When is a bit worth much more than $k_B T \ln 2$?

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Abstract: Physical processes that obtain, process, and erase information involve tradeoffs between information and energy. The fundamental energetic value of a bit of information exchanged with a reservoir at temperature $T$ is $k_B T \ln 2$. This paper investigates the situation in which information is missing about just what physical process is about to take place. The fundamental energetic value of such information can be far greater than $k_B T \ln 2$ per bit.

Ever since Maxwell’s introduction of his famous ‘demon’ who could use information to extract free energy from a system at thermal equilibrium, it has been clear that physical processes involve tradeoffs between energy, information, and entropy [1-2]. Szilard’s 1928 investigation of Maxwell’s demon in terms of a single-particle heat engine showed that the fundamental energetic ‘value’ of a bit of information exchanged with a thermal reservoir at temperature $T$ is $k_B T \ln 2$, where $k_B$ is Boltzmann’s constant [3]. Similarly, Landauer’s principle [4] states that the erasure of a bit of information by interaction with such a reservoir requires energy $k_B T \ln 2$. In many macroscopic situations, however, a bit of information can be worth far more that $k_B T \ln 2$. For example, consider a situation in which an apple is hidden in one of two boxes, and one is allowed to choose only one box, and receive its contents. If one knows which box the apple is in (one bit of information), one can obtain the full energetic value of the apple, say, 100 kilocalories. If one doesn’t know
which box the apple is in, then one can obtain only 50 kilocalories on average. The bit of information about which box the apple is in is worth $50 \approx 10^{25} k_B T \ln 2$ at room temperature. This anecdotal example (there are many others) shows that information can have far greater energetic value than $k_B T \ln 2$ per bit. But are such macroscopic energetic values for bits fundamental? Might there be a way of getting around the apparent wastefulness of ignorance? This paper provides a formal physical and mathematical analysis to show that the answer that missing a bit of information can fundamentally require macroscopic dissipation: there is no way around it. We apply the Kolchinsky-Wolpert theorem [5] to show that, generically, gaining a bit of information about a macroscopic system can allow one to gain large amounts of free energy. Conversely, lacking that bit forces any attempt to harvest free energy to undergo large amounts of dissipation.

The last few decades have seen a revolution in non-equilibrium statistical mechanics [6-18], with the realization that many thermodynamic processes are governed by exact and unexpected relations such as the Jarzynski equality [6] and the Crooks fluctuation theorem [7]. The Kolchinsky-Wolpert theorem [5] is such an exact relation that governs the amount of work dissipated in an isothermal process. It provides a simple formula that allows the comparison between the minimum amount of work dissipated, and the actual amount dissipated. The K-W theorem states that a stochastic process in which a system exchanges energy and entropy with a bath at temperature $T$, the excess dissipated work obeys

$$W_D(r_0) - W_D(q_0) = k_B T (D(r_0\|q_0) - D(r_1\|q_1)).$$

Here, $W_D(r_0)$ is the work dissipated into the environment at temperature $T$ when the initial probabilities for the microstates $x_0$ with energy $E_0(x_0)$ of the system are given by $r_0(x_0)$; $W_D(q_0)$ is the minimum work dissipated for the optimal initial probability distribution $q_0(x_0)$ (under quite general conditions $q_0(x_0)$ has full support [5]); $D(r\|q) = -\sum_x r(x) \ln (q(x)/r(x))$ is the Kullback-Leibler divergence/relative entropy; $r_1(x_1) = \sum_{x_0} r_0(x_0)p(x_1|x_0)$ are the output probabilities when the input probability distribution was $r_0$; $q_1(x_1)$ are the output probabilities given that the input probability distribution was $q_0$. The K-W theorem is straightforward to derive and applies to arbitrarily complicated stochastic processes.

In a companion paper [19] we show that the maximum increase in the free energy of the system obeys a similar equation:

$$\Delta F(r_0) - \Delta F(\hat{q}_0) = -k_B T (D(r_0\|\hat{q}_0) - D(r_1\|\hat{q}_1)).$$
where $\Delta F(p_0)$ is the increase in free energy of the system when the initial distribution over states is $p_0(x_0)$. Here, $\hat{q}_0(x_0)$ is the initial probability distribution that maximizes the increase in free energy over the process (which is a different task from minimizing dissipated work, i.e., $\hat{q}_0 \neq q_0$ in general). All of our results hold equally for the maximum free energy increase as well as minimum dissipated work. In [19] we also show that these results hold for quantum mechanical systems under the operation of completely positive maps, with the quantum K-L divergence exchanged for the classical K-L divergence.

