Fourier decay rate of coin-tossing type measures

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Abstract

The Fourier decay rate of a coin-tossing type measure is investigated. Explicit estimation is given. Our method relies upon a classical result of Hartman and Kershner. As an application, we present a new example of a measure whose Fourier decay rate could be as slowly as we please, and for this measure almost every point is absolutely normal.

Keywords: Fourier transforms, Coin-tossing type measures, Normal numbers

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1. Introduction

Given a Borel probability measure on $\mathbb{R}$, the Fourier-Stieltjes transform of $\mu$ is defined by

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{2\pi itx} d\mu(x), \quad t \in \mathbb{R}. \quad (1.1)$$

The decay rate of $\hat{\mu}(t)$ as $|t| \to \infty$ can provide lots of information about $\mu$. For instance, the absolute continuity and singularity of the measure $\mu$, the equidistribution phenomenon [15] and Roth-type theorem [19] on the support of the measure $\mu$. It is well known that if the Fourier decay rate of a measure is polynomial, that is $|\hat{\mu}(t)| = O(|t|^{-\epsilon})$ for some positive constant $\epsilon$, then the Hausdorff dimension of $\mu$ is no less than $2\epsilon$. Moreover, $\mu$ almost every point is absolutely normal (normal to any integer bases). In fact, the condition of polynomial decay can be weakened to logarithmic decay, see [21]. So one would ask: do there exist some measures which have slower Fourier decay rate, yet to guarantee that $\mu$ almost every point is absolutely normal? We answer this question positively by providing an explicit example.

Although the Fourier decay rate of measure plays a very important role in Fourier analysis, dynamical systems or fractal geometry, it is not easy to calculate, and often even to estimate. Of course, it may also happen that there is no decay at all. For instance, the standard middle-third Cantor measure has no decay rate, i.e., $\hat{\mu}(t) \to 0$ as $|t| \to \infty$, while its asymptotic average of its Fourier transform tends to 0. This has been proved in 1939 by Wiener and Wintner and generalized to more general self-similar measures, see [27, 28].

The first nontrivial examples which can be explicitly estimated are Bernoulli convolutions $\mu_\lambda$, i.e., the distribution of the random sum $\sum \pm \lambda^n$, where the signs are chosen independently.
with probability $\frac{1}{2}$. It is Kershner and N. K. Bary (independently) who first showed that the Fourier decay rate has close relation with the arithmetic property of $\lambda$ [18]. More precisely, the Fourier decay rate of $\mu_{\lambda}$ is logarithmic when $\lambda$ is a rational number, similar results for some classes of algebraic numbers, see [2, 7, 11]. And Erdős [9, 10] proved that the decay rate of $\hat{\mu}_{\lambda}(t)$ is polynomial for a.e. $\lambda$ sufficiently close to one, for more information in this direction, see the survey [24].

Recently there are many interests in understanding the Fourier decay of various measures. For instance, the behaviour of Fourier decay rate of measures which are images of smooth perturbation of self-similar measures [13, 23]. Very recently, by using the discretized sum-product theorem on $\mathbb{R}$, Bourgain and Dyatlov [1] proved that the Patterson-Sullivan measure on the limit set of a co-compact subgroup of $SL(2, \mathbb{Z})$ has polynomial decay rate. For similar results about Furstenberg measures on $SL(2, \mathbb{R})$, see [20], Gibbs measures of the Gauss map, see [15, 16].

In this paper, we are concerned with the coin-tossing type measure $\mu$ which is an infinite convolution on $(0, 1)$ and satisfies

$$\mu = *_{n=1}^{\infty} \left( \frac{1}{2} (1 + \phi(n)) \delta_0 + \frac{1}{2} (1 - \phi(n)) \delta_{2^{-n}} \right), \quad (1.2)$$

where $\phi : \mathbb{N} \to (0, 1)$ is a weight function, and $\delta_x$ is the Dirac measure at $x$. There is an alternative way to interpret the measure $\mu$ from probabilistic viewpoint. Consider the random sum

$$S = \sum_{n=0}^{\infty} X_n 2^{-n},$$

where $\{X_n\}_{n \geq 0}$ are the independent random variables satisfying

$$P(X_n = 0) = \frac{1}{2} (1 + \phi(n)), \quad P(X_n = 1) = \frac{1}{2} (1 - \phi(n)).$$

In fact, $X_n$ can be interpreted as a random variable, which is one if the n-th coin is head (with probability $\frac{1}{2} (1 - \phi(n)))$, and zero otherwise in an infinite coin-tossing experiment. Then $\mu$ is the distribution of $S$, i.e., $\mu(A) = P(\omega : S(\omega) \in A)$.

There are two basic questions to ask: is the measure $\mu$ absolutely continuous or singular (with respect to Lebesgue measure)? Does the Fourier transform of it vanish at infinity (if so, we call it Rajchman measure)? For the first question, the classical Jessen-Wintner law of pure types implies that $\mu$ is either absolutely continuous or singular. In 1942, Salem proved that the above coin-tossing type measure is singular if and only if the series $\sum \phi^2(n)$ is divergent. The starting point (from geometric viewpoint) of Salem is to present a direct and simple construction of a singular strictly increasing function (the previous known examples are usually either not very simple, or their construction is not direct). In the construction, Salem used an elegant idea which was from the well-known Borel’s normal numbers theory. Moreover, he proved that $\phi(n) \to 0$ is a necessary condition of the measure $\mu$ being a Rajchman measure.

