Bayesian Entailment Hypothesis: How Brains Implement Monotonic and Non-monotonic Reasoning

Hiroyuki Kido
Cardiff University, United Kingdom
KidoH@cardiff.ac.uk

Abstract
Recent success of Bayesian methods in neuroscience and artificial intelligence gives rise to the hypothesis that the brain is a Bayesian machine. Since logic, as the laws of thought, is a product and practice of the human brain, it leads to another hypothesis that there is a Bayesian algorithm and data-structure for logical reasoning. In this paper, we give a Bayesian account of entailment and characterize its abstract inferential properties. The Bayesian entailment is shown to be a monotonic consequence relation in an extreme case. In general, it is a sort of non-monotonic consequence relation without Cautious monotony or Cut. The preferential entailment, which is a representative non-monotonic consequence relation, is shown to be maximum a posteriori entailment, which is an approximation of the Bayesian entailment. We finally discuss merits of our proposals in terms of encoding preferences on defaults, handling change and contradiction, and modeling human entailment.

1 Introduction
Bayes’ Theorem plays an important role in various fields such as artificial intelligence, neuroscience, cognitive science, statistical physics and bioinformatics. It lays the foundation of most modern systems in artificial intelligence [Russell and Norvig, 2009]. Recent studies of neuroscience empirically show that Bayesian methods explain several functions of the cerebral cortex [Lee and Mumford, 2003; Knill and Pouget, 2004; George and Hawkins, 2005; Colombo and Seriès, 2012; Funamizu et al., 2016]. It is the outer portion of the brain in charge of higher-order functions such as perception, memory, emotion and thought. These successes of Bayesian methods give rise to the Bayesian brain hypothesis that brain is a Bayesian machine [Friston, 2012; Sanborn and Chater, 2016].

If the Bayesian brain hypothesis is true then it is natural to think that there is a Bayesian algorithm and data-structure for logical reasoning. This is because logic, as the laws of thought, is a product and practice of the human brain. Pursuing a Bayesian account of logical reasoning is important. First, it has a potential to be a mathematical model to explain how the human brain implements logical reasoning. Second, it supports the Bayesian brain hypothesis in an analytical way in terms of logic. Third, it gives a way to critically assess the existing formalisms of logical reasoning. Nevertheless, few research has focused on reformulating logical reasoning in terms of Bayesian perspectives (see Section 4 for discussion).

In this paper, we begin by assuming the posterior distribution over valuation functions. The probability of each valuation function represents how much the state of the world specified by the valuation function is natural, normal or typical. We then assume a causal relation from the valuation function to each sentence. Let \( v \) and \( \alpha \) denote a valuation function and a sentence, respectively. Under the two assumptions, the probability that \( \alpha \) is true, denoted by \( p(\alpha) \), will be shown to have

\[
p(\alpha) = \sum_v p(\alpha, v) = \sum_v p(\alpha|v)p(v).
\]

That is, the probability of any sentence is determined by taking into account all valuation functions probabilistically distributed. Given a set \( \Delta \) of sentences with the same assumptions, we will show to have

\[
p(\alpha|\Delta) = \sum_v p(\alpha|v)p(v|\Delta).
\]

This equation is known as Bayesian learning [Russell and Norvig, 2009]. Intuitively speaking, \( \Delta \) updates the probability distribution over valuation functions, i.e., \( p(v) \) for all \( v \); and then the updated distribution is used to predict the truth value of \( \alpha \). We define a Bayesian entailment, denoted by \( \Delta \models \alpha \), using \( p(\alpha|\Delta) \geq \omega \), as usual.

Several important facts are derived from the simple idea. The Bayesian entailment is shown to be a monotonic consequence relation when \( \omega = 1 \). In general, it is a sort of non-monotonic consequence relation without Cautious monotony or Cut. The preferential entailment [Shoham, 1987], which is a representative non-monotonic consequence relation, is shown to be an approximation of the Bayesian entailment. We derive it from the relationship between maximum a posteriori estimation and Bayesian estimation. These results imply that both monotonic and non-monotonic consequence relations can be seen as Bayesian learning with a fixed probability threshold. We discuss that our proposals have advantages of encoding preferences on defaults, handling change and contradiction, and modeling human entailment.
and contradiction, and being a mathematical model of human entailment.

This paper is organized as follows. Section 2 gives a probabilistic model for a Bayesian entailment. Section 3 focuses on its correctness. We analyze its inferential properties in terms of monotonic and non-monotonic consequence relations. Section 4 concludes with discussion of related work.

## 2 Bayesian Entailment

Let $L$, $P$ and $v$ respectively denote the propositional language, the set of all propositional symbols in $L$, and a valuation function, $v : P \rightarrow \{0, 1\}$, where 0 and 1 mean the truth values, false and true, respectively. To handle uncertainty of states of the world, we assume that valuation functions are probabilistically distributed. Let $V$ denote a random variable over valuation functions. $p(V = v)$ denotes the probability of valuation function $v_i$. It reflects the probability of the state of the world specified by $v_i$. Given two valuation functions $v_1$ and $v_2$, $p(V = v_1) > p(V = v_2)$ represents that the state of the world specified by $v_1$ is more natural, typical or normal than that of $v_2$. When the cardinality of $P$ is $n$, there are $2^n$ possible states of the world. Thus, there are $2^n$ possible valuation functions. It is the case that $0 \leq p(v_i) \leq 1$, for all $i$ such that $1 \leq i \leq 2^n$, and \(\sum_{i=1}^{2^n} p(V = v_i) = 1\).

