POSITIVITY CONSTRAINTS ON ANOMALIES AND
SUPERSYMMETRY

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The relation between the trace and R-current anomalies in 4D supersymmetric theories implies that the $U(1)_R^2$, $U(1)_R$, and $U(1)_R^3$ anomalies which matched in studies of N=1 Seiberg duality satisfy positivity constraints. These constraints are tested in a large number of N=1 supersymmetric gauge theories in the non-Abelian Coulomb phase, and they are satisfied in all renormalizable models with unique anomaly-free R-current, including those with accidental symmetry. Most striking is the fact that the flow of the Euler anomaly coefficient, $a_{UV} - a_{IR}$, is always positive, as conjectured by Cardy.

1 2D theories

This talk is based on our joint papers with D. Anselmi, J. Erlich, D. Freedman and M. Grisaru.

Various positivity constraints on anomalies have been known for long time in the context of two-dimensional quantum field theories (QFT). In this section I briefly remind a few aspects of renormalization group (RG) flow in 2D theories.

Consider a 2D non-critical unitary QFT which flows from the ultraviolet (UV) to the infrared (IR) fixed point. Let $T = T_{zz}$ denote a component of the stress tensor, where $z = x_1 + ix_2$. For simplicity we assume that there is only one coupling constant $g$ in the theory. At a finite distance $x$ we have $\langle T(x)T(0) \rangle = c(g(1/x))/2z^4$. Here $g(1/x)$ is a running coupling constant at the scale $1/|x|$. The central function $c(g(1/x))$ interpolates between the UV value, $c_{UV}$ at $x \to 0$ and the IR value, $c_{IR}$ at $x \to \infty$. It follows from the unitarity that $c(g(1/x)) > 0$, and hence $c_{UV}$, $c_{IR} > 0$.

The same theory can be considered in an external gravitational background with a metric $g_{\mu\nu}$ and a curvature $R$. The presence of external gravitational field induces an external anomaly term in the operator equation for the trace anomaly $T^\mu_\mu = \tilde{c}(g(m))R/12 + \text{internal anomaly}$, where $m$ is an RG scale. Here the internal anomaly term denotes a part of the trace anomaly that depends on the quantum fields. The function $\tilde{c}(g(m))$ flows to $c_{UV}$ at $m \to \infty$ (the ultraviolet) and to $c_{IR}$ at $m \to 0$ (the infrared).

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Zamolodchikov’s c-theorem\cite{3} states that there exists a function \( c(g,x) = c(g(1/x)) + \text{correction terms} \) which i) is monotone decreasing toward the IR, ii) is stationary at the fixed points, iii) coincides with the central charges at the fixed points. The correction terms vanish at the critical points so that \( c_{UV} - c_{IR} > 0 \). Zamolodchikov’s c-function essentially counts the number of physical degrees of freedom. The number of degrees of freedom in the IR is less than in the UV, i.e. the RG flow is irreversible. It also follows from the Zamolodchikov c-theorem that the central charge at a critical point does not depend on the marginal deformations.

A natural question is how much of the above facts generalize to four dimensional theories. In 1988 Cardy\cite{4} conjectured that the Euler anomaly coefficient in the trace of stress tensor in the presence of external gravitational field \( T_\mu = a(g(m))\tilde{R}\tilde{R} + ... \) universally obeys \( a_{UV} - a_{IR} > 0 \). (We shall refer to the inequality \( a_{UV} - a_{IR} > 0 \) as the \( a \)-theorem.) This conjecture has been tested in the perturbation theory\cite{5}. Also, some evidence supporting this conjecture has been found\cite{6} in the SU\((N_c)\) series of SUSY gauge theories with \( N_f \) fundamental quark flavors where the conjectured electric-magnetic duality\cite{7} allows to reformulate a strongly coupled (confining) theory in terms of the weakly coupled magnetic one. The basic techniques for computing the flow of central charges when there is an interacting IR fixed point were developed\cite{1} and applied to the conformal phase for \( 3N_c/2 < N_f < 3N_c \). This approach does not use the electric-magnetic duality conjecture. It has been applied\cite{2} to a large set of supersymmetric theories where new non-perurbative evidence for Cardy’s conjecture has been found. The formulas of refs. [1,2] have been checked\cite{8} by explicit perturbative computation. Finally, Forte and Latorre presented a proof\cite{9} of Cardy’s conjecture. In this talk I focus on the approach of refs. [1,2].

