Quantum Chaos of Bogoliubov Waves for a Bose-Einstein Condensate in Stadium Billiards

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We investigate the possibility of quantum (or wave) chaos for the Bogoliubov excitations of a Bose-Einstein condensate in billiards. Because of the mean field interaction in the condensate, the Bogoliubov excitations are very different from the single particle excitations in a non-interacting system. Nevertheless, we predict that the statistical distribution of level spacings is unchanged by mapping the non-Hermitian Bogoliubov operator to a real symmetric matrix. We numerically test our prediction by using a phase shift method for calculating the excitation energies.

PACS numbers: 05.45.-a, 03.65.Ta, 03.75.-b, 42.50.Vk

In recent years, the realization of Bose-Einstein condensation (BEC) of dilute gases \footnote{1} has opened new opportunities for studying dynamical systems in the presence of many-body interactions. However, most previous investigations have focused on one dimensional or high dimensional separable systems and the dynamics of BEC in nonseparable systems with two or more degrees of freedom have not received much attention \footnote{2}.

In the linear Schrödinger equation, systems with two or more degrees of freedom can be characterized by the statistics of energy levels: the typical distribution of the spacing of neighboring levels is Poisson or Gaussian Ensembles for separable or nonseparable systems respectively \footnote{3}. In the limit of short wavelengths (geometric optics) \footnote{4}, classical trajectories emerge from the linear Schrödinger equation and the two types of quantum statistics have been linked to different classical behaviors: Poisson to regular motion, while Gaussian Ensembles to chaotic motion. It is natural to ask whether these findings for the linear Schrödinger equation can be generalized to other types of wave equations \footnote{5}. The Bogoliubov equation \footnote{6} obtained from the linearization about the ground state of the Gross-Pitaevskii (G-P) equation has a purely real spectrum, and there is also a classical limit in the sense of geometric optics. It therefore makes sense and will be very interesting to explore the relationship between the Bogoliubov level statistics and regularity of the corresponding classical trajectories.

There is, however, an important difference between the two types of equations; while the Schrödinger equation is Hermitian, the Bogoliubov equation is non-Hermitian and its statistics can not readily be predicted by standard random matrix theory. In fact, the Bogoliubov equation belongs to the category of symplectic problems, describing linearized motion about stationary states in nonlinear classical Hamiltonian systems. This can be easily understood by noting that the G-P equation does have a classical Hamiltonian structure (of infinite dimensions, though) \footnote{7} and that the Bogoliubov equation describes excitations about a stationary solution of the G-P equation. The non-Hermiticity of the Bogoliubov equation makes it allowable to have complex eigenvalues in general, which signifies instability of the stationary solution. This will not happen about the ground state (lowest energy state) which is always stable. Therefore, our investigation of the Bogoliubov problem should shed light on the behaviors of motions around stable stationary states in extensive classical Hamiltonian systems.

In this Letter, we investigate the level statistics of Bogoliubov elementary excitation in separable circular as well as nonseparable stadium billiards. These are the excitations of a system of interacting particles in contrast with the modes of non-interacting particles described by the linear Schrödinger equation \footnote{8} \footnote{9}. The classical trajectories of Bogoliubov waves are found to be regular in circular billiards and chaotic in stadium billiards. By mapping the non-Hermitian Bogoliubov operator to a real symmetric matrix, we find the mean field interactions in the condensate do not change the level statistics of Bogoliubov excitations. This surprising result is tested numerically by using a phase shift method for calculating the excitation energy. In the regime of strong interaction and low excitation energy (phonon), we map the Bogoliubov equation to an equivalent Schrödinger equation with Neumann boundary condition and show that the statistics of Bogoliubov levels are the same as that for the Schrödinger equation, although interactions in the condensate do change the average number of levels up to a certain energy.

