Eigenfunctions of Composition Operators on Bloch-type Spaces

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Abstract

Suppose \( \varphi \) is a holomorphic self map of the unit disk and \( C_\varphi \) is a composition operator with symbol \( \varphi \) that fixes the origin and \( 0 < |\varphi'(0)| < 1 \). This work explores sufficient conditions that ensure all holomorphic solutions of Schröder equation for the composition operator \( C_\varphi \) belong to a Bloch-type space \( B_\alpha \) for some \( \alpha > 0 \). The results from composition operators have been extended to weighted composition operators in the second part of this work.

1 Introduction

Let \( \mathcal{D} \) be the unit disk of the complex plane \( \mathbb{C} \). Suppose that \( \mathcal{H}(\mathcal{D}) \) denotes space of holomorphic functions defined on the unit disk. Recall that a holomorphic function \( f \) on \( \mathcal{D} \) said to be in Bloch-type space \( B_\alpha \) for some \( \alpha > 0 \) if

\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.
\]

Under the norm

\[
\|f\|_{B_\alpha} = |f(0)| + \sup_{z \in \mathcal{D}} (1 - |z|^2)^\alpha |f'(z)|,
\]

\( B_\alpha \) becomes a Banach space. From the definition of Bloch-type spaces, it immediately follows that \( B_\alpha \subset B_\beta \) for \( \alpha \leq \beta \) and \( B_\alpha \subset H^\infty \) for \( \alpha < 1 \).

Functions in the Bloch space have been studied extensively by many authors, see [1] and [8]. It has been shown in [8] that the Bloch-type norm for \( \alpha > 1 \) is equivalent to the \( \alpha - 1 \) Lipschitz-type norm:

\[
\|f\|_{B_\alpha} \approx \sup_{z \in \mathcal{D}} (1 - |z|^2)^{\alpha - 1} |f(z)|, \quad f \in B_\alpha, \ \alpha > 1.
\]
Composing functions $f$ in $\mathcal{H}(\mathcal{D})$, with any holomorphic self-map $\varphi$ of $\mathcal{D}$, induces a linear transformation, denoted by $C_\varphi$ and called a composition operator on $\mathcal{H}(\mathcal{D})$:

$$C_\varphi f = f \circ \varphi.$$  

For any $u \in \mathcal{H}(\mathcal{D})$ we define weighted composition operator $uC_\varphi$ on $\mathcal{H}(\mathcal{D})$ as

$$uC_\varphi(f) = (u)(f \circ \varphi).$$

In this work, we study holomorphic solutions $f$ of the Schröder’s equation

$$(C_\varphi)f(z) = \lambda f(z), \quad (1.3)$$

and of the weighted Schröder’s equation

$$uC_\varphi f = \lambda f, \quad (1.4)$$

where $\lambda$ is a complex constant. Assuming $\varphi$ fixes the origin and $0 < |\varphi'(0)| < 1$, Königs in [5] showed that the set of all holomorphic solutions of Eq. (1.3) (eigenfunctions of $C_\varphi$ acting on $\mathcal{H}(\mathcal{D})$) is exactly $\{\sigma^n\}_{n=0}^\infty$, where $\sigma$, principal eigenfunction of $C_\varphi$, is called Königs function of $\varphi$. Following the Königs’s work, Hosokawa and Nguyen in [4] showed that the set of all eigenfunctions of $uC_\varphi$ acting on $\mathcal{H}(\mathcal{D})$ is exactly $\{v\sigma^n\}_{n=0}^\infty$ where $v$ is principal eigenfunction of $uC_\varphi$ and $\sigma$ is the Königs function.

According to a general result of Hammond in [2] if $uC_\varphi$ is compact on any Banach space of holomorphic functions on $\mathcal{D}$ containing the polynomials, all eigenfunctions $v\sigma^n$ belong to the Banach space. Hosokawa and Nguyen in [4] under somewhat strong restrictions on the growths of $u$ and $\varphi$ near the boundary of the unit disk showed that all the eigenfunctions $v\sigma^n$ are eigenfunctions of $uC_\varphi$ acting on the Bloch space $B$. Our goal in this work is to study conditions under which all eigenfunctions $v\sigma^n$ belong to a Bloch-type space $B_\alpha$.

The basic organization of this paper is as follows. We present results concerning to composition operators in Section 3. Theorem 3.1 provides the sufficient conditions that ensure all the eigenfunctions $\sigma^n$ belong to Bloch type spaces $B_\alpha$ for $\alpha < 1$. Similar results for $\alpha = 1$ and $\alpha > 1$ are presented by Theorem 3.2 and 3.3 respectively. Towards the end of this work we prove results concerning to weighted composition operators.
2 Preliminaries

We recall the following criteria from [6, Theorem 2.1] for boundedness of $uC_\varphi$ on Bloch-type spaces $B_\alpha$.

**Theorem 2.1.** Let $u$ be analytic on $D$, $\varphi$ an analytic self-map of $D$ and $\alpha$ be a positive real number.