2 Anomalies in 4D theories

The computation of chiral anomalies of the \( R \)-current and conserved flavor currents is one of the important tools used to determine the non-perturbative infrared behavior of the many supersymmetric gauge theories analyzed during the last few years. The anomaly coefficients are subject to rigorous positivity constraints by virtue of their relation to two-point functions of currents and stress tensors, and to other constraints conjectured in connection with possible four-dimensional analogues of the Zamolodchikov c-theorem. The two-point functions have been considered\cite{10} as central functions whose ultraviolet and infrared limits define central charges of super-conformal theories at the endpoints of the renormalization group flow. The positivity conditions are
reasonably well known from studies of the trace anomaly for field theories in external backgrounds. In supersymmetric theories the trace anomaly of the stress tensor and conservation anomaly of the $R$-current are closely related, which leads to positivity constraints on chiral anomalies.

The theoretical basis for the analysis of anomalies in supersymmetric theories comes from a combination of three fairly conventional ideas, namely

- The close relation between the trace anomaly of a four-dimensional field theory with external sources for flavor currents and stress tensor and the two point correlators $\langle J_\mu(x) J_\nu(y) \rangle$ and $\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle$ and their central charges.
- The close relation in a supersymmetric theory between the trace anomaly $\Theta = T_\mu^\mu$ and the anomalous divergence of the $R$-current $\partial_\mu R^\mu$.
- The fact that anomalies of the $R$-current can be calculated at an infrared superconformal fixed point using 't Hooft anomaly matching. This is the standard procedure, and one way to explain it is to use the all orders anomaly free $S$-current of Kogan, Shifman, and Vainshtein.

We consider a supersymmetric gauge theory containing chiral superfields $\Phi^i_\alpha$ in irreducible representations $R_i$ of the gauge group $G$. To simplify the discussion we assume that the superpotential $W = 0$, but the treatment can be generalized to include non-vanishing superpotential.

We consider a conserved current $J_\mu(x)$ for a non-anomalous flavor symmetry $F$ of the theory, and we add a source $B_\mu(x)$ for the current, effectively considering a new theory with an additional gauged U(1) symmetry but without kinetic terms for $B_\mu$. The source can be introduced as an external gauge superfield $B(x, \theta, \bar{\theta})$ so supersymmetry is preserved. We also couple the theory to an external supergravity background, contained in a superfield $H^a(x, \theta, \bar{\theta})$, but we discuss only the vierbein $e_\mu^a(x)$ and the component $V_\mu(x)$ which is the source for the $R^\mu$ current of the gauge theory.

The trace anomaly of the theory then contains several terms

$$\Theta = \frac{1}{2g^3} \tilde{\beta}(g)(F^a_{\mu\nu})^2 + \frac{1}{32\pi^2} \tilde{b}(g) B_\mu^2 + \frac{\tilde{c}(g)}{16\pi^2} (W_{\mu\nu\rho\sigma})^2 - \frac{a(g)}{16\pi^2} (\bar{R}_{\mu\nu\rho\sigma})^2 + \frac{\tilde{c}(g)}{6\pi^2} V_\mu^2,$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor, $\bar{R}_{\mu\nu\rho\sigma}$ is the dual of the curvature, and $B_{\mu\nu}$ and $V_{\mu\nu}$ are the field strengths of $B_\mu$ and $V_\mu$ respectively. All anomaly coefficients depend on the coupling $g(m)$ at renormalization scale $m$. The first term of (1) is the internal trace anomaly, where $\tilde{\beta}(g)$ is the numerator of the NSVZ beta function $\frac{3}{4} \tilde{\beta}(g) = -g^3 [3T(G) - \sum_i T(R_i)(1 - \gamma_i(g(\mu)))] / 16\pi^2$. Here $T(G)$ and $T(R_i)$ are the Dynkin indices of the adjoint representation of $G$ and the representation $R_i$ of the chiral superfield $\Phi^i_\alpha$, and $\gamma_i/2$ is the anomalous dimension of $\Phi^i_\alpha$.  

