Crossed Burnside rings and cohomological Mackey 2-motives

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Abstract

Balmer and Dell’Ambrogio introduced the pseudo-functor $\mathcal{P}$ from the bicategory of $k$-linear Mackey 2-motives to the bicategory of $k$-linear cohomological Mackey 2-motives over a commutative ring $k$. They showed that $\mathcal{P}$ maps the general Mackey 2-motives to the cohomological Mackey 2-motives by using the ring homomorphism from the crossed Burnside ring of a finite group $G$ over $k$ to the center $ZkG$ of group algebra $kG$ ([BD21, Theorem 5.3]). We study the behavior of motivic decomposition of cohomological Mackey 2-motives as images by $\mathcal{P}$ of motivic decomposition of Mackey 2-motives.

1 Introduction

Let $G$ be a finite group and let $k$ be a commutative ring. Balmer-Dell’Ambrogio theory of Mackey 2-functors and Mackey 2-motives can be applied to a very wide range of research subjects. The theory suggests that the decompositions of many of the categories listed below as abelian categories are all controlled by Mackey 2-motives. Here are some examples:

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In representation theory, \( \mathcal{M}(G) = \mathbb{k}G\text{-mod} \) for the group algebra \( \mathbb{k}G \) of \( G \) over \( \mathbb{k} \).

- In representation theory, derived categories \( \mathcal{M}(G) = \text{D}(\mathbb{k}G) \) of \( \mathbb{k}G\text{-mod} \).
- In equivariant homotopy theory, homotopy categories of \( G \)-spectra \( \mathcal{M}(G) = \text{SH}(G) \).
- In noncommutative geometry, Kasparov categories \( \mathcal{M}(G) = \text{KK}(G) \) of \( G\text{-C}^*\)-algebras.

In order to unify and decompose the various abelian categories \( \mathcal{M}(G) \) mentioned above, Balmer and Dell’Ambrogio came up with the notion of Mackey 2-motives which was inspired by the idea of the plain 1-category of pure motives in algebraic geometry originated by Grothendieck. They figured out that what controls the decomposition of \( \mathcal{M}(G) \) is the motivic decomposition

\[ G \simeq (G, e_1) \oplus (G, e_2) \oplus \cdots \oplus (G, e_n), \]

where \( e_i \) is an idempotent of the endomorphism ring \( \text{End}_{\text{Span}(G,G)}(\text{Id}_G) \) of the identity 1-cell \( \text{Id}_G : G \to G \) in the bicategory \( \text{Span}(G,G) \) ([BD20, Definition 7.1.7]) of \( \mathbb{k} \)-linear Mackey 2-motives, and proved that the ring \( \text{End}_{\text{Span}(G,G)}(\text{Id}_G) \) is isomorphic to the crossed Burnside ring \( B^c_\mathbb{k}(G) \) ([BD20, Theorem 7.4.5]) introduced by the third author ([Yo97]). Moreover, they introduced a theory of cohomological Mackey 2-functors and cohomological Mackey 2-motives in [BD21]. As one of the many wonderful results that will be a result from Balmer-Dell’Ambrogio theory in the future, they introduced pseudo-functor \( P \) which sends the ordinary Mackey 2-motive to the cohomological Mackey 2-motive in [BD21] to analyze in detail the relationship between the ordinary Mackey 2-motive and the cohomological Mackey 2-motive.

The main purpose of this paper is to give further refinements (Theorems 4.5, 4.7, and 4.8) of [BD21, Theorem 5.3] for some \( \mathbb{k} \). The second purpose of this paper is to determine the primitive idempotents of the crossed Burnside ring \( B^c_\mathbb{k}(G) \) of \( G \) over \( \mathbb{Z} \) (Theorem 2.2). Although it is not directly related to the above topics, we will describe the properties (Proposition 1.3 and Corollary 1.4) of the ring homomorphism from \( B^c_\mathbb{k}(G) \) to the center \( \mathbb{Z}\mu_\mathbb{k}(G) \) of the Mackey algebra of \( G \) over \( \mathbb{k} \) introduced by Bouc. To investigate all primitive idempotents of \( B^c_\mathbb{o}(G) \) over a complete discrete valuation ring \( \mathcal{O} \), we apply Green functor theory by Bouc to give a decomposition of \( B^c_\mathbb{o}(G) \) (Proposition 3.5).

The paper is organized as follows: Section 2 is a recollection of definitions and basic results on the crossed Burnside ring. Section 3 is a recollection of...
Bouc’s theory of Green functors. As an application of this, we give the decomposition of $B^c_G(G)$, essentially, it was discussed in [Bo03a]. Section 4 describes the behavior of motivic decomposition of cohomological Mackey 2-motives as images by the several pseudo-functors $P$ of motivic decomposition of Mackey 2-motives. Especially, we provide a condition to determine which primitive idempotent of the crossed Burnside ring is included in kernel of $\rho_G$ introduced by Balmer and Dell’Ambrogio ([BD21, Theorem 5.3]).

In this paper, we fix a finite group $G$ with the identity $e$ and $k$ be a commutative ring with identity.

