Boundedness of Mikhlin Operator in Morrey Space

Dian Maharani\textsuperscript{1}, Jalina Widjaja\textsuperscript{2}, and Marcus Wono Setya Budhi\textsuperscript{3}

Analysis and Geometry Major Group, Mathematics Department, Mathematics and Science Faculty, Bandung Institute of Technology

E-mail: \textsuperscript{1}rani.dian.rani@gmail.com, \textsuperscript{2}jalina@math.itb.ac.id, \textsuperscript{3}wono@math.itb.ac.id

Abstract. A multiplier operator is an operator which maps a function to the product of that function with a certain function. A Fourier multiplier operator is a multiplier operator which uses Fourier transformation of a function. S. G. Mikhlin proved the boundedness of the operator in classical Lebesgue space if the multiplier function (generator of the Fourier multiplier operator) is a bounded function. In this paper, we prove that Fourier multiplier operator is also bounded on classical Morrey space which is greater than the Lebesgue space.

Keyword. Mikhlin Multiplier Theorem, Fourier multiplier operator, Morrey space, bounded operator.

1. Introduction

There are many problems that can be solved using mathematics. We can model some of the problems using differential equations. For example, the heat diffusion in a wire with length $L$ can be modeled as a partial differential equation. We can extend the problem into heat diffusion in a wire with infinite length.

\begin{equation}
  u_t = ku_{xx}, \text{ for all } x \in \mathbb{R}, t > 0, \text{ and for some } k \in \mathbb{R},
\end{equation}

\begin{equation}
  u(x, 0) = \phi(x), x \in \mathbb{R},
\end{equation}

where $u(x, t)$ is the temperature of the wire at some point $x \in \mathbb{R}$ and at the time $t \geq 0$. The function $\phi$ represents the initial condition.

We can represent the solution of the differential equation (1) with the initial condition (2) into operator form, that is

\begin{equation}
  u(x, t) = \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy.
\end{equation}

If

\begin{equation}
  G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},
\end{equation}

then (3) can be written as a convolution:

\begin{equation}
  u(x, t) = \int_{-\infty}^{\infty} \phi(y)G(x - y, t)dy = (G(\cdot, t) * \phi)(x).
\end{equation}
Using one of the Fourier transformation properties, we obtain:

$$\hat{u}(x,t) = (\hat{G} \ast \hat{\phi}) = (2\pi)^{\frac{n}{2}} \hat{G}\hat{\phi}. \tag{5}$$

The operator which maps $\hat{\phi}$ to $\hat{u}(x,t)$ in (5) is called multiplier operator generated by $\hat{G}$. Here, $\hat{G}$ is called Fourier multiplier. There is another Fourier multiplier operator, i.e. Riesz transformation,

$$R_j(f)(x) = \mathcal{F}^{-1}[m_j\hat{f}](x), \tag{6}$$

where $m_j(\xi) = -i\xi_j|\xi|$ and $f \in L^2(\mathbb{R}^n)$, for $j = 1,\ldots,n$. In $\mathbb{R}$, when $j = 1$, we get $m_1(\xi) = -i \text{sign}(\xi)$. Hence (6) become Hilbert transform, i.e.

$$H(f)(x) = \mathcal{F}^{-1}[m_1\hat{f}](x).$$

On 1956, S. G. Mikhlin generalized the Fourier multiplier operator called Mikhlin operator which generalize Riesz transform and another Fourier multiplier operators. He state that it is bounded in Lebesgue space, $L^p(\mathbb{R}^n), 1 < p < \infty$ [1]. The next section presents the previous result about this boundedness obtained by researches. In section 3, we prove the boundedness of this operator in Morrey space.

2. Previous Research

S. G. Mikhlin stated the generalization of the Fourier multiplier operator and proved that it is bounded in Lebesgue space. We called this operator as Mikhlin operator and described it below [1].

**Definition 1** (Mikhlin Operator). Let $N = n + 2, n \in \mathbb{N}$. Moreover, let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be an $N$-times differentiable function such that

$$|\partial^\alpha \xi m(\xi)| \leq C|\xi|^{-|\alpha|} \tag{7}$$

for all $\xi \neq 0, |\alpha| \leq N$. A Mikhlin operator is defined as

$$M(f) = \mathcal{F}^{-1}[mf]$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

**Theorem A** (Mikhlin Multiplier Theorem). Let $M$ be Mikhlin operator. Then $M$ can be extended to a bounded linear operator in lebesgue space, i.e.

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), 1 < p < \infty. \tag{8}$$

The proof of this theorem is using Littlewood-Paley partition of unity and Calderón-Zygmund Decomposition. It has been written by some authors, one of them is Helmut Abels [1]. The detail can be read in thesis written by Dian Maharani [3]. Next, operator $M$ with the above conditions is called Mikhlin operator.

In this paper, we will explore the boundedeness of Mikhlin operator in Morrey space [4]. Firstly, we will introduce the concept of Morrey space.

**Definition 2** (Morrey Space). Let $0 \leq \lambda < n, 1 \leq p < \infty$. Morrey space, denoted as $L^{p,\lambda}(\mathbb{R}^n)$, is defined as

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy < \infty \right\},$$
where $B(x,r)$ is a ball centered in $x \in \mathbb{R}^n$ with radius $r > 0$. Norm of the functions in this space is

$$\sup_{r>0, x \in \mathbb{R}^n} \left( \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} = \|f\|_{L^p,\lambda}(\mathbb{R}^n) < \infty.$$ 

From the definition above, we can see that Morrey space will be same as Lebesgue space when $\lambda = 0$. But when $\lambda \neq 0$, Morrey space is larger than Lebesgue space. In other word, Lebesgue space is subspace of Morrey space.

