Schur $\sigma$-Groups of Scholz-Taussky Type $F$

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Abstract. For finite metabelian 3-groups $M$ with elementary bicyclic commutator quotient $M/M' \cong C_3 \times C_3$, coclass $cc(M) \in \{4, 6\}$, and transfer kernel type $F$, the smallest Schur $\sigma$-groups $S$ with second derived quotient $S/S'' \cong M$ are determined. Evidence is provided of arithmetical realizations of these groups by second 3-class groups $M \cong G_3^\sigma(K) = \text{Gal}(F_p^\infty(K)/K)$, respectively 3-class field tower groups $S \cong G_3^\sigma(K) = \text{Gal}(F_p^\infty(K)/K)$, of imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$.

1. Introduction

A Schur $\sigma$-group $S$ has a balanced presentation $d(S) = r(S)$ with coinciding generator rank $d(S) = \dim_k H^1(S, F_p)$ and relation rank $r(S) = \dim_k H^2(S, F_p)$, and an automorphism $\sigma \in \text{Aut}(S)$ inducing the inversion $x \mapsto x^{-1}$ on the first and second cohomology group, $H^1(S, F_p)$ and $H^2(S, F_p)$. Therefore, $\sigma$ is called generator- and relator-inverting. According to a theorem of Koch and Venkov [6], the Galois group $\text{Gal}(F_p^\infty(K)/K)$ of the maximal unramified pro-$p$ extension of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ must be a Schur $\sigma$-group when $p$ is an odd prime number. Various deeper results on Schur $\sigma$-groups were established by Arrigoni [1].

The focus of the present article is on the smallest odd prime $p = 3$ and finite 3-groups $G$ with elementary bicyclic commutator quotient $G/G' \cong C_3 \times C_3$ and transfer kernel type $F$, according to the terminology of Scholz and Taussky [19] § 2.I.F, p. 36. Since the 3-group $G$ has two generators $G = \langle x, y \rangle$ with $x^3, y^3 \in G'$, it possesses four maximal self-conjugate subgroups,

\begin{equation}
U_1 = \langle x, G' \rangle, \quad U_2 = \langle y, G' \rangle, \quad U_3 = \langle xy, G' \rangle, \quad U_4 = \langle xy^2, G' \rangle,
\end{equation}

with Artin transfer homomorphisms $T_i : G/G' \to U_i/U_i'$, and the transfer kernel type of $G$ is defined by $\kappa(G) = (\kappa_1, \ldots, \kappa_4)$, where $\kappa_i = 0$ if $\ker(T_i) = G/G'$ and $\kappa_i = j$ if $\ker(T_i) = U_j/G'$ with $1 \leq i, j \leq 4$. More precisely, $\kappa(G)$ is an orbit under the action of the symmetric group $S_4$, consisting of equivalent quartets $\kappa \sim \pi^{-1} \kappa \pi$ with $\pi \in S_4$ [3] § 2.2, pp. 475–476]. Type $F$ splits into four subtypes, $F_7, \kappa \sim (3443), F_{11}, \kappa \sim (1143), F_{12}, \kappa \sim (1343), F_{13}, \kappa \sim (3143)$, sharing a common transposition (43). Types $F_{11}, F_{12}$ have a fixed point (1), types $F_{7}, F_{13}$ do not.

The smallest metabelian $\sigma$-groups $M$ of type $F$ have order $\text{ord}(M) = 3^3$, coclass $cc(M) = 4$ and nilpotency class $cl(M) = 5$. The smallest non-metabelian Schur $\sigma$-groups $S$ of type $F$ have order $\text{ord}(S) = 3^{20}$, soluble length $\text{sl}(S) = 3$, coclass $cc(S) = 11$ and nilpotency class $cl(S) = 9$.

Since the order of 3-groups $G$ in the SmallGroups database [2] is bounded by $\text{ord}(G) \leq 3^8$, all the groups under investigation must be constructed by the $p$-group generation algorithm [17, 18, 5], and they will be denoted in the terminology of the ANUPQ-package [4] by the combination of an absolute identifier and one (or several) relative identifier(s) in the shape

\begin{equation}
G \simeq \langle o, i \rangle - \# s; j \text{ with order } o \leq 3^8, \text{ step size } s \leq \nu, \text{ and } i, j \in \mathbb{N},
\end{equation}

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where \( i \) is the absolute identifier in the SmallGroups library, and \( \nu \) is the nuclear rank of \( (o,i) \).

We begin with basic foundations concerning the evolution of transfer kernels in descendant trees in §2 second abelian type invariants in §3 and bifurcation in descendant trees in §4.

In §2 the smallest sporadic metabelian \( \sigma \)-groups \( M \) of type \( F \) outside of coclass trees are identified. They are all of order \( 3^9 \) and coclass 4. By the construction of extremal root paths, suitable non-metabelian Schur \( \sigma \)-groups \( S \) with \( S/S'' \simeq M \) are provided. The smallest of them are of order \( 3^{20} \) and soluble length 3.

In §3 the smallest periodic metabelian \( \sigma \)-groups \( M \) of type \( F \) are identified as vertices of coclass trees \( 10,11,13 \). They are all of order \( 3^{11} \) and coclass 4. Smallest non-metabelian Schur \( \sigma \)-groups \( S \) with \( S/S'' \simeq M \) are of order \( 3^{23} \) and soluble length 3.

In §4 bigger sporadic metabelian \( \sigma \)-groups \( M \) of type \( F \) outside of coclass trees are identified. They are all of order \( 3^{13} \) and coclass 6. Smallest non-metabelian Schur \( \sigma \)-groups \( S \) with \( S/S'' \simeq M \) are of order \( 3^{26} \) and soluble length 3.

2. Evolution (Contraction) of Transfer Kernels in Trees

In §3, and with more depth in §10, we have seen that finite 3-groups with transfer kernel types \( E.6, E.14 \), respectively E.8, E.9, are immediate descendants of skeleton groups with transfer kernel type c.18, respectively c.21. For finite 3-groups with one of the four transfer kernel types \( F \), we have to establish a corresponding overview of possible skeleton types.

**Lemma 1. (Skeleton types.)** Sporadic metabelian 3-groups, as isolated vertices outside of coclass trees, with one of the four transfer kernel types \( F \) are immediate descendants with step size \( s = 2 \) of skeleton groups with transfer kernel type b.10, \( \kappa \sim (0043) \).

Periodic metabelian 3-groups, as terminal vertices of coclass trees, with one of the four transfer kernel types \( F \) are immediate descendants with step size \( s = 1 \) of skeleton groups with three possible transfer kernel types \( d \). In detail:

- **Type F.7**, \( \kappa(K) \sim (4343) \), can only arise from type d.19, \( \kappa(K) \sim (0343) \).
- **Type F.11**, \( \kappa(K) \sim (2243) \sim (1143) \), arises either from type d.23, \( \kappa(K) \sim (0243) \), or from type d.25, \( \kappa(K) \sim (0143) \),
- **Type F.12**, \( \kappa(K) \sim (1343) \sim (3243) \), arises either from type d.19, \( \kappa(K) \sim (0343) \), or from type d.23, \( \kappa(K) \sim (0243) \),
- **Type F.13**, \( \kappa(K) \sim (2343) \sim (3143) \), arises either from type d.19, \( \kappa(K) \sim (0343) \), or from type d.25, \( \kappa(K) \sim (0143) \).

**Proof.** By the antitony principle for the Artin pattern \( 11 \) \( \S \S \) 5.1–5.4, pp. 78–87, a partial (one-dimensional) transfer kernel is inherited by an immediate descendant from its parent, whereas a total (two-dimensional) transfer kernel may contract to any one-dimensional transfer kernel.

For a sporadic isolated vertex of step size \( s = 2 \), both total kernels of type b.10, \( \kappa \sim (0043) \), of the parent may shrink at once, and thus the immediate descendant may have any of the four transfer kernel types \( F \), \( \kappa(K) \sim (4343) \) or \( \kappa(K) \sim (1143) \) or \( \kappa(K) \sim (1343) \) or \( \kappa(K) \sim (2343) \).

