Hölder regularity for fractional $p$-Laplace equations

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MS received 25 December 2022; accepted 2 March 2023

Abstract. We give an alternative proof for Hölder regularity for weak solutions of nonlocal elliptic quasilinear equations modelled on the fractional $p$-Laplacian where we replace the discrete De Giorgi iteration on a sequence of concentric balls by a continuous iteration. This work can be viewed as the nonlocal counterpart to the ideas developed by Tiziano Granucci.

Keywords. Nonlocal operators; weak solutions; Hölder regularity; De Giorgi isoperimetric inequality.

2010 Mathematics Subject Classification. 35K51, 35A01, 35A15, 35R11.

1. Introduction

In this article, we give an alternative proof of local Hölder regularity for weak solutions to fractional elliptic equations modelled on the fractional $p$-Laplacian denoted by

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy = 0.$$ 

The proof is based on the techniques developed in [13] which in turn is based on the ideas developed in [22].

The De Giorgi approach to the proof of Hölder continuity relies on two steps, colloquially referred to as “shrinking lemma” and “measure to pointwise estimate”.

In the local case, the “shrinking lemma” is achieved through an application of De Giorgi isoperimetric inequality. However, the Sobolev–Slobodetskii space of $W^{s,p}$ functions may have “jumps”. As a result, the standard De Giorgi isoperimetric inequality, also sometimes called the “no jump lemma” does not hold for them. Cozzi [7] proved a version of De Giorgi isoperimetric inequality for $s$ close to 1 and employed it to obtain a proof of Hölder regularity for operators whose prototype is the fractional $p$-Laplacian which is stable as $s \to 1$. For $s$ away from 1, Cozzi relied on the so-called “good term” in the energy
estimates. Indeed, Cozzi defined a De Giorgi class for nonlocal operators which has a “good term” on the left-hand side of his energy estimate. It is an open problem whether a better form of De Giorgi isoperimetric inequality can be proved in the fractional case. This problem is avoided in the earlier paper [10] by relying on a logarithmic estimate in addition to a basic Caccioppoli inequality.

On the other hand, the “measure to uniform estimate” is achieved through a De Giorgi iteration of concentric balls in both [10] and [7]. The main novelty of this paper is that we avoid the use of De Giorgi iteration in the proof of Hölder regularity in [10] by using an oscillation theorem in the spirit of [13, 22] which replaces the discrete iteration on an infinite sequence of concentric balls with a continuous iteration procedure.

We state the oscillation theorem below:

**Theorem 1 (Oscillation theorem).** Assume that $u$ is a locally bounded weak solution of (2.2) in the ball $B_R(x_0)$. If

$$
|\{u \leq 0\} \cap B_r| \geq \frac{1}{2}|B_r|,
$$

for some ball $B_r \subset B_{4r} \subset B_R(x_0)$, then

$$
\sup_{B_r} u_+ \leq C_{\kappa} \left( \frac{|\{u > 0\} \cap B_{2r}|}{|B_{2r}|} \right)^{\gamma} \left( \sup_{B_{4r}} u_+ + \text{Tail}(u_+, x_0, 2r) \right) + \kappa \text{Tail}(u_+; x_0, r),
$$

where $\gamma = \frac{p-\delta}{p\delta}$ for some fixed $\delta \in (0, p)$. The constant $\kappa$ is an arbitrary number in $(0, 1]$ and $C_{\kappa} > 0$ depends on $N, p, s, \Lambda, \kappa$ and $\delta$.

**Remark 2.** The constant $C_{\kappa}$ blows up as $\kappa \to 0$. The quantity $\kappa \in (0, 1]$ allows for an interpolation between the local and the nonlocal terms.

Using the oscillation theorem, we prove the following Hölder regularity result.

**Theorem 3.** Assume that $u$ is a locally bounded weak solution of (2.2) in the ball $B_2(x_0) \subset \Omega$. Then there exist constants $\alpha \in (0, 1)$ and $C > 0$ depending only on the data such that for any $y \in B_r(x_0)$, the following estimate holds:

$$
|u(x_0) - u(y)| \leq C \left( \frac{|x_0 - y|}{r} \right)^{\alpha} \left( 2\text{Tail}(u, x_0, r) + 4||u||_{L^\infty(B_{2r}(x_0))} \right).
$$

A second point of interest in this paper is a new De Giorgi isoperimetric inequality for $W^{s,p}$ functions in the spirit of [19]. As expected, the resulting inequality has a “jump term” which limits its applicability in extracting a “shrinking lemma” from it. However, this partially answers a question raised by Cozzi in [8] and we hope that this may lead to a new proof of Hölder regularity in the future.

**Theorem 4 (Fractional De Giorgi isoperimetric inequality).** Let $R > 0, u \in W^{s,p}(B_R)$ with $s \in (0, 1), p \geq 1$ and $k, l \in \mathbb{R}$ be two levels such that $k < l$. Then
\[(l - k) \left| \{ x \in B_R : u(x) \leq k \} \right| \left| \{ x \in B_R : u(x) \geq l \} \right| \leq CR^{N+s} \left| A_k^l \right|^{p-1} \frac{1}{p} \left( \int_{A_k^l} \left( \int_{A_k^l} \frac{|u(x) - k|^p}{|x - y|^{N+sp}} \right)^{1/p} + \left( \int_{A_k^l} \left( \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \right)^{1/p} \right)^{1/p} \right) + CR^{N+s} \left| A_l^+ \right|^{p-1} \frac{1}{p} \left( \int_{A_l^+} \left( \int_{A_l^+} \frac{(l - k)^p}{|x - y|^{N+sp}} \right)^{1/p} \right) \right., \quad (1.3)

where \( A_k^l = \{ x \in B_R : k < u(x) < l \} \), \( A_l^+ = \{ x \in B_R : u(x) > l \} \), and \( A_k^- = \{ x \in B_R : u(x) > l \} \).

1.1 History of the problem

Much of the early work on regularity of fractional elliptic equations in the case \( p = 2 \) was carried out by Silvestre [21], Caffarelli and Vasseur [4], Caffarelli et al. [5] and also by Bass and Kassmann [1, 2, 18]. An early formulation of the fractional \( p \)-Laplace operator was done by Ishii and Nakamura [17] and the existence of viscosity solutions was established. DiCerchio et al. [10] extended the De Giorgi–Nash–Moser framework to study the regularity of the fractional \( p \)-Laplace equation. The subsequent work of Cozzi [7] covered a stable (in the limit \( s \to 1 \)) proof of Hölder regularity by defining a novel fractional De Giorgi class. Explicit exponents for Hölder regularity were found in [3] and other works of interest are found in [6, 9, 16].

2. Notations and preliminaries

In this section, we will fix the notation, provide definitions and state some standard auxiliary results that will be used in subsequent sections.

2.1 Notations

We begin by collecting the standard notation that will be used throughout the paper.

- The number \( N \geq 1 \) denotes the space dimension.
- Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \).
- We shall use the notation

\[
B_\rho(x_0) = \{ x \in \mathbb{R}^N : |x - x_0| < \rho \},
\]

\[
\overline{B}_\rho(x_0) = \{ x \in \mathbb{R}^N : |x - x_0| \leq \rho \}.
\]
Integration with respect to space will be denoted by a single integral \( \int \) whereas integration on \( \Omega \times \Omega \) or \( \mathbb{R}^N \times \mathbb{R}^N \) will be denoted by a double integral \( \int \int \).

The notation \( a \lesssim b \) is shorthand for \( a \leq Cb \), where \( C \) is a universal constant which only depends on the dimension \( N \), exponent \( p \), and the numbers \( \Lambda, s \).

For a function \( u \) defined on \( B_{r}(x_0) \) and any level \( k \in \mathbb{R} \), we write \( w_\pm = (u - k)_\pm \).

We denote \( A_\pm(k) = \{w_\pm > 0\} \); for any ball \( B_r \), we write \( A_\pm(k) \cap (B_r) = A_\pm(k, r) \).

