The Power of Adaptivity in SGD: Self-Tuning Step Sizes with Unbounded Gradients and Affine Variance

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Abstract

We study convergence rates of AdaGrad-Norm as an exemplar of adaptive stochastic gradient methods (SGD), where the step sizes change based on observed stochastic gradients, for minimizing non-convex, smooth objectives. Despite their popularity, the analysis of adaptive SGD lags behind that of non adaptive methods in this setting. Speciﬁcally, all prior works rely on some subset of the following assumptions: (i) uniformly-bounded gradient norms, (ii) uniformly-bounded stochastic gradient variance (or even noise support), (iii) conditional independence between the step size and stochastic gradient. In this work, we show that AdaGrad-Norm exhibits an order optimal convergence rate of $O(\text{poly log}(T)/\sqrt{T})$ after $T$ iterations under the same assumptions as optimally-tuned non adaptive SGD (unbounded gradient norms and affine noise variance scaling), and crucially, without needing any tuning parameters. We thus establish that adaptive gradient methods exhibit order-optimal convergence in much broader regimes than previously understood.

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1 Introduction

Due to its simplicity, an enormous amount of literature, starting by Robbins and Monro [RM51], has sought to understand convergence guarantees for variants of stochastic gradient descent (SGD):

\[ w_{t+1} = w_t - \eta_t g_t, \]

for minimizing a function \( F(\cdot) \) using stochastic gradients \( g_t \) and a step size schedule \( \eta_t \). When the (non-convex) objective function is smooth (i.e., has \( L \)-Lipschitz-continuous gradients) and the stochastic gradients are unbiased and have affine variance\(^1\), i.e.,

\[
\mathbb{E}[g] = \nabla F(w) \quad \text{and} \quad \mathbb{E} \left[ \| g - \nabla F(w) \|^2 \right] \leq \sigma_0^2 + \sigma_1^2 \| \nabla F(w) \|^2 ,
\]

then it is well-known that SGD with a properly-tuned step size (depending on \( L \) and \( \sigma_1 \)) converges to a first-order stationary point with error \( O(\sqrt{T}) \) after \( T \) iterations [GL13; BCN18]. Moreover, [ACDFSW19] showed this rate is tight under these assumptions.

Given these results, it is natural to ask if knowledge of \( L \) and \( \sigma_1 \) is necessary to obtain this optimal rate of convergence. Indeed, this has been the motivation for adaptive step size algorithms such as Adagrad-Norm, where for any parameters \( \eta, b_0 > 0 \), the step size, \( \eta_t \), is given by

\[
\eta_t = \frac{\eta}{b_t}, \quad \text{where} \quad b_t^2 = b_0^2 + \sum_{s=1}^{t} \| g_s \|^2 = b_{t-1}^2 + \| g_t \|^2 .
\]

Ward, Wu, and Bottou [WWB19] showed that AdaGrad-Norm enjoys a \( O(\log(T)/\sqrt{T}) \) convergence rate even when neither \( L \) nor \( \sigma_0 \) are used to tune the step size-schedule. However, their analysis only holds when \( \sigma_1 = 0 \) and the gradients are uniformly upper-bounded – an assumption which is violated even by strongly convex functions such as \( F(w) = \| w \|^2 \). In fact, [LO19, Section 4] suggests that, due to the correlation between \( \eta_t \) and \( g_t \) in the standard AdaGrad-Norm, the assumption that the gradients are uniformly-bounded might be necessary to prove their convergence guarantee. Although some works on similar adaptive SGD algorithms do not require the gradients to be uniformly upper-bounded [LO19; LO20], their analysis only holds when the step-size \( \eta_t \) is (conditionally) independent of the current stochastic gradient \( g_t \), and require subgaussian noise (a condition which forces \( \sigma_1 = 0 \)). However, disentangling \( \eta_t \) from \( g_t \) is detrimental to the normalization scheme, rendering these methods crucially dependent on the knowledge of the Lipschitz constant \( L \) for determining their step size.

Extending these results from the bounded variance setting \( (\sigma_1 = 0) \) to the affine variance setting is important. Indeed, results that hold only for the case of bounded variance effectively require that one has noiseless access to gradients when their magnitudes are large (see Remark 1 for more discussion). As opposed to the non-adaptive SGD setting where this extension is immediate (discussed above), in AdaGrad-Norm (and more generally, in adaptive methods), the bias introduced by the correlation between \( \eta_t \) and \( g_t \) causes this additional variance to be significantly more problematic.

1.1 Contributions, Key Challenges and the Main Insights

We show that AdaGrad-Norm converges to a first-order stationary point with error \( O(\text{polylog}(T)/\sqrt{T}) \) after \( T \) iterations under the same noise assumptions as well-tuned SGD (stochastic gradients are

\(^1\)While the proof of convergence under affine variance is not given explicitly in [GL13], by slightly modifying the step size choice, the analysis given in this work continues to hold with no additional modifications. Indeed, this observation is made explicitly by Bottou, Curtis, and Nocedal [BCN18, Theorem 4.8].
unbiased, with affine variance, as in (1)). Thus, we achieve a convergence rate with optimal dependence on $T$ up to polylogarithmic factors [ACDFS19], even when the step-size sequence is chosen without knowledge of $L, \sigma_0$, or $\sigma_1$. In a sense, this establishes a “best of both worlds” result for adaptive SGD methods, showing that they can converge at the same rate (up to logarithmic factors) as in [GL13] without any hyperparameter tuning of the step-size sequence. Our results show that neither the assumption of uniformly-bounded gradients nor the assumption of uniformly-bounded variance is necessary; thus, adaptive gradient methods exhibit robust performance in much broader regimes than what has been established by prior studies.

Our analysis must overcome two main challenges: (i) possibly unbounded gradients, and (ii) an additional bias term introduced by affine variance. Prior work avoided or circumvented these challenges via additional assumptions. Our work requires several new insights that we believe may be of independent interest. Furthermore, as we state in Remark 14, these insights are broadly applicable to related adaptive algorithms such as coordinate-wise AdaGrad. We outline these below.

**Main Challenge 1: Unbounded gradients.** Prior work by Ward, Wu, and Bottou [WWB19], under uniformly bounded gradients and uniformly bounded variance assumptions, introduce a proxy $\tilde{\eta}_t$ for the step size in (2). Unlike $\eta_t$ (the true step size), this proxy is decorrelated from $g_t$. Furthermore, this proxy scales inversely to (the square root of) the sum of gradients. The boundedness assumption is used to deterministically bound each individual gradient term in the sum, and thus derives a lower-bound of $E[\tilde{\eta}_t] = \Omega(1/\sqrt{T})$. This directly leads to a convergence rate of $\tilde{O}(1/\sqrt{T})$ to a first-order stationary point in their context. Without the bounded gradient and thus, bounded variance assumptions, however, it is unclear if $E[\tilde{\eta}_t]$ scales as $\tilde{O}(1/\sqrt{T})$. Instead of assuming a uniform, deterministic bound on each summand as in the prior approach, we develop techniques of independent interest that permit us to directly bound this sum in expectation.

**Key Insight 1: Recursively-improving inequalities.** We identify two properties satisfied by AdaGrad-Norm (as well as related adaptive algorithms) – bounded iterate steps and norm-squared step decay – which allow us to derive an initial lower bound of $\tilde{\eta}_t = \Omega(1/poly(T))$ which holds with sufficiently high probability, and a corresponding upper bound on the sum of the gradients of $\sum_{t \in [T]} \|\nabla F(w_t)\|^2 = O(T^2 \log(T))$. While this polynomial bound is too loose to result in any convergence rate, it does provide a starting point. Our key technical approach here is a recursion, where in each iteration, we improve both these bounds using a result that shows their product is a subset of time steps,

$$\sum_{t \in [T]} (1 - \sigma_1 \text{bias}_t) \|\nabla F(w_t)\|^2 \leq \mathbb{E} \left[ F(w_t) - F(w_{t+1}) \mid \mathcal{F}_t \right] + \text{const} \mathbb{E} \left[ \tilde{\eta}_t^2 \|g_t\|^2 \mid \mathcal{F}_{t-1} \right],$$

where $\text{const}$ is a constant which scales with $\sigma_0$ and $L$. Whenever $\text{bias}_t > 1/\sigma_1$, then the “negative drift” term from the bounded variance case, $-\tilde{\eta}_t \cdot \|\nabla F(w_t)\|^2$, becomes positive, making the derivation of the invariant upper bound identified above (in Key Insight 1, Lemma 12) a serious challenge. The presence of this bias is the reason that prevents the analysis from the uniformly-bounded variance case to directly extend to the affine variance framework, as happens in the standard SGD analysis of Ghadimi and Lan [GL13] by simply scaling down the step size there by $1/(1+\sigma_1^2)$.

**Key Insight 2: Focus on the “good” times.** To handle this bias, we first restrict our analysis to a subset of time steps, $S_{\text{good}} = \{t \in [T] : \text{bias}_t \leq 1/2\sigma_1\}$, which we refer to as the “good” time steps.
Intuitively, these are the time steps during which the bias term is sufficiently small. As it turns out, the overwhelming majority of time steps are, in fact, “good,” as shown in Lemma 8.

Key Insight 3: Compensating for the “bad” times. Although the overwhelming majority of time steps are “good,” in order to get a convergence rate that depends on $F(w_1) - F^*$, we still have to reason about the “bad” time steps in $S_{\text{good}}$. As it turns out, if the gradient at even one of these bad times is large (say, $\|\nabla F(w_t)\|^2 = T^2(1)$) then our upper bound on $F(w_{t+1}) - F(w_t)$ is prohibitively large, presenting a serious challenge for the convergence analysis. We circumvent this issue using a novel approach that assigns nearby (in terms of time) “good” times to every “bad” one, thereby mitigating the effects of “bad” time steps in the analysis. This compensation insight formalized in Lemma 10, coupled with the fact that “most” time-steps are typically “good,” allows us to overcome the bias term introduced by the affine variance scaling.

1.2 Related Work

The original AdaGrad algorithm was proposed simultaneously in [DHS11; MS10] in the context of online convex optimization over bounded domains. Streeter and McMahan [SM10] were the first to consider a variant of AdaGrad which is often referred to as AdaGrad-Norm, again in the context of online convex optimization over bounded domains. Later, Orabona and Pál [OP15] studied this algorithm in the context of online linear optimization over a potentially unbounded domain. Subsequently, there has been much analysis of AdaGrad and variants, but all assume that either the gradients are bounded [WWB19; GG20; ZSJSL18; DBBU20], or assume that the step-size is uncorrelated with the current stochastic gradient [LO19; LO20]. Very recently, in a setting similar to ours, Jin, Xing, and He [JXH22] established asymptotic almost-sure convergence of the AdaGrad-Norm iterates to first-order stationary points. Unlike our work, they do not provide rates of convergence, and their focus on asymptotics makes their analysis, techniques, and results significantly different from ours.

Beyond the analysis of AdaGrad, there are several studies of related adaptive algorithms such as Adam [KB15] and variants, see [CLSH18; DBBU20; GXYJY21], and AvaGrad [SMBM21]. Finally, [ZCCTYG20] derived convergence rates for several adaptive algorithms, including AdaGrad, RMSProp, and AMSGrad. The analysis in these works assumed uniformly-bounded stochastic gradients; further details in Appendix A.

2 Preliminaries

We study the convergence of stochastic gradient descent with adaptively chosen step sizes for minimizing a non-convex, smooth function $F(\cdot)$ over unbounded domain $\mathbb{R}^d$ with $F^* = \inf_{w \in \mathbb{R}^d} F(w) > -\infty$. In our context, adaptive step sizes are those which depend on the current stochastic gradient, as well as, potentially, those from past iterates. We focus on the AdaGrad-Norm algorithm (2), although our arguments readily extend to the coordinate-wise AdaGrad case (albeit, at a cost of additional dependence on the dimension). We denote $F_t = \sigma \{w_1, g_1, \ldots, w_t, g_t, w_{t+1}\}$ as the sigma algebra generated by the observations of the algorithm after observing the first $t$ stochastic gradients. We assume the following throughout the paper.

Assumption 1 (Unbiased gradients). For each time $t$, the stochastic gradient, $g_t$, is an unbiased estimate of $\nabla F(w_t)$, i.e., $\mathbb{E}[g_t | F_{t-1}] = \nabla F(w_t)$.

Assumption 2 (Affine variance). For fixed constants $\sigma_0, \sigma_1 \geq 0$, the variance of the stochastic gradient $g_t$ at any time $t$ satisfies $\mathbb{E} \left[ \|g_t - \nabla F(w_t)\|^2 | F_{t-1} \right] \leq \sigma_0^2 + \sigma_1^2 \|\nabla F(w_t)\|^2$. 

Remark 1. (Motivation for Affine Variance) This scaling is important for machine learning applications with feature noise (including missing features) [Ful09; KL20], in robust linear regression [XCM08], and generally whenever the model parameters are multiplicatively perturbed by noise (e.g., a multilayer network, where noise from a previous layer multiplies the parameters in subsequent layers). More broadly, restricting to bounded variance (i.e., assuming $\sigma_t^2 = 0$) is equivalent to assuming “noiseless” access to the gradient when the magnitude of the gradient grows (e.g., a strongly convex function); this is because the stochastic gradient is an arbitrarily small perturbation of the true gradient in this regime. Finally as discussed earlier, the analysis for non adaptive SGD is assuming “noiseless” access to the gradient when the magnitude of the gradient grows (e.g., a strongly convex function). That is, for every independent (conditioned on the history $\mathcal{F}_{t-1}$) of the key steps to removing the uniform gradient bound, and may be of independent interest (e.g., useful for refining the convergence rates for strongly convex problems).

Further, we will assume that the function $F(\cdot)$ is $L$-smooth:

**Assumption 3 (L-smoothness).** The function $F(\cdot)$ is $L$-smooth, i.e., has $L$-Lipschitz continuous gradients. That is, for every $w, w' \in \mathbb{R}^d$, $\|\nabla F(w) - \nabla F(w')\| \leq L \|w - w'\|$. A key property of AdaGrad-Norm is that the step-size sequence is tightly controlled:

$$\|w_{t+1} - w_t\| \leq \eta \quad \text{and} \quad \sum_{t \in [T]} \|w_{t+1} - w_t\|^2 \leq \eta \log(\frac{b_2^2}{b_2}).$$

In fact, variations of this observation have been noted for a number of AdaGrad variants [WWB19; DBBU20]. While simple, it is crucially important to our analysis, since, taken together with Assumption 3, it implies that the gradient at time $t$ scales at most polynomially in $t$.

**Lemma 2 (Polynomial control of gradients (informal statement of Lemmas 21 and 24)).** Consider any times $t_1 \leq t_2 \in [T]$ during a run of algorithm (2). Then, deterministically,

$$\|\nabla F(w_{t_2})\| - \|\nabla F(w_{t_1})\| \leq \eta L(t_2 - t_1).$$

Moreover, with probability at least $1 - \delta$, the following bound also holds

$$\|\nabla F(w_{t_2})\| - \|\nabla F(w_{t_1})\| \leq \eta L\sqrt{(t_2 - t_1) \log(\text{poly}(t_2)/\delta)}.$$

As a consequence of Lemma 2, this implies that $\sum_{t \in [T]} \|\nabla F(w_t)\|^2 = T(\|\nabla F(w_1)\| + \eta L T)^2 = \mathcal{O}(T^3)$ deterministically, and an analogous bound of $\mathcal{O}(T^2 \log(T/\delta))$ with probability $1 - \delta$. Of course, Lemma 2 only gives a much weaker control over $\|\nabla F(w_t)\|^2$ than a uniform bound, and has not (to the best of our knowledge) been previously exploited. However loose, this bound nonetheless is one of the key steps to removing the uniform gradient bound, and may be of independent interest (e.g., useful for refining the convergence rates for strongly convex problems).

As mentioned earlier, a key difficulty in analyzing adaptive algorithms is the bias introduced by the correlation between the step size $\eta_t$ and the stochastic gradient $g_t$ at each time $t$. To analyze the convergence of such algorithms, it is useful to introduce the following “decorrelated” step size.