We apply the K-W theorem to the fundamental problem raised in the introduction: When does ignorance of the underlying stochastic process necessarily lead to macroscopically large amounts of dissipation, or require one to forgo obtaining a large increase in free energy?

Consider the following situation. We prepare our system so that the initial probabilities for its microscopic states $x_0$ are $r_0(x_0)$. We then insert our system into a ‘black box’ in contact with a bath at temperature $T$, where the system either undergoes stochastic process $A$ or stochastic process $B$, each of which occurs with probability $1/2$. We don’t know which process will occur in the box. (Below, we generalize to more than two processes, occurring with different probabilities.) We minimize the excess dissipation over all initial probability distributions $r_0$. We then compare this minimum dissipation in the absence of knowing which process takes place with the minimum average dissipation $(1/2)(W_D(q_0^A) + W_D(q_0^B))$ that can be obtained if we do know which process takes place. That is, we calculate the energetic value of the bit of information that tells us whether the underlying process is $A$ or $B$. From the K-W theorem, we see that our goal is to find the initial distribution $r_0$ that minimizes

$$\Delta = (1/2)(D(r_0\|q_0^A) - D(r_1^A\|q_1^A) + D(r_0\|q_0^B) - D(r_1^B\|q_1^B)),\quad (3)$$

where $r_1^{A,B}(x_1)$ are the final probabilities for $x_1$ when the process in the box is $A, B$.

Let the two stochastic processes that can be inside the box be defined by conditional probabilities $p_A(x_1|x_0)$ and $p_B(x_1|x_0)$ for output states $x_1$ given input states $x_0$. The method of Lagrange multipliers yields an equation for the initial distribution $r_0(x_0)$ that minimizes the quantity $\Delta$ in equation (3):

$$- \ln r_0(x_0) + (1/2)(\ln q_0^A(x_0) + \ln q_0^B(x_0)) + (1/2) \sum_{x_1} (\ln r_1^A(x_1) - \ln q_1^A(x_1))p_A(x_1|x_0) + (1/2) \sum_{x_1} (\ln r_1^B(x_1) - \ln q_1^B(x_1))p_B(x_1|x_0) = 0.\quad (4)$$
The distribution \( r_0 \) that solves these equations is not obvious. However, as we’ll now show, the amount of excess dissipation can easily be macroscopic.

As a simple example, suppose that the stochastic process \( A \) always yields the same final distribution \( r_1^A(x_1) \), independent of the initial distribution \( r_0(x_0) \). Similarly, assume that the stochastic process \( B \) always yields the same final distribution \( r_1^B(x_1) \), independent of the initial distribution \( r_0(x_0) \). For example, the process \( A \) could always end up with the system in a thermal state at temperature \( T_A \), while \( B \) always ends up with a thermal state at temperature \( T_B \). In this setting, because the final state of the process is ultimately the same no matter how one prepares the initial state, the final relative entropy drops out of equation (3), simplifying the calculation of the minimum dissipation. Minimizing the dissipation means finding the initial distribution \( r_0 \) that minimizes

\[
\Delta = (1/2) \left( D(r_0 \| q_0^A) + D(r_0 \| q_0^B) \right) = D(r_0 \| \sqrt{q_0^A q_0^B}).
\]  

(5)

Note that \( \sqrt{q_0^A(x) q_0^B(x)} \) is not in general a probability distribution – this is the key point – but it can still be inserted into the formula for the K-L divergence.

Because they are the same for all initial preparations of the state of the system, the final distributions \( r_1^{A,B} \) now drop out of equation (3), and a simple Lagrangian minimization shows that the minimum dissipation occurs when

\[
r_0 = e^{\hat{\Delta}} \sqrt{q_0^A q_0^B}.
\]  

(6)

The minimum dissipation is \( k_B T \hat{\Delta} \), where

\[
\hat{\Delta} = - \ln \left( \sum_x \sqrt{q_0^A(x) q_0^B(x)} \right),
\]  

(7)

to ensure that \( r_0 \) is a properly normalized distribution. In the quantum case, when the optimal initial density matrices are \( \chi_0^A, \chi_0^B \) for the two quantum processes \( A, B \), the minimum dissipation occurs for initial density matrix \( \rho_0 \), where \( \ln \rho_0 = \hat{\Delta} + (1/2)(\ln \chi_0^A + \ln \chi_0^B) \).

The derivation of equations (6-7) immediately gives the generalization to \( K \) processes occurring with probability \( p_k \). Let the optimal initial distribution for the \( k \)’th process be \( q_0^k \). Then we have

\[
r_0 = e^{\hat{\Delta}} \left( (q_0^1)^{p_1} \ldots (q_0^K)^{p_K} \right),
\]  

(8)

where

\[
\hat{\Delta} = - \ln \left( \sum_x \left( \sum_{k=1}^{K} (q_0^k(x))^{p_k} \right) \right).
\]  

(9)
Equations (6), (8) give the initial probability distribution over microstates of the system that minimizes excess dissipation for processes that have a fixed final distribution.

