Some generalisations of the inequalities for positive linear maps

R. Sharma, P. Devi and R. Kumari
Department of Mathematics & Statistics
Himachal Pradesh University
Shimla -5,
India - 171005
email: rajesh.sharma.hpn@nic.in

Abstract. We obtain generalisations of some inequalities for positive unital linear maps on matrix algebra. This also provides several positive semidefinite matrices, and we get some old and new inequalities involving the eigenvalues of a Hermitian matrix.

AMS classification. 15A45, 15A42, 46L53.

Key words and phrases. Positive definite matrices, Positive unital linear maps, tensor product, eigenvalues.
1 Introduction

Let \( \mathbb{M}(n) \) be the \( \mathbb{C}^* \)- algebra of all \( n \times n \) complex matrices. Let \( \Phi : \mathbb{M}(n) \to \mathbb{M}(k) \) be a positive unital linear map [3]. Kadison’s inequality [11] says that for any Hermitian element \( A \) of \( \mathbb{M}(n) \), we have

\[
\Phi(A^2) \geq \Phi(A)^2
\]

or equivalently

\[
\begin{bmatrix}
I & \Phi(A) \\
\Phi(A) & \Phi(A^2)
\end{bmatrix} \geq O.
\]

(1.1)

For more details, generalisations and extensions of this inequality, see Davis [9] and Choi [7, 8].

A complementary inequality due to Bhatia and Davis [2] says that if the spectrum of a Hermitian matrix \( A \) is contained in the interval \([m, M]\), then

\[
\Phi(A^2) - \Phi(A)^2 \leq \left( \frac{M - m}{2} \right)^2 I.
\]

(1.3)

They also proved that

\[
\Phi(A^2) - \Phi(A)^2 \leq (\Phi(A) - mI)(MI - \Phi(A)).
\]

(1.4)

The inequality (1.4) provides a refinement of (1.3). For more details and applications of these inequalities, see [4 – 6].

The Kadison inequality (1.1) is a noncommutative version of the classical inequality

\[
\mathbb{E}(X^2) \geq \mathbb{E}(X)^2,
\]

where \( \mathbb{E}(X) = \int x f(x) \, dx \), \( f(x) \geq 0 \) and \( \int f(x) \, dx = 1 \). Kadison [11] remarks that the standard proof of the corresponding inequalities for scalars do not apply to give simple proofs for linear maps. In case of Kadison’s inequality (1.1), if \( \Phi(A) \) and \( \Phi(A^2) \) commute one can reduce the problem to the real valued case, and the results follow from these considerations.

The inequality (1.5) is subsumed in a more general Jensen’s inequality that says that if \( f \) is a convex function on \((a, b)\) then

\[
f(\mathbb{E}(X)) \leq \mathbb{E}(f(X)).
\]

(1.6)
One generalisation of the Kadison inequality (1.1) is a noncommutative analogue of (1.6) that says that if $A$ is a Hermitian matrix whose spectrum is contained in $(a, b)$ and if $f$ is matrix convex function [3] on $(a, b)$ and $\Phi$ is a positive unital linear map, then

$$f(\Phi(A)) \leq \Phi(f(A)),$$

(1.7)

see Davis [9] and Choi [7]. Bhatia and Sharma [4] have shown that for $2 \times 2$ matrices the inequality (1.7) holds true for all ordinary convex functions on $(a, b)$. It is well known that the same is true for every positive unital linear functional $\varphi : M(n) \to \mathbb{C}$.

The inequality (1.5) is also subsumed in one more general inequality that says that $[\mathbb{E}(X^{i+j-2})]$ is a real symmetric positive semidefinite matrix, see [10]. Here, we study such possible extensions of the noncommutative inequalities (1.2) and (1.4). The proof of the inequality (1.2) involves the Spectral theorem and the properties of tensor product, see [3]. By the Spectral theorem,

$$A = \sum_{k=1}^{n} \lambda_k P_k,$$

where $\lambda_k$ are the eigenvalues of $A$ and $P_k$ the corresponding projections with $\sum_{k=1}^{n} P_k = I$.

