Asymptotics of $L_p$-norms of Hermite polynomials and Rényi entropy of Rydberg oscillator states

A. I. Aptekarev, J.S. Dehesa, P. Sánchez-Moreno, and D. N. Tulyakov

Abstract. The asymptotics of the weighted $L_p$-norms of Hermite polynomials, which describes the Rényi entropy of order $p$ of the associated quantum oscillator probability density, is determined for $n \to \infty$ and $p > 0$. Then, it is applied to the calculation of the Rényi entropy of the quantum-mechanical probability density of the highly-excited (Rydberg) states of the isotropic oscillator.

1. Introduction

A long standing problem in classical analysis and approximation theory is the determination of the weighted $L_p$-norms

$$
\|\rho_n\|_p = \left\{ \int_\Delta [\rho_n(x)]^p \, dx \right\}^{\frac{1}{p}} = \left\{ \int_\Delta [\omega(x)y_n^2(x)]^p \, dx \right\}^{\frac{1}{p}}; \quad p > 0,
$$

where $\{y_n(x)\}$ denotes a sequence of real polynomials orthogonal with respect to the weight function $\omega(x)$ on the interval $\Delta$ so that

$$
\int_\Delta y_n(x)y_m(x)\omega(x) \, dx = \delta_{m,n}; \quad m, n \in \mathbb{N},
$$

and

$$
\rho_n(x) = \omega(x)y_n^2(x).
$$

We call \[12\] Rakhmanov’s probability density of the polynomial $y_n(x)$ since this mathematician discovered in 1997 (see \[20\]) that it governs the asymptotic ($n \to \infty$) behaviour of the ratio $y_{n+1}/y_n$ for general $\omega > 0$ a.e. (positive almost everywhere) on the finite interval $\Delta$. Physically, $\rho_n(x)$ describes the radial probability density of the ground and excited states of the physical systems whose non-relativistic wavefunctions are controlled by the polynomials $y_n(x)$ \[12\].

2000 Mathematics Subject Classification. Primary 11B37, 94A17; Secondary 30E15, 33C45.

Key words and phrases. Asymptotic behaviour of solutions of difference equations, orthogonal polynomials, Hermite polynomials, Rényi entropy, Rydberg states.
The \( L_p \)-norms (1.1) are closely related to the frequency or entropic moments\cite{13,24,30}:

\[
W_p[\rho_n] = \left\langle |\rho_n(x)|^{p-1} \right\rangle = \int_{\Delta} |\rho_n(x)|^p \, dx = \|\rho_n\|^p_p,
\]

the Rényi entropies\cite{23}:

\[
R_p[\rho_n] = \frac{1}{1-p} \ln W_{p-1}[\rho]; \quad p > 0, p \neq 1,
\]

and the Rényi spreading lengths\cite{15}:

\[
L^R_p[\rho_n] = \exp (R_p[\rho_n]) = \|\rho_n\|^p_p^{-\frac{1}{p-1}}.
\]

These quantities have been applied in numerous fields from economics and electrical engineering to chemistry, quantum physics and approximation theory, as summarized in e.g. Refs\cite{6,9,16}. The study of the \( L_p \) norms of orthogonal polynomials is of independent interest in the theory of general orthogonal and extremal polynomials. This problem is connected with the classical research of S.N. Bernstein on the asymptotics of the \( L_p \) extremal polynomials in\cite{5}, that has recently received further development\cite{17,18}. On the other hand, its statement is a generalization of a widely known problem of Steklov on the estimation of the \( L_\infty \) norms of polynomials orthonormal with respect to a positive weight (see\cite{27}). Indeed, for \( p = 1 \) the norms are bounded (they are just equal to 1); however, for \( p = \infty \) (as it has been shown by Rakhmanov\cite{21}) they may grow to infinity. What happens with the boundedness of the \( L_p \)-norms of the Rakhmanov density of the orthogonal polynomials when \( 1 < p < \infty \)? Our work sheds light on this issue for Hermite polynomials.

