Conditional Expectation of a Markov Kernel Given Another with Some Applications in Statistical Inference and Disease Diagnosis

A.G. Nogales

Dpto. de Matemáticas, Universidad de Extremadura
Avda. de Elvas, s/n, 06006–Badajoz, SPAIN.
e-mail: nogales@unex.es
Fax number: +34 924272911

Abstract.
Markov kernels play a decisive role in probability and mathematical statistics theories, and are an extension of the concepts of sigma-field and statistic. Concepts such as independence, sufficiency, completeness, ancillarity or conditional distribution have been extended previously to Markov kernels.
In this paper, the concept of conditional expectation of a Markov kernel given another is introduced, setting its first properties. An application to clinical diagnosis is provided, obtaining an optimality property of the predictive values of a diagnosis test. In a statistical framework, this new probabilistic tool is used to extend to Markov kernels the theorems of Rao-Blackwell and Lehmann-Scheffé. A result about the completeness of a sufficient statistic is obtained in passing by properly enlarging the family of probabilities. As a final statistical scholium, a generalization of a result about the completeness of the family of nonrandomized estimators is given.

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1 Introduction

Markov kernels (also referred to as stochastic kernels or transition probabilities) play an important role in probability theory and mathematical statistics. Indeed, the conditional distribution of one random variable given another is a Markov kernel (here, we use the term random variable as being synonymous of a measurable function between two arbitrary measurable spaces). In fact, as we shall show below, every Markov kernel is the conditional distribution of some random variable given another. A transition matrix in Markov chains theory defines a Markov kernel. Sampling probabilities and posterior distributions in Bayesian inference are Markov kernels. In statistical decision theory, randomized procedures (also named decision rules or, even, strategies) are Markov kernels, while nonrandomized procedures are statistics. It is well known that, in some situations, the optimum procedure is a randomized one: for example, the fundamental lemma of Neyman and Pearson shows how randomization is necessary to obtain a most powerful test; Lehmann (2005) also describes many other statistical situations where the use of randomization is properly justified. Pfanzagl (1994, Example 4.2.2) shows a testing problem where there is no nonrandomized test at least as good as a certain randomized test.

A Markov kernel can also be considered as a generalization of the concepts of σ-field and random variable (or statistic, in a statistical framework).

Well known concepts of the theory of probabilities or mathematical statistics, such as independence, completeness, ancillarity or conditional distribution have been extended to Markov kernels in Nogales (2013a) and Nogales (2013b). The reader is referred to Heyer (1982) for the corresponding extension to the concept of sufficiency in the context of informativity for statistical experiments. Notice that an extension to Markov kernels of the concepts and results of probability and mathematical statistics should not be considered useless, as it is not the extension to Markov kernels (or transitions) of the classical theorems of the product measure and Fubini: it is the version for Markov kernel of these theorems what we need to describe the joint distribution of two random variables $X$ and $Y$ in terms of the marginal distribution of $X$ and the conditional distributions of $Y$ given a value of $X$.

On the other hand, the conditional expectation $E(Y|X)$ of an integrable $n$-dimensional random variable $Y$ given a random variable $X$ is the main tool in the study of the relationship between them; in fact, $y = E(Y|X = x)$ is the so-called regression curve (in a wide sense) of $Y$ on $X$. Basic properties and results on conditional expectations can be found in almost every graduate text in probability theory, after its mathematical introduction in Kolmogorov (1933).

In this paper, we introduce a new probabilistic tool: the conditional expectation for Markov kernels. Its relationship with the concept of conditional distribution of a Markov kernel given another is established. Some basic properties, two examples of calculation of such a conditional expectation, and a representation theorem in terms of conditional expectation for random variables are also given. One of the given examples is applied to clinical diagnosis, where some expectations and conditional expectations can be found in almost every graduate text in probability theory, after its mathematical introduction in Kolmogorov (1933).

As a statistical application, in this paper we make use of such tools to extend to Markov kernels the theorems of Rao-Blackwell and Lehmann-Scheffé. These well known theorems are major milestones of mean unbiased estimation theory, going back to Rao (1945) and Blackwell (1947) regard to the Theorem of Rao-Blackwell, and to Lehmann and Scheffé (1950) regard to the Theorem of Lehmann-Scheffé. The reader is referred to Pfanzagl (1994, p. 105) for a version for statistics of these theorems; it is assured even there that a more general version of the Rao-Blackwell theorem can be proved in the same way for randomized estimators when a sufficient and complete statistics exists. In this paper both results are generalized for randomized estimators when a sufficient and complete Markov kernel is known. Two examples of sufficient Markov kernels associated to any statistic are given; two more similar examples are provided for complete Markov kernels.

Notice also that, as the conditional expectation of a Markov kernel given another is a statistic, a generalization of a result about the completeness of the family of nonrandomized estimators is also obtained.

Finally, looking for an elusive example of application of this generalized version of the Lehmann-
Scheffé Theorem, a result about getting the completeness of a given sufficient statistic has been obtained. The key: to enlarge the family of probabilities adequately.

For ease of reading, the proofs of the results have been collected in the last section of the paper.

2 Basic definitions

The concepts presented in this section can be found in Heyer (1982) (see also Dellacherie and Meyer (1988)) and therefore they will be exposed very briefly, even at risk of being somewhat dense. However the usual notations in this area has been modified. As explained in the Remak 2 below, the concept of Markov kernel is an extension of the concept of random variable (and also of the concept of σ-field) and the notation to be used for operations with Markov kernels, the same that for random variables, tries to highlight this analogy.

In the next, \((\Omega, \mathcal{A}), (\Omega_1, \mathcal{A}_1)\), and so on, will denote measurable spaces. A random variable is a map \(X : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)\) such that \(X^{-1}(A_1) \in \mathcal{A}\), for all \(A_1 \in \mathcal{A}_1\). Its probability distribution (or, simply, distribution) \(P_X\) with respect to a probability measure \(P\) on \(\mathcal{A}\) is the image measure of \(P\) by \(X\), i.e., the probability measure on \(\mathcal{A}_1\) defined by \(P_X(A_1) := P(X^{-1}(A_1))\). We will write \(\times\) instead of \(\otimes\) for the product of σ-fields or measures. \(\mathcal{R}^k\) will denote the Borel σ-field on \(\mathbb{R}^k\).

**Definition 1.** (Markov kernel) A Markov kernel \(M_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)\) is a map \(M_1 : \Omega \times \mathcal{A}_1 \rightarrow [0, 1]\) such that

1. \(\forall \omega \in \Omega, M_1(\omega, \cdot)\) is a probability measure on \(\mathcal{A}_1\),
2. \(\forall A_1 \in \mathcal{A}_1, M_1(\cdot, A_1)\) is \(\mathcal{A}\)-measurable.

**Remarks.** 1) Given two random variables \(X_i : (\Omega, \mathcal{A}, P) \rightarrow (\Omega_i, \mathcal{A}_i), i = 1, 2\), the conditional distribution of \(X_2\) given \(X_1\), when it exists, is a Markov kernel \(M : (\Omega_1, \mathcal{A}_1) \rightarrow (\Omega_2, \mathcal{A}_2)\) such that \(P(X_1 \in A_1, X_2 \in A_2) = \int_{A_1} M(\omega, A_2) dP(\omega)_1\), for all \(A_1 \in \mathcal{A}_1\) and \(A_2 \in \mathcal{A}_2\). We write \(P^{X_2|X_1=\omega_1}(A_2) := M(\omega_1, A_2)\). Reciprocally, every Markov kernel is a conditional distribution; namely, given a Markov kernel \(M_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)\), it is easily checked that

\[
M_1(\omega, A_1) = (P \otimes M_1)^{\pi_1|\pi=\omega}(A_1),
\]

where \(\pi : \Omega \times \Omega_1 \rightarrow \Omega\) and \(\pi_1 : \Omega \times \Omega_1 \rightarrow \Omega_1\) are the coordinatewise projections and \(P \otimes M_1\) stands for the only probability measure on the product space \((\Omega \times \Omega_1, \mathcal{A} \times \mathcal{A}_1)\) such that \((P \otimes M_1)(A \times A_1) = \int_A M_1(\omega, A_1) dP(\omega)\) for all \(A \in \mathcal{A}\) and \(A_1 \in \mathcal{A}_1\).