For the second question, In 1971, Blum and Epstein [5] used the elementary method to prove that $\phi(n) \to 0$ is not only a necessary but also a sufficient condition of $\mu$ being a Rajchman measure, for more information, the reader could see Section 6.7 of [13] or [22]. So there is a natural question remaining: what are explicit Fourier decay rates for various
weight functions $\phi$?

Observe that when $\phi(n) \equiv 0$, $\mu$ degenerates to the Lebesgue measure. Heuristically speaking, the more rapidly of $\phi$ decrease to 0, the more rapidly for the corresponding measure should asymptotically tend to 0. Our main result of this paper is to show that this is indeed the case. As far as we know, there is only one explicit decay rate result, due to Hartman
and Kershner, who proved that $|\hat{\mu}(t)| = O(\log^{-\frac{1}{2}} |t|)$ when $\phi(n) = n^{-\frac{1}{2}}$. We follow Hartman-Kershner method to present some explicit decay rates for general various weight functions $\phi$. Finally, there are some other related investigations about the coin-tossing measures, such as multifractal analysis and Strichartz-type inequalities, see [4] and references therein.

Our main result is the following:

**Theorem 1.1.** Suppose $\phi : \mathbb{N} \to (0, 1)$ is a monotone function decreasing to zero. Let $\mu$ be the corresponding measure defined as (1.2). Then we have

1. When $\frac{\phi(n)}{\phi(n+1)} < 2$ for any $n$, then $|\hat{\mu}(t)| = O(\phi([\gamma_1 \log_2 |t|]))$, where $\gamma_1 \in (0, 1)$ is a constant depending only on $\phi$;

2. When $\frac{\phi(n)}{\phi(n+1)} \geq 2$ for any $n$, then $|\hat{\mu}(t)| = O(|t|^{-\gamma_2})$, where $\gamma_2 \in (0, 1)$ is a constant depending only on $\phi$.

**Remark 1.1.** Typical examples of weight functions $\phi$ in case (1) are $(\log n)^{-\tau_1}, n^{-\tau_2}, \tau_1, \tau_2$ are positive constants, while for case (2) are $\kappa^{-n}, \kappa$ is constant (no smaller than 2). We also note that the constants $\gamma$ can be explicitly provided in the term of $\phi$ (see Section 3). One may expect that the above theorem is valid if 2 is replaced by any integers $a$ in the definition of (1.2). In fact, we will see that if so, the corresponding measure will have no decay rate. Furthermore, our proof is also based on the behaviour of the fractional part of $\{2^{-n}t\}_{n \geq 1}$, here 2 plays a very key role, see Proposition 2.1 in Section 2.

Recall that a real number $x \in [0, 1)$ is said to be normal to the base $b \in \mathbb{N}, b \geq 2$, if $x = 0, x_1x_2 \cdots$ is expanded in the base $b$, every combination of digits occurs with the equal frequency. Equivalently, $\{b^n x\}_{n \geq 1}$ is uniformly distributed modus one. Further, if $x$ is normal for any base $b$, then $x$ is called absolutely normal. The Borel's normal numbers theorem states that Lebesgue measure almost every number in $[0, 1)$ is absolutely normal. In 1964, Kanhane and Salem asked whether the same is true with respect to any Borel measure whose Fourier coefficients vanish at infinity. In 1986, Lyons constructed a counterexample (a variant of coin-tossing type measure) which answers Kahane-Salem problem negatively. Moreover, he gave an almost best possible condition on the rate of decay in order that $\mu$ almost every number is normal (see Section 2). For general measures, Kanhen-Salem problem is false. However, for the specific coin-tossing measures, we shall show that Kahane-Salem problem is positive.

**Theorem 1.2.** Let $\mu$ be a measure as the above Theorem 1.1. Then $\mu$ almost every point is absolutely normal.

**Remark 1.2.** This theorem is implicitly contained in the paper [3], which based on the modified Schmidt’s lemmas and Davenport-Erdős-Leveque criterion. For completeness, in
Section 2 we shall give an alternative proof which is more elementary (originated from Cassels and Schmidt). Our result could imply that the converse of Davenport-Erdős-Leveque criterion does not hold, more precisely, if $\mu$ every point $x$ is absolutely normal, then it may have decay rate as slowly as we please. So there is a natural question: does there exist a measure which has no decay rate, and for which almost every point is absolutely normal? Unfortunately, the proofs of above results do not seem to give any information about the question.

Combining Theorem 1.1 with Theorem 1.2, we immediately obtain

**Corollary 1.1.** There exists a Rajchman measure whose Fourier decay tends to zero as slowly as we please, and for which almost every point is absolutely normal.

