Quadratic–quartic functional equations in RN–spaces

M. Bavand Savadkouhi
Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
e-mail: bavand.m@gmail.com

M. Eshaghi Gordji
Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
e-mail: madjid.eshaghi@gmail.com

Choonkil Park
Department of Mathematics, Hanyang University,
Seoul 133-791, South Korea
e-mail: baak@hanyang.ac.kr

Abstract. In this paper, we obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms

\[ f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 2[f(2x) - 4f(x)] - 6f(y). \]

1. Introduction

The stability problem of functional equations originated from a question of Ulam [33] in 1940, concerning the stability of group homomorphisms. Let \((G_1, .)\) be a group and let \((G_2, \ast, d)\) be a metric group with the metric \(d(., .)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\) such that if a mapping \(h : G_1 \rightarrow G_2\) satisfies the inequality \(d(h(x \ast y), h(x) \ast h(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \rightarrow G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [15] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \(f : E \rightarrow E'\) be a mapping between Banach spaces such that

\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \]

for all \(x, y \in E\) and some \(\delta > 0\). Then there exists a unique additive mapping \(T : E \rightarrow E'\) such that

\[ \|f(x) - T(x)\| \leq \delta \]

for all \(x \in E\). Moreover, if \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in E\), then \(T\) is \(\mathbb{R}\)-linear. In 1978, Th. M. Rassias [27] provided a generalization of the Hyers' theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [10] answered the question for the case \(p > 1\), which was raised by Rassias. This new concept is known as

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Hyers-Ulam-Rassias stability of functional equations (see [1] 2 3 11 16 17 28 29). The functional equation
\begin{equation}
 f(x + y) + f(x - y) = 2f(x) + 2f(y).
\end{equation}

is related to a symmetric bi-additive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.1) is said to be a quadratic mapping. It is well known that a mapping \( f \) between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive mapping \( B \) such that \( f(x) = B(x,x) \) for all \( x \) (see [11]). The bi-additive mapping \( B \) is given by
\begin{equation}
 B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)).
\end{equation}

The Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings \( f : A \to B \), where \( A \) is a normed space and \( B \) is a Banach space (see [32]). Cholewa [5] noticed that the theorem of Skof is still true if relevant domain \( A \) is replaced an abelian group. In [7], Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.1). Grabiec [12] has generalized these results mentioned above. In [26], W. Park and J. Bae considered the following quartic functional equation
\begin{equation}
 f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y) + 6f(y)] - 6f(x).
\end{equation}

In fact, they proved that a mapping \( f \) between two real vector spaces \( X \) and \( Y \) is a solution of (1.3) if and only if there exists a unique symmetric multi-additive mapping \( M : X^4 \to Y \) such that \( f(x) = M(x,x,x,x) \) for all \( x \). It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (1.3), which is called a quartic functional equation (see also [6]). In addition, Kim [19] has obtained the Hyers-Ulam-Rassias stability for a mixed type of quartic and quadratic functional equation.

The Hyers-Ulam-Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [20]-[25]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm \( T_M \).

The aim of this paper is to investigate the stability of the additive-quadratic functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary continuous \( t \)-norms.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [4] 21 22 30 31. Throughout this paper, \( \Delta^+ \) is the space of distribution functions that is, the space of all mappings \( F : \mathbb{R} \cup (-\infty, \infty) \to [0,1] \) such that \( F \) is left-continuous and non-decreasing on \( \mathbb{R} \), \( F(0) = 0 \) and \( F(+\infty) = 1 \). \( D^+ \) is a subset of \( \Delta^+ \) consisting of all functions \( F \in \Delta^+ \) for which \( l^{-} F(+\infty) = 1 \), where \( l^{-} F(x) \) denotes the left limit of the function \( F \) at the point \( x \), that is, \( l^{-} F(x) = \lim_{t \to x^-} F(t) \). The space \( \Delta^+ \) is partially ordered by the usual point-wise ordering of functions, i.e., \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \) in \( \mathbb{R} \). The maximal element for \( \Delta^+ \) in this order is the distribution function \( \varepsilon_0 \) given by
\begin{equation}
 \varepsilon_0(t) = \begin{cases} 
 0, & \text{if } t \leq 0, \\
 1, & \text{if } t > 0.
\end{cases}
\end{equation}

