EXISTENCE AND MULTIPLICITY OF A NONHOMOGENEOUS POLYHARMONIC EQUATION WITH CRITICAL EXPONENTIAL GROWTH IN EVEN DIMENSION

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Abstract. In this paper we study the existence of at least two positive weak solutions for an inhomogeneous fourth order equation with Navier boundary data involving nonlinearities of critical growth with a bifurcation parameter \( \lambda \) in \( \mathbb{R}^{2m} \). We establish here the lower and upper bound for \( \lambda \) which determine multiplicity and non-existence respectively.

1. Introduction

Let \( \Omega \subset \mathbb{R}^{2m} \) be a bounded domain. In this context we study the existence of multiple solutions in \( W^{m,2}_N(\Omega) = \{ u \in W^{m,2}(\Omega) : \Delta^j u = 0 \text{ on } \partial \Omega \text{ for } 0 \leq j < \frac{m}{2} \} \) of the following \( 2m \)-th order problem with Navier boundary condition

\[
\left\{ \begin{array}{l}
(-\Delta)^m u = \mu |u|^p e^{u^2} + \lambda h(x) \quad \text{in } \Omega, \\
u = 0 = \Delta u = \ldots = \Delta^{m-1} u \quad \text{on } \partial \Omega
\end{array} \right.
\]

where \( h \geq 0 \text{ in } \Omega, \|h\|_{L^2(\Omega)} = 1, \lambda > 0, \mu = 1 \text{ if } p > 0 \text{ and } \mu \in (0, \lambda_1(\Omega)) \text{ if } p = 0 \). Also assume that \( \lambda_1(\Omega) \) be the first eigenvalue of \( (-\Delta)^m \) on \( W^{m,2}_N(\Omega) \) with Navier boundary condition and which is strictly positive. The existence of multiple solutions for analogous problems in higher dimension with critical exponent have been studied in [5], [2] for the Dirichlet boundary condition and in [11] for Navier boundary condition. The corresponding problem for second order elliptic equations have been studied in [8] for dimension two, and in [9] for higher dimensions. The critical growth for the nonlinearity is \( u \mapsto |u|^p u, \quad p = \frac{4m}{n-2m}, \) when \( n \geq 2m + 1 \) from the Sobolev embedding in \( \mathbb{R}^n \). In [7] Moser proved the following,

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a bounded domain. There exists a constant \( C_n > 0 \) such that for any \( u \in W^{1,n}_0(\Omega), n \geq 2 \) with \( \|\nabla u\|_{L^n(\Omega)} \leq 1 \), then

\[
\int_{\Omega} e^{\alpha |u|^p} dx \leq C_n |\Omega|,
\]

where

\[
p = \frac{n}{n-1}, \quad \alpha_n := n w_{n-1}^{-1},
\]

and \( w_{n-1} \) is the surface measure of the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). Furthermore the integral on the left hand side can be made arbitrarily large if \( \alpha > \alpha_n \) by appropriate choice of \( u \) with \( \|\nabla u\|_{L^n(\Omega)} \leq 1 \).

The embedding

\[
W^{1,n}_0(\Omega) \ni u \mapsto e^{\alpha |u|^p} \in L^1(\Omega)
\]

is compact for \( \alpha < \alpha_n \) and is not compact for \( \alpha = \alpha_n \).

In [1] Adams extended the above result of Moser to higher order Sobolev spaces. To state the result of Adams we define the following \( m \)-th order derivatives of \( u \in C^m(\Omega) \):

\[
\nabla^m u = \left\{ \begin{array}{ll}
\Delta^m u & \text{for } m \text{ even}, \\
\nabla \Delta^m u & \text{for } m \text{ odd}.
\end{array} \right.
\]

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Furthermore, $\|\nabla^m u\|_p$ is the $L^p$ norm of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also denote $W^{m, \frac{n}{m}}_0(\Omega)$ to be the completion of $C^\infty_0(\Omega)$ under the Sobolev norm

$$\|u\|_{W^{m, \frac{n}{m}}(\Omega)} = \left(\|u\|_{\frac{n}{m}}^m + \sum_{|\alpha|=1}^m \|D^\alpha u\|_{\frac{n}{m}}^m\right)^{\frac{1}{m}}. \tag{1.2}$$

Adams proved the following embedding:

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $m$ is a positive integer less than $n$, then there exists a constant $C_0 = C(n, m) > 0$ such that for any $u \in W^{m, \frac{n}{m}}_0(\Omega)$ with $\|\nabla^m u\|_{\frac{n}{m}} \leq 1$, then

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{m}}) \, dx \leq C_0, \tag{1.3}$$

for all $\beta \leq \beta_{n,m}$

$$\beta_{n,m} = \begin{cases} \frac{n}{w_{n-1}} \left[ \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n-m}{n}} & \text{when } m \text{ is odd,} \\ \frac{n}{w_{n-1}} \left[ \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n-m}{n}} & \text{when } m \text{ is even.} \end{cases}$$

Furthermore, for any $\beta > \beta_{n,m}$, the integral can be made as large as possible by appropriate choice of $u$ with $\|\nabla^m u\|_{\frac{n}{m}} \leq 1$.

**Remark 1.1.** We remark that for the case $n = 2m = 4$, Lu-Yang in [6] and in general Zhao-Chang [12] showed the existence of an explicit sequence for $n = 2m$ to prove the sharpness of the constant in $W^{m, \frac{n}{m}}_0(\Omega)$.

Let $W^{m, \frac{n}{m}}_N(\Omega)$ denote the following subspace of $W^{m, \frac{n}{m}}(\Omega)$:

$$W^{m, \frac{n}{m}}_N(\Omega) = \{ u \in W^{m, \frac{n}{m}}(\Omega) : \Delta^j u = 0, \text{ on } \partial \Omega \text{ for } 0 \leq j \leq [(m-1)/2] \}.$$

Note that $W^{m, \frac{n}{m}}_N(\Omega)$ is strictly contained in $W^{m, \frac{n}{m}}_0(\Omega)$. Therefore,

$$\sup_{u \in W^{m, \frac{n}{m}}_N(\Omega)} \int_{\Omega} \exp(\beta_{n,m} |u|^{\frac{n-m}{n}}) \, dx \leq \sup_{u \in W^{m, \frac{n}{m}}_0(\Omega)} \int_{\Omega} \exp(\beta_{n,m} |u|^{\frac{n-m}{n}}) \, dx.$$

Tarsi [10] later extended Adams’ result for the larger space $W^{m, \frac{n}{m}}_N(\Omega)$. The key step in her work is to embed $W^{m, \frac{n}{m}}_N(\Omega)$ into a Zygmund space. We state her embedding theorem below

**Theorem 1.3.** Let $n > 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there is a constant $C_n > 0$ such that for all $u \in W^{m, \frac{n}{m}}_N(\Omega)$ with $\|\nabla^m u\|_{\frac{n}{m}} \leq 1$, we have

$$\int_{\Omega} e^{|u|^{\frac{n-m}{n}}} \, dx < C_n |\Omega| \quad \forall \beta \leq \beta_{n,m} \tag{1.4}$$

and the constant $\beta_{n,m}$ appearing in (1.4) is sharp and $\beta_{n,m}$ is same as in Theorem 1.2.

**Remark 1.2.** Here we remark that the bilinear form

$$\langle u, v \rangle \mapsto \int_{\Omega} \nabla^m u \cdot \nabla^m v = \begin{cases} \int_{\Omega} \Delta^k u \Delta^k v & \text{if } m = 2k, \\ \int_{\Omega} \nabla(\Delta^k u) \cdot \nabla(\Delta^k v) & \text{if } m = 2k + 1, \end{cases} \tag{1.5}$$

defines a scalar product on $W^{m,2}_0(\Omega)$ and $W^{m,2}_N(\Omega)$. Furthermore if $\Omega$ is bounded this scalar product induces a norm equivalent to (1.2).

