Information Geometry on the Space of Equilibrium States of Black Holes in Higher Derivative Theories

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Abstract

We study the information-geometric properties of two black hole solutions in higher derivative theories of gravity, namely the Deser-Sarioglu-Tekin solution and the Clifton-Barrow black hole. Our investigation is focused on deriving the relevant information metrics and their scalar curvatures on the space of equilibrium states for the corresponding gravitational backgrounds. The analysis is conducted within the framework of Geometrothermodynamics and shows highly non-trivial statistical behavior for both solutions.

KEYWORDS: Information geometry, black holes, phase transitions, higher derivative theories.

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1 Introduction

The black holes are one of the most intriguing stellar objects in the known universe, which are under extensive investigation from both experimental and theoretical point of view. The recent discovery of gravitational waves [1–5] and the forthcoming image of the apparent shape of the Sgr A* black hole [6–8], can only enrich our understanding of these mysterious objects.

In the recent years the puzzling existence of dark matter and dark energy, which cannot be explained neither by Einstein’s General theory of relativity (GR) nor the Standard model of elementary particles, suggest that alternative models have to be kept in mind. From gravitational perspective one can consider modified theories of gravity and in particular higher derivative theories (HDTs), which include contributions from polynomial or non-polynomial functions of the scalar curvature. Most prominent of whom are the so called $f(R)$ theories [9–11]. The $f(R)$ gravity is a whole family of models with a number of predictions, which differ from those of GR. Therefore, there is a great deal of interest in understanding the possible phases and stability of such higher derivative theories and thereof their admissible black hole solutions [12–17].

However, a consistent description of black holes necessarily invokes the full theory of quantum gravity. Unfortunately, at present day, our understanding of such theory is incomplete at best. The latter prompts one to resort to alternative approaches, which promise to uncover many important aspects of quantum gravity and black holes themselves. One of these approaches is called information geometry [18–21].

The framework of information geometry is the essential tool for understanding how classical and quantum information can be encoded onto the degrees of freedom of any physical system. Since geometry studies mutual relations between elements, such as distance and curvature, it provides us with a set of powerful analytic tools to study previously inaccessible features of the systems under consideration. It has emerged from studies of invariant geometrical structure involved in statistical inference, where one defines a Riemannian metric, known as Fisher information metric [18], together with dually coupled affine connections on the manifold of probability distributions.

Information geometry already finds important applications across many branches of modern physics, showing intriguing results. Some of them, relevant to our study, include condensed matter systems [22–35], black holes [36–57] and string theory [39,49,58,59]. Further applications can also be found in [19,21].

When dealing with systems such as black holes, which seems to possess enormous amount of entropy [60–63], one can consider their space of equilibrium states, equipped with a Riemannian metric, known as the Ruppeiner information metric [64]. The latter is a thermodynamic
limit of the above-mentioned Fisher information metric. Although G. Ruppeiner developed
his geometric approach within fluctuation theory, when utilized for black holes, it seems to
capture many of their phase structure features resulting from the dynamics of the underlying
microstates. In this case one implements the entropy as a thermodynamic potential to define
a Hessian metric structure on the state-space statistical manifold with respect to the existing
extensive parameters of the system.

Nevertheless, one can utilize the internal energy (the ADM mass in the case of black holes)
as an alternative thermodynamic potential, which lies at the heart of the Weinhold’s metric
approach [22] to equilibrium thermodynamic states. The resulting Weinhold information met-
ric is conformally related to Ruppeiner’s metric with the temperature $T$ as a conformal factor.
Unfortunately, the resulting statistical geometries coming from both approaches may not often
agree with each other. The reasons for this behavior are still unclear, although several attempts
to resolve the issue have already been suggested [50, 52, 65–67].

In the current paper we are going to study the thermodynamic information spaces of two
static, spherically symmetric black hole solutions in higher derivative theories within the frame-
work of geometric thermodynamics. The first higher derivative solution of interest is derived
by Deser, Sarioglu and Tekin in [68]. The authors of [68] consider a contribution from a non-
polynomial term of the Weyl tensor to Einstein–Hilbert Lagrangian. The second black hole
solution is found by Barrow and Clifton in a modified theory of the type $f(R) = R^{1-\delta}$, with $\delta$
a small real parameter [69]. Due to the fact that there is no known way of defining an ADM
mass for these higher derivative black hole solutions the Weinhold’s method is not applicable
for our considerations. However, we will consider the approaches to the information metric of
Ruppeiner [64], Quevedo [66] and HPEM [70].

This paper is organized as follows. In Section 2 we shortly discuss the basic concepts of
geometrothermodynamics (GTD). In Section 3 we study the GTD information content of the
Deser-Sarioglu-Tekin solution and show that it has a non-trivial phase structure. On the focus
of our investigation in Section 4 is the GTD information metrics and their algebraic invariants
for the Clifton-Barrow black hole solution. Finally, in Section 5, we make a short summary of
our results.

2 Information geometry on the space of equilibrium ther-

modynamic states

Due to the pioneering work of Bekenstein [60] and Hawking [63] we know that any black hole
represents a thermal system with well-defined temperature and entropy. Taking into account
that black holes may posses also charge $Q$ and angular momentum $J$ one can formulate the
analogue to the first law of thermodynamics for black holes such as

$$dM = T dS + \Phi_Q dQ + \Omega dJ.$$  \hspace{1cm} (2.1)

Here $\Phi_Q$ is the charge potential and $\Omega$ is the angular velocity of the event horizon. Equation
(2.1) expresses the conserved ADM mass $M$ as a function of the entropy, $S$, and the other
extensive parameters, describing the macrostates of the black hole. One can equivalently solve
Eq. (2.1) with respect to the entropy.