**Definition 1.1.** ([30]). A mapping \( T : [0,1] \times [0,1] \to [0,1] \) is a continuous triangular norm (briefly, a continuous \( t \)-norm) if \( T \) satisfies the following conditions:
\begin{enumerate}
  \item \( T \) is commutative and associative;
  \item \( T \) is continuous;
  \item \( T(a,1) = a \) for all \( a \in [0,1] \);
  \item \( T(a,b) \leq T(c,d) \) whenever \( a \leq c \) and \( b \leq d \) for all \( a,b,c,d \in [0,1] \).
\end{enumerate}
Typical examples of continuous $t$-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a + b - 1, 0)$ (the Lukasiewicz $t$-norm). Recall (see [13, 14]) that if $T$ is a $t$-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{n+1}^n x_i$ is defined recurrently by $T_{n+1}^1 x_1 = x_1$ and $T_{n+1}^n x_i = T(T_{n+1}^{n-1} x_i)$ for $n \geq 2$. $T_{n+1}^\infty x_i$ is defined as $T_{n+1}^\infty x_{n+i}$. It is known ([14]) that for the Lukasiewicz $t$-norm the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^\infty (1 - x_n) < \infty.$$  

**Definition 1.2.** ([31]). A random normed space (briefly, RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu$ is a mapping from $X$ into $D^+$ such that the following conditions hold:

- (RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
- (RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space $(X, \mu, T_M)$, where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and $T_M$ is the minimum $t$-norm. This space is called the induced random normed space.

**Definition 1.3.** Let $(X, \mu, T)$ be an RN-space.

1. A sequence $\{x_n\}$ in $X$ is said to be convergent to $x$ in $X$ if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
2. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
3. An RN-space $(X, \mu, T)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent to a point in $X$.

**Theorem 1.4.** ([30]). If $(X, \mu, T)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Recently, M. Eshaghi Gordji et al. establish the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [8] and [29]).

In this paper, we deal with the following functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + f(x - y)] + 2[f(2x) - 4f(x)] - 6f(y) \quad (14)$$

on RN-spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of (14).

In Section 2, we investigate the general solution of the functional equation (14) when $f$ is a mapping between vector spaces and in Section 3, we establish the stability of the functional equation (14) in RN-spaces.

## 2. General Solution

We need the following lemma for solution of (14). Throughout this section $X$ and $Y$ are vector spaces.

**Lemma 2.1.** If a mapping $f : X \to Y$ satisfies (14) for all $x, y \in X$, then $f$ is quadratic-quartic.
Proof. We show that the mappings \( g : X \to Y \) defined by \( g(x) := f(2x) - 16f(x) \) and \( h : X \to Y \) defined by \( h(x) := f(2x) - 4f(x) \) are quadratic and quartic, respectively.

Letting \( x = y = 0 \) in (1.4), we have \( f(0) = 0 \). Putting \( x = 0 \) in (1.4), we get \( f(-y) = f(y) \). Thus the mapping \( f \) is even. Replacing \( y \) by \( 2y \) in (1.4), we get

\[
f(2x + 2y) + f(2x - 2y) = 4[f(x + 2y) + f(x - 2y)] + 2[f(2x) - 4f(x)] - 6f(2y)
\]

(2.1)

for all \( x, y \in X \). Interchanging \( x \) with \( y \) in (1.4), we obtain

\[
f(2y + x) + f(2y - x) = 4[f(y + x) + f(y - x)] + 2[f(2y) - 4f(y)] - 6f(x)
\]

(2.2)

for all \( x, y \in X \). Since \( f \) is even, (2.2), one gets

\[
f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 2[f(2y) - 4f(y)] - 6f(x)
\]

(2.3)

for all \( x, y \in X \). It follows from (2.1) and (2.3) that

\[
[f(2(x + y)) - 16f(x + y)] + [f(2(x - y)) - 16f(x - y)] = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)]
\]

for all \( x, y \in X \). This means that

\[
g(x + y) + g(x - y) = 2g(x) + 2g(y)
\]

for all \( x, y \in X \). Therefore, the mapping \( g : X \to Y \) is quartic.

