Spacetime Path Formalism for Massive Particles of Any Spin

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Abstract

Earlier work presented a spacetime path formalism for relativistic quantum mechanics arising naturally from the fundamental principles of the Born probability rule, superposition, and spacetime translation invariance. The resulting formalism can be seen as a foundation for a number of previous parameterized approaches to relativistic quantum mechanics in the literature. Because time is treated similarly to the three space coordinates, rather than as an evolution parameter, such approaches have proved particularly useful in the study of quantum gravity and cosmology. The present paper extends the foundational spacetime path formalism to include massive, non-scalar particles of any (integer or half-integer) spin. This is done by generalizing the principle of translational invariance used in the scalar case to the principle of full Poincaré invariance, leading to a formulation for the nonscalar propagator in terms of a path integral over the Poincaré group. Once the difficulty of the non-compactness of the component Lorentz group is dealt with, the subsequent development is remarkably parallel to the scalar case. This allows the formalism to retain a clear probabilistic interpretation throughout, with a natural reduction to non-relativistic quantum mechanics closely related to the well known generalized Foldy-Wouthuysen transformation.

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I. INTRODUCTION

Reference 1 presented a foundational formalism for relativistic quantum mechanics based on path integrals over parametrized paths in spacetime. As discussed there, such an approach is particularly suited for further study of quantum gravity and cosmology, and it can be given a natural interpretation in terms of decoherent histories 2. However, the formalism as given in 1 is limited to scalar particles. The present paper extends this spacetime path formalism to non-scalar particles, although the present work is still limited to massive particles.

There have been several approaches proposed in the literature for extending the path integral formulation of the relativistic scalar propagator 3, 4, 5, 6 to the case of non-scalar particles, particularly spin-1/2 (see, for example, 7, 8, 9, 10, 11). These approaches generally proceed by including in the path integral additional variables to represent higher spin degrees of freedom. However, there is still a lack of a comprehensive path integral formalism that treats all spin values in a consistent way, in the spirit of the classic work of Weinburg 12, 13, 14 for traditional quantum field theory. Further, most earlier references assume that the path integral approach is basically a reformulation of an a priori traditional Hamiltonian formulation of quantum mechanics, rather than being foundational in its own right.

The approach to be considered here extends the approach from 1 to non-scalar particles by expanding the configuration space of a particle to be the Poincaré group (also known as the inhomogeneous Lorentz group). That is, rather than just considering the position of a particle, the configuration of a particle will be taken to be both a position and a Lorentz transformation. Choosing various representations of the group of Lorentz transformations then allows all spins to be handled in a consistent way.

The idea of using a Lorentz group variable to represent spin degrees of freedom is not new. For example, Hanson and Regge 15 describe the physical configuration of a relativistic spherical top as a Poincaré element whose degrees of freedom are then restricted. Similarly, Hannibal 16 proposes a full canonical formalism for classical spinning particles using the Lorentz group for the spin configuration space, which is then quantized to describe both spin and isospin. Rivas 17, 18, 19 has made a comprehensive study in which an elementary particle is defined as “a mechanical system whose kinematical space is a homogeneous space of the Poincaré group”.

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Rivas actually proposes quantization using path integrals, but he does not provide an explicit derivation of the non-scalar propagator by evaluating such an integral. A primary goal of this paper to provide such a derivation.

Following a similar approach to [1], the form of the path integral for non-scalar particles will be deduced from the fundamental principles of the Born probability rule, superposition, and Poincaré invariance. After a brief overview in Sec. II of some background for this approach, Sec. III generalizes the postulates from [1] to the non-scalar case, leading to a path integral over an appropriate Lagrangian function on the Poincaré group variables.

The major difficulty with evaluating this path integral is the non-compactness of the Lorentz group. Previous work on evaluating Lorentz group path integrals (going back to [20]) is based on the irreducible unitary representations of the group. This is awkward, since, for a non-compact group, these representations are continuous [21] and the results do not generalize easily to the covering group $SL(2, \mathbb{C})$ that includes half-integral spins.

Instead, we will proceed by considering a Wick rotation to Euclidean space, which replaces the non-compact Lorentz group $SO(3, 1)$ by the compact group $SO(4)$ of rotations in four dimensions, in which it is straightforward to evaluate the path integral. It will then be argued that, even though the $SO(4)$ propagator cannot be assumed the same as the true Lorentz propagator, the propagators should be the same when restricted to the common subgroup $SO(3)$ of rotations in three dimensions. This leads directly to considerations of the spin representations of $SO(3)$.

Accordingly, Sec. IV develops the Euclidean $SO(4)$ propagator and Sec. V then considers the reduction to the three-dimensional rotation group and its spin representations. However, rather than using the usual Wigner approach of reduction along the momentum vector [22], we will reduce along an independent time-like four-vector [23, 24]. This allows for a very parallel development to [1] for antiparticles in Sec. VI and for a clear probability interpretation in Sec. VII.

Interactions of non-scalar particles can be included in the formalism by a straightforward generalization of the approach given in [1]. Section VIII gives an overview of this, though full details are not included where they are substantially the same as the scalar case.

Natural units with $\hbar = 1 = c$ are used throughout the following and the metric has a signature of $(- + + +)$. 


II. BACKGROUND

Path integrals were originally introduced by Feynman [25, 26] to represent the non-relativistic propagation kernel \( \Delta(x_1 - x_0; t_1 - t_0) \). This kernel gives the transition amplitude for a particle to propagate from the position \( x_0 \) at time \( t_0 \) to the position \( x_1 \) at time \( t_1 \). That is, if \( \psi(x_0; t_0) \) is the probability amplitude for the particle to be at position \( x_0 \) at time \( t_0 \), then the amplitude for it to propagate to another position at a later time is

\[
\psi(x; t) = \int d^3x_0 \Delta(x - x_0; t - t_0)\psi(x_0; t_0).
\]

A specific path of a particle in space is given by a position function \( q(t) \) parametrized by time (or, in coordinate form, the three functions \( q^i(t) \) for \( i = 1, 2, 3 \)). Now consider all possible paths starting at \( q(t_0) = x_0 \) and ending at \( q(t_1) = x_1 \). The path integral form for the propagation kernel is then given by integrating over all these paths as follows:

\[
\Delta(x_1 - x_0; t_1 - t_0) = \zeta \int D^3q \delta^3(q(t_1) - x_1)\delta^3(q(t_0) - x_0)e^{iS[q]},
\]

where the phase function \( S[q] \) is given by the classical action

\[
S[q] \equiv \int_{t_0}^{t_1} dt L(q(t)),
\]

with \( L(q) \) being the non-relativistic Lagrangian in terms of the three-velocity \( \dot{q} \equiv dq/dt \).

In Eq. (1), the notation \( D^3q \) indicates a path integral over the three functions \( q^i(t) \). The Dirac delta functions constrain the paths integrated over to start and end at the appropriate positions. Finally, \( \zeta \) is a normalization factor, including any limiting factors required to keep the path integral finite (which are sometimes incorporated into the integration measure \( D^3q \) instead).

As later noted by Feynman himself [5], it is possible to generalize the path integral approach to the relativistic case. To do this, it is necessary to consider paths in spacetime, rather than just space. Such a path is given by a four dimensional position function \( q(\lambda) \), parametrized by an invariant path parameter \( \lambda \) (or, in coordinate form, the four functions \( q^\mu(\lambda) \), for \( \mu = 0, 1, 2, 3 \)).

The propagation amplitude for a free scalar particle in spacetime is given by the Feynman propagator

\[
\Delta(x - x_0) = \frac{e^{i p \cdot (x - x_0)}}{p^2 + m^2 - i\epsilon}.
\]
It can be shown (in addition to [5], see also, e.g., [1, 3]) that this propagator can be expressed in path integral form as
\[
\Delta(x - x_0) = \int_{\lambda_0}^{\infty} d\lambda \zeta \int D^4q \delta^4(q(\lambda) - x)\delta^4(q(\lambda_0) - x_0)e^{iS[q]},
\]
where
\[
S[q] \equiv \int_{\lambda_0}^{\lambda} d\lambda' L(\dot{q})(\lambda'),
\]
and \(L(\dot{q})\) is now the relativistic Lagrangian in terms of the four-velocity \(\dot{q} \equiv dq/d\lambda\).

Notice that the form of the relativistic expression differs from the non-relativistic one by having an additional integration over \(\lambda\). This is necessary, since the propagator must, in the end, depend only on the change in position, independent of \(\lambda\). However, as noted in [1], Eq. (3) can be written as
\[
\Delta(x - x_0) = \int_{\lambda_0}^{\infty} d\lambda \Delta(x - x_0; \lambda - \lambda_0),
\]
where the relativistic kernel
\[
\Delta(x - x_0; \lambda - \lambda_0) = \zeta \int D^4q \delta^4(q(\lambda) - x)\delta^4(q(\lambda_0) - x_0)e^{iS[q]}
\]
now has a form entirely parallel with the non-relativistic case. The relativistic kernel can be considered to represent propagation over paths of the specific length \(\lambda - \lambda_0\), while Eq. (4) then integrates over all possible path lengths.

Given the parallel with the non-relativistic case, define the parametrized probability amplitudes \(\psi(x; \lambda)\) such that
\[
\psi(x; \lambda) = \int d^4x_0 \Delta(x - x_0; \lambda - \lambda_0)\psi(x_0; \lambda_0).
\]
Parametrized amplitudes were introduced by Stueckelberg [27, 28], and parametrized approaches to relativistic quantum mechanics have been developed by a number of subsequent authors [23, 29, 30, 31, 32, 33, 34, 35, 36]. The approach is developed further in the context of spacetime paths of scalar particles in [1].

In the traditional presentation, however, it is not at all clear why the path integrals of Eqs. (1) and (2) should reproduce the expected results for non-relativistic and relativistic propagation. The phase functional \(S\) is simply chosen to have the form of the classical action, such that this works. In contrast, [1] makes a more fundamental argument that the exponential form of Eq. (5) is a consequence of translation invariance in Minkowski
spacetime. This allows for development of the spacetime path formalism as a foundational approach, rather than just a re-expression of already known results.

The full invariant group of Minkowski spacetime is not the translation group, though, but the Poincaré group consisting of both translations and Lorentz transformations. This leads one to consider the implications of applying the argument of [1] to the full Poincaré group.

Now, while a translation applies to the position of a particle, a Lorentz transformation applies to its frame of reference. Just as we can consider the position $x$ of a particle to be a translation by $x$ from some fixed origin $O$, we can consider the frame of reference of a particle to be given by a Lorentz transformation $\Lambda$ from a fixed initial frame $I$. The full configuration of a particle is then given by $(x, \Lambda)$, for a position $x$ and a Lorentz transformation $\Lambda$—that is, the configuration space of the particle is also just the Poincaré group. The application of an arbitrary Poincaré transformation $(\Delta x, \Lambda')$ to a particle configuration $(x, \Lambda)$ results in the transformed configuration $(\Lambda'x + \Delta x, \Lambda')$.

A particle path will now be a path through the Poincaré group, not just through spacetime. Such a path is given by both a position function $q(\lambda)$ and a Lorentz transformation function $M(\lambda)$ (in coordinate form, a Lorentz transformation is represented by a matrix, so there are sixteen functions $M_{\mu \nu}(\lambda)$, for $\mu, \nu = 0, 1, 2, 3$). The remainder of this paper will re-develop the spacetime path formalism introduced in [1] in terms of this expanded conception of particle paths. As we will see, this naturally leads to a model for non-scalar particles.

III. THE NON-SCALAR PROPAGATOR

This section develops the path-integral form of the non-scalar propagator from the conception of Poincaré group particle paths introduced in the previous section. The argument parallels that of [1] for the scalar case, motivating a set of postulates that lead to the appropriate path integral form.

