Abstract. We develop a systematic procedure of finding integrable ”relativistic” (regular one–parameter) deformations for integrable lattice systems. Our procedure is based on the integrable time discretizations and consists of three steps. First, for a given system one finds a local discretization living in the same hierarchy. Second, one considers this discretization as a particular Cauchy problem for a certain 2-dimensional lattice equation, and then looks for another meaningful Cauchy problems, which can be, in turn, interpreted as new discrete time systems. Third, one has to identify integrable hierarchies to which these new discrete time systems belong. These novel hierarchies are called then ”relativistic”, the small time step $\hbar$ playing the role of inverse speed of light. We apply this procedure to the Toda lattice (and recover the well–known relativistic Toda lattice), as well as to the Volterra lattice and a certain Bogoyavlensky lattice, for which the ”relativistic” deformations were not known previously.
1 Introduction

The theory of integrable differential–difference, or lattice, systems is by now a well developed
and well understood subject. Nevertheless, some intriguing questions remain open, and the
aim of this paper is to close one of them: what is relativistic Volterra lattice? Let us start
with the necessary background.

Certainly, two most celebrated and well studied integrable lattice systems are the Toda
lattice (TL),
\[ \dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k(b_{k+1} - b_k) \] (1.1)
and the Volterra lattice (VL),
\[ \dot{u}_k = u_k(v_k - v_{k-1}), \quad \dot{v}_k = v_k(u_{k+1} - u_k) \] (1.2)

For readers more familiar with another forms of these systems, we recall that the Newtonian
form of the Toda lattice,
\[ \ddot{x}_k = e^{x_{k+1}} - x_k - e^{x_k} - x_{k-1} \] (1.3)
equivalent also to the Hamiltonian one,
\[ \dot{x}_k = p_k, \quad \dot{p}_k = e^{x_{k+1}} - x_k - e^{x_k} - x_{k-1} \] (1.4)
is recovered from (1.1), if the variables \( a_k, b_k \) are parametrized according to Manakov–
Flaschka formulas
\[ b_k = p_k, \quad a_k = e^{x_{k+1}} - x_k \] (1.5)
while the more convenient form of the Volterra lattice,
\[ \dot{a}_k = a_k(a_{k+1} - a_{k-1}) \] (1.6)
arises from (1.2) upon the re-naming
\[ u_k = a_{2k-1}, \quad v_k = a_{2k} \] (1.7)

These two lattice systems are connected to one another in two different ways. On the one
hand, the Volterra lattice (1.6) is a restriction to the manifold \( b_k = 0 \) of the second flow of
the Toda hierarchy. On the other hand (and this connection will be of a primary interest for
us here) the flows (1.1) and (1.2) are connected by a Miura map, or, better, by two different
Miura maps \( \mathcal{M}_{1,2} : (u, v) \mapsto (a, b) \):
\[ \mathcal{M}_1 : \begin{cases} 
 b_k = u_k + v_{k-1} \\
 a_k = u_kv_k
 \end{cases} \quad \mathcal{M}_2 : \begin{cases} 
 b_k = u_k + v_k \\
 a_k = u_{k+1}v_k
 \end{cases} \] (1.8)
Some time ago a remarkable discovery was made by Ruijsenaars \[ R \] in the area of integrable lattice systems: he found a relativistic generalization of TL (RTL). The corresponding system may be viewed as a regular deformation of TL:

\[
\dot{d}_k = (1 + hd_k)(a_k - a_{k-1}), \quad \dot{a}_k = a_k(d_{k+1} - d_k + ha_{k+1} - ha_{k-1}) \tag{1.9}
\]

(the reason for choosing here \( d \) instead of \( b \) will become clear in the main text). Under the parametrization

\[
d_k = \left( e^{hp_k} - 1 \right) / h, \quad a_k = e^{x_{k+1} - x_k + hp_k} \tag{1.10}
\]

(which is a regular deformation of (1.5)) the equations of motion (1.9) may be presented as

\[
\begin{cases}
\dot{u}_k = u_k(v_k - v_{k-1} + hu_k v_k - hu_{k-1} v_{k-1}) \\
\dot{v}_k = v_k(u_{k+1} - u_k + hu_{k+1} v_{k+1} - hu_k v_k)
\end{cases}
\]

which implies also Newtonian equations of motion

\[
\ddot{x}_k = (1 + h\dot{x}_k)(1 + h\dot{x}_{k+1}) \frac{e^{x_{k+1} - x_k}}{1 + h^2e^{x_{k+1} - x_k}} - (1 + h\dot{x}_{k-1})(1 + h\dot{x}_k) \frac{e^{x_{k} - x_{k-1}}}{1 + h^2e^{x_k - x_{k-1}}} \tag{1.12}
\]

Mathematical structures related to RTL, including Lax representations, multi-Hamiltonian structure, and so on, were further investigated in [BR], [ZTOF], [S1], [S2]. A general approach to constructing ”relativistic” generalizations of integrable lattice systems, applicable to the whole lattice KP hierarchy, was proposed in [GK1].

Paradoxically, up to now nobody seems to know what relativistic Volterra lattice is. We propose here an answer to this question. Let us stress that we are not concerned here with relativistic invariance. Instead, our aim is to construct integrable lattice systems, which are regular one-parameter deformations of VL and are Miura–related to RTL in the same manner as VL is related to TL. Needless to say that the ”relativistic Miura maps” we are looking for have to be regular deformations of the standard ones.

Surprisingly, these Miura maps can be choosen to be identical with the nonrelativistic ones. There exist two different systems that are sent to the RTL by either of two Miura maps (1.8), \( \mathcal{M}_{1,2}(u, v) = (a, d) \), so that ”the relativistic Volterra lattice” actually splits into two systems:

\[
\begin{cases}
\dot{u}_k = u_k(v_k - v_{k-1} + hu_k v_k - hu_{k-1} v_{k-1}) \\
\dot{v}_k = v_k(u_{k+1} - u_k + hu_{k+1} v_{k+1} - hu_k v_k)
\end{cases}
\]

and

\[
\begin{cases}
\dot{u}_k = u_k(v_k - v_{k-1} + hu_{k+1} v_k - hu_k v_{k-1}) \\
\dot{v}_k = v_k(u_{k+1} - u_k + hu_{k+1} v_k - hu_k v_{k-1})
\end{cases}
\]

2
We suspect that our way to finding these systems might be more significant than the systems themselves. Namely, our route is through the theory of integrable time discretizations. It was discovered in [S1] that certain integrable discretization of the usual TL,

\[ \tilde{b}_k = b_k + h(a_k - a_{k-1}), \quad \tilde{a}_k(1 + h\tilde{b}_k) = a_k(1 + h\tilde{b}_{k+1}) \quad (1.15) \]

shares the integrals of motion and the invariant Poisson structure with RTL (upon the change of variables \( b_k = d_k + h a_{k-1} \)). In other words, this discretization belongs to the RTL hierarchy. In [PGR], [S3] it was noticed that this discretization is connected to other one, belonging to the TL hierarchy:

\[ \tilde{b}_k = b_k + h(a_k - \tilde{a}_{k-1}), \quad \tilde{a}_k(1 + h\tilde{b}_k) = a_k(1 + h\tilde{b}_{k+1}) \quad (1.16) \]

These two 1+1-dimensional discrete systems (with one discrete space coordinate and one discrete time) may be seen as resulting from one and the same 2-dimensional discrete system (living on a 2-dimensional lattice) by posing an initial value problem in two different ways. In other words, the Cauchy data for two discretizations are prescribed on two different "discrete curves" on the 2-dimensional lattice. Being very close on the 2-dimensional lattice, the equations (1.15), (1.16) have nevertheless very different properties, such as invariant Poisson brackets, integrals of motion, Lax matrices, etc. – all in one, they belong to different hierarchies.

Thus, our strategy to finding the "relativistic" deformations for continuous time lattice systems (with one discrete space coordinate) may be described as follows.

First, for a given integrable lattice system, one constructs integrable time discretization belonging to the same hierarchy. Such time discretizations appeared first in [AL], [GK2], and a systematic procedure was developed in [S3], [S4]. As a rule, this approach results in nonlocal equations of motion. However, these nonlocal discrete time equations of motion may be brought into a local form with the help of the so-called localizing changes of variables, which were found in [S6] for a large set of examples.

Second, and this is a crucial step in finding new hierarchies, one considers the resulting discrete time system as a system on a two-dimensional lattice, and tries to find new meaningful initial value problems for this two-dimensional lattice. This is close in spirit to the constructions in [PNC], [NPCQ], [PN]. The resulting discrete time systems belong to integrable hierarchies distinct from the original one (this fact is not stressed in the papers just mentioned).

As the third and final step, one has to identify these novel hierarchies, in particular, to find integrals of motion, invariant Poisson structures, and higher flows.

In this paper we first recall how this program could be realized to find RTL (although the actual way to discovering this system was quite different), and then demonstrate how the relativistic Volterra lattice may be derived. In particular, we show that (1.13) and (1.14)
are the simplest flows of the hierarchies, to which the following two explicit discretizations of VL belong:

\[ \tilde{u}_k(1 + h\tilde{v}_{k-1}) = u_k(1 + h\tilde{v}_k), \quad \tilde{v}_k(1 + h\tilde{u}_k) = v_k(1 + h\tilde{u}_{k+1}) \]  

and

\[ \tilde{u}_k(1 + h\tilde{v}_{k-1}) = u_k(1 + h\tilde{v}_k), \quad \tilde{v}_k(1 + h\tilde{u}_k) = v_k(1 + h\tilde{u}_{k+1}) \]  

respectively. One can see that the corresponding constructions may be described as a factorization of RTL, in a complete analogy with the nonrelativistic case. Finally, we show how to generalize these results to some of the Bogoyavlensky lattices.

2 General framework

2.1 Lax equations and representations

Our approach to integrable lattice systems is based on the notion of Lax representations. We consider Lax equations of one of the following types:

\[ \dot{L} = \left[ L, \pi_+(f(L)) \right] = -\left[ L, \pi_-(f(L)) \right] \]  

or

\[ \dot{L}_j = L_j \cdot \pi_+(f(T_{j-1})) - \pi_+(f(T_j)) \cdot L_j = -L_j \cdot \pi_-(f(T_{j-1})) + \pi_-(f(T_j)) \cdot L_j \]  

Let us explain the notations.

Let \( g \) be an associative algebra. One can introduce in \( g \) the structure of Lie algebra in a standard way. Let \( g_+, g_- \) be two subalgebras such that as a vector space \( g \) is a direct sum \( g = g_+ \oplus g_- \). Denote by \( \pi_{\pm} : g \mapsto g_{\pm} \) the corresponding projections. Finally, let \( f : g \mapsto g \) be an Ad–covariant function on \( g \), and let \( L \) stand for a generic element of \( g \). Then (2.1) is a certain differential equation on \( g \).

Further, let \( g = \bigotimes_{j=1}^m g_j \) be a direct product of \( m \) copies of the algebra \( g \). A generic element of \( g \) is denoted by \( L = (L_1, \ldots, L_m) \). We use also the notation

\[ T_j = T_j(L) = L_j \cdot \ldots \cdot L_1 \cdot L_{m+1} \cdot \ldots \cdot L_{j+1} \]  

Then (2.2) is a certain differential equation on \( g \). Such equations are sometimes called Lax triads.

One says that (2.1), resp. (2.2), is a Lax representation of a Hamiltonian flow

\[ \dot{x} = \{H, x\} \]  

4
on a Poisson manifold \((\mathcal{X}, \{\cdot, \cdot\})\), if there exists a map \(L : \mathcal{X} \mapsto g\) (resp. \(L : \mathcal{X} \mapsto g\)) such that the former equations of motion are equivalent to the latter ones. Let us stress that when considering equations (2.1), resp. (2.2) in the role of Lax representation, the letter \(L\) (resp. \(L\)) does not stand for a generic element of the corresponding algebra any more; rather, it represents the elements of the images of the maps \(L : \mathcal{X} \mapsto g\) and \(L : \mathcal{X} \mapsto g\), correspondingly. The elements \(L(x)\), resp. \(L(x)\) (and the map \(L\), resp. \(L\), itself) are called Lax matrices.

2.2 \(r\)–matrix Poisson brackets

Recall that there exist several constructions of Poisson brackets on associative algebras implying the Lax form of Hamiltonian equations of motion. We recall here some of them.

Suppose that \(g\) carries a nondegenerate scalar product \(\langle \cdot, \cdot \rangle\), bi–invariant with respect to the multiplication in \(g\).

Definition 2.1 \([STS]\) A linear \(r\)–matrix bracket on \(g\) corresponding to the operator \(R\) is defined by:

\[
\{\phi, \psi\}_1(L) = \frac{1}{2} \left\langle [R(\nabla\phi(L)), \nabla\psi(L)] + [\nabla\phi(L), R(\nabla\psi(L))], L \right\rangle
\]  

If this is indeed a Poisson bracket, it will denoted by \(PB_1(R)\).

Theorem 2.2 \([STS]\) A sufficient condition for (2.5) to define a Poisson bracket is given by the modified Yang–Baxter equation for the operator \(R\), \(m_{YB}(R; \alpha)\), which reads

\[
[R(u), R(v)] - R([R(u), v] + [u, R(v)]) = -\alpha [u, v] \quad \forall u, v \in g
\]  

Now let \(A_1, A_2, S\) be three linear operators on \(g\), \(A_1\) and \(A_2\) being skew–symmetric:

\[
A_1^* = -A_1, \quad A_2^* = -A_2
\]  

Definition 2.3 \([S2]\) A quadratic \(r\)–matrix bracket on \(g\) corresponding to the triple \(A_1, A_2, S\) is defined by:

\[
\{\phi, \psi\}_2(L) = \frac{1}{2} \left\langle A_1(d'\phi(L)), d'\psi(L) \right\rangle - \frac{1}{2} \left\langle A_2(d\phi(L)), d\psi(L) \right\rangle
\]

\[
+ \frac{1}{2} \left\langle S(d\phi(L)), d'\psi(L) \right\rangle - \frac{1}{2} \left\langle S^*(d'\phi(L)), d\psi(L) \right\rangle
\]  

where we denote for brevity

\[
d\phi(L) = L \cdot \nabla\phi(L), \quad d'\phi(L) = \nabla\phi(L) \cdot L
\]  

If this expression indeed defines a Poisson bracket, we shall denote it by \(PB_2(A_1, A_2, S)\).
In what follows we shall usually suppose the following condition to be satisfied:

\[ A_1 + S = A_2 + S^* = R \]  \hspace{1cm} (2.10)

Then a linearization of \( PB_2(A_1, A_2, S) \) in the unit element of \( g \) coincides with \( PB_1(R) \), and we call the former a *quadratization* of the latter.

**Theorem 2.4** \[S2\] A sufficient condition for (2.8) to be a Poisson bracket is given by the equations (2.10) and

\[ mYB(R; \alpha), \hspace{0.5cm} mYB(A_1; \alpha), \hspace{0.5cm} mYB(A_2; \alpha) \]  \hspace{1cm} (2.11)

One of the most important properties of the \( r \)--matrix brackets is the following one.

**Theorem 2.5** Ad–invariant functions on \( g \) are in involution with respect to the bracket \( PB_1(R) \) and with respect to its quadratizations \( PB_2(A_1, A_2, S) \). The Hamiltonian equations of motion on \( g \) corresponding to an Ad–invariant Hamilton function \( \varphi \), have the Lax form

\[ \dot{L} = \frac{1}{2} [L, R(f(L))] \]  \hspace{1cm} (2.12)

where \( f(L) = \nabla \varphi(L) \) for the linear \( r \)--matrix bracket, and \( f(L) = d\varphi(L) \) for its quadratizations.

