THE GEOMETRY OF BLUNDON’S CONFIGURATION

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Abstract. Denote by $\mathcal{T}(R, r)$ the family of triangles inscribed in the circle of center $O$ with the radius $R$ and circumscribed to the circle of center $I$ with the radius $r$. This defines the Blundon’s configuration. The family $\mathcal{T}(R, r)$ contains only two isosceles triangles $A_{\text{min}}B_{\text{min}}C_{\text{min}}$ and $A_{\text{max}}B_{\text{max}}C_{\text{max}}$, which are extremal for Blundon’s inequalities (1). Some properties of Blundon’s configuration are given Section 2. Applications are presented in the last section where a strong version of Blundon’s inequalities is obtained (Theorem 7).

1. Introduction

Given a triangle $ABC$, denote by $O$ the circumcenter, $I$ the incenter, $N$ the Nagel point, $s$ the semiperimeter, $R$ the circumradius, and $r$ the inradius of $ABC$. W. J. Blundon [7] has proved in 1965 that the following inequalities hold

$$2R^2 + 10Rr - r^2 - 2(R - 2r) \sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r) \sqrt{R^2 - 2Rr}.$$  

(1)

The inequalities (1) are fundamental in triangle geometry because they represent necessary and sufficient conditions (see [7]) for the existence of a triangle with given elements $R, r$ and $s$. The algebraic character of inequalities (1) is discussed in the papers [10] and [11] and an elementary proof to the weak form of (1) is given in [8]. Other results connected to (1) are contained in [13]. We mention that D. Andrica, C. Barbu [2] (see also [1, Section 4.6.5, pp.125-127]) give a direct geometric proof to Blundon’s inequalities by using the Law of Cosines in triangle $ION$. They have obtained the formula

$$\cos \widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R - 2r) \sqrt{R^2 - 2Rr}}.$$  

(2)

Because $-1 \leq \cos \widehat{ION} \leq 1$, obviously it follows that (2) implies (1), showing the geometric character of (1). In the paper [3] other Blundon’s type inequalities are obtained using the same idea and different points instead of points $I, O, N$. If $\phi$ denotes $\min \{ |A - B|, |B - C|, |C - A| \}$, then in the paper [15] is proved the following improvement to (1), $-\cos \phi \leq \cos \widehat{ION} \leq \cos \phi$. A geometric proof to this inequalities is given in the paper [4].

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In Section 2 of the present note we study some geometric properties of the Blundon’s configuration. In the last section we present a strong version of Blundon’s inequalities.

2. The Blundon’s configuration

It is well-known that distance between points $O$ and $N$ is given by

$$\text{ON} = R - 2r.$$ \hfill (3)

The relation (3) reflects geometrically the difference between the quantities involved in the Euler’s inequality $R \geq 2r$. In the book of T. Andreescu and D. Andrica [1, Theorem 1, pp. 122-123] is given a proof to relation (3) using complex numbers. In the paper [5] similar relations involving the circumradius and the exradii of the triangle are proved and discussed.

Denote by $\mathcal{T}(R, r)$ the family of all triangles having the circumradius $R$ and the inradius $r$, inscribed in the circle of center $O$ and circumscribed to the circle of center $I$, where the points $O$ and $I$ are fixed. Let us observe that the inequalities (1) give in terms of $R$ and $r$ the exact interval containing the semiperimeter $s$ for triangles in family $\mathcal{T}(R, r)$.

