Abstract—We examined the effects of inhomogeneity on the dynamics and structural properties using Boolean networks. Two different power-law rank outdegree distributions were embedded to determine the role of hubs. The degree of randomness and coherence of the binary sequence in the networks were measured by entropy and mutual information, depending on the number of outdegrees and types of Boolean functions for the hub. With a large number of outdegrees, the path length from the hub reduces as well as the effects of Boolean function on the hub are more prominent. These results indicate that the hubs play important roles in networks’ dynamics and structural properties. By comparing the effect of the skewness of the two different power-law rank distributions, we found that networks with more uniform distribution exhibit shorter average path length and higher event probability of coherence but lower degree of coherence. Networks with more skewed rank distribution have complementary properties. These results indicate that highly connected hubs provide an effective route for propagating their signals to the entire network.

I. INTRODUCTION

Recent studies of complex networks [1] have shown inhomogeneous connectivities having a small number of highly connected nodes, frequently called as hubs, along with many poorly connected nodes. The inhomogeneity in connectivity has large effects on the property and/or function of the hubs; for example robustness in metabolic, genetic regulatory and neural networks [2]; performance of artificial neural networks [3]; and dynamics of coupled oscillators [4], [5].

The Boolean network [6] is one of the discrete dynamical models for the transcriptional regulatory network and exhibits binary sequences of the state variables that represent expression pattern of the network [6], [7]. Since the state variables in the network are sensitive to inputs from other nodes via directed edges, and affect other nodes, the quality of communication is characterized as the size of mutual information [8]. This mutual information indicates the degree of coherence, synchronization, amount of information content in the state variables, or potential for computational capability of the network [9].

In this study, we show the role of hubs for the emergence of coherence in a Boolean network. Since we embedded power-law rank outdegree distributions in the Boolean networks with an input connectivity of $K_{in}=2$, the model networks have some hubs that integrate many outdegree connections. Because the hubs synchronously transmit their state to the downstream nodes, these nodes are simultaneously affected by single or multiple hubs. The structural condition seems to automatically provide global coherence in the state variables; however, the structural aspects give only the possible effects of the hubs. In fact, we need to consider a type of Boolean function at the hubs and path length from the hubs. We show both the effects of Boolean functions and path length on both the event probability and size of entropy and mutual information.

II. MODEL

Dynamics of the Boolean networks [6], [7] are determined by

$$X_i(t+1) = B_i [X(t)] \quad (i=1,2,\ldots,N),$$

where $X_i(t)$ is the binary state, 0 or 1, of node $i$ at time $t$; $B_i(\cdot)$ is the Boolean function [see Tables I–III] used to simultaneously update the state of node $i$; and $X(t)$ is a binary vector that gives the states of the $N$ nodes in the network [See Fig. 2(a)]. After assigning initial states $X(0)$ to the nodes, the successive states of the nodes are updated by input states and their Boolean function. The dynamical behavior of these networks is represented by the time series of the binary states. The time course follows a transient phase from an initial state until a periodic pattern, known as an attractor, is established.

TABLE I

| Input | Output |
|-------|--------|
| 0 0   | 0 0 1 1 |
| 0 1   | 0 0 1 0 |
| 1 0   | 0 0 1 0 |
| 1 1   | 0 0 1 0 |

TABLE II

| Input | Output |
|-------|--------|
| 0 0   | 0 0 1 1 |
| 0 1   | 0 0 1 1 |
| 1 0   | 0 0 1 0 |
| 1 1   | 0 0 1 0 |

III. NUMERICAL CONDITION

We randomly constructed $10^4$ Boolean networks in each power-law rank distribution [see Fig. 1] with a fixed network...
TABLE III
CONSTANT type Boolean functions with indegree \( K_{in} = 2 \).
Since output probability of one or zero equals 1 or 0 with random binary inputs the functions have the same output entropy, \( H(0) = H(1) = 0 \).

| Input | Output |
|-------|--------|
| 0 0   | 0 1    |
| 1 0   | 0 1    |
| 1 1   | 0 1    |

Since output probability of one or zero equals 1 or 0 with random binary inputs the functions have the same output entropy, \( H(0) = H(1) = 0 \).

size. \(2 \times 10^3\) initial states were applied to each network. Sixteen different Boolean functions [see Tables I–III] were used with equal probabilities. Note that all the generated networks use the same amount of resources since the size of the network is fixed—256 nodes, 512 directed edges, and 16 sets (\(16^2 = 256\)) for all the Boolean functions [see Table VI in Section VI]. We measured entropy (randomness) and mutual information (coherence) of the state variables to characterize the dynamics of the Boolean networks [7], [8], [9] [see Fig. 2].

IV. RESULTS

A. Dynamics

In total, we obtained 137254 and 459240 attractors from type I and type II distributions, respectively. The size of entropy and mutual information were measured from the attractors. Major statistics of dynamical properties are shown in Table IV. By comparing the effect of the skewness of the two different power-law rank distributions, we found that on average networks with more uniform distribution (type II) exhibit higher event probability of coherence, but lower degree of coherence.

B. Path length

To get structural property of propagating route of state variables, we measured two properties:

1) Path length which the average number of the directed edges in the shortest path from a node to all reachable nodes.
2) Average path length which the average number of the path lengths for all the nodes.

