Unary FA-presentable binary relations: transitivity and classification results

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ABSTRACT

Automatic presentations, also called FA-presentations, were introduced to extend finite model theory to infinite structures whilst retaining the solubility of fundamental decision problems. A particular focus of research has been the classification of those structures of some species that admit FA-presentations. Whilst some successes have been obtained, this appears to be a difficult problem in general. A restricted problem, also of significant interest, is to ask this question for unary FA-presentations: that is, FA-presentations over a one-letter alphabet. This paper studies unary FA-presentable binary relations.

It is proven that transitive closure of a unary FA-presentable binary relation is itself unary FA-presentable. Characterizations are then given of unary FA-presentable binary relations, quasi-orders, partial orders, tournaments, directed trees and forests, undirected trees and forests, and the orbit structures of unary FA-presentable partial and complete mappings, injections, surjections, and bijections.

1 INTRODUCTION

Automatic presentations, also known as FA-presentations, were introduced by Khoussainov & Nerode [1] to fulfill a need to extend finite

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model theory to infinite structures while retaining the solubility of interesting decision problems. They have been applied to structures such as orders \([2, 3, 4]\) and algebraic structures \([5, 6, 7]\).

One main avenue of research has been the classification of those structures of some species that admit FA-presentations. Classifications are known for finitely generated groups \([5, \text{Theorem 6.3}]\) and cancellative semigroups \([7, \text{Theorem 13}]\), for integral domains (and more generally for rings with identity and no zero divisors) \([6, \text{Corollary 17}]\), for Boolean algebras \([8, \text{Theorem 3.4}]\), and for ordinals \([4]\).

Broadly speaking, there has been more success in classifying algebraic structures than for combinatorial ones. However, it has been possible to classify certain combinatorial structures that admit \textit{unary} FA-presentations (that is, FA-presentations over a one-letter alphabet), including, for example, bijective functions \([9, \text{Theorem 7.12}]\), equivalence relations \([9, \text{Theorem 7.13}]\), linear orders \([9, \text{Theorem 7.15}]\), and graphs \([9, \text{Theorem 7.16}]\).

In this paper, we study unary FA-presentable binary relations using a new diagrammatic representation for unary FA-presentations that we recently developed. We have already successfully deployed this representation in a study of unary FA-presentable algebras \([10]\). This representation allows us to visualize and manipulate elements of a unary FA-presentable relational structure in a way that is more accessible than the corresponding arguments using languages and automata.

We begin by describing the diagrammatic representation of unary FA-presentations in \(\S\ 3\). We first apply it to prove that, given a unary FA-presentable structure with a binary relation \(R\), then that structure augmented by the transitive closure of \(R\) also admits a unary FA-presentation (\textit{Theorem 4.6}). This result is peculiar to \textit{unary} FA-presentable binary relations, because there exist (non-unary) FA-presentable graphs in which reachability is undecidable \([11, \text{Examples 2.4(iv) \& 3.17}]\). As a corollary of this stronger result, one recovers the previously-known result asserting the regularity of the reachability relation for unary FA-presentable undirected graphs of finite degree \([12, \text{Corollary 7.6}]\).

We then turn to classification results. First, we give a technical theorem that classifies all unary FA-presentable binary relations (\textit{Theorem 5.1}). With the aid of some lemmata, this allows us to classify the unary FA-presentable quasi-orders (\textit{Theorem 5.8}), partial orders (\textit{Theorem 5.10}), and tournaments (\textit{Theorem 5.12}).

We then classify the unary FA-presentable directed trees (\textit{Theorem 6.9}) and forests (\textit{Theorem 6.12}) and then undirected trees and forests (\textit{Theorem 6.13}). Finally, we classify, in terms of their the orbit structures, unary FA-presentable mappings (\textit{Theorem 7.4}), injections (\textit{Theorem 7.6}), and surjections (\textit{Theorem 7.5}), and recover the previously-known characterization of unary FA-presentable bijections (\textit{Theorem 7.7}). These results also characterize the unary FA-presentable partial versions of these types of maps.

2 PRELIMINARIES

The reader is assumed to be familiar with the theory of finite automata and regular languages; see \([13, \text{Chs 2–3}]\) for background reading. The empty word (over any alphabet) is denoted \(\varepsilon\).

\textbf{Definition 2.1.} \ Let \(L\) be a regular language over a finite alphabet \(A\). Define,
for $n \in \mathbb{N}$,
\[ L^n = \{ (w_1, \ldots, w_n) : w_i \in L \text{ for } i = 1, \ldots, n \}. \]

Let $\$ \in A$. The mapping $\text{conv} : (A^*)^n \to ((A \cup \{\$\})^*)^n$ is defined as follows. Suppose
\[
\begin{align*}
  w_1 &= w_{1,1}w_{1,2} \cdots w_{1,m_1}, \\
  w_2 &= w_{2,1}w_{2,2} \cdots w_{2,m_2}, \\
  &\vdots \\
  w_n &= w_{n,1}w_{n,2} \cdots w_{n,m_n},
\end{align*}
\]

where $w_{i,j} \in A$. Then $\text{conv}(w_1, \ldots, w_n)$ is defined to be
\[
\big( w_{1,1}, w_{2,1}, \ldots, w_{n,1} \big) \big( w_{1,2}, w_{2,2}, \ldots, w_{n,2} \big) \cdots \big( w_{1,m_1}, w_{2,m_2}, \ldots, w_{n,m_n} \big),
\]
where $m = \max\{m_i : i = 1, \ldots, n\}$ and with $w_{i,j} = \$ \text{ whenever } j > m_i$.

Observe that the mapping $\text{conv}$ maps an $n$-tuple of words to a word of $n$-tuples.

**Definition 2.2.** Let $A$ be a finite alphabet, and let $R \subseteq (A^n^*)^n$ be a relation on $A^*$. Then the relation $R$ is said to be **regular** if
\[
\text{conv}R = \{ \text{conv}(w_1, \ldots, w_n) : (w_1, \ldots, w_n) \in R \}
\]
is a regular language over $(A \cup \{\$\})^n$.

**Definition 2.3.** Let $S = (\{S\}, R_1, \ldots, R_n)$ be a relational structure. Let $L$ be a regular language over a finite alphabet $A$, and let $\phi : L \to S$ be a surjective mapping. Then $(L, \phi)$ is an **automatic presentation** or an **FA-presentation** for $S$ if, for all relations $R \in \{\{=\}, R_1, \ldots, R_n\}$, the relation
\[
\Lambda^R(R, \phi) = \{ (w_1, w_2, \ldots, w_r) \in L^r : R(w_1 \phi, \ldots, w_r \phi) \},
\]
where $r$ is the arity of $R$, is regular.

If $S$ admits an FA-presentation, it is said to be **FA-presentable**.

If $(L, \phi)$ is an FA-presentation for $S$ and the mapping $\phi$ is injective (so that every element of the structure has exactly one representative in $L$), then $(L, \phi)$ is said to be **injective**.

If $(L, \phi)$ is an FA-presentation for $S$ and $L$ is a language over a one-letter alphabet, then $(L, \phi)$ is a **unary** FA-presentation for $S$, and $S$ is said to be **unary FA-presentable**.

Every FA-presentable structure admits an injective **binary** FA-presentation; that is, where the language of representatives is over a two-letter alphabet; see [1, Corollary 4.3] and [9, Lemma 3.3]. Therefore the class of binary FA-presentable structures is simply the class of FA-presentable structures. However, there are many structures that admit FA-presentations but not unary FA-presentations: for instance, any finitely generated virtually abelian group is FA-presentable [5, Theorem 8], but unary FA-presentable groups must be finite [9, Theorem 7.19]. Thus there is a fundamental difference between unary FA-presentable structures and all other FA-presentable structures.

**Definition 2.4.** If $(L, \phi)$, where $L \subseteq A^*$, is an injective unary FA-presentation for a structure $S$, and $s$ is an element of $S$, then $\ell(s)$ denotes the length of the unique word $w \in L$ with $w \phi = s$. [Notice that $A^\ell(s) = s A^{-1}$ for all elements $s$ of $S$.]
The fact that a tuple of elements \((s_1, \ldots, s_n)\) of a structure \(S\) satisfies a first-order formula \(\theta(x_1, \ldots, x_n)\) is denoted \(S \models \theta(s_1, \ldots, s_n)\).

**Proposition 2.5** ([1, Theorem 4.4]). Let \(S\) be a structure with an FA-presentation \((L, \phi)\). For every first-order formula \(\theta(x_1, \ldots, x_n)\) over the structure, the relation

\[
\Lambda(\theta, \phi) = \{(w_1, \ldots, w_n) \in L^n : S \models \theta(w_1 \phi, \ldots, w_n \phi)\}
\]

is regular.

Proposition 2.5 is fundamental to the theory of FA-presentations and will be used without explicit reference throughout the paper.

The following important result shows that in the case of unary FA-presentations for infinite structures, we can assume that the language of representatives is the language of all words over a one letter alphabet, and also that the map into the domain of the structure is injective:

**Theorem 2.6** ([14, Theorem 3.1]). Let \(S\) be an infinite relational structure that admits a unary FA-presentation. Then \(S\) has an injective unary FA-presentation \((a^*, \psi)\).

We now gather some miscellaneous preliminary results that we will use later in the paper:

**Theorem 2.7** ([9, Theorem 7.13]). Let \(X\) be a set and \(\rho\) an equivalence relation on \(X\). Then \((X, \rho)\) is unary FA-presentable if and only if there are only finitely many infinite \(\rho\)-equivalence classes and there is a bound on the cardinality of the finite \(\rho\)-equivalence classes.

The disjoint union of a family of structures \(S^{(i)} = (S^{(i)}_1, \sigma_1^{(i)}, \ldots, \sigma_n^{(i)})\) with the same signature (where \(i\) ranges over an index set \(I\)) is the structure

\[
\bigcup_{i \in I} S^{(i)}_1 \bigcup_{i \in I} \sigma_1^{(i)} \bigcup_{i \in I} \sigma_1^{(i)} \ldots \bigcup_{i \in I} \sigma_n^{(i)},
\]

where \(\sqcup\) denotes disjoint union as sets.

**Lemma 2.8** ([9, Proposition 7.6(ii)]). The disjoint union of two unary FA-presentable structures with the same signature is unary FA-presentable.

**Lemma 2.9.** The disjoint union of countably many isomorphic copies of a finite structure is unary FA-presentable.

**Proof of 2.9.** Let \(S\) be a finite structure. Suppose the domain \(S\) of \(S\) contains \(n\) elements. Then there is a bijection \(\psi : \{a^0, \ldots, a^{n-1}\} \to S\). For any relation \(\sigma\) of \(S\), the relation \(\Lambda(\sigma, \psi)\) is finite and thus regular.

Define a map \(\phi\) from \(a^*\) to the (domain of) the disjoint union of countably many copies of \(S\) by letting \(a^{n+1} \phi\) be the element corresponding to \(a^i \psi\) in the \(i\)-th copy of \(S\), where \(0 \leq j < n\). Then for any relation \(\sigma\) of the disjoint union,

\[
\Lambda(\sigma, \phi) = (a^n, a^n)^* \Lambda(\sigma, \psi),
\]

and so is regular. Thus \((a^*, \phi)\) is a unary FA-presentation for the the disjoint union of countably many copies of \(S\).
3 PUMPING AND DIAGRAMS

This section develops a diagrammatic representation for unary FA-presentations. Although we only discuss how this representation works for binary relations, it also applies more generally to unary FA-presentations for arbitrary relational structures; see [10, §4] for details.

Let $\mathfrak{S}$ be a structure with a binary relation $R$, and suppose $\mathfrak{S}$ is unary FA-presentable. By Theorem 2.6, there is an injective unary FA-presentable structure $(a^*, \phi)$ for $\mathfrak{S}$. Let $\mathfrak{A}$ be a deterministic 2-tape synchronous automaton recognizing $\Lambda(R, \phi)$. Let us examine the structure of the automaton $\mathfrak{A}$. For ease of explanation, view $\mathfrak{A}$ as a directed graph with no failure states: $\mathfrak{A}$ fails if it is in a state and reads a symbol that does not label any outgoing edge from that state.

Since $\mathfrak{A}$ recognizes words in $\text{conv}(a^2)$, it will only successfully read words lying in $(a, a)^* ((a, \$)^* \cup (\$, a)^*)$. Thus an edge labelled by $(a, a)$ leads to a state whose outgoing edges can have labels $(a, a), (a, \$), (\$, a)$. However, an edge labelled by $(a, \$)$ leads to a state all of whose outgoing edges are labelled by $(a, \$)$. In fact, the determinism of $\mathfrak{A}$ ensures there is at most one such outgoing edge. Similarly, an edge labelled by $(\$, a)$ leads to a state with either no outgoing edges or a single outgoing edge labelled by $(\$, a)$.

Since $\mathfrak{A}$ is deterministic, while it successfully reads pairs $(a, a)$ it follows a uniquely determined path which, if the string of such pairs is long enough, will form a uniquely determined loop. This loop, if it exists, is simple. From various points along this loop and the path leading to it, paths labelled by $(a, \$)$ and $(\$, a)$ may ‘branch off’. In turn, these paths, if they are long enough, lead into uniquely determined simple loops. Figure 1 shows an example.

Let $D$ be a multiple of the lengths of the loops in $\mathfrak{A}$ that also exceeds the number of states in $\mathfrak{A}$.

Let $\mathfrak{A}$ have initial state $q_0$ and transition function $\delta$. Consider a word $uvw \in \text{convL}(\mathfrak{A})$, where $v = b^\beta$ for some $b \in \{a, \$\}^2$ and $\beta \geq D$. Suppose that $(q_0, u) \delta = q$. When $\mathfrak{A}$ is in state $q$ and reads $v$, it completes a loop before finishing reading $v$. So $v$ factorizes as $v'v''v'''$, with $|v''| > 0$, such that $(q, v') \delta = (q, v''v''') \delta = q'$. Assume that $|v'|$ is minimal, so that $q'$ is the first state on the loop that $\mathfrak{A}$ encounters while reading $v$. Assume further that $|v''|$ is minimal, so that $\mathfrak{A}$ makes exactly one circuit around the loop while reading $v''$. Now, by definition, $D$ is a multiple of $|v''|$. Let $m = D/|v''|$. So

![Figure 1](image-url)
\(|v(v')^{m+1}v''| = |v|+D\). By the pumping lemma, \(uv'(v')^{m+1}v''w \in \text{conv} L(\mathcal{A})\).

Consider what this means in terms of the pair \(\bar{p} = (a^{p_1}, a^{p_2})\) such that \(\text{conv}(\bar{p}) = uvw\). Since \(v' \in b^*\), it follows that \(uv'(v')^{m+1}v''w = \text{conv}(a^{p_1+q_1}, a^{p_2+q_2})\), where

\[
q_j = \begin{cases} 
0 & \text{if } p_j \leq |u| \\
D & \text{if } p_j \geq |uv|.
\end{cases}
\]

(Note that either \(p_j \leq |u|\) or \(p_j \geq |uv|\) since \(v \in b^*\) for a fixed pair \(b \in \{a, \$\}^2\)). Therefore we have the following:

**Pumping rule 1.** If the components of a pair in \(\Lambda(R, \phi)\) can be partitioned into those that are of length less than \(l \in \mathbb{N}\) and those that have length at least \(l + D\), then [the word encoding] this pair can be pumped so as to increase by \(D\) the lengths of those components that are at least \(l + D\) letters long and yield another [word encoding a] pair in \(\Lambda(R, \phi)\).

(Notice that this also applies when both components have length at least \(D\); in this case, set \(l = 0\).)

With the same setup as above, suppose \(|v| \geq 2D\). Then \(\mathcal{A}_l\) must follow the loop labelled by \(v''\) starting at \(q'\) at least \(m = D/|v''|\) times. That is, \(v\) factorizes as \(v'(v'')^m\). By the pumping lemma, \(uv'v'' \in \text{conv} L(\mathcal{A}_l)\) and \(|v'v''| = |v| - D\). Therefore, we also have the following:

**Pumping rule 2.** If the components of a pair in \(\Lambda(R, \phi)\) can be divided into those that are of length less than \(l \in \mathbb{N}\) and those that have length at least \(l + 2D\), then [the word encoding] this pair can be pumped so as to decrease by \(D\) the length of those components that are at least \(l + 2D\) letters long and yield another [word encoding a] pair in \(\Lambda(R, \phi)\).

