Electric Current Multipole Moments in Classical Electrodynamics

A. J. Silenko

Institute of Nuclear Problems, Belarusian State University, Minsk 220080, Belarus

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The general theory of electric current multipoles appearing due to the motion of magnetic dipoles and the change in these values or orientations are suggested. Static multipoles, including an anapole, are studied in detail.

§1. Introduction

Electric current multipoles (ECM) occur due to the motion of conductors carrying current and due to the variation of the current strength in conductors. While possessing a number of typical multipole properties, they have some interesting distinctions. Typically, electric charge multipoles occur due to a particular spatial distribution of charge. ECM, as distinct from charge multipoles, are generated only by conductors carrying a current. Noncompensated charges are absent in the rest system of any conductor, and the charge density is equal to zero. There are two types of ECM. In laboratory systems for the ECM occurring due to the motion of conductors carrying a current, the charge density is nonzero. Its appearance is due to the transformation of the charge and current densities as components of a four-vector. Only the charge density occurring in the motion of conductors determines electric multipole moments of the system. For this type of multipole, the electric current dipoles have been known since long ago.\(^1\) (The classical theory of electric current quadrupoles is given in Refs. 2) and 3), while their quantum theory is given in Refs. 4) and 5). The existence of the second type of ECM has been suggested by Miller.\(^6\) Multipoles of this type appear due to the change of current strength in conductors or due to the change of the spatial orientation of conductors carrying a current (or magnetic multipoles). The introduction of a magnetic current is a convenient approach for the description of these multipoles.\(^6\) It is obvious that electric multipoles of both types appear due to the arbitrary motion of conductors carrying current. This is a general case and will be analyzed in this work in terms of classical electrodynamics.

The investigation of ECM is of a great practical importance. The electric current dipole moment occurring due to the motion of a particle having a magnetic dipole moment determines the interaction of the particle’s spin with an electrostatic field. The electric current quadrupole moment (ECQM) appearing due to the orbital motion of nucleons can make a distinct contribution to the total quadrupole moments of nuclei.\(^3\) -\(^5\) As shown in Ref. 7), a moving anapole interacts with an inhomogeneous electric field. The investigation of this phenomenon is of great interest because it concerns the non-contact interaction of the anapole with the static external field. As
shown below, the interaction of an anapole with an inhomogeneous electric field is due to the possession of ECQM by a moving anapole.

§2. The field of electric current multipoles

The electric current multipole moments of microscopic objects are most interesting for the present investigation. For these objects, it is sufficient to investigate the motion of point magnetic dipoles rather than that of conductors of finite sizes carrying a current. In the case of macroscopic objects, when the sizes of conductors carrying a current should be taken into account, the formulae obtained for magnetic dipoles can usually be applied with the substitution of a set of point magnetic dipoles for conductors carrying a current (the magnetic sheets model).

We now obtain the expression for the electric field generated by point magnetic dipoles. The strength for this field is defined by the formula

$$E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t}.$$  (1)

Let \(r\) be a radius-vector of a point at which a point magnetic dipole is found, and let us define the strength of the field at a remote point having radius-vector \(R\), where \(|R| \gg |r|\). Since we analyze systems that can radiate electromagnetic waves, we emphasize that \(|R| \ll \lambda\), where \(\lambda\) is a characteristic wavelength of the radiation. In the reference system where the dipole is at rest, we have the scalar potential \(\phi' = 0\), and in the laboratory system,

$$\phi' = (v \cdot A)/c \equiv (\dot{r} \cdot A)/c,$$  (2)

where \(v\) is the velocity of the translational motion of a magnetic dipole. Since

$$A = \frac{\mu \times r'}{r'^3}, \quad r' = R - r,$$

by expansion of the quantity \(A\) into a series we obtain

$$A = -[\mu \times \nabla] \frac{1}{R} + (r \cdot \nabla)[\mu \times \nabla] \frac{1}{R} - \frac{1}{2} (r \cdot \nabla)(r \cdot \nabla)[\mu \times \nabla] \frac{1}{R} + \cdots + (-1)^{k+1} \frac{1}{k!} (r \cdot \nabla)^k[\mu \times \nabla] \frac{1}{R} + \cdots,$$  (3)

