A Construction of Killing Spinors on $S^n$

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ABSTRACT

We derive simple general expressions for the explicit Killing spinors on the $n$-sphere, for arbitrary $n$. Using these results we also construct the Killing spinors on various AdS×Sphere supergravity backgrounds, including $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$. In addition, we extend previous results to obtain the Killing spinors on the hyperbolic spaces $H^n$.

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1 Introduction

Finding the explicit form of Killing spinors on curved spaces can be an involved task. Often, one merely uses integrability conditions to establish their existence and to determine their multiplicities. In this way it is easy to show that spheres and anti-de Sitter spacetimes preserve all supersymmetries, i.e. they admit the maximum number of Killing spinors. However, one does not obtain explicit solutions by this method. Although establishing their existence is often sufficient, there are situations where it is necessary to know their explicit forms.

There exists a very simple explicit construction of the Killing spinors on $n$-dimensional anti-de Sitter spacetime $\text{AdS}_n$, for arbitrary $n$ \cite{1}. This exploits the fact that $\text{AdS}_n$ can be written in horospherical coordinates, in terms of which the metric takes the simple form

$$ ds^2 = dr^2 + e^{2r} \eta_{\alpha\beta} dx^\alpha dx^\beta, $$

where $\eta_{\alpha\beta}$ is the Minkowski metric in $(n-1)$ dimensions, and the Ricci tensor satisfies $R_{\mu\nu} = -(n-1) g_{\mu\nu}$. It was shown in \cite{1} that the Killing spinors, satisfying $D_\mu \epsilon = \frac{1}{2} \Gamma_\mu \epsilon$, are then expressible as

$$ \epsilon = e^{\frac{1}{2} r} \epsilon_0^+, \quad \text{or} \quad \epsilon = \left( e^{-\frac{1}{2} r} + e^{\frac{1}{2} r} x^\alpha \Gamma_\alpha \right) \epsilon_0^-, $$

where $\epsilon^\pm$ are arbitrary constant spinors satisfying $\Gamma_r \epsilon_0^\pm = \pm \epsilon_0^\pm$. One can alternatively write the two kinds of Killing spinor together in one equation, as

$$ \epsilon = e^{\frac{1}{2} r} \Gamma_r \left( 1 + \frac{1}{2} x^\alpha \Gamma_\alpha (1 - \Gamma_r) \right) \epsilon_0, $$

where $\epsilon_0$ is an arbitrary constant spinor. It is therefore manifest that the number of independent Killing spinors is equal to the number of components in the spinors. (The Killing spinors for $\text{AdS}_4$, written in the standard AdS coordinate system, were obtained in \cite{2}.) It is worth remarking that the horospherical metric (1) can equally well have other spacetime signatures $(p, n-p)$, by taking other signatures $(p, n-p-1)$ for the metric $\eta_{\alpha\beta}$. The isometry group is $SO(p + 1, n-p)$. The case $p = 1$ gives $\text{AdS}_n$, with $SO(2, n-1)$, while $p = 0$ gives the positive-definite hyperbolic metric on $H^n$, with $SO(1, n)$. (Expressions for the Killing spinors on $H^2$ and $H^3$, which are special cases of (3), were given in \cite{3}.) Thus equation (3) gives the Killing spinors on all of the $\text{AdS}_n$ spacetimes, hyperbolic spaces $H^n$, and the other maximally-symmetric spacetimes with $(p, n-p)$ signature.

There is an alternative Killing spinor equation that one can consider when $n$ is even, namely $D_\mu \epsilon = \frac{1}{2} \gamma \Gamma_\mu \epsilon$, where $\gamma$ is the chirality operator, expressed as an appropriate
product over the $\Gamma_{\mu}$, with $\gamma^2 = 1$. We easily see that the solutions of this equation can be written as

$$\epsilon = e^{\frac{1}{2} r \gamma^{\alpha} \Gamma_{\alpha}} \left(1 + \frac{1}{2} \gamma^{x^{\alpha} \Gamma_{\alpha}(1 - i \gamma \Gamma_{r})}\right) \epsilon_0 .$$  

Note that in all the cases above, we considered a “unit radius” AdS$_n$, or $H^n$, etc., given by (1). It is trivial to extend the results to an arbitrary scale size, by replacing (1) by $ds^2 = \lambda^{-2} (dr^2 + e^{2r} \eta_{\alpha\beta} dx^\alpha dx^\beta)$, which has the Ricci tensor $R_{\mu\nu} = -(n - 1) \lambda^2 g_{\mu\nu}$. The Killing spinor equations then become $D_\mu \epsilon = \frac{1}{2} \lambda \Gamma_\mu \epsilon$, etc. It is easily seen that the solutions are given by precisely the same expressions (3), etc., with no modifications whatsoever. (In [1] a different coordinatisation of the general-radius AdS$_n$ was used, in which the expressions for the Killing spinors do depend upon the scale-setting parameter.)

