Elliptic Curves and Algebraic Geometry Approach in Gravity Theory I. The General Approach

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Abstract

Based on the distinction between the covariant and contravariant metric tensor components in the framework of the affine geometry approach and also on the choice of the contravariant components, it was shown that a wide variety of third, fourth, fifth, sixth, seventh - degree algebraic equations exists in gravity theory. This fact, together with the derivation of the algebraic equations for a generally defined contravariant tensor components in this paper, are important in view of finding new solutions of the Einstein’s equations, if they are treated as algebraic ones. Some important properties of the introduced in hep-th/0107231 more general connection have been also proved - it possesses affine transformation properties and it is an equiaffine one. Basic and important knowledge about the affine geometry approach and about gravitational theories with covariant and contravariant connections and metrics is also given with the purpose of demonstrating when and how these theories can be related to the proposed algebraic approach and to the existing theory of gravity and relativistic hydrodynamics.

1 INTRODUCTION

Inhomogeneous cosmological models have been intensively studied in the past in reference to colliding gravitational waves [1] or singularity structure and generalizations of the Bondi - Tolman and Eardley-Liang-Sachs metrics [2, 3]. In these models the inhomogeneous metric is assumed to be of the form [2]

\[ ds^2 = dt^2 - e^{2\alpha(t,r,y,z)} dr^2 - e^{2\beta(t,r,y,z)} (dy^2 + dz^2) \]  (1.1)
(or with \(r \to z\) and \(z \to x\)) with an energy-momentum tensor \(G_{\mu\nu} = k \rho u_\mu u_\nu\) for the irrotational dust. The functions \(\alpha(t, r, y, z)\) and \(\beta(t, r, y, z)\), determined by the Einstein’s equations, are chosen in a special form [4], so that the integrations of (some) of the components of the Einstein’s equations is ensured.

A nice feature of the approach is that in the limit \(t \to \infty\) [5] and under a special choice of the pressure as a definite function of time the metric approaches an isotropic form [4]. Other papers, also following the approach of Szafron-Szekerez are [6,7]. In [7], after an integration of one of the components - \(G^0_1\) of the Einstein’s equations, a solution in terms of an elliptic function is obtained.

In different notations, but again in the framework of the Szafron-Szekerez approach the same integrated in [7] nonlinear differential equation

\[
\left(\frac{\partial \Phi}{\partial t}\right)^2 = -K(z) + 2M(z)\Phi^{-1} + \frac{1}{3}\Lambda \Phi^2
\]  

was obtained in the paper [8] of Kraniotis and Whitehouse. They make the useful observation that (1.2) in fact defines a (cubic) algebraic equation for an elliptic curve, which according to the standard algebraic geometry prescriptions (see [9] for an elementary, but comprehensive and contemporary introduction) can be parametrized with the elliptic Weierstrass function

\[
\rho(z) = \frac{1}{z^2} + \sum_\omega \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]
\]  

and the summation is over the poles in the complex plane. Two important problems immediately arise, which so far have remained without an answer:

1. The parametrization procedure with the elliptic Weierstrass function in algebraic geometry is adjusted for cubic algebraic equations with number coefficients! Unfortunately, equation (1.2) is not of this type, since it has coefficient functions in front of the variable \(\Phi\), which depend on the complex variable \(z\). In view of this, it makes no sense to define ”Weierstrass invariants” as

\[
g_2 = \frac{K^2(z)}{12} \quad ; \quad g_3 = \frac{1}{216}K^3(z) - \frac{1}{12}\Lambda M^2(z) \quad ,
\]  

since the above functions have to be set up equal to the complex numbers \(g_2\) and \(g_3\) (the s. c. Eisenstein series)

\[
g_2 = 60 \sum_{\omega \subset \Gamma} \frac{1}{\omega^4} = \sum_{n,m} \frac{1}{(n + m\tau)^4} \quad ,
\]  

\[
g_3 = 140 \sum_{\omega \subset \Gamma} \frac{1}{\omega^6} = \sum_{n,m} \frac{1}{(n + m\tau)^6}
\]  

and therefore additional equations have to be satisfied in order to ensure the parametrization with the Weierstrass function.
2. Is the Szekerez - Szafron metric the only case, when the parametrization with the Weierstrass function is possible? Closely related to this problem is the following one - is only one of the components of the Einstein’s equation parametrizable with $\rho(z)$ and its derivative?

This series of three papers has the aim to present an adequate mathematical algorithm for finding solutions of the Einstein’s equations in terms of elliptic functions. This approach is based on the clear distinction between covariant and contravariant metric tensor components within the s.c affine geometry approach, which will be clarified further in Section 2. Afterwords, a cubic algebraic equation in terms of the contravariant metric components will be obtained, which according to the general prescription and the algorithm in the previous paper [10] can be parametrized with the Weierstrass function and its derivative. Respectively, if the contravariant components are assumed to be known, then a cubic (or a quartic) algebraic equation with respect to the covariant components can be investigated and parametrized again with the Weierstrass function. Thus it will turn out that the parametrization with the Weierstrass function will be possible not only in the Szafron-Szekeres case, but also in the general case due to the ”cubic” algebraic structure of the gravitational Lagrangian. This is an important point since valuable cosmological characteristics for observational cosmology such as the Hubble’s constant $H(t) = \frac{\dot{R}(t)}{R(t)}$ and the deceleration parameter $q = -\frac{\ddot{R}(t)R(t)}{R^2(t)}$ may be expressed in terms of the Jacobi’s theta function and of the Weierstrass elliptic function respectively [8]. Unfortunately, in the paper [8] the Eisenstein series (1.5-1.6) have not been taken into account, due to which the obtained expression for the metric will be another one and will be modified.

Instead of searching elliptic solutions of the Einstein’s equations for each separate case of a given metric, as in nearly all of the mentioned papers, in this series of papers another approach will be proposed. First, a cubic algebraic equation will be parametrized with respect to one of the contravariant components, following the approach in a previous paper [10]. In the second part, this parametrization will be extended to more than one variable in the multivariable cubic algebraic equation. This will be a substantial and new development, different from the standard algebraic geometry approach, in which only two-dimensional cubic equations are parametrized with the (elliptic) Weierstrass function and its derivative. Finally, in the third part the dependence of the generalized coordinates $X^i = X^i(x_1, x_2, x_3, \ldots, x_n)$ on the complex variable $z$ will be established from a derived system of first-order nonlinear differential equations. The generalized coordinates can be regarded as $n$- dimensional hypersurfaces, defining a transition from an initially defined set of coordinates $x_1, x_2, \ldots, x_n$ on a chosen manifold to another set of the generalized coordinates $X^1, X^2, \ldots, X^n$. Since the covariant metric components $g_{ij}$ also depend on these coordinates, this means that their dependence on the complex variable $z$ will also be known. In other words, at the end of the applied approach, each initially given function $g_{ij}(t, x)$ of the time and space coordinates will be expressed also as $g_{ij}(z)$. The algebraic approach will be applied to the s.c. cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian, but further it will be shown that not only the approach will be applicable in the general case of an arbitrary
contravariant tensor, but also concrete solutions for the metric $g_{ij}(z)$ will be given in the case of specially chosen simple metrics.

The first part of the present paper continues and develops further the approach from the previous paper [10], where a definite choice of the contravariant metric tensor was made in the form of the factorized product $\tilde{g}^{ij} = dX^i dX^j$. The differentials $dX^i$ are assumed to lie in the tangent space $T_X$ of the generalized coordinates. In Section 2 of the present paper some basic facts about the affine geometry approach and the s.c. gravitational theory with covariant and contravariant metrics and connections (GTCCMC) will be reminded, but also some new material, related to relativistic hydrodynamics in the context of GTCCMC is added. The basic and very important idea in this section is to show that GTCCMC are already "incorporated" in the theoretical framework of the already known gravitational theories - as an example the known projective formalism is taken, but at the same time in certain theories (such as the Arnowitt-Deser-Misner 3 + 1 decomposition), certain assumptions are made so that they do not fall within the class of GTCCMC. This is an interesting observation, since it clearly shows that the limiting assumptions can naturally be removed. In the next Section 3 it will be demonstrated briefly how the cubic algebraic equation with respect to the differentials $dX^i$ was derived in [10], but in fact the aim will be to show that depending on the choice of variables in the gravitational Lagrangian or in the Einstein's equations, a wide variety of algebraic equations (of third, fourth, fifth, seventh, tenth-degree) in gravity theory may be treated, if a distinction between the covariant metric tensor components and the contravariant ones is made. This idea, originally set up by Schouten and Schmutzer, was further developed in the papers [13, 14] mainly with the purpose of classification of such more general GTCCMC [13, 14]. Also, in Section 3 the important and new physical notion of a "tensor length scale" is introduced in a natural way within the GTCCMC, and this notion is a generalization of the metrical (scale) function $l(x) = ds^2 = g_{ij} dx^i dx^j$ in usual gravity theory. In Section 4 intersecting algebraic varieties will be proposed as a method for obtaining the known solutions in the standard gravity theory. In Section 5 it will be shown that the previously investigated in [10] under some restrictive assumptions cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian and the Einstein's equations can be written as algebraic ones also in the general case of an arbitrary contravariant tensor $g^{ij}$.

