Weakly Non-ideal Bose Gas: Comments on Critical Temperature Calculations

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A number of calculations have appeared for the Bose-Einstein condensation temperature ($T_c$) of a weakly repulsive dilute Bose gas. After a short survey of previous work, I point out several issues related to these approaches, and outline future calculations.

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I. INTRODUCTION

During the recent upsurge of activity in the field of Bose-Einstein condensation (BEC), some older questions concerning the weakly interacting Bose gas have been revived. One such un-resolved question is the effect of a weak repulsion on the critical temperature $T_c$. Calculations on this question has produced widely dissimilar results: increases and decreases of $T_c$, proportional to $a$, $\sqrt{a}$, $a \ln a$, etc. have been reported by various authors. (Here $a$ is the scattering length.) It is only during the past 2-3 years that the community has been converging toward a consensus.

Considering the diverse approaches to the $T_c$ problem, and the variety of issues that have arisen in this connection, it seems appropriate at this point to attempt to draw some connections between the different approaches, and point out some misconceptions and fine points. The purpose in this article is to review and comment on existing work, rather than to present new calculations.

The model is described by the Hamiltonian

$$\hat{H} = \sum_k c_k \hat{b}_k^\dagger \hat{b}_k + \frac{1}{2V} \sum_{p,q,k} U(k) \hat{b}_{p+k} \hat{b}_{q-k}^\dagger \hat{b}_p^\dagger \hat{b}_q,$$

where $c_k = k^2/2m$ is the free-gas spectrum, and $\hat{b}_k, \hat{b}_k^\dagger$ are bosonic operators. In principle, one is interested in any possible interaction function $U(k)$, but the simplest possible form is a delta function in real space, so that $U(k) = U$ is momentum-independent. To first order $U$ is related to the s-wave scattering length $a$ by $U = 4\pi \hbar^2 a/m$. A dimensionless measure of the interaction is the quantity $an^{1/3}$, where $n = N/V$ is the density. The program is to examine the shift in the critical temperature $T_c$, as compared to a noninteracting gas of the same density, as a function of $an^{1/3}$, for small $an^{1/3}$.

In section II, the background is set by summarizing different calculations and basic results. Section III contains the comments on various approaches to the $T_c$ problem. In the concluding section IV, I suggest some directions for additional calculations.

II. BACKGROUND AND PERSPECTIVE

Early predictions for the shift in transition temperature appear in the work of Lee, Yang and Huang, around forty years ago, as summarized in [1]. Recently, Huang has published [26] a prediction for an increase in $T_c$ due to the inter-particle repulsion, proportional to the square root of the interaction strength, $T_c - T_c^{(0)} \sim \sqrt{an^{1/3}}$. Schakel in a short note [21], and Baym et al in [1], have both discussed the problem with this work; in particular [1] contains an analysis showing how Huang's result is generated spuriously from a truncation of the virial expansion used at mean field level.

A similar spurious result appearing in the literature is that of Toyoda [27], who in 1982 performed mean-field-like calculations deriving a Landau-Ginzburg type thermodynamic function. His prediction was a decrease of the transition temperature, $\Delta T_c \sim -\sqrt{an^{1/3}}$. Again, this is believed to be an artifact of mean field approximations. The present understanding is that there is no mean field effect on the transition temperature, and that higher-order correlations are required to find the leading shift in $T_c$.

A more recent work that also predicts a negative shift in $T_c$ is the canonical-ensemble work of Wilkens et al [20]. This work has been critiqued in [8]. I discuss some related issues in section III E.

In addition, in view of the continuing appearance of negative-shift predictions, I give in section III B a simple argument for the increase of the transition temperature due to the addition of a repulsive interaction. This argument makes use of eq (2), which is derived independent of any particular method of treating interaction effects.

During the past several years, a large body of work has been published by Baym, Laloe and collaborators [1, 3–8], and a number of authors have done followup work [9–11]. These workers predict a positive shift in $T_c$ that to leading order is linear in the scattering length: $\Delta T_c = c_1 an^{1/3} + O(a^2 n^{2/3})$. Attempts to calculate the coefficient $c_1$, however, have continued to give fluctuating results, and this has prompted the Baym group to predict [7] a logarithmic contribution at the next order, i.e., a $-a^2 \ln a$ contribution. The presence of such a non-analyticity might explain the difficulty in any numerical estimate of $c_1$. 