2. Preliminaries

In this section, we give some notations and preliminaries. We use $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some absolute constant $C$. If we need the implied constant $C$ to depend on a parameter, we will indicate this by subscripts; thus, for instance, $O_K$ denotes a quantity bounded in magnitude for some $C_K$ depending only on $K$ ($K$ may be a function).

Throughout this text, we use $\| \cdot \|$ to denote the distance to the nearest integer, $\lceil x \rceil$ the smallest integer larger than $x$, $[1, N]$ the set $\{1, 2, \ldots, N\}$ and $e(x) = e^{2\pi ix}$ respectively.

To begin with, we give an easy lemma which motivates our results.

**Lemma 2.1.** Let $a \geq 3$ be an integer and let

$$
\mu_a = \ast_{n=1}^{\infty} \left( \frac{1}{2} (1 + \phi(n)) \delta_0 + \frac{1}{2} (1 - \phi(n)) \delta_{a^{-n}} \right),
$$

then we have

$$
|\widehat{\mu_a}(t)| \to 0 \quad \text{as} \quad t \to \pm \infty.
$$

Recall that the characterization of $\mu_2(a = 2)$ being a Rajichman measure is $\phi(n) \to 0$. We shall prove that for any weight function $\phi$, $\mu_a$ are not Rajichman measures when $a \geq 3$ is an integer.

**Proof.** By the multiplication rule of Fourier transform of measures, it is easy to see that

$$
\widehat{\mu_a}(t) = \prod_{n=1}^{\infty} \left( \frac{1}{2} (1 + \phi(n)) + \frac{1}{2} (1 - \phi(n)) e(a^{-n}t) \right).
$$

Hence

$$
|\widehat{\mu_a}(t)|^2 = \prod_{n=1}^{\infty} \left( \frac{1}{2} (1 + \phi^2(n)) + \frac{1}{2} (1 - \phi^2(n)) \cos 2\pi a^{-n}t \right)
$$

$$
= \prod_{n=1}^{\infty} \left( \phi^2(n) \sin^2(\pi a^{-n}t) + \cos^2(\pi a^{-n}t) \right)
$$

$$
\geq \prod_{n=1}^{\infty} \cos^2(\pi a^{-n}t).
$$

4
Choosing $t$ along the sequence $(a^k)_{k \geq 1}$, we have

$$|\widehat{\mu}_a(a^k)|^2 \geq \prod_{n=1}^{\infty} \cos^2 \pi a^{-n+k} = \prod_{n=1}^{\infty} \cos^2 \pi a^{-n} \pi \geq \cos^2 \pi a \prod_{n=2}^{\infty} \cos^2 \frac{\pi}{2^n} = \frac{4}{\pi^2} \cos^2 \frac{\pi}{a} > 0.$$ 

Thus we have $|\widehat{\mu}_a(t)| \to 0(t \to \pm \infty).$

\[\square\]

**Remark 2.1.** Indeed, by the well-known Salem-Zygmund theorem [26], we can prove if $a$ is Pisot number, the conclusion also holds.

From the above lemma, so here and below, we only consider the case $a = 2$. The following equation will be repeatedly used

$$\widehat{\mu}(t) = \prod_{n=1}^{\infty} \left( \frac{1}{2} (1 + \phi(n)) + \frac{1}{2} (1 - \phi(n)) \cos(2^{-n}t) \right). \quad (2.1)$$

So

$$|\widehat{\mu}(t)|^2 = \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n}t + \cos^2 \pi 2^{-n}t). \quad (2.2)$$

In order to estimate the Fourier decay rate, we need to study the factors of the infinite product in the right-hand of the above equations, so it reduces to investigate the fraction part of the sequence $\{2^{-n}t\}$.

The basic idea is that the larger of the value $\|2^{-n}t\|$ we have, the smaller of the value of $\cos^2 \pi 2^{-n}t$ (in the second term of equation (2.2)) it is, while the value of $\sin^2 \pi 2^{-n}t$ is large, but $\phi(n)$ is very small. Thus the whole factor $\phi^2(n) \sin^2 \pi 2^{-n}t + \cos^2 \pi 2^{-n}t$ is small.

For convenience, we consider the set

$$A_\phi := \{t : \|t\| < 2^{-K_\phi} \},$$

where $K_\phi$ is a large number which depends on $\phi$. i.e., the set consists of all points $t > 0$ which are within a distance $2^{-K_\phi}$ of an integer. Note that $K_\phi$ will be chosen later and may be taken different values in different contexts.

Let

$$R_n = \left\{ t : |t - \frac{k}{2}| < 2^{-K_\phi - n}, \text{ for some odd } k \right\}$$

denotes the set of all points $t > 0$ which are at a distance at least $\frac{1}{2} - 2^{-K_\phi - n}$ to the nearest integer. Clearly if the point $t$ falls into the set $R_n$, the value of $\cos^2 \pi t$ is small. Here, we have the following lemma.

