Weyl’s Lagrangian in teleparallel form

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Abstract

The main result of the paper is a new representation for the Weyl Lagrangian (massless Dirac Lagrangian). As the dynamical variable we use the coframe, i.e. an orthonormal tetrad of covector fields. We write down a simple Lagrangian – wedge product of axial torsion with a lightlike element of the coframe – and show that this gives the Weyl Lagrangian up to a nonlinear change of dynamical variable. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a metric, differential form, wedge product and exterior derivative. Our result assigns a variational meaning to the tetrad representation of the Weyl equation suggested by J. B. Griffiths and R. A. Newing.

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1 Main result

Throughout this paper we work on a 4-manifold $M$ equipped with prescribed Lorentzian metric $g$. The construction presented in the paper is local so we do not make a priori assumptions on the geometric structure of spacetime $\{M, g\}$. The metric $g$ is not necessarily the Minkowski metric.

The accepted mathematical model for a massless neutrino field is the following complex linear partial differential equation on $M$ known as Weyl's equation:

$$i\sigma^\alpha_{ab}\{\nabla\}_\alpha \xi^a = 0. \quad (1)$$

The corresponding Lagrangian is

$$L_{\text{Weyl}}(\xi) := \frac{i}{2}(\bar{\xi}^\beta \sigma^\alpha_{ab}\{\nabla\}_\alpha \xi^a - \xi^a \sigma^\alpha_{ab}\{\nabla\}_a \bar{\xi}^\beta) * 1. \quad (2)$$

Here $*1$ is the standard volume 4-form (Hodge dual of the scalar 1), $\sigma^\alpha$, $\alpha = 0, 1, 2, 3$, are Pauli matrices, $\xi$ is the unknown 2-component spinor field and $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection defined by formulae (24), (25).

Throughout the paper we will often deal with a situation when a pair of complex fields differs by a constant complex factor of modulus 1. We will say in this case that the two fields are equal modulo $U(1)$ and use the mathematical symbol $\text{mod } U(1)$ to indicate this fact in formulae.

It is well known that Weyl's Lagrangian (2) is $U(1)$-invariant:

$$\xi \, \text{mod } U(1) \equiv \tilde{\xi} \implies L_{\text{Weyl}}(\xi) = L_{\text{Weyl}}(\tilde{\xi}).$$

In view of this we call two spinor fields equivalent if they are equal modulo $U(1)$ and gather spinor fields into equivalence classes according to this relation. We call an equivalence class of spinors nonvanishing if its representatives do not vanish at any point.

The purpose of our paper is to give an alternative, much simpler and geometrically more transparent, representation for the Weyl Lagrangian (2). To this end we introduce instead of the spinor field a different unknown – the so-called coframe. A coframe is a quartet of real covector fields $\vartheta^j$, $j = 0, 1, 2, 3$, satisfying the constraint

$$g_{\alpha\beta} = o_{jk} \vartheta^j_\alpha \vartheta^k_\beta \quad (3)$$

where $o_{jk} = o_{kj} := \text{diag}(1, -1, -1, -1)$. For the sake of clarity we repeat formula (3) giving the tensor indices explicitly and performing summation in the frame indices explicitly: $g_{\alpha\beta} = o_{jk} \vartheta^j_\alpha \vartheta^k_\beta = \vartheta^0_\alpha \vartheta^0_\beta - \vartheta^1_\alpha \vartheta^1_\beta - \vartheta^2_\alpha \vartheta^2_\beta - \vartheta^3_\alpha \vartheta^3_\beta$.

Formula (3) means that the coframe is a field of orthonormal bases with orthonormality understood in the Lorentzian sense. Of course, at every point of the manifold $M$ the choice of coframe is not unique: there are 6 real degrees of freedom in choosing the coframe and any pair of coframes is related by a Lorentz transformation.
At a physical level choosing the coframe as the unknown quantity means that we allow every point of spacetime to rotate and assume that rotations of different points are totally independent. These rotations are described mathematically by attaching to each spacetime point a coframe (= orthonormal basis). The approach in which the coframe plays the role of the dynamical variable is known as teleparallelism (= absolute parallelism). This is a subject promoted by A. Einstein and É. Cartan [1] [2] [3].

The idea of rotating points may seem exotic, however it has long been accepted in continuum mechanics within the so-called Cosserat theory of elasticity [4]. The Cosserat theory of elasticity has been in existence since 1909 and appears under various names in modern applied mathematics literature such as oriented medium, asymmetric elasticity, micropolar elasticity, micromorphic elasticity etc. Cosserat elasticity is closely related to the theory of ferromagnetic materials [5] and the theory of liquid crystals [6] [7]. As to teleparallelism, it is, effectively, a special case of Cosserat elasticity: here the assumption is that the elastic continuum experiences no displacements, only rotations. With regards to the latter it is interesting that Cartan acknowledged [8] that he drew inspiration from the monograph [4] of the Cosserat brothers.

Define the 3-form

\[ T^{ax} := \frac{1}{3} \epsilon_{ijk} \vartheta^j \wedge d \vartheta^k \]  

where \( d \) denotes the exterior derivative. This 3-form is called axial torsion of the teleparallel connection. The geometric meaning of the latter phrase is explained in a concise fashion in Appendix A whereas a detailed exposition of the application of torsion in field theory and the history of the subject can be found in [9] [10]. What is important at this stage is the observation that the 3-form (4) is a measure of deformations generated by rotations of spacetime points.

Note that the 3-form (4) has the remarkable property of conformal covariance: if we rescale our metric and coframe as

\[ g_{\alpha \beta} \rightarrow e^{2h} g_{\alpha \beta} \]  
\[ \vartheta^j \rightarrow e^h \vartheta^j \]

where \( h : M \rightarrow \mathbb{R} \) is an arbitrary scalar function, then our 3-form is scaled as

\[ T^{ax} \rightarrow e^{2h} T^{ax} \]

without the derivatives of \( h \) appearing. The issue of conformal covariance and invariance will be examined in detail in Section 6.

It is tempting to use the 3-form (4) as our Lagrangian but the problem is that we are working in 4-space. In order to turn our 3-form into a 4-form we proceed as follows.

Put

\[ l := \vartheta^0 + \vartheta^3. \]  

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This is a nonvanishing real lightlike covector field. It will eventually (see Section 7) transpire that the covector field has the geometric meaning of neutrino current.

We define our “teleparallel” Lagrangian as

\[ L_{\text{tele}}(\vartheta) := \vartheta \wedge T^{\text{ax}}. \]  

Note that formulae (4), (8), (9) are very simple. They do not contain spinors, Pauli matrices or covariant derivatives. The only concepts used are those of a differential form, wedge product and exterior derivative. Even the metric does not appear in formulae (4), (8), (9) explicitly: it is incorporated implicitly via the constraint (3).