Therefore the above results imply that the problem $(P)$ nonlinearity of critical growth.

**Theorem 1.1.** There exist positive real numbers $\lambda_* \leq \lambda^*$ with $\lambda_*$ independent of $h$ such that the problem $(P)$ has at least two positive solutions for all $\lambda \in (0, \lambda_*)$ and no solution for all $\lambda > \lambda^*$. 
In spite of possible failure of Palais-Smale condition due to the presence of critical exponent we adapt the method of [9] to prove the existence of the first solution by a decomposition of Nehari manifold into three parts. However for the existence of second solution we rely on the refined version of the Mountain-Pass Lemma introduced by Ghoussoub-Preiss [3].

2. Decomposition of Nehari Manifold

Let $f(u) = \mu |u|^p u e^{u^2}$. The corresponding energy functional to the problem $(P)$ is given by

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(u) - \lambda \int_{\Omega} h u$$

where $F(u) = \int_0^u f(s) ds$. As the energy functional is not bounded below on $W^{m,2}_N(\Omega)$, we need to study $J(u)$ on the Nehari manifold

$$\mathcal{M} = \{ u \in W^{m,2}_N(\Omega) \setminus \{0\} : \langle J'(u), u \rangle = 0 \},$$

where $J'(u)$ denotes the Frechet derivative of $J$ at $u$, and $\langle \cdot, \cdot \rangle$ is the inner product. Here we note that $\mathcal{M}$ contains every nonzero solution of the problem $(P)$. We note that for any $u \in W^{m,2}_N(\Omega)$,

$$\langle J'(u), u \rangle = \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f(u) u - \lambda \int_{\Omega} hu,$$

$$\langle J''(u)u, u \rangle = \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f'(u) u^2.$$

Similarly to the method used in [9], we split $\mathcal{M}$ into three parts:

$$\mathcal{M}^0 = \{ u \in \mathcal{M} : \langle J''(u)u, u \rangle = 0 \},$$

$$\mathcal{M}^+ = \{ u \in \mathcal{M} : \langle J''(u)u, u \rangle > 0 \},$$

$$\mathcal{M}^- = \{ u \in \mathcal{M} : \langle J''(u)u, u \rangle < 0 \}.$$

3. Topological Properties of $\mathcal{M}^0, \mathcal{M}^+, \mathcal{M}^-$

Our first aim is to show, for some small $\lambda, \mathcal{M}^0 = \{0\}$. For this let $\zeta > 0$, if $p > 0$ and $\zeta < \frac{\lambda - \mu}{\mu}$ if $p = 0$. Let $\Lambda = \{ u \in W^{m,2}_N(\Omega) : \int_{\Omega} |\nabla u|^2 \leq (1 + \zeta) \int_{\Omega} f'(u) u^2 \}$. Lemma 3.2 implies that $\Lambda \neq \{0\}$. We now assume the following important hypothesis:

$$\lambda > 0, \| h \|_{L^2(\Omega)} = 1, \text{ and}$$

$$\inf_{u \in \Lambda \setminus \{0\}} \left( \mu \int_{\Omega} (p + 2u^2) |u|^{p+2} e^{u^2} - \lambda \int_{\Omega} hu \right) > 0.$$ (3.1)

The condition (3.1) forces $\lambda$ to be suitably small. Indeed we can prove the following.

**Proposition 3.1.** Let

$$\lambda < \mu C_0 \frac{\| u \|_{W^{m,2}_N(\Omega)}}{\| u \|_{W^{m,2}_N(\Omega)}}$$

where $C_0 = \inf_{u \in \Lambda \setminus \{0\}} \int_{\Omega} (p + 2u^2) |u|^{p+2} e^{u^2} > 0$. Then (3.1) holds.

**Proof.** Step 1: $\inf_{u \in \Lambda \setminus \{0\}} \| u \|_{W^{m,2}_N(\Omega)} > 0$.

Assume the contradiction, then there exists a sequence $\{ u_n \} \subset \Lambda \setminus \{0\}$ such that $\| u_n \|_{W^{m,2}_N(\Omega)} \to 0$ as $n \to \infty$. Let $v_n = \frac{u_n}{\| u_n \|_{W^{m,2}_N(\Omega)}}$. Then $\| v_n \|_{W^{m,2}_N(\Omega)} = 1$ and $v_n$ satisfies

$$1 \leq (1 + \zeta) \int_{\Omega} f'(u_n) u_n^2, \quad \forall n.$$ (3.3)

Since $u_n \to 0$ in $W^{m,2}_N(\Omega)$, by Adams’ embedding for the higher order derivative in Theorem 1.3 we get $f'(u_n) \to f'(0)$ in $L^r(\Omega)$ for all $r > 1$. Since $v_n$ is bounded in $W^{m,2}_N(\Omega)$, $v_n$ has a weak
limit say $v$ in $W^{m,2}_N(\Omega)$. Certainly $\|v\|_{W^{m,2}_N(\Omega)} \leq 1$ and up to a subsequence denote it same as $v_n$ which converges strongly to $v$ in $L^r(\Omega)$ for all $r \geq 1$. Hence from (3.3) we get

$$\int \Omega |\nabla v|^2 \leq 1 \leq (1 + \zeta) f'(0) \int \Omega v^2. \tag{3.4}$$

This gives a contradiction if $p > 0$ in which case $f'(0) = 0$. If $p = 0$, by assumption

$$\int \Omega |\nabla v|^2 \geq \lambda_1 \int \Omega v^2 > (1 + \zeta) \mu \int \Omega v^2$$

which gives a contradiction to (3.3) since $f'(0) = \mu$. This proves Step 1.

It is easy to check that using Step 1 and the definition of $\Lambda$:

$$\inf_{u \in \Lambda \setminus \{0\}} \int \Omega (p + 2u^2)|u|^{p+2}e^{u^2} = C_0 > 0. \tag{3.5}$$

**Step 2:** Finally we have,

$$\lambda \left| \int \Omega hu \right| \leq \lambda \|u\|_{L^2(\Omega)} \leq \lambda |\Omega|^{\frac{p+2}{p+4}} \left( \int \Omega |u|^{p+4} \right)^{\frac{1}{p+4}}$$

$$\leq \frac{\lambda |\Omega|^{\frac{p+2}{p+4}}}{\mu(p + 2u^2)|u|^{p+2}e^{u^2}} \left( \mu \int \Omega (p + 2u^2)|u|^{p+2}e^{u^2} \right)$$

$$\leq \left( \frac{\lambda |\Omega|^{\frac{p+2}{p+4}}}{\mu C_0^{\frac{p+2}{p+4}}} \right) \left( \mu \int \Omega (p + 2u^2)|u|^{p+2}e^{u^2} \right).$$

Hence from the above inequality together with (3.2) and (3.5) the proof is complete. \qed

**Lemma 3.1.** Suppose $\lambda > 0$ be such that (3.1) holds. Then $\mathcal{M}^0 = \{0\}$.

**Proof.** Let $u \in \mathcal{M}^0, u \neq 0$. Then we have

$$\int \Omega |\nabla u|^2 = \int \Omega f(u)u + \lambda \int \Omega hu, \tag{3.6}$$

$$\int \Omega |\nabla u|^2 = \int \Omega f'(u)u^2. \tag{3.7}$$

We note that from (3.7)

$$\int \Omega |\nabla u|^2 = \int \Omega f'(u)u^2 < (1 + \zeta) \int \Omega f'(u)u^2$$

it implies that $u \in \Lambda \setminus \{0\}$. From these two expressions we get

$$\lambda \int \Omega hu = \int \Omega (f'(u)u - f(u))u = \mu \int \Omega (p + 2u^2)|u|^{p+2}e^{u^2}$$

which violates the condition (3.1). Therefore $\mathcal{M}^0 = \{0\}$. \qed

Next we are going to discuss the topological properties of $\mathcal{M}^+$ and $\mathcal{M}^-$. Given $u \in W^{m,2}_N(\Omega) \setminus \{0\}$, we define $\xi_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\xi_u(s) = s \int \Omega |\nabla u|^2 - \int \Omega f(su)u. \tag{3.8}$$

The choice of the above function is consequence of the following expression,

$$\langle J'(su), su \rangle = s \left( s \int \Omega |\nabla u|^2 - \int \Omega f(su)u - \lambda \int \Omega hu \right).$$

So, $\xi_u(s) = \lambda \int \Omega hu$ if and only if $su \in \mathcal{M}$, for $s > 0$.