The physical idea, which will be exploited in this paper will be: can such a gravitational theory with a more general contravariant tensor be equivalent to the usual and known to us theory with a contravariant metric tensor, which is at the same time the inverse one of the covariant one? By "equivalence" it is meant that the gravitational Lagrangian in both approaches should be equal, on the base of which the s.c. cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian and the Einstein's equations can be written as algebraic ones also in the general case of an arbitrary contravariant tensor $g^{ij}$.
elementary proof), which is a typical notion, introduced in classical affine geometry [15, 16] and meaning that there exists a volume element, which is preserved under a parallel displacement of a basic \( n \)-dimensional vector \( e \equiv e_{i_1 i_2 \ldots i_n} \). Equivalently defined, \( \tilde{\Gamma}^s_{kl} \) is an equiaffine connection [15, 16] if it can be represented in the form \( \tilde{\Gamma}^s_{ks} = \partial_k l e \), where \( e \) is a scalar quantity. This notion turns out to be very convenient and important, since for such types of connections we can use the known formulae for the Ricci tensor, but with the connection \( \tilde{\Gamma}^s_{kl} \) instead of the usual Christoffel one \( \Gamma^s_{kl} \). Moreover, the Ricci tensor \( \tilde{R}_{ij} \) will again be a symmetric one, i.e. \( \tilde{R}_{ij} = \tilde{R}_{ji} = \partial_k \tilde{\Gamma}^k_{ij} - \partial_i \tilde{\Gamma}^k_{kj} + \tilde{\Gamma}^k_{kl} \tilde{\Gamma}^l_{ij} - \tilde{\Gamma}^m_{ki} \tilde{\Gamma}^k_{jm} \).

In usual gravity theory, the contravariant components are at the same time inverse to the covariant ones and thus the correspondence between "covectors" (in our terminology - these are the "vectors") and the "vectors" (i.e. the contravariant vectors) is being set up, respectively there is correspondence between covariant and contravariant tensors. By "correspondence" it is meant that both these kinds of tensors satisfy the matrix equation \( g_{ij} g^{jk} = \delta^k_i \). However, within the framework of affine geometry, such a correspondence is not necessarily to be established (see again [15-18]) and both tensors have to be treated as different mathematical objects, defined on one and the same manifold. If both components constitute the algebraic variety, satisfying the Einstein’s equations, considered as a set of intersecting multivariable cubic and quartic algebraic surfaces (further instead of cubic surfaces we shall continue to use the terminology "cubic curves"), then one can speak about separate classes of solutions for the covariant metric tensor components and for the contravariant ones.

If one assumes the existence of inverse contravariant metric tensor components \( \tilde{g}^{jk} \) and considers the quadratic system \( g_{ij} \tilde{g}^{jk} = g_{ij} dX^j dX^k = \delta^k_i \) as intersecting with the set of cubic and quartic algebraic Einstein’s equations, then it might be expected that the standardly known solutions of the Einstein’s equations should be recovered. However, this is not yet mathematically proved, neither has this been formulated as a problem. General theorems for intersection of algebraic curves of different (arbitrary) degrees are given in [19, 21, 22].

2 AFFINE GEOMETRY APPROACH AND GRAVITATIONAL THEORIES WITH COVARIANT AND CONTRAVARIANT CONNECTIONS AND METRICS

This section has the purpose to review some of the basic aspects of gravitational theories with covariant and contravariant metrics and connections (GTCCMC), which would further allow the application of algebraic geometry and of the theory of algebraic equations in gravity theory. The section contains also some new material, concerning the application of GTCCMC in relativistic hydrodynamics.

It is known in gravity theory that the knowledge of the metric tensor \( g_{ij} \) determines
the space - time geometry, which means that the Christoffell connection
\[ \Gamma^l_{ik} \equiv \frac{1}{2} g^{ls} (g_{ks,i} + g_{is,k} - g_{ik,s}) \]  
(2.1)
and the Ricci tensor
\[ R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km} \]  
(2.2)
can be calculated.

It is useful to remember also from standard textbooks [24] the s. c. Christoffell connection of the first kind:
\[ \Gamma^i_{;kl} \equiv g^{im} \Gamma^m_{kl} = \frac{1}{2} (g_{ik,l} + g_{il,k} - g_{kl,i}) \]  
(2.3)
obtained from the expression for the zero covariant derivative
\[ 0 = \nabla_l g_{ik} = g_{ik,l} - g_{m(i} \Gamma^m_{k)l} . \]

By contraction of (2.3) with another contravariant tensor field \( \tilde{g}^{is} \), one might as well define another connection:
\[ \tilde{\Gamma}^s_{kl} \equiv g^{is} \Gamma^i_{;kl} = \frac{1}{2} \tilde{g}^{is} (g_{ik,l} + g_{il,k} - g_{kl,i}) \]  
(2.4)
not consistent with the initial metric \( g_{ij} \). Clearly the connection (2.4) is defined under the assumption that the contravariant metric tensor components \( \tilde{g}^{is} \) are not to be considered to be the inverse ones to the covariant components \( g_{ij} \) and therefore \( \tilde{g}^{is} g_{im} = \tilde{f}^s_m(x) \).

In fact, the definition \( \tilde{g}^{is} g_{im} = \tilde{f}^s_m \) turns out to be inherent to gravitational physics. For example, in the projective formalism one decomposes the standardly defined metric tensor (with \( g_{ij} g_{jk} = \delta^k_i \)) as
\[ g_{ij} = p_{ij} + h_{ij} \]  
(2.5)

As a consequence
\[ p_{ij} h^{jk} = 0 \]  
(2.6)
meaning that the contravariant projective metric components \( p^{jk} \) are no longer inverse to the covariant ones \( p_{ij} \).

An example of gravitational theories with more than one connection are the so called theories with affine connections and metrics [13], in which there is one connection \( \Gamma^\gamma_{\alpha\beta} \) for the case of a parallel transport of covariant basic vectors \( \nabla_{e_\beta} e_\alpha = \Gamma^\gamma_{\alpha\beta} e_\gamma \) and a separate connection \( P^\gamma_{\alpha\beta} \) for the contravariant basic vector \( e^\gamma \), the defining equation for which is \( \nabla_{e_\beta} e^\alpha = P^\alpha_{\gamma\beta} e^\gamma \). In these theories, the contravariant vector and tensor fields are assumed to be not the inverse ones to the covariant vector and tensor fields. This implies that
\[ e_\alpha e^\beta = f^\beta_\alpha (x) \neq \delta^\beta_\alpha \]  
(2.8)
for such theories and consequently, a distinction is made between covariant and contravariant metric tensors (and vectors too). Clearly, in the above given case (2.7) of projective gravity, this theory should be considered as a GTCCMC. In the same spirit, since the well-known Arnowitt - Deser - Misner (ADM) (3+1) decomposition of spacetime [44, 45] is built upon the projective transformation (2.5), it might be thought that it should also be considered as such a theory. But in fact, the ADM (3+1) formalism definitely is not an example for this, because due to the special identification of the vector field’s components with certain components of the projective tensor

\[ g_{00} := -(N^2 - N_i N^i) ; \quad g^{00} := \frac{1}{N^2} , \]

\[ g_{ij} := p_{ij} ; \quad g^{ij} := \frac{p^{ij} - N^i N^j}{N^2} , \]

\[ g_{0i} := N_i ; \quad g^{0i} := \frac{N^i}{N^2} , \]

all the contravariant projective tensor components \( p^{\alpha \beta} \) \((\alpha, \beta, \gamma = 0, 1, 2, 3 ; \quad i, j = 1, 2, 3)\) turn out to be the inverse ones to the covariant projective components \( p_{\alpha \gamma} \). Indeed, it follows that

\[ p_{ij} p^{jk} = \delta^k_i \quad ; \quad N_i N^i = N^2 \quad ; \quad N_i N^j = \delta^j_i \quad . \]

In the case of the ADM (3+1) decomposition, such an identification is indeed possible and justified, since the gravitational field possesses coordinate invariance, allowing to disentangle the dynamical degrees of freedom from the gauge ones. But in the case when the tensors \( h_{ij} \) are related with some moving matter (with a prescribed motion) and an observer, ”attached” to this matter ”measures” all the gravitational phenomena in his reference system by means of the projective metric \( p_{ij} \), this will be no longer possible. Then the relation (2.7) will hold, and the resulting theory will be a gravitational theory with covariant and contravariant metrics and connections (GTCCMC). Naturally, if the tensor \( h_{ij} \) in (2.5) and (2.7) is taken in the form \( h_{ij} = \frac{1}{e} u_i u^j \) and if the vector field \( u \) (tangent at each point of the trajectory of the moving matter) is assumed to be non-normalized (i.e. \( e(x) = u_i u^i \neq 1 \)), then one would have to work not within the standard relativistic hydrodynamics theory (where \( p_{ij} = g_{ij} - u_i u_j \) and \( p_{ij} p^{jk} = \delta^k_i - u_i u^k \)), but within the formalism of GTCCMC (where \( p_{ij} p^{jk} = f^k_i = \delta^k_i - \frac{1}{e} u_i u^k \neq \delta^k_i \)). One may wonder why this should be so, since the last two formulae for \( p_{ij} p^{jk} \) for both cases look very much alike, with the exception of the ”normalization” function \( \frac{1}{e} \) in the second formulae. But in what follows it shall become clear that in the first case the right-hand side has a tensor transformation property, while in the second case due to the function \( \frac{1}{e} \) there would be no such property. And this shall turn out to be crucial.

