We prove that an arbitrary function, which is holomorphic on some neighbourhood of 
$z = 0$ in $\mathbb{C}^N$ and vanishes at $z = 0$, whose values are bounded linear operators mapping 
one separable Hilbert space into another one, can be represented as the transfer function 
of some multiparametric discrete time-invariant conservative scattering linear system 
with a Krein space of its inner states.

\section{Introduction}

An arbitrary function, holomorphic on a neighbourhood of $z = 0$ in $\mathbb{C}$, whose values are 
bounded linear operators mapping one separable Hilbert space into another one, can be represented as the transfer function 
of some discrete time-invariant linear system. This fact was established by D.Z. Arov in [3] (see also [4, 7]). Later on, T.Ya. Azizov proved (in a different terminology) that an arbitrary holomorphic operator-valued function on a neighbourhood of $z = 0$ in $\mathbb{C}$ has a $J$-conservative scattering system realization (see [5]). For that, he constructed a $J$-conservative scattering system dilation of an arbitrary discrete time-invariant linear system. Then, using realization mentioned earlier and taking its $J$-conservative scattering system dilation, due to the fact that a transfer function remains the same under dilation of a system, he obtained a desired $J$-conservative realization.

Let us note that there exist other proofs of this $J$-conservative realization theorem (see, e.g., [2] and references there), however in this paper we shall follow, in the main, the same scheme as in Azizov’s proof, for the proof of the multivariable generalization of this theorem.

First, we introduce the notion of multiparametric discrete time-invariant $J$-conservative scattering linear system (or conservative scattering linear system with a Krein space of inner states) which generalizes the notion of multiparametric conservative scattering linear system introduced in [6]. Second, we prove that an arbitrary multiparametric linear system has a $J$-conservative scattering system dilation (the notion of dilation for multiparametric linear systems was introduced in [7]). Next, we use our generalization of the realization theorem mentioned in the very beginning of this paper, to several variables [10], and the fact that the transfer function of a multiparametric linear system remains the same under dilation [9], and prove the main result of this paper on the existence of a multiparametric $J$-conservative scattering system realization for an arbitrary operator-valued function which is holomorphic on some neighbourhood of $z = 0$ in $\mathbb{C}^N$ and vanishes at $z = 0$. 
The organization of this paper is as follows. In Section 1 we give some preliminaries, with exact formulations of the results mentioned above, on $J$-conservative scattering linear systems in the one-parametric discrete case, in the system-theoretical language convenient for the sequel. In Section 2 we introduce multiparametric $J$-conservative scattering linear systems (in the discrete case), and formulate our main theorems. Section 3 contains the proofs of these theorems.

1 Preliminaries on one-parametric $J$-conservative scattering linear systems

In our notation, a one-parametric discrete time-invariant linear system $\alpha$ is given by

$$\alpha : \begin{cases} x(t) = Ax(t-1) + Bu(t-1), \\ y(t) = Cx(t-1) + Du(t-1) \end{cases} \quad (t = 1, 2, \ldots),$$

(1.1)

where for each $t$ vectors $x(t), u(t), y(t)$ belong to separable Hilbert spaces $X, U, Y$, respectively (throughout this paper we consider only such type of spaces when nothing is said especially); $A : X \to X$, $B : U \to X$, $C : X \to Y$, $D : U \to Y$ are bounded linear operators.

Such a form of a system differs from the standard one by the unit shift in the argument of an output signal $y(\cdot)$, that leads, in fact, to the equivalent theory (for more details and motivation of such a notation of a system, see [8]). Thus, for example, the transfer function of a system $\alpha$ of the form (1.1) is given by

$$\theta_\alpha(z) = zD + zC(I_X - zA)^{-1}zB$$

(1.2)
in some neighbourhood of $z = 0$ in $\mathbb{C}$, i.e. differs from the standard one by multiplier $z$ only (here $I_X$ denotes the identity operator on $X$).

The first result mentioned in Section 0 can be formulated now as follows: an arbitrary $L(U, Y)$-valued function $\theta$ holomorphic on some neighbourhood $\Gamma$ of $z = 0$ in $\mathbb{C}$ and vanishing at $z = 0$ can be realized as the transfer function of some system $\alpha$ of the form (1.1), i.e., $\theta(z) = \theta_\alpha(z)$ in some neighbourhood (possibly, smaller than $\Gamma$) of $z = 0$ (here we denote by $L(U, Y)$ the Banach space of all bounded linear operators from a separable Hilbert space $U$ into a separable Hilbert space $Y$).

Let the operator $J \in L(X) := L(X, X)$ be given such that $J = J^* = J^{-1}$ (such a $J$ is said to be a canonical symmetry on $X$). Then $J$ determines on $X$ the new inner product $[x_1, x_2]_J := \langle Jx_1, x_2 \rangle$ (here $\langle \cdot, \cdot \rangle$ stands for a Hilbert space inner product on $X$) which is, in general, indefinite, and the space $X$ with this new inner product has the structure of a Krein space (for more information on Krein spaces see, e.g., [5]).

Let $\alpha = (1; A, B, C, D; X, U, Y)$ be a one-parametric linear system of the form (1.1), and $J \in L(X)$ be a canonical symmetry. Set $J_1 := J \oplus I_U \in L(X \oplus U)$, $J_2 := J \oplus I_Y \in L(X \oplus Y)$. We shall call $\alpha$ a one-parametric $J$-conservative scattering system if the system operator

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(X \oplus U, X \oplus Y)$$
is \((J_1, J_2)\)-unitary, i.e.

\[
G^* J_2 G = J_1, \quad G J_1 G^* = J_2.
\]

In the particular case when \(J = I_X\), a \(J\)-conservative scattering system is a conservative scattering one.

Let us note that one may consider a \(J\)-conservative scattering system \(\alpha = (1; A, B, C, D; X, U, Y)\) as a conservative scattering one, however with a Krein space of its inner states, i.e., the equations of energy balance for \(\alpha\) have the same form as for conservative scattering system with a Hilbert space of inner states, but with \(J\)-metric \([\cdot, \cdot]_J\) instead of Hilbert metric \([\cdot, \cdot]\) for states of a system \(\alpha\):

\[
[x(t), x(t)]_J - [x(t-1), x(t-1)]_J = \|u(t-1)\|^2 - \|y(t)\|^2 \quad (t = 1, 2, \ldots),
\]

and the analogous equation holds for states, inputs and outputs of the conjugate system \(\alpha^* := (1; A^*, C^*, B^*, D^*; X, Y, U)\).

Recall \([3]\) (see also \([4, 7]\)) that a system \(\tilde{\alpha} = (1; \tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{X}, U, Y)\) is called a dilation of a system \(\alpha = (1; A, B, C, D; X, U, Y)\) if \(\tilde{X} \supset X\), and there exist subspaces \(D\) and \(D_\ast\) in \(\tilde{X}\) such that

\[
\tilde{X} = D \oplus X \oplus D_\ast, \quad \tilde{A} D \subset D, \quad \tilde{C} D = \{0\}, \quad \tilde{A}^* D_\ast \subset D_\ast, \quad \tilde{B}^* D_\ast = \{0\},
\]

\[
A = P_X \tilde{A}|X, \quad B = P_X \tilde{B}, \quad C = \tilde{C}|X
\]

(here \(P_X\) stands for the orthogonal projector onto \(X\) in \(\tilde{X}\)). For that, the transfer functions of \(\alpha\) and \(\tilde{\alpha}\) coincide in some neighbourhood of \(z = 0\) in \(\mathbb{C}\).

