SYMMETRIC TENSOR RANK WITH A TANGENT VECTOR: A GENERIC UNIQUENESS THEOREM

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Abstract. Let $X_{m,d} \subset \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order $d$ Veronese embedding of $\mathbb{P}^m$. Let $\tau(X_{m,d}) \subset \mathbb{P}^N$, be the tangent developable of $X_{m,d}$. For each integer $t \geq 2$ let $\tau(X_{m,d},t) \subset \mathbb{P}^N$, be the join of $\tau(X_{m,d})$ and $t-2$ copies of $X_{m,d}$. Here we prove that if $m \geq 2$, $d \geq 7$ and $t \leq 1 + \lfloor (m+d-2)/(m+1) \rfloor$, then for a general $P \in \tau(X_{m,d},t)$ there are uniquely determined $P_1, \ldots, P_{t-2} \in X_{m,d}$ and a unique tangent vector $\nu$ of $X_{m,d}$ such that $P$ is in the linear span of $\nu \cup \{P_1, \ldots, P_{t-2}\}$, i.e. a degree $d$ linear form $f$ (a symmetric tensor $T$ of order $d$) associated to $P$ may be written as

$$f = L_{t-2}^{d-1}L_t + \sum_{i=1}^{t-2} L_i^d, \quad (T = v_1^{\otimes (d-1)}v_t + \sum_{i=1}^{t-2} v_i^{\otimes d})$$

with $L_i$ linear forms on $\mathbb{P}^m$ ($v_i$ vectors over a vector field of dimension $m+1$ respectively), $1 \leq i \leq t$, that are uniquely determined (up to a constant).

1. Introduction

In this paper we want to address the question of the uniqueness of a particular decomposition for certain given homogeneous polynomials. An analogous question can be rephrased in terms of uniqueness of a particular tensor decomposition of certain given symmetric tensors. In fact, given a homogeneous polynomial $f$ of degree $d$ in $m+1$ variables defined over an algebraically closed field $\mathbb{K}$, there is an obvious way to associate a symmetric tensor $T \in S^d(V_\mathbb{K})$, with $\dim(V_\mathbb{K}) = m+1$, to the form $f$. We will always work over an algebraically closed field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) = 0$. Fix integers $m \geq 2$ and $d \geq 3$. Let $j_{m,d} : \mathbb{P}^m \hookrightarrow \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order $d$ Veronese embedding of $\mathbb{P}^m$ and set $X_{m,d} := j_{m,d}(\mathbb{P}^m)$ (we often write $X$ instead of $X_{m,d}$). Let $\mathbb{K}[x_0, \ldots, x_m]$ be the polynomial ring of homogeneous degree $d$ polynomials in $m+1$ variables over $\mathbb{K}$ and let $V_\mathbb{K}^m$ be the dual space of $V_\mathbb{K}$.

Since obviously $\mathbb{P}^m \simeq \mathbb{P}(\mathbb{K}[x_0, \ldots, x_m]) \simeq \mathbb{P}(V_\mathbb{K}^m)$, an element of the Veronese variety $X_{m,d}$ can be interpreted either as the projective class of a $d$-th power of a linear form $L \in \mathbb{K}[x_0, \ldots, x_m]$ or as the projective class of a symmetric tensor $T \in S^d(V_\mathbb{K}^m) \subset (V_\mathbb{K}^m)^\otimes d$ for which there exists $v \in V_\mathbb{K}^m$ s.t. $T = v^{\otimes d}$.

For each integer $t$ such that $1 \leq t \leq N$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^N$ of the union of all $(t-1)$-dimensional linear subspaces spanned by $t$ points of $X$ (the $t$-secant variety of $X$). From this definition one can understand that the generic element of $\sigma_t(X_{m,d})$ can be interpreted either as $[f] = [L_1^d + \cdots + L_t^d] \in$
\( \mathbb{P}(\mathbb{K}[x_0, \ldots, x_m]_d) \) with \( L_1, \ldots, L_t \in \mathbb{K}[x_0, \ldots, x_m]_1 \) or as \( [T] = \sum_{i=1}^{t} v_i^d \subset \mathbb{P}(S^d(V^*_{K})) \) with \( v_1, \ldots, v_t \in V^*_{K} \). For a given form \( f \) (or a symmetric tensor \( T \)), the minimum integer \( t \) for which there exists such a decomposition is called the symmetric rank of \( f \) (or of \( T \)). Finding those \( v_i \)'s, \( i = 1, \ldots, t \) such that \( T = \sum_{i=1}^{t} v_i^d \) with \( t \) the symmetric rank of \( T \), is known as the Tensor Decomposition problem and it is a generalization of the Singular Value Decomposition problem for symmetric matrices (i.e. if \( T \in S^2(V^*_{K}) \)). The existence and the possible uniqueness of the decompositions of a form \( f \) as \( L_1^d + \cdots + L_t^d \) with \( t \) minimal is studied in certain cases in [6], [8], [10], [11].

