The Supergravity Dual of $N = 1$ Super Yang-Mills Theory

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ABSTRACT

We find an exact, $N = 1$ supersymmetric kink solution of 5d gauged supergravity. We associate this solution with the RG flow from $N = 4$ super Yang-Mills theory, deformed by a relevant operator, to pure $N = 1$ super Yang-Mills in the IR. We test this identification by computing various QFT quantities using the supergravity dual: the tension of electric and magnetic strings and the gaugino condensate. As demanded by our identification, our kink solution is a true deformation of $N = 4$, that exhibits confinement of quarks, magnetic screening, and spontaneous chiral symmetry breaking.

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1 Introduction

In previous papers [1, 2] we explored several properties of relevant deformations of strongly-coupled $N = 4$ super Yang-Mills using its dual [3, 4, 5] description in terms of 5d gauged supergravity. In this paper we present a holographic RG flow from $N = 4$ SYM to pure $N = 1$ SYM in the IR. We find agreement with field theory expectations, namely, quarks confine, monopoles are screened, and there is a gaugino condensate.

This paper is the natural continuation of [1, 2], where we studied the supergravity description of RG flows. In particular, in [1] we proved the existence of a kink solution of 5d supergravity that was interpreted as the dual of a mass-deformed $N = 4$ super Yang-Mills, flowing in the IR to a non-supersymmetric, interacting, local conformal field theory (CFT) (see also [6]). We also introduced a c-function and we proved a c-theorem valid for all supergravity flows involving the metric and an arbitrary number of scalars. Such a setup, namely coupled gravity-scalar equations is remarkably versatile, and it can also describe deformations of $N = 4$ flowing to non-conformal, interacting theories.

Most of the solutions asymptotic to the $N = 4$ UV fixed point go to infinity in the space of parameters described by the 5d supergravity. It is natural to interpret these solutions as RG flows to non-conformal IR theories. Examples were considered in [2, 7]. The generic solutions discussed in [2] exhibit different behaviours in the IR, from confinement to screening, associated with an IR singularity that is common to many 10d solutions [8, 9, 11, 12, 13]. In [7] the Coulomb branch of $N = 4$ YM was studied.

In view of the double role of supergravity solutions with an asymptotic AdS region (deformations of an UV fixed point versus the same theory in a different vacuum [14, 13]), we may ask whether the solutions in [2] are true deformations of $N = 4$ YM or simply different states (e.g. spontaneously broken vacua) of the same theory. The case of [8, 9] turned out to correspond to a different vacuum of $N = 4$ YM; the same is certainly true also for [6]. Unlike the solutions in [6, 8, 9], the ones in [2] are expected to describe deformations. However, up to now, no explicit, analytically solvable example of deformations was given in literature. One of the purposes of this paper is to provide such an example, describing the pure $N = 1$ YM case.

Another important question is the reliability of supergravity solutions. Problems were pointed out in [7], after an explicit comparison of supergravity predictions to gauge theory results. In the screening case discussed in [7], the type IIB string goes near the singularity, and any actual computation requires the form of the solution in the IR, where large corrections to the supergravity result are expected. It is then important, to better understand the seriousness of this problem, to consider other solutions where comparison with quantum field theory expectation can be made. The simplest such case, which corresponds to a deformation of the $N = 4$ theory, is the flow to pure $N = 1$ YM. In this paper, we exhibit a solution with all qualitative
features of pure $N = 1$ super YM theory, in particular, quark confinement. Even in our case the
confining string probes a high-curvature singularity; it is therefore encouraging to find that at
least in some cases this problem does not change the qualitative picture (confinement, magnetic
screening, chiral symmetry braking) emerging from field theory. To better control our solutions
we are led to consider supersymmetric examples. The equally interesting $N = 0$ solutions are
certainly less reliable, even though they are easier to study [2].

This paper is organised as follows: in Section 2, we give a brief review of the basic results
in the interpretation of 5d supergravity solutions as RG flows, following [1, 2]. We also discuss
the cases of flows to IR conformal and non-conformal theories, with particular attention to
open problems. Section 3 describes in details the flow to $N = 1$ YM and its properties. The
computation of electric and magnetic strings, and the identification of the supergravity field
describing the gaugino condensate are described. That Section also contains a brief discussion
of the stability of chirally-symmetric versus chiral-symmetry breaking vacua. The Appendix
contains various useful formulae used throughout the paper.

2 RG Flow from 5d Supergravity: a Brief Review

To fix once for all our notations, we will use coordinates where the Anti-de-Sitter space reads

$$ds^2 = dy^2 + e^{2y/R} dx^\mu dx_\mu, \quad \mu = 0, 1, 2, 3.$$  \hspace{1cm} (1)

$R$ is the $AdS$ radius and it will be henceforth set to 1, unless explicitly stated.

The fifth coordinate $y$ of $AdS_5$ has a natural interpretation as an energy scale [3, 10, 17, 18].
As in [1], we associate larger energies with increasing $y$. We look for the supergravity description
of the quantum field theory RG flow that connects, say, the $N = 4$ YM theory to some IR CFT.
It was proposed in [1] to identify this RG flow with solutions of type IIB supergravity that
interpolate between $AdS_5 \times S^5$ at $y = \infty$ and the type IIB background associated with the IR
CFT at $y = -\infty$. We are interested in studying relevant deformations of the $N = 4$ theory. The
explicit dependence on $y$ must therefore break the conformal group $O(4, 2)$ to the 4d Poincaré
group. $y = \infty$ will be considered the UV, while $y = -\infty$ the extreme IR.

In the previous and following discussions, the $N = 4$ theory can be replaced by any other
CFT that admits an $AdS$ description.

Supergravity solutions have a double meaning: they may describe deformations of a CFT,
or, the same theory in a different vacuum [14, 15]. Indeed, the running of coupling constants
and parameters along the RG flow can be induced in the UV theory in two different ways: by
deforming the CFT with a relevant operator, or by giving a nonzero VEV to some operator.
What the case is for a given solution depends on the asymptotic UV behaviour. The CFT
operators that are used as deformations, or that are acquiring a non-zero VEV, can be found by linearising the solution near $y = \infty$. The type IIB supergravity scalar modes $\lambda(y)$ that deform the $AdS_5 \times S^5$ solution at $y = \infty$ are associated with CFT operators $O_\lambda$ $[4, 5]$. The linearised 5d equation of motion for a fluctuation $\lambda(y)$ in the $AdS$-background eq. (1) reads

$$\ddot{\lambda} + 4 \dot{\lambda} = M^2 \lambda,$$

(2)

where the dot means the derivative with respect to $y$. The mass of the supergravity mode is related to the conformal dimension $\Delta$ of the operator $O_\lambda$ by $\Delta = -2 + \sqrt{4 + M^2}$ $[4, 5]$.

Equation (2) has a solution depending on two arbitrary parameters

$$\lambda(y) = Ae^{-(4-\Delta)y} + Be^{-\Delta y}.$$  

(3)

We are interested in the case of relevant operators, where $\Delta \leq 4$. A solution with $A \neq 0$ is (UV) asymptotic to the non-normalizable solution $e^{-(4-\Delta)y} [5]$, and is associated with a deformation of the $N = 4$ theory with the operator $O_\lambda$. On the other hand, a solution asymptotic to the normalizable solution $e^{-\Delta y}$ ($A = 0$) is associated with a different vacuum of the UV theory, where the operator $O_\lambda$ has a non-zero VEV $[14, 15]$.

