Adaptive n-ary Activation Functions for Probabilistic Boolean Logic

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Abstract—Balancing model complexity against the information contained in observed data is the central challenge to learning. In order for complexity-efficient models to exist and be discoverable in high dimensions, we require a computational framework that relates a credible notion of complexity to simple parameter representations. Further, this framework must allow excess complexity to be gradually removed via gradient-based optimization. Our n-ary, or n-argument, activation functions fill this gap by approximating belief functions (probabilistic Boolean logic) using logit representations of probability. Just as Boolean logic determines the truth of a consequent claim from relationships among a set of antecedent propositions, probabilistic formulations generalize predictions when antecedents, truth tables, and consequents all retain uncertainty. Our activation functions demonstrate the ability to learn arbitrary logic, such as the binary exclusive disjunction (p xor q) and ternary conditioned disjunction (c ? p : q ), in a single layer using an activation function of matching or greater arity. Further, we represent belief tables using a basis that directly associates the number of nonzero parameters to the effective arity of the belief function, thus capturing a concrete relationship between logical complexity and efficient parameter representations. This opens optimization approaches to reduce logical complexity by inducing parameter sparsity.

Index Terms—machine learning, activation functions, Boolean logic, unary, binary, ternary, truth functions, belief functions

I. INTRODUCTION

Theoretically optimal learning, i.e. assimilating data into models with rigorous mathematical justification for the resulting uncertainty in predictions, may be comprehensively understood within the Bayesian paradigm of reason. Our investigation into the provenance of prior belief in abstract learning algorithms [1] affirms the foundational elegance of minimizing algorithmic complexity, as measured by encoding information [2]–[4]. This perspective was originally presented by Solomonoff [5]–[8] as Algorithmic Probability (AP). Minimum Description Length (MDL), proposed by Rissanen [9], [10], can be understood as a closely-related optimization approach to identify models that dominate predictions under AP.

Our research into how to suppress model information during training compels the perspective motivating this work: to acquire complexity-efficient models within a limited computational budget, a learning architecture must (1) admit a variety of model complexities that have a clear relationship to parameter representations and (2) differentiable parameter trajectories must connect high-complexity to low-complexity models, since gradient-based optimization is essential to navigate high-dimensions.

Bayesian or probabilistic formulations of Boolean algebra provide a natural domain to satisfy these requirements. In general, if we are given n antecedents, propositions with known truth values, then we can determine the truth value of a consequent by referencing a truth table with $2^n$ entries. When each entry is either true or false, there are $2^n$ distinct Boolean truth functions. If we generalize truth functions to belief functions and truth tables to belief tables [11], then also counting belief tables with entries that are totally uncertain gives $3^{2^n}$ qualitatively distinct belief functions. Thus, increasing n within a feasible limit allows dimensionality to work in our favor because n-ary logic becomes extremely expressive, making it easier to discover useful models. Extending this to the Bayesian paradigm requires describing not only our belief in plausible antecedent states, but also our belief in the consequent probability conditioned on each distinct antecedent realization. Rather than employing fixed elementary unary and binary functions (not, and, or, etc.), wherein each truth table contains only completely certain (true or false) outcomes, we must allow consequent probability within belief tables to be adjusted during training. Then, gradient-based optimization makes the full variety of truth functions discoverable.

A. Our Contributions

We show how to formulate and efficiently parameterize activation functions that are capable of learning arbitrary n-ary probabilistic truth tables, or belief tables, using logit representations of antecedent probabilities, conditional probabilities, and consequent probabilities. This is achieved by implementing an activation function that accept n-arguments per output element. As with typical neural networks, the arguments to the activation function are simply computed from matrix multiply on the previous layer. Each activation output is processed from a parameterized belief table with $2^n$ elements. For modest arities, $n \leq 6$, such belief tables are feasible to store and compute. In fact, the number of parameters only scales approximately linearly with n in this domain, since most parameters are contained in the weight matrix that constructs antecedents from the previous layer. Specifically, for an input width $w_{\text{in}}$ and output width $w_{\text{out}}$, the weight matrix contains $n w_{\text{out}} w_{\text{in}}$ elements, whereas the belief table contains $w_{\text{out}}^{2^n}$.

Our parameterized activation functions are more flexible than the fixed versions proposed by Lowe et al. [12], which developed logit-space activation functions that explicitly encode approximations of fixed binary operations (and, or,
and xnor), while still maintaining the computational efficiency that results from simple compositions of min, max, and abs operations. These compositions also yield well-conditioned gradients for optimization, ensuring that the same gradient magnitudes propagate from consequents to relevant antecedents and belief table elements, rather than the extreme scaling imbalance that direct probabilistic computations can produce. Not only does our approach allow a single activation function to learn any binary operation, it also generalizes to learn any n-ary logic within a single layer.

### Algorithm 1 Adaptive Unary Activation

**Input:** \( y \) is a \( w_{out} \times 1 \) vector of antecedent logits.

Implied by the shape of \( y \), \( w_{out} \) is number of channels.

\( A \) is a \( w_{out} \times 2 \) matrix of belief-table logits.

**Output:** \( z \) is a \( w_{out} \times 1 \) vector of consequent logits.

1: function \( z = \text{unary}(y, A) \)
2: Compute belief-table sums, \( s \), and differences, \( d \),
\[
\begin{bmatrix}
  s & d
\end{bmatrix} = A \begin{bmatrix}
  1 & -1 \\
  1 & 1
\end{bmatrix}.
\]
3: \( z = \frac{1}{2} \left( \max(|y| + s, |d + y|) - \max(|y| - s, |d - y|) \right) \)
4: end function

### Algorithm 2 Adaptive n-ary Activation

**Input:** \( Y \) is a \( w_{out} \times n \) matrix of antecedent logits.

Output width \( w_{out} \) and arity \( n \) are implied by \( Y \).

\( \Theta \) is a \( w_{out} \times 2^n \) matrix of activation parameters.

