Finite-$N$ corrections to the superconformal index of toric quiver gauge theories

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The superconformal index of quiver gauge theories realized on D3-branes in toric Calabi–Yau cones is investigated. We use the AdS/CFT correspondence and study D3-branes wrapped on supersymmetric cycles. We focus on brane configurations in which a single D3-brane is wrapped on a cycle, and we do not take account of branes with multiple wrapping. We propose a formula that gives finite-$N$ corrections to the index caused by such brane configurations. We compare the predictions of the formula for several examples with the results on the gauge theory side obtained by using localization for small sizes of gauge groups, and confirm that the formula correctly reproduces the finite-$N$ corrections up to the expected order.

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1. Introduction

The anti-de Sitter / conformal field theory (AdS/CFT) correspondence [1–3] has been extensively investigated, and a lot of evidence has been found. However, the majority of previous works have concerned the large-$N$ limit. One reason for this is that as the parameter relation

$$N \sim \frac{L^4}{l_p^4}$$  \hspace{1cm} (1.1)

shows, if $N$ is small the Planck length $l_p$ becomes comparable with the AdS radius $L$ and the quantum gravity effect is expected to be relevant. Even so, we may be able to obtain non-trivial evidence for the AdS/CFT correspondence with finite $N$ by analyzing quantities protected by supersymmetry. In this paper we use the superconformal index [4] as such a protected quantity, and study the AdS/CFT correspondence for finite $N$. Even if the quantum gravity correction does not affect the index there is another source of finite-$N$ corrections. The parameter relation in Eq. (1.1) can be rewritten in terms of the D3-brane tension $T_{D3} \sim l_p^{-4}$ as

$$N \sim L^4 T_{D3},$$  \hspace{1cm} (1.2)

and we can regard $N$ as a typical value of the classical action of a D3-brane wrapped on a cycle with scale $L$. Namely, D3-branes wrapped on supersymmetric cycles in the internal space contribute to the superconformal index as a finite-$N$ correction. Such wrapped branes were first studied in Ref. [5] in the context of the AdS/CFT correspondence, and found to correspond to baryonic operators on the gauge theory side. The purpose of this paper is to investigate this correspondence at the level of the superconformal index. We have already performed similar analyses for S-fold theories [6] and
orbifold quiver gauge theories [7]. In this paper we consider toric quiver gauge theories as a natural extension of them.

A toric quiver gauge theory is defined as a gauge theory realized by putting D3-branes in a toric Calabi–Yau three-fold. There is a systematic prescription [8,9] to determine the gauge theory from the toric data of the Calabi–Yau. The holographic dual is type IIB string theory in $\text{AdS}_5 \times \text{SE}_5$, where $\text{SE}_5$ is the Sasaki–Einstein manifold which is the base of the Calabi–Yau cone. This relation provides an infinite number of examples of the AdS/CFT correspondence.

On the gauge theory side, in principle, the index can be calculated with the help of the localization formula, which includes the matrix integral. For a small class of theories, including the $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang–Mills theory and $\mathcal{N} = 2$ circular quiver gauge theories, the matrix integral in the Schur limit has been carried out and analytic formulas obtained [10,11]. However, in general we need to rely on numerical analysis, and it is difficult practically to obtain the index up to the desired order for a gauge group with large rank. In this paper we consider only theories with $\mathcal{N} = 2$ and $\mathcal{N} = 3$, for which we can calculate the index within a realistic time.

On the gravity side we calculate the finite-$\mathcal{N}$ correction as the contribution of wrapped D3-branes. In this paper we take account of brane configurations with single wrapping. Although configurations with multiple wrapping would also contribute to the index, they give higher-order terms than single-wrapping ones, and we neglect them in this paper. Concretely, we propose the following relation:

$$I = I_{\text{KK}} \left( 1 + \sum_I T_{S_I}^{D3} + \cdots \right), \quad (1.3)$$

where $T_{S_I}^{D3}$ is the contribution of a D3-brane wrapped on a supersymmetric three-cycle $S_I$ corresponding to a corner of the toric diagram. The dots represent the contribution of multiple wrapping, which is out of scope of this paper.

This paper is organized as follows. In the next section we summarize basic aspects of toric geometry and the corresponding quiver gauge theories. We define $U(1)$ charges for general toric quiver gauge theories and define the superconformal index in terms of them. In Sect. 3 we propose an explicit expression for the D3-brane contribution $T_{S_I}^{D3}$ in Eq. (1.3). We give a simple prescription to calculate it for each supersymmetric cycle from the toric data depicted as the toric diagram. In Sect. 4 we confirm the correctness of the formula by calculating the index on both sides of the AdS/CFT correspondence. We show the results for $\mathcal{N} = 2$ in the main text, and results for $\mathcal{N} = 3$ in Appendix B. In Sect. 5 we summarize our results. The appendices give additional results and explanations.

2. Toric geometry and quiver gauge theories

2.1. Toric data

By definition, a toric Calabi–Yau $n$-fold has $U(1)^n$ isometry, and we can describe it as $T^n$ fibration over an $n$-dimensional base $B$, which is a closed subset of $\tilde{\mathcal{V}} = \mathbb{R}^n$. The action of $U(1)^n$ is not free and the fiber reduces to a lower-dimensional torus on the boundary of $B$.

The toric diagram of a toric Calabi–Yau three-fold is a convex polygon drawn in the two-dimensional lattice $L_\mathbb{Z} \subset L = \mathbb{R}^2$. We label lattice points on the boundary of the diagram by $I = 1, 2, \ldots, d$ in counter-clockwise order, where $d$ is the perimeter of the diagram. We treat the label $I$ as a cyclic variable with the relation $I + d \approx I$. We label edges on the boundary of the toric diagram by $r \in \mathbb{Z} + 1/2$, which is also treated as a cyclic variable. We assign $r = I + 1/2$ to the edge between vertices $I$ and $I + 1$. Let $V_I$ be the two-dimensional integral vector pointing to the vertex $I$ in $L_\mathbb{Z}$. We also define three-dimensional integral vectors $V_I = (V_I, 1)$. We denote the
three-dimensional lattice $\mathbb{Z}^3$ with $V_I$ defined in it by $V_{\mathbb{Z}} \in \mathcal{V} = \mathbb{R}^3$. $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are dual spaces to each other and the inner product $\mathcal{V} \otimes \tilde{\mathcal{V}} \rightarrow \mathbb{R}$ is defined. The polyhedral cone in $\mathcal{V}$ spanned by $V_I$ is called the toric cone (see Fig. 1(a)).

The toric data expressed as the set of vectors $V_I \in \mathcal{V}_{\mathbb{Z}}$ has two roles. One is to define the base manifold $B$. $B$ is defined as the dual cone of the toric cone by

$$B = \{y \in \tilde{\mathcal{V}} \mid V_I \cdot y \geq 0 \ \forall I\} \quad (2.1)$$

(see Fig. 1(b)). For each $I$ we define the facet $F_I$ by

$$F_I = \{y \in B \mid V_I \cdot y = 0\}. \quad (2.2)$$

Let $R_r$ with $r = I + 1/2$ be the ridge shared by the two adjacent facets $F_I$ and $F_{I+1}$. If a vertex $I$ is not a corner but is on the side between two corners $I_1$ and $I_2$, Eq. (2.2) gives not a plane but a line. We can regard such an $F_I$ as a shrinking facet.

The five-dimensional Sasaki–Einstein manifold $SE_5$ is the subset of the Calabi–Yau cone defined by $\rho = 1$. The radial coordinate $\rho$ in the Calabi–Yau cone is given by $\rho = b \cdot y$, where $b \in \mathcal{V}$ is a particular vector inside the toric cone called the Reeb vector. The cross section of $B$ defined by $b \cdot y = 1$ is a polygon $P$. The Sasaki–Einstein manifold $SE_5$ is given as the $T^3$ fibration over $P$. We denote the edge $F_I \cap P$ and the corner $R_r \cap P$ by $E_I$ and $C_r$, respectively.

The fiber shrinks in a particular manner on the boundary of $P$. The second role of the toric data is to specify how the fiber shrinks on the boundary. At a generic point in $P$ the fiber is $T^3$, which is identified with $\mathcal{V}/\mathcal{V}_{\mathbb{Z}}$, and we can specify a cycle in the $T^3$ by a vector in $\mathcal{V}_{\mathbb{Z}}$. On an edge $E_I$ the cycle $V_I$ shrinks, and at a generic point in $E_I$ the fiber is $T^2$. At a corner $C_r$ two cycles $V_{r+1/2}$ shrink at the same time and the fiber becomes $S^1$. For each edge $E_I$ the fibration over $E_I$ gives a closed three-dimensional manifold, which we denote by $S_I$.

If $I_1$ and $I_2$ are two adjacent corners of the toric diagram and $k := I_2 - I_1 > 1$ then there are $k - 1$ vertices between $I_1$ and $I_2$, and the corresponding cycles $S_I$ are shrinking at the degenerate corners $C_{I_{n+1/2}} = \cdots = C_{I_{n-1/2}}$. This means the existence of $A_k$ type singularity along the $S^1$ fiber over the degenerate corners. Each shrinking three-cycle $S_I$ ($I_1 < I < I_2$) is the direct product of the $S^1$ and a shrinking two-cycle at the singularity.

### 2.2. Quiver gauge theories

There is an algorithm [8,9] to obtain from the toric data the quiver diagram of the gauge theory realized on D3-branes placed at the apex of a toric manifold. We will not explain it in detail but comment only on facts relevant to our analysis. It is known that the number of anomaly-free $U(1)$ global internal symmetries is the same as the perimeter $d$ of the toric diagram. They include

- one superconformal $R$-symmetry,
two mesonic symmetries, and

\( d - 3 \) baryonic symmetries.

The mesonic and baryonic symmetries are also called flavor symmetries. The \( R \)-symmetry and the mesonic symmetries act on the \( SE_5 \) as isometries. If the manifold has a non-Abelian isometry group they are the Cartan part of it.

An \( R \)-symmetry is defined as a symmetry that acts on the supercharges in a specific way. This condition, however, does not fix the \( R \)-charge, and there is an ambiguity to mix the other \( d - 1 \) charges that do not act on supercharges. We can determine the \( R \)-charge appearing in the superconformal algebra by using \( a \)-maximization [12] (or, equivalently, volume minimization [13] on the gravity side.)

For analysis of a general toric quiver gauge theory there is a convenient basis of \( d \) independent charges rather than the basis associated with the classification above. We denote them by \( R_I \) \((I = 1, \ldots, d)\), and each of them corresponds to a vertex on the boundary of the toric diagram. We use a bipartite graph to connect a toric diagram and a corresponding quiver gauge theory, and on the gauge theory side the \( R_I \) are defined with perfect matchings in the bipartite graph [14–16]. We normalize them so that all bi-fundamental fields carry charges 0 or +1. All the \( R_I \) act on the supercharge in the same manner and hence they are all \( R \)-charges with unusual normalization; the supercharges carry \( R_I = \pm 1/2 \). There is a prescription to associate each perfect matching on the graph with a vertex in the toric diagram. For each corner of the toric diagram there is a unique perfect matching and we can uniquely define the corresponding \( R_I \), while for other vertices on the boundary we have more than one perfect matching and we need to choose one of them to define \( R_I \). These \( d \) charges form a basis of the global symmetries, and all charges listed above are linear combination of \( R_I \) in the form

\[
\sum_{I=1}^{d} c_I R_I.
\] (2.3)

For an \( R \)-charge with the standard normalization \( \sum_{I=1}^{d} c_I = 2 \), while for a flavor symmetry \( \sum_{I=1}^{d} c_I = 0 \). An important and convenient property of these charges is that their geometric action on \( SE_5 \) is specified by \( V_I \). Therefore, for a baryonic symmetry, which has no geometric action, the coefficients satisfy \( \sum_{I=1}^{d} c_I V_I = 0 \). This is consistent with the fact that the rank of the baryonic symmetry is \( d - 3 \).

On the gravity side \( R_I \) is the angular momentum associated with the geometric action \( V_I \). A wrapped D3-brane can carry angular momenta even if it stays still due to the coupling to the background RR 4-form potential field \( C_4 \), and the values depend on the gauge choice of \( C_4 \). We can specify the gauge choice by specifying the singular locus of the potential field just like the Dirac string of a Dirac monopole. The singularity of \( C_4 \) is expressed as a three-cycle in \( SE_5 \). \( R_I \) is defined with \( C_4 \) with the singularity on \( S_I \). With this definition we can show that a D3-brane wrapped over \( S_I \) carries

\[
R_I' = N \delta_{II'}.
\] (2.4)

For a vertex \( I \) which is not a corner there exist more than one perfect matching, and hence the definition of \( R_I \) is ambiguous. This is related to the ambiguity of choosing a basis of three-cycles at the singularity. At an \( A_{k-1} \) singularity there are \( k - 1 \) shrinking cycles. We can relate these cycles to simple roots of \( A_{k-1} \) algebra, and we have ambiguity in the choice of the simple roots from the root system. There are \( k! \) bases related by Weyl reflections. Once we fix a basis of shrinking two-cycles we can choose \( R_I \) so that the charge relation in Eq. (2.4) holds.
On the gauge theory side the baryonic symmetry is defined as the anomaly-free subgroup of the abelian group defined by replacing all gauge groups $SU(N)$ by $U(1)$. It has, in general, the form

$$U(1)^{d-3} \times G_{\text{disc}},$$

where $G_{\text{disc}}$ is a discrete group. We neglect $G_{\text{disc}}$ in the main text for simplicity, and provide some analysis of $G_{\text{disc}}$ in Appendix C.

