On massless 4D Gravitons from Asymptotically $AdS_5$ Space-times

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ABSTRACT: We investigate the conditions for obtaining four-dimensional massless spin-2 states in the spectrum of fluctuations around an asymptotically $AdS_5$ solution of Einstein-Dilaton gravity. We find it is only possible to have normalizable massless spin-2 modes if the space-time terminates at some IR point in the extra dimension, far from the UV AdS boundary, and if suitable boundary conditions are imposed at the “end of space.” In some of these cases the 4D spectrum consists only of a massless spin-2 graviton, with no additional massless or light scalar or vector modes. These spin-2 modes have a profile wave-function peaked in the interior of the 5D bulk space-time. Under the holographic duality, they may be sometimes interpreted as arising purely from the IR dynamics of a strongly coupled QFT living on the AdS boundary.
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1. Introduction and Summary

Gravity remains to date the least understood of all known fundamental interactions. Recent cosmological data imply the presence of energy sources in the universe resembling a vacuum energy. This is hard to explain in theories of fundamental physics that successfully incorporate gravity, notably string theory.

String theory provides a perturbatively well defined quantum theory of gravity at energies well below the ten-dimensional Planck scale, but seems to fail to naturally explain today’s acceleration of the universe. Although the theory has been successful in resolving several types of space-time singularities, ubiquitous singularities like the ones appearing inside black holes or at the beginning of the universe have resisted resolution. It is quite plausible that they cannot be resolved with a perturbative treatment of the theory. This is in contrast with the microscopic understanding of black hole entropy in string theory, that was essentially successful.

The resolution of black-hole and cosmological singularities should be correlated with the behavior of the theory at high energy. String theory is strongly coupled at energies approaching the Planck scale, and strong coupling dualities are seemingly unable to help in this direction. Closed string theory is a theory with an effective cutoff at the string scale $M_s$ typically much smaller than the Planck scale $M_P$ in the perturbative regime. It is therefore not a surprise that the perturbative theory breaks down at energies much higher than the effective cutoff.

The successful counting of black-hole microstates [1] led to the AdS/CFT correspondence [2] and its generalizations. This development gave a concrete realization to the correspondence between large-N gauge theories and string theories as advocated much earlier, [3]. An interesting byproduct of this correspondence is the use of gauge-theory dynamics, in order to understand non-perturbative and/or high energy issues in string theory on spaces with boundary. The old matrix model approach could be reinterpreted in this new light as a two-dimensional bulk-boundary correspondence, [4]. One could go further and postulate that observable gravity is described by a four-dimensional large-N (gauge) theory, unifying in such a fashion gravity with the other interactions in four dimensions, [5, 6].

A glimpse of the five-dimensional gravitational setup, associated to four-dimensional conformally invariant gauge theories is provided by the Randall-Sundrum geometry, [7]. The geometry is locally $\text{AdS}_5$, but the boundary of the space has been removed by placing a gravitating 4-brane in $\text{AdS}_5$ supplemented by a $\mathbb{Z}_2$ orbifold identification with a fixed point at the brane position. The dual gauge theory is a cutoff version of the $\mathcal{N}=4$ SYM theory, although at high energies this identification is only qualitative, [8]. On the gravitational side the presence of the UV cutoff allows for a four-dimensional massless graviton, on top of the continuum of the five dimensional KK gravitons. From the four-dimensional gauge theory point of view, the massless graviton is fundamental, and cannot be described by the gauge theory alone. This is consistent with the intuition that bulk particles with wave-functions peaked at the UV brane are fundamental in the dual gauge theory, while particles with wave-functions peaked in the IR are composite bound-states, [8].
In this paper we will systematically investigate the question: can a massless four-dimensional graviton appear in five-dimensional warped geometries that are asymptotically AdS? This is qualitatively distinct from the RS setup, since here the space has a UV boundary with no cut-off, and the standard RS graviton wave-function is non-normalizable. In this sense we are searching for massless composite gravitons, since necessarily the graviton wave-function must have an important support in the IR section of the geometry.

One of the main motivations for this exercise is that we want to imagine that the five-dimensional gravitational theory is a dual version of a four-dimensional large-N theory, and the graviton is a composite of gluons. In this context, the generic relevant bulk fields associated to a gauge theory are the metric, dual to the stress tensor of the YM theory; a scalar \( \Phi \) (the dilaton) associated to the operator \( TR[F_{\mu \nu}F^{\mu \nu}] \); and a pseudoscalar \( A \) (the axion), associated to the operator \( TR[F \wedge F] \). As the axion is understood not to affect the vacuum structure of the gauge theory at large N, because it couples to instantons, we will neglect it for most of the paper.

The question whether one can find a “composite” 4D graviton using arguments from holography was first investigated in [9], where a model was proposed in which the metric is the only bulk field, and the space-time is a slice of \( AdS_5 \) like in RS I. Bulk and boundary mass terms for the five dimensional metric fluctuations were then added to the Einstein-Hilbert action at quadratic level to modify the profile of the graviton zero-mode in the extra dimensions, and it was found that for a particular choice of parameters there is indeed an IR-peaked zero-mode. In our work, we consider more general 5D geometries, characterized by a non trivial dilaton background, i.e. we perform a systematic analysis of the question whether 4D massless graviton modes can appear in 5D Einstein-Dilaton theory (as suggested by a dual gauge theory picture). We avoid some of the drawbacks of [9] by dealing with a generally covariant bulk theory: rather than modifying the graviton dynamics by hand with mass terms, we modify the background geometry by considering a nontrivial scalar field in the bulk. Adding a dilaton is in some sense a minimal modification of the bulk theory, but at the same time it is sufficient to carry on a complete study of the 5D geometries that have 4D flat sections, due to the freedom in choosing the dilaton potential [10]. As we will later argue, our results are general and valid for any other set of bulk fields and actions.

The method we use to analyze the 4D spectrum parallels the analysis of the spectrum of 4D modes in brane-world models with a bulk scalar [11, 12, 13, 14], and it follows the procedure used in cosmological perturbation theory (see [15] for a review): we look at the linearized equations for classical fluctuations of the bulk fields (metric and dilaton, in our case), classify them according to their 4-momentum and representation of the 4D Lorentz group, and look for normalizable solution of the corresponding linearized field equations. In this context, various authors have computed the low energy “glueball” spectra of gauge theories which are dual to deformations of the archetypal \( AdS_5 \times S^5 \) background, and that develop a mass gap and exhibit confinement in the IR [16, 17, 18, 19, 20, 21]. In fact, our philosophy and our techniques are very similar to the kind of constructions generally referred to \( AdS/QCD \) [22, 23] (see [24] for an up-to-date list of references): there, one tries to reproduce the known low energy features of QCD by engineering a
specific “phenomenological” 5D model. In our paper we are not interested in QCD, but the method we follow is similar: we try to study which kinds of 5D setups have a field theory dual that contains a massless spin-2 “glueball” in the low-energy spectrum. Since the backgrounds we study are asymptotically $AdS_5$, the dual theory will be asymptotically conformal (but not asymptotically free) in the UV, although we are not going to analyze its detailed nature.

We also analyze the potential presence of other massless fields, like scalars and vector bosons that may arise from the universal sector of the bulk theory. This is important, as their presence can make the gravitational theory clash with data.

The basic result of our paper can be summarized as follows. Consider an asymptotically $AdS_5$ space-time with metric of the form:

$$ds^2 = a^2(y) \left( dy^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right)$$

This is generic, since any 5D warped space-time with 4D Lorentz invariance can be brought to this form. What we show is:

- **There can be no massless 4D graviton modes unless the space-time “ends” at a finite value of the extra coordinate, $y = y_0$.**

The “end of space” at $y_0$ can correspond either to an IR brane, (analogous to the Planck brane of RSI, as it was the case in [9]), or to a naked singularity. In the latter case we show that:

- **It is possible in some cases to have a massless spin-2 as the only propagating 4D massless degree of freedom with no extra scalars or vectors.**

We believe that the singular case, although at first sight less appealing, may ultimately be more interesting than having a boundary at some arbitrary cut-off point. If we want to provide a mechanism for ending space-time at a finite value of the extra coordinate, it is preferable that this results automatically from the dynamics, rather than from a boundary put by hand at an arbitrary location. Unfortunately, we will see that although this is true for the background (the location and type of singularity are purely determined by the bulk potential plus UV initial conditions), the aspect of the IR dynamics in which we are more interested, i.e. the spectrum of light modes, depends in a crucial way on additional information that we must provide at the singularity, in the form of boundary conditions for the fluctuations, and cannot be deduced from bulk or UV-boundary physics. This ultimately makes our search for an emergent massless graviton only partially successful, since any attempt at resolving the singularity must reproduce those boundary conditions for which massless gravitons are in the spectrum. This makes our results non-universal, and dependent on the details of the dynamics of the singularity.

It should be stressed at this point that gauge theories that confine in the IR and have a five-dimensional bulk dual at all scales are generically expected to have a semiclassical IR singularity that is resolved in the full string theory, [25].

We should mention that there is a theorem that forbids massless composite gravitons in gauge theories, with specific assumptions including relativistic invariance and the existence
of well defined particle states. [26]. There are known exceptions, like the gravitons in AdS$_5$, which violate one or most assumptions. We will discuss this issue and develop its ramifications in the last section.

Our paper is organized as follows. In Section 2 we present our general Einstein-Dilaton model. In Section 3 we perform the linear fluctuation analysis of the various bulk modes, classified according to their 4D mass and Lorentz tensor properties. In Section 4 we tackle the question of the normalizability of modes, and we systematically investigate what kind of models have the desired normalizable zero-mode spectrum. In Section 5 we discuss the issues of boundary conditions for singular space-times. In Section 6 we present a concrete example, in which we derive explicitly the effective four-dimensional coupling of the massless spin-2 zero-modes. In Section 7 we present some ideas on how to circumvent some of the difficulties of this setup, in particular how to evade the “no-go” argument exposed above. Some of the technical details are left to the appendix.

2. Five-Dimensional Dilaton-Gravity

A generic gauge theory in four dimensions has three gauge-invariant operators of lowest dimension: the stress tensor, dual to a metric, $T_{\mu\nu}$ dual to a scalar (the dilaton), and $\text{tr}[F \wedge F]$ dual to a pseudoscalar, the axion. As the axion couples to instantons and these are negligible at large $N$ we will not consider it further.

Driven by these considerations we consider 5D gravity plus a scalar field with a generic potential$^1$,

$$S = \frac{1}{2k^2} \int d^5x \sqrt{-g} \left( R - \partial_A \Phi \partial^A \Phi - V(\Phi) \right), \quad (2.1)$$

and background solutions of the form

$$ds^2 = a(y)^2 \left( dy^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \Phi(y, x^\mu) = \Phi_0(y) \quad (2.2)$$

For later reference, we define the functions $B(y), z(y)$ as:

$$B(y) = \frac{3}{2} \log a(y), \quad z(y) = \frac{\Phi'_0}{(a'/a)}. \quad (2.3)$$

Einstein’s equations take the form:

$$\Phi'^2_0 = -3 \left( \frac{a''}{a} - 2 \frac{a'^2}{a^2} \right) = 2B'' + \frac{4}{3}B'^2, \quad (2.4)$$

$$a^2V(\Phi_0) = -3 \left( \frac{a''}{a} + 2 \frac{a'^2}{a^2} \right) = 2B'' - 4B'^2. \quad (2.5)$$

$^1$with space-time coordinates $x^A = (x^\mu, y)$. The metric has signature $(-+++)$; capital Latin indexes $A, B, \ldots$ are 5D indexes, taking values $\mu, y$; Greek indexes $\mu, \nu, \ldots$ are 4D indexes ranging from $0 \ldots 3$. A primes denote derivative w.r.t. $y$. 

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and they imply the Dilaton field equation, as usual. The 5D curvature scalar\(^2\) is given by:

\[
R = a^{-2} \Phi'^2_0 + \frac{5}{3} V = -8a^{-2} \left( \frac{a''}{a} + \frac{1}{2} \frac{a'^2}{a^2} \right) = -\frac{16}{3} e^{A/3B} (B'^2 - B'') .
\]  

(2.7)

In the rest of this paper we will consider space-times which have an asymptotically \(AdS_5\) region near \(y = 0\), where the metric and scalar field behave as

\[
ds^2 \sim \frac{1}{(ky)^2} \left(dy^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right), \quad \Phi'_0 \sim 0,
\]

(2.8)

where \(k\) is the \(AdS\) curvature scale and it is determined by the value of the scalar potential near \(y = 0\). In the dual field theory language this means that we are considering gauge theories that are UV-complete and do not necessitate a UV cut-off.

We will take the \(y\) coordinate to range from zero to either infinity or to a finite value \(y_0\). Far from the \(AdS\) boundary, the background scalar field will have a nontrivial profile, and the metric will generically differ from the one of \(AdS\).

We can briefly compare this situation with what happens in the RS models, where the 5D metric is \(AdS\) but the space-time is cut-off by a \(\mathbb{Z}_2\) orbifold action at \(y = y_c > 0\). These models supports a normalizable massless spin-2 mode which is localized near \(y_c\) and mediates 4D gravity. In our case instead, we take the space-time to extend all the way to the \(AdS\) boundary. As a consequence, the massless RS mode becomes non-normalizable and is projected out of the spectrum. In the rest of the paper we will investigate under what conditions the space-times under consideration still support a normalizable massless spin-2 mode, which this time will be localized far from the \(AdS_5\) boundary.