**Lemma 2.2.** 1) If $t \notin A_\phi$, then there exists a positive $\delta = \delta_\phi < 1$ such that

$$|\phi^2(n) \sin^2 \pi t + \cos^2 \pi t| < \delta < 1, \text{ for any } n \geq 1.$$  

2) If $t \in R_n$, then

$$|\cos \pi t| \leq \pi 2^{-n-K_\phi}.$$
Proof. It is obvious that
\[ |\phi^2(n) \sin^2 \pi t + \cos^2 \pi t| = |1 - (1 - \phi^2(n)) \sin^2 \pi t| \leq 1 - (1 - \phi^2(1)) \sin^2 \pi 2^{-K_{\phi}} =: \delta < 1. \]
For 2), by the definition of \( R_n \),
\[ |\cos \pi t| \leq |\sin(\pi 2^{-K_{\phi-n}})| \leq \pi 2^{-K_{\phi-n}}. \]

Now let \( t \geq 2 \) be fixed, we assume that \( m \geq 1 \) is the unique integer such that
\[ 2^m \leq t < 2^{m+1}. \]
Without loss of generality, we assume that

(i) There exist exactly \( k \) (\( k \leq m \)) indices for which \( 2^{-n} t \in A_\phi \) among the indices of \( 1, 2, \cdots, m \).

(ii) these \( k \) values of \( n \) consist of \( j \) (\( 1 \leq j \leq k \)) blocks \( B_i \) (\( i = 1, 2, \cdots, j \)), each composed of \( l_i (k \geq l_i \geq 1) \) successive indices. Namely,
\[ \sum_{i=1}^{j} l_i = k. \] \hspace{1cm} (2.3)

(iii) the blocks \( B_i \) are ordered in such a way that the indices contained in \( B_i \) are less than those contained in \( B_{i+1} \) (\( i = 1, 2, \cdots, j - 1 \)).

(iv) we define the \( n_i \) (good index) as the first index which does not fall into \( A_\phi \) after the block \( B_i \). Obviously,
\[ n_i \geq \sum_{s=1}^{i} l_s + 1. \] \hspace{1cm} (2.4)

Figure 1: Behavior of fractional part of \( \{2^{-n} t\}_{n \geq 1} \)

The following combinatorial proposition is useful in the proof of Theorem 1.1.

Proposition 2.1. Let \( n_i, l_i \) be as above, then we have \( 2^{-n_i} t \in R_{l_i} \).
Proof. Without loss of generality, write the block $B_i$ as (in the descending order),

$$B_i = \{n_i - 1, \ldots , n_i - l_i - 1, n_i - l_i\}.$$

By the assumption (i) and (iv), we have

$$2^{-n_i} t \notin A_\phi, 2^{-(n_i-1)} t \in A_\phi, 2^{-(n_i-2)} t \in A_\phi, \ldots , 2^{-(n_i-l_i)} t \in A_\phi.$$

By the definition of $A_\phi$, then

$$\begin{cases}
|2^{-n_i} t - k_0| \geq 2^{-K_\phi} \\
|2^{-(n_i-1)} t - k_1| < 2^{-K_\phi} \\
|2^{-(n_i-2)} t - k_2| < 2^{-K_\phi} \\
\vdots \\
|2^{-(n_i-l_i)} t - k_{l_i}| < 2^{-K_\phi}
\end{cases} \quad (2.5)$$

where $k_j (0 \leq j \leq l_i)$ is the integral part of $2^{-n_i} t$, respectively. We claim that $k_{l_i} = 2^{l_i-1} k_1$. Indeed, consider the two equations

$$2^{-(n_i-1)} t = k_1 + \varepsilon_1 \quad \text{and} \quad 2^{-(n_i-2)} t = k_2 + \varepsilon_2,$$

where $|\varepsilon_1| < 2^{-K_\phi}$, $|\varepsilon_2| < 2^{-K_\phi}$. From the above equations, we have

$$|k_1 - 2k_2| = |\varepsilon_1 - 2\varepsilon_2| < 3 \cdot 2^{-K_\phi} \leq 1. \quad (2.6)$$

Since the left hand side of (2.6) is an integer, it follows that $k_1 = 2k_2$. Similarly, if there are $l_i$ consecutive indices which fall into $A_\phi$, i.e.,

$$|\varepsilon_j| < 2^{-K_\phi} \quad \text{for all} \quad 1 \leq j \leq l_i,$$

then

$$k_{l_i} = 2^{l_i-1} k_1. \quad (2.7)$$

Substituting (2.7) into the last inequality of (2.5), we have

$$|2^{-n_i} t - k_1/2| < 2^{-K_\phi-l_i}. \quad (2.8)$$

On the other hand,

$$|2^{-n_i} t - k_0| \geq 2^{-K_\phi}. \quad (2.9)$$

Combining (2.8) with (2.9), it is easy to see that $k_1$ is odd. Thus

$$2^{-n_i} t \in R_{l_i}.$$ 

\hfill \Box

Armed with the combinatorial Proposition 2.1, we have the following lemma which is useful in the estimate of the Fourier decay.