In this paper, we find an explicit construction of the Killing spinors on $S^n$. (Explicit results for $n = 2$ and $n = 3$ were obtained in [3].) One might think that since AdS$_n$ can be related to $S^n$ by appropriate complexifications of coordinates, it should be possible to obtain expressions for the Killing spinors on $S^n$ that are analogous to those given above. However, things are not quite so simple, because the ability to write the metric on AdS$_n$ in the simple form (1) depends rather crucially on the fact that its isometry group $SO(2, n - 1)$ is non-compact. (One can easily see that (1) has $(n - 1)$ commuting Killing vectors $\partial_\mu$, which exceeds the rank $[(n + 1)/2]$ of the isometry group when $n > 3$. This is not possible for compact groups.) We shall thus present a different construction for the Killing spinors of $S^n$, which, although more complicated, is still explicit, and of an essentially simple structure. Our main result is contained in equation (3) in section 2, which also contains a detailed proof. In section 3 we combine the results for AdS and spheres, to give the explicit expressions for Killing spinors in various AdS$_m \times S^n$ supergravity backgrounds, with $(m, n) = (4,7), (7,4), (5,5), (3,3), (3,2), (2,3), (2,2)$. In appendix A, we collect some useful expressions for the representation and decomposition of Dirac matrices.

## 2 Killing spinors on $S^n$

### 2.1 Results

We begin by writing the metric on a unit $S^n$ in terms of that for a unit $S^{n-1}$ as

$$ds_n^2 = d\theta_n^2 + \sin^2 \theta_n ds_{n-1}^2 ,$$  

(5)
with $ds^2_1 = d\theta_1^2$. This has Ricci tensor given by $R_{ij} = (n - 1) g_{ij}$. We then consider the Killing spinor equation on the unit $n$-sphere, for arbitrary $n$, namely

$$D_j \epsilon = \frac{i}{2} \Gamma_j \epsilon . \quad (6)$$

We shall first present our results for the solutions to this equation, and then present the proof later. We find that the Killing spinors can be written as

$$\epsilon = e^{\frac{i}{2} \theta_n \Gamma_n} \left( \prod_{j=1}^{n-1} e^{-\frac{i}{2} \theta_j \Gamma_{j,j+1}} \right) \epsilon_0 , \quad (7)$$

where $\epsilon_0$ is an arbitrary constant spinor, and the indices on the Dirac matrices are vielbein indices. We use the convention that the $\Gamma$ matrices are Hermitian, satisfying the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2 \delta_{ij}$. Note that here, and in all other analogous formulae in the paper, the factors in the product in (7) are ordered anti-lexigraphically, i.e. starting with the $\theta_{n-1}$ term at the left. Note also that the exponential factors in (7) can be written as

$$e^{\frac{i}{2} \theta_n \Gamma_n} = 1 \cos \frac{1}{2} \theta_n + i \Gamma_n \sin \frac{1}{2} \theta_n , \quad e^{-\frac{i}{2} \theta_j \Gamma_{j,j+1}} = 1 \cos \frac{1}{2} \theta_j - \Gamma_{j,j+1} \sin \frac{1}{2} \theta_j . \quad (8)$$

One can also consider the Killing spinor equation with the opposite sign for the $\Gamma_j$ term, namely

$$D_j \epsilon_- = -\frac{i}{2} \Gamma_j \epsilon_- . \quad (9)$$

The previous solution (7) is easily modified to give solutions of this equation. One finds

$$\epsilon_- = e^{-\frac{i}{2} \theta_n \Gamma_n} \left( \prod_{j=1}^{n-1} e^{\frac{i}{2} \theta_j \Gamma_{j,j+1}} \right) \epsilon_0 . \quad (10)$$

This is immediately verified by noting that (9) is obtained from (7) by changing the sign of the gamma matrices.