We assume that every propositional sentence is a random variable that has a truth value either 0 or 1. For all $\alpha \in L$, $p(\alpha = 1)$ represents the probability that $\alpha$ is true and $p(\alpha = 0)$ that $\alpha$ is false. We assume that $[\alpha]$ denotes the set of all valuation functions in which $\alpha$ is true and $[\alpha]_v$ denotes the truth value under valuation function $v$.

**Definition 1** (Interpretation). Let $\alpha$ be a propositional sentence and $V$ be a valuation function. The conditional probability distribution over $\alpha$ given $V$ is given as follows.

\[ p(\alpha | V) = \frac{[\alpha]_V}{V} \]

\[ p(\alpha | V) = 1 - \frac{[\neg \alpha]_V}{V} \]

As usual, we assume that the truth value is caused only by the valuation functions. $p(\alpha)$ is thus given by

\[ p(\alpha) = \sum_{v_i} p(\alpha, V = v_i) = \sum_{v_i} p(\alpha | V = v_i) p(V = v_i). \]

**Example 1.** Suppose two propositional symbols $a$ and $b$. Consider the following probability distribution over valuation functions, where $p(V = v) \approx p(v)$.

\[ p(V) = (p(v_1), p(v_2), p(v_3), p(v_4)) = (0.5, 0.2, 0, 0.3) \]

The left table in Table 1 shows all $2^2 = 4$ valuation functions and their probability distribution. The right table in Table 1 shows $p(a \lor \neg b | V)$. $p(a \lor \neg b = 1)$ is derived as follows, where $[\alpha]_{V = v}$ is abbreviated to $[\alpha]_v$.

\[ p(a \lor \neg b = 1) = \sum_{i=1}^{4} p(a \lor \neg b = 1 | v_i) p(v_i) \]

\[ = \sum_{i=1}^{4} [a \lor \neg b]_{v_i} p(v_i) \]

\[ = p(v_1) + p(v_3) + p(v_4) = 0.8 \]

| $V_i$ | $p(V)$ | $p(a)$ | $p(\neg b)$ | $p(a \lor \neg b)$ |
|-------|--------|--------|-------------|-----------------|
| $v_1$ | 0.5    | 0      | 0           | 0.5             |
| $v_2$ | 0.2    | 0.1    | 0.9         | 0.8             |
| $v_3$ | 0.1    | 0      | 0.1         | 0.1             |
| $v_4$ | 0.3    | 0      | 0.3         | 0.3             |

$p(a \lor \neg b = 0) = 0.2$ is shown in the same way.

Definition 1 implies that the probability of the truth of a sentence is not primitive. We thus guarantee that it satisfies the Kolmogorov axioms.

**Proposition 1.** Let $\alpha, \beta \in L$. The followings hold.

1. $0 \leq p(\alpha = i), \text{for all } i$.
2. $\sum_{i} p(\alpha = i) = 1$.
3. $p(\alpha \lor \beta = i) = p(\alpha = i) + p(\beta = i) - p(\alpha \land \beta = i)$, for all $i$.

**Proof.** Since any sentence takes a truth value either 0 or 1, it is sufficient for (1) and (2) to show $p(\alpha = 0) + p(\alpha = 1) = 1$ and $0 \leq p(\alpha = 0), p(\alpha = 1) \leq 1$. We have

\[ p(\alpha = 0) = \sum_{v} p(\alpha = 0 | v) p(v) = \sum_{v} (1 - [\neg \alpha]_v) p(v) \]

\[ p(\alpha = 1) = \sum_{v} p(\alpha = 1 | v) p(v) = \sum_{v} [\alpha]_v p(v) \]

(1) is true because $0 \leq p(v) \leq 1$, for all $v$. (2) is true because $p(\alpha = 0) + p(\alpha = 1) = \sum_{v} p(v) = 1$ holds. Let $X$ be $\{[\alpha]_v + [\beta]_v - [\alpha \land \beta]_v\}$. (3) can be developed as follows, where the first expression comes when $i = 0$ and the second when $i = 1$.

\[ \sum_{v} p(v)(1 - [\alpha \lor \beta]_v) = \sum_{v} p(v)(1 - X) \]

\[ \sum_{v} p(v)[\alpha \lor \beta]_v = \sum_{v} p(v)X \]

There are four possible cases. If $[\alpha]_v = [\beta]_v = 0$ then the expression in the bracket of the right expressions turn out to be 0($= 0 + 0 - 0$), if $[\alpha]_v = 0$ and $[\beta]_v = 1$ then 1($= 0 + 1 - 0$), if $[\alpha]_v = 1$ and $[\beta]_v = 0$ then 1($= 1 + 0 - 0$), and if $[\alpha]_v = [\beta]_v = 1$ then 1($= 1 + 1 - 1$). All the results are consistent with $[\alpha \lor \beta]_v$. $\square$

**Proposition 2.** $p(\alpha = 0) = p(\neg \alpha = 1)$ holds, for any $\alpha \in L$.

In what follows, we thus replace $p(\alpha = 0)$ by $p(\neg \alpha = 1)$ and then abbreviate $p(\neg \alpha = 1)$ to $p(\neg \alpha)$, for all $\alpha \in L$.

