently,

$$(d+\varepsilon/2) |\log f(n)| \leq (n-N)\varepsilon/2n |\log \lambda| = \frac{N}{n} \log q.$$

Since $N/n \leq (1+\delta)^{-1}$, the right hand side of the latter expression dominates

$$\left(1 - \frac{1}{1+\delta}\right) \frac{\varepsilon}{2} |\log \lambda| - \frac{1}{1+\delta} \log q,$$

and this is at least

$$\left(1 - \frac{1}{1+\delta}\right) \frac{\varepsilon}{4} |\log \lambda|$$

if we choose $q$ close enough to 1. Since $\log f(n)/n \rightarrow 0$ as $n \rightarrow \infty$, we can ensure that this quantity dominates $(d+\varepsilon/2) |\log f(n)|/n$ for large enough $n$. Suitably redefining $C_0 > 0$, if necessary, will guarantee that $n \geq (1+\delta)N$ is sufficiently large to be sure this is true. From these inequalities it follows that the quasi-Assouad dimension of $\mu$ is at most $d+\varepsilon$ as required.
We now turn to proving the equalities involving the local dimension. For each \( q > 1 \) we have, by (5.1),

\[
(Cq^n)^{-1} P_n(\Delta_n(x)) \leq \mu(B(x, f(n)\lambda^n)) \leq Cq^n P_n(\Delta_n(x))
\]

where \( C > 0 \) depends on \( q \). Thus

\[
\frac{\log q}{\log \lambda} + \liminf_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda} \leq \liminf_{n \to \infty} \frac{\log \mu(B(x, f(n)\lambda^n))}{\log \lambda^n} \leq \liminf_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda} - \frac{\log q}{\log \lambda}.
\]

Since the inequality above holds for all \( q > 1 \), we deduce

\[
\liminf_{n \to \infty} \frac{\log \mu(B(x, f(n)\lambda^n))}{\log \lambda^n} = \liminf_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda}
\]

Given any \( r > 0 \), choose \( n \) such that \( f(n+1)\lambda^{n+1} < r \leq f(n)\lambda^n \). Since \( \log f(n)/n \to 0 \), we deduce

\[
\liminf_{n \to \infty} \frac{\log \mu(B(x, r))}{\log r} = \liminf_{n \to \infty} \frac{\log \mu(B(x, f(n)\lambda^n))}{\log f(n)\lambda^n} = \liminf_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \lambda}.
\]

Thus \( \dim_{loc} \mu(x) \) is as claimed.

The arguments for the upper local dimension are identical and left to the reader. \( \square \)

6. Dimensions of Generalized Regular Measures

Suppose \( \Delta \in \mathcal{F}_N \) has descendant net subinterval \( \Delta' \in \mathcal{F}_{N+n} \). If \( u \in \Lambda_N \) with \( S_u[0,1] \supseteq \Delta \), then there is some word \( w \) such that \( uw \in \Lambda_{n+N} \) and \( S_{uw}[0,1] \supseteq \Delta' \). We call such a word \( w \) a path of level \( n \) (of \( \Delta \)). Clearly,

\[
P_{N+n}(\Delta_{N+n}(x)) \geq \inf \{ p_w : w \text{ path of level } n \} P_N(\Delta_N(x)).
\]

We call \( w \) a left-edge path if \( S_w(0) = 0 \) and a right-edge path if \( S_w(1) = 1 \). Put

\[
\Gamma^L_{\Delta,n} = \sum_{w \text{ left-edge path of } \Delta \text{ of level } n} p_w \quad \text{and} \quad \Gamma^R_{\Delta,n} = \Gamma^R_{[0,1],n}.
\]

We define \( \Gamma^R_{\Delta,n} \) and \( \Gamma^R_{n} \) similarly, and set \( \Gamma_n = \Gamma^L_n + \Gamma^R_n \) and \( \Gamma_{\Delta.n} = \Gamma^R_{\Delta,n} + \Gamma^L_{\Delta,n} \). Note that

\[
\Gamma^L_n = \sum_{w \in \Lambda_n, S_w(0) = 0} p_w = P_n(\Delta_n(0))
\]

hence the constants \( s, t \) introduced in (3.6) are also equal to

\[
s = \liminf_{n \to \infty} \left( \Gamma^L_n \right)^{1/n}, \quad t = \liminf_{n \to \infty} \left( \Gamma^R_n \right)^{1/n}.
\]

We first prove that all \( \Gamma^L_{\Delta,n} \) are comparable to \( \Gamma^L_n \).