### 2.3. Superconformal index

We define the superconformal index by

$$I = \text{tr}_{\text{BPS}} \left[ e^{2\pi i (J+\overline{J})} q^{\frac{3}{2} J} \prod_{l=1}^{d} v_I^{R_l} \right],$$

where the $R_l$ are the $R$-charges defined in the previous subsection and $J$ and $\overline{J}$ are the Lorentz spins. The trace $\text{tr}_{\text{BPS}}$ is the summation over states (operators) saturating the BPS bound

$$\{ \overline{Q}, Q \} = H - 2\overline{J} - \frac{3}{2} r_* \geq 0,$$

where $Q$ is the supercharge with the quantum numbers

$$[J, Q] = 0, \quad [\overline{J}, Q] = -\frac{1}{2} Q, \quad [R_l, Q] = \frac{1}{2} Q.$$  

For Eq. (2.6) to be the superconformal index the fugacities must satisfy

$$\prod_{l=1}^{d} v_I = q^{3}.$$  

Notice that to calculate the index in Eq. (2.6) we do not have to know the superconformal $U(1)_R$ charge $r_*$.  

In the numerical calculation we necessarily introduce a maximum order at which we cut off the infinite series. For this purpose we need to define “the order” for each term in the index as a linear combination of quantum numbers appearing in the index as exponents of fugacities. We can use the linear combination $3\overline{J} + \frac{3}{2} r$ of $\overline{J}$ and an appropriately chosen $R$-charge $r$. A choice of $r$ given by

$$r = \sum_{l=1}^{d} c_l R_l, \quad \sum_{l=1}^{d} c_l = 2 \quad (2.10)$$

is implemented in the calculation by the variable change

$$v_I = u_I q^{\frac{3}{2} c_I} \quad (2.11)$$

with new fugacities $u_I$ constrained by

$$\prod_{l=1}^{d} u_I = 1. \quad (2.12)$$

After this variable change the order agrees with the exponent of the fugacity $q$. We can use an arbitrary set of $c_I$ satisfying the constraint in Eq. (2.10) as long as the $R$-charge is positive for all
BPS operators. This requires \( c_I \geq 0 \), where the equality is allowed only for \( I \) corresponding to the shrinking cycles.

A natural choice for \( r \) with clear physical meaning is the superconformal \( U(1)_R \) charge \( r_\ast \). In that case the order is \( 3J + \frac{3}{2}r_\ast = H + J \) for BPS operators, and up to the spin \( J \) the order is identified with the dimension of operators. However, they are in general irrational, and then we need special treatment in the numerical calculation. To simplify the calculation we adopt \( r_\ast \) with rational coefficients.

3. Index from AdS

3.1. Kaluza–Klein contributions

In the large-\( N \) limit the superconformal index is calculated on the gravity side as the index of the Kaluza–Klein modes, and given by

\[
\mathcal{I}^{\text{KK}} = \text{Pexp} \ i^{\text{KK}},
\]  

(3.1)

where \( \text{Pexp} \) is the plethystic exponential defined in Eq. (A.2) and \( i^{\text{KK}} \) is the single-particle index of the Kaluza–Klein modes. For the simplest example with \( SE_5 = \mathbb{S}^5 \) corresponding to the \( \mathcal{N} = 4 \) supersymmetric \( U(\mathbb{N}) \) Yang–Mills theory \( i^{\text{KK}} \) was calculated in Ref. [4]:

\[
\frac{v_1}{1 - v_1} + \frac{v_2}{1 - v_2} + \frac{v_3}{1 - v_3}.
\]  

(3.2)

(To be precise, this is not the single-particle index of the \( \mathcal{N} = 4 \) theory given in Ref. [4]. We subtracted the contribution of the diagonal \( U(1) \) part of the \( \mathcal{N} = 1 \) vector multiplet while the diagonal parts of the three adjoint \( \mathcal{N} = 1 \) chiral multiplets are left included because we treat adjoint fields as special bi-fundamental fields, for which we cannot define the diagonal part.) In the case of an abelian orbifold \( \mathbb{S}^5/\Gamma \), the contribution of the gravity multiplet is given by Eq. (3.2) again with \( v_I \) replaced by appropriate powers of them. In addition, if the orbifold has singular loci, we need to include the contribution of tensor multiplets localized on the loci. The number of tensor multiplets is \( d - 3 \), and each tensor multiplet contributes to \( i^{\text{KK}} \) by [17]

\[
\frac{w}{1 - w}
\]  

(3.3)

with an appropriate choice of \( w \). From these results it is natural to guess the following formula for a general toric manifold:

\[
\mathcal{I}^{\text{KK}} = \text{Pexp} \left( \sum_{r=1/2}^{d-1/2} \frac{w_r}{1 - w_r} \right),
\]  

(3.4)

where \( r \) is a label for edges of the toric diagram and \( w_r \) are fugacities defined for each edge \( r \). Indeed, the formula in the form of Eq. (3.4) was derived in Refs. [18,19], and the fugacities \( w_r \) are defined in the following way.

Let us focus on a ridge \( r \) of the dual cone shared by two adjacent facets \( F_I \) and \( F_{I+1} \) with \( I = r - 1/2 \). Let \( g_r \in \tilde{V}_\mathbb{Z} \) be the primitive integer vector along the ridge. In general, a choice of a vector in \( \tilde{V} \) defines a coordinate in \( V \) through the inner product. We denote the coordinate defined with \( g_r \) by \( \varphi_r \). The coordinate \( \varphi_r \) has clear geometric meaning. By definition, \( g_r \) is orthogonal to \( V_I \) and \( V_{I+1} \). Therefore, on the corresponding cycles in the \( T^3 \) fiber it takes a constant value, and it is well defined even if these cycles shrink on the ridge. Namely, it is a coordinate parameterizing the \( \mathbb{S}^1 \) fiber on the
ridge. The primitivity of $g_r$ means that the coordinate $\phi_r$ is normalized so that the period of $S^1$ is 1. We define fugacities associated with the coordinates by

$$w_r = \prod_{I=1}^{d} v_I^{g_I g_r},$$

and these are the fugacities appearing in Eq. (3.4).

### 3.2. Wrapped D3-brane contributions

Let us discuss the contribution of D3-branes wrapped on three-cycles.

Analyses of orbifold theories [7,17] show that D3-branes wrapping on shrinking cycles do not contribute to the index. The large-$N$ index is reproduced by taking account of only the Kaluza–Klein contribution of the gravity and tensor multiplets. This fact suggests that we have only to consider D3-branes wrapped on visible cycles corresponding to corners of the toric diagram.

In this paper we only focus on D3-branes with single wrapping, and leave the analysis of branes with multiple wrapping for future work. A technical remark is in order. If there are shrinking cycles, the definition of a basis for the shrinking cycles is ambiguous and the distinction between single wrapping and multiple wrapping becomes unclear. To make the following analysis simple, as much as possible we neglect the wrapping numbers for shrinking cycles by setting the corresponding fugacity to be 1. For more detailed explanation of this point see Appendix C.2, in which we show an example of analysis keeping such fugacities.

With this remark in mind, we propose the following formula for the index of a D3-brane wrapped over a visible cycle $S_I$:

$$i_{S_I}^{V} = m_I v_I^N \text{Pexp} i_{S_I}^{D3}, \quad I \in \{\text{corners}\}. \quad (3.6)$$

The factor $v_I^N$ is the classical contribution obtained from Eq. (2.4). $i_{S_I}^{D3}$ is the single-particle index of the fluctuations on the wrapped D3-brane, and $m_I$ is a numerical factor representing the degeneracy associated with the $U(1)$ holonomy on the wrapped D3-brane. In the following we explain how we can determine $i_{S_I}^{D3}$ and $m_I$ from the toric data.

Let $S_I$ be a visible three-cycle we are focusing on. This means $I$ is a corner of the toric diagram. $S_I$ is a $T^2$ fibration over $E_I$ with the fiber shrinks at two ends. We can identify the $T^2$ fiber with $L/\mathbb{Z}_2$. (Remember that $L$ is the plane with the toric diagram drawn on it and $\mathbb{Z}_2$ is the integer lattice in $L$.) We introduce two coordinates $\phi_r$ and $\phi_{r'}$ with $r = I - \frac{1}{2}$ and $r' = I + \frac{1}{2}$ to parameterize the plane $L$. The two sides sharing $I$ are given by $\phi_r = 0$ and $\phi_{r'} = 0$ (see Fig. 2). We regard the plane $L$ as the covering space of the $T^2$ fiber on $E_I$. The $T^2$ fiber is given as a lattice quotient of the plane as follows.

![Diagram showing two parallelograms used to describe the $T^2$ fiber over $E_I$.](image)
Let $I - k'$ and $I + k$ be the corners adjacent to $I$; $k$ and $k'$ are the lengths of the sides sharing the vertex $I$. We define $m$ and $n$ by

$$m = \varphi_r|_{I+k}, \quad n = \varphi_{r'}|_{I-k'},$$

(3.7)

and consider the parallelogram on $L$ defined by (Fig. 2(a))

$$0 \leq \varphi_r < m, \quad 0 \leq \varphi_{r'} < n.$$

(3.8)

We regard this as a torus by identifying opposite sides. If there are no lattice points in this parallelogram except $\varphi_r = \varphi_{r'} = 0$, this torus is nothing but the fiber at a generic point on $E_I$. The cycle with constant $\varphi_r$ shrinks at the endpoint $C_r$, while the cycle with constant $\varphi_{r'}$ shrinks at the other endpoint $C_{r'}$. As a result, the three-cycle $S_I$ is topologically $S^3$. If the area of the parallelogram is greater than 1 we should perform the orbifolding with $\Gamma$, the discrete group defined by the lattice points in the parallelogram. Then the three-cycle $S_I$ is topologically $S^3/\Gamma$.

The $U(1)$ holonomy on the D3-brane wrapped on $S_I$ is specified by an element of the dual group $\tilde{\Gamma} = \text{Hom}(\Gamma, U(1))$, and the number of elements of $\tilde{\Gamma}$ is the same as $|\Gamma|$, which is the same as the area of the parallelogram. Because we consider a single D3-brane, no fields on the brane couple to the $U(1)$ holonomy and the existence of $|\Gamma|$ values of the holonomy affects the index simply as the overall numerical factor. This gives the degeneracy factor

$$m_I = |\Gamma| = (V_{I+k'} - V_I) \times (V_{I-k} - V_I).$$

(3.9)

We can also take $I - 1$ and $I + 1$ to define the parallelogram (Fig. 2(b)). Namely, instead of Eq. (3.7) we define $m$ and $n$ by

$$m = \varphi_r|_{I+1}, \quad n = \varphi_{r'}|_{I-1}.$$

(3.10)

If $k k' > 1$ this gives a smaller parallelogram. With this minimal choice we obtain a smaller orbifold group $\Gamma'$, which is related to $\Gamma$ by

$$\Gamma' = \Gamma/(\mathbb{Z}_k \times \mathbb{Z}_{k'}).$$

(3.11)

This should give the same $S_I$ as above. Namely, the following relation should hold:

$$S^3/\Gamma = S^3/\Gamma'.$$

(3.12)

This is checked as follows. From the relation in Eq. (3.11), $S^3/\Gamma = (S^3/(\mathbb{Z}_k \times \mathbb{Z}_{k'}))/\Gamma'$ and this is the same as $S^3/\Gamma'$ because $\mathbb{Z}_k$ and $\mathbb{Z}_{k'}$ act on the coordinates $\varphi_r$ and $\varphi_{r'}$ independently and $S^3/(\mathbb{Z}_k \times \mathbb{Z}_{k'})$ is topologically the same as $S^3$.

Even if we use the expression $S_I = S^3/\Gamma'$, the multiplicity $m_I$ is still given by Eq. (3.9) because when we use $\Gamma'$ the $S^3$ before the orbifolding has a type $A_{k-1}$ singular locus if $k > 1$ and type $A_{k'-1}$ singular locus if $k' > 1$. These singular loci cause the additional factors $k$ and $k'$ in the multiplicity factor, and we again obtain $m_I = k k'|\Gamma'| = |\Gamma|$.

Next, let us determine $I_{S_I}^{D3}$, the single-particle index of the fluctuations of the fields on a D3-brane wrapped on $S_I$. In the case of $SE_5 = S^5$ it is [6]

$$f(q, y, w_r, w_{r'}) = 1 - \frac{(1 - q^{-3} w_r w_{r'})(1 - q^3 y)(1 - q^3 y^{-1})}{(1 - w_r)(1 - w_{r'})},$$

(3.13)
where \( w_r \) and \( w_{r'} \) are fugacities associated with the two ridges \( r = I - 1/2 \) and \( r' = I + 1/2 \).

We claim that if \( S_I \) is topologically \( S^3 \) the formula in Eq. (3.13) gives the correct answer even for a general toric manifold for which \( S^3 \) is not always round. The reason is as follows. The index can be regarded as the partition function of the theory defined in \( S^3 \times S^1 \) with appropriate background fields. In general, a supersymmetric partition function in a supersymmetric background depends only on a small number of parameters of the background fields [20]. In the case of spacetime with the topology \( S^3 \times S^1 \) the parameters are nothing but the fugacities, and even for a deformed \( S^3 \) the index should take the same form as the one for the round \( S^3 \) with appropriately chosen fugacities depending on the background fields. The fugacities can be determined as parameters for the complex structure of the manifold and the moduli of the normal bundle following the detailed analysis in Ref. [20]. Fortunately, in the case of the toric theories we are studying here we can skip this analysis. In the basis of charges used in the definition in Eq. (2.6) of the index, all charges are quantized in units of \( 1/2 \) and quantum numbers of BPS states do not change under continuous deformations as long as the deformations respect the supersymmetry and the \( U(1) \) symmetries, and then the index is the same as that for the round sphere.