Let \( K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty) \) be a symmetric measurable function satisfying
\[
\frac{(1 - s)}{\Lambda |x - y|^{N + ps}} \leq K(x, y) \leq \frac{(1 - s)\Lambda}{|x - y|^{N + ps}} \quad \text{for almost all } x, y \in \mathbb{R}^N, \tag{2.1}
\]
for some \( s \in (0, 1) \), \( p > 1 \), \( \Lambda \geq 1 \).

In this paper, we are interested in the regularity theory for the operator \( \mathcal{L} \) defined formally by
\[
\mathcal{L}u = \text{P.V.} \int_{\mathbb{R}^N} K(x, y) |u(x) - u(y)|^{p-2}(u(x) - u(y))dy, \quad x \in \mathbb{R}^N.
\]

### 2.2 Function spaces

Let \( 1 < p < \infty \). We denote by \( p' = p/(p - 1) \) the conjugate exponent of \( p \). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). We define the Sobolev–Slobodeckiĭ space, which is the fractional analogue of Sobolev spaces.

\[
W^{s, p}(\Omega) = \left\{ \psi \in L^p(\Omega) : [\psi]_{W^{s, p}(\Omega)} < \infty \right\}, \quad s \in (0, 1),
\]
where the seminorm \([\cdot]_{W^{s, p}(\Omega)}\) is defined by
\[
[\psi]_{W^{s, p}(\Omega)} = \left( \iint_{\Omega \times \Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

The space when endowed with the norm \( \|\psi\|_{W^{s, p}(\Omega)} = \|\psi\|_{L^p(\Omega)} + [\psi]_{W^{s, p}(\Omega)} \) becomes a Banach space. The space \( W^{s, p}_0(\Omega) \) is the subspace of \( W^{s, p}(\mathbb{R}^N) \) consisting of functions that vanish outside \( \Omega \). We will use the notation \( W^{s, p}_{u_0}(\Omega) \) to denote the space of functions in \( W^{s, p}(\mathbb{R}^N) \) such that \( u - u_0 \in W^{s, p}_0(\Omega) \).

Since the regularity result requires some finiteness condition on the nonlocal tails, we define the tail space as follows:
\[
L^m_\alpha(\mathbb{R}^N) := \left\{ v \in L^m_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|v(x)|^m}{1 + |x|^{N+\alpha}} \, dx < +\infty \right\}, \quad m > 0, \alpha > 0.
\]
We define the nonlocal tail of a function $v$ in the ball $B_R(x_0)$ by
\[
\text{Tail}(v; x_0, R) := \left[ \frac{R^{sp}}{\|B_R(x_0)\|} \int_{\mathbb{R}^N \setminus B_R(x_0)} \frac{|v(x)|^{p-1}}{|x - x_0|^{N+sp}} dx \right]^{\frac{1}{p-1}},
\]
which is a finite number when $v \in W^{s,p}(\mathbb{R}^N)$ or when $v \in L_{sp}^{-1}(\mathbb{R}^N)$.

**Remark 5.** The definition of the tail space may be motivated by the fact that the existence result for the boundary value problem associated to the nonlocal operator
\[
\begin{cases}
Lu = 0 & \text{in } \Omega, \\
u = g & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{2.2}
\]
where $\Omega$ is a bounded domain, is posed in either the space $W^{s,p}(\mathbb{R}^N)$ or $W^{s,p}(\Omega') \cap L_{sp}^{-1}(\mathbb{R}^N)$ for $\Omega'$ satisfying $\Omega \subseteq \Omega'$. The existence and uniqueness in the first of these cases can be proved by the direct method of calculus of variations by considering the corresponding minimization problem with $g$ in $W^{s,p}(\mathbb{R}^N)$. In the second case, the proof of existence for $g \in W^{s,p}(\Omega') \cap L_{sp}^{-1}(\mathbb{R}^N)$ is outlined in [3, Proposition 2.12] by the standard theory of monotone operators, see [20]. The variational theory seems to require stronger assumptions on data $g$.

### 2.3 Definitions

Now, we are ready to state the definition of a weak sub(super)-solution.

**DEFINITION 6**

Let $g \in W^{s,p}(\mathbb{R}^N)$. A function $u \in W^{s,p}(\mathbb{R}^N)$ is said to be a weak solution to (2.2) if $u - g \in W^{s,p}_0(\Omega)$ and
\[
\iint_{C_\Omega} K(x, y)|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y)) = 0,
\]
for all $\phi \in W^{s,p}_0(\Omega)$, where $C_\Omega := (\Omega^c \times \Omega^c)^c = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$.

### 2.4 Auxiliary results

We recall the following well-known lemma concerning the geometric convergence of sequence of numbers (see [12, Lemma 4.1 from Section I] for details).

**Lemma 7.** Let $\{Y_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive numbers, satisfying the recursive inequalities
\[
Y_{n+1} \leq Cb^n Y_n^{1+\alpha},
\]
where $C > 1$, $b > 1$, and $\alpha > 0$ are the given numbers. If

$$Y_0 \leq C^{-\frac{1}{p'}} b^{-\frac{1}{p'}},$$

then $\{Y_n\}$ converges to zero as $n \to \infty$.

We recall the following energy estimate for weak solution to (2.2) whose proof may be found in [10, Theorem 1.4] and [7, Proposition 8.5].

**Theorem 8 (Caccioppoli inequality [10, Theorem 1.4]).** Let $p \in (1, \infty)$ and let $u \in W^{s,p}(\Omega) \cap L^{p-1}_s(\mathbb{R}^N)$ be a weak solution to (2.2). Then, for any $B_R(x_0) \subseteq \Omega$, the following estimate holds:

$$\int_{B_R(x_0) \times B_R(x_0)} K(x, y) |w_\pm(x)\phi(x) - w_\pm(y)\phi(y)|^p \, dx \, dy \leq C \int_{B_R(x_0) \times B_R(x_0)} K(x, y) \max\{w_\pm(x)^p, w_\pm(y)^p\} |\phi(x) - \phi(y)|^p \, dx \, dy + C \int_{B_R(x_0)} w_\pm \phi^p(x) \, dx \left( \sup_{y \in \text{supp}(\phi)} \int \mathbb{R}^N \setminus B_R(x_0) K(x, y) w_\pm^{p-1}(x) \, dx \right),$$

where $w_\pm(x) = (u - k)_\pm$ for any level $k \in \mathbb{R}$, $\phi \in C^\infty_c(B_R)$ and $C > 0$ only depends on $p$.

We state the logarithmic estimate which is essential in obtaining the “shrinking lemma”. Once again, the proof may be found in [10, Theorem 1.3].

**Theorem 9 (Logarithmic estimates [10, Theorem 1.3]).** Let $p \in (1, \infty)$ and let $u \in W^{s,p}(\Omega) \cap L^{p-1}_s(\mathbb{R}^N)$ be a supersolution to (2.2) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Then, for any $B_r \equiv B_r(x_0) \subseteq \Omega$ and any $d > 0$, the following estimate holds:

$$\int_{B_r(x_0)^2} K(x, y) \left| \log \left( \frac{d + u(x)}{d + u(y)} \right) \right|^p \, dx \, dy \leq C r^{N-sp} \left\{ d^{1-p} \left( \frac{r}{R} \right)^{sp} \left[ \text{Tail}(u--; x_0, R) \right]^{p-1} + 1 \right\},$$

where $u_-(x) = \max\{-u(x), 0\}$. $C > 0$ is a constant that depends only on $N, p, s, \Lambda$.

An immediate consequence of the logarithmic estimate is the following estimate.

**COROLLARY 10 [10, Corollary 3.2]**

Let $p \in (1, \infty)$ and let $u \in W^{s,p}(\Omega) \cap L^{p-1}_s(\mathbb{R}^N)$ be a solution to (2.2) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Let $a, d > 0$, $b > 1$ and define

$$v := \min\{(\log(a + d) - \log(u + d))_+, \log(b)\}.$$
Then the following estimate is true, for any $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$,

$$\int_{B_r} |u - (v)_{B_r}|^p \, dx \leq C \left\{ d^{1-sp} \left( \frac{r}{R} \right)^{sp} (\text{Tail}(u_-, x_0, R))^{p-1} + 1 \right\},$$

where $C$ is a constant that depends only on $N, p, s, \Lambda$.