We now show that the excess dissipation $k_B T \Delta$ can be – and typically will be – a macroscopic quantity. Consider thermodynamically reversible processes $A$ and $B$, for which the minimum excess dissipation is zero. Suppose that as above, $A$ takes any initial distribution to the same final distribution, and $B$ does too (the final distribution for $B$ can be different than the one for $A$). In addition, suppose that $A$ is thermodynamically reversible when the initial state of the system is a thermal state $q^A_0(x_0) = \frac{1}{Z(\beta_A)} e^{-\beta_A E_0(x_0)}$, 

where $\beta_A = 1/k_B T_A$. Similarly, suppose that $B$ is thermodynamically reversible for an initial thermal state with the same energy function $E_0$, but a different inverse temperature $\beta_B = 1/k_B T_B$. For example, $A$ could be the optimal process for extracting work from the initial thermal state $q^A_0$ by rapidly changing the system’s energy function/Hamiltonian, putting the system in contact with a bath at temperature $T$, and isothermally varying the Hamiltonian to its desired final form [16-18]; similarly, $B$ could be the optimal process for extracting work from the initial thermal state $q^B_0$. Substituting these thermal distributions into equation (6) shows that the optimal input distribution $r_0$ is that of a thermal state with inverse temperature $(1/2)(\beta_A + \beta_B)$, and equation (7) yields

$$\hat{\Delta} = (1/2)(\ln Z(\beta_A) + \ln Z(\beta_B)) - \ln Z((1/2)(\beta_A + \beta_B)).$$

Because the excess dissipation grows as the the number of subsystems grows, for macroscopic systems $\hat{\Delta}$ can become arbitrarily large. When the system is macroscopic, with $N$ subsystems, and $\beta_A$ and $\beta_B$ differ significantly, the dissipation typically grows proportionally to $N$. For example, the partition function $Z_N(\beta)$ for $N$ non-interacting, identical systems, each with partition function $Z(\beta)$ grows as $Z_N(\beta) = (Z(\beta))^N$. In this case, the excess dissipation $k_B T \hat{\Delta}$ grows as $O(1) N k_B T$. For interacting systems, when $\beta_A$ and $\beta_B$ are close in value, we can expand equation (10) in a Taylor series around $(1/2)(\beta_A + \beta_B)$ to show that the minimum excess dissipation is

$$\hat{\Delta} = \langle (\Delta E)^2 \rangle (\beta_A - \beta_B)^2 / 2,$$

where $\langle (\Delta E)^2 \rangle$ is the variance in energy, which also scales as $N$. 

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Equation (12) is a special instance of the general case where the two processes $A$ and $B$ are very similar, so that the optimal initial distributions $q_0^A \approx q_0^B \approx q_0$. Expanding equation (3) to second order and minimizing yields excess dissipation

$$
\hat{\Delta} = (1/4)\left( \sum_{x_0} \delta q_0(x_0)^2/q_0(x_0) - \sum_{x_1} \delta q_1(x_1)^2/q_1(x_1) \right),
$$

(13)

where $\delta q_0(x_0) = q_0^A(x_0) - q_0^B(x_0)$ and $\delta q_1(x_1) = q_1^A(x_1) - q_1^B(x_1)$. That is, in the infinitesimal regime, the excess dissipation is proportional to the Fisher information distance between the optimal input distributions $q_0^A(x_0), q_0^B(x_0)$, minus the Fisher information distance between the output distributions. $\hat{\Delta}$ is non-negative because of the data processing inequality for Fisher information [20]. In the quantum case the excess dissipation is proportional to the Burres metric distance [21] between input states minus the Burres distance between output states.

Discussion: The Kolchinsky-Wolpert theorem quantifies the amount of excess dissipation that occurs during stochastic processes if one prepares a physical system in the ‘wrong’ initial state, i.e., a state that fails to minimize dissipation. Similarly, [19] quantifies the lost free energy gain when one prepares a physical system in the wrong state. This paper applied these results to the case where one is ignorant of the underlying dynamics of the system. Intuitively, if one doesn’t know what is going to happen, one’s best efforts can be far more wasteful than if one does know what is going to happen. This paper provided a rigorous treatment of this intuition in the case of stochastic processes: even if one lacks only a single bit of information about which stochastic process is going to take place, the best one can do in extracting free energy and minimizing dissipation can be far worse than if one possesses that bit.

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