where \((r \cdot \nabla)^k\) is a product of \(k\) cofactors \((r \cdot \nabla)\), and \(|\mu \times \nabla| f(R) \equiv |\mu \times \nabla f(R)|\). Taking the identity \((v \cdot [\mu \times \nabla]) = ([v \times \mu] \cdot \nabla)\) into account, the electric field strength could be expressed as

$$E = E^{(1)} + E^{(2)} + E^{(3)},$$  (4)

$$E^{(1)} = -\nabla \phi, \quad \phi = \phi^{(1)} + \phi^{(2)} + \cdots, \quad \phi^{(1)} = - (d \cdot \nabla) \frac{1}{R},$$

$$\phi^{(2)} = -(r \cdot \nabla)(d \cdot \nabla) \frac{1}{R}, \quad \cdots, \quad \phi^{(k)} = (-1)^k \frac{1}{(k-1)!} (r \cdot \nabla)^{k-1}(d \cdot \nabla) \frac{1}{R}.$$  (5)
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\[ E^{(2)} = \nabla \times G, \quad G = G^{(0)} + G^{(1)} + G^{(2)} + \cdots, \quad G^{(0)} = \frac{\mu}{cR}, \quad G^{(1)} = \frac{1}{c} \frac{(\mathbf{r} \cdot \nabla) \dot{\mu}}{R}, \]

\[ G^{(2)} = -\frac{1}{2c} (\mathbf{r} \cdot \nabla)(\mathbf{r} \cdot \nabla) \frac{\mu}{R}, \quad \cdots, \quad G^{(k)} = (-1)^{k+1} \frac{1}{ck!} (\mathbf{r} \cdot \nabla)^k \frac{\mu}{R}, \quad (6) \]

\[ E^{(3)} = \nabla \times N, \quad N = N^{(1)} + N^{(2)} + \cdots, \quad N^{(1)} = \frac{1}{c} (\mathbf{v} \cdot \nabla) \frac{\mu}{R}, \]

\[ N^{(2)} = -\frac{1}{c} (\mathbf{r} \cdot \nabla)(\mathbf{v} \cdot \nabla) \frac{\mu}{R}, \quad \cdots, \quad N^{(k)} = (-1)^{k+1} \frac{1}{c(k-1)!} (\mathbf{r} \cdot \nabla)^{k-1} (\mathbf{v} \cdot \nabla) \frac{\mu}{R}, \quad (7) \]

where \( \mathbf{d} = [\mathbf{v} \times \mu]/c \). The formulae (4)–(7) describe an electric field generated both due to the translational motion of a magnetic dipole moment and due to the change of its value or orientation. The indices for \( \phi, G, N \) define the rank of the multipoles.

The quantity \(-\frac{1}{c} \frac{\partial A}{\partial t}\) in formula (1) corresponds to the sum \( E^{(2)} + E^{(3)} \). The potential \( \phi^{(1)} \) coincides with the potential generated by a charged dipole moment \( \mathbf{d} \), and formula (5) is the expansion in terms of multipoles. By separating the quadrupole and contact interactions in the potential \( \phi^{(2)} \), the latter can be expressed as

\[ \phi^{(2)} = \frac{1}{6} Q_{ij} \frac{\partial^2}{\partial X_i \partial X_j} \left( \frac{1}{R} \right) + \frac{1}{6} \tau \frac{\partial^2}{\partial X_k^2} \left( \frac{1}{R} \right), \quad (8) \]

\[ Q_{ij} = 3x_i d_j + 3d_i x_j - 2\delta_{ij} x_k d_k, \quad \tau = 2x_k d_k. \quad (9) \]

As is well known, the values \( \mathbf{d}, Q_{ij} \) and \( \tau \) represent the dipole moment, the quadrupole moment tensor and the mean square of the charge radius, respectively. For a system of charges, we have

\[ \mathbf{d} = \int \rho r dV, \quad Q_{ij} = \int \rho (3x_i x_j - \delta_{ij} r^2) dV, \quad \tau = \int \rho r^2 dV. \quad (10) \]