**Proposition 6.1.** There exists \( c > 0 \) such that \( \Gamma^L_n \leq \Gamma^L_{\Delta,n} \leq c \Gamma^L_n \) for all \( n \) and \( \Delta \). Similarly, \( \Gamma^R_n \) is comparable to \( \Gamma^R_{\Delta,n} \).
Proof. Let $\Delta = [a, b]$ be a net interval of level $N$. Then $\Gamma_{\Delta,n}^L = \sum p_w$, where the sum is over all $w$ where $S_w(0) = 0$ and there is some $u \in \Lambda_N$ such that $uw \in \Lambda_{N+n}$ and $S_u(0) = a$. This means that $w, u$ must satisfy the conditions $r_u \leq \lambda^N$, $r_{uw} > \lambda^N$, $r_{uw} \leq \lambda^{N+n}$ and $r_{(uw)} = r_{uw-} > \lambda^{N+n}$. Since $r_j \geq \lambda$ for all $j$, it follows that $r_w < \lambda^{n-1}$ and $r_{w-} > \lambda^n$. Consequently,

$$\Gamma_{\Delta,n}^L \leq \sum_{S_w(0)=0,} p_w \sum_{r_w < \lambda^{n-1}, r_{w-} > \lambda^n}$$

(6.1)

$$= \sum_{S_w(0)=0,} p_w + \sum_{S_w(0)=0, r_w \leq \lambda^n, r_{w-} > \lambda^n} p_w.$$

Recall that $w \in \Lambda_n$ is equivalent to $w$ satisfying $r_w \leq \lambda^n$ and $r_{w-} > \lambda^n$, thus the left sum in (6.1) is equal to $\Gamma_{\Delta,n}^L$. For the second sum, let $k$ be the minimal integer such that $r^k_0 \leq \lambda$. For fixed $r_{w'} \in (\lambda^n, \lambda^{n-1})$, choose $j = j(w') \in \{1, 2, \ldots, k\}$ such that $r_{w'}r^{j-1}_0 \leq \lambda^n$ and $r_{w'}r^{j-1}_0 \leq \lambda^n$. Let $\overline{0}_j$ be the unique word of length $j$ containing just the letter 0. Then

$$\sum_{j=1}^{k} \sum_{S_{w}(0)=0, r_{w} \in (\lambda^n, \lambda^{n-1}), r_{w-} > \lambda^n} p_w \leq \sum_{j=1}^{k} \sum_{S_{w}(0)=0, r_{w} \in (\lambda^n, \lambda^{n-1}), r_{w-} > \lambda^n} p_{\overline{0}_j} \leq k_p0^{-k} \Gamma_{\Delta,n}^L.$$

We conclude $\Gamma_{\Delta,n}^L \leq (1 + kp_0^{-k}) \Gamma_{\Delta,n}^L$, as required. \hfill $\square$

Definition 6.2. The weighted iterated function system $\{S_j, p_j\}$ is generalized regular if for each $q > 1$,

$$\lim_{n \to \infty} Q_n q^{-n} \sup_{\Delta} \Gamma_{\Delta,n} = 0,$$

where the supremum is taken over all net intervals.

We will also call the self-similar measure $\mu$ associated with $\{S_j, p_j\}$ a generalized regular measure.

Corollary 6.3. A self-similar measure $\mu$ is generalized regular if and only if

$$\lim_{n \to \infty} Q_n q^{-n} \Gamma_{\Delta,n}^L = \lim_{n \to \infty} Q_n q^{-n} \Gamma_{\Delta,n}^R = 0$$

for every $q > 1$.

The notion of ‘generalized regular’ was introduced in the study of non-equicontractive finite type iterated function systems where it was observed that generalized regular implies weakly comparable (see [11, Theorem 4.11]). In fact, this holds in general.

Proposition 6.4. A generalized regular self-similar measure $\mu$ is weakly comparable.

Proof. Fix $q > 1$ and choose $N_0$ so $\sup_{\Delta} \Gamma_{\Delta,n} \leq Q^{-1}_n q^0/2$ for all $n \geq N_0$. Since $P_n(\Delta)$ is finite for all $n$, we can find $c > 0$ such that

$$\frac{1}{c} q^{-k} P_k(\Delta_2) \leq P_k(\Delta_1) \leq c q^k P_k(\Delta_2)$$

whenever $\Delta_1, \Delta_2$ are adjacent net intervals of level $k$ for all $k = 1, \ldots, N_0$. 