2 Idempotents of a crossed Burnside ring

2.1 Crossed Burnside rings

We recall a construction of a crossed Burnside ring of $G$ over $\mathbb{Z}$ from [Bo03a], [OY01], and [Yo97]. Denote by $G^c$ the set $G$, on which $G$ acts by conjugation. The category of crossed $G$-sets is the category of $G$-sets over $G^c$: a crossed $G$-set $(X,\alpha)$ is a pair consisting of a finite $G$-set (i.e. a finite set with a left $G$-action), together with a $G$-map $\alpha$ from $X$ to $G^c$, and a morphism of crossed $G$-sets from $(X,\alpha)$ to $(Y,\beta)$ is a $G$-map $f$ from $X$ to $Y$ such that $\beta \circ f = \alpha$. The crossed Burnside group $B^c_G$ is defined as the Grothendieck group of the category of crossed $G$-sets, for relation given by disjoint union decomposition. Denote by $[X,\alpha]$ the isomorphism class of the crossed $G$-set $(X,\alpha)$. If $(X,\alpha)$ and $(Y,\beta)$ are crossed $G$-sets, then their product is the crossed $G$-set $(X \times Y,\alpha \cdot \beta)$, where $X \times Y$ is the direct product of $X$ and $Y$, with diagonal $G$-action, and $\alpha \cdot \beta$ is the map from $X \times Y$ to $G^c$ defined by $(\alpha \cdot \beta)(x,y) = \alpha(x)\beta(y)$. This product on crossed $G$-sets clearly commutes with disjoint unions, hence it gives a product on the group $B^c_G$. This turns $B^c_G$ into a ring. We call it crossed Burnside ring of $G$. The identity element of this ring is $[\bullet, u_\bullet]$, where $\bullet$ is a $G$-set of cardinality one, and the map $u_\bullet$ sends the unique element of $\bullet$ to the identity of $G$. A transitive crossed $G$-set is isomorphic to $(G/H, m_a)$ for a subgroup $H \leq G$ and a map $m_a$ from $G/H$ to $G^c$ by $m_a(gH) = g^a := gag^{-1}$ for an element $a \in C_G(H)$. Let $\mathcal{P}_G$ denote the set of pairs $(H, a)$ consisting of a subgroup $H$ of $G$ and an element $a \in C_G(H)$. The group $G$ acts by conjugation on $\mathcal{P}_G$, and we denote by $[\mathcal{P}_G]$ a set of representatives of $G$-orbits on $\mathcal{P}_G$. If $(H, a) \in \mathcal{P}_G$, we denote by $[H, a]_G$ or $[(G/H)_a]$ the isomorphism class of the crossed $G$-set $(G/H, m_a)$. It is well known that a set $\{[H, a]_G \mid (H, a) \in [\mathcal{P}_G]\}$ forms a basis of $B^c_k(G)$ ([OY01 (3.1.c)], [Bo03a Corollary 2.2.3]) over $k$. The ring has Burnside ring $B_k(G)$ of $G$ over $k$ with basis $\{[G/H] \mid H \in C(G)\}$, where
C(G) is a complete set of conjugacy classes of subgroups of G, as a subring (see Lemma 2.1). We denote by \( B(G) \) (or \( B^c(G) \)), if the base ring \( k = \mathbb{Z} \).

### 2.2 Some maps between \( B(G) \) and \( B^c(G) \)

We define a ring homomorphism \( \alpha_G^k : B_k^c(G) \to B_k(G) \) by
\[
[[G/U]] \to [G/U]
\]
and define a ring homomorphism \( \iota_G^k : B_k(G) \to B_k^c(G) \) by
\[
[G/U] \mapsto [[G/U]]_c.
\]
Since \( \alpha_G^k \circ \iota_G^k = \text{id}_{B_k(G)} \), the Burnside ring \( B(G) \) is identified with \( \text{Im} \iota_G^k \). Let
\[
\tilde{B}_k(G) := \prod_{H \in C(G)} k.
\]

There exists a ring monomorphism \( \phi^k : B_k(G) \to \tilde{B}_k(G) \) given by
\[
[G/U] \mapsto (\phi^k_H[(G/U)])_{H \in C(G)},
\]
where \( \phi^k_H([(G/U)]) = \text{inv}_H([(G/U)]) = \{ gU \in G/U \mid H \leq gU \} \). The ring homomorphism \( \varepsilon^k_H : kC_G(H) \to k \) with \( H \leq G \) given by
\[
\sum \ell_s x_s \mapsto \sum \ell_s
\]
is called the augmentation map of group algebra \( kC_G(H) \) of the centralizer \( C_G(H) \) of \( H \) in \( G \) over \( k \) (cf. [MS02, Definition 3.2.9]).

We define a ring homomorphism \( \tilde{\alpha}_G^c : \tilde{B}_k^c(G) \to \tilde{B}_k(G) \) where the subring
\[
\tilde{B}_k^c(G) = \left( \prod_{H \leq G} Z_kC_G(H) \right)^G
\]
consists of \( G \)-fixed elements of product ring \( \prod_{H \leq G} Z_kC_G(H) \) of the centers of \( kC_G(H) \), by
\[
(x_H)_{H \leq G} \mapsto (\varepsilon^k_H(x_H))_{H \in C(G)}
\]
and define a ring homomorphism \( \tilde{\iota}_G^c : \tilde{B}_k(G) \to \tilde{B}_k^c(G) \) by
\[
(y_H)_{H \in C(G)} \mapsto (y_H)_{H \leq G},
\]
where \( \tilde{y}_H = y_K \) for a conjugate \( K \in C(G) \) of \( H \) in \( G \). Obviously, \( \tilde{\alpha}_G^k \circ \tilde{\alpha}_G^k = \text{id}_{\tilde{B}_k(G)} \). For a subgroup \( H \leq G \) there is a ring homomorphism \( \varphi^k_H : \tilde{B}_k^c(G) \to \tilde{B}_k^c(G) \) defined by linearity by

\[
\varphi^k_H([D,s]) = \sum_{gD \in (G/D)^H} g \cdot s = \sum_{t \in G} \sharp \{ gD \in (G/D)^H \mid g \cdot s = t \} \cdot t.
\]

The Burnside homomorphism is defined by

\[
\varphi^k = (\varphi^k_H)_{H \in C(G)} : \tilde{B}_k^c(G) \to \tilde{B}_k^c(G).
\]

For simplicity, we will often use symbols \( \varphi, \phi, \alpha_G, \iota_G \) etc. for \( k = \mathbb{Z} \). We provide the following two lemmas.