2.1 The 5d Supergravity: Flow Between CFT’s and the c-Theorem

10d solutions interpolating between CFT’s are difficult to find because of the high non-linearity of the type IIB equations of motion. Perturbation theory around the UV point is not practicable and exact solutions are not easy to construct. If we restrict the number of operators that we may use to deform the UV fixed point, we can consider the 5d gauged supergravity instead of the full 10d theory $[1, 6]$.

The $N = 8$ gauged supergravity $[19]$ is a consistent truncation of type IIB on $S^5$ $[20]$. This means that every solution of the 5d theory can be lifted to a consistent 10d type IIB solution. The operators that can be described using 5d supergravity are enough to mimic many RG flows. In particular, all mass terms of fermions and scalars, which can be described by supergravity modes, are included in the 5d truncated theory. The only relevant operators that cannot be described by this method are cubic terms in the scalars. Five-dimensional gauged supergravity has 42 scalars, which transform under the $N = 4$ YM R-symmetry $SU(4)$ as $\mathbf{1}, \mathbf{20}, \mathbf{10}$. The singlet is associated with the marginal deformation corresponding to a change in the coupling constant of the $N = 4$ theory. The $\mathbf{20}$ is a mode with mass square $M^2 = -4$ and is associated with the symmetric traceless operator $\text{Tr} \phi_i \phi_j, i, j = 1, \ldots, 6$ of dimension 2. The $\mathbf{10}$ has mass square $M^2 = -3$ and is associated with the fermion bi-linear operator $\text{Tr} \lambda_A \lambda_B, A, B = 1, \ldots, 4$.

$^2$Here we are only considering the UV behaviour.
of dimension 3. Using these operators, we can discuss at least all mass deformations that have a supergravity description\(^3\).

The effective 5d Lagrangian, restricted to the 42 scalars \(\lambda_a\), is
\[
L = \sqrt{-g} \left[ -\frac{R}{4} + \frac{1}{2} g^{IJ} \partial_I \lambda_a \partial_J \lambda_b G^{ab} + V(\lambda) \right],
\]
(4)

In all our applications, we will choose a convenient parametrisation where the scalar kinetic term is canonical (\(G^{ab} = \delta^{ab}\)). This is the most general Lagrangian for scalars coupled to gravity. The contribution of the 5d gauged supergravity\(^4\) is to provide the potential \(V(\lambda)\).

\(V(\lambda)\) has a central critical point where all the \(\lambda_a\) vanish with \(SO(6)\) symmetry and \(AdS\) metric: it corresponds to the \(N = 4\) YM theory. It is natural to associate all other \(AdS\) critical points of the potential with IR CFT’s that can be connected to the \(N = 4\) theory by an RG flow. The scalars with a non-zero VEV, \(\lambda_{IR}\), at the critical points indicate which operators have been used to deform the UV theory. Up to now, all critical points with at least \(SU(2)\) symmetry have been classified\(^{[21],[22]}\). There exist three \(N = 0\) theories with symmetry \(SU(3) \times U(1)\), \(SO(5)\) and \(SU(2) \times U(1)^2\). All those IR CFT theories are obtained as mass deformations of the \(N = 4\) theory\(^{[1],[3]}\). Both the first two critical points might be unstable\(^{[6],[23]}\); the third one has not been yet investigated. Moreover, there is an \(N = 1\) point with symmetry \(SU(2) \times U(1)\)\(^{[22]}\), which is stable due to supersymmetry; it is associated with the \(N = 4\) theory deformed with a supersymmetric mass for one of the three \(N = 1\) chiral superfields\(^{[24]}\). The value of the potential at the critical point (determining the cosmological constant of the \(AdS\) space) is associated with the central charges \(c = a\) of the CFT by\(^5\)
\[
c = a \sim (-V(\lambda_{IR}))^{-3/2}.
\]
(5)

In the case of the supersymmetric point \(SU(2) \times U(1)\) the central charge predicted by supergravity agrees with quantum field theory expectation\(^ {24}\).

The supergravity description of the RG flow connecting the \(N = 4\) theory to one of these IR CFT’s was found in\(^{[1]}\) in the form of a kink solution, which interpolates between the two critical points. The form of a 4d Poincaré invariant metric is
\[
ds^2 = dy^2 + e^{2\phi(y)} dx^\mu dx_\mu, \quad \mu = 0, 1, 2, 3.
\]
(6)

We look for solutions that are asymptotic to \(AdS_5\) spaces (but with different radii \(R_{UV}\) and \(R_{IR}\)) both for \(y \to \infty\) and \(y \to -\infty\): \(\phi(y) \to y/R_{UV} + \text{const}\) for \(y \to \infty\), and \(\phi(y) \to y/R_{IR} + \text{const}\) for \(y \to -\infty\). Analogously, \(\lambda(y) \to 0\) for \(y \to \infty\), while \(\lambda(y) \to \lambda_{IR}\) for \(y \to -\infty\). The equations

\(^3\)The only missing state, the R-symmetry singlet \(\sum_{i=1}^6 \text{Tr} \phi_i \phi_i\), is associated with a stringy mode.

\(^4\)Supersymmetry allows to compute \(c\) and \(a\) using the results in\(^{[24]}\).

\(^5\)The equations

\[c = a \sim (-V(\lambda_{IR}))^{-3/2}.
\]
of motion for the scalars and the metric read
\[
\ddot{\lambda}_a + 4\dot{\phi}\dot{\lambda}_a = \frac{\partial V}{\partial \lambda_a},
\]
(7)
\[
6(\dot{\phi})^2 = \sum_a (\dot{\lambda}_a)^2 - 2V.
\]
(8)

With the above boundary conditions and a reasonable shape for the potential, a kink interpolating between critical points always exists \([1]\). The argument is roughly as follows: by eliminating \(\phi\) from one of the equations, the problem reduces to the classical motion of a particle in a potential with never-vanishing, positive damping. Since the particle is rolling down from the top of a hill (the IR CFT), the damping implies that it will stop at the bottom of the hill (the \(N = 4\) point)\(5\).

The interpretation of the kink as a RG flow can be strengthened by exhibiting a candidate c-function \([1]\),
\[
c(y) = \text{const} \left( T_{yy} \right)^{-3/2}.
\]
(9)
constructed with the \(y\) component of the stress-energy tensor, \(T_{yy} = 6(\dot{\phi})^2 = \sum_a (\dot{\lambda}_a)^2 - 2V\). This expression certainly reduces to the central charge eq. (3) at the critical points, where \(\dot{\lambda}_a = 0\). Moreover, it is a straightforward exercise to prove that \(c(y)\) is monotonic, using the equations of motion and the already-mentioned boundary conditions\(6\). Even more simply, \(c(y)\) is inversely proportional to the energy of our classical particle in a potential, and, with a positive damping, energy always decreases. This proof of the c-theorem is valid for all the supergravity flows involving the metric and an arbitrary number of scalars; it generalises to generic Poincaré invariant supergravity solutions in \([27]\).

