Mesoscopic Mechanical Resonators as Quantum Non-Inertial Reference Frames

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An atom attached to a micrometer-scale wire that is vibrating at a frequency \(\sim 100 \text{ MHz} \) and with displacement amplitude \(\sim 1 \text{ nm} \) experiences an acceleration magnitude \(\sim 10^9 \text{ m} \cdot \text{s}^{-2} \), approaching the surface gravity of a neutron star. As one application of such extreme non-inertial forces in a mesoscopic setting, we consider a model two-path atom interferometer with one path consisting of the 100 MHz vibrating wire atom guide. The vibrating wire guide serves as a non-inertial reference frame and induces an in principle measurable phase shift in the wave function of an atom traversing the wire frame. We furthermore consider the effect on the two-path atom wave interference when the vibrating wire is modeled as a quantum object, hence functioning as a quantum non-inertial reference frame. We outline a possible realization of the vibrating wire, atom interferometer using a superfluid helium quantum interference setup.

During the past decade there has been a growing effort to demonstrate nano-to-mesoscale mechanical systems existing in manifest quantum states [1–5]. One motivation is to understand how classical dynamics arises from quantum dynamics for systems with center of mass much larger than that of a single atom, and in particular establish whether the quantum-to-classical transition is solely a consequence of environmentally induced decoherence [6] or perhaps ultimately due to some as yet undiscovered, fundamental ‘collapse’ mechanism [7]. Two important milestones have been the demonstration of a \(\sim 5 \text{ GHz} \) mechanical resonator mode in a single phonon Fock state [8] and the demonstration of an entangled \(\sim 10 \text{ MHz} \) mechanical resonator mode–microwave cavity mode state [9].

A particularly straightforward mechanical geometry is that of a long, thin suspended beam (wire) that is supported at both ends (i.e., doubly clamped). The wire can be driven transversely, exciting its fundamental flexural mode resonance, using several available actuation methods. For the example of a crystalline silicon (Si) wire that is a few micrometers long and a fraction of a micrometer in cross section, the mechanical fundamental flexural frequency is \(\Omega \sim 2\pi \times 100 \text{ MHz} \) [10]. Consider an Si atom or other atom type attached to the surface midway along the length of such a vibrating wire. Suppose that the midpoint displacement amplitude is \(X_0 \sim 1 \text{ nm} \). Then assuming that the midpoint undergoes simple harmonic motion, \(X(t) = X_0 \cos (\Omega t + \varphi_0) \), we have for the maximum acceleration experienced by the attached atom \(\ddot{X}_{\max} = \Omega^2 X_0 \sim 10^9 \text{ m} \cdot \text{s}^{-2} \). This is an unexpectedly large acceleration, \(10^8 \) times larger than the gravitational acceleration \(g \) on the surface of the Earth and closer in magnitude to the surface gravity of a neutron star [11].

In this Letter, we analyze one possible application of these extreme accelerations in a mesoscopic setting; there are undoubtedly other hitherto unexplored applications. In particular, we shall consider a model two-path atom interferometer shown schematically in Fig. 1. The right path consists of a vibrating wire, atom guide segment with micrometer dimensions similar to those described above. The left path is fixed, without a vibrating wire segment. Incident atom wave packets split into left and right wave packet components. The right wave component will accumulate a phase shift relative to the left wave component as a consequence of the vibrating wire functioning effectively as a non-inertial reference frame for the traversing right wave. The right wave component eventually exits the non-inertial frame and recombines with the left wave component. As we shall see below, this results in an in principle detectable fringe visibility for the left-right wave component interference as the vibrating wire amplitude is varied in the nanometer range.

Quantum wave interference due to gravitational and non-inertial forces has been experimentally demonstrated for neutrons [12–15], atoms [16], Cooper-pairs [17], and electrons [18], verifying the equivalence between these forces for quantum matter systems [19]. However, in all of these experiments and accompanying analyses the non-inertial reference frames (and of course gravity) were treated as classical systems and there was no reason to view them otherwise. On the other hand, with mesoscopic mechanical resonators now being prepared and measured in manifest quantum states [8, 9], as described above, it is very natural to consider the consequences for the matter wave interference of the vibrating wire functioning effectively as a quantum non-inertial reference frame [20–23]. We shall therefore consider the effect on the fringe visibility for the left-right wave component interference when the vibrating wire is described quantum-mechanically.