Consider condensed atoms confined in a quarter-stadium shaped trap of area $A$ (with length of the top straight side $L$, radius of the semicircle $R$) and height $d$, where $d << R$ so that lateral motion is negligible and the system is essentially two dimensional \footnote{10}. With only a quarter of a stadium, one is restricted to a single symmetry class of the full problem \footnote{11}. The dynamics of the BEC are described by the G-P equation

\[
\frac{\partial}{\partial t} \psi = -\frac{1}{2} \nabla^2 \psi + gN |\psi|^2 \psi, \tag{1}
\]
where \( g = 2\sqrt{2}a/d \) is the scaled strength of nonlinear interaction, \( N \) is the number of atoms, \( a \) is the s-wave scattering length. The ground state of BEC can be written as \( \psi = \psi_0 (\mathbf{r}) \exp (-i\mu t) \), where \( \mu \) is the chemical potential and \( \psi_0 (\mathbf{r}) \) can be taken as real. The length and the energy are measured in units of \( b = \sqrt{4A/\pi} \) and \( \hbar^2/ma^2 \) respectively, so that the scaled area of billiards is \( \tilde{A} = \pi/4 \). The dynamics of the elementary excitations are obtained by linearizing G-P equation about the ground state \( \psi_0 \) and their energy spectrum is described by the time-independent Bogoliubov equation [6]

\[
L \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}, \quad L = \sigma_z \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}, \quad (2)
\]

where \( \sigma_z \) is the Pauli matrix, \( H_1 = -\frac{1}{2} \nabla^2 + 2gN\psi_0^2 - \mu \), \( H_2 = gN\psi_0^2 \), \( E \) is the Bogoliubov excitation energy, and \( (u, v) \) is the eigenfunction of linear operator \( L \). The ground state wavefunction \( \psi_0 \) (excitation \( (u, v) \)) has the normalization \( \int_{\tilde{A}} \psi_0^2 dx \, dy = 1 \) (\( \int_{\tilde{A}} (u^2 - v^2) dx \, dy = 1 \)) and satisfies the boundary condition \( \psi_{0\alpha\tilde{A}} = 0 \) (\( (u, v)_{\partial\tilde{A}} = 0 \)).

Classical trajectories or rays arise from the Bogoliubov equation in geometric optics approximation [2, 4]. Assume a trial Bogoliubov wave of the form \( (u, v) = (\alpha, \beta) e^{iS} \) and consider a slowly-varying medium \( (\nabla \alpha, \nabla \beta) \) small and a slowly-varying velocity \( (\nabla^2 S) \) small approximation. We obtain the Eikonal equation \( |\nabla S|^2 = p^2 \) with

\[
p = \sqrt{\mu + \sqrt{E^2 + (g N \psi_0^2)^2 - 2g N \psi_0^2}}.
\]

The classical trajectories of the Bogoliubov waves are still governed by the ray equation \( \frac{d}{dt}(p\dot{w}) = \nabla P \), where \( \dot{w} \) is the direction of the trajectory and \( w \) is the arc-length coordinate along the trajectory. Interestingly, the classical trajectories are simply straight lines for non-interacting as well as interacting uniform gases because \( p = \sqrt{E_k} \), the local momentum in both cases. In the regime of strong interaction where the ground state of BEC is nearly uniform, the classical trajectories of Bogoliubov waves are straight lines and undergo elastic specular reflection at the boundary of the billiard. Therefore we predict that the Bogoliubov level statistics are still Poisson in circular billiards and Gaussian Orthogonal Ensembles (GOE) in stadium billiards through quantum classical correspondence.

This prediction is supported by a general argument based on mapping the non-Hermitian Bogoliubov operator \( L \) to a real symmetric matrix. The linear operator \( L \) can be written as \( L = \sigma_z Q \), where \( Q \) is a real symmetric positive definite matrix because the ground state of BEC is thermodynamical stable [5]. The positive definiteness of \( Q \) yields the decomposition \( Q = T^T T \) (\( T \) is a real matrix with nonzero eigenvalues) and the Bogoliubov equation reduces to

\[
T \sigma_z T^T \left( T \begin{pmatrix} u \\ v \end{pmatrix} \right) = E \left( T \begin{pmatrix} u \\ v \end{pmatrix} \right), \quad (3)
\]

Therefore Bogoliubov excitation energy is the eigenvalue of a real symmetric matrix \( T \sigma_z T^T \) and should have GOE distribution in stadium billiards for arbitrary interaction strength [5].

