Čech border homology and cohomology groups
and some applications

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Abstract
In the paper the Čech border homology and cohomology groups of closed pairs of normal spaces are constructed and investigated. These groups give intrinsic characterizations of Čech homology and cohomology groups based on finite open coverings, homological and cohomological coefficients of cyclicity, small and large cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces.

Keywords and Phrases: Čech homology, Čech cohomology, Stone-Čech compactification, remainder, cohomological dimension, coefficient of cyclicity.

Introduction
The investigation and discussion presented in this paper are centered around the following problem:
Find necessary and sufficient conditions under which a space of given class has a compactification whose remainder has the given topological property (cf. [Sm2], Problem I, p.332 and Problem II, p.334).
This problem for different topological invariants and properties was studied by several authors:

• J.M.Aarts [A], J.M.Aarts and T.Nishiura [A-N], Y. Akaike, N. Chinen and K. Tomoyasu [Ak-Chin-T], V.Baladze [B1], M.G. Charalambous [Ch], A.Chigogidze ([Ch1], [Ch2]), H. Freudenthal ([F1],[F2]), K.Morita [Mo], E.G. Skljarenko [Sk], Ju.M.Smirnov ([Sm1]-[Sm5]) and H.De Vries [V] found conditions under which the spaces have extensions whose remainders have given covering and inductive dimensions, and combinatorial properties.

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• The remainders of finite order extensions are defined and investigated by H.Inasaridze ([I_1], [I_2]). Using the results obtained in these papers, H.Inasaridze [I_3], L.Zambakhidze ([Z_1],[Z_2]), and I.Tsereteli [Ts] solved interesting problems of homological algebra, general topology and dimension theory.

• $n$-dimensional (co)homology groups of remainders of precompact spaces are studied by V.Baladze [B_3], V.Baladze and L.Turmanidze [B-Tu].

• A.Calder [C] described $n$-dimensional cohomotopy groups of remainders of Stone-Čech compactifications.

• The characterizations of shapes of remainders of spaces are established in papers of V.Baladze ([B_2],[B_3]), B.J.Ball [Ba], J.Keesling ([K_1], [K_2]), J.Keesling and R.B. Sher [K-Sh].

The present paper is motivated by the general problem mentioned above. Specifically, we study this problem for the properties: Čech (co)homology groups based on finite open covers, coefficients of cyclicity and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces are given groups and given numbers, respectively.

In this paper we define the Čech type covariant and contravariant functors which coefficients in an abelian group $G$,

$$\hat{H}_n^f(\cdot, \cdot; G) : \mathcal{N}_p^2 \rightarrow \mathcal{A}$$

and

$$\hat{H}_n^c(\cdot, \cdot; G) : \mathcal{N}_p^2 \rightarrow \mathcal{A},$$

from the category $\mathcal{N}_p^2$ of closed pairs of normal spaces and proper maps to the category $\mathcal{A}$ of abelian groups and homomorphisms. The construction of these functors is based on all border open covers of pairs $(X,A) \in \text{ob}(\mathcal{N}_p^2)$ (see Definition 1.1 and Definition 1.2).

One of our main results of the paper is the following theorem (see Theorem 2.1). Let $\mathcal{M}_p^2$ be the category of closed pairs of metrizable spaces and proper maps. For each closed pair $(X,A) \in \text{ob}(\mathcal{M}_p^2)$, one has

$$\tilde{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_n^f(X,A; G)$$

and

$$\tilde{H}_n^c(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_n^c(X,A; G),$$

where $\tilde{H}_n^f(\beta X \setminus X, \beta A \setminus A; G)$ and $\tilde{H}_n^c(\beta X \setminus X, \beta A \setminus A; G)$ are Čech homology and cohomology groups based on all finite open covers of $(\beta X \setminus X, \beta A \setminus A)$, respectively (see [E-St], Ch. IX, p.237).

We also consider the border cohomological and homological coefficients of cyclicity $\eta^f_G$ and $\eta^c_G$, border small and large cohomological dimensions $d^f_\infty(X;G)$.
and $D^f_\ell(X;G)$ and prove the following relations (see Theorem 2.3, Theorem 2.5 and Theorem 2.8):

$$
\eta^G_\ell(X,A) = \eta_G(\beta X \setminus X, \beta A \setminus A),
$$

$$
\eta^G_\ell(X,A) = \eta^G(\beta X \setminus X, \beta A \setminus A),
$$

$$
d^f_\ell(X;G) = d_f(\beta X \setminus X;G),
$$

$$
D^f_\ell(X;G) = D_f(\beta X \setminus X;G),
$$

where $\eta_G(\beta X \setminus X, \beta A \setminus A)$, $\eta^G(\beta X \setminus X, \beta A \setminus A)$ and $d_f(\beta X \setminus X;G)$, $D_f(\beta X \setminus X;G)$ are well known cohomological coefficient of cyclicity [No], homological coefficient of cyclicity (see Definition 2.2) and small cohomological dimension, large cohomological dimension [N] of remainders $(\beta X \setminus X, \beta A \setminus A)$ and $\beta X \setminus X$, respectively.

Without any further reference we will use definitions and results from the monographs General Topology [En], Algebraic Topology [E-St] and Dimension Theory [N].

1 On Čech border homology and cohomology groups

In this section we give an outline of a generalization of Čech homology theory by replacing the set of all finite open coverings in the definition of Čech (co)homology group $(\check{H}^p(X,A;G))$, $\check{H}_f^p(X,A;G)$ (see [E-St], Ch.IX, p.237) by a set of all finite open families with compact enclosures. For this aim we give the following definitions.

An indexed family of subsets of set $X$ is a function $\alpha$ from an indexed set $V_\alpha$ to the set $2^X$ of subsets of $X$. The image $\alpha(v)$ of index $v \in V_\alpha$ is denoted by $\alpha_v$. Thus the indexed family $\alpha$ is the family $\alpha = \{\alpha_v\}_{v \in V_\alpha}$. If $|V_\alpha| < \aleph_0$, then we say that $\alpha$ family is a finite family.

Let $V_{\alpha}'$ be a subset of set $V_\alpha$. A family $\{\alpha_v\}_{v \in V_{\alpha}'}$ is called a subfamily of family $\{\alpha_v\}_{v \in V_\alpha}$.

By $\alpha = \{\alpha_v\}_{v \in (V_\alpha,V_{\alpha}')}$ we denote the family consisting of family $\{\alpha_v\}_{v \in V_\alpha}$ and its subfamily $\{\alpha_v\}_{v \in V_{\alpha}'}$.

**Definition 1.1.** (see [Sm4]). A finite family $\alpha = \{\alpha_v\}_{v \in V_\alpha}$ of open subsets of normal space $X$ is called a border cover of $X$ if its enclosure $K_\alpha = X \setminus \bigcup_{v \in V_\alpha} \alpha_v$ is a compact subset of $X$.

**Definition 1.2.** (cf. [Sm4]). A finite open family $\alpha = \{\alpha_v\}_{v \in (V_\alpha,V_{\alpha}^A)}$ is called a border cover of closed pair $(X,A) \in \mathcal{A}^2$ if there exists a compact subset $K_\alpha$ of $X$ such that $X \setminus K_\alpha = \bigcup_{v \in V_\alpha} \alpha_v$ and $A \setminus K_\alpha \subseteq \bigcup_{v \in V_{\alpha}^A} \alpha_v$.

The set of all border covers of $(X,A)$ is denoted by $\operatorname{cov}_\infty(X,A)$. Let $K_\alpha^A = K_\alpha \cap A$. Then the family $\{\alpha_v \cap A\}_{v \in V_{\alpha}^A}$ is a border cover of subspace $A$. 

3
Definition 1.3. Let $\alpha, \beta \in \text{cov}_\infty(X, A)$ be two border covers of $(X, A)$ with indexing pairs $(V_\alpha, V_\alpha^A)$ and $(V_\beta, V_\beta^A)$, respectively. We say that the border cover $\beta$ is a refinement of border cover $\alpha$ if there exists a refinement projection function $p : (V_\beta, V_\beta^A) \rightarrow (V_\alpha, V_\alpha^A)$ such that for each index $v \in V_\beta$ ($v \in V_\beta^A$) $\beta_v \subset \alpha_{p(v)}$.