**Lemma 2.3.**

$$|\widehat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^j \left( \phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)} \right).$$
Proof. Recall the equation (2.2), we have

$$|\hat{\mu}(t)|^2 = \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n}t + \cos^2 \pi 2^{-n}t)$$

$$\leq \delta^{m-k-j} \prod_{i=1}^{j} (\phi^2(n_i) + \pi^2 2^{-2(i+K_\phi)}).$$

The reason of the last inequality is as follows: By assumption (i), there exist exactly $m - k$ values of $n$ ($n = 1, 2, \cdots, m$) such that $2^{-n}t$ is not in $A_\phi$. Among these indices, there are $j$ good indices $n_k, 1 \leq k \leq j$. For these good indices, from Lemma 2.2 and Proposition 2.1, we have

$$\phi^2(n_i) \sin^2 \pi 2^{-n_i}t + \cos^2 \pi 2^{-n_i}t \leq \phi^2(n_i) + \pi^2 2^{-2(i+K_\phi)}.$$ 

The remaining $m - k - j$ ($\geq 0$) factors are dominated by $\delta$ from Lemma 2.2. Finally, all other factors are replaced by 1.

Remark 2.2. We can obtain the lower bound of the Fourier decay along some subsequence. Let $t = 2^m$, we claim that $|\hat{\mu}(2^m)|^2 \geq \frac{4}{\pi^2} \phi^2(m + 1)$. In fact,

$$|\hat{\mu}(2^m)|^2 = \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 2^{-n+m} \pi + \cos^2 2^{-n+m} \pi)$$

$$= \phi^2(m + 1) \prod_{n=m+2}^{\infty} (\phi^2(n) \sin^2 2^{-n+m} \pi + \cos^2 2^{-n+m} \pi)$$

$$\geq \phi^2(m + 1) \prod_{n=2}^{\infty} \cos^2 2^{-n} \pi = \frac{4}{\pi^2} \phi^2(m + 1).$$

In the remainder of this section, we present some known results about the Fourier transform of a measure and the normal numbers theory. First, we recall the well-known Davenport-Erdős-Leveque theorem.

Theorem 2.1. Let $\mu$ be a probability measure on $[0, 1]$ and $\{s_k\}_{k \geq 1}$ a sequence of positive integers with strictly increasing. If for any $h \in \mathbb{Z}\setminus\{0\}$,

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^{N} \hat{\mu}(h(s_n - s_m)) < \infty,$$

then for $\mu$ almost every $x$, the sequence $\{s_kx\}_{k \geq 1}$ is uniformly distributed modulo 1.

Employing the above criterion with Cauchy condensation lemma, it is not hard to prove that if the measure $\mu$ satisfies that

$$\sum_{n=2}^{\infty} \frac{\hat{\mu}(n)}{n \log n} < \infty,$$

then $\mu$ almost all $x$, $\{n_kx\}_{k \geq 1}$ is uniformly distributed mod 1, where $\{n_k\}_{k \geq 1}$ is any lacunary integer sequence. Furthermore, if the above series is divergent, there exists a counterexample which implies the conclusion does not hold (under some mild additional condition), the details see [21].

Next, in order to prove Theorem 1.2, almost in the same spirit of the case 1) of Lemma 2.2, here we need another lemma.
Lemma 2.4. Let $a, b$ be two positive real numbers, then
\[
|a + be(t)| \leq a + b - 4 \min\{a, b\}\|t\|^2.
\]

Proof. Without loss of generality, we may assume that $a \geq b$, otherwise, replace $t$ by $-t$. Thus, it suffices to prove $a + b - |a + be(t)| \geq 4b\|t\|^2$. Using the basic inequality
\[
\sin^2 t \geq \frac{4}{\pi^2}t^2, \quad \text{for all } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.
\]
Since $0 \leq \|t\| \leq \frac{1}{2}$, substituting $t$ for $\pi\|t\|$, we obtain
\[
\sin^2 \pi t = \sin^2 \pi\|t\| \geq 4\|t\|^2. \tag{2.10}
\]

It is easy to get
\[
(a + b + |a + be(t)|)(a + b - |a + be(t)|) = 4ab\sin^2 \pi t. \tag{2.11}
\]
Since $a, b$ are positive,
\[
a + b + |a + be(t)| \leq 4a. \tag{2.12}
\]
By (2.10) and (2.11), we get
\[
(a + b + |a + be(t)|)(a + b - |a + be(t)|) \geq 16ab\|t\|^2 \geq 4a4b\|t\|^2. \tag{2.13}
\]
Combing (2.12) with (2.13), we complete the proof.

Corollary 2.1.
\[
\left| \frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e(2^{-n}t) \right| \leq 1 - 2(1 - \phi(1))\|2^{-n}t\|.
\]

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we shall prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1: When $\frac{\phi(n)}{\phi(n + 1)} < 2$, by Lemma 2.3, we have
\[
|\hat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^{j} (\phi^2(n_i) + \pi^22^{-2(l_i+K\phi)}). \tag{3.1}
\]
By the assumption,
\[
\phi(1 + l_1) < 2^{l_2}\phi(1 + l_1 + l_2). \tag{3.2}
\]
From (2.4) and $\phi(n)$ is non-increasing, then
\[
\phi(n_i) \leq \phi(1 + l_1 + l_2 + \cdots + l_i). \tag{3.3}
\]
Now we estimate the right-hand side of (3.1) by induction on $j$. When $j = 2$, we have
\[
\prod_{i=1}^{2}(\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) = (\phi^2(n_1) + \pi^22^{-2(l_1+K_\phi)})(\phi^2(n_2) + \pi^22^{-2(l_2+K_\phi)}).
\]