Then

$$A^r = \sum_{k=1}^{n} \lambda_k^r P_k , \Phi(A^r) = \sum_{k=1}^{n} \lambda_k^r \Phi(P_k) \quad \text{and} \quad \sum_{k=1}^{n} \Phi(P_k) = I.$$

(1.8)

We augment the technique of the proof [3] of the inequality (1.2) and prove one more generalisation of the Kadison inequality that says that the matrix

$$
\begin{pmatrix}
I & \Phi(A) & \cdots & \Phi(A^r) \\
\Phi(A) & \Phi(A^2) & \cdots & \Phi(A^{r+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\Phi(A^r) & \Phi(A^{r+1}) & \cdots & \Phi(A^{2r})
\end{pmatrix}
$$

is positive semidefinite, for all $r = 0, 1, 2...$ (see Corollary (2.1), below). Our main results (Theorem 2.1 – 2.5) provide some generalisations of the Bhatia-Davis inequality (1.4). It is shown that some recent bounds for the central moments are the special cases of our results. Lower and upper bounds respectively for the largest and smallest eigenvalues of a Hermitian matrix are derived, (Theorem 3.3). Inequalities analogous to Kadison’s inequality (1.2) are obtained (Theorem 3.2-3.3).
2 Main results

**Lemma 2.1.** For $x \geq y$, the $(r+1) \times (r+1)$ matrix $[x^{i+j-1} - yx^{i+j-2}]$ is positive semidefinite (psd), where $1 \leq i, j \leq r+1$, $r = 0, 1, 2, \ldots$.

**Proof.** The matrix $[x^{i+j-2}]$ is psd for all $x \in \mathbb{R}$. Its rank is one and trace is non-negative. Likewise, the matrix $[x-y]$ is psd for $x \geq y$. The Schur product

$$[x^{i+j-2}] \odot [x-y] = [x^{i+j-1} - yx^{i+j-2}]$$

is also psd. ■

**Theorem 2.1.** Let $\Phi : M(n) \rightarrow M(k)$ be a positive unital linear map. Let $A$ be any Hermitian element of $M(n)$ whose spectrum is contained in $[m, M]$. Then

$$[\Phi (A^{i+j-1}) - m\Phi (A^{i+j-2})]_{r+1 \times r+1} \geq O \quad (2.1)$$

and

$$[M\Phi (A^{i+j-2}) - \Phi (A^{i+j-1})]_{r+1 \times r+1} \geq O, \quad (2.2)$$

where $1 \leq i, j \leq r+1$, $r = 0, 1, 2, \ldots$

**Proof.** Using (1.8), we have

$$[\Phi (A^{i+j-1}) - m\Phi (A^{i+j-2})] = \sum_{k=1}^{n} [\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}] \odot \Phi (P_k). \quad (2.3)$$

Also,

$$\sum_{k=1}^{n} [\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}] \odot \Phi (P_k) = \sum_{k=1}^{n} [\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}] \odot \Phi (P_k)$$

where $\odot$ denotes the tensor product of matrices. So,

$$[\Phi (A^{i+j-1}) - m\Phi (A^{i+j-2})] = \sum_{k=1}^{n} [\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}] \odot \Phi (P_k). \quad (2.3)$$

Since $P_k$ is psd, $\Phi (P_k)$ are psd. It follows from Lemma 2.1 that the matrix $[\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}]$ is psd. The tensor product of two psd matrices is psd. Each summand in (2.3) is psd, and so is the sum. So, (2.1) holds true. Likewise, (2.2) follows from the fact that the matrix $[M\lambda_k^{i+j-2} - \lambda_k^{i+j-1}]$ is psd. ■

**Corollary 2.1.** Under the conditions of Theorem 2.1,

$$[\Phi (A^{i+j-2})]_{r+1 \times r+1} \geq O. \quad (2.4)$$
For $A \geq O$, we also have
\[
\left[ \Phi \left( A^{i+j-1} \right) \right]_{r+1 \times r+1} \geq O. \tag{2.5}
\]