2) The concept of Markov kernel extends the concepts of random variable and σ-field. A random variable \(T_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)\) will be identified with the Markov kernel \(M_{T_1} : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)\) defined by

\[
M_{T_1}(\omega, A_1) = \delta_{T_1(\omega)}(A_1) = I_{A_1}(T_1(\omega)),
\]

where \(\delta_{T_1(\omega)}\) denotes the Dirac measure -the degenerate distribution- at the point \(T_1(\omega)\) and \(I_{A_1}\) is the indicator function of the event \(A_1\). The sub-σ-field \(B \subset \mathcal{A}\) will be identified with the Markov kernel \(M_B : (\Omega, \mathcal{A}) \rightarrow (\Omega, B)\) given by \(M_B(\omega, B) = \delta_\omega(B)\).

**Definition 2.** (Image of a Markov kernel) The image (or probability distribution) of a Markov kernel \(M_1 : (\Omega, \mathcal{A}, P) \rightarrow (\Omega_1, \mathcal{A}_1)\) on a probability space is the probability measure \(P^{M_1}\) on \(\mathcal{A}_1\) defined by

\[
P^{M_1}(A_1) := \int_\Omega M_1(\omega, A_1) dP(\omega).
\]

**Remark.** Note that

\[
P^{M_1} = (P \otimes M_1)^{\pi_1}
\]

where \(\pi_1 : \Omega \times \Omega_1 \rightarrow \Omega_1\) denotes the coordinatewise projection onto \(\Omega_1\). So, if \(f : (\Omega_1, \mathcal{A}_1) \rightarrow \mathbb{R}\) is a nonnegative or \(P^{M_1}\)-integrable function,

\[
\int_{\Omega_1} f(\omega_1) dP^{M_1}(\omega_1) = \int_{\Omega_1} \int_{\Omega_1} f(\omega_1) M_1(\omega, d\omega_1) dP(\omega)
= \int_{\Omega \times \Omega_1} f(\omega_1) d(P \otimes M_1)(\omega, \omega_1).
\]
Definition 3. (Composition of Markov kernels) The composition of two Markov kernels $M_1 : (\Omega_1, \mathcal{A}_1) \rightarrow (\Omega_2, \mathcal{A}_2)$ and $M_2 : (\Omega_2, \mathcal{A}_2) \rightarrow (\Omega_3, \mathcal{A}_3)$ is defined as the Markov kernel

$$M_2M_1 : (\Omega_1, \mathcal{A}_1) \rightarrow (\Omega_3, \mathcal{A}_3)$$

given by

$$M_2M_1(\omega_1, A_3) = \int_{\Omega_2} M_2(\omega_2, A_3)M_1(\omega_1, d\omega_2).$$

(b) (Composition of a Markov kernel and a random variable) Let $X_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)$ be a random variable and $M_1 : (\Omega_1, \mathcal{A}_1) \rightarrow (\Omega_1', \mathcal{A}_1')$ a Markov kernel. A new Markov kernel $M_1X_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1', \mathcal{A}_1')$ is defined by means of

$$M_1X_1(\omega, A_1') := M_1(X_1(\omega), A_1').$$

Remark. When $M_{X_1}$ is the Markov kernel corresponding to the random variable $X_1$, we have that $M_{X_1}X_1 = M_1M_{X_1}$.

3  Expectation and conditional expectation for Markov kernels

In this section, of a character rather technical, we first recall the concept of expectation of a Markov kernel and introduce the definition of conditional expectation for Markov kernels, the last being parallel to the definition of conditional expectation for random variables that can be found, for instance, in Heyer (1982, p. 264). Some examples of calculation and some possible applications to clinical diagnoses will be given joint with some properties –even a optimality one– of this new concept.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Onwards, $\mathbb{R}^k$ is supposed to be endowed with its Borel $\sigma$-field $\mathcal{R}^k$.

Definition 4. (Expectation of a Markov kernel) A Markov kernel $M_1 : (\Omega, \mathcal{A}, P) \rightarrow \mathbb{R}^k$ is said to be $P$-integrable if the map $\omega \mapsto \int_{\mathbb{R}^k} xM_1(\omega, dx)$ is $P$-integrable, i.e., if there exists and is finite the integral

$$\int_{\Omega} \int_{\mathbb{R}^k} xM_1(\omega, dx)dP(\omega)$$

or, which is the same, if the distribution $(P \otimes M_1)^{\pi_2}$ has finite mean, where $\pi_2 : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ denotes the second coordinatewise projection. In this case, we define the expectation of the Markov kernel $M_1$ as

$$E_P(M_1) := \int_{\Omega} \int_{\mathbb{R}^k} xM_1(\omega, dx)dP(\omega)$$

Remark. Notice that the previous definition extends standard expectation for a random variable $X : (\Omega, \mathcal{A}, P) \rightarrow \mathbb{R}^k$, as $E_P(M_X) = E_P(X)$.

Definition 5. Let $M_1 : (\Omega, \mathcal{A}, P) \rightarrow \mathbb{R}^k$ be a $P$-integrable Markov kernel. We define a set function $M_1 \cdot P$ on $\mathcal{A}$ by

$$(M_1 \cdot P)(A) := \int_{\Omega} \int_{\mathbb{R}^k} xM_1(\omega, dx)dP(\omega).$$

Note that $M_1 \cdot P \ll P$ and $(M_1 \cdot P)^{M_2} \ll P^{M_2}$, when $M_2 : (\Omega, \mathcal{A}, P) \rightarrow (\Omega_2, \mathcal{A}_2)$ is another Markov kernel.

Definition 6. (Conditional expectation of a Markov kernel given another) Let $M_1 : (\Omega, \mathcal{A}, P) \rightarrow \mathbb{R}^k$ be a $P$-integrable Markov kernel and $M_2 : (\Omega, \mathcal{A}, P) \rightarrow (\Omega_2, \mathcal{A}_2)$ be a Markov kernel. The conditional expectation $E_P(M_1|M_2)$ is defined by:

$$E_P(M_1|M_2) := \frac{d(M_1 \cdot P)^{M_2}}{dP^{M_2}}$$
i.e., \( E_P(M_1|M_2) \) is the (equivalence class of) real measurable function(s) on \((\Omega_2, A_2)\) such that, for every \(A_2 \in A_2\),

\[
\int_{\Omega} M_2(\omega, A_2) \int_{\mathbb{R}^k} xM_1(\omega, dx) dP(\omega) = \int_{A_2} E_P(M_1|M_2) dP(M_2) = \int_{\Omega} \int_{A_2} E_P(M_1|M_2)(\omega_2) M_2(\omega, d\omega_2) dP(\omega).
\]

The next result yields an integral representation of such a conditional expectation. First, we refer the reader to Nogales (2013b) for the definition and existence of the conditional distribution \(P^{M_1|M_2}\) of a Markov kernel \(M_1 : (\Omega, A) \rightarrow (\Omega_1, A_1)\) with respect to another Markov kernel \(M_2 : (\Omega, A, P) \rightarrow (\Omega_2, A_2)\). Namely, it is defined as a Markov kernel \(L : (\Omega_2, A_2) \rightarrow (\Omega_1, A_1)\) such that, for every pair of events \(A_1 \in A_1\) and \(A_2 \in A_2\),

\[
\int_{\Omega} M_1(\omega, A_1) M_2(\omega, A_2) dP(\omega) = \int_{A_2} L(\omega_2, A_1) dP(M_2(\omega_2)) = \int_{\Omega} \int_{A_2} L(\omega_2, A_1) M_2(\omega, d\omega_2) dP(\omega)
\]

**Theorem 1.** Let \(M_1\) and \(M_2\) be two Markov kernels as in the previous definition. Then

\[
E_P(M_1|M_2)(\omega_2) = \int_{\mathbb{R}^k} x P^{M_1|M_2}(\omega_2, dx)
\]

(in the sense that the last integral defines a version of the conditional expectation of \(M_1\) given \(M_2\)). More generally, if \(f : \mathbb{R}^k \rightarrow \mathbb{R}^m\) has nonnegative components or is \(P^{M_1}\)-integrable function, then

\[
E_P(f M_1|M_2)(\omega_2) = \int_{\mathbb{R}^k} f(x) P^{M_1|M_2}(\omega_2, dx),
\]

where \(fM_1\) is the Markov kernel defined by \(fM_1(\omega, C) := M_1(\omega, f^{-1}(C)), \omega \in \Omega, C \in \mathcal{R}^m\).