1. If $0 < \alpha < 1$, then $uC_\varphi$ is bounded on $B_\alpha$ if and only if $u \in B_\alpha$ and
   \[
   \sup_{z \in D} |u(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.
   \]

2. The operator $uC_\varphi$ is bounded on $B$ if and only if
   
   (a) $\sup_{z \in D} |u'(z)| (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} < \infty$
   
   (b) $\sup_{z \in D} |u(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty$.

3. If $\alpha > 1$, then $uC_\varphi$ is bounded on $B_\alpha$ if and only if the following are satisfied.
   
   (a) $\sup_{z \in D} |u'(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} < \infty$
   
   (b) $\sup_{z \in D} |u(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty$.

The following theorem from [6, Theorem 3.1] provides the compactness criterion for $uC_\varphi$ acting on $B_\alpha$.

**Theorem 2.2.** Let $u$ be holomorphic function on $D$ and let $\varphi$ be holomorphic self map of $D$. Let $\alpha$ be a positive real number, and $uC_\varphi$ is bounded on $B_\alpha$.

1. If $0 < \alpha < 1$ then $uC_\varphi$ is compact on $B_\alpha$ if and only if
   \[
   \lim_{|\varphi(z)| \to 1^-} |u(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.
   \]

2. The operator $uC_\varphi$ is compact on $B$ if and only if the following are satisfied.
   
   (a) $\lim_{|\varphi(z)| \to 1^-} |u'(z)| (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} = 0$
   
   (b) $\lim_{|\varphi(z)| \to 1^-} |u(z)| \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0$.

3. If $\alpha > 1$, then $uC_\varphi$ is compact on $B_\alpha$ if and only if the following are satisfied.
(a) \( \lim_{|\varphi(z)| \to 1^-} |u'(z)| \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha}} = 0 \)

(b) \( \lim_{|\varphi(z)| \to 1^-} |u(z)| \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha}} |\varphi'(z)| = 0 \).

Remark 2.1. If we assume \( u \equiv 1 \) in Theorem 2.1 and Theorem 2.2, they provide the criterion for boundness and compactness of composition operators \( C_\varphi \) acting on Bloch-type spaces \( B_\alpha \).

The following two theorems are fundamental for our work. Theorem 2.3 is the famous Kônigs’s Theorem about the solutions to SchrÖder’s equations (see [5] and [7, Chapter 6]).

Theorem 2.3 (Kônigs’s Theorem (1884)). Assume \( \varphi \) is a holomorphic self map of \( D \) such that \( \varphi(0) = 0 \) and \( 0 < |\varphi'(0)| < 1 \). Then the following assertions hold.

i. The sequence of functions
\[
\sigma_k(z) := \frac{\varphi_k(z)}{\varphi'(0)^k},
\]
where \( \varphi_k \) is the \( k \)th iterates of \( \varphi \), converges uniformly on a compact subset of \( D \) to a non-constant function \( \sigma \) that satisfies (1.3) with \( \lambda = \varphi'(0) \).

ii. \( f \) and \( \lambda \) satisfy (1.3) if and only if there is a positive integer \( n \) such that \( \lambda = \varphi'(0)^n \) and \( f \) is a constant multiple of \( \sigma^n \).

The following theorem characterizes all eigenfunctions of a weighted composition operator under some restriction on the symbol (see [3]).

Theorem 2.4. Assume \( \varphi \) is a holomorphic self map of \( D \) and \( u \) is a holomorphic map of \( D \) such that \( u(0) \neq 0, \varphi(0) = 0, 0 < |\varphi'(0)| < 1 \). Then, the following statements hold.

i. The sequence of functions
\[
v_k(z) = \frac{u(z)u(\varphi(z))u(\varphi(\varphi(z)))...u(\varphi_{k-1}(z))}{u(0)^k}
\]
where \( \varphi_k \) is the \( k \)th iterates of \( \varphi \), converges to a non-constant holomorphic function \( v \) of \( D \) that satisfies (1.4) with \( \lambda = u(0) \).

ii. \( f \) and \( \lambda \) satisfy (1.4) if and only if \( f = v\sigma^n \), \( \lambda = u(0)\varphi'(0)^n \), where \( n \) is a non-negative integer and \( \sigma \) is the solution of the Schröder equation (1.3) \( \sigma \circ \varphi = \varphi'(0)\sigma \).
3 Composition operators

In this section, we investigate sufficient conditions that ensure the eigenfunctions $\sigma^n$ of a composition operator belong to $B_\alpha$ for some positive number $\alpha$ and for all positive integers $n$.

**Definition 3.1.** Let us define the *Hyperbolic $\alpha$-derivative* of $\varphi$ at $z \in \mathcal{D}$ by

$$\varphi^{(h_\alpha)}(z) = \frac{(1 - |z|^2)^\alpha \varphi'(z)}{(1 - |\varphi(z)|^2)^\alpha}.$$  

When $\alpha = 1$ then it is simply called the Hyperbolic derivative of $\varphi$ at $z$ and denoted by $\varphi^{(h)}(z)$.