In order to understand this also from another point of view, let us perform a covariant differentiation of both sides of the relation (2.8). Then one can obtain that the two connections are related in the following way [13]

\[ f^i_{j,k} = \Gamma^i_{jk} \quad f^i_l + P^i_{lk} f^l_j \quad ; \quad (f^i_{j,k} = \partial_k f^i_j) \quad . \]
Note also the following important moment - $f^\beta_\alpha (x)$ are considered to be the components of a function. Otherwise, if they are considered to be a (mixed) tensor quantity, the covariant differentiation of the mixed tensor $f^\beta_\alpha (x)$ in the right-hand side of $e_\alpha e^\beta \equiv f^\beta_\alpha (x)$ would give exactly the same expression as in the left-hand side. This would mean that from a mathematical point of view there would be no justification for the introduction of the second covariant connection $P^i_k$ - the covariant differentiation would give a quantity on the left-hand side, which would be identically equal to $\nabla_\gamma f^\beta_\alpha (x)$ for every choice of the two connections $\Gamma^i_{jk}$ and $P^i_k$, including also for the standard case (Einsteinian gravity) $P^i_k = -\Gamma^i_{jk}$. However, in view of the fact that $f^\beta_\alpha (x)$ are related with the description of some moving matter in the Universe, then a tensor transformation law should not be prescribed to them. So they should remain components of a function and consequently, the introduction of the second connection $P^i_k$ is inevitable.

In confirmation of this, it can easily be seen that the quantity $\delta^i_k - h_{ij} h^{jk} \neq \delta^i_k$ in (2.7), which is to be set up equal to $f^i_k (x)$, does not have a tensor transformation property for arbitrarily chosen tensor fields $h_{ij}$. More concretely, it would have such a transformation property if the equality

$$\left( \delta^j_i - \frac{1}{e} h_{ik} h^{kj} \right)'(X) =$$

$$\frac{\partial x^\alpha}{\partial X^i} \frac{\partial X^j}{\partial x^\beta} \left( \delta^\beta_\alpha - \frac{1}{e} h_{\alpha \gamma} h^{\gamma \beta} \right)(X)$$

(2.14)

holds. Now since $h_{ik} h^{kj}$ transforms as a tensor, then the fulfillment of (2.14) would mean that the equality

$$\delta^i_j = \frac{\partial x^\alpha}{\partial X^i} \frac{\partial X^j}{\partial x^\beta} \delta^\beta_\alpha$$

(2.15)

should hold for any derivatives $\frac{\partial x^\alpha}{\partial X^i}$ and $\frac{\partial X^j}{\partial x^\beta}$. But if $t^\alpha_i := \frac{\partial x^\alpha}{\partial X^i}$ and $t^\beta_j := \frac{\partial X^j}{\partial x^\beta}$ are the components of some set of tetrad fields, this would imply that this set is orthonormal, i.e. $\delta^j_i = t^j_i t^\alpha_i \delta^\alpha_\beta$ - a property, which now we shall prove to be not consistent with equality (2.7) $p_{ij} p^{jk} = \delta^k_i - h_{ij} h^{jk} \neq \delta^k_i$. The reason is that (2.7) already implies the existence of a basis of covariant and contravariant basic vector fields $e_i$ and $\tilde{e}^i$, such that $e_i \tilde{e}^i = f^i_i$ - in fact, this will be the essence of a proposition, which shall soon be proved. Also, if $\tilde{e}^i$ is another system of basic fields for which $e_i e^j = \delta^j_i$, then $\tilde{e}^i = f^i_k e^k$ and the orthonormality condition can be written as (for $\alpha = \beta$)

$$\tilde{e}^i = t^\alpha_i t^\alpha_\alpha = \tilde{t}^\alpha_i e_\alpha \tilde{e}^\alpha_\alpha = \tilde{t}^\alpha_i f^\alpha_\alpha f^\alpha_j .$$

(2.16)

But the orthonormality condition is defined and should have one and the same form in all reference frames, including the reference frame $(e_\alpha, \tilde{e}^i)$, in which the components of the tetrad field are $\tilde{t}^\alpha_i$. Consequently $f^\alpha_i f^\alpha_\alpha = 1$, which however is in contradiction with the arbitrariness in determining $f^\alpha_\alpha$. The contradiction is due to the assumption that the tensor transformation property (2.14) holds, and since the expression in (2.14) equals $f^\alpha_\alpha$, it should not transform as a tensor (note also that $f^\alpha_\alpha \neq f^\alpha_i$), at least for the investigated case of the projective transformation (2.5). Also, the contradiction is that (2.15) is fulfilled.
for any \( t_i^a := \frac{\partial x^a}{\partial x^i} \) and \( t^i_j := \frac{\partial x^i}{\partial x_j} \), or in other words - \textit{it should hold in any basis}, but at the same time \textit{we found a basis, in which (2.16) holds and not (2.15)}. For the case of standard relativistic hydrodynamics, although \( f^k_i = \delta^k_i - u_i u^k \neq \delta^k_i \), such a problem of course will not appear because of the unit vector normalization \( u^i u_i = 1 \) for \textit{every vector field}, which is imposed \textit{apriori}.

Now it is easy to understand why and in \textit{what cases the distinction between covariant and contravariant metric components will lead to an inevitable introduction of two different connections} \( \Gamma^k_{ij} \) and \( P^k_{ij} \). For the purpose, let us prove the following statement:

**Proposition 1** If \( e_1, e_2, ..., e_n \) is a basis of covariant vector fields and \( f^\alpha_i \) are certain functions or constants, then a basis of contravariant basic fields \( \tilde{e}^{\alpha_1}, \tilde{e}^{\alpha_2}, ..., \tilde{e}^{\alpha_n} \) can be found so that for each \( i \) and \( \alpha_j \) one has \( e_i \tilde{e}^{\alpha_j} = f^\alpha_i \).

This statement in fact is a generalization of the well-known theorem from differential geometry that if a basis of (covariant) vector fields is given then a dual basis of (contravariant) vector fields can be found, so that the contravariant vector fields are the \textit{inverse ones to the covariant ones}, i.e. \( e_i \tilde{e}^{\alpha_j} = \delta^\alpha_i \).

The proof is very simple, but essentially based on the relation (2.13). If the covariant basic vector fields are given, then the contravariant connection components \( \Gamma^k_{ij} \) will be known too. Since \( f^k_{i,j,k} \) are derivatives of a function, one may take the expression (2.13) \( f^k_{i,j,k} = \Gamma^l_{j,k} f^l_i + P^l_{ik} f^l_j \), which for the moment shall be treated as a system of \( n \cdot \left[ \frac{n(n+1)}{2} \right] \) linear algebraic equations with respect to the (unknown) connection components \( P^l_{ik} \). A solution of this system can be found for the connection components \( P^l_{ik} \). Then the condition for the parallel transport of the contravariant basic vector fields \( \nabla e_\alpha \tilde{e}^\gamma = P^\alpha_{\gamma \beta} \tilde{e}^\beta \) can be written as \( \partial_\beta \tilde{e}^\alpha = P^\alpha_{\gamma \beta} \tilde{e}^\gamma \) and considered as a system of \( n \) ordinary differential equations with respect to the components \( \tilde{e}^\alpha \). From this system, the unique solution for \( \tilde{e}^{\alpha_1}, \tilde{e}^{\alpha_2}, ..., \tilde{e}^{\alpha_n} \) can be found up to integration constants, obtained after the integration of the differential equations.

After proving this proposition, the difference between standard relativistic hydrodynamics and \textit{"modified" relativistic hydrodynamics with a variable length} can be easily understood. In the first case, the right-hand side in \( p_{ij} P^{jk} = \delta_i^k - u_i u^k = f^k_i \neq \delta_i^k \) transforms as a tensor, which is ensured also by normalization property \( u^i u_i = 1 \). Therefore (2.13) and the proposition will not hold, so the contravariant basic vector fields are determined in the standard way \( e_i \tilde{e}^j = \delta_i^j \) and more importantly, they cannot be determined in another way, in spite of the fact that again \( f^k_i \neq \delta_i^k \).