This ability to pump so as to increase or decrease lengths of components of a pair by a constant \(D\) lends itself to a very useful diagrammatic representation of the unary FA-presentation \((a^*, \phi)\). Consider a grid of \(D\) rows and infinitely many columns. The rows, from bottom to top, are \(B[0], \ldots, B[D-1]\). The columns, starting from the left, are \(C[0], C[1], \ldots\). The point in column \(C[x]\) and row \(B[y]\) corresponds to the word \(a^{xD+y}\). For example, in the following diagram, the distinguished point is in column \(C[3]\) and row \(B[2]\) and so corresponds to \(a^{3D+2}\):

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B[0] B[1] B[2] B[3] B[4] B[D-1]
C[0] C[1] C[2] C[3] C[4] C[5] C[6]
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The power of such diagrams is due to a natural correspondence between pumping as in Pumping rules 1 and 2 and certain simple manipulations of pairs of points in the diagram. Before describing this correspondence, we must set up some notation. We will not distinguish between a point in the grid and the word to which it corresponds. The columns are ordered in the obvious way, with \(C[x] < C[x']\) if and only if \(x < x'\).
For any element \( u \in \alpha^* \), let \( b(u) \) be the index of the row containing \( u \) and let \( c(u) \) be the index of the column containing \( u \). For brevity, write \( B[u] \)
for \( B[b(u)] \) and \( C[u] \) for \( C[c(u)] \). Extend the notation for intervals on \( \mathbb{N} \) to intervals of contiguous columns. For example, for \( x, x' \in \mathbb{N} \) with \( x \leq x' \), let \( C[x, x'] \)
denotes the set of elements in columns \( C[x], \ldots, C[x'-1] \), and \( C(x, \infty) \)
denotes the set of elements in columns \( C[x+1], C[x+2], \ldots \).

Define for every \( n \in \mathbb{Z} \) a partial map \( \tau_n : \alpha^* \rightarrow \alpha^* \), where \( \alpha^k \tau_n \)
is defined to be \( \alpha^{k+nD} \) if \( k+nD \geq 0 \) and is otherwise undefined. Notice that if \( n \geq 0 \), the
map \( \tau_n \) is defined everywhere. In terms of the diagram, \( \alpha^k \tau_n \)
is the element obtained by shifting \( \alpha^k \) to the right by \( n \) columns if \( n \geq 0 \) and to the left by
\( -n \) columns if \( n < 0 \). The values of \( k \) and \( n < 0 \) for which \( \alpha^k \tau_n \)
are undefined are precisely those where shifting \( \alpha^k \) to the left by \( -n \) columns would carry it
beyond the left-hand edge of the diagram.

Now, elements of \( \Lambda(R, \phi) \) are pairs of words, and thus can be viewed as
arrows in the diagram.

Consider a pair \((p, q)\) in \( \Lambda(R, \phi) \), viewed as an arrow in the diagram. If
the arrow \((p, q)\) neither starts nor ends in \( \mathbb{N} \), so that both \( p \)
and \( q \) are of
length at least \( D \) and so the word encoding \((p, q)\) can be pumped before
both components in accordance with Pumping rule 1. This corresponds to
shifting both components rightwards by one column. This rightward shifting
of components can be iterated arbitrarily many times to yield new arrows.

Similarly, if the arrow \((p, q)\) starts and ends in non-adjacent columns, then
\( p \) and \( q \) differ in length by at least \( D \) and hence the word encoding \((p, q)\)
can be pumped between these two components in accordance with Pumping
rule 1. This corresponds to shifting the rightmost of \( p \) or \( q \) rightwards by one
column. This rightward shifting of arrows can be iterated arbitrarily many
times to yield new arrows.

Hence we have the following diagrammatic version of Pumping rule 1:

**Arrow rule 1.** Consider an element \((p, q)\) of \( \Lambda(R, \phi) \), viewed as an arrow in
the diagram.

1. If \((p, q)\) neither starts nor ends in \( \mathbb{N} \), then for any \( k \in \mathbb{N} \) the arrow
\((p \tau_k, q \tau_k)\) obtained by shifting \((p, q)\) right by \( k \) columns also lies in \( \Lambda(R, \phi) \).
2. If \( c(p) > c(q) + 1 \), then for any \( k \in \mathbb{N} \) the arrow \((p \tau_k, q)\) obtained by
shifting \( p \) right by \( k \) columns also lies in \( \Lambda(R, \phi) \).
3. If \( c(q) > c(p) + 1 \), then for any \( k \in \mathbb{N} \) the arrow \((p, q \tau_k)\) obtained by
shifting \( q \) right by \( k \) columns also lies in \( \Lambda(R, \phi) \).

On the other hand, if the arrow \((p, q)\) neither starts nor ends in \( \mathbb{N} \),
then \( p \) and \( q \) are both of length at least \( 2D \) and so the word encoding \((p, q)\)
can be pumped before both components in accordance with Pumping rule 2.
This corresponds to shifting both components leftwards by one column. This
leftward shifting can be iterated to yield new arrows for as long as neither \( p \)
or \( q \) lies in \( \mathbb{N} \).

Similarly, if \( c(p) \) and \( c(q) \) differ by at least \( 2 \), then \( p \) and \( q \) differ in length
by at least \( 2D \) and hence the word encoding \((p, q)\) can be pumped between
these two components in accordance with Pumping rule 2. This corresponds
to shifting the rightmost of \( p \) or \( q \) leftwards by one column. This leftward
shift of one end of the arrow can be iterated to yield new arrows for as long
as there are at least two columns between \( p \) and \( q \).

Hence we have the following diagrammatic version of Pumping rule 2:
ARROW RULE 2. Consider an element \((p, q)\) of \(\Lambda(R, \phi)\) viewed as an arrow in the diagram.

1. If \((p, q) \in C[h, \infty)\), then for any \(k \in \mathbb{N}\) with \(0 < k < h\) the arrow \((p\tau_{-k}, q\tau_{-k})\) obtained by shifting \((p, q)\) left by \(k\) columns also lies in \(\Lambda(R, \phi)\).

2. If \(c(p) > c(q) + h\), then for any \(k \in \mathbb{N}\) with \(0 < k < h\) the arrow \((p\tau_{-k}, q)\) obtained by shifting \(p\) left by \(k\) columns also lies in \(\Lambda(R, \phi)\).

3. If \(c(q) > c(p) + h\), then for any \(k \in \mathbb{N}\) with \(0 < k < h\) the arrow \((p, q\tau_{-k})\) obtained by shifting \(q\) left by \(k\) columns also lies in \(\Lambda(R, \phi)\).

An arrow from \(p\) to \(q\), where \(|c(p) - c(q)| \leq 1\) is called a short arrow. Any other arrow is called a long arrow. In both Arrow rules 1 and 2, case 1 applies to both long and short arrows; cases 2 and 3 apply only to long arrows. Hence, as shown in Figure 2, the two ends of a short arrow maintain the same relative position when Arrow rules 1 and 2 are applied; those of a long arrow need not.

4 TRANSITIVE CLOSURE

Consider an FA-presentation \((L, \phi)\) for a relational structure \(S\) that includes a binary relation \(R\). Let \(R_1\) and \(R_2\) be, respectively, the reflexive and symmetric closures of \(R\):

\[
(x, y) \in R_1 \iff ((x, y) \in R) \lor (x = y);
(x, y) \in R_2 \iff ((x, y) \in R) \lor ((y, x) \in R).
\]

Then since \(R_1\) and \(R_2\) are defined by first-order formulae, Proposition 2.5 shows that \(\text{conv}\Lambda(R_1, \phi)\) and \(\text{conv}\Lambda(R_2, \phi)\) are regular.

In contrast, the transitive closure \(R^+\) is not defined by a first-order formula and so Proposition 2.5 does not apply. Indeed, since there exist FA-presentable directed graphs where reachability is undecidable [11, Examples 2.4(iv) & 3.17], \(\text{conv}\Lambda(R^+, \phi)\) is not regular in the case of general FA-presentations.

However, this section is dedicated to showing that, in the particular case of unary FA-presentable structures, \(\text{conv}\Lambda(R^+, \phi)\) is regular. Theorem 4.1 below
proves that \( \text{conv}\Lambda(R^*, \phi) \) is regular, where \( R^* \) is the reflexive and transitive closure of \( R \). The corresponding result for \( R^+ \) can be proved by a similar method. Notice that Theorem 4.1 considerably strengthens both [9, Lemma 7.10], which essentially states (in different language) that the reflexive and transitive closure of a unary FA-presentable unary function is unary FA-presentable, and [12, Corollary 7.6], which states that in a unary FA-presentable undirected graph of finite degree, the reachability relation is regular.

**Theorem 4.1.** Let \( S \) be a structure admitting an injective unary FA-presentation \( (\alpha^*, \phi) \). Let \( R \) be some binary relation in the signature of \( S \). Then \( \Lambda(R^*, \phi) \) is regular, where \( R^* \) denotes the reflexive and transitive closure of \( R \). Hence \( S \) augmented by \( R^* \) is also unary FA-presentable.

**Proof of 4.1.** Suppose the diagram for \( (\alpha^*, \phi) \) has \( D \) rows. The relation \( R \) is binary, so elements of \( \Lambda(R, \phi) \) may be viewed as arrows between points in the diagram. Two points will then lie in \( \Lambda(R^*, \phi) \) if and only if they are linked by a directed path (possibly of length zero). Thus, throughout the proof, we will reason mainly about arrows and directed paths.

Informally, the overall strategy is to break up a path from \( p \) to \( q \) into three parts: a subpath from \( p \) to the minimum column the path visits, a subpath from one vertex of this column to another, and a subpath from this column to \( q \), and then to replace these subpaths with subpaths that are either short or that can be broken into segments between columns at most \( 2D + 2 \) apart in a way that makes the replacement subpath recognizable by an automaton. This constant \( 2D + 2 \) becomes vital only later, in a pumping argument in the proof of Lemma 4.4 below, but it is introduced immediately because certain other constants are defined in terms of it.

For every \( p, q \in \mathbb{C}[0, 2D + 2] \) such that there is a directed path from \( p \) to \( q \), fix some such path \( \alpha_{p, q} \). Let \( k_1 \) be the maximum of the lengths of the various paths \( \alpha_{p, q} \).

**Lemma 4.2.** There is a constant \( k_2 \) with the following property: for all points \( p \) and \( q \) with \( |c(p) - c(q)| \leq 2D + 2 \), if there is a directed path from \( p \) to \( q \) that does not visit the column \( C[0] \), then there is a directed path from \( p \) to \( q \) of length at most \( k_2 \) that does not visit \( C[0] \).

**Proof of 4.2.** For convenience, let \( h = 2D + 2 \). Consider all pairs of distinct points \( p', q' \) in \( C[1, h + 1] \). For each such pair, consider whether there is some \( i \in \mathbb{N}^0 \) such that there is a directed path from \( p'\tau_i \) to \( q'\tau_i \) that does not visit \( C[0] \). If such an \( i \) exists, let \( i(p', q') \) be the minimum such \( i \) and let \( \alpha_{p', q'} \) be a directed path from \( p\tau_i(p', q') \) to \( q\tau_i(p', q') \). Let \( k_2 \) be the maximum of the lengths of the various paths \( \alpha_{p', q'} \).

Now let \( p \) and \( q \) be arbitrary with \( c(p) \leq c(q) \leq c(p) + h \). The case where \( c(q) \leq c(p) \leq c(q) + h \) is similar. Suppose there is a directed path from \( p \) to \( q \) that does not visit \( C[0] \). Let \( p' = p\tau_{c(p) - 1} \) and \( q' = q\tau_{c(p) - 1} \). Then \( p' \) and \( q' \) lie in \( C[1, h + 1] \). Since there is a directed path from \( p \) to \( q \), the quantity \( i(p', q') \) and the path \( \alpha_{p', q'} \) are defined. Furthermore, the minimality of \( i(p', q') \) ensures that \( i(p', q') \leq c(p) - 1 \). Let \( j = c(p) - 1 - i(p', q') \). Then the directed path \( (\alpha_{p', q'}\tau_j) \) is defined by Arrow rule 1 and runs from \( p \) to \( q \). (See Figure 3.) So there is a directed path from \( p \) to \( q \) of length at most \( k_2 \) that does not visit \( C[0] \).
FIGURE 3. A path of length at most \( k_2 \) from \( p \) to \( q \) is obtained by shifting the path \( \alpha_{p',q'} \) to the right by \( j = c(p) - 1 - i(p',q') \) columns. Notice that the path \( \alpha_{p',q'} \) may visit vertices to the left of \( p \tau_{i(p',q')} \) or the right of \( q \tau_{i(p',q')} \); the only thing that is guaranteed is that its length is at most \( k_2 \) and that it does not visit any point of \( C[0] \).

Let \( k = \max\{k_1, k_2\} + 1 \). Let

\[
S = R^{k} = E \cup R \cup (R \circ R) \cup (R \circ R \circ R) \cup \ldots \cup (R \circ \cdots \circ R),
\]

where \( E \) is the equality relation. Then \( S \) is first-order definable and so \( \Lambda(S, \phi) \) is regular.

Define the relation

\[
V = \{(t_0\phi, t_n\phi) : (\exists n \in \mathbb{N})(\exists t_1, \ldots, t_n \in a^*) (\forall i \in \{0, \ldots, n-1\}) (|t_i| < |t_{i+1}| \land (t_i\phi, t_{i+1}\phi) \in S)\}, \tag{4.1}
\]

and dually

\[
V' = \{(t_0\phi, t_n\phi) : (\exists n \in \mathbb{N})(\exists t_1, \ldots, t_n \in a^*) (\forall i \in \{0, \ldots, n-1\}) (|t_{i+1}| < |t_i| \land (t_{i+1}\phi, t_{i}\phi) \in S)\}, \tag{4.2}
\]

Notice that \( V \subseteq R^+ \) and \( V' \subseteq R^+ \), and that (4.1) and (4.2) are not first-order definitions because in the quantifications \( (\exists t_1, \ldots, t_n - 1) \), the number of variables is arbitrary. Notice further that it follows immediately from their definitions that \( V \) and \( V' \) are transitive.

[The definitions of \( V \) and \( V' \) could be formulated using quantification over the domain of \( S \), but (4.1) and (4.2) are notationally more useful since we will reason using the words \( t_i \in a^* \).]

**Lemma 4.3.** The relations \( \Lambda(V, \phi) \) and \( \Lambda(V', \phi) \) are regular.

**Proof of 4.3.** We prove the regularity of \( \Lambda(V, \phi) \); the argument for \( \Lambda(V', \phi) \) is symmetric.

Let \( \mathcal{A} \) be an automaton recognizing \( \text{conv}\Lambda(S, \phi) \). Construct an automaton \( \mathcal{B} \) recognizing \( \text{conv}\Lambda(V, \phi) \) as follows.

The automaton \( \mathcal{B} \) reads words of the form \( \text{conv}(a^m, a^{m+n}) \) for \( m \in \mathbb{N} \), \( n \in \mathbb{N} \); that is, where the word on the left-hand track is shorter than the one on the right-hand track. So the automaton need only read input symbols from \( \{(a, a), (S, a)\} \).

The operation of the automaton \( \mathcal{B} \) consists in running two copies \( \mathcal{A}_1, \mathcal{A}_2 \) of the automaton \( \mathcal{A} \) simultaneously. The copy \( \mathcal{A}_1 \) faithfully simulates the operation of \( \mathcal{A} \). The second copy \( \mathcal{A}_2 \) always follows the \( (a, a) \) transition, regardless of whether the input is \( (a, a) \) or \( (S, a) \). Furthermore, whenever \( \mathcal{A}_1 \) is in a state corresponding to an accept state \( f \) of \( \mathcal{A} \), and \( \mathcal{A}_2 \) is in a state corresponding
to any state \( q \) of \( \mathfrak{A} \), the automaton \( \mathfrak{B} \) can make an \( \epsilon \)-transition so that both \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are in states corresponding to \( q \). The accept states of \( \mathfrak{B} \) are those where \( \mathfrak{A}_1 \) is in a state corresponding to an accept state of \( \mathfrak{A} \), and \( \mathfrak{A}_2 \) is in any state.

Prove that \( L(\mathfrak{B}) \subseteq \text{conv} \Lambda(V, \phi) \) as follows: Suppose \( \mathfrak{B} \) reads \( h_0 \) symbols \((a, a)\) before switching to reading symbols \((\$, a)\). Suppose it makes \( n - 1 \) of the \( \epsilon \)-transitions described above, after having read a total of \( h_1, h_2, \ldots, h_{n-1} \) symbols \((\$, a)\), and that it accepts after having read a total of \( h_n \) symbols.

Now, if \( \mathfrak{B} \) makes an \( \epsilon \)-transition so that \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are both in state \( q \), then if it makes another \( \epsilon \)-transition without reading any input, then it does not change its state, for both \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) continue to have state \( q \). Similarly, if \( \mathfrak{B} \) makes an \( \epsilon \)-transition while reading symbols \((a, a)\), it does not change its state. Therefore \( \mathfrak{B} \) can read the same input by not making any \( \epsilon \)-transitions while reading symbols \((a, a)\), and not making more than one consecutive \( \epsilon \)-transition. Hence we can assume without loss that \( h_0 < h_1 < \ldots < h_n \). For each \( i \in \{0, \ldots, n\} \), set \( t_i = a^{h_i} \).

