Abstract

The problem of radio wave reflection from an optically thick plane monotonous layer of magnetized plasma is considered at present work. The plasma electron density irregularities are described by spatial spectrum of an arbitrary form. The small-angle scattering approximation in the invariant ray coordinates is suggested for analytical investigation of the radiation transfer equation. The approximated solution describing spatial-and-angular distribution of radiation reflected from a plasma layer has been obtained. The obtained solution has been investigated numerically for the case of the ionospheric radio wave propagation. Two effects are the consequence of multiple scattering: change of the reflected signal intensity and anomalous refraction.
The Radiation Transfer at a Layer of Magnetized Plasma With Random Irregularities

N.A.Zabotin A.G.Bronin
G.A.Zhbankov

Rostov State University
194, Stachki Ave., Rostov-on-Don, 344090, Russia
E-mail: zabotin@iphys.rnd.runnet.ru
PACS: 52.35.Hr, 42.25.Fx, 94.20.Bb

January 31, 2022
1 Introduction

Basic goal of the present work consists in derivation and analysis of the transfer equation solution describing spatial-and-angular distribution of radiation reflected from a plane stratified layer of magnetized plasma with random irregularities.

The radiation transfer equation (RTE) in a randomly irregular magnetized plasma was obtained in the work [1] under rather general initial assumptions. In particular, the medium average properties were assumed smoothly varying both in space and in time. In the work [2] the radiation energy balance (REB) equation describing radiation transfer in a plane stratified layer of stationary plasma with random irregularities has been deduced. The invariant ray coordinates, allowing one to take into account, by a natural way, refraction of waves and to represent the equation in the most simple form, were used there. In the work [3] it was shown that the REB equation is a particular case of the radiation transfer equation obtained in [1] and can be deduced from the latter by means of transition to the invariant ray coordinates. REB equation, thus, allows one to investigate influence of multiple scattering in a plane stratified plasma layer on the characteristics of radiation. In particular, it enables one to determine the spatial-and-angular distribution of radiation leaving the layer if the source directivity diagram and irregularity spatial spectrum are known. A few effects which require of wave amplitudes coherent summation for their description (for example, phenomenon of enhanced backscattering) are excluded from consideration. However, the multiple scattering effects are much stronger, as a rule. This is true, in particular, for the ionospheric radio propagation.

The numerical methods of the transfer equation solving developed in the theory of neutron transfer and in the atmospheric optics appear useless for the REB equation analysis. They are adapted, basically, to the solution of one-dimensional problems with isotropic scattering and plane incident wave. In a case of magnetized plasma the presence of regular refraction, aspect-sensitive character of scattering on anisometric irregularities and high dimension of the problem (the REB equation contains two angular and two spatial coordinates as independent variables) complicate construction of the effective numerical algorithm for its solving. In this situation it is expedient to solve the REB equation in two stages. The first stage consists of obtaining of the approximated analytical solution allowing one to carry out the
qualitative analysis of its properties and to reveal of its peculiarities. At the second stage the numerical estimation methods can be applied to the obtained analytical solution. This approach has been realized at our work.

We begin the present paper from a detailed exposition of the invariant ray coordinates concept. Then possibility to use of the small-angle scattering in the invariant coordinates approximation is discussed. Two modifications of the REB equation solution are obtained. The analysis of the obtained solutions (both analytical and numerical) concludes the paper.

2 Invariant ray coordinates and the radiation energy balance equation

It is convenient to display graphically the electromagnetic wave propagation in a plane-stratified plasma layer with the aid of the Poeverlein construction \[4, 5\]. We shall briefly describe it. Let the Cartesian system of coordinates has axis \(z\) perpendicular and the plane \(x0y\) parallel to the plasma layer. We shall name such coordinate system “vertical”. It is assumed that the vector of the external magnetic field \(\vec{H}\) is situated in the plane \(z0y\). Module of radius-vector of any point inside of the unit sphere with centrum in the coordinate origin corresponds to the value of refractive index \(n_i(v, \alpha)\), where \(i = 1\) relates to the extraordinary wave, \(i = \) relates to the ordinary one, \(v = \omega_e^2/\omega^2\), \(\omega_e^2\) is the plasma frequency, \(\omega^2\) is the frequency of a wave, \(\alpha\) is the angle between radius-vector and magnetic field \(\vec{H}\). The refractive index surface corresponding to a fixed value of \(v\) and to all possible directions of the radius-vector represents a rotation body about an axis parallel to vector \(\vec{H}\) (see fig. 1).

Convenience of the described construction (in fact, this is an example of coordinate system in space of wave vectors \(\vec{k}\)) is become evident when drawing the wave trajectory: it is represented by a straight line, parallel to the axis \(z\). This is a consequence of the generalized Snell law, which also requires of equality of the fall angle and exit angle onto/from a layer (\(\theta\)), and constantness of the wave vector azimuth angle (\(\varphi\)). Note, that the crossing point of a wave trajectory with the refractive index surface under given value of \(v\) determines current direction of the wave vector in a layer (it is antiparallel to a radius-vector) and current direction of the group speed vector.
(it coincides with the normal to the refractive index surface). The projection of a wave trajectory onto the plane $x0y$ is a point which radius-vector has module $\sin \theta$ and its angle with relation to axis $x$ equals to $\varphi$. Thus, the coordinates $\theta$ and $\varphi$ define completely the whole ray trajectory shape in a plane layer and outside of it and are, in this sense, invariant on this trajectory.

