EXISTENCE AND NONEXISTENCE OF TRAVELING PULSES IN
A LATERAL INHIBITION NEURAL NETWORK

YIXIN GUO AND AIJUN ZHANG

Department of Mathematics
Drexel University
Philadelphia, PA 19104, USA

(Communicated by Yuan Lou)

Abstract. We study the spatial propagating dynamics in a neural network of excitatory and inhibitory populations. Our study demonstrates the existence and nonexistence of traveling pulse solutions with a nonsaturating piecewise linear gain function. We prove that traveling pulse solutions do not exist for such neural field models with even (symmetric) couplings. The neural field models only support traveling pulse solutions with asymmetric couplings. We also show that such neural field models with asymmetric couplings will lead to a system of delay differential equations. We further compute traveling 1–bump solutions using the system of delay differential equations. Finally, we develop Evans functions to assess the stability of traveling 1–bump solutions.

Contents

1. Introduction 1730
2. Neural network equations and traveling patterns 1731
  2.1. The coupling and gain functions 1732
  2.2. Reflection of traveling pulse solution with respect to the shift \( x_0 \) 1733
  2.3. Non-existence of traveling N-bump solution for symmetric coupling 1733
3. Traveling 1–bump solution for asymmetric coupling and Heaviside gain 1736
4. Traveling 1–bump solution for asymmetric coupling and piecewise linear gain 1738
  4.1. Link between integral equations and delay equations 1739
  4.2. Matching conditions 1741
5. Stability 1744
6. Discussion 1749
Appendix 1751
Acknowledgments 1753
REFERENCES 1753

2010 Mathematics Subject Classification. Primary: 34B05, 34B15, 34D20, 35Q92, 92B20.
Key words and phrases. Neural field model, propagation dynamics, traveling pulses, saddle node bifurcation, delay differential equation, Evans function.
Yixin Guo is supported by NSF grant DMS–1226180.
1. **Introduction.** Neural field models have been widely used to study a diverse range of neurobiological phenomena including wave propagation, spiral waves and Turing patterns. Neuronal waves such as traveling fronts and pulses have been observed in vivo in a number of sensory cortical areas and in vitro experiments using thin brain slices [2, 9, 10, 28, 31, 33, 43]. Traveling fronts can occur in an excitatory neural network when inhibition is blocked [6, 15, 22]. In the presence of slow adaptation, a traveling pulse can occur [1, 11, 20, 31, 41]. In this paper, we focus on the neuronal wave propagation of the neural field model considering asymmetric excitatory and inhibitory synaptic connections of neurons without any slow adaption.

The current paper is devoted to the study of propagation dynamics in a neural network. Consider the following neural field equation,

\[ \tau \frac{\partial u(x, t)}{\partial t} = -u(x, t) + \int_{-\infty}^{\infty} w(x - y)f(u(y, t))dy, \]  

(1)

where \( u \) represents the synaptic input to neurons located at position \( x \) and time \( t \). The coupling function \( w(\cdot) \) determines the connections between neurons which are related to excitation and inhibition. Without loss of generality, we set the synaptic time \( \tau \) to 1. The gain function, \( f(u) \), represents the firing rate of neurons in the network. In the current paper, we focus on a traveling solution that is of the form \( u(x, t) = \phi(x - ct) \), and \( \lim_{\xi \to \pm \infty} \phi(\xi) = 0 \).

The neural field model (1) has been extensively studied by many authors but still far from being well-understood. Previous work, for example, [1, 6, 11, 13, 14, 22, 23, 24, 31, 33, 41, 42], have probed standing and traveling patterns. Many of these work focused on the neural field model (1) with the Heaviside gain, including Amari’s pioneer work [1]. Amari studied dynamics of pattern formation in lateral-inhibition type of neural fields with the Heaviside gain function \( n \) [1]. With an even coupling function and a piecewise linear gain in Equation (1), one of the current authors considered the existence and stability of both traveling front and standing pulse solutions [22, 23, 24]. For traveling pulse solutions, some authors considered a neural field model with pure excitatory coupling and a slow local negative feedback component [26, 31, 32]. One common feature of the aforementioned works is that the coupling function is symmetric, no matter the neural field is pure excitatory or lateral inhibitory. Several studies dealt with spatiotemporal patterns of the neural field model using asymmetric couplings [7, 34, 40, 44]. The authors examined traveling patterns of the neural field model with or without a negative feedback using asymmetric couplings. In these studies, they all set the gain to be the Heaviside function except that [44] investigated pulse propagation of the neural field model with a linear threshold gain and other special assumptions. Traveling pluses to Equation (1) with asymmetric couplings and a more general gain function, rather than the Heaviside step function, have not been studied thoroughly.

In the current study, we focus on the neural field model (1) with piecewise linear gain functions, of which the Heaviside step function is a special case, and asymmetric couplings, generated by translational shifts of symmetric functions. Our primary goal is to investigate the nonexistence, existence and stability of traveling pulses and bifurcations of spatiotemporal patterns. We show that standing pulses may exist for the model with symmetric couplings while traveling N-bump solutions only exist for asymmetric couplings. We further investigate the existence of traveling 1-bump solutions by computing the solutions of a system of delay differential equations.
We can apply an Evans function approach to determine the stability of standing and traveling solutions \([22, 37, 38, 45, 46]\). Evans function gained its name in the series of papers by John W. Evans \([16, 17, 18, 19]\) and has been widely used to study stability of dynamic patterns. For example, in \([37, 38]\), Sandstede used Evans functions to assess nonlinear stability of traveling waves. In \([45]\), Zhang developed an Evans function and investigated asymptotic stability of traveling front solutions for a system of synaptically coupled neuronal networks and in \([46]\) analyzed the existence and exponential stability of traveling wave/pulse solutions of a singularly perturbed system of integral differential equations. In \([12]\), Coombes and Owen applied Evans functions for integral neural field equations with Heaviside firing rate functions. In \([7]\), Bressloff and Wilkerson developed the Evans function and determined the stability of traveling 1-bump solution to Equation (1) with a Heaviside gain function. However, for the neural field model with our piecewise linear gain, we cannot apply the above mentioned approach \([37, 38, 45, 46]\) directly. We will derive the associated Evans function in our specific case and assess the stability of traveling 1-bump solutions of (1) in Section 5.

The remainder of the paper is organized as follows. In Section 2, we introduce the asymmetric coupling generated by a translational shift of a ‘Mexican Hat’, the gain functions, the definition of traveling pulses and reflection of traveling solutions with respect to the shift in the coupling. We also show that there are no traveling pulse solutions to Equation (1) for symmetric coupling functions. In Section 3, we consider the existence of traveling pulses and the saddle-node bifurcations with respect to the shift in the coupling to Equation (1) of Amari type. In Section 4, reformulating the neural field model to a system of delay differential equations, we consider the existence of traveling pulses and the saddle-node bifurcations with respect to the shift and the gain to Equation (1) with a piecewise linear gain. In Section 5, we derive Evans functions to traveling 1-bump solutions found in Section 4 and assess their stability. We discuss technical issues and possible extensions in Section 6. Section 6 contains an appendix with details in proofs of lemmas or parameters unspecified in previous sections.

All numerical calculations and graphs are obtained by mathematical solver including Mathematica, Maple and Matlab.

2. Neural network equations and traveling patterns. We are interested in traveling solutions in the form \(u(x,t) = u(x-ct)\), where \(c \neq 0\). Letting \(\xi = x-ct\), we rewrite the neural field equation (1) as

\[
-cu'\xi = -u(\xi) + \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta. \tag{2}
\]

Let the region of excitation of a traveling solution of (2) to be \(R[u] = \{\xi \in \mathbb{R} | u(\xi) > h\}\), and \(R[u]\) is the union of \(N\) disjoint bounded intervals, where \(N\) is a positive integer. For a traveling pulse solution of (2), there exists \(2N\) \(\xi_j\) such that \(u(\xi_j) = h\) for \(j = 1, ..., 2N\). Due to translational invariance of traveling solutions, we set \(\xi_1\) to 0. Then we define traveling pulse solutions of (2) as follows:

**Definition 2.1 (Traveling pulse solution).** A solution \(u(\xi)\) of (2) is called a traveling \(N\)-bump solution with speed \(c \neq 0\) if it satisfies
\begin{align}
\lim_{\xi \to \pm \infty} u(\xi) &= 0; \\
u(\xi) &= h \text{ only at } \xi_j \text{ for } j = 1, \ldots, 2N; \\
u'(\xi_j) &> 0 \text{ for } j \text{ odd, } u'(\xi_j) < 0 \text{ for } j \text{ even}; \\
u(\xi) &\geq h \text{ for } \xi \in [\xi_{2j-1}, \xi_{2j}], j = 1, 2, \ldots, N, \text{ otherwise } u(\xi) < h,
\end{align}

where \([\xi_{2j-1}, \xi_{2j}]\) for \(j = 1, 2, \ldots, N\) are the \(N\) disjoint bounded intervals such that \(\xi_{j_1} < \xi_{j_2}\) whenever \(j_1 < j_2\), and \(\xi_1 = 0\).

2.1. **The coupling and gain functions.** The ‘Mexican Hat’ coupling function, satisfying the following conditions, has been used in many of the previous work (\([1, 6, 11, 14, 22, 23, 24, 31, 33]\) and the references therein).

\textbf{(H0)} ‘Mexican Hat’ function \(w_0(x)\) satisfies the following:
1. \(w_0 \in L^1(\mathbb{R})\) is symmetric, that is, \(w_0(x) = w_0(-x)\).
2. \(w_0(x)\) is decreasing on \([0, \gamma]\) and increasing on \([\gamma, \infty)\). Moreover, \(w_0(x)\) obtains a negative minimum at \(\gamma\), a positive maximum at 0 and \(\lim_{x \to \infty} w_0(x) = 0\).

In this paper, we introduce asymmetry into the ‘Mexican hat’ function by a horizontal shift. Therefore we focus on coupling functions in the form \(w(x) = w_0(x - x_0)\), for fixed \(x_0 \in \mathbb{R}\). If \(x_0 \neq 0\), it causes the network to be asymmetric and induces direction sensitivity. Our motivation to introduce asymmetry into the ‘Mexican hat’ is that the symmetric coupling supports only the standing pulses and traveling fronts \([22, 23, 24]\) but not traveling pulses at all. Nonexistence of traveling pulses with symmetric couplings is proved in Section 2.3. This shift parameter is the key for traveling pulse solution to exist. The neural field model with similar (or same) asymmetric coupling function were studied in \([7, 44]\). Here we consider a similar mechanism for generating pulses, based on asymmetric excitatory/inhibitory synaptic connections. Figure 1 shows two examples of asymmetric ‘Mexican hat’ coupling function \(w(x) = Ae^{-a|x-x_0|} - Be^{-b|x-x_0|}\), with shift parameter \(x_0 > 0\)(solid black curve) and \(x_0 < 0\) (solid grey curve), where \(A, a, B, b, x_0\) are given constants.

![Figure 1](image)

**Figure 1.** The middle function curve for \(w(x) = Ae^{-a|x|} - Be^{-b|x|}\) is symmetric and the other two are asymmetric for \(w(x-x_0)\). The right solid black curve is for \(x_0 > 0\) and the left grey curve is for \(x_0 < 0\).