In the approach of geometric thermodynamic all extensive parameters of the given black
hole background can be used in the construction of its equilibrium thermodynamic parameter
space. The latter can be equipped with a Riemannian metric in several ways. In particular,
one can introduce Hessian metrics, whose components are calculated as the Hessian of a given
thermodynamic potential. For example, depending on which potential we chose to apply for
the description of the thermodynamic states in equilibrium, one can write the two most popular
thermodynamic metrics, namely the Weinhold information metric [22],
\[ ds^2_W = \partial_a \partial_b M \, dX^a \, dX^b , \]  
(2.2) 
defined as the Hessian of the ADM mass \( M \), and the Ruppeiner information metric [24],
\[ ds^2_R = -\partial_a \partial_b S \, dY^a \, dY^b , \]  
(2.3) 
defined as the Hessian of the entropy \( S \). Here \( X_a[Y_b], a, b = 1, \ldots, n \), collectively denote all of the system’s extensive variables except for \( M[S] \). One can show that both metrics are conformally related to each other via the temperature:
\[ ds^2_W = T \, ds^2_R . \]  
(2.4) 
The importance of using Hessian metrics on the equilibrium manifold is best understood when one considers small fluctuations of the thermodynamic potential. The latter is extremal at each equilibrium point, but the second moment of the fluctuation turns out to be directly related to the components of the corresponding Hessian metric. From statistical point of view one can define a Hessian metrics on a statistical manifold spaned by any type or number of extensive (or intensive) parameters, without necessarily respecting the first law of thermodynamics. This is due to the fact that the Hessian metrics are not Legendre invariant, thus they do not necessarily preserve the geometric properties of the system when a different thermodynamic potential is chosen. However, for Legendre invariant metrics, the first law of thermodynamics follows naturally.

In order to make things Legendre invariant one can start from the \((2n + 1)\)-dimensional thermodynamic phase space \( \mathcal{F} \), spanned by the thermodynamic potential \( \Phi \), the set of extensive variables \( E^a \), and the set of intensive variables \( I^a \), \( a = 1, \ldots, n \). Now, consider a symmetric bilinear form \( \mathcal{G} = \mathcal{G}(Z^A) \) defining a non-degenerate metric on \( \mathcal{F} \) with \( Z^A = (\Phi, E^a, I^a) \), and the Gibbs 1-form \( \Theta = d\Phi - \delta_{ab} I^a dE^b \), where \( \delta_{ab} \) is the identity matrix. If the condition \( \Theta \wedge (d\Theta)^n \neq 0 \) is satisfied, then the triple \( (\mathcal{F}, \mathcal{G}, \Theta) \) defines a contact Riemannian manifold. The Gibbs 1-form is invariant with respect to Legendre transformations by construction, while the metric \( \mathcal{G} \) is Legendre invariant only if its functional dependence on \( Z^A \) does not change under a Legendre transformation. Legendre invariance guarantees that the geometric properties of \( \mathcal{G} \) do not depend on the choice of thermodynamic potential.

On the other hand, one is interested on constructing a viable Riemannian metric \( g \) on the \( n \)-dimensional subspace of equilibrium thermodynamic states \( \mathcal{E} \subset \mathcal{F} \). The space \( \mathcal{E} \) is defined by the smooth mapping \( \phi : \mathcal{E} \rightarrow \mathcal{F} \) or \( E^a \rightarrow (\Phi(E^a), E^a, I^a) \), and the condition \( \phi^* (\Theta) = 0 \). The last restriction leads explicitly to the generalization of the first law of thermodynamics (2.1)
\[ d\Phi = \delta_{ab} I^a dE^b , \]  
(2.5) 
and the condition for thermodynamic equilibrium,
\[ \frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b . \]  
(2.6) 
The natural choice for \( g \) is the pull-back of the phase space metric \( \mathcal{G} \) onto \( \mathcal{E} \), \( g = \phi^* (\mathcal{G}) \). Here, the pull-back also imposes the Legendre invariance of \( \mathcal{G} \) onto \( g \), but there are plenty of Legendre invariant metrics on \( \mathcal{F} \) to choose from. In Ref. [67] it was found that the general metric for the equilibrium state-space can be written in the form
\[ g^{I,I} = \beta_{\Phi} \Phi(E^c) \chi^{ab}_{I,I} \frac{\partial^2 \Phi}{\partial E^b \partial E^c} \, dE^a \, dE^c , \]  
(2.7)
where $\chi^b_a = \chi^{af} \delta^b_f$ is a constant diagonal matrix and $\beta_\Phi \in \mathbb{R}$ is the degree of generalized homogeneity, $\Phi(\lambda^{\beta_1} E^1, \ldots, \lambda^{\beta_N} E^N) = \lambda^{\beta_\Phi} \Phi(E^1, \ldots, E^N)$, $\beta_\alpha \in \mathbb{R}$. In this case the Euler’s identity for homogeneous functions can be generalized in the form:

$$\beta_{ab} E^a \frac{\partial \Phi}{\partial E^b} = \beta_\Phi \Phi,$$

where $\beta_{ab} = \text{diag}(\beta_1, \beta_2, \ldots, \beta_N)$. In the case $\beta_{ab} = \delta_{ab}$ one returns to the standard Euler’s identity. If we choose to work with $\beta_{ab} = \delta_{ab}$, for complicated systems this may lead to non-trivial conformal factor, which is no longer proportional to the potential $\Phi$. On the other hand, if we set $\chi_{ab} = \delta_{ab}$, the resulting metric $g^I$ can be used to investigate systems with at least one first-order phase transition. Alternatively, the choice $\chi_{ab} = \eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$, leads to a metric $g^{II}$, which applies to systems with second-order phase transitions.