Quadratic \( r \)--matrix brackets have interesting and important features when considered on a "big" algebra \( g = \bigotimes_{j=1}^{m} g \). This algebra carries a (nondegenerate, bi–invariant) scalar product

\[ \langle\langle L, M \rangle\rangle = \sum_{k=1}^{m} \langle L_k, M_k \rangle \]

Working with linear operators on \( g \), we use the following natural notations. Let \( A : g \mapsto g \) be a linear operator, let \( (A(L))_i \) be the \( i \)th component of \( A(L) \); then we set

\[ (A(L))_i = \sum_{j=1}^{m} (A)_{ij}(L_j) \]  \hspace{1cm} (2.13)

For a smooth function \( \Phi(L) \) on \( g \) we also denote by \( \nabla_j \Phi, d_j \Phi, d'_j \Phi \) the \( j \)th components of the corresponding objects.

Now let \( A_1, A_2, S \) be linear operators on \( g \) satisfying conditions analogous to (2.7) and to (2.11). One has, obviously:

\[ ((A_1)_{ij})^* = -(A_1)_{ji}, \hspace{1cm} ((A_2)_{ij})^* = -(A_2)_{ji}, \hspace{1cm} (S_{ij})^* = (S^*)_{ji} \]
Then one can define the bracket $\text{PB}_2(A_1, A_2, S)$ on $g$. In components it reads:

$$\{\Phi, \Psi\}_2(u) = \frac{1}{2} \sum_{i,j=1}^{m} \left\langle (A_1)_{ij}(d'_j\Phi), d'_i\Psi \right\rangle - \frac{1}{2} \sum_{i,j=1}^{m} \left\langle (A_2)_{ij}(d_j\Phi), d_i\Psi \right\rangle$$

$$+ \frac{1}{2} \sum_{i,j=1}^{m} \left\langle S_{ij}(d_j\Phi), d'_i\Psi \right\rangle - \frac{1}{2} \sum_{i,j=1}^{m} \left\langle (S^*)_{ij}(d'_j\Phi), d_i\Psi \right\rangle$$

(2.14)

**Theorem 2.6** [S5] Let $g$ be equipped with the Poisson bracket $\text{PB}_2(A_1, A_2, S)$. Suppose that the following relations hold:

$$(A_1)_{i+1,j+1} = -(S)_{i+1,j} = (S^*)_{i,j+1} = -(A_2)_{i,j} \quad \text{for} \quad i \neq j$$

$$(A_1)_{j+1,i,j+1} - (A_2)_{j,j+1} + (S)_{j+1,i} - (S^*)_{j,j+1} = 0 \quad \text{for all} \quad 1 \leq j \leq m$$

Then each map $T_j : g \mapsto g$ (2.3) is Poisson, if the target space $g$ is equipped with the Poisson bracket

$$\text{PB}_2\left((A_1)_{j+1,i+1}, (A_2)_{j,j}, (S)_{j+1,j}\right)$$

Hamilton function of the form $\Phi(L) = \varphi(L_m \cdots L_1)$, where $\varphi$ is an $\text{Ad}$–invariant function on $g$, generates Hamiltonian equations of motion on $g$ having the Lax form:

$$\dot{L}_j = L_j B_{j-1} - B_j L_j, \quad B_j = \frac{1}{2} R_j(d\varphi(T_j))$$

(2.15)

where

$$R_j = (A_1)_{j+1,j+1} + (S)_{j+1,j} = (A_2)_{j,j} + (S^*)_{j,j+1}$$

(In all the formulas the subscripts should be taken (mod $m$)).

We have discussed above the $r$–matrix origin of Lax equations. If one is concerned with a Lax representation of a Hamiltonian flow (2.4) on a Poisson manifold $(\mathcal{X}, \{\cdot, \cdot\})$, then finding an $r$–matrix interpretation for it consists of finding an $r$–matrix bracket on $g$ (or on $\hat{g}$) such that the Lax matrix map $L : \mathcal{X} \mapsto g$ (resp. $L : \mathcal{X} \mapsto \hat{g}$) is a Poisson map.

### 2.3 Factorizations and integrable discretization

A further remarkable feature of the equations (2.1) and (2.2) is a possibility to solve them explicitly in terms of a certain factorization problem in the Lie group $G$ corresponding to $\hat{g}$ [Syl], [STS], [RSTS]. (Actually, this can be done even in a more general situation of hierarchies governed by $R$–operators satisfying the modified Yang–Baxter equation, see [RSTS]). The factorization problem is described by the equation

$$U = \Pi_+(U) \cdot \Pi_-(U), \quad U \in G, \quad \Pi_\pm(U) \in G_\pm$$

(2.16)
where \( G_{\pm} \) are two subgroups of \( G \) with the Lie algebras \( g_{\pm} \), respectively. This problem has a unique solution in a certain neighbourhood of the group unit.

In the situations we consider in the sequel \( G \) will be a matrix group, and we write the adjoint action of the group elements on \( g \) as a conjugation by the corresponding matrices. In this context we write \( \Pi_{\pm}(U) \) for \( (\Pi_{\pm}(U))^{-1} \). Correspondingly, we call \( \text{Ad} \)-covariant functions \( g \mapsto g \) also "conjugation covariant". This notation has an additional advantage of being applicable also to functions \( g \mapsto G \).

Based on the above mentioned explicit solution, the following recipe for integrable discretization was formulated in [S3], [S4]. In all the difference equations below we use the tilde to denote the discrete time shift, so that, for example, \( \tilde{L} = L(t + h) \), if \( L = L(t), t \in \mathbb{R} \).

**Recipe.** Suppose you are looking for an integrable discretization of an integrable system (2.4) allowing a Lax representation of the form (2.1). Then as a solution of your task you may take the difference equation

\[
\tilde{L} = \Pi_+^{-1}(F(L)) \cdot L \cdot \Pi_+(F(L)) = \Pi_-(F(L)) \cdot L \cdot \Pi_+^{-1}(F(L)) \tag{2.17}
\]

with the same Lax matrix \( L \) and some conjugation covariant function \( F : g \mapsto G \) such that

\[ F(L) = I + h f(L) + O(h^2) \]

Analogously, if your system has a Lax representation of the form (2.2) on the algebra \( g \), then you may take as its integrable discretization the difference Lax equation

\[
\tilde{L}_j = \Pi_+^{-1}(F(T_j)) \cdot L_j \cdot \Pi_+(F(T_{j-1})) = \Pi_-(F(T_j)) \cdot L_j \cdot \Pi_+^{-1}(F(T_{j-1})) \tag{2.18}
\]

with \( F \) as above.

Obviously, by construction, the discretizations obtained in this way share the Lax matrix and therefore the integrals of motion with their underlying continuous time systems. Moreover, they share also the invariant Poisson brackets. Indeed, the above mentioned factorization theorems imply that the maps (2.17), (2.18) are the time \( h \) shifts along the trajectories of the corresponding flows (2.1), (2.2) with

\[ f(L) = h^{-1} \log(F(L)) = f(L) + O(h) \]

This, in turn, implies that if all flows of the hierarchy (2.1) [resp. (2.2)] are Hamiltonian with respect to a certain Poisson bracket, then our discretizations are Poisson maps with respect to this bracket. In particular, if the Lax matrices \( L \) [resp. \( L \)] form a Poisson submanifold for some \( r \)-matrix bracket, then this submanifold is left invariant by the corresponding Poisson map (2.17) [resp. (2.18)]. We shall say that our recipe yields discretizations living in the same hierarchies as the underlying continuous time systems.
2.4 Basic algebras and operators

In what follows we fix an algebra \( g \) suited to describe various lattice systems with periodic boundary conditions, as well as certain operators taking part in the corresponding \( r \)-matrix constructions. So, starting from this point the symbols \( g, g_\pm, G, G_\pm, \pi_\pm, \Pi_\pm, R, A_1, A_2, S \) will carry only the following meanings.

The algebra \( g \) is a twisted loop algebra over \( gl(N) \), consisting of Laurent polynomials \( L(\lambda) \) with coefficients from \( gl(N) \) and a natural commutator

\[
[u_\lambda, v_\lambda] = [u, v] \lambda^{j+k},
\]

satisfying an additional condition

\[
\Omega L(\lambda) \Omega^{-1} = L(\omega \lambda), \quad \text{where } \Omega = \text{diag}(1, \omega, \ldots, \omega^{N-1}), \quad \omega = \exp(2\pi i/N)
\]

In other words, elements of \( g \) have the following structure:

\[
L(\lambda) = \sum_{p} \ell^{(p)} E^p
\]

In this formula \( \ell^{(p)} = \text{diag} \left( \ell_1^{(p)}, \ldots, \ell_N^{(p)} \right) \) are diagonal matrices, and

\[
\mathcal{E} = \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad \mathcal{E}^{-1} = \lambda^{-1} \sum_{k=1}^{N} E_{k,k+1}
\]

are the shift matrices (here and below \( E_{jk} \) stands for the matrix whose only nonzero entry is on the intersection of the \( j \)th row and the \( k \)th column and is equal to 1; we set \( E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1} \), and in general understand all subscripts \( \text{(mod } N) \)).

The nondegenerate bi–invariant scalar product on \( g \) is chosen as

\[
\langle L(\lambda), M(\lambda) \rangle = \text{tr}(L(\lambda) \cdot M(\lambda))_0
\]

the subscript 0 denoting the free term of the formal Laurent series. This scalar product allows to identify \( g^* \) with \( g \).

The two subalgebras \( g_+, g_- \) such that \( g = g_+ \oplus g_- \) are chosen as

\[
g_+ = \left\{ \sum_{p \geq 0} \ell^{(p)} E^p \right\}, \quad g_- = \left\{ \sum_{p < 0} \ell^{(p)} E^p \right\}
\]

The corresponding decomposition \( L = \pi_+(L) + \pi_-(L) \) of an arbitrary element \( L \in g \) will be called the generalized LU decomposition.

We denote also by \( g_0 \) the commutative subalgebra of \( g \) consisting of \( \lambda \)–independent diagonal matrices (i.e. of matrices \( \ell^{(0)} \mathcal{E}^0 \)). The linear operator on \( g \) assigning to each element \( L(\lambda) \) \((2.19)\) its free term \( \ell^{(0)} \) will be denoted by \( P_0 \).
The group $G$ corresponding to the twisted loop algebra $g$ is a twisted loop group, consisting of $GL(N)$–valued functions $U(\lambda)$ of the complex parameter $\lambda$, regular in $\C P^1 \setminus \{0, \infty\}$ and satisfying $\Omega U(\lambda) \Omega^{-1} = U(\omega \lambda)$. Its subgroups $G_+$ and $G_-$ corresponding to the Lie algebras $g_+$ and $g_-$, are singled out by the following conditions: the elements $U(\lambda) \in G_+$ are regular in the neighbourhood of $\lambda = 0$, and the elements $U(\lambda) \in G_-$ are regular in the neighbourhood of $\lambda = \infty$ and take in this point the value $U(\infty) = I$. We call the corresponding $\Pi_+ \Pi_-$ factorization the generalized LU factorization.

The basic operator governing the hierarchies of Lax equations, is:

$$R = \pi_+ - \pi_-$$ (2.22)

Denote by $R_0$, $P_0$ its skew–symmetric and its symmetric parts, respectively:

$$R_0 = (R - R^*)/2, \quad P_0 = (R + R^*)/2$$

Obviously, this definition of $P_0$ is consistent with the previous one.

Let the skew–symmetric operator $W$ act on $g_0$ according to

$$W(E_{jj}) = \sum_{k<j} E_{kk} - \sum_{k>j} E_{kk}$$

and on the rest of $g$ according to $W = W \circ P_0$. Finally, define:

$$A_1 = R_0 + W, \quad A_2 = R_0 - W, \quad S = P_0 - W, \quad S^* = P_0 + W$$ (2.23)

These operators will be basic building blocks in the quadratic $r$–matrix brackets appearing below.

3 Reminding the Toda lattice case

3.1 TL

We consider the equations of motion of TL ([1]) under periodic boundary conditions: all subscripts are taken (mod $N$). The phase space of this system is

$$\mathcal{T} = \R^{2N}(b_1, a_1, \ldots, b_N, a_N)$$ (3.1)

(recall that $a_0 \equiv a_N$, $b_{N+1} \equiv b_1$).

There exist three compatible local Poisson brackets on $\mathcal{T}$ such that the system TL is Hamiltonian with respect to each one of them, see [K1]. We adopt once and forever the following conventions: the Poisson brackets will be defined by writing down all nonvanishing
brackets between the coordinate functions; the indices in the corresponding formulas are taken (mod $N$).

The "linear" Poisson structure on $\mathcal{T}$ is defined by the brackets
\[
\{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k
\]
the corresponding Hamilton function for the flow TL is given by:
\[
H_2(a, b) = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k
\]

The "quadratic" Poisson structure has the following definition:
\[
\begin{align*}
\{b_k, a_k\}_2 &= -a_k b_k, & \{a_k, b_{k+1}\}_2 &= -a_k b_{k+1} \\
\{b_k, b_{k+1}\}_2 &= -a_k, & \{a_k, a_{k+1}\}_2 &= -a_{k+1} a_k
\end{align*}
\]

The Hamilton function generating TL in this bracket is:
\[
H_1(a, b) = \sum_{k=1}^{N} b_k
\]

Finally, the "cubic" bracket on $\mathcal{T}$ is given by the relations
\[
\begin{align*}
\{b_k, a_k\}_3 &= -a_k (b_k^2 + a_k), & \{a_k, b_{k+1}\}_3 &= -a_k (b_{k+1}^2 + a_k), \\
\{b_k, b_{k+1}\}_3 &= -a_k (b_k + b_{k+1}), & \{a_k, a_{k+1}\}_3 &= -2 a_k a_{k+1} b_{k+1}, \\
\{b_k, a_{k+1}\}_3 &= -a_k a_{k+1}, & \{a_k, b_{k+2}\}_3 &= -a_k a_{k+1}
\end{align*}
\]

The corresponding Hamilton function of the flow TL is:
\[
H_0(a, b) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k)
\]

The Lax representation of the Toda lattice $[F]$, $[M]$ lives in the algebra $\mathfrak{g}$ introduced in the previous section. We shall work with the Lax matrix
\[
L(a, b, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} b_k E_{k,k} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} = \mathcal{E} + b + a \mathcal{E}^{-1}
\]
where diagonal matrices $a = \text{diag}(a_1, \ldots, a_N)$, $b = \text{diag}(b_1, \ldots, b_N)$ are introduced.
The equations of motion (1.1) are equivalent to the Lax equations

\[ \dot{L} = [L, B] = -[L, A] \]  

(3.9)

with

\[ B = \pi_+(L) = \lambda \sum_{k=1}^N E_{k+1,k} + \sum_{k=1}^N b_k E_{k,k} = \mathcal{E} + b \]  

(3.10)

\[ A = \pi_-(L) = \lambda^{-1} \sum_{k=1}^N a_k E_{k,k+1} = a\mathcal{E}^{-1} \]  

(3.11)

where \( \pi_\pm : g \mapsto g_\pm \) are the projections to the subalgebras \( g_\pm \).

Spectral invariants of the Lax matrix \( L(a, b, \lambda) \) serve as integrals of motion of this system. Note that all Hamilton functions in different Hamiltonian formulations belong to these spectral invariants. For instance,

\[ H_2(a, b) = \frac{1}{2} \left( \text{tr} \, L^2(a, b, \lambda) \right)_0, \quad H_1(a, b) = \left( \text{tr} \, L(a, b, \lambda) \right)_0 \]

where the subscript ”0” is used to denote the free term of the corresponding Laurent series.

