New construction of the brane coproduct and vanishing of cup products on sphere spaces

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Abstract

Using the loop coproduct, Menichi proved that the cup product with the orientation class vanishes for a closed connected oriented manifold with non-trivial Euler characteristic. We generalize this to the sphere spaces, i.e. the mapping spaces from spheres, using two generalizations of the loop coproduct to sphere spaces. One is constructed in this paper and the other in a previous paper of the author.

1 Introduction

In this article, we give a new construction of the brane coproduct, which we call the non-symmetric brane coproduct. Comparing this coproduct with another coproduct constructed in [Wak], we prove the vanishing of some cup products on the cohomology of mapping spaces from spheres.

Chas and Sullivan [CS99] introduced the loop product on the homology $H_*(LM)$ of the free loop space $LM = \text{Map}(S^1, M)$ of a manifold $M$ of dimension $m$. Cohen and Godin [CG04] extended this product to other string operations, including the loop coproduct, whose dual has the form

$$\delta^\vee : H^*(LM) \to H^{*+m}(LM \times LM).$$

Although the loop coproduct is almost trivial by [Tam10], Menichi [Men13] used the loop coproduct to obtain the following vanishing result.

**Theorem 1.1** ([Men13, Theorem 1]). Let $M$ be a connected, closed oriented manifold of dimension $m$, $\omega \in H^m(M)$ its orientation class, and $\chi(M)$ its Euler characteristic. Then, for any $\alpha \in H^{2\omega}(LM)$, we have

$$\chi(M)e_{0}^\omega \omega \cdot \alpha = 0 \in H^{[\alpha]+m}(LM),$$

where $e_0 : LM \to M$ is the evaluation map at the base point $0 \in S^1$.

Moreover, Félix and Thomas [FT09] generalized the loop coproduct to Gorenstein spaces. A Gorenstein space is a generalization of a Poincaré duality space (i.e. a space satisfying Poincaré duality) in an algebraic way. See Definition 3.2 for the definition.
Using the algebraic method due to Félix and Thomas, the author constructed a generalization of the loop coproduct, called the brane coproduct. Here we explain it along with a little generalization. Let $\mathbb{K}$ be a field, $k$ a positive integer and $M$ a $k$-connected space with $H^*(M) = H^*(M; \mathbb{K})$ of finite type. Denote by $S^k M = \text{Map}(S^k, M)$ the mapping space from the $k$-dimensional sphere to $M$. We fix an arbitrary element $\gamma \in \text{Ext}^i_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$, where $C^*(M)$ is the singular cochain algebra on $M$. Then we can construct (the dual of) the brane coproduct

$$\delta^\gamma_\vee : H^*(S^k M \times S^k M) \to H^{*+l}(S^k M).$$

Note that $\gamma$ will be specified under some assumption on $M$, and that we can choose $l$ and $\gamma$ depending on the purpose. See Section 3 for details.

Next we explain the non-symmetric brane coproduct, which will be defined in this article. Assume $M$ is a Poincaré duality space (i.e. a space satisfying Poincaré duality) over $\mathbb{K}$ of dimension $m$. Then we can define the non-symmetric brane coproduct

$$\delta^\gamma_{\text{ns}} : H^*(S^k M \times S^k M) \to H^{*+m}(S^k M).$$

Note that the non-symmetric brane coproduct can be defined for any 1-connected Poincaré duality space, without the assumption of $k$-connectivity. See Section 4 for details.

The non-symmetric brane coproduct $\delta^\gamma_{\text{ns}}$ seems to be non-commutative, from the explicit formula in Theorem 5.1. On the other hand, the brane coproduct $\delta^\gamma_\vee$ is commutative in the sense of Proposition 6.4. In spite of such difference, these coproducts coincide with each other under some assumptions. This coincidence gives some non-trivial relations on $H^*(S^k M)$, which is the main theorem of this article:

**Theorem 1.2.** Let $k$ be a positive integer, $M$ a $k$-connected Poincaré duality space over $\mathbb{K}$ of dimension $m$, and $\omega \in H^m(M)$ its orientation class. Assume

1. $k = 1$ or
2. $k \geq 1$ is odd, the characteristic of $\mathbb{K}$ is zero, and $\text{dim}_{\mathbb{K}} (\bigoplus_n \pi_n(M) \otimes \mathbb{K}) < \infty$.

Then, for any $\alpha \in H^{>0}(S^k M)$, we have

$$\chi(M) \text{ev}_0^* \omega \cdot \alpha = 0 \in H^{[\alpha]+m}(S^k M).$$

**Remark 1.3.** This theorem generalizes Theorem 1.1 due to Menichi, since we do not assume that $M$ is a manifold and $k = 1$. See Remark 6.10 for the reason why we need the assumption $k$ is odd.

We prove the above theorem using the following general result.
Theorem 1.4. Let $M$ be a $k$-connected Poincaré duality space over $\mathbb{K}$ of dimension $m$, $\omega \in H^m(M)$ its orientation class. We fix an arbitrary element $\gamma \in \text{Ext}^m_{C^*}(S^{k-1} M)(C^* (M), C^* (S^{k-1} M))$.

Define $\lambda_\gamma \in \mathbb{K}$ by the equation $c^* \circ (H^* (\gamma))(1) = \lambda_\gamma \omega \in H^m(M)$, where $c: M \to S^{k-1} M$ is the embedding as constant maps. See Section 2 for the definition of the map $H^* (\gamma): H^* (M) \to H^* (S^{k-1} M)$. Then, for any $\alpha \in H^{>0} (S^{k} M)$, we have

$$\lambda_\gamma \mathbf{ev}_0^* \omega \cdot \alpha = 0 \in H^{|\alpha|+m} (S^{k} M).$$

We conjecture that, for any $M$ and $k$ as in Theorem 1.4, there is an element $\gamma$ satisfying $\lambda_\gamma = \chi(M)$. The assumptions [1] and [2] give sufficient conditions for the existence of such $\gamma$.

Throughout this article, $\mathbb{K}$ denotes a field. The characteristic $\text{ch} \mathbb{K}$ of the field $\mathbb{K}$ is zero in Subsection 6.3 and Section 7. In other (sub)sections, $\text{ch} \mathbb{K}$ can be zero or any prime. For a vector space $V$ over $\mathbb{K}$, we denote the dual of $V$ by $V^\vee$. For spaces $X$ and $Y$, we denote the mapping space from $X$ to $Y$ by $Y^X$. For $x \in X$, let $\mathbf{ev}_x: Y^X \to Y$ be the evaluation map at $x$. Denote by $[X, Y]$ the homotopy set of maps from $X$ to $Y$. Base points does not matter since we consider it only when $X$ is 0-connected and $Y$ is 1-connected.

This article is organized as follows. Section 2 contains basic definitions and properties of $\text{Ext}$, which we use in definitions of the brane coproducts. In Section 3 we review the previous construction of the brane coproduct. We define the non-symmetric brane coproduct in Section 4, and, under some assumptions, explicitly compute it in Section 5. In Section 6, we compare two brane coproducts and prove Theorem 1.2 and Theorem 1.4 using explicit construction of shriek maps given in Section 7.

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2 Definition and properties of Ext

Let $A$ be a differential graded algebra (dga), and $M$ and $N$ $A$-modules over a field $\mathbb{K}$ of any characteristic. Then the extension module is defined as $\text{Ext}_A(M, N) = H^*(\text{Hom}_A(P, A))$, where $P$ is a semifree resolution of $M$ over $A$. See [FHT01, Section 6] for details of semifree resolutions. For an element $\alpha \in \text{Ext}_A(M, N)$, we define $H^*(\alpha)$:

$$H^*(\alpha) : H^*(M) \xrightarrow{\sim} H^*(P) \xrightarrow{H^*(\alpha)} H^*(N).$$

This defines a linear map $\text{Ext}_A(M, N) \rightarrow \text{Hom}_{H^*(A)}((H^*(M), H^*(N))$; $\alpha \mapsto H^*(\alpha)$.