Let us now examine the behaviour of our Lagrangian (9) under Lorentz transformations of the coframe:

\[ \vartheta^j \xrightarrow{\Lambda} \tilde{\vartheta}^j = \Lambda^j_k \vartheta^k \]  

where the \( \Lambda^j_k \) are real scalar functions satisfying the constraint

\[ o_{jk} \Lambda^j_r \Lambda^k_s = o_{rs}. \]  

Obviously, transformations (10), (11) form an infinite-dimensional Lie group. Within this group we single out an infinite-dimensional Lie subgroup \( H \) as follows. Put

\[ m := \vartheta^1 + i \vartheta^2. \]  

The subgroup \( H \) is defined by the condition of preservation modulo \( U(1) \) of the complex 2-form \( \vartheta \wedge m \). More precisely, a Lorentz transformation (10), (11) is included in \( H \) if and only if

\[ \vartheta \wedge m \mod U(1) = \tilde{\vartheta} \wedge \tilde{m} \]  

where \( \tilde{\vartheta} = \vartheta^0 + \vartheta^3 \) and \( \tilde{m} = \vartheta^1 + i \vartheta^2 \).

The first main result of our paper is

**Theorem 1** The teleparallel Lagrangian (9) is invariant under the action of the group \( H \).

In view of Theorem 1, we call two coframes equivalent if they differ by a transformation from the subgroup \( H \) and gather coframes into equivalence classes according to this relation.

The second main result of our paper is

**Theorem 2** The equivalence classes of coframes \( \vartheta \) and nonvanishing spinor fields \( \xi \) are in a one-to-one correspondence given by the formula

\[ (\vartheta \wedge m)_{\alpha\beta} \mod U(1) = \sigma_{\alpha\beta a b} \xi^a \xi^b \]  

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where \( l \) and \( m \) are defined by formulae (8) and (12) respectively, \( \vartheta \) and \( \xi \) are arbitrary representatives of the corresponding equivalence classes and \( \sigma_{\alpha\beta} \) are “second order” Pauli matrices (13). Furthermore, under the correspondence (14) we have

\[
L_{\text{tele}}(\vartheta) = -\frac{4}{3}L_{\text{Weyl}}(\xi). \tag{15}
\]

A shorter way of stating Theorem 2 is “the nonlinear change of variable

\[
\text{coframe } \vartheta \quad \longleftrightarrow \quad \text{spinor field } \xi
\]

specified by formula (14) shows that the two Lagrangians, \( L_{\text{tele}}(\vartheta) \) and \( L_{\text{Weyl}}(\xi) \), are the same up to a constant factor”. The only problem with such a statement is that it brushes under the carpet the important question of gauge invariance.

The above results were announced, without a detailed proof, in the short communication [11].

The paper has the following structure. In Section 3 we describe explicitly the gauge group \( H \) which we initially defined implicitly by formula (13). In Sections 4 and 5 we prove Theorems 1 and 2 respectively. In Section 6 we present a modified version of our construction which makes conformal invariance more obvious. The concluding discussion is contained in Section 7.

## 2 Notation and conventions

Our notation follows [11, 12, 13, 14, 15]. In particular, in line with the traditions of particle physics, we use Greek letters to denote tensor (holonomic) indices. We identify differential forms with antisymmetric tensors.

All our constructions are local and occur in a neighbourhood of a given point \( P \in M \). Moreover, we assume that we have a given reference coframe \( \vartheta \) defined in a neighbourhood of \( P \); we need this reference coframe to specify orientation and positive direction of time.

We restrict our choice of local coordinates on \( M \) to those with \( \det \vartheta^i_\alpha > 0 \). This means that we work in local coordinates with specific orientation. In particular, this allows us to define the Hodge star: we define the action of \(*\) on a rank \( r \) antisymmetric tensor \( R \) as

\[
(*R)_{\alpha_{r+1}...\alpha_4} := (r!)^{-1} \sqrt{|\det g|} R^{\alpha_1...\alpha_r} \varepsilon_{\alpha_1...\alpha_4} \tag{16}
\]

where \( \varepsilon \) is the totally antisymmetric quantity, \( \varepsilon_{0123} := +1 \).

The coframe \( \vartheta \) which serves as our dynamical variable is assumed to satisfy

\[
\det \vartheta^i_0 > 0, \tag{17}
\]

and \( \vartheta^0 \cdot \vartheta^0 > 0 \). These assumptions mean that we work with coframes \( \vartheta \) which can be obtained from our reference coframe \( \vartheta^0 \) by proper Lorentz transformations: \( \vartheta^i = \Lambda^i_k \vartheta^k \) where the \( \Lambda^i_k \) are real scalar functions satisfying conditions

\[
o_{ij} \Lambda^i_j \Lambda^j_i = o_{kr}, \quad \det \Lambda^j_i > 0, \quad \Lambda^0_0 > 0.
\]

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We define the forward light cone (at a given point) as the set of covectors of the form $c_0 \vec{\theta}^0$ with $\omega^{\alpha \beta} c_\alpha c_\beta = 0$ and $c_0 > 0$. This implies, in particular, that our covector $l$ defined by formula (8) lies on the forward light cone.

Details of our spinor notation are given in Appendix A of [15]. In particular, the defining relation for Pauli matrices is $\sigma^\alpha_{\alpha \beta} \sigma^\beta_{\alpha \beta} + \sigma^\beta \sigma^\alpha_{\alpha \beta} = 2g^{\alpha \beta} \delta^a_c$.

Consider (at a given point) covectors of the form $\sigma^\alpha_a \epsilon^b_c$, $\xi^a \bar{\xi}^b$, $\xi^a \neq 0$. These covectors are lightlike and the set of all such covectors forms a cone. We assume that this cone is the forward light cone defined above. In other words, we assume that the positive direction of time encoded in our Pauli matrices agrees with the positive direction of time encoded in our coframe.

We define

$$\sigma^\alpha_{\alpha \beta} := (1/2)(\sigma^\alpha_{\alpha \beta} \epsilon^b_c \sigma^\beta_{\alpha \beta} - \sigma^\beta_{\alpha \beta} \epsilon^b_c \sigma^\alpha_{\alpha \beta})$$ (18)

where

$$\epsilon_{ab} = \epsilon_{\dot{a} \dot{b}} = \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(the first spinor index enumerates the rows and the second one the columns). These “second order” Pauli matrices are polarized, i.e. $\star \sigma = \pm i \sigma$ depending on the choice of “basic” Pauli matrices $\sigma_{\alpha \beta}$. Here the explicit formula for the action of the Hodge star on second order Pauli matrices is

$$\star \sigma_{\gamma \delta ab} := \frac{1}{2} \sqrt{|\det g|} \sigma^\alpha_{\alpha \beta} \epsilon_{\alpha \beta \gamma \delta}.$$ (19)

We assume that

$$\star \sigma = -i \sigma.$$ (19)

Note that formula (17) implies

$$\star (l \wedge m) = -i (l \wedge m)$$ (20)

where the covectors $l$ and $m$ are defined by formulae (8) and (12) respectively. We chose the sign in the RHS of (19) so as to agree with (20). In other words, the meaning of condition (19) is that the orientation encoded in our Pauli matrices agrees with the orientation encoded in our coframe.

