Schmidt number of bipartite and multipartite states under local projections

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The Schmidt number is a fundamental parameter characterizing the properties of quantum states, and the local projections are a fundamental operation in quantum physics. We investigate the relation between the Schmidt numbers of bipartite states and their projected states. We show that there exist bipartite positive-partial-transpose (PPT) entangled states of any given Schmidt number. We further construct the notion of joint Schmidt number for multipartite states, and its relation with the Schmidt number of bipartite reduced density operators.

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I. INTRODUCTION

The Schmidt number is a parameter characterizing quantum states. A quantum state is entangled if and only if its Schmidt number is greater than one. Entangled states play the fundamental role in quantum-information applications such as quantum computing and cryptography. One quantum state \( \rho \) is converted into another state \( \sigma \) under the physical environment of local operations and classical communications (LOCC). In spite of the complex mathematical configuration of LOCC, the most basic operation in LOCC is the local projection \( P \). Mathematically we have \( \rho \rightarrow \sigma = (I_A \otimes P_B) \rho (I_A \otimes P_B) \). In this process the Schmidt number is non-increasing, and the decrease of Schmidt number is decided by the local projection. In this paper we begin by recalling the Schmidt number in Definition 1 and the notion of birank. Then we construct the notion of bi-Schmidt number in Eq. (1). We further...
provide the upper bound of entanglement of formation of quantum states in terms of the Schmidt number in (2). The bound is saturated when the states are antisymmetric two-qubit states. Next we recall the definition of direct sum and tensor product of two quantum states, and obtain a few preliminary results in Lemma 3 and 4. The entanglement of the tensor product of two quantum states is investigated in Lemma 5. Next we recall the the positive and copositive maps in Definition 9 and 10. As an application, we show in Lemma 12 that for any bipartite states \( \rho \) and \( \sigma \) with \( \text{SN}(\sigma) \leq \text{SN}(\rho) \), the Schmidt number of the perturbation \( \rho + \epsilon \sigma \) remains \( \text{SN}(\rho) \) for sufficiently small \( \epsilon > 0 \).

The main result of this paper is as follows. We will investigate how the projection influences the Schmidt number of both bipartite and multipartite states. For bipartite states the investigation is carried out in Lemma 15 and 18. As an application we show that every positive-partition-tranpose (PPT) entangled \( \rho \) is of Schmidt number 2 in Corollary 17. It provides an alternative proof for a conjecture in [1]. We further show that the projected state \( \sigma \) can reach any integer smaller than the Schmidt number of \( \rho \) in Lemma 23. As an application of this result, we show that there exist bipartite PPT entangled states of any given integer in Theorem 24. This is based on the preliminary results developed in Lemma 19 and Proposition 20. We also investigate when an entangled state can be projected onto a separable state in terms of their rank. For multipartite states, we introduce the notion of expansion and coarse graining respectively in Definition 26 and 28. We investigate their relation to the Schmidt number of bipartite reduced density operators in Theorem 27 and Lemma 29. We further construct the notion of joint Schmidt number for multipartite states in Definition 30 and 31. We also restrict the joint Schmidt number of a multipartite pure state by the Schmidt numbers of its bipartite reduced density operators in Theorem 32. As an application, we show in Lemma 33 that any multipartite entangled PPT state with Schmidt number at least 3 when regarded as bipartite states, has rank at least 5.

The rest of the paper is organized as follows. In Sec. II we introduce the preliminary definitions, notations and facts used in the paper. They include the Schmidt number in Sec. II A the positive map in Sec. II B and linear algebra in Sec. II C. In Sec. III we show that there exist bipartite PPT entangled states of any given Schmidt number. Next we introduce the notion of expansion and coarse graining of multipartite states in terms of the Schmidt number respectively in Sec. IV A and IV B. We further present the joint Schmidt number for multipartite states, and their relation to bipartite reduced density operator in Sec. IV C.

## II. PRELIMINARIES

Let \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) be the bipartite Hilbert space with \( \text{Dim} \mathcal{H}_A = M \) and \( \text{Dim} \mathcal{H}_B = N \). Since the case \( M = 1 \) or \( N = 1 \) is trivial, we assume \( 2 \leq M \leq N \). We say that \( \rho \) is a \( M \times N \) state when \( \text{rank} \rho_A = M \) and \( \text{rank} \rho_B = N \). We shall work with bipartite quantum states \( \rho \) on \( \mathcal{H} \). We shall write \( I_k \) for the identity \( k \times k \) matrix. We denote by \( \mathcal{R}(\rho) \) and \( \ker \rho \) the range and kernel of a linear map \( \rho \), respectively. From now on, unless stated otherwise, the states will not be normalized. We shall denote by \( \{|i\rangle_A : i = 0, \ldots, M-1\} \) and \( \{|j\rangle_B : j = 0, \ldots, N-1\} \) o. n. bases of \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. The partial transpose of \( \rho \) w. r. t. the system \( A \) is defined as \( \rho^A := \sum_{i,j} |j\rangle \langle i| \otimes \langle |j\rangle |i\rangle \). We say that \( \rho \) is PPT if \( \rho^A \geq 0 \). Otherwise \( \rho \) is NPT, i.e., \( \rho^A \) has at least one negative eigenvalue. We say that two bipartite states \( \rho \) and \( \sigma \) are equivalent under SLOCC if there exists an invertible local operator (ILO) \( A \otimes B \) such that \( \rho = (A^A \otimes B^A)\sigma(A \otimes B) \) \([2]\). In particular, they are locally equivalent when \( A \) and \( B \) are unitary matrices. It is easy to see that any ILO transforms distillable, PPT, entangled, or separable state into the same kind of states. We shall often use ILOs to simplify the density matrices of states. A subspace which contains no product state, is referred to as a completely entangled subspace (CES).

In the following subsections, we respectively introduce the Schmidt number, the positive map, and a few results from linear algebra. In Sec. II A we review the Schmidt number in Definition 1 and introduce the direct sum and tensor product of two bipartite states. In Sec. II B we review the positive and copositive map in Definition 9 and the completely positive and copositive map in Definition 10. We further review the reduction map and investigate a family of \( k \)-positive map. In Sec. II C we review and construct a few results on linear algebra. Lemma 8 shows a corollary in terms of maximally entangled states, when a bipartite state has a given Schmidt number.

### A. Schmidt number

In this subsection we review the definition of Schmidt number \([3]\) and its physical meanings. Then we construct the notion of bi-Schmidt number for PPT states. We also review the B-direct sum of quantum states, entanglement of formation, and quantum channel, and their relation to the Schmidt number.
Definition 1 A bipartite density matrix $\rho$ has Schmidt number $SN(\rho) = k$ if (i) for any decomposition $\{p_i \geq 0, |\psi_i\rangle\}$ of $\rho$, at least one of the vectors $|\psi_i\rangle$ has Schmidt rank at least $k$ and (ii) there exists a decomposition of $\rho$ with all vectors $|\psi_i\rangle$ of Schmidt rank at most $k$.

For example the $M \times N$ pure state has Schmidt number $M$. Another example is that the two-qubit mixed state $\rho = |\alpha\rangle\langle\alpha| + |00\rangle\langle00|$ where $|\alpha\rangle = |00\rangle + |11\rangle$ has Schmidt number two. To understand this fact, we assume that $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ as an arbitrary decomposition of $\rho$. As we shall see in Lemma 13 one can obtain that there is always some $|\psi_i\rangle$ of tensor rank two. Then Definition 1 shows that $SN(\rho) = 2$. It further implies that the Schmidt number of bipartite states does not increase under LOCC. So the Schmidt number is an entanglement monotone for bipartite states. For simplicity we denote $SN(\rho)$ as the Schmidt number of $\rho$. Suppose $\rho = \rho_0 + (1 - p)\beta$ is a quantum state, $\alpha, \beta$ are two states, and $p \in (0, 1)$. It is known that for some $\rho$ we have $SN(\rho) < pSN(\alpha) + (1 - p)SN(\beta)$, e.g., $\alpha = 2(|00\rangle + |11\rangle)(|00\rangle + |11\rangle) + (|00\rangle - |11\rangle + |22\rangle)(|00\rangle - |11\rangle - |22\rangle)$, $\beta = (|00\rangle - |11\rangle - |22\rangle)(|00\rangle - |11\rangle - |22\rangle)$ and $p = 1/2$. On the other hand, suppose that $\rho = (1 - p)\sigma + pI$ is a quantum state and $\sigma$ has Schmidt number two. By Lemma 12 if $p$ is small enough then $\rho$ has $SN(\rho) = 2$. Then we have $SN(\rho) > (1 - p)SN(\sigma) + pSN(I)$. The above two examples imply that the Schmidt number is neither convex nor concave, although Schmidt number is an entanglement monotone. This is different from many known entanglement monotones in quantum information, e.g., the entanglement of formation is convex [1]. Meanwhile, the Schmidt number of the state $\rho = p|\alpha\rangle\langle\alpha| + (1 - p)|\beta\rangle\langle\beta|$ may rely on $p$. An example is $|\alpha\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. We can easily show that $\rho$ is separable if and only if $p = 1/2$, and entangled otherwise. This physical phenomenon mathematically corresponds to the change of Schmidt number of $\rho$ between 1 and 2.

To apply the Schmidt number to PPT states $\rho$, we recall the notion of birank. It has been used to investigate the two-qutrit PPT entangled states of rank four [2]. We denote the pair of integers $(\text{rank}_\rho, \text{rank}_{\rho^T})$ as the birank of $\rho$. Similar to the birank, we denote the pair of integers

$$ (SN(\rho), SN(\rho^T)) \tag{1} $$

as the bi-Schmidt number, namely the BSN of $\rho$. Unlike the birank, the BSN is defined for PPT states only because the Schmidt number is defined only for quantum states. Below is an application of BSN. The proof is by the fact that the partial transpose of a separable state is still separable.

Lemma 2 If $\rho$ is a PPT state and $SN(\rho), SN(\rho^T) \in \{1, 2\}$ then $SN(\rho) = SN(\rho^T)$.

Here is another application of the Schmidt number. We refer to $E_F(\rho)$ as the entanglement of formation (EOF) of the state $\rho$. It is a fundamental entanglement measure for quantum states and has been widely investigated in the past years [4]. However the estimation of the bound of EOF has been an involved problem. Definition 1 implies that

$$ E_F(\rho) \leq \log_2 SN(\rho), \tag{2} $$

i.e., an upper bound of EOF of $\rho$ is $\log_3 SN(\rho)$ ebits. It is known that any quantum state in the 3-dimensional antisymmetric subspace $A$ is locally equivalent to a two-qubit maximally entangled state. So $E_F(\rho) = 1$ ebit and $SN(\rho) = 2$ when $R(\rho) \subseteq A$. It implies that the equality in (2) holds when $R(\rho) \subseteq A$. In this sense, the EOF of antisymmetric states is analytically characterized by their Schmidt number.

Next we investigate the Schmidt number of the collective use of two quantum states. For this purpose we introduce two notions from quantum information. The first notion is the direct sum of two spaces. It plays an important role in many quantum-information problems such as the distillability problem [6] and bipartite unitary operations [7–9]. We shall denote $V \oplus W$ as the ordinary direct sum of two matrices $V$ and $W$, and $V \oplus_B W$ as the direct sum of $V$ and $W$ from the $B$ side (called “B-direct sum”). In the latter case, $V$ and $W$ respectively act on two subspaces $H_A \otimes H_{B'}$ and $H_A \otimes H_{B''}$ such that $H_{B'} \perp H_{B''}$. We shall denote the tensor product of two bipartite states $\rho_{A_1B_1}$ and $\sigma_{A_2B_2}$ as another bipartite state of the system $A_1A_2$ and $B_1B_2$.

