Deformation of locally free actions 
and the leafwise cohomology

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1 Introduction

This is a note of my lectures at “Advanced courses in Foliation” in the research program “Foliation”, which was held at the Centre de Recerca Matemàtica in the May of 2010. In this note, we discuss about the relationship between deformation of actions of Lie groups and the leafwise cohomology of the orbit foliation.

In early 1960’s, Palais [41] proved the local rigidity of smooth actions of compact group. Hence, such actions have no non-trivial deformation. In contrast to compact groups, almost all $\mathbb{R}$-actions (i.e., flows) are not locally rigid, and their bifurcation is an important issue in the theory of dynamical systems. In the last decades, rigidity theory for locally free actions of higher-dimensional non-compact groups have been rapidly developed. The reader can find examples of locally rigid or parameter rigid actions in many papers [3, 7, 8, 9, 10, 45, 46, 16, 21, 29, 30, 33, 38, 39], some of which we will discuss about in this article.

Rigidity problem can be regarded as a special case of deformation problem. In many situations, the deformation space of a geometric structure is described by a system of non-linear partial differential equations. Its linearization defines a cochain complex, so called the deformation complex, and the space of infinitesimal deformation is identified with the first cohomology of this complex. For locally free actions of Lie groups, the deformation complex is realized as the (twisted) leafwise de Rham complex of the orbit foliation.

The reader may wish to develop a general deformation theory of locally free actions in terms of the deformation complex, like the deformation theory of complex manifolds founded by Kodaira and Spencer. However, the leafwise de Rham complex is not elliptic, and it causes two difficulties to develop a fine theory. First, the leafwise cohomology groups are infinite dimensional in general, and they are hard to compute. Second, we can not use the implicit function theorem for maps between Banach spaces because of the lack of a priori estimate. We will focus on the techniques to overcome these difficulties for several explicit examples, instead of developing a general theory.
The main tools for computation of the leafwise cohomology are Fourier analysis, representation theory, and Mayer-Vietoris argument developed by El Kacimi Aloui and Tihami. Matsumoto and Mitsumatsu also developed a technique based on ergodic theory of hyperbolic dynamics. We will discuss about these techniques in Section 4.

For several cases, we can reduce the deformation problem to a linear one without help of any implicit function theorem. The first case is the parameter deformation of abelian actions. In this case, the problem is a linear one. In fact, the deformation space can be naturally identified with the space of infinite deformations. The second case is the parameter rigidity of solvable actions. Although the problem itself is not linear in this case, we can decompose it to the solvability of linear equations for several examples. In Section 5 we will see how to reduce the rigidity problem of such actions to the (almost) vanishing of the first cohomology of the leafwise cohomology.

For general cases, the deformation problem cannot be reduced to a linear one directly. One way to describe the deformation space is to apply Hamilton’s implicit function theorem. Although it needs an estimate on solutions of partial differential equations and it is difficult to establish in general, there are a few examples for which we can apply the theorem. Another way is to use the theory of hyperbolic dynamics. We will give a brief discussion about these techniques in Section 6.

The author recommend the readers to read the survey papers [5] and [36]. The former contains a nice exposition on application of Hamilton’s implicit function theorem to rigidity problem of foliations. The second is a survey on parameter rigidity problem, which is one of the sources of my lecture at the Centre de Recerca Matemàtica.

To end the introduction, the author would like to thank the organizers of the research program “Foliations” at the CRM who invited me to give a lecture in the program. The author is also thankful to the staff of the CRM for their warm hospitality.

2 Locally free actions and their deformation

In this section, we define locally free actions and their infinitesimal correspondent. We also introduce deformation of actions and the several concepts of finiteness of codimension of the conjugacy classes of an action in the space of locally free actions.

2.1 Locally free actions

In this note, we will work in the $C^\infty$-category. So, the term “smooth” means “$C^\infty$” and all diffeomorphisms are of class $C^\infty$. All manifolds and Lie groups will be connected. For manifolds $M_1$ and $M_2$, we denote the space of smooth
maps from $M_1$ to $M_2$ by $C^\infty(M_1, M_2)$. It is endowed with the $C^\infty$ compact-open topology. By $\mathcal{F}(x)$, we denote the leaf of a foliation $\mathcal{F}$ which contains a point $x$.

Let $G$ be a Lie group and $M$ a manifold. We denote the unit element of $G$ by $1_G$ and the identity map of $M$ by $\text{Id}_M$. We say a smooth map $\rho : M \times G \to M$ is a (smooth right) action if

1. $\rho(x, 1_G) = x$ for any $x \in M$, and
2. $\rho(x, gh) = \rho(\rho(x, g), h)$ for any $x \in M$ and $g, h \in G$.

For $\rho \in C^\infty(M \times G, M)$ and $g \in G$, we define a map $\rho^g : M \to M$ by $\rho^g(x) = \rho(x, g)$. Then, $\rho$ is an action if and only if the map $g \mapsto \rho^g$ is a homomorphism from $G$ to the group $\text{Diff}^\infty(M)$ of diffeomorphisms of $M$. By $\mathcal{A}(M, G)$, we denote the subset of $C^\infty(M \times G, M)$ that consists of actions of $G$. It is a closed subspace of $C^\infty(M \times G, M)$. For $\rho \in \mathcal{A}^\infty(M, G)$ and $x \in M$, the set

$$\mathcal{O}_\rho(x) = \{ \rho^g(x) \mid g \in G \}$$

is called the $\rho$-orbit of $x$.

**Example 2.1.** $\mathcal{A}(M, G)$ is non-empty for any $M$ and $G$. In fact, it contains the trivial action $\rho_{\text{triv}}$, which is defined by $\rho_{\text{triv}}(x, g) = x$. For any $x \in M$, $\mathcal{O}_{\rho_{\text{triv}}}(x) = \{ x \}$.

Let us introduce an infinitesimal description of actions. By $\mathfrak{X}(M)$, we denote the Lie algebra of smooth vector fields on $M$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\text{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ be the space of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{X}(M)$. In this note, we identify $\mathfrak{g}$ with the subspace of $\mathfrak{X}(G)$ consisting of vector fields invariant under left translations. For $\rho \in \mathcal{A}(M, G)$, we define the infinitesimal action $I_\rho : \mathfrak{g} \to \mathfrak{X}(M)$ associated with $\rho$ by

$$I_\rho(\xi)(x) = \frac{d}{dt} \rho(x, \exp(t\xi)) \bigg|_{t=0}.$$ 

It is easy to see that $I_\rho$ is an element of $\text{Hom}(\mathfrak{g}, \mathfrak{X}(M))$.

**Proposition 2.2.** Two actions $\rho_1, \rho_2 \in \mathcal{A}(M, G)$ coincide if $I_{\rho_1} = I_{\rho_2}$. If $G$ is simply connected and $M$ is closed, then any $I \in \text{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ is the infinitesimal action associated with some action in $\mathcal{A}(M, G)$.

**Proof.** Take $\rho_1, \rho_2 \in \mathcal{A}(M, G)$. The curve $t \mapsto \rho_i(x, \exp(t\xi))$ is the integral curve of the vector field $I_{\rho_i}(\xi)$ for any $i = 1, 2$, $x \in M$ and $\xi \in \mathfrak{g}$. If $I_{\rho_1} = I_{\rho_2}$, then the uniqueness of the integral curve implies that $\rho_1(x, \exp(t\xi)) = \rho_2(x, \exp(t\xi))$ for any $x \in M$, $t \in \mathbb{R}$, and $\xi \in \mathfrak{g}$. Since the union of one-parameter subgroups of $G$ generates $G$, we have $\rho_1 = \rho_2$.

Suppose that $G$ is simply connected and $M$ is a closed manifold. Let $E$ be a subbundle of $T(M \times G)$ given by

$$E(x, g) = \{(I(\xi)(x), \xi(g)) \in T_{(x,g)}(M \times G) \mid \xi \in \mathfrak{g}\}.$$
For any $\xi, \xi' \in \mathfrak{g}$, we have

$$[(I(\xi), \xi), (I(\xi'), \xi')] = ([I(\xi), I(\xi')], [\xi, \xi']) = (I([\xi, \xi']), [\xi, \xi']).$$

By Frobenius' theorem, the subbundle $E$ is integrable. Let $\mathcal{F}$ be the foliation on $M \times G$ generated by $E$. The space $M \times G$ admits a left action of $G$ by $g \cdot (x, g'g) = (x, gg')$. The subbundle $E$ is invariant under this action. Hence, we have $g \cdot \mathcal{F}(x, g') = \mathcal{F}(x, gg')$. Since $G$ is simply connected and the foliation $\mathcal{F}$ is transverse to the natural fibration $\pi : M \times G \to G$, we can define a smooth map $\rho : M \times G \to M$ so that $\mathcal{F}(x, 1_G) \cap \pi^{-1}(g) = \{(\rho^g(x), g)\}$. Take $x \in M$ and $g, g' \in G$. Then, $(\rho^g(\rho^{g'}(x)), g)$ is contained in $\mathcal{F}(\rho^{g'}(x), 1_G)$. Applying $g'$ from left, it implies that $(\rho^g \circ \rho^{g'}(x), g'g)$ is an element of $\mathcal{F}(\rho^g(x), g')$. Since $\mathcal{F}(\rho^g(x), g') = \mathcal{F}(x, 1_G)$ and $\{\mathcal{F}(\rho^g(x), g')\} = \mathcal{F}(x, 1_G) \cap \pi^{-1}(g'g)$ by the definition of $\rho$, we have $\rho^g \circ \rho^{g'}(x) = \rho^{g'g}(x)$. Therefore, $\rho$ is a right action of $G$. Now it is easy to check that $I_\rho = I$.

We say that an action $\rho \in \mathcal{A}(M, G)$ is locally free if the isotropy group $\{g \in G \mid \rho^g(x) = x\}$ is a discrete subgroup of $G$ for any $x \in M$. By $\mathcal{A}_{LF}(M, G)$, we denote the set of locally free actions of $G$ on $M$. Of course, the trivial action is not locally free unless $M$ is zero-dimensional. The following is a list of basic examples of locally free actions.

**Example 2.3 (Flows).** An locally free $\mathbb{R}$-action is just a smooth flow with no stationary points. Remark that $\mathcal{A}_{LF}(M, \mathbb{R})$ is empty if $M$ is a closed manifold with non-zero Euler characteristic.

**Example 2.4 (The standard action).** Let $G$ be a Lie group, and $\Gamma, H$ be closed subgroup of $G$. The standard $H$-action on $\Gamma \backslash G$ is the action $\rho_\Gamma \in \mathcal{A}(\Gamma \backslash G, H)$ defined by $\rho_\Gamma(\Gamma g, h) = \Gamma(gh)$. The action $\rho$ is locally free if and only if $g^{-1}Hg \cap \Gamma$ is a discrete subgroup of $\Gamma$ for any $g \in G$. In particular, if $\Gamma$ itself is a discrete subgroup of $G$, then $\rho$ is locally free.

**Example 2.5 (The suspension construction).** Let $M$ be a manifold and $G$ be a Lie group. Take a discrete subgroup $\Gamma$ of $G$, a closed subgroup $H$ of $G$, and a right action $\sigma : M \times \Gamma \to M$. We put $M \times \sigma G = M \times G / \langle x, g \rangle \sim (\sigma(x, g^{-1}), \gamma g)$. Then, $M \times \sigma G$ is an $M$-bundle over $\Gamma \backslash G$. We define a locally free action $\rho$ of $H$ on $M \times \sigma G$ by $\rho([(x, g), h]) = (x, gh)$.

We say a homomorphism $I : \mathfrak{g} \to \mathfrak{X}(M)$ is regular if $I(\xi)(x) \neq 0$ for any $\xi \in \mathfrak{g} \backslash \{0\}$ and $x \in M$.

**Proposition 2.6.** An action $\rho \in \mathcal{A}(M, G)$ is locally free if and only if $I_\rho$ is regular.

**Corollary 2.7.** For any $\rho \in \mathcal{A}_{LF}(M, G)$, the orbits of $\rho$ form a smooth foliation. If the manifold $M$ is closed, then the map $\rho(x, \cdot) : G \to \mathcal{O}(x, \rho)$ is a covering for any $x \in M$, where $\mathcal{O}(x, \rho)$ is endowed with the leaf topology.
Proofs of the proposition and the corollary are easy and left to the reader. If $M$ is closed, the set of regular homomorphisms is an open subset of $\text{Hom}(\mathfrak{g}, \mathfrak{x}(M))$. Hence, $\mathcal{A}_LF(M, G)$ is an open subset of $\mathcal{A}(M, G)$ in this case.

Let $\mathcal{F}$ be a foliation on a manifold $M$. We denote the tangent bundle of $\mathcal{F}$ by $T\mathcal{F}$ and $\mathfrak{x}(\mathcal{F})$ be the space of vector fields on $M$ tangent to $\mathcal{F}$. Let $\mathcal{A}_LF(\mathcal{F}, G)$ be the set of locally free actions of a Lie group $G$ whose orbit foliation is $\mathcal{F}$. The subspace $\mathcal{A}_LF(\mathcal{F}, G)$ of $\mathcal{A}_LF(M, G)$ is closed and it consists of actions $\rho$ such that $I_\rho$ is an element of $\text{Hom}(\mathfrak{g}, \mathfrak{x}(\mathcal{F}))$.

### 2.2 Rigidity and deformations of actions

We say that two actions $\rho_1 \in \mathcal{A}(M_1, G)$ and $\rho_2 \in \mathcal{A}(M_2, G)$ on manifolds $M_1$ and $M_2$ are $(C^\infty)$-conjugate (and write $\rho_1 \simeq \rho_2$) if there exists a diffeomorphism $h : M_1 \to M_2$ and an automorphism $\Theta$ of $G$ such that $\rho_2(h) \circ h = h \circ \rho_1^g$ for any $g \in G$. For a given foliation $\mathcal{F}$ on $M$, let $\text{Diff}(\mathcal{F})$ be the set of diffeomorphisms of $M$ which preserves each leaf of $\mathcal{F}$, and $\text{Diff}_0(\mathcal{F})$ be its arc-wise connected component that contains $\text{Id}_M$. We say that two actions $\rho_1, \rho_2 \in \mathcal{A}_LF(\mathcal{F}, G)$ are $(C^\infty)$-parameter-equivalent (and write $\rho_1 \equiv \rho_2$) if they are conjugate by a pair $(h, \Theta)$ such that $h$ is an element of $\text{Diff}_0(\mathcal{F})$. It is easy to see that conjugacy and the parameter-equivalence are equivalence relations.