Recently these quantities have been calculated for polynomials \( y_n(x) \) with arbitrary degree \( n \) by means of the combinatorics-based Bell polynomials in the Hermite\cite{25}, Laguerre\cite{26} and Jacobi\cite{14} cases; see also\cite{8}. However, this methodology is computationally very demanding and analytically inefficient for high and very high values of \( n \).

The aim of this work is the asymptotic (\( n \to \infty \)) determination of the entropic moments of Hermite polynomials, i.e.

\[
W_p[\rho_n] = \int_{-\infty}^{+\infty} \left[ e^{-x^2} H_n^2(x) \right]^p \, dx; \quad p > 0.
\]

The solution of this problem is relevant not only \textit{per se} because it extends previous results\cite{2,4} obtained when \( p \in [0, \frac{1}{2}] \), but also because it paves the way for the evaluation of the \( p \)-th order Rényi entropy of the highly-excited (i.e., Rydberg) states of the physical systems whose radial wavefunctions are controlled by Hermite’s polynomials such as, e.g. the oscillator-like systems.

The structure of the paper is the following. In Section\cite{2} the asymptotics \( L_p \)-norms of the Hermite polynomials is found for \( p > 0 \). In Section\cite{3} these results are applied to evaluate the Rényi entropy of Rydberg states of the quantum harmonic oscillator. Finally, some conclusions are given.
2. $L_p$-norms of Hermite polynomials: Asymptotics ($n \to \infty$)

In this section we find the main term of asymptotics $W_p[r_{n-1}]$ (see (1.4)) when $n \to \infty$. Here we consider Hermite polynomials $H_n(x)$ in the standard normalization

$$H_n(x) = (2x)^n \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n + 1/2)} e^{-x^2}.$$

They satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H_0(x) = 1, \quad H_{-1}(x) = 0,$$

and have the following norm

$$h_n = \int_{-\infty}^{+\infty} \left[H_n^2(x) e^{-x^2}\right] dx = \sqrt{\pi} n! 2^n.$$

To reach the goal we need to have good asymptotics for $H_n(x)$ on $\mathbb{R}$. The first strong asymptotics formulae for Hermite polynomials are due to Plancherel and Rotach [19] (see also [28]). They describe polynomials $H_n$ when $n \to \infty$ in the following subdomain of $\mathbb{R}$:

a) $x = \sqrt{2n + 1} \cosh y, \quad \epsilon \leq y < \infty$;

b) $x = \sqrt{2n + 1 + n^{-\delta} t}, \quad t \in K \subset \mathbb{C}$;

c) $x = \sqrt{2n + 1} \cos y, \quad \epsilon \leq y \leq \pi - \epsilon$.

Using Plancherel–Rotach formulae (and their generalization for Freud weights from [22]) it was obtained [2] for the orthonormal Hermite polynomials $\tilde{H}_{n-1}(x)$ that

$$\int_{-\infty}^{+\infty} \left[\tilde{H}_{n-1}^2(x) e^{-x^2}\right]^p dx = c_p (2n)^{1-p} (1 + o(1)),$$

where

$$c_p = \left(\frac{2}{\pi}\right)^p \frac{\Gamma(p + 1/2)}{\Gamma(p + 1)} \frac{\Gamma(1 - \frac{p}{2})}{\Gamma(1 - \frac{p}{2})}.$$

Using (2.2) and the Stirling’s formula, this expression together with (1.4) produces the following asymptotics for the entropic moment of the orthogonal Hermite polynomial $H_{n-1}(x)$:

$$W_p[r_{n-1}] = c_p h_{n-1}^p (2n)^{1-p} (1 + o(1)) = c_p \pi^n (2n)^{p(n-1)+1/2} e^{-pn} (1 + o(1)) .$$

The restriction on $p$ in (2.4) appeared because the Plancherel–Rotach formulae in (2.3) do not match each other, i.e. subdomains in (2.3) do not intersect. Particularly, there is a gap between zone a) and zone b) in (2.3), which plays an important role for the limit (1.4) when $n \to \infty$. In the asymptotics of (2.4) the main contribution in the left hand side integral gives the part of the integral described in a) of (2.3). The gap between zone a) and zone b) gives the main contribution in the integral for bigger $p$.

