STRICT SOLUTIONS TO STOCHASTIC SEMILINEAR
EVOLUTION EQUATIONS IN M-TYPE 2 BANACH SPACES

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Abstract. This paper is devoted to studying stochastic semilinear evolution equations in Banach spaces of M-type 2. First, we prove existence, uniqueness and regularity of strict solutions. Then, we give an application to stochastic partial differential equations.

1. Introduction. We study the Cauchy problem for a stochastic parabolic semilinear evolution equation

\[
\begin{cases}
    dX + AXdt = [F_1(t) + F_2(X)]dt + G(t)dW(t), & 0 < t \leq T, \\
    X(0) = \xi
\end{cases}
\]

in a complex separable Banach space \(E\) of M-type 2 with norm \(\|\cdot\|\) and the Borel \(\sigma\)-field \(B(E)\). Here,

(i) \(A\) is a densely defined, closed linear operator in \(E\).
(ii) \(W\) is an \(H\)-valued cylindrical Wiener process, which is defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). And, \(H\) is a separable Hilbert space.
(iii) \(F_1\) is a measurable process from \((\Omega_T, \mathcal{P}_T)\) to \((E, B(E))\), where \(\Omega_T = [0, T] \times \Omega\) and \(\mathcal{P}_T\) is the predictable \(\sigma\)-field on \(\Omega_T\).
(iv) \(F_2\) is a measurable function from \((\Omega \times E, \mathcal{F} \times B(E))\) to \((E, B(E))\).
(v) \(G \in L^2([0, T])\), where \(L^2([0, T])\) is given in Definition 2.8 in the next section. Roughly speaking, \(G(t), 0 \leq t \leq T, \) are \(\gamma\)-radonifying operators from \(H\) to \(E\).
(vi) \(\xi\) is an \(E\)-valued \(\mathcal{F}_0\)-measurable random variable.

Stochastic parabolic evolution equations have already been studied since 1970s. The main interest was to construct unique mild solutions, see Brzeźniak [1], Da Prato-Zabczyk [4], van Neerven et al. [15], and references therein. Some researchers, however, were devoted to constructing stronger solutions.

In [4], Da Prato-Zabczyk studied a linear case of (1), say

\[
\begin{cases}
    dX + AXdt = F(t)dt + IdW(t), & 0 < t \leq T, \\
    X(0) = \xi
\end{cases}
\]
in a Hilbert space $H \equiv E$. Here, $W$ is a $Q$-Wiener process on $H$ with a symmetric nonnegative nuclear operator $Q$ in $L(E)$; $F$ is a progressive measurable and integrable process on $[0, T]$; and $I$ is the identity operator in $E$. Under the conditions:

(i) The trace of $Q$ is finite,

(ii) $A^\beta Q^{1/2}$ for some $\frac{1}{2} < \beta \leq 1$ is a Hilbert-Schmidt operator,

existence of strong solutions to (2) has been shown. Here, strong solution means that $X(t) \in D(A), \mathbb{P}_T-$a.s., $\int_0^T \|AX(s)\| ds < \infty, \mathbb{P}-$a.s., and

$$X(t) = \xi + \int_0^t [AX(s) + F(s)] ds + IW(t), \quad t \in [0, T].$$

($\mathbb{P}_T$ is the product of the Lebesgue measure on $[0, T]$ and $\mathbb{P}$.) Clearly, the condition (ii) is restrictive. When $W$ is a standard Brownian motion in $\mathbb{R}^d$, $Q$ becomes the identity matrix of size $d$. Then, (ii) implies that $A$ is a bounded linear operator.

Brzeźniak [1] treated a stochastic linear evolution equation of the form

$$\begin{cases}
    dX + AX dt = F(t) dt + \sum_{j=1}^d B_j X dw^j(t), & 0 < t \leq T, \\
    X(0) = \xi
\end{cases}$$

in an M-type 2 separable Banach space $E$. Here, $B_j$ are unbounded linear operators in $E$; and $w^j$ are independent standard Brownian motions. It has been assumed that

$$\sum_{j=1}^d \|B_j u\|^2_{D_A(\frac{1}{2}, 2)} \leq C_1 \|u\|^2_{D(A)} + C_2 \|u\|^2_{D_A(\frac{1}{2}, 2)}, \quad u \in D(A),$$

where $D_A(\frac{1}{2}, 2) = \{u \in E; \int_0^\infty \|A e^{tA} u\|^2 dt < \infty\}$ and that

$$\xi \in L^2(\Omega; D_A(\frac{1}{2}, 2)).$$

The author then proved existence of a unique solution to (3) in the space:

$$X \in L^2([0, T] \times \Omega; D(A)) \cap C([0, T]; L^2(\Omega; D_A(\frac{1}{2}, 2))).$$

In [11, 13], we have constructed strict solutions to stochastic (autonomous and non-autonomous) linear evolution equations. In addition, these strict solutions possess maximal regularity. Some implications of these results will be presented in the next section.

This paper constructs strict solutions possessing strong regularities to the stochastic semilinear equation (1). Our results can be applied to solve a wide class of stochastic partial differential equations. In order to obtain these results, we use an approach which generally includes the use of Yoshida approximation, analytical semigroups, and fixed point arguments to prove the existence of solutions, and the use of Kolmogorov continuity criteria to show their regularity.

The organization of the paper is as follows. Section 2 is preliminary. In this section, we first review the following concepts: weighted Hölder continuous function spaces, sectorial operators and analytical semigroups, cylindrical Wiener processes, stochastic integrals in Banach spaces of martingale type 2, and Kolmogorov continuity criteria. Then, we introduce the concepts of strict and mild solutions, and some results for linear evolution equations.

Section 3 presents our main results. Existence, uniqueness and regularity of both mild and strict solutions are proved in this section. In Section 4, we present an example of stochastic partial differential equations to illustrate our abstract results.
2. Preliminary. This section reviews some basic notions as well as some implications of our previous results on linear evolution equations.

2.1. Function spaces. For $0 < \sigma < \beta < 1$, denote by $\mathcal{F}^{\beta,\sigma}((0,T];E)$ the space of all $E$-valued continuous functions $F$ on $(0,T]$ with the following three properties:

(i) $t^{1-\beta} F(t)$ has a limit as $t \to 0$.

(ii) $F$ is Hölder continuous with exponent $\sigma$ and weight $s^{1-\beta+\sigma}$, i.e.

$$
\sup_{0<s<t \leq T} \frac{s^{1-\beta+\sigma} \|F(t) - F(s)\|}{(t-s)^\sigma} = \sup_{0<t \leq T} \sup_{0<s<t} \frac{s^{1-\beta+\sigma} \|F(t) - F(s)\|}{(t-s)^\sigma} < \infty.
$$

(iii) $\lim_{t \to 0} w_F(t) = 0$, where $w_F(t) = \sup_{0<s<t} \frac{s^{1-\beta+\sigma} \|F(t) - F(s)\|}{(t-s)^\sigma}$.

It is easy to see that $\mathcal{F}^{\beta,\sigma}((0,T];E)$ is a Banach space with norm

$$
\|F\|_{\mathcal{F}^{\beta,\sigma}(E)} = \sup_{0<t \leq T} t^{1-\beta} \|F(t)\| + \sup_{0<s<t \leq T} \frac{s^{1-\beta+\sigma} \|F(t) - F(s)\|}{(t-s)^\sigma}.
$$

The space $\mathcal{F}^{\beta,\sigma}((0,T];E)$ is called a weighted Hölder continuous function space ([16]). Clearly, for $F \in \mathcal{F}^{\beta,\sigma}((0,T];E)$,

$$
\|F(t)\| \leq \|F\|_{\mathcal{F}^{\beta,\sigma}(E)} t^{\beta-1}, \quad 0 < t \leq T.
$$

Remark 1. (a) The space $\mathcal{F}^{\beta,\sigma}((0,T];E)$ is not a trivial space. The function $F$ defined by $F(t) = t^{\beta-1} f(t), 0 < t \leq T$, belongs to this space, where $f$ is any function in $C^\sigma([0,T];E)$ such that $f(0) = 0$, where $C^\sigma$ is the space of $\sigma$-Hölder continuous functions.

(b) The space $\mathcal{F}^{\beta,\sigma}((a,b];E), 0 \leq a < b < \infty$, is defined in a similar way. For more details, see [16].

2.2. Sectorial operators and analytical semigroups. A densely defined, closed linear operator $A$ is said to be sectorial if it satisfies two conditions:

(i) The spectrum $\sigma(A)$ of $A$ is contained in an open sectorial domain $\Sigma_\varpi$:

$$
\sigma(A) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \varpi\}, \quad 0 < \varpi < \frac{\pi}{2}.
$$

(ii) The resolvent of $A$ satisfies the estimate

$$
\|(\lambda - A)^{-1}\| \leq \frac{M_\varpi}{|\lambda|}, \quad \lambda \notin \Sigma_\varpi
$$

with some $M_\varpi > 0$ depending only on the angle $\varpi$.

Theorem 2.1 ([16]). Let $A$ be a sectorial operator. Then, $(-A)$ generates an analytical semigroup $S(t) = e^{-tA}, 0 \leq t < \infty$, having the following properties:

(i) For any $\theta \geq 0$, there exists $\iota_\theta > 0$ such that

$$
\|A^\theta S(t)\| \leq \iota_\theta t^{-\theta}, \quad 0 < t < \infty.
$$

In particular, there exists $\nu > 0$ such that

$$
\|S(t)\| \leq \iota_\nu t^{-\nu}, \quad 0 < t < \infty.
$$
(ii) For any $0 < \theta \leq 1$, 
$$\|S(t) - I - \theta A^{-\theta}\| \leq \frac{\theta^{-\theta} t^\theta}{\theta}, \quad 0 < t < \infty. \quad (7)$$

Throughout this paper, if not specified, we always assume that $A$ is a sectorial operator. Furthermore, whenever we mention $\iota_\theta, \nu$ or $S(t)$, we mean they come from Theorem 2.1.

2.3. **Cylindrical Wiener processes.** Let us review a central notion to the theory of stochastic evolution equations, say cylindrical Wiener processes on a separable Hilbert space $H$. The following definition is given in [3].

**Definition 2.2.** Let $Q$ be a symmetric nonnegative nuclear operator in $L(H)$. An $H$-valued stochastic process $W$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is called a $Q$-Wiener process if it satisfies the conditions:

(i) $W(0) = 0$ a.s.

(ii) $W$ has continuous sample paths.

(iii) $W$ has independent increments.

(iv) The law of $W(t) - W(s), 0 < s \leq t$, is a Gaussian measure on $H$ with mean 0 and covariance $Q$.

Note that a nuclear operator is a bounded linear operator with finite trace. Here, the trace of $Q$ is defined by

$$\text{Tr}(Q) = \sum_{i=1}^{\infty} \langle Q e_i, e_i \rangle,$$

where $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal basis in $H$.

