Investigating Spherical Images of a Curve According to Type-1 Bishop Frame in Weyl Space Using Prolonged Covariant Derivative

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Abstract

In this study, we investigated spherical images of a curve according to type-1 Bishop frame in three dimensional Weyl space. Further, we expressed the relations among Frenet-Serret and type-1 Bishop frame apparatus. We defined the concepts of general helix, slant helix, spherical curve and also circle by using prolonged covariant derivative in Weyl space. Later, provided that these spherical images satisfy the above definitions, the conditions obtained were expressed in terms of first and second curvatures and hence Bishop curvatures. Additionally, parallel displacement condition of the binormal vector fields of the $n_{1}$ and $n_{2}$ Bishop spherical images of a curve along their own tangent vector fields was discussed.

Keywords: Weyl space, General helix, Slant helix, Spherical curve, Spherical image.
1. Introduction

Bishop frame (or type-1 Bishop frame) was introduced by Bishop (1975). This frame was also named as alternative or parallel frame of the curves. Many researchers used Bishop frame in several spaces, such as Bükçi and Karacan (2008a and 2009), Yılmaz et al. (2010; in Euclidean space), Bükçi and Karacan (2008b; in Lorentzian space), Karacan and Bükçi (2007 and 2008), Yılmaz (2009; in Minkowski 3-space) and Kofoğlu (2020; in Weyl space).

2. Preliminaries

Let \( C: x^i = x^i(s) \) (\( s \) is the arc length parameter of \( C \)) be a curve in three dimensional Weyl space \( W_i \) (\( i = 1, 2, 3 \)). Let us denote Frenet-Serret frame and Bishop (or type-1 Bishop) frame belonging to \( C \) by \( \{ v, y, z \} \) and \( \{ v, n, \xi \} \), respectively. Both of these frames are orthonormal bases.

Frenet-Serret formulas are expressed as
\[
\begin{align*}
\nu^k v^i_k v^j & = \kappa_1 v^i_1 + \kappa_2 v^i_2 \\
\nu^k v^i_2 v^j & = -\kappa_1 v^i_1 + \kappa_2 v^i_2 \\
\nu^k v^i_3 v^j & = -\kappa_2 v^i_2
\end{align*}
\]
where \( \kappa_1 \) and \( \kappa_2 \) are the first and second curvatures of \( C \), respectively.

Derivative formulas of the vector fields of Bishop frame are in the following form:
\[
\begin{align*}
\nu^k v^i_k v^j & = \kappa_1 n^i_1 + \kappa_2 n^i_2 \\
\nu^k v^i_2 n^j & = -\kappa_1 n^i_1 \\
\nu^k v^i_3 n^j & = -\kappa_2 n^i_2
\end{align*}
\]
Here, \( \kappa_1 \) and \( \kappa_2 \) are Bishop curvatures (Bishop, 1975). Their equivalents in Weyl space (Kofoğlu, 2020) are
\[
k_1 = \frac{p}{T_k v^i_k v^j n^j} \quad (j, k, p = 1, 2, 3)
\]
or
\[
k_1 = \frac{p}{2} v^i_k v^j n^j = \kappa_1 n^i_1 \quad (j, k, p = 1, 2, 3)
\]
and
\[
k_2 = \frac{p}{T_k v^i_k v^j n^j} g_{ij}
\]
or
\[
k_2 = \frac{p}{2} v^i_k v^j n^j = \kappa_1 n^i_1 \quad (j, k, p = 1, 2, 3)
\]
where, \( \frac{2}{p} = T_k v^i_k \) is geodesic curvature of the net \( \{ v, y, z \} \) (Tsareva, 1990) and \( c_1 = \frac{p}{2} v^i_1 \) is the geodesic curvature vector field of the net \( \{ v, y, z \} \) (Tsareva, 1990).

Also, \( k_1 = \kappa_1 \cos \theta, \quad k_2 = \kappa_2 \sin \theta, \quad k_1^2 = k_2^2 + k_2^2 \) and \( \kappa_2 = v^k \nu^j_k \theta (\theta = \theta(s)) \) where \( \theta = \kappa(v, n) \) (Kofoğlu, 2020).