**Output:** \( z \) is a \( w_{out} \times 1 \) vector of consequent logits.

1: function \( z = \text{n}_n\text{ary}(Y, \Theta) \)

Remark: Let columns of each matrix be indexed as
\[
Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}
\] and
\[
A^{(i)} = \begin{bmatrix} a_1^{(i)} & a_2^{(i)} & \cdots & a_{2^{n-i}}^{(i)} \end{bmatrix} \text{ for } i \in \{0, 1, \ldots, n\}.
\]
2: Change basis from parameter representations to belief-table logits (\( \otimes \) is the Kronecker product),
\[
A^{(0)} = \Theta \begin{bmatrix} n \otimes_{i=1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{bmatrix}.
\]
3: \( \text{for } i = 1, 2, \ldots, n \text{ do} \)
4: \( \text{for } j = 1, 2, \ldots, 2^{n-i} \text{ do} \)
5: \( a_j^{(i)} = \text{unary} \left( y_{i}, \begin{bmatrix} a_{2j-1} \quad a_{2j} \end{bmatrix}^{(i-1)} \right) \)
6: \( \text{end for} \)
7: \( \text{end for} \)
8: Return output, \( z = A^{(n)} \).
9: end function

Both Algorithm 1 and Algorithm 2 provide simplified forms of these activation functions. The change of basis in Algorithm 2, line 2, which we will examine closely in Section III-C, allows us to match representation sparsity to the number of antecedents that are actually needed to evaluate the result. Thus, this basis forges a direct relationship between encoding complexity and logical complexity through sparsity. Although it is beyond the primary scope of this paper, this association opens optimization approaches to suppress logical complexity by promoting parameter sparsity during training.

We cover background information regarding probabilistic Boolean logic and related work on activation functions in Section II. Section III contains our main derivations, analysis relating sparsity to logical complexity, and a gradient comparison for our approach versus a direct probabilistic computation. We present experimental results in Section IV followed by a brief discussion of future challenges and concluding remarks in Section V.

### II. Background

The study of logical relationships between antecedent claims and consequents is the subject of Boolean algebra. Thus, it may be unsurprising that human biology evolved to replicate elementary Boolean logic operations. Gideon et al. [13] showed that individual human neurons are capable of replicating the behavior of exclusive disjunction, xnor, a capability that is not possible using a single affine transformation followed by either rectified linear units (ReLU), MaxOut [14], or MaxMin [15] activation functions. Moreover, higher-arity logical operations also capture important relationships that are central to reasoning, e.g. the conditioned disjunction [16] is a ternary operation that is so useful for programming, it is written compactly in C as \(( \text{condition} ? \text{result_if_true} : \text{result_if_false} )\) to capture conditional reasoning.

While deep neural networks have demonstrated astonishing feats of pattern recognition and task optimization, we remain curious as to whether more efficient architectures exist that build upon these foundations of logic. The arity, or number of arguments, needed to evaluate a consequent has a clear connection to computational complexity, and therefore motivates a compelling framework for prior belief in abstract learning algorithms. Given this, we pursue an approach that is easily capable of learning diverse logical compositions.

#### A. Remark on Notation

The notation we adopt builds toward an implementation wherein a network layer will act upon an input vector, \( x \), using a linear transformation, \( y = Mx \), to provide activation function inputs, \( z = f(y, \Theta) \), where \( \Theta \) is an array of activation parameters. Although we denote the output as \( z \) to avoid excessive indexing, it is understood to take on the role of \( x \) in the next layer, i.e. \( x^{(t+1)} = z^{(t)} \). To derive the activation function, however, our analysis begins by considering probabilities associated with underlying Boolean random variables, which we represent using Greek letters (\( \xi, \psi, \zeta \)) so that the corresponding logits can be written as the Latin versions (\( x, y, z \)).
B. Probabilistic Boolean Logic

In Boolean algebra, propositions are held to be either false or true, written as 0 or 1, respectively. The typical construction introduces basic operations, including the unary \texttt{not}(\cdot) and the binary functions, \texttt{and}(\cdot, \cdot) and \texttt{or}(\cdot, \cdot). In this context, \textit{binary} refers to the number of arguments, rather than the values each argument may take. \textit{Boolean} refers to values themselves, e.g. 0 or 1. An \textit{n-ary} law of composition, or truth function, accepts \( n \geq 1 \) arguments to compute an outcome, \( f : \{0, 1\}^n \rightarrow \{0, 1\} \). The arguments are called antecedents, denoted elementwise as \( \psi_i \in \{0, 1\} \) for \( i \in [n] \) or as the tuple \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \), and the outcome \( \zeta = f(\psi) \in \{0, 1\} \) is called the consequent.

These tools of deductive reasoning, however, are only justified in contexts of complete certainty, the domains of computer science and mathematical proofs that build and analyze consistent axiomatic foundations using formal definitions and their emergent properties. Bayesian reasoning is an inductive approach, consistent with the laws of probability, to analyze degrees of plausibility among multiple explanations. Incorporating probability into logical analysis of propositions goes back to the work of Good [17] and De Finetti [18], who examined logical combinations of propositions that may be expressed as a composition of negations, conjunctions, and disjunctions, where antecedents retain some degree of uncertainty.

One begins by constructing a probability space over the set of potential outcomes, i.e. \( \Omega = \{0, 1\} \) for a single claim. Thus, our belief in a claim, e.g. that a particular antecedent is true, \( \psi_1 = 1 \), is represented by the probability mass function \( p(\psi_1 = 1) \in [0, 1] \), a Bernoulli random variable. Provided antecedents are independent, we have the usual compositions: negation, \( p(\neg \psi_1) = 1 - p(\psi_1) \); conjunction, \( p(\texttt{and}(\psi_1, \psi_2)) = p(\psi_1)p(\psi_2) \); and disjunction, \( p(\texttt{or}(\psi_1, \psi_2)) = 1 - p(\neg(\psi_1)p(\neg(\psi_2)) \). For our purposes, however, we must also incorporate uncertainty into the truth tables themselves.

Pearl [11] provides a comprehensive reference for probabilistic reasoning, including generalizing truth tables to belief tables to allow consequent uncertainty that is conditioned upon every possible antecedent realization. This general approach forms the joint distribution, including both the consequent and all antecedents, as \( p(\zeta, \psi_1, \psi_2) = p(\zeta | \psi_1, \psi_2)p(\psi_1, \psi_2) \). Marginalizing over all mutually exclusive antecedent states gives the general binary belief function, \( p(\zeta) = p(\zeta | \psi_1, \psi_2)p(\psi_1, \psi_2) + p(\zeta | \neg\psi_1, \neg\psi_2)p(\neg\psi_1, \neg\psi_2) + p(\zeta | \neg\psi_1, \psi_2)p(\neg\psi_1, \psi_2) + p(\zeta | \psi_1, \neg\psi_2)p(\psi_1, \neg\psi_2) \).