In ref. \([27]\) the conditions for a supersymmetric flow were found. As usual, a solution for which the fermionic shifts vanish, automatically satisfies the equations of motion. Moreover, this shortcut reduces the second order equations to first order ones. For a supersymmetric solution, the potential \(V\) can be written in terms of a superpotential, \(W\), as
\[
V = \frac{g^2}{8} \sum_{a=1}^{n} \left| \frac{\partial W}{\partial \lambda_a} \right|^2 - \frac{g^2}{3} |W|^2,
\]
(10)
where \(W\) is one of the eigenvalue of the tensor \(W_{ab}\) defined in \([21]\). The equations of motion reduce to
\[
\dot{\lambda}_a = \frac{g}{2} \frac{\partial W}{\partial \lambda_a},
\]
(11)
\[
\dot{\phi} = -\frac{g}{3} W.
\]
(12)
\(5\)Note that, in this particle interpretation, we are going from IR to UV.

\(6\)From the equations of motion we have \(6\ddot{\phi} = -4\sum_a (\dot{\lambda}_a)^2\) and it is easy to check that, due to the boundary conditions, \(\dot{\phi}\) is always positive. To avoid confusion, note that \(y\) measures the energy in a direction opposite to the one of the RG group: \(y\) increases when the energy increases. \(c(y)\) is however monotonic exactly in the expected way: it decreases going from UV to IR.
It is easy to check that a solution of eq. (12) satisfies also the second order equations (8). Using this result, in [27] the $N = 1$ fixed point $SU(2) \times U(1)$ was studied in detail, finding agreement with quantum field theory expectations.

All these kink solutions generically correspond to deformations of the UV fixed point; this can be explicitly checked for the $SU(2) \times U(1)$ point.

### 2.2 Flow to Non-Conformal Theories

What remains yet to discuss is the most interesting part of the story, but also the subtiest: the case of non-conformal IR theories. Most solutions that start at the $N = 4$ fixed point do not flow to a different critical point, instead, they run away to infinity in configuration space. It is natural to interpret these solutions as RG flows to non-conformal theories [2]. It is indeed expected that most of the mass deformations of $N = 4$ fixed point flow to pure-glue theories in the IR. In [2], we examined a general class of solutions and we provided the general tools for studying the IR behaviour of those theories. Solutions were studied for which the scalar potential becomes irrelevant in the IR. Despite this restriction, those solutions can be called “generic,” since they fill a large subset of the parameter space. In those solutions the IR 5d Einstein metric has a “universal” singular form

$$\quad ds^{2} = dy^{2} + \sqrt{y - a} dx^{\mu} dx_{\mu}, \quad \mu = 0, 1, 2, 3,$$

with a logarithmic behaviour $\lambda_{a} = A_{a} \log(y - a)$ for the scalar fields.

Supersymmetric solutions are not “generic,” since they satisfy eq. (12), which prevents the potential from being irrelevant in the IR. However, modifications of the results of [2] due to supersymmetry are minimal, mostly a slight change in the form of the IR metric, which remains singular; scalar fields maintain a logarithmic behaviour. Supersymmetric solutions associated with the Coulomb branch of the $N = 4$ theory were studied in [7].

In the solutions found in [2], there is a singularity in the IR, whose generic form is common to many ten dimensional solutions from type IIB [3, 4, 10] to type OB [28] and even to non-critical strings [11, 12]. Agreement with the solutions in [8, 4, 14] is expected, since they can be described in the language of $N = 8$ 5d gauge supergravity. Agreement with type O and non-critical strings is unexpected, but it becomes a little less surprising when one realizes that those solutions can be reduced to effective 5d theories of scalars coupled to gravity.

Depending on the value of the parameters $A_{a}$, the solutions may exhibit conformal behaviour, confinement, or screening in the IR. The appropriate case can be determined by computing a Wilson loop. Following [29], the Wilson loop associated with a contour $C$ on the boundary is obtained minimising the world-sheet action of a string living in the background (6) and whose endpoints live on the boundary contour $C$. By considering a rectangular loop $C$, with one side
of length $L$ in the direction $x$ and another one of length $T$ along the time axis, and by choosing the standard embedding $\sigma = x, \tau = t$ we find:

$$S = \int d\tau d\sigma \sqrt{G_{\text{ind}}} = T \int dx T(y) e^{\phi(y)} \sqrt{(\partial_x y)^2 + e^{2\phi(y)}}. \quad (14)$$

Here $T(y)$ is the tension of the fundamental (the case of a quark) or of the D1 string (monopole) in five dimensions, which in general is a non-trivial function of the scalar fields. As discussed in [2] on the basis of naive dimensional reduction from ten dimensions, the tensions can be read from the coefficients of the kinetic term for the NS-NS and R-R antisymmetric tensors, respectively. Since in the five-dimensional supergravity Lagrangian [21] the kinetic term for the antisymmetric tensors is written in the first order formalism

$$\epsilon_{\alpha\beta} B_{I\alpha} \wedge dB_{I\beta} + A_{I\alpha,J\beta} B_{I\alpha} \wedge *B_{J\beta}, \quad (15)$$

the tensions come from the matrix $A_{I\alpha,J\beta}$ in the quadratic term. In particular they receive contribution only from the diagonal entries of $A$, the off-diagonal ones contributing to the mass terms for the B tensors.

The Wilson loop is conveniently studied in the coordinates where the quark-antiquark (or monopole-antimonopole) energy reads,

$$E = S/T = \int dx \sqrt{(\partial_x u)^2 + f(u)}, \quad (16)$$

the change of variable being given by

$$\frac{\partial u}{\partial y} = T(y) e^{\phi(y)}, \quad f(u) = T^2(u) e^{4\phi(u)}. \quad (17)$$

The boundary is now at $u = +\infty$, while the IR region corresponds to small values of $u$.

The Wilson loop behaviour is determined by that of the function $f(u)$ [2, 30]. [2] reviews in detail all the different cases that can occur, here we give only a sketch of the results. If $f(u) \sim u^\gamma$ for small $u$, with $\gamma \geq 2$ the quark-antiquark energy has a power-like behaviour, with exponent depending on $\gamma$. The conformal case is $\gamma = 4$. If $0 \leq \gamma \leq 2$, then the string will find energetically favourable to go straight to the IR region, where to separate the quarks does not cost any energy [30]. When $\gamma \leq 0$, there is an effective barrier that prevents the string from penetrating into the deep IR region; it will live near the minimum of the function $f(u)$. We therefore expect an area law behaviour, i.e. confinement. A similar result holds when the function $f(u)$ does not diverge, but is strictly bounded above zero. The case of a barrier (confinement) is the one where the supergravity computation is more reliable. It is plausible that the minimum of $f(u)$ lives in a region far from the singularity, where supergravity can be still trusted. On the other hand, the case of screening is the least reliable. The string has to
touch the singularity before the quarks can be separated without spending energy; but large corrections to the supergravity result certainly are expected in the deep IR region.

Having settled all the machinery for studying solutions where scalars run to infinity, we may start to discuss their physical meaning. The first important problem is to understand whether a solution corresponds to a deformation or to a different vacuum of the $N = 4$ theory. After initial confusion, the solution in \cite{8, 9} was identified as a different phase of the $N = 4$ theory; the same is certainly true for the solutions in \cite{7}, describing the Coulomb branch of $N = 4$ YM. We should not conclude from this that all the solutions in \cite{2} correspond to vacua of the $N = 4$ theory.