Once the information metric for a given statistical system is constructed one can proceed in calculating its algebraic invariants, i.e. the information curvatures such as the Ricci scalar, the Kretschmann invariant etc. The latter quantities are relevant for extracting information about the phase structure of the system. As suggested by G. Ruppeiner in Ref. [64], the Ricci information curvature $R_I$ is related to the correlation volume of the system. This association follows from the idea that it will be less probable to fluctuate from one equilibrium thermodynamic state to other, if the distance between the points on the statistical manifold, which correspond to these states, increases.

Furthermore, the sign of $R_I$ can be linked to the nature of the interparticle interactions in composite thermodynamical systems [71]. Specifically, if $R_I = 0$, the interactions are absent, and we end up with a free theory (uncorrelated bits of information). The latter situation corresponds to flat information geometry. For positive curvature, $R_I > 0$, the interactions are repulsive, therefore we have an elliptic information geometry, while for negative curvature, $R_I < 0$, the interactions are of attractive nature and an information geometry of hyperbolic type is realized.

Finally, the scalar curvature of the parameter manifold can also be used to measures the stability of the physical system under consideration. In particular, the information curvature approaches infinity in the vicinity of critical points, where phase transition occurs [25]. Even more, the information metric invariants tend to diverge not only at the critical points of phase transitions, but on whole regions of points on the statistical space, called spinodal curves. The latter can be used to discern physical from non-physical situations.

Notice further that in the case of Hessian metrics, in order to ensure global thermodynamic stability of a given macro configuration of the black hole one requires that all the principle minors of the metric tensor be strictly positive definite due to the probabilistic interpretation involved [24]. In any other cases (Quevedo, HPEM, etc) the physical interpretation of the metric components is unclear and one can only impose the convexity condition on the thermodynamic potential, $\partial_a \partial_b \Phi \geq 0$, which is the second law of thermodynamics. Nevertheless, imposing positiveness of the black hole’s heat capacity is mandatory in any cases.

In what follows we are going to study the thermodynamic stability and the phase structure of the Deser-Sarioglu-Tekin solution (3.2) and the Clifton-Barrow background (4.2) with respect to Ruppeiner, Quevedo and HPEM information metrics.
3 State-space information geometry of the Deser-Sarioglu-Tekin solution

3.1 The DST black hole

One starts with the following action (in units $\kappa = 1$)

$$A = \frac{1}{2} \int_M d^4x \sqrt{-g} \left( R + \beta_n |\text{Tr}(C^n)|^{1/n}\right),$$

(3.1)

where $C$ is the Weyl tensor and $\beta_n$ are some real constant coefficients. The spherically symmetric Deser-Sarioglu-Tekin solution [68, 72],

$$ds^2 = -k^2 r^{2(1-p(\sigma))} \left( p(\sigma) - \frac{c}{r^{1/p(\sigma)}} \right) dt^2 + \frac{dr^2}{p(\sigma) - \frac{c}{r^{1/p(\sigma)}}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(3.2)

follows from (3.1) by setting $n = 2$ and $\sigma = \beta_2/\sqrt{3}$. Here, the integration constant $k$ can be eliminated by a proper rescaling of the time coordinate $t$. For convenience we have defined the function $p(\sigma)$ as

$$p(\sigma) = \frac{1 - \sigma}{1 - 4\sigma}.$$  

(3.3)

We also note that the metric (3.2) is not asymptotically flat, unless we are in the case $\sigma = 0$, for which the charge $c > 0$ can be interpreted as the ADM mass of GR ($c = 2M$). Otherwise its physical meaning is unclear, most likely related to effects from dark matter and dark energy.

One can impose several restrictions on the parameters $\sigma$ and $c$ of the solution (3.2) by calculating the independent curvature invariants, i.e. the Ricci scalar,

$$R = -\frac{6\sigma r^{-\frac{3}{1+\sigma^2}}}{(\sigma - 1)(4\sigma - 1)} \left( 3\sigma r^{\frac{3}{1+\sigma^2}} + 4c\sigma - c \right),$$

(3.4)

and the Kretschmann invariant,

$$K = \frac{12r^6(1+\frac{1}{4\sigma^2})}{(\sigma - 1)^2(4\sigma - 1)^2} \left( 3\sigma^2(\sigma(7\sigma - 2) + 4) r^2(\frac{3}{4\sigma^2} + 4) + c^2(1 - 4\sigma)^2(5\sigma^2 + 1) \right.$$

$$-6c\sigma(4\sigma - 1)(\sigma^2 + \sigma + 1) r^{\frac{3}{4\sigma^2} + 4}).$$

(3.5)

Both invariants are singular at $\sigma = 1$ and $\sigma = 1/4$. One can check that $R$ and $K$ also diverge at $c \to \pm\infty$. These points must be excluded from the parameter space of the solution. Interestingly enough the curvatures are finite at $\sigma \to \pm\infty$ (for $r \neq 0$), while at $\sigma \to 0$ one recovers GR with ADM mass $2M = c$,

$$\lim_{\sigma \to 0} R = 0, \quad \lim_{\sigma \to \pm\infty} R = -\frac{3}{2r^6} (4c + 3r^4), \quad \lim_{\sigma \to 1, \frac{1}{4}} R, K = \infty, \quad \lim_{c \to \pm\infty} R, K = \infty,$$

(3.6)

$$\lim_{\sigma \to 0} K = \frac{12c^2}{r^6}, \quad \lim_{\sigma \to \pm\infty} K = \frac{3}{4r^{12}} \left( 80c^2 - 24c^4 r^4 + 21r^8 \right).$$

(3.7)

One can impose further restrictions on the parameters of the solution by preserving the signature of the metric. Looking at Eq. (3.2) this simply means that we have to exclude the interval $1/4 < \sigma < 1$. In the following subsections we show that the information curvature is not real on the interval $1/4 < \sigma < 1$.