**Definition 3.2.** Suppose $\varphi$ is a holomorphic self map of $\mathcal{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$ and $\varphi_m$ is the $m^{th}$ iteration of $\varphi$ for some fixed non-negative integer $m$. Then we say $\varphi$ satisfies condition (A) if there exists a non-negative integer $m$ such that

$$|\varphi^{(h_\alpha)}(\varphi_m(z))| = \frac{(1 - |\varphi_m(z)|^2)^\alpha |\varphi'(\varphi_m(z))|}{(1 - |\varphi_{m+1}(z)|^2)^\alpha} \leq |\varphi'(0)|,$$  

for all $z \in \mathcal{D}$ and for some fixed $\alpha > 0$.

**Remark 3.1.** If $\varphi$ satisfies the condition (A) for some $m$ then it satisfies the condition for all non-negative integers greater than $m$.

The following example provides a family of maps that satisfies condition (A). The example is extracted from [3].

**Example 3.1.** Consider a map $\gamma$ that maps the unit disk univalently to the right half plane. This map is given by

$$\gamma(z) = \frac{1 + z}{1 - z}.$$  

For any $t \in (0, 1)$, define

$$\varphi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}.$$  

It is well known that $\varphi_t$ maps the unit disk into the unit disk for each $t \in (0, 1)$, see [7]. These maps are known as *lens map*. 
Claim 3.1. \( \varphi_t \) satisfies condition \( \text{(A)} \) for \( \alpha = 1 \) and \( m = 0 \). That is to say for all \( t \in (0, 1) \), \( |\varphi_t^{(h)}(z)| \leq |\varphi_t'(0)| \) for all \( z \in D \).

Proof. Clearly, \( \varphi_t(0) = 0 \) and

\[
|\varphi_t'(0)| = \frac{2t |\gamma(z)| |\gamma'(z)|}{|\gamma(z) + 1|^2}.
\]

Since \( \gamma'(z) = \frac{2}{(1-z)^2} \), we see that \( |\varphi_t'(0)| = t \). It is known that image of \( \varphi_t \) touches the boundary of the unit disk non tangentially at 1 and \(-1\). Now put \( w = \gamma(z) = re^{i\theta} \), we see that

\[
|\varphi_t^{(h)}(z)| = \frac{1 - |z|^2}{1 - \frac{|w^t - 1|^2}{|w^t + 1|^2}} 2t \frac{|w^t - 1| |w'|}{|w^t + 1|^2} \frac{1 - |z|^2}{|w^t + 1|^2} |w^t - 1|^2 2t |w^t - 1| |w'|.
\]
On the other hand, we have

\[ |w^t + 1|^2 - |w^t - 1|^2 = (w^t + 1)(\overline{w}^t + 1) - (w^t - 1)(\overline{w}^t - 1) \]
\[ = (w^t + 1)(\overline{w}^t + 1) - (w^t - 1)(\overline{w}^t - 1) \]
\[ = 2(w^t + \overline{w}^t) \]
\[ = 2 \left( e^{it\theta} + e^{-it\theta} \right) \]
\[ = 4 r^t \cos t\theta. \]

And \( w' = \gamma'(z) = \frac{2}{(1 - z)^2} \)

\[ \left| \varphi_t^{(h)}(z) \right| = \frac{1 - |z|^2}{|1 - z|^2} \frac{t \ r^{t-1} |e^{(t-1)\theta}|}{r^t \cos t\theta}. \]

Using \( z = \frac{w - 1}{w + 1} \), we get

\[ \left| \varphi_t^{(h)}(z) \right| = \frac{1 - \left| \frac{w - 1}{w + 1} \right|^2}{1 - \left| \frac{w - 1}{w + 1} \right|^2} \frac{t \ r^{t-1}}{r^t \cos t\theta} \]
\[ = \frac{|w + 1|^2 - |w - 1|^2}{4} \frac{t \ r^{t-1}}{r^t \cos t\theta} \]
\[ = \frac{4 \ r \cos \theta}{r^t \cos t\theta} \frac{t \ r^{t-1}}{r^t \cos t\theta} \]
\[ = \frac{t \cos \theta}{\cos t\theta}. \]

If \( z \in (-1, 1) \) then \( \gamma(z) \in \mathbb{R}_+ \). Therefore \( \theta = 0 \) and so \( |\varphi_t^{(h)}(z)| = t \). On the other hand if \( z \in \mathcal{D} \setminus (-1, 1) \) then \( |\theta| \in (0, \pi/2) \). Hence \( \cos t\theta > \cos \theta > 0 \)
and so \( |\varphi_t^{(h)}(z)| < t \). This completes the proof. \( \square \)

**Remark 3.2.** From the proof of Claim [3.1] we see that \( |\varphi_t^{(h)}(z)| \rightarrow 0 \) as \( z \) approaches the boundary of the unit disk along the real-axis. Hence the composition operator with symbol \( \varphi_t \) is a non-compact operator on \( \mathcal{B} \).

The following proposition provides the sufficient condition that ensures the Königs function belongs to Bloch-type spaces. This proposition plays an important role to prove main theorems.