The ultimate goal is the classification of actions in $\mathcal{A}_LF(M, G)$ or $\mathcal{A}_LF(\mathcal{F}, G)$ up to conjugacy, or parameter-equivalence for given $G$ and $M$, or $F$. Let $\rho_0$ be an action in $\mathcal{A}_LF(M, G)$ and $\mathcal{F}$ be its orbit foliation. We say $\rho_0$ is $(C^\infty)$-rigid if any action in $\mathcal{A}_LF(M, G)$ is conjugate to $\rho_0$. We say $\rho_0$ is $(C^\infty)$-parameter rigid if any action in $\mathcal{A}_LF(\mathcal{F}, G)$ is parameter-equivalent to $\rho_0$.

It is useful to introduce a local version of rigidity. We say $\rho_0$ is locally rigid if there exists a neighborhood $\mathcal{U}$ of $\rho_0$ such that any action in $\mathcal{U}$ is conjugate to $\rho_0$. We also say $\rho_0$ is locally parameter rigid if there exists a neighborhood $\mathcal{U}$ of $\rho_0$ in $\mathcal{A}(\mathcal{F}, G)$ such that any action in $\mathcal{U}$ is parameter-equivalent to $\rho_0$.

As we mentioned in the introduction, local rigidity for compact groups actions was settled in early 1960’s.

**Theorem 2.8 (Palais [11]).** Any action of a compact group on a closed manifold is locally rigid.

For non-compact groups, there are many non-rigid actions. So, it is natural to introduce the concept of deformations of actions. We say that a family $(\rho_\mu)_{\mu \in \Delta}$ of elements of $\mathcal{A}(M, G)$ parametrized by a manifold $\Delta$ is a $C^\infty$ family if the map $\tilde{\rho} : (x, g, \mu) \mapsto \rho_\mu(x, g)$ is a smooth map. By $\mathcal{A}_LF(M, G; \Delta)$, we denote the set of $C^\infty$ family of actions in $\mathcal{A}_LF(M, G)$ parametrized by $\Delta$. Under the identification with $(\rho_\mu)_{\mu \in \Delta}$ and $\tilde{\rho}$, the topology of $C^\infty(M \times G \times \Delta, M)$ induces

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1 For $k \in \mathbb{Z}$, let $\rho_k$ be an action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $S^1$ by $\rho(s, t) = ks + t$. It is easy to see that $\rho_1$ is locally parameter rigid. Of course, all the orbits of $\rho_k$ coincides with $S^1$ for any $k \geq 1$. However, $\rho_k$ is parameter equivalent to $\rho_1$ if and only if $|k| = 1$ (the mapping degree of $\rho_k(\cdot, t)$ must be $\pm 1$). So, $\rho_1$ is locally parameter rigid, but not parameter rigid action.

It is unknown whether any locally parameter rigid locally free action of contractible Lie group on a closed manifold is parameter rigid or not.
a topology on $A_{LF}(M, G; \Delta)$. We say that $(\rho_\mu)_{\mu \in \Delta}$ is a (finite dimensional) deformation of $\rho \in A(M, G)$ if $\Delta$ is an open neighborhood of 0 in a finite dimensional vector space and $\rho_0 = \rho$.

In several cases, actions are not locally rigid, but their conjugacy class is “of finite codimension” in $A_{LF}(M, G)$. Here, we formulate two types of finiteness of codimension. Let $(\rho_\mu)_{\mu \in \Delta} \in A_{LF}(M, G; \Delta)$ be a deformation of $\rho$. We say that $(\rho_\mu)_{\mu \in \Delta}$ is locally complete if there exists a neighborhood $U$ of $\rho$ in $A_{LF}(M, G)$ such that any action in $U$ is conjugate to $\rho_\mu$ for some $\mu \in \Delta$. We also say that $(\rho_\mu)_{\mu \in \Delta}$ is locally transverse if any $C^\infty$ family in $A_{LF}(M, G; \Delta)$ sufficiently close to $(\rho_\mu)_{\mu \in \Delta}$ contains an action conjugate to $\rho$. Roughly speaking, the local completeness means that the space $A(M, G)/\sim$ is locally finite dimensional at the conjugacy class of $\rho$. The local transversality means the family $(\rho_\mu)_{\mu \in \Delta}$ is transverse to the conjugacy class of $\rho$ at $\mu = 0$.

We define analogous concepts for actions in $A_{LF}(F, G)$. Let $F$ be a foliation on a manifold $M$. We say that $(\rho_\mu)_{\mu \in \Delta} \in A_{LF}(M, G; \Delta)$ preserves $F$ if all $\rho_\mu$’s are actions in $A_{LF}(F, G)$. By $A_{LF}(F, G; \Delta)$, we denote the subset of $A_{LF}(M, G; \Delta)$ that consists of families preserving $F$. We call a deformation in $A_{LF}(F, G; \Delta)$ a parameter deformation. Let $(\rho_\mu)_{\mu \in \Delta} \in A_{LF}(F, G; \Delta)$ be a parameter deformation of an action $\rho$. We say that $(\rho_\mu)_{\mu \in \Delta}$ is locally complete in $A_{LF}(F, G)$ if there exists a neighborhood $U$ of $\rho$ in $A_{LF}(F, G)$ such that any action in $U$ is parameter-equivalent to $\rho_\mu$ for some $\mu \in \Delta$. We also say that $(\rho_\mu)_{\mu \in \Delta} \in A_{LF}(F, G; \Delta)$ is locally transverse in $A_{LF}(F, G)$ if any $C^\infty$ family in $A_{LF}(F, G; \Delta)$ sufficiently close to $(\rho_\mu)_{\mu \in \Delta}$ contains an action parameter-equivalent to $\rho$.

3 Rigidy and deformation of flows

The real line $\mathbb{R}$ is the simplest Lie group among non-trivial and connected ones. Recall that any locally $\mathbb{R}$-action is just a smooth flow with no stationary points. In this section, we discuss about rigidity of locally free $\mathbb{R}$-actions as a model case.

3.1 Parameter rigidity of locally free $\mathbb{R}$-actions

Parameter rigidity of a locally free $\mathbb{R}$-action is characterized by the solvability of a partial differential equation.

Theorem 3.1. Let $\rho_0$ be a smooth locally free $\mathbb{R}$-action on a closed manifold $M$ and $X_0$ the vector field generating $\rho_0$. Then, $\rho_0$ is parameter rigid if and only if the equation

$$f = X_0 g + c.$$  \hspace{1cm} (1)

admits a solution $(g, c) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}$ for any given $f \in C^\infty(M, \mathbb{R})$.

The above equation is called the cohomology equation over $\rho_0$.

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2 This terminology is not common. Any suggestion of a better terminology is welcome.
Proof. First, we suppose that $\rho_0$ is parameter-rigid. Let $\mathcal{F}$ be the orbit foliation of $\rho_0$ and take $f \in C^\infty(M, \mathbb{R})$. Since $M$ is closed, $f_1 = f + c_1$ is a positive valued function for some $c_1 > 0$. Let $\rho$ be a flow generated by the vector field $(1/f_1)X_0$. By the assumption, there exists $h \in \text{Diff}_0(\mathcal{F})$ and $c_2 \in \mathbb{R}$ such that $\rho^{c_2} \circ h = h \circ \rho_0$. The diffeomorphism $h$ has the form $h(x) = \rho^{-g(x)}$ with some $C^\infty(M, \mathbb{R})$. So, we have

$$
\rho_0^t(x) = \rho^{c_2 t + g \rho_0^t(x) - g(x)}(x)
$$

for any $x \in M$, and hence, $X_0 = (c_2 + X_0g)X$. Since $X_0 = f_1X$, it implies $f_1 = c_2 + X_0g$. Therefore, the pair $(g, c_2 - c_1)$ is a solution of (1).

Next, we suppose Equation (1) can be solved for any pair $(g, c)$. Take an action $\rho \in A_{LF}(\mathcal{F})$. Let $X$ be the vector field generating $\rho$ and $f$ be the non-zero function satisfying $f \cdot X = X_0$. By assumption, Equation (1) has a solution $(g, c)$ for $f$. Since $f$ is non-zero and $X_0g(x) = 0$ for some $x \in M$, we have $c \neq 0$. Put $h(x) = \rho^{-g(x)}$. Then, we have $\rho^{c t} \circ h = h \circ \rho_0^t$. Since the maps $t \mapsto \rho_0^t(x)$ and $t \mapsto \rho^{c t}(h(x)) = h(\rho_0^t(x))$ are covering maps from $\mathbb{R}$ to $\mathcal{F}(x)$ for any $x \in M$, the map $h$ is a self-covering of $M$. Since $h$ is homotopic to the identity, the map $h$ is a diffeomorphism. Therefore, $\rho$ is equivalent to $\rho_0$. \qed

We say that a point $x \in M$ is a periodic point of a locally free flow $\rho \in A_{LF}(M, \mathbb{R})$ if $\rho^T(x) = x$ for some $T > 0$. The orbit of $x$ is called a periodic orbit. A point $x$ is periodic if and only if the orbit $\mathcal{O}(x, \rho)$ is compact.

**Corollary 3.2.** Let $\rho$ be an action in $A_{LF}(M, \mathbb{R})$. Suppose that $\rho$ admits two distinct periodic orbits. Then, $\rho$ is not parameter rigid.

**Proof.** By the assumption, there exists $x_1, x_2 \in M$ and $T_1, T_2 > 0$ such that $\mathcal{O}(x_1, \rho) \neq \mathcal{O}(x_2, \rho)$ and $\rho^{T_i}(x_i) = x_i$ for each $i = 1, 2$. Choose a smooth function $f$ such that $f \equiv 0$ on $\mathcal{O}(x_1, \rho)$ and $f \equiv 1$ on $\mathcal{O}(x_2, \rho)$. Then, there exists no solution of (1) for $f$. In fact, if $(g, c)$ is a solution, then we have

$$
\frac{1}{T} \int_0^T f \circ \rho^t(x)dt = c
$$

for any $x \in M$ and $T > 0$ with $\rho^T(x) = x$. However, the left-hand side should be 0 or 1 for $x = x_1$ or $x_2$. \qed

There is a classical example of a parameter rigid flow. For $N \geq 1$, we denote the $N$-dimensional torus $\mathbb{R}^N/\mathbb{Z}^N$ by $\mathbb{T}^N$. For $v \in \mathbb{R}^N$, we define a linear flow $R_v$ on $\mathbb{T}^N$ by $R_v^t(x) = x + v$. The vector field $X_v$ corresponding to $R_v$ is a parallel vector field on $\mathbb{T}^N$.

We say that $v \in \mathbb{R}^N$ is Diophantine (or badly approximable) if there exists $\tau > 0$ such that

$$
\inf_{m \in \mathbb{Z}^N \setminus \{0\}} \langle (m, v) \rangle \cdot \|m\|^\tau > 0,
$$

where $\langle ., . \rangle$ and $\| \cdot \|$ are the Euclidean inner product and norm on $\mathbb{R}^N$. When $v$ is Diophantine, we call the flow $R_v$ a Diophantine linear flow and its orbit foliation a Diophantine linear foliation.
Theorem 3.3 (Kolmogorov). The cohomology equation \( (1) \) over a Diophantine linear flow on \( \mathbb{T}^N \) admits a solution for any \( f \in C^\infty(\mathbb{T}^N, \mathbb{R}) \). In particular, any Diophantine linear flow is parameter rigid.

Proof. Take the Fourier expansion
\[
f(x) = \sum_{m \in \mathbb{Z}^N} a_m \exp(2\pi \langle m, x \rangle \sqrt{-1})
\]
of \( f \). Since \( f \) is a smooth function, we have
\[
\sup_{m \in \mathbb{Z}^N} \|m\|^k |a_m| < \infty \tag{2}
\]
for any \( k \geq 1 \).

Fix a Diophantine vector \( v \in \mathbb{R}^N \). Put \( b_0 = 0 \) and
\[
b_m = \frac{a_m}{2\pi \langle m, v \rangle \sqrt{-1}}
\]
for \( m \neq 0 \). Then,
\[
g(x) = \sum_{m \in \mathbb{Z}^N} b_m \exp(2\pi \langle m, x \rangle \sqrt{-1})
\]
is a formal solution of \( f = X_v g + a_0 \). Since \( v \) is Diophantine, there exists \( \tau > 0 \) and \( C > 0 \) such that \( |b_m| \leq C \|m\|^\tau |a_m| \) for any \( m \in \mathbb{Z}^N \). By Equation 2, we have
\[
\sup_{m \in \mathbb{Z}^N} \|m\|^k |b_m| < \infty
\]
for any \( k \geq 1 \). It implies that \( g \) is a smooth function. \( \square \)

Diophantine linear flows are the only known examples of non-trivial parameter rigid flows.

Conjecture 3.4 (Katok). Any parameter rigid flow on a closed manifold is conjugate to a Diophantine linear flow.

Recently, some partial results on the conjecture are obtained.

Theorem 3.5 (F.Rodrigues-Hertz and A.Rodrigues-Hertz [44]). Let \( M \) be a closed manifold with the first Betti number \( b_1 \). If \( \rho \in A_{LF}(M, \mathbb{R}) \) is parameter rigid, then there exists a smooth submersion \( \pi : M \to \mathbb{T}^{b_1} \) and a Diophantine linear flow \( R_t \) on \( \mathbb{T}^{b_1} \) such that \( \pi \circ \rho_t = R_t \circ \pi \).

In particular, if \( b_1 = \dim M \), then \( M \) is diffeomorphic to \( \mathbb{T}^{b_1} \) and \( \rho \) is conjugate to a Diophantine linear flow.

Theorem 3.6 (Forni [17], Kocsard [31], and Matsumoto [37]). Any locally free parameter rigid flow on a three-dimensional closed manifold is conjugate to a Diophantine linear flow on \( \mathbb{T}^3 \).
3.2 Deformation of flows

There is no known example of a locally rigid flow and it is almost hopeless to find it.

**Proposition 3.7.** If \( \rho \in A_{LF}(M, \mathbb{R}) \) is locally rigid, then there exists a neighborhood \( U \) of \( \rho \) such that no \( \rho' \in U \) admits a periodic point.

**Proof.** Let \( U \) be the conjugacy class of \( \rho \). Since \( \rho \) is locally rigid, it is a neighborhood of \( \rho \). For \( \rho' \in A_{LF}(M, \mathbb{R}) \), put

\[
\Lambda(\rho') = \{ \det D\rho'_{x} | x \in M, T \in \mathbb{R}, \rho'(x) = x \}.
\]

It is invariant under conjugacy. Hence, \( \Lambda(\rho') = \Lambda(\rho) \) for any \( \rho' \in U \).