The gain function we consider in this paper is given by

\[f(u) = (\alpha(u - h) + 1)\Theta(u - h),\]
where $\alpha \in \mathbb{R}$, and $\Theta(\cdot)$ is the Heaviside function with threshold $h$, that is, $\Theta(u) = 1$ if $u > 0$, $\Theta(u) = 0$ otherwise (see Figure 2).

![Figure 2. The gain function for $f(u) = (\alpha(u - h) + 1)\Theta(u - h)$.

2.2. Reflection of traveling pulse solution with respect to the shift $x_0$. Assume that $w_0(x)$ satisfies $(H0)$. We denote $w^+(x) = w_0(x - x_0)$ and $w^-(x) = w_0(x + x_0)$ for fixed $x_0 > 0$. Note that $w^+(x) = w^-(x)$.

Lemma 2.2. If $\phi(\cdot)$ such that $u(x, t) = \phi(x - ct)$ is a solution of

$$\frac{\partial u(x, t)}{\partial t} = -u(x, t) + \int_{-\infty}^{\infty} w^+(x - y)f(u(y, t))dy.$$

(4)

Then $v(\tau, z) = \phi(-z + ct\tau)$ is a solution of

$$\frac{\partial v(\tau, z)}{\partial \tau} = -v(\tau, z) + \int_{-\infty}^{\infty} w^-(z - w)f(v(w, \tau))dw.$$

(5)

Proof. This can be verified by letting $x = -z, t = \tau, y = -w$ and $v(\tau, z) = \phi(x - ct) = \phi(-z + ct\tau)$, together with $w^+(x) = w^-(x)$. \hfill \Box

Lemma 2.2 implies that if we have a traveling solution with speed $c$ ($c > 0$) for coupling $w^+$, then there exists a traveling solution with speed $-c$ for $w^-$. Moreover, letting $\eta = z + ct\tau$, $v(\tau, z) = \phi(-\eta)$, which implies that the shape of the traveling solution is the reflection of the previous one (see Figure 3). Due to the reflection, throughout all later sections, we always assume $x_0 \geq 0$ in $w(x) = w_0(x - x_0)$, hence the traveling speed $c > 0$.

2.3. Non-existence of traveling N-bump solution for symmetric coupling. In this subsection, we will prove that for an even coupling function with shifting parameter $x_0 = 0$, that is, the coupling satisfies $(H0)$, there are no traveling $N$-bump solutions for any finite $N$. We prove the non-existence of traveling $N$-bump solution with even coupling by contradiction. Assuming that $u(\cdot)$ is a traveling $N$-bump solution of (2) with a symmetric coupling, as in Definition 2.1 with $u(\xi_j) = h$ for $j = 1, \ldots, 2N$ and $\xi_1 = 0$, we first prove the following lemma.

Lemma 2.3. Let $w(\cdot)$ satisfy $(H0)$ and $\alpha = 0$. If $u$ is a traveling $N$-bump solution to (2), then $\sum_{i=1}^{N} u'(\xi_{2i-1}) = \sum_{i=1}^{N} u'(\xi_{2i})$. 
Figure 3. Reflection of traveling pulse solution for 
\[ w(x) = A e^{-a|x-x_0|} - B e^{-b|x-x_0|} \]
and \( f(u) = (\alpha (u - h) + 1) \Theta(u - h) \)
with \( A = 5, a = 0.42, B = 1, b = 0.1, h = 2.7 \) and \( \alpha = 0.00293 \). The figure to the left is for two traveling 1-bump solutions with positive speeds for \( x_0 = -7.75 \) and the figure to the right is for those reflection ones with negative speeds for \( x_0 = 7.75 \).

Proof. Since \( \alpha = 0 \), for a traveling N–bump solution, the system can be reduced to the following,
\[ -cu'(\xi) = -u(\xi) + \sum_{i=1}^{N} \int_{\xi_{2i-1}}^{\xi_{2i}} w(\xi - \eta) d\eta. \]
\( (6) \)
Let \( W(t) \) be an anti-derivative of \( w(t) \), that is, \( W'(t) = w(t) \). Thus,
\[ -cu'(\xi) = -u(\xi) + \sum_{i=1}^{N} [W(\xi - \xi_{2i-1}) - W(\xi - \xi_{2i})]. \]
\( (7) \)
Together with \( u(\xi_j) = h \) for \( j = 1, ..., 2N \), summing Equation (7) at odd and even nodes \( \xi_j \) respectively, we have
\[ c \sum_{k=1}^{N} u'(\xi_{2k-1}) = Nh - \sum_{k=1}^{N} \sum_{i=1}^{N} [W(\xi_{2k-1} - \xi_{2i-1}) - W(\xi_{2k-1} - \xi_{2i})] \]
and
\[ c \sum_{k=1}^{N} u'(\xi_{2k}) = Nh - \sum_{k=1}^{N} \sum_{i=1}^{N} [W(\xi_{2k} - \xi_{2i-1}) - W(\xi_{2k} - \xi_{2i})]. \]
Thus, to prove the lemma, it suffices to show that
\[ \sum_{k=1}^{N} \sum_{i=1}^{N} [W(\xi_{2k-1} - \xi_{2i-1}) - W(\xi_{2k-1} - \xi_{2i})] = \sum_{k=1}^{N} \sum_{i=1}^{N} [W(\xi_{2k} - \xi_{2i-1}) - W(\xi_{2k} - \xi_{2i})]. \]
Since \( w(\xi) = w(-\xi) \), \( W(\xi) = -W(-\xi) \) for any \( \xi \in \mathbb{R} \). Therefore \( \sum_{k=1}^{N} \sum_{i=1}^{N} W(\xi_{2k} - \xi_{2i}) = 0 \) and \( \sum_{k=1}^{N} \sum_{i=1}^{N} W(\xi_{2k-1} - \xi_{2i-1}) = 0 \). Then to prove the lemma, it suffices to
show that
\[ -\sum_{k=1}^{N} \sum_{i=1}^{N} W(\xi_{2k-1} - \xi_{2i}) = \sum_{k=1}^{N} \sum_{i=1}^{N} W(\xi_{2k} - \xi_{2i-1}), \]
which can be followed from \( W(\xi) = -W(-\xi). \)

\[ \square \]

**Theorem 2.4.** Assume that \( w(\cdot) \) satisfies (H0). There are no traveling \( N \)-bump solutions to equation (2).

**Proof of theorem 2.4.** We first consider the simple case with \( \alpha = 0 \). Suppose that \( u(\xi) \) is a traveling \( N \)-bump solution. By Lemma 2.3, \( \sum_{i=1}^{N} u'(\xi_{2i-1}) = \sum_{i=1}^{N} u'(\xi_{2i}) \), which contradicts with \( \sum_{i=1}^{N} u'(\xi_{2i-1}) > 0 > \sum_{i=1}^{N} u'(\xi_{2i}) \) followed from Definition 2.1.

In the case of \( \alpha \neq 0 \), we show the non-existence of traveling pulse solution with symmetric \( w(\cdot) \) by contradiction. We assume that there exists a traveling pulse solution \( u(\cdot) \) to equation (2) with speed \( c \neq 0 \). In the rest of the proof, we will write \( u \) and \( u' \) as \( u(\xi) \) and \( u'(\xi) \) respectively. Then
\[ -cu' = -u + \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta. \]

Multiplying \( f'(u)u' \) to both sides, we have
\[ -cf'(u)[u']^2 = -f'(u)u'u' + f'(u)w' \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta. \]
Integrating over \(( -\infty, \infty) \), we have
\[ -c \int_{-\infty}^{\infty} f'(u)[u']^2 d\xi = -\int_{-\infty}^{\infty} f'(u)u'ud\xi + \int_{-\infty}^{\infty} f'(u(\xi))u'(\xi) \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta d\xi. \]
Note that
\[ -\int_{-\infty}^{\infty} f'(u)u'ud\xi = -\sum_{k=1}^{N} \int_{\xi_{2k-1}}^{\xi_{2k}} f'(u)u'ud\xi = -\sum_{k=1}^{N} \int_{u(\xi_{2k-1})}^{u(\xi_{2k})} f'(u)udu = 0. \]
Therefore
\[ -c \int_{-\infty}^{\infty} f'(u)[u']^2 d\xi = \int_{-\infty}^{\infty} f'(u(\xi))u'(\xi) \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta d\xi. \]

One can show that \( I := \int_{-\infty}^{\infty} f'(u(\xi))u'(\xi) \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta d\xi = 0 \) by modifying the arguments of Theorem 3.1 in [15]. Since \( |\int_{-\infty}^{\infty} f'(u(\xi))[w'(\xi)]^2 d\xi| \geq |\alpha| \int_{\xi_{1}}^{\xi_{2}} [w'(\xi)]^2 d\xi > 0 \), \( c \) has to be zero, contradicting with \( c \neq 0 \).

Finally, to show that \( I = 0 \), we use the fact that
\[ \int_{-\infty}^{\infty} f'(u(\xi))u'(\xi) \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta d\xi = 0, \]
which is subtracted from \( I \), then we have
\[ I = \int_{-\infty}^{\infty} f'(u(\xi))u'(\xi) \int_{-\infty}^{\infty} w(\xi - \eta)(f(u(\eta)) - f(u(\xi)))d\eta d\xi. \]
If we interchange $\xi$ and $\eta$ in the right hand side of equation (8), and then add to $I$, we have

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\xi - \eta)(f'(u(\eta)) - f(u(\xi)))((f'(u(\xi))u'(\xi) - f'(u(\eta))u'(\eta))d\eta d\xi.$$ 

Let $t = \xi - \eta$ and then

$$I = -\frac{1}{2} \int_{-\infty}^{\infty} w(t) \int_{-\infty}^{\infty} (f(u(\xi - t)) - f(u(\xi)))((f'(u(\xi))u'(\xi) - f'(u(\xi - t))u'(\xi - t))d\xi dt.$$ 

Note that

$$\int_{-\infty}^{\infty} (f(u(\xi - t)) - f(u(\xi)))((f'(u(\xi))u'(\xi) - f'(u(\xi - t))u'(\xi - t))d\xi = \frac{1}{2} (f(u(\xi - t)) - f(u(\xi)))^2\int_{-\infty}^{\infty} = 0,$$

which implies that $I = 0$ as required.

3. Traveling 1–bump solution for asymmetric coupling and Heaviside gain. In the special case when $\alpha = 0$, the gain function is given by $f(u) = \Theta(u - h)$, the Heaviside function. In this section, we consider traveling 1–bump solutions for asymmetric couplings and Heaviside gains. Letting the width of the bump be $\theta$, the neural field equation (2) can be reduced to the following,

$$-cu'(\xi) = -u(\xi) + \int_{0}^{\theta} w(\xi - \eta) d\eta.$$ 

Bressloff and Wilkerson have studied the existence and stability of traveling 1–bump solutions for Heaviside gains with asymmetric couplings to (9) in [7]. In this section, by assuming that $w_0$ satisfies $(H0)$ and $w(x) = w_0(x - x_0)$, we further explore the relation between the speed $c$, the width of the traveling 1–bump pulse, $\theta$, and the shift $x_0$ using (9).

