Convergence of a higher order finite difference scheme for two-phase Stefan problems.

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Abstract. This work establishes a sufficient condition for convergence of a semi-implicit finite difference scheme with variable space grid of fourth order of accuracy in space, previously set by the authors. The scheme is intended to solve one-dimensional two-phase Stefan problems. The proofs of the above-mentioned results are based on estimates which are obtained and subsequently employed in order to find an upper bound of the discretization error both at phases and the interface. The computational behavior of the scheme under the sufficient condition is corroborated by a series of numerical tests.

1. Introduction

Many tasks in science and technology are mathematically modeled through moving boundary problems, where solutions to one or several partial differential equations are sought as well as the moving boundaries of the regions where these are defined. For example, fluid flux through porous media, mass and heat transfer processes involving phase changes and different chemical reactions [1] and [2]. Phase change processes are modelled as moving boundary problems also known as Stefan problems. The isothermal nature of phase change phenomena provides appealing applications such as: thermal insulation, thermo-electric generation in concentrating solar power plants, heat recovery and thermal energy storage for domestic heating of water [3]-[6]. Only few analytic solutions to such problems are known, therefore it is more common to apply numerical methods in order to solve them; the finite difference method (FDM) is a main and fruitful approach in dealing with moving boundary value problems [7]-[15]. The present work aims to establish a sufficient condition for convergence of a novel finite difference scheme (FDS) introduced in [16].

The study of conditions to ensure the consistency, convergence and stability of FDS is a fundamental issue for their use and the numerical implementation of these schemes [17]. In particular, building transient and steady solutions in melting and freezing processes enhances the generation and employment of special technologies in different industry fields [18]. The main result proved in this material states: A sufficient condition for convergence of a semi-implicit FDS with variable space grid of higher \( O(\Delta t + (\Delta x)^4) \) order, which is intended to solve one dimensional two phase Stefan problems. This statement contributes to the development of both the theory and the implementations of FDS in the study of Stefan problems and related questions. Moving boundary problems subject to Stefan conditions are essentially non-linear. Thus, the study of requirements for establishing or comparing the convergence of FDS is a complex task, which lacks of general procedures (as for example Lax-Richtmyer equivalence theorem, von Neumann stability and other conditions [17]-[20] and [21]). Hence, convergence and its rate for difference schemes regarding classical and generalized
problems are often studied by considering their specificity [22]-[24]. Occasionally, arguments are based only on numerical experiments.

The convergence to classical solutions of FDS involving the central scheme of second order has been studied for one-dimensional one-phase problems in several modern works [13], [25]-[29]. Notice as well, that these type of results on convergence play an important role in studying the existence of classical and weak solutions to boundary value problems of all kind. The main tool in order to prove the theorem is based on estimates of the discretization errors for the solution at phases and the interface. These estimates are obtained through a careful and intricate analysis of the approximate solution. It is worth mentioning that we have not found in the literature any recent papers related to the convergence of FDS for classical solutions of Stefan problems.

This material is organized in several sections as follows: section 2 is devoted to the statement of the moving boundary problem and the FDS introduced in [16]; in section 3, a sufficient condition for convergence of the scheme is formulated and proved; in section 4 presents numerical experiments which illustrate the sufficient condition set above, and finally section 5 summarizes the conclusions of the work.

2. Two-phase Stefan problem and the associated difference scheme.

A previous work [16] which studies the temperatures in two phases of a one-dimensional finite domain and the behavior of the interface which divides the domain, is briefly described. The thermal diffusivities within each phase are assumed to be constant and equal to $\alpha_i = k_i/\rho_iC_i$, for $i = 1,2$ which corresponds to phase 1 (2) or liquid(solid). Here, $k_i$ is the thermal conductivity, $\rho_i$ is the density and $C_i$ is the specific heat capacity at phase $i$. We will assume that, $\rho_i = 1$ and $C_i = 1$. The task is mathematically modeled through the following moving boundary value problem

$$\frac{\partial u_1(x,t)}{\partial t} = \alpha_1 \frac{\partial^2 u_1(x,t)}{\partial x^2}, \text{with} \quad 0 < x < \xi(t), 0 < t \leq T,$$

$$\frac{\partial u_2(x,t)}{\partial t} = \alpha_2 \frac{\partial^2 u_2(x,t)}{\partial x^2}, \text{with} \quad \xi(t) < x < L, 0 < t \leq T,$$