To prove that \( h : X \to Y \) is quartic, we have to show that

\[
h(x + 2y) + h(x - 2y) = 4[h(x + y) + h(x - y) + 6h(y)] - 6h(x)
\]

for all \( x, y \in X \). Since \( f \) is even, the mapping \( h \) is even. Now if we interchange \( x \) with \( y \) in the last equation, we get

\[
h(2x + y) + h(2x - y) = 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y)
\]

(2.4)

for all \( x, y \in X \). Thus it is enough to prove that \( h \) satisfies in (2.4). Replacing \( x \) and \( y \) by \( 2x \) and \( 2y \) in (1.4), respectively, we obtain

\[
f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 2[f(4x) - 4f(2x)] - 6f(2y)
\]

(2.5)

for all \( x, y \in X \). Since \( g(2x) = 4g(x) \) for all \( x \in X \),

\[
f(4x) = 20f(2x) - 64f(x)
\]

(2.6)

for all \( x \in X \). By (2.5) and (2.6), we get

\[
f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 32[f(2x) - 4f(x)] - 6f(2y)
\]

(2.7)

for all \( x, y \in X \). By multiplying both sides of (1.4) by 4, we get

\[
4[f(2x + y) + f(2x - y)] = 16[f(x + y) + f(x - y)] + 8[f(2x) - 4f(x)] - 24f(y)
\]

for all \( x, y \in X \). If we subtract the last equation from (2.7), we obtain

\[
h(2x + y) + h(2x - y) = [f(2(2x + y)) - 4f(2x + y)] + [f(2(2x - y)) - 4f(2x - y)]
\]

\[
= 4[f(2(x + y)) - 4f(x + y)] + 4[f(2(x - y)) - 4f(x - y)]
\]

\[
+ 24[f(2x) - 4f(x)] - 6f(2y) - 4f(y)]
\]

\[
= 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y)
\]

for all \( x, y \in X \).

Therefore, the mapping \( h : X \to Y \) is quartic. This completes the proof of the lemma. \( \square \)
Theorem 2.2. A mapping $f : X \to Y$ satisfies (1.4) for all $x, y \in X$ if and only if there exist a unique symmetric multi-additive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that

$$f(x) = M(x, x, x, x) + B(x, x)$$

for all $x \in X$.

Proof. Let $f$ satisfies (1.4) and assume that $g, h : X \to Y$ are mappings defined by

$$g(x) := f(2x) - 16f(x), \quad h(x) := f(2x) - 4f(x)$$

for all $x \in X$. By Lemma 2.1, we obtain that the mappings $g$ and $h$ are quadratic and quartic, respectively, and

$$f(x) = \frac{1}{12} h(x) - \frac{1}{12} g(x)$$

for all $x \in X$.

Therefore, there exist a unique symmetric multi-additive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $\frac{1}{12} h(x) = M(x, x, x, x)$ and $\frac{1}{12} g(x) = B(x, x)$ for all $x \in X$ (see [1, 26]). So

$$f(x) = M(x, x, x, x) + B(x, x)$$

for all $x \in X$. The proof of the converse is obvious. \qed

3. Stability

Throughout this section, assume that $X$ is a real linear space and $(Y, \mu, T)$ is a complete RN-space.

Theorem 3.1. Let $f : X \to Y$ be a maping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ( $\rho(x, y)$ is denoted by $\rho_{x,y}$ ) with the property:

$$\mu f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)(t) \geq \rho_{x,y}(t)$$

for all $x, y \in X$ and all $t > 0$. If

$$\lim_{n \to \infty} T_{i=1}^{\infty}(\rho_{2^n+i-1,x,2^n+i-1,x}(\frac{2^{2n+i}}{4})) + \rho_{2^n+i-1,x,2^n+i-1,x}(\frac{2^{2n+i}}{4}) + \rho_{0,2^n+i-1,x}(\frac{2^{2n+i}}{4}) = 1$$