**Lemma 2.1**

(i) The diagrams

\[
\begin{array}{ccc}
B^c_k(G) & \xrightarrow{\varphi^k} & \tilde{B}^c_k(G) \\
\alpha^k_G & \downarrow & \alpha^k_G \\
B^c_k(G) & \xrightarrow{\phi^k} & \tilde{B}^c_k(G)
\end{array}
\]

are commutative.

(ii) Let \( x \in \tilde{B}^c_k(G) \). If \( \varphi^k(x) = \iota^k_G(y) \) for some \( y \in \tilde{B}^c_k(G) \), then \( \iota^k_G \circ \alpha^k_G(x) = x \).

(Proof) The statement (i) is clear. We prove the statement (ii). Since \( \tilde{\alpha}^k_G \circ \iota^k_G = \text{id}_{\tilde{B}_k(G)} \), it follows from the statement (i) that

\[
\varphi^k \circ \iota^k_G \circ \alpha^k_G(x) = \tilde{\alpha}^k_G \circ \phi^k \circ \alpha^k_G(x) = \iota^k_G \circ \tilde{\alpha}^k_G \circ \varphi^k(x) = i^k_G \circ \alpha^k_G \circ \iota^k_G(y) = i^k_G(y) = \varphi^k(x).
\]

This shows that \( \iota^k_G \circ \alpha^k_G(x) = x \), completing the proof.  

**2.3 Primitive idempotents**

The primitive idempotents of the crossed Burnside algebra follows from a theorem of Bouc ([Bo03a] or [OY01]). The primitive idempotents of \( B_\mathbb{Q}(G) \) have been determined by Gluck ([Gl81]) and the third author ([Yo83]) independently. They are indexed by the conjugacy classes of subgroups of \( G \). We denote by \( \iota^k_G \in \tilde{B}_k(G) \) the primitive idempotent indexed by \( H \leq G \). The primitive idempotents of the Burnside ring \( B(G) \) follows from a theorem of Dress ([Dr69], or [Bo00 Corollary 3.3.6]). We denote by \( C^\infty(G) \) a complete
set of the conjugacy classes of perfect subgroups of $G$. We write $H =_G K$ to denote that a subgroup $H \leq G$ is $G$-conjugate to $K$. The set of elements

$$f_J^G = \sum_{H^\infty =_G J, H \in \mathcal{C}(G)} e_H^G,$$

where $H^\infty$ is the smallest normal subgroup of $H$ for which the quotient is soluble, for $J \in \mathcal{C}^\infty(G)$, is the set of primitive idempotents of $B(G)$ (Be91, Corollary 5.4.8 (Dress)).

The ring homomorphism $\iota_G : B(G) \to B^c(G)$ provides a decomposition of the identity of $B^c(G)$ as a sum of orthogonal idempotents $\iota_G(f_J^G)$, for $J \in \mathcal{C}^\infty(G)$. We will show that the idempotents $\iota_G(f_J^G)$'s are all the primitive idempotents of $B^c(G)$.

The next theorem is one of the main results of this paper and is applied to the proof of Theorem 4.5.

**Theorem 2.2** The set of elements $\iota_G(f_J^G)$, for $J \in \mathcal{C}^{(\infty)}(G)$, is the set of primitive idempotents of $B^c(G)$.

(Proof) Let $x$ be an idempotent of $B^c(G)$. According to [MS02, Corollary 7.2.4], $ZC_G(H)$ with $H \leq G$ contains only trivial idempotents, where $\phi(x) = i_G(y)$ for some $y \in \tilde{B}(G)$. This, combined with Lemma 2.1 (ii), shows that $\iota_G \circ \alpha_G(x) = x$. By this fact, we may identify $x$ with $\alpha_G(x) \in B(G)$. Since the map $\alpha_G : B^c(G) \to B(G)$ is a ring homomorphism, it follows that $\alpha_G(x)$ is an idempotent of $B(G)$. Consequently, the idempotents of $B^c(G)$ are those of $B(G)$. This completes the proof. ■

We may denote by $f_J^G$ a primitive idempotent of $B^c(G)$ indexed by $J \in \mathcal{C}^{\infty}(G)$ from Theorem 2.2 above. We denote by

$$\begin{array}{ccc}
B^c_Q(G) & \xrightarrow{\phi} & \tilde{B}^c_Q(G) \\
\alpha^G_Q & \downarrow & \phi^G_Q \\
B_Q(G) & \xrightarrow{\phi} & \tilde{B}_Q(G)
\end{array} \quad \begin{array}{ccc}
B^c_Q(G) & \xrightarrow{\phi} & \tilde{B}^c_Q(G) \\
\iota^G_Q & \downarrow & \iota^G_Q \\
B_Q(G) & \xrightarrow{\phi} & \tilde{B}_Q(G)
\end{array} \quad (2.2)
$$

the commutative diagrams (2.1) for $k = \mathbb{Q}$.