Consider a pullback diagram

$\begin{array}{ccc}
D & \longrightarrow & E \\
\downarrow & & \downarrow^p \\
A & \longrightarrow & B
\end{array}$

such that $p: E \rightarrow B$ is a fibration and $B$ is 1-connected. Let us recall the linear map

$$p^*: \text{Ext}^l_{C^*(B)}(C^*(A), C^*(B)) \rightarrow \text{Ext}^l_{C^*(E)}(C^*(D), C^*(E))$$

introduced in [FT09, Remark after Theorem 2]. Let $P$ be a semifree resolution of $C^*(A)$ over $C^*(B)$. Then we have a linear map

$$\text{Hom}_{C^*(B)}(P, C^*(B)) \rightarrow \text{Hom}_{C^*(E)}(C^*(E) \otimes_{C^*(B)} P, C^*(E) \otimes_{C^*(B)} C^*(B))$$

by sending $\varphi \in \text{Hom}_{C^*(B)}(P, C^*(B))$ to $\text{id}_{C^*(E)} \otimes \varphi$. Here, $C^*(E) \otimes_{C^*(B)} P$ is a semifree $C^*(E)$ module by [FHT01, Lemma 6.2]. Moreover, the Eilenberg-Moore map $C^*(E) \otimes_{C^*(B)} P \rightarrow C^*(D)$ is a quasi-isomorphism by the Eilenberg-Moore theorem [Smi67, Theorem 3.2]. Hence $C^*(E) \otimes_{C^*(B)} P$ is a semifree resolution of $C^*(D)$ over $C^*(E)$, and the linear map (2.1) induces the required map $p^*$.

The above constructions satisfy naturality in the following sense, which can be proved directly from the definitions.
Proposition 2.2. Consider a diagram

and elements $\alpha \in \text{Ext}^n_{C^{\ast}}(B^\ast(A), C^\ast(B))$ and $\alpha' \in \text{Ext}^n_{C^{\ast}B'}(C^\ast(A'), C^\ast(B'))$. Here $p$ and $p'$ are fibrations and the front and back squares are pullback diagrams. Assume that the elements $\alpha$ and $\alpha'$ are mapped to the same element in $\text{Ext}^n_{C^{\ast}}(B^\ast(A), C^\ast(B'))$ by the morphisms induced by $a$ and $b$, and that the Eilenberg-Moore maps of two pullback diagrams are isomorphisms. Then the following diagram commutes.

3 Review of the previous construction of the brane coproduct

In this section, we review the previous construction of the brane coproduct from [Wak]. Here we explain it in a generalized way, which is necessary for the comparison in Section 6.

First we give a general construction. Let $K$ be a field of any characteristic, $k$ a positive integer, $S$ and $T$ $k$-dimensional manifolds, and $M$ a $k$-connected space. We fix an arbitrary element

$$
\gamma \in \text{Ext}^n_{C^{\ast}(S^{k-1}M)}(C^\ast(M), C^\ast(S^{k-1}M)).
$$

To define the brane coproduct, consider the diagram

$$
\begin{array}{c}
M^S \times T & \overset{\text{comp}}{\leftarrow} & M^S \times M^T & \overset{\text{incl}}{\rightarrow} & M^S \times M^T \\
\downarrow^{\text{res}} & & \downarrow & & \downarrow \\
S^{k-1}M & \overset{c}{\leftarrow} & M,
\end{array}
$$

(3.1)
where the square is a pullback diagram, the map res is the restriction map to the embedded sphere $S^{k-1}$ which comes from the connected sum $S \# T$, and the map $c$ is an embedding as constant maps.

Then the dual
\[
\delta^\vee \gamma : H^* (M^S \times M^T) \to H^{*+n} (M^S \# T)
\]
of the brane coproduct with respect to $\gamma$ is defined as the composition
\[
\text{comp} \circ \text{incl} : H^* (M^S \times M^T) \xrightarrow{\text{incl}^*} H^* (M^S \times_\pi M^T) \xrightarrow{\text{comp}} H^{*+n} (M^S \# T).
\]
Here the shriek map $\text{comp}$ is defined by $\text{comp} = H^* (\text{res} ^* (\gamma))$.

Next we specify the element $\gamma$ under some assumptions, which was considered in [Wak]. Here we use the notion of a Gorenstein space.

**Definition 3.2 ([FHT88])**. Let $m \in \mathbb{Z}$ be an integer. A path-connected topological space $M$ is called a $(\mathbb{K})$-Gorenstein space of dimension $m$ if
\[
\dim \text{Ext}^l_{C^*(M)} (\mathbb{K}, C^*(M)) = \begin{cases} 1, & \text{if } l = m \\ 0, & \text{otherwise}. \end{cases}
\]

For example, a Poincaré duality space over $\mathbb{K}$ is a $\mathbb{K}$-Gorenstein space, and its dimension as a Gorenstein space coincides with the one as a Poincaré duality space. Moreover, the following proposition gives an important example of a Gorenstein space.

**Proposition 3.3 ([FHT88 Proposition 3.4])**. A 1-connected topological space $M$ is a $\mathbb{K}$-Gorenstein space if $\mathbb{K}$ is a field of characteristic zero and $\pi_*(M) \otimes \mathbb{K}$ is finite dimensional.

Now we can specify the element $\gamma$ by the following theorem.

**Theorem 3.4 ([Wak Corollary 3.2])**. Assume $\mathbb{K}$ is a field of characteristic zero. Let $M$ be a $(k - 1)$-connected (and 1-connected) space of finite type such that $\Omega^{k-1} M$ is a Gorenstein space of dimension $\bar{m}$. Then we have an isomorphism
\[
\text{Ext}^l_{C^*(S^{k-1}M)} (C^*(M), C^*(S^{k-1}M)) \cong H^l \cong \mathbb{K}^{\bar{m}} (M)
\]
for any $l \in \mathbb{Z}$.

When $l = \bar{m}$, we have the generator
\[
c_1 \in \text{Ext}^\bar{m}_{C^*(S^{k-1}M)} (C^*(M), C^*(S^{k-1}M)) \cong H^0 (M) \cong \mathbb{K}
\]
up to non-zero scalar multiplication. The brane coproduct $\delta^\vee \gamma$ for the case $\gamma = c_1$ is the brane coproduct constructed in [Wak].
4 New construction of the brane coproduct

In this section, we give a new construction of the brane coproduct, which we call the non-symmetric brane coproduct. This is different from the previous one and we will compare them in Section 6.

Let $K$ be a field of any characteristic, $k$ a positive integer, $T$ a $k$-dimensional manifold with a base point $t_0$, and $M$ a 1-connected Poincaré duality space of dimension $m$. We fix base points $d_0 \in D^k$ and $s_0 \in S^k$ such that $d_0$ is mapped to $s_0$ by the quotient map $D^k \to S^k$. For an element $g \in M^T$, we denote by $M^T_g$ the component of $M^T$ containing $g$.

For $f \in S^k M$ and $g \in M^T$, we define a map $f + g \in M^T$ as follows. Fix an embedded $k$-disk around $t_0$ in $T$. Then we have the quotient map $q : T \to S^k \vee T$, which is given by pinching the boundary of the embedded disk. Since $M$ is path-connected, there is a map $f' \in S^k M$ such that $f'(s_0) = g(t_0)$ and $f'$ is homotopic to $f$ (without preserving base points). Define $f + g$ to be the composition $T \xrightarrow{q} S^k \vee T \xrightarrow{f' + g} M$. Since $M$ is 1-connected, the map $f + g$ is well-defined up to homotopy.

Instead of (3.1), we consider the diagram

\[
\begin{array}{ccc}
M^T_{f+g} & \xleftarrow{\text{comp}} & S^k_f M \times_M M^T_g \\
\downarrow \text{res} & & \downarrow \text{pr}_1 \\
D^k M & \xleftarrow{\iota} & S^k_f M,
\end{array}
\]

(4.1)

where the square is a pullback diagram, the map res is the restriction to the embedded $k$-disk, and the map $\iota$ is the inclusion induced by the quotient map $D^k \to S^k$.