The Killing spinors discussed above exist on $S^n$ for any $n$. When $n$ is even, there is an alternative equation that can also be considered, namely

$$D_j \epsilon = \frac{1}{2} \gamma \Gamma_j \epsilon , \quad (11)$$

where $\gamma$ is the chirality operator formed from the product of the $\Gamma$ matrices, satisfying $\gamma^2 = 1$. In this case, we find that the corresponding Killing spinors can be written as

$$\epsilon = e^{\frac{i}{2} \theta_n \gamma \Gamma_n} \left( \prod_{j=1}^{n-1} e^{-\frac{i}{2} \theta_j \Gamma_{j,j+1}} \right) \epsilon_0 , \quad (12)$$

We may again also consider the Killing spinors satisfying (11) with the sign of the $\Gamma_j$ term reversed, namely

$$D_j \epsilon = -\frac{i}{2} \gamma \Gamma_j \epsilon . \quad (13)$$
The solutions are again obtained by sending $\theta_n \to -\theta_n$, giving

$$\epsilon_- = e^{-\frac{1}{2}\theta_n \gamma \Gamma_n} \left( \prod_{j=1}^{n-1} e^{-\frac{1}{2}\theta_j \Gamma_{j,j+1}} \right) \epsilon_0 \ ,$$

(14)

As in the AdS and $H^m$ cases discussed in section 1, we may again trivially extend the results to an $n$-sphere of arbitrary radius, with metric $ds_n^2 = \lambda^{-2} \left( d\theta_n^2 + \sin^2 \theta_n ds_{n-1}^2 \right)$ and Ricci tensor $R_{ij} = (n-1) \lambda^2 g_{ij}$. The Killing spinor equations are modified to $D_j \epsilon = \frac{i}{2} \lambda \Gamma_j \epsilon$, etc., but again the expressions (7), etc., for the Killing spinors receive no modification whatsoever.

2.2 Proofs

The proofs of these results proceed by substituting our expressions into the corresponding Killing spinor equations. We begin by showing that in the orthonormal basis $e^n = d\theta_n$, $e^a = \sin \theta_n e^{a(n-1)}$, the spin connection for the metric (5) is given by

$$\omega^{ab} = \omega_{(n-1)}^{ab} \ , \quad \omega^{an} = \cos \theta_n e^{a(n-1)} \ ,$$

(15)

where $a \leq n-1$, and $e^{a(n-1)}$ and $\omega_{(n-1)}^{ab}$ are the vielbein and spin connection for $S^{n-1}$. (Note that the index $n$ always denotes the specific value $n$ of the dimension of the $n$-sphere.) Thus we can write the vielbein and spin connection on $S^n$ as

$$e^j = \left( \prod_{k=j+1}^{n} \sin \theta_k \right) d\theta_j \ ,$$

$$\omega^{jk} = \cos \theta_k \left( \prod_{\ell=j+1}^{k-1} \sin \theta_\ell \right) d\theta_j \ , \quad 1 \leq j < k \leq n \ .$$

(16)

The Killing spinor equation (11) can be written as

$$\partial_j \epsilon + \frac{i}{2} \omega^{jk} \Gamma_{k\ell} \epsilon = \frac{i}{2} e^k \Gamma_k \epsilon \ ,$$

(17)

where $\omega^{jk}$ and $e^k$ are the coordinate-index components of $\omega^{k\ell}$ and $e^k$, i.e. $\omega^{k\ell} = \omega^{jk} d\theta_j$ and $e^k = e^k d\theta^j$. These can be read off from (16). Note that the indices on the $\Gamma$ matrices in (17) are vielbein indices.

We now make the following two definitions:

$$U_j^{\ k} \equiv \left( \prod_{\ell=j+1}^{k} e^{-\frac{1}{2}\theta_\ell \Gamma_{\ell,\ell+1}} \right) \Gamma_{j,j+1} \left( \prod_{\ell=j+1}^{k} e^{-\frac{1}{2}\theta_\ell \Gamma_{\ell,\ell+1}} \right)^{-1} \ , \quad k \geq j \ ,$$

(18)

$$V_j \equiv e^{\frac{i}{2}\theta_n \Gamma_n} U_j^{n-1} e^{-\frac{i}{2}\theta_n \Gamma_n} \ ,$$

(19)
where as usual, the factors with the larger $\ell$ values in the product sit to the left of those with smaller $\ell$ values. (Note that if the upper limit on the product is less than the lower limit, then it is defined to be 1.) It is now evident that verifying that the expression (7) gives a solution to the Killing spinor equation (6) amounts to proving that

$$V_j = -i e^j \Gamma_j + \sum_{k>j}^{n} \omega_j^{jk} \Gamma_{jk} .$$

(20)

We prove this by first establishing two lemmata. The first, whose proof is elementary, states that if $X$ and $Y$ are matrices such that $[X,Y] = 2Z$, and $[X,Z] = -2Y$, then

$$e^{\frac{1}{2} \theta X} Y e^{-\frac{1}{2} \theta X} = \cos \theta Y + \sin \theta Z .$$

(21)