**Definition 3 (Decorrelated step sizes).** The decorrelated step size “proxy” at time $t$, which is independent (conditioned on the history $\mathcal{F}_{t-1}$) of $g_t$, is denoted by $\tilde{\eta}_t$ and defined as

$$\tilde{\eta}_t := \frac{\eta}{\sqrt{b_{t-1}^2 + (1 + \sigma_t^2) \|\nabla F(w_t)\|^2 + \sigma_0^2}}.$$

Notice that $\tilde{\eta}_t$ is the natural lower bound on $\mathbb{E} [\eta_t | \mathcal{F}_{t-1}]$ by applying Jensen’s inequality.
3 Motivating our Proof

We have discussed the two main challenges in Section 1.1: unbounded gradients and affine variance. We discuss these in more detail, now that we have the required mathematical definitions from Section 2. Adaptive stochastic gradient methods exhibit two difficulties not present in the non-adaptive regime: (i) Since the step size $\eta_t$ depends on the trajectory of stochastic gradients, one must argue about the scaling of these stochastic gradients, and (ii) the step size is correlated with the current gradient, $g_t$, as well as the past gradients. These manifest themselves as follows: by $L$-smoothness (Assumption 3) and the AdaGrad-Norm algorithm (2), we have that

$$\eta_t \| \nabla F(w_t) \|^2 \leq F(w_t) - F(w_{t+1}) - \eta_t \langle \nabla F(w_t), g_t - \nabla F(w_t) \rangle + \frac{L\eta_t^2}{2} \| g_t \|^2. \tag{6}$$

When $\eta_t$ and $g_t$ are conditionally independent, then the inner product term above is mean-zero. As a consequence, as long as the step size $\eta_t \leq \Omega(1/\sqrt{T})$, (6) immediately implies that

$$\mathbb{E} \left[ \sum_{t \in [T]} \frac{\eta_t^2}{2} \| \nabla F(w_t) \|^2 \right] \leq F(w_1) - F^* + \frac{L\sigma_0^2}{2} \sum_{t \in [T]} \eta_t^2. \tag{7}$$

Moreover, as long as $\eta_t = \Omega(1/\sqrt{T})$, a $O(1/\sqrt{T})$ convergence rate is immediate (see [GL13; BCN18] for details). In contrast, in the adaptive setting, the inner product term of (6) may no longer be mean-zero, since $\eta_t$ depends on $g_t$. While Li and Orabona [LO19] circumvented this issue by studying a step-size sequence which depends on the past but not current gradient, Ward, Wu, and Bottou [WWB19] and Défossez, Bottou, Bach, and Usunier [DBBU20] analyzed adaptive gradient methods by introducing (for the sake of analysis) a step-size proxy (identical to Definition 3 when $\sigma_1 = 0$), $\tilde{\eta}_t = \eta/\sqrt{1+\|\nabla F(w_t)\|^2+\sigma_0^2}$, which is conditionally independent of $g_t$. Using that, (6) can be rewritten as

$$\tilde{\eta}_t \| \nabla F(w_t) \|^2 \leq \mathbb{E} \left[ F(w_t) - F(w_{t+1}) \right| \mathcal{F}_{t-1}] + \mathbb{E} \left[ (\tilde{\eta}_t - \eta) \langle \nabla F(w_t), g_t \rangle \right| \mathcal{F}_{t-1}]

+ \mathbb{E} \left[ \frac{L\eta_t^2}{2} \| g_t \|^2 \right| \mathcal{F}_{t-1}]. \tag{8}$$

As noted in prior work, one can show that $\mathbb{E} \left[ \sum_{t \in [T]} \eta_t^2 \| g_t \|^2 \right] = O(\log(T))$ (note that this need not be true in the non-adaptive setting; see Lemma 23 for a proof in our setting) and thus, for the remainder of this discussion, we focus only on the remaining terms of (8).

Unbounded Gradients: Lower-bounding the step size. Although in the non-adaptive setting, we could simply choose $\eta_t = \Omega(1/\sqrt{T})$, in the adaptive regime it is no longer obvious that such a condition holds. One may observe, however, that by Jensen’s inequality and Definition 3

$$\mathbb{E} [\eta_t] \geq \mathbb{E} [\tilde{\eta}_t] \geq \frac{\eta}{\sqrt{b_0^2 + T\sigma_0^2 + (1 + \sigma_1^2)\mathbb{E} \left[ \sum_{s \in [t]} \| \nabla F(w_s) \|^2 \right]}}. \tag{9}$$

As discussed in Section 1.1, it should be clear by observing (9) that the reason prior studies [WWB19; GG20; ZSJSL18; DBBU20] assumed a uniform upper bound on the gradients is to bound the denominator in (9). This allows one to conclude that both $\eta_t$ and $\tilde{\eta}_t$ scale as $\Omega(1/\sqrt{T})$ in expectation. Since our setting is one where neither the gradients nor the variances are uniformly bounded, new techniques are required to get around this challenge.
Affine Variance: Upper-bounding the bias. The bias term in (8) presents another difficulty in analyzing the rate of convergence in the adaptive setting. Specifically, in the affine variance setting

\[
\mathbb{E}[(\tilde{\eta}_t - \eta_t) \langle \nabla F(w_t), g_t \rangle | F_{t-1}] \leq \frac{\tilde{\eta}_t}{2} (1 + \sigma_1 \text{bias}_t) \|\nabla F(w_t)\|^2 + \frac{2\sigma_0}{\eta} \mathbb{E} \left[ \eta_t^2 \|g_t\|^2 | F_{t-1} \right],
\]

where \(\text{bias}_t := 4\sqrt{\mathbb{E} \left[ \|g_t\|^2 / (b_t^2 - \|g_t\|^2) \right]}\) is the additional bias introduced by the affine variance scaling (see Lemma 5 for the proof). Notice that in the bounded variance setting (i.e., \(\sigma_1 = 0\)), (10) corresponds precisely to the bound obtained by Ward, Wu, and Bottou [WWB19] which was used to derive

\[
\mathbb{E} \left[ \sum_{t \in [T]} \frac{\tilde{\eta}_t}{2} \|\nabla F(w_t)\|^2 \right] \leq F(w_1) - F^* + c_0 \log(\text{poly}(T)),
\]

where \(c_0 = 2\sigma_0 \eta + L\eta^2/2\). This inequality is analogous to (7) and, combined with the lower bound \(\mathbb{E} [\tilde{\eta}] = \Omega(1/\sqrt{T})\), immediately leads to the desired convergence rate. When \(\sigma_1 \leq 1/8\), (10) takes essentially the same form as (11), since, deterministically, \(\mathbb{E} \left[ \|g_t\|^2 / (b_{t-1}^2 + \|g_t\|^2) \right] \leq 1\). When \(\sigma_1 \geq 1/4\), however, the first term of (10) can potentially be quite large and cannot be controlled simply by scaling down the step size. Indeed, this additional bias can be problematic, since the “positive drift” could completely cancel out the “negative drift”, i.e., the \(-\tilde{\eta}_t \|\nabla F(w_t)\|^2\) term, in (8). Handling this combination of negative and positive drifts constitutes our second challenge.

4 Main Results

In this section, we sketch out the key ideas that go into deriving a bound on the convergence rate to a first order stationary point. Our main result is the following:

**Theorem 4** (Informal statement of Theorem 35). The iterates of AdaGrad-Norm satisfy:

\[
\min_{t \in [T]} \|\nabla F(w_t)\|^2 \leq C \log^{13/4}(T) \sqrt{T},
\]

with probability at least \(1 - \delta\), where \(C\) has quadratic dependence on the diameter \(F(w_1) - F^*\) and initial gradient norm \(\|\nabla F(w_1)\|\) and polynomial dependence on \(b_0, \eta, L, \sigma_0\) and \(\sigma_1\).

As highlighted in Section 3, obtaining Theorem 4 has two main obstacles: (1) devising a way to deal with the additional bias, term introduced by the affine variance scaling, and (2) lower bounding the step size proxy (for which, as we discussed, it suffices to upper bound \(\mathbb{E} \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right]\)). We now outline the main ideas needed to overcome each of these.

4.1 Bounding the Bias via a Compensation Argument

As displayed in (10), the affine variance scaling introduces additional bias that our analysis must handle. Indeed, this bound taken together with (8) implies the following lemma.

\footnote{While one could control this term using a batch size of \(\Omega(\sigma_1^2)\), we are interested in the standard setting where the batch size is 1, and the algorithm does not know the parameter \(\sigma_1\).}
Lemma 5. Let us recall the step size proxy, $\tilde{\eta}_t$, from Definition 3. Then, we have that

$$\frac{\tilde{\eta}_t}{2} (1 - \sigma_1 \text{bias}_t) \|\nabla F(w_t)\|^2 \leq \mathbb{E} \left[ F(w_t) - F(w_{t+1}) \middle| \mathcal{F}_{t-1} \right] + c_0 \mathbb{E} \left[ \frac{\|g_t\|^2}{b_{t-1}^2 + \|g_t\|^2} \middle| \mathcal{F}_{t-1} \right],$$

where bias$_t := 4 \sqrt{\mathbb{E} \left[ \|g_t\|^2/(b_{t-1}^2 + \|g_t\|^2) \right]}$ is the additional bias term introduced by the affine variance scaling and $c_0 = 2\sigma_0 \eta + L\eta^2/2$.

By Lemma 5, whenever bias$_t \geq 1/\sigma_1$, we cannot upper bound $\tilde{\eta}_t \|\nabla F(w_t)\|^2$ as we could in the bounded variance case ($\sigma_1 = 0$). To overcome this issue, we utilize the following new ideas.

**Key Idea: Focus on the “good” times.** Note that, as long as bias$_t$ is small, the bound in Lemma 5 is still useful. Hence, instead of summing both sides of the expression in Lemma 5 for all times $t \in [T]$, we need to focus on the good events and separate them from the bad events in which bias$_t > 1/\sigma_1$. To do so, we first formally define the good time instances as follows.

**Definition 6 (“Good” times).** Using the notation from Lemma 5, we call a time $t \in [T]$ “good” if $1 - \sigma_1 \text{bias}_t \geq \frac{1}{2}$, and denote $S_{\text{good}}$ as the set of all such times in the interval $[T]$. Similarly, we call a time $t \in [T]$ “bad” if it is not “good,” and take $S_{\text{good}}^c$ as the set of all bad times.

By this definition, the “good” times are those for which a bound on $\tilde{\eta}_t \|\nabla F(w_t)\|^2$ is preserved. By summing the expression in Lemma 5 over only the “good” times and applying the second inequality from (5), we can derive the following result.

**Lemma 7 (Informal statement of Lemma 25).** Recall the step size proxy of Definition 3 and the notation in Definition 6. With $c_0 = 2\sigma_0 \eta + L\eta^2/2$, we obtain

$$\mathbb{E} \left[ \sum_{t \in S_{\text{good}}} \frac{\tilde{\eta}_t}{4} \|\nabla F(w_t)\|^2 \right] \leq F(w_1) - F^* + c_0 \log(\text{poly}(T)) + \mathbb{E} \left[ \sum_{t \notin S_{\text{good}}} F(w_{t+1}) - F(w_t) \right], \quad (12)$$

The above expression is almost the same as the expression (11) which was obtainable in the bounded variance case. The main differences are: (i) the residual term involving the deviations at the “bad” times, and (ii) the summation over only $S_{\text{good}}$ instead of all times $[T]$. Since most times are typically “good”, as we show in Lemma 8, (ii) is not a serious issue. However, the magnitude of the deviations in “bad” times could be large, casting (i) a more serious hurdle.

**Lemma 8 (Informal statement of Lemma 26).** Let $S_{\text{good}}$ be the set of “good” times from Definition 6. Then, we have that

$$\mathbb{E} \left[ |S_{\text{good}}^c| \right] \leq 64\sigma^2 \log(\text{poly}(T)) \quad \text{and} \quad \mathbb{E} \left[ |S_{\text{good}}^c|^2 \right] \leq \left( 64\sigma^2 (1 + 128\sigma^2) + 2 \right) \log^2(\text{poly}(T)).$$

**Proof sketch.** An alternative condition that is equivalent to the one in Definition 6 is $t$ is “good” if $\mathbb{E} \left[ \eta_t^2 \|g_t\|^2 \middle| \mathcal{F}_{t-1} \right] \leq \frac{\eta^2}{64\sigma^2}$. This alternate condition allows us to argue about the expected number of “bad” times via a pigeonholing argument. Specifically, by the tower rule of expectations and the definition of $\eta_t$, one can show (see Lemma 23 for details).

$$\mathbb{E} \left[ \sum_{t \in [T]} \mathbb{E} \left[ \eta_t^2 \|g_t\|^2 \middle| \mathcal{F}_{t-1} \right] \right] = \mathbb{E} \left[ \sum_{t \in [T]} \eta_t^2 \|g_t\|^2 \right] = \eta^2 \mathbb{E} \left[ \log(\frac{\eta^2}{b_{t-1}^2}) \right] = \eta^2 \log(\text{poly}(T)).$$

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As an aside, using essentially the same arguments, we can show that $|S_{\text{good}}^c|$ satisfies the Bernstein condition with parameter $\text{const} \cdot \log(T)$, which implies that, with high probability, $|S_{\text{good}}^c| \leq \text{const} \cdot \log^2(T)$. 


Hence, if more than $64\sigma_1^2 \log(\text{poly}(T))$ times were “bad” in expectation, then since each bad time leads to $\mathbb{E} \left[ \eta_t^2 \| g_t \|^2 \mid F_{t-1} \right] > \eta_t^2 / 64\sigma_1^2$, we would reach a contradiction to the above bound. □

This result shows that most times are “good.” Hence, replacing the sum over all time instances with summation over good time instances in (12) would not be a major issue, as long as we can ensure that the additional term corresponding to the bad events, i.e., $\mathbb{E} \left[ \sum_{t \notin S_{\text{good}}^t} F(w_{t+1}) - F(w_t) \right]$, would not lead to a vacuous upper bound. Next, we formally show how this goal can be achieved.

Key Idea: Compensating for the “bad” times. Lemma 7 shows that, even when we focus on the good times, we still must argue about the deviations at bad times to obtain a convergence guarantee. In order to address this problem, we begin by rewriting Lemma 7 by: (i) upper bounding the “bad” times using the (potentially quite large) bound obtained from Lemma 5, and (ii) subtracting some of the “good” deviation terms from both sides to compensate for the bad terms.

Henceforth, we associate each “bad” time $t$ with a set of compensating “good” times, denoted by $S_{t}^{\text{comp}}$, such that all compensating sets are disjoint. Further, we denote the union of these sets with $S_{t}^{\text{comp}} := \bigcup_{t \in S_{\text{good}}^t} S_{t}^{\text{comp}}$ and the remaining good time steps with $\bar{S} := S_{\text{good}} \setminus S_{t}^{\text{comp}}$. Hence, immediately from Lemma 7, we derive the following.

Lemma 9 (Informal statement of Lemma 27). In the same setting as Lemma 7, we have that

$$
\mathbb{E} \left[ \sum_{t \in \bar{S}} \frac{\tilde{\eta}_t}{4} \| \nabla F(w_t) \|^2 \right] \leq F(w_1) - F^* + c_0 \log(\text{poly}(T))
+ \mathbb{E} \left[ \sum_{t \notin S_{\text{good}}^t} \frac{(4\sigma_1 - 1)}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S_{t}^{\text{comp}}} \frac{\tilde{\eta}_{t'}}{4} \| \nabla F(w_{t'}) \|^2 \right],
$$

where $\bar{S} := S_{\text{good}} \setminus S_{t}^{\text{comp}}$ are remaining “good” times after compensation, and $c_0 = 2\sigma_0 \eta + Ln^2 / 2$.

The above expression is promising in the following sense. If for every “bad” time $t \in S_{\text{good}}^{t}$ one could find enough compensating “good” times $t' \in S_{t}^{\text{comp}}$ with $\tilde{\eta}_{t'} \| \nabla F(w_{t'}) \|^2$ of the same order as the analogous term for $t$, then the last term in Lemma 9 could be bounded deterministically as a function of the size of the bad set, $|S_{\text{good}}^{t}|$. By Lemma 8, both the size of this set and its square are no more than $O(\text{poly log}(T))$ in expectation. Hence, this bound suffices to recover an expression similar to (11). The next lemma gives insight into how one can select such “competing” times.