Let’s fix a larger Hilbert space $H_1$ such that $H$ is embedded continuously into $H_1$ and the embedding $J: H \to H_1$ is Hilbert-Schmidt (i.e. $\sum_{i=1}^{\infty} \|Je_i\|_{H_1}^2 < \infty$). For example ([6]), one takes $H_1$ to be the closure of $H$ under the norm

$$\|h\|_{H_1} = \left( \sum_{n=1}^{\infty} \frac{\langle h, e_n \rangle_{H_1}^2}{n^2} \right)^{\frac{1}{2}}.$$

For any $h_1 \in H_1$,

$$\langle JJ^* e_m, h_1 \rangle_{H_1} = \langle J^* e_m, J^* h_1 \rangle_H = \sum_{k=1}^{\infty} e_k \langle J^* e_m, e_k \rangle_H \sum_{k=1}^{\infty} e_k \langle J^* h_1, e_k \rangle_H$$

$$= \sum_{k=1}^{\infty} \langle J^* e_m, e_k \rangle_H \langle J^* h_1, e_k \rangle_H = \sum_{k=1}^{\infty} \langle e_m, J e_k \rangle_{H_1} \langle h_1, e_k \rangle_{H_1}$$

$$= \sum_{k=1}^{\infty} \langle e_m, e_k \rangle_{H_1} \langle h_1, e_k \rangle_{H_1} = \|e_m\|_{H_1}^2 \langle h_1, e_m \rangle_{H_1}$$

$$= \frac{1}{m^2} \langle h_1, e_m \rangle_{H_1}, \quad m = 1, 2, \ldots.$$

Thereby, $JJ^* e_m = \frac{1}{m^2} e_m$ for $m = 1, 2, \ldots$. Therefore,

$$\text{Tr}(JJ^*) = \sum_{m=1}^{\infty} \langle JJ^* e_m, e_m \rangle = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$
Thus, $JJ^*$ is a nuclear operator. From this operator, one can define a cylindrical Wiener process ([4]).

**Definition 2.3.** An $H_1$-valued $JJ^*$-Wiener process is called a cylindrical Wiener process on $H$.

2.4. **Stochastic integrals in Banach spaces of M-type 2.**

**Definition 2.4 ([9]).** A Banach space $E$ is said to be of martingale type 2 (or M-type 2), if there is a constant $c(E)$ such that for all $E$-valued martingales $\{M_n\}_n$, it holds true that

$$\sup_n \mathbb{E}\|M_n\|^2 \leq c(E) \sum_{n \geq 0} \mathbb{E}\|M_n - M_{n-1}\|^2,$$

where $M_{-1} = 0$.

It is known that Hilbert spaces are of M-type 2 and that, when $2 \leq p < \infty$, the $L^p$ space is the same.

When $E$ is of M-type 2, stochastic integrals with respect to cylindrical Wiener processes can be constructed in a quite similar way as for the usual Itô integrals. There are several literature dealing with this subject ([1, 2, 5, 7, 14]). Let us explain the construction.

Let $W$ be a cylindrical Wiener process on a separable Hilbert space $H$, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

**Definition 2.5.**

(i) A function $\varphi : [0, T] \to E$ is said to be strongly measurable if it is the pointwise limit of a sequence of simple functions.

(ii) A function $\phi : [0, T] \to L(H; E)$ is said to be $H$-strongly measurable if $\phi(\cdot)h : [0, T] \to E$ is strongly measurable for any $h \in H$.

**Definition 2.6 ($\gamma$-radonifying operators).**

(i) Let $\{\gamma_n\}_{n=1}^\infty$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $H$. A bounded linear operator $\phi$ from $H$ to $E$ is called a $\gamma$-radonifying operator if the Gaussian series $\sum_{n=1}^\infty \gamma_n e_n e_n \in L^2(\Omega', E)$.

(ii) The set of all $\gamma$-radonifying operators is denoted by $\gamma(H; E)$.

According to Definition 2.6, a natural norm in $\gamma(H; E)$ is defined by

$$\|\phi\|_{\gamma(H; E)} = \left[ \mathbb{E}\left( \sum_{n=1}^\infty \gamma_n \phi e_n \right)^2 \right]^{1/2}.$$

It is known that the norm is independent of the orthonormal basis $\{e_n\}_{n=1}^\infty$ and the Gaussian sequence $\{\gamma_n\}_{n=1}^\infty$. Furthermore, the normed space $(\gamma(H; E), \| \cdot \|_{\gamma(H; E)})$ is complete ([14]). The following lemma is used very often in this paper.

**Lemma 2.7.** Let $\phi_1 \in L(E)$ and $\phi_2 \in \gamma(H; E)$. Then, $\phi_1 \phi_2 \in \gamma(H; E)$ and

$$\|\phi_1 \phi_2\|_{\gamma(H; E)} \leq \|\phi_1\|_{L(E)} \|\phi_2\|_{\gamma(H; E)}.$$

**Definition 2.8.** Denote by $\mathcal{I}^2([0, T])$ the class of all adapted processes $\phi : [0, T] \to \gamma(H; E)$ in $L^2((0, T); \gamma(H; E))$, which are $H$-strongly measurable.

For every $\phi \in \mathcal{I}^2([0, T])$, the stochastic integral $\int_0^T \phi(t)dW(t)$ is defined as a limit of integrals of adapted step processes. By a localization argument one can extend stochastic integrals to the class $\mathcal{I}((0, T))$ of all $H$-strongly measurable and adapted processes $\phi : [0, T] \to \gamma(H; E)$ which are in $L((0, T); \gamma(H; E))$ ([14]).
Theorem 2.9. Let $E$ be a Banach space of M-type 2. Let $W$ be a cylindrical Wiener process on a separable Hilbert space $H$. Then, there exists $c(E) > 0$ depending only on $E$ such that
\[
\mathbb{E}\left\| \int_0^T \phi(t) dW(t) \right\|^2 \leq c(E) \| \phi \|^2_{L^2(0,T; \gamma(H;E))}, \quad \phi \in \mathcal{I}^2([0,T]),
\]
here $\| \phi \|^2_{L^2(0,T; \gamma(H;E))} = \int_0^T \| \phi(s) \|^2_{\gamma(H;E)} ds$. In addition, for every $\phi \in \mathcal{I}([0,T])$, the process $\{\int_0^t \phi(s)dW(s), 0 \leq t \leq T\}$ is an $E$-valued continuous local martingale and a Gaussian process.

For the proof, see, e.g., [2, 8].

Lemma 2.10. Let $B$ be a closed linear operator in $E$ and $\phi : [0,T] \to \gamma(H;E)$. If $\phi$ and $B\phi$ belong to $\mathcal{I}^2([0,T])$, then
\[
B \int_0^T \phi(t) dW(t) = \int_0^T B\phi(t) dW(t) \quad \text{a.s.}
\]
The proof of Lemma 2.10 is very similar to one in [4]. So, we omit it.

Theorem 2.11. Let $\zeta$ be an $E$-valued stochastic process on $[0,T]$. Assume that for some $c > 0$ and $\epsilon_i > 0$ ($i = 1, 2$),
\[
\mathbb{E}\|\zeta(t) - \zeta(s)\|^{\epsilon_1} \leq c |t - s|^{1+\epsilon_2}, \quad 0 \leq s, t \leq T.
\]
Then, $\zeta$ has a version whose $\mathbb{P}$-almost all trajectories are Hölder continuous functions with an arbitrarily smaller exponent than $\frac{\epsilon_2}{\epsilon_1}$.

When $\zeta$ is a Gaussian process, one can weaken the condition (8).

Theorem 2.12. Let $\zeta$ be an $E$-valued Gaussian process on $[0,T]$ such that $\mathbb{E}\zeta(t) = 0$ for $t \geq 0$. Assume that for some $c > 0$ and $0 < \epsilon \leq 1$,\[
\mathbb{E}\|\zeta(t) - \zeta(s)\|^{\epsilon} \leq c (t - s)^{\epsilon}, \quad 0 \leq s, t \leq T.
\]
Then, there exists a modification of $\zeta$ whose $\mathbb{P}$-almost all trajectories are Hölder continuous functions with an arbitrarily smaller exponent than $\frac{\epsilon}{2}$.

For the proofs of Theorems 2.11 and 2.12, see, e.g., [4].

2.5. Strict and mild solutions.

Definition 2.13. An adapted $E$-valued continuous process $X$ on $[0,T]$ is called a strict solution of (1) if for $0 < t \leq T$, a.s.
\[
\int_0^T \| F_2(X(s)) \| ds < \infty,
\]
\[
X(t) \in \mathcal{D}(A) \quad \text{and} \quad \left\| \int_0^t A X(s) ds \right\| < \infty,
\]
and
\[
X(t) = \xi - \int_0^t A X(s) ds + \int_0^t \left[ F_1(s) + F_2(X(s)) \right] ds + \int_0^t G(s)dW(s).
\]
Definition 2.14. An adapted $E$-valued continuous process $X$ on $[0,T]$ is called a mild solution of (1) if for $0 < t \leq T$, a.s.
\[ \int_0^T \| S(t-s)F_2(X(s)) \| ds < \infty, \]
and
\[ X(t) = S(t)\xi + \int_0^t S(t-s)[F_1(s) + F_2(X(s))] ds + \int_0^t S(t-s)G(s)dW(s). \]

A strict (mild) solution $X$ on $[0,T]$ is said to be unique if any other strict (mild) solution $\bar{X}$ on $[0,T]$ is indistinguishable from it, i.e.
\[ P\{X(t) = \bar{X}(t) \text{ for every } t \in [0,T]\} = 1. \]

Remark 2. It is easy to see that a strict solution is a mild solution. The inverse is, however, not true in general.

2.6. Linear evolution equations. In this section, we introduce some results for the linear case of (1), say $F_2 \equiv 0$ in $E$. These results can be easily proved by using the method in [10, 11, 13]. Notice that the method includes the use of Yoshida approximation and analytical semigroups to prove the existence of solutions, and the use of Kolmogorov continuity criteria (Theorems 2.11 and 2.12) to show their regularity. To keep the length of this paper reasonably, we omit their proofs.

Let us rewrite (1) as
\[
\begin{cases}
    dX + AXdt = F_1(t)dt + G(t)dW(t), & 0 < t \leq T, \\
    X(0) = \xi.
\end{cases}
\]

Assume that $F_1$ belongs to a weighted Hölder continuous function space:
\[
(F1) \quad \text{For some } 0 < \sigma < \beta < \frac{1}{2}, \quad F_1 \in \mathcal{F}^{\beta,\sigma}((0,T]; E) \text{ a.s. and } E\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 < \infty.
\]

And $G$ satisfies one of the following conditions:
\[
(Ga) \quad \text{For some } 1 - \beta < \delta \leq 1, \quad A^\delta G \in \mathcal{F}^{\beta + \frac{1}{2},\sigma}((0,T]; \gamma(H;E)).
\]
\[
(Gb) \quad G \in \mathcal{F}^{\beta + \frac{1}{2},\sigma}((0,T]; \gamma(H;E)).
\]

The first result is about existence and uniqueness of strict solutions.