**Proposition 3.1.** Assume \( C_\varphi \) is bounded on \( \mathcal{B}_\alpha \) and \( \varphi \) satisfies the condition \( [A] \) for some \( \alpha > 0 \) and for some fixed non-negative integer \( m \). Then \( \sigma \) belongs to \( \mathcal{B}_\alpha \).
Thus, by using the condition (A) repeatedly, we get
\[
(1 - |z|^2)\alpha |\varphi_k'(z)| \leq M(1 - |\varphi(z)|^2)^\alpha, \quad \text{for } z \in \mathcal{D}.
\] (3.1)

For \( m \) given by the assumption, choose non-negative integer \( k \) such that
\[
k > m.
\]
For \( z \in \mathcal{D} \), we have
\[
(1 - |z|^2)\alpha |\varphi_k'(z)| = (1 - |z|^2)\alpha |\varphi'(\varphi_{k-1}(z))\varphi'(\varphi_{k-2}(z)) \ldots \varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z)) \ldots \varphi'(\varphi_{k-2}(z))\varphi'(\varphi_{k-1}(z))|.
\]

By using (3.1),
\[
(1 - |z|^2)\alpha |\varphi_k'(z)| \leq M(1 - |\varphi(z)|^2)^\alpha |\varphi'(\varphi(z)) \ldots \varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z)) \ldots \varphi'(\varphi_{k-2}(z))\varphi'(\varphi_{k-1}(z))|.
\]
Again using (3.1) repeatedly, we get
\[
(1 - |z|^2)\alpha |\varphi_k'(z)| \leq M^m \left( 1 - |\varphi_m(z)|^2 \right)^\alpha |\varphi'(\varphi_m(z)) \ldots \varphi'(\varphi_{k-2}(z))\varphi'(\varphi_{k-1}(z))|
\]
Now using the condition (A) repeatedly, we get
\[
(1 - |z|^2)\alpha |\varphi_k'(z)| \leq M^m |\varphi'(0)^{k-m}| (1 - |\varphi_k(z)|^2)^\alpha.
\]
Thus,
\[
\lim_{k \to \infty} (1 - |z|^2)\alpha \left| \frac{\varphi_k'(z)}{\varphi'(0)^k} \right| \leq \frac{M^m}{|\varphi'(0)^m|} \lim_{k \to \infty} (1 - |\varphi_k(z)|^2)^\alpha \leq \frac{M^m}{|\varphi'(0)^m|}.
\]

This implies that \((1 - |z|^2)^\alpha |\sigma'(z)| \leq \frac{M^m}{|\varphi'(0)^m|}\). Hence, \( \sigma \in \mathcal{B}_\alpha \). \( \blacksquare \)

The following corollary provides a sufficient condition that ensures all the integer powers of the Königs function belong to Bloch-type spaces \( \mathcal{B}_\alpha \) for \( \alpha < 1 \).

**Theorem 3.1.** Suppose \( \alpha < 1 \). If \( C_\varphi \) is bounded on \( \mathcal{B}_\alpha \) and \( \varphi \) satisfies the condition (A), then \( \sigma^n \in \mathcal{B}_\alpha \) for all positive integers \( n \).

**Proof.** From Proposition 3.1, we see that \( \sigma \in \mathcal{B}_\alpha \). Suppose \( \mathbb{H}^\infty \) denotes the space of bounded holomorphic functions on the unit disk \( \mathcal{D} \). Since \( \mathcal{B}_\alpha \subset \mathbb{H}^\infty \) for \( \alpha < 1 \), so there exists a positive constant \( C \) such that \( \|\sigma\|_{\mathbb{H}^\infty} \leq C \).

\[
(1 - |z|^2)^\alpha |(\sigma^n(z))'| = (1 - |z|^2)^\alpha | n \sigma^{n-1}(z) \sigma'(z) |
\leq \|\sigma\|_{\mathcal{B}_\alpha} n |\sigma^{n-1}(z)|
\leq n \|\sigma\|_{\mathcal{B}_\alpha} C^{n-1}.
\]

Hence, \( \sigma^n \in \mathcal{B}_\alpha \) for all positive integers \( n \). \( \blacksquare \)

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The following theorem gives a sufficient condition that ensures all the integer powers of Königs function belong to the Bloch space.

**Theorem 3.2.** Suppose \( \varphi \) is a holomorphic self map of \( D \), \( \varphi(0) = 0 \), \( 0 < |\varphi'(0)| < 1 \). Also, assume that

\[
1 - |z|^2 \leq \frac{2}{1 - |\varphi(z)|^2} |\varphi'(z)| \leq |\varphi'(0)|, \quad \text{for all } z \in D. \tag{3.2}
\]

Then \( C_\varphi \) is bounded on \( B \) and \( \sigma^n \in B \) for all positive integers \( n \).