We can form any higher-arity belief function by specifying the desired conditional probabilities and evaluating it with our belief in the antecedent distribution. When every conditional probability is certain, we recover a logical truth function. For example, exclusive disjunction, \texttt{xor}(\cdot, \cdot), can be obtained by setting \( p(\zeta | \psi_1, \neg\psi_2) = p(\zeta | \neg\psi_1, \psi_2) = 1 \) and \( p(\zeta | \psi_1, \psi_2) = p(\zeta | \neg\psi_1, \neg\psi_2) = 0 \). Substitution with antecedent independence gives \( p(\texttt{xor}(\psi_1, \psi_2)) = p(\psi_1)(1 - p(\psi_2)) + (1 - p(\psi_1)p(\psi_2)) \). This is equivalent to computing the expected consequent probability using our belief in antecedents as the view of expectation, which bears an important connection to how we understand rational measures of information [19]. When we have a set of \( n \) antecedents, a complete belief table must contain consequent probabilities for every possible antecedent realization, the \( n \)-ary Cartesian product \( \{0, 1\}^n \) or vertices of a hypercube in \( n \) dimensions.

C. Approximating Logical Uncertainty

The concept of fuzzy logic was introduced and developed in the seminal work of Zadeh [20], [21], building on his concept of fuzzy sets [22], wherein elements of a set are present on a continuum of \textit{grades of membership}. By building off of the set-theoretic concepts of union and intersection, respectively formulated for grades of membership using \texttt{max} and \texttt{min} operators, Zadeh obtains fuzzy logic versions of the corresponding binary operations (or and \texttt{and}, respectively). Thus, MaxMin activation functions can be understood to approximate logical uncertainty associated with fixed complementary pairs of binary operations.

Our approach is somewhat similar to fuzzy logic, but we derive our activation functions systematically from Log-Sum-Exponential-Max (LSEM) approximations to logit representations of antecedents, truth-table probabilities, and consequents. See Section III-A for details. We observe that LSEM approximations can be applied to any binary truth function and, with modest algebraic manipulation, the Approximate Independent Logit (AIL) formulations presented by Lowe [12] are recovered.

- **\texttt{and}_{AIL}(Y) = \min(y_1, y_2, y_1 + y_2)\**
- **\texttt{or}_{AIL}(Y) = \max(y_1, y_2, y_1 + y_2)\**
- **\texttt{xor}_{AIL}(Y) = \text{sign}(y_1y_2) \min(\text{abs}(y_1), \text{abs}(y_2))\**

Figure 1 visualizes these approximations for comparison to the exact logit-space equivalents. These are not, however, the functions we use in practice. By parameterizing belief tables, our approach becomes general, allowing consequent uncertainty to be adjusted within higher-arity belief functions as merited by the training data.

D. Other Logic-Related Work

Payani and Fekri [23] propose a novel learning paradigm for deep neural networks using differentiable binary operators, \texttt{and}, \texttt{or}, and \texttt{xor}. They show how these elementary operators combine in simple and meaningful ways to form Neural Logic Networks (NLNs) that outperform state-of-the-art neural inductive logic programming (ILP) for benchmark tasks that include addition, multiplication, and sorting.
Binarized neural networks learn models with equivalent Boolean logic representations that can be formally analyzed with logic-based tools, such as SAT solvers. Narodytska et al. [24] analyze the efficiency of architectural choices for these networks and discuss how they affect the performance of logic-based reasoning. They also propose training procedures to obtain simpler network for SAT solvers that avoid sacrificing accuracy on a primary task.

Tavares et al. [25] show that deep learning can incorporate several families of Boolean functions, and relatively small and shallow neural networks find good approximations of these functions. They also discuss the difficulty of learning complex Boolean formulas in Conjunctive Normal Form (CNF), i.e. as a conjunction of several simple clauses. Their analysis shows that increases, fewer input states satisfy all clauses simultaneously a conjunction of several simple clauses. Their analysis shows that since our approach approximates marginalization over an entire belief table, every example isolates relevant belief table parameters and informs model updates. We compare the effects of backpropagation using a direct probabilistic marginalization versus our approach more closely in Section III-D.

III. DERIVATION OF n-ARY ACTIVATION FUNCTIONS

Our objective is to transform an input width of $w_{in}$ neurons, representing plausible truth states for a set of propositions, to $w_{out}$ outputs. To emphasize that our representation of plausible inputs states is just an approximation, and that it depends on the particular inputs to a network, we write input distributions as $q(\xi_j)$ for $j \in [w_{in}]$, rather than $p(\xi_j)$. This also avoids confusion with prior belief or needing to include extra conditional variables to provide the distinction. The probability that $\xi_j = 1$ is simply $q(\xi_j)$ and the probability that $\xi_j = 0$ is $q(\xi_j) = 1 - q(\xi_j)$.

Since the activation function will mimic n-ary belief functions for each output channel, the leading linear transformation must isolate $w_{out} \times n$ antecedents. If we were operating on probabilities directly, we would have to constrain these transformations to convex combinations of inputs, i.e. all matrix entries would need to be nonnegative with rows summing to 1, to ensure that the results represent coherent probabilities. Working in logit space, however, easily avoids this complication,

$$x_j = \log\left(\frac{q(\xi_j)}{q(\xi_j)}\right), \text{ so that } q(\xi_j) = \frac{1}{1 + \exp(-x_j)}.$$ 

Then, any linear transformation $y = Mx$ with $M \in \mathbb{R}^{w_{out} \times w_{in}}$ will provide antecedents with coherent probabilistic interpretations. The matricization, $Y = \text{matrix}(y, w_{out} \times n)$, allows us to identify the consequent index, $k \in [w_{out}]$, with the row and the corresponding argument index, $i \in [n]$, with the column. In principle, the antecedent probabilities become $q(\psi_{ki}) = \left[1 + \exp(-y_{ki})\right]^{-1}$, although these are never actually computed. Likewise, each consequent channel, $z_k$, also has a probability implied by the corresponding logit, $q(\zeta_k) = \left[1 + \exp(-z_k)\right]^{-1}$.