For a periodic isolated vertex of step size \( s = 1 \), the single total kernel of a parent of type d.19, \( \kappa(K) \sim (0343) \), may either contract to type F.12, \( \kappa(K) \sim (1343) \), or type F.13, \( \kappa(K) \sim (2343) \) or type H.4, \( \kappa(K) \sim (3343) \). The latter must be eliminated, when type F is desired.

The single total kernel of a parent of type d.23, \( \kappa(K) \sim (0243) \), may either shrink to type F.12, \( \kappa(K) \sim (3243) \), or type F.11, \( \kappa(K) \sim (2243) \) or type G.16, \( \kappa(K) \sim (1243) \). The latter with two fixed points must be eliminated, when type F is desired.

The single total kernel of a parent of type d.25, \( \kappa(K) \sim (0143) \), may either shrink to type F.11, \( \kappa(K) \sim (1143) \), or type F.13, \( \kappa(K) \sim (3143) \) or type G.19, \( \kappa(K) \sim (2143) \). The latter with two transpositions must be eliminated, when type F is desired.

3. Second Abelian Type Invariants of Imaginary Quadratic Fields

Let \( N(\ell) \) be the nearly homocyclic abelian 3-group of logarithmic order \( \ell \), that is, \( N(2j) = (j,j) \) if \( \ell = 2j \geq 0 \) is even, and \( N(2j+1) = (j+1,j) \) if \( \ell = 2j + 1 \geq 1 \) is odd. Second abelian type invariants \( (\text{ATI}) \, \alpha^{(2)}(k) \) of any algebraic number field \( k \) with 3-class group \( \text{Cl}_3(k) \simeq C_3 \times C_3 \),
second 3-class group $M = G_3^2(k) = \text{Gal}(\mathbb{F}_3^2(k)/k)$, nilpotency class $c = \text{cl}(M)$, coclass $r = \text{cc}(M)$, and capitulation type $F$ are always of the logarithmic shape

$\alpha^{(2)}(k) = [11; (Pol; \text{Com}, A_1), (Cpl; \text{Com}, A_2), (111; \text{Com}, A_3), (111; \text{Com}, A_4)]$

with triplets $A_1$, $A_2$, and dodecuplets $A_3$, $A_4$, and abelian quotient invariants $Pol = N(c)$ of the polarization, $Cpl = N(r + 1)$ of the co-polarization [12, Eqn. (5.1), p. 140], and $\text{Com} = N(c - 1) \times N(r - 1)$ of the commutator subgroup [9, Appendix]. (The defect is zero for type $F$.)

**Definition 1.** Second abelian type invariants $\alpha^{(2)}(k)$ are said to be of the
- **first category** (extreme) if all components of the dodecuplets $A_3$, $A_4$ have at least rank 4,
- **second category** (wild, elevated) if some but not all components of the dodecuplets $A_3$, $A_4$ have at least rank 4,
- **third category** (tame, moderate) if all components of the dodecuplets $A_3$, $A_4$ have rank 3.

Recent investigations by Eric Ahlgvist and Magnus Carlsson suggest that imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with $\alpha^{(2)}(K)$ of the first and second category have infinite 3-class field towers. Unfortunately, their proof cannot be applied to elementary bicyclic $\text{Cl}_3(K)$. So the infinitude of the 3-class field tower for wild and extreme type $F$ of $K$ remains a conjecture.

4. **Bifurcation in Descendant Trees**

The basic genetics of transfer kernel types in §2 can now be applied to describe the mutual position of a finite Schur $\sigma$-group $S$ of type $F$ and category 3 and its metabelianization $M = S/S''$. As opposed to the types $G$ and $H$, where $S$ may be a descendant of $M$ [12], the situation for the types $E$ and $F$ can be described in terms of **root paths** [15, Dfn. 2.2, p. 87, §3, p. 88] by the following definition.

**Definition 2.** The vertex $B$ of the biggest order in the meet of the root paths of $S$ and $M$ is called the **bifurcation** between $S$ and $M$. The vertex $R$ of the smallest order in the root path of $S$ which has the transfer kernel type of $S$ is called the **settled root** of $S$ (since no further contraction of transfer kernels can occur for descendants of $R$). The **fork** between $S$ and $M$ is the union of the path from $R$ to $B$ and the path from $M$ to $B$ (which intersect in $B$).

In ostensibly simplified form, the vertices of the fork are only labelled by the letters of their transfer kernel types, and the edges are labelled by their step sizes. In view of possible arithmetical realizations, we shall restrict our investigations to three main situations,

- Schur $\sigma$-group $S$ with sporadic metabelianization $M$ of coclass $\text{cc}(M) = 4$ in §5
- Schur $\sigma$-group $S$ with periodic metabelianization $M$ of coclass $\text{cc}(M) = 4$ in §6 and
- Schur $\sigma$-group $S$ with sporadic metabelianization $M$ of coclass $\text{cc}(M) = 6$ in §7

**Theorem 1.** Let $S = \text{Gal}(F_3^\infty(K)/K)$ be the Schur $\sigma$-Galois group of the maximal unramified pro-3 extension of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d < 0$, elementary bicyclic 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$, capitulation type $F$, and second abelian type invariants $\alpha^{(2)}(K)$ of category 3. Then the fork between $S$ and $M = S/S'' \simeq \text{Gal}(F_3^2(K)/K)$, in dependence on the first abelian type invariants $\alpha^{(1)}(K)$, is given by

$F \xrightarrow{2} b \stackrel{4}{\leftrightarrow} F $ if $\alpha^{(1)}(K) = [11; 32, 32, 111, 111],

(4.1) \hspace{1cm} F \xrightarrow{1} d \xrightarrow{1} d \xrightarrow{2} b \stackrel{4}{\leftrightarrow} d \xrightarrow{2} d \xrightarrow{4} F $ if $\alpha^{(1)}(K) = [11; 43, 32, 111, 111],

F \xrightarrow{2} b \xrightarrow{2} b \xrightarrow{2} b \xrightarrow{2} b \xrightarrow{4} b \xrightarrow{4} F $ if $\alpha^{(1)}(K) = [11; 43, 43, 111, 111].$

**Proof.** In [15, §3.6, pp. 101–104] the fork was called **fork topology**. The metabelian part of the fork has been described in [15, Cor. 3.1, p. 104] for the two sporadic situations $\alpha^{(1)}(K) = [11; 32, 32, 111, 111]$ with class 5, coclass 4, and $\alpha^{(1)}(K) = [11; 43, 43, 111, 111]$ with class 7, coclass 6, and in [15, Thm. 3.6, p. 103] for the periodic situation $\alpha^{(1)}(K) = [11; 43, 32, 111, 111]$ with class 7, coclass 4. The bifurcation is always the fixed vertex $B = (2187, 64)$. Therefore, the root paths $P$ in [15, pp. 103–104] must be shortened by the omission of the trailing three edges.
(2187, 64) \xrightarrow{2} (243, 3) \xrightarrow{2} (27, 3) \xrightarrow{1} (9, 2) of type b \xrightarrow{2} b \xrightarrow{2} a \xrightarrow{1} a. The non-metabelian part of the fork is a consequence of the extremal root path property of finite Schur \(\sigma\)-groups \(S\) of category 3, and it is symmetric with respect to the types (but not with respect to the step sizes).  

5. REALIZATION OF SPORADIC METABELIAN \(\sigma\)-GROUPS \(G_3^2(K)\) OF COCLASS 4

In Table I, second abelian type invariants \(\alpha^{(2)}(K)\) of imaginary quadratic fields \(K = \mathbb{Q}(\sqrt{d})\) with discriminants in the narrow range \(-5 \cdot 10^5 < d < 0\), bicyclic 3-class group \(C_3(K) \simeq C_3 \times C_3\), second class group \(M = G_3^2(K)\), \(\text{ord}(M) = 3^5\), and type \(F\) are given, ordered by capitulation subtypes \(\varkappa(K)\). **Boldface** font emphasizes exceptional components of the first or second category.