Let $W(\cdot)$ be such that $W'(t) = w(t)$. Thus, \( \int_{0}^{\theta} w(\xi - \eta) d\eta = W(\xi) - W(\xi - \theta) \) and by variational formula of ODE, we have the general solution of (9),

$$u(\xi) = u(0)e^{\xi/c} - \frac{1}{c} e^{\xi/c} \int_{0}^{\xi} e^{-s/c}(W(s) - W(s - \theta)) ds$$

(10)

In order for (10) to qualify as a candidate of a traveling 1–bump solution, $u(\xi)$ must satisfy $u(\pm\infty) = 0$ and $u(0) = u(\theta) = h$. It is trivial to verify $u(\pm\infty) = 0$. The threshold conditions $u(0) = u(\theta) = h$ generate the following system:

$$\begin{cases}
(\theta - \frac{1}{c} \int_{0}^{\theta} e^{-s/c}(W(s) - W(s - \theta)) ds) e^{\theta/c} = h; \\
(\theta - \frac{1}{c} \int_{\infty}^{\theta} e^{-s/c}(W(s) - W(s - \theta)) ds) = 0.
\end{cases}$$

(11)

For a fixed $x_0$, the two unknowns in (11) are the speed $c$ and the width $\theta$. The two equations in system (11) define two curves on the $(\theta, c)$ plane. The exact number of traveling 1–bump solutions for a fixed $x_0$ depends on the number of intersections of the two curves defined by system (11).

An example is shown in Figure 4 for $w(x) = Ae^{-a|x-x_0|} - Be^{-b|x-x_0|}$ with $A = 5, b = 0.1, B = 1, a = 0.42$ and $h = 2.7$. There are two intersections in the left figure for $x_0 = 8.10$. Then there are two traveling 1–bump solutions. The one with larger width and slower speed is stable while the unstable one with smaller width travels with faster speed. The stability will be verified in later section 5.
We define a saddle-node bifurcation with respect to the shift in the neural field model with asymmetric coupling as the following: there is a positive number $x_0^*$, called a saddle–node point, such that

- If $-x_0^* < x_0 < x_0^*$, there are two traveling 1-bump solutions.
- At $x_0 = \pm x_0^*$, there is exactly one traveling 1-bump solution.
- If $x_0 < -x_0^*$ or $x_0 > x_0^*$, there are no traveling 1-bump solutions.

In the same example given in Figure 4, the middle figure (II) shows that there is only one traveling 1-bump solution when $x_0^* \approx 8.122$. As we increase $x_0$ further, there are no more traveling 1-bump solutions (right figure (III) in Figure 4).

![Figure 4](image-url)

**Figure 4.** The plots are for $w(x) = Ae^{-a|x-x_0|} - Be^{-b|x-x_0|}$ with $A = 5$, $b = 0.1$, $B = 1$, $a = 0.42$ and $h = 2.7$. The dashed and solid curves are corresponding to the first and the second equations in (11) respectively with $c$ and $\theta$ as unknowns. Bifurcation occurs at $x_0^* \approx 8.122$. (I) Two Solutions for $x_0 = 8.10$. (II) One solution for $x_0 = 8.122$. (III) No solution for $x_0 = 8.15$.

To further investigate the relations between shift $x_0$ with speed $c$ and width $\theta$, we first prove the following proposition:

**Proposition 1.** If $\hat{W} = W(\infty) - 2W(0) > 0$, then $c$ has the same sign as $x_0$.

**Proof.** By the reflection property of traveling solutions, it suffices to show that $c > 0$ whenever $x_0 > 0$. Without loss of generality, assume that $x_0 > 0$. By Equation (9), we have $-cu'(0) = h + W(0) - W(-\theta)$ and $-cu'(-\theta) = h + W(\theta) - W(0)$. Thus, $c(u'(0) - u'(-\theta)) = W(\theta) + W(-\theta) - 2W(0)$. Since $u'(0) - u'(-\theta) > 0$, $c$ has the same sign as $W(\theta) + W(-\theta) - 2W(0) := G(\theta)$. We claim that $G(\theta)$ only obtains the absolute minimum at 0. Computing the critical points, we set $G'(\theta) = w(\theta) - w(-\theta) = 0$. By observation of Figure 1, there are three intersections for $w(\theta)$ and $w(-\theta)$, which implies that there are three critical points for $G(\theta)$. Among these three critical points, 0 is the only local minimum. Since $G(\pm\infty) = \hat{W} > 0$, 0 is an absolute minimum, as required.

By proposition 1, we generate the plot in II of Figure 5 using the same set of parameters as in Figure 4. The result is consistent with Figure 4. The saddle node bifurcation occurs at $x_0^* = 8.122$, and for a fixed shift $x_0$ such that $0 < x_0 < x_0^*$, there is one larger (slower) traveling 1-bump solution with width and speed corresponding to the points of dashed curve in Figure 5 and one smaller (faster) traveling pulse with width and speed corresponding to the points of solid curve in Figure 5. For $x_0$ greater than $x_0^*$, there is no traveling 1-bump solution. When $x_0$ is approaching...
zero, as it turns out the traveling speed is near zero, which is consistent with the fact that a symmetric coupling does not support traveling pulse. Interestingly, using our reflection property given in Lemma 2.2, the graph of the width versus \( x_0 \) will be a reflection with respect to \( x_0 = 0 \), which leads to a mirrored graph in quadrant II. The graph of the speed versus \( x_0 \) is a reflection with respect to \((0,0)\) that leads to a graph in quadrant III.

![Graph](image)

**Figure 5.** The plots are for \( w(x) = Ae^{-a|x-x_0|} - Be^{-b|x-x_0|} \) with \( A = 5, b = 0.1, B = 1, a = 0.42, \) and \( h = 2.7 \). (I) \( \theta \) versus \( x_0 \). (II) \( c \) versus \( x_0 \).

4. **Traveling 1–bump solution for asymmetric coupling and piecewise linear gain.** In this section and later sections 5 and 6, we assume that \( w(x) = Ae^{-a|x-x_0|} - Be^{-b|x-x_0|} \). One of the main goals of the current study is to find traveling 1–bump solutions of the neural field equation (2), which is the following:

\[
-cu'(\xi) = -u(\xi) + \int_{-\infty}^{\infty} w(\xi - \eta)f(u(\eta))d\eta,
\]

using piecewise linear gain with nonzero \( \alpha \), and nonsymmetric coupling with shift parameter \( x_0 \). The shift parameter \( x_0 \) not only plays a critical role in the existence of the traveling pulse solutions. It also poses new challenges in solving the neural
field equation for traveling pulse solutions. Due to the nonzero shift parameter \(x_0\), we are not able to convert the neural field equation (2) into ordinary differential equations, as demonstrated below.

Let \(\eta' = \eta + x_0\), equation (2) turns into

\[- cu' (\xi) = - u (\xi) + \int_{-\infty}^{\infty} w_0 (\xi - \eta') f (u (\eta' - x_0)) d\eta'. \tag{12}\]

One can still apply Fourier transform to equation (12) to deconvolve the convolution. However, this transformation will lead to a system of delay differential equations, which is a major difference from previous work in finding standing pulse and traveling front solutions (see [22, 23, 24]). We show the link between equation (12) and a system of delay differential equations in detailed derivation in subsection 4.1. Then we will solve this system by deriving matching conditions for the system of delay differential equations.

### 4.1. Link between integral equations and delay equations.

Based on the techniques used in [23], we convert Equation (12) into fifth order delay differential equations (14) and (15), which is a major difference to the case with an even coupling kernel. For a function \(g \in L^1 (\mathbb{R})\), the Fourier transform and its inverse Fourier transform are defined by

\[\mathcal{F}[g (s)] = \int_{\mathbb{R}} g (\xi) e^{-is\xi} d\xi\]

and

\[g (\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[g (s)] e^{is\xi} ds.\]

Note that \(\mathcal{F}[g (\xi - x_0)] = \mathcal{F}[g (\xi)] e^{-isx_0}\), \(\mathcal{F}[e^{-a|\xi|}] = \frac{2a}{a^2 + s^2}\) and \(\mathcal{F}[g^{(n)} (\xi)] = (is)^n \mathcal{F}[g]\). Then we have

\[\mathcal{F}[w] = e^{-isx_0} (\frac{2a}{a^2 + s^2} - B \frac{2b}{b^2 + s^2}) = e^{-isx_0} (\frac{2(Aa - Bb)s^2 + 2(Aab^2 - Bba^2)}{a^2b^2 + (a^2 + b^2)s^2 + s^4}).\]

Using \(\mathcal{F}[-cu' + u] = \mathcal{F}[w * f] = \mathcal{F}[w] \mathcal{F}[f]\), we have

\[(a^2b^2 + (a^2 + b^2)s^2 + s^4) \mathcal{F}[-cu' + u] = e^{-isx_0} (2(Aa - Bb)s^2 + 2(Aab^2 - Bba^2)) \mathcal{F}[f]. \tag{13}\]

For the left of (13), using the linear property of the Fourier Transform with the identities \(\mathcal{F}[u^{(n)}] = (is)^n \mathcal{F}[u]\), we have

\[(a^2b^2 + (a^2 + b^2)s^2 + s^4) \mathcal{F}[-cu' + u] = -\mathcal{F}[(cu^{(v)} - u^{(v)}) - (a^2 + b^2)cu''' + (a^2 + b^2)u'' + a^2b^2cu' - a^2b^2u].\]

For the right of (13), letting \(F(u (\xi - x_0)) = 2(Aa - Bb) \frac{d^2 f (u (\xi - x_0))}{dx^2} - 2ab(AB - Ba) f (u (\xi - x_0))\), we have

\[e^{-isx_0} (2(Aa - Bb)s^2 + 2ab(AB - Ba)) \mathcal{F}[f (u (\xi))] = -\mathcal{F}[F(u (\xi - x_0))].\]

Therefore,

\[cu^{(v)} - u^{(v)} - (a^2 + b^2)cu''' + (a^2 + b^2)u'' + a^2b^2cu' - a^2b^2u = F(u (\xi - x_0)). \tag{14}\]

Note that if \(u (\xi - x_0) < h\), that is, \(\xi < x_0\) or \(\theta + x_0\), we have \(F(u (\xi - x_0)) = 0\). Therefore, for \(\xi < x_0\) or \(\xi > \theta + x_0\),

\[cu^{(v)} - u^{(v)} - (a^2 + b^2)cu''' + (a^2 + b^2)u'' + a^2b^2cu' - a^2b^2u = 0. \tag{15}\]
The corresponding characteristic equation for Equation (15) is given by
\[ c\lambda^5 - \lambda^4 - (a^2 + b^2)c\lambda^3 + (a^2 + b^2)\lambda^2 + a^2b^2c\lambda - a^2b^2 = 0. \] (16)

Solving Equation (16) gives us the zeros \( \{ \pm b, \pm a, \frac{1}{c} \} \) of characteristic equation (16). If \( a, b, \frac{1}{c} \) are distinct, then the general solution of (15) is given by
\[ u(\xi) = c_1e^{b\xi} + c_2e^{-b\xi} + c_3e^{a\xi} + c_4e^{-a\xi} + c_5e^{\frac{1}{c}\xi}. \] (17)

We can also obtain the general solution of (15) for the case that two or three are equal among \( \{ b, a, \frac{1}{c} \} \) by ODE variational formula. Hereafter we only provide the exact form for the case that \( a, b, \frac{1}{c} \) are distinct.