Assume \( n \geq N_0 + 1 \). We proceed by induction on \( n \). Suppose \( \Delta_1, \Delta_2 \) are adjacent net intervals of level \( n \) where, without loss of generality, \( \Delta_1 \) is to the left of \( \Delta_2 \). If \( \hat{\Delta}_j \) is the ancestor of \( \Delta_j \) at level \( n - k \), then \( P_n(\Delta_j) \sim P_{n-k}(\hat{\Delta}_j) \), with constants of comparability depending only on \( k \). Thus we can assume \( \Delta_1, \Delta_2 \) have no common ancestor within \( N_0 \) levels.

For \( j = 1, 2 \), let \( \hat{\Delta}_j \) be the \((n - N_0)\)-level ancestor of \( \Delta_j \). Let \( D_1 \) denote the words \( u \in \Lambda_n - \Lambda_0 \) where \( S_u[0,1] \) contains \( \hat{\Delta}_1 \), but not \( \hat{\Delta}_2 \). Define \( D_2 \) analogously and let \( \mathcal{E} \) denote those \( u \in \Lambda_n - \Lambda_0 \) where \( S_u[0,1] \) contains both \( \hat{\Delta}_1 \) and \( \hat{\Delta}_2 \).

Consider any \( \tau \in \Lambda_n \) with \( S_{\tau}[0,1] \) covering \( \hat{\Delta}_1 \). Then \( \tau = uw \) where \( u \in \Lambda_{n-N_0} \), \( S_u[0,1] \) contains \( \hat{\Delta}_1 \) and \( w \) is a path of level \( N_0 \) of \( \hat{\Delta}_1 \). The word \( u \) belongs to either \( D_1 \) or \( \mathcal{E} \) and in the former case \( w \) is a right edge path. Thus

\[
P_n(\Delta_1) = \sum_{u \in \Lambda_{n-N_0}} p_{uw} = \sum_{\text{w path of level } N_0 \text{ of } \hat{\Delta}_1} p_{uw},
\]

where \( S_{uw}[0,1] \geq \Delta_1 \).

\[
= \sum_{u \in D_1} p_u p_w + \sum_{u \in \mathcal{E}} p_u p_w
\]

\[
\leq \sum_{u \in D_1} p_u \Gamma_{\hat{\Delta}_1-N_0} + \sum_{u \in \mathcal{E}} p_u
\]

\[
\leq \frac{Q^{-1}_{N_0} q^{N_0}}{2} \sum_{u \in D_1} p_u + \sum_{u \in \mathcal{E}} p_u.
\]

Now \( \sum_{u \in D_1} p_u \leq \sum_{u \in D_1} p_u + \sum_{u \in \mathcal{E}} p_u = P_{n-N_0}(\hat{\Delta}_1) \) and \( \sum_{u \in \mathcal{E}} p_u \leq \sum_{u \in \mathcal{E}} p_u + \sum_{u \in D_1} p_u = P_{n-N_0}(\hat{\Delta}_2) \). Hence applying the inductive assumption and using the fact that \( P_n(\Lambda_{n-N_0}) \leq P_n(\Delta_2)Q_{N_0} \), we have

\[
P_n(\Delta_1) \leq \frac{Q^{-1}_{N_0} q^{N_0}}{2} P_{n-N_0}(\hat{\Delta}_1) + P_{n-N_0}(\hat{\Delta}_2)
\]

\[
\leq \left( \frac{Q^{-1}_{N_0} q^{N_0}}{2} c q^{n-N_0} + 1 \right) P_{n-N_0}(\hat{\Delta}_2)
\]

\[
\leq \left( \frac{1}{2} Q^{-1}_{N_0} c q^n + 1 \right) P_n(\Delta_2)Q_{N_0}.
\]

Taking \( c \geq 0 \) sufficiently large, we obtain the desired conclusion that \( P_n(\Delta_1) \leq c q^n P_n(\Delta_2) \).

**Corollary 6.5.** Let \( \{S_j, p_j\} \) be an equicontractive and regular weighted iterated function system, i.e. \( r_j = \lambda \) for all \( j \) and \( p_0 = p_{m-1} = \min_j p_j \). Then the associated self-similar measure is generalized regular and thus weakly comparable.