In the following, we report numerical test of this prediction by developing a phase shift method to calculate the Bogoliubov excitation energy. Notice that the condensate density is nearly uniform [11] in the interior of billiards that yields the planewave forms of the Bogoliubov excitations. Similar to the scattering wave method in linear quantum mechanics [12], the nonuniform condensate density close to the boundary and the hard walls can be taken as a pseudopotential and the scattering by this pseudopotential only induces a phase shift of the excited planewave in the interior of the billiard. The phase shift may be determined by solving the one-dimensional G-P equation with an infinite wall at \( x \leq 0 \). Far from the wall, the Bogoliubov equation has both planewave and exponential solutions for a certain excitation energy. We numerically integrate [13] the one-dimensional Bogoliubov equation with two different initial conditions to eliminate the exponential terms and extract the planewave solutions. The phase shift \( \delta \) is obtained by comparing the numerical solution with the expected sin(\( kx + \delta \)) dependence. The result is shown in Fig. 1 as a function of \( k/\sqrt{g'} \), where \( g' = gN\psi_0^2 \) and \( \psi_0 \) is the condensate wavefunction far from the wall. We see that the phase shift approaches \( \pi/2 \) in the limit of low energy excitation and strong interactions (phonon) and asymptotically approaches zero in the regime of high excitation energy and weak interaction (free particles). As expected, the phase shift is always zero for \( g' = 0 \).

With the phase shift method, we can calculate the Bogoliubov excitation energy in one-dimensional billiards. The reflected planewaves from the boundary walls \( x = 0 \) and \( L \) can be written as \( \phi_0 = C \sin (kx + \delta_k) \) and \( \phi_L = F \sin(-k(x-L) + \delta_k) \), respectively. The continuum of the wavefunction and its derivative in the interior of the billiards require \( \phi_0/\phi'_0 = \phi_L/\phi'_L \) that yields the quantization condition

\[
kL + 2\delta_k = n\pi, \quad (4)
\]

where \( n \) is an integer. Eq. (4) determines wavevector \( k \) and Bogoliubov excitation energy \( E = \sqrt{k^2/2 + 2g'} \).
Table I: Comparison of Bogoliubov excitation energies in one-dimensional billiards using both phase shift ($E_1$) and matrix diagonalization methods ($E_2$). $gN = 1000$.

| $E_1$ | $E_2$ | $E_1$ | $E_2$ | $E_1$ | $E_2$ |
|-------|-------|-------|-------|-------|-------|
| 0     | 2E-12 | 907.4 | 909.7 | 2144.4| 2147.4|
| 105.8 | 102.6 | 1038.7| 1041.1| 2333.0| 2336.0|
| 212.2 | 213.6 | 1175.9 | 1178.3 | 2529.7 | 2532.8 |
| 320.1 | 322.0 | 1319.6 | 1321.9 | 2735.3 | 2738.2 |
| 430.4 | 432.4 | 1470.0 | 1472.3 | 2949.0 | 2952.3 |
| 543.6 | 545.6 | 1627.0 | 1629.7 | 3171.9 | 3175.1 |
| 660.4 | 662.5 | 1791.7 | 1794.5 | 3403.6 | 3406.9 |
| 781.5 | 783.6 | 1964.1 | 1967.0 | 3644.1 | 3647.6 |

For the non-interacting case ($g = 0$), the phase shift $\delta = 0$ and Eq. (4) reduces to $k = n\pi/L$, the quantization condition for a single particle in one-dimensional billiards. As $k$ approaches zero ($E \to 0$), the phase shift $\delta_k \to \pi/2$, therefore $k = 0$ ($E = 0$) is a solution of Eq. (4) that corresponds to the ground state of BEC (the uniform density in the interior of the billiard indicates the $\pi/2$ phase shift).

