Interacting Topological Superconductors and possible Origin of 16n Chiral Fermions in the Standard Model

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Motivated by the observation that the Standard Model of particle physics (plus a right-handed neutrino) has precisely 16 Weyl fermions per generation, we search for (3 + 1)-dimensional chiral fermionic theories and chiral gauge theories that can be regularized on a 3 dimensional spatial lattice when and only when the number of flavors is an integral multiple of 16. All these results are based on the observation that local interactions reduce the classification of certain (4 + 1)-dimensional topological superconductors from \(Z\) to \(Z_{\alpha}\), which means that one of their (3 + 1)-dimensional boundaries can be gapped out by interactions without breaking any symmetry when and only when the number of boundary chiral fermions is an integral multiple of 16.

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In the Standard Model (SM) of particle physics \[27\], when the energy scale is higher than the vacuum expectation value (VEV) of the Higgs field, \(v = 246\) GeV, effectively there are in total 16 massless left-handed chiral (Weyl) fermions in each generation:

\[
H = \int d^3x \sum_{n=1}^{16} \sum_{a=1}^{16} \psi_{a,n}^\dagger i\mathbf{\sigma} \cdot \mathbf{\bar{\sigma}} \psi_{a,n}.
\]

For example, in the first generation \((n = 1)\) there are left chiral fermions \((u_d, d_u)_L, (u_d^c, d_u^c)_R, (e, \nu)_L, (\bar{e}, \bar{\nu})_R\) \[28\], where \(a = 1, 2, 3\) is the color index. In the SO(10) Grand Unified Theory (GUT) \[2, 3\], these fermions couple to the SO(10) gauge field as a 16 dimensional irreducible spinor representation of the SO(10) gauge group, which we will denote by \(\psi \sim 16_L\). Since both the SM and the SO(10) GUT have no known gauge anomaly, it is expected that they can be regularized as a full quantum system on a three dimensional spatial lattice. However, it is well-known that if we want to write down a lattice model for the 16 left fermions in the SM with chiral gauge couplings, we will inevitably also obtain “mirror” fermions, \(\psi' \sim 16_R\) \((16\) right fermions) in the low-energy effective field theory that arises from the same lattice model \[4, 5\].

In order to get around the fermi doubling theorem, one method is to realize the SM on the 3d boundary of a 4d topological insulator (TI) or topological superconductor (TSC) \[6, 7\]. Then the 16 mirror right Weyl fermions will be localized on the other opposite boundary, which is spatially separated from the SM. Fermions at each boundary can naturally have a chiral coupling to the bulk gauge fields. However, this method requires subtle adjustment of the scale of the fourth dimension: if the fourth dimension is too large, the gauge boson in the bulk will be gapless and interfere with the low energy physics of the boundary; on the other hand if the fourth dimension is too small, then the SM suffers from interference with its mirror sector on the other boundary \[8\].

It would be ideal if we can gap out the mirror sector without affecting the left fermions in the SM. Then we can regularize the SM on the 3d boundary of a 4d TI or TSC with a very thin fourth dimension (which makes the bulk generically a 3d system). However, if the mirror sector is gapped out in the standard way, namely they are gapped out by condensing a boson field that couples to the Majorana mass (Cooper pair mass) of the right fermions, \(\psi^a_R i\sigma^y \psi^b_R\), then the same boson field would couple to the left fermions and gap them out as well. Thus we seek a new mechanism to gap out the 16 right fermions with interactions, such that the mirror fermions are gapped while having zero bilinear expectation value, \(\langle \psi^a_R i\sigma^y \psi^b_R \rangle = 0\), for arbitrary flavor indices \(a, b = 1, ..., 16\). We will now drop the primes for notational convenience, and it should be understood that when we attempt to gap out the fermions \(\psi_a\) we intend to gap out the mirror sector while leaving the ordinary fermions gapless.

At this point we would like to make clear that we do not yet have a lattice regularization for the full SM. Nevertheless, we do provide explicit examples of non-abelian chiral gauge theories where the above condition is satisfied if and only if they have 16n Weyl fermions, and we hope that this novel result strongly re-invigorates the search for a lattice regularization of the SM.