It is clear that $\text{cov}_\infty(X, A)$ becomes a directed set with the relation $\alpha \leq \beta$ whenever $\beta$ is a refinement of $\alpha$.

Note that for each $\alpha \in \text{cov}_\infty(X, A)$, $\alpha \leq \alpha$, and if for each $\alpha, \beta, \gamma \in \text{cov}_\infty(X, A)$, $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.

Let $\alpha, \beta \in \text{cov}_\infty(X, A)$ be two border covers with indexing pairs $(V_\alpha, V_\alpha^A)$ and $(V_\beta, V_\beta^A)$, respectively. Consider a family $\gamma = \{ \gamma_v \}_{v \in (\gamma, \gamma^A)}$, where $V_\gamma = V_\alpha \times V_\beta$ and $V_\gamma^A = V_\alpha^A \times V_\beta^A$. Let $v = (v_1, v_2)$, where $v_1 \in V_\alpha$, $v_2 \in V_\beta$. Assume that $\gamma_v = \alpha v_1 \cap \beta v_2$. The family $\gamma = \{ \gamma_v \}_{v \in (\gamma, \gamma^A)}$ is a border cover of $(X, A)$ and $\gamma \geq \alpha, \beta$.

For each border cover $\alpha \in \text{cov}_\infty(X, A)$ with indexing pair $(V_\alpha, V_\alpha^A)$, by $(X_\alpha, A_\alpha)$ denote the nerve $\alpha$, where $A_\alpha$ is the subcomplex of simplexes $s$ of complex $X_\alpha$ with vertices of $V_\alpha^A$ such that $\text{Car}_\alpha(s) \cap A \neq \emptyset$, where $\text{Car}_\alpha(s)$ is the carrier of simplex $s$ (see [E-St], pp.234). The pair $(X_\alpha, A_\alpha)$ is a simplicial pair. Moreover, any two refinement projection functions $p, q : \beta \rightarrow \alpha$ induce contiguous simplicial maps of simplicial pairs $p_\alpha^\beta, q_\alpha^\beta : (X_\beta, A_\beta) \rightarrow (X_\alpha, A_\alpha)$ (see [E-St], pp. 234-235).

Using the construction of formal homology theory of simplicial complexes ([E-St], Ch.VI) we can define the unique homomorphisms

$$p_\alpha^\beta : H_n(X_\beta, A_\beta : G) \rightarrow H_n(X_\alpha, A_\alpha : G)$$

and

$$p_\alpha^\gamma : H^n(X_\alpha, A_\alpha : G) \rightarrow H^n(X_\beta, A_\beta : G),$$

where $G$ is any abelian coefficient group.

Note that $q_\alpha^\beta = 1_{H_n(X_\alpha, A_\alpha : G)}$ and $p_\alpha^\gamma = 1_{H^n(X_\alpha, A_\alpha : G)}$. If $\gamma \geq \beta \geq \alpha$ then

$$p_\gamma^\alpha = p_\beta^\alpha \circ p_\beta^\gamma$$

and

$$p_\alpha^\gamma = p_\beta^\gamma \circ p_\gamma^\alpha.$$ 

Thus, the families

$$\{ H_n(X_\alpha, A_\alpha : G), p_\alpha^\beta, \text{cov}_\infty(X, A) \}$$

and

$$\{ H^n(X_\alpha, A_\alpha : G), p_\alpha^\beta, \text{cov}_\infty(X, A) \}$$

form inverse and direct systems of groups.

The inverse and direct limit groups of above defined inverse and direct systems are denoted by symbols

$$\hat{H}_\infty(X, A : G) = \lim_{\alpha} \{ H_n(X_\alpha, A_\alpha : G), p_\alpha^\beta, \text{cov}_\infty(X, A) \}$$
and
\[ \hat{H}_n^\infty(X, A; G) = \lim_{\to} \{ H^n(X_\alpha, A_\alpha; G), p_\alpha^*: \text{cov}_\infty(X, A) \} \]

and called \( n \)-dimensional \( \hat{\text{Č}} \)ech border homology group and \( n \)-dimensional \( \hat{\text{Č}} \)ech border cohomology group of pair \((X, A)\) with coefficients in abelian group \( G \), respectively.

According to [E-St] a border cover \( \alpha \in \text{cov}_\infty(X, A) \) indexed by \((V_\alpha, V_\alpha^A)\) is called proper if \( V_\alpha^A \) is the set of all \( v \in V_\alpha \) with \( \alpha_v \cap A \neq \emptyset \). The set of proper border covers is denoted by \( \text{Pcov}_\infty(X, A) \). Now define a function
\[ \rho : \text{cov}_\infty(X) \to \text{cov}_\infty(X, A) \]
By definition, for each border cover of \( X \) \( \alpha = \{ \alpha_v \}_{v \in V_\alpha} \)
\[ \rho(\alpha) = \{ \alpha_v \}_{v \in (V_\alpha, V')} \]
where \( V' \) is the set of \( v \in V_\alpha \) for which \( \alpha_v \cap A \neq \emptyset \). It is clear that the family \( \rho(\alpha) \) is a proper border cover and the function \( \rho : \text{cov}_\infty(X) \to \text{Pcov}_\infty(X, A) \) induced by \( \rho \) is one to one. Moreover, if \( \alpha' \leq \alpha \), then \( \rho(\alpha') \leq \rho(\alpha) \).

**Proposition 1.4.** For each pair \((X, A) \in \text{ob}(\mathcal{N}_p^2)\) the set \( \text{Pcov}_\infty(X, A) \) of proper border covers of \((X, A)\) is a cofinal subset of \( \text{cov}_\infty(X, A) \).

**Proof.** Let \( \alpha = \{ \alpha_v \}_{v \in (V_\alpha, V^A)} \) be a border cover of \((X, A)\). Assume that
\[ V' = \{ \alpha_v | \alpha_v \cap A \neq \emptyset, v \in V_\alpha^A \} \]
Consider a family \( \beta = \{ \beta_v \}_{v \in (V_\alpha, V')} \) consisting of subsets
\[ \beta_v = \alpha_v \setminus A, \quad v \in V_\alpha \setminus V' \]
and
\[ \beta_v = \alpha_v, \quad v \in V'. \]
Note that \( \beta \) is a border cover of \((X, A)\) and \( \beta \geq \alpha \).

Consequently, in definitions of \( \hat{\text{Č}} \)ech border homology and cohomology groups of pairs \((X, A) \in \text{ob}(\mathcal{N}_p^2)\) we may replace the set \( \text{cov}_\infty(X, A) \) by the subset \( \text{Pcov}_\infty(X, A) \).

Now we define, for a given proper map \( f : (X, A) \to (Y, B) \) of pairs, the induced homomorphisms
\[ f_\infty^*: \hat{H}_n^\infty(X, A; G) \to \hat{H}_n^\infty(Y, B; G) \]
and
\[ f_\infty^*: \hat{H}_n^\infty(X, A; G) \to \hat{H}_n^\infty(Y, B; G). \]

Let \( \alpha \in \text{cov}_\infty(Y, B) \) be a border cover with index set \( V_\alpha \) and \( K_\alpha = Y \setminus \bigcup_{v \in V_\alpha} \alpha_v \). Consider a family \( \alpha' = \{ f^{-1}(\alpha_v) \}_{v \in V_\alpha} \). Note that
\[ X \setminus \bigcup_{v \in V_\alpha} f^{-1}(\alpha_v) = X \setminus f^{-1}(\bigcup_{v \in V_\alpha} \alpha_v) = X \setminus f^{-1}(Y \setminus K_\alpha) = f^{-1}(K_\alpha). \]

Let \( \alpha'_v = f^{-1}(\alpha_v) \) and \( V_{\alpha'} = V_\alpha \). Since \( f \) is proper, \( f^{-1}(K_\alpha) \) is a compact subset of \( X \).

