An Approach to Yukawa’s Elementary Domain Based on AdS$_5$ Spacetime

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Abstract: The field equations of elementary domains proposed by Yukawa in 1968 are studied from the viewpoint of particle embedded AdS$_5$ spacetime with warp factor. The fifth dimension in AdS$_5$ spacetime is known to produce the branes associated with the energy hierarchy through the warp factor in the spacetime. By the boundary conditions in the fifth dimension, the orbifold $S^1/Z_2$, the particles embedded in this spacetime can not escape to the outside the boundaries, like the particle in an infinite square well potential. This fact adds a new insight to the fields in this spacetime by considering the superposition of the fields for those particles confined in this dimension; that is, the fields living on the IR brane, the low energy boundary of the fifth dimension, is able to obey a difference equation like Yukawa’s elementary domain under suitable conditions. We study those conditions in detail in addition to the construction of particle models in this spacetime.

Keywords: Non-Local Fields, Elementary Domain, AdS Space, Extra Dimension
1 Introduction

The anti-de Sitter space have brought out the various interesting points of view in the modern elementary particle physics since the appearance of the AdS/CFT correspondence [1–3]. In particular, the fifth dimension in AdS$_5$ spacetime with the warp factor has been known to play a significant role to understand the energy hierarchy of respective branes distinguished by the fifth coordinate [4, 5]. According to this hierarchy, the high and low energy ends of the fifth dimension are called respectively as UV brane and IR brane. For the anti-de Sitter space, it has become clear that there exists a relation between the higher-spin gravity theories in the bulk and the bi-local composite field theory in IR brane associated with this hierarchy [6–8]. In the context of this bi-local composite field theory, we had also pointed out the possibility for a timelike non-local structure of the fields observed in the IR brane [9].

The non-local field theories based on the quantum field theory by Yukawa were started from the bi-local field theories in 1949 [10, 11] aimed at overcoming the problems inherent in local field theories. Yukawa asserted the importance of non-local field theories from two points of view: one is the uniform explanation of the variety of elementary particles and the other is to get rid of the divergence problem. Nowadays it becomes common idea to introduce some sort of non-local structure of elementary particles such as the string models, although the importance of Yukawa’s attempts was not so recognized in those days.
Now, roughly speaking, Yukawa’s attempts to construct non-local field theories are classified into two categories: one is the bi-local fields, which was proposed in the beginning as a kind of matrix elements of quantum fields such as
\[ \langle x | \Phi | x' \rangle = \Phi (X, \bar{x}) \neq \phi (x) \delta^4 (x - x') , \]
where \( X = \frac{1}{2} (x + x') \) and \( \bar{x} = x - x' \). The field equation applied to \( \Phi (X, \bar{x}) \) in the early stage was incomplete; and, under subsequent developments, the standard form has come to be
\[ \begin{align*}
\hat{P}^2 + \hat{p}^2 + \kappa^2 \bar{x}^2 + \omega \Phi (X, \bar{x}) &= 0, \quad (\kappa, \omega = \text{const.}) \quad (1.1) \\
\hat{P} \cdot (\kappa \bar{x} + i \hat{p}) \Phi (X, \bar{x}) &= 0, \quad (1.2)
\end{align*} \]
where \( \hat{P} \) and \( \hat{p} \) are momenta conjugate to \( X \) and \( \bar{x} \) respectively \([12, 13]\). Those equations say that the bi-local fields may be considered as a prototype of the string model or the matrix fields. Another approach to the non-local field theories by Yukawa was the theory of elementary domain proposed in 1968, which includes more drastic changes in field equations; he required a difference equation along a timelike direction instead of a differential equation such that \([14, 15]\)
\[ \exp \left( \sum_{\alpha=1}^{4} \lambda_\alpha \epsilon^\alpha_\mu \partial / \partial X_\mu \right) \Psi = \exp (-i \lambda S) \Psi , \quad (1.3) \]
where \( \lambda_\alpha \epsilon^\alpha_\mu \) and \( S \) are respectively vector and scalar dynamical variables constructed out of internal degrees of freedom of the system. Unfortunately, this type of non-local field theories could not progress furthermore due to the lack of guiding principles in addition to that eq.(1.3) allows the ghost solutions arising from a timelike extension of fields. However, if we regard eq.(1.3) as an effective equation of an elementary field lying behind, then the form of difference field equation is not so extraordinary. The purpose of this work is, thus, to find out a meaning of the domain type of field equations as effective one’s based on the field theories in the AdS\(_5\) spacetime.

In a preceding article \([16]\), we had shown that the \( \kappa \)-Minkowski structure based on the AdS\(_5\) spacetime can derive a domain type of field equation in \( M_4 \), though the field equation is consisting of the terms \( e^{\pm \lambda (\partial X)^2} \Psi \) instead of the directional difference term \( e^{\pm \lambda \partial (\partial X)_\nu} \Psi \). In order to get a directional difference type of field equation, a four vector inherent in the particle under consideration is necessary. In \([17]\), we also pointed out that a one-dimensional compact extra space attached to \( M_4 \) spacetime may produce a domain type of field equation in \( M_4 \) due to the infinite square well potential in the fifth dimension. Therefore, in this paper, we study a possibility that the spinning particles embedded in the AdS\(_5\) spacetime with warp factor gives rise to the domain type of effective field in \( M_4 \) through such a dynamics in the fifth dimension.

In the next section, we study toy models of the particles embedded in five-dimensional Minkowski spacetime with a compact moving-extra dimension. In the rest frame of this extra dimension, we require Dirichlet type of boundary conditions for the fields of the particles; on the other hand, in the laboratory system of the extra dimension, we define the end giving the IR brane. Under such a compact moving-extra dimension, those fields observed in the IR brane obtain a timelike extension similar to Yukawa’s elementary domain.
In section 3, the spin-less particles embedded in AdS\(_5\) spacetime are discussed. In section 4, the discussion is made on the case of spinning particle, to which a timelike vector is introduced associated with the spin degrees of freedom of the particle. Finally, section 5 is devoted to the summary and discussion.

In appendices A and B, we give supplementary discussions on the eigenvalue problem in the fifth dimension and the structure of spin representation space, respectively.

2 A domain like wave equation of a particle in a five-dimensional spacetime

We consider a five-dimensional (5D) spacetime \((x^\mu, x^5)\), \((\mu = 0, 1, 2, 3)\) with the metric diag\((\eta_{\mu\nu}) = (- + + + +)\) realized as the limit of vanishing warp factor in the Randall-Sundrum model. Namely, the spacetime is flat, but the fifth dimensional space is the \(S^1/\mathbb{Z}_2\) orbifold, where \(S^1\) is the circle with radius \(R\) parametrized by \(-\pi \leq \theta \leq \pi\); and we set \(x^5 = y = \theta R\), \((L = \pi R)\); at a later point, the extra dimension is reformulated so that it becomes time dependent.

2.1 Periodic structure of Green function for Klein-Gordon like equation

Since the line element of this spacetime is \(ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2\), the free spin-less mass \(m_0\) particle satisfies the Klein-Gordon like equation

\[
[\hat{p}_\mu \hat{p}^\mu + \hat{p}_y^2 + (m_0 c)^2] \Psi(x, y) = 0. 
\]  

In addition to this field equation, we require the following Dirichlet type of boundary conditions:

\[
\Psi(x, y)|_{y=0} = \Psi(x, y)|_{y=L} = 0. \tag{2.2}
\]

Then the dynamics of \(y\) variable comes to be equivalent to that of a particle in the box \(0 \leq y \leq L\) (infinite square well); and so, from eqs.\((2.1)\) and \((2.2)\), the orthonormal complete basis in the \(y\) space are

\[
\phi_n(y) = \frac{\sqrt{2}}{L} \sin(k_n y), \quad (k_n = \frac{n\pi}{L}; n = 1, 2, \cdots). \tag{2.3}
\]

The functions \((2.3)\) satisfy the eigenvalue equation \(\hat{p}_y^2 \phi_n(y) = \lambda_n \phi_n(y)\), \((\hat{p}_y = -i\hbar \partial_y)\) and the periodic conditions \(\phi_n(y) = \phi_n(y + 2\pi R) = -\phi_n(-y)\).

The Green function of the Klein-Gordon like equation \((2.1)\) can be represented by using the the complete basis \(\{\phi_n\}, (n = 0, 1, 2, \cdots)\) and the eigenstates of four momentum operators defined by \(\hat{p}_\mu \hat{p}^\mu = p_\mu p^\mu\), \((\langle p'|p''\rangle = \delta^4(p' - p'')\) ), as

\[
G_{ba} = \langle x_b| \otimes \langle y_b| \left(\hat{p}_\mu \hat{p}^\mu + \hat{p}_y^2 + (m_0 c)^2 - i\epsilon\right)^{-1}|x_a\rangle \otimes |y_a\rangle
\]

\[
= \frac{i}{2\hbar} \int_0^\infty d\tau \langle x_b| \otimes \langle y_b| \left\{\sum_{n=1}^\infty \int d^4p e^{-\frac{i}{\hbar} \frac{1}{2} (p_\mu p^\mu + \lambda_n + (m_0 c)^2 - i\epsilon)} |p\rangle \otimes |\phi_n\rangle \langle \phi_n| \right\} |x_a\rangle \otimes |y_a\rangle
\]

\[
= \frac{i}{2\hbar} \int_0^\infty d\tau \sum_{n=1}^\infty \frac{1}{i} \left(\frac{1}{\sqrt{2\pi \hbar \tau}}\right) e^{\frac{i}{\hbar} \left\{\frac{(x_b - x_a)^2}{2\hbar} - \tau (\lambda_n + (m_0 c)^2 - i\epsilon)\right\}} \phi_n(y_b) \phi_n(y_a) \tag{2.4}
\]
Here, the summation with respect to \( n \) can be rewritten as
\[
\sum_{n=1}^{\infty} e^{-\frac{i}{2\hbar} \tau \phi_n(y_b)} \phi_n(y_a) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\frac{i}{2\hbar} \frac{\hbar k_n^2 \tau}{2}} \sin(k_n y_b) \sin(k_n y_a)
\]
\[
= \frac{1}{iL} \sum_{n=-\infty}^{\infty} e^{-\frac{i}{2\hbar} \frac{\hbar k_n^2 \tau}{2}} \sin(k_n y_b) e^{i k_n y_a}
\]
\[
= \frac{1}{iL} \int dp_y \sum_{n=-\infty}^{\infty} \delta(p_y - \hbar k_n) e^{-\frac{i}{\hbar} \frac{p_y^2 \tau}{2}} \sin \left( \frac{1}{\hbar} p_y y_b \right) e^{i p_y y_a}. \tag{2.5}
\]