In the second case, the situation is just the opposite - the right-hand side of \( p_{ij} P^{jk} = \delta_i^k - \frac{1}{e} u_i u^k = f^k_i \neq \delta_i^k \) transforms not as a tensor because of the \textit{"normalization"} factor \( \frac{1}{e} \), the proposition holds and thus the basic vector fields are determined as \( e_i \tilde{e}^j = f^j_i \). \textit{Therefore, the treatment of relativistic hydrodynamics with \textit{"variable length"} should be within the GTCCMC.}

In the present case, the introduced new connection (2.4) \textit{should not be identified} with the connection \( P^\gamma_{\alpha \beta} \), since the connection \( \tilde{\Gamma}^k_{kl} \equiv \tilde{g}^{ij} \Gamma_{ijkl} \) is introduced by means of modifying the contravariant tensor and not on the base of any separate parallel transport.
Moreover, the connection $\tilde{\Gamma}_{kl}^s$ turns out to be a linear combination of the Christoffel connection components $\Gamma_{\alpha\beta}^\gamma$, and the relation between them is not of the type (2.13). In such a way, there will not be a contradiction with the case when the two connections $\Gamma_{\alpha\beta}^\gamma$ and $\tilde{\Gamma}_{kl}^s$ are not defined as separate ones, since later on, in deriving the cubic algebraic equation in the general case and for the case $\tilde{g}^{ij} = dX^jdX^k$ also, it would be supposed that $\tilde{g}^{is}$ is a tensor. This would mean (from $\tilde{g}^{is}g_{im} \equiv f^s_m(x)$) that $f^s_m(x)$ will also be a (mixed) tensor quantity, and therefore the covariant differentiation of $e_\alpha e_\beta \equiv f_\beta^\alpha(x)$ will not produce any new relation.

3 BASIC ALGEBRAIC EQUATIONS IN GRAVITY THEORY. TENSOR LENGTH SCALE

Now if one applies again the new definition $\tilde{g}^{ij} \equiv dX^idX^j$ of the contravariant tensor with respect to the Ricci tensor, then the following fourth-degree algebraic equation can be obtained

$$ R_{ik} = dX^l \left[ g_{is,t} \frac{\partial (dX^s)}{\partial x^k} - \frac{1}{2} p g_{ik,t} + \frac{1}{2} g_{il,s} \frac{\partial (dX^s)}{\partial x^k} \right] +$$

$$ + \frac{1}{2} dX^l dX^m dX^r dX^s \left[ g_{m[k,t}g_{l],i} + g_{i[l,t}g_{mr,k]} + 2 g_{i[l,t}g_{mr,l]} \right] , \quad (3.1) $$

where $p$ is the scalar quantity

$$ p \equiv \text{div}(dX) \equiv \frac{\partial (dX^l)}{\partial x^l} , \quad (3.2) $$

which "measures" the divergency of the vector field $dX$. The algebraic variety of the equation consists of the differentials $dX^i$ and their derivatives $\frac{\partial (dX^s)}{\partial x^k}$.

In the same spirit, one can investigate the problem whether the gravitational Lagrangian in terms of the new contravariant tensor can be equal to the standard representation of the gravitational Lagrangian. This standard (first) representation of the gravitational Lagrangian is based on the standard Christoffel connection $\Gamma_{ij}^k$ (given by formulae (2.1)), the Ricci tensor $R_{ik}$ (formulae (2.2)) and the other contravariant tensor $\tilde{g}^{ij} = dX^idX^j$

$$ L_1 = -\sqrt{-g} g^{ik} R_{ik} = -\sqrt{-g} dX^idX^k R_{ik} . \quad (3.3) $$

In the second representation the Christoffel connection $\tilde{\Gamma}_{ij}^k$ and the Ricci tensor $\tilde{R}_{ik}$ are "tilda" quantities, meaning that the "tilda" Christoffel connection is determined by formulae (2.4) with the new contravariant tensor $\tilde{g}^{ij} = dX^idX^j$ and the "tilda" Ricci tensor $\tilde{R}_{ik}$ by formulae (2.2), but with the "tilda" connection $\tilde{\Gamma}_{ij}^k$ instead of the usual Christoffel connection $\Gamma_{ij}^k$. Thus the expression for the second representation of the gravitational Lagrangian acquires the form

$$ L_2 = -\sqrt{-g} g^{il} \tilde{R}_{il} = -\sqrt{-g} dX^idX^i \left\{ p \Gamma_{ik}^r g_{kr} dX^k - \Gamma_{ik}^r g_{kr} d^2 X^k - \Gamma_{r(i}^r g_{k)r} d^2 X^k \right\} . \quad (3.4) $$
The condition for the *equivalence of the two representations* (i.e. \( L_1 = L_2 \)) gives a cubic algebraic equation with respect to the algebraic variety of the first differential \( dX^i \) and the second one \( d^2 X^i \) [10]

\[
dX^i dX^l \left( p \Gamma^r_{ik} g_{kr} dX^k - \Gamma^r_{ik} g_{kr} d^2 X^k - \Gamma^r_{l(i} g_{k)r} d^2 X^k \right) - dX^i dX^l R_{il} = 0 \quad .
\] (3.5)

Note the following essential peculiarity of the second representation (3.4) - due to the choice of the ”modified” contravariant tensor, the second quadratic term with the ”tilda” connection in the expression for \( \tilde{R}_{ij} \) is equal to zero

\[
- \sqrt{-g} dX^i dX^k (\tilde{\Gamma}^l_{ik} \tilde{\Gamma}^m_{lm} - \tilde{\Gamma}^m_{il} \tilde{\Gamma}^l_{km}) = - \frac{1}{2} \sqrt{-g} dX^i dX^k dX^l dX^m (-d g_{lm} g^{rs} g_{ks,i} - d g_{ik} dX^r g_{mr,l} +
+ d g_{i d} X^r g_{mr,k} + d g_{km} dX^s g_{ls,i}) - \sqrt{-g} dX^i dX^k dX^l dX^m dX^s dX^r (g_{ks,i} g_{mr,l} - g_{ls,i} g_{mr,k}) = 0.
\] (3.6)

The first differential \( d g_{ij} \) in (3.6) is represented as \( d g_{ij} \equiv \frac{\partial g_{ij}}{\partial x^s} dX^s \equiv \Gamma^r_{s(i} (g_{j)r}) dX^s \).

Following the same approach, in [10] the Einstein’s equations in vacuum for the general case were derived under the assumption that the contravariant metric tensor components are the ”tilda” ones:

\[
0 = \tilde{R}_{ij} - \frac{1}{2} g_{ij} \tilde{R} = \tilde{R}_{ij} - \frac{1}{2} g_{ij} dX^m dX^n \tilde{R}_{mn} =
- \frac{1}{2} p g_{ij} \Gamma^r_{mn} g_{kr} dX^k dX^m dX^n + \frac{1}{2} g_{ij} (\Gamma^r_{km} g_{nr} + \Gamma^r_{m(n} g_{k)r}) d^2 X^k dX^m dX^n +
+ p \Gamma^r_{ij} g_{kr} dX^k - (\Gamma^r_{ik} g_{jr} + \Gamma^r_{j(i} (g_{k)r}) d^2 X^k \quad .
\] (3.7)

This equation represents again a system of cubic equations. In addition, if the differentials \( dX^i \) and \( d^2 X^i \) are known, but not the covariant tensor \( g_{ij} \), the same equation can be considered also as a cubic algebraic equation with respect to the algebraic variety of the metric tensor components \( g_{ij} \) and their first derivatives \( g_{ij,k} \).

It might be thought that the definite choice of the contravariant tensor is a serious restriction, in view of the fact that the second derivatives of the covariant tensor components \( g_{ij,kl} \) are not present in the equation. This is indeed so, because the algebraic structure of the equation is simpler to deal with in comparison with the general case, and so it is easier to implement the algorithm for parametrization, developed in [10]. But there is one argument in favour of this choice (although the case for an arbitrary contravariant tensor is also more important) - since the metric can be expressed as \( ds^2 = l(x) = g_{ij} dX^i dX^j \) (consequently \( dX^i dX^j = l(x) g^{ij} \), the obtained cubic algebraic equations (3.5) and (3.7) can be considered with regard also to the length function \( l(x) \). Since for Einsteining gravity \( g_{ij} g^{jk} = \delta^k_i \) (i.e. \( g^{jk} = \tilde{g}^{jk} = dX^j dX^k \)), then for this case the length function is ”postulated” to be \( l = 1 \). Note that this choice of the contravariant tensor \( \tilde{g}^{jk} \) in the form of a factorized product is a partial (and not a general) choice, but further it shall be shown that the cubic equation for reparametrization invariance of the gravitational
Lagrangian can be written also for a generally chosen tensor \( \tilde{g}^{ik} \). Then from \( g_{ij} \tilde{g}^{ik} = l_i^k \) and \( l_i^k = l \delta_i^k \), the length function is again recovered. The important point here is that the length function can also be obtained as a solution of the cubic equation, and thus in more general theories of gravity solutions with \( l \neq 1 \) may exist. In fact, for a general contravariant tensor \( \tilde{g}^{ij} = l_k^i g^{kj} \neq dX^i dX^j \), it would be natural to propose to call \( l_k^i \) a "tensor length scale", and the previously defined length function \( l(x) \) is a partial case of the tensor length scale for \( l_j^i = l \delta_j^i \). The physical meaning of the notion of tensor length scale is simple - in the different directions (i.e. for different \( i \) and \( j \)) the length scale is different. In particular, some motivation for this comes from Witten’s paper [46], where in discussing some aspects of weakly coupled heterotic string theory (when there is just one string couplings) and the obtained too large bound on the Newton’s constant it was remarked that "the problem might be ameliorated by considering an anisotropic Calabi-Yau with a scale \( \sqrt{\alpha} \) in \( d \) directions and \( \frac{1}{M_{\text{GUT}}} \) in \((6-d)\) directions". For example, this can be realized if one takes

\[
\begin{align*}
  l_i^k &= g_{ij} dX^j dX^k = l_i \delta_i^k & \text{for } i, j, k = 1, \ldots, d, \\
  l_a^b &= g_{ac} dX^c dX^b = l_a \delta_a^b & \text{for } a, b, c = d + 1, \ldots, 6.
\end{align*}
\]