The traveling pulse solution \( u(\xi) \) satisfies \( u(\pm\infty) = 0 \) by definition. For \( c > 0 \), by equation (15), \( u(\xi) \) must have for the following left and right pieces on its appropriate domain where \( u(\xi - x_0) < h \):

For \( \xi \leq x_0 \),
\[ u(\xi) = c_1e^{\frac{1}{c}\xi} + c_2e^{a\xi} + c_3e^{b\xi}, \] (18)
and for \( \xi \geq \theta + x_0 \),
\[ u(\xi) = c_4e^{-a\xi} + c_5e^{-b\xi}. \] (19)

When \( u(\xi - x_0) \) is above threshold \( h \), i.e. \( x_0 \leq \xi \leq \theta + x_0 \), \( u(\xi) \) must satisfy the delay differential equation (14) with delay \( x_0 \) showing in the nontrivial and nonlinear term \( F(u(\xi - x_0)) \). We solve the delay differential equation using Myshkis’ Method of Steps [29]. First, since we have explicit form of \( u(\xi) \) for \( \xi \in [0, x_0] \), we can have explicit form of \( F(u(\xi - x_0)) \) for \( \xi \in [x_0, 2x_0] \). Then equation (14) becomes a regular ODE on \([x_0, 2x_0]\), and we can solve it by variational formula. We continue to repeat this process on the next interval \([2x_0, 3x_0]\) until we reach or go over \( \theta + x_0 \).

The number of steps equals to \( \lceil \frac{\theta}{x_0} \rceil \), where the ceiling function \( \lceil x \rceil \) is given by \( \lceil x \rceil = \text{min}(n \in \mathbb{Z} | n \geq x) \). Obviously, \( \lceil \frac{\theta}{x_0} \rceil \geq 1 \) since \( \theta > 0 \) and \( x_0 > 0 \). Thus, we can define \( u(\cdot) \) piecewisely on \([x_0, \theta + x_0]\),
\[ u(\xi) = u_i(\xi), \quad \xi \in [ix_0, \text{min}\{(i + 1)x_0, \theta + x_0\}], \quad \text{for} \quad i = 1, \ldots, \lceil \frac{\theta}{x_0} \rceil \] (20)
where \( u_i(\xi) \) will be specified in the appendix. In summary, equations (14) can be solved by the following steps:

1. On \((-\infty, x_0]\), the leftmost piece \( u_i(\xi) \) is given by Equation (18).
2. Solve Equation (14) on \( \xi \in [ix_0, \text{min}\{(i + 1)x_0, \theta + x_0\}] \) for \( i = 1, \ldots, \lceil \frac{\theta}{x_0} \rceil \). We first set \( i = 1 \). We repeat the following:
   (i) if \( i \leq \lceil \frac{\theta}{x_0} \rceil \) then solve it by using history data, else exit;
   (ii) set \( i = i + 1 \).
3. On \([\theta + x_0, \infty)\), the rightmost piece \( u_\ell(\xi) \) is given by Equation (19).

Thus, the traveling pulse solution \( u(\xi) \) has the following form:
\[ u(\xi) = \begin{cases} 
   u(\xi), & \xi < x_0 \\
   u_i(\xi), & \xi \in [ix_0, \text{min}\{(i + 1)x_0, \theta + x_0\}], \quad \text{for} \quad i = 1, \ldots, \lceil \frac{\theta}{x_0} \rceil \\
   u_\ell(\xi), & \xi > \theta + x_0.
\end{cases} \] (21)

In (21), \( u_i(\xi) \) and \( u_\ell(\xi) \) are given by (18) and (19) with 5 unknowns \( c_1, c_2, c_3, c_4, \) and \( c_5 \). The piece above threshold is
\[ u_i(\xi) = u_i^p(\xi) + d_{5i-4}e^{a\xi} + d_{5i-3}e^{-a\xi} + d_{5i-2}e^{b\xi} + d_{5i-1}e^{-b\xi} + d_{5i}e^{\frac{1}{c}\xi}, \] (22)
where \( d_n \)s are unknown coefficients and \( u^i_p(z) \) is a particular solution to
\[
 cu^{(i)} - u^{(iv)} - (\alpha^2 + b^2) cu''' + (\beta^2 + b^2) u'' + \alpha^2 b^2 u' - \alpha^2 b^2 u = F(u_{i-1}(\xi - x_0)),
\] for \( i = 1, \ldots, \left[ \frac{h}{x_0} \right] \) starting with \( u_0(\xi) = u_l(\xi) \). We provide exact forms of \( u^i_p(\xi) \) for
\( i=1,2 \) in the appendix. In the next subsection, we will show how to find the unknown constants in the traveling pulse solution for the case with \( c > 0 \) and \( x_0 > 0 \).

4.2. Matching conditions. In order to obtain the explicit form of a traveling pulse, we need to solve a system of equations with same number of equations and unknowns. Let \( M = \left[ \frac{h}{x_0} \right] \). For a traveling pulse described by (21), we have 5\( M + 12 \) unknown constants \( c, c_1, c_2, c_3, c_4, c_5, d_1, \ldots, d_{5(M+1)} \) and \( \theta \). To find these unknowns, we derive a set of matching conditions on the particular points including \( \xi = \theta \) and joint points between pieces of \( u(\xi) \). Since \( u(0) = u(\theta) = h \), we have the following two matching conditions,
\[
 u_l(0) = c_1 + c_2 + c_3 = h \tag{24}
\]
and
\[
 u(\theta) = h. \tag{25}
\]
At \( x_0 \), at which the traveling pulse crosses threshold from below \( h \) to above \( h \), we have five matching conditions(see Appendix for the derivation),
\[
 u_1(x_0) = u_l(x_0), \tag{26}
\]
\[
 u'_1(x_0) = u'_l(x_0), \tag{27}
\]
\[
 u''_1(x_0) = u''_l(x_0), \tag{28}
\]
\[
 u'''_1(x_0) = u'''_l(x_0) + \frac{2}{c} (aA - bB), \tag{29}
\]
\[
 u_{1(iv)}(x_0) = u^v_{1(iv)}(x_0) + \frac{2}{c^2} (aA - bB) + \frac{2}{c} (aA - bB) \alpha u_l(0), \tag{30}
\]
At each \( ix_0 \) for \( i = 2, \ldots, M \) for \( M \geq 2 \), where the pieces of a traveling pulse are connected entirely above threshold, we have five matching conditions by the continuity and smoothness of \( u \) at these points,
\[
 u_{i-1}(ix_0) = u_{i}(ix_0), \tag{31}
\]
\[
 u'_{i-1}(ix_0) = u'_i(ix_0), \tag{32}
\]
\[
 u''_{i-1}(ix_0) = u''_i(ix_0), \tag{33}
\]
\[
 u'''_{i-1}(ix_0) = u'''_i(ix_0), \tag{34}
\]
\[
 u_{i(iv)}(ix_0) = u_{i(iv)}(ix_0). \tag{35}
\]
At \( \theta + x_0 \), where a traveling pulse crosses threshold from above \( h \) to below \( h \), we have five matching conditions(see Appendix for the derivation),
\[
 u_r(\theta + x_0) = u_M(\theta + x_0), \tag{36}
\]
\[
 u'_r(\theta + x_0) = u'_M(\theta + x_0), \tag{37}
\]
\[
 u''_r(\theta + x_0) = u''_M(\theta + x_0), \tag{38}
\]
\[
 u'''_r(\theta + x_0) = u'''_M(\theta + x_0) + \frac{2}{c} (aA - bB), \tag{39}
\]
\[
 u_{r(iv)}(\theta + x_0) = u^v_{r(iv)}(\theta + x_0) + \frac{2}{c^2} (aA - bB) + \frac{2}{c} (aA - bB) \alpha u'(\theta). \tag{40}
\]
In total, we have 5\( M + 12 \) equations with 5\( M + 12 \) unknowns. To solve this system of 5\( M + 12 \) equations numerically, we need a good initial guess to the system.
Ignited by continuation (perturbation) idea, we use the traveling pulse solutions of the neural field equation (2) with the Heaviside gain function as the initial guess. We find an approximate solution to the system (24)–(40) with piecewise linear gain with small α by starting with the solution of Amari type. It should be pointed out that as α increases, the M will change dramatically. For different M, we have different systems to be solved.

The following is an example of a pair of traveling 1–bump solutions for $A = 5$, $a = 0.42$, $B = 1$, $b = 0.1$, $x_0 = 7.75$, $h = 2.7$ and $\alpha = 0.00293$ (see Figure 6). The smaller (faster) traveling 1–bump solution with $c = 18.49$ and $\theta = 15.475$ is given by

$$u(\xi) = \begin{cases} 
   u_1(\xi) = -1.52206e^{0.054\xi} + 4.29000e^{0.11\xi} - 6.79369E^{-2}e^{0.42\xi}, & \text{for } \xi < x_0, \\
   u_1(\xi) = 3.77939 - 3.51372E^{0.42\xi} + 5.75401e^{-0.11\xi} + 1.02441e^{0.054\xi} \\
   \quad -9.77674E^{-1}e^{0.11\xi} + 2.50212E^{-4}e^{0.42\xi} - 6.43533E^{-4}e^{0.054\xi} \\
   \quad -6.82071E^{-3}e^{0.11\xi} - 5.67533E^{-6}e^{0.42\xi}, & \text{for } \xi \in [x_0, 2x_0], \\
   u_2(\xi) = 3.82157 - 2.59462E^{0.42\xi} + 8.18561e^{-0.11\xi} + 1.25591e^{0.054\xi} \\
   \quad -1.25171e^{0.11\xi} + 1.00998E^{-4}e^{0.42\xi} - 1.55135e^{-0.42\xi} \\
   \quad -2.37667E^{-2}e^{0.11\xi} + 1.57023E^{-3}e^{0.054\xi} + 1.93548E^{-3}e^{0.11\xi} \\
   \quad +2.69526E^{-8}e^{0.42\xi} - 5.03215E^{-7}e^{0.054\xi} + 7.61187E^{-6}e^{0.11\xi} \\
   \quad -2.37053E^{-10}e^{0.42\xi}, & \text{for } \xi \in [2x_0, \theta + x_0], \\
   u_r(\xi) = 2.34330E^{0.42\xi} - 2.83158E^{-0.42\xi}, & \text{for } \xi > \theta + x_0,
\end{cases}$$

(41)

where $E$ denotes 10. In (41), two pieces, $u_1(\xi)$ and $u_2(\xi)$, form the part above threshold because $M = \frac{x_0}{x_0} = 2$. $u_1(\xi)$ is connected with $u_1(\xi)$ at $\xi = x_0$ according to (26) to (30). $u_2(\xi)$ is connected with $u_r(\xi)$ at $\xi = x_0 + \theta$ satisfying (36) to (40).