Note that the above diagram is related to the diagram (3.1) in the following way. When $M$ is $k$-connected, we have the diagram

\[
\begin{array}{ccc}
M^T & \xleftarrow{\text{comp}} & S^k M \times_M M^T \\
\downarrow \text{res} & & \downarrow \text{pr}_1 \\
D^k M & \xleftarrow{\iota} & S^k M
\end{array}
\]

(4.2)

where the two squares are pullback diagrams (and hence so is the outer square). In this diagram, the upper square coincides with (4.1) and the outer square coincides with (3.1). We use this diagram to compare the two brane coproducts in Section 6.

We define the dual

\[
\delta^\vee_{ns} : H^*(S^k_f M \times_M M^T_g) \to H^{*+m}(M^T_{f+g})
\]
of the brane coproduct by the composition
\[ \text{comp}_t \circ \text{incl}^*: H^*(M_{f+g}^T) \xrightarrow{\text{incl}^*} H^*(S_f^k M \times_M M_g^T) \xrightarrow{\text{comp}_t} H^{*-m}(S_f^k M \times M_g^T). \]
Here, \( \text{comp}_t \) is the shriek map constructed from the diagram (4.1). In order to define it, we need the corollary of the following proposition.

**Proposition 4.3** ([FT09, Lemma 1]). Let \( F: X \to N \) be a map between 0-connected spaces. Assume that \( N \) is a Poincaré duality space of dimension \( n \). Define a linear map

\[ \Phi: \text{Ext}^{l}_{C^*(N)}(C^*(X), C^*(N)) \to \text{Hom}_K(H^{n-l}(X), H^n(N)) \]

by \( \Phi(\alpha) = H^*(\alpha)|_{H^{n-l}(X)} \). Then \( \Phi \) is an isomorphism.

Then we have the following corollary, which is an analogue of Theorem 3.4 for the case of the non-symmetric brane coproduct.

**Corollary 4.4.** Consider the same assumption with Proposition 4.3. Additionally assume \( l = n \) and \( j = 0 \). Then we have an isomorphism

\[ \text{Ext}^{n}_{C^*(N)}(C^*(X), C^*(N)) \xrightarrow{\cong} H^n(N); \quad \alpha \mapsto H^*(\alpha)(1). \]

Applying Corollary 4.4 to the case \( F = \iota \) and \( n = m \), we have the generator

\[ \iota \in \text{Ext}^{m}_{C^*(M)}(C^*(S_f^k M), C^*(D^k M)) \cong H^m(D^k M) \cong K \]

up to non-zero scalar multiplication. Using this element with the diagram (4.1), we define \( \text{comp}_t = H^*(\text{res}^* (\iota)) \). This completes the definition of the non-symmetric brane coproduct.

Next we give more convenient description of \( \text{comp}_t \). Consider the commutative diagram

\[
\begin{array}{cccccc}
M_{f+g}^T & \xrightarrow{\text{comp}} & S_f^k M \times_M M_g^T & \xrightarrow{\text{incl}} & S_f^k M \times M_g^T \\
pr_2 \downarrow & & \approx & & \\
M_{f+g}^T & \leftarrow & S_f^k M \times_M M_{f+g}^T & \xrightarrow{\rho} & S_f^k M
\end{array}
\]

\[
\begin{array}{cccccc}
ev_{t_0} \downarrow & & \downarrow \text{res} & & \downarrow \text{pr}_1 & & \downarrow \ev_{t_0} \\
D^k M & \xrightarrow{\iota} & S_f^k M \times_M M_{f+g}^T & \leftarrow & S_f^k M
\end{array}
\]

where the front and back square are pullback squares. Here \( \rho: S_f^k M \times_M M_{f+g}^T \xrightarrow{\cong} S_f^k M \times_M M_{f+g}^T \) is defined by \( \rho(\varphi, \psi) = (\varphi, \varphi + \psi) \), which is well-defined since
we are working on the fiber product over $M$. By Proposition 2.2, we have
\[ H^*(\text{res}^*(\iota_1)) \circ \rho^* = H^*(\text{ev}_{t_0}^*(\tilde{\iota})) \] and hence
\[ \text{comp}_i = H^*(\text{ev}_{t_0}^*(\tilde{\iota})) \circ (\rho^*)^{-1}. \] (4.5)

Here $\tilde{\iota} \in \text{Ext}^{m}_{C^*(M)}(C^*(S^k_0 M), C^*(M))$ is the image of $\iota_1$ under the isomorphism induced by $\text{ev}_{t_0}$.

5 Computation of the non-symmetric brane coproduct

In this section, we use the same notation and assumptions as in Section 4. Let $0 \in S^k M$ be the constant map and denote the orientation class of $M$ by $\omega \in H^m(M)$. This section is devoted to the proof of the following formula of the non-symmetric brane coproduct.

**Theorem 5.1.** For the case $f = 0 \in [S^k, M]$, the non-symmetric coproduct
\[ \delta_{\text{ns}}^\vee: H^*(S^k_0 M \times M^T_g) \to H^{*+m}(M^T_g) \]
is described by
\[ \delta_{\text{ns}}^\vee(u \times v) = \text{ev}_{t_0}^*(\omega \cdot c^*(u)) \cdot v, \]
where $u \times v$ denotes the cross product of $u \in H^*(S^k_0 M)$ and $v \in H^*(M^T_g)$, and $c: M \to S^k_0 M$ is the embedding as constant maps.

This is an analogue of [Men13, Theorem 30] in the case of the non-symmetric coproduct. Note that, when $\text{ch} K = 0$, the above formula can be proved easily by using rational models of mapping spaces given in [Ber15].

To prove Theorem 5.1, we need some propositions. First we investigate the map $(\rho^*)^{-1}$ in (1.5). Define $\sigma: M^T_g \to S^k_0 M \times M M^T_g$ by $\sigma(\psi) = (c(\psi(t_0)), \psi)$.

**Proposition 5.2.** For any $x \in H^*(S^k_0 M \times M^T_g)$, we have
\[ (\rho^*)^{-1} x = x \in \text{Ker}(\sigma^*). \]

**Proof.** Let $\tilde{\rho}$ be the homotopy inverse of $\rho$. Then we have $\rho \circ \sigma \simeq \sigma$ and hence $\sigma^*((\rho^*)^{-1} x - x) = \sigma^*(\tilde{\rho}^* x - x) = \sigma^* x - \sigma^* x = 0$. \qed

Next we relate $\text{Ker}(\sigma^*)$ with $H^*(\text{ev}_{t_0}^*(\tilde{\iota}))$.

**Proposition 5.3.** Consider a pullback diagram
\[
\begin{array}{ccc}
E & \xleftarrow{g} & X \\
\downarrow{p} & & \downarrow{q} \\
B & \xleftarrow{f} & A
\end{array}
\]
such that the Eilenberg-Moore map is an isomorphism, and take an element \( \alpha \in \text{Ext}^{C \ast}(B)(C \ast(A), C \ast(B)) \). Let \( \sigma: E \to X \) and \( \tau: B \to A \) be sections of \( g \) and \( f \), respectively, satisfying \( q \circ \sigma = \tau \circ p \). Assume that there is an element \( \tilde{\alpha} \in \text{Ext}^{C \ast}(B)(C \ast(B), C \ast(B)) \) which is mapped to \( \alpha \) by the map induced by \( \tau \).

Then

\[
\text{Ker}(\sigma^\ast) \subset \text{Ker}(H^\ast(p^\ast \alpha)).
\]

Proof. Applying Proposition 2.2 to the following diagram, we have \( H^\ast(p^\ast \alpha) = H^\ast(p^\ast \tilde{\alpha}) \circ \sigma^\ast \), and this proves the proposition.