As shown in [72] the event horizon of the DST black hole has a radius $r_H = (c/p)^p$. One
can then proceed with Wald’s proposal [73] to calculate the black hole entropy. The explicit formula for the entropy of the DST black hole solution is given by [72]:

$$S = \pi \left[ c \left( \frac{3}{\sigma - 1} + 4 \right) \right]^{\frac{2(\sigma - 1)}{4\sigma - 1}} (1 + \varepsilon \sigma), \quad \begin{cases} \varepsilon = +1, & \sigma < \frac{1}{4}, \\ \varepsilon = -1, & \sigma > 1. \end{cases}$$ (3.8)

The Hawking temperature yields

$$T = \frac{1}{4\pi} \left( \frac{c}{p} \right)^{2 - 3p}. \quad (3.9)$$

The singular points of the temperature are $\sigma = 1/4$ and $\sigma = 1$. The temperature has one local extremal point at $(c = 2/3, \sigma = -1/5)$, which is a saddle point. At $\sigma \to -1/5$ one has $T \to 1/(4\pi)$ and the DST temperature doesn’t depend on the charge $c$. Moreover, the heat capacity (3.19) of the DST solution diverges at $\sigma \to -1/5$, which associates this point with a phase transition of the DST black hole.

The equilibrium thermodynamic state-space of the DST solution (3.2) is two-dimensional manifold, spanned by the parameters $(c, \sigma)$. Therefore, regardless of the chosen potential, the metric on this space can be written in the form

$$ds^2 = g_{cc} dc^2 + 2 g_{c\sigma} dc d\delta + g_{\sigma\sigma} d\sigma^2. \quad (3.10)$$

In 2 dimensions all the relevant information about the phase structure is encoded only in the information metric and its Ricci invariant. The latter is proportional to the (only one independent) component of the Riemann curvature tensor,

$$R_I = \frac{2 R_{I,1212}}{g_{cc} g_{\sigma\sigma} - g_{c\sigma}^2}. \quad (3.11)$$

Once the scalar information curvature is obtained, we can identify its singularities as phase transition points, which should be compared to the resulting divergence points in the heat capacity. If a complete match is found one can rely on the considered information metric as suitable for describing the space of equilibrium states for the given black hole solution.

### 3.2 Information curvatures and phase structure of the DST solution

In the following subsections we will apply the Ruppeiner’s, Quevedo’s and HPEM’s geometric approaches to the state-space of the DST solution (3.2). The space of equilibrium states for (3.2) is spanned by the parameters $(c, \sigma)$. Our goal will be to find the regions on the state-space corresponding to global thermodynamic stability of the DST solution and to identify possible critical points of phase transitions by comparing the singularities of the information curvatures to those of the DST heat capacity.

#### 3.2.1 The Ruppeiner information metric

We begin by calculating the Ruppeiner information metric,

$$g_{ab} = -\partial_a \partial_b S(c, \sigma), \quad a, b = 1, 2, \quad (3.12)$$

with the entropy defined in Eq. (3.8). The explicit form of the components is written by

$$g_{cc}^{(e)} = \frac{2 \pi (\sigma - 1) (2 \sigma + 1) (\varepsilon \sigma + 1)}{c^2 (1 - 4 \sigma)^2} A^{\frac{2(\sigma - 1)}{4\sigma - 1}}, \quad (3.13)$$
\[ g_{cc}^{(\varepsilon)} = g_{c} = \frac{2 \pi (\varepsilon (1 - 2 \sigma (\sigma (8 \sigma - 9) + 6)) - 6 (\sigma - 1)(\varepsilon + 1) \ln A - 6 \sigma - 3)}{c (4 \sigma - 1)^3} A^{2(\varepsilon - 1)}, \]

\[ g_{\sigma\sigma}^{(\varepsilon)} = -\frac{6 \pi A^{2(\sigma - 1)}}{(1 - 4 \sigma)^4 (\sigma - 1)} \times \left[ 2 (\varepsilon + 16) \sigma^2 - (13 \varepsilon + 46) \sigma + 2 \varepsilon + 5 
+ 2 (\sigma - 1) (3 (\varepsilon \sigma + 1) \ln A - 2 (5 \varepsilon + 8) \sigma + \varepsilon - 2) \ln A \right], \tag{3.14} \]

where

\[ A = c \left( \frac{3}{\sigma - 1} + 4 \right). \tag{3.15} \]

Imposing global thermodynamic stability (Sylvester’s criterion), \( g_{cc} > 0 \land g_{\sigma\sigma} > 0 \land \det \hat{g} > 0 \), with respect to the Ruppeiner’s information metric together with the positivity of the heat capacity, \( C > 0 \), we can constrain the possible values of the parameters \((c, \sigma)\) in the region \(-0.84 < \sigma < -0.50\) and \(0 < c < 0.033\) (Fig. 1). Outside this region the information metric is not positive definite and the DST solution is thermodynamically unstable globally. The restricted values of the parameters effectively discard the case \( \varepsilon = -1 \), i.e. \( \sigma > 1 \). The boundary curve of the region turns out to be defined by the equation \( \det \hat{g} = 0 \), where the Ruppeiner metric becomes degenerate. The latter property is also depicted by one of the spinodal curves of the Ricci information scalar as shown on Fig. 3.

