Singularity removal in optical instruments without reflections or induced birefringence

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Abstract. The refractive index profiles for some optical instruments such as Eaton lenses or invisible spheres include point singularities where the refractive index approaches infinity. A photorealistic visualization of imperfectly approximating such a singularity with realistic materials is presented and a method involving replacing regions near optical singularities with nonsingular birefringent materials is expanded upon in order to allow transmutations without introducing surface reflections when combining rescaling and transmutation operations. A new method, not derivable through transformation optics, for removing point singularities in optical instruments without introducing birefringence is also introduced. Both methods may prove useful in the design of gradient index optical devices.

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1. Introduction

Singularities, or points where the refractive index approaches 0 or infinity, are a feature of gradient index optical instruments that are manifest when light rays are forced by design to follow increasingly curved trajectories near a critical region in the refractive index profile. Their presence in any optical instrument represents a barrier to fabrication, because materials that exhibit the extreme properties required near a singularity are unavailable. The removal of singularities, or the transmutation thereof by introducing birefringence, is of interest because of the resulting eased materials requirements. The recent first physical realization of an Eaton lens [1], or all-angle retroreflector, was made possible by the ability to mathematically remove a point singularity in the original design, now decades old [2]. This paper is motivated by a previous example of point singularity removal [3], and seeks a formulaic approach to singularity removal. It will be shown that point singularities where the refractive index approaches infinity can always be removed in such a way that the resultant anisotropic index values are always finite and greater than 1, which is useful for the fabrication of devices with dielectrics at optical wavelengths. The method developed herein also ensures that the resulting index profile remains reflection-free even when one polarization is sacrificed to reduce materials requirements. In the second part of this paper, it is further shown that point singularities in positive-index materials can be removed by replacing any singularity and its environs with a small region of negative-index material, obviating any need for birefringence.

All the cases when light rays entering a spherically symmetric device of isotropic refractive index are forced to all bend by the same turning angle will feature a singularity at the origin. Figure 1 illustrates a handful of such cases, the index profiles of which can be calculated by known methods [4] (most have only numerical solutions for the refractive index profile). The strength of the singularity, or the speed at which the index approaches infinity, is related to the turning angle necessary for rays near the singularity, which for the rest of this paper will be assumed to be at the origin of a spherically symmetric region.

As an example of the severe consequences of a singularity on the possibility of fabricating a device, consider an invisible sphere (360° of turning in figure 1). A perfect invisible sphere is shown in figure 2(a), which, being invisible, cannot be seen in the photographic simulation shown. However, imagine that only optical materials with a refractive index in the range of 1 < \( n < 3 \) were available (close to reality, in fact) and the entire invisible sphere were fabricated with this material. In other words, \( n \) is clamped at 3 even if the profile calls for a higher \( n \). As figure 2(b) reveals, only a thin annular region is actually invisible, with the center part of the lens not functioning correctly. The photorealistic images were created from an all-angle panorama of the background scene, which was post-processed to include the optical device [5, 6]. It is clear that removal of singularity, and consequent relaxation of materials requirements to those achievable by dielectrics, is a worthy goal, given the dire situation shown in figure 2.

2. Singularity removal by inducing birefringence

In spherically symmetric devices, transformation optics allows a coordinate change from \( r \) to \( R \), where \( R \) represents a new radial coordinate in a metamaterial [7]–[12], according to the following formula, where \( n(r) \) represents the original function, \( R(r) \) is the transform function...
Figure 1. The turning angle that rays undergo in reflection-free lenses is related to the strength of the singularity at the origin. Three examples of ray traces for various uniform turning angles are shown.

Figure 2. (a) A simulation of an invisible sphere functioning correctly in the Chinese Garden, Singapore. (b) An invisible sphere of diameter \( d \) with perturbed ray trajectories caused by arbitrarily limiting materials inside the sphere to dielectrics of \( 1 < n < 3 \).

\[ \varepsilon = \mu = \left\{ \frac{nr^2}{R^2} \frac{dR}{dr}, n \frac{dr}{dR}, n \frac{dr}{dR} \right\} \]  

In equation (1), the three components shown in braces represent diagonal tensor components in spherical coordinates (the first of the three being the radial coordinate). According to the
Tyc–Leonhardt method [3] for singularity transmutation, singularities that behave like \( n(r) = r^p \) near the origin should be transformed according to the rule \( R(r) = r^{p+1} \). Since many choices for \( R(r) \) are possible, it is now worthwhile to consider why some choices may be more desirable than others.