Since \( B \setminus K_\alpha \subseteq \bigcup_{v \in V_\beta^B} \alpha_v \), the subfamily \( \{ f^{-1}(\alpha_v) \mid v \in V_\alpha^B \} \) is such that \( A \setminus f^{-1}(K_\alpha) \subseteq \bigcup_{v \in V_\alpha^A} f^{-1}(\alpha_v) \). Let \( V_{\alpha'}^A = V_\alpha^B \) and \( K_{\alpha'} = f^{-1}(K_\alpha) \). Note that \( A \setminus K_{\alpha'} \subseteq \bigcup_{v \in V_{\alpha'}^A} f^{-1}(\alpha_v) \). Hence, \( \alpha' = \{ f^{-1}(\alpha_v) \}_{v \in (V_{\alpha'}, V_{\alpha})} \) is a border cover of the pair \( (X, A) \).

It is clear that \( X_{\alpha'} \) is a subcomplex of \( Y_\alpha \) and \( A_{\alpha'} \) is a subcomplex of \( B_\alpha \).

By a symbol \( f_\alpha : (X_{\alpha'}, A_{\alpha'}) \to (Y_\alpha, B_\alpha) \) denote the simplicial inclusion of \( (X_{\alpha'}, A_{\alpha'}) \) into \( (Y_\alpha, B_\alpha) \).

If \( \alpha, \beta \in \text{cov}_\infty(Y, B) \) and \( \beta \geq \alpha \), then the diagrams

\[
\begin{array}{ccc}
H_n(X_{\beta'}, A_{\beta'}; G) & \xrightarrow{f_\beta*} & H_n(X_{\beta}, A_{\beta}; G) \\
\downarrow p_{\alpha*}^{\beta'} & & \downarrow p_{\alpha*}^\beta \\
H_n(X_{\alpha'}, A_{\alpha'}; G) & \xrightarrow{f_\alpha*} & H_n(X_{\alpha}, A_{\alpha}; G)
\end{array}
\]

and

\[
\begin{array}{ccc}
H^n(X_{\alpha}, A_{\alpha}; G) & \xrightarrow{f_\alpha^*} & H^n(X_{\alpha'}, A_{\alpha'}; G) \\
\downarrow p_{\alpha*}^{\beta} & & \downarrow p_{\alpha*}^{\beta'} \\
H^n(X_{\beta}, A_{\beta}; G) & \xrightarrow{f_\beta^*} & H^n(X_{\beta'}, A_{\beta'}; G).
\end{array}
\]

commute.

Thus, for each \( \alpha \in \text{cov}_\infty(Y, B) \), the induced homomorphisms \( f_\alpha* \) and \( f_\alpha^* \) together with function \( \varphi : \text{cov}_\infty(Y, B) \to \text{cov}_\infty(X, A) \) given by formula

\[ \varphi(\alpha) = f^{-1}(\alpha), \alpha \in \text{cov}_\infty(Y, B) \]

form maps

\[ (f_\alpha*, \varphi) : \{ H_n(X_{\alpha'}, A_{\alpha'}), p_{\alpha*}^{\beta'}, \text{cov}_\infty(X, A) \} \to \{ H_n(Y_\alpha, A_{\alpha}), p_{\alpha*}^{\beta}, \text{cov}_\infty(Y, B) \} \]

6
and
\[(f^*_{\alpha}, \varphi) : \{H^n(Y_\alpha, A_\alpha), p^\beta_\alpha, \text{cov}_\infty(Y, B)\} \to \{H^n(X_\alpha', A_\alpha'), \alpha', \text{cov}_\infty(X, A)\}\].

The limits of maps \((f_{\alpha*}, \varphi)\) and \((f^*_{\alpha}, \varphi)\) are denoted by
\[f^\infty : \hat{H}_n^\infty(X, A; G) \to \hat{H}_n^\infty(Y, B; G)\]
and
\[f^\infty : \hat{H}_n^\infty(Y, B; G) \to \hat{H}_n^\infty(X, A; G)\]
and called homomorphisms induced by proper map \(f : (X, A) \to (Y, B)\).

Note that if \(f : (X, A) \to (Y, B)\) is the identity map, then the induced homomorphisms \(f^\infty : \hat{H}_n^\infty(X, A; G) \to \hat{H}_n^\infty(Y, B; G)\) and \(f^\infty : \hat{H}_n^\infty(Y, B; G) \to \hat{H}_n^\infty(X, A; G)\) are the identity homomorphisms. Furthermore, for each proper maps \(f : (X, A) \to (Y, B)\) and \(g : (Y, B) \to (Z, C)\)
\[(g \cdot f)^\infty = g^\infty \cdot f^\infty\]
and
\[(g \cdot f)^\infty = f^\infty \cdot g^\infty.\]

We have the following theorem.

**Theorem 1.5.** There exist the covariant and contravariant functors
\[\hat{\mathcal{H}}_n^\infty(-, -; G) : \mathcal{A}_p^2 \to \mathcal{A}b\]
and
\[\hat{\mathcal{H}}_n^\infty(-, -; G) : \mathcal{A}_p^2 \to \mathcal{A}b\]
given by formulas
\[\hat{\mathcal{H}}_n^\infty(-, -; G)(X, A) = \hat{H}_n^\infty(X, A; G), \quad (X, A) \in \text{ob} (\mathcal{A}_p^2)\]
\[\hat{\mathcal{H}}_n^\infty(-, -; G)(f) = f^\infty, \quad f \in \text{Mor} (\mathcal{A}_p^2)((X, A), (Y, B))\]
and
\[\hat{\mathcal{H}}_n^\infty(-, -; G)(X, A) = \hat{H}_n^\infty(X, A; G), \quad (X, A) \in \text{ob} (\mathcal{A}_p^2)\]
\[\hat{\mathcal{H}}_n^\infty(-, -; G)(f) = f^\infty, \quad f \in \text{Mor} (\mathcal{A}_p^2)((X, A), (Y, B)).\]

**Proof.** The proof follows from above discussion.

We will call the functors \(\hat{\mathcal{H}}_n^\infty(-, -; G)\) and \(\hat{\mathcal{H}}_n^\infty(-, -; G)\) Čech border homology and cohomology functors, respectively.

Now we define boundary and coboundary homomorphisms
\[\partial_n^\infty : \hat{H}_n^\infty(X, A; G) \to \hat{H}_{n-1}^\infty(A; G)\]
and
\[\delta_n^\infty : \hat{H}_{n-1}^\infty(A; G) \to \hat{H}_n^\infty(X, A; G).\]
Let \((X, A) \in \text{ob} (\mathbb{I}_p)\), \(\beta, \alpha \in \text{cov}_\infty (X, A)\) and \(\beta \geq \alpha\). The refinement projection functions induce the unique homomorphisms \(p^\beta_* : H_n (A_\beta; G) \to H_n (A_\alpha; G)\) and \(p^\beta_* : H^n (A_\alpha; G) \to H^n (A_\beta; G)\), which form inverse systems
\[
\{H_n (A_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\} \text{ and } \{H_n (X_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\}
\]
and direct systems
\[
\{H^n (A_\alpha; G), p^\beta_* , \text{cov}_\infty (X, A)\} \text{ and } \{H^n (X_\alpha; G), p^\beta_* , \text{cov}_\infty (X, A)\}.
\]

Let
\[
\hat{H}_n^\infty (A ; G)_{(X, A)} = \lim_{\alpha \to \infty} \{H_n (A_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\},
\]
\[
\hat{H}_n^\infty (X; G)_{(X, A)} = \lim_{\alpha \to \infty} \{H_n (X_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\},
\]
\[
\hat{H}_n^\infty (A ; G) (X, A) = \lim_{\alpha \to \infty} \{H^n (A_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\},
\]
\[
\hat{H}_n^\infty (X; G) (X, A) = \lim_{\alpha \to \infty} \{H^n (X_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\}.
\]

Our main aim is to show that the groups \(\hat{H}_n^\infty (A ; G) (X, A)\), \(\hat{H}_n^\infty (A ; G) (X, A)\) and \(\hat{H}_n^\infty (X; G) (X, A)\) are isomorphical groups.