**Proof.** Boundedness of \( C_\varphi \) on the Bloch space is consequence of Schwarz-Pick theorem. From the hypothesis of the theorem, we have

\[
(1 - |z|^2) \log \frac{2}{1 - |z|}|\varphi'(z)| \leq |\varphi'(0)|(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|}, \quad \text{for all } z \in D. \tag{3.3}
\]

Suppose \( k \) be a positive integer, then

\[
(1 - |z|^2)|\varphi_k'(z)| \log \frac{2}{1 - |z|} = (1 - |z|^2)|\varphi'(z)\varphi'(z)\ldots\varphi'(\varphi_{k-1}(z))| \log \frac{2}{1 - |z|} \\
= (1 - |z|^2) \log \frac{2}{1 - |z|}|\varphi'(z)\varphi'(z)\ldots\varphi'(\varphi_{k-1}(z))|.
\]

By using (3.3), we see that

\[
(1 - |z|^2)|\varphi_k'(z)| \log \frac{2}{1 - |z|} \leq |\varphi'(0)|(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|}|\varphi'(z)\ldots\varphi'(\varphi_{k-1}(z))|.
\]

And using (3.3) repeatedly, we get

\[
(1 - |z|^2)|\varphi_k'(z)| \log \frac{2}{1 - |z|} \leq 2|\varphi'(0)|^k(1 - |\varphi_k(z)|^2) \log \frac{2}{1 - |\varphi_k(z)|} \\
\leq 2|\varphi'(0)|^k(1 - |\varphi_k(z)|) \log \frac{2}{1 - |\varphi_k(z)|}.
\]

Since \( \log x \leq x \) for \( x > 1 \),

\[
(1 - |z|^2)|\varphi_k'(z)| \log \frac{2}{1 - |z|} \leq 4|\varphi'(0)|^k.
\]

Hence,

\[
\lim_{k \to \infty} (1 - |z|^2) \left| \frac{\varphi_k(z)}{\varphi'(0)^k} \right| \log \frac{2}{1 - |z|} = (1 - |z|^2)|\sigma'(z)| \log \frac{2}{1 - |z|} \leq 4, \quad z \in D
\]
which shows that
\[ |\sigma'(z)| \leq \frac{4}{(1 - |z|^2) \log \frac{2}{1 - |z|}}. \] (3.4)

Recall that \( \sigma(0) = 0 \). Now let us get an estimate for \( \sigma \).

\[
|\sigma(z)| = \left| \int_0^1 \sigma'(tz)d(tz) \right| \\
\leq \int_0^1 |\sigma'(tz)d(tz)| \\
\leq \int_0^1 \frac{4}{\log \frac{2}{1 - |tz|^2}} \frac{1}{1 - |tz|^2} d(t|z|) \\
\leq 4 \left[ \log \left( \log \frac{2}{1 - t|z|} \right) \right]_0^1 \\
= 4 \left[ \log \left( \log \frac{2}{1 - |z|} \right) - \log(\log 2) \right]. \] (3.5)

Now by using (3.4) and the estimate above for \( \sigma \), we get

\[
(1 - |z|^2)(\sigma^n(z))' = (1 - |z|^2) n |\sigma^{n-1}(z)\sigma'(z)| \\
\leq 4^n n \left( \log \log \frac{2}{1 - |z|} - \log \log 2 \right)^{n-1} \frac{1}{\log \frac{2}{1 - |z|}}.
\]

Taking limit \( |z| \to 1 \), it is easy to see that the right hand side of the last expression goes to zero. Hence \( \sigma^n \in \mathcal{B} \) for all positive integers \( n \).

Let us recall the Lipschitz-type norm which is equivalent to the usual norm defined for function \( f \in \mathcal{B}_\alpha \), \( \alpha > 1 \):

\[ \|f\|_{\mathcal{B}_\alpha} \equiv \sup_{z \in \mathcal{D}} (1 - |z|^2)^{\alpha-1}|f(z)|. \]

Next, we present results for the Bloch-type spaces, \( \mathcal{B}_\alpha \) for \( \alpha > 1 \). Let us start with the following definition.

**Definition 3.3.** Suppose \( f \in \mathcal{B}_\alpha \) for some \( \alpha > 0 \), then we define the *Bloch number* of \( f \) by \( b_f = \inf_\alpha \{ \alpha : f \in \mathcal{B}_\alpha \} \).

**Proposition 3.2.** Suppose \( \beta > 0 \). Then \( f^n \in \mathcal{B}_{\beta+1} \) for all positive integers \( n \) if and only if \( b_f \) is at most 1.
Proof. Suppose \( f^n \in B_{\beta + 1} \) for all positive integers \( n \). Need to show \( b_f \leq 1 \). On the contrary assume \( b_f > 1 \). Then there exists a positive integer \( n_0 \) such that \( 1 < 1 + \frac{\beta}{n_0} < b_f \). Now in the view of the Lipschitz-type norm, we see that for any fixed positive integer \( M \) there exists \( z \in D \) such that

\[
M \leq (1 - |z|^2)^{\beta/n_0} |f(z)| \leq \left( (1 - |z|^2)^{\beta/n_0} |f(z)| \right)^{n_0} = (1 - |z|^2)^\beta |f(z)|^{n_0},
\]

which shows that

\[
M \leq \sup_{z \in D} (1 - |z|^2)^{\beta} |f(z)|^{n_0} = \|f^{n_0}\|_{B_{\beta + 1}}.
\]

Since \( M \) is an arbitrary positive integer, \( f^{n_0} \notin B_{\beta + 1} \). Which is a contradiction.