The larger (slower) one with $c = 13.877$ and $\theta = 22.965$ is given by

$$u(\xi) = \begin{cases} 
   u_1(\xi) = -7.97946e^{0.072\xi} + 1.07749E^{0.11\xi} - 9.54505E^{-2}e^{0.42\xi}, & \text{for } \xi > x_0, \\
   u_1(\xi) = 3.77939 - 4.50748E^{-0.42\xi} + 9.03815e^{-0.11\xi} - 1.39897E^{-1}e^{0.072\xi} \\
   \quad +8.54299E^{-1}e^{0.11\xi} + 3.01575E^{-4}e^{0.42\xi} - 1.64534E^{-2}e^{0.072\xi} \\
   \quad -3.75117E^{-2}e^{0.11\xi} - 1.11742E^{-5}e^{0.42\xi}, & \text{for } \xi \in [x_0, 2x_0], \\
   u_2(\xi) = 3.82157 - 29.6039e^{-0.42\xi} + 9.95071e^{-0.11\xi} + 1.41206e^{0.072\xi} \\
   \quad -7.31560E^{-1}e^{0.11\xi} + 5.08577E^{-6}e^{0.42\xi} - 2.55475e^{-0.42\xi} \\
   \quad -3.37955E^{-2}e^{-0.11\xi} - 3.48775E^{-3}e^{0.072\xi} - 9.71467E^{-3}e^{0.11\xi} \\
   \quad -3.31893E^{-5}e^{0.42\xi} + 9.1666E^{-5}e^{0.072\xi} + 5.17484E^{-8}e^{0.42\xi} \\
   \quad -6.54009E^{-10}e^{0.42\xi}, & \text{for } \xi \in [2x_0, 3x_0], \\
   u_3(\xi) = 3.82204 - 3.48234E^{-0.42\xi} + 9.87689e^{-0.11\xi} + 2.03681e^{0.072\xi} \\
   \quad -1.35204e^{0.11\xi} + 6.09992E^{-6}e^{0.42\xi} - 1.03721E^{-2}e^{-0.42\xi} - 2.85005E^{-2}e^{-0.11\xi} \\
   \quad +2.15486E^{-3}e^{0.072\xi} + 1.27219E^{-3}e^{0.11\xi} + 5.22069E^{-10}e^{0.42\xi} \\
   \quad -7.10328E^{-2}e^{0.11\xi} + 4.50078E^{-5}e^{0.42\xi} + 1.76034E^{-7}e^{0.072\xi} \\
   \quad +2.3346E^{-8}e^{0.072\xi} + 3.30617E^{-12}e^{0.42\xi} - 2.28171E^{-8}e^{0.072\xi} \\
   \quad -1.06375E^{-7}e^{0.11\xi} - 1.27594E^{-14}e^{0.42\xi}, & \text{for } \xi \in [3x_0, \theta + x_0], \\
   u_r(\xi) = 7.00303E^{5}e^{-0.42\xi} - 8.18949E^{-0.42\xi}, & \text{for } \xi > \theta + x_0.
\end{cases}$$

(42)
Figure 6. Example of two traveling 1–bump solutions with $A = 5$, $a = 0.42$, $B = 1$, $b = 0.1$, $x_0 = 7.75$, $h = 2.7$ and $\alpha = 0.00293$. One is a smaller (faster) bump with width $\theta_s = 15.475$ and $c = 18.49$ (expression given by (41)) while the other a larger (slower) bump with width $\theta_l = 22.965$ and speed $c = 13.877$ (expression given by (42)).

This larger traveling pulse is constructed the same way as the smaller traveling pulse given in (41). The difference is that the larger pulse has three pieces, $u_1(\xi)$, $u_2(\xi)$, and $u_3(\xi)$ for the part above threshold due to its larger width, hence bigger $M = \left\lceil \frac{\theta}{x_0} \right\rceil = 3$.

In both (41) and (42), $u_l(\xi)$ and $u_r(\xi)$ follow their forms given in (18) and (19), respectively. The part in between $u_l(\xi)$ and $u_r(\xi)$ follows in (22). In all our numeric and symbolic computations, we use the default precision of 16 decimal digits in Mathematica. In (41) and (42), we use the scientific notation to represent all the coefficients with $E$ for 10.

We find a family of pairs of large and small traveling pulses as we vary the shift parameter $x_0$. The dashed curves in (I) and (II) of Figure 7 show the speed and width of the coexisting large and small traveling pulses in term of $x_0$ for a fixed set of parameters $A = 5, b = 0.1, B = 1, a = 0.42, h = 2.7$, and $\alpha = 0.001$. As shown in Figure 7, there is a critical $x_0^*$ value where the large and small traveling pulses collide into one. For any $x_0$ greater than $x_0^*$, we no longer have traveling pulse solutions. This saddle node bifurcation in term of shift parameter $x_0$ is similar to what we see in the Heaviside case.

As we change the value of $\alpha$, the saddle–node–node shifts its position too. Compared with $\alpha = 0$, if $\alpha > 0$, $x_0^*$ will become bigger while if $\alpha < 0$, $x_0^*$ will become smaller. Moreover, the width (speed) of the larger traveling pulse increases (decreases) while the width (speed) of the smaller one decreases (increases) as $\alpha$ increases(see Figure 7).

There is a saddle node bifurcation with respect to the gain $\alpha$ for the neural field model with asymmetric coupling if we fixed the shift and all other parameters. Similarly, we can define a saddle–node bifurcation with respect to $\alpha$ as the following:

There is a number $\alpha^*$ in some interval, called a saddle–node point, such that on this interval,

- If $\alpha < \alpha^*$, there are no traveling 1–bump solutions.
- At $\alpha = \alpha^*$, there is exactly one traveling 1–bump solution.
- If $\alpha > \alpha^*$, there are two traveling 1–bump solutions if they exist.

For example, with $A = 5, b = 0.1, B = 1, a = 0.42, h = 2.7$ and $x_0 = 7.75$. The
saddle node bifurcation occurs approximately at $\alpha^* \approx -0.0301$ (see Figure 8). The bifurcation point $\alpha^*$ is computed numerically by continuing the large and small traveling pulses as we vary the parameter value for $\alpha$. While we are not aware of any evidence to show biophysical meaning of a negative gain, this step only serves as the computational exploration of saddle node bifurcations for the mathematical model. There is no implication of any biophysical meaning. Notice that the saddle node bifurcation occurs at a different $\alpha$ value if we change other parameters in the coupling function, or the gain function.

5. Stability. In this section, we will determine the linear stability of traveling 1–bump solution $u(\xi)$ to equation (1). To this end, we let $U(\xi,t) = u(\xi) + \tilde{\phi}(\xi,t)$, and linearize the equation (1) at $u(\xi)$. Then we have

$$\frac{\partial \tilde{\phi}(\xi,t)}{\partial t} = c \frac{\partial \tilde{\phi}(\xi,t)}{\partial \xi} - \tilde{\phi}(\xi,t) + \int_{-\infty}^{\infty} w(\xi - \eta)f'(u(\eta))\tilde{\phi}(\eta,t) d\eta. \quad (43)$$

Letting $\tilde{\phi}(\xi,t) = e^{\lambda t}\phi(\xi)$, the equation (43) becomes

$$\lambda \phi(\xi) = c\psi'(\xi) - \phi(\xi) + \int_{-\infty}^{\infty} w(\xi - \eta)f'(u(\eta))\phi(\eta) d\eta. \quad (44)$$

Note that
The problem associated with operator \( L \) only need to check the point spectrum of \( L \). The essential spectrum of \( T \varphi \) is a simple eigenvalue and the real parts of all other eigenvalues are negative. The stability of traveling pulses with Heaviside gain that is stability of standing pulses with \( c = 0 \) in equation (45) has been investigated in [24]. The stability of traveling pulses with Heaviside gain that is \( \alpha = 0 \) in equation (45) has been shown in [7]. We focus on the case of \( c > 0 \) and \( \alpha \neq 0 \). As in [7], the real parts of essential spectrum of \( \mathcal{L}_0 \) is \(-1\). Due to the compactness of \( T \), the essential spectrum of \( \mathcal{L} \) is the same as \( \mathcal{L}_0 \). The negative real parts of essential spectrum of \( \mathcal{L} \) will not introduce instability. To determine the stability, we only need to check the point spectrum of \( \mathcal{L} \). Hereafter, we focus on an eigenvalue problem associated with operator \( \mathcal{L} \), \( \mathcal{L} \varphi = \lambda \varphi \), or,

\[
c \varphi' - (\lambda + 1) \varphi + \varphi(0) / |u'(0)| w(\xi) + \varphi(\theta) / |u'(\theta)| w(\xi - \theta) + \alpha \int_0^\theta w(\xi - \eta) \varphi(\eta) d\eta = 0. \tag{45}
\]

For simplicity, we rewrite Equation (45) as

\[
\varphi' = \hat{\lambda} \varphi - k_0 w(\xi) - k_0 w(\xi - \theta) - \hat{\alpha} \hat{W}(\xi),
\]

where \( \hat{\lambda} = \lambda + 1 / c \), \( k_0 = \varphi(0) / |u'(0)| \), \( k_0 = \varphi(\theta) / |u'(\theta)| \), \( \hat{\alpha} = \alpha / c \) and \( \hat{W}(\xi) = \int_0^\theta w(\xi - \eta) \varphi(\eta) d\eta \).

By Fourier transform and its inverse we have

\[
\varphi^{(\nu)} - \hat{\lambda} \varphi^{(\nu)} = (a^2 + b^2) u'' + \hat{\lambda} (a^2 + b^2) u'' + a^2 b^2 \varphi' - \hat{\lambda} a^2 b^2 \varphi = F(\varphi(\xi - x_0)), \xi \in [x_0, \theta + x_0],
\]

\[
\int_{-\infty}^{\infty} w(\xi - \eta) f'(u(\eta)) \varphi(\eta) d\eta = \varphi(0) / |u'(0)| w(\xi) + \varphi(\theta) / |u'(\theta)| w(\xi - \theta) + \alpha \int_0^\theta w(\xi - \eta) \varphi(\eta) d\eta.
\]

\[
\mathcal{L} \varphi(\xi) = c \varphi' - \varphi(\theta) / |u'(\theta)| w(\xi - \theta) + \alpha \int_0^\theta w(\xi - \eta) \varphi(\eta) d\eta.
\]

Let \( T \varphi := \hat{\alpha} \hat{W}(\xi) \), and \( \mathcal{L}_0 \varphi(\xi) = c \varphi' - \varphi(\xi) + k_0 w(\xi) + k_0 w(\xi - \theta) \) and then \( \mathcal{L} = \mathcal{L}_0 + T \). We call that the traveling pulse is unstable if there is an eigenvalue whose real part is positive, otherwise we call that the traveling pulse is stable if 0 is a simple eigenvalue and the real parts of all other eigenvalues are negative. The stability of standing pulses with \( c = 0 \) in equation (45) has been investigated in [24]. The stability of traveling pulses with Heaviside gain that is \( \alpha = 0 \) in equation (45) has been shown in [7]. The plots are for \( A = 5, b = 0.1, B = 1, a = 0.42, h = 2.7 \) and \( x_0 = 7.75 \). Examples of traveling pulses with \( \alpha = 0.00293 \) are marked with circle (Example in Equation (41)) and star (Example in Equation (42)) in the plots respectively.

\[
\varphi(\xi) = \hat{\alpha} \hat{W}(\xi), \quad w(\xi) = \varphi(\xi) / |u'(\xi)|, \quad \varphi(0) / |u'(0)|, \quad \varphi(\theta) / |u'(\theta)|, \quad \alpha \int_0^\theta w(\xi - \eta) \varphi(\eta) d\eta.
\]
\[ \varphi^{(v)} - \tilde{\lambda} \varphi^{(iv)} - (a^2 + b^2)w''' + \lambda (a^2 + b^2) \varphi'' + a^2 b^2 \varphi' - \tilde{\lambda} a^2 b^2 \varphi = 0, \xi < x_0 \text{ or } \xi > \theta + x_0, \]

where

\[ F(\varphi(\xi - x_0)) = \hat{\alpha} [2(Aa - Bb) \varphi''(\xi - x_0) - 2ab(Ab - Ba) \varphi(\xi - x_0)]. \]