To check the validity of the phase shift method, we also calculate the Bogoliubov excitation energies using traditional matrix diagonalization method in which the linear operator $L$ is represented as a matrix and the diagonalization process gives the excitation energy $E$. The results are compared with those from the phase shift method in Table 1. We see that the phase shift method gives accurate results for the Bogoliubov excitation energy. The wavefunction of the first three excited states is shown in Fig. 2. Clearly, the excitation wavefunction is described by the planewave $\sin(kx + \delta)$ in the interior of the billiards and drops to zero at the boundary.

In the regime of weak interaction and high excitation energy ($k^2/2 >> 2g'$), the excitations behave like free particles where the spectrum is well understood. Direct calculations of Bogoliubov excitation energies in two-dimensional billiards for arbitrary interaction strength and excitation energy are difficult for both phase shift and matrix diagonalization methods. However, we are more interested in the regime of strong interaction and low excitation energy ($k^2/2 << 2g'$, phonon), where the effect of interactions is essential. In this regime, the phase shift $\delta$ is approximately $\pi/2$ as seen from Eq. 1, which means that the first derivative of the planewave should be zero at the boundary, instead of the zero wavefunction for the single particle case. The Bogoliubov wavefunction far from the wall can be written as $(U_k, V_k) \phi$, where $(U_k, V_k) = \frac{1}{2} (\chi + \chi^{-1}, \chi - \chi^{-1})$, $\chi = \left(\frac{k^2/2}{g^2/2 + 2g'}\right)^{1/4} R$, $\phi$ satisfies the Schrödinger equation with Neumann boundary condition

$$\nabla^2 \phi + k^2 \phi = 0, \quad \frac{\partial \phi}{\partial \bar{n}} = 0,$$

where $\bar{n}$ is the normal direction of the billiard wall. Eq. (5) has an analytical solution $\phi = BJ_{2m}(kr) \cos(2m\theta)$ in quarter-circular billiards, where $J_{2m}(kr)$ is the Bessel function, $m$ is an integer, and $B$ is a normalization constant. The boundary condition is satisfied by requiring $J_{2m}'(kR) = 0$ that determines wavevector $k$ and Bogoliubov excitation energy. No analytical solution is available for stadium billiards and we use the ansatz of the superposition of planewaves $\phi(x, y) = \sum_{j=1}^{M} a_j \cos(k_{jx}x) \cos(k_{jy}y)$ (planewave decomposition method $[14]$), where $k_{jx} = k \cos(\theta_j)$, $k_{jy} = k \sin(\theta_j)$, $M$ is the number of planewaves, $\theta_j = 2j\pi/M$, i.e. the direction angles of the wave vectors are chosen equidistantly.

The ansatz solves Eq. (5) inside the billiards and the boundary condition determines the possible wavevector $k$ and Bogoliubov excitation energy.

The average number of energy levels $\langle N(E) \rangle$ up to $E$ should satisfy a Weyl-like type formula $[15]$

$$\langle N(E) \rangle = \frac{1}{4\pi} \left( A E' + D \sqrt{E'} + C \right),$$

where $g' = gN\sqrt{\phi_0} = 25000$. Solid and dotted lines are from numerical results and Eq. (6), respectively.

Figure 2: Bogoliubov excitation wavefunctions in one-dimensional billiards for $gN = 1000$. Solid lines are from the matrix diagonalization method and dotted lines are from phase shift method $\sin(kx + \delta)$. (a), (b) and (c) represent the first three excitation wavefunctions, respectively.

Figure 3: Plots of the average number of energy levels $\langle N(E) \rangle$ up to energy $E$ in quarter-circular (a) and quarter-stadium (b) billiards. $g' = gN\phi_0 = 25000$. Solid and dotted lines are from numerical results and Eq. (6), respectively.
that is tuned to the blue side of atomic resonance, forward motion could be confined to a closed pattern of light is in the vertical direction to counteract gravity. Transverse motion are still Poisson in circular billiards and GOE in stadium billiards. The statistics of Bogoliubov excitation energy levels effects [16].

