Examples of limits of Frobenius (type) structures: the singularity case

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Abstract

We give examples of families of Frobenius type structures on the punctured plane and we study their limits at the boundary. We then discuss the existence of a limit Frobenius manifold. We also give an example of a logarithmic Frobenius manifold.

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1 Introduction

Let \( w_1, \cdots, w_n \) be positive integers and \( f : (\mathbb{C}^*)^n \to \mathbb{C} \) be the Laurent polynomial defined by

\[
    f(u_1, \cdots, u_n) = u_1 + \cdots + u_n + \frac{1}{u_1^{w_1} \cdots u_n^{w_n}}.
\]

It has been explained in [7] how to attach to \( f \) a canonical Frobenius manifold: the two main ingredients are a Frobenius type structure on a point, that is a tuple

\[
    (E^0, R_0^0, R_0^\infty, g^0)
\]

where \( E^0 \) is a finite dimensional vector space over \( \mathbb{C} \), \( g^0 \) is a symmetric and nondegenerate bilinear form on \( E^0 \), \( R_0^0 \) and \( R_0^\infty \) are two endomorphisms of \( E^0 \) such that \( R_0^\infty + (R_0^\infty)^* = n\text{Id} \) and \( (R_0^0)^* = R_0^0 \) (\( * \) denotes the adjoint with respect to \( g^0 \)) and a pre-primitive and homogeneous section of \( E^0 \), namely a section which is a cyclic vector of \( R_0^0 \) and an eigenvector of \( R_0^\infty \). The canonical solution of the Birkhoff problem for the Brieskorn lattice of \( f \) given by M. Saito’s method yields the required canonical Frobenius type structure. This is the punctual construction. This gives, for \( w_1 = \cdots = w_n = 1 \), the mirror partner of the projective space \( \mathbb{P}^n \) (see [1]), and more generally the mirror partner of the weighted projective space \( \mathbb{P}(1, w_1, \cdots, w_n) \) (see [9] and [3]).

The purpose of these notes is to give analogous results for the deformation \( F : (\mathbb{C}^*)^n \times X \to \mathbb{C} \) of \( f \) defined by

\[
    F(u_1, \cdots, u_n, x) = u_1 + \cdots + u_n + \frac{x}{u_1^{w_1} \cdots u_n^{w_n}}.
\]

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where $X := \mathbb{C}^*$ and then to discuss the existence of a "limit" Frobenius manifold as $x$ approaches 0. This kind of problem is also considered in [4], using another strategy (we will not use the reference [6] at all) and for a different class of functions. Notice however that the case $w_1 = \cdots = w_n = 1$ is common to both papers.

Let us precise the situation: let

$$
G = \frac{\Omega^n(U)[\tau, \tau^{-1}, x, x^{-1}]}{(d_u - \tau d_u F) \wedge \Omega^{n-1}(U)[\tau, \tau^{-1}, x, x^{-1}]} 
$$

be the (Fourier-Laplace transform of the) Gauss-Manin system of $F$ and

$$
G_0 = \frac{\Omega^n(U)[\tau^{-1}, x, x^{-1}]}{(\tau^{-1} d_u - d_u F) \wedge \Omega^{n-1}(U)[\tau^{-1}, x, x^{-1}]} 
$$

be (the Fourier-Laplace transform of) its Brieskorn lattice, where the notation $d_u$ means that the differential is taken with respect to $u$ only. $G$ is equipped with a connection $\nabla$ defined by

$$
\nabla_{\partial_{\tau}} (\omega_i \tau^i) = i \omega_i \tau^{i-1} - F \omega_i \tau^i 
$$

and

$$
\nabla_{\partial_x} (\omega_i \tau^i) = \frac{\partial \omega_i}{\partial x} \tau^i - \frac{\partial F}{\partial x} \omega_i \tau^{i+1}.
$$

In particular, if we put $\theta := \tau^{-1}$, $G_0$ is stable under the action of $\theta^2 \nabla_{\partial_{\theta}}$. One defines in the same way the Gauss-Manin system (resp. the Brieskorn lattice) $G''$ (resp. $G_0''$) of the Laurent polynomial $f$ (see [5] section 4 for details). It turns out that one can solve the Birkhoff problem for $G_0$ on the whole $X$: $G_0$ is a free $\mathbb{C}[\theta, x, x^{-1}]$-module of rank $\mu = 1 + w_1 + \cdots + w_n$ and there exists a basis of $G_0$ in which the matrix of the connection $\nabla$ takes the form

$$
\left( \frac{A_0(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + \left( - \frac{A_0(x)}{\mu \theta} + R \right) \frac{dx}{x},
$$

$A_0(x)$ being a $\mu \times \mu$ matrix with coefficients in $\mathbb{C}[x]$, $A_\infty$ and $R$ being diagonal matrices with constant coefficients (see proposition 3.1.3). This gives a Frobenius type structure on $X$ (see [5] and [8]), that is a tuple

$$
(X, E, R_0, R_\infty, \Phi, \nabla)
$$

where the different objects involved satisfy some natural compatibility relations which can be extended, and this is done in section 4 to a Frobenius type structure with metric (corollary 5.1.1)

$$
\mathbb{F} = (X, E, R_0, R_\infty, \Phi, \nabla, g)
$$

which will be the central object of these notes. It should be emphasized that the metric $g$ plays here a fundamental role. This Frobenius type structure, together with the data of a pre-primitive, homogeneous and $\nabla$-flat form, yields also a Frobenius manifold on $\Delta \times \mathbb{C}^{n-1}, 0$ where $\Delta$ denotes the open disc of radius one, centered at $x = 1$ (see [5], [8]): we will use it first to compare the canonical Frobenius manifolds attached to the different polynomials $F_x := F(\cdot, x)$, $x \in \Delta$, by the punctual construction (see section 5).

The second part of these notes (section 6) is devoted to the study of the limit, as $x$ approaches 0, of the Frobenius type structure $\mathbb{F}$. This limit is defined using Deligne’s canonical extension
\(L^\varphi\) such that the eigenvalues of the residue of \(\nabla \partial_x\) are contained in \([0, 1]\): this lattice is easily described in our situation. The key point is that \(gr^V(\mathcal{L}^\varphi/x\mathcal{L}^\varphi)\), the graded module associated with the Malgrange-Kashiwara \(V\)-filtration at \(x = 0\), yields a Frobenius type structure on a point which can thus be seen as the canonical limit Frobenius type structure (notice that this result is not always true if we consider \(\mathcal{L}^\varphi/x\mathcal{L}^\varphi\) instead of \(gr^V(\mathcal{L}^\varphi/x\mathcal{L}^\varphi)\), that is if we forget the graduation). In order to define a canonical limit Frobenius manifold, we still need a pre-primitive and homogeneous section of this limit Frobenius type structure, see again [8] and the references therein: we show in section 6 an explicit description of this canonical limit Frobenius manifold. In general, that is if there is an \(w_i\) such that \(w_i \geq 2\), the situation is less clear for the following reasons: first, we do not have a general statement saying that one can derive a Frobenius manifold from the canonical limit Frobenius type structure (nevertheless, it should emphasized that we do not assert here that such a limit does not exist); second, even if it happens to be the case, one could get several Frobenius manifolds which can be difficult to compare.

The last section is devoted to logarithmic Frobenius manifolds: if \(w_1 = \cdots = w_n = 1\), we show how to get, with the help now of a suitable extension of \(G_0^0\) at \(x = 0\), a Frobenius type structure with logarithmic pole along \(\{x = 0\}\) in the sense of [10, Definition 1.6], yielding a logarithmic Frobenius manifold. If there exists a weight \(w_i\) such that \(w_i \geq 2\), we have all the tools to define a Frobenius type structure with a logarithmic pole along \(\{x = 0\}\), except the metric: the symmetric bilinear form constructed here is flat but not non-degenerate.

The starting point of this paper is the reference [2] in which A. Bolibruch discusses the properties of the limit of an isomonodromic family of Fuchsian systems. It happens that, in our geometric situation, this family is produced, \(\text{via}\) an inverse Fourier-Laplace transformation, by a solution of the Birkhoff problem for the Brieskorn lattice of a rescaling \(H(u, x) = xf(u)\) of a tame regular morse function \(f\), see [11, Chapitre VI]. This leads naturally to the following question: given a Frobenius type structure on \(X\), what can we expect at the limit? Finally, the reference [4], and I thank C. Sabbah for a discussion about this, explains the choice of the deformation \(F\) made in these notes (the computations for the rescaling \(H\) are similar and easier to the ones performed here). The last section grew up after a discussion with C. Sevenheck who suggested me to work with the natural extensions \(L^\varphi_0\) of \(G_0^0\): I thank him for that.

## 2 The canonical solution of the Birkhoff problem for \(G_0^0\)

### 2.1 Preliminaries

Let \(S_\mu\) be the disjoint union of the sets \[
\left\{ \frac{\ell \mu}{w_i} \mid \ell = 0, \cdots, w_i - 1 \right\}
\]
for \(i = 0, \cdots, n\), where we put \(w_0 = 1\). Its cardinal is equal to \(\mu := 1 + w_1 + \cdots + w_n\). We number the elements of \(S_\mu\) from 0 to \(\mu - 1\) in an increasing way and write

\[S_\mu = \{s_0, s_1, \cdots, s_{\mu - 1}\}\]

(thus, \(s_k \leq s_{k+1}\)). Notice that \(0 \leq \frac{2k}{\mu} < 1\). Define, for \(k = 0, \cdots, \mu - 1\),

\[\alpha_k = k - s_k.\]
Lemma 2.1.1 One has \( s_0 = \cdots = s_n = 0, \ s_{n+1} = \frac{\mu}{\max w_i} \) and \( s_k + s_{\mu+n-k} = \mu \) for \( k \geq n + 1 \).

