CLOSED MODEL CATEGORIES FOR PRESHEAVES OF SIMPLICIAL GROUPOIDS AND PRESHEAVES OF 2-GROUPOIDS

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Abstract. It is shown that the category of presheaves of simplicial groupoids and the category of presheaves of 2-groupoids have Quillen closed model structures. Furthermore, their homotopy categories are equivalent to the homotopy categories of simplicial presheaves and homotopy 2-types, respectively.

1. Introduction

Quillen’s axioms for a closed model categories have proved to be a successful framework for expanding the scope of the tools of homotopy theory. Many categories have been shown to have a useful closed model structure. A central example is the category $S$ of simplicial sets [Qui67]. Further examples related to the present study include the category of simplicial groupoids by Dwyer and Kan [DK84, GJ99], the category of 2-groupoids by Moerdijk and Svensson [MS93], the category of simplicial presheaves by Jardine [Jar87], the category of simplicial sheaves by Joyal [Joy84] and the category of sheaves of simplicial groupoids by Joyal and Tierney [JT91, JT96] and Crans [Cra95], and sheaves of 2-groupoids by Crans [Cra95].

We use techniques developed by Jardine [Jar87] to prove that the category of presheaves of simplicial groupoids and the category of presheaves of 2-groupoids have closed model structures.

Let $C$ be a small site. The adjunction $G : S \rightleftarrows sGd : \bar{W}$ between the loop groupoid functor and the universal cocycle functor induces an adjunction

$$G : S Pre(C) \rightleftarrows sGd Pre(C) : \bar{W}.$$  

A map $f : X \to Y \in sGd Pre(C)$ is defined to be a fibration (weak equivalence) if $\bar{W}(f)$ is a global fibration (topological weak equivalence) in $S Pre(C)$. A map is a cofibration if it has the left lifting property with respect to all maps that are both fibrations and weak equivalences. We show that with these classes of morphisms, $sGd Pre(C)$ is a closed model category.

Similarly, there is an adjunction $G : S \rightleftarrows 2 - Gpd : \bar{W}$, which induces an adjunction

$$G : S Pre(C) \rightleftarrows 2 - Gpd Pre(C) : \bar{W}.$$  

2000 Mathematics Subject Classification. 18G55, 55U35 (18B40, 18F20, 18G30, 55P15).

Key words and phrases. presheaves of simplicial groupoids, presheaves of 2-groupoids, Quillen closed model categories, simplicial presheaves, homotopy 2-types.
As with the category of presheaves of simplicial groupoids above, this adjunction and
the global fibrations and topological weak equivalences in $S\text{Pre}(C)$ induce a closed
model structure on the category of presheaves of 2-groupoids.

The model structures of Joyal and Tierney [JT96] and Crans [Cra95] are simi-
larly obtained from a model structure on simplicial sheaves on $C$, but using different
functors.

We also show that the homotopy category associated to the first category is equiv-
alent to the homotopy category of simplicial presheaves, and that the homotopy
category associated to the latter category is equivalent to the homotopy category
of homotopy 2-types. In fact, the adjunction (1) is a Quillen equivalence and the
adjunction (2) is a Quillen adjunction which induces the desired equivalence of homot-
opy categories. Applications of this theory include homotopy classification results
for $G$-torsors and gerbes [JL06, Jar06a, Jar06b]

2. Preliminaries

A groupoid is a small category in which every morphism is invertible. Let $Gd$
denote the category of groupoids. A simplicial object in a category $C$ is a functor
$\Delta^{op} \rightarrow C$, where $\Delta$ is the category of finite ordinal numbers and order-preserving
maps. A simplicial groupoid is a simplicial object in the category of groupoids that
is levelwise constant. That is, a simplicial groupoid $G$ has $\text{Ob}(G_n) = \text{Ob}(G)$ and
for each ordinal number map $\phi : n \rightarrow m$, the induced map $\phi^* : G_m \rightarrow G_n$ is the
identity map. For each $x, y \in G$, the morphism sets $G_n(x, y)$ give the $n$-simplices of
the simplicial set $G(x, y)$. Let $sGd$ denote the category of simplicial groupoids.

We remark that our simplicial groupoids are groupoids enriched in simplicial sets.
The simplicial groupoids of Joyal and Tierney [JT96] are more general. They are the
groupoids in simplicial sets.

A 2-groupoid is a strict 2-category in which every 1-morphism and every 2-morph
has a (strict) inverse. Let $2 - \text{Gpd}$ be the category of small 2-groupoids and strict
functors.

Given categories $C$ and $D$, a $D$–valued presheaf on $C$ is a functor $C^{op} \rightarrow D$. Mor-
phisms of presheaves are natural transformations. Presheaves of simplicial groupoids
are $sGd$–valued presheaves and presheaves of 2-groupoids are $2 - \text{Gpd}$–valued presheaves.
The category of $D$–valued presheaves on $C$ is denoted $D\text{Pre}(C)$.

A Quillen closed model category $D$ is a category which is equipped with three classes
of morphisms, called cofibrations, fibrations and weak equivalences which together
satisfy the following axioms [Qui67, Qui69, GJ99]. Fibrations (cofibrations) that
are also weak equivalences are called trivial fibrations (trivial cofibrations).

**CM1:** The category $D$ is closed under all finite limits and colimits.
**CM2:** Suppose that the following diagram commutes in $D$:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{h} & & \downarrow{f} \\
Z & & 
\end{array}
$$
If any two of \( f, g \) and \( h \) are weak equivalences, then so is the third.

**CM3:** If \( f \) is a retract of \( g \) and \( g \) is a weak equivalence, fibration or cofibration, then so is \( f \).

**CM4:** Suppose that we are given a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{p} & X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\
V & \xrightarrow{\scriptstyle g} & Y
\end{array}
\]

where \( i \) is a cofibration and \( p \) is a fibration. Then the lifting exists, making the diagram commute, if either \( i \) or \( p \) is also a weak equivalence.

