Every function is the representation function of an additive basis for the integers

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Abstract

Let \( A \) be a set of integers. For every integer \( n \), let \( r_{A,h}(n) \) denote the number of representations of \( n \) in the form \( n = a_1 + a_2 + \cdots + a_h \), where \( a_1, a_2, \ldots, a_h \in A \) and \( a_1 \leq a_2 \leq \cdots \leq a_h \). The function

\[
r_{A,h} : \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}
\]

is the representation function of order \( h \) for \( A \). The set \( A \) is called an asymptotic basis of order \( h \) if \( r_{A,h}^{-1}(0) \) is finite, that is, if every integer with at most a finite number of exceptions can be represented as the sum of exactly \( h \) not necessarily distinct elements of \( A \). It is proved that every function is a representation function, that is, if \( f : \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\} \) is any function such that \( f^{-1}(0) \) is finite, then there exists a set \( A \) of integers such that \( f(n) = r_{A,h}(n) \) for all \( n \in \mathbb{Z} \). Moreover, the set \( A \) can be arbitrarily sparse in the sense that, if \( \varphi(x) \geq 0 \) for \( x \geq 0 \) and \( \varphi(x) \rightarrow \infty \), then there exists a set \( A \) with \( f(n) = r_{A,h}(n) \) and \( \text{card} \{ a \in A : |a| \leq x \} < \varphi(x) \) for all \( x \).

It is an open problem to construct dense sets of integers with a prescribed representation function. Other open problems are also discussed.

1 Additive bases and the Erdős-Turán conjecture

Let \( \mathbb{N}, \mathbb{N}_0, \) and \( \mathbb{Z} \) denote the positive integers, nonnegative integers, and integers, respectively. Let \( A \) be a set of integers. For every positive integer \( h \), we define the sumset

\[
hA = \{a_1 + \cdots + a_h : a_i \in A \text{ for all } i = 1, \ldots, h\}.
\]

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We denote by \( r_{A,h}(n) \) the number of representations of \( n \) in the form \( n = a_1 + a_2 + \cdots + a_h \), where \( a_1, a_2, \ldots, a_h \in A \) and \( a_1 \leq a_2 \leq \cdots \leq a_h \). The function \( r_{A,h} \) is called the representation function of order \( h \) of the set \( A \).

We denote the cardinality of the set \( A \) by \( \text{card}(A) \). If \( A \) is a finite set of integers, we denote the maximum element of \( A \) by \( \max(A) \). For any integer \( t \) and set \( A \subseteq \mathbb{Z} \), we define the translate

\[
t + A = \{ t + a : a \in A \}.
\]

In this paper we consider additive bases for the set of all integers. The set \( A \) of integers is called a basis of order \( h \) for \( \mathbb{Z} \) if every integer can be represented as the sum of \( h \) not necessarily distinct elements of \( A \). The set \( A \) of integers is called an asymptotic basis of order \( h \) for \( \mathbb{Z} \) if every integer with at most a finite number of exceptions can be represented as the sum of \( h \) not necessarily distinct elements of \( A \). Equivalently, the set \( A \) is an asymptotic basis of order \( h \) if the representation function \( r_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{ \infty \} \) satisfies the condition

\[
\text{card} \left( r_{A,h}^{-1}(0) \right) < \infty.
\]

For any set \( X \), let \( \mathcal{F}_0(X) \) denote the set of all functions

\[
f : X \to \mathbb{N}_0 \cup \{ \infty \}
\]

such that

\[
\text{card} \left( f^{-1}(0) \right) < \infty.
\]

We ask: Which functions in \( \mathcal{F}_0(\mathbb{Z}) \) are representation functions of asymptotic bases for the integers? This question has a remarkably simple and surprising answer. In the case \( h = 1 \) we observe that \( f \in \mathcal{F}_0(\mathbb{Z}) \) is a representation function if and only if \( f(n) = 1 \) for all integers \( n \notin f^{-1}(0) \). For \( h \geq 2 \) we shall prove that every function in \( \mathcal{F}_0(\mathbb{Z}) \) is a representation function. Indeed, if \( f \in \mathcal{F}_0(\mathbb{Z}) \) and \( h \geq 2 \), then there exist infinitely many sets \( A \) such that \( f(n) = r_{A,h}(n) \) for every \( n \in \mathbb{Z} \). Moreover, we shall prove that we can construct arbitrarily sparse asymptotic bases \( A \) with this property. Nathanson \[7\] previously proved this theorem for \( h = 2 \) and the function \( f(n) = 1 \) for all \( n \in \mathbb{Z} \).

This result about asymptotic bases for the integers contrasts sharply with the case of the nonnegative integers. The set \( A \) of nonnegative integers is called an asymptotic basis of order \( h \) for \( \mathbb{N}_0 \) if every sufficiently large integer can be represented as the sum of \( h \) not necessarily distinct elements of \( A \). Very little is known about the class of representation functions of asymptotic bases for \( \mathbb{N}_0 \). However, if \( f \in \mathcal{F}_0(\mathbb{N}_0) \), then Nathanson \[13\] proved that there exists at most one set \( A \) such that \( r_{A,h}(n) = f(n) \).