Proof. See [7, p. 2]. \( \square \)

Corollary 2.1.2 One has \( \alpha_0 = 0, \cdots, \alpha_n = n, \ \alpha_{k+1} \leq \alpha_k + 1 \) for all \( k \),
\[ \alpha_k + \alpha_{\mu+n-k} = n \]
for \( k \geq n + 1 \) and
\[ \alpha_k + \alpha_{n-k} = n \]
for \( k = 0, \cdots, n \).

Proof. One has \( \alpha_{k+1} \leq \alpha_k + 1 \) because \((s_k)\) is increasing. The remaining assertions are clear. \( \square \)

2.2 The Birkhoff problem for \( G_0^o \)

Let \( A_0^o \) and \( A_{\infty} \) be the \( \mu \times \mu \) matrices defined by
\[ A_{\infty} = \text{diag}(\alpha_0, \cdots, \alpha_{\mu-1}) \]
and
\[ A_0^o = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ \mu & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 0 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \cdots & \mu & 0 \end{pmatrix}. \]

Example 2.2.1 We will work with the following examples:
(1) \( n = 2 \) and \( w_1 = w_2 = 2 \): one has \( \mu = 5 \) and \( A_{\infty} = \text{diag}(0, 1, 2, \frac{1}{2}, \frac{3}{2}) \).
(2) \( w_1 = \cdots = w_n = 1 \): one has \( \mu = n + 1 \) and \( A_{\infty} = \text{diag}(0, 1, \cdots, n) \).

The following results are shown in [7]:

Lemma 2.2.2 (1) \( G_0^o \) is a free \( \mathbb{C}[\theta] \)-module of rank \( \mu \), equipped with a connection \( \nabla \) with a pole of Poincaré rank less or equal to 1 at \( \theta = 0 \).
(2) The Birkhoff problem for \( G_0^o \) has a solution: there exists a basis \( \omega^o = (\omega_0^o, \cdots, \omega_{\mu-1}^o) \) of \( G_0^o \) over \( \mathbb{C}[\theta] \) in which the matrix of the connection \( \nabla \) takes the form
\[ \left( \frac{A_0^o}{\theta} + A_{\infty} \right) \frac{d\theta}{\theta}. \]

Moreover, the eigenvalues of \( A_{\infty} \) run through the spectrum at infinity of the polynomial \( f \).

We even have a little bit more: the basis \( \omega^o \) constructed in loc. cit. is the canonical solution of the Birkhoff problem given by M. Saito’s method. In particular, it is compatible with the \( V \)-filtration at \( \tau = 0 \) (see the last assertion of [7, Proposition 3.2]). One then has \( A_0^o V_\alpha \subset V_{\alpha+1} \): in other words, if \( (A_0^o)_{ij} \neq 0 \) then \( \alpha_{i-1} \leq \alpha_{j-1} + 1 \). One can moreover endow \( G_0^o \) with a "metric": this is discussed in section 4.
3 The Birkhoff problem for $G_0$

We give in this section the counterpart of the previous results for $G_0$. We will put $u_0 := \frac{1}{u_1 \cdots u_n}$ and $\omega_0 := \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$.

3.1 A natural solution

Define

$$\Gamma_0 = \{(y_1, \cdots, y_n) \in \mathbb{R}^n | y_1 + \cdots + y_n = 1\},$$

$$\Gamma_j = \{(y_1, \cdots, y_n) \in \mathbb{R}^n | y_1 + \cdots + y_{j-1} + (1 - \frac{\mu}{w_j}) y_j + \cdots + y_n = 1\}$$

for $j = 1, \cdots, n$, with

$$\chi_{\Gamma_0} = u_1 \frac{\partial}{\partial u_1} + \cdots + u_n \frac{\partial}{\partial u_n}$$

and, for $j = 1, \cdots, n$,

$$\chi_{\Gamma_j} = u_1 \frac{\partial}{\partial u_1} + \cdots + u_{j-1} \frac{\partial}{\partial u_{j-1}} + (1 - \frac{\mu}{w_j}) u_j \frac{\partial}{\partial u_j} + \cdots + u_n \frac{\partial}{\partial u_n}.$$

Define also, for $g = u_1^{a_1} \cdots u_n^{a_n}$,

$$\phi_{\Gamma_j}(g) = a_1 \cdots + a_{j-1} + (1 - \frac{\mu}{w_j}) a_j + \cdots + a_n$$

and

$$h_{\Gamma_j} = \chi_{\Gamma_j}(F) - F.$$

We thus have $h_{\Gamma_0} = -\mu x u_0$ and $h_{\Gamma_j} = -\frac{\mu}{w_j} u_j$ if $j = 1, \cdots, n$.

Lemma 3.1.1 One has, for any monomial $g$, the equality

$$(\tau \partial_\tau + \phi_{\Gamma_j}(g)) g \omega_0 = \tau h_{\Gamma_j} g \omega_0$$

in $G$. In particular, one has

$$\tau \partial_\tau \omega_0 = \tau h_{\Gamma_0} \omega_0.$$

Proof. Direct computation. \qed

This lemma is the starting point in order to solve the Birkhoff problem for $G_0$, as it has been the starting point to solve the one for $G_0^\mu$ (see [7, section 3]). Put $\omega_1 = u_0 \omega_0$: the equality

$$\tau \partial_\tau \omega_0 = \tau h_{\Gamma_0} \omega_0$$

becomes

$$-\frac{1}{\mu} \tau \partial_\tau \omega_0 = x \tau \omega_1.$$
Iterating the process (the idea is to define $\omega_2 = -\frac{1}{\mu} \omega_1 h_1$ etc...), one gets sections $\omega_1, \ldots, \omega_{\mu-1}$ of $G$ satisfying

$$-\frac{1}{\mu} (\tau \partial_\tau + \alpha_k) \omega_k = \tau \omega_{k+1}$$

for $k = 1, \ldots, \mu - 1$ (we put $\omega_\mu = \omega_0$): this can be done as [7] section 2 and proof of proposition 3.2. We thus have

$$\omega_k = u^{a(k)} \omega_0$$

where the multi-indices $a(k)$ are defined in [7] p. 3.

Define, for $x \in X$,

$$A_0(x) := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ \mu x & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}$$

and

$$R = \text{diag}(0, -1, \ldots, -1, -s_{\mu-1}/\mu, \cdots, -s_{n+1}/\mu)$$

(the entry $-1$ is counted $n$ times).

**Example 3.1.2**

1. If $n = 2$ and $w_1 = w_2 = 2$, one has $R = -\text{diag}(0, 1, 1, 1, 1, 1)$.
2. If $w_1 = \cdots = w_n = 1$, one has $R = -\text{diag}(0, 1, 1, \cdots, 1)$.

**Proposition 3.1.3**

1. $G_0$ is a free $\mathbb{C}[x, x^{-1}, \theta]$-module of rank $\mu$ and $\omega = (\omega_0, \cdots, \omega_{\mu-1})$ is a basis of it.
2. In the basis $\omega$, the matrix of the connection $\nabla$ takes the form

$$\left( \frac{A_0(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + \left( R - \frac{A_0(x)}{\mu \theta} \right) \frac{dx}{x}$$

where $A_\infty$ is the diagonal matrix defined in section 2.

**Proof.**

1. One shows that $G_0$ is finitely generated as in [7] proposition 3.2], with the help of lemma 3.1.1. To show that it is free notice that, again by [7] proposition 3.2], a section of the kernel of the surjective map

$$(\mathbb{C}[x, x^{-1}])^\mu \to G_0 \to 0$$

is given by $\mu$ Laurent polynomials which vanishes everywhere. Let us show (2): the assertion about $\nabla_{\partial_\theta}$ is clear (by definition of the $\omega_k$’s). Recall that the action of $\nabla_{\partial_\theta}$ is defined, for $\eta \in G_0$, by

$$\nabla_{\partial_\theta}(\eta) = -\frac{\partial F}{\partial x} \eta \theta^{-1} + \partial_x(\eta) = -u_0 \eta \theta^{-1} + \partial_x(\eta).$$

An easy computation shows that one has, for $\eta = u_0 u_1^{a_1} \cdots u_n^{a_n} \omega_0$,

$$u_0 \eta = \frac{1}{\mu x} F \eta - \frac{1}{\mu x} \theta \sum_{i=1}^{n} a_i - w_i) \eta.$$
Since $\theta^2\nabla \phi_0$ is induced by the multiplication by $F$, the matrix $\nabla \phi_x$ in the basis $\omega$ takes the form

$$\frac{-A_0(x)}{\mu x \theta} + \frac{1}{\mu x} T$$

where $T$ is the diagonal matrix defined by (apply the process above to $\omega_0$, $\omega_1 = u^{a(1)} \omega_0$, \ldots, $\omega_{\mu-1} = u^{a(\mu-1)} \omega_0$)

$$T_{kk} = \sum_{i=1}^{n} a(k - 1)i - w_i - \alpha_{k-1}.$$ 

Now, one has $\sum_{i=1}^{n} a(k - 1)i = k - 2$ (see [7, section 2]) and $\sum_{i=1}^{n} w_i = \mu - 1$ so that $T_{kk} = k - 1 - \alpha_{k-1} - \mu = s_{k-1} - \mu$. Use now the symmetry property of the et $s_k$’s (see lemma 2.1.1). Of course, $R = T/\mu$.