Using, further, Poisson’s summation rule \(^1\), the right-hand side of eq.(2.5) becomes
\[
\text{r.h.s of eq. (2.5)} = \frac{1}{iL} \int dp_y \frac{L}{\pi \hbar} \sum_{\tau = -\infty}^{\infty} e^{\frac{i}{\hbar} (2rLp_y)} e^{-\frac{i}{\hbar} \frac{p_y^2 \tau}{2}} \sin \left( \frac{1}{\hbar} p_y y_b \right) e^{i p_y y_a}
\]
\[
= \frac{-1}{2\pi \hbar} \sum_{\tau = -\infty}^{\infty} \int dp_y \left[ e^{-\frac{i}{\hbar} \frac{p_y^2 \tau}{2} - (2rL + y_b + y_a)p_y} \right] - e^{-\frac{i}{\hbar} \frac{p_y^2 \tau}{2} - (2rL - y_b + y_a)p_y}
\]
\[
= \sum_{\tau = -\infty}^{\infty} \frac{-1}{\sqrt{2\pi \hbar} \tau} \left[ e^{\frac{i}{\hbar} \frac{(2rL+2y_b)^2}{2\tau}} - e^{\frac{i}{\hbar} \frac{(2rL-2y_b)^2}{2\tau}} \right], \tag{2.6}
\]

where the use has been made of the notations
\[
f_{ba} = f_b - f_a
\]
\[
\tilde{f}_{ba} = \frac{1}{2} (f_b + f_a). \tag{2.7}
\]

Therefore, the Green function \( G_{ba} \) can be represented by \([18, 19]\)
\[
G_{ba} = \frac{i}{2\hbar} \int_{0}^{\infty} d\tau \sum_{\tau = -\infty}^{\infty} \frac{-1}{\sqrt{2\pi \hbar} \tau} \left[ e^{\frac{i}{\hbar} \frac{(2rL+2y_b)^2}{2\tau}} - e^{\frac{i}{\hbar} \frac{(2rL-2y_b)^2}{2\tau}} \right]
\]
\[
\times \left[ e^{\frac{i}{\hbar} \frac{1}{\tau} \left( (\bar{x}_{ba})^2 + (2rL+2y_b)^2 \right)} - e^{\frac{i}{\hbar} \frac{1}{\tau} \left( (\bar{x}_{ba})^2 + (2rL-2y_b)^2 \right)} \right]
\]
\[
= \sum_{\tau = -\infty}^{\infty} \left[ K \left( (\bar{x}_{ba})^2 + (2rL + 2y_b)^2 \right) - K \left( (\bar{x}_{ba})^2 + (2rL - 2y_b)^2 \right) \right], \tag{2.8}
\]

where \(^2\)
\[
K(z) = -\frac{1}{2\hbar} \int_{0}^{\infty} d\tau \left( \frac{1}{\sqrt{2\pi \hbar} \tau} \right)^5 e^{\frac{i}{\hbar} \frac{1}{\tau} \left[ \frac{1}{2} - \tau \left( \frac{(m_0c)^2}{2} \right) \right]} \tag{2.9}
\]

\(^1\)Since \( f(p) \equiv \sum_n \delta (p - \frac{nL}{L}) \) is a periodic function of \( p \) with the period \( \pi \hbar/L \), it can be expanded in the Fourier series of \( e_r(p) = \sqrt{\frac{\pi}{L}} e^{2i \pi r p/\hbar} \) \((r = 0, \pm 1, \pm 2 \ldots \) so that \( f(p) = \sum_{r = -\infty}^{\infty} f_r \delta(r) \). Here, the coefficients \( f_r \) are given by \( f_r = \int_{-\pi \hbar/L}^{\pi \hbar/L} dpe_r(p) \). Substituting the results for the Fourier series, we arrive at the summation rule: \( \sum_r \delta (p - \frac{rL}{L}) = \frac{1}{L} \sum_e e^{2i \pi r p/\hbar} \) \((r = 0, \pm 1, \pm 2 \ldots \)

\(^2\)If we use \( \lim_{\tau \to 0} \int_{0}^{\infty} d\tau \int_{0}^{\infty} \text{e}^{\text{e}^{\text{e}^{-br}}} = -\frac{\sqrt{\pi}}{\sqrt{ab+\text{e}^{ab}}} \int_{0}^{\infty} \text{e}^{-\text{e}^{ab}} \text{e}^{-\sqrt{ab}} \) \((a, b > 0)\), one can write \( K(z) = \frac{i (\text{e}^{(m_0c)^2} / (2\hbar)^5 \int \left[ z \left( \frac{(m_0c)^2}{2\hbar} \right) \right] \right), \) where \( f(x) = (\frac{1}{\sqrt{\pi}} + x) \text{e}^{-x^2/\pi}. \)
It is obvious that the Green function $G_{ba}(\bar{x}_{ba}, \bar{y}_{ba}, \bar{y}_{ba})$ satisfies the boundary conditions $G_{ba}|_{y_a=0,L} = G_{ba}|_{\bar{y}_a=0,L} = 0$ as a result of eq.(2.2). We also note that the $G_{ba}(x_{ba}, \bar{y}_{ba}, \bar{y}_{ba})$ is not invariant under the translational in the fifth dimension due to the presence of $y_{ba}$, the mean value of $y_a$, but invariant under the discrete transformations $\bar{y}_{ba} \rightarrow \bar{y}_{ba} \pm L$ and $\bar{y}_{ba} \rightarrow \bar{y}_{ba} \pm 2L$; and so, one can write

$$G_{ba}(\bar{x}_{ba}, \bar{y}_{ba}, \bar{y}_{ba}) = G_{ba}(\bar{x}_{ba}, \bar{y}_{ba}, \bar{y}_{ba} \pm L) = G_{ba}(\bar{x}_{ba}, \bar{y}_{ba} \pm 2L, \bar{y}_{ba}).$$ \hspace{1cm} (2.10)

Then, the field defined by

$$\Psi(x_b, y_b) = \int d^4x_a \int dy_a G_{ba}(\bar{x}_{ba}, \bar{y}_{ba}, \bar{y}_{ba})\Phi(x_a, y_a)$$ \hspace{1cm} (2.11)

satisfies the Klein-Gordon like equation with the initial source field $\Phi(x_a, y_a)$; that is, $(\hat{p}_\mu \hat{p}_\mu + m^2 c^2)\Psi = \Phi$. By virtue of eq.(2.10), the field $\Psi(x_b, y_b)$ also satisfies the periodic condition $\Psi(x_b, y_b + 2L) = \Psi(x_b, y_b)$.

We now examine the case such that $\Phi(x_a, y_a) = \delta(y_a - y_0)\Phi_0(x_a)$; that is, the initial state is placed on a hypersurface, a 3-brane in this toy model, defined by the condition $y = y_0$. Then, the $\Psi(x_b, y_b)$ becomes an intersection of the bulk field $\Psi(x_b, y_b)$ ($0 \leq y_b \leq L$), which is produced by the source $\Phi_0(x_a)$, with the $y = y_0$ hypersurface. Owing to eq.(2.10), the $\Psi(x_b, y_b)$ has a periodic structure with respect to $y_0$, which characterizes some spacelike extension of those fields. One can, however, expect that if the extra dimension depends on time by some way, then eq.(2.10) will cause a eq.(1.3) type of field equations for $\Psi(x_b, y_0)$.

In this case, the Lorentz covariance in 4D Minkowski spacetime is broken; however, we don’t worry about this point until section 4, where the recovery of the covariance will be discussed. In what follows, to confirm those conjectures on the timelike extension, we try to study two cases of time-dependent extra dimensions.

2.2 Time-dependent models of the fifth dimension

(i) Lorentz boost

We consider two reference systems in 5D spacetime: one is $\Sigma$ system with the coordinates $(x^\mu) = (x^0, x^i, y)$; and the other is $\Sigma'$ system with the coordinates $(x'^\mu) = (x'^0, x'^i, y')$, which moves against the $\Sigma$ system along $y$ direction with a constant velocity $c\beta$. The fifth dimension in the $\Sigma'$ system is a compact space with the proper interval $0 \leq y' \leq L$. On the other side, the fifth dimension in the $\Sigma$ system is moving, and it suffers the Lorentz contraction so that the interval becomes $0 \leq y \leq \tilde{L} = L \sqrt{1 - \beta^2}$. However, since the fifth dimension is a compact space with boundaries, the displacement of the $\Sigma'$ system in $\Sigma$ system must be understood in the covering space $y \equiv y + n\tilde{L}$, ($n = 0, \pm 1, \cdots$); then, the Lorentz boost can be figured in the $\Sigma$ system as figure 1.

We regard from now on that eqs.(2.1)~(2.11) refer the equations in the $\Sigma'$ system, on which the fifth dimension rests; and so, the Green function $G_{ba}'$ in eq.(2.8) is newly written as $G_{ba}'$, and eq.(2.11) should be understood as

$$\Psi'(x'_b, y'_b) = \int d^4x'_a \int dy'_a G_{ba}'(x'_ba, y'_ba, y'_ba)\Phi'(x'_a, y'_a).$$ \hspace{1cm} (2.12)
The fifth dimension is a compact space; and, the Lorentz boost should be understood in a covering space of \( 0 \leq y' \leq L \) extended by the identification \( y' \equiv y + L \) (\( y \equiv y + L \)).