Many of the results that have been proved about asymptotic bases for \( \mathbb{N}_0 \) are negative. For example, Dirac \[2\] showed that the representation function of an asymptotic basis of order 2 cannot be eventually constant. Erdős and Fuchs \[4\] proved that the average value of a representation function of order 2
cannot even be approximately constant, in the sense that, for every infinite set $A$ of nonnegative integers and every real number $c > 0$,

$$\sum_{n \leq N} r_{A,2}(n) \neq cN + o\left(N^{1/4} \log^{-1/2} N\right).$$

Erdős and Turán [3] conjectured that if $A$ is an asymptotic basis of order $h$ for the nonnegative integers, then the representation function $r_{A,h}(n)$ must be unbounded, that is,

$$\limsup_{n \to \infty} r_{A,h}(n) = \infty.$$

This famous unsolved problem in additive number theory is only a special case of the general problem of classifying the representation functions of asymptotic bases of finite order for the nonnegative integers.

2 Two lemmas

We use the following notation. For sets $A$ and $B$ of integers and for any integer $t$, we define the sumset

$$A + B = \{a + b : a \in A, b \in B\},$$

the translation

$$A + t = \{a + t : a \in A\},$$

and the difference set

$$A - B = \{a - b : a \in A, b \in B\}.$$

For every nonnegative integer $h$ we define the $h$-fold sumset $hA$ by induction:

$$0A = \{0\},$$

$$hA = A + (h-1)A = \{a_1 + a_2 + \ldots + a_h : a_1, a_2, \ldots, a_h \in A\}.$$  

The counting function for the set $A$ is

$$A(y, x) = \text{card}\{a \in A : y \leq a \leq x\}.$$  

In particular, $A(-x, x)$ counts the number of integers $a \in A$ with $|a| \leq x$.

Let $[x]$ denote the integer part of the real number $x$.

**Lemma 1** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that $f^{-1}(0)$ is finite. Let $\Delta$ denote the cardinality of the set $f^{-1}(0)$. Then there exists a sequence $U = \{u_k\}_{k=1}^\infty$ of integers such that, for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$f(n) = \text{card}\{k \geq 1 : u_k = n\}$$

and

$$|u_k| \leq \left\lfloor \frac{k + \Delta}{2} \right\rfloor.$$  

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Proof. Every positive integer $m$ can be written uniquely in the form

$$m = s^2 + s + 1 + r,$$

where $s$ is a nonnegative integer and $|r| \leq s$. We construct the sequence

$$V = \{0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

where

$$v_{s^2+s+1+r} = r \quad \text{for } |r| \leq s.$$

For every nonnegative integer $k$, the first occurrence of $-k$ in this sequence is $v_{k^2} = -k$, and the first occurrence of $k$ in this sequence is $v_2^2 = k$.

The sequence $U$ will be the unique subsequence of $V$ constructed as follows. Let $n \in \mathbb{Z}$. If $f(n) = \infty$, then $U$ will contain the terms $v_{s^2+s+1+n}$ for every $s \geq |n|$. If $f(n) = \ell < \infty$, then $U$ will contain the $\ell$ terms $v_{s^2+s+1+n}$ for $s = |n|, |n| + 1, \ldots, |n| + \ell - 1$ in the subsequence $U$, but not the terms $v_{s^2+s+1+n}$ for $s \geq |n| + \ell$. Let $m_1 < m_2 < m_3 < \cdots$ be the strictly increasing sequence of positive integers such that $\{v_{m_k}\}_{k=1}^\infty$ is the resulting subsequence of $V$. Let $U = \{u_k\}_{k=1}^\infty$, where $u_k = v_{m_k}$. Then

$$f(n) = \text{card}(\{k \geq 1 : u_k = n\}).$$

Let $\text{card}(f^{-1}(0)) = \Delta$. The sequence $U$ also has the following property: If $|u_k| = n$, then for every integer $m \notin f^{-1}(0)$ with $|m| < n$ there is a positive integer $j < k$ with $u_j = m$. It follows that

$$\{0, 1, -1, 2, -2, \ldots, n - 1, -(n - 1)\} \setminus f^{-1}(0) \subseteq \{u_1, u_2, \ldots, u_{k-1}\},$$

and so

$$k - 1 \geq 2(n - 1) + 1 - \Delta.$$

This implies that

$$|u_k| = n \leq \frac{k + \Delta}{2}.$$

Since $u_k$ is an integer, we have

$$|u_k| \leq \left[ \frac{k + \Delta}{2} \right].$$

This completes the proof. □

Lemma 1 is best possible in the sense that for every nonnegative integer $\Delta$ there is a function $f: \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $\text{card}(f^{-1}(0)) = \Delta$ and a sequence $U = \{u_k\}_{k=1}^\infty$ of integers such that

$$|u_k| = \left[ \frac{k + \Delta}{2} \right] \quad \text{for all } k \geq 1. \quad (1)$$
For example, if $\Delta = 2\delta + 1$ is odd, define the function $f$ by

$$f(n) = \begin{cases} 
0 & \text{if } |n| \leq \delta \\
1 & \text{if } |n| \geq \delta + 1
\end{cases}$$

and the sequence $U$ by

$$u_{2i-1} = \delta + i,$$
$$u_{2i} = -(\delta + i)$$

for all $i \geq 1$.