**Remark 3.1.4** It follows from the second part of the proposition that $(\alpha_0, \ldots, \alpha_{\mu-1})$ is the spectrum at infinity of any function $F_x := F(\cdot, x)$, $x \in X$, see [7, Proposition 3.2].

### 3.2 Towards the canonical extensions of $G$ at $x = 0$

#### 3.2.1 The $\varphi$-solution

Define $\omega_0^\varphi = \omega_0$ and $\omega_1^\varphi = xu_0 \omega_0^\varphi = x \omega_1$ (such a choice is also natural because $h_{\Gamma_0} = -\mu xu_0$): one has

$$-\frac{1}{\mu} \tau \partial_\tau \omega_0^\varphi = \tau \omega_1^\varphi$$

and gets as above forms $\omega_1^\varphi, \ldots, \omega_{\mu-1}^\varphi$ satisfying

$$-\frac{1}{\mu} (\tau \partial_\tau + \alpha_k) \omega_k^\varphi = \tau \omega_{k+1}^\varphi$$

for all $k = 0, \ldots, \mu - 2$ and

$$-\frac{1}{\mu} (\tau \partial_\tau + \alpha_{\mu-1}) \omega_{\mu-1}^\varphi = x \tau \omega_0^\varphi.$$

One has also

$$\omega^\varphi = \omega P$$

where $P = \text{diag}(1, x, \ldots, x)$. Put, for $x \in X$,

$$A_0^\varphi(x) = P^{-1} A_0(x) P$$

that is

$$A_0^\varphi(x) = 
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \mu x \\
\mu & 0 & 0 & \cdots & 0 & 0 \\
0 & \mu & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \mu & 0 \\
\end{pmatrix}$$
and
\[ R^\varphi = \text{diag}(0, 0, \cdots, 0, s_{n+1}/\mu, \cdots, s_{\mu-1}/\mu) \] 
(0 is counted \((n + 1)\)-times). If \(\mu = n + 1\), one has \(R^\varphi = 0\). \(\overline{A}_0^\varphi\) will denote the value of \(A_0^\varphi(x)\) at \(x = 0\).

**Proposition 3.2.1**

1. \(\omega^\varphi = (\omega_0^\varphi, \cdots, \omega_\mu^\varphi)\) is a basis of \(G_0\).
2. In this basis, the matrix of the connection \(\nabla\) takes the form
\[
\left( \frac{A_0^\varphi(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + (R^\varphi - \frac{A_0^\varphi(x)}{\mu\theta}) \frac{dx}{x}.
\]
3. For \(\theta \neq 0\), the residue matrix of \(\nabla_{\partial_x}\) at \(x = 0\) takes the form
\[ R^\varphi - \frac{\overline{A}_0^\varphi}{\mu\theta}. \]

Its eigenvalues are contained in \([0, 1]\).

**Proof.** (1) and (2) follow from proposition 3.1.3, using lemma 2.1.1 and the fact that \(R^\varphi dx/x = R dx/x + P^{-1}dP\).

(3) follows because \(0 \leq \frac{s_k}{\mu} < 1\). \(\square\)

**Remark 3.2.2**

Using the variable \(\tau := \theta^{-1}\), we find that the matrix of the connection \(\nabla\) takes the form
\[
(-A_0^\varphi(x)\tau - A_\infty) \frac{d\tau}{\tau} + (-A_0^\varphi(x)\tau - A_\infty + H) \frac{dx}{\mu x}
\]
where \(H\) is the diagonal matrix \(\text{diag}(0, 1, \cdots, \mu - 1)\). This can be used, because the entries of \(H\) are integers, to show that the monodromies \(T\) and \(T'\) corresponding respectively to the loops around the divisors \(\{\tau = 0\} \times X\) and \(\mathbb{C} \times \{0\}\) in \(\mathbb{C} \times X\) are related by the formula
\[ T^{-1} = (T')^\mu \]
(see [4, Corollary 6.5 (ii)]).

### 3.2.2 The \(\psi\)-solution

Define now

- \(Q = \text{diag}(1, x, \cdots, x, 1, \cdots, 1)\) \((x\) is counted \(n\)-times),
- \(A_0^\psi(x) = Q^{-1}A_0(x)Q\),
- \(R^\psi = -\text{diag}(0, \cdots, 0, \frac{s_{n-1}}{\mu}, \cdots, \frac{s_{\mu-1}}{\mu})\).
Lemma 3.2.3 (1) Suppose that $\mu \geq n + 2$. One has $(A_0^\psi(x))_{1,\mu} = \mu$,
\[(A_0^\psi(x))_{i+1,i} = \mu\]
if $i \neq n + 1$ and
\[(A_0^\psi(x))_{n+2,n+1} = \mu x.\]
(2) Suppose that $\mu = n + 1$. One has $(A_0^\psi(x))_{1,\mu} = \mu x$ and
\[(A_0^\psi(x))_{i+1,i} = \mu\]
if $i = 1, \ldots, n$.

Proof. Clear. \qed

In the sequel, $\overline{A}_0^\psi$ will denote the value of $A_0^\psi(x)$ at $x = 0$. Note that $\overline{A}_0^\psi$ is regular.

Proposition 3.2.4 (1) $\omega^\psi = \omega Q$ is a basis of $G_0$,
(2) In this basis, the matrix of $\nabla$ takes the form
\[\left(\frac{A_0^\psi(x)}{\theta} + A_\infty\right) \frac{d\theta}{\theta} + \left(R^\psi - \frac{A_0^\psi(x)}{\mu \theta}\right) \frac{dx}{x}.\]
(3) If $\theta \neq 0$, the residue matrix of $\nabla_{\partial_x}$ at $x = 0$ takes the form
\[R^\psi - \frac{A_0^\psi}{\mu \theta}.\]
Its eigenvalues are contained in $]-1, 0]$. 

Example 3.2.5 If $n = 2$ and $w_1 = w_2 = 2$, one has
\[A_0^\psi(x) = 5 \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\]
and
\[\overline{A}_0^\psi = 5 \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.\]

Remark 3.2.6 Let $\mathcal{L}$ (resp. $\mathcal{L}^\phi$, $\mathcal{L}^\psi$) be the $\mathbb{C}[x, \theta, \theta^{-1}]$-submodule of $G$ generated by $\omega$ (resp. $\omega^\phi$, $\omega^\psi$). One has $\mathcal{L}^\phi \subset \mathcal{L}^\psi \subset \mathcal{L}$. Notice that
\[\omega_0 = \omega_0^\phi = \omega_0^\psi\]
and
\[R(\omega_0) = R^\phi(\omega_0^\phi) = R^\psi(\omega_0^\psi) = 0.\]
If $\mu = n + 1$, one has $\mathcal{L}^\phi = \mathcal{L}^\psi$. 

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Remark 3.2.7 (The flat basis) Let $\Delta$ be the open disc in $\mathbb{C}$ of radius 1, centered at $x = 1$. The local basis $\omega^{\text{flat}} := \omega x^{-R}$ ($x \in \Delta$) of $G_0^n$ is flat: the matrix of the connection $\nabla$ takes the form
\[
\left( \frac{A^\text{flat}_0(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} - \frac{A^\text{flat}_0(x)}{\mu \theta} \frac{dx}{x}
\]
where $A^\text{flat}_0 = x^R A_0(x)x^{-R}$. Notice that the matrix $A^\text{flat}_0(x)$ has a limit $\overline{A}^\text{flat}_0$ when $x$ approaches 0: this is due to the fact that the sequence $(s_k)$ is increasing (this is equivalent to the fact that $\alpha_{k+1} \leq \alpha_k + 1$).

4 Duality

We define in this section a non-degenerate, symmetric and flat bilinear form on $G_0$.

4.1

The lattice $G_0^n$ is equipped with a non-degenerate bilinear form
\[
S^0 : G_0^n \times G_0^n \to \mathbb{C}[\theta]^n,
\]
$\nabla^0$-flat and satisfying, for $p(\theta) \in \mathbb{C}[\theta]$,
\[
p(\theta) S^0(\bullet, \bullet) = S^0(p(\theta) \bullet, \bullet) = S^0(\bullet, p(-\theta) \bullet).
\]

The basis $\omega^0 = (\omega^0_0, \cdots, \omega^0_{\mu-1})$ given by lemma 2.2.2 is adapted to $S^0$: one has
\[
S^0(\omega^0_k, \omega^0_j) = \delta_{k,n-j} \theta^n
\]
for $k = 0, \cdots, n$ and
\[
S^0(\omega^0_k, \omega^0_j) = \delta_{k,\mu+n-j} \theta^n
\]
for $k = n+1, \cdots, \mu-1$. In particular, $A_\infty + A_\infty^* = nI$ and $(A^0_0)^* = A^0_0$, where $^*$ denotes the adjoint with respect to $S^0$. All these results can be found in [7, Sect. 4].

We define, in the basis $\omega = (\omega_0, \cdots, \omega_{\mu-1})$ given by proposition 3.1.3
\[
S(\omega_0, \omega_n) = x^{-1} \theta^n,
\]
\[
S(\omega_k, \omega_{n-k}) = x^{-2} \theta^n
\]
for $k = 1, \cdots, n-1$,
\[
S(\omega_k, \omega_{\mu+n-k}) = x^{-1} \theta^n
\]
for $k = n+1, \cdots, \mu-1$ and
\[
S(\omega_i, \omega_j) = 0
\]
only otherwise. Notice that $S$ is constant in the flat basis $\omega^{\text{flat}}$: one has $S(\omega^{\text{flat}}_i, \omega^{\text{flat}}_j) = S^0(\omega^0_i, \omega^0_j)$ for all $i$ and for all $j$ (this follows from the symmetry property of the $s_k$'s). Define now
\[
S : G_0 \times G_0 \to \mathbb{C}[x, x^{-1}, \theta]^n
\]
by linearity, using the rules

\[ p(\theta)(\bullet, \bullet) = S(p(\theta)\bullet, \bullet) = S(\bullet, p(-\theta)\bullet) \]

and

\[ a(x)S(\bullet, \bullet) = S(a(x)\bullet, \bullet) = S(\bullet, a(x)\bullet) \]

for \( p(\theta) \in \mathbb{C}[\theta] \) and \( a(x) \in \mathbb{C}[x, x^{-1}] \).

