RANDOMLY WALKING 1D QUANTUM HARMONIC OSCILLATOR. AVERAGED TRANSITION PROBABILITIES.

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One-dimensional problem for quantum harmonic oscillator with "regular+random" frequency subjected to the external "regular+random" force is considered. Averaged transition probabilities are found.

1 Introduction

There are a lot of works published recently and devoted to quantum chaos, i.e. they consider quantum systems analogous to classical systems that demonstrate the features of chaotic behaviour. Exploration is developed at different directions: investigation of energy levels distribution; derivation and calculation of objects (analogous to classical Lyapunov exponents or KS-entropy) which can witness for chaos in quantum system and others.

We present a new approach for quantum description of above mentioned systems that may be used for the investigation of a problem of multichannel scattering. It is known that scattering with rearrangement (typical for chemical reactions) goes through resonance complex formation which causes chaotic behaviour. It is impossible to predict the way to be followed by a system after leaving the complex because of the small parameters change. As it was shown in our previous works, the scattering process with rearrangement may be described in framework of randomly walking harmonic oscillator model. In this paper we consider one-dimensional case. More simple case has been already introduced earlier.

2 Description of the problem.

We consider the wave function $\Psi_{stc}(t, x)$, describing the state of the system as a random process with the time evolution determined by equation

$$i\partial_t \Psi_{stc} = \hat{H} \Psi_{stc},$$

(1)

where 1D Hamiltonian $\hat{H}$ is quadratic in space variable

$$\hat{H} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \Omega^2(t)x^2 - F(t)x,$$

(2)
while functions $\Omega^2(t)$ and $F(t)$ are random functions of time variable. We suppose that by definition

$$\Omega^2(t) = \Omega_0^2(t) + \sqrt{2\epsilon_1 p_1 f_1(t)} \Theta(t - t_1),$$

$$F(t) = F_0(t) + \sqrt{2\epsilon_2 p_2 f_2(t)} \Theta(t - t_2),$$

(3)

where $\Omega_0^2(t)$ and $F_0(t)$ are deterministic functions while $f_1(t), f_2(t)$ are independent zero mean gaussian random processes with two-point correlations of $\delta$ form:

$$< f_i(t) f_j(t') > = \delta_{ij}(t - t'), \quad i, j = 1, 2.$$  

(4)

Constants $\epsilon_i, i = 1, 2$ control the power of forces $f_i(t), i = 1, 2$, while functions $p_i(t), i = 1, 2$ are nonnegative $p_1, p_2 \geq 0$, which leads to the suggestion of the following asymptotic behaviour

$$\Omega_0(t) \rightarrow +\infty, \quad F_0(t) \rightarrow 0, \quad p_i(t) \rightarrow 0, \quad i = 1, 2.$$  

(5)

which guarantees in the limit $t \rightarrow -\infty$ existence of stationary states $\phi^n(t, x)$

$$\phi^n(t, x) = e^{-\frac{i(n+1/2)\Omega_n t}{\pi}} \phi^n(x),$$

$$\phi^n(x) = \left( \frac{1}{2^n n!} \sqrt{\frac{\Omega_n}{\pi}} \right)^{1/2} e^{-\Omega_n x^2/2} H_n(\sqrt{\Omega_n} x),$$

(6)

where $H_n(x)$ are Hermitian polynomials. At the limit $t \rightarrow +\infty$ there also exist stationary states $\phi^n(t, x)$, which may be got from (6) by simple replacement of $\Omega_n$ by $\Omega_{out}$. Moments $t_1$ and $t_2$ of switching on the noise are chosen to be finite in order to provide the correctness of subsequent constructions. Now we formulate our task: to get the averaged transition probabilities $W_{nm}$ from initial stationary states $\phi^n(t, x)$ to final ones $\phi^m(t, x)$ which come from the evolution described by (1)-(2).

