Regularity conditions for spherically symmetric solutions of
Einstein-nonlinear electrodynamics equations

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In this report, for static spherically symmetric (SSS) solutions of the Einstein
equations coupled to nonlinear electrodynamics (NLE) the regularity conditions at
the center are established. The NLE is derived from a Lagrangian \( \mathcal{L} = \mathcal{L}(\mathcal{F}) \),
depending on the electromagnetic invariant \( \mathcal{F} = F_{\mu \nu} F^{\mu \nu} / 4 \). Regular solutions are
characterized by the finite behavior at the center of the curvature invariants of the
Riemman tensor. For regular SSS metrics, the traceless Ricci (TR) tensor eigenvalue
\( S \), the Weyl tensor eigenvalue \( \Psi_2 \) and the scalar curvature \( R \) are singular–free at the
center. Regular NLE SSS electric solutions, which are characterized by the \( \mathcal{F}(r = 0) = 0 \), approach to the flat or conformally flat de Sitter–Anti de Sitter (regular)
spacetimes at the center; moreover, this family of solutions may exhibit different
asymptotic behavior at spatial infinity such as the Reissner–Nordström (Maxwell)
asymptotic, or present the dS–AdS or other kind of asymptotic. Pure magnetic
NLE SSS solutions shear the single magnetic invariant \( 2\mathcal{F}_m = h_0^2 / r^4 \), thus they are
singular in the magnetic field and may exhibit a regular flat or (A)dS behavior at
the center.

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I. INTRODUCTION

In our previous publication [1] dealing with a static spherically symmetric (SSS) metric in the framework of the Einstein–nonlinear electrodynamics theory, it was established that a linear superposition principle of solutions of different kinds holds. In this report we focus mostly in the determination of the conditions under which a SSS metric is regular (singular–free) at the center (origin of coordinates).

Nowadays “black hole” becomes a term of everyday use; in cinema, television programs, and internet are shown scenes about the evolution, collapse, interaction of black holes, emission of jets of energy, using the resources of numerical mathematical simulation and the abilities of the multimedia professionals. In spite of the large theoretical progress, and astrophysical advances in the black hole physics still remain some incognita about their final singular stage, if any, as predicted by the theory, or a regular behavior allowing to avoid the singular catastrophe. The determination of the regularity conditions is the problem we posed in this report for a certain kind of metrics.

The most famous black holes are described by the static spherically symmetric (SSS) Schwarzschild solution [2], the first exact solution derived by in 1916 for a point mass $m$, and by the Kerr stationary axial symmetric solution [3], derived in 1964, for a rotating mass. The charged point mass solution was reported by Reissner [4], 1916, and Nordström [5], 1918, while the charged rotating mass black hole solution was reported by Newman and collaborators [6]. Vacuum and Maxwell charged black hole solutions are all singular at the center.

