Galois action on Fuchsian surface groups and their solenoids  

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Abstract  
Let $C$ be a complex algebraic curve uniformised by a Fuchsian group $\Gamma$. In the first part of this paper we identify the automorphism group of the solenoid associated with $\Gamma$ with the Belyaev completion of its commensurator $\text{Comm}(\Gamma)$ and we use this identification to show that the isomorphism class of this completion is an invariant of the natural Galois action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ on algebraic curves. In turn this fact yields a proof of the Galois invariance of the arithmeticity of $\Gamma$ independent of Kazhdan’s.  

In the second part we focus on the case in which $\Gamma$ is arithmetic. The list of further Galois invariants we find includes: i) the periods of $\text{Comm}(\Gamma)$, ii) the solvability of the equations $X^2 + \sin^2 \frac{2\pi k}{2}$ in the invariant quaternion algebra of $\Gamma$ and iii) the property of $\Gamma$ being a congruence subgroup.

1 Introduction and statement of results  
We recall that two subgroups $H_1$ and $H_2$ of a group $G$ are said to be commensurable if $H_1 \cap H_2$ has finite index in both $H_1$ and $H_2$ and that the commensurator of a subgroup $H < G$ is the subgroup $\text{Comm}(H)$ consisting of the elements $g \in G$ such that $H$ and $gHg^{-1}$ are commensurable.

Let $C$ be a compact Riemann surface (or, equivalently, a nonsingular complex projective curve) of genus $g \geq 2$ and let $\Gamma \cong \pi_1(C)$ be the Fuchsian surface group uniformising $C$. Here we deal with the commensurator of $\Gamma$ in $\text{PSL}_2(\mathbb{R})$, namely  

$$\text{Comm}(\Gamma) = \{ \alpha \in \text{PSL}_2(\mathbb{R}) : \alpha \Gamma \alpha^{-1} \cap \Gamma \text{ has finite index in both } \Gamma \text{ and } \alpha \Gamma \alpha^{-1} \}$$

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We observe that the group $\Gamma$ is determined by $C$ only up to conjugation in $\text{PSL}_2(\mathbb{R})$ and that commensurable groups have the same commensurator. Another object which depends only on the commensurability class of $\Gamma$ is the Sullivan solenoid $H_C$ (or $\mathcal{H}_C$) associated with $C$ (or with $\Gamma$). These two objects will play a central role in this paper. They are related by the fact that the automorphism group of the solenoid, let us denote it $\text{Aut}(H_C)$ (or $\text{Aut}(\mathcal{H}_C)$), is generated by $\text{Comm}(\Gamma)$ and $\hat{\Gamma}$, the profinite completion of $\Gamma$.

Let $\text{Gal}(\mathbb{C}/\mathbb{Q})$ denote the group of field automorphisms of $\mathbb{C}$, and let $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$. The obvious (Galois) action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ on the coefficients of the defining equations transforms a complex algebraic variety $X$ defined over a subfield $k \subset \mathbb{C}$ into another algebraic variety $X^{\sigma}$ defined over the subfield $k^{\sigma} := \sigma(k)$.

Finding invariants for this action is an important problem in Complex Algebraic Geometry. For instance, the Betti numbers and the profinite completion of the fundamental group are Galois invariants but, in dimension $\geq 2$, the fundamental group itself and the universal cover are not (see e.g. [Se], [MiSu], [Ca], [GJ], [GJT], [GR1], [GR2] and [Go1]). Such phenomena do not occur in dimension one since in this case the Galois action is genus preserving, but Grothendieck’s theory of dessins d’enfants and algebraic curves share similar ideas and goals ([Gro], see also [GG] and [JW]).

Let $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ and let $\Gamma^{\sigma}$ denote the Fuchsian group uniformising the conjugate algebraic curve $C^{\sigma}$. We shall refer to $\Gamma^{\sigma}$ as the Galois conjugate of $\Gamma$ by $\sigma$. We observe that the rule $(\sigma, \Gamma) \mapsto \Gamma^{\sigma}$ defines an action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ on the set of $\text{PSL}_2(\mathbb{R})$-conjugacy classes of Fuchsian surface groups. As it has been said, in this (1-dimensional) case $\Gamma$ and $\Gamma^{\sigma}$ must be isomorphic as abstract groups. However, little seems to be known about the relationship between $\Gamma$ and $\Gamma^{\sigma}$ as subgroups of $\text{PSL}_2(\mathbb{R})$. Note that $\text{Comm}(\Gamma)$ depends precisely on the way $\Gamma$ sits inside $\text{PSL}_2(\mathbb{R})$.

In the first part of this article we study the group $\text{Aut}(H_C)$; we show it to be the Belyaev completion of $\text{Comm}(\Gamma)$ (see the definition in 3.1) and derive some properties which remain invariant under Galois action. In the second part we also look for Galois invariants but we use mainly tools pertaining to the theory of arithmetic groups.

1.1 The automorphism group of a solenoid $\text{Aut}(H_C)$.

The explicit description of the automorphisms of $H_C$ was made by Odden in [Odd1], where he showed that any element $F \in \text{Aut}(H_C)$ can be written as a product $F = \alpha \tau$, with $\alpha \in \text{Comm}(\Gamma)$ and $\tau \in \hat{\Gamma}$ essentially in a unique way. However the author remarks that “(this) theorem does not shed light on its group structure”.

We will consider the completion of $\text{Comm}(\Gamma)$ relative to the topology determined by the finite index subgroups of $\Gamma$ and, following Belyaev [Be], we will make it into a totally disconnected locally compact group which we will denote $\text{Comm}(\Gamma)$. Then we will prove (Theorem 7)

- $\text{Aut}(H_C) \cong \text{Comm}(\Gamma)$ (isomorphism of groups)
Then we will show that there is a natural action of $Gal(\mathbb{C}/\mathbb{Q})$ on $\mathcal{H}_C$ which, for any $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$, induces the following isomorphisms (Theorem 10):

- $\overline{\text{Comm}(\Gamma)} \simeq \text{Comm}(\Gamma^\sigma)$ (isomorphism of topological groups)
- $\dfrac{\text{Comm}(\Gamma)}{\Gamma} \simeq \dfrac{\text{Comm}(\Gamma^\sigma)}{\Gamma^\sigma}$ (isomorphism of sets of co-sets)

In view of a well-known theorem of Margulis [Mar], which states that $\Gamma$ is arithmetic if and only if $[\text{Comm}(\Gamma) : \Gamma] = \infty$, the second of these isomorphisms provides an alternative proof of the following result which Kazhdan ([Kaz]) proved in arbitrary dimension.

- $C$ is uniformised by an arithmetic group if and only if $C^\sigma$ is. (Corollary 11).

In the particular (but generic) case in which $\Gamma$ is non-arithmetic the isomorphism class of the group $\text{Comm}(\Gamma)$ is also a Galois invariant; more precisely we will prove the following result (Theorem 14)

- Suppose that $\Gamma$ is non-arithmetic, then $\text{Comm}(\Gamma) \simeq \text{Comm}(\Gamma^\sigma)$.

The second part of the article will be devoted to the arithmetic case.

### 1.2 The arithmetic case

The main tool to understand the relation between $\Gamma$ and $\Gamma^\sigma$ in the arithmetic case is a theorem by Doi-Naganuma. This result establishes the relation between the quaternion algebra $(k\Gamma, A\Gamma)$ associated with $\Gamma$ and the quaternion algebra $(k\Gamma^\sigma, A\Gamma^\sigma)$ associated with the Galois conjugate group $\Gamma^\sigma$ (see Theorem 16 for the precise statement and 2.1 as well as 5.1 for main notions from the theory of arithmetic Fuchsian groups and quaternion algebras). For instance this theorem states that $k\Gamma^\sigma = (k\Gamma)^\sigma$, hence if $\Gamma$ and $\Gamma'$ are two arithmetic Fuchsian groups whose invariant trace fields are not Galois conjugate, then the curves $C = \mathbb{H}/\Gamma$ and $C' = \mathbb{H}/\Gamma'$ cannot be Galois conjugate.

In this paper we will use this theorem to study the behaviour of the torsion of the group $\text{Comm}(\Gamma)$ under Galois action. To be more precise, let $\mathcal{P}(\Gamma) \subset \mathbb{N}$ denote the set of orders (or periods) of the finite order elements of $\text{Comm}(\Gamma)$. Then we will show that (Theorem 20).

- $\mathcal{P}(\Gamma) = \mathcal{P}(\Gamma^\sigma)$, for any $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$.

In other words the set of periods of $\text{Comm}(\Gamma)$ is another Galois invariant which could tell apart surface groups uniformising curves in different Galois orbits.

C. Maclachlan [Mac1] has described the set $\mathcal{P}(\Gamma) \subset \mathbb{N}$ in terms of the invariant quaternion algebra of $\Gamma$, namely

$$\mathcal{P}(\Gamma) = \{m \in \mathbb{N} : \cos \frac{2\pi}{m} \in k\Gamma \text{ and the field } k\Gamma(e^{2\pi i/m}) \text{ embeds in } A\Gamma\}$$

Using this one can deduce, for instance, that the subset $\mathcal{P}^{\text{odd}}(\Gamma)$ of odd periods admits the following simpler description (Proposition 27).
• \( P^{\text{odd}}(\Gamma) = \{ m \in \mathbb{N} : m \text{ odd and } A\Gamma \text{ contains a square root of } -\sin^2 \frac{2\pi}{m} \} \)

Hence the solvability of the quadratic equations \( X^2 + \sin^2 \frac{2\pi}{2k+1}, \ k \in \mathbb{N} \) is also Galois invariant.

Using this last description of the periods it will not be difficult to construct an explicit family of arithmetic groups \( \Gamma_p, p \) an odd prime, enjoying the property that \( \text{Comm}(\Gamma_p) \) contains an element of odd prime order \( q \) if and only if \( p = q \) (Example 28).

Since algebraic curves uniformised by arithmetic groups are defined over number fields (Proposition 12), in the arithmetic case the invariants of the action of \( \text{Gal}(\mathbb{C}/\mathbb{Q}) \) can be seen as invariants of the action of the more interesting group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \) the absolute Galois group. We will prove the following result (Theorem 29):

• \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts faithfully on the set of \( \text{PSL}_2(\mathbb{R}) \)-conjugacy classes of Fuchsian surface groups. Moreover, this action possesses the following invariants:

1. The isomorphism class of the group \( \text{Comm}(\Gamma) \).
2. The set \( \mathcal{P}(\Gamma) \) of periods of \( \text{Comm}(\Gamma) \).
3. The solvability of the quadratic equations \( X^2 + \sin^2 \frac{2\pi}{2k+1}, \ k \in \mathbb{N} \) in the invariant quaternion algebra \( A\Gamma \).
4. The Galois conjugacy class of the field \( k\Gamma \). (In fact \( k\Gamma^\sigma = (k\Gamma)^\sigma \) for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)).
5. The property of being a congruence subgroup.

2 Preliminaries

2.1 Arithmetic Fuchsian groups

Here we give a brief summary of some aspects of the theory of arithmetic Fuchsian groups relevant to this paper. For details we refer to [MR], chapter 8.2 (but see also the nice survey article [Mac2]).

Let \( k \) be a commutative field of characteristic \( \neq 2 \) and let \( a, b \in k^* \), the group of units in \( k \). We consider a quaternion algebra \( A \), also denoted by \((k, A)\), over \( k \) identified with \( \text{Hilbert symbol } A = (a, b) \). This is the \( k \)-algebra which as a vector space over \( k \) has a basis \( \{\tilde{1}, \tilde{i}, \tilde{j}, \tilde{k}\} \) such that its ring structure is determined by the relations \( \tilde{1} = 1, \tilde{i}^2 = a, \tilde{j}^2 = b \) and \( \tilde{k} = \tilde{i}\tilde{j} = -\tilde{j}\tilde{i} \). For instance, \( (\frac{-1}{1}) \) is the classical Hamilton quaternion field \( \mathbf{H} \) whereas \( (\frac{1,1}{\mathbb{R}}) = M_2(\mathbb{R}) \) (with basis \( \tilde{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tilde{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tilde{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) ). In fact these are the only two quaternion algebras over the field of real numbers up to isomorphism, for it is known that \( (\frac{a,b}{\mathbb{R}}) \) equals \( \mathbf{H} \) if \( a \) and \( b \) are simultaneously negative and equals \( M_2(\mathbb{R}) \) otherwise. The norm of an element \( x = x_0 + x_1\tilde{i} + \ldots \)
Let $x^2 + y^2 + z^2 + w^2 = 1$ with $1$ which is a finitely generated order $O \subset \mathbb{R}$ (possibly after conjugation in $PSL_2(\mathbb{R})$). We will call this group the arithmetic group associated with the quaternion algebra $(k, A)$.

