The equivalence between the Hamiltonian and Lagrangian formulations for the parametrization invariant theories

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Summary.- The link between the treatment of singular Lagrangians as field systems and the canonical Hamiltonian approach is studied. It is shown that the singular Lagrangians as field systems are always in exact agreement with the canonical approach for the parametrization invariant theories.

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1 Introduction

In previous papers [1-4] the Hamilton-Jacobi formulation of constrained systems has been studied. This formulation leads us to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE) as follows:

\[ H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\alpha}) = 0, \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \]  
(1)

where

\[ H'_\alpha = H_\alpha(t_\beta, q_a, p_a), \]
(2)

and \( H_0 \) is defined as

\[ H_0 = p_a w_a + p_\mu q_\mu \big|_{q_\nu = -H_\nu} - L(t, q_\nu, \dot{q}_\nu = w_a), \]
\[ \mu, \nu = n - r + 1, ..., n. \]
(3)

The equations of motion are obtained as total differential equations in many variables as follows:

\[ dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha. \]
(4)

\[ dz = (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha; \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r \]
(5)

where \( z = S(t_\alpha; q_a) \). The set of equations (4,5) is integrable [3,4] if

\[ dH'_0 = 0, \]
\[ dH'_\mu = 0, \quad \mu = n - r + 1, ..., n. \]
(6)

If condition (6,7) are not satisfied identically, one considers them as new constraints and again tests the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates
$q_a$ and $p_a$ as functions of $t_\alpha$, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians $H'_\alpha$ are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$ respectively.

In ref. [5] the singular Lagrangians are treated as field systems. The Euler-Lagrange equations of singular systems are proposed in the form

$$\frac{\partial}{\partial t_\alpha}\left[\frac{\partial L'}{\partial (\partial_\alpha q_a)}\right] - \frac{\partial L'}{\partial q_a} = 0, \quad \partial_\alpha q_a = \frac{\partial q_a}{\partial t_\alpha}, \quad (8)$$

with constraints

$$dG_0 = -\frac{\partial L'}{\partial t} dt, \quad (9)$$

$$dG_\mu = -\frac{\partial L'}{\partial q_\mu} dt, \quad (10)$$

where

$$L'(t_\alpha, \partial_\alpha q_a, \dot{q}_\alpha, q_a) = L(q_a, q_\alpha, \dot{q}_a = (\partial_\alpha q_a)\dot{t}_\alpha), \quad \dot{q}_\mu = \frac{dq_\mu}{dt}, \quad (11)$$

$$G_\alpha = H_\alpha(q_a, t_\beta, p_a = \frac{\partial L}{\partial q_a}). \quad (12)$$

In order to have a consistent theory, one should consider the variations of the constraints (9), (10).

In this paper we would like to study the link between the treatment of singular Lagrangians as field systems and the canonical formalism for the parametrization invariant theories.

## 2 Prametrization invariant theories as singular systems

In ref. [3] the canonical method treatment of the parametrization-invariant theories is studied and will be briefly reviewed here.

Let us consider a system with the action integral as

$$S(q_i) = \int dt \mathcal{L}(q_i, \dot{q}_i, t), \quad i = 1, \ldots, n, \quad (13)$$

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where $\mathcal{L}$ is a regular Lagrangian with Hessian $n$. Parameterize the time $t \rightarrow \tau(t)$, with $\dot{\tau} = \frac{dt}{d\tau} > 0$. The velocities $\dot{q}_i$ may be expressed as
\begin{equation}
\dot{q}_i = q'_i \dot{\tau},
\end{equation}
where $q'_i$ are defined as
\begin{equation}
q'_i = \frac{dq_i}{d\tau}.
\end{equation}
Denote $t = q_0$ and $q_\mu = (q_0, q_i), \mu = 0, 1, ..., n$, then the action integral (13) may be written as
\begin{equation}
S(q_\mu) = \int d\tau i\mathcal{L}(q_\mu, \frac{q'_i}{\dot{\tau}}),
\end{equation}
which is parameterization invariant since $L$ is homogeneous of first degree in the velocities $q'_\mu$ with $L$ given as
\begin{equation}
L(q_\mu, q'_\mu) = i\mathcal{L}(q_\mu, \frac{q'_i}{\dot{\tau}}).
\end{equation}
The Lagrangian $L$ is now singular since its Hessian is $n$.