Now Azizov’s result mentioned in Section \([4]\) can be formulated in the following way. An arbitrary system \(\alpha = (1; A, B, C, D; X, U, Y)\) has a dilation \(\tilde{\alpha} = (1; \tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{X}, U, Y)\) which is a one-parametric \(J\)-conservative scattering system for certain canonical symmetry \(J \in L(\tilde{X})\). As a corollary (see Section \([4]\)), an arbitrary \(L(U, Y)\)-valued function \(\theta\) holomorphic on some neighbourhood \(\Gamma\) of \(z = 0\) in \(\mathbb{C}\) and vanishing at \(z = 0\) can be realized as the transfer function of some system \(\alpha = (1; A, B, C, D; X, U, Y)\) of the form \((1.1)\), which is a one-parametric \(J\)-conservative scattering system for certain canonical symmetry \(J \in L(X)\), i.e., \(\theta(z) = \theta_\alpha(z)\) in some neighbourhood (possibly, smaller than \(\Gamma\)) of \(z = 0\).

## 2 Multiparametric \(J\)-conservative scattering linear systems

Let us recall some definitions from \([8]\) concerning multiparametric discrete time-invariant linear systems. Such a system \(\alpha\) is given by

\[
\alpha : \begin{cases} 
  x(t) = \sum_{k=1}^{N} (A_k x(t - e_k) + B_k u(t - e_k)), \\
  y(t) = \sum_{k=1}^{N} (C_k x(t - e_k) + D_k u(t - e_k)) 
\end{cases} \quad (t \in \mathbb{Z}^N, \ |t| > 0), \tag{2.1}
\]

where for \(t = (t_1, \ldots, t_N) \in \mathbb{Z}^N\) we set \(|t| := \sum_{k=1}^{N} t_k\), for all \(k \in \{1, \ldots, N\}\) we set \(e_k := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^N\) with 1 on the \(k\)-th place, and zeros on other places; for each
t vectors $x(t), u(t), y(t)$ belong to (separable Hilbert) spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y}$, respectively; for all $k \in \{1, \ldots, N\} A_k : \mathcal{X} \to \mathcal{X}, B_k : \mathcal{U} \to \mathcal{X}, C_k : \mathcal{X} \to \mathcal{Y}, D_k : \mathcal{U} \to \mathcal{Y}$ are bounded linear operators. We shall use the short notation $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ where $A, B, C, D$ mean $N$-tuples of operators $A_k, B_k, C_k, D_k$, respectively. The transfer function of a system $\alpha$ of the form (2.1) is given by

$$\theta_\alpha(z) = zD + zC(I_X - zA)^{-1}zB$$

(2.2)

in some neighbourhood of $z = 0$ in $\mathbb{C}^N$, where for $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and an $N$-tuple of operators $T = (T_1, \ldots, T_N)$ we use the notation $zT := \sum_{k=1}^N z_k T_k$. It is clear that system (2.1) is the generalization of system (1.1), and formula (2.2) is the generalization of formula (1.2) for transfer function, to the case of several variables.

Recall that a system $\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is called a dilation of a multiparametric linear system $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ if for each $z \in \mathbb{C}^N$ a system $\tilde{\alpha}_z := (1; z\tilde{A}, z\tilde{B}, z\tilde{C}, z\tilde{D}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a dilation of a one-parametric linear system $\alpha_z := (1; zA, zB, zC, zD; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, i.e., $\tilde{\mathcal{X}} \supset \mathcal{X}$, and there exist subspaces $D_z$ and $D_{*z}$ in $\tilde{\mathcal{X}}$ such that

$$\tilde{\mathcal{X}} = D_z \oplus \mathcal{X} \oplus D_{*z}, \quad z\tilde{A}D_z \subset D_z, \quad z\tilde{C}D_z = \{0\}, \quad (z\tilde{A})^* D_{*z} \subset D_{*z}, \quad (z\tilde{B})^* D_{*z} = \{0\}$$

$$zA = P_X(z\tilde{A})|X, \quad zB = P_X(z\tilde{B}), \quad zC = (z\tilde{C})|X.$$ 

For that, the transfer functions of $\alpha$ and $\tilde{\alpha}$ coincide in some neighbourhood of $z = 0$ in $\mathbb{C}^N$.

Let $T^N := \{\zeta \in \mathbb{C}^N : |\zeta_k| = 1, \ k = 1, \ldots, N\}$ be the $N$-dimensional unit torus.

**Definition 2.1** Let $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and a canonical symmetry $J \in L(\mathcal{X})$ be given. We shall call $\alpha$ a multiparametric $J$-conservative scattering linear system if for each $\zeta \in T^N \alpha_\zeta := (1; \zeta A, \zeta B, \zeta C, \zeta D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a one-parametric $J$-conservative scattering linear system, i.e. for $J_1 = J \oplus I_\mathcal{U} \in L(\mathcal{X} \oplus \mathcal{U}), \ J_2 = J \oplus I_\mathcal{Y} \in L(\mathcal{X} \oplus \mathcal{Y})$ one has

$$(\zeta G)^* J_2 (\zeta G) = J_1, \quad (\zeta G) J_1 (\zeta G)^* = J_2,$$

(2.3)

where $G = (G_1, \ldots, G_N)$ is the $N$-tuple of system operators

$$G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X} \oplus \mathcal{Y} \quad (k = 1, \ldots, N).$$

(2.4)

In the particular case when $J = I_\mathcal{X}$, this notion coincides with the notion of multiparametric conservative scattering linear system introduced in [8]. Let us remark here that another type of multiparametric conservative scattering linear systems for the discrete case was considered by J.A. Ball and T.T. Trent in [8].

By equating corresponding coefficients of trigonometric polynomials in the left and right parts of (2.3), one can easily see that $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a multiparametric $J$-conservative scattering linear system if and only if the following equalities hold:

$$\sum_{k=1}^N G_k^* J_2 G_k = J_1,$$

(2.5)
for any input multisequence

\[ G_k^*J_2G_l = 0 \quad (k \neq l), \quad (2.6) \]
\[ \sum_{k=1}^{N} G_k J_1 G_k^* = J_2, \quad (2.7) \]
\[ G_k J_1 G_l^* = 0 \quad (k \neq l). \quad (2.8) \]

This definition can be also formulated in terms of energy balance equations, i.e. the conservation of energy for a system \( \alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), and for its conjugate
system \( \alpha^* = (N; A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}) \), where for \( N \)-tuples \( T = (T_1, \ldots, T_N) \) of operators \( T_k \in L(H_1, H_2) \) we set \( T^* := (T_1^*, \ldots, T_N^*) \) with \( T_k^* \in L(H_2, H_1), \ k = 1, \ldots, N \), and for the "energy" of states of \( \alpha \) and \( \alpha^* \) use \( J \)-metric \( \langle \cdot, \cdot \rangle_J \) instead of Hilbert metric \( \langle \cdot, \cdot \rangle \). More precisely, the following proposition is valid.

**Proposition 2.2** \( \alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a multiparametric \( J \)-conservative scattering linear system with some canonical symmetry \( J \in L(\mathcal{X}) \) if and only if

(i) for any input multisequence \( \{u(t) : |t| \geq 0\} \) of \( \alpha \) satisfying the condition

\[ \forall n \in \mathbb{N} \quad \sum_{|t|=n} \|u(t)\|^2 < \infty, \]

and its initial states collection \( \{x(t) : |t| = 0\} \) satisfying the condition

\[ \sum_{|t|=0} \|x(t)\|^2 < \infty, \]

one has for any \( n \in \mathbb{N} : \sum_{|t|=n} \|x(t)\|^2 < \infty, \sum_{|t|=n} \|y(t)\|^2 < \infty \), hence for any \( n \in \mathbb{N} \cup \{0\} \) the series \( \sum_{|t|=n} \langle x(t), x(t) \rangle_J \) is absolutely convergent, and

\[ \sum_{|t|=n} \langle x(t), x(t) \rangle_J - \sum_{|t|=n-1} \langle x(t), x(t) \rangle_J = \sum_{|t|=n-1} \|u(t)\|^2 - \sum_{|t|=n} \|y(t)\|^2; \quad (2.9) \]

(ii) the statement analogous to (i) holds for the conjugate system \( \alpha^* \).