**Figure 1.** Lorentz boost to the direction of \( x^5 \) axis

Let us set the coincidence \( \Sigma = \Sigma' \) at \( x^0 = 0 \); then, the Lorentz transformation \( x^\mu = U x'^\mu U^{-1} \) or \( x^{\hat{a}} = U^{-1} x^{x'} U \) is explicitly given by

\[
x^0 = x^0 C_\beta - y S_\beta
\]
\[
y' = y C_\beta - x^0 S_\beta
\]
\[
x^{\hat{a}} = x^i
\]

Further, since

\[
U \left[ \hat{p}_\mu \hat{p}^\mu + (m_0 c)^2 \right] G'_{ba} \Phi'_a = \left[ \hat{p}_\mu \hat{p}^\mu + (m_0 c)^2 \right] G_{ba} \Phi_a
\]

(2.14)

the explicit forms of \( G_{ba} \) and \( \Phi_a \) are obtained by substituting the right-hand sides of eq.(2.13) for \( x^{\hat{a}} \) in \( G'_{ba} \) and in \( \Phi' \).

Now, in our standpoint, the boundary conditions (2.2) are satisfied by the coordinates in the \( \Sigma' \) system; on the other side, the boundary surface corresponding IR brane in AdS\(_5\) spacetime is located at \( y = \tilde{L} \) in the \( \Sigma \) system. Then, by virtue of eq.(2.13), the condition \( \beta_a = \beta_b = \tilde{L} \) gives rise to

\[
x'^0_{ba} = x^0_{ba} C_\beta, \quad x'^i_{ba} = \bar{x}^i_{ba}, \quad y'_{ba} = - x^0_{ba} S_\beta \]

and \( \gamma_{ba} = \tilde{L} C_\beta - \bar{x}^0_{ba} S_\beta \), respectively. Thus, by taking \( d^4 x d y = d^4 x' d y' \) into account, writing \( \Psi_0(x_b) \equiv \Psi(x_b, \tilde{L}) \) and \( \Phi(x_a, \gamma_a) = \delta(y_a - \tilde{L} \Phi_0(x_a) \), the eq.(2.11) in the \( \Sigma \) system can be written as

\[
\Psi_0(x_b) = \int d^4 x_a G'_{ba} \left( \bar{x}^0_{ba} C_\beta, \bar{x}^i_{ba}, - \bar{x}^0_{ba} S_\beta, \tilde{L} C_\beta - \bar{x}^0_{ba} S_\beta \right) \Phi_0(x_a)
\]

(2.15)

Then, with \( L_B = L/S_\beta \), eq.(2.10) leads to

\[
e^{\pm L_B (\tilde{\alpha}_b) b} \Psi_0(x_b) = \int d^4 x_a \left\{ e^{\pm L_B (\tilde{\alpha}_b) b} G_{ba} \right\} \Phi_0(x_a)
\]

\[
= \int d^4 x_a \left\{ e^{\mp L_B (\tilde{\alpha}_b) a} G'_{ba} \left( \bar{x}^0_{ba} C_\beta, \bar{x}^i_{ba}, - \bar{x}^0_{ba} S_\beta, \tilde{L} C_\beta - \bar{x}^0_{ba} S_\beta \mp L \right) \right\} \Phi_0(x_a)
\]

\[
= \int d^4 x_a G_{ba} \left\{ e^{\pm L_B (\tilde{\alpha}_b) a} \Phi_0(x_a) \right\}
\]

\[
= \int d^4 x_a G_{ba} \Phi_0(x_a^0 \pm L_B, x^i_a).
\]

(2.16)
The result is not trivial, since this finite-time displacement equation holds only for the parameter $L_B = L/S\beta$ with $\beta \neq 0$; otherwise, the form of the Green function $\tilde{G}_{ba}$ in eq. (2.16) will be changed.

(ii) rotating $S^1$ circle

As another possible model of moving-extra dimension, let us consider the circle $S^1$ with radius $R$ on which two coordinate systems of angle variables $\{\theta\}$ and $\{\theta'\}$ are defined. First, the $\{\theta\}$ defines the five-dimensional space with the period $2\pi$ by $\theta \equiv \theta + 2\pi$. Secondly, the $\{\theta'\}$ is the angle variable defined by $\theta = \theta' + \Omega x^0$, where $c\Omega$ is a constant angular velocity of the rotation. Each coordinate system is looked from the other system so as to be a rotating frame figure 2.2. We set that $\{\theta\}$ defines a time-independent extra dimension; then, $\{\theta'\}$ becomes a coordinate of a rotating frame depending on time through $\theta' = \theta - \Omega x^0$. The $R$ is defined as the proper radius of the circle $S^1$ observed in the rotating frame. The $(x^\mu, \theta)$ is corresponding to the $\Sigma$ system in the case (i). However, since the $\{\theta'\}$ defines a coordinate in non-inertial system, the $(x'^\mu, \theta')$, $(x'^\mu = x^\mu)$ is not counterpart of the $\Sigma'$ system in the case (i); and, we must be careful to write down the line element in $M_4 \otimes S^1$ spacetime. For the observers on the ring, the line element has the expression

$$ds^2 = dx'^\mu dx'_\mu + (Rd\theta')^2 = -(dx^0)^2 + dx^i dx_i + (dy - \beta dx^0)^2$$  \hspace{1cm} (2.17)

where $dy = Rd\theta$ and $\beta = \Omega R$. The period with respect to $\theta$ respectively gives the circumferences of the ring by $y \equiv y + 2L$, ($L = \pi R$) in the non-rotating frame. In eq. (2.17), $x^0$ is the global time, and the extra dimension depends on this time. On the other side, the $d\tilde{x}^0 \equiv \gamma^{-1}dx^0 + \beta d\tilde{y}$ in the above line element defines a local time depending on extra dimension $\{\tilde{y}\}$, which is fixed on the rotating frame. In the relativistic description of rotating rigid ring (or disk), it is sometimes said that the period in the rotating frame is expressed by $\tilde{y} \equiv \tilde{y} + 2L$, ($L = \pi R$) under the the synchronization of the points on the ring; and so, there arises the contraction of circumference of the rotating ring. This problem has a long history of controversy[20]. However, since we are dealing with the
\[ y \equiv y + L \equiv -y \text{ for fields on } y \text{ space, the circle becomes the orbifold } S^1/Z_2. \]

The Lagrangian of a mass \( m_0 \) particle embedded in this spacetime can be written as
\[ \mathcal{L} = -m_0 c \sqrt{-\left(\frac{ds}{dt}\right)^2}, \]
where the \( \tau \) is a time ordering parameter. The definition of the line element in eq. (2.17) leads to the constraint
\[ K \equiv -(p_0 + \beta p_y)^2 + p_i^2 + p_y^2 + (m_0 c)^2 = 0, \quad (2.18) \]
where \( p_0, p_i \) and \( p_y \) are momenta conjugate to \( x^0, x^i \) and \( y \) respectively. In q-number theory, \( K \Psi = 0 \) gives the free field equation of this particle, which can be reduced to the Klein-Gordon type of equation by the canonical transformation
\[ U \hat{K} \Psi = \left\{ -(\hat{p}_0)^2 + \hat{p}_i^2 + \hat{p}_y^2 + (m_0 c)^2 \right\} \Psi = 0 \quad (\hat{\Psi} = U \Psi), \quad (2.19) \]
where \( U = e^{\hat{K} \beta x^0 \hat{p}_y} \). Hence, the independent solutions of eq. (2.19) in the interval \( 0 \leq y' \leq L \) with the boundary conditions \( \hat{\Psi} \rvert_{y' = 0} = \hat{\Psi} \rvert_{y' = L} = 0 \) will be reduced to those of eqs. (2.1) and (2.2). Since \( U^{-1}x^\mu U = x^\mu \) and \( U^{-1}y' U = y - \beta x^0 \), the solution \( \hat{K} \Psi = 0 \) can be obtained by the inverse transformation \( \Psi(x^\mu, y') = U^{-1} \hat{\Psi}(x^\mu, y') = \hat{\Psi}(x^\mu, y - \beta x^0) \). This means that the Green function \( G_{ba} = (\hat{K}^{-1})_{ba} \) is given by the substitution \( y \rightarrow y - \beta x^0 \) for one in eq. (2.8).

Under those settings, we again regard \( y = 0 \) and \( y = L \) as the places of UV and IR branes respectively; then, the counterpart of eq. (2.15) becomes
\[ \Psi_0(x_b) = \int d^4 x_a G_{ba}(\vec{x}_b, -\beta \vec{x}^0_b, L - \beta \vec{x}^0_b) \Phi_0(x_a) \]
\[ \equiv \int d^4 x_a \tilde{G}_{ba} \Phi_0(x_a). \quad (2.20) \]

Then, in a similar way to derive eq. (2.16), one can obtain
\[ e^{\pm L_R (\partial_0)_{b}} \Psi_0(x_b) = \int d^4 x_a \tilde{G}_{ba} \Phi_0(x_0^a \pm L_R, x^i_a), \quad (2.21) \]
where \( L_R = L/\beta \). The result is the same as eq. (2.16) except the substitution \( L_B \rightarrow L_R \). We hereafter write the equations in the case (i) and those in the case (ii) in common by using \( L' \) instead of using \( L_B \) or \( L_R \).