3 Formal expressions for wave functional and transition probabilities.

The main results of this paper are found on the basis of formal solution of (1)-(2), which may be constructed for arbitrary $\Omega^2(t)$ and $F(t)$ as a functional of solutions of classical equations of motion. Namely, as was shown in, the following representation for solution of (1)-(2) may be written that turns at the limit $t \rightarrow -\infty$ to the stationary state $\phi^n(t, x)$ defined in (6):

$$\Psi_{ste}^{(n)}(t, x) = \frac{1}{\sqrt{r}} \exp \left\{ i \left[ \dot{\eta}(x - \eta) + \frac{\dot{r}}{2r}(x - \eta)^2 + \sigma \right] \right\} \phi^n(\tau, \frac{x - \eta}{r}), \quad n = 1, 2, \ldots, $$

(7)
where $\eta(t)$ is a solution of the classical equation of motion for oscillator with frequency $\Omega(t)$ subjected to external force $F(t)$:

$$\ddot{\eta} + \Omega^2(t)\eta = F(t), \quad \eta(-\infty) = \dot{\eta}(-\infty) = 0,$$

(8)

$\sigma(t)$ is an action functional corresponding to this solution

$$\sigma(t) = \int_{-\infty}^{t} \left[ \frac{1}{2} \dot{\eta}^2 - \frac{1}{2} \Omega^2(t)\eta^2 + F\eta \right] dt',$$

(9)

while $r(t)$ and $\tau(t)$ can be expressed in terms of solution of homogeneous equation corresponding to (8)

$$\ddot{\xi} + \Omega^2(t)\xi = 0, \quad \xi(t) \sim e^{\Omega n t}$$

(10)

in the following way: $\xi(t) = r(t)e^{\gamma(t)}$, $r(t) = |\xi(t)|$, $\tau(t) = \gamma(t)/\Omega n$. Functions $\Psi_{stc}^{(n)}(t,x)$ are in fact functionals for the realisation of random processes $f_1(t)$ and $f_2(t)$. Therefore it is natural to call them "wave functionals".

We are interested in construction of transition probabilities from the states $\Psi_{stc}^{(n)}(t,x)$ to stationary states $\varphi_{out}^{m}(t,x)$ in the limit $t \to +\infty$ averaged over the random processes $f_1(t)$ and $f_2(t)$ realisations. Let’s designate them as $W_{nm}$.

Having defined

$$\Psi_{stc}^{(n)}(t,x) = \sum_{m=0}^{\infty} c_{nm}(t|f_1, f_2)\varphi_{out}^{m}(t,x), \quad W_{nm} = \lim_{t \to +\infty} \langle |c_{nm}|^2 \rangle,$$

(11)

we denoted the averaging over $f_1$ and $f_2$ by the symbol $\langle .. \rangle$. An expression for generating function of coefficients $c_{nm}$ is known (see [5]). Having at hand formal expressions for objects of interest we should turn the averaging procedure to the form convenient for analytical or numerical treatment.

4 Equation for the distribution function.

Functions $\xi(t)$ and $\eta(t)$ defined by equations (8), (10) are random processes. It is more convenient to treat with the other random processes $z_1 = \eta$, $z_2 = \dot{\eta}$, $z_3 = Re(\xi/\xi)$, $z_4 = Im(\xi/\xi)$ which in total we denote briefly as $\vec{z}$. One can find that the Focker-Plank equation for conditional distribution function $P(\vec{z}, t|\vec{c}, t_>): = \langle \delta(\vec{z}(t) - \vec{z}) \rangle_{\vec{z}(t_>) = \vec{c}}$ looks as follows:

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{4} \frac{\partial (K_i P)}{\partial z_i} + (\epsilon_2 p_2 + \epsilon_1 p_1 z_1^2) \frac{\partial^2 P}{\partial z_2^2} + \epsilon_1 p_1 \frac{\partial^2 P}{\partial z_3^2} + 2\epsilon_1 p_1 z_1 \frac{\partial^2 P}{\partial z_2 \partial z_3} \equiv \hat{L}P,$$

(12)
where $K_1 = z_2$, $K_2 = F_0 - \Omega_0^2 z_1$, $K_3 = z_2^2 - z_3^2 - \Omega_0^2$, $K_4 = -2z_3z_4$. It is supplemented with the initial condition $P|_{t=t>} = \delta(\vec{z} - \vec{\zeta})$ and with the requirement that $\int P d\vec{z}$ is finite. Random vector $\vec{\zeta}$ of initial values of trajectories is described by distribution function $R(\vec{\zeta}, t_>)$ which we do not specify without loss of information in final results.