**Proof. Necessity.** Let the collections \( \{u(t) : |t| \geq 0\} \) and \( \{x(t) : |t| = 0\} \) of inputs and states of \( \alpha \) satisfy the assumptions of (i). Apply induction on \( n \). Suppose that for \( n = m - 1 \), where \( m \in \mathbb{N} \), we have \( \sum_{|t|=n} \|x(t)\|^2 < \infty \). Then, by virtue of (2.4),

\[
\sum_{|t|=m} \|x(t)\|^2 + \sum_{|t|=m} \|y(t)\|^2 = \sum_{|t|=m} \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\|^2 \\
= \sum_{|t|=m} \left\| \sum_{k=1}^{N} G_k \begin{pmatrix} x(t - e_k) \\ u(t - e_k) \end{pmatrix} \right\|^2 \\
\leq \max_{t \in \{1, \ldots, N\}} \|G_t\|^2 \cdot \sum_{|t|=m} \left( \sum_{k=1}^{N} \left\| \begin{pmatrix} x(t - e_k) \\ u(t - e_k) \end{pmatrix} \right\|^2 \right) \\
\leq \ N^2 \cdot \max_{t \in \{1, \ldots, N\}} \|G_t\|^2 \cdot \sum_{|t|=m} \sum_{k=1}^{N} \left\| \begin{pmatrix} x(t - e_k) \\ u(t - e_k) \end{pmatrix} \right\|^2
\]
This implies \( \sum_{|t|=n} \|x(t)\|^2 \lesssim \sum_{|t|=n} \|y(t)\|^2 \lesssim \sum_{|t|=n} \|u(t)\|^2 \) for any \( n \in \mathbb{N} \). Therefore, the latter holds for an arbitrary \( n \in \mathbb{N} \). Since for any \( x \in \mathcal{X} \) we have \( \|x\| \leq \|x\|^2 \) (see, e.g., [5]), for any \( n \in \mathbb{N} \) the series \( \sum_{|t|=n} [x(t), x(t)]_J \) converges absolutely. Now, by virtue of (2.1) and from (2.3), (2.4) we get for any \( n \in \mathbb{N} \)

\[
\sum_{|t|=n} [x(t), x(t)]_J + \sum_{|t|=n} \|y(t)\|^2 = \sum_{|t|=n} \left[ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right]_J = \sum_{|t|=n} \left\langle J_2 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\rangle
\]

\[
= \sum_{|t|=n} \left\langle J_2 \sum_{k=1}^N G_k \begin{pmatrix} x(t-e_k) \\ u(t-e_k) \end{pmatrix}, \sum_{j=1}^N G_j \begin{pmatrix} x(t-e_j) \\ u(t-e_j) \end{pmatrix} \right\rangle
\]

\[
= \sum_{|t|=n} \sum_{k=1}^N \sum_{j=1}^N \left\langle G_k^* J_2 G_k \begin{pmatrix} x(t-e_k) \\ u(t-e_k) \end{pmatrix}, \begin{pmatrix} x(t-e_j) \\ u(t-e_j) \end{pmatrix} \right\rangle
\]

\[
= \sum_{|t|=n} \sum_{k=1}^N \left\langle G_k^* J_2 G_k \begin{pmatrix} x(t-e_k) \\ u(t-e_k) \end{pmatrix}, \begin{pmatrix} x(t-e_k) \\ u(t-e_k) \end{pmatrix} \right\rangle
\]

\[
= \sum_{|t|=n-1} \left\langle \sum_{k=1}^N G_k^* J_2 G_k \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\rangle = \sum_{|t|=n-1} \left\langle J_1 \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\rangle
\]

\[
= \sum_{|t|=n-1} \left[ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right]_J = \sum_{|t|=n-1} [x(t), x(t)]_J + \sum_{|t|=n-1} \|u(t)\|^2,
\]

that is equivalent to (2.9), and we have proved the necessity of condition (i). The necessity of condition (ii) is established analogously, by rewriting (2.1) for \( \alpha^* \) and using (2.7) and (2.8).

**Sufficiency.** Let us set for arbitrary \( x_0 \in \mathcal{X} \), \( u_0 \in \mathcal{U} \)

\[
x(t) := \begin{cases} x_0 & \text{for } t = 0, \\ 0 & \text{anywhere else for } |t| = 0,
\end{cases}
\]

\[
u(t) := \begin{cases} u_0 & \text{for } t = 0, \\ 0 & \text{anywhere else for } |t| \geq 0.
\end{cases}
\]

Clearly, the collections \( \{u(t) : |t| \geq 0\} \) and \( \{x(t) : |t| = 0\} \) of inputs and states of \( \alpha \) satisfy the assumptions of (i). Then we can write down for them (2.3), with \( n = 1 \), as follows:

\[
\sum_{k=1}^N [x(e_k), x(e_k)]_J - [x_0, x_0]_J = \|u_0\|^2 - \sum_{k=1}^N \|y(e_k)\|^2,
\]
or equivalently,

\[
\begin{bmatrix}
(x_0 \\
u_0)
\end{bmatrix} = \sum_{k=1}^N \begin{bmatrix}
(x(e_k)) \\
y(e_k)
\end{bmatrix} = J_1 \Rightarrow \begin{bmatrix}
(x_0) \\
(y_0)
\end{bmatrix}.
\]

By using system equations (2.1), we obtain

\[
\begin{align*}
\langle J_1 \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \rangle &= \left\langle \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \right\rangle = \sum_{k=1}^N \left\langle \begin{pmatrix} x(e_k) \\ y(e_k) \end{pmatrix}, \begin{pmatrix} x(e_k) \\ y(e_k) \end{pmatrix} \right\rangle J_2 \\
&= \sum_{k=1}^N \left\langle J_2 \begin{pmatrix} x(e_k) \\ y(e_k) \end{pmatrix}, \begin{pmatrix} x(e_k) \\ y(e_k) \end{pmatrix} \right\rangle = \sum_{k=1}^N \left\langle J_2 G_k \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}, G_k \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \right\rangle \\
&= \left\langle \sum_{k=1}^N G_k^* J_2 G_k \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \right\rangle.
\end{align*}
\]

Since \(x_0 \in \mathcal{X}\), \(u_0 \in \mathcal{U}\) are arbitrary, and the operators \(J_1\) and \(\sum_{k=1}^N G_k^* J_2 G_k\) are bounded and selfadjoint, the latter implies (2.5). Analogously, (ii) implies (2.7).