All spectral invariants turn out to be in involution with respect to each of the Poisson brackets (3.2), (3.4), (3.6). Most directly it follows from the \( r \)-matrix interpretation of the Lax equation (3.9).

**Theorem 3.1** The Lax matrix map \( L(a, b, \lambda) : T \mapsto g \) is Poisson, if \( T \) carries \{\cdot, \cdot\}_1 and \( g \) carries \( \text{PB}_1(R) \), and also if \( T \) carries \{\cdot, \cdot\}_2 and \( g \) carries \( \text{PB}_2(A_1, A_2, S) \).

The first statement is from [AM], the second one – from [S2]. For other versions of such statements (including the \( r \)-matrix interpretation of the cubic bracket) see [DLT1], [OR], [MP].

### 3.2 TL \( \rightarrow \) dTL

In order to find an integrable time discretization for the flow TL, we apply the recipe of the previous section with \( F(L) = I + hL \), i.e. we take as a solution of this problem the map described by the discrete time Lax equation

\[ \bar{L} = B^{-1}LB = AAL^{-1} \quad \text{with} \quad B = \Pi_+(I + hL), \quad A = \Pi_-(I + hL) \]  

(3.12)
Theorem 3.2 \[S3\] (see also \[GK2\]). Consider the change of variables \( T(a, b) \mapsto \mathcal{T}(a, b) \) defined by the formulas

\[
b_k = b_k + h a_{k-1}, \quad a_k = a_k(1 + h b_k) \quad (3.13)
\]

The discrete time Lax equation (3.12) is equivalent to the map \((a, b) \mapsto (\tilde{a}, \tilde{b})\) which in the variables \((a, b)\) is described by the following equations:

\[
\tilde{b}_k = b_k + h(a_k - \tilde{a}_{k-1}), \quad \tilde{a}_k(1 + h \tilde{b}_k) = a_k(1 + h b_{k+1}) \quad (3.14)
\]

**Proof.** The tri–diagonal structure of the matrix \( L \) implies the following bi–diagonal structure for the factors \( B, A \):

\[
B = \Pi_+ \left( I + hL \right) = \sum_{k=1}^{N} (1 + h b_k) E_{k,k} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \quad (3.15)
\]

\[
A = \Pi_- \left( I + hL \right) = I + h \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \quad (3.16)
\]

The formulas (3.13) represent the matrix equation \( I + h T = BA \) which serves as a definition of the matrices \( B, A \). Obviously, these formulas define a local diffeomorphism \( \mathcal{T}(a, b) \mapsto \mathcal{T}(a, b) \). Now, the Lax equation (3.12) is easy to see to be equivalent to \( I + h \tilde{T} = B^{-1}(BA)B = A(BA)A^{-1} \), or

\[
\tilde{B} \cdot \tilde{A} = A \cdot B \quad (3.17)
\]

The equations of motion (3.14) are now nothing but the componentwise form of the latter matrix equation. \( \blacksquare \)

The map (3.14) will be denoted \( \text{dTL} \). Obviously, it serves as a difference approximation to the Toda flow TL (1.1). The construction assures numerous positive properties of this discretization: in the coordinates \((a, b)\) the map \( \text{dTL} \) is Poisson with respect to each one of the Poisson brackets (3.2), (3.4), (3.6), it has the same integrals of motion as the flow TL, etc. Going to the coordinates \((a, b)\) deforms the integrals of motion and the Poisson brackets. Moreover, since the inverse to the map (3.13) is nonlocal, the invariant Poisson brackets become, generally speaking, also nonlocal in the coordinates \((a, b)\). Remarkably, there turn out to exist such linear combinations of the basic invariant Poisson brackets whose pull–backs to the coordinates \((a, b)\) are described by local formulas.

**Theorem 3.3 a)** The pull–back of the bracket

\[
\{ \cdot, \cdot \}_1 + h \{ \cdot, \cdot \}_2 \quad (3.18)
\]
on \( T(a, b) \) under the change of variables (3.13) is the following bracket on \( T(a, b) \):
\[
\{ b_k, a_k \} = -a_k(1 + h b_k), \quad \{ a_k, b_{k+1} \} = -a_k(1 + h b_{k+1})
\] (3.19)

b) The pull–back of the bracket
\[
\{ \cdot, \cdot \}_2 + h\{ \cdot, \cdot \}_3
\] (3.20)
on \( T(a, b) \) under the change of variables (3.13) is the following bracket on \( T(a, b) \):
\[
\{ b_k, a_k \} = -a_k(b_k + h a_k)(1 + h b_k), \quad \{ a_k, b_{k+1} \} = -a_k(b_{k+1} + h a_k)(1 + h b_{k+1})
\]
\[
\{ b_k, b_{k+1} \} = -a_k(1 + h b_k)(1 + h b_{k+1}), \quad \{ a_k, a_{k+1} \} = -a_k a_{k+1}(1 + h b_{k+1})
\] (3.21)

c) The brackets (3.19), (3.21) are compatible. The map \( dT_{\lambda} \) (3.14) is Poisson with respect to both of them.

**Proof.** To prove the theorem, one has, for example, in the (less laborious) case a) to verify the following statement: the formulas (3.19) imply that the nonvanishing pairwise Poisson brackets of the functions (3.13) are
\[
\{ b_k, a_k \} = -a_k(1 + h b_k), \quad \{ a_k, b_{k+1} \} = -a_k(1 + h b_{k+1})
\]
\[
\{ b_k, b_{k+1} \} = -a_k(1 + h b_k)(1 + h b_{k+1}), \quad \{ a_k, a_{k+1} \} = -a_k a_{k+1}(1 + h b_{k+1})
\]
This verification consists of straightforward calculations. ■

The map (3.14) was first found in [HTI], along with the Lax representation, but without discussing its Poisson structure and its place in the continuous time Toda hierarchy.

We can now determine the hierarchy of continuous time lattice equations to which the map (3.14) belongs. Clearly, this is the Toda hierarchy pulled–back under the map (3.13). The previous theorem allows to calculate the corresponding equations of motion in a systematic (Hamiltonian) fashion.

**Theorem 3.4** [K1]. The pull–back of the flow \( T_{\lambda} \) under the change of variables (3.13) is described by the following differential equations:
\[
\dot{b}_k = (a_k - a_{k-1})(1 + h b_k), \quad \dot{a}_k = a_k(b_{k+1} - b_k)
\] (3.22)

**Proof.** To determine the pull–back of the flow \( T_{\lambda} \), we can use the Hamiltonian formalism. An opportunity to apply it is given by the Theorem 3.3. We shall use the statement a) only. Consider the function \( h^{-1}H_1(a, b) = h^{-1}\sum_{k=1}^N b_k \). It is a Casimir of the bracket \( \{ \cdot, \cdot \}_1 \), and generates exactly the flow \( T_{\lambda} \) in the bracket \( h\{ \cdot, \cdot \}_2 \). Hence it generates the flow \( T_{\lambda} \) also in the bracket (3.13). The pull–back of this Hamilton function is equal to \( h^{-1}\sum_{k=1}^N (b_k + h a_{k-1}) \). It remains only to calculate the flow generated by this function in the Poisson brackets (3.19). This results in the equations of motion (3.22). ■
3.3 dTL → explicit dTL

Now consider the equations (3.14) as equations on the 2-dimensional lattice. In other words, we attach the variables $b_k = b_k(t)$, $a_k = a_k(t)$ to the lattice site $(k, t) \in \mathbb{Z} \times h\mathbb{Z}$. A linear change of independent variables $(k, t) \mapsto (k, \tau) = (k, t + kh)$ mixes space and time coordinates, which is equivalent to changing the Cauchy path on the lattice to $\{\tau = 0\} = \{t = -kh\}$. For another instances of such staircase (or sawtooth) Cauchy paths on the lattice, see [PNC], [NPCQ], [FV].

To deal with the new initial-value problem, denote

$$b_k(\tau) = b_k(t) = b_k(\tau - kh), \quad a_k(\tau) = a_k(t) = a_k(\tau - kh) \quad (3.23)$$

Denoting the $h$--shift in $\tau$ still by the tilde, it is easy to see that the variables $a_k = a_k(\tau)$, $b_k = b_k(\tau)$ satisfy the following difference equations:

$$\tilde{b}_k = b_k + h(a_k - a_{k-1}), \quad \tilde{a}_k(1 + h\tilde{b}_k) = a_k(1 + h\tilde{b}_{k+1}) \quad (3.24)$$

These equations serve as an explicit discretization of the flow TL. Indeed, they allow to calculate $(\tilde{a}, \tilde{b})$ explicitly, if $(a, b)$ is known (first $\tilde{b}$, then $\tilde{a}$).

Of course, it is tempting to conclude about the integrability of the system (3.24) from the known integrability of the equations (3.14). However, we do not know any general statements allowing such conclusions, so that the integrability of (3.24) has to be proved independently. Moreover, we are not aware of any systematic procedure allowing to determine invariant Poisson brackets of the map (3.24) starting with the known invariant Poisson brackets of the map (3.14). Hence even the definition of integrability is nontrivial in the case of (3.24).

Our approach to these problems will be based on Lax representations.

Recall that the Lax equations may be considered as compatibility conditions of linear problems. In particular, the discrete time Lax equation (3.17) has two versions:

$$\tilde{B} \tilde{A} = A(BA)A^{-1} = B^{-1}(BA)B$$

The first one is a compatibility condition of two linear problems:

$$BA\psi = \mu\psi, \quad \tilde{\psi} = A\psi \quad (3.25)$$

while the second one is a compatibility condition of

$$BA\psi = \mu\psi, \quad \tilde{\psi} = B^{-1}\psi \quad (3.26)$$

Here $\psi = (\psi_1, \ldots, \psi_N)$ is an auxiliary vector, $\mu$ is a spectral parameter (having nothing in common with the spectral parameter $\lambda$ entering the elements of the algebra $g$ such as $L(a, b, \lambda)$).
These two pairs of linear problems are equivalent, correspondingly, to the following ones:

\[ B\tilde{\psi} = \mu \psi, \quad A\psi = \tilde{\psi} \quad (3.27) \]

and

\[ A\tilde{\psi} = \mu \tilde{\psi}, \quad B\psi = \tilde{\psi} \quad (3.28) \]

We are now in a position to derive from these two pairs of auxiliary linear problems two different Lax representations for the map (3.24). To this end, we assume that the variables \(\psi_k(t)\) are also attached to lattice sites \((k,t)\), and perform in the above equations the change of variables \((k,t) \mapsto (k,\tau) = (k,t + kh)\). We denote, in addition to the variables \(a_k, b_k\) introduced above, also

\[ \psi_k(\tau) = \psi_k(t) = \psi_k(\tau - kh) \]

**Theorem 3.5** The map (3.24) allows the following Lax representation:

\[ \tilde{L}_- = D^{-1}L_-C = C^{-1}L_- \iff \tilde{D}C^{-1} = C^{-1}D \quad (3.29) \]

with the Lax matrix

\[ L_- = (DC^{-1} - I)/h = \left( \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (b_k - h a_{k-1}) E_{kk} + h \lambda \sum_{k=1}^{N} a_k E_{k,k+1} \right) C^{-1} \quad (3.30) \]

where

\[ D = \sum_{k=1}^{N} (1 + h b_k - h^2 a_{k-1}) E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \quad (3.31) \]

\[ C = I - h \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \quad (3.32) \]

**Proof.** The statement may be, of course, easily verified, but we want to show how it can be derived. To this end we start with (3.27). In coordinates these equations read:

\[ (1 + h b_k) \tilde{\psi}_k + h \lambda \tilde{\psi}_{k-1} = \mu \psi_k, \quad \tilde{\psi}_k = \psi_k + h \lambda^{-1} a_k \psi_{k+1} \]

This implies, after the change of variables \((k,t) \mapsto (k,\tau)\):

\[ (1 + h b_k) \tilde{\psi}_k + h \lambda \tilde{\psi}_{k-1} = \mu \psi_k, \quad \tilde{\psi}_k = \psi_k + h \lambda^{-1} a_k \tilde{\psi}_{k+1} \]

After simple manipulations the latter system may be put into the form

\[ (1 + h b_k - h^2 a_{k-1}) \tilde{\psi}_k + h \lambda \tilde{\psi}_{k-1} = \mu \psi_k, \quad \psi_k = \tilde{\psi}_k - h \lambda^{-1} a_k \tilde{\psi}_{k+1} \]

16
In matrix notations the latter equations look like

\[ D \tilde{\psi} = \mu \psi, \quad C \tilde{\psi} = \psi \]

with the matrices \( D, C \) introduced above. The compatibility condition of these two linear problems coincides with (3.29). ■

**Theorem 3.6** The map (3.24) allows the following Lax representation:

\[ \tilde{L}_+ = \tilde{\Delta}^{-1} \mathcal{F} \cdot L_+ \cdot \mathcal{F}^{-1} \tilde{\Delta} \quad (3.33) \]

with the matrices

\[ L_+ = \mathcal{F}^{-1} \left( \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (b_k - h a_{k-1}) E_{kk} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \right) \quad (3.34) \]

\[ \mathcal{F} = I - h \lambda \sum_{k=1}^{N} E_{k+1,k} = I - h \mathcal{E} \quad (3.35) \]

\[ \tilde{\Delta} = \text{diag}(1 + h b_1, \ldots, 1 + h b_N) \quad (3.36) \]

**Proof.** We perform with (3.28) transformations analogous to those of the previous proof. In coordinates (3.28) reads:

\[ \psi_k + h \lambda^{-1} a_k \psi_{k+1} = \mu \tilde{\psi}_k, \quad (1 + h b_k) \tilde{\psi}_k + h \lambda \tilde{\psi}_{k-1} = \psi_k \]

This implies, after the change of variables \((k,t) \mapsto (k,\tau)\):

\[ \psi_k + h \lambda^{-1} a_k \tilde{\psi}_{k+1} = \mu \tilde{\psi}_k, \quad (1 + h b_k) \tilde{\psi}_k + h \lambda \psi_{k-1} = \psi_k \]

Straightforward manipulations imply:

\[ \left( 1 + h b_k - h^2 a_k \frac{1 + h b_k}{1 + h b_{k+1}} \right) \psi_k + h \lambda^{-1} a_k \frac{1 + h b_k}{1 + h b_{k+1}} \psi_{k+1} = \mu (\psi_k - h \lambda \psi_{k-1}), \]

\[ \tilde{\psi}_k = \frac{1}{1 + h b_k} (\psi_k - h \lambda \psi_{k-1}) \]

This may be put in a matrix form:

\[ \mathcal{F}^{-1} P \psi = \mu \psi, \quad \tilde{\psi} = \tilde{\Delta}^{-1} \mathcal{F} \psi \]
with the matrices $F$, $\Delta$ as above, and

$$P = \sum_{k=1}^{N} \left( 1 + h b_k - h^2 a_k \frac{1 + h b_k}{1 + h b_{k+1}} \right) E_{kk} + h \lambda^{-1} \sum_{k=1}^{N} a_k \frac{1 + h b_k}{1 + h b_{k+1}} E_{k,k+1} \quad (3.37)$$

Now the compatibility condition of the latter two linear problems reads:

$$\mathcal{F}^{-1} \tilde{P} = \Delta^{-1} \mathcal{F} \cdot \mathcal{F}^{-1} P \cdot \Delta^{-1} \mathcal{F}$$

In order to bring this into the required form (3.33), notice that the equations of motion (3.24) imply the following simple formula:

$$\tilde{P} = \sum_{k=1}^{N} (1 + h b_k - h^2 a_{k-1}) E_{kk} + h \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \quad (3.38)$$

so that

$$\mathcal{F}^{-1} \tilde{P} = I + h L_+$$

This completes the proof. □

### 3.4 Explicit dTL $\rightarrow$ RTL

Now that we have two different Lax representations for the map (3.24), with the Lax matrices $L_\pm$, the natural question is the one about their relation. It is easy to see that this relation may be described as follows:

$$\Omega_1 L_-(\lambda) \Omega_1^{-1} = L_+^T(\alpha^{-1} \lambda^{-1})$$

where $\alpha = (a_1 a_2 \ldots a_N)^{1/N}$ and

$$\Omega_1 = \text{diag}(1, \alpha^{-1} a_1, \alpha^{-2} a_1 a_2, \ldots, \alpha^{-N+1} a_1 a_2 \ldots a_{N-1})$$

Both Lax matrices for the explicit dTL (3.24) are substantially different from the one for the implicit dTL (3.14) (the latter Lax matrix is the usual Toda one (3.8) with the variables $(a, b)$ parametrized according to (3.13)). The next natural question is: what is the continuous time hierarchy, to which (3.24) belongs? In other words, how can one get continuous time flows with the Lax matrix $L_\pm$, and what is the underlying invariant Poisson structure(s)?