We will need the following embedding result from [7].

**Theorem 11** [7, Lemma 4.6]. Let $N \in \mathbb{N}$ and $0 < \sigma < s < 1$. Let $\Omega \subset \mathbb{R}^N$ be bounded and measurable. Then for any $f \in W^{s,p}(\Omega)$ and $1 \leq q < p$, it holds that

$$\left[ \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^q}{|x - y|^N + q} \right]^{\frac{1}{q}} \leq C_0 |\Omega|^{\frac{p-q}{pq}} (\text{diam}(\Omega))^{s-\sigma} \left[ \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x - y|^N + ps} \right]^{\frac{1}{p}},$$

where $C_0 = \left[ \frac{N(p-q)}{(s-\sigma)pq} |B_1| \right]^{\frac{p-q}{pq}}$.

We have the following local version of the Sobolev–Poincaré-type inequality for which we refer to [11].

**Theorem 12** Let $B_r$ be a ball with radius $r$ and let $s \in (0, 1)$ and $1 \leq p, sp \leq N$ and let $1 \leq q \leq \frac{Np}{N-sp}$. Then for any $f \in W^{s,p}(B_r)$, we have

$$\left( \int_{B_r} \left| \frac{f - (f)_{B_r}}{r^s} \right|^q \right)^{\frac{1}{q}} \leq C(N, s, p) \left( \int_{B_r} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x - y|^N + ps} \right)^{\frac{1}{p}}. \quad (2.3)$$

Here we have taken $q \in (1, \infty)$ is any number in the case $sp = N$.

We will require the following version of Sobolev–Poincaré inequality where the function is zero on a large set.

**Theorem 13** [7, Corollary 4.9]. Let $B_r$ be a ball with radius $r$ and let $s \in (0, 1)$ and $1 \leq p, sp \leq N$. Let $f \in W^{s,p}(B_r)$ and suppose that $u = 0$ on a set $\Omega_0 \subseteq B_r$ with $|\Omega_0| \geq \gamma |B_r|$ for some $\gamma \in (0, 1]$. Then there is $q > p$ such that

$$\left( \int_{B_r} |f(x)|^q \, dx \right)^{\frac{1}{q}} \leq C(N, s, p, \gamma) r^{s-\frac{N(q-p)}{4pq}} \left( \int_{B_r} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x - y|^N + ps} \right)^{\frac{1}{p}}. \quad (2.4)$$

Here we have taken $q = p^* = \frac{Np}{N-sp}$, when $sp < N$ and $q \in [p, \infty)$ when $sp = N$. 
Proof. The proof of Theorem 13 in the case $sp < N$ can be found in [7, Corollary 4.9] and in the case of $sp = N$, we make use of [11, Theorem 6.9] and follow the same strategy of the proof of [7, Corollary 4.9].

2.5 Main results

We prove the following main theorem.

**Theorem 14.** Let $p \in (1, \infty)$ and let $u \in W^{s,p}(\Omega) \cap L_{sp}^{p-1}(\mathbb{R}^N)$ be a solution to (2.2). Then $u$ is locally Hölder continuous in $\Omega$.

3. A quantified boundedness theorem

We prove the following quantified version of local boundedness estimate for weak solution to (2.2). The proof is similar to the one in [10, Theorem 1.1] but we present it here for completeness and in order to properly track the constants.

**Proposition 15.** If $u \in W^{s,p}_{\text{loc}}(\mathbb{R}^N)$ is a weak subsolution to (2.2), $B_R(x_0) \subset \Omega$ and $t \in (0, 1)$. Then

$$\sup_{B_{tR}(x_0)} u \leq C b \beta^{\frac{1}{p}} \Gamma \frac{N}{sp} \frac{1}{sp^{\frac{(p-1)N}{sp}}} \left( \int_{B_R(x_0)} u_{++}^p(x)dx \right)^{\frac{1}{p}}$$

$$+ \kappa \left( R^{sp} \int_{\mathbb{R}^N \setminus B_R} \frac{u_{++}^{p-1}}{|x-x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}}$$

where $\Gamma := \left(1 + \frac{1}{(1-t)^p} + \frac{1}{x^{N+sp}}\right)$. $\beta = \frac{sp}{N-sp}$, $\kappa \in (0, 1)$ is an arbitrary number, and $b > 1$, $C$ are positive numbers depending only on $N, s, p, \Lambda$.

**Proof.** We begin by defining the following quantities:

$$r_j = tR + \frac{1-j}{2j}R, \quad R_0 = R, \quad R_{\infty} = tR,$$

$$\tilde{r}_j = \frac{r_j+r_{j+1}}{2}, \quad B_j = B_{r_j} \quad \text{and} \quad \tilde{B}_j = B_{\tilde{r}_j}.$$ Further, take $\phi_j \in C_0^\infty(\tilde{B}_j)$, $0 \leq \phi_j \leq 1$, $\phi \equiv 1$ on $B_{j+1}$ such that $|\nabla \phi_j| \leq \frac{1}{R_j-R_{j+1}} \leq \frac{2j+2}{(1-t)R}$. Also, for a fixed $\tilde{k} > 0$ to be chosen later, we define

$$k_j = (1-2^{-j})\tilde{k}, \quad \tilde{k}_j = \frac{k_j+k_{j+1}}{2}, \quad w_j = (u-k_j)_+, \quad \tilde{w}_j = (u-\tilde{k}_j)_+.$$

By Caccioppoli inequality (Theorem 8), we get

$$\int_{B_j} \int_{B_j} K(x, y)|\tilde{w}_j(x)\phi_j(x) - \tilde{w}_j(y)\phi_j(y)|^p dxdy \leq \frac{1}{1}$$
\[
\begin{align*}
\leq \int_{B_j} \int_{B_j} K(x, y) \max \{\tilde{w}_j(x), \tilde{w}_j(y)\}^p |\phi_j(x) - \phi_j(y)|^p \, dx \, dy \\
+ \left( \int_{B_j} \tilde{w}_j(y) \phi_j(y) \, dy \right) \left( \sup_{y \in \text{spt } \phi_j} \int_{\mathbb{R}^N \setminus B_j} K(x, y) \tilde{w}_j^{p-1} \, dx \right).
\end{align*}
\]

We first estimate II as follows:

\[
II \overset{(2.1)}{\lesssim} \int_{B_j} \int_{B_j} \max \{\tilde{w}_j(x), \tilde{w}_j(y)\}^p |\phi_j(x) - \phi_j(y)|^p \frac{dx \, dy}{|x - y|^{N + sp}}
\]

\[
\overset{(a)}{\lesssim} \frac{2^{j+2}}{(1 - \eta) R} \left( \int_{B_j} w_j^p(x) \, dx \right) \left( \sup_{x \in \text{spt } \phi_j} \int_{B_j(x_0)} \frac{1}{|x - y|^{N + sp - p}} \, dy \right) \quad (3.2)
\]

\[
\overset{(b)}{\lesssim} \frac{2^{j+2}}{(1 - \eta) R} \left( \int_{B_j} w_j^p(x) \, dx \right) \left( \int_{B_j} w_j^p(x) \, dx \right) \quad (3.2)
\]

\[
\overset{(c)}{\lesssim} \frac{2^{j+2}}{(1 - \eta) R^p} \left( \int_{B_j} w_j^p(x) \, dx \right),
\]

where to obtain (a), we have used that \(|\nabla \phi_j| \leq \frac{2^{j+2}}{(1 - \eta) R} \); in (b), we have used a standard calculation by conversion to polar coordinates that yields

\[
\left( \sup_{x \in \text{spt } \phi_j} \int_{B_j(x_0)} \frac{1}{|x - y|^{N + sp - p}} \, dy \right) \leq R_j^{(1-s)p}
\]

and for (c), we use the fact that \(R_j \leq R\) for all \(j \in \mathbb{N}\).