Next we define a function \(\varphi : \text{cov}_\infty (X, A) \to \text{cov}_\infty (A, \emptyset)\). Let \(\alpha = \{\alpha_v\}_{v \in (V, V^A)} \in \text{cov}_\infty (X, A)\). Assume that \((\varphi (\alpha))_v = \alpha_v \cap A\) for \(v \in V^A\). We have defined the border cover \(\varphi (\alpha) \in \text{cov}_\infty (A, \emptyset)\) indexed by pair \((V, \emptyset)\).

Let \(K_\alpha = X \setminus \bigcup_{v \in V^A} \alpha_v\). Note that
\[
A \setminus (K_\alpha \cap A) = \bigcup_{v \in V^A} (\alpha_v \cap A) = \bigcup_{v \in V^A} (\varphi (\alpha))_v.
\]

It is clear that \(K_\alpha \cap A\) is a compact subset of the subspace \(A\). Thus, \(\varphi (\alpha) \in \text{cov}_\infty (A, \emptyset)\). The defined function is an order preserving function.

It is easy to show that the image of function \(\varphi\) is a cofinal subset of set \(\text{cov}_\infty (A, \emptyset)\). Note that \(A_\alpha = A_{\varphi (\alpha)}\). By \(\varphi : A_{\varphi (\alpha)} \to A_\alpha\) denote this simplicial isomorphism. Hence, the family of pairs \((\varphi_\alpha, \varphi)\) induces a map of inverse systems and direct systems
\[
(\varphi_\alpha, \varphi) : \{H_n (A_\alpha; G), p^\beta_*, \text{cov}_\infty (A, \emptyset)\} \to \{H_n (A_\alpha; G), p^\beta_*, \text{cov}_\infty (X, A)\}
\]
and
\[
(\varphi_\alpha^*, \varphi) : \{H^n (A_\alpha; G), p^\beta_* , \text{cov}_\infty (X, A)\} \to \{H_n (A_\alpha; G), p^\beta_* , \text{cov}_\infty (A, \emptyset)\}.
\]

Let \(\Phi_n = \lim (\varphi_\alpha, \varphi)\) and \(\Phi_n^* = \lim (\varphi_\alpha^*, \varphi)\). Since all homomorphisms \(\varphi_\alpha\) and \(\varphi_\alpha^*\) are isomorphisms, the limit homomorphisms
\[
\Phi_n : H_n^\infty (A ; G) \to H_n^\infty (A ; G)_{(X, A)}
\]

8
and

\[ \Phi^n : \hat{H}_n^\infty(A;G)^{(X,A)} \to \hat{H}_n^\infty(A;G) \]

are isomorphisms.

Let us also define a function \( \psi : \text{cov}_\infty(X,A) \to \text{cov}_\infty(X,\emptyset) \). For each \( \alpha = \{ \alpha_v \}_{v \in (V_\alpha, V_\alpha^A)} \in \text{cov}_\infty(X,A) \) assume that \( (\psi(\alpha))_v = \alpha_v, v \in V_\alpha. \) The family \( \psi(\alpha) \) is indexed by \( (V_\alpha, \emptyset) \) and \( \psi(\alpha) \in \text{cov}_\infty(X,\emptyset). \)

Note that \( X_\alpha = X_{\psi(\alpha)}. \) Let \( \psi_\alpha : X_{\psi(\alpha)} \to X_\alpha \) be a simplicial isomorphism. The family of pairs \( (\psi_\alpha, \psi) \) induce the maps of inverse and direct systems

\[ (\psi_\alpha, \psi) : \{ H_n(X_\alpha; G), p_{\alpha^*}, \text{cov}_\infty(X, \emptyset) \} \to \{ H_n(X_\alpha; G), p_{\alpha^*}, \text{cov}_\infty(X, A) \} \]

and

\[ (\psi_\alpha^*, \psi) : \{ H^n(X_\alpha; G), p_{\alpha^*}, \text{cov}_\infty(X, \emptyset) \} \to \{ H^n(X_\alpha; G), p_{\alpha^*}, \text{cov}_\infty(X, A) \}. \]

Let \( \Psi_n = \lim \downarrow (\psi_\alpha, \psi) \) and \( \Psi^n = \lim \downarrow (\psi_\alpha^*, \psi). \) Since each \( \psi_\alpha^* \) and \( \psi_\alpha \) are isomorphisms, the induced limit homomorphisms

\[ \Psi_n : \hat{H}_n^\infty(X; G) \to \hat{H}_n^\infty(X; G)^{(X,A)} \]

and

\[ \Psi^n : \hat{H}_n^\infty(X; G)^{(X,A)} \to \hat{H}_n^\infty(X; G) \]

are isomorphisms.

There exist the limit sequences

\[ \cdots \to \hat{H}_n^\infty(X; A; G) \xrightarrow{j_n^\infty} \hat{H}_n^\infty(X; G)^{(X,A)} \xrightarrow{i_n^\infty} \hat{H}_n^\infty(A; G)^{(X,A)} \xrightarrow{\partial_{n+1}^\infty} \hat{H}_{n+1}^\infty(X; A; G) \to \cdots \]

and

\[ \cdots \to \hat{H}_n^\infty(X; A; G) \xrightarrow{j_n^\infty} \hat{H}_n^\infty(X; G)^{(X,A)} \xrightarrow{i_n^\infty} \hat{H}_n^\infty(A; G)^{(X,A)} \xrightarrow{\delta_n^\infty} \hat{H}_{n+1}^\infty(X; A; G) \to \cdots \]

generated by the families consisting of homology and cohomology sequences of simplicial pairs \( (X_\alpha, A_\alpha), \alpha \in \text{cov}_\infty(X, A) \), respectively.

Consider the diagrams

\[ \hat{H}_n^\infty(X; A; G) \xrightarrow{\partial_n^\infty} \hat{H}_{n-1}^\infty(A; G)^{(X,A)} \xrightarrow{\Phi_{n-1}} \hat{H}_n^\infty(A; G) \]

and

\[ \hat{H}_n^\infty(A; G) \xrightarrow{\Psi_n} \hat{H}_n^\infty(A; G)^{(X,A)} \xrightarrow{\delta_n^\infty} \hat{H}_{n+1}^\infty(X; A; G) \]
and define the boundary homomorphism of Čech border homology groups and coboundary homomorphism of Čech border cohomology groups as compositions
\[ \partial_n^\infty = (\Phi_{n-1})^{-1} \cdot \partial_n^\infty \]
and
\[ \delta_n^\infty = \delta_n^\infty \cdot (\Psi^n)^{-1} . \]

In this way we arrive to the following theorems.

**Theorem 1.6.** Let \( f : (X, A) \to (Y, B) \) be a proper map. Then hold the following equalities
\[ (f|_A)^\infty \cdot \partial_n^\infty = \partial_n^\infty \cdot f^\infty \]
and
\[ \delta_n^\infty (f|_A)^\infty = f^\infty \cdot \delta_n^\infty . \]

**Proof.** The desired equalities follow from the commutativity of the diagrams.

\[
\begin{array}{ccc}
\hat{H}_n(X, A; G) & \xrightarrow{\partial_n^\infty} & \hat{H}_{n-1}(A; G)_{(X, A)} & \xleftarrow{\Phi_{n-1}} & \hat{H}_{n-1}(A; G) \\
\downarrow \Phi_{\infty} & & \downarrow \Phi_{\infty} \downarrow (f|_A)^\infty & & \downarrow \Phi_{\infty} \\
\hat{H}_n(Y, B; G) & \xrightarrow{\partial_n^\infty} & \hat{H}_{n-1}(B; G)_{(Y, B)} & \xleftarrow{\Phi_{n-1}} & \hat{H}_{n-1}(B; G)
\end{array}
\]

and

\[
\begin{array}{ccc}
\hat{H}_{n-1}(B; G) & \xleftarrow{\Phi_{n-1}^{-1}} & \hat{H}_{n-1}(B; G)_{(Y, B)} & \xrightarrow{\delta_n^\infty} & \hat{H}_n^\infty(Y, B; G) \\
\downarrow (f|_A)^\infty & & \downarrow (f|_A)^\infty \downarrow \delta_n^\infty & & \downarrow f^\infty \\
\hat{H}_{n-1}(A; G) & \xleftarrow{\Phi_{n-1}} & \hat{H}_{n-1}(A; G)_{(X, A)} & \xrightarrow{\delta_n^\infty} & \hat{H}_n^\infty(X, A; G),
\end{array}
\]

where \((f|_A)^\infty\) and \((f|_A)^\infty\) are defined as the appropriate limit homomorphisms.