The following are two examples of calculation.

**Example 1.** Given \(\theta \in [0, 1]\), let \(\Omega = \{0, 1\}\), \(A = \mathcal{P}(\Omega)\) and \(P\) the probability measure on \((\Omega, A)\) assigning probability \(\theta\) to the point 1 and \(1-\theta\) to the point 0. For \(i = 1, 2\), consider the Markov kernel \(M_i : (\Omega, A) \rightarrow (\Omega, A)\) defined by the stochastic matrix

\[
\begin{pmatrix}
p_i & 1-p_i \\
q_i & 1-q_i
\end{pmatrix},
\]

where \(0 \leq p_i, q_i \leq 1\). Then

\[
P^{M_i} \{\{0\}\} = \int_{\{0, 1\}} M_i(\omega, \{0\}) dP(\omega) = (1-\theta)p_1 + \theta q_1
\]

and \(P^{M_i} \{\{1\}\} = (1-\theta)(1-p_1) + \theta (1-q_1)\). Hence,

\[
E_P(M_i) = \int_{\{0, 1\}} \int_{\mathbb{R}} x M_i(\omega, dx) dP(\omega) = P^{M_i}(\{1\}).
\]

Moreover, if \(L := P^{M_2|M_1} : (\Omega, A) \rightarrow (\Omega, A)\), according to Nogales (2013b, Prop. 2), given \(\omega_1, \omega_2 \in \{0, 1\}\),

\[
L(\omega_1, \{\omega_2\}) = \frac{\int_{\{0, 1\}} M_1(i, \omega_1) M_2(i, \omega_2) dP(i)}{\int_{\{0, 1\}} M_1(i, \omega_1) dP(i)} = \frac{(1-\theta)M_1(0, \omega_1) M_2(0, \omega_2) + \theta M_1(1, \omega_1) M_2(1, \omega_2)}{(1-\theta)M_1(0, \omega_1) + \theta M_1(1, \omega_1)}
\]
Hence
\[ L(0, \{1\}) = \frac{(1 - \theta)M_1(0, 0)M_2(0, 1) + \theta M_1(1, 0)M_2(1, 1)}{(1 - \theta)M_1(0, 0) + \theta M_1(1, 0)} = \frac{(1 - \theta)p_1(1 - p_2) + \theta q_1(1 - q_2)}{(1 - \theta)p_1 + \theta q_1} \]
and
\[ L(1, \{1\}) = \frac{(1 - \theta)M_1(0, 1)M_2(0, 1) + \theta M_1(1, 1)M_2(1, 1)}{(1 - \theta)M_1(0, 1) + \theta M_1(1, 1)} = \frac{(1 - \theta)(1 - p_1)(1 - p_2) + \theta (1 - q_1)(1 - q_2)}{(1 - \theta)(1 - p_1) + \theta (1 - q_1)}, \]
while \( L(\omega_1, \{0\}) = 1 - L(\omega_1, \{1\}), \omega_1 = 0, 1. \) Finally, for \( \omega_1 \in \{0, 1\}, \)
\[ E_P(M_2|M_1)(\omega_1) = \int_{\{0, 1\}} xP_{M_2|M_1}(\omega_1, dx) = L(\omega_1, \{1\}). \]

**Subexample 1.1:** (Application to clinical diagnosis) Consider a diagnosis tests \( T \) for a certain disease \( D \). We write \( D = 1 (= 0) \) for an individual having (not having) the disease as determined by a “gold standard” diagnostic procedure, and \( T = 1 (= 0) \) if the diagnostic is positive (negative). There are several terms that are commonly used in this context: \( P(D = 1) \) is called the prevalence of the disease (on a given population), while \( s = P(T = 1|D = 1) \) is the sensitivity of the test and \( e = P(T = 0|D = 0) \) its specificity. The stochastic matrix
\[ M_1 = \begin{pmatrix} p_1 &= e \\ q_1 &= 1 - s \\ & 1 - e \\ s & & \\ & & 1 \end{pmatrix}, \]
describes the transition probabilities from the state \( i \in \{0, 1\} \) (the gold standard test is negative - \( i = 0 \)- or positive - \( i = 1 \)-) to the state \( j \in \{0, 1\} \) (the test \( T \) is negative - \( j = 0 \)- or positive - \( j = 1 \)-). This way, \( M_1 \) becomes a Markov kernel from \( \{0, 1\} \) to \( \{0, 1\} \) and its probability distribution \( P^{M_1} \) satisfies
\[ P^{M_1}([1]) = (1 - \theta)(1 - e) + \theta(1 - s) = P(T = 1), \]
the probability that any given individual of the population receive a positive diagnostic. If \( M_2 \) denotes the gold standard diagnostic test, \( M_2 \) is (identified with) the identity matrix of order 2 (i.e., \( p_2 = 1 \) and \( q_2 = 0 \)). So, analogously, \( P^{M_2}(\{1\}) = P(D = 1) = \theta \). Moreover, if \( L := P^{M_2|M_1} \), according to the example, we have that
\[ E_P(M_2|M_1)(1) = L(1, \{1\}) = \frac{\theta s}{(1 - \theta)(1 - e) + \theta s} = P(D = 1|T = 1) \]
is the so-called predictive positive value \( PPV \) of the diagnosis test \( T \) and, in the same way,
\[ E_P(M_2|M_1)(\{0\}) = L(0, \{1\}) = 1 - \frac{(1 - \theta)e}{(1 - \theta)e + \theta(1 - s)} = 1 - P(D = 0|T = 0), \]
\( P(D = 0|T = 0) \) being the predictive negative value \( PNV \) of \( T \). Now, if
\[ N_1 = \begin{pmatrix} 1 - e & e \\ 1 - s & s \end{pmatrix}, \]
we have that
\[ E_P(N_1) = P^{N_1}(\{1\}) = (1 - \theta)e + \theta s = P(T = 0, D = 0) + P(T = 1, D = 1) \]
is the “accuracy” of \( T \), i.e., the proportion of true diagnostics of \( T \). Moreover
\[ E_P(M_2|N_1)(1) = \frac{\theta s}{\theta s + (1 - \theta)e} = \frac{P(T = 1, D = 1)}{P(T = 0, D = 0) + P(T = 1, D = 1)} \]
is the proportion of positive true diagnostics among all true diagnostics of \( T \). □
Example 2. For $1 \leq i \leq 3$, let $(\Omega_i, A_i, \mu_i)$ be a $\sigma$-finite measure space such that $(\Omega_i, A_i)$ is a
standard Borel space for $X_i$. We assume that the joint distribution of $X = (X_1, X_2, X_3)$ admits a density $f$ with respect to the product measure
$\mu_1 \times \mu_2 \times \mu_3$. We write $f_{ij}$ for the joint $\mu_i \times \mu_j$-density of $(X_i, X_j)$ when $1 \leq i < j \leq 3$, and $f_i$ for the $\mu_i$-density of $X_i$. It is shown in Nogales (2013b, Example 1) that the conditional distributions $M_i = P_{X_i | X_1} : (\Omega_1, A_1) \rightarrow (\Omega_i, A_i)$, $i = 2, 3$, and $L := P_{M_2 | M_3}$ exist, where $P_1 = P_{X_1}$, and that a density of $L(\omega_3, \cdot)$ with respect to $\mu_2$ is the map
$$
\omega_2 \mapsto \int_{\Omega_1} \frac{f_{12}(\omega_1, \omega_2) f_{13}(\omega_1, \omega_3)}{f_1(\omega_1) f_3(\omega_3)} d\mu_1(\omega_1)
$$
$L$ is in fact the conditional distribution of a conditional distribution given another conditional distribution! So, when $(\Omega_2, A_2) = (\mathbb{R}^k, \mathcal{B}^k)$ and $\mu_2$ is the Lebesgue measure, we have that the conditional expectation $E_{P_1}(M_2 | M_3)$ is the map
$$
\omega_3 \mapsto \int_{\mathbb{R}^k} x \int_{\Omega_1} \frac{f_{12}(\omega_1, x_2) f_{13}(\omega_1, \omega_3)}{f_1(\omega_1) f_3(\omega_3)} d\mu_1(\omega_1) dx_2
$$
For instance, let $X = (X_1, X_2, X_3)$ be a trivariate normal random variable with null mean and $P_1$ the marginal distribution of $X_1$. For $i = 2, 3$, consider the Markov kernel $M_i = P_{X_i | X_1}$, the conditional distribution of $X_i$ given $X_1$. It is shown in Nogales (2013b) that the conditional distribution $L := P_{M_2 | M_3}$ of the Markov kernel $M_2$ given $M_3$ with respect to $P_1$ satisfy that $L(x_3, \cdot)$ is the univariate normal distribution of mean $\frac{\sigma_2 \rho_{13} \mu_2}{\sigma_1}$ and variance $\sigma_2^2(1 - \rho_{12}^2 \rho_{13}^2)$, where $\sigma_1$ is the standard deviation of $X_1$ and $\rho_{ij}$ stand for the correlation coefficient of $X_i$ and $X_j$. According to the previous result, the conditional expectation of $M_2$ given $M_3$ is the random variable $x_3 \mapsto \frac{\sigma_2 \rho_{13} \mu_2}{\sigma_1} x_3$.