From now on we assume that $k$ is a number field, that is a finite extension of $\mathbb{Q}$. Let us denote by $R_k$ the ring of integers of $k$. An order in $A$ is a subring $O \subset A$ with $1$ which is a finitely generated $R_k$-module satisfying $O \otimes_{R_k} k = A$. Given an order $O \subset A$ one may consider the group

$$O^1 = \{ x \in O : n_A(x) = 1 \}. $$

We will call this group the norm-1-group of $O$. For instance, if $A = \left( \frac{1,1}{\mathbb{Q}} \right) \cong M_2(\mathbb{Q})$, then $O = M_2(\mathbb{Z})$ is an order such that $O^1 \cong SL_2(\mathbb{Z})$.

If $\tau : k \to \mathbb{C}$ is a field embedding we will denote by $k^\tau$ the image of $k$ in $\mathbb{C}$ and by $A^\tau$ the quaternion algebra $A^\tau := \left( \frac{\tau(a), \tau(b)}{k^\tau} \right)$.

Let us now assume that $A = \left( \frac{a,b}{k} \right)$ satisfies the following conditions:

1. $k$ is a totally real number field (which means that the image $k^\tau$ of any embedding $\tau : k \to \mathbb{C}$ lies in $\mathbb{R}$).

2. There is an isomorphism $\rho : A \otimes_k \mathbb{R} = \left( \frac{a,b}{\mathbb{R}} \right) \cong M_2(\mathbb{R})$. One then says that $A$ is unramified at the infinite place corresponding to identity embedding $id$.

3. $A^\tau \otimes_{k^\tau} \mathbb{R} := \left( \frac{\tau(a), \tau(b)}{\mathbb{R}} \right) \cong \mathbb{H}$, for every $\tau$ different from the inclusion map.

In this case one says that $A$ is ramified at the infinite places $\tau \neq id$ (see 5.1).

In that situation a deep theorem by Borel and Harish-Chandra ensures that if $G < PSL_2(\mathbb{R})$ is a subgroup commensurable to a group of the form $P(\rho(O^1))$, where $O$ is an order of $A$ and $P : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ stands for the canonical projection, then, $G$ is a finite volume Fuchsian group. Groups arising in this way (possibly after conjugation in $PSL_2(\mathbb{R})$) are called arithmetic (Fuchsian) groups (associated with the quaternion algebra $(k, A)$). Note that the condition that $A$ is unramified at the identity and ramified at all embeddings different from the identity is a choice that is not essential for the validity of Borel-Harish-Chandra theorem. It is rather that any quaternion algebra $(k, A)$ with $k$ totally real and the property that there exists exactly one embedding $\tau_0$ such that $A^{\tau_0} \otimes_{k^{\tau_0}} \mathbb{R} \cong M_2(\mathbb{R})$ and $A^{\tau} \otimes_{k^\tau} \mathbb{R} \cong \mathbb{H}$ for all embeddings $\tau \neq \tau_0$ will give rise to Fuchsian groups. The condition is often expressed as

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H}^{[k : \mathbb{Q}]-1},$$

where $[k : \mathbb{Q}]$ denotes the degree of the number field $k$. It is known that an arithmetic group associated with $(k, A)$ fails to be co-compact only when $(k, A) \cong (\mathbb{Q}, M_2(\mathbb{Q}))$. 


The quaternion algebra \((k, A)\) with which a given arithmetic Fuchsian group \(\Gamma\) is associated can be recovered as follows:

\[
k = k\Gamma = \mathbb{Q}\left(\text{trace}(\gamma) : P(\gamma) \in \Gamma^{(2)}\right),
\]

where \(\Gamma^{(2)}\) is the group generated by all squares of elements of \(\Gamma\), and

\[
A = A\Gamma = \{ \sum a_i \gamma_i : a_i \in k\Gamma, P(\gamma_i) \in \Gamma^{(2)} \}.
\]

Since the arithmetic group associated with a given quaternion algebra is only defined up to commensurability equivalence one finds that two arithmetic Fuchsian groups \(\Gamma_1\) and \(\Gamma_2\) are commensurable if and only if \((k\Gamma_1, A\Gamma_1) = (k\Gamma_2, A\Gamma_2)\). Accordingly \(k\Gamma\) and \(A\Gamma\) are referred to as the invariant trace field and the invariant quaternion algebra respectively.

The commensurator of an arithmetic Fuchsian group \(\Gamma\) can be obtained from its invariant quaternion algebra as

\[
\text{Comm}(\Gamma) = A^+/k^* \cong \tilde{P}(A^+),
\]

where \(A^+ = \{ X \in A\Gamma := n(X^\tau) > 0 \text{ for all } \tau \in \text{Hom}(k, \mathbb{R}) \}\) is the group of all elements in \(A\) with totally positive norm and \(\tilde{P}\) stands for the homomorphism obtained by first embedding \(A\) into \(M_2(\mathbb{R})\) via the identity embedding, then dividing each matrix \(X\) by the square root of its determinant to get an element of \(SL_2(\mathbb{R})\) and finally applying the projection \(P : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})\).

We end this section by stating one of the most important theorems in the theory of arithmetic Fuchsian groups:

**Margulis’ theorem:** For a finite volume Fuchsian group \(\Gamma\) the following three conditions are equivalent.

1. \(\Gamma\) is arithmetic.
2. \(\Gamma\) has infinite index in \(\text{Comm}(\Gamma)\).
3. \(\text{Comm}(\Gamma)\) is a dense subgroup of \(PSL_2(\mathbb{R})\) in the usual matrix topology.

**Warning:** Given the invariant quaternion algebra \(A\Gamma = \left(\frac{a, b}{k}\right)\) of the group \(\Gamma\) uniformising an algebraic curve \(C\) and an element \(\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})\) one should not confuse \(A\Gamma^\sigma\) with \(\left(\text{Comm}(\Gamma)^\sigma\right)\). The first one is the invariant quaternion algebra of the Fuchsian group uniformising the algebraic curve \(C^\sigma\) whereas the latter refers to the quaternion algebra \(\left(\frac{\sigma(a), \sigma(b)}{k^\sigma}\right)\). In general these two algebras are different, see Theorem 16.

2.1.1 An explicit example

We now construct an arithmetic group derived from a quaternion algebra over the real field \(k = \mathbb{Q}(\sin \frac{2\pi}{p})\), where \(p \geq 5\) is a prime number. This example will be
revisited later in the paper. We shall start with a simple observation relative to
this field \( k \).

**Lemma 1.** \( k = \mathbb{Q}(\sin\frac{2\pi}{p}) = \mathbb{Q}(\cos\frac{\pi}{2p}). \) In particular \( k \) is a totally real field.

**Proof.** The identity \( \sin\frac{2\pi}{p} = -\cos\frac{(4+p)\pi}{2p} \) shows that \( k \) is the subfield of the
cyclotomic field \( \mathbb{Q}(e^{(4+p)2\pi i/4p}) = \mathbb{Q}(e^{2\pi i/4p}) \) fixed by the complex conjugation,
where the last equality holds because \( 4 + p \) and \( 4p \) are co-prime. The result
follows.

With \( k \) as above, our quaternion algebra is going to be
\[ A = \left( \begin{array}{cc} -1 & b_p \\ \frac{b_p}{k} & \end{array} \right) \text{ where } b_p = \cos(\frac{2\pi}{p}) - 1 + \frac{32}{p^2} \]
(instead of \( 1 - \frac{32}{p^2} \) one can take any other rational number lying between \( \cos\frac{4\pi}{p} \) and \( \cos\frac{2\pi}{p} \)).

Let us check that \( A \) satisfies the three conditions required to apply the Borel–
Harish-Chandra theorem.
1. By Lemma 1 the field \( k \) is totally real.
2. Indeed \( A \otimes_k \mathbb{R} \cong M_2(\mathbb{R}) \), via the isomorphism that sends \( \hat{1}, \hat{i}, \hat{j} \) and \( \hat{k} \) to
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{b_p} & 0 \\ 0 & -\sqrt{b_p} \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -\sqrt{b_p} \\ \sqrt{b_p} & 0 \end{pmatrix}. \]
3. For any \( \tau : k \to \mathbb{C} \) different from the inclusion map we have
\[ b_p^\tau = \tau \left( \cos\frac{2\pi}{p} - 1 + \frac{32}{p^2} \right) = \cos\frac{2l\pi}{p} - 1 + \frac{32}{p^2}; \text{ for some } 2 \leq l \leq (p - 1)/2 \]
which is a negative real number, and therefore \( A^\tau \otimes_k \mathbb{R} = \left( \frac{-1, b_p^\tau}{\mathbb{R}} \right) \cong \mathbb{H} \).

Now, the fact that the ring of integers of \( k = \mathbb{Q}(\cos\frac{\pi}{2p}) \) is \( R_k = \mathbb{Z}[2\cos\frac{\pi}{2p}] \)
(see e.g. [Wa], Proposition 2.16) allows us to write down an obvious explicit order
of \( A \), namely
\[ \mathcal{O} = R_k + R_k \hat{i} + R_k \hat{j} + R_k \hat{k} \]
\[ \cong \left\{ X = \begin{pmatrix} a_1 + a_3\sqrt{b_p} & a_2 - a_4\sqrt{b_p} \\ -a_2 - a_4\sqrt{b_p} & a_1 - a_3\sqrt{b_p} \end{pmatrix} : a_i \in \mathbb{Z}[2\cos\frac{\pi}{2p}] \right\} \]
According to what has been said above, for every prime number \( p \geq 5 \), the group
\[ \Gamma_p = P(\mathcal{O}^1) = \left\{ X \in \mathcal{O} : (a_1^2 - a_3^2b_p) + (a_2^2 - a_4^2b_p) = 1 \right\} \]
(2)
is going to be a co-compact arithmetic Fuchsian group. Moreover, by Selberg’s
lemma suitable finite index subgroups of \( \Gamma_p \) will be surface groups.
2.2 Solenoids

Let \((C, p)\) be a pointed algebraic curve, or, equivalently, a pointed compact Riemann surface, of genus greater than one. We shall denote by \(\Lambda\) the collection of all pointed unramified covers of \((C, p)\) so that \(\lambda \in \Lambda\) stands for a triple \((C_\lambda, p_\lambda; f_\lambda)\) where \(f_\lambda : C_\lambda \to C\) is an unramified cover with \(f_\lambda(p_\lambda) = p\). Endowed with the partial order defined by \(\lambda \leq \mu\) if there is an unramified cover \(f_{\mu,\lambda} : C_\mu \to C_\lambda\) such that \(f_{\mu,\lambda}(p_\mu) = p_\lambda\) and \(f_\lambda \circ f_{\mu,\lambda} = f_\mu\), the set \(\Lambda\) becomes a directed set. The maps \(f_{\mu,\lambda}\) are sometimes called the bonding functions. If we denote by \(o\) the element of \(\Lambda\) representing the triple \((C, p; id.)\) then \(f_{\mu,o} = f_\mu\).

The family \(\{(C_\mu, f_{\mu,\lambda})\}_{\lambda \leq \mu \in \Lambda}\) forms an inverse system of coverings of \(C\). By the solenoid associated with \(C\), which we will denote by \(H_C\) = \(\varprojlim \left\{ C_\lambda \right\}\), we shall mean to the projective limit of the inverse system (3). (Here and in the sequel we refer to [RZ] for generalities on inverse systems and projective limits).

Suppose that we choose another base point \(p' \in C\), then a canonical identification between pointed covers of \((C, p)\) and pointed covers of \((C, p')\) can be obtained by making the triple \((C_\lambda, p_\lambda; f_\lambda)\) correspond to the triple \((C_\lambda, p'_\lambda; f_\lambda)\) where \(p'_\lambda\) is the endpoint of the lift to \((C_\lambda, p_\lambda; f_\lambda)\) of a simple path in \(C\) connecting \(p\) to \(p'\). Under this identification the inverse system (3) remains unchanged. This means that the definition of \(H_C\) is independent of the choice of the base point \(p \in C\).

However, the fact that we work with pointed covers allows the following group theoretic interpretation of \(H_C\). Let \(H\) denote the upper half plane and let us fix a pointed universal cover map \((H, \ast) \to (C, p)\). Then we can canonically identify the fundamental group \(\pi_1(C, p)\) with the group \(\Gamma < \text{PSL}_2(\mathbb{R})\) of deck transformations of this cover and each fundamental group \(\pi_1(C_\lambda, p_\lambda)\) with the subgroup \(\Gamma_\lambda < \Gamma\) of deck transformations of the covering \((H, \ast) \to (C_\lambda, p_\lambda)\). We then have commutative diagrams

\[
\begin{array}{ccc}
\mathbb{H}/\Gamma_\lambda & \xrightarrow{\phi_\lambda} & C_\lambda \\
\downarrow & & \downarrow f_\lambda \\
\mathbb{H}/\Gamma & \xrightarrow{\phi_\lambda} & C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbb{H}/\Gamma_\mu & \xrightarrow{\phi_\mu} & C_\mu \\
\downarrow & & \downarrow f_{\mu,\lambda} \\
\mathbb{H}/\Gamma_\lambda & \xrightarrow{\phi_\lambda} & C_\lambda
\end{array}
\]  

(4)

where the vertical arrows on the left are the obvious projection maps induced by the corresponding inclusions of Fuchsian groups and the horizontal ones are isomorphisms of Riemann surfaces uniquely determined by the choice of \(\phi_\lambda\) and the condition \(\phi_\lambda([\ast]) = p_\lambda\). All this allows us to alternatively rewrite our solenoid as \(H_\Gamma = \varprojlim \mathbb{H}/\Gamma_\lambda\), where \(\Gamma_\lambda\) ranges among all finite index subgroups of \(\Gamma\). The use of \(H_C\) or \(H_\Gamma\) through the paper will tend to depend on whether we want to emphasize the algebraic or the hyperbolic nature of the solenoid.