**Table 1.** 2nd ATI of \(K = \mathbb{Q}(\sqrt{d})\), \(d < 0\), with sporadic \(M = G_3^2(K)\), \(cc(M) = 4\)

| Type \(\alpha^{(2)}(K)\) | \(\varkappa(K) = (3443)\) |
|----------------|------------------|
| \(\langle 11; (32; 2221, A_1), (32; 2221, A_2), (111; 2221, A_3), (111; 2221, A_4) \rangle\) |
| \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
| F.7 124363 | \((3211)^3\) | \((3211)^3\) | \((1^6)^3, (2221)^3, (2211)^6\) | \((2^21^3)^3, (214)^3, (2211)^6\) |
| 225299 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 260515 | \((3321)^3\) | \((3321)^3\) | \((1^6)^3, (3221)^3, (2221)^3, (2211)^3\) | \((1^6)^3, (21^4)^3, (2211)^3, (2211)^3\) |
| 243480 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 43476 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 486264 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| F.11 27156 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 214160 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 394999 | \((3111)^3\) | \((3111)^3\) | \((2111)^9, (2211)^3\) | \((1^5)^3, (2111)^3, (1111)^6\) |
| 477192 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 484004 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| F.12 31908 | \((3111)^4\) | \((3111)^4\) | \((2111)^9, (2211)^3\) | \((2111)^9, (1111)^6\) |
| 135587 | \((3111)^3\) | \((3111)^3\) | \((2211)^3, (2111)^6, (221)^3\) | \((1^5)^3, (2111)^3, (1111)^6\) |
| 160403 | \((3211)^3\) | \((3211)^3\) | \((21^4)^3, (2222)^3, (2211)^3\) | \((1^5)^3, (2211)^3, (2111)^3\) |
| 184132 | \((2111)^3\) | \((3111)^3\) | \((2211)^3, (2111)^6, (221)^3\) | \((2111)^6, (1111)^6\) |
| 189959 | \((3111)^3\) | \((3111)^3\) | \((2211)^3, (2111)^6, (221)^3\) | \((221)^3, (211)^9\) |
| 291220 | \((3111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 454631 | \((3111)^3\) | \((3111)^3\) | \((2111)^9, (2211)^3\) | \((2111)^6, (1111)^6\) |
| 499159 | \((3111)^3\) | \((3111)^3\) | \((2111)^9, (221)^3\) | \((1^5)^3, (2111)^3, (1111)^6\) |
| F.13 67480 | \((4111)^3\) | \((3111)^3\) | \((3211)^3, (2111)^6, (221)^3\) | \((2111)^6, (1111)^6\) |
| 104627 | \((4111)^3\) | \((3111)^3\) | \((2111)^9, (221)^3\) | \((5^3, (2111)^3, (1111)^6\) |
| 167904 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 224580 | \((3211)^3\) | \((3211)^3\) | \((21^4)^3, (15^6)^3, (2211)^3\) | \((1^5)^3, (2211)^3, (2111)^6\) |
| 287155 | \((4111)^3\) | \((3111)^3\) | \((2111)^9, (221)^3\) | \((2111)^6, (1111)^6\) |
| 296407 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 317747 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 344667 | \((4111)^3\) | \((3111)^3\) | \((2211)^3, (2111)^6, (221)^3\) | \((221)^3, (2111)^3, (1111)^6\) |
| 401603 | \((4111)^3\) | \((3111)^3\) | \((221)^3, (211)^9\) | \((221)^3, (211)^9\) |
| 426891 | \((4111)^3\) | \((3111)^3\) | \((2111)^9, (221)^3\) | \((2111)^6, (1111)^6\) |
| 487272 | \((3111)^3\) | \((3111)^3\) | \((2111)^9, (221)^3\) | \((1^5)^3, (2111)^3, (1111)^6\) |

Throughout the sequel, we restrict our investigations to the tame situation of category 3.

The second \(\text{AQI}, \alpha^{(2)}(S) = [11; (32; 2221, A_1), (32; 2221, A_2), (111; 2221, A_3), (111; 2221, A_4)],\) in the Tables 2, 3, 4, 5 contain triplets \(A_1, A_2\) and dodeculept \(A_3, A_4\). In terms of the **fixed**
bifurcation $B = \langle 2187, 64 \rangle$, the tables give a complete overview of all possible Schur $\sigma$-groups $S$ among the descendants of $R = B - \#4; k$ as non-metabelian roots, for any assigned metabelian $\sigma$-group $M = B - \#2; j$ with $M = R/R'' = S/S''$, order $\text{ord}(M) = 3^3$, nilpotency class $\text{cl}(M) = 5$, coclass $\text{cc}(M) = 4$, and transfer kernel types F.11, F.12, F.13, F.7. All second AQI are of category 3 (tame, moderate).

Table 2. Correspondence for sporadic $M = S/S''$, $\text{cc}(M) = 4$, of type F.11

| Type | lo(M) | j | k | lo(S) | sl(S) | # | $\alpha^{(2)}(S)$ |
|------|-------|---|---|-------|-------|---|-----------------|
| F.11 |       | 9 | 36| 127   | 20    | 3 | 81/81          |
|      |       |   |   |       |       |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       | 144| 20| 3     | 27/27 |   |                |
|      |       |   |   |       |       |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       | 172| 20| 3     | 81/81 |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       |   |   |       |       |   |                |
|      |       | 180| 20| 3     | 27/27 |   |                |
|      |       |   |   |       |       |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       | 119| 20| 3     | 81/81 |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       |   |   |       |       |   |                |
|      |       | 139| 20| 3     | 27/27 |   |                |
|      |       |   |   |       |       |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       | 164| 20| 3     | 81/81 |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |
|      |       |   |   |       |       |   |                |
|      |       | 182| 20| 3     | 27/27 |   | $(4111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9$ |

Theorem 2. Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d < 0$, bicyclic 3-class group $C_3(K) \simeq C_9 \times C_9$, second 3-class group $M = G_3^3(K)$, $\text{ord}(M) = 3^3$, capitation type F.11, $\varkappa(K) \sim (1143)$, and first abelian type invariants $\alpha^{(1)}(K) = [11; 32, 32, 111, 111]$. If the second abelian type invariants are of third category, they must have the shape

$$\alpha^{(2)}(K) = [11; 32, 2221, (4111)^3, (32; 2221, (4111)^3), (111; 2221, (221)^3, (211)^9]^2],$$

and the 3-class field tower $F_3^\infty(K)$ has precise length $\ell_3(K) = 3$ and relative degree $3^{20}$ over $K$.

Proof. This is an immediate consequence of Table 2 since all the non-metabelian roots $R = B - \#4; k$ with $k \in \{119, 127, 139, 144, 164, 172, 180, 182\}$ lead to Schur $\sigma$-groups $S$ with soluble length $\text{sl}(S) = 3$, logarithmic order $\text{lo}(S) = 20$, and have the second abelian type invariants in Formula (5.1). □

Example 1. According to Table 1 the quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d \in \{ -27156, -241160, -477192, -484804 \}$ have the capitulation type F.11, $\varkappa(K) \sim (1143)$, and second abelian type invariants of third category in Formula (5.1). According to Theorem 2 they have a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$.