$$u_i(x,0) = f_i(x), i = 1,2; 0 < x \leq L,$$

$$u_1(x,t)|_{x=0} = u_l, 0 \leq t \leq T,$$

$$u_2(x,t)|_{x=L} = u_s, 0 \leq t \leq T,$$

$$u_1(x,t)|_{x=\xi(t)} = u_2(x,t)|_{x=\xi(t)} = u_f, 0 \leq t \leq T,$$

$$\rho_i u_f \frac{d\xi(t)}{dt} = k_2 \frac{\partial u_2(x,t)}{\partial x} \bigg|_{x=\xi(t)} - k_1 \frac{\partial u_1(x,t)}{\partial x} \bigg|_{x=\xi(t)}, 0 < t \leq T,$$

$$\xi(t)|_{t=0} = \xi_0, \quad 0 < \xi_0 < L,$$

where the functions $u_l(0,t), u_s(L,t), u_f, f_1(x), i = 1,2$ are assumed to be continuous and $f_2(x) \leq u_f \leq f_1(x)$. In addition, the classical solutions $u_i(x,t), i = 1,2$ of equations (1)-(8) are considered to be smooth enough. It is important to note that in the Stefan condition given by equation (7), the density appearing on the left hand side strongly depends on the direction of the phase transition as discussed in [16], if $\frac{d\xi(t)}{dt} > 0$ then $i = 1$ and if $\frac{d\xi(t)}{dt} < 0$ then $i = 2$. Note that equation (7) involves the mathematical modeling of the phase change process, taking into account the dimensions of the variables and parameters which describe the physical problem.

In order to simplify the exposition we assume that, $\rho_i = 1$ and $C_i = 1, i = 1,2$.

Immediately the following non classical finite difference scheme (NC-FDS) is used to solve equations (1)-(8).

$$\beta^{(1)}_1 m^{-1} n u_1 + \beta^{(2)}_1 m n u_1 + \rho^{(1)}_1 m+1,n u_1 - m+1,n^{-1} u_1 - 10 m,n^{-1} u_1 - m^{-1} n^{-1} u_1 = 0, \quad 1 \leq m \leq N_t, 1 \leq n \leq N_t$$

(9)
\[ \beta^{(1)}_{2} m^{-1} u_2 + \beta^{(2)}_{2} m n u_2 + \beta^{(1)}_{2} m + 1, n u_2 - m + 1, n - 1 u_2 - 10 m, n - 1 u_2 - m, n - 1 u_2 = 0, \quad N_1 \leq m \leq N_1 + N_2, 1 \leq n \leq N_t, \]

\[ m, n u_i = m f^0_i, \quad i = 1, 2; \]

\[ 0, n u_4 = u_4^0, \]

\[ N_2, n u_2 = u_2^L, \]

\[ N_1, n u_1 = N_1, n u_2 = u_f, \]

\[ \xi_{n+1}^u = \xi_{n}^u + \frac{k_2 \Delta t}{\rho L_f} n, n u_2^{(1)} - \frac{k_1 \Delta t}{\rho L_f} N_1, n u_1^{(1)} \]

\[ \xi(0) = \xi_0, \quad 0 < \xi < L \]

Where

\[ u_i^{(1)} = \frac{j}{12 \Delta x_i^2} \left( -25 m, n u_i + 48 m + j n u_i - 36 m + j n u_i + 16 m + 3j n u_i - 3 \right) \]

Here \( j = -1 \) for \( i = 1 \) (medium 1), and \( j = 1 \) for \( i = 2 \) (medium 2). Where \( \Delta x_i^n \) represents the space step of medium \( i \) at level \( n \) and \( \Delta t \) the time step.

In equations (9)-(17) the functions \( u \) and \( \xi \) are grid functions, \( m^i n g = g(x_m, t_n) \), \( \beta_1^{(1)} = (1 - 12 \lambda_i^n) \), \( \beta_2^{(2)} = (10 + 24 \lambda_i^n) \) and \( \lambda_i^n = \alpha_i \Delta t / (\Delta x_i^n)^2 \) for \( i = 1, 2 \). The equations (9)-(14) for the phases stand for the six-point stencil shown in figure 1.

The operation of the scheme for the interface, from equations (15)-(17) is shown in the stencil according to figure 2.