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Beginning first with a classical description, we approximate the non-inertial frame as a long, thin beam (wire) of length $L$ with hinged boundary conditions, so that for small amplitude transverse displacements ($X_0 \ll L$) the frame coordinates in the lowest, fundamental flexural mode are $X(z, t) = X_0 \sin(\pi z/L) \cos(\Omega t + \varphi_0)$, $0 \leq z \leq L$. We suppose that atoms traversing the frame in the longitudinal, $z$ coordinate direction are described by localized wave packets propagating with uniform group velocity $v = L/T$ relative to the vibrating frame, where $T$ is the atom dwell time in the frame. The atoms are assumed to be confined by a harmonic potential to the frame in their transverse, $x$ coordinate direction that is aligned with the frame flexing $X$ coordinate direction. The bound atom potential is then $V(x, t) = \frac{1}{2}m\omega^2 \left[ x - X_0 \sin(\pi t/T) \cos(\Omega t + \varphi_0) \right]^2$, where both atom and frame coordinates ($x$ and $X$, respectively) are defined relative to a common origin in the assumed inertial lab frame. The various characteristic frequencies are assumed to satisfy $\pi/T \ll \Omega \ll \omega$, so that the atom spends many oscillation cycles in the frame, while it is tightly bound with negligible transverse motion relative to the frame.

Micrometer scale resonators have masses $M \sim 10^{-15}$ kg, while the atom mass $m \sim 10^{-27}$ kg, i.e., $M \gg m$. It is therefore reasonable to neglect the back action of the atom on the classical frame. We assume that the frame is initially excited in its fundamental transverse flexural mode and freely oscillates with negligible change in amplitude (i.e., damping) over the atom dwell time $T$. In terms of the transverse atom $x$ and frame $X$ coordinates, the Schrödinger equation for the atom in the vibrating frame described by the right path wave component $\psi_R(x, t)$ is then

$$i\hbar \frac{\partial \psi_R}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_R}{\partial x^2} + \frac{1}{2}m\omega^2 \left[ x - X_0 \sin(\pi t/T) \cos(\Omega t + \varphi_0) \right]^2 \psi_R. \quad (1)$$

This equation may be straightforwardly solved assuming a Gaussian function form. With the atom entering the frame initially in its transverse ground state: $\psi_R(x, 0) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m\omega x^2}{2\hbar} \right)$, the resulting interference between the left and right path wave components at $t = T$ is

$$\psi_L(x, T)\psi_R(x, T) = \sqrt{\frac{m\omega}{\pi \hbar}} \exp \left( -\frac{m\omega x^2}{\hbar} + i\phi \right), \quad (2)$$

where the accumulated phase difference between the left and right waves is

$$\phi \approx \frac{m\Omega^2X_0^2T}{8\hbar} = \frac{m\Omega^2X_0^2L}{8\hbar v}. \quad (3)$$

Note that the only atom attributes that the phase difference (3) depends on are its inertial mass $m$ and transversal velocity $v$ relative to the vibrating frame; $\omega$ does not appear to leading order as a consequence of the atom assumed to be tightly bound transversely to the frame, i.e., $\omega \gg \Omega$. Putting in some numbers: with $m \sim 10^{-27}$ kg, $\Omega \sim 2\pi \times 10^8$ s$^{-1}$, and $X_0 \sim 10^{-9}$ m, we have $\phi \sim 0.5$ T(μsec). Thus, significant phase shifts result for dwell times in excess of a microsecond.

