DIRAC EQUATION WITH VECTOR AND SCALAR POTENTIALS VIA SUSY QM

E. S. Rodrigues\textsuperscript{(a,b)}, A. F. de Lima\textsuperscript{(a)} and R. de Lima Rodrigues\textsuperscript{β}

\textsuperscript{(a)}Unidade Acadêmica de Física, Universidade Federal de Campina Grande
58109-790 Campina Grande - PB, Brazil

\textsuperscript{β}Unidade Acadêmica de Educação, Universidade Federal de Campina Grande
58.175-000 Cuité - PB, Brazil

\textsuperscript{(b)}Departamento de Física, Instituto Federal de Educação, Ciência e Tecnologia do Sertão de Pernambuco - Campus Salgueiro
Cidade Universitária, 56.000-000 Salgueiro - PE, Brazil

Abstract

In this work, a spin $\frac{1}{2}$ relativistic particle described by a generalized potential containing both the Coulomb potential and a Lorentz scalar potential in Dirac equation is investigated in terms of the generalized ladder operators from supersymmetry in quantum mechanics. This formalism is applied for the generalized Dirac-Coulomb problem which is an exactly solvable potential in relativistic quantum mechanics. We obtain the energy eigenvalues and calculate explicitly the energy eigenfunctions for the ground state and first excited state.

PACS numbers: 03.65.Fd, 03.65.Ge, 11.30.Pb

\textup{e-mails} to ESR is eriverton.rodrigues@ifsertao-pe.edu.br, to AFL is aerlima@df.ufcg.edu.br and to RLR are rafaelr@cbpf.br or rafael@df.ufcg.edu.br.
I. INTRODUCTION

Using first the separation in polar coordinates system for the generalized Dirac equation, in three dimensions, the supersymmetry (SUSY) in quantum mechanics (QM) is investigated. We consider a spin $\frac{1}{2}$ particle in the Coulomb potential, $V_c = -\frac{A_1}{r}$, and a Lorentz scalar potential, added to the mass term in the Dirac equation which may be interpreted as an effective position dependent mass. If the scalar potential is assumed to be created by the exchange of massless scalar mesons, it has the form $V_s = -\frac{A_2}{r}$.

The (3+1) and (1+1) dimensional Dirac equations with both scalar-like and vector-like potentials have been well known in the literature for a long time [1]. Exact solutions for the bound states in this mixed potential can be obtained by the method of separation of variables [2–5] and also by the use of the dynamical algebra $SO(2, 1)$ [6]. In a recent paper the solution of the scattering problem for this potential has been obtained by an analytic method and also by an algebraic method by Vaidya and Silva Souza [7].

We do not think that the solution of the relativistic Coulomb potential with position-dependent mass in the Coulomb field has been correctly solved.

Recently exact solutions have been found for fermions in the presence of a classical background which is a mixing of the time-dependent of a gauge potential and a scalar potential [9]. Also, exactly solvable Eckart scalar and vector potentials in the Dirac equation have been investigated via SUSY QM [10], the $S$-wave Dirac equation has been solved exactly for a single particle with spin and pseudospin symmetry moving in a central Woods-Saxon potential [11], and it was shown that some other cases for which the Dirac equation with classical potentials of vector and scalar natures with spherical asymmetry can be solved exactly [12].

In this work we consider an alternative calculation for the energy spectrum and eigenfunctions of the Dirac equation via a Schrödinger-like wave equation, for the vector and scalar potentials recently studied by Alhaidari [8]. The connection between the Johnson-Lippmann [13, 14] and the generalized Johnson-Lippman operators [15, 16] in a Dirac-Coulomb problem via SUSY QM has been studied in 3-dimensional space. In [13] relations between various algebraic approaches via radial equation are pointed out, and in [14] the bound eigenfunctions and spectra of a Dirac hydrogen atom have been found via $su(1, 1)$ Lie algebra.

Our calculation generalizes the results previously obtained for the relativistic Coulomb
problem. Instead of a direct generalization of Sukumar’s calculation we use the equivalent approach developed by Fukui and Aizawa \[19\] and also by Balantekin \[20, 21\] where shape invariance \[22\] plays an important role. The Coulomb problem in non-relativistic quantum mechanics has been obtained by the use of SUSY shape invariances \[27\]. The special case of the Dirac-Coulomb potential has been treated recently via SUSY quantum mechanics by one of the authors \[28\].