The covariant derivatives of a vector field and a spinor field are defined as

$$\nabla_\alpha v^\beta := \partial_\alpha v^\beta + \Gamma^\beta_{\alpha \gamma} v^\gamma,$$ (21)

$$\nabla_\alpha \xi^a := \partial_\alpha \xi^a + \frac{1}{4} \sigma^a_{\alpha \beta} (\partial_\alpha \sigma^\beta_{\beta c} + \Gamma^\beta_{\alpha \gamma} \sigma^\gamma_{\beta c} \epsilon^h_c) \xi^b$$ (22)

where $\Gamma^\beta_{\alpha \gamma}$ are the connection coefficients. Throughout the main text of the paper we use the Levi-Civita connection and indicate this by curly brackets. That is, for the Levi-Civita connection we write formulae (21), (22) as

$$\{\nabla\}_{\alpha} v^\beta := \partial_\alpha v^\beta + \{\Gamma\}_{\alpha \gamma} v^\gamma,$$ (23)

$$\{\nabla\}_{\alpha} \xi^a := \partial_\alpha \xi^a + \frac{1}{4} \sigma^a_{\alpha \beta} (\partial_\alpha \sigma^\beta_{\beta c} + \{\Gamma\}_{\alpha \gamma} \sigma^\gamma_{\beta c} \epsilon^h_c) \xi^b$$ (24)
\[
\Gamma^{\beta}_{\alpha\gamma} = \left\{ \begin{array}{c}
\beta \\
\alpha \gamma
\end{array} \right\} := \frac{1}{2} g^{\beta\delta} (\partial_\alpha g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\alpha\gamma})
\] (25)

are the Christoffel symbols uniquely determined by the metric. An alternative (teleparallel) connection will be introduced in Appendix A.

In performing subsequent calculations it will be convenient for us to switch from the real coframe \((\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3)\) to the complex coframe \((l, m, n)\) where \(l, m, n\) are given by formulae (8), (12) and

\[
n := \vartheta^0 - \vartheta^3
\] (26)

respectively. Note that in this new notation the constraint (3) takes the form

\[
g = (1/2) (l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m).
\] (27)

The quartet of covectors \((l, m, \bar{m}, n)\) is known as a null tetrad or a Newman–Penrose tetrad [16].

### 3 The gauge group \(H\)

In this section we describe explicitly the gauge group \(H\) which we initially defined implicitly by formula (13).

Consider a Lorentz transformation of the coframe (10) satisfying the defining condition (13) of our group \(H\). (Recall that here the \(\Lambda^{j_k}\) are not assumed to be constant, i.e. they are real scalar functions satisfying (11).) We denote this Lorentz transformation \(\Lambda\).

Condition (14) means that \(\Lambda\) is a composition of two Lorentz transformations:

\[
\Lambda = \Lambda'' \Lambda'
\] (28)

where \(\Lambda'\) is a rotation by a constant angle \(\varphi\) in the \(\vartheta^1, \vartheta^2\)–plane

\[
\begin{pmatrix}
l \\
m \\
n
\end{pmatrix} \xmapsto{\Lambda'}
\begin{pmatrix}
l \\
e^{i\varphi} m \\
n
\end{pmatrix}
\] (29)

and \(\Lambda''\) is a Lorentz transformation preserving the 2-form \(l \wedge m\). Our convention for writing compositions of Lorentz transformations is as follows. When looking at a Lorentz transformation (10) we view the real coframe as a column of height 4 with entries \(\vartheta^k, \ k = 0, 1, 2, 3\), and the the Lorentz transformation itself as multiplication by a real \(4 \times 4\) matrix \(\Lambda_{j_k}\), so the group operation is matrix multiplication with the matrix furthest to the right acting on the coframe first. Say, formula (28) means that \(\Lambda'\) acts on the coframe first.

It is known, see Section 10.122 in [17], that Lorentz transformations preserving the 2-form \(l \wedge m\) admit an explicit description:

\[
\begin{pmatrix}
l \\
m \\
n
\end{pmatrix} \xmapsto{\Lambda''}
\begin{pmatrix}
l \\
m + fl \\
n + f\bar{m} + \bar{f}m + |f|^2 l
\end{pmatrix}
\] (30)
where $f : M \rightarrow \mathbb{C}$ is an arbitrary scalar function. Substituting (29), (30) into (28) we arrive at the explicit formula for an element $\Lambda$ of the group $H$:

\[
\begin{pmatrix}
l \\
m \\
n
\end{pmatrix} \Lambda \begin{pmatrix}
l \\
e^{i\varphi}m + fl \\
n + fe^{-i\varphi}\bar{m} + \bar{f}e^{i\varphi}m + |f|^2l
\end{pmatrix}.
\] (31)

Let us now examine the structure of the group $H$.

The group of rotations in the $\vartheta^1, \vartheta^2$–plane is isomorphic to $U(1)$. Hence further on we will refer to the group of Lorentz transformations of the coframe of the form (29) as $U(1)$. Let us emphasise that the $\varphi$ appearing in formula (29) is a constant, not a function.

Let us denote by $B_2^2(M)$ the group of Lorentz transformations of the coframe preserving the 2-form $l \wedge m$, see formula (30). In choosing the notation $B_2^2$ we follow [17] whereas the “M” indicates dependence on the point of the manifold $M$, i.e. it highlights the fact that the $f$ appearing in formula (30) is a function, not a constant.

Both $U(1)$ and $B_2^2(M)$ are abelian subgroups of $H$. Moreover, it is easy to see that $B_2^2(M)$ is a normal subgroup of $H$, $B_2^2(M) \triangleleft H$, and that $H$ is a semidirect product of $B_2^2(M)$ and $U(1)$, $H = B_2^2(M) \rtimes U(1)$.

The infinite-dimensional Lie group $H$ is itself nonabelian. However, it is very close to being abelian: $H$ contains the infinite-dimensional abelian Lie subgroup $B_2^2(M)$ of codimension 1.

4 Proof of Theorem

Let us rewrite our teleparallel Lagrangian (9) in terms of the complex coframe (8), (12), (26):

\[
L_{\text{tele}}(\vartheta) = \frac{1}{6} l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}).
\] (32)

The group $H$ is a semidirect product of the groups $B_2^2(M)$ and $U(1)$ so in order to check that (32) is invariant under the action of $H$ it is sufficient to check that (32) is invariant under the actions of $B_2^2(M)$ and $U(1)$ separately. $U(1)$-invariance is obvious: just substitute (29) into (32) noting that $\varphi$ is constant. Hence, it remains only to we check that our teleparallel Lagrangian (32) is invariant under the transformation (30).