The fields \( E^{(2)} \) and \( E^{(3)} \) are solenoidal (\( \nabla \cdot E^{(2)} = \nabla \cdot E^{(3)} = 0 \)). For immobile magnetic dipoles (\( \mathbf{v} = 0 \)) \( E^{(1)} = E^{(3)} = 0 \), and their electric field is the field of ECM described in Ref. 6). The value \( \dot{\mu} \) could be nonzero due to the change both of the magnetic moment value and the direction of its orientation. The characteristic feature of this type of ECQM is the possibility of their description through the introduction of the magnetic current of density \( \mathbf{j}^{(m)} = \sum \dot{\mathbf{\mu}} \delta (\mathbf{r} - \mathbf{r}_i^{(m)}) \). This allows us to use the mathematical tools of electrodynamics for magnetic charges. \(^9\)

Formula (6) gives the expansion of the pseudovector potential \( \mathbf{G} \) in terms of degrees \( r/R \). However, in the general case this formula does not allow an expansion in terms of multipoles typical for usual electric currents. In particular, the relation typical for magnetic dipoles,

\[ (\mathbf{r} \cdot \nabla) \frac{\dot{\mu}}{R} = \frac{1}{2} [\mathbf{r} \times \dot{\mathbf{\mu}}] \times \nabla \frac{1}{R}, \]

is not fulfilled for dipole term \( G^{(1)} \) in (6):

\[ (\mathbf{r} \cdot \nabla) \frac{\dot{\mu}}{R} \neq \frac{1}{2} [\mathbf{r} \times \dot{\mathbf{\mu}}] \times \nabla \frac{1}{R}. \]
This is due to the fact that the value $\dot{\mu} = \int j^{(m)} dV$ may not be given in the form of $\dot{\mu} = \int \rho^{(m)} v dV$, similar to the expression $j = \rho v$. However, the appropriate ECM appear in case of a particular configuration of magnetic currents, repeating the configuration of electric currents, generating different magnetic multipoles. The formulae for ECM in this case coincide with the formulae for magnetic multipoles with the substitution $E \rightarrow -H$, $G \rightarrow -A$, $j^{(m)} \rightarrow j$. In particular, in the expansion in terms of multipoles, electric anapole (toroid dipole) moments appear in a natural way. Their existence is suggested in Refs. 10) and 11). The ECM field, accounting for terms of second order in $r$, is described by the expression

$$E = \nabla \times G,$$

$$G_i = -e_{ijk} d'_{j} \frac{\partial}{\partial X_k} \left( \frac{1}{R} \right) + \left[ -\frac{1}{6} e_{ijn} Q'_{nk} + \frac{1}{4\pi} (\delta_{ik} a_j' - \delta_{jk} a_i') \right] \frac{\partial^2}{\partial X_j \partial X_k} \left( \frac{1}{R} \right),$$

(11)

where $d'$, $Q'$ and $a'$ are the electric current dipole, quadrupole and anapole moments, respectively:

$$d' = -\frac{1}{2c} [r \times \dot{\mu}], \quad Q'_{nk} = \frac{1}{c} (e_{nlm} x_k + e_{klm} x_n) \dot{\mu} x_m, \quad a' = -\frac{\pi}{c} \dot{\mu} r^2.$$

(12)

It is interesting to compare (11) and (12) with the corresponding formulae for magnetic multipoles,

$$H = \nabla \times A,$$

$$A_i = -e_{ijk} \mu_j \frac{\partial}{\partial X_k} \left( \frac{1}{R} \right) + \left[ -\frac{1}{6} e_{ijn} M_{nk} + \frac{1}{4\pi} (\delta_{ik} a_j - \delta_{jk} a_i) \right] \frac{\partial^2}{\partial X_j \partial X_k} \left( \frac{1}{R} \right),$$

$$\mu = \frac{1}{2c} \int [r \times j] dV, \quad M_{nk} = -\frac{1}{c} \int (e_{nlm} x_k + e_{klm} x_n) j x_m dV,$$

$$a = -\frac{\pi}{c} \int j r^2 dV,$$

(13)

where $\mu$, $M_{nk}$ and $a$ are the magnetic dipole, quadrupole and anapole moments.