After having read the word \((a, a)^{h_0}(\$, a)^{h_1-h_0}\), the automaton \( \mathfrak{B} \) makes an \( \epsilon \)-transition. So, by the construction of \( \mathfrak{B} \), the first simulated copy of \( \mathfrak{A} \) must have been in an accept state. Hence \((a, a)^{h_0}(\$, a)^{h_1-h_0} \in L(\mathfrak{A})\) and so \((t_0 \phi, t_1 \phi) \in S\).

Immediately after making the \( \epsilon \)-transition after having read a total of \( h_i \) symbols (where \( i \in \{1, \ldots, n-1\} \)), the first simulated copy of \( \mathfrak{A} \) is in the state \( \mathfrak{A} \) enters after having read \((a, a)^{h_i}\). The automaton \( \mathfrak{B} \) reads \((\$, a)^{h_{i+1}-h_i}\) and then makes another \( \epsilon \)-transition or accepts; in either case the first simulated copy of \( \mathfrak{A} \) is in an accept state. Thus \( \mathfrak{A} \) must accept \((a, a)^{h_i}(\$, a)^{h_{i+1}-h_i} \in L(\mathfrak{A})\) and so \((t_i \phi, t_{i+1} \phi) \in S\).

Thus \((t_i \phi, t_{i+1} \phi) \in S\) and \(|t_i| < |t_{i+1}|\) for all \( i \in \{0, \ldots, n-1\} \). So \((t_0 \phi, t_n \phi) \in V\).

Define \( T \) to consist of all pairs \((p\phi, q\phi)\) such that there is a directed path of non-zero length from \( p \) to \( q \) in which all visited points except the first lie in \( C(p, \infty) \). (This entails \( c(q) > c(p) \).

Define \( U \) to consist of all pairs \((p\phi, q\phi)\) such that there is a directed path from \( p \) to \( q \) in which all visited points except the last lie in \( C(q, \infty) \). (This entails \( c(p) > c(q) \).

**Lemma 4.4.** \( T \subseteq V \) and \( U \subseteq V' \).

**Proof of 4.4.** We prove that \( T \subseteq V \). A symmetric argument proves \( U \subseteq V' \).

We are required to show that \((p\phi, q\phi) \in T \implies (p\phi, q\phi) \in V \). We will prove this statement via induction on \( c(q) - c(p) \).

First, if \( c(q) - c(p) \leq 2D + 2 \), then there is a directed path from \( p \) to \( q \) of length at most \( k \) (by the definition of \( k \) in terms of \( k_1 \) and \( k_2 \)). Hence \((p\phi, q\phi) \in S\). Since \(|p| < |q|\), it follows from (4.1) that \((p\phi, q\phi) \in V \).
So suppose \( c(q) - c(p) = h > 2D + 2 \) and that the result holds for all \( p', q' \) with \( c(q') - c(p') < h \). Let \( \alpha \) be a directed path from \( p \) to \( q \) in which all visited points except the first lie in \( C(p, \infty) \), as per the definition of \( T \). There are several cases to consider:

1. The path \( \alpha \) visits some point in \( C(p, q) \). Let \( x \) be the last such point on \( \alpha \) in this range. Let \( \beta \) be the subpath of \( \alpha \) up to and including this visit to \( x \), and let \( \gamma \) be the subpath from \( x \) to \( q \). Then \( \beta \) is a directed path that shows that \( (p\phi, x\phi) \in T \), and \( \gamma \) is a directed path that shows that \( (x\phi, q\phi) \in T \), since the choice of \( x \) ensures that \( \gamma \) never visits \( C[0, x] \) again. Hence, since \( c(x) - c(p) \) and \( c(q) - c(x) \) are both less than \( h \), the induction hypothesis applies to show that \( (p\phi, x\phi) \in V \) and \( (x\phi, q\phi) \in V \). Since \( V \) is transitive, it follows that \( (p\phi, q\phi) \in V \).

2. The path \( \alpha \) does not visit any point in \( C(p, q) \). Then the first arrow in \( \gamma \) is a long arrow that runs from \( p \) to some point \( x \in C(q, \infty) \). Now consider two sub-cases:

   (a) \( c(x) - c(q) \leq 2D + 2 \). Then there is a directed path from \( x \) to \( q \) of length at most \( k_2 \). Hence there is a path from \( p \) to \( q \) of length at most \( k \) (since \( k \geq k_2 + 1 \)). Hence \( (p\phi, q\phi) \in S \). Since \( |p| < |q| \), it follows from (4.1) that \( (p\phi, q\phi) \in V \).

   (b) \( c(x) - c(q) > 2D + 2 \). The subpath \( \beta \) from \( x \) to \( q \) has to visit or bypass the \( 2D + 2 \) columns in \( C(q, x) \). One of two cases must hold: either the subpath \( \beta \) includes some arrow between two points \( s \) and \( t \) with \( c(t) - c(s) > 2 \), or the subpath \( \beta \) visits at least \( D + 1 \) columns between \( C[x] \) and \( C[q] \). Consider each of these sub-sub-cases in turn; in both we will construct a new directed path \( \alpha' \) from \( p \) to \( q \):

   i. The subpath \( \beta \) includes some arrow between points \( s \) and \( t \) with \( c(t) - c(s) > 2 \). Let \( \gamma \) be the subpath of \( \beta \) from \( x \) to \( s \) and let \( \delta \) be the subpath of \( \beta \) from \( t \) to \( q \). Since the path \( \beta \) does not visit any point in \( C[0, q] \) and \( c(q) > 2D + 2 \), the path \( (\gamma)\tau_{-1} \) exists by \textbf{Arrow rule 2}. Let \( \alpha' \) be the path formed by concatenating the arrow from \( p \) to \( xt_{-1} \) (which exists by \textbf{Arrow rule 2}), the path \( (\gamma)\tau_{-1} \), the arrow from \( st_{-1} \) to \( t \) (which exists by \textbf{Arrow rules 1 and 2}), and the path \( \delta \).

   ii. The subpath \( \beta \) visits at least \( D + 1 \) columns between \( C[x] \) and \( C[q] \). Therefore, since there are only \( D \) distinct rows in the diagram, the pigeonhole principle shows that the subpath \( \beta \) visits two points \( s \) and \( t \) (in that order) with \( c(x) \geq c(s) > c(t) \geq c(q) \) and \( b(s) = b(t) \); furthermore, we can choose \( s \) and \( t \) with \( c(s) - c(t) \leq 2D + 2 \). Let \( \gamma \) be the subpath of \( \beta \) from \( x \) to \( s \) and let \( \delta \) be the subpath of \( \beta \) from \( t \) to \( x \). Since the path \( \beta \) does not visit any point in \( C[0, q] \) and \( c(q) > 2D + 2 \), the path \( (\gamma)\tau_{-c(s) - c(t)} \) exists by \textbf{Arrow rule 2}. Let \( \alpha' \) by the path formed by concatenating the arrow from \( p \) to \( xt_{-c(s) - c(t)} \) (which exists by \textbf{Arrow rule 2}), the path \( (\gamma)\tau_{-c(s) - c(t)} \), and the path \( \delta \).

In either sub-sub-case, the path \( \alpha' \) also runs from \( p \) to \( q \), but the first arrow of \( \alpha' \) ends at a point \( x' \) where \( c(x) > c(x') \geq c(x') - 2D - 2 \). So replacing \( \alpha \) by \( \alpha' \) and iterating this process eventually yields a path \( \alpha \) from \( p \) to \( q \) where \( c(x) - c(q) \leq 2D + 2 \), reducing this sub-case to sub-case (a).
Let \[ W = S \cup (S \circ V) \cup (V' \circ S) \cup (V' \circ S \circ V). \] Notice that \( \Lambda(W, \phi) \) is regular and that \( W \subseteq R^* \).

**Lemma 4.5.** Suppose there is a directed path from \( p \) to \( q \). Then \( (p\phi, q\phi) \in W \).

*Proof of 4.5.* Let \( \alpha \) be a directed path from \( p \) to \( q \). Let \( i \in \mathbb{N}^0 \) be maximal such that all points on \( \alpha \) lie in \( C[i, \infty) \). Let \( x \) be the point in \( C[i] \) that \( \alpha \) visits first, and let \( \beta \) be the subpath of \( \alpha \) from \( p \) to \( x \). Let \( y \) be the point in \( C[i] \) that \( \alpha \) visits last, and let \( \gamma \) be the subpath of \( \alpha \) from \( y \) to \( q \).

Then there is a path from \( x \) to \( y \) of length at most \( k \) (by the definition of \( k \) in terms of \( k_1 \) and \( k_2 \)), and so \( (x\phi, y\phi) \in S \). If \( p \neq x \), then \( \beta \) is a path of non-zero length from \( p \) to \( x \), every point of which, except the last, lies in \( C(i, \infty) \) by the choice of \( x \); hence \( (p\phi, x\phi) \in U \). If \( y \neq q \), then \( \gamma \) is a path of non-zero length from \( y \) to \( q \), every point of which, except the first, lies in \( C(i, \infty) \); hence \( (y\phi, q\phi) \in T \).

Therefore, there are four cases:

1. \( p = x \) and \( y = q \). Then \( (p\phi, q\phi) = (x\phi, y\phi) \in S \);
2. \( p \neq x \) and \( y = q \). Then \( (p\phi, q\phi) = (p\phi, y\phi) = (p\phi, x\phi) \circ (x\phi, y\phi) \in U \circ S \);
3. \( p = x \) and \( y \neq q \). Then \( (p\phi, q\phi) = (x\phi, q\phi) = (x\phi, y\phi) \circ (y\phi, q\phi) \in S \circ T \);
4. \( p \neq x \) and \( y \neq q \). Then \( (p\phi, q\phi) = (p\phi, x\phi) \circ (x\phi, y\phi) \circ (y\phi, q\phi) \in U \circ S \circ T \).

Since \( T \subseteq V \) and \( U \subseteq V' \) by Lemma 4.4 and its dual, in each case \( (p\phi, q\phi) \in W \).

We can now complete the proof of Theorem 4.1. As noted above, \( W \subseteq R^* \). By Lemma 4.5, \( R^* \subseteq W \). Therefore \( W = R^* \), and hence \( \Lambda(R^+, \phi) = \Lambda(W, \phi) \) is regular.

As noted above, the regularity of \( \Lambda(R^+, \phi) \) can be proved by a similar method. The proof is slightly more technical, since one has to exclude paths of length zero, but conceptually the same. We state the result in full for completeness:

**Theorem 4.6.** Let \( S \) be a structure admitting an injective unary FA-presentation \((a^+, \phi)\). Let \( R \) be some binary relation in the signature of \( S \). Then \( \Lambda(R^+, \phi) \) is regular, where \( R^+ \) denotes the transitive closure of \( R \). Hence \( S \) augmented by \( R^+ \) is also unary FA-presentable.

**Corollary 4.7.** Let \( S \) be a structure admitting an injective unary FA-presentation \((a^+, \phi)\). Let \( R \) be some binary relation in the signature of \( S \). Let \( Q \) be the equivalence relation generated by \( R \). Then \( \Lambda(Q, \phi) \) is regular. Hence \( S \) augmented by \( Q \) is also unary FA-presentable.

*Proof of 4.7.* Let \( R' \) be the symmetric closure of \( R \). Then \( \Lambda(R', \phi) \) is regular since \( R' \) is first-order definable in terms of \( R \). Since the equivalence relation \( Q \) is the reflexive and transitive closure \((R')^* \) of \( R' \), the relation \( \Lambda(Q, \phi) \) is regular by Theorem 4.1.
5 BINARY RELATIONS

This section is devoted to characterizing unary FA-presentable binary relations (Theorem 5.1), with the aim of subsequently giving useful characterizations of unary FA presentable quasi-orders (Theorem 5.8), partial orders (Theorem 5.10), and tournaments (Theorem 5.12).

These characterizations have a common form. A structure of one of these species is unary FA-presentable if it can be obtained by extending a finite structure of the same species in a particular ‘periodic’ fashion that we call ‘propagation’. The way in which the finite structure extends is determined by a collection of distinguished five-element subsets and the relations holding between these and the rest of the structure. The importance of the subsets having five elements is to ensure that transitivity is preserved when an infinite structure is obtained by propagating a finite one (see the comments following the proof of Lemma 5.6).

5.1 FA-foundational binary relations

Let \( \mu \) be a binary relation defined on a finite set \( Q \). We will consider \( (Q, \mu) \) as a directed graph with vertex set \( Q \) and edges \( \mu \), so that there is at most one directed edge from \( p \) to \( q \) for any \( p, q \in Q \).

Suppose \( Q \) is equipped with a distinguished collection of disjoint subsets \( P_0, \ldots, P_{n-1} \) called seeds, that fulfill the following conditions. Each seed \( P_k \) consists of five elements \( p_1^{(k)}, p_2^{(k)}, p_3^{(k)}, p_4^{(k)}, p_5^{(k)} \) such that the following conditions are satisfied for \( k, l \in \{0, \ldots, n-1\} \) (including the possibility that \( k = l \)) and \( q \in Q' = Q - (P_0 \cup \ldots \cup P_{n-1}) \):

1. \( p_1^{(k)} \mu p_1^{(l)} \iff p_2^{(k)} \mu p_2^{(l)} \iff p_3^{(k)} \mu p_3^{(l)} \iff p_4^{(k)} \mu p_4^{(l)} \iff p_5^{(k)} \mu p_5^{(l)} \). That is, either all or none of the edges in Figure 4(a) run from \( P_k \) to \( P_l \). (There may be other edges between \( P_k \) and \( P_l \) that are not shown in the figure, but either all the edges shown here are present or none are. The same caveat applies to the remaining conditions.) If all of these edges are present, we say there is an \( S_0 \) connection from \( P_k \) to \( P_l \).

2. \( p_1^{(k)} \mu p_2^{(l)} \iff p_2^{(k)} \mu p_3^{(l)} \iff p_3^{(k)} \mu p_4^{(l)} \iff p_4^{(k)} \mu p_5^{(l)} \). That is, either all or none of the edges in Figure 4(b) run from \( P_k \) to \( P_l \). If all these are present, we say there is an \( S_{+1} \) connection from \( P_k \) to \( P_l \).

3. \( p_2^{(k)} \mu p_1^{(l)} \iff p_3^{(k)} \mu p_2^{(l)} \iff p_4^{(k)} \mu p_3^{(l)} \iff p_5^{(k)} \mu p_4^{(l)} \). That is, either all or none of the edges in Figure 4(c) run from \( P_k \) to \( P_l \). If all these edges are present, we say there is an \( S_{-1} \) connection from \( P_k \) to \( P_l \).

4. \( p_1^{(k)} \mu p_3^{(l)} \iff p_1^{(k)} \mu p_4^{(l)} \iff p_1^{(k)} \mu p_5^{(l)} \iff p_2^{(k)} \mu p_2^{(l)} \iff p_2^{(k)} \mu p_3^{(l)} \). That is, either all or none of the edges in Figure 4(d) run from \( P_k \) to \( P_l \). If all these edges are present, we say there is an \( S_{+\infty} \) connection from \( P_k \) to \( P_l \).

5. \( p_3^{(k)} \mu p_1^{(l)} \iff p_4^{(k)} \mu p_1^{(l)} \iff p_5^{(k)} \mu p_1^{(l)} \iff p_4^{(k)} \mu p_2^{(l)} \iff p_5^{(k)} \mu p_2^{(l)} \). That is, either all or none of the edges in Figure 4(e) run from \( P_k \) to \( P_l \). If all these edges are present, we say there is an \( S_{-\infty} \) from \( P_k \) to \( P_l \).

6. \( q \mu p_2^{(k)} \iff q \mu p_3^{(k)} \iff q \mu p_4^{(k)} \iff q \mu p_5^{(k)} \). That is, either all or none of the edges in Figure 4(f) run from \( q \) to \( P_k \). If all these edges are
Figure 4. Conditions on edges between $p_k$, $p_l$, and $q$, where $k, l \in \{0, \ldots, n - 1\}$ and $q \in Q'$.
present, we say there is a $T_{+\infty}$ from $q$ to $P_k$.

7. $p^{(k)}_2 \mu q \iff p^{(k)}_3 \mu q \iff p^{(k)}_4 \mu q \iff p^{(k)}_5 \mu q$. That is, either all or none of the edges in Figure 4(g) run from $P_k$ to $q$. If all these edges are present, we say there is a $U_{-\infty}$ connection from $P_k$ to $q$.

A finite binary relation equipped with such a collection of distinguished subsets is called a unary FA-founded binary relation.

For convenience, define the following additional connections for $q \in Q'$ and $k \in \{0, \ldots, n-1\}$:

1. $q \mu p^{(k)}_1$. That is, the edge in Figure 4(h) runs between $q$ and $p^{(k)}_1$. If this edge is present, we say there is a $T_0$ connection from $q$ to $P_k$.

2. $p^{(k)}_1 \mu q$. That is, the edge in Figure 4(i) runs between $p^{(k)}_1$ and $q$. If this edge is present, we say there is a $U_0$ connection from $P_k$ to $q$.