The second notion from quantum information is the combination of different systems. Let $\rho_{A_iB_i}$ be an $M_i \times N_i$ state of rank $r_i$ acting on the Hilbert space $H_i = H_{A_i} \otimes H_{B_i}$, $i = 1, 2$. Suppose $\rho$ of systems $A_1, A_2$ and $B_1, B_2$ is a state acting on the Hilbert space $H_1 \otimes H_2 = H_{A_1} \otimes H_{B_1} \otimes H_{A_2} \otimes H_{B_2}$. By switching the two middle factors, we can consider $\rho$ as a composite bipartite state acting on the Hilbert space $H_A \otimes H_B$ where $H_A = H_{A_1} \otimes H_{A_2}$ and $H_B = H_{B_1} \otimes H_{B_2}$. In that case we shall write $\rho = \rho_{A_1A_2B_1B_2}$. Let $\text{Tr}_{A_1B_1}\rho = \rho_{A_2B_2}$ and $\text{Tr}_{A_2B_2}\rho = \rho_{A_1B_1}$. So $\rho$ is an $M_1M_2 \times N_1N_2$ state of rank not larger than $r_1r_2$. In particular for the tensor product $\rho = \rho_{A_1A_2} \otimes \rho_{A_2A_1}$, it is easy to see that $\rho$ is an $M_1M_2 \times N_1N_2$ state of rank $r_1r_2$. The above definition can be easily generalized to the tensor product of $N$ states $\rho_{A_iB_i}, i = 1, \ldots, N$. They form a bipartite state on the Hilbert space $H_{A_1} \otimes \cdots \otimes H_{A_N} \otimes H_{B_1} \otimes \cdots \otimes H_{B_N}$.

For simplicity we denote the system $A$ as $A_1, \ldots, A_N$ and denote $B$ as $B_1, \ldots, B_N$. For example, it is known that $SN(\rho^{\otimes 2}) \in \{SN(\rho), SN(\rho^2)\}$, and $SN(\rho^{\otimes 2})$ may reach any integer in the interval $[SN(\rho), SN(\rho^2)]$ when $SN(\rho) = M$. An example is the two-qubit isotropic state [3, Fig. 1]. Now we have
Lemma 3 Suppose $\rho = \alpha \oplus_B \beta$ where $\alpha$ and $\beta$ are both bipartite quantum states. Then

(i) $\text{SN}(\rho) = \max\{\text{SN}(\alpha), \text{SN}(\beta)\}$.

(ii) $\text{SN}(\rho^\otimes n) = \max\{\text{SN}(\alpha \otimes \cdots \otimes \alpha), \text{SN}(\alpha \otimes \cdots \otimes \beta), \cdots, \text{SN}(\beta \otimes \alpha \otimes \cdots \otimes \alpha), \cdots, \text{SN}(\beta \otimes \cdots \otimes \beta)\}$.

Proof. (i) By definition we have $\text{SN}(\rho) \leq \max\{\text{SN}(\alpha), \text{SN}(\beta)\}$. On the other hand we can project $\rho$ onto $\alpha$ and $\beta$ by local projectors. Since the Schmidt number is an entanglement monotone we have $\text{SN}(\rho) \geq \max\{\text{SN}(\alpha), \text{SN}(\beta)\}$.

(ii) The assertion follows from (i). This completes the proof. □

We generalize the Lemma as follows. It is known that any quantum physical operation can be expressed as a completely positive (CP) map $\Lambda(\cdot)$ of the form $\Lambda(\rho) := \sum_i P_i \rho P_i^\dagger$ where $\sum_i P_i^\dagger P_i \leq I$. If the equality holds then the operation is a completely positive trace-preserving (CPTP) map, namely a quantum channel. We construct the relation between quantum operation and Schmidt number.

Lemma 4 Suppose $\rho$ is a bipartite state, and $\Lambda(\cdot) = \sum_i P_i(\cdot)P_i^\dagger$ is a quantum operation such that $(I_\Lambda \otimes \Lambda)\rho = \rho$. Then $\text{SN}(\rho) = \max_i\{\text{SN}(\rho_i)\}$ where $\rho_i = (I_\Lambda \otimes P_i)\rho(I_\Lambda \otimes P_i^\dagger)$.

Proof. By definition we have $\text{SN}(\rho) \leq \max_i\{\text{SN}(\rho_i)\}$. Since the Schmidt number is an entanglement monotone we have $\text{SN}(\rho) \geq \max_i\{\text{SN}(\rho_i)\}$.

If the channel is $\Lambda(\cdot) = P(\cdot)P^\dagger + (I - P)(\cdot)(I - P)^\dagger$ where $P$ is a projector, then Lemma 4 reduces to Lemma 3. Finding out the states $\rho$ satisfying the hypothesis of Lemma 4 is an interesting question. For example, we can assume $\rho$ as the quantum-classical separable state $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$. □

The following Lemma investigates the entanglement of the tensor product of two quantum states.

Lemma 5 Let the integers $m_1, n_1, m_2, n_2 \in \{2, 3\}$, $m_1 + n_1 < 6$ and $m_2 + n_2 < 6$. Suppose $\rho_1$ and $\rho_2$ are $m_1 \times n_1$ and $m_2 \times n_2$ states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$ and $\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$, respectively. $\rho_1 \otimes \rho_2$ is a bipartite state w.r.t the bi-partition $A_1 A_2 : B_1 B_2$.

(i) If either of the two states $\rho_1$ and $\rho_2$ is entangled, then $\rho_1 \otimes \rho_2$ is a NPT state.

(ii) Conversely, if $\rho_1 \otimes \rho_2$ is a PPT state, then both $\rho_1$ and $\rho_2$ are separable states.

Proof. (i) Assume that $\rho_1$ is entangled. It follows from the Peres-Horodecki criterion [11], that the least eigenvalues of $\rho_1^\Gamma A_1$ is negative and the largest eigenvalues of $\rho_1^\Gamma A_2$ is positive. Since the eigenvalues of $(\rho_1 \otimes \rho_2)^\Gamma A_1 A_2 = \rho_1^\Gamma A_1 \otimes \rho_2^\Gamma A_2$ are the pairwise products of eigenvalues of $\rho_1^\Gamma A_1$ and $\rho_2^\Gamma A_2$, there exists a negative eigenvalue in the spectrum of $(\rho_1 \otimes \rho_2)^\Gamma A_1 A_1$. (ii) follows (i) immediately. This completes the proof. □

The Lemma shows that the entanglement of the tensor product implies the entanglement of at least one state in the tensor product. On the other hand, if $\rho_1 + \rho_2$ is a separable state then $\rho_1$ and $\rho_2$ may be both entangled. An example is $\rho_1 = |\alpha_+\rangle\langle \alpha_+|$ and $\rho_2 = |\alpha_-\rangle\langle \alpha_-|$ where $|\alpha_{\pm}\rangle = |11\rangle \pm |22\rangle$. This is different from (ii) which works for the tensor product of two states. Moreover if we want to construct PPT entangled states using the tensor product of two PPT entangled states by Lemma 6 then $\rho_1$ and $\rho_2$ have to be $M \times N$ PPT entangled states where $M, N \geq 3$.

As another application of Schmidt rank, we introduce a subspace containing only highly entangled states [12].

Definition 6 A subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ is said to be a $k$-CES ($k \leq \min\{m, n\}$) if it contains no nonzero Schmidt rank $l$ vectors for $l \leq k$.

For example, if the range of a bipartite state is 1-completely entangled then the state is entangled. This is how the PPT entangled states by unextendible product bases are constructed [13]. The definition of Schmidt number implies

Lemma 7 If $\rho$ is a bipartite quantum state whose $R(\rho)$ is a $k$-CES, then $\text{SN}(\rho) \geq k + 1$.

The Lemma gives a sufficient condition such that $\rho$ is entangled. The condition is not necessary. An example is the two-qubit state $|00\rangle\langle 00| + \langle 00 | + |11\rangle)(00 + |11\rangle).$ One can easily show that the state is entangled and its range is not 1-completely entangled. Hitherto most results shows that estimating the Schmidt number is a hard problem. The following result from [3] provides a method for the estimation in terms of the maximally entangled states.

Lemma 8 For any density matrix $\rho$ with $M = N$ and Schmidt number $k$, we have

$$\max_{\Psi_M} \langle \Psi_M | \rho | \Psi_M \rangle \leq \frac{k}{N},$$ (3)

where we maximize over $M \times M$ bipartite maximally entangled states $|\Psi_M\rangle$. 

An equivalent statement is presented in [14, Proposition 2.4.12]. That is if \( \langle \Psi_M | \rho | \Psi_M \rangle > \frac{k}{2} \) for some maximally entangled state \( | \Psi_M \rangle \) then \( \text{SN}(\rho) > k \). This result can be used to infer the Schmidt number of quantum states. For example let us consider the mixed state \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \), where \( |\psi_i\rangle \) has the maximum Schmidt rank. The greater \( p_1 \) is, the greater \( \langle \Psi_M | \rho | \Psi_M \rangle \) becomes. Then Lemma 8 shows that the Schmidt number of \( \rho \) also increases.

B. Positive map

In this subsection, we investigate the Schmidt number in the view of positive and copositive maps. They play the fundamental roles in operator algebra and have a deep connection with quantum information. For example, the known Peres-Horodecki criterion says that a two-qubit or qubit-qutrit state is separable if and only if its partial transpose is a positive-semidefinite matrix. Here the transpose is a positive but not 2-positive map. In general we define the positive and copositive maps as follows.

**Definition 9** A map \( \phi \in B(M_m(\mathbb{C}), M_n(\mathbb{C})) \) is said to be \( k \)-positive/\( k \)-copositive if the map \( \text{id}_k \otimes \phi/\tau_k \otimes \phi \) is positive, respectively.

Here \( \tau_k \) is the transpose map in \( B(M_k(\mathbb{C}), M_k(\mathbb{C})) \). Denote by \( P_k[m, n]/P^k[m, n] \) the set of all \( k \)-positive/\( k \)-copositive maps in \( B(M_m(\mathbb{C}), M_n(\mathbb{C})) \). Using these definitions we introduce completely positive and completely copositive maps.

**Definition 10** A map \( \phi \in B(M_m(\mathbb{C}), M_n(\mathbb{C})) \) is completely positive/completely copositive if for every positive integer \( k \), \( \phi \) is \( k \)-positive/\( k \)-copositive, respectively. \( \phi \) is said to be decomposable if it is the sum of a completely positive map and a completely copositive map.

With the well known dual cone relation [12, 18] between positive maps and quantum states, the Schmidt number of an \( m \times n \) entangled state \( \rho \) can be rephrased as

\[
\text{SN}(\rho) = \max \{ l : \exists \phi \in P_l \text{ s.t. } \text{Tr}(\rho C^l_\phi) < 0 \} + 1, \quad (4a)
\]

\[
= \min \{ l : \text{Tr}(\rho C^l_\phi) \geq 0 \forall \phi \in P_l \}. \quad (4b)
\]

Here \( C_\phi = \sum_{i,j=1}^m |i\rangle \langle j| \otimes |\phi(i)\rangle |\phi(j)\rangle \) is the Choi matrix of the positive map \( \phi \), and \( C^l_\phi \) is the transpose of \( C_\phi \). We denote the pairing of a quantum state \( \rho \) and a positive map \( \phi \) by \( \langle \rho, \phi \rangle = \text{Tr}(\rho C^l_\phi) \). If \( \langle \rho, \phi \rangle < 0 \), then \( \phi \) is called an entanglement witness by which one can detect whether \( \rho \) is entangled [13]. If such a map exists, then the detected state has Schmidt number at least two. To decide the Schmidt number of \( \rho \), one should continue to test \( \rho \) using \( k \)-positive maps as entanglement witnesses until for certain \( k \), no \( k \)-positive map can serve as an entanglement witness to \( \rho \). Let us illustrate this principle by assuming that \( \rho \) is a \( 3 \times 3 \) entangled PPT state. We will make use of the following result from [20].