Since $\varphi_1$ ([OY01, (4.2)], [Bo03a, 2.3.1]) is a ring homomorphism from $B^c(G)$ to the center $ZZG$ of group ring $ZG$, as a matter of course we see that $\varphi_1(1_{B^c(G)}) = 1$. More precisely, we can obtain an element $x \neq 1_{B^c(G)}$ of $B^c(G)$, where $x$ gives the identity of $ZZG$ as the image of $\varphi_1^Q$. 

6
Corollary 2.3 Let $f^G_J$ be a primitive idempotent of $B^c(G)$ with $J \in C^\infty(G)$. Then
\[
\varphi_1^Q(f^G_J) = \begin{cases} 
1 & J = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

(Proof) Let $H \leq G$. Since
\[
\varphi_1^Q(f^G_J) = \varphi_1^Q(I^G_J(e^G_H)) = \sum_{H \in C(G)} \varphi_1^Q(I^G_J(e^G_H)) = \begin{cases} 
1 & J = 1, \\
0 & \text{otherwise}
\end{cases} \]
from (2.2), we obtain
\[
\varphi_1^Q(f^G_J) = \varphi_1^Q(I^G_J(e^G_H)) = \sum_{H \in C(G)} \varphi_1^Q(I^G_J(e^G_H)) = \begin{cases} 
1 & J = 1, \\
0 & \text{otherwise}
\end{cases}
\]

Let $\mathbb{K}$ be a field of characteristic 0. Suppose that $\mathbb{K}$ is big enough. The primitive idempotents of $B^c_\mathbb{K}(G)$ have been determined by [OY01] and [Bo03a]. They are indexed by the subgroup $H$ of $G$ and irreducible $\mathbb{K}$-character $\theta$ of $C_G(H)$. Let $e^G_H,\theta$ be a primitive idempotent of $B^c_\mathbb{K}(G)$. Then since $e^G_H,\theta$ is given by $e^G_H,\theta := \varphi^{-1}(e^H,\theta)$ where $e^H,\theta$ is a primitive idempotent of $\widetilde{B}^c_k(G)$ ([OY01 Theorem (5.5)]), we have following result which prepares for proving Theorem 4.7.

Lemma 2.4 Let $e^G_H,\theta$ be a primitive idempotent of $B^c_\mathbb{K}(G)$. Then
\[
\varphi_1^K(e^G_H,\theta) = \begin{cases} 
e^G_H & H = 1, \\
0 & \text{otherwise,}
\end{cases}
\]
where $e_\theta$ is a primitive idempotent of $Z\mathbb{K}C_G(H)$ (c.f. [NT88 Theorem 2.22]).

3 Idempotents of a $p$-local crossed Burnside ring

Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic 0, with residue field $k$ of characteristic $p > 0$, and field of fractions is big enough. Review some results from Bouc’s theory to obtain a decomposition of the crossed Burnside ring $B^c_\mathcal{O}(G)$ of $G$ over $\mathcal{O}$, we summarize the basic properties of Green functors and various results on its decomposition.
3.1 Green functors

A Mackey functor $M$ for $G$ with values in the category $\mathbf{k}\text{-mod}$ of $\mathbf{k}$-modules is a bivariant functor $M = (M_*, M^*)$ from the category of finite $G$-sets to $\mathbf{k}\text{-mod}$, with the following two properties:

1. Let $X$ and $Y$ be any finite $G$-sets, and let $i_X$ (resp. $i_Y$) denote the canonical injection from $X$ (resp. $Y$) into $X \sqcup Y$. Then the morphisms
   \[(M_*(i_X), M_*(i_Y)) : M(X) \oplus M(Y) \to M(X \sqcup Y),\]
   \[\left(M^*(i_X) \ M^*(i_Y)\right) : M(X \sqcup Y) \to M(X) \oplus M(Y)\]
   are mutually inverse isomorphisms.

2. Let
   \[
   \begin{array}{ccc}
   X & \xrightarrow{a} & Y \\
   b \downarrow & & \downarrow c \\
   Z & \xrightarrow{d} & W
   \end{array}
   \]
   be a pull-back diagram of finite $G$-sets. Then
   \[
   M_*(b) \circ M^*(a) = M^*(d) \circ M_*(c).
   \]

A Green functor $A$ for $G$ over $\mathbf{k}$ is a Mackey functor for $G$ over $\mathbf{k}$, endowed for any $G$-sets $X$ and $Y$ with $\mathbf{k}$-bilinear maps $A(X) \times A(Y) \to A(X \times Y)$ with the following properties:

1. If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of $G$-sets, then the squares
   \[
   \begin{array}{ccc}
   A(X) \times A(Y) & \xrightarrow{x} & A(X \times Y) \\
   A_*(f \times A_*(g)) \downarrow & & \downarrow A_*(f \times g) \\
   A(X') \times A(Y') & \xrightarrow{x} & A(X' \times Y')
   \end{array}
   \]
   \[
   \begin{array}{ccc}
   A(X) \times A(Y) & \xrightarrow{x} & A(X \times Y) \\
   A^*(f \times A^*(g)) \downarrow & & \downarrow A^*(f \times g) \\
   A(X') \times A(Y') & \xrightarrow{x} & A(X' \times Y')
   \end{array}
   \]
   are commutative.

2. If $X$, $Y$ and $Z$ are $G$-sets, then the square
   \[
   A(X) \times A(Y) \times A(Z) \xrightarrow{(x) \times \text{id}_{A(Z)}} A(X) \times A(Y \times Z) \\
   \xrightarrow{(x) \times (y)} A(X \times Y) \times A(Z) \xrightarrow{\times} A(X \times Y \times Z)
   \]
   is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.
If \( \bullet \) denotes the trivial \( G \)-set of cardinality 1, there exists an element \( \varepsilon_A \in A(\bullet) \), called the unit of \( A \), such that for any \( G \)-set \( X \) and for any \( a \in A(X) \)

\[
A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)
\]

denoting by \( p_X \) (resp. \( q_X \)) the projection from \( X \times \bullet \) (resp. \( \bullet \times X \)) to \( X \).