This work is organized as follows. In section II, from the time independent Dirac equation for a potential in terms of the Coulomb potential and a Lorentz scalar potential we construct the radial equations. In section III, we have achieved the diagonalization of the matrix that appears in the interaction term of the radial Dirac equation, supercharges, supersymmetric Hamiltonian and the ground state energy eigenvalue and eigenfunction. In section IV, using the shape invariance properties we deduce generalised ladder operators to build up the bound state energy eigenvalues and eigenfunctions. The conclusion is presented in section V.

II. TIME INDEPENDENT DIRAC EQUATION

The time independent Dirac equation with vector \(V(r)\) and scalar potentials \(V_S(r)\) may be written in the form

\[
H \Psi = E \psi,
\]

where the Hamiltonian is given by

\[
H = \rho_1 \otimes \sigma \cdot \vec{p} + \rho_3 \otimes 1_{2 \times 2} (M + V_S(r)) + V(r) 1_{4 \times 4},
\]

and we have used a direct product notation in which \(\rho_i\) and \(\sigma_i\), \((i = 1, 2, 3)\) are the Pauli spin matrices obeying \([\rho_i, \sigma_j] = 0\), with \(\hbar = 1 = c\).

The Hamiltonian \(H\) commutes with the total angular momentum

\[
\vec{J} = \vec{L} + \frac{1}{2} \vec{\sigma}.
\]

It also commutes with the Dirac operator

\[
K = \rho_3 (1 + \sigma \cdot \vec{L}),
\]
and with the inversion operator

\[ P = \rho_3 I, \]  

where \( I \) inverts the spatial coordinates and the momenta.

The symmetry group of this Hamiltonian is characterized by the two vector invariants \( J \) and \( K \), precisely as in the nonrelativistic case. A complete set of mutually commuting operators is \( H, J^2, J_3, K, P \) which have the simultaneous eigenvectors \( | E, j, m, \kappa > \) with

\[
\begin{align*}
J^2 | E, j, m, \kappa > &= j(j + 1) | E, j, m, \kappa > \\
J_3 | E, j, m, \kappa > &= m | E, j, m, \kappa > \\
K | E, j, m, \kappa > &= -k | E, j, m, \kappa >
\end{align*}
\]

where \( j \) half integral, \( m = -j, \ldots j \) and \( k = \pm \left( j + \frac{1}{2} \right) \).

To separate variables we introduce two component spinors which are eigenfunctions of \( J^2, J_z, L^2, S^2 \) and are of two types

\[
\phi^{(+)}_{j,m} = \begin{pmatrix} \frac{\ell + \frac{1}{2} + m}{2l+1} \frac{1}{2} Y_{\ell,m-\frac{1}{2}} \\ \frac{\ell + \frac{1}{2} - m}{2l+1} \frac{1}{2} Y_{\ell,m+\frac{1}{2}} \end{pmatrix},
\]

for \( j = \ell + \frac{1}{2}, k = j + \frac{1}{2} > 0 \) and

\[
\phi^{(-)}_{j,m} = \begin{pmatrix} \frac{\ell + \frac{1}{2} - m}{2l+1} \frac{1}{2} Y_{\ell,m-\frac{1}{2}} \\ -\frac{\ell + \frac{1}{2} + m}{2l+1} \frac{1}{2} Y_{\ell,m+\frac{1}{2}} \end{pmatrix},
\]

for \( j = \ell - \frac{1}{2}, k = -(j + \frac{1}{2}) \).

In the above basis one verify that

\[
J^2 \phi^{(\pm)}_{j,m} = j(j + 1) \phi^{(\pm)}_{j,m},
\]

\[
\vec{\sigma} \cdot \vec{n} \phi^{(\pm)}_{j,m} = \phi^{(\pm)}_{j,m},
\]

where \( \vec{n} = \frac{\vec{r}}{r} \), and
\[
(1 + \vec{\sigma} \cdot \vec{L})\phi^{(\pm)}_{jm} = -k\phi^{(\pm)}_{jm},
\]
where \(k = \pm \left( j + \frac{1}{2} \right) \) for \( j = \ell \mp \frac{1}{2} \).