To begin, let $\Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda - \lambda_0)$ be the transition amplitude for a particle to go from the configuration $(x_0, \Lambda_0)$ at $\lambda_0$ to the configuration $(x, \Lambda)$ at $\lambda$. By Poincaré invariance, this amplitude only depends on the relative quantities $x - x_0$ and $\Lambda \Lambda_0^{-1}$. By parameter shift invariance, it only depends on $\lambda - \lambda_0$. Similarly to the scalar case (Eq. (4)), the full
propagator is given by integrating over the kernel path length parameter:

\[ \Delta(x - x_0, \Lambda\Lambda_0^{-1}) = \int_0^\infty d\lambda \Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda). \]  \hspace{1cm} (6)

The fundamental postulate of the spacetime path approach is that a particle’s transition amplitude between two points is a superposition of the transition amplitudes for all possible paths between those points. Let the functional \( \Delta[q, M] \) give the transition amplitude for a path \( q(\lambda), M(\lambda) \). Then the transition amplitude \( \Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda - \lambda_0) \) must be given by a path integral over \( \Delta[q, M] \) for all paths starting at \( (x_0, \Lambda_0) \) and ending at \( (x, \Lambda) \) with the parameter interval \([\lambda_0, \lambda]\).

**Postulate 1.** For a free, non-scalar particle, the transition amplitude \( \Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda - \lambda_0) \) is given by the superposition of path transition amplitudes \( \Delta[q, M] \), for all possible Poincaré path functions \( q(\lambda), M(\lambda) \) beginning at \( (x_0, \Lambda_0) \) and ending at \( (x, \Lambda) \), parametrized over the interval \([\lambda_0, \lambda]\). That is,

\[ \Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda - \lambda_0) = \zeta \int D^4 q \int D^6 M \\
\delta^4(q(\lambda) - x)\delta^6(M(\lambda)\Lambda^{-1} - I)\delta^4(q(\lambda_0) - x_0)\delta^6(M(\lambda_0)\Lambda_0^{-1} - I)\Delta[q, M], \]  \hspace{1cm} (7)

where \( \zeta \) is a normalization factor as required to keep the path integral finite.

As previously noted, the notation \( D^4 q \) in Eq. (7) indicates a path integral over the four path functions \( q^\mu(\lambda) \). Similarly, \( D^6 M \) indicates a path integral over the Lorentz group functions \( M^{\mu\nu}(\lambda) \). While a Lorentz transformation matrix \( \Lambda^{\mu\nu}(\lambda) \) has sixteen elements, any such matrix is constrained by the condition

\[ \Lambda^\alpha_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu = \eta_{\mu\nu}, \]  \hspace{1cm} (8)

where \( [\eta_{\mu\nu}] = \text{diag}(-1, 1, 1, 1) \) is the flat Minkowski space metric tensor. This equation is symmetric, so it introduces ten constraints, leaving only six actual degrees of freedom for a Lorentz transformation. The Lorentz group is thus six dimensional, as indicated by the notation \( D^6 \) in the path integral.

To further deduce the form of \( \Delta[q, M] \), consider a family of particle paths \( q_{x_0, \Lambda_0}, M_{x_0, \Lambda_0} \), indexed by the starting configuration \((x_0, \Lambda_0)\), such that

\[ q_{x_0, \Lambda_0}(\lambda) = x_0 + \Lambda_0q(\lambda) \quad \text{and} \quad M_{x_0, \Lambda_0}(\lambda) = \Lambda_0M(\lambda), \]
where $\tilde{q}(\lambda_0) = 0$ and $\tilde{M}(\lambda_0) = I$. These paths are constructed by, in effect, applying the Poincaré transformation given by $(x_0, \Lambda_0)$ to the specific functions $\tilde{q}$ and $\tilde{M}$ defining the family. (Note how the ability to do this depends on the particle configuration space being the same as the Poincaré transformation group.)

Consider, though, that the particle propagation embodied in $\Delta[q, M]$ must be Poincaré invariant. That is, $\Delta[q', M'] = \Delta[q, M]$ for any $q', M'$ related to $q, M$ by a fixed Poincaré transformation. Thus, all members of the family $q_{x_0, \Lambda_0}, M_{x_0, \Lambda_0}$, which are all related to $\tilde{q}, \tilde{M}$ by Poincaré transformations, must have the same amplitude $\Delta[q_{x_0, \Lambda_0}, M_{x_0, \Lambda_0}] = \Delta[\tilde{q}, \tilde{M}]$, depending only on the functions $\tilde{q}$ and $\tilde{M}$.

Suppose that a probability amplitude $\psi(x_0, \Lambda_0)$ is given for a particle to be at in an initial configuration $(x_0, \Lambda_0)$ and that the transition amplitude is known to be $\Delta[\tilde{q}, \tilde{M}]$ for specific relative configuration functions $\tilde{q}, \tilde{M}$. Then, the probability amplitude for the particle to traverse a specific path $(q_{x_0, \Lambda_0}(\lambda), M_{x_0, \Lambda_0}(\lambda))$ from the family defined by the functions $\tilde{q}, \tilde{M}$ should be just $\Delta[q_{x_0, \Lambda_0}, M_{x_0, \Lambda_0}]\psi(x_0, \Lambda_0) = \Delta[\tilde{q}, \tilde{M}]\psi(x_0, \Lambda_0)$.

However, the very meaning of being on a specific path is that the particle must propagate from the given starting configuration to the specific ending configuration of the path. Further, since the paths in the family are parallel in configuration space, the ending configuration is uniquely determined by the starting configuration. Therefore, the probability for reaching the ending configuration must be the same as the probability for having started out at the given initial configuration $(x_0, \Lambda_0)$. That is,

$$|\Delta[\tilde{q}, \tilde{M}]\psi(x_0, \Lambda_0)|^2 = |\psi(x_0, \Lambda_0)|^2.$$ But, since $\Delta[\tilde{q}, \tilde{M}]$ is independent of $x_0$ and $\Lambda_0$, we must have $|\Delta[q, M]|^2 = 1$ in general.

This argument therefore suggests the following postulate.

**Postulate 2.** For any path $(q(\lambda), M(\lambda))$, the transition amplitude $\Delta[q, M]$ preserves the probability density for the particle along the path. That is, it satisfies

$$|\Delta[q, M]|^2 = 1.$$  \hspace{1cm} (9)

The requirements of Eq. (9) and Poincaré invariance mean that $\Delta[q, M]$ must have the exponential form

$$\Delta[q, M] = e^{iS[\tilde{q}, \tilde{M}]}.$$  \hspace{1cm} (10)
for some phase functional $S$ of the relative path functions

$$
\tilde{q}(\lambda) \equiv M(\lambda_0)^{-1}(q(\lambda) - q(\lambda_0)) \quad \text{and} \quad \tilde{M}(\lambda) \equiv M(\lambda_0)^{-1}M(\lambda).
$$

As discussed in [1], we are actually justified in replacing these relative functions with path derivatives under the path integral, even though the path functions $q(\lambda)$ and $M(\lambda)$ may not themselves be differentiable in general. This is because a path integral is defined as the limit of discretized approximations in which path derivatives are approximated as mean values, and the limit is then taken over the path integral as a whole, not each derivative individually. Thus, even though the individual path derivative limits may not be defined, the path integral has a well-defined value so long as the overall path integral limit is defined.

However, the quantities $\tilde{q}$ and $\tilde{M}$ are expressed in a frame that varies with the $M(\lambda_0)$ of the specific path under consideration. We wish instead to construct differentials in the fixed “laboratory” frame of the $q(\lambda)$. Transforming $\tilde{q}$ and $\tilde{M}$ to this frame gives

$$
M(\lambda_0)\tilde{q}(\lambda) = q(\lambda) - q(\lambda_0) \quad \text{and} \quad M(\lambda_0)\tilde{M}(\lambda)M(\lambda_0)^{-1} = M(\lambda)M(\lambda_0)^{-1}.
$$

Clearly, the corresponding derivative for $q$ is simply $\dot{q}(\lambda) \equiv dq/d\lambda$, which is the tangent vector to the path $q(\lambda)$. The derivative for $M$ needs to be treated a little more carefully. Since the Lorentz group is a Lie group (that is, a continuous, differentiable group), the tangent to a path $M(\lambda)$ in the Lorentz group space is given by an element of the corresponding Lie algebra [37, 38]. For the Lorentz group, the proper such tangent is given by the matrix $\Omega(\lambda) = \dot{M}(\lambda)M(\lambda)^{-1}$, where $\dot{M}(\lambda) \equiv dM/d\lambda$.

Together, the quantities $(\dot{q}, \Omega)$ form a tangent along the path in the full Poincaré group space. We can then take the arguments of the phase functional in Eq. (10) to be $(\dot{q}, \Omega)$. Substituting this into Eq. (10) gives

$$
\Delta(x - x_0, \Lambda \Lambda_0^{-1}; \lambda - \lambda_0) = \zeta \int D^4q \int D^6M \delta^4(q(\lambda) - x)\delta^6(M(\lambda)\Lambda^{-1} - I)\delta^4(q(\lambda_0) - x_0)\delta^6(M(\lambda_0)\Lambda_0^{-1} - I)e^{iS[\dot{q}, \Omega]}, \quad (11)
$$

which reflects the typical form of a Feynman sum over paths.

Now, by dividing a path $(q(\lambda), M(\lambda))$ into two paths at some arbitrary parameter value $\lambda$ and propagating over each segment, one can see that

$$
S[\dot{q}, \Omega; \lambda_1, \lambda_0] = S[\dot{q}, \Omega; \lambda_1, \lambda] + S[\dot{q}, \Omega; \lambda, \lambda_0], \quad (12)
$$
where $S[\dot{q}, \Omega; \lambda', \lambda]$ denotes the value of $S[\dot{q}, \Omega]$ for the path parameter range restricted to $[\lambda, \lambda']$. Using this property to build the total value of $S[\dot{q}, \Omega]$ from infinitesimal increments leads to the following result (whose full proof is a straightforward generalization of the proof given in [1] for the scalar case).

**Proposition A (Form of the Phase Functional).** The phase functional $S$ must have the form

$$S[\dot{q}, \Omega] = \int_{\lambda_0}^{\lambda_1} d\lambda' L[\dot{q}, \Omega; \lambda'],$$

where the parametrization domain is $[\lambda_0, \lambda_1]$ and $L[\dot{q}, \Omega; \lambda]$ depends only on $\dot{q}$, $\Omega$ and their higher derivatives evaluated at $\lambda$.