By the Kupka-Smale theorem (see e.g., [43]), the set \( U \) contains a flow with at most countably many periodic orbits. It implies that \( \rho \) admits at most countably many periodic orbits, and hence, \( \Lambda(\rho) \) is at most countable. However, if \( \Lambda(\rho) \) is non-empty, then small perturbation on a small neighborhood of a periodic orbit can produce a flow \( \rho' \in U \) such that \( \Lambda(\rho') \neq \Lambda(\rho) \).

It is unknown whether any open subset of \( A_{LF}(M, \mathbb{R}) \) contains a flow with a periodic point or not. On the other hand, any open subset of the set of \( C^1 \) flows (with \( C^1 \)-topology) contains a flow with a periodic point. It is just an immediate consequence of Pugh’s \( C^1 \)-closing lemma [42]. The validity of the \( C^\infty \)-closing lemma is a long-standing open problem in the theory of dynamical systems.

The following exercise shows that it is hard to find a locally complete deformation of a flow.

**Exercise 3.8.** Suppose that a flow \( \rho \in A_{LF}(M, \mathbb{R}) \) admits infinitely many periodic orbit. Show that any deformation \( (\rho_{\mu})_{\mu} \in \Delta \) of \( \rho \) is not locally complete.

On the other hand, the Diophantine linear flow admits a locally transverse deformation.

**Theorem 3.9.** Let \( v \in \mathbb{R}^N \) be a Diophantine vector and \( E \subset \mathbb{R}^N \) be its orthogonal complement. Then, the deformation \( (R_{v+\mu})_{\mu} \in E \) of \( R_{v} \) is locally transverse.

The theorem is derived from the following result due to Herman. Fix \( N \geq 2 \) and a point \( x_0 \in T^N \). Let \( \text{Diff}(T^N, x_0) \) be the set of diffeomorphisms of \( T^N \) which fix \( x_0 \).

**Theorem 3.10 (Herman).** Suppose \( v \in \mathbb{R}^N \) is Diophantine. Then, there exists a neighborhood \( U \) of \( X_v \) in \( \mathcal{X}(T^N) \), a neighborhood \( V \) of \( \text{Id}_{T^N} \) in \( \text{Diff}(T^N, x_0) \), and smooth map \( \bar{w} : U \rightarrow \mathbb{R}^N \) which satisfy the following property: For any \( Y \in U \), there exists a unique diffeomorphism \( h \in V \) such that \( Y = h_{*}(X_v) + X_{\bar{w}(Y)} \).

**Proof.** We give only a sketch of proof here. See e.g., [43] for details. We define a smooth map \( \Phi : \text{Diff}(T^N, x_0) \times \mathbb{R}^N \rightarrow \mathcal{X}(T^N) \) by \( (h, w) \rightarrow h_{*}(X_v) + X_w \). The theorem is an immediate consequence of the Nash-Moser inverse function theorem if we can apply it to \( \Phi \) at \( (\text{Id}_{T^N}, v) \). The Nash-Moser inverse
function theorem requires the existence of the inverse of the differential $D\Phi$ not only at one point but also on a neighborhood. It is possible to show $D\Phi_{(\text{Id}_{TN},v)}$ is invertible if Equation (1) can be solved for any $f$. Moreover, the solvability of (1) implies that $D\Phi$ is invertible at any point close to $(\text{Id}_{TN},v)$. Hence, we can apply the Nash-Moser inverse function theorem.

Proof of Theorem 3.9. Let $U$, $V$, and $\bar{w}$ be the neighborhoods and the map in Herman’s theorem. For $\rho \in A(T^N,\mathbb{R})$, we denote the vector field generating $\rho$ by $Y_\rho$. Take neighborhoods $U$ of 0 in $E$ and $V$ of $(Rv+\mu)\in E$ in $A_{LF}(T^N,\mathbb{R};E)$, and a constant $\delta > 0$ such that $(1+c)Y_{\rho_{\mu}} \in U$ for any $(\rho_{\mu})\in E$, $\mu \in U$, and $c \in (-\delta,\delta)$. For $(\rho_{\mu})\in E$, we define map $\Psi_{(\rho_{\mu})} : U \times (-\delta,\delta) \to \mathbb{R}^N$ by $\Psi_{(\rho_{\mu})}(\mu,c) = \bar{w}(1+c)\cdot Y_{\rho_{\mu}}$. It is a smooth map and smoothly depends on $(\rho_{\mu})\in E$. By the uniqueness of the choice of $h \in \text{Diff}_0(T^N)$ in Herman’s theorem, we have $\Psi_{(Rv+\mu)}(\mu,c) = (1+c)\mu + cv$. In particular, $\Psi_{(Rv+\mu)}$ is a local diffeomorphism at $(\mu,c) = (0,0)$. Now, the usual inverse function theorem implies that if $(\rho_{\mu})\in E$ is sufficiently close to $(Rv+\mu)\in E$ then there exists $(\mu_*,c_*) \in U \times (-\delta,\delta)$ such that $\Psi_{(\rho_{\mu})}(\mu_*,c_*) = 0$. Hence, there exists $h_* \in V$ which conjugates $R_v$ with $\rho_{\mu_*}$.

The above family $(Rv+\mu)\in E$ is the best possible in the following sense.

Exercise 3.11. Let $(\rho_{\mu})\in \Delta \in A_{LF}(T^N,\mathbb{R};\Delta)$ be a deformation of $R_v$ for $v \in \mathbb{R}^N$. Show that if the dimension of $\Delta$ is less than $N-1$, then $(\rho_{\mu})\in \Delta$ is not a locally transverse deformation.

4 The leafwise cohomology

As we saw in the previous section, the cohomology equation plays an important role in the rigidity problem of locally free $\mathbb{R}$-actions. For actions of general Lie groups, the solvability of the equation is generalized to the almost vanishing of the first leafwise cohomology of the orbit foliation. In this section, we give the definition of the leafwise cohomology and show some of its basic properties. We also compute the cohomology for several examples.

4.1 The definition and some basic properties

Let $F$ be a foliation on a manifold $M$. As before, we denote the tangent bundle of $F$ by $TF$. We also denote the dual bundle of $TF$ by $T^*F$. For $k \geq 0$, let $\Omega^k(F)$ be the space of smooth sections of $\wedge^kT^*F$. Each element of $\Omega^*(F)$ is called a leafwise $k$-form.

By Frobenius’ theorem, if $X, Y \in \mathfrak{X}(F)$, then $[X, Y] \in \mathfrak{X}(F)$. Hence, we can
define the leafwise differential $d^*_F : \Omega^k(F) \to \Omega^{k+1}(F)$ by

$$(d^*_F \omega)(X_0, \cdots, X_k) = \sum_{0 \leq i \leq k} (-1)^i X_i \left( \omega(X_0, \cdots, \hat{X}_i, \cdots, X_k) \right) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k)$$

for $X_0, \cdots, X_k \in \mathcal{X}(F)$. Same as the usual exterior differential, the leafwise differential satisfies $d^*_{F+1} \circ d^*_F = 0$. For $k \geq 0$, the $k$-th leafwise cohomology group $H^k(F)$ is the $k$-th cohomology group of the cochain complex $(\Omega^\ast(F), d_F)$.

**Example 4.1.** $H^0(F)$ is the space of smooth functions which are constant on each leaf of $F$. Hence, if $F$ has a dense leaf, then $H^0(F) \simeq \mathbb{R}$.

**Example 4.2.** Suppose that $F$ is a one-dimensional orientable foliation on a closed manifold $M$. Let $X_0$ be a vector field generating $F$. Take $\omega_0 \in \Omega^1(F)$ such that $\omega_0(X_0) = 1$. Then, we have $d^*_F g = (X_0 g) \cdot \omega_0$ for $g \in \Omega^0(F) = C^\infty(M, \mathbb{R})$. Since $d^*_F$ is the zero map, the cohomology equation $[\ ]$ is solved for any $f \in C^\infty(M, \mathbb{R})$ if and only if $H^1(F) \simeq \mathbb{R}$. In this case, $[\omega_0]$ is a generator of $H^1(F)$.

There are two important homomorphisms whose target is $H^\ast(F)$. The first is a homomorphism from the de Rham cohomology group. Let $\Omega^k(M)$ and $H^k(M)$ be the space of (usual) smooth $k$-forms and the $k$-th de Rham cohomology group of $M$. By Frobenius’ theorem, the restriction of a closed (resp. exact) $k$-form to $\otimes^k T_F$ defines a $d_F$-closed (resp. exact) leafwise $k$-form. So, the restriction map $r : \Omega^k(M) \to \Omega^k(F)$ induces a homomorphism $r_* : H^k(M) \to H^k(F)$.

The second is a homomorphism from the cohomology of a Lie algebra when $F$ is the orbit foliation of a locally free action. Let us recall the definition of the cohomology group of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra. For $k \geq 0$, we define the differential $d^k_{\mathfrak{g}} : \wedge^k \mathfrak{g}^\ast \to \wedge^{k+1} \mathfrak{g}^\ast$ by $d^0_{\mathfrak{g}} = 0$ and

$$(d^k_{\mathfrak{g}} \alpha)(\xi_0, \cdots, \xi_k) = \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_0, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_k)$$

for $k \geq 1$ and $\xi_0, \cdots, \xi_k \in \mathfrak{g}$. The $k$-th cohomology group $H^k(\mathfrak{g})$ is the $k$-th cohomology group of the chain complex $(\wedge^\ast \mathfrak{g}^\ast, d^k_{\mathfrak{g}})$.

**Exercise 4.3.** $H^1(\mathfrak{g})$ is isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Suppose that $F$ is the orbit foliation of a locally free action $\rho$ of a Lie group $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $I_\rho \in \text{Hom}(\mathfrak{g}, \mathcal{X}(M))$ be the infinitesimal action associated with $\rho$. Then, $I_\rho$ induces a homomorphism $\iota_\rho : \wedge^\ast \mathfrak{g}^\ast \to \Omega^\ast(F)$ by

$$\iota_\rho(\alpha)_x(I_\rho(\xi_1), \cdots, I_\rho(\xi_k)) = \alpha(\xi_1, \cdots, \xi_k)$$

for any $\alpha \in \wedge^k \mathfrak{g}^\ast$, $\xi_1, \cdots, \xi_k \in \mathfrak{g}$, and $x \in M$. Since the map $\iota_\rho$ commutes with the differentials, it induces a homomorphism $(\iota_\rho)_* : H^\ast(\mathfrak{g}) \to H^\ast(F)$. 

11
Proposition 4.4. The homomorphism $(\iota_\rho)_*: H^1(\mathfrak{g}) \to H^1(\mathcal{F})$ between the first cohomology groups is injective when $M$ is a closed manifold.

Proof. Fix $\alpha \in \text{Ker} d^l_\mathfrak{g}$ such that $(\iota_\rho)_*(\alpha) = 0$. Then, there exists $\xi \in C^\infty(M, \mathbb{R})$ such that $\iota_\rho(\xi) = d^l_\mathfrak{g} \xi$. For $\xi \in \mathfrak{g}$, let $\Phi_\xi$ be the flow on $M$ generated by $I_\rho(\xi)$. For any $\xi \in \mathfrak{g}$, $T > 0$, and $x \in M$,

$$\alpha(\xi) \cdot T = \int (\Phi_\xi(x))_{0 \leq t \leq T} (\iota_\rho(\xi) - \Phi_\xi(x)) = g \circ \Phi_\xi(x) - g(x).$$

Since the last term is bounded and $T$ is arbitrary, we have $\alpha(\xi) = 0$ for any $\xi \in \mathfrak{g}$. Therefore, $\alpha = 0$.

Example 4.5. Let $\mathcal{F}$ be the orbit foliation of a Diophantine linear flow $R_\omega$ on $T^N$. By Theorem 3.3, $H^1(\mathcal{F})$ is isomorphic to $\mathbb{R}$. The above proposition implies that $H^1(\mathcal{F})$ is generated by the dual $\omega_\nu$ of the constant vector field $X_\nu$. The form $\omega_\nu$ is the restriction of a usual 1-form. So, $H^1(\mathcal{F}) = \text{Im } \iota_* = \text{Im } r_*$. In particular, the map $r_*$ is not injective for $N \geq 2$.

The vanishing of the first leafwise cohomology of the orbit foliation implies the existence of invariant volume.

Proposition 4.6 (dos Santos [46]). Let $G$ be a simply connected Lie group and $\mathcal{F}$ be a foliation on an orientable closed manifold $M$. If $H^1(\mathcal{F}) \cong H^1(\mathfrak{g})$, then any $\rho \in \mathcal{A}_{\mathcal{F}}(\mathcal{F}, G)$ preserves a smooth volume, i.e. there exists a smooth volume $\nu$ on $M$ such that $(\rho^\nu)^* \nu = \nu$ for any $\nu \in G$.

Proof. Fix an action $\rho \in \mathcal{A}_{\mathcal{F}}(\mathcal{F}, G)$ and a smooth volume form $\nu$ on $M$. We define a leafwise one-form $\omega \in \Omega^1(\mathcal{F})$ by $L_X \nu = \omega(X) \cdot \nu$ for any $X \in \mathfrak{X}(\mathcal{F})$. Then,

$$(d\omega(X,Y)) \cdot \nu = \{X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y])\} \nu$$

$$= L_X(L_Y \nu) - L_Y(L_X \nu) - L_{[X,Y]} \nu$$

$$= 0$$

for any $X, Y \in \mathfrak{X}(\mathcal{F})$. Since $H^1(\mathcal{F}) = \text{Im } (\iota_\rho)_*$ by assumption and Proposition 4.4, there exists a smooth function $f$ on $M$ and $\alpha \in \mathfrak{g}^*$ such that $\omega = \iota_\rho(\alpha) + df$.

Define a new volume form $\nu_f$ on $M$ by $\nu_f = e^{-f} \cdot \nu$. It satisfies

$$(L_{\iota_\rho(\xi)}) \nu_f = \iota_\rho(\alpha)(I_\rho(\xi)) \cdot \nu_f = \alpha(\xi) \cdot \nu_f$$

for any $\xi \in \mathfrak{g}$. Since $M$ is a closed manifold, $\alpha(\xi)$ must be zero. It implies that $\rho$ preserves the volume $\nu_f$. 