Let \( \tau(X) \subset \mathbb{P}^N \) be the tangent developable of \( X \), i.e. the closure in \( \mathbb{P}^N \) of the union of all embedded tangent spaces \( T_pX, P \in X \). Obviously \( \tau(X) \subset \sigma_2(X) \) (and \( X \) is integral). Since \( d \geq 3 \), the variety \( \tau(X) \) is a divisor of \( \sigma_2(X) \) ([5], Proposition 3.2). An element in \( \tau(X_{m,d}) \) can be described both as \( [f] \in \mathbb{P}(\mathbb{K}[x_0, \ldots, x_m]_d) \) for which there exists two linear forms \( L_1, L_2 \in \mathbb{K}[x_0, \ldots, x_m] \) such that \( f = L_1^t - L_2^t \), and as \( [T] \in \mathbb{P}(S^d(V^*_{K})) \) for which there exists two vectors \( v_1, v_2 \in V^*_{K} \) such that \( T = v_1^d - v_2^d \) ([5], [4]).

Fix integral positive-dimensional subvarieties \( A_1, \ldots, A_s \subset \mathbb{P}^N, s \geq 2 \). The join \([A_1, A_2]\) is the closure in \( \mathbb{P}^N \) of the union of all lines spanned by a point of \( A_1 \) and a different point of \( A_2 \). If \( s \geq 3 \) define inductively the join \([A_1, \ldots, A_s]\) by the formula \([A_1, \ldots, A_s] := ([A_1, \ldots, A_{s-1}], A_s]\). The join \([A_1, \ldots, A_s]\) is an integral variety and \( \dim([A_1, \ldots, A_s]) \leq \min\{N, s-1 + \sum_{i=1}^{s} \dim(A_i)\} \). The integer \( \min\{N, s-1 + \sum_{i=1}^{s} \dim(A_i)\} \) is called the expected dimension of the join \([A_1, \ldots, A_s]\). Obviously \([A_1, \ldots, A_s] = [A_{\sigma(1)}, \ldots, A_{\sigma(s)}]\) for any permutation \( \sigma : \{1, \ldots, s\} \to \{1, \ldots, s\} \).

The secant variety \( \sigma_t(X), t \geq 2 \), is the join of \( t \) copies of \( X \). For each integer \( t \geq 3 \) let \( \tau(X, t) \subset \mathbb{P}^N \) be the join of \( \tau(X) \) and \( t-2 \) copies of \( X \). We recall that \( \min\{N, t(m+1)-2\} \) is the expected dimension of \( \tau(X, t) \), while \( \min\{N, t(m+1)-1\} \) is the expected dimension of \( \sigma_t(X) \). In the range of triples \((m, d, t)\) we will meet in this paper both \( \tau(X, t) \) and \( \sigma_t(X) \) have the expected dimensions and hence \( \tau(X, t) \) is a divisor of \( \sigma_t(X) \). An element in \( \tau(X_{m,d}, t) \) can be described both as \([f] \in \mathbb{P}(\mathbb{K}[x_0, \ldots, x_m]_d) \) for which there exist linear forms \( L_1, \ldots, L_t \in \mathbb{K}[x_0, \ldots, x_m]_1 \) such that \( f = L_1^t - L_2^t + \sum_{i=1}^{t-2} L_i^d \), and as \([T] \in \mathbb{P}(S^d(V^*_{K})) \) for which there exist \( v_1, \ldots, v_t \in V^*_{K} \) such that \( T = v_1^d - v_2^d + \sum_{i=1}^{t-2} v_i^d \).

After [3], it is natural to ask the following question.

**Question 1.** Assume \( d \geq 3 \) and \( \tau(X, t) \neq \mathbb{P}^N \). Is a general point of \( \tau(X, t) \) in the linear span of a unique set \( \{P_0, P_1, \ldots, P_{t-2}\} \) with \( \langle P_0, P_1, \ldots, P_{t-2} \rangle \in \tau(X) \times X^{t-2} \)?

For non weakly \((t-1)\)-degenerate subvarieties of \( \mathbb{P}^N \) the corresponding question is true by [8], Proposition 1.5. Here we answer it for a large set of triples of integers \((m, d, t)\) and prove the following result.