The common property of ECM is the coincidence of their fields at great distances with the fields of corresponding to charge multipoles.

Note the principle difference between the two types of ECM. The existence of ECM, appearing due to the translational motion of conductors carrying a current, is related to the appearance of a nonzero density of charges:

$$\rho = \frac{j_0 \cdot v'}{c^2 (1 - v'^2/c^2)^{1/2}} = \frac{j \cdot v'}{c^2},$$

(14)

where $v'$ is the velocity of a conductor, $j_0$ is the current density in the rest system for a conductor carrying a current, and $j$ is the current density in the laboratory system, being a sum of conduction and convection currents, $\rho v'$. The charges appearing are real, since the relation $\nabla E = 4\pi \rho$ is satisfied. Note that the summed-up charge equals zero, as follows from the law of charge conservation, due to arbitrary motion of a conductor carrying a current in any reference system. The charge density equals zero in the entire space for ECM appearing due to the change of value or...
direction of magnetic moments. These moments occur due to a nonzero value \( \frac{1}{2\pi} \frac{\partial H}{\partial t} \), which might be called the magnetic displacement current, by analogy to electric displacement current. The nonzero density of effective magnetic charges, defined by a formula similar to formula (14), occurs due to the motion of the ECM of this type. However, magnetic charges and currents are just effective, and not real, since ECM are described by just the Maxwell equations valid for \( \rho^{(m)} = 0, \ j^{(m)} = 0 \).

An essential distinction between the multipoles investigated in Refs. 1)–5) and Miller multipoles\(^6\) is related to the type of Lagrangian and Hamiltonian of the interaction. The field of ECM of the first type is caused by the appearance of a nonzero charge density, and the scalar potential of this field is also nonzero. Therefore, external charges interact in the same way with the fields of charge and current electric moments, and the Lagrangians of the interaction in the two cases are equal, \( \mathcal{L}_{\text{int}} = -e\phi \) (\( e \) is an external charge). The situation is different for multipoles of second type (Miller multipoles) having unique electrodynamic properties. The scalar potential of their field is equal to zero, and the Lagrangian of the interaction of an external charge with the field of a multipole also equals zero for a charge at rest, \( \mathcal{L}_{\text{int}} = -\frac{1}{c}(v \cdot A) \to 0 \), when the velocity of the charge \( v \to 0 \). However, despite this circumstance, the charge and the Miller multipoles effectively interact. The force equal to \( \mathbf{F} = \frac{d\mathbf{p}}{dt} = e\mathbf{E} \) acts on the charge and affects the kinetic momentum \( \mathbf{p} \) of the charge, i.e., causes its motion. The expressions for \( \mathcal{L}_{\text{int}} \) and \( \mathbf{F} \) agree because the constancy of the generalized momentum \( \mathbf{P} = \frac{\partial\mathcal{L}}{\partial \dot{t}} = \mathbf{p} + e\mathbf{A}/c \) follows from the Lagrange equation,

\[
\frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{v}} = \frac{d\mathbf{P}}{dt} = \frac{\partial\mathcal{L}}{\partial \mathbf{r}},
\]

and the kinetic momentum \( \mathbf{p} = mv/\sqrt{1 - v^2/c^2} \) varies due to the variation of the vector potential \( \mathbf{A} \). Hence, though the Lagrangian and Hamiltonian of the interaction of the external resting charge with the Miller multipole are equal to zero, their derivatives are nonzero, and this charge begins to undergo motion in the Miller multipole field in the same way as it would move in the field of the corresponding charge multipole. The motion of the charge in the field of ECM of both types does not differ from its motion in the charge multipole field with a moment of the same value.