There is the following relation among the vector fields of these two frames (Kofoğlu, 2020):
\[
\begin{pmatrix}
v^i_1 \\
v^i_2 \\
v^i_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\nu^i_1 \\
\nu^i_2 \\
\nu^i_3
\end{pmatrix}
\]

3. The Expression of Special Curves in Weyl Space

**Definition 1.** Let \( C \) be a be a curve in three dimensional Weyl space. \( C \) is called a general helix if the tangent vector field \( v = v^i \) of \( C \) has constant angle \( \varphi \) with some fixed vector field \( u = u^i \), i.e.,
\[
g_{ij} v^i u^j = \cos \varphi = \text{constant}
\]
where \( g_{ij} u^i u^j = 1 \).

Using Şemin (1983), we can express the condition to be a general helix in the following form:

**Theorem 1.** \( C \) is a general helix if and only if
\[
\frac{k_2}{k_1} = \text{constant}
\]
where \( k_1 \) and \( k_2 \) are the first and second curvatures of \( C \).

With the help of Izumiya and Takeuchi (2004) and Kula et al. (2010) and using prolonged covariant derivative, the following proposition can be given:

**Proposition 1.** If \( C \) is a slant helix, then
\[
\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}} \left( \frac{v^k \nu^j_k}{\kappa_2} \right) = \text{constant}
\]
is satisfied.

**Proof.** Let \( C \) be a slant helix. Then \( C \) is a curve with \( k_2 \neq 0 \) and \( \left( \frac{v^k \nu^j_k}{\kappa_2} \right) \) is a constant function.

Let \( \tilde{C} : y^i = y^i(\varphi) \) be the spherical image of principal normal vector field \( v^i \) of \( C \) (\( \varphi \) is the arc length parameter of \( \tilde{C} \)). Then, using Frenet frame the following equalities are satisfied:
\[
\begin{align*}
\nu^k \nu^j_k y^i & = \nu^k \nu^j_k v^i \\
\left( \frac{v^k \nu^j_k}{\kappa_2} \right) \alpha & = -\kappa_1 v^i + \kappa_2 v^i_3 \\
\nu^i_1 & = -\frac{\kappa_1}{\kappa_1 + \kappa_2} v^i_1 + \frac{\kappa_2}{\kappa_1 + \kappa_2} v^i_3
\end{align*}
\]
where $\vec{v}$ is the tangent vector field of $C$ and $a = a(s) = \sqrt{\kappa_1^2 + \kappa_2^2}$.

The prolonged covariant derivative of $\vec{v}^i$ in the direction of $v$ is

$$\frac{\partial^k \vec{v}^k}{\partial s^i} = \left(\frac{\partial^k \vec{v}^k}{\partial s^i}\right)\alpha = \left(\frac{\partial^k \vec{v}^k}{\partial s^i}\right) = \frac{k_2 k_1^2}{(\kappa_1^2 + \kappa_2^2)^2}\left(p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right)\frac{v^i}{\kappa_1}$$

$$+ \frac{k_1^2}{(\kappa_1^2 + \kappa_2^2)^2}\left(p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right)\frac{v^i}{\kappa_1}$$

where $k_1$ is the first curvature of $C$, $\vec{v}$ is the principal normal vector field of $C$ and $\vec{v}^i$ is the geodesic curvature vector field of the net ($\vec{v}, \vec{v}, \vec{v}$).

The geodesic curvature of the spherical image of the principal normal vector field of $v$ is

$$k_2^2 = g_{ij}^2 \vec{v}^i \vec{v}^j = \bar{k}_2^2 = \left[p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right]^2 - \frac{k_1^2}{(\kappa_1^2 + \kappa_2^2)^2} + 1.$$ (10)

Let us denote the first term of in the right hand side of the above equality by $\sigma^2$. Then, we get

$$\bar{k}_2^2 = \sigma^2 + 1.$$ (11)

Since the spherical image of the principal normal vector field $v$ is a part of a circle in $S^2$, $\bar{k}_1$ must be non-zero constant and so

$$\sigma^2 = \left[p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right]^2 - \frac{k_1^2}{(\kappa_1^2 + \kappa_2^2)^2} + 1.$$ (12)

is a constant function.

Here $\kappa_1 = g_{ij} \left(p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right) \frac{v^i}{\kappa_1} = 2$ and $\kappa_1 = g_{ij} \left(p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right) \frac{v^i}{\kappa_1} = -\frac{2}{\kappa_1}$. □

Further, by means of Kofoğlu (2020), the following theorem is valid:

**Theorem 2.** Let $C$ be a curve which has non-zero Bishop curvatures in $W_3$. $C$ is a slant helix if and only if $\frac{k_1}{k_2}$ is constant.