Clearly, the functional $L[\dot{q}, \Omega; \lambda]$ plays the traditional role of the Lagrangian. The simplest non-trivial form for this functional would be for it to depend only on $\dot{q}$ and $\Omega$ and no higher derivatives. Further, suppose that it separates into uncoupled parts dependent on $\dot{q}$ and $\Omega$:

$$L[\dot{q}, \Omega; \lambda] = L_q[\dot{q}; \lambda] + L_M[\Omega; \lambda].$$

The path integral of Eq. (11) then factors into independent parts in $q$ and $M$, such that

$$\Delta(x - x_0, \Lambda \Lambda_0^{-1}; \lambda - \lambda_0) = \Delta(x - x_0; \lambda - \lambda_0)\Delta(\Lambda \Lambda_0^{-1}; \lambda - \lambda_0).$$

(13)

If we take $L_q$ to have the classical Lagrangian form

$$L_q[\dot{q}; \lambda] = L_q(\dot{q}(\lambda)) = \frac{1}{4} \dot{q}(\lambda)^2 - m^2,$$

for a particle of mass $m$, then the path integral in $q$ can be evaluated to give

$$\Delta(x - x_0; \lambda - \lambda_0) = (2\pi)^{-4} \int d^4 p e^{i p \cdot (x-x_0)} e^{-i (\lambda - \lambda_0)(p^2 + m^2)}. $$

(14)

Similarly, take $L_M$ to be a Lorentz-invariant scalar function of $\Omega(\lambda)$. $\Omega$ is an antisymmetric matrix (this can be shown by differentiating the constraint Eq. (8)), so the scalar $\text{tr}(\Omega) = \Omega^\mu_{\mu} = 0$. The next simplest choice is

$$L_M[\Omega; \lambda] = L_M(\Omega(\lambda)) = \frac{1}{2} \text{tr}(\Omega(\lambda)\Omega(\lambda)^T) = \frac{1}{2} \Omega^{\mu\nu}(\lambda)\Omega_{\mu\nu}(\lambda).$$

**Postulate 3.** For a free non-scalar particle of mass $m$, the Lagrangian function is given by

$$L(\dot{q}, \Omega) = L_q(\dot{q}) + L_M(\Omega),$$

(10)
where
\[ L_q(\dot{q}) = \frac{1}{4} \dot{q}^2 - m^2 \]
and
\[ L_M(\Omega) = \frac{1}{2} \text{tr}(\Omega\Omega^T). \]

Evaluating the path integral in $M$ is complicated by the fact that the Lorentz group is not compact, and integration over the group is not, in general, bounded. The Lorentz group is denoted $SO(3,1)$ for the three plus and one minus sign of the Minkowski metric $\eta$ in the defining pseudo-orthogonality condition Eq. (8). It is the minus sign on the time component of $\eta$ that leads to the characteristic Lorentz boosts of special relativity. But since such boosts are parametrized by the boost velocity, integration of this sector of the Lorentz group is unbounded. This is in contrast to the three dimensional rotation subgroup $SO(3)$ for the Lorentz, which is parameterized by rotation angles that are bounded.

To avoid this problem, we will Wick rotate \[39\] the time axis in complex space. This replaces the physical $t$ coordinate with $it$, turning the minus sign in the metric to a plus sign, resulting in the normal Euclidean metric $\text{diag}(1,1,1,1)$. The symmetry group of Lorentz transformations in Minkowski space then corresponds to the symmetry group $SO(4)$ of rotations in four-dimensional Euclidean space. The group $SO(4)$ is compact, and the path integration over $SO(4)$ can be done \[20\].

Rather than dividing into boost and rotational parts, like the Lorentz group, $SO(4)$ instead divides into two $SO(3)$ subgroups of rotations in three dimensions. Actually, rather than $SO(3)$ itself, it is more useful to consider its universal covering group $SU(2)$, the group of two-dimensional unitary matrices, because $SU(2)$ allows for representations with half-integral spin \[38, 40, 41\]. (The covering group $SU(2) \times SU(2)$ for $SO(4)$ in Euclidean space corresponds to the covering group $SL(2,\mathbb{C})$ of two-dimensional complex matrices for the Lorentz group $SO(3,1)$ in Minkowski space.)

Typically, Wick rotations have been used to simplify the evaluation of path integrals parametrized in time, like the non-relativistic integral of Eq. \[11\]. In this case, replacing $t$ by $it$ results in the exponent in the integrand of the path integral to become real. Unlike this case, the exponent in the integrand of a spacetime path integral remains imaginary, since the Wick rotation does not affect the path parameter $\lambda$. Nevertheless, the path integral can be evaluated, giving the following result (proved in the Appendix).
Proposition B (Evaluation of the SO(4) Path Integral). Consider the path integral

\[ \Delta(\Lambda_E \Lambda_{E_0}^{-1}; \lambda - \lambda_0) = \eta_E \int D^6 M_E \delta^6(M_E(\lambda) \Lambda_E^{-1} - I) \delta^6(M_E(\lambda_0) \Lambda_{E_0}^{-1} - I) \]

\[ \exp \left[ i \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{2} \text{tr}(\Omega_E(\lambda')\Omega_E(\lambda')^T) \right] \]

over the six dimensional group SO(4) \( \sim SU(2) \times SU(2) \), where \( \Omega_E(\lambda') \) is the element of the Lie algebra so(4) tangent to the path \( M_E(\lambda) \) at \( \lambda' \). This path integral may be evaluated to get

\[ \Delta(\Lambda_E \Lambda_{E_0}^{-1}; \lambda - \lambda_0) = \sum_{\ell_A, \ell_B} \exp^{-i(\Delta m_A^2 + \Delta m_B^2)(\lambda - \lambda_0)} (2\ell_A + 1)(2\ell_B + 1)\chi^{(\ell_A, \ell_B)}(\Lambda_E \Lambda_{E_0}^{-1}) \]

where the summation over \( \ell_A \) and \( \ell_B \) is from 0 to \( \infty \) in steps of 1/2, \( \Delta m_\ell^2 = \ell(\ell + 1) \) and \( \chi^{(\ell_A, \ell_B)} \) is the group character for the \((\ell_A, \ell_B)\) SU(2) \( \times \) SU(2) group representation.

The result of Eq. (16) is in terms of the representations of the covering group SU(2) \( \times \) SU(2). A (matrix) representation \( L \) of a group assigns to each group element \( g \) a matrix \( D^{(L)}(g) \) that respects the group operation, that is, such that \( D^{(L)}(g_1g_2) = D^{(L)}(g_1)D^{(L)}(g_2) \).

The character function \( \chi^{(L)} \) for the representation \( L \) of a group is a function from the group to the reals such that

\[ \chi^{(L)}(g) \equiv \text{tr}(D^{(L)}(g)). \]

The group SU(2) has the well known spin representations, labeled by spins \( \ell = 0, 1/2, 1, 3/2, \ldots \) \([40, 41]\) (for example, spin 0 is the trivial scalar representation, spin 1/2 is the spinor representation and spin 1 is the vector representation). A \((\ell_A, \ell_B)\) representation of SU(2) \( \times \) SU(2) then corresponds to a spin-\( \ell_A \) representation for the first SU(2) component and a spin-\( \ell_B \) representation for the second SU(2) component.

Of course, it is not immediately clear that this result for SO(4) applies directly to SO(3, 1). In some cases, it can be shown that the evolution propagator for a non-compact group is, in fact, the same as the propagator for a related compact group. Unfortunately, the relationship between SO(4) and SO(3, 1) (in which an odd number, three, of the six generators of SO(4) are multiplied by \( i \) to get the boost generators for SO(3, 1)) is such that the evolution propagator of the non-compact group does not coincide with that of the compact group \([42]\).
Nevertheless, SO(4) and SO(3,1) both have compact SO(3) subgroups, which are isomorphic. Therefore, the restriction of the SO(4) propagator to its SO(3) subgroup should correspond to the restriction of the SO(3,1) propagator to its SO(3) subgroup. This will prove sufficient for our purposes. In the next section, we will continue to freely work with the Wick rotated Euclidean space and the SO(4) propagator as necessary. To show clearly when this is being done, quantities effected by Wick rotation will be given a subscript $E$, as in Eq. (16).

IV. THE EUCLIDEAN PROPAGATOR

For a scalar particle, one can define the probability amplitude $\psi(x; \lambda)$ for the particle to be at position $x$ at the point $\lambda$ in its path [1, 27, 28]. For a non-scalar particle, this can be extended to a probability amplitude $\psi(x, \Lambda; \lambda)$ for the particle to be in the Poincaré configuration $(x, \Lambda)$, at the point $\lambda$ in its path. The transition amplitude given in Eq. (7) acts as a propagation kernel for $\psi(x, \Lambda; \lambda)$:

$$\psi(x, \Lambda; \lambda) = \int d^4x_0 \int d^6\Lambda_0 \Delta(x - x_0, \Lambda\Lambda_0^{-1}; \lambda - \lambda_0)\psi(x_0, \Lambda_0; \lambda_0).$$

The Euclidean version of this equation has an identical form, but in terms of Euclidean configuration space quantities:

$$\psi(x_E, \Lambda_E; \lambda) = \int d^4x_{E0} \int d^6\Lambda_{E0} \Delta(x_E - x_{E0}, \Lambda_E\Lambda_{E0}^{-1}; \lambda - \lambda_0)\psi(x_{E0}, \Lambda_{E0}; \lambda_0). \quad (17)$$

Using Eq. (13), substitute into Eq. (17) the Euclidean scalar kernel (as in Eq. (14), but with a leading factor of $i$) and the SO(4) kernel (Eq. (16)), giving

$$\psi(x_E, \Lambda_E; \lambda) = \sum_{\ell_A, \ell_B} \int d^4x_{E0} \int d^6\Lambda_{E0}$$

$$\Delta^{(\ell_A,\ell_B)}(x_E - x_{E0}; \lambda - \lambda_0)\chi^{(\ell_A,\ell_B)}(\Lambda_{E0}\Lambda_{E0}^{-1})\psi(x_{E0}, \Lambda_{E0}; \lambda_0), \quad (18)$$

where

$$\Delta^{(\ell_A,\ell_B)}(x_E - x_{E0}; \lambda - \lambda_0) \equiv i(2\pi)^{-4} \int d^4p_E e^{ip_E \cdot (x_E - x_{E0})} e^{-i(\lambda - \lambda_0)(p_E^2 + m^2 + \Delta m^2_A + \Delta m^2_B)}.$$

Since the group characters provide a complete set of orthogonal functions [40], the function $\psi(x_{E0}, \Lambda_{E0}; \lambda_0)$ can be expanded as

$$\psi(x_{E0}, \Lambda_{E0}; \lambda_0) = \sum_{\ell_A, \ell_B} \chi^{(\ell_A,\ell_B)}(\Lambda_{E0})\psi^{(\ell_A,\ell_B)}(x_{E0}; \lambda_0).$$
Substituting this into Eq. (18) and using
\[ \chi^{(\ell,\ell_B)}(\Lambda_E) = \int d^6\Lambda_{E0} \chi^{(\ell,\ell_B)}(\Lambda_E\Lambda_{E0}^{-1})\chi^{(\ell,\ell_B)}(\Lambda_{E0}) \]
(see [40]) gives
\[ \psi(x_E, \Lambda_E; \lambda) = \sum_{\ell_A,\ell_B} \chi^{(\ell_A,\ell_B)}(\Lambda_E)d^{(\ell_A,\ell_B)}(x_E - x_{E0}; \lambda - \lambda_0)\psi^{(\ell_A,\ell_B)}(x_{E0}; \lambda_0). \]  
(19)

The general amplitude \(\psi(x_E, \Lambda_E; \lambda)\) can thus be expanded into a sum of terms in the various \(SU(2) \times SU(2)\) representations, the coefficients \(\psi^{(\ell_A,\ell_B)}(x_E; \lambda_0)\) of which each evolve separately according to Eq. (19). As is well known, reflection symmetry requires that a real particle amplitude must transform according to a \((\ell, \ell)\) or \((\ell_A, \ell_B) \oplus (\ell_B, \ell_A)\) representation. That is, the amplitude function \(\psi(x_E, \Lambda_E; \lambda)\) must either have the form
\[ \psi(x_E, \Lambda_E; \lambda) = \chi^{(\ell,\ell)}(\Lambda_E)\psi^{(\ell,\ell)}(x_E; \lambda) \]
or
\[ \psi(x_E, \Lambda_E; \lambda) = \chi^{(\ell_A,\ell_B)}(\Lambda_E)d^{(\ell_A,\ell_B)}(x_E; \lambda) + \chi^{(\ell_B,\ell_A)}(\Lambda_E)\psi^{(\ell_B,\ell_A)}(x_E; \lambda). \]