Let $k, l \in \{0, \ldots, n-1\}$. Notice that every possible edge from $P_k$ to $P_l$ is part of exactly one connection $S_{-\infty}, S_{-1}, S_0, S_{+1}, S_{+\infty}$. Thus the set of edges from $P_k$ to $P_l$ is made up of a union (possibly empty) of these connections. Similarly, the set of edges from $q \in Q'$ to $P_k$ is made up of a union of $T_0$ and $T_{+\infty}$, and similarly the set of edges from $P_k$ to $q$ is made up of a union of $U_0$ and $U_{-\infty}$.

5.2 Propagating an FA-founded binary relation

Extend $Q$ to an infinite set as follows. For each $k \in \{0, \ldots, n-1\}$, let $\overline{P}_k = \{p^{(k)}_i : i \in \mathbb{N}\}$. Let

$$\overline{Q} = Q' \cup \bigcup_{k=0}^{n-1} \overline{P}_k.$$ 

That is, to obtain $\overline{Q}$ from $Q$, each seed $P_k$ of $Q$ is extended to an infinite subset $\overline{P}_k$.

We now describe how to extend $\mu$ to a binary relation $\overline{\mu}$ on $\overline{Q}$. Define $\overline{\mu}$ as follows: first, for $p, q \in Q'$ as follows: $p \overline{\mu} q$ if $p \mu q$. For any $k, l \in \{0, \ldots, n-1\}$ and $q \in Q'$:

1. If $p^{(k)}_i \mu p^{(l)}_1$, then $p^{(k)}_i \overline{\mu} p^{(l)}_1$ for all $i \in \mathbb{N}$. That is, if there is an edge $(p^{(k)}_i, p^{(l)}_1) \in \mu$, then all the edges shown in Figure 5(a) are present. [We will discuss why some edges are shown as bold and some as normal weight later in this subsection. There may be other edges between $\overline{P}_k$ and $\overline{P}_l$, not shown in this figure. These remarks apply to the other cases below.] If all these edges are present, we say there is an $S_0$ connection from $\overline{P}_k$ to $\overline{P}_1$.

2. If $p^{(k)}_1 \mu p^{(l)}_2$, then $p^{(k)}_1 \overline{\mu} p^{(l)}_{i+1}$ for all $i \in \mathbb{N}$. That is, if there is an edge $(p^{(k)}_1, p^{(l)}_{i+1}) \in \mu$, then all the edges shown in Figure 5(b) are present. If all these edges are present, we say there is an $S_{+1}$ connection from $\overline{P}_k$ to $\overline{P}_1$.

3. If $p^{(k)}_2 \mu p^{(l)}_1$, then $p^{(k)}_{i+1} \overline{\mu} p^{(l)}_1$ for all $i \in \mathbb{N}$. That is, if there is an edge $(p^{(k)}_{i+1}, p^{(l)}_1) \in \mu$, then all the edges shown in Figure 5(c) are present. If all these edges are present, we say there is an $S_{-1}$ connection from $\overline{P}_k$ to $\overline{P}_1$.
Figure 5. Extension of $\mu$ to $\bar{\pi}$, where $q \in Q'$ and $k, l \in \{0, \ldots, n-1\}$. 
4. If \( p_1^{(k)} \mu p_3^{(l)} \), then \( p_1^{(k)} \mu p_3^{(l)} \mu p_{1+j}^{(k)} \) for all \( i, j \in \mathbb{N} \) with \( j \geq 2 \). That is, if there is an edge \( (p_1^{(k)}, p_3^{(l)}) \in \mu \), then all the edges shown in Figure 5(d) are present. (For clarity, only edges that both start and end in the scope of this diagram are shown. The same applies to the following diagrams.) If all these edges are present, we say there is an \( S_{+\infty} \) connection from \( \overline{P}_k \) to \( \overline{P}_1 \).

5. If \( p_3^{(k)} \mu p_1^{(l)} \), then \( p_3^{(k)} \mu p_1^{(l)} \mu p_{1+i}^{(l)} \) for all \( i, j \in \mathbb{N} \) with \( j \geq 2 \). That is, if there is an edge \( (p_3^{(k)}, p_1^{(l)}) \in \mu \), then all the edges shown in Figure 5(e) exist. If all these edges are present, we say there is an \( S_{-\infty} \) connection from \( \overline{P}_k \) to \( \overline{P}_1 \).

6. If \( q \mu p_1^{(k)} \), then \( q \mu p_1^{(k)} \mu p_1^{(k)} \) for all \( i \in \mathbb{N} \) with \( i \geq 2 \). That is, if there is an edge \( (q, p_1^{(k)}) \in \mu \), then all the edges shown in Figure 5(f) are present. If all these edges are present, we say there is an \( T_{+\infty} \) connection from \( q \) to \( \overline{P}_1 \).

7. If \( p_2^{(k)} \mu q \), then \( p_2^{(k)} \mu q \) for all \( i \in \mathbb{N} \) with \( i \geq 2 \). That is, if there is an edge \( (p_2^{(k)}, q) \in \mu \), then all the edges shown in Figure 5(g) are present. If all these edges are present, we say there is an \( U_{-\infty} \) connection from \( \overline{P}_k \) to \( q \).

For convenience, define the following additional connections for \( q \in Q' \) and \( k \in \{1, \ldots, n\} \):

1. If \( q \mu p_1^{(k)} \), then \( q p_1^{(k)} \mu q \). That is, the edge in Figure 5(h) runs from \( q \) to \( p_1^{(k)} \). If this edge is present, we say that there is a \( T_0 \) connection from \( q \) to \( \overline{P}_k \). Notice that a \( T_0 \) connection from \( q \) to \( P_k \) and a \( T_0 \) connection from \( q \) to \( \overline{P}_k \) consist of the same edge.

2. If \( p_1^{(k)} \mu q \), then \( p_1^{(k)} \mu q \). That is, the edge in Figure 5(i) runs from \( p_1^{(k)} \) to \( q \). If this edge is present, we say there is a \( U_0 \) connection from \( \overline{P}_k \) to \( q \). Notice that a \( U_0 \) connection from \( P_k \) to \( q \) and a \( U_0 \) connection from \( \overline{P}_k \) to \( q \) consist of the same edge.

Observe that each edge in any diagram in Figure 4 also appears in the corresponding diagram in Figure 5. That is, \( p \mu q \) if and only if \( p \mu q \) for any \( p, q \in Q \). Thus \( \overline{\mu} \) genuinely extends \( \mu \). (The bold edges in Figure 5 are those present in \( \mu \).)

Notice further that a \( S_{\sigma} \) connection from \( P_k \) to \( P_1 \) gives rise to an \( S_{\sigma} \) connection from \( \overline{P}_k \) to \( \overline{P}_1 \) for any \( \sigma \in (-\infty, -1, 0, +1, +\infty) \). Similarly, a \( T_{\sigma} \) connection from \( q \in Q' \) to \( P_1 \) gives rise to a \( T_{\sigma} \) connection from \( q \) to \( \overline{P}_1 \) for \( \sigma \in (0, +\infty) \), and a \( U_{\sigma} \) connection from \( P_k \) to \( q \in Q' \) gives rise to a \( U_{\sigma} \) connection from \( q \) to \( \overline{P}_1 \) for \( \sigma \in (-\infty, 0) \).

Let \( k, l \in \{0, \ldots, n-1\} \). Notice that every possible edge from \( \overline{P}_k \) to \( \overline{P}_l \) is part of exactly one connection \( S_{-\infty}, S_{-1}, S_0, S_1, S_{+\infty} \). Thus the set of edges from \( \overline{P}_k \) to \( \overline{P}_1 \) is made up of a union (possibly empty) of these connections. Similarly, the set of edges from \( q \in Q' \) to \( \overline{P}_k \) is made up of a union of \( T_0 \) and \( T_{+\infty} \) and similarly the set of edges from \( \overline{P}_k \) to \( q \) is made up of a union of \( U_0 \) and \( U_{-\infty} \).

5.3 Characterization of binary relations

**Theorem 5.1.** A binary relation is unary FA-presentable if and only if it can be obtained by propagating a unary FA-foundational binary relation.

**Proof of 5.1. First part.** Let \((Q, \overline{\mu})\) be a binary relation obtained by propagating a unary FA-foundational binary relation. Retain notation from Subsections 5.1...
and 5.2. Notice first that if \((\overline{Q}, \overline{\mu})\) is finite (which can happen if the number of seeds \(n = 0\)), it is unary FA-presentable. So assume \((\overline{Q}, \overline{\mu})\) is infinite, which requires \(n > 0\).

Define a representation map \(\phi : a^* \rightarrow \overline{Q}\) as follows. Elements of \(Q'\) are represented by the words \(a^{i_0}, \ldots, a^{i_{|Q'|-1}}\). The elements \(p_i^{(k)} \in P_k\), where \(k \in \{0, \ldots, n-1\}\) and \(i \in \mathbb{N}\) are represented by words of the form \(a^{i_0 + ni + k}\). That is, given a word \(a^j\) with \(j \geq |Q'|\), the set \(P_k\) to which \(a^j\) belongs is determined by the remainder of dividing \(j - |Q'|\) by \(n\), and the subscript \(i\) is determined by the (integer) quotient of \(j - |Q'|\) by \(n\).

An automaton recognizing conv\(\Lambda(\overline{\mu}, \phi)\) functions as follows: while reading each of its two tracks, it stores either the length of the word read up to a maximum length of \(|Q'|\), or the length of the word modulo \(n\). Thus, for the input word on each track, the automaton knows either which element of \(Q'\) is represented by the input word, or which of the various \(P_k\) contains the element represented by the input word. The automaton also stores the difference in lengths between the two input words, up to a maximum difference of \(\pm (2n + |Q'|)\). In particular, therefore, in the case when both input words represent elements \(p_i^{(k)}\) and \(p_j^{(l)}\), the automaton knows whether \(j - i\) is less than or equal to \(-2\), equal to \(-1\), \(0\), or \(1\), or at least \(2\). In the case when one word represents \(q \in Q'\) and the other \(p_i^{(k)}\), the automaton knows whether the subscript \(i\) is \(1\) or at least \(2\).

We claim that this bounded amount of stored information is enough for the automaton to decide whether the two input words represent elements related by \(\overline{\mu}\).

First, if both words represent elements of \(Q'\), the automaton can accept if and only if the two elements (which it stores) are related by \(\overline{\mu}\).

Second, if the element represented by the left-hand input word is \(q \in Q'\) and the other element is \(p_i^{(k)}\), the automaton accepts either if \(i = 1\) and there is a \(\overline{T}_0\) connection between \(q\) and \(\overline{P}_k\), or if \(i \geq 2\) and there is a \(\overline{T}_{+\infty}\) connection between \(q\) and \(\overline{P}_k\). Recall that the automaton stores the element \(q\), the index \(k\), and whether the subscript \(i\) is \(1\) or at least \(2\).

The case where the element represented by the right-hand input word is \(q \in Q'\) and the other element is \(p_i^{(k)}\) is similar.

Third, if the element represented by the left-hand input word is \(p_i^{(k)}\) and the other is \(p_j^{(l)}\), then the automaton must accept if and only if

- \(j - i \geq 2\) and there is a \(\overline{T}_{+\infty}\) connection between \(\overline{P}_k\) and \(\overline{P}_l\), or
- \(j - i = 1\) and there is a \(\overline{T}_{+1}\) connection between \(\overline{P}_k\) and \(\overline{P}_l\), or
- \(j - i = 0\) and there is a \(\overline{T}_0\) connection between \(\overline{P}_k\) and \(\overline{P}_l\), or
- \(j - i = -1\) and there is a \(\overline{T}_{-1}\) connection between \(\overline{P}_k\) and \(\overline{P}_l\), or
- \(j - i \leq 2\) and there is a \(\overline{T}_{-\infty}\) connection between \(\overline{P}_k\) and \(\overline{P}_l\).

Recall that the automaton knows sufficient information about \(j - i\) and knows the indices \(k\) and \(l\).

Second part. Suppose that \((\overline{Q}, \overline{\mu})\) is a unary FA-presentable binary relation. If \(\overline{Q}\) is finite, it is a unary FA-foundational binary relation with \(n = 0\), as defined in Subsection 5.1.

So assume without loss of generality that \(\overline{Q}\) is infinite and let \((a^*, \phi)\) be an injective unary FA-presentation for \((\overline{Q}, \leq)\). Suppose the diagram for \((a^* \phi)\)
has D rows. Each element of $Λ(\vec{µ}, \phi)$ is an arrow between two points in the diagram.

Let $Q'$ be those points represented by words in the leftmost column $C[0]$ of the diagram; that is, $Q' = \{C[0]\}$. For $k = 0, \ldots, D - 1$ and $i \in \mathbb{N}$, let $p_i^{(k)}$ be the element represented by the unique word in the row $B[k]$ and column $C[i]$; that is, $p_i^{(k)} = (B[k] \cap C[i])\phi$. Let $\overrightarrow{P}_k = (B[k] - C[0])\phi$ and $\overrightarrow{P}_k = (B[k] \cap C[1, 5])\phi$.

As a consequence of Arrow rules 1 and 2, for any $\sigma = \{-\infty, -1, 0, +1, +\infty\}$ and $k, l \in \{0, \ldots, D - 1\}$, either there is an $\overrightarrow{S}_\sigma$ connection from from $\overrightarrow{P}_k$ to $\overrightarrow{P}_l$, or no edge that is part of an $\overrightarrow{S}_\sigma$ connection runs from $\overrightarrow{P}_k$ to $\overrightarrow{P}_l$. Similarly, for any $q \in Q'$ and $\kappa \in \{0, \ldots, D - 1\}$, either there is a $\overrightarrow{T}_{+\infty}$ connection from $q$ to $\overrightarrow{P}_k$, or no edge that is part of an $\overrightarrow{T}_{+\infty}$ connection runs from $\overrightarrow{P}_k$ to $\overrightarrow{P}_k$, and either there is a $\overrightarrow{T}_{-\infty}$ connection from $\overrightarrow{P}_k$ to $\overrightarrow{P}_k$, or no edge that is part of an $\overrightarrow{T}_{-\infty}$ connection runs from $\overrightarrow{P}_k$ to $\overrightarrow{P}_k$.

Let $Q = Q' \cup \bigcup_{k=0}^{D-1} P_k$, and let $\mu$ be the restriction of $\vec{µ}$ to $Q$. Then $(Q, \mu)$ is a finite binary relation equipped with distinguished subsets $P_0, \ldots, P_{D-1}$.

Furthermore, the conditions on connections between the sets $P_k$ hold (as restrictions of the conditions on connections between the sets $\overrightarrow{P}_k$ in $(Q, \vec{µ})$) by the observations in the previous paragraph. Hence $(Q, \mu)$ is a unary FA-fundamental binary relation with seeds $P_0, \ldots, P_{D-1}$. It is easy to see that propagating $(Q, \mu)$ yields $(\overrightarrow{Q}, \vec{µ})$.

5.4 Preservation of properties

Preparatory to the characterization results in the next section, we prove that various properties are preserved in passing from $(Q, \mu)$ to $(\overrightarrow{Q}, \vec{µ})$ and vice versa. The key to several of the proofs is the following result:

**Lemma 5.2.** For any $x, y \in \overrightarrow{Q}$, there exist $x', y' \in Q$ such that:

1. if $x = y$, then $x' = y'$;
2. the map $x \mapsto x'$ and $y \mapsto y'$ is an isomorphism between the induced substructures \{x, y\} and \{x', y'\}.

**Proof of 5.2.** Consider three cases separately, depending on whether none, one, or both of $x$ and $y$ lie in $Q$:

1. Suppose both $x$ and $y$ lie in $Q$. Then let $x' = x$ and $y' = y$; there is nothing to prove.
2. Suppose only one of $x$ and $y$ lies in $Q$; without loss of generality, assume $x \in Q'$ and $y \in \overrightarrow{Q} - Q'$. Then $y = p_i^{(k)} \in P_k$ for some $k = 0, \ldots, n - 1$ and $i \in \mathbb{N}$. Let $x' = x$ and $y' = p_i^{(k)}$. Then the following are equivalent:

   (1) $x' \mu y'$;
   (2) there is an $T_{+\infty}$ connection from $x'$ to $P_k$;
   (3) there is an $T_{+\infty}$ connection from $x$ to $\overrightarrow{P}_k$;
   (4) $x' \overrightarrow{y}$. Similarly, by $U_{-\infty}$ and $\overrightarrow{U}_{-\infty}$ connections from $P_k$ to $x$ and $\overrightarrow{P}_k$ to $x$, we see that $y' \mu x'$ if and only if $y \overrightarrow{µ} x$. Similarly, by considering $S_0$ and $\overrightarrow{S}_0$ connections from $P_k$ to $P_k$ and $\overrightarrow{P}_k$ to $\overrightarrow{P}_k$, we see that $y' \mu x'$ if and only if $y \overrightarrow{µ} x$. Hence in this second case the map is an isomorphism of induced substructures.