Now, for arbitrary \(x_1, x_2 \in \mathcal{X}\), \(u_1, u_2 \in \mathcal{U}\), and \(k, j \in \{1, \ldots, N\}\) \((k \neq j)\) set

\[
x(t) := \begin{cases} x_1 & \text{for } t = e_k - e_j, \\
x_2 & \text{for } t = 0, \\
0 & \text{anywhere else for } |t| = 0,
\end{cases}
\]

\[
u(t) := \begin{cases} u_1 & \text{for } t = e_k - e_j, \\
u_2 & \text{for } t = 0, \\
0 & \text{anywhere else for } |t| \geq 0.
\end{cases}
\]

Clearly, the collections \(\{u(t) : |t| \geq 0\}\) and \(\{x(t) : |t| = 0\}\) of inputs and states of \(\alpha\) satisfy the assumptions of (i). Then we can write down for them (2.3), with \(n = 1\), as follows:

\[
\sum_{l=1}^N [x(e_k - e_j + e_l) x(e_k - e_j + e_l)]_J + \sum_{l: \, l \neq k} \left[ x(e_l), x(e_l) \right]_J - \left[ x_1, x_1 \right]_J - \left[ x_2, x_2 \right]_J
\]

\[
= \|u_1\|^2 + \|u_2\|^2 - \sum_{l=1}^N \|y(e_k - e_j + e_l)\|^2 - \sum_{l: \, l \neq k} \|y(e_l)\|^2,
\]

or equivalently,

\[
\sum_{l=1}^N \left\langle \begin{pmatrix} x(e_k - e_j + e_l) \\ y(e_k - e_j + e_l) \end{pmatrix}, \begin{pmatrix} x(e_k - e_j + e_l) \\ y(e_k - e_j + e_l) \end{pmatrix} \right\rangle J_2 + \sum_{l: \, l \neq k} \left[ \begin{pmatrix} x(e_l) \\ y(e_l) \end{pmatrix}, \begin{pmatrix} x(e_l) \\ y(e_l) \end{pmatrix} \right] J_2
\]

\[
= \left\langle \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} \right\rangle J_1 + \left\langle \begin{pmatrix} x_2 \\ u_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} \right\rangle J_1.
\]

By using system equations (2.1), we obtain

\[
\sum_{l: \, l \neq j} \left\langle J_2 G_l \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, G_l \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} \right\rangle
\]
In this section we will use the results from [1, 8, 9] on the Agler–Schur class $S_N(U, \mathcal{Y})$. This class consists of all $L(U, \mathcal{Y})$-valued functions

$$\theta(z) = \sum_{t \in \mathbb{Z}_+^N} z^t \hat{\theta}_t$$

that is equivalent to

$$\langle \sum_{l=1}^N G_l^* J_2 G_l \left( \frac{x_1}{u_1} \right), \left( \frac{x_1}{u_1} \right) \rangle + \langle \sum_{l=1}^N G_l^* J_2 G_l \left( \frac{x_2}{u_2} \right), \left( \frac{x_2}{u_2} \right) \rangle$$

By (2.3) established formerly, we obtain:

$$2 \text{Re} \langle G^*_j J_2 G_k \left( \frac{x_2}{u_2} \right), \left( \frac{x_1}{u_1} \right) \rangle = 0.$$ 

One can substitute $-i\bar{x}_2$ instead of $x_2$, and $-i\bar{u}_2$ instead of $u_2$, and obtain

$$2 \text{Im} \langle G^*_j J_2 G_k \left( \frac{\bar{x}_2}{\bar{u}_2} \right), \left( \frac{x_1}{u_1} \right) \rangle = 0.$$ 

Since $x_1, x_2, \bar{x}_2 \in \mathcal{X}, u_1, u_2, \bar{u}_2 \in \mathcal{U}$ can be taken arbitrary, (2.4) follows. Analogously, (ii) implies (2.8).

The proof is complete. □

Now let us formulate two main theorems of this paper.

**Theorem 2.3** An arbitrary system $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of the form (2.1) has a dilation $\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, \tilde{\mathcal{U}}, \tilde{\mathcal{Y}})$, which is a multiparametric $J$-conservative scattering system for certain canonical symmetry $J \in L(\tilde{\mathcal{X}})$.

**Theorem 2.4** An arbitrary $L(U, \mathcal{Y})$-valued function $\theta$ holomorphic on some neighbourhood $\Gamma$ of $z = 0$ in $\mathbb{C}^N$ and vanishing at $z = 0$ can be realized as the transfer function of some system $\hat{\alpha} = (N; \hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{\mathcal{X}}, \hat{\mathcal{U}}, \hat{\mathcal{Y}})$ of the form (2.1), which is a multiparametric $J$-conservative scattering system for certain canonical symmetry $J \in L(\hat{\mathcal{X}})$, i.e., $\theta(z) = \theta_\alpha(z)$ in some neighbourhood (possibly, smaller than $\Gamma$) of $z = 0$.

### 3 Proofs of the main results

In this section we will use the results from [1, 3, 4] on the Agler–Schur class $S_N(U, \mathcal{Y})$. This class consists of all $L(U, \mathcal{Y})$-valued functions

$$\theta(z) = \sum_{t \in \mathbb{Z}_+^N} z^t \hat{\theta}_t$$
(here $\mathbb{Z}_+^N := \{ t \in \mathbb{Z}^N : t_k \geq 0, \ k = 1, \ldots, N \}$, for $z \in \mathbb{C}^N$ and $t \in \mathbb{Z}_+^N$ as usually $z^t := \prod_{k=1}^N z_k^t$, and $\hat{\theta}_t$ are Taylor coefficients of $\theta$), which are holomorphic on the open unit polydisk $D^N := \{ z \in \mathbb{C}^N : |z_k| < 1, \ k = 1, \ldots, N \}$ and satisfy the following condition: for any separable Hilbert space $\mathcal{H}$, any $N$-tuple $T = (T_1, \ldots, T_N)$ of commuting contractions on $\mathcal{H}$, and any positive real $r < 1$ one has

$$\|\theta(rT)\| < 1,$$

where

$$\theta(rT) = \theta(rT_1, \ldots, rT_N) := \sum_{t \in \mathbb{Z}_+^N} r^{|t|}T^t \otimes \hat{\theta}_t \in L(\mathcal{H} \otimes \mathcal{U}, \mathcal{H} \otimes \mathcal{Y}),$$

$T^t := \prod_{k=1}^N T_{k}^{t_k}$, and the series in (3.1) converges in operator norm.

**Lemma 3.1** For an arbitrary multiparametric linear system $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ there exist separable Hilbert spaces $\mathcal{M}_k$, with canonical symmetries $J^{(k)} \in L(\mathcal{M}_k)$, and holomorphic $L(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k)$-valued functions $F_k$ on $\mathbb{D}^N (k = 1, \ldots, N)$ such that $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$I_{\mathcal{X} \oplus \mathcal{U}} - (\lambda G)^*(zG) = \sum_{k=1}^N (1 - \lambda_k z_k) F_k(\lambda)^* J^{(k)} F_k(z).$$

**Proof.** Set $L_G(z) := zG$, $\varepsilon := \sup_T \| \sum_{k=1}^N T_k \otimes G_k \|$ where this supremum is taken over all $N$-tuples of commuting contractions $T = (T_1, \ldots, T_N)$ on a common separable Hilbert space $\mathcal{H}$. If $\varepsilon \leq 1$ then $L_G \in S_N(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$, and by Theorem 2.6 of [1] the assertion of Lemma 3.1 follows with $J^{(k)} = I_{\mathcal{M}_k}$ ($k = 1, \ldots, N$).

