δ-Dynamic chromatic number of Helm graph families

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Abstract: An $r$-dynamic coloring of a graph $G$ is a proper coloring $c$ of the vertices such that $|c(N(v))| \geq \min\{r, d(v)\}$, for each $v \in V(G)$, where $N(v)$ and $d(v)$ denote the neighborhood and the degree of $v$, respectively. The $r$-dynamic chromatic number of a graph $G$ is the minimum $k$ such that $G$ has an $r$-dynamic coloring with $k$ colors. In this paper, we obtain the $\delta$-dynamic chromatic number of middle, total, and central of helm graph, where $\delta = \min_{v \in V(G)} \{d(v)\}$.

Keywords: $r$-dynamic coloring; middle graph; total graph; central graph

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1. Introduction

Throughout this paper all graphs are finite and simple. The $r$-dynamic chromatic number was first introduced by Montgomery (2001). An $r$-dynamic coloring of a graph $G$ is a map $c$ from $V(G)$ to a set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v$.

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\( v \in V(G), |c(N(v))| \geq \min\{r, d(v)\} \), where \( N(v) \) denotes the set of vertices adjacent to \( v \) and \( d(v) \) is its degree. The \( r \)-dynamic chromatic number of a graph \( G \), written \( \chi_d(G) \), is the minimum \( k \) such that \( G \) has an \( r \)-dynamic proper \( k \)-coloring. The 1-dynamic chromatic number of a graph \( G \) is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in Ahadi, Akbari, Dehghan, and Ghanbari (2012), Akbari, Ghanbari, and Jahanbakam (2009, 2010), Alishahi (2012), Lai, Montgomery, and Poon (2003). There are many upper bounds and lower bounds for \( \chi_d(G) \) in terms of graph parameters. For example,

For a graph \( G \) with \( \Delta(G) \geq 3 \), Lai et al. (2003) proved that \( \chi_d(G) \leq \Delta(G) + 1 \). An upper bound for the dynamic chromatic number of a \( d \)-regular graph \( G \) in terms of \( \chi(G) \) and the independence number of \( G \), \( \alpha(G) \), was introduced in Dehghan and Ahadi (2012). In fact, it was proved that \( \chi_d(G) \leq \chi(G) + 2 \log_2 \alpha(G) + 3 \). Taherkhani gave in (2016) an upper bound for \( \chi_2(G) \) in terms of the chromatic number, the maximum degree \( \Delta \) and the minimum degree \( \delta \), i.e. \( \chi_2(G) - \chi(G) \leq \left( \frac{\Delta \epsilon}{\delta} \log \left( 2e \left( \Delta^2 + 1 \right) \right) \right) \).

Li, Yao, Zhou, and Broersma proved in (2009) that the computational complexity of \( \chi_d(G) \) for a 3-regular graph is an NP-complete problem. Furthermore, Liu and Zhou (2008) showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with maximum degree 3, is NP-complete.

In this paper, we study \( \chi_r(G) \) when \( r \) is \( \delta \), the minimum degree of the graph. We find the \( \delta \)-dynamic chromatic number for middle, total, and central graph of helm graph.

2. Results

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The middle graph (Michalak, 1981) of \( G \), denoted by \( M(G) \) is defined as follows. The vertex set of \( M(G) \) is \( V(G) \cup E(G) \). Two vertices \( x, y \) of \( M(G) \) are adjacent in \( M(G) \) in case one of the following holds: (i) \( x, y \) are in \( E(G) \) and \( x, y \) are adjacent in \( G \). (ii) \( x \) is in \( V(G) \), \( y \) is in \( E(G) \), and \( x, y \) are incident in \( G \).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The total graph (Michalak, 1981) of \( G \), denoted by \( T(G) \) is defined in the following way. The vertex set of \( T(G) \) is \( V(G) \cup E(G) \). Two vertices \( x, y \) of \( T(G) \) are adjacent in \( T(G) \) in case one of the following holds: (i) \( x, y \) are in \( V(G) \) and \( x \) is adjacent to \( y \) in \( G \). (ii) \( x, y \) are in \( E(G) \) and \( x, y \) are adjacent in \( G \). (iii) \( x \) is in \( V(G) \), \( y \) is in \( E(G) \) and \( x, y \) are incident in \( G \).

The central graph (Vernold Vivin, 2007) \( C(G) \) of a graph \( G \) is obtained from \( G \) by subdividing each edge of \( G \) exactly once and then joining each pair of vertices of the original graph which were previously non-adjacent.

The helm graph \( H_n \) is the graph obtained from an \( n \)-wheel graph by adjoining a pendent edge at each node of the cycle. Where \( V(H_n) = \{v\} \cup \{v_i, v_{i+1}, \ldots, v_{n-1}\} \cup \{u_i, u_{i+1}, \ldots, u_{n-1}\} \) and \( E(H_n) = \{e_i; 1 \leq i \leq n - 1\} \cup \{e'_i; 1 \leq i \leq n - 1\} \cup \{s_i; 1 \leq i \leq n - 1\} \), where \( e_i \) is the edge \( vv_i \) (\( 1 \leq i \leq n - 1\)), \( e'_i \) is the edge \( vv_{i+1} \) (\( 1 \leq i \leq n - 2\)) and \( s_i \) is the edge \( vu_i \) (\( 1 \leq i \leq n - 1\)). This notation is valid through the entire paper.