Dependancy among the random variables is shown in Figure 1 using a Bayesian network, a directed acyclic graphical model. Sentence $\alpha$ has a directed edge only from valuation function $V$. It represents that the valuation function is the direct cause of the truth value of the sentence. The dependence between $V$ and another sentence $\beta_i$ is the same as $\alpha$. Only $\beta_i$ is colored grey. It means that $\beta_i$ is assumed to be observed, which is in contrast to the other nodes assumed to
be predicted or estimated. The box surrounding \( \beta_i \) is a plate. It represents that there are \( N \) sentences \( \beta_1, \beta_2, \ldots, \beta_N \) to which there are directed edges from \( V \). Given the dependency, the conditional probability of \( \alpha \) given \( \beta_1, \beta_2, \ldots, \beta_N \) is given as follows.

\[
p(\alpha|\beta_1, \beta_2, \ldots, \beta_N) = \frac{p(\alpha, \beta_1, \beta_2, \ldots, \beta_N)}{p(\beta_1, \beta_2, \ldots, \beta_N)} = \frac{\sum_v p(\alpha|v) \prod_{i=1}^N p(\beta_i|v)}{\sum_v p(v) \prod_{i=1}^N p(\beta_i|v)}
\]

Example 2 (Continued).

\[
p(-a|a \lor b, -a \lor b) = p(v_1) = \frac{0.5}{0.8} = 0.625
\]

Now, we want to investigate logical properties of \( p(\alpha|\beta_1, \beta_2, \ldots, \beta_N) \). We thus define a consequence relation between \( \{\beta_1, \beta_2, \ldots, \beta_N\} \) and \( \alpha \).

**Definition 2** (Bayesian entailment). Let \( \alpha \in \mathcal{L}, \Delta \subseteq \mathcal{L} \) and \( \omega \) satisfy \( 0 \leq \omega \leq 1 \). \( \alpha \) is a Bayesian entailment of \( \Delta \) with respect to \( \omega \), denoted by \( \Delta \models_{\omega} \alpha \), if \( p(\alpha|\Delta) \geq \omega \) or \( p(\Delta) = 0 \).

Condition \( p(\Delta) = 0 \) guarantees that \( \alpha \) is a Bayesian entailment of \( \Delta \) when \( p(\alpha|\Delta) \) is undefined due to division by zero. It happens when \( \Delta \) has no models, i.e., \( [\Delta] = \emptyset \), or zero probability, i.e., \( p(\Delta) = 0 \), for all \( v \in [\Delta] \), \( \models_{\omega} \alpha \) is a special case of Definition 2. It holds when \( p(\Delta) \geq \omega \). We call Definition 2 Bayesian entailment because \( p(\alpha|\Delta) \) can be developed as follows.

\[
p(\alpha|\Delta) = \frac{\sum_v p(\alpha|v) p(\Delta|v) p(\Delta)}{p(\Delta)} = \frac{\sum_v p(\alpha|v) p(v|\Delta) p(\Delta)}{p(\Delta)}
\]

The last expression is often called *Bayesian learning* where \( \Delta \) updates the distribution over valuation functions, i.e., \( P(v) \), and truth values of \( \alpha \) is predicted using the updated distribution. Therefore, the Bayesian entailment allows us to see logical consequences as machine learning predictions.

### 3 Correctness

#### 3.1 Monotonic Consequence Relation

Recall that propositional entailment \( \Delta \models \alpha \) is defined as follows: For all valuation functions \( v \), if \( \Delta \) is true in \( v \) then \( \alpha \) is true in \( v \). The Bayesian entailment \( \models_{\omega} \) works in a similar way as the propositional entailment. The only difference is that the Bayesian entailment ignores valuation functions with zero probability. The valuation functions with zero probability represent impossible states of the world.

**Theorem 1.** Let \( \alpha \in \mathcal{L} \) and \( \Delta \subseteq \mathcal{L} \). \( \Delta \models_{\omega} \alpha \) holds if and only if for all valuation functions \( v \) such that \( p(v) \neq 0 \), if \( \Delta \) is true in \( v \) then \( \alpha \) is true in \( v \).