We notice that for any finite index subgroup of \(K < \Gamma\) the family of finite index subgroups of \(K\) provides a co-final family of finite index subgroups of \(\Gamma\) which means that \(H_\Gamma \cong H_K\). In other words, the solenoid \(H_\Gamma\) depends only on the commensurability class of \(\Gamma\).
In the same vein we observe that in defining the solenoid we can restrict ourselves to the collection of finite index normal subgroups, since they already form a co-final family, and so we can view $\mathcal{H}_C$ as the projective limit of pointed Galois covers of $C$, that is

$$\mathcal{H}_C \equiv \mathcal{H}_\Gamma := \lim_{\Gamma \rhd f; \Gamma} \mathbb{H}/\Gamma$$

(5)

where the notation $\rhd f$ stands for finite index normal subgroup.

We recall that the profinite completion of $\Gamma$ is precisely the group

$$\hat{\Gamma} := \lim_{\Gamma \rhd f; \Gamma} \Gamma/\Gamma$$

(6)

and that each of the finite groups $\Gamma/\Gamma_\lambda$ can be viewed as the automorphism group of the covering $\mathbb{H}/\Gamma_\lambda \to \mathbb{H}/\Gamma$ which through the identifications in (4) can be further identified to $\text{Aut}(C_\lambda, f_\lambda)$, the group of automorphisms of the equivalent covering $f_\lambda : C_\lambda \to C$. This means that the group $\hat{\Gamma}$ is isomorphic to the algebraic fundamental group of $C$, usually denoted $\pi_1^{alg}(C, p) := \lim_{\lambda} \text{Aut}(C_\lambda, f_\lambda)$. As the similarity of the expressions (5) and (6) suggests, there is a natural action of the group $\hat{\Gamma}$ on the solenoid $\mathcal{H}_C$ which will play an important role later on.

It is sometimes convenient to choose a co-final sequence of finite index normal subgroups

$$\cdots \Gamma_{n+1} < \Gamma_n < \Gamma_{n-1} < \cdots < \Gamma_1 = \Gamma; \quad \text{with} \quad \bigcap_n \Gamma_n = \{1\}.$$  

and regard $\mathcal{H}_\Gamma$ (resp. $\hat{\Gamma}$) as the projective limit of the Riemann surfaces $\mathbb{H}/\Gamma_n$ (resp. the groups $\Gamma/\Gamma_n$). Typically one chooses $\Gamma_n$ be the standard characteristic sequence of subgroups of $\Gamma$ defined as

$$\Gamma_n = \text{intersection of all subgroups of } \Gamma \text{ of index } \leq n.$$  

(7)

So from now on we will use freely the following alternative notation

$$\mathcal{H}_\Gamma = \lim_{\Gamma/\Gamma_n} \mathbb{H}/\Gamma_n = \lim_{\Gamma/\Gamma_n} C_n = \mathcal{H}_C$$

and

$$\hat{\Gamma} = \lim_{\Gamma/\Gamma_n} \Gamma/\Gamma_n \equiv \lim_{\Gamma/\Gamma_n} \text{Aut}(C_n, f_n) = \pi_1^{alg}(C, p),$$

(8)

where the algebraic curves $C_n$ and the bonding functions $f_{n,n-1} : C_n \to C_{n-1}$ and $f_n : C_n \to C$ correspond to the Riemann surfaces $\mathbb{H}/\Gamma_n$ and the obvious projection maps $\pi_{n,n-1} : \mathbb{H}/\Gamma_n \to \mathbb{H}/\Gamma_{n-1}$ and $\pi_n : \mathbb{H}/\Gamma_n \to \mathbb{H}/\Gamma$ through the identifications in (4).

The group $\hat{\Gamma}$ may also be seen as the standard Cauchy sequence completion of $\Gamma$ with respect to the non-Archimedean metric $d$ defined by the formula
\[d(\gamma_1, \gamma_2) = \frac{1}{n} \quad \text{if} \quad \gamma_2^{-1}\gamma_1 \in \Gamma_n \setminus \Gamma_{n+1}\]

Usually this completion is realized as follows:
\[\hat{\Gamma} = \{\tau = (\tau_n)_n \in \prod_n (\Gamma/\Gamma_n) : \tau_n \equiv \tau_{n-1} (mod \, \Gamma_{n-1})\} \quad (9)\]

The solenoid \(H^{\hat{\Gamma}}\) can be canonically identified with the quotient of the product \(H \times \hat{\Gamma}\) under the action of \(\Gamma\) defined by
\[\gamma(z, \tau) = (\gamma(z), \tau \gamma^{-1})\]

This quotient space we shall denote by \(H \times_{\Gamma} \hat{\Gamma}\) and the identification can be realized as follows (see [Odd], Theorem 3.5):
\[H \times_{\Gamma} \hat{\Gamma} \equiv \mathcal{H}_\Gamma \quad \text{and} \quad (\tau_n(z) \in \mathbb{H}/\Gamma_n)\]  

Both the hyperbolic metric \(d_h\) on \(\mathbb{H}\) and the non-archimedean metric \(\hat{d}\) on \(\hat{\Gamma}\) are \(\Gamma\)-invariant and so the corresponding product metric on \(\mathbb{H} \times \hat{\Gamma}\) descends to a well-defined metric \(d_H\) on \(\mathcal{H}_\Gamma\) that induces the inverse-limit topology. In explicit terms
\[d_H([z_1, \tau_1], [z_2, \tau_2]) = \inf_{g \in \Gamma} \max \left\{d_h(gz_2, g\gamma^{-1}), \hat{d}(\tau_2, \tau_1 g^{-1})\right\}\]

Let \(r = 1/n\) be smaller than the injectivity radius of \(\mathbb{H}/\Gamma\) (or, equivalently, the minimum of the translation length of the elements in \(\Gamma \setminus \{\text{Id}\}\)); then the ball of radius \(r\) in \(\mathcal{H}_\Gamma\) centered at a point \([z, \tau]\) equals \(D \times T\) where \(D\) is the hyperbolic disc of radius \(r\) centered at \(z\) and \(T\) is the coset \(\tau \Gamma_{n+1}\). We see that \(\mathcal{H}_\Gamma\) carries a complex structure with respect to the \(z\)-variable. Specifically, a continuous function \(f(z, \tau)\) defined on an open set of \(\mathcal{H}_\Gamma\) will be said to be holomorphic if it is so with respect to the \(z\)-variable. As a topological space \(\mathcal{H}_\Gamma\) is compact and connected, but not path connected. There are natural projections
\[
\begin{array}{ccc}
\mathcal{H}_C & \equiv & \mathbb{H} \times_{\Gamma} \hat{\Gamma} \\
\pi_1 & & \pi_2 \\
\mathcal{C} & \equiv & \mathbb{H}/\Gamma \\
& & \hat{\Gamma}/\Gamma \\
\end{array}
\]

where \(\hat{\Gamma}/\Gamma\) stands for the set of left co-sets. The fibers of \(\pi_1\) are totally disconnected compact sets identifiable to \(\hat{\Gamma}\) and the fibres of \(\pi_2\), called the leaves, are the path connected components, each of them being a dense subset of \(\mathcal{H}_C\). They are parametrized by \(\hat{\Gamma}/\Gamma\). The leaf
\[\mathcal{B}_C = \mathcal{B}_\Gamma := \pi_2^{-1}(\{1\}) = \mathbb{H} \times_{\Gamma} \{1\}\]

will be called the base leaf and it is conformally equivalent to \(\mathbb{H}\).

Riemann surface solenoids were introduced by D. Sullivan in [Sul] and were then studied by several authors, including Sullivan himself, Biswas, Markovic, Nag, Odden, Fenner, Šarić and several others ([BN], [MaSa], [Odd], [PS]).
2.2.1 Automorphisms of a solenoid

We start with a couple of observations

**Lemma 2.** Let \( \{ \Gamma_n \} \) be the standard sequence of characteristic subgroups introduced in \([4]\). Then

1. An element \( \alpha \in PSL_2(\mathbb{R}) \) lies in \( \text{Comm}(\Gamma) \) if and only if for any index \( n \) there is a finite index subgroup \( K_n < \Gamma_n \) such that \( \alpha K_n \alpha^{-1} \) is also a finite index subgroup of \( \Gamma_n \).

2. Let \( \alpha \in \text{Comm}(\Gamma) \). There is an increasing sequence of positive integers \( a_n = a_n(\alpha) \) such that \( \alpha \Gamma_{a_n} \alpha^{-1} < \Gamma_n \).

**Proof.**

(1) Since \( \text{Comm}(\Gamma_n) = \text{Comm}(\Gamma) \), it is enough to prove it for \( \Gamma_1 = \Gamma \). In one direction this is proved by letting \( K \) be \( \Gamma \cap \alpha^{-1} \Gamma \alpha \). For the converse statement note that now by hypothesis \( \Gamma \cap \alpha \Gamma \alpha^{-1} \) contains \( \alpha K \alpha^{-1} \) which is a finite index subgroup of both \( \Gamma \) and \( \alpha \Gamma \alpha^{-1} \), hence \( \Gamma \cap \alpha \Gamma \alpha^{-1} \) must also be a finite index subgroup of both \( \Gamma \) and \( \alpha \Gamma \alpha^{-1} \).

(2) follows by choosing \( a_n \) such that \( \Gamma_{a_n} \) is contained in \( K_n \).

The first part of Lemma 2 allows us to regard the elements \( \alpha \in \text{Comm}(\Gamma) \) as automorphisms (i.e. holomorphic bijections) of \( H \Gamma \) in the following way. Choose a finite index subgroup \( K < \Gamma \) such that \( \alpha K \alpha^{-1} < \Gamma \). Then the automorphism induced by \( \alpha \) results as composition of the following three isomorphisms: Start with the identification \( H \Gamma \equiv H K \), then compose with the obvious isomorphism \( H K \sim \cong H \alpha K \alpha^{-1} \) and then with the identification \( H \alpha K \alpha^{-1} \equiv H \Gamma \). In more explicit terms:

\[
H_{\Gamma} \equiv H_{K} \stackrel{\sim}{\to} H_{\alpha K \alpha^{-1}} \equiv H_{\Gamma} \]

(11)

where \( \tau' \in \hat{K}, \gamma \in \Gamma \) and \( \tau = \tau' \gamma \). (Here we are using the identification \( \hat{K} / K \cong \hat{\Gamma} / \Gamma \), which is a direct consequence of the elementary identity \( \Gamma / K \cong \hat{\Gamma} / \hat{K} \).

Note that on the base leaf \( \alpha \) acts simply by the formula \( \alpha[z, \text{Id}] = [\alpha z, \text{Id}] \), thus \( \alpha \) is base leaf preserving. Conversely, let \( F : H_{\Gamma} \to H_{\Gamma} \) be an automorphism preserving the base leaf \( B_{\Gamma} \equiv \mathbb{H} \) then, clearly, \( F(z, \text{Id}) = [\alpha z, \text{Id}] \) for some \( \alpha = \alpha_F \in PSL_2(\mathbb{R}) \) and it turns out that, in fact, \( \alpha \in \text{Comm}(\Gamma) \). This result can be found in \([\text{Odd}]\), Corollary 4.8 and \([\text{BN}], \text{Proposition} 4.12\).

On the other hand an element \( \tau \in \hat{\Gamma} \) acts as an automorphism of \( H_{\Gamma} \) simply by the rule

\[
\tau([z, \tau']) = [z, \tau \tau']
\]

(12)

This formula defines a transitive action of \( \hat{\Gamma} \) on the set of leaves which agrees with the standard group action of \( \hat{\Gamma} \) on \( \hat{\Gamma} / \Gamma \). This action is base leaf preserving only when \( \tau \in \Gamma \). In other words, one has

\[
\hat{\Gamma} \cap \text{Comm}(\Gamma) = \Gamma
\]

(13)
Now, let \( F \in \text{Aut}(H_\Gamma) \) be an arbitrary automorphism of \( H_\Gamma \) and suppose that \( F \) sends the base leaf \( \pi^{-1}(1) \) to another leaf \( \pi^{-1}(\tau) \), then \( \tau^{-1} \circ F \) will be a base leaf preserving automorphism, thus we see that

\[
\text{Aut}(H_\Gamma) = \hat{\Gamma} \cdot \text{Comm}(\Gamma) = \text{Comm}(\Gamma) \cdot \hat{\Gamma}
\]

where the second equality holds because the product of the two subgroups is a group.