Oppenheimer et al. [7], and [8], at the end of the 30’s studied the collapse of a massive spherically symmetric star and came to the conclusion that in the process (time dependent) of approaching to its critical surface, the star will increase indefinitely its curvature, and that the light radiated by an imploding star will be red–shifted, and, as it reaches its critical radius, the redshift will become infinite and the star will disappear from the observer sight; it becomes black behind the event horizon. The static spherically symmetric representation of the Schwarzschild metric exhibits an “apparent singularity” or “Schwarzschild singularity” at the Schwarzschild radius $r = 2m$; various attempt were done, by means of new coordinates, to remove the “apparent singularity.” A real success was achieved by the introduction of null coordinates by Finkelstein [9], 1959, Kruskal [10], 1960, and Szekeres [11], 1960, for the maximal extension of the Schwarzschild metric and the interpretation of the Schwarzschild radius as the surface bound to the event horizon. Later, in 1963, Kerr reported his rotating black hole solution; this discovery catapulted a period of intense activity in the field of black hole theory. To get an insight into black holes Penrose introduced spinors in the description of spacetimes, 1960 and 1972, see [12], and developed the so called Penrose diagram procedure. Hawking [13] in the beginning of the 70’s, formulated the thermodynamics of black holes. Meanwhile, in differential geometry various tools become of
common use. Newmann and Penrose introduced in general relativity their tetrade formalism. Petrov [14], 1966, published the algebraic classification of the Riemman and Weyl tensors which allows a deeper understanding of the algebraic spacetime structures. Plebański [15], 1964, reported the classification of the energy–momentum tensor. It was only until recently that regular black holes irrupted in the scene in a formal way supported by the nonlinear electrodynamics (NLE). Ayón–Beato and García published the first regular static spherically symmetric charged solution [16] described in terms of NLE potentials; this contribution caused a big impulse in the study of regular solutions in Einstein gravity; these authors found also the nonlinear electrodynamics source (magnetic potential) [17] to the Bardeen model [18] reported in 1968; the first metric structure fitting the tensor energy conditions exhibiting curvature regularity everywhere. Born and Infeld (BI) [20], 1934, formulated nonlinear electrodynamics (NLE) to provide the electron with a spatial environment supporting a charge with finite self–energy. Plebański [21], 1970, published a generalization of the BI NLE. Later García, Salazar, and Plebański [22] reported all Petrov type D solutions to Einstein–Born–Infeld electrodynamics allowing for stationary and axial symmetries, among them the static spherically symmetric solution; they also developed NLE allowing for duality rotations [23] and studied the birefringence properties of the theory [24]. Fradkin and Tseytlin [25], and Gibbons and Rasheed [26] established that BI–NLE surges at low energy limit of the string theory; this fact leads to a renewed interest on BI–NLE. In the last decade various families of the SSS solutions of Einstein–NLE equations has been reported (the number of articles is large), and attempts to derive regular stationary rotating solutions have been undertaken by various researchers without notorious successes. In Einstein theory to establish the regularity of solutions one studies the behavior of the Riemann tensor invariants of the gravitational field. The decomposition of the Riemann tensor into its Lorenzian irreducible parts gives rise to the conformal Weyl (CW) tensor $C_{\alpha\beta\gamma\delta}$, the traceless Ricci tensor $S_{\alpha\beta}$, and the scalar curvature $R$. Petrov classification gives a full account of the invariant properties of the CW tensor, see [19]. Plebański classification [15] deals with the algebraic properties of $S_{\alpha\beta}$. The first step in the study of the regularity of solutions is the analysis of the invariant properties of the TR tensor $S^\mu_\nu = R^\mu_\nu - R\delta^\mu_\nu/4$ due to its relation with the energy–momentum (EM) tensor $T^\mu_\nu$, namely, $S^\mu_\nu = T^\mu_\nu - T\delta^\mu_\nu/4$. Next, one establishes the conditions that allow for regularity of the CW tensor invariants and finally, the behavior of the scalar curvature. For the SSS metric there are only three independent Riemann curvature tensor invariants; the CW tensor eigenvalue $\Psi_2$, the TR tensor eigenvalue $S$, and the scalar $R$. The content of this article is organized as follows: Section II is devoted to the general structural description of NLE. In section III the Einstein–NLE equations for a SSS metric are derived explicitly together with the Ricci eigenvalue $S$, the scalar curvature $R$, and the Weyl $\Psi_2$ invariants. In section IV the conditions for the regularity at the center of a electric NLE SSS metric are established and a theorem is formulated: Regular NLE SSS electric solutions, which are characterized by the $F(r = 0)$, asymptotically approach to the flat or conformally flat de Sitter–Anti de Sitter (regular) spacetimes at the center. Section V deals with magnetic metrics characterized by singular magnetic field invariant and (possible) regular curvature invariants. In section VI the general linear superposition property of these Einstein–pure electric and magnetic solution is shown. This article ends with some Final Remarks.
II. GENERAL NONLINEAR ELECTRODYNAMICS

We follow the standard notation and conventions presented in [19]. Electrodynamics of any kind are constructed on the electromagnetic anti symmetric field tensor $F_{\mu\nu}$ and its dual tensor $F^*_{\alpha\beta}$

$$F_{\mu\nu} = 2A_{[\nu,\mu]}, \quad F^*_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}F^{\mu\nu},$$

(2.1)

where $\epsilon_{\alpha\beta\mu\nu}$ is the totally anti-symmetric Levi–Civita pseudo–tensor. These tensors determine the invariants $\mathcal{F}$ and $\mathcal{G}$, namely

$$\mathcal{F} = F_{\mu\nu}F^{\mu\nu}/4, \quad \mathcal{G} = F_{\mu\nu}F^{*\mu\nu}/4,$$

(2.2)

The general nonlinear electrodynamics in Einstein gravitational theory is derived from Riemman–Hilbert action constructed with the curvature scalar $R$, the Lagrangian $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$, and a $\Lambda$–term if any,

$$S = \int \sqrt{-g}(R - \mathcal{L} - \Lambda)d^4x.$$  

(2.3)

The Einstein equations arising from the variation of this action are

$$E^\mu_\nu := R^\mu_\nu - R \delta^\mu_\nu/2 + \Lambda \delta^\mu_\nu - \kappa T^\mu_\nu = 0,$$