Let \( i : A \to X \) and \( j : X \to (X, A) \) be the inclusion maps.

**Theorem 1.7.** Let \((X, A) \in ob(\mathcal{A}_p^2)\). Then the Čech border cohomology sequence

\[
\cdots \to \hat{H}_{n-1}^\infty(A; G) \xrightarrow{\delta_n^\infty} \hat{H}_n^\infty(A, G) \xrightarrow{j^\infty} \hat{H}_n^\infty(X, A; G) \xrightarrow{i^\infty} \hat{H}_n^\infty(X, G) \xrightarrow{i^\infty} \hat{H}_n^\infty(A; G) \to \cdots
\]
is exact while the Čech border homology sequence

\[ \cdots \rightarrow \hat{H}_{n-1}^\infty(A;G) \rightarrow \hat{H}_n^\infty(X,A;G) \rightarrow \hat{H}_n^\infty(X,G) \rightarrow \hat{H}_n^\infty(A;G) \rightarrow \cdots \]

is partially exact.

Proof. One can prove this theorem analogously to the corresponding theorem of the classical Čech theory [E-St]. \( \square \)

**Theorem 1.8.** Let \((X,A) \in ob(\mathcal{M}_p^2)\) and \(G\) be an abelian group. If \(U\) is open in \(X\) and \(\bar{U} \subset \text{int}A\), then the inclusion map \(i : (X \setminus U, A \setminus U) \rightarrow (X,A)\) induces isomorphisms

\[ i_*^\infty : \hat{H}_n^\infty(X \setminus U, A \setminus U) \rightarrow \hat{H}_n^\infty(X,A;G) \]

and

\[ j^* \infty : \hat{H}_n^\infty(X,A;G) \rightarrow \hat{H}_n^\infty(X \setminus U, A \setminus U) \]

Proof. Let \(\text{cov}'_\infty(X,A)\) be the subset of \(\text{cov}(X,A)\) consisting of all covers \(\alpha = \{\alpha_v\}_{v \in (\alpha, \beta)}\) with property:

- if \(\alpha_v \cap U \neq \emptyset\), then \(v \in V_{\alpha}^A\) and \(\alpha_v \subset A\).

First we prove that \(\text{cov}'_\infty(X,A)\) is cofinal in \(\text{cov}_\infty(X,A)\). Let \(\alpha = \{\alpha_v\}_{v \in (\alpha, \beta)}\) be a border cover of \((X,A)\) with enclosure \(K_\alpha\). Let \(V'\) be a set such that \(V' \cap V_{\alpha} = \emptyset\) and there exists a bijective function between \(V_{\alpha}^A\) and \(V'\). Let \(v \in V_{\alpha}^A\). The correspondence element of \(v\) in \(V'\) denote by \(v'\). Define the border cover \(\gamma = \{\gamma_v\}_{v \in (\alpha_\gamma \cup V', V_{\alpha}^A \cup V')} \in \text{cov}_\infty(X,A)\). Let

\[ \gamma_v = \alpha_v \setminus \bar{U}, \quad v \in V_{\alpha} \]

and

\[ \gamma_{v'} = \alpha_v \cap \text{int}A, \quad v' \in V'. \]

It is clear that \(\gamma\) is a border cover of \((X,A)\) with enclosure \(K_\alpha\) and \(\gamma \geq \alpha\).

We prove that \(i^{-1}(\text{cov}_\infty(X,A))\) is cofinal in \(\text{cov}_\infty(X \setminus U, A \setminus U)\). Let \(\beta = \{\beta_v\}_{v \in (\alpha, \beta)}\) be a border cover of \((X \setminus U, A \setminus U)\) with enclosure \(K_\beta\). Define a border cover \(\alpha = \{\alpha_v\}_{v \in (\alpha, \beta)} \in \text{cov}_\infty(X,A)\).

Let

\[ \alpha_v = \beta_v \cup U. \]

The family \(\alpha = \{\alpha_v\}_{v \in (\alpha, \beta)}\) is a border cover of \((X,A)\) with enclosure \(K_\beta\).

Let \(\gamma \in \text{cov}'_\infty(X,A)\) be a border cover such that \(\gamma \geq \alpha\). It is clear that \(i^{-1}(\gamma) \geq \beta = i^{-1}(\alpha)\).

Note that

\[ X_\alpha = (X \setminus U)_{\beta} \cup A_\alpha \]

and

\[ (A \setminus U)_{\beta} = (X \setminus U)_{\beta} \cap A_\alpha. \]

11
As in [E-St] we can prove that there exist isomorphisms
\[ i_\alpha^*: H_n((X \setminus U)_\beta, (A \setminus U)_\beta; G) \rightarrow H_n(X_\alpha, A_\alpha; G) \]
and
\[ i_\alpha^* : H^n(X_\alpha, A_\alpha; G) \rightarrow H^n((X \setminus U)_\beta, (A \setminus U)_\beta; G). \]

The conclusion of the theorem is a consequence of these isomorphisms.

**Theorem 1.9.** If \( X \) is a compact space, then for each \( n \neq 0 \),

\[ \check{H}_n^\infty(X; G) = 0 = \check{H}_n^n(X; G) \]

and

\[ \check{H}_0^\infty(X; G) = G = \check{H}_0^n(X; G). \]

**Proof.** Let \( \alpha \in \text{cov}_\infty(X) \) be the border cover of \( X \) consisting of empty set. It is clear that \( \alpha \) is a refinement of any border cover of \( X \). The set \( \{\alpha\} \) is a cofinal subset of \( \text{cov}_\infty(X) \). Consider the inverse system \( \{H_n(X_\alpha; G), p^{\alpha*}_\alpha, \{\alpha\}\} \) and direct system \( \{H^n(X_\alpha; G), p^{\alpha**}_\alpha, \{\alpha\}\} \). We have

\[
\lim_{\leftarrow} \{H_n(X_\alpha; G), p^{\alpha*}_\alpha, \{\alpha\}\} = \check{H}_n^\infty(X; G) = H_n(X_\alpha; G)
\]

and

\[
\lim_{\rightarrow} \{H^n(X_\alpha; G), p^{\alpha**}_\alpha, \{\alpha\}\} = \check{H}_n^n(X; G) = H^n(X_\alpha; G).
\]

The nerve \( X_\alpha \) consists of one vertex. Using the methods of proofs of results VI.3.8 and VI.4.3 of [E-St] we than conclude that

\[ \check{H}_n^\infty(X; G) = 0 = \check{H}_n^n(X; G) \]

and

\[ \check{H}_0^\infty(X; G) = G = \check{H}_0^n(X; G). \]

Thus, Čech border homology (cohomology) functors \( \check{H}_n^\infty(-, -; G) : \mathcal{A}_p^2 \rightarrow \mathcal{G} \) satisfy the Steenrod-Eilenberg type axioms (cf.[E-St]): axiom of natural transformation, axiom of partially exactness (axiom of exactness), axiom of excision and axiom of dimension; but they do not satisfy the proper homotopy axiom.

The above obtained results yield the next theorem.