**CM5:** Any map \( f : X \to Y \) may be factored:

(a) \( f = p \cdot i \) where \( p \) is a fibration and \( i \) is a trivial cofibration, and

(b) \( f = q \cdot j \) where \( q \) is a trivial fibration and \( j \) is a cofibration.

A **right proper closed model category** \( \mathcal{D} \) is a closed model category satisfying the following axiom:

**P1:** the class of weak equivalences is closed under base change by fibrations. In other words, given a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g_*} & Y \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{\scriptstyle g} & W
\end{array}
\]

of \( \mathcal{D} \) with \( p \) a fibration, if \( g \) is a weak equivalence then so is \( g_* \).

Suppose that \( \mathcal{C} \) and \( \mathcal{D} \) are two closed model categories. We call a functor \( F : \mathcal{C} \to \mathcal{D} \) a **left Quillen functor** if \( F \) is a left adjoint and preserves cofibrations and trivial cofibrations. We call a functor \( U : \mathcal{D} \to \mathcal{C} \) a **right Quillen functor** if \( U \) is a right adjoint and preserves fibrations and trivial fibrations.

Suppose that \( (F,U,\varphi) \) is an adjunction from \( \mathcal{C} \) to \( \mathcal{D} \). That is, \( F \) is a functor \( \mathcal{C} \to \mathcal{D} \), \( U \) is a functor \( \mathcal{D} \to \mathcal{C} \), and \( \varphi \) is a natural isomorphism \( \mathcal{D}(FC,D) \to \mathcal{C}(C,UD) \) expressing \( U \) as a right adjoint of \( F \). We call \( (F,U,\varphi) \) a **Quillen adjunction** if \( F \) is a left Quillen functor (cf. [Hov99]). A Quillen adjunction \( (F,U,\varphi) : \mathcal{C} \to \mathcal{D} \) is called a **Quillen equivalence** if and only if, for all cofibrant \( X \) in \( \mathcal{C} \) and fibrant \( Y \) in \( \mathcal{D} \), a map \( f : FX \to Y \) is a weak equivalence in \( \mathcal{D} \) if and only if \( \varphi(f) : X \to UY \) is a weak equivalence in \( \mathcal{C} \) (cf. [Hov99]).

### 3. Presheaves of simplicial groupoids

Dwyer and Kan [DK84, GJ99] show that with the following definitions of weak equivalence, fibration and cofibration, the category \( s\text{Gd} \) of simplicial groupoids satisfies the axioms for a closed model category.

A map \( f : G \to H \) of simplicial groupoids is said to be a **weak equivalence of** \( s\text{Gd} \) if

(1) the morphism \( f \) induces an isomorphism \( \pi_0G \cong \pi_0H \), and
(2) each induced map \( f : G(x, x) \to H(f(x), f(x)), x \in \text{Ob}(G) \) is a weak equivalence of simplicial groups (or of simplicial sets).

A map \( g : H \to K \) of simplicial groupoids is said to be a \textit{fibration} of \( sGd \) if

(1) the morphism \( g \) has path lifting property in the sense for every object \( x \) of \( H \) and morphism \( \omega : g(x) \to y \) of the groupoids \( K_0 \), there is a morphism \( \hat{\omega} : x \to z \) of \( H_0 \) such that \( g(\hat{\omega}) = \omega \), and

(2) each induced map \( g : H(x, x) \to K(g(x), g(x)), x \in \text{Ob}(H) \) is a fibration of simplicial groups (or of simplicial sets).

A \textit{cofibration} of simplicial groupoids is defined to be a map which has the left lifting property with respect to all morphisms of \( sGd \) which are both fibrations and weak equivalences.

Let \( C \) be a fixed small Grothendieck site. That is, a small category with a Grothendieck topology. There is an adjunction between the loop groupoid functor \( G : S \to sGd \) and the universal cocycle functor \( W \) \cite{GJ99, Lemma V.7.7}. By applying these functors pointwise to simplicial presheaves and presheaves of simplicial groupoids, one obtains functors

\[
G : SPre(C) \rightleftarrows sGdPre(C) : W
\]

So there is

**Proposition 3.1.** The functor \( G : SPre(C) \to sGdPre(C) \) is left adjoint to the functor \( W \).

A map \( f : X \to Y \) in the category \( sGdPre(C) \) is said to be a \textit{fibration} if the induced map \( W(f) : WX \to WY \) is a global fibration in the category \( SPre(C) \) in the sense of \cite{Jar87}. A map \( g : Z \to U \) in the category \( sGdPre(C) \) is said to be a \textit{weak equivalence} if the induced map \( W(g) : WZ \to WU \) is a topological weak equivalence in the category \( SPre(C) \) in the sense of \cite{Jar87}. A \textit{cofibration} in the category \( sGdPre(C) \) is a map of presheaves of simplicial groupoids which has the left lifting property with respect to all fibrations and weak equivalences. Say that a map of presheaves of simplicial groupoids \( f \) is a \textit{trivial fibration} if it is both a fibration and a weak equivalence; a map \( g \) is a \textit{trivial cofibration} if it is both a cofibration and a weak equivalence.

Here we follow Jardine \cite{Jar87}. The site \( C \) is small, so that there is a cardinal number \( \alpha \) such that \( \alpha \) is larger than the cardinality of the set of subsets \( PMor(C) \) of the set of morphisms \( Mor(C) \) of \( C \). A simplicial presheaf \( X \) is said to be \( \alpha \)-\textit{bounded} if the cardinality of each \( X_n(U), U \in C, n \geq 0 \), is smaller than \( \alpha \).

A map \( p : X \to Y \) in the category \( SPre(C) \) is a global fibration if and only if it has the right lifting property with respect to all trivial cofibrations \( i : U \to V \) since there exist the
adjoint diagrams:

\[
\begin{array}{cccc}
GU & 
\rightarrow & G \\
G(i) & \downarrow & q \\
GV & 
\rightarrow & H \\
\end{array}
\quad
\begin{array}{cccc}
U & 
\rightarrow & \overline{WG} \\
\downarrow & i & \downarrow (\overline{W(q)}) \\
V & 
\rightarrow & \overline{WH} \\
\end{array}
\]  \quad (D)

For each \( W \in \mathcal{C} \), \( GV(W)_n \) is the free groupoid on generators \( x \in V(W)_{n+1} \) subject to some relations, and \( \text{Ob}(GV(W)) = V(W)_0 \), so the cardinality of each \( \text{Mor}(GV(W)_n) \), \( n \geq 0 \) and \( \text{Ob}(GV(W)) \) is smaller than \( \beta = \max(2^\alpha, \infty) \). We also call the presheaf of simplicial groupoids \( GV \) is \( \beta \)-bounded.