Equation (3) is to be compared with the measured gravitational phase difference between two interfering neutron paths \(12\): $\phi = mgaL/(hv)$ \(19\), where $m$ is the neutron mass, $g_a$ is the component of gravity in the plane of the neutron paths, $A$ is the area enclosed by the two paths, and $v$ is the neutron velocity. With $\Omega^2X_0$ identified as the acceleration magnitude and $X_0L$ identified as the effective area magnitude in (3), we can see more clearly the equivalence between the acceleration and gravity induced phase shifts.

Interestingly, the measured accumulated phase shift in Ref. \(12\) is of the same order of magnitude as the estimated phase shift following from Eq. (3). This can be understood from the fact that, while the vibrating wire acceleration magnitude is eight orders of magnitude larger than $g \approx 10$ ms$^{-2}$, the effective area to atom velocity ratio magnitude is about eight orders of magnitude smaller than the corresponding ratio for the neutron interferometer. However, in contrast to the much larger, $\sim 10$ centimeter scale neutron interferometer where the acceleration due to gravity is of course well described classically, the micrometer scale, non-inertial vibrating wire frame can also in principle be prepared in a manifest quantum state.

A possibly more fundamental way to understand the phase difference (3) follows from the original observation of de Broglie \(24\) that the phase of a particle’s wave function can be directly expressed in terms of the proper time along the path of the particle. In particular, we...
have 19, 25:
\[ \phi = -\frac{mc^2}{h} \left( \int_{t=0}^{t=T} d\tau_R - \int_{t=0}^{t=T} d\tau_L \right) , \]  
(4)
where \( c \) is the speed of light in vacuum and \( \tau_{L(R)} \) is the proper time elapsed for the atom traveling along the left (right) interferometer path. Since the frame velocity \( \dot{X}_{\text{max}} = \Omega X_0 \approx 1 \text{ ms}^{-1} \) for the above considered parameters, we have \( |v_{L(R)}(t)| \ll c \) and hence \( d\tau_{L(R)} = \sqrt{1 - (v_{L(R)}(t)/c)^2} dt \approx (1 - 1/2 (v_{L(R)}(t)/c)^2) dt \), where \( v_{L(R)}(t) \) is the left (right) path atom velocity relative to the lab frame. Substituting in the approximation \( v_R(t) \approx -X_0 \Omega \sin(\pi t/T) \sin(\Omega t + \varphi_0) \), valid for the condition \( \pi/T \ll \Omega \ll \omega \), Eq. (4) then gives the same result as Eq. (3). Thus, the phase difference (3) can be viewed as a consequence of the “twins paradox” 19, 25, where the right path wave packet bound to its nonmetallic, vibrating frame “ages” less than the left path wave packet during the elapsed lab coordinate time interval \( T \).

Moving on now to treating the vibrating frame as a quantum system, the Schrödinger equation for the composite atom-frame wave function \( \Phi_R(x, t) \) is:
\[ i\hbar \frac{\partial \Phi_R}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 \Phi_R}{\partial x^2} + \frac{1}{2} M \Omega^2 X^2 \Phi_R + \frac{1}{2} m \omega^2 [x - X \sin(\pi t/T)]^2 \Phi_R , \]  
(5)
where we neglect the coupling between the frame and its dissipative environment and \( M \) is the effective motional mass of the frame in the fundamental flexural mode. The atom is assumed to enter the frame initially in its transverse ground state: \( \psi_R(x, 0) = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m \omega x^2}{2 \hbar} \right) \), with \( \Phi_R(x, 0) = \psi_R(x, 0) \psi_R(X, 0) \) for some initial prepared frame state \( \psi_R(X, 0) \).