The answer is known for both Lax formulations. Introduce the phase space

$$\mathcal{R} = \mathbb{R}^{2N}(d_1, a_1, \ldots, d_N, a_N) \quad (3.39)$$
Theorem 3.7 [S2] The Lax matrix map \( L_-(a, d, \lambda) : \mathcal{R} \mapsto g \), where

\[
L_- = (DC^{-1} - I)/h = \left( \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} d_k E_{kk} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \right) C^{-1} \tag{3.40}
\]

\[
D(a, d, \lambda) = \sum_{k=1}^{N} (1 + hd_k)E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.41}
\]

\[
C(a, d, \lambda) = I - h\lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \tag{3.42}
\]

is Poisson, if \( \mathcal{R} \) is equipped with the Poisson bracket

\[
\{d_k, a_k\}_1 = -a_k, \quad \{a_k, d_{k+1}\}_1 = -a_k, \quad \{d_k, d_{k+1}\}_1 = h\lambda_a_k \tag{3.43}
\]

and \( g \) is equipped with PB\(_1\)(R), and also if \( \mathcal{R} \) is equipped with the Poisson bracket

\[
\{d_k, a_k\}_2 = -a_k d_k, \quad \{a_k, d_{k+1}\}_2 = -a_k d_{k+1} \tag{3.44}
\]

\[
\{d_k, d_{k+1}\}_2 = -a_k, \quad \{a_k, a_{k+1}\}_2 = -a_k a_{k+1}
\]

and \( g \) is equipped with PB\(_2\)(A\(_1\), A\(_2\), S). The hierarchy of continuous time flows is of the usual form

\[
\dot{L}_- = [L_-, \pm\pi_\pm f(L_-)] \tag{3.45}
\]

The "first" flow of the hierarchy, corresponding to \( f(L) = L \), coincides with RTL (3.9) and allows a Lax representation with the auxiliary matrix

\[
\pi_+(L_-) = \sum_{k=1}^{N} (d_k + h\lambda_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.46}
\]

The map

\[
\tilde{L}_- = D^{-1}L_-D = C^{-1}L_-C
\]

is interpolated by the flow of this hierarchy with \( f(L) = h^{-1}\log(I + hL) \).

Actually, an additional information is available from [S2]. The evolution of the factors \( C, D \) for the flows of the RTL hierarchy (3.45) is described by the Lax triads

\[
\dot{D} = \pm D \cdot \pi_\pm(f(C^{-1}D)) \mp \pi_\pm(f(DC^{-1})) \cdot D
\]

\[
\dot{C} = \pm C \cdot \pi_\pm(f(C^{-1}D)) \mp \pi_\pm(f(DC^{-1})) \cdot C
\]
In particular, for the flow RTL (1.9) the corresponding auxiliary matrices in the Lax triads are (3.46) and
\[
\pi_+ \left( (C^{-1}D - I)/h \right) = \sum_{k=1}^{N} (d_k + h a_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.47}
\]

For the case of the quadratic bracket an $r$-matrix interpretation in $g \otimes g$ is possible. Namely, the Lax matrix map $(D, C^{-1}) : R \mapsto g = g \otimes g$ is Poisson, if $R$ is equipped with the bracket $\{\cdot, \cdot\} + h \{\cdot, \cdot\}_2$, i.e.
\[
\{d_k, a_k\} = -(1 + hd_k) a_k, \quad \{a_k, d_{k+1}\} = -(1 + hd_{k+1}) a_k, \quad \{a_k, a_{k+1}\} = -h a_k a_{k+1} \tag{3.48}
\]

and $g \otimes g$ is equipped with $h \text{PB}_2(A_1, A_2, S)$ with the operators
\[
A_1 = \begin{pmatrix} A_1 & -S \\ S^* & A_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} A_2 & -S^* \\ S & A_2 \end{pmatrix}, \quad S = \begin{pmatrix} S & S \\ S & -S^* \end{pmatrix} \tag{3.49}
\]

Theorem 2.6 then assures that the monodromy matrix maps $DC^{-1}$ and $C^{-1}D$ are Poisson, if the target $g$ is equipped with $h \text{PB}_2(A_1, A_2, S)$, which implies the Poisson property for the Lax matrix maps $L_- = (DC^{-1} - I)/h$ and $(C^{-1}D - I)/h$, if the target $g$ is equipped with $\text{PB}_1(R) + h \text{PB}_2(A_1, A_2, S)$.

For the hierarchy attached to the Lax matrix $L_+$, somewhat less detailed information is available.

**Theorem 3.8** [BR], [GK1] The hierarchy of Lax equations related to the Lax matrix
\[
L_+(a, d, \lambda) = (I - h \mathcal{E})^{-1} \left( \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} d_k E_{kk} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \right) \tag{3.50}
\]
is given by the formula
\[
\dot{L}_+ = [L_+, \pm \pi_{\pm}(f(L_+)) + \sigma(f(L_+))] \tag{3.51}
\]
where $\sigma$ is a linear map from $g$ into the set of diagonal matrices defined as
\[
\sigma \left( \sum_p E^{(p)} \right) = \sum_{p<0} h^{-p} \epsilon^{(p)} \tag{3.52}
\]

In particular, the ”first ” flow of this hierarchy corresponding to $f(L) = L$ coincides with RTL (1.9) and allows the Lax representation
\[
\dot{L}_+ = [\mathcal{A}, L_+] \tag{3.53}
\]
with the auxiliary matrix

\[ A = \pi_-(L_+) - \sigma(L_+) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} - \hbar \sum_{k=1}^{N} a_k E_{kk} \]

It has to be mentioned that each of the formulations above has its specific advantages. Theorem 3.7 has an advantage of including the RTL hierarchy into the standard framework of the lattice KP hierarchy. On the other hand, Theorem 3.8, found for RTL in [BR], has an advantage of being a particular case of a much more general construction, due to [GK1], which delivers a "relativistic" generalization of the whole lattice KP hierarchy. An \( r \)-matrix interpretation of the latter result is unknown (see, however, [OR] for a linear bracket in the open–end case).

So, upon the identification

\[ d_k = b_k - \hbar a_{k-1} \]

the explicit discrete time Toda map (3.24) belongs to the relativistic Toda hierarchy.

Remarks.

1. In the coordinates \((a, b)\) the equations of motion (1.9) take the form

\[ \dot{b}_k = a_k (1 + \hbar b_k) - a_{k-1} (1 + \hbar b_{k-1}), \quad \dot{a}_k = a_k (b_{k+1} - b_k + \hbar a_{k+1} - \hbar a_k) \quad (3.51) \]

2. In the coordinates \((a, b)\) the linear Poisson bracket (3.43) of the RTL formally coincides (incidentally) with the linear Poisson bracket (3.2) of the usual Toda lattice:

\[ \{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k \quad (3.52) \]

Note also that in the coordinates \((a, d)\) the quadratic Poisson structure (3.44) formally coincides with the quadratic Poisson bracket (3.4) for the usual Toda lattice.

3. The third ("cubic") invariant Poisson structure is known for the RTL hierarchy, how-
ever its $r$–matrix interpretation is unknown:

\[
\begin{align*}
\{d_k, a_k\}_3 &= -a_k(d_k^2 + a_k + hda) \\
\{a_k, d_{k+1}\}_3 &= -a_k(d_{k+1}^2 + a_k + hda_{k+1}) \\
\{d_k, d_{k+1}\}_3 &= -a_k(d_k + d_{k+1} + hda_{k+1}) \\
\{a_k, a_{k+1}\}_3 &= -a_ka_{k+1}(2d_{k+1} + hda_k + hda_{k+1}) \\
\{d_k, a_{k+1}\}_3 &= -a_ka_{k+1}(1 + hda_k) \\
\{a_k, d_{k+2}\}_3 &= -a_ka_{k+1}(1 + hda_{k+2}) \\
\{a_k, a_{k+2}\}_3 &= -hda_ka_{k+1}a_{k+2}
\end{align*}
\]

4 Volterra and relativistic Volterra lattices

4.1 VL

We consider here the equations of motion of VL (1.2) under periodic boundary conditions, so that the corresponding phase space is:

\[
V = \mathbb{R}^{2N}(u_1, v_1, \ldots, u_N, v_N)
\]

There exist two compatible local Poisson brackets on $V$ invariant under the flow VL:

\[
\{u_k, v_k\}_2 = -u_kv_k, \quad \{v_k, u_{k+1}\}_2 = -v_ku_{k+1}
\]

and

\[
\begin{align*}
\{u_k, v_k\}_3 &= -u_kv_k(u_k + v_k), \\
\{v_k, u_{k+1}\}_3 &= -v_ku_{k+1}(v_k + u_{k+1}) \\
\{u_k, u_{k+1}\}_3 &= -u_kv_ku_{k+1}, \\
\{v_k, v_{k+1}\}_3 &= -v_ku_{k+1}v_{k+1}
\end{align*}
\]

The corresponding Hamilton functions for the flow VL are equal to

\[
H_1(u, v) = \sum_{k=1}^{N} u_k + \sum_{k=1}^{N} v_k
\]

and

\[
H_0(u, v) = \sum_{k=1}^{N} \log(u_k) \quad \text{or} \quad H_0(u, v) = \sum_{k=1}^{N} \log(v_k)
\]
(the difference of the latter two functions is a Casimir of $\{\cdot, \cdot\}_3$).

The Lax representation of VL we use here lives in $\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g}$ and was introduced in [K1] (where the system (1.2) was called "modified Toda lattice"), see also [S5]. Consider the following two matrices:

$$
U(u, v, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} u_k E_{k,k} = \mathcal{E} + u
$$

(4.6)

$$
V(u, v, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} = I + v\mathcal{E}^{-1}
$$

(4.7)

These formulas define the "Lax matrix" $(U, V) : \mathcal{V} \mapsto \mathfrak{g} \otimes \mathfrak{g}$. The flow (1.2) is equivalent to either of the following Lax equations in $\mathfrak{g} \otimes \mathfrak{g}$:

$$
\begin{cases}
\dot{U} = UB_2 - B_1 U \\
\dot{V} = VB_1 - B_2 V
\end{cases}
\quad \text{or} \quad
\begin{cases}
\dot{U} = A_1 U - UA_2 \\
\dot{V} = A_2 V - VA_1
\end{cases}
$$

(4.8)

with the matrices

$$
B_1 = \pi_+(UV), \quad B_2 = \pi_+(VU), \quad A_1 = \pi_-(UV), \quad A_2 = \pi_-(VU)
$$

(4.9)

so that

$$
\begin{align*}
B_1 &= \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_{k-1}) E_{kk} \\
B_2 &= \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_k) E_{kk} \\
A_1 &= \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1} \\
A_2 &= \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k E_{k,k+1}
\end{align*}
$$

(4.10-4.13)

As a consequence, the matrices

$$
T_1(u, v, \lambda) = U(u, v, \lambda)V(u, v, \lambda), \quad T_2(u, v, \lambda) = V(u, v, \lambda)U(u, v, \lambda)
$$

(4.14)

satisfy the usual Lax equations in $\mathfrak{g}$:

$$
\dot{T} = [T, \pm \pi_\pm(T)]
$$

(4.15)

The Lax equations (4.8), (4.15) may be given an $r$–matrix interpretation.
Theorem 4.1  The Lax matrix map \( (U, V) : \mathcal{V} \mapsto g \otimes g \) is Poisson, if \( \mathcal{V} \) is equipped with the bracket \( \{ \cdot, \cdot \}_2 \) and \( g \otimes g \) is equipped with the bracket \( \text{PB}_2(A_1, A_2, S) \) corresponding to the operators \( A_1, A_2, S \) defined in (3.4). The maps \( T_{1,2} : g \otimes g \mapsto g \),

\[
T_1(U, V) = UV, \quad T_2(U, V) = VU
\]

are Poisson, if the target space \( g \) is equipped with the bracket \( \text{PB}_2(A_1, A_2, S) \).

It is easy to see that the matrices \( T_1, T_2 \) formally coincide with the Lax matrix (3.8) of the Toda lattice, with the coordinates \((a, b)\) given by the corresponding formula for the Miura maps in (1.8). The Poisson property of the monodromy map is therefore reflected in the (first half) of the following statements about the Miura maps.

Theorem 4.2  The Miura maps \( M_{1,2} : \mathcal{V} \mapsto \mathcal{T} \) are Poisson, if \( \mathcal{V} \) carries the bracket \( \{ \cdot, \cdot \}_2 \) (4.2) and \( \mathcal{T} \) carries the bracket \( \{ \cdot, \cdot \}_2 \) (3.4), and also if \( \mathcal{V} \) carries the bracket \( \{ \cdot, \cdot \}_3 \) (4.3) and \( \mathcal{T} \) carries the bracket \( \{ \cdot, \cdot \}_3 \) (3.6).