The new mechanism described in the previous paragraph becomes possible if the bulk 4d state satisfies the following criteria:

1. The interacting bulk system should not have the global charge \(U(1)\) symmetry \(\psi_a \rightarrow e^{i\theta} \psi_a\), since this symmetry is anomalous once the boundary chiral fermions are coupled to the gauge fields in the SM. Thus according to the anomaly matching condition \[10, 11\], if the boundary chiral fermions can be gapped by interaction, this system necessarily breaks this \(U(1)\) symmetry. This implies that the bulk system is a topological superconductor instead of topological insulator.

2. Each 3d boundary of the 4d TSC has \(k\) chiral Weyl fermions \((k\) is a divisor of 16), and the 4d TSC has a
Since time-reversal is antiunitary, this symmetry guarantees that no fermion bilinear term is allowed at the boundary, i.e. all terms of the form $i\bar{\gamma}_a\gamma_b$ are odd under $Z_2^T$. Thus without interaction, the boundary of this 1$d$ TSC is always degenerate, although the bulk is gapped and nondegenerate. However, Fidkowski and Kitaev demonstrated that a time-reversal invariant four-fermion interaction can gap out the 0$d$ boundary without spontaneously breaking the $Z_2^T$ (the 0$d$ boundary is gapped with $\langle i\bar{\gamma}_a\gamma_b \rangle = 0$), when and only when the system has 8$n$ copies of such a 1$d$ TSC. This implies that the 1$d$ time-reversal symmetry protected TSC has only $Z_8$ classification under interaction.

The work in Ref. [13] [16] was soon generalized to 2$d$ TSC with a 1$d$ boundary [17] [20]. For example, Ref. [17] studied a 2$d$ TSC with both $Z_2$ and time-reversal-symmetry, whose gapped bulk is simply a $p \pm ip$ TSC with $p_x + ip_y$ pairing for spin-up fermion $c_1$, and $p_x - ip_y$ pairing for spin-down fermion $c_4$. The 1$d$ boundary of this TSC has a 1$d$ nonchiral Majorana fermion with Hamiltonian:

$$H = \int dx \left( \chi_L i\partial_x \chi_L - \chi_R i\partial_x \chi_R \right).$$

(3)

$\chi_L$ and $\chi_R$ are Bogoliubov quasiparticles of $c_1$ and $c_4$, respectively. On the 1$d$ boundary the $Z_2$ and $Z_2^T$ transformations act as the following:

$$Z_2 : \chi_L \rightarrow \chi_L, \quad \chi_R \rightarrow -\chi_R,$$

$$Z_2^T : \chi_L \rightarrow \chi_R, \quad \chi_R \rightarrow \chi_L.$$  

(4)

With these symmetries, it is straightforward to verify that for arbitrary numbers of the boundary Eq. (3) any fermion bilinear mass term is forbidden. For example, $\bar{\chi}\chi = 2i\chi_L\chi_R$ is forbidden by the $Z_2$ symmetry. Ref. [17] showed that although all the fermion bilinear mass terms are forbidden by symmetry at the boundary, when there are 8$n$ copies of this $p \pm ip$ TSC, a particular four fermion interaction term which preserves both $Z_2$ and $Z_2^T$ will still gap out the boundary without degeneracy, namely the 1$d$ boundary is gapped with $\langle \bar{\chi}_a\chi_b \rangle = 0$ for arbitrary flavor index $a, b$.

Ref. [21] [22] studied the classification of 3$d$ TI/TSC under interaction. For example, Ref. [22] studied a 3$d$ TI with U(1) and time-reversal symmetry whose 2$d$ boundary is described by the Hamiltonian

$$H = \int d^2x \psi^\dagger (i\sigma^x \partial_x + i\sigma^y \partial_y) \psi.$$  

(5)

The U(1) and time-reversal symmetry act as:

$$U(1) : \psi \rightarrow e^{i\theta} \psi, \quad Z_2^T : \psi \rightarrow \sigma^y \psi^\dagger,$$