Next, we put
\[
\Psi = \begin{pmatrix}
\frac{iG_{\ell j}}{r} \phi^\ell_{jm} \\
F_{\ell j} \vec{\sigma} \cdot \vec{n} \phi^\ell_{jm}
\end{pmatrix},
\]
where
\[
\phi^\ell_{jm} = \phi^{(\pm)}_{jm},
\]
for \( j = \ell \pm \frac{1}{2} \). The operator \( \vec{\sigma} \cdot \vec{n} \) when expressed in this representation, as was done in (10), has the property of reversing the sign of \( \kappa \). The result which can be easily established by means of this operator relation
\[
[1 + \vec{\sigma} \cdot \vec{L}, \vec{\sigma} \cdot \vec{n}]_{+} = 0.
\]

Next, using this relation we have
\[
K\Psi = -k\Psi.
\]

Further, using the relations
\[
\vec{\sigma} \cdot \vec{p} \frac{f(r)}{r} \phi^\ell_{jm} = -\frac{i}{r} \left( \frac{df}{dr} + \frac{kf}{r} \right) \vec{\sigma} \cdot \vec{n} \phi^\ell_{jm},
\]
and
\[
\vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot \vec{n} f(r) \frac{\phi^\ell_{jm}}{r} = -\frac{i}{r} \left( \frac{df}{dr} - \frac{kf}{r} \right) \phi^\ell_{jm},
\]
we get the radial equations
\[
\frac{dG_{\ell j}}{dr} + \frac{k}{r} G_{\ell j} - (E + M + V_S(r) + V(r)) F_{\ell j} = 0,
\]
\[
\frac{dF_{\ell j}}{dr} = \frac{k}{r} F_{\ell j} + (E - M - V_S(r) + V(r)) G_{\ell j} = 0,
\]
where \( k \) is the eigenvalue of the Dirac operator \( K \).
Defining

\[ \Phi = \begin{pmatrix} G_{\ell j} \\ F_{\ell j} \end{pmatrix}, \]

\[ \Lambda = \pm \kappa \rho_3 - r V_S(r) \rho_1 - i r V(r) \rho_2, \]

\[ \vec{k} \cdot \vec{\rho} = M \rho_1 + i E \rho_2, \]

the Dirac radial equation for the hydrogen atom may be written in the form

\[ \left[ \frac{d}{dr} + \frac{\Lambda}{r} - \vec{k} \cdot \vec{\rho} \right] \Phi = 0. \]

Next we show that as the Dirac equation becomes a Schrödinger-like wave equation, with the vector and scalar potentials.

III. ENERGY SPECTRUM AND EIGENFUNCTIONS VIA SUSY

Now, let \( S \) be the operator which diagonalises the matrix that appears in the interaction term. The matrix \( \Lambda \) may be diagonalized by using the result

\[ S^{-1} \Lambda S = \lambda \rho_3, \]

where

\[ \lambda^2 = \kappa^2 - A_1^2 + A_2^2 \]

and

\[ S = \begin{pmatrix} c & d \frac{A_1 - A_2}{\lambda + |\kappa|} \\ c \frac{A_1 + A_2}{\lambda + |\kappa|} & d \end{pmatrix}. \]

Defining

\[ \Phi = S \hat{\Phi}, \]

we get

\[ \left[ \frac{d}{dr} + \frac{\lambda}{r} \rho_3 - \vec{k} \cdot \vec{\rho} \right] \hat{\Phi} = 0. \]
where

\[ 2 \frac{d}{c} \lambda (| \kappa | + \lambda) \hat{k}_- = (| \kappa | + \lambda)^2 (M + E) - (A_1 - A_2)^2 (M - E) \]

\[ 2 \frac{d}{c} \lambda (| \kappa | + \lambda) \hat{k}_+ = (| \kappa | + \lambda)^2 (M - E) - (A_1 + A_2)^2 (M + E) \]

\[ \hat{k}_3 = \frac{EA_1 + MA_2}{\lambda}. \quad (26) \]

Thus, we get

\[ - \hat{k}_- \hat{k}_+ = \frac{a^2}{\lambda^2} \]

\[ \hat{k}^2 M^2 = k^2 = M^2 - E^2 \quad (27) \]

where

\[ M^2 a^2 = (EA_1 + MA_2)^2 - k^2 \lambda^2, \quad (28) \]

which provides a relation between the SUSY and an additional constant of motion in the Dirac-Coulomb problem will be reported separately.