Assuming one of the above two forms, shift the particle mass to \(m'^2 = m^2 + 2\Delta m^2\) or \(m'^2 = m^2 + 2\Delta m^2_{\ell_A} + 2\Delta m^2_{\ell_B}\), so that
\[ \psi(x_E, \Lambda_E; \lambda) = \int d^4x_0, \int d^6\Lambda_0 \chi^{(L)}(\Lambda_E\Lambda_0^{-1})\Delta(x_E - x_{E0}; \lambda - \lambda_0)\psi(x_{E0}, \Lambda_0; \lambda_0), \]
where \(\Delta\) here is (the Euclidean version of) the scalar propagator of Eq. (14), but now for the shifted mass \(m'\), and \((L)\) is either \((\ell, \ell)\) or \((\ell_A, \ell_B)\). That is, the full kernel must have the form
\[ \Delta^{(L)}(x_E - x_{E0}, \Lambda_E\Lambda_0^{-1}; \lambda - \lambda_0) = \chi^{(L)}(\Lambda_E\Lambda_0^{-1})\Delta(x_E - x_{E0}; \lambda - \lambda_0). \]  
(20)

As is conventional, from now on we will use four-dimensional spinor indices for the \((1/2, 0) \oplus (0, 1/2)\) representation and vector indices (also four dimensional) for the \((1, 1)\) representation, rather than the \(SU(2) \times SU(2)\) indices \((\ell_A, \ell_B)\) (see, for example, [41]). Let \(\mathcal{D}^{(\ell)}(\Lambda_E)\) be a matrix representation of the \(SO(4)\) group using such indices. Define correspondingly indexed amplitude functions by
\[ \psi^{(\ell)}(x_E; \lambda) \equiv \int d^6\Lambda_E \mathcal{D}^{(\ell)}(\Lambda_E)\psi(x_E, \Lambda_E; \lambda) \]
(21)
(note the \textit{double} indexing of \(\psi\) here).

These \(\psi^{\prime \prime}_{l}\) are the elements of an algebra over the \(SO(4)\) group for which, given \(x_{E}\) and \(\lambda\), the \(\psi(x_{E}, \Lambda_{E}; \lambda)\) are the \textit{components}, “indexed” by the group elements \(\Lambda_{E}\) (see Section III.13 of \([40]\)). The product of two such algebra elements is (with summation implied over repeated up and down indices)

\[
\psi^{\prime \prime}_{l}(x_{E}; \lambda)\psi^{\prime}_{2 l}(x_{E}; \lambda) = \int d^{6}\Lambda_{E1} \int d^{6}\Lambda_{E2} D^{\prime \prime}_{l}(\Lambda_{E1}) D^{\prime}_{l}(\Lambda_{E2}) \psi_{1}(x_{E}, \Lambda_{E1}; \lambda) \psi_{2}(x_{E}, \Lambda_{E2}; \lambda)
\]

\[
= \int d^{6}\Lambda_{E} D^{\prime \prime}_{l}(\Lambda_{E}) \int d^{6}\Lambda_{E1} \psi_{1}(x_{E}, \Lambda_{E1}; \lambda) \psi_{2}(x_{E}, \Lambda_{E1}^{-1}\Lambda_{E}; \lambda)
\]

\[
= (\psi_{1}\psi_{2})^{\prime \prime}_{l}(x_{E}; \lambda),
\]

where the second equality follows after setting \(\Lambda_{E2} = \Lambda_{E1}^{-1}\Lambda_{E}\) from the invariance of the integration measure of a Lie group (see, for example, \([37]\), Section 4.11, and \([40]\), Section III.12—this property will be used regularly in the following), and the product components \((\psi_{1}\psi_{2})(x_{E}, \Lambda_{E}; \lambda)\) are defined to be

\[
(\psi_{1}\psi_{2})(x_{E}, \Lambda_{E}; \lambda) = \int d^{6}\Lambda_{E} \psi_{1}(x_{E}, \Lambda_{E}'; \lambda) \psi_{2}(x_{E}, \Lambda_{E}'^{-1}\Lambda_{E}; \lambda).
\]

Now substitute Eq. (17) into Eq. (21) to get

\[
\psi^{\prime \prime}_{l}(x_{E}; \lambda) = \int d^{6}\Lambda_{E} \int d^{4}x_{E0} \int d^{6}\Lambda_{E0} D^{\prime \prime}_{l}(\Lambda_{E}) \Delta(x_{E} - x_{E0}, \Lambda_{E} \Lambda_{E0}^{-1}; \lambda - \lambda_{0}) \psi(x_{E0}, \Lambda_{E0}; \lambda_{0}).
\]

Changing variables \(\Lambda_{E} \rightarrow \Lambda_{E0}^{l} \Lambda_{E0}\) then gives

\[
\psi^{\prime \prime}_{l}(x_{E}; \lambda) = \int d^{4}x_{0} \left[ \int d^{6}\Lambda_{E} D^{\prime \prime}_{l}(\Lambda_{E}) \Delta(x_{E} - x_{0}, \Lambda_{E}; \lambda - \lambda_{0}) \right]
\]

\[
\int d^{6}\Lambda_{E0} D^{\prime}_{l}(\Lambda_{E0}) \psi(x_{E0}, \Lambda_{E0}; \lambda_{0})
\]

\[
= \int d^{4}x_{0} \Delta^{\prime \prime}_{l}(x_{E} - x_{0}; \lambda - \lambda_{0}) \psi^{\prime}_{l}(x_{0}; \lambda_{0}),
\]

where the kernel for the algebra elements \(\psi^{\prime \prime}_{l}(x_{E}; \lambda)\) is thus

\[
\Delta^{\prime \prime}_{l}(x_{E} - x_{E0}; \lambda - \lambda_{0}) = \int d^{6}\Lambda_{E} D^{\prime \prime}_{l}(\Lambda_{E}) \Delta(x_{E} - x_{E0}, \Lambda_{E}; \lambda - \lambda_{0}).
\]

Substituting Eq. (20) into this, and using the definition of the character for a specific representation, \(\chi(\Lambda_{E}) \equiv \text{tr}(D(\Lambda_{E}))\), gives

\[
\Delta^{\prime \prime}_{l}(x_{E} - x_{E0}; \lambda - \lambda_{0}) = \left[ \int d^{6}\Lambda_{E} D^{\prime \prime}_{l}(\Lambda_{E}) D^{\prime}_{l}(\Lambda_{E}) \right] \Delta(x_{E} - x_{E0}; \lambda - \lambda_{0}).
\]
Use the orthogonality property

\[ \int d^6 \Lambda_E D^\ fraudulent{I}'(\Lambda_E)D^\ fraudulent{I}(\Lambda_E) = \delta^{\ fraudulent{I}}\ fractional{I}, \]

where the $SO(4)$ integration measure has been normalized so that $\int d^6 \Lambda_E = 1$ (see [40], Section 11), to get

\[ \Delta^{\ fraudulent{I}}(x_E - x_{E0}; \lambda - \lambda_0) = \delta^{\ fraudulent{I}}\ fractional{I}(x_E - x_{E0}; \lambda - \lambda_0). \tag{22} \]

The $SO(4)$ group propagator is thus simply $\delta^{\ fraudulent{I}}\ fractional{I}$. As expected, this does not have the same form as would be expected for the $SO(3,1)$ Lorentz group propagator. However, as argued at the end of Sec. III, the propagator restricted to the compact $SO(3)$ subgroup of $SO(3,1)$ is expected to have the same form as for the $SO(3)$ subgroup of $SO(4)$. So we turn now to the reduction of $SO(3,1)$ to $SO(3)$.

V. SPIN

In traditional relativistic quantum mechanics, the Lorentz-group dependence of non-scalar states is reduced to a rotation representation that is amenable to interpretation as the intrinsic particle spin. Since, in the usual approach, physical states are considered to have on-shell momentum, it is natural to use the 3-momentum as the vector around which the spin representation is induced, using Wigner’s classic “little group” argument [22].

However, in the spacetime path approach used here, the fundamental states are not naturally on-shell, rather the on-shell states are given as the time limits of off-shell states [1]. Further, there are well-known issues with the localization of on-shell momentum states [43, 44]. Therefore, instead of assuming on-shell states to start, we will adopt the approach of [23, 24], in which the spin representation is induced about an arbitrary timelike vector. This will allow for a straightforward generalization of the interpretation obtained in the spacetime path formalism for the scalar case [1].

First, define the probability amplitudes $\psi^{\ fraudulent{I}}(x; \lambda)$ for a given Lorentz group representation similarly to the correspondingly indexed amplitudes for $SO(4)$ representations from Sec. IV. Corresponding to such amplitudes, define a set of ket vectors $|\psi\rangle I$, with a single Lorentz-group representation index. The $|\psi\rangle I$ define a vector bundle (see, for example, [38]), of the same dimension as the Lorentz-group representation, over the scalar-state Hilbert space.
The basis position states for this vector bundle then have the form \(|x; \lambda\rangle_l\), such that

\[ \psi^\prime I(x; \lambda) = G^\prime I I_l(x; \lambda) |\psi\rangle_I, \]

with summation assumed over repeated upper and lower indices and \(G\) being the invariant matrix of a given Lorentz group representation such that

\[ D^I \mathcal{G} D = D \mathcal{G} D = G, \]

for any member \(D\) of the representation, where \(D^I\) is the Hermitian transpose of the matrix \(D\). For the scalar representation, \(G\) is 1, for the (Weyl) spinor representation it is the Dirac matrix \(\beta\) and for the vector representation it is the Minkowski metric \(\eta\).

In the following, \(G\) will be used (usually implicitly) to “raise” and “lower” group representation indices. For instance,

\[ l^\prime \langle x' ; \lambda| \equiv G l^\prime l_l \langle x ; \lambda|, \]

so that

\[ \psi^\prime I(x; \lambda) = l^\prime \langle x; \lambda| \psi \rangle_I. \] (23)

The states \(|x; \lambda\rangle_I\) are then normalized so that

\[ l^\prime \langle x' ; \lambda|x; \lambda\rangle_I = \delta^I l \delta^4 (x' - x), \] (24)

that is, they are orthogonal at equal \(\lambda\).

Consider an arbitrary Lorentz transformation \(M\). Since \(\psi(x, \Lambda; \lambda)\) is a scalar, it should transform as \(\psi'(x', \Lambda'; \lambda) = \psi(M^{-1}x', M^{-1}\Lambda'; \lambda)\). In terms of algebra elements,

\[
\begin{align*}
\psi^R I(x'; \lambda) &= \int d^6\Lambda' \mathcal{D}^R I (\Lambda') \psi(M^{-1}x', M^{-1}\Lambda'; \lambda) \\
&= \int d^6\Lambda \mathcal{D}^R I (M) \mathcal{D}^R I (\Lambda) \psi(M^{-1}x', \Lambda; \lambda) \\
&= \mathcal{D}^R I (M) \psi^R I (M^{-1}x; \lambda).
\end{align*}
\] (25)

Let \(\hat{U}(\Lambda)\) denote the unitary operator on Hilbert space corresponding to the Lorentz transformation \(\Lambda\). Then, from Eq. (23),

\[ \psi^R I(x'; \lambda) = l^\prime \langle x'; \lambda| \psi' \rangle_I = l^\prime \langle x'; \lambda| \hat{U}(\Lambda) |\psi\rangle_I. \]
This and Eq. (25) imply that
\[ \hat{U}(\Lambda)^{-1}|x'; \lambda\rangle_l = |\Lambda^{-1}x'; \lambda\rangle_v [D(\Lambda)^{-1}]^\nu_l, \]
or
\[ \hat{U}(\Lambda)|x; \lambda\rangle_l = |\Lambda x; \lambda\rangle_v D^\nu_l(\Lambda). \tag{26} \]
Thus, the \(|x; \lambda\rangle_l\) are localized position states that transform according to a representation of the Lorentz group.