If \( X \) and \( Y \) are \( G \)-sets, if \( a \in A(X) \) and \( b \in A(Y) \), set

\[
a \times^p b = A_*(t)(b \times a) \in A(X \times Y),
\]

where \( t \) is a \( G \)-map defined by \( Y \times X \to X \times Y; (y, x) \mapsto (x, y) \). Define the center \( Z(A) \) of \( A \) by

\[
Z(A)(X) = \{ a \in A(X) \mid \forall Y, \forall b \in A(Y), a \times b = a \times^p b \}
\]

for a \( G \)-set \( X \) \([12.1] \). The functor \( Z(A) \) is a sub-Green functor of \( A \).

If \( e \in Z(A)(\bullet) \) is an idempotent, we denote by \( e \times A \) the subfunctor of \( A \) defined for a \( G \)-set \( X \) by

\[
(e \times A)(X) = e \times A(X) \subset A(X).
\]

Then \( e \times A \) is a sub-Green functor of \( A \), with \( e = e \times \varepsilon_A \in (e \times A)(\bullet) \) as unit.

We denote by \( W(H) \) the quotient group \( N_G(H)/H \) for a subgroup \( H \leq G \).

Let \( R \) be a ring in which every prime divisor of \( |G| \) is invertible, except for \( p \) which is not invertible. The primitive idempotents of the Burnside ring \( B_R(G) \) follows from a theorem of Dress \([Bd69], \text{or} \ [Bo00, \text{Corollary} 3.3.6] \). We write \( O^p(G) \) to denote that the smallest normal subgroup of \( G \) for which the quotient is \( p \)-group. A group \( J \) is \( p \)-perfect if \( O^p(J) = J \). We denote by \( C^p(G) \) a complete set of the conjugacy classes of \( p \)-perfect subgroups of \( G \). The set of elements

\[
f^G_J = \sum_{O^p(H)=GJ, H \in C(G)} \varepsilon^G_H,
\]

for \( J \in C^p(G) \), is the set of primitive idempotents of \( B_R(G) \) \([Be91, \text{Corollary} 5.4.8 (\text{Dress})] \).

For notations such as \( \text{Ind}_N^G \), \( \text{Inf}_N^N \), etc. we follows that of Bouc’s book \([Bo97] \).

**Theorem 3.1** \([Bo97, \text{Proposition} 12.1.11]\) Let \( R \) be a ring in which every prime divisor of \( |G| \) is invertible, except for \( p \) which is not invertible. Let
A be Green functor for $G$ over $R$. Then there are isomorphisms of Green functors

$$A \simeq \bigoplus_{J \in C^{p}(G)} f^{G}_{J} \times A \quad (3.3)$$

and

$$f^{G}_{J} \times A \simeq \text{Ind}^{G}_{N_{G}(J)} \text{Inf}^{N_{G}(H)}_{W(J)} \left( j^{W(J)}_{1} \times (\text{Res}^{G}_{N_{G}(J)} A) J \right). \quad (3.4)$$

### 3.2 Dress construction

We summarize the Dress construction of Green functors introduced by Bouc ([Bo03b]). See also [OY04].

Let $S$ be a finite $G$-set. If $M$ is a Mackey functor for $G$ over $/CZ$, then the Mackey functor $M_{S}$ is the bivariant functor defined on the finite $G$-set $Y$ by

$$M_{S}(Y) = M(Y \times S).$$

If $f : Y \rightarrow Z$ is a map of $G$-sets, then

$$(M_{S})_{*}(f) = M_{*}(f \times \text{id}_{S}), \quad (M_{S})^{*}(f) = M^{*}(f \times \text{id}_{S}).$$

Then $M_{S}$ is a Mackey functor for $G$ and $M_{S}(\bullet) \cong M(S)$.

A $G$-monoid is a monoid endowed with a left $G$-action by monoid automorphisms. A morphism of $G$-monoids is a $G$-equivariant monoid homomorphism. A crossed $G$-monoid is a pair $(S, \varphi)$, where $S$ is a $G$-monoid, and $\varphi : S \rightarrow G^{c}$ is a morphism of $G$-monoids.

**Proposition 3.2** [Bo03b] Let $(S, \varphi)$ be a crossed $G$-monoid. If $A$ is a Green functor for $G$ over $\mathbb{k}$, let $A_{S}$ denote the Mackey functor obtained by the Dress construction from the $G$-set $S$. If $X$ and $Y$ are finite $G$-sets, defined a product map $\times_{S} : A_{S}(X) \otimes \mathbb{k} A_{S}(Y) \rightarrow A_{S}(X \times Y)$ by

$$\forall a \in A_{S}(X), \forall b \in A_{S}(Y), a \otimes b \mapsto a \times_{S} b = A(\sigma)(a \times b),$$

where $\sigma : X \times S \times Y \times S \rightarrow X \times Y \times S$ sending $(x, s, y, s')$ to $(x, \varphi(s)y, ss')$. Moreover, denote by $\varepsilon_{A_{S}}$ the element $A_{S}(f)(\varepsilon_{A})$ of $A(\varepsilon_{A}) \cong A_{S}(\bullet)$, where $f$ is the map sending the unique element of $\bullet$ to the identity element of $S$. Then $A_{S}$ is a Green functor for $G$ over $\mathbb{k}$.