If \( S_I \) is an orbifold \( S^3/\Gamma \) we need to perform the orbifold projection. Let \((\varphi, \varphi')\) be the coordinates \((\varphi_r, \varphi_{r'})\) at the lattice point in the parallelogram corresponding to \( g \in \Gamma \). The single-particle index on \( S_I = S^3/\Gamma \) is given by

\[
I_{S^3}^{3D} = \mathcal{P}_\Gamma f(q, y, w_r, w_{r'}) \equiv \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(q, y, \omega_g w_r, \omega_g w_{r'}), \tag{3.14}
\]

where \( \omega_k = \exp \frac{2\pi i}{k} \). This formula reproduces the results of the orbifold theories in Ref. [7].

As a simple consistency check let us confirm that two descriptions of \( S_I \) with the two orbifold groups \( \Gamma \) and \( \Gamma' \) related by Eq. (3.11) give the same single-particle index. We should confirm the relation

\[
\mathcal{P}_\Gamma f(q, y, w_r, w_{r'}) = \mathcal{P}_{\Gamma'} f(q, y, w_r, w_{r'}). \tag{3.15}
\]

Because of the relation in Eq. (3.11) we can decompose \( \mathcal{P}_\Gamma \) as \( \mathcal{P}_\Gamma = \mathcal{P}_{\Gamma'}\mathcal{P}_{\mathbb{Z}_k \times \mathbb{Z}_{k'}} \) where \( \mathcal{P}_{\mathbb{Z}_k \times \mathbb{Z}_{k'}} \) is defined by

\[
\mathcal{P}_{\mathbb{Z}_k \times \mathbb{Z}_{k'}} f(q, y, w_r, w_{r'}) = \frac{1}{kk'} \sum_{i=0}^{k-1} \sum_{j=0}^{k'-1} f(q, y, \omega_k^i w_r, \omega_{k'}^j w_{r'}). \tag{3.16}
\]

It is sufficient to show that

\[
f(q, y, w_r, w_{r'}) = \mathcal{P}_{\mathbb{Z}_k \times \mathbb{Z}_{k'}} f(q, y, w_r, w_{r'}). \tag{3.17}
\]

By using the explicit form of the function \( f \) in Eq. (3.13) we can easily confirm Eq. (3.17).

### 4. Examples

In this section we calculate the superconformal index on the gravity side according to the formula in Eq. (3.6) and compare the obtained results with those numerically calculated on the gauge theory side by the localization method summarized in Appendix A to confirm that our prescription does work correctly. Due to machine power limitations we take \( N = 2 \) and 3. We will show only the results for \( N = 2 \) in this section; the results for \( N = 3 \) are shown in Appendix B.
4.1. $T^{1,1}$ (conifold)

First, we consider $SE_5 = T^{1,1}$. $T^{1,1}$ is the base of the conifold, and the corresponding boundary theory is the so-called Klebanov–Witten theory [21]. The toric diagram, the bipartite graph, and the quiver diagram are shown in Fig. 3, and the matter contents of this theory are shown in Table 1.

Corresponding to four vertices of the toric diagram we have four $R$-charges $R_I$ ($I = 1, 2, 3, 4$). The conifold and $T^{1,1}$ have the isometry $(SU(2)_A \times SU(2)_B)/\mathbb{Z}_2 \times U(1)_{r^*}$. The $U(1)_{r^*}$ factor is the superconformal $R$-symmetry generated by

$$ r = \frac{1}{2} (R_1 + R_2 + R_3 + R_4). \quad (4.1) $$

The Cartan generators $F_A$ and $F_B$ of the $SU(2)$ factors are

$$ F_A = R_1 - R_3, \quad F_B = R_2 - R_4. \quad (4.2) $$

The remaining linear combination of $R_I$ is the baryonic charge

$$ B = \frac{1}{N} (R_1 - R_2 + R_3 - R_4), \quad (4.3) $$

which does not act on $T^{1,1}$ geometrically. On the gauge theory side this can be regarded as the non-diagonal part of the $U(1) \times U(1)$ symmetry defined by replacing the $SU(N)$ gauge groups by $U(1)$. Because the theory is non-chiral this symmetry is anomaly free. See Table 1 for the charge assignments for these generators. Corresponding to the charges $F_A, F_B$, and $B$, we introduce the fugacities $u, v$, and $\zeta$ by the relation

$$ v_1^{R_1} v_2^{R_2} v_3^{R_3} v_4^{R_4} = q^{\frac{1}{2} r} u^{F_A} v^{F_B} \zeta^B. \quad (4.4) $$

The canonical variables $v_I$ are expressed with new variables by

$$ v_1 = q^{\frac{3}{4}} u \zeta^{-\frac{1}{4}}, \quad v_2 = q^{\frac{3}{4}} v, \quad v_3 = q^{\frac{3}{4}} \zeta^\frac{3}{4} u, \quad v_4 = q^{\frac{3}{4}} v \zeta^\frac{3}{4}. \quad (4.5) $$

---

**Table 1.** The matter contents and charge assignments for the Klebanov–Witten theory (the conifold).

| Fields | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $r_*$ | $F_A$ | $F_B$ | $NB$ |
|--------|-------|-------|-------|-------|-------|-------|-------|------|
| $X^1_1(A_1)$ | 1     | 0     | 0     | 0     | 1/2   | 1     | 0     | 1    |
| $X^1_2(B_1)$ | 0     | 1     | 0     | 0     | 1/2   | 0     | 1     | −1   |
| $X^2_2(A_2)$ | 0     | 0     | 1     | 0     | 1/2   | −1    | 0     | 1    |
| $X^2_3(B_2)$ | 0     | 0     | 0     | 1     | 1/2   | 0     | −1    | −1   |
Due to the $SU(2)_A \times SU(2)_B$ flavor symmetry the index can be written in terms of $SU(2)$ characters $\chi^B_n = \chi_n(u)$ for $SU(2)_A$ and $\chi^I_n = \chi_n(v)$ for $SU(2)_B$ as well as the $SU(2)$ spin character $\chi^I_n = \chi_n(v)$, where the functions $\chi_n(c)$ are defined by

$$\chi_n(c) = \frac{c^{n+1} - c^{-(n+1)}}{c - c^{-1}}. \quad (4.6)$$

On the gauge theory side we define the index for the sector with a specific baryon number $B$ by the expansion with respect to $\zeta$:

$$\mathcal{I}^{\text{gauge}} = \sum_{B \in \mathbb{Z}} \mathcal{I}^{\text{gauge}}_B, \quad \mathcal{I}^{\text{gauge}}_B \propto \zeta^B. \quad (4.7)$$

Instead of defining $\mathcal{I}^{\text{gauge}}_B$ as the coefficients in the $\zeta$ expansion, we include $\zeta^B$ in $\mathcal{I}^{\text{gauge}}_B$.

On the gravity side the baryonic charge corresponds to the wrapping number of D3-branes [22]. Remember that $H_3(T^{1,1} \mathbb{Z}) = \mathbb{Z}$ and the wrapping sectors are labeled by a single integer $B$, which is identified with the baryonic charge. By combining Eqs. (2.4) and (4.3) we can determine the wrapping number of each cycle $S_i$. Equivalently, we can read off the charges as the exponents of $\zeta$ from the classical factors

$$v^N_1 = q^{3N} u^N \zeta, \quad v^N_2 = \frac{q^{3N} v^N}{\zeta}, \quad v^N_3 = \frac{q^{3N} \zeta}{u^N}, \quad v^N_4 = \frac{q^{3N}}{v^N \zeta}. \quad (4.8)$$

By using these we can determine brane configurations contributing to the index of a specific wrapping sector and the orders of their contributions. For example, let us consider the $B = 1$ sector. There are two single-wrapping brane configurations $S_1$ and $S_3$ with $B = 1$. They give $\mathcal{O}(q^{3N})$ contributions to the index. Furthermore, we also have multiple-wrapping configurations like $S_1 + S_2 + S_3$. If we naively assume that the contribution is given as the product of constituent contributions, this gives $\mathcal{O}(q^{2N})$ terms. The expected orders of the corrections obtained in this way for different wrapping numbers are shown in Table 2.

Now, let us start the comparison of the results on the gravity side and those on the gauge theory side. We consider the $N = 2$ case here; see Appendix B for the results for $N = 3$. Let us first consider the $B = 0$ sector. The relation expected from the order estimation of the multiple-wrapping D3-brane contribution is

$$\mathcal{I}^{\text{gauge}}_0 = \mathcal{I}^{\text{KK}} + \mathcal{O}(q^{3N}). \quad (4.9)$$

| $B$ | Lowest order, (config.) | Higher order (config.) |
|-----|-------------------------|------------------------|
| $-2$ | $\frac{q^{3N}}{\zeta}$, $(S_2, S_4)$ | $q^{3N}$, $(S_1 + 2S_2, \ldots)$ |
| $-1$ | $q^{3N}$, $(S_1, S_4)$ | $q^{3N}$, $(S_1 + S_2, \ldots)$ |
| 0   | $1$                      | $q^{3N}$, $(S_1 + S_2, \ldots)$ |
| 1   | $q^{3N}$, $(S_1, S_3)$ | $q^{3N}$, $(S_1 + S_2, \ldots)$ |
| 2   | $q^{3N}$, $(S_1, S_4)$ | $q^{3N}$, $(S_1 + S_2, \ldots)$ |
On the gauge theory side the numerical analysis for \( N = 2 \) gives
\[
\mathcal{I}_{0}^{\text{gauge}} = 1 + \chi_1^u \chi_1^v q^3 + (2 - \chi_2^u - \chi_2^v + 2 \chi_2^u \chi_2^v)q^3
+ (-2 \chi_1^u \chi_1^v - \chi_3^u \chi_1^v - \chi_1^u \chi_3^v + 2 \chi_3^u \chi_3^v + 2(1 + \chi_2^u \chi_2^v)\chi_4^j)q^9 + \mathcal{O}(q^6). \tag{4.10}
\]

On the gravity side \( \mathcal{I}^{\text{KK}} \) was first calculated in Ref. [23], and is given by Eq. (3.4) with the fugacities
\[
w_1 = \frac{q^3}{u}, \quad w_2 = \frac{q^3}{u}, \quad w_3 = \frac{q^3}{v}, \quad w_4 = \frac{q^3}{uv}. \tag{4.11}
\]

These are invariant under the \( \mathbb{Z}_2 \) action \((u, v) \rightarrow (-u, -v)\). This is consistent with the fact that the \( \mathbb{Z}_2 \) quotient group in the isometry group is generated by \( e^{\pi i(F_A + F_B)} \). Furthermore, these do not contain the baryonic fugacity \( \zeta \), and this is consistent with the fact that the Kaluza–Klein modes do not have wrapping numbers. Equation (3.4) gives
\[
\mathcal{I}^{\text{KK}} = (\ldots \text{terms identical with Eq. (4.10)} \ldots )
+ (2 \chi_1^u \chi_1^v - \chi_3^u \chi_1^v - \chi_1^u \chi_3^v + 3 \chi_3^u \chi_3^v)q^9 + \mathcal{O}(q^6). \tag{4.12}
\]

Equations (4.10) and (4.12) satisfy the relation in Eq. (4.9).

Next, let us consider the \( B \neq 0 \) sectors. We have two sectors \( B = \pm 1 \) with the contribution of single-wrapping branes. To extract the brane contributions we consider the ratio \( \mathcal{I}_B / \mathcal{I}^{\text{KK}} \) rather than \( \mathcal{I}_B \). \( S_1 \) and \( S_3 \) carry \( B = 1 \) and \( S_2 \) and \( S_4 \) carry \( B = -1 \). The expected relations are
\[
\frac{\mathcal{I}_1^{\text{gauge}}}{\mathcal{I}^{\text{KK}}} = (\mathcal{I}_{S_1}^{D_{3}} + \mathcal{I}_{S_3}^{D_{3}}) + \mathcal{O}(q^6),
\frac{\mathcal{I}_{-1}^{\text{gauge}}}{\mathcal{I}^{\text{KK}}} = (\mathcal{I}_{S_2}^{D_{3}} + \mathcal{I}_{S_4}^{D_{3}}) + \mathcal{O}(q^6). \tag{4.13}
\]

These two relations are not independent due to a symmetry. The toric diagram (Fig. 3(a)) and the bipartite graph (Fig. 3(b)) have the \( \mathbb{Z}_4 \) rotational symmetry. The counter-clockwise \( \pi / 2 \) rotation maps the vertex \( I \) to \( I + 1 \) and the charge \( R_I \) to \( R_{I+1} \). The charges \( r_s, F_A, F_B \), and \( B \) are mapped as
\[
r_s \rightarrow r_s, \quad F_A \rightarrow F_B, \quad F_B \rightarrow -F_A, \quad B \rightarrow -B. \tag{4.14}
\]

Therefore, the index is invariant under
\[
(q, y, u, v, \zeta) \rightarrow (q, y, v, 1/u, 1/\zeta), \tag{4.15}
\]
and this transforms the two relations in Eq. (4.13) to each other. We focus on the \( B = 1 \) sector. The calculation on the gauge theory side with \( N = 2 \) gives
\[
\frac{\mathcal{I}_1^{\text{gauge}}}{\mathcal{I}^{\text{KK}}} = \zeta \left[ \chi_2^u q^3 + (-\chi_1^u \chi_1^v + (-1 + \chi_2^u)\chi_4^j)q^9
+ (1 - \chi_2^u - \chi_2^v - \chi_2^u \chi_2^v + \chi_1^u \chi_1^v \chi_1^j \chi_4^j + (-1 + \chi_2^u)\chi_4^j)q^9
+ (2 \chi_1^u \chi_1^v - \chi_3^u \chi_1^v - \chi_1^u \chi_3^v - \chi_3^u \chi_3^v
+ (1 - \chi_2^u - 3 \chi_2^v + 3 \chi_2^u \chi_2^v)\chi_4^j + (-1 + \chi_2^u)\chi_4^j)q^6
\right].
\]
As mentioned above, this is written in terms of $\text{SU}(2)$ characters $\chi_n^u$ and $\chi_n^v$. On the gravity side there are two contributions:

$$\mathcal{I}_{S_1}^{D_3} = v_1^N \text{Pexp} (q, y, w_{1+}, w_{1+}), \quad \mathcal{I}_{S_1}^{D_3} = v_3^N \text{Pexp} (q, y, w_{2+}, w_{3+}).$$

Although neither of them respects the $\text{SU}(2)_A$ flavor symmetry, the sum of two contributions becomes a linear combination of $\text{SU}(2)_A$ characters by the mechanism explained in detail in Ref. [6]. The result for $N = 2$ is

$$\mathcal{I}_{S_1}^{D_3} + \mathcal{I}_{S_1}^{D_3} = \zeta \left[ \ldots \text{terms identical with Eq. (4.16)} \ldots \right] + (\chi_4^u - \chi_0^u + \chi_2^v + 2\chi_2^u \chi_2^v - \chi_6^u \chi_2^v + \chi_4^u \chi_2^4 - \chi_8^u \chi_4^4)
+ (1 - \chi_2^u - \chi_2^v \chi_1^3) X_2^4 + (-1 + \chi_2^v) \chi_4^4 q^{15} + O(q^{17}) \right].$$

Again, the result is consistent with the expected relation in Eq. (4.13).