When \( G \) is a simplicial group there is a pullback diagram

\[
\begin{array}{ccc}
G & \xrightarrow{i} & \overline{WG} \\
\downarrow & & \downarrow q \\
* & \xrightarrow{s} & \overline{WG} \\
\end{array}
\]

where \( q \) is a fibration of simplicial sets [GJ99, Lemma V.4.1], \( G \) is the fibre over the unique vertex \( * \in \overline{WG} \). \( G \) is a simplicial group, so \( G \) is a Kan complex [GJ99, Lemma I.3.4]. \( \overline{WG} \) is a Kan complex [GJ99, Corollary V.6.8], so is \( WG \), then for any vertex \( v \in G \) there exists a long exact sequence

\[
\ldots \rightarrow \pi_n(G, v) \xrightarrow{i_*} \pi_n(WG, v) \xrightarrow{q_*} \pi_n(\overline{WG}, *) \xrightarrow{\partial} \pi_{n-1}(G, v) \rightarrow \ldots 
\]

by Lemma I.7.3 in [GJ99]. \( WG \) is contractible [GJ99, Lemma V.4.6], so \( \pi_n(WG, v) = 0, n \geq 1 \); and \( \pi_0(WG) = 0 \), since for any two vertices \( a, b \in WG_0 = G_0 \), there exists a 1-simplex \( (s_0b, b^{-1}a) \in WG_1 = G_1 \times G_0 \), s.t., \( d_1(s_0b, b^{-1}a) = b, d_0(s_0b, b^{-1}a) = a \).

Then

\[
\pi_n(G, v) = \pi_{n+1}(\overline{WG}, *), n \geq 1 \\
\pi_0G = \pi_1(\overline{WG}, *)
\]

For an ordinary groupoid \( H \), it’s standard to write \( \pi_0H \) for the set of path components of \( H \). By this one means that

\[
\pi_0H = \text{Ob}(H) / \sim
\]

where there is a relation \( x \sim y \) between two objects of \( H \) if and only if there is a morphism \( x \rightarrow y \) in \( H \).

If now \( A \) is a simplicial groupoid, all of the simplicial structure functors \( \theta^* : A_n \rightarrow A_m \) induce isomorphisms \( \pi_0A_n \cong \pi_0A_m \). We shall therefore refer to \( \pi_0A_0 \) as the set of path components of the simplicial groupoid \( A \), and denote it by \( \pi_0A \). When \( A \) is a simplicial groupoid, \( \text{Ob}(A) = (\overline{WA})_0, \text{Mor}(A_0) = (\overline{WA})_1 \), so \( \pi_0A \cong \pi_0(\overline{WA}) \).

Choose a representative \( x \) for each \([x] \in \pi_0A\), the inclusion

\[
i : \bigsqcup_{[x] \in \pi_0A} A(x, x) \rightarrow A
\]
is a homotopy equivalence of simplicial groupoids, and the induced map

$$\mathcal{W}(i) : \mathcal{W}(\bigcup_{[x] \in \pi_0 A} A(x, x)) \to \mathcal{W}A$$

is a weak equivalence of simplicial sets. $\mathcal{W}$ preserves disjoint unions, $\mathcal{W}(\bigcup_{[x] \in \pi_0 A} A(x, x)) = \bigcup_{[x] \in \pi_0 A} \mathcal{W}(A(x, x))$ [GJ99, p. 303,304].

$$\pi_n(\mathcal{W}(\bigcup_{[x] \in \pi_0 A} A(x, x)), x) = \pi_n(\mathcal{W}(A(x, x)), *) \cong \pi_{n-1}(A(x, x), v), \quad n \geq 2, \quad v \in A(x, x)_0$$

so one obtains

$$\pi_n(A(x, x), v) \cong \pi_{n+1}(\mathcal{W}A, x), \quad n \geq 1, \quad x \in \text{Ob}(A), \quad v \in A(x, x)_0, \quad \pi_0(A(x, x)) \cong \pi_1(\mathcal{W}A, x).$$

According to the definition of topological weak equivalence of simplicial presheaves in [Jar87] and the relations between simplicial groupoids and simplicial sets, we can give an explicit description of the weak equivalences of presheaves of simplicial groupoids.

For any presheaf of simplicial groupoids $X$ and any object $U \in \mathcal{C}$ and $x \in \text{Ob}(X(U))$, $X(U)(x, x)$ is a simplicial group. Associated to this presheaf of simplicial groupoids $X$ on $\mathcal{C}$ and $* \in X(U)(x, x)_0$ is a presheaf $\pi_n^{\text{simp}}(X|_U, x, *) (n \geq 1)$ on the comma category $\mathcal{C} \downarrow U$, the presheaf of simplicial homotopy groups of $X|_U$, based at $*$, which is defined by

$$(\mathcal{C} \downarrow U)^{\text{op}} \to \text{Grp}$$

$$\varphi : V \to U \mapsto \pi_n(X(V)(x_V, x_V), *_{V})$$

where $x_V$ and $*_{V}$ are the images of $x$ and $*$ in $X(V)$ under the map $X(U) \to X(V)$ which is induced by $V \to U$, respectively. The simplicial homotopy group $\pi_n(X(V)(x_V, x_V), *_{V})$ exists since the simplicial group $X(V)(x_V, x_V)$ is a Kan complex [GJ99, Lemma I.3.4].