3. Suppose neither $x$ nor $y$ lies in $Q$. Then $x, y \in \overrightarrow{Q} - Q'$ and hence $x = p_i^{(k)}$
and \( y = p_j^{(l)} \) for some \( k, l \in \{0, \ldots, n - 1\} \) and \( i, j \in \mathbb{N} \). Let \( x' = p_3^{(k)} \) and let

\[
y' = \begin{cases} 
p_1^{(l)} & \text{if } i \leq j - 2, \\
p_2^{(l)} & \text{if } i = j - 1, \\
p_3^{(l)} & \text{if } i = j, \\
p_4^{(l)} & \text{if } i = j + 1, \\
p_5^{(l)} & \text{if } i \geq j + 2. 
\end{cases} \tag{5.1}
\]

Then the following are equivalent: (1) \( x' \mu y' \); (2) there is an \( S_\sigma \) connection from \( P_k \) to \( P_l \), where

\[
\sigma = \begin{cases} 
\infty & \text{if } i \leq j - 2, \\
+1 & \text{if } i = j - 1, \\
0 & \text{if } i = j, \\
-1 & \text{if } i = j + 1, \\
-\infty & \text{if } i \geq j + 2;
\end{cases}
\]

(3) there is an \( \overline{S}_\sigma \) connection from \( \overline{P}_k \) to \( \overline{P}_l \); (4) \( x \mu y \) (by the choice of \( y' \) in (5.1)). Similarly, \( y' \mu x' \) if and only if \( y \mu x \). Finally, considering the presence or absence of an \( \overline{S}_\sigma \) connection from \( \overline{P}_k \) to \( \overline{P}_l \) shows that \( x' \mu x' \) if and only if \( x \mu x \); similarly, \( y' \mu y' \) if and only if \( y \mu y \). Hence in this third case the map is an isomorphism of induced substructures.

Since reflexivity, symmetry, and anti-symmetry are defined by axioms over two variables, the following three lemmata follow easily from Lemma 5.2.

**Lemma 5.3.** The unary FA-foundational binary relation \((Q, \mu)\) is reflexive if and only if the propagated binary relation \((\overline{Q}, \overline{\mu})\) is reflexive.

**Lemma 5.4.** The unary FA-foundational binary relation \((Q, \mu)\) is symmetric if and only if the propagated binary relation \((\overline{Q}, \overline{\mu})\) is symmetric.

**Lemma 5.5.** The unary FA-foundational binary relation \((Q, \mu)\) is anti-symmetric if and only if the propagated binary relation \((\overline{Q}, \overline{\mu})\) is anti-symmetric.

The next result is the analogue of Lemmata 5.3–5.5 for transitivity.

**Lemma 5.6.** The unary FA-foundational binary relation \((Q, \mu)\) is transitive if and only if the propagated binary relation \((\overline{Q}, \overline{\mu})\) is transitive.

**Proof of 5.6.** Suppose first that \((\overline{Q}, \overline{\mu})\) is transitive. Then \((Q, \mu)\) is transitive since \(\mu\) is the restriction of \(\overline{\mu}\) to \(Q\).

Now suppose that \((Q, \mu)\) is transitive. Showing that \((\overline{Q}, \overline{\mu})\) is transitive requires consideration of many cases similar to each other. Suppose there is an edge from \(p_k^{(i)}\) to \(p_l^{(j)}\) and an edge from \(p_i^{(l)}\) to \(p_j^{(m)}\). To prove transitivity, it is necessary to show that there is an edge from \(p_k^{(i)}\) to \(p_j^{(m)}\). The different cases arise from the various possible connections in which these edges lie.

Every case proceeds in the same way: the edge from \(p_k^{(i)}\) to \(p_l^{(j)}\) and the edge from \(p_i^{(l)}\) to \(p_j^{(m)}\) lie in connections \(S_\rho\) and \(S_\sigma\). By the definition of propagation, there is a \(S_\rho\) connection from \(P_k\) to \(P_l\) and an \(S_\sigma\) connection from \(P_i\) to \(P_j\). Transitivity in \((Q, \mu)\) then forces one or more connections \(S_\tau\) to hold between \(P_k\) and \(P_m\), and propagation then requires \(\overline{S}_\tau\) to hold between...
The cases involving one or two edges to or from elements of $Q'$ are similar.

We will limit ourselves to proving one exemplary case in full detail and summarizing the others. Consider the case where the edge from $p_h^{(k)}$ to $p_i^{(1)}$ lies in an $\mathcal{S}_\infty$ connection and the edge from $p_i^{(1)}$ to $p_j^{(m)}$ lies an $\mathcal{S}_{-1}$ connection. That is, $h + 2 \leq i$ and $i - 1 = j$. Hence $j \geq h + 1$.

By the definition of propagation, in $(Q, \mu)$ there is an $\mathcal{S}_\infty$ connection from $P_k$ to $P_1$ and an $\mathcal{S}_{-1}$ connection from $P_1$ to $P_m$. In particular, there are edges from $p_1^{(k)}$ to $p_3^{(1)}$, from $p_3^{(1)}$ to $p_2^{(m)}$, from $p_1^{(k)}$ to $p_4^{(1)}$, and from $p_4^{(1)}$ to $p_3^{(m)}$. Hence, by the transitivity of $(Q, \mu)$, there are edges from $p_1^{(k)}$ to $p_2^{(m)}$, and from $p_1^{(k)}$ to $p_3^{(m)}$. Therefore there are $\mathcal{S}_{+\infty}$ and $\mathcal{S}_{+1}$ connections from $P_k$ to $P_m$.

Therefore, by the definition of propagation, there are $\mathcal{S}_{+\infty}$ and $\mathcal{S}_{+1}$ connections from $\mathcal{P}_k$ to $\mathcal{P}_m$. Thus there are edges from $p_h^{(k)}$ to $p_j^{(m)}$ for $j = h + 1$ and for $j \geq h + 2$, and thus for all $j \geq h + 1$.

Hence the transitivity condition holds when the first edge from $p_h^{(k)}$ to $p_i^{(1)}$ lies in an $\mathcal{S}_\infty$ connection from $\mathcal{P}_k$ to $\mathcal{P}_1$ and the second lies edge from $p_i^{(1)}$ to $p_j^{(m)}$ in an $\mathcal{S}_{-1}$ connection from $\mathcal{P}_1$ to $\mathcal{P}_m$.

The various cases are summarized in Table 1.

We remark on two further exemplary cases: if there is a $T_0$ connection from $q \in Q'$ to $P_1$ and a $U_0$ connection from $P_1$ to $s \in Q'$, then transitivity means there is a single edge (a ‘$Q'$-edge’) from $q$ to $s$. If there is a $T_0$ connection from $q \in Q'$ to $P_1$ and a $U_{-\infty}$ connection from $P_1$ to $s \in Q'$, then transitivity does not enforce an edge from $q$ to $s$.

Table 1. Connections enforced by the transitivity of $(Q, \mu)$. Row labels show the connection between $P_k$ (or an element of $Q'$) and $P_l$ (or an element of $Q'$), column labels show the connection between $P_l$ (or an element of $Q'$) and $P_m$ (or an element of $Q'$). The cell in a particular row and column shows the connection(s) between $P_k$ (or an element of $Q'$) and $P_m$ (or an element of $Q'$) enforced by transitivity. The notation $S_*$ abbreviates $S_{-\infty}, S_{-1}, S_0, S_1, S_{+1}, S_{+\infty}$; the notation $S_{+1,\infty}$ abbreviates $S_{+1}, S_{+\infty}$; the notation $S_{-\infty, -1}$ abbreviates $S_{-\infty}, S_{-1}$. A table for the connections required for transitivity of $(Q, \mu)$ can be obtained from this table by replacing each $S_{\sigma}, T_{\sigma}, U_{\sigma}$ or $U_{\sigma}$ by the corresponding $S_{\sigma}, T_{\sigma}, U_{\sigma}$ or $U_{\sigma}$.
applying transitivity in \((Q, \leq)\) to get an edge from \(p_i^{(k)}\) to \(p_5^{(m)}\) and hence an \(S_{+\infty}\) connection from \(P_k\) to \(P_1\).

5.5 Orders

Equipped with the lemmata from the previous subsection, we can now characterize unary FA-presentable quasi-orders and partial orders. Recall that a quasi-order is a binary relation that is reflexive and transitive. The following characterization follows immediately from the lemmata in the previous subsection.

**Proposition 5.7.** A quasi-order is unary FA-presentable if and only if it can be obtained by propagating a unary FA-foundational quasi-order.

**Proof of 5.7.** This is immediate from Theorem 5.1 and Lemmata 5.3 and 5.6.

However, we can improve Proposition 5.7 to the following result:

**Theorem 5.8.** A quasi-order is unary FA-presentable if and only if it can be obtained by propagating a unary FA-foundational quasi-order in which every seed \(P_k\) is either an anti-chain (with none of the \(p_i^{(k)}\) being comparable), an ascending chain (with \(p_1^{(k)} < p_2^{(k)} < p_3^{(k)} < p_4^{(k)} < p_5^{(k)}\)), a descending chain (with \(p_1^{(k)} > p_2^{(k)} > p_3^{(k)} > p_4^{(k)} > p_5^{(k)}\)), or a strongly connected component (with \(p_i^{(k)} \leq p_j^{(k)}\) for all \(i, j \in \{1, \ldots, 5\}\).

**Proof of 5.8.** Notice that in one direction the result has already been proven: propagating such a unary FA-foundational quasi-order yields a unary FA-presentable quasi-order by Proposition 5.7.

Therefore let \((Q, \leq)\) be a unary FA-presentable quasi-order and let \((a^*, \phi)\) be an injective unary FA-presentation. Follow the second part of the proof of Theorem 5.1 to obtain a unary FA-foundational quasi-order \((Q, \leq)\), with distinguished sets \(P_k\), that, when propagated, yields \((Q, \leq)\). Consider some distinguished set \(P_k\) and the corresponding \(\overline{P}_k\).

Suppose there is an \(\overline{S}_{+\infty}\) connection from \(\overline{P}_k\) to itself. Then there is an edge from \(p_1^{(k)}\) to \(p_3^{(k)}\). Let \(a^1, a^m \in a^*\) be such that \(a^1\phi = p_1^{(k)}\) and \(a^m\phi = p_3^{(k)}\).

By the definition of \(\overline{P}_k\) (in the proof of Theorem 5.1), \(b(a^1) = b(a^m) = k\) and \(c(a^1) = 1\) and \(c(a^m) = 3\). Thus \(m = 1 + 2D\). That is, \((a^1, a^{1+2D}) \in \Lambda(\leq, \phi)\).

By Pumping rule 2, \((a^1, a^{1+D}) \in \Lambda(\leq, \phi)\). Notice that \(b(a^{1+D}) = b(a^1)\) and \(c(a^{1+D}) = c(a^1) + 1 = 2\). Thus there is an edge from \(a^1\phi = p_1^{(k)}\) to \(a^{1+D}\phi = p_2^{(k)}\). Hence there is an \(\overline{S}_{+1}\) connection from \(\overline{P}_k\) to itself.

Similarly, one can show that if there is an \(\overline{S}_{-\infty}\) from \(\overline{P}_k\) to itself, then there is an \(\overline{S}_{-1}\) connection from \(\overline{P}_k\) to itself.

If there is an \(\overline{S}_{+1}\) connection from \(\overline{P}_k\) to itself, there is an \(\overline{S}_{+\infty}\) connection from \(\overline{P}_k\) to itself as a consequence of transitivity. Similarly, if there is an \(\overline{S}_{-1}\) connection from \(\overline{P}_k\) to itself, there is an \(\overline{S}_{-\infty}\) connection from \(\overline{P}_k\) to itself as a consequence of transitivity.

Thus from \(\overline{P}_k\) to itself, either there are both \(\overline{S}_{+1}\) and \(\overline{S}_{+\infty}\) connections or there are neither, and similarly for \(\overline{S}_{-1}\) and \(\overline{S}_{-\infty}\) connections. Thus there are four possibilities:

1. No connections \(\overline{S}_{-\infty}, \overline{S}_{-1}, \overline{S}_{+1}, \overline{S}_{+\infty}\) from \(\overline{P}_k\) to itself. Then \(\overline{P}_k\) and thus \(P_k\) are antichains.
2. Connections $S_{-\infty}$ and $S_{-1}$ from $P_k$ to itself, but neither $S_{+1}$ nor $S_{+\infty}$. Then $P_k$ and thus $P_k$ are descending chains.
3. Connections $S_{+\infty}$ and $S_{+1}$ from $P_k$ to itself, but neither $S_{-1}$ nor $S_{-\infty}$. Then $P_k$ and thus $P_k$ are ascending chains.
4. All connections $S_{-\infty}, S_{-1}, S_{+1}, S_{+\infty}$ from $P_k$ to itself. Then $P_k$ and thus $P_k$ are strongly connected components.

5. The preceding result yields the following decomposition result for unary FA-presentable quasi-orders:

**Corollary 5.9.** Every unary FA-presentable quasi-order decomposes as a finite disjoint union of trivial quasi-orders, countably infinite ascending chains, countably infinite descending chains, countably infinite anti-chains, and countably infinite strongly connected components.

We can now characterize unary FA-presentable partial orders:

**Theorem 5.10.** A partial order is unary FA-presentable if and only if it can be obtained by propagating a unary FA-foundational partial order in which every distinguished set $P_k$ is either an anti-chain (with none of the $p_{1}^{(k)}$ being comparable), an ascending chain (with $p_{1}^{(k)} < p_{2}^{(k)} < p_{3}^{(k)} < p_{4}^{(k)} < p_{5}^{(k)}$), or a descending chain (with $p_{1}^{(k)} > p_{2}^{(k)} > p_{3}^{(k)} > p_{4}^{(k)} > p_{5}^{(k)}$).

**Proof of 5.10.** This is immediate from Theorem 5.8, Lemma 5.5, and the observation that no partial order contains a strongly connected component.

We also have a decomposition result for unary FA-presentable partial orders, analogous to Corollary 5.9:

**Corollary 5.11.** Every unary FA-presentable partial order decomposes as a finite disjoint union of trivial partial orders, countably infinite ascending chains, countably infinite descending chains, and countably infinite anti-chains.

### 5.6 Tournaments

Recall that $(X, \rightarrow)$ (where $\rightarrow$ is a binary relation on $X$) is a tournament if (when viewed as a directed graph) every pair of distinct vertices is connected by a single directed edge, and there is no edge from a vertex to itself. That is, for every $x, y \in X$ with $x \neq y$, either $x \rightarrow y$ or $y \rightarrow x$ (but not both), and $x \not\rightarrow x$ for every $x \in X$. Since this is an axiom over two variables, the following characterization of unary FA-presentable tournaments is an easy consequence of Lemma 5.2.

**Theorem 5.12.** A tournament is unary FA-presentable if and only if it can be obtained by propagating a unary FA-foundational tournament.

In order to give decomposition result in the spirit of Corollaries 5.9 and 5.11, we need some terminology:

**Definition 5.13.** A countably infinite tournament $(X, \rightarrow)$, where $X = \{x_i : i \in \mathbb{N}\}$, is said to be:

1. complete ascending if $x_i \rightarrow x_j$ for all $i < j$;
2. **complete descending** if \( x_i \leftarrow x_j \) for all \( i < j \);
3. **near-complete ascending** if \( x_i \leftarrow x_{i+1} \) for all \( i \), and also \( x_i \rightarrow x_j \) for all \( i < j-1 \);
4. **near-complete descending** if \( x_i \rightarrow x_{i+1} \) for all \( i \), and also \( x_i \leftarrow x_j \) for all \( i < j-1 \).

**Corollary 5.14.** *Every unary FA-presentable tournament decomposes as a finite disjoint union of trivial tournaments, countably infinite complete ascending tournaments, countably infinite complete descending tournaments, countably infinite near-ascending tournaments, and countably infinite near-descending tournaments.***

**Proof of 5.14.** By Theorem 5.12, any unary FA-presentable tournament \((X, \rightarrow)\) is obtained by propagating a unary FA-foundational tournament. Then \((X, \rightarrow)\) is the finite disjoint union of the finite set \( Q' \) and the various \( \overline{P}_k \). For any \( k \), consider the connections that can run from \( \overline{P}_k \) to \( \overline{P}_k \). Clearly the presence of an \( S_0 \) connection is incompatible with \((X, \rightarrow)\) being a tournament. Again from \((X, \rightarrow)\) being a tournament, we see that there is either an \( S_{-\infty} \) connection or an \( S_{+\infty} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \) (but not both). Similarly, there is either an \( S_{-1} \) connection or an \( S_{+1} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \).