**Proof.** We show that \( \Delta \models_{\omega} \alpha \) holds if and only if there is a valuation function \( v \) such that \( p(v) \neq 0 \), \( \Delta \) is true in \( v \), and \( \alpha \) is false in \( v \). From Definition 2, \( \Delta \models_{\omega} \alpha \) holds if and only if \( p(\Delta) \neq 0 \) and \( p(\alpha|\Delta) \neq 1 \) hold. From Definition 1, \( p(\Delta) \neq 0 \) holds if and only if there is a valuation function \( v^* \) such that \( p(v^*) \neq 0 \) and \( \Delta \) is true in \( v^* \). \( \Rightarrow \) This has been proven so far. \( \Rightarrow \) \( \Delta \models_{\omega} \alpha \) holds due to \( v^* \). Since \( p(\Delta) = \sum_v p(v) [\Delta]_{v} \neq 1 \), there is \( v \in [\Delta] \setminus [\alpha] \) such that \( p(v) \neq 0 \). \( \Delta \) is true in \( v \) and \( \alpha \) is false in \( v \).

It is known that monotonic consequence relations can be characterized by the three properties: Reflexivity, Monotony and Cut. Let \( \models_{\omega} \subseteq \mathcal{L} \times \mathcal{F} \). Those properties on \( \models_{\omega} \) are defined as follows, where \( \alpha, \beta \in \mathcal{L} \) and \( \Delta \subseteq \mathcal{L} \).

- **Reflexivity**: \( \forall \Delta \forall \alpha, \Delta, \alpha \models_{\omega} \alpha \)
- **Monotony**: \( \forall \Delta \forall \alpha, \beta, \models_{\omega} \alpha \models_{\omega} \beta \)
- **Cut**: \( \forall \Delta \forall \alpha \forall \beta, \text{if } \Delta, \alpha \models_{\omega} \beta \text{ then } \Delta, \beta \models_{\omega} \alpha \)

Monotonicity states that if \( \alpha \) is a consequence of \( \Delta \) then \( \beta \) is also a consequence of \( \Delta \) as well. Cut states that an addition of any consequence of \( \Delta \) to \( \Delta \) does not reduce any consequence of \( \Delta \). The Bayesian entailment \( \models_{\omega} \) is monotonic in the sense that it satisfies all of the properties.

**Theorem 2.** The Bayesian entailment \( \models_{\omega} \) satisfies Reflexivity, Monotony and Cut.

**Proof.** (Reflexivity) It is true because \( \models_{\omega} \subseteq \models_{\omega+1} \) holds. (Monotony) Since \( \Delta \models_{\omega} \alpha \) holds, \( [\Delta] \leq [\alpha] \) or \( p(v) = 0 \) holds, for all \( v \in [\Delta] \setminus [\alpha] \). For all \( v \notin [\Delta] \setminus [\alpha] \), it is thus true that \( v \notin [\Delta] \) holds and \( v \in [\alpha] \). Therefore, \( p(\alpha|\Delta, \beta) = \sum_v p(v) [\Delta]_{v}[\alpha]_{v}[\beta]_{v} p(\Delta) = \sum_v p(v) [\Delta]_{v} [\alpha]_{v} [\beta]_{v} p(v) = 1 \).

Here, we have excluded all \( v \in [\Delta] \setminus [\alpha] \) because of \( p(v) = 0 \). (Cut) Since \( \Delta \models_{\omega} \alpha \) holds, \( [\Delta] \leq [\beta] \) or \( p(v) = 0 \) holds, for all \( v \in [\Delta] \setminus [\beta] \). Since \( \Delta \models_{\omega} \alpha \) holds, \( [\Delta, \beta] \subseteq [\alpha] \) or \( p(v) = 0 \) holds, for all \( v \in [\Delta, \beta] \setminus [\alpha] \). Let \( \Delta = ([\Delta] \setminus [\beta]) \cup ([\Delta, \beta] \setminus [\alpha]) \). For all \( v \notin \Delta \), it is thus true that if \( v \in [\Delta] \) holds then \( v \in [\beta] \) and \( v \in [\alpha] \) hold. We thus have \( p(\alpha|\Delta, \beta) = \sum_v p(v) [\Delta]_{v} [\alpha]_{v} [\beta]_{v} p(\Delta) = \sum_v p(v) [\Delta]_{v} [\alpha]_{v} [\beta]_{v} p(\Delta) = 1 \).

However, the Bayesian entailment \( \models_{\omega} \) is not monotonic when it is with probability \( \omega \) where \( 0.5 < \omega < 1 \). The Bayesian entailment \( \models_{\omega} \) satisfies Reflexivity, but does not satisfy Monotony and Cut.
Table 2: Counter-examples of Monotony (left) and Cut (right).

|       | p(V) | a | b |
|-------|------|---|---|
| v₁    | 0    | 0 | 0 |
| v₂    | 1 − ω| 0 | 1 |
| v₃    | 1 − ω| 1 | 0 |
| v₄    | 2ω − 1| 1 | 1 |

Proof. (Reflexivity) Obvious from ≡ ⊆ ≡ ⊆ ω. (Monotony) We show a counter-example. Given the set {a, b} of propositional symbols, consider the probability distribution over valuation functions shown in Table 2. Note that \( \sum_v p(v) = 1 \) holds. It is the case that

\[
p(a) = p(v₃) + p(v₄) = (1 - ω) + (2ω - 1) = ω
\]

\[
p(a|b) = \frac{p(v₄)}{p(v₂) + p(v₄)} = \frac{2ω - 1}{(1 - ω) + (2ω - 1)} = \frac{2ω - 1}{ω}.
\]