**Theorem 2.1** Let \( n \geq 8 \). The \( \delta \)-dynamic chromatic number of the middle graph of a helm of order \( 2n - 1 \) is \( \chi_d(M(H_n)) = n \).

**Proof** By the definition of middle graph, \( V(M(H_n))) = V(H_n) \cup E(H_n) = \{v\} \cup \{v_i; 1 \leq i \leq n - 1\} \cup \{u_i; 1 \leq i \leq n - 1\} \cup \{e_i; 1 \leq i \leq n - 1\} \cup \{e'_i; 1 \leq i \leq n - 1\} \cup \{s_i; 1 \leq i \leq n - 1\} \). The vertices \( v \) and \( \{e'_i; 1 \leq i \leq n - 1\} \) induce a clique of order \( K_n \) in \( M(H_n) \). Thus, \( \chi_d(M(H_n)) \geq n \).

Consider the following \( n \)-coloring of \( M(H_n) \):

\[ \]
For $1 \leq i \leq n - 1$, assign the color $c_i$ to $e_i$ and assign the color $c_{i'}$ to $v_i$. For $1 \leq i \leq n - 1$, assign the color $c_{i'}$ to $u_i$, $\deg(u_i) = \delta(M(H_n)) = 2$. For $1 \leq i \leq n - 1$, assign to $e_i'$ one of the allowed colors—such color exists, because $\deg(u_i) = 8$. For $1 \leq i \leq n - 1$, if any, assign to vertex $v_i$ one of the allowed colors—such color exists, because $\deg(v_i) = 4$. For $1 \leq i \leq n - 1$, if any, assign to vertex $s_i$ one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V(M(H_n))$. Thus, this coloring is a $\delta$-dynamic coloring. Hence, $\chi_H(M(H_n)) \leq n$. Therefore, $\chi_H(H_n) = n$, $\forall n \geq 8$. □

**Theorem 2.2** Let $n \geq 9$. The $\delta$-dynamic chromatic number of the total graph of a helm of order $2n - 1$ is $\chi_H(T(H_n)) = n$.

**Proof** By the definition of total graph $V(T(H_n)) = V(H_n) \cup E(H_n) = \{v\} \cup \{e_i: 1 \leq i \leq n - 1\} \cup \{s_i: 1 \leq i \leq n - 1\} \cup \{e_i': 1 \leq i \leq n - 1\} \cup \{e'_i: 1 \leq i \leq n - 1\}$, assign to vertex $s_i$ one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V(M(H_n))$. Thus, this coloring is a $\delta$-dynamic coloring. Hence, $\chi_H(T(H_n)) \geq n$. Consider the following $n$-coloring of $T(H_n)$:

For $1 \leq i \leq n - 1$, assign the color $c_{1}$ to $e_{1}$ and assign the color $c_{n}$ to $v$. For $1 \leq i \leq n - 1$, assign to $u_i$ one of the allowed colors—such color exists, because $\delta(u_i) = $\deg(u_i) = 2$. For $1 \leq i \leq n - 1$, assign to $e_i'$ one of the allowed colors—such color exists, because $\deg(u_i) = 8$. For $1 \leq i \leq n - 1$, if any, assign to vertex $v_i$ one of the allowed colors—such color exists, because $\deg(v_i) = 4$. For $1 \leq i \leq n - 1$, if any, assign to vertex $s_i$ one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V(M(H_n))$. Thus, this coloring is a $\delta$-dynamic coloring. Hence, $\chi_H(T(H_n)) \leq n$. Therefore, $\chi_H(T(H_n)) = n$, $\forall n \geq 9$. □

**Theorem 2.3** Let $n \geq 4$. The $\delta$-dynamic chromatic number of the central graph of a helm of order $2n - 1$ is $\chi_H(C(H_n)) = 2n - 1$.

**Proof** By the definition of central graph, subdividing each edge of $H_n$ exactly once and then joining each pair of vertices of $H_n$ which were non-adjacent. Let $V(C(H_n)) = V(H_n) \cup E(H_n) = \{v\} \cup \{e_i: 1 \leq i \leq n - 1\} \cup \{u_i: 1 \leq i \leq n - 1\} \cup \{s_i: 1 \leq i \leq n - 1\} \cup \{e'_i: 1 \leq i \leq n - 1\}$, assign to vertex $s_i$ one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V(M(H_n))$. Thus, this coloring is a $\delta$-dynamic coloring. Hence, $\chi_H(T(H_n)) \geq n$. Consider the following $n$-coloring of $C(H_n)$:

For $1 \leq i \leq n - 1$, assign the color $c_{1}$ to $v_i$ and assign the color $c_{n}$ to $v$. For $1 \leq i \leq n - 1$, assign to $u_i$ one of the allowed colors—such color exists, because $\deg(u_i) = $\deg(u_i) = 2$. For $1 \leq i \leq n - 1$, assign to $e_i'$ one of the allowed colors—such color exists, because $\deg(u_i) = 8$. For $1 \leq i \leq n - 1$, if any, assign to vertex $v_i$ one of the allowed colors—such color exists, because $\deg(v_i) = 4$. For $1 \leq i \leq n - 1$, if any, assign to vertex $s_i$ one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V(M(H_n))$. Thus, this coloring is a $\delta$-dynamic coloring. Hence, $\chi_H(T(H_n)) \leq n$. Therefore, $\chi_H(T(H_n)) = n$, $\forall n \geq 9$. □
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