The canonical method [1-4] leads us to obtain the set of Hamilton-Jacobi partial differential equations as follows:
\begin{align}
H_0' &= p_\tau - L(q_0, q_i, \dot{q}_0, \dot{q}_i = w_i) + p_i^r q'_i + \\
P_\tau q_0 \big|_{p_\tau=-H_\tau=0} &= p_\tau = \frac{\partial S}{\partial \tau}, \\
H_t' &= p_t + H_t = 0, \\
p_t = \frac{\partial S}{\partial t},
\end{align}
where $H_t$ is defined as
\begin{equation}
H_t = -\mathcal{L}(q_i, w_i) + p_i^r w_i.
\end{equation}
Here, $p_i^r$ and $p_t$ are the generalized momenta conjugated to the generalized coordinates $q_i$ and $t$ respectively.

The equations of motion are obtained as total differential equations in many variables as follows:
\begin{equation}
dq^i = \frac{\partial H_0'}{\partial p_i} d\tau + \frac{\partial H_t'}{\partial p_i} dq^0 = \frac{\partial H_t'}{\partial p_i} dq^0,
\end{equation}
\[
\frac{dp_i}{dt} = -\frac{\partial H'}{\partial q_i} d\tau + \frac{\partial H'}{\partial q_i} dq^0 = -\frac{\partial H'}{\partial q_i} d\tau,
\]
\[
\frac{dp_t}{dt} = -\frac{\partial H'}{\partial q_0} d\tau + \frac{\partial H'}{\partial q_0} dq^0 = 0.
\]

Since
\[
dH' = dp_t + H_t,
\]

vanishes identically, this system is integrable and the canonical phase space coordinates \(q_i\) and \(p_i\) are obtained in terms of the time \((q_0 = t)\).

Now, let us look at the Lagrangian (17) as a field system. Since the rank of the Hessian matrix is \(n\), this Lagrangian can be be treated as a field system in the form
\[
q_i = q_i(\tau, t),
\]
thus, the expression
\[
q'_i = \frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t} \dot{t},
\]
can be replaced in eqn. (17) to obtain the modified Lagrangian \(L'\):
\[
L' = i\mathcal{L}(q_i, \frac{1}{t}(\frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t} \dot{t})).
\]

Making use of eqn (8), we have
\[
\frac{\partial L'}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L'}{\partial (\frac{\partial q_i}{\partial \tau})} \right) - \frac{\partial}{\partial \tau} \left( \frac{\partial L'}{\partial (\frac{\partial q_i}{\partial t})} \right) = 0.
\]

Calculations show that eqn. (28) leads to well-known Lagrangian equation as
\[
\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial (\frac{\partial q_i}{\partial t})} \right) = 0.
\]

Using eqn. (20), we have
\[
H_t = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial q_i} q_i.
\]

In order to have a consistent theory, one should consider the total variation of \(H_t\). In fact
\[
dH_t = -\frac{\partial \mathcal{L}}{\partial t} dt.
\]
Making use of eq. (10), one finds
\[ dH_t = -\frac{\partial L'}{\partial t} d\tau. \] (32)
Besides, the quantity \( H_0 \) is identically satisfied and does not lead to cons-
triants.
One should notice that equations (21,22) are equivalent to equations (28,29).