Theorem 2.15. Let $(F1)$ and $(Ga)$ be satisfied. Then, there exists a unique strict solution $X$ of (9). Furthermore, $X$ possesses the regularity
\[ AX \in C((0,T]; E) \text{ a.s.} \]
and satisfies the estimate
\[
E\|X(t)\|^2 + t^2E\|AX(t)\|^2 \leq C[E\|\xi\|^2 + E\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 t^{2\beta} + \|A^\delta G\|_{\mathcal{F}^{\beta + \frac{1}{2},\sigma}(\gamma(H;E))}^2 \times \{t^{2\beta} + t^{2(\beta + \delta)}\}^\gamma, \quad 0 \leq t \leq T,
\]
where $C > 0$ is some constant depending only on the exponents.

The second result is about the regularity of strict solutions for more regular initial value, say $\xi \in D(A^\beta)$. 

Theorem 2.16. Let (F1) and (Ga) be satisfied. Assume that $\xi \in \mathcal{D}(A^\beta)$ a.s. Then, the strict solution $X$ of (9) has the space-time regularity
\[ X \in C([0,T]; \mathcal{D}(A^\beta)) \cap C^{\gamma_1}([0,T]; E) \quad \text{a.s.,} \]
\[ AX \in C^{\gamma_2}([\epsilon,T]; E) \quad \text{a.s.} \]
for any $0 < \gamma_1 < \beta$, $0 < \gamma_2 < \beta + \delta - 1$, $0 < \gamma_2 \leq \sigma$ and $0 < \epsilon \leq T$. In addition, $X$ satisfies the estimate
\[ \mathbb{E}\|A^\beta X(t)\|^2 \leq C[e^{-2\nu t}\mathbb{E}\|A^\beta \xi\|^2 + \mathbb{E}\|F_1\|_{p,\sigma,\gamma(H;E)}^2 \gamma(H;E)] + \|A^\delta G\|_{p,\alpha,\gamma(H;E)}^2 \gamma(H;E)], \quad 0 \leq t \leq T \]
with some $C > 0$ depending only on the exponents.

The final result is about the regularity of mild solutions of (9). For mild solutions, we only need the time regularity (Gb) of $G$.

Theorem 2.17. Let (F1) and (Gb) be satisfied. Suppose further that $\xi \in \mathcal{D}(A^\beta)$ a.s. Then, there exists a unique mild solution $X$ of (9) possessing the regularity
\[ X \in C^{\alpha}([0,T]; E) \quad \text{a.s.,} \quad 0 \leq \alpha < \beta, \]
and
(i) When $\beta \geq \frac{1}{4}$, for any $0 < \epsilon < T$, $\frac{1}{4} \leq \theta \leq \beta$, $0 \leq \gamma < \frac{1}{2} - \theta$ and $\gamma \leq \sigma$,
\[ A^\theta X \in C^{\gamma}([\epsilon,T]; E) \quad \text{a.s.} \]
(ii) When $\beta < \frac{1}{4}$, for any $0 < \epsilon < T$, $0 \leq \theta \leq \beta$, $0 \leq \gamma < \theta$ and $\gamma \leq \sigma$,
\[ A^\theta X \in C^{\gamma}([\epsilon,T]; E) \quad \text{a.s.} \]
In addition, $X$ satisfies the estimate
\[ \mathbb{E}\|A^\beta X(t)\|^2 \leq C[e^{-2\nu t}\mathbb{E}\|A^\beta \xi\|^2 + \mathbb{E}\|F_1\|_{p,\sigma,\gamma(H;E)}^2 \gamma(H;E)] + \|G\|_{p,\alpha,\gamma(H;E)}^2 \gamma(H;E)], \quad 0 \leq t \leq T. \]
Here $C$ is some positive constant depending only on $\nu_\theta (\theta \geq 0)$ in (5) and the exponents; and $\nu$ is defined by (6).

3. Main results. In this section, we construct strict solutions to (1). First, we prove existence of a unique local mild solution to (1), provided that either (Ga) or (Gb) takes place. Then, we show that the mild solution is a strict solution under certain conditions.

3.1. Mild solutions. First, we consider the case where (Ga) holds true. Let’s fix $\eta, \beta, \sigma$ such that
\[ \left\{ \begin{array}{l}
\max\{0, 2\eta - \frac{1}{2}\} < \beta < \eta < \frac{1}{2}, \\
0 < \sigma < \beta < \frac{1}{2},
\end{array} \right. \]
Assume that
(F2a) $F_2 : \mathcal{D}(A^\eta) \to E$ and satisfies a Lipschitz condition of the form
\[ \|F_2(x) - F_2(y)\| \leq c_{F_2}\|A^\eta(x - y)\| \quad \text{a.s.,} \quad x, y \in \mathcal{D}(A^\eta), \]
where $c_{F_2} > 0$ is some constant.
Theorem 3.1. Let (F1), (F2a) and (Ga) be satisfied. Assume that \( \xi \in D(A^\beta) \) a.s. such that \( \mathbb{E}\|A^\beta\xi\|^2 < \infty \). Then, (1) possesses a unique local mild solution \( X \) in the function space:

\[
X \in C([0,T_{loc}]; D(A^\alpha)) \cap C([0,T_{loc}]; D(A^\beta)) \quad \text{a.s.}
\]

for any \( 0 < \epsilon < T_{loc} \), \( 0 < \gamma < \min\{\beta + \delta - 1, \frac{1+2\beta}{4} - \eta\} \), where \( T_{loc} \) is some positive constant in \([0,T]\) depending on the exponents, \( \mathbb{E}\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 \), \( \mathbb{E}\|F_2(0)\|^2 \), \( \mathbb{E}\|A^\beta\xi\|^2 \), and \( \|A^\beta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))} \). Furthermore, \( X \) satisfies the estimates

\[
\mathbb{E}\|A^\beta X(t)\|^2 \leq C\mathbb{E}\|A^\beta \xi\|^2 + C\mathbb{E}\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 [1 + t^{2(1-\eta)}] + C\mathbb{E}\|F_2(0)\|^2 t^{2(1-\eta)} + C\mathbb{E}\|F_2(0)\|^2 t^{2(1+\beta-2\eta)} + C\mathbb{E}\|F_2(0)\|^2 t^{2(1+\beta-2\eta)} + C\mathbb{E}\|F_2(0)\|^2 t^{2(1+\beta-2\eta)},
\]

and

\[
\mathbb{E}\|A^\beta X(t)\|^2 \leq C\mathbb{E}\|F_2(0)\|^2 t^{2(1-\eta)} + C\mathbb{E}\|A^\beta \xi\|^2 + C\mathbb{E}\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 [t^{2(1-\eta)} + t^{2(1+\beta-2\eta)}] + C\mathbb{E}\|F_2(0)\|^2 t^{2(1-\eta)} + C\mathbb{E}\|F_2(0)\|^2 t^{2(1+\beta-2\eta)} + C\mathbb{E}\|F_2(0)\|^2 t^{2(1+\beta-2\eta)},
\]

on \([0,T_{loc}]\) with some \( C > 0 \) depending only on the exponents.

Proof. We use the fixed point theorem for contractions to prove existence and uniqueness of a local mild solution.

For each \( T_{loc} \in (0,T] \), set an underlying space as follows. Denote by \( \Xi(T_{loc}) \) the set of all \( E \)-valued processes \( Y \) on \([0,T_{loc}]\) such that \( Y(0) \in D(A^\beta) \) a.s. and

\[
\sup_{0 \leq t \leq T_{loc}} t^{2(\gamma-\beta)} \mathbb{E}\|A^\beta Y(t)\|^2 \leq \sup_{0 \leq t \leq T_{loc}} \mathbb{E}\|A^\beta Y(t)\|^2 < \infty.
\]

Up to indistinguishability, \( \Xi(T_{loc}) \) is a Banach space with norm

\[
\|Y\|_{\Xi(T_{loc})} = \left[ \sup_{0 < t \leq T_{loc}} t^{2(\gamma-\beta)} \mathbb{E}\|A^\beta Y(t)\|^2 + \sup_{0 \leq t \leq T_{loc}} \mathbb{E}\|A^\beta Y(t)\|^2 \right]^{\frac{1}{2}}.
\]

Let’s fix \( \kappa > 0 \) such that

\[
\frac{\kappa^2}{2} > \max\{C_1, C_2\},
\]

where \( C_1 \) and \( C_2 \) will be fixed later. Consider a subset \( \mathcal{Y}(T_{loc}) \) of \( \Xi(T_{loc}) \) which consists of all processes \( Y \in \Xi(T_{loc}) \) such that

\[
\max \left\{ \sup_{0 < t \leq T_{loc}} t^{2(\gamma-\beta)} \mathbb{E}\|A^\beta Y(t)\|^2, \sup_{0 \leq t \leq T_{loc}} \mathbb{E}\|A^\beta Y(t)\|^2 \right\} \leq \kappa^2.
\]

Obviously, \( \mathcal{Y}(T_{loc}) \) is a nonempty closed subset of \( \Xi(T_{loc}) \).

For \( Y \in \mathcal{Y}(T_{loc}) \), define a function \( \Phi Y \) on \([0,T_{loc}]\) by

\[
\Phi Y(t) = S(t)\xi + \int_0^t S(t-s)[F_1(s) + F_2(0)]ds + \int_0^t S(t-s)G(s)dW(s).
\]

Our goal is then to verify that there exists \( T_{loc} > 0 \) such that \( \Phi \) is a contraction mapping from \( \mathcal{Y}(T_{loc}) \) into itself, and that the fixed point of \( \Phi \) is the desired solution of (1). For this purpose, we divide the proof into several steps.
Step 1. Let \( Y \in \mathcal{Y}(T_{loc}) \). Let us verify that
\[ \Phi Y \in \mathcal{Y}(T_{loc}). \]

For \( \beta \leq \theta < \frac{1}{2} \), (14) and (15) give
\[
t^2(\theta-\beta)E\|A^\theta \Phi Y(t)\|^2 \\
\leq 3t^2(\theta-\beta)E\|A^\theta S(t)\|^2 + \left\| \int_0^t A^\theta S(t-s)(F_1(s) + F_2(Y(s)))ds \right\|^2 \\
+ \left\| \int_0^t A^\theta S(t-s)G(s)dW(s) \right\|^2 \\
\leq 3t^2(\theta-\beta)\|A^\theta S(t)\|^2 E\|A^\beta\xi\|^2 \\
+ 6t^2(\theta-\beta)E\left\| \int_0^t A^\theta S(t-s)F_1(s)ds \right\|^2 \\
+ 6t^2(\theta-\beta)E\left\| \int_0^t A^\theta S(t-s)F_2(Y(s))ds \right\|^2 \\
+ 3c(E)t^2(\theta-\beta)\int_0^t \|A^\theta S(t-s)G(s)\|^2_\gamma(H;E)ds.
\]