Using Şemin (1983), we can write the following proposition:

**Proposition 2.** If $C$ is a spherical curve,

$$\frac{k_2}{k_1} + \frac{v^i \vec{V}^i}{\frac{k_1}{k_2}} \left(\frac{1}{k_2} \frac{1}{k_1}\right) = 0 \quad (l = 1,2,3)$$

is satisfied.

**Proof.** Let $C$ be a spherical curve. If we choose center of the sphere as origin, the position vector at any point of $C$ satisfies the following relation:

$$[g_{ij}x^i x^j]^2 = R^2.$$ (13)

Here $R$ is the radius of the sphere and it is constant.

Taking prolonged covariant derivative of $g_{ij}x^i x^j = R$ in the direction of $v$, we get

$$g_{ij}v^i x^j = 0$$ (14)

where $v$ is the tangent vector field of $C$.

Taking prolonged covariant derivative of (14) in the direction of $v$, we have

$$g_{ij} \left(p^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}\right) x^j + 1 = 0$$

$$g_{ij}k_1 v^j x^j + 1 = 0$$

$$g_{ij} v^j x^j = -\frac{1}{k_1}$$ (15)

where $g_{ij} v^j x^j = 1$, $v$ is the principal normal vector field of $C$ and $\kappa_1$ is the first curvature of $C$.

Taking prolonged covariant derivative of (15) in the direction of $v$, we obtain

$$-k_2 g_{ij} v^j x^j = \frac{v^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}}{1}$$

$$-g_{ij} v^j x^j = \frac{1}{k_2} \frac{v^k \vec{v}^k_{k_2} \frac{k_2}{\kappa_1}}{1}$$

(16)

where $v$ is the binormal vector field of $C$, $k_2$ is the second curvature of $C$, $g_{ij} v^j x^j = 0$ and $g_{ij} v^j x^j = 0$.

Again taking prolonged covariant derivative of (16) in the direction of $v$, we have

$$k_2^2 g_{ij} v^j x^j = v^i \vec{V}^i \left(\frac{1}{k_2} \frac{1}{k_1}\right)$$

(17)

where $g_{ij} v^j x^j = 0$.

Using (15) in (17), we get

$$\frac{k_2}{k_1} + \frac{v^i \vec{V}^i}{\frac{k_1}{k_2}} \left(\frac{1}{k_2} \frac{1}{k_1}\right) = 0$$

(18)

which concludes the proof. □

With the help of Nomizu and Yano (1974) and Özdeğer and Şentürk (2002), the following definition and proposition can be formulated:

**Definition 2.** $C$ is called a circle if there exists a vector field $z^i$ and a positive constant $k$ such that
where \( g_{ij} u^i u^j = 1 \).

**Proposition 3.** If \( \mathcal{C} \) is a circle, the equation
\[
v^i \hat{\nabla} (v^k \hat{\nabla}^i v^j) + g_{ij} (v^k \hat{\nabla} v^j) (v^k \hat{\nabla}^i v^j) v^i = 0
\]
is satisfied. Conversely, if \( \mathcal{C} \) satisfies the above equation, \( \mathcal{C} \) is either a geodesic or a circle.

### 4. About The Spherical Images of a Curve in Weyl Space

**Definition 3.** Let \( \mathcal{C} \) be a curve in \( W_2 \). If we translate the first vector field of type-1 Bishop frame to the center \( O \) of the unit sphere \( S^2 \), we obtain a spherical image \( \hat{\mathcal{C}} : u^i = u^i(\hat{s}) \) (\( \hat{s} \) is the arc length parameter of \( \hat{\mathcal{C}} \)). \( \hat{\mathcal{C}} \) is called tangent Bishop spherical image or indicatrix of the curve \( \mathcal{C} \).

In order to investigate the relations between type-1 Bishop and Frenet-Serret invariants we take the prolonged covariant derivative of \( u^i \) in the direction of \( v \), we have
\[
\frac{d}{ds} v^k \hat{\nabla} u^i = k_i n^i + k,n^i
\]
(19)
\[
\frac{d}{ds} (v^k \hat{\nabla} u^i) a = k_i n^i + k, n^i
\]
(20)
\[
\frac{d}{ds} \bar{v}^i a = k_i n^i + k, n^i
\]
(21)
where \( \bar{v}^i \) is the tangent vector field of \( \hat{\mathcal{C}} \), \( g_{ij} \bar{v}^i \bar{v}^j = 1 \) and \( a = a(s) \).

