TROPICAL LAURENT SERIES, THEIR TROPICAL ROOTS, AND LOCALIZATION RESULTS FOR THE EIGENVALUES OF NONLINEAR MATRIX FUNCTIONS

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Abstract. Tropical roots of tropical polynomials have been previously studied and used to localize roots of classical polynomials and eigenvalues of matrix polynomials. We extend the theory of tropical roots from tropical polynomials to tropical Laurent series. Our proposed definition ensures that, as in the polynomial case, there is a bijection between tropical roots and slopes of the Newton polygon associated with the tropical Laurent series. We show that, unlike in the polynomial case, there may be infinitely many tropical roots; moreover, there can be at most two tropical roots of infinite multiplicity. We then apply the new theory by relating the inner and outer radii of convergence of a classical Laurent series to the behavior of the sequence of tropical roots of its tropicalization. Finally, as a second application, we discuss localization results both for roots of scalar functions that admit a local Laurent series expansion and for nonlinear eigenvalues of regular matrix valued functions that admit a local Laurent series expansion.

1. Introduction

Tropical algebra is a branch of mathematics developed in the second half of the twentieth century. The adjective tropical was coined in honour of the Brazilian mathematician Imre Simon, who in 1978 introduced a min-plus structure on \( \mathbb{N} \cup \{\infty\} \) to be used in automata theory [20]. In the following decades, researchers started using the nomenclature we are now familiar with. For instance, Cuninghame-Green and Meijer introduced the term max-algebra in 1980 [9], while max-plus appeared in the nineties and early two-thousands [3, 15].

Even though tropical algebra has found many other applications, in computational mathematics a preeminent one has been to approximately and cheaply locate the roots of polynomials or, more in general, the eigenvalues of matrix polynomials. Indeed, having a cheap initial estimate of these values is often very important for numerical methods: for instance, contour integral algorithms heavily rely on knowing inclusion region for the eigenvalues [1, 2, 5]; in 2009, Betcke proposed a tropical algebra based diagonal scaling for a matrix polynomial \( P \) to improve the conditioning of its eigenvalues \( \lambda_j \) near a target eigenvalue \( \lambda \), which requires the knowledge of \( |\lambda| \) [4]; the Ehrlich–Aberth method for the polynomial eigenvalue problem needs a starting point [7] which can also be estimated via tropical algebra. In 1996, Bini proposed an algorithm to compute polynomial roots based on Acherb’s method and

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the Newton polygon, but without the formalism of tropical algebra [6]. In 2009, Gaubert and Sharify showed that the tropical roots of a scalar polynomial can give bounds on its “standard” roots [10]. Later, Noferini, Sharify and Tisseur [17] proposed similar results for matrix-valued polynomials, and compared their bounds with the ones in [8, 16].

The goal of the present paper is to generalize those results to the case of meromorphic (matrix-valued) functions expressed as Laurent series. The article is structured as follows: Section 2 is devoted to extending the notion of tropical polynomials to tropical Laurent series and to defining the concept of tropical roots in these settings. In Section 3 we consider how tropical roots can give bounds on the eigenvalues of meromorphic matrix-valued functions, just like in the polynomial case. Finally, in Section 4 we show how to update the infinite Newton polygon of a tropical Laurent function under specific circumstances, and how the results of the previous section may benefit future contour integral eigensolvers.

## 2. Algebra and analysis of tropical Laurent series

### 2.1. Tropical algebra

The max-plus semiring \( \mathbb{R}_{\max} \) is the algebraic structure composed of the set \( \mathbb{R} \cup \{-\infty\} \) equipped with the max operation \( \oplus \) as addition, and the standard addition \( \otimes \) as multiplication, respectively. The zero and unit elements of this semiring are \(-\infty\) and 0.

Similarly, one defines the max-times semiring \( \mathbb{R}_{\max,\times} \) as the set of nonnegative real numbers \( \mathbb{R}^+ \) equipped with the max operation \( \oplus \) as addition and the standard multiplication \( \otimes \) as multiplication, respectively. Under these settings, the zero and unit element are 0 and 1. \( \mathbb{R}_{\max,\times} \) is isomorphic with \( \mathbb{R}_{\max} \) thanks to the map \( x \mapsto \log x \), with the usual convention \( \log(0) = -\infty \). We refer with the word tropical to any of these two structures, as many other researchers have done before us.

### 2.2. From tropical polynomials to tropical series

A max-plus tropical polynomial \( tp \) is a function of a variable \( x \in \mathbb{R}_{\max} \) of the form \( tp(x) := \bigoplus_{j=0}^d \log b_j \otimes x^{\otimes j} = \max_{0 \leq j \leq d} (\log b_j + jx) \).

where \( d \) is a nonnegative integer and \( b_0, \ldots, b_d \in \mathbb{R}^+ \). The degree of \( tp(x) \) is \( d \) if \( b_d \neq 0 \). The finite tropical roots \( \log \alpha \) are the points where the maximum is attained at least twice, which correspond to the non-differentiable points of the function \( tp(x) \). For such roots, the multiplicity can be defined as the jump of the derivative at \( \log \alpha \), that is \( \lim_{\varepsilon \to 0} \frac{dtp}{dx} |_{\log \alpha + \varepsilon} - \frac{dtp}{dx} |_{\log \alpha - \varepsilon} \). The multiplicity of \(-\infty\) as a root of \( tp \) is given by \( \inf \{ j \mid b_j \neq 0 \} \).

Similarly, a max-times tropical polynomial is the function \( txp(x) := \bigoplus_{j=0}^d b_j \otimes x^{\otimes j} = \max_{0 \leq j \leq d} (b_jx^j) \),

with \( x, b_0, \ldots, b_d \in \mathbb{R}^+ \) and again the tropical roots \( \alpha \) are the points of non-differentiability of the function above; this time the multiplicity is the jump of the derivative of the logarithm of the polynomial at the root. The tropical roots of a max-times tropical polynomial with coefficients \( b_0, \ldots, b_d \) are the exponents of the tropical roots of the max-plus tropical polynomials with coefficients...
log \log_{b_0} \ldots \log_{b_d}, including by convention \(-\infty =: \log 0\), and have the same multiplicities: this follows immediately if one observes that for any \(x \in \mathbb{R}^+\) it holds
\[
\max_{0 \leq j \leq d} (\log b_j + j \log x) = \log \max_{0 \leq j \leq d} (b_j \cdot x^j).
\]

A tropical version of the fundamental theorem of algebra holds, implying that a tropical polynomial of degree \(d\) has \(d\) tropical roots counting multiplicities \([9]\).

Using the Newton polygon, the computation of the tropical roots can be recast as a geometric problem which amounts to computing the upper convex hull of the points \((j, \log b_j)\) (see, for example, the introduction of \([19]\)). This can be done in \(O(d)\) operations using the Graham scan algorithm, since the points are already sorted by their abscissa (see Section 2.3) \([11]\).

We are concerned with a generalization of tropical polynomials that can be helpful in studying analytic functions, similarly to the role played by tropical polynomials in locating the roots (or eigenvalues) of (possibly matrix-valued) functions.

**Definition 2.1 (Tropical Laurent series).** Let \((b_j)_{j \in \mathbb{Z}}\) be a sequence of elements of \(\mathbb{R}^+\), indexed by integers and not all zero. The associated max-times tropical Laurent series is
\[
times f(x) := \sup_{j \in \mathbb{Z}} (b_j x^j),
\]
with \(x \in \mathbb{R}^+\) and it takes values in \(\mathbb{R}^+ \cup \{+\infty\}\). Similarly, the associated max-plus tropical Laurent series is a function of variable \(x \in \mathbb{R} \cup \{-\infty\}\)
\[
t(x) := \sup_{j \in \mathbb{Z}} (\log |b_j| + jx),
\]
taking values in \(\mathbb{R} \cup \{\pm \infty\}\). In addition, let \(f(\lambda) = \sum_{j \in \mathbb{Z}} b_j \lambda^j\) be a complex function defined by a Laurent series. Then \(t \times f(x) = \sup_{j \in \mathbb{Z}} (|b_j| x^j)\) is the max-times tropicalization of \(f(\lambda)\), while \(t f(x) := \sup_{j \in \mathbb{Z}} (\log |b_j| + jx)\) is its max-plus tropicalization.

**Remark 2.2.** There are two obvious differences in Definition 2.1 with respect to the polynomial case. The first is the use of supremum in lieu of maximum; this is necessary because there might be points \(x\) where the value of \(t \times f(x)\) is not attained by any of the polynomial function \(b_j x^j\). In addition, note that \(t \times f(x)\) (\(t f(x)\), respectively) may not be well-defined on the whole \(\mathbb{R}_{\max, x} \times (\mathbb{R}_{\max, x} \text{, respectively})\). The next sections are going to clarify these aspects.

The next lemma is a well-known result of convex analysis \([18, \text{Theorem 5.5}]\). We decided to recall it to make this section self-contained.

**Lemma 2.3.** Let \(g_j(x)\) be a set of real-valued functions all defined and convex on the same interval \(\Omega \subseteq \mathbb{R}\), and indexed over some non-empty set \(I\), possibly infinite or even uncountable. Then, the largest domain of definition of the function
\[
g : D \to \mathbb{R}, \quad x \mapsto g(x) = \sup_{j \in I} g_j(x)
\]
is an interval \(D \subseteq \Omega \subseteq \mathbb{R}\); moreover, \(g(x)\) is convex on \(D\).

**Definition 2.4.** Given a sequence of elements in \(\mathbb{R}^+ \{b_j\}_{j \in \mathbb{Z}}\), or equivalently a Laurent series with non-negative coefficients \(\sum_{j \in \mathbb{Z}} b_j x^j\), we denote by \(I := \{j \in \mathbb{Z} : b_j > 0\}\) the set of indices corresponding to the nonzero elements.
Proposition 2.5. Let \((b_j)_{j \in \mathbb{Z}}\) be a sequence of elements of \(\mathbb{R}^+\), indexed by integers and not all zero. Let \(I\) be its nonzero elements as in Definition 2.4. Then the following statements are true:

1. The domain of the function
   \[ \tau: f : D \rightarrow \mathbb{R}^+, \quad x \mapsto \tau f(x) = \sup_{j \in I} b_j x^j \]
   is an interval \(D \subseteq \mathbb{R}^+\); moreover, \(\tau f(x)\) is convex on \(D\).

2. The domain of the function \(f\), restricted to \(\mathbb{R}\),
   \[ \tau f : D_+ \rightarrow \mathbb{R}, \quad x \mapsto \tau f(x) = \sup_{j \in I} (\log b_j + jx) \]
   is an interval \(D_+ \subseteq \mathbb{R}\); moreover, \(\tau f(x)\) is convex on \(D_+\).

Proof. This easily follows from Lemma 2.3. Indeed:

1. For any \(j \in I\), \(b_j x^j\) is convex on \(\mathbb{R}^+\) (note \(j \in I \Rightarrow b_j > 0\)).
2. For any \(j \in I\), \(\log b_j \in \mathbb{R}\). Hence, \(\log b_j + jx\) is convex, on \(\mathbb{R}\).

