I. INTRODUCTION

The quantum de Finetti theorems [1–4] and post selection theorem [5] have a unique role in the analysis of quantum information processing tasks. These theorems give us a way to reduce permutation invariant quantum states to a more structured state, called the quantum de Finetti state. In general, we say that a state is of de Finetti-type if it is a convex combination of i.i.d. (independent identical distributed) states. de Finetti states are usually much easier to handle than general states due to their simple structure and therefore a reduction to such states can simplify calculations and proofs of various quantum information processing tasks. Indeed, one of the famous applications of reductions to de Finetti states is a proof which states that in order to prove security of quantum key distribution against general attacks it is sufficient to consider attacks on individual signals [5]. Other applications include quantum tomography [6] or the quantum reverse Shannon coding [7].

It is thus interesting to see whether such a theorem is unique for quantum theory or can be proven for more general theories. More specifically, we are interested in a theorem which will allow us to do a similar de Finetti reduction when considering conditional probability distributions. Any physical system can be described by a conditional probability distribution $P_{A|X}$ where $X$ is the input, or the measurement of the system, and $A$ is the output. In particular, $P_{A|X}(a|x)$ is the probability for outcome $a$ given that a measurement $x$ was made. Note that the system may have as many inputs and outputs as required and therefore we do not restrict the structure of the system by describing it as a conditional probability distribution.

Conditional probability distributions can be used to describe the operational behaviour of physical systems, including systems that might not conform with quantum theory, such as non-signalling systems. Consider for example a system shared by two parties, $P_{AB|XY}$, where $X$ and $A$ are, respectively, the input and output of the first party and $Y$ and $B$ of the second. We then say that the system is non-signalling if it cannot be used to communicate, i.e., the output of one party is independent of the input of the other. The famous PR-box [8] is an example for a non-quantum bipartite system which can be written as a (non-signalling) conditional probability distribution.

One of the main motivations for a de Finetti reduction for conditional probability distributions in general, and non-signalling systems in particular, is device independent cryptography [8, 9]. In device independent cryptography we consider the devices as black boxes, about which we know nothing. The security of protocols based on such systems can therefore rely only on the observed statistics. When analysing these protocols the quantum de Finetti theorems cannot be applied for they depend on the dimension of the quantum systems, which is unknown in device independent protocols. Several different non-signalling de Finetti theorems have been established recently [10–12], but they cannot be applied to these tasks either.

In this paper we prove a general de Finetti reduction theorem, from which we can derive several more specialized statements that are of interest for applications. The most simple and straightforward corollary is a de Finetti reduction which can be applied to permutation invariant conditional probability distributions.

The second corollary is a reduction which can be applied to a family of systems which is relevant for cryptographic protocols based on the CHSH inequality [13] or the chained Bell inequalities [14, 15]. In this corollary we can connect any system out of this family of systems to a special non-signalling de Finetti system $P_{AB|XY}^{CHSH}$. In contrast to the non-signalling de Finetti theorems of [11, 12] we do not assume anything about the internal structure of $P_{AB|XY}$, i.e., we do not need to assume non-signalling between any of the subsystems.

Up to date, almost all known device independent cryptographic protocols are based on the CHSH inequality or the more general chained Bell inequalities. For this reason we pay specific attention to systems which are relevant for such protocols. However, our theorem can be applied also to other families of systems which might be useful in future protocols. As an example of an application of our theorem we prove that for protocols which
are based on the violation of the CHSH and chained Bell inequalities it is sufficient to consider the case where Alice and Bob share our de Finetti system $\tau_{AB|XY}$. We do this by bounding the distance between two channels which act on conditional probability distributions.

The remainder of this paper is structured as follows. In Section II we present and explain the general de Finetti theorem and its corollaries. We then show in Section III how our de Finetti reductions can be used in applications and, in particular, how they can simplify security proofs of non-signalling cryptography. The proof for the de Finetti reduction theorem is given in the Appendix. We first explain an easier to follow proof that holds for systems with a CHSH symmetry (Appendix A) and then provide the general proof.

II. RESULTS

We start with some basic definitions. $A$ and $X$ denote random variables over $a \in \{0,1,...,l-1\}^n$ and $x \in \{0,1,...,m-1\}^n$ respectively. An $n$-partite system $P_{A|X}$ is a conditional probability distribution if for every $x \sum_a P_{A|X}(a|x) = 1$ and for every $a, x P_{A|X}(a|x) \geq 0$.

When we consider two different systems $P_{A|X}$ and $Q_{A|X}$ it is understood that both systems are over the same random variables $X$ and $A$. The set $\{1,2,...,n\}$ is denoted by $[n]$.

For stating the de Finetti reduction theorem we will need the following definitions.

Definition 1. Given a system $P_{A|X}$ and a permutation $\pi$ of its subsystems, we denote by $P_{A|X} \circ \pi$ the system which is defined by

$$\forall a, x \quad (P_{A|X} \circ \pi)(a|x) = P_{\pi(a)|\pi(x)}(a|x) .$$

An $n$-partite system $P_{A|X}$ is permutation invariant if for any permutation $\pi$

$$P_{A|X} = P_{A|X} \circ \pi .$$

As mentioned in the previous section, we say that a system is a de Finetti system if it is a convex combination of i.i.d. systems. Formally,

Definition 2. A de Finetti system $\tau_{A|X}$ is a system of the form

$$\tau_{A|X} = \int Q^{\otimes n}_{A_1|X_1} dQ_{A_1|X_1}$$

where $x_1 \in \{0,1,...,m-1\}$, $a_1 \in \{0,1,...,l-1\}$, $dQ_{A_1|X_1}$ is some measure on the space of 1-party systems and $Q^{\otimes n}_{A_1|X_1}$ is a product of $n$ identical 1-party systems $Q_{A_1|X_1}$. We are now ready to state the different de Finetti reductions. For simplicity we start by giving the first corollary of the more general theorem (Theorem 4). This corollary is the reduction for conditional probability distributions, which connects general permutation invariant systems to a specific de Finetti system.

Corollary 3 (de Finetti reduction for conditional probability distributions). There exists a de Finetti system $\tau_{A|X}$ where $x \in \{0,1,...,m-1\}^n$ and $a \in \{0,1,...,l-1\}^n$ such that for every permutation invariant system $P_{A|X}$

$$\forall a, x \quad P_{A|X}(a|x) \leq (n+1)^m(l-1) \tau_{A|X}(a|x) .$$

The de Finetti system $\tau_{A|X}$ is an explicit system which we construct in the proof of the general theorem in Appendix B. This theorem holds for every permutation invariant system $P_{A|X}$, not necessarily quantum or non-signalling. Note however that according to Definition 1 we consider permutations which permute the 1-party subsystems of $P_{A|X}$.

Corollary 3 is relevant for the case in which one considers permutation invariant conditional probability distributions $P_{A|X}$. However, if the systems one considers have additional symmetries $\mathcal{S}$ then we can prove a better de Finetti reduction — a reduction with a smaller factor and a special de Finetti system with the same symmetries $\mathcal{S}$.

In the following we consider a specific family of symmetries — symmetries between different inputs and outputs of the subsystems of $P_{A|X}$. Formally, the types of symmetries that we consider are described, among other things, by a number $d \leq m(l-1)$ which we call the degrees of freedom of the symmetry (see Appendix B for details and formal definition of the symmetries). The general theorem then reads:

Theorem 4 (de Finetti reduction for conditional probability distributions with symmetries $\mathcal{S}$). There exists a de Finetti system $\tau_{A|X}^\mathcal{S}$ where $x \in \{0,1,...,m-1\}^n$ and $a \in \{0,1,...,l-1\}^n$ such that for every permutation invariant system $P_{A|X}$ with symmetry $\mathcal{S}$ (with $d$ degrees of freedom)

$$\forall a, x \quad P_{A|X}(a|x) \leq (n+1)^d \tau_{A|X}^\mathcal{S}(a|x) .$$

For the case of no symmetry we have $d = m(l-1)$ from which Corollary 3 stated before follows. The proof of this theorem is given in Appendix B.

The symmetries $\mathcal{S}$ that we consider are of particular interest when considering cryptographic protocols which are based on non-signalling systems. For example, the systems which are being used in protocols which are

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1 Since we permute $a$ and $x$ together this is exactly as permuting the subsystems.

2 In contrast to systems $P_{AB|XY}$ which can also be permuted as $(P_{A|B|X|Y} \circ \pi)(ab|xy) = P_{AB|XY}(\pi(a)\pi(b)|\pi(x)\pi(y))$, as is usually the case in cryptographic tasks. For dealing with such systems we will consider a different reduction, stated as Corollary 3.
There exists a non-signalling de Finetti system $P_{AB|XY}$ which can be used to simplify such protocols. A test $\mathcal{T}$ appears to be rather restrictive, i.e. $d = 1$ for the CHSH symmetry. For pedagogical reasons, we also present a self-contained proof including an explicit construction of the system $\tau_{AB|XY}^{\text{CHSH}}$ in Appendix A.

Although the assumption about the symmetry of the systems in Corollary 6 appears to be rather restrictive, the statement turns out to be useful for applications. In the next section we show how the de Finetti reductions in general are useful and, in particular, how Corollary 6 can be used to simplify the analysis of protocols which are based on the violation of the CHSH inequality.