**Proof.** Under these assumptions the edge paths of level \( n \) are the words \((0)^n\) and \((1)^n\). Thus \( \Gamma_n = p_0^n + p_{m-1}^n \). The equicontractive assumption ensures that \( Q_n \leq (\min_j p_j)^{-n} = p_0^{-n} = p_{m-1}^{-n} \), thus such an IFS is generalized regular and hence weakly comparable. \( \square \)
In our next result, we prove that all generalized regular measures satisfying the asymptotic gap weak separation property have quasi-Assouad dimension equal to the maximum upper local dimension.

**Theorem 6.6.** Suppose \( \mu \) is a generalized regular self-similar measure, with support \([0, 1]\), that satisfies the asymptotic gap weak separation condition. Then
\[
\dim qA \mu = \max \{\dimloc \mu(0), \dimloc \mu(1)\} = \max \{\dimloc \mu(x) : x \in \text{supp} \mu\}.
\]

**Proof.** Without loss of generality, assume \( \max \{\dimloc \mu(0), \dimloc \mu(1)\} = \dimloc \mu(0) \), which by Proposition 3.9 is equal to
\[
d = \log s / \log \lambda \text{ where } s = \liminf (\Gamma_n^L)^{1/n}.
\]

First, we will verify that for small enough \( \varepsilon > 0 \) and every \( \delta > 0 \) there are constants \( C, C_0 \), depending on \( \varepsilon, \delta \), such that if \( r \leq R^{1+\delta} \leq R \leq C_0 \), then for all \( x \),
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left( \frac{R}{r} \right)^{d(1+\varepsilon)}.
\]

Consequently, \( \dim qA \mu \leq d \).

Fix \( \delta, \varepsilon > 0 \) and \( x \in [0, 1] \). Assume \( r \leq R^{1+\delta} \leq R \leq C_0 \) where \( C_0 \) will be specified later. Choose integers \( N, n \) such that \( f(N+1)\lambda^{N+1} < R \leq f(N)\lambda^{N} \) and \( \lambda^n \leq r < \lambda^{n-1} \). By (3.2),
\[
\mu(B(x, r)) \geq \mu(B(x, \lambda^n)) \geq P_n(\Delta_n(x))
\]
and since the measure \( \mu \) is weakly comparable,
\[
\mu(B(x, R)) \leq \mu(B(x, f(N)\lambda^{N})) \leq Cq^NP_n(\Delta_N(x))
\]
for \( C \) depending on \( q \), see (5.1).

We see that
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{Cq^NP_n(\Delta_N(x))}{P_n(\Delta_n(x))} \leq Cq^nQ_n^{-N}.
\]
As \( \mu \) is generalized regular, \( \Gamma_m^LQ_m^{-m} \to 0 \) as \( m \to \infty \). Thus, we can choose \( N_1 \) such that if \( m = n - N \geq \delta N_1 \) then \( Q_m \leq q^m(\Gamma_m^L)^{-1} \). Therefore
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{q^mCq^n}{\Gamma_m^L} = Cq^n(\Gamma_m^L)^{-1}.
\]

As noted in Proposition 3.9 \( s = \liminf (\Gamma_n^L)^{1/n} \in (0, 1) \). Thus we can choose \( N_2 \) so that if \( m \geq \delta N \) for some \( N \geq N_2 \), then
\[
\Gamma_m^L \geq s^{m(1+\varepsilon)/2}.
\]
We require that \( C_0 \) be so small that if \( R \leq C_0 \), then \( R \leq f(N)\lambda^{N} \) for \( N \geq \max(N_1, N_2) \). Hence for \( r \leq R^{1+\delta} \leq R \leq C_0 \) we have
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq Cq^n\lambda^{-m(1+\varepsilon)/2}.
\]
As \( s = \lambda^{d} \), for any fixed \( \varepsilon > 0 \), there exists \( c > 0 \) such that
\[
(6.2) \quad \left( \frac{R}{r} \right)^{d(1+\varepsilon)} \geq \lambda^{(N-n+2)d(1+\varepsilon)} f(N+1)^{d(1+\varepsilon)} = cf(N+1)^{d(1+\varepsilon)}s^{-m(1+\varepsilon)}.
\]
We note that
\[ C q^m s^{-m(1+\varepsilon)/2} \leq c f(N + 1)^{d(1+\varepsilon)} s^{-m(1+\varepsilon)} \]
if and only if
\[ C' q^n f(N + 1)^{-d(1+\varepsilon)} \leq s^{-m\varepsilon/2}, \]
where \( C' \) is the appropriate constant. Taking logarithms, this is equivalent to
\[ \frac{1}{m} (\log C' + n \log q - d(1+\varepsilon) \log f(N + 1)) \leq \varepsilon |\log s|/2. \]
Now, \( m = n - N \geq \delta n/(1+\delta) \), so \((n \log q)/m \leq (1+\delta)(\log q)/\delta\) and \((1/m) \log f(N + 1) \to 0\). Thus with a suitable choice of \( q \) close to 1 (depending on \( \varepsilon, \delta \)) and large enough \( N \), we can achieve this inequality. With this further constraint on \( C_0 \) it then follows that for a suitable constant \( c \), we have
\[ \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq c \left( \frac{R}{r} \right)^{d(1+\varepsilon)} \text{ for all } r \leq R^{1+\delta} \leq R \leq C_0, \]
and this implies \( \dim_{qA} \mu \leq d(1+\varepsilon) \) for all \( \varepsilon > 0 \).