**Proof.** Adding (2.1) and (2.2), we immediately get (2.4). The inequality (2.5) is a special case of (2.1); $m = 0$. ■

For $\lambda_k \geq m > 0$, $k = 1, 2, ..., n$, the matrix $\left[ 1 - \frac{m}{\lambda_k} \right]$ is psd and so is the matrix
\[
\left[ \lambda_k^{i+j-2} \circ \left[ 1 - \frac{m}{\lambda_k} \right] = \lambda_k^{i+j-2} - m\lambda_k^{i+j-3} \right]. \tag{2.6}
\]
Using arguments similar to those used in the proof of Theorem 2.1, we have the following theorem.

**Theorem 2.2.** Let $A$, $\Phi$, $m$ and $M$ be as in Theorem 2.1. If $A$ is positive definite, then
\[
\left[ \Phi \left( A^{i+j-2} - m\Phi \left( A^{i+j-3} \right) \right) \right]_{r+1 \times r+1} \geq O \tag{2.7}
\]
and
\[
\left[ M\Phi \left( A^{i+j-3} \right) - \Phi \left( A^{i+j-2} \right) \right]_{r+1 \times r+1} \geq O. \tag{2.8}
\]

**Lemma 2.2.** For $x \geq y \geq z$, the $(r+1) \times (r+1)$ matrix $\left[ y^{i+j-2} (x-y) (y-z) \right]$ is psd, $1 \leq i, j \leq r+1$, $r = 0, 1, 2, ....$

**Proof.** It is enough to note that the Schur product
\[
\left[ y^{i+j-2} (x-y) (y-z) \right] = \left[ y^{i+j-2} \right] \circ \left[ (x-y) (y-z) \right] \tag{2.9}
\]
is psd. ■

**Theorem 2.3.** Under the condition of Theorem 2.1, we have
\[
\left[ \Phi(A^{i+j-2}(A-mI)(M-I-A)) \right]_{r+1 \times r+1} \geq O. \tag{2.10}
\]

**Proof.** Using the Lemma 2.2, we see that the matrix
\[
\left[ \lambda_k^{i+j-2} \circ \left[ (\lambda_k - m) (M - \lambda_k) \right] = \lambda_k^{i+j-2} (\lambda_k - m) (M - \lambda_k) \right]
\]
is psd. The proof now follows on using arguments similar to those used in the proof of Theorem 2.1. ■

**Theorem 2.4.** Let $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ be a positive unital linear map. Let $A$ be any Hermitian element of $\mathbb{M}(n)$ with distinct eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_k$. Then
\[
\left[ \Phi(A^{i+j-2}(A - \lambda_j I)(A - \lambda_j I)) \right]_{r+1 \times r+1} \geq O. \tag{2.11}
\]
Proof. Since all the $\lambda_k$ ($k = 1, 2, ..., n$) lies outsides ($\lambda_{j-1}, \lambda_j$), $j = 2, 3, ... k$, we have $(\lambda_k - \lambda_{j-1})(\lambda_k - \lambda_j) \geq 0$. Therefore, the Schur product

$$[\lambda_k^{i+j-2}] \circ [(\lambda_k - \lambda_{j-1})(\lambda_k - \lambda_j)] = [\lambda_k^{i+j-2} (\lambda_k - \lambda_{j-1})(\lambda_k - \lambda_j)]$$

is psd. The proof now follows on using arguments similar to those used in the proof of Theorem 2.1. ■

Theorem 2.5. Let $A$, $\Phi$, $m$ and $M$ be as in Theorem 2.1. If $A$ is positive definite, then

$$[\Phi (A^{i+j-3}(mI - mI)(M - A))]_{r+1 	imes r+1} \geq O. \quad (2.12)$$

Proof. The arguments are similar to the proof of the above theorem. Note that for $x \geq y \geq z$ and $y > 0$, we have $x + z - xzy^{-1} - y \geq 0$. So, the Schur product

$$[\lambda_k^{i+j-2}] \circ [(m + M - mM\lambda_k^{-1} - \lambda_k)] = [\lambda_k^{i+j-3} (\lambda_k - m)(M - \lambda_k)]$$

is psd. ■

3 Special Cases

If $A$ and $B$ are positive definite matrices then the block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is psd if and only if $A \geq XB^{-1}X^*$, see [3]. Using this result we can discuss various special cases of inequalities derived above. We demonstrate some of these cases here.