If $\Delta = 2\delta$ is even, define $f$ by

$$f(n) = \begin{cases} 
0 & \text{if } -\delta \leq n \leq \delta - 1 \\
1 & \text{if } n \geq \delta \text{ or } n \leq -\delta - 1
\end{cases}$$

and the sequence $U$ by $u_1 = \delta$ and

$$u_{2i} = \delta + i,$$
$$u_{2i+1} = -(\delta + i)$$

for all $i \geq 1$. In both cases the sequence $U$ satisfies (1).

The set $A$ is called a **Sidon set of order $h$** if $r_{A,h}(n) = 0$ or 1 for every integer $n$. If $A$ is a Sidon set of order $h$, then $A$ is a Sidon set of order $j$ for all $j = 1, 2, \ldots, h$.

**Lemma 2** Let $A$ be a finite Sidon set of order $h$ and $d = \max\{|a| : a \in A\}$. If $|c| > (2h - 1)d$, then $A \cup \{c\}$ is also a Sidon set of order $h$.

**Proof.** Let $n \in h(A \cup \{c\})$. Suppose that

$$n = a_1 + \cdots + a_j + (h-j)c = a'_1 + \cdots + a'_\ell + (h-\ell)c,$$

where

$$0 \leq j \leq \ell \leq h,$$
$$a_1, \ldots, a_j, a'_1, \ldots, a'_\ell \in A,$$

and

$$a_1 \leq \cdots \leq a_j \quad \text{and} \quad a'_1 \leq \cdots \leq a'_\ell.$$  

If $j < \ell$, then

$$|c| \leq |(\ell-j)c|$$
$$= |a'_1 + \cdots + a'_\ell - (a_1 + \cdots + a_j)|$$
$$\leq (\ell + j)d$$
$$\leq (2h - 1)d$$
$$< |c|,$$

which is absurd. Therefore, $j = \ell$ and $a_1 + \cdots + a_j = a'_1 + \cdots + a'_\ell$. Since $A$ is a Sidon set of order $j$, it follows that $a_i = a'_i$ for all $i = 1, \ldots, j$. Consequently, $A \cup \{c\}$ is a Sidon set of order $h$. □
3 Construction of asymptotic bases

We can now construct asymptotic bases of order $h$ for the integers with arbitrary representation functions.

**Theorem 1** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that the set $f^{-1}(0)$ is finite. Let $\varphi : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function such that $\lim_{x \to \infty} \varphi(x) = \infty$. For every $h \geq 2$ there exist infinitely many asymptotic bases $A$ of order $h$ for the integers such that

$$r_{A,h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z},$$

and

$$A(-x,x) \leq \varphi(x)$$

for all $x \geq 0$.

**Proof.** By Lemma 1 there is a sequence $U = \{u_k\}_{k=1}^\infty$ of integers such that

$$f(n) = \text{card}\left(\{k \geq 1 : u_k = n\}\right)$$

for every integer $n$.

Let $h \geq 2$. We shall construct an ascending sequence of finite sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ such that, for all positive integers $k$ and for all integers $n$,

(i)$$r_{A_k,h}(n) \leq f(n),$$

(ii)$$r_{A_k,h}(n) \geq \text{card}\left(\{i : 1 \leq i \leq k \text{ and } u_i = n\}\right),$$

(iii)$$\text{card}(A_k) \leq 2k,$$

(iv)$$A_k \text{ is a Sidon set of order } h-1.$$ Conditions (i) and (ii) imply that the infinite set

$$A = \bigcup_{k=1}^{\infty} A_k$$

is an asymptotic basis of order $h$ for the integers such that $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$.