Radiation of an arbitrary point source of electromagnetic waves within the solid angle $\theta \div \theta + d\theta; \varphi \div \varphi + d\varphi$ corresponds to the energy flow in the $\vec{k}$-space inside of a cylindrical ray tube parallel to axis $z$ with cross section $\sin \theta d(\sin \theta) d\varphi = \sin \theta \cos \theta d\theta d\varphi$. In case of regular (without random irregularities) plasma layer this energy flow is conserved and completely determined by the source directivity diagram:

$$P(z, \vec{\rho}, \theta, \varphi) = P_0(\vec{\rho}, \theta, \varphi)$$

where $P$ is energy flow density in the direction determined by angles $\theta, \varphi$ through the point $\vec{\rho}$ on some base plane situated outside of the layer parallel to it (in the ionosphere case it is convenient to choose the Earth’s surface as the base plane), $z$ is distance from the base plane (height in the ionosphere case). We shall assume in the present paper that function $z(v)$ is monotonous in the region of wave propagation and reflection. If random irregularities are absent and source of radiation is point, variable $\vec{\rho}$ in (1) is superfluous, as the matter of fact, since unequivocal relation between it and angles of arrival of a ray $\theta, \varphi$ exists. When scattering is present the radiation energy redistributes over angular variables $\theta, \varphi$ and in space what is described by variable $\vec{\rho}$. The value of $P$ satisfies in this case to the equation of radiation energy balance [2, 3]:

$$\frac{d}{dz}P(z, \theta, \varphi, \vec{\rho}) = \int \{ -P(z, \vec{\rho}, \theta, \varphi) \sin \theta \cos \theta C^{-1}(z; \theta, \varphi) \cdot \sigma \left[ \alpha_0(\theta, \varphi), \beta_0(\theta, \varphi); \alpha(\theta', \varphi'), \beta(\theta', \varphi') \right] \sin \alpha(\theta', \varphi') \left| \frac{\partial(\alpha, \beta)}{\partial(\theta, \varphi)} \right| +$$

$$+ P(z, \vec{\rho} - \Phi(z; \theta', \varphi'; \theta, \varphi, \theta', \varphi') \sin \theta' \cos \theta' C^{-1}(z; \theta', \varphi'). \cdot \sigma \left[ \alpha_0(\theta', \varphi'), \beta_0(\theta', \varphi'); \alpha(\theta, \varphi), \beta(\theta, \varphi) \right] \sin \alpha(\theta, \varphi) \left| \frac{\partial(\alpha, \beta)}{\partial(\theta, \varphi)} \right| \} d\theta' d\varphi'$$

where $C(z; \theta, \varphi)$ is cosine of a ray trajectory inclination angle corresponding to the invariant angles $\theta$ and $\varphi; |\partial(\alpha, \beta)/\partial(\theta, \varphi)|$ is Jacobean of transition from angular coordinates $\theta, \varphi$ to the wave vector polar and azimuth angles $\alpha$ and $\beta$ in the “magnetic” coordinate system (which axis $0z$ is parallel to the magnetic field);

$$\sigma \left[ \alpha_0(\theta, \varphi), \beta_0(\theta, \varphi); \alpha(\theta', \varphi'), \beta(\theta', \varphi') \right] \equiv \sigma[\theta, \varphi; \theta', \varphi']$$
is scattering differential cross section describing intensity of the scattered wave with wave vector coordinates $\alpha, \beta$ in magnetic coordinate system (corresponding invariant coordinates are $\theta', \varphi'$) which arises at interaction of the wave with wave vector coordinates $\alpha_0, \beta_0$ (invariant coordinates $\theta, \varphi$) with irregularities. Vector function $\Phi(z; \theta', \varphi'; \theta, \varphi)$ represents the displacement of the point of arrival onto the base plane of a ray which has angular coordinates $\theta'$ and $\varphi'$ after scattering at level $z$ with relation to the point of arrival of an incident ray with angular coordinates $\theta, \varphi$. It is essential that in a plane-stratified medium the function $\Phi$ is determined only by smoothed layer structure $v(z)$ and does not depend on the scattering point horizontal coordinate and also on coordinate $\rho$ of the incident and scattered rays. Note also that ratio $\Phi(z; \theta, \varphi; \theta', \varphi') = -\Phi(z; \theta', \varphi'; \theta, \varphi)$ takes place.

It is possible to check up that equation (2) satisfies to the energy conservation law: when integrating over all possible for level $z$ values of $\theta, \varphi$ and all $\rho$ its right side turns into zero. It is natural since in absence of true absorption the energy inside the plasma layer does not collected.

Analyzing expression for the scattering differential cross section in a magnetized plasma (see, for example, [6]), it is easy to be convinced that the following symmetry ratio takes place:

$$\sigma[\theta, \varphi; \theta', \varphi'] n^2 \cos \vartheta' = \sigma[\theta', \varphi'; \theta, \varphi] n'^2 \cos \vartheta$$  \hfill (3)

where $\vartheta$ is angle between the wave vector and group speed vector, $n$ is refractive index. Using (3) the equation (2) can be presented as follows:

$$\frac{d}{dz} P(z, \rho, \theta, \varphi) = \int Q(z; \theta, \varphi; \theta', \varphi') \left\{ P(z, \rho - \Phi(z; \theta', \varphi'; \theta, \varphi), \theta', \varphi') - P(z, \rho, \theta, \varphi) \right\} d\theta' d\varphi'$$  \hfill (4)

where $Q(z; \theta, \varphi; \theta', \varphi') = \sigma(\theta, \varphi; \theta', \varphi') C^{-1}(z, \theta, \varphi) \sin \theta ' |d\Omega_k/d\Omega|$, and quantity $\tilde{Q}(z; \theta, \varphi; \theta', \varphi') \equiv Q(z; \theta, \varphi; \theta', \varphi') \sin \theta \cos \theta$ is symmetric with relation to rearrangement of shaded and not shaded variables. The equation REB in the form (4) has the most compact and perfect appearance. It is clear from physical reasons that (4) has to have a unique solution for given initial distribution $P_0(\rho, \theta, \varphi)$. The obtained equation can be directly used for numerical calculation of the signal strength spatial distribution in presence of scattering. However, as it was noted at introduction already, this approach leads to essential difficulties. Subsequent sections describe the method of construction of the energy balance equation approximated analytical solution.
3 Small-angle scattering approximation in the invariant ray coordinates

Let us consider the auxiliary equation of the following kind, which differs from (4) only by absence of the dash over variable $\omega$ marked by arrow:

$$\frac{d}{dz} P(z, \tilde{\rho}, \omega) = \int Q(z; \omega; \omega') \left\{ P(z, \tilde{\rho} + \tilde{\Phi}(z; \omega'; \omega), \omega) - P(z, \tilde{\rho}, \omega) \right\} d\omega'$$

(5)

where designation $\omega = \{\theta, \varphi\}$, $d\omega = d\theta d\varphi$ has been used for the sake of compactness. Equation (5) can be easily solved analytically by means of Fourier transformation over variable $\tilde{\rho}$. The solution has the following form:

$$\tilde{P}(z, \tilde{q}, \omega) = P_0(\tilde{q}, \omega) S(z, 0; \tilde{q}, \omega)$$

(6)

where $P_0(\tilde{q}, \omega)$ is the Fourier image of the energy flow density of radiation passing the layer in absence of scattering and the value of $S$ is defined by the expression

$$S(z_2, z_1, \tilde{q}, \omega) = \exp \left\{ \int_{z_1}^{z_2} dz' \int d\omega' Q(z'; \omega; \omega') \left[ e^{i\tilde{q}\tilde{\Phi}(z'; \omega; \omega')} - 1 \right] \right\}$$

(7)

One should note that integration over $z$ in this and subsequent formulae, in fact, corresponds to integration along the ray trajectory with parameters $\theta, \varphi$. The area of integration over $\omega'$ includes rays which reflection level $h_r(\omega') > z$.