and

$$\lim_{n \to \infty} \rho_{2^n,x,2^n,y}(2^{2n}t) = 1$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\mu f(2x) - 16f(x) - Q_1(x)(t) \geq T_{i=1}^{\infty}(\rho_{2^{-i-1,x,2^{-i-1,x}}}(\frac{2^i}{4}) + \rho_{2^{-i-1,x,2^{-i-1,x}}}(2^i t) + \rho_{0,2^{-i-1,x}}(\frac{3\cdot 2^i t}{4}))$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.1), we obtain

$$\mu f(3x) - 6f(2x) + 15f(x)(t) \geq \rho_{x,x}(t)$$

for all $x \in X$. Letting $y = 2x$ in (3.1), we get

$$\mu f(4x) - 4f(2x) + 8f(x) - 4f(-x)(t) \geq \rho_{x,2x}(t)$$

(3.6)
for all $x \in X$. Putting $x = 0$ in (3.1), we obtain

$$
\mu_{f(y)} - 3f(-y)(t) \geq \rho_{0,y}(t)
$$

(3.7)

for all $y \in X$. Replacing $y$ by $x$ in (3.7), we see that

$$
\mu_{f(x)} - 3f(-x)(t) \geq \rho_{0,x}(t)
$$

(3.8)

for all $x \in X$. It follows from (3.6) and (3.8) that

$$
\mu_{f(4x) - 4f(3x) + 4f(2x) + 4f(x)}(t) \geq \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right)
$$

(3.9)

for all $x \in X$. If we add (3.5) to (3.9), then we have

$$
\mu_{f(4x) - 20f(2x) + 64f(x)}(t) \geq \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right).
$$

(3.10)

Let

$$
\psi_{x,x}(t) = \rho_{x,x}\left(\frac{t}{4}\right) + \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right)
$$

(3.11)

for all $x \in X$. Then we get

$$
\mu_{f(4x) - 20f(2x) + 64f(x)}(t) \geq \psi_{x,x}(t)
$$

(3.12)

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$
\mu_{g(2x) - 4g(x)}(t) \geq \psi_{x,x}(t)
$$

(3.13)

for all $x \in X$. Thus we have

$$
\mu_{g(2x)}(t) \geq \psi_{x,x}(2^t)
$$

(3.14)

for all $x \in X$ and all $t > 0$. Hence

$$
\mu_{\frac{2^{k+1}x}{2^{k+1}}} \left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^k x, 2^k x}(2^{k+1}t)
$$

(3.15)

for all $x \in X$ and all $k \in \mathbb{N}$. This means that

$$
\mu_{\frac{2^{k+1}x}{2^{k+1}}} \left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^k x, 2^k x}(2^{k+1}t)
$$

(3.16)

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}$, it follows

$$
\mu_{\frac{2^{k}x}{2^k}}(t) \geq T_{k=0}^{n-1} \left(\mu_{\frac{2^{k+1}x}{2^{k+1}}} \left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=0}^{n-1} \left(\psi_{2^k x, 2^k x}(2^{k+1}t)\right)
$$

(3.17)

$$
= T_{k=1}^{n-1} \left(\psi_{2^i x, 2^i x}(2^{i}t)\right)
$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{2^{k}x}{2^k}\}$, we replace $x$ with $2^m x$ in (3.17) to obtain that

$$
\mu_{\frac{2^{m+2}x}{2^{m+2}}} \left(\frac{t}{2^{m+2}}\right) \geq T_{k=1}^{m-1} \left(\psi_{2^{m+2} x, 2^{m+2} x}(2^{m+2}t)\right).
$$

(3.18)

Since the right hand side of the inequality (3.18) tends to 1 as $m$ and $n$ tend to infinity, the sequence $\{\frac{2^{k}x}{2^k}\}$ is a Cauchy sequence. Thus we may define $Q_1(x) = \lim_{m \to \infty} \frac{2^{m}x}{2^{m}}$ for all $x \in X$. Now we show that $Q_1$ is a quadratic mapping. Replacing $x, y$ with $2^n x$ and $2^n y$ in (3.1), respectively, we get

$$
\mu_{g(2x+y) + g(2y-x) - 4g(x+y) - 4g(x-y) - 2g(2x) + 8g(x) + 6g(y)}(t) \geq \rho_{2^n x, 2^n y}(2^{2n}t).
$$

(3.19)

Taking the limit as $n \to \infty$, we find that $Q_1$ satisfies (1.4) for all $x, y \in X$. By Lemma 2.1, the mapping $Q_1 : X \to Y$ is quadratic.