### 3.3 Decomposition of a crossed Burnside ring over $p$-local ring

Let $X$ be a $G$-set. We denote by $b(X)$ the Grothendieck group of the category of $G$-sets over $X$ : a $G$-set $(Y, \alpha)$ over $X$ is a pair consisting of a finite $G$-set
Let \((Y, \phi)\) be a \(G\)-set, except for \(p\) which is not invertible. Let \((Y, \phi)\) be a \(G\)-set over \(X\). If \(f : X \to X'\) is a morphism of \(G\)-sets, then we put \(b_*(((Y, \phi)) = (Y, f \circ \phi)\). If \(f : X' \to X\) is a morphism of \(G\)-sets, then we denote by \(b^*((Y, \phi))\) the pull-back \((Y', \phi')\) of \((Y, \phi)\) along \(f\), obtained by filling the cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{\alpha} & Y \\
\phi' \downarrow & & \downarrow \phi \\
X' & \xrightarrow{f} & X.
\end{array}
\]

If \(E = (U, \phi)\) (resp. \(F = (V, \psi)\)) is a \(G\)-set over \(X\) (resp. over \(Y\)), we denote by \(E \times F\) the \(G\)-set \((U \times V, \phi \times \psi)\) over \(X \times Y\). Then the product \(\times\) can be extended by linearity to a product from \(b(X) \otimes \mathbb{Z} b(Y)\) to \(b(X \times Y)\).

**Proposition 3.3** [Bo97] Proposition 2.4.3] With those notations above, \(b = (b_*, b^*)\) is a Green functor for \(G\) over \(\mathbb{Z}\).

We have a Green functor which gives a crossed Burnside ring \(B^c_k(G)\). Set \(\mathbb{k} b(X) = \mathbb{k} \otimes \mathbb{Z} b(X)\) for a \(G\)-set \(X\). The next proposition follows from Proposition 3.2.

**Proposition 3.4** [Bo93b] Theorem 5.1] Let \(\mathbb{k} b = (\mathbb{k} b_*, \mathbb{k} b^*)\) be the Burnside Green functor for \(G\) over \(\mathbb{k}\) and \(G^e := (G^c, \text{id}_{G^c})\) be the crossed \(G\)-monoid. Then \(\mathbb{k} b_{G^e}\) is a Green functor for \(G\) over \(\mathbb{k}\) and \(\mathbb{k} b_{G^e}(\bullet) \cong B^c_k(G)\).

We have a decomposition of a crossed Burnside ring \(B^c_{\mathcal{O}}(G)\) of \(G\) over \(\mathcal{O}\).

**Proposition 3.5** Let \(R\) be a ring in which every prime divisor of \(|G|\) is invertible, except for \(p\) which is not invertible. Let \(\{f^G_J \mid J \in \mathcal{O}^p(G)\}\) the set of primitive idempotents of \(B_{R^G}(G)\). Let \(R_{G^e}\) be the Green functor for \(G\) over \(R\). Then there is an isomorphism of Green functors

\[
R_{G^e} \cong \bigoplus_{J \in \mathcal{O}^p(G)} f^G_J \times R_{G^e} \quad (3.5)
\]

and

\[
f^G_J \times R_{G^e} \cong \text{Ind}_{N_G(J)}^{G}(\text{Inf}_{W(J)}^{N_G(H)}(f^W_J \times (\text{Res}_{N_G(J)}^{G} R_{G^e})^J)) \quad (3.6)
\]

In particular, there are ring isomorphisms

\[
B^c_{\mathcal{O}}(G) \cong \bigoplus_{J \in \mathcal{O}^p(G)} \iota^G_J(f^G_J)B^c_{\mathcal{O}}(G) \quad (3.7)
\]

and

\[
\iota^G_J(f^G_J)B^c_{\mathcal{O}}(G) \cong \iota^G_{W(J)}(f^W_J)B^c_{\mathcal{O}}(W(J)) \quad (3.8)
\]
(Proof) By Theorem 3.1 and Proposition 3.4, we have isomorphisms (3.5) and (3.6) of Green functors. By evaluation at the trivial $G$-set of those isomorphisms of Green functors, we have isomorphisms (3.7) and (3.8) of rings as with Corollary 5.7.6 of [Bo00]. ■

A Bouc’s $\mathcal{O}$-algebra

$A(G) := B^*_\mathcal{O}(G) \iota^\mathcal{O}_G(f^G_1)$

which contains all primitive idempotents $e \in B^*_\mathcal{O}(G)$ such that $\iota^\mathcal{O}_G(f^G_1)e = e$ is introduced by Bouc ([Bo03a, 3.2.3]). He has determined the primitive idempotents of $A(G)$ ([Bo03a, 3.2.11]). They are indexed by the $p$-blocks of $\mathbb{Z}_kG$. We denote by $i_G$ the primitive idempotent of $A(G)$ corresponding to a $p$-block $i \in \mathbb{Z}_kG$ for a group $G$. We have a decomposition

$B^*_\mathcal{O}(G) \cong \bigoplus_{J \in \mathcal{O}(G)} A(W(J))$

into the direct sum of ideals by (3.7) of Proposition 3.5. The set of elements $i_{W(J)} \in A(W(J))$, for $J \in \mathcal{O}(G)$ and a $p$-block $i \in \mathbb{Z}_kW(J)$, is the set of primitive idempotents of $B^*_\mathcal{O}(G)$. Moreover, we have an expression

$1 = \sum_{J \in \mathcal{O}(G), i \in \mathbb{Z}_kW(J) : \text{p-block}} i_{W(J)} \quad (3.9)$

as the sum of primitive idempotents of the crossed Burnside ring $B^*_\mathcal{O}(G)$ of $G$ over a discrete valuation ring $\mathcal{O}$. By the construction of $A(W(J))$ for $J \in \mathcal{O}(G)$, we have the following lemma.