Theorem 3. An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d < 0$, elementary bicyclic 3-class group $C_3(K)$, second 3-class group $M = G_3^3(K)$, $\text{ord}(M) = 3^3$, capitation type F.12, $\varkappa(K) \sim (1343)$, and second abelian type invariants of third category

$$\alpha^{(2)}(K) = [11; 32, 2221, (311)^3], (32; 2221, (311)^3), (111; 2221, (221)^3, (211)^9]^2],$$

has a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$. For second abelian type invariants

$$\alpha^{(2)}(K) = [11; 32, 2221, (4111)^3], (32; 2221, (311)^3), (111; 2221, (221)^3, (211)^9]^2],$$

the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

Proof. This is an immediate consequence of Table 3 since the unique non-metabelian roots $R = B - \#4; k$ with $k \in \{157, 160\}$ which lead to Schur $\sigma$-groups $S$ with soluble length $\text{sl}(S) = 4$ have the second abelian type invariants in Formula (5.3). □

Example 2. According to Table 1 the single quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -291220$ has capitulation type F.12, $\varkappa(K) \sim (1343)$, and second abelian type invariants of third category (5.2). According to Theorem 3 it has a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$. 
Table 3. Correspondence for sporadic \( M = S/S'' \), \( cc(M) = 4 \), of type F.12

| Type \( j \) | \( k \) | \( lo(S) \) | \( sl(S) \) | \#  | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( \sigma(S) = (1343) \) |
|-----------|-----|-------|-------|----|--------|--------|--------|--------|------------------|
| F.12      | 9   | 43    | 20    | 3  | 27/27  | (3111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 143 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 170 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 187 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 130 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 146 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 175 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 190 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 126 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 162 | 23, 26, 29 | 4  | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 194 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 116 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 113 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 157 | 23, 26, 29 | 4  | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 185 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 188 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 160 | 23, 26 | 4     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |
|           | 195 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)³, (211)⁹ | (221)³, (211)⁹ |

Theorem 4. An imaginary quadratic field \( K = \mathbb{Q}(\sqrt{d}) \) with fundamental discriminant \( d < 0 \), elementary bicyclic 3-class group \( Cl_3(K) \), second 3-class group \( M = G_3^2(K) \), \( \text{ord}(M) = 3^9 \), capitulation type F.13, \( \sigma(K) \sim (3143) \), and second abelian type invariants of third category

\[
\alpha(2)(K) = [11; (32; 221, (3111)^3), (32; 2221, (3111)^3), (111; 2221, (221)^3, (211)^9]^2
\]

has a 3-class field tower \( F_3^K \) of precise length \( \ell_3(K) = 3 \). For second abelian type invariants

\[
\alpha(2)(K) = [11; (32; 2221, (4111)^3), (32; 2221, (3111)^3), (111; 2221, (221)^3, (211)^9]^2
\]

Table 4. Correspondence for sporadic \( M = S/S'' \), \( cc(M) = 4 \), of type F.13

| Type \( j \) | \( k \) | \( lo(S) \) | \( sl(S) \) | \#  | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( \sigma(S) = (3143) \) |
|-----------|-----|-------|-------|----|--------|--------|--------|--------|------------------|
| F.13      | 9   | 41    | 132   | 3  | 27/27  | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 147 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 177 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 185 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 122 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 114 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 167 | 23, 26, 29 | 4  | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 192 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 112 | 23, 26, 29 | 4  | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 136 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 158 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 171 | 20    | 3     | 27/27 | (3111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 115 | 23, 26 | 4     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
|           | 137 | 20    | 3     | 27/27 | (4111)² | (3111)² | (221)², (211)⁹ | (221)², (211)⁹ |
the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

**Proof.** This is an immediate consequence of Table[8] since the unique non-metabelian roots $R = B - \#4; k$ with $k \in \{112, 115\}$ which lead to Schur $\sigma$-groups $S$ with soluble length $sl(S) = 4$ have the second abelian type invariants in Formula[5],[5]. □

**Example 3.** According to Table[8] the quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with discriminants $d \in \{-167,064, -296,407, -317,747, -401,603\}$ have the capitulation type $F.13$, $\kappa(K) = (3143)$, and second abelian type invariants of third category [5],[5]. According to Theorem[5] no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$. However, outside of the range in Table[8] we found two quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with discriminants $d \in \{-731,867, -803,591\}$, capitulation type $F.13$, $\kappa(K) = (3143)$, and second abelian type invariants of third category [6],[6]. According to Theorem[5] they have a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$.

**Table 5.** Correspondence for sporadic $M = S/S''$, $cc(M) = 4$, of type F.7

| Type | $lo(M)$ | $j$ | $k$ | $lo(S)$ | $sl(S)$ | $\#$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $\kappa(S) = (3443)$ |
|------|---------|-----|-----|---------|---------|------|-------|-------|-------|-------|-----------------|
| F.7  |         |     |     |         |         |      |       |       |       |       |                 |
| 9    | 55      | 121 | 20  | 3       | 27/27   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 131 | 20  | 3       | 27/27   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 165 | 26, 29, 32, 38 | 4 | 27/27 | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 196 | 20  | 3       | 27/27   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
| 9    | 56      | 123 | 26, 29, 32, 35, 41 | 4 | 18/18 | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 142 | 20  | 3       | 18/18   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 169 | 20  | 3       | 27/27   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 128 | 26, 29, 32 | 4 | 18/18 | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 145 | 20  | 3       | 18/18   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |
|      |         | 174 | 20  | 3       | 27/27   | (3111)^3 | (3111)^3 | (221)^3 | (211)^9 | (221)^3 | (211)^9 |

**Theorem 5.** For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with discriminant $d < 0$, elementary bicyclic 3-class group $C_3(K)$, second 3-class group $M = G_3^2(K)$, $ord(M) = 3^9$, and capitulation type $F.7$, $\kappa(K) \sim (3443)$, the second abelian type invariants of third category must have the shape

$\alpha^{(2)}(K) = [11; (32, 222, (3111)^3), (32, 222, (3111)^3), (3111)^3, (3111)^3, (221)^3, (211)^9, (221)^3, (211)^9, (221)^3, (211)^9]$.

and the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

**Proof.** This is an immediate consequence of Table[8] since the unique non-metabelian roots $R = B - \#4; k$ with $k \in \{123, 128, 165\}$ which lead to Schur $\sigma$-groups $S$ with soluble length $sl(S) = 4$ also have the common second abelian type invariants in Formula[5],[5]. □

**Example 4.** According to Table[8] the quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with discriminants $d \in \{-225,299, -343,380, -423,476, -486,264\}$ have the capitulation type $F.7$, $\kappa(K) = (3443)$, and second abelian type invariants of third category [5],[5]. According to Theorem[5] no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$.

**Corollary 1.** The logarithmic order of the Schur $\sigma$-group $S = Gal(F_3^\infty(K)/K)$ is generally $lo(S) = 20$ for a 3-class field tower of length $\ell_3(K) = 3$, and it depends on the capitulation type $\kappa(K)$,

$lo(S) \in \{23, 26, 29\}$ for type $F.12$ or $F.13$,

$lo(S) \in \{26, 29, 32, 35, 38, 41\}$ for type $F.7$.

in the case of a tower of length $\ell_3(K) = 4$.

**Proof.** This is a common consequence of the Tables[2],[3],[4],[5] □
6. Realization of Periodic Metabelian $\sigma$-Groups $G_3^2(K)$ of Coclass 4

Table 6 gives second abelian type invariants $\alpha^{(2)}(K)$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants in the bigger range $-10^6 < d < 0$, bicyclic 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$, second 3-class group $M = G_3^2(K)$, ord$(M) = 3^{11}$, and type $F$, ordered by capitulation subtypes $\varkappa(K)$. **Boldface** font indicates categories 1 and 2.

| Type  | $\alpha^{(2)}(K)$ | $\varkappa(K)$ = (3443) |
|-------|-------------------|-------------------------|
| F.7   | $\alpha^{(2)}(K)$ | $\varkappa(K)$ = (3443) |
| 469816 | (5211)$^3$         | (3111)$^4$             |
| 643011 | (221)$^3$         | (3111)$^4$             |
| 797556 | (221)$^3$         | (3111)$^4$             |
| F.11  | $\varkappa(K)$ = (1143) |
| 469787 | (5211)$^3$         | (221)$^3$, (211)$^9$ |
| F.12  | $\varkappa(K)$ = (1343) |
| 249371 | (221)$^3$         | (221)$^3$, (211)$^9$ |
| 278427 | (3111)$^3$         | (221)$^3$, (211)$^9$ |
| 382123 | (221)$^3$         | (221)$^3$, (211)$^9$ |
| F.13  | $\varkappa(K)$ = (3143) |
| 159208 | (4211)$^3$         | (221)$^3$, (211)$^9$ |
| 262628 | (5211)$^3$         | (221)$^3$, (211)$^9$ |
| 273284 | (4211)$^3$         | (221)$^3$, (211)$^9$ |
| 551112 | (5211)$^3$         | (221)$^3$, (211)$^9$ |
| 940943 | (4211)$^3$         | (211)$^9$, (221)$^3$ |
| 947463 | (4211)$^3$         | (211)$^9$, (221)$^3$ |

Throughout the sequel, we restrict our investigations to the tame situation of category 3.