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Stoof’s renormalization-group analysis [12, 13] of the three-dimensional boson gas has led to a prediction [13, 14] of an increase of the transition temperature proportional to $a[\ln a]$, i.e., a non-analyticity at leading order. A leading non-analytic behavior would actually explain even better the uncertain results obtained in attempts to calculate the coefficient $c_1$, based on the assumption of linear shift $\Delta T_c \sim c_1 a n^{1/3}$.

An older attack on the $T_c$ problem is that of Kanno [18, 19]. In a series of papers, Kanno introduced a “quasilinear” canonical transformation approach to the many-body problem, which he used to calculate (among other things) the free energy and hence transition temperature of the boson gas. Since Kanno’s work has mostly been ignored by the current $T_c$ community, I give a description of his work in section III F. I also show how some of his derivations can be performed in a more transparent way.

Schakel’s work on developing an effective theory [22] of the weakly non-ideal Bose gas has led to the prediction [23] of a shift in the transition temperature that is strictly linear in the interaction. The connection of this calculation with other approaches has not yet been explored in detail. Some issues are pointed out in section III D.

Our recent report [2] addresses the question of $T_c$ within a quasiparticle framework. Other than working out details of a quasiparticle description (ultraviolet divergence, effect of quasiparticle lifetime), we also provide a non-selfconsistent method of regularizing the infrared divergences in the description of the transition point. This results in a shift of $T_c$ approximately $\propto a\sqrt{|\ln a|}$. The procedure is uncontrolled; however, this result points strongly to a non-analytic contribution in the dependence of $T_c$ on the interaction.

Other relevant calculations include Kleiner’s “five-loop” determination [31] of the coefficient $c_1$ mentioned above, assuming a purely linear leading shift, and the “linear $\delta$-expansion” calculations [32] of de Souza Cruz et al. Kleiner has questioned [31] the $\delta$-expansion method, but a detailed analysis of the relationship of this approach to others seems to be lacking in the literature.

A. Mean Field

For a momentum-independent potential, the two mean field contributions to the self-energy (Hartree and Fock) are equal, $\Sigma^{(1)}(k,\omega) = Un + Un = 2Un$. Since the self-energy is momentum-independent, it can simply be absorbed into a redefinition of the chemical potential.

$$E_k^{(1)} = \frac{k^2}{2m} - \mu + \Sigma^{(1)}(k, E_k^{(1)}) = \frac{k^2}{2m} - \mu'$$

$$N = N_0 + \frac{V}{\lambda^3} g_{3/2}(e^\beta \mu')$$

Here $\mu' = \mu - \Sigma^{(1)} = \mu - 2Un$. The transition now simply corresponds to $\mu' = 0$ instead of $\mu = 0$, and $T_c$ remains unchanged. The lesson here is that a momentum-dependent self-energy is required for a shift in $T_c$.

In second-order calculations, (e.g., in [2],) it is usual to have the 1st-order self-energy already absorbed in the chemical potential, i.e., to use $\mu'$ as the chemical potential. This allows one to exclude those second-order diagrams which can be obtained by 1st-order self-energy insertions into 1st-order diagrams.

Spatially extended interaction. In a real system, of course, the description of the interaction in terms of a single parameter ($U_0$ or $a$) is valid only for very weak interactions or at very low temperatures/densities; if these conditions are not satisfied one has to take into account the detailed shape of the potential $V(x) = V_0\sigma(x)$ which is not a delta function in real space (not a constant in momentum space) any more. In such a case we get a momentum-dependent Fock term, and hence a contribution to $T_c$ from the first-order self-energy. Assuming a Gaussian $V(x)$ of width $a$ and height $V_0$, one gets a correction to the spectrum that can be incorporated as an effective mass $m^* = m[1 - V_0 a^2 \mu n/3]^{-1}$, resulting in a negative shift [16]:

$$\frac{\Delta T_c}{T_c^{(0)}} = -\frac{1}{3} \frac{mnV_0a^2}{T_c^{(0)}}.$$  

The negative shift at higher densities is present in liquid helium and presumably in other Bose fluids. However, for weak interactions, this shift is less important than the leading $O(a)$ effect, which is the current theoretical challenge.