**Lemma 11** Every 2-positive or 2-copositive map in \( B(M_3(\mathbb{C}), M_3(\mathbb{C})) \) is decomposable.

The Lemma implies that for any 2-positive map \( \phi \in B(M_3(\mathbb{C}), M_3(\mathbb{C})) \), \( \phi = \phi_1 + \phi_2 \) and \( \phi_1/\phi_2 \) is completely positive/completely copositive, respectively. Then \( \text{Tr}(\rho C^l_\phi) = \text{Tr}(\rho C^l_{\phi_1}) + \text{Tr}(\rho C^l_{\phi_2}) = \text{Tr}(\rho C^l_{\phi_1}) + \text{Tr}(\rho^T(C^l_{\phi_2})^T) \geq 0 \) because all matrices involved are positive. By [10] every \( 3 \times 3 \) PPT entangled state \( \rho \) has Schmidt number 2 since no 2-positive map can serve as an entanglement witness to them. As another application of [4], next we show that the Schmidt number is stable under perturbation.

**Lemma 12** For any bipartite states \( \rho \) and \( \sigma \) with \( \text{SN}(\sigma) \leq \text{SN}(\rho) \), the Schmidt number of the perturbation \( \rho + \epsilon \sigma \) remains \( \text{SN}(\rho) \) for sufficiently small \( \epsilon > 0 \).

**Proof.** For any \( l \geq \text{SN}(\rho) \geq \text{SN}(\sigma) \), we have \( \text{tr}(\rho C^l_\sigma) \geq 0 \forall \phi \in P_l \) and \( \text{tr}(\sigma C^l_\sigma) \geq 0 \forall \phi \in P_l \) by equation (11). Therefore \( \text{tr}((\rho + \epsilon \sigma) C^l_\sigma) = \text{tr}(\rho C^l_\phi) + \epsilon \text{tr}(\sigma C^l_\phi) \geq 0 \forall \phi \in P_l \) for any non-negative \( \epsilon \). On the other hand, taking \( l = \text{SN}(\rho) - 1 \), there exists a positive map \( \psi \in P_l \) such that \( \text{tr}(\rho C^l_\psi) < 0 \) by equation (1a). Choosing a sufficiently small \( \epsilon \), we also have \( \text{tr}((\rho + \epsilon \sigma) C^l_\psi) = \text{tr}(\rho C^l_\psi) + \epsilon \text{tr}(\sigma C^l_\psi) < 0 \). Hence by equation (11) we have \( \text{SN}(\rho + \epsilon \sigma) \) remains \( \text{SN}(\rho) \) for sufficiently small \( \epsilon \). \( \square \)

A similar property holds for quantum entanglement. That is, if \( \rho \) is entangled, then \( \rho + \epsilon \sigma \) remains entangled for sufficiently small \( \epsilon > 0 \). This fact can be proved by using the entanglement witness.
Let us recall the reduction map \( \Lambda(\alpha) = (\text{Tr} \alpha)I - \alpha \) for any positive semidefinite matrix \( \alpha \) [21]. Let \( \Lambda_A \) and \( \Lambda_B \) be the maps respectively acting on the system \( A \) and \( B \). One can show

\[
\begin{align*}
\Lambda_A(\rho) &= I_A \otimes \rho_B - \rho, \\
\Lambda_B(\rho) &= \rho_A \otimes I_B - \rho,
\end{align*}
\]

for any bipartite state \( \rho \). The reduction map is a positive but not completely positive (PNCP) map. If both matrices in (5) are semidefinite positive then we say that \( \rho \) satisfies the reduction criterion. Otherwise \( \rho \) violates the reduction criterion, i.e., one of the two matrices in (5) is not semidefinite positive. It is known that if the reduction criterion is violated then \( \rho \) is distillable [21]. The reduction criterion is weaker than the PPT criterion.

C. Linear algebra

In this subsection we review and construct a few results on linear algebra used throughout the paper. We have seen in Definition 11 that computing the Schmidt number of a quantum state requires the investigation of all decompositions the state. The following result provides the closed formula for the decomposition [22].

Lemma 13 Let \( \rho \) be a quantum state and the spectral decomposition \( \rho = \sum_i p_i |\alpha_i\rangle \langle \alpha_i| \) such that \( p_i > 0 \) and the \( |\alpha_i\rangle \) are pairwise orthonormal states. Then any decomposition \( \rho = \sum_{j=1}^m q_j |b_j\rangle \langle b_j| \) with \( q_j > 0 \) satisfies \( \sqrt{q_j} |b_j\rangle = \sum_i u_{ij} \sqrt{p_i} |a_i\rangle \) for an order-\( m \) unitary matrix \( [u_{ij}] \).

The lemma will be used in the proof of Lemma 15 studying the Schmidt number of quantum states and their projection. The next result is used for detecting the Schmidt number of bipartite states in Lemma 16.

Lemma 14 Suppose \( |\psi\rangle \) and \( |\varphi\rangle \) are two bipartite states in \( \mathcal{H}_A \otimes \mathcal{H}_B \). There exists a nonzero state \( |\gamma\rangle \in \mathcal{H}_A \) or \( \mathcal{H}_B \) such that the two states \( \langle \gamma | \psi \rangle \) and \( \langle \gamma | \varphi \rangle \) in \( \mathcal{H}_B \) or \( \mathcal{H}_A \) are proportional, and one of them is nonzero.

Proof. Suppose \( \{|a_j\rangle\}_{j=1,\ldots,M} \) and \( \{|b_j\rangle\}_{j=1,\ldots,N} \) are respectively two orthonormal basis in \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Without loss of generality, we assume that \( |\psi\rangle \) is not parallel to \( |\phi\rangle \). We write the Schmidt decomposition as

\[
|\psi\rangle = \sum_{j=1}^L c_j |a_j, b_j\rangle \quad \text{where} \quad c_j \neq 0, \quad L \leq M \leq N, \quad \text{and} \quad |\varphi\rangle = \sum_{j=1}^M \sum_{k=1}^N d_{jk} |a_j, b_k\rangle.
\]

If some \( d_{jk} \neq 0 \) when \( L < j \) or \( L < k \), then we choose \( |\gamma\rangle = |a_j\rangle \) or \( |\gamma\rangle = |b_k\rangle \), and the assertion holds. If all \( d_{jk} = 0 \) when \( L < j \) or \( L < k \), we can find two complex number \( x, y \) such that the nonzero state \( x|\psi\rangle + y|\varphi\rangle \) has Schmidt number strictly less than \( L \). Choose \( |\gamma\rangle \in \text{span}\{|a_1\rangle, \ldots, |a_L\rangle\} \) and \( \langle \gamma|x|\psi\rangle + y|\varphi\rangle \rangle = 0 \). Then the two states \( \langle \gamma | \psi \rangle \) and \( \langle \gamma | \varphi \rangle \) in \( \mathcal{H}_B \) are proportional, and \( \langle \gamma | \psi \rangle \) is nonzero. So the assertion holds. This completes the proof. \( \square \)

Note that the space in which the state \( |\gamma\rangle \) belongs to cannot be fixed. An example is that \( |\psi\rangle = |00\rangle + |11\rangle \) and \( |\phi\rangle = |01\rangle + |12\rangle \). One can show that no \( |\gamma\rangle \in \mathcal{H}_A \) satisfies the assertion. On the other hand one can choose \( |\gamma\rangle = |0\rangle \in \mathcal{H}_B \).

III. SCHMIDT NUMBER OF BIPARTITE STATES

In this section we investigate the Schmidt number of bipartite states under local projections. Bipartite entangled states are the fundamental resources in quantum computing and cryptography. For this purpose bipartite states are converted into Bell states with a smaller Schmidt number under local projections asymptotically. This is the well-known entanglement distillation or purification [4]. Next, bipartite states are entangled if and only if they have Schmidt number greater than one. Deciding whether a state is entangled is the well-known separability problem. One may detect the entanglement by locally projecting the target state onto another state with smaller dimensions. The local projections play important roles in both issues. We begin by proposing a preliminary Lemma on the Schmidt number and local projections.

Lemma 15 Let \( \rho \) be an \( M \times N \) entangled state, \( k \in [1, M - 1] \) an integer, \( P \) a matrix of rank \( M - k \), and \( \sigma = (P \otimes I_B)\rho(P^\dagger \otimes I_B) \) the projected state. Then

\[
(i) \quad \max\{1, \text{SN}(\rho) - k\} \leq \text{SN}(\sigma) \leq \min\{\text{SN}(\rho), M - k\}.
\]
(ii) We have \( \rho = \sum |\psi_j\rangle\langle\psi_j| \), where
\[
|\psi_j\rangle = \sum_{i=1}^{\text{SN}(\sigma)} |a_{j,i}, b_{j,i}\rangle + \sum_{i=1}^{k} |z_i, y_{j,i}\rangle, \quad \mathcal{R}(P) = \text{span}\{|a_{j,i}\rangle\}, \quad |z_i\rangle \perp |z_j\rangle, \text{ and} \\
|z_i\rangle \perp P \text{ for all } i, j.
\]
(iii) If \( \text{SN}(\rho) = M \), then \( \text{SN}(\sigma) = M - k \).
Below we further assume that \( \rho \) is PPT. Then
(iv)
\[
\max\{1, \text{SN}(\rho^\Gamma) - k\} \leq \text{SN}(\sigma^\Gamma) \leq \min\{\text{SN}(\rho^\Gamma), M - k\}.
\]
(v) If \( k = \text{SN}(\sigma) = 1 \), then \( \text{SN}(\rho) = \text{SN}(\rho^\Gamma) = 2 \).
(vi) If \( k = \min\{\text{SN}(\rho), \text{SN}(\rho^\Gamma)\} - s \), and \( \text{SN}(\rho) \not= \text{SN}(\rho^\Gamma) \), then \( \max\{\text{SN}(\sigma), \text{SN}(\sigma^\Gamma)\} \geq s + 1 \).
(vii) If \( k = M - 2 \) or \( M - 1 \), then \( \text{SN}(\sigma) = \text{SN}(\sigma^\Gamma) \in \{1, 2\} \).
(viii) If \( \text{SN}(\rho) = \text{SN}(\rho^\Gamma) \), then \( \text{SN}(\sigma) - \text{SN}(\sigma^\Gamma) \in [-k, k] \).

**Proof.** (i) Since the Schmidt number of quantum states is invariant up to local invertible operators, we may assume that \( P \) is a projector. Let \( P = \sum_{i=1}^{M-k} |v_i\rangle\langle v_i| \) and \( \{|v_1\rangle, \ldots, |v_M\rangle\} \) an orthonormal basis of \( \mathcal{H}_A \). Let \( \rho = \sum_{j} |\psi_j\rangle\langle\psi_j| \) where
\[
|\psi_j\rangle = \sum_{i=1}^{M} |v_i, u_{ij}\rangle \text{ and } |u_{ij}\rangle \text{ are nonnormalized vectors.}
\]
We have
\[
|\alpha_j\rangle := (P \otimes I_B)|\psi_j\rangle = \sum_{i=1}^{M-k} |v_i, u_{ij}\rangle := |\psi_j\rangle - |\beta_j\rangle,
\]
where
\[
|\beta_j\rangle = \sum_{i=M-k+1}^{M} |v_i, u_{ij}\rangle.
\]
Using Lemma 13 we may assume that \( |\alpha_j\rangle \) are pairwise orthogonal, and we do not change the expression of \( \rho \) since there is no confusion. Since \( \sigma = \sum_j |\alpha_j\rangle\langle\alpha_j| \), we can find a unitary matrix \( W = |\beta_{jl}\rangle \) such that for any \( k \) the pure state \( \sum_{j} w_{jl}|\alpha_j\rangle \) has Schmidt rank at most \( \text{SN}(\sigma) \). Hence
\[
\rho = \sum_{j} (|\alpha_j\rangle + |\beta_j\rangle)(|\alpha_j\rangle + \langle\beta_j|) = \sum_{i=1}^{(M-k)} \left( \sum_{j} w_{jl}(|\alpha_j\rangle + |\beta_j\rangle) \right) \left( \sum_{j} w_{jl}(|\alpha_j\rangle + \langle\beta_j|) \right).
\]

The definition of Schmidt number and Lemma 10 imply that \( \text{SN}(\rho) \leq \text{SN}(\sigma) + k \). Since \( \sigma \) is nonzero we always have \( \text{SN}(\sigma) \geq 1 \). So we have proved the lower bound in (4).