Let us transform now equation (4) by the following way:

$$\frac{d}{dz} P(z, \tilde{\rho}, \omega) = \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \tilde{\rho} + \tilde{\Phi}(z; \omega'; \omega), \omega) - P(z, \tilde{\rho}, \omega) \right\} +$$

$$+ \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \tilde{\rho} + \tilde{\Phi}(z; \omega; \omega'), \omega') - P(z, \tilde{\rho} + \tilde{\Phi}(z; \omega; \omega'), \omega) \right\}$$

(8)

Its solution will be searched for in the form

$$P(z, \tilde{\rho}, \omega) = \tilde{P}(z, \tilde{\rho}, \omega) + X(z, \tilde{\rho}, \omega)$$

(9)

Thus, auxiliary equation (5) allows to present the solution of the equation (4) in the form (9). This is an exact representation while some approximated expressions for quantities $\tilde{P}$ and $X$ are not used.
By substituting (9) into the equation (4) one can obtain the following equation for the unknown function $X$:

$$
\frac{d}{dz}X(z, \vec{\rho}, \omega) = \int d\omega' Q(z; \omega; \omega') \left\{ \tilde{P} \left[ z, \tilde{\rho} + \tilde{\Phi}(z; \omega; \omega'), \omega' \right] - \tilde{P} \left[ z, \tilde{\rho} + \tilde{\Phi}(z; \omega; \omega'), \omega \right] + \int d\omega' Q(z; \omega; \omega') \left\{ X \left[ z, \tilde{\rho} + \tilde{\Phi}(z; \omega; \omega'), \omega' \right] - X(z, \tilde{\rho}, \omega) \right\} \right\}.
$$

(10)

We shall assume now that the most probable distinction of angles $\omega'$ and $\omega$ is small. The heuristic basis for this assumption is given by analysis of the Poeverlein construction (fig. 1). It is easy to be convinced examining the Poeverlein construction that scattering near the reflection level even for large angles in the wave vector space entails small changes of the invariant angles $\theta, \varphi$. This is especially true for irregularities strongly stretched along the magnetic field (in this case the edges of scattered waves wave vectors form circles shown in fig. 1 as patterns A and B). One should note also that the changes of invariant angles $\theta, \varphi$ are certainly small if scattering with small change of a wave vector direction takes place. This situation is typical for irregularity spectra, in which irregularities with scales more than sounding wave length dominate. Thus, the small-angle scattering approximation in the invariant coordinates has wider applicability area than common small-angle scattering approximation.

Scattering with small changes of $\theta, \varphi$ entails small value of $|\tilde{\Phi}|$. That follows directly both from sense of this quantity and from the fact what $|\tilde{\Phi}(z, \omega, \omega)| = 0$. Let us make use of that to carry out expansion of quantity $X$ at the right side of the equation (11) into the Taylor series with small quantities $\omega' - \omega$ and $|\tilde{\Phi}|$. Note that making similar expansion of function $P$ at the initial equation (4) would be incorrect since function $P$ may not to have property of continuity. For example, in case of a point source, $P_0$ is a combination of $\delta$-functions. As it will be shown later, the function $X$ is expressed through $P_0$ by means of repeated integration and, hence, differentiability condition fulfils much easier for it.

Leaving after expansion only small quantities of the first order, we obtain the following equation in partial derivatives:

$$
\frac{\partial}{\partial z}X(z, \tilde{\rho}, \omega) - A_\omega(z, \omega) \frac{\partial}{\partial \omega}X(z, \tilde{\rho}, \omega) + A_{\tilde{\rho}}(z, \omega) \frac{\partial}{\partial \tilde{\rho}}X(z, \tilde{\rho}, \omega) = f(z, \tilde{\rho}, \omega)
$$

(11)
where
\[ A_\omega(z, \omega) = \int d\omega' Q(z; \omega; \omega') (\omega' - \omega); \]
\[ A_\rho(z, \omega) = \int d\omega' Q(z; \omega; \omega') \Phi(z, \omega, \omega'); \]
\[ f \left( z, \vec{\rho}, \omega \right) = \int d\omega' Q(z; \omega; \omega'). \]
\[ \cdot \left\{ \widetilde{P} \left[ \left. z, \vec{\rho} + \Phi(z; \omega, \omega') \right|, \omega \right] - \widetilde{P} \left[ \left. z, \vec{\rho} + \Phi(z; \omega, \omega'), \omega \right| \right] \right\} \]

Here is the characteristic system for the equation (11):
\[ \frac{dX}{dz} = f \left( z, \vec{\rho}, \omega \right); \]
\[ \frac{d\vec{\rho}}{dz} = A_\rho(z, \omega); \]
\[ \frac{d\omega}{dz} = -A_\omega(z, \omega) \]
and initial conditions for it at \( z = 0 \):
\[ X = 0; \quad \vec{\rho}' = \vec{\rho}; \quad \omega = \omega_0. \]

It is necessary to emphasize the distinction between quantity \( \vec{\rho}' \), which is a function of \( z \), and invariant variable \( \vec{\rho} \).