(2.4)

where the energy–momentum electromagnetic tensor, for $\mathcal{L}(\mathcal{F}, \mathcal{G})$, is given by

$$T_{\mu\nu} = -\mathcal{L} g_{\mu\nu} + \frac{d\mathcal{L}}{d\mathcal{F}} F_{\mu\sigma} F^\sigma_\nu + \frac{d\mathcal{L}}{d\mathcal{G}} F_{\mu\sigma} F^{*\sigma}_\nu.$$

(2.5)

The electromagnetic field equations are:

the Bianchi identity

$$F_{[\mu\nu,\lambda]} = 0 \equiv F^{*\mu\nu;\nu} = 0,$$

(2.6)

and the field equations

$$
\left( \frac{d\mathcal{L}}{d\mathcal{F}} F^{\mu\nu} + \frac{d\mathcal{L}}{d\mathcal{G}} F^{*\mu\nu} \right)_\nu = 0 \\
\equiv [\sqrt{-g}(\frac{d\mathcal{L}}{d\mathcal{F}} F^{\mu\nu} + \frac{d\mathcal{L}}{d\mathcal{G}} F^{*\mu\nu})]_\nu = 0.
$$

(2.7)

In this work we shall restrict the study to the case $\mathcal{L} = \mathcal{L}(\mathcal{F})$. Maxwell theory is based on a linear relation between the Lagrangian $\mathcal{L}$ and the electromagnetic invariant $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}/4$, such that $\mathcal{L} = \mathcal{F} \rightarrow L_\mathcal{F} = 1$ in the whole spacetime. For physically relevant nonlinear electrodynamics one imposes on $\mathcal{L}$ and $\mathcal{F}$ the Maxwell weak field limits or “Maxwell asymptotic”:

$$(\mathcal{L} \rightarrow \mathcal{F}, \quad \mathcal{L}_{\mathcal{F}} := \frac{d\mathcal{L}}{d\mathcal{F}} \rightarrow 1, \text{ for small } \mathcal{F}, \ (\mathcal{F} \rightarrow 0)), \quad$$

(2.8)

in accordance with Born and Infeld [20], see also [23], i.e., in a weak field limit $\mathcal{F}: \mathcal{F}_{NLE} \rightarrow \mathcal{F}$, and $\mathcal{L}_{NLE} \rightarrow \mathcal{L}$, such that $\mathcal{L} \rightarrow \mathcal{F}$; $\mathcal{L}_{\mathcal{F}} \rightarrow 1$. Moreover, the NLE theory ought to fulfill the energy conditions: the local energy has to be positive and the energy flux has to be carried by a timelike four vector.
III. EINSTEIN–NLE EQUATIONS FOR A STATIC SPHERICALLY SYMMETRIC METRIC

In this section we derive the Einstein equations coupled to a NLE determined for a Lagrangian $\mathcal{L}$ depending on the electromagnetic invariant $\mathcal{F}$ and also the curvature invariants for the SSS metric given in Schwarzschild coordinates $\{\theta, r, \phi, t\}$

$$ds^2 = r^2 d\theta^2 + \frac{r^2}{Q(r)} dr^2 + r^2 \sin^2 \theta d\phi^2 - \frac{Q(r)}{r^2} dt^2. \quad (3.1)$$

The Einstein equations coupled to a matter field tensor $T_{\mu\nu}$ and a cosmological constant $\Lambda$, $\kappa = 1$, are

$$E_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = 0. \quad (3.2)$$

The evaluation of the Einstein tensor $G_{\mu\nu}$, and the curvature scalar $R$, for the metric (3.1), yields

$$G_{\theta\theta} = \frac{1}{2} \ddot{Q} r^2 - \frac{\dot{Q}}{r^3} + \frac{Q}{r^4}, \quad G_{t\theta} = \dot{Q} \frac{r}{r^3} - \frac{Q}{r^4} - \frac{1}{r^2},$$

$$R = \frac{2 \ddot{Q}}{r^2} - \frac{\dot{Q}}{r^2}. \quad (3.3)$$

It allows, from the point of view of the eigenvalue problem, for two different double eigenvalues $\lambda_\theta = \lambda_\phi = G_{\theta\theta} = G_{\phi\phi}$, and $\lambda_r = \lambda_t = G_{t\theta} = G_{r\phi}$. Thus the related energy–momentum tensor may describe electrodynamics, see for instance Lichnerowicz [27]. Other way to arrive at this conclusion is by means of the search for the eigenvalues of the traceless Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu}$, which amounts to

$$S_{\mu\nu} = S \left( \delta_{\theta\theta} \delta_{\nu\nu} - \delta_{\theta\phi} \delta_{\nu\phi} + \delta_{\phi\phi} \delta_{\mu\nu} - \delta_{\mu\phi} \delta_{\nu\theta} \right),$$

$$S = \frac{\dot{Q}}{4 r^3} - \frac{\ddot{Q}}{r^3} + \frac{Q}{r^4} + \frac{1}{2 \frac{r^2}{r^2}}. \quad (3.4)$$