Now we are ready to state the following lemma.

**Lemma 3.2.** For every $u \in W^{m,2}_N(\Omega) \setminus \{0\}$ there exists a unique $s_u = s_u(u) > 0$ such that $\xi_u(.)$ has its maximum at $s_u$ with $\xi_u(s_u) > 0$. Also there holds $s_u \in \Lambda \setminus \{0\}$. 

Lemma 3.3. Let \( \lambda \) be such that (3.1) holds. Then, for every \( u \in W^{2,2}(\Omega) \setminus \{0\} \), there exists a unique \( s_0(s)(u) > 0 \) such that \( s_0(s)u \in M^- \) and \( J(s_0(s)u) = \max_{s \leq s_0} J(su) \forall s \in [s_0, \infty), s \neq s_0 \). Also if we assume \( \int_\Omega hu > 0 \), then there exists a unique \( s_+ = s_+(u) > 0 \) such that \( s_+u \in M^+ \). In particular \( s_+ < s_0 \) and \( J(s_+u) \leq J(su) \) for all \( s \in [0, s_0] \).

Proof. Define the functional \( \rho_u : [0, \infty) \to \mathbb{R} \) by \( \rho_u(s) = J(su) \). Then it is easy to verify that \( \rho_u \in C^2([0, \infty], \mathbb{R}) \cap C((0, \infty), \mathbb{R}) \). Then we have

\[
\rho_u'(s) = \xi_u(s) - \lambda \int_\Omega hu, \quad \rho_u''(s) = \xi_u'(s), \quad \forall t > 0.
\]

Now from (3.1) and (3.12) we have,

\[
\xi_u(s) - \lambda \int_\Omega hu = \frac{1}{s_0} \left\{ \mu \int_\Omega (p + 2(s_0u)^2)|s_0u|^{p+2}e^{(s_0u)^2} - \lambda \int_\Omega h(s_0u) \right\} > 0
\]

Since \( \xi_u(.) \) is strictly decreasing in \( (s_0, \infty) \) and \( \lim_{t \to \infty} \xi_u(s) = -\infty \), there exists a unique \( s_- = s_-(u) > s_0 \) such that \( \xi_u(s_-) = \lambda \int_\Omega hu \). That is \( s_-u \in M^- \). One has \( s_+ > s_0 \) and \( \rho_u'(s) < 0 \), we get \( s_-u \in M^- \).

On the other hand when \( \int_\Omega hu > 0 \) we have \( \lim_{s \to 0^+} \xi_u(s) < 0 \) and which gives for \( s \) close to \( 0 \), \( \xi_u(s) - \lambda \int_\Omega hu < 0 \). Hence there exists a unique \( s_+ \) such that \( \xi_u(s_+) = \lambda \int_\Omega hu \) which implies \( s_+u \in M^+ \). From the graph we see that \( \xi_u(.) \) is strictly decreasing in \( (0, s_0) \). Hence we have \( s_+u \in M^+ \).

And the remaining properties of \( s_-, s_+ \) can be proved by analyzing the identity \( \rho_u(s) = \xi_u(s) - \lambda \int_\Omega hu \). \( \square \)
Remark 3.1. If we define the positive cone $\mathcal{P} = \{ u \in W^{m,2}_N(\Omega) : \int_{\Omega} hu > 0 \}$ in $W^{m,2}_N(\Omega)$. Then we note that $\mathcal{M}^+ \subset \mathcal{P}$.

The next corollary shows some topological properties of $\mathcal{M}^+, \mathcal{M}^-$.

**Corollary 3.1.** Let $S_{W^{m,2}_N(\Omega)} = \{ u \in W^{m,2}_N(\Omega) : \|u\|_{W^{m,2}_N(\Omega)} = 1 \}$. Then there exists a diffeomorphism $S^+ : S_{W^{m,2}_N(\Omega)} \rightarrow \mathcal{M}^-$ defined by $S^+(u) = s_+(u)u$. Also $\mathcal{M}^+$ is homeomorphic to $S_{W^{m,2}_N(\Omega)} \cap \mathcal{P}$.

**Proof.** The function $S^+$ is continuous because $s_+$ is continuous as an application of implicit function theorem applied to $(s,u) \rightarrow \xi_u(s) - \lambda \int_{\Omega} hu$ and we deduce the continuity of $(S^+)^{-1}$ by the fact that $(S^+)^{-1}(w) = \frac{w}{\|w\|}$. In a similar manner we can prove that $\mathcal{M}^+$ is homeomorphic to $S_{W^{m,2}_N(\Omega)} \cap \mathcal{P}$. \hfill □

Relying on the embedding of $W^{m,2}_N(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ and using the estimate $F(s) \leq \frac{|s|^p}{2p}(e^{s^2} - 1)$, for all $s \in \mathbb{R}$ we have the following lemma on the lower bound and upper bound.

**Lemma 3.4.** There exists $C_1, C_2 > 0$ such that

$$-C_2 \lambda^{2p+8} \geq \theta_0 \geq -C_1 \lambda^{\frac{p+4}{2}}.$$

Where, $\theta_0 = \inf\{ J(u) : u \in \mathcal{M} \}$.

**Proof.** We prove the case of the lower bound. Let $u \in \mathcal{M}$ then from the definition,

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla^m u|^2 - \int_{\Omega} F(u) - \lambda \int_{\Omega} hu$$

$$= \int_{\Omega} \left[ \frac{1}{2} f(u) u - F(u) \right] - \frac{\lambda}{2} \int_{\Omega} hu.$$

We note that a simple calculation gives

$$F(t) \leq \frac{\mu |t|^p}{2} (e^{t^2} - 1), \quad \text{for all } s \in \mathbb{R}.\tag{3.13}$$

Using (3.13) we get

$$J(u) \geq \frac{\mu}{2} \int_{\Omega} \left( (u^2 - 1)e^{u^2} + 1 \right) - \frac{\lambda}{2} \int_{\Omega} hu$$

$$\geq \frac{c\mu}{2} \int_{\Omega} |u|^{p+4} - \frac{\lambda}{2} \int_{\Omega} hu,\tag{3.14}$$

since $(s^2 - 1)e^{s^2} + 1 \geq cs^4$ for some $c > 0$, for all $s \in \mathbb{R}$. By an application of Holder inequality we get

$$\int_{\Omega} hu \leq |\Omega|^{\frac{p+2}{p+4}} \|u\|_{L^{p+4}(\Omega)}.\tag{3.15}$$