5 Averaged transition probabilities.

Averaged values of objects that are local in $\vec{z}(t)$ are obtained by simple integration with weighting function $P(\vec{z}, t)$ from (12). Functionals in consideration $W_{nm}$ are nonlocal in $\vec{z}(t)$, but they have a special form which allows to reduce the averaging procedure to solving some parabolic differential equation. Namely, one can write down the following representation for averaged transition probabilities at arbitrary time $t$

$$\langle |c_{nm}|^2 \rangle = \int d\vec{\zeta} R(\vec{\zeta}, t_>) \int d\vec{z} H_{nm}(\vec{z}) Q_{nm}(\vec{z}, t),$$

(13)

Functions $H_{nm}(\vec{z})$ are obtained from specific form of generating function for coefficients $c_{nm}$. Functions $Q_{nm}(\vec{z}, t)$ include $\vec{\zeta}$ as a parameter and are the solutions of the following problem

$$\frac{\partial Q_{nm}}{\partial t} (\vec{z}, t) = (\hat{L} - V_{nm}) Q_{nm}(\vec{z}, t),$$

$$Q_{nm}(\vec{z}, t) \to \delta(\vec{z} - \vec{\zeta}), \quad Q_{nm}(\vec{z}, t) \to 0, \quad ||\vec{z}|| \to \infty,$$

(14)

where $V_{nm} = p_{nm}z_3$, $p_{00} = p_{01} = 1$, $p_{10} = p_{11} = 3$. Operator $\hat{L}$ is defined in (12). Formulas (13)-(14) are exact and give probabilities $W_{nm}$ at the limit $t \to +\infty$.

Let’s suppose that random forces $f_1$ and $f_2$ act with a constant power after switching on and then are switched off at the moment $t_c$. Let’s also assume that $t_c$ is large enough to allow the replacement of $Q_{nm}(\vec{z}, t)$ by the stationary limit $Q_{nm}^*(\vec{z}) \equiv \lim_{t \to +\infty} Q_{nm}(\vec{z}, t)$. In such case one can obtain the following expression for averaged probabilities $W_{nm}$

$$W_{nm} = \Omega_{in}^{P_{nm}} \int d\xi_1 d\xi_2 d\xi_3 \tilde{Q}_{nm}^*(\xi_1, \xi_2, \xi_3) \bar{H}_{nm}(\xi_1, \xi_2, \xi_3),$$

(15)

where function $\tilde{Q}_{nm}^*(z_1, z_2, z_3)$ satisfies the stationary problem

$$-z_2 \frac{\partial}{\partial z_1} + \Omega_{out}^2 z_1 \frac{\partial}{\partial z_1} + (z_2^2 + \Omega_{out}^2) \frac{\partial}{\partial z_2} + (\epsilon_2 + \epsilon_1 z_1^2) \frac{\partial^2}{\partial z_2^2} + \epsilon_1 \frac{\partial^2}{\partial z_3^2} +$$

$$+ \epsilon_2 \frac{\partial^2}{\partial z_3^2} + \epsilon_3 \frac{\partial^2}{\partial z_4^2} +$$
\[
+2\varepsilon_1 z_1 \frac{\partial^2}{\partial z_2 \partial z_3} Q_{nm}(z) + (2 - p_{nm}) z_3 Q_{nm}(z) = 0. \tag{16}
\]

and there are representations for the first several functions \(\bar{H}_{nm}\):