### 4.2. $T^{2,2}$ (complex cone over $F_0$)

The next example we discuss is $SE_3 = T^{2,2}$. As is obvious from the toric diagram shown in Fig. 4 this is a $\mathbb{Z}_2$ orbifold of $T^{1,1}$. The corresponding Calabi–Yau cone is a $\mathbb{Z}_2$ orbifold of the conifold, which is often referred to as the complex cone over $F_0$. The isometry group is $(\text{SU}(2)_A/\mathbb{Z}_2) \times (\text{SU}(2)_B/\mathbb{Z}_2) \times U(1)_r$. Again, we have four $R$-charges $r_i$ ($i = 1, 2, 3, 4$) corresponding to the corners of the toric diagram. We define charges $r_\ast, F_1, F_2$, and $B$ in the same way as in the conifold case:

$$r_\ast = \frac{1}{2}(R_1 + R_2 + R_3 + R_4),$$
$$F_A = R_1 - R_3, \quad F_B = R_2 - R_4, \quad B = \frac{1}{N}(R_1 - R_2 + R_3 - R_4).$$

The charge assignments are shown in Table 3.

Corresponding to the charges in Eq. (4.19) we define the fugacities $u, v$, and $\xi$ by

$$v_1^{R_1} v_2^{R_2} v_3^{R_3} v_4^{R_4} = \zeta^{2r_\ast + F_A + F_B}. \quad (4.20)$$
Table 3. Matter contents and charge assignments for the $F_0$ model.

| Fields | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $r_e$ | $F_A$ | $F_B$ | NB |
|-------|------|------|------|------|------|------|------|----|
| $X^1_{i2}$ | 1 | 0 | 0 | 0 | 1/2 | 1 | 0 | 1 |
| $X^1_{i2}$ | 0 | 0 | 1 | 0 | 1/2 | −1 | 0 | 1 |
| $X^2_{i3}$ | 0 | 1 | 0 | 0 | 1/2 | 0 | 1 | −1 |
| $X^2_{i3}$ | 0 | 0 | 0 | 1 | 1/2 | 0 | −1 | −1 |
| $X^3_{i4}$ | 0 | 0 | 1 | 0 | 1/2 | −1 | 0 | 1 |
| $X^3_{i4}$ | 1 | 0 | 0 | 0 | 1/2 | 1 | 0 | 1 |
| $X^4_{i4}$ | 0 | 0 | 0 | 1 | 1/2 | 0 | −1 | −1 |
| $X^3_{i4}$ | 0 | 1 | 0 | 0 | 1/2 | −1 | 0 | 1 |

Table 4. The lowest- and higher-order contributions to the indices of the D3-branes with wrapping numbers $B \in \mathbb{Z}$, and the corresponding brane configurations for $T^{2,2}$.

| $B$ | Lowest order, (config.) | Higher order, (config.) |
|-----|-------------------------|------------------------|
| −2  | $q^{2N}$, $(2S_2, \ldots)$ | $q^{2N}$, $(2S_2, \ldots)$ |
| −1  | $q^{2N}$, $(2S_1 + 2S_2, \ldots)$ | $q^{2N}$, $(2S_1 + 2S_2, \ldots)$ |
| 0   | $q^{2N}$, $(S_2, S_4)$ | $q^{2N}$, $(S_2, S_4)$ |
| 1   | $q^{2N}$, $(S_1 + 2S_2, \ldots)$ | $q^{2N}$, $(S_1 + 2S_2, \ldots)$ |
| 2   | $q^{2N}$, $(S_1, \ldots)$ | $q^{2N}$, $(S_1, \ldots)$ |

The canonical variables $v_I$ expressed in terms of the new ones are

$$v_1 = q^{3/4} u \zeta u, \quad v_2 = q^{3/4} v_1 \zeta, \quad v_3 = q^{3/4} \zeta u, \quad v_4 = q^{3/4} v_1 \zeta. \quad (4.21)$$

These relations are identical to those in Eq. (4.5). The ridge fugacities are given by

$$w_1 = v_2 v_3, \quad w_1 + v_2 = v_2^2 v_3, \quad w_2 + v_3 = v_2^2 v_4, \quad w_3 + v_4 = v_2^2 v_2. \quad (4.22)$$

These are the squares of the corresponding variables in Eq. (4.11). This is a reflection of the fact that $T^{2,2}$ is the $\mathbb{Z}_2$ quotient of $T^{1,1}$. These are invariant under two $\mathbb{Z}_2$ actions $u \to -u$ and $v \to -v$ corresponding to the quotient group $\mathbb{Z}_2^2$ in the isometry group.

According to the general rule there is one continuous baryonic $U(1)$ symmetry. In addition, we have the non-trivial discrete factor $G_{\text{disc}} = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this section we neglect this; see Appendix C.1 for some analysis with these discrete factors. Again, we define the index $T_B^{\text{gauge}}$ of the sector with a specific baryonic charge $B$ by the expansion

$$T_B^{\text{gauge}} = \sum_{B \in \mathbb{Z}} T_B^{\text{gauge}}, \quad T_B^{\text{gauge}} \propto \zeta^B. \quad (4.23)$$

For small values of $B$ we show in Table 4 the expected order of the corrections due to wrapped branes. We focus on the sectors with $B = 0$ and $B = \pm 1$. The expected relations are as follows:

$$T_0^{\text{gauge}} = T_0^{\text{KK}} + \mathcal{O}(q^{2N}),$$

$$T_{-1}^{\text{gauge}} / T_0^{\text{KK}} = T_{S_2}^{D_3} + T_{S_4}^{D_3} + \mathcal{O}(q^{2N}),$$

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\[ \frac{T^{\text{gauge}}}{T^{\text{KK}}} = T^{D^3}_{S_1} + T^{D^3}_{S_3} + \mathcal{O}(q^2 N). \]  

(4.24)

We check these relations for the \( N = 2 \) case. See also Appendix B for the analysis with \( N = 3 \). We use the \( SU(2) \) characters \( \chi_n^{\mu}, \chi_n^{\nu}, \) and \( \chi_n^{j} \) defined in Sect. 4.1 to write down indices.

On the gauge theory side the localization formula gives, for \( B = 0 \),

\[ T^{\text{gauge}}_0 = 1 + \left( 5 \chi_2^{\nu} \chi_2^{\nu} - \chi_2^{\nu} - \chi_2^{\nu} + 5 \right) q^3 + \mathcal{O}(q^7). \]  

(4.25)

On the gravity side, Eq. (3.4) with the fugacities in Eq. (4.22) gives

\[ T^{\text{KK}} = 1 + \left( \chi_2^{\nu} \chi_2^{\nu} - \chi_2^{\nu} - \chi_2^{\nu} + 1 \right) q^3 + \mathcal{O}(q^9). \]  

(4.26)

Comparing Eqs. (4.25) and (4.26), we find that the first relation in Eq. (4.24) holds.

Next, let us consider the sectors with \( B = \pm 1 \). As in the conifold case the toric diagram and the bipartite graph have \( \mathbb{Z}_4 \) rotational symmetry, and the \( B = +1 \) and \( B = -1 \) sectors are related by the variable change in Eq. (4.15). We consider only the \( B = +1 \) sector. On the gauge theory side we obtain

\[ \frac{T^{\text{gauge}}}{T^{\text{KK}}} = \frac{\chi_2^{\nu} q^3 + 2 (\chi_2^{\nu} - 1) \chi_1^{j} q^3}{\chi_2^{\nu} q^3 + 2 (\chi_2^{\nu} - 1) \chi_1^{j} q^3} + 2 \left( \frac{3 \chi_4^{\nu} \chi_2^{\nu} + \chi_2^{\nu} + 2 \chi_2^{\nu} - 1 + (\chi_2^{\nu} - 1) \chi_2^{\nu}}{q^3 + \mathcal{O}(q^9)} \right). \]  

(4.27)

On the gravity side the index for each supersymmetric cycle is given by

\[ T^{D^3}_{S_1} = 2 \gamma^N \exp \left[ \frac{f(q,y,w_{j-1/2}^{1/2},w_{j+1/2}^{1/2}) + f(q,y,-w_{j-1/2}^{1/2},-w_{j+1/2}^{1/2})}{2} \right]. \]  

(4.28)

The sum of the two contributions \( T^{D^3}_{S_1} \) and \( T^{D^3}_{S_3} \) for the \( N = 2 \) case is

\[ T^{D^3}_{S_1} + T^{D^3}_{S_3} = \xi \left( \ldots \text{terms identical with Eq. (4.27)} \ldots \right) + 2 \left( -\chi_2^{\nu} - 2 \chi_2^{\nu} - 1 + (\chi_2^{\nu} - 1) \chi_2^{\nu} \right) q^3 + \mathcal{O}(q^9). \]  

(4.29)

We find that the last relation in Eq. (4.24) holds.

4.3. \( Y^{2,1} \) (complex cone over \( dP_1 \))

There is a family of \( SE_5 \) denoted by \( Y^{p,q} \). We consider \( Y^{2,1} \) as the simplest example. The corresponding Calabi–Yau cone is the complex cone over the first del Pezzo surface (\( dP_1 \)). The toric diagram, the bipartite graph, and the quiver diagram are shown in Fig. 5.

Generally, the isometry of \( Y^{p,q} \) is \( SU(2) \times U(1) \times U(1) \), and the toric diagram has \( d = 4 \). For \( Y^{2,1} \) we define generators of the flavor symmetry by

\[ F_1 = R_2 - R_4, \quad F_2 = \frac{1}{2} (R_1 - R_3), \quad B = \frac{1}{N} (2 (R_2 + R_4) - R_1 - 3 R_3). \]  

(4.30)

\( F_1 \) is the Cartan part of the \( SU(2) \) isometry and \( F_2 \) generates one of the \( U(1) \) isometries. For the \( U(1)_R \) charge we take

\[ r = \frac{1}{2} (R_1 + R_2 + R_3 + R_4). \]  

(4.31)
Fig. 5. (a) Toric diagram of $Y^{2,1}$. (b) The corresponding bipartite graph. (c) The corresponding quiver diagram.

Table 5. Matter contents and charge assignments for the $dP_1$ model.

| Fields | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $r$ | $F_1$ | $F_2$ | $NB$ |
|--------|-------|-------|-------|-------|-----|-------|-------|------|
| $X_{12}$ | 0 | 0 | 1 | 0 | 1/2 | 0 | $-1/2$ | $-3$ |
| $X_{13}$ | 0 | 1 | 0 | 0 | 1/2 | 1 | 0 | 2 |
| $X_{23}$ | 0 | 0 | 0 | 1 | 1/2 | $-1$ | 0 | 2 |
| $X_{34}$ | 0 | 1 | 1 | 0 | 1 | 1 | $-1/2$ | $-1$ |
| $X_{24}$ | 0 | 0 | 1 | 1 | 1 | $-1$ | $-1/2$ | $-1$ |
| $X_{31}$ | 1 | 0 | 0 | 0 | 1/2 | 0 | $1/2$ | $-1$ |
| $X_{31}$ | 0 | 1 | 0 | 0 | 1/2 | 1 | 0 | 2 |
| $X_{32}$ | 0 | 0 | 0 | 1 | 1/2 | $-1$ | 0 | 2 |
| $X_{13}$ | 1 | 0 | 0 | 0 | 1/2 | 0 | $1/2$ | $-1$ |
| $X_{22}$ | 1 | 0 | 0 | 0 | 1/2 | 0 | $1/2$ | $-1$ |

for simplicity of the index calculation, rather than the one in the superconformal algebra [24]:

$$r_* = (-3 + \sqrt{13})R_1 + \frac{16 - 4\sqrt{13}}{3}(R_2 + R_4) + \frac{-17 + 5\sqrt{13}}{3}R_3.$$  (4.32)

The charge assignments are given in Table 5.