Let \(K_\phi\) be large enough so that
\[
\phi^2(2) + \pi^24^{-1-K_\phi} + \pi^24^{-K_\phi} \leq 1 \quad \text{and} \quad \pi2^{-K_\phi} \leq 1.
\]

Then by (3.2) and (3.3), and using the monotonic property of \(\phi\), it follows that
\[
\prod_{i=1}^{2}(\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) = (\phi^2(n_1) + \pi^22^{-2(l_1+K_\phi)})(\phi^2(n_2) + \pi^22^{-2(l_2+K_\phi)})
\leq (\phi^2(2) + \pi^24^{-1-K_\phi} + \pi^24^{-K_\phi})\phi^2(1 + l_1 + l_2 + \pi4^22^{-2(l_1+l_2+2K_\phi)}
\leq \phi^2(1 + l_1 + l_2) + \pi^22^{-2(l_1+l_2+K_\phi)}.
\]

Here, we may choose
\[
K_\phi = K_{\phi,1} := \frac{1}{2} \log_2 \frac{5\pi^2}{4(1 - \phi^2(1))}.
\]

By induction, together with (2.3) and (3.2), then
\[
\prod_{i=1}^{j}(\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) \leq \phi^2(1 + l_1 + l_2 + \cdots + l_j) + \pi^22^{-2(l_1+l_2+\cdots+l_j+K_\phi)}
\leq \phi^2(1 + k) + \pi^22^{-2(k+K_\phi)}
\leq (1 + \frac{\pi^2}{4K_\phi \phi^2(1)})\phi^2(k).
\]

Hence,
\[
|\hat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^{j}(\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) \leq C\delta^{m-k-j}\phi^2(k),
\]

where \(C = 1 + \pi^24^{-K_\phi}\phi^{-2}(1)\) is a constant depending on \(\phi\).

Now, we estimate the inequality (3.4) under the condition that
\[
m - j - k \geq 0 \quad \text{and} \quad j \leq k \leq m.
\]

Let \(0 < \gamma_1 := \frac{-\log_2\delta}{2} < 1\), we distinguish two cases,

(a) if \(k \geq \gamma_1 m\), from (3.5) and \(\phi\) is non-increasing, it is easy to see that
\[
\delta^{m-k-j}\phi^2(k) \leq \delta^0\phi^2([\gamma_1 m]) = \phi^2([\gamma_1 m]).
\]

(b) if \(k < \gamma_1 m\), we have
\[
\delta^{m-k-j}\phi^2(k) \leq \delta^{m-k\gamma_1}\phi^2(k) \leq \delta^{m-k}/\gamma_1\phi^2(k) = \phi^2([\gamma_1 m]) \frac{\phi^2(k)\delta^m}{\phi^2([\gamma_1 m])\delta^\gamma_1}.
\]
By the assumption of $0 < \phi(n) < 2\phi(n + 1)$, and the definition of $\gamma_1$, we obtain
\[
\frac{\phi^2(k)\delta^m}{\phi^2([\gamma_1m])\delta^k} \leq \frac{\phi^2(k)\delta^{[\gamma_1m]-1}}{\phi^2([\gamma_1m])\delta^k} = \delta^{[\gamma_1m]-k} \cdot \frac{\phi^2(k)}{\phi^2([\gamma_1m])}\delta^{-\frac{1}{\gamma_1}}.
\]
Together (a) with (b), we have
\[
|\hat{\mu}(t)|^2 \leq 4\phi^2([\gamma_1m]).
\]
Recall that
\[
2^m \leq t < 2^{m+1},
\]
therefore, we obtain that
\[
|\hat{\mu}(t)| \leq 2\phi([\gamma_1 \log_2 |t|]). \tag{3.6}
\]
Now we turn to the proof of the second part of Theorem 1.1. Similarly, we have
\[
\prod_{i=1}^2 (\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) = (\pi^22^{-2(l_1+K_\phi)} + \phi^2(n_1))(\pi^22^{-2(l_2+K_\phi)} + \phi^2(n_2))
\]
\[
\leq (\pi^42^{-4K_\phi} + \pi^22^{-2K_\phi}\phi^2(1) + \pi^22^{-2(1+K_\phi)}\phi^2(1))2^{-2(1+l_1+l_2)} + \phi^2(1 + l_1 + l_2)
\]
\[
\leq 2^{-2(l_1+l_2)} + \phi^2(1 + l_1 + l_2).
\]
The above inequalities have used the non-increasing property of $\phi$ and the assumption (1) of Theorem 1.1, and the fact of $\phi$ is a weight function ($0 < \phi < 1$).