4.2  \( \text{VL} \rightarrow \text{dVL} \)

To find an integrable time discretization for the flow \( \text{VL} \), we apply the recipe of Sect. 2.3 with \( F(T) = I + hT \), i.e. we consider the map described by the discrete time "Lax triads"

\[
\begin{align*}
\tilde{U} &= B_1^{-1}UB_2, \\
\tilde{V} &= B_2^{-1}VB_1
\end{align*}
\]

or

\[
\begin{align*}
\tilde{U} &= A_1UA_2^{-1} \\
\tilde{V} &= A_2VA_1^{-1}
\end{align*}
\]

(4.16)

with

\[
B_1 = \Pi_+(I + hUV), \quad B_2 = \Pi_+(I + hVU), \quad A_1 = \Pi_-(I + hUV), \quad A_2 = \Pi_-(I + hVU)
\]

Theorem 4.3  Consider the change of variables \( \mathcal{V}(u,v) \mapsto \mathcal{V}(u,v) \) defined by the formulas

\[
\begin{align*}
u_k &= u_k(1 + hv_{k-1}), \\
v_k &= v_k(1 + hu_k)
\end{align*}
\]

(4.17)

The discrete time Lax equations (1.16) are equivalent to the map \((u,v) \mapsto (\tilde{u}, \tilde{v})\) which in coordinates \((u, v)\) is described by the following equations:

\[
\begin{align*}
\tilde{u}_k(1 + h\tilde{v}_{k-1}) &= u_k(1 + hv_k), \\
\tilde{v}_k(1 + h\tilde{u}_k) &= v_k(1 + hu_{k+1})
\end{align*}
\]

(4.18)
Proof. The formulas (4.17) allow to find the factors $B_{1,2}$, $A_{1,2}$ in a closed form. Indeed, with the help of (4.17) we can represent (4.6), (4.7) as

$$U = \sum_{k=1}^{N} u_k (1 + h v_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$  (4.19)

$$V = I + \lambda^{-1} \sum_{k=1}^{N} v_k (1 + h u_k) E_{k,k+1}$$  (4.20)

From these formulas we derive:

$$I + h U V = h \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} \left( (1 + h u_k)(1 + h v_{k-1}) + h^2 u_{k-1} v_{k-1} \right) E_{kk}$$

$$+ h \lambda^{-1} \sum_{k=1}^{N} u_k v_k (1 + h u_k)(1 + h v_{k-1}) E_{k,k+1}$$

Obviously, this matrix may be factorized as $I + h U V = B_1 A_1$, where

$$B_1 = \sum_{k=1}^{N} (1 + h u_k)(1 + h v_{k-1}) E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}$$  (4.21)

$$A_1 = I + h \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1}$$  (4.22)

Similarly, we find:

$$I + h V U = h \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} \left( (1 + h u_k)(1 + h v_k) + h^2 u_k v_{k-1} \right) E_{kk}$$

$$+ h \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k (1 + h u_k)(1 + h v_k) E_{k,k+1}$$

which may be factorized as $I + h V U = B_2 A_2$, where

$$B_2 = \sum_{k=1}^{N} (1 + h u_k)(1 + h v_k) E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}$$  (4.23)

$$A_2 = I + h \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k E_{k,k+1}$$  (4.24)

Now the discrete time Lax equations $B_1 \tilde{U} = U B_2$, $B_2 \tilde{V} = V B_1$ (or their $A$–analogs) immediately imply the equations of motion (4.18).
Comparing two pairs of formulas (4.21), (4.22) and (4.23), (4.24) for dVL with the formulas (3.15), (3.16), we see immediately that the Miura maps $M_{1,2} : V(u, v) \mapsto T(a, b)$ are conjugated by the changes of variables (4.17), (3.13) with the maps $M_{1,2} : V(u, v) \mapsto T(a, b)$ given by:

$$
M_1 : \begin{cases}
1 + h b_k = (1 + h u_k)(1 + h v_{k-1}) \\
a_k = u_k v_k
\end{cases}
M_2 : \begin{cases}
1 + h b_k = (1 + h u_k)(1 + h v_k) \\
a_k = u_{k+1} v_k
\end{cases}
(4.25)
$$

The map (4.18), denoted hereafter by dVL, serves as a difference approximation to the flow VL (1.2). By construction, in the variables $(u, v)$ this map shares the integrals of motion and the invariant Poisson structures with the flow VL. In the coordinates $(u, v)$ the integrals of motion and the Poisson brackets become deformed, the latter ones become nonlocal, since the inverse of the map (4.17) is nonlocal. Nevertheless, we have the following local invariant Poisson bracket for the map dVL:

**Theorem 4.4** [K1] The pull–back of the bracket

$$
\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3
$$

on $V(u, v)$ under the change of variables (4.17) is the following bracket on $V(u, v)$:

$$
\{u_k, v_k\} = -u_k v_k (1 + h u_k)(1 + h v_k),
\{v_k, u_{k+1}\} = -v_k u_{k+1} (1 + h v_k)(1 + h u_{k+1})
(4.27)
$$

The map (4.18) is Poisson with respect to the bracket (4.27). The maps $M_{1,2}$ (4.25) are Poisson, if $V(u, v)$ carries the bracket (4.27) and $T(a, b)$ carries the bracket (3.21).

**Proof** – by a direct verification. ■

The map (4.18) was found in [THO], [HTI], where, however, its place in the continuous time Volterra hierarchy and its Poisson structure were not elaborated. The Miura maps (4.25) were found in [K1] without connecting them to the discretization problem, and also in [HT] in the discretization context, but without Poisson properties.

The previous theorem allows also to calculate the equations of motion of continuous time hierarchy, to which the map dVL belongs.

**Theorem 4.5** [K1] The pull–back of the flow VL under the map (4.17) is described by the following equations of motion:

$$
\dot{u}_k = u_k (1 + h u_k)(v_k - v_{k-1}),
\dot{v}_k = v_k (1 + h v_k)(u_{k+1} - v_k)
(4.28)
$$

26
Proof. Since the flow VL has a Hamilton function $h^{-1} \log H_0(u,v)$ in the Poisson bracket $\{\cdot,\cdot\}_3$, and this function is a Casimir of the bracket $\{\cdot,\cdot\}_2$, we conclude that this function generates the flow VL also in the bracket (4.26). This means that the pull–back of the flow VL is a Hamiltonian flow in the bracket (4.27) with the Hamilton function

$$h^{-1} \sum_{k=1}^{N} \log \left( u_k (1 + h v_{k-1}) \right)$$

Calculating the corresponding equations of motion, we arrive at the equations (4.28).

4.3 $dVL \rightarrow$ explicit $dVL$

We use the same trick as in the case of $dTL$, in order to extract from $dVL$ the explicit discretization for VL. Namely, we attach the variables $u_k = u_k(t)$, $v_k = v_k(t)$ to the lattice site $(k,t) \in \mathbb{Z} \times h\mathbb{Z}$, and then perform the change of independent variables $(k,t) \mapsto (k,\tau) = (k,t + kh)$. We denote

$$u_k(\tau) = u_k(t) = u_k(\tau - kh), \quad v_k(\tau) = v_k(t) = v_k(\tau - kh) \quad (4.29)$$

Denoting the $h$–shift in $\tau$ still by the tilde, we immediately derive from (4.18) the following difference equations for the variables $u_k$, $v_k$:

$$\tilde{u}_k(1 + hv_{k-1}) = u_k(1 + hv_k), \quad \tilde{v}_k(1 + h\tilde{u}_k) = v_k(1 + h\tilde{u}_{k+1}) \quad (4.30)$$

This is clearly an explicit discretization, since, knowing $(u,v)$, it allows to calculate explicitly first $\tilde{u}$, and then $\tilde{v}$.

We find now Lax representations for the map (4.30). As in the case of the $dTL$, there exist two versions thereof.

Theorem 4.6 The map (4.30) allows the following Lax representation:

$$\tilde{U} = C^{-1}UC_2, \quad \tilde{V}C^{-1} = C_2^{-1}V \quad (4.31)$$

with the matrices

$$U(u,v,\lambda) = \sum_{k=1}^{N} u_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \quad (4.32)$$

$$V(u,v,\lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} \quad (4.33)$$

$$C(u,v,\lambda) = I - h\lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1} \quad (4.34)$$

$$C_2(u,v,\lambda) = I - h\lambda^{-1} \sum_{k=1}^{N} \tilde{u}_{k+1} v_{k} E_{k,k+1} \quad (4.35)$$
The Lax representation (4.31) implies also

\[ \tilde{L}_- = C^{-1}L_-C \iff \tilde{U} \tilde{V} \tilde{C}^{-1} = C^{-1}UV \]  

(4.36)

with the Lax matrix

\[ L_- = UV \tilde{C}^{-1} \]  

(4.37)

**Proof.** We start with the following Lax representation of the dVL in \( g \otimes g \):

\[ \tilde{U} = A_1 U A_2^{-1}, \quad \tilde{V} = A_2 V A_1^{-1} \]

with the matrices \((4.19), (4.20), (4.22), (4.24)\). These Lax equations may be considered as the compatibility condition of the following linear problems:

\[
\begin{align*}
U\phi &= \mu \psi \\
V\psi &= \mu \phi
\end{align*}
\]

\[
\begin{align*}
\tilde{\phi} &= A_2 \phi \\
\tilde{\psi} &= A_1 \psi
\end{align*}
\]

We shall call the first pair the spectral linear problem, and the second pair the evolutionary linear problem. In coordinates the above equations may be presented as

\[
\begin{align*}
\begin{cases}
\phi_k + h \lambda^{-1} \nu_k \psi_{k+1} = \mu \psi_k \\
\lambda \phi_{k-1} - \lambda \phi_k = \mu \phi_k
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\phi_k + h \lambda^{-1} \nu_k \psi_{k+1} = \mu \psi_k \\
\lambda \phi_{k-1} - \lambda \phi_k = \mu \phi_k
\end{cases}
\end{align*}
\]

Upon the change of variables \((k, t) \mapsto (k, \tau)\) this implies:

\[
\begin{align*}
\begin{cases}
\phi_k + h \lambda^{-1} \nu_k \psi_{k+1} = \mu \psi_k \\
\lambda \phi_{k-1} - \lambda \phi_k = \mu \phi_k
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\phi_k + h \lambda^{-1} \nu_k \psi_{k+1} = \mu \psi_k \\
\lambda \phi_{k-1} - \lambda \phi_k = \mu \phi_k
\end{cases}
\end{align*}
\]

After simple manipulations (substitute into the first pair of equalities the values of \(\phi_{k-1}\) and \(\psi_k\) obtained from the second pair of equalities) this can be brought into the form

\[
\begin{align*}
\begin{cases}
\phi_k + h \lambda^{-1} \nu_k \psi_{k+1} = \mu \psi_k \\
\lambda \phi_{k-1} - \lambda \phi_k = \mu \phi_k
\end{cases}
\end{align*}
\]

or in the matrix form

\[
\begin{align*}
\begin{cases}
U \phi = \mu \psi \\
V \tilde{\psi} = \mu \phi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
C_2 \tilde{\phi} = \phi \\
C \tilde{\psi} = \psi
\end{cases}
\end{align*}
\]

where we introduced the matrices \(U, V, C, C_2\) from \((4.32)-(4.35)\). The compatibility condition of these linear problems coincides with (4.31). \(\blacksquare\)
Theorem 4.7 The map (4.30) allows the following Lax representation:

\[
\begin{align*}
\mathcal{F}^{-1}\bar{U}\Delta_2 &= (\bar{\Delta}_1^{-1}\bar{\Delta}_3^{-1}\mathcal{F}) \cdot \mathcal{F}^{-1}U \Delta_2 \cdot (\mathcal{F}^{-1}\bar{\Delta}_1\bar{\Delta}_2) \\
\bar{\Delta}_2^{-1}\bar{V} &= (\bar{\Delta}_2^{-1}\bar{\Delta}_1^{-1}\mathcal{F}) \cdot \Delta_2^{-1}V \cdot (\mathcal{F}^{-1}\Delta_3\Delta_1)
\end{align*}
\]  

(4.38)

with the matrices \(U, V, F\) from (4.32), (4.33), (3.35), and

\[
\Delta_1 = \text{diag}(1 + hu_k), \quad \Delta_2 = \text{diag}(1 + hv_k)
\]

(4.39)

\[
\Delta_3 = \text{diag}(1 + hv_{k-1})
\]

(4.40)

The Lax representation (4.38) implies also

\[
\bar{L}_+ = (\bar{\Delta}_1^{-1}\bar{\Delta}_3^{-1}\mathcal{F}) \cdot L_+ \cdot (\mathcal{F}^{-1}\Delta_3\Delta_1)
\]

(4.41)

with the Lax matrix

\[
L_+ = \mathcal{F}^{-1}UV
\]

(4.42)

Proof. This time we start with the following Lax representation of the map dVL:

\[
\bar{U} = B_1^{-1}UB_2, \quad \bar{V} = B_2^{-1}VB_1
\]

with the matrices (1.19), (4.20), (4.21), (4.23). These Lax equations are the compatibility conditions of the following linear problems:

\[
\begin{align*}
\bar{U}\phi &= \mu\bar{\psi} \\
\bar{V}\bar{\psi} &= \mu\phi
\end{align*}
\]

(4.43)

\[
\begin{align*}
B_1\bar{\psi} &= \psi
\end{align*}
\]

(4.44)

(it turns out to be more convenient to take the tilded versions of the equations in the first pair). In components the above equations take the form:

\[
\begin{align*}
\bar{u}_k(1 + h\bar{v}_{k-1})\bar{\phi}_k + \lambda\bar{\phi}_{k-1} &= \mu\bar{\psi}_k \\
\bar{v}_k + \lambda^{-1}\bar{v}_k(1 + h\bar{u}_k)\bar{\psi}_{k+1} &= \mu\bar{\phi}_k
\end{align*}
\]

(4.45)

\[
\begin{align*}
(1 + hu_k)(1 + hv_k)\bar{\phi}_k + h\lambda\bar{\phi}_{k-1} &= \phi_k \\
(1 + hv_{k-1})(1 + hu_k)\bar{\psi}_k + h\lambda\bar{\psi}_{k-1} &= \psi_k
\end{align*}
\]

After the change of variables \((k, t) \mapsto (k, \tau)\) this takes the form:

\[
\begin{align*}
\bar{u}_k(1 + hv_{k-1})\bar{\phi}_k + \lambda\bar{\phi}_{k-1} &= \mu\bar{\psi}_k \\
\bar{v}_k + \lambda^{-1}\bar{v}_k(1 + h\bar{u}_k)\bar{\psi}_{k+1} &= \mu\bar{\phi}_k
\end{align*}
\]

(4.46)

\[
\begin{align*}
(1 + hu_k)(1 + hv_k)\bar{\phi}_k + h\lambda\bar{\phi}_{k-1} &= \phi_k \\
(1 + h\bar{v}_{k-1})(1 + hu_k)\bar{\psi}_k + h\lambda\bar{\psi}_{k-1} &= \psi_k
\end{align*}
\]
Now we use the equations of motion (4.30) to bring the first pair (the spectral problem) into the form

\[
\begin{align*}
\tilde{u}_k(1 + h\tilde{v}_k)\tilde{\phi}_k + \lambda \tilde{\phi}_{k-1} &= \mu \tilde{\psi}_k \\
\tilde{\psi}_k + \lambda^{-1} \tilde{v}_k(1 + h\tilde{u}_{k+1})\tilde{\phi}_{k+1} &= \mu \tilde{\phi}_k
\end{align*}
\] (4.43)

The second pair (the evolutionary problem) may be presented as

\[
\begin{align*}
(1 + h\tilde{u}_k)(1 + h\tilde{v}_k)\tilde{\phi}_k &= \phi_k - h\lambda \tilde{\phi}_{k-1} \\
(1 + h\tilde{v}_{k-1})(1 + h\tilde{u}_k)\tilde{\psi}_k &= \psi_k - h\lambda \tilde{\psi}_{k-1} \iff \tilde{\phi} = \Delta_2^{-1}\Delta_1^{-1}\mathcal{F}\phi \\
\tilde{\psi} &= \Delta_1^{-1}\Delta_3^{-1}\mathcal{F}\psi
\end{align*}
\] (4.44)

Now we transform the equations describing the spectral problems. From the first equations of the two pairs in (4.43), (4.44) it is easy to derive:

\[\phi_k - (1 + h\tilde{v}_k)\tilde{\phi}_k = h\mu \tilde{\psi}_k\]

and using this in the first equation in (4.43), we bring the latter into the form

\[\tilde{u}_k(1 + h\tilde{v}_k)\tilde{\phi}_k + h\lambda (1 + h\tilde{v}_{k-1})\tilde{\phi}_{k-1} = \mu (\tilde{\psi}_k - h\lambda \tilde{\psi}_{k-1})\] (4.45)

From the second equation in (4.44) we find the value

\[\frac{1}{1 + h\tilde{v}_k}(\tilde{\psi}_k + \lambda^{-1} \tilde{v}_k \tilde{\psi}_{k+1}) = \mu \tilde{\phi}_k\] (4.46)

Now we can put the spectral problems in (4.45), (4.46) into the matrix form:

\[
\begin{align*}
\mathcal{F}^{-1}U\Delta_2 \tilde{\phi} &= \mu \tilde{\psi} \\
\Delta_2^{-1}\mathcal{V}\tilde{\psi} &= \mu \tilde{\phi}
\end{align*}
\] (4.47)

The compatibility conditions of the linear problems (4.44), (4.47) coincide with (4.41).
4.4 Explicit dVL → RVL

Now we have obtained two Lax matrices for the explicit dVL:

\[
L_-(u, v, \lambda) = UVC^{-1} \quad \text{and} \quad L_+(u, v, \lambda) = F^{-1}UV
\]

The relation between them is the following:

\[
\Omega_1 U(\lambda) \Omega_2^{-1} = V^T(\lambda'), \quad \Omega_2 V(\lambda) \Omega_1^{-1} = U^T(\lambda') \quad \Omega_1 C(\lambda) \Omega_1^{-1} = F^T(\lambda')
\]

so that

\[
\Omega_1 L_-(\lambda) \Omega_1^{-1} = L_+(\lambda')
\]

where \( \lambda' = \alpha^{-1} \lambda^{-1} \), \( \alpha = (u_1v_1...u_Nv_N)^{1/N} \) and

\[
\Omega_1 = \text{diag}(1, \alpha^{-1}u_1v_1, \alpha^{-2}u_1u_2v_2, \ldots, \alpha^{-N+1}u_1v_{N-1}u_Nv_N)
\]

\[
\Omega_2 = \text{diag}(1, \alpha^{-1}v_1u_2, \alpha^{-2}v_1u_2v_2u_3, \ldots, \alpha^{-N+1}v_1u_2...v_{N-1}u_N)
\]

We postulate that the hypothetic relativistic Volterra lattice (RVL) lives in the hierarchy associated to these Lax matrices, and it remains to identify this hierarchy. (The above relation between \( L_\pm \) assures that the both Lax matrices generate one and the same hierarchy). We start with the Lax matrix \( L_- \).