The asymptotic description of Hermite polynomials in the subdomains covering all $\mathbb{R}$ was obtained not so long ago. In 1999 Deift et al [11] (see also [10]) have obtained the global asymptotic portrait of polynomials orthogonal with respect to exponential weights by means of the powerful matrix Riemann-Hilbert method.
As a corollary for Hermite polynomials $H_n(\sqrt{2nz})$, they obtained asymptotics as $n \to \infty$ and $z$ belongs to

\begin{equation}
\begin{align*}
a) & \quad |z| \geq 1 + \delta; \\
b) & \quad 1 - \delta \leq |z| \leq 1 + \delta; \\
c) & \quad |z| \leq 1 - \delta,
\end{align*}
\end{equation}

for small $\delta > 0$. Evidently, there are no gaps between zones and Deift et al’s asymptotics can be used for obtaining asymptotics of (1.4) for bigger $p$.

Recently a new approach for obtaining the global asymptotic portrait of orthogonal polynomials has appeared [29]. Contrary to the matrix Riemann-Hilbert method which starts from the weight of orthogonality, the starting point in [29] is the recurrence relation which characterizes the orthogonal polynomials. The application of this approach to Hermite polynomials brought an asymptotic description in the following subdomains of $\mathbb{R}$:

\begin{equation}
\begin{align*}
a) & \quad x^2 \in [2n + n^{\frac{1}{4}} + \theta; \infty); \\
b) & \quad x^2 \in [2n - n^{\frac{1}{4}} + \theta; 2n + n^{\frac{1}{4}} + \theta]; \\
c) & \quad x^2 \in [0; 2n - n^{\frac{1}{4}} + \theta],
\end{align*}
\end{equation}

for $\theta \in (0; \frac{\pi}{4})$. It is worth noting that zones a) and c) in (2.7) are wider than zones a) and c) in (2.8); in its turn zone b) in (2.7) is wider than b) in (2.8). In these zones we take $\theta < \frac{1}{6}$ for Hermite polynomials. Then, it follows from Theorem 5 of [29] that:

\begin{equation}
\begin{align*}
in a) & \quad H_{n-1}(x) = \frac{1}{\sqrt{2}} \left( x + \sqrt{x^2 - 2n} \right)^{n-\frac{1}{2}} \exp \left( \frac{x^2 - n - x\sqrt{x^2 - 2n}}{2} \right) (1 + o(1)), \\
in c) & \quad H_{n-1}(x) = \sqrt{2} \left( \sqrt{2n} \right)^{n-\frac{1}{2}} \exp \left( \frac{x^2 - n}{2} \right) \\
& \times \cos \left( \frac{n - 1}{2} \right) \arcsin \left( \sqrt{1 - \frac{x^2}{2n}} - \frac{x\sqrt{2n - x^2}}{2} - \frac{\pi}{4} + o(1) \right) (1 + o(1)), \\
in b) & \quad H_{n-1}(x) = \frac{\sqrt{2\pi}}{\sqrt{x^2 - n}} \exp \left( \frac{x^2}{4} + o(1) \right) \Ai \left( -\frac{\sqrt{2}}{2} z + o \left( n^{-\frac{1}{2}} \right) \right) (1 + o(1)),
\end{align*}
\end{equation}

where $z := \frac{2n}{x^2} - x^2$, and $\Ai$ denotes the Airy function (see [1], page 367).

Using these asymptotics we can obtain the main result of this section.