Proposition 2.5 does not specify the nature of the interval \(D\), which can be open, closed, or semiopen (either side being open/closed): indeed, examples can be constructed for all four cases. It can also happen that \(D = \mathbb{R}^+\), that \(D\) is empty, or that \(D\) is a single point. In addition, if \(\tau f(x)\) and \(f(x)\) are, resp., the max-times and the max-plus tropicalization of the same (classical) function \(f(x)\), then \(D = [a, b]\) is the domain of \(\tau f(x)\) if and only if \(D_+ := [\log a, \log b]\) is the domain of \(f(x)\). (Similar statements hold when \(D\) is not closed but open or semiopen.)

An immediate consequence of Proposition 2.5 is that

\[ Y := \{y \in D^o : \tau f(x)\text{ is not differentiable at } x = y\} \Rightarrow \#Y \leq \aleph_0. \]

where \(D^o\) is the interior of \(D\). In particular, \(\tau f(x)\) (as any convex function of a real variable) has left and right derivative everywhere in its domain of definition \(D\) and it is differentiable almost everywhere in the interior of \(D\), except for at most countably many points at which the left and right derivative differ. On the other hand, if \(\#I \geq 2\) then it is clear that \(Y\) may be not empty. Indeed, we will define the set of tropical roots of \(\tau f(x)\) as \(Y \cup (D \setminus D^o)\), together (possibly) with \(\{0\}\). Via the logarithmic map, an analogous definition holds for the tropical roots of \(f(x)\).

Definition 2.6 (Tropical roots of Laurent series). Consider a max-plus Laurent series

\[ \tau f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \tau f(x) := \sup_{j \in \mathbb{Z}} (\log b_j + jx) \]

as a real valued function defined on some interval \(D_+\). A point \(\log \alpha \in D_+\) is said to be a tropical root of \(f(x)\) if:

1. either \(\tau f(x)\) is not differentiable at \(\log \alpha \in \mathbb{R}\), and in this case the multiplicity of \(\log \alpha\) is the size of the jump of the derivative at \(\log \alpha\), i.e.,
   \[ m := \frac{d}{dx} \tau f(\log \alpha) - \frac{d}{dx} \tau f(\log \alpha); \]

2. or \(\log \alpha = -\infty\) if \(b_j = 0\) for \(j \leq 0\) and in this case the multiplicity is given by \(\inf\{j \mid b_j \neq 0\}\);

3. or \(\log \alpha \in \mathbb{R}\) is a finite endpoint of \(D_+\), and in this case the multiplicity of \(\log \alpha\) is \(m = \infty\).
Similarly consider $t_\alpha f(x) := \sup_{j \in \mathbb{Z}} (b_j x^j)$ as a real valued function defined on some interval $D$. Then, $\alpha \in D$ is a tropical root of $t_\alpha f(x)$ with multiplicity $m$ if $\log \alpha$ is a tropical root of $\sup_{j \in \mathbb{Z}} (\log b_j + jx)$ with multiplicity $m$.

As expected, Definition 2.6 falls back to the polynomial case when $t_\alpha f(x)$ is a polynomial: if $t_\alpha f(x)$ is a polynomial, then $D = [0, \infty]$, hence the second subcase falls back to 0 being a root of $t_\alpha f(x)$, while the third one never happens. In addition note that the multiplicity of a tropical root may be infinite. This happens if $\alpha$ is nonzero, is an endpoint of $D$, and belongs to $D$, that is, if $t_\alpha f(\alpha) \in \mathbb{R}$ but either $t_\alpha f(x) = +\infty$ for $x > \alpha$ or $t_\alpha f(x) = +\infty$ for $x < \alpha$. Analogous observations can be made for $tf(x)$.

Before continuing to expose our theory, we give some examples to gain further insight on the different cases in the definition of tropical roots.

**Example 2.7.** Consider $f(\lambda) = -\log(1 - \lambda)$ and its Taylor expansion in $[-1, 1]$. Then

$$f(\lambda) = \sum_{j=1}^{\infty} \frac{x^j}{j} \Rightarrow t_\alpha f(x) = \sup_{j \geq 1} \left( \frac{x^j}{j} \right) = \begin{cases} x & \text{if } x \leq 1; \\ \infty & \text{if } x > 1. \end{cases}$$

The domain of $t_\alpha f(x)$ is $D = [0, 1]$. The right endpoint $\alpha_1 = 1 \in D$ is a tropical root of infinite multiplicity. The point $\alpha_0 = 0$ is a simple root, because $b_j = 0$ for $j \leq 0$. They are the only tropical roots because $t_\alpha f(x)$ is differentiable everywhere else in $D$. If we had considered $g(\lambda) = 1 - \log(1 - \lambda)$, then $t_\alpha g(x)$ would not have had $\alpha_0 = 0$ as a tropical root, even though $D = [0, 1]$: in fact, the special condition for 0 to be a tropical root would not be satisfied, as for an endpoint $a$ to be a root $\log a$ needs to be finite.

**Example 2.8.** Let $H_j = \sum_{k=1}^{j} k^{-1}$ denote the $j$-th harmonic number and consider

$$f(\lambda) = \sum_{j=1}^{\infty} e^{H_j} x^j \Rightarrow t_\alpha f(x) = \sup_{j \geq 1} (e^{H_j} x^j).$$

The domain of $t_\alpha f(x)$ is $D = [0, 1]$. As in the previous example, $\alpha_0 = 0$ is a root with multiplicity 1 because $b_j = 0$ for $j \leq 0$. The points of nondifferentiability are

$$\alpha_j = e^{-1/j}, \quad j = 1, 2, 3, \ldots$$

which have all multiplicity 1, and accumulate at 1. In Figure 1 we plotted $t_\alpha f(x)$ and $\alpha_j$. Note that $t_\alpha f(x) = +\infty$ if and only if $x \geq 1$. However, 1 $\notin D$, so 1 itself is not a tropical root.

**Example 2.9.** Let

$$f(\lambda) = \sum_{j=0}^{\infty} e^{j^2} x^j \Rightarrow t_\alpha f(x) = \sup_{j \geq 0} (e^{j^2} x^j) \equiv +\infty.$$ 

In this case, the domain of $t_\alpha f(x)$ as a real function is empty, and hence there are no tropical roots.

We now report an example of a tropical series $t_\alpha f(x)$ that has tropical roots accumulating on the right boundary, and also a largest tropical root.
Example 2.10. Let \( b_j := \prod_{k=1}^{j} (e^{2^{-k}} - 1) \), and
\[
f(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j \Rightarrow \text{sup}_{j \geq 0} b_j x^j.
\]
The domain of \( t_x f(x) \) is \( D = [0, e] \). The right endpoint \( e \) is therefore a tropical root (while 0 is not); by a direct computation we see that the points of nondifferentiability are \( \alpha_j = \exp(1 - 2^{-j}) \) for \( j = 1, 2, 3, \ldots \). The \( \alpha_j \) have \( e \) as accumulation point.

Remark 2.11. From now on we will assume that 0 is never a tropical root of \( t_x f(x) \). On one hand, this is not a restriction, because if 0 is a root of multiplicity \( m \), then it means \( b_j = 0 \) for \( j < m \) and thus we can consider a shifted version of \( t_x f(x) \). On the other, it greatly simplifies the exposition, because in this case \( \alpha \) is a tropical root if it is a nondifferentiable point or a closed endpoint of \( D \).

As a practical example, the function of Example 2.7 can be modified to remove the zero root dividing it by \( \lambda \), and thus yielding
\[
\hat{f}(\lambda) = \sum_{j=0}^{\infty} \lambda^j (j + 1).
\]
For this modified function, the domain of \( t_x \hat{f}(x) \) is still \( D = [0, 1] \). However, in this case there is no longer a tropical root at zero, but the only tropical root is \( e \), still with infinite multiplicity.

Before discussing the connection between tropical Laurent series \( t_x f(x) \) and the roots of the related function \( f(\lambda) \), we shall see how the relation between tropical roots and the associated Newton polygon extends from the polynomial setting to the Laurent series case. In addition, in Example 2.8 and 2.10 we have witnessed that the non differentiability points converge to the right endpoint of \( D \). This does not happen by chance, and it will be the second topic of the next section.

Since once again it is clear that one can map a max-times Laurent series to a max-plus Laurent series by taking the logarithm of the coefficients (and the roots are also mapped under the logarithm), we will for convenience focus on max-times Laurent series to analyse the connection with the Newton polygon.
2.3. Asymptotic behaviour of tropical roots and infinite Newton polygons. The goals of this subsection are multiple. First, we show that all the tropical roots are isolated, except for two possible accumulation points. Then we prove a result that mirrors Definition 2.6: \( t \times f(x) \in \mathbb{R}^+ \) if \( x \in [\alpha_{-\infty}, \alpha_{+\infty}] \), where \( \alpha_{\pm \infty} \) will be defined in Definition 2.14. Finally, we define the Newton polygon, which can be used to compute the tropical roots with their multiplicities. Even though it is a standard tool for the roots of tropical polynomials, it requires few more technicalities in the Laurent case: some properties are less obvious because we are dealing with a possibly infinite, albeit countable, number of roots.

We begin with a preliminary lemma: even though the tropical roots may be countably infinite, we show that they are isolated, except possibly for two accumulation points.

Lemma 2.12. Let \( t \times f(x) \) be a max-times Laurent series with non-differentiable points \( (\alpha_k)_{k \in T'} \), indexed by \( T' \subseteq \mathbb{Z} \). Then all the \( \alpha_k \) are isolated with the only possible accumulation points being \( \sup_{k \in T'} \alpha_k \) and \( \inf_{k \in T'} \alpha_k \). In particular, without loss of generality we can assume that the \( \alpha_k \) are sorted in a non-decreasing way, i.e., that if \( j > k \) then \( \alpha_j > \alpha_k \).

Proof. Observe that we can write:

\[
(2.2) \quad t \times f(x) := \sup_{j \geq 0} g_j(x), \quad g_j(x) := \max_{k \in \mathbb{Z}, |k| \leq j} (b_k x^k).
\]

In other words, \( t \times f(x) \) can be approximated from below by the functions \( g_j(x) \). Therefore we consider the non-differentiable points of \( g_j(x) \) and see how they change as we let \( j \to \infty \).

First, note that \( g_0(x) \) is the constant function \( g_0(x) \equiv b_0 \). In general, the function \( g_j(x) \) is obtained by \( g_{j-1}(x) \) adding at most two more monomial functions to the set whose supremum is taken; more precisely, one function is added if and only if \( b_j \neq 0 \) and the other if and only if \( b_{-j} \neq 0 \). The functions to be added have a larger exponent (in absolute value) than all the others considered before. We claim that this automatically introduces at most two new non-differentiable points, which will become the leftmost and the rightmost ones. Hence, there are at most two accumulation points, that (if present) are indeed \( \sup_{k \to \pm \infty} \alpha_k \) and \( \inf_{k \to \pm \infty} \alpha_k \), proving the statement.

It remains to prove the claim. For the sake of simplicity, we consider only what happens at the right hand side of our domain: the proof for the left hand side is analogous. There are two cases: either the new non-differentiable point, say \( \alpha \), is already the rightmost one, or it superposes or lies on the left of one or more previous non-differentiable points. In the first scenario, a new non-differentiable point \( \alpha \) is created; in the latter, every non-differentiable point larger than \( \alpha \) disappears, and \( \alpha \) becomes the rightmost one. In Figure 2 we plotted these two cases while building \( g_2(x) \): in 2a the new point is already the rightmost one, while in 2b it superposes with the previous point.