For periodic metabelianizations $M$ on coclass trees, we need more space for relative identifiers in chains of descendants. Since all second AQTs to be considered for finite soluble length are of third category (tame, moderate), the second abelian quotient invariants,

$$\alpha^{(2)}(S) = \{11; (43; 3321, A_1), (32; 3321, A_2), (111; 3321, A_3), (111; 3321, A_4)\},$$

in the Tables 7, 8, 9, 10 contain triplets $A_1$, $A_2$ and fixed dodecuplets $A_3 = A_4 = \{(221)\)^3, (211)\)^9\}$, excluded from the table in order to provide more space. The tables give a complete overview of all possible Schur $\sigma$-groups $S$ among the descendants of $R = B - #4; k_1 - #2; k_2 - #4; k_3$ as non-metabelian roots, for any assigned metabelian $\sigma$-group $M = B - #2; j_1 - #1; j_2 - #1; j_3$ with $M = R/R'' = S/S''$, order ord$(M) = 3^{11}$, nilpotency class cl$(M) = 7$, coclass cc$(M) = 4$, and transfer kernel type $F.13$, $F.12$, $F.11$, $F.7$.

**Theorem 6.** For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d < 0$, bicyclic 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$, second 3-class group $M = G_3^2(K)$, ord$(M) = 3^{11}$, capitulation type F.13, $\varkappa(K) \sim (3143)$, and any among the second abelian type invariants of third category,

(6.1) $\alpha^{(2)}(K) = \{11; (43; 3321, (4211)^3), (32; 3321, (3111)^3), (111; 3321, (221)^3, (211)^9\}^2\}$,

(6.2) $\alpha^{(2)}(K) = \{11; (43; 3321, (5211)^3), (32; 3321, (3321)^3), (111; 3321, (221)^3, (211)^9\}^2\}$,

(6.3) $\alpha^{(2)}(K) = \{11; (43; 3321, (4211)^3), (32; 3321, (4111)^3), (111; 3321, (221)^3, (211)^9\}^2\}$,

the 3-class field tower $F_{3^\infty}(K)$ has two possible lengths $\ell_3(K) \in\{3, 4\}$.

**Proof.** This is an immediate consequence of Table 7 since for each of the three patterns in Formulas (6.1), (6.2), (6.3) there exist Schur $\sigma$-groups $S$ with soluble lengths sl$(S) \in\{3, 4\}$. \qed
Example 5. According to Table 6 the quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-159208, -262628, -273284, -551112\}$ have the capitulation type F.13, $\mathcal{A}(K) = (3143)$, and 2nd ATI of third category $[6.1]$ for $d = -159208$, $[6.2]$ for $d \in \{-262628, -551112\}$, $[6.3]$ for $d = -273284$. According to Theorem 8, no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$. However, disclosure of the skeleton type d.19, $j_1 \in \{39, 44\}$, is enabled for $[6.2]$, and d.25, $j_1 \in \{57, 59\}$, for $[6.3]$.

Table 7. Correspondence for periodic $M = S/S''$, $cc(M) = 4$, of type F.13

| Type lo(M) | $j_1$ | $j_2$ | $j_3$ | # | $k_1$ | $k_3$ | lo(S) | sl(S) | # | $\alpha^{(2)}(S)$ | $A_1$ | $A_2$ |
|------------|-------|-------|-------|---|-------|-------|-------|-------|---|-------------|------|------|
| F.13 11 | 39 | 7 | 3 | 1/25 | 117 | 24; 19, 26 | 29; 32, 41 | 3; 4 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 138 | 11, 15, 16 | 23 | 3 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 162 | 12, 13, 17 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 184 | 19, 24, 26 | 23 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 39 | 7 | 8 | 1/25 | 117 | 3; 5, 7 | 29; 32, 41 | 3; 4 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 138 | 1, 5, 9 | 23 | 3 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 162 | 2, 6, 7 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 184 | 3, 5, 7 | 23 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 44 | 1 | 2 | 1/25 | 114 | 6; 2, 7 | 29; 32, 41 | 3; 4 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 136 | 3, 5, 7 | 23 | 3 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 159 | 3, 5, 7 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 189 | 21, 22, 26 | 23 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 44 | 1 | 9 | 1/25 | 114 | 12; 13, 17 | 29; 32, 35 | 3; 4 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 136 | 19, 24, 26 | 23 | 3 | 3/27 | (5211)$^3$ | (3111)$^3$ |
| 11 | 159 | 19, 24, 26 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 189 | 1, 5, 9 | 23 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 57 | 1 | 2 | 1/15 | 124 | 10, 15, 17 | 26 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 124 | 20, 22, 27 | 23 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 168 | 3, 9, 5 | 26; 32 | 4 | 3/15 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 5, 9, 12 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 16, 22, 25 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 28, 32, 42 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 57 | 1 | 4 | 1/15 | 124 | 2, 4, 9 | 23 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 124 | 12, 14, 16 | 26 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 168 | 4, 12, 11 | 26; 29 | 4 | 3/15 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 2, 7, 17 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 19, 23, 29 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 197 | 33, 37, 41 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 59 | 6 | 3 | 1/15 | 129 | 1, 5, 9 | 26 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 129 | 20, 24, 25 | 23 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 173 | 6, 14, 2 | 26; 29 | 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 6, 8, 13 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 17, 22, 30 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 32, 34, 40 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 59 | 6 | 4 | 1/15 | 129 | 3, 8, 4 | 29; 32, 35 | 4 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 129 | 11, 15, 16 | 23 | 3 | 3/27 | (4211)$^3$ | (4111)$^3$ |
| 11 | 173 | 4, 12, 8 | 26; 29 | 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 3, 7, 10 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 16, 24, 25 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
| 11 | 198 | 28, 36, 42 | 29 | 4 | 3/45 | (4211)$^3$ | (3111)$^3$ |
In the Tables 7, 8, 9, 10 the Schur $\sigma$-groups $S$ have the settled root $R = B - \#4; k_1 - \#2; k_2 - \#4; k_3$ with some unique relative identifier $k_2 = k_2(k_1)$, metabelianization $M \simeq B - \#2; j_1 - \#1; j_2 - \#1; j_3$, and 2$^\text{nd}$ AQI $\alpha^{(2)}(S) = [11; (43; 3321, A_1), (32; 3321, A_2), (111; 3321, A_3), (111; 3321, A_4)]$ with $A_3 = A_4 = (221)^3, (211)^3$. The relative identifier $k_3$ cannot be given explicitly, since the unique $\sigma$-descendant of $B - \#4; k_1$ is constructed directly by means of the $p$-covering group, without using the $p$-group generation algorithm.

Table 8. Correspondence for periodic $M = S/S''$, $cc(M) = 4$, of type F.12

| Type $\text{lo}(M)$ | $j_1$ | $j_2$ | $j_3$ | $k_1$ | $k_2$ | $k_3$ | $\text{lo}(S)$ | $\text{sl}(S)$ | $\#$ | $\alpha^{(2)}(S)$ | $\text{cc}(S) = (1343)$ |
|-------------------|------|------|------|------|------|------|-------------|-------------|-----|-----------------|-----------------|
| 11 54 8 2 1/25    | 120  | 13, 17, 12 | 26, 29 | 4 | 3 | $3/27$ | (4211)$^3$ | (3111)$^3$ |
| 11 40 20, 22, 27  | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 40 38, 40, 45  | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 40 56, 58, 63  | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 166 20, 22, 27 | 176  | 29 | 3 | 2 | $27$ | (4211)$^3$ | (4111)$^3$ |
| 11 54 8 4 1/25    | 120  | 20, 25, 24 | 26, 29 | 4 | 3 | $3/27$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 10, 15, 17 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 28, 33, 35 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 73, 78, 80 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 166 10, 15, 17 | 176  | 26 | 3 | 2 | $27$ | (4211)$^3$ | (4111)$^3$ |
| 11 54 8 6 1/25    | 120  | 19, 27, 23 | 26, 35 | 4 | 3 | $3/27$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 12, 14, 16 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 30, 32, 34 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 75, 77, 79 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 166 12, 14, 16 | 176  | 26 | 3 | 2 | $27$ | (4211)$^3$ | (4111)$^3$ |
| 11 54 8 8 1/25    | 120  | 3, 4, 8 | 26, 32 | 4 | 3 | $3/27$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 2, 4, 9    | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 47, 49, 54 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 140 65, 67, 72 | 140  | 29 | 4 | 3 | $81$ | (4211)$^3$ | (3111)$^3$ |
| 11 166 2, 4, 9    | 176  | 23 | 3 | 2 | $27$ | (4211)$^3$ | (4111)$^3$ |
| 11 176 1, 5, 9    | 176  | 26 | 3 | 2 | $27$ | (4211)$^3$ | (4111)$^3$ |
Theorem 7. For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d < 0$, bicyclic 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$, second 3-class group $M = G_3^2(K)$, ord($M$) = 311, capitulation type F.12, $\kappa(K) \sim (1343)$, and any among the second abelian type invariants of third category, 