### 2.3 Initial states in IR brane

Now, the next task is to determine the initial state \( \Phi_0(x_a) \) on the boundary surface corresponding to IR brane in the case of AdS-Spacetime, which is given by \( y = L \) for the rotating extra dimension. The initial states are given arbitrary in principle; however, we assume that the initial state satisfies \( (\hat{p}_\mu \hat{p}^\mu + (mc)^2) \Phi_0 = 0, \) where \( m \sim m_0 e^{-kL} \ll m_0 \), and the \( k \) is a dimensional parameter inherent in the AdS-Spacetime. Then we can put the initial state as \( \Phi_0(x) = \delta (\hat{p}^2 + (mc)^2) \phi_p(x), (m \approx 0) \). Here, the \( \delta \)-function allows two

---

background extra-dimensional spacetime instead of rigid ring, we don’t worry about this problem.
branches of solutions, since its argument is a quadratic form of \( \{ \hat{p}^\mu \} \). A simple choice of such a branch is to put \( \phi_p(x) = \theta(\hat{p}^0)\phi_0(x) \); that is, that

\[
\Phi_0(x) = \frac{1}{2\sqrt{\hat{p}^2 + (mc)^2}} \left( \hat{p}^0 - \sqrt{\hat{p}^2 + (mc)^2} \right) \phi_0(x), \tag{2.22}
\]

from which the following follows

\[
\Phi_0(x^0 \pm L', x^i) = e^{\pm \frac{i}{\hbar} L' \hat{p}^0} \Phi_0(x) = e^{\pm \frac{i}{\hbar} L' \sqrt{\hat{p}^2 + (mc)^2}} \Phi_0(x). \tag{2.23}
\]

Another interesting choice of the branch is given by putting \( \phi_p(x) = e^{-\frac{i}{\hbar} \hat{p}^+ x^-} \phi_0(x^+, x_-) \), \( (p^+ = \text{const.} > 0) \), where \( a^\pm = \frac{1}{\sqrt{2}}(a^0 \pm a^3) \) and \( a_\perp = (a^1, a^2) \) are the light-cone combinations of variables. Then, we obtain that

\[
\Phi_0(x) = \left[ \hat{p}^+ - \frac{1}{2p^+} \right] e^{-\frac{i}{\hbar} \hat{p}^+ x^-} \phi_0(x^+, x_-) \tag{2.24}
\]

which leads to

\[
\Phi_0(x^0 \pm L', x^i) = e^{\pm \frac{i}{\hbar} L' \frac{p^+ + 1}{2p^+}(\hat{p}_\perp^2 + (mc)^2)} \Phi_0(x^0, x^i). \tag{2.25}
\]

We note that the operators acting on the initial state \( \Phi_0 \) in the right hand sides of eqs.\((2.23)\) and \((2.25)\) contain space derivatives only; on this account, substituting eq.\((2.23)\) \( (\text{eq.} \text{(2.25)}) \) for eq.\((2.16) \ (\text{eq.} \text{(2.21)}) \), one can derive

\[
e^{\pm L'(\partial_0)} \Psi_0(x_b) = e^{\pm \frac{i}{\hbar} \hat{S}} \Psi_0(x_b), \tag{2.26}
\]

where

\[
\hat{S} = \begin{cases} 
L' \sqrt{\hat{p}^2 + (mc)^2} & \text{, } (\phi_p(x) = \theta(\hat{p}^0)\phi_0(x)) \\
\frac{L'}{\sqrt{2}} \left[ p^+ + \frac{1}{2p^+} (\hat{p}_\perp^2 + (mc)^2) \right] & \text{, } (\phi_p(x) = e^{-\frac{i}{\hbar} \hat{p}^+ x^-} \phi_0(x^+, x_-))
\end{cases}. \tag{2.27}
\]

As emphasized previously, eq.\((2.26)\) is not trivial, since eq.\((2.26)\) does not hold for an arbitrary \( L' \). The structure of eq.\((2.26)\) is nothing but the one of eq.\((1.3)\). We can regard a linear combination of \( \pm \) signs in eq.\((2.26)\) such as sinh\((L'(\partial_0))\Psi_0 = -i \sin(\frac{1}{\hbar} \hat{S})\Psi_0 \) as field equation; a significant combination should be chosen by asymptotic conditions on \( \Psi_0 \), etc.. Further, since the operator \( \hat{W}_\pm \equiv e^{\pm L' \partial_0} - e^{\mp i \hat{S}} \) commutes with \( \hat{p}_\mu \), the resultant field equation \( \hat{W}_\pm \Psi_0 = 0 \) is invariant under the translation; in other words, the four momentum conserve in spite of the time dependence of the extra dimension. This is due to the reason that the starting point of the difference equation \((2.26)\) is free.

It should also noticed that if the \( m^2 \) is negligible small, the initial state \((2.22)\) gain a scaling dimension \( 2 - w \) provided that \( \phi_0(x) \) has the scaling dimension \( w \); that is, \( \phi_0(\lambda x) = \lambda^{-w}\phi_0(x) \) is satisfied. This fact may be meaningful, when we deduce the form of \( \Phi_0(x) \) in consideration of some CFT structures in IR brane.
3 Spin-less particles in AdS$_5$ spacetime

In this section, we discuss the spin-less particles embedded in the AdS$_5$ spacetime characterized by the line element with the warp factor:

$$ds^2 = e^{-2ky}g_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

(3.1)

where the fifth dimension is assumed to be $S^1/Z_2$ having the coordinate $0 \leq y \leq L$.

3.1 The case of time-independent extra dimension

Before going to the problems of time-dependent extra dimension, it is worthwhile to study the case of time-independent extra dimension to make clear the relation between the inner product of states under some boundary conditions and the operator ordering.

The action $S = -m_0c \int \sqrt{-g}ds^2$ for a spin-less particle with the proper mass $m_0$ gives, as usual, the constraint

$$\mathcal{K}_0 \equiv e^{2ky}p_\mu p^\mu + p_y^2 + (m_0c)^2 = 0.$$  

(3.2)

In q-number theory, the substitution $\mathcal{K}_0(p_\mu, p_y) \rightarrow \hat{\mathcal{K}}_0(\hat{p}_\mu, \hat{p}_y)$ causes no problems of operator ordering; and, the inner products $(\mathcal{K}_0)_{ij} = \int d^4x \int_0^L dy \Psi^*_i \mathcal{K}_0 \Psi_j$ define matrix elements of a hermitian operator under the boundary conditions $\Psi|_{y=0} = \Psi|_{y=L} = 0$. Though the field equation $\hat{\mathcal{K}}_0 \Psi = 0$ is a q-number counterpart of eq.(3.2), its explicit form depends on the representations of $\{\Psi\}$ space.

To deal with the field equation in AdS$_5$ spacetime, it is convenient to use the variable $z = e^{ky}, (1 \leq z \leq L = e^{kL})$ rather than $y$. Indeed, since

$$\hat{p}_y^2 = z^2 \left\{ (\hat{p}_z - i\frac{\hbar k}{2z})^2 - \frac{(\hbar k)^2}{4z^2} \right\}, \quad \left( \hat{p}_z = -i(\hbar k) \frac{\partial}{\partial z} \right),$$  

(3.3)

the field equation $\hat{\mathcal{K}}_0 \Psi = 0$, where $\hat{\mathcal{K}}_0 = e^{-2ky}\hat{\mathcal{K}}_0$, contains power potentials of $z$ only. Furthermore, under the substitution $\Psi = \sqrt{z} \hat{\Psi}$, there arises the canonical transformation

$$\hat{\mathcal{K}}_0 = z^{1/2}\hat{\mathcal{K}}_0 z^{-1/2} = \hat{p}_\mu p^\mu + \hat{p}_z^2 - (\hbar k)^2 \frac{\Delta}{z^2},$$  

(4.4)

where

$$\Delta = \left( \frac{1}{2} \right)^2 - \left( \frac{m_0c}{\hbar k} \right)^2.$$  

(5.5)

Another possibility for the q-number counterpart of $\mathcal{K}_0$ is to put $\hat{\mathcal{K}}_0 = \frac{1}{\sqrt{g}} \hat{\mathcal{K}}_0 (\sqrt{-g}g^{\mu\nu} \hat{p}_\mu)$, where $g$ is the determinant of the metric in the expression $ds^2 = g_{\mu\nu}dx^\mu dx^\nu, (\mu, \nu = 0, 1, 2, 3, 5; x^5 = y)$ for eq.(3.1). In this case, since we get $\hat{\mathcal{K}}_0 = \hat{p}^2 + e^{-2ky}p_y \left( e^{-4ky}p_y \right) + e^{-2ky}(m_0c)^2 = \hat{p}^2 - (\hbar k)^2 \left( \frac{2}{\sqrt{z}} - \frac{1}{z} \right)^2 - (\hbar k)^2 \frac{\Delta}{z^2}$ with $\Delta = (\frac{2z}{\sqrt{z}})^2 - (\frac{m_0c}{\hbar k})^2$, the $\hat{\mathcal{K}}_0 = z^{3/2}\hat{\mathcal{K}}_0 z^{-3/2}$ is again reduced to the form of eq.(3.4). However, we don’t adopt this type of operator ordering, since the Laplacian is not obvious to extend its form to the case of spinning particle in addition that the covariance between $M_4$ and $S^1/Z_2$ is not required.
Thus, the resultant field equation $\hat{K}\tilde{\Psi} = 0$ in $\{\tilde{\Psi}\}$ space, becomes a simple Klein-Gordon type of equation with a power potential. The operator $\hat{K}_0$ is hermitian under the boundary conditions $\tilde{\Psi}\big|_{z=1} = \tilde{\Psi}\big|_{z=L_z} = 0$ obviously; and, the matrix elements of $\hat{K}_0$ in $\{\tilde{\Psi}\}$ space are related to those of $\hat{K}_0$ in $\{\Psi\}$ space by

$$
(K_0)_{ij} = \int d^4x \int_0^L dy \tilde{\Psi}_i^\dagger \hat{K}_0 \tilde{\Psi}_j = \frac{1}{\kappa} \int d^4x \int_1^{L_z} dz \tilde{\Psi}_i^\dagger \hat{K}_0 \tilde{\Psi}_j
$$

(3.6)

; that is, that the description in $\{\Psi\}$ space and that in $\{\tilde{\Psi}\}$ space are equivalent. The eq.(3.6) also explains the reason why the ordering $e^{-2ky}\hat{K}_0$ is preferable.

In the field equation $\hat{K}_0\tilde{\Psi} = 0$, the $V(z) \equiv -(hk)^2\Delta z$ plays the role of an effective potential in $z$-space, the sign of which is decided by the ratio between $m_0$ and $k$. If we assume that the large mass hierarchy is realized by $M_{W} \simeq e^{-kL}M_{P}$, where $M_{W}$ and $M_{P}$ are respectively a mass in TeV scale and the Planck mass, then the spacetime parameter will be estimated as $kL \simeq 35$. The $k^2L$ is proportional to the scalar curvature in AdS$_5$ spacetime; and, its scale is usually set to be $k \sim l_{P}^{-1}$, the scale of inverse Planck length. The effective potential $V(z)$, then, becomes negative for that the Compton wave length of the particle $\lambda \equiv \frac{k}{mc}$ is larger than the twice Planck length $2l_{P}$; that is, for $m_0 \lesssim \frac{1}{2}M_{P} = \frac{hk}{2c}$.