3 Classical fields as constrained systems

In the following sections we would like to study the Hamiltonian and the
Lagrangian formulations for classical field systems and demonstrating the
equivalence between these two formulations for the reparametrization in-
variant fields.
A classical relativistic field \( \phi_i = \phi_i(\vec{x}, t) \) in four space-time dimensions
may be described by the action functional
\[ S(\phi_i) = \int dt \int d^3 x \{ L(\phi_i, \partial_\mu \phi_i) \}, \quad \mu = 0, 1, 2, 3, \quad i = 1, 2, \ldots, n, \] (33)
which leads to the Euler-Lagrange equations of motion as
\[ \frac{\partial L}{\partial \phi_i} - \partial_\mu \left[ \frac{\partial L}{\partial (\partial_\mu \phi_i)} \right] = 0. \] (34)
One can go over from the Lagrangian description to the Hamiltonian
description by using the definition
\[ \pi_i = \frac{\partial L}{\partial \dot{\phi}_i}, \] (35)
then canonical Hamiltonian is defined as
\[ H_0 = \int d^3 x (\pi_i \dot{\phi}_i - L). \] (36)
The equations of motion are obtained as
\[ \dot{\pi}_i = -\frac{\partial H_0}{\partial \phi}, \] (37)
\[ \dot{\phi}_i = \frac{\partial H_0}{\partial \pi_i}. \] (38)
4 Reparametrization invariant fields

In analogy with the finite dimensional systems, we introduce the reparametrization invariant action for the field system as

\[ S = \int d\tau \int L_R d^3 x, \]  

where

\[ L_R = i\mathcal{L}(\phi_i, \partial_{\mu} \phi_i). \]  

Following the canonical method \([1-4]\), we obtain the set of [HJPDE] as

\[ H'_0 = \pi_{\tau} + \pi_{i}^{(\tau)} \frac{d\phi_i}{d\tau} + \pi_{t} \frac{dt}{d\tau} - L_R = 0, \quad \pi_{\tau} = \frac{\partial S}{\partial \tau}, \]  

\[ H'_t = \pi_t + H_t = 0, \quad \pi_t = \frac{\partial S}{\partial t}, \]  

where \(H_t\) is defined as

\[ H_t = -\mathcal{L}(\phi_i, \partial_{\mu} \phi_i) + \pi_{i}^{(\tau)} \frac{d\phi_i}{dt}, \]  

and \(\pi_{i}^{(\tau)}, \pi_t\) are the generalized momenta conjugated to the generalized coordinates \(\phi_i\) and \(t\) respectively.

The equations of motion are obtained as

\[ d\phi_i = \frac{\partial H'_0}{\partial \pi_i} d\tau + \frac{\partial H'_t}{\partial \pi_i} dt = \frac{\partial H'_t}{\partial \pi_i} dt, \]  

\[ d\pi^i = -\frac{\partial H'_0}{\partial \phi_i} d\tau - \frac{\partial H'_t}{\partial \phi_i} dt = -\frac{\partial H'_t}{\partial \phi_i} dt, \]  

\[ d\pi_t = -\frac{\partial H'_0}{\partial t} d\tau - \frac{\partial H'_t}{\partial t} dt = 0. \]

Now the Euler-Lagrangian equation for the field system reads as

\[ \frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\phi_i_{,\mu})} \right) = 0. \]  

Again as for the finite dimensional systems, equations \((44,45)\) are equivalent to equations \((47)\) for field systems.
5 Conclusion

As it was mentioned in the introduction, if the rank of the Hessian matrix for discrete systems is $n-r$; $0 < r < n$, then the systems can be treated as field systems [5]. The treatment of Lagrangians as field systems is always in exact agreement with the Hamilton-Jacobi treatment for reparametrization invariant theories. The equations of motion (21, 22) are equivalent to the equations of motion (28, 29). Besides the the variations of constraints (31) and (32) are identically satisfied and no further constraints arise.

In analogy with the finite dimensional systems, it is observed that the Lagrangian and the Hamilton-Jacobi treatments for the reparametrization invariant fields are in exact agreement.

References

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