We find, on adding (2.7) and (2.8), that if $A > 0$ then

$$\begin{bmatrix} \Phi(A^{-1}) & I & \cdots & \Phi(A^{-1}) \\ I & \Phi(A) & \cdots & \Phi(A^r) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(A^{-1}) & \Phi(A^r) & \cdots & \Phi(A^{2r-1}) \end{bmatrix}$$

is psd. So,

$$\begin{bmatrix} \Phi(A^{-1}) & I \\ I & \Phi(A) \end{bmatrix} \geq O.$$

This gives Choi [7] inequality

$$\Phi(A^{-1}) \geq \Phi(A)^{-1},$$
see [3].

For \( r = 1 \), Theorem 2.1 says that

\[
\begin{bmatrix}
\Phi(A) - mI & \Phi(A^2) - m\Phi(A) \\
\Phi(A^2) - m\Phi(A) & \Phi(A^3) - m\Phi(A^2)
\end{bmatrix} \geq O
\]

and

\[
\begin{bmatrix}
MI - \Phi(A) & M\Phi(A) - \Phi(A^2) \\
M\Phi(A) - \Phi(A^2) & M\Phi(A^2) - \Phi(A^3)
\end{bmatrix} \geq O.
\]

For \( \Phi(A) > mI \), we therefore have

\[
\Phi(A^3) \geq m\Phi(A^2) + (\Phi(A^2) - m\Phi(A)) (\Phi(A) - mI)^{-1} (\Phi(A^2) - m\Phi(A)) \tag{3.1}
\]

and for \( \Phi(A) < MI \), we have

\[
\Phi(A^3) \leq M\Phi(A^2) - (M\Phi(A) - \Phi(A^2)) (MI - \Phi(A))^{-1} (M\Phi(A) - \Phi(A^2)). \tag{3.2}
\]

For the corresponding commutative cases of the inequalities (3.1) and (3.2),

\[
\mathbb{E}(X^3) \geq m\mathbb{E}(X^2) + \frac{(\mathbb{E}(X^2) - m\mathbb{E}(X))^2}{\mathbb{E}(X) - m}
\]

and

\[
\mathbb{E}(X^3) \leq M\mathbb{E}(X^2) - \frac{(M\mathbb{E}(X) - \mathbb{E}(X))^2}{M - \mathbb{E}(X)},
\]

see [14]. The bounds for \( \mathbb{E}(X^2) \) in terms of \( \mathbb{E}(X^{-1}) \) and \( \mathbb{E}(X) \) are derived in [13] on using derivatives. These inequalities follow from our Theorem 2.2 for \( r = 1 \). Likewise, the bounds for \( \mathbb{E}(X^4) \) in [15] follow from our Theorem 2.3 for \( r = 1 \). The inequalities for the central moments [13 – 15] also follow from our more general results. Let \( \varphi : \mathbb{M}(n) \rightarrow \mathbb{C} \) be a positive unital linear functional. Let \( B = A - \varphi(A)I \), \( a = m - \varphi(A) \) and \( b = M - \varphi(A) \). It follows from Theorem 2.1 that

\[
[\varphi(B^{i+j-1}) - a\varphi(B^{i+j-2})]_{r+1 \times r+1} \geq O \tag{3.3}
\]

and

\[
[b\varphi(B^{i+j-2}) - \varphi(B^{i+j-1})]_{r+1 \times r+1} \geq O. \tag{3.4}
\]

From inequalities (3.3) and (3.4), we also have

\[
[\varphi(B^{i+j-2})]_{r+1 \times r+1} \geq O.
\]
Likewise, we can discuss the corresponding inequalities for functionals related to Theorem 2.2-2.5.