3 The group structure of \( \text{Aut}(H_\Gamma) \)

Turning to the identity (14) above we should mention that, in fact, Odden has shown (see Theorem 4.13 and Corollary 4.14 in [Odd]) that the rule \((\tau, \alpha) \to \tau \alpha\) yields a bijection between the set \( \hat{\Gamma} \times \text{Comm}(\Gamma) \) (where \( \Gamma \) acts by the rule \( \gamma(\tau, \alpha) = (\tau \gamma^{-1}, \gamma \alpha) \)) and \( \text{Aut}(H_\Gamma) \). The author warns however that “(this) theorem does not shed light on its group structure”. This group structure will turn out to be a certain completion of \( \text{Comm}(\Gamma) \) whose description is the goal of this section.

3.1 The Belyaev completion of \( \text{Comm}(\Gamma) \).

A Hecke pair is a pair of groups \((G, H)\) where \( H \) is a subgroup of \( G \) such that \( H \) and \( gHg^{-1} \) are commensurable for all \( g \in G \). If \( H \) is residually finite the natural left action of \( G \) on the sets of cosets

\[
B := \{ xK : x \in G \text{ and } K \text{ commensurable with } H \}
\]
determines a faithful permutation representation \( G \hookrightarrow \text{Symm}(B) \).

Endowed with the the topology of pointwise convergence arising from the discrete topology on \( B \) the group \( \text{Symm}(B) \) of permutations of \( B \) becomes a Hausdorff topological group (see [He], see also [KLQ] and [RW]). As \( xK \) varies in \( B \) the collection of stabilisers

\[
\text{Symm}(B)_{xK} = \{ \phi \in \text{Symm}(B) : \phi(xK) = xK \}
\]

provides a subbase of neighbourhoods of the identity for this topology. Accordingly, the collection of stabilisers

\[
G_{xK} = \text{Symm}(B)_{xK} \cap G = xKx^{-1}, \text{ with } xK \in B
\]
is a subbase of neighbourhoods of the identity for the induced topology on \( G \).

Here we will be concerned with the Hecke pair \((\text{Comm}(\Gamma), \Gamma)\) where, as before, \( \Gamma \) stands for the Fuchsian group uniformising a compact Riemann surface of genus \( g \geq 2 \). We will denote by \( \text{Comm}(\Gamma) \) the closure of (the image of) \( \text{Comm}(\Gamma) \) in \( \text{Symm}(B) \) and we will refer to this group as the Belyaev completion of \( \text{Comm}(\Gamma) \).

A feature of this completion is that it contains the group \( \hat{\Gamma} \) as the closure of \( \Gamma \). This can be seen by considering the following chain of topological groups

\[
\Gamma < \prod_n (\Gamma/\Gamma_n) < \prod_n \text{Symm}(\Gamma/\Gamma_n) < \text{Symm}(B^*) < \text{Symm}(B)
\]
where $B^* = \{ x\Gamma_n : x \in \text{Comm}(\Gamma), n \in \mathbb{N} \} \subset B$. Since $\prod_n (\Gamma/\Gamma_n)$ is compact, hence closed, the closure of $\Gamma$ in $\text{Symm}(B)$ agrees with its closure in $\prod_n (\Gamma/\Gamma_n)$, but this is one of the definitions of $\hat{\Gamma}$; see [Be].

Similarly, since $\text{Symm}(B^*)$ is clearly a closed subset of $\text{Symm}(B)$, the closures of $\text{Comm}(\Gamma)$ in $\text{Symm}(B^*)$ and $\text{Symm}(B)$ agree.

The next proposition collects the main properties of $\text{Comm}(\Gamma)$.

**Proposition 3.**

1. $\text{Comm}(\Gamma)$ is a locally compact subgroup of $\text{Symm}(B^*) < \text{Symm}(B)$ such that the closure $\overline{H}$ of each subgroup $H$ commensurable with $\Gamma$ is a compact open subgroup.

2. $\Gamma$ can be identified to $\hat{\Gamma}$.

3. $\text{Comm}(\Gamma) = \hat{\Gamma} \cdot \text{Comm}(\Gamma) = \text{Comm}(\Gamma) \cdot \hat{\Gamma}$

4. $\text{Comm}(\Gamma)/\hat{\Gamma} \equiv \text{Comm}(\Gamma)/\Gamma$

**Proof.** (1) is the content of Theorem 7.1 in [Be].

(2) is the comment preceding this proposition.

(3) Let $x \in \text{Comm}(\Gamma)$. Applying part (1) to $H = \Gamma$ we see that $x\Gamma \cap \text{Comm}(\Gamma)$ is non-empty, so there is some $\alpha \in \text{Comm}(\Gamma)$ such that $\alpha = x\tau$ for some $\tau \in \hat{\Gamma}$, hence $x \in \text{Comm}(\Gamma) \cdot \hat{\Gamma} = \hat{\Gamma} \cdot \text{Comm}(\Gamma)$.

(4) follows from (3). \[\square\]

**Remark 4.** ([KLQ], Example 2.4). We observe that although $\text{Comm}(\Gamma)$ is complete (being locally compact), the group $\text{Symm}(B)$ itself is not. For instance the sequence of functions

$$
\phi_n(xK) = \begin{cases} 
  xK & \text{if } xK \neq \Gamma_i, \text{ with } 1 \leq j \leq n \\
  \Gamma_{i+1} & \text{if } xK = \Gamma_i, \text{ with } i < n \\
  \Gamma_1 & \text{if } xK = \Gamma_n
\end{cases}
$$

converges to the shift map $\Gamma_i \rightarrow \Gamma_{i+1}$ which is not a bijection. In fact in [Be] the completion of $G$ is defined to be the closure $\overline{G}$ of $G$ in $\text{Map}(G)$, the semigroup of maps of $G$, and it is then shown that $G < \text{Symm}(B)$.

### 3.2 The isomorphism $\text{Aut}(\mathcal{H}_C) \cong \text{Comm}(\Gamma)$

**Lemma 5.** The inclusion $\text{Comm}(\Gamma) < \text{Aut}(\mathcal{H}_\Gamma)$ induces the following natural bijection between sets of cosets

$$
\text{Comm}(\Gamma)/\Gamma_n \simeq \text{Aut}(\mathcal{H}_{\Gamma})/\hat{\Gamma}_n \\
x\Gamma_n \rightarrow x\hat{\Gamma}_n
$$

**Proof.** When $n = 1$, $\Gamma_n = \Gamma$, and the result follows from (13) and (14). But the result holds for any $n$ because $\text{Comm}(\Gamma_n) = \text{Comm}(\Gamma)$ and $\mathcal{H}_{\Gamma_n} = \mathcal{H}_{\Gamma}$. \[\square\]

The bijections $\text{Comm}(\Gamma)/\Gamma_n \simeq \text{Aut}(\mathcal{H}_{\Gamma})/\hat{\Gamma}_n$ permit us to identify the sets $B^* = \{ x\Gamma_n : x \in \text{Comm}(\Gamma), n \in \mathbb{N} \}$ and $\hat{B} = \{ F\hat{\Gamma}_n : F \in \text{Aut}(\mathcal{H}_{\Gamma}), n \in \mathbb{N} \}$ and therefore to transfer the natural action of $\text{Aut}(\mathcal{H}_{\Gamma})$ on $\hat{B}$ to an action on $B^*$. 

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Lemma 6.  
1. The action of Aut(HΓ) on B∗ introduced above is given by the formula
\[
\text{Aut}(HΓ) \times B^* \rightarrow B^*
\]
\[
(F, x\Gamma_n) \mapsto \alpha x\Gamma_n
\]
where \(\alpha = \alpha(F, x, n) \in \text{Comm}(\Gamma)\) is such that \(\alpha^{-1}F \in x\hat{\Gamma}_n x^{-1}\).

2. This action is faithful and so it yields a permutation representation of Aut(HΓ) in Symm(B∗) < Symm(B).

3. Its restriction to Comm(Γ) is the standard action \((\alpha, x\Gamma_n) \rightarrow \alpha x\Gamma_n\).

Proof. 1) The condition imposed on \(\alpha\) only means that \(Fx\hat{\Gamma}_n = \alpha x\hat{\Gamma}_n\).

2) and 3) are obvious.

This lemma together with Proposition 3 allows us to regard both groups Comm(Γ) and Aut(HΓ) as subgroups of Symm(B∗).

Theorem 7.

\(\text{Comm}(\Gamma) = \text{Aut}(HΓ)\)

Proof. Let us denote by \(j : \text{Aut}(HΓ) \hookrightarrow \text{Symm}(B^*)\) the permutation representation described in Lemma 6. What we have to prove is that \(\text{Comm}(\Gamma) = j(\text{Aut}(HΓ))\).

By the third part of this lemma we only need to show that 1) \(j(\text{Aut}(HΓ))\) is a closed subset of \(\text{Symm}(B^*)\) and 2) \(j(\text{Comm}(\Gamma))\) is dense in it.

1) \(j(\text{Aut}(HΓ))\) is a closed subset of \(\text{Symm}(B^*)\).

We start with the observation that Comm(Γ) is countable. Indeed in the non-arithmetic case Comm(Γ) is still discrete and in the arithmetic case even the whole invariant quaternion algebra \(A\Gamma\) is countable. So let

\[\text{Comm}(\Gamma) = \{x_1 = 1, x_2, \cdots, x_n, \cdots\}\]

be an enumeration of its elements. Then we can inductively construct a co-final subsequence \(\Gamma_{a_1} > \Gamma_{a_2} > \cdots > \Gamma_{a_n} > \cdots\) of our standard sequence of characteristic subgroups \(\{\Gamma_n\}\) enjoying the property

\[\Gamma_{a_n} < \bigcap_{i,j \leq n} x_i\Gamma_j x_i^{-1}\]

Let now \(f \in \text{Symm}(B^*)\) be an element in the closure of \(j(\text{Aut}(HΓ))\). We want to show that \(f\) lies in fact in \(j(\text{Aut}(HΓ))\). In order to do that, for any \(n \in \mathbb{N}\), we consider the following neighbourhoods of \(f\) in \(\text{Symm}(B^*)\):

\[V(f, n) = \{h \in \text{Symm}(B^*) : h(x_i\Gamma_{a_i}) = f(x_i\Gamma_{a_i}); i,j \leq n\} = f \bigcap_{i,j \leq n} \text{Symm}(B^*)_{\Gamma_{a_i}}\]

Let \(F_n \in \text{Aut}(HΓ)\) such that \(j(F_n) \in V(f, n) \cap j(\text{Aut}(HΓ))\). By Lemma 6 \(j(F_n)(\Gamma_{a_n}) = \alpha_n \Gamma_{a_n}\) with \(\alpha_n \in \text{Comm}(\Gamma)\) such that \(\alpha_n^{-1}F_n \in \hat{\Gamma}_{a_n} \hat{\Gamma}_{a_{n-1}}\). Therefore \(j(F_n)(\Gamma_{a_{n-1}}) = \alpha_n \Gamma_{a_{n-1}}\). But, on the other hand, \(j(F_n)(\Gamma_{a_{n-1}}) = f(\Gamma_{a_{n-1}}) = \)
\[ j(F_{n-1}) \Gamma_{a_{n-1}} = \alpha_{n-1} \Gamma_{a_{n-1}}. \] The conclusion is that \( \alpha_{n-1}^{-1} \alpha_n \in \Gamma_{a_{n-1}}. \) Thus, we can write
\[
\begin{align*}
\alpha_2 &= \alpha_1 \gamma_1, & \gamma_1 &\in \Gamma_{a_1} \\
\alpha_3 &= \alpha_2 \gamma_2 = \alpha_1 \gamma_1 \gamma_2, & \gamma_2 &\in \Gamma_{a_2} \\
&\vdots & &\vdots \\
\alpha_i &= \alpha_1 \gamma_1 \gamma_2 \cdots \gamma_{i-1}, & \gamma_{i-1} &\in \Gamma_{a_{i-1}}
\end{align*}
\]

Now, put \( \tau_i = \gamma_1 \gamma_2 \cdots \gamma_i \pmod{\Gamma_{a_{i+1}}} \). Then, clearly, the sequence \( \tau = (\tau_i)_i \) defines an element of \( \hat{\Gamma} = \varprojlim \Gamma/\Gamma_{a_i}. \) We claim that \( f = j(F) \), where \( F = \alpha_1 \tau \in \text{Aut}(\mathcal{H}_C). \)

Indeed, \( f \) agrees with \( j(F) \) at the points \( \Gamma_{a_n} \in \mathcal{B}^* \) because on the one hand
\[
F(\Gamma_{a_n}) = j(F_{n})(\Gamma_{a_n}) = \alpha_n \Gamma_{a_n} \text{ and on the other hand}
\]
\[
\alpha_n^{-1} F = \alpha_n^{-1} \alpha_1 \tau = (\tau_{n-1}^{-1} \tau_i)_i \equiv (\gamma_n \gamma_{n+1} \cdots \gamma_{n+k-1} \pmod{\Gamma_{a_{n+k}}})_k \in \hat{\Gamma}_{a_n}.
\]