3. Convergence of the NC-FDS.
In this section we find a sufficient condition for the convergence of the NC-FDS equations (9)-(17) to the classical solution of equations (1)-(8).
Since existence and uniqueness of classical solution of equations (1)-(8) is ensured (see [25]), the strategy for finding a sufficient condition consists in suitably estimating the discretization error (for temperatures and interface) under a certain hypothesis, which involves a ratio of time and space step in the grid and some physical characteristics of the problem. Then, convergence of equations (9)-(17) is shown by establishing that the discretization error estimate approaches zero when the mesh is refined. In this analysis, we consider not only the values of the variables at the nodes of the mesh but also in points outside the nodes for the previous time, which are computed using linear interpolation (see figure 2). Pay attention to the fact that the scheme is of the front-tracking type, implicit with variable grid and fourth order in the spatial variable. Let us consider the discretization errors of the temperatures for each phase, which are defined at each node by

$$m_n e_l = m_n U_i - m_n U_l$$

(18)

where $U$ is the exact solution of our problem and $u$ is the approximate solution.

In the following, we will analyze convergence of the finite difference scheme in equations (9)-(17), or equivalently $m_n e_l \to 0$ when $\Delta x_i^n \to 0, \Delta t \to 0$, at each phase.

As in [16] acting on the exact solution we obtain

$$\beta_i^{(1)} m_{n+1, i} U_i + \beta_i^{(2)} m_{n, i} U_i + \beta_i^{(1)} m_{n-1, i} U_i - 10 m_{n-1, i} U_i - m_{n+1, i} U_i - m_{n-1, i} U_i + O(\Delta t^2)$$

$$+ O(\Delta t \cdot (\Delta x_i)^4) = 0$$

(19)

From equations (9), (10), (18) and (19) it follows

$$\beta_i^{(1)} m_{n+1, i} e_i + \beta_i^{(2)} m_{n, i} e_i + \beta_i^{(1)} m_{n-1, i} e_i - 10 m_{n-1, i} e_i - m_{n+1, i} e_i - m_{n-1, i} e_i + O(\Delta t^2)$$

$$+ O(\Delta t \cdot (\Delta x_i)^4) = 0$$

(20)

Errors at level $n$ must be minimized with respect to errors at level $n-1$, thus equation (20) is written as

$$\beta_i^{(2)} m_{n+1, i} e_i = -\beta_i^{(1)} (m_{n+1, i} e_i + m_{n-1, i} e_i) + 10 m_{n-1, i} e_i + m_{n+1, i} e_i + m_{n-1, i} e_i + O(\Delta t^2)$$

$$+ O(\Delta t \cdot (\Delta x_i)^4)$$

(21)

By applying absolute value and triangle inequality to equation (21). It is observed that the discretization error can be estimated adequately when $\lambda_i \geq 1/12$. In effect

$$(10 + 24\lambda_i^2)|m_{n, i} e_i| \leq (12\lambda_i^2 - 1)|m_{n+1, i} e_i| + (12\lambda_i^2 - 1)|m_{n-1, i} e_i| + 10|m_{n-1, i} e_i|$$

$$+ |m_{n+1, i} e_i| + |m_{n-1, i} e_i| + O(\Delta t^2 + \Delta t \cdot (\Delta x_i)^4)$$

(22)

Let us put,

$$e_i(q) = \max_{1 \leq m \leq N_i} |m_{q, i} e_i|, \quad 1 \leq q \leq N_i \quad (i = 1, 2)$$

(23)

Then, the substitution of equation (23) into equation (22) yields

$$e_i(n) \leq e_i(n-1) + O(\Delta t^2 + \Delta t \cdot (\Delta x_i)^4)$$

(24)

Doing the iterations from $n = 1$ to $n = N_i$, to obtain

$$e_i(N_i) \leq N_i \cdot O(\Delta t^2 + \Delta t \cdot (\Delta x_i)^4) = N_i \cdot \Delta t \cdot O(\Delta t + (\Delta x_i)^4) = t_{max} \cdot O(\Delta t + (\Delta x_i)^4)$$

(25)

Where it is observed that $e_i(N_i) \to 0$ when $\Delta x_i \to 0, \Delta t \to 0$. This is, what we wanted to demonstrate for the phases.