We now estimate III as follows:

\[
III \overset{(a)}{\lesssim} \frac{2^{j(N + sp)}}{(1 - \eta)^{N + sp}} \left( \int_{B_j} \frac{w_j^p(y)}{(k_j - \tilde{k}_j)^{p-1}} \, dy \right) \left( \int_{\mathbb{R}^N \setminus B_j} \frac{w_j^{p-1}}{|x - x_0|^{N + sp}} \, dx \right)
\]

\[
\overset{(b)}{\lesssim} \frac{2^{j(N + sp + p - 1)}}{k^{p-1}(1 - \eta)^{N + sp} R_j^{sp}} \left( \int_{B_j} w_j^p(y) \, dy \right) \left( \frac{1}{R_j^{sp}} \int_{\mathbb{R}^N \setminus B_j} \frac{u_j^{p-1}}{|x - x_0|^{N + sp}} \, dx \right),
\]

where in (a), we have used the fact that \(\tilde{w}_j \leq \frac{w_j}{(k_j - \tilde{k}_j)^{p-1}}\), and also since \(x \in \mathbb{R}^N \setminus B_j\) and \(y \in \text{spt } \phi_j = \tilde{B}_j\), we have

\[
\frac{|y - x_0|}{|x - y|} \leq 1 + \frac{|x - x_0|}{|x - y|} \leq 1 + \frac{r_j}{r_j - \tilde{r}_j} \leq \frac{2^{j+3}}{(1 - \eta)}.
\]

To obtain (b), we use the definitions in (3.1).

\textbf{Estimate for I.} In order to estimate I, we note that \(k_{j+1} - \tilde{k}_j = \frac{\tilde{k}_j}{2^{j+2}}\) and

\[
\int_{B_j} |(u - \tilde{k}_j) + \phi_j|^p \, dx \geq (k_{j+1} - \tilde{k}_j)^{p - p} \int_{B_{j+1}} w_j^p(x) \, dx,
\]
which together implies
\[
\left( \frac{\tilde{k}}{2j+2} \right)^{p^*-p} \int_{B_{j+1}} w_{j+1}^p(x) \, dx \leq \int_{B_j} |\tilde{w}_j \phi_j|^{p^*} \, dx. \tag{3.4}
\]

We now proceed with the estimate of I. Define \( g := \tilde{w}_j \phi_j \), then for \( s p < N \), we can apply (2.3) to get
\[
\left( \int_{B_j} |g - \bar{g}|^{p^*} \, dx \right)^{\frac{1}{p^*}} \lesssim \left( \int_{B_j} \left( \int_{B_j} \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}} \, dy \right)^{\frac{1}{p}} \right)^{\frac{1}{p^*}}, \tag{3.5}
\]
where \( p^* := \frac{Np}{N-sp} \) and \( \bar{g} := \frac{1}{B_j} \int g(x) \, dx \). By triangle inequality, we have
\[
\left( \int_{B_j} |g|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq \left( \int_{B_j} |g - \bar{g}|^{p^*} \, dx \right)^{\frac{1}{p^*}} + \left( \int_{B_j} |g| \, dx \right)^{\frac{1}{p^*}}. \tag{3.6}
\]
Hence
\[
A_{j+1}^p := \int_{B_{j+1}} w_{j+1}^p \, dx \tag{3.7}
\]
\[
\leq \left( \frac{2j+2}{\tilde{k}} \right)^{p^*-p} \int_{B_j} |\tilde{w}_j \phi_j|^{p^*} \, dx \tag{3.4}
\]
\[
\leq \left( \frac{2j+2}{\tilde{k}} \right)^{p^*-p} \left\{ \left( \int_{B_j} |g - \bar{g}|^{p^*} \, dx \right) \left( \int_{B_j} |g| \, dx \right)^{p^*} \right\} \tag{3.6}
\]
\[
\lesssim \left( \frac{2j+2}{\tilde{k}} \right)^{p^*-p} \left\{ \left( \frac{r_j^{sp}}{r_j^N} \int_{B_j} \int_{B_j} \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}} \, dy \, dx \right)^{\frac{p^*}{p}} + \left( \int_{B_j} |g| \, dx \right)^{\frac{p^*}{p}} \right\} \tag{a}
\]
\[
\lesssim \left( \frac{2j+2}{\tilde{k}} \right)^{p^*-p} \left\{ \left( \frac{r_j^{sp}}{r_j^N} \int_{B_j} \int_{B_j} \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}} \, dy \, dx \right)^{\frac{p^*}{p}} + \left( \int_{B_j} \int_{B_j} \frac{|g|^p}{|x - y|^{N+sp}} \, dy \, dx \right)^{\frac{p^*}{p}} \right\} + A_{j+1}^{p^*} \tag{b}
\]
\[
\leq \left( \frac{2j+2}{\tilde{k}} \right)^{p^*-p} \left\{ \left( \frac{r_j^{sp}}{r_j^N} \int_{B_j} \int_{B_j} \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}} \, dy \, dx \right)^{\frac{p^*}{p}} + \left( \int_{B_j} \int_{B_j} \frac{|g|^p}{|x - y|^{N+sp}} \, dy \, dx \right)^{\frac{p^*}{p}} \right\} + A_{j+1}^{p^*} \tag{3.8}
\]
where to obtain (a), we made use of (3.5) along with Hölder’s inequality and to obtain (b), we made use of the Caccioppoli inequality along with the definition of \( A_j \).

**Estimate for II.** We further estimate II from (3.2) to get
\[
\frac{r_j^{sp}}{r_j^N} \int_{B_j} w_j^p \, dx \leq \frac{2j+2}{(1-t)R^*} \left( \int_{B_j} w_j^p (x) \, dx \right) \leq \frac{2(j+2)^p}{(1-t)^p} A_j^p, \tag{3.9}
\]
where we recall \( r_j \leq R \) for all \( j \in \mathbb{N} \).
**Estimate for III.** In order to estimate III appearing in (3.3), we proceed as follows:

\[
\frac{r_j^{sp}}{r_j^N} \cdot \text{III} \leq \frac{r_j^{sp}}{\tilde{r}_j} \cdot \frac{2^j (N+sp+p-1)}{k^p} \left( \int_{B_j} w_j^p (y) dy \right) \cdot \left( R^{sp} \int_{\mathbb{R}^N \setminus B_{R_j}} \frac{u_0^{p-1}}{|x - x_0|^{N+sp}} dx \right) \cdot \left( \frac{2^j (N+sp+p-1)}{\kappa^{p-1}} \int_{B_j} w_j^p (y) dy \right) \cdot \left( \frac{2^j (N+sp+p-1)}{\kappa^{p-1}} \right)^{A_j^p},
\]

(3.10)

where to obtain (a), we make the choice

\[
\tilde{k} \geq \frac{1}{\kappa} \left( R^{sp} \int_{\mathbb{R}^N \setminus B_{R_j}} \frac{u_0^{p-1}}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p^2-1}}.
\]

Substituting (3.9) and (3.10) in (3.7), we get

\[
A_{j+1} \leq C \left( \frac{2^j + 2}{\tilde{k}} \right)^{\frac{p^*}{p}} \frac{2^j (N+sp+p-1)}{p^*} \left( \frac{(1-p)^p}{p^2} \right)^{A_j^p} \left( \frac{1}{(1-t)} + \frac{1}{(1-t)^{N+sp}} \right)^{\frac{p^*}{p^2}} A_j^p,
\]

where we also use the fact that \( \kappa \in (0, 1) \) and \( C \) is a constant only depending on \( s, p, N \).