**Theorem 1.10.** Let \((X, A, B)\) be a triple of normal space \( X \) and its closed subsets \( A \) and \( B \) with \( B \subset A \). Then the Čech border homology sequence

\[
\cdots \leftarrow \check{H}_{n-1}^\infty(A, B; G) \leftarrow \check{H}_n^\infty(X, A; G) \leftarrow \check{H}_n^\infty(X, B; G) \leftarrow \check{H}_n^\infty(A, B; G) \leftarrow \cdots
\]
and the Čech border cohomology sequence

\[ \cdots \to \hat{H}^{n-1}_\infty(A, B; G) \xrightarrow{\delta^n_\infty} \hat{H}^n_\infty(X, A; G) \xrightarrow{j^n_\infty} \hat{H}^n_\infty(X, B; G) \xrightarrow{i^n_\infty} \hat{H}^n_\infty(A, B; G) \to \cdots \]

are partially exact and exact, respectively. Here \( \partial^n_\infty = j^{n-1}_\infty \cdot \delta^n_\infty \), \( \delta^n_\infty = \delta^n_\infty \cdot j^n_\infty \) and \( j^n_\infty \) and \( i^n_\infty \) are the homomorphisms induced by the inclusion maps \( j^\prime : A \to (A, B) \), \( i^\prime : (A, B) \to (X, B) \) and \( j : (X, B) \to (A, B) \).

Proof. The proof is similar to the proof of the corresponding Theorems 10.2 and 10.2c of [E-St] (see Ch. I, §10).

2 On some applications of Čech border homology and cohomology groups

Now we are mainly interested in the following problem: how to characterize the Čech border homology and cohomology groups, coefficients of cyclicity, and cohomological dimensions of remainders of Stone-Čech compactifications of spaces.

Our main result about the connection between Čech (co)homology groups of remainders and Čech border (co)homology groups of spaces is.

**Theorem 2.1.** Let \( (X, A) \in \text{ob}(\mathcal{M}_2^\beta) \) and let \( (\beta X, \beta A) \) be the pair of Stone-Čech compactifications of \( X \) and \( A \). Then

\[
\hat{H}^n_\beta(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^n_\infty(X, A; G)
\]

and

\[
\check{H}^n_\beta(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^n_\infty(X, A; G).
\]

**Proof.** Let \( \alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})} \) and \( \alpha' = \{\alpha'_w\}_{w \in (W_\alpha, W_{\alpha'}^{\beta A \setminus A})} \) be the closed covers of pairs \( (\beta X \setminus X, \beta A \setminus A) \) and \( \alpha \geq \alpha' \). By Lemma 4 of [Sm4] there exist open swellings \( \beta_1 = \{\beta_1^v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})} \) and \( \beta' = \{\beta'_w\}_{w \in (W_\alpha, W_{\alpha'}^{\beta A \setminus A})} \) of \( \alpha \) and \( \alpha' \) in \( \beta X \), respectively. Assume that \( \alpha_v \subseteq \alpha'_w \), \( k = 1, 2, \cdots, m_v \). Let

\[
\beta_v = \beta_1^v \cap \left( \bigcap_{k=1}^{m_v} \beta'_w \right), \quad v \in V_\alpha.
\]

Note that \( \alpha_v \subseteq \beta_v \subseteq \beta_1^v \) for each \( v \in V_\alpha \). It is clear that \( \beta = \{\beta_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})} \) is a swelling of \( \alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})} \) and \( \beta \geq \beta' \).

The swelling in \( \beta X \) of closed cover \( \alpha \) of \( (\beta X \setminus X, \beta A \setminus A) \) is denoted by \( s(\alpha) \). Let \( S \) be the set of all swellings of such kind.

Now define an order \( \geq \) in \( S \). By definition,

\[
s(\alpha') \geq s(\alpha) \iff s(\alpha') \geq s(\alpha) \land \alpha' \geq \alpha.
\]
It is clear that $S$ is directed by $\geq$. Let $((\beta X \setminus \alpha X, (\beta A \setminus A)_{\alpha})$ be the nerve of $s(\alpha) \in S$ and $p_{s(\alpha),\alpha}^{s(\alpha)\prime}$ be the projection simplicial map induced by the refinement $\alpha' \geq \alpha$. Consider an inverse system

$$\{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha),\alpha}^{s(\alpha)\prime}, S\}$$

and a direct system

$$\{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha),\alpha}^{s(\alpha)'}, S\}.$$

Let $\varphi : S \to \text{cov}^\uparrow((\beta X \setminus X, (\beta A \setminus A)_{\alpha})$ be the function in the set of closed finite covers of pair $(\beta X \setminus X, (\beta A \setminus A)$ given by formula

$$\varphi(s(\alpha)) = \alpha, \quad s(\alpha) \in S.$$  

Note that $\varphi$ is an increasing function and

$$\varphi(S) = \text{cov}^\uparrow((\beta X \setminus X, (\beta A \setminus A).$$

For each index $s(\alpha) \in S$, we have

$$H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_\alpha; G)$$

and

$$H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_\alpha; G).$$

It is known that for normal spaces the Čech (co)homology groups based on finite open covers and on finite closed covers are isomorphic. By Theorems 3.14 and 4.13 of ([E-St], Ch.VIII) we have

$$\tilde{H}_n((\beta X \setminus X, (\beta A \setminus A)_{s(\alpha)}; G) \approx \lim_{\leftarrow} \{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha),\alpha}^{s(\alpha)\prime}, S\} \quad (1)$$

and

$$\tilde{H}^n((\beta X \setminus X, (\beta A \setminus A)_{s(\alpha)}; G) \approx \lim_{\rightarrow} \{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha),\alpha}^{s(\alpha)'}, S\} \quad (2)$$

For each swelling $s(\alpha) = \{s(\alpha)_v\}_{v \in (\beta X, V_{\alpha}^{\beta A \setminus A})} \in S$, the family

$$s(\alpha) \wedge X = \{s(\alpha)_v \cap X\}_{v \in (\beta X, v_{\alpha}^{\beta A \setminus A})}$$

is a border cover of $(X, A)$.

Let $\psi : S \to \text{cov}_\infty(X, A)$ be the function defined by formula

$$\psi(s(\alpha)) = s(\alpha) \wedge X, \quad s(\alpha) \in S.$$  

The function $\psi$ increases and $\psi(S)$ is a cofinal subset of $\text{cov}_\infty(X, A)$. Note that the correspondence

$$((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}) \to (X_{s(\alpha) \wedge X}, A_{s(\alpha) \wedge X}) : s(\alpha)_v \to s(\alpha)_v \cap X, \quad v \in V_\alpha$$

is a border cover of $(X, A)$.
induces an isomorphism of pairs of simplicial complexes. Thus, for each \( s(\alpha) \in S \), we have the isomorphisms

\[
H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n(X_{s(\alpha) \wedge X}, A_{s(\alpha) \wedge X}; G)
\]

and

\[
H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n(X_{s(\alpha) \wedge X}, A_{s(\alpha) \wedge X}; G).
\]

By Theorems 3.15 and 4.13 of ([E-St], Ch.VIII), we have

\[
\hat{H}_n^\infty(X, A; G) = \lim_{\longrightarrow} \{ H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)^*}, S \} \tag{3}
\]

and

\[
\hat{H}_n^\infty(X, A; G) = \lim_{\longrightarrow} \{ H_n((\beta X \setminus X, \beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)^*}, S \}. \tag{4}
\]

From (1), (2), (3) and (4) it follows that

\[
\hat{H}_n^\infty(X, A; G) = \hat{H}_n^f(\beta X \setminus X, \beta A \setminus A; G)
\]

and

\[
\hat{H}_n^\infty(X, A; G) = \hat{H}_n^f(\beta X \setminus X, \beta A \setminus A; G).
\]

The cohomological coefficient of cyclicity \( \eta_G(X, A) \) of pair \((X, A)\) was defined by S.Novak [N] and M.F.Bokstein [Bo]. Dually one can define the homological coefficient of cyclicity \( \eta^G(X, A) \) of pair \((X, A)\).

