Cohomology and MP Spacetimes

Richard Atkins
richard.atkins@twu.ca
Department of Mathematics
Trinity Western University
7600 Glover Road
Langley, BC, V2Y 1Y1 Canada

Abstract

This paper pursues a cohomological formulation for gravitation in which gravity might be expressed in terms of a gravitational potential, much in the spirit of electrodynamics. To this end we introduce a cochain complex consisting of \((n+2)\)-tensors, symmetric in two indices and skew-symmetric in the remaining \(n\) indices. The cohomology of the complex is shown to be isomorphic to the Čech cohomology of an appropriately defined sheaf of functions. Furthermore, it is demonstrated that in isotropic coordinates MP spacetimes as well as the Schwarzschild black hole may be thusly represented, that is, by means of the coboundary of a potential, as defined by the differential operator of the complex.
1 Introduction

The field strength in classical electrodynamics is given by a symmetric-free two-tensor $F = F_{\mu\nu}$, defining an element in the second de Rham cohomology group. The motion of a charged particle is then determined by

$$\ddot{x}^\mu = -k F_{\nu}^{\mu} \dot{x}^\nu$$

for some constant $k$, with respect to a flat background structure. The associated quantum field theory rests upon this cohomological foundation in that the dynamical variables to be quantized are the electrodynamic potentials, whose coboundaries in the de Rham complex give the field strength. This leads us to surmise whether a quantum description of gravitation demands a similar representation for an appropriately chosen cohomology theory defined, at least, for a significant class of spacetimes.

A free particle in a gravitational field follows a geodesic:

$$\ddot{x}^\mu = -\Gamma_{\nu\lambda}^{\mu} \dot{x}^\nu \dot{x}^\lambda$$

The form of these equations of motion suggests, by analogy to equation (1), the interpretation of the connection as the gravitational field strength. We introduce a flat metric $\eta = diag(+1, -1, -1, -1)$ with respect to the coordinates $x^\mu$ and define $\Gamma_{\mu\nu\lambda} = \eta^{\mu\tau} \Gamma_{\nu\lambda}^{\tau}$. The connection can be uniquely decomposed into a sum of a completely symmetric part $\Gamma_{(\mu\nu\lambda)}$ and a symmetric-free part $F_{\mu\nu\lambda}$

$$\Gamma_{\mu\nu\lambda} = \Gamma_{(\mu\nu\lambda)} + F_{\mu\nu\lambda}$$

This paper develops a cohomology theory for symmetric-free tensors of the form

$$S_{\mu_1...\mu_n\nu_1\nu_2}$$

which are skew-symmetric in the $\mu$ indices and symmetric in the $\nu$ indices. It is shown that in isotropic coordinates, the symmetric-free part of the field strength of Majumdar-Papapetrou (MP) spacetimes (cf. [4], [8], [9], [10]) is the coboundary of a potential in the associated cochain complex. Since $\Gamma_{\nu\lambda}^{\mu}$ is not a tensorial object the existence of such potentials is dependent upon the choice of coordinates considered;
indeed, such a characterization of the connection will hold only in a very restricted
class of coordinate systems, if at all. Thus this formulation introduces a kind of gauge
fixation of the diffeomorphism invariance inherent in general relativity.

In the following section a cochain complex \((K^*(M), d_K)\) is defined for manifolds
\(M\) endowed with a flat connection (cf. [1], [2], [17]). The associated cohomology
groups \(H^q_K(\mathbb{R}^n)\) for Euclidean space is determined by application of the Poincaré
Lemma. The cohomology \(H^*_K(M)\) for a general manifold \(M\) is then identified with
the Čech cohomology of the sheaf of local affine functions using spectral sequences
(cf. [3], [5], [6]). Section 3 defines a second cochain complex \((G^*(M), d_G)\) isomorphic
to \((K^*(M), d_K)\) which relates more directly to gravitation. In the final section, we
consider the MP class of spacetime solutions along with the Schwarzschild black hole
and show that they may be written in terms of a gravitational potential in the manner
described above.

## 2 Cohomology of the \(K^*(M)\) complex

In this section we define a cochain complex \((K^*(M), d_K)\), which will be intermediary
to the cohomology theory associated to gravitation, to be developed in the next
section. We begin with some basic definitions and conventions.