$(S_{\mu\nu}) = S \text{ diag } (1, -1, 1, -1)$. Remarkable is the relation $G_{\theta\theta} - G_{t\theta} = 2S$. Therefore, according to the Plebański [15] classification of matter tensors; this algebraic structure corresponds to electrodynamics, $[(11)(1, 1)] \sim [2 S - 2 T]_{(11)}$, no matter if it is nonlinear or linear (Maxwell), see also, Stephani et al. [19], Chapter 5, and Chapter 13, §13.4. Consequently, static spherically symmetric metrics allow, besides the vacuum with $\Lambda$ solutions, $[(111, 1)] \sim [4 T]_{(1)}$, only electrodynamics solutions to Einstein equations coupled to linear [27] (Maxwell) or nonlinear electrodynamics [15].

In a certain sense, we are facing a theorem of uniqueness of classes of electrodynamics solutions; for each electromagnetic invariant Lagrangian $\mathcal{L}$, related to the electromagnetic invariant $\mathcal{F}$, the solution is unique. Due to the established existence of two double eigenvalues different in pairs there is no room for fluids; any attempt to accommodate in the above Schwarzschild metric (3.1) other kind of fields, different from electrodynamics or vacuum, is spurious, although in the literature one finds ”solutions”–let us call them better space–time models–for anisotropic fluids. This is the reason why we avoid to use the fluid terminology in a place where there is no room for it.
A. Nonlinear electrodynamics for $\mathcal{L}(\mathcal{F})$

This metric structure allows for an energy–momentum (EM) tensor $T_{\mu\nu}$ associated to the electromagnetic field (EF) tensor $F_{\mu\nu}$, $F_{\mu\nu} = 2A_{[\mu,\nu]}$. For Einstein–NLE equations with $\mathcal{L}(\mathcal{F})$, the only non vanishing EF components are the electric $F_{rt}$ and magnetic $F_{\theta\phi}$ fields, therefore the electromagnetic field tensors are

$$F_{\mu\nu} = 2F_{rt}\delta^t_\mu\delta^r_\nu + 2F_{\theta\phi}\delta^\theta_\mu\delta^{\phi}_\nu$$  \hspace{1cm} (3.5)

$$F^{\mu\nu} = -2F_{rt}\delta^\mu_r\delta^\nu_t + 2\frac{F_{\theta\phi}}{r^4 \sin^2(\theta)} \delta^\mu_\theta \delta^\nu_\phi.$$  \hspace{1cm} (3.6)

Consequently, the electromagnetic invariant $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}/4$ is given by

$$\mathcal{F} = -\frac{1}{2} (F_{rt})^2 + \frac{1}{2} \left(\frac{F_{\theta\phi}}{r^4 \sin^2(\theta)}\right)^2 = \mathcal{F}_e + \mathcal{F}_m,$$

$$\mathcal{F}_e = -\frac{1}{2} (F_{rt})^2, \quad \mathcal{F}_m = \frac{1}{2} \left(\frac{F_{\theta\phi}}{r^4 \sin^2(\theta)}\right)^2.$$  \hspace{1cm} (3.7)

The Bianchi identities $F_{[\mu\nu;\alpha]} = 0$ and the EF equations

$$\left(\frac{d\mathcal{L}}{d\mathcal{F}}F^{\mu\nu}\right)_{;\nu} = 0 \rightarrow (\sqrt{-g}\mathcal{L}_\mathcal{F}F^{\mu\nu})_{;\nu} = 0$$

$$\zeta \frac{\partial}{\partial x^\nu} \left( r^2 \sin \theta \mathcal{L}_\mathcal{F} F^{\mu\nu} \right) = 0,$$  \hspace{1cm} (3.8)

where $\frac{d\mathcal{L}}{d\mathcal{F}} =: \mathcal{L}_\mathcal{F}$, lead to the electric field equations

$$\mathcal{L}_\mathcal{F} F_{rt} = -\frac{q_0}{r^2} \rightarrow F_{rt}(r^2 \frac{d\mathcal{L}}{dr} - q_0 \frac{d}{dr} F_{rt}) - 2\frac{q_0 h_0^2}{r^5} = 0, $$

$$\mathcal{F}_e = -\frac{1}{2} (F_{rt})^2,$$  \hspace{1cm} (3.9)

and to the magnetic field equations

$$F_{\theta\phi} = h_0 \sin \theta, \quad \mathcal{F}_m = \frac{1}{2} \frac{h_0^2}{r^4}. $$  \hspace{1cm} (3.10)

