Generalized hybrid momentum maps and reduction by symmetries of
forced mechanical systems with inelastic collisions

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Abstract
This paper discusses reduction by symmetries for autonomous and non-autonomous forced mechanical systems with inelastic collisions. In particular, we introduce the notion of generalized hybrid momentum map and hybrid constants of the motion to give general conditions on whether it is possible to perform symmetry reduction for Hamiltonian and Lagrangian systems subject to non-conservative external forces and non-elastic impacts, as well as its extension to time-dependent mechanical systems subject to time-dependent external forces and time-dependent inelastic collisions. We illustrate the applicability of the method with examples and numerical simulations.

Key words: Hybrid systems, Lagrangian and Hamiltonian systems, reduction by symmetries, time-dependent mechanical systems.

1 Introduction

Hybrid systems are dynamical systems with continuous-time and discrete-time components on its dynamics. This class of dynamical systems are capable of modelling several physical systems, such as multiple UAV (unmanned aerial vehicles) systems [36] and legged robots [45], among many others [21], [44]. Simple hybrid systems are a class of hybrid systems introduced in [30], denoted as such because of their simple structure. A simple hybrid system is characterized by a tuple $\mathcal{H} = (D, X, S, \Delta)$, where $D$ is a smooth manifold, $X$ is a smooth vector field on $D$, $S$ is an embedded submanifold of $D$ with co-dimension 1, and $\Delta : S \to D$ is a smooth embedding. This type of hybrid system has been mainly employed for the understanding of locomotion gaits in bipeds and insects [3], [26], [45]. In the situation where the vector field $X$ is associated with a mechanical system (Lagrangian or Hamiltonian), alternative approaches for mechanical systems with nonholonomic and unilateral constraints have been considered in [9], [13], [14], [27], [29].

The reduction of mechanical systems with symmetries plays a fundamental role in understanding the many important and interesting properties of these systems. Given a Hamiltonian on a symplectic manifold on which a Lie group acts symplectically, Marsden-Weinstein Reduction Theorem [40] states that under certain conditions one can reduce the phase space to another symplectic manifold by “dividing out” by the symmetries. In addition, the trajectories of the Hamiltonian on the original phase space determine the corresponding trajectories on the reduced space.

The key idea is that when a dynamical system exhibits a symmetry, it produces a conserved quantity for the system, and one can reduce the degrees of freedom in the dynamics by making use of these conserved quantity. One of the classical reduction by symmetry procedures in mechanics is the Routh reduction method [22], [1]. During the last few years there has been a growing interest in Routh reduction, mainly motivated by physical applications –see [19], [25],

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[33], [34] and references therein. Furthermore, Routh reduction for hybrid systems has been studied and applied in the field of bipedal locomotion [3]. The reduced simple hybrid system is called simple hybrid Routhian system [5]. A hybrid scheme for Routh reduction for simple hybrid Lagrangian systems with cyclic variables can be found in [5] and [11]. Symplectic reduction for hybrid Hamiltonian systems has been studied in [4], and extended to Poisson reduction in [18], to time-dependent systems in [12] and for forced Lagrangian systems in [41]. These approaches considered elastic collisions only (i.e., the momentum map is preserved in the impact). However, to the best of our knowledge, the hybrid analogue for symmetry reduction in mechanical systems subject to external forces and inelastic collisions has not been considered in the literature. This is important for real applications since external forces allow to describe friction, dissipation, as well as control forces or certain non-holonomic constraints. Moreover, in practice, collisions are usually inelastic, unlike the collisions considered in the previously mentioned work.

This paper considers symmetry reduction of simple hybrid mechanical systems, both time-independent and time-dependent, via Routh reduction. As it was studied in [15] (see also [37,16]), the reduction of a forced continuous system requires considering a group of symmetries which leaves invariant both the Lagrangian (or Hamiltonian) function and the external force. Regarding the reduction of simple hybrid systems, Ames and Sastry [5,4] had considered the so-called hybrid momentum maps, i.e., a momentum map which is preserved both by the continuous and the discrete dynamics. Here, we consider a more general class of hybrid systems for which the impact is inelastic, i.e., the impact map can change the value of the momentum map (see Examples 25 and 32). This will lead to the existence of a reduced space for each interval of time between to subsequent impacts. Additionally, we considered hybrid systems which are forced, which requires characterizing the impact map can change the value of the momentum map (see Examples 25 and 32). This will lead to the existence of a reduced space for each interval of time between to subsequent impacts. Additionally, we considered hybrid systems which are forced, which requires characterizing the impact. However, to the best of our knowledge, the hybrid analogue for symmetry reduction in mechanical systems subject to external forces and inelastic collisions has not been considered in the literature. This is important for real applications since external forces allow to describe friction, dissipation, as well as control forces or certain non-holonomic constraints. Moreover, in practice, collisions are usually inelastic, unlike the collisions considered in the previously mentioned work.

The main contributions of the present paper are the following:

(i) The relation between the hybrid flows for forced hybrid Hamiltonian and Lagrangian systems. This is shown in Propositions 1 and 11
(ii) The reduction by symmetries in the Hamiltonian framework (Theorem 20) and its Lagrangian counterpart.
(iii) In particular, to show Theorem 20 we introduced the concept of generalized hybrid momentum maps to translate the dynamics from one switching surface into another. In addition, Proposition 19 shows the relation between the isotropy subgroups before and after an inelastic collision.
(iv) In Examples 25 and 32, we show that generalized momentum maps are not hybrid momentum maps (in the sense of [5]), allowing the latter reduction in the elastic situation solely.
(v) Propositions 26 and 28 extend Propositions 1 and 11 to non-autonomous systems. In particular, to hybrid systems with switching surfaces and impact maps depending explicitly on time. This is done by using the geometric framework of cosymplectic manifolds for time-dependent systems.
(vi) Propositions 26 and 28 extend Propositions 1 and 11 to non-autonomous systems. In particular, to hybrid systems with switching surfaces and impact maps depending explicitly on time. This is done by using the geometric framework of cosymplectic manifolds for time-dependent systems.
(vii) Finally, Theorem 29 shows the reduction by symmetries for time-dependent hybrid forced Hamiltonian systems, and we further develop the reduction procedure for time-dependent hybrid forced Lagrangian systems.

This paper is structured as follows. Section 2 provides preliminary knowledge on mechanical systems subject to external forces and Routh reduction. Section 3 introduces the hybrid forced mechanical systems from a Hamiltonian and Lagrangian description, respectively, and how their hybrid flows are related. In addition, we introduce the notion of hybrid constants of motion which is further employed in Section 4 where we show the main results of the paper. In particular, we introduce generalized hybrid momentum maps to show in Theorem 20 the reduction by symmetries of forced mechanical systems with inelastic collisions (i.e., impacts that can modify the value of the momentum map) in both the Hamiltonian and the Lagrangian formalisms. In addition, in Subsection 4.1 we take special attention to the case of cyclic coordinates. Finally, Section 5 extends the reduction procedure to time-dependent systems by using a cosymplectic framework for non-autonomous systems. We employ the reduction procedures in some examples, in particular, a rolling disk with dissipation and hitting a fixed and moving wall in Examples 25 and 32, respectively. The types of symmetries of a forced Hamiltonian or Lagrangian system and their associated constants of the motion are recalled in Appendix A.

2 Routh reduction for mechanical systems subject to external forces

We begin by introducing some definitions about mechanical systems subject to external forces and Routh reduction.

Throughout this paper, let Q be an n-dimensional differentiable manifold, which represents the configuration space of a dynamical system. Let $T_qQ$ and $T^*_qQ = (T_qQ)^*$ denote the tangent and cotangent spaces of Q at the point $q \in Q$. Let $\tau_Q : TQ \rightarrow Q$ and $\pi_Q : T^*Q \rightarrow Q$ be its tangent bundle and its cotangent bundle, respectively; namely, $TQ = \bigcup_{q \in Q} T_qQ$ and $T^*Q = \bigcup_{q \in Q} T^*_qQ$, with the canonical projections $\tau_Q : (q^i, \dot{q}^i) \mapsto (q^i)$ and $\pi_Q : (q^i, p_i) \mapsto (q^i)$. Here $(q^i)$ ($1 \leq i \leq n$) are local coordinates in Q, and $(q^i, \dot{q}^i)$ and $(q^i, p_i)$ are their induced coordinates in $TQ$ and $T^*Q$, respectively.
Let $M$ and $N$ be smooth manifolds. For each $p$-form $\alpha$ and each vector field $X$ on $M$, $\iota_X \alpha$ denotes the interior product of $\alpha$ by $X$, and $\mathcal{L}_X \alpha$ denotes the Lie derivative of $\alpha$ with respect to $X$. For a smooth map $F : M \to N$, its tangent map $TQF : TM \to N$ will indistinctly be called its pushforward and denoted by $F_*$. Unless otherwise stated, sum over paired covariant and contravariant indices will be understood.

2.1 Geometric formulation of Lagrangian and Hamiltonian systems

The dynamics of a mechanical system can be determined by the Euler-Lagrange equations associated with a Lagrangian function $L : TQ \to \mathbb{R}$. A mechanical Lagrangian is given by
\[
L(q, \dot{q}) = K(q, \dot{q}) - V(q),
\]
where $K : TQ \to \mathbb{R}$ is the kinetic energy and $V : Q \to \mathbb{R}$ is the potential energy. The kinetic energy is given by
\[
K(q, \dot{q}) = \frac{1}{2} \sum_{i,j} (\dot{q})^i \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \dot{q}^j.
\]
where the symbol $\mathcal{M}_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$.