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The various external anomalies are contained in the three coefficients $\tilde{b}(g)$, $\tilde{c}(g)$ and $a(g)$. The free field (i.e. one-loop) values of $\tilde{c}$ and $a$ have been known for many years. In a free supersymmetric gauge theory with $N_v = \dim G$ gauge multiplets and $N_\chi$ chiral multiplets these values one has $c_{UV} = (3N_V + N_\chi)/24$ and $a_{UV} = (9N_V + N_\chi)/48$. If $T_i^j$ is the flavor matrix for the current $J_\mu(x)$ which is the $\bar{\theta}\theta$ component of the superfield $\Phi_{\alpha i}^a T_i^j \Phi_{\alpha j}^a$, and $\dim R_i$ is the dimension of the representation $R_i$, the free-field value of $\tilde{b}$ is $b_{UV} = \sum_{i,j} (\dim R_i) T_j^i T_i^j$. The subscript $UV$ indicates that the free-field values are reached in the ultraviolet limit of an asymptotically free theory.

The coefficients $\tilde{b}(g)$, $\tilde{c}(g)$ and $a(g)$ can be shown to be related to the correlation functions $\langle J_\mu(x) J_\nu(0) \rangle$, $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle$ and $\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) T_{\alpha\beta}(y) \rangle$ respectively. Various positivity constarints for $\tilde{b}(g)$, $\tilde{c}(g)$ and $a(g)$ follow from this relation.

In a supersymmetric theory in the external U(1) gauge and supergravity backgrounds discussed above, the divergence of the $R^\mu$ current and the trace of the stress tensor are components of a single superfield. Therefore the supersymmetry partner of the trace anomaly $\Theta$ of (1) is

$$\partial_\mu (\sqrt{g} R^\mu) = -\frac{1}{3g^3} \tilde{\beta}(g)(F \bar{F}) - \frac{\tilde{b}(g)}{48\pi^2} (B \bar{B}) + \frac{\tilde{c}(g) - a(g)}{24\pi^2} R \bar{R} + \frac{5a(g) - 3\tilde{c}(g)}{9\pi^2} (V \bar{V})$$

(2)

where $R$ and $\bar{R}$ on the right hand side are the curvature tensor and its dual. The ratio $-2/3$ between the first two terms of (1) and (2) is well known in global supersymmetry, but the detailed relation of the anomaly coefficients of the gravitational section was first derived in [1] by evaluating the appropriate components of the curved superspace anomaly equation

$$\bar{D}^\alpha J_{\alpha\alpha} = \frac{1}{24\pi^2} (\bar{c} W^2 - a \Xi)$$

(3)

where $J_{\alpha\alpha}$, $W^2$ and $\Xi$ are the supercurrent, super-Weyl, and super-Euler superfields respectively. This equation shows that all gravitational anomalies are described by the two functions $\tilde{c}(g)$ and $a(g)$, and this is also the reason why the coefficients of the third and fifth terms of (2) are related.

The infrared central charges $b_{IR}$, $c_{IR}$ and $a_{IR}$ are determined as the infrared limit (at $m \to 0$) of $\tilde{b}(g)$, $\tilde{c}(g)$ and $a(g)$.

These central charges are related to the conventional $U(1)_R F^2$, $U(1)_R$ and $U(1)_3^R$ anomalies. It is useful to derive this relation using the formalism of the all-orders anomaly-free $S^\mu$ current [4]. The external anomalies of this current can be clearly seen to agree in the infrared limit with those of the $R^\mu$ current.
which is in the same multiplet as the stress tensor, and thus part of the $N = 1$ superconformal algebra of the infrared fixed point theory.