(6.4) $\alpha^{(2)}(K) = [11; (43; 3321, (4211)^3), (32; 3321, (3111)^3), (111; 3321, (221)^3, (211)^9)^2],$

(6.5) $\alpha^{(2)}(K) = [11; (43; 3321, (5211)^3), (32; 3321, (3111)^3), (111; 3321, (221)^3, (211)^9)^2],$

(6.6) $\alpha^{(2)}(K) = [11; (43; 3321, (4211)^3), (32; 3321, (4111)^3), (111; 3321, (221)^3, (211)^9)^2],$

the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

Proof. This is an immediate consequence of Table 8 since for each of three patterns in Formulas (6.4), (6.5), (6.6) there exist Schur $\sigma$-groups $S$ with soluble lengths $sl(S) \in \{3, 4\}$. Incidentally, $\square$

Example 6. According to Table 9 the quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-278427, -382123\}$ have the capitulation type F.12, $\kappa(K) = (1343)$, and 2nd ATI of third category (6.4) for $d = -382123$, (6.5) for $d = -278427$. According to Theorem 7 no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$. However, the skeleton type is d.19, $j_1 \in \{39, 44\}$, for (6.5).

Table 9. Corresponence for periodic $M = S/S''$, cc($M$) = 4, of type F.11

| Type | lo($M$) | $j_1$ | $j_2$ | $j_3$ | # | $k_1$ | $k_3$ | lo($S$) | sl($S$) | # | $\alpha^{(2)}(S)$ | $A_1$ | $A_2$ |
|------|---------|------|------|------|---|-------|-------|--------|--------|---|-----------------|-------|-------|
| F.11 | 11      | 54   | 8    | 1    | 1/25 | 120 | 15, 16; 11 | 26; 29 | 4   | 3/27         | (5211)^4 | (3111)^4 |
|      | 11      | 140  | 19, 24, 26 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 37, 42, 44 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 55, 60, 62 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 19, 24, 26 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 176  | 20, 24, 25 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 54   | 8    | 9    | 1/25 | 120 | 1, 5, 9 | 26; 32 | 4   | 3/27 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 3, 5, 7 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 48, 50, 52 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 140  | 66, 68, 70 | 29 | 3   | 3/81 | (5211)^3 | (3111)^3 |
|      | 11      | 166  | 3, 5, 7 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 176  | 10, 14, 18 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 57   | 1    | 1    | 1/15 | 124 | 1, 6, 8 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 124  | 21, 23, 25 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 168  | 1, 14, 8 | 26; 29 | 4   | 3/15 | (5211)^3 | (3111)^3 |
|      | 11      | 197  | 3, 4, 10 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |
|      | 11      | 197  | 14, 21, 30 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |
|      | 11      | 197  | 34, 36, 43 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |
|      | 11      | 129  | 10, 14, 18 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 129  | 21, 22, 26 | 23 | 3   | 3/27 | (4211)^3 | (4111)^3 |
|      | 11      | 173  | 1, 10, 13 | 26; 32 | 4   | 3/27 | (5211)^3 | (3111)^3 |
|      | 11      | 198  | 2, 4, 12 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |
|      | 11      | 198  | 18, 21, 29 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |
|      | 11      | 198  | 33, 39, 44 | 29 | 3   | 3/45 | (5211)^3 | (3111)^3 |

Theorem 8. An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d < 0$, elementary bicyclic 3-class group $\text{Cl}_3(K)$, second 3-class group $M = G_3^2(K)$, ord($M$) = 311, capitulation type F.11, $\kappa(K) \sim (1143)$, and second abelian type invariants of third category

(6.7) $\alpha^{(2)}(K) = [11; (43; 3321, (4211)^3), (32; 3321, (4111)^3), (111; 3321, (221)^3, (211)^9)^2]$
has a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$. For second abelian type invariants
(6.8) $\alpha^{(2)}(K) = [11; (43; 3321, (5211)^3), (32; 3321, (3111)^3), (111; 3321, (221)^3, (211)^9)^2],
the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.  

Proof. This is an immediate consequence of Table 2 since the settled roots $R$ arising from the unique non-metabelian $B = \#4; k_1$ with $k_1 \in \{120, 168, 173\}$ which lead to Schur $\sigma$-groups $S$ with soluble length $sl(S) = 4$ have the second abelian type invariants in Formula (6.8). □

Example 7. According to Table 4 the quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -469787$ has the capitulation type $F.11, \sigma(K) = (1143)$, and second abelian type invariants of third category (6.8). According to Theorem 8 no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$. However, outside of the range in Table 6 we found quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-2005880, -2554868, -3283223, -3937999\}$, capitulation type $F.11, \sigma(K) = (1143)$, and second abelian type invariants of third category (6.8). According to Theorem 8 they have a 3-class field tower $F_3^\infty(K)$ of precise length $\ell_3(K) = 3$.

Theorem 9. For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d < 0$, elementary bicyclic 3-class group $C_{13}(K)$, second 3-class group $M = G_3^2(K)$, $\text{ord}(M) = 3^{11}$, and capitulation type $F.7, \sigma(K) \sim (3443)$, the second abelian type invariants of third category must have the shape
(6.9) $\alpha^{(2)}(K) = [11; (43; 3321, (4111)^3), (32; 3321, (3111)^3), (111; 3321, (221)^3, (211)^9)^2],$
and the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

Proof. This is an immediate consequence of Table 10 since the settled roots $R$ arising from the unique non-metabelian $B = \#4; k_1$ with $k_1 \in \{136, 138, 184\}$ which lead to Schur $\sigma$-groups $S$ with soluble length $sl(S) = 4$ also have the common second abelian type invariants in Formula (6.8).

Table 10. Correspondence for periodic $M = S/S''$, $cc(M) = 4$, of type F.7

| Type lo(M) | $j_1$ | $j_2$ | $j_3$ | # | $k_1$ | $k_3$ | lo(S) | sl(S) | # | $\alpha^{(2)}(S)$ | $A_1$ | $A_2$ |
|------------|-----|-----|-----|---|-----|-----|-----|-----|---|-----------------|-----|-----|
| $\sigma(S) = (3443)$ |
| F.7 | 11 | 39 | 7 | 5 | 1/25 | 117 | 2, 4, 9 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 138 | 3, 4, 8 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 162 | 1, 5, 9 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 184 | 12, 14, 16 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 39 | 7 | 6 | 1/25 | 117 | 20, 22, 27 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 138 | 12, 17, 13 | 26; 29 | 3; 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 162 | 10, 14, 18 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 184 | 10, 15, 17 | 29 | 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 44 | 1 | 5 | 1/25 | 114 | 1, 5, 9 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 136 | 2; 4, 9 | 26; 29, 32 | 3; 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 159 | 2, 4, 9 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 189 | 12, 13, 17 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 44 | 1 | 6 | 1/25 | 114 | 10, 14, 18 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 136 | 20, 27, 22 | 26; 29 | 3; 4 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 159 | 20, 22, 27 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |
| 11 | 189 | 10, 14, 18 | 26 | 3 | 3/27 | (4211)$^3$ | (3111)$^3$ |

Example 8. According to Table 6 the quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-643011, -797556\}$ have the capitulation type $F.7, \sigma(K) = (3443)$, and second abelian type invariants of third category (6.9). According to Theorem 8 no sharp decision about the length of the 3-class field tower $F_3^\infty(K)$ can be drawn, since there are two possibilities $\ell_3(K) \in \{3, 4\}$.
Corollary 2. The logarithmic order of the Schur σ-group \( S = \text{Gal}(F_3^\infty/K) \) is \( \text{lo}(S) \in \{23, 26, 29\} \) for a 3-class field tower of length \( \ell_3(K) = 3 \), and it depends on the capitulation type \( \kappa(K) \).

\[
\text{lo}(S) \in \begin{cases} 
\{26, 29, 32, 35, 41\} & \text{for type } F.13, \\
\{26, 29, 32, 35\} & \text{for type } F.12, \\
\{26, 29, 32\} & \text{for type } F.11, \\
\{29, 32\} & \text{for type } F.7, 
\end{cases}
\]

in the case of a tower of length \( \ell_3(K) = 4 \).