Suppose that $\varepsilon > 1$. Then $\varepsilon^{-1} L_G \in S_N(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$. By Theorem 2.6 of [1], there exist separable Hilbert spaces $\mathcal{M}_k^+$ and holomorphic $L(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k^+)$-valued functions $H_k^+$ on $\mathbb{D}^N (k = 1, \ldots, N)$ such that $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$I_{\mathcal{X} \oplus \mathcal{U}} - (\varepsilon^{-1} \cdot \lambda G)^*(\varepsilon^{-1} \cdot zG) = \sum_{k=1}^N (1 - \lambda_k z_k) H_k^+ (\lambda)^* H_k^+(z).$$

Setting $F_k^+(z) := \varepsilon H_k^+(z)$ for $z \in \mathbb{D}^N, k = 1, \ldots, N$, we obtain $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$\varepsilon^2 I_{\mathcal{X} \oplus \mathcal{U}} - (\lambda G)^*(zG) = \sum_{k=1}^N (1 - \lambda_k z_k) F_k^+(\lambda)^* F_k^+(z).$$

(3.3)

Since $\varepsilon^{-1} I_{\mathcal{X} \oplus \mathcal{U}} \in S_N(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{U})$, again by Theorem 2.6 of [1], there exist separable Hilbert spaces $\mathcal{M}_k^-$ and holomorphic $L(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k^-)$-valued functions $H_k^-$ on $\mathbb{D}^N (k = 1, \ldots, N)$ such that $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$(1 - \varepsilon^{-2}) I_{\mathcal{X} \oplus \mathcal{U}} = \sum_{k=1}^N (1 - \lambda_k z_k) H_k^- (\lambda)^* H_k^-(z).$$
Setting $F_k^-(z) := \varepsilon H_k^-(z)$ for $z \in \mathbb{D}^N$, $k = 1, \ldots, N$, we obtain $\forall \lambda \in \mathbb{D}^N$, $\forall z \in \mathbb{D}^N$:

$$(\varepsilon^2 - 1)I_{X \in \mathcal{U}} = \sum_{k=1}^{N} (1 - \bar{\lambda}_k z_k) F_k^-(\lambda) F_k^-(z). \quad (3.4)$$

Set $\mathcal{M}_k := \mathcal{M}_k^+ \oplus \mathcal{M}_k^-$, and according to this orthogonal decomposition define $F_k : \mathbb{D}^N \to \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k)$ by

$$F_k(z) := \begin{pmatrix} F_k^+(z) \\ F_k^-(z) \end{pmatrix} \quad (z \in \mathbb{D}^N),$$

and $J^{(k)} := I_{\mathcal{M}_k^+} \oplus (-I_{\mathcal{M}_k^-}) \in \mathcal{L}(\mathcal{M}_k^+ \oplus \mathcal{M}_k^-) = \mathcal{L}(\mathcal{M}_k)$ for $k = 1, \ldots, N$. By subtracting (3.4) from (3.3) for each $\lambda \in \mathbb{D}^N$ and $z \in \mathbb{D}^N$, we obtain (3.2), that completes the proof. □

As a by-product of Lemma 3.1, we obtain the following result.

**Proposition 3.2** For an arbitrary $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function $\theta$ holomorphic on some neighbourhood $\Gamma$ of $z = 0$ in $\mathbb{C}^N$ and vanishing at $z = 0$ there exist separable Hilbert spaces $\mathcal{M}_k$, with canonical symmetries $J^{(k)} \in \mathcal{L}(\mathcal{M}_k)$, and holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{M}_k)$-valued functions $H_k$ on a neighbourhood $\Gamma_0 \subset \Gamma$ of $z = 0$ in $\mathbb{C}^N$ ($k = 1, \ldots, N$) such that $\forall \lambda \in \Gamma_0, \forall z \in \Gamma_0$

$$I_{\mathcal{U}} - \theta(\lambda)^\ast \theta(z) = \sum_{k=1}^{N} (1 - \bar{\lambda}_k z_k) H_k(\lambda)^\ast J^{(k)} H_k(z). \quad (3.5)$$

**Proof**. By Theorem 1 of [10], $\theta$ can be realized as the transfer function of some multiparametric linear system $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, i.e., $\theta(z) = \theta_\alpha(z)$ in some neighbourhood $\Omega \subset \Gamma$ of $z = 0$ in $\mathbb{C}^N$. If $\tilde{u}(z) = \sum_{t \in \mathbb{Z}_+^N} z^t u(t)$ is a $\mathcal{U}$-valued function holomorphic on some neighbourhood of $z = 0$ (here $u(\cdot)$ is some input multisequence of $\alpha$, which has the support in $\mathbb{Z}_+^N$) then (see [3]) one can write down the so-called $Z$-transform of $\alpha$:

$$\hat{\alpha} : \begin{cases} \hat{x}(z) = zA\hat{x}(z) + zB\hat{u}(z), \\
\hat{y}(z) = zC\hat{x}(z) + zD\hat{u}(z), \end{cases} \quad (3.6)$$

with holomorphic functions

$$\hat{x}(z) = (I_{\mathcal{X}} - zA)^{-1} zB\hat{u}(z), \quad (3.7)
\hat{y}(z) = \theta_\alpha(z)\hat{u}(z) \quad (3.8)$$

on some neighbourhood $\Omega_0$ of $z = 0$ in $\mathbb{C}^N$. We can consider without a loss of generality that $\Omega_0 \subset \Omega$.

Set $\Gamma_0 := \Omega_0 \cap \mathbb{D}^N$, and let $u_1, u_2 \in \mathcal{U}$ be arbitrary. Then by using Lemma 3.1 and equalities (3.6), (3.7), (3.8) twice, for $\tilde{u}(\cdot) \equiv u_1$ and for $\tilde{u}(\cdot) \equiv u_2$, we have for all $\lambda \in \Gamma_0, z \in \Gamma_0$:

$$\langle (I_{\mathcal{U}} - \theta(\lambda)^\ast \theta(z)) u_1, u_2 \rangle = \langle u_1, u_2 \rangle_{\mathcal{U}} - \langle \theta(z) u_1, \theta(\lambda) u_2 \rangle_{\mathcal{Y}}$$

$$= \left\langle \begin{pmatrix} \hat{x}_1(z) \\ u_1 \end{pmatrix}, \begin{pmatrix} \hat{x}_2(\lambda) \\ u_2 \end{pmatrix} \right\rangle_{\mathcal{X} \oplus \mathcal{U}} - \left\langle \begin{pmatrix} \hat{x}_1(z) \\ \hat{y}_1(z) \end{pmatrix}, \begin{pmatrix} \hat{x}_2(\lambda) \\ \hat{y}_2(\lambda) \end{pmatrix} \right\rangle_{\mathcal{X} \oplus \mathcal{Y}} \quad (\mathcal{X} \oplus \mathcal{Y})$$
\begin{align*}
&= \left\langle \left( \hat{x}_1(z) \atop u_1 \right), \left( \hat{x}_2(\lambda) \atop u_2 \right) \right\rangle_{\mathcal{X} \oplus \mathcal{U}} - \left\langle zG \left( \hat{x}_1(z) \atop u_1 \right), \lambda G \left( \hat{x}_2(\lambda) \atop u_2 \right) \right\rangle_{\mathcal{X} \oplus \mathcal{Y}} \\
&= \left\langle (I_\mathcal{X} \oplus \mathcal{U}) - (\lambda G)^* (zG) \left( \hat{x}_1(z) \atop u_1 \right), \left( \hat{x}_2(\lambda) \atop u_2 \right) \right\rangle_{\mathcal{X} \oplus \mathcal{U}} \\
&= \sum_{k=1}^{N} (1 - \hat{\lambda}_k z_k) F_k(\lambda)^* J^{(k)} F_k(z) \left( \hat{x}_1(z) \atop u_1 \right), \left( \hat{x}_2(\lambda) \atop u_2 \right) \right\rangle_{\mathcal{X} \oplus \mathcal{U}} \\
&= \sum_{k=1}^{N} (1 - \hat{\lambda}_k z_k) \left( (I_\mathcal{X} - \lambda \mathbf{A})^{-1} \lambda \mathbf{B} \right)^* F_k(\lambda)^* J^{(k)} F_k(z) \left( (I_\mathcal{X} - z \mathbf{A})^{-1} z \mathbf{B} \right) u_1, u_2 \right]\rangle_{\mathcal{U}}.
\end{align*}

Setting
\[ H_k(z) := F_k(z) \left( (I_\mathcal{X} - z \mathbf{A})^{-1} z \mathbf{B} \right) \] (\( z \in \Gamma_0 \)),
we obtain (3.5). □

**Proof of Theorem 2.3.** Let, as in the proof of Lemma 3.1, \( L_G(z) := zG, \varepsilon := \sup_T \| \sum_{k=1}^{N} T_k \otimes G_k \| \) where this supremum is taken over all \( N \)-tuples of commuting contractions \( T = (T_1, \ldots, T_N) \) on a common separable Hilbert space \( \mathcal{H} \), and \( G = (G_1, \ldots, G_N) \) be the \( N \)-tuple of operators (2.4) corresponding to a given multiparametric linear system \( \alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \).