We introduce new fugacities $u, v,$ and $\zeta$ by

$$v_1^{R_1}v_2^{R_2}v_3^{R_3}v_4^{R_4} = q^{3/2}ru^{F_1}v^{F_2}\zeta^B.$$  (4.33)

The fugacities $v_I$ in terms of the new ones are

$$v_1 = \frac{q^{3}v_1}{\zeta^\frac{1}{2}}, \quad v_2 = \frac{q^{2}u\zeta^\frac{3}{2}}{v_2}, \quad v_3 = \frac{q^{3}}{v_3\zeta^\frac{3}{2}}, \quad v_4 = \frac{q^{2}\zeta^2}{u}.$$  (4.34)

We have one continuous baryonic $U(1)$ symmetry. We can easily confirm on the gauge theory side that it is the full anomaly-free baryonic symmetry, and there is no discrete factor in this example.

On the gauge theory side we define the index $I_B^{\text{gauge}}$ of the sector with a specific baryonic charge $B$ by the expansion

$$I^{\text{gauge}} = \sum_{B \in \mathbb{Z}} I_B^{\text{gauge}}, \quad I_B^{\text{gauge}} \propto \zeta^B.$$  (4.35)

On the gravity side the expected orders of corrections due to wrapped D3-branes are shown in Table 6. There are three sectors $B = -1, -3,$ and $2$ that receive corrections from single-wrapping brane configurations. We focus on these three sectors and the $B = 0$ sector. The expected relations for these sectors are

$$I_0^{\text{gauge}} = I^{\text{KK}} + O(q^{9/2}N),$$

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Table 6. The lowest- and higher-order contributions to the indices of the D3-branes with the wrapping numbers $B \in \mathbb{Z}$, and the corresponding brane configurations for $Y^2.1$.

| $B$ | Lowest order, (config.) | Higher order, (config.) |
|-----|-------------------------|------------------------|
| $-3$ | $q^2N$, $(S_1)$ | $q^2N$, $(3S_1)$ |
| $-2$ | $q^2N$, $(2S_1)$ | |
| $-1$ | $q^2N$, $(S_1)$ | $q^2N$, $(S_1 + S_2, \ldots)$ |
| $0$ | $1$ | $q^2N$, $(2S_1 + S_2)$ |
| $1$ | $q^2N$, $(S_1 + S_2, \ldots)$ | |
| $2$ | $q^2N$, $(S_2, S_1)$ | $q^3N$, $(2S_1 + 2S_2, \ldots)$ |
| $3$ | $q^2N$, $(S_1 + 2S_2, \ldots)$ | |

\[
\frac{T_{-1}^{\text{gauge}}}{T_{\text{KK}}} = T_{S_1}^{D3} + O(q^2N),
\]
\[
\frac{T_{-3}^{\text{gauge}}}{T_{\text{KK}}} = T_{S_3}^{D3} + O(q^2N),
\]
\[
\frac{T_{2}^{\text{gauge}}}{T_{\text{KK}}} = T_{S_2}^{D3} + T_{S_4}^{D3} + O(q^3N). \tag{4.36}
\]

We check these relations for $N = 2$. See also Appendix B for the analysis with $N = 3$.

For the $B = 0$ sector the localization formula with $N = 2$ gives

\[
T_{0}^{\text{gauge}} = 1 + v\chi_{1}^{\mu}q^{\frac{9}{2}} + \left(\frac{\chi_{2}^{\mu}}{v} - \frac{\chi_{1}^{\mu}}{v}\right)q^{\frac{15}{2}} + (10v^{2}\chi_{2}^{\mu} - v^{2})q^{\frac{9}{2}} + O(q^{\frac{21}{2}}). \tag{4.37}
\]

We use the $SU(2)$ characters $\chi_{n}^{\mu} = \chi_{n}(u)$ and $\chi_{n}'^{\mu} = \chi_{n}(v)$. On the gravity side the contribution of the Kaluza–Klein modes is given by Eq. (3.4) with the fugacities

\[
w_{1} = v_{1}^{2}, \quad w_{1+1} = v_{2}^{2}, \quad w_{2+1} = v_{3}^{2}, \quad w_{3+1} = v_{4}^{2}. \tag{4.38}
\]

The result is

\[
T_{\text{KK}}^{\text{gauge}} = 1 + v\chi_{1}^{\mu}q^{\frac{9}{2}} + \left(\frac{\chi_{2}^{\mu}}{v} - \frac{\chi_{1}^{\mu}}{v}\right)q^{\frac{15}{2}} + (2v^{2}\chi_{2}^{\mu} - v^{2})q^{\frac{9}{2}} + O(q^{6}), \tag{4.39}
\]

and we find the first relation in Eq. (4.36).

For the $B = -1$ sector, the localization formula with $N = 2$ gives

\[
\frac{T_{-1}^{\text{gauge}}}{T_{\text{KK}}} = \frac{1}{\xi} \left[3vq^{2} + 3\chi_{1}^{\mu}q^{2} + \left(\frac{3\chi_{2}^{\mu}}{v} + 3v\chi_{1}^{\prime}\right)q^{3} + O(q^{2})\right]. \tag{4.40}
\]

On the gravity side the corresponding brane contribution is

\[
T_{S_1}^{D3} = 3v_{1}^{2} P \exp \left(\frac{1}{3} \sum_{k=0}^{2} f(q, v, \omega_{2}^{k}w_{1}^{\frac{1}{2}}, \omega_{3}^{k}w_{1}^{\frac{1}{2}})\right) \tag{4.41}
\]

\[
= \frac{(N=2)}{\xi} \left[\ldots \text{terms identical with Eq. (4.40)} \ldots + 3\chi_{2}^{\prime} q^{2} + O(q^{2})\right].
\]

These relations are consistent with the second relation in Eq. (4.36).
For the $B = -3$ sector, the localization formula with $N = 2$ gives

$$\frac{\mathcal{I}_{-3}^{\text{gauge}}}{\mathcal{I}_{\text{KK}}} = \frac{1}{\zeta^3} \left[ q^3 + \left( v + \frac{\chi_1^\prime}{v} \right) q^3 - \chi_1^u q^{15} + \left( 10v^3 - \frac{1}{v} + \frac{\chi_2^2}{v} \right) q^9 + \mathcal{O}(q^{21}) \right]. \quad (4.42)$$

On the gravity side we obtain

$$\mathcal{I}_{S_3}^{D^3} = v_3^N \exp \left( f(q, y, w_{2+\frac{1}{2}}, w_{3+\frac{1}{2}}) \right)$$

$$\equiv (N=2) \frac{1}{\zeta^3} \left[ (\ldots \text{terms identical with Eq. (4.42)} \ldots) + \left( v^3 - \frac{1}{v} + \frac{\chi_2^2}{v} \right) q^9 + \mathcal{O}(q^{21}) \right]. \quad (4.43)$$

We see that the third relation in Eq. (4.36) holds.

For the $B = 2$ sector, the gauge theory result is

$$\frac{\mathcal{I}_{2}^{\text{gauge}}}{\mathcal{I}_{\text{KK}}} = \zeta^2 \left[ 2\chi_2^u q^\frac{1}{2} + 2 \left( \chi_2^u - 1 \right) \chi_1^\prime q^3 - 2v\chi_1^u q^{15} \right. \right.$$

$$+ 2 \left. \left( -\chi_2^u - 3 + \left( \chi_2^u - 1 \right) \chi_2^\prime \right) q^9 + 2 \left( -\frac{\chi_1^u}{v} + \chi_1^u \chi_1^\prime \right) q^{21} \right. \right.$$

$$+ 2 \left. \left( -v^2 \chi_2^u + 5v^2 \chi_4^u + 4v^2 + \left( \chi_2^u - 1 \right) \chi_3^\prime \right) q^{27} + \mathcal{O}(q^{37}) \right]. \quad (4.44)$$

On the gravity side the contributions of $S_2$ and $S_4$ are

$$\mathcal{I}_{S_2}^{D^3} = 2v_2^N \exp \left( \frac{\sum_{\pm f(q, y, \pm w_{1+\frac{1}{2}}, \pm w_{2+\frac{1}{2}})}}{2} \right). \quad (4.45)$$

$$\mathcal{I}_{S_4}^{D^3} = 2v_4^N \exp \left( \frac{\sum_{\pm f(q, y, \pm w_{3+\frac{1}{2}}, \pm w_{4+\frac{1}{2}})}}{2} \right). \quad (4.46)$$

These are summed to

$$\mathcal{I}_{S_2}^{D^3} + \mathcal{I}_{S_4}^{D^3} = 2\zeta^2 \chi_N(u)q^{\frac{1}{2}N} + \ldots$$

$$\equiv (N=2) \zeta^2 \left[ (\ldots \text{terms identical with Eq. (4.44)} \ldots) \right.$$

$$+ 2 \left. \left( -\chi_2^u v^2 - v^2 + \left( \chi_2^u - 1 \right) \chi_3^\prime \right) q^{27} + \mathcal{O}(q^{37}) \right]. \quad (4.47)$$

By comparing Eqs. (4.44) and (4.47), we find that the last relation in Eq. (4.36) holds.

### 4.4. $L^{1,2,1}$ (suspended pinch point)

The final example is $L^{1,2,1}$, the base of the suspended pinch point. The toric diagram, the bipartite graph, and the quiver diagram are shown in Fig. 6.

Because the toric diagram has a vertex on the boundary which is not a corner (vertex 1 in Fig. 6(a)), the manifold has the corresponding shrinking cycle. In this section we neglect the wrapping number on the shrinking cycle by setting the corresponding fugacity to be 1. See Appendix C.2 for an analysis with it taken into account. In this section we take account of only four charges, $R_2$, $R_3$, $R_4$, and $R_5$, and
Fig. 6. (a) Toric diagram of \( L^{1,2,1} \). (b) The corresponding bipartite graph. (c) The corresponding quiver diagram.

Table 7. Matter contents and charge assignments for the \( L^{1,2,1} \) model. The charges \( R_1, R'_1 \), and \( \tilde{B} \) are related to the shrinking cycle and are defined in Appendix C.2.

| Fields | \( R_1 \) | \( R'_1 \) | \( R_2 \) | \( R_3 \) | \( R_4 \) | \( R_5 \) | \( r \) | \( F_1 \) | \( F_2 \) | \( NB \) | \( \tilde{NB} \) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( X_{11} \) | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | -2 | 0 | 0 |
| \( X_{12} \) | 0 | 1 | 0 | 0 | 0 | 1 | 1/2 | 0 | 1 | -1 | -1 |
| \( X_{21} \) | 1 | 0 | 1 | 0 | 0 | 0 | 1/2 | 0 | 1 | 1 | 1 |
| \( X_{23} \) | 0 | 0 | 0 | 0 | 1 | 0 | 1/2 | -1 | -1 | 2 | 0 |
| \( X_{32} \) | 0 | 0 | 0 | 1 | 0 | 0 | 1/2 | 1 | -1 | -2 | 0 |
| \( X_{31} \) | 1 | 0 | 0 | 0 | 0 | 1 | 1/2 | 0 | 1 | -1 | 1 |
| \( X_{33} \) | 0 | 1 | 1 | 0 | 0 | 0 | 1/2 | 0 | 1 | 1 | -1 |

Associated with the corners of the toric diagram. For convenience we define the following four linear combinations:

\[
r = \frac{1}{2}(R_2 + R_3 + R_4 + R_5),
\]
\[
F_1 = R_3 - R_4,
\]
\[
F_2 = R_2 - R_3 - R_4 + R_5,
\]
\[
B = \frac{1}{N}(R_2 - 2R_3 + 2R_4 - R_5); \tag{4.48}
\]

\( r \) is not the \( R \)-charge \( r_* \) appearing in the superconformal algebra, which is given by \([25–27]\)

\[
r_* = \frac{1}{\sqrt{3}}(R_2 + R_5) + \left(1 - \frac{1}{\sqrt{3}}\right)(R_3 + R_4). \tag{4.49}
\]

The charge assignments for the charges defined above are shown in Table 7.

In this example we have \( \mathbb{Z}_2 \) symmetry acting on the toric diagram as the permutation of the vertices \( (1, 2, 3, 4, 5) \rightarrow (1, 5, 4, 3, 2) \). On the gauge theory side this is the charge conjugation mapping a bi-fundamental field \( X_{ab} \) to \( X_{ba} \); \( r_* \) and \( F_2 \) are even under this charge conjugation while \( F_1 \) and \( B \) are odd.

The gauge theory is non-chiral and the baryonic symmetry is obtained by replacing the \( SU(N) \) gauge groups by \( U(1) \) and removing the diagonal \( U(1) \). There are two baryonic charges, and \( B \) defined in Eq. (4.48) is one of them. The other (shown as \( \tilde{B} \) in Table 7) is associated with the shrinking cycle and we neglect it in this section. We have no discrete factor in the baryonic symmetry.