More exactly, we have

$$s_{\text{min}}^2 = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}$$

and

$$s_{\text{max}}^2 = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$ 

The triangles in the family $\mathcal{T}(R, r)$ are situated ”between” two extremal triangles $A_{\text{min}}B_{\text{min}}C_{\text{min}}$ and $A_{\text{max}}B_{\text{max}}C_{\text{max}}$ determined by $s_{\text{min}}$ and $s_{\text{max}}$. These triangles are isosceles with respect to the vertices $A_{\text{min}}$ and $A_{\text{max}}$. Indeed, according to formula (2), the triangle in the family $\mathcal{T}(R, r)$ with minimal semiperimeter corresponds to the equality case $\cos \overrightarrow{IO}N = 1$, i.e. the points $I, O, N$ are collinear and $I$ and $N$ belong to the same ray with the origin $O$. Let $G$ and $H$ be the centroid and the orthocenter of triangle. Taking in to account the well-known property that points $O, G, H$ belong to Euler’s line of triangle, this implies that $O, I, G$ must be collinear, hence in this case triangle $ABC$ is isosceles. In similar way, the triangle in the family $\mathcal{T}(R, r)$ with maximal semiperimeter corresponds to the equality case $\cos \overrightarrow{IO}N = -1$, i.e. the points $I, O, N$ are collinear and $O$ is situated between $I$ and $N$. Using again the Euler’s line of the triangle, it follows that triangle $ABC$ is isosceles.

We call the Blundon’s configuration, the geometric situation in Figure 1.
THEOREM 1. The family $\mathcal{F}(R, r)$ contains only two isosceles triangles, i.e. the extremal triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$.

Proof. The triangle $ABC$ in $\mathcal{F}(R, r)$ is isosceles with $AB = AC$ if and only if $OI$ is perpendicular to $BC$. Because $B_{\min}C_{\min}$ and $B_{\max}C_{\max}$ are perpendicular to $OI$, the conclusion follows. □

In what follows we will determine some elements of the isosceles triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$.

We have $A_{\min}D = R - OD = R - (OI - r)$, where the point $D$ is defined in Figure 1. It follows

$$A_{\min}D = h_{\min} = R + r - OI = R + r - \sqrt{R^2 - 2Rr}. \quad (4)$$

Similarly, we have

$$A_{\max}E = h_{\max} = R + r + OI = R + r + \sqrt{R^2 - 2Rr}. \quad (5)$$

REMARK 1. Because $OD \geq 0$, it follows $OI \geq r$ and we get

$$R \geq r(1 + \sqrt{2}), \quad (6)$$

i.e.

$$r \leq (\sqrt{2} - 1)R.$$
This is a short geometric proof to the A. Emmerich inequality [9], true for every non-acute triangle.

Consider \( a_m = B_{\min}C_{\min}, b_m = A_{\min}B_{\min} = A_{\min}C_{\min}, K_m = \frac{a_m h_{\min}}{2} \) the area of triangle \( A_{\min}B_{\min}C_{\min} \). We have

\[
R = \frac{a_m b_m^2}{4K_m} = \frac{b_m^2}{2h_{\min}},
\]

therefore

\[
2Rh_{\min} = b_m^2 = h_{\min}^2 + \frac{a_m^2}{4},
\]

hence

\[
a_m^2 = 4h_{\min}(2R - h_{\min}). \tag{7}
\]

From equations (4) and (7) it follows

\[
a_m^2 = 4r \left( 2R - r + 2\sqrt{R^2 - 2Rr} \right). \tag{8}
\]

Denote \( a_M = B_{\max}C_{\max}, b_M = A_{\max}B_{\max} = A_{\max}C_{\max}, \) and let \( K_M = \frac{a_M h_{\max}}{2} \) be the area of triangle \( A_{\max}B_{\max}C_{\max} \). We have

\[
R = \frac{a_M b_M^2}{4K_M} = \frac{b_M^2}{2h_{\max}},
\]

hence

\[
2Rh_{\max} = b_M^2 = h_{\max}^2 + \frac{a_M^2}{4}.
\]

From here we obtain

\[
a_M^2 = 4h_{\max}(2R - h_{\max}). \tag{9}
\]

Using the equations (5) and (9) it follows

\[
a_M^2 = 4r \left( 2R - 2r + 2\sqrt{R^2 - 2Rr} \right). \tag{10}
\]

Combining the equations (8) and (10) we obtain

\[
a_m^2 + a_M^2 = 8r(2R - r) \quad \text{and} \quad a_m a_M = 4r\sqrt{r^2 + 4Rr}.
\]