A special case of Corollary 2.1 says that for every Hermitian matrix $A$,
\[
\begin{bmatrix}
\Phi (A^2) & \Phi (A^3) \\
\Phi (A^3) & \Phi (A^4)
\end{bmatrix} \geq O.
\]

We prove a refinement of this inequality for positive definite matrices in the following theorem.

**Theorem 3.1.** Let $A$, $\Phi$ and $m$ be as in Theorem 2.1. If $A \geq m > 0$, then
\[
\begin{bmatrix}
\Phi (A^2) & \Phi (A^3) \\
\Phi (A^3) & \Phi (A^4)
\end{bmatrix} \geq 2m \begin{bmatrix}
\Phi (A) & \Phi (A^2) \\
\Phi (A^2) & \Phi (A^3)
\end{bmatrix} - m^2 \begin{bmatrix} I & \Phi (A) \\
\Phi (A) & \Phi (A^2) \end{bmatrix} \geq O. \quad (3.5)
\]

**Proof.** The second inequality (3.5) follows from the fact that for $0 < m \leq \lambda_j$, the Schur product
\[
\begin{bmatrix}
m(2\lambda_j - m) & m\lambda_j(2\lambda_j - m) \\
m\lambda_j(2\lambda_j - m) & m\lambda_j^2(2\lambda_j - m)
\end{bmatrix} = \begin{bmatrix}
(2\lambda_j - m) & (2\lambda_j - m) \\
(2\lambda_j - m) & (2\lambda_j - m)
\end{bmatrix} \otimes \begin{bmatrix} m & m\lambda_j \\
m\lambda_j & m\lambda_j^2 \end{bmatrix}
\]
is psd.

Likewise, the first inequality (3.5) follows from the fact that the matrix
\[
\begin{bmatrix}
(\lambda_j - m)^2 & \lambda_j(\lambda_j - m)^2 \\
\lambda_j(\lambda_j - m)^2 & \lambda_j^2(\lambda_j - m)^2
\end{bmatrix} = \begin{bmatrix} 1 & \lambda_j \\
\lambda_j & \lambda_j^2 \end{bmatrix} \otimes \begin{bmatrix} (\lambda_j - m)^2 & (\lambda_j - m)^2 \\
(\lambda_j - m)^2 & (\lambda_j - m)^2 \end{bmatrix}
\]
is psd. ■

The matrix
\[
\begin{bmatrix}
\Phi (A^2) & \Phi (A) \\
\Phi (A) & \Phi (A)
\end{bmatrix}
\]
is not always psd. It is here interesting to note the following theorem.

**Theorem 3.2.** Let $\Phi$ be as in Theorem 2.1. For $A > 0$, we have
\[
\begin{bmatrix}
\Phi (A^2) & \Phi (A) \\
\Phi (A) & \Phi (A - \log A)
\end{bmatrix} \geq O.
\]

**Proof.** We have
\[
\begin{bmatrix}
\Phi (A^2) & \Phi (A) \\
\Phi (A) & \Phi (A - \log A)
\end{bmatrix} = \sum \begin{bmatrix}
\lambda_j^2 & \lambda_j \\
\lambda_j & \lambda_j - \log \lambda_j
\end{bmatrix} \otimes \Phi (P_j).
\]
For $x > 0$, $x - \log x \geq 1$. So the Schur product
\[
\begin{bmatrix}
\lambda_j^2 & \lambda_j \\
\lambda_j & \lambda_j - \log \lambda_j
\end{bmatrix}
= \begin{bmatrix}
\lambda_j^2 & \lambda_j \\
\lambda_j & 1
\end{bmatrix}
\circ
\begin{bmatrix}
1 & 1 \\
1 & \lambda_j - \log \lambda_j
\end{bmatrix}
\]
is psd. \hfill \blacksquare

In this context, one can easily obtain the following inequalities
\[
\begin{bmatrix}
\Phi (\log M I - \log A) & \Phi (\log M A - A \log A) \\
\Phi (\log M A - A \log A) & \Phi (\log M A^2 - A^2 \log A)
\end{bmatrix} \geq O
\]
and
\[
\begin{bmatrix}
\Phi (\log A - \log m I) & \Phi (A \log A - \log m A) \\
\Phi (A \log A - \log m A) & \Phi (A^2 \log A - \log m A^2)
\end{bmatrix} \geq O.
\]

Bounds for eigenvalues in terms of the entries of the matrix have been studied extensively in literature, [13 – 16]. A special case of our Theorem 2.1 gives inequalities related to extreme eigenvalues.