We define new fugacities by

\[
\prod_{I=1}^{5} v_I^{R_I} = q^{\frac{1}{2} r} u^{F_1} v^{F_2} \zeta B, \tag{4.50}
\]
Table 8. The lowest- and higher-order contributions to the indices of the D3-branes with the wrapping numbers \( B \in \mathbb{Z} \), and the corresponding brane configurations for \( L^{1,2,1} \).

| \( B \) | Lowest order, (config.) | Higher order, (config.) |
|---|---|---|
| -2 | \( q^{3N}, (S_1) \) | \( q^{3N}, (2S_1) \) |
| -1 | \( q^{3N}, (S_2) \) | \( q^{3N}, (S_2 + S_1) \) |
| 0 | 1 | \( q^{3N}, (S_3 + S_4, \ldots) \) |
| 1 | \( q^{3N}, (S_2) \) | \( q^{3N}, (S_4 + S_5) \) |
| 2 | \( q^{3N}, (S_4) \) | \( q^{3N}, (2S_2) \) |

and \( v_I \) are given in terms of the new variables by

\[
v_1 = 1, \quad v_2 = q^{\frac{3}{4}} v_\zeta, \quad v_3 = \frac{q^{\frac{1}{2}} u}{v_\zeta}, \quad v_4 = \frac{q^{\frac{1}{2}} \zeta}{uv}, \quad v_5 = \frac{q^{\frac{3}{4}} v}{\zeta}; \quad (4.51)
\]

\( v_1 \) is the fugacity associated with the shrinking cycle and is set to be 1.

We define the index \( I^\text{gauge}_B \) of the sector with a specific baryonic charge \( B \) by the expansion

\[
I^\text{gauge}_B = \sum_{B \in \mathbb{Z}} I^\text{gauge}_B, \quad I^\text{gauge}_B \propto \zeta^B. \quad (4.52)
\]

The brane configurations and expected orders of their contribution are shown in Table 8 for \(-2 \leq B \leq 2\). The expected relations obtained from the order estimation in Table 8 are

\[
I^\text{gauge}_0 = I^\text{KK} + \mathcal{O}(q^{\frac{3}{2}N}), \quad I^\text{gauge}_1 = I^\text{D3} + \mathcal{O}(q^{\frac{1}{2}N}), \quad I^\text{gauge}_{-2} = I^\text{D3} + \mathcal{O}(q^{\frac{1}{2}N}), \\
I^\text{gauge}_2 = I^\text{D3} + \mathcal{O}(q^{\frac{1}{2}N}), \quad I^\text{gauge}_{-1} = I^\text{D3} + \mathcal{O}(q^{\frac{1}{2}N}). \quad (4.53)
\]

Let us confirm the relations in Eq. (4.53) for \( N = 2 \). See also Appendix B for the analysis with \( N = 3 \).

For the \( B = 0 \) sector on the gauge theory side we obtain

\[
I^\text{gauge}_0 = 1 + \left( v^2 + \frac{2}{v_\zeta} \right) q^{\frac{3}{2}} + \left( uv + \frac{v}{u} \right) q^\frac{9}{4} + \left( 4v^4 + \frac{5}{v_\zeta^2} + 2 \right) q^3 + \mathcal{O}(q^{\frac{15}{4}}). \quad (4.54)
\]

On the gravity side, by using Eq. (3.4) with the fugacities

\[
w_{1 + \frac{1}{2}} = w_{1 + 1} = v_3 v_4, \quad w_{2 + \frac{1}{2}} = v_1 v_4 v_5^2, \quad w_{3 + \frac{1}{2}} = v_1 v_2 v_5, \quad w_{4 + \frac{1}{2}} = v_1 v_2^2 v_3 \quad (4.55)
\]

we obtain the index

\[
I^\text{KK} = (\ldots \text{terms identical with Eq. (4.54)} \ldots) + \left( 2v^4 + \frac{5}{v_\zeta^2} + 2 \right) q^3 + \mathcal{O}(q^{\frac{15}{4}}). \quad (4.56)
\]
These two indices are consistent with the first relation in Eq. (4.53).

Let us discuss the \( B \neq 0 \) sectors. Because \( \mathcal{I}_B \) and \( \mathcal{I}_{-B} \) are related by charge conjugation we only consider \( B > 0 \) sectors.

For the \( B = 1 \) sector we have, on the gauge theory side,
\[
\frac{\mathcal{I}_{\text{gauge}}}{\mathcal{I}_{\text{KK}}} = 2\zeta \left[ v^2 q^2 + \frac{v}{u} q^2 + \left( \frac{1}{u^2} - 1 + v^2 \chi'_{n_1} \right) q^3 - \frac{v^3}{u} q^{15} + \mathcal{O}(q^9) \right], \tag{4.57}
\]
where \( \chi'_{n_1} = \chi_{n_1}(y) \), and on the gravity side,
\[
\mathcal{I}_{S_2}^{D3} = 2v_2^N \exp\left(f(q,y,w_{1+\frac{1}{2}},w_{2+\frac{1}{2}})\right)
\overset{(N=2)}{=} 2\zeta \left[ \ldots \text{terms identical with Eq. (4.57)} \ldots \right] + \left( \frac{1}{u^2v} - \frac{v^3}{u} \right) q^{15} + \mathcal{O}(q^9). \tag{4.58}
\]
These two are consistent with the second relation in Eq. (4.53).

For the \( B = 2 \) sector we have, on the gauge theory side,
\[
\frac{\mathcal{I}_{\text{gauge}}}{\mathcal{I}_{\text{KK}}} = \zeta^2 \left[ \frac{1}{u^2v^2} q^2 + \frac{v}{u} q^2 + \left( -\frac{1}{u^2} + 3v^4 + \frac{\chi'_{1}}{u^2v^2} \right) q^3 + \mathcal{O}(q^{15}) \right], \tag{4.59}
\]
and on the gravity side,
\[
\mathcal{I}_{S_4}^{D3} = v_4^N \exp\left(f(q,y,w_{3+\frac{1}{2}},w_{4+\frac{1}{2}})\right)
\overset{(N=2)}{=} \zeta^2 \left[ \ldots \text{terms identical with Eq. (4.59)} \ldots \right] + \left( -\frac{1}{u^2} + v^4 + \frac{\chi'_{1}}{u^2v^2} \right) q^3 + \mathcal{O}(q^{15}). \tag{4.60}
\]
These two are consistent with the fourth relation in Eq. (4.53).

5. Conclusions

We have investigated the superconformal index of \( \mathcal{N} = 1 \) quiver gauge theories realized on D3-branes in toric Calabi–Yau manifolds. The holographic dual of such a quiver gauge theory is type IIB string theory in \( AdS_5 \times SE_5 \), where \( SE_5 \) is a toric Sasaki–Einstein manifold. A D3-brane wrapped on a supersymmetric three-cycle in \( SE_5 \), which corresponds to a baryonic operator in the gauge theory, contributes to the index as a finite-\( N \) correction. We proposed the formula in Eq. (3.6) for the correction to the index due to such a D3-brane and fluctuation modes of massless fields on the brane. Equation (3.6) is a natural generalization of similar formulas proposed in Ref. [6] for S-folds and in Ref. [7] for orbifolds. Similarly to these previous cases a wrapped D3-brane has topology \( S^3 / \Gamma \), where \( \Gamma \) is an abelian group. A difference is that for a toric manifold the \( S^3 \) is in general not round. Even so, the formula is still quite simple thanks to the fact that the index depends on the background geometry through only a small number of parameters.

The formula is applicable to general toric quiver gauge theories. Starting from the toric data of the \( SE_5 \) we can easily calculate the corrections induced by D3-branes with single wrapping. We did not take account of D3-branes with multiple wrapping. We confirmed that the formula works correctly for several examples (\( SE_5 = T^{1,1}, T^{2,2}, Y^{2,1}, \) and \( L^{1,2,1} \)) by comparing the index obtained from
the formula with the result of numerical calculation using the localization method. The errors are consistent with the interpretation that they are due to branes with multiple wrapping.

The formula consists of three factors: the degeneracy factor \( m_I \), the classical factor \( v^N_I \), and the excitation factor \( \text{Pexp} i^{D3} \). The degeneracy factor \( m_I \) was interpreted as the degeneracy of states on the wrapped D3-brane due to the presence of different gauge holonomies. Because we considered only D3-branes with single wrapping the theory on the D3-brane is \( U(1) \) gauge theory consisting only of neutral fields. Then holonomies do not couple to any excitation modes on the brane, and different holonomies simply give an overall numerical factor \( m_I \).

Part of the degeneracy factor is associated with torsion cycles or shrinking cycles, and we can turn on fugacities coupling to them. In the main text we neglected wrapping on such cycles by setting the corresponding fugacity to be 1. See Appendix C for preliminary analyses in which we introduce fugacities to see the refined structure of the index in two examples.

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Appendix A. Localization formula
A toric quiver gauge theory has gauge group \( SU(N)^nV \) and bi-fundamental chiral multiplets \( X_{ab} \) belonging to the representation \( (N, \overline{N}) \) of \( SU(N)_a \times SU(N)_b \). The superconformal index is given by

\[
I^{\text{gauge}} = \prod_{a=1}^{nV} \int d\mu^{(a)} \text{Pexp} i^{\text{gauge}},
\]  

where Pexp is the plethystic exponential defined by

\[
\text{Pexp} \left( \sum_k c_k x_1^{n_{1,k}} \cdots x_p^{n_{p,k}} \right) = \prod_k (1 - x_1^{n_{1,k}} \cdots x_p^{n_{p,k}})^{-c_k},
\]  

and \( \int d\mu^{(a)} \) is the integral over the \( SU(N)_a \) gauge fugacities \( z^{(a)}_i \) \((i = 1, \ldots, N)\) defined by

\[
\int d\mu^{(a)} = \frac{1}{N!} \int \frac{dz_i^{(a)}}{2\pi iz_i^{(a)}} \prod_{i > j} (z_i^{(a)} - z_j^{(a)}).
\]  

The gauge fugacities are constrained by

\[
\prod_{i=1}^{N} z_i^{(a)} = 1.
\]  

The single-particle index \( i^{\text{gauge}} \) is the sum of contributions of vector multiplets \( V^a \) and bi-fundamental chiral multiplets \( X_{ab} \):

\[
i^{\text{gauge}} = \sum_{a=1}^{nV} i[V^a] + \sum_{ab} i[X_{ab}],
\]
where the contribution of the $SU(N)_a$ vector multiplet $V_a$ is
\[
i[V_a] = -\left(\frac{yq^3}{1-yq^3} + \frac{y^{-1}q^3}{1-y^{-1}q^3}\right)\chi^{(a)}_{\text{adj}}, \tag{A.6}\]
and the contribution of the chiral multiplet $X_{ab}$ is
\[
i[X_{ab}] = \frac{\chi[X_{ab}] - q^3\overline{\chi}[X_{ab}]}{(1-yq^3)(1-y^{-1}q^3)}. \tag{A.7}\]
$\chi[X_{ab}]$ is defined by
\[
\chi[X_{ab}] = \prod_{I} \langle R_j | X_{ab} | (a) \rangle \chi^{(a)}_{\text{fund}} \chi^{(b)}_{\text{fund}}, \tag{A.8}\]
and $\overline{\chi}$ is defined from this by replacing all fugacities with their inverse. $\chi^{(a)}_{\text{fund}}$, $\overline{\chi}^{(b)}_{\text{fund}}$, and $\chi^{(a)}_{\text{adj}}$ are the $SU(N)_a$ characters of the fundamental, the anti-fundamental, and the adjoint representations, respectively:
\[
\chi^{(a)}_{\text{fund}} = \sum_{i=1}^{N} z_i^{(a)}, \quad \overline{\chi}^{(a)}_{\text{fund}} = \sum_{i=1}^{N} \frac{1}{z_i^{(a)}}, \quad \chi^{(a)}_{\text{adj}} = \chi^{(a)}_{\text{fund}}\overline{\chi}^{(a)}_{\text{fund}} - 1. \tag{A.9}\]

Appendix B. Results for $N = 3$

In this appendix we show the results for $N = 3$ for the models analyzed in Sect. 4 for $N = 2$.