The solutions in \cite{2, 8, 9, 10, 31} are associated with operators that may reasonably have a non-zero VEV in the $N = 4$ theory. It is reasonable to associate solutions were the dilaton (and/or axion) runs \cite{8, 9, 31} with vacua with non-zero $\langle \text{Tr } F^2 \rangle$, and solutions where the scalars in the 20 run \cite{7} with the Coulomb branch of $N = 4$ YM, where $\langle \text{Tr } (\phi_i \phi_j) \rangle \neq 0$. It is instead difficult to image solutions where the 10 runs as vacua of the $N = 4$ theory with non-zero fermionic condensate. It is quite plausible that many of the solutions with non-zero 10 correspond to deformations of the UV fixed point rather than different vacua. Since, in general, we do not know the full solution connecting the UV with the singular IR region, we can not explicitly check this claim. Here, we simply note that the description of a different vacuum requires a UV-normalizable solution, with asymptotic behaviour as in eq. (3). This would require a fine tuning ($A = 0$); obviously, this cannot be the generic situation. Note also that the case of a running dilaton considered in \cite{8, 9}, which turned out to correspond to a vacuum of the theory, is quite special. For operators of dimension 4, equation (3) becomes,

$$\lambda(y) = A + Be^{-4y}$$

The non-normalizable solution, the one associated with a true deformation of the UV theory, here becomes a constant. In the case of the dilaton, a non-normalizable solution can be always re-absorbed into a shift of the $N = 4$ coupling constant. This means that every solution where only the dilaton (and/or axion) runs is UV-asymptotic to the normalizable solution; therefore, it corresponds to a different vacuum of the $N = 4$ theory. Completely different is the case of dimension-three operators, where both behaviours in eq. (3) are allowed. In this paper we provide an explicit example of solution associated with a deformation of the $N = 4$ theory.

The UV behaviour in the supersymmetric case can always be determined using equations (12). In this paper we will consider a supersymmetric case, the RG flow from $N = 4$ to pure $N = 1$ YM, which corresponds to the running of fields in the 10; it can be explicitly recognised as

\footnote{We can understand which operator gets a VEV or appears as a deformation by looking at the supergravity fields that run in $y$.}
a deformation of the UV theory. We will prove that quarks confine and monopoles are screened
by computing a Wilson loop. Once we are in the the $N = 1$ theory, the existence of a fermionic
condensate is no more to be regarded as suspect; it is instead expected, since pure $N = 1$ YM
has a gluino condensate. We will check the existence of the condensate by playing with the
double role of supergravity solutions, deformations versus VEV. We will find a supersymmetric
solution of the equations of motion depending on two scalars $m$ and $\sigma$, the first being associated
with the mass term which breaks $N = 4$ to $N = 1$ pure YM, the second being associated with
the gluino condensate operator $\mathcal{O}$. By checking the asymptotic UV behaviour, $m$ turns out to be
associated with a deformation of the $N = 4$ theory, while $\sigma$ with a VEV. This means that our
solution describes an RG flow to the $N = 1$ pure YM in a vacuum with non-zero condensate.

We must spend a few words on the reliability of these singular solutions. Since the curvature
and the kinetic terms for the scalars typically diverge in the IR region, large corrections to
supergravity may be expected. An explicit comparison between supergravity and quantum field
theory has been made in [7] for the case of the $N = 4$ Coulomb branch and several discrepancies
were pointed out. By itself, this result does not immediately implies that all the other solutions
describing confinement should be considered as suspect. In a certain sense, screening is more
sensitive to infrared physics than confinement: any amount of string tension, no matter how
small, will destroy it.

Our contribution to the debate about reliability of supergravity solutions is to present a
pure $N = 1$ YM example, where all qualitative features expected in the quantum field theory,
namely confinement and a gaugino condensate, can be explicitly found in the supergravity side.
We must also point out that the singularity of the metric eq. (13) might be an artifact of the 5d
Einstein-frame metric. Such a singularity may appear also when the higher-dimensional metric
is regular. For instance, write $R^9$ with flat metric in polar coordinates as $R^+ \times S^3 \times R^5$, perform
a dimensional reduction to $R^+ \times R^5$, and rescale to the 5d Einstein frame: the resulting metric
has the same singularity (at $r = 0$) as in eq. (13). Naturally, when the singularity is an artifact
of the dimensional reduction, the supergravity computation of the string tension is completely
justified. This is the case, for instance, of the non-conformal backgrounds in ref. [32].

3 The $N = 1$ Theory

In this Section we will discuss the properties of the five-dimensional supergravity solution cor-
responding to a deformation of $N = 4$ Super Yang-Mills theory with a supersymmetric mass
term for the three fermions in the chiral $N = 1$ multiplets. In $N = 1$ notation, this is a mass

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8 These fields appear as $\mathbf{6}$ and $\mathbf{1}$ in the decomposition $\mathbf{10} \rightarrow \mathbf{1} + \mathbf{6} + \mathbf{\bar{3}}$ of $SU(4)$ under $SU(3) \times U(1)$.
term for the three chiral superfields $X_i$

$$\int d^2 \theta m_{ij} \text{Tr} \, X_i X_j + \text{c.c.}, \quad (19)$$

where $m_{ij}$ is a complex, symmetric matrix.

The theory flows in the IR to pure $N = 1$ Yang-Mills, which confines.

To obtain the standard $N = 1$ pure Yang-Mills with fixed scale $\Lambda$, we need a fine tuning of the UV parameters, in which the mass $m$ diverges while the 't Hooft coupling constant, $x$, goes to zero as an (inverse) logarithm of $m$. This is outside the regime of validity of supergravity, which requires a large $x$, but we may still expect to find from supergravity the qualitative properties of the theory.

### 3.1 Mass Deformation

The supersymmetric mass term for the chiral multiplets, $m_{ij}$, transforms as the 6 of $SU(3)$, and the corresponding supergravity mode appears in the decomposition of the $\mathbf{10} \rightarrow \mathbf{1} + \mathbf{6} + \mathbf{3}$ of $SU(4)$ under $SU(3) \times U(1)$. In principle, a generic non-zero VEV for $m_{ij}$ will induce non-zero VEVs for other scalars as well, due to the existence of linear couplings of $m$ to other fields in the potential. However, if we further impose $SO(3)$ symmetry, by taking an $m_{ij}$ proportional to the identity matrix, a simple group theory exercise shows that all remaining fields can be consistently set to zero.

The five-dimensional action for the scalars, $\mathbf{27}$,

$$L = \sqrt{-g} \left[ -\frac{R}{4} - \frac{1}{24} \text{Tr} \,(U^{-1} \partial U)^2 + V(U) \right], \quad (20)$$

is written in terms of a $27 \times 27$ matrix $U$, transforming in the fundamental representation of $E_6$ and parametrising the coset $E_6/USp(8)$. In a unitary gauge, $U$ can be written as $U = e^X$, $X = \sum_a \lambda_a T_a$, where $T_a$ are the generators of $E_6$ that do not belong to $USp(8)$. This matrix has exactly 42 real independent parameters, which are the scalars of the supergravity theory.