**Figure 1:** The "island of global thermodynamic stability" (the shaded region) for the DST solution in the Ruppeiner’s case. One is confined in the region \(-0.84 < \sigma < -0.50\) and \(0 < c < 0.033\). For values of the parameters outside this region the information metric is not positive definite and the DST solution becomes unstable from thermodynamic standpoint. The shaded region represents the physically admissible values of \((c, \sigma)\) for which the probability of fluctuating between different macrostate is well defined. The boundary of the region is defined by the equation \( \det \hat{g} = 0 \), which is also one of the spinodal curves of the DST Ricci information scalar as depicted on Fig. 3.

In order to identify possible critical points of phase transitions we calculate the Ruppeiner information curvature:

\[ R_I^{(\varepsilon)} = \frac{3 (4 \sigma - 1)}{2 \pi (\sigma - 1)^2} \frac{D}{B^2} A^{-2(\sigma - 1)}, \tag{3.16} \]

where the coefficient \( B \) reads

\[ B = 6 (\varepsilon \sigma + 1) (3 (\varepsilon \sigma + 1) \ln A + 8 \varepsilon \sigma^2 - 4 (3 \varepsilon + 2) \sigma + \varepsilon - 4) \ln A \]
and $D$ yields

$$D = 12 (\varepsilon + 1) \times \left[ 32 \varepsilon (\varepsilon + 4) \sigma^5 - 24 \varepsilon (5 \varepsilon + 8) \sigma^4 + 2 (\varepsilon (25 \varepsilon - 36) + 32) \sigma^3 \\
+ 8 (\varepsilon (2 \varepsilon + 11) - 15) \sigma^2 - 6 \varepsilon (\varepsilon + 2) \sigma + \varepsilon (\varepsilon + 6) + 12 \sigma + 17 \right] \\
- \left[ \varepsilon^3 (2 (2 \sigma - 1) (2 \sigma (8 \sigma (13 \sigma - 7) - 91) + 45) \sigma^2 + 1) \\
+ 3 \varepsilon (4 \sigma (8 \sigma (2 \sigma (8 \sigma - 5) - 15) + 17) - 7) + 9) + 12 (8 \sigma (\sigma (2 \sigma - 9) + 3) + 5) \\
+ 2 \varepsilon^2 (8 + \sigma (303 + 4 \sigma (113 + 8 \sigma (\sigma (15 + 16 \sigma) - 60))) - 78)) \right] \ln A \\
+ 3 (1 + \varepsilon \sigma)^3 \ln^2 A \left[ (17 + 8 \sigma (21 + 16 (\sigma - 3) \sigma) + \varepsilon (2 \sigma (49 + 2 \sigma (4 \sigma (4 \sigma - 3) - 33)) - 9)) \\
+ 54 (\sigma - 1) (1 + \varepsilon \sigma)^3 \ln^3 A. \right]$$

(3.18)

The first two noticeable points, at which the information curvature $R_I$ diverges, are $\sigma \to 1$ and $\sigma \to 1/4$. The other divergences occur at $c \to 0$, $c \to \pm \infty$. In the interval $1/4 < \sigma < 1$ the information scalar $R_I^{(c)}$ is not real and this interval has to be discarded. For large values of the charge $c$ there is a spinodal curve for the DST solution, which is shown in Fig. 2. All singular points, at which the scalar curvature diverges, lie outside the “island of global thermodynamic stability” form Fig. 1. For small values of the charge $c$, one finds a second spinodal curve, shown in Fig. 6, which encloses the island of global thermodynamic stability and is defined by the set of points where $\det \hat{g} = 0$.

(a) The DST spinodal curve in 2d  (b) The DST spinodal curve in 3d

**Figure 2:** The DST spinodal curve for large values of the charge $c$. a) The curve begins around $c = 700$ and spreads out to infinity in two branches. It lies outside the region of thermodynamical stability (Figure 1) of the DST solution. b) The information curvature $R_I^{(c)}$ for large values of the parameter $c$. On the spinodal curve the curvature diverges. Everywhere else the geometry is flat or almost flat, except at $c \to 0$, where $R_I^{(c)}$ also diverges.
Figure 3: The spinodal curve of the DST solution in the Ruppeiner’s case near the origin. It is defined by the set of points where $\text{det} \hat{g} = 0$, where $\hat{g}$ is the Ruppeiner metric. The curve also encloses the island of global thermodynamic stability for the DST solution, hence separating the physical from unphysical situations.

In order to see if the singularities of the Ruppeiner scalar curvature correspond to true phase transitions we calculate the heat capacity, $C = T \partial S / \partial T$, of the DST solution:

$$C = \frac{2 \pi (\sigma - 1) \left(\frac{\sigma - 1}{4 \sigma - 1}\right)^{\frac{2(\sigma - 1)}{4 \sigma - 1}} \varepsilon^{\frac{2(\sigma - 1)}{4 \sigma - 1}} (\varepsilon + 1)}{5 \sigma + 1}.$$  \hspace{1cm} (3.19)

It is singular at $\sigma = 1/4$ and $\sigma = -1/5$ and tends to zero at $\sigma = 1$, while being not real in the interval $1/4 < \sigma < 1$. Although $\sigma = 1/4$ and $\sigma = 1$ seem to correspond to true critical points, the Ruppeiner information curvature is not singular at $\sigma = -1/5$. The latter mismatch shows that the Ruppeiner approach is not an appropriate choice for the description of the equilibrium state-space of the DST solution.