**Theorem 3.3.** Let $A$ be any Hermitian element of $\mathbb{M}(n)$. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest eigenvalues of $A - \varphi(A)I$. Then the cubic equation
\[
x^3 + \frac{\beta_1}{\gamma} x^2 + \frac{\beta_2}{\gamma} x + \frac{\beta_3}{\gamma} = 0 \tag{3.6}
\]
is positive or negative according as $x = \mu_{\max}$ or $x = \mu_{\min}$, where
\[
\begin{align*}
\beta_1 &= -\varphi (B^4) \varphi (B^3) - (\varphi (B^2))^2 \varphi (B^3) + \varphi (B^2) \varphi (B^5) \\
\beta_2 &= -\varphi (B^3) \varphi (B^5) + (\varphi (B^4))^2 \varphi (B^2) - (\varphi (B^2))^2 \varphi (B^4) \\
\beta_3 &= 2\varphi (B^2) \varphi (B^3) \varphi (B^4) - (\varphi (B^2))^2 \varphi (B^5) - (\varphi (B^3))^3 \\
\gamma &= (\varphi (B^3))^2 - \varphi (B^2) \varphi (B^4) + (\varphi (B^2))^3.
\end{align*}
\]

**Proof.** It follows from (3.3), that
\[
\begin{vmatrix}
-a & \varphi (B^2) & \varphi (B^4) - a \varphi (B^2) \\
\varphi (B^2) & \varphi (B^3) - a \varphi (B^2) & \varphi (B^4) - a \varphi (B^3) \\
\varphi (B^3) - a \varphi (B^2) & \varphi (B^4) - a \varphi (B^3) & \varphi (B^5) - a \varphi (B^4)
\end{vmatrix} \geq 0. \tag{3.7}
\]
Also, $\lambda_{\min} I \leq A \leq \lambda_{\max} I$. So, $a = \lambda_{\min} - \varphi(A) = \mu_{\min}$. Expanding the determinant (3.7) we see that the expression (3.6) is non-positive. Likewise, the inequality (3.4) implies that (3.6) is non-negative for $b = \mu_{\max}$. \hfill \blacksquare
Example: Let

\[ A = \begin{bmatrix} 3 & -3\sqrt{2} & -9 \\ -3\sqrt{2} & -6 & -3\sqrt{2} \\ -9 & -3\sqrt{2} & 3 \end{bmatrix}. \]

The estimates of Wolkowicz and Styan [16] gives \( \lambda_{\min} \leq -\sqrt{48} = -6.928 \) and \( \lambda_{\max} \geq 6.928 \) and estimates of Sharma et al. [14] gives \( \lambda_{\min} \leq -\sqrt{96} = -9.792 \) and \( \lambda_{\max} \geq 9.792 \). Our Theorem 3.3 shows that the \( \lambda_{\min} \) is less than equal to smallest root of \( x^3 - 144x = 0 \). So, \( \lambda_{\min} \leq -12 \). Likewise, \( \lambda_{\max} \geq 12 \). The eigenvalues of \( A \) are \(-12, 0, 12\).