It may be noted that Eq. (25) can be rewritten as

\[ A^+ \hat{G} = \hat{k}_- \hat{F}, \quad A^- \hat{F} = -\hat{k}_+ \hat{G} \quad (29) \]

where

\[ A^\pm = \pm \frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda}. \quad (30) \]

These relations are similar to the relations between the two components of the eigenfunctions of a supersymmetric Hamiltonian [17]

\[ \mathcal{H} = [Q, Q^\dagger]_+ = QQ^\dagger + Q^\dagger Q \]

\[ \mathcal{H} = \begin{pmatrix} A^+ A^- & 0 \\ 0 & A^- A^+ \end{pmatrix} = \begin{pmatrix} \mathcal{H}_- & 0 \\ 0 & \mathcal{H}_+ \end{pmatrix} \]

\[ Q^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \quad [Q, \mathcal{H}]_- = 0 = [Q^\dagger, \mathcal{H}]_-, \quad (31) \]
where $Q$ and $Q^\dagger$ are the supercharges satisfying the anticommutation relations of the SUSY algebra. The supersymmetric partner Hamiltonians $\mathcal{H}_\pm$ satisfy the following eigenvalue equations: $\mathcal{H}_\pm \psi_{(n)} = E_{(n)} \psi_{(n)}$.

Next, Eq. (29) may be written as

$$A^- A^+ \hat{G} = \frac{a^2}{\lambda^2} M^2 \hat{G}, \quad A^+ A^- \hat{F} = \frac{a^2}{\lambda^2} M^2 \hat{F}$$

which shows that every eigenvalue of $A^+ A^-$ is also an eigenvalue of $A^- A^+$ except when $A^- \hat{F} = 0$. This condition gives the lower component for the "ground state" wave function

$$\hat{F}_0 = r^\lambda e^{-\frac{(E_0 A_1 + M A_2)^r}{\lambda}}$$

and

$$\frac{E_0}{M} = \pm \left(1 - \frac{A_1^2 + A_2^2}{\lambda^2 + A_1^2} + \frac{A_1^2 A_2^2}{(\lambda^2 + A_1^2)^2}\right)^{\frac{1}{2}} - \frac{A_1 A_2}{\lambda^2 + A_1^2},$$

where $E_0$ is the ground state energy eigenvalue, for $a^2 = 0$.

IV. EXCITED STATES VIA SHAPE INVARIANCE CONDITION

The shape-invariant SUSY partner potentials are similar in shape and differ only in the parameters that appear in them. More specifically, if $V_-(x; a_1)$ is any potential, adjusted to have zero ground state energy $E^{(0)}_-=0$, its SUSY partner $V_+(x; a_1)$ must satisfy the requirement

$$V_+(x; a_1) = V_-(x; a_2) + R(a_2), \quad a_2 = f(a_1),$$

where $a_1$ is a set of parameters, $a_2$ a function of the parameters $a_1$ and $R(a_2)$ is a remainder independent of $x$. Then, starting with $V_1 = V_-(x; a_2)$ and $V_2 = V_+(x; a_1) = V_1(x; a_2) + R(a_2)$ in (35), one constructs a hierarchy of Hamiltonians

$$H_n = -\frac{1}{2} \frac{d^2}{dx^2} + V_-(x; a_n) + \sum_{s=2}^{n} R(a_s),$$

where $a_s = f^s(a_1)$, i.e., the function $f$ applied $s$ times. In view of Eqs. (35) and (36), we have

8
\[ H_{n+1} = -\frac{1}{2} \frac{d^2}{dx^2} + V_-(x; a_{n+1}) + \sum_{s=2}^{n+1} R(a_s) \]  
\[ = -\frac{1}{2} \frac{d^2}{dx^2} + V_+(x; a_{n}) + \sum_{s=2}^{n} R(a_s). \]  

Comparing (36), (37) and (38), we immediately note that \( H_n \) and \( H_{n+1} \) are SUSY partner Hamiltonians with identical energy spectra except for the ground state level

\[ E^{(0)}_n = \sum_{s=2}^{n} R(a_s) \]  

of \( H_n \), which follows from Eq. (36) and the normalization that for any \( V_-(x; a) \), \( E^{(0)}_n = 0 \).