Let \(M\) be a manifold and \(\nabla\) a flat connection on \(M\). \(\nabla\) acts on a covariant \(n\)-tensor
\(C = C_{\mu_1\cdots\mu_n}\) on \(M\) by introducing an index to the left:

\[
(\nabla C)_{\mu_1\cdots\mu_{n+1}} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_{n+1}} = dx^{\mu_1} \otimes \nabla_{\partial/\partial x^{\mu_1}} (C)
\]

This shall be expressed more conveniently in terms of indices by

\[
(\nabla C)_{\mu_1\cdots\mu_{n+1}} = \nabla_{\mu_1} C_{\mu_2\cdots\mu_{n+1}}
\]

Define skew-symmetrization \(s(C)\) by

\[
s(C)_{\mu_1\cdots\mu_n} = C_{[\mu_1\cdots\mu_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \sgn(\sigma) C_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}
\]

where the sum ranges over all permutations \(\sigma\) on \(n\) letters, \(S_n\). Thus, \(s(s(C)) = s(C)\).

Define \(d\) by

\[
(d\nabla C)_{\mu_1\cdots\mu_{n+1}} = \nabla_{[\mu_1} C_{\mu_2\cdots\mu_{n+1}]} \mu_{n+1}
\]
Since $\nabla$ is flat, $d_{\nabla}^{2} = 0$.

Let $\text{Sym}^{n}(M)$ denote the symmetric sections of $T^{n,0}M$, the tensor product of $n$ copies of the cotangent bundle. The module $K^{n}(M)$ over the ring of smooth functions $\Omega^{0}(M)$ on $M$ is defined, for $n \geq 2$, to be the submodule of

$$\Omega^{n}(M) \otimes_{\Omega^{0}(M)} \text{Sym}^{1}(M)$$

consisting of tensor fields $T_{\mu_{1} \cdots \mu_{n}\nu}$ satisfying the condition

$$T_{[\mu_{1} \cdots \mu_{n}\nu]} = 0 \quad \text{(2)}$$

$\Omega^{n}(M)$, as usual, denotes the module of $n$-forms on $M$. Set $K^{0}(M) = \Omega^{0}(M)$ and $K^{1}(M) = \text{Sym}^{2}(M)$. The cochain complex $(K^{*}(M), d_{K})$ is defined to be

$$0 \rightarrow K^{0}(M) \xrightarrow{\nabla^{2}} K^{1}(M) \xrightarrow{d_{K}} K^{2}(M) \xrightarrow{d_{K}} K^{3}(M) \xrightarrow{d_{K}} \cdots$$

The cohomology of the complex is

$$H^{q}_{K}(M) = \frac{\ker d_{K}}{\text{im} d_{K}}$$

which naturally inherits a grading from $K^{*}(M)$.

**Theorem 1**

$$H^{q}_{K}(\mathbb{R}^{n}) = \begin{cases} \mathbb{R}^{n+1} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Proof:**

We work in coordinates of $\mathbb{R}^{n}$ for which covariant differentiation with respect to $\nabla$ coincides with partial differentiation.

Let $q = 0$. $f \in K^{0}(\mathbb{R}^{n})$ is a cocycle iff $\partial_{\mu}\partial_{\nu}f = 0$ for all $\mu$ and $\nu$. In this case, $f$ has the form $f = a_{\mu}x^{\mu} + b$ for some real constants $a_{\mu}$ and $b$. Therefore $H_{K}^{0}(\mathbb{R}^{n}) = \mathbb{R}^{n+1}$.

Suppose $q = 1$. Let $T = T_{\mu\nu} \in K^{1}(\mathbb{R}^{n})$ be a cocycle. Then $\partial_{\mu}T_{\nu\lambda} = \partial_{\nu}T_{\mu\lambda}$. By the Poincaré Lemma, there exist functions $f_{\mu}$ defined on $\mathbb{R}^{n}$ such that $T_{\mu\nu} = \partial_{\mu}f_{\nu}$. Since $T$ is symmetric, $\partial_{\mu}f_{\nu} = \partial_{\nu}f_{\mu}$. Thus there exists $f \in K^{0}(\mathbb{R}^{n})$ such that $f_{\mu} = \partial_{\mu}f$, by the Poincaré Lemma again. Therefore $T_{\mu\nu} = \partial_{\mu}\partial_{\nu}f$ and so $T = d_{K}f$. This shows that $H_{K}^{1}(\mathbb{R}^{n}) = 0$. 

