Neutrino oscillation experiments
and the neutrino mass spectrum

S.M. Bilenky
Joint Institute for Nuclear Research, Dubna, Russia, and
INFN, Sezione di Torino, Via P. Giuria 1, I–10125 Torino, Italy

C. Giunti
INFN, Sezione di Torino, and Dipartimento di Fisica Teorica, Università di Torino,
Via P. Giuria 1, I–10125 Torino, Italy

and

W. Grimus
Institute for Theoretical Physics, University of Vienna,
Boltzmanngasse 5, A–1090 Vienna, Austria

Abstract
All the possible schemes of neutrino mixing with four massive neutrinos
inspired by the existing experimental indications in favor of neutrino
mixing are considered in a model-independent way. Assuming that in
short-baseline experiments only one mass-squared difference is relevant,
it is shown that the scheme with a neutrino mass hierarchy is not com-
patible with the experimental results. Only two schemes with two pairs
of neutrinos with close masses separated by a mass difference of the
order of 1 eV are in agreement with the results of all experiments. One
of these schemes leads to possibly observable effects in $^3$H and $(\beta\beta)_{0\nu}$
experiments.

---

1 Talk presented by S.M. Bilenky at the XVII International Conference on Neutrino Physics and Astrophysics, Helsinki, June 1996.
The determination of the values of the neutrino masses and mixing angles is the key problem of today’s experimental neutrino physics. The effects of neutrino masses and mixing are searched in more than 60 different experiments ($^3$H $\beta$-spectrum, ($\beta\beta$)$_{0\nu}$ decay, neutrino oscillations, solar neutrinos).

At present there exist three indications in favor of neutrino mixing. The first indication comes from the solar neutrino experiments. Assuming the Standard Solar Model [1] prediction for the solar neutrino fluxes, the data of four solar neutrino experiments (Homestake [2], Kamiokande [3], GALLEX [4] and SAGE [5]) can be explained by neutrino mixing with $\Delta m^2 \sim 10^{-5}$ eV$^2$ [6], in the case of resonant MSW transitions, or with $\Delta m^2 \sim 10^{-10}$ eV$^2$ [7], in the case of vacuum oscillations ($\Delta m^2$ is the neutrino mass-squared difference).

The second indication in favor of neutrino mixing comes from the data of the Kamiokande [8], IMB [9] and Soudan [10] atmospheric neutrino experiments. These data can be explained by $\nu_\mu \leftrightarrow \nu_\tau$ or $\nu_\mu \leftrightarrow \nu_e$ oscillations with $\Delta m^2 \sim 10^{-2}$ eV$^2$ [8].

Finally, indications in favor of $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_e$ oscillations with $\Delta m^2 \sim 1$ eV$^2$ were found in the LSND experiment [11].

In order to incorporate these three different scales of $\Delta m^2$ in a coherent scheme for neutrino mixing, it is necessary to have (at least) four massive neutrinos. We will consider here all the possible mixing schemes of four massive neutrinos with mass-squared differences relevant for the explanation of the results of the solar, atmospheric and LSND neutrino experiments. We will take also into account the limits on the neutrino oscillation parameters obtained in reactor and accelerator experiments on the search for neutrino oscillations.

We will show that only two schemes with two pairs of neutrinos with close masses separated by a mass difference of the order of 1 eV, which is relevant for the oscillations observed in the LSND experiment, are compatible with the results of all neutrino oscillation experiments.

Let us consider two groups of neutrinos $\nu_1, \ldots, \nu_r$ and $\nu_{r+1}, \ldots, \nu_n$, with close masses $m_1 \leq \ldots \leq m_r$ and $m_{r+1} \leq \ldots \leq m_n$, and let us assume that in short-baseline neutrino oscillation experiments

$$\frac{\Delta m_{ii}^2 L}{2p} \ll 1 \quad \text{for} \quad i \leq r \quad \text{and} \quad \frac{\Delta m_{ii}^2 L}{2p} \ll 1 \quad \text{for} \quad i \geq r + 1,$$