If \( A(\theta, x) \) denotes the matrix of the covariant derivative \( \nabla_{\partial\theta} \), \( \nabla_{\partial\theta} \) will denote the covariant derivative whose matrix is \(-A(-\theta, x)\) in the same basis. We will say that \( S \) is \( \nabla\text{-flat} \) if

\[ \partial_{\theta}S(\varepsilon, \eta) = S(\nabla_{\partial\theta}(\varepsilon), \eta) + S(\varepsilon, \nabla_{\partial\theta}(\eta)) \]

and

\[ \partial_xS(\varepsilon, \eta) = S(\nabla_{\partial_x}(\varepsilon), \eta) + S(\varepsilon, \nabla_{\partial_x}(\eta)). \]

Keep the notations of section 3 and put \( C_0 = -A_0(x)/\mu x, C_\infty = R/x, C_0^\psi = -A_0^\psi(x)/\mu x \) etc...

**Lemma 4.1.1** (1) One has

\[ (A_0(x))^* = A_0(x), \quad A_\infty + A_\infty^* = nI, \quad C_0^* = C_0, \]

where \( * \) denotes the adjoint with respect to \( S \). Same results for \( A_0^{\text{flat}}(x), C^{\text{flat}}_0(x), C_0^\phi, A_0^\psi(x), C_0^\psi \).

(2) The bilinear form \( S \) is \( \nabla\text{-flat} \).

**Proof.** (1) The first equality follows from the definition of \( A_0(x) \) and from the definition of \( S \). For the second, use moreover the symmetry property of the numbers \( \alpha_k \) (see lemma [2.1.2]). The third equality is then clear and (2) follows from (1) and from the definition of \( S \) and \( R \). \( \square \)

We have in particular

\[ x \partial_xS(\varepsilon, \eta) = S(R(\varepsilon), \eta) + S(\varepsilon, R(\eta)) \]

and this give a symmetry property for the matrix \( R \). Notice also that \( S \) is the only \( \nabla\text{-flat} \) bilinear form which restricts to \( S^0 \) for \( x = 1 \). Last, notice that the coefficient of \( \theta^n \) in \( S(\varepsilon, \eta), \varepsilon, \eta \in G_0 \), depends only on the classes of \( \varepsilon \) and \( \eta \) in \( G_0/\theta G_0 \). We will denote it by \( g([\varepsilon], [\eta]) \). This defines a nondegenerate bilinear form on \( G_0/\theta G_0 \) (see [11], p. 211).

5 Application to the construction of Frobenius type structures

and Frobenius manifolds

5.1 Frobenius type structures

In our situation, a Frobenius type structure on \( X \) is a tuple (see also [5] and [8])

\[ (X, E, \nabla, R_0, R_\infty, \Phi, g) \]

where
• $E$ is a $\mathbb{C}[x, x^{-1}]$-free module,

• $R_0$ and $R_\infty$ are $\mathbb{C}[x, x^{-1}]$-linear endomorphisms of $E$,

• $\Phi : E \to \Omega^1(X) \otimes E$ is a $\mathbb{C}[x, x^{-1}]$-linear map,

• $g$ is a metric on $E$, i.e a $\mathbb{C}[x, x^{-1}]$-bilinear form, symmetric and nondegenerate,

• $\nabla$ is a connection on $E$,

these object satisfying the relations

$$\nabla^2 = 0, \; \nabla(R_\infty) = 0, \; \Phi \wedge \Phi = 0, \; [R_0, \Phi] = 0,$$

$$\nabla(\Phi) = 0, \; \nabla(R_0) + \Phi = [\Phi, R_\infty],$$

$$\nabla(g) = 0, \; \Phi^* = \Phi, \; R_0^* = R_0, \; R_\infty + R_\infty^* = r \text{Id}$$

for a suitable constant $r$, $^*$ denoting as above the adjoint with respect to $g$.

Keep the notations of section 3.1. The basis $\omega$ gives an extension of $G_0$ as a trivial bundle $\mathcal{G}$ on $\mathbb{P}^1 \times X$ (the module of its global sections is generated by $\omega_0, \cdots, \omega_{\mu-1}$) equipped with a connection with logarithmic pole at $\tau := \theta^{-1} = 0$ and pole of Poincaré rank less or equal to one at $\theta = 0$. Define $E := i_{\{\theta=0\}} \mathcal{G}$, $E_\infty := i_{\{\tau=0\}} \mathcal{G}$ ($E$ and $E_\infty$ are canonically isomorphic) and, for $i, j = 0, \cdots, \mu - 1$, $[]$ denoting the class in $E$,

• $R_0[\omega_i] := [\theta^2 \nabla_{\partial_\theta} \omega_i],$

• $g([\omega_i], [\omega_j]) := \theta^{-n} S(\omega_i, \omega_j),$

• $\Phi_\xi[\omega_i] := [\theta \nabla_{\partial_\xi} \omega_i]$ for any vector field $\xi$ on $X$.

The connection $\nabla$ and the endomorphism $R_\infty$ are defined analogously, using the restriction $E_\infty$, $[]$ denoting now the class in $E_\infty$,

• $R_\infty[\omega_i] := [-\nabla_{\partial_\tau} \omega_i],$

• $\nabla_\xi[\omega_i] := [\nabla_{\partial_\xi} a].$

**Corollary 5.1.1** The tuple $(X, E, R_0, R_\infty, \Phi, \nabla, g)$ is a Frobenius type structure on $X := \mathbb{C}^*$.  

**Proof.** This follows from proposition 3.1.3 and lemma 4.1.1 (see [11] Chapitre V, 2)).
5.2 Frobenius manifolds "in family"

Recall that $\Delta$ denotes the open disc in $\mathbb{C}$ of radius 1, centered at $x = 1$. Corollary 5.1.1 gives also an analytic Frobenius type structure

$$\mathcal{F} = (\Delta, E_{an}^0, R_{\infty}^0, \Phi_{an}, \nabla^{an}, g^{an})$$

on the simply connected domain $\Delta$. Let $\omega_0^{an}$ be the class of $\omega_0$ in $E_{an}$: $\omega_0^{an}$ is $\nabla^{an}$-flat because $R(\omega_0) = 0$ (see remark 3.2.6). The universal deformations and the period maps that we will consider are the ones defined in [5] and [8].

Lemma 5.2.1 (1) The Frobenius type structure $\mathcal{F}$ has a universal deformation

$$\tilde{\mathcal{F}} = (N, \tilde{E}_{an}^0, \tilde{R}_{\infty}, \tilde{\Phi}_{an}, \tilde{\nabla}^{an}, \tilde{g}^{an})$$

parametrized by $N := \Delta \times (\mathbb{C}^{\mu-1}, 0)$.

(2) The period map defined by the $\nabla^{an}$-flat extension of $\omega_0^{an}$ to $\tilde{\mathcal{F}}$ is an isomorphism which makes $N$ a Frobenius manifold.

Proof. (1) We can use the adaptation of [8] Theorem 2.5] given in [5] Section 6] because $\omega_0^{an}, R_{\infty}^{an}(\omega_0^{an}), \cdots, (R_{\infty}^{an})^{\mu-1}(\omega_0^{an})$ generate $E_{an}$ and because $u_0 := 1/u_1^{\mu_1} \cdots u_n^{\mu_n}$ is not equal to zero in $E_{an}$. (2) follows from (1) (see e.g. [8] Theorem 4.5]).

The previous construction can be also done in the same way "point by point" (see [7] and [8] and the references therein) and, as quoted in the introduction, this is the classical point of view. Indeed, let $x \in \Delta$ and put $\tilde{F}_x := F(., x)$. One can attach to the Laurent polynomial $F_x$ a Frobenius type structure on a point (see section 6.2 below) $\tilde{\mathcal{F}}_x^{pt}$, a universal deformation $\tilde{\mathcal{F}}_x^{pt}$ of it and finally, with the help of the section $\omega_0$, a Frobenius structure on $M := (\mathbb{C}^{\mu}, 0)$. We will call it "the Frobenius structure attached to $F_x$". Let $\mathcal{F}_x$ (resp. $\tilde{\mathcal{F}}_x$) be the germ of $\mathcal{F}$ (resp. $\tilde{\mathcal{F}}$) at $x \in \Delta$ (resp. $(x, 0)$).

Proposition 5.2.2 (1) The deformations $\tilde{\mathcal{F}}_x$ and $\tilde{\mathcal{F}}_x^{pt}$ are isomorphic.

(2) The period map defined by the flat extension of $\omega_0^{an}$ to $\tilde{\mathcal{F}}_x$ is an isomorphism. This yields a Frobenius structure on $M$ which is isomorphic to the one attached to $F_x$.