\( ω > \frac{2ω - 1}{ω} \) holds if and only if \( (ω - 1)^2 > 0 \) holds. It is thus always true when \( 0.5 < ω < 1 \). Therefore, \( \not{≈} \) a but \( b \not{≈} a \) hold. (Cut) We show a counter-example. Consider the probability distribution over valuation functions shown in Table 2. Note that \( \sum_v p(v) = 1 \) holds. It is the case that

\[
p(a) = p(v₃) + p(v₄) = ω(1 - ω) + ω² = ω
\]

\[
p(a ∧ b|a) = \frac{p(v₄)}{p(v₃) + p(v₄)} = \frac{ω²}{ω(1 - ω) + ω²} = ω
\]

\[
p(a ∧ b) = p(v₄) = ω²
\]

\( ω > ω² \) is always true when \( 0.5 < ω < 1 \). Therefore, \( \not{≈} \) a and \( a \not{≈} a ∧ b \) hold, but \( b \not{≈} a ∧ b \) hold.

Example 3. Let \( ω = 0.8 \) in the left table in Table 2. Monotony does not hold because \( p(a) = 0.8 \) and \( p(a|b) = 0.75 \) hold. Let \( ω = 0.8 \) in the right table in Table 2. Cut does not hold because \( p(a) = 0.8 \), \( p(a ∧ b|a) = 0.8 \) and \( p(a ∧ b) = 0.64 \) hold.

3.2 Non-monotonic Consequence Relation

It is known that there are at least four core properties characterizing non-monotonic consequence relations: Supraclassicality, Reflexivity, Cautious monotony and Cut. Let \( \vdash \subseteq Pow(ℒ) × ℒ \) be a consequence relation on \( ℒ \). They are formally defined as follows, where \( α, β ∈ ℒ \) and \( Δ ⊆ ℒ \).

- Supraclassicality: \( ∀Δ∀α, \text{ if } Δ ⊢ α \text{ then } Δ ⊢ α \)
- Reflexivity: \( ∀Δ∀α, Δ, α \vdash α \)
- Cautious monotony: \( ∀Δ∀α∀β, \text{ if } Δ ⊢ α \text{ and } Δ ⊢ α \text{ then } Δ, β ⊢ α \)
- Cut: \( ∀Δ∀α∀β, \text{ if } Δ ⊢ β \text{ and } Δ, β ⊢ α \text{ then } Δ ⊢ α \)

We have already discussed reflexivity and cut. Supraclassicality states that the consequence relation extends the monotonic consequence relation. Cautious monotony states that if \( α \) is a consequence of \( Δ \) then it is a consequence of supersets of \( Δ \) as well. However, it is weaker than monotony because the supersets are restricted to consequences of \( Δ \). Consequence relations satisfying those properties are often called a cumulative consequence relation [Brewka et al., 1997].

Theorem 4. Let \( ω \) satisfy \( 0.5 < ω < 1 \). The Bayesian entailment \( ≡ω \) satisfies Supraclassicality and Reflexivity, but does not satisfy Cautious monotony and Cut.

Proof. (Reflexivity & Cut) See Theorem 3. (Supraclassicality) This is obvious from \( ≡ ⊆ ≡ ⊆ ω \). (Cautious monotony) It is enough to show a counter-example. Given set \( \{a, b\} \) of atomic propositions, consider again the distribution over valuation functions shown in the left table in Table 2. We have

\[
p(a) = p(v₃) + p(v₄) = (1 - ω) + (2ω - 1) = ω
\]

\[
p(b) = p(v₂) + p(v₄) = (1 - ω) + (2ω - 1) = ω
\]

\[
p(a|b) = \frac{p(v₄)}{p(v₂) + p(v₄)} = \frac{2ω - 1}{ω}.
\]

\( ω < \frac{2ω - 1}{ω} \) holds if and only if \( (ω - 1)^2 \leq 0 \) holds. It is always false when \( 0.5 < ω < 1 \) holds.

Theorem 4 shows that, in general, the Bayesian entailment \( ≡ω \) is not cumulative. A natural criticism against the Bayesian entailment is that it is inadequate as a non-monotonic consequence relation due to the lack of Cautious monotony and Cut. Indeed, Gabbay [Gabbay, 1985] considers, on the basis of his intuition, that non-monotonic consequence relations satisfy at least Cautious monotony, Reflexivity and Cut. However, it is controversial because of unintuitive behavior of Cautious monotony and Cut in extreme cases. A consequence relation \( ∼ \) with Cut, for instance, satisfies \( ∼ x_{N+1} \) when it satisfies \( ∼ xᵢ \) and \( xᵢ ∼ x_{i+1} \), for all \( i(1 ≤ i ≤ N) \). This is very unintuitive when \( N \) is large.

Brewka [Brewka et al., 1997] in fact points out the infinite transitivity as a weakness of Cut.