Proof. This is a common consequence of the Tables 7, 8, 9, 10.

7. Realization of Sporadic Metabelian σ-Groups \( G_3^2(K) \) of Coclass 6

In Table 11 second abelian type invariants \( \alpha^{(2)}(K) \) of imaginary quadratic fields \( K = \mathbb{Q}(\sqrt{d}) \) with discriminants in the most extensive range \(-10^7 < d < 0\), bicyclic 3-class group \( \text{Cl}_3(K) \cong C_3 \times C_3 \), second 3-class group \( M = G_3^2(K) \), \( \text{ord}(M) = 3^{13} \), and type F are given, ordered by capitulation subtypes \( \kappa(K) \).

**Boldface** font indicates the categories 1 and 2.

**Table 11.** 2nd ATI of \( K = \mathbb{Q}(\sqrt{d}) \), \( d < 0 \), with sporadic \( M = G_3^2(K) \), cc\( (M) = 6 \)

| Type   | \( \alpha^{(2)}(K) \) | \( \kappa(K) = (3443) \) | \( \kappa(K) = (1143) \) | \( \kappa(K) = (1343) \) | \( \kappa(K) = (3143) \) |
|--------|------------------------|-----------------------|------------------------|------------------------|------------------------|
| -d     | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) |
| F.7    | \( (5211)^3 \) \( (4211)^3 \) \( (2111)^3, (2111)^3, (221)^3 \) \( (2111)^6, (1111)^6 \) \( (221)^3, (211)^9 \) | | | | |
| F.11   | \( (5211)^3 \) \( (4211)^3 \) \( (2111)^3, (221)^3, (211)^9 \) | \( (211)^9, (211)^9 \) \( (221)^3, (211)^9 \) | | | |
| F.12   | \( (5211)^3 \) \( (4211)^3 \) \( (2111)^9, (221)^3 \) | \( (111)^{13}, (2111)^3, (1111)^6 \) | | | | |
| F.13   | \( (5211)^3 \) \( (4211)^3 \) \( (2111)^9, (221)^3, (211)^9 \) | | \( (221)^3, (211)^9 \) | | | |

Throughout the sequel, we restrict our investigations to the tame situation of category 3.

In the Tables 12, 13, 14, 15 the Schur σ-groups \( S \) have the settled root \( R = B - #4; k_1 - #2; k_2 - #4; k_3 \) with some unique relative identifier \( k_2 = k_2(k_1) \), metabelianization \( M \cong B - #2; 33 - #2; 25 - #2; j \), and 2nd AQL \( \alpha^{(2)}(S) = [11; (43; 3332, A_1), (43; 3332, A_2), (111; 3332, A_3), (111; 3332, A_4)] \).

**Theorem 10.** Let \( K = \mathbb{Q}(\sqrt{d}) \) be an imaginary quadratic field with fundamental discriminant \( d < 0 \), elementary bicyclic 3-class group \( \text{Cl}_3(K) \cong C_3 \times C_3 \), capitulation type F.7, \( \kappa(K) \cong (3443) \), and first abelian type invariants \( \alpha^{(1)}(K) = [11; (43; 3332, A_1), (43; 3332, A_2), (111; 3332, A_3), (111; 3332, A_4)] \).

If the second abelian type invariants are of third category, they must have the shape

\begin{equation}
\alpha^{(2)}(K) = [11; (43; 3332, (4211)^3), (43; 3332, (4211)^3), (111; 3332, (221)^3, (211)^9)^2]
\end{equation}

and the 3-class field tower \( F_3^\infty(K) \) has precise length \( \ell_3(K) = 3 \) and relative degree \( 3^{26} \) over \( K \).
Table 12. Correspondence for sporadic \( M = S/S'' \), \( cc(M) = 6 \), of type F.7

| Type \( lo(M) \) | \( j \) | \( k_1 \) | \( k_3 \) | \( lo(S) \) | \( sl(S) \) | \# | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( \varkappa(S) = (3443) \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| F.7 | 13 | 59 | 148 | 8 | 26 | 3 | 1/45 | (4211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |
| | 13 | 60 | 179 | 8 | 26 | 3 | 1/45 | (4211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |
| | 13 | 62 | 148 | 24 | 26 | 3 | 1/45 | (4211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |

Proof. This is an immediate consequence of Table 12 since all the non-metabelian roots \( R = B - \#4; k_1 \) with \( k_1 \in \{148, 179\} \) lead to Schur \( \sigma \)-groups \( S \) with transfer kernel type F.7, \( \varkappa(K) = (3443) \), whose soluble length is \( sl(S) = 3 \), logarithmic order \( lo(S) = 26 \), and which have the second abelian type invariants in Formula (7.1).

Example 9. According to Table 12 the quadratic fields \( K = \mathbb{Q}(\sqrt{d}) \) with discriminants \( d \in \{-5053191, -8723023\} \) satisfy the assumptions of Theorem 10. Thus, their 3-class field tower has precise length \( \ell_3(K) = 3 \) and relative degree \( [F_3^∞(K) : K] = 3^{20} \).

Table 13. Correspondence for sporadic \( M = S/S'' \), \( cc(M) = 6 \), of type F.13

| Type \( lo(M) \) | \( j \) | \( k_1 \) | \( k_3 \) | \( lo(S) \) | \( sl(S) \) | \# | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( \varkappa(S) = (3143) \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| F.13 | 13 | 45 | 148 | 7 | 26 | 3 | 1/45 | (5211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |
| | 13 | 51 | 179 | 7 | 32 | 4 | 1/45 | (5211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |
| | 13 | 54 | 148 | 180 | 26 | 3 | 1/45 | (5211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |
| | 13 | 56 | 148 | 3 | 26 | 3 | 1/45 | (5211)^3 | (4211)^3 | (221)^3, (211)^9 | (221)^3, (211)^9 |

Theorem 11. Let \( K = \mathbb{Q}(\sqrt{d}) \) be an imaginary quadratic field with fundamental discriminant \( d < 0 \), elementary bicyclic 3-class group \( Cl_3(K) \simeq C_3 \times C_3 \), capitulation type F.13, \( \varkappa(K) \sim (3143) \), and first abelian type invariants \( \alpha^{(1)}(K) = \{11; 43, 43, 111, 111\} \). If the second abelian type invariants are of third category, they must either have the shape

\[
\alpha^{(2)}(K) = \{11; (43; 3332, (5211)^3), (43; 3332, (4211)^3), (111; 3332, (221)^3, (211)^9)^{\prime}\}
\]
and the 3-class field tower $F_3^\infty(K)$ has precise length $\ell_3(K) = 3$ and relative degree $3^{26}$ over $K$, or the shape
\[(7.3) \quad \alpha^{(2)}(K) = [11; (43, 3332, (4211)^3)^3, (43, 3332, (4211)^3), (111; 3332, (221)^3, (211)^9)^2]\]
and the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

**Proof.** This is an immediate consequence of Table 11 since only the non-metaabelian root $R = B - \#4; k_1$ with $k_1 = 179$ leads to Schur $\sigma$-groups $S$ with transfer kernel type F.13, $\chi(K) = (3143)$, and soluble length $sl(S) = 4$, and these have second abelian type invariants in Formula (7.3). □

**Example 10.** According to Table 11 the quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -6863219$ satisfies Formula (7.2) of Theorem 11. Thus, its 3-class field tower has exact length $\ell_3(K) = 3$ and relative degree $[F_3^\infty(K) : K] = 3^{26}$.