On account of (4), (5) and (6),
\[
t^2(\theta-\beta)E\|A^\theta \Phi Y(t)\|^2 \\
\leq 3t^2(\theta-\beta)E\|A^\beta\xi\|^2 + 6t^2(\theta-\beta)E\|F_1\|^2_{L^2(H;E)} \left\| \int_0^t (t-s)^{-\theta}s^{\beta-1}ds \right\|^2 \\
+ 6t^2(\theta-\beta)E\left\| \int_0^t (t-s)^{-\theta}\|F_2(Y(s))\|ds \right\|^2 \\
+ 3c(E)t^2(\theta-\beta)\int_0^t \|A^\theta\delta\|^2\|S(t-s)\|^2\|A^\beta G(s)\|^2_\gamma(H;E)ds \\
\leq 3t^2(\theta-\beta)E\|A^\beta\xi\|^2 + 6t^2(\theta-\beta)E\|F_1\|^2_{L^2(H;E)} \left\| \int_0^t (t-s)^{-\theta}s^{\beta-1}ds \right\|^2 \\
+ 6t^2(\theta-\beta)\int_0^t (t-s)^{-\theta}\|F_2(Y(s))\|^2ds \\
+ 3c(E)t^2\|A^\beta\delta\|^2\|A^\beta G\|^2_{L^2(H;E)} \left\| \int_0^t e^{-2\nu(t-s)}s^{2\beta-1}ds \right\|^2 \\
= 3t^2(\theta-\beta)E\|A^\beta\xi\|^2 + 6t^2\|F_1\|^2_{L^2(H;E)}B(\beta, 1-\theta)^2 \\
+ 6t^2\|F_2(Y(s))\|^2ds \\
+ 3c(E)t^2\|A^\beta\delta\|^2\|A^\beta G\|^2_{L^2(H;E)} \left\| \int_0^t e^{-2\nu(t-s)}s^{2\beta-1}ds \right\|^2 < \infty,
\]
where \( B(\cdot, \cdot) \) is the Beta function defined by
\[
B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1}ds, \quad x, y > 0,
\]
and
\[
\chi(\theta, \beta, \nu) = \sup_{0 \leq t < \infty} t^2(\theta-\beta)\int_0^t e^{-2\nu(t-s)}s^{2\beta-1}ds < \infty.
\]
Notice that we have just used this identity
\[
\int_0^t (t-s)^{-\theta} s^{\beta-1} ds = t^{\beta-\theta} B(\beta, 1-\theta), \quad t > 0.
\]
On the other hand, due to (F2a) and (14),
\[
\mathbb{E}[F_2(Y(t))]|^2 \leq \mathbb{E}[c F_2 \|A^\theta Y(t)\| + \|F_2(0)\|^2]
\leq 2(c F_2 \mathbb{E}[A^\theta Y(t)]^2 + \mathbb{E}[F_2(0)]^2)
\leq 2(c F_2^2 t^{2(\beta-\eta)} + \mathbb{E}[F_2(0)]^2), \quad 0 < t \leq T_{loc}.
\]
(16)

Thereby,
\[
t^{2(\theta-\beta)} \mathbb{E}[\|A^\theta \Phi Y(t)\|^2] \leq 3c^2 F_2^2 \mathbb{E}[A^\theta \xi]^2 + 6c^2 \mathbb{E}[F_1]^2 F_{\beta,\sigma}(E) B(\beta, 1-\theta)^2
\]
\[+ 12c^2 F_2^4 \int_0^t (t-s)^{-2\theta} s^{2(\beta-\eta)} ds + 3c(E) F_{\theta,\sigma}(E) A^{\theta-\delta} ||A^\delta G||_{\beta,\sigma}^2 (1 + 2\beta-2\eta) t^{2(\beta-\eta)} + \frac{12c^2 \mathbb{E}[F_2(0)]^2}{1-2\theta},\]
(17)
here we used the identity
\[
\int_0^t (t-s)^{-2\theta} s^{2(\beta-\eta)} ds = t^{1-2\theta+2\beta-2\eta} B(1 + 2\beta-2\eta, 1-2\theta).
\]

We apply these estimates with \(\theta = \eta\) and \(\theta = \beta\) as follows. Put
\[
C_1 = 3c^2 F_2^2 \mathbb{E}[A^\theta \xi]^2 + 6c^2 B(\beta, 1-\eta)^2 \mathbb{E}[F_1]^2 F_{\beta,\sigma}(E)
+ 3c(E) F_{\theta,\sigma}(E) A^{\theta-\delta} ||A^\delta G||_{\beta,\sigma}^2 (1 + 2\beta-2\eta) t^{2(\beta-\eta)} + \frac{12c^2 \mathbb{E}[F_2(0)]^2}{1-2\theta},
\]
(18)
\[
C_2 = 3c^2 F_2^2 \mathbb{E}[A^\theta \xi]^2 + 6c^2 B(\beta, 1-\eta)^2 \mathbb{E}[F_1]^2 F_{\beta,\sigma}(E)
+ 3c(E) F_{\theta,\sigma}(E) A^{\theta-\delta} ||A^\delta G||_{\beta,\sigma}^2 (1 + 2\beta-2\eta) t^{2(\beta-\eta)} + \frac{12c^2 \mathbb{E}[F_2(0)]^2}{1-2\theta},
\]
and take \(T_{loc}\) to be any constant satisfying
\[
\begin{cases}
 c F_2^2 K^2 B(1 + 2\beta - 2\eta, 1 - 2\eta) T_{loc}^{2(1-\eta)} + \frac{\mathbb{E}[F_2(0)]^2}{1-2\eta} T_{loc}^{2(1-\beta)} \leq \frac{\kappa^2 - C_1}{12\eta}\,
 c F_2^2 K^2 B(1 + 2\beta - 2\eta, 1 - 2\eta) T_{loc}^{2(1-\eta)} + \frac{\mathbb{E}[F_2(0)]^2}{1-2\beta} T_{loc}^{2(1-\beta)} \leq \frac{\kappa^2 - C_2}{12\beta},
\end{cases}
\]
(19)
Then, by using (13) and (17), we have
\[ t^{2(\theta-\beta)} \mathbb{E}\|A^{\theta} \Phi Y(t)\|^2 \]
\[ \leq C_1 + 12c_2^2c_F^2 \kappa^2 B(1 + 2\beta - 2\eta, 1 - 2\eta) t^{2(1-\eta)} \]
\[ + \frac{12c_2^2 \mathbb{E}\|F_2(0)\|^2}{1 - 2\eta} t^{2(1-\beta)} \leq \kappa^2, \quad 0 < t \leq T_{loc}, \]
and
\[ \mathbb{E}\|A^{\beta} \Phi Y(t)\|^2 \]
\[ \leq C_2 + 12c_2^2c_F^2 \kappa^2 B(1 + 2\beta - 2\eta, 1 - 2\beta) t^{2(1-\eta)} \]
\[ + \frac{12c_2^2 \mathbb{E}\|F_2(0)\|^2}{1 - 2\beta} t^{2(1-\beta)} \leq \kappa^2, \quad 0 < t \leq T_{loc}. \]

We have thus shown that
\[ \max \left\{ \sup_{0 < t \leq T_{loc}} t^{2(\theta-\beta)} \mathbb{E}\|A^{\theta} \Phi Y(t)\|^2, \sup_{0 \leq t \leq T_{loc}} \mathbb{E}\|A^{\beta} \Phi Y(t)\|^2 \right\} \leq \kappa^2. \]

In addition, it is clear that \( \Phi Y(0) = \xi \in \mathcal{D}(A^{\beta}) \) a.s. Hence,
\[ \Phi Y \in \mathcal{Y}(T_{loc}). \]

**Step 2.** Let us show that \( \Phi \) is a contraction mapping, provided that \( T_{loc} > 0 \) is any constant satisfying (19). Let \( Y_1, Y_2 \in \mathcal{Y}(T_{loc}) \) and \( 0 \leq \theta < \frac{1}{2} \). It follows from (5) and (15) that
\[ t^{2(\theta-\beta)} \mathbb{E}\|A^{\theta}(\Phi Y_1(t) - \Phi Y_2(t))\|^2 \]
\[ = t^{2(\theta-\beta)} \mathbb{E}\left\| \int_0^t A^{\theta}S(t-s)[F_2(Y_1(s)) - F_2(Y_2(s))] ds \right\|^2 \]
\[ \leq t^{2(\theta-\beta)} \mathbb{E}\left\| \int_0^t A^{\theta}S(t-s)\|F_2(Y_1(s)) - F_2(Y_2(s))\| ds \right\|^2 \]
\[ \leq t^{4(\theta-\beta)} \mathbb{E}\left\| \int_0^t (t-s)^{-\theta} \|F_2(Y_1(s)) - F_2(Y_2(s))\| ds \right\|^2. \]

Thanks to (F2a) and (12),
\[ t^{2(\theta-\beta)} \mathbb{E}\|A^{\theta}(\Phi Y_1(t) - \Phi Y_2(t))\|^2 \]
\[ \leq c_{F_2}^2 t^{4(\theta-\beta)} \mathbb{E}\left\| \int_0^t (t-s)^{-\theta} \|A^{\theta}(Y_1(s) - Y_2(s))\| ds \right\|^2 \]
\[ \leq c_{F_2}^2 t^{4(\theta-\beta)} \mathbb{E}\left\| \int_0^t (t-s)^{-2\theta} \|A^{\theta}(Y_1(s) - Y_2(s))\|^2 ds \right\|^2 \]
\[ = c_{F_2}^2 t^{4(\theta-\beta)} \mathbb{E}\left\| \int_0^t (t-s)^{-2\theta} \|A^{\theta}(Y_1(s) - Y_2(s))\|^2 ds \right\|^2 \]
\[ \leq c_{F_2}^2 t^{4(\theta-\beta)} \mathbb{E}\left\| \int_0^t (t-s)^{-2\theta} s^{2(2\beta-\eta)} \|Y_1 - Y_2\|_{\mathbb{E}(T_{loc})} ds \right\|^2 \]
\[ = c_{F_2}^2 t^{4(\theta-\beta)} B(1 + 2\beta - 2\eta, 1 - 2\theta) t^{2(1-\eta)} \|Y_1 - Y_2\|_{\mathbb{E}(T_{loc})}^2 \]
\[ \leq c_{F_2}^2 t^{4(\theta-\beta)} B(1 + 2\beta - 2\eta, 1 - 2\theta) T_{loc}^{2(1-\eta)} \|Y_1 - Y_2\|_{\mathbb{E}(T_{loc})}^2. \]
Applying these estimates with \( \theta = \eta \) and \( \theta = \beta \), we conclude that
\[
\|\Phi Y_1 - \Phi Y_2\|_{\mathcal{E}(T_{\text{loc}})}^2 = \sup_{0 \leq t \leq T_{\text{loc}}} \|2^{(\eta - \beta)}E[A^n(\Phi Y_1(t) - \Phi Y_2(t))]\|^2
\]
\[
+ \sup_{0 \leq t \leq T_{\text{loc}}} E[\|A^\beta(\Phi Y_1(t) - \Phi Y_2(t))\|_2^2]
\]
\[
\leq \epsilon^2 T_{\text{loc}}^2 [2\beta(1 + 2\beta - 2\eta, 1 - 2\eta) + \epsilon_0^2 B(1 + 2\beta - 2\eta, 1 - 2\beta)]
\]
\[
\times T_{\text{loc}}^{2(1-\eta)}\|Y_1 - Y_2\|_{\mathcal{E}(T_{\text{loc}})}^2.
\]
It then follows from (19) and (20) that \( \Phi \) is contraction on \( \Upsilon(T_{\text{loc}}) \).