On the other hand, it is known that the Schmidt number is non-increasing under the local operations and classical communications \( 3\). So \( \text{SN}(\sigma) \leq \text{SN}(\rho) \). Besides, the inequality \( \text{SN}(\sigma) \leq M - k \) follows from the fact that \( P \) has rank \( M - k \). We have proved (i).

(ii) It suffices to prove \( \mathcal{R}(P) = \text{span}\{|a_{j,i}\rangle\} \). The inclusion \( \mathcal{R}(P) \supseteq \text{span}\{|a_{j,i}\rangle\} \) is evident. If the inclusion is strict, then \( P > \text{rank } \mathcal{A} \).

On the other hand Since \( (P \otimes I_B)\rho(\rho^\Gamma \otimes I_B) = \sigma \), we have \( P \rho A P^\Gamma = \sigma A \). Since \( \text{rank } \sigma A = M \) we have rank \( P = \text{rank } \mathcal{A} \). We have a contradiction and thus \( \mathcal{R}(P) = \{a_{j,i}\}\} \).

(iii) The assertions both follow from the proof of (i).
(iv) The assertion follows from (i) by replacing \( \rho \) by \( \rho^\Gamma \).
(v) Since \( k = 1 \) and \( \text{SN}(\sigma) = 1 \), (i) implies \( 1 \leq \text{SN}(\rho) \leq 2 \), and (iv) implies \( 1 \leq \text{SN}(\rho^\Gamma) \leq 2 \). Since \( \rho \) and \( \rho^\Gamma \) are both separable or not, we have proved the assertion.

(vi) The assertion follows from (i).
(vii) The assertion follows from (i).
(viii) The assertion follows by summing up (3) and minus (1). This completes the proof. \( \Box \)
It follows from (6) that $SN(\sigma) = 2$. First we prove (12). Since $\rho$ is a 2 × 3 state and still 1-undistillable, it is separable. So we have $SN(\sigma) = 1 < SN(\rho) = 2 = M - k$. The first inequality in (6) may be also strict. First we give an example of NPT $\rho$ and $M = N = 3$. An example is the antisymmetric state $\rho = \sum_{j,k=0,j<k} (\langle jk| - |kj\rangle)$. Up to ILOs we may assume the projector $P = |0\rangle\langle 0| + |1\rangle\langle 1| + (a|0\rangle + b|1\rangle)\langle 2|$ where $a,b$ are complex numbers. Then $(P \otimes I_2)\rho(P^\dagger \otimes I_2)$ is an NPT two-qubit state for any $a,b$. So it is entangled, and $SN(\rho) = SN(\sigma) = 2$. Below is an example of PPT state, where $k = 1$ and $SN(\rho) = 2$. Note that these two states also saturate the last equality in (6).

**Example 16** Let $\rho = \alpha \oplus \beta$ be a PPT entangled state, where $\alpha$ and $\beta$ are both $3 \times 3$ PPT entangled states, $R(\alpha_A) = R(\alpha_B) = \text{span}\{|1\rangle, |2\rangle, |3\rangle\}$ and $R(\beta_A) = R(\beta_B) = \text{span}\{|4\rangle, |5\rangle, |6\rangle\}$. It follows from Lemma 3 and Corollary 17 that $SN(\rho) = SN(\alpha) = SN(\beta) = 2$.

Let $P$ be a projector of rank five on $H_A$. We can express $P$ as $P = \sum_{i=1}^{6} |a_i\rangle\langle i|$, where $|a_1\rangle, \ldots, |a_6\rangle$ span a 5-dimensional subspace in $\mathbb{C}^6$. Hence either $|a_1\rangle, |a_2\rangle, |a_3\rangle$ or $|a_4\rangle, |a_5\rangle, |a_6\rangle$ span a 3-dimensional subspace in $\mathbb{C}^6$. Let $\sigma = (P_A \otimes I_B)\rho(P_A \otimes I_B)$. We have

$$\sigma = \left( \sum_{i=1}^{3} |a_i\rangle\langle i| \right)_A \alpha \left( \sum_{i=1}^{3} |i\rangle\langle a_i| \right)_A \oplus_B \left( \sum_{i=4}^{6} |a_i\rangle\langle i| \right)_A \beta \left( \sum_{i=4}^{6} |i\rangle\langle a_i| \right)_A.$$

(11)

So either the first state or the second state in (11) is still a $3 \times 3$ PPT entangled state. It follows from Lemma 3 and Corollary 17 that $SN(\sigma) = 2 = SN(\rho)$.

In Lemma 15 (iii), one can generate quantum states of Schmidt number $M - k$ using rank $M - k$ projections from a Schmidt number $M$ state. The converse of (iii) does not hold. An example is the normalized antisymmetric projector on the $3 \times 3$ subspace. This is an entangled state. Further we propose an example of separable state. Consider a $2 \times 3$ PPT state $\rho$ with any rank 1 projection, we have $SN(\rho) = 1 < M$ and $SN(\sigma) = 1 = M - k$.

Interestingly, Lemma 15 provides an alternative proof for a Conjecture in [1], see the Corollary below. An alternative proof using positive maps can be found in [20].

**Corollary 17** Let $\rho$ be a $3 \times 3$ state. Then

(i) every PPT entangled $\rho$ is of Schmidt number 2;

(ii) every Schmidt-number-3 $\rho$ is an NPT state. Moreover, for any matrix $P,Q \in M_3(\mathbb{C})$ with $\text{rank}(P) = \text{rank}(Q) = 2$, the projected states $(P \otimes I_3)\rho(P^\dagger \otimes I_3)$ or $(I_3 \otimes Q)\rho(I_3 \otimes Q^\dagger)$ are NPT states. So $\rho$ is distillable.

**Proof.** (i) This assertion follows Lemma 15 (i), in which we set $M = N = 3$ and $k = 1$. Then we have $SN(\rho) \leq SN(\sigma) + 1$. Note that $\sigma$ is a $2 \times 3$ PPT state which is also separable [23].

(ii) The first assertion follows easily from (i). WLOG, assume that the projected states $\sigma = (P \otimes I_3)\rho(P^\dagger \otimes I_3)$ is a PPT state. So $\sigma$ is a separable state, hence it violates the inequality $SN(\sigma) \geq SN(\rho) - k = 2$. The last assertion follows from the fact that any $2 \times N$ NPT states are distillable. This completes the proof.

The projected states may not be NPT even if the original state is NPT. For example, for any rank-one $P$ the state $(P \otimes I_B)\rho(P^\dagger \otimes I_B)$ is a separable state. It is an open problem to find out when the projected state is NPT, and it relates to the well-known distillability problem. Next we consider the relation between the Schmidt numbers of the two tensors of a bipartite state and the two copies of its projected state.

**Lemma 18** If $\rho$ and $\sigma$ are introduced in Lemma 17 then

$$SN(\sigma^{\otimes 2}) \leq \min\{SN(\rho^{\otimes 2}), (M-k)^2\},$$

$$SN(\rho^{\otimes 2}) \leq SN(\sigma)^2 + 2kSN(\sigma) + k^2.$$

(12)

(13)

**Proof.** First we prove (12). Since $\sigma = (P \otimes I_B)\rho(P^\dagger \otimes I_B)$, we can project $\rho^{\otimes 2}$ onto $\sigma^{\otimes 2}$. Hence $SN(\sigma^{\otimes 2}) \leq SN(\rho^{\otimes 2})$. It follows from (9) that $SN(\sigma) \leq M - k$. So $\sigma$ is the convex sum of pure states of Schmidt rank at most $M - k$. So $\sigma^{\otimes 2}$ is the convex sum of pure states of Schmidt rank at most $(M - k)^2$. We have $SN(\sigma^{\otimes 2}) \leq (M - k)^2$. So (12) holds. Next (13) follows from the fact $SN(\rho) \leq SN(\sigma) + k$, which is from Lemma 15 (i) and (ii). This completes the proof.

The Lemma shows that the Schmidt number of the tensor product of the same state is bounded by that of the tensor product of its projected states. One may similarly extend the Lemma to the tensor product of...
Lemma 19 Let $\rho = \alpha_{A_1B_1} \otimes \beta_{A_2B_2}$ be a bipartite state on the system $A_1A_2$ and $B_1B_2$.

(i) If neither of the range of the states $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ contains any product state, then $\SN(\rho) > 2$, and any decomposition of $\rho$ consists of pure states of Schmidt rank at least three.

(ii) In (i) if $\SN(\rho) = 3$, then $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$ where

$$|\psi_i\rangle = |a_i\rangle_{A_1A_2}|b_i\rangle_{B_1B_2} + |c_i\rangle_{A_1A_2}|d_i\rangle_{B_1B_2} + |e_i\rangle_{A_1A_2}|f_i\rangle_{B_1B_2},$$

(14) is a bipartite state of Schmidt number three. For any $i$, the spaces $R((\rho_{A_1A_2})$ and $R((\rho_{B_1B_2})$ both have no product state.

(iii) If $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ are both two-qutrit PPT entangled states of rank four, then $\SN(\rho) = 4$.

Proof. Since the range of the state $\alpha_{A_1B_1}$ does not contain any product state, $\alpha_{A_1B_1}$ is entangled. So $\rho$ is also entangled and has Schmidt number at least two. Since the range of $\alpha_{A_1B_1}$ does not contain any product state, the pure state in any decomposition of $\rho$ is a bipartite entangled state.

We disprove the assertion. Suppose there is a decomposition of $\rho$ containing a Schmidt-rank-two bipartite pure entangled state, i.e., $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$ where

$$|\psi_i\rangle = |a_i\rangle_{A_1A_2}|b_i\rangle_{B_1B_2} + |c_i\rangle_{A_1A_2}|d_i\rangle_{B_1B_2}.$$  

(15) It follows from Lemma 14 that there exists a nonzero state $|\gamma\rangle \in H_{A_1}$ (or $H_{A_2}$) such that the two states $|\gamma|a_i\rangle$ and $|\gamma|c_i\rangle$ in $H_{A_2}$ (or $H_{A_1}$) are proportional, and one of them is nonzero. Hence $|\gamma|\psi_i\rangle$ is a product state of the system $A_2$ (or $A_1$) and $B_1B_2$. By tracing out system $A_1B_1$ (or $A_2B_2$), we obtain that the range of $\beta_{A_2B_2}$ (or $\alpha_{A_1B_1}$) contains a product state. It is a contradiction with the assumptions. So we have $\SN(\rho) > 2$, and any decomposition of $\rho$ consists of pure states of Schmidt rank at least three.

(ii) The first assertion follows from (i). Using (15) we shall regard $|a_i\rangle$, $|c_i\rangle$, $|e_i\rangle$ as an arbitrary basis of $R((\rho_{A_1A_2})$, and $|b_i\rangle$, $|d_i\rangle$, $|f_i\rangle$ as an arbitrary basis of $R((\rho_{B_1B_2})$. To prove the second assertion, it suffices to show that for any $i$, the states $|a_i\rangle$, $|b_i\rangle$, $|c_i\rangle$, $|d_i\rangle$, $|e_i\rangle$, $|f_i\rangle$ all have Schmidt number greater than one. We have three cases.