(6)

which forbid all the fermion bilinear mass terms at the 2$d$ boundary for arbitrary copies of the system, namely the classification at the noninteracting level is $Z$. Nevertheless, Ref. [22] argued that a U(1) and $Z_2^T$ invariant
short range interaction reduces the classification of this
3d TI to Z_8, namely when there are 8n copies of Eq. 4
at the 2d boundary, interaction can gap out the boundary
without spontaneously generating any fermion bilinear mass term. Ref. 22 argued that many 3d TIs and
TSCs with different symmetries have similar interaction-reduced classifications, and it is almost universally true
that when the 2d boundary has 16 2d Majorana fermions,
the boundary can be trivially gapped out by interaction.

In this work we will generalize the works summarized
above to four spatial dimensions. Before making con-
nection to the SM, we will first study a simple example of
4d TSC whose classification is reduced by interaction.
In particular we will study a 4d TSC whose boundary
contains two flavors of (3 + 1)d chiral fermions:

\[
H = \int d^3x \sum_{a=1}^2 \psi_a^\dagger (i\sigma \cdot \hat{\nabla}) \psi_a. \tag{7}
\]

We define the following U(1), Z_2 and time-reversal sym-
metry on \(\psi_a\):

\[
U(1) : \psi_a \rightarrow [e^{i\pi y}]_{ab} \psi_b,
\]

\[
Z_2 : \psi_a \rightarrow (\tau^y)_{ab} \psi_b,
\]

\[
Z_T^2 : \psi_a \rightarrow K \sigma^y (\tau^y)_{ab} \psi_b, \tag{8}
\]

where \(K\) is a complex conjugation. For the bulk state,
we can use the same bulk band structure introduced for
the 4d quantum Hall state in Ref. 23, 24:

\[
H = \sum_{a=1}^2 \sum_k \psi_{k,a}^\dagger \left( \sum_{i=1}^4 \Gamma_i \sin(k_i) \right) \psi_{k,a} + m \psi_{k,a}^\dagger \Gamma_5 \left( \sum_{i=1}^4 \cos(k_i) - 3 \right) \psi_{k,a}. \tag{9}
\]

Where \(\Gamma_{1,2,3} = \sigma^{1,2,3} \otimes \rho^3, \Gamma_4 = 1_{2 \times 2} \otimes \rho^2, \Gamma_5 = 1_{2 \times 2} \otimes \rho^2,\) where \(\rho^2\) are another set of Pauli matrices. The 3d
boundary of this theory has precisely two flavors of chiral
fermions Eq. 4.

As long as we preserve the \(U(1) \times Z_2 \times Z_T^2\) symmetry,
the 3d boundary can never be gapped without interaction
for arbitrary copies of this system, because the only
fermion bilinear mass terms that can gap out the bound-
ary are the Cooper pair operators: \(\psi_a^\dagger \sigma^\dagger \psi_b + H.c.\) which
inherently break at least one of the symmetries. Thus this
4d TSC has a Z classification with the \(U(1) \times Z_2 \times Z_T^2\)
symmetry, at the noninteracting level.

In the following we will argue that short range interac-
tions can reduce the classification of this 4d TSC to Z_8:
local four-fermion interactions can gap out 8 copies of
Eq. 4 (i.e. 16 chiral fermions at the 3d boundary) without
generating a nonzero expectation value for any fermion
bilinear mass operator. Since a weak short range interac-
tion is irrelevant, the interaction has to have a magnitude
larger than some nonzero critical value in order to suc-
cessfully gap out the fermions.

Directly studying strong four-fermion interactions is
difficult, so we will follow the same logic as in Ref. 22:
we will first manually break a subgroup of the \(U(1) \times Z_2 \times
Z_T^2\) symmetry by condensing an order parameter that
transforms nontrivially under these symmetries. Then
we will proliferate the defects of the condensate to restore
the broken symmetry. After proliferating the defects, the
order parameter becomes disordered and can be safely
integrated out. This generates an effective four-fermion interaction at low energy.