When substituting (30) into (32) we will get an expression which is a sum of two terms:

- term without derivatives of the function $f$, and
- term with derivatives of the function $f$.

\footnote{The group $B_2^2$ can, in fact, be characterised as the nontrivial abelian subgroup of the Lorentz group. See Appendix B in [11] for details.}
Looking at our original formula (9) we see that the term without derivatives of the function \( f \) does not change the teleparallel Lagrangian because our transformation (30) preserves the covector field \( l \) and because axial torsion is an irreducible piece of torsion (i.e. the 3-form (4) is invariant under rigid Lorentz transformations). So it only remains to check that the term with derivatives of the function \( f \) vanishes. The term in question is

\[
(1/6) \ l \wedge (-\bar{m} \wedge df \wedge l - m \wedge d\bar{f} \wedge l)
\]

which is clearly zero. □

5 Proof of Theorem 2

The gauge group \( H \) allows us to gather coframes into equivalence classes: we call two coframes equivalent if they differ by a transformation from \( H \). We will now establish the geometric meaning of these equivalence classes of coframes.

Let us first fix a spacetime point \( x \in M \) and examine in detail the geometric meaning of the group \( B^2 \). We initially defined \( B^2 \) as the group of Lorentz transformations preserving the 2-form \( l \wedge m \). The complex nonzero antisymmetric tensor \( l \wedge m \) is polarized (see (20)) and has the additional property \( \det(l \wedge m) = 0 \). It is easy to see (and this fact was extensively used in [11, 12, 13, 14, 15]) that such a tensor can be written in terms of a nonzero spinor \( \xi \) as

\[
(l \wedge m)_{\alpha \beta} = \sigma_{\alpha \beta ab} \xi^a \xi^b \tag{33}
\]

with the spinor defined uniquely up to sign. Thus, the group \( B^2 \) can be reinterpreted as the group of Lorentz transformations preserving a given nonzero spinor \( \xi \) and the equivalence classes of coframes are related to this spinor according to formula (33). Here the relationship between an equivalence class of coframes and a nonzero spinor is one-to-two because formula (33) allows us to change the sign of \( \xi \).

Remark 1 One can use the above observation to formulate an alternative definition of a spinor: a spinor is a coset of the Lorentz group with respect to the subgroup \( B^2 \). In using this definition one, however, has to decide whether to use left or right cosets as \( B^2 \) is not a normal subgroup of the Lorentz group.

Remark 2 In \( \text{SL}(2, \mathbb{C}) \) notation the group \( B^2 \) is written in a particularly simple way: \( B^2 = \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \bigg| \ f \in \mathbb{C} \right\} \).

Let us now allow dependence on the spacetime point \( x \in M \). Then the group \( B^2(M) \) is the group of Lorentz transformations preserving a given nonzero spinor field \( \xi \), with the equivalence classes of coframes related to the spinor field according to formula (33). Here the relationship between an equivalence class of coframes and a nonvanishing spinor field remains one-to-two.
Finally, let us switch from the group $B^2(M)$ to $H = B^2(M) \rtimes U(1)$. This means that in our definition of equivalence classes of coframes we allow $l \wedge m$ to be multiplied by a constant complex factor of modulus 1, so formula (33) turns into (14). Here the relationship between an equivalence class of coframes and a nonvanishing spinor field becomes one-to-infinity because formula (14) allows us to multiply the nonvanishing spinor field $\xi$ by a constant complex factor of modulus 1; note that this eliminates the difference between $\xi$ and $-\xi$. It remains only to gather nonvanishing spinor fields $\xi$ into equivalence classes as described in the beginning of Section 1 and we arrive at a one-to-one correspondence between equivalence classes of coframes and nonvanishing spinor fields given by the explicit formula (14).

In the remainder of this section we perform the nonlinear change of variable

$$\text{spinor field } \xi \quad \longrightarrow \quad \text{coframe } \vartheta$$

and show that $L_{\text{Weyl}}(\xi)$ turns into $-\frac{3}{4} L_{\text{tele}}(\vartheta)$. In order to simplify calculations we observe that we have freedom in our choice of Pauli matrices. It is sufficient to prove formula (15) for one particular choice of Pauli matrices, hence it is natural to choose Pauli matrices in a way that makes calculations as simple as possible. Note that this trick is not new: it was, for example, extensively used by A. Dimakis and F. Müller-Hoissen [18, 19, 20].

We choose Pauli matrices

$$\sigma_{\alpha ab} = \tilde{\vartheta}^j_{\alpha} s_{jab}$$

where

$$s_{jab} = \begin{pmatrix} s_{0ab} \\ s_{1ab} \\ s_{2ab} \\ s_{3ab} \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ -1 & 0 \end{pmatrix}.$$ (35)

Let us stress that in the statement of Theorem 2 Pauli matrices are not assumed to be related in any way to the coframe $\vartheta$. We are just choosing the particular Pauli matrices (34), (35) to simplify calculations in our proof.

Note that the matrices (34), (35) satisfy all the conditions listed in Section 2.

We now calculate explicitly the corresponding second order Pauli matrices:

$$\sigma_{\alpha\beta ab} = \frac{1}{2} (\overline{\vartheta}^j \wedge \overline{\vartheta}^k)_{\alpha\beta} s_{jkab}$$

(36)
where

\[
s_{jkab} = \begin{pmatrix}
0 & s_{01ab} & s_{02ab} & s_{03ab} \\
s_{10ab} & 0 & s_{12ab} & s_{13ab} \\
s_{20ab} & s_{21ab} & 0 & s_{23ab} \\
s_{30ab} & s_{31ab} & s_{32ab} & 0
\end{pmatrix} := \begin{pmatrix}
0 & (1, 0) & (i, 0) & (0, -1) \\
(-1, 0) & (0, 1) & (0, i) & (0, -1) \\
(-i, 0) & (0, -i) & (0, -1) & (0, 0) \\
(0, 1) & (1, 0) & (i, 0) & (0, -i)
\end{pmatrix}.
\]  

(37)

Substituting (8), (12) and (36), (37) into the equation (14) we see that this equation can be easily resolved for \( \xi \) giving

\[
\xi^a \mod U(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]  

(38)

Formula (38) may seem strange: we are proving Theorem 2 for a general non-vanishing spinor field \( \xi \) but ended up with formula (38) which is very specific. However, there is no contradiction here because we chose Pauli matrices specially adapted to the coframe and, hence, specially adapted to the corresponding spinor field.