Moreover, since \( \hat{\Gamma}_{a_n} < x_i \hat{\Gamma} x_i^{-1} \) for \( n \geq i, j \), the above relation also proves that
\[
F(x_i \Gamma_j) = \alpha_n x_i \Gamma_j = f(x_i \Gamma_j) = f(x_i \Gamma_j) \text{ and we conclude that } j(F) \text{ agrees with } f \text{ at all points of } \mathcal{B}^*, \text{ as claimed.}
\]

2) \( j(\text{Comm}(\Gamma)) \) is dense in \( j(\text{Aut}(\mathcal{H}_F)). \)

Our subbasis of neighbourhoods of the identity in \( j(\text{Aut}(\mathcal{H}_F)) \) consists of the sets \( j(T_n T_n^{-1}), T \in \text{Aut}(\mathcal{H}_F) \); but since, by \([\text{[4]}]\), each \( T \) equals to a product of the form \( T = x \tau' \), with \( x \in \text{Comm}(\Gamma) \) and \( \tau' \in \hat{\Gamma} \), this subbasis can be rewritten as \( \{ j(x \Gamma_n x^{-1}) : x \in \text{Comm}(\Gamma) \} \). This makes it clear that any set in this subbasis contains one of the form \( \Gamma_m \) for sufficiently large \( m \). It follows that if \( F \in \text{Aut}(\mathcal{H}_F) \) any neighbourhood of \( j(F) \) contains a smaller one of the form \( V_m = j(F \cdot \Gamma_m) \). Thus, what we need to see is that for each \( m \) the intersection \( V_m \cap j(\text{Comm}(\Gamma)) \) is nonempty. Now, setting \( F = \alpha \tau \), with \( \alpha \in \text{Comm}(\Gamma) \) and \( \tau = (\tau_i)_i \in \hat{\Gamma} \) we see that, as before, \( (\alpha \tau_m)^{-1} F = \tau^{-1} \tau_m \in \Gamma_m \). This means that \( j(\alpha \tau_m) \in V_m \cap j(\text{Comm}(\Gamma)) \) as required.

\[ \square \]

4 Galois action on solenoids and commensurators

4.1 The group \( \text{Comm}(\Gamma) \) is Galois invariant

Let \( Gal(\mathbb{C}/\mathbb{Q}) \) denote the group of field automorphisms of \( \mathbb{C} \), and let \( \sigma \in Gal(\mathbb{C}/\mathbb{Q}) \). The obvious action of \( Gal(\mathbb{C}/\mathbb{Q}) \) on complex algebraic curves transforms pointed unramified covers of \((C, p)\) into pointed unramified covers of \((C^\sigma, p^\sigma)\) thereby yielding a natural bijection
\[
\sigma : \mathcal{H}_C \to \mathcal{H}_{C^\sigma} \\
(q_n) \to (q_n^\sigma)
\]
where \( \mathcal{H}_C \) is regarded as the limit of the inverse system \( \{ (C_n, f_{n,n-1}) \}_{n} \) introduced in \([\text{[3]}]\) so that \( q_n \) stands for a point in \( C_n \) such that \( f_{n,n-1}(q_n) = q_{n-1} \).

Our first goal is to show that although this bijection is far from being an isomorphism (even far from being continuous) it induces an isomorphism between the corresponding automorphism groups \( \text{Aut}(\mathcal{H}_C) \) and \( \text{Aut}(\mathcal{H}_{C^\sigma}) \). In order to do that we first need to write the elements of \( \text{Aut}(\mathcal{H}_C) \) in a convenient way.
Notice that to any increasing subsequence of positive integers \( \{a_n\}_n \) there corresponds a co-final subsequence of standard characteristic subgroups \( \{\Gamma_{a_n}\}_n \) and algebraic curves \( C_{a_n} \) which still define the solenoid, that is \( \mathcal{H}_C = \varprojlim C_{a_n} \). Let

\[
F = \{F_n\}_n : \{(C_{a_n}, f_{a_n,a_{n-1}})\}_n \rightarrow \{(C_n, f_{n,n-1})\}_n
\]

be a morphism of inverse systems. By that we mean (see \[HZ\]) that each \( F_n \) is a rational (or, equivalently, holomorphic) map between \( C_{a_n} \) and \( C_n \) such that for any \( n \geq 2 \) the following diagram commutes

\[
\begin{array}{ccc}
C_{a_n} & \xrightarrow{F_n} & C_n \\
\downarrow f_{a_n,a_{n-1}} & & \downarrow f_{n,n-1} \\
C_{a_{n-1}} & \xrightarrow{F_{n-1}} & C_{n-1}
\end{array}
\]

Clearly, in this situation, \( F \) defines a holomorphic map \( F : \mathcal{H}_C \rightarrow \mathcal{H}_C \). Our next result states that any automorphism of \( \mathcal{H}_C \) can be written in this way.

**Proposition 8.** Every automorphism \( F \) of \( \mathcal{H}_C \) is induced by a morphism of inverse limits as in \[15\].

**Proof.** We deal with the two different kinds of automorphisms separately.

1. \( F = \tau = (\tau_n)_n \in \hat{\Gamma} = \varprojlim \Gamma/\Gamma_n \).

From the expression of the automorphism \( \tau \) on the model of the solenoid \( \mathbb{H} \times \hat{\Gamma} \) given in \[12\] and the identification between the two models of the solenoid made in \[10\] one infers that in this case \( F \) is the automorphism induced by the morphism of inverse limits \( \{\tau_n\}_n : \{((\mathbb{H}/\Gamma_n, \pi_n))\}_n \rightarrow \{((\mathbb{H}/\Gamma_1, \pi_1))\}_n \) or its corresponding algebraic version \( \{F_n\}_n : \{C_n, f_n\}_n \rightarrow \{C_n, f_n\}_n \).

2. \( F = \alpha \in \text{Comm}(\Gamma) \).

Let \( \{\Gamma_{n_0}\} \) the co-final family of subgroups of \( \Gamma \) whose existence is guaranteed in part 2) of Lemma 2 and consider the holomorphic surjections \( A_n : \mathbb{H}/\Gamma_{n_0} \rightarrow \mathbb{H}/\Gamma_n \) defined by the following composition of maps

\[
\mathbb{H}/\Gamma_{n_0} \xrightarrow{\alpha} \mathbb{H}/\alpha \Gamma_{n_0} \xrightarrow{\alpha^{-1}} \mathbb{H}/\Gamma_n
\]

Clearly, these maps define a morphism of inverse systems \( \{A_n\} : \{((\mathbb{H}/\Gamma_{n_0}, \pi_{n_0}))\} \rightarrow \{((\mathbb{H}/\Gamma_1, \pi_1))\} \) which yields a holomorphic map \( A : \mathcal{H}_C \rightarrow \mathcal{H}_C \) which coincides with \( \alpha \) on the base leaf and, by continuity, on the whole solenoid. Now we only need to replace the holomorphic maps \( A_n : \mathbb{H}/\Gamma_{n_0} \rightarrow \mathbb{H}/\Gamma_n \) and \( \pi_{n_0,n-1} : \mathbb{H}/\Gamma_n \rightarrow \mathbb{H}/\Gamma_{n-1} \) by their corresponding rational maps \( F_n : C_{a_n} \rightarrow C_n \) and \( f_{n,n+1} : C_n \rightarrow C_{n-1} \).

3. Finally, for an arbitrary \( F \in \text{Aut}(\mathcal{H}_C) \) we can use \[14\] to write \( F \) in the form \( F = \alpha \circ \tau \) and then the conclusion follows by composition of the two previous cases. \( \square \)

**Proposition 9.** (1) For any \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \) the bijection \( \sigma : \mathcal{H}_C \rightarrow \mathcal{H}_C^\sigma \) introduced at the beginning of this section induces a group isomorphism

\[
\begin{array}{ccl}
\text{Aut}(\mathcal{H}_C) & \rightarrow & \text{Aut}(\mathcal{H}_C^\sigma) \\
F & \mapsto & F^\sigma := \sigma \circ F \circ \sigma^{-1}
\end{array}
\]
(2) This isomorphism maps $\pi_1^{alg}(C, p) \equiv \hat{\Gamma}$ onto $\pi_1^{alg}(C^\sigma, p^\sigma) \equiv \hat{\Gamma}^\sigma$.

(3) If the pair $(C, p)$ is defined over a number field one can replace in the above statements the group $\text{Gal}(\mathbb{C}/\mathbb{Q})$ by the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

**Proof.** (1) By Proposition 8 we can assume that $F \in \text{Aut}(H_C)$ is given by a morphism of inductive limits of the form

$$F = \{F_n\}_n : \left\{ \left( C_{an}, f_{an} \right) \right\}_n \to \left\{ \left( C_n, f_n \right) \right\}_n$$

and then $F^\sigma$ will be the automorphism of $H_C^\sigma$ induced by

$$F^\sigma = \{F^\sigma_n\}_n : \left\{ \left( C_{an}^\sigma, f_{an}^\sigma \right) \right\}_n \to \left\{ \left( C_n^\sigma, f_n^\sigma \right) \right\}_n$$

where $F^\sigma_n = \sigma \circ F_n \circ \sigma^{-1}$. We point out that each map $F^\sigma_n : C_{an}^\sigma \to C_n^\sigma$ is a rational (i.e. holomorphic) map, in fact the rational map whose defining equations are obtained by applying $\sigma$ to the defining equations of the rational map $F_n : C_{an} \to C_n$. We also observe that since $\{(C_{an}, p_{an})\}_n$ is a co-final family of pointed coverings of $(C, p)$ the family $\{(C_{an}^\sigma, p_{an}^\sigma)\}_n$ is a co-final family of coverings of $(C^\sigma, p^\sigma)$. We thus conclude that $H_C^\sigma$ is the projective limit of the inverse system $\left\{ \left( C_{an}^\sigma, f_{an}^\sigma \right) \right\}_n$ and that $F^\sigma$ induces an automorphism of $H_C^\sigma$.

(2) Clearly, if in the discussion above $F \in \pi_1^{alg}(C, p) = \varprojlim \text{Aut}(C_n, f_n)$, then $F^\sigma \in \pi_1^{alg}(C^\sigma, p^\sigma) = \varprojlim \text{Aut}(C_n^\sigma, f_n^\sigma)$, as stated.

(3) This statement follows from the observation that if $(C, p)$ is defined over $\mathbb{Q}$ then, the covering pointed curves $(C_{an}, p_{an})$ as well as the automorphisms $F_n$ in the proof of the part (2) above are also defined over $\overline{\mathbb{Q}}$ (see [Go]) and so the Galois conjugates $C_{an}^\sigma, p_{an}^\sigma$ and $F^\sigma_n$ of $C_{an}, p_{an}$ and $F_n$ make perfect sense when $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

We can now prove that the isomorphism class of the group $\overline{\text{Comm}(\Gamma)}$ is preserved under Galois action.

**Theorem 10.** Let $C$ be an algebraic curve uniformized by a Fuchsian group $\Gamma$.

Then the action of a Galois element $\sigma$ on the solenoid $H_C$ induces the following isomorphisms

1. $\overline{\text{Comm}(\Gamma)} \cong \overline{\text{Comm}(\Gamma^\sigma)}$ (isomorphism of topological groups).
2. $\overline{\text{Comm}(\Gamma)} \cong \overline{\text{Comm}(\Gamma^\sigma)}$ (isomorphism of sets of co-sets).
3. $\overline{\text{Comm}(\Gamma)} \cong \overline{\text{Comm}(\Gamma^\sigma)}$ (isomorphism of sets of co-sets).

**Proof.** (1) That $\sigma$ induces an isomorphism between these two groups follows from a combination of Theorem 7 and the first part of Proposition 9. The topological side follows from the second part of Proposition 9 which shows that this isomorphism sends the subbasis of neighbourhoods $\{\Gamma_n\}_n$ into the subbasis of neighbourhoods $\{\Gamma_n^\sigma\}_n$.

(2) follows from the second part of Proposition 9.

(3) follows from the previous statement and Proposition 9 part 4. 

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From here we can easily obtain the following result due to Kazhdan\cite{Kaz}

**Corollary 11.** \(\Gamma\) is arithmetic if and only if \(\Gamma^\sigma\) is arithmetic.

**Proof.** The result follows from the third part of Theorem \ref{thm:10} together with Margulis’ theorem.

The general result that complex varieties uniformised by arithmetic groups are defined over number fields is attributed to Baily and Borel \cite{BB} (although in their paper only the fact that they are quasi-projective varieties seems to be explicitly stated and perhaps reference to the work of Shimura is needed). In the case of dimension 1 we are concerned with here, we can use Corollary \ref{cor:11} to give the following simple proof.

**Proposition 12.** Let \(C\) be an algebraic curve uniformised by an arithmetic group \(\Gamma\). Then \(C\) is defined over a number field.