There are therefore four cases to consider, depending on which connections \( S_{-\infty} \) or \( S_{+\infty} \) and \( S_{-1} \) or \( S_{+1} \) are present:

- Suppose an \( S_{-\infty} \) connection and an \( S_{-1} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \) are present. Then [the substructure induced by] \( \overline{P}_k \) is a countably infinite complete descending tournament.
- Suppose an \( S_{+\infty} \) connection and an \( S_{+1} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \) are present. Then \( \overline{P}_k \) is a countably infinite complete ascending tournament.
- Suppose an \( S_{-\infty} \) connection and an \( S_{+1} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \) are present. Then \( \overline{P}_k \) is a countably infinite near-complete descending tournament.
- Suppose an \( S_{+\infty} \) connection and an \( S_{-1} \) connection from \( \overline{P}_k \) to \( \overline{P}_k \) are present. Then \( \overline{P}_k \) is a countably infinite near-complete ascending tournament.

This completes the proof.

6 **trees & forests**

This section is devoted to characterizing unary FA-presentable directed and undirected trees and forests. For our purposes, a directed tree is simply a directed graph that can be obtained by taking a tree and assigning a direction to each edge.

The characterization results describe unary FA-presentable trees as those that can be obtained, via a construction we call attachment, from finite trees and from two species of infinite trees that we will define shortly: shallow stars and periodic paths.

**Definition 6.1.** Let \((G, \gamma)\) and \((T, \eta)\) be directed graphs, and let \( g \in G \) and \( t \in T \) be distinguished vertices. The result of attaching \((T, \eta)\) at \( t \) to the vertex \( g \) of \((G, \gamma)\) is the graph obtained by taking the disjoint union of the graphs \((G, \gamma)\) and \((T, \eta)\) and identifying the vertices \( g \) and \( t \).
We now introduce the two species of infinite trees used in the characterization results.

**Definition 6.2.** First we define a **template**, which comprises a quadruple \((T, \eta, t_0, t_1)\), where \((T, \eta)\) is a finite directed tree with vertex set \(T\), edge set \(\eta\), and \(t_0, t_1 \in T\) are distinguished vertices with \(t_0\) being a leaf vertex.

Let \((T, \eta, t_0, t_1)\) be a template. Consider the graph \(S(T, \eta, t_0, t_1)\) obtained by taking the disjoint union of countably many copies \((T^j, \eta^j, t_0^j, t_1^j)\) and amalgamating vertices related by the equivalence relation generated by

\[
\mu = \{(t_0^j, t_0^{j+1}) : j \in \mathbb{N}_0\}.
\]

In the case where \(t_0 = t_1\), the graph \(S(T, \eta, t_0, t_1)\) is the tree obtained by amalgamating all the leaf vertices \(t_0^j = t_1^j\) into a single vertex. In this case, we call the resulting graph a **shallow star**, the amalgamated vertex is called the **centre** of \(S(T, \eta, t_0, t_1)\), and an edge incident on the centre is called a **ray** of \(S(T, \eta, t_0, t_1)\). Notice that the centre is the unique vertex of infinite degree in \(S(T, \eta, t_0, t_1)\). Notice that either all the rays start at the centre, in which case \(S(T, \eta, t)\) is said to be **outward**, or all the rays end at the centre, in which case \(S(T, \eta, t)\) is said to be **inward**. (See the example in Figure 6.)

In the case where \(t_0 \neq t_1\), the graph \(S(T, \eta, t_0, t_1)\) is the tree obtained by amalgamating the vertex \(t_0^{j+1}\) in the \((j+1)\)-th copy of the template with \(t_1^j\) in the \(j\)-th copy. In this case, we call the resulting graph a **periodic path**. Notice that there is a unique simple path \(\beta\) in \((T, \eta, t_0, t_1)\) from \(t_0\) to \(t_1\). Let \(\beta^j\) be the corresponding path in \((T^j, \eta^j, t_0^j, t_1^j)\). Then in \(S(T, \eta, t_0, t_1)\) the concatenation of the paths \(\beta^j\) form a infinite simple path. This path is called the **spine** of \(S(T, \eta, t_0, t_1)\). Notice that the spine is infinite in only one direction; in the other direction it begins at \(t_0^{(0)}\), which is called the base of \(S(T, \eta, t_0, t_1)\). (See the example in Figure 7.)

**Lemma 6.3.** Let \((G, \gamma)\) be graph admitting an FA-presentation (respectively, unary FA-presentation) \([L, \phi]\), and let \(Q\) be an equivalence relation on \(G\) such that \(\Lambda(Q, \phi)\)
Lemma 6 implies the existence of four other arrows (dashed in the diagram) that mean the graph cannot be a tree.

is regular. Then the graph \((G_Q, \gamma_Q)\) formed by amalgamating all vertices related by \(Q\) is also FA-presentable (respectively, unary FA-presentable).

**Proof of 6.3.** Let \(\gamma' = Q \circ \gamma \circ Q\); then \(\Lambda(\gamma', \phi)\) is regular and so \((G, \gamma')\) is FA-presentable (respectively, unary FA-presentable). It is easy to see that \(Q\) is a congruence on \((G, \gamma')\) and that \((G_Q, \gamma_Q)\) is obtained by factoring \((G, \gamma')\) by \(Q\). Hence \((G_Q, \gamma_Q)\) is FA-presentable (respectively, unary FA-presentable) [11, Corollary 3.7(iii)].

**Lemma 6.4.** Let \((G, \gamma)\) and \((T, \eta)\) be unary FA-presentable directed graphs, and let \(g \in G\) and \(t \in T\). The graph obtained by attaching \((T, \eta)\) at \(t\) to \(g\) is also unary FA-presentable.

**Proof of 6.4.** The disjoint union \((S, \sigma)\) of \((G, \gamma)\) and \((T, \eta)\) admits is unary FA-presentable by **Lemma 2.8**. Let \(R\) be the relation \(\{(g, t)\}\) and let \(Q\) be the equivalence relation it generates. Since \(R\) is finite, \(\Lambda(R, \phi)\) is regular. So, by **Corollary 4.7**, \(\Lambda(Q, \phi)\) is regular. The graph obtained attaching \((T, \eta)\) at \(t\) to \(g\) is then unary FA-presentable by **Lemma 6.3**.

**Lemma 6.5.** Shallow stars and periodic paths are is unary FA-presentable.

**Proof of 6.5.** Let \((T, \eta, t_0, t_1)\) be a template. Follow the proof of **Lemma 2.9** to obtain a unary FA-presentation \((a^*, \phi)\) for the disjoint union of countably many copies of \((T, \eta, t_0, t_1)\). Let \(p\) and \(q\) be such that \(a^p\phi\) is the first copy of the vertex \(t_0\) and \(a^q\phi\) the first copy of the vertex \(t_1\). Then, by the proof of **Lemma 2.9**, \[\Lambda(\mu, \phi) = \{a^{q+kn}, a^{p+(k+1)n} : k \in \mathbb{N}\}\] (where \(\mu\) is as defined in (6.1)) and so is regular. Let \(Q\) be the equivalence relation generated by \(\mu\); then \(\Lambda(Q, \phi)\) is regular by **Corollary 4.7**. Hence \(S(T, \eta, t_0, t_1)\) is unary FA-presentable by **Lemma 6.3**.

Before stating and proving the characterization theorem for unary FA-presentable directed trees, we need the following technical lemmata:

**Lemma 6.6.** Let \((a^*, \phi)\) be an injective unary FA-presentation for a directed tree. Then in the diagram, there cannot be a long arrow between two points in \(C[1, \infty)\). Equivalently, any long arrow must either start or end in \(C[0]\).

**Proof of 6.6.** Suppose that \((a^*, \phi)\) is an injective unary FA-presentation for a directed tree \(T\) and that there is a long arrow from \(p\) to \(q\) with \(p, q \in C[1, \infty]\). Consider the case when \(c(q) > c(p)\); the other case is similar. Since \(c(q) −

![Figure 8](https://via.placeholder.com/150)
c(p) > 1, Arrow rule 1 shows that there are arrows from p to (q)τ₂, from p to (q)τ₄, from (p)τ₂ to (q)τ₄, and from (p)τ₂ to (q)τ₂, as illustrated in Figure 8. Thus in the graph there is an undirected cycle (p)φ → ((q)τ₂)φ ← ((p)τ₂)φ → ((q)τ₄)φ ← (p)φ, which contradicts T being a tree.

Lemma 6.7. Let (α*, φ) be an injective unary FA-presentation for a directed tree. Let p ∈ C[1, ∞). Then pφ has degree at most 4D.

Proof of 6.7. Suppose pφ has degree greater than 4D and let k = c(p). There are exactly 4D points in C[0] ∪ C[k−1] ∪ C[k] ∪ C[k+1], so there is some point q in C[1, k−2] ∪ C[k+2, ∞) such that there is an arrow between p and q (in some direction). This is a long arrow since |c(p) − c(q)| ≥ 2, and this contradicts Lemma 6.6.

Lemma 6.8. Let (α*, φ) be an injective unary FA-presentation for a directed tree. The following are equivalent:

1. The diagram contains a long arrow.
2. Some vertex of the graph has infinite degree.
3. Some vertex of the graph has degree greater than 3D.

Proof of 6.8. (1 ⇒ 2) Suppose there is a long arrow between p and q in some direction. Assume without loss of generality that c(q) ≥ c(p) + 2. Then by Arrow rule 1, there are arrows between p and qτₙ for all n. Hence pφ has infinite degree.

(2 ⇒ 3) This is trivial.

(3 ⇒ 1) Suppose pφ has degree greater than 3D. Let k = c(p). Then there is some arrow between p and a vertex q outside C[k−1, k+1] (since this set contains 3D elements). So |c(p) − c(q)| ≥ 2 and thus the arrow between p and q is a long arrow.

Theorem 6.9. A directed tree is unary FA-presentable if and only if it is isomorphic to a tree obtained by starting from a finite directed tree and attaching to it finitely many shallow stars (at their centres) and finitely many periodic paths (at their bases).

Proof of 6.9. In one direction, the proof is easy. A graph of the prescribed form is unary FA-presentable by Lemmata 6.4, 6.5.

The other direction of the proof is much longer and more complex. Let (α*, φ) be an injective unary FA-presentation for a directed tree (T, η). Suppose the diagram for (α*, φ) has D rows.

Since (T, η) is a tree, the arrows in the diagram also form a tree. Thus there is a unique simple (undirected) path between any two points in the diagram.

Let K consist of C[0] ∪ C[1] together with all points that lie on simple paths starting and ending in C[0] ∪ C[1]. Then the subgraph induced by K is a (connected) finite graph. This will be part of the finite tree in the statement of the theorem.

Suppose there is an arrow between p and q’, where |c(p) − c(q’)| ≥ 2. Then either p or q’ lies in C[0] by Lemma 6.6. Without loss of generality, suppose p ∈ C[0]. By Arrow rules 2 and 1, there is an arrow between p and q’τₙ for all n ≥ −c(q’) + 2. Let m ≥ −c(q’) + 2 be such that q’τₙ ∈ K for all n ≥ m. Let q = q’τₙ. Replace K by K ∪ {qτₙ : −c(q) + 2 ≤ l < 0}. Notice that K remains connected. (The reasoning in this paragraph and the next is illustrated in Figure 9.)
Then for all \( n \geq 0 \) the point \( q \tau_n \) does not lie in \( K \) and there is an arrow between \( p \) and \( q \tau_n \). The aim is to show that the edges corresponding to these arrows are the rays of a shallow star with centre \( p \).

**Lemma 6.10.** For any simple undirected path \( \alpha \) starting at \( q \tau_n \), where \( n \geq 0 \), and not including the arrow from \( p \) to \( q \tau_n \):

1. The path \( \alpha \) does not visit any point of \( K \).
2. For every \( m \) with \( m \geq -n \), the map \( \tau_m \) is defined for every vertex of \( \alpha \), and \( (\alpha)\tau_m \) is a path in the diagram.
3. The path \( \alpha \) has length at most \( D \).

**Proof of 6.10.**

1. Suppose first that \( \alpha \) visits some element \( x \in K \). Without loss of generality, assume \( x \) is the first point of \( K \) that \( \alpha \) visits. Let \( \alpha' \) be the part of \( \alpha \) starting at \( q \tau_n \) and ending at \( x \). Since \( K \) is a connected set, there is a path \( \beta \) from \( x \) to \( p \) wholly within \( K \). So the edge from \( p \) to \( q \tau_n \) and the paths \( \alpha' \) and \( \beta \) form a cycle in the diagram, which contradicts \((T,\eta)\) being a tree. So the path \( \alpha \) does not visit any point of \( K \).

2. Suppose that for some \( m \geq -n \), the map \( \tau_m \) is not defined for some point of \( \alpha \). (Since \( \tau_m \) is always defined when \( m \) is positive, we know immediately that \( m \) is negative.) Note that if \( \alpha \) included a long arrow, it would have to visit \( C[0] \), which would contradict part 1 since \( C[0] \subseteq K \). So \( \alpha \) consists only of short arrows. For each \( t \), let \( r_t \) be the \( t \)-th point visited by \( \alpha \). So, since \( \tau_m \) is undefined for some point on \( \alpha \), there is some \( s \) such that \( c(r_s) < -m \). Since \( \alpha \) consists only of short arrows, \( c(r_{t+1}) \geq c(r_t) - 1 \) for all \( t \). Hence there is some \( s' \) such that \( c(r_{s'}) = -m + 1 \). Let \( \alpha' \) be the subpath of \( \alpha \) from \( q \tau_n \) up to the first point lying in \( C[r_{s'}] \). Then \( \tau_m \) is defined for every point on \( \alpha' \) and the path \((\alpha')\tau_m \) exists by Arrow rule 2, starts at \( q \tau_{n+m} \) and ends at some point in \( C[1] \), which contradicts part 1 applied to the path \( \alpha' \) since \( C[1] \subseteq K \). (This reasoning is illustrated Figure 10.)

Suppose that \((\alpha)\tau_m \) is not a path for some \( m \geq -n \). Arrow rule 1 shows that \((\alpha)\tau_m \) is a path for all \( m \geq 0 \). Arrow rule 2 shows that \((\alpha)\tau_m \) is a path unless \( \tau_m \) shifts some point of \( \alpha \) to \( C[0] \). But in this case, we can choose \( \alpha' \) as in the previous paragraph so that \((\alpha')\tau_m \) ends in \( C[1] \) and get the same contradiction. So \((\alpha)\tau_m \) is a path in the diagram.
Lemma C

subgraph has bounded degree. Furthermore, the original graph is obtained
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edges in this induced subgraph are all either short or run between points in
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, which is a path in the diagram by part 2. Let α1 be the
subpath of α from qτn to y; let α2 be the subpath of (α)τm from qτn+m to
m
xτm. Then the paths α1 and α2 and the arrows between p and qτn and
between ατqτn+m form a cycle, which contradicts (T, η) being a tree.
So α cannot have length greater than D.

By Lemma 6.10(2), for any n ∈ N0, β is a simple path starting at q that
does not include the arrow from p to q if and only if (β)τn is a simple path
starting at qτn that does not include the arrow from p to qτn.

Let F consist of all points lying on simple paths starting at q that do not
include the arrow from p to q. (The set F includes q itself.) By Lemma 6.10(3),
each of these paths has length at most D. By Lemma 6.10(1), none of these
paths visits K, and in particular does not visit C[0]. Hence, by Lemma 6.7,
the degree of any point in F is bounded by 4D and hence F consists of only
finitely many elements. By Lemma 6.10(2), every elements of F lies in C[2, ∞],
so Fτn is defined for all n ∈ N0. By the previous paragraph, all the induced
subgraphs Fτn are isomorphic and so, together with the arrows between p to
qτn, form a shallow star with center p, and this shallow star is attached at its
centre pφ to some vertex of the finite tree.

Since this shallow star contains every point qτn, it is clear that there can be
at most D such shallow stars.