A Lagrangian $L$ is said to be regular if $\det \mathcal{M} \neq 0$, where
\[
\mathcal{M} = (\mathcal{M}_{ij}) := \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \text{ for all } i, j \text{ with } 1 \leq i, j \leq n.
\]
The equations describing the dynamics of the system are given by the Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i},
\]
with $i = 1, \ldots, n$; a system of $n$ second-order ordinary differential equations. If $L$ is regular, the Euler-Lagrange equations induce a vector field $X_L : TQ \to T(Q(TQ))$ describing the dynamics of the Lagrangian system, given by
\[
X_L(q^i, \dot{q}^i) = \left( q^i, \dot{q}^i; q^i, \mathcal{M}_{ij} \left( \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \dot{q}^j \right) \right).
\]

We denote by $\mathbb{FL} : TQ \to T^*Q$ the Legendre transformation associated with $L$; that is, $\mathbb{FL} : TQ \to T^*Q$, $(q, \dot{q}) \mapsto (q, p := L/\dot{q})$. The map $\mathbb{FL} : TQ \to T^*Q$ relates velocities and momenta. In fact, the Legendre transformation connects Lagrangian and Hamiltonian formulations of mechanics. We say that the Lagrangian is hyperregular if $\mathbb{FL}$ is a diffeomorphism between $TQ$ and $T^*Q$ (this is always the case for mechanical Lagrangians). If $L$ is hyperregular, one can work out the velocities $\dot{q} = \dot{q}(q, p)$ in terms of $(q, p)$ and define the Hamiltonian function (the “total energy”) $H : T^*Q \to \mathbb{R}$ as $H(q, p) = p^i \dot{q}^i(q, p) - L(q, \dot{q}, p)$, where we have used the inverse of the Legendre transformation to express $\dot{q} = \dot{q}(q, p)$. The evolution vector field corresponding to the Hamiltonian $H$, denoted by $X_H$, is defined by
\[
X_H = \frac{\partial H}{\partial p^i} \partial q^i - \frac{\partial H}{\partial q^i} \partial p^i,
\]
and its integral curves are solutions of Hamilton’s equations $\dot{q}^i = \frac{\partial H}{\partial p^i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$. A Hamiltonian is said to be mechanical (resp. kinetic) if its associated Lagrangian is mechanical (resp. kinetic).

2.2 Geometric formulation of forced mechanical systems

An external force is geometrically interpreted as a semibasic 1-form on $T^*Q$ (see [15] and [20] for instance). A Hamiltonian system with external forces, so called forced Hamiltonian system, is given by the pair $(H, F)$ determined by a Hamiltonian function $H : T^*Q \to \mathbb{R}$ and a semibasic 1-form $F$ on $T^*Q$ locally described as $F = F_1(q,p)dq^i$. Let $\omega_Q = -d\theta_Q$ be the canonical symplectic form of $T^*Q$, where locally $\theta_Q = p^i dq^i$ and $\omega_Q = dq^i \wedge dp_i$. The dynamics of the forced Hamiltonian system is given by the vector field $X_{H,F}$, defined by
\[
\iota_{X_{H,F}} \omega_Q = dH + F.
\]
If $X_H$ is the Hamiltonian vector field for $H$, that is, $\iota_{X_H} \omega_Q = dH$ and $Z_F$ is the vector field defined by $\iota_{Z_F} \omega_Q = F$, then we have $X_{H,F} = X_H + Z_F$. Locally, these vector fields can be written as
\[
X_H = \frac{\partial H}{\partial p^i} \partial q^i - \frac{\partial H}{\partial q^i} \partial p^i,
\]
\[
X_{H,F} = \frac{\partial H}{\partial p^i} \partial q^i - \left( \frac{\partial H}{\partial q^i} + F_i \right) \partial p^i.
\]

The Poincaré-Cartan 1-form on $TQ$ associated with the Lagrangian function $L : TQ \to \mathbb{R}$ is defined by $\theta_L = S^\ast (dL)$ where $S^\ast$ is the adjoint operator of the vertical endomorphism on $TQ$, which is locally defined by $S = dq^i \otimes \frac{d}{dq^i}$, where the symbol $\otimes$ denotes a tensorial product. The Poincaré-Cartan 2-form is $\omega_L = -d\theta_L$, so locally $\omega_L = dq^i \wedge d\left( \frac{d}{dq^i} \right)$. One can easily verify that $\omega_L$ is symplectic if and only if $L$ is regular (see [1]). The energy of the system is given by $E_L = \frac{\partial L}{\partial q^i} \frac{d}{dq^i} - L$.

In the tangent bundle, one external force is also represented by a semibasic 1-form $F^L$ on $TQ$, locally given by $F^L = F_1^L(q,p)dq^i$. A forced Lagrangian system is determined by the pair $(L, F^L)$ and its dynamics is given by
\[
\iota_{X_{L,F^L}} \omega_L = dE_L + F^L.
\]

The forced Euler-Lagrange vector field $X_{L,F^L}$ is a SODE and its integral curves satisfy the forced Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -F_1^L(q, \dot{q}), \text{ } i = 1, \ldots, n.
\]
If $L$ is regular, the forced Euler-Lagrange vector field is given by
\[
X_{L,F^L}(q^i, \dot{q}^i) = \left( q^i, \dot{q}^i; \mathcal{M}_{ij} \left( -F_1^L + \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \dot{q}^j \right) \right).
\]
Let \((L, F^L)\) be a forced hyperregular Lagrangian system and let \((H, F)\) be its associated forced Hamiltonian system, i.e., \(E_L = H \circ F^L\). In the un-forced case we can relate \(X_{L,F^L}\) and \(X_{H,F}\) as follows:

**Proposition 1** The tangent map of \(F^L\) maps \(X_{L,F^L}\) onto \(X_{H,F}\). In other words \((T F^L) X_{L,F^L} = X_{H,F}\), where \((T F^L) : T(TQ) \to T(T^*Q)\). In particular, the flow of \(X_{L,F^L}\) is mapped onto the flow of \(X_{H,F}\).

**Proof.** The evolution vector field \(X_{H,F}\) is characterized by 
\[
\iota_{X_{H,F}} \omega_Q = dH + F. \quad \text{Observe that}
\]
\[
(F^L)^* (\iota_{X_{H,F}} \omega_Q) = (F^L)^* (dH + F) = (F^L)^* dH + (F^L)^* F = d((F^L)^* H) + (F^L)^* F = dE_L + F^L = \iota_{X_{L,F^L}} \omega_{L,F}.
\]

This means that
\[
\iota_{X_{L,F^L}} \omega_{L,F} = (F^L)^* (\iota_{X_{H,F}} \omega_Q) = (F^L)^* \iota_{X_{H,F}} (F^L)^* \omega_Q = \iota_{(F^L)^* ((F^L)^* \iota_{X_{H,F}} \omega_Q)} = \iota_{(F^L)^* \omega_{L,F}}.
\]

This last implies \(X_{L,F^L} = (F^L)^* X_{H,F}\), that is, \((F^L)* X_{L,F^L} = X_{H,F}\). \(\square\)

### 2.3 Routh reduction for forced mechanical systems

There exists a large class of systems for which the Lagrangian (or Hamiltonian) does not depend on some of the generalized coordinates. Such coordinates are called cyclic and the corresponding generalized momenta are easily checked to be constants of the motion - see [1], [22]. Routh’s reduction is a classical reduction technique which takes advantage of the conservation laws to define a reduced Lagrangian function, the so-called Routhian function, such that, when the conservation of momenta is taken into account, the solutions of the Euler-Lagrange equations for the Routhian are in correspondence with the solutions of the Euler-Lagrange equations for the original Lagrangian.

Routh’s reduction can be extended to forced systems as follows [22]. Suppose that the configuration space is of the form \(Q = Q_1 \times Q_2\), where we denote an element \(q^i \in Q\) by \(q^i = (q^i, q^j)\), with \(q^i \in Q_1\) and \(q^j \in Q_2\), for \(j = 2, \ldots, n\).

Let \(L(q^1, q^1, q^j, \dot{q}^j)\) be a hyperregular Lagrangian with cyclic coordinate \(q^1\), that is, \(\frac{\partial L}{\partial \dot{q}^1} = 0\) and let \(F_i\) be a non-conservative force such that \(F_i\) is independent of \(q^i\) for all \(i = 1, \ldots, n\) and \(F_1(q^1, \ldots, q^n) = 0\). Fundamental to reduction is the notion of a momentum map \(J_F : TQ \to \mathfrak{g}^*\), which makes explicit the conserved quantities in the system. Here \(\mathfrak{g}\) is the Lie algebra associated with the Lie group of symmetries \(G\), and \(\mathfrak{g}^*\) denotes its dual as vector space. In the framework we are considering here, \(J_L(q^1, \dot{q}^1, q^j, \dot{q}^j) = \frac{\partial L}{\partial \dot{q}^j}\).

Fix a value of the momentum \(\mu = \frac{\partial L}{\partial \dot{q}^1}\). Since \(L\) is hyperregular, the last equation admits an inverse, and allows us to write \(\dot{q}^1 = f(q^1, \ldots, q^n, \dot{q}^2, \ldots, \dot{q}^n, \mu)\). Consider the function
\[
R^\mu_F(q^j, \dot{q}^j) = (L - \dot{q}^1 \mu)\bigg|_{\mu},
\]
where the notation \(|_{\mu}\) means that we have used the relation \(\mu = \frac{\partial L}{\partial \dot{q}^1}\) to replace all the appearances of \(\dot{q}^1\) in terms of \((q^j, \dot{q}^j)\) and the parameter \(\mu\). The function \(R^\mu_F\) is called Routhian function. Similarly we define the reduced force \(F_\mu\) as \(F_\mu := F|_{\mu}\).

**Remark 2** The previous definition of Routhian function is the classical one considered in the literature as in [22] and [43]. A more general notion of Routhian can be given in terms of the so-called mechanical connection (see [39]), when \(L\) is kinetic, or even more general by using any connection on the principal bundle \(Q \to Q/G\). Indeed, let \(A : TQ \to \mathfrak{g}\) be a connection one form, and let us denote by \(A_q(\cdot) = (\mu, A(\cdot))\) the 1-form on \(Q\) obtained by contraction with \(\mu \in \mathfrak{g}^*\). Then one can define the Routhian function as \(R^\mu = L - A_q : TQ \to \mathbb{R}\). Similarly one can define the reduced force \(F_\mu\) by contraction with \(\mu \in \mathfrak{g}^*\). A review of the unforced Routh reduction in these more general settings can be found in [39] (see also [33]).

**Remark 3** Note that it may happen, even in the hyperregular case, that there is no Routhian function as one function generating the reduced dynamics but there is a family of functions much as it happens with the Hamiltonian as one try to employ the Legendre transformation for singular Lagrangians. An example of such a case can be found in [25] (see Example 3.4). Nevertheless in this paper we shall restrict to the case of mechanical Lagrangians (i.e., kinetic minus potential energy), where the last situation cannot happen.