Lemma 10. Recall the step size proxy $\tilde{\eta}_t$ from Definition 3. For any time $t \in [T]$ and set $S_{t}^{\text{comp}} \subset [T]$ such that (i) $t > \max(S_{t}^{\text{comp}})$ and (ii) $|S_{t}^{\text{comp}}| = n_{\text{comp}} := 8 [4\sigma_1 - 1]$, we have

$$
\frac{4\sigma_1 - 1}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S_{t}^{\text{comp}}} \frac{\tilde{\eta}_{t'}}{4} \| \nabla F(w_{t'}) \|^2 \leq \frac{\eta_t^2 Ln_{\text{comp}}}{8} (t - \min(S_{t}^{\text{comp}})).
$$

The above result serves as our guide for constructing the set $S_{t}^{\text{comp}}$ to upper-bound the residual term from Lemma 9. Indeed, it tells us that, in order to bound the deviation $\tilde{\eta}_t \| \nabla F(w_t) \|^2$ at a bad time $t$, we should not simply pick arbitrary “good” times to offset this deviation. Instead, we should pick times which are as close as possible (in time) to $t$. Perhaps surprisingly, it suffices to use the deviations at a constant $n_{\text{comp}} = O(\sigma_1)$ number of “good” times to compensate for the deviation of $t$. Importantly, these “good” times which we choose must come earlier in time than $t$.  

10
Lemma 11. There exists a construction of $S_{\text{comp}} = \bigcup_{t \in S_{\text{good}}} S_{s_{\text{comp}}}^t$, where $S_{s_{\text{comp}}}^t$ denotes the compensating “good” times for a bad time $t \in S_{\text{good}}^c$ (disjoint from other $S_{s_{\text{comp}}}^t$), satisfying $|S_{s_{\text{comp}}}^t| \leq n_{\text{comp}} := 8[4\sigma_1 - 1]$ and $t > \max(S_{s_{\text{comp}}}^t)$, where one of the these holds:

1. $|S_{s_{\text{comp}}}^t| = n_{\text{comp}}$ and $t - \min(S_{s_{\text{comp}}}^t) \leq n_{\text{comp}} \cdot |S_{\text{good}}^c|

2. $|S_{s_{\text{comp}}}^t| < n_{\text{comp}}$ and $t \leq n_{\text{comp}} \cdot |S_{\text{good}}^c|

By condition 1 of Lemma 11 combined with Lemma 10, the deviation at a “bad” time $t$ can always be bounded by $O(|S_{\text{good}}^c|)$ whenever there are enough times to compensate for it. Whenever there are not enough compensating times for $t$, condition 2 of Lemma 11 implies that this time $t$, and thus also the associated deviation (as we discussed above), must be bounded by $O(|S_{\text{good}}^c|)$. Therefore, the total deviation cannot be more than $O(|S_{\text{good}}^c|^2)$, which is $O(\log^2(T))$ in expectation. Through these observations, we obtain our desired bound, the analogue of (11).

Lemma 12 (Informal statement of Lemma 30). Let the set $S_{\text{comp}}$ from Lemma 9 be chosen as in Lemma 11. Then, denoting $\tilde{S} := S_{\text{good}} \setminus S_{\text{comp}}$ as the set of “good” times after compensation,

$$\mathbb{E} \left[ \sum_{t \in \tilde{S}} \frac{\hat{\eta}_t}{\sqrt{t}} \| \nabla F(w_t) \|^2 \right] \leq F(w_1) - F^* + c_1 \cdot \log^2(T).$$

(13)

With Lemma 12 in place, we are very close to obtaining a $\tilde{O}(1/\sqrt{T})$ convergence rate. Indeed, if we knew deterministically that $\tilde{\eta}_t = \Omega(1/\sqrt{T})$, then substituting in Lemma 12, we could conclude that $\mathbb{E} \left[ \sum_{t \in \tilde{S}} \| \nabla F(w_t) \|^2 \right] = O(\sqrt{T} \log^2(T))$. This would immediately imply that $\mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^2 \right] = \tilde{O}(1/\sqrt{T})$ by lower bounding the average by the minimum, and noting that $|\tilde{S}| = \Omega(T)$ with high probability (an easy consequence of Lemmas 8 and 11). However, since $\tilde{\eta}_t$ is a random variable which can be significantly smaller than $1/\sqrt{T}$ on some sample paths, deriving the required bound is challenging. Below, we formally show how we address this.

4.2 Bounding the Expected Sum of Gradients via Recursive Improvement

As mentioned above, to finalize our convergence result, we need to show that $\mathbb{E} [\tilde{\eta}_t] = \Omega(1/\sqrt{T})$. However, the naive bound one can derive for $\mathbb{E} [\tilde{\eta}_t]$ as an immediate corollary of Lemma 2 is worse...
than what we require. We show instead that we can start with a loose lower bound on $\tilde{\eta}_t$ which holds with sufficiently high probability and recursively improve it to obtain our desired bound on $E[\tilde{\eta}_t]$.

**Key Idea: Recursively-improving inequalities.** We initialize the recursion with an upper bound on $E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right] \leq O(T^2 \log(T))$ from Lemma 2, and use this to derive a lower bound on $\tilde{\eta}_t$ with high probability (Step 1). Next we use the upper bound on the expected sum of products, $\sum \tilde{\eta}_t \|\nabla F(w_t)\|^2$ (the caveat being that the sum is over most but not all of the time indices), from Lemma 12 to decrease the upper bound on $E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right]$ (Step 3). This iteration is now recursed ad infinitum, resulting in Lemma 13. Crucial to this iteration is the observation that the upper bound in Lemma 12 remains unchanged even as the lower bound on $\tilde{\eta}_t$ and upper bound on $E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right]$ evolve – hence, we term Lemma 12 as the “invariant upper bound” property. While this description gives the main intuition, using this requires more care (see Steps 2 and 3) because the relation between $\tilde{\eta}_t$ and $\sum_{t \in [T]} \|\nabla F(w_t)\|^2$ is over all times, whereas the upper bound in Lemma 12 contains only the “good” times not used for compensation.

**Step 1: Lower bounding $\tilde{\eta}_t$.** We start with an upper bound on the expected sum of gradients, $E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right] \leq c_2 T^x \log^y(h(T))$, where $c_2$ is a sufficiently large constant, $h(T)$ is a polynomial function of $T$, and $x$ and $y$ are parameters which can initially, as a consequence of Lemma 2, be chosen as $x = 2$ and $y = 1$. This directly implies an analogous bound on $E \left[ \tilde{\eta}_t \right]$ (recall that $b_t$ is defined in (2)) through (4). Thus, one immediately obtains, through Markov’s inequality, a loose upper bound on $b_{T-1}^2 \leq c_2 \log^y(h(T))$ which holds with probability at least $1 - O(\log^y(h(T))/T^\gamma_1)$ (where we set $\gamma_1 = (4-x)/3$ and $\gamma_2 = 2(y-1)/3$). Thus, taking $E_T(\delta)$ to be this high probability event, and applying the deterministic bound on $\|\nabla F(w_t)\|$ from Lemma 2, we obtain a lower bound for each $\tilde{\eta}_t$ whenever $E_T(\delta)$ is true, which we use to obtain:

$$E \left[ \sum_{t \in S} \tilde{\eta}_t \|\nabla F(w_t)\|^2 \right] \geq E \left[ \sum_{t \in S} \tilde{\eta}_t \|\nabla F(w_t)\|^2 1\{E_T(\delta)\} \right] \geq \frac{\eta E \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 1\{E_T(\delta)\} \right]}{\sqrt{2c_2 T^{x+\gamma_1} \log^{\gamma_2}(h(T))}}. \tag{14}$$

**Step 2: Bounding the “good” terms.** To remove the indicator function in the lower bound, one can use the fact that $E \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 1\{E_T(\delta)\} \right] = E \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] (1 - 1\{E_T(\delta)^C\})$ and the polynomial upper bound that we have on the gradients sum from Lemma 2 together with an upper bound on the failure probability of $E_T(\delta)^C$. Moreover, we importantly use the “invariant” upper bound on $E \left[ \sum_{t \in S} \tilde{\eta}_t \|\nabla F(w_t)\|^2 \right]$ from Lemma 12 together with the lower bound on this same quantity from (14) to conclude that

$$E \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] \leq \frac{c_2}{2} T^{x+2} \log^{y+3}(h(T)). \tag{15}$$

Note that this is almost an improved bound on $E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right]$. However, the summation range in (15) is a subset of $[T]$ that almost has the same size.

**Step 3: Bounding the “bad” terms.** It remains only to bound $E \left[ \sum_{t \in S^\text{comp}} \|\nabla F(w_t)\|^2 \right]$. Recall that, by construction of $S^\text{comp}$ in Lemma 11, $|S^\text{comp}| \leq n_{\text{comp}} \cdot |S^\text{good}|$. Further, by the result in Lemma 2, each $\|\nabla F(w_t)\|^2 = O(T \log(T))$ with probability at least $1 - T^{-2}$, and $O(T^2)$ deterministically.
Hence, by using Lemma 8 to bound the expected size of $|S_{\text{good}}^c|$, we obtain that
\[ \mathbb{E} \left[ \sum_{t \in \tilde{S}} \| \nabla F(w_t) \|^2 \right] \leq \frac{c_2^2}{2} T^{\frac{x+2}{3}} \log \frac{T}{\beta} (h(T)). \] (16)

Thus, by combining the results of (15) and (16), we conclude that $\mathbb{E} \left[ \sum_{t \in [T]} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^{(x+2)/3} \log^{y+5/3} (h(T))$. We may thus use this improved bound recursively in place of the original choice of $x$ and $y$ from Step 1. The conclusion of this “recursive improvement” argument is that $\mathbb{E} \left[ \sum_{t \in [T]} \| \nabla F(w_t) \|^2 \right] = \tilde{O}(T)$, which, by Jensen’s inequality, implies $\mathbb{E} [\hat{\eta}] = \tilde{\Omega}(1/\sqrt{T})$. This result is summarized in Lemma 13.

**Lemma 13** (Informal statement of Lemma 31). Suppose that, for some parameters $x \in [1, 4]$, $y \geq 1$, $h(T)$ a polynomial function of $T$, and sufficiently large constant $c_2$ independent of $T$, $\mathbb{E} \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^x \log^{y} (h(T))$, Then, the following tighter bound also holds:
\[ \mathbb{E} \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^{\frac{x+2}{3}} \log \frac{T}{\beta} (h(T)). \] (17)

In particular, as a consequence of Lemma 2,
\[ \mathbb{E} \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^{\frac{x+2}{3}} \log \frac{T}{\beta} (h(T)) \quad \text{and} \quad \mathbb{E} [\hat{\eta}] \geq \tilde{\Omega}(1/\sqrt{T}). \] (18)

### 4.3 Wrapping Up

With these two bounds in place, obtaining the convergence result Theorem 4 is immediate. Indeed, we note that Lemma 12 gives us (essentially) the same bound as (10) (modulo the summation over the set $\tilde{S}$ instead of all times $[T]$). Therefore, we may apply (essentially) the same Hölder’s inequality argument as in [WWB19], replacing their application of the uniform gradient bound with the result of Lemma 13, and taking extra care that our summation from Lemma 12 is over a random set $\tilde{S}$. We give the full proof of this theorem in Appendix G.

**Remark 14.** While we focus in this paper on the convergence rate of one particular adaptive SGD method, our methods are not overly specialized to AdaGrad-Norm. Indeed, using nearly identical arguments per coordinate, we can obtain similar $\tilde{O}(1/\sqrt{T})$ convergence rates under the similar assumptions for coordinate-wise AdaGrad, albeit with an additional polynomial dependence on $d$.

### 5 Conclusion

In this paper, we extended the analysis of AdaGrad-Norm to the setting where the gradients are possibly unbounded and the noise variance scales affinely. We showed that under these conditions, together with the standard smoothness assumption, the iterates of AdaGrad-Norm reach a first-order stationary point of a nonconvex function with an error of $\mathcal{O}(\text{poly log(T)}/\sqrt{T})$. For studying this more general setting, we developed several novel analytical techniques, including recursively improving inequalities to handle unbounded gradients, and a compensation argument to deal with bias induced by the affine variance condition.
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Additional Details on Related Work

Ghadimi and Lan [GL13] were the first to study the convergence of SGD for optimizing a non-convex, smooth objective function. They proved that a properly-tuned SGD converges to a first-order stationary point at rate $O(1/\sqrt{T})$, if the step sizes are chosen as $\eta_t = \min\left\{1/(1+\sigma_t^2)L, \tilde{D}/\sigma_0\sqrt{T}\right\}$ for a constant $\tilde{D} > 0$. Further, Arjevani, Carmon, Duchi, Foster, Srebro, and Woodworth [ACDFSW19] later proved that the $O(1/\sqrt{T})$ convergence rate is unimprovable for any algorithm with only first-order oracle access, assuming the function is non-convex, Lipschitz, and the stochastic gradients are unbiased with bounded variance.

The original AdaGrad algorithm was proposed simultaneously by Duchi, Hazan, and Singer [DHS11] and McMahan and Streeter [MS10] whereas Streeter and McMahan [SM10] were the first to consider a variant of AdaGrad referred to as AdaGrad-Norm. Ward, Wu, and Bottou [WWB19] analyzed AdaGrad-Norm for minimizing a smooth, non-convex function with uniformly-bounded gradients. This was the first work to show that AdaGrad-Norm converges at essentially the same rate as SGD, but without the need to know the smoothness constant (albeit under the restrictive assumption that the gradients are uniformly upper-bounded). In a simultaneous work, Li and Orabona [LO19] studied a variant of AdaGrad-Norm where step size $\eta_t$ is conditionally independent of the current stochastic gradient $g_t$, unlike in the standard AdaGrad setting. They provided a similar convergence guarantee without needing a uniform upper-bound on the stochastic gradients, but requiring that the noise have bounded support and additionally requiring knowledge of the smoothness parameter $L$ to tune their step sizes. In a followup work [LO20], the same authors proved high-probability convergence of a class of adaptive algorithms (including their variant of AdaGrad-Norm, as well as coordinate-wise AdaGrad with momentum) under the assumption of subgaussian noise. Note that, like the earlier result, their step sizes needed to be tuned with knowledge of the smoothness parameter, and further needed to be conditionally independent of the current gradient. Slightly more recently, Gadat and Gavra [GG20] studied the asymptotic convergence of AdaGrad (as well as and RMSProp), where their analysis requires uniform gradient bounds as well as uniform bounds on the 2nd and 4th moments of the gradient noise. Zou, Shen, Jie, Sun, and Liu [ZSJSL18] studied a weighted version of coordinate-wise AdaGrad with momentum, where they assumed the gradients were uniformly bounded. Défossez, Bottou, Bach, and Usunier [DBBU20] later improved upon these results with respect to the dependence on the momentum parameter.

Beyond the analysis of AdaGrad, a large number of recent works have studied the convergence of other adaptive algorithms, all of which are based on the assumption of uniformly-bounded stochastic gradients. For instance, Chen, Liu, Sun, and Hong [CLSH18] studied the convergence of a class of Adam-like algorithms (originally introduced by Kingma and Ba [KB15]). Later, building on the results of Ward, Wu, and Bottou [WWB19], Défossez, Bottou, Bach, and Usunier [DBBU20] improved on this analysis of Adam with respect to the dependence on the momentum parameter and range of valid hyperparameters. Guo, Xu, Yin, Jin, and Yang [GXYJY21] provide an alternate analysis of a class of Adam-like algorithms for different momentum parameter scaling. Savarese, McAllester, Babu, and Maire [SMBBM21] studied “delayed” versions of Adam (as well as a new algorithm they called AvaGrad), which makes the step sizes $\eta_t$ conditionally independent of the current stochastic gradient, $g_t$. 


B Preliminaries

Here, we provide proofs for claims from Section 2, as well as some auxiliary results and notation. We additionally state some definitions that will be useful for proving our results.