\[
\bar{H}_{00}(\xi_1, \xi_2, \xi_3) = \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{|\xi_0(t_1)|\sqrt{\Sigma(\xi_3)}} \exp \left\{ -\frac{\Omega_{out}\Omega_{in}^2}{\Sigma(\xi_3)} [\xi_3(\xi_1 + \mu_1) - \xi_2 - \mu_2]^2 \right\},
\]

\[
\bar{H}_{01}(\xi_1, \xi_2, \xi_3) = \frac{2\Omega_{out}\Omega_{in}^2}{\Sigma(\xi_3)} [\xi_2 - \xi_1 \xi_3 - \xi_3 \mu_1 + \mu_2] \bar{H}_{00}(\xi_1, \xi_2, \xi_3),
\]

\[
\mu_1 = -d_5 + \sqrt{\frac{2\nu}{\Omega_{in}}} (d_1 \cos \beta + d_2 \sin \beta)
\]

\[
\mu_2 = -d_6 + \sqrt{\frac{2\nu}{\Omega_{in}}} (d_3 \cos \beta + d_4 \sin \beta)
\]

\[
\Sigma(\xi_3) = \frac{\Omega_{in}\Omega_{out}}{(1 - \rho)} \left\{ (d_1^2 + d_2^2)(1 + \rho) - 2\sqrt{\rho} \left[ (d_1^2 - d_2^2) \cos \delta + 2d_1d_2 \sin \delta \right] \right\} \xi_3^2 + 2 \left[ -(d_2d_4 + d_1d_3)(1 + \rho) + 2\sqrt{\rho} \left[ (d_1d_4 + d_2d_3) \sin \delta + (d_1d_3 - d_2d_4) \cos \delta \right] \right] \xi_3 + \left[ (d_3^2 + d_4^2)(1 + \rho) + 2\sqrt{\rho} \left[ (d_3^2 - d_4^2) \cos \delta - 2d_3d_4 \sin \delta \right] \right], \tag{17}
\]

where \(\delta = \delta_1 + \delta_2\). For regular functions \(\xi_0(t)\) and \(\eta_0(t)\), defined by equations

\[
\dot{\xi}_0 + \Omega_0^2(t)\xi_0 = 0, \quad \xi_0(t) \rightarrow_{-\infty} e^{i\Omega_{out}t}, \quad \xi_0(t) = \xi_{01}(t) + i\xi_{02}(t).
\]

\[
\dot{\eta}_0 + \Omega_0^2(t)\eta_0 = F_0(t), \quad \eta_0(-\infty) = \dot{\eta}_0(-\infty) = 0
\]

we have used the following representations

\[
\xi_0(t) \rightarrow_{+\infty} C_1 e^{i\Omega_{out} t} + C_2 e^{-i\Omega_{out} t}, \quad C_1 = |C_1| e^{i\delta_1}, \quad C_2 = |C_2| e^{i\delta_2},
\]

\[
\eta_0(t) = \frac{1}{\sqrt{2\Omega_{in}}} (\xi_0 d^* + \xi_0^* d), \quad d(t) = \frac{i}{\sqrt{2\Omega_{in}}} \int_{-\infty}^{t} \xi_0(t') F_0(t') dt',
\]

Also we have used designations

\[
\rho = \left| \frac{C_2}{C_1} \right|^2, \quad d = \lim_{t \rightarrow +\infty} d(t) = \sqrt{\rho} e^{i\beta}.
\]
\[ d_1 = \xi_01(t_c), \ d_2 = \xi_02(t_c), \ d_3 = \dot{\xi}_01(t_c), \ d_4 = \dot{\xi}_02(t_c), \]
\[ d_5 = \eta_0(t_c), \ d_6 = \dot{\eta}_0(t_c). \]

Obtained formula (15) gives an approximate value for \( W_{nm} \). It is necessary to emphasise that it is true only for finite \( \epsilon_1, \epsilon_2 \), while the reducing \( \epsilon_1 \) or \( \epsilon_2 \) leads to increasing the time \( (t_c - t_>) \) necessary for setting the stationary distribution.

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