Proof. Note first that $\tilde{\mathcal{F}}_x^{pt}$ is a deformation of $\mathcal{F}_x$: this follows from the fact that $u_0$ does not belong to the Jacobian ideal of $f$: see [5] section 7]. Better, $\tilde{\mathcal{F}}_x^{pt}$ is a universal deformation of $\mathcal{F}_x$ because $\mathcal{F}_x$ is a deformation of $\mathcal{F}_x^{pt}$. This gives (1) because, by definition, two universal deformations of a same Frobenius type structure are isomorphic. (2) is then clear.

As a consequence, the universal deformations $\tilde{\mathcal{F}}_x^{pt}$, $x \in \Delta$, are the germs of a same section, namely $\tilde{\mathcal{F}}$. Thus, the Frobenius structure attached to $F_{x_1}$, $x_1 \in \Delta$, can be seen as an analytic continuation of the one attached to $F_{x_0}$, $x_0 \in \Delta$.

6 Limits

Our goal is now to define a canonical limit, as $x$ approaches 0, of the Frobenius type structure constructed in corollary 5.1.1. We will use Deligne’s canonical extensions.
6.1 Résumé: the canonical extensions at \( x = 0 \)

Recall the lattices defined \( \mathcal{L}^\varphi \) and \( \mathcal{L}^\psi \) defined in remark 3.2.6. Put

\[
\mathcal{L}^\varphi := \mathcal{L}^\varphi / x\mathcal{L}^\varphi.
\]

This is a free \( \mathbb{C}[\theta, \theta^{-1}] \)-module of rank \( \mu \), equipped with a connection \( \nabla_{\partial_0} \) (induced by \( \nabla_{\partial_0} \)). In the sequel, \( \mathcal{L}^\varphi \) will denote the basis of \( \mathcal{L}^\varphi \) induced by \( \omega^\varphi \). Recall also that \( A_0^\varphi(x) \) at \( x = 0 \): it is a \( \mu \times \mu \) Jordan matrix. The following theorem summarizes the results obtained in the previous sections:

**Theorem 6.1.1**

1) \( \mathcal{L}^\varphi \) is equipped with a connection

\[
\nabla : \mathcal{L}^\varphi \to \theta^{-1}\Omega_{\mathbb{C}^* \times \mathbb{C}}(\log(\mathbb{C}^* \times \{0\})) \otimes \mathcal{L}^\varphi.
\]

The matrix of \( x\nabla_{\partial_x} \) in the basis \( \omega^\varphi \) takes the form

\[
-\frac{A_0^\varphi(x)}{\mu\theta} + \text{diag}(0, \ldots , 0, \frac{s_{n+1}}{\mu}, \ldots , \frac{s_{\mu-1}}{\mu})
\]

and the one of \( \nabla_{\partial_0} \) takes the form

\[
\frac{A_0^\varphi(x)}{\theta^2} + A_\infty + \theta.
\]

2) \( x\nabla_{\partial_x} \) induces a map on \( \mathcal{L}^\varphi \) whose matrix, in the basis \( \mathcal{L}^\varphi \), takes the form

\[
-\frac{A_0^\varphi}{\mu\theta} + \text{diag}(0, \ldots , 0, \frac{s_{n+1}}{\mu}, \ldots , \frac{s_{\mu-1}}{\mu}).
\]

Its eigenvalues are contained in \([0, 1]\).

3) The matrix of \( \nabla_{\partial_0} \), acting on \( \mathcal{L}^\varphi \), takes the form, in the basis \( \mathcal{L}^\varphi \),

\[
\frac{A_0^\varphi}{\theta^2} + A_\infty + \theta.
\]

\( \square \)

**Corollary 6.1.2** \( \mathcal{L}^\varphi \) is Deligne’s canonical extension of the bundle \( G \) to \( \mathbb{C}^* \times \mathbb{C} \) such that the eigenvalues of the residue of \( \nabla_{\partial_x} \) are contained in \([0, 1]\).

Analogous statements for \( \mathcal{L}^\psi := \mathcal{L}^\psi / \theta \mathcal{L}^\psi \) (replace \([0, 1]\) by \([1, 0]\)). \( \mathcal{L}^\varphi \) is the space of the ”vanishing cycles” and \( \mathcal{L}^\psi \) is the one of the ”nearby cycles”. More generally, and after a base change of matrix \( x^r I, r \in \mathbb{Z} \), the lattice \( \mathcal{L}^\varphi \) (resp. \( \mathcal{L}^\psi \)) gives the canonical extensions whose eigenvalues are contained in \([k, k+1]\) (resp. \([k, k+1]\)).
6.2 Limits of Frobenius type structures

Ideally, the limit of our Frobenius type structure as \( x \) approaches 0 should be a Frobenius type structure on a point that is a tuple

\[
(E^{\text{lim}}, R_0^{\text{lim}}, R_{\infty}^{\text{lim}}, g^{\text{lim}})
\]

where \( E^{\text{lim}} \) is a finite dimensional vector space over \( \mathbb{C} \), \( g^{\text{lim}} \) is a symmetric and nondegenerate bilinear form on \( E^{\text{lim}} \), \( R_0^{\text{lim}} \) and \( R_{\infty}^{\text{lim}} \) being two endomorphisms of \( E^{\text{lim}} \) satisfying \( (R_0^{\text{lim}})^* = R_0^{\text{lim}} \) and \( R_{\infty}^{\text{lim}} + (R_{\infty}^{\text{lim}})^* = r \text{Id} \) for a suitable complex number \( r \), * denoting the adjoint with respect to \( g \).

It turns out that our limit will be defined with the help of the graded module \( gr^V(\mathcal{L}^\phi) \) associated with the Kashiwara-Malgrange \( V \)-filtration at \( x = 0 \).

6.2.1 The \( V \)-filtration at \( x = 0 \)

Recall the basis \( \omega^\phi = (\omega^\phi_0, \cdots, \omega^\phi_{\mu-1}) \) of \( G_0 \) over \( \mathbb{C}[\theta, x, x^{-1}] \) (it is thus also a basis of \( G \) over \( \mathbb{C}[\theta, \theta^{-1}, x, x^{-1}] \)) defined in section 3.2.1. Put \( v(\omega^\phi_0) = \cdots = v(\omega^\phi_n) = 0 \) and, for \( k = n+1, \cdots, \mu-1 \), \( v(\omega^\phi_k) = s_k/\mu \). Define, for \( 0 \leq \alpha < 1 \),

\[
V^\alpha G = \sum_{\alpha \leq v(\omega^\phi_k)} \mathbb{C}\{\theta, \theta^{-1}\} \omega^\phi_k + x \sum_{\alpha > v(\omega^\phi_k)} \mathbb{C}\{\theta, \theta^{-1}\} \omega^\phi_k,
\]

\[
V^{\alpha >} G = \sum_{\alpha < v(\omega^\phi_k)} \mathbb{C}\{\theta, \theta^{-1}\} \omega^\phi_k + x \sum_{\alpha \geq v(\omega^\phi_k)} \mathbb{C}\{\theta, \theta^{-1}\} \omega^\phi_k
\]

and \( V^{\alpha + p} G = x^p V^\alpha G \) for \( p \in \mathbb{Z} \) and \( \alpha \in [0, 1] \). This gives a decreasing filtration \( V^\bullet \) of \( G \) by \( \mathbb{C}\{\theta, \theta^{-1}\}\)-submodules such that

\[
V^\alpha G = \mathbb{C}[\theta, \theta^{-1}] < \mathcal{L}^\phi_k v(\omega^\phi_k) = \alpha > V^{\alpha >} G.
\]

Notice that \( \mathcal{L}^\phi = V^0 G \) and that \( \mathcal{L}^\phi = V^0 G/V^1 G \). We will put \( H^\alpha := V^\alpha G/V^{\alpha >} G \) and \( H = \bigoplus_{\alpha \in [0, 1]} H^\alpha \).

**Lemma 6.2.1**

(1) For each \( \alpha \), \( (x\nabla_{\partial_x} - \alpha) \) is nilpotent on \( H^\alpha \).

(2) Let \( N \) be the nilpotent endomorphism of \( H \) which restricts to \( (x\nabla_{\partial_x} - \alpha) \) on \( H^\alpha \). Its Jordan blocks are in one to one correspondence with the maximal constant sequences in \( \mathcal{S}_w \) and the corresponding sizes are the same.

(3) Let \( e_k \) be the class of \( \omega^\phi_k \) in \( H \). Then \( e = (e_0, \cdots, e_{\mu-1}) \) is a basis of \( H \) over \( \mathbb{C}[\theta, \theta^{-1}] \).

**Proof.** (1) It suffices to prove the assertion for \( \alpha \in [0, 1] \). By theorem 6.1.1, we have

\[
x\nabla_{\partial_x} \omega^\phi_k = -\frac{1}{\theta} \omega^\phi_{k+1}
\]

for \( k = 0, \cdots, n - 1 \) and \( x\nabla_{\partial_x} \omega^\phi_n \in V^{\alpha >} G \). Moreover, for \( k = n + 1, \cdots, \mu - 2 \) we have

\[
(x\nabla_{\partial_x} - \frac{s_k}{\mu}) \omega^\phi_k = -\frac{1}{\theta} \omega^\phi_{k+1}
\]
and this is equal to 0 in \( H^v(\omega_k^\mu) \) if \( s_{k+1} > s_k \). Last,

\[
(x \nabla_{\theta_x} - \frac{s_{\mu-1}}{\mu})\omega_{\mu-1}^\phi \in x \sum_{v(\omega_{\mu-1}^\phi) \geq v(\omega_k^\mu)} \mathbb{C}\{x\} \omega_k^\phi \in V^{>s_{\mu-1}/\mu} G.
\]

(2) follows from (1) and (3) follows from the definition of \( V^* \).

\[\Box\]

**Remark 6.2.2** Let \( B \) be the matrix of \( N \) in the basis \( e \): we have \( B_{i,j} = 0 \) if \( i \neq j + 1 \), \( B_{i+1,i} = -\frac{1}{\theta} \) if \( \alpha_i = \alpha_{i-1} + 1 \) (equivalently if \( s_i = s_{i-1} \)) and \( B_{i+1,i} = 0 \) if \( \alpha_i = \alpha_{i-1} + 1 \).