**Lemma 15.** For any sequence \( \{a_s\}_{s=0}^{\infty} \) such that \( a_0 > 0 \) and \( a_s \geq 0 \) for all \( s \),

\[
\sum_{t=0}^{T} \frac{a_t}{\sum_{s=0}^{t} a_s} \leq 1 + \log \left( \sum_{t=0}^{T} a_t \right) - \log (a_0)
\]

*Proof.* The base case of \( T = 0 \) holds with equality. Let us now assume that the claim holds at \( T \). Then, we have that

\[
\sum_{t=0}^{T+1} \frac{a_t}{\sum_{s=0}^{t} a_s} \leq 1 + \log \left( \sum_{t=0}^{T} a_t \right) - \log(a_0) + \frac{a_{T+1}}{\sum_{s=0}^{T+1} a_s} \\
\leq 1 + \log \left( \sum_{t=0}^{T} a_t \right) - \log(a_0) + \log \left( \frac{\sum_{s=0}^{T+1} a_s}{\sum_{s=0}^{T} a_s} \right) \\
= 1 + \log \left( \sum_{t=0}^{T+1} a_t \right) - \log(a_0),
\]

where the first inequality holds by the induction hypothesis, and the second because of the fact \( x < -\log(1-x) \) (where \( \log(\cdot) \) denotes the natural logarithm).

Our analysis will focus on adaptive gradient algorithms with a particularly convenient structure, which we refer to as the *Bounded Step-Size Property*

**Definition 16 (\( \beta_{\text{step}} \)-Bounded Step-Size Property).** We say that an optimization algorithm has \( \beta_{\text{step}} \)-Bounded Step-Sizes if, for any pair of adjacent iterates \((w_t, w_{t+1})\) generated by the algorithm, the following inequality holds deterministically:

\[
\|w_{t+1} - w_t\| \leq \beta_{\text{step}}.
\]

Another convenient property of the algorithms we study is what we call the *Decay Property:*

**Definition 17 (\( (\beta_{\text{decay}}, b_0) \)-Decay Property).** We say that an optimization algorithm satisfies the \( (\beta_{\text{decay}}, b_0) \)-Decay Property if the iterate sequence \( \{w_t\}_{t \in [T]} \) satisfies the following inequality deterministically:

\[
\sum_{t=1}^{T} \|w_{t+1} - w_t\|^2 \leq \beta_{\text{decay}} \cdot \log \left( 1 + \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_0^2} \right).
\]

We observe that these property is satisfied by a number of interesting adaptive gradient algorithms.

**Observation 18.** AdaGrad-Norm has \( \eta \)-Bounded Step-Sizes and \( (\eta^2, b_0) \)-Decay. The first follows since for any time \( t \geq 0 \),

\[
\|w_{t+1} - w_t\| = \eta \frac{\|g_t\|}{\sqrt{b_{t-1}^2 + \|g_t\|^2}} \leq \eta.
\]

The second is an immediate consequence of Lemma 15.
Observation 19. Coordinate-wise AdaGrad (with coordinate-dependent step sizes
\[ \tilde{\eta}_{t,i} := \frac{\eta}{\sqrt{b_{t-1,i}^2 + (g_{t,i})^2}} \]
has \( \eta \cdot \sqrt{d} \)-Bounded Step-Sizes and \((d\eta^2, b_0)\)-Decay. The first follows since since \( |w_{t+1,i} - w_{t,i}| \leq \eta \) for every coordinate \( i \in [d] \). The second follows by applying Lemma 15 to the sum of \( |w_{t+1,i} - w_{t,i}|^2 = \eta^2 b_{t-1,i}^2 + (g_{t,i})^2 \) for each coordinate.

Remark 20. We note here that many of the lemmas to follow could be stated in more generality by using Definitions 16 and 17. For simplicity and to showcase our ideas in the simplest manner, we will state everything in the context of the AdaGrad-Norm algorithm (2).

By Assumption 3 and Definition 16, we also have the following simple, but quite useful, facts, which give us crude but, crucially, polynomial (in \( T \)) bound on \( \| \nabla F(w_t) \| \):

Lemma 21. Consider any optimization algorithm which satisfies Definition 16 running on an \( L \)-smooth objective function \( F \). Then, for any times \( t_2 \geq t_1 \),
\[ \| \nabla F(w_{t_2}) - \nabla F(w_{t_1}) \| \leq \eta L (t_2 - t_1). \]
In particular, this implies that
\[ \| \nabla F(w_t) \| \leq \| \nabla F(w_1) \| + \eta L t \]

Proof. The proof follows by first applying the triangle inequality and using a telescoping sum to bound
\[ \| \nabla F(w_{t_2}) - \nabla F(w_{t_1}) \| \leq \| \nabla F(w_{t_2}) - \nabla F(w_{t_1}) \| = \left\| \sum_{s = t_1}^{t_2-1} \nabla F(w_{s+1}) - \nabla F(w_s) \right\|, \]
then noting that, for each \( s \in [t_1, t_2] \), by Assumption 3 and Definition 16,
\[ \| \nabla F(w_{s+1}) - \nabla F(w_s) \| \leq L \| w_{s+1} - w_s \| \leq L \cdot \eta. \]

The above bound on \( \| \nabla F(w_t) \| \) is quite useful, since it guarantees a polynomial (in \( T \)) bound for \( \| \nabla F(w_t) \| \). However, note that this bound is much more crude than the bound assumed by Ward, Wu, and Bottou [WWB19] and Défossez, Bottou, Bach, and Usunier [DBBU20] (where they assumed \( \| \nabla F(w_t) \|^2 \leq B < \infty \) for every \( t \)). It turns out that, on “nice” sample paths, a significantly tighter bound can be derived. Intuitively, these sample paths are those for which the quantity \( b_T^2 = b_0^2 + \sum_{t=1}^T \| \eta t \|^2 \) is bounded by a polynomial in \( T \).

Definition 22 (Nice event). For any time \( s \in [T] \) and failure probability \( \delta \in (0, 1] \), we define the following “nice event”:
\[ \mathcal{E}_s(\delta) = \left\{ \begin{array}{c} b_T^2 \leq b_0^2 + \frac{s \sigma_0^2}{\delta} + (1 + \sigma_1^2) \mathbb{E} \left[ \sum_{t \in [s]} \| \nabla F(w_t) \|^2 \right] \end{array} \right\}. \]
We note that, by construction, Markov’s inequality tells us that this event occurs with probability at least $1 - \delta$, i.e., $\Pr[\mathcal{E}_s(\delta)^c] \leq \delta$. Further, taking
\[
f(s) = 2 + \frac{\sigma_2^2 s}{b_0^2} + \frac{(1 + \sigma_1^2)s}{b_0^2} (\|\nabla F(w_1)\| + \eta L s)^2,
\]

it follows (by upper bounding $\mathbb{E} \left[ \sum_{t \in [s]} \|\nabla F(w_s)\|^2 \right] \leq s (\|\nabla F(w_1)\| + \eta L s)^2$ by Lemma 21) that, whenever $\mathcal{E}_s(\delta)$ is true, we have that $b_2^2/b_0^2 \leq f(s)/\delta$.

As we will soon see, bounding the quantity $\sum_{t \in [T]} \|g_t\|^2/b_2^2$ will be crucial in many parts of our analysis. Under the “nice” events from Definition 22, this quantity can be easily controlled:

**Lemma 23.** For any choice of initialization $b_0^2 > 0$, and on any sample path,
\[
\sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \leq \log \left( \frac{b_T^2}{b_0^2} \right).
\]

Further, assuming that the “nice event” (19) ($\mathcal{E}_s(\delta)$) is true at time $s \in [T]$, and taking $f(\cdot)$ as in (20),
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \big| F_s \right] \leq \log \left( \frac{f(T)}{\delta} \right).
\]

In particular, since $\mathcal{E}_0(1)$ is always true, the above implies that
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \right] \leq \log(f(T)),
\]

Additionally, when $\mathcal{E}_T(\delta)$ (the nice event at time $T$) is true,
\[
\sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \leq \log \left( \frac{f(T)}{\delta} \right).
\]

**Proof.** (21) follows immediately from Lemma 15, where we choose $a_0 = b_0^2$ and $a_t = \|g_t\|^2$ for each $t > 0$. To show (22), we note that, on any sample path, by (21) and Jensen’s inequality,
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \big| F_s \right] \leq \log \left( \frac{b_T^2}{b_0^2} \right) \leq \log \left( 1 + \frac{s}{1} \frac{\|g_t\|^2}{b_0^2} + \frac{\mathbb{E} \left[ \sum_{t=s+1}^{T} \|g_t\|^2 \big| F_s \right]}{b_0^2} \right).
\]

To bound this term above, first observe that, as noted in (4), Assumptions 1 and 2 imply that
\[
\mathbb{E} \left[ \|g_t\|^2 \big| F_{t-1} \right] \leq \sigma_0^2 + (1 + \sigma_1^2) \|\nabla F(w_t)\|^2.
\]

Further, when (19) ($\mathcal{E}_s(\delta)$) is true at time $s$, we have that, by Lemma 21,
\[
\frac{1}{b_0^2} \sum_{t=1}^{s} \|g_t\|^2 \leq \frac{s \sigma_0^2 + (1 + \sigma_1^2)}{b_0^2} \mathbb{E} \left[ \sum_{t \in [s]} \|\nabla F(w_t)\|^2 \right] \leq \frac{s \sigma_0^2 + (1 + \sigma_1^2)s (\|\nabla F(w_1)\| + \eta L s)^2}{b_0^2}.
\]

Combining the above bounds, we conclude that
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2 - 1 + \|g_t\|^2} \big| F_s \right] \leq \log \left( 1 + \frac{T \sigma_0^2 + (1 + \sigma_1^2)T (\|\nabla F(w_1)\| + \eta L T)^2}{b_0^2} \right) \leq \log \left( \frac{f(T)}{\delta} \right),
\]
as claimed. Finally, observe that (23) and (24) follow immediately from (22), taking $s = 0$ (noting that $\mathcal{E}_0(1)$ is true deterministically) and $s = T$, respectively. □
With the above construction in place, we are ready to give a slightly stronger bound for 
\( \| \nabla F(w_t) \|^2 \), improving upon Lemma 21 (with high probability) in many interesting regimes.

**Lemma 24.** Consider any time \( t \in [T] \) during a run of the algorithm, where the algorithm was initialized at a starting point \( w_1 \), and is currently at iterate \( w_t \). Then,

\[
\| \nabla F(w_t) \|^2 \leq 2 \| \nabla F(w_1) \|^2 + 2\eta^2 L^2 t \cdot \log \left( \frac{b_t^2}{b_0^2} \right)
\]

and additionally, assuming that \( \mathcal{E}_t(\delta) \) from Definition 22 is true, and taking \( f(\cdot) \) as in (20), then

\[
\| \nabla F(w_t) \|^2 \leq 2 \| \nabla F(w_1) \|^2 + 2\eta^2 L^2 t \cdot \log \left( \frac{f(t)}{\delta} \right).
\]

**Proof.** The proof follows effectively from the same arguments used to prove Lemma 21, only using the improved bound from Lemma 23 in place of Observation 18. Indeed, using the same decomposition, and applying Cauchy-Schwarz, we have that

\[
\| \nabla F(w_t) \|^2 \leq 2 \| \nabla F(w_1) \|^2 + 2L^2 \eta^2 \left( \sum_{s=1}^{t} \frac{\| g_s \|^2}{b_0^2 + \sum_{k=1}^{s} \| g_k \|^2} \right)^2.
\]

\[
\leq 2 \| \nabla F(w_1) \|^2 + 2L^2 \eta^2 t \sum_{s=1}^{t} \frac{\| g_s \|^2}{b_0^2 + \sum_{k=1}^{s} \| g_k \|^2}
\]

\[
\leq 2 \| \nabla F(w_1) \|^2 + 2L^2 \eta^2 t \log \left( \frac{b_t^2}{b_0^2} \right)
\]

where the first inequality follows from the decomposition used in the proof of Lemma 21, the second follows by Cauchy-Schwarz, and the third from Lemma 15.

The second claim follows immediately from the above, combined with Lemma 23.

□

### C Deriving the Starting Point

Here, we provide the proof for the starting point of our analysis, Lemma 5, from Section 4.

**Lemma 5.** Let us recall the step size proxy, \( \hat{\eta}_t \), from Definition 3. Then, we have that

\[
\frac{\eta_t}{2} \left( \sigma_1 - \text{bias}_t \right) \| \nabla F(w_t) \|^2 \leq \mathbb{E} \left[ F(w_t) - F(w_{t+1}) \mid \mathcal{F}_{t-1} \right] + c_0 \mathbb{E} \left[ \frac{\| g_t \|^2}{b_{t-1}^2 + \| g_t \|^2} \mid \mathcal{F}_{t-1} \right],
\]

where \( \text{bias}_t := 4 \sqrt{\mathbb{E} \left[ \| g_t \|^2 / (b_{t-1}^2 + \| g_t \|^2) \mid \mathcal{F}_{t-1} \right]} \) is the additional bias term introduced by the affine variance scaling and \( c_0 = 2\sigma_0 \eta_t + L\eta_t^2 / 2 \).

**Proof.** We will begin by using our assumption of \( L \)-smoothness, along with the definition of the algorithm, to get the bound:

\[
F(w_{t+1}) - F(w_t) \leq \langle \nabla F(w_t), w_{t+1} - w_t \rangle + \frac{L}{2} \| w_{t+1} - w_t \|^2
\]

\[
= -\eta_t \langle \nabla F(w_t), g_t \rangle + \frac{L\eta_t^2}{2} \| g_t \|^2
\]

\[
= -\eta_t \| \nabla F(w_t) \|^2 - \eta_t \langle \nabla F(w_t), g_t - \nabla F(w_t) \rangle + \frac{L\eta_t^2}{2} \| g_t \|^2
\]

\[
= -\eta_t \| \nabla F(w_t) \|^2 + \eta_t \langle \nabla F(w_t), g_t \rangle + \frac{L\eta_t^2}{2} \| g_t \|^2
\]

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Now, as noted in \cite{WWB19}, the inner product term is *not* zero in expectation, since \( \eta_t \) depends on \( g_t \). Hence, we introduce a step size proxy \( \tilde{\eta}_t \) from Definition 3, which *is* independent of \( g_t \) (conditioned on \( F_{t-1} \)). This choice, unlike \( \tilde{\eta}_t \), satisfies:

\[
\mathbb{E} \left[ \tilde{\eta}_t \langle \nabla F(w_t), g_t - \nabla F(w_t) \rangle \mid F_{t-1} \right] = 0
\]

Hence, by taking expectations of our first inequality and adding and subtracting this mean-zero quantity, we have that

\[
\mathbb{E} \left[ F(w_{t+1}) \mid F_{t-1} \right] - F(w_t) \leq -\tilde{\eta}_t \|\nabla F(w_t)\|^2 - \mathbb{E} \left[ (\eta_t - \tilde{\eta}_t) \langle \nabla F(w_t), g_t \rangle \right] + \mathbb{E} \left[ \frac{L\eta_t^2}{2} \|g_t\|^2 \mid F_{t-1} \right]
\]

We will now focus on bounding the second term. Observe that, denoting \( a = b_{t-1}^2 + \|g_t\|^2 \) and \( b = b_{t-1}^2 + (1 + \sigma_t^2) \|\nabla F(w_t)\|^2 + \sigma_0^2 \),

\[
\left| \eta_t - \tilde{\eta}_t \right| = \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{ab}} = \frac{b - a}{\sqrt{ab}(\sqrt{a} + \sqrt{b})}
\]