Moreover, it is easily checked that $P_1^{M_2} = P_{X_2}$ and $E_{P_1}(M_2) = E_P(X_2)$. □

Note that
\begin{align*}
\int_{\Omega_2} E_P(M_1 | M_2) dP^{M_2} &= \int_{\Omega_2} \int_{\mathbb{R}^k} x P^{M_1 | M_2}(\omega_2, dx) dP^{M_2}(\omega_2) \\
&= \int_{\Omega_2} \int_{\mathbb{R}^k} xdP^{M_2 \times M_1}(\omega_2, x) \\
&= \int_{\mathbb{R}^k} xd(P^{M_2 \times M_1})^{\pi}(x),
\end{align*}
where $\pi : \Omega_2 \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is the coordinatewise projection and $M_2 \times M_1 : (\Omega, A) \rightarrow (\Omega_2 \times \mathbb{R}^k, A_2 \times \mathcal{B}^k)$ satisfies $(M_2 \times M_1)(\omega, A_2 \times A_1) = M_2(\omega, A_2) \cdot M_1(\omega, A_1)$, $A_i \in A$, $i = 1, 2$. But $(P^{M_2 \times M_1})^{\pi} = P^{M_1}$. Hence
\begin{align*}
E_{P^{M_2}}(E_P(M_1 | M_2)) &= \int_{\Omega_2} E_P(M_1 | M_2) dP^{M_2} = \int_{\mathbb{R}^k} xdP^{M_1}(x) = E_P(M_1).
\end{align*}

This way we obtain the following corollary, which generalizes a known property of usual conditional expectations.

Corollary 1. Let $M_1$ and $M_2$ be two Markov kernels as in the previous Definition. Then
\begin{align*}
E_{P^{M_2}}(E_P(M_1 | M_2)) &= E_P(M_1).
\end{align*}

We can have a representation of conditional expectations for Markov kernels in terms of conditional expectations for random variables.

Theorem 2. If $M_1$ is $P$-integrable, $E_P(M_1 | M_2) = E_{P^{M_1}}(\tilde{M}_1 | \pi_2)$ where $\tilde{M}_1 : (\Omega \times \Omega_2, A \times A_2) \rightarrow \mathbb{R}^k$ is defined by $\tilde{M}_1(\omega, \omega_2) := \int_{\mathbb{R}^k} x M_1(\omega, dx)$, and $\pi_2$ is the second coordinatewise projection on $\Omega \times \Omega_2$.

As a consequence of this representation theorem and Jensen’s Inequality, we have the next result.
Corollary 2. For every $Z \in \mathcal{L}^2(\Omega_2, A_2, (P \otimes M_2)\pi_2)$, we have that

$$\|\bar{M}_1 - EP(M_1|M_2)\|^2 \leq \|\bar{M}_1 - Z\|^2,$$

i.e.,

$$\int_{\Omega \times \Omega_2} (\bar{M}_1(\omega, \omega_2) - EP(M_1|M_2)(\omega_2))^2 d(P \otimes M_2)(\omega, \omega_2) \leq \int_{\Omega \times \Omega_2} (\bar{M}_1(\omega, \omega_2) - Z(\omega_2))^2 d(P \otimes M_2)(\omega, \omega_2),$$

**Subexample 1.1 (cont.): (Application to clinical diagnosis) Applying the preceding Corollary to Subexample 1.1, writing $a = Z(0)$ ($a$ could represent the probability that the decision 0 is taken, i.e., the test $T$ discards the disease) and $b = Z(1)$ ($b$ could represent the probability that the decision 1 is taken, i.e., the test $T$ confirms the disease), we obtain the following interpretation of predictive values of a diagnostic test $T$:

$$(1 - PNV, PPV) = \arg \min_{(a,b) \in \mathbb{R}^2} \{(1-a)^2 e + b^2(1-e)(1-\theta) + [a^2(1-s) + (1-b)^2 s] \theta\}.$$ 

Notice that, for a non-ill individual (i.e., when $D = 0$), the right decision will be $(a_0, b_0) = (1, 0)$, and $(1-a)^2 e + b^2(1-e)$ is a weighted squared distance between $(a, b)$ and the optimal point $(1, 0)$ on $\{D = 0\}$; the weights are $e = P(T = 0|D = 0)$ and $1-e = P(T = 1|D = 0)$ for the discrepancy between $a$ and $a_0 = 1$, and that of $b$ and $b_0 = 0$, respectively, as can be expected. Analogously, for an ill individual (i.e., when $D = 1$), the right decision is $(a_1, b_1) = (0, 1)$, and $a^2(1-s) + (1-b)^2 s$ is also a properly weighted squared distance between $(a, b)$ and the optimal point $(0, 1)$ on $\{D = 1\}$.

Notice finally that, in the daily clinical practice, it is not known whether $D = 0$ or $D = 1$ and we should choose $(a, b)$ in such a way that its simultaneous distance to $(1, 0)$ on $\{D = 0\}$ and to $(0, 1)$ on $\{D = 1\}$ reach a minimum; obviously, this simultaneous squared distance is weighted according to the sizes of the subpopulations $\{D = 0\}$ and $\{D = 1\}$. □

4 Some statistical applications: extension to Markov kernels of the Rao-Blackwell and the Lehmann-Scheffé theorems

Now, we position ourselves in a statistical context. Let $(\Omega, A, \mathcal{P})$ be a statistical experiment (i.e., $\mathcal{P}$ is a family of probability measures on the measurable space $(\Omega, A)$).