The second lemma states that

$$U_j^k = \sec \theta_{k+1} \omega_j^{jk,k+1} \Gamma_{j,k+1} + \sum_{\ell>j}^{k} \omega_j^{j\ell} \Gamma_{j\ell} , \quad k \geq j .$$

(22)

We prove this by induction. From the definition (18), we know that $U_j^j = \Gamma_{j,j+1}$, which clearly satisfies (22) since $\omega_j^{j,j+1} = \cos \theta_{j+1}$. Assuming then that (22) holds for a specific $k \geq j$, we will have that

$$U_j^{k+1} = e^{-\frac{1}{2} \theta_{k+1} \Gamma_{k+1,k+2}} U_j^k e^{\frac{1}{2} \theta_{k+1} \Gamma_{k+1,k+2} + \sum_{\ell>j}^{k} \omega_j^{j\ell} \Gamma_{j\ell} ,}$$

(23)

where we have made use of the fact that the $\Gamma_{j\ell}$ in the last term all commute with $\Gamma_{k+1,k+2}$, since $\ell \leq k$. The first term can be evaluated using lemma 1, giving

$$U_j^{k+1} = \sec \theta_{k+1} \omega_j^{jk,k+1} \left( \cos \theta_{k+1} \Gamma_{j,k+1} + \sin \theta_{k+1} \Gamma_{j,k+2} \right) + \sum_{\ell>j}^{k+1} \omega_j^{j\ell} \Gamma_{j\ell} ,$$

(24)

Now, it follows from (16) that $\omega_j^{j,k+2} = \cos \theta_{k+2} \tan \theta_{k+1} \omega_j^{j,k+1}$. Using this, we then obtain (22) with $k$ replaced by $k+1$, completing the inductive proof.

Having established the lemmata, we can substitute the expression $U_j^{n-1}$ from (22) into the definition of $V_j$ given in (18), giving

$$V_j = \sec \theta_{n} \omega_j^{jn} e^{\frac{1}{2} \theta_{n} \Gamma_{n}} \Gamma_{jn} e^{-\frac{1}{2} \theta_{n} \Gamma_{n} + \sum_{\ell>j}^{n-1} \omega_j^{j\ell} \Gamma_{j\ell} ,}$$

(25)
where we have used lemma 1 to derive the second line. Since $e_j^i = \tan \theta_i \omega_j^i$, as can be seen from (18), it follows that (25) gives (20). This completes the proof that (11) satisfies the Killing spinor equation (10). An essentially identical proof shows that (12) satisfies the alternative Killing spinor equation (11) in even dimensions.

3 Killing spinors on $\text{AdS} \times \text{Sphere}$

An application of the formulae obtained in this paper is to construct the explicit forms of the Killing spinors in the full $D$-dimensional spacetime of a supergravity theory that admits an $\text{AdS}_m \times S^n$ solution, where $D = m + n$. Consider, for example, the $\text{AdS}_4 \times S^7$ solution of $D = 11$ supergravity. This is obtained by taking $F_{\mu \nu \rho \sigma} = 6m \epsilon_{\mu \nu \rho \sigma}$ with $\mu = 0, 8, 9, 10$, implying that the Ricci tensors on $\text{AdS}_4$ and $S^7$ satisfy $R_{\mu \nu} = -12m^2 g_{\mu \nu}$ and $R_{mn} = 6m^2 g_{mn}$ respectively [4]. The Killing spinors $\epsilon$ must satisfy

$$0 = \delta \psi_M = D_M \epsilon - \frac{1}{288} (\hat{\Gamma}_{MNPQR} F^{NPQR} - 8F_{MNPQ} \hat{\Gamma}^{NPQ}) \epsilon .$$

(26)

Using the appropriate decomposition of Dirac matrices given in appendix A, this implies that on $\text{AdS}_4$ and $S^7$ we must have

$$\text{AdS}_4 : \quad D_\mu \epsilon_{\text{AdS}} = i m \gamma \Gamma_\mu \epsilon_{\text{AdS}} ,$$

$$S^7 : \quad D_j \eta = \frac{1}{2} m \Gamma_j \eta$$

(27)

with $j = 1, \ldots, 7$. From the results obtained in this paper we find that the Killing spinors on $\text{AdS}_4 \times S^7$ can be written as

$$\text{AdS}_4 \times S^7 : \quad \epsilon = e^{\frac{1}{2}r} \gamma_\Gamma \left( 1 + \frac{1}{2} x^a(\hat{\gamma} \hat{\Gamma}_a + \hat{\Gamma}_a \hat{\Gamma}_a) \right) e^{\frac{1}{2} \theta_j \hat{\gamma} \Gamma_j} \left( \prod_{j=1}^6 e^{-\frac{1}{2} \theta_j \hat{\gamma} \hat{\Gamma}_j} \right) \epsilon_0 ,$$

(28)

where $\hat{\gamma} \equiv -\frac{i}{24} e^{\mu \nu \rho \sigma} \hat{\Gamma}_{\mu \nu \rho \sigma} = \gamma \otimes 1$ is a “pseudo chirality operator,” and $\epsilon_0$ is an arbitrary 32-component constant spinor in $D = 11$. Note that the explicit numerically-assigned indices refer to the seven directions on the 7-sphere.