From this, we conclude that

\[
\left| \frac{\eta_t - \tilde{\eta}_t}{\eta} \right| = \frac{(1 + \sigma_t^2) \|\nabla F(w_t)\|^2 + \sigma_0^2 - \|g_t\|^2}{\sqrt{ab(\sqrt{a} + \sqrt{b})}} \leq \frac{(|\nabla F(w_t)| - \|g_t\|)(|\nabla F(w_t)| + \|g_t\|) + \sigma_0^2 + \sigma_t^2 \|\nabla F(w_t)\|^2}{\sqrt{ab(\sqrt{a} + \sqrt{b})}} \leq \frac{\|g_t - \nabla F(w_t)\| + \sqrt{\sigma_0^2 + \sigma_t^2 \|\nabla F(w_t)\|^2}}{\sqrt{b_{t-1}^2 + \|g_t\|^2}} \leq \frac{\|g_t - \nabla F(w_t)\| + \sqrt{\sigma_0^2 + \sigma_t^2 \|\nabla F(w_t)\|^2}}{\sqrt{b_{t-1}^2 + (1 + \sigma_t^2) \|\nabla F(w_t)\|^2 + \sigma_0^2}}
\]

Plugging this bound into the above, and taking expectation with respect to the filtration at \( t - 1 \), we have shown that

\[
\mathbb{E} \left[ F(w_{t+1}) \mid F_{t-1} \right] - F(w_t) \leq -\tilde{\eta}_t \|\nabla F(w_t)\|^2 + \eta \mathbb{E} \left[ \frac{\|g_t - \nabla F(w_t)\| \|\nabla F(w_t)\| \|g_t\|}{\sqrt{b_{t-1}^2 + \|g_t\|^2}} \mid F_{t-1} \right] + \eta \mathbb{E} \left[ \frac{\sqrt{\sigma_0^2 + \sigma_t^2 \|\nabla F(w_t)\|^2}}{\sqrt{b_{t-1}^2 + (1 + \sigma_t^2) \|\nabla F(w_t)\|^2 + \sigma_0^2}} \mid F_{t-1} \right] + \frac{L\eta_t^2}{2} \mathbb{E} \left[ \frac{\|g_t\|^2}{\|g_t\|^2} \mid F_{t-1} \right]
\]

We will now show that the second and third terms above have the same upper bound. Focus on the second term above, we apply Hölder’s inequality and the affine variance assumption to conclude that

\[
\mathbb{E} \left[ \frac{\|g_t - \nabla F(w_t)\| \|\nabla F(w_t)\| \|g_t\|}{\sqrt{b_{t-1}^2 + \|g_t\|^2}} \mid F_{t-1} \right]
\]
To conclude, we can bound the second term above, using the inequality which shows that the second and third terms have exactly the same upper bound. Combining these expressions, we arrive at the claimed inequality.

\[
\mathbb{E} \left[ \frac{\|g_t\|^2}{\|b_{t-1}^2 \|= \frac{\sqrt{\|g_t\|^2}}{b_{t-1}^2} + 1 + \sigma_0^2 + \sigma_1^2 \|\nabla F(w_t)\|^2} + \sigma_0^2 \right] \mathbb{E} \left[ \frac{\|g_t\|^2}{\|b_{t-1}^2 + \|g_t\|^2} \right] \leq \mathbb{E} \left[ \frac{\|g_t\|^2}{\|b_{t-1}^2 + \|g_t\|^2} \right]
\]

Now, focusing on the third term, by Jensen’s inequality to the concave function \(\sqrt{\cdot}\), we know that

\[
\mathbb{E} \left[ \frac{\|g_t\|^2}{\sqrt{b_{t-1}^2} + \|g_t\|^2} \right] = \mathbb{E} \left[ \frac{\|g_t\|^2}{b_{t-1}^2 + \|g_t\|^2} \right] \leq \mathbb{E} \left[ \frac{\|g_t\|^2}{b_{t-1}^2 + \|g_t\|^2} \right]
\]

which show that the second and third terms have exactly the same upper bound. Combining these expressions and rearranging, we find

\[
\mathbb{E} \left[ F(w_{t+1}) \mid F_{t-1} \right] - F(w_t) \leq -\tilde{\eta} \|\nabla F(w_t)\|^2
\]

\[
+ 2\eta \sqrt{\sigma_0^2 + \sigma_1^2 \|\nabla F(w_t)\|^2} \|\nabla F(w_t)\|
\]

\[
+ \frac{L\eta^2}{2} \mathbb{E} \left[ \frac{\|g_t\|^2}{b_{t-1}^2 + \|g_t\|^2} \right] \leq -\tilde{\eta} \|\nabla F(w_t)\|^2
\]

To conclude, we can bound the second term above, using the inequality \(ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2\), choosing \(a = \frac{\sqrt{\|g_t\|^2}}{b_{t-1}^2 + \|g_t\|^2} \) and \(b = \frac{2\sqrt{\eta}}{b_{t-1}^2 + \|g_t\|^2} \). After grouping the resulting expressions, we arrive at the claimed inequality.

\[
\mathbb{D} \text{ Most Times are (Typically) Good}
\]

Here, we provide proofs regarding properties and consequences of the “good” times (Definition 6) from Section 4.

**Lemma 25.** Recalling the step size proxy of Definition 3 and the notation in Definition 6, we obtain

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{S}_{\text{good}}} \frac{\tilde{\eta}_t}{4} \|\nabla F(w_t)\|^2 \right]
\]
\begin{align*}
&\leq F(w_1) - F^* + c_0\mathbb{E}\left[\sum_{t \in S_{\text{good}}} \frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2}\right] + \mathbb{E}\left[\sum_{t \notin S_{\text{good}}} F(w_{t+1}) - F(w_t)\right] \\
&\leq F(w_1) - F^* + c_0 \log(f(T)) + \mathbb{E}\left[\sum_{t \notin S_{\text{good}}} \frac{4\sigma_1 - 1}{2}\eta_t \left\|\nabla F(w_t)\right\|^2\right],
\end{align*}

where \(c_0 = 2\sigma_0\eta + L\eta^2/2\).

**Proof.** The proof is an easy consequence of Lemma 5 together with the fact that \(\{t \in S_{\text{good}}\} \in \mathcal{F}_{t-1}\). Indeed, by construction of \(S_{\text{good}}\), whenever \(t \in S_{\text{good}}\), we have that

\[1 - 4\sigma_1 \sqrt{\mathbb{E}\left[\frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2} \mid \mathcal{F}_{t-1}\right]} \geq \frac{1}{2}.
\]

Therefore, Lemma 5 implies that, whenever \(t \in S_{\text{good}}\),

\[
\mathbb{E}\left[F(w_{t+1}) - F(w_t) \mid \mathcal{F}_{t-1}\right] \leq -\frac{\eta}{4} \frac{\left\|\nabla F(w_t)\right\|^2}{\sqrt{b_{t-1}^2 + (1 + \sigma_1^2) \left\|\nabla F(w_t)\right\|^2 + \sigma_0^2}} + \left(2\sigma_0\eta + \frac{L\eta^2}{2}\right) \mathbb{E}\left[\frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2} \mid \mathcal{F}_{t-1}\right] \tag{25}
\]

Summing this expression over all “good” times \(t \in S_{\text{good}}\), recalling that \(\{t \in S_{\text{good}}\} \in \mathcal{F}_{t-1}\), and applying the tower rule of expectations, we find that the LHS of the resulting expression can be written more simply as:

\[
\sum_{t \in [T]} \mathbb{E}\left[F(w_{t+1}) - F(w_t) \mid \mathcal{F}_{t-1}\right] 1\{t \in S_{\text{good}}\} = \sum_{t \in [T]} \mathbb{E}\left[F(w_{t+1}) - F(w_t)\right] 1\{t \in S_{\text{good}}\} \mid \mathcal{F}_{t-1}\]

\[= \sum_{t \in [T]} \mathbb{E}\left[(F(w_{t+1}) - F(w_t)) 1\{t \in S_{\text{good}}\}\right]
\]

\[= \mathbb{E}\left[\sum_{t \in S_{\text{good}}} F(w_{t+1}) - F(w_t)\right].
\]

Thus, applying the same argument tower rule argument as above to the RHS of (25) after summing over all \(t \in S_{\text{good}}\), and rearranging, we obtain

\[
\mathbb{E}\left[\sum_{t \in S_{\text{good}}} \frac{\etas}{4} \left\|\nabla F(w_t)\right\|^2\right] \leq \mathbb{E}\left[\sum_{t \in S_{\text{good}}} F(w_{t+1}) - F(w_t)\right] + \frac{\eta^2(4\sigma_0\eta + L)}{2} \mathbb{E}\left[\sum_{t \in S_{\text{good}}} \frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2}\right].
\]

Observing that, by adding and subtracting \(\mathbb{E}\left[\sum_{t \notin S_{\text{good}}} F(w_t) - F(w_{t+1})\right]\) to the above expression, and by upper bounding \(F(w_1) - \mathbb{E}[F(w_T)] \leq F(w_1) - F^*\), we obtain the first inequality.

To obtain the second inequality, we note that, since \(\{t \notin S_{\text{good}}\} \in \mathcal{F}_{t-1}\), we may use the same arguments as presented earlier, along with the observation that, since \(\left\|g_t\right\|^2/(b_{t-1}^2 + \left\|g_t\right\|^2) \leq 1\) deterministically, \(1 - 4\sigma_1 \sqrt{\mathbb{E}\left[\frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2} \mid \mathcal{F}_{t-1}\right]} \geq 1 - 4\sigma_1\), to conclude that, whenever \(t \notin S_{\text{good}}\),

\[
\mathbb{E}\left[F(w_{t+1}) - F(w_t) \mid \mathcal{F}_{t-1}\right] \leq \frac{\eta}{2} (4\sigma_1 - 1) \left\|\nabla F(w_t)\right\|^2 \sqrt{b_{t-1}^2 + (1 + \sigma_1^2) \left\|\nabla F(w_t)\right\|^2 + \sigma_0^2} + \left(2\sigma_0\eta + \frac{L\eta^2}{2}\right) \mathbb{E}\left[\frac{\left\|g_t\right\|^2}{b_{t-1}^2 + \left\|g_t\right\|^2} \mid \mathcal{F}_{t-1}\right].
\]
Summing and taking expectations of the above expression, using the resulting expression to bound
\[ \mathbb{E} \left[ \sum_{t \notin S_{\text{good}}} F(w_{t+1}) - F(w_t) \right], \]
and using Lemma 23 to bound \[ \mathbb{E} \left[ \sum_{t=1}^{T} \frac{\|g_t\|^2}{b_t^2} \right] \leq \log(f(T)), \]
we reach the desired inequality.

\[ \text{Proof.} \]

\[ \text{Lemma 26. Let } S_{\text{good}} \text{ be the set of "good" times from Definition 6. Then, we have that}^{4} \]
\[ \mathbb{E} \left[ |S_{\text{good}}^c| \right] \leq 64\sigma_1^2 \log(f(T)) \text{ and } \mathbb{E} \left[ |S_{\text{good}}^c|^2 \right] \leq \left( 64\sigma_1^2(1 + 128\sigma_1^2) + 2 \right) \log^2(T^2f(T)), \]
where \( f(\cdot) \) is as defined in (20).

\[ \text{Proof.} \]

We first prove the first inequality. Note that, if \( t \notin S_{\text{good}} \), then \( \mathbb{E} \left[ \|g_t\|^2/b_t^2 \mid F_{t-1} \right] > 1/64\sigma_1^2 \) by construction. Conveniently, this lower bound tells us that, for each time \( t \in [T] \),
\[ \mathbb{E} \left[ \|g_t\|^2/b_t^2 \right] \geq \mathbb{E} \left[ \mathbb{E} \left[ \|g_t\|^2/b_t^2 \mid F_{t-1} \right] 1 \{ t \notin S_{\text{good}} \} \right] \geq \frac{1}{64\sigma_1^2} \mathbb{E} \left[ 1 \{ t \notin S_{\text{good}} \} \right]. \]

Now, summing the above expression over all times \( t \in [T] \), and applying Lemma 23, we find that
\[ \log(f(T)) \geq \sum_{t \in [T]} \mathbb{E} \left[ \|g_t\|^2/b_t^2 \right] \geq \frac{1}{64\sigma_1^2} \mathbb{E} \left[ \sum_{t \in [T]} 1 \{ t \notin S_{\text{good}} \} \right] \geq \frac{1}{64\sigma_1^2} \mathbb{E} \left[ |S_{\text{good}}^c| \right], \]
as claimed.

Now, observe that, for that first result, we only used our guarantee on \( \mathbb{E} \left[ \sum_{t \in [T]} \|g_t\|^2/b_t^2 \right] \). However, Lemma 23 tells us much more. Indeed, assuming that \( \mathcal{E}_s(\delta) \) (the nice event from Definition 22) is true for some \( s \in [T] \)
\[ \sum_{t=s+1}^{T} \mathbb{E} \left[ 1 \{ t \notin S_{\text{good}} \} \mid \mathcal{F}_s \right] \leq 64\sigma_1^2 \log(f(T)/s), \tag{26} \]
where the above follows by noting (similarly as before), for every \( t > s \),
\[ \mathbb{E} \left[ \|g_t\|^2/b_t^2 \mid \mathcal{F}_s \right] = \mathbb{E} \left[ \mathbb{E} \left[ \|g_t\|^2/b_t^2 \mid F_{t-1} \right] \mid \mathcal{F}_s \right] \]
\[ \geq \mathbb{E} \left[ \mathbb{E} \left[ \|g_t\|^2/b_t^2 \mid F_t \right] 1 \{ t \notin S_{\text{good}} \} \mid \mathcal{F}_s \right] \geq \frac{1}{64\sigma_1^2} \mathbb{E} \left[ 1 \{ t \notin S_{\text{good}} \} \mid \mathcal{F}_s \right]. \]

Summing the above expression and upper bounding the resulting LHS using Lemma 23 (where we assume that \( \mathcal{E}_s(\delta) \) is true) yields (26). We can use this bound as follows: since \( |S_{\text{good}}^c|^2 = \left( \sum_{t \in [T]} 1 \{ t \notin S_{\text{good}} \} \right)^2 \), we may expand this expression and apply the tower rule of expectations to observe that
\[ \mathbb{E} \left[ |S_{\text{good}}^c|^2 \right] = \mathbb{E} \left[ |S_{\text{good}}^c| \right] + 2 \sum_{t_1=1}^{T} \mathbb{E} \left[ 1 \{ t_1 \notin S_{\text{good}} \} \mathbb{E} \left[ \sum_{t_2=t_1+1}^{T} 1 \{ t_2 \notin S_{\text{good}} \} \mid F_{t_1} \right] \right]. \]

\( ^4 \)As an aside, using essentially the same arguments, we can show that \( |S_{\text{good}}^c| \) satisfies the Bernstein condition with parameter \( \text{const} \cdot \log(T) \), which implies that, with high probability, \( |S_{\text{good}}^c| \leq \text{const} \cdot \log^2(T) \).
As a consequence, can show the existence of a true bound for sum of normalized gradients in expectation for "good" times $S_{\text{good}} = \{t \in [T] \mid (\ast) \leq 1/2\}$

$$\mathbb{E} \left[ \sum_{t \in S_{\text{good}}^c} \tilde{\eta}_t \|\nabla F(w_t)\|^2 \right] \leq F_0 - F^* + c_1 \cdot \log^2(T)$$

Figure 1: A flow chart of the main ideas underlying the compensation argument used to prove Lemma 30

By (26), we additionally know that, for each time $t_1$,

$$\mathbb{E} \left[ \mathbb{1}_{\{t_1 \notin S_{\text{good}}\}} \mathbb{E} \left[ \sum_{t_2 = t_1 + 1}^{T} \mathbb{1}_{\{t_2 \notin S_{\text{good}}\}} \mid \mathcal{F}_{t_1} \right] \right] \leq 64\sigma_1^2 \log(f(T)/\delta) \mathbb{E} \left[ \mathbb{1}_{\{t_1 \notin S_{\text{good}}\}} \right] + T \Pr[\mathcal{E}_{t_1}^c(\delta)^c].$$

As a result, since $\Pr[\mathcal{E}_{t_1}^c(\delta)^c] \leq \delta$ by construction, and choosing $\delta = 1/T^2$, we conclude that

$$\mathbb{E} \left[ |S_{\text{good}}^c|^2 \right] \leq (1 + 128\sigma_1^2 \log(T^2 f(T))) \mathbb{E} \left[ |S_{\text{good}}^c| \right] + 2$$

$$\leq (64\sigma_1^2(1 + 128\sigma_1^2) + 2) \log^2(T^2 f(T)),$$

as claimed.