The EM tensor $T^{\mu}_{\nu}$ and its trace $T$ can be given as

$$T^{\mu}_{\nu} = \left(-\mathcal{L} + \mathcal{L}_\mathcal{F} h_0^2/r^4 \right) \left(\delta^\mu_\theta \delta^\theta_\nu + \delta^\mu_\phi \delta^\phi_\nu\right)$$

$$- \left(\mathcal{L} + \mathcal{L}_\mathcal{F} F_{rt}^2 \right) \left(\delta^\mu_t \delta^t_\nu + \delta^\mu_\theta \delta^\theta_\nu\right)$$

$$T := T^{\mu}_{\mu} = -4\mathcal{L} + 4\mathcal{L}_\mathcal{F} (\mathcal{F}_e + \mathcal{F}_m),$$  \hspace{1cm} (3.11)

from where is apparent that its eigenvalues fulfill $\lambda_\theta = \lambda_\phi$, and $\lambda_r = \lambda_t$, see \cite{27}. Change

Notice that in the general electromagnetic (dyonic) case the derivative $\frac{d\mathcal{L}}{d\mathcal{F}} = \frac{d\mathcal{L}}{dr} / \frac{d\mathcal{F}}{dr}$, sing in $-\frac{d\mathcal{F}}{dr} = F_{rt} \frac{dF_{rt}}{dr} + 2\frac{h_0^2}{r^4}$, which makes the Einstein equations quite involved.
The nontrivial components of the Einstein–NLE equations are $E^r_r = E^t_t$ and $E^\phi_\phi$, hence the Einstein equations reduce to

$$E^r_r = E^t_t = \frac{\dot{Q}}{r^3} - \frac{Q}{r^4} - \frac{1}{r^2} + \Lambda + \mathcal{L} + \frac{d\mathcal{L}}{dF} (F_{rt})^2 = 0,$$

$$E^\phi_\phi = E^\phi_\phi = \frac{\dot{Q}}{2r^2} - \frac{Q}{r^4} + \frac{1}{r^2} + \Lambda + \frac{d\mathcal{L}}{dF} \frac{h_0^2}{r^4} = 0,$$

(3.12)

Using the EF equation $\mathcal{L}_F F_{rt} = -q_0/r^2$, one may isolate, via subtraction, $q_0 F_{rt}$ and $\mathcal{L}$ as

$$q_0 F_{rt} (r) = -\frac{\dot{Q}}{2} + 2 \frac{\dot{Q}}{r} - 2 \frac{Q}{r^2} - 1 + \frac{h_0^2}{r^2} \mathcal{L}_F,$$

$$\mathcal{L} (r) = -\frac{\dot{Q}}{2r^2} + \frac{\dot{Q}}{r^3} - \frac{Q}{r^4} - \Lambda + \frac{h_0^2}{r^4} \mathcal{L}_F.$$

(3.13)

to determine $F_{rt}(r)$, $\mathcal{L}(r)$, and the structural function $Q(r)$.

The EM conservation equation $T^{\mu\nu}_{\quad ;\nu} = 0$ leads to the condition

$$r^2 \frac{d}{dr} \mathcal{L}(r) - \frac{d}{dr} (q_0 F_{rt} (r)) + 2 \frac{h_0^2}{r^3} \mathcal{L}_F = 0,$$

(3.14)

which becomes an identity–Bianchi identity–by using the Einstein equations (3.13).

**B. The three curvature invariants of the SSS metric: $S$, $\Psi_2$, and $R$**

The invariant characterization of the algebraic properties of the gravitational–matter field, as it has been widely detailed above, begins with the determination of the eigenvalues of the TR tensor $S^\mu_\nu$. In the studied case $S^\mu_\nu$ amounts to $(S^\mu_\nu) = S \text{ diag}(1, -1, 1, -1) = \text{ diag}(\lambda_\theta, \lambda_r, \lambda_\phi, \lambda_t)$, where

$$S = \frac{\dot{Q}}{4r^2} - \frac{\dot{Q}}{r^3} + \frac{Q}{r^4} + \frac{1}{2r^2} = \frac{1}{2} \mathcal{L}_F \left( F_{rt}^2 + \frac{h_0^2}{r^4} \right),$$

$$= -\frac{q_0}{2r^2} F_{rt} + \frac{1}{2} \mathcal{L}_F \frac{h_0^2}{r^4}.$$

(3.15)