Taking the norm of both sides of (21), we get
\[
a = \sqrt{k^2 + k_2^2}.
\]
(22)

Let us take \( a = \sqrt{k_i^2 + k_2^2} \). Hence, we obtain
\[
\frac{d}{ds} \bar{v}^i = \frac{k_1}{\sqrt{k_i^2 + k_2^2}} n^i + \frac{k_2}{\sqrt{k_i^2 + k_2^2}} n^i
\]
(23)
where \( k_1 = g_{ij} c^i n^j \) and \( k_2 = g_{ij} c^i n^j \).

Taking the prolonged covariant derivative of (23) in the direction of \( v \), we have
\[
\frac{d}{ds} v^k \hat{\nabla} \bar{v}^i = \frac{d}{ds} (v^k \hat{\nabla} \bar{v}^i) a
\]
\[
= \frac{k^2}{(k_i^2 + k_2^2)^{3/2}} \left( v^k \hat{\nabla}^i k_1 \right) \frac{n^i}{k_2}
\]
\[
+ \frac{k_2}{(k_i^2 + k_2^2)^{3/2}} \left( v^k \hat{\nabla}^i k_2 \right) \frac{n^i}{k_1}
\]
\[
- \frac{k_i^2 + k_2^2}{k_i^2} v^i
\]
(24)
and
\[\bar{\mathcal{K}}_2 = \frac{1}{(k_i^2 + k_2^2)^3} \left[ \frac{k_2^2}{k_2^2 + k_2^2} \right] \left( v^k \hat{\nabla}^i k_2 \right) \frac{k_1}{k_i^2}
\]
\[
+ \frac{3}{k_i^2} \left( v^k \hat{\nabla}^i k_2 \right) \left( v^k \hat{\nabla}^i k_1 \right)
\]
(30)
where \( k_2 \) is the second curvature of \( \tilde{C} \) and \( g_{ij}\tilde{v}_i\tilde{v}_j = 1 \), \( k_1 = g_{ij}c^i\eta^j \) and \( k_2 = g_{ij}c^i\eta^j \).

**Corollary 1.** Let \( \tilde{C} \) be the tangent Bishop spherical image of \( C \). If \( \frac{k_1}{k_2} \) = constant, then the tangent Bishop spherical image \( u^i = u^i(\tilde{s}) \) is a circle.

**Proof.** If we use Proposition 3, we have

\[
\begin{align*}
\mathbf{v}^i(1 & \mathbf{v}^k \mathbf{v}_k)^{\tilde{s}} \\
= \left[ k_2^2 \left( \frac{k_1^2}{(k_2^2 + k_2^2)^{3/2}} \right) \left( \frac{k_1^2}{k_2^2} \right) \right]^{\tilde{s}} \\
+ \frac{k_2^2}{(k_2^2 + k_2^2)^{3/2}} & \left[ k_2 \left( \frac{k_1^2}{k_2^2} + k_2 \left( \frac{k_1^2}{k_2^2} \right)^2 \right) \right]^{\tilde{s}} \\
- & \frac{k_2^2}{(k_2^2 + k_2^2)^{3/2}} \left[ k_2 \left( \frac{k_1^2}{k_2^2} + k_2 \left( \frac{k_1^2}{k_2^2} \right)^2 \right) \right]^{\tilde{s}} \\
- & \frac{1}{(k_2^2 + k_2^2)^{3/2}} \left[ k_2 \left( \frac{k_1^2}{k_2^2} + k_2 \left( \frac{k_1^2}{k_2^2} \right)^2 \right) \right]^{\tilde{s}} \\
- & \frac{k_1 k_2}{(k_2^2 + k_2^2)^{3/2}} \left[ k_2 \left( \frac{k_1^2}{k_2^2} + k_2 \left( \frac{k_1^2}{k_2^2} \right)^2 \right) \right]^{\tilde{s}} \\
\end{align*}
\]

(31)

and

\[
\begin{align*}
\mathbf{g}_{ij} & \left( \frac{1}{1} \mathbf{v}^k \mathbf{v}_k \right)^{\tilde{s}} \\
= & \left( k_2^2 \frac{k_1^2}{(k_2^2 + k_2^2)^{3/2}} \right) \left( \frac{k_1^2}{k_2^2} \right) \left( \frac{k_1^2}{k_2^2} \right)^2 + (k_2^2 + k_2^2) \right]^{\tilde{s}} \\
\end{align*}
\]

(32)

where \( k_1 = g_{ij}c^i\eta^j \) and \( k_2 = g_{ij}c^i\eta^j \).