**Theorem 2.1.** Let $H_n(x)$ be the Hermite polynomials with the standard normalization (2.7). Then the frequency or entropic moments $W_p[\rho_{n-1}]$, given by Eq. (1.4), have for $n \to \infty$ the following asymptotic values

\begin{equation}
W_p[\rho_{n-1}] = \begin{cases}
c_p \pi^p (2n)^{p(n-1)+1/2} e^{-pn} (1 + o(1)), & p < 2, \\
2(2n)^{2n-\frac{2}{p}} e^{-2n} (\ln(n) + O(1)), & p = 2, \\
2C_p 2^{-p} (2n)^{p(n-\frac{2}{p})-\frac{1}{2}} e^{-pn} (1 + o(1)), & p > 2,
\end{cases}
\end{equation}

where the constant $c_p$ is defined in (2.5) and the constant $C_p$ is equal to

$$C_p = \int_{-\infty}^{+\infty} \left[ \frac{2\pi}{\sqrt{2}} \Ai \left( -\frac{\sqrt{2}}{2} z \right) \right]^p dz.$$
We note that the first asymptotic formula in the right hand side of (2.10) coincides with (2.6), but now it holds true in the maximal range of $p$ (when $p = 2$, then $c_p = \infty$); let us also highlight that the main term of the asymptotics is growing. Moreover, the smaller terms contain a constant which depends on $p$, and when $p \to 0$ this constant tends to infinity; however, our formula is correct for any small fixed $p > 0$. We also note that the leading term of all three formulae in the right hand side of (2.10) match each other when $p \to 2$.

**Proof.** Doing identical transformations and some evident asymptotic estimates, we have from (2.9) that:

\[(2.11)\]

in a) $H_{n-1}^2(x)e^{-x^2} = \frac{(2n)^{n-1}}{2} e^{-n} \exp \left( (2n - 1) \arccosh \frac{x}{\sqrt{2n}} - x\sqrt{x^2 - 2n} \right) \times \left( \frac{x}{2n} - 1 \right)^{-1} (1 + o(1))$;

in c) $H_{n-1}^2(x)e^{-x^2} = (2n)^{n-1} e^{-n} \times \left( 1 - \sin \left( (2n - 1) \arcsin \sqrt{1 - \frac{x^2}{2n}} - x\sqrt{2n - x^2} + o(1) \right) \right) \times \left( 1 - \frac{x^2}{2n} \right)^{\frac{1}{2}} (1 + o(1))$;

in b) $H_{n-1}^2(x)e^{-x^2} = (2n)^{n-\frac{3}{2}} e^{-n} \frac{2\pi}{\sqrt{2}} \left( \frac{x}{\sqrt{2n}} \right)^{2(n-\frac{3}{2})} \exp \left( \frac{2n - x^2}{2} \right) \times \text{Ai}^2 \left( -\frac{\sqrt{2}}{2} z + o \left( n^{-\frac{1}{2}} \right) \right) (1 + o(1))$.

Now we start to estimate the integral in (1.4). We consider the interval of integration $[0, \infty)$ (since the integral is even) and split it in the subintervals a), b), c) as in (2.8). And we split the interval b) in (2.8) into three subintervals:

\[(2.12)\]

b$_1$) $x^2 \in \left[ 2n - n^{\frac{1}{3} + \theta}; 2n - Mn^{\frac{1}{3}} \right]$;

b$_2$) $x^2 \in \left[ 2n - Mn^{\frac{1}{3}}; 2n + Mn^{\frac{1}{3}} \right]$;

b$_3$) $x^2 \in \left[ 2n + Mn^{\frac{1}{3}}; 2n + n^{\frac{1}{3} + \theta} \right]$.

Thus, we have splitted $x \in [0, \infty)$ on five zones (see Figure 2.1).

\[0 \quad \sqrt{2n - n^{\frac{1}{3} + \theta}} \quad \sqrt{2n - Mn^{\frac{1}{3}}} \quad \sqrt{2n} \quad \sqrt{2n + Mn^{\frac{1}{3}}} \quad \sqrt{2n + n^{\frac{1}{3} + \theta}} \quad \infty \]