The belief table that defines the activation function must articulate the probability of a consequent state $\zeta_k = 1$ for each realization of antecedents, i.e. storing $p(\zeta_k | \psi_{ki})$ where the tuple $\psi_{ki} = (\psi_{k1}, \psi_{k2}, \ldots, \psi_{kn})$ varies over all values in $\{0, 1\}^n$. In order to build toward n-ary activations, we begin with the unary case, $n = 1$. To propagate our belief from $q(\psi_{k1})$ to $q(\zeta_k)$, we have the elementary marginalization,

$$q(\zeta_k) = p(\zeta_k | \neg \psi_{k1}) q(\neg \psi_{k1}) + p(\zeta_k | \psi_{k1}) q(\psi_{k1}). \quad (1)$$

Just as we operate on logit representations of antecedents and consequences, we also use logits to represent belief-table entries. In the unary case, the belief table is a matrix $A \in \mathbb{R}^{w_{out} \times 2}$ so that the two elements in row $k$, $a_{k0}$ and $a_{k1}$, indicate

$$p(\zeta_k | \neg \psi_{k1}) = \frac{1}{1 + e^{-a_{k0}}}, \quad p(\zeta_k | \psi_{k1}) = \frac{1}{1 + e^{-a_{k1}}} \quad (2)$$

Going forward, we will call elements $a_{kj}$ of the belief table $A$ the belief table logits. Combining Equations (1) and (2), using logit representations of both antecedents and consequents, gives the general unary logit-space marginalization,

$$z_k = \log\left(\frac{e^{a_{k0} - a_{k1} - y_{k1}} + e^{-a_{k0} + a_{k1} + y_{k1}}}{1 + e^{a_{k0} - a_{k1} + y_{k1}}}ight) - \log\left(\frac{e^{-a_{k0} + a_{k1} + y_{k1}} + e^{-a_{k0} - a_{k1} - y_{k1}}}{1 + e^{-a_{k0} - a_{k1} + y_{k1}}}\right). \quad (3)$$

Equation (3) shows that even this unary case is too complicated to be used directly, and would defeat the convenience of logit-space activation functions. Both log and exp are also relatively expensive operations that require numerical library implementations; they are not implemented as single assembly operations, whereas the alternatives we will use (max, min, and abs) are often are. Implementing such activation functions in a large neural network would become prohibitive if we had to repeatedly use many expensive operations.

A. Log-Sum-Exponential Max Approximation

Fortunately, these constructions allow a simple and efficient approximation, the Log-Sum-Exponential Max (LSEM)

$$\log\left(\sum_{i=1}^{n} \exp(x_i)\right) \approx \max_{i=1}^{n} x_i.$$ 

The approximation error is bound by $0 \leq \log\left(\sum_{i=1}^{n} \exp(x_i)\right) - \max_{i=1}^{n} x_i \leq \log(n)$ and we can apply maximization in steps, using only a couple terms at a time, without introducing any compounded approximation errors. When we apply LSEM approximation to Equation (3), and simplify results with the identity $\max(a, b) = (a + b + |a - b|)/2$, we have

$$z_k = \frac{1}{2} \left[\max(|y_{k1}| + s_k, |d_k + y_{k1}|) - \max(|y_{k1}| - s_k, |d_k - y_{k1}|)\right], \quad (4)$$

where $s_k = a_{k0} + a_{k1}$ and $d_k = a_{k1} - a_{k0}$. For obvious reasons, we call $s$ the sum and $d$ the difference. Algorithm 1 shows the basic implementation.

1We typically use 1-based indexing, but for this work it is easier to see the relationship between the column index and the antecedent state if count from zero in binary.
Figure 2 illustrates the LSEM approximation acting on an antisymmetric pair of logits (left) as well as several resulting unary activation functions (right), shown with marker symbols, and compares them to the exact marginalization logits (solid lines). We remark that ReLU activation corresponds to a unary LSEM approximation wherein output symmetry of belief is only broken when the antecedent has a positive logit.

\begin{align*}
q(\mathcal{C}_k | (\psi_k)_i^{n+1}) &= q(\mathcal{C}_k | -\psi_k, (\psi_k)_i^{n+1}) q(-\psi_k) + q(\mathcal{C}_k | \psi_k, (\psi_k)_i^{n+1}) q(\psi_k), \quad (7)
\end{align*}

where \( i = 2, \ldots, n. \) When \( i = n, \) we interpret \((\psi_k)_i^n\) as the empty tuple, i.e. \( q(\mathcal{C}_k | (\psi_k)_i^n) = q(\mathcal{C}_k). \)

Since the conditional probabilities we compose in each subsequent marginalization serve the same role as a lower-arity belief table, we can also represent them as matrices of logits, \( A^{(i)} \in \mathbb{R}^{n_{\text{out}} \times 2^{n-1}} \), where the superscript \( (i) \) indicates the number of antecedents that have been marginalized thus far. Specifically, for \( i \in [n], \) we have

\[ a_{kj}^{(i)} = \log \left( \frac{q(\mathcal{C}_k | (\psi_k)_i^{n+1})}{q(-\mathcal{C}_k | (\psi_k)_i^{n+1})} \right), \text{ with } j = \text{index}((\psi_k)_i^n). \]

Thus, \( A^{(0)} \) indicates that the initial belief tables have not marginalized any antecedents. Algorithm 2 shows how Equations (6) and (7) can be implemented easily. If we have \( n \) arguments, the first marginalization acts on \( 2^n - 1 \) edges of a hypercube, the second acts on \( 2^{n-2} \) edges of the previous result, and so forth until the last argument marginalizes over a single edge.

While the total number of unary marginalizations, \( w_{\text{out}}(2^n - 1) \), may seem prohibitive, this computation is reasonable for modest values of \( n. \) Although our logic experiments in Section IV-A only test up to \( n = 6, \) during this research we successfully tested transformations with \( w_{\text{in}} = w_{\text{out}} = 32 \) and \( n = 8 \) without a significant computational burden. At these specifications, the number of parameters in the preceding matrix \( M \) is exactly equal to the number of parameters needed to construct the belief table, \( A^{(0)} \). In practice, however, we anticipate \( n = 4 \) being both reasonably expressive and computationally efficient.