Let $\pi_n(X|_U, x, *)$ be the associated sheaf of the presheaf $\pi_n^{\text{simp}}(X|_U, x, *)$, i.e., $\pi_n(X|_U, x, *) = L^2\pi_n^{\text{simp}}(X|_U, x, *)$. Then $\pi_n(X|_U, x, *)$ is a sheaf of groups which is abelian if $n \geq 2$. The sheaves $\pi_0(X|_U, x)$ and $\pi_0(X)$ of path components are defined similarly.

A map $f : X \to Y$ of presheaves of simplicial groupoids is a \textit{weak equivalence} if it induces isomorphisms of sheaves

$$f_* : \pi_n(X|_U, x, *) \cong \pi_n(Y|_U, f_* x, f_* *), \quad n \geq 1, \quad U \in \mathcal{C}, \quad x \in \text{Ob}(X(U)), \quad * \in X(U)(x, x)_0$$

$$f_* : \pi_0(X|_U, x) \cong \pi_0(Y|_U, f_* x).$$

In view of Proposition 1.18 in [Jar87], the weak equivalences are just same as the combinatorial weak equivalences in [Jar87]. Since the weak equivalences are given
by the isomorphisms between sheaves, thus, Proposition 1.11 in \cite{Jar87} implies (or directly from CM2 for the category $\text{SPre}(\mathcal{C})$)

**Lemma 3.2.** Given maps of presheaves of simplicial groupoids $f : X \to Y$ and $g : Y \to Z$, if any two of $f$, $g$, or $g \circ f$ are weak equivalences, then so is the third.

**Lemma 3.3.** The functor $X \mapsto \overline{W}G(X)$ preserves weak equivalences of simplicial presheaves.

*Proof.* When $T$ is a simplicial set, the natural simplicial map $\eta : T \to \overline{W}G(T)$ is a weak equivalence of simplicial sets \cite[Theorem V.7.8]{GJ99}. So the map $X \to \overline{W}G(X)$ is a pointwise weak equivalence of simplicial presheaves, then it is a weak equivalence.

There exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \overline{W}G(X) \\
\downarrow f & & \downarrow \overline{W}G(f) \\
Y & \xrightarrow{\eta_Y} & \overline{W}G(Y)
\end{array}
$$

where both $\eta_X$ and $\eta_Y$ are weak equivalences, if $f : X \to Y$ is a weak equivalences, so is $\overline{W}G(f)$ by the CM2 of the closed model category $\text{SPre}(\mathcal{C})$. \hfill \Box

**Lemma 3.4.** The functor $G : \text{SPre}(\mathcal{C}) \to \text{sGdPre}(\mathcal{C})$ preserves cofibrations and weak equivalences.

*Proof.* The adjoint diagrams (D) imply that the functor $G$ preserves cofibrations. Lemma 3.3 implies that $G$ preserves weak equivalences. \hfill \Box

**Lemma 3.5.** The category $\text{sGdPre}(\mathcal{C})$ has all pushouts, and is hence cocomplete. The class of cofibrations in $\text{sGdPre}(\mathcal{C})$ is closed under pushout.

*Proof.* The category $\text{sGd}$ has all pushouts and is cocomplete, so is the category $\text{sGdPre}(\mathcal{C})$, since we can take the pushout and colimit pointwise. The second statement is obvious. \hfill \Box

There exists a Kan $Ex^\infty$ functor from $\text{SPre}(\mathcal{C})$ to $\text{SPre}(\mathcal{C})$, such that $Ex^\infty X$ is locally fibrant for any simplicial presheaf $X$ and the canonical map $\nu : X \to Ex^\infty X$ is a pointwise weak equivalence. \cite{Jar96}.

Fix a Boolean localization $\varphi : \text{Shv}(\mathcal{B}) \to \mathcal{E}$, and consider the functors

$$
\text{SPre}(\mathcal{C}) \xrightarrow{L^2} \text{SE} \xrightarrow{\varphi^*} \text{SShv}(\mathcal{B}),
$$

where $L^2$ is the associated sheaf functor. In \cite{Jar96} Jardine proves that the topological weak equivalence between simplicial presheaves \cite{Jar87} coincides with the local weak equivalence $\varphi^*L^2$, i.e., a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is a topological weak equivalence if the induced map $\varphi^*L^2 : \varphi^*L^2 Ex^\infty X \to \varphi^*L^2 Ex^\infty Y$ is a pointwise weak equivalence.
Notice that there is a commutative diagram

\[
\begin{array}{ccc}
s\text{GdPre}(C) & \xrightarrow{L^2} & s\text{Gd}\mathcal{E} \\
\downarrow \varphi^* & & \downarrow \varphi^* \\
S\text{Pre}(C) & \xrightarrow{L^2} & S\mathcal{E}
\end{array}
\begin{array}{ccc}
s\text{GdShv}(\mathcal{B}) & \xrightarrow{\varphi^*} & S\text{Shv}(\mathcal{B}) \\
\downarrow \varphi^* & & \downarrow \varphi^* \\
\downarrow \varphi^* & & \downarrow \varphi^*
\end{array}
\]

\(\varphi^* G\) is locally fibrant simplicial presheaf for any presheaf of simplicial groupoid \(G\), so a map \(f : G \to H\) of presheaves of simplicial groupoids on \(C\) is a weak equivalence if the induced map \(\varphi^* L^2 : \varphi^* L^2 G \to \varphi^* L^2 H\) is a pointwise weak equivalence.

**Proposition 3.6.** Trivial cofibrations of presheaves of simplicial groupoids are closed under pushout.

**Proof.** Suppose that

\[
\begin{array}{ccc}
G & \xrightarrow{i} & C \\
\downarrow & & \downarrow \\
H & \xrightarrow{i'} & D
\end{array}
\]

is a pushout in the category \(s\text{GdPre}(C)\). \(i\) is a trivial cofibration, then \(i'\) is a cofibration by Lemma 3.5.