**E  Compensating for “Bad” Time-Steps**

Here, we provide proofs for the compensation arguments presented in Section 4.

**Lemma 27.** In the same setting as Lemma 25, for any set $S_{\text{comp}}^c := \cup_{t \in S_{\text{good}}} S_{\text{comp}}^c[t] \subseteq S_{\text{good}}$ (where $S_{\text{comp}}^c[t]$ denotes the compensating set for a bad time $t$ which is disjoint from all other $S_{\text{comp}}^c[t]$), we
Figure 2: A possible configuration of each bad time \( \tau_{i}^{\text{bad}} \in S_{\text{good}}^{c} \) and the associated compensating good times \( S_{\text{comp}}^{i} \) from the greedy construction in Lemma 11 on the interval \([T]\). Observe that, by this greedy construction, \( \tau_{i}^{\text{bad}} \) has the largest \( n_{\text{comp}} \) “good” times in its compensation set, \( S_{\text{comp}}^{i} \). The remaining compensation sets are built greedily from the largest time to the smallest. Hence, \( \tau_{i}^{\text{bad}} \) has only a single compensating time, and all smaller bad times have no compensating times. Finally, note that the number of “bad” times, \( |S_{\text{good}}^{c}| \), is typically quite small relative to \( T \) (see Lemma 26), even though it is not depicted as such in the above figure.

We emphasize, however, that the proof of Lemma 10 does not rely on the notions of “good” or “bad” times in \( S_{\text{comp}}^{i} \). Let us begin by proving that, for any times \( t \geq t' \),

\[
\frac{\tilde{\eta}}{4} \left\| \nabla F(w_t) \right\|^2 - \eta_t' \left\| \nabla F(w_{t'}) \right\|^2 \leq \frac{\eta^2 L \eta_{\text{comp}}}{8} (t - t').
\] (27)
The claim is trivial when \( t' = t \), so we focus on the case when \( t' < t \). Let us denote \( a = b^2_{t-1} + (1 + \sigma_1^2) \| \nabla F(w_t) \|^2 + \sigma_0^2 \) and \( b = b^2_{t-1} + (1 + \sigma_1^2) \| \nabla F(w_{t'}) \|^2 + \sigma_0^2 \). Then, observe that

\[
\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{ab}} = \frac{b - a}{\sqrt{ab} (\sqrt{a} + \sqrt{b})}.
\]

Therefore, we can observe that the step sizes are sufficiently close, since

\[
\frac{\tilde{\eta}_t - \tilde{\eta}_{t'}}{\eta} \leq \frac{1}{\sqrt{b^2_{t-1} + (1 + \sigma_1^2) \| \nabla F(w_t) \|^2 + \sigma_0^2}} - \frac{1}{\sqrt{b^2_{t-1} + (1 + \sigma_1^2) \| \nabla F(w_{t'}) \|^2 + \sigma_0^2}} = \frac{(1 + \sigma_1^2)(\| \nabla F(w_{t'}) \|^2 - \| \nabla F(w_t) \|^2)}{\sqrt{ab} (\sqrt{a} + \sqrt{b})} = \frac{(1 + \sigma_1^2)(\| \nabla F(w_{t'}) \| - \| \nabla F(w_t) \|)(\| \nabla F(w_{t'}) \| + \| \nabla F(w_t) \|)}{\sqrt{ab} (\sqrt{a} + \sqrt{b})} \leq \frac{(1 + \sigma_1^2) \| \nabla F(w_{t'}) \| - \| \nabla F(w_t) \|}{\sqrt{ab}} \leq \frac{\eta L (t - t')}{\| \nabla F(w_t) \| \| \nabla F(w_{t'}) \|}
\]

where the last line follows by Lemma 21. We will now use this observation in order to prove the claimed inequality. We will proceed considering two cases.

In the first case, if \( \| \nabla F(w_t) \| > 2 \eta L(t - t') \), then by Lemma 21, \( \| \nabla F(w_{t'}) \| \geq \| \nabla F(w_t) \| - \eta L(t - t') \geq 1/2 \| \nabla F(w_t) \| \). This implies that

\[
\frac{1}{4} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \tilde{\eta}_{t'} \| \nabla F(w_{t'}) \|^2 \leq \frac{1}{4} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \tilde{\eta}_{t'} \| \nabla F(w_t) \| - \eta L(t - t') \|^2 \leq \frac{1}{4} \| \nabla F(w_t) \|^2 - \tilde{\eta}_{t'} \| \nabla F(w_t) \| \leq \frac{\eta^2 L(t - t')}{4 \| \nabla F(w_t) \|} \leq \frac{\eta^2 L(t - t')}{2}
\]

In the alternative case, when \( \| \nabla F(w_t) \| \leq 2 \eta L(t - t') \), then

\[
\frac{1}{4} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \tilde{\eta}_{t'} \| \nabla F(w_{t'}) \|^2 \leq \frac{1}{4} \tilde{\eta}_t \| \nabla F(w_t) \|^2 \leq \frac{\eta^2 L(t - t')}{2}
\]

where the first inequality follows by lower bounding the second term by zero, and the second by definition of \( \tilde{\eta}_t \), and the third by assumption. Thus, we obtain exactly the same bound in both cases, which establishes (27).

Now, we use (27) to prove the claim. Indeed, since, by construction, \( |S| = n_{\text{comp}} \geq 8(4 \sigma_1 - 1) \), so

\[
\frac{4 \sigma_1 - 1}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S} \frac{\tilde{\eta}_{t'}}{4} \| \nabla F(w_{t'}) \|^2 \leq \frac{1}{4} \sum_{t' \in S} \left( \frac{\tilde{\eta}_t}{4} \| \nabla F(w_t) \|^2 - \tilde{\eta}_{t'} \| \nabla F(w_{t'}) \|^2 \right).
\]

Therefore, using (27) to bound the above, and recalling that \( |S| = n_{\text{comp}} \), we conclude that

\[
\frac{4 \sigma_1 - 1}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S} \frac{\tilde{\eta}_{t'}}{4} \| \nabla F(w_{t'}) \|^2 \leq \frac{\eta^2 L n_{\text{comp}}}{8} (t - \text{min}(S)),
\]

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as claimed.

Lemma 11. There exists a construction of $S_{\text{comp}}^c = \bigcup_{t \in [S_{\text{good}}^c]} S_{[t]}^{c_{\text{comp}}}$, where $S_{[t]}^{c_{\text{comp}}}$ denotes the compensating “good” times for a bad time $t \in S_{\text{good}}^c$ (disjoint from other $S_{[t]}^{c_{\text{comp}}}$), satisfying $|S_{[t]}^{c_{\text{comp}}}| \leq n_{\text{comp}} := 8 \lfloor 4\sigma_1 - 1 \rfloor$ and $t > \max(S_{[t]}^{c_{\text{comp}}})$, where one of the these holds:

1. $|S_{[t]}^{c_{\text{comp}}}| = n_{\text{comp}}$ and $t - \min(S_{[t]}^{c_{\text{comp}}}) \leq n_{\text{comp}} \cdot |S_{\text{good}}^c|

2. $|S_{[t]}^{c_{\text{comp}}}| < n_{\text{comp}}$ and $t \leq n_{\text{comp}} \cdot |S_{\text{good}}^c|

Proof.

Constructing $S_{\text{comp}}^c$. We begin by giving a detailed description of our greedy construction of $S_{\text{comp}}^c$ which was briefly described in Section 4. To begin, let us denote $\tau_{[i]}^{\text{bad}}$ as the $i$th largest time in $S_{\text{good}}^c$. For notational simplicity, we will abuse our notation and refer to $S_{[t]}^{c_{\text{comp}}}$ and $\tau_{[i]}^{\text{bad}}$ interchangeably as the set of compensating “good” times for $\tau_{[i]}^{\text{bad}}$. We will iteratively construct each $S_{[i]}^{c_{\text{comp}}}$ for each $i \in [\lfloor S_{\text{good}}^c \rfloor]$, starting with $i = 1$. Let us denote

$$S_{[i]}^{\text{eligible}} = \{ t \in S_{\text{good}} | t < \min\{\tau_{[i]}^{\text{bad}}, \min(S_{[i-1]}^{c_{\text{comp}}})\} \}$$

as the set of eligible compensating “good” times for $\tau_{[i]}^{\text{bad}}$. Intuitively, these are the set of “good” times smaller than $\tau_{[i]}^{\text{bad}}$ which have not been used to compensate for larger bad times $\tau_{[i']}^{\text{bad}} > \tau_{[i]}^{\text{bad}}$. Note that we take $\min(S_{[i]}^{c_{\text{comp}}}) = +\infty$ and $\min(\emptyset) = -\infty$ so that (i) $S_{[i]}^{\text{eligible}}$ consists of every “good” time which is smaller than $\tau_{[i]}^{\text{bad}}$, and (ii) if $S_{[i-1]}^{c_{\text{comp}}} = \emptyset$, then there are no eligible times for $\tau_{[i]}^{\text{bad}}$, i.e., $S_{[i]}^{\text{eligible}} = \emptyset$.

We may then choose the “compensating” set $S_{[i]}^{c_{\text{comp}}}$ for $\tau_{[i]}^{\text{bad}}$ as the largest (at most) $n_{\text{comp}}$ times in $S_{[i]}^{\text{eligible}}$. It is clear by this construction that $S_{[i]}^{c_{\text{comp}}} \cap S_{[i']}^{c_{\text{comp}}} = \emptyset$ for every $i \neq i' \in [\lfloor S_{\text{good}}^c \rfloor]$.

We will further take $i^*$ to be the smallest index in $[\lfloor S_{\text{good}}^c \rfloor]$ such that $|S_{[i]}^{c_{\text{comp}}}| < n_{\text{comp}}$. Intuitively, this is the index of the largest “bad” time $\tau_{[i]}^{\text{bad}}$ which is not fully compensated.

Establishing the properties of $S_{\text{comp}}^c$. Note that, as required, each $|S_{[i]}^{c_{\text{comp}}}| \leq n_{\text{comp}}$ and $\tau_{[i]}^{\text{bad}} > \max(S_{[i]}^{c_{\text{comp}}})$ by the construction of $S_{\text{comp}}^c$ described above. Further, note that since $i^*$ is chosen as the smallest index for which $|S_{[i^*]}^{c_{\text{comp}}}| < n_{\text{comp}}$, it must be the case that $|S_{[i]}^{c_{\text{comp}}}| = n_{\text{comp}}$ for every $i < i^*$, and $|S_{[i]}^{c_{\text{comp}}}| < n_{\text{comp}}$ for every $i \geq i^*$. Therefore, to reason about the two conditions, we need to consider only the cases (i) $\tau_{[i]}^{\text{bad}} > \tau_{[i^*]}^{\text{bad}}$ and (ii) $\tau_{[i]}^{\text{bad}} \leq \tau_{[i^*]}^{\text{bad}}$.

Case 1: Let us first consider a bad time $\tau_{[i]}^{\text{bad}} > \tau_{[i^*]}^{\text{bad}}$. Clearly, $|S_{[i]}^{c_{\text{comp}}}| = n_{\text{comp}}$. By the greedy construction of the compensating sets, observe that

$$\left| \left( \max(S_{[i]}^{c_{\text{comp}}}, \tau_{[i]}^{\text{bad}}) \right) \cap S_{\text{good}} \right| \leq (i - 1) \cdot n_{\text{comp}}. \quad (28)$$

Indeed, these are the times in $S_{\text{good}}$ associated with a compensating set $S_{[i]}^{c_{\text{comp}}}$ for a larger “bad” time $\tau_{[i']}^{\text{bad}} > \tau_{[i]}^{\text{bad}}$. If there were any more “good” times on this interval, then they would have been assigned to $S_{[i]}^{c_{\text{comp}}}$ by definition of our greedy procedure. Next, note that

$$\left| \left[ \min(S_{[i]}^{c_{\text{comp}}}, \max(S_{[i]}^{c_{\text{comp}}})) \right] \cap S_{\text{good}} \right| = n_{\text{comp}}. \quad (29)$$
These times corresponding to the $n_{\text{comp}}$ times in $S_{i[i]}^{\text{comp}}$. Indeed, by the greedy construction of our compensating sets, $\max(S_{i[i+1]}^{\text{comp}}) < \min(S_{i[i]}^{\text{comp}})$ for every $i' \in [S_{i[i]}^{c}]$, and the procedure always chooses the largest “good” times available in $S_{i[i]}^{c}$, so no other good times can lie on this interval. Finally, we observe that

\[
\left[ \min(S_{i[i]}^{\text{comp}}, \tau_{i[i]}^{\text{bad}}) \right) \cap S_{\text{good}}^{c} \leq |S_{\text{good}}^{c}| - i, \quad (30)
\]

corresponding to the at most $|S_{\text{good}}^{c}| - i$ bad times $\tau_{i[i]}^{\text{bad}} < \tau_{i[i]}^{\text{bad}}$. Combining Eqs. (28) to (30), we conclude that $\tau_{i[i]}^{\text{bad}} - \min(S_{i[i]}^{\text{comp}}) \leq n_{\text{comp}} \cdot |S_{\text{good}}^{c}|$.

Case 2: We now consider the case when $\tau_{i[i]}^{\text{bad}} \leq \tau_{i[i]}^{\text{bad}}$. Clearly, $|S_{i[i]}^{\text{comp}}| < n_{\text{comp}}$. Since we need only to show that $\tau_{i[i]}^{\text{bad}}$ is upper bounded by $n_{\text{comp}} \cdot |S_{\text{good}}^{c}|$, it suffices to show this for $\tau_{i[i]}^{\text{bad}}$. Our arguments will follow in a similar spirit as Case 1. Indeed, using exactly the same arguments used to establish Eqs. (28) and (29), we know that

\[
\left[ \min(S_{i[i]}^{\text{comp}}, \tau_{i[i]}^{\text{bad}}) \right) \cap S_{\text{good}}^{c} \leq i^* \cdot n_{\text{comp}}. \quad (31)
\]

Further, by the greedy construction of the compensating sets, since $|S_{i[i]}^{\text{comp}}| < n_{\text{comp}}$, it must be the case that

\[
\left[ 1, \min(S_{i[i]}^{\text{comp}}) \right) \cap S_{\text{good}}^{c} = \emptyset, \quad (32)
\]

since otherwise, any remaining elements could have been added to $S_{i[i]}^{\text{comp}}$. Therefore, since

\[
\left[ 1, \tau_{i[i]}^{\text{bad}} \right) \cap S_{\text{good}}^{c} = |S_{\text{good}}^{c}| - i^*, \quad (33)
\]

we conclude by Eqs. (31) to (33) that $\tau_{i[i]}^{\text{bad}} \leq n_{\text{comp}} \cdot |S_{\text{good}}^{c}|$, as claimed.

**Lemma 29.** If $S_{i[i]}^{\text{comp}}$ is constructed as in Lemma 11, then the “residual” term from Lemma 27 can be bounded as follows:

\[
\mathbb{E} \left[ \sum_{t \in S_{\text{good}}^{c}} \frac{(4\sigma_t - 1)}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S_{i[i]}^{\text{comp}}} \frac{\tilde{\eta}_{t'}^2}{4} \| \nabla F(w_{t'}) \|^2 \right] 
\leq 128\eta\sigma_t^2 \| \nabla F(w_1) \| \log(f(T))
+ \eta^2 L n_{\text{comp}} \left( \frac{n_{\text{comp}}}{8} + 2 \right) (64\sigma_t^2 (1 + 128\sigma_t^2) + 2) \log^2(T^2 f(T)).
\]

**Proof.** Borrowing the notation from the proof of Lemma 11, we will use $\tau_{i[i]}^{\text{bad}}$ to denote the $i$th largest “bad” time in $S_{\text{good}}^{c}$, and, abusing notation slightly, use $S_{i[i]}^{\text{comp}}$ and $S_{\tau_{i[i]}^{\text{bad}}}^{\text{comp}}$ interchangeably to denote the compensating “good” times for $\tau_{i[i]}^{\text{bad}}$. Further, we take $i^*$ to be the index of the first “bad” time $\tau_{i[i]}^{\text{bad}}$ which cannot be fully compensated, i.e., $|S_{i[i]}^{\text{comp}}| < n_{\text{comp}}$. Using this notation, we may rewrite the residual term from Lemma 27 in the following convenient manner:

\[
\mathbb{E} \left[ \sum_{t \in S_{\text{good}}^{c}} \frac{(4\sigma_t - 1)}{2} \tilde{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{t' \in S_{i[i]}^{\text{comp}}} \frac{\tilde{\eta}_{t'}^2}{4} \| \nabla F(w_{t'}) \|^2 \right]
\]

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and we can take

\[ F(B) \leq \sum_{i \leq t} \left( \frac{4\sigma_1 - 1}{2} \hat{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{\nu \in S^\text{comp}} \tilde{\eta}_\nu \| \nabla F(w_\nu) \|^2 \right) \]

+ \mathbb{E} \left[ \sum_{i \geq t+1} \frac{4\sigma_1 - 1}{2} \hat{\eta}_t \| \nabla F(w_t) \|^2 \right].