From equations (8) and (10) we get the inequality \( a_M < a_m \). Also, we have

\[
\cos A_{\min} = 2\cos^2 \frac{A_{\min}}{2} - 1 = 2 \cdot \frac{h_{\min}^2}{b_m^2} - 1 = \frac{h_{\min}}{R} - 1, \tag{11}
\]

and similarly

\[
\cos A_{\max} = 2\cos^2 \frac{A_{\max}}{2} - 1 = 2 \cdot \frac{h_{\max}^2}{b_m^2} - 1 = \frac{h_{\max}}{R} - 1. \tag{12}
\]
Theorem 2. The following relations hold:

\[ \sin \frac{A_{\text{max}}}{2} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{R} - \frac{2r}{R}} \]  
(13)

and

\[ \sin \frac{A_{\text{min}}}{2} = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{R} - \frac{2r}{R}}. \]  
(14)

Proof. Using formulas (12) and (5), we have successively

\[
\begin{align*}
\sin^2 \frac{A_{\text{max}}}{2} &= 1 - \cos A_{\text{max}} = 1 - \frac{h_{\text{max}}}{R} = 1 - \frac{R + r + \sqrt{R^2 - 2Rr}}{2R} \\
&= \frac{R - r - \sqrt{R^2 - 2Rr}}{2R} = \frac{2R^2 - 2Rr - 2R \sqrt{R^2 - 2Rr}}{4R^2} = \left( \frac{R - \sqrt{R^2 - 2Rr}}{2R} \right)^2,
\end{align*}
\]

and the formula (13) follows.

In similar way, using formulas (11) and (4), we obtain

\[
\begin{align*}
\sin^2 \frac{A_{\text{min}}}{2} &= 1 - \cos A_{\text{min}} = 1 - \frac{h_{\text{min}}}{R} = 1 - \frac{R + r - \sqrt{R^2 - 2Rr}}{2R} \\
&= \frac{R - r + \sqrt{R^2 - 2Rr}}{2R} = \frac{2R^2 - 2Rr + 2R \sqrt{R^2 - 2Rr}}{4R^2} = \left( \frac{R + \sqrt{R^2 - 2Rr}}{2R} \right)^2,
\end{align*}
\]

and we get the formula (14). \(\square\)

The results in Theorem 1 and Theorem 2 clarify with different proofs the results contained in Theorems 1-2 in the paper [14].

3. Consequences for Blundon’s inequalities

In this section we give some applications in the spirit of papers [6] and [12]. We begin with the following auxiliary result.

Lemma 3. Let \( P \) be a point situated in the interior of the circle \( \mathcal{C}(O; R) \). If \( P \neq O \), then the function \( A \mapsto PA \) is strictly increasing on the semicircle \( \overline{M_0M_1} \), where the points \( M_0, M_1 \) are the intersection of \( OP \) with the circle \( \mathcal{C} \) such that \( P \in (OM_0) \).

Proof. Without loss of generality, we can assume that \( O \) is the origin of the coordinates system \( xOy \) and \( P \) is situated on the positive half axis. In this case we have \( P(x_0, 0), x_0 > 0, A(R \cos t, R \sin t), t \in [0, \pi] \), and

\[ PA^2 = (R \cos t - x_0)^2 + (R \sin t)^2 = R^2 + x_0^2 - 2Rx_0 \cos t. \]

Because the cosine function is strictly decreasing on the interval \([0, \pi]\) and \( x_0 > 0 \) we obtain that the function \( A \mapsto PA^2 \) is strictly increasing, and the conclusion follows. \(\square\)
THEOREM 4. In the Blundon’s configuration, the function $A \mapsto \angle BAC$ is strictly increasing on the semicircle $A_{\text{max}}A_{\text{min}}$.