**Lemma 3.6** Let $i_{W(J)}$ be a primitive idempotent of $B^*_\mathcal{O}(G)$, for $J \in \mathcal{O}(G)$ and $p$-block $i \in \mathbb{Z}_kW(J)$. Then

$\varphi_1^\mathcal{O}(i_{W(J)}) = \begin{cases} i & J = 1, \\ 0 & \text{otherwise.} \end{cases}$

### 4 Images of a motivic decomposition by pseudo-functor $\mathcal{P}_k$

Recall the pseudo-functor $\mathcal{P}_k$ from the $\mathbb{k}$-linear bicategory $\text{Mack}_k := (\mathbb{k}\text{Span}^rt)^\flat$ of Mackey 2-motives (see [BD21, Recollection 2.2]) to the $\mathbb{k}$-linear bicategory $\text{Mack}^\text{coh}_k := (\text{biperm}^rt)^\flat$ of cohomological Mackey 2-motives (see [BD21, Definition 4.18]). Balmer and Dell’Ambrogio showed the following theorem.
Theorem 4.1 [BD21, Theorem 5.3] For every finite group $G$, there is a well-defined surjective morphism of commutative rings $\rho_G : B^r_k(G) \to Z(kG)$ sending a basis element $[H,a]_G$ to $\sum_{x \in [G/H]} x a$. The pseudo-functor $P_{/CZ}$ maps the general Mackey 2-motive $\bigoplus_i (G_i, e_i)$ to the cohomological Mackey 2-motive $\bigoplus_i (G_i, \rho_G(e_i))$, where $(G, 0) \cong 0$ in both bicategories.

Remark 4.2 The ring homomorphism $\rho_G$ above is same as $\varphi_1$ in [OY01, (4.2)] and $z_1$ in [Bo03a, 2.3.1]. The map $\rho_G$ is not only a surjective ring homomorphism, but also essentially a special case of the homonymous one studied in [BD20, CH. 7.5].

Although not directly necessary for discussions in this paper, we would like to mention here the relation between the ring homomorphism $\rho_G$ and the Mackey algebra of $G$.

Thévenaz and Webb introduced the Mackey algebra $\mu_k(G)$ of $G$ over $k$ in [TW95]. We denote by $Y_k(G)$ the endomorphism algebra $\end{kG}(k\Omega_G)$ of permutation $kG$-module $k\Omega_G$ generated by the left $G$-set $\Omega_G = \coprod_{H \leq G} G/H$ ([Bo97, Proposition 12.3.2], [Ro15, Definition 2.9]). Thévenaz and Webb also introduced the cohomological Mackey algebra $\co\mu_k(G)$ of $G$ over $k$. They pointed out that $\co\mu_k(G)$ is isomorphic to $Y_k(G)$ and there exists a natural projection $p_k : \mu_k(G) \to Y_k(G)$ ([TW95, Section 16]). The ring homomorphism

$$\zeta_k : B^r_k(G) \to Z\mu_k(G); [L, a]_G \mapsto \sum_{U \leq G} \sum_{w \in [L/G/U]} t^U_{L \cap U} w^{-1} a w r^U_{L \cap U}$$

is introduced by Bouc ([Bo03a, Proposition 4.4.2]). Moreover, he showed that the existence of an isomorphism

$$\iota_k : Z_kG \to Z\co\mu_k(G), \quad (4.10)$$

the details of the mapping are due to Rognerud ([Ro15, Proposition 3.4]):

the ring isomorphism $\iota_k$ is defined by

$$Z_kG \ni z = \sum_{x \in G} \lambda_x x \mapsto \sum_{H \leq G} \sum_{g \in [H \setminus G/H]} \left( \sum_{x \in H} \lambda_g x \right) t^H_{H \cap H \cap H \cap H \cap H} \in Z\co\mu_k(G).$$
Proposition 4.3 The homomorphism $\zeta_k$ is surjective.

(Proof) The natural map $p_k : \mu_k(G) \to Y_k(G) \cong \text{co}\mu_k(G)$ induces a ring homomorphism
$$p'_k : Z\mu_k(G) \to Z\text{co}\mu_k(G).$$
Since the surjectivity of $\varphi^1_k$ and the commutative diagram

\[
\begin{array}{ccc}
B^c_k(G) & \xrightarrow{\zeta_k} & Z\mu_k(G) \\
\downarrow{\nu^1_k} & & \downarrow{p'_k} \\
Z_kG & \xrightarrow{i_k} & Z\text{co}\mu_k(G)
\end{array}
\] (4.11)

shows that $p'_k$ is surjective, we see that $\zeta_k$ is surjective from the commutativity of (4.11) again. ■

We recall the Burnside Green functor $\mathbb{k}b$ as used in Proposition 3.4. Bouc introduced that $\mathbb{k}b(\Omega G \times \Omega G)$ has a ring structure which is isomorphic to $\mu_k(G)$ ([Bo97, Proposition 4.5.1]). See also [Ro15, Remark 2.5].

Corollary 4.4 There exists a commutative diagram

\[
\begin{array}{ccc}
B^c_k(G) & \xrightarrow{\zeta_k} & Z\mu_k(G) \\
\downarrow{\varphi^1_k} & & \downarrow{p'_k} \\
Z_kG & \xrightarrow{i_k} & Z\text{co}\mu_k(G)
\end{array}
\] (4.12)

of algebras.