We have now proved that every tropical root \( \alpha_k \) of \( t \times f(x) \) is either isolated or an endpoint of the domain. In addition, Lemma 2.12 guarantees that the roots can be indexed as \( (\alpha_k)_{k \in T} \) with \( T \subseteq \mathbb{Z} \cup \{-\infty, +\infty\} \), and \( T' \subseteq T \). Here, we make the choice that the indices \( -\infty \) (resp. \( +\infty \)) is used if and only if the left (resp. right) endpoint is both accumulation point and a tropical roots. With this
choice, we can assume, without loss of generality, that the roots are sorted and the inherited topology from $\mathbb{R}$ is respected. The nature of $T$ may vary considerably. It may be either bounded or unbounded above, and either bounded or unbounded below, depending on whether the sequence of tropical roots is bi-infinite, infinite only on the left, infinite only on the right, or finite. For example, if $t_{x}f(x)$ is a polynomial, then $T$ is clearly finite. The next proposition proves that if $t_{x}f(x)$ has a finite number of roots, then the cardinality of $I$ (the set of indices corresponding to nonzero coefficients) is finite.

**Proposition 2.13.** Let $t_{x}f(x) = \sup_{j \in \mathbb{Z}} b_{j}x^{j}$ be a tropical Laurent series. Let $I := \{j \in \mathbb{Z} : b_{j} > 0\}$ and let $(\alpha_{k})_{k \in T}$ be tropical roots, with $T \subset \mathbb{Z}$. Then the set of indices $I$ is bounded above if and only if $T$ bounded above and the domain $D$ of $t_{x}f(x)$ is not closed on the right. Similarly, $I$ is bounded below if and only if $T$ is bounded below and $D$ is not closed on the left, or the left endpoint of $D$ is zero and $b_{0} \neq 0$.

**Proof.** We first prove the case when $I$ is bounded above. If $I$ is bounded above, i.e., $b_{j} = 0$ definitely for $j$ large enough, then $T$ is bounded above and $t_{x}f(x) \in \mathbb{R}^{+}$ for every $x$ large enough, because

$$t_{x}f(x) = \sup_{j \in I} b_{j}x^{j} = \max_{j \in I} b_{j}x^{j}.$$ 

Conversely, assume that $T$ is bounded above and that $D$ is not closed on the right, i.e., there is no rightmost tropical root with infinite multiplicity. Let $\alpha_{p}$ be the rightmost tropical root and let $j_{p} \in I$ be the index where the value $t_{x}f(\alpha_{p})$ is attained for the last time. Then, by definition of tropical root, for every $x > \alpha_{p}$ it holds

$$b_{j_{p}}x^{j_{p}} \geq b_{j}x^{j},$$

for every $j > 0$, which implies that $t_{x}f(x) \in \mathbb{R}^{+}$ for $x$ large enough, and therefore $D$ needs to be unbounded on the right. Equivalently, we can rewrite the previous
that if \( T(2.3) \) equation as

\[
x^{j-j_p} \leq \frac{b_{j_p}}{b_j}.
\]

Now assume that \( I \) is not bounded above. Then there exist infinite values of \( j \) such that \( b_j > 0 \) and \( j > j_p \). This yields a contradiction, because \( \lim_{x \to \infty} x^{j-j_p} = \infty \), but \( (2.3) \) implies this value is bounded above by a constant for every \( x \geq a \).

The other statement follows by a similar reasoning, with the only exception of zero which needs to be treated separately. If zero is in the domain, then \(-\infty\) is a tropical root (and therefore \( T \) is not bounded below), unless \( b_0 \neq 0 \). In the latter case, zero is not a root and all the proof follows by the same argument as above.

We are now ready to extend the definition \( \alpha_{\pm \infty} \), covering any possible \( T \). Note that if \( T \) is bounded above then \( \sup T \in T \) and if \( T \) is bounded below then \( \inf T \in T \).

**Definition 2.14.** Given a sequence \((\alpha_k)_{k \in T}\) of tropical roots we define \( \alpha_{\pm \infty} \) as elements of \( \mathbb{R}^+ \cup \{+\infty\} \) as follows:

\[
\alpha_{+\infty} = \begin{cases} 
\sup_{k \in T} \alpha_k, & \text{if } \sup(T) = +\infty, \\
+\infty, & \text{otherwise;}
\end{cases}
\alpha_{-\infty} = \begin{cases} 
\inf_{k \in T} \alpha_k, & \text{if } \inf(T) = -\infty, \\
0, & \text{otherwise.}
\end{cases}
\]

By definition, if \( t_{\alpha} f(x) \) is well defined as a real-valued function on \( D = [a, b] \), and \( a > 0 \), then \( a \) and \( b \) are tropical roots of infinite multiplicity. A priori, \( t_{\alpha} f(x) \) may also have infinite roots and it may happen, for example, that \( \lim_{k \to +\infty} \alpha_k = b \): indeed, Lemma 2.12 does not exclude this scenario.

Before proving the main result of this section, we state the definition of the Newton polygon for Laurent series and show that the usual result linking the tropical roots with the slopes of the Newton polygon holds true for the isolated ones. To keep the exposition concise, we consider max-times tropical series: it is clear how to adapt the definition to the max-plus case via the exponential map.

**Definition 2.15 (Newton polygon).** Let \( t_{\alpha} f(x) = \sup_{j \in \mathbb{Z}} b_j x^j \) be a max-times tropical Laurent series. We define the Newton polygon \( \mathcal{N}_{t_{\alpha} f} \) as the upper convex hull of \( \{(j, \log b_j) \mid b_j \neq 0\}\_{j \in \mathbb{Z}} \) and we denote with

\[
\mathcal{K}_{t_{\alpha} f} := \{j \mid (j, \log b_j) \in \mathcal{N}_{t_{\alpha} f}\}
\]

the vertices of the Newton polygon for \( t_{\alpha} f(z) \).

**Remark 2.16.** Note that the term “polygon” is used with a slight abuse of notation in Definition 2.15, since the Newton polygon could comprise an infinite number of vertices (and become what’s sometimes known as apeirogon). Nevertheless, we prefer to use this nomenclature for consistency with the polynomial case.

**Lemma 2.17 (Newton Polygon for isolated roots).** Let \( t_{\alpha} f(x) \) be a max-times tropical Laurent series, and \( \mathcal{N}_{t_{\alpha} f} \) its Newton polygon. Then for each segment connecting two vertices \((j_k, \log b_{j_k})\) and \((j_{k+1}, \log b_{j_{k+1}})\) with \( j_k \neq j_{k+1} \) we have a tropical root \( \alpha_k \) corresponding to a point of non-differentiability, with finite multiplicity \( m_k \), where:

\[
\alpha_k = \left( \frac{b_{j_k}}{b_{j_{k+1}}} \right)^{\frac{1}{j_{k+1} - j_k}}, \quad m_k := j_{k+1} - j_k.
\]

Equivalently, the isolated roots of \( t_{\alpha} f(x) \) are the exponential of the opposite of the slopes of the corresponding segments.
Proof. By Lemma 2.12, all the tropical roots are isolated. Hence, if $\alpha_k$ is a non-differentiable point for $t_\alpha f(x)$, then there exists $\varepsilon > 0$ such that
\[
 t_\alpha f(x) = \begin{cases} 
 b_j x^{j_k} & \alpha_k - \varepsilon \leq x \leq \alpha_k \\
 b_{j_k+1} x^{j_{k+1}} & \alpha_k \leq x \leq \alpha_k + \varepsilon
\end{cases}
\]
for some $j_k \leq j_{k+1}$. The indices are determined by checking which functions (locally) realize the supremum $t f(x) = \sup_{k \in \mathbb{Z}} \log b_j + j x$; by definition of upper convex hull, they thus correspond to the vertices of the Newton polygon. Indeed, the non-differentiability at $x = \alpha_k$ happens if and only if the segment connecting $(j_k, \log b_{j_k})$ and $(j_{k+1}, \log b_{j_{k+1}})$ is an edge of the Newton Polygon; this is equivalent to imposing that all points of the form $(j, \log b_j)$ for $j \notin \{j_k, j_{k+1}\}$ fall below the line extending the segment:
\[
 \log(b_j) < \log b_{j_k} + \frac{j-j_k}{j_{k+1}-j_k} (\log b_{j_{k+1}} - \log b_{j_k}), \quad \forall j \notin \{j_k, j_{k+1}\}.
\]
Using $\log \alpha_k = -(\log b_{j_{k+1}} - \log b_{j_k})/(j_{k+1} - j_k)$ we have the equivalent statement
\[
 \log b_j - \log b_{j_k} + (j-j_k) \log \alpha_k < 0 \iff \frac{b_j}{b_{j_k}} < \frac{\alpha_k}{\alpha_k^j},
\]
which is precisely the sought condition.

By equating the two expressions at $\alpha_k$ we obtain
\[
 \alpha_k = \left( \frac{b_{j_k}}{b_{j_{k+1}}} \right)^{\frac{1}{j_{k+1}-j_k}}, \quad m_k = j_{k+1} - j_k,
\]
where $m_k$ is the jump of the derivative of the max-plus version of the tropical polynomial.

\[ \square \]

**Theorem 2.18.** Let $t_\alpha f(x) = \sup_{j \in \mathbb{Z}} (b_j x^j)$ be a max-times Laurent series, and $(\alpha_k)_{k \in T}$ be the sequence of tropical roots. Let $\alpha_{\pm \infty}$ be defined as in Definition 2.14.

Then,
\[
 x \in ]\alpha_{-\infty}, \alpha_{+\infty}[ \Rightarrow t_\alpha f(x) \in \mathbb{R}^+
\]
and
\[
 x \notin [\alpha_{-\infty}, \alpha_{+\infty}] \Rightarrow t_\alpha f(x) = \infty.
\]

Furthermore, $t_\alpha f(\alpha_{+\infty}) \in \mathbb{R}^+$ if and only if $\alpha_{+\infty}$ is a tropical root. Respectively, $t_\alpha f(\alpha_{-\infty}) \in \mathbb{R}^+$, if and only if $\alpha_{-\infty}$ is a tropical root.

**Proof.** We only prove the claim for $\alpha_{+\infty}$, i.e., we argue that $t_\alpha f(x)$ is finite if $x < \alpha_{+\infty}$ and large enough, and that $t_\alpha f(x)$ is infinite if $x > \alpha_{+\infty}$: the claim for $\alpha_{-\infty}$ admits an identical proof, so we omit it. We analyse separately three possible cases.

1. **There is a largest tropical root and it has finite multiplicity.** First, note that $\alpha_{+\infty} = +\infty$, hence the second part of the thesis is vacuously true. The fact that $t_\alpha f(x) \in \mathbb{R}^+$ for $x$ large enough follows immediately from Proposition 2.13, because $b_j = 0$ for $j > 0$ large enough. Here we have $t_\alpha f(\alpha_{+\infty}) = +\infty$ by convention.

2. **There is a largest tropical root with infinite multiplicity** In this case the statement of the theorem coincides with Definition 2.6, because a root has infinite multiplicity if and only if it is a closed endpoint of $D$, where $t_\alpha f(x)$ is a well-defined function. Hence $t_\alpha f(\alpha_{+\infty}) \in \mathbb{R}^+$. 