\section*{C. Basis for Sparse Representations of Effective Arity}

We can construct the belief-table logits with a change of basis that allows the number of stored nonzero parameters to reflect the number of arguments that are actually needed to evaluate the function. This allows us to prefer lower-arity logic during training by optimization methods that increase parameter sparsity. For an \( n \)-ary activation function with \( w_{\text{out}} \) channels, we store a parameter matrix, \( \Theta \in \mathbb{R}^{w_{\text{out}} \times 2^n} \), and form a basis \( B \) from Kronecker products to obtain the belief table matrix,

\[ A^{(0)} = \Theta B \quad \text{where} \quad B = \bigotimes_{i=1}^{n} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad (8) \]

The \( i^{th} \) factor in the Kronecker product indicates how corresponding columns in \( \Theta \) will affect the \( i^{th} \) dimension of a belief table, viewed as a hypercube. This effectively separates parameters into those that have a symmetric impact, i.e. those that multiply the first row of factor \( i, \) and those that have an antisymmetric impact, multiplying the second. It is not an accident that the matrix in Algorithm 1 that generates the sum and difference vectors, symmetric and antisymmetric
terms, is exactly twice the inverse of one of these factors. By setting all parameters that have an asymmetric impact on a given antecedent to zero, we create a belief function that is indifferent to the state of that antecedent and, thus, has lower effective arity. The following theorem formalizes this claim. The proof is contained in Appendix A.

1) Theorem: Irrelevant Antecedents and Parameter Zeros: Let $p(\zeta_k \mid \psi_i)$ represent the $k$th $n$-ary belief table. Evaluating the tuple of antecedents, $\psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{ikn})$, at any vertex of the hypercube, $\psi_i \in \{0, 1\}^n$, gives the probability of the consequent $\zeta_k$. Let $a_k \in \mathbb{R}^n$ be the row vector of logit representations, i.e. $a_{kj} = \logit(p(\zeta_k \mid \psi_i))$ where the column index $j$ maps to $\psi_k = \text{bits}(j)$, counting from zero. The subset of irrelevant antecedents is defined as

$$I = \{\psi_{ki} \mid p(\zeta_k \mid \psi_i) = p(\zeta_k \mid (\psi_{k1}, \ldots, \neg \psi_{ki}, \ldots, \psi_{kn}))$$

for all $\psi_i \in \{0, 1\}^n$.

Using the change of basis, Equation (8), if we have $\psi_{ki} \in I$ then $\theta_{kj} = 0$ for all $\theta_{kj} = 1$.

2) Change of Basis Examples: A single soft and operation, using logit magnitudes $\alpha$ in the belief table, can be represented with a binary activation function as $A^{(0)} = \alpha [-1 -1 -1 1]$. The change of basis gives parameters $\Theta = \alpha [-\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}]$. If we represent the same soft and as a ternary activation function on arguments 1 and 3, we obtain the same nonzeros, but interspersed among zeros for all antisymmetric parameters in argument 2, i.e. when $\text{bit}(2) = 1$ or $j = \{2, 3, 6, 7\}$, $\theta = \alpha [-\frac{1}{2} \frac{1}{2} 0 0 \frac{1}{2} \frac{1}{2} 0 0]$.

An immediate consequence of this theorem is that the number of nonzero parameters, $n_{nz}$, is bounded by the effective arity, $m = n - |I|$, as $n_{nz} \leq 2^m$. Curiously, if we extend this reasoning to derive the complexity of any belief function in this basis, we find that some functions are simpler than we might expect. Table I shows several binary activation functions.

| Operation | $a_{i0}$ | $a_{i1}$ | $a_{i10}$ | $a_{i11}$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ |
|-----------|---------|---------|----------|----------|-------------|-------------|-------------|-------------|
| true      | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| arg1      | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| not2      | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| xor       | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| relu1     | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| relu2     | 0       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| imply     | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| and       | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |
| or        | 1       | 1       | 1        | 0        | 0           | 0           | 0           | 0           |

Note the difference between the canonical formulation of material implication, imply or $p \rightarrow q$, and the formulation employing a stronger notion of uncertainty, imply*. For imply*, the outcome is $q$ if the premise $p$ is true, otherwise the outcome is true. In contrast, the more reserved imply* remains uncertain when the premise is false, and sustains a reduction in representation complexity as a result. Another hybrid function, and*, also reduces representation complexity in comparison to both and and or by only reporting agreement between both arguments, otherwise remaining uncertain.

D. Gradient Comparison

The gradient structure of an activation function can have a critical impact on practical trainability. For an activation function to be used in modern deep neural networks, it must avoid the problem of vanishing gradients. When we examine the gradient structure of the unary LSEM in Algorithm 1, i.e. Equation (4), we note that each maximization argument is simply a signed sum of an antecedent logit and two belief table logits. This means that when we subtract both maxima, the signs either cancel or double, and then the result is divided by two. Thus, $z_k$ is effectively a sparse signed sum over $y_{k1}$, $a_{k0}$, and $a_{k1}$, i.e. each term is multiplied by either $-1$, $0$, or $1$. Given loss $J$, backpropagation easily gives gradient magnitudes

$$\frac{\partial J}{\partial y_{k1}}, \frac{\partial J}{\partial a_{k0}}, \frac{\partial J}{\partial a_{k1}} \in \{0, \frac{\partial J}{\partial a_{k2}}\}.$$ 

Further, if $\frac{\partial J}{\partial a_{k0}} \neq 0$ then $\frac{\partial J}{\partial a_{k1}} = 0$ and vice versa. To see this, note that if $|y| + s \geq |d + y|$ then we must have $|y| - s = 2|y| - |(y + s)| \leq 2|y| - |d + y| \leq |d - y|$, causing sign cancellation to occur for either $a_{k0}$ or $a_{k1}$, since both an $s$ and $d$ term survive. Similarly, mapping $(s, y) \mapsto (-s, -y)$ shows that $|y| - s \geq |d - y|$ then $|y| + s \leq |d - y|$, also causing one cancellation. It is also possible for both $|d + y| > |y|$ and $|d - y| > |y| - s$. Let us define signs, $\sigma_+ = \pm 1$ and $\sigma_+ = \mp 1$, so that $|d + y| = \sigma_+(d + y)$ and $|d - y| = \sigma_-(d - y)$. Then $|d + y| + |d - y| = (\sigma_+ + \sigma_-)d + (\sigma_+ - \sigma_-)y > 2|y|$. This only holds if the signs are equal, $\sigma_+ = \sigma_-$, and $|d| > |y|$, thus causing $d$ dependence to vanish from the difference so that both $\frac{\partial J}{\partial a_{k0}} = \frac{\partial J}{\partial a_{k1}} = 0$.