Recall that, as explained e.g. in [8], the RS models have a dual holographic interpretation as a 4D CFT with a UV cut-off, living on the UV brane. According to this interpretation, 5D normalizable modes that are localized near the UV brane correspond to elementary states in the 4D theory, while modes that are localized far from the UV brane are non-elementary. In particular, in the holographic interpretation of the RS models, the graviton is always treated as elementary, and the Einstein action is part of the fundamental action defined at the UV cut-off. In the holographic description of our setup the graviton is not part of the fundamental degrees of freedom, which consist in an asymptotically conformal field theory at strong coupling. A massless spin-2 mode localized far from the boundary of \(AdS\) is instead to be interpreted as arising from the IR dynamics.

3. Linear Fluctuations

In this section we derive the equations for the linear fluctuations around the background introduced above. We derive the perturbation equations by varying the action, eq. (2.1),

\[
-\frac{3}{2} R = g^{AB} G_{AB} = g^{AB} \left[ \partial_A \Phi_0 \partial_B \Phi_0 - \frac{1}{2} g_{AB} \left( g^{CD} \partial_C \Phi_0 \partial_D \Phi_0 + V \right) \right] = a^{-2} \left( -\frac{3}{2} \Phi'^2_0 - \frac{5}{2} a'^2 V \right) .
\]

(2.6)

\(^2\)this can be obtained from the trace of Einstein’s equation:
expanded up to second order in the fluctuation. This is useful to determine the normalization of the fluctuations. This kind of analysis has appeared in various works, see e.g. [11, 12, 13, 14]. However, we found that some subtleties that arise when dealing with 4D-massless modes have been overlooked, and this case necessitates separate treatment. A detailed analysis of this case is performed in Appendix A.2. The details of this procedure can be found in Appendix A, here we will give only the salient points.

A generic perturbation of the metric and the dilaton around the background (2.2) can be written as

\[ ds^2 = a^2(y) (\eta_{AB} + h_{AB}) \]

\[ = a^2(y) \left[ (1 + 2\phi) dy^2 + 2A_\mu dydx^\mu + (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \right] , \quad (3.1) \]

\[ \Phi = \Phi_0(y) + \chi , \quad (3.2) \]

where \( \phi \), \( A_\mu \), \( h_{\mu\nu} \) and \( \chi \) are functions of \( y, x^\mu \).

Under a 5D diffeomorphism, \( (\delta y = \xi^5, \delta x_\mu = \xi_\mu) \), the fluctuations defined by eq. (3.1) transform as:

\[ \delta h_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu - 2\eta_{\mu\nu} \frac{a'}{a} \xi^5 , \quad (3.3) \]

\[ \delta A_\mu = -\xi'_\mu - \partial_\mu \xi^5 , \quad (3.4) \]

\[ \delta \phi = -\xi^5'' - \frac{a'}{a} \xi^5 , \quad (3.5) \]

\[ \delta \chi = -\Phi'_0 \xi^5 . \quad (3.6) \]

Due to gauge invariance, not all of these perturbations are dynamical: the usual counting of degrees of freedom for gravitational theories implies that the metric and dilaton fluctuation, \( (h_{AB}, \chi) \), contain 16 components, out of which 5 are eliminated by gauge transformation and another 5 can be eliminated through the non-dynamical components of Einstein’s equations. We are left with a total of 6 degrees of freedom, which correspond to a 5D massless spin-2 plus a scalar.

Expanding eq. (2.1) to second order around the background (2.2) we obtain the following action for the perturbations:

\[ S^{(2)} = \frac{1}{2k_5^2} \int d^4 x dy a^3(y) \left[ L_{\text{ein}}^{(2)} - \frac{1}{4} h'^{\rho\sigma} h'^{\rho\sigma} + \frac{1}{4} (h')^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \]

\[ - \partial_\mu \chi \partial^\mu \chi - \chi'^2 - \frac{1}{2} a^2 \partial_\mu V \chi^2 - \partial^\mu \phi (\partial^\nu h_{\mu\nu} - \partial_\mu h) \]

\[ + 2\Phi'_0 \phi' \chi + \Phi'_0 h' \chi + 4\Phi'_0 \phi ' + 2\Phi'_0 A^\mu \partial_\mu \chi \left. \right] - (a^3 A^\mu)' \left[ \partial_\mu h - \partial^\mu h_{\mu\nu} \right] + (a^3)' \left[ -2A_\mu \partial^\mu \phi - 2\phi' - \phi h' \right] , \quad (3.7) \]

where from now on \( h \equiv h_{\mu\nu} \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and

\[ L_{\text{ein}}^{(2)} = -\frac{1}{4} \partial_\mu h_{\rho\sigma} \partial^\rho h'^{\sigma} + \frac{1}{2} \partial^\mu h_{\rho\sigma} \partial^\nu h'^{\rho\sigma} - \frac{1}{2} \partial^\mu h \partial_\mu h'^{\rho} + \frac{1}{4} \partial^\mu h \partial_\mu h , \quad (3.8) \]

is the quadratic part of the 4D Einstein-Hilbert Lagrangian.
3.1 Classification of Physical Fluctuations

Varying the action (3.7) w.r.t. $h_{\mu\nu}, A_\mu, \phi$ and $\chi$, we obtain field equations which correspond to the linearized Einstein-Dilaton field equations, and are reported in Appendix A. We are interested in solutions that correspond to particles of a definite mass in four dimensions and that are normalizable in the fifth dimension (finite 4D kinetic term after dimensional “reduction”), i.e. solutions of the form

$$\Omega(y,x^\mu) = \Psi_k(y) \omega_k(x^\mu), \quad \Box_k \omega_k = -k^2 \omega_k,$$

(3.9)

where $\Omega(y,x^\mu)$ is any of the fields under consideration. Due to the linear character of the equations we can consider each mode, labeled by 4-momentum $k_\mu$, separately. One has to separate two cases: massless modes ($k_\mu k^\mu = 0$) and massive ones ($k_\mu k^\mu = -m^2 \neq 0$). This distinction is needed if we want to classify the modes according to transformation properties w.r.t. the 4D Lorentz group. This analysis is carried out in detail in Appendix A.

**Massive Sector**

In the massive sector, the gauge-invariant propagating fields are a transverse, traceless tensor $h^{TT}_{\mu\nu}$ (a massive spin-2 in 4D) and a scalar field $\zeta$, for a total of $(5 + 1 = 6)$ degrees of freedom. The scalar field $\zeta$ is defined as

$$\zeta = \psi - \frac{\chi}{z}, \quad \psi = \frac{1}{6} \left( h^\mu_\mu - \frac{\partial^\mu \partial^\nu h_{\mu\nu}}{m^2} \right),$$

(3.10)

where $z(y)$ is defined in eq. (2.3). The field equations are:

$$h^{TT'}_{\mu\nu} + 3 \frac{d'}{a} h^{TT'}_{\mu\nu} + \Box_4 h^{TT}_{\mu\nu} = 0,$$

(3.11)

$$\zeta'' + \left( 3 \frac{d'}{a} + 2 \frac{z'}{z} \right) \zeta' + \Box_4 \zeta = 0.$$

(3.12)

Notice that the tensor equation is the same as the massless scalar field eq. in the 5D background (2.2).

**Massless Sector**

In the massless sector, the six physical degrees of freedom are divided as follows: a transverse tensor and vector, $h^{sTT}_{\mu\nu}$ and $A^{sT}_\mu$, which are also spatially-transverse (2 components each), plus two massless scalar fields $\zeta_1$ and $\zeta_2$. In an appropriate gauge the corresponding fluctuations appearing in (3.1) take the form:

$$h_{\mu\nu} = h^{sTT}_{\mu\nu} + 2 \eta_{\mu\nu} \psi,$$

(3.13)

$$A_\mu = A^{sT}_\mu,$$

(3.14)

with the transverse vector and tensor obeying:

$$h^{sTT'}_{\mu} = \partial^\mu h^{sTT}_{\mu} = \partial^i h^{sTT}_i = 0,$$

$$\partial^\mu A^{sT}_\mu = \partial^i A^{sT}_i = 0.$$

(3.15)
The field equations for these fields and the remaining scalar fluctuations $\phi$ and $\chi$ are:

\[
(h^{sTT}_{\mu\nu})'' + 3\frac{a'}{a}(h^{sTT}_{\mu\nu})' = 0; \tag{3.16}
\]
\[
(a^3A^{sT}_{\mu})' = 0; \tag{3.17}
\]
\[
\phi = -2\psi; \tag{3.18}
\]
\[
\zeta_1' = 0, \quad \left(\frac{a^4}{a^2}\zeta_2\right)' = -2a^3\zeta_1, \tag{3.19}
\]

where $\zeta_1$ and $\zeta_2$ are given by the linear combinations:

\[
\psi = \zeta_1 + \zeta_2, \quad \chi = z\zeta_2, \tag{3.20}
\]

and we have of course $\Box_4h^{sTT}_{\mu\nu} = \Box_4A^{sT}_{\mu} = \Box_4\zeta_1 = \Box_4\zeta_2 = 0$.

### 3.2 Effective Actions and Normalizability

Next, we determine the normalization conditions for the modes found in the previous section. To do this we substitute in the action (3.7) the fields whose $y$-profile satisfies the field equations for a given mass.

#### 3.2.1 Tensor Modes

The equation obeyed by the transverse traceless tensors modes (both in the massless and massive case) is eq (3.11):

\[
h^{TT''}_{\mu\nu} + 3\frac{a'}{a}h^{TT'}_{\mu\nu} + \Box_4h^{TT}_{\mu\nu} = 0. \tag{3.21}
\]

Separating $h^{TT}_{\mu\nu}$ into a purely spatial part and a $y$-dependent profile:

\[
h_{\mu\nu} = h(y)h^{(4)TT}(x), \quad \Box_4h^{(4)TT}_{\mu\nu} = m^2h^{(4)TT}_{\mu\nu}, \tag{3.22}
\]

we get the equation for the profile $h(y)$:

\[
h'' + 3\frac{a'}{a}h' + m^2h = 0. \tag{3.23}
\]

In the massive case there are five independent polarizations, while in the massless case we gauged away all but two polarizations (the ones of a physical massless spin-2 in 4D).

It is convenient to introduce the function $B(y)$ and the wave-function $\psi_t(y)$:

\[
a(y) = e^{-\frac{3}{4}B(y)}, \quad \psi_t(y) = e^{-B(y)}h(y). \tag{3.24}
\]

Then (3.23) becomes a Schrödinger-like equation for $\psi_t(y)$:

\[
-\psi_t'' + V_t(y)\psi_t = m^2\psi_t, \quad V_t = B'^2 - B''. \tag{3.25}
\]

Notice that the potential $V_t$ is proportional to the 5D curvature scalar, see eq. (2.7):

\[
V_t = -\frac{3}{16}a^2R. \tag{3.26}
\]
To get the normalization measure, we insert the ansatz \( h_{\mu\nu} = h(y)h^{(4)}_{\mu\nu}(x) \) into (3.7):

\[
S^{TT} = \frac{1}{2k_5^2} \int d^4x dy a^3(y) \left[ -\frac{1}{4} h(y)^2 \partial_\rho h^{(4)}_{\mu\nu} \partial^\rho h^{(4)}_{\mu\nu} - \frac{1}{4} h'(y)^2 h^{(4)}_{\mu\nu} h^{(4)}_{\mu\nu} \right] \\
= \frac{1}{2k_5^2} \int dy a^3(y) h(y)^2 \int d^4x \left[ -\frac{1}{4} \partial_\rho h^{(4)}_{\mu\nu} \partial^\rho h^{(4)}_{\mu\nu} - m^2 h^{(4)}_{\mu\nu} h^{(4)}_{\mu\nu} \right].
\]

(3.27)

In the second step we have integrated by parts and used (3.23). From the expression above it is clear that the mode corresponds to a normalizable 4D mode if \( a^{3/2}h(y) \) is square-integrable or, in terms of the wave function defined by (3.24), if

\[
\int dy |\psi(y)|^2 < \infty.
\]

(3.28)

That is, the solution of eq. (3.24) must be normalizable in the usual quantum-mechanical norm.