(3) There is no largest tropical root. There are two subcases.

- If $\alpha_{+\infty} = +\infty$, then the set of tropical roots is unbounded above. Let $x \in \mathbb{R}^+$ be sufficiently large, then this implies that there exists $k$ such that $\alpha_{k-1} \leq x \leq \alpha_k$, and hence

$$0 < b_{jk-1}x^{jk-1} \leq t_k f(x) \leq b_{jk} x^{jk} < +\infty,$$

while the second implication is vacuously true. Similarly to the first case, $t_k f(\alpha_{+\infty}) = +\infty$ by convention.

- Assume now $\alpha_{+\infty} \in \mathbb{R}^+$. If $x = \alpha_{+\infty} - \varepsilon$ for $\varepsilon > 0$ and not too large, then there exist two tropical roots $\alpha_{k-1} < x < \alpha_k$ and we can conclude that $t_k f(x) \in \mathbb{R}^+$ arguing similarly to the first subcase; assume now that $x > \alpha_{+\infty}$, and write $x = \alpha_{+\infty}(1 + \varepsilon)$, with $\varepsilon > 0$; we define $c_j := b_j \alpha_j^{+\infty}$. Observe that for all $j$ it holds

$$t_k f(x) \geq b_j x^j = b_j (\alpha_{+\infty} (1 + \varepsilon))^j = c_j (1 + \varepsilon)^j.$$

Since the sequence of tropical roots is increasing, its limit for the index tending to $+\infty$ is an upper bound. Hence, by Lemma 2.17 we have that for all $k$ such that $\alpha_k$ exists

$$\left( \frac{b_{jk}}{b_{jk+1}} \right)^{1/j} < \alpha_{+\infty} \Leftrightarrow c_{jk} < c_{jk+1}.$$

Therefore, $c_{jk}$ is also an increasing sequence, and in particular $c_{jk} > c_{jk+1}$ for all $k \geq \ell$. As a consequence, fixing any $\ell$ such that $\alpha_\ell$ is defined,

$$t_k f(x) \geq b_j x^j > c_{jk} (1 + \varepsilon)^j$$

whose limit when $k \to +\infty$ is $+\infty$. Hence, $t_k f(x) = +\infty$ for all $x > \alpha_{+\infty}$, and $\alpha_{+\infty}$ is on the right boundary of $D$. Since it cannot be a tropical root (otherwise we would have a largest tropical root contradicting the assumption), we have $t_k f(\alpha_{+\infty}) = +\infty$. □

**Theorem 2.19** (Newton polygon for Tropical Laurent series). Let $t_k f(x)$ be a tropical Laurent series and let $N_{t_k f}$ be its Newton polygon. Then $t_k f(x)$ has a largest finite tropical root $\alpha_p$ of infinite multiplicity if and only if $N_{t_k f}$ has a rightmost segment of infinite length and vertex $(j_p, \log b_{j_p})$, with

$$\log \alpha_p = \inf_{i \in I} \sup_{j \geq 1} \left( -\frac{\log b_j - \log b_i}{j - i} \right).$$

Similarly, $t_k f(x)$ has a smallest finite tropical root $\alpha_f$ of infinite multiplicity if and only if $N_{t_k f}$ has a leftmost segment of infinite length and vertex $(j_f, b_{j_f})$, with

$$\log \alpha_f = \sup_{i \in I} \inf_{j \leq 1} \left( \frac{-\log b_i - \log b_j}{i - j} \right).$$

**Proof.** Assume $t_k f(x)$ has a largest tropical root of infinite multiplicity $\alpha_p$. By Definition 2.6 this corresponds to $D$ being bounded above and closed at its upper endpoint. In addition, recall that the set of indices $I := \{j : b_j > 0\}$ is not bounded above thanks to Proposition 2.13. Furthermore, the sequence $T$ of the indices of the tropical roots is bounded above by Theorem 2.18: in particular, there exists a rightmost point of non-differentiability, say, $\alpha_{p-1}$, which is in correspondence to a segment on the Newton polygon by Lemma 2.17. We label its rightmost segment by $(j_p, \log b_{j_p})$. Then the rightmost convex hull of the points $(j, \log b_j)$ must have
a rightmost infinite segment\(^1\). It follows from the definition of convex hull that the slope of this rightmost segment must be equal to the supremum of all the slopes of the segments through \((j, \log b_j)\) and \((j_p, \log b_{j_p})\), where the supremum is taken over all the (infinitely many) values of \(j > j_p\) such that \(b_j > 0\). Moreover, for any \(j_p < i < j\), it must be
\[
\frac{\log b_j - \log b_{j_p}}{j - j_p} \leq \frac{\log b_j - \log b_i}{j - i},
\]
as otherwise \(\alpha_{p-1}\) is not the penultimate tropical root. Thus, the opposite of the rightmost slope is
\[
\inf_{j \geq j_p} \left( -\frac{\log b_j - \log b_{j_p}}{j - j_p} \right) = \inf \sup_{i \geq j} \left( -\frac{\log b_j - \log b_i}{j - i} \right) =: \log \alpha_p.
\]
The case of a smallest tropical root of infinite multiplicity is dealt with analogously.

\(^1\)To visualize an example of such rightmost infinite segment, see Figure 5 in Section 3.3.

As noted in the introduction, many researchers have used the tropical roots of \(t \times p(x)\) to retrieve bounds on the localization of the roots of a scalar polynomial \(p(\lambda)\) or the eigenvalues of a matrix polynomial \(P(\lambda)\) \([17, 19]\). In the next section, we will generalize this approach to include (matrix-valued) Laurent series. Albeit technical, the results of the present section allow us to make this step very naturally.

3. Tropicalization of functions analytic on an annulus

The aim of this section is to relate the tropical roots of Laurent series with the eigenvalues of matrix-valued functions analytic on an annulus. More precisely, given the Laurent series
\[
F(\lambda) := \sum_{j \in \mathbb{Z}} B_j \lambda^j, \quad B_j \in \mathbb{C}^{m \times m},
\]
we consider \(F(\lambda) : \Omega \to \mathbb{C}^{n \times n}\), where \(\Omega\) is the largest annulus where the sum defining \(F(\lambda)\) is convergent. We denote this set as \(\Omega := \hat{A}(R_1, R_2)\), and the radii \(R_1\) and \(R_2\) are uniquely determined by the decay rate of \(\|B_j\|\) for \(j \to \pm \infty\).

**Lemma 3.1.** Let \(F(\lambda) = \sum_{j \in \mathbb{Z}} B_j \lambda^j\) be a Laurent series with \(B_j \in \mathbb{C}^{m \times m}\), and \(\|\cdot\|\) any matrix norm. Then, the series is convergent in the annulus \(\Omega := \hat{A}(R_1, R_2)\), where
\[
R_2^{-1} = \limsup_{j \to \infty} \|B_j\|^{1/j}, \quad R_1 = \limsup_{j \to \infty} \|B_{-j}\|^{1/j},
\]
where we employ the usual convention \(\infty^{-1} = 0\).

**Proof.** We argue that the radii of convergence for \(F(\lambda)\) are the same as those of the scalar series \(f(\lambda) := \sum_{i \in \mathbb{Z}} b_i \lambda^i\), where \(b_i = \|B_i\|\), and that this holds independently of the choice of the norm \(\|\|\). This follows by observing that for all \(\lambda\)
\[
\|F(\lambda)\| \leq |f(\lambda)| \leq c_n \cdot \sum_{i,j=1}^n |F(\lambda)_{ij}| \leq c_n \cdot C_n \cdot \|F(\lambda)\|,
\]
where \(c_n, C_n\) are some positive constants (both allowed to depend on \(n\)) and we have used the fact that all matrix norms are equivalent. \(\square\)
Note that the definition covers Taylor series, for which \( R_1 = 0 \) and \( R_2 = \infty \).

For matrix polynomials, where \( \Omega = \mathbb{C} \), the number of eigenvalues is always determined by the size \( n \) of the matrix coefficients \( B_j \) and the degree \( d \); in contrast, a Laurent series \( F(\lambda) \) may have zero, finite or countably infinite eigenvalues inside \( \Omega \). We also assume that \( F(\lambda) \) is regular, i.e., \( \det F(\lambda) \neq 0 \). This implies that the eigenvalues of \( F(\lambda) \) coincide with the roots of the scalar function \( \det F(\lambda) \). Given an arbitrary, but fixed, subordinate matrix norm \( \| \cdot \| \), to the matrix Laurent series \( F(\lambda) \) we associate the max-times Laurent series

\[
(3.2) \quad t_x F(x) := \sup_{j \in \mathbb{Z}} (\| B_j \| x^j) = \sup_{j \in \mathbb{Z}} (b_j x^j).
\]

While a priori the tropical roots of \( t_x F(x) \) depend on the choice of the norm, we will prove in Theorem 3.4 that the quantities \( \alpha_{\pm\infty} \) do not, and are related to the radii of convergence of \( F(\lambda) \). It is clear that a max-plus tropicalization can be defined analogously, and we omit the details.

3.1. Tropical roots and radii of convergence. The extension of the definition of tropical roots from tropical polynomials to tropical Laurent series has been discussed in Section 2.2. We noted that most roots can still be defined as the points of non-differentiability of the piecewise polynomial function \( t_x F(x) \), and can be computed through the Newton polygon (see Theorem 2.19). This may pose a computational challenge, as we need to compute the convex hull of a countable set of points; we discuss how to perform this task in detail in Theorem 3.13. We denote the ratio between two consecutive roots with

\[
(3.3) \quad \delta_j = \frac{\alpha_j}{\alpha_{j+1}} < 1, \quad \text{for } j \in \mathbb{Z}.
\]

Intuitively, one can visualize \( \delta_j \) on the associated Newton polygon. The smaller \( \delta_j \) is, the spikier the polygon is in \((j_k, \log B_{jk})\). We illustrate this below with two scalar examples.

Example 3.2. Consider the function

\[
f(\lambda) = \frac{15}{(1 - 3\lambda)(\lambda - 2)},
\]

which is holomorphic in \( \hat{A}(1/3, 2) \). It is not difficult to see that

\[
f(\lambda) = 6(\cdots + \frac{1}{3\lambda^2} + \frac{1}{\lambda} + \frac{1}{2} + \frac{\lambda}{4} + \cdots) = \sum_{j \in \mathbb{Z}} b_j \lambda^j
\]

where \( b_j = 2 \cdot 3^{j+2} \) if \( j < 0 \), and \( b_j = 3 \cdot 2^{-j} \) for \( j \geq 0 \). The (truncated) Newton polygon is given in Figure 3 and we can easily detect that there are only two roots, \( \alpha_{-\infty} = 1/3 \) and \( \alpha_{\infty} = 2 \), using the characterization from Lemma 2.17. In addition, one can check that \( t_x f(\alpha_{+\infty}) = 3 \) and \( t_x f(\alpha_{-\infty}) = 18 \), hence \( t_x f(x) \) is a well-defined real-valued function in the domain \( D = [\alpha_{-\infty}, \alpha_{\infty}] \), in concordance with both Definition 2.6 and Theorem 2.18.
Example 3.3. We consider again the function

\[ f(\lambda) = \sum_{j=1}^{\infty} e^{H_j \lambda^j} \Rightarrow t_f(x) = \sup_{j \geq 0} (e^{H_j x^j}). \]

defined in Example 2.8. We have that the domain of \( f(\lambda) \) is the open disk \( \Omega = D(1) \), while the domain \( D \) of \( t_f(x) \) is \( D = [0, 1] \). In Figure 4 we plotted its truncated Newton polygon. There we can see that the slopes converge from below to 0, in concordance with the fact that the nonzero roots are \( \alpha_j = e^{-1/j} \rightarrow 1 \), as \( j \) goes to infinity.