**Figure 2.1.** Zones of $\mathbb{R}_+$, which gives different contribution to the integral, depending on $p$.

We recall, that $\theta$ is a fixed small number, such that $0 < \theta < \frac{1}{6}$ and the constant $M$ will be chosen depending on $p$. 
Making the change of variables $\frac{x}{\sqrt{2n}} = t$ in (2.11), we obtain for the integrals along the interval a) in (2.8):

$$I_a = \int_0^\infty \left( H_{n-1}^2(x)e^{-x^2} \right)^p dx = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn/2p}$$

$$\times \int_0^\infty \exp \left[ p(2n-1)\arccosh t - 2ntp\sqrt{t^2 - 1} + o(1) \right] \frac{dt}{(t^2 - 1)^{\frac{p}{2}}},$$

and for the interval c) in (2.8):

$$I_c = \int_0^{\sqrt{2n-Mn^{\frac{1+\theta}{2}}}} \left( H_{n-1}^2(x)e^{-x^2} \right)^p dx = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn}$$

$$\times \int_0^{1+\frac{1}{2}n^{\theta-\frac{1}{2}}+\epsilon_n} \left[ 1 - \sin \left( (2n-1)\arcsin \sqrt{1-t^2 - 2nt\sqrt{1-t^2}} \right) + o(1) \right]^p \frac{dt}{(1-t^2)^{\frac{p}{2}}},$$

where $\epsilon_n = o\left(n^{\theta-\frac{1}{2}}\right)$.

Then we pass to the integrals along b)-(2.8). The idea to split interval b) into three subintervals (2.12) was because we are intending to use in the subintervals $b_1$ and $b_3$ in (2.12) the asymptotics of the Airy function from b)-(2.11); in $b_2$-(2.12) we shall use the explicit expression for the Airy function. Noticing that for $n \to \infty$, we have from definition of $z$ in (2.9)

$$z = \frac{2n}{x^\frac{3}{2}} \Rightarrow x \simeq \frac{2n}{z} - \frac{z}{2\sqrt{2n}} \Rightarrow dx = -\frac{dz}{2\sqrt{2n}}$$

Thus,

$$I_{b_1} = \int_{\sqrt{2n-Mn^{\frac{1}{2}}}} \left( H_{n-1}^2(x)e^{-x^2} \right)^p dx$$

$$\simeq (2n)^{p(n-\frac{3}{2})-\frac{1}{2}} e^{-pn} \frac{1}{2} \int_M^{n^\theta} \left[ 1 + \sin \left( \frac{2}{3}z^\frac{3}{2} \right) \right]^p z^{-\frac{5}{2}} dz,$$
\[ I_{b_3} = \sqrt{2n + Mn^{\frac{3}{4}}} \int_{\sqrt{2n + Mn^{\frac{3}{4}}}}^\infty \left( H_{n-1}^2(x)e^{-x^2} \right)^p \ dx \]

(2.14) \[ \simeq (2n)^{p(n-\frac{2}{3})-\frac{1}{2}} e^{-pn^2} \int_M^{n^\theta} \exp \left( -\frac{2}{3}pz^{\frac{3}{2}} \right) z^{-\frac{2}{3}} \ dx \]

\[ I_{b_2} = \sqrt{2n - Mn^{\frac{3}{4}}} \int_{\sqrt{2n - Mn^{\frac{3}{4}}}}^\infty \left( H_{n-1}^2(x)e^{-x^2} \right)^p \ dx \]

(2.15) \[ \simeq (2n)^{p(n-\frac{2}{3})-\frac{1}{2}} e^{-pn^2} \int_{-M}^{M} \left[ \frac{2\pi}{\sqrt{2}} \exp \left( -\frac{z\sqrt{2}}{2} \right) \right] \ dx. \]

The symbol \( \simeq \) means that the ratio of the left and right hand sides tends to unity.

Now we can analyse the contributions of the various \( p \)-depending parts of the integral of the left hand side of (2.10), when \( n \to \infty \). We note, that all \( o(1) \) terms in our asymptotic analysis are differentiable, therefore they will not make contributions in our further estimates of the integrals.