**Theorem 1.** Fix integers \( m \geq 2 \) and \( d \geq 6 \). If \( m \leq 4 \), then assume \( d \geq 7 \). Set \( \beta := \lfloor (m+d-2)/(m+1) \rfloor \). Assume \( 3 \leq t \leq \beta + 1 \). Let \( P \) be a general point of \( \tau(X, t) \). Then there are uniquely determined points \( P_1, \ldots, P_{t-2} \in X \) and \( Q \in \tau(X) \) such that \( P \in \{P_1, \ldots, P_{t-2}, Q\} \), i.e. (since \( d > 2 \)) there are uniquely determined points \( P_1, \ldots, P_{t-2} \in X \) and a unique tangent vector \( \nu \) of \( X \) such that \( P \in \{P_1, \ldots, P_{t-2}\} \cup \nu \).
In terms of homogeneous polynomials Theorem 1 may be rephrased in the following way.

**Theorem 2.** Fix integers \( m \geq 2 \) and \( d \geq 6 \). If \( m \leq 4 \), then assume \( d \geq 7 \). Set \( \beta := \lfloor \frac{(m+d-2)}{(m+1)} \rfloor \). Assume \( 3 \leq t \leq \beta+1 \). Let \( P \) be a general point of \( \tau(X,t) \) and let \( f \) be a homogeneous degree \( d \) form in \( K[x_0, \ldots, x_m] \) associated to \( P \). Then \( f \) may be written in a unique way

\[
f = L_{i-t-1}^{d-1}L_t + \sum_{i=1}^{t-2}L_i^d
\]

with \( L_i \in K[x_0, \ldots, x_m]_1, 1 \leq i \leq t \).

In the statement of Theorem 2 the form \( f \) is uniquely determined only up to a non-zero scalar, and (as usual in this topic) “uniqueness” may allow not only a permutation of the forms \( L_1, \ldots, L_{t-2} \), but also a scalar multiplication of each \( L_i \).

In terms of symmetric tensors Theorem 1 may be rephrased in the following way.

**Theorem 3.** Fix integers \( m \geq 2 \) and \( d \geq 6 \). If \( m \leq 4 \), then assume \( d \geq 7 \). Set \( \beta := \lfloor \frac{(m+d-2)}{(m+1)} \rfloor \). Assume \( 3 \leq t \leq \beta+1 \). Let \( T \) be a general point of \( \tau(X,t) \) and let \( T \in S^d(V_k^*) \) be a symmetric tensor associated to \( P \). Then \( T \) may be written in a unique way

\[
T = v_{i-t-1}^{(d-1)}v_t + \sum_{i=1}^{t-2}v_i^{\otimes d}
\]

with \( v_i \in V_k^*, 1 \leq i \leq t \).

As above, in the statement of Theorem 3 the tensor \( T \) and the vectors \( v_i \)'s are uniquely determined only up to non-zero scalars.

To prove Theorem 1 and hence Theorems 2 and 3 we adapt the notion and the results on weakly defective varieties described in [6]. It is easy to adapt [6] to joins of different varieties instead of secant varieties of a fixed variety if a general tangent hyperplane is tangent only at one point (7). However, a general tangent space of \( \tau(X) \) is tangent to \( \tau(X) \) along a line, not just at the point of tangency. Hence a general hyperplane tangent to \( \tau(X,t), t \geq 3 \), is tangent to \( \tau(X,t) \) at least along a line. We prove the following result.

**Theorem 4.** Fix integers \( m \geq 2 \) and \( d \geq 6 \). If \( m \leq 4 \), then assume \( d \geq 7 \). Set \( \beta := \lfloor \frac{(m+d-2)}{(m+1)} \rfloor \). Assume \( t \leq \beta+1 \). Let \( P \) be a general point of \( \tau(X,t) \). Let \( P_1, \ldots, P_{t-2} \in X \) and \( Q \in \tau(X) \) be the points such that \( P \in \langle \{P_1, \ldots, P_{t-2}, Q\} \rangle \). Let \( \nu \) be the tangent vector of \( X \) such that \( Q \) is a point of \( \langle \nu \rangle \setminus \nu_{\text{red}} \). Let \( H \subset P^N \) be a general hyperplane containing the tangent space \( T_P \tau(X,t) \) of \( \tau(X,t) \). Then \( H \) is tangent to \( X \) only at the points \( P_1, \ldots, P_{t-2}, \nu_{\text{red}} \), the scheme \( H \cap X \) has an ordinary node at each \( P_i \), and \( H \) is tangent to \( \tau(X) \setminus X \) only along the line \( \langle \nu \rangle \).

2. **Preliminaries**

**Notation 1.** Let \( Y \) be an integral quasi-projective variety and \( Q \in Y_{\text{reg}} \). Let \( \{kQ, Y\} \) denote the \((k-1)\)-th infinitesimal neighborhood of \( Q \) in \( Y \), i.e. the closed subscheme of \( Y \) with \( (T_Q)^\mathbb{K} \) as its ideal sheaf. If \( Y = P^m \), then we write \( kQ \) instead of \( \{kQ, P^m\} \). The scheme \( \{kQ, Y\} \) will be called a \( k \)-point of \( Y \). We also say that a 2-point is a double point, that a 3-point is a triple point and a 4-point is a quadruple point.
We give here the definition of a $(2, 3)$-point as it is in [3], p. 977.