Let us consider the discretization errors at the interface, which are defined through

$$e_n = \xi_n^U - \xi_n^I$$

(26)

In this case, when $\Delta x_i \to 0, \Delta t \to 0$, we must show that $e_n \to 0$. Developing this part, we have for equation (20) considering the exact solution that
\[ \xi_{n+1}^U = \xi_n^U + \frac{k_2 \Delta t}{L_f \rho} n_i n u_2^{(i)} - \frac{k_1 \Delta t}{L_f \rho} n_i n u_1^{(i)} + O(\Delta t^2) + O(\Delta t \cdot (\Delta x_i)^4) \]  

(27)

Substituting equation (27) in equation (26) and taking equation (17) into account we obtain

\[ e_{n+1} = e_n + \frac{k_2 \Delta t}{12 \Delta x^2 L_f \rho} (-25 n_i n e_2 + 48 n_i+1 n e_2 - 36 n_i+2 n e_2 + 16 n_i+3 n e_2 - 3 n_i+4 n e_2) \]

\[ - \frac{k_1 \Delta t}{12 \Delta x^2 L_f \rho} (-25 n_i n e_1 + 48 n_i-1 n e_1 - 36 n_i-2 n e_1 + 16 n_i-3 n e_1 - 3 n_i-4 n e_1) + O(\Delta t^2) \]

\[ + O(\Delta t \cdot (\Delta x_i)^4) \]

(28)

By applying absolute value and triangle inequality to equation (28), keeping in mind equation (23), considering \( \Delta x_i \leq \Delta x \) and \( k = \max(k_1, k_2) \), we have

\[ |e_{n+1}| \leq |e_n| + \frac{32k \Delta t}{3 \Delta x^2 L_f \rho} (e_2(n) + e_1(n)) + O(\Delta t^2) + O(\Delta t \cdot (\Delta x_i)^4) \]

(29)

Doing the iterations from \( n = 1 \) to \( n = N_t - 1 \), to obtain

\[ |e_{N_t}| \leq N_t \cdot (O(\Delta t^2) + O(\Delta t \cdot (\Delta x_i)^4)) + \frac{32k \Delta t}{3 \Delta x^2 L_f \rho} \sum_{p=1}^{N_t-1} (e_2(p) + e_1(p)) \]

(30)

Considering \( E = \max(e_2(n), e_1(n)) \) also, in the phase \( E(N_t - 1) \leq t_{\text{max}} \cdot O(\Delta t + (\Delta x_2)^4) \), keeping in mind that \( N_t \cdot \Delta t = t_{\text{max}} \) and from the condition for \( \lambda_i = \frac{\alpha_i \Delta t}{(\Delta x_i)^2} \geq 1/12 \) we have that \( \Delta t / \Delta x_2 = O(\Delta x_2) \), we have

\[ |e_{N_t}| \leq t_{\text{max}} \cdot (O(\Delta x_1^4) + O(\Delta t)) + \frac{64k(t_{\text{max}})^2}{3 L_f \rho} O(\Delta x_2 + \Delta x_3^2) \]

(31)

Hence \( |e_{N_t}| \rightarrow 0 \), when \( \Delta x_1, \Delta t \rightarrow 0 \). This is, what we wanted to demonstrate for the interface.

Considering the \( \sup norm \| m,n \|_\infty \equiv \sup_{1 \leq m \leq M, 1 \leq n \leq N} \| m,n \|_\infty \) and with the appropriate assumptions on the existence, uniqueness and smoothness of the classical solution to equations (1)-(8), it follows the convergence of the proposed difference scheme.

**Theorem:** Sufficient condition for convergence of the FDS of order \( O((\Delta x_2)^4) \). Suppose \( u_1(x, t) \in C^{1,0,5}(0, \xi(t) \times 0, T) \), \( u_2(x, t) \in C^{1,0,5}(\xi(t), L \times 0, T) \), \( \xi(t) \in C^5(0, T) \) and \( m,n u_1, m,n u_2, \xi, 0 \leq m \leq M, 0 \leq n \leq N \) are solutions of the problem equations (1)-(8) and the difference scheme of equations (9)-(17) respectively, then a sufficient condition for the convergence in the \( \sup norm \| - \|_\infty \) of the NC-FDS to the solution of the differential problem is that

\[ \lambda_i^n \geq 1/12 \]

(32)

where \( \lambda_i^n = \alpha_i \Delta t / (\Delta x_i^2) \) for \( i = 1, 2 \), and \( \Delta x_i^n \) stands for the space step in each medium at level \( n \). The NC-FDS is of order \( O(\Delta t + (\Delta x_2)^4) \).