We rewrite the iterative inequality as

\[
\frac{A_{j+1}}{k} \leq C b^j \kappa^{-1} \left( \frac{1}{\Gamma} \right)^{p^*} \left( \frac{A_j}{k} \right)^{1+\beta},
\]

where we have defined \( b := 2 \frac{(N+sp+p-1)}{p(N-sp)} + sp \frac{sp}{N-sp} \) so that \( b \geq 1 \);

\[
\Gamma := \left( 1 + \frac{1}{(1-t)^p} + \frac{1}{(1-t)^{N+sp}} \right) \quad \text{and} \quad \beta := \frac{p^*}{p} - 1 = \frac{sp}{N-sp}.
\]

We are now in a position to apply Theorem 7 which implies \( A_j \rightarrow 0 \) as \( j \rightarrow \infty \) if we choose

\[
\frac{A_0}{k} \leq \left( C \kappa \frac{1}{p^2} \Gamma \right)^{-\frac{1}{p^2}} b - \frac{1}{p^2}.
\]

Thus, we may choose

\[
\tilde{k} = \kappa \left( R^{sp} \int_{\mathbb{R}^N \setminus B_{R_j}} \frac{u_0^{p-1}}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p^2-1}} + C b^p \kappa^{-1} \left( \frac{1}{p^2} \right)^{\frac{N}{sp^2}} A_0,
\]

where we also use the fact that \( \kappa \in (0, 1) \) and \( C \) is a constant only depending on \( s, p, N \).
to conclude that

\[
\sup_{B_tR(x_0)} u \leq b^\frac{1}{p} \Gamma^{\frac{N}{np^2}} \mathcal{H}^{-(p-1)N} \left( \int_{B_{R}(x_0)} u^p_+(x)dx \right)^{\frac{1}{p}} + \mathcal{H} \left( R^{sp} \int_{\mathbb{R}^n \setminus B_tR} \frac{u^{p-1}_+}{|x-x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}}.
\]

\[\square\]

4. An iterated boundedness result

In this section, we prove an iterated boundedness result.

**Theorem 16.** Let \( t \in (0, 1) \). If \( u \in W^{s,p}_{\text{loc}}(\mathbb{R}^N) \) is a weak subsolution to (2.2), \( B_R(x_0) \subset \Omega \) and \( \delta > 0 \) and \( \mathcal{H} \in (0, 1] \), then

\[
\sup_{B_{\delta tR}(x_0)} u \leq C \mathcal{H} \frac{\Gamma^{N}}{\Gamma^s} \left( \int_{B_{R}(x_0)} u^\delta_+(x)dx \right)^{\frac{1}{\delta}} + \mathcal{H} \text{Tail}(u_+, x_0, tr),
\]

where \( C \) depends on \( N, s, p, \delta \) and \( t \).

**Proof.** Note that we only need to prove the result for \( \delta < p \) as the other case follows directly from Theorem 15 and Holder’s inequality.

Let \( R_0 > 0 \) be fixed and \( t \in (0, 1) \) such that \( tR_0 < R_0 < R \). Define the following quantities:

\[
R_i := tR_0 + (1-t)R_0 \sum_{j=1}^{i} \frac{1}{2^j},
\]

\[
t_i := \frac{R_i}{R_{i+1}} = \frac{tR_0 + (1-t)R_0 \sum_{j=1}^{i} \frac{1}{2^j}}{tR_0 + (1-t)R_0 \sum_{j=1}^{i+1} \frac{1}{2^j}}.
\]

Therefore,

\[
1 - t_i = 1 - \frac{R_i}{R_{i+1}} = \frac{(1-t)R_0}{2^i+1} \frac{1}{tR_0 + (1-t)R_0 \sum_{j=1}^{i+1} \frac{1}{2^j}} = \left( \frac{1-t}{t + (1-t) \sum_{j=1}^{i+1} \frac{1}{2^j}} \right) \frac{1}{2^i+1}.
\]

Hence

\[
\frac{1}{1-t_i} \leq \frac{2^{i+1}}{(1-t)}.
\]
In consequence,
\[
\Gamma_i \leq \left(1 + \frac{2^p(i+1)}{(1-t)p} + \frac{2(N+sp)(i+1)}{(1-t)^{N+sp}}\right)
\]
\[
\leq C2^{i(N+sp+p)} \left(1 + \frac{1}{(1-t)p} + \frac{1}{(1-t)^{N+sp}}\right). \quad (4.1)
\]

We apply Theorem 15 with \(t\) and \(R\) replaced with \(t_i\) and \(R_i\) to obtain
\[
\sup_{B_{R_i}} u \leq Cb \frac{N}{i^{sp}} \kappa^{-(p-1)N} sp \left(\int_{B_{R_{i+1}}} u_+^p(x)dx\right)^{\frac{1}{p}}
\]
\[
+ \kappa \left( R_{i+1}^{sp} \int_{\mathbb{R}^n \setminus B_{R_i}} \frac{u_+^{p-1}(x)}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}}.
\]

We apply Young’s inequality to the first term with exponents \(\frac{p}{p-\delta}\) and \(\frac{p}{\delta}\) to obtain
\[
\sup_{B_{R_i}} u \leq \eta \sup_{B_{R_i+1}} u + \frac{C}{\eta^{\frac{p}{p-\delta}}} b \frac{N}{i^{sp}} \kappa^{-(p-1)N} sp \left(\int_{B_{R_{i+1}}} u_+^{\delta}(x)dx\right)^{\frac{1}{\delta}}
\]
\[
+ \kappa \left( R_{i+1}^{sp} \int_{\mathbb{R}^n \setminus B_{R_i}} \frac{u_+^{p-1}(x)}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}},
\]

where \(\eta \in (0, 1)\) is to be chosen. Using (4.1), we obtain
\[
\sup_{B_{R_i}} u \leq \eta \sup_{B_{R_{i+1}}} u + \tilde{C} d^i \left(\int_{B_{R_{i+1}}} u_+^{\delta}(x)dx\right)^{\frac{1}{\delta}}
\]
\[
+ \kappa \left( R_{i+1}^{sp} \int_{\mathbb{R}^n \setminus B_{R_i}} \frac{u_+^{p-1}(x)}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}}, \quad (4.2)
\]

where
\[
\tilde{C} = \frac{C}{\eta^{\frac{p}{p-\delta}}} b \frac{N}{i^{sp}} \kappa^{-(p-1)N} sp \left(1 + \frac{1}{(1-t)p} + \frac{1}{(1-t)^{N+sp}}\right)^{\frac{N}{sp}}, \quad \text{and} \quad d = 2^{\frac{(N+sp+p)N}{sp}}.
\]

Also observe that
\[
\int_{B_{R_{i+1}}} u_+^{\delta}(x)dx \leq 2^N \int_{B_{R_0}} u_+^{\delta}(x)dx. \quad (4.3)
\]

On the other hand,
\[
\left( R_{i+1}^{sp} \int_{\mathbb{R}^n \setminus B_{R_i}} \frac{u_+^{p-1}(x)}{|x - x_0|^{N+sp}} dx \right)^{\frac{1}{p-1}} \leq \frac{1}{t^{p-1}} \text{Tail}(u_+; x_0, t R_0). \quad (4.4)
\]
Substituting (4.3) and (4.4) in (4.2), we get
\[
\sup_{B_{R_i}} u \leq \eta \sup_{B_{R_{i+1}}} u + \tilde{C} d^i \left( \int_{B_{R_0}} u_+^\delta (x) \, dx \right)^{\frac{1}{2}} + \kappa \frac{1}{t^{\frac{sp}{p-1}}} \text{Tail}(u_+; x_0, t R_0),
\]
where
\[
\tilde{C} = \frac{C}{\eta^{p-2}} b^{\frac{p}{2p-2}} \xi^{-(p-1)N_{sp}} \left( 1 + \frac{1}{(1-t)^p} + \frac{1}{(1-t)^N + \nu} \right)^{\frac{N}{sp}}.
\]
and \(d = 2 \frac{(N + sp) N}{sp^2} \).