The theorems of Rao-Blackwell and Lehmann-Scheffé are central results of unbiased point estimation theory. We pursue in this section a version in the Markov kernel framework.

The concepts defined in the preceding sections can be extended to a statistical framework in a standard way. The concept of sufficiency for Markov kernels is introduced in Heyer (1982, p.163). Recall that, given a Markov kernel $M_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)$ and $P \in \mathcal{P}$, the conditional probability $P(A|M_1)$ of an event $A \in \mathcal{A}$ given $M_1$ is defined as the Radon-Nikodym derivative $d(I_A \cdot P)^{M_1}/dP^{M_1}$, where $I_A : P$ denotes the measure defined on $A$ by $(I_A \cdot P)(B) = P(A \cap B)$. In other words, $P(A|M_1)$ is the (equivalence class of) real random variable(s) on $(\Omega_1, \mathcal{A}_1)$ such that, for every $A_1 \in \mathcal{A}_1$,

$$\int_{A_1} M_1(\omega, A_1)dP(\omega) = \int_{A_1} P(A|M_1)dP^{M_1}$$

\begin{equation}
= \int_{\bar{\Omega}_1} \int_{A_1} P(A|M_1)(\omega_1) M_1(\omega, d\omega_1)dP^{M_1}(\omega)
\end{equation}

**Definition 7.** (Sufficiency of a Markov kernel) A Markov kernel $M_1 : (\Omega, \mathcal{A}) \rightarrow (\Omega_1, \mathcal{A}_1)$ is said to be sufficient if, for every $A \in \mathcal{A}$, there exists a common version $f_A : (\Omega_1, \mathcal{A}_1) \rightarrow [0, 1]$ to the conditional probabilities $P(A|M_1)$, $P \in \mathcal{P}$.
Remarks. 1) The previous definition generalizes that of a sufficient statistic in the sense that a statistic \( T_1 \) is sufficient if and only if the corresponding kernel \( M_{T_1}(\omega, A_1) = \delta_{T_1(\omega)}(A_1) \) is sufficient. Also, a sub-\( \sigma \)-field \( B \subset A \) is sufficient if and only if its corresponding kernel \( M_B : (\Omega, A) \rightarrow (\Omega, B) \), defined by \( M_B(\omega, B) := \delta_{\omega}(B) \), is also.

2) Theorem 22.3 of Heyer (1982) shows that a Markov kernel \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_1, A_1) \) is sufficient if and only if the \( \sigma \)-field \( \pi_1^{-1}(A_1) \) is sufficient in the statistical experiment \( (\Omega \times \Omega_1, A \times A_1, \{P \otimes M_1 : P \in \mathcal{P}\}) \), where \( \pi_1 \) denotes the coordinatewise projection over \( \Omega_1 \).

3) (Sufficiency of Markov kernels when densities are available) Suppose that \( \mathcal{P} \) is dominated by a \( \sigma \)-finite measure \( \mu \) on \( (\Omega, A) \). Typically the Lebesgue measure in the absolute continuous case and the counting measure in the discrete case. Let \( f_P \) be a \( \mu \)-density of \( P \in \mathcal{P} \). Let \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_1, A_1) \) be a Markov kernel and suppose that \( m_1 : (\Omega \times \Omega_1, A \times A_1) \rightarrow [0, \infty[ \) is a measurable function such that, for every \( \omega \in \Omega \), \( m_1(\omega, \cdot) \) is a \( \mu_1 \)-density of the probability measure \( M_1(\omega, \cdot) \), where \( \mu_1 \) is a \( \sigma \)-finite measure on \( (\Omega_1, A_1) \). It is readily shown that

\[
\frac{d(P \otimes M_1)}{d(\mu \times \mu_1)}(\omega,\omega_1) = m_1(\omega,\omega_1) \cdot f_P(\omega).
\]

According to the previous remark and the factorization theorem, the Markov kernel \( M_1 \) is sufficient if and only if there exist a measurable function \( h : (\Omega \times \Omega_1, A \times A_1) \rightarrow [0, \infty[ \) and, for each \( P \in \mathcal{P} \), a measurable function \( g_P : (\Omega_1, A_1) \rightarrow [0, \infty[ \) such that

\[
m_1(\omega,\omega_1) \cdot f_P(\omega) = g_P(\omega_1) \cdot h(\omega,\omega_1), \quad \forall \omega,\omega_1.
\]

Here we introduce two examples, one discrete and one continuous, of sufficient Markov kernels not associated to statistics.

Example 3. Let \( \Omega = \{1, 2, 3\} \), \( A \) the \( \sigma \)-field of all subsets of \( \Omega \), and \( \mathcal{P} := \{P_\theta : \theta \in [0, 1]\} \), where \( P_\theta \) assigns probability \( \theta/3 \) to the points 1 and 2 and probability \( 1 - 2\theta/3 \) to the point 3. The Markov kernel \( M : (\Omega, A) \rightarrow (\Omega, A) \) defined by the stochastic matrix

\[
\begin{pmatrix}
1/3 & 2/3 & 0 \\
1/3 & 2/3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

is sufficient and is not associated to any statistic. \( \square \)

Example 4. Let \( (\Omega, A) = (\mathbb{R}^+, \mathcal{R}^+) \) and \( \mathcal{P} = \{P_\theta : \theta = 0, 1, 2, \ldots\} \), where \( dP_\theta(x) = I_{[\theta, \theta+1]}(x) \, dx \). For \( x \geq 0 \), we denote by \( M(x, \cdot) \) the uniform distribution on the interval \( [\lfloor x \rfloor, \lfloor x \rfloor + 1] \), where \( \lfloor x \rfloor \) stands for the integer part of \( x \). The Markov kernel \( M : (\Omega, A) \rightarrow (\Omega, A) \) is sufficient and is not associated to any statistic. \( \square \)

Let us recall from Nogales (2013a) the generalization of the concept of completeness to Markov kernels.

Definition 8. (Completeness of Markov kernels) A Markov kernel \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_1, A_1) \) is said to be complete (respectively, boundedly complete) if, for every (respectively, bounded) real statistic \( f : (\Omega_1, A_1, \{P^{M_1} : P \in \mathcal{P}\}) \rightarrow \mathbb{R} \),

\[
E_{P^{M_1}} f = 0, \quad \forall P \in \mathcal{P} \quad \implies \quad f = 0, \quad P^{M_1}\text{-almost surely}, \quad \forall P \in \mathcal{P}.
\]

Remarks. 1) A Markov kernel \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_1, A_1) \) is (respectively, boundedly) complete if and only if the \( \sigma \)-field \( \pi_1^{-1}(A_1) \) on the statistical experiment \( (\Omega \times \Omega_1, A \times A_1, \{P \otimes M_1 : P \in \mathcal{P}\}) \) is also, where \( \pi_1 \) denotes the coordinatewise projection over \( \Omega_1 \), which in turn is equivalent to the (bounded) completeness of \( \pi_1 \) (see Nogales (2103a)). Moreover, if \( M_1 \) is the Markov kernel corresponding to a statistic \( T_1 \), then \( M_1 \) is (boundedly) complete if and only if \( T_1 \) is also.