Now give the following definitions and results.

**Definition 2.2.** Let \( G \) be an abelian group and \( n \) nonnegative integer. A border (co)homological coefficient of cyclicity of pair \((X, A)\) \( \in \text{ob}(\mathcal{A}_p^2) \) with respect to \( G \) denoted by \( \eta_G(X, A) \) is \( n \), if \((\hat{H}_n^\infty(X, A; G) = 0) \) \( \hat{H}_m^\infty(X, A; G) = 0 \) for all \( m > n \) and \((\hat{H}_n^\infty(X, A; G) \neq 0) \) \( \hat{H}_n^\infty(X, A; G) \neq 0 \).

Finally, \( \eta_G(X, A) = \infty \) \( \eta_G(X, A) = \infty \) if for every \( m \) there is \( n \geq m \) with \((\hat{H}_m^\infty(X, A; G) \neq 0) \) \( \hat{H}_m^\infty(X, A; G) \neq 0 \).

**Theorem 2.3.** For each pair \((X, A)\) \( \in \text{ob}(\mathcal{A}_p^2) \),

\[
\eta_G(X, A) = \eta_G(\beta X \setminus X, \beta A \setminus A)
\]

and

\[
\eta^G(X, A) = \eta^G(\beta X \setminus X, \beta A \setminus A).
\]

**Proof.** This is an immediate consequence of Theorem 2.1. Indeed, let \( \eta_G(\beta X \setminus X, \beta A \setminus A) = n \). Then for each \( m > n \), \( \hat{H}_m^n(\beta X \setminus X, \beta A \setminus A; G) = 0 \) and \( \hat{H}_m^n(\beta X \setminus X, \beta A \setminus A; G) \neq 0 \). From the isomorphism

\[
\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_f^n(X, A; G)
\]

it follows that \( \hat{H}_m^n(X, A; G) = 0 \) for each \( m > n \), and \( \hat{H}_m^n(X, A; G) \neq 0 \). Thus, \( \eta_G(X, A) = n = \eta_G(\beta X \setminus X, \beta A \setminus A) \).

Analogously we can prove equality \( \eta^G(X, A) = \eta^G(\beta X \setminus X, \beta A \setminus A) \). \[\square\]
The theory of cohomological dimension has become an important branch of dimension theory since A. Dranishnikov solved P.S. Alexandrov’s problem and developed the theory of extension dimension ([D], [D-Dy]).

Our next aim is to study some questions of theory of cohomological dimension. In particular, we now give a description of cohomological dimension of remainder of Stone-Čech compactification of metrizable space.

Following Y. Kodama (see the appendix of [N]) and T. Miyata [Mi] we give the following definition.

**Definition 2.4.** The border small cohomological dimension $d^f_{\infty}(X; G)$ of normal space $X$ with respect to group $G$ is defined to be the smallest integer $n$ such that, whenever $m \geq n$ and $A$ is closed in $X$, the homomorphism $i_{A, \infty}^* : \hat{H}^m_{\infty}(X; G) \to \hat{H}^m_{\infty}(A; G)$ induced by the inclusion $i : A \to X$ is an epimorphism.

The border small cohomological dimension of $X$ with coefficient group $G$ is a function $d^f_{\infty} : \mathcal{N} \to \mathbb{N} \cup \{0, +\infty\} : X \to n$, where $d^f_{\infty}(X; G) = n$ and $\mathcal{N}$ is the set of all positive integers.

**Theorem 2.5.** Let $X$ be a metrizable space. Then the following equality

$$d^f_{\infty}(X; G) = d_f(\beta X \setminus X; G)$$

holds, where $d_f(\beta X \setminus X; G)$ is the small cohomological dimension of $\beta X \setminus X$ (see [N], p.199).

**Proof.** Let $A$ be a closed subset of $X$. Assume that $d_f(\beta X \setminus X; G) = n$. Then for each $m \geq n$ the homomorphism $i_{A, X, \infty}^* : \hat{H}^m_f(\beta X \setminus X; G) \to \hat{H}^m_f(\beta A \setminus A; G)$ is an epimorphism. Consider the following commutative diagram

$$\begin{array}{ccc}
\hat{H}^m_{\infty}(X; G) & \cong & \hat{H}^m_f(\beta X \setminus X; G) \\
\downarrow & & \downarrow \\
\hat{H}^m_{\infty}(A; G) & \cong & \hat{H}^m_f(\beta A \setminus A; G). \\
\end{array}$$

(5)

It is clear that the homomorphism

$$i_{A, \infty}^* : \hat{H}^m_f(X; G) \to \hat{H}^m_f(A; G)$$

also is an epimorphism for each $m \geq n$. Thus,

$$d^f_{\infty}(X; G) \leq n = d_f(\beta X \setminus X; G).$$

(6)

Let $d^f_{\infty}(X; G) = n$. To see the reverse inequality, let $B$ be a closed subset of $\beta X \setminus X$ and let $m \geq n$.

Consider an open in $\beta X \setminus X$ neighbourhood $U$ of $B$. There exists an open neighbourhood $V$ of $B$ in $\beta X \setminus X$ such that $\bar{V}\setminus X \subset U$. By Lemma 5 of
Let $\alpha \in H^m(N(\hat{\alpha}); G)$. There is a closed finite cover $\tilde{\alpha}$ of $\alpha$ in $X$ such that $\beta \tilde{\alpha} \cap X$ represents the element $\hat{\beta}$. Using Lemma 4 of [Sm4] we can find an open set $\mathcal{W}$ in $\beta \tilde{\alpha}$ such that $\mathcal{W} \supseteq \beta \tilde{\alpha} \cap X$. It is clear that $\beta \mathcal{W} \cap (\beta \tilde{\alpha} \cap X) \subseteq U$. Let $A = \beta \mathcal{W} \cap X$. It is clear that $\beta A = \beta \mathcal{W}$. We have

$$W^{\beta X} = W \cap X^{\beta X} \subseteq W^{\beta X} \cap X^{\beta X} \subseteq W^{\beta X} = \beta \mathcal{W}.$$ 

Consequently, $\beta A = W^{\beta X} \cap X^{\beta X} = \beta \mathcal{W}$. This shows that

$$B \subset \beta A \cap (\beta \tilde{\alpha} \cap X) \subset U.$$ 

Hence, we have

$$B \cap \beta A \cap X \subset U.$$ 

Thus, for each closed set $B$ of $\beta \tilde{\alpha} \cap X$ and its open neighbourhood $U$ in $\beta \tilde{\alpha} \cap X$, there exists a closed subset $A$ in $X$ such that $B \subset \beta A \cap X \subset U$.

Let $a \in H^3(B; G)$. There is a closed finite cover $\alpha$ of $B$ such that an element $a_\alpha \in H^m(N(\alpha); G)$ represents the element $a$.

Using Lemma 4 of [Sm4] we can find the swellings $\hat{\alpha}$ and $\tilde{\alpha}$ of $\alpha$ in $B$ and $\beta \tilde{\alpha} \cap X$, respectively, such that $\hat{\alpha}_\beta = \tilde{\alpha}$. Let $U$ be the union of elements of $\hat{\alpha}$. There is a closed set $A$ of $X$ with $B \subset \beta A \cap X \subset U$. The nerves $N(\alpha)$, $N(\hat{\alpha})$ and $N(\tilde{\alpha})$ are isomorphic. We can assume that

$$H^m(N(\alpha); G) = H^n(N(\hat{\alpha}); G) = H^n(N(\tilde{\alpha}) \cap (\beta \tilde{\alpha} \cap X); G) = H^n(N(\beta A \cap X); G).$$

Hence, the element $a_\alpha$ also belongs to the group $H^m(N(\beta A \cap X); G)$. Consequently, it represents some element $b$ of $H^m(\beta A \cap X; G)$. The inclusion $i_A : A \to X$ induces an epimorphism $i^*_A : \hat{H}^m(X; G) \to \hat{H}^m(A; G)$. From diagram (5) it follows that the homomorphism $i^*_{\beta A \cap X} : \hat{H}^m(\beta \tilde{\alpha} \cap X; G) \to \hat{H}^m(\beta A \cap X; G)$ is an epimorphism. Consequently, there is an element $c \in \hat{H}^m(\beta \tilde{\alpha} \cap X; G)$ such that $i^*_{\beta A \cap X}(c) = b$. The homomorphism $j^*_B : \hat{H}^m(\beta A \cap X; G) \to \hat{H}^m(B; G)$ induced by the inclusion $j_B : B \to \beta A \cap X$ satisfies the condition $j^*_B(b) = a$. From equality $i_{\beta A \cap X} = j_B$ it follows that $i^*_B(c) = a$.