We write the general solutions of (46) and (47), respectively, as the following:

\[ \varphi_-(\xi) = \begin{cases} c_1 e^{a \xi} + c_2 e^{b \xi} + c_3 e^{c \xi} - \lambda \xi / \hat{\lambda}, & \xi \notin \{a, b\} \\ c_1 e^{a \xi} + c_2 e^{b \xi} + c_3 e^{c \xi}, & \xi = a, \\ c_1 e^{a \xi} + c_2 e^{b \xi} + c_3 e^{c \xi}, & \xi = b. \end{cases} \tag{48} \]

- For \( \xi \in (-\infty, x_0] \), we can have

\[ \varphi_-(\xi) = \varphi_1(x_0), \tag{49} \]

and

\[ \varphi'_-(\xi) = \varphi'_1(x_0). \tag{50} \]

Then differentiating both sides to Equation (45), we have that

\[ \varphi''(\xi) = \tilde{\lambda} \varphi'(\xi) - k_0 w'(\xi) - k_0 w'(\xi - \theta) - \hat{\alpha} \bar{W}''(\xi). \]

Note that

\[ \bar{W}''(\xi) = [\int_{\xi}^{\xi - t} w(t) \varphi(\xi - t) dt]' = w(\xi - \theta) \varphi(\theta) - w(\xi) \varphi(0) + \int_{\xi}^{\xi - t} w(t) \varphi'(\xi - t) dt, \]

and then \( \bar{W}''(x_0) = 0 \) and \( \bar{W}''(\theta + x_0) = 0 \). Therefore, we have

\[ \varphi''_-(x_0) - \varphi''_1(x_0) = -2k_0(aA - bB). \tag{51} \]

We have

\[ \varphi'''(\xi) = \tilde{\lambda} \varphi''(\xi) - k_0 w''(\xi - \theta) - \hat{\alpha} \bar{W}'''(\xi). \]

Note that

\[ \bar{W}'''(\xi) = w'(\xi - \theta) \varphi(\theta) - w''(\xi) \varphi(0) + w(\xi - \theta) \varphi'(\theta) - w(\xi) \varphi'(0) + \int_{\xi}^{\xi - t} w(t) \varphi''(\xi - t) dt, \]

and then \( \bar{W}'''(x_0) = -2k_0(Aa - Bb) \varphi(0) \) and \( \bar{W}'''(\theta + x_0) = 2(Aa - Bb) \varphi(\theta) \). Thus,

\[ \varphi'''_-(x_0) - \varphi'''_1(x_0) = 2(aA - bB)(-\tilde{\lambda}k_0 + \hat{\alpha} \varphi(0)), \tag{52} \]

We have

\[ \varphi'''(\xi) = \tilde{\lambda} \varphi'''(\xi) - k_0 w'''(\xi - \theta) - \hat{\alpha} \bar{W}'''(\xi). \]
Note that
\[ W''''(\xi) = w''''(\xi - \theta)\varphi(\theta) - w''''(\xi)\varphi(0) + w'(\xi - \theta)\varphi'(\theta) - w'(\xi)\varphi'(0) \]
\[ + w(\xi - \theta)\varphi''(\theta) - w(\xi)\varphi''(0) + \int_{\xi}^{\xi - t} w(t)\varphi'''(\xi - t)dt, \]
and then
\[ W''''(x_0 -) - W''''(x_0 +) = -(w'(x_0 -) - w'(x_0 +))\varphi'(0) = -2(Aa - Bb)\varphi'(0) \]
and
\[ W''''(\theta + x_0) - W''''(x_0 +) = 2(Aa - Bb)\varphi'(\theta). \]

Then
\[ \varphi_1(\theta + x_0) - \varphi_1(x_0) = 2(aA - bB)(-\hat{\lambda}^2k_0 + \hat{\lambda}\hat{\alpha}\varphi(0) + \hat{\alpha}\varphi'(0)) - 2(Aa^3 - Bb^3)k_0, \] (53)

At \( ix_0 \) for \( i = 2, ..., M \) and \( M \geq 2 \), we have five matching conditions,
\[ \varphi_{i-1}(ix_0) = \varphi_i(ix_0), \] (54)
\[ \varphi_{i-1}'(ix_0) = \varphi_i'(ix_0), \] (55)
\[ \varphi_{i-1}''(ix_0) = \varphi_i''(ix_0), \] (56)
\[ \varphi_{i-1}'''(ix_0) = \varphi_i'''(ix_0), \] (57)
\[ \varphi_{i-1}^{(iv)}(ix_0) = \varphi_i^{(iv)}(ix_0). \] (58)

At \( \theta + x_0 \), following the same arguments as \( x_0 \), we have five matching conditions,
\[ \varphi_+(\theta + x_0) = \varphi_M(\theta + x_0), \] (59)
\[ \varphi'_+(\theta + x_0) = \varphi_M'(\theta + x_0), \] (60)
\[ \varphi''_M(\theta + x_0) - \varphi''_+(\theta + x_0) = -2k_0(aA - bB), \] (61)
\[ \varphi'''_M(\theta + x_0) - \varphi'''_+(\theta + x_0) = -2\hat{\lambda}k_0(aA - bB) - 2\hat{\alpha}(aA - bB)\varphi(\theta), \] (62)
\[ \varphi^{(iv)}_M(\theta + x_0) - \varphi^{(iv)}_+(\theta + x_0) = 2(aA - bB)(-\hat{\lambda}^2k_0 + \hat{\lambda}\hat{\alpha}\varphi(\theta) + \hat{\alpha}\varphi'(\theta)) - 2(Aa^3 - Bb^3)k_0. \] (63)

Let \( v = (c_1, c_2, c_3, d_1, ..., d_{5(M+1)}, c_4, c_5)^T \). Equations (49)-(63) generate a matrix \( D(\lambda) \) such that \( D(\lambda)v = 0 \). Then, we have that \( \lambda \) is an eigenvalue of \( L \) if and only if \( \det(D(\lambda)) = 0 \). Note that 0 is an eigenvalue of \( L \). Thus, the Evans function is defined by \( E(\lambda) := \det(D(\lambda)) = 0 \), which can be used to find eigenvalues and thus assess stability of traveling pluses.

We point out that Evans function has three different forms for \( \lambda = ac - 1 \) (i.e. when \( \hat{\lambda} = a \) in (48)), \( \lambda = bc - 1 \) (i.e. \( \hat{\lambda} = b \) in (48)), or \( \lambda \notin \{ac - 1, bc - 1\} \) (i.e. \( \hat{\lambda} \notin \{a, b\} \) in (48)) because \( \varphi_+ \) takes three different expressions according to (48). To first determine whether \( ac - 1, \) or \( bc - 1, \) is an eigenvalue, we can compute the Evans function using its correct forms at \( \lambda = ac - 1 \) (the second expression for \( \varphi_+ \)), and \( \lambda = bc - 1 \) (the third expression for \( \varphi_+ \)), respectively. If \( E(ac - 1) \neq 0 \) (or \( E(bc - 1) \neq 0 \)), then \( ac - 1 \) (or \( bc - 1 \)) is not an eigenvalue of \( L \). Then we search the eigenvalues on the complex plane using the corresponding Evans function derived by taking the first expression of \( \varphi_+ \) in (48), which imposes the condition that \( \lambda \notin \{ac - 1, bc - 1\} \).

To search the eigenvalues that are different from \( ac - 1 \) and \( bc - 1 \), for the larger(slower) bump traveling pulse solution and the smaller(faster) one, we use graphical contour plots of real parts and imaginary parts of Evans function \( E(\lambda) \) with level zero. The intersections of the two contour plots indicate the eigenvalues. Recall that there are two traveling 1–bump solutions for the set of parameters \( A = 5; a = 0.42; B = 1; b = 0.1 \) , \( x_0 = 7.75 \), and \( \alpha = 0.00293 \). The smaller (faster) one with \( c = 18.49 \) and \( \theta_s = 15.475 \) is unstable because we can obtained a real eigenvalue between 0 and 1 (\( \lambda_2 \) in Figure 9). Notice that we exclude \( \lambda_0 = bc - 1 = 0.849018129 \) on Figure 9. \( ac - 1 \) is out of the range and not shown on Figure 9.
Figure 9. Contour plots of $\text{Re}(E(\lambda)) = 0$ (dashed lines) and $\text{Im}(E(\lambda)) = 0$ (solid lines) associated with traveling pulse with the smaller bump on $[-1, 3] \times [-1, 1]$ with $A = 5, b = 0.1, B = 1, a = 0.42, h = 2.7, x_0 = 7.75$ and $\alpha = 0.00293$. The plot is for $E(\lambda)$ with $\lambda \not\in \{ac - 1, bc - 1\}$ in (48), so the point $\lambda_3 = bc - 1 = 0.849018129$ is excluded in the plot. $\lambda_2 > 0$ implies that the traveling pulse is unstable.

The larger (slower) one with $c = 13.877$ and $\theta_l = 22.965$ is stable because the real parts of all non-zero eigenvalues are negative (see Figure 10). The actual numerical results show there are no any real parts of eigenvalues bigger than 0 in Figure 10. We exclude the points with $\lambda_3 = 0.387736864$ or $\lambda_4 = 4.828494827$ in the Figures 10, where $\lambda = ac - 1$ or $bc - 1$, that is, $\lambda = a$ or $b$. At these two particular points, we have to compute their corresponding Evans function with their corresponding setups for $\varphi_-(\xi)$ in (48). We have that $E(0.387736864) \neq 0$ and $E(4.828494827) \neq 0$, which verify that $0.387736864$ and $4.828494827$ are not eigenvalues.

In Figure 11, we show the contour plots of Evans function $E(\lambda)$ to a pair of traveling pulse solutions for $A = 5, b = 0.1, B = 1, a = 0.42, \alpha = 0.0002, h = 2.7$ and $x_0 = 8.1$. By observation of Figure 11 together with Figure 10 and Figure 9, there is only one nonzero real eigenvalue for each traveling pulse solution. The plot in the left of Figure 11 is for stable larger pulse with width $\theta = 20.35328857210932$, speed $c = 15.95820230800241$ and $\lambda_1^l = -0.0719$. The plot in the right is for unstable small pulse with width $\theta = 18.48776823587279$, speed $c = 17.11901722028035$ and $\lambda_1^s = 0.0850$.

As a logical next step to investigate the stability of traveling pulses, we trace how the eigenvalues for the larger and smaller traveling pulses change as we vary the shift parameter $x_0$. Both the larger and smaller pulses continue to coexist until they reach the saddle traveling pulse at $x_0^s$ as shown in section 4. Their eigenvalues $\lambda_1^l$ and $\lambda_1^s$ continue in the fashion that $\lambda_1^l$ remains negative and $\lambda_1^s$ is always positive before $x_0$ reaches $x_0^s$. As shown in Figure 12, numerical computation demonstrates that while $x_0$ goes to $x_0^s$, the negative eigenvalue $\lambda_1^l$ for the larger traveling pulse and the positive $\lambda_1^s$ for the smaller traveling pulse both move toward 0. There are only zero eigenvalues at the saddle point $x_0^s$. In Figure 12, the three different curves of eigenvalues, for different $\alpha$ values, give us comparison on the saddle point location.