In $D = 11$ supergravity there is also a solution $\text{AdS}_7 \times S^4$. An analogous calculation gives the result that the Killing spinors in this background can be written as

$$\text{AdS}_7 \times S^4 : \quad \epsilon = e^{\frac{1}{2}r} \gamma_\Gamma \left( 1 + \frac{1}{2} x^a(\hat{\gamma} \hat{\Gamma}_a + \hat{\Gamma}_a \hat{\Gamma}_a) \right) e^{\frac{1}{2} \theta_j \hat{\gamma} \Gamma_j} \left( \prod_{j=1}^3 e^{-\frac{1}{2} \theta_j \hat{\gamma} \hat{\Gamma}_j} \right) \epsilon_0 ,$$

(29)

Note that as implied by (27), the $\text{AdS}_4$ and $S^7$ have different radii, which are related by the eleven-dimensional field equations. However, as noted before, in our coordinatisation the Killing spinors are independent of the scale sizes.
where $\hat{\gamma} \equiv \hat{\Gamma}_{12345} = \mathbf{1} \otimes \gamma$, and all numerically-assigned indices refer to the four directions on $S^4$. Again, $\epsilon_0$ is an arbitrary 32-component constant spinor in $D = 11$.

As another explicit example let us look at Type IIB supergravity on $\text{AdS}_5 \times S^5$. The gravitino transformation rules are

$$0 = \delta \psi_M = D_M \epsilon + \frac{1}{1920} \hat{\Gamma}^{NPQRS} \hat{\Gamma}_M F_{NPQRS} \epsilon, \quad (30)$$

where $\epsilon$ is a ten dimensional spinor of positive chirality, satisfying

$$\hat{\Gamma}_0 \ldots \hat{\Gamma}_9 \epsilon = \epsilon. \quad (31)$$

Choosing now $F_{\mu \nu \rho \lambda} = 4m \epsilon_{\mu \nu \rho \lambda}$ and $F_{ijklm} = 4m \epsilon_{ijklm}$, equation (30) reduces to

$$D_M \epsilon - \frac{1}{2} m (\sigma_1 \otimes 1 \otimes \mathbf{1}) \hat{\Gamma}_M \epsilon = 0, \quad (32)$$

where we are using the (odd,odd) decomposition of Dirac matrices given in appendix A. With the ansatz

$$\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_{\text{AdS}} \otimes \eta, \quad (33)$$

for a spinor of positive chirality, we obtain the equations for the $\text{AdS}_5$ and $S^5$ subspaces:

$$\text{AdS}_5: \quad D_\mu \epsilon_{\text{AdS}} = \frac{1}{2} m \Gamma_\mu \epsilon_{\text{AdS}},$$

$$\text{S}^5: \quad D_j \eta = \frac{i}{2} m \Gamma_j \eta, \quad (34)$$

which are the standard Killing spinor equations. Putting the $\text{AdS}$ and $S^n$ results together, we obtain the explicit expression for the Killing spinors on $\text{AdS}_5 \times S^5$

$$\text{AdS}_5 \times S^5: \quad \epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x^\alpha \left( i \hat{\gamma} \hat{\Gamma}_\alpha + \hat{\Gamma}_r \hat{\Gamma}_\alpha \right) \right) e^{-\frac{1}{2} \theta_5 \hat{\gamma} \hat{\Gamma}_5} \left( \prod_{j=1}^{4} e^{-\frac{1}{2} \theta_j \hat{\gamma} \hat{\Gamma}_{j,ji+1}} \right) \epsilon_0, \quad (35)$$

where $\epsilon_0$ is an arbitrary 32-component constant spinor of positive chirality, and $\hat{\gamma} \equiv \hat{\Gamma}_{12345} = -\sigma_2 \otimes 1 \otimes 1$, where the numerical indices lie in $S^5$.