where $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, $L$ is the distance between the neutrino source and detector and $p$ is the neutrino momentum. In this case, only the neutrino mass-squared difference $\Delta m^2 \equiv m_n^2 - m_1^2$ is relevant for short-baseine neutrino oscillations and the amplitude of the transition $\nu_\alpha \to \nu_\beta$ ($\nu_\alpha$ and $\nu_\beta$ are any active or sterile neutrinos) is given by

$$A_{\nu_\alpha \to \nu_\beta} \simeq e^{-iE_it} \left\{ \delta_{\alpha\beta} + \sum_{i \geq r+1} U_{\beta i} U_{\alpha i}^* \left[ \exp \left( -i \frac{\Delta m_{ii}^2 L}{2p} \right) - 1 \right] \right\},$$

where $U$ is the $n \times n$ unitary mixing matrix. From Eq.(2), using the unitarity of the mixing matrix, for the oscillation probabilities we obtain (for details see Ref.[12])

$$P_{\nu_\alpha \to \nu_\beta} = \frac{1}{2} A_{\alpha \beta} \left( 1 - \cos \frac{\Delta m_{ii}^2 L}{2p} \right), \quad (\alpha \neq \beta),$$

(3)
\[ P_{\nu_\alpha \to \nu_\alpha} = 1 - \sum_{\beta \neq \alpha} P_{\nu_\alpha \to \nu_\beta} = 1 - \frac{1}{2} B_{\alpha;\alpha} \left( 1 - \cos \frac{\Delta m^2 L}{2p} \right) , \]  

where the oscillation amplitudes \( A_{\alpha;\beta} (\alpha \neq \beta) \) and \( B_{\alpha;\alpha} \) are given by

\[ A_{\alpha;\beta} = 4 \left| \sum_{i \geq r+1} U_{\beta i} U^*_i \right|^2 = 4 \left| \sum_{i \leq r} U_{\beta i} U^*_i \right|^2 , \]

\[ B_{\alpha;\alpha} = \sum_{\beta \neq \alpha} A_{\alpha;\beta} = 4 \sum_{i \geq r+1} |U_{\alpha i}|^2 \left( 1 - \sum_{i \geq r+1} |U_{\alpha i}|^2 \right) 
= 4 \sum_{i \leq r} |U_{\alpha i}|^2 \left( 1 - \sum_{i \leq r} |U_{\alpha i}|^2 \right) . \]

We will apply now the formulas (3)-(6) to the case \( n = 4 \) and to all possible values of \( r \).

Let us start with the case of a neutrino mass hierarchy:

\[ m_1 \ll m_2 \ll m_3 \ll m_4 , \]  

with with \( \Delta m_{21}^2 \) and \( \Delta m_{32}^2 \) relevant for the suppression of the flux of solar neutrinos and for the atmospheric neutrino anomaly, respectively. Using the formulas (3)-(6) (with \( n = 4 \) and \( r = 3 \)), for the oscillation amplitudes we obtain

\[ A_{\alpha;\beta} = 4|U_{\beta 4}|^2 |U_{\alpha 4}|^2 , \]

\[ B_{\alpha;\alpha} = 4|U_{\alpha 4}|^2 (1 - |U_{\alpha 4}|^2) . \]

We will consider the range \( 0.3 \text{eV}^2 \leq \Delta m^2 \leq 10^4 \text{eV}^2 \), which covers the sensitivity of all short-baseline experiments. At any fixed value of \( \Delta m^2 \), from the exclusion plots of the Bugey [13], CDHS [14] and CCFR [15] disappearance experiments we have \( B_{\alpha;\alpha} \leq B_{\alpha;\alpha}^0 (\alpha = e, \mu) \). The values of \( B_{e,e} \) and \( B_{\mu,\mu}^0 \) can be obtained from the corresponding exclusion curves. With the help of Eq.(9) we find that the elements \( |U_{\alpha 4}|^2 \) must satisfy one of the two inequalities

\[ |U_{\alpha 4}|^2 \leq a_0^0 \quad \text{or} \quad |U_{\alpha 4}|^2 \geq 1 - a_0^0 \quad (\alpha = e, \mu) , \]

where (see Ref.[16])

\[ a_0^0 = \frac{1}{2} \left( 1 - \sqrt{1 - B_{\alpha;\alpha}^0} \right) . \]