**Definition 1.** Let \( q \subset K[x_0, \ldots, x_m] \) be the reduced ideal of a simple point \( Q \subset Y \). We say that \( Z \) is a representative ideal if \( Z \) is a point of a (2, 3)-point if it is the zero-dimensional scheme whose representative ideal is \((q^3 + l^2)\).

**Remark 1.** Notice that \( 2Q \subset Z(Q, L) \subset 3Q \).

We recall the notion of weak non-defectivity for an integral and non-degenerate projective variety \( Y \subset P^r \) (see [6]). For any closed subscheme \( Z \subset P^r \) set:

\[
(1) \quad H(-Z) := \{ |I_Z| \}
\]

**Notation 2.** Let \( Z \subset P^r \) be a zero-dimensional scheme such that \( \{2Q, Y\} \subset Z \). For all \( Q \in Z_{\text{red}} \). Fix \( H \in H(-Z) \) where \( H(-Z) \) is defined in [1]. Let \( H_c \) be the closure in \( Y \) of the set of all \( Q \in Y_{\text{reg}} \) such that \( t_Q Y \subset H \).

The contact locus \( H_Z \) of \( H \) is the union of all irreducible components of \( H_c \) containing at least one point of \( Z_{\text{red}} \).

We use the notation \( H_Z \) only in the case \( Z_{\text{red}} \subset Y_{\text{reg}} \).

Fix an integer \( k \geq 0 \) and assume that \( \sigma_{k+1}(Y) \) doesn’t fill up the ambient space \( P^r \). Fix a general \((k+1)\)-uple of points in \( Y \) i.e. \((P_0, \ldots, P_k) \in Y^{k+1} \) and set

\[
(2) \quad Z := \bigcup_{i=0}^k \{2P_i, Y\}.
\]

The following definition of weakly \( k \)-defective variety coincides with the one given in [6].

**Definition 2.** A variety \( Y \subset P^r \) is said to be weakly \( k \)-defective if \( \dim(H_Z) > 0 \) for \( Z \) as in (2).

In [6], Theorem 1.4, it is proved that if \( Y \subset P^r \) is not weakly \( k \)-defective, then \( H_Z = Z_{\text{red}} \) and that \( \text{Sing}(Y \cap H) = (\text{Sing}(Y) \cap H) \cup Z_{\text{red}} \) for a general \( Z = \bigcup_{i=0}^k \{2P_i, Y\} \) and a general \( H \in H(-Z) \). Notice that \( Y \) is weakly 0-defective if and only if its dual variety \( Y^* \subset P^{r*} \) is not a hypersurface.

In [7] the authors considered also the case in which \( Y \) is not irreducible and hence its joins have as irreducible components the joins of different varieties.

**Lemma 1.** Fix an integer \( y \geq 2 \), an integral projective variety \( Y, L \in \text{Pic}(Y) \) and \( P \in Y_{\text{reg}} \). Set \( x := \dim(Y) \). Assume \( h^0(Y, I_{yL}) \otimes L = h^0(Y, L) - (x+y) \). Fix a general \( F \in |I_{yL} \otimes L| \). Then \( P \) is an isolated singular point of \( F \).

**Proof.** Let \( u : Y' \to Y \) denote the blowing-up of \( Y \) at \( P \) and \( E := u^{-1}(P) \) the exceptional divisor. Since \( \dim(Y) = x \), we have \( E \cong P^{x-1} \). Set \( R := u^*(L) \). For each integer \( t \geq 0 \) we have \( u_*(R(-tE)) \cong I_{tL} \otimes L \). Thus the push-forward \( u_* \) induces an isomorphism between the linear system \( |R(-tE)| \) on \( Y' \) and the linear system \( |I_{tL} \otimes L| \) on \( Y \). Set \( M := R(-yE) \). Since \( O_{Y'}(E) \cong O_E(-1) \) (up to the identification of \( E \) with \( P^{x-1} \)), we have \( R(-tE)|E \cong O_E(t) \) for all \( t \in \mathbb{N} \). Consider on \( Y' \) the exact sequence:

\[
(3) \quad 0 \to M(-E) \to M \to O_E(y) \to 0
\]

Our hypothesis implies that \( h^0(Y, I_{yL} \otimes L) = h^0(Y, L) - (x+y) \). Thus our assumption implies \( h^0(Y', M(-E)) = h^0(Y', R) - (x+y) = h^0(Y', R) - (x+y-1) - (x+y) = h^0(Y', M) - h^0(E, O_E(y)) \). Thus [3] gives the surjectivity of the restriction map.
\[ \rho : H^0(Y', M) \to H^0(E, M|_E). \] Since \( y \geq 0 \), the line bundle \( M|_E \) is spanned. Thus the surjectivity of \( \rho \) implies that \( M \) is spanned at each point of \( E \). Hence \( M \) is spanned in a neighborhood of \( E \). Bertini’s theorem implies that a general \( F' \in |M| \) is smooth in a neighborhood of \( E \). Since \( F \) is general and \( |M| \cong |I_{gP} \otimes L|, P \) is an isolated singular point of \( F \).