In both the examples above, the tropical roots not only converge to (or are) the endpoints of the domain \( D \) of \( t_f(x) \), but the latter coincide with the radii of convergence of \( f(\lambda) \). Once again, this does not happen by chance, as explained in the following theorem.
Theorem 3.4. Consider the $n \times n$ Laurent matrix-valued function $F(\lambda) = \sum_{i \in \mathbb{Z}} B_i \lambda^i$, holomorphic in the open annulus $\Omega = \mathcal{A}(R_1, R_2)$, where $R_1$ and $R_2$ are defined in (3.1). Let $(\alpha_k)_{k \in \mathbb{T}}$ be the sequence of distinct tropical roots of the tropicalization $t_x F(x)$ and let $\alpha_{\pm \infty}$ be the quantities of Definition 2.14. Then $R_2 = \alpha_{+\infty}$ and $R_1 = \alpha_{-\infty}$.

Proof. Thanks to Lemma 3.1, the series is convergent in $\mathcal{A}(R_1, R_2)$, defined as in (3.1). Then, we may equivalently consider $f(\lambda)$ and the proof becomes similar to the one of Theorem 2.18. As usual, we prove this statement only for $R_2$ as the argument for $R_1$ is identical.

1. There is a largest tropical root and it has finite multiplicity. We have $\alpha_{+\infty} = +\infty$, and from Proposition 2.13 it follows $b_j = 0$ for $j > 0$ and large enough. Hence $R_2 = +\infty$.

2. There is a largest tropical root with infinite multiplicity, which we denote by $\alpha_p = S_2$. By definition of tropical roots it holds

$$b_j \alpha_p^j \leq b_{jp} \alpha_p^{j_p}$$

therefore

$$\frac{b_j}{b_{jp}} \leq S_2^{(j_p - j)}.$$

This implies that $F(\lambda)$ is well-defined for every $|\lambda| \leq S_2$ by (3.1), therefore $S_2 \leq R_2$. We now claim that $S_2 \geq R_2$. Let $(b_j)_{j \in \mathbb{N}}$ be the subsequence of $(b_j)_{j \in \mathbb{N}}$ obtained by only keeping the indices corresponding to a curve that attains the supremum in the definition of $t_x F(x)$. Then

$$\frac{1}{R_2} \geq \limsup_{j \to \infty} [b_j]^{1/j} \geq \limsup_{k \to \infty} b_{jk}^{1/j_k}.$$

Let $j_{k'} = p$ be the index corresponding to the last tropical root with infinite multiplicity $\alpha_p$. Then for all $k > k'$ we have

$$x^{j_{k'}} b_{jk} \geq x^p b_p,$$

for every $x \geq \alpha_p$ by definition of tropical root. Hence,

$$b_{jk} \geq b_p S_2^{p - j_{k'}} \Rightarrow b_{jk}^{1/j_k} \geq \left(\frac{b_p S_2}{S_2}ight)^{1/j_k}.$$

It follows that

$$\frac{1}{R_2} \geq \limsup_{k \to \infty} \left(\frac{b_p S_2}{S_2}ight)^{1/j_k} = \frac{1}{S_2}.$$

3. There is no largest tropical root. If there is no largest tropical root, then $\alpha_{+\infty} = \lim_{j \to +\infty} \alpha_j$. If $\alpha_{+\infty} \in \mathbb{R}^+$, then one can argue as in case (2) above and show that $R_2 = \alpha_{+\infty}$. If $\alpha_{+\infty} = +\infty$, then the sequence of tropical roots is unbounded. But if it is unbounded, then again by the proof of case (2) given any tropical root $\alpha_k$ the power series defining $F(\lambda)$ must converge for all $|\lambda| < e^{\alpha_k}$, hence $R_2 = +\infty$. \qed
3.2. Eigenvalue localization for Laurent series. In this section we generalize to matrix-valued Laurent series the localization theorems derived in [17] for matrix-valued polynomials. We are going to follow the steps of [17]; thus, we simply state those results whose proof literally does not change, and we prove the ones that need some adjustments due to the infinite number of indices. As expected, the core idea is Rouché’s Theorem, which we recall for clarity.

**Theorem 3.5 (Rouché’s Theorem).** Let $S, Q : \Omega_0 \to \mathbb{C}^{n \times n}$ be meromorphic matrix-valued functions, where $\Omega_0$ is an open connected subset of $\mathbb{C}$. Assume that $S(x)$ is nonsingular for every $x$ on the simple closed curve $\Gamma \subset \Omega_0$. Let $\| \cdot \|$ be any operator norm on $\mathbb{C}^{n \times n}$ induced by a vector norm on $\mathbb{C}^n$. If $\|S(x)^{-1}Q(x)\| < 1$ for every $x \in \Gamma$, then $\det(S + Q)$ and $\det(S)$ have the same number of zeros minus poles inside $\Gamma$, counting multiplicities.

When we will apply Rouché’s Theorem to meromorphic functions $F(\lambda)$, hence their Laurent expansions will only have a finite amount of negative indices. Under these settings, $\Omega_0$ will be the disk $\mathcal{D}(R_2)$, and, as in [17], we are using a given tropical root $\alpha_j$ as a parameter scaling, with $\lambda = \alpha_j \mu$ and $\tilde{F}(\mu)$ equal to

$$
(3.4) \quad (t_x F(\alpha_j))^{-1} F(\lambda) = \left(\|B_{k_{j-1}}\|\alpha_j^{k_{j-1}}\right)^{-1} F(\alpha_j \mu) = \sum_{i=0}^{d} \tilde{B}_i \mu^i =: \tilde{F}(\mu),
$$

where

$$
(3.5) \quad \tilde{B}_i = \left(\|B_{k_{j-1}}\|\alpha_j^{k_{j-1}}\right)^{-1} B_i \alpha_j^i.
$$

**Lemma 3.6** ([17, Lemma 2.2]). The norms of the coefficients $\tilde{B}_i$ (3.5) of the scaled function have the following properties:

$$
\|\tilde{B}_i\| \leq \begin{cases} 
\delta_j^{k_{j-1}-i} & \text{if } i < k_{j-1}, \\
1 & \text{if } k_{j-1} < i < k_j, \\
\delta_j^{i-k_j} & \text{if } i > k_j,
\end{cases}
\|\tilde{B}_{k_{j-1}}\| = \|\tilde{B}_{k_j}\| = 1.
$$

With the goal of invoking Theorem 3.5, we decompose $\tilde{F}(\mu)$ as the sum of

$$
(3.6) \quad S(\mu) = \sum_{i=k_{j-1}}^{k_j} \tilde{B}_i \mu^i, \quad Q(\mu) = \sum_{i \not\in \{k_{j-1}, \ldots, k_j\}} \tilde{B}_i \mu^i.
$$

We use the notation $\kappa(A) := \|A\|\|A^{-1}\|$ to denote the condition number of any nonsingular matrix $A$. Thanks to the following results, we can localize the eigenvalues of $S(\lambda)$.

**Lemma 3.7** ([17, Lemma 2.3]). Let $P(\lambda) = \sum_{j=0}^{d} B_j \lambda^j$ with $B_0, B_d \neq 0$ be a regular matrix polynomial. Then every eigenvalue of $P(\lambda)$ satisfies

$$(1 + \kappa(B_0))^{-1} \alpha_1 \leq |\lambda| \leq (1 + \kappa(B_d)) \alpha_q,$$

where $\alpha_1, \alpha_q$ are the smallest and largest finite tropical roots, respectively. Furthermore, if $B_0$ and $B_d$ are invertible, both inequalities are strict.

**Lemma 3.8** ([17, Lemma 2.4]). Let $m_j := k_j - k_{j-1}$. Then, if $B_{k_{j-1}}$ and $B_{k_j}$ are nonsingular, then the $n(k_j - k_{j-1})$ nonzero eigenvalues of $S(\lambda)$ are located in the open annulus $\mathcal{A}((1 + \kappa(B_{k_{j-1}}))^{-1}, 1 + \kappa(B_{k_j}))$. 

Proof. If \( k_{j-1} \geq 0 \), then the lemma is in fact equivalent to [17, Lemma 2.4]. Otherwise, consider the polynomial \( \tilde{S}(\lambda) = \lambda^{-k_{j-1}} S(\lambda) \), which we can apply the mentioned above to. This implies that \( \tilde{S}(\lambda) \) has \( nm_j \) nonzero eigenvalues in \( \tilde{A}((1 + \kappa(B_{k_{j-1}}))^{-1}, 1 + \kappa(B_{k_j})) \), which yields that \( S(\lambda) \) has \( nm_j \) nonzero eigenvalues and \( nk_{j-1} \) poles in zero. □

**Lemma 3.10** ([17, Lemma 2.6]). Given \( c, \delta > 0 \) such that \( \delta \leq (1 + 2c)^{-2} \), the quadratic polynomial

\[
p(r) = r^2 - \left( 2 + \frac{1 - \delta}{\delta(1 + c)} \right)r + \frac{1}{\delta}
\]

has two real roots

\[
f := f(\delta, c) = \frac{(1 + 2c)\delta + 1 - \sqrt{\delta^2 - (1 - (1 + 2c)^2\delta)}}{2(1 + c)}, \quad g = (\delta f)^{-1},
\]

with the properties that

(i) \( 1 < 1 + c \leq f \leq g \),

(ii) \( \frac{1}{f-1} + \frac{1}{g-1} = \frac{1}{c} \).
Theorem 3.11. Let \( \ell^- > -\infty \) and let \( F(\lambda) = \sum_{j=\ell^-}^{\infty} B_j \lambda^j \) be a regular, meromorphic Laurent function, analytic in the open annulus \( \Omega := \hat{\Lambda}(R_1, R_2) \). For every \( j \in \mathbb{Z} \), let \( f_j = F(\delta_j, \kappa(B_k_j)) \), where \( f(\delta, c) \) is defined as in Lemma 3.10, and \( g_j = (\delta_j f_j)^{-1} \). Then:

1. If \( \delta_j \leq (1 + 2\kappa(B_k_j))^{-2} \), then \( F(\lambda) \) has exactly \( nk_j \) eigenvalues minus poles inside the disk \( \mathcal{D}((1 + 2\kappa(B_k_j))\alpha_j) \) and it does not have any eigenvalue inside the open annulus \( \hat{\Lambda}((1 + 2\kappa(B_k_j))\alpha_j, (1 + 2\kappa(B_k_j))^{-1}\alpha_{j+1}) \).