**Step 3.** Given
\[
0 < \gamma < \min \left\{ \beta + \delta - 1, \frac{1 + 2\beta}{4} - \eta \right\}.
\]

Let us prove existence of a mild solution \( X \) to (1) on \([0, T_{\text{loc}}]\) in the function space
\[
X \in C^\gamma([\epsilon, T_{\text{loc}}]; D(A^\eta)) \cap C([0, T_{\text{loc}}]; D(A^\beta)) \quad \text{a.s.}
\]
for any \( 0 < \epsilon < T_{\text{loc}} \), where \( T_{\text{loc}} \) is a constant satisfying (19).

Thanks to the fixed point theorem, Step 2 implies that there exists \( X \in \Upsilon(T_{\text{loc}}) \) such that \( X = \Phi X \), i.e. \( X \) is a mild solution on \([0, T_{\text{loc}}]\).

We now prove that \( X \) belongs to the space in (22). For this purpose, we divide \( X \) into two parts: \( X(t) = X_1(t) + I_2(t) \), where
\[
X_1(t) = S(t)\xi + \int_0^t S(t-s)[F_1(s) + F_2(X(s))]ds,
\]
and
\[
I_2(t) = \int_0^t S(t-s)G(s)dW(s).
\]

By Theorem 2.16, we have
\[
I_2(t) + \int_0^t A I_2(s)ds = \int_0^t G(s)dW(s), \quad 0 \leq t \leq T_{\text{loc}},
\]
and \( I_2 \in C^\gamma([0, T_{\text{loc}}]; D(A)) \) a.s.

Let \( 0 < \epsilon < T_{\text{loc}} \). Since
\[
C^\gamma([0, T_{\text{loc}}]; D(A)) \subset C^\gamma([\epsilon, T_{\text{loc}}]; D(A^\eta)) \cap C([0, T_{\text{loc}}]; D(A^\beta)),
\]
what we need is to prove that
\[
X_1 \in C^\gamma([\epsilon, T_{\text{loc}}]; D(A^\eta)) \cap C([0, T_{\text{loc}}]; D(A^\beta)) \quad \text{a.s.}
\]

For the proof, we use the Kolmogorov continuity theorem. For \( 0 < s < t \leq T_{\text{loc}} \),

by the semigroup property,
\[
X_1(t) - X_1(s) = S(t-s)S(s)\xi + S(t-s) \int_s^t S(s-r)[F_1(r) + F_2(X(r))]dr
\]
\[
- X_1(s) + \int_s^t S(t-r)[F_1(r) + F_2(X(r))]dr
\]
\[
= [S(t-s) - I]X_1(s) + \int_s^t S(t-r)[F_1(r) + F_2(X(r))]dr.
\]
Let $\frac{1}{2} < \rho < 1 - \eta$. Due to (4), (5), (7) and (23), we have
\[
\|A^n[X_1(t) - X_1(s)]\|
\leq \|S(t - s) - I\|A^{-\rho}\|A^{n+\rho}X_1(s)\|
+ \int_s^t \|A^nS(t - r)\|\|F_1(r)\| + \|F_2(X(r))\|dr
\leq \frac{t_1 - \rho(t - s)^\rho}{\rho} \|A^{n+\rho}\|\bigg|S(s)\xi + \int_0^s (s - r)[F_1(r) + F_2(X(r))]dr\bigg|
+ \eta \int_s^t (t - r)^{-\eta}\|F_1(r)\| + \|F_2(X(r))\|dr
\leq \frac{t_1 - \rho(t - s)^\rho}{\rho} \|A^{n+\rho - \beta}\|\|S(s)\|\|A^\beta\xi\|
+ \frac{t_1 - \rho(t - s)^\rho}{\rho} \int_0^s (s - r)^{-\eta - \rho}S(s - r)\|F_1(r)\|dr
+ \frac{t_1 - \rho(t - s)^\rho}{\rho} \int_0^s (s - r)^{-\eta - \rho}S(s - r)\|F_2(X(r))\|dr
+ \eta \int_s^t (t - r)^{-\eta}\|F_1(r)\|dr + \eta \int_s^t (t - r)^{-\eta}\|F_2(X(r))\|dr
\leq \frac{t_1 - \rho(t - s)^\rho}{\rho} \|A^{n+\rho - \beta}\|\|S(s)\|\|A^\beta\xi\|
+ \frac{t_1 - \rho(t - s)^\rho}{\rho} \|F_1\|_{\mathcal{B}^\beta(E)}\|S(s)\|\|A^\beta\xi\|
+ \eta \int_s^t (t - r)^{-\eta}\|F_2(X(r))\|dr
\leq \frac{t_1 - \rho(t - s)^\rho}{\rho} \|A^{n+\rho - \beta}\|\|S(s)\|\|A^\beta\xi\|
+ \frac{t_1 - \rho(t - s)^\rho}{\rho} \|F_1\|_{\mathcal{B}^\beta(E)}\|S(s)\|\|A^\beta\xi\|
+ \eta \int_s^t (t - r)^{-\eta}\|F_2(X(r))\|dr.
\]
Dividing $\beta - 1$ as $\beta - 1 = (\eta + \rho - 1) + (\beta - \eta - \rho)$ and putting $r = s + u(t - s)$, it is seen that
\[
\int_s^t (t - r)^{-\eta - \rho}dr \leq \int_s^t (t - r)^{-\eta}(r - s)\|F_2(X(r))\|dr.
\]
Hence,

\[ \| A^\alpha [X_1(t) - X_1(s)] \| \]
\[ \leq \frac{t_1 - \rho t_\eta + \beta}{\rho} \| A^\beta \xi \|_{s^\beta - \eta - \rho} (t - s)^\rho \]
\[ + \left[ \frac{t_1 - \rho t_\eta + \beta}{\rho} B(\beta, 1 - \eta - \rho) + t_\eta B(\eta + \rho, 1 - \eta) \right] \]
\[ \times \| F_1 \|_{\mathcal{F}^{\rho, \sigma}} \| s^\beta - \eta - \rho (t - s)^\rho \]
\[ + \frac{t_1 - \rho t_\eta + \beta}{\rho} (t - s)^\rho \int_0^s (s - r)^{-\eta - \rho} \| F_2(X(r)) \| dr \]
\[ + t_\eta \int_s^t (t - r)^{-\eta} \| F_2(X(r)) \| dr, \quad 0 < s < t \leq T_{loc}. \]

Taking expectation of the square of the both hand sides of the above inequality, it follows that

\[ \mathbb{E} \| A^\alpha [X_1(t) - X_1(s)] \|^2 \]
\[ \leq \frac{4 t_1^2 - \rho^2 t_\eta - \beta}{\rho^2} \mathbb{E} \| A^\beta \xi \|^2_{s^{2(\beta - \eta - \rho)}} (t - s)^{2\rho} \]
\[ + 4 \left[ \frac{t_1 - \rho t_\eta + \beta}{\rho} B(\beta, 1 - \eta - \rho) + t_\eta B(\eta + \rho, 1 - \eta) \right]^2 \]
\[ \times \mathbb{E} \| F_1 \|^2_{\mathcal{F}^{\rho, \sigma}} \| s^{2(\beta - \eta - \rho)} (t - s)^{2\rho} \]
\[ + \frac{4 t_1^2 - \rho^2 t_\eta - \beta}{\rho^2} (t - s)^{2\rho} \mathbb{E} \left[ \int_0^s (s - r)^{-\eta - \rho} \| F_2(X(r)) \| dr \right]^2 \]
\[ + 4 t_\eta^2 \mathbb{E} \left[ \int_s^t (t - r)^{-\eta} \| F_2(X(r)) \| dr \right]^2, \quad 0 < s < t \leq T_{loc}. \]

Since

\[ \left[ \int_0^s (s - r)^{-\eta - \rho} \| F_2(X(r)) \| dr \right]^2 \]
\[ = \left[ \int_0^s (s - r)^{-\eta - \rho} (s - r)^{-\eta - \rho} \| F_2(X(r)) \| dr \right]^2 \]
\[ \leq \int_0^s (s - r)^{-\eta - \rho} dr \int_0^s (s - r)^{-\eta - \rho} \| F_2(X(r)) \|^2 dr \]
\[ = \frac{s^{1 - \eta - \rho}}{1 - \eta - \rho} \int_0^s (s - r)^{-\eta - \rho} \| F_2(X(r)) \|^2 dr \]

and

\[ \left[ \int_s^t (t - r)^{-\eta} \| F_2(X(r)) \| dr \right]^2 \leq (t - s) \int_s^t (t - r)^{-2\eta} \| F_2(X(r)) \|^2 dr, \]
we have
\[
\mathbb{E}[|A^\eta|X_1(t) - X_1(s)|^2] \leq \frac{4t^2_\eta^2}{\rho^2} \mathbb{E}[|A^\beta\xi|^2 s^{2(\beta - \eta - \rho)}(t - s)^{2\rho}]
\]
\[
+ 4 \left[ \frac{t_1^{-\rho}t_\eta + \rho B(\beta, 1 - \eta - \rho)}{\rho} + \epsilon_\eta B(\eta + \rho, 1 - \eta) \right]^2
\]
\[
\times \mathbb{E}[\|F_1\|_{\mathbb{F}^\beta}(s)]^2 s^{2(\beta - \eta - \rho)}(t - s)^{2\rho}
\]
\[
+ \frac{4t_1^2 - \rho^2 s^{1 - \eta - \rho}}{(t - s)^{2\rho}} \int_0^s (s - r)^{-\eta - \rho} \mathbb{E}[\|F_2(X(r))\|^2] dr
\]
\[
+ 4t_\eta^2(t - s) \int_s^t (t - r)^{-2\eta} \mathbb{E}[\|F_2(X(r))\|^2] dr.
\]