**Example 6.2.3** (1) \( n = 2 \) and \( w_1 = w_2 = 2 \): one has \( S_w = (0, 0, 0, 5^2, 5^2) \) so that \( N \) has one Jordan block of size 3 and one Jordan block of size 2. Its matrix in the basis \( e \) is

\[
-\frac{1}{\theta} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

(2) \( w_1 = \cdots = w_n = 1 \): one has \( S_w = (0, \cdots, 0) \) so that \( N \) has one Jordan block of size \( \mu = n + 1 \). Its matrix in the basis \( e \) is

\[
-\frac{1}{\theta} \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\cdots \ & \cdots \ & \cdots \ & \cdots \ & \cdots \ & \cdots \\
0 & 0 & \cdots & 1 & 0 \end{pmatrix}.
\]

**Corollary 6.2.4** The filtration \( V^* \) is the Kashiwara-Malgrange filtration at \( x = 0 \).

**Proof.** By lemma 6.2.1 our filtration satisfies all the characterizing properties of the Kashiwara-Malgrange filtration at \( x = 0 \).

\[\Box\]

### 6.2.2 The canonical limit Frobenius type structure

The module \( H \) is free over \( \mathbb{C}[\theta, \theta^{-1}] \) and is equipped with a connection \( \nabla \) whose matrix in the basis \( e \) takes the form

\[
\left( \frac{[A_0]}{\theta} + [A_\infty] \right) \frac{d\theta}{\theta}
\]

where \([A_0] = -\mu \theta B \) (as above, \( B \) is the matrix of \( N \) in the basis \( e \)) and where \([A_\infty] \) is the diagonal matrix with eigenvalues \((\alpha_0, \cdots, \alpha_{\mu-1})\). Let \( H_0 \) be the \( \mathbb{C}[\theta] \)-submodule of \( H \) generated by \( e \) and define

\[
k : H_0 \times H_0 \to \mathbb{C}[\theta] \theta^n
\]

by

\[
k(e_k, e_{n-k}) = \theta^n
\]

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for $k = 0, \cdots, n$,
\[ k(e_k, e_{\mu+n-k}) = \theta^n \]
for $k = n+1, \cdots, \mu - 1$ and $k(e_i, e_j) = 0$ otherwise. Last, put $E^{lim} = H_0/\theta H_0$. $E^{lim}$ is thus equipped with two endomorphisms $R_0^{lim}$ and $R_\infty^{lim}$ and with a non-degenerate bilinear form $g^{lim}$: if $\overline{e_i}$ denotes the class of $e_i$ in $E^{lim}$, $R_0^{lim}$ (resp. $R_\infty^{lim}$) is the endomorphism of $E^{lim}$ whose matrix is $[A_0]$ (resp. $[A_\infty]$) in the basis $\overline{e}$ and $g^{lim}$ is obtained from $k$ as in the end of section 4.

**Theorem 6.2.5** The tuple
\[ (E^{lim}, R_0^{lim}, R_\infty^{lim}, g^{lim}) \]
is a Frobenius type structure on a point.

**Proof.** It is enough to show that $(R_0^{lim})^* = R_0^{lim}$ and that $R_\infty^{lim} + (R_\infty^{lim})^* = nId$. To make the proof readable, we write $e_i$ instead of $\overline{e}_i$, $R_0$ instead of $R_0^{lim}$ etc... We will repeatedly use lemma 2.1.1, corollary 2.1.2 and lemma 6.2.1.

(a) Let us show first that $R_\infty + (R_\infty)^* = nId$:
(i) assume $0 \leq i \leq n$: we have $g(R_\infty(e_i), e_j) = \alpha_i g(e_i, e_j)$ so that
\[ g(R_\infty(e_i), e_j) = \alpha_i \]
if $i + j = n$ and $g(R_\infty(e_i), e_j) = 0$ otherwise. In the same way,
\[ g(e_i, R_\infty(e_j)) = \alpha_j \]
if $i + j = n$ and $g(e_i, R_\infty(e_j)) = 0$ otherwise. If $i + j = n$, one has $\alpha_j = \alpha_{n-i} = n - i - s_{n-i} = n - i = n - \alpha_i$.
(ii) Assume now $n + 1 \leq i \leq \mu - 1$: we have
\[ g(R_\infty(e_i), e_j) = \alpha_i \]
if $i + j = \mu + n$ and $g(R_\infty(e_i), e_j) = 0$ otherwise. In the same way,
\[ g(e_i, R_\infty(e_j)) = \alpha_j \]
if $i + j = \mu + n$ and $g(e_i, R_\infty(e_j)) = 0$ otherwise. If $i + j = \mu + n$, one has
\[ \alpha_j = \alpha_{\mu+n-i} = \mu + n - i - s_{\mu+n-i} = \mu + n - i + (s_i - \mu) = n - \alpha_i. \]

(b) Let us show now that $(R_0)^* = R_0$.
(i) Assume that $0 \leq i \leq n - 1$. If $i + j + 1 = n$ (thus $0 \leq j \leq n - 1$), one has
\[ g(R_0(e_i), e_j) = g(e_{i+1}, e_j) = 1 \]
and
\[ g(e_i, R_0(e_j)) = g(e_i, e_{j+1}) = 1. \]
If $i + j + 1 \neq n$, one always has
\[ g(R_0(e_i), e_j) = g(e_i, R_0(e_j)) = 0. \]
(ii) Assume that \( i = n \). Then \( g(R_0(e_n), e_j) = 0 \) because \( R_0(e_n) = 0 \) by lemma 6.2.1 and \( g(e_n, R_0(e_j)) = 0 \) because \( e_0 \) does not belong to the image of \( R_0 \).

(iii) Assume \( n + 1 \leq i \leq \mu - 1 \).

If \( i + 1 + j = \mu + n \) then \( s_{i+1} = s_i \) if and only if \( s_{j+1} = s_j \) (because \( s_{\mu+n-i} = \mu - s_i \)). If it is the case, one has

\[
  g(R_0(e_i), e_j) = g(e_{i+1}, e_j) = 1
\]

and

\[
  g(e_i, R_0(e_j)) = g(e_i, e_{j+1}) = 1.
\]

If \( i + 1 + j = \mu + n \) but \( s_{i+1} \neq s_i \) (and thus \( s_{j+1} \neq s_j \)), one has \( R_0(e_i) = 0 \) and \( R_0(e_j) = 0 \) so that

\[
  g(R_0(e_i), e_j) = g(e_i, R_0(e_j)) = 0.
\]

If \( i + 1 + j \neq \mu + n \), one always has

\[
  g(R_0(e_i), e_j) = g(e_i, R_0(e_j)) = 0.
\]

\[\square\]

**Remark 6.2.6** The conclusion of the previous theorem is not always true if we do not consider the graded module \( H := \text{gr}^V(\mathcal{L}^\varphi/\mathcal{L}^\varphi) \). Indeed, if we work directly on \( \overline{\mathcal{L}}^\varphi := \mathcal{L}^\varphi/\mathcal{L}^\varphi \) we can define, in the same way as above, the tuple

\[
  (E, R_0, R_\infty, G)
\]

where

- \( E = \overline{\mathcal{L}}^\varphi/\mathcal{L}^\varphi \),
- \( G \) is the symmetric and nondegenerate bilinear form on \( E \) defined by \( G(e'_k, e'_{n-k}) = 1 \) for \( k = 0, \ldots, n \), \( G(e'_k, e'_{\mu+n-k}) = 1 \) for \( n + 1, \ldots, \mu - 1 \) and \( G(e'_i, e'_j) = 0 \) otherwise, \( e'_k \) denoting the class of \( \omega^e_k \) in \( E \),
- \( R_0 \) (resp. \( R_\infty \)) is the endomorphism of \( E \) whose matrix is \( \overline{A}^0 \) (resp. \( A_\infty \)) in the basis \( e' = (e'_0, \ldots, e'_{\mu-1}) \).

The point is that this tuple is a Frobenius type structure on a point if and only if \( \mu = n + 1 \): for instance, if \( \mu \geq n + 2 \), we have \( G(R_0(e'_n), e'_{\mu-1}) = 1 \) but \( G(e'_n, R_0(e'_{\mu-1})) = 0 \) so that \( (R_0)^* \neq R_0 \).

This symmetry default shows that the tuple \( (E, R_0, R_\infty, G) \) is not a Frobenius type structure. The case \( \mu = n + 1 \) is directly checked.

### 6.3 Pre-primitive sections "at the limit"

We will say that an element \( e \) of a \( \mu \)-dimensional vector space \( E \) over \( \mathbb{C} \), equipped with two endomorphisms \( A \) and \( B \), is a pre-primitive section of the triple \( (E, A, B) \) if \( (e, A(e), \ldots, A^{\mu-1}(e)) \) is a basis of \( E \) over \( \mathbb{C} \) and that \( e \) is homogeneous if it is an eigenvector of \( B \). Let

\[
  (E^{\lim}_0, R_0^{\lim}, R_\infty^{\lim}, G^{\lim})
\]

be the limit Frobenius structure given by theorem 6.2.5. Recall that \( \omega_0^e \) denotes the class of \( \omega_0^e \) in \( E^{\lim}_0 \).
Lemma 6.3.1 (1) $\overline{e}_0$ is a homogeneous section of the triple $(E^\text{lim}, R_0^\text{lim}, R_\infty^\text{lim})$.
(2) $\overline{e}_0$ is a pre-primitive section of the triple $(E^\text{lim}, R_0^\text{lim}, R_\infty^\text{lim})$ if and only if $\mu = n + 1$. If $\mu \geq n + 2$, this triple has no pre-primitive section at all.