3
Now consider $q > 1$ and suppose $T = T_{\mu_1 \cdots \mu_q \nu} \in K^q(\mathbb{R}^n)$ is a cocycle:

$$\partial_{[\mu_1} T_{\mu_2 \cdots \mu_q] \nu} = 0$$

By the Poincaré Lemma, there exist $(q - 1)$-forms $A_\nu = A_{\mu_1 \cdots \mu_{q-1} \nu}$, skew-symmetric in the $\mu$ indices, such that $T_{\mu_1 \cdots \mu_q \nu} = \partial_{[\mu_1} A_{\mu_2 \cdots \mu_q] \nu}$. However, $A = A_\nu = A_{\mu_1 \cdots \mu_{q-1} \nu}$ is not necessarily an element of $K^{q-1}(\mathbb{R}^n)$ since the condition

$$s(A) = A_{[\mu_1 \cdots \mu_{q-1}] \nu} = 0$$

is not guaranteed. In order to remedy this we make use of the freedom available in the choice of $A$. Observe that

$$ds(A) = s(\nabla A) = s(d \nabla A) = s(T) = 0$$

Therefore, $s(A) = dB$ for some $(q - 1)$-form $B$, by applying the Poincaré Lemma once more. Define $A' = A - d \nabla B$. Then

$$s(A') = s(A) - s(d \nabla B) = s(A) - dB = 0$$

Furthermore,

$$d_K A' = d_K (A) - d_K (d \nabla B) = T - d^2 \nabla B = T$$

Therefore $A' \in K^{q-1}(\mathbb{R}^n)$ is a preimage of $T$ under $d_K$ and so $H_K^q(\mathbb{R}^n) = 0$.

Next, we seek to relate the $K^*(M)$ cohomology to the more familiar Čech cohomology. Let $\text{Aff}$ denote the sheaf on $M$ whose sections over an open subset $U \subseteq M$ is the kernel of the map $\nabla^2 : \Omega^0(U) \rightarrow \text{Sym}^2(U)$. That is, $\text{Aff}$ is the sheaf of local affine functions. $K^q$ shall denote the sheaf whose sections over $U$ is $K^q(U) = K^q(U)$. 

**Theorem 2** $H_K^n(M) \cong H^n(M, \text{Aff})$, for all $n \geq 0$.

**Proof:**

The demonstration is similar to the spectral sequence argument used to prove the Čech-de Rham isomorphism.
Let $\mathcal{U} = \{U_i : i \in I\}$ be a good cover of $M$. Consider the double complex $F = \bigoplus F^{p,q}$, where

$$F^{p,q} = \mathcal{C}^{p}(\mathcal{U}, \mathcal{K}^q) = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} K^q(U_{\alpha_0 \alpha_1 \cdots \alpha_p})$$

and

$$U_{\alpha_0 \alpha_1 \cdots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$$

$F$ is equipped with two differential operators, $d_K : F^{p,q} \rightarrow F^{p,q+1}$ and $\delta : F^{p,q} \rightarrow F^{p+1,q}$, the Čech coboundary operator defined by

$$(\delta \omega)_{\alpha_0 \alpha_1 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}}|_{U_{\alpha_0 \alpha_1 \cdots \alpha_{p+1}}}$$

for $\omega \in F^{p,q}$. Since the $\mathcal{K}^q$ are fine sheaves, the $E'_1$ term of the second spectral sequence is

$$E'^{p,q}_1 = H'^{p,q}_d = \begin{cases} K^q(M) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

The $E'_2$ term is

$$E'^{p,q}_2 = H'^{p,q}_d H_\delta = \begin{cases} H'^q_K(M) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $E'^{p,q}_2 = E'^{p,q}_\infty$,

$$H'^n_K(M) \cong H'^n_D(F)$$

where the right hand side is the cohomology of the double complex.

By the previous theorem, the first spectral sequence has $E_1$ term

$$E^{p,q}_1 = H^{p,q}_{d_K} = \begin{cases} \mathcal{C}^p(\mathcal{U}, \text{Aff}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

The $E_2$ term is

$$E^{p,q}_2 = H^{p,q}_d H_{d_K} = \begin{cases} H^p(\mathcal{U}, \text{Aff}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

$E^{p,q}_2 = E^{p,q}_\infty$ and so,

$$H^n(\mathcal{U}, \text{Aff}) \cong H^n_D(F)$$
From the isomorphisms (3) and (4),
\[ H^n_K(M) \cong H^n(U, \text{Aff}) \]

Taking the direct limit of \( H^n(U, \text{Aff}) \) yields the desired result.
\[ \square \]

3 Cohomology of the \( G^*(M) \) complex

A second cochain complex \((G^*(M), d_G)\) isomorphic to \((K^*(M), d_K)\) is introduced, with which we may give a cohomological description of certain spacetimes.