We shall not discuss here the philosophy and details of the computation of the potential and kinetic terms, since the have been extensively described in several previous papers $[6, 21, 22]$. In the Appendix, the interested reader can find a summary of the relevant formulae and the explicit parametrisation for the coset manifold representative we used. The result for the diagonal scalar $m_{ij}$ is

$$L = \sqrt{-g} \left\{ -\frac{R}{4} + \frac{1}{2} (\partial m)^2 - \frac{3}{8} \left[ 3 + \left( \cosh \frac{2m}{\sqrt{3}} \right)^2 + 4 \cosh \frac{2m}{\sqrt{3}} \right] \right\}, \quad (21)$$

$^9$Unlike other examples $[27]$, the second order perturbation in $m$, which amounts to the trace of the scalar mass terms, is associated with a stringy state and does not force us to introduce a second scalar field in the supergravity flow.

$^{10}$In the following we will always set the coupling constant $g$ equal to 2, so that the scalar potential in the $N = 8$ vacuum, where all scalars have zero VEV, is normalised as $V(N = 8) = -3$. 

10
where $R$ is the curvature of $AdS_5$.

We want to find a solution of 5d gauged supergravity with one asymptotically-$AdS_5$ region, corresponding to the UV $N = 4$ super Yang-Mills theory. The ansatz for the 4-d Poincaré invariant metric is

$$ds^2 = dy^2 + e^{2\phi(y)} dx^\mu dx_\mu, \quad \mu = 0, 1, 2, 3,$$

with $\phi \sim y/R + \text{const}$ when $y \to \infty$. We also assume that our scalar only depends on the radial coordinate. The boundary conditions for the scalars is that they approach the $SO(6), N = 8$ invariant stationary point: for $y \to \infty$, $m$ must vanish.

The solution we are looking for should correspond to an $N = 1$ supersymmetric flow on the dual field theory side. As discussed in [21, 22], it is possible to determine the number of supersymmetries of a certain supergravity configuration by inspection of the eigenvalues of the tensor $W_{ab}$ in the scalar potential (see the Appendix for the definition of the tensors below)

$$V = -\frac{1}{8} \left[ 2W_{ab}W^{ab} - W_{abcd}W^{abcd} \right].$$

The number of supersymmetries is given by the number of eigenvalues for which eq. (10) is valid [27].

In our case, where only the scalar $m$ is considered, $W_{ab}$ has two different eigenvalues

$$W_1 = -\frac{1}{4} \left( 5 \cosh \frac{2m}{\sqrt{3}} + 1 \right),$$

$$W_2 = -\frac{3}{4} \left( \cosh \frac{2m}{\sqrt{3}} + 1 \right),$$

with degeneracy 6 and 2 respectively. It is easy to check that only the second one satisfies eq. (10). This corresponds to an $N = 1$ supersymmetric gauge theory.

Supersymmetry also makes it possible to reduce the problem of solving the second order equations of motion to solving equations for the fermion shifts, i.e. first order equations, with $W = -\frac{3}{4} \left( \cosh \frac{2m}{\sqrt{3}} + 1 \right)$. They read

$$\dot{m} = -\frac{\sqrt{3}}{2} \sinh \frac{2m}{\sqrt{3}},$$

$$\dot{\phi} = \frac{1}{2} \left( 1 + \cosh \frac{2m}{\sqrt{3}} \right).$$

These equations can be solved exactly, and give

$$\phi(y) = \frac{1}{2} \left( y + \log[2 \sinh(y - C_1)] \right),$$

$$m(y) = \frac{\sqrt{3}}{2} \log \left[ \frac{1 + e^{-(y-C_1)}}{1 - e^{-(y-C_1)}} \right].$$
The metric has a singularity at $y = C_1$

$$ds^2 = dy^2 + |y - C_1| dx^\mu dx_\mu. \quad (30)$$

Around this point $m$ behaves as

$$m \sim -\frac{\sqrt{3}}{2} \log(y - C_1) + \text{const.} \quad (31)$$

Notice that, although singular, this is not the universal behaviour found in [2]: indeed, because of supersymmetry, it is not possible to ignore the potential in the equations of motion. On the other hand, it is easy to see that this solution corresponds to a true deformation of the gauge theory. Indeed, $m$ approaches the boundary in the UV ($y \to \infty$) as $m \sim e^{-y}$, which is the required behaviour of a deformation (see eq.(3)).

As we said before, we expect the gauge theory to exhibit confinement in the IR. In order to prove confinement for our solution, we need to check that the Wilson loop has an area law behaviour. As discussed in the previous section, to compute the Wilson loop we need to minimise the action for a string whose endpoints are constrained on a contour $C$ on the boundary. For the background (22) and with the choice of variables of eq.(17), this reads

$$E = S/T = \int dx \sqrt{(\partial_x u)^2 + f(u)}. \quad (32)$$

As already explained in the previous Section (see also [2] for a more detailed discussion), confinement depends on the behaviour for small $u$ of the function

$$f(u) = T^2(u)e^{4\phi(u)}. \quad (33)$$

In our example the tension of the fundamental strings and of the D1-strings, $T(u)$, are, respectively

$$T_{F_1}^2 = 4 \left( \cosh \frac{4m}{\sqrt{3}} + \cosh \frac{2m}{\sqrt{3}} \right), \quad (34)$$
$$T_{D_1}^2 = 8 \left( \cosh \frac{m}{\sqrt{3}} \right)^2, \quad (35)$$

so that the asymptotic behaviour of the corresponding functions $f(u)$ is

$$f_{(q\bar{q})}(u) \sim 1, \quad f_{(m\bar{m})}(u) \sim |u - C_1|. \quad (36)$$

According to the discussion in the previous section, this is indeed what we need to have quark confinement (i.e. a linear potential) and monopole screening.

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11 We consider a rectangular loop $C$ as in eq. (14) and we also factorize the trivial dependence in $t$.

12 The explicit form of the matrix $A$ from which the tension can be read is given in the Appendix.
3.2 Gaugino Condensate

Another feature of $N = 1$ Super Yang Mills is the presence of a gaugino condensate. It is then natural to ask whether this feature can be found in the supergravity solution. The key point is again the double meaning of the solutions in AdS: true deformations of the gauge theory correspond to non-normalizable solutions of the supergravity equation of motion, while different vacua correspond to normalizable solutions.

In the decomposition $10 \rightarrow 1 + 6 + 3$ of $SU(4)$ under $SU(3) \times U(1)$, a scalar $SU(3)$-singlet appears: $\sigma$. It corresponds to a bilinear operator in the gaugino fields $\tilde{\gamma}^{\alpha \beta \gamma \delta \epsilon} \frac{1}{c^{(3)}} \sigma_{\gamma \delta \epsilon} \tau^{\alpha \beta}$.

It is tempting to interpret a non-zero VEV for this scalar as the gaugino condensate. To substantiate this interpretation, we must look for solutions where both the fields $m$ and $\sigma$ are given a non-zero VEV and check whether they have the right UV asymptotic behaviour: $m$ has to be asymptotic to the non-normalizable solution $e^{-(4-\Delta)y}$ of eq. (3), and $\sigma$ to the normalizable one $e^{-\Delta y}$. It is easy to see that this is indeed the case.