Two integrals in the right-hand side of (24) can be estimated as follows. Thanks to (16),

\[
\int_0^s (s - r)^{-\eta - \rho} \mathbb{E}[\|F_2(X(r))\|^2] dr \leq 2 \int_0^s (s - r)^{-\eta - \rho} [c_F^2 \kappa r^{2(\beta - \eta)}] + \mathbb{E}[\|F_2(0)\|^2] dr
\]
\[
= 2c_F^2 \kappa^2 B(1 + 2\beta - 2\eta, 1 - \eta - \rho) s^{1 + 2\beta - 3\eta - \rho} + \frac{2\mathbb{E}[\|F_2(0)\|^2]}{1 - \eta - \rho} s^{1 - \eta - \rho},
\]

and

\[
\int_s^t (t - r)^{-2\eta} \mathbb{E}[\|F_2(X(r))\|^2] dr \leq 2 \int_s^t (t - r)^{-2\eta} [c_F^2 \kappa r^{2(\beta - \eta)}] + \mathbb{E}[\|F_2(0)\|^2] dr
\]
\[
= 2c_F^2 \kappa^2 \int_s^t (t - r)^{-2\eta} r^{2(\beta - \eta)} dr + \frac{2\mathbb{E}[\|F_2(0)\|^2]}{1 - 2\eta} (t - s)^{1 - 2\eta}.
\]

The latter integral can be evaluated by dividing $2(\beta - \eta)$ as $2(\beta - \eta) = (\beta - \frac{1}{2}) + (\frac{1}{2} + \beta - 2\eta)$ and putting $r = s + u(t - s)$:

\[
\int_s^t (t - r)^{-2\eta} r^{2(\beta - \eta)} dr \leq \int_s^t (t - r)^{-2\eta} (r - s)^{\beta - \frac{1}{2}} t^{\beta + \frac{1}{2} - 2\eta} dr
\]
\[
= B(\frac{1}{2} + \beta, 1 - 2\eta) t^{\beta + \frac{1}{2} - 2\eta} (t - s)^{\frac{1}{2} + \beta - 2\eta}.
\]
By virtue of (24), (25), (26) and (27),
\[ E\| A^\eta [X_1(t) - X_1(s)] \|^2 \]
\[ \leq \frac{4t_1^2 - \rho^2 + \beta}{\rho^2} E\| A^{\beta} \xi \|^2 2(\beta - \eta - \rho)(t - s)^{2\rho} \]
\[ + 4 \left[ \left( 1 - \rho^2 \eta + \rho B(\beta, 1 - \eta - \rho) + \epsilon B(\eta, 1 - \eta) \right) \right]^2 \]
\[ \times E\| F_1 \|_{\beta, \sigma(\mathcal{E})}^2 2(\beta - \eta - \rho)(t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times \max\{ \epsilon^{(1 + \beta - 2\eta - \rho)}, T_{loc}^{2(1 + \beta - 2\eta - \rho)} \} (t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times T_{loc}^{2(1 - \eta - \rho)} (t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times (t - s)^{2(1 - \eta)}, \quad 0 < s < t \leq T_{loc}. \]

In particular,
\[ E\| A^\eta [X_1(t) - X_1(s)] \|^2 \]
\[ \leq \frac{4t_1^2 - \rho^2 + \beta}{\rho^2} E\| A^{\beta} \xi \|^2 2(\beta - \eta - \rho)(t - s)^{2\rho} \]
\[ + 4 \left[ \left( 1 - \rho^2 \eta + \rho B(\beta, 1 - \eta - \rho) + \epsilon B(\eta, 1 - \eta) \right) \right]^2 \]
\[ \times E\| F_1 \|_{\beta, \sigma(\mathcal{E})}^2 2(\beta - \eta - \rho)(t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times \max\{ \epsilon^{(1 + \beta - 2\eta - \rho)}, T_{loc}^{2(1 + \beta - 2\eta - \rho)} \} (t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times T_{loc}^{2(1 - \eta - \rho)} (t - s)^{2\rho} \]
\[ + \frac{8t_1^2 - \rho^2 + \beta}{\rho^2} E\| F_2(0) \|^2 \]
\[ \times (t - s)^{2(1 - \eta)}, \quad \epsilon \leq s < t \leq T_{loc}. \]

Since this estimate holds true for any \( \frac{1}{2} < \rho < 1 - \eta \), and since \( 1 < \frac{3}{2} + \beta - 2\eta < 2(1 - \eta) \), Theorem 2.11 provides that for \( 0 < \alpha < \frac{1 + 2\beta}{4} - \eta \),
\[ X_1 \in \mathcal{C}^\alpha ([\epsilon, T_{loc}]; \mathcal{D}(A^\eta)) \quad \text{a.s.} \]

As a consequence, by (21),
\[ X_1 \in \mathcal{C}^\gamma ([\epsilon, T_{loc}]; \mathcal{D}(A^\eta)) \quad \text{a.s.} \]

Since \( A^\beta X_1 = A^{\beta-\eta} A^\eta X_1 \), \( A^\beta X_1 \) is continuous on \( (0, T_{loc}) \).

It now remains to prove that \( A^\beta X_1 \) is continuous at \( t = 0 \). From Theorem 2.16 the process
\[ A^\beta [S(\cdot) \xi + \int_0^\cdot S(\cdot - s) F_1(s) ds] \]
is continuous at $t = 0$. Meanwhile, (5) and (16) give
\[
\mathbb{E}\left\| A^\beta \int_0^t S(t-s)F_2(X(s))ds \right\|^2 \\
\leq \mathbb{E}\left[ \int_0^t \| A^\beta S(t-s)\| \| F_2(X(s)) \| ds \right]^2 \\
\leq \ell_2^2 t \mathbb{E}\left[ \int_0^t (t-s)^{-2\beta} \| F_2(X(s)) \| ds \right]^2 \\
\leq \ell_2^2 t \int_0^t (t-s)^{-2\beta} \| F_2(X(s)) \|^2 ds \\
\leq 2^2 \beta T \int_0^t (t-s)^{-2\beta} [c^2 \kappa^2 s^{2(\beta-\eta)} + \mathbb{E} \| F_2(0) \|^2] ds \\
= 2^2 \beta T \mathbb{E} \| F_2(0) \|^2 \frac{1}{1-2\beta} \rightarrow 0 \quad \text{as } t \rightarrow 0.
\]

Therefore, there exists a decreasing sequence $\{t_n, n = 1, 2, 3, \ldots\}$ converging to 0 such that
\[
\lim_{n \rightarrow \infty} A^\beta \int_0^{t_n} S(t_n-s)F_2(X(s))ds = 0 \quad \text{a.s.}
\]

By the continuity of $A^\beta \int_0^t S(\cdot-s)F_2(X(s))ds$ on $(0,T_{loc})$, it implies that
\[
\lim_{t \rightarrow 0} A^\beta \int_0^t S(t-s)F_2(X(s))ds = 0 \quad \text{a.s.,}
\]
i.e. $A^\beta \int_0^t S(\cdot-s)F_2(X(s))ds$ is continuous at $t = 0$. In this way, we conclude that
\[
A^\beta X_1 = A^\beta \left[ S(\cdot)\xi + \int_0^\cdot S(\cdot-s)F_1(s)ds \right] \\
+ A^\beta \int_0^\cdot S(\cdot-s)F_2(X(s))ds
\]
is continuous at $t = 0$.

**Step 4.** Let us prove the estimate (10). We have
\[
A^\beta X(t) = A^\beta \left[ S(t)\xi + \int_0^t S(t-s)F_1(s)ds + \int_0^t S(t-s)G(s)dW(s) \right] \\
+ A^\beta \int_0^t S(t-s)F_2(X(s))ds \\
= A^\beta X_1(t) + A^\beta X_4(t), \quad 0 \leq t \leq T_{loc}.
\]

On account of Theorem 2.16, it is easily seen that
\[
\mathbb{E}\| A^\beta X_3(t) \|^2 \leq \rho_1 \| \mathbb{E} A^\beta \xi \|^2 + \mathbb{E}\| F_1 \|_{\mathcal{F}^\beta,\sigma(E)}^2 \\
+ \| A^\beta G \|_{\mathcal{F}^\beta+\frac{1}{2} \sigma(H;E)}^2 \|_{\mathcal{F}^{2\beta}}, \quad 0 \leq t \leq T_{loc},
\]
where $\rho_1$ is a positive constant depending only on the exponents. Meanwhile, (18) and (28) give that there exists $\rho_2 > 0$ depending only on the exponents such that

$$
E\|A^\beta X(t)\|^2 \leq \rho_2 [E\|A^\beta \xi\|^2 + E\|F_1\|_2^{3\beta,\sigma(E)} + \|A^\delta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))}]^2(1-\eta) + \rho_2 E\|F_2(0)\|^2 t^{2(1-\beta)}, \quad 0 \leq t \leq T_{loc}.
$$

Thus,

$$
\begin{align*}
E\|A^\beta X(t)\|^2 &\leq 2E\|A^\beta X_3(t)\|^2 + 2E\|A^\beta X_4(t)\|^2 \\
&\leq C[E\|A^\beta \xi\|^2 + E\|F_1\|_2^{3\beta,\sigma(E)}][1 + t^{2(1-\eta)}] + C\|A^\delta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))} t^{2\beta + 2(1-\eta)} + C E\|F_2(0)\|^2 t^{2(1-\beta)}, \quad 0 \leq t \leq T_{loc},
\end{align*}
$$

with some $C > 0$ depending only on the exponents.

**Step 5.** Let us prove the estimate (11). By using (4), (5), (6) and (13), we have

$$
\begin{align*}
E\|A^n X(t)\|^2 &= E\|A^n S(t)\xi + \int_0^t A^n S(t-s)s F_1(s)ds + \int_0^t A^n S(t-s)G_s dW(s) + \int_0^t A^n S(t-s)F_2(X(s))ds\|^2 \\
&\leq 4E\|A^n S(t)\xi\|^2 + 4E\left[\int_0^t \|A^n S(t-s)\|| F_1(s)||ds\right]^2 \\
&\quad + 4c(E) \int_0^t \|A^n S(t-s)\|^2 \|A^\delta G(s)\|^2_{\gamma(H;E)} ds \\
&\quad + 4\eta E\left[\int_0^t (t-s)^{-\eta} \|F_2(X(s))\||ds\right]^2 \\
&\quad + 4\eta E\left[\int_0^t (t-s)^{-\eta} \|F_2(X(s))\||ds\right]^2 \\
&\quad + 4c(E) \int_0^t \|A^n - B^{\beta,1-\eta}\|^2 \|A^\delta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))} \int_0^t e^{-2\nu(t-s)} s^{2\beta-1} ds \\
&\leq 4\eta E\|A^n \xi\|^2 t^{-2(\eta-\beta)} + 4\eta E\|A^\delta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))} \int_0^t e^{-2\nu(t-s)} s^{2\beta-1} ds \\
&\quad + 4\eta t \int_0^t (t-s)^{-2\eta} \|F_2(X(s))\|^2 ds \\
&\quad + 4c(E) \int_0^t \|A^n - B^{\beta,1-\eta}\|^2 \|A^\delta G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H;E))} \\
&\quad \times \sup_{0 \leq t < \infty} \int_0^t e^{-2\nu(t-s)} s^{2\beta-1} ds, \quad 0 < t \leq T_{loc}.
\end{align*}
$$
On account of (16),

\[ 4 \varepsilon_t^2 t \int_0^t (t-s)^{-2\eta} \mathbb{E}\|F_2(X(s))\|^2 ds \]

\[ \leq 8 \varepsilon_t^2 \int_0^t (t-s)^{-2\eta} [c_{F_2}^2 \kappa^2 s^{2(\beta-\eta)} + \mathbb{E}\|F_2(0)\|^2] ds \]

\[ = 8 \varepsilon_t^2 \left[ c_{F_2}^2 \kappa^2 (1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} + \frac{\mathbb{E}\|F_2(0)\|^2}{1-2\eta} t^{2(1-\eta)} \right]. \]