(3.8)\hspace{1cm}(3.9)

Note also the justification for the name "tensor length scale" - if \( l_k^i \) is a tensor quantity, so will be the "modified" contravariant tensor \( \tilde{g}^{ij} = l_i^k g^{kj} \), and consequently in accord with section 2 there will be no need for the introduction of a new covariant connection \( P_{ij}^b \). And this is indeed the case, because the relation between the two connections \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^k_{ij} \) is given by formulae (2.4) \( \tilde{\Gamma}^k_{il} := \tilde{g}^{is} g_{sm} \Gamma^m_{kl} \). In other words, since these two connections are not considered to be "separately introduced" and so they do not depend on one another by means of the equality (2.13), this particular investigated case does not fall within the classification of spaces with covariant and contravariant metrics and connections (Table I in a previous paper [47]). This is an important "terminological" clarification, since it turns out that it is possible to have a theory with (separate) covariant and contravariant metrics, but not (with separate) connections as well. And such a theory is fully consistent from a mathematical point of view, as demonstrated above. However, at this point it is important to clarify what is meant by "a theory with (separate) covariant and contravariant metrics" - it should be understood only with respect to the metrics \( g_{\mu\nu} \) and \( \tilde{g}^{is} \). But if we take the contravariant metric \( \tilde{g}^{is} \) (and ignore for the moment the metric \( g_{\mu\nu} \)), then from the equality \( \tilde{g}_{ij} \tilde{g}^{jk} = \delta_i^k \) one can determine an inverse to the contravariant metric \( \tilde{g}^{is} \) new covariant metric \( \overline{g}_{ij} \), and consequently, the following new contravariant connection \( \Gamma^i_{kl} \) can also be determined

\[
\Gamma^i_{kl} := \tilde{g}^{is} \Gamma_{ijkl} = \tilde{g}^{is} \overline{g}_{im} \Gamma^m_{kl} = \frac{1}{2} \tilde{g}^{is} (\overline{g}_{ik,l} + \overline{g}_{il,k} - \overline{g}_{kl,i}).
\]

(3.10)

Evidently, with respect to the metric \( \overline{g}_{ij} \) (and its inverse contravariant one \( \tilde{g}^{jk} \)), we have the usual gravitational theory with the contravariant \( \overline{\Gamma}_{\alpha\beta} \) and covariant \( P_{\alpha\beta} \) connections

\[
\nabla_{e_\beta} e^\alpha = \overline{\Gamma}_{\alpha\beta} e^\gamma; \quad \nabla_{e_\beta} e^\alpha = P_{\gamma\beta} e^\gamma; \quad P_{\gamma\alpha} = -\overline{\Gamma}_{\alpha\beta}.
\]

(3.11)
However, although with respect to the metrics \( g_{\mu \nu} \) and \( \tilde{g}^{ij} \) and the connections \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^s_{kl} := \tilde{g}^{is}g_{im}\Gamma^m_{kl} \) (3.4) the theory is with covariant and contravariant metrics (only), connections \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^s_{kl} \) can be determined (by means of the additional metric \( \tilde{g}_{\mu \nu} \)) in the following way

\[
\nabla_{\epsilon_j}e_\alpha = \Gamma^\gamma_{\alpha \beta}e_\gamma; \quad \nabla_{\epsilon_j}e^\alpha = \tilde{\Gamma}^\alpha_{\gamma \beta}e^\gamma ,
\]

so that with respect to these connections the theory can be considered a GTCCMC. This also means that Table I in [47] correctly does not account for theories with different covariant and contravariant metrics only, because the different GTCCMC are in principle with different covariant and contravariant metrics and with different connections.

The purpose of the present paper further will be: how can one extend the proposed in [10] approach for the definitely determined contravariant metric components to the case of a generally defined contravariant tensor \( \tilde{g}^{ij} \neq dX^i dX^j \)?

4 INTERSECTING ALGEBRAIC VARIETIES AND STANDARD (EINSTEINIAN) GRAVITY THEORY

A more general theory with the definition of the contravariant tensor as \( \tilde{g}^{ij} \equiv dX^i dX^j \) should contain in itself the standard gravitational theory with \( g_{ij}g^{jk} = \delta_i^k \). From a mathematical point of view, this should be performed by considering the intersection [19] of the cubic algebraic equations (3.7) with the system of \( n^2 \) quadratic algebraic equations for the algebraic variety of the \( n \) variables

\[
g_{ij}dX^j dX^k = \delta_i^k \quad .
\]

In its general form \( g_{ij}\tilde{g}^{jk} = \delta_i^k \) with an arbitrary contravariant tensor \( \tilde{g}^{jk} \), this system can also be considered together with the Einstein’s ”algebraic” system of equations, which in the next section shall be derived for a generally defined contravariant tensor. From an algebro-geometric point of view, this is the problem about the intersection of the Einstein’s algebraic equations with the system of \( n^2 \) (linear) hypersurfaces for the \( \binom{n}{2} + n \) contravariant variables, if the covariant tensor components are given. Since the derived Einstein’s algebraic equations are again cubic ones with respect to the contravariant metric components, this is an analogue to the well-known problem in algebraic geometry about the intersection of a (two-dimensional) cubic curve with a straight line. However, in the present case the straight line and the cubic curve are multi-dimensional ones, which is a substantial difference from the standard case.

The standardly known solutions of the Einstein’s equations can be obtained as an intersection variety of the Einstein’s algebraic equations with the system \( g_{ij}\tilde{g}^{jk} = \delta_i^k \). However, the strict mathematical proof that such an intersection will give the known solutions is still lacking.
It might seem that the system of equations (4.1) does not have solutions with respect to \( g_{ij} \) (and thus no solutions of the Einstein’s equations can be found for the standard case), since the determinant \( \det || dX^i dX^j ||_{i,j=1 \ldots n} = 0 \) equals to zero! In another paper it will be proved that such a matrix operator system of equations [20] \( Y_{ij} g^{ik} = \delta^i_r \) with unknown variables \( Y_{ij} \equiv g_{ij} \) (which is not a system of linear algebraic equations, but instead a system of matrix equations) can be transformed to a system of linear algebraic equations \( \tilde{A}_{ij} \tilde{Y}^j = T_i \) (\( T_i \) is a vector-column). This system always has a solution at least for some of the variables - the others may be determined arbitrarily. Therefore, solutions will exist and will be well-determined even in the case of a zero determinant.

5 ALGEBRAIC EQUATIONS FOR A GENERAL CONTRAVARIANT METRIC TENSOR

Let us write down the algebraic equations for all admissible parametrizations of the gravitational Lagrangian for the generally defined contravariant tensor \( \tilde{g}^{ij} \), following the same prescription as in section 3, where the equality of the two representations of the gravitational Lagrangian has been supposed:

\[
\tilde{g}^{ik} \tilde{g}^{jl}_{\delta_{s}^{l}} (\Gamma^r_{ik} g_{rs})_{,l} + \tilde{g}^{ik} \tilde{g}^{js}_{\delta_{mr}^{s}} g_{pr} g_{qs} (\Gamma^r_{ik} \Gamma^p_{lm} - \Gamma^p_{il} \Gamma^q_{km}) - R = 0 \quad \text{(5.1)}
\]

This equation is again a cubic algebraic equation with respect to the algebraic variety of the variables \( \tilde{g}^{ij} \) and \( \tilde{g}^{ij}_{,k} \), and the number of variables in the present case is much greater than in the previous case for the contravariant tensor \( \tilde{g}^{ij} \equiv dX^i dX^j \). At the same time, this equation is a fourth-degree algebraic equation with respect to the covariant metric tensor \( g_{ij} \) and its first and second partial derivatives. With respect to the algebraic variety of all the variables \( \tilde{g}^{ij} \), \( \tilde{g}^{ij}_{,k} \), \( g_{ij} \), \( g_{ij,k} \), \( g_{ij,kl} \), the above algebraic equation is of seventh order and with coefficient functions, due to the presence of the terms with the affine connection \( \Gamma^q_{ik} \) and its derivatives, which contain the contravariant tensor \( g^{ij} \) and \( g^{ij}_{,k} \).