Solving the characteristic system we obtain
\[ \omega = \omega(z, \omega_0), \quad \vec{\rho}' = \vec{\rho} - \int_{z_0}^{z} d\omega' A_{\vec{\rho}}[z', \omega(z', \omega_0)] \]
where \( z_0 \) is \( z \) coordinate of the base plane. It follows that
\[ X(z_0, \vec{\rho}, \omega) = \int_{z_0}^{\omega} dz' f \left\{ \left. \left. z', \vec{\rho} - \int_{z'}^{z_0} dz'' A_{\vec{\rho}} [z'', \omega(z'', \omega_0)], \omega(z', \omega_0) \right| \right\} \quad (12) \]

Generally, expression (12) gives the exact solution of the equation (11). However, since we are already within the framework of the invariant coordinate small-angle scattering approximation which assumes small value of \( A_\omega(z, \omega) \), it is possible to simplify the problem a little. Assuming \( A_\omega \approx 0 \) and omitting index 0 at invariant coordinates \( \omega \), we are coming to the following approximate representation for function \( X \):
\[ X(z_0, \vec{\rho}, \omega) = \int_{0}^{z_0} dz' \left\{ \widetilde{P} \left[ \left. z', \vec{\rho} + \Phi(z'; \omega; \omega') + \vec{D}(z_0, z', \omega'), \omega \right| \right] - \widetilde{P} \left[ \left. z', \vec{\rho} + \Phi(z'; \omega; \omega') + \vec{D}(z_0, z', \omega), \omega \right| \right] \right\} \quad (13) \]
where \[ \vec{D}(z_2, z_1, \omega) = \int_{z_1}^{z_2} dz' \int d\omega' Q(z'; \omega; \omega') \Phi(z', \omega, \omega'). \]

Thus, in the invariant coordinate small-angle scattering approximation the solution of the REB equation (4) is represented as a sum of two terms (see (9)), the first of which is

\[
\tilde{P}(z_0, \vec{\rho}, \omega) = \frac{1}{(2\pi)^2} \int d^2q \; P_0(\vec{q}, \omega) \cdot \exp \left\{ i\vec{q}\vec{\rho} + \int_{z_0}^{z_0} dz' \int d\omega' Q(z'; \omega; \omega') \left[ e^{i\vec{q}\Phi(z'; \omega; \omega')} - 1 \right] \right\}
\]

where \( \frac{1}{(2\pi)^2} \int d^2q \; P_0(\vec{q}, \omega) \exp(i\vec{q}\vec{\rho}) = P_0(\vec{\rho}, \omega), \) and the second one is given by expression (13).

The solution can be presented in the most simple form if one uses again the smallness of quantity \( \mid \vec{\rho} \mid \) and expands the second exponent in the formula (14) into a series. Leaving after expansion only small quantities of the first order, one can obtain:

\[
P(z_0, \vec{\rho}, \omega) \cong P_0 \left[ \vec{\rho} + \vec{D}(z_0, 0, \omega), \omega \right] + \int_{z_0}^{z_0} dz' \int d\omega' Q(z'; \omega; \omega') \cdot \tilde{P} \left[ z', \vec{\rho} + \vec{\Phi}(z'; \omega; \omega') + \vec{D}(z_0, z', \omega) + \vec{D}(z', 0, \omega'), \omega' \right] - \tilde{P} \left[ z', \vec{\rho} + \vec{\Phi}(z'; \omega; \omega') + \vec{D}(z_0, 0, \omega), \omega \right].
\]

The last operation is the more precise the faster value of \( P_0(\vec{q}, \omega) \) decreases under \( |\vec{q}| \to \infty. \) The solution of the radiation energy balance equation obtained in the present section in the form (9), (14), (13), or in the form (15), expresses the spatial-and-angular distribution of radiation intensity passing layer of plasma with scattering through the spatial-and-angular distribution of the incident radiation, that is, in essence, through the source directivity diagram.

4 Alternative approach in solving the REB equation

The REB equation solving method stated in the previous section is based on representation of quantity \( P(z, \vec{\rho}, \omega) \) as a sum of the singular part \( \tilde{P}(z, \vec{\rho}, \omega) \)
and the regular one \( X(z_0, \vec{p}, \omega) \). Regularity of the \( X(z_0, \vec{p}, \omega) \) has allowed one to use the expansion into the Taylor series over variables \( \vec{p} \) and \( \omega \) at the equation (10) right side and to transform the integral-differential equation (10) into the first order partial derivative differential equation (11).

However, the stated approach is not the only possible. The REB equation can be transformed right away using Fourier-representation of the function \( P(z, \vec{p}, \omega) \):

\[
P(z, \vec{p}, \omega) = \frac{1}{(2\pi)^2} \int d^2q P(z, \vec{q}, \omega) \exp(i\vec{q}\vec{p})
\]

(16)

Substitution of (16) into (4) gives the following equation for quantity \( P(z, \vec{q}, \omega) \):

\[
\frac{d}{dz} P(z, \vec{q}, \omega) = \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \vec{q}, \omega') \exp \left[ i\vec{q}\vec{\Phi}(z; \omega; \omega') \right] - P(z, \vec{q}, \omega) \right\}
\]

(17)

The quantity \( P(z, \vec{q}, \omega) \) is a differentiable function even when \( P(z, \vec{p}, \omega) \) has peculiarities. Therefore, in the invariant coordinate small-angle scattering approximation it is possible to use the following expansion:

\[
P(z, \vec{q}, \omega') \approx P(z, \vec{q}, \omega) + \frac{\partial P(z, \vec{q}, \omega)}{\partial \omega} (\omega' - \omega).
\]

(18)

Substituting (18) in (17) we obtain the partial derivative differential equation

\[
\frac{\partial}{\partial z} P(z, \vec{q}, \omega) - \tilde{A} (z, \vec{q}, \omega) \frac{\partial}{\partial \omega} P(z, \vec{q}, \omega) - P(z, \vec{q}, \omega) \tilde{S} (z, \vec{q}, \omega) = 0
\]

(19)

where

\[
\tilde{S} (z, \vec{q}, \omega) = \int d\omega' Q(z; \omega; \omega') \left[ e^{i\vec{q}\vec{\Phi}(z; \omega; \omega')} - 1 \right]
\]

\[
\tilde{A} (z, \vec{q}, \omega) = \int d\omega' Q(z; \omega; \omega') \exp \left[ i\vec{q}\vec{\Phi}(z; \omega; \omega') \right] (\omega' - \omega).
\]