**Proof.** Let \(\sigma \in Gal(\mathbb{C})\). By Corollary \ref{cor:11} the group \(\Gamma^\sigma\) uniformising the algebraic curve \(C^\sigma\) is also arithmetic. Now, by a theorem of Takeuchi \cite[Theorem 2.1]{Ta2} there are only finitely many arithmetic surface groups of any given genus. Therefore, as \(\sigma\) varies in \(Gal(\mathbb{C})\), only finitely many isomorphism classes of curves \(C^\sigma\) are obtained. This means that \(C\) is defined over a number field (see \cite{Go}).

Unfortunately, the bijections \(\sigma : \mathcal{H}_C \rightarrow \mathcal{H}_{C^\sigma}\) will not preserve the base leaf in general and so there is no a priori reason why the isomorphism \(\text{Comm}(\Gamma) \simeq \text{Comm}(\Gamma^\sigma)\) above should preserve any properties of the commensurator. However we will show that:

1. The periods of \(\text{Comm}(\Gamma)\) are Galois invariant, and
2. In the non-arithmetic case even the isomorphism class of the group \(\text{Comm}(\Gamma)\) is also Galois invariant.

The proof of 1) will be carried out in Section \ref{sec:5} which is devoted to arithmetic groups. The proof of 2) is the content of the next subsection.

### 4.2 The non-arithmetic case

We next show that for non-arithmetic groups the isomorphism class of the group \(\text{Comm}(\Gamma)\) is also Galois invariant. In order to do that we need to recall the notion of *semiregular (or uniform)* covers. These are coverings of compact Riemann surfaces \(f : C \rightarrow C'\) such that all points of \(C\) within any given fibre have the same multiplicity. This is equivalent to saying that \(f : C \rightarrow C'\) corresponds to the projection \(\mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Gamma'\) induced by an inclusion of \(\Gamma\) in another Fuchsian group \(\Gamma'\) (see e.g. \cite{GG}). Note that \(\Gamma'\) will be the uniformising group of \(C'\) only when \(f\) is unramified. Otherwise it will be a *Fuchsian group of signature* \((g' ; m_1, \cdots, m_r)\) where \(g'\) is the genus of \(C'\) and \(m_1, \cdots, m_r\) the multiplicities of the branching values \(x'_1, \cdots, x'_r \in C'\) of the morphism \(f\). We will also need the following straightforward implication of Margulis’ theorem.
Lemma 13. 1. Let $\Gamma$ be a non-arithmetic Fuchsian surface group. Then $\text{Comm}(\Gamma)$ is the largest Fuchsian group containing $\Gamma$.

2. Let $C$ be an algebraic curve uniformised by a non-arithmetic Fuchsian group $\Gamma$. Then the uniform covering corresponding to the obvious projection $\mathbb{H}/\Gamma \to \mathbb{H}/\text{Comm}(\Gamma)$ can be recognised as the one of highest degree among all uniform coverings $f : C \to C'$ with source $C$ and arbitrary target $C'$.

Proof. (1) By Margulis' theorem $\text{Comm}(\Gamma)$ is a Fuchsian group. Now, if a group $\Gamma_1$ containing $\Gamma$ is Fuchsian then $\Gamma$ must have finite index in $\Gamma_1$. Thus, we have $\Gamma < \Gamma_1 < \text{Comm}(\Gamma_1) = \text{Comm}(\Gamma)$.

(2) Follows directly from (1).

Theorem 14. Let $C$ be an algebraic curve uniformised by a non-arithmetic Fuchsian group $\Gamma$. Then

$$\text{Comm}(\Gamma) \simeq \text{Comm}(\Gamma^\sigma)$$  
(isomorphic as abstract groups)

Proof. Let $f : C \to C'$ be the uniform cover of highest degree given by Lemma 13. Then, the Galois conjugate covering $f^\sigma : C^\sigma \to C'^\sigma$ is also a uniform cover of highest degree. By Corollary 11 the group $\Gamma^\sigma$ is also non-arithmetic and by Lemma 13 these two coverings correspond to the inclusions $\Gamma < \text{Comm}(\Gamma)$ and $\Gamma^\sigma < \text{Comm}(\Gamma^\sigma)$ respectively. Moreover, suppose that $C'$ has genus $g'$ and $f$ has $r$ branching values with multiplicities $m_1, \cdots, m_r$, then the same holds for $C'^\sigma$ and $f^\sigma$. This means that the groups $\text{Comm}(\Gamma)$ and $\text{Comm}(\Gamma^\sigma)$ are both Fuchsian groups with the same signature $(g'; m_1, \cdots, m_r)$ and therefore isomorphic.

5 Arithmetic groups

In this final section we will study more closely the effect of Galois action on the invariant quaternion algebra $(k\Gamma, A\Gamma)$ of an arithmetic Fuchsian surface group $\Gamma$, that is, an arithmetic Fuchsian groups which uniformises an algebraic curve $C \cong \mathbb{H}/\Gamma$. In view of Proposition 12 we can and we will replace the group $\text{Gal}(C/Q)$ with the absolute Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$. We have seen in Corollary 11 that for any $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ the Galois conjugate curve $C^\sigma$ is isomorphic to a Riemann surface $\mathbb{H}/\Gamma^\sigma$ where $\Gamma^\sigma$ is again arithmetic. We would like to know how the quaternion algebras $(k\Gamma, A\Gamma)$ and $(k\Gamma^\sigma, A\Gamma^\sigma)$ are related.

This relation has been described by Doi and Naganuma in [DN] (see also [MiSu]). In order to present their results we need to recall some facts from the theory of algebras over number fields.

5.1 Classification of quaternion algebras over number fields

Let $k$ be a totally real number field. The embeddings $\tau \in \text{Hom}(k, \mathbb{C}) (= \text{Hom}(k, \mathbb{R}))$ are called infinite or archimedean places of $k$. As already mentioned, the quaternion algebra $A = \left( \frac{a, b}{k} \right)$ is called unramified at the infinite place $\tau$ if $A^\tau \otimes_{k^\tau} \mathbb{R}$ is
isomorphic to $M_2(\mathbb{R})$, where $A^\sigma = \left( \frac{\tau(a), \tau(b)}{k_p} \right)$. Otherwise (i.e. if $A^\tau \otimes k^\sigma \cong H_k$) $A$ is called ramified at $\tau$. We denote by $\text{Ram}_{\infty}A$ the set of all infinite places at which $A$ is ramified. Our condition on $A$ is that $A$ is unramified exactly at $\tau = \text{id}$. Let $R_k$ denote the ring of integers in $k$ and let $p$ be a prime ideal in $R_k$. Then $p$ defines a non-archimedean absolute value on $k$, which is unique up to equivalence. The prime ideals in $R_k$ are also called finite or non-archimedean places of $k$. Let $k_p$ denote the completion of $k$ with respect to this absolute value and $\tau_p : k \to k_p$ be the corresponding embedding. We can consider the quaternion algebra $A^{(\tau_p)} = A \otimes_k k_p = \left( \frac{\tau_p(a), \tau_p(b)}{k_p} \right)$ over $k_p$. There are two possibilities for $A^{(\tau_p)}$: either $A^{(\tau_p)} \cong M_2(k_p)$ or $A^{(\tau_p)}$ is a division algebra which is then uniquely determined up to isomorphism. In the first case we say that $A$ unramified at the finite place $p$ whereas in the second case we call $A$ ramified at the finite place $p$. By $\text{Ram}(A)$ we will refer to the set of all finite places at which $A$ is ramified. We will further set $\text{Ram}(A) = \text{Ram}_{\infty}A \cup \text{Ram}_fA$.

Finally, we note that given a number field $k$ and an automorphism $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$, Galois conjugation defines an obvious ring isomorphism between $R_k$ and $R_k\sigma$ such that the rule $p \to p^\sigma$

provides a bijection between prime ideals of $R_k$ and prime ideals of $R_k\sigma$ which induces a bijection between $\text{Ram}_fA$ and $(\text{Ram}_fA)^\sigma := \{ p^\sigma : p \in \text{Ram}_fA \}$. We observe in passing that $(\text{Ram}_fA)^\sigma = \text{Ram}_fA^\sigma$.

The following facts are well-known (see e.g. [MR], Theorem 7.3.6, page 236):

1. $\text{Ram}(A)$ is a finite set of even cardinality
2. Let $T$ be a finite set of (archimedean and/or non-archimedean) places of $k$ with even cardinality, then there exists a quaternion algebra $A$ over $k$ such that $\text{Ram}(A) = T$. This quaternion algebra is uniquely determined up to isomorphism.

5.2 Doi-Naganuma’s theorem

Let $(k, A) = (k \Gamma, A \Gamma)$ be the quaternion algebra associated with an arithmetic surface group and let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The structure of the pair $(k \Gamma^\sigma, A \Gamma^\sigma)$ has been described by Doi and Naganuma in [DN] (see also [MiSu]).

We first recall some definitions. Let $\mathcal{O}$ be a maximal order in $A$. We define the group of totally positive units of $\mathcal{O}$ as $\mathcal{O}^+ = A^+ \cap \mathcal{O}^*$. Let $a$ be an ideal in $R_k$. The principal congruence subgroup of level $a$ is the group $G^+_\mathcal{O}(a) := \{ x \in \mathcal{O}^+ | x - 1 \in a\mathcal{O} \} < GL_2^+(\mathbb{R})$.

We will write $\Gamma^+_\mathcal{O}(a) = \tilde{P}(G^+_\mathcal{O}(a))$ where $\tilde{P} : GL_2^+(\mathbb{R}) \to PSL_2(\mathbb{R})$ stands for the obvious projection map.
Theorem 15. ([DN]) Let \( \Gamma_0^+(a) \) be a torsion-free principal congruence subgroup in \( \mathbb{A} \) and let \( C \) be an algebraic curve analytically isomorphic to the Riemann surface \( \mathbb{H}/\Gamma_0^+(a) \). Then, for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), the conjugate curve \( C^\sigma \) is analytically isomorphic to \( \mathbb{H}/\Gamma_0^+(a^\sigma) \) where \( \mathcal{O}' \) is a maximal order in the quaternion algebra \( (k', A') \) determined, up to isomorphism, by the three following properties:

1. \( k' = k^\sigma \).
2. \( \text{Ram}_f A' = (\text{Ram}_f A)^\sigma \).
3. The only archimedean place at which \( A' \) is unramified is the one corresponding to the identity on \( k' \).

This implies the following

Theorem 16. Let \( \Gamma \) be an arithmetic Fuchsian surface group and let \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). The associated quaternion algebra \( (k^\sigma \Gamma, A^\sigma \Gamma) \) is determined, up to isomorphism, by the following three properties:

1. \( k^\sigma \Gamma = (k\Gamma)^\sigma \).
2. \( \text{Ram}_f A^\sigma \Gamma = (\text{Ram}_f A\Gamma)^\sigma \).
3. The only archimedean place at which \( A^\sigma \Gamma \) is unramified is the one corresponding to the identity on \( k^\sigma \).

Proof. Let \( \mathcal{O} \) be an order in \( A\Gamma \) such that \( \Gamma \) is commensurable to \( \text{P}(\rho(\mathcal{O}^1)) \). We may assume that \( \mathcal{O} \) is a maximal order.

Let \( \Gamma_2 = \Gamma_0^+(a) \) be a torsion-free principal congruence subgroup as in Theorem 15 and set \( \Gamma_{12} = \Gamma \cap \Gamma_2 \). Then the algebraic curve \( C_{12} = \mathbb{H}/\Gamma_{12} \) is simultaneously an unramified cover of \( C = \mathbb{H}/\Gamma \) and \( C_2 = \mathbb{H}/\Gamma_2 \), and therefore \( C_{12}^\sigma \) is simultaneously an unramified cover of \( C^\sigma \) and \( C_2^\sigma \). Since quaternion algebras associated with arithmetic groups are only defined up to commensurability we conclude that \( A\Gamma = A\Gamma_{12} = A\Gamma_2 \) and \( A^\sigma \Gamma = A^\sigma \Gamma_{12} = A^\sigma \Gamma_2 \). Now apply Theorem 15.

Corollary 17. The notation being as above, let \( k \) be a normal extension of \( \mathbb{Q} \). Then \( A^\sigma \Gamma = A\Gamma \) if and only if \( (\text{Ram}_f A\Gamma)^\sigma = \text{Ram}_f A\Gamma \).

Corollary 18. Suppose that \( k\Gamma = \mathbb{Q} \). Then \( A^\sigma \Gamma = A\Gamma \). In particular \( \text{Comm}(\Gamma^\sigma) = \text{Comm}(\Gamma) \).

Example 19. Let \( \Gamma_p \), \( p \) odd prime, the family of groups constructed in [2,1,1]. As for two different prime numbers \( p \) and \( q \) the fields \( \mathbb{Q}(\cos(2\pi/p)) \) and \( \mathbb{Q}(\cos(2\pi/q)) \) are not Galois conjugate, Theorem 16 implies that no pair of surface subgroups \( \Gamma_p^* < \Gamma_p \) and \( \Gamma_q^* < \Gamma_q \) can be Galois conjugate.
5.3 Congruence property as a Galois invariant

Let $\tilde{P} : GL^+_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ be the obvious projection. A subgroup of $G < GL^+_2(\mathbb{R})$ (resp. $\Gamma < PSL_2(\mathbb{R})$) is called a congruence subgroup in $A$ if $G$ (resp. $\tilde{P}^{-1}(\Gamma)$) contains some principal congruence subgroup $G^+_\mathcal{O}(a)$ for some maximal order $\mathcal{O}$ and some ideal $a$ of $\mathcal{O}$.