Since Algorithm 2 is a composition of several calls to Algorithm 1, each intermediate result may only have, at most, one nonzero derivative with respect to a preceding partially marginalized belief table. That is, if $\frac{\partial a_{i10}}{\partial a_{i11}} = \pm 1$ then $\frac{\partial a_{i10}}{\partial a_{i11}^{-1}} = 0$ and vice versa. Thus, the magnitude of the loss gradients is constant or zero through the backpropagation sequence to all antecedents and the gradient with respect to belief table entries contains, at most, a single nonzero. It easily follows that the change of basis, Equation (8), transmits the same gradient magnitude to all stored parameters for the belief table.

In contrast, if we directly marginalized probabilities in Equations (6) and (7), then backpropagation on the last marginalization would give

$$\frac{\partial J}{\partial q(\zeta_k \mid \psi_{kn})} = \frac{\partial J}{\partial q(\zeta_k)} q(\psi_{kn}),$$

$$\frac{\partial J}{\partial q(\zeta_k \mid \neg \psi_{kn})} = \frac{\partial J}{\partial q(\zeta_k)} (1 - q(\psi_{kn})), \quad \text{and}$$

$$\frac{\partial J}{\partial q(\psi_{kn})} = \frac{\partial J}{\partial q(\zeta_k)} (q(\zeta_k \mid \psi_{kn}) - q(\zeta_k \mid \neg \psi_{kn})).$$

Since the marginalizations can be computed in any order, analogous formulas hold for any $\psi_{ki}$ for $i \in [n]$. The point is the magnitude of each gradient is attenuated by each
marginalization, either multiplying by a probability \((\in [0, 1])\)
or by a difference of probabilities \((\in [-1, 1])\), and potentiallycreating a vanishing gradient problem. If we composed severalbelief functions in this form, the gradient with respect toinitial belief tables could be orders of magnitude smaller thanwith respect to final belief tables. In contrast, our activationfunctions eliminate this issue by ensuring that equivalentgradient magnitudes are transmitted through the activationfunction.

IV. NUMERICAL EXPERIMENTS

Our primary objective is to forge a path toward logicalcomplexity suppression in support of principled uncertaintyquantification in automatic learning algorithms. While ouractivation functions promote that goal, reaching it will stillrequire additional advances that are beyond the scope of thiscontribution. Although our activation functions may reducethe number of layers needed to express logical relationshipsbetween latent antecedents and consequents, the number ofparameters within fixed layers increases, approximately linearlywith \(n\) for \(n \leq 6\), to retain the same input and outputwidths. Without having new training approaches thatefficiently induce sparsity and thereby reduce the number ofeffective parameters, a fair comparison to other activationfunctions is both difficult and somewhat premature.

These experiments simply demonstrate that our activationfunctions perform as intended and are already competitivewith other activation functions on standard networks andlearning problems. The first set of numerical experimentsexamines the ability of a variety of activation functions torecover truth functions with a range of arities and shows thatour activation functions do indeed learn high-arity logicefficiently. We then compare activation functions on a simpleConvolutional Neural Network (CNN) for MNIST [26] as wellas a standard CNN for CIFAR-10 [27]. Since we do not yethave the tools to efficiently constrain the number of effectivedegrees of freedom in parameters, we frame CNN comparisonsin terms of layers with equal output widths, i.e. all activationfunctions compute the same number of output neurons inrespective networks. This probes how efficiently each neurontransfers relevant information. Appendix A contains a tablefor each experiment with the number of parameters used ineach architecture.

A. Learning Higher-Arity Logic

This set of experiment is designed to investigate how easilyvarious activation functions are able to learn random truthfunctions. Each ground truth consists of 32 independent inputvariables and 32 output variables and a fixed ground truthfunction arity \(\gamma\). Each output function is generated by selectinga random subset of \(\gamma\) distinct inputs, providing antecedents,and a random truth function is then generated by drawing2\(^\gamma\) truth-table entries selected from \({0, 1}\) uniformly. Withthese mappings, we can generate data sets by drawing 32independent inputs and applying the 32 corresponding truthfunctions per training case. True and false values are replacedwith maximum and minimum logits, \(\pm 6.91\), corresponding toprobabilities 0.999 and 0.001.

In order to compare a variety of activation functions withdifferent effective arities, we need to take fair measures toensure that the number of inputs needed to compute eachoutput can be composed in a hidden layer sequence. For anactivation function with effective arity \(n\), the highest numberof input arguments that could have been considered by hiddenlayer \(\ell\) is \(n^\ell\). Thus, hidden layer \(\ell\) must have at least \(32[\frac{n}{\ell}]\)elements and, counting the final layer, we need \(L = [\log_n(\gamma)]\)layers to obtain outputs with enough computational pathsways to at least match the number of input dependencies thatare actually present in the ground truth.

Despite taking these measures, we know that such compositionsare not capable of representing the complete variety ofhigher-arity truth functions, which is one advantage of simplyusing a higher-arity layer. For example, there are \(2^{2^3} = 65,536\)distinct quaternary truth functions. If we attempt to replicatethese using only compositions of 3 binary operations (denoted\(0_1, 0_2,\) and \(0_3\)) of the form

\[
(x_1 \circ_1 x_2) \circ_3 (x_3 \circ_2 x_4), \quad (x_1 \circ_1 x_3) \circ_3 (x_2 \circ_2 x_4),
\]
or

\[
(x_1 \circ_1 x_4) \circ_3 (x_2 \circ_2 x_3),
\]

we will find there are only 1,208 distinct functions. Thus, boththis counting argument and our experimental results, shownin Table II, demonstrate that simple compositions of low-arityactivation functions do not always have sufficient flexibility toreconstruct high-arity logic.

Each set of comparisons is performed for a range of groundtruth arities, \(n = 2, 3, \ldots, 6\). Training, validation, and testdatasets contain 5000, 2500, and 2500 cases, respectively.Every ground truth arity uses the same training, validation, andtest sets for all activation functions and all training trials. Tosupport reproducibility and illuminate the variety of outcomes,we run 12 trials for each paired ground-truth and activationfunction. Every trial begins by seeding the random numbergenerator with the trial index before initializing networkparameters and beginning training. Training proceeds with10 epochs of ADAM optimization [28] and the epoch withoptimal validation loss is saved. Figure 3 shows the range oftest results from the cross-validation optimum.