We finally remark that our technique can be extended to study inequalities involving normal matrices. Choi [7, 8] showed that for any normal element \( A \) of \( M(n) \),

\[ \Phi(A) \Phi(A^*) \leq \Phi(A^*A), \quad \Phi(A^*) \Phi(A) \leq \Phi(A^*A). \]

Equivalently

\[ \begin{bmatrix} I & \Phi(A) \\ \Phi(A^*) & \Phi(A^*A) \end{bmatrix} \succeq 0. \]

Note that the matrix

\[ \begin{bmatrix} 1 & \lambda_j & |\lambda_j|^2 \\ \bar{\lambda}_j & |\lambda_j|^2 & \overline{\lambda}_j |\lambda_j|^2 \\ |\lambda_j|^2 & \lambda_j |\lambda_j|^2 & |\lambda_j|^4 \end{bmatrix} \]

is psd. Therefore, for any positive unital linear map \( \Phi \) and for every normal matrix \( A \in M(n) \), we have

\[ \begin{bmatrix} I & \Phi(A) & \Phi(A^*A) \\ \Phi(A^*) & \Phi(AA^*) & \Phi(A^*A^2) \\ \Phi(A^*A) & \Phi(A^*A^2) & \Phi(A^*A^2A^2) \end{bmatrix} \succeq 0. \quad (3.8) \]

So, for every positive unital linear functional \( \varphi \), we have

\[ \varphi(|B|^4) \geq \frac{|\varphi(B|B|^2)|^2}{\varphi(|B|^2)} + \left( \varphi(|B|^2) \right)^2, \quad (3.9) \]

where \( B = A - \varphi(A)I \) and \( |B|^2 = B^*B \). The inequality (3.8) is a noncommutative analogue, and (3.9) is a complex analogue of the classical inequality [12]

\[ \mathbb{E}[Y^4] \geq \frac{\mathbb{E}[Y^3]^2}{\mathbb{E}[Y^2]} + \mathbb{E}[Y^2]^2, \]
where $Y = X - \mathbb{E}[X]$.

**Acknowledgements.** The authors are grateful to Prof. Rajendra Bhatia for the useful discussions and suggestions, and I.S.I. Delhi for a visit in January 2015 when this work had begun. The support of the UGC-SAP is also acknowledged.

**References**

[1] R. Bhatia, Matrix Analysis, Springer Verlag New York, 2000.

[2] R. Bhatia, C. Davis, A better bound on the variance, Amer. Math. Monthly 107 (2000) 353-357.

[3] R. Bhatia, Positive Definite Matrices, Princeton University Press, 2007.

[4] R. Bhatia, R. Sharma, Some inequalities for positive linear maps, Linear Algebra Appl. 436 (2012) 1562-1571.

[5] R. Bhatia, R. Sharma, Positive linear maps and spreads of matrices, Amer. Math. Monthly 121 (2014) 619-624.

[6] R. Bhatia, R. Sharma, Positive linear maps and spreads of matrices-II, Linear Algebra Appl. 491 (2016) 30-40.

[7] M.D. Choi, A Schwarz inequality for positive linear maps on $C^*$- algebras, Illinois J. Math. 18 (1974) 565-574.

[8] M.D. Choi, Some assorted inequalities for positive linear maps on $C^*$- algebras, J. Operator Theory 4 (1980) 271-285.

[9] C. Davis, A Schwarz inequality for convex operator functions, Proc. Amer. Math. Soc. 8 (1957) 42-44.

[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 2013.

[11] R.V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. Math. 56 (1952) 494-503.
[12] K. Pearson, Mathematical contributions to the theory of evolution, XIX: Second supplement to a memoir on skew variation, Philos. Trans. Roy. Soc. London Ser. 216(A) (1916) 429-457.

[13] R. Sharma, Some more inequalities for Arithmetic Mean, Harmonic Mean and Variance, Journal of Mathematical Inequalities 2(1) (2008) 109-114.

[14] R. Sharma, R. Bhandari, M. Gupta, Inequality related to the Cauchy-Schwarz inequality, Sankhya 74(A) (2012) 101-111.

[15] R. Sharma, R. Kumar, R. Saini, G. Kapoor, Bounds on spread of matrices related to fourth central moment, Bull. Malays. Math. Sci. Soc. DOI 10.1007/540840-015-0267-1.

[16] H. Wolkowicz, G.P.H Styan, Bounds for eigenvalues using traces, Linear Algebra Appl. 29 (1980) 471-506.