If we regard the pair \((R^\mu_F, F_\mu)\) as a new forced Lagrangian system in the variables \((q^j, \dot{q}^j)\), then the solutions of the forced Euler-Lagrange equations for \((R^\mu_F, F_\mu)\) are in correspondence with those for \((L, F)\) when one takes into account the relation \(\mu = \frac{\partial L}{\partial \dot{q}^1}\). More precisely:

(a) Any solution of the forced Euler-Lagrange equations for \((L, F)\) with momentum \(\mu = \frac{\partial L}{\partial \dot{q}^1}\) projects onto a solution of the forced Euler-Lagrange equations for \((R^\mu_F, F_\mu)\), given by \(\frac{\partial L}{\partial \dot{q}^j} \frac{d}{d\tau} = \frac{\partial R^\mu_F}{\partial \dot{q}^j} = -(F_\mu)\), for \(j = 2, \ldots, n\). These equations will be referred as forced
Routh equations and they induce a vector field $X^F_H : TQ_2 \rightarrow T(TQ_2)$ describing the dynamics of the reduced system, called Routhian vector field.

(b) Conversely, any solution of forced Routh equations for $(R^F_H, F, \mu)$ can be lifted to a solution of the forced Euler-Lagrange equations for $(L, F)$ with $\mu = \frac{\partial L}{\partial q^\top}$.

3 Simple hybrid forced Hamiltonian systems and hybrid constants of motion

Roughly speaking, the term hybrid system refers to a dynamical system which exhibits both continuous and discrete time behaviours. In the literature, one finds slightly different definitions of hybrid system depending on the specific class of applications of interest. For simplicity, and following [30] and [10], we will restrict ourselves to the so-called simple hybrid mechanical systems in Hamiltonian form.

Simple hybrid systems [30] are characterized by the 4-tuple $\mathcal{H} = (D, X, S, \Delta)$, where $D$ is a smooth manifold called the domain, $X$ is a smooth vector field on $D$, $S$ is an embedded submanifold of $D$ with co-dimension 1 called the switching surface, and $\Delta : S \rightarrow D$ is a smooth embedding called the impact map. $S$ and $\Delta$ are also referred to as the guard and the reset map, respectively. The triple $(D, S, \Delta)$ is called a hybrid manifold.

The dynamics associated with a simple hybrid system is described by an autonomous system with impulse effects as in [45]. We denote by $\Sigma_{\mathcal{H}}$ the simple hybrid dynamical system generated by $\mathcal{H}$, that is,

$$\Sigma_{\mathcal{H}} : \begin{cases} \dot{v}(t) = X(v(t)), & \text{if } v^-(t) \notin S, \\ \dot{v}^+(t) = \Delta(v^-(t)), & \text{if } v^-(t) \in S, \end{cases}$$

where $v : I \subset \mathbb{R} \rightarrow D$, and $v^-, v^+$ denote the states immediately before and after the times when integral curves of $X$ intersect $S$ (i.e., pre and post impact of the solution $v(t)$ with $S$), namely

$$\lim_{\tau \rightarrow t^+-} x(\tau) := \lim_{\tau \rightarrow t^-} x(\tau).$$

Remark 4 A solution of a simple hybrid system may experience a Zeno state if infinitely many impacts occur in a finite amount of time. To exclude these types of situations, we require the set of impact times to be closed and discrete, as in [45], so we will assume implicitly throughout the remainder of the paper that $\Sigma(S) \cap S = \emptyset$ and that the set of impact times is closed and discrete.

Definition 5 A simple hybrid system $\mathcal{H} = (D, X, S, \Delta)$ is said to be a simple hybrid forced Hamiltonian system if it is determined by $\mathcal{H}_F := (T^*Q, X_{H,F}, S_H, \Delta_H)$, where $X_{H,F} : T^*Q \rightarrow T(T^*Q)$ is the Hamiltonian forced vector field associated with the forced Hamiltonian system $(H, F)$ (see Subsection 2.1), $S_H$ is the switching surface, a submanifold of $T^*Q$ with co-dimension one, and $\Delta_H : S_H \rightarrow T^*Q$ is the impact map, a smooth embedding.

The simple hybrid forced dynamical system generated by $\mathcal{H}_F$ is given by

$$\Sigma_{\mathcal{H}_F} : \begin{cases} \dot{v}(t) = X_{H,F}(v(t)), & \text{if } v^-(t) \notin S_H, \\ \dot{v}^+(t) = \Delta_H(v^-(t)), & \text{if } v^-(t) \in S_H, \end{cases}$$

where $v(t) = (q(t), p(t)) \in T^*Q$.

Alternatively, $\Delta_H$ could be described by an impulsive external force appearing only on the instant on the impact (see [29, 27, 28, 13] and references therein).

Remark 6 Note that it is possible to consider more general definitions of simple hybrid forced Hamiltonian system. Indeed, it suffices to take $D$ a manifold with a certain geometric structure (e.g., a symplectic, Poisson, Jacobi or contact structure), and then take $X_H$ the Hamiltonian vector field that this structure defines for a Hamiltonian function $H$ on $D$, and take $F$ a semibasic 1-form on $D$.

Example 7 Suppose that $(Q, g)$ is a Riemannian manifold. Then, the switching map could be the tangent sphere bundle $S = \{(q, \dot{q}) \in TQ \mid g(q, q) = 1\}$ (see [31] for instance).

Definition 8 A simple hybrid forced Lagrangian system is a simple hybrid system determined by $\mathcal{L}_F := (TQ, X_{L,F,e}, S_L, \Delta_L)$, where $X_{L,F,e} : TQ \rightarrow T(TQ)$ is the forced Lagrangian vector field associated with the forced Lagrangian system $(L, F, \mu)$, $S_L$ the switching surface, a submanifold of $TQ$ with co-dimension one, and $\Delta_L : S_L \rightarrow TQ$ the impact map as defined before.

Definition 9 A hybrid flow for $\mathcal{H}_F$ is a tuple $\chi_{\mathcal{H}_F} = (\Lambda, \beta, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite (or infinite) indexing set,
- $\beta = \{I_i\}_{i \in \Lambda}$ a set of intervals, called hybrid intervals, where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i + 1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, $N$ finite, with $\tau_1, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$,
- $\mathcal{C} = \{c_i\}_{i \in \Lambda}$ is a collection of solutions for the vector field $X_{H,F}$ specifying the continuous-time dynamics, i.e., $c_i = X_{H,F}(c_i(t))$ for all $i \in \Lambda$, and such that for each $i, i + 1 \in \Lambda$, (i) $c_i(\tau_{i+1}) \in S_H$, and (ii) $\Delta_H(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$.

Similarly, it is possible to define a hybrid flow $\chi_{\mathcal{L}_F}$ for a simple hybrid forced Lagrangian system $\mathcal{L}_F$. The relation between both hybrid flows is given by the following result, based on the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case achieved via the fiber derivative $\mathcal{F}H$ that we will define below.
Definition 10 The fiber derivative of a Hamiltonian is the map $\mathbb{F}H: T^*Q \to TQ$ defined by

$$\alpha_q \cdot \mathbb{F}H(\beta_q) = \left. \frac{d}{dt} \right|_{t=0} H(\beta + t\alpha),$$

$\alpha_q, \beta_q \in T^*_q Q$ which in local coordinates is $\mathbb{F}H(q,p) = (q, \dot{q} = (q, \frac{\partial H}{\partial q}(q,p)))$. We say that $H$ is regular if $\mathbb{F}H$ is a local diffeomorphism and that $H$ is hyperregular if $\mathbb{F}H$ is a (global) diffeomorphism. Equivalently, $H$ is regular (resp. hyperregular) if $\mathbb{F}H$ is a local (resp. global) isomorphism of fibre bundles.

Proposition 11 If $\chi^{\mathbb{F}F}(t) = (\Lambda, \varepsilon, \varepsilon)$ is a hybrid flow for $H_F$, $S_L = \mathbb{F}H(S_H)$, and $\Delta_L$ is defined in such a way that $\mathbb{F}H \circ \Delta H = \Delta_L \circ \mathbb{F}H |_{S_L}$, then $\chi^{\mathbb{F}F}(t) = (\Lambda, \varepsilon, \varepsilon(H)(\varepsilon))$ with $(\mathbb{F}H)(\varepsilon) = \{((\mathbb{F}H)(\varepsilon))_i\}_{i \in \Lambda}$.

Proof. If $c_i(t)$ is an integral curve of $X_{H,F}$, $c_i(t) = (\mathbb{F}H \circ c_i(t)$ is an integral curve for $X_{L,F}$. In this way, if we consider a solution $c_0(t)$ with initial value $c_0 = (q_0, p_0)$ defined on $[\tau_0, \tau_1]$, then $c_0(t)$ is a solution with initial value $c_0 = (q_0, q_0)$ defined on $[\tau_0, \tau_1]$. Likewise for a solution $c_1(t)$ defined on $[\tau_1, \tau_2]$, we get a corresponding solution $c_1(t)$ defined on the same hybrid interval $[\tau_1, \tau_2]$. Proceeding inductively, one finds $c_i(t)$ defined on $[\tau_i, \tau_{i+1}]$. It only remains to check that $c_1(t)$ satisfies $c_i(\tau_{i+1}) = c_{i+1}(\tau_{i+1})$, but using the properties of $\mathbb{F}H$,

(i) $c_1(\tau_{i+1}) = (\mathbb{F}H \circ c_0)(\tau_{i+1}) = \mathbb{F}H(c_1(\tau_{i+1}))$ and given that $c_0(\tau_{i+1}) \in S_H$ then $c_1(\tau_{i+1}) \in S_L$.

(ii) $\Delta_L(\tau_{i+1}) = \Delta_L \circ \mathbb{F}H \circ c_1(\tau_{i+1}) = \mathbb{F}H \circ \Delta_L \circ c_1(\tau_{i+1}) = \mathbb{F}H(c_{i+1}(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$.

Definition 12 Let $(H, F)$ be a forced Hamiltonian system on $T^*Q$, with Hamiltonian forced vector field $X_{H,F}$. Let $(T^*Q, X_{H,F}, S_H, \Delta_H)$ be a hybrid forced Hamiltonian system. Then, a function $f$ on $T^*Q$ is called a hybrid constant of the motion if

(i) it is a constant of the motion for $(H, F)$, i.e., $X_{H,F}(f) = 0$.

(ii) it is left invariant by the impact map, namely, $f \circ \Delta_H = f \circ i$, where $i : S_H \to T^*Q$ denotes the canonical inclusion.

The definition of hybrid constant of the motion for a simple hybrid forced Lagrangian system is completely analogous.

4 Generalized hybrid momentum maps and symplectic reduction of simple hybrid forced mechanical systems

Definition 13 Let $G$ be a Lie group and $Q$ a smooth manifold. A left-action of $G$ on $Q$ is a smooth map $\psi : G \times Q \to Q$ such that $\psi(e, q) = g$ and $\psi(h, \psi(g, q)) = \psi(hg, q)$ for all $g, h \in G$ and $q \in Q$, where $e$ is the identity of the group $G$ and the map $\psi_g : Q \to Q$ given by $\psi_g(q) = \psi(g, q)$ is a diffeomorphism for all $g \in G$.