3.3 Preferential Entailment

The preferential entailment [Shoham, 1987] is a well-known approach to a non-monotonic consequence relation. It is defined on a preferential structure \( ⟨ℒ, ∼⟩ \), where \( ℒ \) is a set of valuation functions and \( ∼ \) is an irreflexive and transitive relation on \( ℒ \). \( v₁ ∼ v₂ \) represents that \( v₁ \) is preferable to \( v₂ \) in the sense that the world identified by \( v₁ \) is more normal, typical or natural than the one identified by \( v₂ \). Given a preferential structure \( ⟨ℒ, ∼⟩ \), \( α \) is a preferential entailment of \( Δ \), denoted by \( Δ ∼(⟨ℒ, ∼⟩) α \), if \( α \) is true in all \( ∼\)-maximal models of \( Δ \).

Now, we introduce several concepts to show the relationship between the preferential entailment and the Bayesian entailment. \( v_{MAP} \) is said to be a maximum a posteriori (MAP) estimate if it satisfies

\[ v_{MAP} = \arg \max_v p(v|Δ). \]

1This definition is not absolute. The authors [Krau et al., 1990] define it as a relation satisfying Reflexivity, Left logical equivalence, Right weakening, Cut and Cautious monotony.

2For the sake of simplicity, we do not adopt the common practice in logic that \( v₂ ∼ v₁ \) denotes \( v₁ \) is preferable to \( v₂ \).

3\( ∼ \) has to be smooth (or stuttered) [Krau et al., 1990] so that a maximal model certainly exists.
Let $\Delta$ be an order-preserving map of $V \rightarrow \{0,1\}$ into a valuation function. It means that there is a single state of the world that is very normal, natural or typical. Under the assumption, we have $p(V|\Delta) \approx 1$ if $V = v_{MAP}$ and 0 otherwise, where $\approx$ denotes an approximation. We now have

$$p(\alpha|\Delta) = \sum_v p(\alpha|\Delta)p(v|\Delta) \approx p(\alpha|v_{MAP}).$$

$p(\alpha|v_{MAP}) \in \{0,1\}$ holds. We thus define a maximum a posteriori entailment as follows.

**Definition 3** (Maximum a posteriori entailment). Let $\alpha \in \mathcal{L}$ and $\Delta \subseteq \mathcal{L}$. $\alpha$ is a maximum a posteriori entailment of $\Delta$, denoted by $\Delta \approx_{MAP} \alpha$, if $p(\alpha|v_{MAP}) = 1$ or $p(\Delta) = 0$, where $v_{MAP} = \arg\max_v p(v|\Delta)$.

Given two ordered sets $(S_1, \leq_1)$ and $(S_2, \leq_2)$, a function $f$ is said to be an order-preserving (or isotope) map of $(S_1, \leq_1)$ into $(S_2, \leq_2)$ if $x \leq_1 y$ implies that $f(x) \leq_2 f(y)$, for all $x, y \in S_1$. The next theorem relates the maximum a posteriori entailment to the preferential entailment.

**Theorem 5.** Let $(\mathcal{V}, \succ)$ be a preferential structure and $p : \mathcal{V} \rightarrow [0,1]$ be a probability mass function over $\mathcal{V}$. If $p$ is an order-preserving map of $(\mathcal{V}, \succ)$ into $[0,1]$, then $\Delta \rightarrow_{(\mathcal{V}, \succ)} \alpha$ implies $\Delta \approx_{MAP} \alpha$.

**Proof.** It is obviously true when $\Delta$ has no model. Let $v^*$ be a $\succ$-maximal model of $\Delta$. It is sufficient to show $p(\alpha|v^*) = 1$ and $v^* = \arg\max_v p(v|\Delta)$. Since $\Delta \rightarrow_{(\mathcal{V}, \succ)} \alpha$, $\alpha$ is true in $v^*$. Thus, $p(\alpha|v^*) = [\alpha]_{v^*} = 1$. We have

$$\arg\max_v p(v|\Delta) = \arg\max_v p(\Delta|v)p(v) = \arg\max_v [\Delta]_v p(v).$$

Since $\Delta$ is true in $v^*$, $[\Delta]_{v^*} = 1$. Since $p$ is order-preserving, if $v_1 \succ v_2$ then $p(v_1) \geq p(v_2)$, for all $v_1, v_2$. Thus, if $v$ is $\succ$-maximal then $p(v)$ is maximal. Therefore, $v^* = \arg\max_v [\Delta]_v p(v)$ holds. \qed

**Example 4.** Suppose the probability distribution over valuation functions shown in the table in Figure 2 and the preferential structure $(\{v_1, v_2, v_3, v_4\}, \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_4, v_2)\})$. The transitivity of $\succ$ is depicted in the graph shown in Figure 2. As shown in the graph, the probability mass function is an order-preserving map of $(\mathcal{V}, \succ)$ into $[0,1]$.