Iterating (4.5), we obtain
\[
\sup_{B_{R_0}} u \leq \eta^i \sup_{B_{R_i}} u + \tilde{C} \left( \int_{B_{R_0}} u_+^\delta (x) \, dx \right)^{\frac{1}{2}} + \kappa \frac{1}{t^{\frac{sp}{p-1}}} \text{Tail}(u_+; x_0, t R_0) \sum_{j=0}^{i-1} (\eta)^j.
\]
Now, we choose \(\eta\) such that \(d \eta = \frac{1}{2}\) and take the limits as \(i \to \infty\) to obtain the estimate
\[
\sup_{B_{R_0}} u \leq \tilde{C} \left( \int_{B_{R_0}} u_+^\delta (x) \, dx \right)^{\frac{1}{2}} + \kappa \frac{1}{t^{\frac{sp}{p-1}}} \text{Tail}(u_+; x_0, t R_0).
\]
Now, redefine \(\kappa\) by setting \(\kappa \to \kappa t^{\frac{sp}{p-1}}\) to finish the proof. \(\Box\)

5. A sharp De Giorgi isoperimetric inequality

Proof of Theorem 4. We let \(g\) denote the following truncated function:
\[
g = \begin{cases} 
\min\{u, l\} - k & \text{if } u > k, \\
0 & \text{if } u \leq k.
\end{cases}
\]
Let us denote
\[
A_k^- := \{x \in B_R : u(x) < k\}, \quad A_k^+ := \{x \in B_R : u(x) > l\}, \\
A_k^l := \{x \in B_R : k < u(x) < l\}.
\]
Then we have the following two estimates:
\[
\int_{B_R} |g - (g)_{B_R}| \, dx = \int_{B_R \setminus A_k^-} |g - (g)_{B_R}| \, dx + \int_{A_k^-} |(g)_{B_R}| \, dx \geq |A_k^-| |(g)_{B_R}| 
\]
and
\[
\int_{B_R} |g| \, dx = \int_{A_k^+} (l - k) \, dx + \int_{A_k^l} |g| \, dx \geq (l - k) |A_k^+|.
\]
Thus, combining the two estimates (5.1) and (5.2), we obtain

\[(l - k)|A^+_l| \leq \int \frac{|g|}{2|A_k|} dx \leq \int \frac{|g - (g)_{B_R}|}{|B_R|} dx + |B_R||g|_{B_R} \]

\[ \leq 2|g|_{B_R} \int \frac{|g - (g)_{B_R}|}{B_R} dx \]

\[ \leq C \frac{|R_N|}{|A_k|} \int \frac{|g - (g)_{B_R}|}{B_R} dx. \]

To the last inequality, we apply the Poincaré inequality (2.3) in \( W^{\frac{2}{s-1}} \) to obtain

\[(l - k)|A^+_l| - |A^-_k| \leq C(2 - s)R^{N + \frac{s}{2}} \iint_{B_R \times B_R} \frac{|g(x) - g(y)|}{|x - y|^{N + \frac{s}{2}}} dxdy. \]

At this point, we write \( B_R \times B_R \) as a union of nine sets, viz., \( A^+_l \times A^+_l, A^+_l \times A^-_l, A^+_l \times A^-_l, A^-_l \times A^+_l, A^-_l \times A^-_l, A^+_l \times A^+_l, A^+_l \times A^-_l \) and \( A^-_l \times A^-_l \). We discard the sets \( A^+_l \times A^+_l \) and \( A^-_l \times A^-_l \) from the analysis since \( |g(x) - g(y)| = 0 \) on these two sets. Also, by symmetry considerations, the analysis for the pair \( A^+_l \times A^-_l \) and \( A^-_l \times A^+_l \) is the same. The same goes for the pair \( A^-_l \times A^-_l \) and \( A^+_l \times A^-_l \); and for the pair \( A^-_l \times A^+_l \) and \( A^+_l \times A^-_l \). Thus, we may write

\[(l - k)|A^+_l||A^-_k| \leq C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{|g(x) - g(y)|}{|x - y|^{N + \frac{s}{2}}} dxdy \]

\[+ C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{|g(x) - g(y)|}{|x - y|^{N + \frac{s}{2}}} dxdy \]

\[+ C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{|g(x) - g(y)|}{|x - y|^{N + \frac{s}{2}}} dxdy \]

\[+ C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{|g(x) - g(y)|}{|x - y|^{N + \frac{s}{2}}} dxdy. \]  \( (5.3) \)

We make the following elementary observations:

- On \( A^-_l \), we have \( g = 0 \).
- On \( A^+_l \), we have \( g = l - k \).
- On \( A^-_l \), we have \( g = u - k \).

As a result, \( (5.3) \) becomes

\[(l - k)|A^+_l||A^-_k| \leq C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{l - k}{|x - y|^{N + \frac{s}{2}}} dxdy \]

\[+ C(2 - s)R^{N + \frac{s}{2}} \iint_{A^-_l \times A^+_l} \frac{u - k}{|x - y|^{N + \frac{s}{2}}} dxdy \]
\[ + C(2-s)R^{N+\frac{s}{2}} \iint_{A_k^l \times A_k^l} \frac{|l - u(x)|}{|x - y|^{N+s}} \, dx \, dy \]
\[ + C(2-s)R^{N+\frac{s}{2}} \iint_{A_k^l \times A_k^l} \frac{|u(x) - u(y)|}{|x - y|^{N+s}} \, dx \, dy. \]  

(5.4)

We obtain the desired inequality (1.3) as soon as we apply Theorem 11 to each of the terms on the right-hand side of (5.4). For each of the term, we employ the inclusion of $W^{\frac{1}{2}, 1}$ in $W^{s, p}$ to complete the proof. □

6. Proof of oscillation theorem

The proof of Theorem 1 relies on the iterated boundedness estimate proved in Theorem 16. The proof is motivated from [14]. Also, see [15, Theorem 4.9] for an argument in the same spirit.

\textit{Proof (Proof of oscillation theorem).} We write $B_r(x_0)$ as $B_r$ for any $r > 0$. From Theorem 16, for any $\kappa \in (0, 1)$, we have

\[ \sup_{B_R} u \leq \frac{C_{\kappa}}{R^{N/\delta}} \left( \int_{B_{2R}} u_+^\delta(x) \, dx \right)^{\frac{1}{\delta}} + \kappa \text{Tail}(u_+, x_0, R). \]  

(6.1)

We will estimate the first term in (6.1). Using Hölder’s inequality, we have

\[ \left( \int_{B_{2R}} u_+^\delta(x) \, dx \right)^{\frac{1}{\delta}} \leq \left( \int_{B_{2R}} u_+^p(x) \, dx \right)^{\frac{1}{p}} |\{u > 0\} \cap B_{2R}|^{\frac{p-\delta}{p\delta}}. \]  

(6.2)

We further estimate the first factor on the right-hand side of (6.2) using Theorem 13 applied with $q = p$ and the hypothesis 1.1 to get

\[ \int_{B_{2R}} u_+^p \, dx \leq CR^{sp} \iint_{B_{2R} \times B_{2R}} \frac{|u_+(x) - u_+(y)|^p}{|x - y|^{N+sp}} \, dx \, dy. \]  

(6.3)

Finally, we estimate the $W^{s, p}$ seminorm in (6.3) by Caccioppoli’s inequality in Theorem 8 followed by Young’s inequality for the Tail term:

\[ \iint_{B_{2R} \times B_{2R}} \frac{|u_+(x) - u_+(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \leq \frac{C}{R^{sp}} \left( \|u_+\|_{L^p(B_{4R})} + \|u_+\|_{L^1(B_{4R})} \text{Tail}(u_+, x_0, 2R)^{p-1} \right) \]
\[ \leq CR^{N-sp}(\sup_{B_{4R}} u_+)^p + CR^{N-sp}\text{Tail}(u_+, x_0, 2R)^p. \]  

(6.4)
Substituting the expressions (6.2), (6.3) and (6.4) in (6.1), we receive the following scale-invariant oscillation theorem:

\[ \sup_{B_R} u \leq C_n \frac{R^N}{R^{N}} \left( \frac{u + \text{Tail}(u_+, x_0, 2R)}{R^p} \right) + \nu \text{Tail}(u_+, x_0, R),\]

which is (1.2).

\[ \square \]

7. Proof of Hölder regularity

In this section, we will prove the local Hölder regularity for weak solutions of fractional $p$-Laplace type operators. We mainly follow the proof in [10]. The main novelty of this proof is that we do not rely on De Giorgi iteration for oscillation decay. As a result, the main difference from the proof in [10] is in the proof of the so-called “measure to uniform estimate”. However, we write all the steps here in order to present a fully self-contained proof.

7.1 Oscillation decay

Let us define the following quantities. For $j \in \mathbb{N}$, let $0 < r < R/2$, for some $R$ such that $B_R(x_0) \subset \Omega$ and

\[ r_j = \sigma^j \frac{r}{2}, \quad \sigma \in \left(0, \frac{1}{4}\right) \quad \text{and} \quad B_j := B_{r_j}(x_0). \]

Further, define

\[ \frac{1}{2} \omega(r_0) = \frac{1}{2} \omega \left(\frac{R}{2} \right) := \text{Tail}(u, x_0, R/2) + 2 ||u||_{L^\infty(B_R(x_0))} \]

and

\[ \omega(\rho) = \left(\frac{\rho}{r_0}\right)^{\alpha} \omega(r_0), \]

for some $\alpha < \frac{sp}{p-1}$ and $\rho < r$. The quantities $\sigma$ and $\alpha$ will be fixed in the course of the proof.