Substituting (24) and (38) into (2) we get

\[
L_{\text{Weyl}}(\xi) = i \left( \xi^c g_{ab} g^{ac} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \xi^d - \xi^a g_{ab} \sigma^b \psi_{\dot{c}} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \right) \right) \right) + 1
\]

\[
= i \left( \sigma^a \sigma^b \partial^c \partial^d \sigma^a \sigma^b \psi_{\dot{c}} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \right) \right) + 1
\]

\[= i \left( \sigma^a \sigma^b \partial^c \partial^d \sigma^a \sigma^b \psi_{\dot{c}} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \right) \right) + 1.
\]

We now write down the spinor summation indices explicitly:

\[
L_{\text{Weyl}}(\xi) = i \left( \xi^c g_{ab} g^{ac} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \xi^d - \xi^a g_{ab} \sigma^b \psi_{\dot{c}} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \right) \right) \right) + 1
\]

\[
= i \left( \sigma^a \sigma^b \partial^c \partial^d \sigma^a \sigma^b \psi_{\dot{c}} \left( \partial_a \sigma^b \psi_{\dot{c}} + \{ \Gamma^b_{\alpha \gamma} \sigma^\gamma g_{\dot{c}} \} \right) \right) + 1
\]

Note that the terms with \( a = 1, \dot{c} = \dot{1} \) and \( b = \dot{1}, c = 1 \) cancelled out. Finally,
we substitute explicit formulae (34), (35) for our Pauli matrices which gives us

\[
L_{\text{Weyl}}(\xi) = \frac{i}{8} \left( l^\alpha (-\bar{m}_\beta) \{ \nabla \}_\alpha m^\beta + \bar{m}^\alpha (-m_\beta) \{ \nabla \}_\alpha l^\beta + \bar{m}^\alpha l^\beta \{ \nabla \}_\alpha m^\beta \\
- l^\alpha (-m_\beta) \{ \nabla \}_\alpha \bar{m}^\beta - m^\alpha (-\bar{m}_\beta) \{ \nabla \}_\alpha l^\beta - m^\alpha l^\beta \{ \nabla \}_\alpha \bar{m}^\beta \right) \star 1
\]

\[
= \frac{i}{16} \left( (m \wedge \bar{m})^{\alpha \beta} \{ \nabla \}_\alpha l_\beta - (l \wedge \bar{m})^{\alpha \beta} \{ \nabla \}_\alpha m_\beta + (l \wedge m)^{\alpha \beta} \{ \nabla \}_\alpha \bar{m}_\beta \right) \star 1
\]

\[
= \frac{i}{16} \left( (m \wedge \bar{m})^{\alpha \beta} (dl)_\alpha \beta - (l \wedge \bar{m})^{\alpha \beta} (dm)_\alpha \beta + (l \wedge m)^{\alpha \beta} (d\bar{m})_{\alpha \beta} \right) \star 1
\]

\[
= \frac{i}{8} \left( [*(m \wedge \bar{m})] \wedge dl - [*(l \wedge \bar{m})] \wedge dm + [*(l \wedge m)] \wedge d\bar{m} \right).
\]

But \(*((l \wedge m) = -i(l \wedge m) (\text{see (20)}) and \(*(m \wedge \bar{m}) = +i(l \wedge n) so the above formula becomes

\[
L_{\text{Weyl}}(\xi) = -\frac{1}{8} (l \wedge n \wedge dl - l \wedge \bar{m} \wedge dm - l \wedge m \wedge d\bar{m}).
\]

Comparing with (32) we arrive at (15). □

6 Conformal invariance

Until now we kept the metric fixed but now we shall scale the metric as (5) and the Pauli matrices as

\[
\sigma_\alpha \mapsto e^h \sigma_\alpha.
\]

(39)

Recall that here \( h : M \to \mathbb{R} \) is an arbitrary scalar function. Let us also scale the spinor field as

\[
\xi \mapsto e^{-(3/2)h} \xi.
\]

(40)

It is well known that the Weyl Lagrangian (2) is invariant under the transformation (5), (39), (40).

Examination of formulae (14), (18) shows that the transformation (5), (39), (40) induces the following transformation of the complex coframe (8), (12), (26):

\[
\begin{pmatrix}
  l \\
  m \\
  n
\end{pmatrix} \mapsto
\begin{pmatrix}
  e^{-2h}l \\
  e^h m \\
  e^{4h} n
\end{pmatrix}
\]

(41)

Of course, it is easy to check directly that our teleparallel Lagrangian (32) is invariant under the transformation (11).

The transformation (11) is a composition of two commuting transformations: a conformal rescaling of the coframe (6) and a Lorentz boost

\[
\begin{pmatrix}
  \vartheta^0 \\
  \vartheta^3
\end{pmatrix} \mapsto
\begin{pmatrix}
  \cosh 3h & -\sinh 3h \\
  -\sinh 3h & \cosh 3h
\end{pmatrix}
\begin{pmatrix}
  \vartheta^0 \\
  \vartheta^3
\end{pmatrix}.
\]

12
The presence of a Lorentz boost in this argument is somewhat unnatural so we suggest below a modified version of our teleparallel Lagrangian, one for which conformal invariance is self-evident. Recall that our original teleparallel Lagrangian $L_{\text{tele}}(\vartheta)$ was defined by formula (9) or, equivalently, in terms of the complex coframe, by formula (32).

Put

$$
\tilde{L}_{\text{tele}}(\vartheta, s) := sL_{\text{tele}}(\vartheta) = sl \wedge T^a = (s/6) l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}) \quad (42)
$$

where $s : M \to (0, +\infty)$ is a scalar function. The function $s$ will play the role of an additional dynamical variable. In view of (7) the Lagrangian (42) does not change if we scale the coframe as (6), the metric as (5) and the scalar $s$ as $s \to e^{-3h}s$. Hence, the Lagrangian (42) is conformally invariant and, moreover, this conformal invariance is quite obvious.

Let us now examine the properties of the Lagrangian (42) for fixed metric. Of course, it is invariant under the action of the group $H$ which was described implicitly in Section 1 and explicitly in Section 3 (see formula (31)). However, it is also invariant under the transformation

$$
\begin{pmatrix}
  l \\
  m \\
  n \\
  s
\end{pmatrix} \mapsto \begin{pmatrix}
  e^{-k}l \\
  m \\
  e^k n \\
  e^k s
\end{pmatrix}
$$

where $k : M \to \mathbb{R}$ is an arbitrary scalar function. The transformation (42) is a composition of two transformations: a Lorentz boost

$$
\begin{pmatrix}
  \vartheta^0 \\
  \vartheta^3
\end{pmatrix} \mapsto \begin{pmatrix}
  \cosh k & - \sinh k \\
  - \sinh k & \cosh k
\end{pmatrix} \begin{pmatrix}
  \vartheta^0 \\
  \vartheta^3
\end{pmatrix}
$$

and a rescaling of the scalar $s$, $s \to e^k s$. We will denote the infinite-dimensional Lie group of transformations (43) by $J(M)$.