Letting the limit as $n \to \infty$ in (3.17), we get (3.4) by (3.11).
Finally, to prove the uniqueness of the quadratic mapping $Q_1$ subject to (3.4), let us assume that there exists another quadratic mapping $Q_1'$ which satisfies (3.4). Since $Q_1(2^n x) = 2^n Q_1(x)$, $Q_1'(2^n x) = 2^n Q_1'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (3.4), it follows that

\[
\mu_{Q_1(x)} - Q_1'(x)(2t) = \mu_{Q_1(2^n x)} - Q_1'(2^n x)(2^{2n+1} t)
\]

\[
\geq T(\mu_{Q_1(2^n x)} - g(2^n x)(2^{2n} t), \mu_{Q_1'(2^n x)} - Q_1'(2^n x)(2^{2n} t))
\]

\[
\geq T(T_{i=1}^{\infty}(\rho_{2^n-1,2^n} + \frac{2^{2n+1} t}{4}) + \rho_{0,2^n-1,2^n} + \frac{2^{2n+1} t}{4})
\]

\[
+ \rho_{0,2^n-1,2^n}(\frac{3.2^{2n+1} t}{4}))
\]

\[= \mu_{Q_1(x)} - Q_1'(x)(2t) \geq T_{i=1}^{\infty}(\rho_{2^n-1,2^n} + \frac{2^{2n+1} t}{4}) + \rho_{0,2^n-1,2^n} + \frac{2^{2n+1} t}{4})
\]

\[+ \rho_{0,2^n-1,2^n}(\frac{3.2^{2n+1} t}{4}))
\]

(3.20)

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (3.20), we conclude that $Q_1 = Q_1'$.

\[\square\]

**Theorem 3.2.** Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ( $\rho(x, y)$ is denoted by $\rho_{x, y}$ ) with the property:

\[\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x, y}(t)
\]

(3.21)

for all $x, y \in X$ and all $t > 0$. If

\[\lim_{n \to \infty} T_{i=1}^{\infty}(\rho_{2^n-1,2^n} + \frac{2^{4n+3} t}{4}) + \rho_{0,2^n-1,2^n} + \frac{2^{4n+3} t}{4} = 1
\]

(3.22)

and

\[\lim_{n \to \infty} \rho_{2^n x, y}(2^{4n} t) = 1
\]

(3.23)

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q_2 : X \to Y$ such that

\[\mu_{f(2x)-4f(x)-Q_2(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^n-1,2^n} + \frac{2^{3} t}{4}) + \rho_{0,2^n-1,2^n} + \frac{3.2^{3} t}{4})
\]

(3.24)

for all $x \in X$ and all $t > 0$.

**Proof.** Putting $y = x$ in (3.21), we obtain

\[\mu_{f(3x) - 6f(2x)+15f(x)}(t) \geq \rho_{x, x}(t)
\]

(3.25)

for all $x \in X$. Letting $y = 2x$ in (3.21), we get

\[\mu_{f(4x)-4f(3x)+8f(x)}-4f(-x)}(t) \geq \rho_{x, 2x}(t)
\]

(3.26)

for all $x \in X$. Putting $x = 0$ in (3.21), we obtain

\[\mu_{f(y)} - 3f(-y)}(t) \geq \rho_{0, y}(t)
\]

(3.27)

for all $y \in X$. Replacing $y$ by $x$ in (3.27), we get

\[\mu_{f(x)} - 3f(-x)}(t) \geq \rho_{0, x}(t)
\]

(3.28)

for all $x \in X$. It follows from (3.6) and (3.28) that

\[\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq \rho_{x, 2x}(t) + \rho_{0, x}(\frac{3t}{4})
\]