Conversely, suppose \( b_f \leq 1 \). Since \( B_\alpha \subset B \) for all \( \alpha \leq 1 \), then clearly \( f \in B \). For any fixed \( \beta > 0 \) and for any fixed positive integer \( n \),

\[
(1 - |z|^2)^{\beta + 1} |(f^n)'(z)| = (1 - |z|^2)^{\beta + 1} |nf^{n-1}(z)f'(z)|
\]

\[
= n(1 - |z|^2)f'(z)((1 - |z|^2)^\beta |f^{n-1}(z)|)
\]

\[
\leq n\|f\|_B (1 - |z|^2)^\beta \left( \|f\|_B \log \frac{1}{1 - |z|} \right)^{n-1}
\]

\[
= n(\|f\|_B)^n (1 - |z|^2)^\beta \left( \log \frac{1}{1 - |z|} \right)^{n-1}.
\]

The last expression goes to zero as \( |z| \to 1 \). This shows that \( f^n \in B_{\beta + 1} \) for all positive integers \( n \).

\[ \Box \]

**Theorem 3.3.** Suppose \( \varphi \) is a holomorphic self map of \( D \), \( \varphi(0) = 0 \), \( 0 < |\varphi'(0)| < 1 \), and also assume \( \alpha > 1 \). If \( |\varphi^{(h)}(z)| \leq |\varphi'(0)| \) for all \( z \in D \) then \( C_\varphi \) is bounded on \( B_\alpha \) and \( \sigma^n \in B_\alpha \) for all positive integers \( n \).

**Proof.** Since \( |\varphi^{(h)}(z)| \leq |\varphi'(0)| \) for all \( z \in D \), from Proposition \[3.1\] \( \sigma \in B \). So \( b_f \leq 1 \). Therefore the result follows from Proposition \[3.2\]. \[ \Box \]

### 4 Weighted Composition operator

Let us recall that if \( u \) is a holomorphic function of the unit disk, and \( \varphi \) is a holomorphic self map of the unit disk then the Schröder equation for weighted composition operator is given by

\[
u(z)f(\varphi(z)) = \lambda f(z),
\]

(4.1)
where \( f \in \mathcal{H}(D) \) and \( \lambda \) is a complex constant.

Let us also recall that if \( u(0) \neq 0, \varphi(0) = 0, 0 < |\varphi'(0)| < 1 \) then the solutions of (4.1) are given by Theorem 2.4. The principal eigenfunction corresponding to the eigenvalue \( u(0) \) is denoted by \( v \) and all other eigenfunctions are of the form \( v\sigma^n \) where \( \sigma \) is the Königs function of \( \varphi \) and \( n \) is a positive integer. Hosokawa and Nguyen [4] studied equation (4.1) in the Bloch space and obtained the following result.

**Theorem 4.1.** Assume \( \varphi \) is a holomorphic self map of \( D \) with \( \varphi(0) = 0 \) and \( 0 < |\varphi'(0)| < 1 \), and \( u \) is holomorphic map of \( D \) such that \( u(0) \neq 0 \). Let us also assume that \( uC_\varphi \) is bounded on \( \mathcal{B} \). For \( 0 < r < 1 \), set

\[
M_r(\varphi) := \sup_{|z|=r} |\varphi(z)|, \quad a_r := \sup_{|z|=r} (|u'(z)\varphi(z)| + |u(z)\varphi'(z)|).
\]

Suppose that

(i) \( \lim_{r \to 1} \log(1 - r) \log M_r(\varphi) = \infty \).

(ii) \( \log |a_r| < \epsilon \log(1 - r) \log M_r(\varphi) \),

where \( \epsilon > 0 \) is a constant satisfying \( \epsilon \log \|\varphi\|_\infty > -1 \).

Then, \( v\sigma^n \in \mathcal{B} \) for all non-negative integer \( n \).

Here we investigate the properties of weight \( u \) and symbol \( \varphi \) of weighted composition operators \( uC_\varphi \) that ensure \( v\sigma^n \) belongs to Bloch-type spaces \( \mathcal{B}_\alpha \) for some \( \alpha > 0 \) and for all non negative integer \( n \). Let us begin with following remark.

**Remark 4.1.** Suppose \( f \) is a holomorphic function defined on \( D \). If \( \|f'\|_\infty < M \) for some \( M > 0 \) then

\[
|f(z) - f(0)| = \left| \int_0^1 zf'(tz)dt \right| \\
\leq \int_0^1 |z f'(tz)|dt \\
\leq M \int_0^1 |z|dt
\]

If \( f \) also satisfies \( f(0) = 0 \), then \( \|f\|_\infty \leq M \).