In the first case, we assume that $|a_i\rangle$, $|c_i\rangle$ and $|e_i\rangle$ are product states. Let $|a_i\rangle = w_1, w_2\rangle$, $|c_i\rangle = x_1, x_2\rangle$ and $|e_i\rangle = y_1, y_2\rangle$. The second assertion is trivial when for $j = 1$ or 2, two of the states $|w_j\rangle$, $|x_j\rangle$ and $|y_j\rangle$ are proportional, or all of the three states are linearly independent. The only unsolved case is that for $j = 1$ and 2, any two of $|w_j\rangle$, $|x_j\rangle$ and $|y_j\rangle$ are linearly independent and all of the three states are linearly dependent. According to Lemma 14 there exists a nonzero state $|\gamma\rangle \in H_{B_1}$ or $H_{B_2}$ such that the two states $|\gamma|d\rangle$ and $|\gamma|f\rangle$ in $H_{B_2}$ or $H_{B_1}$ are proportional, and one of them is nonzero. Let $|z\rangle \perp |w_1\rangle$ or $|w_2\rangle$, and $|z\rangle$ is not orthogonal to $|y_1\rangle$, $|z_1\rangle$ or $|y_2\rangle$, $|z_2\rangle$. Then $\langle z | \langle \gamma | \psi_i\rangle$ is a product state. We trace out $\rho_{A_1B_1}$ by using the state $|z\rangle|\gamma\rangle$ as a state in the trace. Then one can show the second assertion, since the range of the state $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ does not contain any product state.

Next we assume that $|a_i\rangle$ and $|c_i\rangle$ are product states, and $|e_i\rangle$ is an entangled state. If $|e_i\rangle + x|a_i\rangle + y|c_i\rangle$ is a product state for some complex numbers $x, y$, then we have proved the assertion in the first case. So $|e_i\rangle + x|a_i\rangle + y|c_i\rangle$ is an entangled state for any $x, y$. It implies that there is a state $|z\rangle \in H_{A_1}$ or $H_{A_2}$ such that $\langle z | \langle e_i\rangle \neq 0$ and $\langle z | \langle a_i\rangle = (\langle z | \langle c_i\rangle = 0$. By tracing out one of $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$, we can obtain that the range of the other state contains product states. It is a contradiction with the assumption. So we have proved the second assertion.

We assume that $|a_i\rangle$ is a product state, and $|c_i\rangle$ and $|e_i\rangle$ are both entangled states. If $|e_i\rangle + x|a_i\rangle + y|c_i\rangle$ is a product state for some complex numbers $x, y$ then we have proved the assertion in the last two cases. So $|e_i\rangle + x|a_i\rangle + y|c_i\rangle$ is an entangled state for any $x, y$. One can similarly show that $|e_i\rangle + x|a_i\rangle + y|e_i\rangle$ is an entangled state for any $x, y$. Lemma 15 implies that there is a state $|\gamma\rangle \in H_{B_1}$ or $H_{B_2}$ such that the two states $|\gamma|d\rangle$ and $|\gamma|f\rangle$ in $H_{B_2}$ or $H_{B_1}$ are proportional, and one of them is nonzero. We have $\langle \gamma | \psi_i\rangle = |a_i\rangle \otimes |b_i\rangle + |g_i\rangle \otimes |h_i\rangle$, where $|g_i\rangle$ is the linear combination of $|c_i\rangle$ and $|e_i\rangle$. So $|g_i\rangle$ is an entangled state. We can find a state $|h\rangle \in H_{A_1}$ or $H_{A_2}$ such that $\langle h | a_i\rangle = 0$ and $\langle h | g_i\rangle \neq 0$. So $R(\rho_{A_1B_1})$ or $R(\rho_{A_2B_2})$ contains a product state $|h\rangle |g_i\rangle \otimes |h_i\rangle$. It is a contradiction with the assumption. So we have proved the second assertion.

One can similarly prove the second assertion by exchanging the systems $A_1A_2$ and $B_1B_2$.

(iii) It is known that neither of the range of the states $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ contains any product state. Further we can choose that $|a_i\rangle$ and $|c_i\rangle$ have Schmidt rank two, because $R(\rho_{A_1A_2})$ is a 3-dimensional subspace of $C^3 \otimes C^3$. Next if there is a state $|\alpha\rangle \in H_{A_1}$ or $H_{A_2}$ orthogonal to $|a_i\rangle$, $|c_i\rangle$ and $|e_i\rangle$ at the same time, then $R(\rho_{A_1A_2}) \subset \{|a\rangle \otimes C^3$. So $R(\rho_{A_1A_2})$ contains a product state and it is a contradiction with (ii). Hence there is no state orthogonal to $|a_i\rangle$, $|c_i\rangle$ and $|e_i\rangle$ at the same time. It implies that if there is a state $|\alpha\rangle \in H_{A_1}$ or $H_{A_2}$ orthogonal to $|a_i\rangle$, $|c_i\rangle$, then there is a product state in $R(\rho_{A_2B_2})$ or $R(\rho_{A_2B_1})$. It is a contradiction with (ii). So such $|\alpha\rangle$ does not exist. We shall use these facts below.
It follows from Lemma 14 that there exists a nonzero state \(|\gamma\rangle \in \mathcal{H}_{B_1} \) or \(\mathcal{H}_{B_2} \) such that the two states \(\langle\gamma | d_i\rangle\) and \(\langle\gamma | f_i\rangle\) in \(\mathcal{H}_{B_1} \) or \(\mathcal{H}_{B_2} \) are proportional, and one of them is nonzero. We have \(\langle\gamma | \psi_i\rangle = |a_i\rangle \otimes \langle \gamma | b_i\rangle + |g_i\rangle \otimes |h_i\rangle\), where \(|g_i\rangle\) is the linear combination of \(|a_i\rangle\) and \(|e_i\rangle\). We can find a state \(|\psi\rangle \in \mathcal{H}_{A_1} \) or \(\mathcal{H}_{A_2} \) such that \(\langle h | a_i\rangle = 0\) and \(\langle h | g_i\rangle \neq 0\). So \(\mathcal{R}(\alpha_{A_1B_1})\) or \(\mathcal{R}(\beta_{A_2B_2})\) contains a product state \(|h_i\rangle \otimes |h_i\rangle\). It is a contradiction with the assumption. So we have proved the second assertion. This completes the proof.

Next we generalize Lemma 19(i) to the tensor product of many bipartite states.

**Proposition 20** Let \(\rho = \otimes_{j=1}^n \alpha_{A_jB_j}\) be a bipartite state of systems \(A_1 \cdots A_n : B_1 \cdots B_n\), where \(\alpha_{A_jB_j}\) are bipartite states of the system \(A_jB_j, j = 1, \cdots, n\), respectively. Suppose neither of \(\mathcal{R}(\alpha_{A_jB_j})\) contains any product state. Then \(\text{SN}(\rho) > n\), and any decomposition of \(\rho\) consists of pure states of Schmidt rank at least \(n + 1\).

**Proof.** By the definition of Schmidt number, it suffices to prove the second assertion. That is any decomposition of \(\rho\) consists of pure states of Schmidt rank at least \(n + 1\). Suppose it is wrong. Let \(\rho = \sum_i |\psi_i\rangle \langle \psi_i|\) where

\[
|\psi_1\rangle = |a_1\rangle_{A_1} \cdots |a_n\rangle_{A_n} |b_1\rangle_{B_1} \cdots |b_n\rangle_{B_n} + \cdots + |a_k\rangle_{A_1} \cdots |a_n\rangle_{A_n} |b_k\rangle_{B_1} \cdots |b_n\rangle
\]

is a bipartite pure state of Schmidt rank \(k \leq n\). Lemma 14 implies that there exists a nonzero state \(|\gamma\rangle\) in \(\mathcal{H}_{A_1}\) such that the two states \(\langle\gamma | a_1\rangle\) and \(\langle\gamma | a_2\rangle\) in \(\mathcal{H}_{A_2} \cdots \mathcal{H}_{A_n}\) are proportional, and one of them is nonzero. Let \(\gamma' \in \mathcal{H}_{B_1}\) be a state such that \(|\varphi\rangle := \langle \gamma', \gamma | \psi_1\rangle \neq 0\). So \(|\varphi\rangle\) is a bipartite pure state of Schmidt rank \(k - 1 \leq n - 1\). Next using Lemma 14 again, we can find a state \(|\delta, \delta'\rangle\) in \(\mathcal{H}_{A_2B_2}\) such that \(\langle \delta, \delta' | \varphi\rangle \neq 0\) and has Schmidt rank at most \(n - 2\). Continuing in the same vein we can finally find a product state \(|\alpha\rangle\) in \(\mathcal{H}_{A_1} \cdots \mathcal{H}_{A_{n-1}} \mathcal{H}_{B_1} \cdots \mathcal{H}_{B_{n-1}}\) such that \(\langle \alpha | \psi_1\rangle \in \mathcal{H}_{A_1B_1}\) is nonzero and has Schmidt rank at most one. So it is a product state in \(\mathcal{R}(\rho_{A_nB_n})\). This is a contradiction with the assumption. So we have proved \(\text{SN}(\rho) > n\). This completes the proof.

The result implies that there exists a PPT entangled state of Schmidt number \(n\), where \(n\) can be greater than any given integer. The state has equal birank \((r, r)\) for some integer \(r\). Moreover, we can obtain a PPT entangled state of an arbitrary Schmidt number by the upcoming Lemma 23 from the aforementioned statement.

### A. Approximation by Schmidt number

Different quantum states may play the same role in quantum-information tasks. Their similarity decides how they play in the tasks. The similarity of quantum states can be characterized by many quantum-information quantities, such as the fidelity, entanglement measure and equivalence under LOCC. In this subsection, we investigate the similarity between two quantum states in terms of their Schmidt number. First of all we present the following definitions.

**Definition 21** Let \(\rho\) be an \(M \times N\) entangled state, and \(k \in [1, M - 1]\) an integer. We define two quantities:

\[
\text{SN}_{\max}(\rho, k) := \max_P \{\text{SN}(\sigma)\},
\]

\[
\sigma = (P \otimes I_B)\rho(P^\dagger \otimes I_B), \quad \text{Dim ker}(P) = k;
\]

\[
\text{SN}_{\min}(\rho, k) := \min_P \{\text{SN}(\sigma)\},
\]

\[
\sigma = (P \otimes I_B)\rho(P^\dagger \otimes I_B), \quad \text{Dim ker}(P) = k.
\]

The two quantities in Definition 21 can be estimated in a few special cases. If \(k = M - 1\) then \(\sigma\) is separable. We have \(\text{SN}_{\max}(\rho, M - 1) = \text{SN}_{\min}(\rho, M - 1) = 1\). If \(k = M - 2\) then we have \(\text{SN}_{\max}(\rho, M - 2), \text{SN}_{\min}(\rho, M - 2) \in [1, 2]\). One may similarly prove that \(\text{SN}_{\max}(\rho, 1), \text{SN}_{\min}(\rho, 1) \in [\text{SN}(\rho) - 1, \text{SN}(\rho)]\). Lemma 15 (i) implies that

\[
\max\{1, \text{SN}(\rho) - k\} \leq \text{SN}_{\min}(\rho, k) \leq \text{SN}_{\max}(\rho, k) \leq \min\{\text{SN}(\rho), M - k\}.
\]

The condition by which \(1 = \text{SN}_{\min}(\rho, k)\) or \(\text{SN}(\rho) - k = \text{SN}_{\min}(\rho, k)\) holds is in Lemma 15 (ii). If \(\text{SN}_{\max}(\rho, k) = \text{SN}(\rho)\) for some \(k\), then the space consisting all projected \(\sigma\) best approximates \(\rho\) in terms of Schmidt number. It is difficult in general to determine whether such a best approximation exists for an arbitrary \(\rho\). The equalities depend on the dimensions \((M, N)\) as well as the pair \(\text{SN}(\rho, k)\). To illustrate, let \(k = 1\) and pick \(\rho\) from the set of all \(3 \times 3\) PPT states. By Corollary 17 we know that \(\text{SN}(\rho) = 2\). Hence \(1 = \text{SN}_{\max}(\rho, 1) < \text{SN}(\rho) = 2\) since every \(2 \times 3\) PPT states are separable. Consider \(\rho\) from the set of all \(3 \times 3\) PPT states, then either \(\text{SN}(\rho) = 2\) or \(\text{SN}(\rho) = 3\). If \(\text{SN}(\rho) = 3\), by Corollary 17 the projected states are NPT entangled states. Thus we have 2 = \(\text{SN}_{\max}(\rho, 1) < \text{SN}(\rho) = 3\). If \(\text{SN}(\rho) = 2\),
consider the antisymmetric state \( \rho = \sum_{j,k=0,j<k}^N (|j\rangle - |k\rangle)(\langle j| - \langle k|) \). Choose a projector \( P = |0\rangle\langle 0| + |1\rangle\langle 1| \). Then \( (P \otimes I_2)\rho(P^\dagger \otimes I_2) \) is entangled. The next Lemma shows the relation between the Schmidt number of a quantum state and its projection in terms of Definition 21.