Under the condition \( \frac{k_1}{k_2} \) = constant, using (31) and (32), we obtain

\[
\begin{align*}
\mathbf{v}^i(1 \mathbf{v}^k \mathbf{v}_k)^{\tilde{s}} + \mathbf{g}_{ij} \left( \frac{1}{1} \mathbf{v}^k \mathbf{v}_k \right)^{\tilde{s}} \left( \frac{1}{1} \mathbf{v}^k \mathbf{v}_k \right) = 0.
\end{align*}
\]

(33)

Hence, we see that, the tangent Bishop spherical image \( u^i = u^i(\tilde{s}) \) of \( C \) is a circle. □

**Definition 4.** Let \( C \) be a curve in \( W_3 \). If we translate the second vector field of type-1 Bishop frame to the center \( O \) of the unit sphere \( S^2 \), we obtain a spherical image \( \tilde{F} : f^i = f^i(\tilde{s}) \) (\( \tilde{s} \) is the arc length parameter of \( \tilde{F} \)), \( \tilde{F} \) is called \( \eta \) Bishop spherical image or indicatrix of the curve \( C \).

The relations between type-1 Bishop and Frenet-Serret invariants are obtained by taking the prolonged covariant derivative of \( f^i \) in the direction of \( v_1 \), we have

\[
\begin{align*}
\mathbf{v}^k \mathbf{v}_k f^i & = -k_1 \frac{k_2}{k_1} \\
\left( \frac{\mathbf{v}^k \mathbf{v}_k f^i}{k_1} \right) b & = -k_1 \frac{k_2}{k_1} \\
\frac{\mathbf{v}^k \mathbf{v}_k f^i}{k_1} b & = -k_1 \frac{k_2}{k_1} \\
\end{align*}
\]

(34)

(35)

(36)

where \( \tilde{v}^i \) is the tangent vector field of \( \tilde{F} \), \( g_{ij}\tilde{v}^i\tilde{v}^j = 1 \) and \( b = b(s) \).

Taking the norm of both sides of (36), we get

\[
\begin{align*}
\mathbf{b} & = \langle k_i \rangle \\
\end{align*}
\]

(37)

Let us take \( b = -k_1 \). In this case, we obtain

\[
\begin{align*}
\tilde{v}_i & = v_i \\
\end{align*}
\]

(38)

Taking the prolonged covariant derivative of (38) in the direction of \( v_1 \), we have

\[
\begin{align*}
\mathbf{v}^k \mathbf{v}_k \tilde{v}^i & = \left( \frac{\mathbf{v}^k \mathbf{v}_k v_i}{k_1} \right) b = \frac{\mathbf{v}^k \mathbf{v}_k v_i}{k_1} \\
\mathbf{M}_1 \tilde{v}^i = -k_1 \frac{k_2}{k_1} \frac{n^i}{k_2} \\
\mathbf{M}_1 \tilde{v}^i = -n^i \frac{k_2}{k_1} n^i \\
\end{align*}
\]

(39)

(40)

(41)

Taking the norm of both sides of (41), we get

\[
\begin{align*}
\mathbf{M}_1 & = \left( 1 + \left( \frac{k_2}{k_1} \right)^2 \right) - \left( 1 + \left( \frac{g_{ij}\eta^i\eta^j}{k_1} \right)^2 \right) \\
\end{align*}
\]

(42)

and

\[
\begin{align*}
\tilde{v}_i & = -\frac{1}{\mathbf{M}_1} n^i \frac{k_2}{k_1} n^i \\
\end{align*}
\]

(43)

where \( \tilde{v}^i \) is the principal normal vector field of \( \tilde{F} \), \( g_{ij}\tilde{v}^i\tilde{v}^j = 1 \) and \( \tilde{M}_1 \) is the first curvature of \( \tilde{F} \).