If the connection is assumed to be the “tilda connection” \( \tilde{\Gamma}^r_{kl} \equiv dX^i dX^s \Gamma_{i;kl} \), then the same equation can be regarded as a sixth-degree equation with respect to the algebraic variety of \( dX^i \) and its derivatives.

Similarly, the Einstein’s equations can be written as a system of third-degree algebraic equations with respect to the (generally chosen) contravariant variables and their derivatives

\[
0 = \tilde{R}_{ij} - \frac{1}{2} g_{ij} \tilde{R} =
\]

\[
= \tilde{g}^{tr} (\Gamma_{r;ij})_{,l} + \tilde{g}^{tr}_{,l} \tilde{g}_{ij} + \tilde{g}^{tr}_r \tilde{g}^{ms}_l (\Gamma^r_{lr} \Gamma_{s;lm} - \Gamma_{sl} \Gamma^r_{lm}) -
\]

\[
- \frac{1}{2} g_{ij} \tilde{g}^{nm} \tilde{g}^{lr}_{,s} \Gamma^r_{mn} g_{rs} - \frac{1}{2} g_{ij} \tilde{g}^{nm} \tilde{g}^{ls}_{,r} \Gamma^r_{mn} g_{rs}_{,l} -
\]

\[
- \frac{1}{2} g_{ij} \tilde{g}^{nm} \tilde{g}^{ls}_{,r} \Gamma^r_{mn} g_{qs} (\Gamma^q_{nk} \Gamma^p_{lm} - \Gamma^p_{nl} \Gamma^q_{km}) \quad \text{(5.2)}
\]
Interestingly, the same system of equations can be considered as a system of fifth-degree equations with respect to the covariant variables (which is the difference from the previous case). The mathematical treatment of fifth-degree equations is known since the time of Felix Klein’s famous monograph [25], published in 1884. A way for resolution of such equations on the base of earlier developed approaches by means of reducing the fifth-degree equations to the so called modular equation has been presented in the more recent monograph of Prasolov and Solov’yev [9]. Some other methods for solution of third-, fifth- and higher-order algebraic equations have been given in [26, 27]. A complete description of elliptic, theta and modular functions has been given in the old monographs [28, 29]. Also, solutions of $n$-th degree algebraic equations in theta-constants [30] and in special functions [31] are interesting in view of the not yet proven hypothesis in the paper by Kraniotis and Whitehouse [8] that "all nonlinear solutions of general relativity are expressed in terms of theta-functions, associated with Riemann-surfaces".

The basic knowledge about the parametrization of a cubic algebraic equation with the Weierstrass function and its derivative, which shall be extensively used in the subsequent parts of this paper, is given in almost all basic textbooks on elliptic functions [9, 11, 32, 33] and many others. However, the most complete, detailed and exhaustive knowledge about elliptic functions and automorphic forms is contained in the two two-volume books [34, 35] of Felix Klein and Robert Fricke, written more than 100 years ago. More specific and advanced topics on elliptic curves from a mathematical point of view such as the group of rational points, cubic curves over finite fields, families of elliptic curves and torsion points and etc. are contained in the monographs [36, 37]. A very understandable exposition of the classical topics on cubic algebraic curves and at the same time the most contemporary issues such as the Mordell’s and Dirichlet’s theorems and $L$ functions, modular forms and theories of Eichler-Shimura are given in the book of Knapp [38], which can be used for first acquaintance in these topics. A consistent, modern and full exposition of elliptic curves in the language of modern mathematics is given in the (two consequent) monographs of Silverman [39, 40]. A classical and very understandable exposition of the relation of elliptic curves with modular forms is given in [41], also in [42]. From a modern standpoint the relation of elliptic curves with number theory and modular forms is given in the review articles of Cohen and Don Zagier in [43], also introductory knowledge on hyperelliptic integrals, compact Riemann surfaces and Abelian varieties are presented in the review article by Bost also in [43].

Two other important problems can be pointed out with reference to algebraic equations in gravity theory:

1. One can find solutions of the system of Einstein’s equations not as solutions of a system of nonlinear differential equations, but as elements of an algebraic variety, satisfying the Einstein’s algebraic equations. The important new moment is that this gives an opportunity to find solutions of the Einstein’s equations both for the components of the covariant metric tensor $g_{ij}$ and for the contravariant ones $\tilde{g}^{jk}$. This means that solutions may exist for which $g_{ij}\tilde{g}^{jk} \neq \delta^k_i$. In other words, a classification of the solutions of the Einstein’s equations can be performed in an entirely new and nontrivial manner - under a given contravariant tensor, the covariant tensor and its derivatives have to be found from
the algebraic equation, or under a given covariant tensor, the contravariant tensor and its derivatives can be found.

2. The condition for the zero - covariant derivative of the covariant metric tensor \( \nabla_k g_{ij} = 0 \) and of the contravariant metric tensor \( \nabla_k \tilde{g}^{ij} = 0 \) can be written in the form of the following cubic algebraic equations with respect to the variables \( g_{ij}, g_{ij,k} \) and \( \tilde{g}^{ls} : \)

\[
\nabla_k g_{ij} \equiv g_{ij,k} - \tilde{\Gamma}^l_{k(i} g_{j)l} = g_{ij,k} - \tilde{g}^{ls} \Gamma_{s;k(i} g_{j)l} = 0 \tag{5.3}
\]

and

\[
0 = \nabla_k \tilde{g}^{ij} = \tilde{g}^{ij}_k + \tilde{g}^{r(i} \tilde{g}^{j)s} \Gamma_{r;sk} \tag{5.4}
\]

The first equation (5.3) is linear with respect to \( \tilde{g}^{ls} \) and quadratic with respect to \( g_{ij}, g_{ij,k} \), while the second equation (5.3) is linear with respect to \( g_{ij}, g_{ij,k} \) and quadratic with respect to \( \tilde{g}^{ls} \).

Since the treatment of the above cubic algebraic equations is based on singling out one variable, let us rewrite equation (5.1) for the effective parametrization of the gravitational action for the case of diagonal metrics \( g_{\beta\beta} \) and \( \tilde{g}^{\alpha\alpha} \), singling out the variable \( \tilde{g}^{44} : \)

\[
A(\tilde{g}^{44})^3 + B_\alpha (\tilde{g}^{44})^2 \tilde{g}^{\alpha\alpha} + C_{\alpha\alpha} \tilde{g}^{44} \tilde{g}^{\alpha\alpha} + (\Gamma^\alpha_{44} g_{\alpha\alpha}) \tilde{g}^{44} g_{\gamma\gamma} + D_{\alpha\gamma} \tilde{g}^{44} \tilde{g}^{\alpha\alpha} g^{\gamma\gamma} + F_{\alpha\gamma} = 0 \tag{5.5}
\]

where the coefficient functions \( A, B_\alpha, C_{\alpha\alpha}, D_{\alpha\gamma} \) and the free term \( F_{\alpha\gamma} \) denotes an expression, depending on the covariant metric tensor and the affine connection. In (5.5) the Greek indices run the values \( \alpha, \beta, \gamma = 1, 2, 3, 4 \), while all the other indices run from 1 to 4.

The representation (5.5) of the cubic equation is the starting point for the parametrization with the Weierstrass function, which will be performed elsewhere, following the algorithm in the paper [10]. In the second part of this paper, this would be performed for the case a multivariable cubic algebraic equation (although again within the framework of the factorizing approximation \( \tilde{g}^{ij} \equiv dX^i dX^j \)) and this is entirely different from the standardly known case in algebraic geometry of parametrization of a two-dimensional cubic algebraic equation in its parametrizable form.

6 DISCUSSION

In this paper we continued the investigation of cubic algebraic equations in gravity theory, which has been initiated in a previous paper [10].

Unlike in the paper [10], where the treatment of cubic algebraic equations has been restricted only to the choice of the contravariant tensor \( \tilde{g}^{ij} = dX^i dX^j \), in the present paper it has been demonstrated that under a more general choice of \( \tilde{g}^{ij} \), there is a wide variety of algebraic equations of various order, among which an important role play the cubic equations. Their derivation is based on two important initial assumptions:
1. The covariant and contravariant metric components are treated independently, which is a natural approach in the framework of affine geometry [15 - 18].

2. Under the above assumption, the gravitational Lagrangian (or Ricci tensor) should remain the same as in the standard gravitational theory with inverse contravariant metric tensor components.

It will be proved in Appendix A that the new connection \( \tilde{\Gamma}_{ij}^k = \frac{1}{2} dX^k dX^l (g_{jl,i} + g_{il,j} - g_{ij,l}) \) has again an affine connection transformation property, provided that a complicated system of nonlinear differential equations are satisfied. This system is expected to have a broad class of solutions.