To see that we have equality, we first note that from Proposition 3.3 we have
\[ \frac{\mu(B(0, \lambda^N))}{\mu(B(0, f(n)\lambda^n))} \geq \frac{\mu(\Delta_N(0))}{\mu(\Delta_n(0))} \geq A^{-n} \frac{P_N(\Delta_N(0))}{P_n(\Delta_n(0))} \geq A^{-n} \frac{\Gamma_n^L J(N)^{\log A}}{\Gamma_n^L}. \]

Along a suitable subsequence, \((n_j)\), we have \((\Gamma_{n_j}^L)^{1/n} \leq s^{1-\varepsilon}\). Thus for \( N \) sufficiently large we have
\[ \frac{\mu(B(0, \lambda^N))}{\mu(B(0, f(n_j)\lambda^{n_j}))} \geq A s^{(1+\varepsilon)N} f(N)^{\log A \frac{n_j^{N+1}}{\log n_j}} = A s^{N-n_j + \varepsilon (N+n_j)}, \]
while
\[ \left( \frac{\lambda^N}{f(n_j)\lambda^{n_j}} \right)^d = s^{N-n_j} f(n_j)^{-d}. \]

Similar reasoning to above shows that if \( \gamma > 0 \), then
\[ \left( \frac{\lambda^N}{f(n_j)\lambda^{n_j}} \right)^{d(1-\gamma)} \leq A \frac{\mu(B(0, \lambda^N))}{\mu(B(0, f(n_j)\lambda^{n_j}))} \]
for sufficiently large \( N \). That implies \( \dim_{qA} \mu \geq s(1-\gamma) \) for all \( \gamma > 0 \) and hence \( \dim_{qA} \mu = s \).

The generalized regular condition in Theorem 6.6 is not necessary. Consider the IFS \( S_0(x) = x/3, S_1(x) = x/3 + 1/3, S_2(x) = x/3 + 2/3 \), with probabilities \( p_0 = p_2 = 2/5 \), \( p_1 = 1/5 \) and associated self-similar measure \( \mu \). This is a comparable, but not generalized regular, iterated function system. Note that \( Q_n(x) \leq 5^n \) for all \( x \) and \( n \). Thus Theorem 5.3 yields that \( \dim_{qA} \mu \leq \log 5/\log 3 \), which coincides with \( \dim_{loc} \mu(1/2) \), the maximum upper local dimension. Since the quasi-Assouad dimension is always an upper bound on the upper local dimension of the measure, we have equality here.

It would be desirable to know if all weakly comparable measures \( \mu \) have the property that \( \dim_{qA} \mu = \sup \{ \dim_{loc} \mu(x) \ : \ x \in [0,1] \} \).
Theorem 6.6 implies, in particular, that a generalized regular measure satisfying the asymptotic gap weak separation condition and having full support has finite quasi-Assouad dimension. The next example illustrates that it need not have finite Assouad dimension.

6.1. An example with finite quasi-Assouad and infinite Assouad dimension. Consider the IFS $S_j(x) = x/3 + d_j$ where $d_0 = 0$, $d_1 = 1/6$, $d_2 = 1/3$, $d_3 = 2/3$ and probabilities $p_0 = p_1 = p_3 = 1/6$, $p_2 = 1/2$. This IFS is equicontractive, finite type, regular and of full support. Thus it is generalized regular and hence weakly comparable. Applying Theorem 6.6 gives $\dim_{qA} \mu = \dim_{\text{loc}} \mu(0) = \log 4/\log 3$.