2. For any \( j < s \), if \( \delta_j \leq (1 + 2\kappa(B_k_j))^{-2} \) and \( \delta_s \leq (1 + 2\kappa(B_k_j))^{-2} \), then \( F(\lambda) \) has exactly \( n(k_s - k_j) \) eigenvalues inside the closed annulus \( \hat{\Lambda}((1 + 2\kappa(B_k_j))^{-1}\alpha_{j+1}, (1 + 2\kappa(B_k_j))\alpha_s) \).

Proof. As a preliminary step, note that for a fixed \( \delta \geq 1 \) and \( \delta \leq (1 + 2c)^{-2} \), the function \( f(\delta, c) \) of Lemma 3.10 is increasing and attains its maximum, which is \( 1 + 2c \), at \( \delta = (1 + 2c)^{-2} \). Therefore it holds \( f(\delta_j, \kappa(B_k_j)) \leq 1 + 2\kappa(B_k_j) \) for \( \delta_j \leq (1 + 2\kappa(B_k_j))^{-2} \).

1. Assume that \( \delta_j \leq (1 + 2\kappa(B_k_j))^{-2} \) and partition \( \tilde{F}(\mu) \) as in (3.6). Consider now \( r \) such that

\[
1 + \kappa(B_k_j) < r < 1/\delta_j.
\]

Note that \( r \) is well defined because \( \delta_j \leq (1 + 2\kappa(B_k_j))^{-2} < (1 + \kappa(B_k_j))^{-1} \).

Thanks to Lemma 3.8, we have that \( S(\mu) \) is nonsingular on the circle \( \Gamma_r = \{ \mu \in \mathbb{C} : |\mu| = r \} \). As soon as we can apply Rouché’s Theorem 3.5 to \( \tilde{F}(\mu) = S(\mu) + Q(\mu) \) and \( \Gamma_r \), we will have concluded. Therefore we have to check that \( ||S(\mu)^{-1}Q(\mu)|| < 1 \) for all \( \mu \in \Gamma_r \). Since

\[
\delta_j-1 < 1 < 1 + \kappa(B_k_j) < r < 1/\delta_j,
\]

we can simply apply the bounds retrieved in Lemma 3.9. It follows

\[
||S(\mu)^{-1}Q(\mu)|| \leq ||S(\mu)^{-1}||||Q(\mu)||
\]

\[
\leq \frac{r^{-k_j} (r-1) \kappa(B_k_j)}{r-1 - \kappa(B_k_j)(1-r^{-m_j})} \left( \frac{\delta_j^{-1}r^{k_j-1}}{r-\delta_j^{-1}} + \frac{\delta_j r^{k_j+1}}{1-\delta_j r} \right).
\]

The latter bound is less than 1 if

\[
\frac{\delta_j^{-1}r^{-m_j}}{r-\delta_j^{-1}} + \frac{\delta_j r}{1-\delta_j r} < \frac{r-1 - \kappa(B_k_j)(1-r^{-m_j})}{(r-1) \kappa(B_k_j)},
\]

or equivalently, if

\[
\frac{\delta_j r}{1-\delta_j r} < \frac{r-1 - \kappa(B_k_j)}{(r-1) \kappa(B_k_j)} + r^{-m_j} \left( \frac{1}{r-1} - \frac{\delta_j^{-1}}{r-\delta_j^{-1}} \right).
\]

Since \( \frac{1}{r-1} > \frac{\delta_j^{-1}}{r-\delta_j^{-1}} \), the last inequality holds when \( \frac{\delta_j r}{1-\delta_j r} < \frac{r-1 - \kappa(B_k_j)}{(r-1) \kappa(B_k_j)} \), which is equivalent to \( p(r) < 0 \), where \( p(x) \) is the polynomial introduced in Lemma 3.10 with \( \delta = \delta_j \) and \( c = \kappa(B_k_j) \). Lemma 3.10 assures us that \( p(r) \) is negative for the values of \( r \) such that

\[
f_j < r < g_j,
\]

given that \( f_j \) and \( g_j \) are the two roots of \( p(x) \). The same lemma also tells us that \( f_j \geq 1 + \kappa(B_k_j) \) and \( g_j \leq (\delta_j)^{-1} \), hence (3.9) is sharper than (3.8). In particular, note that for any \( |\mu| = r \) where \( r = f_j \) or \( r = g_j \)
the upper bound for $\|S(\mu)^{-1}Q(\mu)\|$ is equal to 1. Therefore, such $\mu$ belongs to the domain of analiticity, and we have $R_1 \leq f_j < g_j \leq R_2$. Finally, by Rouché’s Theorem this implies that $S(\mu)$ and $\bar{F}(\mu)$ have the same number of eigenvalues minus poles, i.e., $nk_j$, inside the disk $\mathcal{D}(r)$ for any $r$ such that $f_j < r < g_j$. Furthermore, there are no eigenvalues in the open annulus $\mathcal{A}(f_j, g_j)$, because the quantity of eigenvalues minus poles is constant and there cannot be poles in $\mathcal{A}(f_j, g_j)$. The thesis then follows from the scaling $\lambda = \mu \alpha_j$ and from the preliminary point.

(2) By the proof of the previous item, if $\delta_j \leq (1 + 2 \kappa(B_{k_j}))^{-2}$ then $F(\lambda)$ has $nk_j$ eigenvalues minus poles inside $D((1 + 2 \kappa(B_{k_j}))\alpha_j)$ and no eigenvalues in $\mathcal{A}((1 + 2 \kappa(B_{k_j}))\alpha_j, (1 + 2 \kappa(B_{k_j}))^{-1}\alpha_j+1)$. A similar statement holds if $\delta_j^* \leq (1 + 2 \kappa(B_{k_j}))^{-2}$. Given that poles cannot lie there, this implies that $F(\lambda)$ has exactly $n(k_{j_x} - k_j)$ eigenvalues inside the closed annulus $\mathcal{A}((1 + 2 \kappa(B_{k_j}))^{-1}\alpha_{j+1}, (1 + 2 \kappa(B_{k_j}))\alpha_s)$. □

When the non-zero coefficients with negative (resp. positive) indices are a finite number, we may also find an exclusion (resp. inclusion) disc centered at zero. This may be seen as a generalization of Lemma 3.7.

**Theorem 3.12.** Let $F(\lambda) = \sum_{j=-\infty}^{\infty} B_j \lambda^j$ be a (classic) Laurent series. When $\ell^- < -\infty$ and $B_{\ell^-}$ is non singular, then $F(\lambda)$ has $\ell^-$ eigenvalues minus poles at zero, and the other eigenvalues of $F(\lambda)$ satisfy

$$\left(1 + \kappa(B_{\ell^-})\right)^{-1} \alpha_1 \leq |\lambda|$$

Similarly, if $\ell^+ < \infty$ and $B_{\ell^+}$ is non singular, then every eigenvalue satisfies

$$|\lambda| \leq \left(1 + \kappa(B_{\ell^-})\right)\alpha_q,$$

where $\alpha_q$ is the maximum tropical root.

**Proof.** The proof follows the ideas in [12, Lemma 4.1]. We consider the case $\ell^+ < \infty$ first. The definition of $\alpha_q$ implies that:

$$\|B_i\| \leq \alpha_q^{\ell^+ - i}\|B_{\ell^+}\|, \quad i \leq \ell^+.$$ 

Assume for a contradiction that there is an eigenvalue $|\lambda| > \left(1 + \kappa(B_{\ell^-})\right)\alpha_q$. Then, we may consider a normalized eigenvector $\|x\| = 1$, and write

$$\|F(\lambda)x\| \geq \|B_{\ell^+}|\lambda|^{\ell^+} x\| - \sum_{i<\ell^+} \|B_i\||\lambda|^i$$

$$\geq |\lambda|^{\ell^+} \left(\|B_{\ell^+}^{-1}\|^{-1} - \sum_{i<k} \|B_i\||\lambda|^{i-\ell^+}\right)$$

$$\geq |\lambda|^{\ell^+} \left(\|B_{\ell^+}^{-1}\|^{-1} - \sum_{i<k} \|B_{\ell^+}\|\alpha_q^{\ell^+ - i}|\lambda|^{i-\ell^+}\right)$$

$$= |\lambda|^{\ell^+} \|B_{\ell^+}^{-1}\|^{-1} \left(1 - \kappa(B_{\ell^+}) \sum_{i<k} \frac{\alpha_q}{|\lambda|^{\ell^+ - i}}\right)$$

$$= |\lambda|^{\ell^+} \|B_{\ell^+}^{-1}\|^{-1} \left(1 - \kappa(B_{\ell^+}) \frac{\alpha_q}{|\lambda| - \alpha_q}\right),$$

which is a contradiction.
where the last inequality follows assuming that $|\lambda| > (1 + \kappa(B_{\ell-})) \alpha_q$, as we did. Hence, we have the sought upper bound for $|\lambda|$. If we have $\ell^- > -\infty$ and $B_{\ell^-}$ nonsingular, we can rewrite $F(\lambda)$ as

$$F(\lambda) = \lambda^{\ell^-} P(\lambda),$$

where $P(\lambda)$ is a Taylor series (or matrix polynomial) with $\det P(0) \neq 0$. Hence, we conclude that $F(\lambda)$ has an eigenvalue (or pole, depending on the sign of $\ell^-$) of the desired algebraic multiplicity. Applying the above reasoning to $F(\frac{1}{\lambda})$ yields the lower bound for the remaining eigenvalues. □

3.3. Practical computation of the Newton polygon. Given a tropical Laurent series, computing its Newton polygon may be challenging, as the customary Graham Scan algorithm [11] could require an infinite number of comparisons.

We now present two strategies for addressing this problem. First, we may assume that the norms of the coefficients $B_j$ can be computed easily and cheaply (which is indeed true, for example, for the $1$ and the $\infty$ operator norms) and we show that the finite truncation of the Newton polygon “converge” (in an appropriate sense) to the infinite one. Second, we consider the case where the Laurent series is the modification of a known function, of which only a finite number of coefficients are altered. This is not unrealistic: for example, in the context of delay differential equations it is typical to deal with functions that are polynomials in both $\lambda$ and $e^\lambda$ [13, 14].

**Theorem 3.13** (Truncation of the Newton polygon). Let $F: \Omega \to \mathbb{C}^{n \times n}$ be a Laurent function on $\Omega = \mathcal{A}(R_1, R_2)$ and let $t_x F(x)$ be its tropicalization. Let $t_x F_\delta(x) = \bigoplus_{0 \leq k \leq d} \|B_k\| x^k$ be any “right-finite” truncation and let $C_2 > 0$ be a constant such that $\|B_k\| < C_2 R_2^{-k}$ for any $k > 0$. Then for any two consecutive indices $i, j \in \mathcal{K}_{t_x F_\delta}$ such that $\|B_j\| \geq \|B_i\| R_2^{-(j-i)}$, let $\ell_2$ be defined as

$$\ell_2 := \frac{(j-i) \log C + i \log \|B_j\| - j \log \|B_i\|}{(j-i) \log R_2 + \log \|B_j\| - \log \|B_i\|}$$

and suppose moreover that for all $j + 1 \leq k \leq \ell_2$ it holds

$$\|B_k\| \leq \|B_j\| \frac{b^k}{b^{j-i}} \|B_i\|^{\frac{1}{b-1}}. \quad (3.10)$$

Then $i, j$ are also consecutive indices in $\mathcal{K}_{t_x F}$. Similarly, let $t_x F_{-\delta}(x) = \bigoplus_{0 \leq k \leq d} \|B_{-k}\| x^{-k}$ be a “left-finite” truncation and let $C_1 > 0$ be a constant such that $\|B_{-k}\| < C_1 R_1^{-k}$ for any $k > 0$. Then for any two consecutive indices $-i, -j \in \mathcal{K}_{t_x F_{-\delta}}$ such that $\|B_{-j}\| \geq \|B_{-i}\| R_1^{-(1-i)}$, let $\ell_1$ be defined as

$$\ell_1 := \frac{(i-j) \log C_1 + i \log \|B_{-j}\| - j \log \|B_{-i}\|}{(i-j) \log R_1 + \log \|B_{-j}\| - \log \|B_{-i}\|}$$

and suppose moreover that for all $j + 1 \leq k \leq \ell_1$ it holds

$$\|B_{-k}\| \leq \|B_{-j}\| \frac{b^k}{b^{j-i}} \|B_{-i}\|^{\frac{1}{b-1}}. \quad (3.11)$$

Then $-i, -j$ are also consecutive indices in $\mathcal{K}_{t_x F}$. 