2) (Completeness of Markov kernels when densities are available) Suppose that \( \mathcal{P} \) is dominated by a \( \sigma \)-finite measure \( \mu \) on \( (\Omega, A) \). Let \( f_P \) be a \( \mu \)-density of \( P \in \mathcal{P} \). Let \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_1, A_1) \) be a
Markov kernel and suppose that \( m_1 : (\Omega \times \Omega_1, \mathcal{A} \times \mathcal{A}_1) \to [0, \infty] \) is a measurable function such that, for every \( \omega \in \Omega \), \( m_1(\omega, \cdot) \) is a \( \mu_1 \)-density of the probability measure \( M_1(\omega, \cdot) \), where \( \mu_1 \) is a \( \sigma \)-finite measure on \((\Omega_1, \mathcal{A}_1)\). It is readily shown that

\[
\frac{d(P \otimes M_1)}{d(\mu \times \mu_1)}(\omega, \omega_1) = m_1(\omega, \omega_1) \cdot f_P(\omega).
\]

According to the previous remark, the Markov kernel \( M_1 \) is complete if and only if for every statistic \( f : (\Omega_1, \mathcal{A}_1) \to \mathbb{R} \) we have that

\[
\int_{\Omega \times \Omega_1} f(\omega_1)m_1(\omega_1, \omega) f_P(\omega)d(\mu \times \mu_1)(\omega, \omega_1) = 0, \ \forall P \in \mathcal{P} \text{ } \implies \text{ } f = 0, \ (P \otimes M_1)^{\ast 1} - \text{c.s.}, \forall P \in \mathcal{P}.
\]

Here we present two examples of complete Markov kernels not associated to statistics.

**Example 5.** Let \( \Omega = \{1, 2\} \), \( \mathcal{A} = \mathcal{P}(\Omega) \) and \( \mathcal{P} := \{P_\theta : \theta \in [0, 1]\} \), where \( P_\theta \) assigns probability \( \theta \) to the point 1 and probability \( 1 - \theta \) to the point 2. The Markov kernel \( M : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A}) \) defined by the stochastic matrix

\[
\begin{pmatrix}
 p & 1 - p \\
 q & 1 - q
\end{pmatrix}
\]

is complete for \( p, q \in [0, 1] \) when \( p \neq q \), and it is not associated to any statistic unless \( p, q \in \{0, 1\} \).

**Example 6.** Let \( \Omega = \mathbb{R}^+ \), \( \mathcal{A} = \mathcal{P}(\Omega) \) and \( \mathcal{P} := \{P_\theta : \theta > 0\} \), where \( P_\theta \) denotes the exponential distribution of parameter \( \theta \). For \( x > 0 \), we denote by \( M(x, \cdot) \) the uniform distribution on the interval \([x, x + 1]\). The Markov kernel \( M : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A}) \) is complete and is not associated to any statistic.

Now we are ready to obtain a first extension to Markov kernels of the theorem of Lehmann-Scheffé. \[\text{Theorem[6]}\] yields a more general result. First, recall that an statistic \( T : (\Omega, \mathcal{A}, \mathcal{P}) \to \mathbb{R}^k \) is said to be an unbiased estimator of a function \( f : \mathcal{P} \to \mathbb{R}^k \) whenever \( E_P(T) = f(P) \), for all \( P \in \mathcal{P} \). \( T \) is said to be a minimum variance estimator of \( f \) if it is unbiased and has less variance than any other unbiased estimator of \( f \).

Let \( M_1 : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Omega_1, \mathcal{A}_1) \) a Markov kernel and \( T : (\Omega, \mathcal{A}, \mathcal{P}) \to \mathbb{R}^k \) be a statistic. We say that \( T \) is a measurable function of \( M_1 \) if there exists a measurable map \( S : (\Omega_1, \mathcal{A}_1) \to \mathbb{R}^k \) such that \( M_1 = M_2 \).

**Theorem 3.** Assuming the previous notations, let us suppose that \( M_1 : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Omega_1, \mathcal{A}_1) \) is a sufficient and complete Markov kernel and \( T : (\Omega, \mathcal{A}, \mathcal{P}) \to \mathbb{R}^k \) is an unbiased estimator of a function \( f : \mathcal{P} \to \mathbb{R}^k \). If \( T \) is a measurable function of \( M_1 \), then it is the minimum variance unbiased estimator of \( f \).

Now let us recall the definition of unbiased (randomized) estimator.

**Definition 9.** (Unbiased estimator) An unbiased estimator of a function \( f : \mathcal{P} \to \mathbb{R}^k \) is a \( \mathcal{P} \)-integrable Markov kernel \( M : (\Omega, \mathcal{A}, \mathcal{P}) \to (\mathbb{R}^k, \mathcal{R}^k) \) such that

\[
E_P(M) := \int_{\Omega} \int_{\mathbb{R}^k} xM(\omega, dx)dP(\omega) = f(P), \ \forall P \in \mathcal{P}
\]

**Theorem 4.** Let \( M_1 : (\Omega, \mathcal{A}, \mathcal{P}) \to \mathbb{R}^k \) and \( M_2 : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Omega_2, \mathcal{A}_2) \) be Markov kernels. If \( M_2 \) is sufficient, then there exists a regular conditional probability \( P^{M_1|M_2} \) of \( M_1 \) given \( M_2 \) which is independent of \( P \in \mathcal{P} \). There exists also a common version of the conditional expectations \( E_P(M_1|M_2) \), \( P \in \mathcal{P} \); it will be denoted \( E(M_1|M_2) \).

The next theorem extend to Markov kernels the Rao-Blackwell theorem.

**Theorem 5.** (Theorem of Rao-Blackwell generalized) Let \( M_1 : (\Omega, \mathcal{A}, \mathcal{P}) \to \mathbb{R}^k \) be an estimator of \( f : \mathcal{P} \to \mathbb{R} \) and \( M_2 : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Omega_2, \mathcal{A}_2) \) be a sufficient Markov kernel for \( \mathcal{P} \). Then \( E(M_1|M_2) \) is an estimator of \( f \) with less convex risk than \( M_1 \). If the loss function is strictly convex then, given \( P \in \mathcal{P} \), the risk at \( P \) of \( E(M_1|M_2) \) is strictly less than the risk at \( P \) of \( M_1 \) unless \( E(M_1|M_2) \) is a Markov kernel, where \( M_1 \) is defined as in \( \text{Theorem[2]} \). Finally, if \( M_1 \) is unbiased, so is \( E(M_1|M_2) \).
Remark. Since \( E(M_1 | M_2) \) is a statistic, this theorem shows that the class of non-randomized unbiased estimators of \( f \) is complete in the sense that, for every randomized unbiased estimator \( M_1 \) of \( f \), there exists a non-randomized unbiased estimator \( E(M_1 | M_2) \) with less convex risk than \( M_1 \). Note that this assertion remains true if the assumption of unbiasedness is dropped. This result generalizes a similar assertion remains true if the assumption of unbiasedness is dropped. This result generalizes a similar

Theorem 6. (Theorem of Lehmann-Scheffé generalized) Let \( M_1 : (\Omega, A, \mathcal{P}) \rightarrow \mathbb{R}^k \) be an unbiased estimator of \( f : \mathcal{P} \rightarrow \mathbb{R}^k \) and \( M_2 : (\Omega, A, \mathcal{P}) \rightarrow (\Omega_2, A_2) \) be a sufficient and complete Markov kernel for \( \mathcal{P} \). Then \( E(M_1 | M_2) \) is the estimator of \( f \) which minimizes the convex risk among all unbiased estimators of \( f \).

5 Looking for an example of application of the generalized Lehmann-Scheffé Theorem

As a Markov kernel is both an extension of the concepts of random variable and \( \sigma \)-field, Theorem 6 can be considered as an unification of the statistics and \( \sigma \)-fields versions of the Lehmann-Scheffé Theorem.

But, beyond that, to give full meaning to this result, a sufficient and complete Markov kernel \( M \) that is not associated to any statistic will be desirable. To this end, I wonder if the following general procedure can work: We start with a sufficient Markov kernel \( M \) not associated to any statistic --two examples has been provided above- and construct a greater family of probabilities for which \( M \) is complete and remains still sufficient. This way, if \( M \) has finite mean \( f \), the statistic \( E(M_1 | M) \) would be the estimator of \( f \) which minimizes the convex risk among all unbiased estimators of \( f \).