A particular interesting choice of the proper mass of the particle is to put $m_0 = \frac{1}{2}\frac{hk}{c} = \frac{1}{2}M_{P}$, to which the effective potential $V(z)$ vanishes. In this case, the field equation $\hat{K}_0\tilde{\Psi} = 0$ is reduced to eq.(2.1) in the previous section by reading $(k^{-1}z, 0, L_z)$ as $(y, m_0, L)$. Then, the Green function (2.8) is again applicable to the field $\tilde{\Psi}(x, z)$ for the particle with $m_0 = \frac{1}{2}M_{P}$. This is a result of fine tuning of the parameters, but the form of (2.8) approximately holds even for the particle with $m_0 c^2 \lesssim$ GUT energy (appendix A). In what follows, we confine our attention to this particular interesting case only. The next task is to extend this model in AdS$_5$ spacetime to one with time-dependent extra dimension.

3.2 The case of time-dependent extra dimension

We, here, study a spin-less particle embedded in the AdS$_5$ type of spacetime with the warp factor, in which the extra dimension is inclined to a timelike direction so that

$$
ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu + (dy - \beta e^{-ky}dx^0)^2,
$$

(3.7)

This line element is reduced to eq.(3.1) and eq.(2.17) in the respective limit of $\beta \to 0$ and $k \to 0$; and, we regard that the coordinate $y$ in eq.(3.7) represents the orbifold $S^1/Z_2$ for the fields having the identification $y \equiv y + L \equiv -y$. As with the models in the previous section, we don’t care that the line element (3.7) breaks the Lorentz symmetry in $(x^{\mu})$ space until the next section, where the recovery of this symmetry will be discussed.

Now, we set the action for a spin-less mass $m_0$ particle by using a modified form of the above line element so that

$$
S = -m_0c \int \left( e^{-2ky}dx^2 - (dy - \beta e^{-ky}(dx^0 + du))^2 \right),
$$

(3.8)

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where the \( u \) is an auxiliary variable of the particle introduced by a technical reason. From this action, one can derive the constraints
\[
\begin{align*}
\mathcal{K}_\beta &= e^{2ky} \left\{ -(p_0 - p_u)^2 + p_d^i \right\} + \left\{ p_y^2 + (m_0 e)^2 \right\} = 0, \\
\phi_\beta &= p_u + \beta e^{-ky} p_y = 0,
\end{align*}
\]
where \( p_u \) is the momentum conjugate to \( u \) \(^5\). Here, we have written the ordering of each term in eq.(3.9) by taking into account the form in q-number theory. In the other words, we define the q-number counterparts of those constraints by the substitution \((p_\mu, p_y, p_u) \to (\hat{p}_\mu, \hat{p}_y, \hat{p}_u)\) keeping the ordering of the right-hand side in eq.(3.9) and eq.(3.10).

In terms of \( z = e^{ky} \), one can write \( e^{-ky} \hat{p}_y = \hat{p}_z \), \( \hat{p}_y^2 = \text{r.h.s of eq.}(3.3) \) and so on. Further, according to the line of thought in the previous subsection, we should define \( \hat{\mathcal{K}}(\beta) = e^{-2ky} \hat{\mathcal{K}}_\beta \) so that
\[
\begin{align*}
\hat{\mathcal{K}}(\beta) &= \left\{ -(\hat{p}_0 - \hat{p}_u)^2 + \hat{p}_d \hat{p}_d^i \right\} + \left( \hat{p}_z - i \frac{\hbar k}{2z} \right)^2 - (\hbar k)^2 \frac{\Delta}{z^2}, \\
\hat{\phi}_\beta &= \hat{p}_u + \beta \hat{p}_z.
\end{align*}
\]
Then the equations \( \hat{\mathcal{K}}(\beta) \hat{\Psi} = 0 \) and \( \hat{\phi} \hat{\Psi} = 0 \) look like to play the roles of the field equation and its supplementary condition, respectively. Since, however, those equations are not compatible, we first carry out the unitary transformations \( \hat{\mathcal{K}}'_{\beta} = U \hat{\mathcal{K}}_{\beta} U^{-1} \) and \( \hat{\phi}'_\beta = U \hat{\phi}_\beta U^{-1} \) by \( U = e^{-\frac{\hbar k}{2z} \beta u} \), which lead to
\[
\begin{align*}
\hat{\mathcal{K}}'_{\beta} &= \hat{p}_\mu \hat{p}_\mu + \left( \hat{p}_z - i \frac{\hbar k}{2z} \right)^2 - (\hbar k)^2 \frac{\Delta}{z^2}, \\
\hat{\phi}'_\beta &= \hat{p}_u + \beta \hat{p}_z.
\end{align*}
\]
Therein, the \( \hat{p}_u \) is removed from \( \hat{\mathcal{K}}'_{\beta} \); and, it is contained only in \( \hat{\phi}'_\beta \); and so, in this stage, we can eliminate \( \hat{\phi}'_\beta \) by regarding \( \hat{p}_u \equiv -\beta \hat{p}_z \) as the definition of \( \hat{p}_u \). In relation to this reinterpretation, hereafter, we have to read \( U = e^{\frac{\hbar k}{2z} \beta u} \).

Lastly, by means of the canonical transformation \( \hat{\mathcal{K}}_\beta = z^{1/2} \hat{\mathcal{K}}'_\beta z^{-1/2} \), we arrive at
\[
\hat{\mathcal{K}}_\beta = \hat{p}_\mu \hat{p}_\mu + \hat{p}_z^2 - (\hbar k)^2 \frac{\Delta}{z^2}.
\]
Accordingly, the field equation \( \hat{\mathcal{K}}_\beta \hat{\Psi} = 0 \) in \{\Psi\} space can be represented as \( \hat{\mathcal{K}}_\beta \hat{\Psi} = 0 \) in \{\hat{\Psi}\} space. Further, remembering \( \hat{\Psi} = \sqrt{\hat{\mathcal{K}}} \Psi' \), \( (\Psi' = U \Psi) \), one can verify the relation between the matrix elements in \{\Psi\} space and those in \{\hat{\Psi}\} space.
\[
(\mathcal{K}_\beta)_{ij} = \int d^4 x \int_0^L dy \Psi_i^* \hat{\mathcal{K}}_{\beta} \Psi_j = \frac{1}{k} \int d^4 x \\int_1^{L_x} dz \Psi_i^* \hat{\mathcal{K}}_{\beta} \hat{\Psi}_j.
\]
The resultant field equation \( \hat{\mathcal{K}}_\beta \hat{\Psi} = 0 \) coincides with the \( \hat{\mathcal{K}}_\beta \hat{\Psi} = 0 \) in the previous subsection, in which the effective potential term again vanishes for \( \Delta = 0 \). Since we are dealing with this case, the field equation is reduced to Klein-Gordon type of equation (2.1).

\(^5\)The equation (3.10) says that the \( u \) is a variable causing the displacement \( z \to z - \beta ku \).
Then, applying the boundary conditions $\hat{\Psi}(x, 1) = \hat{\Psi}(x, L_z) = 0$, the independent basis in $z$ space become $\phi^b_\theta(z) = \sqrt{\frac{2}{L(z)}} \sin \left( \frac{\pi n(z-1)}{L(z)} \right), \ (L(z) = L_z - 1, n = 1, 2, \cdots)$.

The Green function $\hat{G}_{ba}$ for $\hat{\Psi}(x, z)$ is, thus, obtained by substituting $(z, L(z))$ for $(y, L)$ in eq.(2.8) and eq.(2.9). Then the field $\hat{\Psi}(x_b, z_b)$ with the initial field $\Phi(x_a, z_a)$ can be written as

$$
\hat{\Psi}(x_b, z_b) = \int d^4x_a \int dz_a \hat{G}(\bar{x}_ba, \bar{z}_ba, \bar{z}_ba) \Phi(x_a, z_a).
$$  \hfill (3.17)

Under the substitutions $\Psi' = z_b^{-1/2} \hat{\Psi}_b$ and $\Phi' = z_a^{-1/2} \Phi_a$, this equation becomes

$$
\Psi'(x_b, z_b) = \int d^4x_a \int dz_a \frac{\bar{z}_a}{z_b} \hat{G}(\bar{x}_ba, \bar{z}_ba, \bar{z}_ba) \Phi'(x_a, z_a).
$$  \hfill (3.18)

Further, applying the inverse transformation $U_b^{-1} = (e^{-\frac{i}{\hbar} \beta a_0 \hat{P}_a})_b$ to $\Psi'_b$, we obtain

$$
\Psi(x_b, z_b) = \int d^4x_a \int dz_a \frac{z_a}{z_b - k\beta x_b^0} \hat{G}(\bar{x}_ba, \bar{z}_ba - k\beta x_b^0, \bar{z}_ba - \frac{1}{2} k\beta x_b^0) \Phi'(x_a, z_a).
$$  \hfill (3.19)

The last step is to fix $z_a$ and $z_b$ to $z = L_z$, the value of $z$ on the IR end. Then writing $\Phi'(x_a, z_a) = \delta(z_a - L_z) \Phi_0(x_a)$ and $\Phi_0(x_b) = \sqrt{1 - k\beta x_b^0 / L_z} \Psi(x_b, z_b = L_z)$, the eq.(3.19) is reduced to a simple form

$$
\Psi_0(x_b) = \int d^4x_a \hat{G}(\bar{x}_ba, -k\beta x_b^0, L_z - \frac{1}{2} k\beta x_b^0) \Phi_0(x_a).
$$  \hfill (3.20)

Since the Green function $\hat{G}$ have the periods $2L(z)$ and $L(z)$ with respect to the second and the third arguments respectively, one can derive the following difference type of equation:

$$
e^{\pm L'(\bar{\alpha})_b} \Psi_0(x_b) = \int d^4x_a \hat{G}(\bar{x}_ba, -k\beta x_b^0, L_z - \frac{1}{2} k\beta x_b^0) \Phi_0(x_a^0 \pm L', x_a^1),
$$  \hfill (3.21)

where $L' = 2L(z)/k\beta$. As discussed in the previous section, this equation reads to a domain type of field equations by assuming the initial field $\Phi_0(x_a)$ such as eqs.(2.22) and (2.24).