Figure 3 shows the differences in the both path lengths. Nodes with higher outdegrees have shorter path lengths.

Theoretical relationship between the number of outdegrees, \(X\) of a starting node and path length, \(i\) can be written as [For details, see Section VI.]

\[
i = \log_{K_{in}} \left( \frac{255}{X} + 1 \right).
\]

We obtained good relationship between numerical results and Eq. (2).
2.5 3 3.5 4 4.5 5 5.5

Path length

Rank

1 2 3 4 5 6 7 8 9

(a)

1 2 3 4 5 6 7 8 9

Outdegree from a starting node

0 2 4 6 8 10

1 10 100 2 5 50 20

1st 2nd 4th 9th 5th 1st 2nd

(type I)

(type II)

(k = 2)

(k = 3)

(k = 10)

Average path length for type II = 7.73
Average path length for type I = 10.4

1 2 3 4 5

Proportion of positive entropy

(type I)

(type II)

High Output connectivity Low

Path length

Long Short

Fig. 3. (a) Relationships between the rank of the hubs [see Fig. 1] and the path length. (b) Dependence of path length on the number of outdegree from a starting node. The dotted lines show the relationship of Eq. (2) [see Section VI]. Both path lengths are obtained from $10^4$ generated networks with $N = 256$.

C. Rank Dependent Dynamics

Together with the network structural condition, we show the dependence of rank distributions on entropy and mutual information in Figs. 4 and 5. We classified different 16 Boolean functions into three types (AND-OR, XOR, and CONSTANT types) based on the input–output relationships [10], [11]. The dependence of the Boolean function on entropy and mutual information is prominent in higher ranked hubs on both the outdegree distribution styles. The Boolean functions with larger entropy tend to have larger mutual information. These results have weak dependence but clear tendencies, and suggest that the collective (global) coherence in the state variables of the networks is subject to the style of upstream (local) conditions, including the number of outdegrees from a hub and assignment of the Boolean functions.

V. CONCLUSIONS

We have summarized the results in Table V. A critical control parameter for synchronizability of coupled oscillators in complex networks is coupling strength or path length among the oscillators [4], [5]. Since our Boolean networks consist of the same amount of resources, some nodes integrate many edges, while others have less. As shown in Fig. 3, with a larger number of outdegree from hubs, the path length from the hubs reduces. The shorter path length can result in a stronger coupling strength between the hubs and other nodes. Because of the differences between type I and type II distributions, the higher ranked hub in type I distribution have an advantage to transmit their signals [see Figs. 4(a) and 5(a)], and the hubs in type II distribution have longer path lengths and show smaller dependences [see Figs. 4(b) and 5(b)]. On the other hand, type II distributions exhibit higher event probability of entropy and mutual information and a large number of attractors [see Tables IV and V]. The resultant dynamical properties are due to the more decentralized topology that can lead to the presence of some mid-rage hubs. The hubs permit local and/or weak coherences to emerge in the networks.

The higher indegree $K_{in} > 2$ provides higher density of the edges as well as a narrower range of the path length [see Fig. 3], and therefore the dependence of the outdegree from a hub will become less prominent.

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VI. APPENDIX

Since the networks are randomly constructed with $K_{in} = 2$, all the nodes have two downstream nodes, on average, regardless of outdegree distribution styles. The total number of downstream nodes from a starting node $T(i)$ can be written as

$$T(i) = \sum_{n=0}^{i} 2^n - 2^i - 1,$$

(3)

where $i$ is the number of steps (path length) from the starting node. With increasing $i$, the total number of downstream nodes increases. When the starting node has $X$ outdegree,

$$XT(i) = X(2^i - 1).$$

(4)

Since the size of network is limited to 256 in the paper, the total number will reach to the 255.

$$X(2^i - 1) = 255,$$

(5)

where 255 indicates the exclusion of a starting node from the network. Finally, we can obtain the number of steps (path length),

$$i = \log_2\left(\frac{255}{X} + 1\right).$$

(6)

When $X = 1$, the $i$ takes 8, meaning that 8 steps are required to reach other nodes in the same network on average [see Fig. 3(b)]. For general form of Eq. (6),

$$i = \log_{K_{in}}\left(\frac{(N - 1)(K_{in} - 1)}{X} + 1\right),$$

(7)

where $K_{in}$ is indegree of the Boolean network and $N$ is the number of nodes in the network.

| Rank | $H$=XOR | $H$=AND-OR | $H$=CONSTANT |
|------|----------|------------|--------------|
| 1    | 3725     | 4995       | 1220         |
| 2    | 3736     | 5042       | 1222         |
| 3    | 3696     | 5029       | 1275         |
| 4    | 3710     | 5020       | 1270         |
| 5    | 3705     | 5095       | 1200         |
| 6    | 3725     | 4996       | 1279         |
| 7    | 3756     | 5006       | 1238         |
| 8    | 3768     | 5012       | 1220         |
| 9    | 3815     | 4951       | 1234         |

Table VI

Number of networks out of $10^3$ networks (type I) are indicated. Three different types of Boolean functions [see Tables I–III] at the hubs are used with equal probability. "H" in the table denotes the hubs Boolean function [see Fig. 2(b)].

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