### 3.3 Vector Modes

In this sector there are only zero-modes, and no massive ones. In the gauge we are using, the transverse vector is purely contained in \( A_\mu \), since we gauged away the vector mode from \( h_{\mu\nu} \). From eq. (3.17) we obtain the profile of \( A_\mu \):

\[
A_\mu(x, y) = \frac{1}{a^3(y)} A^{(4)}_\mu(x).
\]

(3.29)

From the action (3.7) we obtain the normalization condition:

\[
S^{(V)} = \frac{1}{2k_5^2} \int d^4x dy a^3(y) \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] = \frac{1}{2k_5^2} \int dy a^{-3}(y) \int d^4x \left[ -\frac{1}{4} F^{(4)}_{\mu\nu} F^{\mu\nu(4)} \right]
\]

(3.30)

So we have a normalizable 4D massless vector if

\[
\int dy a^{-3}(y) < \infty
\]

(3.31)

### 3.4 Scalar Modes - Massive Case

The relevant scalar quantity is \( \zeta(y, x) \), defined in (3.10). In fact, when we use an ansatz that solves the 5D equations, \( \zeta(y, x) \) is the only scalar mode that appears in the actions. Decomposing \( \zeta(x, y) \) in modes with a given non-zero 4D mass we get, from eq. (3.12), the equation for the profile:

\[
\zeta'' + \left( 3 \frac{a'}{a} + 2 \frac{z'}{z} \right) \zeta' + m^2 \zeta = 0.
\]

(3.32)

Defining:

\[
C(y) = \log z(y), \quad \psi_s(y) = e^{-G(y)} \zeta(y), \quad G = B - C,
\]

we get a Schrödinger equation for the wave-function \( \psi_s \), like eq. (3.28), except for the substitution \( B(y) \to G(y) \) in the potential:

\[
-\psi_s'' + V_s(y) \psi_s = m^2 \psi_s, \quad V_s = G'^2 - G''.
\]

(3.34)
As was shown in [14], the quadratic action in the scalar sector reduces to an action for the gauge-invariant field $\zeta$ alone:

$$S^{(S)} = \frac{1}{2k_5^2} \int d^4x dy a^3(y) z^2(y) [\zeta'^2 + (\partial_\mu \zeta)^2] = \frac{1}{2k_5^2} \int d^4x dy e^{-2G(y)} [\zeta'^2 + (\partial_\mu \zeta)^2].$$

(3.35)

It follows that, as in the case of the tensor modes, the normalization condition in terms of $\psi_s(y)$ involves the trivial measure:

$$\int dy |\psi_s(y)|^2 < \infty. \quad (3.36)$$

### 3.5 Scalar Modes - Massless Case

In the gauge used in Section 3.1 and discussed in detail in Appendix A.2, where the tensor and vector massless fluctuations propagate two physical components each, the scalar metric fluctuations appearing in eq (3.1) are given by:

$$\chi, \; h_{\mu\nu} = 2\eta_{\mu\nu} \psi, \; \phi = -2\psi, \; A_\mu = 0. \quad (3.37)$$

The linear combinations $\zeta_1 = \psi - \chi/z$ and $\zeta_2 = z\chi$ satisfy eqs. (3.14). These equations can be easily solved in terms of two independent integration “constants” $F(x), G(x)$, which play the role of the two 4D scalar massless degrees of freedom:

$$\zeta_1 = F(x); \quad \zeta_2 = \frac{a'}{a^3} \int a^3(y) F(x). \quad (3.38)$$

To see how these modes are normalized in 4D terms, we insert (3.37) in the action (3.7), and after using the appropriate zero-mode scalar field equations, found in Appendix A.2 (namely eqs. (A.44), (A.45), (A.49) and (A.50)), the action reduces to a purely 4D kinetic term:

$$S_{kin}^{(S)} = \frac{1}{2k_5^2} \int d^4x dy a^3(y) [-6\partial^\mu \psi \partial_\mu \psi - \partial_\mu \chi \partial^\mu \chi]. \quad (3.39)$$

The fields appearing in eq. (3.39) are not yet canonically normalized in four dimensional terms, since they have a nontrivial $y$-dependence. To determine the actual normalization we must use the explicit form of the solution, eqs. (3.38). The result in terms of the $y$-independent fields $F(x), G(x)$ is not particularly illuminating. In the next Section we will use only its asymptotic forms close to the boundaries of space-time to determine whether the corresponding modes are indeed normalizable.

### 4. Hunting for Zero-modes

After setting the stage with the analysis of the previous section, we proceed to the main purpose of this paper: asking whether it is possible, in this setup, to have modes that resemble massless 4D gravitons, i.e. normalizable solutions of eq. (3.23) with $m^2 = 0$. If the answer is positive, at energies below the mass of the first massive excitation we can truncate the theory to an effective 4D theory of the massless states, by integrating over $y$. 
The requirement of normalizability is equivalent to asking that these modes have a finite 4D kinetic term (a finite value of the four-dimensional Planck scale) after the $y$-coordinate has been integrated out. Moreover, we would like to have zero-modes of spin-2 only, with no additional massless vectors or scalars.

The spectrum of fluctuations depends essentially on the form of the scale factor, described by $B(y)$. This function determines the potential entering the tensor and scalar perturbation equations, eqs. (3.25) and (3.34), as well as the respective normalizability conditions. The scale factor depends on the solution of the background Einstein’s equations, that in turn depends on the dilaton potential, entering the original action, (2.1). One could ask which kind of dilaton potentials generate solutions with a spectrum like the one we are looking for. However, in the 5D Einstein-Dilaton theory there is a huge simplification [10]: under certain conditions, discussed in the next subsection, given an arbitrary function $a(y)$, we are automatically guaranteed that there exists a dilaton potential $V_a(\Phi)$ such that a space-time with scale factor $a(y)$ is a solution of Einstein’s equations. Therefore we can parametrize our model by a choice of $a(y)$, or equivalently $B(y)$, and reverse-engineer $V(\Phi)$ at the end. This is much simpler, since the potentials in eqs. (3.25) and (3.34) and the normalization conditions are express directly as functions of the scale factor. At the end, we have to check that the solution we obtained does not contain pathologies, e.g. violate some positive energy condition.

To summarize, the strategy we follow is the following: we investigate in what geometries, parametrized by $B(y)$, the spin-2 Schrödinger equation, (3.25), admits normalizable zero-energy eigenstates, and at the same time the spin-0 and spin-1 massless modes, given by eqs. (3.38) and (3.29), respectively, are not normalizable.

We will restrict our analysis to asymptotically $AdS_5$ space-times, since in this case our setup can be given a holographic interpretation in terms of an asymptotically conformal 4D field theory. If we relax this assumption, an example of a spectrum like the one we are looking for can be found e.g. in the RSII model[7]: there is no dilaton, just a cosmological constant, and the scale factor and tensor potential take the form

$$B(y) = 3 \frac{k^2}{4} \log (\epsilon + k|y|), \quad V_t(y) = 15 \frac{k^2}{4} \left(\epsilon + k|y|\right)^2 - 3 \frac{k^2}{4} \delta(y)$$  \hspace{1cm} (4.1)

This potential supports a massless zero-mode graviton with wave-function peaked around $y = 0$, given by (as we will see in the next subsections):

$$\psi_{RS}(y) = e^{-B(y)} = \frac{1}{(\epsilon + k|y|)^{3/2}}$$  \hspace{1cm} (4.2)

This corresponds, in the holographic interpretation, to an elementary graviton, added to the CFT (cut-off at an UV energy scale $k/\epsilon$) in the UV of the theory. In our case, we are more interested in the possibility of realizing the graviton as a non-elementary state of the CFT, therefore we need the wave-function to vanish in the UV. In fact, if we remove the UV cut-off from the RS model, $\epsilon \to 0$, and let the space-time extend all the way to the $AdS_5$ boundary, we can see immediately from eq. (4.2) that the RS zero-mode becomes non-normalizable in the norm (3.28). On the other hand, the second independent zero-energy solution of eq. (3.25) behaves near the boundary as $y^{5/2}$, and can give rise to a
normalizable zero mode, depending on the large $y$ behavior of the scale factor. We will analyze this question in detail in the rest of the paper. We classify the geometries according to their properties at large $y$: the range of $y$ is either infinite, or the $y$ coordinates stops at some “end of space” point $y_0$, where there could be either a singularity or a boundary. Before starting this analysis of the zero-modes, we briefly review what is the degree of arbitrariness we can use in choosing the geometry through the function $B(y)$.

### 4.1 Energy Conditions and Constraints on $B(y)$

One of the features of the 5D model we are considering is that one can give a general classification of the possible behavior of fluctuations for the metrics of the form (2.2) without having first to specify the dilaton potential: the spectrum depends on a single function $B(y)$, that can be chosen fairly arbitrarily. In fact, given any choice of scale factor $a(y)$, or equivalently any function $B(y)$, so long as the condition:

$$\Phi_0'' = 2B'' + \frac{4}{3}B'^2 > 0,$$

(4.3)

is satisfied, we can always find a potential $V(\Phi)$ such that Einstein’s equations are solved by that choice of $B(y)$\[10\]. To see this, do a coordinate transformation so that the metric takes the form:

$$ds^2 = dr^2 + e^{-4B(r)/3} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \frac{dy}{dr} = e^{2B(y)/3}.$$

(4.4)

Then (2.4),(2.5) can be rewritten as (a dot denotes derivative w.r.t. $r$):

$$\dot{\Phi}_0^2(r) = 2\ddot{B}(r), \quad V(\Phi_0(r)) = 2\ddot{B}(r) - \frac{16}{3}\dot{B}^2(r).$$

(4.5)

If $\dot{\Phi}_0 \neq 0$, $\Phi_0 = \Phi_0(r)$ can be inverted to give $r = r(\Phi_0)$, and we can define a “superpotential” $W(\Phi) = (4/3)\ddot{B}(r(\Phi))$. From (4.3) we then obtain $V$ as a function of $\Phi$ as

$$V(\Phi) = \frac{9}{4} \left( \frac{\partial W}{\partial \Phi} \right)^2 - 3W^2.$$  

(4.6)

If $\dot{\Phi}_0$ vanishes at some isolated points, we can repeat the above procedure piecewise in every range of the $r$ coordinate in which $\dot{\Phi}_0(r) \neq 0$. So, if needed we can relax the condition (4.3) to include the equality sign. Thus, in what follows we can think of $B(y)$ as a function that we can choose at will, provided (4.3) is satisfied in the whole range of $y$ under consideration.

From the first of eqs. (4.3), we see that the condition (4.3) is equivalent to asking that the function $B(r)$ have non-negative second derivative. In terms of $B(y)$ this means that

$$\text{the function } \ddot{B}(r) \text{ must be non-decreasing.}$$

(4.7)

In terms of the conformal coordinate $y$ that we use throughout this paper, the function $B(r)$ is given explicitly by:

$$\ddot{B}(r) = B'(y)\dot{y} = B'(y) \exp[2B(y)/3].$$

(4.8)
Notice that condition (4.3) is nothing but the Null Energy Condition (NEC) for the space-times in consideration. The dilaton stress tensor is

\[ T_{AB} = \partial_A \Phi \partial_B \Phi - \frac{1}{2} g_{AB} \left( \partial^C \Phi \partial_C \Phi + V(\Phi) \right). \] (4.9)

The NEC requires that for any null vector \( v^A \), \( T_{AB} v^A v^B \geq 0 \) (see e.g. [8]). From eq. (4.9) this means, for our background \( \Phi = \Phi_0(y) \),

\[ (v^y)^2 (\Phi')^2 \geq 0. \] (4.10)

In order to obtain more general space-times that violate the NEC (if one were willing to do so), one would have to include “ghost” scalar fields, with opposite sign kinetic term. In this case the sign of the first term in (4.9) would be negative, and one would be able to consider space-times in which (4.3) is not satisfied. We will not consider this possibility any further in this work.

Let us look at the NEC in the case of asymptotically AdS space-time. Here, \( ds^2 \sim y^{-2} (dy^2 + dx_{\mu}^2) \) as \( y \sim 0 \), and we have

\[ B(y) \sim \frac{3}{2} \log y, \quad \dot{B}(0) \sim \frac{3}{2}. \] (4.11)

In particular, \( B'(y) \) cannot vanish at any finite positive value of \( y \), otherwise \( \dot{B} \) would also vanish there (see eq. (4.8)) and it would decrease from 3/2 to zero. Since \( B'(0) > 0 \), this implies that \( B'(y) > 0 \) for any finite \( y \), i.e. \( B(y) \) must be monotonically increasing. This is a necessary condition for the NEC to be satisfied.

One may wonder whether one could construct more general 5D background, using different kinds of bulk matter than just a single scalar field. For example, one could consider a generic 5D “perfect fluid”, with stress tensor of the form (if we limit ourselves to space-times that preserve 4D Lorentz invariance)

\[ T_{AB} = \begin{pmatrix} W \\ T_{\eta \nu} \end{pmatrix}. \] (4.12)

Different “equations of state” \( T = T(W) \) give rise to different solutions, exactly as in cosmological models. Einstein’s equations are

\[ 6 \left( \frac{a'}{a} \right)^2 = W, \quad 3 \frac{a''}{a} = T, \] (4.13)

and the conservation law obtained from these two equations is

\[ W' + 2 \frac{a'}{a} (W - 2T) = 0. \] (4.14)

For example, a linear relation \( T = \omega W \) gives rise to power-law solutions, \( W \propto a^{2(2\omega - 1)} \) and \( a(y) \propto y^{1/(1-2\omega)} \). Patching together different solutions in different \( y \) regions, we can obtain piece-wise continuous bulk geometries, with a different power-law in each region. We can
also do the same thing smoothly, by adding various different components characterized by different $\omega$’s: we would have transitions between different regions in each of which a different component dominates. From eq. (4.12) one can see that the null energy condition requires $\omega < 1$. Matter that violate this inequality is analogous to “phantom-like” matter in usual 4D cosmology, one particular realization being the “ghost” dilaton we mentioned above.

The parametrization (4.12) is in principle more general than having just a dilaton. However, the equations for the tensor fluctuations, eq (3.11), depend only on the scale factor, and we have seen that already with a single scalar field we can obtain any warped metric of the form (2.2) (therefore the most general $SO(3,1)$-preserving 5D metric) that does not violate the NEC. Therefore the results we obtain in the Einstein-Dilaton model for the spectrum of spin-2 modes are completely model-independent: they depend only on the geometry of the solution, no matter what is the specific matter or field content of the model. These details have an influence on the scalar sector of the fluctuations, whose corresponding equations (3.12), (3.19) contain the non-universal function $z(y)$.

4.2 Spin 2

We first look for spin-2 zero-modes, i.e. a transverse traceless tensor with two propagating d.o.f., whose profile satisfies eq (3.25) with $m^2 = 0$:

$$-\psi''_t + (B'^2 - B''_t) \psi_t = 0.$$  (4.15)

The two independent solutions are:

$$\psi_{t}^{UV}(y) = e^{-B(y)}, \quad \psi_{t}^{IR}(y) = e^{-B(y)} \int dy' e^{2B(y')},$$  (4.16)

where the labels indicate the region where the wave-function is peaked. The question of finding zero modes reduces to asking whether any of these solutions is normalizable for a given choice of the function $B(y)$.

In an asymptotically $AdS$ space-time, $B(y) \sim \frac{3}{2} \log ky$ as $y \sim 0$, and the two tensor modes behave for small $y$ as:

$$\psi_{t}^{UV}(y) \sim y^{-3/2}, \quad \psi_{t}^{IR}(y) \sim y^{5/2}.$$  (4.17)

This immediately makes $\psi_{t}^{UV}$ non-normalizable, and excludes it from the spectrum. Notice that this mode is the one that survives in RSII, where $AdS$ is cut-off before reaching the boundary. In the holographic description this is the mode that is relevant in the $UV$, and couples as a source to the CFT stress tensor. The other mode instead vanishes close to the boundary, and can be interpreted holographically as a mode that arises in the $IR$.