B. $T_c$ in Liquid Helium

As a highly correlated liquid, the properties of liquid helium are not expected to be described well by the dilute weakly-interacting Bose gas model. However, the deviation of the lambda transition temperature (2.2K), from the BEC temperature of an ideal gas with the same density (3.1K), can be mostly explained using a simple effective mass description. In other words, the experimental $T_c$ is reproduced quite closely by replacing the helium atomic mass $m$ by a quasiparticle effective mass $m^*$ in the ideal gas expression $T_c^{(0)} = (2\pi/m\zeta(3/2)^{2/3}) n^{2/3}$.

The effective mass in the liquid exceeds the bare mass, because of the inertia of the medium which has to make way for any atom to move. This results in a lowering of $T_c$. The effect is essentially equivalent to that described in the last section for a momentum-dependent interaction.

Feynman [28] calculated a partition function for liquid helium that takes into account the effective mass enhancement described above. Using this partition function, Chapline calculated [29] a $T_c$ quite close to the actual lambda point of helium. This is equivalent to using the effective mass in the ideal gas expression for $T_c^{(0)}$.

According to current understanding, therefore, one expects the following variation of $T_c$ with $n^{1/3}a$: an initial
linear-like increase, followed by a maximum and then a decrease due to the effective-mass mechanism described above, until eventually $\Delta T_c$ becomes negative. This kind of dependence has been seen in the Monte-Carlo calculation of Grütter, Ceperley, Laloe [6], and in Reppy’s experiments [30] described below.

C. Relevant Experiments

One could think of studying the $T_c$ problem experimentally, e.g., in the context of trapped atomic Bose gases. Unfortunately, trap effects dominate in the dependence of $T_c$ on the interaction for these systems. Turning on a repulsion for a trapped Bose gas causes an expansion of the gas, reducing the density and hence decreasing the transition temperature $T_c \propto n^{2/3}$ [33]. In contrast to the intrinsic effect under review here, trapping effects on $T_c$ are found at mean-field level already.

A different experimental realization of the weakly non-ideal Bose gas is the $^4$He-Vycor system explored by Reppy et al [30]. Here the small-$an^{1/3}$ regime is reached by enormously decreasing the $^4$He density. Reppy’s investigation of the transition temperature shows a linear-like increase of $T_c$ at low $an^{1/3}$, followed by a maximum and a decrease to values lower than $T_c^{(0)}$. The general behavior is satisfying, but for the initial increase, the data is not yet good enough to choose between $\sim a$, $\sim a \ln a$, or $\sim a^{\sqrt{a}}$-like behavior.

III. COMMENTS ON VARIOUS APPROACHES

A. Baym and collaborators

The work of Baym, Laloe and collaborators [1, 3–8] establishes definitely that a repulsive interaction increases $T_c$, and that the dependence of $\Delta T_c$ on the interaction is of linear order, as opposed to quadratic or $\sim \sqrt{a}$.

One should note, however, that the Baym group’s work relies heavily on power-counting and dimensional arguments. This kind of argument typically does not catch logarithmic corrections. The possibility of leading non-analytic dependence of $\Delta T_c$ on the interaction, therefore, needs to be further examined.

It has been suggested [24] that the large-$N$ calculation [4] is a reliable indicator that the leading shift is purely linear, because of intuition gained from other systems where large-$N$ calculations capture all logarithms. I would like to point out, however, that the $N \neq 2$ models belong to a different universality class, and have different excitation spectra as compared to the usual ($N = 2$) Bose gas. More care is therefore required in interpreting results from the large-$N$ approach.

The Baym-Laloe collaboration seems to have been the first to relate the single-particle spectrum at $T_c$ to the shift $\Delta T_c$. Another noteworthy aspect is the use of Ursell operators [1, 5], and a corresponding diagram technique, to describe particle correlations. (As far as I can tell, for this problem, the Ursell operator technique does not give any extra information compared to the usual Green function techniques.)

B. Direction of Shift, Infrared Spectrum

Once the shift in $T_c$ is related to the modification of the single-particle spectrum, the direction of the shift can be argued using well-known facts about the spectrum of the Bose gas.

We will use the following expression for the shift, as appearing in [2]:

$$\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3n} \int \frac{d^3k}{(2\pi)^3} \left[ f_\epsilon(\xi_k) - f_\epsilon(\epsilon_k) \right].$$

Here $f_\epsilon$ is the Bose function at the critical point; $f_\epsilon(x) = (e^{\epsilon/kT_c} - 1)^{-1}$; and $\xi_k$ is the quasiparticle spectrum at $T_c$, i.e., $\xi_k = \epsilon_k + \Sigma(k; \epsilon_k; T = T_c)$.