**Proposition 20.** Being uniformized by a congruent subgroup is a Galois invariant.

**Proof.** Let $C$ be an algebraic curve isomorphic to $\mathbb{H}/\Gamma$ where $\tilde{P}^{-1}(\Gamma)$ contains some principal congruence subgroup $G^+_{\mathcal{O}}(a)$.

Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $C$ is covered by the curve $C_1 := \mathbb{H}/\Gamma^+_{\mathcal{O}}(a)$ and so $C^\sigma$ is covered by $C_1^\sigma$. This implies that the uniformizing group of $C^\sigma$, which we have denoted throughout $\Gamma^\sigma$, contains the uniformizing group of $C^+_1$, which by Theorem 15 is a principal congruence subgroup of the form $\Gamma^+_{\mathcal{O}}(a^\sigma)$. This means that $\Gamma^\sigma$ is a congruence subgroup, as was to be seen.

We mention another direct consequence of Theorem 15.

**Corollary 21.** Let $k$ be a normal extension of $\mathbb{Q}$ and $A$ a quaternion algebra over $k$ with following properties:

1. For any two maximal orders $\mathcal{O}$ and $\mathcal{O}'$ in $A$ there exists $x \in A$ such that $\mathcal{O}' = x^{-1}\mathcal{O}x$.

2. $\text{Ram}_f(A)$ is invariant under $\text{Gal}(k/\mathbb{Q})$.

Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $a$ be an ideal in $R_k$ such that $a^\sigma = a$. Then, the curve $C(a) = \mathbb{H}/\Gamma^+_{\mathcal{O}}(a)$ is isomorphic to its Galois conjugate curve $C(a)^\sigma$.

**Example 22.** All the conditions of Corollary 21 are satisfied in the case where $k = \mathbb{Q}$ and $\Gamma^+_{\mathcal{O}}(a)$ is a principal congruence subgroup in a division quaternion algebra $A$ over $\mathbb{Q}$. This only reflects the fact that in this case the curves $C(a)$ are defined over $\mathbb{Q}$ (see e.g. [EI], 2.3).

**Example 23.** Consider the number field $k = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where $\zeta_7 = \exp(2\pi i/7)$, that is $k = \mathbb{Q}(\cos 2\pi/7)$ is the totally real subfield of the cyclotomic field of 7th roots of unity. Let $A$ be the quaternion algebra over $k$ which is unramified at the infinite place corresponding to the identity embedding and ramified at the two other infinite places and such that $\text{Ram}_f(A) = \emptyset$. This uniquely determines $A$ up to isomorphism (see [ET], 5.4). The detailed study of this algebra carried out by Elkies in the last section of [EI] (see also [ET], 5.3) allows us to illustrate the previous results in this case.

Set $c = \zeta_7 + \zeta_7^{-1} = 2 \cos 2\pi/7$ so that $k = \mathbb{Q}(c)$ and $R_k = \mathbb{Z}[c]$. Then (EI, 4.4):

1. $A = (\frac{c,c}{k})$ and there is an embedding $\rho : A \otimes_k \mathbb{R} \simeq M_2(\mathbb{R})$ given by the basis $\bar{1} = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, $\bar{i} = \begin{pmatrix} \sqrt{c} & 0 \\ 0 & -\sqrt{c} \end{pmatrix}$, $\bar{j} = \begin{pmatrix} 0 & \sqrt{c} \\ -\sqrt{c} & 0 \end{pmatrix}$ and $\bar{ij} = \begin{pmatrix} 0 & -\sqrt{c} \\ \sqrt{c} & 0 \end{pmatrix}$.
(2) The ring
\[ O = \mathbb{Z}[c[i, j]] = \mathbb{Z}[c]i + \mathbb{Z}[c]j + \mathbb{Z}[c]ij, \]
where \( j^2 = \frac{1}{2}(1 + c^2 + c + 1) \), is a maximal order in \( A \) and the group \( P(O^1) \), which in this case agrees with \( P(O^+) \), is isomorphic to the triangle Fuchsian group \( \Delta = \Delta(2, 3, 7) \).

(3) There exists only one prime ideal \( p_7 \) above \( p = 7 \). The Galois group \( \text{Gal}(k/\mathbb{Q}) \) fixes \( p_7 \). The same is true for every prime \( p \equiv \pm 2, \pm 3 \mod 7 \): There exists only one prime ideal \( p_\sigma \) above \( p \). Every automorphism of \( k \) fixes \( p_\sigma \).

(4) Above each prime \( p \equiv \pm 1 \mod 7 \) there are three different prime ideals \( p_{1,p}, p_{2,p}, \) and \( p_{3,p} \). These three prime ideals form a Galois orbit; that is, after a possible renumeration \( p_{2,p} = p_{1,p}^\sigma \) and \( p_{3,p} = p_{1,p}^{\sigma^2} \), where \( \sigma \) is the generator of the Galois group \( \text{Gal}(k/\mathbb{Q}) \) (of order three).

Now, let \( q \) be a prime ideal in \( \mathbb{R} \) and denote by \( \Gamma = \Gamma^O_0(q) \) the corresponding principal congruence subgroup so that \( A = \text{AG} \). Let \( C(q)^\sigma \cong \mathbb{H}/\Gamma^\sigma \) be the corresponding algebraic curve. We claim that for every \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) the curve \( C(q)^{\sigma^\sigma} \cong \mathbb{H}/\Gamma^\sigma \) agrees with the curve \( C(q^\sigma) \) uniformized by the principal congruence subgroup \( \Gamma^O_0(q^\sigma) \) of \( A \). This is because Corollary \([17]\) implies that \( \text{AG} = \Delta \) and Theorem \([17]\) tells us that \( \Gamma^\sigma = \Gamma^O_0(q^\sigma) \) where \( \mathcal{O}^\sigma \) is a maximal order in \( A \). A computation of the narrow class number \( h_\infty \) associated with \( A \) (see \([MR]\), p. 221) gives \( h_\infty = 1 \) which means that there is only one conjugacy class of maximal orders, hence up to conjugation by an element \( x \in A \) the order \( \mathcal{O}^\sigma \) agrees with \( \mathcal{O} \). This allows us to draw the following conclusions:

(a) If \( q \) lies above a rational prime \( p = 7 \) or \( p \equiv \pm 2, \pm 3 \mod 7 \) for every \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) the curve \( C(p)^\sigma \) is isomorphic to \( C(p) \). This only reflects the fact that these curves are defined over \( \mathbb{Q} \) (although see Remark \([24]\) below).

(b) If \( q = p_\sigma \) is a prime ideal above a rational prime \( p \equiv \pm 1 \mod 7 \), then the \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \)-orbit of \( C(p)^\sigma \) consists of three non-isomorphic curves \( C(p_\sigma) = C(p_{1,p}), C(p_{2,p}), C(p_{3,p}) \). Each of them remains invariant under the action of \( \text{Gal}(\mathbb{Q}/\mathbb{Q})/k) \) and, as above, this implies that they can be defined over the number field \( k \).

One way to see that these three curves are pairwise non-isomorphic or, equivalently, that the uniformizing groups \( \Gamma(p_{i,p}) \) are not conjugate in \( \text{PSL}_2(\mathbb{R}) \) is as follows: The triangle group \( \Delta \) is known to be a maximal Fuchsian group. This implies that the groups \( \Gamma_i = \Gamma(p_{i,p}) \) are not only normal in \( \Delta \) but that, in fact, their normalizers \( N(\Gamma_i) \) agree with \( \Delta \). Now, if we had \( \Gamma_i = x\Gamma_jx^{-1} \) for some \( x \in \text{PSL}_2(\mathbb{R}) \), then \( \Gamma_i \) would be contained in the maximal triangle groups \( \Delta \) and \( x\Delta x^{-1} \), hence we would have \( N(\Gamma_i) = \Delta = x\Delta x^{-1} \). This would imply that the element \( x \) lies in the normalizer of \( \Delta \) which, by maximality, agrees with \( \Delta \). This, in turn, would yield that \( \Gamma_i = \Gamma_j \), a contradiction.

These examples are geometrically interesting as they provide examples of Riemann surfaces of genus \( > 1 \) with maximal number of automorphisms (see for instance \([JW]\)).
Remark 24. If $C$ is an arbitrary curve defined over a number field, the invariance of the isomorphism class of $C$ under an absolute Galois group $\text{Gal}(\mathbb{Q}/k)$ only means that the so-called field of moduli of the curve is contained in $k$. Fortunately when, as in this case, the uniformizing group is normally contained in a triangle group the field of moduli is also a field of definition \cite{Wl}.

5.4 The torsion of $\text{Comm}(\Gamma)$ is Galois invariant

We can use Theorem 16 also to prove the equality $\mathcal{P}(\Gamma) = \mathcal{P}(\Gamma^\sigma)$ between the sets of periods of the commensurator of $\Gamma$ and the commensurator of $\Gamma^\sigma$. In order to do so, we first recall (see \cite{MR} Theorem 8.4.4) that the commensurator of $\Gamma$ in $\text{PSL}_2(\mathbb{R}) \cong \text{PGL}_2(\mathbb{R})$ is $\tilde{P}(A^\Gamma)$.

5.4.1 Maclachlan’s characterization of torsion in $\text{Comm}(\Gamma)$.

Following the work of Chinburg and Friedman \cite{CF}, C. Maclachlan (\cite{Mac1}, see also \cite{MR}, Lemma 12.5.6) showed the following result:

**Proposition 25.** $\text{Comm}(\Gamma)$ contains an element of order $m \geq 3$ if and only if the following properties hold:

i) $\cos \frac{2\pi}{m}$ lies in the invariant trace field $k\Gamma$.

ii) There is an embedding of $k\Gamma$-algebras $\varphi : k\Gamma(e^{\pi i/m}) \hookrightarrow A\Gamma$.

In that case $z = 1 + e^{2\pi i/m} \in A\Gamma$ provides such a finite order element.

Actually, in Lemma 12.5.6 of \cite{MR} this result is stated for $P(A^*)$ instead of $\text{Comm}(\Gamma) = P(A^\Gamma)$ but the result holds just as well for $P(A^\Gamma)$ because the element $z = 1 + e^{2\pi i/m}$ lies in $P(A^\Gamma)$, since its image in $A\Gamma$, $\varphi(z) = \left( \frac{1 + \cos \frac{2\pi}{m}}{\sin \frac{2\pi}{m}}, \frac{\sin \frac{2\pi}{m}}{1 + \cos \frac{2\pi}{m}} \right)$, has positive determinant. Note that with respect to other embeddings the positivity of the norm is automatically satisfied, as the norm form in Hamiltonian quaternions is a positive definite quadratic form.

**Theorem 26.** Let $\mathcal{P}(\Gamma) \subset \mathbb{N}$ denote the set of orders (or periods) of finite order elements of $\text{Comm}(\Gamma)$. Then $\mathcal{P}(\Gamma) = \mathcal{P}(\Gamma^\sigma)$, for any $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$.

**Proof.** $P(A\Gamma^\sigma)$ always contains an element of order 2. So, we need to prove that if $(k\Gamma, A\Gamma)$ satisfies the conditions i) and ii) of the above Proposition 25 then so does $(k\Gamma^\sigma, A\Gamma^\sigma)$.

This clearly holds for i) since by the Doi-Naganuma’s theorem $k\Gamma^\sigma = (k\Gamma)^\sigma$.

In order to prove that this is also the case for ii) we first recall a criterion due to Brauer, Hasse and Noether for embedding quadratic field extensions into quaternion algebras (see \cite{MR} Theorem 7.3.3) and \cite{CF}:

- A quadratic field extension $K$ of a number field $k$ can be embedded into a quaternion algebra $A$ over $k$ if and only if every archimedean or non-archimedean place of $k$ ramified in $A$ either ramifies or remains prime in $K$. 

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This applied to our situation tells us that what we need to prove is that if this condition is satisfied for the quaternion algebra \((k, A) = (k \Gamma, A \Gamma)\) and the quadratic extension \(K = k \Gamma(e^{2\pi i/m})\) of \(k \Gamma\) then so is for the quaternion algebra \((k \Gamma^\sigma, A \Gamma^\sigma)\) and the quadratic extension \(K^\sigma = k \Gamma^\sigma(e^{2\pi i/m})\) of \(k \Gamma^\sigma\).