Since most parameters in each network are contained withinthe linear transformation matrices, the number of parameterscales roughly linearly with \(n\). See Section V.

Because we know that the ground truth is generated byselecting only 1 out of 32 inputs per antecedent, we would likeregularization to compel most of the linear coefficients to zero.To do this, we adaptively adjust \(L_1\) regularization to cancel asmall fraction of the near-largest gradient. Specifically, we sortthe average gradient magnitudes computed from ADAM andselect the 15/16-largest element from the array. This allows us toset the \(L_1\) weight to cancel a small fraction, 0.1, times thismagnitude, thus leaving the largest gradients nearly unchangedwhile pushing parameters with much smaller gradients to zero.

Results are plotted in Figure 3 for several activation functions.

B. MNIST CNN

We developed this activation function with the intention ofsupporting more information-efficient processing pathways. To
that end, we notice that typical CNNs are structured so that adjacent strides overlap. Using a filter of width $f$ with $c_{in}$ input channels and $c_{out}$ output channels, each application of the filter has a computational complexity $c_{out}c_{in}f^2$. If the layer has a spatial width of $x$ and we use a stride distance of $s$, then each filter must be applied $x^2/s^2$ times, disregarding edge effects with appropriate padding. If this is followed by a $2 \times 2$ max pooling operation, the number of spacial elements reduces by a factor of 4. Thus, the total number of fused multiply-accumulations (FMAs) is $c_{out}c_{in}f^2x^2s^{-2}$. Taking $f = 3$ and $s = 1$ results in $9c_{out}c_{in}x^2$ operations per layer.

We would like to forge an approach where adjacent neurons are efficiently summarized, touching each only once with a small filter. Our activation functions, however, also require $n \times c_{out}$ antecedents to obtain $c_{out}$ consequent channels. Thus, using $f = 2$ and $s = 2$ and dropping the max pooling operation, we obtain the same size of output, but with a computational factor of $nc_{out}c_{in}x^2$. Even though our activation functions also have $2^n c_{out}$ parameters, the complexity is still dominated by the convolution, provided $n \leq 6$.

Since simple networks can obtain highly accurate predictions on the MNIST [26] dataset with only a fraction of examples, each experiment only uses 6000 training images, 1200 validation images, and 1200 test images. This is enough to resolve differences in expressivity for the activation functions we compare. We determine the number of channels in each layer by multiplying the channels from the previous layer by 4 until we hit a maximum, $w_{\text{max}}$. Because we are interested in how efficiently higher-arity activation functions capture latent variables, we choose relatively low limits, $w_{\text{max}} \in \{4, 6, 8, 12, 16, 24\}$. Note that 4 is the minimum number of Boolean variables that are capable of distinguishing 10 outcomes. As before, we perform 12 trials, seeded by the trial index, using 10 epochs of ADAM. The test set is evaluated from the cross-validation optimum and the distribution is plotted in Figure 4. Unsurprisingly, higher arity activation functions make the best use of width-restricted networks.

C. CIFAR-10 CNN

This set of experiments uses a standard CNN architecture for CIFAR-10 [27] consisting of 3 convolutions, each with
3×3 filters, stride 1, and interleaved by an ArgMaxAbs (AMA) layer,

\[ \text{ama}(X) = \arg \max_{x_{ij}} |x_{ij}|. \]

Note that for ReLU activation, AMA is identical to max-pooling, but it is more suitable for the other activation functions by preserving strong signed signals. The first convolution computes \( w_1 \) channels and the second and third compute \( 2w_1 \) channels. The last convolution is followed by a dense layer with \( 2w_1 \) channels. We run tests for various widths, \( w_1 \in \{16, 32, 64\} \). Finally, we have a dense layer to 10 softmax outputs.

Each experiment uses 40000 training images, 10000 validation images, and 10000 test images. We use a fixed \( L_1 \) regularization weight, \( w_1 = 0.5 \), as well as a fixed \( L_2 \) weight, \( w_2 = 0.01 \), for all experiments. Again, we perform 12 trials, seeded by the trial index, using 10 epochs of ADAM. The test set is evaluated from the cross-validation optimum and the distribution is plotted in Figure 5.

![Figure 5](image_url)

**Fig. 5.** These test demonstrate that our adaptive logic activation functions can perform better than alternative activation functions in some cases. The best average results were obtained from ternary logic, yet we know that quaternary logic is capable of representing all the same outcomes. This suggests that efficient training methods that properly exploit sparsity remain to be understood.

V. Discussion and Summary

Since high-arity functions are capable of subsuming low-arity functions as well as ReLU and AIL activation functions, we are convinced that the results we have obtained thus far do not demonstrate the full potential of high-arity logic. The challenge remains to efficiently compel these functions to mimic simple logic when possible. Doing so would allow us to use layers with a maximum expected arity and naturally reveal the latent complexity needed during training.

Solving this challenge will require more sophisticated machinery than an ad hoc regularization strategy to deal with the increase in parameter dimensionality. An \( n \)-ary layer with \( w_{\text{in}} \) inputs and \( w_{\text{out}} \) outputs requires \( nw_{\text{out}}w_{\text{in}} \) parameters in the linear transformation, \( y = Mx \), and \( w_{\text{out}}2^n \) activation parameters in \( \Theta \). We expect the increase in parameters to be dominated by the matrix \( M \) in practice. For small \( n \) and even moderate widths, the number of parameters grows linearly with \( n \). For example, taking \( w_{\text{in}} = w_{\text{out}} = 32 \) and \( n = 1, 2, 3, 4, 5, \) and 6, the number of parameters in one layer grows as \( P = 1088, 2176, 3328, 4608, 5120, \) and 6176. The exponential term only has a small impact until \( n > \log_2(w_{\text{in}}) \).

Yet, even linear parameter growth can be problematic if we do not have a robust mechanism to limit parameter complexity as \( n \) increases.

Even without a method to efficiently handle increased parameters for higher-arity logic, both binary and ternary activation functions still provide a useful increase in network expressivity. Both of these activation functions consistently outperform Relu, MaxMin, and MaxAIL activation functions on the simple networks we tested. Further, since our ternary activation functions open the ability to learn conditional reasoning in a single layer, this simple change may open a new domain of learning capacity and improve our ability to extract complex predictions from data.