The expression used by Baym et al [1, 3], $\Delta T_c \propto \int dkU(k)/[k^2 + U(k)]$, is the high-temperature (zero Matsubara frequency) version of (2). Noting that the major contribution to (2) comes from the infrared, one can write approximately

$$\Delta T_c \propto \int_0^{\text{cutoff}} dk \: k^2 [\epsilon_k^{-1} - \xi_k^{-1}].$$

This equation might introduce or leave out logarithmic corrections; however the following argument is unchanged for (2).

If the quasiparticle spectrum $\xi_k$ is “harder” than the bare spectrum $\epsilon_k = k^2/2m$, i.e., if $\xi_k$ is sub-quadratic in $k$ or quadratic with an effective mass $m^* < m$, then the $\epsilon_k$ integral dominates and the shift is positive. On the other hand if the spectrum at the transition point is “softened” by the interaction, then eq (3) [ or (2)] leads to a decrease in the transition temperature.

Since the weakly interacting Bose gas is known to have a linear (Bogoliubov) infrared spectrum below the critical temperature, and a quadratic spectrum above $T_c$, it is extremely reasonable to assume, even without any RG or perturbation-theoretic calculations, that the spectrum at the critical point should be something intermediate between linear and quadratic. As pointed out above, this means a positive shift in the transition temperature.

The long-wavelength spectrum at the transition point should be $\xi_k \sim k^{2-\eta}$, where $\eta$ is the anomalous dimension. The present system falls in the same universality class as the XY model, or the $N = 2$ quantum rotor model. For this universality class, large-$N$ and $\epsilon$-expansion calculations give respectively $\eta \approx 0.14$ and $\eta \approx 0.02$, and a recent Monte Carlo calculation [35] has produced the value $\eta \approx 0.038$.

Second-order treatments of the self-energy (e.g., the Baym group’s [1, 3] and our [2] work) over-estimates the
modification of the spectrum, giving a \( k^{3/2} \) infrared behavior (\( \eta = 0.5 \)). On the other hand, some other treatments (such as Stoof’s [13] and Schakel’s [22, 23]) neglect the modification altogether (\( \eta = 0 \)).

C. Renormalization group treatment (Stoof)

The RG flow equations derived by Bijlsma & Stoof [13] allow one to numerically locate the transition point. Under RG flow, the physical effective chemical potential flows to positive and negative values, if one starts out respectively in the Bose-condensed and the symmetry-unbroken phase. The critical point is therefore identified by finding parameters for which the chemical potential

\[
\mu_c \approx \frac{a}{\lambda} \ln a / \lambda
\]

The differences between Stoof’s and the Baym group’s results merit further study. Stoof has suggested [14] that an improved RG calculation, including the momentum dependence of the effective two-body interaction

\[
\mu_c \approx (a / \lambda) \ln (a / \lambda).
\]

D. Schakel: IR regularization

In treating his effective-action theory near the transition point, Schakel [22, 23] is also faced with infrared (IR) divergences. One of the remarkable features of this work is an innovative analytic continuation procedure used to regulate IR problems.

Baym et al have remarked [1] that Schakel’s prescription (\( \zeta \)-function regularization) must be incorrect because it gives an incorrect result for the compressibility of the ideal gas. This argument, unfortunately, is derived from an equation that Schakel derived explicitly for the nontrivial broken-symmetry ground state [23, 24] of an interacting gas (\( \mu > 0 \)). The same equations are not applicable to the noninteracting case (\( \mu \leq 0 \)) for which the structure of the vacuum is different (more trivial). The situation is therefore more complicated than indicated in Baym et al’s note. In response, Schakel has provided a more detailed explanation and justification [25] for the \( \zeta \)-function continuation procedure.

It seems to me that the regularization procedure itself is quite possibly sound. My concern with Schakel’s derivation is different — just below the transition point (broken-symmetry phase), Schakel uses a quadratic spectrum. A more accurate treatment would use the phonon-like spectrum \( E_k \approx \sqrt{\epsilon_k^2 + 2 U n_0(T) c_k} \). The effect such a spectrum would have on the thermodynamic potential or on \( T_c \) is not clear. (Redoing Schakel’s \( T_c \) calculation with a phonon-like spectrum seems quite difficult.)