The five-dimensional Lagrangian for the fields $m$ and $\sigma$ now becomes

$$ L = \sqrt{-g} \left\{ -\frac{R}{4} + \frac{1}{2} (\partial m)^2 + \frac{1}{2} (\partial \sigma)^2 + \frac{3}{8} \left[ \left( \cosh \frac{2m}{\sqrt{3}} \right)^2 + 4 \cosh \frac{2m}{\sqrt{3}} \cosh 2\sigma - (\cosh 2\sigma)^2 + 4 \right] \right\}. \quad (37) $$

The tensor $W_{ab}$ in the potential has one eigenvalue, $W_2 = \frac{3}{4} \left( \cosh \frac{2m}{\sqrt{3}} + \cosh 2\sigma \right)$, with degeneracy 2, satisfying eq. (10). Thus, we are still dealing with solutions describing $N = 1$ supersymmetric RG flows

$$ \phi(y) = \frac{1}{2} \log \left[ 2 \sinh (y - C_1) \right] + \frac{1}{6} \log \left[ 2 \sinh (3y - C_2) \right], \quad (38) $$

$$ m(y) = \sqrt{3} \frac{2}{2} \log \left[ \frac{1 + e^{-(y-C_1)}}{1 - e^{-(y-C_1)}} \right], \quad (39) $$

$$ \sigma(y) = \frac{1}{2} \log \left[ \frac{1 + e^{-(3y-C_2)}}{1 - e^{-(3y-C_2)}} \right]. \quad (40) $$

The UV is at $y = +\infty$, and it is immediate to see that $m$ and $\sigma$ have the desired asymptotic behaviour for $y \rightarrow \infty$:

$$ m \sim e^{-y}, \quad \sigma \sim e^{-3y}. \quad (41) $$

The solution we have found corresponds, therefore, to a supersymmetric mass deformation of $N = 4$ SYM that gives rise to a RG flow to a $N = 1$ SYM vacuum with a gaugino condensate.

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13 A non-zero VEV for this field alone was studied in [1, 6]. It leads to a non-supersymmetric, conformal IR fixed point.

14 Note that the IR contributions to operator condensates found in [3] vanish in this case.
Finally one may ask whether the presence of the condensate affects the IR behaviour of the solution, in particular confinement. The form of the tensions for the F1 and D1-string in the presence of both \( m \) and \( \sigma \) is

\[
T^2_{F1} = 4 \left[ \cosh \frac{4m}{\sqrt{3}} + \cosh \left( \frac{2m}{\sqrt{3}} + 2\sigma \right) \right],
\]

\[
T^2_{D1} = 8 \left[ \cosh \left( \frac{m}{\sqrt{3}} - \sigma \right) \right]^2.
\]

It is easy to see from these formulae that the asymptotic behaviours of the functions \( f(q,\bar{q})\) and \( f(m,\bar{m})\) in eq.(36) remain unchanged: the solution is still confining.

Notice that the only assumption we make is that the integration constants \( C_1, C_2 \) satisfy the inequality \( C_2 \leq 3C_1 \).

It is interesting to notice that our solution depends on two constants of integration. This may signal that our supergravity background describes a two-parameter deformation of pure \( N = 1 \) super Yang-Mills. After all, the construction in ref. [34] shows that such deformation exists, and is given by a suitable compactification to four dimensions of a 6d \( (2,0) \) superconformal theory.

Notice also that eqs. (38,39,40) describe chiral-symmetry preserving \( (\sigma = 0) \) as well as chiral-symmetry breaking flows \( (\sigma \neq 0) \). The chirality-preserving flow is unstable in the sense that any nonzero \( \sigma \) in the UV (i.e. at large positive \( y \)), no matter how small, flows in the IR \( (y = C_1) \) to a finite, nonzero \( \sigma = -\log[\tanh(3C_1/2 - C_2/2)] \). This behaviour agrees with field theory expectations [33, 34].

Moreover, let us point out that a nonzero condensate \( \sigma \) implies the existence of domain walls, interpolating between vacua with different values of \( \sigma \). Since the tension of these domain walls scales as \( N \), instead of \( N^2 \), in the large-N limit [33], they do not appear as classical supergravity solitons, but rather as D-branes [34]. This is consistent with the vanishing of the potential barrier between chiral-symmetry breaking vacua, evident from our explicit solution. To find which quantum soliton or D-brane plays the role of domain wall in our solution is an interesting problem that we shall leave open (for the time being, at least).

Finally, we would like to remark that the parametrisation used in eq.(40) allows also to describe the \( N = 0 \) \( SU(3) \) critical point. Denoting by \( R_{IR} \) the radius of the Anti-de-Sitter space corresponding to the IR fixed point, the square mass of the fluctuation of the field \( m \) around the \( SU(3) \) point is \(- (40/9) R_{IR}^{-2} \), which violates the unitarity bound [35]. This supports the suggestion that the \( SU(3) \) critical point is unstable [23]. Our parametrisation differs from that obtained in [3], which we used in [2]. Thus the scalar potential in Section 3 of [2] has to be substituted with eq.(21). This change does not affect the general results discussed in that paper.

\footnote{See eq.(A.12) for the explicit form of the coset manifold representative.}
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Appendix: Conventions and Useful Formulae for the Lagrangian

In this Appendix we discuss some of the details of the computations that led to the results in the paper. We will present our results for the case where both the fields \( m \) and \( \sigma \) are turned on. The case where only \( m \) is considered can be easily recovered by setting \( \sigma \) equal to zero. This Appendix is not meant to be self-contained, since the tools for computing the potential have been already described in details in [21] and recently reviewed in [6, 22]. The reader may refer to the above-mentioned papers for more details.

Parametrisation of the Coset Manifold

The five-dimensional Lagrangian for the scalars is written in terms of the \( 27 \times 27 \) matrix \( U \), parametrising the coset \( E_6/USp(8) \), as

\[
L = \sqrt{-g} \left[ \frac{R}{4} - \frac{1}{24} \text{Tr} \left( U^{-1} \partial U \right)^2 + V(U) \right].
\]

To evaluate it, one has first to choose a parametrisation for the coset manifold representative, \( U = \exp X \), that gives canonical kinetic terms for the scalars. The precise form of the matrix \( X \) is obtained using the global and local symmetries of the problem, and the fact that \( U \) maps an element of the representation \( 27 \) of \( E_6 \) into itself. The general parametrisation for \( U \) has been worked out in [21]. In the gauged theory, only the group \( SU(4) \times SL(2; R) \) is a symmetry of the Lagrangian. The 42 scalars then decompose according to

\[
42 \rightarrow 20'(0) + 10(-2) + \overline{10}(0) + 1(4) + 1(-4),
\]

while the vectors in the \( 27 \) decompose as

\[
27 \rightarrow 15(0) + 6(2) + 6(-2).
\]

The subscripts denote the charges of the \( U(1) \) factor in \( SL(2; R) \). The \( 27 \) of \( E_6 \) is represented by a couple of antisymmetric symplectic-traceless indices \( A, B \), running from 1 to 8, and, in the
\( SU(4) \times SL(2; R) \) basis, it decomposes as

\[
27 \rightarrow 15 + (6, 2), \tag{A.4}
\]

\[
z^{AB} \rightarrow \begin{pmatrix} z_{IJ} \\ z^I_\alpha \end{pmatrix}, \tag{A.5}
\]

with \( I, J = 1, \ldots, 6 \) indices of \( SU(4) \) and \( \alpha = 1, 2 \) indices of \( SL(2; R) \). The variation of the vector \( z^{AB} \) under \( E_6(6) \) is:

\[
\delta z_{IJ} = -\Lambda^M_I z_{MJ} - \Lambda^M_J z_{IM} + \sum_{\alpha} \Sigma^{M\alpha}_{IJ} z_{\alpha}, \tag{A.6}
\]

\[
\delta z^I_\alpha = \Lambda^I_\alpha z^M + \Lambda^\alpha_\beta z^I_\beta + \sum_{\alpha} \Sigma^{MN\alpha}_{IM} z_{MN}, \tag{A.7}
\]