From (3.14) and (3.15) we get,

$$J(u) \geq \frac{c\mu}{2} \|u\|_{L^{p+4}(\Omega)}^{p+4} - \left( \frac{\lambda |\Omega|^{\frac{p+2}{p+4}}}{2} \|u\|_{L^{p+4}(\Omega)} \right).\tag{3.16}$$

By considering the global minimum of the function

$$\omega(x) = \left( \frac{c\mu}{2} \right) x^{p+4} - \left( \frac{\lambda |\Omega|^{\frac{p+2}{p+4}}}{2} \right) x,$$

It can be shown that

$$J(u) \geq -C_1 \lambda^{\frac{p+4}{2}}.$$\hfill □

In a similar fashion we can prove the upper bound for $J$. \hfill □
As a consequence of Lemma 3.1 we have:

**Lemma 3.5.** Let \( \lambda \) and \( h \) satisfy (3.1). Given \( u \in \mathcal{M} \setminus \{0\} \) there exists \( \delta > 0 \) and a differentiable function \( s : \{w \in W^{m,2}_N(\Omega) : \|w\|_{W^{m,2}_N(\Omega)} < \delta \} \rightarrow \mathbb{R} \), with

\[
s(0) = 1, s(w)(u-w) \in \mathcal{M}, \quad \forall \|w\|_{W^{m,2}_N(\Omega)} < \delta
\]

and

\[
\langle s'(0), v \rangle = \frac{2 \int_{\Omega} \nabla m u \cdot \nabla m v - \int_{\Omega} (f'(u)u + f(u))v - \lambda \int_{\Omega} hv}{\int_{\Omega} |\nabla m u|^2 - \int_{\Omega} f'(u)u^2}
\]

**Proof.** We define the function \( G : \mathbb{R} \times W^{m,2}_N(\Omega) \rightarrow \mathbb{R} \) by,

\[
G(s,w) = s \int_{\Omega} |\nabla m (u-w)|^2 - \int_{\Omega} f(s(u-w))(u-w) - \lambda \int_{\Omega} h(u-w).
\]

Then \( G \in C^1(\mathbb{R} \times W^{m,2}_N(\Omega); \mathbb{R}) \) and since \( u \in \mathcal{M} \) it implies \( G(1,0) = \int_{\Omega} |\nabla m u|^2 - \int_{\Omega} f(u)u - \lambda \int_{\Omega} hv = 0 \). Also \( G_s(1,0) \neq 0 \), indeed \( G_s(1,0) = \int_{\Omega} |\nabla m u|^2 - \int_{\Omega} f'(u)u^2 \neq 0 \) thanks to Lemma 3.1. Then by the Implicit Function Theorem, there exists \( \delta > 0, s : \{w \in W^{m,2}_N(\Omega) : \|w\| < \delta \} \rightarrow \mathbb{R} \) of class \( C^1 \) that satisfies:

\[
G(s(w), w) = 0 \text{ for all } w \in W^{m,2}_N(\Omega), \|w\|_{W^{m,2}_N(\Omega)} < \delta,
\]

\[
s(0) = 1.
\]

Also

\[
0 = s(w)G(s(w), w)
\]

\[
= \int_{\Omega} (s(w)|\nabla m (u-w)|^2 - \int_{\Omega} f(s(w)(u-w))(s(w)(u-w)) - \lambda \int_{\Omega} h(s(w)(u-w)),
\]

that is \( s(w)(u-w) \in \mathcal{M} \) for all \( w \in W^{m,2}_N(\Omega) \) with \( \|w\| < \delta \). Now if we differentiate the identity \( G(s(w), w) = 0 \) with respect to \( w \), we get

\[
0 = \langle G_s(s(w), w) + G_w(s(w), w), v \rangle \text{ for all } v \in W^{m,2}_N(\Omega).
\]

Putting \( w = 0 \) in the above identity

\[
0 = \langle G_s(1,0)s'(0) + G_w(1,0), v \rangle = G_s(1,0)(s'(0), v) + \langle G_w(1,0), v \rangle
\]

and we deduce from above

\[
\langle s'(0), v \rangle = -\frac{\langle G_w(1,0), v \rangle}{G_s(1,0)}
\]

\[
= \frac{2 \int_{\Omega} \nabla m u \cdot \nabla m v - \int_{\Omega} (f'(u)u + f(u))v - \lambda \int_{\Omega} hv}{\int_{\Omega} |\nabla m u|^2 - \int_{\Omega} f'(u)u^2}.
\]

\( \square \)

4. **Local Minimum of \( J \) in \( W^{m,2}_N(\Omega) \)**

We are now in a situation to prove the existence of a minimizer for \( J \) and hence we guarantee the existence of first solution.

Since \( \mathcal{M} \) is a closed set of \( W^{m,2}_N(\Omega) \), hence a complete metric space. Now \( J \) is bounded below on \( \mathcal{M} \). By the Ekeland’s Variational Principle there exists a sequence \( \{u_n\} \subset \mathcal{M} \setminus \{0\} \) satisfying:

\[
J(u_n) < \theta_0 + \frac{1}{n}, \quad J(v) \geq J(u_n) - \frac{1}{n} \|v - u_n\|_{W^{m,2}_N(\Omega)} \quad \forall v \in \mathcal{M}
\]

**Proposition 4.1.** Let \( \lambda \) and \( h \) satisfy (3.1). Then

\[
\lim_{n \rightarrow \infty} \|J'(u_n)\|_{(W^{m,2}_N(\Omega))^{-1}} = 0.
\]
Proof. We proceed in a few steps. With the help of Lemma 3.4 we’ve \( \lim_{n\to\infty} \|u_n\|_{W_{N}^{m,2}} > 0 \).

Claim 1: \( \lim_{n\to\infty} \int_{\Omega} (p + 2u_n^2)|u_n|^{p+2}e^{u_n^2} > 0 \).

If possible let’s assume that for a subsequence of \( \{u_n\} \), which is still denoted by \( \{u_n\} \), we have

\begin{equation}
\lim_{n\to\infty} (p + 2u_n^2)|u_n|^{p+2}e^{u_n^2} \to 0 \quad \text{as} \quad n \to \infty
\end{equation}

Here we note that \( u_n \to 0 \) in \( L^q(\Omega) \) for all \( q \in [1, \infty) \) using (4.2), and if \( p > 0 \),

\[ \int_{\Omega} f(u_n)u_n = \mu \int_{\Omega} |u_n|^{p+2}e^{u_n^2} \to 0 \quad \text{as} \quad n \to \infty. \]

Therefore we have \( \int_{\Omega} f(u_n)u_n \to 0, \int_{\Omega} hu_n \to 0 \) as \( n \to \infty \). Which imply that \( \|u_n\|_{W_{N}^{m,2}} \to 0 \) as \( n \to \infty \) because \( \{u_n\} \subset \mathcal{M} \) hence a contradiction to the fact that \( \lim_{n\to\infty} \|u_n\|_{W_{N}^{m,2}} > 0 \). Similar argument for \( p = 0 \).

Claim 2: \( \lim_{n\to\infty} \{\int_{\Omega} |\nabla^m u|^2 - \int_{\Omega} f'(u_n)u_n^2\} > 0 \).

Let the claim doesn’t hold. Then for a subsequence \( \{u_n\} \) we have

\[ \int_{\Omega} |\nabla^m u|^2 - \int_{\Omega} f'(u_n)u_n^2 = o_n(1). \]

From this and the fact \( \lim_{n\to\infty} \|u_n\|_{W_{N}^{m,2}} > 0 \) we deduce that,

\[ \lim_{n\to\infty} f'(u_n)u_n^2 > 0. \]

Therefore we have \( u_n \in \Lambda \setminus \{0\} \) for large \( n \). Since \( \{u_n\} \subset \mathcal{M} \) we get

\[ o_n(1) = \lambda \int_{\Omega} hu_n + \int_{\Omega} [f(u_n) - f'(u_n)u_n]u_n \]

\[ = -\mu \int_{\Omega} (p + 2u_n^2)|u_n|^{p+2}e^{u_n^2} + \lambda \int_{\Omega} hu_n, \]

which contradicts (3.1). This completes the proof of the claim.