Let L = (aD)*F. Then L = \bigcup_{n ∈ N0} Fτn. The language L consists of all
the points corresponding to vertices of the shallow star centered at pφ except
the point p itself. Since F is finite, L is regular. Since there are at most D
different shallow stars, the language M′ formed by the union of the languages
L corresponding to the various shallow stars is regular. Thus the language
M = a* − M′ consisting of words that either lie outside these shallow stars or
are the centres of the shallow stars, is regular. Hence the subgraph induced
by Mφ is also unary FA-presentable. Notice that the arrows corresponding to
dges in this induced subgraph are all either short or run between points in
K. Thus a bounded number of arrows corresponding to edges in this induced
subgraph start or end at any point of M and thus every vertex of this induced
subgraph has bounded degree. Furthermore, the original graph is obtained

3. Suppose that α has length greater than D. Since there are only D rows,
there are distinct points x and y with b(x) = b(y). Interchanging x and y
if necessary, assume c(x) < c(y). Let m = c(y) − c(x). Then y = xτm. So y
also lies on ατm, which is a path in the diagram by part 2. Let α1 be the
subpath of α from qτn to y; let α2 be the subpath of (α)τm from qτn+m to
m
xτm. Then the paths α1 and α2 and the arrows between p and qτn and
between p and qτn+m form a cycle, which contradicts (T, η) being a tree.
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does not include the arrow from p to q if and only if (β)τn is a simple path
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Let F consist of all points lying on simple paths starting at q that do not
include the arrow from p to q. (The set F includes q itself.) By Lemma 6.10(3),
each of these paths has length at most D. By Lemma 6.10(1), none of these
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so Fτn is defined for all n ∈ N0. By the previous paragraph, all the induced
subgraphs Fτn are isomorphic and so, together with the arrows between p to
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Since this shallow star contains every point qτn, it is clear that there can be
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Let L = (aD)*F. Then L = \bigcup_{n ∈ N0} Fτn. The language L consists of all
the points corresponding to vertices of the shallow star centered at pφ except
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different shallow stars, the language M′ formed by the union of the languages
L corresponding to the various shallow stars is regular. Thus the language
M = a* − M′ consisting of words that either lie outside these shallow stars or
are the centres of the shallow stars, is regular. Hence the subgraph induced
by Mφ is also unary FA-presentable. Notice that the arrows corresponding to
dges in this induced subgraph are all either short or run between points in
K. Thus a bounded number of arrows corresponding to edges in this induced
subgraph start or end at any point of M and thus every vertex of this induced
subgraph has bounded degree. Furthermore, the original graph is obtained

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Figure 11. The path $\alpha$ starting at $p$ visits two points $x_k$ and $x_l$ lying in the same row. The subpath $\beta$ is the part of $\alpha$ from $p$ to $x_k$; the subpath $\gamma$ is the part of $\alpha$ from $x_k$ to $x_l$. A small part of $\alpha$ is not included in either $\beta$ or $\gamma$. Translating $\gamma$ to the right through multiples of $m$ and concatenating the result gives an infinite path starting at $p$. We define the infinite periodic track to be the concatenation of $(\gamma)\tau_m, (\gamma)\tau_{2m}, \ldots$. The points on $\beta$ and $\gamma$ will be added to $K'$.

by attaching at most $D$ shallow stars to this subgraph.

Therefore we have reduced to case of a unary FA-presentable directed tree $(T, \eta)$ whose vertices all have bounded degree. So we now assume that $(\alpha^*, \phi)$ is a unary FA-presentation for such a graph. By Lemma 6.7, the diagram for $(\alpha^*, \phi)$ contains only short arrows. We define $K$ as before, to consist of $C[0] \cup C[1]$ together with all points that lie on simple paths starting and ending in $C[0] \cup C[1]$.

Consider any path $\alpha$ starting from some point in $p \in K$ and otherwise only visiting points outside $K$. Suppose this path visits points of at least $D + 1$ different columns in $C[p, \infty)$. Then $\alpha$ visits at least one point in every column in $C[p, c(p) + D]$. For each $l \in \{c(p), \ldots, c(p) + D\}$, let $x_l$ be the first point $\alpha$ visits in $C[l]$. Since $\alpha$ consists only of short arrows, it visits some point in each column in $C[p, l]$ before it visits $x_l$. Hence $\alpha$ visits $x_k$ before $x_l$ whenever $k < l$. Since there are only $D$ rows, there exist $k$ and $l$ with $k < l$ such that $b(x_k) = b(x_l)$.

Let $\beta$ be the subpath of $\alpha$ from $p$ to $x_k$ and let $\gamma$ be the subpath from $x_k$ to $x_l$. Let $m = l - k = c(x_l) - c(x_k)$. Then $x_l = x_k \tau_m$. Since $\gamma$ is a subpath of $\alpha$, it lies entirely outside $K$ and so does not visit $C[0]$. Hence the path $(\gamma)\tau_n$ is defined and present for all $n \in \mathbb{N}^0$ by Arrow rule 1. So the concatenation of $(\gamma)\tau_m, (\gamma)\tau_{2m}, \ldots$ is an infinite path starting at $x_l$ and formed by ‘periodic’ repetitions of a translation of $\gamma$. (See Figure 11.) We will call these paths the infinite periodic tracks. Notice that this infinite periodic track does not include $\gamma$ or $\beta$.

The aim is to show that the edges corresponding to the arrows in this infinite periodic track form the spine of a periodic path. Notice that there can be at most $D$ distinct infinite periodic tracks, since each one must visit at least one vertex in each column to the right of its starting-point (because there are only short arrows in this diagram). Furthermore, these infinite periodic tracks will be disjoint.

First of all, we are going to define a finite set $L$ of points. These points will correspond to the vertices that do not form part of an infinite periodic path. We need to ensure everything ‘non-periodic’ lies in $L$. First of all, we will gather all the points lying on the various paths $\beta$ and $\gamma$ into a set $K'$, then we will deal with any other points in that might be connected to $K'$ by paths that do not give rise to infinite periodic tracks.

Let $\beta$ and $\gamma$ range over all possible values obtained from paths $\alpha$ as de-
scribed above. Then let \( K' \) consist of the points on the various paths \( \beta \) and \( \gamma \) except the last point of \( \gamma \) (which is also the first point on \( (\gamma)\tau_m \)). The set \( K' \) is finite since there are at most \( D \) infinite periodic tracks and each path \( \beta \) is of bounded length. Notice that all the infinite periodic tracks start from a point adjacent to a point in \( K' \) (and indeed visit no points of \( K' \)).

Let \( x \) be some point that is connected to \( K \) by a path that does not include any point of an infinite periodic track. Then \( x \) is connected to some point \( s \in K \) by a simple path \( \alpha \) of bounded length, since otherwise it would visit at least \( D+1 \) columns of \( C[s, \infty) \) and so would give rise to an infinite periodic track by the reasoning above and thus would include at least the first point (the point \( x_L \), which is the first point of \( \gamma\tau_m \)) of this path. Therefore all such points \( x \) are connected to \( K \) by simple paths of bounded length. Since the vertices of \( (T, \eta) \) have bounded degree by assumption, there can only be finitely many such points. Let \( K'' \) be the set of these points. Let \( L = K \cup K' \cup K'' \), and let \( L' \) be \( L \) together with the starting points of the infinite periodic tracks (each of which is adjacent to some point of \( K' \subseteq L \). Notice that \( L \) and \( L' \) are connected sets, and that any point not in \( L \) is connected to some point in \( K' \subseteq L \) by a unique simple path that visits at least one element of an infinite periodic track.

**Lemma 6.11.** Let \( \delta \) be a path starting at a point on some \( (\gamma)\tau_m \) (for some \( h \in \mathbb{N} \)) and not including any edge of \( (\gamma)\tau_m \) or \( \gamma \). (Here \( \gamma \) is as in the definition of an infinite periodic track.) That is, \( \delta \) ‘branches off’ from the infinite periodic track at some point on \( (\gamma)\tau_m \). Then:

1. The path \( \delta \) does not visit any point of \( L \).
2. For every \( j \) with \( j > -h \), the path \( (\delta)\tau_{(h+j)m} \) is defined and present in the diagram.
3. The path \( \delta \) has length at most \( D^2 + D \).

**Proof of 6.11.** 1. Suppose that \( \delta \) visits some point \( y \) of \( L \). Without loss of generality, assume \( y \) is the first point of \( L \) that \( \delta \) visits. Now, if \( \delta \) branches off from the infinite track at any point except its first point (the first point of \( (\gamma)\tau_m \)), then the second point on \( \delta \) must lie outside \( L \). (Since \( L \), by definition, contains no points connected to \( K \subseteq L \) by a path including a point of an infinite track.) On the other hand, if \( \delta \) branches off at the first point of \( (\gamma)\tau_m \), then the first arrow on \( L \) is not the last arrow on \( \gamma \), and so again the second point of \( \delta \) lies outside \( L \). Therefore we obtain a cycle by following the path \( \delta \) to \( y \), then the path in \( L \) back to the start of the infinite track, then back along the infinite path to the start of \( \delta \). This is a contradiction, and so \( \delta \) cannot visit any point of \( L \).

2. Suppose that for some \( j > -h \), the path \( (\delta)\tau_{(h+j)m} \) is not defined. Since \( \delta \) consists only of short arrows, we can use the reasoning in the proof of Lemma 6.10 to take a shorter path \( \delta' \) such that \( (\delta')\tau_{(h+j)m} \) is defined, present in the diagram, and ends at some point in \( C[I] \subseteq L \). (See Figure 12.) Since \( \delta \) does not include any edge of any \( (\gamma)\tau_m \) or \( \gamma \), neither does \( \delta' \). Hence \( (\delta')\tau_{(h+j)m} \) does not include any edge of \( \gamma \) or \( (\gamma)\tau_m \). This contradicts part 1 applied to \( (\delta')\tau_{(h+j)m} \).

3. Suppose \( \delta \) has length greater than \( D^2 + D \). Let \( \delta \) branch off from the infinite periodic track at a point \( q \). Since \( \delta \) includes no point from \( C[0] \subseteq L \), the path \( (\delta)\tau_{jm} \) is defined for all \( j \in \mathbb{N}^0 \). Since the infinite track is made up of \( (\gamma)\tau_m \), \((\gamma)\tau_{2m} \), \ldots , it follows that \( q\tau_{jm} \) lies on the infinite track for all \( j \in \mathbb{N}^0 \). Then since \( \delta \) has length greater than \( D^2 + D \) and there are only
Lemma Arrow

Figure 12. We suppose there is some path \( \delta \) and \( j > -h \) such that \( (\delta)_{\tau_{(h+j)m}} \) is undefined. Then \( \tau_{(h+j)m} \) shifts part of \( \delta \) off the left-hand side of the diagram. Since \( \delta \) consists only of short arrows, we can choose \( \delta' \) to be that initial part of \( \delta \) that is shifted by \( \tau_{(h+j)m} \) to end in \( C[1] \).

D rows and \( m \leq D + 1 \) there exist two points \( x \) and \( y \) on \( \delta \) such that \( b(x) = b(y) \) and \( c(x) \equiv c(y) \pmod{m} \). Without loss of generality, suppose \( c(x) < c(y) \). Let \( j = (c(y) - c(x))/m \). Then \( x\tau_{jm} = y \), and so \( y \) also lies on the path \( (\delta)_{\tau_{jm}} \), which is defined and present in the diagram by Arrow rule 1. Let \( \alpha_1 \) be the subpath of \( \delta \) from \( q \) to \( y \); let \( \alpha_2 \) be the subpath of \( (\delta)_{\tau_{jm}} \) from \( q\tau_{jm} = y \). Then \( \alpha_1, \alpha_2 \), and the part of the infinite periodic track between \( q \) and \( q\tau_{jm} \) form a cycle, which is a contradiction. So \( \delta \) cannot have length greater than \( D^2 + D \).

For \( i \in \mathbb{N} \), let \( F_i \) consist of all the points connected to \( (\gamma)_{\tau_{im}} \) by any simple path that does not include any edges of \( (\gamma)_{\tau_{im}} \) or \( (\gamma)_{\tau_{(i-1)m}} \). (That is, \( F_i \) consists of the points lying on paths that branch off from the infinite periodic track at some point on \( (\gamma)_{\tau_{im}} \).) Notice that \( F_i \) contains all points of \( (\gamma)_{\tau_{(im)}} \). By Lemma 6.11(3), each such path is of bounded length. Since the graph is of bounded degree, each \( F_i \) is finite. Furthermore, by Lemma 6.11(3), \( F_i \tau_{jm} = F_{i+1} \) (where \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \) with \( -i < j \)), and the subgraphs induced by each \( F_i \phi \) are isomorphic.

Let \( (S, \sigma) \) be the subgraph induced by the vertices \( F_1 \phi \). Let \( s_0 \) be the vertex corresponding to the first vertex on \( (\gamma)_{\tau_m} \), Let \( s_1 \) be the vertex corresponding to the last vertex on \( (\gamma)_{\tau_m} \). Then since the subgraphs induced by the \( F_i \phi \) are isomorphic, the infinite periodic track \( (\gamma)_{\tau_m}, (\gamma)_{\tau_{2m}}, \ldots \) is mapped by \( \phi \) to the spine of the infinite periodic path \( P(S, \sigma, s_0, s_1) \). It is clear that \( P(S, \sigma, s_0, s_1) \) is attached at its base to some vertex in the rest of the graph.

It has already been established that there are only finitely many infinite periodic paths, so the graph must be made up of finitely many periodic paths attached to the finite subgraph induced by \( L\phi \). This completes the proof in this direction.

Equipped with a characterization of unary FA-presentable directed trees, we now turn to characterizing the unary FA-presentable directed forests:

**Theorem 6.12.** A countable directed forest is unary FA-presentable if and only if:

1. It has only finitely many infinite components, each of which is a unary FA-presentable directed tree.
2. There is a bound on the size of its finite components.
**Proof of 6.12.** Let \((a^*, \phi)\) be an injective unary FA-presentation for a directed forest \((T, \eta)\). Let \(\zeta\) be the equivalence relation generated by \(\eta\); then \(\Lambda(\zeta, \phi)\) is regular by Corollary 4.7 and so \((a^*, \phi)\) is a unary FA-presentation for \((T, \eta, \zeta)\). Notice that \(\zeta\) is the undirected reachability relation on \((T, \eta)\) and so its equivalence classes are the connected components of \((T, \eta)\), which are directed trees. By Theorem 2.7, there are finitely many infinite components and a bound on the cardinality of the finite components. Consider some infinite component \(U\). Notice that the set of elements in \(U\) first-order definable in terms of \(\zeta\) and some \(u \in U\). So the language \(K\) of words in \(a^*\) that represent elements of \(U\) (that is, \(K = U\phi^{-1}\)) is regular. Thus \((K, \phi|_K)\) is a unary FA-presentation for the component \(U\). Since \(U\) was arbitrary, every infinite component is unary FA-presentable. This completes one direction of the proof.

Let \((T, \eta)\) be a countable directed forest that has only finitely many infinite components, each of which is a unary FA-presentable directed tree, and with a bound on the size of its finite components.

Consider the finite components. Since each is a directed tree and there is a bound on their cardinalities, there are only finitely many isomorphism types amongst them. Let \((P_1, \pi_1), \ldots, (P_p, \pi_p)\) be those finite components whose isomorphism types appear only finitely many times among the finite components of \((T, \eta)\). Suppose there are \(q\) different isomorphism types that appear infinitely often. For \(i \in \{1, \ldots, q\}\), choose a representive \((Q_i, \kappa_i)\) of each isomorphism class. Let \((R_1, \rho_1), \ldots, (R_r, \rho_r)\) be the infinite components.

For each \(i\), the union of countably many copies of the finite directed tree \((Q_i, \kappa_i)\) is unary FA-presentable by Lemma 2.9. The union \((Q, \kappa)\) of the \(q\) forests thus obtained is unary FA-presentable by iterated application of Lemma 2.8. Thus the directed forest \((T, \eta)\), which is the union of \((Q, \kappa)\) and the various \((P_i, \pi_i)\) and \((R_i, \rho_i)\) is unary FA-presentable by iterated application of Lemma 2.8.

Finally, we can apply the characterization of unary FA-presentable directed forests to obtain a characterization of unary FA-presentable [undirected] forests:

**Theorem 6.13.** A forest is unary FA-presentable if and only if it can be obtained from a unary FA-presentable directed forest by changing directed edges to undirected edges (that is, by replacing the edge relation with its symmetric closure).

**Proof of 6.13.** First, notice that if \((T, \eta)\) is a unary FA-presentable directed forest, then the symmetric closure \(\sigma\) of \(\eta\) is first-order definable in terms of \(\eta\). Thus \((T, \sigma)\) is also unary FA-presentable.

Let \((T, \eta)\) be forest admitting an injective unary FA-presentation \((a^*, \phi)\). The edge relation \(\eta\) is symmetric. Define a new relation \(\eta'\) as follows

\[
(s, t)\eta' \iff ((s, t) \in \eta) \land (\ell(s) < \ell(t)).
\]

Then \((T, \eta')\) is a directed graph. For every pair of elements \(s, t\) that are connected by an (undirected) edge in \((T, \eta)\) (that is, both \((s, t)\) and \((t, s)\) are in \(\eta\), exactly one of \(\ell(s) < \ell(t)\) or \(\ell(t) < \ell(s)\) holds, and thus there is either a directed edge from \(s\) to \(t\) in \((T, \eta')\) (that is, \((s, t) \in \eta'\)) or an edge from \(t\) to \(s\) in \((T, \eta')\) (that is, \((t, s) \in \eta'\)). So \((T, \eta')\) is an undirected forest. Furthermore, \(\Lambda(\eta', \phi)\) is regular since a finite automaton can compare the lengths of its two input words. Thus \((T, \eta')\) is unary FA-presentable. It is clear that making \(\eta'\) symmetric yields \((T, \eta)\).
In this final section, we apply the results of previous sections, particularly § 6, to classify the orbit structures of unary FA-presentable maps and more generally partial maps.