While the transform itself preserves the behavior of the original index profile as far as ray behavior is concerned, for practical reasons devices are not typically fabricated with all components of \( \varepsilon \) and \( \mu \) intact. To reduce the materials requirements for fabrication, one polarization is often sacrificed, a technique that has been employed in practically every transformation-optics-designed device ever fabricated. This is possible because the propagation of one polarization (transverse magnetic (TM)) on any origin-cutting slice in a spherically symmetric device is influenced only by components \( \varepsilon_r \), \( \varepsilon_\phi \) and \( \mu_\theta \) and the other (transverse electric (TE)) by \( \mu_r \), \( \mu_\phi \) and \( \varepsilon_\theta \). This also implies that for the TM polarization, ray trajectories are determined by the quantities \( \sqrt{\varepsilon_r \mu_\theta} \) and \( \sqrt{\varepsilon_\phi \mu_\theta} \) instead of individual tensors [13]. If fabrication is to take place at optical wavelengths, then dielectrics require \( \mu_r = 1 \) while achieving \( \mu_\phi = 1 \) for two out of the three components. Note that this operation does not necessarily require the rescaling factor to be a constant since \( \sqrt{\varepsilon_r \mu_\theta} \) and \( \sqrt{\varepsilon_\phi \mu_\theta} \) are unaltered for TM waves (and similarly for TE waves). If \( \mu_r \) is then arbitrarily set to \( \mu_r = 1 \) for implementation in dielectrics, then correct operation will be maintained for only one of the two polarizations,

\[
\varepsilon = \left\{ \frac{n^2 r^2}{R^2}, n^2 \left( \frac{dr}{dR} \right)^2, n^2 \left( \frac{dR}{dr} \right)^2 \right\}.
\]

The rescaling operation can result in reflections if care is not taken in choosing the function \( R(r) \), because correct behavior is only guaranteed if all space is rescaled. Since the device is finite in extent and the region outside any device (air) cannot be rescaled, reflections will always be introduced unless the rescaling factor approaches 1 at the outer surface of the device. Many different choices of \( R(r) \) can achieve this but can be difficult to find in practice; the following rules always work.

First, consider the situation before rescaling. Given an index profile of \( n \propto r^p \) for \(-1 < p \leq 0\), choose a polynomial \( R = \sum_{n=1}^{m} C_n r^{p+n} \) where \( m \) is sufficiently large to meet the conditions \( \sum_{n=1}^{m} C_n = 1 \) and \( \sum_{n=1}^{m} (p+n)C_n = 1 \) with \( C_1 = 0.5 \). The resulting anisotropy at the origin will always be \((p+1)^{-2}\), which means greater anisotropy is required to transmute stronger singularities. Consider an Eaton lens where the singularity has the behavior \( n \propto r^{-0.5}; \) the function \( R(r) = 0.5\sqrt{r} + 0.5r^{3/2} \) meets the requirements set out above for \( m = 2 \). The anisotropy at the origin is always 4 since \( p = -0.5 \) for the Eaton lens, which is easily seen because

\[
\varepsilon_\phi = 4\sqrt{\varepsilon_r} = 4.
\]

For any device, the actual values of the two unequal tensor components at the origin depend on the function \( n(r) \), and can be adjusted by changing \( C_1 \) and increasing \( m \) if necessary to still meet...
Figure 3. (a) A transform (before rescaling) yielding a constant angular tensor component \([1, 3]\). (b) A transform yielding convergence with \(n(r)\) at the outer boundary with all components always finite and greater than 1.

Figure 4. Ray trajectories for (a) a standard Eaton lens, (b) a lens transformed as in figures 3(a) and (c) a lens transformed as in figure 3(b).

The above requirements. For example, in an Eaton lens where

\[
R(r) = 0.25\sqrt{r} + r^{3/2} - 0.25r^{5/2}, \quad \frac{\varepsilon_\phi}{\varepsilon_r} = \frac{8\sqrt{2}}{2\sqrt{2}} = 4. 
\]

It is possible for components at the origin to always be greater than 1 and finite, which is important for implementation in dielectrics. Figure 3 shows two possibilities for an Eaton lens, illustrating the differences at the outer boundary (where \(r = 1\)) between a transform used by Ma et al to fabricate the first Eaton lens [1] (and see surface reflections) and the transform \(R(r) = 0.5\sqrt{r} + 0.5r^{3/2}\). Corresponding ray trajectories are shown in figure 4.