## 4 A possible model of spinning particle in AdS$_5$ spacetime

Until now, the time dependence of extra dimension has been introduced without regard to the Lorentz invariance of the action through the substitutions such as $dy \rightarrow dy - \beta e^{-ky}dx^0$. A simple way to recover the Lorentz invariance is to replace $dx^0$ by $V_\mu dx^\mu$, where $V_\mu$ is a timelike Lorentz vector constructed out of the dynamical variables inherent in the particles under consideration. As a model of such a particle, we study a spinning particle characterized by line element

$$
ds^2 = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu + (dy - \beta e^{-ky} V_\mu dx^\mu)^2,
$$  \hfill (4.1)

where $V_\mu \equiv \tilde{\zeta}_\gamma \mu \theta$, $(\tilde{\zeta} = \zeta^T \gamma^0)$ is the bilinear representation of a Lorentz vector by Majorana spinors $\theta^\alpha = (\theta^\alpha)^* \gamma^\alpha = (\zeta^\alpha)^* \gamma^\alpha$, $(\alpha = 1, 2, 3, 4)$, and the $\gamma^\mu$, $(\mu = 0, 1, 2, 3)$ are $\gamma$
matrices in Majorana representation. The \((\theta, \zeta)\) may be either Grassmann variables or ordinary variables. We set the action of this particle, by taking (3.8) into account, so that

\[
S = \int \left[ -\kappa c \sqrt{-e^{2ky}dx^2 - \{dy - \beta e^{-ky}(V \cdot dx + du)\}^2} - (-1)^{\epsilon}i\hbar \tilde{\zeta}^\alpha \, d\theta^\alpha \right],
\]

where \(-i\hbar \tilde{\zeta}^\alpha\) is the momentum conjugate to \(\theta^\alpha\). We put simply \(\kappa\) as a constant, although it may be a scalar function constructed out of \((\theta, \zeta)\) in general. The \(\epsilon\) is the Grassmann parity defined so that \(\epsilon = 1\) for Grassmann \((\theta, \zeta)\) and \(\epsilon = 0\) for otherwise. In what follows, we confine our attention to the case of Grassmann variables, since we are interested in a finite dimensional spin representation. In q-number theory, thus, we require the anti commutator \(\{\tilde{\zeta}^\alpha, \theta^\beta\} = \delta^{\alpha \beta}\); more details on the spin representations in the present model will be given in appendix B.

The action \(S\) derives the simultaneous constraints

\[
\mathcal{K}_S \equiv e^{2ky} (p - V p_a)^2 + \left\{ \mu^2 + (\kappa c)^2 \right\} = 0, \quad (4.3)
\]

\[
\phi_S \equiv p_a + \beta e^{-ky} p_y = 0. \quad (4.4)
\]

Under the decomposition \(V^\mu = V^\mu_\parallel + V^\mu_\perp\), where \(V^\mu_\parallel = -p^\mu (p \cdot V)\), \(V^\parallel_\perp = V^2 - V^\parallel_\parallel^2\), and \(p^\mu = p^\mu / \sqrt{-p^2}\), \((p^2 = -1)\), the \(\mathcal{K}_S\) can also be written as

\[
\mathcal{K}_S = e^{2ky} (p - V p_a)^2 + V_\perp^2 \beta^2 \mu^2 + \left\{ \mu^2 + (\kappa c)^2 \right\}. \quad (4.5)
\]

We, here, regard that the \(V^2_\perp = V^2 - V^\parallel_\parallel^2\) takes a constant associated with the representation of \((\theta, \zeta)\) in q-number theory. In other words, the explicit form of \(\mathcal{K}_S\) is determined for respective spin representations.

Now, similarly in the previous section, we define the operator \(\hat{\mathcal{K}}_S = e^{-2ky} \hat{\mathcal{K}}_S\) in the q-number theory such that

\[
\hat{\mathcal{K}}_S = (\hat{p} - V_\parallel \hat{p}_a)^2 + e^{-2ky} \left\{ (V_\perp^2 \beta^2 + 1) \hat{\mu}^2 + (\kappa c)^2 \right\}. \quad (4.6)
\]

Then, with consideration for \([x \cdot V_\parallel, \hat{p}^\mu_\parallel] = [x \cdot V_\parallel, V^\mu_\parallel] = 0\) and \([x \cdot V_\parallel, \hat{p}^\mu_\perp] = i\hbar V^\mu_\parallel\) one can remove the \(V_\parallel \hat{p}_a\) in \(\hat{\mathcal{K}}_S\) by the transformation \(\hat{\mathcal{K}}'_S = U \hat{\mathcal{K}}_S U^{-1}\) with \(U = e^{i\hat{p} \cdot (x \cdot V_\parallel)}\); and, the \(\hat{\phi}_S\) stays unchanged under this transformation so that \(\hat{\phi}'_S = U \hat{\phi}_S U^{-1} = \hat{\phi}_S\). Thus using the variable \(z = e^{ky}\) as usual, the operators \(\hat{\mathcal{K}}'_S\) and \(\hat{\phi}'_S\) become

\[
\hat{\mathcal{K}}'_S = \hat{p}^2 + A \left\{ \left( \hat{p}_z - \frac{i \hbar k}{2z} \right)^2 + (\hbar k)^2 \Delta' \right\}, \quad (4.7)
\]

\[
\hat{\phi}'_S = \hat{p}_a + \beta \hat{p}_z, \quad (4.8)
\]

where \(A = V^2_\perp \beta^2 + 1\) and

\[
\Delta' = \left( \frac{1}{2} \right)^2 - \left( \frac{\kappa c}{\sqrt{Ahk}} \right)^2. \quad (4.9)
\]
In this stage, we eliminate $\hat{\delta}'_{S}$ by reading $\hat{p}_{a} \equiv -\beta \hat{p}_{z}$ as the definition of $\hat{p}_{a}$; after that, applying the transformation $\hat{K}_{S} = z^{1/2} \hat{K}'_{(S)} z^{-1/2}$, we finally arrive at the expression

$$
\hat{K}_{S} = \hat{p}^{2} + A \left\{ p_{z}^{2} - (\hbar k)^{2} \Delta'_{z} \right\} .
$$

(4.10)

If we, here, use the variable $z' = z/\sqrt{A}$, then the field equation $\hat{K}_{(S)} \Psi = 0$ will coincide with $\hat{K}_{\beta} \tilde{\Psi} = 0$ in the previous section. Hereafter, we again consider the case $\Delta' = 0$ realized for $\kappa/\sqrt{A} = 1/\hbar M_{P}$, where as shown in appendix B, the operator $A$ takes the eigenvalues $1, 3\beta^{2} + 1, 6\beta^{2} + 1, \cdots$ respectively in the spin $0, 1/2, 1, \cdots$ representation spaces.

The Green function $(\hat{K}_{S}^{-1})_{ba}$ in $\{ \Psi(x, z') \}$ space, has the same form as eq.(2.8) with eq.(2.9) by replacing $0 \leq y \leq L$ by $1 \leq z \leq L_{z}$, or by $A^{-1/2} \leq z' \leq L_{z} A^{-1/2}$. The transition from $\Phi_{a}$ to $\tilde{\Psi}_{b}$ is, thus, written as

$$
\tilde{\Psi}(x_{b}, z_{b}) = \frac{1}{\sqrt{A}} \int d^{4}x_{a} \int dz_{a} \tilde{G} \left( \bar{x}_{ba}, \bar{z}_{ba}/\sqrt{A}, \bar{z}_{ba}/\sqrt{A} \right) \Phi(x_{a}, z_{a}),
$$

(4.11)

where $\Psi(x, z) = \Psi(x, z' = z/\sqrt{A})$ etc. Then, going to the description by $\Psi' = \tilde{\Psi}/\sqrt{z}$, $\Phi' = \bar{\Phi}/\sqrt{z}$, and carrying out the transformation $\Psi = U^{-1} \Psi'$ with $U^{-1} = e^{-x/2 \hat{V}}(\hat{x}) \beta \hat{p}_{z}$, we obtain

$$
\Psi(x_{b}, z_{b}) = \frac{1}{\sqrt{A}} \int d^{4}x_{a} \int dz_{a} \frac{z_{a}}{z_{b} - k\beta(x \cdot V_{\parallel})_{b}} \times \tilde{G} \left( \bar{x}_{ba}, \bar{z}_{ba} - k\beta(x \cdot V_{\parallel})_{b}, \bar{z}_{ba} - \frac{1}{2}k\beta(x \cdot V_{\parallel})_{b} \right) \Phi'(x_{a}, z_{a}).
$$

(4.12)

The next step is to fix $z_{a}$ and $z_{b}$ to $z = L_{z}$; then, setting $\Phi'(x_{a}, z_{a}) = \delta(z_{a} - L_{z}) \Phi_{0}(x_{a})$ and $\Psi_{0}(x_{b}) = \sqrt{1 - k\beta(x \cdot V_{\parallel})_{b}/L_{z}} \Psi(x_{b}, z_{b} = L_{z})$, the eq.(4.12) becomes

$$
\Psi_{0}(x_{b}) = \frac{1}{\sqrt{A}} \int d^{4}x_{a} \tilde{G} \left( \bar{x}_{ba}, -k\beta(x \cdot V_{\parallel})_{b}, \frac{L_{z} - \frac{1}{2}k\beta(x \cdot V_{\parallel})_{b}}{\sqrt{A}} \right) \Phi_{0}(x_{a}),
$$

(4.13)

By definition, the Green function has the periodicities $L_{z'} (= L_{z}/\sqrt{A})$ and $2L_{z'}$ with respect to the second and the third arguments in $\tilde{G}$ respectively, one can verify

$$
e^{\pm L'(V_{\parallel})_{b} \Psi_{0}(x_{b})} = \frac{1}{\sqrt{A}} \int d^{4}x_{a} \tn{G} \left( \bar{x}_{ba}, -k\beta(x \cdot V_{\parallel})_{b}, \frac{L_{z} - \frac{1}{2}k\beta(x \cdot V_{\parallel})_{b}}{\sqrt{A}} \right)
$$