However, the DST solution is thermodynamically stable only for positive values of the heat capacity. The region of positive heat capacity is shown in Fig. 4 and it contains the region of global thermodynamic stability defined by the Sylvester’s criterion for positive definite information metric (Fig. 1).
Figure 4: The region of positive heat capacity for the DST solution is locked between $-1 < \sigma < -1/5$ and arbitrary $c > 0$.

One can see that the GR limit, $\sigma \to 0$, is excluded. At that limit the heat capacity (3.19) reduces to the Schwarzschild case, $C = -2 \pi c^2$, with $c = 2M$, which is thermodynamically unstable.

3.2.2 The Quevedo and HPEM information metrics

The Quevedo information metric on the equilibrium state-space of the DST solution is given by

$$ds_Q^2 = \beta_S S \left( \partial_\sigma^2 S d\sigma^2 - \partial_c^2 S dc^2 \right) = \tilde{g}_{cc} dc^2 + \tilde{g}_{\sigma\sigma} d\sigma^2.$$  

(3.20)

One can find the degree of generalized homogeneity, $\beta_S$, directly from the Euler’s theorem for homogeneous functions (2.8):

$$c \frac{\partial S}{\partial c} + \sigma \frac{\partial S}{\partial \sigma} = \beta_S S.$$  

(3.21)

The latter equation for $\beta_S$ leads to the following components of the information metric:

$$\tilde{g}_{cc} = - \left( c \partial_c S + \sigma \partial_\sigma S \right) \partial_c^2 S, \quad \tilde{g}_{\sigma\sigma} = \left( c \partial_c S + \sigma \partial_\sigma S \right) \partial_\sigma^2 S.$$  

(3.22)

Contrary to Ruppeiner’s metric in Quevedo’s case we do not have clear physical interpretation of the components of the metric, therefore the Sylvester’s criterion does not apply. However one can impose only the convexity condition $\partial_\sigma \partial_\sigma S \geq 0$ (the second law of thermodynamics).

On Fig. 5 is depicted the intersection in the parameter space between the convexity condition and the region of positive heat capacity for the DST solution.
Figure 5: The intersection in the parameter space between the convexity condition $\partial_b \partial_b S \geq 0$ and the region of positive heat capacity for the DST solution. One is confined in the region $-1/2 < \sigma < -1/5$ and $0 < c < \infty$.

Next we find the Quevedo information curvature,

$$\tilde{R}_I = \frac{g_{cc} \left( \partial_\sigma g_{cc} \partial_\sigma g_{\sigma \sigma} + (\partial_\sigma g_{\sigma \sigma})^2 \right) + g_{\sigma \sigma} \left( \partial_\sigma g_{cc} \partial_\sigma g_{\sigma \sigma} - 2 g_{cc} \left( \partial_\sigma^2 g_{cc} + \partial_\sigma^2 g_{\sigma \sigma} \right) \right) + (\partial_\sigma g_{cc})^2}{2 g_{cc}^2 g_{\sigma \sigma}}.$$  \hspace{1cm} (3.23)

Although Eq. (3.23) is a complicated function of $c$ and $\sigma$, the singularities of the information curvature can be depicted graphically as shown on Fig. 6. One notices 3 distinctive singularities at $\sigma = -1$, $-1/2$, $1/4$ and several spinodal curves. However the Quevedo information curvature is regular at $\sigma = -1/5$. The latter mismatch of the singularities between the Quevedo invariant and the heat capacity suggests that the Quevedo information approach is also not appropriate for the description of the equilibrium state-space of the DST solution.

Figure 6: Density plot of the Quevedo information curvature $\tilde{R}_I$. There are 3 distinctive singularities at $\sigma = -1$, $-1/2$, $1/4$ and several spinodal curves. However the Quevedo information curvature is not singular at $\sigma = -1/5$, where the DST heat capacity diverges.

As an alternative to Quevedo information metric one can use the HPEM metric proposed in [70], the difference here being in the conformal factor. In this case we will use the entropy
instead of the ADM mass, thus
\[
ds^2_{HPEM} = c \frac{\partial_c S}{(\partial^2 S)^3} \left( -\partial^2_c dS \partial_c^2 + \partial^2_c S d\sigma^2 \right). \tag{3.24}
\]

On Fig. 7 is shown the density plot of the HPEM information curvature. Although it is not plagued with spinodal curves, we note that it is still regular at \( \sigma = -1/5 \), thus unable to account for the singularities of the heat capacity (3.19).

![Density plot of the HPEM information curvature](image)

**Figure 7:** Density plot of the HPEM information curvature \( \tilde{\mathcal{R}_I} \), which is singular at \( \sigma = -1, -1/2, 1/4 \), but regular at \( \sigma = -1/5 \).

The mismatch of the singularities between the three different information approaches and the heat capacity (3.19) of the DST black hole suggests that this solution is highly non-trivial from thermodynamic standpoint and further investigation beyond GTD is necessary in order to capture all relevant features of its space of equilibrium states.