The graph of a partial map \( f : X \rightarrow X \) is the directed graph with vertex set \( X \) and edge set \( \{(x, (x)f) : x \in X, (x)f \text{ is defined}\} \). Two elements \( x, y \in X \) lie in the same orbit if there exist \( m, n \in \mathbb{N}^0 \) such that \( (x)f^m = (y)f^n \). They lie in the same strong orbit if there exist \( m, n \in \mathbb{N}^0 \) such that \( (x)f^m = y \) and \( (y)f^n = x \). In terms of the graph, \( x \) and \( y \) lie in the same orbit if they are connected by an undirected path; \( x \) and \( y \) lie in the same strong orbit if there is a directed path from \( x \) to \( y \) and a directed path from \( y \) to \( x \). Hence the orbits of the partial map are the connected components of the graph; the strong orbits of the partial map are the strongly connected components of the graph. Our characterization results are all stated in these terms.

We start by characterizing unary FA-presentable maps and partial (Theorem 7.4). Starting from this result, we then obtain characterizations of unary FA-presentable injections and partial injections (Theorem 7.6), surjections and partial surjections (Theorem 7.5), and bijections and partial bijections (Theorem 7.7). Naturally, the characterization for bijections is equivalent to the previously known one [9, Theorem 7.12], albeit in a very different form.

Like the characterization result for directed trees (Theorem 6.9), the characterization results for orbit structures of unary FA-presentable partial maps are stated in terms of attaching periodic paths and shallow stars to finite graphs. We need to specify certain special types of periodic paths and shallow stars, and also define some related terms:

**Definition 7.1.** Retain notation from Definition 6.2.

An inward rooted tree is a tree with a distinguished vertex, called the root, towards which all its edges are oriented.

A shallow star \( S(T, \eta, t_0, t_1) \) is inwardly oriented if every edge of the template graph \( (T, \eta, t_0, t_1) \) is oriented towards the distinguished vertex \( t_0 = t_1 \). That is, every edge of \( S(T, \eta, t_0, t_1) \) is oriented towards the centre vertex.

A periodic path \( S(T, \eta, t_0, t_1) \) is inwardly oriented if every edge of the template graph \( (T, \eta, t_0, t_1) \) is oriented towards the distinguished vertex \( t_0 \). That is, every edge of \( P(T, \eta, t_0, t_1) \) is oriented towards the base.

A periodic path \( S(T, \eta, t_0, t_1) \) is outwardly oriented if every edge of the template graph \( (T, \eta, t_0, t_1) \) is oriented towards the distinguished vertex \( t_1 \). That is, every edge of \( P(T, \eta, t_0, t_1) \) is oriented towards the unbounded direction of the spine.

An inward path is the periodic path \( P([(t_0, t_1); ([t_1, t_0]), t_0, t_1]) \) (where the template has two vertices \( t_0 \) and \( t_1 \) and a single edge from \( t_1 \) to \( t_0 \)). Notice that an inward path is an inwardly oriented periodic path.

An outward path is the periodic path \( P([(t_0, t_1); ([t_0, t_1]), t_0, t_1]) \) (where the template has two vertices \( t_0 \) and \( t_1 \) and a single edge from \( t_0 \) to \( t_1 \)). Notice that an outward path is an outwardly oriented periodic path.

A bi-infinite path is a directed path with vertex set \( \{v_i : i \in \mathbb{Z}\} \) and edge relation \( \{(v_i, v_{i+1}) : i \in \mathbb{Z}\} \); this is isomorphic to the path obtained by attaching an outward path to an inward path at their base vertices.

**Theorem 7.2.** A map with a single orbit is unary FA-presentable if and only if its orbit can be obtained in one of the following ways:
figure 13. Possible example orbits of a unary FA-presentable maps and partial maps: (a) is finite, (b) contains a cycle (arising from case 1 in Theorem 7.2), (c) contains an infinite outwardly oriented periodic path (arising from case 2 in Theorem 7.2), and (d) contains a vertex of outdegree 0 (where the map is undefined, arising from case 3 in Theorem 7.3).

1. Start with a finite directed cycle. First, attach finitely many inward rooted finite trees (at their roots) to the cycle. To any vertices of the resulting finite graph, attach finitely many inwardly oriented periodic paths (at their bases) and finitely many inwardly oriented shallow stars (at their centres).

2. Start with an inward rooted finite tree. To the root of the tree, attach one outwardly oriented periodic path (at its base). To any vertices of the resulting graph, attach finitely many inwardly oriented periodic paths (at their bases) and finitely many inwardly oriented shallow stars (at their centres).

[Figure 13 shows some examples of orbits described in Theorem 7.3. Notice that if no periodic paths or shallow stars are attached case 1 gives a finite graph. Case 2 requires that an outwardly oriented periodic path is attached and so always yields an infinite graph.]

Proof of 7.2. First part. Let \((a^*, \phi)\) be a unary FA-presentation for \((X, f)\), where \(f : X \to X\) has only one orbit. Let \(Q\) be the transitive closure of \(f\); then \(\Lambda(Q, \phi)\) is regular by Theorem 4.6. Define a relation \(R\) on \(X\) by

\[
(x, y) \in R \iff ((x, y) \in Q) \land ((y, x) \in Q).
\]

Then two distinct elements of \(X\) are related by \(R\) if and only if they lie in the same strong orbit of \(f\). Notice that \((x, x) \in R\) if and only if \((x)f^m = x\) for some \(m > 0\). Thus the strong orbits of \(f\) are the \(R\)-classes plus singleton orbits for all elements not in some \(R\)-class.

Notice that any \(R\)-classes must form a directed cycle. There cannot be two distinct \(R\)-classes, for otherwise they would be connected by some path, and then some vertex on this path would have outdegree 2, which is impossible. So there is either a unique \(R\)-class or no \(R\)-class. Deal with these cases separately.
There is a unique R-class. Since $\Lambda(R, \phi)$ is regular, we can factor $(X, f)$ by
$R$ to get a unary FA-presentable map $(X, f')$, which is essentially the map
$(X, f)$ with all the points of the R-class merged to a single point $z$. Notice
that $(z) f' = z$.

Consider the graph $(X', f')$. Remove the single edge from $z$ to itself.
The resulting graph is a directed tree (with all edges oriented towards $z$)
and is unary FA-presentable. So by Theorem 6.9, this tree consists of a
finite tree with finitely many periodic paths and shallow stars attached.
Since all the edges are oriented towards $z$, this element $z$ must lie in the
finite graph (since the orientation of edges along the spine of a periodic
paths is periodic). So all the periodic decorated paths and all the shallow
stars are inwardly oriented.

Thus the original graph $(X, f)$ must consist of these same inwardly ori-
tented periodic paths and inwardly oriented shallow stars attached to finite
inward rooted trees, which are in turn attached at their roots to the finite
cycle that forms the unique R-class of $f$. Thus case 1 in the statement of the
theorem holds.

There is no R-class. Then the graph $(X, f)$ is a tree and so by Theorem
6.9, this tree consists of a finite tree $F$ with finitely many periodic paths
and shallow stars attached. Since all vertices have outdegree 1, all the
shallow stars are inwardly oriented. Since every vertex has outdegree 1,
all edges of $F$ must be oriented towards a particular vertex, where a single
outwardly oriented periodic path must be attached. All the other periodic
paths attached must be inwardly oriented, again by the fact that all vertices
have outdegree 1. Thus case 2 in the statement of the theorem holds.

Second part. If $(X, \eta)$ is a graph as described in the theorem statement, then
every vertex of $(X, \eta)$ has outdegree exactly 1. Thus we can define a map
$f : X \to X$ by letting $(x)f$ be the terminal vertex of the unique edge starting at
$x$. It is clear that $(X, \eta)$ is the graph of $(X, f)$. Furthermore, $(X, \eta)$, and hence
$(X, f)$ is unary FA-presentable by Lemmata 6.4 and 6.5.

Theorem 7.3. A partial map with a single orbit is unary FA-presentable if and only
if its orbit can be obtained in as described in case 1 or 2 of Theorem 7.2 or in the
following way:

3. Start with an inward rooted finite tree. To any vertices of this finite graph, attach
finitely many inwardly oriented periodic paths (at their bases) and finitely many
inwardly oriented shallow stars (at their centres).

Proof of 7.3. First part. Let $(\alpha^*, \phi)$ be a unary FA-presentation for $(X, f)$, where
$f : X \to X$ has only one orbit. Extend $f$ to a complete map $f' : X \to X$ by defining
$$(x)f' = \begin{cases} (x)f & \text{if } (x)f \text{ is defined} \\ x & \text{otherwise}. \end{cases}$$

From the graph perspective $(X, f')$ is formed by taking the graph $(X, f)$ and
adding a loop at every vertex of outdegree 0.

Notice that the support of $f$ is first-order definable and so $(X, f')$ is also
unary FA-presentable. Furthermore, $(X, f')$ also has only one orbit. So the
graph $(X, f')$ is as described in Theorem 7.2. If $(X, f)$ and $(X, f')$ are identical,
the proof is complete. So assume that $(X, f)$ and $(X, f')$ are distinct. Then at
least one loop is added to the graph \((X, f)\) to form \((X, f')\). So case 1 of Theorem 7.2 applies, with the initial cycle being a loop at a single vertex. Removing this (unique) loop to recover \((X, f)\) gives a graph obtained as described in case 3 of the theorem statement.

**Second part.** If \((X, \eta)\) is a graph as described in the statement, then every vertex of \((X, \eta)\) has outdegree 0 or 1. Thus we can define a partial map \(f : X \to X\) by letting \((x)f\) be the terminal vertex of the unique edge starting at \(x\), if such an edge exists, and otherwise leaving \((x)f\) undefined. It is clear that \((X, \eta)\) is the graph of \((X, f)\). Furthermore, \((X, \eta)\), and hence \((X, f)\) is unary FA-presentable by Lemmata 6.4 and 6.5.

With this characterization of individual orbits, the characterization of the orbit structures of unary FA-presentable maps now follows quickly:

**Theorem 7.4.** A map (respectively, partial map) is unary FA-presentable if and only if the following conditions hold:

1. There is a bound on the size of the finite orbits.
2. There are finitely many infinite orbits.
3. Each orbit is unary FA-presentable and so as described in Theorem 7.2 (respectively, Theorem 7.3).

**Proof of 7.4.** Let \((X, f : X \to X)\) be a unary FA-presentable map (respectively, partial map). If \(X\) is finite, there is nothing to prove. So assume \(X\) is infinite and let \((a^*, \phi)\) be a unary FA-presentation for \((X, f)\). Let \(Q\) be the equivalence relation generated by \(f\). Since \(L(Q, \phi)\) is regular by Corollary 4.7, \(Q\) must have finitely many infinite equivalence classes and a bound on the size of its finite equivalence classes by Theorem 2.7. But the equivalence classes are simply the orbits of \(f\). It remains to observe that since the membership relation of each of the equivalence classes is first-order definable, the set of words representing elements of any orbit is regular, and thus the map (respectively, partial map) \(f\) restricted to any orbit is unary FA-presentable, and hence the restriction of \(f\) to each such infinite orbit is thus as described in Theorem 7.2 (respectively, Theorem 7.3).

In the other direction, the result follows by applying Lemmata 2.8 and 2.9 in a manner similar to the proof of Theorem 6.12.

We can now characterize unary FA-presentable surjections, injections, and bijections.

**Theorem 7.5.** A surjective map is unary FA-presentable if and only if the following conditions hold:

1. There is a bound on the size of the finite orbits, and every finite orbit is a cycle.
2. There are finitely many infinite orbits, and each can be obtained in one of two ways:
   
   (a) Start with a finite directed cycle. First, attach finitely many inward rooted finite trees (at their roots) to the cycle. To every leaf vertex, and possibly to other vertices, of the resulting finite graph, attach finitely many inward infinite paths (at their bases).
figure 14. Possible example orbits of unary FA-presentable surjections and partial surjections: (a) is finite, (b) contains a cycle (arising from case 2(a) in Theorem 7.5), (c) contains an infinite outwardly oriented periodic path (arising from case 2(b) in Theorem 7.5), and (d) contains a vertex with outdegree 0 (where the map is undefined arising from case 2(c) in Theorem 7.5).

(b) Start with an inward rooted finite tree. To the root of the tree, attach root one outwardly oriented infinite path (at its base). To every leaf vertex, and possibly to other vertices, of the resulting graph, attach finitely many inwardly oriented periodic paths (at their bases).

A partial surjective map is unary FA-presentable if and only if the following conditions hold: its finite orbits are as described in condition 1, and its infinite orbit are obtained either as described in case 2(a) or 2(b) above or in the following way:

2. (c) Start with an inward rooted finite tree. To every leaf vertex, and possibly to other vertices, of the resulting graph, attach finitely many inwardly oriented periodic paths (at their bases).

Proof of 7.5. Complete surjective maps. Let $(X, f : X \to X)$ be a unary FA-presentable surjective map. Then its orbits are as described in Theorems 7.4 and 7.2. Every vertex of the graph $(X, f)$ has indegree at least 1. Every finite orbit must therefore be a cycle.

Consider some infinite orbit. Suppose first that this orbit contains a cycle (case 1 of Theorem 7.2). Every vertex that does not lie on this cycle must lie on an infinite inward path, by induction using the fact that every vertex has indegree at least one. Thus there can be no shallow stars attached, all attached periodic paths must be inward infinite paths, and at least one inward infinite path must be attached to every leaf vertex of the finite graph.

Suppose now that this orbit does not contain a cycle (case 2 of Theorem 7.2). Then every vertex must lie on an infinite inward path, by induction using the fact that every vertex has indegree at least one. Thus there can be no shallow stars attached, a single outward infinite path must be attached to the
root of the finite tree, all other attached periodic paths must be inward infinite path, and at least one inward infinite path must be attached to every leaf vertex of the initial finite tree.

In the other direction, a graph of the form described in the statement is the graph of a unary FA-presentable map by Theorem 7.4. Furthermore, every vertex of such a graph has outdegree exactly 1 and indegree at least 1 and hence is the graph of a surjection.

**Partial surjective maps.** The strategy is essentially the same as the proof Theorem 7.3, so we only sketch the proof. Let \((X, f : X \to X)\) be a unary FA-presentable surjective partial map. Extend the map to a complete map \(f'\) by defining \((x) f' = x\) whenever \((x) f\) is undefined. Note that this preserves unary FA-presentability and surjectivity. Any infinite orbit where \(f'\) does not coincide with \(f\) contains a loop and so case 2(a) applies. Removing this loop yields a graph that can be obtained as per case 2(c).

**Theorem 7.6.** An injective map is unary FA-presentable if and only if its orbits satisfy the following conditions:

1. There is a bound on the size of the finite orbits, and every finite orbit is a cycle.
2. There are finitely many infinite orbits, each being either an outwardly oriented infinite path or a bi-infinite path.

A partial injective map is unary FA-presentable if and only if its orbits satisfy the following conditions:

3. There is a bound on the size of the finite orbits, and every finite orbit is a cycle or a finite path.
4. There are finitely many infinite orbits, each being either an outwardly oriented infinite path, an inwardly oriented infinite path or a bi-infinite path.

**Proof of 7.6.** Let \((X, f : X \to X)\) be a unary FA-presentable injective map (respectively, partial injective map). Then its orbits are as described in Theorems 7.4 and 7.2 (respectively, Theorems 7.4 and 7.3). In particular, there is a bound on the size of the finite orbits and finitely many infinite orbits.

Now, every vertex of the graph \((X, f)\) has outdegree 1 (respectively, at most 1) and indegree at most 1. It follows that every finite orbit must be a cycle (respectively, a cycle or a finite path), and every infinite orbit either an outward (respectively, outward or inward) infinite path or a bi-infinite path.

In the other direction, a graph of the form described in the statement is the graph of a unary FA-presentable map (respectively, partial map) by Theorem 7.4. Furthermore, every vertex of such a graph has outdegree 1 (respectively, at most 1) and indegree at most 1 and hence is the graph of an injection (respectively, partial injection).
THEOREM 7.7. A bijective map is unary FA-presentable if and only if its orbits satisfy the following conditions:

1. There is a bound on the size of the finite orbits, and every finite orbit is a cycle.
2. There are finitely many infinite orbits, each being a bi-infinite path.

A partial bijection is unary FA-presentable if and only if its orbits satisfy the following conditions:

1. There is a bound on the size of the finite orbits, and every finite orbit is a cycle.
2. There are finitely many infinite orbits, each being an inward infinite path or a bi-infinite path.

Proof of 7.7. Let \((X, f : X \to X)\) be a unary FA-presentable bijection (respectively, partial bijection). Then in particular \(f\) is injective and so its orbits are as described in Theorem 7.6. Every vertex of the graph \((X, f)\) has outdegree 1 (respectively, at most 1) and indegree 1 and no infinite orbit can consist of a outward infinite path. Every infinite orbit is thus a bi-infinite path (respectively, an inward infinite path or a bi-infinite path).

In the other direction, a graph of the form described in the statement is the graph of a unary FA-presentable map (respectively, partial map) by Theorem 7.4. Furthermore, every vertex of such a graph has outdegree exactly 1 (respectively, at most 1) and indegree exactly 1 and hence is the graph of a bijection (respectively, partial bijection).

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