If \( \varepsilon \leq 1 \) then \( L_G \in S_N(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}) \), and by Theorem 4.2 of [2], system \( \alpha \) has a conservative dilation \( \tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, \tilde{\mathcal{U}}, \tilde{\mathcal{Y}}) \), i.e. a \( J \)-conservative one with \( J = I_{\tilde{\mathcal{X}}} \).

Suppose now that \( \varepsilon > 1 \). Applying Lemma 3.1 to \( \alpha \), we have the existence of separable Hilbert spaces \( \mathcal{M}_k \) with canonical symmetries \( J^{(k)} \in L(\mathcal{M}_k) \), and holomorphic \( \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k) \)-valued functions \( F_k \) on \( \mathbb{D}^N (k = 1, \ldots, N) \) such that (3.2) holds. Let us define these spaces \( \mathcal{M}_k \), operators \( J^{(k)} \), and functions \( F_k \) (\( k = 1, \ldots, N \)) exactly as in the proof of Lemma 3.1, i.e., \( \mathcal{M}_k := \mathcal{M}_k^+ \oplus \mathcal{M}_k^- \), \( J^{(k)} := I_{\mathcal{M}_k} \oplus (-I_{\mathcal{M}_k}) \in L(\mathcal{M}_k^+ \oplus \mathcal{M}_k^-) = L(\mathcal{M}_k) \),

\[ F_k(z) := \begin{pmatrix} F_k^+(z) \\ F_k^-(z) \end{pmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}_k^+ \oplus \mathcal{M}_k^-) \] (\( z \in \mathbb{D}^N \)),

so that (3.3) and (3.4) hold. Set \( \mathcal{M}_k^\pm := \bigoplus_{k=1}^{N} \mathcal{M}_k^\pm, \mathcal{M} := \bigoplus_{k=1}^{N} \mathcal{M}_k = \mathcal{M}^+ \oplus \mathcal{M}^-, J_{\mathcal{M}} := \bigoplus_{k=1}^{N} J^{(k)} \in L(\bigoplus_{k=1}^{N} \mathcal{M}_k) = L(\mathcal{M}) \),

\[ F^\pm(z) := \begin{pmatrix} F_1^\pm(z) \\ \vdots \\ F_N^\pm(z) \end{pmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}^\pm), \quad F(z) := \begin{pmatrix} F_1(z) \\ \vdots \\ F_N(z) \end{pmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}) \] (\( z \in \mathbb{D}^N \)),

\[ P_k^\pm := P_{\mathcal{M}_k^\pm} \in L(\mathcal{M}^\pm), \quad P_k := P_{\mathcal{M}_k} \in L(\mathcal{M}) \] (\( k = 1, \ldots, N \)). Then \( \forall z \in \mathbb{C}^N \), \( \varepsilon^2 \mathbf{P} = \bigoplus_{k=1}^{N} z_k I_{\mathcal{M}_k^\pm} \in L(\mathcal{M}^\pm), \varepsilon^2 \mathbf{P} = \bigoplus_{k=1}^{N} z_k I_{\mathcal{M}_k} \in L(\mathcal{M}) \).

It follows from (3.3) that \( \forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N \)
\[ F^+(0) F^+(0) = F^+(0) F^+(z) = F^+(\lambda) F^+(0) = \varepsilon^2 I_{\mathcal{X} \oplus \mathcal{U}}. \] (3.9)
In particular, $F^+(0)$ is a bounded and boundedly invertible operator, and $F^+(0)(\mathcal{X} \oplus \mathcal{U})$ is a closed lineal, i.e. a subspace in $\mathcal{M}^+$. Analogously, it follows from (3.4) that $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$F^+(0)^*F^+(0) = F^-(0)^*F^-(0) = (z^2 - 1)I_{\mathcal{X} \oplus \mathcal{U}}. \quad (3.10)$$

In particular, $F^-(0)$ is a bounded and boundedly invertible operator, and $F^-(0)(\mathcal{X} \oplus \mathcal{U})$ is a closed lineal, i.e. a subspace in $\mathcal{M}^-$. It follows from (3.9) and (3.10) that $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$

$$(F^+(\lambda) - F^+(0))^*(F^+(\lambda) - F^+(0)) = F^+(\lambda)^*F^+(\lambda) - F^+(0)^*F^+(0). \quad (3.11)$$

Taking into account (3.9), rewrite (3.3) as

$$(\lambda P^+ F^+(\lambda))^*(z P^+ F^+(\lambda)) = \begin{pmatrix} F^+(\lambda) - F^+(0) \end{pmatrix}^* \begin{pmatrix} F^+(z) - F^+(0) \end{pmatrix}. \quad (3.12)$$

Taking into account (3.10), rewrite (3.4) as

$$(\lambda P^- F^-(\lambda))^*(z P^- F^-(\lambda)) = (F^-(\lambda) - F^-(0))^*(F^- z - F^-(0)). \quad (3.13)$$

Now, adding (3.13) to (3.12), we get $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$:

$$(\lambda P F(\lambda))^*(z P F(z)) = \begin{pmatrix} F(\lambda) - F(0) \end{pmatrix}^* \begin{pmatrix} F(z) - F(0) \end{pmatrix}. \quad (3.14)$$

and subtracting (3.13) from (3.12), we get $\forall \lambda \in \mathbb{D}^N, \forall z \in \mathbb{D}^N$:

$$(\lambda P F(\lambda))^*J(\mathcal{M})(z P F(z)) = \begin{pmatrix} F(\lambda) - F(0) \end{pmatrix}^* \begin{pmatrix} J_{\mathcal{M}_\lambda} \end{pmatrix} \begin{pmatrix} F(z) - F(0) \end{pmatrix}. \quad (3.15)$$

It follows from (3.14) that there exists unique unitary operator

$$U : \bigvee_{z \in \mathbb{D}^N} z P F(z)(\mathcal{X} \oplus \mathcal{U}) \rightarrow \bigvee_{z \in \mathbb{D}^N} \begin{pmatrix} F(z) - F(0) \end{pmatrix} (\mathcal{X} \oplus \mathcal{U}) \quad (3.16)$$

such that $\forall z \in \mathbb{D}^N$:

$$U(z P)F(z) = \begin{pmatrix} F(z) - F(0) \end{pmatrix}. \quad (3.17)$$

It follows from (3.9) and (3.10) that

$$F(0)^*J(\mathcal{M})F(0) = I_{\mathcal{X} \oplus \mathcal{U}}, \quad (3.18)$$
i.e. \( F(0) \in L(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}) \) is a \((I_{\mathcal{X} \oplus \mathcal{U}}, J_\mathcal{M})\)-semiunitary operator. Moreover, \( F(0)(\mathcal{X} \oplus \mathcal{U}) = F^+(0)(\mathcal{X} \oplus \mathcal{U}) \oplus F^-(0)(\mathcal{X} \oplus \mathcal{U}) \) is a closed lineal, i.e. a subspace in \( \mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^- \). In addition, from \( \text{(3.9)} \) and \( \text{(3.10)} \) one can see that \( \forall z \in \mathbb{D}^N \):

\[
F(0)^*(F(z) - F(0)) = 0,
\]

hence

\[
(F(z) - F(0))(\mathcal{X} \oplus \mathcal{U}) \subset \mathcal{M} \ominus F(0)(\mathcal{X} \oplus \mathcal{U}).
\]