Remark that the converse of the proposition does not hold. In fact, there is an easy counterexample. The linear flow associated with a rational vector preserves the standard volume of the torus. However, the first leafwise cohomology of the orbit foliation is infinite dimensional since all points of the torus are periodic.
4.2 Computation by the Mayer-Vietoris argument

Let \( \mathcal{F} \) be a foliation on a manifold \( M \). By \( \mathcal{F}|_U \), we denote the restriction of \( \mathcal{F} \) to an open subset \( U \) of \( M \). More precisely, the leaf \( (\mathcal{F}|_U)(x) \) is the connected component of \( \mathcal{F}(x) \cap U \) which contains \( x \). For \( k \geq 0 \), we define a sheaf \( \Omega^k_{\mathcal{F}}(U) = \Omega^k(\mathcal{F}|_U) \) and \( H^k_{\mathcal{F}}(U) = H^k(\mathcal{F}|_U) \). For open subsets \( U_1 \) and \( U_2 \) of \( M \), we can show the Mayer-Vietoris exact sequence

\[
\cdots \xrightarrow{\delta^k} H^k_{\mathcal{F}}(U_1 \cup U_2) \xrightarrow{i_1^*} H^k_{\mathcal{F}}(U_1) \oplus H^k_{\mathcal{F}}(U_2) \xrightarrow{i_1^* + i_2^*} H^k_{\mathcal{F}}(U_1 \cap U_2) \xrightarrow{\delta^k} H^{k+1}_{\mathcal{F}}(U_1 \cup U_2) \xrightarrow{i_2^*} \cdots
\]

as the usual de Rham cohomology.

Let us compute the leafwise cohomology of several foliations of suspension type, using the above exact sequence. Let \( \mathcal{F} \) be a foliation on a manifold \( M \). Suppose that a diffeomorphism \( h \) of \( M \) satisfies \( h(\mathcal{F}(x)) = \mathcal{F}(hx) \) for any \( x \in M \). By \( M_h \), we denote the mapping torus \( M \times \mathbb{R}/(x, t + 1) \sim (hx, t) \). The product foliation \( \mathcal{F} \times \mathbb{R} \) on \( M \times \mathbb{R} \) induces a foliation \( \mathcal{F}_h \) on \( M_h \). The foliation \( \mathcal{F}_h \) is called the suspension foliation of \( \mathcal{F} \).

Take an open cover \( M_h = U_1 \cup U_2 \) such that \( U_1 = M \times (0, 1) \) and \( U_2 = M \times (-1/2, 1/2) \). Then, the natural projection from \( U_i \) to \( M \) induces an isomorphism between \( H^*(\mathcal{F}) \) and \( H^*_U(\mathcal{F}_h) \). Similarly, \( H^*_U(\mathcal{F}_h) \) is naturally isomorphic to \( H^*(\mathcal{F}) \oplus H^*(\mathcal{F}) \). Under these identifications, the map \( i^* \) is given by \( i^*(a, b) = (a - b, a - h^*(b)) \) for \( (a, b) \in H^*(\mathcal{F}) \oplus H^*(\mathcal{F}) \). Hence, we have

\[
\text{Ker} i^* \simeq \text{Ker}(I - h^*), \quad \text{Im} i^* \simeq H^*(\mathcal{F}) \oplus \text{Im}(I - h^*).
\]

The Mayer-Vietoris exact sequence implies

\[
H^*(\mathcal{F}_h) \simeq \text{Ker} i^* \oplus \text{Im} \delta^{s-1} \simeq \text{Ker} i^* \oplus [H^{s-1}(\mathcal{F}) \oplus H^{s-1}(\mathcal{F})]/\text{Im} i^{s-1}.
\]

Therefore,

\[
H^*(\mathcal{F}_h) \simeq \text{Ker}(I - h^*) \oplus [H^{s-1}(\mathcal{F})/\text{Im}(I - h^{s-1})]. \quad (3)
\]

We compute \( H^1(\mathcal{F}_h) \) for two explicit examples. The first is a direct generalization of Theorem 3.3. Suppose that \( v_1, \ldots, v_p \in \mathbb{R}^N \) are linearly independent. We define the linear action \( \rho \in \mathcal{A}(\mathbb{T}^N, \mathbb{R}^p) \) by \( \rho(t_1, \ldots, t_p)(x) = x + \sum_{i=1}^p t_i v_i \). We say the action \( \rho \) is Diophantine if there exists \( \tau > 0 \) such that

\[
\inf_{m \in \mathbb{Z}^N \setminus \{0\}} \| \langle (m, v_1), \ldots, (m, v_k) \rangle \| \cdot \| m \|^\tau > 0.
\]

Its orbit foliation is called a Diophantine linear foliation.

**Theorem 4.7** ([14], see also [2]). Let \( \mathcal{F} \) be a \( p \)-dimensional Diophantine linear foliation on \( \mathbb{T}^N \). Then, \( H^*(\mathcal{F}) \simeq H^*(\mathbb{T}^p) \).

**Proof.** Proof is by induction of \( p \). The case \( p = 1 \) is a consequence of Theorem 3.3 and Example 4.1. Suppose that the theorem holds for \( p - 1 \). Let \( \mathcal{F} \) be a \( p \)-dimensional Diophantine linear foliation on \( \mathbb{T}^N \). Then, there exists a
(p − 1)-dimensional Diophantine linear foliation \( F' \) on \( T^{N−1} \) such that \( F_\rho \) is diffeomorphic to the suspension foliation of \( F' \) by a translation \( h \) of \( T^N \). By the assumption of induction, \( H^*(F') = \text{Im}(\iota_{\rho'})^* \). Since the translation \( h \) preserves each element of \( \text{Im} \iota_{\rho'} \), the map \( h_* \) is the identity map. By (3) and the assumption of induction,

\[
H^k(F) \simeq H^k(F') \oplus H^{k−1}(F') \simeq H^k(T^{p−1}) \oplus H^{k−1}(T^{p−1}).
\]

Therefore, \( H^*(F) \simeq H^*(T^p) \).

Another example is the suspension of the stable foliation of a hyperbolic toral automorphism. Let \( A \) be an integer valued matrix with \( \det A = 1 \). We define a diffeomorphism \( F_A \) on \( T^2 \) by \( F_A(x + Z^2) = Ax + Z^2 \). Suppose that eigenvalues \( \lambda, \lambda^{-1} \) of \( A \) satisfies \( \lambda > 1 > \lambda^{-1} > 0 \). Let \( E^s \) be the eigenspace of \( \lambda^{-1} \) and \( F^s \) be the foliation on \( T^2 \) given by \( F^s(x) = x + E^s \). Since \( F_A(F^s(x)) = F^s(F_A(x)) \), the foliation \( F^s \) induces the suspension foliation \( F_A \) on the mapping torus \( M_A \).

Theorem 4.8 \([14] \). \( H^1(F_A) \simeq \mathbb{R} \).

Proof. It is known that \( F^s \) is a Diophantine linear foliation. So, we have \( H^0(F^s) \simeq H^1(F^s) \simeq \mathbb{R} \). By a direct computation, we can check that \( F_A^* = I \) on \( H^0(F^s) \) and \( F_A^* = \lambda^{-1} \cdot I \) on \( H^1(F^s) \). The isomorphism (3) implies \( H^1(F^s) \simeq \mathbb{R} \). \( \square \)

El Kacimi-Alaoui and Tihami also computed the first leafwise cohomology group for the suspension foliation of higher dimensional hyperbolic toral automorphisms. See [14].

As the usual de Rham cohomology, the Mayer-Vietoris sequence is generalized to a spectral sequence. Let \( U = \{U_i\} \) be a locally finite open cover of \( M \). By the same construction as the Čech-de Rham complex (see e.g., [6]), we obtain a double complex \( (C^*(U, \Omega^*_F), d_F, \delta) \), where

\[
C^p(U, \Omega^q_F) = \bigoplus_{i_1 < \cdots < i_p} \Omega^q_F(U_{i_1} \cap \cdots \cap U_{i_p})
\]

and \( \delta : C^*(U, \Omega^*_F) \to C^{*+1}(U, \Omega^*_F) \) is a natural linear map induced by inclusions. Moreover, we can show that the sequence

\[
0 \to \Omega^0(F) \to C^0(U, \Omega^0_F) \xrightarrow{\partial} C^1(U, \Omega^0_F) \xrightarrow{\partial} C^2(U, \Omega^0_F) \to \cdots
\]

is exact. The following theorem is proved by the standard method.

Theorem 4.9 (El Kacimi Alaoui and Tihami [14]). There exists a spectral sequence \( \{E^{r, *}_r\} \) such that \( E^1_0 = C^0(U, \Omega^0_F) \), \( E^2_0 = H^0(U, \Omega^0_F) \), and \( \{E^{r, *}_r\} \) converges to \( H^*(F) \).

You can find several applications of the spectral sequence in [14].
4.3 Other examples

In this subsection, we give several examples of foliations whose first leafwise cohomology is computed by other methods.

Fix \( p \geq 1 \) and a cocompact lattice \( \Gamma \) of \( SL(p+1,\mathbb{R}) \). Put \( M_\Gamma = \Gamma \backslash SL(p+1,\mathbb{R}) \). By \( A \), we denote the subset of \( SL(p+1,\mathbb{R}) \) that consists of positive diagonal matrices. It is a closed subgroup of \( SL(p+1,\mathbb{R}) \) isomorphic to \( \mathbb{R}^p \).

The Weyl chamber flow is the action \( \rho : A_LF(M_\Gamma, A) \) given by \( \rho(\Gamma x, a) = \Gamma(xa) \).

Let \( \mathcal{A}_p \) be the orbit foliation of this action.

**Theorem 4.10** (Katok and Spatzier [29]). If \( p \geq 2 \), \( H^1(\mathcal{A}_p) \cong \mathbb{R}^p \).

The key features of the proof are the decay of matrix coefficients of the regular \( L^2 \)-representation and the hyperbolicity of \( A \)-action. Remark that Katok and Spatzier proved a similar result for a wide class of Lie groups of real-rank more than one.

As an application of the above theorem, we compute another foliation on \( M_\Gamma \). Let \( P \) be the subgroup of \( SL(p+1,\mathbb{R}) \) that consisting of upper triangular matrices with positive diagonals. It naturally acts on \( M_\Gamma \) from right. Let \( \mathcal{F}_p \) be the orbit foliation of this action.

**Theorem 4.11.** If \( p \geq 2 \), \( H^1(\mathcal{F}_p) \cong \mathbb{R}^p \).

*Proof.* Let \( E_{ij} \) be the square matrix of size \( (p+1) \) whose \((i,j)\)-entry is one and the other entries are zero. For \( i, j = 1, \ldots, p+1 \), we define flows \( \Phi_{ij} \) and \( \Psi_{ij} \) on \( M_\Gamma \) by \( \Phi^t_{ij}(\Gamma g) = \Gamma g \exp(t(E_{ii} - E_{jj})) \) and \( \Psi^t_{ij}(\Gamma g) = \Gamma g \exp(tE_{ij}) \).

Let \( X_{ij} \) and \( Y_{ij} \) be the vector fields on \( M_\Gamma \) which correspond to \( \Phi_{ij} \) and \( \Psi_{ij} \), respectively. Remark that \( \mathcal{A}_p \) is generated by \( X_{ij} \)'s and \( \mathcal{F}_p \) is generated by \( X_{ij} \)'s and \( Y_{ij} \)'s with \( i < j \).

Take a \( d\mathcal{F}_p \)-closed 1-form \( \omega \in \Omega^1(\mathcal{F}_p) \). The restriction of \( \Omega \) to \( TA_p \) is \( d\mathcal{A}_p \)-closed. By Theorem 1.10 there exists \( h \in C^\infty(M_\Gamma, \mathbb{R}) \) such that \( (\omega - dh)(X_{ij}) \) is a constant function for any \( i, j = 1, \ldots, p+1 \).

We put \( \omega' = \omega + dh \) and show \( \omega'(Y_{ij}) = 0 \) for any \( i < j \). Fix \( i, j, k = 1, \ldots, p+1 \) so that \( i < j \) and \( k \neq i, j \). Since \( [X_{ik}, Y_{ij}] = Y_{ij} \) and \( d\mathcal{F}_p(X_{ik}, Y_{ij}) = 0 \), we have \( X_{ik}(\omega'(Y_{ij})) = \omega'(Y_{ij}) \). It implies that \( \omega'(Y_{ij})(\Phi^t_k(x)) = e^t\omega'(Y_{ij})(x) \) for any \( t \in \mathbb{R} \) and \( x \in M_\Gamma \).

By the compactness of \( M_\Gamma \), \( \omega'(Y_{ij}) \) is constantly zero. Therefore, any \( d\mathcal{F}_p \)-closed 1-form is cohomologous to the constant form which vanishes at \( Y_{ij} \) for any \( i < j \). \( \Box \)

For \( p = 1 \), the Weyl chamber flow is an \( \mathbb{R} \)-action, and it is naturally identified with the geodesic flow of a two-dimensional hyperbolic orbifold. It admits infinitely many periodic points and hence, \( H^1(\mathcal{A}_1) \) is infinite dimensional. By contrast, the following theorem asserts that \( H^1(\mathcal{F}_1) \) is finite dimensional. Let \( \rho_\Gamma \) be the natural right action of \( P \) on \( \Gamma \backslash SL(2,\mathbb{R}) \). We denote the Lie algebra of \( SL(2,\mathbb{R}) \) by \( \mathfrak{sl}_2(\mathbb{R}) \). Let \( \iota_{\mathfrak{sl}_2} : H^1(\mathfrak{sl}_2(\mathbb{R})) \to H^1(\mathcal{F}_1) \) and \( r_* : H^1(M) \to H^1(\mathcal{F}_1) \) be homomorphisms defined in Section 4.1.
Theorem 4.12 (Matsumoto and Mitsumatsu [38]). The map

$$(ιρ)_* ⊕ r_* : H^1(\mathfrak{sl}_2(\mathbb{R})) ⊕ H^1(Γ\backslash SL(2, \mathbb{R})) → H^1(F_1)$$

is an isomorphism.

Kanai [27] prove the corresponding result for general simple Lie groups of real-rank one. In the both results, the key feature of the proof is the hyperbolicity of $A$-subaction.

For the above examples, the hyperbolic behavior of the $A$-action is fundamental to the computation of the leafwise cohomology. In the last example below, the action has no hyperbolic nature and the leafwise cohomology is computed by purely representation-theoretic method. Let $Γ$ be a cocompact lattice of $SL(2, \mathbb{C})$ and put

$$u(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

for $z ∈ \mathbb{C}$. We define an action $ρ ∈ A_{LF}(Γ\backslash SL(2, \mathbb{C}), \mathbb{C})$ by $ρ(Γx, z) = Γ(xu(z))$. Let $F$ be the orbit foliation of $ρ$.

Theorem 4.13 (Mieczkowski [39]). The image of $d^0_F$ is a closed subspace of $\ker d^1_F$ and there exists a subspace $H$ of $\ker d^1_F$ such that $H ≃ H^1(M)$ and

$$\ker d^1_F = \text{Im} d^0_F ⊕ \text{Im} ι_ρ ⊕ H.$$ 

In particular, $H^1(F) ≃ \mathbb{R}^2 ⊕ H^1(M)$.