\( \tilde{v}^i \) is the binormal vector field of \( \tilde{F} \) and it is defined in the form:

\[
\tilde{v}^i = c_{ijk} \tilde{v}^j \tilde{v}^k \\
\]

(44)

If (38) and (43) are used in (44), we have

\[
\begin{align*}
\tilde{v}^i & = -\frac{1}{\mathbf{M}_1} n^i + \frac{k_2}{\mathbf{M}_1} k_1 n^i \\
\end{align*}
\]

(45)

Taking the prolonged covariant derivative of (45) in the direction of \( v_1 \), we get
\[ v_{k}^{1}v_{3}^{i} \left( \frac{v_{k}^{1}v_{3}^{i}}{v_{k}^{1}} \right) b = -\bar{M}_{2}v_{i}^{j}(-k_{1}) \]

\[ = -\left( \frac{v_{k}^{1}v_{k}^{1}}{M_{1}} \right) n_{i}^{j} + \left( \frac{v_{k}^{1}v_{k}^{1}}{M_{1}} \right) k_{2} n_{i}^{j} + \frac{1}{M_{1}} \left( \frac{v_{k}^{1}v_{k}^{1}}{k_{1}} \right) n_{i}^{j} \]

and multiplying (46) by \( g_{ij}v_{i}^{j} \), we obtain

\[ \bar{M}_{2} = -\frac{1}{[M_{1}]^{2}k_{1}} \left( \frac{v_{k}^{1}v_{k}^{1}}{k_{1}} \right) = -\frac{k_{1}}{k_{1}^{2} + k_{2}^{2}} v_{k}^{1}v_{k}^{1} \]

or

\[ \bar{M}_{2} = -\frac{g_{ij}c^{n}n_{i}^{j}}{[g_{ij}c^{n}n_{i}^{j}]} \left( \frac{v_{k}^{1}v_{k}^{1}}{g_{ij}c^{n}n_{i}^{j}} \right) \bar{v}_{k}^{i} \]

where \( \bar{M}_{2} \) is the second curvature of \( \bar{F} \).

\[ \text{Corollary 2. Let } \bar{F} \text{ be a Bishop spherical image of } C. \text{ If } \frac{k_{1}}{k_{2}} = \text{constant}, \text{ then } n_{i}^{j} \text{ Bishop spherical image } f^{i} = f^{i}(\bar{s}) \text{ is a circle.} \]

\[ \text{Proof. By using Proposition 3, we get} \]

\[ v_{k}^{1}v_{k}^{1} \left( \frac{v_{k}^{1}v_{k}^{1}}{v_{k}^{1}} \right) = v_{k}^{1}v_{k}^{1} \left( \frac{v_{k}^{1}v_{k}^{1}}{v_{k}^{1}} \right) \]

\[ = (v_{k}^{1}v_{k}^{1}k_{1}) n_{i}^{j} + (v_{k}^{1}v_{k}^{1}k_{2}) n_{i}^{j} \]

\[ - (k_{1}^{2} + k_{2}^{2}) v_{i}^{j} \]

and

\[ g_{ij} \left( v_{k}^{1}v_{k}^{1} \right) \left( v_{k}^{1}v_{k}^{1} \right) = g_{ij} \left( v_{k}^{1}v_{k}^{1} \right) \left( v_{k}^{1}v_{k}^{1} \right) \]

\[ = (k_{1}^{2} + k_{2}^{2}) v_{i}^{j} \]

Summing (49) and (50), we have

\[ \frac{1}{v_{k}^{1}v_{k}^{1}} v_{k}^{1}v_{k}^{1} + g_{ij} \left( v_{k}^{1}v_{k}^{1} \right) \left( v_{k}^{1}v_{k}^{1} \right) \]

\[ = \left( v_{k}^{1}v_{k}^{1}k_{1} \right) n_{i}^{j} + \left( v_{k}^{1}v_{k}^{1}k_{2} \right) n_{i}^{j} \]