We construct the sets $A_k$ by induction. Since the set $f^{-1}(0)$ is finite, there exists an integer $d_0$ such that $f(n) \geq 1$ for all integers $n$ with $|n| \geq d_0$. If $u_1 \geq 0$, choose a positive integer $c_1 > 2hd_0$. If $u_1 < 0$, choose a negative integer $c_1 < -2hd_0$. Then

$$|c_1| > 2hd_0.$$
Let
\[ A_1 = \{-c_1, (h-1)c_1 + u_1\}. \]
The sumset \( hA_1 \) is the finite arithmetic progression
\[
hA_1 = \{-hc_1 + (hc_1 + u_1)i : i = 0, 1, \ldots, h\}
\[
= \{-hc_1, u_1, hc_1 + 2u_1, 2hc_1 + 3u_1, \ldots, (h-1)hc_1 + hu_1\}.
\]
Then \(|n| \geq hc_1 > d_0\) for all \( n \in hA_1 \setminus \{u_1\} \), and so
\[
r_{A_1,h}(n) = 1 \leq f(n)
\]
for all \( n \in hA_1 \). It follows that \( r_{A_1,h}(n) \leq f(n) \) for all \( n \in \mathbb{Z} \). The set \( A_1 \) is a Sidon set of order \( h \), hence also a Sidon set of order \( h - 1 \). Thus, the set \( A_1 \) satisfies conditions (i)–(iv).

We assume that for some integer \( k \geq 2 \) we have constructed a set \( A_{k-1} \) satisfying conditions (i)–(iv). If
\[
r_{A_{k-1},h}(u_k) = \text{card} \left( \{i : 1 \leq i \leq k \text{ and } u_i = u_k\} \right) - 1 < f(u_k).
\]
We shall construct a Sidon set \( A_k \) of order \( h - 1 \) such that
\[
\text{card}(A_k) = \text{card}(A_{k-1}) + 2
\]
and
\[
r_{A_k,h}(n) = \begin{cases} 
r_{A_{k-1},h}(n) + 1 & \text{if } n = u_k \\
r_{A_{k-1},h}(n) & \text{if } n \in hA_{k-1} \setminus \{u_k\} \\
1 & \text{if } n \in hA_k \setminus (hA_{k-1} \cup \{u_k\}). \end{cases} \tag{2}
\]
Define the integer
\[
d_{k-1} = \max \left( \{|a| : a \in A_{k-1} \cup \{u_k\}\} \right). \tag{3}
\]
Then
\[
A_{k-1} \subseteq [-d_{k-1}, d_{k-1}].
\]
If \( u_k \geq 0 \), choose a positive integer \( c_k \) such that \( c_k > 2hd_{k-1} \). If \( u_k < 0 \), choose a negative integer \( c_k \) such that \( c_k < -2hd_{k-1} \). Then
\[
|c_k| > 2hd_{k-1}. \tag{4}
\]
Let
\[
A_k = A_{k-1} \cup \{-c_k, (h-1)c_k + u_k\}.
\]
Then
\[
\text{card}(A_k) = \text{card}(A_{k-1}) + 2 \leq 2k.
\]

We shall assume that \( u_k \geq 0 \), hence \( c_k > 0 \). (The argument in the case \( u_k < 0 \) is similar.) We decompose the sumset \( hA_k \) as follows:

\[
hA_k = \bigcup_{r+i+j=h, \ r,i,j \geq 0} (r(h-1)c_k + ru_k - ic_k + jA_{k-1}) = \bigcup_{r=0}^h B_r,
\]

where

\[
B_r = r(h-1)c_k + ru_k + \bigcup_{i=0}^{h-r} (-ic_k + (h - r - i)A_{k-1}).
\]

If \( n \in B_r \), then there exist integers \( i \in \{0, 1, \ldots, h-r\} \) and \( y \in (h-r-i)A_{k-1} \) such that

\[
n = r(h-1)c_k + ru_k - ic_k + y.
\]

Since

\[
|y| \leq (h-r-i)d_{k-1},
\]

it follows that

\[
n \geq r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1}
\]

and

\[
n \leq r(h-1)c_k + ru_k - ic_k + (h-r-i)d_{k-1}.
\]

Let \( m \in B_{r-1} \) and \( n \in B_r \) for some \( r \in \{1, \ldots, h\} \). There exist nonnegative integers \( i \leq h-r \) and \( j \leq h-r+1 \) such that

\[
n - m \geq (r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1})
\]

\[
- ((r-1)(h-1)c_k + (r-1)u_k - jc_k + (h-r+1-j)d_{k-1})
\]

\[
= (h-1+j-i)c_k + u_k - (2h-2r-i-j+1)d_{k-1}
\]

\[
\geq (h-1-i)c_k - 2hd_{k-1}.
\]

If \( r \geq 2 \), then \( i \leq h-2 \) and inequality (4) implies that

\[
n - m \geq c_k - 2hd_{k-1} > 0.
\]

Therefore, if \( m \in B_{r-1} \) and \( n \in B_r \) for some \( r \in \{2, \ldots, h\} \), then \( m < n \).