**Lemma 22** \( \text{SN}_{\text{max}}(\rho, k) = \text{SN}(\rho) \) holds for some \( k \) if and only if \( \text{SN}_{\text{max}}(\rho, 1) = \text{SN}(\rho) \).

**Proof.** The “if” part is trivial. It suffices to prove the “only if” part. Suppose \( \text{SN}_{\text{max}}(\rho, k) = \text{SN}(\rho) \). Since the Schmidt number does not increase under LOCC, we have

\[
\text{SN}_{\text{max}}(\rho, k) \leq \cdots \leq \text{SN}_{\text{max}}(\rho, 1) \leq \text{SN}(\rho).
\]

(20)

So the assertion holds. This completes the proof. \( \Box \)

Note that \( \text{SN}_{\text{max}}(\rho, k) \) may not equal \( \max_Q \{\text{SN}(\sigma, \sigma = (I_A \otimes Q)\rho(I_A \otimes Q^\dagger), \text{Dim ker}(Q) = k\} \). An example is \( \rho = |\psi\rangle\langle \psi| + |03\rangle\langle 03| \), and \( |\psi\rangle = |00\rangle + |11\rangle + |22\rangle \), \( k = 1 \), \( M = 3 \) and \( N = 4 \). One can show that \( \text{SN}_{\text{max}}(\rho, 1) = 2 \) and \( \max_Q \{\text{SN}(\sigma, \sigma = (I_A \otimes Q)\rho(I_A \otimes Q^\dagger), \text{Dim ker}(Q) = 1\} = 3 \). In general, we have the following Lemma.

**Lemma 23** Let \( \rho \) be an \( M \times N \) entangled state, \( P \) and \( Q \) two nonzero projectors respectively on \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Then

(i) the following three integer sets are the same,

\[
\{\text{SN}(\sigma) : \sigma = (P \otimes I)\rho(P^\dagger \otimes I), \forall P \neq 0\} = \{\text{SN}(\sigma) : \sigma = (I \otimes Q)\rho(I \otimes Q^\dagger), \forall Q \neq 0\} = \{1, 2, \ldots, \text{SN}(\rho)\}.
\]

(21)

(ii) For any \( P \) there exists a \( Q \) such that

\[
\text{SN}((P \otimes I)\rho(P^\dagger \otimes I)) = \text{SN}(I \otimes Q)\rho(I \otimes Q^\dagger)).
\]

(22)

**Proof.** (i) Consider the set \( A_k = \{\text{SN}(\sigma) : \sigma = (P \otimes I)\rho(P^\dagger \otimes I), \text{Dim ker} P \leq k\} \). By Lemma 15(i), we obtain \( A_1 = \{\text{SN}(\rho) - 1, \text{SN}(\rho)\} \) or \( A_1 = \{\text{SN}(\rho)\} \). Denote by \( P_k \) a projector with \( \text{Dim ker} P_k = k \). Since any projection \( P_k \) can be written into \( P_k = P_k P_{k-1} \), we have \( A_k = \{\text{SN}(\sigma_k) : \sigma_k = (P_k \otimes I)\sigma_k-1(P_k^\dagger \otimes I), \sigma_k-1 \in A_{k-1}\} \). Hence the set difference \( A_k \setminus A_{k-1} \) is either an empty set or a set of single number by Lemma 15(i). Using induction one has \( A_{M-1} = \{1, \ldots, \text{SN}(\rho)\} \). Similarly, we have the set \( B_k = \{\text{SN}(\sigma) : \sigma = (I \otimes Q)\rho(I \otimes Q^\dagger), \text{Dim ker} Q \leq k\} \) and \( B_{N-1} = \{1, \ldots, \text{SN}(\rho)\} = A_{M-1} \).

(ii) is an immediate consequence of (i). \( \Box \)

We also conjecture that for \( k = 1, \ldots, M - 1 \), the integer set \( \{\text{SN}(\sigma) : \sigma = (P \otimes I_B)\rho(P^\dagger \otimes I_B), \text{Dim ker}(P) = k\} \) is exactly the set of consecutive integers \( \{\text{SN}_{\text{min}}(\rho, k), \ldots, \text{SN}_{\text{max}}(\rho, k)\} \). The conjecture holds when \( k = M - 1, M - 2 \) and 1, as shown by the argument below. From Proposition 20 and Lemma 22 we obtain a main result of this paper.

**Theorem 24** For any integer \( r \), there exists a bipartite PPT entangled state of Schmidt number \( r \).

**IV. SCHMIDT NUMBER OF MULTIPARTITE STATES**

Multipartite quantum states have a more complicated structure than that of bipartite states and have been extensively investigated in past years. For example the well-known \( n \)-partite Greenberger-Horne-Zeilinger (GHZ) state \( \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}) \) is the generalization of Bell state. It has been realized in experiments for small \( n \) with a high fidelity and play an important role in quantum computing. In this section we generalize the notion of Schmidt number to multipartite states. The tensor rank of an \( N \)-partite quantum state \( |\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \) is defined as the minimum integer \( r \) such that there exist \( r \) product states \( |a_{j,1}, \ldots, a_{j,N}\rangle \) and \( |\psi\rangle = \sum_{j=1}^r |a_{j,1}, \ldots, a_{j,N}\rangle \). For example the \( n \)-partite GHZ state has tensor rank two. Now Definition 11 can be generalized to multipartite states as follows.

**Definition 25** A multipartite density matrix \( \rho \) has Schmidt number \( k \) if (i) for any decomposition of \( \rho \), \( \{p_i > 0, |\psi_i\rangle\} \) at least one of the vectors \( |\psi_i\rangle \) has at least tensor rank \( k \) and (ii) there exists a decomposition of \( \rho \) with all vectors \( |\psi_i\rangle \) of tensor rank at most \( k \).
For example, the three-qubit mixed state \( \rho = |\alpha\rangle\langle\alpha| + |000\rangle\langle000| \) where \( |\alpha\rangle = |000\rangle + |111\rangle \) has Schmidt number two. To understand this fact, we assume that \( \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \) as an arbitrary decomposition of \( \rho \). Using Lemma 13 one can obtain that there is always some \( |\psi_i\rangle \) of tensor rank two. Then Definition 26 shows that \( \text{SN}(\rho) = 2 \), and that the Schmidt number of multipartite states does not increase under LOCC. So the Schmidt number is also an entanglement measure for multipartite states. Evidently, Definition 26 reduces to Definition 1 for bipartite states \( \rho \). For simplicity we will regard tensor rank and Schmidt number as the same notion and use only Schmidt number. Further, the Schmidt number for bipartite and multipartite states are both invariant under ILOs. It is known that the Schmidt number is non-increasing under the local operations and classical communications [3]. So the Schmidt number is an entanglement monotone. Hence, the exact transformation under LOCC from a bipartite state \( |\psi\rangle \) of smaller Schmidt rank to \( |\varphi\rangle \) of bigger Schmidt rank is impossible. On the other hand, the transformation may be asymptotically realized by distilling EPR pairs from \( |\psi\rangle \) and then preparing \( |\varphi\rangle \). Third, it is known that for bipartite pure states \( |\varphi\rangle \) we have \( \text{SN}(|\varphi\rangle^{\otimes n}) = n \text{SN}(|\varphi\rangle) \). For multipartite pure states \( |\psi\rangle \), we have \( \text{SN}(|\psi\rangle^{\otimes n}) \leq n \text{SN}(|\psi\rangle) \) and the inequality is strict for the multiqubit W state \( |\psi\rangle \) and integers \( n > 1 \).

In the following subsections we construct and investigate three quantities of multipartite states, namely the expansion, coarse graining and joint Schmidt number. Their definitions are respectively given in Definition 26, 28 and 31. The expansion describes the global states whose reduced density operators are the target multipartite states. The coarse graining constructs multipartite states from the known ones by combining systems. The joint Schmidt number is another Schmidt number of multipartite states and different from Definition 25. The main results are given in Theorem 27, Lemma 29, Theorem 32 and Lemma 33. These establish the connection between the Schmidt number, local ranks of reduced density operators and global multipartite states.

### A. Expansion

In this subsection we investigate the Schmidt number of multipartite states and their reduced density operators. We review the notion of expansion which works for the well-known quantum marginal problem.

**Definition 26** If \( \rho_A \) and \( \rho_B \) are the reduced density operators of a quantum state \( \rho_{AB} \), then we say that \( \rho_{AB} \) is an expansion of \( \rho_A \) and \( \rho_B \).

The expansion of a quantum state describes the global physical environment when the quantum state is regarded as a local state. When \( \rho_{AB} \) is a pure state, it is also called the purification of \( \rho_A \) and \( \rho_B \) in literatures. For example if \( \rho_A = \rho_B = \frac{1}{2} I_2 \) then any two-qubit maximally entangled state \( \rho_{AB} \) is the expansion of \( \rho_A \) and \( \rho_B \). Some \( \rho_A \) and \( \rho_B \) do not have any purification (or even expansion). Using the definition we have

\[ (i) \text{ The Schmidt number of } \rho_{ABC} \text{ is not smaller than the Schmidt number of } \rho_{AB}, \rho_{AC} \text{ and } \rho_{BC}. \]

\[ (ii) \rho_{AB} \text{ has Schmidt number at most } k \text{ if and only if there is a tripartite state } \rho_{ABC} \text{ of Schmidt number at most } k. \]

**Theorem 27** (i) The Schmidt number of \( \rho_{ABC} \) is not smaller than the Schmidt number of \( \rho_{AB}, \rho_{AC} \) and \( \rho_{BC} \).

(ii) \( \rho_{AB} \) has Schmidt number at most \( k \) if and only if there is a tripartite state \( \rho_{ABC} \) of Schmidt number at most \( k \).

(iii) Suppose \( |\psi\rangle_{ABC} \) is the purification of \( \rho_{AB} \). Then

\[
\min\{\text{SN}(\rho_{AB}) \cdot \text{rank} \rho_{AB}, \ \text{rank} \rho_A \cdot \text{rank} \rho_B\} \\
\geq \text{SN}(|\psi\rangle_{ABC}) \\
\geq \max\{\text{rank} \rho_{AB}, \ \text{rank} \rho_A, \ \text{rank} \rho_B\} \\
\geq \text{SN}(\rho_{AB}).
\]

(iv) If \( \rho_{AB} \) is a PPT state, then the first two equalities in (23) hold simultaneously if and only if \( \text{rank} \rho_A \cdot \text{rank} \rho_B = \text{rank} \rho_{AB} \) or \( \text{SN}(\rho_{AB}) = 1 \), i.e. \( \rho_{AB} \) is a separable state.