Next, we consider the diagram

\[
\begin{array}{ccc}
M^T_g & \xleftarrow{pr_2} & S^k_0 M \times_M M^T_g \\
\downarrow{ev_{t_0}} & & \downarrow{pr_1} \\
M & \xleftarrow{c} & S^k_0 M.
\end{array}
\]

Note that the maps \( \sigma: M^T_g \to S^k_0 M \times_M M^T_g \) and \( c: M \to S^k_0 M \), are sections of \( pr_2 \) and \( ev_{t_0} \), respectively. Recall from (4.5) that we are using \( \tilde{\iota} \) to compute the non-symmetric brane coproduct.

**Corollary 5.4.** Under the above notation, we have

\[
\text{Ker}(\sigma^\ast) \subset \text{Ker}(H^\ast(ev_{t_0}^\ast(\tilde{\iota}))).
\]

Proof. By Corollary 4.4 the map \( c \) induces an isomorphism

\[
\text{Ext}^{m}_{C^\ast(M)}(C^\ast(M), C^\ast(M)) \xrightarrow{\cong} \text{Ext}^{m}_{C^\ast(M)}(C^\ast(S^k_0 M), C^\ast(M)).
\]

Thus we obtain \( \tilde{\alpha} \) as in the assumption of Proposition 5.3 and hence it proves the corollary.

By Proposition 5.2 and Corollary 5.4 Theorem 5.1 reduces to the following simple proposition.
Proposition 5.5. Consider a pullback diagram

\[
\begin{array}{c}
E \\
\downarrow p \\
B \\
\downarrow f \\
A
\end{array}
\begin{array}{c}
\rightarrow X \\
\downarrow q \\
\rightarrow A
\end{array}
\]

such that the Eilenberg-Moore map is an isomorphism, and an element \( \alpha \in \text{Ext}_{C^*(B)}(C^*(A), C^*(B)) \). Then the composition \( H^*(A \times E) \xrightarrow{incl^*} H^*(X) \xrightarrow{H^*(p^* \alpha)} H^*(E) \) satisfies

\[
H^*(p^* \alpha) \circ \text{incl}^*(u \times v) = p^*(H^*(\alpha)(u)) \cdot v
\]

for any \( u \in H^*(A) \) and \( v \in H^*(E) \).

Proof. Consider the diagram

\[
\begin{array}{c}
B \times E \\
\downarrow pr_1 \\
B \\
\downarrow f \\
A
\end{array}
\begin{array}{c}
\leftarrow E \\
\downarrow p \\
A \\
\downarrow pr_1 \\
B
\end{array}
\begin{array}{c}
\leftrightarrow X \\
\downarrow q \\
A
\end{array}
\begin{array}{c}
\leftarrow g \\
\downarrow f \times id \\
A \times E
\end{array}
\begin{array}{c}
\rightarrow g \\
\downarrow f \times id \\
A \times E
\end{array}
\begin{array}{c}
\leftarrow (p, id) \\
\downarrow f \times id \\
B \times E
\end{array}
\begin{array}{c}
\rightarrow (q, g) \\
\downarrow f \times id \\
B \times E
\end{array}
\]

By Proposition 2.2, we have

\[
H^*(p^* \alpha) \circ \text{incl}^* = (p, id)^* \circ (H^*(pr_1^* \alpha)).
\]

Since the fibration \( pr_1 \) is very simple, we can prove

\[
H^*(pr_1^* \alpha)(u \times v) = H^*(\alpha)(u) \times v
\]

by a direct computation from the definition.

Now we give a proof of Theorem 5.1 using the above corollary and propositions.

Proof of Theorem 5.1. By 4.3, we have

\[
\delta_{ns}^\vee(u \times v) = H^*(ev_{t_0}^*(\tilde{\iota})) \circ (\rho^*)^{-1} \circ \text{incl}^*(u \times v).
\]

By Proposition 5.2 and Corollary 5.4 we have

\[
H^*(ev_{t_0}^*(\tilde{\iota})) \circ (\rho^*)^{-1} = H^*(ev_{t_0}^*(\tilde{\iota})).
\]

Thus

\[
\delta_{ns}^\vee(u \times v) = H^*(ev_{t_0}^*(\tilde{\iota})) \circ \text{incl}^*(u \times v),
\]

and hence Proposition 5.5 proves the theorem.
6 Comparison of two brane coproducts

In this section, we compare the two brane coproducts. As an application, we prove Theorem 1.4.

6.1 Proof of Theorem 1.4

In this subsection, we prove Theorem 1.2.

Let $\mathbb{K}$ be a field of any characteristic, $k$ a positive integer, and $M$ a $k$-connected Poincaré duality space of dimension $m$. We fix an arbitrary element

$$\gamma \in \text{Ext}^m_{C^\ast(S^kM)}(C^\ast(M), C^\ast(S^{k-1}M)).$$

Then we have the brane coproduct

$$\delta^\gamma_m: H^\ast(S^kM \times S^kM) \to H^\ast(S^kM)$$

for the case $S = T = S^k$ by the construction given in Section 3.

Remark 6.1. The degree $m$ of the element $\gamma$ is different from the degree $\bar{m}$ of $c_1$ in Theorem 3.4. These degrees coincide under the assumption (2) of Theorem 1.2 (see Remark 6.10). This case will be treated in Subsection 6.3 and Section 7.

To compare $\delta^\gamma_m$ with $\delta^\nu_m$, we relate $\gamma$ with $\nu$. As in Theorem 1.4, define $\lambda_\gamma \in \mathbb{K}$ by the equation

$$c^\ast \circ (H^\ast(\gamma))(1) = \lambda_\gamma \omega \in H^m(M),$$

where $\omega$ is the orientation class of $M$.

Proposition 6.2. Under the above notation, we have

$$\text{res}^\ast(\gamma) = \lambda_\gamma \nu_1 \in \text{Ext}^m_{C^\ast(D^kM)}(C^\ast(S^kM), C^\ast(D^kM)),$$

where $\text{res}^\ast$ is the lift along the lower pullback square in 4.2. Moreover, this implies

$$\delta^\nu_m = \lambda_\gamma \delta^\gamma_m: H^\ast(S^kM \times S^kM) \to H^\ast(S^kM).$$

Proof. Let $\omega \in H^m(M) \cong H^m(D^kM)$ be the orientation class. Recall from Corollary 4.4 that $\nu_1$ is characterized by $H^\ast(\nu_1)(1) = \omega$. Hence it is enough to prove $H^\ast(\text{res}^\ast(\gamma))(1) = \lambda_\gamma \omega$.

Let $\eta: P \tilde{\to} C^\ast(M)$ be a semifree resolution of $C^\ast(M)$ over $C^\ast(S^{k-1}M)$, and $u \in P$ a cocycle such that $\eta(u) = 1$. Take a representative $\varphi \in \text{Hom}_{C^\ast(D^kM)}(P, C^\ast(S^{k-1}M))$ of $\gamma$. Then we have $[\varphi(u)] = H^\ast(\gamma)(1) \in H^m(S^{k-1}M)$. By definition, $H^\ast(\text{res}^\ast(\gamma))$ is represented by the chain map $\text{id}_{C^\ast(D^kM)} \otimes \varphi$ in

$$\text{Hom}_{C^\ast(D^kM)}(C^\ast(D^kM) \otimes_{C^\ast(S^{k-1}M)} P, C^\ast(D^kM) \otimes_{C^\ast(S^{k-1}M)} C^\ast(S^{k-1}M)).$$

Hence we have $H^\ast(\text{res}^\ast(\gamma))(1) = [\text{id}_{C^\ast(D^kM)} \otimes \varphi](1 \otimes u) = c^\ast[\varphi(u)] = \lambda_\gamma \omega \in H^m(M)$ under the identification $H^m(M) = H^m(D^kM)$. This proves the proposition.
Next we consider the commutativity of the coproduct $\delta^\vee$. Let $\tau: S^{k-1}M \to S^{k-1}M$ be the map induced from the orientation reversing map on $S^{k-1}$, satisfying $\tau^2 = \text{id}$. Then $\tau$ induces the map

$$\tau^* = \text{Ext}_{\tau^*}(\text{id}, \tau^*): \text{Ext}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \to \text{Ext}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)).$$

By the definition of $\lambda_\gamma$, we have

$$\lambda_\gamma = \lambda_{\tau^*\gamma}. \quad (6.3)$$

The coproduct is commutative in the following sense.