In the case \( r = 1 \) we have \( m \in B_0 \) and \( n \in B_1 \). If \( i \leq h-2 \), then (4) implies that

\[
n - m \geq (h-1-i)c_k - 2hd_{k-1} \geq c_k - 2hd_{k-1} > 0
\]

and (5) implies that

\[
n \geq (h-1-i)c_k + u_k - (h-1-i)d_{k-1} > c_k - hd_{k-1} > 0.
\]

If \( r = 1 \) and \( i = h-1 \), then \( n = u_k \). Therefore, if \( m \in B_0 \) and \( n \in B_1 \), then \( m < n \) unless \( m = n = u_k \). It follows that the sets \( B_0, B_1 \setminus \{u_k\}, B_2, \ldots, B_h \) are pairwise disjoint.
Let $n \in B_r$ for some $r \geq 1$. Suppose that $0 \leq i \leq j \leq h - r$, and that
\[ n = r(h - 1)c_k + ru_k - ic_k + y \quad \text{for some } y \in (h - r - i)A_{k-1} \]
and
\[ n = r(h - 1)c_k + ru_k - jc_k + z \quad \text{for some } z \in (h - r - j)A_{k-1}. \]
Subtracting these equations, we obtain
\[ z - y = (j - i)c_k. \]
Recall that $|a| \leq d_{k-1}$ for all $a \in A_{k-1}$. If $i < j$, then
\[ c_k \leq (j - i)c_k = z - y \leq |y| + |z| \leq (2h - 2r - i - j)d_{k-1} < 2hd_{k-1} < c_k, \]
which is impossible. Therefore, $i = j$ and $y = z$. Since $0 \leq h - r - i \leq h - 1$ and $A_{k-1}$ is a Sidon set of order $h - 1$, it follows that
\[ r_{A_{k-1},h-r-i}(y) = 1 \]
and so
\[ r_{A_k,h}(n) = 1 \leq f(n) \quad \text{for all } n \in (B_1 \setminus \{u_k\}) \cup \bigcup_{r=2}^{h} B_r. \]
Next we consider the set
\[ B_0 = hA_{k-1} \cup \bigcup_{i=1}^{h} (-ic_k + (h - i)A_{k-1}). \]
For $i = 1, \ldots, h$, we have
\[ c_k > 2hd_{k-1} \geq (2h - 2i + 1)d_{k-1} \]
and so
\[ \max (-ic_k + (h - i)A_{k-1}) \leq -ic_k + (h - i)d_{k-1} < -(i - 1)c_k + (h - i + 1)d_{k-1} \leq \min (-ic_k + (h - i)A_{k-1}). \]
Therefore, the sets $-ic_k + (h - i)A_{k-1}$ are pairwise disjoint for $i = 0, 1, \ldots, h$. In particular, if $n \in B_0 \setminus hA_{k-1}$, then
\[ n \leq \max (-c_k + (h - 1)A_{k-1}) \leq -c_k + (h - 1)d_{k-1} < -d_{k-1} \leq -d_0 \]
and $f(n) \geq 1$. Since $A_{k-1}$ is a Sidon set of order $h - 1$, it follows that
\[ r_{A_k,h}(n) = 1 \leq f(n) \quad \text{for all } n \in \bigcup_{i=1}^{h} (-ic_k + (h - i)A_k) = B_0 \setminus hA_{k-1}. \]
It follows from (3) that for any \( n \in B_0 \setminus hA_{k-1} \) we have
\[
n < -d_{k-1} \leq u_k,
\]
and so \( u_k \notin B_0 \setminus hA_{k-1} \). Therefore,
\[
r_{A_k,h}(u_k) = r_{A_{k-1},h}(u_k) + 1,
\]
and the representation function \( r_{A_k,h} \) satisfies the three requirements of (2).

We shall prove that
\[
A_k = A_{k-1} \cup \{-c_k, (h-1)c_k + u_k\}.
\]
is a Sidon set of order \( h - 1 \). Since \( A_{k-1} \) is a Sidon set of order \( h - 1 \) with
\[
d_{k-1} \geq \max\{|a| : a \in A_{k-1}\},
\]
and since
\[
c_k > 2hd_{k-1} > (2(h - 1) - 1)d_{k-1},
\]
Lemma 2 implies that \( A_{k-1} \cup \{-c_k\} \) is a Sidon set of order \( h - 1 \).

Let \( n \in (h - 1)A_k \). Suppose that
\[
n = r(h-1)c_k + ru_k - ic_k + x
\]
\[
= s(h-1)c_k + su_k - jc_k + y,
\]
where
\[
0 \leq r \leq s \leq h - 1,
\]
\[
0 \leq i \leq h - 1 - r,
\]
\[
0 \leq j \leq h - 1 - s,
\]
\[
x \in (h - 1 - r - i)A_{k-1},
\]
and
\[
y \in (h - 1 - s - j)A_{k-1}.
\]
Then
\[
|x| \leq (h - 1 - r - i)d_{k-1}
\]
and
\[
|y| \leq (h - 1 - s - j)d_{k-1}.
\]
If \( r < s \), then \( j \leq h - 2 \) and
\[
(h-1)c_k \leq (s-r)(h-1)c_k + (s-r)u_k
\]
\[
= (j-i)c_k + x - y
\]
\[
\leq (j-i)c_k + (2h - 2 - r - s - i - j)d_{k-1}
\]
\[
\leq (h-2)c_k + 2hd_{k-1}
\]
\[
< (h-1)c_k,
\]
which is absurd. Therefore, \( r = s \) and

\[-ic_k + x = -jc_k + y \in (h - 1 - r)(A_k \cup \{-c_k\}).\]

Since \( A_k \cup \{-c_k\} \) is a Sidon set of order \( h - 1 \), it follows that \( i = j \) and that \( x \) has a unique representation as the sum of \( h - 1 - r - i \) elements of \( A_k \). Thus, \( A_k \) is a Sidon set of order \( h - 1 \).