Now let us show that the subspace \( \mathcal{K}_0 := \mathcal{M} \ominus F(0)(\mathcal{X} \oplus \mathcal{U}) \) in \( \mathcal{M} \) is a Krein space with respect to the metric \([\cdot, \cdot]_J_0\) induced by the canonical symmetry \( J_0 := P_{\mathcal{K}_0} J_\mathcal{M}|_{\mathcal{K}_0} \). By Theorem I.7.16 of \( \text{[5]} \), in order that \( J_0 \) is a canonical symmetry on \( \mathcal{K}_0 \), it is necessary and sufficient that any \( h \in \mathcal{M} \) has a \( J_\mathcal{M}\)-orthogonal projection onto \( \mathcal{K}_0 \), i.e. a vector \( h_0 \in \mathcal{K}_0 \) such that \( J_\mathcal{M}(h - h_0) \perp \mathcal{K}_0 \). For an arbitrary \( h \in \mathcal{M} \) set \( h_0 := h - J_\mathcal{M}F(0)F(0)^*h \). Since, due to \( \text{(3.13)} \),

\[
F(0)^*h_0 = F(0)^*h - F(0)^*J_\mathcal{M}F(0)F(0)^*h = 0,
\]

we get \( h_0 \in \mathcal{M} \ominus F(0)(\mathcal{X} \oplus \mathcal{U}) = \mathcal{K}_0 \). For an arbitrary \( g \in \mathcal{K}_0 \) we have:

\[
\langle J_\mathcal{M}(h - h_0), g \rangle = \langle J_\mathcal{M}^2F(0)F(0)^*h, g \rangle = \langle F(0)F(0)^*h, g \rangle = 0.
\]

Thus, \( h_0 \) is a desired \( J_\mathcal{M}\)-orthogonal projection of \( h \) onto \( \mathcal{K}_0 \), and we have proved that \( J_0 \) is a canonical symmetry on \( \mathcal{K}_0 \) (i.e., \( \mathcal{K}_0 \) is a Krein space with respect to the metric \([\cdot, \cdot]_{J_0}\)).

Further,

\[
\bigvee_{z \in \mathbb{D}^N} z\Psi F(z)(\mathcal{X} \oplus \mathcal{U}) \subset \mathcal{M},
\]

\[
\bigvee_{z \in \mathbb{D}^N} \left( \frac{F(z) - F(0)}{z \Phi} \right)(\mathcal{X} \oplus \mathcal{U}) \subset \mathcal{K}_0 \oplus \mathcal{X} \oplus \mathcal{Y}.
\]

Define on the space \( \mathcal{K}_I := \mathcal{K}_0 \oplus \mathcal{X} \oplus \mathcal{Y} \) the canonical symmetry \( J_I := J_0 \oplus I_{\mathcal{X} \oplus \mathcal{Y}} \in L(\mathcal{K}_I) \). Due to \( \text{(3.13)} \), the operator \( U \), defined by \( \text{(3.10)}, \text{(3.17)} \), can be considered as a bounded and boundedly invertible on its domain \((J_\mathcal{M}, J_I)\)-isometric operator from \( \mathcal{M} \) to \( \mathcal{K}_I \), whose domain and range are given by

\[
\mathcal{D}(U) = \bigvee_{z \in \mathbb{D}^N} z\Psi F(z)(\mathcal{X} \oplus \mathcal{U}), \quad \mathcal{R}(U) = \bigvee_{z \in \mathbb{D}^N} \left( \frac{F(z) - F(0)}{z \Phi} \right)(\mathcal{X} \oplus \mathcal{U}).
\]

By Theorem V.2.18 of \( \text{[5]} \), there exist a separable Hilbert space \( \mathcal{K}_{II} \), a canonical symmetry \( J_{II} \in L(\mathcal{K}_{II}) \), and a \((\hat{J}_1, \hat{J}_2)\)-unitary operator \( \hat{U} : \mathcal{K}_{II} \oplus \mathcal{M} \to \mathcal{K}_{II} \oplus \mathcal{K}_I \), with

\[
\hat{J}_1 := J_{II} \oplus J_\mathcal{M} \in L(\mathcal{K}_{II} \oplus \mathcal{M}), \quad \hat{J}_2 := J_{II} \oplus J_I \in L(\mathcal{K}_{II} \oplus \mathcal{K}_I)
\]

such that \( \hat{U} \) is an extension of \( U \), i.e. \( P_{\mathcal{K}(U)} \hat{U} | \mathcal{D}(U) = U \).

Since for any \( z \in \mathbb{D}^N \) we have

\[
(F(z) - F(0))(\mathcal{X} \oplus \mathcal{U}) \subset \mathcal{M} \ominus F(0)(\mathcal{X} \oplus \mathcal{U}) = \mathcal{K}_0,
\]
we get for any \( z \in \mathbb{D}^N \):
\[
F(z) = \begin{pmatrix} F(z) - F(0) \\ F(0) \end{pmatrix} \in L(\mathcal{X} \oplus \mathcal{U}, \mathcal{M}) = L(\mathcal{X} \oplus \mathcal{U}, \mathcal{K}_0 \oplus F(0)(\mathcal{X} \oplus \mathcal{U})).
\]