Four further analogous examples that arise in lower-dimensional supergravities are

$$\text{AdS}_3 \times S^3: \quad \epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x^\alpha \left( -i \hat{\gamma} \hat{\Gamma}_\alpha + \hat{\Gamma}_r \hat{\Gamma}_\alpha \right) \right) e^{-\frac{1}{2} \theta_3 \hat{\gamma} \hat{\Gamma}_3} \left( \prod_{j=1}^{2} e^{-\frac{1}{2} \theta_j \hat{\gamma} \hat{\Gamma}_{j,ji+1}} \right) \epsilon_0, \quad (36)$$

$$\text{AdS}_3 \times S^2: \quad \epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x^\alpha \left( 1 \hat{\gamma} \hat{\Gamma}_\alpha + \hat{\Gamma}_r \hat{\Gamma}_\alpha \right) \right) e\frac{1}{2} \theta_2 \hat{\gamma} \hat{\Gamma}_2 e^{-\frac{1}{2} \theta_1 \hat{\gamma} \hat{\Gamma}_{12}} \epsilon_0, \quad (37)$$

$$\text{AdS}_2 \times S^3: \quad \epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x^\alpha \left( i \hat{\gamma} \hat{\Gamma}_\alpha + \hat{\Gamma}_r \hat{\Gamma}_\alpha \right) \right) e\frac{1}{2} \theta_3 \hat{\gamma} \hat{\Gamma}_3 e^{-\frac{1}{2} \theta_1 \hat{\gamma} \hat{\Gamma}_{12}} \epsilon_0, \quad (38)$$

$$\text{AdS}_2 \times S^2: \quad \epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x^\alpha \left( \hat{\gamma} \hat{\Gamma}_\alpha + \hat{\Gamma}_r \hat{\Gamma}_\alpha \right) \right) e\frac{1}{2} \theta_2 \hat{\gamma} \hat{\Gamma}_2 e^{-\frac{1}{2} \theta_1 \hat{\gamma} \hat{\Gamma}_{12}} \epsilon_0, \quad (39)$$

or

$$\epsilon = e^{\frac{1}{2} r \hat{\gamma} \hat{\Gamma}_r} \left( 1 + \frac{1}{2} x \left( \hat{\gamma} \hat{\Gamma}_x + \hat{\Gamma}_r \hat{\Gamma}_x \right) \right) e\frac{1}{2} \theta_2 \hat{\gamma} \hat{\Gamma}_2 e^{-\frac{1}{2} \theta_1 \hat{\gamma} \hat{\Gamma}_{12}} \epsilon_0, \quad (40)$$
where the Dirac matrices are the ones appropriate to the total spacetime dimension in each case. In the case where one or other space in the factored product is even dimensional, \( \hat{\gamma} \) is the pseudo chirality operator given by the appropriate product of the hatted Dirac matrices in the even-dimensional factor. For this reason, there are two possibilities in the AdS\(_2 \times S^2\) example, reflecting the two possibilities for the Dirac matrix decomposition given in appendix A. The first corresponds to taking \( \hat{\gamma} \) to be the pseudo chirality operator in AdS\(_2\), and the second to taking it instead to be in S\(_2\). In the case of AdS\(_3 \times S^3\), \( \hat{\gamma} \equiv \hat{\Gamma}^{123} = -i \sigma_2 \otimes 1 \otimes 1 \), where the numerical indices lie in S\(_3\), while \( \hat{\tilde{\gamma}} \equiv \frac{1}{6} \epsilon_{\mu\nu\rho} \hat{\Gamma}^{\mu\nu\rho} = \sigma_1 \otimes 1 \otimes 1 \). In all the examples, \( \epsilon_0 \) is an arbitrary constant spinor in the total space. It will be subject to a chirality condition in the AdS\(_3 \times S^3\) example, if the \( D = 6 \) supergravity is chosen to be the minimal chiral theory, and \( \hat{\tilde{\gamma}} \) can then be replaced by \( \hat{\gamma} \) in the expression for the Killing spinors.

4 Discussion

In this paper, we have obtained explicit expressions for the Killing spinors on \( S^n \) for all \( n \). We then used the results to obtain the full Killing spinors on various AdS\(_m \times S^n\) spacetimes that arise as solutions in supergravity theories. These are of considerable interest owing to the conjectured duality relation to conformal theories on the AdS boundaries. One further application of these results is to construct the Killing vectors, and conformal Killing vectors, from appropriate bilinear products \( \epsilon' \Gamma_i \epsilon \) of Killing spinors. As discussed in \([3]\), products where the Killing spinors \( \epsilon' \) and \( \epsilon \) on \( S^n \) either both satisfy \([3]\) or both satisfy \([5]\) give Killing vectors, while products where one satisfies \([3]\) and the other satisfies \([4]\) give conformal Killing vectors. In general, it is necessary to use both of the Killing-vector constructions in order to obtain all the Killing vectors on \( S^n \). At large \( n \) there is a considerable redundancy in the construction, since the number of Killing spinors grows exponentially with \( n \), while the number of Killing vectors grows only quadratically with \( n \). In certain low dimensions, there is a more elegant exact spanning of the Killing vectors using this construction, such as for \( S^7 \) where the antisymmetric products \( \bar{\epsilon}^\alpha \Gamma_i \epsilon^\beta \) of the eight Killing spinors \( \epsilon^\alpha \) give the 28 Killing vectors of \( SO(8) \) \([4]\).