### III. APPLICATIONS TO CRYPTOGRAPHY

To illustrate the use of the de Finetti reductions stated in the previous section, we start by considering the following simplified application. Let $\mathcal{T}$ be a test which interacts with a system $P_{A|X}$ and outputs “success” or “fail” with some probabilities. We denote by $\Pr_{\text{fail}}(P_{A|X})$ the probability that $\mathcal{T}$ outputs “fail” after interacting with $P_{A|X}$. We consider permutation invariant tests, defined as follows.

**Definition 7.** A test $\mathcal{T}$ is permutation invariant if for all systems $P_{A|X}$ and all permutations $\pi$ we have

$$\Pr_{\text{fail}}(P_{A|X}) = \Pr_{\text{fail}}(P_{A|X} \circ \pi).$$

Using the de Finetti reduction in Corollary 6 we can prove upper bounds of the following type:

**Lemma 8.** Let $\mathcal{T}$ be a permutation invariant test. Then for every system $P_{A|X}$

$$\Pr_{\text{fail}}(P_{A|X}) \leq (n + 1)^{m(l-1)} \Pr_{\text{fail}}(\tau_{A|X}).$$

The proof of this lemma is given in Appendix C.

The importance of the de Finetti reductions is already obvious from this simplified example — if one wishes to prove an upper bound on the failure probability of the test $\mathcal{T}$, instead of proving it for systems $P_{A|X}$ it is sufficient to prove it for the de Finetti system $\tau_{A|X}$ and “pay” for it with the additional polynomial factor of $(n+1)^{m(l-1)}$. Since the de Finetti system has a nice i.i.d. structure this can highly simplify the calculations of the bound. Moreover, when considering security proofs one

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3 Here a permutation acts on the bipartite system as $(P_{AB|XY} \circ \pi)(ab|xy) = P_{AB|XY} (\pi(a)\pi(b)|\pi(x)\pi(y))$.

4 Intuitively, in the CHSH symmetry there is only one degree of freedom, i.e. $d = 1$, since we are only free to choose one value $p$ when defining the basic CHSH system given in Figure 1. Less symmetry implies more degrees of freedom.
usually finds that the bound on $\Pr_{\text{fail}}(\tau_{A|X})$ is exponentially small in $n$. If this is indeed the case then the polynomial factor of $(n+1)^m(l-1)$ will not affect the bound. That is, an exponentially small bound on $\Pr_{\text{fail}}(\tau_{A|X})$ implies an exponentially small bound on $\Pr_{\text{fail}}(\tau_{A|X})$.

While the notion of a test as discussed above allows for a simple treatment of cases where the failure can be directly defined as an event, it is unfortunately not directly applicable to security proofs for general cryptographic protocols, such as quantum key distribution. In order to prove security one usually needs to establish an upper bound on the distinguishing advantage between the applied protocol and an ideal protocol. Formally we describe the protocols by channels which act on the system and bound the distinguishing advantage between the two channels.

When considering quantum protocols this distinguishing advantage is given by the diamond norm \[13]. The distance between two channels $E$ and $F$ which act on quantum states $\rho_A$ is given by $\|E - F\|_\diamond = \max_{\rho_{AC}} \| (E - F) \otimes 1_{AC} \rho_{AC} \|$, where $\rho_{AC}$ is a purification of $\rho_A$ and $\| \cdot \|$ is the trace distance. Informally the idea is that in order to distinguish two channels we are not only allowed to choose the input state to the channels $\rho_A$, but also keep to ourselves a purifying system $\rho_C$.

Although the definition of the diamond norm includes a maximisation over all states $\rho_{AC}$, using the quantum post selection theorem, it was proven that when considering permutation invariant channels it is sufficient to calculate the distance for a specific quantum de Finetti state \[5]. Motivated by this we prove here a similar bound on a distance analogous to the diamond norm for channels which act on conditional probability distributions.

We consider here channels of the form $E : \{P_{A|X}\} \to \{P_K\}$ which interact with conditional probability distributions $P_{A|X}$ and output a classical bit string $k \in \{0,1\}^t$ of some length $t \geq 0$ with some probability $P_K(k)$. The probability distribution of the output depends on the channel $E$ itself and is given by the following definition.

**Definition 9.** The probability that a channel $E$ will output a string $k \in \{0,1\}^t$ when interacting with $P_{A|X}$ is

$E_K(k) = \sum_x \Pr_E(x) \sum_{a|x(x,a)=k} P_{A|X}(a|x)$

where $\Pr_E(x)$ is the probability that the channel $E$ will input $x$ to $P_{A|X}$ and $E(a,x)$ is the function according to which the output of the channel is determined. Analogously,

$E_{K|C}(k|c) = \sum_x \Pr_E(x) \sum_{a|x(x,a)=k} P_{A|XC}(a|xc) \cdot$

The connection between the channel and the system is illustrated in Figure 2

Before considering the distance between two such channels we need the following definition.

**Definition 10.** An extension\[7] of a system $P_{A|X}$ is a system $P_{AC|XZ}$ such that

$\forall z \in Z : P_{A|X}(a|x) = \sum_c P_{AC|XZ}(ac|xz)$.

We say that an extension $P_{AC|XZ}$ is non-signalling if the second marginal, $P_{C|Z}$ is also properly defined, i.e., it does not depend on $x$.

For simplicity (and since it is the relevant scenario for cryptography) we consider here only non-signalling extensions $P_{AC|XZ}$ of $P_{A|X}$.

**Definition 11.** The distance between two channels $E, F : \{P_{A|X}\} \to \{P_K\}$ according to the diamond norm is

$\|E - F\|_\diamond = \max_{P_{AC|XZ}} \| (E - F) \otimes 1_{AC} (P_{AC|XZ}) \|_1$,

where the maximisation is over all systems $P_{A|X}$ and all possible extensions of them and

$E \otimes 1_{P_{AC|XZ}} = E \otimes 1_{P_{A|XC}} \cdot P_{C|Z} = E_{K|C} \cdot P_{C|Z}$.

$F \otimes 1_{P_{AC|XZ}}$ is defined in a similar way.

A more explicit expression for the diamond norm is given in Equation (22) in Appendix \[14].

As in Definition 7 we say that a channel is permutation invariant if for all permutations $\pi$, $E(P_{A|X}) = E(P_{A|X} \circ \pi)$. In a similar manner we can also consider channels which are $S$ invariant.

**Definition 12.** We say that a mapping $\mu$ of $(a,x)$ to $(a',x')$ respects the symmetry $S$ if for every system $P_{A|X}$ with this symmetry $P_{A|X} = P_{A|X} \circ \mu$.

A channel $E$ is $S$ invariant if for every $P_{A|X}$ and every mapping $\mu$ which respects $S$ we have $E(P_{A|X}) = E(P_{A|X} \circ \mu)$.

\[5\] In quantum physics, a purification is a special case of an extension.
For example, when considering the CHSH symmetry in Definition 5 one such possible mapping \( \mu \) may map \((a_i, b_i, x_i, y_i) = (0, 0, 0, 0) \) to \((a_i, b_i, x_i, y_i) = (1, 0, 1, 1) \) for every \( i \in [n] \) (see Figure 1).

Using these concepts and the de Finetti reduction given in Theorem 1 we can prove the following bound on the diamond norm.

**Theorem 13.** For any two permutation invariant and \( \mathcal{S} \) invariant channels \( \mathcal{E}, \mathcal{F} : \{P_{A|X}\} \to \{P_K\} \)

\[
\|\mathcal{E} - \mathcal{F}\|_\diamond \leq (n + 1)^d \max_{\tau_{\mathcal{S}|AC}^S} \| (\mathcal{E} - \mathcal{F}) \otimes \mathbb{I} \|_1
\]

where \( d \) is the number of degrees of freedom of \( \mathcal{S} \) and \( \tau_{\mathcal{S}|AC}^S \) is a non-signalling extension of the de Finetti system \( \tau_{A,X}^S \).

The proof is given in Appendix C.

In particular, for the case of CHSH symmetry we get the following corollary.

**Corollary 14.** For any two permutation invariant and CHSH invariant channels \( \mathcal{E}, \mathcal{F} : \{P_{AB|XY}\} \to \{P_K\} \)

\[
\|\mathcal{E} - \mathcal{F}\|_\diamond \leq (n + 1)^4 \max_{\tau_{\mathcal{S}|ABC}^{CHSH}} \| (\mathcal{E} - \mathcal{F}) \otimes \mathbb{I} \|_1
\]

where \( \tau_{\mathcal{S}|ABC}^{CHSH} \) is a non-signalling extension of the de Finetti system \( \tau_{AB|XY}^{CHSH} \).

Corollary 14 implies that when proving security of cryptographic protocols based on the CHSH inequality it is sufficient to consider the case where Alice and Bob share the de Finetti system \( \tau_{AB|XY}^{CHSH} \). However, one still needs to take into account all possible non-signalling extensions of this bipartite system to a tripartite system \( \tau_{ABC|XYZ}^{CHSH} \) that includes the adversary, as can be seen from the maximisation over \( \tau_{ABC|XYZ}^{CHSH} \). These type of proofs can be done, as for example in [18].