The proposed approach allows to treat the Einstein’s equations as algebraic equations, and thus to search for separate classes of solutions for the covariant and contravariant metric tensor components. It can be supposed also that the existence of such separate classes of solutions might have some interesting and unexplored until now physical consequences. Some of them will be demonstrated in reference to theories with extra dimensions, but no doubt the physical applications are much more numerous.

Also, it has been shown that the "transition" to the standard Einsteinian theory of gravity can be performed by investigating the intersection with the corresponding algebraic equations.

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7 APPENDIX A: SOME PROPERTIES OF THE NEWLY INTRODUCED CONNECTION \( \tilde{\Gamma}_{ij}^k = \frac{1}{2} dX^k dX^l (g_{jl,i} + g_{il,j} - g_{ij,l}) \)

A1: A PROOF OF THE AFFINE TRANSFORMATION LAW FOR THE CONNECTION \( \tilde{\Gamma}_{ij}^k \)

A1.1 THE NECESSARY CONDITION
Next we proceed with the proof that the defined (in the preceding paper [10] also) connection
\[ \tilde{\Gamma}^k_{ij} = \frac{1}{2} dX^k dX^l (g_{jl,i} + g_{il,j} - g_{ij,l}) = dX^k dX^r g_{sr}(X) \Gamma^s_{ij}(X) \] (A1)
has the transformation property of an affine connection under the coordinate transformations \( X^i = X^i(x^1, x^2, ..., x^n) \).

From the defining equation (A1) for \( \tilde{\Gamma}^k_{ij} \), the tensor transformation property for \( g'_{ij}(X) \)
\[ g'_{ij}(X) = \frac{\partial x^k}{\partial X^l} \frac{\partial x^l}{\partial X^i} g_{kl}(x) \] , (A2)
the affine transformation law for the "usual" connection \( \Gamma^k_{ij} \)
\[ \Gamma^k_{ij}(X) = \Gamma^m_{np}(x) \frac{\partial X^k}{\partial x^m} \frac{\partial x^m}{\partial X^i} \frac{\partial x^p}{\partial X^j} + \frac{\partial^2 x^m}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^m} \] (A3)
and from the expressions for the differentials \( dX^k \) and \( dX^r \) we may write down
\[ \tilde{\Gamma}^k_{ij}'(X) = dX^k(X) dX^r(X) g'_{sr}(X) \Gamma^s_{ij}(X) = \]
\[ = \Gamma^m_{np}(x) \frac{\partial X^k}{\partial x^m} \frac{\partial x^m}{\partial X^i} \frac{\partial x^p}{\partial X^j} g_{sr}(x) dx^\alpha dx^\beta + \frac{\partial^2 x^m}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^m} g_{sr}(x) dx^\alpha dx^\beta . \] (A4)

On the other hand, if \( \tilde{\Gamma}^k_{ij}(X) \) is an affine connection, then it should satisfy the affine connection transformation law (A3)
\[ \tilde{\Gamma}^k_{ij}'(X) = \Gamma^m_{np}(x) \frac{\partial X^k}{\partial x^m} \frac{\partial x^m}{\partial X^i} \frac{\partial x^p}{\partial X^j} + \frac{\partial^2 x^m}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^m} . \] (A5)

Making use of the defining equation (A1) (but in terms of the initial coordinates \( x^1, x^2, ..., x^n \)),
the above expression can be written also as
\[ \tilde{\Gamma}^k_{ij}'(X) = \Gamma^m_{np}(x) \frac{\partial X^k}{\partial x^m} \frac{\partial x^m}{\partial X^i} \frac{\partial x^p}{\partial X^j} g_{m\beta}(x) dx^\alpha dx^\beta + \frac{\partial^2 x^m}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^m} \] . (A6)

Clearly, if \( \tilde{\Gamma}^k_{ij}(X) \) is an affine connection, from the R. H. S. of (A5) and (A7) it would follow that the following relation has to be satisfied
\[ dx^\alpha dx^\beta g_{m\beta}(x) \frac{\partial^2 x^m}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^m} - \frac{\partial^2 x^\alpha}{\partial X^i \partial X^j} \frac{\partial X^k}{\partial x^\alpha} = 0 , \] (A8)

which in fact represents the necessary condition for the definition of the connection \( \tilde{\Gamma}^k_{ij}(X) \)
as an affine connection. It can easily be proved that in case of commuting operators of differentiation \( \frac{\partial}{\partial x^i} \) and \( \frac{\partial}{\partial x^j} \) (in the general case, however, they do not commute), equation (A8) is fulfilled.
Now let us investigate the two-dimensional case, but when the assumption about the commutation of the derivatives is dropped out. Then equation \( (A8) \) on the integral curves \( dx^1 = C_1 \) and \( dx^2 = C_2 \) can be written as (the relation \( \frac{\partial x^p}{\partial X^s} \frac{\partial X^s}{\partial x^t} = \delta^p_t \) is also taken into account)

\[
\frac{\partial X^k}{\partial x^l} \left\{ (C_1^2 g_{11} + C_1 C_2 g_{12} - 1) M^{lij} - (C_2^2 g_{22} + C_1 C_2 g_{12} - 1) M^{lij} + \right.
\]
\[
+ (C_1^2 g_{12} + C_1 C_2 g_{22}) M^{lij} - (C_2^2 g_{12} + C_1 C_2 g_{11}) M^{lij} \right\} = 0 , \tag{A9}
\]

where \( M^{lij} \) and \( M^{lij} \) \((k, n = 1 \text{ or } 2)\) are the introduced notations for the expressions

\[
M^{lijk} := \frac{\partial x^k}{\partial X^i} \frac{\partial^2 x^k}{\partial X^i \partial X^j} ; \quad M^{lijk} := \frac{\partial x^k}{\partial X^i} \frac{\partial^2 x^n}{\partial X^i \partial X^j} . \tag{A10}
\]

Now interchanging the functions \( x^1 \leftrightarrow x^2 \) in \( (A9) \) and substracting the derived equation from \( (A9) \), one can obtain

\[
\frac{\partial X^k}{\partial x^l} \left\{ (C_1^2 g_{11} - C_2^2 g_{22}) T^{lij} + \right.
\]
\[
+ \left[ g_{12}(C_1^2 + C_2^2) + C_1 C_2 (g_{11} + g_{22}) \right] M^{lij} \left\{ (M^{lij} - M^{lij}) \right\} = 0 , \tag{A11}
\]

where \( T^{lij} \) is an introduced notation for

\[
T^{lij} := \frac{\partial x^2}{\partial X^i} \frac{\partial^2 x^1}{\partial X^i \partial X^j} - \frac{\partial x^1}{\partial X^i} \frac{\partial^2 x^2}{\partial X^i \partial X^j} . \tag{A12}
\]

It can easily be checked that

\[
T^{ijl} := T^{lij} - T^{jil} = \frac{\partial}{\partial X^i} \left\{ \{x^1, x^2\}_X, X^i \right\} , \tag{A13}
\]

where \( \{x^1, x^2\}_X, X^i \) is the notation for the s.c. "one-dimensional Poisson bracket"

\[
\{x^1, x^2\}_X, X^i := \frac{\partial x^1}{\partial X^j} \frac{\partial x^2}{\partial X^i} - \frac{\partial x^1}{\partial X^i} \frac{\partial x^2}{\partial X^j} . \tag{A14}
\]

It can be proved that

\[
M^{lijk} = \frac{\partial}{\partial X^i} \left[ \frac{\partial x^k}{\partial X^i} \frac{\partial x^k}{\partial X^j} \right] ; M^{lijk} = 0 . \tag{A15}
\]
Since we would like to obtain a relation by combining all the components of eq. (A15) in the two-dimensional case, we can write down equation (A11) with interchanged indices \( l \leftrightarrow j \). Subtracting the obtained equation from (A11) and taking into account the "antisymmetric" relations (A13) and (A15), one obtains the simple equation

\[
\frac{\partial X^k}{\partial x^l} (C_1^2 g_{11} - C_2^2 g_{22}) \frac{\partial}{\partial X^i} \{x^1, x^2\}_{X^i, X^i} = 0 .
\] (A16)

Therefore in the general two-dimensional case of non-commuting operators of differentiation, the "modified" connection \( \tilde{\Gamma}^k_{ij} \) has affine transformation properties in each one of the following cases

1. If the generalized coordinates \( X^1 \) and \( X^2 \) do not depend on \( x^1 \).
2. If the Poisson bracket \( \{x^1, x^2\}_{X^i, X^i} \) is constant on the integral curves \( dx^1 = C_1 \) and \( dx^2 = C_2 \).
3. If the following relation is fulfilled for the metric components \( g_{11} \) and \( g_{22} \) and for the (arbitrary) constants \( C_1 \) and \( C_2 \)

\[
C_1^2 g_{11} - C_2^2 g_{22} = 0 .
\] (A17)