4 State-space information geometry of the Clifton-Barrow solution

4.1 The Clifton-Barrow black hole

One considers an \( f(R) \) theory of the following type
\[
A = \int_M d^4 x \sqrt{-g} \frac{R^{1+\delta}}{16 \pi G^{1+\delta}}, \tag{4.1}
\]
where \( \delta \) is a small real parameter. The Clifton-Barrow solution [69],
\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{4.2}
\]
is the static spherically symmetric solution of this higher derivative theory. Here, the metric coefficients \( A(r) \) and \( B(r) \) are given by \( (G = 1) \)
\[
A(r) = r^{\frac{2\delta}{\delta-1}} \left( 1 - \frac{c}{r^{\frac{2\delta}{\delta-1}} - 2(\delta+1)} \right), \tag{4.3}
\]
and
\[
B(r) = \frac{(4\delta^2 - 2\delta + 1)(2\delta(\delta+1) - 1)}{(\delta-1)^2 \left( 1 - \frac{c}{r^{\frac{2\delta}{\delta-1}} - 2(\delta+1)} \right)}, \tag{4.4}
\]
where $c > 0$ is a real constant related to contributions from dark matter and dark energy, while only at $\delta = 0$ one can interpret the parameter $c$ as the ADM mass of GR ($c = 2M$).

To exclude any nonphysical values of the parameters one finds the Ricci scalar of the CB solution,

$$R = \frac{6\, \delta \,(\delta + 1)}{(2\, \delta \,(\delta + 1) - 1)} \frac{1}{r^2},$$

which is singular at $\delta = (-1 \pm \sqrt{3})/2$, so as the second algebraic invariant.

The event horizon of the CB black hole is located at $r_h = c^{1-\delta}/(1-2\delta+4\delta^2)$. Using Wald’s proposal one can calculate the entropy of the CB solution [72]:

$$S = \frac{\pi \,(\delta + 1)}{G} \frac{6\, v(\delta)}{G} \delta,$$

where

$$w(\delta) = \frac{2\,(\delta - 1)^2}{4\, \delta^2 - 2\, \delta + 1}, \quad v(\delta) = \frac{\delta \,(\delta + 1)}{2\, \delta \,(\delta + 1) - 1}.$$ (4.7)

The corresponding temperature is ($G = 1$):

$$T = \frac{2\, \delta \,(\delta + 1) - 1}{4\pi} \, \sqrt{\frac{6\, \delta - 3}{2\, \delta (\delta + 1) - 1 - 2}},$$ (4.8)

and the heat capacity, $C = T \partial S/\partial T$,

$$C = \frac{\pi \, 3^\delta \,(\delta - 1)^2}{\delta} \left(\frac{1}{2\, \delta \,(\delta + 1) - 1} + 1\right)^{\delta+1} \frac{2\,(\delta - 1)^2}{C^2 - 2\, \delta^2 + 1}. $$ (4.9)

As expected in the limit $\delta \to 0$ the entropy, the temperature and the heat capacity of the CB solution reduce to their values in GR. The temperature and the heat capacity have no local extrema. On Fig. 8 is depicted the density plot of the heat capacity, which is not defined for values in the region $(-1 - \sqrt{3})/2 < \delta < -1$ and $0 < \delta < (-1 + \sqrt{3})/2$. Outside these regions $C$ is regular everywhere. The points $\delta = (-1 \pm \sqrt{3})/2$ are excluded due to Eq. (4.5).

**Figure 8:** The density plot of the heat capacity for the CB solution ($G = 1$). The heat capacity is not real for values between $(-1 - \sqrt{3})/2 < \delta < -1$ and $0 < \delta < (-1 + \sqrt{3})/2$. Outside these regions $C$ is regular everywhere. The points $\delta = (-1 \pm \sqrt{3})/2$ are excluded due to Eq. (4.5).
4.2 Information curvatures and phase structure of the CB black hole

In the following subsections we will apply the Ruppeiner’s, Quevedo’s and HPEM’s geometric approaches to the state-space of the CB solution (4.2). The space of equilibrium states for (4.2) is spanned by the parameters \((c, \delta)\). Our goal will be to find the regions on the state-space corresponding to global thermodynamic stability of the CB solution and to identify possible critical points of phase transitions by comparing the singularities of the information curvatures to those of the CB heat capacity.

4.2.1 The Ruppeiner information metric

The Ruppeiner information metric for the CB solution (4.2) is given by

\[
 ds_R^2 = -\frac{\partial^2 S}{\partial E^a \partial E^b} dE^a dE^b = g_{cc} dc^2 + 2 g_{c\delta} dc d\delta + g_{\delta\delta} d\delta^2, \tag{4.10}
\]

where its components read

\[
 g_{cc} = -\pi (\delta + 1) (w - 1) w c^{w(\delta)-2} \left( \frac{6 v}{G} \right)^{\delta}, \tag{4.11}
\]

\[
 g_{c\delta} = g_{\delta c} = -\frac{\pi c^{w-1}}{G} \left( \frac{6 v}{G} \right)^{\delta} \left[ \delta (\delta + 1) w v' + (\delta + 1) w' (w \ln c + 1) + w \left( (\delta + 1) \ln \left( \frac{6 v}{G} \right) + 1 \right) \right], \tag{4.12}
\]

\[
 g_{\delta\delta} = -\frac{\pi c^{w}}{G} \left( \frac{6 v}{G} \right)^{\delta} \left[ \delta (\delta^2 - 1) v'^2 + \delta (\delta + 1) \ln^2 6 + \ln 36 + \frac{1}{v} \left( 2 v' \left( \delta + 1 \right) w' \ln c + 2 \delta + (\delta + 1) \delta \ln \left( \frac{6 v}{G} \right) + 1 \right) + \delta (\delta + 1) v'' \right] + \left( w' \left( (\delta + 1) w' \ln c + (\delta + 1) \ln 36 + 2 (\delta + 1) \ln \left( \frac{v}{G} \right) + 2 \right) + (\delta + 1) w'' \ln c + \ln \left( \frac{v}{G} \right) \left( (\delta + 1) \ln \left( \frac{36 v}{G} \right) + 2 \right) \right]. \tag{4.13}
\]

The information metric and the heat capacity are positive definite simultaneously only in the region \(0.75 < \delta < 1.34\) and \(0 < c < 0.052\) (Fig. 9).
The "island of global thermodynamic stability" (the shaded region) for the CB solution in the Ruppeiner’s case. One is confined in the region $0.75 < \delta < 1.34$ and $0 < c < 0.052$. For values of the parameters outside this region the information metric and the heat capacity may not be positive definite and the CB solution becomes thermodynamically unstable.