**Proposition 6.4.**

$$\delta^\vee_\gamma (\alpha \times \beta) = (-1)^{|\alpha||\beta|} \delta^\vee_{\tau^*\gamma}(\beta \times \alpha)$$

The proposition is proved by the same method with the commutativity of the brane coproduct $\delta^\vee_c$ [Wak, Theorem 1.5]. Note that we used the equation $\tau^*c = (-1)^{\bar{m}}c$ [Wak, Equation (7.11)] to prove $\delta^\vee_c (\alpha \times \beta) = (-1)^{|\alpha||\beta|+\bar{m}} \delta^\vee_c (\beta \times \alpha)$.

**Proof of Theorem 1.4.** Since the fibration $ev_0: S^kM \to M$ has a section $c: M \to S^kM$, we have a decomposition $H^{>0}(S^kM) \cong H^{>0}(M) \oplus \text{Ker}(c^*)$. When $\alpha \in H^{>0}(M)$, we have $\alpha \omega = 0 \in H^{|\alpha|+m}(M) = 0$. Hence we assume $\alpha \in \text{Ker}(c^*)$. Then, by Theorem 6.1 we have

$$\delta^\vee_{\text{ns}}(\alpha \times 1) = ev_0^*(\omega \cdot c^*(\alpha)) \cdot 1 = 0$$
$$\delta^\vee_{\text{ns}}(1 \times \alpha) = ev_0^*(\omega \cdot c^*(1)) \cdot \alpha = ev_0^*\omega \cdot \alpha.$$

Moreover, we have

$$\lambda_\gamma \delta^\vee_{\text{ns}}(\alpha \times 1) = \delta^\vee_c (\alpha \times 1) = \pm \delta^\vee_{\tau^*\gamma}(1 \times \alpha) = \pm \lambda_\gamma \delta^\vee_{\text{ns}}(1 \times \alpha)$$

by (6.3), Proposition 6.4 and Proposition 6.2. These equations prove the theorem. 

6.2 Proof of Theorem 1.2 (1)

In this subsection, we prove Theorem 1.2 under the assumption (1). As a preparation of the proof, we investigate the map $\Phi$ in Proposition 4.3.

As in Proposition 4.3, let $X$ be a 0-connected space, $N$ a Poincaré duality space of dimension $n$, and $F: X \to N$ a map. We denote the orientation class of $N$ by $\omega_N \in H^n(N)$ and the fundamental class by $[N] \in H_n(N)$. Then we have $\langle \omega_N, [N] \rangle_N = 1$, where $\langle -, - \rangle_N: H^*(N) \otimes H_*(N) \to \mathbb{K}$ denotes the pairing.
Proposition 6.5. Fix arbitrary elements \( x \in H_{n-l}(X) \) and \( \nu \in H^j(X) \). Let \( \beta_x: H^{n-l}(X) \to H^n(N) \) be the linear map defined by \( \beta_x(\varphi) = \langle \varphi, x \rangle_X \omega_N \) for \( \varphi \in H^{n-l}(X) \). Using the isomorphism \( \Phi \) in Proposition 4.3, we define

\[
\alpha_x = \Phi^{-1}(\beta_x) \in \text{Ext}_{C^*(N)}^{j}(C^*(X), C^*(N)).
\]

Then the element \( H^*(\alpha_x)(\nu) \in H^{l+j}(N) \) is the unique element which satisfies

\[
\langle \psi, H^*(\alpha_x)(\nu) \cap [N] \rangle_N = (-1)^{(n-l-j)} \langle F^*\psi \cdot \nu, x \rangle_X
\]

for any \( \psi \in H^{n-l-j}(N) \).

Proof. Since the cap product \(- \cap [N]\) is an isomorphism by the Poincaré duality, such element is uniquely determined. Since \( H^*(\alpha_x) \) is \( H^*(N) \)-linear, we have \( \psi \cdot H^*(\alpha_x)(\nu) = (-1)^{(n-l-j)} H^*(\alpha_x)(F^*\psi \cdot \nu) \). Using this equation, we can prove (6.6) by a straightforward calculation.

Now we begin the proof of Theorem 1.2(1). Let \( M \) be a 1-connected Poincaré duality space of dimension \( m \). Here we write \( LM = S^1 \times M \) as usual. Recall that

\[
\Delta_l \in \text{Ext}_C^{m}(M \times M)(C^*(M), C^*(M \times M)) \cong \mathbb{K}
\]

is the generator, which is defined up to non-zero scalar multiplication.

Proposition 6.7. The element \( H^*(\Delta_l)(1) \in H^m(M \times M) \) is the diagonal class, i.e. the Poincaré dual of the homology class \( \Delta_l[M] \in H^m(M \times M) \). In particular, we have

\[
\Delta^* \circ (H^*(\Delta_l))(1) = \chi(M)\omega \in H^*(M).
\]

Proof. Since \( M \times M \) is also a Poincaré duality space, we can apply Proposition 6.5 for the case \( F = \Delta, n = 2m, l = m, j = 0, x = [M], \) and \( \nu = 1 \). Since \( \Delta_l \) is defined up to non-zero scalar multiplication, we may assume \( \Delta_l = (-1)^m \alpha_{[M]} \).

By (6.6), we have

\[
\langle \psi, H^*(\Delta_l)(1) \cap [M^2] \rangle_{M^2} = \langle \Delta^*\psi \cdot 1, [M] \rangle_M = \langle \psi, \Delta^*[M] \rangle_{M^2}
\]

for any \( \psi \in H^m(M^2) \), and hence \( H^*(\Delta_l)(1) \cap [M^2] = \Delta^*[M] \).

It is well-known that the diagonal class satisfies the required property (c.f. e.g. [MS74], pp. 127–129, Section 11).

Now we have the following theorem using the above lemma.

Theorem 6.8 (Theorem 1.2(1)). Let \( M \) be a 1-connected Poincaré duality space over \( \mathbb{K} \) and denote its orientation class by \( \omega \in H^m(M) \). Then, for any \( \alpha \in H^{n-m}(LM) \), we have

\[
\chi(M)\text{ev}_0^\omega \cdot \alpha = 0 \in H^{n-m}(LM).
\]

Proof. Apply Theorem 1.4 and Proposition 6.7.

Remark 6.9. This theorem generalizes [Men13, Theorem 1] in the sense that our theorem can be applied to Poincaré duality spaces, not only manifolds.
6.3 Proof of Theorem 1.2(2)

In this section, we prove Theorem 1.2 under the assumption (2).

Let $k$ be a positive odd integer and $M$ a $k$-connected Poincaré duality space over $K$ of dimension $m$. Assume $\text{ch} K = 0$ and $\dim_{K} (\bigoplus_{n} \pi_{n}(M) \otimes K) < \infty$.

First we explain why we assume $k$ is odd in the assumption (2) in Theorem 1.2.

Remark 6.10. Let $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ be bases of $\bigoplus_{n} \pi_{2n}(M) \otimes K$ and $\bigoplus_{n} \pi_{2n-1}(M) \otimes K$, respectively. Then we have the following.

- $\chi(M) \neq 0$ if and only if $p = q$. See Theorem 7.13 for details.
- Define $a_i = |x_i|$ and $b_j = |y_j|$. By [FHT88, Proposition 5.2], we have $m = \dim M = \sum_j b_j + \sum_i (1 - a_i)$. By the same formula, we have $\bar{m} = \dim \Omega^{k-1}M = \begin{cases} m - (q - p)(k - 1), & \text{if } k \text{ is odd,} \\ -m - (k - 2)p + kq, & \text{if } k \text{ is even.} \end{cases}$

Thus, except for rare exceptions, $\bar{m}$ coincides with $m$ if and only if $k$ is odd and $p = q$.