A. Summary

We introduced a framework to learn arbitrary belief functions that accept \( n \) antecedent arguments. This allows training to adjust the underlying logic to not only coherently account for antecedent uncertainty, but also each conditional consequent uncertainty, which is more consistent with Bayesian reasoning. By representing all probabilities (antecedents, belief table conditional probabilities, and consequents) as logits, we are able to systematically derive computationally efficient approximations. Moreover, the resulting gradients are well-suited to optimization, because backpropagation transmits full gradient magnitudes to relevant antecedents and belief table parameters. We also showed how a particular basis allows sparse representations to replicate lower arity logic using the same number of nonzeros as if we had used a lower arity activation function.

Our approach allows dimensionality to work in our favor. Although implementing \( n \) argument logic requires \( 2^n \) parameters per activation channel, the resulting function becomes capable of distinguishing \( 3^{2^n} \) qualitatively distinct functions, thus dramatically increasing the ability of an architecture to discover useful predictions and avoiding relying on complicated compositions that may be difficult to adjust in order to correct isolated predictions.

We showed that our activation functions are capable of learning random truth functions of 6 arguments within a single layer. Our experiments with MNIST and CIFAR-10 datasets show that even just ternary logic may provide a useful increase in expressivity over alternative activation functions. Given that Boolean logic provides a mathematically rigorous framework for analyzing the relationships between propositions, we believe that a soft logic framework, generalizing binary truth function to \( n \)-ary belief functions, will prove to be a powerful tool in abstract learning algorithms. Since our activation functions associate sparsity with logical complexity, this provides an important pathway to suppress logical complexity during training, which we believe has a fundamental relationship with rigorously justified prior belief and few-shot learning.

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APPENDIX

PROOF OF IRRELEVANT ZEROS

We begin by considering the last antecedent. We can partition the truth table as \( a_k = [\alpha_0 \quad \alpha_1] \), so that \( \alpha_0 \in \mathbb{R}^{2^n-1} \) and \( \alpha_1 \in \mathbb{R}^{2^n-1} \) give consequent logits when the last antecedent is false and true, respectively. Since \( a_k = \theta_k B \), we can also partition the parameters using matching sizes as \( \theta_k = [\varphi_0 \quad \varphi_1] \). Examining \( B \) easily shows \( \alpha_0 = \varphi_0 - \varphi_1 \) and \( \alpha_1 = \varphi_0 + \varphi_1 \). From the definition of irrelevant antecedents, the last antecedent is irrelevant if and only if \( \alpha_0 = \alpha_1 \). It immediately follows that this is equivalent to \( \varphi_1 = 0 \), i.e. all columns indexed by \( j \) for which \( \text{bit}_n(j) = 1 \).

We can apply this result to any antecedent by permuting the order of antecedents as follows. Let \( \text{permute}(\cdot) \) be the desired permutation on \( n \)-bit sequences. We can form a permutation matrix \( P \) that maps all \( 2^n \) original columns, indexed by \( j \), to permuted columns, indexed by \( j' \), as \( P_{jj'} = 1 \) for all \( \text{bits}(j') = \text{permute}(\text{bits}(j)) \). Denoting all permuted representations using primes, we have

\[
a_k' = a_k P = [\theta_k P] \left[ B^T BP \right] = \theta_k' B'.
\]

Since the structure of the Kronecker product for \( B \) gives the element-wise formula \( B_{ij} = (-1)^{(\text{bit}_i(j)) \land \text{bit}_i(j')} \), applying the same permutation to rows and columns of \( B \) merely changes the order of or compositions in the exponent, so \( B' = B \). Thus, any antecedent reordering applied as a permutation to \( a_k \) is represented by the same permutation to \( \theta_k \).

If \( \psi_{ki} \in I \), permuting \( i \leftrightarrow n \) implies that \( \theta_k' = 0 \) for all permuted columns, index by \( j' \), with \( \text{bit}_n(j') = 1 \), which means in the original order \( \theta_k = 0 \) for all \( \text{bit}_i(j) = 1 \).

NETWORK PARAMETERS

| Arity | Relu | MinMax | MaxAIL | 2-Ary | 3-Ary | 4-Ary | 6-Ary |
|-------|------|--------|--------|-------|-------|-------|-------|
|       | 1024 | 1024   | 2048   | 2176  | 3232  | 4068  | 8192  |
|        | 4096 | 4096   | 8192   | 8736  | 9378  | 9088  | 8192  |
|        | 4096 | 4096   | 8192   | 8736  | 9378  | 9088  | 8192  |
|        |      |        |        |       |       |       |       |
|       | 12264| 11264  | 22528  | 22528 | 25296 | 23552 | 23552 |
|        | 11264| 11264  | 22528  | 22528 | 25296 | 23552 | 23552 |
|        |      |        |        |       |       |       |       |

TABLE II
NUMBER OF PARAMETERS IN LOGIC NETWORKS

| Channel Limit | 4    | 6    | 8    | 12   | 16   | 24   |
|---------------|------|------|------|------|------|------|
| Relu          | 312  | 604  | 992  | 2056 | 3504 | 6656 |
| MaxMin        | 312  | 604  | 992  | 2056 | 3504 | 6656 |
| MaxAIL        | 584  | 1148 | 1904 | 2992 | 6848 | 13072|
| 2-Ary         | 664  | 1260 | 2048 | 4208 | 7120 | 13440|
| 3-Ary         | 1016 | 1916 | 3104 | 6344 | 10736| 20224|
| 4-Ary         | 1448 | 2584 | 4304 | 8096 | 14624| 27376|

TABLE III
NUMBER OF PARAMETERS IN MNIST CNNs

| Activation | CIFAR-10 CNNs |
|------------|---------------|
|            | 4             | 6             | 8             | 12            | 16             | 24            |
| Relu       | 80112         | 179688        | 318944        |                |                |                |
| MaxMin     | 80112         | 179688        | 318944        |                |                |                |
| MaxAIL     | 199904        | 358896        | 637248        |                |                |                |
| 2-Ary      | 160352        | 359568        | 638144        |                |                |                |
| 3-Ary      | 240592        | 539448        | 957344        |                |                |                |
| 4-Ary      | 321280        | 720000        | 1277440       |                |                |                |

TABLE IV
NUMBER OF PARAMETERS IN CIFAR-10 CNNs

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