Let \( \tau \in S_{n+1} \) be the cyclic permutation \((1 \ 2 \ 3 \ \cdots \ n+1)\). The \( i \)th power \( \tau^i \) of \( \tau \) is the cycle given by \( \tau^i(k) = k+i \mod n+1 \), where the remainders upon division by \( n+1 \) are \( \{1, \ldots, n+1\} \). For instance, \( \tau^2(1) = 3, \tau^2(2) = 4, \ldots, \tau^2(n-1) = n+1, \tau^2(n) = 1 \) and \( \tau^2(n+1) = 2 \).

For \( n \geq 2 \), define the \( \Omega^0(M) \)-module \( G^n(M) \) to be the submodule of \( \Omega^{n-1}(M) \otimes \Omega^0(M) \text{Sym}^2(M) \) consisting of tensor fields \( S_{\mu_1 \cdots \mu_{n+1}} \) satisfying the condition
\[ \sum_{i=0}^{n} (-1)^{in} S_{\tau^i(1) \cdots \tau^i(n+1)} = 0 \]
Explicitly,
\[ S_{\mu_1 \cdots \mu_{n+1}} + (-1)^n S_{\mu_2 \cdots \mu_{n+1} \mu_1} + S_{\mu_3 \cdots \mu_{n+1} \mu_1 \mu_2} + (-1)^n S_{\mu_4 \cdots \mu_{n+1} \mu_1 \mu_2 \mu_3} + \]
\[ S_{\mu_5 \cdots \mu_{n+1} \mu_2 \mu_3 \mu_4} + \cdots + (-1)^n S_{\mu_{n+1} \mu_1 \cdots \mu_n} = 0 \quad (5) \]

Define \( \phi^n : K^n(M) \to G^n(M) \) by
\[ \phi^n(T)_{\mu_1 \cdots \mu_{n-1} \nu \lambda} = \frac{1}{2}(T_{\mu_1 \cdots \mu_{n-1} \nu \lambda} + T_{\mu_1 \cdots \mu_{n-1} \lambda \nu}) \]
for \( T = T_{\mu_1 \cdots \mu_{n-1} \nu \lambda} \in K^n(M) \). Let \( \psi^n : G^n(M) \to K^n(M) \) be defined by
\[ \psi^n(S)_{\mu_1 \cdots \mu_n \nu} = \frac{2n}{n+1} S_{[\mu_1 \cdots \mu_n] \nu} \]
for $S = S_{\mu_1 \cdots \mu_n \nu} \in G^n(M)$. Set $G^0(M) = K^0(M) = \Omega^0(M)$, $G^1(M) = K^1(M) = Sym^2(M)$ and define $\phi^n : K^n(M) \to G^n(M)$ and $\psi^n : G^n(M) \to K^n(M)$ to be the identity for $n = 0, 1$. Henceforth, we will drop the superscript ”$n$” for these mappings.

In what follows, it shall be shown that $\phi$ and $\psi$ are inverse to each other.

**Proposition 3** $\psi \circ \phi = Id_{K^n}$.

**Proof:**

Let $T = T_{\mu_1 \cdots \mu_{n+1}} \in K^n(M)$, for $n \geq 2$. Denote $S = \phi(T) \in G^n(M)$. Then

$$
\psi \circ \phi(T)_{\mu_1 \cdots \mu_{n+1}} = \psi(S)_{\mu_1 \cdots \mu_{n+1}} = \frac{2n}{n + 1} S_{[\mu_1 \cdots \mu_n] \mu_{n+1}} = \frac{2n}{n + 1} \frac{1}{n!} \sum_{\sigma \in S_n} sg(\sigma) S_{\sigma(1) \cdots \mu_{n+1}} = \frac{2n}{n + 1} \frac{1}{n!} \sum_{\sigma \in S_n} sg(\sigma) \frac{1}{2} \left( T_{\sigma(1) \cdots \mu_{n+1}} + T_{\mu(1) \cdots \mu_{n+1}} \right)
$$

$$
= \frac{n}{n + 1} \frac{1}{n!} \left( n! T_{\mu_1 \cdots \mu_{n+1}} + \frac{1}{n} P \right)
$$

where

$$
P = \sum_{i=1}^{n} (-1)^{n-i} \sum_{\sigma \in S_n} sg(\sigma) T_{\sigma(1) \cdots \mu_{n+1}}
$$

$$
= - \sum_{\tau \in S_{n+1}} sg(\tau) T_{\mu(1) \cdots \mu_{n+1}} + \sum_{\sigma \in S_n} sg(\sigma) T_{\mu(1) \cdots \mu_{n+1}}
$$