The nature of the phase after proliferating the defects
depends on the quantum numbers and spectrum of the
defects. The desired fully symmetric, gapped and non-
degareate state is only possible when the defects in the
condensate have a trivial spectrum. We will analyze three
different types of order parameters and defects, and all
these defects suggest that a symmetric, fully gapped, and
nondegenerate boundary state is only possible when there
are 16 chiral fermions at the 3d boundary.

Let us first spontaneously break the U(1) symmetry by
condensing an O(2) “superfluid” order parameter at the
3d boundary:

\[
\tilde{\phi} = (\text{Re}[\psi^\dagger \sigma^y \tau^x \psi], \text{Re}[\psi^\dagger \sigma^y \tau^z \psi]). \tag{10}
\]

This superfluid order parameter gaps out the chiral
fermions and breaks the U(1) symmetry, but preserves
the Z_2 and Z_T^2 symmetries in Eq. 8. The broken U(1)
symmetry can be restored by proliferating the vortex lines of the O(2) order parameter in Eq. 10 (Fig. 2).
Proliferation of the vortex line can be systematically de-
scribed in the dual formalism. However, we have to be
careful with the core of the vortex line, since it is the sin-
gularity of the O(2) order parameter, and the fermions
may become gapless along the vortex line. In our current
case, the vortex line of this O(2) order parameter traps
1d nonchiral Majorana fermion modes that are localized
at the vortex line. Upon solving the Dirac equation in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Illustration of topological defects: (a) monopole, (b)
vortex line, (c) domain wall. Without interaction, all these
defects are nontrivial, i.e. they have degenerate/gapless spec-
tra. However, interactions make all these defects gapped
and nondegenerate, thus after proliferating these defects, the
3d boundary enters a symmetric, fully gapped and nondege-
nerate phase.}
\end{figure}
the vortex background, we find that these modes are described by the Hamiltonian of Eq. 3 and their transformation properties under the residual $Z_2$ and $Z_2^T$ symmetries are precisely those given in Eq. 4. Thus as we already argued, without interaction, this 1d system cannot be gapped without degeneracy, for arbitrary copies of this system.

However, based on the results in Ref. 17–20, we know that for 8 copies of this system, a $Z_2$ and $Z_2^T$ invariant short range interaction at the vortex line can gap out these 1d Majorana modes without degeneracy, i.e. without spontaneously breaking the $Z_2$ and $Z_2^T$ symmetry. Thus when and only when the 3d boundary has 16 chiral fermions can we gap out the boundary without generating a fermion bilinear mass term.

We can also analyze other different types of defects of the system. For example, we can temporarily break the $Z_2$ symmetry in Eq. 8 by condensing the Ising order parameter

$$\phi = \text{Im}[\psi^d \sigma^y \psi],$$

(11)

which only breaks the $Z_2$ symmetry but preserves the $U(1)$ and $Z_2^T$. The domain wall of the $Z_2$ order is a 2d manifold (Fig. 2b), and presumably proliferating the domain wall can restore the $Z_2$ symmetry. However, just like the previous paragraph, the $Z_2$ domain wall may carry gapless fermion modes. By solving the Dirac equation at the domain wall directly, we can see that there are two flavors of gapless 2d Majorana fermions described by Eq. 3 that are localized at the domain wall. These 2d Majorana fermions have the residual $U(1) \times Z_2^T$ symmetry, which act on the domain wall states precisely in the same way as Eq. 6. And as long as the $U(1) \times Z_2^T$ symmetry is preserved, all the fermion bilinear mass terms at the domain wall are forbidden, for arbitrary copies of the systems. Thus without interaction at the domain wall, proliferation of the 2d domain wall will not lead to a trivially gapped phase. However, it was demonstrated in Ref. 22 that 8 copies of Eq. 6 (i.e. 8 copies of our current system) with $U(1) \times Z_2^T$ symmetry can be gapped out without degeneracy by local interactions. Thus once again a 3d system with 16 chiral fermions becomes special: when there are 16 chiral fermions at the 3d boundary, we can obtain a fully gapped nondegenerate state by proliferating the $Z_2$ domain walls.