The characteristic system

\[
\frac{d\omega}{dz} = - \tilde{A} (z, \vec{q}, \omega), \quad \frac{dP}{dz} = \tilde{S} (z, \vec{q}, \omega) P(z, \vec{q}, \omega)
\]

(20)

with initial conditions \( P = P_0(\vec{q}, \omega) \), \( \omega = \omega_0 \) at \( z = 0 \) has the following solution:

\[
P(z, \vec{p}, \omega) = \frac{1}{(2\pi)^2} \int d^2q P_0(\vec{q}, \omega_0) \exp \left\{ i\vec{q}\vec{p} + \int_0^z dz' \tilde{S} [z', \vec{q}, \omega(z', \vec{q}, \omega_0)] \right\}
\]

(21)
This solution of the REB equation turns into the expression (14) for $\tilde{P}$ when $\tilde{A} (z, \vec{q}, \omega) \rightarrow 0$. But the latter limit transition corresponds to the invariant coordinate small-angle scattering approximation used in the previous section under derivation of (13) and subsequent expressions. Let us note, however, that in (21), in contrast with (9), any additional terms do not appear. It allows one to assume that in used approximation the ratio

$$X(z, \vec{\rho}, \omega) \ll P(z, \vec{\rho}, \omega)$$

is fulfilled. Additional arguments to the benefit of this assumption will be presented in the following section.

5 Analysis of the solution of the REB equation

We shall show, first of all, that the obtained solution satisfies to the energy conservation law. For this purpose it is necessary to carry out integration of the left and right sides of (15) over $\omega$ and $\vec{\rho}$ multiplied them previously by $\sin \theta \cos \theta$. The area of integration over angles is defined by the condition that both wave $\omega$ and wave $\omega'$ achieve the same level $z$ (since at level $z$ their mutual scattering occurs). To satisfy this condition one should add factors $\Theta [h_r(\omega) - z]$ and $\Theta [h_r(\omega') - z]$ to the integrand expression, where $\Theta(x)$ is the Heaviside step function, $h_r(\omega)$ is the maximum height which can be reached by a ray with parameters $\theta, \varphi$. Now integration can be expanded over all possible values of angles, i.e., over interval $0 \div \pi/2$ for $\theta$ and over interval $0 \div 2\pi$ for $\varphi$. Then, (13) becomes

$$\int_0^z P(\omega) \sin \theta \cos d\omega = \int P_0(\omega) \sin \theta \cos d\omega +$$

$$+ \int_0^z d\omega' \int d\omega \Theta [h_r(\omega) - z'] \Theta [h_r(\omega') - z'] \tilde{Q} (z'; \omega, \omega') [P_0(\omega') - P_0(\omega)]$$

where $P(\omega), P_0(\omega)$ is a result of integration of $P(z_0, \vec{\rho}, \omega)$ and $P_0(\vec{\rho}, \omega)$ correspondingly over variable $\vec{\rho}$.

Due to antisymmetry of the integrand expression with relation to rearrangement of shaded and not shaded variables, the last term in (23) is equal
to zero. Thus, equation (23) reduces to

$$\int P(z_0, \tilde{\rho}, \omega) \sin \theta \cos \theta d\omega d^2\tilde{\rho} = \int P_0(\tilde{\rho}, \omega) \sin \theta \cos \theta d\omega d^2\rho$$

(24)

expressing the energy conservation law: the radiation energy full flow through the base plane remains constant regardless of scattering, as it should be in case of real (dissipative) absorption absence. It is not difficult to check that parity (24) is valid for the exact solution in the form (9) and also for the solution in the form (21).

With relation to the solution in the form (9) the carried out discussion discovers one curious peculiarity. It appears that the radiation energy complete flow through the base plane is determined by the first term (\(\tilde{P}\)). The second one (\(X\)) gives zero contribution to the complete energy flow.

Let us investigate in more detail the structure of quantity \(X(z, \tilde{\rho}, \omega)\) in the invariant coordinate small-angle scattering approximation. Proceeding to the Fourier-representation in the expression (13) produces

\[
X(z_0, \tilde{q}, \omega) = \int_0^{z_0} dz' \int d\omega' \mathcal{Q}(z', \omega, \omega') \left[ \tilde{P}(z', \tilde{q}, \omega') - \tilde{P}(z', \tilde{q}, \omega) \right] \cdot \exp \left\{ i\tilde{q} \left[ \tilde{\Phi}(z'; \omega, \omega') + \tilde{D}(z_0, 0, \omega) \right] \right\}
\]

Employing regularity of function \(\tilde{P}(z, \tilde{q}, \omega)\), the last expression can be written as

\[
X(z_0, \tilde{q}, \omega) = \int_0^{z_0} dz' \frac{\partial}{\partial \omega} \frac{\tilde{P}(z', \tilde{q}, \omega)}{\tilde{\omega}} \tilde{A}(z', \tilde{q}, \omega) \exp \left[ i\tilde{q} \tilde{D}(z_0, z', \omega) \right]
\]

where quantity \(\tilde{A}(z, \tilde{q}, \omega)\) is defined by (19). Thus, it becomes evident that limit transition \(\tilde{A}(z, \tilde{q}, \omega) \to 0\) entails also \(X(z_0, \tilde{\rho}, \omega) \to 0\). This property has been established in previous section with the aid of comparison of two variants of the REB equation solution. Now we can see that its presence is determined by structure of quantity \(X(z_0, \tilde{\rho}, \omega)\).

Results of the present section give the weighty ground to believe that the radiation spatial-and-angular distribution is determined basically by the first term in the solution (9). The second term represents the amendment to
the solution which can be neglected in the invariant coordinate small-angle
scattering approximation. This statement validity can be checked under
detailed research of properties of the obtained REB equation approximated
solutions by numerical methods.