3. \( \tau(X, t) \) is not weak defective

In this section we fix integers \( m \geq 2, d \geq 3 \) and set \( N = \binom{m+d}{m} - 1 \) and \( X := X_{m,d} \). The variety \( \tau(X) \) is 0-weakly defective, because a general tangent space of \( \tau(X) \) is tangent to \( \tau(X) \) along a line. Terracini’s lemma for joins implies that a general tangent space of \( \tau(X, t) \) is tangent to \( \tau(X, t) \) at least along a line (see Remark 5). Thus \( \tau(X, t) \) is weakly 0-defective. To handle this problem and prove Theorem 1, we introduce another definition, which is tailor-made to this particular case. As in [5] we want to work with zero-dimensional schemes on \( X \), not on \( \tau(X) \) or \( \tau(X, t) \). We consider \( X = j_{m,d}(\mathbb{P}^m) \) and the 0-dimensional scheme \( Z \subset X \) which is the image (via \( j_{m,d} \)) of the general disjoint union of \( t-2 \) double points and one \((2,3)\)-point of \( \mathbb{P}^m \), in the case of [5] (see Definition 1). We will often work by identifying \( X \) with \( \mathbb{P}^m \), so e.g. notice that \( H(-0) \) is just \( |O_{\mathbb{P}^m}(d)| \).

**Remark 2.** Fix \( P \in X \) and \( Q \in T_PX \setminus \{P\} \). Any two such pairs \((P, Q)\) are projectively equivalent for the natural action of \( Aut(\mathbb{P}^m) \). We have \( Q \in \tau(X)_{reg} \) and \( T_Q \tau(X) \supset T_PX \). Set \( D := \langle \{P, Q\} \rangle \). It is well-known that \( D \setminus \{P\} \) is the set of all \( O \in \tau(X)_{reg} \) such that \( T_Q \tau(X) = T_O \tau(X) \) (e.g. use that the set of all \( g \in Aut(\mathbb{P}^m) \) fixing \( P \) and the line containing \( P \) associated to the tangent vector induced by \( Q \) acts transitively on \( T_PX \setminus D \).

**Definition 3.** Fix a general \( (O_1, \ldots, O_{l-2}, O) \in (\mathbb{P}^m)^{l-1} \) and a general line \( L \subset \mathbb{P}^m \) such that \( O \in L \). Set \( Z := Z(O, L) \cup \bigcup_{i=1}^{l-2} 2O_i \). We say that the variety \( \tau(X, t) \) is not drip defective if \( \dim(H_Z) = 0 \) for a general \( H \in |I_Z(d)| \).

We are now ready for the following lemma.

**Lemma 2.** Fix an integer \( t \geq 3 \) such that \((m+1)t < n\). Let \( Z_1 \subset \mathbb{P}^m \) be a general union of a quadruple point and \( t-2 \) double points. Let \( Z_2 \) be a general union of \( 2 \) triple points and \( t-2 \) double points. Fix a general disjoint union \( Z = Z(O, L) \cup \bigcup_{i=1}^{l-2} 2P_i \) where \( Z(O, L) \) is a \((2,3)\)-point as in Definition 1 and \( O \) and \( \{P_1, \ldots, P_{l-2}\} \subset \mathbb{P}^m \) are general. Assume \( h^1(\mathbb{P}^m, I_{Z_1}(d)) = h^1(\mathbb{P}^m, I_{Z_2}(d)) = 0 \). Then:

(i) \( h^1(\mathbb{P}^m, I_Z(d)) = 0 \);

(ii) \( \tau(X, t) \) is not drip defective;

(iii) a general \( H \in \mathcal{H}(-Z) \) has an ordinary quadratic singularity at each \( P_i \).

**Proof.** Set \( W := 3O \cup \bigcup_{i=1}^{l-2} 2P_i \). The definition of a \((2,3)\)-point gives that \( Z(O, L) \subset 3O \). Thus \( Z \subset W \subset Z_2 \). Hence \( h^1(\mathbb{P}^m, I_Z(d)) \leq h^1(\mathbb{P}^m, I_{Z_2}(d)) = 0 \). Hence part (i) is proven.