(3.29)
for all \( x \in X \). If we add (3.25) to (3.29), then we have

\[
\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \rho_{x,2x}(\frac{t}{4}) + \rho_{x,2x}(t) + \rho_{o,x}(\frac{3t}{4}).
\] (3.30)

Let

\[
\psi_{x,x}(t) = \rho_{x,2x}(t) + \rho_{o,x}(\frac{3t}{4})
\] (3.31)

for all \( x \in X \). Then we get

\[
\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t)
\] (3.32)

for all \( x \in X \) and all \( t > 0 \). Let \( h : X \to Y \) be a mapping defined by \( h(x) := f(2x) - 4f(x) \). Then we conclude that

\[
\mu_{h(2x)-16h(x)}(t) \geq \psi_{x,x}(t)
\] (3.33)

for all \( x \in X \). Thus we have

\[
\mu_{\frac{h(2x)}{2^{16}},h(x)}(t) \geq \psi_{2^{16}x,2^{16}x}(2^{4(k+1)}t)
\] (3.35)

for all \( x \in X \) and all \( k \in \mathbb{N} \). This means that

\[
\mu_{\frac{h(2x)}{2^{16}},h(x)}(t) \geq \psi_{2^{16}x,2^{16}x}(2^{4(k+1)}t)
\] (3.36)

for all \( x \in X \), \( t > 0 \) and all \( k \in \mathbb{N} \). By the triangle inequality, from 1 \( > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} \), it follows

\[
\mu_{\frac{h(2x)}{2^{16}},h(x)}(t) \geq T_{k=0}^{n-1} \left( \mu_{\frac{h(2x)}{2^{16}},h(x)}\left( \frac{t}{2^{k+1}} \right) \right) \geq T_{k=0}^{n-1} \left( \psi_{2^{16}x,2^{16}x}(2^{4(k+1)}t) \right)
\]

\[
= T_{i=1}^{n} \left( \psi_{2^{1-i}x,2^{1-i}x}(2^{3i}t) \right)
\] (3.37)

for all \( x \in X \) and all \( t > 0 \). In order to prove the convergence of the sequence \( \frac{h(2^n x)}{2^{16}} \), we replace \( x \) with \( 2^{n} x \) in (3.37) to obtain that

\[
\mu_{\frac{h(2^{n+m}x)}{2^{16}},h(x)}(t) \geq T_{i=1}^{n} \left( \psi_{2^{1-i}x,2^{1-i}x}(2^{3i+4m}t) \right). \] (3.38)

Since the right hand side of the inequality (3.38) tends to 1 as \( m \) and \( n \) tend to infinity, the sequence \( \left\{ \frac{h(2^n x)}{2^{16}} \right\} \) is a Cauchy sequence. Thus we may define \( Q_2(x) = \lim_{n \to \infty} \frac{h(2^n x)}{2^{16}} \) for all \( x \in X \). Now we show that \( Q_2 \) is a quartic mapping. Replacing \( x, y \) with \( 2^n x \) and \( 2^n y \) in (3.21), respectively, we get

\[
\mu_{h(2x+y)+h(2x-y)-4h(x+y)-4h(x-y)-2h(2x)+8h(x)+6h(y)}(t) \geq \rho_{2^n x,2^n y}(2^{4n}t).
\] (3.39)

Taking the limit as \( n \to \infty \), we find that \( Q_2 \) satisfies (1.4) for all \( x, y \in X \). By Lemma 2.1 we get that the mapping \( Q_2 : X \to Y \) is quartic.

Letting the limit as \( n \to \infty \) in (3.37), we get (3.24) by (3.31).