---

3Dilaton matter is described by a stress tensor of the form (4.12) with:

$$W = \frac{1}{2} ((\Phi')^2 - a^2 V(\Phi)),$$

$$T = -\frac{1}{2} ((\Phi')^2 + a^4 V(\Phi)).$$
The constant of integration in $\psi^{IR}_t(y)$ is fixed by normalizability around $y = 0$, so that the correct definition of the profile of this mode is:

$$\psi^{IR}_t(y) = e^{-B(y)} \int_0^y dy' e^{2B(y')}.$$  \hspace{1cm} (4.18)

Whether this function is normalizable or not depends uniquely on the behavior of $B(y)$ for large $y$. There are two possibilities:

- the $y$-coordinate extends to $+\infty$.
- space-time ends at a finite value $y = y_0$.

In the first case the resulting space-time is regular, in the second case there is either a singularity or a boundary at $y = y_0$.

4.2.1 Regular Space-times

Suppose the $y$-coordinate ranges from 0 to infinity. Then one can show immediately that the wave-function (4.18) is not normalizable, whatever is the large $y$ behavior of $B(y)$: $\psi^{IR}_t(y)$ always diverges as $y \to +\infty$:

- if $B(y) \to \text{const.}$, $\psi^{IR}_t(y) \sim y$

- if $B(y) \to -\infty$, then $\int_0^y \exp[2B(y')]$ must either diverge to $+\infty$ or go to a strictly positive constant as we take $y \to +\infty$, since it is a monotonically increasing function starting at zero at $y = 0$. In either case, the product of that integral with $\exp[-B(y)]$ diverges at $+\infty$.

- If $B(y) \to +\infty$ then the integral of $\exp[2B(y')]$ diverges much faster as $y \to +\infty$ than the pre-factor $\exp[-B(y)]$ goes to zero, making the product diverge.

These arguments show that the only case in which one can have a massless spin-2 in 4D is if the space terminates at some $y = y_0$. This is the case we will consider in the rest of this section.

4.2.2 Singular Spacetimes

The “end of space” singularity can be caused by $B(y)$ going to $+\infty$ (space-time shrinks to zero size) or to $-\infty$ (the scale factor blows up) at $y = y_0$, or by a divergence in $B'$ or $B''$ while $B(y_0)$ stays finite (in this case the metric remains finite but the curvature blows up). We will analyze these cases separately.

- $B(y)$ diverges as power-law

Suppose that around some $y_0 > 0$

$$B(y) \sim \frac{c}{(y_0 - y)^\beta}, \quad \beta > 0$$ \hspace{1cm} (4.19)

\[^4\text{this case includes } B(y) \to 0\]
where \( c \) is a constant that can be positive or negative. Let us first check what constraints we get from condition (4.3): around \( y_0 \), we have:

\[
\dot{B} \sim \frac{c^3}{(y_0 - y)^{3+1}} e^{(2c/3)(y_0-y)^{-3}},
\]

where we have used the expression of \( \dot{B} \) appearing in eq. (4.8). Suppose \( c < 0 \): then \( \dot{B} \to 0 \) as \( y \to y_0 \). Since around \( y = 0 \) \( \dot{B} \sim 3k/2 > 0 \), \( \dot{B} \) cannot be increasing everywhere, thus violating the condition (4.7). So we must take \( c > 0 \). But in this case one can immediately see that the integral in (4.13) diverges faster than the prefactor goes to zero.

\[
B(y) \text{ diverges logarithmically}
\]

Consider now the case that \( B(y) \) diverges logarithmically:

\[
B(y) \sim -\alpha \log(y_0 - y), \quad y \to y_0.
\]

Now we have

\[
\dot{B} \sim \alpha(y_0 - y)^{-2a/3-1}, \quad y \to y_0.
\]

We must require \( \alpha > 0 \) in order to ensure that \( \dot{B} \) does not decrease in some region. In this case we do find normalizable zero-modes in a certain range of \( \alpha \): around \( y_0 \) we have:

\[
\psi^I_{IR}(y) \sim (y_0 - y)^{\alpha} \left( \text{const} + (y_0 - y)^{-2\alpha+1} \right)
\]

which is normalizable if \( \alpha < 3/2 \). In this case, however, the spectrum of the model is not completely specified, and the existence of the zero mode requires special boundary conditions to be imposed at the singularity. This will be discussed in the next section.

With the warp factor behaving as in (4.21), the graviton potential \( V_t \) at \( y_0 \) behaves as

\[
V_t = B'^2 - B'' \sim \frac{\alpha^2 - \alpha}{(y_0 - y)^2} \quad y \sim y_0
\]

so it goes to \( +\infty \) if \( 1 < \alpha < 3/2 \), and to \( -\infty \) if \( 0 < \alpha < 1 \). According to eq. (2.7), the 5D scalar curvature goes like

\[
R \propto -e^{4/3} B V_t \sim -\frac{\alpha^2 - \alpha}{(y_0 - y)^{(2+4\alpha/3)}}
\]

so it goes either to \( -\infty \) (\( 1 < \alpha < 3/2 \)) or \( +\infty \) (\( 0 < \alpha < 1 \)). For the limiting case \( \alpha = 1 \) one must look at the subleading behavior at the singularity (the potential will diverge less dramatically, as \( (y_0 - y)^{-1} \)).

\[
B(y) \text{ finite}
\]

There is another possibility, namely that the space terminates at a point \( y = y_0 \) where the scale factor is finite but one of its derivatives diverges. This is still a curvature singularity, as one can see from eq. (2.7), and it means that we have to terminate our space-time at this point.
Suppose that as $y \sim y_0$

$$B(y) \sim B_0 + c(y_0 - y)^\beta \quad \beta > 0 \quad (4.26)$$

In order to have a singularity, there must be some derivative of $B$ that blows up at $y_0$, so we will assume $\beta < 2$.

In this case the tensor zero-mode (4.18) is clearly normalizable, as it approaches a finite value as $y \to y_0$.

**4.2.3 Regular Spacetimes with IR Boundary**

A final possibility that we have to mention is the case where there is no singularity at all, but rather there is a boundary at an arbitrary position $y = y_0$ where the space is cut-off. In this case all the modes of various spins that grow in the IR are automatically normalizable, and the spectrum is fully determined by the boundary conditions one imposes at $y = y_0$. This case avoids all the problems related to having to resolve the singularity, which is something that one eventually will want to do. However, from the point of view of the dual theory, the presence of a boundary at an arbitrary position means that we put an IR cut-off at an arbitrary energy scale. This IR modification is something that cannot be reconstructed from the knowledge of the fundamental theory and its dynamics, which on the other hand are encoded in the bulk geometry. On the other hand, the space ending at a singularity also signals some strong IR dynamics in the dual theory, but this IR modification is purely a consequence of the “microscopic” dynamics: the presence and the location of the singularity can be deduced from the defining parameters of the theory, which on the gravity side are encoded in the bulk dilaton potential and UV boundary conditions for the dilaton and the scale factor. This is the same reason why the authors of [24] use singular spaces rather than cut-off branes in the holographic approach to QCD: the singularity is seen as a dynamical, rather than ad hoc, way to terminate the space and having some nontrivial IR dynamics.

**4.3 Spin 1**

In this sector, according to eq. (3.17) we have a vector of the form:

$$A^\mu_T(x, y) = A^{(4)}_\mu(x) a^{-3}(y).$$

The actual presence of this mode as part of the spectrum depends on whether or not the normalization condition (3.31) is satisfied.

**UV behavior**

In the case we are considering, where the space has an asymptotic $AdS_5$-like boundary at $y \to 0$, we have $a^{-3}(y) \sim y^3$ as $y \sim 0$, so the integral in eq. (3.31) converges in the UV. To see whether the vector mode is there or not we must consider what happens in the IR.
IR behavior

In the cases where we found a normalizable massless spin-2 mode, i.e. the space has a curvature singularity at \( y = y_0 \), let us see what happens to the spin-1 normalization condition in the infrared:

- **\( B(y) \) diverges logarithmically**

  We have seen in the discussion of the spin-2 IR behavior that positivity of the dilaton kinetic energy and normalizability of the spin-2 zero-mode requires that close to \( y_0 \):

  \[
  B(y) \sim -\alpha \log(y_0 - y), \quad 0 < \alpha < 3/2. \tag{4.28}
  \]

  The spin-1 normalization condition, (3.31), becomes:

  \[
  \int_{y_0}^{y_0} dy \frac{1}{(y_0 - y)^{2\alpha}} < \infty, \tag{4.29}
  \]

  that is, \( \alpha < 1/2 \). So, in the range \( 1/2 < \alpha < 3/2 \) we find a spin-2 zero-mode but no spin-1 zero modes.

- **\( B(y) \) finite**

  In this case, condition (3.31) is clearly satisfied, and the massless spin-1 is normalizable.

4.4 Spin 0

There are two independent scalar zero-modes, parametrized in terms of the two functions \( F(x) \) and \( G(x) \) introduced in Section 3.5. From eqs. (3.20) and (3.38), the independent metric and dilaton scalar fluctuations are given by:

\[
\psi = \left( 1 - 2 \frac{a'}{a^4} \int^y a^3 \right) F(x) + \frac{a'}{a^4} G(x), \tag{4.30}
\]

\[
\chi = z(y) \frac{a'}{a^4} G(x) - 2z(y) \left( \frac{a'}{a^4} \int^y a^3 \right) F(x). \tag{4.31}
\]

To check normalizability we have to insert the above expressions in eq. (3.39).

UV behavior

We first check which of the two modes \( F, G \) is normalizable around \( y = 0 \). For small \( y \), \( a(y) \sim y^{-1} \) and eqs. (4.30), (4.31) become:

\[
\psi(x, y) \sim -3F(x) + y^2 G(x), \quad \chi(x, y) \sim -4z(y)F(x) + y^2 z(y)G(x) \tag{4.32}
\]

It is useful to write the expression for \( z(y) \) directly in terms of \( B(y) \). From eqs. (2.3) and (2.4) we have:

\[
\frac{z^2(y)}{4} = \frac{9}{4} \left( \frac{2B''}{B'^2} + \frac{4}{3} \right). \tag{4.33}
\]
In pure AdS we have $3B'' = -2B'^2$, and $z(y)$ vanishes identically. Therefore, the small $y$ behavior of the depends on the subleading terms in $B(y)$. Assuming, for small $y$,

$$B(y) \sim \frac{3}{2} \log ky + cy\gamma, \quad \gamma \geq 0 \quad (4.34)$$

where $c$ a constant, we find

$$z(y) \sim y^{\gamma/2}. \quad (4.35)$$

Inserting eqs. in the reduced action, (3.39), we see that the dominant contribution to the integral at $y = 0$ has the form

$$S[F,G] \sim \int_0^y dy a^3 \left[ -54(\partial_\mu F)^2 + 36y^2 \partial_\mu F \partial_\nu G - 6y^4 (\partial_\nu G)^2 \right], \quad (4.36)$$

from which we see immediately that the $F$-mode is non-normalizable in the UV, whereas the $G$-mode is.

**IR Behavior**

Having eliminated the $F$-mode due to its UV non-normalizability, we are left with fields $\chi$ and $\psi$ of the form

$$\psi = \frac{a'}{a^4}(y)G(x), \quad \chi = z(y)\frac{a'}{a^4}(y)G(x) \quad (4.37)$$

In what follows we consider only the cases where we found a normalizable spin-2 zero-mode to exist.

- **$B(y) \sim -\alpha \log(y - y_0), \quad 0 < \alpha < 3/2$**

  In this case, close to $y = y_0$, $z(y) \sim const$ and we find:

  $$S[G] \sim \int_{y_0}^y dy a^3 \left( \frac{a'}{a^4} \right)^2 (\partial G)^2 \sim \int_{y_0}^y (y_0 - y)^{2\alpha} \left[ \frac{(y_0 - y)^{4/3\alpha - 2}}{(y_0 - y)^{16/3\alpha}} (\partial G)^2 \right]$$

  $$\sim \int_0^y dy \frac{(\partial G)^2}{(y_0 - y)^{2\alpha + 2}}. \quad (4.38)$$

  The integral diverges and the $G$-mode is not normalizable in the IR.

To treat the situation when the scale factor at $y_0$ is finite but its first or second derivative diverge, it is useful to consider two separate cases:

- **$B(y) \sim B_0 + B_1(y_0 - y) + c(y_0 - y)^\beta, \quad 1 < \beta < 2$**

  In this case, close to $y_0$, $a'/a \propto B_1$ is finite but $a''/a$ diverges. From (2.3) and (2.4) we see that the function $z(y)$ can be written as:

  $$z^2 = 6 - \frac{a''}{a^2} \sim (y_0 - y)^{\beta - 2}, \quad y \sim y_0, \quad (4.39)$$

  We cannot have $B_1 = 0$ since this would imply $B' \to 0$ as $y \to y_0$ which, together with the finiteness of $B(y_0)$, violates condition (4.7).
and the effective action for the mode $G(z)$ is

$$S[G] \simeq \int_{y_0}^{y_0} dy a^3 \left( \frac{a'}{a^4} \right)^2 z^2(z)(\partial G)^2 \sim \int_{y_0}^{y_0} dy (y_0 - y)^{\beta - 2}(\partial G)^2,$$

which is finite at $y_0$ since we are assuming $\beta < 2$. Therefore in this case one of the two scalar modes is normalizable.