E. Canonical Ensemble: Role of \( N_0(T_c) \).

In Wilkens et al’s paper [20], calculations have been performed using a canonical ensemble of \( N \) particles, and the shift \( \Delta T_c \) obtained is linear and negative in the scattering length. It has been pointed out [8] that the error lies in evaluating quantities (derivatives) at \( N_0 = 0 \), instead of evaluating them at the expectation value of \( N_0(T_c) \). Mueller et al [8] have also used finite-size scaling ideas in combination with a first-order calculation in the canonical ensemble, and obtain a positive linear shift.

The idea that the zero-momentum state occupancy \( N_0 \) becomes “microscopic” at and above the transition needs to be clarified. One finds \( \lim_{N \to 0} (N_0 / N) = 0 \) for \( T \geq T_c \), where \( N \) is the total number of particles, but not necessarily \( N_0 \sim O(1) \). In fact, for both the ideal and the interacting Bose gas, \( N_0(T_c) \sim N^{2/3} \), which is large compared to 1 though small compared to \( N \).

In view of the continuing appearance of the \( N_0(T_c) \sim O(1) \) misconception, I give here a simple derivation of \( N_0(T_c) = \gamma N^{2/3} \), together with an expression for the coefficient \( \gamma \), for the ideal-gas case.

Let us consider the ideal-gas relationship \( N = N_0 + \frac{V}{\hbar} \sqrt{3/2} e^{-\alpha} \), where \( \alpha = -\beta \mu \), and \( g_\alpha \) refers to a Bose-Einstein integral function [37]. Since \( \alpha \) is small near the transition, one can use the Robinson expansion [37] for \( g_{3/2} \). Also, inverting \( N_0 = 1 / (e^{\beta \mu} + 1) \) one gets \( \alpha \approx 1 / N_0 \). Combining these yields, near the transition,

\[
N_0^{3/2} + \left[ \frac{V}{\hbar^3} \sqrt{\frac{3}{2}} - N \right] \sqrt{N_0} - \frac{V}{\hbar^3} (2 \sqrt{\pi}) \approx 0.
\]

Since \( n \hbar^3 = \zeta(3/2) \) at the transition, the coefficient of \( \sqrt{N_0} \) needs to vanish. Therefore

\[
N_0(T_c) \approx \left[ \frac{2 \sqrt{\pi}}{\zeta(3/2)} \right]^{2/3} N^{2/3}.
\]

The \( k = 0 \) occupancy \( N_0 \) can actually be calculated for all \( T \), using an extra Lagrange multiplier \( \mu \) conjugate to \( N \), in addition to usual chemical potential \( \mu \) coupling to \( N \). Using this “two chemical potentials” formalism [34], one can show that for the ideal gas (\( N_0 = 1 \)) occurs at a temperature \( T \approx 2.6 T_c^{(0)} \), well above the transition point.

For the interacting-gas case, an equation corresponding to (4) is difficult to derive, but finite-size scaling arguments [8, 17] show that \( N_0 \sim N^{2/3} \) continues to hold, even though the nonideal Bose gas belongs to a different universality class.
The definition of the transition point used in ref. [20] is in terms of the probability

\[ P_n(T, N) \propto \text{tr} \left[ \hat{\delta}_{N_0,n} e^{-\beta H} \delta_{\hat{\gamma},N} \right] \]  

(5)

for having a condensate occupancy \( N_0 = n \). The claim is that \( P_n(T, N) \) decreases monotonically with \( n \) for all temperatures above the transition, and has a peak at nonzero \( n \) for in the condensed phase, and therefore the transition should be defined as the thermodynamic limit of the zero of \( D(T, N) = P_0(T, N) - P_1(T, N) \).

The correspondence between the canonical particle-counting statistics, and the grand-canonical ideas of average occupancies, is not trivial. One important point to note is that a monotonically decreasing \( P_n \) does not imply a zero expectation value for \( N_0 \), even though \( P_n \) has a maximum at \( n = 0 \). (For example, using \( P_n \propto e^{-\alpha n} \) gives \( \langle N_0 \rangle = \sum_n n P_n = 1/[e^\alpha - 1] \).) The stipulation of a monotonically decreasing \( P_n \) therefore does not imply a zero or \( \mathcal{O}(1) \) value of \( N_0 \) at the transition.

### F. Kanno’s Approach: Using Modified Thermodynamic Potential

Kanno developed an unusual approach [18] to study many-body systems, and also used this approach to calculate Bose gas properties [19].