Thus, we get

\[ E_1^n = E_{n+1} = \ldots = E^{(0)}_{n+1} = \sum_{s=2}^{n+1} R(a_s), \quad n = 1, 2, \ldots \]  

and

\[ \psi_1^{(n)} \propto A_1^+(x; a_1)A_2^+(x; a_2) \ldots A_n^+(x; a_n)\psi^{(0)}_{n+1}(x; a_{n+1}). \]

Equations (40) and (41), succinctly express the SUSY algebraic generalization, for various shape-invariant potentials of physical interest [22, 24], of the method of constructing energy eigenfunctions \( \psi^{(n)}_{osc} \) for the usual ID oscillator problem. Indeed, when \( a_1 = a_2 = \ldots = a_n = a_{n+1} \), we obtain  
\[ \psi^{(n)}_{osc} \propto (a^+)^n \psi^{(0)}_1, \quad A_1^+ = \ldots = A_n^+ = a^+, \quad \psi^{(0)}_1 = \psi^{(0)}_{n+1} = \psi^{(0)}_1 \propto e^{-\frac{\omega^2} {2}}, \]

where \( \omega \) is the angular frequency.

The shape invariance has an underlying algebraic structure and may be associated with Lie algebra [21]. Now, we present our own application of the shape invariant method outlined above. Consider next the supersymmetric partner Hamiltonians given by

\[ \mathcal{H}_- = A^+ A^- = -\frac{d^2}{dr^2} + \lambda (\lambda - 1) \frac{1}{r^2} - \frac{2(EA_1 + MA_2)}{r} + \frac{(EA_1 + MA_2)^2}{\lambda^2} \]
\[ \equiv -\frac{d^2}{dr^2} + V_-(r, \lambda) \]  

and
\[ \mathcal{H}_+ = A^-A^+ = -\frac{d^2}{dr^2} + \frac{\lambda (\lambda + 1)}{r^2} - \frac{2(EA_1 + MA_2)}{r} + \frac{(EA_1 + MA_2)^2}{\lambda^2} \]
\[ \equiv -\frac{d^2}{dr^2} + V_+(r, \lambda). \quad (43) \]

These SUSY partner potentials are shape invariant, since
\[ V_+(r, \lambda) = V_-(r, \lambda + 1) + R(\lambda + 1), \quad (44) \]
where
\[ R(\lambda + 1) = \frac{(EA_1 + MA_2)^2}{\lambda^2} - \frac{(EA_1 + MA_2)^2}{(\lambda + 1)^2}. \quad (45) \]

Hence
\[ A^-A^+(\lambda) = A^+(\lambda + 1)A^-\lambda + 1) + R(\lambda + 1). \quad (46) \]

Following the approach of Fukui and Aizawa [19] and also of Balantekin [20, 21], we define the following ladder operators
\[ B^-\lambda) = T^\dagger\lambda)A^-\lambda), \quad B^+\lambda) = (B^-)^\dagger\lambda), \quad (47) \]
where \( T(\lambda) \) is a translation operator defined by
\[ T(\lambda) = e^{\frac{\partial}{\partial \lambda}}, \quad (48) \]
with \( T^\dagger(\lambda) = e^{-\frac{\partial}{\partial \lambda}} \), and get
\[ [B^-\lambda), B^+\lambda)] = R(\lambda). \quad (49) \]

It is easy to verify that
\[ [A^+(\lambda)A^-\lambda), (B^+)^n(\lambda)] = \sum_{i=1}^{n} R(\lambda + i)(B^+)^n(\lambda). \quad (50) \]
Hence the energy eigenvalues of the Hamiltonian \( \mathcal{H}_- \) are given by
\[ E_{-}^{(n)} = \sum_{i=1}^{n} R(\lambda + i), \]
\[ = \frac{-E^2\gamma^2(E)}{(\lambda + n)^2} + \frac{E^2\gamma^2(E)}{\lambda^2}, \quad \gamma_n(E) = A_1 + \frac{MA_2}{E_n}. \] (51)