We further emphasise that this does not imply that Alice and Bob’s system is a convex combination of i.i.d systems when including the adversary’s knowledge, but only from Alice and Bob’s point of view. This is in contrast to the stronger result achieved by the quantum post selection theorem [5]. However, due to the no-go theorems given in [20, 21] we know that such a stronger result is not possible in the more general scenario that we consider here.

Two additional remarks are in order. First, to use this corollary we must consider protocols which are invariant under the CHSH symmetry (and therefore the channel describing them will also be invariant under the relevant mappings). Fortunately, this invariance can be ensured by performing an additional step in the beginning of the protocol, called depolarisation [22]. The depolarisation procedure will not affect the correctness of the protocol and will make it invariant under the appropriate mappings \( \mu \). Such depolarisation procedures can also be constructed for other types of protocols such as protocols which are based on the chained Bell inequalities.

Second, note that since the system \( \tau_{CHSH}^{AB|XY} \) is not quantum but non-signalling, this result cannot be applied in a trivial manner to proofs where it is assumed that Alice and Bob’s statistics is restricted by quantum theory.

**CONCLUDING REMARKS AND OPEN QUESTIONS**

In this paper we introduced a general de Finetti-type theorem from which various more specialised variants can be derived. Interestingly, such theorems can be formulated even without relying on assumptions regarding the non-signalling conditions between the subsystems or the underlying dimension. In the general theorem, Theorem 1 we can also see how additional symmetries of the systems can affect the factor in the de Finetti reduction. This suggests that the same relationship might exist in the quantum variant of the theorem as well.

As an example for an application we showed how our theorems can be used to simplify the analysis of cryptographic protocols. Specifically, we explained how our theorem can be used in device independent protocols in which the parties are not assumed to be restricted by quantum theory. We hope that this approach will also be useful for quantum device independent information processing protocols in the future.

The techniques used to prove our theorems are different from the techniques used in previous papers to establish general de Finetti theorems. We therefore hope that our techniques will shed new light on de Finetti reductions in general as well as applications in device independent scenarios.

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[1] R. Hudson and G. Moody, Probability Theory and Related Fields 33, 343 (1976).

[2] G. Raggio and R. Werner, Helvetica Physica Acta 62,
Appendix A: A direct proof of the de Finetti reduction for systems with CHSH symmetry

In this section we give a direct proof of Corollary 13. The proof for the general theorem, Theorem 14, is given in the next section. Corollary 13 can also be proven easily by using the general theorem, however, we choose to give also this direct proof to help the interested reader understand the main ideas and insights of the proof which are already present in this simpler version.

In order to prove Corollary 13 we construct a specific de Finetti system \( \tau^\text{CHSH}_{AB|XY} \). As our de Finetti system we choose
\[
\tau^\text{CHSH}_{AB|XY} = \int dQ_{A_1B_1|X_1Y_1} \mathbf{Q}_{A_1B_1|X_1Y_1} \text{ to be a convex combination of systems } \mathbf{Q}_{A_1B_1|X_1Y_1} \text{ where } \mathbf{Q}_{A_1B_1|X_1Y_1} \text{ is the basic system given in Figure 1.}
\]

As a density measure we choose \( dQ \) to be uniform over all systems \( \mathbf{Q}_{A_1B_1|X_1Y_1} \) of this form, i.e., we integrate uniformly over different values of \( p \in [0, \frac{1}{2}] \).

We can write \( \tau^\text{CHSH}_{AB|XY} \) explicitly by using the following notation. For every \( a, b, x, y \in \{0, 1\}^n \) denote by \( N_{\text{CHSH}} \) the number of indices \( m \in [n] \) for which the foursome \( (a_m, b_m, x_m, y_m) \) fulfils the CHSH condition, i.e. \( a_m \oplus b_m = x_m \cdot y_m \). \( n - N_{\text{CHSH}} \) is then the number of indices for which the foursome \( (a_m, b_m, x_m, y_m) \) does not fulfil the CHSH condition.

We can now formally define \( \tau^\text{CHSH}_{AB|XY} \):

**Definition 15.** \( \tau^\text{CHSH}_{AB|XY} \) is the non-signalling system defined by
\[
\tau^\text{CHSH}_{AB|XY}(ab|xy) = \int dQ_{A_1B_1|X_1Y_1}(ab|xy) \mathbf{Q}_{A_1B_1|X_1Y_1} = \int_0^{\frac{1}{2}} \left( \frac{1}{2} - p \right)^n p^{n-N_{\text{CHSH}}} 2 dp .
\]

The de Finetti system is non-signalling since the systems \( \mathbf{Q}_{A_1B_1|X_1Y_1} \) are non-signalling for every value of \( p \) (see Figure 1).

**Lemma 16.** \( \forall a, b, x, y \quad \tau^\text{CHSH}_{AB|XY}(ab|xy) = 2^{-n} \left( \binom{n}{N_{\text{CHSH}}} \right)^{-1} \frac{1}{(n+1)} \).
For every permutation invariant system we can change the value of an entry. In how many such different ways can we change (a, b, x, y) such that both will have the same value as the given entry? Formally, we would like to have a lower bound on $P_{AB|XY}(ab|xy)$ due to the symmetry of $P_{AB|XY}$. Since the sum of all entries with particular inputs $x, y$ is 1 this will give us a bound on $P_{AB|XY}(ab|xy)$.

**Proof.** The integral above can be solved explicitly:

$$
\tau_{AB|XY}^{CHSH}(ab|xy) = \int_0^\frac{1}{2} \left( \frac{1}{2} - p \right)^{N_{CHSH}} p^{(n-N_{CHSH})} 2dp
$$

$$
= 2^{-n} \int_0^1 (1-q)^{N_{CHSH}} q^{(n-N_{CHSH})} dq
$$

$$
= 2^{-n} B(n-N_{CHSH}+1, N_{CHSH}+1)
$$

$$
= 2^{-n} \binom{n}{N_{CHSH}}^{-1} \frac{1}{n+1},
$$

where $B$ is the Beta function. Recall that $N_{CHSH}$ is a functions of $a, b, x$ and $y$ although we do not write it explicitly.

The following lemma gives us an upper bound on any entry $P_{AB|XY}(ab|xy)$ of every permutation invariant system $P_{AB|XY}$ with the CHSH symmetry.

**Lemma 17.** For every permutation invariant system $P_{AB|XY}$ with the CHSH symmetry

$$
P_{AB|XY}(ab|xy) \leq 2^{-n} \binom{n}{N_{CHSH}}^{-1}.
$$

The idea behind the proof of this lemma is to bound the value of a specific entry $P_{AB|XY}(ab|xy)$ by counting how many entries $P_{AB|XY}(\tilde{a}\tilde{b}|xy)$ must have the same value as $P_{AB|XY}(ab|xy)$ due to the symmetry of $P_{AB|XY}$. Since the sum of all entries with particular inputs $x, y$ is 1 this will give us a bound on $P_{AB|XY}(ab|xy)$.

**Proof.** Given $a, b, x, y$ imagine that we are placing a colored ball above each foursome $(a_i, b_i, x_i, y_i)$ as in Figure 3. If the foursome fulfills the CHSH condition we label it with a blue ball, otherwise with a red ball. With this picture in mind, the CHSH symmetry as in Definition 2 actually says that by changing two balls of the same color we do not change the value according to the probability distribution $P_{AB|XY}$.

Given a specific entry $P_{AB|XY}(ab|xy)$ we would like to know how many entries with the same inputs $x, y$ have to have the same value as the given entry. Formally, we would like to have a lower bound on

$$
\mathcal{N}(a, b, x, y) = \left| \{(\tilde{a}, \tilde{b}) \in \{0, 1\}^n \times \{0, 1\}^n | P_{AB|XY}(\tilde{a}\tilde{b}|xy) = P_{AB|XY}(ab|xy) \} \right|.
$$

How small can $\mathcal{N}(a, b, x, y)$ be? Or in other words, in how many ways can we change $a$ and $b$ while getting an entry $P_{AB|XY}(\tilde{a}\tilde{b}|xy)$ with the same value? We now prove

$$
\mathcal{N}(a, b, x, y) \geq 2^n \binom{n}{N_{CHSH}}.
$$

One way of changing $(a, b, x, y)$ to $(\tilde{a}, \tilde{b}, x, y)$ without changing the value of the entry is to change $(a, b, x, y)$ to $(\tilde{a}, \tilde{b}, x, y)$ such that both will have the same sequence of colored balls. For example, in Figure 3 we can change $(a_1, b_1, x_1, y_1) = (0, 0, 0, 0)$ to $(\tilde{a}_1, \tilde{b}_1, x_1, y_1) = (1, 1, 0, 0)$ since they have the same inputs $(x_1, y_1) = (0, 0)$ and both will be denoted by a blue ball (therefore according to the symmetry this change will not affect the overall value of the entry). In how many such different ways can we change $a$ and $b$? For every index $i \in [n]$ and every input bits $x_i, y_i$. 