To prove part (ii) of the lemma we need to prove that \( \dim(H_Z) = 0 \) for a general \( H \in \mathcal{H}(-Z) \). Since \( W \not\supseteq Z_1 \) and \( h^1(\mathbb{P}^m, I_{Z_1}(d)) = 0 \), we have \( \mathcal{H}(-W) \neq 0 \). Since \( W_{red} = Z_{red} \) and \( Z \subset W \), to prove parts (ii) and (iii) of the lemma it is sufficient to prove \( \dim((H_W)_c) = 0 \) for a general \( H_W \in \mathcal{H}(-W) \), where \( W \) is as above and \((H_W)_c\) is as in Notation 2. Assume that this is not true, therefore:
Lemma 5. Fix integers $\alpha := \lfloor \frac{m^2+1}{m+1} \rfloor$ and $\beta := \lfloor \frac{m+2}{m+1} \rfloor$. Set $Z_i := \bigcup_{i=1}^{t} 2P_i$ and $Z' := 3O \cup Z_3$.

(a) Here we assume the existence of a positive dimensional component $J_i \subset (H_W)_c$ containing one of the $P_i$'s, say for example $J_{t-2} \supset P_{t-2}$. Thus a general element of $|I_W(d)|$ is singular along a positive-dimensional irreducible algebraic set containing $P_{t-2}$. Let $w : M \to \mathbb{P}^m$ denote the blowing-up of $\mathbb{P}^m$ at the points $O, P_1, \ldots, P_{t-3}$. Set $E_0 := w^{-1}(O)$ and $E_i := w^{-1}(P_i)$, $1 \leq i \leq t-3$. Let $A$ be the only point of $M$ such that $w(A) = P_{t-2}$. For each integer $y \geq 0$ we have $w_*(I_yA \otimes w^*(O_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \cdots - 2E_{t-3})) = I_{\mathbb{Z}^t \cup yO_{t-2}}(d)$. Applying Lemma 4 to the variety $M$, the line bundle $w^*(O_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \cdots - 2E_{t-3})$, the point $A$ and the integer $y = 2$ we get a contradiction.

(b) Here we prove the non-existence of a positive-dimensional $T \subset (H_W)_c$ containing $O$. Let $w_1 : M_1 \to \mathbb{P}^m$ denote the blowing-up of $\mathbb{P}^m$ at the points $P_1, \ldots, P_{t-2}$. Set $E_j := w_1^{-1}(P_j)$, $1 \leq j \leq t-2$. Let $B \in M_1$ be the only point of $M_1$ such that $w_1(B) = O$. For each integer $y \geq 0$ we have $w_1_*(I_yB \otimes w_1^*(O_{\mathbb{P}^m}(d))(-2E_1 - \cdots - 2E_{t-2})) = I_{\mathbb{Z}^t \cup yO_{t-2}}(d)$. Since $h^1(\mathbb{P}^m, I_{Z_2}(d)) = 0$ and $|I_{Z_2}(d)| \subset |I_{Z_2}(d)|$, by Lemma 4 with $y = 3$ we get a contradiction.

In [3], Lemmas 5 and 6, we proved the following two lemmas:

**Lemma 3.** Fix integers $m \geq 2$ and $d \geq 5$. If $m \leq 4$, then assume $d \geq 6$. Set $\alpha := \lfloor \frac{m^2+1}{m+1} \rfloor$. Let $Z_i \subset \mathbb{P}^m$, $i = 1, 2$, be a general union of $i$ triple points and $\alpha - i$ double points. Then $h^1(I_{Z_i}(d)) = 0$.

**Lemma 4.** Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \frac{m+2}{m+1} \rfloor$. Let $Z \subset \mathbb{P}^m$ be a general union of one quadruple point and $\beta - 1$ double points. Then $h^1(I_Z(d)) = 0$.

We will use the following set-up.

**Notation 3.** Fix any $Q \in \tau(X) \setminus X$. For $d \geq 3$ the point $Q$ uniquely determines a point $B \in X$ and (up to a non-zero scalar) a tangent vector $\nu$ of $X$ with $\nu_{\text{red}} = \{B\}$. We have $Q \in \langle \nu \rangle \setminus \{B\}$ and $T_Q \tau(X)$ is tangent to $\tau(X) \setminus X$ exactly along the line $\langle \nu \rangle = \langle \{B, Q\} \rangle$. Let $O \in \mathbb{P}^m$ be the only point such that $j_{n,d}(O) = B$. Let $u_O : \widetilde{X} \to \mathbb{P}^m$ be the blowing-up of $O$. Let $E := u_O^{-1}(O)$ denote the exceptional divisor. For all integers $x, e$ set $O_{\widetilde{X}}(x, eE) := u^*(O_{\mathbb{P}^m}(x))(eE)$. Let $\mathcal{H}$ denote the linear system $|O_{\widetilde{X}}(d, -3E)|$ on $\widetilde{X}$.