We shall present just one simple example here, for the case of \( S^2 \). From the matrix expression \([11]\) in the appendix, we find that from the Killing spinors \( \epsilon = \Omega_2 \begin{pmatrix} a \\ b \end{pmatrix} \) and
\[ \epsilon' = \Omega_2 \begin{pmatrix} a' \\ b' \end{pmatrix}, \]
we obtain the Killing vectors
\[
K = K^i \partial_i = E_j \epsilon'^\dagger \Gamma_j \epsilon
\]
\[
= (b'b' - a\bar{a}') \frac{\partial}{\partial \theta_1} + i (a\bar{b}' e^{-i\theta_1} - b' e^{i\theta_1}) \frac{\partial}{\partial \theta_2} + (a\bar{b}' e^{-i\theta_1} + b' e^{i\theta_1}) \cot \theta_2 \frac{\partial}{\partial \theta_1},
\]
where \( E_j \) are the components of the inverse vielbein \( E_j = E^i_j \partial_i \).
Choosing different values for the constants \( a, b, a', b' \) spans the complete set of three Killing vectors of \( SO(3) \).

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**A Dirac matrices and their decomposition on product spaces**

It is useful in general to represent the Dirac matrices in terms of the \( 2 \times 2 \) Pauli matrices \( \{ \sigma_1, \sigma_2, \sigma_3 \} \) as follows. In even dimensions \( D = 2n \), we have
\[
\Gamma_1 = \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]
\[
\Gamma_2 = \sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]
\[
\Gamma_3 = \sigma_3 \otimes \sigma_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]
\[
\Gamma_4 = \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]
\[
\Gamma_5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes \mathbf{1},
\]
\[
\cdots \cdots,
\]
\[
\Gamma_{2n-1} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1,
\]
\[
\Gamma_{2n} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2,
\]
\[
(38)
\]
In odd dimensions \( D = 2n + 1 \), we use the above construction for the Dirac matrices of \( 2n \) dimensions, and take
\[
\Gamma_{2n+1} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3.
\]
\[
(39)
\]
When performing Kaluza-Klein reductions, it is necessary to decompose the Dirac matrices of \( D \) dimensions in terms of those of the lower-dimensional spacetime \( M_m \), and the internal space \( K_n \), whose respective dimensions \( m \) and \( n \) add up to \( D \). There are four
cases that arise, namely \((m, n) = (\text{even,odd}), (\text{odd,even}), (\text{even,even})\) and \((\text{odd,odd})\). If we denote the Dirac matrices of the spacetime \(M_m\) by \(\Gamma_\mu\), and those of the internal space \(K_n\) by \(\Gamma_i\), then the Dirac matrices \(\hat{\Gamma}_A\) of \(M_m \times K_n\) can be written as:

\[
\begin{align*}
\text{(even,odd)}: \quad \hat{\Gamma}_\mu &= \Gamma_\mu \otimes 1, & \hat{\Gamma}_i &= \gamma \otimes \Gamma_i, \\
\text{(odd,even)}: \quad \hat{\Gamma}_\mu &= \Gamma_\mu \otimes \gamma, & \hat{\Gamma}_i &= 1 \otimes \Gamma_i, \\
\text{(even,even)}: \quad \hat{\Gamma}_\mu &= \Gamma_\mu \otimes 1, & \hat{\Gamma}_i &= \gamma \otimes \Gamma_i, \\
\text{or} & & \hat{\Gamma}_i &= 1 \otimes \Gamma_i, \\
\text{(odd,odd)}: \quad \hat{\Gamma}_\mu &= \sigma_1 \otimes \Gamma_\mu \otimes 1, & \hat{\Gamma}_i &= \sigma_2 \otimes 1 \otimes \Gamma_i.
\end{align*}
\]

Note that in the final case the extra Pauli matrices \(\sigma_1\) and \(\sigma_2\) are needed in order to satisfy the Clifford algebra, in view of the fact that the Dirac matrices of \(D\) dimensions are twice the size of the simple tensor products of those in \(M_m\) and \(K_n\). Note also in this case that the chirality operator in the total space is \(\sigma_3 \otimes 1 \times 1\).