Now, we will use Lemma 10 to bound the first term above. We will use the trivial bound for the second term: by Definition 3 and Lemma 21, we may bound each term inside of the sum of the second expression above as:

\[ \tilde{\eta}_t \| \nabla F(w_t) \|^2 \leq \frac{\eta}{\sqrt{1 + \sigma_1^2}} \| \nabla F(w_t) \| \leq \frac{\eta}{\sqrt{1 + \sigma_1^2}} (\| \nabla F(w_t) \| + \eta L t^\text{bad}) \]

These two bounds described above, together with Lemma 11 and the fact that each \( i \leq |S^c_{\text{good}}| \), imply that:

\[ \mathbb{E} \left[ \sum_{i \leq t} \left( \frac{4\sigma_1 - 1}{2} \hat{\eta}_t \| \nabla F(w_t) \|^2 - \sum_{\nu \in S^\text{comp}} \tilde{\eta}_\nu \| \nabla F(w_\nu) \|^2 \right) \right] \]

\[ \leq \mathbb{E} \left[ \sum_{i \leq t} \eta^2 L_n^{\text{comp}} (T_{\text{bad}} - \min(S^\text{comp})) \right] + \mathbb{E} \left[ \sum_{i \geq t+1} \eta \frac{4\sigma_1 - 1}{2} \sqrt{1 + \sigma_1^2} (\| \nabla F(w_t) \| + \eta L t^\text{bad}) \right] \]

\[ \leq \eta^2 L_n^{\text{comp}} \mathbb{E} \left[ |S^c_{\text{good}}|^2 \right] + 2\eta \mathbb{E} \left[ |S^c_{\text{good}}| \right] + \eta L_n^{\text{comp}} \mathbb{E} \left[ |S^c_{\text{good}}| \right] \]

\[ \leq \eta^2 L_n^{\text{comp}} \left( \frac{n^{\text{comp}}}{8} + 2 \right) \mathbb{E} \left[ |S^c_{\text{good}}|^2 \right] + 2\eta \| \nabla F(w_t) \| \mathbb{E} \left[ |S^c_{\text{good}}| \right] \].

Applying the bounds on \( |S^c_{\text{good}}| \) from Lemma 26 yields the claimed bound.

\[ \square \]

**Lemma 30.** Let the set \( S^\text{comp} \) from Lemma 27 be chosen as in Lemma 11. Then, taking \( \tilde{S} := S_{\text{good}} \setminus S^\text{comp} \) as the remaining “good” times after compensation, we have that

\[ \mathbb{E} \left[ \sum_{i \in \tilde{S}} \frac{\hat{\eta}_i}{4} \| \nabla F(w_i) \|^2 \right] \leq F(w_1) - F^* + c_1 \cdot \log^2(T^2 f(T)), \]

where we can take

\[ c_1 = 2\eta (\sigma_0 + 64\sigma_1^2 \| \nabla F(w_1) \|) + \frac{L_n^2}{2} \left( 1 + n^{\text{comp}} \left( \frac{n^{\text{comp}}}{4} + 4 \right) (64\sigma_1^2(1 + 128\sigma_1^2) + 2) \right), \]

and \( n^{\text{comp}} \) is the constant, maximum size of a single compensating set specified in Lemma 11.

**Proof.** The result follows immediately by combining Lemmas 27 and 29. Note that this result, up to logarithmic factors, takes essentially the same form as in the uniformly-bounded setting (11). \( \square \)

**F Bounding the Expected Sum of Gradients via Recursive Improvement**

Here, we provide a proof for the recursive improvement argument presented in Section 4.
Result of compensation argument: Bound for sum of normalized true gradients in expectation for contracted “good times” $S_{\text{good}} \setminus S_{\text{comp}}$

$$E\left[\sum_{t \in S_{\text{good}} \setminus S_{\text{comp}}} \frac{x}{c_{\text{2}}} \|\nabla F(w_t)\|^2\right] \leq F_0 - F^* + c_1 \cdot \log^2(T)$$

Bound sum of stochastic gradients w.p. almost surely

$$E\left[\sum_{t \in [T]} \|\nabla F(w_t)\|^2\right] \leq c_2 \cdot T^{\frac{x}{2}} \cdot \log^{\frac{x}{3}}(T)$$

Initialization:

$$x \leftarrow 2, y \leftarrow 1$$

Bound sum of unnormalized true gradients in expectation for the “good” (non-compensating) terms

$$E\left[\sum_{t \in S_{\text{good}} \setminus S_{\text{comp}}} \|\nabla F(w_t)\|^2\right] \leq \frac{x}{c_{\text{2}}} \cdot T^{\frac{x}{2}} \cdot \log^{\frac{x}{3}}(T)$$

Bound sum of unnormalized true gradients in expectation for the “bad” (or compensating) terms

$$E\left[\sum_{t \in S_{\text{good}} \setminus S_{\text{comp}}} \|\nabla F(w_t)\|^2\right] \leq \frac{x}{c_{\text{2}}} \cdot T^2 \cdot \log^2(T)$$

Bound sum of unnormalized true gradients in expectation for all terms

$$E\left[\sum_{t \in [T]} \|\nabla F(w_t)\|^2\right] \leq c_2 \cdot T^x \cdot \log^{\frac{x}{3}}(T)$$

Recursive improvement argument. Observe that $x \rightarrow 1$ and $y \rightarrow 5/2$.

Update $x \leftarrow 2 + \frac{x}{3}$ and $y \leftarrow 5 + \frac{y}{2}$

Conclusion of recursive improvement: Can remove the bounded gradient assumption

$$E\left[\sum_{t \in [T]} \|\nabla F(w_t)\|^2\right] \leq c_2 \cdot T^x \cdot \log^{\frac{x}{3}}(T)$$

Figure 3: A flow chart of the main ideas underlying the “Recursive Improvement” argument of Lemma 31.
F.1 Main Ideas

Lemma 31. Suppose that
\[
E \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^x \log^y (T^2 f(T)),
\]
where \( x \in [1, 4], y \geq 1, \) and
\[
c_2 = \max \left\{ 1, 16(b_0^2 + \sigma_0^2 + 1 + 32\sigma_1^2) (\| \nabla F(w_1) \|^2 + \eta^2 L^2) + \sigma_0^2, 512 \left( \frac{F(w_1) - F^* + \text{const}}{\eta} \right)^2 \right\}
\]

Then, in fact, the following tighter bound also holds:
\[
E \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T^\frac{x+1}{2} \log^\frac{y+1}{2} (T^2 f(T)).
\] (34)

In particular, as a consequence of Lemma 21,
\[
E \left[ \sum_{t=1}^{T} \| \nabla F(w_t) \|^2 \right] \leq c_2 T \log^\frac{1}{2} (T^2 f(T)).
\] (35)

The main idea of the proof is to recursively improve our upper bound on the “normalized” expected sum of gradients from Lemma 30 in expectation combining it with a lower bound on the step size proxy with high (enough) probability obtained from Markov’s inequality and an invariant upper bound provided in Lemma 12. Recall that \( \bar{\eta}_T = \eta \sqrt{b_{T-1}^2 + (1+\sigma_1^2) \| \nabla F(w_t) \|^2 + \sigma_0^2} \), thus to provide a lower bound on the step size proxy we will focus on upper bounding \( b_{T-1} \). In particular taking the expectation, we have that:
\[
E [b_{T-1}^2] = b_0^2 + \sum_{t=1}^{T-1} E [\| g_t \|^2] \leq b_0^2 + (T - 1)\sigma_0^2 + (1 + \sigma_1^2) \sum_{t=1}^{T-1} E [\| \nabla F(w_t) \|^2],
\] (36)

where the above follows by applying Assumptions 1 and 2. Thus, to obtain an upper bound for \( E [b_{T-1}^2] \), we must have a bound for \( E \left[ \sum_{t \in [T-1]} \| \nabla F(w_t) \|^2 \right] \) – the quantity we wish to bound! This highlights the motivation for applying the following improving idea recursively. We begin with a crude (polynomial in \( T \)) upper bound for \( E \left[ \sum_{t \in [T]} \| \nabla F(w_t) \|^2 \right] \), and recursively improve this bound via the interlaced inequalities described above. Repeating this process infinitely many times ultimately obtains the desired upper bound on the expected sum of the gradients.

Proof of Lemma 31.

Step 1: Lower bounding the step size proxy. Recall that Lemma 30 gives an upper bound on \( E \left[ \sum_{t \in S_{\text{good}} \setminus S_{\text{comp}}} \bar{\eta}_t \| \nabla F(w_t) \|^2 \right] \). Using the “nice event” \( \mathcal{E}_T(\delta) \) from Definition 22 with a sufficiently small failure probability \( \delta = (b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{\gamma_1} (T^2 f(T))/T^{\gamma_2} \), where \( \gamma_1, \gamma_2 \geq 0 \) are arbitrary parameters satisfying \( \gamma_1 + x \geq 2 \) and \( \gamma_1 \leq 2 \), we can ensure that the step size proxy \( \bar{\eta}_t \) is sufficiently small. Indeed, these insights allow us to prove Lemma 32, which tells us that:
\[
E \left[ \sum_{t \in \tilde{S}} \bar{\eta}_t \| \nabla F(w_t) \|^2 \right] \geq E \left[ \sum_{t \in S} \bar{\eta}_t \| \nabla F(w_t) \|^2 1 \{ \mathcal{E}_T(\delta) \} \right]
\]
While the above translates the bound in Lemma 30 into a more interpretable form, the presence of \(1\{\mathcal{E}_T(\delta)\}\) makes the above bound not immediately useful. However, by construction, \(\mathcal{E}_T(\delta)\) happens with probability at least \(1 - \delta\). Our choice of \(\delta\) will allow us to show that, effectively, the above upper bound is still true with the indicator removed.

In order to “remove” the indicator from the expectation above, we will need to show that, when \(\mathcal{E}_T(\delta)\) is false, \(\sum_{t \in \mathcal{S}} \| \nabla F(w_t) \|^2 \) cannot be too large. Recall that we have two main tools to upper bound the size of this sum: Lemma 21, which gives a deterministic upper bound of \(O(T^3)\), and Lemma 24, which gives a high-probability upper bound of \(\tilde{O}(T^2)\). These insights allow us to prove Lemma 33, which tells us that

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{S}} \| \nabla F(w_t) \|^2 \right] \geq \eta \mathbb{E} \left[ \frac{\sum_{t \in \mathcal{S}} \| \nabla F(w_t) \|^2 \mathbb{1}\{\mathcal{E}_T(\delta)\}}{\sqrt{2c_2 T^{x+\gamma_1} \log^2(T^2 f(T))}} \right].
\]

(37)

Step 2: Bounding the “good” terms. With the indicator removed from the above expression, we are now ready to use Lemma 30 together with (37) and (38) to obtain a bound on the expected size of the gradients at the good times:

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{S}} \| \nabla F(w_t) \|^2 \right] \leq \left( 4\sqrt{2c_2 T^{x+\gamma_1} \log^2(T^2 f(T))} \right) \frac{F(w_1) - F^* + \text{const} \log^2(T^2 f(T))}{\eta} + \frac{c_2}{4} T^{\max(2-\gamma_1,1)} \log^{y-\gamma_2+1}(T^2 f(T))
\]

\[
\leq \frac{c_2}{4} T^{x+\gamma_1/2} \log^{2+\gamma_2/2}(T^2 f(T)) + \frac{c_2}{4} T^{\max(2-\gamma_1,1)} \log^{y-\gamma_2+1}(T^2 f(T)),
\]

where the second inequality follows by upper bounding \(T^2 f(T) \leq T^2 f(T)\) and \(4\sqrt{2(F(w_1) - F^* + \text{const})}/\eta \leq \sqrt{c_2}/4\). Hence, by choosing \(\gamma_1 = (4-x)/3\) and \(\gamma_2 = 2(y-1)/3^5\), we conclude that

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{S}} \| \nabla F(w_t) \|^2 \right] \leq \frac{c_2}{2} T^{x+\gamma_2} \log^{2+\gamma_2}(T^2 f(T)).
\]

(39)

Step 3: Bounding the “bad” terms. To conclude the argument, we will need to bound the remaining terms, \(\mathbb{E} \left[ \sum_{t \notin \mathcal{S}} \| \nabla F(w_t) \|^2 \right]\). Intuitively, these terms are not problematic for the sake of this argument, since (i) \(\mathbb{E} \left[ \left| S^c \right| \right] = \mathbb{E} \left[ |S^c_{\text{good}} \cup S^c_{\text{comp}}| \right] \leq (1 + n_{\text{comp}}) \mathbb{E} \left[ |S^c_{\text{good}}| \right] \lesssim \log(f(T))\) by construction of \(S_{\text{comp}}\) (Lemma 11) and by our control on the “good” set in Lemma 26, and since (ii) each term \(\| \nabla F(w_t) \|^2\) can be bounded with high probability by \(O(T \log(f(T)))\) by Lemma 24. These arguments are formalized in Lemma 34, which tells us that

\[
\mathbb{E} \left[ \sum_{t \notin \mathcal{S}} \| \nabla F(w_t) \|^2 \right] \leq \frac{c_2}{2} T^{x+\gamma_2} \log^{2+\gamma_2}(T^2 f(T)).
\]

(40)

\footnote{Note that these choices of \(\gamma_1, \gamma_2\) are valid since \(\gamma_1, \gamma_2 \geq 0\) since \(x \leq 4\) and \(y \geq 1\), and since \(x \geq 1\), \(\gamma_1 \leq 2\) and \(x + \gamma_1 \geq 2\).}
Thus, we arrive at (34) by combining the results of (39) and (40). To obtain (35), simply note that we may initialize (34) with \( x = 2 \) and \( y = 1 \) by Lemma 24. Given this initialization, we may invoke our improved bound on the expected sum of gradients (34) recursively, concluding that we may take \( x = 1 \) and \( y = 5/2 \), as claimed.