Set for all \( k \in \{1, \ldots, N\} \) \( \tilde{P}_k := \delta_{ik} I_{K_{II}} \oplus P_k \in L(K_{II} \oplus \mathcal{M}), \) where \( \delta_{ik} \) denotes the Kronecker symbol,
\[
\tilde{G}_k := \tilde{U} \tilde{P}_k (I_{K_{II} \oplus K_0} \oplus F(0)) \in L(K_{II} \oplus K_0 \oplus \mathcal{X} \oplus \mathcal{U}, K_{II} \oplus K_I)
\]
Clearly, \( I_{K_{II} \oplus K_0} \oplus F(0) \in L(K_{II} \oplus K_0 \oplus \mathcal{X} \oplus \mathcal{U}, K_{II} \oplus K_I) \) is a \((J_1, \tilde{J}_1)\)-unitary operator, where
\[
J_1 := J_{II} \oplus J_0 \oplus I_{\mathcal{X} \oplus \mathcal{U}} \in L(K_{II} \oplus K_0 \oplus \mathcal{X} \oplus \mathcal{U}),
\]
for any \( \zeta \in \mathbb{T}^N \) \( \zeta \tilde{P} \in L(K_{II} \oplus \mathcal{M}) \) is a \( \tilde{J}_1\)-unitary operator, and \( \tilde{U} \in L(K_{II} \oplus \mathcal{M}, K_{II} \oplus K_I) \) is a \((\tilde{J}_1, J_2)\)-unitary operator, where
\[
J_2 := \tilde{J}_2 = J_{II} \oplus J_0 \oplus I_{\mathcal{X} \oplus \mathcal{Y}} \in L(K_{II} \oplus K_I) = L(K_{II} \oplus K_0 \oplus \mathcal{X} \oplus \mathcal{Y}).
\]
Therefore, for any \( \zeta \in \mathbb{T}^N \) the operator \( \zeta \tilde{G} \) is \((J_1, J_2)\)-unitary.
Consider the following partitioning of \( \tilde{G}_k (k = 1, \ldots, N) \):
\[
\tilde{G}_k = \begin{pmatrix} \tilde{A}_k & \tilde{B}_k \\ \tilde{C}_k & \tilde{D}_k \end{pmatrix} : (K_{II} \oplus K_0) \oplus (\mathcal{X} \oplus \mathcal{U}) \to (K_{II} \oplus K_0) \oplus (\mathcal{X} \oplus \mathcal{Y}).
\]
Then \( \tilde{\alpha} := (\mathcal{N}; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; K_{II} \oplus K_0, \mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}) \) is a multiparametric \((J_{II} \oplus J_0)\)-conservative scattering system.
Since, by virtue of (3.17), \( \forall z \in \mathbb{D}^N \)
\[
z \tilde{G} \begin{pmatrix} F(z) - F(0) \\ I_{\mathcal{X} \oplus \mathcal{U}} \end{pmatrix} = \tilde{U}(z \tilde{P}) \begin{pmatrix} I_{K_{II} \oplus K_0} & 0 \\ 0 & F(0) \end{pmatrix} \begin{pmatrix} F(z) - F(0) \\ I_{\mathcal{X} \oplus \mathcal{U}} \end{pmatrix} = \tilde{U}(z \tilde{P})F(z) = U(z \tilde{P})F(z) = \begin{pmatrix} F(z) - F(0) \\ z \tilde{G} \end{pmatrix},
\]
we have for all \( z \in \mathbb{D}^N \):
\[
\begin{align*}
z \tilde{A}(F(z) - F(0)) + z \tilde{B} &= F(z) - F(0), \\
z \tilde{C}(F(z) - F(0)) + z \tilde{D} &= z \tilde{G}.
\end{align*}
\]
Therefore, in some neighbourhood \( \Omega \subset \mathbb{D}^N \) of \( z = 0 \) the resolvent \( (I_{K_{II} \oplus K_0} - z \tilde{A})^{-1} \) is well-defined and holomorphic, and we have in this neighbourhood:
\[
F(z) - F(0) = (I_{K_{II} \oplus K_0} - z \tilde{A})^{-1}z \tilde{B},
\]
\[
z \tilde{G} = z \tilde{D} + z \tilde{C}(I_{K_{II} \oplus K_0} - z \tilde{A})^{-1}z \tilde{B} = \theta_\alpha(z).
\]
Therefore, from (3.19) we obtain:
\[
\begin{align*}
\forall z \in \Omega_0 \quad z \tilde{D} &= z \tilde{G} \\
\forall z \in \Omega_0, \forall n \in \mathbb{N} \cup \{0\} \quad z \tilde{C}(z \tilde{A})^nz \tilde{B} &= 0.
\end{align*}
\]
As it was shown in the proof of Theorem 4.2 of [9] (see also Subsection V.3.1 of [5]), the latter means that the system \( \tilde{\alpha} := (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X} = K_{II} \oplus K_0 \oplus \mathcal{X}, \mathcal{U}, \mathcal{V}) \) which is determined by the system operators \( \tilde{G}_k \) coinciding with the system operators \( \tilde{\alpha} \) \((k = 1, \ldots, N)\), however with another block partitioning of these operators, is a dilation of \( \alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{V}) \). Defining a canonical symmetry of \( \tilde{X} \):

\[
J := J_{II} \oplus J_0 \oplus I_{\mathcal{X}} \in L(K_{II} \oplus K_0 \oplus \mathcal{X}) = L(\tilde{X}),
\]

we obtain that \( \tilde{\alpha} \) is a desired \( J \)-conservative scattering system dilation of a given system \( \alpha \).

The proof is complete. □

**Proof of Theorem 2.4.** Let \( \theta \) be a given \( L(\mathcal{U}, \mathcal{V}) \)-valued function holomorphic on a neighbourhood \( \Gamma \) of \( z = 0 \) in \( \mathbb{C}^N \) and vanishing at \( z = 0 \). Then, by Theorem 1 of [10], there exists a realization \( \alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{V}) \) of this function, i.e., \( \theta(z) = \theta_\alpha(z) \) in some neighbourhood \( \Gamma_0 \subset \Gamma \) of \( z = 0 \). By Theorem 2.3 of this paper, there exists a multi-parametric \( J \)-conservative scattering system \( \tilde{\alpha} := (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X}, \mathcal{U}, \mathcal{V}) \), with a canonical symmetry \( J \in L(\tilde{X}) \), which is a dilation of \( \alpha \). By Proposition 3.8 of [9], the transfer functions of a multiparametric linear system of the form (2.1) and of its dilation coincide in some neighbourhood of \( z = 0 \) in \( \mathbb{C}^N \). Hence, \( \theta_\tilde{\alpha}(z) = \theta_\alpha(z) = \theta(z) \) in some neighbourhood \( \Omega_0 \subset \Gamma_0 \) of \( z = 0 \). Thus, the system \( \dot{\alpha} := \tilde{\alpha} \) is a desired \( J \)-conservative scattering system realization of \( \theta \).

The proof is complete. □

**Acknowledgement.** I am thankful to D.Z. Arov for suggesting this problem.

**References**

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc.* Topics in Operator Theory: Ernst D. Hellinger Memorial Volume (L. de Branges, I. Gohberg, and J. Rovnyak, eds.), Oper. Theory and Appl., 48 (1990), 47–66, Birkhäuser-Verlag, Basel.

[2] D. Alpay, A. Dijksma, J. Rovnyak, H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces.* Oper. Theory: Adv. and Appl., 96 (1997), Birkhäuser-Verlag, Basel.

[3] D.Z. Arov, *Scattering theory with dissipation of energy.* Dokl. Akad. Nauk SSSR, 216 (1974), no. 4, 713–716 (Russian); English translation with addenda: Sov. Math. Dokl., 15 (1974), 848–854.

[4] D.Z. Arov, *Passive linear stationary dynamic systems.* Sibirsk. Math. Zh., 20 (1979), no. 2, 211–228 (Russian); English translation: Siberian Math. J., 20 (1979), 149–162.

[5] T.Ya. Azizov, I.S. Iohvidov, *Fundamentals of the Theory of Linear Operators in Spaces with an Indefinite Metric.* Nauka, Moscow, 1986 (Russian).
[6] J.A. Ball, T.T. Trent, *Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna–Pick interpolation in several variables*. J. Funct. Anal., 157 (1998), 1–61.

[7] H. Bart, I. Gohberg, M.A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*. Oper. Theory: Adv. and Appl., 1 (1979), Birkhäuser-Verlag, Basel.

[8] D.S. Kalyuzhniy, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case*. J. Operator Theory, 43 (2000), no. 2, 427–460.

[9] D.S. Kalyuzhniy, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case, II: Existence of conservative dilations*. Integral Equations Operator Theory, 36 (2000), no. 1, 107–120.

[10] D.S. Kalyuzhniy, *On the notions of dilation, controllability, observability, and minimality in the theory of dissipative scattering linear nD systems*. Proceedings CD of the Fourteenth International Symposium of Mathematical Theory of Networks and Systems (MTNS 2000), June 19–23, 2000, Perpignan (France), 6 pp.

Department of Higher Mathematics
Odessa State Academy of Civil Engineering and Architecture
Didrihson str. 4, Odessa, 65029, Ukraine

2000 Mathematics Subject Classification: 47A20, 47A48, 47B50, 93C35