Definition 14 Let $\psi : G \times Q \to Q$ be a Lie group action. $\psi$ is said to be a free action if it has no fixed points, that is, $\psi_g(q) = q$ implies $g = e$. The Lie group action $\psi$ is said to be a proper action if the map $\psi : G \times Q \to Q \times Q$ given by $\psi(g, q) = (q, \psi(q, g))$, is proper, that is, if $K \subset Q \times Q$ is compact, then $\psi^{-1}(K)$ is compact.

We recall that for $q \in Q$, the isotropy (or stabilizer or symmetry) group of $\psi$ at $q$ is given by $G_q := \{g \in G | \psi_q(g) = q\} \subset G$. Since $\psi_q(g)$ is a continuous $G_q = \psi^{-1}_q(q)$ is a closed subgroup and hence a Lie subgroup of $G$.

Consider a Lie group action $\psi : G \times Q \to Q$ of some Lie group $G$ on the manifold $Q$ and assume $\psi$ is a free and proper action. These conditions ensure that the quotient of a smooth manifold by the action is a smooth manifold [1, 35]. Let $g$ be the Lie algebra of $G$ and $g^*$ its dual as vector space. There is a natural lift $\psi_T^Q$ of the action $\psi$ to $T^*Q$, the cotangent lift, defined by $(\psi_T^Q(q, p))(q, p) = (T^*\psi^{-1}(q, p))$.

In particular $\psi_T^Q$ enjoys the following properties [1, 15]:

(i) It preserves the canonical 1-form on $T^*Q$, that is, $(\psi_T^Q)^*\omega_Q = \theta_Q$ for all $g \in G$. Therefore, it is a symplectic action, i.e., $(\psi_T^Q)^*\omega_Q = \omega_Q$ for all $g \in G$.

(ii) It admits an $Ad^*$-equivariant momentum map $J : T^*Q \to g^*$ given by $J(\alpha_q)(\xi) = (\xi_q, \theta_Q)(\alpha_q) = \alpha_q(\xi_Q(q))$ for each $\xi \in g$. Here $\xi_Q$ is the infinitesimal generator of the action of $\xi \in g$ on $Q$ and $\xi_Q$ is the generator of the lifted action on $T^*Q$.

Remark 15 One could consider a general action $\Phi : G \times T^*Q \to T^*Q$ of $G$ on $T^*Q$, not necessarily lifted from an action of $G$ on $Q$. However, in order for the map $J$, defined as above, to be an $Ad^*$-equivariant momentum map, it is required that $\Phi$ preserves $\theta_Q$ (see [1, Theorem 4.2.10]). Then, it is easy to show (e.g., by direct computation in local coordinates) that $\Phi$ is a necessarily lifted action.

Definition 16 Denote by $\{\phi^X_t\}$ the flow of a vector field $X$ on $Q$. We can define the complete lift $X^* = X^* = X^* + p_j \frac{\partial}{\partial p_j}$ of $\{\phi^X_t\}$ to $T^*Q$ whose flow is the cotangent lift of $\{\phi^X_t\}$ (see [16,46,37] for instance). In local coordinates, it is given by $X^* = X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^i}{\partial q^j}$.

Let us first introduce the symplectic reduction for the forced Hamiltonian systems $(H,F)$. If the Hamiltonian $H$ is $G$-invariant, the subgroup $G_F$ of $G$ such that $H$ and $F$ are both $G_F$-invariant can be described as follows. For each $\xi \in g$, consider the real-valued function $J^\xi : T^*Q \to \mathbb{R}$ given by $J^\xi(\alpha_q) = \langle J(\alpha_q), \xi \rangle$, that is $J^\xi = \iota_{\xi_Q}^* \theta_Q$. Let $\xi \in g$, then $J^\xi$ is a conserved quantity for $X_{H,F}$ if and only if $F(\xi_Q) = 0$ (see [15,37]) and $\xi$ leaves $F$ invariant if and only if $\iota_{\xi_Q}^* dF = 0$. In addition, the vector subspace of $g$ given by $g_F = \{\xi \in g : F(\xi_Q) = 0, \iota_{\xi_Q}^* dF = 0\}$ is a Lie subalgebra of $g$. Observe that, for each $\xi \in g_F$, $\xi_Q$ is
a symmetry of the forced Hamiltonian system \((H,F)\) (see Appendix A).

Let \(G_F \subset G\) be the Lie subgroup generated by \(g_F\), let \(J_F : T^*Q \to g_F^*\) be the reduced momentum map with \(\mu \in g_F^*\), a regular value of \(J_F\), and let us denote by \((G_F)_\mu\) the isotropy subgroup in \(\mu\). Since the \(G\)-action on \(T^*Q\) is free and proper by hypothesis and \(\mu\) is a regular value, obviously the \((G_F)_\mu\)-action on \(J^{-1}(\mu)\) is free and proper as well, and thus \(J^{-1}(\mu)/(G_F)_\mu\) is a smooth manifold [35]. We have that [15,37]:

(i) \(J_F^{-1}(\mu)\) is a submanifold of \(T^*Q\) and \(X_{H,F}\) is tangent to it.

(ii) The reduced space \(M_\mu := J_F^{-1}(\mu)/(G_F)_\mu\) is a symplectic manifold, whose symplectic structure \(\omega_\mu\) is uniquely determined by \(\pi_\mu^*\omega_\mu = i_\mu^*\omega_Q\), where \(\pi_\mu : J_F^{-1}(\mu) \to M_\mu\) and \(i_\mu : J_F^{-1}(\mu) \to T^*Q\) denote the canonical projection and the canonical inclusion, respectively.

(iii) \(H\) induces a reduced function \(H_\mu : M_\mu \to \mathbb{R}\) defined by \(H_\mu \circ \pi_\mu = H \circ i_\mu\).

(iv) \(F\) induces a reduced 1-form \(F_\mu\) on \(M_\mu\), uniquely determined by \(\pi_\mu^*F_\mu = i_\mu^*F\).

(v) The forced Hamiltonian vector field \(X_{H,F}\) projects onto \(X_{M_F,F_\mu}\).

**Remark 17** In order to obtain a reduced Hamiltonian function \(H_\mu\) and a reduced external force \(F_\mu\), both \(H\) and \(F\) need to be, independently, \(G_F\)-invariant. The conditions that each \(\xi_\mu\) has to satisfy for this to occur are stronger than the ones required for being a symmetry of the forced Hamiltonian (see Subsection A). As a matter of fact, we can weaken this requirements and reduce \(\alpha_{H,F} := \Delta H + F\) instead of \(H\) and \(F\) separately. Suppose that \(H\) is a \(\psi^{T^*Q}\)-invariant, i.e., \(\alpha_{H,F}(\xi_\mu) = \xi_\mu(H) + F(\xi_\mu) = 0\) for every \(\xi \in g\). In other words, \(\xi_\mu\) is a symmetry of the forced Hamiltonian for every \(\xi \in g\) (since the action \(\psi^{T^*Q}\) leaves \(\theta_\mu\) invariant, \(\xi_\mu \theta_\mu = 0\)). Let \(G_\mu\) be the isotropy group of \(G\) in \(\mu\), where \(\mu \in g^*\) is a regular value of \(J\). Then,

(i) \(J_F^{-1}(\mu)\) is a submanifold of \(T^*Q\) and \(X_{H,F}\) is tangent to it.

(ii) The reduced space \(M_\mu := J_F^{-1}(\mu)/(G_F)_\mu\) is a symplectic manifold, and its symplectic structure \(\omega_\mu\) is uniquely determined by \(\pi_\mu^*\omega_\mu = i_\mu^*\omega_Q\).

(iii) \(\alpha_{H,F}\) induces a reduced 1-form \(\alpha_\mu_{H,F}\) on \(M_\mu\), uniquely determined by \(\pi_\mu^*\alpha_\mu_{H,F} = i_\mu^*\alpha_{H,F}\).

(iv) The forced Hamiltonian vector field \(X_{H,F} = X_{\alpha_{H,F}}\) projects onto \(X_{\alpha_\mu_{H,F}}\), where \(i_{X_{\alpha_\mu_{H,F}}} \omega_\mu = \alpha_\mu_{H,F}\).

Next, we extend the symplectic reduction for forced Hamiltonian systems to simple hybrid forced Hamiltonian systems with symmetries. Consider a simple hybrid forced Hamiltonian system \(H_F = (T^*Q, X_{H,F}, S_H, \Delta_H)\). To perform a hybrid reduction one needs to impose some compatibility conditions between the action and the hybrid system (see [4] and [5]). By a hybrid action on the simple hybrid forced Hamiltonian system \(H_F\) we mean a Lie group action \(\psi : G \times Q \to Q\) such that

- \(H\) is invariant under \(\psi^{T^*Q}\), i.e., \(H \circ \psi^{T^*Q} = H\),
- \(\psi^{T^*Q}\) restricts to an action of \(G\) on \(S_H\),
- \(\Delta_H\) is equivariant with respect to the previous action, namely

\[ \Delta_H \circ \psi^{T^*Q}_{|S_H} = \psi^{T^*Q}_{|S_H} \circ \Delta_H. \]

**Definition 18** A momentum map \(J\) will be called a generalized hybrid momentum map for \(H_F\) if, for each regular value \(\mu_\pm\) of \(J\),

\[ \Delta_H \left( J^{-1}(\mu_-) \right) \subset J^{-1}(\mu_+), \]

for some regular value \(\mu_+\). In other words, for every point in the switching surface such that the momentum before the impact takes a value of \(\mu_-\), the momentum will take a value \(\mu_+\) after the impact. That is, the switching map translates the dynamics from one level set of the momentum map into another.

Consider \(H_F = (T^*Q, X_{H,F}, S_H, \Delta_H)\) equipped with a hybrid action \(\psi\) such that \(H\) and \(F\) are \(G_F\)-invariant with \(G_F \subset G\) being the Lie subgroup generated by \(g_F\) and \(J_F : T^*Q \to g_F^*\) the reduced momentum map, which is also assumed to be a generalized hybrid momentum map. Then, for each \(\xi \in g\), \(J_F^\xi = \langle J,F,\xi \rangle\) is a hybrid constant of the motion.