**Proof.** Observe that for $k > 0$ the points $(k, \log \|B_k\|)$ lie below the line

$$L_1: y = \log C_2 - x \log R_2,$$
due to the fact that \( \log \| B_k \| \leq C_2 R_2^{-k} \). In addition, let

\[
L_2 : y = \log \| B_j \| + (x - j) \frac{\log \| B_j \| - \log \| B_i \|}{j - i}
\]

be the line containing the segment between \((i, \log \| B_i \|)\) and \((j, \log \| B_j \|)\). Now we want to prove that if this segment belongs to \( \mathcal{N}_{\varepsilon R_2} \), then it belongs to \( \mathcal{N}_{\varepsilon F} \). In order to do that, we have to show that all the other points \((k, \log \| B_k \|)\) lie below \( L_2 \). If \( j + 1 \leq k \leq \ell_2 \), then this condition is equivalent to (3.10). On the other hand, if \( k \geq \ell_2 \), then we can prove that \( L_2 \) lies above \( L_1 \), and therefore above \((k, \log \| B_k \|)\). We can do this by pointing out that

\[
L_2(k) - L_1(k) \geq L_2(\ell_2) - L_1(\ell_2) = 0.
\]

Indeed, for the second equality we have

\[
(j - i)(L_2(\ell_2) - L_1(\ell_2)) = (j - i) \log \| B_j \| + (\ell_2 - j)(\log \| B_j \| - \log \| B_i \|)
+ (i - j) \log C + \ell_2(j - i) \log R_2
= (i - j) \log C + (j - \ell_2) \log \| B_i \|
+ (\ell_2 - i) \log \| B_j \| + \ell_2(j - i) \log R_2
= (i - j) \log C + j \log \| B_i \| - i \log \| B_j \|
+ \ell_2((j - i) \log R_2 + \log \| B_j \| - \log \| B_i \|) = 0
\]

by the definition of \( \ell_2 \). For the first inequality, it is sufficient to prove that the slope of \( L_2 \) is larger than the slope of \( L_1 \). This is equivalent to

\[-(j - i) \log R_2 \leq \log \| B_j \| - \log \| B_i \|\]

which is true because we assumed \( \| B_j \| \geq \| B_i \| R_2^{(j - i)} \). The proof for the left truncation is identical, providing changing the slope of \( L_1 \) from \(- \log R_2 \) to \( \log R_1 \).

We now consider a different situation, where we assume to be given the tropical roots \((\alpha_j)_{j \in T}\) of some tropical Laurent series for \( t \_ \_ f(x) \), and we want to determine the tropical roots of the modified function \( t \_ \_ g(x) := t \_ \_ (f(x) + p(x)) \), where \( t \_ \_ p(x) \) is a Laurent polynomial.

An example of this form can be constructed by considering \( t \_ \_ g(x) = t \_ \_ (e^x + p(x)) \); the Newton polygon of the exponential function can be described explicitly by computing the convex hull of the nodes \((j, \log b_j)\), for \( b_j = \frac{1}{j!} \). The polynomial \( t \_ \_ p(x) \) only modifies a finite number of nodes in the tropical roots. A natural question is whether it is possible to determine the new tropical roots \((\tilde{\alpha}_j)_{j \in T}\) with a finite number of comparisons.

The most natural way to address the problem is to generalize the Graham Scan algorithm [11]. To simplify the description, we may assume that \( p(\lambda) = \gamma \) is a constant. Indeed, the case \( p(\lambda) = \gamma \lambda^j \) can be dealt with by shifting the Newton polygon to the left of \( j \) positions, and the general case can be seen as a composition of a finite number of updates with monomials of this form.

Our objective is, given \( I \) the set of non-zero indices of the Laurent series as in Definition 2.4, and the mapping \( j \mapsto b_j \), to construct the modified index set \( \hat{I} \) that contains the nodes of the updated upper convex hull. As we will prove, this new set may be expressed in two ways:
Figure 5. Lines considered by the Graham Scan algorithm in Example 3.14. The algorithm does not terminate within a finite number of slope comparisons.

- We may have that $I$ and $\hat{I}$ only differ by a finite number of elements; hence, we can express $\hat{I}$ by listing such modifications.
- The set $\hat{I}$ can be obtained from $I$ by dropping all the indices to the right (resp. to the left) of 0, and adding 0.

Intuitively, we may proceed as follows:
- If $\log \gamma$ is below the segment defining the Newton polygon in 0, we stop, and we do not include $(0, \log \gamma)$ in the nodes. Otherwise, we include 0 as a node in $\hat{I}$, and move to the next point.
- Starting from $i = 1$, we compare the slope of the segment connecting $(0, \log \gamma)$ to $(j_i, \log b_{j_i})$, with $(j_{i+1}, \log b_{j_{i+1}})$. If the latter slope is smaller, or $j_i$ is the last index in $I$, then we stop.
- Otherwise, we remove the index $j_i$ from $I$, and consider the next indices in the previous bullet point.

The above procedure deals with the points on the right of $(0, \log \gamma)$, and the same algorithm must be repeated for the indices on the left.

As long as $I$ only comprises a finite number of points, the previous algorithm is feasible, and terminates within $O(\#I)$ comparisons. The new tropical polygon can be described by a set of indices $\hat{I}$, obtained with a finite number of additions and/or removals. However, if the cardinality of $I$ is infinite, in general this algorithm may not terminate, and could require to perform an infinite number of comparisons.

Example 3.14. Consider the tropical series associated with the Laurent series

$$f(\lambda) = \sum_{j=1}^{\infty} e^{\frac{j-1}{j}} \lambda^j.$$ 

Clearly, the nodes of the Newton polygon are given by $(j, 1 - \frac{1}{j})$; it is easy to verify that all these nodes belong to the upper convex hull, as illustrated in Figure 5; the convex hull is represented by the dashed red line. If we modify this function by considering

$$g(\lambda) = e + f(\lambda),$$ 

then we need to add the node $(0, 1)$ to the convex hull. If we connect this node to $(j, 1 - \frac{1}{j})$ we obtain a sequence of lines of increasing slope, that converges to the horizontal line $y = 1$. However, the Graham Scan algorithm does not terminate in finite time. The final convex hull correspond to $\{y \leq 1\}$, and is depicted by the blue solid line. The lines considered by the Graham Scan algorithm are reported in dotted green.
We remark that, for this example, the final set $\hat{I}$ is easy to describe: it is composed only by the index 0, which corresponds to a tropical root of infinite multiplicity.

It turns out that the case described in the example is essentially the only one that can “go wrong”. If we detect this situation early, we may adopt the standard Graham Scan procedure for all other cases. This is guaranteed by the following result.

**Theorem 3.15.** Let $(j, \log b_j)$ be the nodes of a Newton polygon, $j \in I \subseteq \mathbb{Z}$; similarly, let $\hat{I}$ be the set of indices corresponding to vertices of the Newton polygon for

$$
\hat{b}_j = \begin{cases}
\gamma & \text{if } j = 0 \\
b_j & \text{otherwise}.
\end{cases}
$$

Then, if any of the following conditions hold, the Graham Scan algorithm applied to the right of 0 terminates within a finite number of steps.

(i) $\alpha_\infty = \infty$  
(ii) $\alpha_\infty < \infty$ and $\limsup_{j \in I} |\log(b_j) - j \log \alpha_\infty| > \log \gamma$.

If instead $\alpha_\infty < \infty$, but condition (ii) is not satisfied, then $\hat{I} \cap \{j \geq 0\} = \{0\}$, and the root at zero has infinite multiplicity.

**Proof.** Let us assume that $\alpha_\infty = \infty$. If there is a finite number of tropical roots, the result is clearly true. Otherwise, $\alpha_\infty = \infty$ implies that the tropical roots are unbounded. In particular, the Graham Scan algorithm proceeds comparing the slopes defined by the sequences:

$$
s_i := \frac{\log b_{j_i} - \log \gamma}{j_i}, \quad t_i := \frac{\log b_{j_{i+1}} - \log b_j}{j_{i+1} - j_i},
$$

and stops as soon as $s_i \geq t_i$. We note that, as long as this condition does not hold, the sequence $s_i$ is increasing, since the “next” node $(j_{i+1}, \log b_{j_{i+1}})$ is above the line passing through $(0, \log \gamma)$ and $(j_i, \log b_j)$. This situation is visible, for instance, in Figure 5, where none of the lines satisfy the condition, and hence the slopes keep increasing. In contrast, the sequence $t_i$ converges to $-\infty$. Hence, there exists a finite index $i$ where $s_i \geq t_i$, and where the condition is satisfied.

It remains to consider the case $\alpha_\infty < \infty$. In this setting, it is not restrictive to assume that $\log \alpha_\infty = 0$, and therefore $\alpha_\infty = 1$: we just need to modify the function $t_x f(x)$ by scaling the variable as $t_x f(\alpha_\infty^{-1} x)$, which yields the Laurent coefficients $b_j \alpha_\infty^{-j}$.

With this choice, the slope of the segments in the tropical roots converges to 0, and the plot is “asymptotically flat”. In addition, all the slopes are non-negative, and therefore the sequence of log coefficients $b_j$ are non-decreasing. We now define $\xi := \lim_{j \to \infty} \log b_j$, and we distinguish two cases, one where $\xi > \log \gamma$, and the other where $\xi \leq \log \gamma$.

If $\xi > \log \gamma$, then there must be at least one point $(j, \log b_j)$ such that the slope of the segment connecting $(0, \log \gamma)$ to it is strictly positive, and this also holds for all the following points, since the $b_j$ are non-decreasing. We can now use the same argument as before: the sequences of slopes $s_i$ and $t_i$ are such that $s_i$ is non-decreasing as long as the stopping condition is not satisfied, and $t_i \to 0$. Hence, the algorithm terminates in a finite number of steps.
Otherwise, if $\xi \leq \log \gamma$, the horizontal line starting from $(0, \log \gamma)$ is above all other points, but any other line passing through the same point and with negative slope necessarily intersects the previous Newton polygon. Hence, 0 is a tropical root of infinite multiplicity, and all nodes with positive indices need to be removed. The statement follows by rephrasing the claim on the original $b_j$, for a generic $\log \alpha_{\infty} \neq 0$.