5 Parameter deformation

Now, we come back to the study of the deformation of locally free actions. In this section, we discuss about the parameter rigidity and the existence of locally complete orbit-preserving deformations.

5.1 The canonical one-form

Let $G$ be a simply connected Lie group and $g$ be its Lie algebra. To simplify the presentation, we assume that $G$ is linear, i.e., a closed subgroup of $GL(N, \mathbb{R})$ with some large $N ≥ 1$. Then, each element of $g$ is naturally identified with a square matrices of size $N$.

Fix a foliation $F$ on a closed manifold $M$. A $g$-valued leafwise 1-form $ω ∈ Ω^1(F) ⊗ g$ is called regular if $ω_x : T_xF → g$ is a linear isomorphism for any $x ∈ M$. Since the infinitesimal action $I_ρ$ associated with $ρ ∈ A_{LF}(F, G)$ is regular, i.e., the map $(I_ρ)_x : g → T_xF$ is an isomorphism, it induces a regular 1-form $ω_ρ ∈ Ω^1(F) ⊗ g$ by $(ω_ρ)_x = (I_ρ)_x^{-1}$. We call $ω_ρ$ the canonical 1-form of $ρ$. 

16
Lemma 5.1. Let $\xi_1, \cdots, \xi_p$ be a basis of $\mathfrak{g}$ and $\alpha_1, \cdots, \alpha_p$ be its dual basis of $\mathfrak{g}^*$. Then, we have
\[
\omega_\rho = \sum_{i=1}^p \iota_\rho(\alpha_i) \otimes \xi_i,
\]
where $\iota_\rho : \mathfrak{g}^* \to \Omega^1(\mathcal{F})$ is the homomorphism defined in Section 4.1.

Proof. For $\xi = \sum_{i=1}^p c_i \xi_i$, we have $\omega_\rho((I_\rho)_{\mathcal{F}}(\xi)) = \xi$ by definition. On the other hand,
\[
\sum_{i=1}^p \iota_\rho(\alpha_i)((I_\rho)_{\mathcal{F}}(\xi_i)) \otimes \xi_i = \sum_{i=1}^p c_i \iota_\rho(\alpha_i)((I_\rho)_{\mathcal{F}}(\xi_i)) \otimes \xi_i = \sum_{i=1}^p c_i \xi_i = \xi.
\]

The group of automorphisms of $G$ acts (from left) on $A_{LF}(\mathcal{F}, G)$ by $(\Theta \cdot \rho)(x, g) = \rho(\Theta^{-1}(g))$.

Exercise 5.2. Show $\omega_{\Theta \cdot \rho} = \Theta_\ast \omega_\rho$, where $\Theta_\ast : \mathfrak{g} \to \mathfrak{g}$ is the differential of $\Theta$.

The following proposition characterizes the canonical 1-form.

Proposition 5.3. A $\mathfrak{g}$-valued leafwise 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$ is the canonical 1-form of some action in $A_{LF}(\mathcal{F}, G)$ if and only if it is a regular 1-form which satisfies the equation
\[
d_\mathcal{F} \omega + [\omega, \omega] = 0,
\]
where $[\omega, \omega]$ is a $\mathfrak{g}$-valued leafwise two-form defined by $[\omega, \omega]_\mathcal{F}(v, w) = [\omega(v), \omega(w)]$ for any $v, w \in T_x \mathcal{F}$.

Proof. Fix a basis $\xi_1, \cdots, \xi_k$ of $\mathfrak{g}$. Let $\{c_{ij}^k\}$ be the structure constants of $\mathfrak{g}$, i.e.,
\[
[\xi_i, \xi_j] = \sum_{k=1}^k c_{ij}^k \xi_k.
\]
Take a regular 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$. Let $X_i$ be a nowhere-vanishing vector field in $\mathfrak{X}(\mathcal{F})$ given by $X_i(x) = \omega_x^{-1}(\xi_i)$. Then,
\[
(d_\mathcal{F} \omega + [\omega, \omega])(X_i, X_j) = X_i(\omega(X_j)) - X_j(\omega(X_i)) - \omega([X_i, X_j]) + [\omega(X_i), \omega(X_j)]
\]
\[
= -\omega([X_i, X_j]) + \sum_k c_{ij}^k \xi_k
\]
\[
= -\omega([X_i, X_j]) + \sum_k c_{ij}^k \omega(X_k)
\]
\[
= \omega(\sum_k c_{ij}^k \omega(X_k) - [X_i, X_j]).
\]
Since $\omega$ is regular, $d_\mathcal{F} \omega + [\omega, \omega] = 0$ if and only if $[X_i, X_j] = \sum_k c_{ij}^k X_k$ for any $i, j$. The latter condition is equivalent to that the linear map $\xi_i \mapsto \sum X_i$ is a homomorphism between Lie algebras. Hence, $d_\mathcal{F} \omega + [\omega, \omega] = 0$ if and only if there exists $\rho \in A_{LF}(\mathcal{F}, G)$ such that $I_\rho(\xi_i) = X_i$, equivalently, $\omega_\rho(X_i(x)) = \xi_i = \omega(X_i(x))$ for any $i$. \qed
The following proposition describe how the canonical 1-form is transformed under parameter-equivalence of actions. We denote the constant map from \( M \) to \( \{1_G\} \) by \( b_{1_G} \).

**Proposition 5.4.** An action \( \rho \in A_{LF}(\mathcal{F}, G) \) is equivalent to \( \rho_0 \) if and only if there exists a smooth map \( b : M \rightarrow G \) homotopic to \( b_{1_G} \) and an endomorphism \( \Theta : G \rightarrow G \) such that

\[
\omega_\rho = b^{-1} \cdot \Theta \cdot \omega_{\rho_0} \cdot b + b^{-1} d_x b.
\] (5)

To prove the proposition, we need to introduce cocycles over an action. Let \( H \) be another Lie group and \( h \) be its Lie algebra. Fix an action \( \rho_0 \in A_{LF}(\mathcal{F}, G) \). We say that \( a \in C^\infty(M \times G, H) \) is a \((H\text{-valued})\) cocycle over \( \rho_0 \) if \( a(x, 1_G) = 1_H \) and \( a(x, gg') = a(x, g) \cdot a(\rho_0^*(x), g') \) for any \( x \in M \) and \( g, g' \in G \). For a cocycle \( a \), we define the canonical 1-form \( \omega_a \in \Omega^1(\mathcal{F}) \otimes h \) of \( a \) by

\[
(\omega_a)_x(X) = \frac{d}{dt} a(x, \exp(t \omega_{\rho_0}(x)))|_{t=0}.
\]

**Lemma 5.5.** Two cocycles \( a_1 \) and \( a_2 \) over \( \rho_0 \) coincide if \( \omega_{a_1} = \omega_{a_2} \).

**Proof.** For \( i = 1, 2 \), we define \( \Phi_i : M \times H \times G \rightarrow M \times H \) by \( \Phi_i((x, (h), g)) = (\rho_0^*(x), h \cdot a(x, g)) \). It is easy to see that \( \Phi_i \) is a locally free action and

\[
I_{\Phi_i}(\xi)(x, h) = (I_{\rho_0}(x), h \cdot \omega_{a_i}(I_{\rho_0}(\xi)(x)) \in T_x M \times h \cdot \omega \cong T(x, g)(M \times H).
\]

If \( \omega_{a_1} = \omega_{a_2} \), then \( \Phi_1 = \Phi_2 \). It implies \( a_1 = a_2 \). \( \square \)

Each action in \( A_{LF}(\mathcal{F}, G) \) defines a \( G \)-valued cocycle naturally.

**Lemma 5.6.** For any \( \rho \in A_{LF}(\mathcal{F}, G) \), there exists a unique \( G \)-valued cocycle \( a \) over \( \rho_0 \) which satisfies \( \rho(x, a(x, g)) = \rho_0(x, g) \) for any \( x \in M \) and \( g \in G \). Moreover, \( a(x, \cdot) : G \rightarrow G \) is a diffeomorphism for any \( x \in M \) and \( \omega_a \) is equal to the canonical 1-form of \( \rho \).

**Proof.** For any \( x \in M \), the maps \( \rho_0(x, \cdot), \rho(x, \cdot) : G \rightarrow \mathcal{F}(x) \) are coverings with \( \rho_0(x, 1_G) = \rho(x, 1_G) = x \). Since \( G \) is simply connected, there exists a unique diffeomorphism \( a_x \) of \( G \) such that \( \rho(x, a_x(g)) = \rho_0(x, g) \). Put \( a(x, g) = a_x(g) \). It is easy to see that the map \( a \) satisfies \( \rho(x, a(x, gg')) = \rho(x, a(x, g) \cdot a(\rho_0(x, g), g')) \). By the uniqueness of \( a_x \), we have \( a(x, gg') = a(x, g) \cdot a(\rho_0(x, g), g') \), and hence, \( a \) is a cocycle.

Take the differential of the equation \( \rho(x, a(x, \exp(t\xi))) = \rho_0(x, \exp(t\xi)) \) at \( t = 0 \) for \( \xi \in g \). Then, we have \( I_{\rho} (\omega_{\rho_0}(I_{\rho_0}(\xi))) = I_{\rho_0}(\xi) \). Since \( I_{\rho_0} \) is regular and \( (\omega_{\rho_0})(x) = (I_{\rho_0})^{-1} \), we have \( \omega_a = \omega_\rho \). \( \square \)

**Lemma 5.7.** Let \( \rho_1 \) and \( \rho_2 \) be actions in \( A_{LF}(\mathcal{F}, G) \), \( h \) a diffeomorphism in \( \text{Diff}_0(\mathcal{F}) \), and \( \Theta \) an endomorphism of \( G \). If \( \rho_2 \Theta(g) \circ h = h \circ \rho_1^g \) for any \( g \in G \), then \( h \) is a diffeomorphism and \( \Theta \) is an automorphism.
Proof. If the differential $\Theta_{\ast} : g \to g$ is not an automorphism, then $\hat{h}(\hat{F}(x)) = \{ \rho_2^{\Theta(g)}(h(x)) \mid g \in G \}$ is a strict subset of $F(h(x)) = F(x)$ by Sard’s theorem. Since $h$ is homotopic to the identity, it is surjective. It implies that $h(\hat{F}(x)) = F(x)$ for any $x \in M$, and hence, $\Theta_{\ast}$ must be an automorphism of $g$. Since $G$ is simply connected, $\Theta$ is an automorphism of $G$.

The maps $\rho_1(x, \cdot)$ and $h \circ \rho_1(x, \Theta(\cdot)) = \rho_2(h(x), \cdot)$ are covering maps from $G$ to $\mathcal{F}(x)$. It implies that $h$ is a self-covering of $M$. Since $h$ is homotopic to the identity, $h$ is a diffeomorphism.

Now, we are ready to prove Proposition 5.4.

Proof of Proposition 5.4. For an endomorphism $\Theta : G \to G$ and $b \in C^\infty(M, G)$, we define a cocycle $a_{b, \Theta}$ over $\rho_0$ by $a_{b, \Theta}(x, g) = b(x)^{-1} \cdot \Theta(g) \cdot b(\rho_0^b(x))$. Let $\omega_{b, \Theta}$ be its canonical 1-form. By a direct calculation, we have

$$\omega_{b, \Theta} = b^{-1} \Theta_{\ast} \omega_{\rho_0} b + b^{-1} d_x b.$$  

Suppose that $\rho$ is equivalent to $\rho_0$. Let $h$ be a diffeomorphism in $\text{Diff}_0(\hat{F})$ and $\Theta$ an automorphism of $G$ such that $\rho^{\Theta(g)} \circ h = h \circ \rho_0^g$ for any $g \in G$. Since $h$ is homotopic to $\text{Id}_M$ through diffeomorphisms preserving each leaf of $\mathcal{F}$, we can take a smooth map $b : M \to G$ homotopic to $b_{1_{\mathcal{F}}}$ such that $h(x) = b(x)^{-1} b(\rho_0^g(x))$. Then, $\rho^{b(x)^{-1} \Theta(g)}(x) = \rho^{b(x)^{-1} \rho_0^g(x)}(x)$, and hence, $\rho_0(x, g) = \rho(x, a_{b, \Theta}(x, g))$. By Lemma 5.6, the cocycle $a_{\rho}$ corresponding to $\rho$ is equal to $a_{b, \Theta}$. It implies $\omega_{a_{\rho}} = \omega_{b, \Theta}$, and hence, $\omega_{\rho} = b^{-1} \Theta_{\ast} \omega_{\rho_0} b + b^{-1} d_x b$.

Suppose that the equation (5) holds for some $\Theta$ and $b$. Since $\omega_{a_{\rho}} = \omega_{\rho} = \omega_{b, \Theta}$, the cocycle $a_{\rho}$ corresponding to $\rho$ coincides with $a_{b, \Theta}$. It implies that $\rho(x, a_{b, \Theta}(x, g)) = \rho_0(x, g)$. Put $h(x) = \rho^{b(x)^{-1}}$. Then, we have $\rho^{\Theta(g)} \circ h = h \circ \rho_0^g$. By Lemma 5.7, $\Theta$ is an automorphism and $h$ is a diffeomorphism.

The above interpretation in terms of leafwise 1-form can be done for general cocycles.

Proposition 5.8 (Matsumoto and Mitsumatu [38]). Let $G, H$ be simply connected Lie groups and $g, h$ be their Lie algebras. Let $\rho_0$ be a locally free action of $G$ on a closed manifold $M$ and $\mathcal{F}$ be the orbit foliation of $\rho_0$.

1. A 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes h$ is the canonical 1-form of some $H$-valued cocycle over $\rho_0$ if and only if $d_x \omega + [\omega, \omega] = 0$.

2. Let $a_1, a_2$ be $H$-valued cocycles over $\rho_0$, $b : M \to H$ be a smooth map homotopic to $b_{1_H}$, and $\Theta$ be an endomorphism of $H$. Then, the equation

$$a_2(x, g) = b(x)^{-1} \cdot \Theta(a_1(x, g)) \cdot b(\rho_0^b(x))$$

holds if and only if

$$\omega_{a_2} = b^{-1} (\Theta_{\ast} \omega_{a_1}) b + b^{-1} d_x b,$$

where $\omega_{a_i}$ is the canonical 1-form of the cocycle $a_i$.  