Let \(\mu_- , \mu_+ \in g_F^*\) be two regular hybrid values of \(J_F\), which means that they are regular values of both \(J_F\) and \(J_F|_{S_H}\). When we combine this definition with the condition (1), we obtain that the following diagram commutes:

\[ \begin{array}{ccc} J_F^{-1}(\mu_-) & \overset{J_F|_{S_H}}{\longrightarrow} & J_F^{-1}(\mu_+) \\ & \downarrow_{\psi^{T^*Q}_{|S_H}} \searrow_{\Delta_H} & \downarrow_{J_F^{-1}(\mu_+)} \\ T^*Q & \rightarrow & S_H \end{array} \]

where \(J_F^{-1}(\mu)\) and \(J_F|_{S_H}^{-1}(\mu)\) are embedded submanifolds of \(T^*Q\) and \(S_H\), respectively. The hook arrows \(\Rightarrow\) in the diagram denote the corresponding canonical inclusions.

For each \(\mu \in g^*\), we denote by \(G_\mu\) be the isotropy subgroup of \(G\) in \(\mu\) under the co-adjoint action, namely, \(G_\mu = \{ g \in G | Ad_g^* \mu = \mu \} \).

**Proposition 19** Let \(H_F = (T^*Q, X_{H,F}, S_H, \Delta_H)\) be a hybrid forced Hamiltonian system, let \(\psi : G \times Q \to Q\) be a Lie group action of a connected Lie group \(G\) on \(Q\). If \(\Delta_H\) is equivariant with respect to \(\psi^{T^*Q}\), and \(\mu_- , \mu_+\) are regular
values of \( J \) such that \( \Delta_H (J|_{S_H^1}(\mu_-)) \subset J^{-1}(\mu_+) \), then \( G_{\mu_-} = G_{\mu_+} \).

**Proof.** Let \( g \in G_{\mu_-} \). Then,

\[
J \circ \Delta_H (J|_{S_H^1}(\mu_-)) = J \circ \Delta_H \circ \psi^T_Q (J|_{S_H^1}(\mu_-)) = J \circ \psi^T_Q \circ \Delta_H (J|_{S_H^1}(\mu_-)) = Ad_{\mu_-}^0 \circ J \circ \Delta_H (J|_{S_H^1}(\mu_-)),
\]

where we have used the equivariance of \( J \) and \( \Delta_H \), so \( g \in G_{\mu_-} \), and hence \( G_{\mu_-} \) is a Lie subgroup of \( G_{\mu_+} \).

Now, observe that \( G_{\mu} \) has the same dimension, for each \( \mu \in g^* \). Therefore, the identity components of \( G_{\mu_-} \) and \( G_{\mu_+} \) coincide. If we assume that \( G \) is connected, \( G_{\mu_-} \) and \( G_{\mu_+} \) are equal to their identity components, so \( G_{\mu_-} = G_{\mu_+} \). □

**Theorem 20** Let \( \mathcal{H}_F = (T^*Q, X_{H,F}, S_H, \Delta_H) \) be a hybrid forced Hamiltonian system. Let \( \psi: G \times Q \to Q \) be a hybrid action of a connected Lie group \( G \) on \( Q \). Suppose that \( H \) and \( F \) are \( G \)-invariant and assume that \( J_F \) is a generalized hybrid momentum map. Consider a sequence \( \{\mu_i\} \) of regular values of \( J_F \), such that \( \Delta_H (J|_{S_H^1}(\mu_i)) \subset J^{-1}(\mu_{i+1}) \). Let \( (G_F)_{\mu_i} = (G_F)_{\mu_0} \) be the isotropy subgroup in \( \mu_i \) under the co-adjoint action. Then,

(i) \( J_F^{-1}(\mu_i) \) is a submanifold of \( T^*Q \) and \( X_{H,F} \) is tangent to it.

(ii) The reduced space \( M_{\mu_i} := J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \) is a symplectic manifold, whose symplectic structure \( \omega_{\mu_i} \), is uniquely determined by \( \pi^*_\mu_i \omega_{\mu_i} = \iota_{\mu_i}^* \omega_Q \), where \( \pi_{\mu_i} : J_F^{-1}(\mu_i) \to M_{\mu_i} \) and \( \iota_{\mu_i} : J_F^{-1}(\mu_i) \to T^*Q \) denote the canonical projection and the canonical inclusion, respectively.

(iii) \( (H,F) \) induces a reduced forced Hamiltonian system \( (H_{\mu_i}, F_{\mu_i}) \) on \( M_{\mu_i} \), given by \( H_{\mu_i} \circ \pi_{\mu_i} = H \circ \iota_{\mu_i} \) and \( \pi^*_\mu_i F_{\mu_i} = \iota_{\mu_i}^* F \). Moreover, the forced Hamiltonian vector field \( X_{H,F} \) projects onto \( X_{H_{\mu_i}, F_{\mu_i}} \).

(iv) \( J_F |_{S_H^1}(\mu_i) \subset S_H \) reduces to a submanifold of the reduced space \( S_{H_{\mu_i}} \subset J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \).

(v) \( \Delta_H|_{J^{-1}(\mu_i)} \) reduces to a map \( \Delta_{H_{\mu_i}} : (S_{H_{\mu_i}})_{\mu_i} \to J_F^{-1}(\mu_{i+1})/(G_F)_{\mu_0} \).

Therefore, after the reduction procedure, we get a sequence of reduced simple hybrid forced Hamiltonian systems \( \{\mathcal{H}_{\mu_i}\} \), where \( \mathcal{H}_{\mu_i} = (J_F^{-1}(\mu_i)/(G_F)_{\mu_0}, X_{H_{\mu_i}, F_{\mu_i}}, (S_{H_{\mu_i}})_{\mu_i}, (\Delta_{H_{\mu_i}})_{\mu_i}) \).

\[
\cdots \to J_F^{-1}(\mu_i) \to J_F^{-1}(\mu_i)_{G_{\mu_0}} \to J_F^{-1}(\mu_i)_{G_{\mu_0}} \to \cdots
\]

\[
\cdots \to J_F^{-1}(\mu_i)_{G_{\mu_0}} \to (S_{H_{\mu_i}})_{\mu_i} \to \Delta_{H_{\mu_i}} \to J_F^{-1}(\mu_{i+1})_{G_{\mu_0}} \to \cdots
\]

**Proof.** See [1,42] for a proof of the first two assertions. The third statement was proven in [15] (see also [37]).

Since the \((G_F)_{\mu_0}\)-action restricts to a free and proper action on \( S_H, (S_H)_{\mu_i} = J_F |_{S_H^1}(\mu_i)/(G_F)_{\mu_0} \) is a smooth manifold. Clearly, it is a submanifold of \( J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \). Since \( \Delta_H \) is equivariant, it induces an embedding \( (\Delta_{H_{\mu_i}})_{\mu_i} : (S_{H_{\mu_i}})_{\mu_i} \to J_F^{-1}(\mu_{i+1})/(G_F)_{\mu_0} \). □

The reduction picture in the Lagrangian side can now be obtained from the Hamiltonian one by adapting the scheme developed in [33]. In the same fashion as in the Hamiltonian side, by a hybrid action on the simple hybrid Lagrangian system \( \mathcal{Z}_F = (TQ, X_{L,F}, S_L, \Delta_L) \) we mean a Lie group action \( \psi: G \times Q \to Q \) such that

- \( L \) is invariant under \( \psi^T_Q \), i.e., \( L \circ \psi^T_Q = L \).
- \( \psi^T_Q \) restricts to an action of \( G \) on \( S_L \).
- \( \Delta_L \) is equivariant with respect to the previous action, namely \( \Delta_L \circ \psi^T_Q |_{S_L} = \psi^T_Q \circ \Delta_L \).

where \( \psi^T_Q \) is the tangent lift of the action \( \psi \) to \( TQ \), defined by \( (g,(q,q)) \mapsto (T\psi g(q,q)) \).

The key idea is that, since \( \psi^T_Q \) is a hybrid action under which \( L \) is invariant, the Legendre transformation \( \mathcal{F} L \) is a diffeomorphism such that:

- It is equivariant with respect to \( \psi^T_Q \) and \( \psi^T_Q \).
- It preserves the level sets of the momentum map, that is, \( \mathcal{F} L((J_F^{-1}(\mu_i)) = J_F^{-1}(\mu_i) \).
- It relates the symplectic structures, that is, \( (\mathcal{F} L)^* \omega_Q = \omega_{L} \), meaning that, \( \mathcal{F} L \) is a symplectomorphism.

It follows that the map \( \mathcal{F} L \) reduces to a symplectomorphism \( \mathcal{F} L_{\text{red}} \) between the reduced spaces. Consider the Lie subalgebra \( \mathfrak{g}_{FL} = \{ \xi \in g : L^*(\xi_Q) = 0, \ i_{\xi_Q}^L F^L = 0 \} \) of \( g \), and let \( G_{FL} \) be the Lie subgroup it generates. Then, the following diagram commutes:

\[
\begin{array}{cccc}
(TQ, S_L, \Delta_L) & \xrightarrow{\mathcal{F} L} & (T^*Q, S_H, \Delta_H) \\
\text{Red.} & & \text{Red.} \\
(M_{\mu_0}^H, (S_L)_{\mu_i}, (\Delta_L)_{\mu_i}) & \xrightarrow{(\mathcal{F} L)_{\text{red}}} & (M_{\mu_0}^H, (S_H)_{\mu_i}, (\Delta_H)_{\mu_i})
\end{array}
\]

where we have used the notation \( M_{\mu_0}^H := (J_L)^{-1}(\mu_i)/(G_{FL})_{\mu_0} \) and \( M_{\mu_0}^H := J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \).

**Lemma 21** If \( (L, F^L) \) is the Lagrangian counterpart of \( (H, F) \) (i.e., \( E_L = H \circ \mathcal{F} L \) and \( F^L = \mathcal{F} L^* F \)), then \( G_{FL} = G_F \).
Proof. For any $g \in G_F$, 
\[
\begin{align*}
\left(\psi^T_g\right)^* F^L &= \left(\psi^T_g\right)^* \mathcal{F}^L \circ F^* L = \left(\mathcal{F}^L \circ \psi^T_g\right)^* F \\
&= \left(\psi^L_g \circ \mathcal{F}^L\right)^* F = \mathcal{F}^L \circ \left(\psi^T_g\right)^* F \\
&= \mathcal{F}^L \circ F^L = F^L,
\end{align*}
\]
where we have used the equivariance of $\mathcal{F}$, so $g \in G_{FL}$. Similarly one can show that, for any $g \in G_{FL}$, $g \in G_F$.