Proof. Obvious, except the last assertion: this follows from the fact that if $\mu \geq n + 2$, $R_0^\text{lim}$ has at least two Jordan blocks for the same eigenvalue 0 (see lemma 6.2.1).

Corollary 6.3.2 $\overline{e}_0$ is a pre-primitive and homogeneous section of the limit Frobenius type structure $(E^\text{lim}, R_0^\text{lim}, R_\infty^\text{lim}, g^\text{lim})$ if and only if $\mu = n + 1$.

6.4 A canonical limit Frobenius manifold

What do we need to construct a Frobenius manifold? In general, a Frobenius type structure and a pre-primitive and homogeneous section of it: the main point is that these two objects give a unique (up to isomorphism) Frobenius manifold, see for instance [5] and the references to B. Dubrovin and B. Malgrange therein.

We assume here that $\mu = n + 1$: theorem 6.2.5 gives a canonical limit Frobenius type structure and corollary 6.3.2 a pre-primitive and homogeneous section of it, so, as explained above, we get in this case a canonical (limit) Frobenius manifold. We can give in this case a precise description of it: recall that

$$[A_0] = (n + 1) \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \ A_\infty = \text{diag}(0, 1, \cdots, n).$$

Let $x = (x_1, \cdots, x_\mu)$ be a system of coordinates on $M = (\mathbb{C}^n, 0)$.

Lemma 6.4.1 There exists a unique tuple of matrices $(\tilde{A}_0(x), A_\infty, \tilde{C}_1(x), \cdots, \tilde{C}_\mu(x))$ such that $\tilde{A}_0(0) = [A_0]$, $(\tilde{C}_i)_1 = -1$ for all $i = 1, \cdots, \mu$ and satisfying the relations

$$\frac{\partial \tilde{C}_i}{\partial x_j} = \frac{\partial \tilde{C}_j}{\partial x_i},$$

$$[\tilde{C}_i, \tilde{C}_j] = 0$$

$$[\tilde{A}_0(x), \tilde{C}_i] = 0$$

$$\frac{\partial \tilde{A}_0}{\partial x_i} + \tilde{C}_i = [A_\infty, \tilde{C}_i]$$

for all $i, j = 1, \cdots, \mu$. Precisely,

(a) $\tilde{C}_1 = -I$,

(b) $\tilde{C}_2 = -J$ where $J$ denotes the nilpotent Jordan matrix of order $\mu$,

(c) $\tilde{C}_i = -J^{i-1}$ for all $i = 1, \cdots, \mu$.

(d) $\tilde{A}_0(x) = -x_1 \tilde{C}_1 - x_2 \tilde{C}_2 + x_3 \tilde{C}_3 + 2x_4 \tilde{C}_4 + \cdots + (\mu - 2)x_\mu \tilde{C}_\mu$. 
Proof. It is clear that the given matrices satisfy the required relations. To show unicity, note that the matrices Ĉ_i are determined by their first column because the matrix A_0(0) is regular, v_0 is pre-primitive and the matrices Ĉ_i commute with A_0(x).

This lemma means the following: the connection \( \overrightarrow{\nabla} \) on the bundle \( \tilde{E} = O_M \otimes E^{\text{lim}} \) whose matrix is

\[
\left( \frac{A_0(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + \sum \frac{\partial \mu}{i=1} \tilde{C}_i dx_i
\]

in the basis \( \tilde{\epsilon} = (\tilde{e}_0, \cdots, \tilde{e}_{\mu-1}) = (1 \otimes \tilde{e}_0, \cdots, 1 \otimes \tilde{e}_{\mu-1}) \) of \( \tilde{E} \) is flat. The matrices \( \tilde{C}_i \) are the matrices of the covariant derivatives \( \nabla_{\partial x_i} \). \( \overrightarrow{\nabla} \) will denote the connection on \( \tilde{E} \) whose matrix is zero in the basis \( \tilde{\epsilon} \): \( \tilde{\epsilon} \) is thus the \( \overrightarrow{\nabla} \)-flat extension of \( \epsilon \). We get in this way a Frobenius type structure on \( M \),

\[
(M, \tilde{E}, \overrightarrow{\nabla}, \tilde{R}_0, \tilde{R}_\infty, \tilde{\Phi}, \tilde{g}).
\]

Corollary 6.4.2 Assume that \( \mu = n + 1 \).

1) The period map

\[
\varphi_{\tilde{e}_0} : TM \to \tilde{E},
\]

\( \varphi_{\tilde{e}_0}(\xi) = -\tilde{\Phi} \xi (\tilde{e}_0) \), is an isomorphism and \( \tilde{e}_0 \) is a homogeneous section of \( \tilde{E} \), that is an eigenvector of \( \tilde{R}_\infty \).

2) The section \( \tilde{e}_0 \) defines, through the period map \( \varphi_{\tilde{e}_0} \) a Frobenius structure on \( M \) which makes \( M \) a Frobenius manifold for which:

(a) the coordinates \( (x_1, \cdots, x_\mu) \) are \( \nabla \)-flat: one has \( \nabla \partial x_i = 0 \) for all \( i = 1, \cdots, \mu \),

(b) the product is constant in flat coordinates: \( \partial x_i \star \partial x_j = \partial x_{i+j-1} \) if \( i + j - 1 \leq \mu \), 0 otherwise,

(c) the potential \( \Psi \) is a polynomial of degree less or equal to 3: \( \Psi = \sum a_{ij} x_i x_j x_{\mu+2-i-j} \), up to a polynomial of degree less or equal to 2,

(d) the Euler vector field is \( E = x_1 \partial x_1 + (n + 1) \partial x_2 - x_3 \partial x_3 - \cdots - (n-1) x_{n+1} \partial x_{n+1} \),

(e) the potential \( \Psi \) is, up to polynomials of degree less or equal to 2, Euler-homogeneous of degree 4 - \( \mu \)

\[
E(\Psi) = (4 - \mu) \Psi + G(x_1, \cdots, x_\mu)
\]

where \( G \) is a polynomial of degree less or equal to 2.

Proof. (1) Follows from the choice of the first columns of the matrices \( \tilde{C}_i \): indeed, the period map \( \varphi_{\tilde{e}_0} \) is defined by \( \varphi_{\tilde{e}_0}(\partial x_i) = -\tilde{\Phi}_i(\tilde{e}_0) = \tilde{e}_{i-1} \), and it is of course an isomorphism. Last, \( \tilde{e}_0 \) is homogeneous because \( \epsilon_0 \) is so. Let us show (2): the isomorphism \( \varphi_{\tilde{e}_0} \) brings on \( TM \) the structures on \( \tilde{E} \): (a) follows from the fact that the first column of the matrices \( \tilde{C}_i \) are constant and (b) from the fact that the matrices \( \tilde{C}_i \) are constant because, by the definition of the product, \( \varphi_{\tilde{e}_0}(\partial x_i \star \partial x_j) = \tilde{C}_i(\tilde{C}_j(\tilde{e}_0)) \); (c) follows from (b) because, in flat coordinates,

\[
g(\partial x_i \star \partial x_j, \partial x_k) = \frac{\partial^3 \Psi}{\partial x_i \partial x_j \partial x_k}.
\]

Last, (d) follows from the definition of \( A_0(x) \) and (e) is a consequence of (c) and (d). \( \Box \)
Of course, the period map can be an isomorphism for other choices of the first columns of the matrices \( C_i \): whatever happens, the resulting Frobenius manifolds will be isomorphic to the one given by the corollary. Indeed, the Frobenius type structure

\[
(M, \tilde{E}, \tilde{\nabla}, \tilde{R}_0, \tilde{R}_\infty, \tilde{\Phi}, \tilde{g})
\]

is a universal deformation of the limit Frobenius type structure \((E^\lim, R^\lim_0, R^\lim_\infty, g^\lim)\) given by theorem 6.2.5 (see [8]). We will thus call the Frobenius manifold given by the corollary the canonical limit Frobenius manifold.

**Remark 6.4.3** If \( \mu \geq n + 2 \), that is if there exists an \( w_i \) such that \( w_i \geq 2 \), we still have a canonical limit Frobenius type structure, but no pre-primitive section of it so that the results in [8] do not apply. In particular, we do not know if one can find matrices as lemma 6.4.1 (this problem is not obvious, even for the simplest examples, see for instance example 2.2.1 (1)), that is if the limit Frobenius type structure and the form \( \bar{e}_0 \) (or any other) give as above a (limit) Frobenius manifold through the period map. Even if it happens to be the case, the previous construction gives then a lot of (limit) Frobenius manifolds and there is no way to compare them (we do not have any kind of unicity here).

### 7 Logarithmic Frobenius type structures and logarithmic Frobenius manifolds: an example and some remarks

Proposition 3.1.3 suggests that we are not so far from a logarithmic Frobenius type structure in the sense of [10, Definition 1.6] and one could expect at the end a logarithmic Frobenius manifold, see [10, Definition 1.4]. Some remarks are in order.