**Proposition 4.1.** Assume \( \varphi \) is a univalent holomorphic self map of the unit disk with \( \varphi(0) = 0 \) and \( 0 < |\varphi'(0)| < 1 \), and \( \sigma \) is Königs function of \( \varphi \). Then, \( \sigma \) is bounded if and only if there is a positive integer \( k \) such that \( \|\varphi_k\|_\infty < 1 \).
Proof. Suppose $\sigma$ is bounded. Since $\varphi$ is univalent, $\sigma$ is also univalent (see [7], page 91). Since $\sigma$ is bounded univalent map, there is a positive integer $k$ such that $\|\varphi_k\|_\infty < 1$ (see [7]).

Conversely suppose there is a positive integer $k$ such that $\|\varphi_k\|_\infty < 1$. Since we have $\varphi(\varphi(z)) = \varphi'(0)\sigma(z)$,

$$\sigma(\varphi_k(z)) = \sigma(\varphi_k(\varphi_{k-1}(z))) = \varphi'(0)\sigma(\varphi_{k-1}(z)) = \varphi'(0)^k\sigma(z).$$

Clearly left hand side is bounded and therefore $\sigma$ is also bounded, which completes the proof.

\[\square\]

Theorem 4.2. Assume $\varphi$ is a univalent holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ satisfying $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for all $z \in D$ and for some fixed $\alpha < 1$. If $u$ is holomorphic map of $D$ such that $u(0) \neq 0$ and $\|u\|_\infty < \infty$ then $uC_\varphi$ is bounded on $B_\alpha$ and $\varphi u^n \in B_\alpha$ for all non-negative integers $n$.

Proof. Since $\|u\|_\infty < \|u\|_\infty + |u(0)| < \infty$ and $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$, $uC_\varphi$ is bounded on $B_\alpha$ for some $\alpha < 1$.

Since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, using Proposition 3.1, we see that $\sigma \in B_\alpha$, $\alpha < 1$ and hence bounded. Since $\varphi$ is univalent, $\sigma$ is univalent. Consequently, there exists a non-negative integer $k$ such that $\|\varphi_k\|_\infty < 1$. Composing $\varphi_{k-1}$ on both sides of the Schröder equation (4.1) from right,

$$u(\varphi_{k-1}(z))f(\varphi_k(z)) = \lambda f(\varphi_{k-1}(z)).$$

(4.2)

The left hand side in the equation above is bounded and so is $f \circ \varphi_{k-1}$. Now differentiating both side of (4.2), we get that

$$u'(\varphi_{k-1}(z)) \varphi'_{k-1}(z) f(\varphi_k(z)) + u(\varphi_{k-1}(z)) f'(\varphi_k(z)) \varphi'_k(z) = \lambda f'(\varphi_{k-1}(z)) \varphi'_{k-1}(z).$$

Multiplying both sides by $(1 - |z|^2)\alpha$ and using boundedness of $\|u\|_\infty, \|u\|_\infty, f \circ \varphi_k$ and $f' \circ \varphi_k$, we see that there exists a constant $M$ such that

$$(1 - |z|^2)^\alpha |\lambda f'(\varphi_{k-1}(z)) \varphi'_{k-1}(z)| \leq M(1 - |z|^2)^\alpha (|\varphi'_{k-1}(z)| + |\varphi'_k(z)|).$$

(4.3)

Right hand side of the above equation is uniformly bounded and therefore the left hand side is bounded. Again, let us compose $\varphi_{k-2}$ on (4.1), to get

$$u(\varphi_{k-2}(z))f(\varphi_{k-1}(z)) = \lambda f(\varphi_{k-2}(z)).$$
Let us differentiate above expression and then multiply by \((1-|z|^2)^\alpha\) on both sides. Then, the use of \((1.2)\) and \((1.3)\) shows that \((1-|z|^2)^\alpha|f'(\varphi_{k-2}(z))\varphi'_{k-2}(z)|\) is bounded.

Continuing this process, we see that that sup\(_{z \in \mathcal{D}}\)\((1-|z|^2)^\alpha|f'(z)|\) is bounded and hence \(f \in \mathcal{B}_\alpha\). From Theorem 2.4 we know that any holomorphic \(f\) satisfying \((4.1)\) is of the form \(v\sigma^n\) for some positive integer \(n\), so \(v\sigma^n \in \mathcal{B}_\alpha\) for all non-negative integers \(n\). This completes the proof.

The following two theorems give us the sufficient conditions that ensure \(v\sigma^n\) belong to Bloch-type spaces \(\mathcal{B}_\alpha\) for some \(\alpha > 1\) and for all non-negative integers \(n\).

**Theorem 4.3.** Let \(\varphi\) be a holomorphic self map of the unit disk with \(\varphi(0) = 0\) and \(0 < |\varphi'(0)| < 1\), and \(u\) is holomorphic map of \(\mathcal{D}\) such that \(u(0) \neq 0\). Let \(\beta\) be a fixed positive number and assume

\[
|u(z)|\frac{(1-|z|^2)\beta}{(1-|\varphi(z)|^2)^\beta} \leq |u(0)|, \quad \text{for all} \ z \in \mathcal{D}.
\]

Then the following statements are true.

i. If \(|\varphi^{(h\alpha)}(z)| \leq |\varphi'(0)| \) for all \(z \in \mathcal{D}\) and for some \(\alpha < 1\), then \(v\sigma^n \in \mathcal{B}_{\beta+1}\) for all non-negative integers \(n\).

ii. If \(|\varphi^{(h)}(z)| \leq |\varphi'(0)| \) for all \(z \in \mathcal{D}\), then \(v\sigma^n \in \mathcal{B}_{p+1}\), for some \(p > \beta\) and for all non-negative integers \(n\).