**A2: THE CONNECTION \( \tilde{\Gamma}^k_{ij} \) AS AN EQUIAFFINE CONNECTION**

We have to prove that the connection \( \tilde{\Gamma}^k_{ij} \) for \( j = k \) can be represented in the form of a gradient of a scalar quantity, i. e.

\[
\tilde{\Gamma}^k_{ij} = \partial_i \ln e .
\] (A18)

In the approximation \( (dX^i)_k = 0 \) one can prove that the connection \( \tilde{\Gamma}^k_{ij} \) is indeed an equiaffine one, since one can set up

\[
\ln e \equiv \frac{1}{2} dX^k dX^s g_{ks} .
\] (A19)

The more complicated and interesting task is to prove that even in the case \( (dX^i)_k \neq 0 \), the connection \( \tilde{\Gamma}^k_{ij} \) will again be an equiaffine one. For the purpose, note that

\[
\tilde{\Gamma}^k_{ik} = \frac{1}{2} dX^k dX^s g_{ks,i} = \frac{1}{2} dX^k dX^s g_{r(i} \Gamma^r_{k)i} = W_i
\] (A20)

and consequently \( \tilde{\Gamma}^k_{ik} \) will be an equiaffine connection if the scalar quantity \( e \) can be determined as a solution of the differential equation

\[
\partial_i \ln e = W_i
\] (A21)

as

\[
e = g(X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n) e^{\int W_i(x_1, .., x_n) dx^i} .
\] (A22)
Note that the function $g$ depends on all variables $X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n$ with the exception of $X_i$, while the function $W_i$ depends on all the variables, including also $X_i$.

Unfortunately, the proof at this stage will be incomplete, since $e$ will depend on the choice of the variable $X_i$, which should not happen with a scalar quantity. Consequently, it should be proved that the function $g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n)$ can be determined in a proper way (so that for every choice of $W_i$) the expression (A22) for $e$ would be a scalar quantity. Until we have not proved it, we shall denote the L.H.S. of (A22) with $e^{(i)}$.

Let us differentiate both sides of (A22) for $e \equiv e^{(i)}$ and $e \equiv e^{(j)}$ by $X^j$ and $X^i$ respectively ($i \neq j$). We shall write down only the first equation, since the second one is obtained from the first after a change of the indices $i \iff j$.

\[
\frac{\partial e^{(i)}}{\partial X^j} = \frac{\partial \ln g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n)}{\partial X^j} e^{(i)} +
\]
\[
+ g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n) e^{\int \frac{\partial W_i}{\partial X^j} dx^i} . \quad (A23)
\]

Now differentiate again the derived equation (A23) for $\frac{\partial e^{(i)}}{\partial X^j}$ by $X^i$ and the other equation for $\frac{\partial e^{(j)}}{\partial X^i}$ by $X^j$. Taking into account also that $\frac{\partial e^{(i)}}{\partial X^i} = e^{(i)} W_i$, applying again (A23) and defining summation over the indices $i$ and $j$, the result will be

\[
\sum_{i,j} \frac{\partial}{\partial X^j} \left( \frac{\partial e^{(i)}}{\partial X^i} \right) = grad [\ln g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n)] (e \cdot W) +
\]
\[
+ \sum_{i,j} \left[ \frac{\partial W_i}{\partial X^j} \frac{\partial e^{(i)}}{\partial X^i} - \frac{\partial W_{ij}}{\partial X^j} e^{(i)} \right] + (e \cdot W) \triangle \ln g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n) , \quad (A24)
\]

where $(e \cdot W)$ denotes a scalar product and the following notation has been introduced

\[
\tilde{W}_{ij} \equiv W_i \frac{\partial \ln g(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n)}{\partial X^j} . \quad (A25)
\]

Again, the second equation will be the same as (A24), but with $i \iff j$. Subtracting the two equations and taking into account the formulae for $graddive = \sum_{i,j} \frac{\partial}{\partial X^i} \left( \frac{\partial e^{(i)}}{\partial X^j} \right)$, one can derive

\[
\sum_{i,j} \left[ \frac{\partial W_i}{\partial X^j} \frac{\partial e^{(i)}}{\partial X^i} - \frac{\partial W_j}{\partial X^i} \frac{\partial e^{(j)}}{\partial X^j} \right] -
\]
\[
- \sum_{i,j} \left[ \frac{\partial \tilde{W}_{ij}}{\partial X^j} e^{(i)} - \frac{\partial \tilde{W}_{ji}}{\partial X^i} e^{(j)} \right] = 0 . \quad (A26)
\]

But it can be written also

\[
\sum_{i,j} \left[ \frac{\partial W_i}{\partial X^j} \frac{\partial e^{(i)}}{\partial X^i} - \frac{\partial \tilde{W}_{ij}}{\partial X^j} e^{(i)} \right] =
\]
\[
= \sum_{i,j} \left[ \frac{\partial W_i}{\partial X^j} \frac{\partial e^{(i)}}{\partial X^j} - e^{(j)} \frac{\partial W_i}{\partial X^j} \frac{\partial \ln g}{\partial X^j} - (e \cdot \mathbf{W}) \Delta \ln g \right].
\]

(A27)

Taking the above expression into account, equation (A26) acquires the form

\[
\sum_{i,j} \left[ \frac{\partial W_i}{\partial X^j} \frac{\partial e^{(i)}}{\partial X^j} - e^{(j)} \frac{\partial W_i}{\partial X^j} \frac{\partial \ln g}{\partial X^j} \right] + \\
+ \sum_{i,j} \left[ e^{(j)} \frac{\partial W_j}{\partial X^i} \frac{\partial \ln g}{\partial X^i} - e^{(i)} \frac{\partial W_i}{\partial X^j} \frac{\partial \ln g}{\partial X^j} \right] = 0.
\]

(A28)

Further we shall assume that each term in the sum is zero, i.e. the equation is fulfilled for each \(i\) and \(j\). Substituting expressions (A22) for \(e^{(i)}\) and \(e^{(j)}\) and (A23) for \(\frac{\partial e^{(i)}}{\partial X^j}\) and \(\frac{\partial e^{(j)}}{\partial X^i}\), differentiating the obtained expression by \(X^i\) and making use again of (A22) and (A23), the following simple differential equation can be obtained:

\[
W_{j,i} \frac{\partial \ln g(X_1,..,X_{j-1},X_{j+1},..,X_n)}{\partial X^i} + (W_{j,ii} - W_{i,j} W_{j,i} - \\
- \frac{W_{i,ji} W_{j,ii}}{W_{i,j}}) + W_{j,i} e^{\int \left( \frac{\partial^2 W_j}{\partial X^{i2}} - \frac{\partial W_j}{\partial X^i} \right) dX^j} = 0.
\]

(A29)

The first case, when this equation will be satisfied will be

\[
W_{j,i} = \frac{\partial W_j}{\partial X^i} = 0 \quad \Rightarrow \quad W_j = f(X_1,..,X_{i-1},X_{i+1},..,X_n)
\]

(A30)

Since this will be fulfilled for every \(i\), then \(W_j\) should be a constant, which of course is a very rare and special case.

The second, more realistic case is when the function \(g\) is a solution of the differential equation (A23) for every \(i\) and \(j\) \((i \neq j)\):

\[
g(X_1,..,X_{j-1},X_{j+1},..,X_n) = F(X_1,..,X_{i-1},X_{i+1},..,X_n) e^{\int \tilde{Q}(X_1,..,X_n) dX^i},
\]

(A31)

where

\[
\tilde{Q}(X_1,..,X_n) \equiv \left( W_{i,j} + \frac{W_{i,ji}}{W_{i,j}} - \frac{W_{j,ii}}{W_{j,ji}} \right) - e^{\int \left( \frac{\partial^2 W_j}{\partial X^{i2}} - \frac{\partial W_j}{\partial X^i} \right) dX^j}.
\]

(A32)

Since the function \(g(X_1,..,X_{j-1},X_{j+1},..,X_n)\) on the L. H. S. of (A31) does not depend on the variable \(X_j\), then for each \(j\) the unknown function \(F(X_1,..,X_{i-1},X_{i+1},..,X_n)\) can be obtained after differentiating both sides of (A31) by \(X^j\). Thus the function \(F\) is a solution of the following differential equation

\[
0 = \frac{\partial F(X_1,..,X_{i-1},X_{i+1},..,X_n)}{\partial X^j} e^{\int \tilde{Q} dX^i} + F(X_1,..,X_{i-1},X_{i+1},..,X_n) e^{\int \frac{\partial \tilde{Q}}{\partial X^i} dX^i}.
\]

(A33)

This precludes the proof that the function \(g\) in (A22) can be determined in such a way that \(e^{(i)}\) would be indeed a scalar quantity and therefore \(e \equiv e^{(i)}\). Throughout the whole proof, we assumed that \(W_i\) determined by (A20), is a vector. This of course should be proved in the same way, in which it was proved that the connection \(\tilde{\Gamma}_{ij}^k\) has affine transformation properties.
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