In the range of \( \Delta m^2 \) considered here \( a_e^0 \) and \( a_\mu^0 \) are small \((a_e^0 \lsim 4 \times 10^{-2}, a_\mu^0 \lsim 10^{-1})\). The large values of \( |U_{e4}|^2 \) and \( |U_{\mu 4}|^2 \) are excluded by the solar and atmospheric neutrino data. In fact, for the neutrino mass spectrum (7) we have (see Refs.[12,17])

\[ P_{\nu_e \to \nu_e}^\odot \geq |U_{e4}|^4 , \]

\[ P_{\nu_\mu \to \nu_\mu}^{\text{atm}} \geq |U_{\mu 4}|^4 . \]

If \( |U_{\alpha 4}|^2 \geq 1 - a_0^0 (\alpha = e, \mu) \), the probabilities \( P_{\nu_e \to \nu_e}^\odot \) and \( P_{\nu_\mu \to \nu_\mu}^{\text{atm}} \) are close to one and the problems of solar and atmospheric neutrinos cannot be explained by neutrino oscillations.
Thus, the only possibility is

\[ |U_{e4}|^2 \leq a_0^e \quad \text{and} \quad |U_{\mu 4}|^2 \leq a_0^\mu . \]  

(14)

Let us consider now \( \nu_\mu \leftrightarrow \nu_e \) oscillations. From Eqs. (8) and (14) we have

\[ A_{\mu;e} = 4|U_{e4}|^2|U_{\mu 4}|^2 \leq 4a_0^e a_0^\mu . \]  

(15)

Therefore, the upper bound for the amplitude \( A_{\mu;e} \) is quadratic in the small quantities \( a_0^e, a_0^\mu \), and \( \nu_\mu \leftrightarrow \nu_e \) oscillations must be strongly suppressed.

In Fig. 4 the limit (15) is presented as the curve passing through the circles. The 90% CL exclusion regions found in the \( \bar{\nu}_e \) disappearance Bugey experiment and in the \( \nu_\mu \to \nu_e \) appearance BNL E776 \[18\] and KARMEN \[19\] experiments are limited in Fig. 4 by the dashed, dot-dashed and dot-dot-dashed curves, respectively. The shadowed region in Fig. 4 is the region of the parameters \( \Delta m^2 \) and \( A_{\mu;e} \) which is allowed by the LSND experiment. It is seen from Fig. 4 that the region allowed by LSND is inside of the regions that are forbidden by the results of all the other experiments. Thus, we come to the conclusion that a mass hierarchy of four neutrinos is not compatible with the results of all neutrino oscillation experiments.

In a similar manner one can demonstrate that all possible schemes with mass spectra in which three masses are clustered and one mass is separated from the cluster by the \( \sim 1 \) eV gap needed for the explanation of the LSND data are not compatible with the results of all neutrino oscillation experiments.

Now we are left only with two possible neutrino mass spectra in which the four neutrino masses appear in two pairs separated by \( \sim 1 \) eV:

\[ \begin{align*}
\text{(A)} & \quad \begin{array}{c}
\text{atm} \\
\text{solar}
\end{array} \\
& \begin{array}{c}
m_1 < m_2 \ll m_3 < m_4 , \\
\text{LSND}
\end{array} \\
\text{(B)} & \quad \begin{array}{c}
\text{solar} \\
\text{atm}
\end{array} \\
& \begin{array}{c}
m_1 < m_2 \ll m_3 < m_4 . \\
\text{LSND}
\end{array}
\end{align*} \]  

(16)

(17)

From Eq. (8) (with \( n = 4 \) and \( r = 2 \)), for these schemes we have

\[ B_{\alpha;\alpha} = 4c_\alpha(1 - c_\alpha) \quad (\alpha = e, \mu) , \]  

(18)

with

\[ c_\alpha \equiv \sum_{i=1,2} |U_{\alpha i}|^2 \quad (\alpha = e, \mu) . \]  