Proof. We use the well-know relation $\sin \frac{A}{2} = \frac{r}{IA}$. From Lemma 3 with $P = I$, the function $A \mapsto IA$ is strictly decreasing on the semicircle $A_{\text{max}}A_{\text{min}}$. Therefore, for two points $A_1, A_2 \in \overparen{A_{\text{max}}A_{\text{min}}}$ in this order, we have $IA_1 > IA_2$. Therefore $\sin \frac{A_1}{2} = \frac{r}{IA_1} < \frac{r}{IA_2} = \sin \frac{A_2}{2}$, implying $\angle B_1A_1C_1 < \angle B_2A_2C_2$. □

From the Law of Sines, for a triangle in the family $\mathcal{T}(R, r)$, we have $a = 2R \sin A$. Using the relation $r = (s - a) \tan \frac{A}{2}$ we obtain

$$s = \frac{r + a \tan \frac{A}{2}}{\tan \frac{A}{2}} = \frac{r + 2R \sin A \tan \frac{A}{2}}{\tan \frac{A}{2}},$$

i.e. the semiperimeter $s$ depends only on the angle $A$.

**Figure 2.** The distribution of triangles in the family $\mathcal{T}(R, r)$

On the other hand, from the relations $bc = \frac{4rRs}{a}$ and $b + c = 2s - a$, it follows that $b, c$ are the roots of the quadratic equation

$$x^2 - (2s - a)x + \frac{4rRs}{a} = 0,$$

that is

$$2s - a \pm \sqrt{4s^2 - 4as + a^2 - \frac{16rRs}{a}}.$$
The above computations show that a triangle in the family $\mathcal{T}(R, r)$ is perfectly determined up to a congruence by the angle $A$. In this way, we obtain the distribution of triangles in the family $\mathcal{T}(R, r)$ (see Figure 2).

**COROLLARY 5.** The distribution of triangles in the family $\mathcal{T}(R, r)$ is in pairs $(\Delta ABC, \Delta A'B'C')$ such that triangles $ABC$ and $A'B'C'$ are congruent and symmetric with respect to the diameter $OI$.

**COROLLARY 6.** In the Blundon’s configuration, the function $A \mapsto BC$ is strictly increasing on the arc $A_{\max}A_0$, and strictly decreasing on the arc $A_0A_{\min}$, where $A_0$ is the point on the semicircle $A_{\max}A_{\min}$ such that $\angle B_0A_0C_0 = \frac{\pi}{2}$.

**THEOREM 7.** (The strong version of Blundon’s inequality) In the Blundon’s configuration, the function $A \mapsto s(A)$, is strictly decreasing on the arc $A_{\max}B_{\min}$, where $s(A)$ denotes the semiperimeter of triangle $ABC$, that is we have the inequalities
\[ s(A_{\max}) \geq s(A) \geq s(B_{\min}). \]

**Proof.** Clearly, $s(A_{\max}) = s_{\max}$, the semiperimeter of triangle $A_{\max}B_{\max}C_{\max}$, and $s(A_{\min}) = s_{\min}$, the semiperimeter of triangle $A_{\min}B_{\min}C_{\min}$. When $A$ moves on the arc $A_{\max}B_{\min}$ from $A_{\max}$ to $B_{\min}$, the angle $\angle ION$ strictly decreases from $\pi$ to 0, i.e., the function $A \mapsto \angle ION$ is strictly decreasing. Assume that we have the order $A_{\max}, A_1, A_2, B_{\min}$. From formula (2) we obtain $s^2(A_1) > s^2(A_2)$, and the conclusion follows. □

The area $K$ of a triangle $ABC$ in the family $\mathcal{T}(R, r)$ is a function of angle $A$, and we have the formula $K = K(A) = rs(A)$, where $s(A)$ is given in (15). The following consequence of Theorem 7 is the strong version of the result in [12, Theorem 1].

**COROLLARY 8.** In the Blundon’s configuration, the function $A \mapsto K(A)$ is strictly decreasing on the arc $A_{\max}B_{\min}$, strictly increasing on the arc $B_{\min}C_{\max}$, and strictly decreasing on $C_{\max}A_{\min}$, where $K(A)$ denotes the area of triangle $ABC$.

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