A potential puzzle concerns that fact that, as a quantum object, the frame also has a phase and hence there may be an ambiguity concerning the part of the phase that “belongs” to the atom. This puzzle is resolved by noting that only the atoms are detected at the interferometer output, so that the frame state must be traced over in the interference term, which can be expressed as (cf. the classical frame interference Eq. (2))
\[ \psi_L^*(x, T) \psi_R(x, T) \text{Tr} \left( \hat{U}_R(T) \hat{\rho}_{\text{frame}}(0) \hat{U}_L^*(T) \right) = \sqrt{\frac{m \omega}{\pi \hbar}} e^{-\frac{m \omega x^2}{2 \hbar}} \left\langle e^{i\phi} \right\rangle . \]  
(6)
Here, we allow for the possibility that the frame is initially in a mixed state, while the unitary operators \( \hat{U}_{L(R)}(T) \) implement the frame evolution over the time interval \( T \) when the atom is either attached (\( R \)) or not attached (\( L \)) to the frame. Formally, we write the accumulated phase difference as an average, \( \left\langle e^{i\phi} \right\rangle \), reflecting the fact that the frame is now in a quantum state.

Equations (5) and (6) may be straightforwardly solved by transforming (5) to instantaneous normal mode coordinates such that (3) becomes separable, and decomposing the initial frame state \( \hat{\rho}_{\text{frame}}(0) \) in terms of a coherent state basis \( |\alpha\rangle \), where \( \alpha = \sqrt{M\Omega/(2\hbar)} X + iP/\sqrt{2MH\Omega} \). Assuming Gaussian solutions and taking advantage of the separation of frequency scales, \( \pi/T \ll \Omega \ll \omega \), we obtain
\[ \left\langle e^{i\phi} \right\rangle \approx \frac{1}{\pi} \int d^2 \alpha \left| \alpha \right| \left| \alpha \right\rangle \langle \alpha | e^{i\omega \dot{M} T / (4M)} \right| e^{i \frac{\Omega}{m \omega^2} (\Omega - \omega) T} . \]  
(7)

Consider the example of a frame initially in a displaced coherent state: \( \hat{\rho}_{\text{frame}}(0) = |\Psi_R(0)\rangle \langle \Psi_R(0)| = |\alpha_0\rangle \langle \alpha_0 | \), where \( \alpha_0 = \sqrt{M\Omega/(2\hbar)} X + iP(0)/\sqrt{2MH\Omega} \), with \( X(0) = X_0 \cos \varphi_0 \) and \( P(0) = -M\Omega X_0 \sin \varphi_0 \). Equation (7) then gives
\[ \left\langle e^{i\phi} \right\rangle \approx e^{i \frac{\omega \dot{M} T}{m \omega^2} |\alpha_0|^2 \Omega T + i \frac{\Omega}{m \omega^2} (\Omega - \omega) T} = e^{i \frac{\omega \dot{M} T}{m \omega^2} + i \frac{\omega \dot{M} T}{m \omega^2} (\Omega - \omega) T} . \]  
(8)

Up to typically negligible back action terms, this coincides with the classical frame accumulated phase difference (3), as we would expect for a coherent frame state.

As another example, consider a frame initially in a Fock state: \( \hat{\rho}_{\text{frame}}(0) = |N\rangle \langle N| , N = 0, 1, 2, \ldots \). Equation (7) then gives
\[ \left\langle e^{i\phi} \right\rangle \approx e^{-i \frac{\omega \dot{M} T}{m \omega^2} \sin(\beta M/2) \sin(\beta \hbar/2 - im \omega \dot{M} T / (8M))} . \]  
(9)

Note that Eq. (9) can be obtained from Eq. (8) simply by replacing the frame coherent state amplitude modulus squared \( |\alpha_0|^2 \) with the frame Fock state number \( N \). However, in contrast to the leading order accumulated phase difference for the coherent state and classical oscillating frame, the phase difference for the Fock state frame depends on the frame mass \( M \).

As our final example, we consider a frame initially in a thermal state: \( \hat{\rho}_{\text{frame}}(0) = Z^{-1} \sum_{N=0}^{\infty} e^{-\beta M (N+1/2)|N\rangle \langle N| . \) In this case, Eq. (7) gives
\[ \left\langle e^{i\phi} \right\rangle \approx e^{-i \frac{\omega \dot{M} T}{m \omega^2} \sin(\beta M/2) \sin(\beta \hbar/2 - im \omega \dot{M} T / (8M))} . \]  
(10)

The interference is suppressed at sufficiently large temperatures \( \beta^{-1} \gg 4MH/(mT) \), i.e., the thermally fluctuating reference frame induces dephasing.