The Ruppeiner information curvature is given by

$$R_I = \frac{(4\delta^2 - 2\delta + 1)G^{1+\delta}}{\pi(\delta - 1)^2(\delta + 1)} \frac{c^{-\frac{2(\delta - 1)^2}{2\delta^2 - 2\delta + 1}}}{M(c, \delta)} - \frac{6\delta(\delta + 1)}{2\delta(\delta + 1) - 1},$$

where the functions $M(c, \delta)$ and $N(c, \delta)$ are too lengthy to be written explicitly here. However, a density plot of $R_I$ is shown in Fig. 10.

The Ruppeiner information curvature is regular everywhere outside the regions $\frac{-(1 - \sqrt{3})}{2} < \delta < -1$ and $0 < \delta < \frac{-(1 + \sqrt{3})}{2}$. By comparing the results from Fig. 8 and Fig. 10 one concludes that there is a complete match between the heat capacity of the CB solution and the Ruppeiner information curvature. Therefore we can be confident that the Ruppeiner’s approach is appropriate for the description of the equilibrium state-space of the CB black hole.

**Figure 9:** The "island of global thermodynamic stability" (the shaded region) for the CB solution in the Ruppeiner’s case. One is confined in the region $0.75 < \delta < 1.34$ and $0 < c < 0.052$. For values of the parameters outside this region the information metric and the heat capacity may not be positive definite and the CB solution becomes thermodynamically unstable.

**Figure 10:** A density plot of the Ruppeiner information curvature for the CB solution. We have a complete match between the Ruppeiner information scalar and the CB heat capacity from Fig. 8.
4.2.2 The Quevedo and HPEM information metrics

The Quevedo and HPEM information metrics for the CB solution are given by

\[ ds_{Q}^{2} = (c \partial_c S + \delta \partial_\delta S) (\partial_\delta^2 S d\delta^2 - \partial_c^2 S dc^2) , \]

and

\[ ds_{HPEM}^{2} = c \frac{\partial_c S}{(\partial_\delta^2 S)^3} (-\partial_c^2 S dc^2 + \partial_\delta^2 S d\delta^2) . \]

The convexity condition, \( \partial_a \partial_b S \geq 0 \), is fulfilled for \( \delta < 0 \), while the heat capacity (4.9) is positive only for \( \delta > (-1 + \sqrt{3})/2 \), thus no common intersection. However, locally one can impose one or the other. Due to their lengthy form the expressions for the scalar curvatures will be studied only graphically as depicted on Fig. 11 and Fig. 12.

![Figure 11](image1)

**Figure 11:** A density plot of the Quevedo information curvature for the CB solution. It does contain the singularities of the heat capacity, but it is also plagued with additional singular points such as \( \delta = 1 \) and additional spinodal curves.

![Figure 12](image2)

**Figure 12:** A density plot of the HPEM information curvature for the CB solution. Although it lacks the spinodal curves from the Quevedo’s case the singularity at \( \delta = 1 \) is still present.

Both HPEM and Quevedo curvatures cover the singularities of the heat capacity, but they also have an additional singular point at \( \delta = 1 \), which is regular for the heat capacity \( C \). Further
investigation of the nature of the additional singularity at $\delta = 1$ is necessary, which does not fall within GTD. At this point it suffices to say that the Ruppeiner’s approach completely match the result of the CB heat capacity and it can be used for the description of the space of equilibrium states for the CB black hole.

5 Conclusion

The formalism of geometric thermodynamics indicates that phase transitions would occur at those points where the thermodynamic information curvature $R_I$ is singular. Near critical points the underlying micro-dynamics becomes strongly correlated and the equilibrium thermodynamic considerations are no-longer applicable. In this case one expects that a more general approach than geometric thermodynamics should hold.

In this paper we considered three different geometric thermodynamics approaches, namely those of Ruppeiner, Quevedo and HPEM, in order to probe the phase structure of black holes in higher derivative theories. We derived the corresponding information metrics and curvatures for the Deser-Sarigölu-Tekin background and the Clifton-Barrow black solution.

The geometrothermodynamic analysis of the space of equilibrium states for the DST solution in all cases (Ruppeiner, Quevedo and HPEM) revealed several critical points and spinodal curves at which the information curvatures are singular. However, the singular point $\sigma = -1/5$, where the DST heat capacity diverges, found no match in any of the three approaches. The later signals that the DST solution is highly non-trivial from thermodynamic standpoint and further analysis, concerning the thermodynamical stability of the solution, which goes beyond GTD, is required.

On the other hand, the heat capacity of the CB black hole and its Ruppeiner information curvature match completely. The latter makes one confident that the Ruppeiner’s approach is the right choice for the description of the equilibrium state-space of the CB solution. By imposing the criteria for global thermodynamic stability we found that the CB solution is globally stable from thermodynamic standpoint only in the parameter region $0.75 < \delta < 1.34$ and $0 < c < 0.052$ (Fig. 9). Even more, global thermodynamic stability confirms that $\delta$ should be a small real number. In CB case the Quevedo and the HPEM information approaches failed to produce viable metrics on the space of equilibrium states.

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