SMASH NILPOTENCE ON UNIRULED 3-FOLDS

RONNIE SEBASTIAN

Abstract. Voevodsky has conjectured that numerical and smash equivalence coincide on a smooth projective variety. We prove this conjecture holds for uniruled 3-folds and for one dimensional cycles on products of Kummer surfaces.

1. Introduction

Throughout this article we work over an algebraically closed field $k$ and with algebraic cycles with rational coefficients.

Let $X$ be a smooth and projective variety over $k$. In [Voe95], Voevodsky defines a cycle $\alpha$ to be smash nilpotent if the cycle $\alpha^n := \alpha \times \alpha \ldots \times \alpha$ on the variety $X^n := X \times X \ldots \times X$ is rationally equivalent to 0. It is trivial to see that a smash nilpotent cycle is numerically trivial, Voevodsky conjectured that the converse also holds. Voevodsky, [Voe95], and Voisin, [Voi94], prove that a cycle which is algebraically trivial is smash nilpotent.

Kimura, [Kim05, Proposition 6.1], proved that a morphism between finite dimensional motives of different parity is smash nilpotent. Thus, if an algebraic cycle can be viewed as a morphism between motives of different parities, then it is smash nilpotent. In [KS09], the authors use this fact to prove that skew cycles on an abelian variety are smash nilpotent. A cycle $\beta$ is called skew if it satisfies $[-1]^* \beta = -\beta$. In [KS09] such cycles are expressed as morphisms between motives of different parity, using the fact that the motive of an abelian variety has a Chow-Kunneth decomposition,

$$h(A) = \bigoplus_{i=0}^{\dim A} h^i(A)$$

and the motives $h^i(A)$ for $i$ odd are oddly finite dimensional.

In [Seb13] it is proved that for one dimensional cycles on a variety dominated by a product of curves, smash equivalence and numerical equivalence coincide. The same result can be deduced from [Mar08] and [Her07], where it is shown that for a smooth projective curve $C$, for any adequate equivalence relation, $[C]_i = 0$ implies that $[C]_{i+1} = 0$, for $i \geq 2$. Here $[C]_i$ denotes the Beauville component of the curve $C$ in its Jacobian satisfying $[n]_*[C]_i = n^i[C]_i$. If we combine this with [KS09], where it is shown that

2010 Mathematics Subject Classification. 14C25.
Key words and phrases. Algebraic cycles, smash nilpotence.
[C]_3 = 0 modulo smash equivalence, then one can deduce the results in Seb13.

If we take the Chow ring of an abelian variety modulo algebraic equivalence and go modulo the subring generated by the cycles in the preceding paragraphs under the Pontryagin product, intersection product and Fourier transform, then there are no nontrivial examples of higher dimensional cycles (dim > 1) for which Voevodsky’s conjecture holds.

The purpose of this article is to write down some more examples for which this conjecture holds. The main theorems in this article are

**Theorem 1.** Let $X$ be uniruled 3-fold. Then numerical and smash equivalence coincide for cycles on $X$.

**Theorem 2.** Let $K_i$, $i = 1, 2, \ldots, N$ be Kummer surfaces. Then numerical and smash equivalence coincide for one dimensional cycles on $X := K_1 \times K_2 \times \cdots \times K_N$.

The proof of the above theorems use Lemma 3, which implies the following. If numerical and smash equivalence coincide on a smooth and projective variety $Y$, then they coincide on $\tilde{Y}$, which is obtained by blowing up $Y$ along a smooth subvariety of dimension $\leq 2$.

**Acknowledgements.** We thank Najmuddin Fakhruddin for useful discussions.

### 2. Smash equivalence and blow ups

Let $Y$ be a smooth variety and $i : X \hookrightarrow Y$ be a smooth and closed subvariety. Let $f : \tilde{Y} \to Y$ denote the blow-up of $Y$ along $X$.

**Lemma 3.** If numerical and smash equivalence coincide for elements in $\text{CH}_i(X)$ for $i \leq r$ and $\text{CH}_r(Y)$, then they coincide for elements in $\text{CH}_r(\tilde{Y})$.

**Proof.** Consider the Cartesian square

$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow g & & \downarrow f \\
\tilde{X} & \xrightarrow{\tilde{i}} & \tilde{Y}
\end{array}$

Then [Ful97, Proposition 6.7] says that there is an exact sequence

$$0 \to \text{CH}_r(X) \to \text{CH}_r(\tilde{X}) \oplus \text{CH}_r(Y) \to \text{CH}_r(\tilde{Y}) \to 0$$