First, we notice that the integral part of \( I_a \) in the right hand side of (2.13) is exponentially small, and there exist constants \( \alpha, c > 0 \), such that this integral is estimated as \( O \left( n^\alpha \exp \left( -cn^2 \right) \right) \), and for \( I_a \) we have

\[ I_a = \frac{(2n)^{p(n-1)+\frac{1}{2}} e^{-pn^2} - \theta}{2p e^{n^2}} O \left( n^\alpha \exp \left( -cn^2 \right) \right). \]

Therefore this part is negligible for (2.10).

Second, we notice that the integral parts of \( I_{b_3} \) and \( I_{b_2} \) in the right hand side of (2.14) and (2.15), respectively, are \( O(1) \) and we have

\[ I_{b_3}, I_{b_3} = (2n)^{p(n-\frac{2}{3})-\frac{1}{2}} e^{-pn^2} - \theta O(1). \]

When \( p < 2 \), the integral parts of \( I_c \) and \( I_{b_1} \) behave as \( O(1) \) and \( O \left( n^{\frac{3}{2}} e^{-\frac{n}{2}} \right) \)

and we have

\[ I_c = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn} O(1), \]

\[ I_{b_1} = (2n)^{p(n-\frac{2}{3})-\frac{1}{4}} e^{-pn} \left[ \frac{p_{n}\left( \frac{p_{n}}{2} \right)}{O \left( n^{\frac{3}{2}} e^{-\frac{n}{2}} \right)} \right], \]

for \( 0 < p < 2 \). Therefore, when \( p < 2 \), only \( I_c \) gives contribution in (2.10) and we have proved, that (2.4) is valid for \( 0 < p < 2 \).

When \( p = 2 \), then both integral parts \( I_c \) and \( I_{b_1} \) have the same logarithmic rate of growth \( O(\ln n) \). Computing the constant in \( O \) we obtain in a non-trivial way that

\[ \int_0^\infty \left( H_{n-1}^2(x)e^{-x^2} \right)^2 \ dx = (2n)^{2n^2-\frac{2}{4}} e^{-2n} (\ln(n) + O(1)), \] for \( p = 2 \).
Finally, for $p > 2$ as we see from (2.13)-(2.15), the integral over $b)-(2.8)$ dominates in (2.10). Thus taking $M \to \infty$, we obtain
\[
\int_{-\infty}^{\infty} \left[H_{n-1}^2(x)e^{-x^2}\right]^p dx = (2n)^{p(n-\frac{3}{2})-\frac{p}{2}} \left(\int_{-\infty}^{\infty} \frac{2\pi}{\sqrt{2}} Ai^2 \left(-\frac{z^{\frac{3}{2}}}{2}\right)\right)^p (1 + o(1)) ; p > 2.
\]
\[\square\]

3. Renyi entropy of Rydberg oscillator states

In this section Theorem 2.1 is applied to obtain the Renyi entropy of the Rydberg states of the one-dimensional harmonic oscillator, described by the quantum-mechanical potential $V(x) = \frac{1}{2}x^2$. It is in this energetic region where the transition from classical to quantum correspondence takes place. The physical solutions of the Schrödinger equation for the harmonic oscillator system (see e.g., [12]), are given by the wavefunctions characterized by the energies $E_n = n + \frac{1}{2}$ and the quantum probability densities
\[
\tilde{\rho}_n(x) = \frac{1}{\sqrt{\pi n!2^n}} e^{-x^2} \tilde{H}_n^2(x),
\]
where $\tilde{H}_n(x)$ denotes the orthonormal Hermite polynomials of degree $n$. The degree $n = 0, 1, 2, \ldots$ labels the energetic level.

The entropic moments of these densities $W_p[\rho_n]$ are expressed in terms of the entropic moments of the Hermite polynomials as
\[
(3.1) W_p[\tilde{\rho}_n] = \frac{1}{\pi^\frac{3}{2}n!p2^n} W_p[\rho_n].
\]
Thus, the entropic moments of the harmonic oscillator states are given by the entropic moments of the orthonormal Hermite polynomials. Consequently, according to equation (1.3) the Renyi entropy of the harmonic oscillator for both ground and excited states is given by the concomitant Renyi entropy of the involved orthonormal Hermite polynomials.