- $B(y) \sim B_0 + c(y - y_0)^\beta$, $0 < \beta < 1$

In this case both $a'/a$ and $a''/a$ diverge. Now we have:

$$z(y)^2 \sim (y_0 - y)^{-\beta},$$

and the effective action is again

$$S[G] \simeq \int_{y_0}^{y_0} dy a^3 \left( \frac{a'}{a^4} \right)^2 z^2(z)(\partial G)^2 \sim \int_{y_0}^{y_0} dy (y_0 - y)^{\beta - 2}(\partial G)^2;$$

but the last integral diverges, since $\beta < 1$, and there are no normalizable scalar modes in this case.

### 4.5 Summary and Discussion

The results of this section are summarized in Table 1. As announced in the introduction, we found that only if the fifth dimension terminates at a finite value of $y$ it is possible to have normalizable graviton zero-modes. In this case, the spectrum is purely discrete, since the tensor Schrödinger’s equation (3.25) is defined on the interval $(0, y_0)$. Therefore, there is a mass gap (generically of order $y_0^{-1}$) that separates the zero-mode from the massive KK modes. This is unlike what happens e.g. in RSII [7], where the KK tower is a continuum starting at $m^2 = 0$. To have a phenomenologically acceptable scenario, the masses of KK gravitons must be larger than of order $mm^{-1}$, unless these modes have suppressed coupling to matter compared with the zero-mode. We will discuss these issues in a concrete example in Section 6.

The end of space can be either a boundary at an arbitrary location, or a singularity. In the latter case the position of the singularity is not arbitrary, but it is determined by the background field equations and boundary conditions on the scale factor and dilaton. Depending on the type of singularity, we can have different kinds of normalizable massless modes, as can be seen in table 1. Notice that when a mode does not appear it is always because it is a non-normalizable 4D state, therefore its would-be 4D kinetic term is infinite and the mode decouples from the low energy sector. This is unlike what was happening in [6]: there, there was a massless scalar mode which had zero, rather than infinite, kinetic term, and did not appear at all in the action at quadratic level. This mode is expected to become strongly coupled when interactions are included. In our case instead there is no danger of having strongly coupled light scalar modes in the interacting theory.

Some of the cases analyzed here cannot be taken too seriously: take for example the logarithmic case, with $\alpha > 1/2$: if we look at the amplitude of the metric perturbation, we
$y \in (0, \infty)$ $y \in (0, y_0)$

| $B(y_0)$ | $-\alpha \log(y_0 - y)$ | finite + $(y_0 - y)^{\beta}$ |
|----------|--------------------------|-----------------------------|
| $0 < \alpha < 1/2$ | $1/2 < \alpha < 1$ | $0 < \beta < 1$ |
| $1 < \alpha < 3/2$ | $1 < \beta < 2$ | |

| Spin-2 | $(-\alpha \log(y_0 - y))^{\infty}$ | $(y_0 - y)^{\infty}$ |
| Spin-1 | $(-\alpha \log(y_0 - y))^{\infty}$ | $(y_0 - y)^{\infty}$ |
| Spin-0 | $(-\alpha \log(y_0 - y))^{\infty}$ | $(y_0 - y)^{\infty}$ |

| $V_t(y_0)$ | $-1/(y_0 - y)^2$ | $-1/(y_0 - y)^2$ | $+1/(y_0 - y)^2$ | $+\infty$ | $-\infty$ |
| $\psi_t(y)$ | $(y_0 - y)^{\alpha}$ | $(y_0 - y)^{1-\alpha}$ | $1/(y_0 - y)^{\alpha-1}$ | finite | finite |
| $R(y_0)$ | $+\infty$ | $+\infty$ | $-\infty$ | $-\infty$ | $+\infty$ |

Table 1: 4D massless spectrum as a function of the IR behavior of $B(y)$. A line $-$ or a circle $\circ$ indicate absence or presence, respectively, of the corresponding normalizable zero-mode. For the parameter ranges not shown in the table there is no normalizable spin-2 zero-mode. The last three lines indicate the behavior of the Shrödinger potential, the tensor mode wave-function and the scalar curvature near the end of the $y$-coordinate range.

find that the total metric is

$$g_{\mu\nu} = a^2 \left( \eta_{\mu\nu} + \left[ a^{-3/2} \psi_t(y) \right] h_{\mu\nu}(x) \right) \sim a^2 \left( \eta_{\mu\nu} + (y_0 - y)^{1-2\alpha} h_{\mu\nu}(x) \right),$$

so for $\alpha > 1/2$ the fluctuations blow up w.r.t. the background metric near the singularity and the perturbation theory expansion breaks down. This does not happen for $0 < \alpha < 1/2$ as well as for the finite scale factor cases, in which the perturbation always stays small.

5. Dealing with the Singularity

In the previous section we encountered different kinds of singularities, with different behavior of the curvature scalars (see Table 1). It would be desirable to study which of these singularities are intrinsically pathological, and which can be cured with a physically meaningful resolution.

As sometimes happen in AdS/CFT, the presence of a singularity in the bulk signals the occurrence of some nontrivial infrared dynamics of the dual theory [1], which once taken properly into account should produce a regular space-time [2]. This is expected to be generic in confining theories [3]. Equivalently, one can hope that the IR singularity will be regularized in a full string theory realization. In any case one should check that the results one obtains do not depend too much on the details of the regularization. On the field theory side this means that what happens to the IR of the theory can be reconstructed from the detailed dynamics of the microscopic degrees of freedom, encoded holographically in the bulk action and boundary conditions close to the $AdS$ boundary, where the bulk theory is under control.

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6This is analogous to the idea that given the fundamental QCD Lagrangian at some UV scale, one could in principle calculate the entire IR spectrum and dynamics without any further assumptions about the low energy physics.
In our case unfortunately the situation is not so simple. In some of the cases we consider the gravitational fluctuations stay finite all the way to the singularity, and bulk perturbation theory does not breakdown. However, we will see that the bulk dynamics plus boundary conditions are not enough to determine the low energy spectrum, but some additional information about the IR must be given as input, in the form of specific boundary conditions at the singularity. This makes the question of regularization subtle, since only for specific regularizations the results we achieve will be recovered. Therefore, our initial program of obtaining a massless spin-2 particle as part of the low energy spectrum of a well defined UV theory is only partially successful.

5.1 Boundary Conditions

The simplest example we found that allows spin-2 zero-modes is the one in which the space has an AdS-like boundary at \( y \sim 0 \) and \( B(y) \) diverges logarithmically as \( B \sim -\alpha \log(y-y_0) \) close to some point \( y = y_0 \): for \( 0 < \alpha < 3/2 \), equation (3.25) admits normalizable solutions with \( m^2 = 0 \). This is not the end of the story, however: consider again the eigenvalue equation for generic \( m^2 \):

\[
-\psi'' + \left( B'^2 - B'' \right) \psi = m^2 \psi. \tag{5.1}
\]

Close to the singularity at \( y = y_0 \), we can ignore the \( m^2 \)-term and the equation simplifies to

\[
\psi'' + \frac{\alpha^2 - \alpha}{(y_0 - y)^2} \psi \sim 0, \tag{5.2}
\]

which is solved by simple power-laws:

\[
\psi \sim c_1(y_0 - y)^\alpha + c_2(y_0 - y)^{1-\alpha}, \quad y \sim y_0. \tag{5.3}
\]

The crucial observation is that, in the range of \( \alpha \) in which we found zero-mode solutions, \( 0 < \alpha < 3/2 \), both terms in eq. (5.3) are square-integrable. This means that the spectrum is not yet determined by the data given so far, but we need to further specify boundary conditions at the singularity. Unless we do so, we would find that eq. (5.1) admits solutions for \( m^2 \) equal to any complex number\(^7\), and that the corresponding eigenfunctions are not orthogonal when inserted back into the action. To make the problem well posed, i.e. to ensure that the operator we are considering has a discrete, non-negative spectrum and orthogonal eigenfunctions, we have to specify boundary conditions at the singularity. It is this choice of boundary conditions that ultimately determines the spectrum\(^8\). In the case at hand, the boundary value of the eigenfunctions is not well defined (it is either zero or infinity, unless \( \alpha = 1 \)), but boundary conditions can be replaced by conditions on the

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\(^7\)More technically, we are dealing with a second order differential operator which is not self-adjoint, since we are taking its domain to be too large. To make it self-adjoint we need to restrict it on functions that obey specific boundary conditions at \( y \sim y_0 \). This is necessary since it is self-adjointness that guarantees discreteness of the spectrum and orthogonality of the eigenfunctions associated to different eigenvalues. See e.g. [33] for technical details.

\(^8\)These extra boundary conditions are not necessary at \( y \sim 0 \): there the two solutions behave as \( y^{5/2} \) and \( y^{-3/2} \), only the first one of which is normalizable: the normalizability requirement is enough to restrict the choice to the solution that vanishes at \( y = 0 \).
asymptotic behavior around $y = y_0$, i.e. by fixing the coefficients $c_1$ and $c_2$ that appear in eq. (5.3) [33].

From the discussion above, it is clear that a zero eigenvalue exists as part of the spectrum only if the right boundary conditions are imposed: these are the ones such that the zero-mode solution (4.18) satisfies them.

Next, we analyze what happens if we start from the boundary conditions for which the spectrum contains a spin-2 zero mode, and we slightly perturb them. Consider the asymptotic behavior around $y = y_0$ given by (5.3). For a specific value $c_1/c_2 = r_0$ the Schrödinger equation (5.1) admits a zero-energy solution. Now we study the same problem with a slightly modified boundary condition:

$$\frac{c_1}{c_2} = r_0 + \epsilon, \quad \epsilon \ll 1. \quad (5.4)$$

As in usual perturbation theory, we look for solutions of (5.1) of the form:

$$\psi(y) = \psi^{(0)} + \psi^{(1)} \quad (5.5)$$

where $\psi^{(0)}$ is the zero-mode, satisfying unperturbed asymptotic conditions (in practice $\psi^{(0)} \equiv \psi^{IR}$), and $\psi^{(1)}$ is taken to be small. Then, the equation for $\psi_1(y)$ is, to lowest order:

$$-\psi^{(1)''} + \left( B'^2 - B'' \right) \psi^{(1)} = m^2 \psi^{(0)}. \quad (5.6)$$

This is solved, formally, by

$$\psi^{(1)}(y) = m^2 \int dy' G_0(y, y') \psi^{(0)}(y'), \quad (5.7)$$

where $G_0(y, y')$ is the Green’s function of the one-dimensional problem with zero-eigenvalue:

$$G_0(y, y') = \frac{1}{2} \left[ \psi^{IR}(y) \psi^{UV}(y') \theta(y' - y) + \psi^{IR}(y) \psi^{UV}(y') \theta(y - y') \right], \quad (5.8)$$

and $\psi^{UV}$ and $\psi^{IR}$ are the independent solutions of the homogeneous equation, and are explicitly given by (4.16) [9].

Close to the singularity, $y \sim y_0$ only the first part of the Green’s function contributes, so the perturbed solution behaves asymptotically as:

$$\psi \sim \psi^{IR}(y) + m^2 \psi^{UV}(y) \int dy' |\psi^{IR}(y')|^2 = c_1^{(0)}(y_0 - y)^\alpha + c_2^{(0)}(y_0 - y)^{1-\alpha} + m^2(y_0 - y)^\alpha \quad (5.9)$$

One can verify immediately that (5.8) satisfies

$$-\partial_y^2 G_0(y, y') + \left( B'^2 - B'' \right) G_0(y, y') = \delta(y - y'),$$

using the Wronskian relation $\psi^{IR} \psi^{UV} - \psi^{IR} \psi^{UV} = 1$. The ordering is fixed requiring that the solution (5.7) is normalizable in the UV. There is an ambiguity from the fact that we can always add to $\psi_1$ a multiple of $\psi_0$ and still get a solution, since the latter is a zero-mode, but this shift is proportional to the unperturbed solution and it is therefore trivial. It can be fixed requiring e.g. that the unperturbed solution is normalized to one.
where $c_1^{(0)}$ and $c_2^{(0)}$ are the expansion coefficients of the unperturbed solution, satisfying by assumption $c_1^{(0)}/c_2^{(0)} = r_0$, and we have used the explicit form of $\psi^{UV}(y)$ in the last equality. Now we impose the perturbed boundary conditions, eq. (5.4) and find that the mass eigenvalue is given, to leading order, by:

$$m^2 = \epsilon c_2^{(0)}. \quad (5.10)$$

This result shows that a small deformation away of the boundary conditions lifts the zero-mode to a light massive mode. Nevertheless the choice of boundary condition is not continuous in the $\epsilon \to 0$ limit: we have seen in Section 4 that the massless and massive bulk fluctuations contain different number of degrees of freedom, at least for in the interesting cases, when no scalar modes are normalizable: for example, in the logarithmic case, there are at most four physical degrees of freedom in the massless sector, while as we have seen it takes a total of six to have a light massive multiplet. This means that the boundary conditions that give rise to a massless graviton are special, and once they are tuned that way the situation is completely stable, since it corresponds to a reduced number of physical modes.

One last comment. If we want to fix boundary conditions using the asymptotic behavior in (5.3), for any given scale factor $a(y)$ it is not clear how to specify the “zero-mode-friendly” boundary conditions a priori, and in a purely local way, since it appears that the subleading behavior of the function in eq. (4.18) (corresponding to the coefficient $c_1$, if $\alpha > 1/2$) carries information about the behavior of the $a(y)$ away from the singularity. An alternative, and equivalent way to impose boundary conditions is to relate the leading UV and IR behaviors: the condition

$$\lim_{y \to 0} \left[ e^{-2B} (e^B \psi)' \right] = \lim_{y \to y_0} \left[ e^{-2B} (e^B \psi)' \right], \quad (5.11)$$

or equivalently

$$\lim_{y \to 0} \left[ a^3 h' \right] = \lim_{y \to y_0} \left[ a^3 h' \right], \quad (5.12)$$

is sufficient to fix the spectrum, and it is obeyed by the zero-mode wave-function. It has the advantage that it does not require detailed knowledge of the scale factor in the bulk.