Finally, to prove the uniqueness of the quartic mapping \( Q_2 \) subject to (3.24), let us assume that there exists a quartic mapping \( Q'_2 \) which satisfies (3.24). Since \( Q_2(2^n x) = 2^n Q_2(x) \) and
\[ Q'_2(2^n x) = 2^{4n} Q'_2(x) \] for all \( x \in X \) and \( n \in \mathbb{N} \), from (3.24), it follows that
\[
\mu_{Q_2(x)} - Q'_2(x)(2t) = \mu_{Q_2(2^n x) - Q'_2(2^n x)}(2^{4n+1} t)
\]
\[
\geq T(\mu_{Q_2(2^n x) - h(2^n x)}(2^{4n} t), \mu_{h(2^n x) - Q'_2(2^n x)}(2^{4n} t))
\]
\[
\geq T(T_{\infty}(\rho_{2^n x - h(2^n x)}(\frac{2^{4n+3} t}{4}) + \rho_{2^n x - 2(n+1)}(\frac{2^{4n+3} t}{4})
\]
\[
+ \rho_{0,2^n x - 1,2^n x - 1}(\frac{2^{4n+3} t}{4})), T_{\infty}(\rho_{2^n x - h(2^n x)}(\frac{2^{4n+3} t}{4})
\]
\[
+ \rho_{2^n x - 1,2^n x - 1}(\frac{2^{4n+3} t}{4})))
\]
(3.40)
for all \( x \in X \) and all \( t > 0 \). By letting \( n \to \infty \) in (3.40), we get that \( Q_2 = Q'_2 \). \]

**Theorem 3.3.** Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there is \( \rho : X \times X \to D^+ \) (\( \rho(x,y) \) is denoted by \( \rho_{x,y} \)) with the property:
\[
H_{f(2x+y)+f(2x-y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.41}
\]
for all \( x, y \in X \) and all \( t > 0 \). If
\[
\lim_{n \to \infty} T_{\infty}(\rho_{2^n x - 1,2^n x - 1}(\frac{2^{4n+3} t}{4})))
\]
\[
= 1
\]
\[
= \lim_{n \to \infty} T_{\infty}(\rho_{2^n x - 1,2^n x - 1}(\frac{2^{4n+3} t}{4})), \rho_{2^n x - 1,2^n x - 1}(\frac{2^{4n+3} t}{4}))
\]
(3.42)
and
\[
\lim_{n \to \infty} \rho_{2^n x,2^n y}(2^{4n} t) = 1 = \lim_{n \to \infty} \rho_{2^n x,2^n y}(2^{4n} t)
\]
(3.43)
for all \( x, y \in X \) and all \( t > 0 \), then there exist a unique quadratic mapping \( Q_1 : X \to Y \) and a unique quartic mapping \( Q_2 : X \to Y \) such that
\[
H_{f(x)-Q_1(x)-Q_2(x)}(t)
\]
\[
\geq T_{\infty}(\rho_{2^n x,2^n x}(\frac{2^{4n+3} t}{4})))
\]
(3.44)
for all \( x \in X \) and all \( t > 0 \). \]

**Proof.** By Theorems 3.1 and 3.2, there exist a quadratic mapping \( Q'_1 : X \to Y \) and a quartic mapping \( Q'_2 : X \to Y \) such that
\[
H_{f(2x)-16f(x)-Q'_1(x)}(t) \geq T_{\infty}(\rho_{2^n x,2^n x}(\frac{2^{4n+3} t}{4})))
\]
and
\[
H_{f(2x)-4f(x)-Q'_2(x)}(t) \geq T_{\infty}(\rho_{2^n x,2^n x}(\frac{2^{4n+3} t}{4})))
\]
for all \( x \in X \) and all \( t > 0 \). So it follows from the last inequalities that
\[
H_{f(x)-Q'_1(x)-Q'_2(x)}(t)
\]
\[
\geq T_{\infty}(\rho_{2^n x,2^n x}(\frac{2^{4n+3} t}{4})))
\]
and
\[
H_{f(x)-Q'_2(x)}(t) \geq T_{\infty}(\rho_{2^n x,2^n x}(\frac{2^{4n+3} t}{4})))
\]
for all \( x \in X \) and all \( t > 0 \). So it follows from the last inequalities that
for all $x \in X$ and all $t > 0$. Hence we obtain (3.46) by letting $Q_1(x) = -\frac{1}{12}Q_1'(x)$ and $Q_2(x) = \frac{1}{12}Q_2'(x)$ for all $x \in X$. The uniqueness property of $Q_1$ and $Q_2$, are trivial. □

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