However, $\mu$ is not doubling and consequently, the Assouad dimension of $\mu$ is infinite. To see this, one can show that $1/2$ is the boundary point of two level $n$ net intervals for each level $n$. The two intervals have the same length, $3^{-n}/2$. Using the techniques developed in [10] it can be shown that the $\mu$-measure of the right interval is at most $c_1 4^{-n}$ for some constant $c_1 > 0$, while the left interval has measure at least $c_2 4^{-n}$ for some $c_2 > 0$. With $R = (\frac{4}{3})3^{-n}$, $r = (\frac{4}{3})3^{-n}$ and $x_n$ the midpoint of the right net interval of level $n$, we have $\mu(B(x_n, R)) \geq c_2 4^{-n}$ and $\mu(B(x_n, r)) \leq c_1 4^{-n}$, while $R/r = 3$, which proves $\mu$ is not doubling.

The purpose of the final example is to show that an equicontractive, regular measure without full support need not have finite quasi-Assouad dimension.

6.2. Equicontractive regular self-similar measure without full support. Consider the iterated function system with $S_j(x) = x/5 + d_j$ where $d_0 = 0$, $d_1 = 1/10$, $d_2 = 2/5$, $d_3 = 4/5$ and probabilities $p_0 = p_1 = p_3 = 1/6$, $p_2 = 1/2$. This IFS is equicontractive, finite type and regular, but the self-similar set is clearly not the full interval $[0,1]$. Indeed, the subintervals $(3/10,2/5)$ and $(3/5,4/5)$ are in the complement of the self-similar set.

As explained in [10], we can associate with each net interval of a finite type IFS a finite tuple called the characteristic vector. The characteristic vector contains all the information needed to essentially determine the measure of the net interval given that of its parent net interval. In fact, finite type is characterized by the property that there are only finitely many of these so-called characteristic vectors and, furthermore, each net interval $\Delta_n$ of level $n$ can be uniquely identified by the $(n+1)$-tuple of characteristic vectors, $(\gamma_j)_{j=0}^n$, where $\gamma_n$ is the characteristic vector of $\Delta_n$, $\gamma_{n-1}$ is the characteristic vector of its parent net interval, $\Delta_{n-1}$, etc.

This IFS has six characteristic vectors, which we label as $1, 2, 3a, 3b, 3c, 4$. Any net interval of level $n - 1$ with characteristic vector $3a, 3b, 3c$ has four children, each of length $5^{-n}/2$. From left to right these are $3a, 3b, 4, 3c$, where $4$ and $3c$ are separated by a gap. Fix $\varepsilon > 0$ and take any large $N$ and $n = \lceil(1 + \delta)N\rceil$. Consider the level $n$ net interval, $\Delta_n$, identified with the tuple,

$$(1, 3a, 3a, \ldots, 3a, 3b, 3b, \ldots, 3b),$$

and its ancestor $\Delta_N$ of level $N$. Let $x_n$ denote the midpoint of $\Delta_n$.

Using the techniques of [10] it can be shown that

$$\mu(\Delta_N) \sim \left\| \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 0 \\ 0 & 1/2 \\ 1/6 \end{bmatrix} \right\|^N \sim 2^{-N},$$
while
\[
\mu(\Delta_n) \sim \left\| \begin{bmatrix} 1/6 & 1/6 \\ 0 & 1/2 \end{bmatrix} \right\|^N \begin{bmatrix} 1/6 & 1/6 \\ 0 & 0 \end{bmatrix}^{n-N} \sim 6^{-n},
\]
where the symbol \( \sim \) means bounded above and below by some constant multiple. Taking \( R = 5^{-N/2} \) and \( r = 5^{-n}/4 \) it follows that
\[
\frac{\mu(B(x_n, R))}{\mu(B(x_n, r))} \geq c_1 \frac{2^{-N}}{6^{-n}} \geq c_1 \frac{6^{(1+\delta)N}}{2^N} = c_1 3^{N \cdot \delta N},
\]
while \( R/r \leq c_2 5^{\delta N} \). Consequently, \( \dim_{qA} \mu = \infty \).

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