The set \( A_k \) satisfies conditions (i) – (iv). It follows by induction that there exists an infinite increasing sequence \( A_1 \subseteq A_2 \subseteq \cdots \) of finite sets with these properties, and that \( A = \cup_{k=1}^{\infty}A_k \) is an asymptotic basis of order \( h \) with representation function \( r_{A,h}(n) = f(n) \) for all \( n \in \mathbb{Z} \).

Let \( A_0 = \emptyset \), and let \( K \) be the set of all positive integers \( k \) such that \( A_k \neq A_{k-1} \). Then

\[ A = \cup_{k \in K} A_k = \cup_{k \in K} \{-c_k, (h-1)ck\}. \]

For \( k \in K \), the only constraints on the choice of the number \( c_k \) in the construction of the set \( A_k \) were the sign of \( c_k \) and the growth condition

\[ |c_k| > 2hd_{k-1} \quad \text{for all integers } k \in K. \]

We shall prove that we can construct the asymptotic basis \( A \) with counting function \( A(-x, x) \leq \varphi(x) \) for all \( x \geq 0 \). Since \( \varphi(x) \to \infty \) as \( x \to \infty \), for every integer \( k \geq 0 \) there exists an integer \( w_k \) such that

\[ \varphi(x) \geq 2k \quad \text{for all } x \geq w_k. \]

We now impose the following additional constraint: Choose \( c_k \) such that

\[ |c_k| \geq w_k \quad \text{for all integers } k \in K. \]

Then

\[ A_1(-x, x) = 0 \leq \varphi(x) \quad \text{for } 0 \leq x < |c_1| \]

and

\[ A_1(-x, x) \leq 2 \leq \varphi(x) \quad \text{for } x \geq |c_1| \geq w_1. \]

Suppose that \( k \geq 2 \) and the set \( A_{k-1} \) satisfies \( A_{k-1}(-x, x) \leq \varphi(x) \) for all \( x \geq 0 \). Since

\[ A_k \cap (-|c_k|, |c_k|) = A_{k-1} \cap (-|c_k|, |c_k|), \]

it follows that

\[ A_k(-x, x) = A_{k-1}(-x, x) \leq \varphi(x) \quad \text{for } 0 \leq x < |c_k| \]

and

\[ A_k(-x, x) \leq 2k \leq \varphi(x) \quad \text{for } x \geq |c_k| \geq w_k. \]

This proves by induction that \( A_k(-x, x) \leq \varphi(x) \) for all \( k \) and \( x \). Since \( \lim_{k \to \infty} |c_k| = \infty \), it follows that for any nonnegative integer \( x \) we can choose \( c_k \) so that \( |c_k| > x \) and

\[ A(-x, x) = A_k(-x, x) \leq \varphi(x). \]
For every integer $k \in K$ we had infinitely many choices for the integer $c_k$ to use in the construction of the set $A_k$, and so there are infinitely many asymptotic bases $A$ with the property that $r_A(n) = f(n)$ for all $n \in \mathbb{Z}$. This completes the proof. □

4 Sums of pairwise distinct integers

Let $A$ be a set of integers and $h$ a positive integer. We define the sumset $h \wedge A$ as the set consisting of all sums of $h$ pairwise distinct elements of $A$, and the restricted representation function

\[ \hat{r}_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{ \infty \} \]

by

\[ \hat{r}_{A,h}(n) = \text{card}\left( \{ \{a_1, \ldots, a_h\} \subseteq A : a_1 + \cdots + a_h = n \text{ and } a_1 < \cdots < a_h \} \right) . \]

The set $A$ of integers is called a restricted asymptotic basis of order $h$ if $h \wedge A$ contains all but finitely many integers, or, equivalently, if $\hat{r}_{A,h}^{-1}(0)$ is a finite subset of $\mathbb{Z}$.

The following theorem can also be proved by the method used to prove Theorem 1.

**Theorem 2** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{ \infty \}$ be a function such that $f^{-1}(0)$ is a finite set of integers. Let $\varphi : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function such that $\lim_{x \to \infty} \varphi(x) = \infty$. For every $h \geq 2$ there exist infinitely many sets $A$ of integers such that

\[ \hat{r}_{A,h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z} \]

and

\[ A(-x,x) \leq \varphi(x) \]

for all $x \geq 0$.