Since $X$ is a smooth subvariety of $Y$, we have that $\tilde{X} \xrightarrow{\partial} X$ is the projective bundle associated to the locally free sheaf $\mathcal{I}_X/\mathcal{I}_X^2$ on $X$. Thus, every element $\beta \in \text{CH}_r(\tilde{X})$ may be expressed as the sum

$$\beta = \sum_{i=0}^{d-1} c_1(\mathcal{O}(1))^i \cap g^* g_*(\beta \cap c_1(\mathcal{O}(1))^{d-1-i}),$$
where $\mathcal{O}(1)$ is the tautological bundle on $\tilde{X}$. If we assume that numerically trivial elements in $CH_i(X)$ are smash nilpotent for $i \leq r$, then the above formula shows that numerically trivial elements in $CH_r(\tilde{X})$ are smash nilpotent. If numerical and smash equivalence coincide for elements in $CH_r(Y)$, then the above exact sequence would show that these coincide for elements in $\tilde{Y}$ as well. □

The following is a standard result which we include for the benefit of the reader.

**Lemma 4.** Let $X$ be a smooth projective variety and let $h : Y \to X$ be a dominant morphism. If numerical and smash equivalence coincide for cycles on $Y$, then they coincide for cycles on $X$.

**Proof.** Let $l \in CH^1(Y)$ be a relatively ample line bundle. The relative dimension of $h$ is $r := \dim(Y) - \dim(X)$ and define $d$ by $h^*(l^r) =: d[Y]$. Then by the projection formula, we have $\forall \alpha \in CH_*(X)$

$$h^*(l^r \cdot h^*\alpha) = d\alpha$$

If $\alpha$ is a numerically trivial cycle on $X$, then $l^r \cdot h^*\alpha$ is a numerically trivial cycle on $Y$ and so is smash nilpotent. The above equation shows that $\alpha$ is smash nilpotent. □

### 3. Examples

#### 3.1. Uniruled 3-folds.

**Definition 5.** By a uniruled 3-fold we mean a smooth projective variety $X$ for which there is a dominant rational map $\varphi : S \times \mathbb{P}^1 \dashrightarrow X$ for some smooth projective surface $S$.

**Proof of Theorem [1].** Since $X$ is projective and $Y := S \times \mathbb{P}^1$ is normal, $\varphi$ can be defined on an open set $U$ whose complement has codimension $\geq 2$. Let $X \hookrightarrow \mathbb{P}^n$ be a closed immersion, composing this with $\varphi$ we get a morphism $g : U \to \mathbb{P}^n$. Let $L$ denote the pullback of $\mathcal{O}(1)$ along $g$. Since $Y \setminus U$ has codimension $\geq 2$, there is a unique line bundle on $Y$ which restricts to $L$, we denote this also by $L$. As $Y$ is smooth and codimension $Y \setminus U$ is $\geq 2$, the restriction map $H^0(Y, L) \to H^0(U, L)$ is an isomorphism, see, for example [Har77, Chapter 3, Ex 3.5]. Let $V \subset H^0(Y, L)$ be the subspace of global sections $g^*H^0(\mathbb{P}^n, \mathcal{O}(1))$. Let $J \subset L$ be the subsheaf generated by $V$, then $I = J \otimes L^{-1}$ is an ideal sheaf such that $Y \setminus \text{Supp}(I) = U$ and $V$ is contained in the image of the map $H^0(Y, I \otimes L) \to H^0(Y, L)$ and it generates $I \otimes L$.

We want to apply the principalization theorem to the ideal sheaf $I$. In characteristic 0, see [Kol07, Theorem 3.21], and in positive characteristic, see [Cut09, Theorem 1.3]. We get a morphism $f : Y' \to Y$ which is obtained as a composite of smooth blow-ups, such that $f^*I$ is a locally principal ideal sheaf and $f$ is an isomorphism on $f^{-1}(U)$. The subspace $f^*V \subset H^0(Y', f^*I \otimes f^*L)$ defines a map $Y' \to \mathbb{P}^n$ which extends $g$. Thus, we get a dominant morphism...
Y' \to X. As S is a surface, numerical and smash equivalence coincide for cycles on S and so for cycles on Y. Since Y' is obtained from Y by blowing up at smooth centers and \( \dim(Y) = 3 \), numerical and smash equivalence coincide for Y' using Lemma 3. Finally, use Lemma 4 to get the same result for X. \( \square \)

3.2. Kummer surfaces. Let \( Y \) be an abelian surface and let \( X \) be the set of 2 torsion points. These are exactly the fixed points for the involution \( x \mapsto x^{-1} \) on \( Y \). This involution lifts to an involution of \( \tilde{Y} \) which we denote \( \tilde{i} \) and the quotient \( \tilde{Y}/\tilde{i} \) is the Kummer surface associated to \( Y \). We denote this surface by \( K \) and by \( \pi \) the quotient map \( \tilde{Y} \to K \).