The heart of the matter for this proof is the weak equivalence. Both \(L^2\) and \(\varphi^*\) are left adjoint functors, so the functor \(\varphi^* L^2\) preserves the pushout

\[
\begin{array}{ccc}
\varphi^* L^2 G & \xrightarrow{\varphi^* L^2(i)} & \varphi^* L^2 C \\
\downarrow & & \downarrow \\
\varphi^* L^2 H & \xrightarrow{\varphi^* L^2(i')} & \varphi^* L^2 D
\end{array}
\]

The map \(\varphi^* L^2(i)\) is a pointwise weak equivalence and pointwise cofibration, so for any \(U \in \mathcal{B}\), the diagram

\[
\begin{array}{ccc}
\varphi^* L^2 G(U) & \xrightarrow{\varphi^* L^2(i)} & \varphi^* L^2 C(U) \\
\downarrow & & \downarrow \\
\varphi^* L^2 H(U) & \xrightarrow{\varphi^* L^2(i')} & \varphi^* L^2 D(U)
\end{array}
\]

is a pushout in the category \(s\text{Gd}\). Since the category \(s\text{Gd}\) is a closed model category in which the map \(\varphi^* L^2(i)\) is a trivial cofibration, the map \(\varphi^* L^2(i') : \varphi^* L^2 C(U) \to \varphi^* L^2 D(U)\) is also a trivial cofibration. Then \(\varphi^* L^2(i') : \varphi^* L^2 C \to \varphi^* L^2 D\) is a pointwise weak equivalence, and thus \(i' : C \to D\) is a weak equivalence in the category \(s\text{GdPre}(C)\). \(\square\)
Given a trivial cofibration $i : A \to B$ in the category $\mathbf{SPre}(\mathcal{C})$, suppose that

$$
\begin{array}{ccc}
GA & \to & C \\
G(i) \downarrow & & \downarrow i' \\
GB & \to & D
\end{array}
$$

is a pushout in the category $\mathbf{sGdPre}(\mathcal{C})$. The map $G(i)$ is a trivial cofibration by Lemma 3.4, then the map $i'$ is a trivial cofibration.

**Lemma 3.7.** Every map $f : X \to Y$ of presheaves of simplicial groupoids may be factored

$$
\begin{array}{ccc}
X & \to & Y \\
\downarrow i & & \downarrow p \\
Z & \to &
\end{array}
$$

where $i$ is a trivial cofibration and $p$ is a fibration.

**Proof.** We use the transfinite small object argument. Choose a cardinal number $\gamma > 2^\beta$, and define a functor $F : \gamma \to \mathbf{sGdPre}(\mathcal{C}) \downarrow Y$ on the partially ordered set $\gamma$ by setting $F(0) = f : X \to Y, F(s) : X(s) \to Y$ such that

1. $X(0) = X$,
2. $X(t) = \lim_{\overset{\to}{s < t}} X(s)$ for all limit ordinals $t < \gamma$, and
3. the map $X(s) \to X(s + 1)$ is defined by the pushout diagram

$$
\begin{array}{ccc}
\bigsqcup_D GU_D & \to & X(s) \\
\downarrow & & \downarrow \\
\bigsqcup_D GV_D & \to & X(s + 1)
\end{array}
$$

where the index $D$ refers to a set of representatives for all diagrams

$$
\begin{array}{ccc}
GU_D & \to & X(s) \\
\downarrow & & \downarrow F(s) \\
GV_D & \to & Y
\end{array}
$$

such that $Gi_D : GU_D \to GV_D$ is induced by $i_D : U_D \to V_D$, where $i_D$ is a trivial cofibration in $\mathbf{SPre}(\mathcal{C})$ with $V_D \alpha$-bounded.

Then $GV_D$ is $\beta$-bounded. Let

$$X(\gamma) = \lim_{\overset{\to}{t < \gamma}} X(t)$$
and consider the induced factorization of $f$

\[
\begin{array}{ccc}
X & \xrightarrow{i_{\gamma}} & X(\gamma) \\
\downarrow{f} & & \downarrow{F(\gamma)} \\
Y & \xrightarrow{j} & Y \\
\end{array}
\]

Then $i_{\gamma}$ is a trivial cofibration, since it is a filtered colimit of such. Also, for any diagram

\[
\begin{array}{ccc}
GU & \xrightarrow{G_{i}} & X(\gamma) \\
\downarrow{G_{i}} & & \downarrow{F(\gamma)} \\
GV & \xrightarrow{F} & Y \\
\end{array}
\]

such that $GV$ is $\beta$–bounded and $Gi$ is a trivial cofibration, the map $GU \to X(\gamma)$ must factor through some $X(n) \to X(\gamma), n < \gamma$, for otherwise $GU$ has too many subobjects. □

For each object $U$ of $\mathcal{C}$, the $U$–sections functor $X \to X(U)$ has a left adjoint $\mathcal{U} : S \to SPre(\mathcal{C})$ which sends the simplicial set $Y$ to the simplicial presheaf $Y_{U}$, which is defined by

\[
Y_{U}(V) = \prod_{\varphi : V \to U} Y.
\]

Then a simplicial presheaves map $q : Z \to X$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $S \subset \Delta^{n}_{U}$ of subobjects of the $\Delta^{n}_{U}, U \in \mathcal{C}, n \geq 0$ [Jar87, p. 68]. So a map $p : G \to H$ of presheaves of simplicial groupoids is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $GS \subset G\Delta^{n}_{U}$ of subobjects of the $G\Delta^{n}_{U}, U \in \mathcal{C}, n \geq 0$. A transfinite small object argument, as in Lemma 3.7, shows that

**Lemma 3.8.** Every map $g : Z \to W$ of presheaves of simplicial groupoids may be factored

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & W \\
\downarrow{j} & & \downarrow{q} \\
M & \xrightarrow{q} & W \\
\end{array}
\]

where $j$ is a cofibration and $q$ is a trivial fibration.