Let us now consider the Rydberg states of the oscillator system; that is, the states with high and very high values of $n$. Then, taking into account equations (2.10) and (3.1), we obtain the asymptotic ($n \to \infty$) values
\[
(3.2) W_p[\tilde{\rho}_{n-1}] = \begin{cases} 
    c_p (2n)^{\frac{3}{2}} (1 + o(1)), & p < 2, \\
    2\pi^{-2}(2n)^{-\frac{1}{2}} (\ln(n) + O(1)), & p = 2, \\
    2\frac{C_p}{(2\pi)^p} (2n)^{-\frac{1}{2}(p+1)} (1 + o(1)), & p > 2,
\end{cases}
\]
for the entropic moments of the Rydberg oscillator states. Finally, it is straightforward to have the expressions for the Renyi entropy of the Rydberg states as follows from Eqs. (3.2) and (1.3).

Figure 3.1 shows the values of the Renyi entropy $R_p[\rho_n]$ for $p = \frac{3}{2}, p = 2$ and $p = 3$, as a function of $n$ from $n = 100$ to $n = 10^{12}$. Notice that in all the cases the Renyi entropy increases with $n$. This indicates that the spreading of these states increases with $n$. Moreover, for the values of $p$ considered, after some initial intersections, the Renyi entropy also increases when $p$ decreases, for very
large values of \( n \), \((n > 10^7)\). This is also the observed behaviour when the Rényi entropy is exactly calculated for low and moderate values of \( n \) (see e.g. [25]). Then, we can conclude that the observed intersections come from the differences between the asymptotic and the exact values of this quantity.

4. Conclusions

In this work, we have shown that the Rényi entropy of the one-dimensional harmonic oscillator is exactly equal to the Rényi entropic integral of the involved orthonormal Hermite polynomials. Then, we have calculated the Rényi entropy of the highly excited states of the oscillator system by use of the asymptotics \((n \to \infty)\) of the \( L_p \)-norms of the Hermite polynomials \( H_n(x) \) which control the corresponding wavefunctions. Remark that no recourse to the quasi-classical approximation has been done. The asymptotics of the \( L_p \)-norms of \( H_n(x) \) was determined by extending some sophisticated ideas and techniques extracted from the modern approximation theory [2, 29]. This research opens the way to investigate the asymptotics of the multivariate Hermite polynomials, what would allows one to compute the Rényi entropy of the Rydberg states of the harmonic oscillator of arbitrary dimensionality.

Acknowledgements

AIA and DT are partially supported by the grant RFBR 11-01-12045 OFIM. AIA is partially supported by the grant RFBR 11-01-00245 and the Chair Excellence Program of Universidad Carlos III Madrid, Spain and Bank Santander. DT are partially supported by the grant RFBR 10-01-00682. JSD and PSM are very grateful for partial support to Junta de Andalucía (under grants FQM-4643 and FQM-2445) and Ministerio de Ciencia e Innovación under project FIS2011-24540. JSD and PSM belong to the Andalusian research group FQM-0207.
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Keldysh Institute for Applied Mathematics, Russian Academy of Sciences and Moscow State University, Moscow, Russia

E-mail address: apteka@keldysh.ru

Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, Granada, Spain

Instituto “Carlos I” de Física Teórica y Computacional, Universidad de Granada, Granada, Spain

E-mail address: dehesa@ugr.es

Departamento de Matemática Aplicada, Universidad de Granada, Granada, Spain

Instituto “Carlos I” de Física Teórica y Computacional, Universidad de Granada, Granada, Spain

E-mail address: pablos@ugr.es

Keldysh Institute for Applied Mathematics, Russian Academy of Sciences and Moscow State University, Moscow, Russia

E-mail address: dnt@mail.nnov.ru