Since the statement of Theorem 1.2 is trivial when $\chi(M) = 0$, we are interested only in the case $\chi(M) \neq 0$, i.e. $p = q$. Moreover, since we will compare two brane coproducts, their degrees $m$ and $\bar{m}$ must coincide. Hence we may assume $k$ is odd. This explains why the assumption (2) in Theorem 1.2 is natural.

Now we give a proposition, which is a key to prove Theorem 1.2.

**Proposition 6.11.** Under the assumption (2) in Theorem 1.2, there exists an element $\gamma \in \text{Ext}^{\bar{m}}_{C^{\ast}((S^{k-1}M), C^{\ast}(S^{k-1}M))}$ such that $c^{\ast} \circ (H^{\ast}(\gamma))(1) = \chi(M) \omega \in H^{\ast}(M)$.

We defer the proof of the proposition to Section 7. Applying the proposition and Theorem 1.4 we have (2) of Theorem 1.2.

**Theorem 6.12 (Theorem 1.2(2)).** Under the assumption (2) in Theorem 1.2, we have $\chi(M)ev_{0}^{\ast}\omega \cdot \alpha = 0 \in H^{|\alpha| + m}(S^{k}M)$ for any $\alpha \in H^{>0}(S^{k}M)$.

Hence the rest of this article is devoted to the proof of Proposition 6.11.

7 Models of shriek maps

In this section, we give a proof of Proposition 6.11. As a preparation of the proof, we explicitly construct a model of the shriek map $c_{i}$ when the coefficient
is a field $\mathbb{K}$ of characteristic zero. By \[\text{this},\] it is enough to construct a non-trivial element in $\text{Ext}^n_{\mathbb{K}}(S^{k-1}M,\mathbb{K})$. In Subsection \[\text{this},\] we construct a candidate of the shriek map, whose non-triviality is proved in Subsection \[\text{this},\] under some assumptions.

The construction is a generalization of the ones in \[\text{this},\] and \[\text{this},\], which treat only the case $k = 1$. Note that, in \[\text{this},\] Proposition 6.2, the shriek map is explicitly constructed when $k$ is even and the minimal Sullivan model is pure, which is much simpler than the one in this section.

Throughout this section, we assume $\text{ch}\mathbb{K} = 0$ and make full use of rational homotopy theory. See \[\text{this},\] for basic definitions and theorems.

For a graded vector space $V$, we define a graded vector space $s^kV$ by $(s^kV)^n = V^{n+k}$. For an element $v \in V$, we denote the corresponding element by $s^kv \in s^kV$. For simplicity, we write $sV = s^1V$.

Let $(\wedge V, d)$ be a Sullivan algebra satisfying $\dim V < \infty$ and $V^1 = 0$. We fix a basis $z_1, \ldots, z_r$ of $V$ such that $dz_{i+1} \in \wedge V(t)$, where $V(t) = \text{span}_{\mathbb{K}}\{z_1, \ldots, z_t\}$.

### 7.1 Construction of a chain map

In this subsection, we give an explicit construction of a candidate of the shriek map for $k \geq 1$. The construction is completely analogous to the one in \[\text{this},\].

In this subsection, we assume $V^{\leq k} = 0$ additionally. Write $S^{k-1} = \wedge V \otimes s^{k-1}V$ and $D^k = D^kV = \wedge V \otimes s^{k-1}V \otimes s^kV$. Here we define two Sullivan algebras $(S^{k-1}, d)$ and $(D^k, d)$, and two linear maps $\sigma : V \to S^{k-1}$ and $\tau : V \to D^k$. Note that $(S^{k-1}, d)$ and $(D^k, d)$ are models of $S^{k-1}M$ and $D^kM$, respectively.

Let $\tilde{s}^{k-1} : S^{k-1} \to S^{k-1}$ be the derivation defined by $\tilde{s}^{k-1}(v) = s^{k-1}v$ and $\tilde{s}^{k-1}(s^{k-1}v) = 0$. By an abuse of notation, we write $\tilde{s}^{k-1}$ simply by $s^{k-1}$. Similarly we define the derivation $s^k : D^k \to D^k$. Note that these derivations are not equal to the compositions of $s^1$ (e.g. $s^{k-1} \neq s^1 \circ \cdots \circ s^1$).

First we define the differentials $d$ on $S^{k-1}$ and $D^k$ in the case $k = 1$. Then $(S^0, d)$ is just the tensor product $(\wedge V, d) \otimes \mathbb{K}$. The dga $(D^1, d)$ is a relative Sullivan algebra over $(\wedge V, d) \otimes \mathbb{K}$, defined by the formula $d(sz) = 1 \otimes z - z \otimes 1 - \sum_{n=1}^{\infty} \frac{(sz)^n}{n!}(z \otimes 1)$ inductively on $t$ (see \[\text{this},\] Section 15 (c)) or \[\text{this},\] Appendix A for details). Then, for $v \in V$, we set $\sigma v = 1 \otimes v \otimes 1$ and $\tau v = -\sum_{n=1}^{\infty} \frac{(sv)^n}{n!}(v \otimes 1)$, which satisfy $dv = \sigma v + \tau v$.

Next we consider the case $k \geq 2$. Define the differential $d$ on $S^{k-1}$ by the formula $d^{k-1}v = (-1)^{k-1}s^{k-1}dv$. Set $\sigma v = s^{k-1}v$, $\tau v = (-1)^{k}s^{k}dv$. Then we define the relative Sullivan algebra $(D^k, d)$ over $(S^{k-1}, d)$ by the formula $d^k = \sigma v + \tau v$. See \[\text{this},\] Section 5) for details. By the following proposition, we can use $S^{k-1}$ and $D^k$ to construct the shriek map $\epsilon_1$.

#### Proposition 7.1 (\[\text{this},\] Proposition 5.1). Let $M$ be a $k$-connected space and $(\wedge V, d)$ be its Sullivan model. Then the above algebras $S^{k-1}$ and $D^k$ are Sullivan models of $S^{k-1}M$ and $D^kM$. In particular, we have

$$\text{Ext}^n_{C^*}(S^{k-1}M, C^*(S^{k-1}M)) \cong H^*(\text{Hom}_{S^{k-1}M}(D^k, S^{k-1}M))$$
Moreover, we define $S^{k-1}(t) = \wedge V(t) \otimes \wedge s^{k-1} V(t)$ and $D^k(t) = \wedge V(t) \otimes \wedge s^{k-1} V(t) \otimes s^k V(t)$. Then we have $\sigma : V(t) \rightarrow S^{k-1}(t)$ and $\tau : V(t) \rightarrow D^k(t-1)$.

Next we give a construction of shriek maps.

**Definition 7.2.** For $t = 0, \ldots, n - 1$ define a $\mathbb{K}$-linear map

$$\Phi : \text{Hom}_{S^{k-1}(t-1)} (D^k(t-1), S^{k-1}(t-1)) \rightarrow \text{Hom}_{S^{k-1}(t)} (D^k(t), S^{k-1}(t))$$

of odd degree as follows.

1. In the case $|z| + k - 1$ is odd, for $f \in \text{Hom}_{S^{k-1}(t-1)} (D^k(t-1), S^{k-1}(t-1))$, define

   $$\Phi(f) \in \text{Hom}_{S^{k-1}(t)} (D^k(t), S^{k-1}(t))$$

   by

   $$\Phi(f)(\nu) = \sigma z_t \cdot f(\nu) - (-1)^{|f|} f(\tau z_t \cdot \nu), \quad \Phi(f)(\nu \cdot (sz_t)^l) = 0$$

   for $\nu \in \wedge s V(t-1)$ and $l \geq 1$.

2. In the case $|z| + k - 1$ is even, for $f \in \text{Hom}_{S^{k-1}(t-1)} (D^k(t-1), S^{k-1}(t-1))$, define $\Phi(f)$ by

   $$\Phi(f)(\nu \cdot sz_t) = (-1)^{|f|+[\nu]} f(\nu), \quad \Phi(f)(\nu) = 0$$

   for $\nu \in \wedge s V(t-1)$.