### F.2 Technical Lemmas

**Lemma 32** (Polynomial control of step sizes). Suppose that:

\[
E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right] \leq c_2 T^x \log^y(h(T))
\]

for some \( x, y \geq 1 \), and \( c_2 \geq \max\{1, (1 + \sigma_1^2)(\|\nabla F(w_1)\| + \eta L)^2 + \sigma_0^2\} \) and \( h(T) \geq 2 \). Recalling the “nice” event \( \mathcal{E}_T(\delta) \) from Definition 22, where we choose \( \delta = (b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{y - \gamma_2}(h(T))/T^{\gamma_1} \) for any \( \gamma_1, \gamma_2 \) satisfying \( \gamma_1 + x \geq 2, \gamma_1, \gamma_2 \geq 0, \) and \( \gamma_1 \leq 2 \). Then, recalling the step size proxy from Definition 3, \( \tilde{n}_t \), we have that

\[
E \left[ \sum_{t \in S} \tilde{n}_t \|\nabla F(w_t)\|^2 \right] \geq \frac{\eta \mathbb{E} \left[ \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2 \mathbb{1}_{\{\mathcal{E}_T(\delta)\}} \right]}{\sqrt{2c_2 T^{x + \gamma_1} \log^{y + \gamma_2}(h(T))}}.
\]

**Proof.** Let us assume that \( \mathcal{E}_T(\delta) \) (the “nice” event from Definition 22) is true. By our assumed bound on \( E \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right] \), and since \( c_2 \geq 1 \), by choice of \( \delta \) we know that

\[
b^2_T \leq b_0^2 + \frac{T \sigma_0^2 + (1 + \sigma_1^2)c_2 T^x \log^y(h(T))}{\delta} \leq c_2 T^{x + \gamma_1} \log^{y + \gamma_2}(h(T)).
\]

Then,

\[
\frac{\eta}{\tilde{n}_t} = \sqrt{b_{t-1}^2 + (1 + \sigma_1^2)\|\nabla F(w_t)\|^2 + \sigma_0^2} \\
\leq \sqrt{c_2 T^{x + \gamma_1} \log^{y + \gamma_2}(h(T)) + (1 + \sigma_1^2)(\|\nabla F(w_1)\| + \eta L)^2 + \sigma_0^2} \\
\leq \sqrt{2c_2 T^{x + \gamma_1} \log^{y + \gamma_2}(h(T))},
\]

where the first inequality follows since \( b_{t-1}^2 \leq b_T^2 \) and by Lemma 21, and the second since \( x + \gamma_1 \geq 2 \) and \( c_2 \geq (1 + \sigma_1^2)(\|\nabla F(w_1)\| + \eta L)^2 + \sigma_0^2 \).

Noting that \( E \left[ \sum_{t \in \mathcal{S}} \tilde{n}_t \|\nabla F(w_t)\|^2 \right] \geq E \left[ \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2 \mathbb{1}_{\{\mathcal{E}_T(\delta)\}} \right] \), and using the lower bound derived above, we obtain the claimed lower bound. \( \square \)

**Lemma 33** (Removing \( \mathbb{1}_{\{\mathcal{E}_T(\delta)\}} \)). In the same setting as in Lemma 32, assuming additionally that \( \sigma_2^2 / 4 \geq 4(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2)(\|\nabla F(w_1)\|^2 + \eta^2 L^2) \) and \( h(T) \geq T^2 f(T) \),

\[
E \left[ \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2 \mathbb{1}_{\{\mathcal{E}_T(\delta)\}} \right] \geq E \left[ \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2 \right] - \frac{c_2}{4} T^{\max(2 - \gamma_1, 1)} \log^{y - \gamma_2 + 1}(h(T))
\]
Proof. Now, in order to “remove” the indicator from the expectation, we will need to show that, when \( \mathcal{E}_T(\delta) \) is false, \( \sum_{t \in S} \|\nabla F(w_t)\|^2 \) cannot be too large. Recall that we have two main tools to upper bound the size of this sum: Lemma 21, which gives a deterministic upper bound of \( O(T^3) \), and Lemma 24, which gives a high-probability upper bound of \( \mathcal{O}(T^2) \). To exploit this “lighter” regime of Lemma 24, it will be useful to introduce the following event:

\[
\mathcal{E}' = \mathcal{E}_T(\delta)^c \cap \{ b_T^2 \leq c_2 T^{x+2} \log^{\gamma_2}(h(T)) \}.
\]

By definition of \( \mathcal{E}' \), \( \mathcal{E}' \subset \mathcal{E}_T(\delta)^c \), so

\[
\Pr[\mathcal{E}'] \leq \Pr[\mathcal{E}_T(\delta)] \leq \delta = \frac{(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{y-\gamma_2}(h(T))}{T^{\gamma_1}}.
\]

Additionally, using Markov’s inequality and the assumed upper bound on \( \mathbb{E} \left[ \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \right] \),

\[
\Pr[\mathcal{E}_T(\delta)^c \cap (\mathcal{E}')^c] = \Pr[b_T^2 > c_2 T^{x+2} \log^{\gamma_2}(h(T))] \leq \frac{(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{y-\gamma_2}(h(T))}{T^2}.
\]

Hence, decomposing \( 1 \{ \mathcal{E}_T(\delta) \} = 1 - 1 \{ \mathcal{E}' \} - 1 \{ \mathcal{E}_T(\delta)^c \cap (\mathcal{E}')^c \} \), and upper bounding \( \sum_{t \in [T]} \|\nabla F(w_t)\|^2 \) using the high probability bound of Lemma 24 under \( \mathcal{E}' \), and using the deterministic bound of Lemma 21 under \( \mathcal{E}_T(\delta)^c \cap (\mathcal{E}')^c \), we have that

\[
\mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \mathbb{1} \{ \mathcal{E}_1 \} \right] \geq \mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] - 2T(\|\nabla F(w_1)\|^2 + \eta^2 L^2 T \log(f(T)/\delta)) \Pr[\mathcal{E}_2] - 2T(\|\nabla F(w_1)\|^2 + \eta^2 L^2 T^2) \Pr[\mathcal{E}_1^c \cap \mathcal{E}_2^c] \\
\geq \mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] - 2T(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{y-\gamma_2}(h(T)) \|\nabla F(w_1)\|^2 \left( \frac{1}{T^{\gamma_1}} + \frac{1}{T^2} \right) - 2T(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2) \log^{y-\gamma_2}(h(T)) \eta^2 L^2 \left( T^{1-\gamma_1} \log(h(T)) + 1 \right),
\]

where in the last inequality, we assume that \( h(T) \geq f(T)/\delta \geq T^2 f(T) \). Now, since \( \gamma_1 \leq 2 \) by assumption, we may simplify the above to conclude that

\[
\mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \mathbb{1} \{ \mathcal{E}_1 \} \right] \\
\geq \mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] - 4(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2)(\|\nabla F(w_1)\|^2 + \eta^2 L^2) T^{\max(2-\gamma_1, 1)} \log^{y-\gamma_2+1}(h(T)).
\]

By our assumption that \( c_2/4 \geq 4(b_0^2 + \sigma_0^2 + 1 + \sigma_1^2)(\|\nabla F(w_1)\|^2 + \eta^2 L^2) \), the claimed bound is immediate. \( \square \)
Lemma 34 (Bounding gradients at the “bad” times). Under the same conditions as Lemmas 32 and 33, we have that

$$\mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] \leq \frac{c_2}{2} T^{\frac{3}{16} + \frac{1}{2}} \log^\frac{9}{2} (h(T)),$$

where we additionally assume that \(c_2/2 \geq 2(\|\nabla F(w_1)\|^2 + \eta^2 L^2) (64\sigma_1^2 + 2)\).

Proof. There are two key insights of this proof. The first is that for each time \(t\), with high probability, we have that \(\|\nabla F(w_t)\|^2 \leq O(T \log(h(T)))\), by Lemma 24. The second insight is that \(\mathbb{E} [|S^c|] \leq O(\log(h(T)))\), which is a simple consequence of the fact that \(|S^c|\) is at most a constant factor larger that \(|S^c_{\text{good}}|\) by construction (Lemma 11), and since \(\mathbb{E} [|S^c_{\text{good}}|] \leq O(h(T))\) by Lemma 26. Let us now show how to combine these insights.

To derive the claimed bound, we will consider the “nice” event \(E_T(\delta)\), where \(\delta\) is a parameter to be chosen shortly. Now, by construction of \(S^{\text{comp}}\) in Lemma 11, \(|S^c| = |S^c_{\text{good}} \cup S^{\text{comp}}| \leq (n_{\text{comp}} + 1)|S^c_{\text{good}}|\). Therefore, using our bound for \(\mathbb{E} [|S^c_{\text{good}}|]\) from Lemma 26 together with the bounds for \(\|\nabla F(w_t)\|^2\) from Lemmas 21 and 24, we have that

$$\mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] \leq \|\nabla F(w_1)\|^2 (128\sigma_1^2 \log(f(T)) + 2T\delta) + \eta^2 L^2 T \left(128\sigma_1^2 \log^2(f(T)/\delta) + 2T^2\delta\right).$$

Therefore, choosing \(\delta = 1/T^2\), we conclude that

$$\mathbb{E} \left[ \sum_{t \in S} \|\nabla F(w_t)\|^2 \right] \leq 2(\|\nabla F(w_1)\|^2 + \eta^2 L^2)(64\sigma_1^2 + 2) \log^2(T^2 f(T))T.$$ 

Since \(c_2/2 \geq 2(\|\nabla F(w_1)\|^2 + \eta^2 L^2) (64\sigma_1^2 + 2)\), \(h(T) \geq T^2 f(T)\), and \(x, y \geq 1\), we the claimed bound follows from the above. \(\square\)

G  Obtaining the Convergence Rate for AdaGrad-Norm

Here, we provide a proof for the main result of this paper, a proof of convergence for the AdaGrad-Norm algorithm.

Theorem 35. With probability at least \(1 - \delta\), the AdaGrad-Norm algorithm (2) for any choice of parameters \(\eta, b_0^2 > 0\) satisfies:

$$\min_{t \in [T]} \|\nabla F(w_t)\|^2 \leq \frac{\tilde{C}_T}{\sqrt{\delta^3 T}}.$$
where
\[
\tilde{C}_T = \frac{2\sqrt{2}(F(w_1) - F^* + c_1 \log^2(T^2f(T)))}{\eta} \sqrt{(\sigma_0^2 + 2c_2(1 + \sigma_1^2)) \log^{1/2}(T^2f(T))} \\
+ \sqrt{2}
(128\sigma_1^2(n_{comp} + 1) \log(f(T)))
\]
c_1 is the constant defined in Lemma 30, c_2 is the constant defined in Lemma 31, n_{comp} is the (constant) maximum number of compensating times from Lemma 11, and f(\cdot) is the (polynomial) function defined in (20).

Proof. Now that we know that \(\mathbb{E}\left[\sum_{t \in \mathcal{S}} \tilde{\eta}_t \|\nabla F(w_t)\|^2\right] = \mathcal{O}(\log^2(T))\) by Lemma 30 and that \(\mathbb{E}\left[\sum_{t \in [T]} \|\nabla F(w_t)\|^2\right] = \tilde{O}(T)\) by Lemma 31, we have all of the tools we need to obtain our claimed convergence rate. Indeed, we can first use the result of Lemma 31 to obtain a uniform lower bound on the step size proxies \(\tilde{\eta}_t\) in expectation. To see this, let us denote
\[
\tilde{\eta}_T := \eta \sqrt{b_{\tilde{\eta}^{-1}} + \sigma_0^2 + (1 + \sigma_1^2) \sum_{t \in [T]} \|\nabla F(w_t)\|^2},
\]
and observe that \(\tilde{\eta}_t \geq \tilde{\eta}_T\) for every \(t \in [T]\), deterministically. Thus, by Hölder’s inequality,
\[
\mathbb{E}\left[\sum_{t \in \mathcal{S}} \tilde{\eta}_t \|\nabla F(w_t)\|^2\right] \geq \mathbb{E}\left[\tilde{\eta}_T \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2\right] \geq \frac{\mathbb{E}\left[\left(\sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2\right)^{2/3}\right]^{3/2}}{\sqrt{\mathbb{E}\left[\left(1/\tilde{\eta}_T\right)^2\right]}}.
\tag{41}
\]
To further lower bound (41), we may first upper bound the denominator using our bound from Lemma 31 together with the definition of \(\tilde{\eta}_T\) and (4):
\[
\eta^2 \mathbb{E}\left[\left(1/\tilde{\eta}_T\right)^2\right] \leq T\sigma_0^2 + 2(1 + \sigma_1^2) \mathbb{E}\left[\sum_{t \in [T]} \|\nabla F(w_t)\|^2\right] \leq T\sigma_0^2 + 2c_2(1 + \sigma_1^2)T \log^{3/2}(T^2f(T)).
\]
Focusing now on lower bounding the numerator of (41),
\[
\mathbb{E}\left[\left(\sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2\right)^{2/3}\right] = \mathbb{E}\left[|\tilde{S}|^{2/3} \left(\frac{1}{|\tilde{S}|} \sum_{t \in \mathcal{S}} \|\nabla F(w_t)\|^2\right)^{2/3}\right] \geq \mathbb{E}\left[|\tilde{S}|^{2/3} \min_{t \in [T]} \|\nabla F(w_t)\|^{2/3}\right],
\]
where the lower bound above follows since the average is always larger than the minimum. If it were the case that \(\tilde{S} = [T]\), then, at this point, we would essentially be done with our proof. However, since \(\tilde{S}\) is a random set, we must take some additional care. Because \(|\tilde{S}|\) is \(\mathcal{O}(\log(T))\) in expectation by Lemma 26, this is only a minor technicality. Indeed,
\[
\mathbb{E}\left[|\tilde{S}|^{2/3} \min_{t \in [T]} \|\nabla F(w_t)\|^{2/3}\right] \geq \left(\frac{T}{2}\right)^{2/3} \mathbb{E}\left[\min_{t \in [T]} \|\nabla F(w_t)\|^{2/3} \mathbbm{1}\{|\tilde{S}| \geq T/2\}\right].
\]
Therefore, collecting the results we have derived so far into a lower bound on (41), and applying the result of Lemma 30 to upper bound this same expression, we have obtained the following upper bound:
\[
\mathbb{E}\left[\min_{t \in [T]} \|\nabla F(w_t)\|^{2/3} \mathbbm{1}\{|\tilde{S}| \geq T/2\}\right] \leq \left(\frac{C_T}{\sqrt{T}}\right)^{2/3},
\tag{42}
\]
where

\[ C_T := \frac{2(F(w_1) - F^* + c_1 \log^2(T^2 f(T)))}{\eta} \sqrt{\left(\sigma_0^2 + 2c_2(1 + \sigma_1^2)\right) \log^{5/2}(T^2 f(T))}. \]

To conclude, we will translate the bound in (42) into one on \( \mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \right] \). Begin by writing

\[
\mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \right] = \mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \mathbb{1} \{ |\tilde{S}| \geq T/2 \} \right] + \mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \mathbb{1} \{ |\tilde{S}^c| \geq T/2 \} \right] \\
\leq \left( \frac{C_T}{\sqrt{T}} \right)^{\frac{2}{3}} + \| \nabla F(w_1) \|^{4/3} \mathbb{P} \left[ |\tilde{S}^c| \geq T/2 \right].
\]

where the inequality follows since \( \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \leq \| \nabla F(w_1) \|^{4/3} \). The above failure probability can be easily upper bounded via Markov’s inequality:

\[
\mathbb{P} \left[ |\tilde{S}^c| \geq T/2 \right] \leq \frac{2(n_{\text{comp}} + 1) \mathbb{E} \left[ |S^c_{\text{good}}| \right]}{T} \leq \frac{128 \sigma_1^2 (n_{\text{comp}} + 1) \log(f(T))}{T},
\]

where we used the fact that, by Lemma 11, \( |\tilde{S}^c| = |S^c_{\text{good}} \cup S^{\text{comp}}| = (n_{\text{comp}} + 1)|S^c_{\text{good}}| \), along with Lemma 26. The above bound combined with (42) thus gives

\[
\mathbb{E} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} \right] \leq \left( \frac{C_T}{\sqrt{T}} \right)^{\frac{2}{3}} + \frac{128 \sigma_1^2 (n_{\text{comp}} + 1) \log(f(T))}{T} \leq \left( \frac{\tilde{C}_T}{\sqrt{T}} \right)^{\frac{2}{3}},
\]

where

\[
\tilde{C}_T := \left( C_T^2 + 128 \sigma_1^2 (n_{\text{comp}} + 1) \log(f(T)) \right)^{\frac{2}{3}} \leq \sqrt{2} \left( C_T + \left( 128 \sigma_1^2 (n_{\text{comp}} + 1) \log(f(T)) \right) \right)^{\frac{3}{2}}.
\]

Hence, by a final application of Markov’s inequality, we obtain, for any \( \delta \in (0, 1) \),

\[
\mathbb{P} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} > \frac{\tilde{C}_T}{\sqrt{3^3 T}} \right] = \mathbb{P} \left[ \min_{t \in [T]} \| \nabla F(w_t) \|^{4/3} > \frac{1}{\delta} \left( \frac{\tilde{C}_T}{\sqrt{T}} \right)^{\frac{2}{3}} \right] \leq \delta.
\]

as claimed. \( \square \)