(19)

From the results of reactor and accelerator disappearance experiments it follows that the parameters \( c_\alpha \) must satisfy one of the two inequalities

\[ c_\alpha \leq a_0^\alpha \quad \text{or} \quad c_\alpha \geq 1 - a_0^\alpha \quad (\alpha = e, \mu) , \]  

(20)

where \( a_0^\alpha \) is given by Eq. (11).
Taking into account the solar and atmospheric neutrino data, in the schemes with a mass spectrum of type (A) or (B) there is only one possibility:

\[
\begin{align*}
(A) & \quad c_e \leq a^0_e \quad \text{and} \quad c_\mu \geq 1 - a^0_\mu , \\
(B) & \quad c_e \geq 1 - a^0_e \quad \text{and} \quad c_\mu \leq a^0_\mu .
\end{align*}
\]

(21)\hspace{1cm} (22)

Let us consider now $\nu_\mu \leftrightarrow \nu_e$ oscillations. Using the Cauchy-Schwarz inequality, from Eq.(4) (with $n = 4$ and $r = 2$) and Eq.(19), for both schemes (A) and (B) we find

\[
A_{\mu,e} = 4 \left| \sum_{i=1,2} U_{ei} U^*_{\mu i} \right|^2 \leq 4 c_e c_\mu .
\]

(23)

From Eqs.(21)–(23) it follows that the upper bound for $A_{\mu,e}$ is only linear in the small quantities $a^0_e$ (in the scheme (A)) and $a^0_\mu$ (in the scheme (B)). Since $a^0_e \gtrsim 5 \times 10^{-3}$ and $a^0_\mu \gtrsim 8 \times 10^{-3}$ for all values of $\Delta m^2$, in the case of both schemes (A) and (B) the limit (23) is compatible with the results of the LSND experiment.

The schemes (A) and (B) lead to different consequences for the experiments on the measurement of the neutrino mass through the investigation of the end-point part of the $^3\text{H}$ $\beta$-spectrum and for the experiments on the search for neutrinoless double-$\beta$ decay ($^{0\nu} \beta\beta$). In fact, we have

\[
\begin{align*}
(A) & \quad \sum_{i=3,4} |U_{ei}|^2 \geq 1 - a^0_e , \\
(B) & \quad \sum_{i=3,4} |U_{ei}|^2 \leq a^0_e .
\end{align*}
\]

(24)\hspace{1cm} (25)

From Eq.(24) it follows that in the case of the scheme (A) the neutrino mass measured in $^3\text{H}$ experiments practically coincides with the “LSND mass” $m_4$:

\[
m_{\nu_e}(^3\text{H}) \simeq m_4 .
\]

(26)

If the scheme (B) is realized in nature and $m_1$ is very small, the mass measured in $^3\text{H}$ experiments is at least two order of magnitude smaller than $m_4$.

If massive neutrinos are Majorana particles, ($^{0\nu} \beta\beta$) decay is possible. In the scheme (A), the effective neutrino mass that is measured in ($^{0\nu} \beta\beta$) decay is equal to

\[
|\langle m \rangle| \simeq \left| \sum_{i=3,4} U^2_{ei} \right| m_4 .
\]

(27)

We have

\[
|\langle m \rangle| \simeq m_4 \sqrt{1 - 4 |U_{e1}|^2 (1 - |U_{e4}|^2) \sin^2 \phi} ,
\]

(28)

where $\phi$ is the difference of the phases of the elements $U_{e3}$ and $U_{e4}$. Depending on the value of the phase $\phi$, the quantity $|\langle m \rangle|$ has a value in the range

\[
2 |U_{e4}|^2 - 1 \leq |\langle m \rangle| \lsim m_4 .
\]

(29)
The upper and lower bounds in Eq. (29) correspond, respectively, to the cases of equal and opposite CP parities of $\nu_3$ and $\nu_4$ (for details see Ref. [12]).

In conclusion, we have shown that the results of the experiments on the search of neutrino oscillations allow us to obtain model-independent information on the spectrum of neutrino masses. Only two possible types of spectra with four massive neutrinos grouped in two pairs with close masses, separated by a mass difference of the order of 1 eV, can accommodate the results of all the present-day neutrino oscillation experiments.