Appendix B.1. $T^{1,1}$ (conifold)

Let us confirm the relations in Eqs. (4.9) and (4.13) for $N = 3$. For the relation in Eq. (4.9) the indices appearing on the left- and right-hand sides are given by
\[
\mathcal{I}_0^{\text{gauge}} = 1 + \chi_1^u \chi_1^v q^3 + (2 - \chi_2^u - \chi_2^v + 2 \chi_3^u \chi_3^v)q^3 \\
+ (2 \chi_1^u \chi_1^v - \chi_2^u \chi_2^v - \chi_3^u \chi_3^v + 3 \chi_5^u \chi_5^v)q^9 \\
+ (2 - 2 \chi_2^u + \chi_4^u - 2 \chi_2^v + \chi_4^v - 2 \chi_4^u \chi_4^v - 2 \chi_5^u \chi_5^v \\
+ 4 \chi_4^u \chi_4^v + 2(\chi_4^u \chi_3^v + \chi_5^u \chi_3^v)\chi_1^f)q^6 + \mathcal{O}(q^{15}), \tag{B.1}\]
\[
\mathcal{I}_0^{\text{KK}} = (\ldots \text{terms identical with Eq. (B.1)} \ldots) \\
+ (6 - 2 \chi_2^u + \chi_4^u - 2 \chi_2^v + \chi_4^v + 4 \chi_4^u \chi_2^v - 2 \chi_4^u \chi_2^v \\
- 2 \chi_2^u \chi_5^v + 5 \chi_5^u \chi_5^v)q^6 + \mathcal{O}(q^{15}), \tag{B.2}\]
and these satisfy Eq. (4.9). The first relation of Eq. (4.13) is confirmed for $N = 2$ with the results
\[
\frac{\mathcal{I}_0^{\text{gauge}}}{\mathcal{I}_0^{\text{KK}}} = \zeta \left[ \chi_1^u q^3 + (\chi_1^u - \chi_2^u \chi_1^v + (-\chi_1^u + \chi_5^u)\chi_1^f)q^{15} \\
+ (\chi_1^u - \chi_3^u - 2 \chi_4^u \chi_1^v + 2 \chi_1^u \chi_1^f + (-\chi_1^u + \chi_5^u)\chi_2^f)q^{21} \\
+ (\chi_3^v - \chi_4^u \chi_3^v + (\chi_1^u - \chi_3^u - \chi_5^u \chi_2^v + 3 \chi_5^u \chi_2^v)\chi_1^f + (-\chi_1^u + \chi_5^u)\chi_3^f)q^{27} \\
+ (\chi_3^v - \chi_5^u + 3 \chi_4^u \chi_2^v + \chi_1^u \chi_4^v - \chi_5^u \chi_2^v - \chi_7 \chi_4^v) \right].
\]
Again, the second relation in Eq. (4.24) can be checked by the transformation of fugacities in terms identical with Eq. (B.3).

\[I_{S_1}^{D_3} + I_{S_3}^{D_3} = \zeta \left[ \ldots \text{terms identical with Eq. (B.3)} \ldots \right.\]
\[+ (-2\chi_{1}^\nu - 2\chi_{2}^\mu - 2\chi_{3}^\mu + 1) q^3 + 2 (\chi_{1}^\nu + \chi_{2}^\mu + \chi_{3}^\mu) q^2 + \mathcal{O}(q^0),\]
\[T_{0}^{\text{gauge}} = 1 + (\chi_{2}^\mu - \chi_{2}^\nu - x_{2}^\nu + 1) q^0 + 4 (\chi_{1}^\mu + \chi_{3}^\mu) q^2 + \mathcal{O}(q^0),\]
\[T_{0}^{\text{KK}} = \ldots \text{terms identical with Eq. (B.5)} \ldots + O(q^2) + \mathcal{O}(q^6),\]
and we can check that the first relation in Eq. (4.24) holds. To confirm the last relation we have
\[
\frac{T_{1}^{\text{gauge}}}{T_{KK}} = \zeta \left[ 2x_{3}^\mu q^3 + 2 \left( x_{3}^\mu - x_{1}^\mu \right) x_{1}^\mu q^1 + 2 \left( x_{3}^\mu - x_{1}^\mu \right) x_{2}^\mu + \left( x_{3}^\mu - x_{1}^\mu \right) x_{3}^\mu q^0 + \mathcal{O}(q^0) \right]
\[+ 2 \left( 3x_{4}^\mu x_{1}^\mu + 3x_{6}^\mu x_{3}^\mu \right) + 3 \chi_{2}^\mu \left( x_{1}^\mu + x_{2}^\mu \right)
\[+ (\chi_{1}^\mu x_{2}^\mu + \chi_{1}^\mu - x_{3}^\mu) x_{1}^\mu - (\chi_{1}^\mu - x_{3}^\mu) x_{3}^\mu + O(q^0) \right],\]
\[T_{S_1}^{D_3} + T_{S_3}^{D_3} = \zeta \left[ \ldots \text{terms identical with Eq. (B.7)} \ldots \right.\]
\[+ 2 \left( (x_{1}^\mu x_{2}^\mu + x_{1}^\mu - x_{3}^\mu) x_{1}^\mu - x_{3}^\mu) x_{3}^\mu q^2 + \mathcal{O}(q^3) \right].\]

Again, the second relation in Eq. (4.24) can be checked by the transformation of fugacities in Eq. (4.15).
Appendix B.3. \( Y^{2,1} \) (complex cone over \( dP_3 \))

We check the relations in Eq. (4.36) for the \( N = 3 \) case except for the last one. For the last case the calculation on the gauge theory side takes a considerable time for an ordinary laptop computer and we could not finish the computation. The first relation in Eq. (4.36) can be confirmed by

\[
I^{\text{gauge}}_0 = 1 + v \chi_1^u q^9 + \left( \frac{\chi_3^u}{v} - \frac{\chi_1^u}{v} \right) q^{15} + (2v^2 \chi_2^u - v^2) q^9 + (\chi_4^u - 1) q^6
- (v^3 \chi_1^u - 11v^3 \chi_3^u) q^{22} + O(q^{29}), \tag{B.9}
\]

\[
I^{KK} = (\ldots \text{terms identical with Eq. (B.9)} \ldots) + (3v^3 \chi_3^u - v^3 \chi_3^u) q^{22} + O\left( q^{15} \right). \tag{B.10}
\]

For the second relation in Eq. (4.36) we can check that this actually holds:

\[
\frac{I^{\text{gauge}}}{I^{KK}_1} = \frac{1}{\xi} \left[ 3v^2 q^9 + 3v^2 \chi_1^u q^3 + 3\left( \frac{\chi_3^u}{v^2} + \frac{3}{v^2} \chi_3^f \right) q^{15} + \frac{3 \chi_3^u}{v^2} q^9 
+ 3 \left( -v^2 \chi_1^f - v^2 \chi_2^f \right) q^{21} + O(q^6) \right], \tag{B.11}
\]

\[
\frac{I^{\text{gauge}}_{D3}}{I^{KK}_{D3}} = \frac{1}{\xi} \left[ (\ldots \text{terms identical with Eq. (B.11)} \ldots) + 3 \left( \frac{\chi_4^u}{v^2} - v^2 \chi_1^f + \frac{3}{v^2} \chi_3^f \right) q^{21} + O(q^6) \right]. \tag{B.12}
\]

For the third relation in Eq. (4.36), explicit calculation shows that

\[
\frac{I^{\text{gauge}}}{I^{KK}_{D3}} = \frac{1}{\xi^3} \left[ \left( \frac{\chi_3^u}{v^2} \right) q^9 + \left( \frac{1}{v^2} + \frac{\chi_3^f}{v^2} \right) q^{15} - \frac{\chi_1^u}{v^2} q^9 + \left( \frac{v^5 + \chi_3^f - \frac{1}{v^2} \chi_3^f}{v^2} \right) q^{21}
+ \left( -v^2 \chi_2^f + 10v^2 - \frac{\chi_1^f}{v^2} + \frac{\chi_3^f}{v^2} \right) q^{27} + O(q^{29}) \right], \tag{B.13}
\]

\[
\frac{I^{\text{gauge}}}{I^{KK}_{D3}} = \frac{1}{\xi} \left[ (\ldots \text{terms identical with Eq. (B.13)} \ldots)
+ \left( -v^2 \chi_2^f + \frac{9}{v^2} \chi_1^f - \frac{\chi_3^f}{v^2} \right) q^{27} + O(q^{15}) \right]. \tag{B.14}
\]

Appendix B.4. \( L^{1,2,1} \) (suspended pinch point)

The first relation in Eq. (4.53) is confirmed for \( N = 3 \) by comparing the following:

\[
I^{\text{gauge}}_0 = 1 + (v^2 + \frac{2}{v^2}) q^3 + \left( uv + \frac{v}{u} \right) q^3 + (2v^4 + \frac{5}{v^4} + 2) q^3
+ \left( uv^3 + \frac{2u}{v} + \frac{v^3}{u} + \frac{2}{uv} \right) q^{15}
+ \left( 2u^2 v^2 + \frac{2u^2}{v^2} + 5v^6 + 10v^6 + 5v^2 + \frac{5}{v^2} \right) q^9 + O(q^{21}), \tag{B.15}
\]

\[
I^{KK} = (\ldots \text{terms identical with Eq. (B.15)} \ldots)
\]
\begin{equation}
+ \left( 2u^2v^2 + \frac{2v^2}{u^2} + 3v^6 + \frac{10}{v^6} + 5v^2 + \frac{5}{v^2} \right) q^9 + O(q^{21}). \tag{B.16}
\end{equation}

The second relation in Eq. (4.53) is confirmed by comparing the following:

\begin{equation}
\mathcal{T}_{\text{gauge}}^{1,KK} = 2\zeta \left[ v^3 q^9 + \frac{v^2}{u} q^3 + \left( \frac{v}{u^2} - v + v^3 \chi_1^f \right) \frac{q^{15}}{u} + \left( \frac{1}{u^3} - \frac{v^4}{u} \right) q^2 \right.
+ \left. \left( -v^3 - \frac{1}{v} + v^3 \chi_2^f \right) q^3 + O(q^6) \right], \tag{B.17}
\end{equation}

\begin{equation}
\mathcal{T}_{S_2}^{D3} = 2\zeta \left[ \ldots \text{terms identical with Eq. (B.17)} \ldots \right.
+ \left. \left( \frac{1}{u^4} - \frac{v^3}{u} - \frac{1}{v} + v^3 \chi_2^f \right) q^2 + O(q^6) \right]. \tag{B.18}
\end{equation}

The fourth relation in Eq. (4.53) is confirmed by comparing the following:

\begin{equation}
\mathcal{T}_{\text{gauge}}^{2,KK} = \zeta^2 \left[ \frac{1}{u^3} q^9 + \frac{1}{u} q^3 + \left( \frac{1}{u^2} + \frac{v^3}{u} + \chi_1^f \right) \frac{q^{15}}{u^3} \right.
+ \left. \left( 3v^6 - \frac{1}{u^2} \right) q^2 + O(q^{21}) \right], \tag{B.19}
\end{equation}

\begin{equation}
\mathcal{T}_{S_4}^{D3} = \zeta^2 \left[ \ldots \text{terms identical with Eq. (B.19)} \ldots \right.
+ \left. \left( v^6 - \frac{1}{u^2} \right) q^2 + O(q^{21}) \right]. \tag{B.20}
\end{equation}

**Appendix C. Refined baryonic charges**

We have seen in the main text that we have two types of cycles: visible cycles and vanishing cycles. Correspondingly, we can divide the $U(1)$ baryonic symmetries into two classes. We also found in an example ($T^{1,1}$) that the discrete factor may appear in the baryonic symmetry. In the main text we considered only the $U(1)$ symmetries associated with visible cycles, and neglected the others by setting the corresponding fugacities to 1. In this appendix we give some analysis with the full baryonic symmetry taken into account.

**Appendix C.1. Discrete baryonic symmetry in the $T^{2,2}$ model**

In Sect. 4.2 we mentioned that the baryonic symmetry of the $T^{2,2}$ model has the non-trivial discrete factor $G_{\text{disc}} = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this appendix we discuss how the corresponding charges appear on the gravity side.

Let us first explicitly derive the baryonic symmetry including the discrete factor. On the gauge theory side, the baryonic symmetry is obtained by replacing the four $SU(N)$ gauge groups by $U(1)$. Because the theory is chiral the $U(1)^4$ symmetry is broken to its subgroup by anomaly. We denote an element of $U(1)^4$ as $(\xi_1, \xi_2, \xi_3, \xi_4)$. Let $\beta_i$ ($i = 1, 2, 3, 4$) be the $U(1)_i$ charges, which are classically conserved. We use the labelling shown in Fig. 4(c). The conservation law is broken by the instanton effect as

\begin{equation}
\Delta \beta_i = \frac{1}{N} \times 2N \times n_{i+1} - \frac{1}{N} \times 2N \times n_{i-1} = 2n_{i+1} - 2n_{i-1}, \tag{C.1}
\end{equation}
Table C1. Baryonic charges of the elementary baryonic operators for the $F_0$ model. $B$ denotes the $U(1)$ charge, and $b_1$ and $b_2$ denote the $\mathbb{Z}_2$ charges.

| Operators | $B$ | $b_1$ | $b_2$ |
|-----------|-----|-------|-------|
| $B_{12}$  | +1  | 1     | 1     |
| $B_{23}$  | −1  | 0     | 1     |
| $B_{34}$  | +1  | 0     | 0     |
| $B_{41}$  | −1  | 1     | 0     |

where $n_i$ are the $SU(N)_i$ instanton numbers. For an element $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ to be anomaly free, the following relation must hold for arbitrary $n_i$:

$$1 = \zeta_1^{\Delta n_1} \zeta_2^{\Delta n_2} \zeta_3^{\Delta n_3} \zeta_4^{\Delta n_4} = \left( \frac{\zeta_4}{\zeta_2} \right)^{2n_1} \left( \frac{\zeta_1}{\zeta_3} \right)^{2n_2} \left( \frac{\zeta_2}{\zeta_4} \right)^{2n_3} \left( \frac{\zeta_3}{\zeta_1} \right)^{2n_4}. \quad (C.2)$$

Namely, $\zeta_i$ must satisfy

$$\frac{\zeta_2^2}{\zeta_4^2} = \frac{\zeta_3^2}{\zeta_1^2} = 1. \quad (C.3)$$

We can set $\zeta_4 = 1$ with the decoupled diagonal $U(1)$. Then the solution of Eq. (C.3) is

$$\zeta_1 = \sigma_1 \zeta, \quad \zeta_2 = \sigma_2, \quad \zeta_3 = \zeta, \quad \zeta_4 = 1, \quad (C.4)$$

where $\zeta \in U(1)$ and $\sigma_1, \sigma_2 = \pm 1$. This means that the anomaly-free baryonic symmetry is $U(1) \times \mathbb{Z}_2^2$. $\zeta$ is identified with the fugacity defined in Eq. (4.20). In addition, we can introduce $\mathbb{Z}_2$-valued fugacities $\sigma_1$ and $\sigma_2$ to define the index.

The action of each $\mathbb{Z}_2$ symmetry on bi-fundamental fields is read off from Eq. (C.4). They act non-trivially only on baryonic operators. Corresponding to the four arrows in the quiver diagram in Fig. 4(c) we have four elementary baryonic operators with dimension $3^4/N$:

$$B_{12} = \text{det} X_{12}, \quad B_{23} = \text{det} X_{23}, \quad B_{34} = \text{det} X_{34}, \quad B_{41} = \text{det} X_{41}, \quad (C.5)$$

and these carry the $\mathbb{Z}_2$ charges shown in Table C1. (We have neglected the superscripts of the bi-fundamental fields, which play no role here.)