\hspace{1cm}
$$\times \Phi_{0}(x_{a} \pm L' V_{\parallel}) \tag{4.14}
$$

where $L' = 2\sqrt{A}L_{z}/(k\beta V_{\parallel}^{2}) = 2L_{z}/(k\beta V_{\parallel}^{2})$, and the $V_{\parallel}^{2}$ can be treated as a number given by the spin representation under consideration (appendix B). Therefore, in a similar sense in eq.(2.22), if we assume the initial fields $\{ \Phi_{0} \}$ so that

$$
\Phi_{0}(x) = \delta(\hat{p}^{2} + (mc)^{2}) \theta(\hat{p}^{0}) \phi_{0}(x), \quad (m = e^{-kL_{z}\kappa/\sqrt{A}}),
$$

(4.15)

then with the aid of $V_{\parallel} \cdot \partial = V \cdot \partial$, we arrive at a domain type of field equation

$$
e^{\pm L'(V \cdot \partial)_{b} \Psi_{0}(x_{b})} = e^{\mp L'(V' \sqrt{\hat{p}^{2} + (mc)^{2} + V \cdot \hat{p}^{0}})_{b} \Psi_{0}(x_{b})} \tag{4.16}
$$

This is nothing but the equation, which is wanted to derive based on the particle model embedded in AdS_{5} spacetime. The key is a periodic structure of the Green function.
5 Summary and discussion

In this paper, we have studied the possibility to derive a Yukawa’s domain type of field equation as an effective field equation for the particles embedded in the AdS$_5$ spacetime with a warp factor. The keys to get such an effective theory are threefold: the first is a periodic structure of Green function of the particle in the AdS$_5$ spacetime with respect to the fifth dimensional variable $y$; the second is time-dependent modifications of the fifth dimension, the third is the choice of the initial states in the IR brane.

In more detail, the fifth dimension in the AdS$_5$ spacetime with the warp factor is set as a one dimensional orbifold, in which the particles suffer an infinite square-well potential under the Dirichlet type of boundary conditions at both ends of the fifth dimension. Then there arises a periodic structure to the Green function in a similar way to the problem of the particle in a box. As a result, a local field equation in the bulk, causes a non-local effective field equation on the IR brane. To set up the local field equation in the bulk, however, the warp factor causes a problem of operator ordering. According to the prescription used in this paper, those local field equations are reduced to simple Klein-Gordon type of equations in $M_5$ under a fine tuning of parameters corresponding to $m_0 = \frac{1}{2} M_P$, where $m_0$ is a proper mass of the particle at UV brane. This means that some non-local feature of field equation on the IR brane is expected for the particle with the mass $m = m_0 e^{-kL}$, which corresponds to $\sim 3.7$ TeV/c$^2$ for $kL \sim 35$.

Secondly, we have imposed the mixing between the time $t$ and the $y$ in the argument of the Green function so that the periodicity of the Green function with respect to the fifth dimensional variable is copied onto the time variable. In other words, we have introduced a time-dependent extra dimension or a moving-extra dimension. There, we have studied two cases: the extra dimension under the Lorentz boost and the orbifold based on rotating $S^1$ circle. In both cases, the Green function acquired desirable properties to derive non-local field equations on the IR brane. In the third step, we have discussed choice of the initial fields in the IR brane to realize Yukawa’s domain type of field equations for those fields restricted to that brane. Then, we could show that a fairly wide class of initial states satisfying the on mass-shell equation led to the domain type of field equations.

Those domain type of field equations are, however, difference equations with respect to the world time $t$ itself; and so, those equations break the Lorentz covariance in the IR brane. In order to recover the covariance, in the last step, we have extended the model to those of spinning particles. One can construct a timelike vector out of those spin degrees of freedom; and so, the directional difference can be set along this vector. As for the spin degrees of freedom, we have studied the case of Majorana spinor of Grassmann variables. Then, according to a similar procedure with the case of spin-less particles, we could again derive a domain type of field equation, to which the Lorentz covariance is ensured. There are, however, many ways introducing the spin degrees of freedom including the case of bosonic spinors. Those model constructions associated with the choice of initial states determines the respective forms of the phase operator $S$ in eq.(1.3).

Lastly, we stress the points in getting a domain type of field equation: (1) one degrees of freedom of the particle is bounded by a infinite square well type of potential, (2) the
boundaries of this potential are moving. Those points (1) and (2) are rather easy to realize in a flat spacetime as the examples in section 2. This suggests that there arises a similar effect in a low energy physics in 3-dimensional space. In addition to investigate of a deeper meaning of Yukawa’s theory of elementary domain, the applications of the present formalism to low energy physics may also be interesting future problem.

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A On the solutions for $\Delta \neq 0$

The $\tilde{K}_0$ in eq.(3.4) has the form $\tilde{K}_0 = \hat{p}^2 + (hk)^2 \mathcal{M}^2$, where $\mathcal{M}^2 = \left(-\frac{d^2}{dz^2} - \frac{\Delta}{z}\right)$. We, here, discuss the eigenvalue equation $(hk)^2 \mathcal{M}^2 \phi_\lambda = \lambda \phi_\lambda$ without requiring the condition $\Delta = 0$; however, we assume that the proper mass $m_0$ is less than or equal to the GUT energy scale. Using the dimensionless variable $\lambda = \frac{(hk)}{\tilde{\lambda}}$, the eigenvalue equation becomes the Bessel’s equation

$$\frac{d^2 u(z)}{dz^2} + \frac{1 - 2\alpha}{z} \frac{du(z)}{dz} + \left(\beta^2 + \frac{\alpha^2 - \nu^2}{z^2}\right) u(z) = 0 \quad (A.1)$$

with $\alpha = \frac{1}{2}$, $\beta^2 = \tilde{\lambda}$ and $\nu^2 = \left(\frac{m_0 c}{E_P}\right)^2$. The independent solutions of this equation have the form $u(z) = Z_\nu(\beta z)$, where $Z_\nu(x)$ is one of $\{J_\nu(x), N_\nu(x), H^{(1)}_\nu(x), H^{(2)}_\nu(x)\}$. By assumption, $\nu^2 \approx 10^{-8}$, $(E_P = M_P c^2)$, we can put $\nu = 0$. Thus, one can take $\sqrt{z} J_0 \left(\sqrt{\tilde{\lambda}} z\right)$ and $\sqrt{z} N_0 \left(\sqrt{\tilde{\lambda}} z\right)$ as the independent solutions of this equation; and so,

$$\phi_\lambda(z) = \sqrt{\tilde{z}} \left(A J_0(\sqrt{\tilde{\lambda}} z) + B N_0(\sqrt{\tilde{\lambda}} z)\right), \quad (A.2)$$

where $A$ and $B$ are constants. The boundary condition $\phi_\lambda(1) = 0$ at UV end allows us to write the $\phi_\lambda(z)$ as

$$\phi_\lambda(z) = N \sqrt{\frac{\pi \sqrt{\tilde{\lambda}} z}{2}} \left\{N_0(\sqrt{\tilde{\lambda}}) J_0(\sqrt{\tilde{\lambda}} z) - J_0(\sqrt{\tilde{\lambda}}) N_0(\sqrt{\tilde{\lambda}} z)\right\} \quad (A.3)$$

$$= -N' \sqrt{\frac{\pi \sqrt{\tilde{\lambda}} z}{2}} \left\{\cos(\delta_\lambda) J_0(\sqrt{\tilde{\lambda}} z) + \sin(\delta_\lambda) N_0(\sqrt{\tilde{\lambda}} z)\right\}, \quad (A.4)$$

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where \(\cos(\delta_\lambda) = -N_0(\sqrt{\lambda})/R, \sin(\delta_\lambda) = J_0(\sqrt{\lambda})/R\) with \(R = \sqrt{N_0(\sqrt{\lambda})^2 + J_0(\sqrt{\lambda})^2}\).

Here, by taking the asymptotic forms of Bessel functions \(^6\) into account, eq.(A.4) leads to

\[
\phi_\lambda(z) \sim \phi_\lambda^{\text{asympt}}(z) \equiv -N' \cos\left(\sqrt{\lambda}z - \frac{\pi}{4} - \delta_\lambda\right), \quad (\sqrt{\lambda}z \gg 1).
\] (A.5)

Then, if we apply the boundary condition at IR end \(\phi_\lambda(L_z) = 0, (L_z = e^{kL})\), we obtain the condition \(\sqrt{\lambda}L_z - \frac{\pi}{4} - \delta_\lambda = (n - \frac{1}{2})\pi, (n = 1, 2, \cdots)\), which is still complex equation of \(\lambda\) due to the \(\lambda\) dependence of \(\delta_\lambda\). This condition also implies that the asymptotic form \(\phi_\lambda^{\text{asympt}}(z)\) is realized under the condition \((z/L_z)n\pi \gg 1\). In practice, each solution \(\phi_\lambda_n(z)\) soon arrives at the asymptotic form \(\phi_\lambda^{\text{asympt}}(z)\) according as \(z/L_z\) increase even for small \(n\). This suggests that \(\phi_\lambda^{\text{asympt}}(z)\) is a useful approximation of \(\phi_\lambda(z)\), though this approximation spoils the boundary condition at IR end. In order to recover the boundary condition at IR end, let us adjust anew the \(\delta_\lambda\) as a parameter so that \(\phi_\lambda^{\text{asympt}}(1) = 0\) holds. Then we obtain another condition \(\sqrt{\lambda} - \frac{\pi}{4} - \delta_\lambda = (l - \frac{1}{2})\pi, (l = 0, \pm 1, \cdots)\), from which the following follows:

\[
\sqrt{\lambda}z - \frac{\pi}{4} - \delta_\lambda = \frac{(n-l)\pi}{L(z)}(z-1) + (l - \frac{1}{2})\pi, \quad (L(z) = L_z - 1).
\] (A.6)

Thus one can write

\[
\phi_n(z) = \phi_\lambda^{\text{asympt}}(z)\big|_{\text{adjusted} \delta_\lambda} = \pm N' \sin\left\{\frac{n\pi}{L(z)}(z-1)\right\}, \quad (n = 1, 2, \cdots).
\] (A.7)

After the normalization \(\pm N' \rightarrow \sqrt{2/(L(z))}\), the \(\{\phi_n(z)\}\) just coincides with the complete basis in eq.(2.3) under the substitution \((y, L) \rightarrow (z-1, L(z))\). Those basis obey \(\mathcal{M}^2 \phi_n(z) \simeq \left\{\frac{n\pi}{L(z)}\right\}^2 \phi_n(z), (n = 1, 2, \cdots)\); here, the approximate equality becomes the exact equality in the asymptotic regions such as \(V(z) = 0 (z \gg \sqrt{\lambda} = \frac{1}{2})\). The independent basis (A.7), thus, satisfy the boundary conditions under the sacrifice such that the eigenvalue equation for those basis hold in the asymptotic region of \(J_0\) and \(N_0\) in spite of that they satisfy the boundary conditions are satisfied exactly.