**Remark 3.** When $d \geq 4$, the line bundle $O_{\widetilde{X}}(d, -3E)$ is very ample, $u_* (O_{\widetilde{X}}(d, -3E)) = I_{\mathbb{Z}O} (1)$, $h^0(\widetilde{X}, O_{\widetilde{X}}(d, -3E)) = \binom{m+d}{m} - \binom{m+2}{m}$ and $h^i(\widetilde{X}, O_{\widetilde{X}}(d, -3E)) = 0$ for all $i > 0$.

**Lemma 5.** Fix integers $m \geq 2$ and $d \geq 5$. If $m \leq 4$, then assume $d \geq 6$. Set $\alpha := \lfloor \frac{m^2+1}{m+1} \rfloor$. Fix an integer $t$ such that $3 \leq t \leq \alpha$. The linear system $\mathcal{H}$ on $\widetilde{X}$ is not $(t-3)$-weakly defective. For a general $O_1, \ldots, O_{t-2} \in \widetilde{X}$ a general $H \in |\mathcal{H}(-2O_1 - \cdots - 2O_{t-2})|$ is singular only at the points $O_1, \ldots, O_{t-2}$ which are ordinary double points of $H$. 

Proof. Fix general $O_1, \ldots, O_{t-2} \in \bar{X}$. Fix $j \in \{1, \ldots, t-2\}$ and set $Z' := 3O_j U \bigcup_{i \neq j} 2O_i$, $Z'' := \bigcup_{i=1}^{t-2} 2O_i$ and $W := 3O_j \cup \bigcup_{i \neq j} 2O_i$. We have $u_t(I_{Z'}(d, -3E)) \cong I_{W, O_j}(1)$. The case $i = 2$ of Lemma 3 gives $h^1(I_Z(d, -3E)) = 0$. Lemma 4 applied to a blowing-up of $\mathbb{P}^m$ at $\{O_1, O, O_t, \ldots, O_{t-2}\} \setminus \{O_j\}$ shows that a general $H \in \mathcal{H}(Z)$ has as an isolated singular point at $O_j$. Since this is true for all $j \in \{1, \ldots, t-2\}$, $\mathcal{H}$ is not $(t-3)$-weakly defective (just by the definition of weak defectivity). The second assertion follows from the first one and [6, Theorem 1.4]. ⊓⊔

Now we can apply Lemmas 2, 3, 4 and 5 and get the following result.

**Theorem 5.** Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor (m+d-2)/(m+1) \rfloor$. Fix an integer $t$ such that $3 \leq t \leq \beta + 1$. Then $\tau(X, t)$ is not drip defective.

Proof. Fix general $P_1, \ldots, P_{t-2}, O \in \mathbb{P}^m$ and a general line $L \subset \mathbb{P}^m$ such that $O \in L$. Set $Z := Z(O, L) \cup \bigcup_{i=2}^{t-2} 2P_i$, $W := 3O \cup \bigcup_{i=1}^{t-2} 2P_{i-2}$, $W' := 3O \cup 3O_i \cup \bigcup_{i=2}^{t-2} 2P_{i-2}$ and $W'' := 4O \cup \bigcup_{i=2}^{t-2} 2P_{i-2}$. Take $O_i \in \bar{X}$ such that $u_t(O_i) = P_i$, $1 \leq i \leq t-2$. Since $u_{O_i}(I_{2O_i \cup \cdots \cup 2O_{t-2}}(d, -4E)) \cong I_W(d)$, Lemma 4 gives $h^1(I_{2O_i \cup \cdots \cup 2O_{t-2}}(d, -4E)) = 0$. Since $Z(O, L) \subset 3O$, the case $y = 3$ of Lemma 5 applied to the blowing-up of $\mathbb{P}^m$ at $O_1, \ldots, O_{t-2}$ shows that a general $H \in [I_W(d)]$ has an isolated singularity at $O$ with multiplicity at most 3. ⊓⊔

Recall that $\text{Sing}(\tau(X)) = X$ and that for each $Q \in \tau(X) \setminus X$ there is a unique $O \in X$ and a unique tangent vector $\nu$ to $X$ at $O$ such that $Q \in \langle \nu \rangle$ and that $\langle \nu \rangle \setminus \{O\}$ is the contact locus of the tangent space $T_Q \tau(X)$ with $\tau(X) \setminus X$.

Let $P$ be a general point of $\tau(X, t)$, i.e. fix a general $(P_1, \ldots, P_{t-2}, Q) \in X^{t-2} \times \tau(X)$ and a general $P \in \langle\langle P_1, \ldots, P_{t-2}, Q\rangle\rangle$.