By a straight-forward calculation, the linear map $\Phi$ is a chain map of odd degree. In other words, the map $\Phi$ satisfies $d\Phi = -\Phi d$.

Hence we define chain maps

$$\varphi_t \in \text{Hom}_{S^{k-1}(t)} (D^k(t), S^{k-1}(t))$$

by $\varphi_0 = \text{id}_\mathbb{K}$ and $\varphi_{t+1} = \Phi(\varphi_t)$, inductively.

### 7.2 The pure case with $k$ odd

Next we investigate the above map in the case $(\wedge V, d)$ is pure and $k$ is odd.

**Definition 7.3** ([FHT01 Section 3.2 (a)]). A Sullivan algebra $(\wedge V, d)$ is pure if $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subset \wedge V^{\text{even}}$.

Here we apply the above construction for the case the basis $z_1, \ldots, z_n$ is given by the sequence $x_1, \ldots, x_p, y_1, \ldots, y_q$, where $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ are (arbitrary) bases of $V^{\text{even}}$ and $V^{\text{odd}}$, respectively. That is, $z_i = x_i$ for $1 \leq i \leq p$ and $z_{p+j} = x_j$ for $1 \leq j \leq q$. In this case, we can write $\tau y_j = (-1)^k \sum \alpha_{ji} s^k x_i$ for some elements $\alpha_{ji} \in S^{k-1}$. Note that $\alpha_{ji} \in \wedge V^{\text{even}}$ when $k \geq 2$, and $\alpha_{ji} \in \wedge V^{\text{even}} \otimes \wedge V^{\text{even}}$ when $k = 1$.

Let $\mu : S^{k-1} \rightarrow \wedge V$ be the multiplication map when $k = 1$, and the map defined by $\mu(v) = v$ and $\mu(s^{k-1} v) = 0$ when $k \geq 2$. Then we have

$$\mu(\alpha_{ji}) = \frac{\partial(dy_j)}{\partial x_i} \in \wedge V^{\text{even}}. \quad (7.4)$$
Lemma 7.5. For any integer \(i\) with \(0 \leq i \leq p\) and any subset \(I \subset [i]\), we have

\[
\varphi_i(s^k x_{[p] \setminus I}) = \begin{cases} 
1, & \text{if } I = \emptyset, \\
0, & \text{if } I \neq \emptyset.
\end{cases}
\]

Moreover, we have the following formulas for \(\varphi_{p+j}\) for \(0 \leq j \leq q\).

Proposition 7.6. Let \(j\) be an integer with \(0 \leq j \leq q\) and \(I \subset [p]\) a subset. Write \(n = l(I)\) and \(I = \{i_1, \ldots, i_n\}\) with \(i_1 < \cdots < i_n\). Then the element \(\varphi_{p+j}(s^k x_{[p] \setminus I}) \in S^{k-1}(j)\) satisfies the following.

1. If \(n = 0\), then we have \(\varphi_{p+j}(s^k x_{[p]}) = \sigma_y[j]\).

2. If \(n < j\), then the element \(\varphi_{p+j}(s^k x_{[p] \setminus I})\) is contained in the ideal \((\sigma_y[k], \ldots, \sigma_y[n]) \subset S^{k-1}(j)\).

3. If \(n \geq j\), then we have

\[
\varphi_{p+j}(s^k x_{[p] \setminus I}) = \begin{cases} 
(-1)^{|I|+pj} \det((\alpha_{t,i})_{1 \leq t, i \leq j}), & \text{if } n = j, \\
0, & \text{if } n > j.
\end{cases}
\]

Proof. We prove the formulas by induction on \(j\). The case \(j = 0\) is already proved in Lemma 7.5. Assume that \(j \geq 1\) and we already have the formulas for \(\varphi_{p+j-1}\).

By Definition 7.5 we have

\[
\varphi_{p+j}(s^k x_{[p] \setminus I}) = \Phi(\varphi_{p+j-1})(s^k x_{[p] \setminus I})
= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p] \setminus I}) + (-1)^{p+j-1} \varphi_{p+j-1}(\tau y_j \cdot s^k x_{[p] \setminus I})
= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p] \setminus I}) + (-1)^{p+j-1} \varphi_{p+j-1} \left( \sum_{1 \leq i \leq p} \alpha_{j,i} \cdot s^k x_i \cdot s^k x_{[p] \setminus I} \right)
= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p] \setminus I}) + (-1)^{p+j} \sum_{1 \leq r \leq j} (-1)^{j-r} \alpha_{j,i,r} \cdot \varphi_{p+j-1}(s^k x_{[p] \setminus I})
\]

(7.7)

where \(I_r = I \setminus \{i_r\}\).

First we prove (1). Since \(|s^k x_i|\) is odd, we have \(\tau y_j \cdot s^k x_{[p]} = (-1)^k \sum \alpha_{j,i} \cdot s^k x_i \cdot s^k x_{[p]} = 0\). Hence we have

\[
\varphi_{p+j}(s^k x_{[p]}) = \Phi(\varphi_{p+j-1})(s^k x_{[p]})
= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p]}) + \varphi_{p+j-1}(\tau y_j \cdot s^k x_{[p]})
= \sigma y_j \cdot \varphi_{p+j-1} = \sigma y_j.
\]
Next we prove \([2]\). Assume \(n < j\). Then, for any \(r\), we have \(\varphi_{p+j-1}(s^k x_{[p]} \backslash I_r) \in (\sigma y_1, \ldots, \sigma y_{j-1})\) by the induction hypothesis, since \(l(I_r) = n - 1 < j - 1\). Thus we have \(\varphi_{p+j}(s^k (x_{[p]} \backslash I_r)) \in (\sigma y_1, \ldots, \sigma y_j)\) by (7.7).

Finally we prove \([3]\). Assume \(n \geq j\). Since \(l(I) = n > j - 1\), we have \(\varphi_{p+j-1}(s^k x_{[p]} \backslash I) = 0\) by the induction hypothesis. Hence (7.7) reduces to the equation

\[
\varphi_{p+j}(s^k x_{[p]} \backslash I) = (-1)^{p+j} \sum_{1 \leq r \leq j} (-1)^{i-r} \alpha_{j,i-r} \cdot \varphi_{p+j-1}(s^k x_{[p]} \backslash I_r) .
\] (7.8)

If \(n > j\), since \(l(I_r) = n - 1 > j - 1\), we have \(\varphi_{p+j-1}(s^k x_{[p]} \backslash I_r) = 0\) and hence \(\varphi_{p+j}(s^k x_{[p]} \backslash I) = 0\) by (7.8). This proves (3) in the case \(n > j\).

Next we assume \(n = j\). Let \(M_{u,r}\) be the minor determinants of the \(j \times j\) matrix \(A = (\alpha_{i,j})_{1 \leq i \leq j, 1 \leq j, r \leq j}\), i.e. \(M_{u,r} = \det ((\alpha_{i,s})_{s \neq r})\). Since \(|I_r| = j - 1\), we have \(\varphi_{p+j-1}(s^k x_{[p]} \backslash I_r) = (-1)^{|I_r| + p(j-1)} M_{j,r}\) by the induction hypothesis. Hence, by (7.8), we have

\[
\varphi_{p+j}(s^k x_{[p]} \backslash I) = (-1)^{p+j} \sum_{1 \leq r \leq j} (-1)^{i-r} \alpha_{j,i-r} \cdot (-1)^{|I_r| + p(j-1)} M_{j,r}
\]

\[
= (-1)^{|I| + pj} \sum_{1 \leq r \leq j} (-1)^{j+r} M_{j,r}
\]

\[
= (-1)^{|I| + pj} \det((\alpha_{t,i})_{1 \leq t, r \leq j}).
\]

This proves (3) in the case \(n = j\). \(\square\)

**Proposition 7.9.** If \(\varphi \in \text{Hom}_{\mathcal{S}^{k-1}}(D^k, \mathcal{S}^{k-1})\) is a chain map satisfying \(\varphi(s^k x_{[p]}) = \sigma y_{[q]}\), then we have

\([\varphi] \neq 0 \in \text{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}).\)