Therefore,

\[ \mathbb{E}\|A^\eta X(t)\|^2 \]

\[ \leq 4 [\varepsilon_t^2 \beta \mathbb{E}\|A^\beta \xi\|^2 + \varepsilon_t^2 B(\beta, 1-\eta) \mathbb{E}\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 t^{2(\beta-\eta)} ] \]

\[ + 8 \varepsilon_t^2 \left[ c_{F_2}^2 \kappa^2 (1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} + \frac{\mathbb{E}\|F_2(0)\|^2}{1-2\eta} t^{2(1-\eta)} \right] \]

\[ + 4 c(E) \varepsilon_t^2 \|A^\eta \|_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))}^2 \]

\[ \times \sup_{0 \leq t < \infty} \int_0^t e^{-2\eta(t-s)} s^{2\beta-1} ds, \quad 0 < t \leq T_{loc}. \]

In view of (13) and (18), it is easily seen that there exists \( C > 0 \) depending only on the exponents such that for \( 0 < t \leq T_{loc} \),

\[ \mathbb{E}\|A^\eta X(t)\|^2 \]

\[ \leq C [\mathbb{E}\|A^\beta \xi\|^2 + \mathbb{E}\|F_1\|_{\mathcal{F}^{\beta,\sigma}(E)}^2 t^{2(\beta-\eta)} ] + C \mathbb{E}\|F_2(0)\|^2 t^{2(1-\eta)} \]

\[ + C \sup_{0 \leq t < \infty} \int_0^t e^{-2\eta(t-s)} s^{2(\beta-1)} ds + t^{2(1+\beta-2\eta)} \|A^\sigma G\|^2_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))} \]

Thus, (11) has been verified.

**Step 6.** Let us prove uniqueness of local mild solutions. Let \( \tilde{X} \) be any other local mild solution to (1) on the interval \([0, T_{loc}]\) which belongs to the space \( \Xi(T_{loc}) \).

The formula

\[ \tilde{X}(t) = S(t)\xi + \int_0^t S(t-s)[F_2(\hat{X}(s)) + F_1(s)] ds + \int_0^t S(t-s)G(s)dW(s) \]

jointed with

\[ X(t) = S(t)\xi + \int_0^t S(t-s)[F_2(X(s)) + F_1(s)] ds + \int_0^t S(t-s)G(s)dW(s) \]

yields that

\[ X(t) - \hat{X}(t) = \int_0^t S(t-s)[F_2(X(s)) - F_2(\hat{X}(s))] ds, \quad 0 \leq t \leq T_{loc}. \]

We can then repeat the arguments in Step 2 to deduce that for \( 0 < \overline{T} \leq T_{loc} \),

\[ \|X - \hat{X}\|^2_{\Xi(\overline{T})} \]

\[ \leq c_{F_2}^2 [\varepsilon_t^2 B(1+2\beta-2\eta, 1-2\eta) + \varepsilon_t^2 B(1+2\beta-2\eta, 1-2\beta)] \]

\[ \times \overline{T}^{2(1-\eta)} \|X - \hat{X}\|^2_{\Xi(\overline{T})}. \]

Let \( \overline{T} \) be a positive constant such that

\[ c_{F_2}^2 [\varepsilon_t^2 B(1+2\beta-2\eta, 1-2\eta) + \varepsilon_t^2 B(1+2\beta-2\eta, 1-2\beta)] \overline{T}^{2(1-\eta)} < 1. \]
Then, (29) implies that \( X = \bar{X} \) a.s. on \([0, \bar{T}]\).

Repeating the same procedure with initial time \( \bar{T} \) and initial value \( X(\bar{T}) = \bar{X}(\bar{T}) \), we derive that \( X(\bar{T} + t) = \bar{X}(\bar{T} + t) \) a.s. for \( 0 \leq t \leq \bar{T} \). This means that \( X = \bar{X} \) a.s. on a larger interval \([0, 2\bar{T}]\).

We continue this procedure by finite times, the extended interval can cover the given interval \([0, T_{loc}]\). Therefore, \( X = \bar{X} \) a.s. on \([0, T_{loc}]\). Thanks to the continuity of \( X \) and \( \bar{X} \) on \([0, T_{loc}]\), they are indistinguishable.

The proof of the theorem is now complete. \( \square \)

Next, we consider the case where (Gb) holds true. Assume further that

\[(F2b) \quad F_2 : \mathcal{D}(A^{\beta}) \to E \text{ satisfies a Lipschitz condition of the form} \]

\[\|F_2(x) - F_2(y)\| \leq c_{F_2}\|A^{\beta}(x - y)\| \quad \text{a.s., } x, y \in \mathcal{D}(A^{\beta}),\]

where \( c_{F_2} > 0 \) is some constant.

**Theorem 3.2.** Let (F1), (F2b) and (Gb) be satisfied. Assume that \( \xi \in \mathcal{D}(A^{\beta}) \) a.s. such that \( E\|A^{\beta}\xi\|^2 < \infty \). Then, (1) possesses a unique local mild solution \( X \) in the function space:

\[X \in \mathcal{C}^\gamma([\epsilon, T_{loc}]; \mathcal{D}(A^{\beta})) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^{\theta})) \quad \text{a.s.}\]

for any \( 0 \leq \gamma < \min\{\frac{1}{2} - \beta, \beta\} \), \( 0 < \epsilon < T \) and \( 0 \leq \theta < \beta \), where \( T_{loc} \) is some positive constant in \([0, T]\) depending only on the exponents and \( E\|F_1\|_{\mathcal{L}(E)}^2, E\|F_2(0)\|^2 \), \( E\|A^{\beta}\xi\|^2 \), \( \|G\|_{\mathcal{L}(\mathcal{D}(A^{\beta}))}^2 \). Furthermore, \( X \) satisfies the estimate

\[E\|A^{\beta}X(t)\|^2 \leq CE\|F_2(0)\|^2(1-\beta) + C[e^{-2\nu t}E\|A^{\beta}\xi\|^2 + E\|F_1\|_{\mathcal{L}(E)}^2]
\]

\[\quad \quad + E\|G\|_{\mathcal{L}(\mathcal{D}(A^{\beta}))}^2(1 + t^{2(1-\beta)})], \quad 0 \leq t \leq T_{loc}\]

with some \( C > 0 \) depending only on the exponents.

The proof for Theorem 3.2 is very similar to one for Theorem 3.1. Therefore, we omit it here.

**Remark 3.** In Theorem 3.1, we assume that the function \( G \) has space-time regularity, meanwhile \( G \) only has time regularity in Theorem 3.2. These regularities of \( G \) implies the existence of mild solutions. Another condition, say the boundedness of \( G \), which also implies the existence of mild solutions to (1) in Hilbert spaces can be found in [12]. The space-time regularity of \( G \) is necessary for the construction of strict solutions in the next section.

### 3.2. Strict solutions

We are now ready to construct strict solutions to (1).

**Theorem 3.3** (Existence of strict solutions). Let (F1), (F2a) and (Ga) be satisfied. Let \( \xi \in \mathcal{D}(A^{\beta}) \) a.s. such that \( E\|A^{\beta}\xi\|^2 < \infty \). Assume further that there exists \( \rho > 0 \) such that \( F_2(x) \in \mathcal{D}(A^{\rho}) \) for \( x \in \mathcal{D}(A^{\theta}) \) and

\[E[\sup_{x \in \mathcal{D}(A^{\theta})}\|A^{\rho}F_2(x)\|] < \infty.\]

Then, (1) possesses a unique strict solution \( X \) on \([0, T_{loc}]\), where \( T_{loc} \) is some positive constant in \([0, T]\) depending only on the exponents and \( E\|F_1\|_{\mathcal{L}(E)}^2, E\|F_2(0)\|^2 \), \( E\|A^{\beta}\xi\|^2 \), \( \|G\|_{\mathcal{L}(\mathcal{D}(A^{\beta}))}^2 \). Furthermore, \( X \) has the regularity

\[X \in \mathcal{C}^\gamma([\epsilon, T_{loc}]; \mathcal{D}(A^{\theta})) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^{\beta})) \quad \text{a.s.}\]
for any $0 < \epsilon < T_{loc}$, $0 \leq \gamma < \min\{\beta + \delta - 1, \frac{1 + 2\beta}{4} - \eta\}$ with the estimate
\[
E\|AX(t)\|^2 \leq C\|\xi\|^2 t^{-2} + C\|F_1\|^2_{\mathcal{F}^{\beta,\sigma}(E)} t^{2(\beta - 1)} + C\|A^\delta G\|^2_{\mathcal{F}^{\beta + \frac{1}{2} \sigma}(\gamma(H;E))} t^{(\beta + \delta - 1)},
\]
with some $C > 0$ depending on the exponents.

Proof. Thanks to Theorem 3.1, (1) has a unique local mild solution $X$ in the function space:
\[
X \in C^\gamma([\epsilon, T_{loc}]; \mathcal{D}(A^\eta)) \cap C([0, T_{loc}]; \mathcal{D}(A^\beta)) \quad \text{a.s.}
\]
for any $0 < \epsilon < T_{loc}$, $0 \leq \gamma < \min\{\beta + \delta - 1, \frac{1 + 2\beta}{4} - \eta\}$.

First, let us show that $X$ is a local strict solution of (1). We divide $X$ into two parts as
\[
X(t) = \left[ S(t)\xi + \int_0^t S(t-s)F_1(s)ds + \int_0^t S(t-s)G(s)dW(s) \right]
\]
\[
+ \int_0^t S(t-s)F_2(X(s))ds
\]
\[
= X_1(t) + X_2(t), \quad 0 \leq t \leq T_{loc}.
\]
Since all the assumptions of Theorem 2.15 are satisfied, it is possible to see that
\[
X_1(t) = \xi + \int_0^t F_1(s)ds - \int_0^t AX_1(s)ds \quad (31)
\]
\[
+ \int_0^t G(s)dW(s), \quad 0 < t \leq T_{loc}.
\]
Furthermore,
\[
E\|AX_1(t)\|^2 \leq C\|\xi\|^2 t^{-2} + C\|F_1\|^2_{\mathcal{F}^{\beta,\sigma}(E)} t^{2(\beta - 1)} + C\|A^\delta G\|^2_{\mathcal{F}^{\beta + \frac{1}{2} \sigma}(\gamma(H;E))} t^{(\beta + \delta - 1)}, \quad 0 < t \leq T_{loc}
\]
with some $C > 0$ depending on the exponents.