4. Numerical results.

In this section, the sufficient condition from theorem is illustrated. Let us consider a classical Stefan problem. An insulated vessel along its length contains a two-phase mixture of a substance whose ends are held at fixed temperatures.

The initial temperature profile used for this example shown in table 1 is a step-function like profile, where the temperature of the liquid(solid) is 1.0(−0.40). These temperatures were calculated for small time values, which is why the temperatures shown in table 1 are close to the initial temperatures at each phase. In this example, a value of \( \Delta t = 6.6667 \times 10^{-5} \) and initial values of \( \Delta x_1 = \Delta x_2 = 0.01 \) were chosen. A thermal diffusivity of \( \alpha_1 = k_1 = 0.15(\alpha_2 = k_2 = 0.20) \) was used for the liquid(solid) phase, hence \( \rho_1 = 1, C_1 = 1 \) for \( i = 1, 2 \), while \( \xi(0) = 0.20, T_f = 0 \) and \( L_f = 2.0 \) in this system of units. The above values were chosen so the sufficient condition shown in theorem is met by \( \lambda_1(\lambda_2) \), which are \( \lambda_1 = 0.10000(\lambda_2 = 0.13333) \) in this example.
Table 1. Temperatures at three adjacent nodes to the interface, for the liquid and solid phases when the sufficient condition holds.

|    | \(u_{N_1-3}\) | \(u_{N_1-2}\) | \(u_{N_1-1}\) | \(u_{N_1}\) | \(u_{N_1+1}\) | \(u_{N_1+2}\) | \(u_{N_1+3}\) |
|----|----------------|----------------|----------------|-------------|-------------|-------------|-------------|
| 1  | 0.99990        | 0.99393        | 0.86293        | 0.00000     | -0.35271    | -0.39782    | -0.39990    |
| 2  | 0.98227        | 0.89653        | 0.59553        | 0.00000     | -0.24779    | -0.36115    | -0.39286    |
| 3  | 0.93137        | 0.77916        | 0.45882        | 0.00000     | -0.18990    | -0.31313    | -0.37132    |

Each row corresponds to a different time value of the simulation. As observed in Table 1, the solution lies within the expected temperature range. On the other hand, when the sufficient condition of the theorem is not satisfied, some temperature values are observed to lie outside the expected range, as shown in Table 2.

Table 2. Temperatures at three adjacent nodes to the interface, for the liquid and solid phases when the sufficient condition does not hold.

|    | \(u_{N_1-3}\) | \(u_{N_1-2}\) | \(u_{N_1-1}\) | \(u_{N_1}\) | \(u_{N_1+1}\) | \(u_{N_1+2}\) | \(u_{N_1+3}\) |
|----|----------------|----------------|----------------|-------------|-------------|-------------|-------------|
| 1  | 0.99998        | 1.00057        | 0.92414        | 0.00000     | -0.37319    | -0.40046    | -0.39999    |
| 2  | 0.00006        | 0.99568        | 0.86163        | 0.00000     | -0.35037    | -0.39896    | -0.40005    |
| 3  | 0.99986        | 0.98731        | 0.80918        | 0.00000     | -0.33070    | -0.39608    | -0.40002    |

In the example shown in Table 2, the time increment and mesh size are kept as before, but the thermal diffusivities used are \(\alpha_1 = k_1 = 0.075\) (\(\alpha_2 = k_2 = 0.10\)) which give values of \(\lambda_1 = 0.05\) (\(\lambda_2 = 0.06667\)) that do not satisfy the sufficient condition of the theorem. As can be observed in Table 2, some of the temperature values lie outside the expected range, illustrating a numerical instability as a consequence of the theorem.