#### 7.1 Logarithmic Frobenius type structures

A Frobenius type structure with logarithmic pole along \( \{ x = 0 \} \) (for short, a logarithmic Frobenius type structure) is a tuple

\[
(E^{\log}, \{0\}, \nabla, R_0, R_\infty, \Phi, g)
\]

where

- \( E^{\log} \) is a \( \mathbb{C}[x] \)-free module,
- \( R_0 \) and \( R_\infty \) are \( \mathbb{C}[x] \)-linear endomorphisms of \( E^{\log} \),
- \( \Phi : E^{\log} \to \Omega^1(\log(\{ x = 0 \})) \otimes E^{\log} \) is a \( \mathbb{C}[x] \)-linear map,
- \( g \) is a metric on \( E^{\log} \), i.e a \( \mathbb{C}[x] \)-bilinear form, symmetric and non-degenerate,
- \( \nabla \) is a connection on \( E^{\log} \) with logarithmic pole along \( \{ x = 0 \} \),

these object satisfying the compatibility relations of section 5.1. One can also define in an obvious way a logarithmic Frobenius type structure without metric.
The main point is to construct $E^{log}$: in our situation, it will be obtained from an extension of $G_0$ as a free $\mathbb{C}[x,\theta]$-module (and not from a canonical extension of $G$ as before). As pointed out by C. Sevenheck, we can use for instance the $\mathbb{C}[x,\theta]$-submodule of $G_0$ generated by $\omega_{0}^{\varphi},\cdots,\omega_{\mu-1}^{\varphi}$ which is a lattice “in $x^n$” of $G_0$. We will denote it by $L_0^\varphi$. Let $L_\infty^\varphi$ be the $\mathbb{C}[x,\tau]$-module generated by $\omega_{0}^{\varphi},\cdots,\omega_{\mu-1}^{\varphi}$ (as usual $\tau = \theta^{-1}$).

**Lemma 7.1.1** $L_0^\varphi$ is equipped with a connection

$$\nabla: L_0^\varphi \to \theta^{-1}\Omega_{\mathbb{C}\times\Delta}(\log((\{0\} \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}))) \otimes L_0^\varphi.$$ 

The matrix of $x\nabla_{\partial_0}$ in the basis $\omega^\varphi$ of $L_0^\varphi$ takes the form

$$-\frac{A_0^\varphi(x)}{\mu\theta} + \text{diag}(0,\cdots,0,\frac{s_{n+1}}{\mu},\cdots,\frac{s_{\mu-1}}{\mu})$$

and the one of $\nabla_{\partial_0}$ takes the form

$$\frac{A_0^\varphi(x)}{\theta^2} + A_\infty^\varphi.$$

**Proof.** Follows from proposition 3.2.1.

Define $E^{log} = L_0^\varphi/\theta L_0^\varphi$. One could imagine that the counterpart of corollary 5.1.1: indeed, define, for $i = 0,\cdots,\mu-1$,

- $R_0[\omega_i^\varphi] := [\theta^2\nabla_{\partial_0}\omega_i^\varphi],$
- $\Phi_\xi[\omega_i^\varphi] := [\theta\nabla_{\partial_0}\omega_i^\varphi]$ for any logarithmic vector field $\xi \in \text{Der}(\log\{x = 0\}),$

and, using the restriction of $L_\infty^\varphi$ to $\tau = 0$,

- $R_\infty[\omega_i^\varphi] := [-\nabla_{x\partial_0}\omega_i^\varphi],$
- $\nabla_\xi[\omega_i^\varphi] = [\nabla_{\partial_0}\omega_i^\varphi]$ for any logarithmic vector field $\xi \in \text{Der}(\log\{x = 0\}).$

In order to define the 'metric’, recall that (see section 4)

$$S(\omega_0^\varphi, \omega_0^\varphi) = \theta^n,$$

$$S(\omega_k^\varphi, \omega_{n-k}^\varphi) = \theta^n$$

for $k = 1,\cdots, n-1$, 

$$S(\omega_k^\varphi, \omega_{\mu+n-k}^\varphi) = x^n\theta^n$$

for $k = n+1,\cdots, \mu-1$ and 

$$S(\omega_i^\varphi, \omega_j^\varphi) = 0$$

otherwise. Extend $S$ to $L_0^\varphi$: as above, we get a flat bilinear symmetric form $g$ on $E^{log}$,

$$g([\omega_i^\varphi], [\omega_j^\varphi]) := \theta^{-n}S(\omega_i^\varphi, \omega_j^\varphi).$$

The main point is that, of course, $g$ which is not non-degenerate, unless $\mu = n+1.$
Corollary 7.1.2  (1) The tuple \((E^{\log}, \{0\}, R_0, R_\infty, \Phi, \nabla, g)\) is a logarithmic Frobenius type structure if \(\mu = n + 1\).

(2) The tuple \((E^{\log}, \{0\}, R_0, R_\infty, \Phi, \nabla)\) is a logarithmic Frobenius type structure without metric if \(\mu \geq n + 2\).

Proof. The previous lemma finally gives a \(\log(\{x = 0\}) - trTLEP\)-structure (see [10, Definition 1.8]) if \(\mu = n + 1\) and we use the 1-1 correspondence between such structures and logarithmic Frobenius type structures given by [10, Proposition 1.10].

\[\square\]

Remark 7.1.3  (1) We can also consider the lattice \(L_0\) (resp. \(L_0^{\psi}\)), defined using the basis \(\omega\) (resp. \(\omega^{\psi}\)). We have

\[L_0^{\tilde{\psi}} \subset L_0^{\psi} \subset L_0.\]

Anyway, we will see below that, even if we forget the metric, the 'good' one to consider is \(L_0^{\tilde{\psi}}\).

(2) In [4], and for \(w_1 = \cdots = w_n = 1\), another (and more intrinsic) extension of \(G_0\) is considered, built with the help of logarithmic vector fields. I don't know for the moment how to compare it with \(L_0^{\tilde{\psi}}\) and if one can define in this way extensions of \(G_0\) if there exists an \(w_i \geq 2\).

7.2 Construction of a logarithmic Frobenius manifold

A manifold \(M\) is a Frobenius manifold with logarithmic poles along the divisor \(D\) (for short a logarithmic Frobenius manifold) if \(\text{Der}_M(\log D)\) is equipped with a metric, a multiplication and two (global) logarithmic vector fields (the unit \(e\) for the multiplication and the Euler vector field \(E\)), all these objects satisfying the usual compatibility relations (see [10, Definition 1.4]). We can also define a Frobenius manifold with logarithmic poles without metric: in this case, we still need a flat, torsionless connection, a symmetric Higgs field (that is a product) and two global logarithmic vector fields as before. Of course \(D = \{x = 0\}\) in what follows. According to T. Reichelt [10, Theorem 1.12] the construction in section [8] can be adapted to get a logarithmic Frobenius manifold from a logarithmic Frobenius type structure: let \((E^{\log}, D, R_0, R_\infty, \Phi, \nabla, g)\) be a logarithmic Frobenius type structure, \(\omega\) be a section of \(E^{\log}\). Define

\[\varphi_\omega : \text{Der}_M(\log D) \rightarrow E^{\log},\]

by

\[\varphi_\omega(\xi) := -\Phi_\xi(\omega).\]

One says that \(\omega\) satisfies

- (IC) if \(\varphi_\omega|_0\) is injective,
- (GC) if \(\omega|_0\) and its images under iteration of the maps \(\Phi_\xi|_0, \xi \in \text{Der}_M(\log D)\), and \(R_0|_0\) generate \(E^{\log}|_0\),
- (EC) if \(\omega\) is an eigenvector of \(R_\infty\).

We will say that a section of \(E^{\log}\) is log-pre-primitive (resp. homogeneous) if its restriction to \(M - D\) is \(\nabla\)-flat and if it satisfies conditions (IC), (GC) (resp. (EC)). We now come back to the logarithmic Frobenius type structure given by corollary [7.1.2].

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Lemma 7.2.1 The (class of the) form $\omega^\varphi_0$ in $E^{\log}$ is log-pre-primitive and homogeneous.

Proof. Follows from proposition 3.2.1: the flatness is given by the fact that $R^\varphi(\omega^\varphi_0) = 0$; conditions (IC) and (GC) hold because the matrix of $\Phi_{x\partial_x}$ is $-\frac{A_0(x)}{\mu}$; last, we have $A_\infty(\omega^\varphi_0) = 0$ and this gives (EC). 

Remark 7.2.2 Assume that $\mu \geq n + 2$: the section $\omega_0$ in $L_0$ is flat but does not satisfy (IC) and the section $\omega^\psi_0$ in $L^\psi_0$ is flat but does not satisfy (GC). In the former case, the only section which satisfies (IC), (EC) and (GC) is $\omega_1$ but this one is not flat; in the latter case, the only section which satisfies (IC), (EC) and (GC) is $\omega^\psi_{n+1}$ but this one is not flat. This explains why we work with $L^\varphi_0$.

Corollary 7.2.3 The log-pre-primitive and homogeneous section $\omega^\varphi_0$ together with the logarithmic Frobenius type structure $(E^{\log}, \{0\}, R_0, R_\infty, \Phi, \triangledown, g)$ define a logarithmic Frobenius manifold if $\mu = n + 1$ and a logarithmic Frobenius manifold without metric if $\mu \geq n + 2$.

Proof. Follows now from [10, theorem 1.12]. 

If $\mu = n+1$, one could expect an explicit description of the logarithmic Frobenius manifold obtained, as in section 6.4. Unfortunately, it is much more difficult and, except some trivial cases, I do not have results in this direction.

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