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**FIG. 3.** Partition to CHSH quartets. If the foursome $(a_i, b_i, x_i, y_i)$ fulfills the CHSH condition it is denoted by a blue ball, otherwise by a red ball.
there are exactly two $a_i, b_i$ for which the CHSH conditions holds (i.e. blue ball) and two for which it does not (red ball). Therefore there are exactly $2^n$ different pairs of strings $(\tilde{a}, \tilde{b})$ such that $(a, b, x, y)$ and $(\tilde{a}, \tilde{b}, x, y)$ have the same sequence of colored balls and therefore $P_{AB|XY}(\tilde{a}b|xy) = P_{AB|XY}(ab|xy)$.

Changing $a$ and $b$ in different ways than the way given above will necessarily change the colors sequence. However, we can still prove using the permutation invariance of $P_{AB|XY}$ that for some specific changes the value of the entry will still stay the same. The specific changes that we consider are determined by permutations of the colored balls.

In order to understand how every different permutation of the balls $\pi$ is realised as a permutation on $x, y, a, b$ consider the example drawn in Figure 4. On the left side we see a permutation of the balls from Figure B. We start by filling up the columns for which there is no change in the color of the ball with the original columns. Then we pair the permuted balls such that each blue ball is replaced with a red ball, and permute the columns according to this paring. The permutation in the figure, for example, is just the permutation of indices $(1, 2)$ and $(4, 7)$. In general, every permutation of the balls can be described by such pairing and between every two different permutation we will have at least one index which will be permuted in one of them and not in the other.

For every permutation $\pi$ as described above we have

$$P_{AB|XY}(ab|xy) = P_{AB|XY}(\pi(a)\pi(b)|\pi(x)\pi(y))$$

(A1)

since the system permutation invariant. We will now show that due to the CHSH symmetry we also have

$$P_{AB|XY}(\pi(a)\pi(b)|\pi(x)\pi(y)) = P_{AB|XY}(\tilde{a}\tilde{b}|xy)$$

(A2)

where $\tilde{a}_\pi = a$ and $\tilde{b}_\pi$ is derived from $b$ by negation of all the bits which are being permuted in $\pi$.

To see that Equation (A2) is correct recall that the permutation $\pi$ permuted two columns $i, j$ only if for one of them the CHSH condition holds and for the other not. Therefore, if for example we had $a_i \oplus b_i = x_i \cdot y_i$ (and the index $i$ was permuted by $\pi$) then $(\pi(a))_i \oplus (\pi(b))_i \neq (\pi(x))_i \cdot (\pi(y))_i$. By definition $(\tilde{a}_\pi)_i = a_i$ and $(\tilde{b}_\pi)_i = \overline{b_i}$ and therefore we also know that $a_i \oplus (\tilde{b}_\pi)_i \neq x_i \cdot y_i$. Combining this with the CHSH symmetry and proceeding in the same way for all the indices that $\pi$ permutes, we get Equation (A2).

Combining Equations (A1) and (A2) we get

$$P_{AB|XY}(ab|xy) = P_{AB|XY}(\tilde{a}\tilde{b}|xy) .$$

Any different permutation $\pi$ will result in a different $\tilde{b}_\pi$ and therefore for any different permutation $\pi$ we have a different entry $P_{AB|XY}(\tilde{a}, \tilde{b}, x, y)$ with the same value as the original entry $P_{AB|XY}(a, b, x, y)$. Since there are $\binom{n}{N_{CHSH}}$ different permutations of the balls we have $\binom{n}{N_{CHSH}}$ different ways of changing $(a, b, x, y)$ to $(\tilde{a}, \tilde{b}, x, y)$.

We can now answer our original question and bound $\mathcal{N}(a, b, x, y)$. We can combine both of the ways given above to change $a$ and $b$ without changing the value of the entry according to $P_{AB|XY}$ (with or without changing the colors sequence). This implies that in total there are at least $2^n \times \binom{n}{N_{CHSH}}$ different ways of changing $a$ and $b$ and we can conclude that

$$\mathcal{N}(a, b, x, y) \geq 2^n \binom{n}{N_{CHSH}} .$$

(A3)

Since for all $x, y$ $\sum_{a, b} P_{AB|XY}(ab|xy) = 1$, we get from Equation (B18) the following bound on the value of $P_{AB|XY}(ab|xy)$:

$$P_{AB|XY}(ab|xy) \leq 2^{-n} \binom{n}{N_{CHSH}}^{-1} .$$
We can now prove Corollary 6 directly.

A direct proof of Corollary 6. By combining Lemma 16 and Lemma 17 we get Corollary 6.

The above proof for systems which have the CHSH symmetry can be also applied to systems which have the symmetry induced by the more general chained Bell inequalities in a similar way. Since the number of measurements of the basic systems $Q_{A_i|X_1}$ does not play a role in the structure of our de Finetti system (see Lemma 16) the same bounds exactly will hold for systems with the chained Bell inequality symmetry.

Appendix B: Proof of the general de Finetti reduction

In this section we prove our most general de Finetti reduction, given in Theorem 4. The proof proceeds along the same lines as the direct proof of Corollary 6 in the previous section. We start by explaining the types of symmetries $\mathcal{S}$ that we deal with and how to construct the appropriate de Finetti system $\tau_{A_1|X}^S$. We then give a lower bound on the entries of the de Finetti system, analogously to Lemma 16, and an upper bound on the entries of a permutation invariant system $P_{A|X}$ with the symmetry $\mathcal{S}$, analogously to Lemma 17. Using these two bounds we get Theorem 4.

Symmetries and de Finetti systems

A symmetry $\mathcal{S}$ is a set of conditions. We say that a system $P_{A|X}$ has a symmetry $\mathcal{S}$ if it fulfills all of these conditions.

For any symmetry $\mathcal{S}$ that we consider we define a different de Finetti system $\tau_{A_1|X}^S$ of the form $\int Q_{A_1|X_1}^S dQ_{A_1|X_1}$. When defining such a de Finetti system for a specific type of symmetry $\mathcal{S}$ we are free to choose the measure $dQ_{A_1|X_1}$ as we like. The key idea is to choose the structure of the systems $Q_{A_1|X_1}$ on which we integrate in such a way that it “encodes” the symmetry $\mathcal{S}$ that we consider.

For example, assume we consider a family of systems $P_{A|X}$, with $a, x \in \{0, 1\}^n$, which has the following type of symmetry $\mathcal{S}$:

$$\forall i \in [n] \quad \forall a_\tau, x \quad P_{A|X}(a_\tau 0, |x) = P_{A|X}(a_\tau 1, |x)$$

(that is, given $a_\tau = a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ and $x$ the probability that the $i$'th bit $a_i$ will be 0 or 1 is the same). We then want $Q_{A_1|X_1}$ to have the following property:

$$\forall x_1 \in \{0, 1\} \quad Q_{A_1|X_1}(0|x_1) = Q_{A_1|X_1}(1|x_1)$$

and we say that $Q_{A_1|X_1}$ encodes the symmetry $\mathcal{S}$.

For the more general treatment it will be easier to start by defining the allowed structure of the system $Q_{A_1|X_1}$ and from it deduce the different types of symmetries and systems that we consider.

Allowed $Q_{A_1|X_1}$ systems

Consider a 1-party system $Q_{A_1|X_1}$ where $x_1 \in \{0, 1, \ldots, m-1\}$ and $a_1 \in \{0, 1, \ldots, l-1\}$. We can think about $Q_{A_1|X_1}$ as $m$ vectors of size $l$. We call each of the $l$ long vectors an input vector, since it describes the probability distribution of the outputs, given a specific input (see Figure 5). Defining a system $Q_{A_1|X_1}$ then reduces to defining its input vectors.

Keeping in mind that we will need to integrate over $Q_{A_1|X_1}$ to get a de Finetti system, we fill in the entries $Q_{A_1|X_1}(a|x)$ of the input vectors with different parameters $\{p_1, p_2, \ldots, p_d\}$, while making sure that the sum of the

![FIG. 5. A system $Q_{A_1|X_1}$ with $m=2$ inputs and $l=3$ outputs](image-url)
we give 2 examples for allowed $Q_{A_1|X_1}$ systems. In other words, the only thing needed for our theorem to hold is a pair $A \in S$. To the left, the two input vectors are permutations of one another. To the right, in each input vector we have a different independent parameter.

![FIG. 6. Different ways of filling an input vector of length 3 with $d$ parameters](image)

![FIG. 7. Two examples for allowed $Q_{A_1|X_1}$ systems. To the left, the two input vectors are permutations of one another. To the right, in each input vector we have a different independent parameter.](image)

input vector is 1 for every value of the parameters $p_i$. The number of parameters $d$ that we use to define $Q_{A_1|X_1}$ quantifies the number of degrees of freedom that $Q_{A_1|X_1}$ has, and it is bounded by $(l-1)m$. Figure 6 shows two different ways of filling in an input vector of length 3 with parameters.

We can now define a specific set of allowed systems $Q_{A_1|X_1}$.

**Definition 18.** A system $Q_{A_1|X_1}$ is said to be allowed if given any two of its input vectors, they are either a permutation of one another or they have a completely different set of parameters.

In Figure 7 we give 2 examples for allowed $Q_{A_1|X_1}$ systems with 2 inputs and 3 outputs.