**Theorem 4.8** The triples \((U, C^{-1}, V)\) form a Poisson Lax matrix map \( V(u, v) \mapsto g \otimes g \otimes g \), if \( V(u, v) \) carries the Poisson bracket

\[
\{u_k, v_k\}_2 = -u_kv_k, \quad \{v_k, u_{k+1}\}_2 = -v_ku_{k+1}
\]

and \( g \otimes g \otimes g \) carries the Poisson bracket \( \text{PB}_2(A_1, A_2, S) \) defined by the operators

\[
A_1 = \begin{pmatrix}
A_1 & -S & -S \\
S^* & A_1 & S^* \\
S^* & -S & A_1
\end{pmatrix}, \quad
A_2 = \begin{pmatrix}
A_2 & -S^* & -S^* \\
S & A_2 & -S^* \\
S & S & A_2
\end{pmatrix}, \quad
S = \begin{pmatrix}
S & S & S \\
S & -S^* & -S^* \\
S & -S & -S^*
\end{pmatrix}
\]

The Lax matrix maps \( L_- = UVC^{-1}, \ VC^{-1}U, \ C^{-1}UV : V(u, v) \mapsto g \) are Poisson, if the target \( g \) is equipped with \( \text{PB}_2(A_1, A_2, S) \). The hierarchy of continuous time flows is described by the Lax triads

\[
\dot{U} = \pm U \cdot \pi_\pm \left(f(VC^{-1}U)\right) \mp \pi_\pm \left(f(UVC^{-1})\right) \cdot U
\]

\[
\dot{C} = \pm C \cdot \pi_\pm \left(f(C^{-1}UV)\right) \mp \pi_\pm \left(f(UVC^{-1})\right) \cdot C
\]

\[
\dot{V} = \pm V \cdot \pi_\pm \left(f(C^{-1}UV)\right) \mp \pi_\pm \left(f(VC^{-1}U)\right) \cdot V
\]
As a consequence, the evolution of the matrix \( L = UVC^{-1} \) is described by the standard Lax equation
\[
\dot{L} = [L, \pm \pi_{\pm}(f(L))]\]

In particular, the "first" flow corresponding to \( f(L) = L \), coincides with the RVL (1.13), and the auxiliary matrices in the Lax triads are given by
\[
\pi_{+}(UVC^{-1}) = \sum_{k=1}^{N}(u_k + v_{k-1} + hu_{k-1}v_{k-1})E_{kk} + \lambda \sum_{k=1}^{N}E_{k+1,k} \tag{4.53}
\]
\[
\pi_{+}(C^{-1}UV) = \sum_{k=1}^{N}(u_k + v_{k-1} + hu_k v_k)E_{kk} + \lambda \sum_{k=1}^{N}E_{k+1,k} \tag{4.54}
\]
\[
\pi_{+}(VC^{-1}U) = \sum_{k=1}^{N}(u_k + v_k + hu_k v_k)E_{kk} + \lambda \sum_{k=1}^{N}E_{k+1,k} \tag{4.55}
\]

The map (4.30) belongs to this hierarchy (in particular, is Poisson with respect to the bracket (4.48)) and corresponds to \( f(L) = h^{-1} \log(I + hL) \).

Proof. The first statement is proved with the help of the tensor notations for the quadratic \( r \)-matrix brackets, along the same lines as the proof of analogous statements in [S2], [S3]. All other statements, except the last one, are consequences of the first one and Theorem 2.6. To identify the place of the map (4.30) in this hierarchy, we have, referring to Theorem 4.6, to prove the following relations:
\[
C^{-1} = \Pi_{-}(I + hUVC^{-1}), \quad C^{-1}_2 = \Pi_{-}(I + hVC^{-1}U), \quad \bar{C}^{-1} = \Pi_{-}(I + hC^{-1}UV)
\]

Clearly, it is enough to prove only the first of them, but it follows immediately from
\[
I + hUVC^{-1} = (C + hUV)C^{-1}
\]

and
\[
C + hUV = \sum_{k=1}^{N}(1 + hu_k + hv_{k-1})E_{kk} + h\lambda \sum_{k=1}^{N}E_{k+1,k} \in G_+, \quad C^{-1} \in G_-
\]

The proof is complete.

Turning to the case of the Lax matrix \( L_+ \), we have the following results.

Theorem 4.9 The hierarchy associated with the Lax matrix
\[
L_+ = F^{-1}UV
\]
is described by the following Lax triads:
\[
\dot{U} = A_1 U - U A_2, \quad \dot{V} = A_2 V - V A_0
\]
which have to be supplemented by the identity
\[
\dot{F} = 0 = A_1 F - F A_0
\]
so that for the matrix \( L_+ = F^{-1} U V \) there holds the Lax equation
\[
\dot{L}_+ = [A_0, L_+]
\]
Here
\[
A_0 = \pi_-(f(L_+)) - \sigma(f(L_+))
\]
and the matrices \( A_{1,2} \) are uniquely defined by the condition of compatibility of the equations of the Lax triads (4.56), (4.57). In particular, the "first" flow of the hierarchy corresponding to \( f(L) = L \), coincides with RVL (1.13), and for this flow
\[
A_0 = \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1} - h \sum_{k=1}^{N} u_k v_k E_{kk}
\]
\[
A_1 = \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k E_{k,k+1} - h \sum_{k=1}^{N} u_{k-1} v_{k-1} E_{kk}
\]
\[
A_2 = \lambda^{-1} \sum_{k=1}^{N} u_k v_{k+1} E_{k,k+1} - h \sum_{k=1}^{N} u_k v_{k} E_{kk}
\]
Proof goes by a direct check. We define \( A_0 \) by (4.59), then put \( A_1 = F A_0 F^{-1} \in g_0 \oplus g_- \). The matrix \( A_2 \in g_0 \oplus g_- \) may be defined in two ways to make either of two Lax triads in (4.56) consistent. These two definitions have to be consistent, because the choice (4.59) assures the validity of the Lax equation (4.58) for the matrix \( L_+ \), due to Theorem 3.8.

Comparing the Lax matrices \( L_\pm \) for the RTL hierarchy (see (3.40), (3.50)) and for the RVL hierarchy (see (4.36), (4.41)), we see that the Miura map between two hierarchies formally coincides with the Miura map between the TL hierarchy and the VL hierarchy, because it is equivalent to identifying \( L(a,d,\lambda) \) with \( U(u,v,\lambda)V(u,v,\lambda) \), where \( L,U,V \) are the matrices from the nonrelativistic theories. This Miura map is
\[
M_1(u,v) = (a,d) : \begin{cases} 
  d_k = u_k + v_{k-1} \\
  a_k = u_kv_k 
\end{cases}
\]
This Miura map is Poisson with respect to the invariant quadratic Poisson brackets, since these also formally coincide with the invariant quadratic Poisson brackets of the nonrelativistic hierarchies. Recall that also the cubic Poisson bracket (3.53) is known for the RTL
hierarchy. It turns out that its pull–back by the Miura map is given by the following nice formulas:

\[
\begin{align*}
\{u_k, v_k\}_3 &= -u_k v_k (u_k + v_k + h u_k v_k), \\
\{u_k, u_{k+1}\}_3 &= -u_k v_k u_{k+1} (1 + h u_k), \\
\{v_k, u_{k+1}\}_3 &= -v_k u_{k+1} v_{k+1} (1 + h v_{k+1}) \\
\{v_k, u_{k+2}\}_3 &= -h v_k u_{k+1} v_{k+1} u_{k+2}
\end{align*}
\] (4.63)

Finally, let us comment on the twin RVL flow (1.14). Obviously, this system, as well as the corresponding Miura map \(M_2\), may be obtained upon the renaming \(u_k \rightarrow v_k, v_k \rightarrow u_{k+1}\). The Lax matrices for the corresponding hierarchy arise from the ones previously concerned upon change \(UV\) to \(VU\). The explicit discretization of the VL living in this hierarchy reads:

\[
\bar{u}_k (1 + h \bar{v}_{k-1}) = u_k (1 + h \bar{v}_k), \quad \bar{v}_k (1 + h u_k) = v_k (1 + h u_{k+1})
\] (4.64)

(here, knowing \(u, v\), one calculates first \(\bar{v}\), and then \(\bar{u}\)).

5 Bogoyavlensky lattices

5.1 BL2

A class of systems called Bogoyavlensky lattices (although discovered several times independently before his papers [B], see the references in [S5], and also [K1]) serve as generalizations of the Volterra lattice. We consider here only one representative of this class, which is however typical.

The integrable lattice system considered in the present section will be called BL2. Its phase space (in the periodic case) is

\[
\mathcal{B} = \mathbb{R}^{3N}(u_1, v_1, w_1, \ldots, u_N, v_N, w_N)
\] (5.1)

The equations of motion read:

\[
\begin{align*}
\dot{u}_k &= u_k (v_k + w_k - v_{k-1} - w_{k-1}) \\
\dot{v}_k &= v_k (u_{k+1} + w_k - u_{k-1} - w_{k-1}) \\
\dot{w}_k &= w_k (u_{k+1} + v_{k+1} - u_k - v_k)
\end{align*}
\] (5.2)

This system is Hamiltonian with the local Poisson brackets on \(\mathcal{B}\)

\[
\begin{align*}
\{u_k, v_k\}_2 &= -u_k v_k, & \{v_k, u_{k+1}\}_2 &= -v_k u_{k+1} \\
\{u_k, w_k\}_2 &= -u_k w_k, & \{w_k, u_{k+1}\}_2 &= -w_k u_{k+1} \\
\{v_k, w_k\}_2 &= -v_k w_k, & \{w_k, v_{k+1}\}_2 &= -w_k v_{k+1}
\end{align*}
\] (5.3)
and the Hamilton function

\[ H_1(u, v, w) = \sum_{k=1}^{N} u_k + \sum_{k=1}^{N} v_k + \sum_{k=1}^{N} w_k \]  

(5.4)

Notice that the system BL2 is more familiar in the form

\[ \dot{a}_k = a_k(a_{k+1} + a_{k+2} - a_{k-1} - a_{k-2}) \]

which arises upon the re-naming of variables

\[ u_k = a_{3k-2}, \quad v_k = a_{3k-1}, \quad w_k = a_{3k} \]

We use here the Lax representation of BL2 living in \( g = g \otimes g \otimes g \) and introduced in [S5]. Consider the following three matrices:

\[
U(u, v, w, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} u_k E_{k,k} = \mathcal{E} + u
\]  

(5.5)

\[
V(u, v, w, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} = I + v \mathcal{E}^{-1}
\]  

(5.6)

\[
W(u, v, w, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} w_k E_{k,k+1} = I + w \mathcal{E}^{-1}
\]  

(5.7)

These formulas define the “Lax matrix” \((U, V, W)\) : \( \mathcal{B} \mapsto g \otimes g \otimes g \). The flow (5.2) is equivalent to either of the following Lax equations in \( g \otimes g \otimes g \):

\[
\begin{align*}
\dot{U} &= UB_3 - B_1 U \\
\dot{V} &= VB_1 - B_2 V \\
\dot{W} &= WB_2 - B_3 W
\end{align*}
\]  

or

\[
\begin{align*}
\dot{U} &= A_1 U - U A_3 \\
\dot{V} &= A_2 V - V A_1 \\
\dot{W} &= A_3 V - V A_2
\end{align*}
\]  

(5.8)

with the matrices

\[
B_1 = \pi_+(UWV), \quad B_2 = \pi_+(VUW), \quad B_3 = \pi_+(WVU)
\]  

(5.9)

\[
A_1 = \pi_-(UWV), \quad A_2 = \pi_-(VUW), \quad A_3 = \pi_-(WVU)
\]  

(5.10)

so that

\[
B_1 = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_{k-1} + w_{k-1}) E_{kk}
\]  

(5.11)

\[
B_2 = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_k + w_{k-1}) E_{kk}
\]  

(5.12)
The Lax equations (5.8) allow the following $r$–matrix interpretation.

**Theorem 5.1** The Lax matrix map $(U,V,W) : B \mapsto g \otimes g \otimes g$ is Poisson, if $B$ is equipped with the bracket $\{ , \}$ and $g \otimes g \otimes g$ is equipped with the bracket $\text{PB}_2(A_1, A_2, S)$ corresponding to the operators $A_1, A_2, S$ defined in (5.43). The monodromy maps $T_{1,2,3} : g \otimes g \otimes g \mapsto g$,

$$T_1(U,V,W) = UWV, \quad T_2(U,V,W) = VUV, \quad T_3(U,V,W) = WVU$$

are Poisson, if the target space $g$ is equipped with the bracket $\text{PB}_2(A_1, A_2, S)$.