6. A Concrete Example

In this section we consider an explicit example which gives rise to IR-emergent massless gravitons. The fifth dimension terminates at a singularity at $y = y_0$, and we derive explicitly the special boundary conditions to be imposed at that point in such a way that a spin-2 massless mode is in the 4D spectrum.

Consider the simplest example when $B(y)$ has AdS-like asymptotics in the UV and logarithmically divergent in the IR:

$$B(y) = \frac{3}{2} \log ky - \alpha \log(1 - y/y_0), \quad 0 < \alpha < 3/2. \quad (6.1)$$
The background metric reads:

$$ds^2 = \left(1 - \frac{y}{y_0}\right)^{\frac{4}{3}} \left(dy^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \quad (6.2)$$

The space-time is asymptotically $AdS_5$ near $y = 0$, with $AdS$ curvature scale given by $k$. We saw in the previous sections that, provided suitable boundary conditions are imposed at the $y = y_0$ singularity, the massless spectrum in 4D contains just one spin-2 mode (if $1/2 < \alpha < 3/2$) or one spin-2 and one spin-1 mode (if $0 < \alpha < 1/2$). The massless graviton is given by:

$$h_{\mu\nu}(y, x) = a^{-3/2(y)} \psi_t(y) h^{(4)}_{\mu\nu}(x), \quad (6.3)$$

where the profile wave-function is

$$\psi_t(y) = e^{-B(y)} \int_0^y e^{2B(y)} = y_0(ky_0)^{3/2} F(y/y_0), \quad (6.4)$$

and $F(z)$ is a dimensionless function of $z = y/y_0$:

$$F(z) = \frac{(1 - z)^\alpha}{z^{3/2}} \int_0^z dz' \frac{z'^3}{(1 - z')^{2\alpha}}. \quad (6.5)$$

It behaves asymptotically as:

$$F_\alpha(z) \sim z^{5/2} \quad z \sim 0, \quad (6.6)$$

$$F_\alpha(z) \sim c_1(1 - z)^\alpha + c_2(1 - z)^{1-\alpha} \quad z \sim 1. \quad (6.7)$$

Near the singularity, at $z = 1$, we have:

$$h = a^{-3/2} \psi_t \sim \left[c_1 + O(1 - z)\right] + (1 - z)^{1-2\alpha} \left[c_2 + O(1 - z)\right] \quad (6.8)$$

We see that if $\alpha > 1/2$ the metric perturbation diverges close to the singularity, so strictly speaking a perturbative treatment is not fully justified. In the following we assume $\alpha < 1/2$: in this case $h(y)$ goes to a constant $c_1$ and the profile $\psi$ vanishes close to $y_0$.

Expanding eq. (6.3) near $y/y_0 = 1$ we can get the ratio $c_1/c_2$, which a posteriori determines what boundary condition we have to impose so that the zero-mode is in the spectrum (recall from the previous section that the boundary condition can be set precisely by specifying this ratio.) We find that the boundary conditions that one needs to impose to keep the zero-mode in the spectrum are:

$$c_1 = 3(1 - \alpha)(2 - \alpha)(2 - 3\alpha)(1 - 2\alpha)c_2 \quad (6.9)$$

Now suppose that “we” (i.e. the visible SM sector) are confined on a 3-brane at a fixed position $y = y_B$ in the geometry just considered. We are going to work in the probe-brane approximation, ignoring the backreaction of the SM brane on the geometry. Then, 4D gravity between brane-localized sources will be mediated by the graviton zero-mode. Higher KK modes of mass $m$ will contribute only below distances $1/m$. This mass scale is set by the parameter $y_0$, i.e. generically we will have $m^{-1} \sim y_0$, therefore if $y_0$ is small...
enough these modes will not contribute to the gravitational attraction at large distances. Below we will make this argument more precise.

Next, we determine the effective 4D gravitational coupling constant $G_N$ on the brane. The action for brane-matter is

$$S_{brane} = \int_{y=y_B} L_{brane}. \tag{6.10}$$

The 4D energy-momentum $T_{\mu\nu}$ couples canonically to the induced metric $\hat{g}_{\mu\nu}$, and is defined by

$$T_{\mu\nu} = -\frac{1}{\sqrt{-\hat{g}}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{6.11}$$

To linear order in the fluctuations, $\hat{g}_{\mu\nu} = a^2(y_B)(\eta_{\mu\nu} + h_{\mu\nu})$, so the brane source term, expanded to linear order in $h_{\mu\nu}$, has the form:

$$S_{brane} = \int \delta \hat{L}_{brane} \delta \hat{g}_{\mu\nu} = \int \sqrt{-\hat{g}} T_{\mu\nu} (a^2 h^{(4)}_{\mu\nu}). \tag{6.12}$$

The physical coordinates measured on the brane, in which the induced metric is just $\eta_{\mu\nu}$, differ from the bulk coordinate $x^\mu$ by a rescaling, $x^\mu \rightarrow a^{-2}(y_B)x^\mu$. After this rescaling, the effective 4D action including the brane source is:

$$S = \frac{1}{2k_5^2} \int dy d^4 x \frac{a^3(y)}{a^2(y_B)} \left( \partial h_{\mu\nu}(y, x) \right)^2 + \int_{y=y_B} d^4 x h_{\mu\nu}(y_B, x) T^{\mu\nu}(x)$$

$$= \frac{1}{2k_5^2 a^2(y_B)} \int dy \psi^2(y) \int d^4 x \left( \partial h^{(4)}_{\mu\nu} \right)^2 + a^{-3/2}(y_B) \psi(y_B) \int d^4 x h^{(4)}_{\mu\nu} T^{\mu\nu}(x), \tag{6.13}$$

where in the last line we have used eq. (6.3). Normalizing canonically the 4D graviton $h^{(4)}_{\mu\nu}(x)$ we can read off from eq. (6.13) the effective 4D Newton’s constant:

$$(8\pi G_N)^{1/2} = k_5 a^{-1/2}(y_B) \psi(y_B) \left( \int_{y_0}^{y_B} \psi^2(y) \right)^{1/2} = k_5 \sqrt{k} \frac{z_B^{1/2}}{(1-z_B)^{\alpha/3}} \frac{F(z_B)}{\mathcal{N}}, \tag{6.14}$$

where $z_B = y_B/y_0$ and $\mathcal{N} = \left( \int_0^1 F^2 \right)^{1/2}$. An example of the behavior of $G_N$ as a function of $z_B$ is displayed in figure 1. For small $z_B$ (i.e. if the brane is close to the $AdS$ boundary), $G_N$ vanishes as $z_B^3$, so gravity effectively decouples close to the boundary. In the dual interpretation, this corresponds to the fact that the graviton, being a composite, ceases to exist at high energy, and its interactions become soft in the UV.

Using the small $z$ behavior (6.6) in eq. (6.14), we can see that the effective 4D Plank scale, $M_p = 1/\sqrt{(8\pi G_N)}$, felt on a brane close to the $AdS$ boundary is:

$$M_p^2 \sim \frac{M^3}{k} \left( \frac{y_0}{y_B} \right)^6. \tag{6.15}$$

This expression does not depend on this specific example, as the UV behavior of the zero-mode wave-function is independent of what happens in the IR. So if we assume we live on a brane close to the UV boundary, the low-energy phenomenology of all the models we are
Considering has the same universal behavior, and will start to differ at energies when the lowest KK states become relevant. This will happen roughly at energies of order $1/y_0$, so the value of $y_0$ is constrained by small scale gravity measurement to be less than say 1mm. This can be easily achieved by taking, e.g. (if one is thinking about having the weak scale as the fundamental Planck scale in the bulk) $M_5 \sim k \sim 1/y_B \sim 1\, TeV$ and $y_0^{-1} \sim 2\, GeV$.

![Graph](Image)

**Figure 1:** Four Dimensional Newton’s Constant as a function of the position of the visible brane, for $\alpha = 0.25$

### 7. Conclusion and Possible Generalizations

We have investigated the presence of a massless four-dimensional graviton from a 5D covariant theory. This may also be interpreted as a massless spin-2 glueball in the spirit of bulk/boundary correspondence. Our (general) result is that in an asymptotically AdS$_5$ space-time with 4D Lorentz covariance, one cannot have normalizable massless 4D spin-2 modes unless the space-time terminates at either a brane or at a singularity. We worked in the context of Einstein-Dilaton gravity, but our result is completely general: the linearized equation for the tensor polarization is sensitive only to the background metric, and one can obtain any physically reasonable (i.e. not containing ghosts) background solution by choosing the dilaton potential appropriately. Therefore our result for the spin-2 modes does not depend at all on the field content of the 5D theory. One could go even further, and allow for space-times that do not satisfy the positivity condition of Sec. 4.1, as will be the case if we include scalar fields with “wrong” sign kinetic term, or the equivalent of a “phantom” perfect fluid matter in the bulk. This would give rise to more general possibilities for the background solution, which are not necessarily pathological: for example, if
the kinetic term of some scalar is controlled by the vev of another field, and has the wrong sign only in some limited region of space-time.

On the positive side, we did find many examples of solutions with normalizable massless spin-2 modes. They contain either an IR boundary, or a IR singularity. In both cases one is faced with the question of how to determine boundary conditions. The usual choices (e.g. Neumann b.c.) do not give a massless graviton. Despite appearances, the singular case seems more appealing than the “hard wall” boundary. From the AdS/CFT perspective, it provides a dynamical way to end space-time, which can be completely encoded in the UV data of the theory. Also, in many cases the singularity renders most or all the unwanted “extra” polarizations, that are usually present in a higher-dimensional theory, non-normalizable.

The presence of the singularity raises various questions, and one starts wondering about possible resolutions that preserve the interesting boundary conditions. It is unlikely that the usual methods - terminating the space-time with an Euclidean black hole horizon, or going to higher dimension and adding vanishing cycles - will work, since they usually lead to Neumann b.c, for which there is no graviton zero-mode\(^\text{10}\).

As the theory living in the bulk is supposed to be coming from a string theory, the singularity should be resolved in the full theory. One can ask whether the kinds of solutions we studied in this work admit a realization within string theory. There are several examples of string theoretical constructions in ten dimensions that give rise to asymptotically AdS\(_5\), singular backgrounds when they are reduced to D=5. Some of the resulting five-dimensional geometries fit in the class that, according to the analysis of this paper, potentially admit massless spin-2 modes as the only low-energy states in D=4. To quote one example, one of the solutions of 5D supergravity considered in \(^\text{35}\) (called “model B” in that paper) has an asymptotically AdS region\(^\text{11}\), and a singularity at finite \(y\) close to which the scale factor behaves like in our eq. (4.21), with \(\alpha = 1/2\): it therefore belongs in the class that, according to our analysis (see Section 4.5), admit a massless graviton localized in the IR. This geometry is a solution of a particular supergravity theory in 5D which comes from a truncation of D=10 supergravity on a background that arises as the near-horizon geometry of D3-branes uniformly distributed over an \(S^3\) \(^\text{30}\). Given that there are string theoretical configurations that give rise to similar singularities to the ones we considered, it would be interesting to study in these examples what kind of boundary conditions are obtained once the full string theoretical resolution of the singularity is reduced to the 5D effective theory.

Our result, that no massless four-dimensional gravitons arise in generic warped extra-dimensional models with AdS\(_5\) asymptotics, reflects the Weinberg-Witten no-go theorem \(^\text{26}\) in the dual field theory. This forbids the occurrence of massless spin-2 states in a 4D theory with a Lorentz-covariant stress tensor. This theorem is based on several

\(^{10}\) We can use for example the argument in \(^\text{34}\): the zero-mode wave equation can be written as \((a^3 h')' = 0\), so if \(a^3 h'\) is non-zero somewhere, it will be non-zero everywhere. Now, the normalizable zero-mode, eq. (4.18), has \(a^3 h' \neq 0\) at the AdS boundary, so it cannot satisfy Neumann b.c. \(h' = 0\) at any point with a finite value of the scale factor.

\(^{11}\) The particular setup considered in \(^\text{35}\) excised that region as in RS, and did not extend all the way to the AdS boundary, in order to render the constant graviton wave-function normalizable.
assumptions, and relaxing one or more of these assumptions is expected to give different results. It would be interesting to understand in which way the cases when we have indeed found a massless spin-2 state circumvent the theorem. One of the assumptions of the theorem, that gravity couples universally to the 4D field theory stress tensor, is certainly not satisfied in our models, since according to the standard AdS/CFT paradigm, the “fundamental” 4D field theory stress tensor is sourced by the non-normalizable graviton zero-mode. There are other assumptions we can relax to find more generic scenarios that could avoid the difficulties we found in this work, and may allow to consider non-singular space-times. We can envision at least three possible directions, all of which can be easily addressed in our formalism. They all correspond to relaxing some assumption of the Weinberg-Witten theorem.

\textit{Give up } \sigma^2 = 0

If we look for massive, rather than massless, spin-2 normalizable modes, the arguments in section 4 do not apply: these modes are of course present generically in our model no matter what the IR behavior of the scale factor, so we can deal with regular space-times that do not end. It is easy to obtain a purely discrete spectrum, for example choosing \( B(y) \sim y^2 \) in the IR, which is clearly allowed by the positivity condition. This gives a quadratic potential for the tensor fluctuations, and the requirement of normalizability at \( y = 0 \) and \( y = +\infty \) are enough to determine the spectrum without need of any extra boundary condition. The spectrum will look like the one needed for “linear confinement” in \( AdS/QCD \) \cite{37}. Of course, if we allow a mass for the graviton, we face a different class of phenomenological problems. The minimal requirement for the model to have some chance is that the graviton mass is tiny and there is large mass gap between the ground state (massive graviton) and the first excited KK states (in order for the theory to not become strongly coupled at low energies). This does not seem impossible to achieve, as we are not aware of any general argument that forbids it, but it is technically difficult. The main obstacle is the positivity requirement \( (1.7) \): if that is lifted, and we allow “wrong sign” kinetic terms for bulk scalars even in some small region, then most likely it will not be difficult getting such a spectrum, but in this case one can presumably adjust things so that the ground state is massless after all.