In the meantime, (5) gives that
\[
\int_0^t \|AS(t-s)F_2(X(s))\|ds
\]
\[
= \int_0^t \|A^{1-\rho}S(t-s)\|\|A^\rho F_2(X(s))\|ds
\]
\[
\leq \frac{t(1-\rho)}{\rho} \int_0^t (t-s)^{\rho-1} \sup_{x \in \mathcal{D}(A^\eta)} \|A^\rho F_2(x)\|ds
\]
\[
= \frac{t(1-\rho)}{\rho} \sup_{x \in \mathcal{D}(A^\eta)} \|A^\rho F_2(x)\|t^\rho < \infty \quad \text{a.s.,} \quad 0 \leq t \leq T_{loc}.
\]
Thereby, $\int_0^t AS(t-s)F_2(X(s))ds$ is well-defined on $[0, T_{loc}]$.

Since $A$ is closed,
\[
AX_2 = \int_0^t AS(t-s)F_2(X(s))ds, \quad 0 \leq t \leq T_{loc},
\]
and
\[ \mathbb{E}[\|AX_2(t)\|^2] \leq \frac{t^2 \rho^2}{\rho^2} \mathbb{E}[\sup_{x \in \mathcal{D}(A^\rho)} \|A^\rho F_2(x)\|^2 t^{2\rho}], \quad 0 \leq t \leq T_{loc}. \] (33)
Thus,
\[
\frac{dX_2(t)}{dt} = \frac{d}{dt} \int_0^t S(t-s)F_2(X(s))ds \\
= F_2(X(t)) - \int_0^t AS(t-s)F_2(X(s))ds \\
= F_2(X(t)) - AX_2(t), \quad 0 \leq t \leq T_{loc}.
\]
This implies that
\[ X_2(t) = \int_0^t F_2(X(s))ds - \int_0^t AX_2(s)ds, \quad 0 \leq t \leq T_{loc}. \] (34)
Combining (31) and (34) yields that \( X = X_1 + X_2 \) is a local strict solution of (1) on \([0, T_{loc}]\).

Let us now prove the estimate (30). Since
\[ \mathbb{E}[\|AX(t)\|^2] \leq 2\mathbb{E}[\|AX_1(t)\|^2 + 2\mathbb{E}[\|AX_2(t)\|^2], \]
(30) follows from (32) and (33). We complete the proof.

4. Examples. This section gives an example to illustrate our results. Consider a nonlinear problem
\[
\begin{aligned}
\frac{\partial X(x,t)}{\partial t} &= \sum_{i,j=1}^n \frac{\partial}{\partial x_j} [a_{ij}(x) \frac{\partial}{\partial x_i} u] + f_1(x) + t^{\beta-1} f_2(t) \varphi(x) \\
&\quad + t^{\beta-\frac{1}{2}} g(t) W(t) \quad \text{in } \mathbb{R}^n \times (0,T), \\
X(x,0) &= X_0(x) \quad \text{in } \mathbb{R}^n.
\end{aligned}
\] (35)
Here,
\begin{itemize}
  \item[(Ex1):] \( W \) is a cylindrical Wiener process on a separable Hilbert space \( H \). And, \( W \) is defined on a complete filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \).
  \item[(Ex2):] \( a_{ij} \in L^\infty(\mathbb{R}^n, \mathbb{R}) \) for \( 1 \leq i, j \leq n \). In addition, there exists \( a_0 > 0 \) such that
  \[ \sum_{i,j=1}^n a_{ij}(x)z_iz_j \geq a_0\|z\|_{\mathbb{R}^n}^2, \quad z = (z_1, \ldots, z_n) \in \mathbb{R}^n, \) a.e. \( x \in \mathbb{R}^n \).
  \item[(Ex3):] \( \varphi \in H^{-1}(\mathbb{R}^n) \) and \( f_1 \) is measurable function from a domain \( \mathcal{D}(f_1) \) of \( H^{-1}(\mathbb{R}^n) \) to \( \mathbb{R} \).
  \item[(Ex4):] \( f_2 \in C^\sigma([0,T]; \mathbb{R}) \) and \( g \in C^\sigma([0,T]; |H^{-1}(\mathbb{R}^n)|) \). In addition, \( f_2(0) = 0 \) and \( g(0) = 0 \) for some \( 0 < \sigma < \beta < \frac{1}{2} \).
  \item[(Ex5):] \( X_0 \) is an \( \mathcal{F}_0 \)-measurable random variable.
\end{itemize}

Let \( A \) be a realization of the differential operator \( -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} [a_{ij}(x) \frac{\partial}{\partial x_i}] + 1 \) in \( (E, \| \cdot \|) = (H^{-1}(\mathbb{R}^n), \| \cdot \|_{H^{-1}(\mathbb{R}^n)}) \). Thanks to [16, Theorem 2.2], the realization \( A \) is a sectorial operator of \( H^{-1}(\mathbb{R}^n) \) with domain \( \mathcal{D}(A) = H^1(\mathbb{R}^n) \). As a consequence, \( -A \) generates an analytical semigroup on \( H^{-1}(\mathbb{R}^n) \).
Using $A$, the equation (35) is formulated as a problem of the form (1) in the Hilbert space $H^{-1}(\mathbb{R}^n)$, where $F_1$, $F_2$ and $G$ are defined as follows. The functions $F_1 : (0, T) \to H^{-1}(\mathbb{R}^n)$ and $G : (0, T) \to \gamma(H; H^{-1}(\mathbb{R}^n))$ are defined by

$$F_1(t) = t^{\beta-1}f_2(t)\phi(x), \quad G(t) = t^{\beta-\frac{1}{2}}g(t).$$

Remark 1 provides that

$$F_1 \in \mathcal{F}^{\beta,\sigma}((0, T]; H^{-1}(\mathbb{R}^n)), \quad G \in \mathcal{F}^{\beta+\frac{1}{2},\sigma}((0, T]; \gamma(H; H^{-1}(\mathbb{R}^n))).$$

In the meantime, $F_2$ is defined by

$$F_2(u) = u + f_1(u).$$

Assume further that

(Ex6): For some $c > 0$ and max${\{0, 2\eta - \frac{1}{2}\}} < \beta < \eta$, $\mathcal{D}(f_1) = \mathcal{D}(A^\rho)$ and

$$\|f_1(u) - f_1(v)\| \leq c\|A^\rho(u - v)\|, \quad u, v \in \mathcal{D}(A^\rho).$$

It is easily seen that the assumptions (F1), (F2a) and (Ga) are satisfied. According to Theorem 3.3, we have the following result.

**Theorem 4.1.** Let (Ex1)-(Ex6) be satisfied. Suppose further that there exists $\rho > 0$ such that

$$\mathbb{E}[\sup_{x \in \mathcal{D}(A^\rho)} \|A^\rho F_2(x)\|] < \infty.$$

Let $X_0 \in \mathcal{D}(A^\beta)$ a.s. such that $\mathbb{E}[\|A^\beta X_0\|^2] < \infty$. Then, (35) possesses a unique local strict solution $X$ on some interval $[0, T_{\text{loc}}]$. Furthermore, $X$ has the regularity

$$X \in \mathcal{C}([\epsilon, T_{\text{loc}}]; \mathcal{D}(A^\rho)) \cap \mathcal{C}([0, T_{\text{loc}}]; \mathcal{D}(A^\beta)) \quad \text{a.s.}$$

for any $0 < \epsilon < T_{\text{loc}}$, $0 \leq \gamma < \min\{\beta + \delta - 1, \frac{1+2\beta}{4} - \eta\}$, and satisfies the estimate

$$\mathbb{E}[\|AX(t)\|^2] \leq C\mathbb{E}[\|X_0\|^2]t^{-2} + C\|F_1\|^2_{\mathcal{F}^{\beta,\sigma}(H^{-1}(\mathbb{R}^n))}t^{2(\beta-1)} + C\|A^\rho G\|^2_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}(\gamma(H; H^{-1}(\mathbb{R}^n)))}t^{2(\beta+\delta-1)} + C\sup_{x \in \mathcal{D}(A^\rho)} \|A^\rho F_2(x)\|t^{2\rho}, \quad 0 < t \leq T_{\text{loc}},$$

where $C > 0$ is a constant depending on the exponents.

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**REFERENCES**

[1] Z. Brzeźniak, Stochastic partial differential equations in M-type 2 Banach spaces, *Potential Anal.*, 4 (1995), 1–45.
[2] Z. Brzeźniak, On stochastic convolution in Banach spaces and applications, *Stochastics Stochastics Rep.*, 61 (1997), 245–295.
[3] R. F. Curtain and P. L. Falb, Stochastic differential equations in Hilbert space, *J. Differential Equations*, 10 (1971), 412–430.
[4] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
[5] E. Dettweiler, On the martingale problem for Banach space valued stochastic differential equations, *J. Theoret. Probab.*, 2 (1989), 159–191.
[6] M. Hairer, An introduction to stochastic PDEs, preprint, (2009) [arXiv:0907.4178].
[7] B. H. Haak and J. M. A. M. van Neerven, Uniformly $\gamma$-radonifying families of operators and the stochastic Weiss conjecture, *Oper. Matrices* 6 (2012), 767–792.
[8] O. Martin, Uniqueness for stochastic evolution equations in Banach spaces, *Dissertationes Math. (Rozprawy Mat.)*, 426 (2004), 63 pp.
[9] G. Pisier, Probabilistic methods in the geometry of Banach spaces, in Probability and Analysis, Lecture Notes in Math., vol. 1206, Springer, Berlin, 1986, 167–241.
[10] T. V. Ta, Existence results for linear evolution equations of parabolic type, Commun. Pure Appl. Anal. 17 (2018), 751–785.
[11] T. V. Ta, Non-autonomous stochastic evolution equations in Banach spaces of martingale type 2: strict solutions and maximal regularity, Discrete Contin. Dyn. Syst., 37 (2017), 4507–4542.
[12] T. V. Ta, Note on abstract stochastic semilinear evolution equations, J. Korean Math. Soc., 54 (2017), 909–943.
[13] T. V. Ta, Y. Yamamoto and A. Yagi, Strict solutions to stochastic linear evolution equations in M-type 2 Banach spaces, Funkcial. Ekvac., 61 (2018), 191–217.
[14] J. M. A. M. van Neerven, M. C. Veraar and L. Weis, Stochastic evolution equations in UMD Banach spaces, J. Funct. Anal., 255 (2008), 940–993.
[15] J. M. A. M. van Neerven, M. C. Veraar and L. Weis, Stochastic maximal $L^p$-regularity, Ann. Probab. 40 (2012), 788–812.
[16] A. Yagi, Abstract Parabolic Evolution Equations and their Applications, Springer-Verlag, Berlin, 2010.

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