**Lemma 3.9.** For the commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow{s} & & \downarrow{p} \\
V & \xrightarrow{V} & Y \\
\end{array}
\]

where $i$ is a trivial cofibration and $p$ is a fibration in the category $sGdPre(\mathcal{C})$, there exists a lifting $s$. 
Proof. Suppose that \( i : U \to V \) is a trivial cofibration. Then \( i \) has a factorization
\[
\begin{array}{ccc}
U & \xrightarrow{j} & W \\
\downarrow{i} & & \downarrow{q} \\
V & &
\end{array}
\]
where \( q \) is a fibration and \( j \) is a trivial cofibration which has the left lifting property with respect to all fibrations by the construction in the proof of Lemma 3.7. But then \( q \) is a trivial fibration, and so the lifting exists in the diagram
\[
\begin{array}{ccc}
U & \xrightarrow{j} & W \\
\downarrow{i} & & \downarrow{q} \\
V & \xrightarrow{1_V} & V
\end{array}
\]
It follows that \( i \) is a retract of \( j \), so that \( i \) has the left lifting property with respect to all fibrations. \( \square \)

**Theorem 3.10.** The category \( s\text{GdPre}(\mathcal{C}) \), with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.

**Proof.** The category \( s\text{Gd} \) is closed under all finite limits and colimits. So taking the limits and colimits pointwise, the category \( s\text{GdPre}(\mathcal{C}) \) is also closed under all finite limits and colimits. This is CM1. CM2 is Lemma 3.2 CM3 is trivial. The first part of CM4 is Lemma 3.9, the second part is the definition of a cofibration. CM5(1) is Lemma 3.7, CM5(2) is Lemma 3.8 \( \square \)

**Remark 3.11.** Fibrations (trivial fibrations) in the category \( s\text{GdPre}(\mathcal{C}) \) have the right lifting property with respect to all maps \( G(i) : GU \to GV \) induced by maps \( i : U \to V \) where \( i \) is a trivial cofibration (cofibration) in the category \( \text{SPre}(\mathcal{C}) \) and \( V \) is \( \alpha \)-bounded. So the category \( s\text{GdPre}(\mathcal{C}) \) is cofibrantly generated.

**Lemma 3.12.**
1. The functor \( \overline{W} : s\text{GdPre}(\mathcal{C}) \to \text{SPre}(\mathcal{C}) \) preserves fibrations and weak equivalences.
2. A map \( K \to \overline{WX} \in \text{SPre}(\mathcal{C}) \) is a weak equivalence if and only if its adjoint \( GK \to X \in s\text{GdPre}(\mathcal{C}) \) is a weak equivalence.

**Proof.** (1) This is implied by the definitions of fibration and weak equivalence.
(2) There is a commutative diagram
\[
\begin{array}{ccc}
K & \xrightarrow{} & \overline{WGK} \\
\downarrow & & \downarrow \\
\overline{WX} & &
\end{array}
\]
where the map \( K \to \overline{WGK} \) is a pointwise weak equivalence \([GJ99\text{, Theorem V.7.8(3)}]\). So the map \( K \to \overline{WX} \) is a weak equivalence if and only if the map \( \overline{WGK} \to \overline{WX} \) is a weak equivalence, i.e., the map \( GK \to X \) is a weak equivalence. \( \square \)
Corollary 3.13. The functor $G$ and $\overline{W}$ induce an equivalence of homotopy categories

$$\text{Ho}(\text{sGdPre}(\mathcal{C})) \simeq \text{Ho}(\text{SPre}(\mathcal{C}))$$

Proof. Lemma 3.12 implies that the natural maps $\varepsilon : GWK \to K$ and $\eta : X \to \overline{WG}X$ are weak equivalences for all presheaves of simplicial groupoids $K$ and simplicial presheaves $X$. □

Corollary 3.14. The adjunction $G : \text{SPre}(\mathcal{C}) \rightleftarrows \text{sGdPre}(\mathcal{C}) : \overline{W}$ is a Quillen equivalence.

Proof. It follows from Theorem 3.10, Proposition 3.1, Lemma 3.4, Lemma 3.12 and the definition of a Quillen adjunction. □

Theorem 3.15. The category $\text{sGdPre}(\mathcal{C})$ is right proper.

Proof. Given a pullback diagram in $\text{sGdPre}(\mathcal{C})$

$$
\begin{array}{ccc}
X & \xrightarrow{g_*} & Y \\
\downarrow & & \downarrow^p \\
Z & \xrightarrow{g} & W
\end{array}
$$

with $p$ a fibration and $g$ a weak equivalence, there exists a pullback diagram in $\text{SPre}(\mathcal{C})$

$$
\begin{array}{ccc}
\overline{WX} & \xrightarrow{\overline{W}g_*} & \overline{WY} \\
\downarrow & & \downarrow^{\overline{W}p} \\
\overline{WZ} & \xrightarrow{\overline{W}g} & \overline{WW}
\end{array}
$$

$\overline{W}$ preserves fibrations and weak equivalences, hence $\overline{W}p$ is a fibration and $\overline{W}g$ is a weak equivalence in $\text{SPre}(\mathcal{C})$. $\text{SPre}(\mathcal{C})$ is proper, so $\overline{W}g_*$ is a weak equivalence as well, hence the map $g_*$ is a weak equivalence. So the axiom $\text{P1}$ holds. □

4. Presheaves of 2-groupoids

Moerdijk and Svensson show that [MS93], with the following definitions of weak equivalence, fibration and cofibration, the category $\textbf{2-Gpd}$ of 2-groupoids satisfies the axioms for a closed model category.

A map $\varphi : \mathcal{A} \to \mathcal{B}$ of 2-groupoids is said to be a weak equivalence of $\textit{2-Gpd}$ if

(1) for every object $b$ of $\mathcal{B}$ there exists an object $a$ of $\mathcal{A}$ and an arrow $\varphi(a) \to b$;
(2) for any two objects $a, a'$ in $\mathcal{A}$, $\varphi$ induces an equivalence of categories (groupoids)

$$\varphi_{a,a'} : \text{Hom}_\mathcal{A}(a, a') \to \text{Hom}_\mathcal{B}(\varphi(a), \varphi(a')).$$

A map $\psi : \mathcal{B} \to \mathcal{A}$ of 2-groupoids is said to be a (Grothendieck) fibration of $\textit{2-Gpd}$ if for any arrow $f : b_1 \to b_2$ in $\mathcal{B}$ and any arrows $g : a_0 \to \psi(b_1)$ and $h : a_0 \to \psi(b_2)$, any deformation $\alpha : h \Rightarrow \psi(f) \circ g$ can be lifted to a deformation $\tilde{\alpha} : \tilde{h} \Rightarrow f \circ \tilde{g}$ in $\mathcal{B}$ (in the sense that $\psi(\tilde{\alpha}) = \alpha, \psi(\tilde{h}) = h$ and $\psi(\tilde{g}) = g$).
A cofibration of 2-groupoids is defined to be a map which has the left lifting property with respect to all morphisms of \(2\text{-Gpd}\) which are both fibrations and weak equivalences.