Thus the inclusion $i_B : B \to \beta \tilde{\alpha} \cap X$ also induces an epimorphism $i^*_B : \hat{H}^m(\beta \tilde{\alpha} \cap X; G) \to \hat{H}^m(B; G)$. Hence, we obtain

$$d_f(\beta X \cap X; G) \leq \alpha = d^f(\beta X \cap X; G). \quad (6)$$

From the inequalities (6) and (7) it follows that

$$d^f(\beta X \cap X; G) = d_f(\beta X \cap X; G).$$

\[\square\]

**Theorem 2.6.** Let $A$ be a closed subspace of a normal space $X$. Then

$$d^f(X; G) \leq d^f(A; G).$$

$$d^f(X; G) \leq d^f(A; G).$$
Proof. Let $B$ be an arbitrary closed subset of $A$ and $j_B : B \to A$, $i_A : A \to X$ and $k_B : B \to X$ be the inclusion maps. Note that $k_B = i_A \cdot j_B$. The induced homomorphisms $k_{B,\infty} : H^m_\infty(X;G) \to H^m_\infty(B;G)$, $i_{A,\infty} : H^m_\infty(X;G) \to H^m_\infty(A;G)$ and $j_{B,\infty} : H^m_\infty(A;G) \to H^m_\infty(B;G)$ satisfy the equality $k_{B,\infty} = j_{B,\infty} \cdot i_{A,\infty}$.

Let $n = d_f^\infty(X;G)$. For each $m \geq n$, the homomorphisms $k_{B,\infty} : H^m_\infty(X;G) \to H^m_\infty(B;G)$ and $i_{A,\infty} : H^m_\infty(X;G) \to H^m_\infty(A;G)$ are epimorphisms. Hence, the homomorphism $j_{B,\infty} : H^m_\infty(A;G) \to H^m_\infty(B;G)$ is also an epimorphism for each $m \geq n$. Thus, $d_f^\infty(A;G) \leq n = d_f^\infty(X;G)$.

Corollary 2.7. For each closed subspace $A$ of a metrizable space $X$,

$$d_f^\infty(A;G) \leq d_f(\beta X \setminus X;G).$$

Definition 2.8. The border large cohomological dimension $D_f^\infty(X;G)$ of normal space $X$ with respect to group $G$ is defined to be the largest integer $n$ such that $H^m_\infty(X,A;G) \neq 0$ for some closed set $A$ of $X$.

The border large cohomological dimension of $X$ with coefficient group $G$ is a function $D_f^\infty : \mathcal{N} \to \mathbb{N} \cup \{0, +\infty\} : X \to n$, where $D_f^\infty(X;G) = n$ and $N$ is the set of all positive integers.

Theorem 2.9. For each metrizable space $X$, one has

$$D_f^\infty(X;G) = D_f(\beta X \setminus X;G),$$

where $D_f(\beta X \setminus X;G)$ is the large cohomological dimension of $\beta X \setminus X$ (see [N], p.199).

Proof. Let $D_f(\beta X \setminus X;G) = n$. Consider an arbitrary closed subspace $A$ of $X$. The remainder $\beta A \setminus A$ is a closed subset of $\beta X \setminus X$. By the assumption, we have $H^m(\beta X \setminus X, \beta A \setminus A;G) = 0$ for each $m > n$. Theorem 2.1 implies that $H^m_\infty(X,A;G) = 0$ for each $m > n$ and $A \subset X$. Thus,

$$D_f^\infty(X;G) \leq n = D_f(\beta X \setminus X;G).$$

(8)

Let $D_f^\infty(X;G) = n$. Assume that $D_f(\beta X \setminus X;G) = n_1 > n$. Then there is a closed set $B$ in $\beta X \setminus X$ such that $H^{m_1}(\beta X \setminus X,B;G) \neq 0$. Using Lemma 4 of [Sm4] and the proof of Theorem 2.5 we can show that there is a closed set $A$ of $X$ such that $B \subset \beta A \setminus A$, and $H^{m_1}(\beta X \setminus X, \beta A \setminus A;G) \neq 0$. By Theorem 2.1 $H^{m_1}(X,A;G) \neq 0$. But it is not possible because $D_f^\infty(X;G) = n$. Therefore, $n_1 \leq n$. Thus,

$$D_f(\beta X \setminus X;G) \leq n = D_f^\infty(X;G)$$

(9)

The inequalities (8) and (9) imply

$$D_f^\infty(X;G) = D_f(\beta X \setminus X;G).$$

□
Theorem 2.10. If $A$ is a closed subset of normal space $X$, then
\[ D_f^f(A; G) \leq D_f^f(X; G). \]

Proof. By Theorem 1.10, for each closed set $B$ of $A$, there is the exact Čech border cohomological sequence
\[ \cdots \rightarrow \hat{H}^{m-1}_\infty(A; G) \xrightarrow{\delta^m} \hat{H}^m_\infty(X, A; G) \xrightarrow{j^*_\infty} \hat{H}^m_\infty(X, B; G) \xrightarrow{i^*_\infty} \hat{H}^m_\infty(A, B; G) \rightarrow \cdots \]

It is clear that, if $m > D_f^f(X; G)$, then $\hat{H}^m_\infty(X, A; G) = \hat{H}^m_\infty(X, B; G) = 0$. Consequently, $\hat{H}^m_\infty(A, B; G) = 0$. Thus, we have
\[ D_f^f(A; G) \leq D_f^f(X; G). \]

Corollary 2.11. For each closed subspace $A$ of metrizable space $X$, one has
\[ D_f^f(A; G) \leq D_f^f(\beta X \setminus X; G). \]

Theorem 2.12. If $X$ is a normal space then
\[ d_f^f(X; G) \leq D_f^f(X; G). \]

Proof. Let $A$ be a closed subset of normal space $X$. Consider the exact Čech border cohomological sequence of pair $(X, A)$
\[ \cdots \rightarrow \hat{H}^{m-1}_\infty(A, B; G) \xrightarrow{\delta^m} \hat{H}^m_\infty(X, A; G) \xrightarrow{j^*_\infty} \hat{H}^m_\infty(X, B; G) \xrightarrow{i^*_\infty} \hat{H}^m_\infty(A, B; G) \rightarrow \cdots \]

Let $m > D_f^f(X; G)$. Note that $j^*_\infty : \hat{H}^m_\infty(X; G) \rightarrow \hat{H}^m_\infty(A; G)$ is an epimorphism. Hence,
\[ d_f^f(X; G) \leq D_f^f(X; G). \]

Corollary 2.13. For each metrizable space $X$, one has
\[ d_f(\beta X \setminus X; G) \leq D_f^f(X; G) \]
and
\[ d_f^f(X; G) \leq D_f(\beta X \setminus X; G). \]

Remark 2.14. The results of this paper also hold for spaces satisfying the compact axiom of countability. Recall that a space $X$ satisfies the compact axiom of countability if for each compact subset $B \subset X$ there exists a compact subset $B' \subset X$ such that $B \subset B'$ and $B'$ has a countable or finite fundamental systems of neighbourhoods (see Definition 4 of [Sm4], p.143). A space $X$ is complete in the sense of Čech if and only if it is $G_\delta$ type set in some compact extension. Each locally metrizable spaces, complete in the seance of Čech spaces [C] and locally compact spaces satisfy the compact axiom of countability.
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