5 Open problems

Let $X$ be an abelian semigroup, written additively, and let $A$ be a subset of $X$. We define the $h$-fold sumset $hA$ as the set consisting of all sums of $h$ not necessarily distinct elements of $A$. The set $A$ is called an asymptotic basis of order $h$ for $X$ if the sumset $hA$ consists of all but at most finitely many elements of $X$. We also define the $h$-fold restricted sumset $h \wedge A$ as the set consisting of all sums of $h$ pairwise distinct elements of $A$. The set $A$ is called a restricted asymptotic basis of order $h$ for $X$ if the restricted sumset $h \wedge A$ consists of all but at most finitely many elements of $X$. The classical problems of additive number theory concern the semigroups $\mathbb{N}_0$ and $\mathbb{Z}$.  

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There are four different representation functions that we can associate to every subset \( A \) of \( X \) and every positive integer \( h \). Let \((a_1, \ldots, a_h)\) and \((a'_1, \ldots, a'_h)\) be \( h \)-tuples of elements of \( X \). We call these \( h \)-tuples equivalent if there is a permutation \( \sigma \) of the set \( \{1, \ldots, h\} \) such that \( a'_{\sigma(i)} = a_i \) for all \( i = 1, \ldots, h \). For every \( x \in X \), let \( r_{A,h}(x) \) denote the number of equivalence classes of \( h \)-tuples \((a_1, \ldots, a_h)\) of elements of \( A \) such that \( a_1 + \cdots + a_h = x \). The function \( r_{A,h} \) is called the unordered representation function of \( A \). This is the function that we studied in this paper. The set \( A \) is an asymptotic basis of order \( h \) if \( r_{A,h}(0) \) is a finite subset of \( X \).

Let \( R_{A,h}(x) \) denote the number of \( h \)-tuples \((a_1, \ldots, a_h)\) of elements of \( A \) such that \( a_1 + \cdots + a_h = x \). The function \( R_{A,h} \) is called the ordered representation function of \( A \).

Let \( \hat{r}_{A,h}(x) \) denote the number of equivalence classes of \( h \)-tuples \((a_1, \ldots, a_h)\) of pairwise distinct elements of \( A \) such that \( a_1 + \cdots + a_h = x \), and let \( \hat{R}_{A,h}(x) \) denote the number of \( h \)-tuples \((a_1, \ldots, a_h)\) of pairwise distinct elements of \( A \) such that \( a_1 + \cdots + a_h = x \). These functions are called the unordered restricted representation function of \( A \) and the ordered restricted representation function of \( A \), respectively. The two restricted representation functions are essentially identical, since \( \hat{R}_{A,h}(x) = h! \hat{r}_{A,h}(x) \) for all \( x \in X \).

In the discussion below, we use only the unordered representation function \( r_{A,h} \), but each of the problems can be reformulated in terms of the other representation functions.

For every countable abelian semigroup \( X \), let \( F(X) \) denote the set of all functions \( f : X \to \mathbb{N}_0 \cup \{\infty\} \), and let \( F_0(X) \) denote the set of all functions \( f : X \to \mathbb{N}_0 \cup \{\infty\} \) such that \( f^{-1}(0) \) is a finite subset of \( X \). Let \( F_c(X) \) denote the set of all functions \( f : X \to \mathbb{N}_0 \cup \{\infty\} \) such that \( f^{-1}(0) \) is a cofinite subset of \( X \), that is, \( f(x) \neq 0 \) for only finitely many \( x \in X \), or, equivalently,

\[
\text{card}(f^{-1}(\mathbb{N} \cup \{\infty\})) < \infty.
\]

Let \( R(X,h) \) denote the set of all \( h \)-fold representation functions of subsets \( A \) of \( X \). If \( r_{A,h} \) is the representation function of an asymptotic basis \( A \) of order \( h \) for \( X \), then \( r_{A,h}^{-1}(0) \) is a finite subset of \( X \), and so \( r_{A,h} \in F_0(X) \). Let \( R_0(X,h) \) denote the set of all \( h \)-fold representation functions of asymptotic bases \( A \) of order \( h \) for \( X \). Let \( R_c(X,h) \) denote the set of all \( h \)-fold representation functions of finite subsets of \( X \). We have

\[
R(X,h) \subseteq F(X),
\]

\[
R_0(X,h) \subseteq F_0(X),
\]

and

\[
R_c(X,h) \subseteq F_c(X),
\]

In the case \( h = 1 \), we have, for every set \( A \subseteq X \),

\[
r_{A,1}(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A,
\end{cases}
\]

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and so
\[ R(X, 1) = \{ f : X \to \{0, 1\} \}, \]
\[ R_0(X, 1) = \{ f : X \to \{0, 1\} : \text{card}(f^{-1}(0)) < \infty \}, \]
and
\[ R_c(X, 1) = \{ f : X \to \{0, 1\} : \text{card}(f^{-1}(\mathbb{N} \cup \{\infty\})) < \infty \}. \]