For the SSS metric, the Weyl curvature invariant $C^2 := C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$ amounts to $C^2 = 48 \Psi_2^2$, where $\Psi_2$ fits

$$-12 r^4 \Psi_2 = r^2 \dot{Q} - 6 r \dot{Q} + 12 Q - 2 r^2.$$

(3.16)

From the point of view of Petrov classification, the $\Psi$'s are related to the eigenvalues $\lambda$ of the eigenbivector equation $C_{abcd} X^{cd} = \lambda X_{ab}$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$. According to the Table 4.2 of [19], the studied metric is of Petrov type D, with eigenvalues $\lambda_1 = \lambda_2 = -2 \Psi_2$, while the remaining $\Psi$'s vanish. For the Newman–Penrose null tetrad basis used, see [1].

The scalar Riemann curvature $R$, is given by

$$r^2 R = 2 - \dot{Q}, \quad R = 4 \mathcal{L} - 4 \mathcal{L}_F (\mathcal{F}_e + \mathcal{F}_m) + 4 \Lambda.$$

(3.17)
IV. REGULARITY CONDITIONS FOR ELECTRICALLY CHARGED SSS METRICS

For a SSS Schwarzschild metric, in the electrically charged case, the invariant eigenvalue $S$, (3.15), fulfills the relations

$$ r^2 S = -r^2 \mathcal{F} \mathcal{L}_r = -q_0 F_{rt}/2, \quad (4.1) $$

in the pure electric and magnetic cases the subscripts $e$ and $m$ in $\mathcal{F}$ are omitted. From (4.1) one establishes at the center $r \to 0$ that: if the invariant $S$ is a finite quantity (FQ), then $F_{rt}$ is zero, $F_{rt} = 0$,

$$ \lim_{r \to 0} r^2 S = -\frac{q_0}{2} \lim_{r \to 0} F_{rt} \to \lim_{r \to 0} F_{rt} = 0; \lim_{r \to 0} \mathcal{F} = 0, $$

$$ \lim_{r \to 0} S \to \text{FQ} \to \lim_{r \to 0} (\mathcal{F} \mathcal{L}_r) \to \text{FQ}; \lim_{r \to 0} (\mathcal{L}_r) \to \text{FQ or 0}. \quad (4.2) $$

The condition $\lim_{r \to 0} \mathcal{F} = 0$ at the center evidences the NLE character of the theory.

Since, in the charged case, the invariant relation

$$ R(q_0; h_0 = 0) = 4\Lambda + 4\mathcal{L} + 4S \quad (4.3) $$

holds, then, for regular solutions, the invariant curvature $R$ ought to be finite at the origin: $\lim_{r \to 0} R \to \text{FQ}$, hence, together with $\lim_{r \to 0} S \to \text{FQ}$, from (4.3) one gets the finiteness of $\mathcal{L}$; $\lim_{r \to 0} \mathcal{L} \to \text{FQ}$. Moreover, in the electric case, from the equation (3.17) one gets $\lim_{r \to 0} \dot{Q} \to 2$. It remains to establish the behavior of $Q$ and $\dot{Q}$ at the origin: from the regularity of $S$ (3.16) and $\Psi_2$ (3.19), one obtains $\lim_{r \to 0} Q \to 0$, and $\lim_{r \to 0} r \dot{Q} \to 0$, which yields

$$ \text{the correct } \lim_{r \to 0} \dot{Q} \to 0, \text{ or } \lim_{r \to 0} \dot{Q} \to \text{“mass”}, $$

in full agreement with the limits at the center of the two regular non-charged SSS flat spacetime and dS–AdS metrics;

$$ \text{Flat : } Q = r^2, \frac{Q}{r^2} = 1, \lim_{r \to 0}\{Q, \dot{Q}, \ddot{Q}\} \to \{0, 0, 2\}, $$

$$ \text{dS–AdS : } Q = r^2 - \Lambda r^4/3, \lim_{r \to 0}\{Q, \dot{Q}, \ddot{Q}\} \to \{0, 0, 2\}. \quad (4.4) $$

These results can be gathered in the form of a theorem:

The Einstein–NLE theory allows for regular electric SSS solutions with regular curvature invariants at the origin $\lim_{r \to 0}\{S, \Psi_2, R\} \to \text{FQ’s}$ if the following regularity conditions at the center hold:

$$ \lim_{r \to 0}\{F_{rt}, \mathcal{F}\} = 0, \lim_{r \to 0}\{\mathcal{F} \mathcal{L}_r, \mathcal{L}_r, \mathcal{L}\} \to \text{FQ’s} $$

$$ \lim_{r \to 0}\{Q, \dot{Q}, \ddot{Q}\} \to \{0, 0, 2\}. \quad (4.5) $$

Regular NLE SSS electric solutions approach to the flat or conformally flat dS–AdS (regular) spacetimes at the center.
A. Maxwell asymptotic at infinity

In the Maxwell theory the only SSS electric solution is Reissner–Nordström (RN) spacetime determined by a single component $F_{rt} = -q_0/r^2$, hence $\mathcal{F} = -q_0^2/(2r^4) = \mathcal{L}$ in the whole spacetime, with all its curvature invariants singular at the origin.