Now we proof the theorem. Let’s assume \( \|J'(u_n)\|_{(W_{N}^{m,2})^*} > 0 \) for all large \( n \) (otherwise obvious).

Now we define \( u = u_n \in \mathcal{M} \) and \( w = \delta J'(u_n) \) (by Riesz representation theorem, we identify \( J'(u_n) \) as an element in \( W_{N}^{m,2}(\Omega) \) still denote \( J'(u_n) \)) for \( \delta > 0 \) small. Therefore we can apply Lemma 3.5 for \( w \) small we get \( s_n(\delta) = s \left[ \delta \frac{J'(u_n)}{\|J'(u_n)\|} \right] > 0 \) such that

\[ w_\delta = s_n(\delta) \left[ u_n - \frac{J'(u_n)}{\|J'(u_n)\|} \right] \in \mathcal{M}. \]

Now from (4.1) and a Taylor expansion we have:

\[ \frac{1}{n} \|w_\delta - u_n\| \geq J(u_n) - J(w_\delta) \]

\[ = (1 - s_n(\delta))J'(w_\delta),u_n + \delta s_n(\delta) \left( J'(w_\delta), \frac{J'(u_n)}{\|J'(u_n)\|} \right) + o(\delta). \]

Dividing by \( \delta > 0 \) and taking limit as \( \delta \to 0 \) we get:

\[ \frac{1}{n} (1 + |s_n'(0)||u_n|) \leq -s_n(0),J'(u_n),u_n + \|J'(u_n)\| = \|J'(u_n)\|. \]

Hence

\[ \|J'(u_n)\| \leq \frac{1}{n} (1 + s_n'(0)||u_n||). \]

We complete the proof by using, \( |s_n'(0)| \) is uniformly bounded on \( n \) by (3.17) and using the Claim 2. \( \square \)

Theorem 4.2. Let \( \lambda, h \) satisfy (3.1). Then there exists a nonnegative function \( u_0 \in \mathcal{M}^+ \) such that \( J(u_0) = \inf_{u \in \mathcal{M} \setminus \{0\}} J(u) \). Moreover, \( u_0 \) is a local minimum for \( J \) in \( W_{N}^{m,2}(\Omega) \).
Proof. Let \( \{ u_n \} \) be a sequence which minimizes \( J \) on \( \mathcal{M} \setminus \{ 0 \} \) as in (4.1).

**Step 1:** \( \lim \inf_{n \to \infty} \int_{\Omega} hu_n > 0 \) and hence \( u_n \in \mathcal{M}^+ \). Indeed \( u_n \in \mathcal{M} \) and making some suitable adjustments

\[
J(u_n) = \frac{p}{2(p+2)} \int_{\Omega} |\nabla^m u|^2 + \int_{\Omega} \left( \frac{1}{p+2} f(u_n)u_n - F(u_n) \right) \]

(4.3)

Thanks to Lemma 3.4 there exists \( C > 0 \). Now we note that \( F(t) < \frac{1}{p+2} f(t)t \) for all \( t \in \mathbb{R} \). Therefore we’ve from (4.3), to make the inequality consistent with sign that

\[
\lim \inf_{n \to \infty} \int_{\Omega} hu_n > 0.
\]

**Step 2:** \( \lim \sup_{n \to \infty} \| u \|_{W^{m,2}_N(\Omega)} < \infty \).

**Case 1.** If \( p > 0 \) then by the means of Sobolev embedding we derive boundedness of \( \{ u_n \} \) in \( W^{m,2}_N(\Omega) \). Using the fact from (4.3)

\[
\int_{\Omega} |\nabla^m u|^2 \leq \lambda \int_{\Omega} hu_n.
\]

**Case 2.** If \( p = 0 \) by using the fact that \( \frac{1}{p+2} f(t) t \geq Ct^4 \) for all \( t \in \mathbb{R} \) and for some \( C > 0 \) we deduce that \( \{ u_n \} \) is a bounded sequence in \( L^2(\Omega) \). And this gives that \( \{ F(u_n) \} \) is a bounded sequence in \( L^1(\Omega) \) using (4.3) and hence \( \{ u_n \} \) is a bounded sequence in \( W^{m,2}_N(\Omega) \).

**Step 3:** Existence of \( u_0 \in \mathcal{M}^+ \).

From the previous step up to a subsequence, \( u_n \to u_0 \) in \( W^{m,2}_N(\Omega) \). Now from the Proposition 2.2 we note that \( \{ f(u_n)u_n \} \) is a bounded sequence in \( L^1(\Omega) \). Therefore from Vitali’s convergence theorem (for details see Lemma 8.3 in \([8]\)) we get that

\[
\int_{\Omega} f(u_n) \phi \to \int_{\Omega} f(u_0) \phi, \text{ for all } \phi \in W^{m,2}_N(\Omega).
\]

Hence \( u_0 \) will solve \((P)\), in particular \( u_0 \in \mathcal{M} \). Here we note that \( u_0 \neq 0 \) as \( h \neq 0 \) that is \( u_0 \in \mathcal{M} \setminus \{ 0 \} \). We see that \( \theta_0 \leq J(u_0) \). From (4.3) we get by using Fatou’s Lemma that \( \theta_0 = \lim \inf_{n \to \infty} J(u_n) \geq J(u_0) \). Therefore \( u_0 \) minimizes \( J \) on \( \mathcal{M} \setminus \{ 0 \} \). Now we have to show \( u_0 \in \mathcal{M}^+ \). From the existence of \( s^-(u_0) \) and \( s^+(u_0) \) in Lemma 3.3 and using the fact \( J(s^+(u_0)) \) we get \( u_0 \in \mathcal{M}^+ \).

**Step 4:** \( u_0 \) is a local minimum for \( J \) in \( W^{m,2}_N(\Omega) \).

We see that \( s^+(u_0) = 1 \) because \( u_0 \in \mathcal{M}^+ \) from Step 3. Also we have from the (3.3) we have

\[
s^+(u_0) = 1 < s^+(u_0)
\]

Now by the continuity of \( s^+(u_0) \), for sufficiently small \( \delta > 0 \)

\[
1 < s^+(u_0 - w), \text{ } \forall w \in W^{m,2}_N(\Omega), \| w \|_{W^{m,2}_N(\Omega)} < \delta.
\]

Now by the Lemma 3.5 for \( \delta > 0 \) small enough if necessary, let \( s : \{ w \in W^{m,2}_N(\Omega) : \| w \| < \delta \} \to \mathbb{R} \) such that \( s(w)(u_0 - w) \in \mathcal{M} \) and \( s(0) = 1 \). Whenever \( s(w) \to 1 \) when \( \| w \| \to 0 \), we can assume that

\[
s(w) < s^+(u_0 - w), \text{ } \forall w \in W^{m,2}_N(\Omega), \| w \| < \delta.
\]

Hence we get \( s(w)(u_0 - w) \in \mathcal{M}^+ \) using the above inequality and Lemma 3.3. Again by using the Lemma 3.3 we see,

\[
J(s(u_0 - w) \geq J(s(w)(u_0 - w)) \geq J(u_0), \text{ } \forall s \in [0,s^+(u_0 - w)].
\]

Hence from (4.4) we observe that \( J(u_0 - w) \geq J(u_0) \) for every \( \| w \|_{W^{m,2}_N(\Omega)} < \delta \). This shows that \( u_0 \) is a local minimizer.