§3. The interaction of electric current multipoles with the external electric field

The two types of ECM differ by the character of their interaction with the external electric field. The Lagrangian and Hamiltonian of the interaction of resting multipoles of the second type (Miller multipoles) with the electric field are equal to zero, because these multipoles have no electric charge. However, here we also observe the unique properties of these multipoles. Under the influence of the electric field they begin to move, and their motion has no distinction from the motion of the corresponding charge multipoles.\(^6\)

The interaction of multipoles of the first type with electric fields also has some
interesting effects. The charge density, occurring due to the motion of conductors carrying a current and described by formula (14), defines the type of the Lagrangian of the interaction:

$$L_{\text{int}} = -d_i \frac{\partial \phi}{\partial X_i} - \frac{1}{6} Q_{ij} \frac{\partial^2 \phi}{\partial X_i \partial X_j} - \frac{1}{6} \tau \frac{\partial^2 \phi}{\partial X_i^2}.$$  \hspace{1cm} (15)

Here $X_i$ is the coordinate of the center of the current system, and the values $d_i, Q_{ij}$ and $\tau$ are given by formulae (10), where $\rho$ is defined by formula (14) (with the substitution of $v$ for $v'$). Note that $\partial^2 \phi/\partial X_i^2 \equiv \Delta \phi = -4\pi \rho_{\text{ext}}(0)$, where $\rho_{\text{ext}}$ is the external charge density. In this case, the distance to the current element is given by $r' = \mathbf{R} + \mathbf{r}$ and $v' \equiv \dot{r}' = \mathbf{V} + \mathbf{v}$, where $\mathbf{v} \equiv \dot{\mathbf{r}}$, $\mathbf{V} \equiv \dot{\mathbf{R}}$. If $v'$ is substituted by $V + v$, then (10) and (14) allows us to the relationships:

$$d_i = \frac{V_k}{c^2} \int j_k x_i dV + \frac{1}{c^2} \int j_k v_k x_i dV = d^{(1)}_i + d^{(2)}_i,$$

$$Q_{ij} = \frac{V_k}{c^2} \int j_k (3x_i x_j - \delta_{ij} r^2) dV + \int j_k v_k (3x_i x_j - \delta_{ij} r^2) dV = Q^{(1)}_{ij} + Q^{(2)}_{ij},$$

$$\tau = \frac{V_k}{c^2} \int j_k r^2 dV + \int j_k v_k r^2 dV = \tau^{(1)} + \tau^{(2)}.$$  \hspace{1cm} (16)

The first terms in (16) define ECM appearing only due to the motion of magnetic multipoles with velocity $\mathbf{V}$. The second terms describe ECM differing from zero in the rest system of the particle. Let us first consider the first terms. We transform them, using the definitions of magnetic multipole moments (13). For the anapole moment, the following expression (see Ref. 13)) is equivalent to (13):

$$a = \frac{2\pi}{c} \int (\mathbf{j} \cdot \mathbf{r}) r dV.$$  \hspace{1cm} (17)

It follows from (13) and (17) that the formula for the anapole moment can also be given in the form

$$a = \frac{2\pi}{3c} \int [\mathbf{r} \times [\mathbf{r} \times \mathbf{j}]] dV.$$  \hspace{1cm} (18)

Allowing for the situation in which the mean of the time derivative of the value varying in certain limits is equal to zero, and $j_k = \rho v_k \equiv \dot{\rho} x_k$, we obtain

$$\langle \frac{d}{dt} (\rho x_k x_i x_j) \rangle = \langle \dot{\rho} x_k x_i x_j \rangle + \langle j_k x_i x_j \rangle + \langle x_k j_i x_j \rangle + \langle x_k x_i j_j \rangle = 0,$$

$$\langle \frac{d}{dt} (\rho x_k x_i) \rangle = \langle \dot{\rho} x_k x_i \rangle + \langle j_k x_i \rangle + \langle x_k j_i \rangle = 0.$$