19
We can extend Propositions 5.3 and 5.4 to the case that $G$ may not be a linear group. In this case, $b^{-1}(\Theta_\omega b)\theta$ is replaced by the adjoint $\text{Ad}_{b^{-1}}\Theta_\omega b$, and $b^{-1}d\omega b$ is replaced by the pull-back $b^t\theta_G$ of the Maurer-Cartan form $\theta_G \in \Omega^1(G) \otimes \mathfrak{g}$, where $\theta_G(\xi(x)) = \xi$ for any $\xi \in \mathfrak{g}$.

### 5.2 Parameter deformation of $\mathbb{R}^p$-actions

Let $M$ be a closed manifold and $\mathcal{F}$ a foliation on $M$. Recall that the first cohomology group of $\mathbb{R}^p$ as a Lie algebra is $\mathbb{R}^p$. Let $\rho$ be an action in $\mathcal{A}_{\mathcal{F}}(\mathbb{R}^p)$, $\tau_\rho : \mathbb{R}^p \to \Omega^1(\mathcal{F})$ is the natural homomorphism induced by $I_\rho$, and $\omega_\rho$ be the canonical 1-form. Since $\text{Im}(\tau_\rho)_* \simeq \mathbb{R}^p$, Lemma 5.1 implies

$$\text{Im}(\tau_\rho)_* \otimes \mathbb{R}^p = \{ \Theta_\omega \mid \Theta \text{ is an endomorphism of } G \}.$$

Identify the abelian group $\mathbb{R}^p$ and the group of positive diagonal matrices of size $p$ and apply Propositions 5.3 and 5.4 for $\mathbb{R}^p$-actions. Then, we obtain the following correspondence between actions in $\mathcal{A}(\mathcal{F}, \mathbb{R}^p)$ and $\mathbb{R}^p$-valued leafwise 1-forms.

**Proposition 5.9.** A $\mathbb{R}^p$-valued leafwise 1-form is the canonical 1-form of an action in $\mathcal{A}_{\mathcal{F}}(\mathbb{R}^p)$ if and only if it is regular and closed. Two actions $\rho_1, \rho_2 \in \mathcal{A}_{\mathcal{F}}(\mathbb{R}^p)$ are parameter-equivalent if and only if the cohomology class $[\omega_{\rho_2}]$ is contained in $\text{Im}(\tau_{\rho_1})_*$. 

As a corollary, we obtain a generalization of Theorem 5.1.

**Theorem 5.10** (Matsumoto and Mitsumatsu [58], see also [45]). Let $\rho$ be a locally free $\mathbb{R}^p$-action on a closed manifold and $\mathcal{F}$ be its orbit foliation. Then $\rho$ is parameter-rigid if and only if $H^1(\mathcal{F}) \simeq \mathbb{R}^p$.

For example, Diophantine linear actions on $\mathbb{T}^N$ (Theorem 4.7) and the Weyl chamber flow (Theorem 4.10) are parameter rigid. Miezczkowski’s action on $M_T = \Gamma \backslash SL(2, \mathbb{C})$ is also parameter rigid when $H^1(M_T)$ is trivial.

What happens for Miezczkowski’s example when $H^1(M_T)$ is non-trivial? The following theorem asserts the existence of locally complete and locally transverse parameter deformation parametrized by an open subset of $H^1(M_T)$.

**Theorem 5.11.** Let $\mathcal{F}$ be a foliation on a closed manifold $M$ and $\rho$ be an action in $\mathcal{A}_{\mathcal{F}}(\mathbb{R}^p, \mathbb{R}^p)$. Suppose that $\text{Im}(\tau_\rho)_* \otimes \mathbb{R}^p$ is closed and there exists a finite dimensional subspace $H$ of $\text{Ker} \ d\omega_\rho$ such that $\text{Ker} \ d\omega_\rho = \text{Im} \ d\omega_\rho \oplus \text{Im} \tau_\omega \oplus H$. Then, there exists an open neighborhood $\Delta$ of $0$ in $H \otimes \mathbb{R}^p$ and a locally complete and locally transverse parameter deformation $(\rho_\mu)_{\mu \in \Delta} \in \mathcal{A}(\mathcal{F}, \mathbb{R}^p; \Delta)$ of $\rho$.

**Proof.** Let $\omega_\rho$ be the canonical 1-form of $\rho$ and $\Delta$ be the set of 1-forms $\mu \in H \otimes \mathbb{R}^p$ such that $\omega_\rho + \mu$ is a regular 1-form. For each $\mu \in \Delta$, there exists the unique action $\rho_\mu \in \mathcal{A}_{\mathcal{F}}(\mathcal{F}, \mathbb{R}^p)$ whose canonical 1-form is $\omega_\rho + \mu$. The set $\Delta$ is an open neighborhood of $0$ and the family is a parameter deformation of $\rho$.
Let us prove the locally completeness of the deformation. Let \( \pi_H : \text{Ker } d^F \rightarrow H \) be the projection associated with the splitting \( \text{Ker } d^F = \text{Im } d^F \oplus \text{Im } \iota_p \oplus H \). It induces a projection \( \pi_H^\otimes p : \text{Ker } d^F \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R} \). It is continuous and the set \( \mathcal{U} = \{ \rho' \in A_{LF}(F, \mathbb{R}) | \pi_H^\otimes p(\omega - \omega_\mu) \in \Delta \} \)

is an open subset of \( A_{LF}(F, \mathbb{R}) \). For \( \rho' \in \mathcal{U} \) with \( \pi_H^\otimes p(\omega_\mu) = \mu \), the cohomology class \( [\omega - (\omega_\mu + \mu)] \) is contained in \( \text{Im}(\iota_\rho)_* \). By Theorem 5.11, \( \rho' \) is parameter-equivalent to \( \rho_\mu \). Therefore, \( (\rho_\mu)_{\mu \in \Delta} \) is a locally complete deformation.

Next, we show the local transversality. If a family \( (\rho_\mu)_{\mu \in \Delta} \) is sufficiently close to the original family \( (\rho_\mu)_{\mu \in \Delta} \), then \( \{ \pi_H^\otimes p(\omega_\mu) | \mu \in \Delta \} \) is a neighborhood of \( 0 \) in \( H \otimes \mathbb{R} \). Hence, \( [\omega_\mu - \omega_\rho] \in \text{Im}(\iota_\rho)_* \) for some \( \mu^* \in \Delta \). By Theorem 5.11 again, \( \rho_\mu^* \) is parameter-equivalent to \( \rho \). Therefore, \( (\rho_\mu)_{\mu \in \Delta} \) is a locally transverse deformation.

### 5.3 Parameter rigidity of some non-abelian actions

As we saw in the previous subsection, the equations in Propositions 5.3 and 5.4 are linear equations for \( \mathbb{R}^p \)-actions. For general case, the equations are nonlinear and it is unclear whether an action \( \rho \) is parameter rigid or not even if we know \( H^1(F) = \text{Im}(\iota_\rho)_* \) for the orbit foliation \( F \). However, we can reduce the parameter rigidity to the triviality of \( H^1(F) \) for several actions of solvable groups.

The first example is an action of three-dimensional Heisenberg group

\[
H = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \bigg| x_1, x_2, x_3 \in \mathbb{R} \right\}.
\]

We denote the Lie algebra of \( H \) by \( h \).

**Theorem 5.12** (dos Santos [46]). Let \( F \) be a foliation on a closed manifold \( M \). If \( H^1(F) \simeq H^1(g) \), then actions in \( A_{LF}(F, H) \) are parameter rigid.

In [46], dos Santos also proved the theorem for higher-dimensional Heisenberg groups and constructed examples which satisfy the assumption of the theorem. Recently, Maruhashi [35] generalized dos Santos’ results to general simply connected nilpotent Lie groups.

**Proof.** Let

\[
\xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

be a basis of \( h \) and \( \alpha_1, \alpha_2, \alpha_3 \) be its dual basis. Fix \( \rho_0 \in A_{LF}(M, H) \) and put \( \eta_i = \iota_{\rho_0}(\alpha_i) \) for each \( i \). Since

\[
[\xi_1, \xi_2] = \xi_3, \quad [\xi_1, \xi_3] = [\xi_2, \xi_3] = 0,
\]

21
In particular, \( \text{Im}(\iota_{p_0}) \simeq H^1(\mathfrak{h}) \) is generated by \([\eta_1]\) and \([\eta_2]\). For \( \omega = \sum_{i=1}^{3} \omega_i \otimes \xi_i \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h} \), the equation \( d\mathcal{F}\omega + [\omega, \omega] = 0 \) is equivalent to

\[
d\mathcal{F}\eta_1 = d\mathcal{F}\eta_2 = 0, \quad d\mathcal{F}\eta_3 = \eta_2 \wedge \eta_1.
\]

By a direct calculation, we can show that the form \( b^{-1} \omega' b + b^{-1} d\mathcal{F} b = \omega' \) for a suitable choice of \( \omega' \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h} \).

Next, we will make \( \omega' \) into a 1-form in \( \text{Im}(\iota_{p_0}) \). Since \( d\mathcal{F}\omega_1 = d\mathcal{F}\omega_2 = 0 \) by Equation (6) and \( H^1(\mathcal{F}) = \text{Im}(\iota_{p_0}) \), by assumption, there exists \( b_1, b_2 \in C^\infty(\mathcal{M}, \mathbb{R}) \) and \( c_{ij}, i,j = 1,2 \in \mathbb{R}^4 \) such that \( \omega_i = c_{i1} \eta_1 + c_{i2} \eta_2 + d\mathcal{F} b_i \) for each \( i \). Put

\[
b(x) = \begin{pmatrix} 1 & b_1(x) & 0 \\ 0 & 1 & b_2(x) \\ 0 & 0 & 1 \end{pmatrix}.
\]

By a direct calculation, we can show that the form \( \omega' = \sum_{i,j=1,2} c_{ij} \eta_j \otimes \xi_i + \omega'_3 \otimes \xi_3 \) satisfies

\[
b^{-1} \omega' b + b^{-1} d\mathcal{F} b = \omega'
\]

for a suitable choice of \( \omega'_3 \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h} \).

Finally, we take an endomorphism \( \Theta \) of \( H \) such that \( \Theta_*(\xi_j) = c_{1j} \xi_1 + c_{2j} \xi_2 + c'_{j} \xi_3 \) for \( j = 1, 2 \). It satisfies \( \Theta_*(\xi_3) = \Theta_*(\xi_1, \xi_2) = (c_{11} c_{22} - c_{12} c_{21}) \xi_3 \). Hence,

\[
\Theta_\sigma\omega_{p_0} = \Theta_*(\sum_{j=1}^{3} \eta_j \otimes \xi_j) = \omega''.
\]
The equations (7), (8), and (9) imply
\[ \omega_{\rho} = (bb')^{-1}\Theta_0\omega_{\rho_0}(bb') + (bb')^{-1}d_F(bb'). \]

By Proposition 5.4 the action \( \rho \) is equivalent to \( \rho_0 \).

The second example is an action of a two-dimensional solvable group
\[ GA = \{ \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix} \mid u, t \in \mathbb{R} \}. \]

Let \( A \) be an element of \( SL(2, \mathbb{R}) \) such that the eigenvalues \( \lambda, \lambda^{-1} \) are real and \( \lambda > 1 \). Let \( F_A \) be a diffeomorphism of \( \mathbb{T}^2 \) given by \( F_A(z + \mathbb{Z}^2) = Az + \mathbb{Z}^2 \) and let \( M_A \) be the mapping torus
\[ M_A = \mathbb{T}^2 \times \mathbb{R}/(x, s + \log \lambda) \sim (F_A(x), s). \]

We define an action \( \rho_A \in \mathcal{A}_{LF}(M_A, GA) \) by
\[ \rho_A([x, s], \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix}) = [x + (e^t u) \cdot v, s + t], \]
where \( v \) is the eigenvector associated with \( \lambda^{-1} \). Remark that the orbit foliation \( \mathcal{F} \) of \( \rho_A \) is diffeomorphic to the second example in Section 4.2.

**Theorem 5.13** (Matsumoto-Mitsumatsu [38]). The action \( \rho_A \) is parameter rigid.

**Proof.** The Lie algebra \( ga \) of \( GA \) has a basis
\[ \xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

Let \( \alpha_1, \alpha_2 \) be the dual basis of \( ga^* \). We put \( \eta_i = \iota_{\rho_A}(\alpha_i) \). Then, \([\xi_1, \xi_2] = \xi_2\), and hence,
\[ d_F\eta_1 = d_F\eta_2 + \eta_1 \wedge \eta_2 = 0. \]

In particular, we have \( \text{Im}(\iota_{\rho_A})_* = [\eta_1] \).

Take \( \rho \in \mathcal{A}_{LF}(F, GA) \). Let \( \omega_{\rho_A} \) and \( \omega_{\rho} \) be the canonical 1-forms of \( \rho_A \) and \( \rho \). Then, \( \omega_{\rho_A} = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2 \) and \( \omega_{\rho} = \omega_1 \otimes \xi_1 + \omega_2 \otimes \xi_2 \) for some \( \omega_1, \omega_2 \in \Omega^1(\mathcal{F}) \). Since \( \omega_{\rho} \) satisfies the equation \( d_F\omega_{\rho} + [\omega_{\rho}, \omega_{\rho}] = 0 \), the form \( \omega_1 \) is closed. By Theorem 4.8 \( H^1(\mathcal{F}) = \text{Im}(\iota_{\rho_A})_* = \mathbb{R}[\eta_1] \). Hence, there exists \( c_1 \in \mathbb{R} \) and \( b_1 \in C^\infty(M_A, GA) \) such that \( \omega_1 = c_1\eta_1 + d_Fb \). By Proposition 4.6 \( \rho \) preserves a smooth volume naturally. As a (not immediate) consequence of this fact, we can obtain \( c_1 = 1 \) (see [38] p.1863–1864 for detail). Put \( \omega' = \eta_1 \otimes \xi_1 + e^{b_1}\omega_2 \otimes \xi_2 \) and
\[ b = \begin{pmatrix} e^{b_1} & 0 \\ 0 & 1 \end{pmatrix}. \]
Then, by a direct calculation, we have \( b^{-1}\omega' b + b^{-1}d_x b = \omega_1 \otimes \xi_1 + \omega_2 \otimes \xi_2 = \omega_p \). Take \( f, g \in C^\infty(M, \mathbb{R}) \) such that \( e^{b_1}\omega_2 = f\eta_1 + g\eta_2 \). Since \( d_x \omega' + [\omega', \omega'] = 0 \), the pair \((f, g)\) satisfies

\[
X_1 g = X_2 f,
\]

where \( X_i = I_{\rho_A}(\xi_i) \).