**Step 5:** A positive local minimum for \( J \). If \( u_0 \geq 0 \) then we get the positivity by using the strong maximum principle. In case if \( u_0 \not\geq 0 \) then we consider \( \tilde{u}_0 = s^+(u_0)|u_0| > 0 \in \mathcal{M}^+ \) and also from the definition \( \rho_{u_0}(s) = \rho_{|u_0|}(s) \) for all \( s > 0 \). Therefore we get \( s^+(|u_0|) = s^+(u_0) \) and from the
definition of \( s_+ \) we deduce \( s_+(u_0) \leq s_+|u_0| \). Hence from Step 4, \( s_+|u_0| \geq 1 \). Therefore by Lemma 3.3 we get \( J(u_0) \leq J(|u_0|) \). Now using the assumption \( h \geq 0 \) in \( \Omega \), we have \( J(|u_0|) \leq J(u_0) \) and which implies that \( u_0 \) minimizes \( J \) on \( \mathcal{M} \setminus \{0\} \). Hence by repeating the same argument as in Step 4 we get the desired result. 

\[ \square \]

5. Existence of The Second Solution

The existence of the second solution for \((P)\) depends on whether we can apply some version of Mountain Pass Lemma. We wish to look for a solution of the form \( v_1 = v + u_0 \) where \( u_0 \) is the local minimum for the functional \((2.1)\). Then we see that \( u_1 \) will solve \((P)\) whenever \( v \) solves the following equation:

\[
(P_1) \quad \begin{cases} 
(-\Delta)^m v = f(v + u_0) - f(u_0) & \text{in } \Omega, \\
v = \Delta v = 0 = \ldots = \Delta^{m-1} u & \text{on } \partial \Omega.
\end{cases}
\]

We can write the above PDE as following

\[
(\tilde{P}) \quad \begin{cases} 
(-\Delta)^m 2v = \tilde{f}(x,v) & \text{in } \Omega, \\
v = \Delta v = 0 = \ldots = \Delta^{m-1} v & \text{on } \partial \Omega,
\end{cases}
\]

by introducing the function \( \tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R} \) and we define by

\[
\tilde{f}(x,s) = f(s + u_0(x)) - f(u_0(x)) \quad \text{if } s \geq 0,
\]

\[
= 0 \quad \text{otherwise.}
\]

The energy functional corresponding to \((\tilde{P})\) is \( J_{u_0} : W^{m,2}_N(\Omega) \to \mathbb{R} \) defined by

\[
J_{u_0}(v) = \frac{1}{2} \int_{\Omega} |\nabla^m v|^2 - \int_{\Omega} \tilde{F}(x,v) dx,
\]

where \( \tilde{F}(x,s) = \int_0^s \tilde{f}(x,t) dt \). Now onwards, we denote \( J_{u_0} \) by \( J_0 \). These type of functionals were studied by [12], [2]. We now state the Generalized Mountain Pass Lemma that was introduced by Ghoussoub-Preiss [3].

**Definition 5.1.** Let \( H \) be a closed subspace of the Banach Space \( W^{m,2}_N(\Omega) \). We say that a sequence \( \{v_n\} \subset W^{m,2}_N(\Omega) \) is a Palais-Smale sequence for \( J_0 \) at the level \( c \) around \( H \) if:

(i) \( \lim_{n \to \infty} \text{dist}(v_n, H) = 0 \)

(ii) \( \lim_{n \to \infty} J_0(v_n) = c \)

(iii) \( \lim_{n \to \infty} \|J_0(v_n)\|_{W^{m,2}_N(\Omega)^-1} = 0 \).

And we say such a sequence a \((PS)_{H,c}\) sequence.

**Remark 5.1.** In case \( H = W^{m,2}_N(\Omega) \), the above definition coincides with the usual Palais-Smale sequence at the level \( c \).

**Lemma 5.1.** Let \( H \subset W^{m,2}_N(\Omega) \) be a closed set, \( c \in \mathbb{R} \). Assume \( \{v_n\} \subset W^{m,2}_N(\Omega) \) be a \((PS)_{H,c}\) sequence. Then (upto a subsequence), \( v_n \rightharpoonup v_0 \) in \( W^{m,2}_N(\Omega) \), and

\[
\lim_{n \to \infty} \int_{\Omega} \tilde{f}(x,v_n) = \int_{\Omega} \tilde{f}(x,v_0), \quad \lim_{n \to \infty} \int_{\Omega} \tilde{F}(x,v_n) = \int_{\Omega} \tilde{F}(x,v_0).
\]

**Proof.** From the fact that \( \{v_n\} \) is a \((PS)_{H,c}\) sequence we have:

\[
\frac{1}{2} \int_{\Omega} |\nabla^m v_n|^2 - \int_{\Omega} \tilde{F}(x,v_n) = c_0 + o_n(1),
\]

\[
\int_{\Omega} \nabla^m v_n \cdot \nabla^m \phi - \int_{\Omega} \tilde{f}(x,v_n) \phi \leq o_n(1) \|\phi\|_{W^{m,2}_N(\Omega)}, \quad \forall \phi \in W^{m,2}_N(\Omega).
\]
Now we claim that,
\textbf{Claim:} \( \sup_n \|v_n\|_{W^{m,2}_N(\Omega)} < \infty, \sup_n \int_\Omega \tilde{f}(x, v_n) < \infty. \)
Given any \( \varepsilon > 0 \) there exists \( s_\varepsilon > 0 \) such that
\begin{equation}
\int_\Omega \tilde{F}(x, s) \leq \varepsilon \tilde{f}(x, s) \quad \text{for all } |s| \geq s_\varepsilon.
\end{equation}
Using (5.2) and (5.4), we see
\begin{align*}
\frac{1}{2} \int_\Omega |\nabla v_n|^2 & \leq \int_{\Omega \cap \{|v_n| \leq s_\varepsilon\}} \tilde{F}(x, v_n) + \int_{\Omega \cap \{|v_n| \geq s_\varepsilon\}} \tilde{F}(x, v_n) + c + o_n(1) \\
& \leq \int_{\Omega \cap \{|v_n| \leq s_\varepsilon\}} \tilde{F}(x, v_n) + \varepsilon \int_\Omega \tilde{f}(x, v_n) v_n + c + o_n(1) \\
& \leq C\varepsilon + \varepsilon \int_\Omega \tilde{f}(x, v_n) v_n.
\end{align*}
Now from (5.5) we obtain,
\begin{align*}
\int_\Omega \tilde{f}(x, v_n) v_n & \leq \int_\Omega |\nabla v_n|^2 + o_n(1) \|v_n\|_{W^{m,2}_N(\Omega)} \\
& \leq 2C\varepsilon + 2\varepsilon \int_\Omega \tilde{f}(x, v_n) v_n + o_n(1) \|v_n\|_{W^{m,2}_N(\Omega)}
\end{align*}
by substituting \( \phi = v_n \) in (5.3).
Hence by choosing \( \varepsilon \) small enough if needed we get
\begin{equation}
\int_\Omega \tilde{f}(x, v_n) v_n \leq \frac{2C\varepsilon}{1-2\varepsilon} + o_n(1) \|v_n\|_{W^{m,2}_N(\Omega)}.
\end{equation}
We conclude the claim using (5.6), (5.3) and also \( \sup_n \int_\Omega \tilde{f}(x, v_n) v_n < \infty. \)
Since \( \{v_n\} \subset W^{m,2}_N(\Omega) \) is bounded, up to a subsequence, \( v_n \rightharpoonup v_0 \) in \( W^{m,2}_N(\Omega) \), for some \( v_0 \in W^{m,2}_N(\Omega). \)
To prove (5.1) we consider \( A \) to be a 2m dimensional Lebesgue measure of a set \( A \subset \mathbb{R}^{2m}. \)
Let \( C = \sup_n \int_\Omega |\tilde{f}(x, v_n)| v_n < \infty \) from the above claim. Given \( \varepsilon > 0 \), we define
\[\mu_\varepsilon = \max_{x \in \Omega, |s| \leq \frac{C\varepsilon}{1-2\varepsilon}} |\tilde{f}(x, s)| s.\]
Then, for any \( A \subset \Omega \) with \( |A| \leq \frac{\varepsilon}{2C\varepsilon} \), we have
\begin{align*}
\int_A |\tilde{f}(x, v_n)| & \leq \int_{A \cap \{|v_n| \geq \frac{C\varepsilon}{1-2\varepsilon}\}} |\tilde{f}(x, v_n)| v_n + \int_{A \cap \{|v_n| \leq \frac{C\varepsilon}{1-2\varepsilon}\}} |\tilde{f}(x, v_n)| \\
& \leq \frac{\varepsilon}{2} + \mu_\varepsilon |A| \leq \varepsilon.
\end{align*}
Hence \( \{\tilde{f}(x, v_n)\} \) is an equi-integrable family in \( L^1(\Omega) \) and so is \( \{\tilde{F}(x, v_n)\} \) (we note that \( |\tilde{F}(x, t)| \leq C_1 |\tilde{f}(x, t)| \) for all \( x \in \Omega, t \in \mathbb{R} \), for some \( C_1 > 0 \)). By applying the Vitali’s convergence theorem we get conclude the lemma.