Thus, from equations (27), (28), (32) and (51), the energy eigenvalues associated to the lower component \( \hat{F}^n \) are given by

\[ E^{(n)} = \sqrt{\frac{M^2}{1 + \frac{\gamma_n^2}{(\sqrt{k^2 - \gamma_n^2} + n)^2}}}, \quad n = 0, 1, 2, \ldots. \] (52)

Solving the last equation we get

\[ \frac{E_n}{M} = -\frac{A_1A_2}{(\hat{n}^2 + A_1^2)} \pm \left[ \frac{A_2^2A_2^2}{(\hat{n}^2 + A_1^2)^2} + \frac{\hat{n}^2 - A_2^2}{\hat{n}^2 + A_1^2} \right]^{\frac{1}{2}} \] (53)

where \( \hat{n} = n_r + \lambda \) and \( n_r = 0, 1, 2, \ldots \). The corresponding eigenfunctions associated to the lower component are given by

\[ \hat{F}_n(r) = (B^+)^n \hat{F}_0(r) \] (54)

where \( \hat{F}_0 \) is given in Eq. (33).

From the previous equation we can determine the first excited state of the eigenfunction. Using

\[ B^+(\lambda) = T(\lambda) A^+(\lambda) \] (55)

and according to the equations (30), (47) and (48) found

\[ B^+ (\lambda) = \left( \frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda} \right) e^{\frac{\phi}{r^2}}. \] (56)

Hence, for the first excited state we get the corresponding eigenfunction associated to the lower component

\[ F_1(r) = B^+ \hat{F}_0(r) = \left( \frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda} \right) e^{\frac{\phi}{r^2}} r^\lambda e^{-\left(\frac{E_n A_1 + MA_2}{\lambda}\right)r}, \] (57)

so,
\begin{equation}
F_1(r) = N_1 \left( \frac{1}{r} - \frac{EA_1 + MA_2}{\lambda(\lambda + 1)} \right) r^{\lambda+1} e^{-\frac{(EA_1+MA_2)r}{\lambda+1}},
\end{equation}

where \( N_1 \) is a normalization constant for the first excited state.

V. CONCLUSION

In the non-relativistic supersymmetric quantum mechanics formalism, ladder operators are constructed to obtain the complete set of the bound state energy spectrum and eigenfunctions for a relativistic potential given by a Coulomb-like and a Lorentz scalar potential in the Dirac equation; one adopts polar coordinates. These new generalized ladder operators obtained via supersymmetry shape-invariant Hamiltonians in quantum mechanics can be reduced to the ones of the Coulomb potential, which provides us the exact relativistic energy. These Hamiltonians, \( \mathcal{H}_- \) and \( \mathcal{H}_+ \), are SUSY-partner Hamiltonians with identical energy spectra, except for the ground state level, satisfying two Schrödinger-like wave equations, with the vector and scalar potentials.

The vastly simplified algebraic treatment within the framework of the primary SUSY algebra, in terms of conserved operator (Runge-Lenz operator) in the full functional space for the nonrelativistic Coulomb problem with spin has been investigated [26]. The primary SUSY with Johnson-Lippmann [29] operator for the Dirac-Coulomb problem is not possible [13].

Our results on the connection between the generalized Johnson-Lippmann operator and SUSY for the bound eigenstates of the generalized relativistic Coulomb problem with position dependent mass will be reported separately.