Beside Morrey space, there are several spaces we use in this paper, i.e. Schwartz space [5] and tempered distribution space [3]. We extend the definition of Fourier transform and its inverse in tempered distribution space [2].

From Theorem A, using the properties of Fourier transformation, we can rewrite Mikhlin operator as convolution [1], i.e.

$$M(f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = (k * f)(x),$$

where function $k$ has properties describe below.

**Proposition B.** Let $M$ be as in Theorem A. Then there is a locally integrable, continuously differentiable function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ with compact support which satisfies

$$|k(z)| \leq C|z|^{-n}, \quad |\nabla k(z)| \leq C|z|^{-n-1},$$

for all $z \neq 0$ such that

$$M(f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy$$

for all $x \notin \text{supp } f, f \in L^2(\mathbb{R}^n)$.

Since the Fourier transformation is extended to any function in tempered distribution space, then the last properties can be extended into any function in tempered distribution space too.

### 3. Boundedness of Mikhlin Operator in Morrey Space

**Theorem 3 (Boundedness of Mikhlin Operator in Morrey Space).** Let $M$ be Mikhlin operator. Then $M$ can be extended to a bounded linear operator in Morrey space, i.e.

$$M : L^p,\lambda(\mathbb{R}^n) \to L^p,\lambda(\mathbb{R}^n), \quad 1 < p < \infty.$$  

**Proof.** Let $z \in \mathbb{R}^n$, $r > 0$, and $f \in L^{p,\lambda}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \leq \lambda < n$. We can write $f = f_1 + f_2$ with $f_1 = f \cdot \chi_{B(z,2r)}$. Since $f \in L^{p,\lambda}(\mathbb{R}^n)$, hence $f_1 \in L^{p,\lambda}(\mathbb{R}^n)$. Meanwhile, $f_2$ is a function in tempered distribution space. We can use the boundedness of $M$ in $L^p$ (Theorem A) such that we get:

$$\|M(f_1)\|_{L^p(B(z,r))} \leq \|M(f_1)\|_{L^p(\mathbb{R}^n)} \leq C_1 \|f_1\|_{L^p(\mathbb{R}^n)} = C_1 \left( \int_{\mathbb{R}^n} |f_1(x)|^p dx \right)^{\frac{1}{p}}$$

$$= C_1 \left( \int_{B(z,2r)} |f(x)|^p dx \right)^{\frac{1}{p}} = C_1 r^{\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

(12)
for $C_1, C'_1 \in \mathbb{R}$. Let $x \in B(z, r)$. Using Proposition B, we get

$$|M(f_2)(x)| = \left| \int_{\mathbb{R}^n} k(x-y) f_2(y) \, dy \right|$$

$$\leq \int_{\mathbb{R}^n} |k(x-y)||f_2(y)| \, dy$$

$$\leq \int_{\mathbb{R}^n} C_2 \frac{|f_2(y)|}{|x-y|^n} \, dy$$

$$= C'_2 \int_{B^c(z,2r)} |f(y)| \int_{|x-y|}^{\infty} t^{-n-1} \, dt \, dy,$$

(13)

for $C_2, C'_2 \in \mathbb{R}$. From (13), by Fubini's Theorem and Holder inequality, we get:

$$|M(f_2)(x)| \leq C'_2 \int_r^{\infty} t^{-n-1} \left( \int_{B(x,t)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \left( \int_{B(x,t)} 1^q \, dy \right)^{\frac{1}{q}} \, dt$$

$$= C'_2 \int_r^{\infty} t^{-n-1} \left( \int_{B(x,t)} |f(y)|^p \, dy \right)^{\frac{1}{p}} t^{\frac{n}{p}} \, dt$$

$$= C'_2 \|f\|_{L^p(\mathbb{R}^n)} \int_r^{\infty} t^{-n-1 + \frac{n}{p} + \frac{\lambda}{p}} \, dt$$

$$= C'_2 \|f\|_{L^p(\mathbb{R}^n)} r^{\frac{\lambda-n}{p}}.$$  

Consequently,

$$\|M(f_2)\|_{L^p(B(z,r))} = \left( \int_{B(z,r)} |M(f_2)(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{B(z,r)} C'_2 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\lambda-n}{p}} \, dx \right)^{\frac{1}{p}}$$

$$= C'_2 \|f\|_{L^p(\mathbb{R}^n)}^\lambda r^{\frac{\lambda-n}{p}} r^{\frac{n}{p}}$$

$$= C'_2 \|f\|_{L^p(\mathbb{R}^n)}^\lambda r^{\frac{n}{p}}.$$  

(14)

From (12) and (14),

$$\|M(f)\|_{L^p(B(z,r))} = \|M(f_1)\|_{L^p(B(z,r))} + \|M(f_2)\|_{L^p(B(z,r))}$$

$$\leq C'_1 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{1}{p}} + C'_2 \|f\|_{L^p(\mathbb{R}^n)}^\lambda r^{\frac{n}{p}}$$

$$= C \|f\|_{L^p(\mathbb{R}^n)}^{\frac{1}{p}}.$$  

(15)

Since (15) is held for any $r > 0$, then

$$\|M(f)\|_{L^p(\mathbb{R}^n)} = \sup_{r > 0, z \in \mathbb{R}^n} r^{-\frac{n}{p}} \|M(f)\|_{L^p(B(z,r))} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$  

□
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