Now, for any future-pointing, timelike, unit vector \(n\) \((n^2 = -1 \text{ and } n^0 > 0)\) define the standard Lorentz transformation
\[ L(n) \equiv R(n)B(|n|)R^{-1}(n), \]
where \(R(n)\) is a rotation that takes the \(z\)-axis into the direction of \(n\) and \(B(|n|)\) is a boost of velocity \(|n|\) in the \(z\) direction. Then \(n = L(n)e\), where \(e \equiv (1, 0, 0, 0)\).

Define the Wigner rotation for \(n\) and an arbitrary Lorentz transformation \(\Lambda\) to be
\[ W(\Lambda, n) \equiv L(\Lambda n)^{-1} \Lambda L(n), \tag{27} \]
such that \(W(\Lambda, n)e = e\). That is, \(W(\Lambda, n)\) is a member of the little group of transformations that leave \(e\) invariant. Since \(e\) is along the time axis, its little group is simply the rotation group \(SO(3)\) of the three space axes.

Substituting the transformation \(\Lambda = L(\Lambda n)W(\Lambda, n)L(n)^{-1}\), into Eq. (26) gives
\[ \hat{U}(\Lambda)|x; \lambda\rangle_l = |\Lambda x; \lambda\rangle_v [D(\Lambda n)W(\Lambda, n)L(n)^{-1}]^\nu_l. \]
Defining
\[ |x, n; \lambda\rangle_l^{(W)} \equiv |x; \lambda\rangle_v [\mathcal{L}(n)]^\nu_l, \tag{28} \]
where \(\mathcal{L}(n) \equiv D(L(n))\), we see that \(|x, n; \lambda\rangle_l^{(W)}\) transforms under \(\hat{U}(\Lambda)\) as
\[ \hat{U}(\Lambda)|x, n; \lambda\rangle_l^{(W)} = |\Lambda x, \Lambda n; \lambda\rangle_v^{(W)} [D(W(\Lambda, n))]^\nu_l, \tag{29} \]
that is, according to the Lorentz representation subgroup given by \(D(W(\Lambda, n))\), which is isomorphic to some representation of the rotation group.
The irreducible representations of the rotation group (or, more exactly, its covering group \( SU(2) \)) are just the spin representations, with members given by matrices \( D_{\sigma'}^{\sigma} \), where the \( \sigma \) are spin indices. Let \( |\psi\rangle_{\sigma} \) be a member of a Hilbert space vector bundle indexed by spin indices. Then there is a linear, surjective mapping from \( |\psi\rangle_l \) to \( |\psi\rangle_{\sigma} \) given by

\[
|\psi\rangle_{\sigma} = |\psi\rangle_l u^l_{\sigma},
\]

where

\[
(u^l_{\sigma'})^* u^l_{\sigma} = \delta_{\sigma' \sigma}.
\] (30)

The isomorphism between the rotation subgroup of the Lorentz group and the rotation group then implies that, for any rotation \( W \), for all \( |\psi\rangle_l \),

\[
|\psi\rangle_l u^l_{\sigma'}[D(W)]^{\sigma'}_{\sigma} = |\psi\rangle_l[D(W)]^{l'}_{l} u^l_{\sigma'}
\]

(with summation implied over repeated \( \sigma \) indices, as well as \( l \) indices) or

\[
u^l_{\sigma'}[D(W)]^{\sigma'}_{\sigma} = [D(W)]^{l'}_{l} u^l_{\sigma'},
\] (31)

where \( D(W) \) is the spin representation matrix corresponding to \( W \).

Define

\[
|x, n; \lambda\rangle_{\sigma} \equiv |x, n; \lambda\rangle_l^{(W)} u^l_{\sigma}.
\] (32)

Substituting from Eq. (28) gives

\[
|x, n; \lambda\rangle_{\sigma} = |x, \lambda\rangle_l u^l_{\sigma}(n).
\] (33)

where

\[
u^l_{\sigma}(n) \equiv [\mathcal{L}(n)]_{l}^{l'} u^l_{\sigma}.
\] (34)

Then, under a Lorentz transformation \( \Lambda \), using Eqs. (29) and (31),

\[
\hat{U}(\Lambda)|x, n; \lambda\rangle_{\sigma} = |\Lambda x, \Lambda n; \lambda\rangle_l^{(W)} [D(W(\Lambda, n))]^{l'}_{l} u^l_{\sigma}
\]

\[
= |\Lambda x, \Lambda n; \lambda\rangle_l^{(W)} u^l_{\sigma'}[D(W(\Lambda, n))]^{\sigma'}_{\sigma}
\]

\[
= |\Lambda x, \Lambda n; \lambda\rangle_{\sigma'}[D(W(\Lambda, n))]^{\sigma'}_{\sigma},
\]

that is, \( |x, n; \lambda\rangle_{\sigma} \) transforms according to the appropriate spin representation.

Now consider a past-pointing \( n \) (\( n^2 = -1 \) and \( n^0 < 0 \)). In this case, \( -n \) is future pointing so that \( -n = L(-n)e \), or \( n = L(-n)(-e) \). Taking \( L(-n) \) to be the standard Lorentz
transformation for past-pointing \( n \), it is thus possible to construct spin states in terms of the future-pointing \(-n\). However, since the spacial part of \( n \) is also reversed in \(-n\), it is conventional to consider the spin sense reversed, too. Therefore, define

\[
v^l_{\sigma}(n) \equiv (-1)^{j+\sigma} u^l_{-\sigma}(-n),
\]

for a spin-\( j \) representation, and, for past-pointing \( n \), take

\[
| x, n; \lambda \rangle_{\sigma} = | x; \lambda \rangle_{l} v^l_{\sigma}(n).
\]

The matrices \( u^l_{\sigma} \) and \( v^l_{\sigma} \) are the same as the spin coefficient functions in Weinberg’s formalism in the context of traditional field theory \(12\) (see also Chapter 5 of \(41\)). Note that, from Eq. (31), using Eq. (27),

\[
u^l_{\sigma'}(\Lambda n)[D(W(\Lambda, n))]^\sigma_{\sigma'} = [D(W(\Lambda, n))]^\sigma_{\sigma'} u^l_{\sigma} = [L(\Lambda n)^{-1} D(\Lambda) L(n)]^\sigma_{\sigma'} u^l_{\sigma},
\]

so, using Eq. (34),

\[
u^l_{\sigma'}(\Lambda n)[D(W(\Lambda, n))]^\sigma_{\sigma'} = [D(\Lambda)]^l_{l'} u^l_{\sigma}(n).
\]

Using this with Eq. (35) gives

\[
[D(\Lambda)]^l_{l'} v^l_{\sigma}(n) = (-1)^{\sigma-\sigma'} u^l_{\sigma'}(\Lambda n)[D(W(\Lambda, n))]^{-\sigma'_{\sigma}}.
\]

Since

\[
(-1)^{\sigma-\sigma'} D(W)^{-\sigma'_{\sigma}} = [D(W)^{\sigma}_{\sigma'}]^*,
\]

(which can be derived by integrating the infinitesimal case), this gives,

\[
u^l_{\sigma'}(\Lambda n)[D(W(\Lambda, n))]^{\sigma'}_{\sigma} = [D(\Lambda)]^l_{l'} v^l_{\sigma}(n).
\]

As shown by Weinberg \(12, 41\), Eqs. (36) and (37) can be used to completely determine the \( u \) and \( v \) matrices, along with the usual relationship of the Lorentz group scalar, spinor and vector representations to the rotation group spin-0, spin-1/2 and spin-1 representations.

Since, from Eqs. (31) and (34),

\[
u^l_{\sigma'}(n)^* u^l_{\sigma}(n) = [L(n)]^l_{l'}^* (u^l_{\sigma'})^* [L(n)]^l_{l'} u^l_{\sigma} = \delta^l_{\sigma'_{\sigma}},
\]

20
Eqs. (24) and (33) give
\[ \sigma\langle x', n; \lambda|x, n; \lambda\rangle = \delta\sigma\delta^4(x' - x) \] (38)
(and similarly for past-pointing \( n \) with \( v^t\sigma \)), so that, for given \( n \) and \( \lambda \), the \( |x, n; \lambda\rangle \) form an orthogonal basis. However, for different \( \lambda \), the inner product is
\[ \sigma\langle x, n; \lambda|x_0, n; \lambda_0\rangle = \Delta\sigma(x - x_0; \lambda - \lambda_0), \] (39)
where \( \Delta\sigma(x - x_0; \lambda - \lambda_0) \) is the kernel for the rotation group. As previously argued, this should have the same form as the Euclidean kernel of Eq. (22), restricted to the rotation subgroup of \( SO(4) \). That is
\[ \Delta\sigma(x - x_0; \lambda - \lambda_0) = \delta\sigma\Delta(x - x_0; \lambda - \lambda_0). \] (40)

As in Eq. (6), the propagator is given by integrating the kernel over \( \lambda \):
\[ \Delta\sigma(x - x_0) = \delta\sigma\Delta(x - x_0), \]
where (using Eq. (14))
\[ \Delta(x - x_0) = \int_{\lambda_0}^{\infty} d\lambda \Delta(x - x_0; \lambda - \lambda_0) = -i(2\pi)^{-4} \int d^4p \frac{e^{ip\cdot(x-x_0)}}{p^2 + m^2 - i\epsilon}, \]
the usual Feynman propagator \([1]\). Defining
\[ |x, n\rangle \equiv \int_{\lambda_0}^{\infty} d\lambda |x, n; \lambda\rangle \]
then gives
\[ \sigma\langle x, n|x_0, n; \lambda_0\rangle = \Delta\sigma(x - x_0). \] (41)

Finally, we can inject the spin-representation basis states \( |x, n; \lambda\rangle \) back into the Lorentz group representation by
\[ |x, n; \lambda\rangle_l \equiv |x, n; \lambda\rangle_{\sigma}u^*_l(n), \]
(and similarly for past-pointing \( n \) with \( v_l^t\sigma \)). Substituting Eq. (33) into this gives
\[ |x, n; \lambda\rangle_l = |x; \lambda\rangle_{l'} P^{l'}_{l}(n), \] (42)
where
\[ P^{l'}_{l}(n) \equiv u^{l'}_{\sigma}(n)u_l^\sigma(n)^* = v_l^\sigma(n)v_l^\sigma(n)^* \] (43)
(the last equality following from Eq. (35)). Using Eqs. (38) and (39), the kernel for these states is

\[ \langle x, n | x_0, n; \lambda_0 \rangle_l = P_l^e (n) \Delta (x - x_0; \lambda - \lambda_0). \]

However, using Eqs. (36) and (37), it can be shown that the states \(|x, n; \lambda\rangle_l\) transform like the states \(|x; \lambda\rangle_l\):

\[ \hat{U} (\Lambda) |x, n; \lambda\rangle_l = |x, n; \lambda\rangle_{l'} \mathcal{D} (\lambda). \]

Taking

\[ |x, n\rangle_l \equiv \int_{\lambda_0}^{\infty} d\lambda |x, n; \lambda\rangle_l \]

and using Eq. (41) gives the propagator

\[ \langle x, n | x_0, n; \lambda_0 \rangle_l = P_l^e (n) \Delta (x - x_0). \] (44)

Now, the states \(|x, n; \lambda\rangle_l\) do not span the full Lorentz group Hilbert space vector bundle of the states \(|x; \lambda\rangle_l\), but they do span the subspace corresponding to the rotation subgroup. Therefore, using Eq. (42) and the idempotency of \(P_l^e (n)\) as a projection matrix,

\[ |x, n\rangle_l = \int d^4 x_0 |x_0, n; \lambda_0\rangle_l \langle x_0, n, \lambda_0 | x_0, n; \lambda_0 \rangle_l^* = \int d^4 x_0 P_l^e (n) \Delta (x - x_0)^* P_l^e (n) |x_0, n; \lambda_0\rangle_l \]

\[ = \int d^4 x_0 P_l^e (n) \Delta (x - x_0)^* |x_0; \lambda_0\rangle_l. \] (45)

VI. PARTICLES AND ANTIPARTICLES

Because of Eq. (41), the states \(|x, n\rangle_\sigma\) allow for a straightforward generalization of the treatment of particles and antiparticles from [1] to the non-scalar case. As in that treatment, consider particles to propagate from the past to the future while antiparticles propagate from the future into the past [27, 28, 45]. Therefore, postulate non-scalar particle states \(|x_+, n\rangle_\sigma\) and antiparticle states \(|x_-, n\rangle_\sigma\) as follows.