The symmetry $S$ behind $Q_{A_1|X_1}$

When considering a specific system $Q_{A_1|X_1}$, it is easy to say which set of conditions it fulfills, i.e., which symmetry $S$ it encodes. For example, the system to the right in Figure 7 encodes the following symmetry of a system $P_{A|X}$ with $a \in \{0,1,2\}^n$ and $x \in \{0,1\}^n$,

$$\forall i \in [n] \quad \forall x, a \in \mathbb{F}_2 \quad P_{A|X}(a_0 | x) = P_{A|X}(a_1 | x)$$

More generally, the symmetry $S$ can be constructed from $Q_{A_1|X_1}$ as follows.

**Definition 19.** Given a system $Q_{A_1|X_1}$ as above the symmetry $S$ is defined by the following symmetry conditions:

For all $i \in [n]$, for all $x, a$ and $x', a'$ where $a' = a_1 \ldots a_{i-1} a'_{i+1} a_{i+2} \ldots a_n$ and $x'$ is defined in a similar way, if $Q_{A_1|X_1}(a_i | x) = Q_{A_2|X_1}(a'_i | x')$ then $P_{A|X}(a | x) = P_{A|X}(a' | x')$.

That is, if we change the pair $(a_i, x_i)$ to some $(a'_i, x'_i)$ of the “same type” according to $Q_{A_1|X_1}$, then the probability according to $P_{A|X}$ does not change.

In Definition 19 we started from the system $Q_{A_1|X_1}$ and derived the symmetry $S$. However, given a set of conditions $S$ one can also try to construct a system $Q_{A_1|X_1}$ which fulfills them. For every symmetry $S$ for which a system $Q_{A_1|X_1}$ can be constructed such that the condition in Definition 19 holds our proof can be applied. In other words, the only thing needed for our theorem to hold is a pair $(S, Q_{A_1|X_1})$ with the desired relationship.

The de Finetti system — integration over $Q_{A_1|X_1}$

Given a specific structure of $Q_{A_1|X_1}$ as previously described, we can now perform the integration over a tensor product of $n$ such systems and get a de Finetti system. As mentioned before, we are free to choose the measure $dQ_{A_1|X_1}$, with which we perform the integration.

For simplicity, we only consider $Q_{A_1|X_1}$ systems in which all the input vectors are permutations of one another (recall Definition 18). It will later become clear, that if we have more independent input vectors, then we can just multiply the different integrals by one another. In general, due to our proof technique, our entire proof can be applied independently for each set of permuted input vectors and then combined in the end to one proof by multiplying the results.
FIG. 8. Input vector with $l = 6$ and $d = 2$. In this example, $t_1 = 2$, $t_2 = 3$ and $t_3 = 1$. We then have $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{3} (1 - 2p_1)$.

In the rest of the proof, we use the following notations. Given the system $Q_{A_i|X_i}$ we denote by $t_i$, for $0 \leq i \leq d$, the number of times the parameter $p_i$ appears in each input vector of $Q_{A_i|X_i}$. In addition, we define $t_{d+1}$ to be the number of times the “unfree” entry appears in the input vector. Using this notation, we can set the range of the parameter $p_i$ to be $[0, c_i = \frac{1}{t_i} \left(1 - \sum_{j<t_i} t_j p_j \right)]$. As an example, consider the input vector in Figure 8, $p_1$ appears two times and therefore $t_1 = 2$ and $c_1 = \frac{1}{2}$. Indeed, in order for this input vector to be a valid probability distribution we must have $p_1 \in [0, \frac{1}{2}]$. For $p_2$ we have $t_2 = 3$ and $c_2 = \frac{1}{3} (1 - 2p_1)$, and $t_3 = 1$.

Next we define the following “coloring” function:

$$
C(a_j, x_j) = \begin{cases} k & Q_{A_i|X_i}(a_j|x_j) = p_k \\ d + 1 & \text{otherwise} \end{cases} \quad (B1)
$$

For every pair of strings $(a, x)$, where $a \in \{0, 1, \ldots, l-1\}^n$ and $x \in \{0, 1, \ldots, m-1\}^n$, we denote by $N_i$ the number of indices $j \in [n]$ for which $C(a_j, x_j) = i$. Using this definition we have $N_{d+1} = n - \sum_{j=1}^{d} N_j$.

Using all the notation above, we can now define our measure to be

$$
dQ_{A_i|X_i} = \frac{dp_1}{c_1} \frac{dp_2}{c_2} \cdots \frac{dp_d}{c_d}
$$

and use it to define our de Finetti-type system.

**Definition 20.** For any symmetry $S$ and the matching system $Q_{A_i|X_i}$ as above, the de Finetti system $\tau^S_{A_i|X_i}$ is defined by

$$
\tau^S_{A_i|X_i}(a|x) = \int Q^n_{A_i|X_i}(a|x) dQ_{A_i|X_i} \equiv \\
\int_0^{c_1} \frac{dp_1}{c_1} \int_0^{c_2} \frac{dp_2}{c_2} \cdots \int_0^{c_d} \frac{dp_d}{c_d} \prod_{j=1}^{N_i} p_1^{N_{d+1}} p_2^{N_{d+1}} \cdots p_d^{N_{d+1}} \left[ \frac{1}{t_{d+1}} \left(1 - \sum_{j=1}^{d} t_j p_j \right) \right]^{n-\sum_{j=1}^{d} N_j}.
$$

**Lower bounding the de Finetti system**

The following lemma is the analogous of Lemma 16 in the previous section.

**Lemma 21.** The following lower bound on $\tau^S_{A_i|X_i}(a|x)$ holds

$$
\forall a, x \quad \tau^S_{A_i|X_i}(a|x) \geq \prod_{j=1}^{d+1} \left( \frac{1}{t_j} \right)^{N_j} \left( \frac{n}{N_1, \ldots, N_{d+1}} \right)^{n-1} \frac{1}{(n+1)^d}
$$

where $(N_1, \ldots, N_{d+1})$ is a multinomial coefficient.

Before we continue to the proof of Lemma 21, note that although we have chosen a specific ordering of the parameters in the integration in Definition 20, this ordering does not affect the bound in Lemma 21. Moreover, this bound is optimal in the sense that there is always at least one pair of strings $(a, x)$ for which the equality is reached, and this pair is independent of the chosen order of the integration.

---

6 Note that the $N_i$’s are functions of the strings $a$ and $x$.

7 Remember that the $c_i$’s are functions of other parameters, therefore $c_1 \cdots c_d$ is not a constant and not even symmetric regarding the different parameters.
Proof. In the proof we use the following formula:

\[
\forall c > 0 \forall n, N \in \mathbb{N}, N \leq n \Rightarrow \int_{0}^{c} \frac{dp}{c} p^{N(c-p)^{n-N}} = e^{n} \int_{0}^{1} q^{N(1-q)^{(n-N)}} dq = e^{n} B(n - N + 1, N + 1) = e^{n} \left(\frac{n}{N}\right)^{-1} \frac{1}{n + 1}
\]

(B2)

where B is the Beta function. We also need the following identities:

\[
t_{i} \cdot c_{i} = 1 - \sum_{j < i} t_{j} p_{j}
\]

(B3)

\[
1 - \sum_{j < i} t_{j} p_{j} = t_{i-1} (c_{i-1} - p_{i-1})
\]

(B4)

\[
\left(\sum_{j=1}^{i} N_{j}\right) \cdot \left(N_{1}, \ldots, N_{i}, n - \sum_{j=1}^{i} N_{j}\right) = \left(N_{1}, \ldots, N_{i+1}, n - \sum_{j=1}^{i+1} N_{j}\right)
\]

(B5)

We start by proving the following by induction:

\[
\begin{align*}
\int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d}} \frac{dp_{d}}{c_{d}} p_{1}^{N_{1}} \cdots p_{d}^{N_{d}} [t_{d} (c_{d} - p_{d})]^{n - \sum_{j=1}^{d} N_{j}} &\geq \\
\quad \prod_{j=1}^{d} \left(\frac{1}{t_{j}}\right) N_{j} \left(N_{1}, \ldots, n - \sum_{j=1}^{d} N_{j}\right)^{-1} \frac{1}{(n + 1)^{d-1}}
\end{align*}
\]

(B6)

Base case, \(d = 1\):

\[
\int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} p_{1}^{N_{1}} [t_{1} (c_{1} - p_{1})]^{n - N_{1}} = \left(\frac{1}{t_{1}}\right)^{N_{1}} \left(n \right)^{-1} \frac{1}{n + 1}
\]

This follows from Equation (B2) while noting that for the first index we have \(c_{1} = \frac{1}{t_{1}}\) by definition.