The Einstein–NLE electric solutions allowing for a Maxwell weak field limit (small) $\mathcal{F}$, $\{\mathcal{L} \to \mathcal{F}, \mathcal{L}_F \to 1\}$, reduce to the Reissner–Nordström (RN) spacetime. Let us analyze in detail the content of the equation (4.1); at spatial infinity $r \to \infty$, the Einstein–Maxwell asymptotic solution should be the Reisner–Nordström solution, determined by $F_{rt} = -q_0^2/(2r^4)$, which substituted in the equation (4.1) gives

$$r^2 F_{rt} \mathcal{L}_F = -q_0 \to \mathcal{L} \to 1 \to \mathcal{L} \to \mathcal{F},$$

i.e., (4.1) gives rise to the correct Maxwell linear condition or, assuming the limiting Maxwell character of the electrodynamics, $\mathcal{L}_F \to 1$, from (4.1) one gets

$$r^2 F_{rt} \mathcal{L}_F = -q_0 \to F_{rt} \to -q_0/r^2,$$

i.e., the electric field for a central charge in the Maxwell theory.

Therefore, there is a subclass of electric SSS metrics that approaches to the Maxwell Reissner–Nordström electric solution at infinity. These solutions exhibit at infinity the finite Maxwell asymptotic ($\mathcal{L} \to \mathcal{F}, \mathcal{L}_F \to 1$ for weak field $\mathcal{F}$), together with the Maxwell field limits: $F_{rt} \to -q_0/r^2, \mathcal{F} \to -q_0^2/(4r^4)$.

B. Analyticity of the field functions

The description of physically relevant fields is expected to be done by well-behaved (analytical) functions. Nevertheless, if one has in mind the description of $\mathcal{L}(\mathcal{F})$, in general, one may find troubles in expressing $r(\mathcal{F})$ because of the possible appearance of transcendent equations. In the case of “regular at the origin and Maxwell at infinity” solutions, in general, the graph of $F_{rt}(r)$ begins from zero in the origin, evolves (grows up or decreases), reaches its maxima and minima, and again, at spatial infinity approaches (from above or below) to zero; the existence of a extremum, where $dF_{rt}/dr = 0$, in the $F_{rt}(r)$ graph points on the appearance of a returning point or a cusp in the parametric plot of $\mathcal{L}(\mathcal{F})$, the graph of $\mathcal{L}(\mathcal{F})$ corresponds to a multiple-valued relation. Nevertheless, this multiple-valued property of $\mathcal{L}(\mathcal{F})$ is not an impediment for the existence of analytic gravitational–electric “regular at the center and Maxwell at infinity” solutions. This lack of analyticity in the relation $\mathcal{L}(\mathcal{F})$ is not worse than the infinity at the origin of the magnetic invariant $\mathcal{F}_m$.

V. MAGNETIC STATIC SPHERICALLY SYMMETRIC METRICS

All SSS gravitational fields coupled to pure magnetic NLE possess a common field with component $F_{\theta\phi} = h_0 \sin \theta$ and a singular at the origin magnetic field invariant of the form $2 \mathcal{F}_m = h_0^2/r^4$; therefore one should strictly call the Einstein–NLE magnetic solutions singular ones, nevertheless the associated gravitational field may show regular behavior of the curvature invariant, thus one may have solutions with singular behavior in the magnetic
field invariants but regular in the curvature invariants, i.e., semi regular or singular–regular hybrid. Moreover, any magnetic solution to NLE, is determined by a single first order differential equation for $Q(r)$ arising from the $E_t$ equation of (3.12)

$$\mathcal{L}(r) = \frac{1}{r^2} + \frac{Q}{r^3} - \frac{\dot{Q}}{r^3} - \Lambda; \text{integrating}$$

$$\rightarrow Q(r) = r^2 - 2mr - \frac{\Lambda}{3}r^4 - r \int r^2 \mathcal{L}(r) \, dr, \quad (5.1)$$

characterized by

$$S(3.4) = -\frac{r}{4} \frac{d\mathcal{L}}{dr}, \quad \Psi_2 = \Psi_2(3.16),$$

$$R = 4\mathcal{L} - 4\mathcal{L}_r F_m + 4\Lambda = 4\mathcal{L} + r \frac{d\mathcal{L}}{dr} + 4\Lambda. \quad (5.2)$$