**Postulate 4.** Normal particle states \(|x_+, n\rangle_\sigma\) are such that

\[ \langle x_+, n | x_0, n; \lambda_0 \rangle_\sigma = \theta (x_0^0 - x_0^0) \Delta_{+}^\sigma (x - x_0) = \theta (x_0^0 - x_0^0) \Delta_{+}^\sigma (x - x_0) \]

and antiparticle states \(|x_-, n\rangle_\sigma\) are such that

\[ \langle x_-, n | x_0, n; \lambda_0 \rangle_\sigma = \theta (x_0^0 - x_0^0) \Delta_{-}^\sigma (x - x_0) = \theta (x_0^0 - x_0^0) \Delta_{-}^\sigma (x - x_0) \].
where \( \theta \) is the Heaviside step function, \( \theta(x) = 0 \), for \( x < 0 \), and \( \theta(x) = 1 \), for \( x > 0 \), and

\[
\Delta_{\pm}^\sigma(x - x_0) = \delta^\sigma\delta(2\pi)^{-3} \int d^3p \ (2\omega_p)^{-1} e^{i[\mp\omega_p(x^0 - x_0^0)+p\cdot(x - x_0)]},
\]

with \( \omega_p \equiv \sqrt{p^2 + m^2} \).

Note that the vector \( n \) used here is timelike but otherwise arbitrary, with no commitment that it be, e.g., future-pointing for particles and past-pointing for antiparticles.

This division into particle and antiparticle paths depends, of course, on the choice of a specific coordinate system in which to define the time coordinate. However, if we take the time limit of the end point of the path to infinity for particles and negative infinity for antiparticles, then the particle/antiparticle distinction will be coordinate system independent.

In taking this time limit, one cannot expect to hold the 3-position of the path end point constant. However, for a free particle, it is reasonable to take the particle 3-momentum as being fixed. Therefore, consider the state of a particle or antiparticle with a 3-momentum \( p \) at a certain time \( t \).

**Postulate 5.** The state of a particle (\( + \)) or antiparticle (\( - \)) with 3-momentum \( p \) is given by

\[
|t, p_{\pm}, n\rangle_\sigma \equiv (2\pi)^{-3/2} \int d^3x \ e^{i(\mp\omega_p t + p\cdot x)}|t, x_{\pm}, n\rangle_\sigma.
\]

Now, following the derivation in [1], but carrying along the spin indices, gives

\[
|t, p_+, n\rangle_\sigma = (2\omega_p)^{-1} \int_{-\infty}^t dt_0 \ |t_0, p_+, n; \lambda_0\rangle_\sigma \quad \text{and}
\]

\[
|t, p_-, n\rangle_\sigma = (2\omega_p)^{-1} \int_t^{+\infty} dt_0 \ |t_0, p_, n; \lambda_0\rangle_\sigma,
\]

where

\[
|t, p_{\pm}, n; \lambda_0\rangle_\sigma \equiv (2\pi)^{-3/2} \int d^3x \ e^{i(\mp\omega_p t + p\cdot x)}|t, x, n; \lambda_0\rangle_\sigma.
\]

Since

\[
\sigma' \langle t', p'_{\pm}, n; \lambda_0|t, p_{\pm}, n; \lambda_0\rangle_\sigma = \delta^{\sigma'}\delta(\mp(t' - t))\delta^3(p' - p),
\]

we have, from Eq. (46),

\[
\sigma' \langle t, p_{\pm}, n|t_0, p_{0\pm}, n; \lambda_0\rangle_\sigma = (2\omega_p)^{-1} \delta^{\sigma'}\delta(\pm(t - t_0))\delta^3(p - p_0).
\]

Defining the time limit particle and antiparticle states

\[
|p_{\pm}, n\rangle_\sigma \equiv \lim_{t \to \pm\infty} |t, p_{\pm}, n\rangle_\sigma,
\]

(48)
then gives
\[ \sigma' \langle p_\pm, n | t_0, p_0, n_\pm; \lambda_0 \rangle_\sigma = (2\omega_p)^{-1}\delta_{\sigma'}(p - p_0), \]
for any value of \( t_0 \).

Further, writing
\[ |t_0, p_\pm, n; \lambda_0 \rangle_\sigma = (2\pi)^{-1/2} e^{i\omega_p t_0} \int dp^0 e^{i p^0 t_0} |p, n; \lambda_0 \rangle_\sigma, \]
where
\[ |p, n; \lambda_0 \rangle_\sigma = (2\pi)^{-2} \int d^4 x e^{i p \cdot x} |x, n; \lambda_0 \rangle_\sigma \]
is the corresponding 4-momentum state, it is straightforward to see from Eq. (46) that the time limit of Eq. (48) is
\[ |p_\pm, n \rangle_\sigma \equiv \lim_{t \to \pm \infty} |t, p_\pm, n \rangle_\sigma = (2\pi)^{1/2}(2\omega_p)^{-1} |\pm \omega_p, p_\pm, n; \lambda_0 \rangle_\sigma. \]

Thus, a normal particle (+) or antiparticle (−) that has 3-momentum \( p \) as \( t \to \pm \infty \) is on-shell, with energy \( \pm \omega_p \). Such on-shell particles are unambiguously normal particles or antiparticles.

For the on-shell states \( |p_\pm, n \rangle_\sigma \), it now becomes reasonable to introduce the usual convention of taking the on-shell momentum vector as the spin vector. That is, set \( n_{p_\pm} \equiv (\pm \omega_p, p) / m \) and define
\[ |p_\pm \rangle_\sigma \equiv |p_\pm, n_{p_\pm} \rangle_\sigma \]
and
\[ |t, p_\pm \rangle_\sigma \equiv |t, p_\pm, n_{p_\pm} \rangle_\sigma, \]
so that
\[ |p_\pm \rangle_\sigma = \lim_{t \to \pm \infty} |t, p_\pm \rangle_\sigma. \]

Further, define the position states
\[ |x_+ \rangle_t \equiv (2\pi)^{-3/2} \int d^3 p e^{i (\omega_p x^0 - p \cdot x)} |x^0, p_+ \rangle_\sigma u_\sigma(n_{p_+})^* \]
and
\[ |x_- \rangle_t \equiv (2\pi)^{-3/2} \int d^3 p e^{i (\omega_p x^0 - p \cdot x)} |x^0, p_- \rangle_\sigma v_\sigma(n_{p_-})^*. \]

Then, working the previous derivation backwards gives
\[ \nu' (x_\pm | x_0; \lambda_0)_{t_1} = \theta(\pm (x^0_0 - x^0_0)) \Delta_{\pm}^{\nu'}(x - x_0), \]

24
where
\[ \Delta_\pm^\prime_i(x - x_0) \equiv (2\pi)^{-3} \int d^3p \, P'^\prime_i(n\rho) (2\omega_p)^{-1} e^{i\omega_p(x^0 - x_0^0) - p \cdot (x - x_0)} . \]

Now, it is shown in [12, 41] that the covariant non-scalar propagator
\[ \Delta'^\prime_i(x - x_0) = -i(2\pi)^{-4} \int d^4p \, P'^\prime_i(p/m) \frac{e^{ip \cdot (x - x_0)}}{p^2 + m^2 - i\varepsilon} , \]
in which \( P'^\prime_i(p/m) \) has the polynomial form of \( P'^\prime_i(n) \), but \( p \) is not constrained to be on-shell, can be decomposed into
\[ \Delta'^\prime_i(x - x_0) = \theta(x^0 - x_0^0) \Delta_+'^\prime_i(x - x_0) + \theta(x^0_0 - x^0) \Delta_-'^\prime_i(x - x_0) + Q'^\prime_i \left( -i \frac{\partial}{\partial x} \right) i\delta^4(x - x_0) , \]
where the form of \( Q'^\prime_i \) depends on any non-linearity of \( P'^\prime_i(p/m) \) in \( p^0 \). Then, defining
\[ |x\rangle_i \equiv \int d^4x_0 \Delta'^\prime_i(x - x_0)^*|x_0\rangle_0 \lambda_0 \langle \nu , \]
\( |x_+\rangle_i \) and \( |x_-\rangle_i \) can be considered as a particle/antiparticle partitioning of \( |x\rangle_i \), in a similar way as the partitioning of \( |x, n\rangle \sigma \) into \( |x, n_+\rangle \sigma \) and \( |x, n_-\rangle \sigma \):
\[ \theta(\pm(x^0 - x_0^0))^\prime_i(x|x_0; \lambda_0) = \theta(\pm(x^0 - x_0^0)) \Delta'^\prime_i(x - x_0) \]
\[ = \theta(\pm(x^0 - x_0^0)) \Delta_\pm'^\prime_i(x - x_0) \]
\[ = ^\prime_i(x_{\pm}|x_0; \lambda_0) . \]

Because of the delta function, the term in \( Q'^\prime_i \) does not contribute for \( x \neq x_0 \).

The states \( |x, n\rangle_i \) and \( |x\rangle_i \) both transform according to a representation \( D'^\prime_i \) of the Lorentz group, but it is important to distinguish between them. The \( |x, n\rangle_i \) are projections back into the Lorentz group of the states \( |x, n\rangle \sigma \) defined on the rotation subgroup, in which that subgroup is obtained by uniformly reducing the Lorentz group about the axis given by \( n \).

The \( |x\rangle_i \), on the other hand, are constructed by inverse-transforming from the momentum states \( |t, p_\pm\rangle \sigma \), with each superposed state defined over a rotation subgroup reduced along a different on-shell momentum vector.