For stationary currents, $\rho = 0$. By dropping the angular brackets and allowing for the symmetry with regard to the indices $i, j$, we find

$$j_k x_i x_j = \frac{2}{3} (j_k x_i x_j - x_k j_i x_j) = \frac{2}{3} \epsilon_{ikl} [\mathbf{r} \times \mathbf{j}]_{[i} x_{j]}$$

$$= \frac{1}{3} \epsilon_{ikl} ([\mathbf{r} \times \mathbf{j}]_{[i} x_{j]} + [\mathbf{r} \times \mathbf{j}]_{[i} x_{j]} + [\mathbf{r} \times \mathbf{j}]_{[i} x_{j]} - [\mathbf{r} \times \mathbf{j}]_{[i} x_{j]}).$$
Hence, 
\[ \frac{1}{c} \left\langle \int j_k x_i x_j dV \right\rangle = \frac{1}{3} e_{ikl} M_{ij} + \frac{1}{2\pi} (\delta_{kj} a_i - \delta_{ij} a_k). \]

Similarly we can obtain 
\[ \frac{1}{c} \left\langle \int j_k x_i dV \right\rangle = e_{ikl} \mu_l. \]

In view of the symmetry with regard to the indices \( i, j \) it is easily found that 
\[ d^{(1)} = \frac{1}{c} [V \times \mu], \quad Q_{ij}^{(1)} = \frac{1}{2c} (e_{ikl} V_k M_{ij} + e_{jkl} V_k M_{li}) + \frac{1}{4\pi c} (3a_i V_j + 3a_j V_i - 2\delta_{ij} a \cdot V), \]
\[ \tau^{(1)} = -\frac{1}{\pi c} a \cdot V. \] (19)

The formulae (19) for ECM caused by the motion of magnetic dipoles and magnetic quadrupoles have been obtained in Refs. 1, 2) and 3), respectively. The formulae (19) show as well that ECQM also appear as a result of the anapole motion. The interaction of the moving anapole with an electrostatic field is a very important nontrivial effect first found by Afanasiev. 7) In view of the fact that in Ref. 7) the more specific case of the anapole formed by point magnetic dipoles was considered, the formulae obtained there agree with (19). We point out that the field of the anapole has no effect on the motion of external charges, because its strength is equal to zero \( (E = H = 0) \). At the same time, the discrepancy of the vector potential from zero leads to the Aharonov-Bohm effect. 13) As is known, the anapole interacts with an external current at contact, and with an alternate electric field at a distance. 14)

When transferring to the quantum mechanical description, the anapole moment of a particle should be expressed through the pseudovector \( I \) of the particle spin. According to the formula \( a = \frac{2}{5} I \) 12) the anapole moment also becomes a pseudovector, and the interaction of the anapole with the electric field is \( P \)-odd.

The appearance of ECQM for a moving anapole and its interaction with an electrostatic field may lead to the observation and measurement of the electron anapole moment in experiments with atoms. The \( P \)-odd interaction of the electron anapole moment with the electrostatic field of a nucleus is proportional to the nucleus charge, and the presence of this interaction can be observed in precise experiments with heavy atoms, similar to the experiment in which the anapole moment of \( ^{133} \)Cs was first detected. 15)

As seen from the formulae (19), the moving anapole interacts at contact with external charges. This interaction is proportional to the value of \( \tau \). It is also \( P \)-odd, and, in particular, contributes to the \( P \)-nonconservation in atoms.

The second terms in (16) describe nonzero ECM for the particles at rest. In the absence of the \( T \)-invariance violation, the dipole moments (including the electric current moments) of atoms and nuclei are equal to zero. The ECQM of atoms and nuclei are nonzero, and for nuclei, as shown in Refs. 3)–5), they are not small. Their contribution to the total quadrupole moments of some nuclei is on the order of a few percent. The evaluations show that for the nucleus \( ^{133} \)Cs, the charge/current quadrupole moments are on the same order of magnitude. 5) We note that the sum of the charge and current quadrupole moments is determined experimentally. The
formulae for $Q^{(2)}_{ij}$ and $\tau^{(2)}$ are more convenient to be written for point magnetic dipoles. These formulae agree with the real physical pattern of this phenomenon for atoms and nuclei. For this purpose, the replacement $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{r}'''$, where $|\mathbf{r}'''| \ll |\mathbf{r}|$, is sufficient. Then the vector $\mathbf{r}$ represents the position of point magnetic dipoles, $\mathbf{v} \equiv \dot{\mathbf{r}}$, and the small vector $\mathbf{r}'''$ represents the position of current elements constituting these dipoles. Also $j = \rho \dot{\mathbf{r}}$. The integrals in (16) are equal to

$$\frac{1}{c^2} \int j_k v_k x_i dV = \frac{1}{c} \sum e_{ikl} v_k \mu_l, \quad \frac{1}{c^2} \int j_k v_k x_i x_j dV = \frac{1}{c} \sum (e_{ikl} x_j + e_{jkl} x_i) v_k \mu_l. \tag{20}$$