The substitution of $\mathcal{L}$ from (5.1) into the right hand side of $S(3.4)$ in (5.2) yields the identity $S(3.4) = S(3.4)$. The regularity of $\Psi_2(3.16)$, and $R(3.17)$ of the magnetic metric requires

$$\lim_{r \to 0} \{Q, \dot{Q}, \ddot{Q}\} \to \{0, 0, 2\},$$

for $m = 0$. For the regularity of $S$, and $R(5.2)$ at $r \to 0$ one has to establish the regular behavior of $rd\mathcal{L}/dr$ and $\mathcal{L}$ at the origin;

$$\lim_{r \to 0} \{\mathcal{L}, r \frac{d\mathcal{L}}{dr}\} \to \{FQ, FQ'\}$$

under the fulfilling of these conditions, one may consider that a magnetic solution at the center approaches to the flat or (A)dS spacetimes, although one has to recall the singularity of the magnetic field invariant $F_m$ there.

VI. LINEAR SUPERPOSITION OF SSS SOLUTIONS OF THE EINSTEIN–NLE EQUATIONS

Incidentally, since the Einstein–NLE differential equations, in both pure electric and magnetic cases, depend linearly on the structural function $Q(r)$, then for each solution $Q_i$ one determines $\mathcal{L}_i$ and the electric field $F_{rt_i}$. Their linear superposition gives rise to a new enlarged total solution $Q_T = \sum Q_i$ such that $\mathcal{L}_T = \sum \mathcal{L}_i$, and $F_{rt_T} = \sum F_{rt_i}$; characterized by the curvature invariants $S_T = \sum S_i$, $\Psi_2T = \sum \Psi_{2i}$, and $R_T = \sum R_i$. In the pure magnetic case, the field $F_{rt}$ ought to be replaced by the single magnetic component $F_{\theta\phi} = h_0 \sin \theta$. The spacetimes allowing for regular curvature invariants at the center can be thought of as immersed in a dS–AdS or in a flat universe.

VII. FINAL REMARKS

Although black holes have been discovered and described theoretically more than half a century ago, it has been only until recently that they have been experimentally found
indirectly on September 14, 2015. The Laser Interferometer Gravitational–Wave Observatory (LIGO) detected the gravitational wave GW150914,(LIGO Virgo collaboration) emitted by a binary system of rotating black holes many thousand light years ago; this first wave detection showed indirectly the existence of spinning black holes. New investigations in experimental black hole physics have been undertaken since that discovery. The list of publications since that date is quite large, in this respect, we cite some articles published recently in Phys. Rev. Letters:

Clovecko et al. [29] “used the spin–precession waves propagating on the background of the spin supercurrents between two Bose–Einstein condensates of magnons... as an experimental tool simulating the properties of the black–and white–hole horizons.”

Hughes et al. [30] studied “the coalescence of two black holes which generates gravitational waves that carry detailed information about the properties of those black holes and their binary configuration.”

Nair et al. [31] presented “a study of whether the gravitational–wave events detected so far by the LIGO–Virgo scientific collaborations can be used to probe higher-curvature corrections to general relativity.”

Yang et al. [32] showed “that if migration traps develop in the accretion disks of active galactic nuclei (AGNs) and promote the mergers of their captive black holes, the majority of black holes within disks will undergo hierarchical mergers—with one of the black holes being the remnant of a previous merger.”

Pook et al. [33] found “strong numerical evidence for a new phenomenon in a binary black hole spacetime, namely, the merger of marginally outer trapped surfaces (MOTSs). By simulating the head-on collision of two nonspinning unequal mass black holes, we observe that the MOTS associated with the final black hole merges with the two initially disjoint surfaces corresponding to the two initial black holes.”

Baumgarte et al. [34] “numerically investigated the threshold of black-hole formation in the gravitational collapse of electromagnetic waves in axisymmetry.”

Coates et al. [35] studied “black hole area quantization in the context of gravitational wave physics.”

Abbott et al. (LIGO Scientific Collaboration and the Virgo Collaboration) [36] presented “a search for subsolar mass ultracompact objects in data obtained during Advanced LIGO’s second observing run. In contrast to a previous search of Advanced LIGO data from the first observing run, this search includes the effects of component spin on the gravitational waveform.”

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