### Table 14. Correspondence for sporadic $S/S''$, $cc(M) = 6$, of type F.12

| Type | lo($M$) | $j$ | $k_3$ | lo($S$) | $sl(S)$ | $\#$ | $\alpha^{(2)}(S)$ | $\chi(S) = (1343)$ |
|------|--------|-----|-------|---------|--------|-----|------------------|------------------|
| F.12 | 13     | 47  | 148   | 22      | 29     | 3   | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 39     | 32  |       |         |        |     | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 179    | 22  | 26    | 3       | 1/45   |     | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 39     | 26  |       |         | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 28     | 35  |       |         | 1/45   |     | 1/45 (4211)^3    | (221)^3 (211)^9  |
|      | 179    | 4   | 26    | 3       | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 28     | 26  |       |         | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 13     | 55  | 148   | 2       | 29     | 3   | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 14     | 29  |       |         | 1/45   |     | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 179    | 2   | 26    | 3       | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 40     | 26  |       |         | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 13     | 57  | 148   | 16      | 29     | 3   | 1/45 (4211)^3    | (211)^3 (221)^9  |
|      | 19     | 32  |       |         | 1/45   |     | 1/45 (4211)^3    | (221)^3 (211)^9  |
|      | 179    | 16  | 26    | 3       | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |
|      | 44     | 26  |       |         | 1/45   |     | 1/45 (5211)^3    | (221)^3 (211)^9  |

**Theorem 12.** Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with fundamental discriminant $d < 0$, elementary bicyclic 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$, capitulation type F.12, $\chi(K) \sim (1343)$, and first abelian type invariants $\alpha^{(1)}(K) = [11; 43, 43, 111, 111]$. If the second abelian type invariants are of third category, they must either have the shape
\[(7.4) \quad \alpha^{(2)}(K) = [11; (43, 3332, (5211)^3), (43, 3332, (4211)^3), (111; 3332, (221)^3, (211)^9)^2]\]
and the 3-class field tower $F_3^\infty(K)$ has precise length $\ell_3(K) = 3$ and relative degree $3^{26}$ over $K$, or the shape
\[(7.5) \quad \alpha^{(2)}(K) = [11; (43, 3332, (4211)^3), (43, 3332, (4211)^3), (111; 3332, (221)^3, (211)^9)^2]\]
and the 3-class field tower $F_3^\infty(K)$ has two possible lengths $\ell_3(K) \in \{3, 4\}$.

**Proof.** This is an immediate consequence of Table 14 since only the non-metaabelian root $R = B - \#4; k_1$ with $k_1 = 148$ leads to Schur $\sigma$-groups $S$ with transfer kernel type F.12, $\chi(K) = (3143)$, and soluble length $sl(S) = 4$, and these have second abelian type invariants in Formula (7.6). □

**Example 11.** According to Table 14 the quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -8751215$ satisfies Formula (7.4) of Theorem 12. Thus, its 3-class field tower has exact length $\ell_3(K) = 3$ and relative degree $[F_3^\infty(K) : K] = 3^{26}$. However, the quadratic field $K = \mathbb{Q}(\sqrt{d})$
with discriminant \(d = -423640\) satisfies Formula (7.6) of Theorem 12 and no sharp decision about the length of the 3-class field tower \(F_3^\infty(K)\) can be drawn, since there are two possibilities \(\ell_3(K) \in \{3, 4\}\).

Table 15. Correspondence for sporadic \(M = S/S''\), \(cc(M) = 6\), of type F.11

| Type | \(lo(M)\) | \(j\) | \(k_1\) | \(k_3\) | \(lo(S)\) | \(sl(S)\) | \# | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
|------|-----------|--------|--------|--------|--------|--------|---|--------|--------|--------|--------|
| F.11 | 13        | 40     | 1      | 32     | 4      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 33     | 29     | 3      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 179    | 1      | 29     | 3      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 33     | 29     | 3      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 179    | 18     | 29     | 3      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 35     | 32     | 4      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      | 13        | 42     | 148    | 18     | 29     | 3      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |
|      |           |        | 35     | 32     | 4      | 1/45   | (5211)^3 | (4211)^3 | (2211)^9 | (2211)^9 | (2211)^9 |

Theorem 13. Let \(K = \mathbb{Q}(\sqrt{d})\) be an imaginary quadratic field with fundamental discriminant \(d < 0\), elementary bicyclic 3-class group \(C_3(K) \cong C_3 \times C_3\), capitulation type F.11, \(\alpha(K) \sim (1143)\), and first abelian type invariants \(\alpha^{(1)}(K) = [11; 43, 43, 111, 111]\). If the second abelian type invariants are of third category, they must have the shape

\[
\alpha^{(2)}(K) = [11; (43; 3332), (5211)^3, (43; 3332, (4211)^3), (111; 3332, (2211)^3, (211)^9]^2
\]

and the 3-class field tower \(F_3^\infty(K)\) has two possible lengths \(\ell_3(K) \in \{3, 4\}\).

Proof. This is an immediate consequence of Table 15 since both non-metabelian roots \(R = B - \#4; k_1\) with \(k_1 \in \{148, 179\}\) lead to Schur \(\sigma\)-groups \(S\) with transfer kernel type F.11, \(\alpha(K) = (1143)\), whose soluble length is \(sl(S) = 4\), and which have the second abelian type invariants in Formula (7.6). \(\square\)

Example 12. According to Table 11 the quadratic fields \(K = \mathbb{Q}(\sqrt{d})\) with discriminants \(d \in \{-4838891, -8439815\}\) satisfy the assumptions of Theorem 13. Thus, no sharp decision about the length of the 3-class field tower \(F_3^\infty(K)\) can be drawn, since there are two possibilities \(\ell_3(K) \in \{3, 4\}\).

Corollary 3. The logarithmic order of the Schur \(\sigma\)-group \(S = \text{Gal}(F_3^\infty(K)/K)\) is generally \(lo(S) \in \{32, 35\}\) for a 3-class field tower of length \(\ell_3(K) = 3\), and it depends on the capitulation type \(\alpha(K)\),

\[
lo(S) \in \begin{cases} 32, 35 & \text{for type F.12,} \\ 32 & \text{for type F.11 or F.13,} \end{cases}
\]

in the case of a tower of length \(\ell_3(K) = 4\).

Proof. This is a common consequence of the Tables 12, 13, 14, 15. \(\square\)

8. Historical remarks

The goal of this article was the investigation of the transfer kernel type F [19, § 2.I.F., p. 36]. Scholz and Taussky also defined transfer kernel type D [19, § 2.I.D., p. 35] and transfer kernel type E [19, § 2.I.E., p. 36]. The unique Schur \(\sigma\)-groups \(S\) of type D are metabelian and have order \(\text{ord}(S) = 3^3\), soluble length \(sl(S) = 2\), coclass \(cc(S) = 2\) and nilpotency class \(cl(S) = 3\). They were identified by Scholz and Taussky by means of annihilator ideals and Schreier polynomials [14]. The smallest non-metabelian Schur \(\sigma\)-groups \(S\) of type E have order \(\text{ord}(S) = 3^3\), soluble length \(sl(S) = 3\), coclass \(cc(S) = 3\) and nilpotency class \(cl(S) = 5\). They were identified by Bush and Mayer by means of polycyclic power commutator presentations [3]. Their metabelianizations
\( M = S/S'' \) of order \( \text{ord}(S) = 3^7 \), coclass \( \text{cc}(S) = 2 \) and nilpotency class \( \text{cl}(S) = 5 \) were already known to Scholz and Taussky in terms of annihilator ideals and Schreier polynomials, but these authors erroneously claimed that these metabelianizations \( M \) are 3-class field tower groups of imaginary quadratic number fields, which is impossible since \( d(M) = 2 < 3 = r(M) \).

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