One can further highlight the relationship of the \( |x\rangle_i \) to the momentum in the position representation by the formal equation (using Eq. (45))
\[ |x\rangle_i = \int d^4x_0 \, P'^\prime_i \left( \frac{im^{-1} \partial}{\partial x} \right) \Delta(x - x_0)^*|x_0; \lambda_0 \rangle_\nu = |x, im^{-1} \partial/\partial x_i \rangle_i = P'^\prime_i \left( \frac{im^{-1} \partial}{\partial x} \right) |x\rangle_i . \]
The \( |x\rangle_i \) correspond to the position states used in traditional relativistic quantum mechanics, with associated on-shell momentum states \( |p_\pm\rangle \). However, we will see in the next section that the states \( |x, n\rangle_i \) provide a better basis for generalizing the scalar probability interpretation discussed in [1].
VII. ON-SHELL PROBABILITY INTERPRETATION

Similarly to the scalar case [1], let $H^{(j,n)}$ be the Hilbert space of the $|x,n;\lambda_0\rangle_\sigma$ for the spin-$j$ representation of the rotation group and a specific timelike vector $n$, and let $H_t^{(j,n)}$ be the subspaces spanned by the $|t,\mathbf{x},n;\lambda_0\rangle_\sigma$, for each $t$, forming a foliation of $H^{(j,n)}$. Now, from Eq. (47), it is clear that the particle and antiparticle 3-momentum states $|t,p_\pm,n;\lambda_0\rangle_\sigma$ also span $H_t^{(j,n)}$. Using these momentum bases, states in $H_t^{(j,n)}$ have the form

$$|t,\psi_\pm,n;\lambda_0\rangle_\sigma = \int d^3p \psi^\sigma_\sigma(p)|t,p_\pm,n;\lambda_0\rangle_\sigma,$$

for matrix functions $\psi$ such that $\text{tr}(\psi^\dagger\psi)$ is integrable. Conversely, it follows from Eq. (49) that the probability amplitude $\psi^\sigma_\sigma(p)$ is given by

$$\psi^\sigma_\sigma(p) = (2\omega_p)^\sigma \langle p_\pm,n|t,\psi_\pm,n;\lambda_0\rangle_\sigma.$$

Let $H_t'^{(j,n)}$ be the space of linear functions dual to $H_t^{(j,n)}$. Via Eq. (53), the bra states $\sigma\langle p_\pm|$ can be considered as spanning subspaces $H_t'^{(j,n)}$ of the $H_t^{(j,n)}$, with states of the form

$$\sigma\langle \psi_\pm,n| = \int d^3p \psi^\sigma_\sigma(p)^* \langle p_\pm,n|.$$

The inner product

$$(\psi_1,\psi_2) \equiv \sigma\langle \psi_1\pm,n|t,\psi_2\pm,n;\lambda_0\rangle_\sigma = \int \frac{d^3p}{2\omega_p} \psi_1^\sigma_\sigma(p)^* \psi_2^\sigma_\sigma(p)$$

gives

$$(\psi,\psi) = \int \frac{d^3p}{2\omega_p} \sum_{\sigma\sigma} |\psi^\sigma_\sigma(p)|^2 \geq 0,$$

so that, with this inner product, the $H_t'^{(j,n)}$ actually are Hilbert spaces in their own right.

Further, Eq. (49) is a bi-orthonormality relation with the corresponding resolution of the identity (see [46] and App. A.8.1 of [47])

$$\int d^3p (2\omega_p)|t,p_\pm,n;\lambda_0\rangle_\sigma \sigma\langle p_\pm,n| = 1.$$

The operator $(2\omega_p)|t,p_\pm,n;\lambda_0\rangle_\sigma \sigma\langle p,n_\pm|$ represents the quantum proposition that an on-shell, non-scalar particle or antiparticle has 3-momentum $p$.

Like the $\psi_{\mathbf{l}}'$ discussed in Sec. [IV] for the Lorentz group, the $\psi^\sigma_\sigma$ form an algebra over the rotation group with components $\psi(p,B)$, where $B^\sigma_\sigma$ is a member of the appropriate representation of the rotation group, such that

$$\psi^\sigma_\sigma(p) = \int d^3B B^\sigma_\sigma \psi(p,B),$$

(54)
with the integration taken over the 3-dimensional rotation group. Unlike the Lorentz group, however, components can also be reconstructed from the $\psi_{\sigma'}_{\sigma}(p)$ by

$$
\psi(p, B) = \beta^{-1}(B^{-1})_{\sigma'}_{\sigma}\psi_{\sigma'}_{\sigma}(p)
$$

(55)

where

$$
\beta \equiv \frac{1}{2j + 1} \int d^3B,
$$

for a spin-$j$ representation, is finite because the rotation group is closed. Plugging Eq. (55) into the right side of Eq. (54) and evaluating the integral does, indeed, give $\psi_{\sigma'}_{\sigma}(p)$, as required, because of the orthogonality property

$$
\int d^3B B^{\sigma'}_{\bar{\sigma}}(B^{-1})_{\sigma'}_{\bar{\sigma}} = \beta \delta^{\sigma'}_{\bar{\sigma}} \delta_{\sigma \bar{\sigma}}
$$

(see [40], Section 11). We can now adjust the group volume measure $d^3B$ so that $\beta = 1$.

The set of all $\psi(p, B)$ constructed as in Eq. (55) forms a subalgebra such that each $\psi(p, B)$ is uniquely determined by the corresponding $\psi_{\sigma'}_{\sigma}(p)$ (see [40], pages 167ff). We can then take $|\psi(p, B)|^2 = |(B^{-1})_{\sigma'}_{\sigma}\psi_{\sigma'}_{\sigma}(p)|^2$ to be the probability density for the particle or antiparticle to have 3-momentum $p$ and to be rotated as given by $B$ about the axis given by the spacial part of the unit timelike 4-vector $n$. The probability density for the particle or antiparticle in 3-momentum space is

$$
\int d^3B |\psi(p, B)|^2 = \psi_{\sigma'}_{\sigma}(p)\psi_{\sigma'}_{\sigma}(p)^*
$$

with the normalization

$$
(\psi, \psi) = \int \frac{d^3p}{2\omega_p} \psi_{\sigma'}_{\sigma}(p)^\ast \psi_{\sigma'}_{\sigma}(p) = 1.
$$

Next, consider that $|t, x, n; \lambda_0\rangle_{\sigma}$ is an eigenstate of the three-position operator $\hat{X}$, representing a particle localized at the three-position $x$ at time $t$. From Eq. (53), and using the inverse Fourier transform of Eq. (50) with Eq. (51), its three momentum wave function is

$$
(2\omega_p)^{\sigma'}\langle p_{\pm}, n|t, x; \lambda_0\rangle_{\sigma} = (2\pi)^{-3/2}\delta^{\sigma'}_{\sigma}e^{i(\pm\omega_p t - p\cdot x)}.
$$

(56)

This is just a plane wave, and it is an eigenfunction of the operator

$$
e^{\pm i\omega_p t} \frac{\partial}{\partial p} e^{\mp i\omega_p t},
$$
which acts as the identity on the spin indices and is otherwise the traditional momentum representation $i \partial / \partial p$ of the three-position operator $\hat{X}$, translated to time $t$.

This result exactly parallels that of the scalar case [1]. Note that this is only so because of the use of the independent vector $n$ for reduction to the rotation group, rather than the traditional approach of using the three-momentum vector $p$. Indeed, it is not even possible to define a spin-indexed position eigenstate in the traditional approach, because, of course, the momentum is not sharply defined for such a state [23, 24].

On the other hand, consider the three-position states $|x \pm \rangle_t$ introduced at the end of Sec. VII. Even though these are Lorentz-indexed, they only span the rotation subgroup. Therefore, we can form their three-momentum wave functions in the $\sigma(p \pm)$ bases. Using Eqs. (52) and (49),

$$(2 \omega_p)^\sigma(p \pm |x \pm \rangle_t) = (2 \pi)^{-3/2} w_t^\sigma(n_p)^* e^{i(\pm \omega_p t - p \cdot x)}. \tag{57}$$

At $t = 0$, up to normalization factors of powers of $(2 \omega_p)$, this is just the Newton-Wigner wave function for a localized particle of non-zero spin [43]. It is an eigenfunction of the position operator represented as

$$w_t \sigma'(n_p)^* e^{i \omega_p t} \frac{\partial}{\partial p} e^{-i \omega_p t} w_t' \sigma'(n_p) \tag{58}$$

for the particle case, with a similar expression using $v_t$ in the antiparticle case. Other than the time translation, this is essentially the Newton-Wigner position operator for non-zero spin [43].

Note that Eq. (56) is effectively related to Eq. (57) by a generalized Foldy-Wouthuysen transformation [48, 49]. However, in the present approach it is Eq. (56) that is seen to be the primary result, with a natural separation of particle and antiparticle states and a reasonable non-relativistic limit, just as in the scalar case [1].

VIII. INTERACTIONS

It is now straightforward to extend the formalism to multiparticle states and introduce interactions, quite analogously to the scalar case [1]. In order to allow for multiparticle states with different types of particles, extend the position state of each individual particle with a particle type index $\nu$, such that

$$\langle x', \nu' ; \lambda | x, \nu ; \lambda \rangle_t = \delta_t^\nu \delta_t^{\nu'} \delta^4(x' - x).$$
Then, construct a basis for the Fock space of multiparticle states as symmetrized/antisymmetrized products of $N$ single particle states:

$$|x_1, \nu_1, \lambda_1; \ldots; x_N, \nu_N, \lambda_N \rangle_{t_1 \ldots t_N} \equiv (N!)^{-1/2} \sum_{\text{perms } P} \delta_P |x_{P_1}, \nu_{P_1}; \lambda_{P_1} \rangle_{t_{P_1}} \cdots$$

where the sum is over permutations $P$ of $1, \ldots, N$, and $\delta_P$ is +1 for permutations with an even number of interchanges of fermions and −1 for an odd number of interchanges.

Define multiparticle states $|x_1, \nu_1; \ldots; x_N, \nu_N \rangle_{t_1 \ldots t_N}$ as similarly symmetrized/antisymmetrized products of $|x \rangle_1$ states. Then,

$$|x'_{t_1} \ldots t_N \rangle_{x'N} \equiv \sum_{\text{ perms } P} \delta_P \prod_{i=1}^{N} \delta_{P_i} \Delta_{P_i}^{x'}(x'_P - x_i),$$

where each propagator is also implicitly a function of the mass of the appropriate type of particle. Note that the use of the same parameter value $\lambda_0$ for the starting point of each particle path is simply a matter of convenience. The intrinsic length of each particle path is still integrated over separately in $|x_1, \nu_1; \ldots; x_N, \nu_N \rangle_{t_1 \ldots t_N}$, which is important for obtaining the proper particle propagator factors in Eq. (59). Nevertheless, by using $\lambda_0$ as a common starting parameter, we can adopt a similar notation simplification as in [1], defining

$$|x_1, \nu_1; \ldots; x_N, \nu_N; \lambda_0 \rangle_{t_1 \ldots t_N} \equiv |x_1, \nu_1, \lambda_0; \ldots; x_N, \nu_N, \lambda_0 \rangle_{t_1 \ldots t_N}.$$ 

It is also convenient to introduce the formalism of creation and annihilation fields for these multiparticle states. Specifically, define the creation field $\hat{\psi}^\dagger_I(x, \nu; \lambda)$ by

$$\hat{\psi}^\dagger_I(x, \nu; \lambda)|x_1, \nu_1, \lambda_1; \ldots; x_N, \nu_N, \lambda_N \rangle_{t_1 \ldots t_N} = |x, \nu, \lambda; x_1, \nu_1, \lambda_1; \ldots; x_N, \nu_N, \lambda_N \rangle_{t_1 \ldots t_N}$$

with the corresponding annihilation field $\hat{\psi}^I(x, \nu; \lambda)$ having the commutation relation

$$[\hat{\psi}^\dagger (x', \nu' ; \lambda), \hat{\psi}^I (x, \nu ; \lambda_0)]_+ = \delta_{\nu'} \Delta^I_{\nu}(x' - x ; \lambda - \lambda_0),$$

where the upper − is for bosons and the lower + is for fermions. Further define

$$\hat{\psi}^I (x, \nu) \equiv \int_{\lambda_0}^{\infty} d\lambda \hat{\psi}^I (x, \nu ; \lambda),$$

so that

$$[\hat{\psi}^\dagger (x', \nu' ; \lambda), \hat{\psi}^I (x, \nu ; \lambda_0)]_+ = \delta_{\nu'} \Delta^I_{\nu}(x' - x),$$

29
which is consistent with the multi-particle inner product as given in Eq. (59). Finally, as in [1], define a special adjoint \( \hat{\psi}^\dagger \) by

\[
\hat{\psi}^\dagger_{\lambda}(x, \nu) = \hat{\psi}^\dagger_{\lambda}(x, \nu; \nu_0) \quad \text{and} \quad \hat{\psi}^\dagger_{\lambda}(x, \nu; \lambda_0) = \hat{\psi}^\dagger_{\lambda}(x, \nu),
\]

which allows the commutation relation to be expressed in the more symmetric form

\[
[\hat{\psi}^\dagger_{\nu}(x', \nu'), \hat{\psi}^\dagger_{\lambda}(x, \nu)]_\mp = \delta_{\nu'}^\nu \Delta^\nu_l(x' - x).
\]

We can now readily generalize the postulated interaction vertex operator of [1] to the non-scalar case.