The last equality follows from the fact that if

$$
(\tau(1), ..., \tau(n+1)) = (\sigma(1), ..., \sigma(i-1), n+1, \sigma(i), ..., \sigma(n))
$$

for $\tau \in S_{n+1}$ and $\sigma \in S_n$ then $sg(\tau) = (-1)^{n-i+1} sg(\sigma) = -(-1)^{n-i} sg(\sigma)$. Hence

$$
P = -(n+1)! T_{[\mu_1 \cdots \mu_{n+1}]} + n! T_{\mu_1 \cdots \mu_{n+1}}
$$

$$
= n! T_{\mu_1 \cdots \mu_{n+1}}
$$

7
by property (2) for elements \( T \in K^n(M) \). Substituting this into the expression for \( \psi \circ \phi(T) \) obtained above gives

\[
\psi \circ \phi(T)_{\mu_1 \cdots \mu_{n+1}} = \frac{n+1}{n+1} \left( n!T_{\mu_1 \cdots \mu_{n+1}} + \frac{1}{n}n!T_{\mu_1 \cdots \mu_{n+1}} \right) = T_{\mu_1 \cdots \mu_{n+1}}
\]

\[\square\]

**Lemma 4** Let \( S = S_{\mu_1 \cdots \mu_n \nu} \in G^{n+1}(M) \), for \( n \geq 1 \). Then

\[
S_{\mu_1 \cdots \mu_n \nu} = \frac{n+1}{n+2} \left( S_{[\mu_1 \cdots \mu_n, \nu]} + S_{[\mu_1 \cdots \mu_n, \nu]} \right)
\]

**Proof:**

For notational simplicity we suppress the \( \mu \)s and write \( S = S_{1 \cdots n \nu} \). Let \( \tau \in S_n \) be the cyclic permutation \((1 \ 2 \ 3 \ \cdots \ n)\). Then

\[
2nS_{1 \cdots n \nu} = \sum_{i=0}^{n-1} (-1)^{i(n+1)} \left( S_{\nu \tau^i(1) \cdots \tau^i(n) \mu} + S_{\tau^i(1) \cdots \tau^i(n) \mu \nu} \right)
\]

Property (5) for elements of \( G^{n+1} \) takes the form

\[
S_{\tau^i(1) \cdots \tau^i(n) \mu \nu} = (-1)^n S_{\tau^i(2) \cdots \tau^i(n) \mu \nu} \tau^i(1) - S_{\tau^i(3) \cdots \tau^i(n) \mu \nu} \tau^i(1) \tau^i(2) + \cdots + (-1)^n S_{\nu \tau^i(1) \cdots \tau^i(n) \mu} \nu \tau^i(1) \tau^i(2) \cdots \tau^i(n) \nu \lambda \nu \lambda \lambda \lambda (6)
\]

Similarly, with the indices \( \nu \) and \( \lambda \) switched,

\[
S_{\nu \tau^i(1) \cdots \tau^i(n) \mu \nu} = (-1)^n S_{\nu \tau^i(2) \cdots \tau^i(n) \mu \nu} \nu \tau^i(1) - S_{\nu \tau^i(3) \cdots \tau^i(n) \mu \nu} \nu \tau^i(1) \nu \tau^i(2) + \cdots + (-1)^n S_{\nu \tau^i(3) \cdots \tau^i(n) \mu \nu} \nu \tau^i(1) \nu \tau^i(2) \cdots \nu \tau^i(n) \lambda \lambda \nu \lambda \lambda \lambda (7)
\]

In the sum \( S_{\tau^i(1) \cdots \tau^i(n) \mu \nu} + S_{\tau^i(1) \cdots \tau^i(n) \mu \nu} \) all the middle terms on the right hand side of (6) and (7) cancel leaving only the end terms. Thus

\[
2n(-1)^n S_{1 \cdots n \nu \lambda}
\]

\[= \sum_{i=0}^{n-1} (-1)^{i(n+1)} \left( S_{\nu \tau^i(1) \cdots \tau^i(n) \mu} + S_{\tau^i(1) \cdots \tau^i(n) \nu} \lambda \tau^i(1) \right) + (\nu \leftrightarrow \lambda)
\]