Let \( \Theta \) be an endomorphism of \( GA \). Then, \( \Theta_*(\xi_1) = \xi_1 \) and \( \Theta_*(\xi_2) = c_2 \cdot \xi_2 \) for some \( c_2 \in \mathbb{R} \). For \( b' \in C^\infty(M, GA) \) of the form \( b'(x) = \begin{pmatrix} 1 & h(x) - c_2 \\ 0 & 1 \end{pmatrix} \), we have

\[
(b')^{-1}\Theta_*(\omega_{b'}) b' + (b')^{-1}d_x b' = \eta_1 \otimes \xi_1 + [(h + X_1 h)\eta_1 + (X_2 h - c_2)\eta_2] \otimes \xi_2.
\]

Hence, the equivalence of \( \rho \) and \( \rho_A \) is reduced to the solvability of the equation

\[
\begin{cases}
  f = h + X_1 h \\
  g = X_2 h - c_2.
\end{cases}
\]

In fact, the following proposition guarantees the solvability, and it completes the proof.

**Proposition 5.14** (Mitsumatsu-Matsumoto [38]). If smooth functions \( f, g \) satisfies the equation (10), then the equation (11) has a solution \((h, c_2)\).

\[\square\]

The group \( GA \) is naturally isomorphic to the subgroup of \( SL(2, \mathbb{R}) \) which consists of upper triangular matrices by the map

\[
\theta : \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^{\frac{t}{2}} & e^{\frac{t}{2}} u \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.
\]

Let \( \Gamma \) be a cocompact lattice of \( SL(2, \mathbb{R}) \) and put \( M_\Gamma = \Gamma \setminus SL(2, \mathbb{R}) \). We define an action \( \rho_\Gamma \in A\mathcal{L}(\rho_\Gamma, GA) \) by \( \rho_\Gamma(\Gamma x, g) = \Gamma(x \cdot \theta(g)) \). It is just the second example in Section 4.3. In [38], Matsumoto and Mitsumatsu showed an analogue of Proposition 5.14 for \( \rho_\Gamma \).

**Proposition 5.15.** Let \( \xi_1, \xi_2 \) be the basis of \( ga \) given in the proof of Theorem 5.13. Put \( X = I_{\rho_\Gamma}(\xi_1) \) and \( S = I_{\rho_\Gamma}(\xi_2) \). Then, if smooth functions \( f, g \in C^\infty(M_\Gamma, \mathbb{R}) \) satisfies \( Sg = Xf \), then the equation

\[
\begin{cases}
  f = h + Xh \\
  g = Sh + c.
\end{cases}
\]

has a solution \((h, c) \in C^\infty(M_\Gamma, \mathbb{R}) \times \mathbb{R} \).

When \( H^1(M_\Gamma) \) is trivial, we have \( H^1(F) \simeq \mathbb{R} \) by Theorem 4.12. In this case, we can prove the parameter rigidity of \( \rho_\Gamma \) by an argument similar to the above.

**Theorem 5.16** (c.f., [38]). When \( H^1(M_\Gamma) \) is trivial, then \( \rho_\Gamma \) is parameter rigid.
5.4 A complete deformation for actions of $GA$

Let $\Gamma$ be a cocompact lattice of $SL(2, \mathbb{R})$ and put $M_\Gamma = \Gamma \backslash SL(2, \mathbb{R})$. Let $\rho_\Gamma \in A_{LF}(M_\Gamma, GA)$ be the action given by $\rho_\Gamma(\Gamma x, g) = \Gamma(x \cdot g)$, which is discussed about in the last paragraph of the previous subsection. It is natural to ask whether $\rho_\Gamma$ is parameter rigid or not when $H^1(M_\Gamma)$ is non-trivial.

Let $F$ be the orbit foliation of $\rho_\Gamma$. First, we determine the space of infinitesimal parameter deformations in terms of the leafwise cohomology. Recall that the space $A_{LF}(F, GA)$ is identified with the solution of the non-linear equation

$$d_F \omega + [\omega, \omega] = 0.$$  \hspace{1cm} (14)

in $\Omega^1(F) \otimes \mathfrak{g}a$. Two actions are parameter equivalent with trivial automorphism if and only if the equation

$$\omega_2 = b^{-1} \omega_1 b + b^{-1} d_F b,$$ \hspace{1cm} (15)

admits a smooth solution $b : M_\Gamma \rightarrow GA$, where $\omega_1$ and $\omega_2$ are the canonical 1-forms of actions. Let $\omega_0$ be the canonical 1-form of $\rho_\Gamma$. Put $\omega_t = \omega_0 + t \omega$ and $b_t = \exp(t \beta)$ with $\omega \in \Omega^1(F) \otimes \mathfrak{g}a$ and $\beta \in \Omega^0(F) \otimes \mathfrak{g}a$. Substitute $\omega_t$ and $b_t$ into the above equations and take the first order term with respect to $t$. Then, we obtain the linearized equations

$$d^0_\rho \omega = 0,$$ \hspace{1cm} (14L)

$$\omega_2 - \omega_2 = d^0_\rho \beta,$$ \hspace{1cm} (15L)

where the linear map $d^k_\rho : \Omega^k(F) \otimes \mathfrak{g}a \rightarrow \Omega^{k+1}(F) \otimes \mathfrak{g}a$ for $k = 1, 2$ is given by

$$d^0_\rho \beta = [\omega_0, \beta] + d_F \beta,$$

$$d^1_\rho \omega = d_F \omega + [\omega, \omega_0] + [\omega_0, \omega].$$

We call the quotient space $\text{Ker} d^1_\rho / \text{Im} d^0_\rho$ the space of infinitesimal parameter deformations of $\rho_\Gamma$ and we denote it by $H^1(\rho_\Gamma, F)$.

**Proposition 5.17.** $H^1(\rho_\Gamma, F) \simeq H^1(M)$.

**Proof.** Fix a basis

$$\xi_X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \xi_S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \xi_U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $SL(2, \mathbb{R})$. The standard right $SL(2, \mathbb{R})$-action on $M_\Gamma$ induces vector fields $X, S, \text{and } U$ which correspond to $\xi_X, \xi_S$ and $\xi_U$. Let $\eta, \sigma, \nu$ be the dual 1-forms of $X, S$ and $U$, respectively. Then, $\Omega^1(F)$ is generated by $\eta$ and $\sigma$ as a $C^\infty(M_\Gamma, \mathbb{R})$-module. Notice that $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is $d_{\rho_\Gamma}$-closed, then $d_F \omega_X = 0$ and $d_F \omega_S = -(\omega_X(X) + \omega_S(S)) \eta \wedge \sigma$.

First, we claim that $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is $d_{\rho_\Gamma}$-exact if and only if $\omega_X$ is $d_F$-exact. For $\varphi \in C^\infty(M_\Gamma, \mathbb{R})$, we have $d^0_{\rho_\Gamma}(\varphi \otimes \xi_X) = (d_F \varphi) \otimes \xi_X - \varphi \otimes \xi_S.$
Hence, if $\omega$ is $d_{pr}$-exact then $\omega_X$ is $d_{F}$-exact. Suppose that $\omega_X$ is $d_{F}$-exact. Take $\varphi \in C^\infty(M_T, \mathbb{R})$ such that $d_{F}\varphi = \omega_X$. By replacing $\omega$ with $\omega + d_{pr}^0(\varphi \otimes \xi_X)$, we may assume that $\omega_X = 0$. Put $\omega_S = f\eta + g\sigma$. Since $\omega$ is $d_{pr}$-closed, we have $Sf = Xg$. Proposition 5.18 implies that there exists $h \in C^\infty(M_T, \mathbb{R})$ and $c \in \mathbb{R}$ such that $f = h + X_T h$ and $g = S_T h - c$. Hence, $\omega = d_{pr}^0(-c \otimes \xi_X + h \otimes \xi_S)$. It completes the proof of the claim.

By the claim, $H^1(\rho, \mathcal{F})$ is isomorphic to

$$\{[\omega_X] \in H^1(\mathcal{F}) \mid d_{pr}(\omega_X \otimes \xi_X + \omega_S \otimes \xi_S) = 0\}.$$ 

So, it is sufficient to show that for any $d_{F}$-closed $1$-form $\omega_X \in \Omega^1(\mathcal{F})$, there exists $\omega_S \in \Omega^1(\mathcal{F})$ such that $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is $d_{pr}$-closed. Fix a Riemannian metric on $M_T$ such that $(X_T, (S_T + U_T)/2, (S_T - U_T)/2)$ is an orthonormal framing of $TM_T$. By Theorem 4.12, there exists $f_0 \in C^\infty(M_T, \mathbb{R})$ such that $\omega_X + d_{F} f_0$ extends to a harmonic $1$-form with respect to the metric. Replacing $\omega$ with $\omega + d_{pr}(f_0 \otimes \xi_X)$, we may assume that $\omega_X$ is the restriction of a harmonic form $\omega_h$ to $TM_T$. Put $\omega_h = f\eta + g\sigma + h\nu$. Since $\omega_h$ is harmonic and $M_T$ is compact (it implies $\mathcal{L}(S_T - U_T)\omega_X = 0$), we can show the equations $2f = (S - U)g$ and $2Y f = -(S + U)g$. Then, it is easy to check that $d_{pr}(\omega_X \otimes \xi_X + (-g\eta + f\sigma) \otimes \xi_S) = 0.

One may expect the existence of a complete deformation whose parameter space of an open subset of $H^1(M) \simeq H^1(\rho_T, \mathcal{F})$. It is done by the author of this note.

**Theorem 5.18** (Asaoka, in preparation). There exists an open subset $\Delta_T$ of $H^1(M_T)$ containing $0$ and a parameter deformation $(\rho_{\mu})_{\mu \in \Delta} \in \mathcal{A}(M_T, \mathcal{F}; \Delta_T)$ of $\rho_T$ such that

1. if $\rho_{\mu}$ is equivalent to $\rho_{\nu}$ then $\mu = \nu$, and
2. any $\rho \in \mathcal{A}_{L_F}(\mathcal{F}, \mathcal{G}_A)$ is equivalent to $\rho_{\mu}$ for some $\mu \in \Delta_T$.

**Corollary 5.19** (Asaoka \cite{3}). When $H^1(M_T)$ is non-trivial, then $\rho_T$ is not parameter rigid.

Construction of the deformation $(\rho_{\mu})_{\mu \in \Delta_T}$ is essentially done in \cite{3}. Remark that the proof does not use the computation of $H^1(\rho_T, \mathcal{F})$. It heavily depends on the ergodic theory of hyperbolic dynamics, especially on the existence of the Margulis measure, and the deformation theory of low dimensional Anosov systems. To obtain the smoothness of the family, we also use the smooth dependence of the Margulis measure, in some sense, with respect to the parameter.

It is natural to expect the corresponding result holds for $SL(2, \mathbb{C})$. However, the corresponding action for $SL(2, \mathbb{C})$ is locally parameter rigid.

**Theorem 5.20** (Asaoka \cite{14}). Let $\Gamma$ be a cocompact lattice of $SL(2, \mathbb{C})$ and $\mathcal{G}_A$ be the subgroup of $SL(2, \mathbb{C})$ which consists of upper triangular matrices. Then, the standard $\mathcal{G}_A$ action on $\Gamma \setminus SL(2, \mathbb{C})$ is locally parameter rigid.
6 Deformation of orbits

In this section, we discuss about deformations which may not preserve the orbit foliation. The equations we need to solve are non-linear even for $\mathbb{R}^p$-actions, as we investigated the deformation of linear flows on tori in Section 3. The main techniques to describe such deformations are the linearization and the Nash-Moser type theorems. The former reduces the problem to the computation of the bundle-valued leafwise cohomology. The latter allows us to construct solutions of the original non-linear problem from the linear one.

6.1 Infinitesimal deformation of foliations

To know deformations of a given locally free actions, it is natural to investigate deformations of the orbit foliation. In this subsection, we describe the space of infinitesimal deformations of a foliation in terms of the leafwise cohomology.

Let $\mathcal{F}$ be a foliation on a manifold $M$. To simplify, we assume that $\mathcal{F}$ admits a complementary foliation $\mathcal{F}^\perp$, i.e., it is transverse to $\mathcal{F}$ and satisfies $\dim \mathcal{F} + \dim \mathcal{F}^\perp = \dim M$. The normal bundle $TM/\mathcal{T}\mathcal{F}$ of $\mathcal{T}\mathcal{F}$ can be naturally identified with the tangent bundle $\mathcal{T}\mathcal{F}^\perp$ of $\mathcal{F}^\perp$. By $\pi^\perp$, we denote the projection from $TM = \mathcal{T}\mathcal{F} \oplus \mathcal{T}\mathcal{F}^\perp$ to $\mathcal{T}\mathcal{F}^\perp$. Let $\Omega^k(\mathcal{F};\mathcal{T}\mathcal{F}^\perp)$ be the space of $\mathcal{T}\mathcal{F}^\perp$-valued leafwise $k$-forms. We define the differential $d^k_{\mathcal{F}} : \Omega^k(\mathcal{F};\mathcal{T}\mathcal{F}^\perp) \to \Omega^{k+1}(\mathcal{F};\mathcal{T}\mathcal{F}^\perp)$ by

$$(d^k_{\mathcal{F}} \omega)(X_0, \ldots, X_k) = \sum_{0 \leq i \leq k} (-1)^i \pi^\perp (X_i, \omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

It satisfies $d^k_{\mathcal{F}} \circ d^k_{\mathcal{F}} = 0$. We denote the quotient $\text{Ker} d^k_{\mathcal{F}} / \text{Im} d^{k-1}_{\mathcal{F}}$ by $H^k(\mathcal{F};\mathcal{T}\mathcal{F}^\perp)$.

Suppose that the foliation $\mathcal{F}$ is $p$-dimensional. For $\omega \in \Omega^1(\mathcal{F};\mathcal{T}\mathcal{F}^\perp)$, we define a $p$-plane field $E_\omega$ on $M$ by

$$E_\omega(x) = \{ v + \omega(v) \mid v \in \mathcal{T}_x \mathcal{F} \}.$$

It gives a one-to-one correspondence between $\mathcal{T}\mathcal{F}^\perp$-valued leafwise 1-forms and $p$-plane field transverse to $\mathcal{T}\mathcal{F}^\perp$. By a direct computation on a local coordinate adapted to the pair $(\mathcal{F}, \mathcal{F}^\perp)$, we obtain the following criterion for the integrability of $E_\omega$.

**Lemma 6.1.** The $p$-plane field $E_\omega$ generates a foliation if and only if $\omega$ satisfies the equation

$$d_F \omega + [\omega, \omega] = 0.$$
By a direct computation on a local coordinate adapted to the pair \((F, F^\perp)\) again, we have
\[
\lim_{t \to 0} \frac{1}{t} \omega_t = d_{F^\perp}^0 \beta.
\]
So, one can regard the cohomology group \(H^1(F; TF^\perp)\) as the space of infinitesimal deformation of the foliation \(F\). We say that a foliation \(F\) infinitesimally rigid if \(H^1(F; TF^\perp) = \{0\}\).