Thus, having incorporated into our original teleparallel Lagrangian (9) an additional dynamical variable, the positive scalar function $s$, we have acquired an additional gauge degree of freedom. The new (extended) gauge group is

$$
\tilde{H} = H \ltimes J(M) = (B^2(M) \ltimes U(1)) \ltimes J(M) = (B^2(M) \ltimes J(M)) \ltimes U(1) = B^2(M) \ltimes (J(M) \times U(1))
$$

where the symbol “$\ltimes$” stands for the semidirect product with the normal subgroup coming first. The action of $\tilde{H}$ preserves the 2-form $l \wedge m$ modulo $U(1)$ and modulo rescaling by a positive scalar function.

We have established the following analogue of Theorem 1.

**Theorem 3** The modified teleparallel Lagrangian (42) is invariant under the action of the group $\tilde{H}$.
In view of Theorem 3 we call two sets of dynamical variables “coframe + positive scalar” equivalent if they differ by a transformation from the group $\tilde{H}$ and gather sets of dynamical variables into equivalence classes according to this relation. The following is an analogue of Theorem 2.

**Theorem 4** The equivalence classes of coframes $\vartheta$ and positive scalars $s$ on the one hand and nonvanishing spinor fields $\xi$ on the other are in a one-to-one correspondence given by the formula

$$s \left( l \wedge m \right)_{\alpha\beta} \overset{\text{mod } U(1)}{=} \sigma_{\alpha\beta a b} \xi^a \xi^b \quad (44)$$

where $l$ and $m$ are defined by formulae (8) and (12) respectively, $\vartheta$, $s$ and $\xi$ are arbitrary representatives of the corresponding equivalence classes and $\sigma_{\alpha\beta}$ are “second order” Pauli matrices (18). Furthermore, under the correspondence (44) we have

$$L_{\text{tele}}(\vartheta, s) = -\frac{4}{3} L_{\text{Weyl}}(\xi). \quad (45)$$

The proof of the first part of Theorem 4 (formula (44)) is essentially a repetition of the proof of the first part of Theorem 2; take argument from the beginning of Section 4 and add one gauge degree of freedom.

As to the second part of Theorem 4 (formula (45)), it simply follows from the second part of Theorem 2 (formula (15)). Indeed, when we replace (14) by (44) the spinor field scales as $\xi \mapsto \sqrt{s} \xi$. But

$$-\frac{4}{3} L_{\text{Weyl}}(\sqrt{s} \xi) = -\frac{4}{3} s L_{\text{Weyl}}(\xi) \overset{\text{by (14)}}{=} s L_{\text{tele}}(\vartheta) \overset{\text{by (14)}}{=} L_{\text{tele}}(\vartheta, s)$$

giving us (45).

### 7 Discussion

Throughout the paper we dealt with Weyl’s Lagrangian (2) as opposed to Weyl’s equation (1) For Weyl’s Lagrangian we found a simple teleparallel representation (9). If one wishes to rewrite Weyl’s equation in teleparallel form then one has to vary the action with respect to the coframe $\vartheta$ and the resulting teleparallel representation of Weyl’s equation does not turn out to be that simple, the reason being that in performing the variation one has to maintain the metric constraint (3) because we agreed (see first paragraph of Section 1) to keep the metric fixed (prescribed). The corresponding calculations are carried out in Appendix B.

The teleparallel representation of Weyl’s equation was first derived by Griffiths and Newing [21]. Our contribution is the teleparallel representation of Weyl’s Lagrangian and observation that for the Lagrangian things become much simpler.

Now, formula (15) (as well as its generalised version (19)) holds for any Lorentzian metric so when using this formula there is really no need in assuming the metric to be fixed. Say, one can vary the action with respect to
the metric \( g \) to derive the teleparallel representation of the neutrino energy–momentum tensor. The calculations are quite straightforward but we do not perform them in this paper for the sake of brevity.

Let us now examine the geometric meaning of the covector field \( l \) defined by formula (8). If we choose Pauli matrices in the special way (34), (35) we get (38) which immediately implies

\[
\ell_\alpha = \sigma_{\alpha a} \xi^a \bar{\xi}^\beta.
\] (46)

Formula (46) remains true for any choice of Pauli matrices because its RHS has an invariant meaning. More specifically, the RHS of (46) is the well-known expression for the neutrino current. In light of this it is not surprising that our field equations imply that the divergence of \( l \) is zero, see formula (59) in Appendix B.

The main issue with our model is that our Lagrangian (9) (as well as its generalised version (42)) is not invariant under rigid Lorentz transformations of the coframe. In the remainder of this section we sketch out a way of dealing with this issue.

Consider the Lagrangian

\[
L(\vartheta, s) := s \| \mathcal{T}^\text{ax} \|^2 * 1 \tag{47}
\]

where \( s : M \to (0, +\infty) \) is a scalar function which plays the role of an additional dynamical variable. This Lagrangian is Lorentz invariant and is a special case of a general quadratic Lorentz invariant Lagrangian (a general one contains squares of all three irreducible pieces of torsion). The special feature of the Lagrangian (47) is that it is conformally invariant: it does not change if we rescale the coframe as (6) and the scalar \( s \) as \( s \mapsto e^{-2h} s \).

Of course, a positive scalar \( s \) is equivalent to a positive density \( \rho \): \( \rho = s \sqrt{|\det g|} \). Thus, having the scalar function \( s \) as a dynamical variable is equivalent to having the density \( \rho \) as a dynamical variable. Thinking in terms of an unknown density \( \rho \) is more natural from the physical viewpoint. However, in this paper we will stick with the scalar \( s \).

We vary the action \( S(\vartheta, s) := \int L(\vartheta, s) \) with respect to the scalar \( s \) and with respect to the coframe \( \vartheta \) subject to the metric constraint (3), which gives us the Euler–Lagrange field equations. The fundamental difference between our original conformally invariant Lagrangian (42) and the new conformally invariant Lagrangian (47) is that the latter is quadratic in torsion, hence the field equations for (47) will be second order.

Suppose now that the metric is Minkowski. It turns out that in this case one can construct an explicit solution of the field equations for (47). This construction goes as follows.

Let \( l \neq 0 \) be a constant real lightlike covector lying on the forward light cone and let \( \vartheta \) be a constant coframe such that \( l \perp \vartheta^1, l \perp \vartheta^2 \); here “constant” means “parallel with respect to the Levi-Civita connection induced by the Minkowski metric”. Then, of course,

\[
l = c(\vartheta^0 + \vartheta^3) \tag{48}
\]
where \( c > 0 \) is some constant (compare with formula (53)). Put

\[
\begin{pmatrix}
\vartheta^0 \\
\vartheta^1 \\
\vartheta^2 \\
\vartheta^3
\end{pmatrix} := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\varphi & \pm \sin 2\varphi & 0 \\
0 & \mp \sin 2\varphi & \cos 2\varphi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\vartheta^0 \\
\vartheta^1 \\
\vartheta^2 \\
\vartheta^3
\end{pmatrix}, \quad s = \text{const} > 0 \tag{49}
\]

where \( \varphi := \int l \cdot dx \) and \( x^\alpha \) are local coordinates. Straightforward calculations show that this coframe \( \vartheta \) and scalar \( s \) are indeed a solution of the field equations for (47). We call this solution a plane wave with momentum \( l \). The upper sign in (49) corresponds to the massless neutrino and lower sign corresponds to the massless antineutrino. Note that we can distinguish the neutrino from the antineutrino without resorting to negative energies. Note also that we automatically get only one type of neutrino (left-handed) and one type of antineutrino (right-handed).