From these equations it is possible to find the form of the matrix \( X \) in the \( SU(4) \times SL(2; R) \) basis:

\[
X = \begin{pmatrix} -2\Sigma^M_I \Lambda^N_J & \sum_{\alpha} \Sigma^{M\alpha}_{IJ} \\ 2\sum_{\alpha} \Sigma^{MN\alpha}_{IJ} & \Lambda^I_\alpha \Lambda^\alpha_\beta + \sum_{\alpha} \Sigma^{N\alpha}_{IJ} \end{pmatrix}. \tag{A.8}
\]

Here \( \Lambda^M_I \) and \( \Lambda^\alpha_\beta \) are real and traceless matrices, that are the generators of \( SU(4) \) and \( SL(2; R) \), while \( \sum_{\alpha} \Sigma^{MN\alpha} \) is real and totally antisymmetric in the indices \( MNI \). The physical states are given by the non-compact generators of \( E_6(6) \), which correspond to the symmetric parts of \( \Lambda^M_I \) and \( \Lambda^\alpha_\beta \), and the self-dual part of \( \sum_{\alpha} \Sigma^{MN\alpha} \). The self-duality condition for \( \sum_{\alpha} \Sigma^{MN\alpha} \) is \[21]\]

\[
\Sigma_{IJK \alpha}^{\pm} = \pm \frac{1}{6} \epsilon_{\alpha\beta} \epsilon_{IJKLMN} \Sigma_{LMN \beta}^{\pm}. \tag{A.9}
\]

In a unitary \( USp(8) \) gauge, \( U \) can be written as

\[
U = \exp X, \tag{A.10}
\]

where \( X = \sum \lambda_a T_a \) is given only by the 42 non compact generators of \( E_6 \); these 42 independent parameters correspond to the 42 scalars.

We are interested in the scalars \( m \) and \( \sigma \) which are in the \( 6_{(2,-2)} \) of \( SU(4) \) and transform as a singlet and a \( 6 \) of the \( SU(3) \subset SU(4) \). Turning them on breaks \( SU(4) \) to \( SU(3) \times U(1) \). We choose the following decomposition of the \( 27 \) in \( SU(3) \times U(1) \) basis \[3]\:

\[
27 \rightarrow \begin{pmatrix} (1,0), 3(4,0), \bar{3}(-4,0), 8(0,0), 3(-2,2), \bar{3}(2,2), 3(-2,-2), \bar{3}(2,-2) \end{pmatrix}, \tag{A.11}
\]

where the first index gives the charges under the \( U(1) \subset SU(4) \) and the second the charges under the other \( U(1) \subset SL(2; R) \). With this choice of basis, the matrix \( X \) has the following

\[\text{The indices } M, N, \ldots \text{ are raised with the invariant tensor } \epsilon_{IJKLMN} = \epsilon^{IJKLMN}. \Sigma_{IJK}^{\alpha} = \frac{1}{6} \epsilon_{\alpha\beta} \epsilon^{IJKLMN} \Sigma_{LMN \beta}^{\pm}.\tag{21}\]
where $i, \bar{i} = 1, \ldots, 3$ are complex $SU(3)$ indices.

**Vielbein and Gamma Matrices**

To compute the potential and the tensions of the fundamental and D1-string we need to know the vielbein $V_{AB}^{ab}$, where $a, b$ are a couple of antisymmetric symplectic-traceless indices $a, b = 1, \ldots, 8$, representing the $\mathbf{27}$ of $USp(8)$. This field, being an element of $E_6/USp(8)$, carries both the indices $A, B = 1, \ldots, 8$ of the $\mathbf{27}$ of $E_6$ and the indices $a, b = 1, \ldots, 8$ of the $\mathbf{27}$ of $USp(8)$. According to ref. [21], one can go from the $E_6$ basis to the $USp(8)$ using the $SO(7)$ gamma matrices, $\Gamma$.

In the complex basis we are using, the vielbein is obtained by multiplying on the right the matrix $U$ by the following vector of gamma matrices:

$$\left( \gamma_{ij}, \epsilon_{ijk}\gamma_{jk}, \epsilon_{ijk}\gamma_{jk}, \frac{\gamma_i(1 - \Gamma_0)}{4}, \frac{\gamma_i(1 + \Gamma_0)}{4}, \frac{\gamma_i(1 + \Gamma_0)}{4}, \frac{\gamma_i(1 - \Gamma_0)}{4} \right).$$

(A.13)

The complex gamma matrices, $\gamma_j$ and $\bar{\gamma}_j$, have a simple expression in terms of the real matrices $\Gamma$ (here we used the definition of the $SO(7)$ gamma matrices of ref. [27]):

$$\gamma_j = \frac{\Gamma_{j+3} + i\Gamma_j}{\sqrt{2}},$$

(A.14)

$$\bar{\gamma}_j = \frac{\Gamma_{j+3} - i\Gamma_j}{\sqrt{2}}.$$  

(A.15)

where $j, \bar{j} = 1, 2, 3$.

\textsuperscript{17}Notice that the parametrisation we obtained differs slightly from that given in \textsuperscript{3}
Potential

The scalar potential is expressed in terms of the two $USp(8)$ tensors

\begin{align}
W_{abcd} &= \epsilon^{\alpha\beta} \eta^{IJ} V_{I\alpha a} V_{J\beta b}, \\
W_{ac} &= -i (\Gamma_0)^{db} W_{abcd},
\end{align}

as \cite{21}

\begin{align}
V &= -\frac{1}{8} \left[ 2W_{ab} W^{ab} - W_{abcd} W^{abcd} \right],
\end{align}

where the $USp(8)$ indices are raised and lowered with the matrix $\Gamma_0$, as shown in ref. [21]. Once the vielbein are known, its evaluation is a lengthy but straightforward computation. Plugging $V_{AB}^{ab}$ in the expressions (A.13),(A.15) and performing some gamma-matrix algebra one gets:

\begin{align}
V &= -\frac{3}{8} \left[ \left( \cosh \frac{2m}{\sqrt{3}} \right)^2 + 4 \cosh \frac{2m}{\sqrt{3}} \cosh 2\sigma - (\cosh 2\sigma)^2 + 4 \right].
\end{align}

As already mentioned in the paper, the eigenvalues of the tensor $W_{ab}$ are related to the number of supersymmetries of the RG flows. In our case $W_{ab}$ reads

\begin{align}
W_{ab} &= -\frac{1}{16} \left[ (18I + J) \cosh \frac{2m}{\sqrt{3}} + (6I - J) \cosh 2\sigma \right],
\end{align}

and it has two eigenvalues

\begin{align}
W_1 &= -\frac{1}{4} \left( 5 \cosh \frac{2m}{\sqrt{3}} + \cosh 2\sigma \right), \\
W_2 &= -\frac{3}{4} \left( \cosh \frac{2m}{\sqrt{3}} + \cosh 2\sigma \right),
\end{align}

with degeneracy 6 and 2 respectively. In eq.(A.20), $I$ is the $(8 \times 8)$ identity matrix and $J = 2 \text{diag}(1,1,-3,1,1,1,-3,1)$.