In Fig. 3, we plot \( \langle \frac{E}{s} \rangle \) using planewave decomposition method. We have computed the Bogoliubov excitation energies in circular billiards \( (R = 1, L = 0) \) using the analytical solution and in stadium billiards \( (R = L = \sqrt{1/(1 + 4/\pi)}) \) using planewave decomposition method. The area and the perimeter of the billiard, and \( C \) is a constant related to the geometry and topology of the billiard boundary. Eq. (6) is only valid in the semiclassical limit \( E' = k^2 \gg \left( \frac{\Delta}{\gamma} \right)^2 \) that yields the Bogoliubov energy \( \frac{E'}{\sqrt{\gamma}} \gg \frac{\Delta}{\gamma} \left( 1 + \frac{\gamma}{\Delta} \right)^{1/2} \). In the limit of large interaction constant \( (E'/\gamma' \ll 1) \), we have \( \langle N(E) \rangle \approx \frac{1}{\pi^2} \left( A \left( \frac{E'}{\gamma'} - \frac{E'}{2\gamma'} \right) + D \sqrt{\frac{E'}{\gamma'}}\right) \) with the condition \( \frac{\Delta}{\gamma} \ll \frac{E'}{\sqrt{\gamma'}} \ll \sqrt{\gamma'} \).

We have computed the Bogoliubov excitation energies in circular billiards \( (R = 1, L = 0) \) using the analytical solution and in stadium billiards \( (R = L = \sqrt{1/(1 + 4/\pi)}) \) using planewave decomposition method. In Fig. 3, we plot \( \langle N(E) \rangle \) in both circular and stadium billiards. The agreement between the numerical results and the corresponding Weyl-like formula (Eq. (6)) is satisfactory. We unfold the spectrum formed by the energies \( E_n \), i.e., we evaluate Eq. (6) for each \( E_n \) in order to obtain the new energies \( \tilde{E}_n = \langle N(E_n) \rangle \). Note that the integer part of \( \tilde{E}_n \) is about \( n \) and, as a result, the corresponding mean level spacing is characterized through \( \langle s \rangle = \sum (\tilde{E}_{n+1} - \tilde{E}_n) / \langle N \rangle \approx 1 \). The resulting level spacing distributions \( P(s) \) are shown in Fig. 4. Clearly, the statistics of Bogoliubov excitation energy levels spacing are still Poisson in circular billiards and GOE in stadium billiards.

Experimentally, the system can be realized by confining BEC in two-dimensional optical billiards. Atoms could be trapped in a one dimensional optical lattice that is in the vertical direction to counteract gravity. Transverse motion could be confined to a closed pattern of light that is tuned to the blue side of atomic resonance, forming a repulsive barrier for the atoms. This pattern can be created by rapid scanning of a beam as was demonstrated for ultracold atoms. However, for the case of a BEC it would be better to create a static patterns, which can be accomplished using a liquidcrystal spatial light modulator [13]. The interaction strength may be adjusted by the confinement in the vertical direction, or by the number of atoms in each node of the standing wave. The Bogoliubov excitation energy may be measured using Raman transition between two hyperfine ground states, denoted [1] and [2] respectively. The condensate would be formed in state [1]. Two co-propagating Raman beams would drive a transition to state [2] and the number of atoms in that state would be measured as a function of the frequency difference in the beams. The coupling efficiency to the excited states must depend upon the spatial profiles of the beams, requiring more detailed analysis [13].

We acknowledge the support from the NSF, Quantum Optics Initiative US Navy - Office of Naval Research, Grant No. N00014-03-1-0639, and the R. A. Welch foundation, MGR also acknowledges supports from the Sid W. Richardson Foundation. We thank L.E. Reichl and E.J. Heller for helpful comments.

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