**The formalism.**

Kanno’s calculation of thermodynamic quantities involves a “quasilinear” canonical transformation [18] of the free-particle operators \( \hat{b} (\hat{b}^\dagger) \):

\[
\hat{b}_i = \hat{\gamma}_i + \sum_{j,k,l} A_i(j;k,l)\hat{\gamma}_j^\dagger \hat{\gamma}_k^\dagger \hat{\gamma}_l + \sum_{j,k,l,m,n} A_i(j;k,l,m,n)\hat{\gamma}_j^\dagger \hat{\gamma}_k \hat{\gamma}_m \hat{\gamma}_n + \ldots ,
\]  

(6)

resulting in a Hamiltonian having a particular (diagonal-like) form. The formalism involves examining the properties of the coefficients \( A \), and the coefficients appearing in the quasi-diagonal hamiltonian. Applying this formalism to the weakly repulsive bose gas [19], one can derive the Bogoliubov spectrum, entropy and pressure above and below the transition, and many other quantities.

Kanno does not provide direct connections to the usual language of perturbative many-body theory. I would like to point out that many of his physical results, if not all, are equivalent to results obtained in more mainstream approaches. For example, the quantities \( A(p) \) and \( B(p) \), appearing in the second paper of the series [18], are equivalent to the normal and anomalous self-energies that appear in modern descriptions.

The shift in \( T_c \).

Kanno’s defines [19] a modified thermodynamic potential

\[
\Omega'(N_0, \mu', T) = -\frac{1}{\beta} \ln \text{tr} e^{-\beta (\hat{H} - \mu' \hat{N}')} ,
\]

with an unusual Lagrange multiplier \( \mu' \) conjugate to the number of non-condensate particles, \( N' = N - N_0 \), as opposed to the usual chemical potential which is conjugate to the total particle number \( N \). It follows that

\[
N - N_0 = N' = - \frac{\partial \Omega'(N_0, \mu', T)}{\partial \mu'} .
\]

(7)

Once \( \Omega' \) is calculated in some approximation, comparing this relation to \( N - N_0 = V g_{3/2}(e^{\beta \mu})/\lambda^3 \) for the free bose gas at the transition point yields the shift \( \Delta T_c \), in terms of the quantity \( \mu'(T_c) \).

The quantity \( \mu' \) itself is obtained from a consideration of the free energy \( F(N_0, N, T) = \Omega'(N_0, \mu', T) + \mu' N' \). The actual value of \( N_0 \) chosen by the system is the one that minimizes \( F(N_0, N, T) \), i.e.,

\[
0 = \frac{\partial F}{\partial N_0} \bigg|_{T=T_c} = \frac{\partial \Omega'}{\partial N_0} \bigg|_{T=T_c} - \mu'(T_c) .
\]

(8)

This equation allows one to express \( \mu'(T_c) \) in terms of the interaction strength.

*Computing \( \Omega' \) in more usual language.*

As advertised in section II, I show here how the modified thermodynamic potential \( \Omega' \) (which is the quantity relevant to the \( T_c \) problem) can be obtained from a diagrammatic second-order perturbative calculation.

The Feynman rules for a diagrammatic perturbative calculation of the modified thermodynamic potential \( \Omega'(\mu', N_0, T) \) are very similar to those for the usual \( \Omega(\mu, T) \). One has to use a momentum-dependent chemical potential \( \mu_k = \mu'(1 - \delta_{k,0}) \), and separate out \( k = 0 \) from the momentum-sums at each stage. Two first order and two second-order diagrams need to be calculated:

\[
\Omega' \approx \Omega'^{(0)} + \begin{array}{c}
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\bullet
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\end{array} .
\]

Other second-order diagrams are excluded by considering the chemical potential to be renormalized upto first order, as discussed in section II A.
The zeroth and first order terms can be evaluated exactly. UV divergences appearing at second order can be removed by eliminating the bare potential $U$ in favor of the 2-body $t$-matrix, as done for the self-energy in [2]. The result is

$$\Omega' = -\frac{V}{\beta \lambda^2} g_{0/2}(e^{\beta \mu'}) + \frac{t}{V} N^2 + 4t^2 N_0 \int \int \frac{f(\epsilon_p - \epsilon_q) f(\epsilon_q - \epsilon_p - q)}{\epsilon_p - \epsilon_q - \epsilon_p - q} \left( \epsilon_p - \epsilon_q - \epsilon_p - q \right) + 2t^2 V \int \int \frac{f(\epsilon_p - \epsilon_q) f(\epsilon_q - \epsilon_p - q)}{\epsilon_p - \epsilon_q - \epsilon_p - q} \left( \epsilon_p - \epsilon_q - \epsilon_p - q \right) . \quad (9)$$

The expression for $\Omega'$ obtained in Ref [19] is essentially the same as eq (9).