First, the right context is fixed. Let \( (\Omega, A, \mathcal{P}) \) be a statistical experiment dominated by a \( \sigma \)-finite measure \( \mu \). Let \( P^* \) be a privileged dominating probability (i.e. \( P^* \) is a probability measure on \( (\Omega, A) \) such that \( \mathcal{P} \ll P^* \) and is of the form \( P^* = \sum_n 2^{-n} P_{\omega_n} \) for some countable dominating subfamily \( \{P_{\omega_n} : n \geq 1\} \) of \( \mathcal{P} \)). A positive response to the following question would allow us to resolve the posed problem, starting with the sufficient Markov kernel of Example 4.

Question 1. Let \( M : (\Omega, A, \mathcal{P}) \rightarrow (\Omega', A') \) be a sufficient Markov kernel and denote \( \hat{\mathcal{P}} \) the family of all probability measures \( \hat{P} \) on \( A \) such that \( \hat{P} \ll P^* \) and \( M \) is sufficient for the extended family \( \mathcal{P} \cup \{P\} \).

(a) \( M \) is sufficient and complete for \( \hat{\mathcal{P}} \)?

(b) Let \( n, k \in \mathbb{N} \) and suppose \( (\Omega', A') = (\mathbb{R}^n, \mathbb{R}^n) \). If \( M \) has finite moments of order \( k \) and \( \hat{P}_k \) denotes the subfamily of \( \hat{\mathcal{P}} \) preserving this property of \( M \), then \( M \) is sufficient and complete for \( \hat{P}_k \)?

Since, according to Heyer (1982, Theorem 22.3) (resp. Remark 1 of Definition 8), a Markov kernel \( M : (\Omega, A, \mathcal{P}) \rightarrow (\Omega', A') \) is sufficient (resp. complete) if and only if the projection statistic

\[
\pi' : (\omega, \omega') \in (\Omega \times \Omega', A \times A', \{P \otimes M : P \in \mathcal{P}\}) \mapsto \omega' \in (\Omega', A')
\]

is sufficient (resp. complete), one might expect an answer to this question from the following -- interesting in itself-- result,

Theorem 7. Let \( S : (\Omega, A, \mathcal{P}) \rightarrow (\Omega', A') \) be a sufficient statistic and \( \hat{\mathcal{P}} \) the family of all probabilities \( \hat{P} \) on \( (\Omega, A) \) such that \( \hat{P} \ll P^* \) and \( S \) is sufficient for \( \mathcal{P} \cup \{P\} \).

(a) Then \( S \) is sufficient and complete for the extended family \( \hat{\mathcal{P}} \).

(b) Suppose now that \( (\Omega', A') = (\mathbb{R}^k, \mathbb{R}^k) \), and \( P^* \in \mathcal{P} \) (or there exists \( P' \in \mathcal{P} \) such that \( \mathcal{P} \ll P' \)). If \( S \) has finite moment of order \( n \in \mathbb{N} \) and \( \mathcal{P}_n \) denotes the set of the probability measures \( \hat{P} \in \mathcal{P} \) such that \( S \) has finite \( \hat{P} \)-moment of order \( n \), then \( S \) is also sufficient and complete for \( \mathcal{P}_n \).

Now we wonder if Question 1 actually becomes a consequence of the previous theorem. For that to be true, it would be enough that the extended family of \( \{P \otimes M : P \in \mathcal{P}\} \) should coincide with \( \{\hat{P} \otimes M : \hat{P} \in \hat{\mathcal{P}}\} \), something that does not seem very clear. So Question 1 remains unsolved but, in the attempt, Theorem 6 has seen the light.
6 Proofs

Proof of Theorem 1 First note that there exists a regular conditional probability $P_{M_1|M_2}$ (see Nogales (2013b)). It will be enough to show that, given $A_2 \in A_2$,

$$\int_{\Omega} M_2(\omega, A_2) \int_{\mathbb{R}^k} x M_1(\omega, dx) dP(\omega) = \int_{A_2} \int_{\mathbb{R}^k} x P_{M_1|M_2}(\omega_2, dx) dP_{M_2}(\omega_2)$$

But by definition of $P_{M_1|M_2}$, for all $A_1, A_2$,

$$\int_{\Omega} M_1(\omega, A_1) M_2(\omega, A_2) dP(\omega) = \int_{A_2} P_{M_1|M_2}(\omega_2, A_1) dP_{M_2}(\omega_2)$$

e.i.,

$$\int_{\Omega} M_2(\omega, A_2) \int_{\mathbb{R}^k} I_{A_1}(x) M_1(\omega, dx) dP(\omega) = \int_{A_2} \int_{\mathbb{R}^k} I_{A_1}(x) P_{M_1|M_2}(\omega_2, dx) dP_{M_2}(\omega_2)$$

It follows in a standard way that, for any nonnegative or $P_{M_1}$-integrable measurable function $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$,

$$\int_{\Omega} M_2(\omega, A_2) \int_{\mathbb{R}^k} f(x) M_1(\omega, dx) dP(\omega) = \int_{A_2} \int_{\mathbb{R}^k} f(x) P_{M_1|M_2}(\omega_2, dx) dP_{M_2}(\omega_2)$$

which gives the proof. □

Proof of Theorem 2 Recall that $P_{M_2} = (P \otimes M_2)^{\pi_2}$. Now we define a Markov kernel $\hat{M}_1 : (\Omega \times \Omega_2, A \times A_2) \rightarrow \mathbb{R}^k$ by $\hat{M}_1((\omega, \omega_2), B) = M_1(\omega, B)$; $\hat{M}_1$ is the extension to $\Omega \times \Omega_2$ of $M_1$.

We will prove that $(P \otimes M_2)^{\hat{M}_1|\pi_2}$ is a regular conditional $P$-probability of $M_1$ given $M_2$. We will use the following result from Nogales (2013b): “If $T_2 : (\Omega, A) \rightarrow (\Omega_2, A_2)$ is a random variable and $K_2(\omega, A_2) = \delta_{T_2(\omega)}(A_2)$ is its corresponding Markov kernel then, writing $P_{M_1|T_2} := P_{M_1|K_2}$, we have $P_{M_1|T_2}(\cdot, A_1) = E_{P_{M_2}}(M_1(\cdot, \cdot)|T_2)$.” Applying this result in the probability space $(\Omega \times \Omega_2, A \times A_2, P \otimes M_2)$, we have that, for $\omega_2 \in \Omega_2$ and $B \in \mathcal{R}^k$,

$$P_{M_2}^{\hat{M}_1|\pi_2}(\omega_2, B) = E_{P \otimes M_2}(\hat{M}_1(\cdot, B)|\pi_2 = \omega_2)$$

Hence, given $A_2 \in A_2$,

$$\int_{A_2} (P \otimes M_2)^{\hat{M}_1|\pi_2 = \omega_2}(B) dP_{M_2}(\omega_2) = \int_{A_2} (P \otimes M_2)^{\hat{M}_1|\pi_2 = \omega_2}(B) dP_{M_2}(\omega_2)$$

which proves that

$$P_{M_2}^{\hat{M}_1|\pi_2} = P_{M_1|M_2}$$

Moreover, (2) can be rewritten in the form

$$\int_{\mathbb{R}^k} I_B(x)(P \otimes M_2)^{\hat{M}_1|\pi_2 = \omega_2}(dx) = E_{P \otimes M_2} \left( \int_{\mathbb{R}^k} I_B(x) M_1(\cdot, dx) \bigg| \pi_2 = \omega_2 \right)$$