Certainly \( J_0(0) = 0 \) and \( v = 0 \) is a local minimum for \( J_0 \). Also we have
\[\lim_{s \to \infty} J_0(sv) = -\infty \quad \text{for any } v \in W^{m,2}_N(\Omega) \setminus \{0\}.
\]
Hence we can fix \( e \in W^{m,2}_N(\Omega) \setminus \{0\} \) such that \( J_0(e) < 0. \) Now we define the mountain pass level
\[c_0 = \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} J_0(\gamma(s)).\]
Where \( \Gamma = \{ \gamma \in C([0,1], W^{m,2}_N(\Omega)) : \gamma(0) = 0, \gamma(1) = e \}. \) Then from the definition of \( c_0 \) it follows \( c_0 \geq 0. \) Define \( R_0 = \|e\|_{W^{m,2}_N(\Omega)} \), we note that \( \inf \{ J_0(v) : \|v\|_{W^{m,2}_N(\Omega)} = R \} = 0 \) for all \( R \in (0, R_0). \)
And we now let \( H = W^{m,2}_N(\Omega) \) if \( c_0 > 0 \) and \( H = \{ \|v\|_{W^{m,2}_N(\Omega)} = \frac{R_0}{2} \} \) if \( c_0 = 0. \) We now state as now the lemma giving upper bound for \( c_0 \)
Lemma 5.2. The upper bound of the Mountain Pass level is below

\[
(5.7) \quad c_0 < \frac{(4\pi)^m m!}{2} = \beta_{2m,m}. 
\]

Proof. Without loss of generality we can assume that the unit ball \( B_0(1) \subset \Omega \). For any \( \epsilon > 0 \) we define

\[
(5.8) \quad \hat{\tau}_n(x) := \begin{cases} 
\sqrt{\frac{1}{2M} \log n + \frac{1}{\sqrt{2M \log n}} \sum_{\gamma=1}^{m-1} (1-k|x|^2)\sum_{\gamma=1}^{m} \frac{1}{\gamma} |x|, & |x| \in [0, \frac{1}{\sqrt{n}}) \\
-\sqrt{\frac{2}{M \log n}} \log |x|, & |x| \in \left[\frac{1}{\sqrt{n}}, 1\right), \\
\chi_n(x), & |x| \in \left[1, \infty\right) 
\end{cases} 
\]

where

\[
M = \frac{(4\pi)^m (m-1)!}{2}, \quad \chi_n \in C^\infty_0(\Omega), \quad \chi_n|_{\partial B_1(0)} = \frac{\chi_n|_{\partial \Omega}}{1}. \quad \text{and} \quad \chi_n|_{\partial B_1(0)} = 0. 
\]

Furthermore, for \( \gamma = 1, 2, \ldots, m-1, D^\gamma \chi_n|_{\partial B_1(0)} = (-1)^{(\gamma-1)!} \sqrt{\frac{2}{M \log n}} \Delta^\gamma \chi_n|_{\partial \Omega} = 0 \) for \( j = 0, 1, 2, \ldots, [(m-1)/2] \) and \( \chi_n, \nabla \chi_n, \Delta \chi_n \) are all \( O\left(\frac{1}{\sqrt{M \log n}}\right) \). Then, \( \hat{\tau}_n \in W^{m,2}_N(\Omega) \). Now we normalize \( \hat{\tau}_n \), setting

\[
\tau_n := \frac{\hat{\tau}_n}{\|\hat{\tau}_n\|_{W^{m,2}_N(\Omega)}} \in W^{m,2}_N(\Omega). 
\]

Suppose (5.7) is not true. This means that, for some \( s_n > 0 \) (see [4]),

\[
J_0(s_n \tau_n) = \sup_{s>0} J_0(s \tau_n) \geq \frac{(4\pi)^m m!}{2} \quad \forall n. 
\]

Hence

\[
(5.9) \quad \frac{s_n^2}{2} - \int_{\Omega} \hat{f}(x, s_n \tau_n) \frac{(4\pi)^m m!}{2} \quad \forall n. 
\]

It follows that \( \frac{d}{ds} J_0(s \tau_n) = 0 \) at the point of maximum \( s = s_n \) for \( J_0 \), we get

\[
(5.10) \quad s_n^2 = \int_{\Omega} \hat{f}(x, s \tau_n)(s_n \tau_n). 
\]

Now we note that from the definition of \( \hat{f} \) we see that \( \inf_{x \in \Omega} \hat{f}(x, s) \geq e^{s^2} \) for \( |s| \) large. Then from (5.9) we get for sufficiently large \( n \)

\[
\frac{s^2}{2} \geq \int_{\{|x| \leq \frac{1}{\sqrt{n}}\}} \hat{f}(x, s_n \tau_n)(s_n \tau_n) \geq \int_{\{|x| \leq \frac{1}{\sqrt{n}}\}} e^{s^2 \tau_n^2}(s_n \tau_n) \\
\geq e^{s^2 \log n} \frac{s_n}{\sqrt{2M}} \sqrt{\log n} \frac{\alpha_{2m}}{n^m} \\
= \frac{\alpha_{2m}}{\sqrt{2M}} e^{s^2} \left( \frac{\alpha_{2m}}{n^m} \right) \log n s_n (\log n)^{\frac{1}{2}}, 
\]

where \( \alpha_{2m} \) is the volume of the unit ball in \( \mathbb{R}^{2m} \). Using the fact \( s^2 \geq (4\pi)^m m! \) from (5.9) and (5.11) it follows that \( s_n \) is bounded and also \( s^2_n \to (4\pi)^m m! \). Also from (5.11) we note

\[
s_n \geq \frac{\alpha_{2m}}{\sqrt{2M}} (\log n)^{\frac{1}{2}}, \quad \text{for all large} \ n 
\]

which gives the contradiction. \( \square \)

We now prove the theorem regarding the existence of second solution.