6 Technique of numerical calculation of multiple scattering effects

We shall proceed from the obtained solution (15) where only the first term
has been retained. In the considered approximation the multiple scattering
results in deformation of the radiation field reflected by a plasma layer
without change of a kind of function describing intensity spatial and angular
distribution. If only single ray with parameters \( \theta_0 (\hat{\rho}), \varphi_0 (\hat{\rho}) \) comes into each
point \( \hat{\rho} \) onto the base plane when reflecting from a regular plasma layer (it
will be so, if the source is dot-like and the frequency is less than critical one
for this layer), for function \( P_0 \) it is possible to use expression of a kind

\[
P_0 (\tilde{\rho}, \theta, \varphi) = \tilde{P}_0 (\hat{\rho}) \delta [\cos \theta - \cos \theta_0 (\hat{\rho})] \delta [\varphi - \varphi_0 (\hat{\rho})]
\]  

(25)

where quantity \( \tilde{P}_0 (\hat{\rho}) \) has the meaning of energy flow at the point \( \hat{\rho} \)
in absence of scattering. Substituting of (25) into (15) and making integration
over angles one can obtain for the energy flow at the point \( \hat{\rho} \)

\[
\tilde{P} (\hat{\rho}) = \tilde{P}_0 \left[ \hat{\rho} + \vec{D} (\theta_1, \varphi_1) \right] \cdot \\
\cdot \left\{ 1 - \theta_{0\hat{\rho}} \left[ \hat{\rho} + \vec{D} (\theta_1, \varphi_1) \right] \vec{D}_\theta (\theta_1, \varphi_1) \right\} \\
\cdot \left\{ 1 - \varphi_{0\hat{\rho}} \left[ \hat{\rho} + \vec{D} (\theta_1, \varphi_1) \right] \vec{D}_\varphi (\theta_1, \varphi_1) \right\} - \\
- \theta_{0\hat{\rho}} \left[ \hat{\rho} + \vec{D} (\theta_1, \varphi_1) \right] \vec{D}_\theta (\theta_1, \varphi_1) \varphi_{0\hat{\rho}} \left[ \hat{\rho} + \vec{D} (\theta_1, \varphi_1) \right] \vec{D}_\varphi (\theta_1, \varphi_1) \right\}^{-1}
\]  

(26)

where \( \vec{D} (\theta, \varphi) \equiv \vec{D} (z_0, 0; \omega) \), subscripts \( \tilde{\rho}, \theta, \varphi \) mean derivatives with corresponding variables, and \( \theta_1, \varphi_1 \) represent new arrival angles of a ray.

Expression (26) uses explicit dependencies of arrival angles \( \theta_0, \varphi_0 \) of a
ray reflected from a regular plasma layer on position \( \hat{\rho} \) which are usually
unknown. As a rule, the dependence of coordinates \( \hat{\rho} \) on \( \theta_0 \) and \( \varphi_0 \) can be
expressed in an explicit form: \( \hat{\rho} = \tilde{\rho}_0 (\theta_0, \varphi_0) \), where \( \tilde{\rho}_0 (\theta, \varphi) \) - point of arrival
onto the base plane of a ray with invariant angles \( \theta \) and \( \varphi \). Expressing \( \theta_{0\hat{\rho}} \)
and $\varphi_0$, in (26) via $\frac{\partial \tilde{\rho}}{\partial \theta}$ and $\frac{\partial \tilde{\rho}}{\partial \varphi}$, we obtain new representation for quantity $\tilde{P} (\tilde{\rho})$:

$$\tilde{P} (\tilde{\rho}) = \tilde{P}_0 \left[ \tilde{\rho} + \tilde{D} (\theta_1, \varphi_1) \right] \left| \frac{\partial (\rho_0 x, \rho_0 y)}{\partial (\theta, \varphi)} \right| \left| \left( \frac{\partial (\rho_0 x - D_x, \rho_0 y - D_y)}{\partial (\theta, \varphi)} \right) \right|^{-1}$$

New arrival angles $\theta_1$ and $\varphi_1$ of a ray can be found by solving the algebraic equation system

$$\tilde{\rho} - \tilde{\rho}_0 (\theta_1, \varphi_1) + \tilde{D} (\theta_1, \varphi_1) = 0$$

(27)

According to (26), an observer being at point $\tilde{\rho}$ discovers two effects connected with scattering in a plasma layer: change of the wave arrival angles and change of the received signal intensity. The analytical results of the present work are valid in a common case of magnetized plasma. However, we shall consider below the case of an isotropic plasma linear layer, as it displays the task basic features at relative simplicity of numerical calculations. The simplification is mainly due to possibility to define the ray trajectories in the analytical form. For a numerical estimation of the effects described by the main term of the REB equation solution it is necessary, first of all, to concretize the kind of the $\Delta N/N$ irregularity spectrum. We shall choose the spectrum of the following kind:

$$F (\tilde{\kappa}) = C_A \left( 1 + \kappa^2_\perp / \kappa^2_{o\perp} \right)^{-\nu/2} \tilde{\delta} (\kappa_\parallel)$$

(29)

where $\kappa_\perp$ and $\kappa_\parallel$ are vector $\tilde{\kappa}$ (irregularity spatial harmonic) components orthogonal and parallel correspondingly to the magnetic field force lines, $\kappa_{o\perp} = 2\pi / l_{0\perp}$, $l_{0\perp}$ is the spectrum external scale, $\tilde{\delta} (x)$ is the Dirac delta-function,

$$C_A = \delta_R^2 \frac{\Gamma (\nu/2)}{2\pi \kappa_{o\perp}^2} \left[ \Gamma \left( \frac{\nu - 2}{2} \right) - 2 \left( \frac{R\kappa_{o\perp}}{2} \right)^{(\nu-2)/2} K_{(\nu-2)/2} \left( R\kappa_{o\perp} \right) \right]^{-1}$$

is a normalization constant, $\Gamma (x)$ is the Gamma-function, $K_\beta (z)$ is the Macdonald function [7]. The $\delta_R$ quantity characterizes the level of irregularities $\Delta N/N$. In mathematical theory of random fields it corresponds to the structural function of the irregularity field for the scale length $R$ [8]. The
considered model of a power-like small-scale irregularity spectrum is used in many areas of modern physics. Both at the ionospheric F region and at the tokamak plasma the irregularities are strongly stretched along the magnetic field. We preserve this feature of irregularities even when dealing with isotropic plasma model. Let the plasma layer with a linear dependence of electron density on depth \( (dz/dv = H) \) be located at distance \( h_0 \) from the radiation source. The scattering cross-section for isotropic plasma is \[ \sigma = \frac{\pi}{2} k_0^4 v^2 F(\Delta \kappa) \] (30)