We expand the index by using two $\mathbb{Z}_2$-valued baryonic charges together with the integer baryonic charge:

$$I_{gauge} = \sum_{B \in \mathbb{Z}} \sum_{b_1, b_2 = 0, 1} \sigma_1^{b_1} \sigma_2^{b_2} T_{(B; b_1, b_2)}^{\text{gauge}}. \quad (C.6)$$

Unlike $\zeta$ we do not include $\sigma_1$ and $\sigma_2$ in the definition of $T_{(B, j)}^{\text{gauge}}$.

As mentioned in Sect. 4.2, the theory has the $\mathbb{Z}_4$ symmetry rotating the toric diagram and the bipartite graph. This acts on the quiver diagram as $\mathbb{Z}_4$ rotation, and we can read off its action on the baryonic charges from Eq. (C.4) as

$$B \rightarrow B' = -B, \quad b_1 \rightarrow b_1' = b_2 + B, \quad b_2 \rightarrow b_2' = b_1. \quad (C.7)$$

The four elementary baryonic operators are transformed as $B_{a, -1, a} \rightarrow B_{a, a+1}$. 

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Let us first consider the sector with charges \((B; b_1, b_2) = (0, 0, 0)\). The dimension of a product of baryonic operators contributing to this sector is at least \(3N\), and the expected relation between indices is
\[
\mathcal{I}^\text{gauge}_{\langle 0, 0, 0 \rangle} = \mathcal{I}^\text{KK} + \mathcal{O}(q^{3N}). \tag{C.8}
\]
Thanks to the refinement, the order of the expected error is larger than the previous one in Eq. (4.24). Indeed, the result of numerical calculation for \(N = 2\) is
\[
\mathcal{I}^\text{gauge}_{\langle 0, 0, 0 \rangle} = 1 + (\chi^u_2 \chi^v_2 - \chi^u_2 - \chi^v_2 + 1) q^3 \\
\quad + (2 \chi^u_2 \chi^v_2 - 2 \chi^u_2 \chi^v_2 - \chi^u_2 - 2 \chi^u_2 \chi^v_2 + 6 \chi^u_2 \chi^v_2 + 5 \chi^u_2 - \chi^v_2 + 5 \chi^v_2 + 6) q^6 \\
\quad + \mathcal{O}(q^{9/2}). \tag{C.9}
\]
and the corresponding Kaluza–Klein mode contribution is
\[
\mathcal{I}^\text{KK} = (\ldots \text{terms identical with Eq. (C.9)} \ldots) \\
\quad + (2 \chi^u_2 \chi^v_2 - 2 \chi^u_2 \chi^v_2 - \chi^u_2 - 2 \chi^u_2 \chi^v_2 + 2 \chi^u_2 \chi^v_2 + \chi^u_2 - \chi^v_2 + \chi^v_2 + 2) q^6 \\
\quad + \mathcal{O}(q^9). \tag{C.10}
\]
From these relations we find that Eq. (C.8) holds.

Next, we consider baryonic sectors with \(B = \pm 1\). Let us focus on the \(B = 1\) sectors. Two baryonic operators \(B_{12}\) and \(B_{14}\) carry charges \((+1; 1, 1)\) and \((+1; 0, 0)\). The corresponding brane configurations are D3-branes wrapped on the three-cycles \(S_1\) or \(S_3\). Note that D3-branes wrapped on \(S_1\) and \(S_3\) can be continuously deformed to each other, and we cannot relate two cycles with two baryonic sectors by one. Instead, we should interpret the existence of the two sectors as the existence of two different values of the \(U(1)\) holonomy on a wrapped D3-brane with the topology \(S^3/\mathbb{Z}_2\). Because the holonomy does not couple with any fluctuation modes on the single D3-brane two sectors must give the same index. This is consistent with the relation
\[
\mathcal{I}^\text{gauge}_{\langle 1, 0, 0 \rangle} = \mathcal{I}^\text{gauge}_{\langle 1, 1, 1 \rangle}, \tag{C.11}
\]
which follows from the \(\mathbb{Z}_4\) symmetry.

The lowest-order contribution to the sector with \((B; \sigma_1, \sigma_2) = (1; 0, 0)\) comes from \(B_{34}\), and on the gravity side this corresponds to a brane wrapping on the \(S_1\) or \(S_3\). At the order of \(q^{15N}\), operators like \(B_{34}B_{23}^2\) contribute to the index, and this corresponds to multiple-wrapping configurations. Therefore, the expected relation is
\[
\frac{\mathcal{I}^\text{gauge}_{\langle 1, 0, 0 \rangle}}{\mathcal{I}^\text{KK}} = \frac{1}{2} (\mathcal{I}^\text{D3}_{S_1} + \mathcal{I}^\text{D3}_{S_3}) + \mathcal{O}(q^{15N}), \tag{C.12}
\]
where the factor \(1/2\) is inserted to remove the degeneracy factor \(m_I = 2\) for two possible values of the holonomy because the sector corresponds to one of them.

The calculation on the gauge theory side with \(N = 2\) gives
\[
\frac{\mathcal{I}^\text{gauge}_{\langle 1, 0, 0 \rangle}}{\mathcal{I}^\text{KK}} = \zeta \left[ \chi^u_2 q^2 + (\chi^v_2 - 1) \chi^l_1 q^3 + (-\chi^u_2 - 2 \chi^v_2 - 1 + (\chi^u_2 - 1) \chi^v_2) q^6 \\
+ ((\chi^v_2 \chi^v_2 - \chi^u_2 - \chi^v_2 + 1) \chi^l_1 + (\chi^u_2 - 1) \chi^v_3) q^6 \right].
\]

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Fig. C1. Web-diagram of the $L^{1,2,1}$ model: (a) wrapping numbers defined with a resolution of overlapping legs; (b) wrapping numbers after two downgoing legs are swapped.

\begin{align*}
+ (5x^u_6x^v_4 - x^u_4x^v_2 - x^u_4x^v_4 - x^u_4 - 4x^u_2x^v_2 + 5x^u_2x^v_4 + 5x^u_2 + 5x^u_2 \\
+ 7x^v_2 + x^v_4 + 6 + (-x^u_2 + 1) x^v_2 + (x^v_2 - 1) x^v_4) \frac{15}{q^2} + O(q^{33}) \Big].
\end{align*}

On the gravity side we have

$$\frac{1}{2} (I_{S_1}^{D3} + I_{S_3}^{D3}) = \xi \left[ \ldots \text{terms identical with Eq. (C.13)} \ldots \right]$$

\begin{align*}
+ (-x^u_4x^v_2 - x^u_4x^v_4 + x^u_2x^v_2 - x^u_4 + 2x^v_2 + x^v_4 + 1 \\
+ (-x^u_2 + 1) x^v_2 + (x^v_2 - 1) x^v_4) \frac{15}{q^2} + O(q^{33}) \right],
\end{align*}

and we find that Eq. (C.12) holds.

Before ending this subsection we comment on the fact that there are two $\mathbb{Z}_2$ factors in the homology of $T^{2,2}$:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}_2, \quad H_2 = \mathbb{Z}, \quad H_3 = \mathbb{Z} \times \mathbb{Z}_2, \quad H_4 = 0, \quad H_5 = \mathbb{Z}. \quad (C.15)$$

It seems natural to identify the torsion subgroup of the homology with the discrete part of the baryonic symmetry. We leave detailed analysis of this problem for future work.

**Appendix C.2. Shrinking cycle in the $L^{1,2,1}$ model**

As is mentioned in Sect. 4.4 we have two perfect matchings for the vertex 1 in the toric diagram of $L^{1,2,1}$ (Fig. 6(a)), and correspondingly we have two $R$-charges $R_1$ and $R'_1$ associated with this vertex. The charge assignments of these charges are shown in Table 7. They are not independent, but are related by

$$R'_1 = R_5 - R_1 + R_2, \quad (C.16)$$

and we have ambiguity in the definition of $R_5$. In the main text we neglect this ambiguity by setting $v_1 = 1$. In this appendix we perform an analysis with this taken into account.

For the following arguments the web diagrams are useful. A web diagram consists of semi-infinite lines perpendicular to edges of the toric diagram (Fig. C1).

(In the arguments below, only external lines are relevant and the central part is expressed as a blob in the figure.) We label external lines by $r = 1/2, 1 + 1/2, \ldots$ just like edges. These lines express NS5-branes in the five-brane realization of the gauge theory. A D3-brane wrapped on a three cycle $S_I$ is mapped to an open D-string from the external line $r = I - 1/2$ to the external line $r = I + 1$ [28,29]. If there is more than one edge between two adjacent corners $I_1$ and $I_2 = I_1 + k$, the corresponding $k$
external lines are parallel to each other. In the conformal limit in which all the external lines meet at a point these parallel lines coincide, and the corresponding cycles shrink. To make these cycles visible we need to resolve the singularity by separating parallel lines from each other. The permutation group $S_k$ acting on these lines is identified with the Weyl group associated with the $A_{k-1}$ singularity acting on the shrinking cycles at the singularity.

In the case of the $L^{1,2,1}$ model we have two parallel external lines corresponding to the $A_1$ singularity. We focus on the shrinking cycle $S_1$ and the visible cycles $S_5$ and $S_2$ intersecting with $S_1$. Let $n_5$, $n_1$, and $n_2$ be the wrapping numbers for these cycles. In the web-diagram these numbers are interpreted as the numbers of open strings (Fig. C1(a)). Let us see what happens when we swap the two parallel external lines $1/2$ and $1 + 1/2$. This corresponds to the Weyl reflection of the $A_1$. The wrapping numbers $n_5$ and $n_2$ are kept unchanged while $n_1$ is non-trivially transformed (Fig. C1(b)):

$$n_1 \rightarrow n'_1 = n_5 - n_1 + n_2. \quad (C.17)$$

We can easily check that this generates $\mathbb{Z}_2$. This relation is identical to the relation of $R$-charges in Eq. (C.16).

The four charges $r$, $F_1$, $F_2$, and $B$ defined in Eq. (4.48) do not include either $R_1$ or $R'_1$ and are invariant under the Weyl reflection. As the additional charge associated with the shrinking cycle it is convenient to use the Cartan generator of the $A_1$ algebra, which is odd under the Weyl reflection. It is

$$\tilde{B} = \frac{1}{N}(R_1 - R'_1). \quad (C.18)$$

We define the corresponding fugacity $\tilde{\zeta}$ by

$$\prod_{l=1}^{5} v_I^{R_I} = q^{3} r_1 v_1 f_1 v_2 f_2 \tilde{B} \tilde{B}. \quad (C.19)$$

The fugacities $v_I$ are given by

$$v_1 = \frac{2}{\tilde{\zeta}} \pi, \quad v_2 = \frac{q^3 v_1}{\tilde{\zeta} \pi}, \quad v_3 = \frac{q^3 u}{v_1 \pi}, \quad v_4 = \frac{q^3 \tilde{\zeta}}{uv}, \quad v_5 = \frac{q^3 v}{\frac{1}{\zeta} \pi}. \quad (C.20)$$

A D3-brane wrapped on the shrinking cycle $S_1$ carries $\tilde{B} = 2$ and its Weyl reflection carries $\tilde{B} = -2$. These correspond to the root vectors of $A_1$. A D3-brane wrapped on the visible cycle $S_2$ carries $\tilde{B} = -1$ and its Weyl reflection $S_2 + S_1$ carries $\tilde{B} = 1$. Namely, these two form the fundamental representation of $A_1$. Both these cycles contribute to $\mathcal{I}^{D3}_{S_2}$ in Eq. (4.58), and the refinement by introducing $\tilde{\zeta}$ splits it into two contributions, $\mathcal{I}^{D3}_{S_2} \propto v_2^N \propto \frac{1}{\zeta}$ and $\mathcal{I}^{D3}_{S_2+S_1} \propto v_2^N v_1^N \propto \tilde{\zeta}$. Therefore, the degeneracy factor 2 in Eq. (4.58) is replaced by the $A_1$ character $\chi_1(\tilde{\zeta}) = \frac{1}{\zeta} + \frac{1}{\tilde{\zeta}}$:

$$\mathcal{I}^{D3}_{S_2} \rightarrow \mathcal{I}^{D3}_{S_2} + \mathcal{I}^{D3}_{S_2+S_1} = \frac{1}{2} \chi_1(\tilde{\zeta}) \mathcal{I}^{D3}_{S_2}. \quad (C.21)$$

This is also the case for the other adjacent visible cycle $S_5$ and its Weyl reflection $S_5 + S_1$:

$$\mathcal{I}^{D3}_{S_5} \rightarrow \mathcal{I}^{D3}_{S_5} + \mathcal{I}^{D3}_{S_5+S_1} = \frac{1}{2} \chi_1(\tilde{\zeta}) \mathcal{I}^{D3}_{S_5}. \quad (C.22)$$

By numerical calculation on the gauge theory side we can find the same refinement. Namely, the overall factor 2 on the gauge theory side is also replaced by the character $\tilde{\zeta}_1$, and the second and fifth relations in Eq. (4.53) still hold after the refinement.
In general, if there are \( k \) parallel external lines, the factor \( k \) in the degeneracy factor is replaced by the fundamental \( A_{k-1} \) character for one of the adjacent cycles and by the anti-fundamental \( A_{k-1} \) character for the other adjacent cycle.

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