**Proof.** Let \(I \subset \mathcal{S}^{k-1}\) be the ideal generated by \(x_1 \otimes 1, \ldots, x_p \otimes 1, y_1 \otimes 1, \ldots, y_q \otimes 1, \sigma x_1, \ldots, \sigma x_p\). Note that \(d(I) \subset I\) since \((\wedge V, d)\) is pure. Consider the evaluation map

\[
\text{ev} : \text{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}) \otimes \text{Tor}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}/I) \to \text{Tor}_{\mathcal{S}^{k-1}}(\mathcal{S}^{k-1}, \mathcal{S}^{k-1}/I) \cong \wedge(\sigma y_1, \ldots, \sigma y_q).
\]

Using \(D^k\) as a resolution of \((\wedge V, d)\) over \(\mathcal{S}^{k-1}\), we have elements \([\varphi] \in \text{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1})\) and \([s^k x_{[p]} \otimes 1] \in \text{Tor}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}/I)\). Then we have

\[
\text{ev}([\varphi] \otimes [s^k x_{[p]} \otimes 1]) = \sigma y_{[q]} \neq 0 \in \wedge(\sigma y_1, \ldots, \sigma y_q).
\]

This proves the proposition. \(\square\)

**Corollary 7.10.** Assume \(p \leq q\), i.e. \(\dim V^\text{even} \leq \dim V^\text{odd}\). Then there is a chain map \(\varphi \in \text{Hom}_{\mathcal{S}^{k-1}}(D^k, \mathcal{S}^{k-1})\) such that

1. \([\varphi] \neq 0 \in \text{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}).\)
\[(2) \quad \mu \circ \varphi(1) = \begin{cases} \det \left( \frac{\partial (dy_j)}{\partial x_i} \right)_{1 \leq i, j \leq p} \in \wedge V^{\text{even}}, & \text{if } p = q, \\
0, & \text{if } p < q. \end{cases} \]

**Proof.** Define \( \varphi = (-1)^{(p+3)k} \varphi_{2p} \). By Proposition 7.6 (1) and Proposition 7.9, we have \( [\varphi] \neq 0 \in \text{Ext}_{S^{k-1}}(\wedge V, S^{k-1}) \). If \( p = q \), then, by (7.4) and Proposition 7.6, we have \( \mu \circ \varphi(1) = \det \left( \frac{\partial (dy_j)}{\partial x_i} \right) \). If \( p < q \), by Proposition 7.6 (2), we have \( \mu \circ \varphi(1) = 0 \) since \( \sigma y_j \in \text{Ker} \mu \).

**Remark 7.11.** We can generalize the non-triviality of the chain map \( \varphi = \varphi_{\dim V} \) using the method and notion given in [Wak16]. Let \((\wedge V, d)\) be a semi-pure Sullivan algebra, i.e. \( \dim V < \infty \) and \( d(V^{\text{even}}) \) is contained in the ideal \( \wedge V \cdot V^{\text{even}} \) generated by \( V^{\text{even}} \). Take bases \( x_1, \ldots, x_p \) and \( y_1, \ldots, y_q \) of \( V^{\text{even}} \) and \( V^{\text{odd}} \), respectively. By induction on \( \dim V \), we have \( \varphi(s^k x_1) = \sigma y_1 \cdot \ldots \cdot \sigma y_q \) along with \( \varphi(\nu) = 0 \) for any \( \nu \in (s^k y_1, \ldots, s^k y_q) \subset D^k \). The first equation \( [\varphi] \neq 0 \in \text{Ext}_{S^{k-1}}(\wedge V, S^{k-1}) \), since Proposition 7.9 also holds for a semi-pure Sullivan algebra.

### 7.3 Proof of Proposition 6.11

In this subsection, we prove Proposition 6.11 using the chain map in Corollary 7.10.

**Definition 7.12 ([FHT01, Section 32]).** A 1-connected space \( M \) is **rationally elliptic** if \( \dim_K (\bigoplus_n H^n(M) \otimes K) < \infty \) and \( \dim_K (\bigoplus_n \pi_n(M) \otimes K) < \infty \).

First we recall a fundamental theorem on rationally elliptic space.

**Theorem 7.13 ([FHT01 Proposition 32.16]).** Let \( M \) be a rationally elliptic space. Then we have

- \( \chi(M) \geq 0 \) and
- \( \dim_K (\bigoplus_n \pi_{2n}(M) \otimes K) \leq \dim_K (\bigoplus_n \pi_{2n-1}(M) \otimes K) \).

Moreover, the following conditions are equivalent:

1. \( \chi(M) > 0 \).
2. \( \dim_K (\bigoplus_n \pi_{2n}(M) \otimes K) < \dim_K (\bigoplus_n \pi_{2n-1}(M) \otimes K) \).
3. The minimal Sullivan model \((\wedge V, d)\) of \( M \) is pure, \( \dim V^{\text{even}} = \dim V^{\text{odd}} = p \), and \( dy_1, \ldots, dy_p \) is a regular sequence in \( \wedge V^{\text{even}} \), where \( V^{\text{odd}} = \text{span}_K \{y_1, \ldots, y_p\} \).

Using the theorem with the construction given in Subsection 7.2, we have the following proposition.
Proposition 7.14. Let $M$ be a rationally elliptic space satisfying the conditions in Theorem 7.13 and $(\wedge V, d)$ its minimal Sullivan model. Write $V^{\text{even}} = \text{span}_{\mathbb{K}} \{x_1, \ldots, x_p\}$ and $V^{\text{odd}} = \text{span}_{\mathbb{K}} \{y_1, \ldots, y_p\}$. Then we have
\[
\det \left( \left( \frac{\partial(dy_j)}{\partial x_i} \right)_{1 \leq i, j \leq p} \right) \neq 0 \in H^*(\wedge V) \cong H^*(M).
\]

Proof. By Corollary 7.10 for $k = 1$, we have a chain map $\varphi \in \text{Hom}_{\wedge V \otimes 2}(\wedge V, \wedge V \otimes 2)$ such that $[\varphi] \neq 0 \in \text{Ext}^m_{\wedge V \otimes 2}(\wedge V, \wedge V \otimes 2)$ and $\mu \circ \varphi(1) = \det \left( \frac{\partial(dy_j)}{\partial x_i} \right) \in \wedge V^{\text{even}}$.

Since $\mu$ is a model of $\Delta: M \to M \times M$, we have
\[
\Delta^* \circ (H^*(\varphi))(1) = [\mu \circ \varphi(1)] = \det \left( \frac{\partial(dy_j)}{\partial x_i} \right).
\]

Since $\text{Ext}^m_{\wedge V \otimes 2}(\wedge V, \wedge V \otimes 2) \cong \text{Ext}^m_{C^*(M \times M)}(C^*(M), C^*(M \times M)) \cong \mathbb{K}$, we have $[\varphi] = \Delta!$ (up to scalar multiplication). Hence by Proposition 6.7 we have
\[
\det \left( \frac{\partial(dy_j)}{\partial x_i} \right) = \chi(M) \omega.
\]

Since $\chi(M) \neq 0$ by (3) of Theorem 7.13 this proves the proposition. 

Remark 7.15. The proposition also follows from [Smi82, Proposition 3]. Here we give an alternative proof using an idea coming from string topology.

Now we give a proof of Proposition 6.11, which completes the proof of Theorem 1.2.

Proof of Proposition 6.11. Since the statement is trivial when $\chi(M) = 0$, we may assume $\chi(M) \neq 0$. Then, by Theorem 7.13 the minimal Sullivan model $(\wedge V, d)$ of $M$ satisfies (3). Take $\varphi \in \text{Hom}_{\wedge S^{k-1}}(D^k, S^{k-1})$ by Corollary 7.10. Then we have $c^* \circ (H^*(\varphi))(1) \neq 0 \in H^*(\wedge V) \cong H^*(M)$ by Proposition 7.14. Thus $\gamma = [\varphi]$ satisfies the equation (after multiplication of a non-zero scalar, if necessary).

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