The summing-up in (20) is performed over all point magnetic dipoles. By averaging over time, and in view of the fact that the mean values are equal to zero for the derivatives of the quantities varying within finite limits with respect to time, we find

$$\langle [\mathbf{v} \times \mathbf{\mu}] \rangle = -\langle [\mathbf{r} \times \dot{\mathbf{\mu}}] \rangle,$$

$$(e_{ikl} x_j + e_{jkl} x_i) v_k \mu_l$$

$$= \left\langle -\frac{1}{2} \left( [\mathbf{r} \times \mathbf{v}] j \mu_j + [\mathbf{r} \times \mathbf{v}] j \mu_j \right) + \delta_{ij} [\mathbf{r} \times \mathbf{v}] \cdot \mathbf{\mu} - \frac{1}{2} (x_i [\mathbf{r} \times \dot{\mathbf{\mu}}] j + x_j [\mathbf{r} \times \dot{\mathbf{\mu}}] i) \right\rangle. \tag{21}$$

Using formulae (16), (20) and (21) and dropping angular brackets, we find

$$d^{(2)} = -\frac{1}{c} \sum [\mathbf{r} \times \dot{\mathbf{\mu}}],$$

$$Q^{(2)}_{ij} = \frac{1}{2c} \sum \left( 3 [\mathbf{r} \times \mathbf{v}] i \mu_j + 3 [\mathbf{r} \times \mathbf{v}] j \mu_i - 2 \delta_{ij} [\mathbf{r} \times \mathbf{v}] \cdot \mathbf{\mu} + 3 x_i [\mathbf{r} \times \dot{\mathbf{\mu}}] j + 3 x_j [\mathbf{r} \times \dot{\mathbf{\mu}}] i \right),$$

$$\tau^{(2)} = \frac{2}{c} \sum (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{\mu}. \tag{22}$$

By introducing the orbital moment $l = [\mathbf{r} \times \mathbf{p}] = m_{rel} [\mathbf{r} \times \mathbf{v}]$ ($m_{rel} = m/\sqrt{1 - v^2/c^2}$ is the relativistic mass) we can transform (22) as follows:

$$Q^{(2)}_{ij} = -\frac{1}{2m_{rel} c} \sum \left\{ 3 l_i \mu_j + 3 l_j \mu_i - 2 \delta_{ij} l \cdot \mathbf{\mu} + 3 m_{rel} (x_i [\mathbf{r} \times \dot{\mathbf{\mu}}] j + x_j [\mathbf{r} \times \dot{\mathbf{\mu}}] i) \right\},$$

$$\tau^{(2)} = \frac{2}{m_{rel} c} \sum (l \cdot \mathbf{\mu}). \tag{23}$$

The value $\dot{\mathbf{\mu}}$ in the formulae (22) and (23) for atoms (nuclei) is determined by the electron (nucleon) spin precession. For states with the orbital moment $l \neq 0$, the terms containing $\dot{\mathbf{\mu}}$ can be neglected.\textsuperscript{2},\textsuperscript{3} In this case the formulae for ECQM coincide with those given in these works. For $l = 0$ (s-states), the terms containing $\dot{\mathbf{\mu}}$ should be taken into account in the calculations.

The term proportional to $\dot{\mathbf{\mu}}$ in formula (23) for ECQM agrees, to the accuracy of the factor $3/2$, with the expression (12) for the quadrupole moment created by
the magnetic current \( j^{(m)} = \sum_i \dot{\mu}_i \delta(r - r_i) \). The presence of this additional factor in (23) is not surprising, because when averaging over time, as done in the derivation of formulae (22) and (23), we determine the mean field at a fixed point in space. In this case, \( \langle \frac{\partial A}{\partial t} \rangle = \langle \frac{dA}{dt} \rangle = 0 \), and there is no contribution of the Miller multipoles to the mean field of the current system. In addition, the Miller multipoles do not contribute to the charge density whose distribution determines the static quadrupole moments. However, we can say that the static ECQM of the magnetic dipole system is the sum of the quadrupole moments caused by the orbital motion of magnetic dipoles and by the variation of the magnetic dipole moment values and their orientations.