8
\[ \begin{aligned}
&= \sum_{i=0}^{n-1} (-1)^{i(n+1)} \left( S_{\nu \tau^i(1) \cdots \tau^i(n) \lambda} + (-1)^{n+1} S_{\nu \tau^i(2) \cdots \tau^i(n) \tau^i(1) \lambda} \right) + (\nu \leftrightarrow \lambda) \\
&= \sum_{i=0}^{n-1} (-1)^{i} S_{\nu \tau^i(1) \cdots \tau^i(n) \lambda} + \sum_{i=0}^{n-1} (-1)^{(i+1)}(-1)^{n+1} S_{\nu \tau^i(2) \cdots \tau^i(n) \tau^i(1) \lambda} + (\nu \leftrightarrow \lambda) \\
&= 2 \sum_{i=0}^{n-1} (-1)^{i} S_{\nu \tau^i(1) \cdots \tau^i(n) \lambda} + (\nu \leftrightarrow \lambda)
\end{aligned} \]

Dividing by 2 and then adding \(2(-1)^n S_{1 \cdots n\nu \lambda}\) to both sides gives

\[\begin{aligned}
(n+2)(-1)^n S_{1 \cdots n\nu \lambda} &= (-1)^n S_{1 \cdots n\nu \lambda} + \sum_{i=0}^{n-1} (-1)^i S_{\nu \tau^i(1) \cdots \tau^i(n) \lambda} + (\nu \leftrightarrow \lambda) \\
&= S_{\nu 1 \cdots n\lambda} + (-1)^n S_{1 \cdots n\nu \lambda} + (-1)^{n+1} S_{\nu 2 \cdots n1\lambda} + S_{\nu 3 \cdots n12\lambda} + \\
&\quad (-1)^{n+1} S_{\nu 4 \cdots n123\lambda} + \cdots + (-1)^{n+1} S_{\nu n12 \cdots n-1\lambda} + (\nu \leftrightarrow \lambda) \\
&= S_{\nu 1 \cdots n\lambda} + (-1)^n S_{1 \cdots n\nu \lambda} + S_{2 \cdots n\nu 1\lambda} + (-1)^n S_{3 \cdots n\nu 12\lambda} + \\
&\quad S_{4 \cdots n\nu 123\lambda} + \cdots + (-1)^n S_{n\nu 12 \cdots n-1\lambda} + (\nu \leftrightarrow \lambda)
\end{aligned}\]

Consider the identity

\[\begin{aligned}
S_{[1 \cdots n\nu] \lambda} &= \frac{1}{n+1} \left( S_{[1 \cdots n\nu] \lambda} + (-1)^n S_{[2 \cdots n\nu] 1\lambda} + \cdots + (-1)^n S_{[n1 \cdots n-1\nu] \lambda} \right) \\
&= \frac{1}{n+1} \left( S_{1 \cdots n\nu \lambda} + (-1)^n S_{2 \cdots n\nu 1\lambda} + \cdots + S_{n\nu 12 \cdots n-1\lambda} + (-1)^n S_{\nu 1 \cdots n\lambda} \right) \\
&= \frac{(-1)^n}{n+1} \left( S_{\nu 1 \cdots n\lambda} + (-1)^n S_{1 \cdots n\nu \lambda} + S_{2 \cdots n\nu 1\lambda} + \cdots + (-1)^n S_{n\nu 12 \cdots n-1\lambda} \right)
\end{aligned}\]

Substitute this into the above expression for \((n+2)(-1)^n S_{1 \cdots n\nu \lambda}\) to obtain

\[S_{1 \cdots n\nu \lambda} = \frac{n+1}{n+2} \left( S_{[1 \cdots n\nu] \lambda} + S_{[1 \cdots n\lambda] \nu} \right)\]

\[\square\]

**Proposition 5** \(\phi \circ \psi = Id_{G^*}\)
Proof:
Let $S = S_{\mu_1 \cdots \mu_n \nu \lambda} \in G^{n+1}(M)$, for $n \geq 1$. Then

$$
\phi \circ \psi(S)_{\mu_1 \cdots \mu_n \nu \lambda} = \frac{1}{2} (\psi(S)_{\mu_1 \cdots \mu_n \nu \lambda} + \psi(S)_{\mu_1 \cdots \mu_n \lambda \nu}) \\
= \frac{n+1}{n+2} (S_{[\mu_1 \cdots \mu_n \nu] \lambda} + S_{[\mu_1 \cdots \mu_n \lambda] \nu}) \\
= S_{\mu_1 \cdots \mu_n \nu \lambda}
$$

where the last equality follows from the lemma.