### B Spin representation space

In this paper, the gamma matrices characterized by \(\{\gamma_{\mu}, \gamma_{\nu}\} = \eta_{\mu\nu}\) in the Majorana representation are given in the following form:

\[
\gamma^0 = i\rho_2 \otimes \sigma_1, \quad \gamma^1 = \rho_1 \otimes \sigma_0, \quad \gamma^2 = \rho_2 \otimes \sigma_2, \quad \gamma^3 = \rho_3 \otimes \sigma_0.
\] (B.1)

\(^6\) As the asymptotic forms of Bessel functions, the following leading terms are used:

\[
J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \quad N_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) \quad (x \gg 1)
\]
where both of $\rho_i$ and $\sigma_i (i = 1, 2, 3)$ are usual Pauli matrices, and $\sigma_0$ is the $2 \times 2$ unit matrix. Since those $\gamma^\mu$'s are real component matrices, we may deal with $(\theta, \zeta)$ as tow sets of real four-component spinors.

The action (4.2) says that the $-i \hbar \tilde{\zeta}^\alpha$ and $\theta^\alpha$, ($\alpha = 1, 2, 3, 4$) are canonical pairs; and, in the q-number theory, we require the commutation rules

$$\{ \tilde{\zeta}^\alpha, \theta^\beta \} \equiv \tilde{\zeta}^\alpha \theta^\beta - (-1)^\gamma \theta^\beta \tilde{\zeta}^\alpha = \delta^{\alpha \beta}, \quad \{ \theta^\alpha, \theta^\beta \} = \{ \tilde{\zeta}^\alpha, \tilde{\zeta}^\beta \} = 0 . \quad (B.2)$$

The generators of Lorentz transformation for the intrinsic spins become

$$S_{\mu \nu} = \tilde{\zeta} \sigma_{\mu \nu} \theta, \quad \left( \sigma_{\mu \nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \right), \quad (B.3)$$

by which the infinitesimal transformations are written as

$$\delta \theta = \frac{i}{2} \delta \omega^{\mu \nu} [S_{\mu \nu}, \theta] = (-1)^\epsilon \frac{i}{2} \delta \omega^{\mu \nu} \sigma_{\mu \nu} \theta, \quad (B.4)$$

$$\delta \tilde{\zeta} = \frac{i}{2} \delta \omega^{\mu \nu} [S_{\mu \nu}, \tilde{\zeta}] = -(-1)^\epsilon \frac{i}{2} \delta \omega^{\mu \nu} \tilde{\zeta} \sigma_{\mu \nu}. \quad (B.5)$$

Under those transformations, one can verify that $\tilde{\zeta} \theta$ and $V_\mu = \tilde{\zeta} \gamma_\mu \theta$ transform respectively as scalar and vector quantities; that is, that $\delta(\tilde{\zeta} \theta) = 0$ and $\delta V_\mu = (-1)^\epsilon \frac{i}{2} \delta \omega^{\mu \alpha} (V_\rho \eta_{\alpha \mu} - V_\sigma \eta_{\rho \mu})$.

Now, the definition of $S_{\mu \nu}$ implies that the ground state, the spin 0 state, and its adjoint state should be defined by $\theta|0\rangle = 0$, $\bar{\zeta}|0\rangle = 0$ and $(\bar{0}| \bar{0} = 1$. Then the states $| \alpha \rangle = \tilde{\zeta}^\alpha |0\rangle$ and $\langle \tilde{\zeta}^\alpha | \beta \rangle = \delta_{\alpha \beta}$, $\langle \tilde{\zeta}^\alpha | \bar{\zeta}^\alpha | \beta \rangle = (\gamma_\mu)_{\alpha \beta}$, and $\langle \alpha | S_{\mu \nu} | \beta \rangle = (\sigma_{\mu \nu})_{\alpha \beta}$ and so on. Further, because of that $V_\mu^2 |0\rangle = V_\mu^2 |0\rangle = 0$, $V_\mu^2 | \alpha \rangle = 4| \alpha \rangle$, and $V_\mu^2 | \bar{\alpha} \rangle = -| \bar{\alpha} \rangle$, one can say that $A|0\rangle = |0\rangle$ and $A|\alpha \rangle = (3\beta^2 + 1)|0\rangle$ for $A = V_\mu^2 \beta^2 + 1$ with $V^2 = V_\parallel^2 - V_\perp^2$. We may continue this procedure to the higher spin states; for example, for $\tilde{\zeta}^\alpha \tilde{\zeta}^\beta |0\rangle$, the spin 2 state, we obtain $A^2 \tilde{\zeta}^\alpha \tilde{\zeta}^\beta |0\rangle = (6\beta^2 + 1) \tilde{\zeta}^\alpha \tilde{\zeta}^\beta |0\rangle$ e.t.c. In any case, the eigenvalues of $A$ are greater than 1, and they remain in a finite set of numbers provided that $\theta$ and $\zeta$ are Grassmann variables.

We finally comment on the Weyl spinors $(\xi, \eta)$ related to $(\theta, \zeta)$ by

$$\theta = \begin{pmatrix} (\xi + \xi^*) \\ -i\sigma_3 (\xi - \xi^*) \end{pmatrix}, \quad \zeta = \begin{pmatrix} (\eta + \eta^*) \\ -i\sigma_3 (\eta - \eta^*) \end{pmatrix}. \quad (B.6)$$

A little calculation leads to

$$V^\mu = \tilde{\zeta} \gamma^\mu \theta = 2 \left( \tilde{\eta}^\dagger \sigma^\mu \tilde{\xi} - (-1)^\epsilon \xi^\dagger \sigma^\mu \tilde{\eta} \right), \quad (B.7)$$

where $\xi = U \tilde{\xi}$ and $\eta = U \tilde{\eta}$ with $U = \sqrt{2}(\sigma_0 + i\sigma_1)$. Further, one can also verify $[\tilde{\zeta}^\alpha, \eta^\beta] = (-1)^\epsilon \delta^{\alpha \beta}$ etc.; and so, the above argument about the spin representations can be traced in terms of the Weyl spinors $(\xi, \eta)$.

References

[1] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231 [hep-th/9711200][inSPIRE].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. B 428 (1998) 105 [hep-th/9802109][inSPIRE].

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253 [hep-th/9802150][inSPIRE].

[4] L. Randall and R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. 83 (1999) 3370 [hep-th/9905221][inSPIRE].

[5] L. Randall and R. Sundrum, *An Alternative to Compactification*, Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064][inSPIRE].

[6] I. R. Klebanov and A. M. Polyakov, *AdS dual of the critical O(N) vector model*, Phy. Lett. B 550 (2002) 213 [hep-th/0210114][inSPIRE].

[7] R. de Mello Koch, A. Jevicki, K. Jin and J. Rodrigues, *AdS$_4$/CFT$_3$ construction from collective fields*, Phys. Rev. D 83 (2011) 025006 [arXiv:1008.0633][inSPIRE].

[8] S. R. Das and A. Jevicki, *Large-N collective fields and holography*, Phys. Rev. D 68 (2003) 044011 [hep-th/0304093][inSPIRE].

[9] K. Aouda, S. Naka, and H. Toyoda, *Bi-Local Fields in AdS$_5$ spacetime*, JHEP 10 (2016) 090 [arXiv:1603.09542][inSPIRE].

[10] H. Yukawa, *Quantum Theory of Non-Local Fields Part I. Free Fields*, Phys. Rev. 77 (1950) 219 [inSPIRE].

[11] H. Yukawa, *Quantum Theory of Non-Local Fields. Part II. Irreducible Fields and their Interaction*, Phys. Rev. 80 (1950) 1047 [inSPIRE].

[12] T. Takabayasi, *Oscillator Model for Particles Underlying Unitary Symmetry*, Nuovo Cim. 33 (1964) 668.

[13] T. Goto, S. Naka and K. Kaminura, *On the Bi-Local Model and String Model*, Prog. Theor. Phys. Suppl. 67 (1979) 69 [inSPIRE].

[14] Y. Katayama and H. Yukawa, *Field Theory of Elementary Domains and Particles. I*, Prog. Theor. Phys. Suppl. 41 (1968) 1 [inSPIRE].

[15] Y. Katayama, I. Umemura and H. Yukawa, *Field Theory of Elementary Domains and Particles. II*, Prog. Theor. Phys. Suppl. 41 (1968) 22 [inSPIRE].

[16] S. Naka, H. Toyoda, T. Takanashi, and E. Umezawa, *Noncommutative Spacetime Realized in AdS$_{n+1}$ Space*, PTEP 2014 (2014) 043B03 [arXiv:1311.1364][inSPIRE].

[17] K. Aouda, N. Kanda, S. Naka, and H. Toyoda, *Deformed Extra Dimension* in Proceedings of the 6th CST-MISC Joint Symposium on Particle Physics, Soryushiron Kenkyu Vol26 (2017).

[18] M. Goodman, *Path integral solution to the infinite square well*, Am. J. Phys. 49 (1981) 843.

[19] W. Janke and H. Kleinert, *Summing Paths for a Particle in a Box*, Letter al Nuovo Ciment 25 (1979) 297 [inSPIRE].

[20] K. Kassner, *Spatial geometry of the rotating disk and its non-rotating counterpart*, Am. J. Phys. 80 (2012) 722 [arXiv:1109.2488][inSPIRE].

[21] H. Bateman, *Higher Transcendental Functions Volume II*, McGRAW-HILL BOOK COMPANY New York, U.S.A. (1953), pg. 13.