There is an adjunction [MS93]:

\[
G : S \rightleftarrows 2\text{-Gpd} : W
\]

where the functor \(W\) is the functor \(N\) in [MS93] and the functor \(G\) is the Whitehead 2-groupoid functor \(W\) in [MS93]:

\[
W(X) = W(|X|, |X^{(1)}|, |X^{(0)}|).
\]

By applying these functors pointwise to simplicial presheaves and presheaves of 2-groupoids, one obtains functors

\[
G : S\text{Pre}(C) \rightleftarrows 2\text{-GpdPre}(C) : W
\]

and there is

**Proposition 4.1.** The functor \(G : S\text{Pre}(C) \rightarrow 2\text{-GpdPre}(C)\) is left adjoint to the functor \(W\).

A map \(f : X \rightarrow Y\) in the category \(2\text{-GpdPre}(C)\) is said to be a **fibration** if the induced map \(W(f) : WX \rightarrow WY\) is a global fibration in the category \(S\text{Pre}(C)\). A map \(g : Z \rightarrow U\) in the category \(2\text{-GpdPre}(C)\) is said to be a **weak equivalence** if the induced map \(W(g) : WZ \rightarrow WU\) is a weak equivalence in the category \(S\text{Pre}(C)\). A cofibration in the category \(2\text{-GpdPre}(C)\) is a map of presheaves of 2-groupoids which has the left lifting property with respect to all fibrations and weak equivalences. Say that a map of presheaves of 2-groupoids \(f\) is a **trivial fibration** if it is both a fibration and a weak equivalence; a map \(g\) is a **trivial cofibration** if it is both a cofibration and a weak equivalence.

Our development follows that for presheaves of simplicial groupoids in the previous section. We will omit proofs that are essentially the same.

A map \(q : G \rightarrow H\) in the category \(2\text{-GpdPre}(C)\) is a fibration if and only if it has the right lifting property with respect to all maps \(G(i) : GU \rightarrow GV\) induced by the maps \(i : U \rightarrow V\) where \(i\) is a trivial cofibration in the category \(S\text{Pre}(C)\) and \(V\) is \(\alpha\)-bounded, since there exist two adjoint diagrams similar to the diagrams \(D\).

For each \(S \in C\), \(\text{Ob}(GV(S)) = V(S)_0\), \(\text{Mor}(GV(S))\) and \(2\text{-cell}(GV(S))\) are free generated by \(V(S)_1\) and \(V(S)_2\), subject to some relations, respectively. So the cardinality of objects, morphisms and 2-cells of 2-groupoid \(GV(S)\) is smaller than \(\beta = \text{max}(2^\alpha, \infty)\), where \(\alpha\) is a bound of the simplicial presheaf \(V(S)\). We also call the presheaf of 2-groupoids \(GV\) is \(\beta\)-bounded.

For each 2-groupoid \(G\) and each object \(x\) of \(G\), there are natural isomorphisms [MS93 Proposition 2.1(iii)]:

\[
\begin{align*}
\pi_0(WG) & \cong \pi_0(G), \\
\pi_1(WG, x) & \cong \pi_1(G, x), \\
\pi_2(WG, x) & \cong \pi_2(G, x), \\
\pi_i(WG, x) & \cong 0 \ (i > 2).
\end{align*}
\]
According to the definition of topological weak equivalence of simplicial presheaves in [Jar87] and the above relations, we can give an explicit description of weak equivalence of presheaves of 2-groupoids.

For any presheaf of 2-groupoids $X$ and any object $U \in \mathcal{C}$ and $x \in \text{Ob}(X(U))$, there is a presheaf on the comma category $\mathcal{C} \downarrow U$, called the presheaf of homotopy groups of $X|_U$, based at $x$, which is defined by

$$\mathcal{C} \downarrow U \to \text{Grp}$$

$$\varphi : V \to U \mapsto \pi_i(X(V), x_V), i = 1, 2$$

where $x_V$ is the image of $x$ in $X(V)$ under the map $X(U) \to X(V)$ which is induced by $V \to U$.

Let $\pi_i(X|_U, x), i = 1, 2$ be the associated sheaves of the above presheaves. The sheaf $\pi_0(X)$ of path components is defined similarly.

A map $f : X \to Y$ of presheaves of 2-groupoids is a weak equivalence if it induces isomorphisms of sheaves

$$f_* : \pi_i(X|_U, x) \cong \pi_i(Y|_U, f x), i = 1, 2; U \in \mathcal{C}, x \in \text{Ob}(X(U))$$

$$f_* : \pi_0(X) \cong \pi_0(Y).$$

In parallel with the corresponding arguments for presheaves of simplicial groupoids, we have

**Lemma 4.2.** Given maps of presheaves of 2-groupoids $f : X \to Y$ and $g : Y \to Z$, if any two of $f, g$, or $g \circ f$ are weak equivalences, then so is the third.