where \( k_0 = \omega/c, F(\kappa) \) is the irregularity spatial spectrum and \( \Delta \kappa \) is the scattering vector. For the case of infinitely stretched irregularities the scattering vector longitudinal and transversal components is defined by the expressions

\[ \Delta \kappa_\parallel = k_0 n (\cos \alpha - \cos \alpha'); \]
\[ \Delta \kappa^2_\perp = k_0^2 n^2 \left[ (\sin \alpha \cos \beta - \sin \alpha' \cos \beta')^2 + (\sin \alpha \sin \beta - \sin \alpha' \sin \beta')^2 \right] \]

where notation of the first section is used and isotropic plasma refractive index is \( n = \sqrt{1 - v} \). In a linear layer of isotropic plasma the vector \( \Phi \) components can be presented as

\[ \Phi_x (v; \theta_k, \varphi_k, \theta'_k, \varphi'_k) = f \left( \theta'_k \right) \cos \varphi'_k - f \left( \theta_k \right) \cos \varphi_k; \]
\[ \Phi_y (v; \theta_k, \varphi_k, \theta'_k, \varphi'_k) = f \left( \theta'_k \right) \sin \varphi'_k - f \left( \theta_k \right) \sin \varphi_k; \]

where

\[ f \left( \theta_k \right) = 2H n \sin \theta_k \left( \sqrt{1 - n^2 \sin^2 \theta_k} + n \cos \theta_k \right) + h_0 \sin \theta_k / \sqrt{1 - n^2 \sin^2 \theta_k}, \]

angles \( \theta_k, \varphi_k, \theta'_k, \varphi'_k \) are current polar and azimuth angles of the wave vectors of incident and scattered waves correspondingly at the "vertical" coordinate system.

Because of \( \delta \)-function presence in the irregularity spectrum \( (29) \), the numerical estimation of expression for \( \tilde{D} (\theta, \varphi) \) is reduced to calculation of double integral over \( v \) and one of angles. It is convenient to proceed to integration
over angle $\beta'$ at the "magnetic" coordinate system. As result we obtain

\[
D_x = \frac{1}{2} \pi k_0^3 H^2 C_A \int_0^{\cos^2 \theta} dv' v'^2 \sqrt{\frac{1-v'}{\cos^2 \theta - v'}} \int_0^{2\pi} d\beta \{ \sin \alpha_1 \cos \beta \}
\]

\[\cdot \left[ 2 \left( v + W_1^2 (1+v) + W_1 \sqrt{1-v} \right) + \frac{h_0}{H} (v + W_1^2 (1-v))^{-1/2} \right] - \]

\[\cos \varphi \left[ 2 \frac{\sin \theta}{\sqrt{1-v}} \left( \cos \theta \pm \sqrt{\cos^2 \theta - v} \right) + \frac{h_0}{H} \frac{\sin \theta}{\sqrt{1-v} \cos \theta} \right] \cdot \left[ 1 + 4k_0^2 \rho_{01}^{-2} (1-v) \sin^2 \alpha_1 \sin^2 \left( \frac{\beta - \beta_1}{2} \right) \right]^{-v/2} +
\]

\[\text{+ similar term in which } \alpha_1, \beta_1 \text{ have been replaced by } \alpha_2, \beta_2 \}
\]

where $W_1 = \cos \alpha_1 \cos \gamma - \sin \alpha_1 \sin \beta \sin \gamma$, angles $\alpha_1, \beta_1$ and $\alpha_2, \beta_2$ are polar and azimuth angles of the incident wave wave vector at the "magnetic" coordinate system for the trajectory ascending and descending branches correspondingly. These are connected to the invariant angles by the relations

\[
\cos \alpha_{1,2} = \pm \cos \gamma \sqrt{\frac{\cos^2 \theta - v}{1-v}} + \frac{\sin \theta}{\sqrt{1-v}} \sin \varphi \sin \gamma;
\]

\[
\sin \alpha_{1,2} = \sqrt{1 - \cos^2 \alpha_{1,2}};
\]

\[
\sin \beta_{1,2} \sin \alpha_{1,2} = \frac{\sin \theta}{\sqrt{1-v}} \sin \varphi \cos \gamma \pm \sqrt{\frac{\cos^2 \theta - v}{1-v}} \sin \gamma;
\]

\[
\cos \beta_{1,2} \sin \alpha_{1,2} = \frac{\sin \theta}{\sqrt{1-v}} \cos \varphi.
\]

where the top signs correspond to the subscript 1, the bottom ones correspond to the subscript 2. Expression for $D_y$ is derived from the expression for $D_x$ by replacement of underlined factors $\sin \alpha_{1,2} \cos \beta$ with $(\cos \alpha_{1,2} \sin \gamma + \sin \alpha_{1,2} \sin \beta \cos \gamma)$ and $\cos \varphi$ with $\sin \varphi$ correspondingly. Two terms in (31) correspond to the trajectory ascending and descending branches of a ray with coordinates $\theta, \varphi$.

The equation system (28) numerical solving was performed using the Newton’s globally converging method described, for example, in [9].
7 Calculation results for isotropic plasma

The calculations were carried out for the following set of parameters: \( h_0 = 150 \) km, \( H = 100 \) km, \( \nu = 2.5 \), \( l_{0\perp} = 10 \) km, \( R = 1 \) km, \( \delta R = 3 \cdot 10^{-3} \) (ionospheric irregularity level characteristic of night quiet conditions), frequency \( f = 5 \) MHz, the angle of irregularity inclination ("magnetic field" inclination) \( \gamma = 25^\circ \). Thus we mean the conditions of the ionosphere sounding from the Earth surface. The intensity attenuation and change of the arrival angles of a signal reflected from a layer (with relation to those values when reflecting from the same plasma layer without irregularities) were calculated for area which sizes were 800 km along the \( y \) axis (the magnetic meridian direction) and 400 km along the \( x \) axis. The radiation source was in the coordinate origin. The calculation results are represented in figures 2 - 4. On the contour map of fig. 2 the constant level lines of the received signal intensity attenuation (in dB) are shown. The signal attenuation calculated value is symmetric with relation to both the \( y \) axis (i.e. magnetic meridian plane) and the \( x \) axis with accuracy determined by numerical calculation errors. The result is not trivial: there is asymmetry in the problem conditions due to the irregularity inclination. The central symmetry is required by the reciprocity theorem \([4, 5]\). Thus the obtained numerical solution is in accordance with the electromagnetic field general properties. This is an additional argument to the benefit of the conclusion about the primary significance of the first term in the REB equation approximate solution \((15)\).