Acknowledgments

RLR wishes to thank the staff of the CBPF and UAE-CES-UFCG for the facilities and are also grateful to J. A. Helayel Neto for interesting discussions and suggestions. ESR and AFL would like to acknowledge the Post-Graduate Coordination in Physics of the UAF-UFCG by incentives and encouragement. RLR is also grateful for interesting discussions with Arvind Narayan Vaidya (In memory), whose advises and encouragement were fundamental.
References

[1] R. K. Su, and Z. Q. Ma J. Phys. A: Math. Gen. 19 (1986) 1739.
[2] W. Greiner, Relativistic Quantum Mechanics, Springer Verlag, New York (1992).
[3] R. S. Tutik J. Phys. A: Math. Gen. 25 (1992) L413.
[4] S. I. Ikhadir, O. Mustafa, R. Sever Hadronic J. 16 57 (1993).
[5] S. K. Bose, A. Schulze-Halberg and M. Singh Phys. Lett. A287 (2001) 321.
[6] S. Panchanan, B. Roy, R. Roychoudhury J. Phys. A 28 (1995) 6467.
[7] A. N. Vaidya and L. E. Silva Souza J. Phys. A: Math. Gen. 35 (2002) 6489.
[8] A. D. Alhaidari Phys. Lett. A322 (2004) 72.
[9] C. Y. Chen Phys. Lett. A399 (2005) 283.
[10] Xia Zou, Liang-Zhong Yi, Chung-Seng Jia Phys. Lett. A346 54 (2005).
[11] Jian-You Guo, Zong-Qiang Sheng Phys. Lett. A338 (2005) 90; H. Bíla, V. Jakubský, M. Znojil Phys. Lett. A338, (2006) 421; Jian-You Guo, Zong-Qiang Sheng Phys. Lett. A350 (2006) 425.
[12] A. de Sousa Dutra and M. Hott, Phys. Lett. A356 (2006) 215.
[13] A. A. Stahlhofen Hlev. Phys. Acta 70 (1997) 372.
[14] R. P. Martínez-y-Romero, H. N. Núñez-Yépez and A. L. Salas-Brito Phys. Lett. A339 (2005) 259.
[15] A. Leviatan Phys. Rev. Lett. 92, 202501 (2004).
[16] Tamar T. Khachidze and Anzor A. Khelashvili, Supersymmetry in Dirac equation for generalized Coulomb potential, arXiv:hep-th/0701259
[17] E. Witten Nucl. Phys. B185 (1981) 513; See also, for example, R. de Lima Rodrigues The quantum mechanics SUSY algebra: an introduction review, Monograph CBPF-MO-03/01, www.biblioteca.cbpf.br/index_2.html (2001), hep-th/0205017, and references therein.
[18] C. V. Sukumar J. Phys. A: Math. Gen. 18 (1985) L697.
[19] T. Fukui and N. Aizawa Phys. Lett. A180 (1993) 308.
[20] A. B. Balantekin Phys. Rev. A57 (1998) 4188.
[21] A. B. Balantekin, M. A. Cândido Ribeiro and A. N. F. Aleixo, J. Phys. A: Math. Gen. 32 (1999) 2785; A. N. F. Aleixo, A. B. Balantekin and M. A. Cândido Ribeiro, J. Phys. A: Math. Gen. 33 (2000) 3173; A. N. F. Aleixo, A. B. Balantekin and M. A. Cândido Ribeiro, J. Phys.
[22] L. E. Gendenshtein *Zh. Eksp. Fis. Piz. Red.* **38** (1983) 299 [JETP Lett. (1983) 356].

[23] F. Cooper, J. N. Ginocchio and A. Khare, *Phys. Rev.* **D36** (1987) 2458.

[24] R. Dutt, A. Khare and U. P. Sukhatme, *Am. J. Phys.* **56** (1988) 163; J. W. Dabrowska, A. Khare and U. Sukhatme, *J. Phys. A: Math. Gen.* **21**, L195-L200 (1988); A. Lahiri, P. Kumar Roy, B. Bagchi, *J. Phys. A: Math. Gen.* **22** (1988) 100.

[25] F. Cooper, A. Khare, U. Sukhatme, *Phys. Rep.* **251** (1995) 267.

[26] R. D. Tangerman, J. A. Tjon *Phys. Rev.* **A48** (1993) 1089.

[27] E. Drigo Filho and M. A. Cândido Ribeiro, *Phys. Scripta* **64** (2001) 348.

[28] R. de Lima Rodrigues *Phys. Lett.* **A326** (2004) 42.

[29] M. H. Johnson, B. A. Lippmann *Phys. Rev.* **A78**, (1950) 329.