**Lemma 4.3.** The functor $X \mapsto \overline{WG}(X)$ preserves weak equivalences of simplicial presheaves.

*Proof.* When $T$ is a simplicial set, there are isomorphisms [MS93]

$$\pi_0(\overline{GT}) \cong \pi_0(GT) \cong \pi_0(T),$$

$$\pi_i(\overline{GT}, t_0) \cong \pi_i(GT, t_0) \cong \pi_i(T, t_0) (i = 1, 2), t_0 \in T_0.$$  

$$\pi_i(\overline{GT}, t_0) = 0 (i > 2).$$

so there exist isomorphisms of sheaves

$$\pi_0(\overline{GX}) \cong \pi_0(GX) \cong \pi_0(X),$$

$$\pi_i(\overline{GX}|_U, x) \cong \pi_i(GX|_U, x) \cong \pi_i(X|_U, x) (i = 1, 2) U \in \mathcal{C} x \in X(U)_0.$$  

and $\pi_i(\overline{GX}|_U, x) = 0 (i > 2).$  

**Lemma 4.4.** The functor $G : S\text{Pre}(\mathcal{C}) \to 2-\text{Gpd}\text{Pre}(\mathcal{C})$ preserves cofibrations and weak equivalences.

**Lemma 4.5.** The category $2-\text{Gpd}\text{Pre}(\mathcal{C})$ has all pushouts, and is hence cocomplete. The class of cofibrations in $2-\text{Gpd}\text{Pre}(\mathcal{C})$ is closed under pushout.
Notice that there is a commutative diagram

\[
\begin{array}{ccc}
2 \mathbf{-Gpd Pre(C)} & \xrightarrow{L^2} & 2 \mathbf{-Gpd C} & \xrightarrow{\varphi^*} 2 \mathbf{-Gpd Shv(B)} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
\mathbf{S Pre(C)} & \xrightarrow{L^2} & \mathbf{S C} & \xrightarrow{\varphi^*} & \mathbf{S Shv(B)}
\end{array}
\]

\(WG\) is a locally fibrant simplicial presheaf for any presheaf of 2-groupoids \(G\), so a map \(f: G \to H\) of presheaves of 2-groupoids on \(C\) is a weak equivalence if the induced map \(\varphi^* L^2 : \varphi^* L^2 G \to \varphi^* L^2 H\) is a pointwise weak equivalence.

**Proposition 4.6.** Trivial cofibrations of presheaves of 2-groupoids are closed under pushout.

Given a trivial cofibration \(i: A \to B\) in the category \(\mathbf{S Pre(C)}\), suppose that

\[
\begin{array}{ccc}
GA & \xrightarrow{i} & C \\
\downarrow G(i) & & \downarrow i' \\
GB & \xrightarrow{i} & D
\end{array}
\]

is a pushout in the category \(2-\mathbf{Gpd Pre(C)}\). Then the map \(i'\) is a trivial cofibration.

**Lemma 4.7.** Every map \(f: X \to Y\) of presheaves of 2-groupoids may be factored

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow p \\
Z & & W
\end{array}
\]

where \(i\) is a trivial cofibration and \(p\) is a fibration.

A map \(p: G \to H\) of presheaves of 2-groupoids is a trivial fibration if and only if it has the right lifting property with respect to all inclusions \(GS \subset G\Delta^n_U\) of subobjects of the \(G\Delta^n_U, U \in C, n \geq 0\). A transfinite small object argument, as in Lemma 3.7, shows that

**Lemma 4.8.** Every map \(g: Z \to W\) of presheaves of 2-groupoids may be factored

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & W \\
\downarrow j & & \downarrow q \\
M & & \quad
\end{array}
\]

where \(j\) is a cofibration and \(q\) is a trivial fibration.

**Lemma 4.9.** For the commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow & & \downarrow p \\
V & \xrightarrow{j} & Y
\end{array}
\]
where \(i\) is a trivial cofibration and \(p\) is a fibration in the category \(2\text{-GpdPre}(\mathcal{C})\), there exists a lifting \(s\).

**Theorem 4.10.** The category \(2\text{-GpdPre}(\mathcal{C})\), with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.

**Lemma 4.11.**

1. The functor \(\overline{W} : 2 \rightarrow \text{GpdPre}(\mathcal{C}) \rightarrow \text{SPre}(\mathcal{C})\) preserves fibrations and weak equivalences.
2. The functors \(G\) and \(\overline{W}\) induce adjoint functors \(G : \text{Ho(SPre}(\mathcal{C})) \rightarrow \text{Ho(2 \rightarrow \text{GpdPre}(\mathcal{C}))} : \overline{W}\) at the level of homotopy categories.

**Proof.** (1) This is implied by the definitions of fibration and weak equivalence.

(2) The functors \(\overline{W}\) and \(G\) both preserve weak equivalences ((1) of this Lemma and Lemma 4.4), they localize to functors of homotopy categories. The triangular identities for the unit and counit will still hold after localization. □

**Corollary 4.12.** The adjunction \(G : \text{SPre}(\mathcal{C}) \rightleftarrows 2 \rightarrow \text{GpdPre}(\mathcal{C}) : \overline{W}\) is a Quillen adjunction.

**Proof.** This follows from Theorem 4.10, Proposition 4.1, and Lemma 4.4. □

Define the category \(2\text{-types}\text{SPre}(\mathcal{C})\) of homotopy 2-types to be the full subcategory of \(\text{Ho}(\text{SPre}(\mathcal{C}))\) given by those simplicial presheaves with sheaves \(\pi_i(X|_U, x) = 0\) for any integer \(i > 2\), any object \(U \in \mathcal{C}\) and any basepoint \(x \in X(U)_0\).

**Theorem 4.13.** The functors \(G\) and \(\overline{W}\) induce an equivalence of homotopy categories \(\text{Ho}(2 \rightarrow \text{GpdPre}(\mathcal{C})) \simeq 2\text{-types}\text{SPre}(\mathcal{C})\)

**Proof.** For a simplicial presheaf \(X\), the natural map \(\eta : X \rightarrow \overline{W}G(X)\) is a weak equivalence if and only if \(\pi_i(X|_U, x) = 0\) for all \(i > 2\), \(U \in \mathcal{C}\) and \(x \in X(U)_0\). For any presheaf of 2-groupoids \(K\), \(\pi_i(\overline{W}K|_U, *) = 0\), for all \(i > 2\), \(U \in \mathcal{C}\), and \(* \in \text{Ob}(K(U))\), and the natural map \(\varphi : GW(K) \rightarrow K\) is a weak equivalence. □

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