In the following we make use of a principal connection on the bundle $Q \to Q/G$ to make some further identifications. If the class of a mechanical Lagrangian, a natural choice is the so-called mechanical connection [39]. Let $\mathcal{A}: TQ \to \mathfrak{g}$ be a connection one form, and let us denote by $\mathcal{A}_\mu(\cdot) = \langle \mu, \mathcal{A}(\cdot) \rangle$ the 1-form on $Q$ obtained by contraction with $\mu \in \mathfrak{g}$. Building on the well-known results on cotangent bundle reduction [38] it is possible to show that there is an identification 
\[
J^1_F(\mu_i)/(G_F)_{\mu_0} \simeq \left(\mathcal{T}^*(Q/G) \times Q/G \right) Q/(G_F)_{\mu_0}.
\]

This identification is a symplectomorphism when the space on the right hand side of (2) is endowed with the symplectic structure $\varpi^1_{\mathcal{T}Q} + \varpi^2_{\mathcal{B}_\mu}$, where $\varpi_1$ and $\varpi_2$ are the canonical projections, and $\mathcal{B}_\mu$ is the so-called magnetic term, obtained from the reduction of $d\mathcal{A}_\mu$ to $Q/(G_F)_{\mu_0}$. For details, see [33,38].

For the reduction in the Lagrangian side, one needs an additional regularity condition, sometimes referred to as $G$-regularity. Precisely, one has the following definition [34].

**Definition 22** Consider a $G_{FL}$-invariant forced Lagrangian system $(L, F^L)$ on $TQ$ (i.e., $L \circ \psi^T_g = L$ and $(\psi^T_g)^* F^L = F^L$ for every $g \in G$) and let $\varphi^1_{\mathcal{T}Q}$ be the infinitesimal generator associated to the canonical lifted action. Then, $(L, F^L)$ is said to be $G_{FL}$-regular if, for each $\varphi_0 \in TQ$, the map $(\varphi_0)_{F^L}: \mathfrak{g}_F \to \mathfrak{g}_F^*$, $\xi \mapsto (J_L)_{F^L}(\varphi_0 + \xi\mathcal{Q}(q))$ is a diffeomorphism.

Essentially, the $G_{FL}$-regularity demands regularity “with respect to the subgroup variables”. Hereafter, the pair $(L, F^L)$ will be assumed to be $G_{FL}$-regular, so that there is an identification $(J_L)^{-1}(\mu_i)/(G_{FL})_{\mu_0} \simeq (\mathcal{T}(Q/G) \times Q/G) Q/(G_{FL})_{\mu_0}$ [see 19 for instance].

The reduced dynamics on this space can be interpreted as the Lagrangian dynamics of some regular Lagrangian subjected to a gyroscopic force (arising from the magnetic term) by working in the more general class of magnetic Lagrangians [32], which in the present situation need to be extended in order to include external forces. Magnetic Lagrangian systems are a broad family of Lagrangian systems on which the Lagrangian function might be independent of some of the velocities. A force term given by a 2-form can also appear in these systems. Since the Routh reduction yields, in general, a reduced system which is not a standard Lagrangian system, magnetic Lagrangian systems provide a quite convenient framework for carrying out Routh reduction. The extension to magnetic Lagrangian systems allows to carry out Routh reduction by stages [34]. The role of the reduced Lagrangian function is played by the so-called Routhian$^1$, which is defined as (the reduction of) the $(G_F)_{\mu_0}$-invariant function $R^L_\mu = L - A_\mu$, restricted to $(J_L)^{-1}(\mu_i)$. The next diagram summarizes the situation:

\[
\begin{array}{c}
\mathcal{T}^*(Q/G) \\
\mathcal{F}^L \\
\mathcal{T}(Q/G) \times Q/G \\
\mathcal{F}^L \\
\mathcal{T}^*(Q/G) \times Q/G \\
\mathcal{F}^L \\
\end{array}
\]

4.1 A particular case: cyclic coordinates

When $(L, F^L)$ is $S^1$-invariant, one recovers the classical notion of a cyclic coordinate (the case $G = \mathbb{R}$ is analogous; if $G$ is a product of $S^1$ or $\mathbb{R}$ one can iterate the procedure). Since $G = S^1$ is abelian, $G_{\mu_1} = G$ for every $\mu_1 \in \mathfrak{g}^*$. The reduced space $(J_L)^{-1}(\mu_i)/(G_F)_{\mu_0}$ can be identified with $T(Q/S^1)$. Similarly, the reduced switching surface $(S_L)_{\mu_1}$ can be identified with a submanifold of $T(Q/S^1)$, and the impact map can be identified with a map $(\Delta_L)_{\mu_1}: (S_L)_{\mu_1} \to T(Q/S^1)$.

If the forced Lagrangian system $(L, F^L)$ has a cyclic coordinate $\theta$, i.e., $L$ is a function of the form $L(\theta, x, \dot{x})$, and $F$ is of the form $F(\theta, x, \dot{x}) = F_x(\theta, x, \dot{x})\dot{x}$. The conservation of the momentum map $(J_L)_{F} = \mu_1$ reads $\partial L/\partial \theta = \mu_1$.

This relation can be used to write $\dot{\theta}$ as a function of the remaining non-cyclic coordinates and their velocities, and the fixed regular value of the momentum map $\mu_1$, namely, $\theta = \theta(x, \dot{x}, \mu_1))$. It is worth noting that this is the stage where the $G_F$-regularity of $(L, F^L)$ is used, in order to guarantee that $\dot{\theta}$ can be expressed in terms of $x, \dot{x}$ and $\mu_1$. If the connection on the bundle $Q \to Q/S^1 = M$ is chosen to be the canonical flat connection, then the Routhian and the reduced external force can be written as

\[
\begin{align*}
R^\mu_\mu(x, \dot{x}) &= \left[ L(\dot{\theta}, x, \dot{x}) - \mu_1 \dot{\theta} \right]_{\theta = \theta(x, \dot{x}, \mu_1)} \quad (3) \\
F^L_\mu(x, \dot{x}) &= F^L(\dot{\theta}, x, \dot{x}) \left|_{\theta = \theta(x, \dot{x}, \mu_1)} \right.
\end{align*}
\]

Here the notation means that $\dot{\theta}$ is expressed as a function of $(x, \dot{x}, \mu_1)$. Observe that (3) coincides with the classical definition of the Routhian [43]. Moreover, since the connection is flat, no magnetic terms appear in the reduced dynamics.

The value of the momentum map will, in general, be modified in the (non-elastic)collisions with the switching surface.

---

$^1$ Note the difference with the Hamiltonian reduction.
Therefore, the reduced Hamiltonian $H_{\mu_i}$ and the reduced external force $F_{\mu_i}$ will have to be defined in each $I_i$, where $\tilde{\vartheta} = \{I_i\}_{i \in \Lambda}$ is the hybrid interval, and they will depend on the value of the momentum $\mu_i$ after the collision at time $\tau_i$. It is worth noting that this also affects the way the impact map $\Delta_H$ is reduced.

Let us denote: (1) $\mu_i$ the momentum of the system in $I_i = [\tau_i, \tau_{i+1}]$, (2) $(\Delta_H)_{\mu_i}$ the reduction of $\Delta_H|_{J^{-1}(\mu_i)}$, and (3) $(S_H)_{\mu_i}$ the reduction of $S_H$, so there is a sequence of reduced simple hybrid forced Hamiltonian systems:

\[
\begin{align*}
\text{Coll.} & \quad [\tau_0, \tau_1] \quad \text{Red.} \quad (T^*(Q/\mathbb{S}^1) \times Q/\mathcal{G}_F)_{\mu_0}, \chi_{\mu_0}, (\Delta_H)_{\mu_0}, (S_H)_{\mu_0}, (\Delta_H)_{\mu_0}) \\
\text{Coll.} & \quad [\tau_1, \tau_2] \quad \text{Red.} \quad (T^*(Q/\mathbb{S}^1) \times Q/\mathcal{G}_F)_{\mu_0}, \chi_{\mu_0}, (\Delta_H)_{\mu_1}
\end{align*}
\]

Here “Coll.” and “Red.” stand for collision and reduction, respectively.

Since the momentum, generally, changes with the collisions the reconstruction procedure will be more challenging. In order to make use a reduced solution to reconstruct the original dynamics, the reduced hybrid data have to be computed after each collision. That is, once the reduced solution for the time interval between two collision events, say between $t = \tau_i$ and $t = \tau_{i+1}$, has been obtained, this solution has to be reconstructed to obtain the new momentum after the collision at $\tau_{i+1}$. After that, this new momentum has to be used in order to build a new reduced hybrid system whose solution should be obtained until the next collision event at $\tau_{i+2}$, and so forth. As usual, in order to reconstruct the hybrid flow from the reduced hybrid flow, one has to integrate the regular value at each stage in the previous diagram, using the solution of the reduced simple hybrid forced Hamiltonian system. Essentially, this is tantamount to imposing the momentum constraint on the reconstructed solution.

More precisely, suppose that $\chi_{\mathcal{H}_F}^{\mu_i}(q_0) = (\Lambda, \beta, \xi^{\mu_i})$ is a hybrid flow of $\mathcal{L}_F^{\mu_i}$. Then we can construct a hybrid flow $\mathcal{L}_F(q_0(\tau_0)) = (\Lambda, \beta, \xi)$ of $\mathcal{L}_F$ by constructing the flow recursively between two collisions. Writing $e^{\mu_i}_{\xi}(t) = (x_{\mu_i}, \dot{x}_{\mu_i})$, we define $c_\xi(t) = (x_{\mu_i}, \dot{x}_{\mu_i}, \theta_{\mu_i}, \hat{\theta}_{\mu_i})$ recursively as follows. Assume that we have a mechanical Lagrangian of the form $L(x, \dot{x}, \theta) = \dot{q}^T Mq - V(q)$, where

\[
\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}
\]

and the mass matrix is $M = \begin{pmatrix} M_{\theta,x}(x) & M_{\theta,x}(x) \\ M_{\theta,x}(x) & M_{\theta}(x) \end{pmatrix}$.