A key feature of the above transformation procedure is that although optical transformations are naturally impedance-matching, this technique preserves impedance matching at the outer device boundary even after the rescaling procedure of equation (2) is carried out. This is because all tensor components of both \(\varepsilon\) and \(\mu\) are equal to 1 at the outer device boundary both before and after rescaling, which is not possible, for instance, in the transform of figure 3(a) if rescaled so that all components of \(\mu\) are equal to 1. In addition to giving the ability to control the tensor component values at the origin, this procedure may
find use in the future design of gradient index devices containing a singularity, especially if implementation is to take place with dielectrics.

3. Singularity removal with negative refraction

In some cases, it may be more advantageous to remove a singularity by incorporating a region of negative refractive index rather than transforming the singularity to a birefringent region. This method is an alternative transmutation technique that cannot be derived by transformation optics. It involves reversing the rays’ curvature near a singularity by exploiting the fact that an interface between a positive and a negative index material of similar magnitude is reflection-free. The method can be explained with the use of the invisible sphere as an example (the refractive index profile is given in [6]).

Given a refractive index profile, a large sample of ray trajectories can be traced from the outer boundary of the device towards the singularity, stopping short when any ray touches the boundary of the inner region in which the singularity is to be replaced with a region filled with the material of negative refractive index. Define this inner boundary at \( r = B \). For each ray in the sample, there will be a required turning angle \( \chi \) within the region \( r < B \) that would normally be accomplished by the singularity and its environs. A large sample of turning angles can be calculated or simulated as a function of the impact factor \( b \) (which is defined as the shortest distance between the ray and the origin at closest approach if it continued in a straight line at its entry angle instead of turning by an angle \( \chi \)). After a function \( \chi(b) \) is assembled either analytically or numerically (by taking a large sample of rays for various \( b \) and examining the required turning angle for each), it should then be altered to account for refraction at the boundary between the two regions where negative and positive refractive indices meet. The inner region’s refractive index distribution \( n = -n_i \) can then be calculated [4] by the following implicit formula, where \( \chi(b) \) represents turning angles towards the origin as a function of the impact factor \( b \) (after accounting for refraction at the positive/negative index interface), \( \rho = r n_i / B n_{outer} \), and \( n_{outer} \) is the positive index of the outer portion of the device at the positive/negative index boundary.

\[
\ln \left( \frac{n_i}{n_{outer}} \right) = \frac{1}{\pi} \int_{B \rho}^{B} \frac{\chi(b) db}{B \sqrt{b^2 / B^2 - \rho^2}}.
\]  

(3)

The results of applying this procedure to an invisible sphere are shown in figure 5 for two different boundaries, \( B = 0.5 \) and \( B = 1 \) (the entire sphere replaced) and corresponding ray trajectories are shown in figure 6. The invisible sphere’s functionality is maintained in both cases with no surface reflections or induced birefringence. In the case of \( B = 1 \), the turning angle function after alteration for refraction is \( \chi(b) = 2\pi - 4 \cos^{-1} b \) and \( n_{outer} = 1 \). For that case, the materials requirements are changed by this procedure from an impossible \( 1 < n < \infty \) to a nearly practical \(-4 < n < -1 \). The negative index profile has an analytical solution:

\[
n(r) = \frac{2}{3 r^2} - \frac{1}{3 u^{1/3}} \frac{u^{1/3}}{3 r^4} \quad \text{where} \quad u = r^6 + 54 r^8 + 6\sqrt{3} (r^{14} + 27 r^{16}).
\]  

(4)

Such values of negative index can be achieved by photonic crystals over narrow bands at some wavelengths, and in some situations may be preferable to introducing birefringence, as described in the above procedure.

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Figure 5. (a) The region \( r < 0.5 \) has been replaced with a negative index medium to remove a singularity. (b) The entire invisible sphere has been replaced with a negative index medium.

Figure 6. (a) Original invisible sphere containing singularity. (b) Invisible sphere with singularity removed by replacing the region \( r < 0.5 \) with a negative index material. (c) Negative index invisible sphere with no singularity.

4. Conclusion

In conclusion, in this paper two methods have been developed for singularity removal, which aids in relaxing the materials requirements for fabrication. By introducing birefringence, it has been demonstrated that a singularity can be transmuted in such a way that the resulting tensor components of permittivity can be finite and greater than 1 with no surface reflections even after a rescaling operation is carried out for operation with one polarization to further relax materials requirements. In this case, the singularity of the index becomes a singularity of geometry, since at the origin the anisotropy is undefined. In the second method, however, it has been found that a singularity can be entirely eliminated through the use of negative index materials. This essentially introduces a discontinuity that has no analogue with systems of particles in potential wells, for example. It cannot be found with transformation optics. The transformation is not conformal, and ray trajectories sometimes cross in the interior space.

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