Comparison & remarks.

The parameter $\mu'(T_c)$ corresponds to the infrared cutoff used in the the non-selfconsistent regularization of [2]. Eq (8) gives the same result for the parameter $\mu'(T_c)$ as that obtained in [2] for the chemical potential $\mu$ using $\mu = \Sigma(0,0)$. The reason for this is that $\partial F/\partial N_0$ represents the energy of a zero-momentum quasiparticle with respect to the chemical potential, i.e., $\Sigma(0,0) - \mu$. Therefore eq (8) is equivalent to the usual condition $\Sigma(0,0) = \mu$ for the Bose-condensation point.

The shift $\Delta T_c$ obtained using eqs (7), (8), (9) is the same as that obtained by our non-selfconsistent regularization scheme in [2].

The distinctive idea in Kanno’s work is the introduction of a modified thermodynamic function $\Omega'$ such that $-\partial \Omega'/\partial \mu'$ replaces the right side in the criticality condition $N = (V/\lambda^2) \zeta(\frac{3}{2})$. This bypasses the issue of self-consistency and the modification of the spectrum at $T_c$, and the use of the chemical potential as a cutoff follows naturally.

IV. SUMMARY, FURTHER CALCULATIONS

To summarise, this article reviews aspects of the calculation of the $T_c$ shift due to a small repulsive interaction in a Bose gas. This question has generated a number of calculations involving diverse techniques; the present discussion is a first step toward developing an organized overview of the different approaches.

The issue that needs to be resolved urgently, in my opinion, is the possibility of a logarithm-like correction to the leading shift in $T_c$. Presumably, a better understanding of the spectrum at the critical point (both at the infrared and the ultraviolet) should help settle this issue. Two directions need to pursued in this regard.

First, as suggested by Stoof, an improved RG calculation, taking into account the momentum dependence of the effective interaction, would improve the ultraviolet behavior of the spectrum in his formalism. Also, it might be worth retaining the wavefunction renormalization (critical index $\eta$), which changes the infrared spectrum at $T_c$ from $k^4$ to $k^{2-\eta}$. This is neglected in [13] on the grounds that $\eta$ is small.

A second approach suggests itself when we note that the quasiparticle treatments [1–3] have all calculated self-energies in terms of the two-body $t$-matrix $t = 4\pi a/m$. This quantity describes all possible collisions between two particles, without taking into account the fact that the surrounding medium has an effect on these collisions. Many-body corrections arising from the surrounding gas can be treated by using the many-body $T$-matrix, which is the sum of ladder diagrams. The two-body $t$-matrix has no temperature-dependence, while the temperature-dependence of the effective potential (vanishing at $T_c$) is captured by the many-body $T$-matrix [13, 15, 36]. Stoof suggests [14] that this might explain why the quasiparticle analysis (perturbative treatment of spectrum modification) over-adjusts the infrared spectrum.

Determining the self-energies and quasiparticle spectra in terms of the many-body $T$-matrix has been attempted [34], using the relations provided by Shi & Griffin [36] between the bare potential $U$, the $t$-matrix, and the many-body $T$-matrix. Our attempt was unsuccessful due to the difficulty of disentangling multiple frequency and momentum inter-dependences at second-order.

Finally, I mention some other work that would be interesting to the $T_c$ community.

- One possibility is to use finite-size scaling ideas, in conjunction with second-order perturbation theory in the grand-canonical ensemble. This would be a complementary to the calculation of Mueller & Baym [8], which uses finite-size scaling with first-order perturbation theory in the canonical ensemble.

- It would be nice to see Schakel’s calculation of the thermodynamic potential, just below $T_c$, done with the spectrum treated more properly, either with a Bogoliubov-like spectrum or with the $E_k(T_c) \sim k^{2-\eta}$ spectrum.

- As mentioned in section II, a detailed comparison is required between the $\delta$-expansion work [32] and other approaches to $T_c$.

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