The third type of scenario we consider is an ordered phase at the 3d boundary that breaks both $U(1)$ and $Z_2$ symmetry, while keeping the $Z_2^T$ symmetry unbroken. In this phase both the O(2) vector and $Z_2$ order parameters in the previous two scenarios condense. The defect that can restore all the symmetries after its proliferation is a “hedgehog” like monopole (Fig. 2b), which is a vortex line penetrating the $Z_2$ domain wall. Based on the results in Ref. 25, there is a Majorana fermion zero mode at the core of the hedgehog monopole, which transforms trivially under time-reversal symmetry, $Z_2^T : \gamma \rightarrow \gamma$. Based on the results in Ref. 13–16, we know that with 8 copies of the system Eq. 7 a local interaction that preserves the $Z_2^T$ interaction can gap out all the 8 Majorana zero modes at the monopole without degeneracy. Then condensing the monopoles can restore all the symmetries and lead to a fully gapped and nondegenerate state at the 3d boundary.

In the analysis above we argued that 8 flavors of the 4d TSC with 16 chiral fermions at its 3d boundary become trivial under interaction. In fact this interaction can have a much larger symmetry than the assumed $U(1) \times Z_2 \times Z_2^T$. For example, the $U(1)$ and $Z_2$ order parameters considered previously (Eq. 10, 11) can be combined together to form an $SU(2)$ vector,

$$\bar{\psi} = \text{Re}[\psi^d (\sigma^y \otimes i\tau^y \tau^z) \psi].$$

(12)

The four-fermion interaction that guarantees the triviality of the hedgehog monopole of this $SU(2)$ vector can have a global symmetry at least as large as $SO(7)$, under which the 8 flavors of 4d TSC transform as an 8-dimensional real spinor representation. This $SO(7)$ symmetry contains the obvious subgroup $SO(6)$, which is locally isomorphic to $SU(4)$, and this in turn contains the obvious subgroup $SU(3)$. Under a hypothetical breaking of $SO(7) \rightarrow SU(4)$, the spinor representation would decompose as $8 \rightarrow 4 \oplus 4$.

In the analysis above we have ignored the gauge fields. We assume that the four fermion interactions play the main role in gapping out the fermions. We can “gauge” the global symmetries of the 4d TSC by coupling the fermions to the gauge fields. For example, we can couple 8 flavors of the 4d TSC discussed in this paper to $SO(7) \times SU(2)$ gauge fields, and one of the 3d boundaries is gapped out by short range interactions. Since the boundary fermions are gapped out without generating any fermion bilinear mass or breaking any symmetry, the gauge fields will not be massive due to Higgs mechanism. As long as the scale of the fourth dimension is small enough, the only low energy degrees of freedom left are 16 chiral fermions on the other boundary coupled to $SO(7) \times SU(2)$ gauge fields. This theory has no gauge anomaly, but as far as we know, there is no other simple way of realizing this chiral gauge theory on a 3d lattice. For example, in a system called “Weyl semimetal” 20, we can realize 8 left and 8 right chiral fermions at different momenta in the 3d Brillouin zone. After a particle-hole transformation, right Weyl fermions will become left fermions, but there would be no $SO(7) \times SU(2)$ continuous symmetry that mixes the chiral fermions at different momenta. Thus the 16 chiral fermions realized in this conventional way cannot be coupled to $SO(7) \times SU(2)$ gauge fields on the lattice. The new mechanism studied in this paper provides a regularization of this chiral gauge theory.

Our results suggest a possible direction of realizing the SM on a 3d lattice. If we want to realize the SM in the
same way, *i.e.* realize the SM on one boundary of a 4d TSC and gap out its mirror sector by local interactions, then we need to argue that the classification of the 4d TSC with $SU(3) \times SU(2) \times U(1)$ symmetry is reduced by short range interactions. We do not have a full argument for this conclusion yet. However, this conclusion is consistent with the fact that the SM with a right-handed neutrino has 16 chiral fermions in each generation. Recently, Ref. [24] has also proposed that the mirror sector of the SM can be gapped out by interactions, but the importance of the flavor number 16 was not pointed out. Based on the observation in this paper, we conclude that the mirror sector can be gapped out by interaction without interfering with the SM only when there are 16 chiral fermions per generation.

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