Here, we choose $K_\phi$ large enough so that
\[
\pi^42^{-4K_\phi} + \pi^22^{-2K_\phi}\phi^2(1) + \pi^22^{-2(1+K_\phi)}\phi^2(1) < 1.
\]
In fact, we can choose $K_\phi = K_{\phi,2} := 3$. Then, by induction we have
\[
\prod_{i=1}^j (\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)}) \leq 2^{-2(l_1+l_2+\cdots+l_j)} + \phi^2(1 + l_1 + l_2 + \cdots + l_j)
\]
\[
= 2^{-2k} + \phi^2(1 + k) \leq (1 + \phi^2(1))2^{-2k}. \tag{3.7}
\]
By Lemma 2.3 and (3.7), it is obvious that
\[
|\hat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^j (\phi^2(n_i) + \pi^22^{-2(l_i+K_\phi)})
\]
\[
\leq C_1\delta^{m-k-j}\cdot 2^{-2k}, \tag{3.8}
\]
where $C_1 = (1 + \phi^2(1))$ is a constant depending on $\phi$.

Next we estimate the above inequality (3.8) under the condition that
\[
m - j - k > 0 \text{ and } j \leq k \leq m.
\]
Let $0 < \gamma_2 := \frac{-\log_2 \delta}{2} < 1$, we distinguish two cases
(a) if \( k \geq \gamma_2 m \), since \( 0 < \delta < 1, m - j - k \geq 0 \), we have
\[
\delta^{m-k-j-2k} \leq \delta^{0} 2^{-2\gamma_2 m} = 2^{-2\gamma_2 m}.
\]

(b) if \( k < \gamma_2 m \), we have
\[
\delta^{m-k-j-2k} \leq \delta^{m-2k-j-2k} \leq \delta^{m-k/\gamma_2 - 2k} \leq 2^{-2\gamma_2 m} \delta^{k/\gamma_2 - 2k} \leq 2^{-2\gamma_2 m}.
\]

Combining (a) with (b), we have
\[
|\hat{\mu}(t)|^2 \leq C_2 2^{-2\gamma_2 m}.
\]

Let \( t \) be large enough, and assume that \( m \geq 1 \) be the unique integer such that
\[
2^m \leq t < 2^{m+1}.
\]

Therefore, we obtain that
\[
|\hat{\mu}(t)| \leq C_2 |t|^{-\gamma_2}.
\] (3.9)

Together (3.6) and (3.9), we obtain the results in Theorem 1.1.

**Proof of theorem 1.2**: We consider two cases: \( \log_b \log_2 \in \mathbb{Q} \) and \( \log_b \log_2 \notin \mathbb{Q} \).

Case one: \( \log_b \log_2 \in \mathbb{Q} \). It is well-known that \( x \) is normal for base \( b \) if and only if \( x \) is normal for base 2 provided \( \log_b \log_2 \in \mathbb{Q} \), so it suffices to prove that \( \mu \) every every \( x \) is normal for base 2, Since \( \phi \) tends to 0 at infinity, and from the aforementioned probability interpretation of the \( \mu \), the random variables \( X_n \) will have the same probability asymptotically. So the conclusion is easy to see from the strong law of large numbers.

Case two: \( \log_b \log_2 \notin \mathbb{Q} \). By Theorem 2.1, it suffices to prove the following
\[
\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\hat{\mu}(h(b^n - b^m))| = O(N^{2-\eta}),
\]
for any non-zero integer \( h \), where \( \eta < 1 \) is a positive constant. It is easy to see that
\[
\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\hat{\mu}(h(b^n - b^m))| \leq 2 \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} |\hat{\mu}(h(b^k - 1)b^n)| \leq 2N + 2 \sum_{k=1}^{N-1} \sum_{n=0}^{N-1} |\hat{\mu}(h(b^k - 1)b^n)|.
\]

Thus, for any \( h \in \mathbb{Z} \setminus \{0\} \), we only need to prove that
\[
\sum_{n=0}^{N-1} |\hat{\mu}(hb^n)| = O(N^{1-\eta}).
\]

For our specific coin-tossing measure, recall that
\[
|\hat{\mu}(h2^k)|^2 = \prod_{n=1}^{\infty} (\phi^2(n + k) \sin^2 \pi h 2^{-n} + \cos^2 \pi h 2^{-n}),
\]

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so if \( b = b_0 2^\tau \), where \( 2 \nmid b_0 \), and \( \tau \) is a nonnegative integer, we have

\[
|\hat{\mu}(hb^n)|^2 = |\hat{\mu}(hb_0^n 2^{m\tau})|^2 = \prod_{n=1}^{\infty} (\phi^2(n + m\tau) \sin^2 \pi hb_0^n 2^{-n} + \cos^2 \pi hb_0^n 2^{-n}) \\
\leq \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi hb_0^n 2^{-n} + \cos^2 \pi hb_0^n 2^{-n}) \\
\leq |\hat{\mu}(hb_0^n)|^2.
\]

Similarly, we could use the same procedure on \( h \), then it reduces to prove

\[
\sum_{n=0}^{N-1} \prod_{k=1}^{\infty} \left( \frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e(hb^n 2^{-k}) \right) = O(N^{1-\eta})
\]

under the restricted condition of \( 2 \nmid b, 2 \nmid h \).