The main detail in fig. 2 is the region having the shape of ellipse with half-axes of 300 km and 60 km where significant intensity attenuation (up to 15 dB) takes place. Outside of this region some increase of the signal intensity (in comparison with its value in absence of irregularities) is observed. This is quite natural result because at an nonabsorbing medium the complete radiation energy flow is conserved and scattering results only in its spatial redistribution. At larger distance from the source the intensity change aspires to zero. The transition from the region of the reflected signal attenuation to the region of amplification is of sharp character. That is, probably, a consequence of approximation used under transformation of the expression \((14)\) to the expression \((15)\) fist term. One can expect that retaining of the higher order terms in the series expansion will result in smoothing of the above transition.

The second of multiple scattering effects (i.e. change of the arrival angles)
is illustrated if figs. 3 and 4. In fig. 3 the contour lines show the absolute value of the polar arrival angle $\theta$ alteration. In fig. 3 the alteration of azimuth angle is presented. One can see that distortion of the polar angle reaches of 5 and maximum alteration of the azimuth angle is $90^\circ$.

Both effects are observed in experiment. The intensity reduction of the vertical sounding signal reflected from the ionosphere should be interpreted by the observer as an additional collisionless mechanism of radio wave attenuation. This phenomenon is observed under natural conditions bringing the increased values of the effective collision frequency [10, 11]. There are the weighty grounds to believe that it is connected to development of small-scale irregularities in the ionosphere. In particular, it is displayed stronger at night time and can reach of 10-15 dB [12]. The latter figure is in accordance with our calculation results. The experimental data on the arrival angle change of a wave on a short line (when transmitter to receiver distance is about 100 km) are reported in [12].

8 Conclusion

In the present work the heuristic basis for use of the invariant coordinate small-angle scattering approximation under solving of the RTE for a magnetized plasma layer is considered. Within the framework of this approximation two versions of the analytical solution have been obtained. They describe spatial-and-angular distribution of radiation reflected from a monotonous plasma layer with small-scale irregularities.

The physical conclusions about influence of the multiple scattering effects in a layer of plasma on the spatial-and-angular characteristics of radiation are possible on the basis of detailed numerical research of the obtained solutions. Such research has been carried out in the present work for the case of isotropic plasma.

It was shown that the main term of the REB equation solution for the radiation reflected from a plasma layer with random irregularities describes two effects: the signal intensity change (attenuation for the normal sounding) and the arrival angle change. Both effects are observed in the experiments on the ionosphere radio sounding. The first one is known as anomalous attenuation of radio waves at natural conditions. Note, that under the ionosphere heating experiments another kind of anomalous attenuation is observed: its
mechanism is based on the mode transformation under scattering, not on the multiple scattering \[13\]. The effect of arrival angle change can be interpreted as a mechanism of additional refraction in the ionosphere and also has experimental confirmation. These two effects numerical estimations obtained in the present work for parameters of a plasma layer and irregularity spectrum typical for mid-latitude ionosphere are in accordance with experimental data.

The considered effects can be observed not only at the ionosphere radio sounding but also at sounding by electromagnetic radiation of other kinds of plasma with random irregularities both in natural and in laboratory conditions.

Acknowledgments. The work was carried out under support of Russian Foundation of Basic Research (grants No. 94-02-03337 and No. 96-02-18499).

References

[1] Bronin A.G. and Zabotin N.A., Sov. Phys. JETP 75, 633 (1992).

[2] Zabotin N.A., Izvestiya Vysshich Uchebnykh Zavedenii, Radiofizika 36, 1075 (1993).

[3] Bronin A.G. and Zabotin N.A., Izvestiya Vysshich Uchebnykh Zavedenii, Radiofizika 36, 1163 (1993).

[4] Ginzburg V.L., ”Propagation of Electromagnetic Waves in Plasmas” (Pergamon Press, New York, 1964), p.682.

[5] Budden K.G. ”Radio waves in the ionosphere” (Cambridge University Press, London, 1961), p.669.

[6] ”Electrodinamics of Plasma”. (Edited by A.I.Akhiezer) (Nauka, Moscow, 1974), p.704.

[7] ”The directory on special functions”. (Edited by Abramovits M. and Stigun I.) (Nauka, Moscow, 1979), p.830.

[8] Rytov S.M., Kravtsov Yu.A. and Tatarskii V.I., ”Introduction to statistical radiophysics”. (Nauka, Moscow, 1978), vol.II, p.463.
[9] Dennis J. and Shnabel R., "Numerical methods of unconditional minimization and solving of the nonlinear equations". (Mir, Moscow, 1988), p.470.

[10] Setty C.S.G.K., Nagpal O.P. and Dhawan V.K., Indian J. Pure and Appl. Phys. 9, 519 (1971).

[11] Vodolazkin V.I., Danilkin N.P., Denisenko P.F. and Faer Yu.N., Geomagnetism and Aeronomy 23, 25 (1983).

[12] Baulch R.N.E., Butcher E.C., Devlin J.C. and Hammer P.R., J. Atmos. and Terr. Phys. 46, 895 (1984).

[13] Robinson T.R. "The heating of the high latitude ionosphere by high power radio waves". (North-Holland Physics Publ., Amsterdam, 1989), p.131.
Figure captions

Fig. 1. A schematic plot of Poeverlein’s construction for ordinary waves. The refractive index surfaces for several values of $v (v = \omega_p^2/\omega^2)$ are shown. The ray trajectories in this "$k$-space" are represented by straight dashed lines parallel to $z$ axis.

Fig. 2. Contour map showing the relative alteration of the intensity (in dB) of the signal reflected from the ionospheric plasma layer due to multiple scattering. Radiation source is in the coordinate origin.

Fig. 3. Contour map showing the relative alteration of the polar arrival angle (in degrees) of the signal reflected from the ionospheric plasma layer due to multiple scattering.

Fig. 4. Same as in fig. 3, but for the azimuth arrival angle.
This figure "fig1.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9803032v1
This figure "fig2.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9803032v1
This figure "fig3.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9803032v1
This figure "fig4.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9803032v1