(v) If \( \rho_{AB} \) is a PPT state then the three equalities in (23) hold simultaneously if and only if \( \text{rank} \rho_A = \text{rank} \rho_B = 1 \).

(vi) If \( \rho_{AB} = |\psi\rangle \langle \psi|_{A_1B_1} \otimes \sum_i |ii\rangle |ii|_{A_2B_2} \) is a bipartite NPT state where \( |\psi\rangle = \sum_j |jj\rangle, A = A_1A_2, B = B_1B_2 \), then the last equality in (23) holds. If \( \rho_{AB} \) has rank one then all three equalities in (23) hold.

**Proof.** (i) Let \( \rho_{ABC} = \sum_i |\psi_i\rangle \langle \psi_i| \) where each \( |\psi_i\rangle \) has Schmidt number at most \( k := \text{SN}(\rho_{ABC}) \). So the pure states \( |i\rangle |j\rangle \) has Schmidt number at most \( k \). Since \( \rho_{AB} = T_C \rho_{ABC} = \sum_j |j\rangle \langle j| \rho_{AB} \otimes |j\rangle \langle j|_C \), the assertion on \( \rho_{AB} \) holds. The other assertions can be proved similarly.

(ii) The “if” part follows from (i). To prove the “only if” part, suppose \( \rho_{AB} = \sum_j |\psi_j\rangle \langle \psi_j|_{AB} \) where each \( |\psi_j\rangle \) has Schmidt number at most \( k \). Then \( \rho_{ABC} = \sum_j |\psi_j\rangle \langle \psi_j|_{AB} \otimes |j\rangle \langle j|_C \) is an expansion of \( \rho_{AB} \) and has Schmidt number at most \( k \).
(iii) Suppose $\rho_{AB} = \sum_{j=1}^r |\alpha_j\rangle\langle\alpha_j|_{AB}$ satisfies that $\text{SN}(\alpha_j) \leq \text{SN}(\rho_{AB})$. Without loss of generality, we may assume that the first $r := \text{rank}\rho_{AB}$ states $|\alpha_1\rangle, \ldots, |\alpha_r\rangle$ are linearly independent, and any $|\alpha_j\rangle$ is in the span of them. It is known that $|\psi\rangle_{ABC} = \sum_{j=1}^r |\alpha_j, u_j\rangle$ where the $|u_j\rangle$'s form a set of o. n. basis in $\mathbb{C}^d$ [24, Eq. (9.66)]. Hence

$$\text{SN}(|\psi\rangle_{ABC}) \leq \sum_{j=1}^r \text{SN}(\alpha_j) \leq r \cdot \text{SN}(\rho_{AB}).$$

Next the inequality $\text{rank } \rho_A \text{ rank } \rho_B \geq k := \text{SN}(|\psi\rangle_{ABC})$ follows from the definition of tensor rank. So we have proved the first inequality in (23). Let $\rho_{AB} = \sum_{i=1}^l |\alpha_i\rangle\langle\alpha_i|$ such that the $|\alpha_i\rangle$ are linearly independent. Then $|\psi\rangle_{ABC} = \sum_{i=1}^l |\alpha_i, l\rangle$, and thus $k \geq r$. Next the assertion $\text{SN}(|\psi\rangle_{ABC}) \geq \max\{\text{rank } \rho_A, \text{rank } \rho_B\}$ follows by writing $|\psi\rangle_{ABC}$ as the bipartite state of systems $A : BC$ and $B : AC$. So we have proved the second inequality in (23). To prove the third inequality $\text{rank } \rho_A \geq \text{SN}(\rho_{AB})$ in (23), we notice that $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ where each bipartite pure state $|\psi_i\rangle$ is an $M \times N$ state where $M \leq \text{rank } \rho_A$ and $N \leq \text{rank } \rho_B$. So the inequality holds.

(iv) The "if" part can be verified straightforwardly. Next we prove the "only if" part. Since $\rho_{AB}$ is a PPT state, then $\text{rank } \rho_{AB} \geq \max\{\text{rank } \rho_A, \text{rank } \rho_B\}$ [25]. Hence the assumption of the "only if" part is equivalent to

$$\begin{align*}
\min\{\text{SN}(\rho_{AB}) \cdot \text{rank } \rho_A, \text{rank } \rho_A \cdot \text{rank } \rho_B\} &= \text{SN}(|\psi\rangle_{ABC}) \\
&= \text{rank } \rho_{AB}.
\end{align*}$$

If $\min\{\text{SN}(\rho_{AB}) \cdot \text{rank } \rho_A, \text{rank } \rho_A \cdot \text{rank } \rho_B\} = \text{SN}(\rho_{AB}) \cdot \text{rank } \rho_A$ then one obtains $(\text{SN}(\rho_{AB}) - 1) \cdot \text{rank } \rho_B = 0$. Hence $\rho_{AB}$ is separable. On the other hand if $\min\{\text{SN}(\rho_{AB}) \cdot \text{rank } \rho_A, \text{rank } \rho_A \cdot \text{rank } \rho_B\} = \text{rank } \rho_A \cdot \text{rank } \rho_B$ then it is obvious that $\text{rank } \rho_A \cdot \text{rank } \rho_B = \text{rank } \rho_{AB}$.

(v) The assertion follows from (iv), [24] and $\text{rank } \rho_{AB} \geq \max\{\text{rank } \rho_A, \text{rank } \rho_B\}$.

(vi) The assertion can be verified straightforwardly using Lemma 3 because the states $|\psi\rangle_{A_1B_1} \otimes |jj\rangle_{A_2B_2}$ are orthogonal each other. This completes the proof.

When $k = 2$, assertion (ii) gives a necessary and sufficient condition for whether $\rho$ has Schmidt number at most two. Besides the equality $\text{SN}(\rho_{ABC}) = \text{SN}(\rho_{AB}) = \text{SN}(\rho_{BC}) = \text{SN}(\rho_{AC})$ may hold for some $\rho_{ABC}$. An example is the three-qubit state $|000\rangle + |a, a, a\rangle$ where $|a\rangle = |0\rangle + |1\rangle$. It is possible that

$$\text{SN}(\rho) > \sum_{1 \leq j_1 < j_2 \leq n} \text{SN}(\rho_{A_{j_1} A_{j_2}}) + \sum_{1 \leq j_1 < j_2 < j_3 \leq n} \text{SN}(\rho_{A_{j_1} A_{j_2} A_{j_3}}) + \cdots + \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq n} \text{SN}(\rho_{A_{j_1} \cdots A_{j_{n-1}}}).$$

For example, the inequality holds when $\rho$ is the $d$-level Greenberger-Horne-Zeilinger state $\sum_{j=1}^d |jj \cdots j\rangle$ when $d$ is sufficiently big. The reason is that any $k$-partite reduced density operator $\sigma$ of $\rho$ is a separable state, i.e., $\sigma = \sum_{j} \langle j|_A \cdots \langle j|_D |j\rangle_A \cdots |j\rangle_D$. Hence $\text{SN}(\sigma) = 1$. All together we have $\sum_{k=2}^{n-1} \binom{n}{k} = \sum_{k=2}^{n-1} (-1)^{n-k} \binom{n-1}{k} = 2^n - (n-1)$ number of terms. If each system has dimension $d_k > 2^n - n - 2$, then any $d$-level GHZ state with $d > 2^n - n - 2$ will satisfy the inequality. Since the Schmidt number is a multipartite entanglement measure, (26) shows the monogamy relation for some states.

In Theorem (iii), we have shown the relation between the Schmidt number, the rank and the purification of a bipartite state. The known inequality $\text{rank } \rho_A \cdot \text{rank } \rho_B \geq \text{rank } \rho_{AB}$ holds for any state $\rho_{AB}$. Eq. (23) gives the inequality $\text{rank } \rho_A \cdot \text{rank } \rho_B \geq \text{SN}(|\psi\rangle_{ABC}) \geq \text{rank } \rho_{AB}$ which is stronger than the known inequality. In assertion (iv), if the state $\rho_{AB}$ is not PPT then it may still make the first two equalities in (23) hold. For example $\rho_{AB}$ is the bipartite pure entangled state. A more complicated example is the mixed entangled state $\rho_{AB} = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$ where $|\alpha\rangle = |11\rangle + |22\rangle$ and $|\beta\rangle = |33\rangle + |44\rangle$. One can verify that the first two equalities in (23) holds since $\text{rank } \rho_A = \text{rank } \rho_B = 4$, $\text{SN}(\rho_{AB}) = \text{rank } \rho_{AB} = 2$ and $\text{SN}(|\psi\rangle_{ABC}) = 4$. On the other hand, the second equality in (23) fails when $|\alpha\rangle = |01\rangle + |10\rangle$ and $|\beta\rangle = |00\rangle$. One can show that $\text{SN}(|\psi\rangle_{ABC}) = 3 > \text{SN}(\rho_{AB}) = \text{rank } \rho_{AB} = 2$. It is an interesting question to investigate when the last equality in (23) holds.

For any tripartite state $|\psi\rangle_{ABC}$, if we regard it as a bipartite state over the split of systems $A$ and $BC$, then we obtain $\text{rank } \rho_A = \text{rank } \rho_{AB}$. Similarly one obtains $\text{rank } \rho_B = \text{rank } \rho_{AC}$, and $\text{rank } \rho_C = \text{rank } \rho_{AB}$. So only three of the six parameters $\text{rank } \rho_A, \text{rank } \rho_B, \text{rank } \rho_C, \text{rank } \rho_{AB}, \text{rank } \rho_{AC}, \text{rank } \rho_{BC}$ are independent. In fact we have chosen the
three parameters \( \rho_A, \rho_B \) and rank \( \rho_{AB} \) in \( [23] \). The other two parameters \( SN(\rho_{AB}) \) and \( SN(|\psi\rangle_{ABC}) \) are also independent from the three parameters. On the other hand the six parameters of a mixed tripartite state may be independent from each other, and the investigation is more complicated. For readers’ reference, the relation between the ranks of global and local systems for the entropy has been recently investigated \( [24] \).

B. Coarse graining

In this subsection we investigate the Schmidt number of multipartite states in terms of its coarse graining. The latter is defined as follows.

**Definition 28** (i) Let \( \rho \) be an \( n \)-partite quantum state of systems \( A_1, \cdots, A_n \). If we partition the systems into \( m \) disjoint parties \( B_1, \cdots, B_m \) then we obtain a new \( m \)-partite quantum state \( \sigma \). We denote \( \sigma \) as a coarse graining of \( \rho \).

(ii) The multipartite PPT states are defined as the states any bipartition of whom is a PPT state. We denote \( \rho^\Gamma_j \) as the partial transpose w. r. t. system \( A_j \).

For example if \( |\psi\rangle = |000\rangle + |111\rangle \), \( B_1 = A_1 \), and \( B_2 = A_2A_3 \), then \( |\phi\rangle = |\psi\rangle = |00\rangle + |13\rangle \) where \( |0\rangle_{B_2} = |00\rangle_{A_2A_3} \) and \( |3\rangle_{B_2} = |11\rangle_{A_2A_3} \). The following claim is clear from the definition.

**Lemma 29** (i) The Schmidt number of a multipartite pure state is not smaller than that of its coarse graining.

(ii) The multipartite state \( \rho \) and its partial transpose \( \rho^\Gamma_j \) are simultaneously separable or not.