Lemma 17. For the quantities defined above, it holds that

\[ \text{osc}_{B_{r_j}} u \equiv \sup_{B_{r_j}} u - \inf_{B_{r_j}} u \leq \omega(r_j), \quad \text{for all } j = 0, 1, 2, \ldots \quad (7.1) \]

Proof. The proof is by induction. The statement is true for $j = 0$ trivially. Let us assume that the statement is true for all $i$ from 0 to $j$. We will prove the truth of the statement for $j + 1$. It is true that one of the two alternatives hold.

Alternative 1:

\[ \left| 2B_{r_{j+1}} \cap \left\{ u \geq \inf_{B_{r_j}} u + \omega(r_j)/2 \right\} \right| \geq \frac{|2B_{r_{j+1}}|}{2}, \quad (7.2) \]
Alternative 2: \[ |2B_{r_{j+1}} \cap \left\{ u \leq \inf_{B_{r_j}} u + \omega(r_j)/2 \right\}| \geq \frac{|2B_{r_{j+1}}|}{2}. \] (7.3)

At this point, in the first alternative, i.e., when (7.2) holds, we define \( u_j := u - \inf_{B_{r_j}} u \), and in the second alternative, i.e., when (7.3) holds, we define \( u_j := \omega(r_j) - (u - \inf_{B_{r_j}} u) \).

As a consequence, in both the cases, it holds that \( u_j \geq 0 \) on \( B_{r_j} \) and

\[ |2B_{r_{j+1}} \cap \{ u_j \geq \omega(r_j)/2 \}| \geq \frac{1}{2} |2B_{r_{j+1}}| \.

Tail decay. We claim that

\[ (\text{Tail}(u_j; x_0, r_j))^{p-1} \leq C \sigma^{-a(p-1)}(\omega(r_j))^{p-1}, \text{ for all } j = 0, 1, 2, \ldots, \] (7.4)

where the constant \( C \) depends on \( N, p, s, \alpha \), but not \( \sigma \). The proof is the same as the proof of [10, inequality (5.6)], however we repeat it here for completeness. It is easy to see that

\[ \sup_{B_{r_i}} |u_j| \leq 2\omega(r_i) \text{ for } i = 0, 1, \ldots, j, \]

so that

\[
(Tail(u_j; x_0, r_j))^{p-1} = r_j^{sp} \sum_{i=1}^{j} \int_{B_{r_{i-1}} \setminus B_{r_i}} \frac{|u_j(x)|^{p-1}}{|x-y|^{N+sp}} dx + r_j^{sp} \int_{\mathbb{R}^N \setminus B_0} \frac{|u_j(x)|^{p-1}}{|x-y|^{N+sp}} dx
\]

\[
\leq r_j^{sp} \sum_{i=1}^{j} \sup_{x \in B_{r_{i-1}} \setminus B_{r_i}} |u_j(x)|^{p-1} \int_{B_{r_{i-1}} \setminus B_{r_i}} \frac{1}{|x-y|^{N+sp}} dx + r_j^{sp}
\]

\[
\int_{\mathbb{R}^N \setminus B_0} \frac{|u_j(x)|^{p-1}}{|x-y|^{N+sp}} dx \leq C \sum_{i=1}^{j} \left( \frac{r_j}{r_i} \right)^{sp} \omega(r_i)^{p-1},
\]

where the expression \( G \) has been estimated as

\[
\int_{\mathbb{R}^N \setminus B_0} \frac{|u_j(x)|^{p-1}}{|x-y|^{N+sp}} dx \leq Cr_0^{-sp} \sup_{B_0} |u|^{p-1} + Cr_0^{-sp} (\omega(r_0))^{p-1}
\]

\[+ C \int_{\mathbb{R}^N \setminus B_0} \frac{|u(x)|^{p-1}}{|x-x_0|^{N+sp}} dx
\]

\[\leq Cr_1^{-sp} (\omega(r_0))^{p-1}.
\]
As a result, we have

\[
(Tail(u_j; x_0, r_j))^{p-1} \leq C \sum_{i=1}^{j} \left( \frac{r_j}{r_i} \right)^{sp} \omega(r_i)^{p-1}
\]

\[
= C \left( \omega(r_0) \right)^{p-1} \left( \frac{r_j}{r_0} \right)^{\alpha(p-1)} \sum_{i=1}^{j} \left( \frac{r_j}{r_i} \right)^{\alpha(p-1)} \left( \frac{r_j}{r_i} \right)^{sp-\alpha(p-1)}
\]

\[
= C \left( \omega(r_j) \right)^{p-1} (sp-\alpha(p-1)) \sum_{i=1}^{j} \sigma_{i}^{sp-\alpha(p-1)}
\]

\[
\leq \left( \omega(r_j) \right)^{p-1} \frac{\sigma}{1 - \sigma^{sp-\alpha(p-1)}}
\]

\[
\leq \frac{4^{sp-\alpha(p-1)}}{\log(4)(sp - \alpha(p-1))} \sigma^{sp-\alpha(p-1)} \left( \omega(r_j) \right)^{p-1}.
\]

For the last inequality, we require \( \sigma \leq \frac{1}{4} \) and \( \alpha(p - 1) < sp \).

**Shrinking lemma.** The proof of the shrinking lemma is the same as the proof of [10, inequality (5.9)], however we repeat it here for completeness. Given that (7.4) holds, we define

\[
v := \min \left\{ \left\lfloor \log \left( \frac{\omega(r_j)/2 + d}{u_j + d} \right) \right\rfloor, k \right\}, \quad k > 0.
\]

Applying Theorem 10 to the function \( v \) with \( a = \omega(r_j)/2 \) and \( b = \exp(k) \), we obtain

\[
\int_{2B_{r_{j+1}}} |v - (v)_{2B_{r_{j+1}}}|^{p} dx \leq C \left\{ d^{1-sp} \left( \frac{r_{j+1}}{r_j} \right)^{sp} \left( \text{Tail}(u_j; x_0, r_j) \right)^{p-1} + 1 \right\}.
\]

Using (7.4), the above estimate becomes

\[
\int_{2B_{r_{j+1}}} |v - (v)_{2B_{r_{j+1}}}|^{p} dx \leq C \left\{ d^{1-sp} \sigma^{sp-\alpha(p-1)} (\omega(r_j))^{p-1} + 1 \right\}.
\]

Now choosing \( d = \epsilon \omega(r_j) \) and

\[
\epsilon := \sigma^{sp-\alpha(p-1)}, \quad (7.5)
\]

we get

\[
\int_{2B_{r_{j+1}}} |v - (v)_{2B_{r_{j+1}}}|^{p} dx \leq C,
\]
where the constant $C$ only depends on data $\frac{sp}{p-1}$ and $\alpha$.

Now, notice that

$$k = \int_{2B_{j+1} \cap \{u \geq \omega(r_j)/2\}} k \, dx = \int_{2B_{j+1} \cap \{v = 0\}} k \, dx \leq 2 \int_{2B_{j+1}} (k - v) \, dx$$

$$= 2(k - (v)_{2B_{j+1}}).$$

(7.7)

Integrating (7.7) over the set $2B_{j+1} \cap \{v = k\}$ results in

$$\frac{|2B_{j+1} \cap \{v = k\}|}{|2B_{j+1}|} k \leq \frac{2}{|2B_{j+1}|} \int_{2B_{j+1} \cap \{v = k\}} (k - (v)_{2B_{j+1}}) \, dx$$

$$\leq \frac{2}{|2B_{j+1}|} \int_{2B_{j+1} \cap \{v = k\}} |v - (v)_{2B_{j+1}}| \, dx \leq C,$$

(7.8)

where in the last inequality, we used (7.6). Now, we choose

$$k = \log \left( \frac{\omega(r_j) + 2\epsilon \omega(r_j)}{6\epsilon \omega(r_j)} \right) = \log \left( \frac{1 + 2\epsilon}{6\epsilon} \right) \simeq \log \left( \frac{1}{\epsilon} \right).$$

(7.9)

Finally, with (7.9) in (7.8), we get

$$\frac{|2B_{j+1} \cap \{u_j \leq 2\epsilon \omega(r_j)\}|}{|2B_{j+1}|} \leq \frac{c_{\log}}{\log \left( \frac{1}{\sigma} \right)},$$

(7.10)

where the constant $c_{\log}$ depends only on data and the quantity $\frac{sp}{p-1} - \alpha$ through (7.5).