When $\alpha = -0.0002$, the eigenvalue curve shrinks a little and $x_0^s$ shifts to the left of the curve for Amari case $\alpha = 0$, while the curve for $\alpha = 0.0002$ stretches and $x_0^s$ shifts to the right.
Figure 10. The larger bump traveling pulse solution is for $A = 5$, $b = 0.1$, $B = 1$, $a = 0.42$, $h = 2.7$, $x_0 = 7.75$ and $\alpha = 0.00293$. The plot is for $E(\lambda)$ with $\lambda \notin \{ac - 1, bc - 1\}$ in (48), so the points $\lambda_3 = bc - 1 = 0.387736864$ and $\lambda_4 = ac - 1 = 4.828494827$ are excluded in the plots. By observation, $\lambda_1 \in (-1,0)$ and $\lambda_2 = 0$ are two real eigenvalues. (I) Contour plots of $\text{Re}(E(\lambda)) = 0$ (dashed lines) and $\text{Im}(E(\lambda)) = 0$ (solid lines) associated with the larger bump on $[-1, 2.3] \times [-1, 1]$. (II) Contour plots of $\text{Re}(E(\lambda)) = 0$ (dashed lines) and $\text{Im}(E(\lambda)) = 0$ (solid lines) associated with the larger bump on $[2.3, 6] \times [-1, 1]$.

Figure 11. The contour plots of Evans function $E(\lambda)$ to a pair of traveling pulse solutions for $A = 5, b = 0.1, B = 1, a = 0.42, \alpha = 0.0002, h = 2.7$ and $x_0 = 8.1$.

6. Discussion. In this paper, we proved the nonexistence of traveling N–bump solutions for neural field models with symmetric couplings and piecewise linear gains. Motivated by the nonexistence, we consider a neural field model with asymmetric couplings. As expected, the asymmetry in the coupling function induces a major difference in finding traveling pulse solutions, comparing with previous method in finding standing pulse and traveling front solutions of the neural field model. Due to the asymmetry in the coupling function, converting the neural field model leads
Figure 12. The plot is the eigenvalue $\lambda^l_1 < 0$ or $\lambda^l_1 > 0$ versus $x_0$ to (44) with traveling pulse solution for $A = 5, b = 0.1, B = 1, \alpha = 0.42$ and $h = 2.7$. The inner black solid curve is for $\alpha = -0.0002$. The grey solid curve in the middle is for $\alpha = 0$. The outer dashed curve is for $\alpha = 0.0002$. At the saddle, the eigenvalue is 0.

To a system of delayed differential equations, instead of a system of ordinary differential equations. Using the Myshkis’ Method of Steps [29], we solved the system of delayed differential equations and showed the co-existence of two traveling one-bump solutions for a given pair of coupling and gain functions. Even though the Myshkis’ step method is effective in find traveling one-bump solutions, it also has limitations, especially for the large traveling pulse. The width of the large traveling pulse increases as the shift parameter $x_0$ in the asymmetric coupling function decreases. Then the number of steps, $M = \lceil \frac{\theta}{\pi} \rceil$, will increase accordingly, which results in a system of many more equations for which we need to solve to obtain a traveling pulse solution. When $x_0$ is near zero, the step number $M$ has infinity as its limit, which will invalidate our method to find the traveling pulse solution because it is impossible to numerically solve a system of infinite number of equations.

After showing the existence of traveling 1-bump pulses, we further develop Evans functions for traveling 1-bump solutions, to assess their stability. Numerical computations of the eigenvalues, using Evans functions, suggest that the traveling pulse with larger width and slower speed is stable while the unstable one has smaller width and travels with faster speed. Although the results obtained here are primarily for $M = 2$ and 3 steps, they may be extended to traveling pulse solutions that require larger finite number of steps.

The goal of the current work is to develop a method, combining both analytic and computational techniques, to find traveling pulse solutions and access their stability. While we focus on finding traveling 1-bump solutions in the current paper, our method does not prevent further extension on traveling 2-bump, 3-bump pulse, and solutions with finite number of bumps. The caveat is that we will need to deal with an equation system of a much larger scale in terms of number of equations, unknowns and Myshkis steps. Such a system is much more difficult to solve due to more complicated technical issues. For this reason, we feel that it would be more fruitful to move toward rigorous proofs of the existence of traveling pulse solutions. We will present our theoretical results toward this direction elsewhere as they are beyond the scope of this paper.
Since Amari’s seminal paper [1], researchers have used the neural field model to study neurobiological phenomena, such as working memory, visual hallucinations, orientation tuning, motion perception and EEG rhythms [4, 5, 21, 36, 35, 39]. Similar nonlocal integro-differential equation models have been used to study chemical front motion and spatial dispersal of cells or organisms [3, 25, 30]. However, to the best of our knowledge, there is no clear evidence to show the mathematical models presented in [3, 25, 30] are connected to neural networks. On the other hand, we are not aware of any appealing applications of the neural field model in other areas, such as engineering. It would be interesting to investigate possible applications of the neural field equation, as a mathematical tool, to model physical processes other than neurobiological phenomena.

Appendix.

Lemma 6.1.

\[ u_t(x_0) = u_1(x_0), \quad \text{(64)} \]
\[ u_t'(x_0) = u_4'(x_0), \quad \text{(65)} \]
\[ u_t''(x_0) = u_5''(x_0), \quad \text{(66)} \]
\[ u_t'''(x_0) = u_1'''(x_0), \quad \text{(67)} \]
\[ u^{(iv)}_t(x_0) = u_1^{(iv)}(x_0) + \frac{2}{c}(aA - bB), \quad \text{(68)} \]

Proof. Equality (64) is due to the continuity of traveling 1-bump solution to

\[ -cu'(z) = -u(z) + \int_0^\theta w(z - y)f(u(y))dy. \quad \text{(69)} \]

By (69), (64) implies (65). Let \( D(z) = \int_0^\theta w(z - y)f(u(y))dy = \int_{z-\theta}^z w(t)f(u(z - t))dt \). Then we have

\[ D'(z) = w(z)f(u(0^+)) - w(z - \theta)f(u(\theta^-)) + \alpha \int_{z-\theta}^z w(t)u'(z - t)dy \]
\[ = w(z) - w(z - \theta) + \alpha \int_{z-\theta}^z w(t)u'(z - t)dy, \]
\[ D''(z) = w'(z) - w'(z - \theta) + \alpha w(z)u'(0^+) - \alpha w(z - \theta)u'(\theta^-) + \alpha \int_{z-\theta}^z w(t)u''(z - t)dy, \]
and

\[ D'''(z) = w''(z) - w''(z - \theta) + \alpha w(z)u''(0^+) - \alpha w(z - \theta)u''(\theta^-) + \alpha \int_{z-\theta}^z w(t)u'''(z - t)dy. \]

So we have \( D'(x_0^+) = D'(x_0^-) = 0, D''(x_0^+) = D''(x_0^-) = \frac{2}{c}(aA - bB) \) and \( D'''(x_0^+) = D'''(x_0^-) = \frac{2}{c}(aA - bB)\alpha u'_1(0) \). Differentiate (69) \( k \) times and we have

\[ -cu^{(k+1)}(z) = -u^{(k)}(z) + D^{(k)}(z), \quad \text{(70)} \]

where \((k)\) indicates the \(n\)-th derivative for \(k = 1, 2, 3\).

Then Equalities (66)-(68) are obtained by subtraction at \( x_0^+ \) and \( x_0^- \) of Equation (70). \( \square \)
Lemma 6.2.

\begin{align}
  u_r(\theta + x_0) &= u_M(\theta + x_0), \\
  u'_r(\theta + x_0) &= u'_M(\theta + x_0), \\
  u''_r(\theta + x_0) &= u''_M(\theta + x_0), \\
  u'''_r(\theta + x_0) &= \frac{2}{c^2}(aA - bB), \\
  u^{(iv)}_r(\theta + x_0) &= u^{(iv)}_M(\theta + x_0) + \frac{2}{c}(aA - bB)u'(	heta).
\end{align}

Proof. It can be proved by the arguments similar to those in previous lemma.

Next in this appendix, we show how to solve the equation (14) for \( c > 0, a, b, \frac{1}{c} \) are not identical. Consider a linear fifth–order nonhomogeneous differential equation of the form

\[ cu^{(iv)} - u^{(iv)} - (a^2 + b^2)cu'' + (a^2 + b^2)u'' + a^2b^2cu' - a^2b^2u = G(z). \]  

Let \( u_p \), be any particular solution of the nonhomogeneous linear differential equation (76), then the general solution to Equation (76) is given by

\[ u(z) = u_p + d_1e^{az} + d_2e^{-az} + d_3e^{bz} + d_4e^{-bz} + d_5e^{\frac{z}{z}}, \]

where the \( d_i, i = 1, 2, ..., 5 \) are arbitrary constants. Then to obtain the traveling pulse solution, it suffices to find \( u_p \) and determine the \( d_i \)s in \( u(z) \).

First we have \( u(z) = u_1(z) = c_2e^{az} + c_3e^{bz} + c_1e^{\frac{z}{z}} \) for \( z < x_0 \) and \( u(z) = u_r(z) = c_4e^{-az} + c_5e^{-bz} \) for \( z > x_0 + \theta \). To get \( u(z) \) in the middle \([x_0, x_0 + \theta]\), that is \( u_i(z) \) for \( i = 1 : M \), we will use Mikhaylov’s Method of Steps.

Note that \( u_i \) satisfies (76) with \( G(z) = F(u(z) - x_0) = F(u_{i-1}(z - x_0)) \) for \( i = 1 : M \) with \( u_0(z) = u(z) \). To compute \( u_1 \), note that \( F(u_0(z)) = \tilde{m}_0 + \tilde{m}_1e^{az} + \tilde{m}_2e^{bz} + \tilde{m}_3e^{\frac{z}{z}} \),

where

\[ \tilde{m}_0 = 2ab(-Ab + Ba)(-ah + 1), \]
\[ \tilde{m}_1 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
\[ \tilde{m}_2 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
and
\[ \tilde{m}_3 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}. \]

To get \( u_p(z) \) in \( u_1(z) \), it reduces to find the particular solution of (76) with \( G(z) \in \{e^{az}, e^{bz}, e^{\frac{z}{z}}\} \). By collecting the coefficients of similar terms in \( u_p(z) \), \( u_1 \) is given by the following:

\[ u_1(z) = m_0 + m_1ze^{az} + m_2ze^{bz} + m_3ze^{\frac{z}{z}} + d_1e^{az} + d_2e^{-az} + d_3e^{bz} + d_4e^{-bz} + d_5e^{\frac{z}{z}}, \]

where

\[ m_0 = 2(-Ab + Ba)(1 - ah), \]
\[ m_1 = \frac{(a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2)c}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
\[ m_2 = \frac{(a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2)c}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
and
\[ m_3 = \frac{(a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2)c}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}. \]