Now, $\{a \lor \neg b\} \rightarrow_{(\mathcal{V}, \succ)} \neg b$ holds because $\neg b$ is true in all $\succ$-maximal models of $\{a \lor \neg b\}$, i.e., $v_1$. Meanwhile, $\{a \lor \neg b\} \approx_{MAP} \neg b$ holds because $p(\neg b|v_1) = 1$ holds where $v_1 = \arg\max_v p(v|a \lor \neg b)$. However, $\{a\} \rightarrow_{(\mathcal{V}, \succ)} \neg b$ holds because $\neg b$ is false in a $\succ$-maximal model of $a \lor \neg b$, i.e., $v_4$. In contrast, $\{a\} \approx_{MAP} \neg b$ holds because $p(\neg b|v_3) = 1$ holds where $v_3 = \arg\max_v p(v|a)$.

The equivalence relation between the maximum a posteriori entailment and the preferential entailment is obtained by restricting the preferential structure to a total order.

**Theorem 6.** Let $(\mathcal{V}, \succ)$ be a totally-ordered preferential structure and $p : \mathcal{V} \rightarrow [0,1]$ be a probability mass function over $\mathcal{V}$. If $p$ is an order-preserving map of $(\mathcal{V}, \succ)$ into $[0,1]$, then $\Delta \rightarrow_{(\mathcal{V}, \succ)} \alpha$ if and only if $\Delta \approx_{MAP} \alpha$.

**Proof.** Same as Theorem 5. The only difference is that such model $v^*$ exists uniquely. \qed

### 4 Discussion and Conclusions

There are a lot of attempts to combine logic and probability theory, e.g., [Adams, 1998; van Fraassen, 1981; van Fraassen, 1983; Morgan, 1983; Cross, 1993; Leblanc, 1979; Leblanc and Morgan, 1983; Pearl, 1991; Goosen, 1979; Reasoner and Domingo, 2006; Thimm, 2013]. Their common interest is not the notion of truth preservation but rather probability preservation, where the uncertainty of the conclusion preserves the uncertainty of the premises. They are different from ours as they presuppose and extend the classical logical consequence.

Besides the preferential entailment, various other semantics for non-monotonic consequence relations have been proposed such as plausibility structure [Friedman and Halpern, 1996], possibility structure [Dubois and Prade, 1990; Benferhat et al., 2003, ranking structure [Goldszmidt and Pearl, 1992] and $\varepsilon$-semantics [Adams, 1975; Pearl, 1989]. The common idea of the first three approaches is that $\Delta$ entails $\alpha$ if $[\Delta \land \alpha] \geq [\Delta \land \neg \alpha]$ holds given preference relation $\succeq$. However, the difficulty associated with determining the preference is discussed by Brewka [Brewka et al., 1997] as follows: Perhaps, the greatest technical challenge left for circumscription and model preference theories in general is how to encode preferences among abnormalities or defaults. Given a default $d$, we think that its preference can be defined by $\sum_{v \in d} p(v|\Delta)$, i.e., the sum of the posterior probabilities of the models in which the default is true. The benefit of this approach is that the preferences among defaults can be dynamically encoded and updated within probabilistic inference in accordance with observation $\Delta$. The idea of $\varepsilon$-semantics is
that $\Delta$ entails $\alpha$ if $p(\alpha|\Delta)$ is close to one given a probabilistic knowledge base that quantifies the strength of the causal relation or dependency between sentences. It differs from our basic idea, depicted in Figure 1, that any sentences are conditionally independent given a model. This fact enables us to mathematically handle interactions between models and sentences within probabilistic inference. It is different from the approaches [Adams, 1975; Pearl, 1989; Hawthorne, 2007; Hawthorne and Makinson, 2007] handling the interactions outside their probabilistic inference.

A natural criticism against our work is that the Bayesian entailment is inadequate as a non-monotonic consequence relation due to the lack of Cautious monotony and Cut. However, we showed that the preferential entailment satisfying Cautious monotony and Cut is shown to correspond to the maximum a posteriori entailment that is an approximation of the Bayesian entailment. It tells us that Cautious monotony and Cut are ideal under the special condition that a state of the world exists deterministically. They are not ideal under the general perspective that states of the world are probabilistically distributed.

The Bayesian entailment is flexible to extend. For example, a possible extension of Figure 1 is a hidden Markov model shown in Figure 3. It has a valuation variable and a sentence(s) variable, for each time step $t$ where $1 \leq t \leq N$. Entailment $\Delta_1, \ldots, \Delta_N \models_{\omega} \alpha_N$ defined in accordance with Definition 2 concludes $\alpha_N$ by taking into account not only the current observation $\Delta_N$ but also the previous states of the world $V_{N-1}$ updated by all of the past observations $\Delta_1, \ldots, \Delta_{N-1}$. It is especially useful when observations are contradictory, ambiguous or easy to change.

Our hypothesis is that the Bayesian entailment can be a mathematical model of how human brains implement an entailment. It is supported by the three reasons. First, as shown in Section 2, it is built on a very simple probabilistic model based on Bayes’ Theorem. Second, as discussed in Section 3, it has general properties as a monotonic and a non-monotonic consequence relation. Third, growing evidences for the consequence relation. Third, growing evidences for the Bayesian brain hypothesis are emerging from the studies of neuroscience. The British Journal for the Philosophy of Science, 63:697–723, 2012.