Suppose now that we are seeking solutions which are not necessarily plane waves. This can be done using perturbation theory. In the language of spinors, perturbation means that we assume the spinor field to be of the form “slowly varying spinor \( \times e^{-i\varphi} \)”. We claim that application of a perturbation argument reduces the quadratic (in torsion) Lagrangian (47) to the linear (in torsion) Lagrangian (42). At the most basic level this can be explained as follows. Note that for a plane wave we have the following two identities: \( T^{ax} = \pm \frac{4}{3} s l \) and \( l = c (\vartheta^0 + \vartheta^3) \) (compare the latter with (48)). Thus, for a plane wave we have

\[
T^{ax} = \pm \frac{4}{3} c * (\vartheta^0 + \vartheta^3). \tag{50}
\]

We now linearize (in torsion) the quadratic Lagrangian (47) about the point (50). We get, up to a constant factor, the linear Lagrangian (42).

The bottom line is that we believe that the true Lagrangian of a massless neutrino field is the quadratic Lagrangian (47). The linear Lagrangian (42) (which is equivalent to Weyl’s Lagrangian (2)) arises only if one adopts the perturbative approach.

The detailed analysis of the quadratic Lagrangian (47) will be the subject of a separate paper. Elements of this analysis have been performed in [22, 23].

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A  Brief introduction to teleparallelism

Given a coframe \( \vartheta \), we introduce a covariant derivative \( \nabla \) such that \( \nabla \vartheta = 0 \). We repeat this formula giving frame and tensor indices explicitly: \( \nabla_{\alpha} \vartheta^\beta = 0 \). We then rewrite the formula in even more explicit form:

\[
\partial_\alpha \vartheta^\beta_{\beta} - \Gamma^\gamma_{\alpha\beta} \vartheta^\gamma_{\gamma} = 0 \tag{51}
\]
where $|\Gamma|^\gamma_{\alpha\beta}$ are the connection coefficients. Note that formula (51) has three free indices $j$, $\alpha$, $\beta$ running through the values $0, 1, 2, 3$. Note also that the connection coefficient $|\Gamma|^\gamma_{\alpha\beta}$ has three indices $\alpha$, $\beta$, $\gamma$ running through the values $0, 1, 2, 3$. Hence, (51) can be viewed as a system of 64 inhomogeneous linear algebraic equations for the determination of the 64 unknown connection coefficients $|\Gamma|^\gamma_{\alpha\beta}$. It is easy to see that its unique solution is

\[
|\Gamma|^\gamma_{\alpha\beta} = a_{ik} g^{\gamma\delta} \partial_\delta \partial_\alpha \partial_\beta^k.
\]

(52)

The corresponding connection is called **teleparallel**. When writing the teleparallel covariant derivative and connection coefficients we use the “modulus” sign to distinguish these from the Levi-Civita covariant derivative and connection coefficients for which we use curly brackets.

Thus, we have two different connections: the Levi-Civita connection used in the main text of the paper and the teleparallel connection used in this appendix. Both are metric compatible: $\{\nabla\} g = |\nabla| g = 0$. The Levi-Civita connection is uniquely determined by the metric whereas the teleparallel connection is uniquely determined by the coframe. For the Levi-Civita connection torsion is zero whereas for the teleparallel connection curvature is zero. Thus, in a sense, the Levi-Civita and teleparallel connections are antipodes.

“Teleparallelism” stands for “distant parallelism”. What is meant here is that the result of parallel transport of a vector (or a covector) does not depend on the choice of curve connecting the two points. This fact can be expressed in even simpler terms as follows. Suppose we have two covectors, $u$ and $v$, at two different points, $P$ and $Q$, of our manifold (spacetime) $M$. We need to establish whether $u$ and $v$ are parallel. To do this, we use the coframe as a basis and write $u = a_j \vartheta^j$, $v = b_j \vartheta^j$. By definition, the covectors $u$ and $v$ are said to be parallel if $a_j = b_j$.

Formula (52) allows us to evaluate torsion of the teleparallel connection:

\[
T^\gamma_{\alpha\beta} := |\Gamma|^\gamma_{\alpha\beta} - |\Gamma|^\gamma_{\beta\alpha} = a_{ik} g^{\gamma\delta} \partial_\delta (\partial_\alpha \vartheta^k - \partial_\beta \vartheta^k) = a_{ik} g^{\gamma\delta} \vartheta^j (d\vartheta^k)_{\alpha\beta}
\]

where $d$ denotes the exterior derivative. Lowering the first tensor index gives a neater representation $T_{\gamma\alpha\beta} = a_{ik} \vartheta^j (d\vartheta^k)_{\alpha\beta}$. Dropping tensor indices altogether we get

\[
T = a_{ik} \vartheta^j \otimes d\vartheta^k.
\]

(53)

It is known that torsion decomposes into three irreducible pieces called tensor torsion, vector torsion and axial torsion. (Vector torsion is sometimes called trace torsion.) In this paper we use only the axial piece. Axial torsion has a very simple meaning: it is the totally antisymmetric piece $T^{ax}_{\alpha\beta\gamma} = \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha})$. Substituting (53) into this general formula we arrive at (43).

Of course, there is much more to teleparallelism than the elementary facts sketched out above. Modern reviews of the physics of teleparallelism can be found in [25, 26, 27, 28, 29, 30].
B  Weyl’s equation in teleparallel form

In this appendix we write down explicitly the Euler–Lagrange field equations resulting from the variation of the action

\[ S_{\text{tele}} := \int L_{\text{tele}} = \int l \wedge T^{\text{ax}} = \frac{1}{3} p_i o_{jk} \int \vartheta^i \wedge \vartheta^j \wedge d \vartheta^k \]  

(54)

with respect to the coframe \( \vartheta \) subject to the metric constraint (3). Here by \( p_i \) we denote the quartet of constants \( p_i := (1 \ 0 \ 0 \ 1) \).

The variation of the coframe is given by the formula

\[ \delta \vartheta^j_k = F^j_k \vartheta^k \]  

(55)

where the \( F^j_k \) are real scalar functions satisfying the antisymmetry condition

\[ F^j_k = -F^k_j. \]  

(56)

Condition (56) ensures that the variation of the RHS of (3) is zero. Of course, the \( \Lambda^j_k \) appearing the RHS of (10) are expressed via the \( F^j_k \) as

\[ \Lambda^j_k = \delta^j_k + F^j_k + \frac{1}{2} F^j_l F^l_k + \ldots \]

(exponential series), or, in matrix notation, \( \Lambda = e^F \). Hence, the matrix-function \( F \) is the linearization of the Lorentz transformation \( \Lambda \) about the identity.