First, we claim that

\[
\sum_{n=0}^{2^r-1} \prod_{k=1}^{\infty} \left( \frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e(hb^n 2^{-k}) \right) = O(2^{r(1-\eta_0)}).
\]

Assume that \( l \) is the largest integer such that \( b \equiv 1 \pmod{2^l} \), by the elementary number theory, \( \{b^n, 0 \leq n \leq 2^r - 1\} \) runs modulo \( 2^{l+r} \) through all residue classes which are congruent to \( 1 \) modulo \( 2^l \). Since \( 2 \nmid h \), \( \{hb^n, 0 \leq n \leq 2^r - 1\} \) runs modulo \( 2^{l+r} \) through all residue classes which are congruent to \( h \) modulo \( 2^l \). If we expand \( hb^n \) by base \( 2 \),

\[
hb^n = \sum_{k=0}^{\infty} d_k 2^k, \quad d_k \in \{0, 1\},
\]

then the string with length \( r \)

\[
d_0(n)d_1(n) \ldots d_{l+r-1}(n)
\]

(3.10)
take exactly one (and only one) of the possible sets of values as \( n \) runs from 0 to \( 2^r - 1 \) (this is first discovered by Cassels, see [6]).

Next, we classify the indices \( n(0 \leq n \leq 2^r - 1) \) into two cases according to the fractional part of \( hb^n 2^{-k} \). For convenience, setting \( \mathcal{P} = \{(0,1), (1,0)\} \) (we call \( (0,1), (1,0) \) regular pairs). We divide the index into two cases, \( I \) and \( II \). For \( I \), the index \( n \) belongs to \( I \) if in the string (3.10) there are at least \( \varepsilon r(0 < \varepsilon < \frac{1}{2}) \) pairs of \( (d_k,d_{k+1}) \) belong to \( \mathcal{P} \); For \( II \), the remaining indices \( n \).

For case \( I \), if \( (d_k,d_{k+1}) \in \mathcal{P} \), clearly,

\[
\|hb^n 2^{-k-2}\| > \frac{1}{4}.
\]

By Corollary 2.1, we have

\[
\sum_{n \in I} \prod_{k=1}^{\infty} \left( \frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e(hb^n 2^{-k}) \right) \leq 2^r \delta_2^r = 2^{r(1+\varepsilon \log \delta_2)} = 2^{r(1-m)}, \quad (3.11)
\]

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where $\delta_2 := \frac{1}{2} + \frac{1}{2}\phi(1), \eta_1 := -\varepsilon \log(\frac{1}{2} + \frac{1}{2}\phi(1)) \in (0, 1)$.

For $II$, we claim that the number of $n$ such that there are less than $\varepsilon r$ regular pairs in the string (3.10) is at most $2^{2r}$. It is easy to see that we only show that the claim under the restriction condition of $k$ is odd. We know that combination of the string in the string (3.10) such that there are exactly $s(s < \varepsilon r)$ indices belong to $P$ is

$$\left(\frac{|r|}{s}\right)^{2^{s}2^{\frac{|r|}{s} - s}} = \left(\frac{|r|}{s}\right)^{2^{\frac{|r|}{s}}}. $$

Thus, the number of $n$ such that there are less than $\varepsilon r$ indices $k$ regular pairs can not exceed

$$\sum_{s=0}^{\lceil \varepsilon r \rceil} \left(\frac{|r|}{s}\right)^{2^{\frac{|r|}{s}}} \leq C \exp(h(\varepsilon)r)2^{\frac{|r|}{s}} \leq 2^{(\frac{1}{2} + \rho(\varepsilon))r},$$

where the second inequality is by Stirling’s formula and $h(t) = -t \log t - (1-t) \log (1-t)$, $\rho(\varepsilon)$ is a small constant which depending on $\varepsilon$. Note that $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. Consequently,

$$\sum_{n \in II} \prod_{k=1}^{\infty} |\left(\frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e(hb^{n}2^{-k})\right) | \leq 2^{(\frac{1}{2} + \rho(\varepsilon))r} := 2^{r(1-\eta_2)}, \quad (3.12)$$

where $\eta_2 := \frac{1}{2} - \rho(\varepsilon)$.

Together (3.11) with (3.12), then we obtain the claim.

For general $N$, writing

$$N = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_k}, r_1 < r_2 < \cdots < r_k.$$

Divide $[1, N]$ into $k$ part $[0, 2^{r_1}], [2^{r_1}, 2^{r_1} + 2^{r_2}], \cdots, [2^{r_1} + 2^{r_2} + \cdots + 2^{r_{k-1}}, N]$. For every part, they have the common form $[N_0, N_0 + 2^r]$, from the above claim, we have

$$\sum_{2^{r} > N_0 > i \geq N_0} |\hat{\mu}(hb^i)| = \sum_{2^{r} > i \geq 0} |\hat{\mu}(hb^{N_0}b^i)| = O(2^{(1-\eta_2)}) = O(N^{(1-\eta_0)}). \quad (3.13)$$

Clearly

$$k \leq \log(N + 1) - 1,$$

thus

$$\sum_{l=0}^{N-1} |\hat{\mu}(hb^l)| \leq CN^{(1-\eta_0)} \log(N + 1) = O(N^{1-\eta}). \quad (3.14)$$

Combining (3.13) with (3.14), so the proof is complete.

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