### 5.2 BL2 $\rightarrow$ dBL2

To find an integrable time discretization for the flow BL2, we again apply the recipe of Sect. 2.3 with $F(T) = I + hT$, i.e. we consider the map described by the discrete time "Lax triads"

$$\begin{align*}
\begin{cases}
\bar{U} = B_3^{-1}UB_3 & \text{or} & \bar{U} = A_1UA_3^{-1} \\
\bar{V} = B_2^{-1}VB_1 \\
\bar{W} = B_3^{-1}WB_2
\end{cases}
\end{align*}$$

(5.14)

with

$$B_1 = \Pi_+(I + hUWV), \quad B_2 = \Pi_+(I + hVUV), \quad B_3 = \Pi_+(I + hWVU)$$

$$A_1 = \Pi_-(I + hUWV), \quad A_2 = \Pi_-(I + hVUV), \quad A_3 = \Pi_-(I + hWVU)$$

**Theorem 5.2** Consider the change of variables $\mathcal{B}(u,v,w) \mapsto \mathcal{B}(u,v,w)$ defined by the formulas

$$\begin{align*}
u_k &= u_k(1 + hw_{k-1})(1 + hv_{k-1}) \\
v_k &= v_k(1 + hu_k)(1 + hw_{k-1}) \\
w_k &= w_k(1 + hv_k)(1 + hu_k)
\end{align*}$$

(5.15)
The discrete time Lax equations (5.14) are equivalent to the map \((u, v, w) \mapsto (\tilde{u}, \tilde{v}, \tilde{w})\) which in coordinates \((u, v, w)\) is described by the following equations:

\[
\tilde{u}_k(1 + h\tilde{v}_{k-1})(1 + h\tilde{w}_{k-1}) = u_k(1 + h v_k)(1 + h w_k) \\
\tilde{v}_k(1 + h\tilde{u}_k)(1 + h\tilde{w}_{k-1}) = v_k(1 + h u_{k+1})(1 + h w_k) \\
\tilde{w}_k(1 + h\tilde{u}_k)(1 + h\tilde{v}_k) = w_k(1 + h u_{k+1})(1 + h v_{k+1})
\] (5.16)

**Proof** is parallel to the proof of Theorem 4.3. The formulas (5.15) allow to find the factors \(B_{1,2,3}, A_{1,2,3}\) in a closed form. Indeed, with the help of (5.15) we can represent (5.5)–(5.7) as

\[
U = \sum_{k=1}^{N} u_k(1 + h w_{k-1})(1 + h v_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} 
\] (5.17)

\[
V = I + \lambda^{-1} \sum_{k=1}^{N} v_k(1 + h u_k)(1 + h w_{k-1})E_{k,k+1} 
\] (5.18)

\[
W = I + \lambda^{-1} \sum_{k=1}^{N} w_k(1 + h v_k)(1 + h u_k)E_{k,k+1} 
\] (5.19)

From these formulas we derive the following expressions for the factors \(B_i = \Pi_+(I + hT_i)\) and \(A_i = \Pi_-(I + hT_i), i = 1, 2, 3:\n
\[
B_1 = \sum_{k=1}^{N} (1 + h u_k)(1 + h w_{k-1})(1 + h v_{k-1})E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} 
\] (5.20)

\[
B_2 = \sum_{k=1}^{N} (1 + h v_k)(1 + h u_k)(1 + h w_{k-1})E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} 
\] (5.21)

\[
B_3 = \sum_{k=1}^{N} (1 + h w_k)(1 + h v_k)(1 + h u_k)E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} 
\] (5.22)

\[
A_1 = I + h \lambda^{-1} \sum_{k=1}^{N} \left( w_{k-1}v_k + u_k v_k + u_k w_k + h w_{k-1}u_k v_k + h u_k v_k w_k \right) E_{k,k+1} \\
+ h \lambda^{-2} \sum_{k=1}^{N} u_k w_k v_{k+1}(1 + h v_k)(1 + h w_k)(1 + h u_{k+1})E_{k,k+2} 
\] (5.23)

\[
A_2 = I + h \lambda^{-1} \sum_{k=1}^{N} \left( u_k w_k + v_k w_k + v_k u_{k+1} + h u_k v_k w_k + h v_k w_k u_{k+1} \right) E_{k,k+1} \\
+ h \lambda^{-2} \sum_{k=1}^{N} v_k u_{k+1} w_{k+1}(1 + h w_k)(1 + h u_{k+1})(1 + h v_{k+1})E_{k,k+2} 
\] (5.24)
\[ A_3 = I + h\lambda^{-1} \sum_{k=1}^{N} (v_k u_{k+1} + w_k v_{k+1} + w_k v_{k+1} + h v_k w_{k+1} + h w_k u_{k+1} v_{k+1}) E_{k,k+1} \]

\[ + h\lambda^{-2} \sum_{k=1}^{N} w_k v_{k+1} u_{k+2} (1 + h u_{k+1}) (1 + h v_{k+1}) (1 + h w_{k+1}) E_{k,k+2} \] (5.25)

Now the equations of motion (5.16) follow directly from the \( B \)-version of the discrete time Lax triads (5.14).

The map (5.16), denoted hereafter by \( dBL2 \), serves as a difference approximation to the flow \( BL2 \) (5.2). According to the general properties of discretizations of Sect. 2.3, this map in the coordinates \((u, v, w)\) is Poisson with respect to the bracket (5.3). Unfortunately, this bracket becomes highly nonlocal in the coordinates \((u, v, w)\). After the re-naming

\[ u_k = a_{3k-2}, \quad v_k = a_{3k-1}, \quad w_k = a_{3k} \]

the map \( dBL2 \) turns into

\[ \tilde{a}_k (1 + h a_{k-1}) (1 + h a_{k-2}) = a_k (1 + h a_{k+1}) (1 + h a_{k+2}) \]

and in this form it appeared for the first time in [THO], and then in [PN], [S4].

**5.3 \( dBL2 \rightarrow \) explicit \( dBL2 \)**

Following the scheme used already for \( dTL \) and \( dVL \), we extract an explicit discretization for the flow \( BL2 \) from the map \( dBL2 \). To this end we denote

\[ u_k(\tau) = u_k(t) = u_k(\tau - kh) \]
\[ v_k(\tau) = v_k(t) = v_k(\tau - kh) \]
\[ w_k(\tau) = w_k(t) = w_k(\tau - kh) \] (5.26)

These variables satisfy the following difference equations with respect to the discrete time \( \tau \):

\[ \tilde{u}_k (1 + h v_{k-1}) (1 + h w_{k-1}) = u_k (1 + h v_k) (1 + h w_k) \]
\[ \tilde{v}_k (1 + h \tilde{u}_k) (1 + h w_{k-1}) = v_k (1 + h \tilde{u}_{k+1}) (1 + h w_k) \]
\[ \tilde{w}_k (1 + h \tilde{u}_k) (1 + h \tilde{v}_k) = w_k (1 + h \tilde{u}_{k+1}) (1 + h \tilde{v}_{k+1}) \] (5.27)

(5.28)

This is indeed an explicit discretization, because, knowing \((u, v, w)\), one calculates explicitly \( \tilde{u}, \tilde{v}, \tilde{w} \) (in this order). We now turn to finding a Lax representation for this map. Unlike the previous cases, we could find reasonable formulas only starting from the \( B \)-version of the Lax representation for \( dBL2 \).
Theorem 5.3 The map (5.27) allows the following Lax representation:

\[
\begin{align*}
\mathcal{F}^{-1} \tilde{U} \Delta_2 \Delta_3 &= (\tilde{\Delta}_1^{-1} \Delta_4^{-1} \Delta_5^{-1} \mathcal{F}) \cdot \mathcal{F}^{-1} \tilde{U} \Delta_2 \Delta_3 \cdot (\mathcal{F}^{-1} \Delta_1 \tilde{\Delta}_2 \tilde{\Delta}_3) \\
\tilde{\Delta}_2^{-1} \tilde{V} &= (\tilde{\Delta}_2^{-1} \Delta_1^{-1} \Delta_5^{-1} \mathcal{F}) \cdot \Delta_2^{-1} \tilde{V} \cdot (\mathcal{F}^{-1} \Delta_5 \Delta_1 \tilde{\Delta}_1) \\
\tilde{\Delta}_3^{-1} \tilde{W} &= (\tilde{\Delta}_3^{-1} \Delta_2^{-1} \Delta_1^{-1} \mathcal{F}) \cdot \Delta_3^{-1} \tilde{W} \cdot (\mathcal{F}^{-1} \Delta_5 \Delta_1 \tilde{\Delta}_2)
\end{align*}
\]  

(5.29)

with the matrices

\[
U = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} u_k E_{k,k} 
\]

(5.30)

\[
V = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} 
\]

(5.31)

\[
W = I + \lambda^{-1} \sum_{k=1}^{N} w_k E_{k,k+1} 
\]

(5.32)

and

\[
\Delta_1 = \text{diag}(1 + h u_k), \quad \Delta_2 = \text{diag}(1 + h v_k), \quad \Delta_3 = \text{diag}(1 + h w_k) 
\]

(5.33)

\[
\Delta_4 = \text{diag}(1 + h v_{k-1}), \quad \Delta_5 = \text{diag}(1 + h w_{k-1}) 
\]

(5.34)

The Lax representation (5.29) implies also

\[
\mathcal{L}_+ = (\tilde{\Delta}_1^{-1} \Delta_4^{-1} \Delta_5^{-1} \mathcal{F}) \cdot \mathcal{L}_+ \cdot (\mathcal{F}^{-1} \Delta_5 \Delta_1 \tilde{\Delta}_1) 
\]

(5.35)

with the Lax matrix

\[
\mathcal{L}_+ = (\mathcal{F}^{-1} U \Delta_2 \Delta_3)(\Delta_3^{-1} W)(\Delta_2^{-1} V) = \mathcal{F}^{-1} U \Delta_2 W \Delta_5^{-1} V 
\]

(5.36)

Proof. We start with the following Lax representation of the map dBL2:

\[
\tilde{U} = B_1^{-1} U B_3, \quad \tilde{V} = B_2^{-1} V B_1, \quad \tilde{W} = B_3^{-1} W B_2
\]

with the matrices (5.17)–(5.19), (5.20)–(5.22). These Lax equations are the compatibility conditions of the following linear problems:

\[
\begin{align*}
\tilde{U} \phi &= \mu \psi \\
\tilde{V} \psi &= \mu \chi \\
\tilde{W} \chi &= \mu \phi
\end{align*}
\]

\[
\begin{align*}
B_3 \phi &= \phi \\
B_1 \psi &= \psi \\
B_2 \chi &= \chi
\end{align*}
\]
(it turns out to be more convenient to take the tilded version of the spectral problem, i.e. of the equations in the first triple). In components the equations of the spectral linear problem take the form:

\[
\begin{align*}
\bar{u}_k(1 + h\bar{w}_{k-1})(1 + h\bar{v}_{k-1})\bar{\phi}_k + \lambda\bar{\phi}_{k-1} &= \mu\bar{\psi}_k \\
\bar{\psi}_k + \lambda^{-1}\bar{v}_k(1 + h\bar{u}_k)(1 + h\bar{w}_{k-1})\bar{\psi}_{k+1} &= \mu\bar{\chi}_k \\
\bar{\chi}_k + \lambda^{-1}\bar{w}_k(1 + h\bar{v}_k)(1 + h\bar{u}_k)\bar{\chi}_{k+1} &= \mu\bar{\phi}_k
\end{align*}
\]

while the equations of the evolutionary linear problem take the form

\[
\begin{align*}
(1 + h\bar{u}_k)(1 + h\bar{v}_k)(1 + h\bar{w}_{k-1})\bar{\phi}_k + h\lambda\bar{\phi}_{k-1} &= \phi_k \\
(1 + h\bar{v}_{k-1})(1 + h\bar{w}_{k-1})(1 + h\bar{u}_k)\bar{\psi}_k + h\lambda\bar{\psi}_{k-1} &= \psi_k \\
(1 + h\bar{w}_{k-1})(1 + h\bar{u}_k)(1 + h\bar{v}_k)\bar{\chi}_k + h\lambda\bar{\chi}_{k-1} &= \chi_k
\end{align*}
\]

After the change of variables \((k, t) \mapsto (k, \tau)\) the spectral equations take the form:

\[
\begin{align*}
\bar{u}_k(1 + h\bar{v}_{k-1})(1 + h\bar{w}_{k-1})\bar{\phi}_k + \lambda\bar{\phi}_{k-1} &= \mu\bar{\psi}_k \\
\bar{\psi}_k + \lambda^{-1}\bar{v}_k(1 + h\bar{u}_k)(1 + h\bar{w}_{k-1})\bar{\psi}_{k+1} &= \mu\bar{\chi}_k \\
\bar{\chi}_k + \lambda^{-1}\bar{w}_k(1 + h\bar{v}_k)(1 + h\bar{u}_k)\bar{\chi}_{k+1} &= \mu\bar{\phi}_k
\end{align*}
\]

and the evolutionary equations take the form

\[
\begin{align*}
(1 + h\bar{u}_k)(1 + h\bar{v}_k)(1 + h\bar{w}_k)\bar{\phi}_k + h\lambda\bar{\phi}_{k-1} &= \phi_k \\
(1 + h\bar{v}_{k-1})(1 + h\bar{w}_{k-1})(1 + h\bar{u}_k)\bar{\psi}_k + h\lambda\bar{\psi}_{k-1} &= \psi_k \\
(1 + h\bar{w}_{k-1})(1 + h\bar{u}_k)(1 + h\bar{v}_k)\bar{\chi}_k + h\lambda\bar{\chi}_{k-1} &= \chi_k
\end{align*}
\]

or, in the matrix notations,

\[
\begin{align*}
\bar{\phi} &= \Delta_3^{-1}\Delta_2^{-1}\Delta_1^{-1}F\phi \\
\bar{\psi} &= \Delta_1^{-1}\Delta_5^{-1}\Delta_4^{-1}F\psi \\
\bar{\chi} &= \Delta_2^{-1}\Delta_1^{-1}\Delta_6^{-1}F\chi
\end{align*}
\]

(5.38)
Now we use the equations of motion (5.27) to bring the spectral equations into the form

\[
\begin{cases}
    u_k(1 + hv_k)(1 + hw_k)\tilde{\phi}_k + \lambda \tilde{\phi}_{k-1} = \mu \tilde{\psi}_k \\
    \tilde{\psi}_k + \lambda^{-1}v_k(1 + hw_k)(1 + h\tilde{u}_{k+1})\tilde{\psi}_{k+1} = \mu \tilde{\chi}_k \\
    \tilde{\chi}_k + \lambda^{-1}w_k(1 + h\tilde{u}_{k+1})(1 + h\tilde{\upsilon}_{k+1})\tilde{\chi}_{k+1} = \mu \tilde{\phi}_k
\end{cases}
\]  

(5.39)

Next, we transform the spectral equations further with the help of the evolutionary ones. From the first equations in (5.37), (5.39) we derive:

\[
\phi_k - (1 + hv_k)(1 + hw_k)\tilde{\phi}_k = h \mu \tilde{\psi}_k
\]

and using this in the first equation in (5.39), we bring the latter into the form

\[
u_k(1 + hv_k)(1 + hw_k)\tilde{\phi}_k + h\lambda(1 + hv_{k-1})(1 + hw_{k-1})\tilde{\phi}_{k-1} = \mu(\tilde{\psi}_k - h\lambda \tilde{\psi}_{k-1})
\]

(5.40)

From the second equation in (5.37) we find the expression

\[
(1 + hw_k)(1 + h\tilde{u}_{k+1})\tilde{\psi}_{k+1} = \frac{1}{1 + hv_k}(\tilde{\psi}_{k+1} - h\lambda \tilde{\psi}_k)
\]

and plugging this into the second equation in (5.39), we find:

\[
\frac{1}{1 + hv_k}(\tilde{\psi}_k + \lambda^{-1}v_k\tilde{\psi}_{k+1}) = \mu \tilde{\chi}_k
\]

(5.41)

Similarly, from the third equation in (5.37) we find the expression

\[
(1 + h\tilde{u}_{k+1})(1 + h\tilde{\upsilon}_{k+1})\tilde{\chi}_{k+1} = \frac{1}{1 + hw_k}(\tilde{\chi}_{k+1} - h\lambda \tilde{\chi}_k)
\]

and plugging this into the third equation in (5.39), we find:

\[
\frac{1}{1 + hw_k}(\tilde{\chi}_k + \lambda^{-1}w_k\tilde{\chi}_{k+1}) = \mu \tilde{\phi}_k
\]

(5.42)

Putting the equations (5.40), (5.41), (5.42) into the matrix notations, we finally find:

\[
\begin{cases}
    F^{-1}U \Delta_2 \Delta_3 \tilde{\phi} = \mu \tilde{\psi} \\
    \Delta_2^{-1}V \tilde{\psi} = \mu \tilde{\chi} \\
    \Delta_3^{-1}W \tilde{\chi} = \mu \tilde{\phi}
\end{cases}
\]

(5.43)

The compatibility conditions of (5.38), (5.43) coincide with (5.29).
5.4 Explicit dBL2 → ”relativistic” BL2

Our general philosophy and the previous theorem suggest the matrix

\[ L_+ = \mathcal{F}^{-1}U \Delta_2 W \Delta_2^{-1}V \] (5.44)

as the Lax matrix of the ”relativistic” BL2 hierarchy. The spirit of [GK1], however, would suggest

\[ L_+ = \mathcal{F}^{-1}UWV \] (5.45)

and this is the choice we actually follow in the next Theorem. However, there is no conflict between these two suggestions, because changing

\[ W \rightarrow \Delta_2 W \Delta_2^{-1} \]

amounts solely to the simple change of variables

\[ w_k \rightarrow w_k \frac{1 + hv_k}{1 + hv_{k+1}} \] (5.46)

(not touching \( u_k, v_k \)). In other words, in order to make the explicit discretization (5.27) belonging to the ”relativistic” BL2 hierarchy defined below, one needs to perform the change of variables (5.46). It turns out that the continuous time flows look more symmetric, if the Lax matrix is choosen as in (5.45), and the discrete time equations (5.46) look more symmetric by the choice (5.44) for the Lax matrix.