Oscillation decay. Finally, to get the oscillation decay, we shall use Theorem 1. This is the only part of our proof of Hölder regularity which departs from that in [10].

In the second alternative, we have $u_j = \omega(r_j) - u + \inf_{B_{r_j}} u$, we set $v = 2\epsilon \omega(r_j) - u_j$. Then we see that

$$\{2B_{j+1} : v \leq 0\} = \{2B_{j+1} : u \leq \inf_{B_{r_j}} u - 2\epsilon \omega(r_j) + \omega(r_j)\}$$

$$\supseteq \{2B_{j+1} : u \leq \inf_{B_{r_j}} u + \omega(r_j)/2\},$$

provided $2\epsilon \leq \frac{1}{2}$.

In particular, we have
which says that \( v \) satisfies hypothesis (1.1) of Theorem 1 from which we obtain

\[
\sup_{B_{r_j+1}} v \leq C \frac{\left( \frac{2B_{r_j+1} \cap \{ v \geq 0 \}}{|2B_{r_j+1}|} \right)}{2^{B_{r_j+1}}} \left( \sup_{B_{r_j+1}} v + \text{Tail}(v_+; x_0, 2r_j+1) \right) + \kappa \text{Tail}(v_+; x_0, r_j+1). \tag{7.11}
\]

From the definition of \( v \), we see that \( v \leq 2\epsilon \omega(r_j) \) on \( B_{r_j} \supseteq 4B_{r_j+1} \), which is used in (7.11) along with (7.10) to obtain

\[
\sup_{B_{r_j+1}} u \leq \omega(r_j) + \inf_{B_{r_j}} u - 2\epsilon \omega(r_j) + C \left( \frac{c \log \log \left( \frac{1}{\sigma} \right)}{\log \left( \frac{1}{\sigma} \right)} \right)^\beta \left( \epsilon \omega(r_j) + \text{Tail}(v_+; x_0, 2r_j+1) \right) + \kappa \text{Tail}(v_+; x_0, r_j+1). \tag{7.12}
\]

We first estimate the Tail term as follows:

\[
\text{Tail}(v_+; x_0, r_j+1)^{p-1} \overset{(a)}{\leq} C\left( r_j+1 \right)^{sp} \int_{B_j \setminus B_{r_j+1}} \frac{v_+^{p-1}}{|x-x_0|^{N+ps}} dx + C\sigma^{sp} \text{Tail}(|uj| + 2\epsilon \omega(r_j); x_0, r_j)^{p-1} \leq C(\epsilon \omega(r_j))^{p-1} + C\sigma^{sp} \text{Tail}(u_j; x_0, r_j)^{p-1} \leq C \left( 1 + \frac{\sigma^{sp-\sigma(p-1)}}{\epsilon^{p-1}} \right) (\epsilon \omega(r_j))^{p-1} \leq C(\epsilon \omega(r_j))^{p-1}, \tag{7.13}
\]

where we obtain (a) by splitting the integral and using the bound \( v \leq |uj| + 2\epsilon \omega(r_j) \) in \( \mathbb{R}^N \); to obtain (b), we note that \( v \leq 2\epsilon \omega(r_j) \) in \( B_{r_j} \) and \( \text{Tail}(2\epsilon \omega(r_j); x_0, r_j)^{p-1} \leq (\epsilon \omega(r_j))^{p-1} \); to obtain (c), we made use of (7.4) and finally to obtain (d), we recall the choice of \( \epsilon \) from (7.5).

Noting that \( \text{Tail}(v_+; x_0, 2r_j+1) \overset{\leq}{\leq} \text{Tail}(v_+; x_0, r_j+1) \) and substituting (7.13) in (7.12), we get

\[
\sup_{B_{r_j+1}} u \leq \omega(r_j) + \inf_{B_{r_j}} u - 2\epsilon \omega(r_j) + 2C\kappa \left( \frac{c \log \log \left( \frac{1}{\sigma} \right)}{\log \left( \frac{1}{\sigma} \right)} \right)^\beta + C\kappa \epsilon \omega(r_j).
\]
First, we choose $\kappa$ sufficiently small so that
\[
\sup_{B_{r_{j+1}}} u - \inf_{B_{r_{j+1}}} u \leq \omega(r_j) + \inf_{B_{r_j}} u - \inf_{B_{r_{j+1}}} u - 2\varepsilon \omega(r_j) + 2C \frac{\log(\frac{r_{j+1}}{r_j})}{\log(\frac{1}{\sigma})} \beta.
\]
(7.14)

Now, choosing $\sigma$ sufficiently small in (7.14), there exists a universal constant $\theta \in (0, 1)$ such that
\[
\text{osc}_{B_{r_{j+1}}} u \leq (1 - \theta) \omega(r_j) = (1 - \theta) \left(\frac{r_j}{r_{j+1}}\right)^{\alpha} \omega(r_{j+1}) \leq \omega(r_{j+1}),
\]
holds once we choose $\alpha > 0$ small enough to satisfy $(1 - \theta)\sigma^{-\alpha} < 1$.

The case when first alternative from (7.2) holds can be handled analogously. \hfill \Box

7.2 Proof of Hölder regularity

Proof of Theorem 14. Given (7.1), it is a short step to see that for any $0 < \rho < \frac{r_0}{2}$, we have
\[
\text{osc}_{B_{\rho}} u \leq C \left(\frac{\rho}{r_0}\right)^{\alpha} \omega(r_0).
\]

To see this, observe that there exists $j \in \mathbb{N}$ such that $r_{j+1} < \rho < r_j$, so that
\[
\text{osc}_{B_{\rho}} u \leq \text{osc}_{B_{r_j}} u \leq \omega(r_j) = \left(\frac{r_j}{r_0}\right)^{\alpha} \omega(r_0) = \left(\frac{r_{j+1}}{r_0}\right)^{\alpha} \omega(r_0) \leq \left(\frac{\rho}{\sigma r_0}\right)^{\alpha} \omega(r_0) \leq C \left(\frac{\rho}{r_0}\right)^{\alpha} \omega(r_0).
\]
(7.15)

Now, let $x_0 \in \Omega$ be fixed and choose $r$ such that $B_{2r}(x_0) \subset \Omega$. Let $y \in B_r(x_0)$ be any point and denote by $d = |x_0 - y|$ the distance between $x_0$ and $y$. First, suppose that $d < r/2$, then there exists $\rho < r/2$ such that $\rho = |x_0 - y|$ and by (7.15), we get
\[
|u(x_0) - y| \leq \text{osc}_{B_{\rho}} u \leq C \left(\frac{\rho}{r}\right)^{\alpha} \omega(r) = C \left(\frac{|x_0 - y|}{r}\right)^{\alpha} \omega(r).
\]

On the other hand, if $d \geq r/2$, then
\[
\frac{|u(x_0) - u(y)|}{|x_0 - y|} \leq \frac{2||u||_{L^\infty(B_r(x_0))}}{r}.
\]
From the definition of $\omega(r) = 2 \text{Tail}(u, x_0, r) + 2\|u\|_{L^\infty(B_{2r})}$, we get the desired regularity

$$|u(x_0) - u(y)| \leq C \left( \frac{|x_0 - y|}{r} \right) \alpha \left( 2 \text{Tail}(u, x_0, r) + 4\|u\|_{L^\infty(B_{2r}(x_0))} \right),$$

for any $y \in B_r(x_0)$ and a universal $\alpha \in (0, 1)$. □

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