References

[1] J.N. Bahcall and M.H. Pinsonneault, Rev. Mod. Phys. 67, 781 (1995); S. Turck-Chièze et al., Phys. Rep. 230, 57 (1993); V. Castellani et al., preprint INFNFE-10-96 (e-Print Archive: astro-ph/9606180); A. Dar and G. Shaviv, Nucl. Phys. B (Proc. Suppl.) 48, 335 (1996).

[2] B.T. Cleveland et al., Nucl. Phys. B (Proc. Suppl.) 38, 47 (1995).

[3] Y. Fukuda et al., Phys. Rev. Lett. 77, 1683 (1996).

[4] GALLEX Coll., Phys. Lett. B 357, 237 (1995).

[5] J.N. Abdurashitov et al., Phys. Lett. B 328, 234 (1994).

[6] GALLEX Coll., Phys. Lett. B 285, 390 (1992); X. Shi, D.N. Schramm and J.N. Bahcall, Phys. Rev. Lett. 69, 717 (1992); P.I. Krastev and S.T. Petcov, Phys. Lett. B 299, 99 (1993); N. Hata and P.G. Langacker, Phys. Rev. 50, 632 (1994); G.L. Fogli and E. Lisi, Astropart. Phys. 2, 91 (1994); G. Fiorentini et al., Phys. Rev. D 49, 6298 (1994); L.M. Krauss, E. Gates and M. White, Phys. Rev. D 51, 2631 (1995).

[7] V. Barger, R.J.N. Phillips, and K. Whisnant, Phys. Rev. Lett. 69, 3135 (1992); P.I. Krastev and S.T. Petcov, Phys. Rev. Lett. 72, 1960 (1994).

[8] Y. Fukuda et al., Phys. Lett. B 335, 237 (1994).

[9] R. Becker-Szendy et al., Nucl. Phys. B (Proc. Suppl.) 38, 331 (1995).

[10] E.A. Peterson, Talk presented at the XVII International Conference on Neutrino Physics and Astrophysics, Helsinki, June 1996.

[11] C. Athanassopoulos et al., Phys. Rev. Lett. 75, 2650 (1995); preprint LA-UR-96-1326 (e-Print Archive: nucl-ex/9605001).

[12] S.M. Bilenky, C. Giunti, C.W. Kim and S.T. Petcov, preprint SISSA 35/96/EP (e-Print Archive: hep-ph/9604364), to be published in Phys. Rev. D.

[13] B. Achkar et al., Nucl. Phys. B 434, 503 (1995).

[14] F. Dydak et al., Phys. Lett. B 134, 281 (1984).
[15] I.E. Stockdale et al., Phys. Rev. Lett. 52, 1384 (1984).

[16] S.M. Bilenky, A. Bottino, C. Giunti and C.W. Kim, Phys. Lett. B 356, 273 (1995); Phys. Rev. D 54 (1996), 1881.

[17] S.M. Bilenky, C. Giunti and W. Grimus, preprint UWTPh-1996-42 (e-Print Archive: hep-ph/9607372).

[18] L. Borodovsky et al., Phys. Rev. Lett. 68, 274 (1992).

[19] J. Kleinfeller, Nucl. Phys. B (Proc. Suppl.) 48, 207 (1996).
Figure 1: The plane of the parameters $A_{\mu,e}$ and $\Delta m^2$ that characterize $\nu_\mu \rightleftharpoons \nu_e$ oscillations. The shadowed regions limited by the solid curves are allowed at 90% CL by the LSND $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ experiment. The regions excluded at 90% CL by the Bugey $\bar{\nu}_e$ disappearance experiment and by the BNL E776 and KARMEN $\nu_\mu \rightarrow \nu_e$ experiments are bounded by the dashed, dash-dotted and dash-dot-dotted curves, respectively. The curve passing through the circles limits the exclusion region that is obtained with Eq.(15) taking into account the results of reactor and accelerator disappearance experiments in the case of a neutrino mass hierarchy.