Let \( Y_i, i = 1, 2 \) be abelian surfaces and let \( K_i, i = 1, 2 \) be the associated Kummer surfaces. Let \( X_i \) be the set of 2 torsion points in \( Y_i \). Similarly, we have the varieties \( \tilde{Y}_i \) and there is a dominant projective map \( \tilde{Y}_1 \times \tilde{Y}_2 \to K_1 \times K_2 \).

The map \( \tilde{Y}_1 \times \tilde{Y}_2 \to Y_1 \times Y_2 \) may be factored as the composite of two blow ups

\[
\tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_1 \times Y_2 \to Y_1 \times Y_2
\]

the first along the surface \( X_1 \times Y_2 \) and the second along the surface \( \tilde{Y}_1 \times X_2 \). Applying Lemma 3 to both these blow ups, we get that numerical and smash equivalence coincide for one dimensional cycles on \( \tilde{Y}_1 \times \tilde{Y}_2 \) and so using Lemma 4 they coincide on \( K_1 \times K_2 \).

**Proof of Theorem 2.** We recall a result from [Seb13] which we need. Let \( N \geq 3 \) be an integer and let \( C \) be a smooth projective curve with a base point \( c_0 \). Let \( \Delta_C \) denote the diagonal embedding \( C \hookrightarrow C^N \). Let \( p_{ij} : C^N \to C^N \) denote the map which leaves the \( i \)th and \( j \)th coordinates intact and the other coordinates are changed to \( c_0 \), for example, \( p_{12}(x_1, x_2, \ldots, x_N) = (x_1, x_2, c_0, c_0, \ldots, c_0) \). Then there are rational numbers \( q_{ij} \) such that

\[
(3.1) \Delta_C \sim_{sm} \sum_{i \neq j} q_{ij} p_{ij}^*(\Delta_C).
\]

Let \( X := K_1 \times K_2 \ldots \times K_N \) be a product of Kummer surfaces. Fix base points \( e_i \in K_i \), and define (we abuse notation here) \( p_{ij} : X \to X \) in the same way as above, using these base points. We remark that if we work modulo algebraic equivalence, for any cycle \( \alpha \in CH_*(X) \), the cycle \( p_{ij}(\alpha) \) is independent of the choice of these base points. Hence, the same is true modulo smash equivalence.

Let \( D \hookrightarrow X \) be a reduced and irreducible one dimensional subvariety. Let \( C \to D \) be its normalization and denote the composite map by \( f : C \to X \). If we let \( p_i \) denote the projection from \( X \) to \( K_i \) and let

\[
\pi := (p_1 \circ f) \times (p_2 \circ f) \ldots (p_n \circ f),
\]

then...
then we get \( f_*([C]) = \pi_*(\Delta_C) \). Using equation (3.1), we get that modulo smash equivalence

\[
[D] = f_*([C]) = \pi_*(\Delta_C) = \sum_{i \neq j} q_{ij} \pi_*(p_{ij}) = \sum_{i \neq j} q_{ij} p_{ij} \pi_*(\Delta_C) = \sum_{i \neq j} q_{ij} p_{ij}([D])
\]

In particular, we get for any one dimensional cycle \( \alpha \),

\[
\alpha = \sum_{i \neq j} q_{ij} p_{ij} \pi_*([\alpha])
\]

As we have seen above, on a product of two Kummer surfaces, numerical and smash equivalence coincide for one dimensional cycles. If \( \alpha \) is numerically trivial, each \( p_{ij} \pi_*([\alpha]) \) is numerically trivial and so smash nilpotent. Thus, \( \alpha \) is smash nilpotent. \( \square \)

**References**

[Cut09] Cutkosky, *Resolution of singularities for 3-folds in positive characteristic*, American Journal of Mathematics **131** (2009), 59–127.

[Ful97] W. Fulton, *Intersection theory*, Springer, 1997.

[Har77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, no. 52, Springer, 1977.

[Her07] F. Herbaut, *Algebraic cycles on the jacobian of a curve with a linear system of given dimension*, Compositio Math. **143** (2007), 883–899.

[Kim05] S. I. Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), 173–201.

[Kol07] Janos Kollar, *Lectures on resolutions of singularities*, Princeton University Press, 2007.

[KS09] B. Kahn and R. Sebastian, *Smash-nilpotent cycles on abelian 3-folds*, Math. Res. Lett. **16** (2009), 1007–1010.

[Mar08] G. Marini, *Tautological cycles on jacobian varieties*, Collect. Math. **59** (2008), 167–190.

[Seb13] Ronnie Sebastian, *Smash nilpotent cycles on products of curves*, Compositio Math. **149** (2013), 1511–1518.

[Voe95] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero*, Internat. Math. Res. Notices **4** (1995), 187–198.

[Voi94] C. Voisin, *Remarks on zero-cycles of self-products of varieties. moduli of vector bundles (sanda, 1994; kyoto, 1994)*, Lecture Notes in Pure and Appl. Math. **179** (1994), 265–285.