Using \( k_{1} = k_{1} \cos \theta \) and \( k_{2} = k_{1} \sin \theta \), we obtain

\[ v_{k}^{1}v_{k}^{1}k_{1} = v_{k}^{1}v_{k}^{1} \left( k_{1} \cos \theta \right) \]

\[ = \left( v_{k}^{1}v_{k}^{1}k_{1} \right) \cos \theta - k_{1} \left( v_{k}^{1}v_{k}^{1} \right) \sin \theta \]

and

\[ v_{k}^{1}v_{k}^{1}k_{2} = v_{k}^{1}v_{k}^{1} \left( k_{1} \sin \theta \right) \]

\[ = \left( v_{k}^{1}v_{k}^{1}k_{1} \right) \sin \theta + k_{1} \left( v_{k}^{1}v_{k}^{1} \right) \cos \theta \]

where \( \theta = \theta(s) = \arccot \frac{k_{1}}{k_{2}} \) and \( v_{k}^{1}v_{k}^{1} \theta = \frac{v_{k}^{1}v_{k}^{1}k_{1}}{1 + \frac{k_{1}^{2}}{k_{2}^{2}}} \).

It is known that \( k_{1} = \sqrt{k_{1}^{2} + k_{2}^{2}} \). In this case, we obtain

\[ v_{k}^{1}v_{k}^{1}k_{1} = v_{k}^{1}v_{k}^{1} \sqrt{k_{2}^{2}} \left( \frac{k_{1}^{2}}{k_{2}^{2}} + 1 \right) \]

\[ = \frac{1}{\sqrt{k_{1}^{2} + k_{2}^{2}} - k_{2} \sin \theta \left( \frac{k_{1}^{2}}{k_{2}^{2}} + \frac{k_{2}^{2}}{k_{2}^{2}} \right)} )} \]

\[ \cdot \left( k_{2}k_{1} \left( v_{k}^{1}v_{k}^{1} \cos \theta \left( \frac{k_{1}^{2}}{k_{2}^{2}} + \frac{k_{2}^{2}}{k_{2}^{2}} \right) \right) \]

\[ + k_{2}k_{1} \left( v_{k}^{1}v_{k}^{1} \right) \right) \]

If \( k_{2} = \text{constant} \), we have \( v_{k}^{1}v_{k}^{1} \theta = 0 \) and \( v_{k}^{1}v_{k}^{1} = 0 \). This means \( v_{k}^{1}v_{k}^{1}k_{1} = 0 \) and \( v_{k}^{1}v_{k}^{1}k_{2} = 0 \). Using these results in (51),

\[ v_{k}^{1}v_{k}^{1} \left( v_{k}^{1}v_{k}^{1} \right) + g_{ij} \left( v_{k}^{1}v_{k}^{1} \right) \left( v_{k}^{1}v_{k}^{1} \right) = 0 \]

is obtained. So, it is shown that \( \bar{F} : f^{i} = f^{i}(\bar{s}) \) is a circle. \( \square \)

\[ \text{Theorem 3. Let } \bar{F} : f^{i} = f^{i}(\bar{s}) \text{ be a Bishop spherical image of } C. \text{ If } \bar{F} \text{ is a general helix, then} \]

\[ \frac{k_{1}^{2}}{(k_{1}^{2} + k_{2}^{2})^{3/2}} \left( \frac{v_{k}^{1}v_{k}^{1}k_{2}}{k_{1}} \right) = \text{constant} \]

is valid.

\[ \text{Theorem 4. Let } \bar{F} : f^{i} = f^{i}(\bar{s}) \text{ be a Bishop spherical image of } C. \text{ If } \bar{F} \text{ is a slant helix, then} \]

\[ \frac{k_{1}^{2}}{(k_{1}^{2} + k_{2}^{2})^{3/2}} \left( \frac{v_{k}^{1}v_{k}^{1}k_{2}}{k_{1}} \right) = \text{constant}. \]

Since \( \bar{F} : f^{i} = f^{i}(\bar{s}) \) is a spherical curve, by using Proposition 2, we can state the following theorem:

\[ \text{Theorem 5. Let } \bar{F} : f^{i} = f^{i}(\bar{s}) \text{ be a Bishop spherical image of } C. \text{ In this case, the following equation} \]

\[ \frac{k_{1}^{2}}{(k_{1}^{2} + k_{2}^{2})^{3/2}} \left( \frac{v_{k}^{1}v_{k}^{1}k_{2}}{k_{1}} \right) - \frac{k_{1}k_{2}}{\sqrt{k_{1}^{2} + k_{2}^{2}}} = \text{constant}. \]

is valid.