Let us deal first with the non-archimedean places. If \(p \in \text{Ram}_f(\overline{\mathbb{Q}})\) ramifies (resp. remains prime) in \(K = k \Gamma(e^{2\pi i/m})\) then so does \(p^\sigma \in \text{Ram}_f(\overline{A \Gamma}) = \text{Ram}_f(\overline{A \Gamma^\sigma})\) in \(K^\sigma\). This can be seen as follows. Let \(\mathfrak{P}\) be a prime ideal in \(K\) lying over \(p\) and let \(\tau : K \rightarrow K\) be the non-trivial \(k\)-automorphism of \(K\). Then \(p\) is ramified or remains prime in \(K\) if and only if \(\mathfrak{P}^\tau = \mathfrak{P}\). The non-trivial \(k^\sigma\)-automorphism of \(K^\sigma\) is then \(\tau^\sigma\) and, clearly, \((\mathfrak{P}^\sigma)^{\tau^\sigma} = \mathfrak{P}^\sigma\). Hence \(p^\sigma\) also remains prime or is ramified in \(K^\sigma\).

As for the archimedean places we recall that an archimedean place \(v\) of a totally real number field \(k\) is ramified in \(K/k\) if the embedding \(v : k \rightarrow \mathbb{C}\) extends to an embedding \(w : K \rightarrow \mathbb{C}\) whose image is not a subfield of the reals. In our case the fields \(K = k \Gamma(e^{2\pi i/m})\) and \(K^\sigma = (k \Gamma^\sigma)(e^{2\pi i/m})\) are totally imaginary extensions of the totally real fields \(k \Gamma\) and \(k \Gamma^\sigma\) so the ramification condition obviously holds. The proof is done. 

When \(m\) is odd these two conditions in Proposition 25 can be formulated in the following simper manner

**Proposition 27.** Let \(m\) be an odd number. Then \(\text{Comm}(\Gamma)\) contains an element of order \(m\) if and only if \(A \Gamma\) contains a square root of \(-\sin^2 \frac{2\pi}{m}\). In particular \(A \Gamma\) contains a square root of \(-\sin^2 \frac{2\pi}{m}\) if and only if \(A \Gamma^\sigma\) does.

**Proof.** We must show that, for \(m\) odd, the above conditions i) and ii) are equivalent to the existence of an element \(X \in A \Gamma\) such that \(X^2 = -\sin^2 \frac{2\pi}{m} \in k \Gamma \subset A \Gamma\).

In one direction this is easy. If the first condition is satisfied then first of all \(\sin^2 \frac{2\pi}{m} = 1 - \cos^2 \frac{2\pi}{m} \in k \Gamma\) and, moreover, \(k \Gamma(e^{2\pi i/m}) = k \Gamma(i \sin \frac{2\pi}{m})\). Now, if in addition, there is an embedding \(\varphi : k \Gamma(i \sin \frac{2\pi}{m}) \rightarrow A \Gamma\) then \(X = \varphi(i \sin \frac{2\pi}{m})\) will provide the required square root.

Conversely, if such \(X\) exists then \(\cos \frac{2\pi}{m} = 1 - 2 \sin^2 \frac{2\pi}{m} \in k \Gamma\). This means that \(k \Gamma\) contains the field \(\mathbb{Q}(\cos \frac{2\pi}{m})\). But this is precisely the Galois subextension of the field \(\mathbb{Q}(e^{2\pi i/m})\) fixed by complex conjugation. Now, \(m\) being odd, \(\mathbb{Q}(e^{2\pi i/m}) = \mathbb{Q}(\cos \frac{2\pi}{m})\), hence \(\mathbb{Q}(\cos \frac{2\pi}{m}) = \mathbb{Q}(\cos \frac{2\pi}{m})\) and so the first condition is satisfied. This in turn implies that \(k \Gamma(e^{2\pi i/m}) = k \Gamma(i \sin \frac{2\pi}{m})\), as before, and now simply sending \(i \sin \frac{2\pi}{m}\) to \(X\) gives an embedding of \(k \Gamma(e^{2\pi i/m})\) in \(A \Gamma\), which is the second condition.

**Example 28.** Let \(\Gamma_p, p\) odd prime, the family of groups constructed in 2.1.1. Then \(\text{Comm}(\Gamma_p)\) contains an element of odd prime order \(q\) if and only if \(q = p\). In particular, the groups \(\text{Comm}(\Gamma_p)\) and \(\text{Comm}(\Gamma_p)\) are not isomorphic, if \(p \neq q\).

These claims can be settled as follows:

By construction (see 2.1.1) \(k \Gamma_p = \mathbb{Q}(\sin \frac{2\pi}{m})\). Thus, by Proposition 27, in order to prove that \(\text{Comm}(\Gamma_p)\) has an element of order \(p\) it is enough to observe that \(A \Gamma^p\) contains some root of \(-1\), namely \(X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).
Now suppose that \( q \neq p \). Then \( \sin^2 \frac{2\pi}{q} = 1 - \cos^2 \frac{2\pi}{q} \) does not lie in \( \mathbb{Q}(\sin \frac{2\pi}{p}) = \mathbb{Q}(\cos \frac{2\pi}{4p}) \) (see Lemma 1). This is because otherwise \( \cos^2 \frac{2\pi}{q} \) would lie in the intersection field \( \mathbb{Q}(e^{2\pi i/q}) \cap \mathbb{Q}(e^{2\pi i/4p}) \) which is equal to \( \mathbb{Q} \) since \( q \) and \( 4p \) are co-prime (see e.g. [Wa], Proposition 2.4). Now from Proposition 27 we deduce that \( \text{Comm}(\Gamma_p) \) cannot contain elements of order \( q \).

5.5 Final result

Although, there are only finitely many arithmetic surface groups of given genus ([Ta2], Theorem 2.1) the group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) is going to act faithfully on them. Our final theorem records this fact and collects the main invariants we have found for this action.

**Theorem 29.** \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) acts faithfully on the set of isomorphy classes of arithmetic surface groups \( \Gamma \) and this action has the following invariants:

1. The isomorphism class of the group \( \text{Comm}(\Gamma) \).
2. The Galois conjugacy class of \( k\Gamma \). (In fact \( k\Gamma^\sigma = (k\Gamma)^\sigma \) for any \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \)).
3. The set \( P(\Gamma) \) of periods of \( \text{Comm}(\Gamma) \).
4. The solvability of the quadratic equations \( X^2 + \sin^2 \frac{2\pi}{q} + 1 \), \( k \in \mathbb{N} \), in the invariant quaternion algebra \( A_{\Gamma} \).
5. The property of being a congurence subgroup.

**Proof.** That the action transforms arithmetically uniformised curves into themselves is the content of Corollary 11. Faithfulness is a consequence of the result proved in [GJ] that the action is faithful on the set of curves uniformised by subgroups of any given triangle group together with the fact that there are plenty of triangle groups which are arithmetic [Ta1].

As for the three listed invariants, 1) is the first part of Theorem 16, 2) is the first part of Theorem 16, 3) is Theorem 28, 4) is Proposition 27 and 5) is Proposition 20.

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**References**

[BB] W. L. Baily, A. Borel, Compactification of Arithmetic Quotients of Bounded Symmetric Domains. Annals of Mathematics Second Series, Vol. 84, No. 3, 442-52 (1966).
[Be] V. V. Belyaev. Locally finite groups containing a finite inseparable subgroup, Siberian Math. J. 34 (1993), no. 2, 218-232.

[BN] I. Biswas, S. Nag. Limit constructions over Riemann surfaces and their parameter spaces, and the commensurability group actions, Selecta Math. (N.S.) 6 , no.2, 185-224 (2000).

[Ca] F. Catanese. Kodaira fibrations and beyond: methods for moduli theory. Jpn. J. Math. 12 (2017), no. 2, 91-174.

[CF] T. Chinburg, E. Friedman. An embedding theorem for quaternion algebras. J. London. Math. Soc., 60, 33-44 (1999)

[DN] K. Doi and H. Naganuma, On the algebraic curves uniformized by arithmetical automorphic functions, Ann. Math. 86 (1967), no. 3, 449–460.

[El1] N.D. Elkies. Shimura curve computations, Algorithmic number theory (Portland, OR, 1998), 1-47, Lecture Notes in Comput. Sci., 1423, Springer, Berlin, 1998.

[El2] N.D. Elkies. The Klein quartic in number theory. The eightfold way:The Beauty of Klein’s quartic, ed. Sylvio Levi. Math. Sci. Res. Inst. Publ., 35, Cambridge Univ. Press, Cambridge (1999), 51-101.

[GG] E. Girono, G. González-Diez. Introduction to Compact Riemann Surfaces and Dessins d’ Enfants. Cambridge University Press (2012). London Mathematical Society Student Texts (79).

[Go] G. González-Diez. Variations on Belyi’s theorem, Q. J. Math. 57 (2006), no. 3, 339-354.

[Go1] G. González-Diez. Galois action on universal covers of Kodaira Fibrations. Duke Mathematical Journal (to appear). DOI:10.1215/00127094-2019-0078

[GJ] G. González-Diez, A. Jaikin-Zapirain. The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces. Proc. London Math. Soc. (2015) 111 (4): 775-796.

[GJT] G. González-Diez, G. Jones and D. Torres-Teigell. Arbitrarily large Galois orbits of non-homeomorphic surfaces. Journal European Journal of Mathematics. Vol 4 (2018) 223-241.

[GR1] G. González-Diez and S. Reyes-Carocca, The arithmeticity of a Kodaira fibration is determined by its universal cover, Comment. Math. Helv., 90 (2015), 429-434.

[GR2] G. González-Diez, S. Reyes-Carocca, Sebastín Families of Riemann surfaces, uniformization and arithmeticity. Trans. Amer. Math. Soc. 370 (2018), no. 3, 1529-1549.
A. Grothendieck, Esquisse d’un programme, 1984 geometric Galois actions (eds L. Schneps and P. Lochak), London Mathematical Society Lecture Note Series 242 (Cambridge University Press, Cambridge, 1999) 547.

Gareth A. Jones and Jürgen Wolfart, Dessins d’Enfants on Riemann Surfaces, Monographs in Mathematics, Springer-Verlag, 2016.

S. Kaliszewski, M. B. Landstad, J. Quigg; Hecke C*-algebras, Schlichting completions and Morita equivalence. Proc. Edinb. Math. Soc. (2) 51 (2008), no. 3, 657-695.

D. Kazhdan. On arithmetic varieties II. Israel Journal of Mathematics. 44, no. 2, 139-159 (1983).

C. Maclachlan. Existence and non-existence of torsion in maximal arithmetic Fuchsian groups. Groups Complex. Cryptol. 1 (2009), no. 2, 287-295.

C. Maclachlan. Introduction to arithmetic Fuchsian groups. Topics on Riemann surfaces and Fuchsian groups (Madrid, 1998), 29-41, London Math. Soc. Lecture Note Ser., 287, Cambridge Univ. Press, Cambridge, 2001.

C. Maclachlan, A.W. Reid. The arithmetic of hyperbolic 3-manifolds. Graduate Texts in Mathematics, 219. Springer-Verlag, New York, 2003.

G.A. Margulis. Discrete Subgroups of Semi-simple Lie Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 40. Springer-Verlag, Berlin, 2000.

James S. Milne and Junecue Suh, Nonhomeomorphic conjugates of connected Shimura varieties, Amer. Journ. Math 132 (2010), no. 3, 731-750.

C. Odden. The baseleaf preserving mapping class group of the universal hyperbolic solenoid. Trans. Amer. Math. Soc. 358 (2006), no. 6, 2637-2650

James S. Milne and Junecue Suh, Nonhomeomorphic conjugates of connected Shimura varieties, Amer. Journ. Math 132 (2010), no. 3, 731-750.

R. C. Penner, D. Šarić. Teichmüller theory of the punctured solenoid. Geometriae Dedicata March 2008, Volume 132, Issue 1, 179-212 (2008).

C. D. Reid, P. R. Wesolek. Homomorphisms into totally disconnected, locally compact groups with dense image. Forum Math. 31 (2019), no 3, 685-701.

L. Ribes, P. Zalesskii, Profinite groups, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 40. Springer-Verlag, Berlin, 2000.

J-P. Serre, Examples des variétés projectives conjuguées non homéomorphes, C.R. Acad. Sci. Paris 258, 4194-4196 (1964).

D. Sullivan. Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers, Milnor Festschrift, Topological Methods in Modern Mathematics (Ed.L. Goldberg and A. Phillips) (1993), 543-563, Publish or Perish.
[Ta1] K. Takeuchi. Arithmetic triangle groups. J. Math. Soc. Japan, Volume 29, Number 1 (1977), 91-106.

[Ta2] K. Takeuchi. Arithmetic Fuchsian groups with signature (1; e), J. Math. Soc. Japan 35, (1983), 381-404.

[Wa] L. C. Washington. Introduction to Cyclotomic Fields, Graduate Texts in Mathematics, 83, Springer-Verlag, 1977.

[Wo] J. Wolfart. The ‘obvious’ part of Belyi’s theorem and Riemann surfaces with many automorphisms. Geometric Galois actions, 1, 97-112, London Math. Soc. Lecture Note Ser., 242, Cambridge Univ. Press, Cambridge, 1997.