**Postulate 6.** An interaction vertex, possibly occurring at any position in spacetime, with some number \( a \) of incoming particles and some number \( b \) of outgoing particles, is represented by the operator

\[
\hat{V} \equiv g_{\nu_1 \cdots \nu_a t_1 \cdots t_b} \int d^4x \prod_{i=1}^{a} \hat{\psi}^\dagger_{\nu_i}(x, \nu_i) \prod_{j=1}^{b} \hat{\psi}^{t_j}(x, \nu_j),
\]

where the coefficients \( g_{\nu_1 \cdots \nu_a t_1 \cdots t_b} \) represent the relative probability amplitudes of various combinations of indices in the interaction and \( \hat{\psi}^\dagger \) is the special adjoint defined in Eq. (60).

Given a vertex operator defined as in Eq. (61), the interacting transition amplitude, with any number of intermediate interactions, is then

\[
G(x_1', \nu_1'; \ldots; x_{N'}, \nu_{N'}; x_1, \nu_1; \ldots; x_N, \nu_N)_{\nu_1' \cdots \nu_{N'} t_1 \cdots t_N} = \langle x_1, \nu_1; \ldots; x_N, \nu_N | \hat{G} | x_1, \nu_1; \ldots; x_{N'}, \nu_{N'}; \lambda_0 \rangle_{t_1 \cdots t_N},
\]

where

\[
\hat{G} \equiv \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \hat{V}^m = e^{-i\hat{V}}.
\]

Each term in this sum gives the amplitude for \( m \) interactions, represented by \( m \) applications of \( \hat{V} \). The \((m!)^{-1}\) factor accounts for all possible permutations of the \( m \) identical factors of \( \hat{V} \).

Clearly, we can also construct on-shell multiparticle states \( |p_1^\pm, \nu_1'; \ldots; p_{N'}^\pm, \nu_{N'}; \sigma_1, \ldots, \sigma_{N'} \rangle \) and \( |t_1, p_1^\pm, \nu_1; \ldots; t_N, p_N^\pm, \nu_N; \lambda_0, \sigma_1, \ldots, \sigma_N \rangle \) from the on-shell particle and antiparticle states \( |p_\pm, \sigma \rangle \) and \( |t, p_\pm, \lambda_0 \rangle \). Using these with the operator \( \hat{G} \):

\[
G(p_1^\pm, \nu_1'; \ldots; p_{N'}^\pm, \nu_{N'}; p_1^\pm, \nu_1; \ldots; p_N^\pm, \nu_N)_{\sigma_1, \ldots, \sigma_{N'}} = \prod_{i=1}^{N'} 2\omega_{p_i} \langle p_1^\pm, \nu_1'; \ldots; p_{N'}^\pm, \nu_{N'}; \hat{G} | t_1, p_1^\pm, \nu_1; \ldots; t_N, p_N^\pm, \nu_N; \lambda_0 \rangle_{\sigma_1, \ldots, \sigma_N},
\]
results in a sum of Feynman diagrams with the given momenta on external legs. Note that use of the on-shell states requires specifically identifying external lines as particles and antiparticles. For each incoming and outgoing particle, + is chosen if it is a normal particle and − if it is an antiparticle. (Note that “incoming” and “outgoing” here are in terms of the path evolution parameter \( \lambda \), not time.)

The inner products of the on-shell states for individual incoming and outgoing particles with the off-shell states for interaction vertices give the proper factors for the external lines of a Feynman diagram. For example, the on-shell state \(|p'\rangle_\sigma\) is obtained in the \(+\infty\) time limit and thus represents a final (i.e., outgoing in time) particle. If the external line for this particle starts at an interaction vertex \(x\), then the line contributes a factor

\[
(2\omega_{p'})^\sigma' \langle p'\mid x; \lambda_0 \rangle_t = (2\pi)^{-3/2}e^{i(+\omega_{p'}x^0-p'\cdot x)}u_{l'}(p')^*.
\]

For an incoming particle on an external line ending at an interaction vertex \(x'\), the factor for this line is (assuming \(x'^0 > t\))

\[
(2\omega_p)^\nu \langle x'\mid t, p_+; \lambda_0 \rangle_\sigma = (2\pi)^{-3/2}e^{i(-\omega_p x'^0+p\cdot x')}u_{l'}(p).
\]

Note that this expression is independent of \(t\), so we can take \(t \rightarrow -\infty\) and treat the particle as \textit{initial} (i.e., incoming in time). The factors for antiparticles are similar, but with the time sense reversed. Thus, the effect is to remove the propagator factors from external lines, exactly in the sense of the usual LSZ reduction \[50\].

Now, the formulation of Eq. \[63\] is still not that of the usual scattering matrix, since the incoming state involves initial particles but final antiparticles, and vice versa for the outgoing state. To construct the usual scattering matrix, it is necessary to have multi-particle states that involve either all initial particles and antiparticles (that is, they are composed of individual asymptotic particle states that are all consistently for \(t \rightarrow -\infty\)) or all final particles and antiparticles (with individual asymptotic states all for \(t \rightarrow +\infty\)). The result is a formulation in terms of the more familiar scattering operator \(\hat{S}\), which can be expanded in a Dyson series in terms of a time-dependent version \(\hat{V}(t)\) of the interaction operator. The procedure for doing this is exactly analogous to the scalar case. For details see \[1\].
IX. CONCLUSION

The extension made here of the scalar spacetime path approach \cite{1} begins with the argument in Sec. II on the form of the path propagator based on Poincaré invariance. This motivates the use of a path integral over the Poincaré group, with both position and Lorentz group variables, for computation of the non-scalar propagator. Once the difficulty with the non-compactness of the Lorentz group is overcome, the development for the non-scalar case is remarkably parallel to the scalar case.

A natural further generalization of the approach, particularly given its potential application to quantum gravity and cosmology, would be to consider paths in curved spacetime. Of course, in this case it is not in general possible to construct a family of parallel paths over the entire spacetime, as was done in Sec. III. Nevertheless, it is still possible to consider infinitesimal variations along a path corresponding to arbitrary coordinate transformations. And one can certainly construct a family of “parallel” paths at least over any one coordinate patch on the spacetime manifold. The implications of this for piecing together a complete path integral will be explored in future work.

Another direction for generalization is to consider massless particles, leading to a complete spacetime path formulation for Quantum Electrodynamics. However, as has been shown in previous work on relativistically parametrized approaches to QED (e.g., \cite{51}), the resulting gauge symmetries need to be handled carefully. This will likely be even more so if consideration is further extended to non-Abelian interactions. Nevertheless, the spacetime path approach may provide some interesting opportunities for addressing renormalization issues in these cases \cite{1}.

In any case, the present paper shows that the formalism proposed in \cite{1} can naturally include non-scalar particles. This is, of course, critical if the approach is to be given the foundational status considered in \cite{1} and the cosmological interpretation discussed in \cite{2}.
APPENDIX: EVALUATION OF THE SO(4) PATH INTEGRAL

Proposition. Consider the path integral

\[ \Delta(\Lambda_E \Lambda_{E0}^{-1}; \lambda - \lambda_0) = \zeta_E \int D^6 M_E \delta^6(M_E(\lambda) \Lambda_E^{-1} - I) \delta^6(M_E(\lambda_0) \Lambda_{E0}^{-1} - I) \]

\[ \exp \left[ i \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{2} \text{tr}(\Omega_E(\lambda') \Omega_E(\lambda')^T) \right] \]

over the six dimensional group SO(4) ~ SU(2) × SU(2), where Ω_E(λ') is the element of the Lie algebra so(4) tangent to the path M_E(λ) at λ'. This path integral may be evaluated to get

\[ \Delta(\Lambda_E \Lambda_{E0}^{-1}; \lambda - \lambda_0) = \sum_{\ell_A, \ell_B} \exp^{-i(\Delta m^2_A + \Delta m^2_B)(\lambda - \lambda_0)} (2\ell_A + 1)(2\ell_B + 1) \chi^{(\ell_A, \ell_B)} (\Lambda_E \Lambda_{E0}^{-1}), \quad (A.1) \]

where the summation over \( \ell_A \) and \( \ell_B \) is from 0 to \( \infty \) in steps of 1/2, \( \Delta m^2 = \ell(\ell + 1) \) and \( \chi^{(\ell_A, \ell_B)} \) is the group character for the \((\ell_A, \ell_B)\) SU(2) × SU(2) group representation.

Proof. Parametrize a group element \( M_E \) by a six-vector \( \theta \) such that

\[ M_E = \exp \left( \sum_{i=1}^{6} \theta_i J_i \right), \]

where the \( J_i \) are so(4) generators for SO(4). Then \( \text{tr}(\Omega_E \Omega_E^T) = \dot{\theta}^2 \), where the dot denotes differentiation with respect to \( \lambda \). Dividing the six generators \( J_i \) into two sets of three SU(2) generators, the six-vector \( \theta \) may be divided into two three-vectors \( \theta_A \) and \( \theta_B \), parametrizing the two SU(2) subgroups. The path integral then factors into two path integrals over SU(2):

\[ \Delta(\Lambda_E \Lambda_{E0}^{-1}; \lambda - \lambda_0) = \zeta_E^{1/2} \int D^3 W_A \delta^3(W_A(\lambda) B_A^{-1} - I) \delta^6(W_A(\lambda_0) B_{A0}^{-1} - I) \exp \left[ i \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{2} \dot{\theta}_A^2 \right] \]

\[ \times \zeta_E^{1/2} \int D^3 W_B \delta^3(W_B(\lambda) B_B^{-1} - I) \delta^6(W_B(\lambda_0) B_{B0}^{-1} - I) \exp \left[ i \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{2} \dot{\theta}_B^2 \right], \]

where \( \Lambda_E = B_A \otimes B_B \) and \( \Lambda_{E0} = B_{A0} \otimes B_{B0} \).

The SU(2) path integrals may be computed by expanding the exponential in group
characters \[20, 52\]. The result is

\[
\zeta_E^{1/2} \int D^3 W \delta^3(W(\lambda)B^{-1} - I)\delta^6(W(\lambda_0)B_0^{-1} - I) \exp \left[ i \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{2} \dot{\theta}^2 \right] = \sum_{\ell} \exp^{-i\Delta m_0^2(\lambda-\lambda_0)(2\ell + 1)} \chi(\ell) (BB_0^{-1}) \, \tag{A.2}
\]

where \(\chi(\ell)\) is the character for the spin-\(\ell\) representation of \(SU(2)\) and the result includes the correction for integration “on” the group space, as given by Kleinert \[52\]. The full \(SO(4)\) path integral is then given by the product of the two factors of the form Eq. \(\tag{A.1}\), which is just Eq. \(\tag{A.1}\), since \[40\]

\[
\chi(\ell_{A, \ell_B}) (\Lambda E \Lambda_E^{-1}) = \chi(\ell_A) (B_A B_A^{-1}) \chi(\ell_B) (B_B B_B^{-1}).
\]

\[\square\]

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