Notice that the theorem could be immediately extended to the case of Neumann boundary conditions instead of Dirichlet boundary conditions [27]. Theorem is illustrated in figures 3, 4 and 5 for a system with Neumann boundary conditions. The system size is \(L = 1\) and the initial interface position is \(\xi(0) = 0.20\). The initial temperature profile is a step-function like profile, where the liquid has a temperature of 1.0 for \(0 < x_1 < 0.20\) and the solid phase has a temperature of 0.20 for \(0.20 < x < 1.0\). High thermal diffusion constants \(\alpha_1 = k_1 = 1.5\) and \(\alpha_2 = k_2 = 2.0\) are chosen in order to facilitate the use of coarse grids and still satisfy the sufficient condition of the theorem. Coarse grids are used to compare the convergence speed between the fourth order scheme and the classical one. In figure 3, this is illustrated by showing the interface dynamics for two examples where the latent heat of fusion is \(L_f = 2.0\) in figure 3.a and \(L_f = 2.5\) in figure 3.b. In both cases, a small quantity of nodes is used within the liquid and solid phases, with \(N_1 = 19\) and \(N_2 = 35\).
Figure 3. Comparison between the higher order and classical schemes with the solution [8] over a semi-infinite domain, for the time evolution of the interface with a) $L_f = 2.0$ and b) $L_f = 2.5$

The solutions obtained with each scheme are compared to the solution for a semi-infinite domain with Neumann boundary conditions [8]. As it can be observed from figure 4, for small time values, the higher order scheme shows a better approach to the solution [8], than the classical scheme. Due to finite size effects, only small time values can be used for comparison with the solution [8]. For the same examples shown in figure 3, the greater convergence speed of the higher order scheme is also illustrated in figure 4 by comparing the solutions for the temperature evolution at $x = 0.25$ within a small time window.

Figure 4. Time evolution for the temperature at $x = 0.25$. Finite difference schemes $O(\Delta x^4)$ and $O(\Delta x^2)$ compared with the solution [8] for a) $L_f = 2.0$ and b) $L_f = 2.5$
For large time values, i.e. close to the steady regime, it has been shown that exact analytical expressions exist for the interface position \([13] \text{ and } [27]\), and these stationary values are strongly dependent on the boundary conditions. In figure 5 two examples are shown, where Neumann boundary conditions are imposed on a system of size \(L = 1\) m. By using conservation of energy, it was shown in \([27]\) that the interface position reaches a stationary value that depends on the initial temperature profile, latent heat of fusion, the specific heat capacity and density of each phase. Values for this thermodynamic variables are taken from \([13]\), and correspond to Aluminum. Given that Neumann boundary conditions imply that energy must be conserved, the order of approximation of the FDS becomes extremely important at large time values. The example shown in figure 5, shows the time evolution of the interface position once the steady state is reached. Numerical simulations become time consuming because preserving energy requires a considerable increment in the number of nodes in order to approach properly the stationary value. The maximum number of nodes used, were \(N_1 = 1101 (N_2 = 1651)\) in figure 5.a and \(N_1 = 1651 (N_2 = 1101)\) in figure 5.b, and \(\Delta t = 0.5s\) in each example. Thermal diffusivities at each phase were obtained from the thermodynamic data of \([13]\) as \(\alpha_1 = k_1/\rho_1C_1\) \((\alpha_2 = k_2/\rho_2C_2)\), where \(C_1 (C_2)\) are the specific heat capacities and \(\rho_1 (\rho_2)\) the densities at each phase. With the chosen time increment \(\Delta t\), number of nodes and values of the thermodynamic variables for Aluminum, the sufficient condition of theorem is met at every time step of the simulations. As it can be observed in figure 5, the classical scheme overestimates the interface position close to the stationary state, whereas the higher order scheme approaches this value from below on figure 5.a on a melting case, and from above in a solidification scenario as shown in figure 5.b; notice that, the classical scheme overestimates the interface position in both cases.

\[
\begin{align*}
\alpha_1 &= k_1/\rho_1C_1 \quad (\alpha_2 = k_2/\rho_2C_2) \\
\rho_1 &\quad (\rho_2) \\
C_1 &\quad (C_2)
\end{align*}
\]

5. Conclusion
The present work considers a novel non-classical finite different scheme with variable space grid for a two-phase one-dimensional Stefan problem. A sufficient condition on the step-sizes of the mesh for the convergence of the scheme has been established and proved. In addition, a strategy based on the study of the discretization error is developed in order to prove the convergence of finite difference schemes for moving boundary value problems. A series of numerical experiments have been...
conducted in order to validate the previous statements. These experiments indicate a greater accuracy of the novel scheme in capturing the steady-state solution of the problem.

Finally, the presented procedure can be performed for the study of the well-posedness of different moving boundary value problems and associated mathematical models. The assertions presented here are a contribution to the theory for designing efficient algorithms to solve moving boundary problems with Stefan conditions.

6. References

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