Induction hypothesis for \(d - 1\):

\[
\int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d-1}} \frac{dp_{d-1}}{c_{d-1}} p_{1}^{N_{1}} \cdots p_{d-1}^{N_{d-1}} [t_{d-1} (c_{d-1} - p_{d-1})]^{n - \sum_{j=1}^{d-1} N_{j}} \geq \\
\quad \prod_{j=1}^{d-1} \left(\frac{1}{t_{j}}\right) \left(N_{1}, \ldots, n - \sum_{j=1}^{d-1} N_{j}\right)^{-1} \frac{1}{(n + 1)^{d-1}}
\]

(B7)

Inductive step:

\[
\begin{align*}
\int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d}} \frac{dp_{d}}{c_{d}} p_{1}^{N_{1}} \cdots p_{d}^{N_{d}} [t_{d} (c_{d} - p_{d})]^{n - \sum_{j=1}^{d} N_{j}} = \\
\quad \int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d-1}} \frac{dp_{d-1}}{c_{d-1}} p_{1}^{N_{1}} \cdots p_{d-1}^{N_{d-1}} \int_{0}^{c_{d}} \frac{dp_{d}}{c_{d}} p_{d}^{N_{d}} [t_{d} (c_{d} - p_{d})]^{n - \sum_{j=1}^{d-1} N_{j} - N_{d}} = \\
\quad \int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d-1}} \frac{dp_{d-1}}{c_{d-1}} p_{1}^{N_{1}} \cdots p_{d-1}^{N_{d-1}} \int_{0}^{c_{d-1}} \frac{dp_{d-1}}{c_{d-1}} p_{d-1}^{N_{d-1}} [t_{d-1} (c_{d-1} - p_{d-1})]^{n - \sum_{j=1}^{d-1} N_{j} - N_{d}} \\
\quad \times t_{d}^{n - \sum_{j=1}^{d-1} N_{j} - N_{d}} c_{d}^{n - \sum_{j=1}^{d-1} N_{j}} \left(n - \sum_{j=1}^{d-1} N_{j}\right)^{-1} \frac{1}{n - \sum_{j=1}^{d-1} N_{j} + 1} = \\
\quad \left(\frac{1}{t_{d}}\right)^{N_{d}} \left(n - \sum_{j=1}^{d-1} N_{j}\right)^{-1} \frac{1}{n - \sum_{j=1}^{d-1} N_{j} + 1} \times \\
\quad \int_{0}^{c_{1}} \frac{dp_{1}}{c_{1}} \cdots \int_{0}^{c_{d-1}} \frac{dp_{d-1}}{c_{d-1}} p_{1}^{N_{1}} \cdots p_{d-1}^{N_{d-1}} \left(1 - \sum_{j=1}^{d-1} t_{j} p_{j}\right)^{n - \sum_{j=1}^{d-1} N_{j}}
\end{align*}
\]

(B8)

(B9)

(B10)
For every permutation invariant system $P_{A|X}$ with symmetry $S$ where we used Equation (B13) to get from (B10) to (B11), the induction hypothesis (B7) to get from (B11) to (B12) and Equation (B5) in the last line. Next, we prove the lemma by using Equation (B6):

$$t_{A|X}(a|x) = \int_{0}^{c_{d}} \prod_{j=1}^{d} \left( \frac{1}{t_{j}} \right)^{N_{j}} \left( \frac{n}{N_{1}, \ldots, N_{d}} \right)^{-1} \frac{1}{(n+1)^{d}}$$

where we used Equation (B4) to get from (B14) to (B15) and Equation (B6) to get from (B15) to (B16).

Upper bounding a permutation invariant system $P_{A|X}$ with symmetry $S$

The following lemma gives us an upper bound on any permutation invariant system $P_{A|X}(a|x)$ with the symmetry $S$. This lemma is the analogous of Lemma 17 in the previous section.

**Lemma 22.** For every permutation invariant system $P_{A|X}(a|x)$ with symmetry $S$ we have

$$\forall a, x \quad P_{A|X}(a|x) \leq \prod_{j=1}^{d+1} \left( \frac{1}{t_{j}} \right)^{N_{j}} \left( \frac{n}{N_{1}, \ldots, N_{d}, N_{d+1}} \right)^{-1}$$

The idea behind the proof is identical to the idea behind the proof of Lemma 17. We bound the value of a specific entry $P_{A|X}(a|x)$ by counting how many entries $P_{A|X}(a|x)$ in the same input vector must have the same value as $P_{A|X}(a|x)$ due to the symmetry of $P_{A|X}$. Since the sum of any input vector is 1 this will give us a bound on $P_{A|X}(a|x)$.

**Proof.** For our counting arguments we use here the same notation of the coloring function $C$ given in Equation (B3) and the definition of $N_{i}$ thereafter. That is, for any $a \in \{0, 1, \ldots, l-1\}^{n}$ and $x \in \{0, 1, \ldots, m-1\}^{n}$, we denote by $N_{i}$ the number of indices $j \in [n]$ for which $C(a_{j}, x_{j}) = i$. We can imagine this as placing a colored ball for each pair...
In the previous section, we have two different ways of changing the color sequence of the balls. Consider the example drawn in Figure 9. The coloring of the pairs \((a_j, x_j)\) according to the structure of \(Q_{A|X}\) to the right. Here we have \(n = 8\), \(N_1 = 4\), \(N_2 = 2\) and \(N_3 = 2\).

![Figure 9](image_url)

**FIG. 9.** The coloring of the pairs \((a_j, x_j)\) according to the structure of \(Q_{A|X}\) to the right. Here we have \(n = 8\), \(N_1 = 4\), \(N_2 = 2\) and \(N_3 = 2\).

\[
\begin{array}{cccccccc}
\pi(a) : & 3 & 1 & 2 & 4 & 5 & 6 & 7 & 8 \\
\pi(x) : & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\pi(j) : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
p_1 & p_2 & 1 - 2p_1 - p_2 & p_1 & 1 - 2p_1 - p_2 & p_1 & p_1 & p_2 \\
\end{array}
\]

**FIG. 10.** The permutation \(\pi\).

\((a_j, x_j)\) as in Figure 9. With this picture in mind, the symmetry \(S\) actually says that by changing two balls of the same color we do not change the value according to the probability distribution \(P_{A|X}\).

Let \(\mathcal{N}(a, x) = |\{\tilde{a} \in \{0, 1, \ldots, l - 1\}^n | P_{A|X}(\tilde{a}|x) = P_{A|X}(a|x)\}|\). In how many ways can we change \(a\) while not changing the value of the entry according to \(P_{A|X}\)? We now prove

\[
\mathcal{N}(a, x) \geq \prod_{j=1}^{d+1} t_j N_j \left( N_1, \ldots, N_d, N_{d+1} \right).
\]

As in the proof of Lemma 17 in the previous section, we have two different ways of changing \(a\) to \(\tilde{a}\): with and without changing the color sequence of the balls.

Indeed, the first possible way to change \(a\) without changing the value of the entry is to change a pair \((a_j, x_j)\) to a pair \((\tilde{a}_j, x_j)\) of the same color (note that we do not change \(x_j\) since we want to stay in the same input vector of \(P_{A|X}\), i.e., not to change the input \(x\)). In the example of Figure 9 we can change the first pair \((a_1, x_1) = (3, 0)\) to \((\tilde{a}_1, x_1) = (0, 0)\), for example. How many different strings \(\tilde{a}\) can we create this way? In each input vector of \(Q_{A|X}\) we have \(t_j\) entries of the \(j\)th color and we can choose a entry with this color for each one of the \(N_j\) indices. Therefore, there are exactly \(\prod_{j=1}^{d+1} t_j N_j\) different strings \(\tilde{a}\) with the same color sequence as \(a\) and hence, according to the symmetry \(S\), with the same value \(P_{A|X}(a, x) = P_{A|X}(\tilde{a}, x)\).

Changing \(a\) in different ways than the way given above will necessarily change the colors sequence. However, we can still prove, using the permutation invariance of \(P_{A|X}\), that for some specific changes the value of the entry will stay the same. The specific changes that we consider are derived by permutations of the colored balls.

In order to understand how every different permutations of the balls \(\pi\) is realised as a permutation on \(x\) and \(a\) consider the example drawn in Figure 10. On the left side we see a permutation of the balls from Figure 9. We start by filling up the columns for which there is no change in the color of the ball with the original columns. Then we fill in the blank columns in such a way that each of the original columns appears once. The permutation in the figure for example, is just the permutation of indices \((3, 4)\) and \((6, 7, 8)\). In general, there might be several ways to choose the permutation on \(x\) and \(a\), but they are all equivalent for our purpose and therefore we can just choose one.

The important thing to note is that between every two different permutations of the balls we always have at least one index in which we have a different colored ball in the end. That is, we can write that for every \(\pi, \pi' \neq \pi\), there exists \(j \in [n]\) such that \(C(\pi(a)_j, \pi(x)_j) \neq C(\pi'(a)_j, \pi'(x)_j)\). We use this to construct from every permutation \(\pi\) a different string \(\tilde{a}_\pi\) for which \(P_{A|X}(a|x) = P_{A|X}(\tilde{a}_\pi|x)\), as follows. For any index \(j \in [n]\) that \(\pi\) permutes we change \(a_j\) to (some) \(\tilde{a}_{j, \pi}\) such that

\[
C(\tilde{a}_{j, \pi}, x_j) = C(\pi(a)_j, \pi(x)_j).
\]

This is always possible since the input vectors of \(Q_{A|X}\) are permutations of one another, i.e., if \(C(\pi(a)_j, \pi(x)_j) = k\) then there must be some \(a'_j\) for which \(C(a'_j, x_j) = k\).

---

8 We mention the input vectors of \(Q_{A|X}\) here just for simplicity. What we really mean is that we have \(t_j\) symmetry conditions, but these were “constructed” from \(Q_{A|X}\) in Definition 19.