First note that $(J_L)F(x, \dot{x}, \theta, \dot{\theta}) = \frac{\partial}{\partial \theta}(x, \dot{x}, \theta, \dot{\theta}) = M_{\theta,x}(x) \dot{x} + M_{\theta}(x) \dot{\theta}$. Then it is easy to see that

\[
\begin{align*}
\hat{\theta}_{\mu_i}(t) &= M^{-1}_{\theta}(\theta_{\mu_i}(t)) (\mu_i - M_{\theta,x}(x_{\mu_i}) \dot{x}_{\mu_i}(t)), \\
\theta_{\mu_i}(t) &= \Delta^*_{\mu_i}(e^{\mu_i-1}(\tau_i)) + \int_{\tau_i}^{t-\tau_i} \hat{\theta}_{\mu_i}(s) ds,
\end{align*}
\]

where $t \in [\tau_i, \tau_{i+1}]$ and $(\Delta^*_{\mu_i})_{\mu_i}(e^{\mu_i-1}(\tau_i))$ is the $\theta$-component of $(\Delta^*_{\mu_i})_{\mu_i}(e^{\mu_i-1}(\tau_i))$. Note that, at each step, one has to reconstruct with the corresponding momenta $\mu_i$ in equation (4) and reduce again the dynamics after the collision with a new momenta $\mu_{i+1}$ as conserved quantity.

**Remark 23** A generalized hybrid momentum map is called a hybrid momentum map if $\Delta_H$ preserves the momentum map. In other words, $J$ is a hybrid momentum map if the diagram

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{J} & S_H \\
\downarrow & \searrow \searrow & \downarrow J^* \Delta_H \\
\end{array}
\]

commutes (see [5], [4]). If the momentum map is hybrid, the reduction of a simple hybrid forced Hamiltonian system $\mathcal{H}_F = (T^*Q, X_{H,F}, S_H, \Delta_H)$, with initial value of the momentum map $\mu_0$, yields a single reduced simple hybrid forced Hamiltonian system $\mathcal{H}_F^{\mu_0} = \langle J^* \mu_0 \rangle((G_F)_{\mu_0}, X_{H_{\mu_0}}, \chi_{\mu_0}, (S_H)_{\mu_0}, (\Delta_H)_{\mu_0})$. Additionally, $J^*F$ is a hybrid constant of the motion for each $\xi \in \mathcal{G}_F$.

**Remark 24** In a simple hybrid Hamiltonian system with a mechanical Hamiltonian function the impact can be obtained from the Newtonian impact equation (see [6] for instance). That is, $\Delta_H(q, p) = (q, P_q(p))$, where $P_q : T_q^*Q \to T_q^*Q$ is given by

\[
P_q(p) = p - (1 + e) \frac{\langle\langle p, d\mathbf{h}_q \rangle\rangle_q}{||d\mathbf{h}_q||^2_q} d\mathbf{h}_q,
\]

with $|| \cdot ||_q$ denoting the corresponding norm on $T_q^*Q$, and $\langle\langle \cdot, \cdot \rangle\rangle_q$ is the inner-product on the vector space $T_q^*Q$ defined through the kinetic energy of the system as

\[
\langle\langle \alpha, \beta \rangle\rangle_q = \sum_{j=1}^{\dim(Q)} \alpha_j \beta_j M^{-1}_{ij}(q), \text{ being } M(q) \text{ the inertia matrix associated with the mechanical system under study.}
\]

The parameter $0 \leq e \leq 1$ is the coefficient of restitution (for instance, $e = 1$ corresponds with elastic impacts and $e = 0$ with inelastic impacts). The switching surface is also defined through the inner product as

\[
S_H = \{ (q, p) \in T^*Q : h(q) = 0 \text{ and } \langle\langle p, d\mathbf{h}_q \rangle\rangle_q \leq 0 \}.
\]

Note that the analytical expressions for the switching surface and the impact map depend on the chosen metric. Hence,
by choosing different metrics one can obtain different expressions for the impact map and switching surface, which could help to obtain invariant expressions for $S_H$ and $\Delta_H$ for a given action.

Similarly, in a simple hybrid Lagrangian system with a mechanical Lagrangian function $L = \tilde{q}^T M(q)\dot{q} - V(q)$, the impact can be obtained from the Newtonian impact equation $P : TQ \rightarrow TQ$ given by

$$P(q, \dot{q}) = \dot{q} - (1 + e) \frac{dh_q \dot{q}}{dh_q M(q)}^{-1} d\dot{q},$$

where $M(q)$ is the inertial matrix for the Lagrangian system, $h$ a function describing the switching surface as a submanifold of $Q$ and $e$ the coefficient of restitution. The switching surface is $S_L = \{ (q, \dot{q}) : TQ \mid h(q) = 0 \text{ and } d\dot{q} \dot{q} \leq 0 \}$.

**Example 25 (Rolling disk with dissipation hitting fixed walls)** Consider a homogeneous circular disk of radius $R$ and mass $m$ moving in the vertical plane $xOy$ (see [27, Example 8.2], and also [29, Example 3.7]). Let $(x, y)$ be the coordinates of the centre of the disk and $\theta$ the angle between a point of the disk and the axis $Oy$. The dynamics of the system is determined by the Hamiltonian $H$ on $T^*(\mathbb{R}^2 \times S^1)$ given by

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2mR^2}p_\theta^2,$$

which is hyperregular since it is mechanical. The system is subject to external forces given by $F(x, y, \theta, p_x, p_y, p_\theta) = F_x dx + F_y dy$ where $F_x = -\frac{2c}{m}(p_x x y - p_y y^2)$, $F_y = \frac{2c}{m}(p_y x^2 y - p_x y^2)$, for some constant $c > 0$. Note that $F_\theta = 0$.

Forced Hamilton equations of motion are

$$\dot{p}_x = \frac{2c}{m}(p_x x y - p_y y^2), \quad \dot{p}_y = -\frac{2c}{m}(p_x x y - p_y y^2), \quad \dot{p}_\theta = 0,$$

$$\dot{x} = \frac{1}{m} p_x, \quad \dot{y} = \frac{1}{m} p_y, \quad \dot{\theta} = \frac{1}{m} p_\theta.$$

Using the Legendre transformation we can obtain the Lagrangian and external force $F^L$. The Lagrangian function $L : T(\mathbb{R}^2 \times S^1) \rightarrow \mathbb{R}$ is given by

$$L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \left[ \frac{1}{2m} (\dot{x}^2 + \dot{y}^2 + k_\theta^2 \dot{\theta}^2) \right]$$

and $F^L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = F'_L dx + F'_L dy$ is an external force given by $F'_L = -2c(\dot{x} y - \dot{y} x), \quad F'_\theta = 2c(\dot{x} y^2 - \dot{y} x^2)$.

The forced Euler-Lagrange equations for the free motion of the disk are

$$m\ddot{x} = -2c(y x \dot{y} - \dot{x} y^2), \quad m\ddot{y} = 2c(x \dot{y}^2 - \dot{x} y x), \quad \ddot{\theta} = 0.$$

Consider the Lie group action of $S^1 \times S^1$ on $Q$ given by $(x, y, \theta) \mapsto (\cos \alpha x - \sin \alpha y, \sin \alpha x + \cos \alpha y, \theta + \beta)$.

Note that $L$ and $F^L$ are invariant under the lifted action on $TQ$. The corresponding momentum map is $(J_L)_F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = (m\dot{x} y - m\dot{y} x, mk^2 \dot{\theta})$.

By introducing polar coordinates $L$ and $F^L$ become

$$L(\theta, r, \dot{\theta}, \dot{r}, \dot{\phi}) = \frac{m}{2}(r^2 + r^2 \dot{\theta}^2 + k_\theta^2 \dot{\theta}^2),$$

$$F'^L(\theta, r, \dot{\theta}, \dot{r}) = 2cr^3 \dot{\theta},$$

respectively. The forced Euler-Lagrange equations (in polar coordinates) are

$$\ddot{r} = \left( r - \frac{2c}{m} \right) \dot{\theta}, \quad m \frac{d(r^2 \dot{\theta})}{dt} = 0, \quad mk^2 \dot{\theta} = 0.$$

The momentum map is now written $(J_L)_F(r, \dot{r}, \theta, \dot{\theta}, \dot{\phi}) = (mr^2 \dot{\theta}, mk^2 \dot{\theta})$. By observing the forced Euler-Lagrange equations in polar coordinates, it is clear that $(J_L)_F$ is preserved. Considering $\mu_1 = mr^2 \dot{\theta}$ and $\mu_2 = mk^2 \dot{\theta}$ (i.e., $\dot{\theta} = \frac{\mu_1}{mr^2}$ and $\dot{\phi} = \frac{\mu_2}{mk^2}$), the Routhian and the reduced force take the form

$$R^L_{\mu}(r, \dot{r}) = \frac{m}{2} \dot{r}^2 - \frac{\mu_1^2}{2mr^2} - \frac{\mu_2^2}{2mk^2}, \quad F^L_{\mu} = \frac{2cr}{m} \mu_1 dr,$$

and the reduced forced Euler-Lagrange equations for the Routhian $R^L_{\mu}$ and reduced external force $F^L_{\mu}$ are given by

$$\ddot{r} = \frac{\mu_1^2}{m^2r^3} - \frac{2cr}{m^2} \mu_1.$$

Suppose that there are two rough walls at the axis $y = 0$ and at $y = h$, where $h = \alpha R$ for some constant $\alpha > 1$. Assume that the impact with a wall is such that the disk rolls without sliding and that the change of the velocity along the $y$-direction is characterized by an elastic constant $e$. When the disk hits one of the walls, the impact map is given by (see [27, Example 8.2], and also [29, Example 3.7])

$$(\dot{x}^-, \dot{y}^-, \dot{\theta}^-) \mapsto \left( \frac{R^2 \dot{x}^- + k_\theta^2 R \dot{\theta}^-}{k_\theta^2 + R^2}, -e \dot{y}^-, \frac{R \dot{x}^- + k_\theta^2 \dot{\theta}^-}{k_\theta^2 + R^2} \right).$$

where the switching surface is given by

$S = \{ (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) | y = R \text{ or } y = h - R, \text{ and } \dot{x} = R \dot{\theta} \}.$

For the sake of simplicity, let us assume that $e = 1$. It is worth noting that, despite the fact this corresponds to an elastic collision, the momentum map will not be preserved in the impact.

One can check that $(J_L)_F$ is a generalized hybrid momentum map but not a hybrid momentum map, i.e., $(J_L)_F(q_1, q^-_1) = (q^-_1, q^-_1, q^-_1).$
\((J_L)_{F}(q_2, \dot{q}_2^+)\) implies that \((J_L)_{F}(q_1, \dot{q}_1^+) = (J_L)_{F}(q_2, \dot{q}_2^+)\) but \((J_L)_{F}(q_1, \dot{q}_1^-) \neq (J_L)_{F}(q_1, \dot{q}_1^-)\). Indeed, if \(x, y, \dot{x}, \dot{y}, \dot{\vartheta}, \dot{\vartheta}^-\) \(\in S\), then

\[
(J_L)_{F} \circ \Delta(x, y, \dot{x}, \dot{y}, \dot{\vartheta}, \dot{\vartheta}^-) = \left(-my^2 - m\dot{y}^- \dot{x} + \frac{k^2 \dot{\vartheta}^- - R\dot{x}^-}{k^2 + R^2}\right) = \left(-mR\dot{\vartheta}^- - mRk^2 \dot{\vartheta}^-\right),
\]

where in the last step we have used that \(y = R\) and \(\dot{x}^- = R\dot{\vartheta}^-\) (for the wall at \(y = h\) the result is analogous).