$\square$

**Corollary 6** $\phi$ and $\psi$ are inverse maps and $K^n(M)$ is canonically isomorphic to $G^n(M)$ for all $n$.

We define $d_G : G^*(M) \rightarrow G^*(M)$ by

$$
d_G = \phi \circ d_K \circ \psi
$$

Equivalently, $d_G$ is the unique map that makes the following diagram commutative

\[\begin{array}{cccc}
\cdots & \xrightarrow{d_K} & K^n(M) & \xrightarrow{d_K} & K^{n+1}(M) & \xrightarrow{d_K} & \cdots \\
\psi & \uparrow & \phi & \uparrow & \phi & \uparrow & \phi \\
\cdots & \xrightarrow{d_G} & G^n(M) & \xrightarrow{d_G} & G^{n+1}(M) & \xrightarrow{d_G} & \cdots
\end{array}\]

In particular, if $S = S_{\mu \nu} \in G^1(M)$ then

$$
(d_G S)_{\mu \nu \lambda} = \frac{1}{2} \nabla_{\mu} S_{\nu \lambda} - \frac{1}{4} (\nabla_{\nu} S_{\mu \lambda} + \nabla_{\lambda} S_{\mu \nu})
$$

$(K^*(M), d_K)$ and $(G^*(M), d_G)$ are naturally isomorphic cochain complexes with inverse cochain maps $\phi$ and $\psi$. A trivial consequence is

**Theorem 7** $H^*_G(M) \cong H^*_K(M)$
4 MP Spacetimes

We now show that the symmetric-free part of the Levi-Civita connection of MP spacetimes and the Schwarzschild solution may be represented by means of a gravitational potential in the \((G^*(M), d_G)\) complex. As mentioned in the introduction, since the Levi-Civita connection is not a tensor this procedure is coordinate dependent and has the effect of selecting, within the full diffeomorphism group, those coordinate systems for which a gravitational potential exists, if in fact it does. The gravitational field is then viewed as a field propagating on a flat background structure defined by such a preferred coordinate system.

The MP spacetimes are a class of analytic solutions to the field equations of general relativity in an electromagnetic field given by

\[
\begin{align*}
    ds^2 &= H^{-2} dt^2 - H^2 d\vec{x}^2, \\
    A_\mu &= \delta_\mu t \alpha(H^{-1} - 1)
\end{align*}
\]

where \(\vec{x} = (x^1, x^2, x^3)\), \(\alpha = \pm 2\) and \(H = H(\vec{x})\) is harmonic in the \(\vec{x}\) variables. For a charge \(q\), \(\alpha = -2 \text{sign}(q)\) and \(H = 1 + \frac{G_N M}{|\vec{x}|}\) this gives the Extreme Reissner-Nordström black hole.

The Schwarzschild solution may also be expressed in isotropic coordinates as

\[
    ds^2 = \left(1 + \frac{\omega/4}{\rho}\right)^2 \left(1 - \frac{\omega/4}{\rho}\right)^{-2} dt^2 - \left(1 - \frac{\omega/4}{\rho}\right)^4 d\vec{x}^2
\]

where

\[
    r = |\vec{x}| = \frac{(\rho - \frac{\omega}{4})^2}{\rho}
\]

More generally, consider any metric in isotropic coordinates of the form

\[
    ds^2 = f(H) dt^2 - g(H) d\vec{x}^2
\]

with Levi-Civita connection \(\Gamma^\mu_{\nu\lambda}\), where \(f\) and \(g\) are arbitrary smooth functions of a single variable and \(H = H(\vec{x})\). We associate to the coordinates a metric

\[
    \eta = \text{diag}(+1, -1, -1, -1)
\]
with which indices are raised and lowered. Define the symmetric-free part $F_{\mu\nu\lambda}$ of 
$\Gamma_{\mu\nu\lambda} = \eta_{\mu\tau}\Gamma^{\tau}_{\nu\lambda}$ by 
$$F_{\mu\nu\lambda} = \Gamma_{\mu\nu\lambda} - \Gamma_{(\mu\nu\lambda)}$$
where
$$\Gamma_{(\mu_1\mu_2\mu_3)} = \frac{1}{3!} \sum_{\sigma \in S_3} \Gamma_{\mu_\sigma(1)\mu_\sigma(2)\mu_\sigma(3)}$$

Then $F_{(\mu\nu\lambda)} = 0$ and since $F_{\mu\nu\lambda}$ is symmetric in the $\nu, \lambda$ indices,
$$F_{\mu\nu\lambda} + F_{\nu\lambda\mu} + F_{\lambda\mu\nu} = 0$$
This is property (5) and so $F = F_{\mu\nu\lambda} \in G^2(M)$.