9 Again, as in the previous footnote, what we really mean is that this holds according to the symmetry \(S\).
We are now left to show that \( P_{A|X}(a, x) = P_{A|X}(\tilde{a}_\pi, x) \). Since \( P_{A|X} \) is permutation invariant, we have

\[
P_{A|X}(a|x) = P_{A|X}(\pi(a)|\pi(x))
\]

and from the symmetry \( S \) (recall Definition 19 and Equation B17) we get

\[
P_{A|X}(\pi(a)|\pi(x)) = P_{A|X}(\tilde{a}_\pi|x) .
\]

Combining these two equations together we get \( P_{A|X}(a, x) = P_{A|X}(\tilde{a}_\pi, x) \) as desired.

Since for every two different permutations of the balls we always have at least one index in which we have a different colored ball in the end, we get different \( \tilde{a}_\pi \)'s from different permutations \( \pi \). There are exactly \((N_1, \ldots, N_d, N_{d+1})\) different permutations of the balls, and therefore the same number of different \( \tilde{a}_\pi \) when proceeding this way.

We can now answer our original question and bound \( N(a, x) \). We can combine both of the ways given above to change \( a \) without changing the value of the entry according to \( P_{A|X} \). This implies that in total, there are at least

\[
\prod_{j=1}^{d+1} l_j^{N_j} \Big( N_1, \ldots, N_d, N_{d+1} \Big) \text{ different ways of changing } a \text{ and we can conclude that}
\]

\[
N(a, x) \geq \prod_{j=1}^{d+1} l_j^{N_j} \Big( N_1, \ldots, N_d, N_{d+1} \Big). \tag{B18}
\]

Since for all \( x \sum_a P_{A|X}(a|x) = 1 \), we get from Equation (B18) the following bound on the entry value \( P_{A|X}(a|x) \):

\[
P_{A|X}(a|x) \leq \prod_{j=1}^{d+1} \left( \frac{1}{l_j} \right)^{N_j} \left( N_1, \ldots, N_d, N_{d+1} \right)^{-1}. \tag{B19}
\]

By combining Lemma 21 and Lemma 22 we get Theorem 4.

**Deriving the corollaries from the general theorem**

As mentioned before, for every symmetry \( S \) for which a system \( Q_{A_1|X_1} \) can be construct such that the condition in Definition 19 holds our proof can be applied. In order to derive the corollaries we just need to describe the type of symmetry that we consider and the relevant \( Q_{A_1|X_1} \) that we use to construct the de Finetti system.

In Corollary 8 for example the systems \( P_{A|X} \) that we consider have no special symmetry. Therefore we can derive this corollary from Theorem 4 by choosing \( Q_{A_1|X_1} \) without any internal symmetry (see for example Figure 11). In this case we have \( d = m(l - 1) \) degrees of freedom, hence we get Corollary 8.

For deriving Corollary 6 we use the system \( Q_{A_1, B_1|X_1, Y_1} \) given in Figure 1 for this system we have \( d = 1 \) and therefore Corollary 6 follows.

**Appendix C: Proofs of Section III**

Post selecting a permutation invariant system from a de Finetti system

In order to derive the results in this section we will use an alternative formulation of the de Finetti reductions given in the main text. We explain and derive this alternative formulation only for Corollary 8 but it can be applied analogously also to the other de Finetti reductions.

**Lemma 23.** There exists a de Finetti system \( \tau_{A|X} \) where \( x \in \{0, 1, \ldots, m - 1\}^n \) and \( a \in \{0, 1, \ldots, l - 1\}^n \) and a non-signalling extension of it to a larger system \( \tau_{AC|XZ} \) such that for every permutation invariant system \( P_{A|X} \) there exists a measurement \( z \) and an outcome of this measurement \( c_z \) for which

\[
\forall a, x \quad \tau_{AC|XZ}(a, c_z|x, z) = \frac{1}{(n+1)^{m(l-1)}} P_{A|X}(a|x).
\]
FIG. 12. Post selecting system $P_{A|X}$ from an extension of $\tau_{A|X}$. If the outcome of the measurement $z$ is $c_z$ then the post measurement system is $P_{A|X}$.

This lemma states that there exists a de Finetti system $\tau_{A|X}$ and a non-signalling extension of it $\tau_{AC|XZ}$ such that any permutation invariant system $P_{A|X}$ can be post selected from it with probability $\geq \frac{1}{(n+1)^m(l-1)}$. When we say that $P_{A|X}$ can be post selected we mean that there exists an input $z$ to $\tau_{AC|XZ}$ and an output of this measurement $c_z$ such that with probability $\tau_{C|Z}(c_z|z) \geq \frac{1}{(n+1)^m(l-1)}$ the post-measurement system is $P_{A|X}$ (see Figure 12). Note that we consider a specific extension $\tau_{AC|XZ}$ of the system $\tau_{A|X}$, and by choosing different inputs $z$ we can post select different systems $P_{A|X}$.

It is easy to see how to derive Lemma 23 from Corollary 3 by using the formalism introduced in [18, 23] of partitions of a conditional probability distribution. We repeat here the relevant statements.

Definition 24. A partition of a system $P_{A|X}$ is a family of pairs $\left\{ (p_c, P_{c|A|X}) \right\}_c$ where $p_c \geq 0$, $\sum_c p_c = 1$ and the systems $P_{c|A|X}$ are such that

$$ P_{A|X} = \sum_c p_c \cdot P_{c|A|X}. $$

Lemma 25 (Lemma 9 in [18]). Given a system $P_{A|X}$, there exists a partition with element $\left( p_c, P_{c|A|X} \right)$ if and only if

$$ \forall a, x \quad p_c \cdot P_{c|A|X}(a|x) \leq P_{A|X}. $$

Lemma 26 (Lemma 3.2 in [23]). Given a system $P_{A|X}$ let $Z$ be the set of all partitions $\left\{ (p_{c_z}, P_{c_z|A|X}) \right\}_{c_z}$ of $P_{A|X}$. There exists an extension system $P_{AC|XZ}$ of $P_{A|X}$ and an input $z$ to it such that

$$ \forall a, x \quad P_{AC|XZ}(a, c_z|x, z) = p_{c_z} \cdot P_{c_z|A|X}(a|x). $$

Moreover, the system $P_{AC|XZ}$ does not allow signalling between the $A/X$ and the $C/Z$ interfaces.

Using the lemmas above and Corollary 3 we can now prove Lemma 23.

Proof. The above lemmas together with Corollary 3 imply that in our case for any permutation invariant system $P_{A|X}$, $\left( n+1 \right)^m(l-1) P_{A|X}$ is an element of a partition of $\tau_{A|X}$. Moreover, there exists a system $\tau_{AC|XZ}$ and an input $z$ such that with probability $\frac{1}{(n+1)^m(l-1)}$ the post-measurement system is $P_{A|X}$:

$$ \forall a, x \quad \tau_{AC|XZ}(a, c_z|x, z) = \frac{1}{(n+1)^m(l-1)} P_{A|X}, $$

i.e., Lemma 23 holds. \[\square\]

\[10\] In the usual cryptographic setting this means a non-signalling condition between Alice and Eve.
A proof of Lemma \textsuperscript{8}

We repeat here Lemma \textsuperscript{8} for convenience.

**Lemma.** Let $\mathcal{T}$ be a permutation invariant test as in Definition \textsuperscript{7} and $\Pr_{\text{fail}}(P_{A|X})$ be the probability that $\mathcal{T}$ outputs “fail” on $P_{A|X}$. Then for any system $P_{A|X}$

$$\Pr_{\text{fail}}(P_{A|X}) \leq (n + 1)^{m(l-1)}\Pr_{\text{fail}}(\tau_{A|X}) .$$

**Proof.** We follow here a similar proof given in \textsuperscript{24} while using the quantum post se lection theorem \textsuperscript{8},

First, since the test $\mathcal{T}$ is permutation invariant it is sufficient to consider only permutation invariant systems. To see this recall that for any system $P_{A|X}$ and permutation $\pi$ we have $\Pr_{\text{fail}}(P_{A|X}) = \Pr_{\text{fail}}(P_{A|X} \circ \pi)$ according to Definition \textsuperscript{7}. Therefore we also have by linearity

$$\Pr_{\text{fail}}(P_{A|X}) = \frac{1}{n!} \sum_{\pi} \Pr_{\text{fail}}(P_{A|X} \circ \pi) = \Pr_{\text{fail}} \left( \frac{1}{n!} \sum_{\pi} P_{A|X} \circ \pi \right). \quad (C1)$$

The system $\frac{1}{n!} \sum_{\pi} P_{A|X} \circ \pi$ is permutation invariant and therefore without loss of generality we can consider only permutation invariant systems.