The value \( \tau \) defined by formula (23) describes the contribution of the electrostatic current contact interaction (ECCI) to the value of the total root-mean-square radius of the system. The ECCI does not depend on \( \dot{\mu} \) in the system of magnetic dipoles. The classical expression for the ECCI, to the accuracy of the constant factor, agrees with the quantum mechanical expression obtained in Ref. 4).

§4. Conclusion

The general formulae for the ECM appearing in a system of conductors carrying a current or magnetic dipoles have been obtained in terms of classical electrodynamics. There are electric current multipole moments of the two types: 1) moments appearing due to the translational motion of conductors; 2) moments appearing due to the current strength variation in conductors or due to the change of the magnetic dipole orientation. Multipoles of the first type are created by electric charges appearing due to the motion of conductors carrying a current (magnetic multipoles). ECM of this type appear both due to the motion of particles having magnetic multipole moments and in the rest system of compound particles. The most important consequence of their existence is sufficiently large ECQM values for some nuclei in comparison with their total quadrupole moments and the appearance of ECQM for a moving anapole, leading to \( P \)-nonconservation in the interaction of particles having an anapole moment with an external electrostatic field. For multipoles of the second type, electric charges are absent, and effective magnetic currents can be introduced in the description of their field. In an electric field, the motion of ECQM of the both types in identical to the motion of the electric charge multipoles. External charges move identically in the ECM field and in the field of the electric charge multipoles. When averaging the electric field over time, only multipoles of the first type remain. The relationships obtained for them can be transformed by introducing an orbital moment. In particular, as a result, the dependence of ECQM on both the orbital motion of magnetic dipoles and the variation of their moment values or the change of their spatial orientations arises. The orbital motion of magnetic dipoles causes the appearance of ECCI, influencing the value of the charge root-mean-square radius of the system.
References

1) J. Frenkel, Z. Phys. 37 (1926), 243.
2) A. J. Silenko, Izv. VUZ Fiz. (Russian Phys. Rep.) 34 (1991), No. 7, 53.
3) A. J. Silenko, J. of Phys. B25 (1992), 1661.
4) A. J. Silenko, Izv. VUZ Fiz. (Russian Phys. Rep.) 38 (1995), No. 4, 66.
5) A. J. Silenko, Phys. Atom. Nucl. 60 (1997), 361.
6) M. A. Miller, Usp. Fiz. Nauk (Sov. Phys.-Usp.) 142 (1984), 147.
7) G. N. Afanasiev, Fiz. Elem. Chast. At. Yadra (Particl. and Nucl.) 24 (1993), 512.
8) I. E. Tamm, Electricity Theory Fundamentals (Nauka, Moscow, 1989), in Russian.
9) V. I. Strajev and L. M. Tomilchik, Electrodynamics for Magnetic Charge (Nauka i technika, Minsk, 1975), in Russian.
10) M. A. Miller, Radiophys. Quant. Electron. 29 (1986), 747.
11) V. M. Dubovik, L. A. Tosunyan and V. V. Tugushev, Sov. Phys. -JETP 63 (1986), 344.
12) O. P. Sushkov, V. V. Flambaum and I. B. Khriplovich, Sov. Phys. -JETP 60 (1985), 873.
13) G. N. Afanasiev, Fiz. Elem. Chast. At. Yadra (Particl. and Nucl.) 21 (1990), 172.
14) V. M. Dubovik and A. A. Cheshkov, Fiz. Elem. Chast. At. Yadra (Particl. and Nucl.) 5 (1974), 791.
15) M. C. Noecker, B. P. Masterson and C. E. Wieman, Phys. Rev. Lett. 61 (1988), 310.