4. Applications

In this section, we discuss a few applications of our theory. Throughout this section, the radii of inclusion obtained numerically are only reported with a couple of significant digits, for improved readability.

4.1. Updating the tropical roots. We begin by considering the problem of finding approximate inclusion sets for the eigenvalues of a Laurent series (or its zeros, for scalar problems). We assume that a known (scalar, for simplicity) Taylor or Laurent series is perturbed in a finite number of coefficients, and we leverage the results in Theorem 3.15.

Example 4.1. We consider

$$g(\lambda) = e^\lambda + p(\lambda), \quad p(\lambda) := 12\lambda - \frac{\lambda^2}{5} + 12\lambda^3 - 0.04\lambda^4 + 10^{-3}\lambda^5 - 0.002\lambda^6.$$  

The Newton polygon for $e^\lambda$ is composed of all the nodes $(i, -\log i!)$, for $i \in \mathbb{Z}$. Hence, we may apply Theorem 3.15 to compute the Newton polygon for $g(\lambda)$, and then rely on Theorem 3.11 to construct inclusion results. The computation of the convex hull yields 3 segments with well-separated slopes according to Theorem 3.11, as visible in Figure 5. The exponential of their negative slopes yields (approximately) the tropical roots $\alpha_1 \approx 0.0769$, $\alpha_2 \approx 1.0337$, and $\alpha_3 \approx 12.2446$. Theorem 3.11 and Theorem 3.12 yield the following inclusions:

- The open disc of radius $r \approx 0.038$ and centered at zero, corresponding to the slope of the first segment of the Newton polygon, does not contain any root, as predicted by Theorem 3.12.
- Similarly, the annulus $A(0.038, 0.46)$ does not contain any root; in addition, there is exactly one root in the annulus $A(0.17, 0.46)$.
- Finally, the annulus $A(2.40, 5.27)$ does not contain roots, and there are exactly two roots in the annulus $A(0.46, 2.40)$.

These inclusions are displayed in the right plot of Figure 6. The roots in that Figure have been approximated by running a few steps of Newton’s iteration starting from a fine grid.

We now consider a variation of Example 4.1 where we deal with a “proper” meromorphic function, i.e., not just a Taylor series.

Example 4.2. Let

$$f(\lambda) = e^\lambda + e^{\lambda^2} - 1,$$

and let $f_n(\lambda) = \sum_{j=-n}^{n} b_j \lambda^j$ be a truncation of $f(\lambda)$ expressed as a Laurent series. Consider the function

$$g(\lambda) = f_n(\lambda) + p(\lambda)$$
where \( p(\lambda) \) is the Laurent polynomial defined by
\[
p(\lambda) := e^6\lambda^{-9} + e^{12}\lambda^{-3} + e + e^2\lambda^2 + e^{-10}\lambda^4 + e^{-14}\lambda^5 + e^{-20}\lambda^5.
\]
Similarly to Example 4.1, the Newton Polygon for \( f_n(\lambda) \) is easily determined, and is composed by the nodes of coordinates \((i, -\log(|i|!))\), with \(|i| \leq n\). Here we set \( n = 45 \) and computed numerically the updated Newton polygon using Theorem 3.15. This yields a Newton polygon with fewer nodes, with the following inclusion/exclusion annuli:

* The annulus \( A_1 := \tilde{A}(0.05, 0.17) \) does not contain any root, and the disc inside it contains 34 roots minus poles.
* The annulus \( A_2 = \tilde{A}(0.79, 3.41) \) does not contain any root, and exactly 6 roots are contained between \( A_1 \) and \( A_2 \).

4.2. Estimating the number of nodes in contour integrals. As we hinted in the introduction, tropical roots are useful if we desire to solve an eigenvalue problem through the means of a contour integral algorithm, because they give us the location of the target set \( \Omega \) where to look. One of the most important parameters of these
procedures is the quadrature rule and the number of quadrature points \( N \) to be used. The choice often falls on the trapezoidal rule, because it is easy to implement and has an exponential drop of the error on elliptical contours. However, as far as we know and as this paper is being written, there is currently no way to determine an “optimal” \( N \) a priori so that the backward error of the eigenpairs is below a given threshold.

In this section we show how the results of Theorem 3.11 can be used to justify a particular choice of \( N \). In 2016 Van Barel and Kravanja showed a relationship between contour integrals of holomorphic functions and filter functions [21]. More specifically, let \( \Omega = D(r) \) be the target set and

\[
A_p = \frac{1}{2\pi i r} \int_{\partial \Omega} z^p F(z)^{-1} \, dz \approx \frac{1}{2\pi i r} \sum_{j=0}^{N-1} w_j F(z_j)^{-1} =: \tilde{A}_0
\]

be the basic contour integral for Beyn’s algorithm and its approximation with the quadrature rule of points and weights \((z_j, w_j)\) [5]. They showed that when we use the trapezoidal rule, we can associate the filter function

\[
b_p(z) = \frac{z^p}{1 - \left(\frac{z}{\tau}\right)^N}
\]

to \( A_p \), where \( b_0(z) \) approximates the function

\[
\chi_0(z) = \begin{cases} 
1 & \text{if } z \in \Omega, \\
0 & \text{if } z \notin \Omega.
\end{cases}
\]

Note that for \( \varepsilon \ll 1 \), it follows \(|b_0(z)| < \varepsilon \) approximately when \(|z| > \varepsilon^{-1/N} \). In general they proved that any eigenvalue \( \lambda \) of \( F(z) \) that lies outside \( \Omega \) becomes an element of noise for the approximation of \( A_p \) of order \( O(|b_0(\lambda)|) \). This has two immediate consequences: if there are eigenvalues \( \lambda \) near \( \Omega \), then the quadrature rule needs a large number \( N \) of quadrature points to reduce \(|b_0(z)| \) and hence the noise; on the other hand, if there are no eigenvalues near \( \Omega \), then there is no need of a large \( N \), with an obvious saving of computational time. Therefore Theorem 3.11 is a perfect tool for this task: the annulus of exclusion can help us set an ideal parameter \( N \). The next examples will serve as a clarification.

**Example 4.3.** Consider a matrix polynomial \( P(\lambda) = \sum_{j=0}^4 B_j \lambda^j \) generated with the Matlab commands

```matlab
rng(42); n = 20;
B0 = randn(n); B1 = 1e5*randn(n);
B2 = randn(n); B3 = 1e-2*randn(n); B4 = 1e3*randn(n);
```

The associated tropical polynomial \( t_{\alpha} P(x) = \max_{1 \leq j \leq 4} \|B_j\|/x^j \) has two roots

\[
\alpha_1 = \|B_0\|/\|B_1\| \approx 10^{-5}, \quad \alpha_2 = (\|B_1\|/\|B_4\|)^{1/3} \approx 4.8
\]

of multiplicity 1 and 3, respectively. It holds that \( \delta_1 < (1 + 2\kappa(B_0))^{-2} \approx 10^{-5} \), hence there are 20 eigenvalues in \( D((1+2\kappa(B_0))\alpha_1) \), and no eigenvalues in \( \tilde{A}((1+2\kappa(B_0))\alpha_1, (1+2\kappa(B_0))^{-1}\alpha_2) \approx \tilde{A}(0.0012, 0.039) \). If we are interested in the eigenvalues inside \( \Omega = D((1+2\kappa(B_0))\alpha_1) \), then we can set \( N = 10 \), given that

\[
b_0(z) = \frac{1}{1 - \left(\frac{z}{\tau}\right)^2} \approx O(10^{-16})
\]
When $|z| \approx 0.039$. Define now the backward error of an eigenpair $(\lambda, v)$ as
\[
\eta(\lambda, v) = \frac{\|F(\lambda)v\|_2}{\|F\|_\Omega \|v\|_2},
\]
with
\[
\|F\|_\Omega := \sup_{z \in \Omega} \|F(z)\|_2.
\]
Then, a basic implementation of Beyn’s algorithm returns a backward error of approximately $10^{-15}$ for all the 20 eigenpairs.

**Example 4.4.** Consider again the polynomial $P(\lambda)$ of Example 4.3 and the function $F(\lambda) = \lambda I + e^{-\lambda}B$, where $B \in \mathbb{R}^{20 \times 20}$ is a randomly generated matrix and $I$ is the identity of the same size. Using the tools developed in this section, we want to find the tropical roots of $t_\times (P(x) + F(x)) =: t_\times G(x)$. Note that $t_\times G(x) = \sup_{j \in \mathbb{N}} \|C_j\|_2 x^j$ with
\[
\begin{align*}
C_0 &= B_0 + \frac{C}{j!}, \quad \text{for } j = 0, 2, 3, 4, \\
C_1 &= B_1 + C + I, \\
C_j &= \frac{C}{j!}, \quad \text{for } j > 4.
\end{align*}
\]
In Figure 8 we plotted the Newton polygon for $t_\times G(x)$ (using the spectral norm). There we can see the first three tropical roots
\[
\begin{align*}
\alpha_1 &= \|C_0\|_2 / \|C_1\|_2 \approx 2 \cdot 10^{-5} \\
\alpha_2 &= (\|C_1\|_2 / \|C_4\|_2)^{1/3} \approx 4.8 \\
\alpha_3 &= (\|C_4\|_2 / \|C_{21}\|_2)^{1/17} \approx 21.3
\end{align*}
\]
By applying the same reasoning of Example 4.3, we know there still are 20 eigenvalues in $\Omega = \mathcal{D}((1 + 2 \kappa(C_0))\alpha_1) := \mathcal{D}(r)$ and no eigenvalues in the annulus $\mathcal{A}((1 + 2 \kappa(C_0))\alpha_1, (1 + 2 \kappa(C_0))^{-1}\alpha_2) \approx \mathcal{A}(0.004, 0.025)$. Hence we can set $N = 18$ so that $b_0(z) = \mathcal{O}(10^{-15})$ for $|z| \approx 0.025$. Once again, a basic implementation of Beyn’s algorithm under these settings return all 20 eigenvalues with a backward error of order $\mathcal{O}(10^{-15})$. 

![Figure 8](image-url)
5. Conclusion

In this work we generalized the concept of tropical roots from polynomials to functions expressed as Taylor or Laurent series. We showed that in order to have a consistent theory, the tropical roots of $t \times f(x)$ include, possibly, not only the points of nondifferentiability but also the extrema of the interval where $t \times f(x)$ is defined as a real-valued function (and zero).

In the second part we proved that the eigenvalues localization theorem stated for matrix polynomials in [17] holds true for meromorphic Laurent functions, under minor adjustments. Furthermore, we showed how to update the Newton polygon of a Laurent function $t \times f(x)$ to obtain the Newton polygon of $t \times (f(x) + p(x))$, where $p(x)$ is a polynomial, and we proved that the strategy terminates almost surely. Finally, we hinted at a new possible application, where the presence or absence of eigenvalues in the neighbourhood of a target set $\Omega$ can help determine the optimal number of quadrature points for contour integral eigensolvers.

Tropical roots of polynomials are already beneficial for scaling and localization in the context of polynomial eigenvalue problems and scalar polynomial equation solvers. We hope that this work may initiate the use of tropical roots of meromorphic functions for similar purposes.

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