The non-zero Christoffel symbols for $ds^2 = f(H)dt^2 - g(H)d\vec{x}^2$ are
$$\Gamma_{tjt} = \Gamma_{ttj} = \frac{1}{2} \partial_j \log f(H) \quad 1 \leq j \leq 3$$
$$\Gamma_{jtt} = -\frac{1}{2} \partial_j f(H) / 2g \quad 1 \leq j \leq 3$$
$$\Gamma_{jkk} = \Gamma_{kkj} = -\frac{1}{4} \partial_j \log g(H) \quad 1 \leq j \neq k \leq 3$$
$$\Gamma_{jkk} = \frac{1}{2} \partial_j v(H) \quad 1 \leq j \neq k \leq 3$$

Here we have denoted $x^j$ and $x^k$ simply by $j$ and $k$, respectively.

Let $u$ be a solution to
$$u' = -\frac{2f'}{3f} \frac{2f'}{3g}$$
and set
$$v = \frac{4}{3} \log g$$

The symmetric-free part $F_{\mu\nu\lambda}$ of $\Gamma_{\mu\nu\lambda}$ is
$$F_{tjt} = F_{ttj} = -\frac{1}{4} \partial_j u(H) \quad 1 \leq j \leq 3$$
$$F_{jtt} = \frac{1}{2} \partial_j u(H) \quad 1 \leq j \leq 3$$
$$F_{jkk} = F_{kkj} = -\frac{1}{4} \partial_j v(H) \quad 1 \leq j \neq k \leq 3$$
$$F_{jkk} = \frac{1}{2} \partial_j v(H) \quad 1 \leq j \neq k \leq 3$$
and all other terms equal to zero. Let $A = A_{\mu\nu} \in G^1(M)$ be defined by
$$A = u(H)dt^2 + v(H)d\vec{x}^2$$
It is straightforward to verify that

$$F = d_G A$$

That is, $F$ is the coboundary of the gravitational potential $A$ in the $G^*(M)$ complex.
References

[1] Bott R., Tu, L.: Differential Forms in Algebraic Topology. Springer 1982

[2] Bredon, G.E.: Sheaf Theory. Mc-Graw-Hill 1967

[3] Leray, J.: Structure de l’anneau d’homologie d’une représentation. C. R. Acad. Sci. Paris 222, 1419-1422 (1946)

[4] Majumdar, S.D.: A class of exact solutions of Einstein’s field equations. Phys. Rev 72, 390-398 (1947)

[5] Massey, W.: Exact couples in algebraic topology. Annals Math. 56, 363-396 (1952)

[6] McCleary, J.: A User’s Guide to Spectral Sequences. Cambridge University Press, 2000

[7] Nördstrom, G.: On the Energy of the Gravitational Field in Einstein’s Theory. Proc. Kon. Ned. Akad. Wet. 20, 1238-1245 (1918)

[8] Ortin, T.: Gravity and Strings. Cambridge University Press, 2004

[9] Papapetrou, A.: A static solution of the equations of the gravitational field for an arbitrary charge distribution. Proc. Royal Irish Acad. 51, 191-205 (1947)

[10] Poisson, E.: A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge University Press, 2004

[11] Reissner, H.: Über die Eigengravitation des elektrischen Felds nach der Einsteinschen Theorie. Ann. Physik 50, 106-120 (1916)

[12] Schwarzschild, K.: Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. Sitzungsber. Deutsch. Akad. Wiss. Berlin, Kl. Math.-Phys. Technik 189-196 (1916)
[13] Serre, J.-P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier \textbf{6}, 1-42 (1956)

[14] Switzer, R.M.: Algebraic Topology: Homology and Homotopy. Springer-Verlag, 1975

[15] Wald, R.M.: General Relativity. University of Chicago Press, 1984

[16] Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups. Springer, 1983

[17] Weibel, C.: An introduction to homological algebra. Cambridge University Press, 1994