In this paper we proved that
\[ R_0(\mathbb{Z}, h) = \mathcal{F}_0(\mathbb{Z}) \quad \text{for all } h \geq 2. \]

Nathanson [8] has extended this result to certain countably infinite groups and semigroups. Let \( G \) be any countably infinite abelian group such that \( \{2g : g \in G\} \) is infinite. For the unordered restricted representation function \( \hat{r}_{A,2} \), we have
\[ R_0(G, 2) = \mathcal{F}_0(G). \]

More generally, let \( S \) is any countable abelian semigroup such that for every \( s \in S \) there exist \( s', s'' \in S \) with \( s = s' + s'' \). In the abelian semigroup \( X = S \oplus G \), we have
\[ R_0(X, 2) = \mathcal{F}_0(X). \]

If \( \{12g : g \in G\} \) is infinite, then \( R_0(X, 2) = \mathcal{F}_0(X) \) for the unordered representation function \( r_{A,2} \).

The following problems are open for all \( h \geq 2 \):

1. Determine \( R_0(\mathbb{N}_0, h) \). Equivalently, describe the representation functions of additive bases for the nonnegative integers. This is a major unsolved problem in additive number theory, of which the Erdős-Turán conjecture is only a special case.

2. Determine \( R(\mathbb{Z}, h) \). In this paper we computed \( R_0(\mathbb{Z}, h) \), the set of representation functions of additive bases for the integers, but it is not known under what conditions a function \( f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \) with \( f^{-1}(0) \) infinite is the representation function of a subset \( A \) of \( X \). It can be proved that if \( f^{-1}(0) \) is infinite but sufficiently sparse, then \( f \in R(\mathbb{Z}, h) \).

3. Determine \( R(\mathbb{N}_0, h) \). Is there a simple list of necessary and sufficient conditions for a function \( f : \mathbb{N}_0 \to \mathbb{N}_0 \) to be the representation function of some set of nonnegative integers?

4. Determine \( R_c(\mathbb{Z}, h) \). Describe the representation functions of finite sets of integers. If \( A \) is a finite set of integers and \( t \) is an integer, then for the translated set \( t + A \) we have
\[ r_{t+A,h}(n) = r_{A,h}(n - ht) \]
for all integers \( n \). This implies that if \( f(n) \in R_c(\mathbb{Z}, h) \), then \( f(n - ht) \in R_c(\mathbb{Z}, h) \) for every integer \( b \), so it suffices to consider only finite sets \( A \) of nonnegative integers with \( 0 \in A \), and functions \( f \in \mathcal{F}_c(\mathbb{N}_0, h) \) with \( f(0) = 1 \).
5. Determine $R_0(G, 2)$, $R(G, 2)$, and $R_c(G, 2)$ for the infinite abelian group $G = \oplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$. Note that $\{2g : g \in G\} = \{0\}$ for this group.

6. Determine $R_0(G, h)$ and $R(G, h)$, where $G$ is an arbitrary countably infinite abelian group and $h \geq 2$.

7. There is a class of problems of the following type. Do there exist integers $h$ and $k$ with $2 \leq h < k$ such that

$$R(\mathbb{Z}, h) \neq R(\mathbb{Z}, k)?$$

We can easily find sets of integers to show that that $R_0(\mathbb{N}_0, h) \neq R_0(\mathbb{N}_0, k)$. For example, let $A = \mathbb{N}$ be the set of all positive integers, and let $h \geq 1$. Then $r_{\mathbb{N}, h}(0) = 0$ and $r_{\mathbb{N}, h}(h) = 1$. If $B$ is any set of nonnegative integers and $k > h$, then either $r_{B, k}(0) = 1$ or $r_{B, k}(h) = 0$, and so $r_{\mathbb{N}, h} \not\in R_0(\mathbb{N}_0, k)$. Is it true that

$$R_0(\mathbb{N}_0, h) \cap R_0(\mathbb{N}_0, k) = \emptyset$$

for all $h \neq k$?

8. By Theorem 1, for every $h \geq 2$ and every function $f \in F_0(\mathbb{Z})$, there exist arbitrarily sparse sets $A$ of integers such that $r_{A, h}(n) = f(n)$ for all $n$. It is an open problem to determine how dense the sets $A$ can be. For example, in the special case $h = 2$ and $f(n) = 1$, Nathanson [7] proved that there exists a set $A$ such that $r_{A, 2}(n) = 1$ for all $n$, and $\log x \ll A(-x, x) \ll \log x$. For an arbitrary representation function $f \in F_0(\mathbb{Z})$, Nathanson [6] constructed an asymptotic basis of order $h$ with $A(-x, x) \gg x^{1/(2h-1)}$. In the case $h = 2$, Cilleruelo and Nathanson [1] improved this to $A(-x, x) \gg x^{\sqrt{2}-1}$.

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