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It follows that, for a nonnegative or integrable measurable function $f : \mathbb{R}^k \to \mathbb{R}^m$,
\[ \int_{\mathbb{R}^k} f(x)(P \otimes M_2)\hat{M}_1\pi_2 = \omega_2 \, dx = E_{P \otimes M_2} \left( \int_{\mathbb{R}^k} f(x)M_1(\cdot, dx) \mid \pi_2 = \omega_2 \right) \]
In particular, for $m = k$ and $f(x) = x$,
\[ \int_{\mathbb{R}^k} x(P \otimes M_2)\hat{M}_1\pi_2 = \omega_2 \, dx = E_{P \otimes M_2} \left( \int_{\mathbb{R}^k} xM_1(\cdot, dx) \mid \pi_2 = \omega_2 \right) \]
Using (3), we obtain
\[ E_P(M_1|M_2) = E_{P \otimes M_2}(\hat{M}_1|\pi_2). \]
\[ \square \]

**Proof of Theorem 3.** Let $T : (\Omega, \mathcal{A}) \to \mathbb{R}^k$ be an arbitrary unbiased estimator of $f$ and denote $\widetilde{T}'(\omega, \omega_1) := T'(\omega)$. Hence $\widetilde{T}'$ is an unbiased estimator of $f$ in the statistical experiment $(\Omega \times \Omega_1, \mathcal{A} \times \mathcal{A}_1, \{P \otimes M_1 : P \in \mathcal{P}\})$. Since the coordinatewise projection $\pi_1$ is sufficient, there exists a version of the conditional expectation $X'$ of $\widetilde{T}$ given $\pi_1$ which is independent of $P \in \mathcal{P}$. The Rao-Blackwell theorem shows that $X' \circ \pi_1$ has less covariance matrix than $T'$.

Since $M_T = M_S M_1$, we have that, for all Borel set $B \in \mathcal{R}^k$ and all $\omega \in \Omega$,
\[ I_B(T(\omega)) = \int_{\Omega_1} I_B(S(\omega_1))M_1(\omega, d\omega_1) \]
Hence, for all $\omega \in \Omega$, $S = T(\omega)$, $M_1(\omega, \cdot)$-a.s. It follows that
\[ \tilde{T}(\omega, \omega_1) = (S \circ \pi_1)(\omega, \omega_1), \quad \{P \otimes M_1 : P \in \mathcal{P}\} - \text{a.s.} \]
where $\tilde{T}(\omega, \omega_1) = T'(\omega)$, for all $\omega \in \Omega$. So, $S$ is a conditional expectation of $\tilde{T}$ given $\pi_1$ for all $P \in \mathcal{P}$.

The completeness of $\pi_1$ shows that $S \circ \pi_1 = X' \circ \pi_1$, $\{P \otimes M_1 : P \in \mathcal{P}\}$-a.s., and this finish the proof. $\square$

**Proof of Theorem 4.** According to Heyer (1982, Theorem 22.3), $M_2$ is sufficient if and only if the coordinatewise projection $\pi_2 : (\Omega \times \Omega_2, \mathcal{A} \times \mathcal{A}_2, \{P \otimes M_2 : P \in \mathcal{P}\}) \to (\Omega_2, \mathcal{A}_2)$ is sufficient. Landers and Rogge (1972, Theorem 7) shows the existence of a common regular conditional probability on $\mathcal{R}^k$ given $\pi_2$. The result follows from this fact and the following representation of the conditional distribution of $M_1$ given $M_2$ obtained in the proof of Theorem 2.
\[ P^{M_1|M_2}(\omega_2, B) = (P \otimes M_2)\hat{M}_1\pi_2(\omega_2, B) = E_{P \otimes M_2}(\hat{M}_1(\cdot, B)|\pi_2 = \omega_2) \]
The second assertion follows from this and Theorem 1. $\square$

**Proof of Theorem 5.** $E(M_1|M_2)$ is well defined by the previous theorem and it is an unbiased estimator of $f$ by Corollary 1. Moreover, if $W : \mathcal{P} \times \mathcal{R}^k \to [0, \infty]$ is a convex loss function (i.e., $W(P, \cdot)$ is a convex function for every $P \in \mathcal{P}$) then applying the Jensen inequality (see Pfanzagl (1994, Theorem 1.10.11)), we obtain from Theorem 4 that
\[ W(P, E_P(M_1|M_2)) = W \left( P \int_{\mathbb{R}^k} xP^{M_1|M_2}(\cdot, dx) \right) \leq \int_{\mathbb{R}^k} W(P, x)P^{M_1|M_2}(\cdot, dx) = E_P(W(P, M_1)|M_2), \quad P^{M_2} - \text{a.s.} \]
where $W(P, M_1)$ denotes the kernel $W(P, \cdot)M_1$. The result follows by integration with respect to $P^{M_2}$. Corollary 1 completes the proof in the unbiased case. $\square$

**Proof of Theorem 6.** If the Markov kernel $M'_1 : (\Omega, \mathcal{A}) \to \mathcal{R}^k$ is an arbitrary unbiased estimator of $f$ then, according to the previous theorem, $X'_1 := E(M'_1|M_2)$ is a nonrandomized unbiased
estimator of $f$ with less convex risk than $M'_1$. Moreover $X_1 := E(M_1|M_2)$ is an unbiased estimator of $f$; so $E_{PM_2}(X_1 - X'_1) = 0$ for all $P \in \mathcal{P}$. Since $M_2$ is complete, we have that $X_1 = X'_1$, $\{P^{M_2} : P \in \mathcal{P}\}$-a.s. So $X_1$ has less convex risk than $M'_1$. □

Proof of Theorem 7. (a) By the sufficiency of $S$, $P^*$ being a privileged dominating probability, we can write the density of $P \in \mathcal{P}$ with respect to $P^*$ in the form $g_P \circ S$ for some suitable non-negative $\mathcal{A}'$-measurable function $g_P$. Such a factorization is also valid for $\hat{P} \in \hat{\mathcal{P}}$, as these properties of $S$ and $P^*$ remain valid for $P \cup \{\hat{P}\}$. Therefore, $S$ is sufficient for the extended family $\hat{\mathcal{P}}$.

To prove the $\hat{\mathcal{P}}$-completeness of $S$, a reasoning by reduction to the absurd will be made by assuming the existence of a probability $\hat{P}_0 \in \hat{\mathcal{P}}$ and a random variable $f : (\Omega', \mathcal{A}') \to \mathbb{R}$ such that $E_{\hat{P}_0 S}(f) = 0$, $\forall \hat{P} \in \hat{\mathcal{P}}$, and $\hat{P}_0 S(f \neq 0) > 0$.

Without loss of generality we can suppose that $\hat{P}_0 S(f > 0) > 0$. Since $\hat{P}_0 \ll P^{*S}$, $\alpha := P^{*S}(f > 0) > 0$. So $g_1 := \frac{1}{\alpha} I_{\{f > 0\}}$ defines a $P^{*S}$-density of a probability measure on $\mathcal{A}'$ and $g_1 \circ S$ becomes a $P^*$-density of a probability measure $P_1$ on $\mathcal{A}$. So $P_1 \in \hat{\mathcal{P}}$ and $dP^{S}_1 = g_1 dP^{*S}$. Moreover,

$$E_{P_1 S}(f) = \frac{1}{\alpha} \int_{\Omega'} f \cdot I_{\{f > 0\}} dP^{*S} > 0,$$

a contradiction which finishes the proof of (a).

(b) The sufficiency is obtained in the same way as in (a). To prove completeness, taking $k = 1$ for simplicity, we just have to check that $P_1 \in \hat{\mathcal{P}}_n$. But

$$E_{P_1}(|S|^n) = \frac{1}{\alpha} \int_{\Omega} |S(\omega)|^n \cdot I_{\{f > 0\}}(S(\omega)) dP^*(\omega) \leq \frac{1}{\alpha} E_{P^*}(|S|^n) < \infty. \quad \square$$

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