Theorem 5.1. Given a local minimum \( u_0 \) of \( J \) in \( W^{m,2}_N(\Omega) \), there exists a point \( v_0 \in W^{m,2}_N(\Omega) \) with \( v_0 > 0 \) in \( \Omega \), such that \( J_0(v_0) = 0 \).
Proof. From Lemma 5.7 we have \( c_0 \in \left(0, \frac{(4\pi)^m m!}{2} \right) \). Consider \( \{v_n\} \) be a Palais-Smale sequence for \( J_0 \) at the level \( c_0 \) around \( H \) (such a \((PS)_{H,c_0}\) sequence exists \([3]\)). Then up to a subsequence \( v_n \to v_0 \) in \( W^{2,2}_N(\Omega) \) for some \( v_0 \in W^{2,2}_N(\Omega) \) by Lemma (5.1) and (5.1) holds. We can easily check that \( v_0 \) is a solution of \((P)\) and therefore a critical point of \( J_0 \). It remains to show that \( v_0 \) is not a trivial solution.

**Case I.** \( c_0 = 0, v_0 = 0 \). We note that \( H = \{\|v\|_{W^{2,2}_N(\Omega)} = \frac{d_0}{2}\} \) in this case. As \( \{v_n\} \) is a \((PS)_{H,c_0}\) sequence we have \( v_n \to 0 \) strongly in \( W^{2,2}_N(\Omega) \). From the fact that \( \text{dist}(v_n, H) = 0 \) and \( H \) is closed we conclude that \( v_n \in H \) and which implies that \( v_0 \in H \) and \( v_0 \) is different from 0.

**Case II.** \( c_0 \in \left(0, \frac{(4\pi)^m m!}{2} \right) \), \( v_0 = 0 \). Using the fact that \( J_0(v_n) \to c_0 \) we see that for given any \( \epsilon > 0, \|v_n\|_{W^{2,2}_N(\Omega)}^2 \leq (4\pi)^m m! - \epsilon \) for all large \( n \). Let \( 0 < \delta < \frac{q}{(4\pi)^m m!} \) and \( q = \frac{(4\pi)^m m!}{(4\pi)^m m! - \epsilon} > 1 \). We have

\[
\int_{\Omega} |\tilde{f}(x, v_n)v_n|^q \leq C \int_{\Omega} e^{((1+\delta)q\|v_n\|^2)} \left( \frac{v_n^2}{\|v_n\|^2} \right)^2,
\]

since \( \sup_{x \in \Omega} |\tilde{f}(x, s)| \leq Ce^{(1+\delta)s^2} \), for all \( s \in \mathbb{R} \), for some \( C > 0 \). Now from the Tarsi’s embedding \([1.4]\) we get that \( \sup_{x \in \Omega} \int_{\Omega} |\tilde{f}(x, v_n)v_n|^q < \infty \) since \( (1+\delta)q\|v_n\|^2 \leq (4\pi)^m m! \). Also by Vitali’s convergence theorem we get \( \int_{\Omega} \tilde{f}(x, v_n)v_n \to 0 \) as \( n \to \infty \) since \( v_n \to 0 \) pointwise almost everywhere in \( \Omega \). Which implies

\[
o_n(1)\|v_n\|_{W^{2,2}_N(\Omega)} = (J_0(v_n), v_n) = \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 - \int_{\Omega} \tilde{f}(x, v_n)v_n
= \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 + o_n(1)
\]

which contradicts the fact \( \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 \to c_0 \) as \( n \to \infty \). Therefore \( v_0 \) is not identically 0 in \( \Omega \). And positivity of \( v_0 \) comes from the fact that \( \tilde{f}(x, s) \geq 0 \) for all \( (x, s) \in \Omega \times \mathbb{R} \) and using the maximum principle. \( \square \)

### 6. Proof of Theorem 1.1

Define \( \lambda_* = \mu C_0^{\frac{p+2}{p+1}} |\Omega|^{-\frac{p+2}{p+1}} \) where \( C_0 \) is same as in the Proposition (3.1). Then condition (3.1) is true whenever \( 0 < \lambda < \lambda_* \). From the Theorem 4.2 and 5.1 we show the existence of at least two positive solutions for \((P)\).

Let \( \phi_1 \) be the eigen function of \((-\Delta)^m \) on \( W^{2,2}_N(\Omega) \). Define

\[
\lambda^* = \left(\frac{\lambda_1}{p+1}\right)^{\frac{p+1}{p+2}} \left(\frac{\int_{\Omega} \phi_1}{\int_{\Omega} h \phi_1}\right).
\]

We prove that there is no solution of \((P)\) when \( \lambda > \lambda^* \). Assume that \( u_\lambda \) be a solution of \((P)\). By multiplying \( \phi_1 \) with \((P)\) and performing integration by parts over \( \Omega \), we get

\[
\int_{\Omega} (-\Delta)^m u_\lambda \phi_1 = \int_{\Omega} f(u_\lambda) \phi_1 + \lambda \int_{\Omega} h \phi_1
\]

implies

\[
(6.1) \quad \lambda \int_{\Omega} h \phi_1 = \int_{\Omega} (\lambda_1 u_\lambda - f(u_\lambda)) \phi_1
\]

We see that \( \lambda_1 t - f(t) \leq \lambda_1 - \mu t^{p+1} = \Theta(t) \) for all \( t > 0 \). The global maximum for the function \( \Theta \) is \( p \left( \frac{\lambda_1}{p+1} \right) \) on \((0, \infty)\). Then from (6.1) and the definition of \( \lambda^* \) we get \( \lambda \leq \lambda^* \). This completes Theorem 1.1.

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References

[1] D. R. Adams, A sharp inequality of J. Moser for higher order derivatives. Ann. Math. 128 (1988), 385-398.
[2] F. Bernis, J. Garcia-Azorero and I. Peral, Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order, Advances in Differential Equations, Volume 1, 2 (1996), 219-240.
[3] N. Ghoussoub and D. Preiss, A general mountain pass principle for locating and classifying critical points, Ann. Inst. H. Poincare- Anal. non lineaire, 6 (1989), 819-851.
[4] Nguyen Lam and Guozhen Lu, Existence of nontrivial solutions to polyharmonic equations with sub-critical and critical exponential growth, Discrete and Continuous Dynamical Systems, Volume 32, 6 (2012), 2187-2205.
[5] Dengfeng Lu, Existence and multiplicity results for critical growth polyharmonic elliptic systems, Mathematical Methods in the Applied Sciences, 37 (2014), 581-596.
[6] Guozhen Lu and Yanyan Yang, Adams’ inequality for bi-laplacian and extremal functions in dimension four, Advances in Mathematics, 220 (2009), 1135-1170.
[7] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. Jour., 20 (1971), 1077-1092.
[8] S. Prashanth and K. Sreenadh, Multiplicity of of solutions to a nonhomogeneous elliptic equation in $\mathbb{R}^2$, Differential and Integral equations, Volume 18, 6 (2005), 681-698.
[9] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincare- Anal. non lineaire, 9 (1992), 281-304.
[10] Cristina Tarsi, Adams’ inequality and limiting Sobolev embeddings into Zygmund spaces, Potential Analysis, 37 (2012), 353-385.
[11] Yajing Zhang, Positive solutions of semilinear biharmonic equations with critical Sobolev exponents, Nonlinear Analysis 75 (2012) 5567.
[12] Liang Zhao and Yuanyuan Chang, Minmax level estimate for a singular quasilinear polyharmonic equation in $\mathbb{R}^{2m}$, J. Differential Equations 254 (2013), 2434-2464.

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