Kinetic Term for the $B$ Fields

Finally, we give the explicit form of the matrix $A_{I\alpha,J\beta}$ in eq.(15). The kinetic terms for the 12 antisymmetric tensors $B_{\mu\nu}^{I\alpha}$ was given in [21]

\begin{align}
-\frac{1}{8} B_{\mu\nu} B^{\mu\nu} + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \eta_{IJ} \epsilon_{\alpha\beta} B_{\mu\nu}^{I\alpha} D_{\rho} B^{J\beta}_{\sigma\tau},
\end{align}

with $B_{\mu\nu}^{ab} = B_{\mu\nu}^{I\alpha} V_{I\alpha}^{ab}$. The form in eq.(15) is obtained when we explicit the dependence on the vielbein in the first term, and we define the matrix

\begin{align}
A_{I\alpha,J\beta} = V_{I\alpha}^{ab} V_{J\beta ab}.
\end{align}
In our example it has the form:

\[
\begin{pmatrix}
\cosh^2 \left( \frac{m}{\sqrt{3}} - \sigma \right) & 0 & 0 & -\sinh^2 \left( \frac{m}{\sqrt{3}} - \sigma \right) \\
0 & \sinh^2 \frac{2m}{\sqrt{3}} + \cosh^2 \left( \frac{m}{\sqrt{3}} + \sigma \right) & \cosh^2 \left( \frac{m}{\sqrt{3}} + \sigma \right) & 0 \\
0 & \cosh^2 \left( \frac{m}{\sqrt{3}} + \sigma \right) - \cosh^2 \frac{2m}{\sqrt{3}} & \sinh^2 \frac{2m}{\sqrt{3}} + \cosh^2 \left( \frac{m}{\sqrt{3}} + \sigma \right) & 0 \\
-\sinh^2 \left( \frac{m}{\sqrt{3}} - \sigma \right) & 0 & 0 & \cosh^2 \left( \frac{m}{\sqrt{3}} + \sigma \right)
\end{pmatrix}
\]

(A.25)

References

[1] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, JHEP 9812 (1998) 022, hep-th/9810126.

[2] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, JHEP 9905 (1999) 026, hep-th/9903020.

[3] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys.Lett. B428 (1998) 105, hep-th/9802109.

[5] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[6] J. Distler and F. Zamora, Adv. Theor. Math. Phys. 2 (1998) 1405, hep-th/9810206.

[7] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Continuous distributions of D3-branes and gauged supergravity, hep-th/9906194. A. Brandhuber and K. Sfetsos, Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories, hep-th/9906201. I. Chepelev and R. Roiban, A note on correlation functions in AdS$_5$/SYM$_4$ correspondence on the Coulomb branch, hep-th/9906224.

[8] A. Kehagias and K. Sfetsos, Phys. Lett. B454 (1999) 270, hep-th/9902125.

[9] S. S. Gubser, Dilaton-driven confinement, hep-th/9902155.

[10] N. R. Constable and R. C. Myers, Exotic Scalar States in the AdS/CFT Correspondence, hep-th/9905081.

[11] A. Polyakov, Nucl. Phys. Proc. Suppl. 68 (1998) 1, hep-th/9711002; Int. J. Mod. Phys. A14 (1999) 645, hep-th/9809054.
[12] A. Armoni, E. Fuchs and J. Sonnenschein, JHEP 9906 (1999) 027, hep-th/9903090.

[13] S. Nojiri and S. D. Odintsov, Phys.Lett. B449 (1999) 39, hep-th/9812017; Phys.Lett. B458 (1999) 226, hep-th/9904036.

[14] V. Balasubramanian, P. Kraus, A. Lawrence and S. Trivedi, Phys. Rev. D59 (1999) 104021, hep-th/9808017.

[15] I. R. Klebanov and E. Witten, AdS/CFT Correspondence and Symmetry Breaking, hep-th/9905104.

[16] A. Peet and J. Polchinski, Phys. Rev. D59 (1999) 065011, hep-th/9809022.

[17] M. Porrati and A. Starinets, Phys. Lett. B454 (1999) 77, hep-th/9903085.

[18] V. Balasubramanian and P. Kraus, Spacetime and the Holographic Renormalization Group, hep-th/9903190.

[19] M. Günyaydin, L.J. Romans and N.P. Warner, Phys. Lett. B154 (1985) 268; M. Pernici, K. Pilch and P. van Nieuwenhuizen, Nucl. Phys. B 259 (1985) 460.

[20] B. de Wit and H. Nicolai, Nucl. Phys. B281 (1987) 211; B. de Wit, H. Nicolai and N. P. Warner, Nucl. Phys. B255 (1984) 29; H. Nastase, D. Vaman and P. van Nieuwenhuizen, Consistent Nonlinear KK Reduction of 11d Supergravity on $AdS_7 \times S_4$ and Self-Duality in Odd Dimensions, hep-th/9905075.

[21] M. Günyaydin, L.J. Romans and N.P. Warner, Nucl. Phys. B272 (1986) 598.

[22] A. Khavaev, K. Pilch and N. P. Warner, New Vacua of Gauged N=8 Supergravity, hep-th/9812033.

[23] D.Z. Freedman, private communication.

[24] A. Karch, D. Lüst and A. Miemiec, Phys. Lett. B454 (1999) 265, hep-th/9810254.

[25] M. Henningson and K. Skenderis, JHEP 9807 (1998) 023, hep-th/9806087; S. S. Gubser, Einstein manifolds and conformal field theories, hep-th/9807164.

[26] D. Anselmi, D. Z. Freedman, M. T. Grisaru, A. A. Johansen, Phys. Lett. B394 (1997) 329, hep-th/9608123; Nucl. Phys. B256 (1998) 543, hep-th/970804; D. Anselmi, J. Erlich, D. Z. Freedman and A. Johansen, Phys.Rev. D57 (1998) 7570, hep-th/9711033.
[27] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, *Renormalization Group Flows from Holography–Supersymmetry and a c-Theorem*, hep-th/9904017.

[28] J. A. Minahan, JHEP 9904 (1999) 007, hep-th/9902074.

[29] J. M. Maldacena, Phys. Rev. Lett. 80 (1998) 4859, hep-th/9803002; S. Rey and J. Yee, *Macroscopic Strings as Heavy Quarks of Large N Gauge Theory and Anti-de Sitter Supergravity*, hep-th/9803001.

[30] S. Rey, S. Theisen and J.T. Yee, Nucl. Phys. B527 (1998) 171, hep-th/9803133; A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, Phys. Lett. B434 (1998) 36, hep-th/9803137.

[31] H. Liu and A. A. Tseytlin, *Dilaton - fixed scalar correlators and AdS5xS5 - SYM correspondence*, hep-th/9906151.

[32] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, hep-th/9803131.

[33] A. Kovner and M. Shifman, Phys. Rev. D56 (1997) 2396, hep-th/9702174; ibidem 7978, hep-th/9706089.

[34] E. Witten, Nucl. Phys. B507 (1997) 658; hep-th/9706109.

[35] P. Breitenlohner and D.Z. Freedman, Ann. Phys. 144 (1982) 249.