**Theorem 5.4** The hierarchy associated with the Lax matrix (5.45) is described by the following Lax triads:

\[ \dot{U} = A_1 U - U A_3, \quad \dot{V} = A_2 V - V A_0, \quad \dot{W} = A_3 W - W A_2 \] (5.47)

which have to be supplemented by the identity

\[ \dot{\mathcal{F}} = 0 = A_1 \mathcal{F} - \mathcal{F} A_0 \] (5.48)

so that for the matrix \( L_+ = \mathcal{F}^{-1}UWV \) there holds the Lax equation

\[ \dot{L}_+ = [A_0, L_+] \] (5.49)

Here

\[ A_0 = \pi_-(f(L_+)) - \sigma(f(L_+)) \] (5.50)
and the matrices $A_{1,2,3}$ are uniquely defined by the condition of compatibility of the equations of the Lax triads (5.47), (5.48). In particular, for the "first" flow of the hierarchy corresponding to $f(L) = L$, one has:

\[ A_0 = \lambda^{-1} \sum_{k=1}^{N} (w_{k-1} v_k + u_k v_k + u_k w_k + h u_k w_{k-1} v_k) E_{k,k+1} \]

\[ + \lambda^{-2} \sum_{k=1}^{N} u_k w_k v_{k+1} E_{k,k+2} - h \sum_{k=1}^{N} \gamma_k E_{kk} \]  

(5.51)

\[ A_1 = \lambda^{-1} \sum_{k=1}^{N} (w_{k-1} v_k + u_k v_k + u_k w_k + h u_k w_{k+1} v_k) E_{k,k+1} \]

\[ + \lambda^{-2} \sum_{k=1}^{N} u_k w_k v_{k+1} E_{k,k+2} - h \sum_{k=1}^{N} \gamma_{k-1} E_{kk} \]  

(5.52)

\[ A_2 = \lambda^{-1} \sum_{k=1}^{N} (u_k w_k + v_k w_k + v_k u_{k+1} + h u_k v_{k+1} w_k) E_{k,k+1} \]

\[ + \lambda^{-2} \sum_{k=1}^{N} v_k u_{k+1} w_{k+1} E_{k,k+2} - h \sum_{k=1}^{N} \gamma_k E_{kk} \]  

(5.53)

\[ A_3 = \lambda^{-1} \sum_{k=1}^{N} (v_k u_{k+1} + w_k u_{k+1} + w_k v_{k+1} + h w_k u_{k+1} v_{k+1}) E_{k,k+1} \]

\[ + \lambda^{-2} \sum_{k=1}^{N} w_k v_{k+1} u_{k+2} E_{k,k+2} - h \sum_{k=1}^{N} \gamma_k E_{kk} \]  

(5.54)

where

\[ \gamma_k = w_{k-1} v_k + u_k v_k + u_k w_k + h u_{k-1} w_{k-1} v_k + h u_k w_{k+1} v_k + h u_k w_k v_{k+1} \]  

(5.55)

The equations of motion of the "first" flow read:

\[ \dot{u}_k = u_k (v_k + w_k - v_k - w_k + h w_k v_{k+1} - h w_{k-1} v_k + h \gamma_k - h \gamma_{k-1}) \]

\[ \dot{v}_k = v_k (w_k + u_{k+1} - w_k - u_k + h u_k w_k - h u_{k-1} w_{k-1} + h \gamma_{k+1} - h \gamma_k) \]  

(5.56)

\[ \dot{w}_k = w_k (u_{k+1} + v_{k+1} - u_k - v_k + h u_{k+1} v_{k+1} - h u_k v_k + h \gamma_{k+1} - h \gamma_k) \]

Proof is analogous to the proof of Theorem 11.9. We define $A_0$ by (5.50) in accordance with the prescription of [GK1], then put $A_1 = \mathcal{F} A_0 \mathcal{F}^{-1} \in g_0 \oplus g_-$. The matrices $A_{2,3} \in g_0 \oplus g_-$ are uniquely defined by the condition of consistency of the Lax triads in (5.47). The results for the "first" flow (5.56) (= "relativistic" BL2) are merely the specialization of this construction for $f(L) = L$ and follow by a direct calculation.

The previous theorem allows to find the equations of motion of all flows of the hierarchy, and to calculate its integrals. Unfortunately, we still do not know its Hamiltonian structure, so that the problem of its integrability in the Liouville–Arnold sense remains open.
6 Conclusion

We discussed in this paper an approach to finding new integrable hierarchies based on the time–discretization. Namely, performing the discretization of a known integrable lattice system and then changing the Cauchy problem for the discrete time equation, we arrive at another discrete time equation which belongs, generally speaking, to an integrable hierarchy distinct from the one we started with. This new hierarchy is a regular deformation of the initial one, the deformation parameter being the time–step of the discretization.

In the Toda lattice case the new hierarchy turns out to be the relativistic Toda one, the deformation parameter acquiring the meaning of the inverse speed of light. This suggests to call in general the new hierarchies obtained by this procedure ”relativistic” ones. It would be interesting to find out, whether this notation allows some physical justification.

In any case, we have now a program of finding ”relativistic” deformations of a large number of known lattice hierarchies. In this connection the collection of local discretizations in [S6] may serve as a starting point. For the systems belonging to the lattice KP hierarchy, another procedure of finding ”relativistic” deformations was suggested in [GK1], which has a priori nothing in common with our one. However, in the examples of the present paper, both procedures lead to similar results. It would be important to find out whether there are some deeper principles behind this coincidence.

7 Acknowledgements

Y.S. gratefully acknowledges the stimulating correspondence with Boris Kupershmidt. On the early stage of this work he also benefited from discussions with Frank Nijhoff.
References

[AL] M.Ablowitz, J.Ladik. A nonlinear difference scheme and inverse scattering. *Stud. Appl. Math.* 55 (1976) 213–229; On solution of a class of nonlinear partial difference equations. *Stud. Appl. Math.* 57 (1977) 1–12.

[AM] M.Adler, P. van Moerbeke. Completely integrable systems, Kac–Moody algebras and curves. *Adv. Math.* 38 (1980) 267–317.

[B] O.I.Bogoyavlensky. Some constructions of integrable dynamical systems. *USSR Math. Izv.*, 31 (1988) 47–75; Integrable dynamical systems associated with the KdV equation. *USSR Math. Izv.*, 31 (1988) 435–454; The Lax representation with a spectral parameter for certain dynamical systems. *USSR Math. Izv.*, 32 (1989) 245–268.

[BR] M.Bruschi, O.Ragnisco. Recursion operator and Bäcklund transformations for the Ruijsenaars–Toda lattices. *Phys. Lett. A* 129 (1988) 21–25; Lax representation and complete integrability for the periodic relativistic Toda lattice. *Phys. Lett. A* 134 (1989) 365–370; The periodic relativistic Toda lattice: direct and inverse problem. *Inv. Probl.* 5 (1989) 389–405.

[D] P.A.Damianou. Master symmetries and $R$–matrices for the Toda lattice. *Lett. Math. Phys.* 20 (1990) 101–112; Multiple Hamiltonian structures for Toda type systems. *J. Math. Phys.* 35 (1994) 5511–5541.

[DLT1] P.Deift, L.-Ch.Li, and C.Tomei. Matrix factorizations and integrable systems. *Commun. Pure Appl. Math.* 42 (1989) 443–521.

[FT2] L.D.Faddeev, L.A.Takhtajan. *Hamiltonian methods in the theory of solitons*, Springer (1987).

[FV] L.D.Faddeev, A.Yu.Volkov. Hirota equation as an example of integrable symplectic map. *Lett. Math. Phys.* 32 (1994) 125–136.

[F] H.Flaschka. On the Toda lattice I. *Phys. Rev. B* 9 (1974) 1924–1925; On the Toda lattice II. Inverse scattering solution. *Progr. Theor. Phys.* 51 (1974) 703–716.
[GK1] J.Gibbons, B.A.Kupershmidt. Relativistic analogs of basic integrable systems. – In: *Integrable and superintegrable systems*, Ed. B.A.Kupershmidt, World Scientific, 1990, 207–231.

[GK2] J.Gibbons, B.A.Kupershmidt. Time discretizations of lattice integrable systems. *Phys. Lett. A*, **165** (1992) 105–110.

[H] R.Hirota. Nonlinear partial difference equations. I–V. *J. Phys. Soc. Japan* **43** (1977) 1423–1433, 2074–2078, 2079–2086; **45** (1978) 321–332; **46** (1978) 312–319.

[H1] R.Hirota. Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Japan* **50** (1981) 3785–3791.

[HT] R.Hirota, S.Tsujimoto. Conserved quantities of a class of nonlinear difference–difference equations. *J. Phys. Soc. Japan* **64** (1995) 3125–3127; RIMS Kokyuroku **868** (1994) 31; RIMS Kokyuroku **933** (1995) 105.

[HTI] R.Hirota, S.Tsujimoto, T.Imai. Difference scheme of soliton equations. In: *"Future directions of nonlinear dynamics in physical and biological systems"*, Eds. P.L.Christiansen, J.C.Eilbeck, and R.D.Parmentier (Plenum, 1993), 7–15.

[KM] M.Kac, P. van Moerbeke. On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices. *Adv. Math.* **16** (1975) 160–169.

[K1] B.A. Kupershmidt. Discrete Lax equations and differential–difference calculus. *Asterisque* **123** (1985).

[K2] B.A.Kupershmidt. Infinitely–precise space–time discretizations of the equation $u_t + uu_x = 0$. In: *Algebraic Aspects of Integrable Systems. In Memory of Irene Dorfman*, Birkhäuser (1996) 205–216.

[LP] L.-C.Li, S.Parmentier. Nonlinear Poisson structures and $r$–matrices. *Commun. Math. Phys.* **125** (1989) 545–563.

[M] S.V.Manakov. On the complete integrability and stochastization in discrete dynamical systems. *Zh. Exp. Theor. Phys.* **67** (1974) 543–555.

[MP] C.Morosi, L.Pizzocchero. $R$–matrix theory, formal Casimirs and the periodic Toda lattice. *J. Math. Phys.*, **37** (1996) 4484–4513.
[MV] J. Moser, A. P. Veselov. Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Commun. Math. Phys.* **139** (1991) 217–243.

[OR] W. Oevel, O. Ragnisco. $R$–matrices and higher Poisson brackets for integrable systems. *Physica A* **161** (1989) 181–220.

[OFZR] W. Oevel, B. Fuchssteiner, H. Zhang, and O. Ragnisco. Mastersymmetries, angle variables and recursion operator of the relativistic Toda lattice. *J. Math. Phys.* **30** (1989) 2664–2676.

[NPCQ] F. W. Nijhoff, V. G. Papageorgiou, H. W. Capel, and G. R. W. Quispel. The lattice Gel’fand–Dikii hierarchy. *Inv. Problems* **8** (1992) 597–621.

[PNC] V. G. Papageorgiou, F. W. Nijhoff, and H. W. Capel. Integrable mappings and nonlinear integrable lattice equations. *Phys. Lett. A* **147** (1990) 106–114; H. W. Capel, F. W. Nijhoff, and V. G. Papageorgiou. Complete integrability of Lagrangian mappings and lattices of KdV type. *Phys. Lett. A*, **155** (1991) 377–387.

[PN] V. G. Papageorgiou, F. W. Nijhoff. On some integrable discrete time systems associated with the Bogoyavlensky lattices. *Physica A* **228** (1996) 172–188.

[PGR] V. Papageorgiou, B. Grammaticos, and A. Ramani. Orthogonal polynomial approach to discrete Lax pairs for initial boundary value problems of the QD algorithm. *Lett. Math. Phys.* **34** (1995) 91–101.

[RSTS] A. G. Reyman, M. A. Semenov-Tian-Shansky. Group theoretical methods in the theory of finite dimensional integrable systems. In: *Encyclopaedia of mathematical science, v.16: Dynamical Systems VII*, Springer (1994) 116–225.

[R] S. N. M. Ruijsenaars. Relativistic Toda systems. *Commun. Math. Phys.* **133** (1990) 217–247.

[STS] M. A. Semenov-Tian-Shansky. What is a classical $r$–matrix? *Funct. Anal. Appl.* **17** (1983) 259–272; Classical $r$–matrices, Lax equations, Poisson Lie groups and dressing transformations. *Lecture Notes Phys.* **280** (1987) 174–214.

[S1] Yu. B. Suris. Generalized Toda chains in discrete time. *Algebra i Anal.* **2** (1990) 141–157; Discrete–time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices. *Phys. Lett. A* **145** (1990) 113–119.
Yu.B.Suris. On the bi–Hamiltonian structure of Toda and relativistic Toda lattices. *Phys. Lett. A* **180** (1993) 419–429.

Yu.B.Suris. Bi–Hamiltonian structure of the *qd* algorithm and new discretizations of the Toda lattice. *Phys. Lett. A* **206** (1995) 153–161.

Yu.B.Suris. A discrete–time relativistic Toda lattice. *J. Phys. A: Math. and Gen.* **29** (1996) 451–465; Integrable discretizations of the Bogoyavlensky lattices. *J. Math. Phys.* **37** (1996) 3982–3996.

Yu.B.Suris. Nonlocal quadratic Poisson algebras, monodromy map, and Bogoyavlensky lattices. *J. Math. Phys.* **38** (1997) 4179–4201.

Yu.B.Suris. Integrable discretizations for lattice systems: local equations of motion and their hamiltonian properties. – solv-int/9709003.

W.W.Symes. The *QR* algorithm and scattering for the finite nonperiodic Toda lattice. *Physica D* **4** (1982) 275–280.

S.Tsujimoto, R.Hirota, and S.Oishi. An extension and discretization of Volterra equation. *Techn. Report IEICE, NLP 92-90.*

A.P.Veselov. Integrable systems with discrete time and difference operators. *Funct. Anal. Appl.* **22** (1988) 1–13.

H.Zhang, G.-Zh.Tu, W.Oevel, and B.Fuchssteiner. Symmetries, conserved quantities, and hierarchies for some lattice systems with soliton structure. *J. Math. Phys.* **32** (1991) 1908–1918.