**Proof of Theorem 4**. Fix a general $P \in \tau(X, t)$, say $P \in \langle\langle P_1, \ldots, P_{t-2}, Q\rangle\rangle$ with $(P_1, \ldots, P_{t-2}, Q)$ general in $X^{t-2} \times \tau(X)$. Terracini’s lemma for joins ([1], Corollary 1.10) gives $T_P \tau(X, t) = \langle T_{P_1}X \cup \cdots T_{P_{t-2}}X \cup T_Q \tau(X) \rangle$. Let $O$ be the point of $\mathbb{P}^m$ such that $Q \in T_{j=0}(O)X$. Let $\mathcal{H}'$ (resp. $\mathcal{H}''$) be the set of all hyperplane $H \subset \mathbb{P}^N$ containing $T_Q \tau(X)$ (resp. $T_P \tau(X, t)$). We may see $\mathcal{H}'$ and $\mathcal{H}''$ as linear systems on the blowing-up $\bar{X}$ of $\mathbb{P}^m$ at $O$. Take $O_i \in \bar{X}$, $1 \leq i \leq t-2$, such that $P_i = u(O_i)$ for all $i$. We have $\mathcal{H}'' = \mathcal{H}'(-2P_1 - \cdots - 2P_{t-2})$ and $\mathcal{H} \subseteq \mathcal{H}'$, where $\mathcal{H}$ is defined in Notation 3. Since $(P_1, \ldots, P_{t-2})$ is general in $X^{t-2}$ for a fixed $Q$ and $\mathcal{H} \subseteq \mathcal{H}'$, Lemma 5 gives that a general $H \in \mathcal{H}''$ intersects $X$ in a divisor which, outside $O$, is singular only at $P_1, \ldots, P_{t-2}$ and with an ordinary node at each $P_i$. Now assume $P \in \langle\langle P_1', \ldots, P_{t-2}', Q'\rangle\rangle$ for some other $(P_1', \ldots, P_{t-2}', Q') \in X^{t-2} \times \tau(X)$. Since $P$ is general in $\tau(X, t)$ and $\tau(X, t)$ has the expected dimension, the $(t-1)$-ple $(P_1', \ldots, P_{t-2}', Q')$ is general in $X^{t-2} \times \tau(X)$. Hence $H \cap X$ is singular at each $P_i'$, $1 \leq i \leq t-2$, and with an ordinary node at each $P_i'$. Since $O$ is not an ordinary node of $H \cap X$, we get $\{P_1, \ldots, P_{t-2}\} = \{P_1', \ldots, P_{t-2}'\}$. Thus $O = O'$. Hence $H$ is tangent to $\tau(X)_{reg}$ exactly along the line $\langle\langle Q, O\rangle\rangle \setminus \{O\}$. Hence $Q' \notin \langle\langle Q, O\rangle\rangle$. Assume $Q \neq Q'$. Since $P$ is general in $\tau(X, t)$, then $P \notin \tau(X, t-1)$. Hence $Q' \notin \langle\langle P_1, \ldots, P_{t-2}\rangle\rangle$ and $Q \notin \langle\langle P_1, \ldots, P_{t-2}\rangle\rangle$. Thus $\langle\langle P_1, \ldots, P_{t-2}, Q\rangle\rangle \cap \langle\langle P_1, \ldots, P_{t-2}, Q'\rangle\rangle = \langle\langle P_1, \ldots, P_{t-2}\rangle\rangle$ if $Q \neq Q'$. Since $P \in \langle\langle P_1, \ldots, P_{t-2}, Q\rangle\rangle \cap \langle\langle P_1, \ldots, P_{t-2}, Q'\rangle\rangle$, we got a contradiction. ⊓⊔

**Proof of Theorem 4**. The case $t = 2$ is well-known and follows from the following fact: for any $O \in X$ and any $Q \in T_O X \setminus \{O\}$ the group $G_O := \{g \in$
\( \text{Aut}(\mathbb{P}^n) : g(O) = O \) acts on \( T_O X \) and the stabilizer \( G_{O,Q} \) of \( Q \) for this action is the line \( \langle \{O,Q\} \rangle \), while \( T_O X \setminus \langle \{O,Q\} \rangle \) is another orbit for \( G_{O,Q} \). Thus we may assume \( t \geq 3 \). Fix a general \( P \in \tau(X,t) \) and a general hyperplane \( H \supset T_P \tau(X,t) \).

If \( H \) is tangent to \( \tau(X) \) at a point \( Q' \in \tau(X) \setminus X \), then it is tangent along a line containing \( Q' \). Let \( E \in X \) be the only point such that \( Q' \in T_E X \). We get \( T_E X \subset \tau(X,t) \) and that \( H \cap T_E X \) is larger than the double point \( 2E \subset X \). Theorem \( \dagger \) gives that \( Q, Q' \) and \( E \) are collinear, i.e. \( H \) is tangent only along the line \( \nu \). \( \square \)

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