**Proof.**

i. From definition of \(v_k\) (see Theorem 2.4), we have

\[
(1-|z|^2)^\beta|v_k(z)| = (1-|z|^2)^\beta \frac{|u(z)u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^k}
\leq (1-|\varphi(z)|^2)^\beta \frac{|u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}}
\leq 1.
\]

Hence \((1-|z|^2)^\beta|v(z)| = \lim_{k \to \infty} (1-|z|^2)^\beta|v_k(z)| \leq 1\). Since \(z\) is arbitrary,

\[
\sup_{z \in \mathcal{D}} (1-|z|^2)^\beta|v(z)| < \infty.
\]
On the other hand the assumption $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ and Proposition $3.1$ implies that $\sigma^n \in B_\alpha \subset \mathbb{H}^\infty$ for all non-negative integer $n$. Hence,

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)\beta |v(z)\sigma^n(z)| < \infty$$

for all non-negative integers $n$. Considering the equivalent norm (see (1.2)), we see that $v\sigma^n \in B_\beta + 1$ for all non-negative integers $n$.

ii. From the proof of (i), we see that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)\beta |v(z)| < \infty. \quad (4.4)$$

On the other hand, since $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$, Proposition $3.1$ implies that $\sigma \in B$ so there exists $M > 0$ such that

$$|\sigma(z)| \leq M \log \frac{2}{1 - |z|^2}. \quad (4.5)$$

Now using equations (4.4) and (4.5), we have

$$(1 - |z|^2)^\beta |v(z)\sigma^n(z)| = (1 - |z|^2)^\beta |v(z)| \{ (1 - |z|^2)^{p-\beta} |\sigma^n(z)| \} \leq CM (1 - |z|^2)^{p-\beta} \left( \log \frac{2}{1 - |z|^2} \right)^n$$

for some constant $C > 0$. Now taking limit $|z| \to 1$, we get that the last expression goes to zero. Hence, $v\sigma^n \in B_{p+1}$ for all non-negative integers $n$.

\[ \square \]

**Theorem 4.4.** Let $\varphi$ be a holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and $u$ is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$. Suppose that $\beta$ is a positive integer and

i. $|u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\beta} \log \frac{2}{(1 - |z|^2)^p} \log \frac{2}{(1 - |\varphi(z)|^2)^p} \leq |u(0)|$, for all $z \in \mathbb{D}$

ii. $|\varphi^{(h)}(z)| \frac{\log \frac{2}{1 - |z|}}{\log \frac{2}{1 - |\varphi(z)|}} \leq |\varphi'(0)|$, for all $z \in \mathbb{D}$.

Then $v\sigma^n \in B_{\beta + 1}$ for all non-negative integers $n$. 

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Proof. From definition of $v_k$ on\(2.4\) and the condition \((i)\), we have

\[
(1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} |v_k(z)| \leq (1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} \frac{|u(z)u(\varphi(z)) \ldots u(\varphi_{k-1}(z))|}{|u(0)|^k} \\
\leq (1 - |\varphi(z)|^2)^\beta \log \frac{2}{(1 - |\varphi(z)|)^\beta} \frac{|u(\varphi(z)) \ldots u(\varphi_{k-1}(z))|}{|u(0)|^k} \\
\leq (1 - |\varphi_k(z)|^2)^\beta \log \frac{2}{(1 - |\varphi_k(z)|)^\beta} \\
\leq 2^\beta (1 - |\varphi_k(z)|)^\beta \log \frac{2}{(1 - |\varphi_k(z)|)^\beta}.
\]

Since \(\log x \leq x\), for \(x > 1\)

\[
(1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} |v_k(z)| \leq 2^{\beta + 1}.
\]

So taking limit \(k\) approaches to \(\infty\), we see that

\[
(1 - |z|^2)^\beta |v(z)| \leq \frac{2^{\beta + 1}}{\log \frac{2}{1 - |z|}}. \tag{4.6}
\]

On the other hand, since \(\varphi\) satisfies condition \((ii)\), equation \(3.5\) of Theorem 3.2 says that there exists \(K > 0\) such that

\[
|\sigma(z)| \leq K \log \frac{2}{1 - |z|}. \tag{4.7}
\]

Now using \(4.6\) and \(4.7\), we get

\[
(1 - |z|^2)^\beta |v(z)\sigma^n(z)| \leq \frac{2^{\beta + 1}K^n}{\log \frac{2}{1 - |z|}} \left(\log \log \frac{2}{1 - |z|}\right)^n.
\]

Clearly right hand side of the above equation goes to 0 as \(|z| \to 1\). Using the norm defined on \((1.2)\), \(v\sigma^n \in B_{\beta + 1}\) for all non negative integers \(n\).

\[\square\]

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