Just as for the classical frame interference, it is interesting to determine whether the quantum frame interference follows from a more fundamental “twins paradox” description, where the proper time of the atom wave packet traversing the vibrating wire frame now develops a quantum uncertainty as a result of the frame being in a quantum state. Formally, the averaged interference term in (6) might be expressed as [cf. Eq. (4)]:
\[ \left\langle e^{i\phi} \right\rangle = \left\langle T e^{-i \frac{\omega \dot{M} T}{m \omega^2} \left[ \int_{t=0}^{t=T} dt R(d\tau_R / dt) - 2 \right]} \right\rangle . \]  
(11)
where $\mathcal{T}$ denotes lab coordinate time ordering and $\hat{\tau}_R$ denotes a quantum proper time operator. A proper time, path integral formulation of $\mathcal{T}$ along the lines of Ref. [11] may be possible, guided by the requirement that the formulation must reproduce expression [7] in the non-relativistic limit. In this respect, treating mesoscopic mechanical resonators as quantum reference frames may yield, via the equivalence principle, possible insights concerning the nature of quantum gravity at low energies [20], in particular the effect of quantum fluctuating space-time on matter wave interference [27, 25].

In the remaining discussion, we outline possible methods for realizing the mesoscopic, vibrating wire interferometer. As was discussed, to realize a phase difference $\phi \sim 1$ with an atom and an acceleration of $10^8 \, g$, one needs a wire frame dwell time of $\sim 1 \, \mu s$. For a nanomechanical resonator with a length of approximately $1 \, \mu m$, this requires an atom velocity of $1 \, \text{ms}^{-1}$ or less. Alternatively, electrons, given their much smaller mass, would require a dwell time approximately 1000 times longer [see Eq. (3)], and hence a velocity of $< 10^{-3} \, \text{ms}^{-1}$ to realize a substantial phase difference in a similar length. As a consequence, atoms appear to be much more favorable for a possible experiment.

Fig. 2 shows a scheme for a possible superfluid interferometer device used to detect quantum phase differences in superfluid helium [29–31]. One arm of the interferometer is interrupted by a 100 nm diameter aperture in a thin silicon nitride membrane, and the other arm a suspended nanochannel mechanical resonator [32] or nanopipe [33] with diameter-to-length dimensions $250 \, \text{nm} \times 1 \, \mu m$. The quantum phase difference at critical velocities through sub-micron apertures is typically $\sim 2\pi \times 10$, which yields a superfluid velocity in the suspended channel of $\sim 1 \, \text{ms}^{-1}$. The quantum phase that is generated through the large acceleration of the suspended channel will produce a mass current through the aperture, which modifies the apparent critical velocity that is measured with an external diaphragm. The nanochannel diameter should be sufficiently narrow to avoid the excitation of transverse acoustic modes at the drive frequency: the speed of first sound in superfluid $^3\text{He}$ is $c_{\text{He}} = 240 \, \text{ms}^{-1}$, yielding an acoustic wavelength $\lambda = c_{\text{He}}/f = 2.4 \, \mu m$ for $f = 100 \, \text{MHz}$. It appears possible to integrate a circuit with these elements on the surface of a chip [34]. Other possible realizations could involve cold atomic clouds or Bose-Einstein condensates steered on the surface of atom-chips [35], or guided through hollow fiber-optic atomic and optical waveguides [26].

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Fig. 2. Superfluid interferometer setup: $D$ is the diaphragm which is used to both impose a chemical potential through hydrostatic pressure and measure the resulting superflow, $A$ is the 100 nm diameter aperture used to monitor the quantum phase difference, $\text{SNC}$ is the suspended nanochannel filled with superfluid $^3\text{He}$ moving through the channel at $1 \, \text{ms}^{-1}$ and accelerating transversely at $10^8 \, g$, driven into motion with an external drive field.
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