Cross, 1993] Charles B. Cross. From worlds to probabilities: A probabilistic semantics for modal logic. Journal of Philosophical Logic, 22:169–192, 1993.

Dubois and Prade, 1990] Didier Dubois and Henri Prade. Readings in uncertain reasoning, chapter An Introduction to Possibilistic and Fuzzy Logics, pages 742–761. Morgan Kaufmann Publishers Inc., San Francisco, USA, 1990.

Friedman and Halpern, 1996] Nir Friedman and Joseph Y. Halpern. Plausibility measures and default reasoning. In Proc. 13th National Conf. on Artif. Intell., pages 1297–1304, 1996.

Friston, 2012] Karl Friston. The history of the future of the bayesian brain. Neuroimage, 62-248(2):1230–1233, 2012.

Funamizu et al., 2016] Akihiro Funamizu, Bernd Kuhn, and Kenji Doya. Neural substrate of dynamic bayesian inference in the cerebral cortex. Nature Neuroscience, 19:1682–1689, 2016.

Gabbay, 1985] Dov M. Gabbay. Theoretical Foundations for Non-monotonic Reasoning in Expert Systems. Springer-Verlag, 1985.

George and Hawkins, 2005] Dileep George and Jeff Hawkins. A hierarchical bayesian model of invariant pattern recognition in the visual cortex. In Proc. Int. Joint Conf. on Neural Networks, pages 1812–1817, 2005.

Goldszmidt and Pearl, 1992] Moisés Goldszmidt and Judea Pearl. Rank-based systems: A simple approach to belief revision, belief update, and reasoning about evidence and actions. In Proc. 3rd Int. Conf. on Principles of Knowledge Representation and Reasoning, pages 661–672, 1992.

Goosens, 1979] William K. Goosens. Alternative axiomatizations of elementary probability theory. Notre Dame Journal of Formal Logic, 20:227–239, 1979.

Hawthorne and Makinson, 2007] James Hawthorne and David Makinson. The quantitative/qualitative watershed for rules of uncertain inference. Studia Logica, 86:247–297, 2007.

Hawthorne, 2007] James Hawthorne. Nonmonotonic conditionals that behave like conditional probabilities above a threshold. Journal of Applied Logic, 5:625–637, 2007.

Knill and Pouget, 2004] David C. Knill and Alexandre Pouget. The bayesian brain: the role of uncertainty in...
neural coding and computation. *Trends in Neurosciences*, 27:712–719, 2004.

[Kraus et al., 1990] Sarit Kraus, Daniel Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44(1-2):167–207, 1990.

[Leblanc and Morgan, 1983] Hugues Leblanc and Charles G. Morgan. Probabilistic semantics for intuitionistic logic. *Notre Dame Journal of Formal Logic*, 24:161–180, 1983.

[Leblanc, 1979] Hugues Leblanc. Probabilistic semantics for first-order logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 25:497–509, 1979.

[Lee and Mumford, 2003] Tai Sing Lee and David Mumford. Hierarchical bayesian inference in the visual cortex. *Journal of Optical Society of America*, 20:1434–1448, 2003.

[Morgan, 1983] Charles G. Morgan. *Probabilistic Semantics for Propositional Modal Logics*, pages 97–116. New York, NY: Haven Publications, essays in epistemology and semantics edition, 1983.

[Pearl, 1989] Judea Pearl. Probabilistic semantics for nonmonotonic reasoning: a survey. In *Proc. 1st Int. Conf. on Principles of Knowledge Representation and Reasoning*, pages 505–516, 1989.

[Pearl, 1991] Judea Pearl. *Probabilistic Semantics for Nonmonotonic Reasoning*, pages 157–188. Cambridge, MA: The MIT Press, philosophy and AI: essays at the interface edition, 1991.

[Richardson and Domingos, 2006] Matthew Richardson and Pedro Domingos. Markov logic networks. *Machine Learning*, 62:107–136, 2006.

[Russell and Norvig, 2009] Stuart Russell and Peter Norvig. *Artificial Intelligence : A Modern Approach, Third Edition*. Pearson Education, Inc., 2009.

[Sanborn and Chater, 2016] Adam N. Sanborn and Nick Chater. Bayesian brains without probabilities. *Trends in Cognitive Sciences*, 20:883–893, 2016.

[Shoham, 1987] Yoav Shoham. Nonmonotonic logics: Meaning and utility. In *Proc. 10th Int. Joint Conf. on Artif. Intell.*, pages 388–393, 1987.

[Thimm, 2013] Matthias Thimm. Inconsistency measures for probabilistic logics. *Artif. Intell.*, 197:1–24, 2013.

[van Fraassen, 1981] Bas van Fraassen. Probabilistic semantics objectified: I. postulates and logics. *Journal of Philosophical Logic*, 10:371–391, 1981.

[van Fraassen, 1983] Bas van Fraassen. Gentlemen’s wagers: Relevant logic and probability. *Philosophical Studies*, 43:47–61, 1983.