Gaugino fields are denoted by $\lambda^a(x)$, $a = 1, \ldots, \dim G$, and scalar and fermionic components of $\Phi^\alpha_i(x)$ by $\phi^\alpha_i(x)$ and $\psi^\alpha_i(x)$ respectively. The canonical $R^\mu$ current (which is the partner of the stress tensor), and the matter Konishi currents $K^\mu_i$ for each representation are

$$R^\mu = \frac{1}{2} \bar{\lambda}^a \gamma^\mu \gamma^5 \lambda^a - \frac{1}{6} \sum_i \bar{\psi}_i^a \gamma^\mu \gamma^5 \psi_i^a + \frac{2}{3} \sum_i \bar{\phi}_i \, D^\mu \phi_i^a$$

$$K^\mu_i = \frac{1}{2} \sum_i \bar{\psi}_i^a \gamma^\mu \gamma^5 \psi_i^a + \sum_i \bar{\phi}_i \, D^\mu \phi_i^a. \quad (4)$$

Conservation of both currents is spoiled by a classical violation for any non-vanishing superpotential $W$ and a chiral anomaly. There is then a (flavor singlet) all-order conserved current ($\partial_\mu S^\mu = 0$)

$$S^\mu = R^\mu + \frac{1}{3} \sum_i (\gamma^i_\mu - \gamma_i) K^\mu_i \quad (5)$$

where $\gamma^i_\mu$ stand for the infrared values of anomalous dimensions of the chiral fields.

In physical correlators the infrared limit can be associated with large distance behavior. Therefore in the infrared (large distance) limit of correlators with an insertion of $R_\mu = S_\mu - \frac{1}{3} \sum_i (\gamma^i_\mu - \gamma_i) K^\mu_i$ the contribution of the Konishi current decreases faster than the contribution of the $S_\mu$ current which has no anomalous dimension. Thus the $S^\mu$ and $R^\mu$ operators and their correlators agree in the long distance limit, as is required at the superconformal $IR$ fixed point. In the free $UV$ limit $\gamma_i \to 0$ and the appropriate correlators can be computed perturbatively.

Because the current $S^\mu$ is exactly conserved without internal anomalies, ‘t Hooft anomaly matching \cite{7} can be applied to calculate the anomalies of its matrix elements with other exactly conserved currents, such as $\partial_\mu \langle S^\mu T^{\rho\sigma} T^{\lambda\tau} \rangle$. By using the fact that $S$ and $R$ coincide at long distances we have the chain of equalities

$$\partial \langle RTT \rangle_{IR} = \partial \langle STT \rangle_{IR} = \partial \langle STT \rangle_{UV} \quad (6)$$

where the last term simply includes the one loop graphs of the current $S$ and gives the $U(1)_R$ anomaly coefficient quoted in the literature. Similar arguments justify the conventional calculation of of $U(1)_R FF$ and $U(1)_{3R}$ anomalies.
3 Formulas for central charges

The previous discussion enables us to write simple formulae for the quantities $b_{IR}$, $c_{IR}$ and $a_{IR}$

\[ b_{IR} = 3 \sum_{ij} (\dim R_i)(1 - r_i)T^j_i T^i_j \]
\[ c_{IR} = \frac{1}{32} (4\dim G + \sum_i (\dim R_i)(1 - r_i)(5 - 9(1 - r_i)^2) \]
\[ a_{IR} = \frac{3}{32} (2\dim G + \sum_i (\dim R_i)(1 - r_i)(1 - 3(1 - r_i)^2)) \].

The corresponding UV quantities are obtained from (7) by replacing $r_i \to 2/3$.

In the presence of accidental symmetry the formulas for the infrared values of $a$, $b$ and $c$ have to be modified. The appearance of accidental symmetry is associated with decoupling in the infrared of a primary gauge invariant chiral composite field $M$. In this case the infrared $R$ current can be determined as an infrared limit of a linear combination $R^{IR}_\mu = S_\mu + A_\mu$, where $A_\mu = \lambda J^{(M)}_\mu$, and $J^{(M)}_\mu$ is the current for the components of the superfield $M$. The coefficient $\lambda$ is fixed by the condition that $R = 2/3$ for the field $M$. The corrected infrared values of the central charges are

\[ b_{IR} = b^{(0)}_{IR} + 3 T^j_i T^i_j \left( r_M - \frac{2}{3} \right) \]
\[ a_{IR} = a^{(0)}_{IR} + \frac{\dim M}{96} (2 - 3r_M)^2(5 - 3r_M) \]
\[ c_{IR} = c^{(0)}_{IR} + \frac{\dim M}{384} (2 - 3r_M)[(7 - 6r_M)^2 - 17] \].