**Example 6.2.** Let \(F\) be the orbit foliation of a Diophantine linear action in \(A_{LF}(\mathbb{T}^N, \mathbb{R}^p)\). Since \(TF^\perp\) is a trivial bundle, Theorem 4.7 implies
\[
H^1(F; F^\perp) \cong H^1(F) \otimes \mathbb{R}^{N-p} \cong \mathbb{R}^{N-p}.
\]
In particular, \(F\) is not infinitesimally rigid.

**Exercise 6.3.** Let \(F\) be the suspension foliation associated to a hyperbolic automorphism on \(T^2\), which is defined in Section 4.2. Show that \(F\) is infinitesimally rigid using a Mayer-Vietoris argument as in Section 4.2.

**Example 6.4** (Kononenko [32], Kanai [27]). Let \(A_p\) be the orbit foliation of the Weyl chamber flow, which is defined in Section 4.3. If \(p \geq 2\), then \(A_p\) is infinitesimally rigid.

### 6.2 Hamilton’s criterion for local rigidity

Let \(F\) be a foliation on a closed manifold \(M\) and \(F^\perp\) be its complementary foliation. We say that \(F\) is locally rigid if any foliation \(F'\) sufficiently close to \(F\) is diffeomorphic to \(F\).

Using Hamilton’s implicit function theorem for non-linear exact sequence [24, Section 2.6], one obtain the following criterion for local rigidity of a foliation.

**Theorem 6.5** (Hamilton [25]). Suppose that there exist continuous linear operators \(\delta^k : \Omega^{k+1}(F; TF^\perp) \to \Omega^k(F; TF^\perp)\) for \(k = 1, 2\), an integer \(r \geq 1\), and a sequence \(\{C_s\}_{s \geq 1}\) of positive real numbers such that

1. \(d_{F^\perp}^0 \circ \delta^0 + \delta^1 \circ d_{F^\perp}^1 = \text{Id}\),

2. \(\|\delta^0 \omega\|_s \leq C_s \|\omega\|_{s+r}\) and \(\|\delta^1 \sigma\|_s \leq C_s \|\sigma\|_{s+r}\) for any \(s \geq 1\), \(\omega \in \Omega^1(F; TF^\perp)\), and \(\sigma \in \Omega^2(F; TF^\perp)\), where \(\|\cdot\|_s\) is the \(C^s\)-norm on \(\Omega^k(F; TF^\perp)\).

Then, \(F\) is locally rigid.

Moreover, we can choose the diffeomorphism \(h\) in the definition of locally rigidity so that it is close to the identity map.

**Theorem 6.6** (El Kacimi Alaoui and Nicolau [16]). Let \(F_A\) be the suspension foliation related to a hyperbolic toral automorphism, which is given in Section 4.2. Then, \(F_A\) satisfies Hamilton’s criterion above. In particular, it is locally rigid.

With the parameter rigidity of the action \(\rho_A\) (Theorem 5.13), we obtain
Corollary 6.7 (Matsumoto and Mitsumatsu [38]). The action $\rho_A$ is locally rigid.

In [16] and [38], they also proved the corresponding results for higher dimensional hyperbolic toral automorphisms.

It is unknown whether the orbit foliation of the Weyl chamber flow satisfies Hamilton’s criterion or not. However, Katok and Spatzier proved the rigidity of the orbit foliation by another method.

Theorem 6.8 (Katok and Spatzier [30]). The orbit foliation $A_p$ of the Weyl chamber flow is locally rigid if $p \geq 2$.

With the parameter rigidity of the Weyl chamber flow (Theorem 4.10) we obtain

Corollary 6.9. The Weyl chamber flow is locally rigid if $p \geq 2$.

6.3 Existence of locally transverse deformations

Although deformation theory is well-developed for transversely holomorphic foliations, ([11, 12, 13, 15, 18, 19, 20]), there is no general deformation theory for smooth foliations with non-trivial infinitesimal deformation so far since we can not apply Hamilton’s criterion in this case. However, there are several actions for which we can find a locally transverse deformation. One example is a Diophantine linear flow, which we discussed in Section 3. In this subsection, we give two more examples.

The first example is a codimension-one Diophantine linear action. By $\text{Diff}_0(S^1)$, we denote the set of orientation-preserving diffeomorphisms of $S^1$. Let $F$ be a codimension-one foliation on $T^{p+1}$ which is transverse to $T^p \times \{s\}$ for any $s$. For each $i = 1, \ldots, p$, we can define a holonomy map $f_i \in \text{Diff}_0(S^1)$ of $F$ along the $i$-th coordinate. The family $(f_1, \cdots, f_p)$ is pairwise commuting. On the other hand, when a pairwise commuting family $(f_1, \cdots, f_p)$ in $\text{Diff}_0(S^1)$ is given, then the suspension construction gives a codimension-one foliation on $F$, which is transverse to $T^p \times \{s\}$ for any $s \in S^1$. Two foliations are diffeomorphic to each other if the corresponding families $(f_1, \cdots, f_p)$ and $(g_1, \cdots, g_p)$ are conjugate, i.e., there exists $h \in \text{Diff}_0(S^1)$ such that $g_i \circ h = h \circ f_i$ for any $i = 1, \cdots, p$. So, the local rigidity problem of $F$ is reduced to the problem on a pairwise commuting family $(f_1, \cdots, f_p)$.

For $f \in \text{Diff}_0(S^1)$, the rotation number $\tau(f) \in S^1$ is defined by

$$\left(\lim_{n \to \infty} \frac{\tilde{f}^n(0)}{n}\right) + \mathbb{Z},$$

where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a lift of $f$. It is known that the map $\tau : \text{Diff}_0(S^1) \to S^1$ is continuous (see e.g., [28, Proposition 11.1.6]). For $\theta \in S^1$, let $r_\theta$ be the rotation defined by $r_\theta(x) = x + \theta$. 29
\textbf{Theorem 6.10} (Moser \cite{40}). Let \((f_1, \cdots, f_p)\) be a pairwise commuting family in \(\text{Diff}_0(S^1)\). Suppose that there exists \(\tau > 0\) such that
\[
\inf_{m \in \mathbb{Z}\backslash\{0\}} \left( \max_{i=1, \cdots, p} \|m \cdot \tau(f_i)\| \right) |m|^{\tau} > 0,
\]
where \(\|m \cdot \tau(f_i)\| = |\exp(2\pi \tau(f_i)\sqrt{-1}) - 1|\). Then, there exists \(h \in \text{Diff}_0(S^1)\) such that \(f_i \circ h = h \circ \tau(f_i)\) for any \(i = 1, \cdots, p\).

As a consequence of the theorem, we can show the existence of locally transverse deformation of codimension-one Diophantine linear action.

\textbf{Theorem 6.11.} Let \(\rho\) be the linear action of \(\mathbb{R}^p\) on \(T^{p+1}\) determined by linearly independent vectors \(v_1, \cdots, v_p \in \mathbb{R}^{p+1}\). Take \(w \in \mathbb{R}^{p+1}\) so that \(v_1, \cdots, v_p, w\) is a basis of \(\mathbb{R}^{p+1}\) and we define a \(C^\infty\) family of actions \((\rho_s)_{s \in \mathbb{R}^p} \in \mathcal{A}_{LF}(T^{p+1}, \mathbb{R}^p; \mathbb{R}^p)\) by
\[
\rho_s^t(x) = x + \sum_{i=1}^p t_i(v_i + s_iw),
\]
for \(x \in M, t = (t_1, \cdots, t_p), s = (s_1, \cdots, s_p) \in \mathbb{R}^p\). If the linear action \(\rho_0\) is Diophantine, then \((\rho_s)_{s \in \mathbb{R}^p}\) is locally transverse at \(s = 0\).

\textbf{Exercise 6.12.} Prove the theorem. One way to do it may be a modification of the proof of Theorem 3.9. One can prove the local transversality of the orbit foliation by the continuity of the rotation number and Moser’s theorem, in stead of Herman’s theorem. The local transversality of action will follow from the parameter rigidity of a Diophantine linear actions.

The second example is a \(\mathbb{R}^2\)-action on \(\Gamma \backslash SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) by commuting parabolic elements. Put
\[
u^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \nu^t_\mu = \exp\left(t \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}\right).
\]
Remark that \(\nu_0^t = \nu^t\).

Let \(\Gamma\) be an irreducible cocompact lattice of \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) and put \(M_\Gamma = \Gamma \backslash (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))\). For \((\mu, \lambda) \in \mathbb{R}^2\), we define an action \(\rho_{\mu, \lambda} \in \mathcal{A}_{LF}(M_\Gamma, \mathbb{R}^2)\) by
\[
\rho_{\mu, \lambda}(\Gamma(x, y), (s, t)) = \Gamma(xu^s_\mu, yu^t_\lambda).
\]
Let \(\mathcal{F}\) be the orbit foliation of \(\rho_{0,0}\).

\textbf{Theorem 6.13} (Mieczkowski \cite{39}). \(H^1(\mathcal{F}) \simeq \mathbb{R}^2\). In particular, the action \(\rho_{0,0}\) is parameter rigid.

One may wish to prove the local transversality of the deformation \((\rho_{\mu, \lambda})(\mu, \nu) \in \mathbb{R}^2\) of \(\rho_{0,0}\) like Diophantine linear actions. Unfortunately, we can not apply the techniques for Diophantine linear actions because of the non-linearity of the space \(SL(2, \mathbb{R})\). Damjanović and Katok developed a new Nash-Moser-type scheme and they obtained the local transversality.
Theorem 6.14 (Danjanović and Katok [10]). The deformation \((\rho_{\mu, \lambda}, \mu, \lambda) \in \mathbb{R}^2\) of \(\rho_{0,0}\) is locally transverse.

In [39] and [10], they also show the parameter rigidity and the existence of a transverse deformation for another actions, which generalize the above results.

### 6.4 Transverse geometric structures

In this subsection, we sketch another method to describe deformations of the orbit foliation which is not locally rigid.

Fix a torsion-free cocompact lattice \(\Gamma\) of \(PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}\). It acts on the hyperbolic plane \(\mathbb{H}^2\) naturally and \(\Sigma = \Gamma \backslash \mathbb{H}^2\) is a closed surface of genus \(g \geq 2\). Let \(T(\Sigma)\) be the Teichmüller space of \(\Sigma\) (see e.g. [26, Chapter 4] for the definition and basic properties). It can be realized as a set of homomorphisms \(\mu\) from \(\Gamma\) to \(PSL(2, \mathbb{R})\) whose image \(\Gamma_{\mu}\) is a cocompact lattice. It is known that \(T(\Sigma)\) has a natural structure of \((6g-6)\)-dimensional smooth manifold.

Let \(P\) be the subgroup of \(PSL(2, \mathbb{R})\) which consists of upper triangular matrices. For each \(\mu\) in \(T(\Sigma)\), we define an action \(\rho_{\mu} \in A_{LF}(\Gamma_{\mu} \backslash PSL(2, \mathbb{R}), P)\) by \(\rho_{\mu}(\gamma_{\mu} \cdot p) = \gamma_{\mu}(x \cdot p)\). The action is the essentially same one as in Sections 5.3 and 5.4. Let \(F_{\mu}\) be the orbit foliation of \(\rho_{\mu}\). To simplify notation, we put \(\rho_{\Gamma} = \rho_{\text{Id}_{\Gamma}}\) and \(F_{\Gamma} = F_{\text{Id}_{\Gamma}}\).

It is well-known that the foliation \(F_{\Gamma}\) is not locally rigid. In fact, \(M_{\mu_1}\) is diffeomorphic to \(M_{\mu_2}\) for any \(\mu_1, \mu_2 \in T(\Sigma)\). However, \(F_{\mu_1}\) is diffeomorphic to \(F_{\mu_2}\) if and only if \(\Gamma_{\mu_1}\) is conjugate to \(\Gamma_{\mu_2}\) as a subgroup of \(PSL(2, \mathbb{R})\). Hence, the family \(\{F_{\mu}\}_{\mu \in T(\Sigma)}\) gives a non-trivial deformation of \(F_{\Gamma}\). Ghys proved that this is the only possible one.

**Theorem 6.15** (Ghys [22]). Any two-dimensional foliation on \(M_\Gamma\) sufficiently close to \(F_{\Gamma}\) is diffeomorphic to \(F_{\mu}\) for some \(\mu \in T(\Sigma)\).

He also proved the global rigidity.

**Theorem 6.16** (Ghys [23]). If a two-dimensional foliation \(F\) on \(M_\Gamma\) has no closed leaves, then \(F\) is diffeomorphic to \(F_{\mu}\) for some \(\mu \in T(\Sigma)\).

The orbit foliation of a locally free \(P\)-action has no closed leaf. Hence, we obtain

**Corollary 6.17.** For any \(\rho \in A_{LF}(M_\Gamma, P)\), there exists \(\mu \in T(\Sigma)\) such that the orbit foliation of \(\rho\) is diffeomorphic to \(F_{\mu}\).

The basic idea of the proof is to find a transverse projective structure of the foliation. Once it is shown, it is not so hard to show that \(F\) is diffeomorphic to \(F_{\mu}\) for some \(\mu\). Ghys constructed the transverse projective structure by using the theory of hyperbolic dynamical systems. Kononenko and Yue [33] gave an alternative proof of Theorem 6.15. They proved the \(C^3\) conjugacy of foliations. However, we can recover the \(C^\infty\) conjugacy from their result with the regularity theorem of conjugacies between Anosov flows by de la Llave and Moriyón [34].
cohomology of the lattice $\Gamma$, which is closely related to the leafwise cohomology of $\mathcal{F}_\Gamma$ valued in the symmetric two-forms on the normal bundle of $TF$. So, it may possible to reduce Theorem \[6.15\] to the vanishing of the bundle-valued leafwise cohomology.

Modifying the construction of a complete parameter deformation of $\rho_\Gamma$ (Theorem \[5.20\]), we obtain a complete deformation of $\rho_\Gamma$.

**Theorem 6.18** (Asaoka, in preparation). *There exists an open subset $\Delta$ of $\mathcal{T}(\Sigma) \times H^1(M_\Gamma)$ and a $C^\infty$ family $(\rho_\mu)_{\mu \in \Delta} \in A_{LF}(M_\Gamma, P)$ such that any $\rho \in A_{LF}(M_\Gamma, P)$ is conjugate to $\rho_\mu$ for some $\mu \in \Delta$.***

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