Substituting (55) into (54) we get

\[ 3 \delta S_{\text{tele}} = p_i o_{jk} \int (p^i \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k + F^j_l \vartheta^i \wedge \vartheta^k \wedge d \vartheta^l + F^j_k \vartheta^i \wedge \vartheta^l \wedge d \vartheta^k + \vartheta^i \wedge \vartheta^j \wedge d F^k_l \wedge \vartheta^l) \]

where \( d F^k_l \) is the gradient of the scalar function \( F^k_l \). Upon contraction with \( o_{jk} \) the second and third terms in the integrand cancel out in view of (56) (that this would happen was clear a priori because axial torsion is invariant under rigid Lorentz transformations) so the above formula becomes

\[ 3 \delta S_{\text{tele}} = \int (p^i o_{ik} F^j_i \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k + p_k o_i \vartheta^k \wedge \vartheta^i \wedge d F^j_i \wedge \vartheta^j) \]

where \( p^i := o^{ij} p_j \). Integration by parts and antisymmetrization in \( i, j \) gives

\[ 6 \delta S_{\text{tele}} = \int F^j_i (p^i o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - p^j o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - 2 p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j). \]

Thus, our field equations are

\[ p^i o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - p^j o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - 2 p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j) = 0. \]  

(57)

The field equations (54) are, of course, equivalent to

\[ * [p^i o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - p^j o_{ik} \vartheta^j \wedge \vartheta^l \wedge d \vartheta^k - 2 p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j)] = 0. \]  

(58)
The advantage of the representation \( \text{(58) } \) is that the left-hand sides of \( \text{(58) } \) are scalars and not 4-forms as in \( \text{(57) } \). We denote the left-hand sides of \( \text{(58) } \) by \( G^{ij} \). Note the antisymmetry \( G^{ij} = -G^{ji} \).

We will now rewrite our field equations \( \text{(58) } \) in more compact form in terms of the complex coframe \( \text{(8), (12), (26). } \)

We note first that \( G^{12} = 4\{\nabla\}_\alpha l_\alpha \). Thus, our field equations \( \text{(58) } \) imply

\[
\{\nabla\}_\alpha l_\alpha = 0. \tag{59}
\]

Note that the scalar \( G^{03} \) also has a clear geometric meaning: \( G^{03} = 3^* L_{tele} \).

Put
\[
q_j := (0 \ 1 \ i \ 0), \quad r_j := (1 \ 0 \ 0 \ -1),
\]
\[
A_{jk} := p_j q_k - p_k q_j, \quad B_{jk} := p_j r_k - p_k r_j - q_j \bar{q}_k + q_k \bar{q}_j, \quad C_{jk} := r_j \bar{q}_k - r_k \bar{q}_j.
\]

The antisymmetric matrices \( \text{Re} \ A, \text{Im} \ A, \text{Re} \ B, \text{Im} \ B, \text{Re} \ C, \text{Im} \ C \) are linearly independent, therefore the system of 6 real equations \( \text{(58) } \) is equivalent to the system of 3 complex equations

\[
A_{ij} G^{ij} = 0, \quad B_{ij} G^{ij} = 0, \quad C_{ij} G^{ij} = 0.
\]

Straightforward calculations show that \( A_{ij} G^{ij} \) is zero for any coframe \( \vartheta \) (this is actually a consequence of Theorem 1), hence our real field equations \( \text{(58) } \) are equivalent to the pair of complex equations

\[
B_{ij} G^{ij} = 0, \quad C_{ij} G^{ij} = 0. \tag{60}
\]

As the systems \( \text{(58) and (60) } \) are equivalent and as equation \( \text{(59) } \) is a consequence of \( \text{(58) } \), equation \( \text{(59) } \) is also a consequence of \( \text{(60) } \). Hence we can extend the system \( \text{(60) } \) by adding equation \( \text{(59) } \): the system \( \text{(60) } \) is equivalent to the system \( \text{(60), (59) } \). The advantage of having \( \text{(59) } \) as a separate equation is that it simplifies subsequent calculations.

We now examine our system of field equations \( \text{(60), (59) } \). Straightforward calculations with account of \( \text{(60) } \) give

\[
B_{ij} G^{ij} = -8 \bar{m}_\alpha v_\alpha, \quad C_{ij} G^{ij} = 8 i n_\alpha \bar{v}_\alpha
\]

where
\[
v_\alpha := \{\nabla\}_\beta (l \wedge m)_{\alpha \beta} - m_\beta \{\nabla\}_\alpha l_\beta. \tag{61}
\]

Thus, our system of field equations \( \text{(60), (59) } \) is equivalent to

\[
\bar{m}_\alpha v_\alpha = 0, \quad n_\alpha v_\alpha = 0 \tag{62}
\]

and \( \text{(59)} \). But \( \text{Re}(\bar{m}_\alpha v_\alpha) = 2\{\nabla\}_\alpha l_\alpha \), so \( \text{(59)} \) is a consequence of \( \text{(62) } \). Hence, \( \text{(62) } \) is the full system of field equations. It is equivalent to the original system of field equations \( \text{(58) } \).

It is easy to see that for any coframe \( \vartheta \) we have

\[
m_\alpha v_\alpha = 0, \quad l_\alpha v_\alpha = 0 \tag{63}
\]
so the pair of scalar complex equations (62) is equivalent to the complex covector equation

\[ v = 0. \]  

(64)

Recall that the LHS of this equation is defined by formula (61).

Equation (64) is the compact “tetrad” representation of the Weyl equation found by Griffiths and Newing [21]. Griffiths and Newing derived (64) directly from Weyl’s equation (11), without examining the Weyl Lagrangian (2).

Let us have a closer look at equation (64) so as to establish the actual number of independent “scalar” equations contained in it and the actual number of independent “scalar” unknowns. It would seem that (64) is a system of 4 complex “scalar” equations (4 being the number of components of the covector \( v \)) for 6 real “scalar” unknowns (6 being the dimension of the Lorentz group). But we already know that we a priori have identities (63) so equation (64) is equivalent to the pair of scalar complex equations (62). It is also easy to see that \( v \) is invariant under the action of the transformation (30), hence the set of solutions to equation (64) is invariant under this transformation which means that we are dealing with a pair of complex “scalar” unknowns (see argument in the beginning of Section 5). Thus, equation (64) is a system of 2 complex “scalar” equations for 2 complex “scalar” unknowns, as expected of the Weyl equation.

Note that the scalar \( \bar{m}_\alpha \nu_\alpha \) appearing in the LHS of (62) is also invariant under the action of the transformation (30) and can be written down explicitly as \( \bar{m}_\alpha \nu_\alpha = 2\{\nabla\}_\alpha \nu^\alpha - \frac{2}{3} \ast L_{\text{tele}} \).
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