Here we denoted by $b^{(0)}_{IR}$, $a^{(0)}_{IR}$ and $c^{(0)}_{IR}$ the expressions for $b$, $a$ and $c$ given by equations (7), and $r_M$ stands for the $S$ charge of the chiral field $M$, specifically the sum of the $S$ charges of its elementary constituents. Here $T^j_i$ stands for the flavor generator associated with $b$.

In what follows we mainly focus on the positivity constraint $a_{UV} - a_{IR} > 0$. As explained above, the gravitational effective action depends on the functions $a$ and $c$. It is natural to assume that a candidate $C$-function measuring the irreversibility of the RG flow may be a universal model independent linear combination $C = ua + vc$. With our formulas for the infrared values of $c$ and $a$ it is easy to show $a_{UV} - a_{IR} > 0$ is the only universal $a$-theorem candidate, so that $C = a$. 

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It is easy to check that in the absence of accidental symmetries in the models with a unique non-anomalous \( S_\mu \) current \( a_{UV} - a_{IR} \geq 0 \) if \( r_i \leq 5/3 \) for all \( \Phi^i \). Remarkably, \( r_i \leq 5/3 \) for all renormalizable models, so the \( a \)-theorem is always satisfied. Most of the positivity conditions, especially the \( a \)-theorem, can be verified essentially by inspection of the tables of \( R \)-charges presented in the literature on the various models. Actually, in many cases one can prove that \( r_i < 5/3 \) as a consequence of asymptotic freedom in absence of accidental symmetry (i.e. when all \( r_i \geq 1/3 \)).

An interesting situation is a flow between two non-trivial conformal theories. A conformal fixed point is characterized by the values of \( b, c \) and \( a \). One may consider the flow \( a_{UV} - a_{IR} \) for a theory which interpolates between two interacting conformal fixed points. Such an interpolation may be obtained by deforming a superconformal theory with a relevant operator which generates an RG flow driving the theory to another superconformal fixed point. (Such a deformation may be, for example due to higgsing the gauge group.) Since we know the conformal theories at both ultraviolet and infrared limits of this interpolating theory, the computation simply requires subtraction of the end-point central charges.

In refs. [1,2] the positivity constraints (including \( a_{UV} - a_{IR} > 0 \)) have been checked for most of renormalizable models with uniquely determined \( S_\mu \) current known to the authors by that time. The list of models includes

- Models with one type of irreducible representation: the \( SU(N_c) \) series, \( SO(N_c) \) series, \( Sp(2N_c) \) series, Pouliot \( Spin(7) \) model, Distler-Karch models with exceptional groups, \( Spin(7) \) Pouliot model with \( N_f \) spinors, \( Q_i, G_2 \) with \( N_f \) 7, \( E_7 \) Distler-Karch model: 4 fundamentals, \( Q_i, E_6 \) Distler-Karch model (I): 6 fundamentals, \( Q_i, E_6 \) Distler-Karch model (II): \( 3 \times (27 + \overline{27}) \) fundamentals, \( Q_i, F_4 \) Distler-Karch model: 5 fundamentals, \( Q_i, Q_i, F_4 \) Distler-Karch model: 4 fundamentals, \( Q_i, Spin(8) \) Distler-Karch model: \( 4 \times (8_v + 8_c + 8_s) \) fundamentals.

- Models with two types of irreps with uniquely determined \( S \) current. This set of models includes Kutasov-Schwimmer-type models for SU groups, for SO and Sp gauge groups, and the Pouliot \( Spin(7) \) model with \( N_c + 4 \) flavors in \( 8 \) and singlets.

It is worth emphasizing that our approach does not work for non-renormalizable models because it is based on an interpolation between the IR critical theory and the UV free theory.
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