\[ \text{Theorem 6. Let } \bar{F} : f^{i} = f^{i}(\bar{s}) \text{ be a Bishop spherical image of } C. \text{ If } \frac{k_{2}}{k_{1}} = \text{constant}, \text{ then } \bar{v} \text{ is parallel translated along } \bar{v}. \]

\[ \text{Proof. From (46), it is known that} \]
Taking the norm of both sides of (62), we have
\[ c = \mp k_2. \]
Let us choose \( c = -k_2. \) Then, we obtain
\[ \bar{v}' = v^i. \]
By taking prolonged covariant derivative of (64) in the direction of \( v' \), we have
\[ v'^k v_k \bar{v}'^l = (\bar{v}'^k v_k^i) c = v'^k v_k^i \]
\[ H_1 \bar{v}'^2(-k_2) = k_1 n^i + k_2 n^i \]
\[ H_1 \bar{v}'^2 = -\frac{k_1 n^i}{k_2} - n^i. \]
By taking the norm of both sides of (67), we get
\[ H_1 = \frac{1}{H_1} \left( \frac{k_1}{k_2} \right)^2 + 1 = \left[ \frac{g_{ij} c' n^i}{g_{ij} c' n^j} \right]^2 + 1 \]
and
\[ \tilde{v}'^i = -\frac{1}{H_1} \frac{k_1 n^i}{k_2} - \frac{1}{H_1} n^i. \]
where \( \tilde{v}'^i \) is the principal normal vector field of \( \tilde{G} \), \( g_{ij} \tilde{v}'^i \tilde{v}'^j = 1 \) and \( H_1 \) is the first curvature of \( \tilde{G} \).
\[ \tilde{v}'_3^i \] is the binormal vector field of \( \tilde{G} \) and it is expressed as follows:
\[ \tilde{v}'_3^i = \epsilon_{ijk} \tilde{v}'^j \tilde{v}'^k. \]
By using (64) and (69) in (70), we have
\[ \tilde{v}'^i = -\frac{1}{H_1} \frac{k_1 n^i}{k_2} - \frac{1}{H_1} n^i. \]
Taking the prolonged covariant derivative of (71) in the direction of \( v'_1 \), we obtain
\[ v'^k v_k^i \bar{v}'^3 = (\bar{v}'^k v_k^i) c = -H_2 \bar{v}'^2(-k_2) \]
\[ = -\left( v'^k v_k^i \frac{1}{H_1} k_1 \right) \frac{k_2 n^i}{H_2} \]
\[ -\frac{1}{H_1} \left( v'^k v_k^i \frac{k_1}{H_1} \right) n^i \]
where \( \bar{v}'^i \) is the tangent vector field of \( \tilde{G} \), \( g_{ij} \bar{v}'^i \bar{v}'^j = 1 \) and \( c = c(s) \).
Theorem 7. Let \( g'^i = g'^i(\tilde{s}) \) be \( \frac{\eta}{2} \) Bishop spherical image of \( C \). If \( \tilde{G} \) is a general helix, then

\[
\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}} k_1 k_2 \left( \frac{v^k \nabla_k}{\sqrt{k_1^2 + k_2^2}} \right) = \text{constant}
\]

is valid.

Since \( \tilde{G} : g'^i = g'^i(\tilde{s}) \) is spherical curve, with the help of Proposition 2, we can state the following theorem:

Theorem 8. Let \( \tilde{G} : g'^i = g'^i(\tilde{s}) \) be \( \frac{\eta}{2} \) Bishop spherical image of \( C \). If \( \tilde{G} \) is a slant helix, then

\[
\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}} \left( \frac{v^k \nabla_k}{\sqrt{k_1^2 + k_2^2}} \right) = \text{constant}
\]

is valid.

5. Conclusions

In this work, we have defined some special curves, such as general helix, slant helix, spherical curve and circle in Weyl space, by using prolonged covariant derivative. Later, we have examined the spherical images of a curve and the conditions to be special curves of these images. Further, we expressed the relations among the vector fields of Frenet-Serret and Bishop frame belonging to the images. We have seen that, all of the concepts discussed can be written in terms of Bishop curvatures. Earlier, the equivalents in Weyl space of Bishop curvatures were obtained. Thus, they were used in this discussion. Finally, parallel displacement condition for binormal vector fields of \( \frac{\eta}{2} \) and \( \frac{\eta}{\eta} \) spherical images was formulated.

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