Assuming one can generate the correct hierarchy between the ground state and the excited states, one would have a very interesting model, which at low energy reduces to massive gravity, and whose UV completion and covariantizations are controllable. One can then explore unambiguously the questions such as strong coupling regime and deviation from GR in the field of large sources \cite{27, 28, 29}.

Another possibility connected to this line of reasoning, is generating a tiny graviton mass via "boundary" interactions with another universe, as was recently advocated in \cite{38, 39}. Indeed, in such a context the stress tensor is not conserved due to \( 1/N^2 \) effects, and this generates a mass for the graviton in a controllable fashion.

\textit{Give up Normalizability}
We can also ask whether we can obtain long-lived 4D spin-2 massless resonances, rather than one-particle states. Lifting the normalizability requirement allows once more to include regular space-times, with infinite range of $y$. One way this could work is having a space that looks like one of our “good” singular examples almost all the way to the singularity, and patch it continuously with some regular space that extends all the way to $y = +\infty$. This will turn the infinite potential wall (or well), felt at the singularity by the tensor wave-function, into a wall of finite height (or well of finite depth). Then the tensor Schrödinger potential appearing in eq. (3.25) would have to approach zero at $y = \infty$ as we need a continuum of light states for the resonance to decay. In this case we need the resonance to be very narrow, and to have significantly enhanced coupling to matter w.r.t. the other continuum modes.

Give up Lorentz Invariance

We have chosen to consider a 5D space-time with exact 4D Lorentz symmetry. If we give up this requirement the Schrödinger potential felt by the tensor fluctuations changes, as seen for example in [40] in the context of a pure $AdS$ bulk. To start with, one could limit the analysis to maximally symmetric 4D space-times, and consider 5D metrics that are conformally $R \times (A)dS_4$.

There are other phenomenological issues that one must clarify in our model. The first concerns the nonlinear interaction of the graviton zero-mode. It is not obvious a priori that, in the models we consider, these reproduce the graviton self-interactions of four-dimensional General Relativity, and this requirement is an additional constraint that realistic models should satisfy. For example, in this model, the coefficient $\kappa_g$ of the cubic self-interaction of zero-mode graviton, and the coupling constant governing the interaction of gravity with 4D matter (the effective Newton’s constant $G_N$), are two independent parameters: the first depends only on the bulk geometry, whereas the second depends also on the position of the brane. A phenomenologically viable model can result only if these two parameters are related in the same way as in ordinary 4D Einstein’s gravity, where $G_N = \kappa_g^2$. This relation can always be recovered by placing the observable brane at an appropriate position, and it would be interesting to see if there is some model in which this is achieved naturally. One could worry about higher order interactions, but the one arising from the cubic gravitational self-interaction is really the only constraint, as this is the only nonlinear term of the Einstein’s equation that is actually tested by direct gravity measurements.

Related to the question of higher order interactions is the fact that, above energies when it is possible to probe the extra dimension, the equivalence principle will generically be violated, because coupling of the zero-mode to the visible sector is universal only insofar the latter is confined at a fixed value of the $y$ coordinate. We will leave these issues for discussion in future work.

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APPENDIX

A. Quadratic Action and Linearized Perturbations

We define the fluctuations around the background (2.2) as:
\[
    ds^2 = a^2(y) \left( \eta_{AB} + h_{AB} \right)
    = a^2(y) \left[ (1 + 2\phi) dy^2 + 2A_\mu dy dx^\mu + (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \right],
\]
\[
    \Phi = \Phi_0(y) + \chi
\]
where $\phi$, $A_\mu$, $h_{\mu\nu}$ and $\chi$ are functions of $y, x^\mu$.

To write the action for these fluctuations at quadratic order it is convenient to use the conformal properties of the curvature scalar: for a metric $g_{AB} = e^{-A}\tilde{g}_{AB}$ we have:
\[
    \sqrt{-\tilde{g}}R = e^{-(D/2-1)A} \sqrt{-g} \left[ \tilde{R} + (D-1)\tilde{g}^{AB}\tilde{\nabla}_A\tilde{\nabla}_B \right] - \frac{(D-2)(D-1)}{4} \tilde{g}^{AB}\tilde{\nabla}_A\tilde{\nabla}_B \tilde{A} \tilde{A} \tilde{B} \tilde{B}.
\]

Specializing to $D = 5$, defining $e^{-A} = a^2$ and neglecting a total derivative we can rewrite the action, eq. (2.1), in the form
\[
    S = \frac{1}{2k_5^2} \int d^4x dy a^3(y) e^{-3/2 A} \sqrt{-g} \left[ \tilde{R} + 3\tilde{g}^{AB}\partial_A \partial_B \Phi - \tilde{g}^{AB}\partial_A \Phi \partial_B \Phi - e^{-A}V(\Phi) \right]
\]
Now we expand this action to quadratic order in the fluctuations (3.1), using $\tilde{g}_{AB} = \eta_{AB} + h_{AB}$ and expanding $\sqrt{-g}$ and $\tilde{R}$ around a flat background:
\[
    \tilde{g}_{AB} = \eta_{AB} + h_{AB}, \quad \tilde{g}^{AB} = \eta^{AB} - h^{AB} + h^{AC}h^B_C;
\]
\[
    \sqrt{-g} = 1 + \frac{1}{2}h - \frac{1}{4} \left( h_{AB}h^{AB} - \frac{1}{2}h^2 \right), \quad h \equiv h^A_A;
\]
\[
    \tilde{R}_{AB} = \partial^C \partial_{(A}h_{B)C} - \frac{1}{2} \partial^C \partial_{[A}h_{B]C} - \frac{1}{2} \partial_A \partial_B h
\]
\[
    + \frac{1}{2} h^{CD} \partial_A h_{BD} - h^{CD} \partial_C \partial_{(A}h_{B)D} + \frac{1}{4} (\partial_A h_{CD})(\partial_B h^{CD}) + \frac{1}{2} (\partial_D h^C_B)(\partial_D h_{AC})
\]
\[
    - \frac{1}{2} (\partial_D h^C_B)(\partial_C h_{AD}) - (\partial_D h^{CD})(\partial_{(A}h_{B)C}) + \frac{1}{2} \partial_D (h^{DC} \partial_C h_{AB})
\]
\[
    - \frac{1}{4} (\partial^C h)(\partial_C h_{AB}) + \frac{1}{2} (\partial^C h)(\partial_{(A}h_{B)C}.
\]
Indexes are contracted with $\eta_{AB}$; the last expression can be found e.g. in [41]. Plugging into (A.3), using the background equations and some integration by parts, one gets:
\[
    S = \frac{1}{2k_5^2} \int d^4x dy a^3(y)
\]
\[
    \left[ -\frac{1}{4} (\partial^C h_{AB})(\partial_C h^{AB}) + \frac{1}{2} (\partial^B h_{AB})(\partial_C h^{AC}) - \frac{1}{2} (\partial_A h)(\partial_C h^{AC}) + \frac{1}{4} (\partial_A h)(\partial^A h)
\]
\[
    + \Phi_0 h^{\prime} + 2 \Phi_0 h^{Ay} \partial_A \chi - \partial_A \chi \partial^A \chi - \frac{1}{2} a^2 \partial_\phi V \chi^2
\]
\[
    - (a^3(y))' \left[ \partial^C h_{Ay} h^C_A - (h^{Ay})' h_{Ay} + \frac{1}{2} \partial^A h_{Ay} \right].
\]

(A.5)
Using the decomposition (A.1), this becomes:

\[
S^{(2)} = \frac{1}{2k_5^2} \int d^4 x dy \, a^3(y) \left[ L^{(2)}_{\text{ein}} - \frac{1}{4} h^{\rho\sigma} h_{\rho\sigma} + \frac{1}{4} (h')^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \\
- \partial_\mu \partial^\mu \chi - \chi'^2 - \frac{1}{2} a^2 \partial_\mu^2 V \chi^2 - \partial^\mu \phi (\partial^\nu h_{\mu\nu} - \partial_\mu h) \\
+ 2 \Phi'_0 \phi' + \Phi'_0 h' \chi + 4 \Phi'_0 \phi' \chi + 2 \Phi'_0 A^\mu \partial_\mu \chi \\
\left. - (a^3 A^\mu)' [\partial_\mu h - \partial^\mu h_{\mu\nu}] \\
+ (a^3)' [-2 A_\mu \partial^\mu \phi - 2 \phi h' - 4 \phi'] \right],
\]

where from now on \( h \equiv h^\mu_\mu \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and

\[
L^{(2)}_{\text{ein}} = -\frac{1}{4} \partial^\mu h_{\rho\sigma} \partial_\rho h^{\sigma\mu} + \frac{1}{2} \partial^\mu h_{\rho\sigma} \partial_\rho h^{\sigma\mu} - \frac{1}{2} \partial^\mu h \partial_\rho h^{\rho\mu} + \frac{1}{4} \partial^\mu h \partial_\rho h^{\rho\mu},
\]

is the quadratic part of the 4D Einstein-Hilbert Lagrangian.

The field equations derived from this action are:

\[
(\mu \nu) \quad h''_{\mu\nu} + 3 \frac{a'}{a} h'_{\mu\nu} + \Box h_{\mu\nu} - 2 \partial^\rho \partial_\rho h_{\mu\nu} + 2 \partial_\mu \partial_\nu h + 2 \partial_\rho \partial_\rho \phi - 2 a^{-3} (a^3 \partial_\mu A_\nu)' \\
+ \eta_{\mu\nu} \left[ -h'' - 3 \frac{a'}{a} h' - \Box h - 2 \Box \phi + 6 \frac{a'}{a} \phi' + 6 \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi \\
- 2 a^{-3} (a^3 \Phi'_0 \chi)' + 2 a^{-3} (a^3 \partial_\mu A^\mu)' + \partial^\rho \partial^\rho h_{\rho\sigma} \right] = 0; \quad (A.6)
\]

\[
(\mu y) \quad \Box A_\mu - \partial^\rho h_{\nu\rho} \\
+ \partial_\mu \left[ -6 \frac{a'}{a} \phi + 2 \Phi'_0 \chi + h' - \partial^\nu A_\nu \right] = 0; \quad (A.7)
\]

\[
(y y) \quad \Box h + \partial^\rho \partial^\rho h_{\mu\nu} - 3 \frac{a'}{a} h' + 6 \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi \\
+ 6 \frac{a'}{a} \partial^\rho A_\nu - 2 a^{-3} (a^3 \Phi'_0 \chi)' + 4 \Phi'_0 \chi' = 0; \quad (A.8)
\]

\[
(Dil) \quad \chi'' + 3 \frac{a'}{a} \chi' + \Box \chi - \frac{1}{2} a^2 \partial_\mu^2 V \chi \\
- 2 a^{-3} (a^3 \Phi'_0 \phi)' + \Phi'_0 \phi' + \frac{1}{2} \Phi'_0 h' - \Phi'_0 \partial^\mu A_\mu = 0. \quad (A.9)
\]

### A.1 Massive Modes

This case was analyzed in detail in \cite{14}. If \( k^2 \neq 0 \) one can further decompose \( A_\mu \) and \( h_{\mu\nu} \) in irreducible representations of the 4D Lorentz group in the following way:

\[
A_\mu = \partial_\mu W + A^T_\mu, \quad \partial^\mu A^T_\mu = 0 \quad (A.10)
\]

\[
h_{\mu\nu} = 2 \eta_{\mu\nu} \psi + 2 \partial_\mu \partial_\nu E + 2 \partial_\rho V^T_{\nu\rho} + h^{TT}_{\mu\nu}, \quad (A.11)
\]
with $\partial^\mu V^T_\mu = \partial^\mu h^{TT}_\mu = h^{TT}_\mu = 0$; the fields $W, \psi$ and $E$ are additional Lorentz-scalars, to be added to the already defined scalar fluctuations $\phi$ and $\chi$ (see eq. (A.1); the fields $A^T_\mu$ and $V^T_\mu$ are Lorentz-vectors. If $k^2 \neq 0$ this decomposition is unique, i.e. one can invert the above relations and write all these fields as functions of $A_\mu$ and $h_{\mu\nu}$. The gauge transformation properties of these fields can be read-off from (3.3-3.4): under 

$$
(\delta x^\mu, \delta y) = (\xi^\mu, \xi^5) \equiv (\xi^T_\mu + \partial^\mu \xi, \xi^5), \quad \partial^\mu \xi^T_\mu = 0
$$

(A.14) 

the fields defined in (A.12-A.13) change according to:

$$
\delta \psi = -\frac{a'}{a} \xi^5, \quad \delta E = -\xi, \quad \delta V^T_\mu = \xi^T_\mu, \quad \delta h^{TT}_{\mu\nu} = 0, \quad \delta W = -\frac{a'}{a} \xi^5, \quad \delta A^T_\mu = \xi^T_\mu.
$$

(A.15)

The field equations (A.8-A.11) can split in separate equations involving the scalar, vector and TT-tensor only:

- tensor modes:

$$
h^{TT\nu}_{\mu\nu} + 3\frac{a'}{a} h^{TT\nu}_{\mu\nu} + \Box h^{TT}_{\mu\nu} = 0;
$$

(A.16)

- vector modes:

$$
\Box (A^T_\mu - V^T_\mu) = 0, \quad \left( a^3 (A^T_\mu - V^T_\mu) \right)' = 0;
$$

(A.17)

- scalar modes:

$$
0 = \psi'' + 3\frac{a'}{a} \psi' - \frac{a'}{a} \phi' - \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi + \frac{1}{3} a^{-3} \left( a^3 \Phi_0 \chi \right)',
$$

(A.18)

$$
0 = \phi + 2\psi - (W - E')' - 3\frac{a'}{a} (W - E'),
$$

(A.19)

$$
0 = \psi' - \frac{a'}{a} \phi + \frac{1}{3} \Phi_0 \chi,
$$

(A.20)

$$
0 = \Box \psi + 4\frac{a'}{a} \psi' - \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi - \frac{a'}{a} \Box (W - E') - \frac{1}{3} a^{-3} \left( a^3 \Phi_0 \chi \right)' - \frac{2}{3} \Phi_0 \chi',
$$

(A.21)

$$
0 = \chi'' + 3\frac{a'}{a} \chi' + \Box \chi - \frac{1}{2} a^2 \partial_\phi^2 V \chi - 2a^{-3} \left( a^3 \Phi_0 \phi \right)' + \Phi_0 \phi' + 4\Phi_0 \psi' - \Phi_0 \Box (W - E').
$$

(A.22)

Notice that only the combinations $A^T_\mu - V^T_\mu$ and $W - E'$ appear in the field equations. Also, notice that there are no massive vector modes.