Next we define the following probabilities. Let $\Pr_{\text{fail} \land c_2}(\tau_{AC|XZ})$ be the probability that the second part of the system, $\tau_{C|Z}$ is measured with $z$ and the output is $c_2$ and that the first part of the system, $\tau_{A|X}$ fails the test $\mathcal{T}$ at the same time. That is,

$$\Pr_{\text{fail} \land c_2}(\tau_{AC|XZ}) = \Pr_{\text{fail}}(\tau_{A|X}) \cdot \tau_{C|Z}(c_2|z).$$

In a similar way we define $\Pr_{\text{fail}|c_2}(\tau_{AC|XZ})$ to be the probability the $\tau_{A|X}$ fails the test $\mathcal{T}$ given that $c_2$ is the outcome measurement of $\tau_{C|Z}$. According to probability theory we have

$$\Pr_{\text{fail}|c_2}(\tau_{AC|XZ}) = \frac{\Pr_{\text{fail} \land c_2}(\tau_{AC|XZ})}{\tau_{C|Z}(c_2|z)} \leq \frac{\Pr_{\text{fail}}(\tau_{A|X})}{\tau_{C|Z}(c_2|z)}$$

since it is always true that $\Pr_{\text{fail} \land c_2}(\tau_{AC|XZ}) \leq \Pr_{\text{fail}}(\tau_{A|X})$.

Lemma \textsuperscript{23} implies that $\tau_{C|Z}(c_2|z) \geq \frac{1}{(n+1)^{m(l-1)}}$ and that $\Pr_{\text{fail}|c_2}(\tau_{AC|XZ}) = \Pr_{\text{fail}}(P_{A|X})$ (given that the outcome measurement was $c_2$, the post measurement system is $P_{A|X}$). All together we get $\Pr_{\text{fail}}(P_{A|X}) \leq (n + 1)^{m(l-1)}\Pr_{\text{fail}}(\tau_{A|X})$ as required.

**Diamond norm for conditional probability distributions**

We start by writing the diamond norm defined in Definition \textsuperscript{11} in an explicit way:

$$\|\mathcal{E} - \mathcal{F}\|_0 = \max_{P_{AC|XZ}} \| (\mathcal{E} - \mathcal{F}) \otimes \mathbb{I}(P_{AC|XZ}) \|_1$$

$$= \max_{P_{AC|XZ}} \| E_{K|C} \cdot P_{C|Z} - F_{K|C} \cdot P_{C|Z} \|_1$$

$$= \max_{P_{AC|XZ}} \frac{1}{2} \sum_k \max_z \sum_c P_{C|Z}(c|z) |E_{K|C}(k|c) - F_{K|C}(k|c)|$$

$$= \max_{P_{AC|XZ}} \frac{1}{2} \sum_k \max_z \sum_c P_{C|Z}(c|z) \times$$

$$\sum_x \Pr_{\mathcal{F}}(x) \sum_{a|x} P_{A|XC}(a|xc) - \sum_x \Pr_{\mathcal{F}}(x) \sum_{a|x} P_{A|XC}(a|xc),$$

where the third equality is due to the explicit form of the trace distance previously given in \textsuperscript{17}, \textsuperscript{18}.

Using this explicit form of the diamond norm we can now prove the following lemma.
Lemma 27. For every two permutation invariant and $S$ invariant (as in Definition 12) channels $\mathcal{E}, \mathcal{F} : \{P_{A|X}\} \to \{P_K\}$ where $P_K$ is a probability distribution over $k \in \{0, 1\}^t$ for some $t > 0$

$$\forall P_{AC|XZ} \quad \| (\mathcal{E} - \mathcal{F}) \otimes 1(\tau_{AC|XZ}^{P_{AC|XZ}}) \|_1 \leq (n + 1)^d \| (\mathcal{E} - \mathcal{F}) \otimes 1(\tau_{AC|XZ}^{P_{AC|XZ}}) \|_1$$

where $\tau_{AC|XZ}^{P_{AC|XZ}}$ is a non-signalling extension of $\tau_{A|X}^S$ which depends on the specific system $P_{AC|XZ}$.

Proof. First, as in the previous proof and Equation (C1), since the channels are permutation invariant and $S$ invariant it is sufficient to consider systems $P_{A|X}$ which are permutation invariant and have the symmetry $S$.

Given a specific system $P_{AC|XZ}$, according to Lemma 20, we can see this extension as a set of convex decompositions of $P_{A|X}$. That is, every possible input $z$ indicates a specific decomposition $\{(p_{cz}, P_{A|X}^{cz})\}_{cz}$ such that $p_{cz} = P_{C|Z}(c_z | z)$ and $P_{A|X}^{cz}(a | x) = P_{AC|XZ}(a, c_z | x, z)$. Since this is a convex decomposition of $P_{A|X}$ we also have

$$\forall z \sum_c p_c \cdot P_{A|X}^{c} = P_{A|X}. \quad \text{(C3)}$$

We now use the set of decompositions of $P_{A|X}$ to construct a set of decompositions of the de Finetti system $\tau_{A|X}^S$. Combining Theorem 4 with Lemma 25 and Lemma 26 we know that there exists a non-signalling system $R_{A|X}$ such that

$$\tau_{A|X}^S = \frac{1}{(n + 1)^d} P_{A|X} + \left(1 - \frac{1}{(n + 1)^d} \right) R_{A|X}$$

$$= \frac{1}{(n + 1)^d} \sum_c p_c \cdot P_{A|X} + \left(1 - \frac{1}{(n + 1)^d} \right) R_{A|X}.$$

where the second equality is due to Equation (C3). For every $z$ this defines a decomposition $\{(\frac{1}{(n + 1)^d}, p_{cz}, P_{A|X}^{cz})\}_{cz} \cup \{(1 - \frac{1}{(n + 1)^d}, R_{A|X})\}$ of $\tau_{A|X}^S$. That is, this defines an extension $\tau_{AC|XZ}^{P_{AC|XZ}}$ of $\tau_{A|X}^S$ where $C' = C \cup \{c'\}$.

This connection between the extensions $P_{AC|XZ}$ and $\tau_{AC|XZ}^{P_{AC|XZ}}$ allow us to get the bound on the trace distance and prove the lemma:

$$\| (\mathcal{E} - \mathcal{F}) \otimes 1(\tau_{AC|XZ}^{P_{AC|XZ}}) \|_1 = \frac{1}{2} \sum_k \max_z \sum_{c \in C'} \tau_{C'|Z}^{P_{AC|XZ}}(c | z) \times$$

$$\times \left| \sum_x \Pr_x(x) \sum_{a|\mathcal{E}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc) - \sum_x \Pr_x(x) \sum_{a|\mathcal{F}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc) \right|$$

$$= \frac{1}{2} \sum_k \max_z \left| \sum_{c \in C} \tau_{C'|Z}^{P_{AC|XZ}}(c | z) \times$$

$$\times \left| \sum_x \Pr_x(x) \sum_{a|\mathcal{E}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc) - \sum_x \Pr_x(x) \sum_{a|\mathcal{F}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc) \right| +$$

$$+ \left(1 - \frac{1}{(n + 1)^d} \right) \times$$

$$\times \left| \sum_x \Pr_x(x) \sum_{a|\mathcal{E}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc') - \sum_x \Pr_x(x) \sum_{a|\mathcal{F}(a,x) = k} \tau_{A|XC'}^{P_{AC|XZ}}(a | xc') \right|$$
\begin{align*}
&\geq \frac{1}{2} \sum_{k} \max_{z} \sum_{c \in C} \tau_{C'|Z}(c|z) \times \\
&\times \left| \sum_{x} \Pr_{E}(x) \sum_{a|E(a,x)=k} \tau_{A|XC'}(a|xc) - \sum_{x} \Pr_{F}(x) \sum_{a|F(a,x)=k} \tau_{A|XC'}(a|xc) \right| \\
&= \frac{1}{2} \sum_{k} \max_{z} \sum_{c \in C} \frac{1}{(n+1)^d} \cdot P_{C|Z}(c|z) \times \\
&\times \left| \sum_{x} \Pr_{E}(x) \sum_{a|E(a,x)=k} P_{A|XC}(a|xc) - \sum_{x} \Pr_{F}(x) \sum_{a|F(a,x)=k} P_{A|XC}(a|xc) \right| \\
&= \frac{1}{(n+1)^d} \| (E - F) \otimes 1(A|X) \|_1 \quad (C4)
\end{align*}

where in order to get the second equality we divide the sum over \( C' = C \cup \{c'\} \) to the sum over \( C \) and then additional part of the partition \( c' \). The next inequality is then correct since

\[
\left(1 - \frac{1}{(n+1)^d}\right) \left| \sum_{x} \Pr_{E}(x) \sum_{a|E(a,x)=k} \tau_{A|XC}(a|xc) - \sum_{x} \Pr_{F}(x) \sum_{a|F(a,x)=k} \tau_{A|XC}(a|xc) \right| \geq 0
\]

and the two last equalities are due to the specific decomposition of \( \tau_{A|X}^S \) that we defined and the definition of the trace distance. The lemma then follows from Equation (C4). \hfill \Box

Theorem 13 now easily follows from Lemma 27.

Proof of Theorem 13 Using Lemma 27

\[
\|E - F\|_\diamond = \max_{P_{AC|XZ}} \| (E - F) \otimes 1(A|X) \|_1 \\
\leq (n+1)^d \max_{\tau_{AC|XZ}} \| (E - F) \otimes 1(\tau_{AC|XZ}) \|_1 \\
\leq (n+1)^d \max_{\tau_{AC|XZ}} \| (E - F) \otimes 1(\tau_{AC|XZ}) \|_1
\]

where \( \tau_{AC|XZ} \) is a non-signalling extension of \( \tau_{A|X}^S \). \hfill \Box