(Proof) The left square is commutative by (4.11) and the right square is also commutative by (2) of [Ro15, Theorem 2.12]. ■

We will return to the discussion of Mackey 2-motive. In case of $k = \mathbb{Z}$, we have a refinement of Theorem 1.1 by Balmer and Dell’Ambrogio.

Theorem 4.5 For every finite group $G$, the pseudo-functor $\mathcal{P}_Z$ maps the Mackey 2-motive

$$G \simeq (G, \overline{\mu}^G_1) \oplus (G, \overline{\mu}^G_{J_2}) \oplus \cdots \oplus (G, \overline{\mu}^G_{J_m})$$ (4.13)

to the cohomological Mackey 2-motive

$$G \simeq (G, \rho_G(\overline{\mu}^G_1)) \oplus (G, \rho_G(\overline{\mu}^G_{J_2})) \cdots \oplus (G, \rho_G(\overline{\mu}^G_{J_m}))$$ (4.14)

$$\simeq (G, 1) \oplus (G, 0) \oplus \cdots \oplus (G, 0)$$ (4.15)

$$\simeq (G, 1),$$ (4.16)

where $\{1, J_2, \ldots, J_m\} = C^\infty(G)$.
(Proof) By an argument of motivic decomposition in the proof of [BD20, 7.5.4] and Theorem 2.2 we have an equivalence (4.13). Then we have the rest of equivalences of cohomological Mackey 2-motive from [BD21, Theorem 5.8] and Corollary 2.3. The proof of the theorem is complete. ■

Example 4.6 (Alternating group $A_5$) Let $G$ be the alternating group $A_5$ of 5-letters. Since $C^\infty(G) = \{1, G\}$, Theorem 2.2 shows that $\{f_{G1}, f_{GG}\}$ is the set of primitive idempotents of $B^r(G)$. Moreover, by an example [OY01, 6.5 (F)], we explicitly determine those elements as follows.

\[
\begin{align*}
\overline{f_{G1}} & = [A_4, \varepsilon] + [D_{10}, \varepsilon] + [S_3, \varepsilon] - [C_3, \varepsilon] - 2[C_2, \varepsilon] + [1, \varepsilon], \\
\overline{f_{GG}} & = [A_5, \varepsilon] - [A_4, \varepsilon] - [D_{10}, \varepsilon] - [S_3, \varepsilon] + [C_3, \varepsilon] + 2[C_2, \varepsilon] - [1, \varepsilon].
\end{align*}
\]

It is easy to see that $\rho_G(\overline{f_{G1}}) = 1$ and $\rho_G(\overline{f_{GG}}) = 0$. Therefore the pseudo-functor $P_Z$ maps the Mackey 2-motive

\[
\mathcal{K} \approx (G, \overline{f_{G1}}) \oplus (G, \overline{f_{GG}})
\]

to the cohomological Mackey 2-motive

\[
\begin{align*}
\mathcal{K} & \approx (G, \rho_G(\overline{f_{G1}})) \oplus (G, \rho_G(\overline{f_{GG}})) \\
& \approx (G, 1) \oplus (G, 0) \\
& \approx (G, 1).
\end{align*}
\]

Let $\mathbb{K}$ be a field of characteristic 0 and suppose that $\mathbb{K}$ is big enough. We denote by $\text{Irr}(\mathbb{K}G)$ the set of all irreducible characters of $G$. Lemma 2.4 gives a refinement of [BD21, Theorem 5.3] for $\mathbb{K} = \mathbb{K}$.

**Theorem 4.7** For every finite group $G$, the pseudo-functor $P_\mathbb{K}$ maps the Mackey 2-motive

\[
\mathcal{K} \approx \bigoplus_{H \in \text{C}(G), \theta \in \text{Irr}(\mathbb{K}CG(H))} (G, e_{H, \theta})
\]

to the cohomological Mackey 2-motive

\[
\mathcal{K} \approx \bigoplus_{H \in \text{C}(G), \theta \in \text{Irr}(\mathbb{K}CG(H))} (G, \rho_G(e_{H, \theta}))
\]

\[
\approx \bigoplus_{\theta \in \text{Irr}(\mathbb{K}G)} (G, \rho_G(e_{1, \theta}))
\]

\[
\approx \bigoplus_{\theta \in \text{Irr}(\mathbb{K}G)} (G, e_\theta).
\]
Let \( \mathcal{O} \) be a complete discrete valuation ring of characteristic 0, with residue field \( k \) of characteristic \( p > 0 \), and field of fractions is big enough. Lemma 3.6 gives a refinement of [BD21, Theorem 5.3] for \( k = \mathcal{O} \).

**Theorem 4.8** For every finite group \( G \), the pseudo-functor \( \mathcal{P}_\mathcal{O} \) maps the Mackey 2-motive

\[
G \simeq \bigoplus_{J \in C^p(G), i \in Z k_{W}(J) : p\text{-block}} (G, i_{W(J)}) \quad (4.27)
\]

to the cohomological Mackey 2-motive

\[
G \simeq \bigoplus_{J \in C^p(G), i \in Z k_{W}(J) : p\text{-block}} (G, \rho_G(i_{W(J)})) \quad (4.28)
\]

\[
\simeq \bigoplus_{i \in Z k_{W}(1) : p\text{-block}} (G, \rho_G(i_{W(1)})) \quad (4.29)
\]

\[
\simeq \bigoplus_{i \in Z k_{G} : p\text{-block}} (G, i) \quad (4.30)
\]

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