### B Some low-dimensional examples

In this appendix, we give explicit matrix expressions for the Killing spinors on the spheres \(S^2\), \(S^3\), \(S^4\) and \(S^5\). These examples arise in the near-horizon structures of Reißner-Nordstrøm black holes, dyonic strings, M5-branes and D3-branes respectively. In each case, we may write the expression \((7)\) for the Killing spinors on \(S^n\) as \(\epsilon = \Omega_n \epsilon_0\). For \(S^2\), taking \(\Gamma_i = \sigma_i\), where \(\sigma_i\) are the usual Pauli matrices, we find

\[
\Omega_2 = \left( \begin{array}{cc} e^{\frac{i}{2}\theta_1} \cos \frac{1}{2} \theta_2 & e^{\frac{i}{2}\theta_1} \sin \frac{1}{2} \theta_2 \\ -e^{-\frac{i}{2}\theta_1} \sin \frac{1}{2} \theta_2 & e^{-\frac{i}{2}\theta_1} \cos \frac{1}{2} \theta_2 \end{array} \right).
\]

To avoid clumsy expressions later, we may define \(t_k = e^{\frac{i}{2}\theta_k}\), \(\bar{t}_k = e^{-\frac{i}{2}\theta_k}\), \(c_k = \cos \frac{1}{2} \theta_k\), \(s_k = \sin \frac{1}{2} \theta_k\). The matrix \(\Omega_2\) thus becomes

\[
\Omega_2 = \left( \begin{array}{cc} \bar{t}_1 c_2 & t_1 s_2 \\ -\bar{t}_1 s_2 & t_1 c_2 \end{array} \right).
\]

For \(S^3\), \(S^4\) and \(S^5\) we obtain

\[
\Omega_3 = \left( \begin{array}{cc} \bar{t}_1 t_3 c_2 & -i t_1 t_3 s_2 \\ -i \bar{t}_1 t_3 s_2 & t_1 \bar{t}_3 c_2 \end{array} \right),
\]

\[
\begin{align*}
\Omega_4 &= \left( \begin{array}{cccc}
\bar{t}_1 \bar{t}_3 c_2 c_4 & \bar{t}_1 t_3 c_2 s_4 & -i t_1 \bar{t}_3 s_2 s_4 & -i t_1 \bar{t}_3 c_4 s_2 \\
-t_1 \bar{t}_3 c_2 s_4 & \bar{t}_1 t_3 c_2 c_4 & -i t_1 \bar{t}_3 c_4 s_2 & i t_1 \bar{t}_3 s_2 s_4 \\
i t_1 t_3 s_2 s_4 & -i \bar{t}_1 t_3 c_4 s_2 & t_1 t_3 c_2 c_4 & -t_1 t_3 c_2 s_4 \\
-i \bar{t}_1 t_3 c_4 s_2 & -i t_1 t_3 s_2 s_4 & t_1 t_3 c_2 c_4 & t_1 t_3 c_2 s_4
\end{array} \right),
\end{align*}
\]
\[ \Omega_5 = \begin{pmatrix} \bar{t}_1 \bar{t}_3 \bar{t}_5 c_2 c_4 & -i \bar{t}_1 \bar{t}_3 \bar{t}_5 c_2 s_4 & -t_1 \bar{t}_3 \bar{t}_5 s_2 s_4 & -it_1 \bar{t}_3 \bar{t}_5 c_4 s_2 \\ -i \bar{t}_1 \bar{t}_3 \bar{t}_5 c_2 s_4 & \bar{t}_1 \bar{t}_3 \bar{t}_5 c_2 c_4 & -it_1 \bar{t}_3 \bar{t}_5 c_4 s_2 & -t_1 \bar{t}_3 \bar{t}_5 s_2 s_4 \\ -\bar{t}_1 t_3 t_5 s_2 s_4 & -i \bar{t}_1 t_3 t_5 c_4 s_2 & t_1 t_3 t_5 c_2 c_4 & -it_1 t_3 t_5 c_2 s_4 \\ -i \bar{t}_1 t_3 t_5 c_4 s_2 & -\bar{t}_1 t_3 t_5 s_2 s_4 & -it_1 t_3 t_5 c_2 s_4 & t_1 t_3 t_5 c_2 c_4 \end{pmatrix} . \] (45)

In these examples we have used the representations of Dirac matrices given in equations (38) and (39) of appendix A.

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