We explain the coarse graining from the point of view of quantum information. In a multipartite state \( |\psi\rangle \), some of the \( n \) systems can be combined so that they perform collective operation, and create more quantum correlation quantitatively and qualitatively in \( |\psi\rangle \). So the coarse graining of \( |\psi\rangle \) represent different entanglement structure from \( |\psi\rangle \). The coarse graining has been used to investigate the geometric measure of entanglement \( [27] \).

C. Joint Schmidt number

In this subsection we construct another version of Schmidt number of multipartite states. This is different from Definition \( [28] \), namely the joint Schmidt number (JSN). We begin by reviewing the version of pure multipartite states constructed in \( [28] \).

**Definition 30** If the multipartite state \( |\phi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \) has Schmidt number \( s_j \) under the bi-partition \( \mathcal{H}_i \otimes (\otimes_{j \neq i} \mathcal{H}_j) \), then we say that \( |\phi\rangle \) has joint Schmidt number \( JSN(|\phi\rangle) = (s_1, \ldots, s_n) \).

For example, the genuinely entangled multiqubit state has joint Schmidt number \( (2, \cdots, 2) \). Essentially, the definition arises in the different bi-partitions of the systems. To generalize it to mixed multipartite states \( \rho \), we denote \( JSN(\rho) \) as the joint Schmidt number of \( \rho \). Given two \( n \)-partite states \( \rho \) and \( \sigma \) with \( JSN(\rho) = (s_1, \ldots, s_n) \) and \( JSN(\sigma) = (t_1, \ldots, t_n) \), we say that \( \sigma \) dominates \( \rho \) and denote it by \( JSN(\rho) \leq JSN(\sigma) \) if \( s_i \leq t_i \) for \( i = 1, \ldots, n \). So two tuples \( (s_1, \cdots, s_n) \) and \( (t_1, \cdots, t_n) \) are equal when they dominate each other.

**Definition 31** The multipartite state \( \rho \) in the system \( \prod_{i=1}^n A_i \) has joint Schmidt number \( s_j \) under the system bipartition of \( A_i : \prod_{j \neq i} A_i \). If in addition there exists a decomposition \( \rho = \sum_i |\phi_i\rangle \langle \phi_i| \) with all \( JSN(|\phi_i\rangle) \leq (s_1, \ldots, s_n) \), then we say the decomposition is a balanced decomposition.

For example, the three-qubit state \( |\psi\rangle |\psi\rangle + |000\rangle |000\rangle \) has joint Schmidt number \( (2, 2, 2) \) where \( |\psi\rangle = |001\rangle + |010\rangle + |100\rangle \). The definition implies that the multipartite state is separable if and only if it has a balanced decomposition with joint Schmidt number \( (1, \ldots, 1) \). Furthermore, for any local operators \( V = \otimes_{j=1}^n V_j \), one can show that \( JSN(V \rho V^\dagger) \leq JSN(\rho) \). Hence the joint Schmidt number is a multipartite entanglement monotone and is physically meaningful. This is similar to the role of Schmidt number for bipartite states. We further investigate the mathematical relation of them.

**Theorem 32** (i) Let \( |\psi\rangle \) be a multipartite state of \( JSN(|\psi\rangle) = (s_1, \ldots, s_n) \). Then \( \max_{j=1,\ldots,n} \{ s_j \} \leq SN(|\psi\rangle) \leq \min_{j=1,\ldots,n} \{ \frac{\prod_{i\neq j}^n s_i}{s_j} \} \).

(ii) If \( |\psi\rangle \) is separable under \( (n-1) \) many bi-partitions, then \( |\psi\rangle \) is separable.
Lemma 33. We propose the following statement.

(ii) The assertion follows from (i) immediately. This completes the proof.

The bound in Theorem 32 (i) is tighter than that in [28, Theorem 4.2], which says \(\text{SN}(\rho) \leq \prod_{i=1}^{n} s_i\). For example consider the tripartite state \(|\psi\rangle = |111\rangle + |122\rangle + |213\rangle + |224\rangle\). One can verify that \(\text{SN}(\psi) = 4\) and \(\text{JSN}(\psi) = (2, 2, 4)\). So \(\text{SN}(\psi) = s_1 s_2 < s_1 s_2 s_3 = 16\). On the other hand, any 4-partite pure state \(|\varphi\rangle_{A_1 A_2 A_3 A_4}\) can be regarded as a tripartite state, say \(|\alpha\rangle_{A_1 A_2 \perp A_3 A_4}\) in terms of Definition 28. If \(\text{JSN}(|\varphi\rangle) = (s_1, s_2, s_3, s_4)\) then \(\text{JSN}(|\alpha\rangle) = (s_1, s_2, s_3)\). So Lemma 29 says that \(\text{SN}(\psi) \geq \text{SN}(\alpha)\), and Theorem 32 says that \(s_1 s_2 \geq \text{SN}(\alpha)\). Hence

\[
\begin{align*}
\text{min}\{\text{SN}(\psi) s_1 s_2\} &\geq \text{SN}(\alpha).
\end{align*}
\]

The condition of \((n - 1)\) many bipartitions in Theorem 32 (ii) is necessary. Indeed a multipartite state \(|\psi\rangle\) may be entangled if its \((n - 2)\) many bipartitions are all separable. An example is the tripartite state \(|\psi\rangle = |000\rangle + |110\rangle\). In spite of Theorem 32 (ii), the biseparability via all bi-partitions does not imply the separability of multipartite mixed states. An example is the 3-qubit PPT entangled state \(\rho = I - \sum_{j=1}^{4} |a_i, b_j, c_i\rangle \langle a_i, b_j, c_i|\) where \(|\{a_i, b_j, c_i\}\rangle\) is a 3-qubit UPB. One can show that \(\text{JSN}(\rho) = (1, 1, 1)\), and \(\rho\) has Schmidt rank two. Since \(\text{SN}(\rho) = 2 > 1^{3/1} = 1\), Theorem 32 (i) cannot be generalized to mixed states.

In fact, any multipartite PPT state of rank at most three, or any non-three-qubit and non-two-qutrit PPT state of rank four is separable [29]. Thus it has joint Schmidt number \((1, 1, \cdots, 1)\). On the other hand, \(\rho\) does not have a balanced decomposition, because \(\rho\) is entangled. One can verify that for any \(j = 1, 2, 3), \rho^j\) is still a PPT entangled state of rank four, and satisfies \(\text{JSN}(\rho^j) = \text{JSN}(\rho) = (1, 1, 1)\) and \(\text{SN}(\rho^j) = \text{SN}(\rho) = 2\). For general entangled states we propose the following statement.

Lemma 33. Let \(\rho\) be a multipartite entangled PPT state of rank four. Then

(i) \(\rho\) and its partial transpose w. r. t. any systems, when regarded as bipartite states, all have Schmidt number two.

(ii) If \(\rho\) is not a two-qutrit state then \(\text{SN}(\rho) = (1, \cdots, 1)\).

(iii) Any multipartite entangled PPT state with Schmidt number at least 3 when regarded as bipartite states, has rank at least 5.

Proof. (i) It is known that any entangled PPT state \(\rho\) of rank four is either a three-qubit or a two-qutrit state [29]. The assertion holds when \(\rho\) is a two-qutrit state by Corollary 17. On the other hand if \(\rho\) is a three-qubit state, then \(\text{JSN}(\rho) = (1, 1, 1)\) [29]. So \(\rho\) is the convex sum of product states over the bipartition of spaces \(\mathcal{H}_1 : \mathcal{H}_{2,3}\). So the assertion also holds.

(ii) The assertion can be proved by the argument similar to that of (i).

(iii) Immediate from (i). This completes the proof.

Lemma 33 (iii) restricts the rank of desired states whose Schmidt number is different from that of its partial transpose. So far there is no example or proof for the existence of such states.

V. PROBLEMS

In this section we introduce some open problems on the Schmidt number. Let \(\rho\) be a bipartite state, \(P\) a projector on \(\mathcal{H}_A\), and \(P^\perp\) the orthogonal projector to \(P\). Let \(\alpha = (P \otimes I)\rho(P \otimes I)\) and \(\beta = (P^\perp \otimes I)\rho(P^\perp \otimes I)\). Then it is natural that \(\text{SN}(\rho) \leq \text{SN}(\alpha) + \text{SN}(\beta)\). However it is generally wrong and we give a counterexample. Let \(\rho = |\psi\rangle \langle \psi| + |\varphi\rangle \langle \varphi| + |\omega\rangle \langle \omega|\) where \(|\psi\rangle = |11\rangle + |22\rangle, |\varphi\rangle = |33\rangle + |44\rangle + |55\rangle\), and \(|\omega\rangle = |33\rangle - |44\rangle + |66\rangle\). Let \(P = |1\rangle \langle 1| + |3\rangle \langle 3| + |4\rangle \langle 4|\). One can verify that \(\alpha\) and \(\beta\) are both separable states. We claim that \(\text{SN}(\rho) = 3\) and thus the inequality is wrong. To
prove the claim, we note that the maximal Schmidt rank of any state in \( \mathcal{R}(\rho) \) is three, then the claim follows from the definition of Schmidt number and Lemma 13.

Lemma 22 shows that if \( \SN_{\min}(\rho, k) = \SN(\rho) \) or \( \SN_{\max}(\rho, k) = \SN(\rho) \) for some \( k \), then the minimum \( k \) is one. On the other hand \( \SN_{\min}(\rho, k) = \SN_{\max}(\rho, k) = 1 \) when \( k = M - 1 \). However

**Conjecture 34** (i) What is the maximum \( j \), such that \( \SN_{\max}(\rho, j) = \SN(\rho) \)?

(ii) What is the minimum \( k \), such that \( \SN_{\max}(\rho, k) = 1 \)?

**Conjecture 35** (i) There exists a PPT state \( \rho \) such that \( \SN(\rho) > \SN(\rho^\Gamma) \).

(ii) Such \( \rho \) exists in \( M \times N \) system where \( 3 \leq M \leq N \) and \( MN \geq 12 \). The simplest \( \rho \) is a \( 3 \times 4 \) PPT state of BSN (2, 3).

(iii) If the simplest \( \rho \) in (ii) exists then \( \SN(\rho^{\otimes 2}) \) has BSN (4, 9).

(iv) If (i) holds then there exists \( \rho \) constructed from a UB (\{ \( a_j, b_j \) \}, i.e., \( \rho = I - \sum |a_j, b_j\rangle \langle a_j, b_j| \).

Since Schmidt number is an entanglement measure, the equality \( \SN(\rho) = \SN(\rho^\Gamma) \) would imply that \( \rho \) and \( \rho^\Gamma \) have the same entanglement. However, to find an example for Conjecture 35 (ii), one has to find a \( 3 \times 4 \) entangled PPT state with Schmidt number 3 [20]. No concrete example has been given in the literature yet. The existence of a \( 3 \times 4 \) PPT state \( \rho \) with \( \SN(\rho) = 3 \) is equivalent to the existence of an indecomposable 2-positive map in \( B(M_2(\mathbb{C}), M_2(\mathbb{C})) \). Note that if such a state exists, then it may provide a candidate for an example for Conjecture 35. One need to further check \( \SN(\rho^\Gamma) = 2 \) besides \( \SN(\rho) = 3 \).

**Conjecture 36** For any positive integer \( L \), there is a PPT state \( \rho \) such that \( |\SN(\rho) - \SN(\rho^\Gamma)| \geq L \).

**Conjecture 37** If \( \SN(\rho) \geq \SN(\sigma) \), then \( \SN(\rho^{\otimes 2}) \geq \SN(\sigma^{\otimes 2}) \).

If the conjecture holds, then \( \SN(\rho^{\otimes 2^n}) \geq \SN(\sigma^{\otimes 2^n}), \forall n \geq 1 \) provided \( \SN(\rho) \geq \SN(\sigma) \).

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