Let us consider the scalar modes. Not all the equations are independent: indeed eq. (A.18) is a consequence of (A.20). One can solve the system eliminating $\phi$ using (A.20) and $\Box (W - E')$ using (A.21) in terms of $\psi$ and $\chi$. Substituting their expressions in (A.22) only on the following linear combination of $\psi$ and $\chi$ appears:

$$
\zeta = \psi - \frac{\chi}{z}, \quad z \equiv \frac{a\Phi_0}{a'},
$$

(A.23)
and equation (A.22) reduces to [14]:

\[
\zeta'' + \left(3 \frac{a'}{a} + 2 \frac{z'}{z}\right) \zeta' + \Box \zeta = 0,
\]

(A.24)

The fact that only a linear combination of \(\psi\) and \(\chi\) is determined by the field equations is a consequence of the gauge freedom \(\delta y = \xi^5(x, y)\): it can be used to set either \(\psi\) or \(\chi\) to an arbitrary function, as one can see from (3.6) and the first of (A.15), and only the gauge invariant combination \(\zeta\) has a physical meaning.

### A.2 Massless Modes

When looking for solutions of the form (3.9) with \(k^2 = 0\), the decompositions (A.12, A.13) become ambiguous: since all the terms on the l.h.s. (except the one proportional to \(\eta_{\mu\nu}\) are transverse and traceless. Equivalently, it is impossible to solve for \(E, W, V_{\mu}\) and \(\psi\) in terms of \(h_{\mu\nu}\) and \(A_{\mu}\), since the resulting expressions would contain inverse powers of \(\Box\), e.g. \(W = \Box^{-1} \partial^\mu A_\mu\). What we need is a decomposition of vectors and tensor along \(k_\mu\) and three other independent 4-vectors. If \(k_\mu\) is timelike, as in the previous subsection, one can take as a basis \(k_\mu\) and three spacelike vectors orthogonal to it, resulting in the decompositions (A.12, A.13). If \(k_\mu\) is null, on the other hand, the basis can be taken as \((k^+, k^-, k_i^\mu)\), \(i = 1, 2\), defined by the relations:

\[
k^+\equiv k_\mu, \quad k^- k^\mu = 1, \quad k^- k^\mu = 0, \quad k_\mu k^\mu = k_\mu k^\mu = \delta^i j \quad (A.25)
\]

One has the completeness relation:

\[
\eta_{\mu\nu} = k^+_{\mu} k^-_{\nu} + k^-_{\mu} k^+_{\nu} + k^i_{\mu} k^i_{\nu} \delta_{ij}. \quad (A.26)
\]

The expansions of a vector and a symmetric tensor along this basis take the form

\[
A_\mu = k^+_{\mu} A^+ + k^-_{\mu} A^- + \sum_{i=1,2} k^i_{\mu} A^i, \quad (A.27)
\]

\[
h_{\mu\nu} = k^+_{\mu} k^+_{\nu} h^{++} + 2 k^+_{(\mu} k^-_{\nu)} h^{+-} + k^-_{\mu} k^-_{\nu} h^{--} + \sum_{i=1,2} 2 k^+_i k^i_{(\mu} h^{+i} + \sum_{i=1,2} 2 k^-_i k^i_{(\mu} h^{-i} + \sum_{i,j=1,2} k^i_{(\mu} k^j_{\nu)} h^{ij}. \quad (A.28)
\]

Under an infinitesimal diffeomorphism with parameters \((\xi^\mu, \xi^5)\) we have:

\[
\delta h_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu - 2 \eta_{\mu\nu} \frac{a'}{a} \xi^5. \quad (A.29)
\]

It is useful to split \(h_{\mu\nu}\) in two parts, as

\[
h_{\mu\nu} = \tilde{h}_{\mu\nu} + 2 \eta_{\mu\nu} \psi, \quad (A.30)
\]

such that \(\tilde{h}_{\mu\nu}\) and \(\psi\) transform separately under 4D and 5D diffeomorphisms, that is:

\[
\delta \tilde{h}_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad \delta \psi = -\frac{a'}{a} \xi^5. \quad (A.31)
\]
From eq. (A.26), $\eta_{\mu\nu}$ contains only $+-$ and $ij$ components, so the only components that can appear in the definition of $\psi$ are $h^{++}$ and $h^{ij}$. These are also the only components in eq. (A.28) that transform under a $\xi_5$-diffeomorphism:

$$\delta h^{++} = -2a'd'_{\xi_5}, \quad \delta h^{ij} = -2a'd^{ij}_{\xi_5}. \quad (A.32)$$

For $\psi$ to transform correctly, it must be of the form $\psi = h^{i}_i/4 + A(h^{++} - h^{i}_i/2)$, for some real number $A$, since the second term is invariant; since $h^{++}$ transforms non-trivially under $\xi_\mu$-diffeomorphisms, we must take $A = 0$. So we have:

$$\psi = \frac{1}{4}h^{i}_i, \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{i}_i. \quad (A.33)$$

Notice that, in a decomposition like that of eq. (A.28), the $ij$ part of $\tilde{h}_{\mu\nu}$ is traceless.

In order to solve the field equations we first fix the gauge for $A_{\mu}$ and $\tilde{h}_{\mu\nu}$. Under a gauge transformation, we have:

$$\delta(\partial^{\mu}\tilde{h}_{\mu\nu}) = -\Box \xi_\nu - \partial_\nu \partial^{\rho} \xi_\rho, \quad \delta\tilde{h}^{\mu}_\mu = -2\partial_\nu \partial^{\rho} \xi_\rho, \quad \delta(\partial^{\mu}A_{\mu}) = -\partial^{\mu}\xi^{\mu}_5 - \Box \xi_5 \quad (A.34)$$

therefore

$$\delta \left( \partial^{\mu}\tilde{h}_{\mu\nu} - \frac{1}{2}\partial_\nu \tilde{h}^{\mu}_\mu \right) = -\Box \xi_\nu, \quad (A.35)$$

and we can set, by an appropriate choice of $\xi_\nu$ and $\xi_5$:

$$\partial^{\mu}\tilde{h}_{\mu\nu} - \frac{1}{2}\partial_\nu \tilde{h}^{\mu}_\mu = 0. \quad (A.36)$$

$$\partial^{\mu}A_{\mu} = 0. \quad (A.37)$$

In terms of the decompositions (A.28, A.28), this gauge corresponds to $A^- = h^{-+} = h^{-i} = 0$.

The gauge choice (A.36, A.37) is left intact by gauge parameters that satisfy $\Box \xi_\mu = \Box \xi_5 = 0$. This implies that $\xi_\mu$ and $\partial_\mu \xi_5$ can also be decomposed in a “light-cone” basis as:

$$\xi_\mu = k^{+}_\mu \xi^+ + k^-_\mu \xi^- + k^i_\mu \xi^i, \quad \partial_\mu \xi_5 = k^{+}_\mu \xi^5. \quad (A.38)$$

We can use this residual gauge freedom, to set $\tilde{h}^{++} = \tilde{h}^{-+} = \tilde{h}^{+i} = 0$ and $A^+ = 0$. This additional choice implies $\partial^{\mu}\tilde{h}_{\mu\nu} = \tilde{h}^{\mu}_\mu = 0$.

To summarize, we can fix the following gauge:

$$A_{\mu} = A^{T}_{\mu}, \quad \partial^{\mu}A_{\mu} = 0 \quad (A.39)$$

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + 2\eta_{\mu\nu} \psi, \quad \partial^{\mu}h_{\mu\nu} = 2\partial_\nu \psi, \quad h^{\mu}_\mu = 8\psi \quad (A.40)$$

$$\tilde{h}^{TT}_{\mu\nu} = \sum_{i,j=1,2} k^{ij}_\mu k^{ij}_\nu \left( h_{ij} - \frac{1}{2}\delta_{ij} h^{l}_l \right), \quad A^{T}_{\mu} = \sum_{i=1,2} k^{i}_\mu A^i. \quad (A.41)$$

Notice that $h^{TT}_{\mu\nu}$ and $A^{T}_{\mu}$ have components only along $i, j$, i.e. the directions orthogonal to the spatial momentum, so they have only two independent components each. This gauge
choice corresponds to keeping only the physical helicity states of the massless spin-2 and spin-1 fields.

Using these gauge conditions, and the fact that we are considering zero-modes of the 4D wave operator, the field equations \((A.8\text{-}A.11)\) become:

\[
\begin{align*}
(\mu\nu) & \quad h_{\mu\nu}'' + 3 \frac{a'}{a} h_{\mu\nu}' + \partial_\mu \partial_\nu (4\psi + 2\phi) - 2 \left( a^3 \partial_\mu A^T_{\nu} \right)' \\
+ \eta_{\mu\nu} \left[ -6\psi'' - 18 \frac{a'}{a} \psi' + 6 \frac{a'}{a} \phi' + 6 \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi \right] - 2a^{-3} \left( a^3 \Phi_0' \chi \right)' &= 0 \quad (A.42) \\
(\mu y) & \quad \partial_\mu \left[ -6 \frac{a'}{a} \phi + 2 \Phi_0' \chi + 6\psi' \right] = 0 \quad (A.43) \\
(yy) & \quad -12 \frac{a'}{a} \psi' + 3 \left[ a'' + 2 \left( \frac{a'}{a} \right)^2 \right] \phi - a^{-3} \left( a^3 \Phi_0' \chi \right)' + 2 \Phi_0' \chi = 0 \quad (A.44) \\
(Dil) & \quad \chi'' + 3 \frac{a'}{a} \chi' - \frac{1}{2} a^2 \partial_\mu^2 V \chi \\
-2a^{-3} \left( a^3 \Phi_0' \phi \right)' + \Phi_0' \phi' + 4 \Phi_0' \psi' = 0. \quad (A.45)
\end{align*}
\]

Taking the trace of eq. \((A.42)\) we see that the two lines of that equation have to vanish individually. The vanishing of the first line can be written in the “light-cone” basis as:

\[
\begin{align*}
(\mu\nu) & \quad \eta_{\mu\nu} \left[ -6 \frac{a'}{a} \psi' + 6 \frac{a'}{a} \phi' + 6 \left( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 \right) \phi \right] = 0 \quad (A.46) \\
(\mu y) & \quad \partial_\mu \left[ -6 \frac{a'}{a} \phi + 2 \Phi_0' \chi + 6\psi' \right] = 0 \quad (A.47) \\
(yy) & \quad -12 \frac{a'}{a} \psi' + 3 \left[ a'' + 2 \left( \frac{a'}{a} \right)^2 \right] \phi - a^{-3} \left( a^3 \Phi_0' \chi \right)' + 2 \Phi_0' \chi = 0 \quad (A.48) \\
2\psi + \phi &= 0 \quad (A.49)
\end{align*}
\]

The three terms in the equation above are independent, so have to vanish separately. This yields equations for the transverse tensor and vector zero-modes, and an equation for the longitudinal component of the vector:

\[
\begin{align*}
(\mu\nu) & \quad h_{\mu\nu}'' + 3 \frac{a'}{a} h_{\mu\nu}' = 0 \quad (A.47) \\
(a^3 A^T_{\mu})' &= 0 \quad (A.48) \\
2\psi + \phi &= 0 \quad (A.49)
\end{align*}
\]

The last equation eliminates one out of the three scalars \((\chi, \psi, \phi)\), so we are left with a total of six propagating fields, which is the correct number of degrees of freedom in our setup.

Eqs. \((A.46)\) and \((A.47)\) give immediately the tensor and vector profiles in the \(y\)-direction. To get the scalar modes wave-functions we need a bit more work. First, consider eq. \((A.43)\): if the 4-momentum is not strictly zero, the quantity in square brackets has to vanish. Combined with eq. \((A.49)\), this gives

\[
\psi' = -2 \frac{a'}{a} \psi - \frac{1}{3} \Phi_0' \chi. \quad (A.50)
\]
Substituting into eq. (A.44) and using repeatedly eq. (A.49), we arrive at:

\[ \psi = -\frac{1}{2} \chi - \frac{1}{2} \left( \frac{a}{a'} \chi \right)' \]  

(A.51)

As a consistency check, one can easily show that the two equations above imply the last remaining scalar equation, (A.45).

After some manipulations, the two equations (A.50, A.51) can be shown to be equivalent to the system:

\[ \zeta' = 0, \quad \left( \frac{a^4}{d^4} \zeta' \right)' = -2a^3 \zeta_1 \]  

(A.52)

\[ \zeta_1 \equiv \psi - \chi, \quad \zeta_2 \equiv \chi \]  

(A.53)

where \( z(y) \) is given by (2.3).

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