Similarly we have \( u(z) = u_2(z) \) on \([x_0, 3x_0]\) with \( G(z) = F(u_1(z - x_0)) \). Plugging \( u_1 \) into \( F(u_1(z - x_0)) \), we have

\[ F(u_1(z - x_0)) = \tilde{m}_0 + (\tilde{m}_1z + \tilde{p}_1)e^{az} + (\tilde{m}_2z + \tilde{p}_2)e^{bz} + (\tilde{m}_3z + \tilde{p}_3)e^{\frac{z}{z}} + \tilde{p}_4e^{-az} + \tilde{p}_5e^{-bz}, \]

where

\[ \tilde{m}_0 = 2ab(-Ab + Ba)(1 + \alpha(-h + m_01)), \]
\[ \tilde{m}_1 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
\[ \tilde{m}_2 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}, \]
and
\[ \tilde{m}_3 = \frac{1}{a^2c_2\alpha_1^2 + ab(-Ab + Ba)c_2}. \]
\[ \hat{n}_1 = \alpha m_1 [(2(-Aa + Bb))a^2 - 2ab(-Ab + Ba)], \]
\[ \hat{n}_2 = \alpha m_2 [(2(-Aa + Bb))b^2 - 2ab(-Ab + Ba)], \]
\[ \hat{n}_3 = \alpha m_3 [(2(-Aa + Bb))c^2 - 2ab(-Ab + Ba)], \]
\[ \hat{p}_1 = (2(-Aa + Bb))\alpha (d_1 a^2 + 2am_1) - 2ab(-Ab + Ba)d_1, \]
\[ \hat{p}_2 = (2(-Aa + Bb))\alpha (d_2 b^2 + 2bm_2) - 2ab(-Ab + Ba)d_2, \]
\[ \hat{p}_3 = (2(-Aa + Bb))\alpha (d_3 c^2 + 2cm_3) - 2ab(-Ab + Ba)d_3, \]
\[ \hat{p}_4 = (2(-Aa + Bb))\alpha d_4 a^2 - 2ab(-Ab + Ba)d_4, \]
and
\[ \hat{p}_5 = (2(-Aa + Bb))\alpha d_5 b^2 - 2ab(-Ab + Ba)d_5. \]

Similarly, to find \( u_p^2(z) \), it reduces to find the particular solution of (76) with \( G(z) \in \{1, e^{az}, e^{bz}, e^{cz}, e^{-az}, e^{-bz}, ze^{az}, ze^{bz}, ze^{cz} \} \). The \( u_2(z) \) is of the following form:
\[ u_2(z) = m_{02} + (n_1 z^2 + p_1 z)e^{az} + (n_2 z^2 + p_2 z)e^{bz} + (n_3 z^2 + p_3 z)e^{cz} + p_4 ze^{-az} + p_5 ze^{-bz} + p_6 ze^{-cz} + d_6 e^{az} + d_7 e^{bz} + d_8 e^{cz} + d_9 e^{-az} + d_{10} e^{-bz} \]
where \( m_{02} = -2(-Ab + Ba)(1 + \alpha (-h + m_{01})) \), and we can have \( n_i \) for \( i = 1, 2, 3 \) and \( p_j \) for \( j = 1, \ldots, 5 \) by collecting the coefficients in the \( u_p \) with \( G(z) = F(u_1(z - x_0)) \).

Theoretically, we can repeat the procedure to compute \( u_i(z) \) for \( i = 1 : M \) with any \( M \). In reality, it is too lengthy to compute \( u_p(z) \) of (76) if \( G(z) \) involves too many exponential terms. The increase of the number \( M \) of middle pieces will increase the order of \( G(z) \) with five more terms each time, which leads to five undetermined coefficients added with five more lengthy matching equations in the system. Numerically, we can deal with the case up to \( M = 3 \) in the current paper.

**Acknowledgments.** We thank the anonymous referees for many constructive comments that helped us improve the exposition.

**REFERENCES**

[1] S. Amari, *Dynamics of pattern formation in lateral-inhibition type neural fields*, *Biol. Cybernet.*, 27 (1977), 77–87.
[2] L. Bai, X. Huang, Q. Yang and J.-Y. Wu, *Spatiotemporal patterns of an evoked network oscillation in neocortical slices: Coupled local oscillators*, *J. Neurophysiol.*, 96 (2006), 2528–2538.
[3] P. W. Bates and G. Zhao, *Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal*, *J. Math. Anal. Appl.*, 332 (2007), 428–440.
[4] G. B. Ermentrout and J. D. Cowan, *A mathematical theory of visual hallucination patterns*, *Biol. Cybern.*, 34 (1979), 137–150.
[5] P. C. Bressloff, J. D. Cowan, M. Golubitsky, P. J. Thomas and M. L. Wiener, *Geometric visual hallucinations, Euclidean symmetry and the functional architecture of striate cortex*, *Phil. Trans. R. Soc. B.*, 356 (2001), 299–330.
[6] P. C. Bressloff and S. E. Folias, *Front bifurcations in an excitatory neural network*, *SIAM J. Appl. Math.*, 65 (2004), 131–151.
[7] P. C Bressloff and J. Williserson, *Traveling pulses in a stochastic neural field model of direction selectivity*, *Frontiers in Computational Neuroscience*, 6 (2012).
[8] P. C. Bressloff, *Spatiotemporal dynamics of continuum neural fields: Invited Topical review*, *J. Phys. A.*, 45 (2012), 033001, 109pp.
[9] Y. Chagnac-Amitai and B. W. Connors, *Synchronized excitation and inhibition driven by intrinsically bursting neurons in neocortex*, *J. Neurophysiol.*, 62 (1989), 1149–1162.
[10] R. D. Chervin, P. A. Pierce and B. W. Connors, *Periodicity and directionality in the propagation of epileptiform discharges across discharges across neocortex*, *J. Neurophysiol.*, 60 (1988), 1695–1713.
[11] S. Coombes, G. J. Lord and M. R. Owen, Waves and bumps in neuronal networks with axo-dendritic synaptic interactions, *Phys. D.*, **178** (2003), 219–241.
[12] S. Coombes and M. R. Owen, Evans functions for integral neural field equations with Heaviside firing rate function, *SIAM J. Appl. Dyn. Syst.*, **3** (2004), 574–600.
[13] M. Enculescu, A note on traveling fronts and pulses in a firing rate model of a neuronal network, *Physica D.*, **196** (2004), 362–386.
[14] G. B. Ermentrout, Reduction of conductance-based models with slow synapses to neural nets, *J. Math. Biol.*, **6** (1994), 679–695.
[15] G. B. Ermentrout and J. B. McLeod, Existence and uniqueness of travelling waves for a neural network, *Proc. Roy. Soc. Edinburgh Sect. A.*, **123** (1993), 461–478.
[16] J. W. Evans, Nerve axon equations, I: Linear approximations, *Indiana Univ. Math. J.*, **21** (1972), 877–955.
[17] J. W. Evans, Nerve axon equations, II: Stability at rest, *Indiana Univ. Math. J.*, **22** (1972), 75–90.
[18] J. W. Evans, Nerve axon equations, III: Stability of the nerve impulse, *Indiana Univ. Math. J.*, **24** (1975), 1169–1190.
[19] J. W. Evans, Nerve axon equations, IV: The stable and unstable impulse, *Indiana Univ. Math. J.*, **24** (1975), 1169–1190.
[20] S. E. Folias and P. C. Bressloff, Stimulus-locked waves and breathers in an excitatory neural network, *SIAM J. Appl. Math.*, **65** (2005), 2067–2092.
[21] M. A. Geise, *Dynamic Neural Field Theory for Motion Perception*, Dordrecht: Kluwer, 1999.
[22] Y. Guo, Existence and stability of traveling fronts in a lateral inhibition neural network, *SIAM J. on Applied Dynamical Systems*, **11** (2012), 1543–1582.
[23] Y. Guo and C. C. Chow, Existence and stability of standing pulses in neural networks: I. existence, *SIAM J. on Applied Dynamical Systems*, **4** (2005), 217–248.
[24] Y. Guo and C. C. Chow, Existence and stability of standing pulses in neural networks: II. stability, *SIAM J. on Applied Dynamical Systems*, **4** (2005), 249–281.
[25] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vickers, The evolution of dispersal, *J. Math. Biol.*, **47** (2003), 483–517.
[26] Z. P. Kilpatrick, S. E. Folias and P. C. Bressloff, Traveling pulses and wave propagation failure in inhomogeneous neural media, *SIAM J. Appl. Dyn. Syst.*, **7** (2008), 161–185.
[27] K. Kishimoto and S. Amari, Existence and stability of local excitations in homogeneous neural fields, *J. Math. Biol.*, **7** (1979), 303–318.
[28] N. Laaris, G. C. Carlson and A. Keller, Thalamic-evoked synaptic interactions in barrel cortex revealed by optical imaging, *J. Neurosci.*, **20** (2000), 1529–1537.
[29] A. D. Myshkis, Differential equations, ordinary with distributed arguments, *Encyclopaedia of Mathematics*, Vol. 3, Kluwer Academic Publishers, Boston, 1989, 144–147.
[30] D. J. Pinto and G. B. Ermentrout, Spatially structured activity in synaptically coupled neuronal networks: I. Traveling fronts and pulses, *SIAM J. Appl. Math.*, **62** (2001), 206–225.
[31] D. J. Pinto, R. K. Jackson and C. E. Wayne, Existence and stability of traveling pulses in a continuous neuronal network, *SIAM J. Appl. Dyn. Syst.*, **4** (2005), 954–984.
[32] D. J. Pinto, S. L. Patrick, W. C. Huang and B. W. Connors, Initiation, propagation and termination of epileptiform activity in rodent neocortex in vitro involve distinct mechanisms, *J. Neurosci.*, **25** (2005), 8131–8140.
[33] D. J. Pinto, W. Troy and T. Kneezel, Asymmetric activity waves in synaptic cortical systems, *SIAM J. Appl. Dyn. Syst.*, **8** (2009), 1218–1233.
[34] P. A. Robinson, C. J. Rennie, J. J. Wright, H. Bahrnamali, E. Gordon and D. I. Rowe D, Prediction of electroencephalographic spectra from neurophysiology, *Phys. Rev. E.*, **63** (2001), 021903.
[35] D. J. T. Liley, P. J. Cadusch and M. P. Dafilis, A spatially continuous mean field theory of electrocortical activity, *Network*, **13** (2002), 67–113.
[36] B. Sandstede, *Stability of travelling waves*, in *Handbook of Dynamical Systems*, B. Fiedler, ed., North–Holland, Amsterdam, **2** (2002), 983–1055.
[37] B. Sandstede, Evans functions and nonlinear stability of travelling waves in neuronal network models, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **17** (2007), 2693–2704.
[38] D. C. Somers, S. Nelson and M. Sur, An emergent model of orientation selectivity in cat visual cortical simple cells, *J. Neurosci.*, **15** (1995), 5448–5465.
[40] W. C. Troy, Traveling waves and synchrony in an excitable large-scale neuronal network with asymmetric connections, *SIAM J. Appl. Dyn. Syst.*, 7 (2008), 1247–1282.

[41] H. R. Wilson and J. D. Cowan, Excitatory and inhibitory interactions in localized populations of model neurons, *Biophys. J.*, 12 (1972), 1–24.

[42] H. R. Wilson and J. D. Cowan, A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue, *Kybernetic*, 13 (1973), 55–80.

[43] J. Y. Wu, L. Guan and Y. Tsau, Propagating activation during oscillations and evoked responses in neocortical slices, *J. Neurosci.*, 19 (1999), 5005–5015.

[44] X. Xie and M. Giese, Nonlinear dynamics of direction-selective recurrent neural media, *Phys. Rev. E*, 65 (2002), 051904, 11pp.

[45] L. Zhang, On stability of traveling wave solutions in synaptically coupled neuronal networks, *Differential Integral Equations*, 16 (2003), 513–536.

[46] L. Zhang, Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks, *J. Differential Equations*, 197 (2004), 162–196.

Received April 2015; revised June 2016.

*E-mail address*: yixin@math.drexel.edu  
*E-mail address*: zhangai@tigermail.auburn.edu