In polar coordinates, for \(\theta = \arctan(y/x)\), we have

\[
\dot{r}^+ = \frac{1}{1 + (y/x)^2} \left(\frac{\dot{y}^+ x - y\dot{x}^+}{x^2}\right),
\]

\[
\dot{r}^- = \frac{1}{r^2} \left(-\dot{y}^- x - y\dot{x}^+ + \frac{R^2 (\dot{\vartheta}^- - 2r \sin \theta \cos \theta \dot{\vartheta}^-)}{R^2 + k^2}\right) \quad (5)
\]

and

\[
\dot{\vartheta}^- = \frac{R \dot{x}^- + k^2 \dot{\vartheta}^-}{k^2 + R^2} = \frac{R \cos \dot{\vartheta}^- - \sin \dot{\vartheta} \dot{\vartheta}^-}{k^2 + R^2} + \frac{k^2 \dot{\vartheta}^-}{k^2 + R^2} \quad (6)
\]

The switching surface can be written as

\[
S = \{(r, \dot{r}, \dot{\vartheta}, \dot{\vartheta}) | r \sin \theta = R or r \sin \theta = h - R,
\]

\[
and \ \dot{r} \cos \theta - r \dot{\theta} \sin \theta = R \dot{\vartheta}\}\}
\]

Let \((\mu_1^+, \mu_2^-)\) and \((\mu_1^-, \mu_2^+)\) be the value of the momentum map before and after the impact, respectively. We can write \(\dot{\vartheta}^+ = \mu_1^+ / mv^2\) and \(\dot{\vartheta}^- = \mu_2^- / mk^2\), so the reduced switching map is

\[
\dot{r}^- \mapsto (2 \cos^2 \theta - 1)\dot{r}^- - 2r \sin \theta \cos \theta \frac{\mu_1}{mv^2},
\]

with the relations

\[
\mu_1^+ = -\mu_1^- \quad \mu_2^+ = \mu_2^-.
\]

which are derived from Eqs. (5) and (6), respectively. The reduced switching surface can be written as

\[
S_{(\mu_1^+, \mu_2^-)} = \{(r, \dot{r}) | r \sin \varphi = R or r \sin \varphi = h - R,
\]

\[
and \ \dot{r} \cos \varphi - r \frac{\mu_1}{mv^2} \sin \varphi = R \frac{\mu_2}{mk^2},
\]

for some \(\varphi \in [0, 2\pi]\) \}

5 Extension to the reduction for time-dependent forced mechanical systems with symmetries

Next we extend the reduction of simple hybrid forced Hamiltonian systems to the case in which the Hamiltonian, the external force and the switching surface depend explicitly on time. The geometric framework that we will employ for non-autonomous mechanics will be a cosymplectic manifold [7]. In particular, in this section we extend the results in [12] to forced mechanical systems.

Consider a time-dependent forced Lagrangian \(L : \mathbb{R} \times TQ \rightarrow \mathbb{R}\) with a time-dependent external force denoted by \(F^L\), a semibasic 1-form on \(\mathbb{R} \times TQ\). Let us denote by \(\mathbb{F}L : \mathbb{R} \times TQ \rightarrow \mathbb{R} \times T^*Q\) the Legendre transformation associated with \(L\), i.e., the map \((t, q, \dot{q}) \mapsto (t, q, p = \partial L / \partial \dot{q})\). Hereinafter, assume that the Lagrangian is hyperregular, \(^2\) i.e., that \(\mathbb{F}L\) is a diffeomorphism. This permits to work out the velocities \(\dot{q}\) in terms of \((t, q, p)\) by means of the inverse of \(\mathbb{F}L\), and to define the Hamiltonian function \(H : \mathbb{R} \times T^*Q \rightarrow \mathbb{R}\) as \(H(t, q, p) = \langle p, \dot{q}(t, q, p) \rangle - L(t, q, \dot{q}(t, q, p))\), and the external force \(F\) such that \(\mathbb{F}L(F^L) = F\), a semibasic 1-form on \(\mathbb{R} \times T^*Q\).

In order to characterize the dynamics of a non-autonomous forced Hamiltonian system, consider the manifold \(\mathbb{R} \times T^*Q\) equipped with the canonical cosymplectic structure \(\omega = dq^i \wedge dp_i, \eta = dt\), where \((q^i)\) are local coordinates on \(Q\), and \((t, q^i, p_i)\) are the induced coordinates on \(\mathbb{R} \times T^*Q\). There is an unique vector field \(\mathcal{R}\) on \(\mathbb{R} \times T^*Q\) such that \(\eta(\mathcal{R}) = 1\) and \(\mathcal{R} \omega = 0\), called the Reeb vector field which in coordinates reads, \(\mathcal{R} = \dot{\vartheta}\). The cosymplectic structure defines an isomorphism between vector fields and 1-forms on \(\mathbb{R} \times T^*Q\). In addition, with every smooth function \(f\) on \(\mathbb{R} \times T^*Q\), one can associate a Hamiltonian vector field \(X_f\), such that \(\mathcal{R} f = 0\) and \(\mathcal{R} \omega = df - \mathcal{R}(f)\). We can also define the evolution vector field \(X_{f,t} = X_f + \mathcal{R}\).

Given a time-dependent Hamiltonian function \(H(t, q, p)\) and a time-dependent external force \(F(t, q, p)\), the forced Hamiltonian vector field \(X_{H, F}\) is the vector field on \(\mathbb{R} \times T^*Q\) defined by \(X_{H, F} = X_H + Z_F\), where

\[
\iota_{X_H} \omega_Q = dH - R(H) \eta, \ i_{X_H} \eta = 0,
\]

\[
\iota_{Z_F} \omega_Q = F - F(R) \eta, \ i_{Z_F} \eta = 0.
\]

\(^2\) In particular, this always holds for mechanical Lagrangians.
The **forced evolution vector field** corresponding to the forced Hamiltonian system \((H, F)\), denoted by \(X_{H,F,t}\), is given by
\[
X_{H,F,t} = \mathbb{R} + X_{H,F} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \left( \frac{\partial H}{\partial q^i} + F_i \right) \frac{\partial}{\partial p_i}.
\]

A similar procedure can be used in the Lagrangian framework. A hyperregular Lagrangian defines a cosmectic structure on \(\mathbb{R} \times TQ\), given by \(\omega_L = \mathcal{F}L^* (dq_i \wedge dp_i) = dq^i \wedge d \left( \frac{\partial L}{\partial \dot{q}^i} \right)\), \(\eta = dt\) (with a slight abuse of notation, the same symbol \(\eta = dt\) will be used for two different 1-forms on different manifolds). Let \(\mathcal{R}_L\) denote the associated Reeb vector field.

The **energy function** \(E_L : \mathbb{R} \times TQ \to \mathbb{R}\) is given by \(E_L (t, q, \dot{q}) = (\mathcal{F}L(q, \dot{q})) \cdot L(t, q, \dot{q})\), from which one can compute the Hamiltonian forced vector field \(X_{H,F,t}\) associated to \(E_L\) and \(F^L\) via the Lagrangian cosmectic structure. This leads to a forced evolution vector field \(X_{H,F,t} = X^L_{t} + \mathcal{R}_L\). Finally, the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case for forced time-dependent systems is achieved via \(\mathcal{F}L\) as follows.

**Proposition 26** The tangent map of \(\mathcal{F}L\) maps \(X^L_{t}\) onto \(X_{H,F,t}\). In other words, \((T\mathcal{F}L)X^L_{t} = X_{H,F,t}\), where \((T\mathcal{F}L) : T(TQ) \to T(T^*Q)\). In particular, the flow of \(X^L_{t}\) is mapped onto the flow of \(X_{H,F,t}\).

**Proof.** The evolution vector field \(X_{H,F,t}\) is characterized by \(\iota_{X_{H,F,t}} \omega_Q = dH - \mathcal{R}(H)\eta\), \(\iota_{X_{H} \omega_Q = 0}\), and \(\iota_{Z_{F^L} \omega_Q = F - \mathcal{R}(F)\eta}\) and \(\iota_{Z_{F} \eta = 0}\). Observe that
\[
(\mathcal{F}L)^*(\iota_{X_{H,F} \omega_Q}) = (\mathcal{F}L)^*(dH - \mathcal{R}(H)\eta) = (\mathcal{F}L)^*(\iota_{X_{H} \omega_Q}) = \delta/(\mathcal{F}L)^*H) - \mathcal{R}(\mathcal{F}L)^*H)\eta = \delta(E_L) - \mathcal{R}(E_L)\eta = \iota_{X^L_{t} \omega_L}.
\]

This means that
\[
\iota_{X^L_{t} \omega_L} = (\mathcal{F}L)^*(\iota_{X_{H,F} \omega_Q}) = \iota_{(\mathcal{F}L)^{-1}}(\iota_{X_{H} \omega_Q}) = \iota_{(\mathcal{F}L)^{-1}}(\iota_{X_{H} \omega_Q}).
\]

Similarly, one can show that \(\iota_{X^L_{t}} \eta = \iota_{(\mathcal{F}L)^{-1}}(\iota_{X_{H} \eta})\). This implies that \(X^L_{t} = (\mathcal{F}L)^{-1}(X_{H,F})\), that is, \((\mathcal{F}L)_* X^L_{t} = X_{H,F}\).

On the other hand, observe that
\[
(\mathcal{F}L)^*(\iota_{Z_{F^L} \omega_Q}) = (\mathcal{F}L)^*(F - \mathcal{R}(F)\eta) = (\mathcal{F}L)^*(F) - (\mathcal{F}L)^*(\mathcal{R}(F)\eta) = F^L - \mathcal{R}(F^L)\eta = \iota_{Z_{F} \omega_L}.
\]

which implies that
\[
\iota_{Z_{F^L} \omega_L} = (\mathcal{F}L)^*(\iota_{Z_{F} \omega_L}) = \iota_{(\mathcal{F}L)^{-1}}(\mathcal{R}(F^L)\eta) = \iota_{(\mathcal{F}L)^{-1}}(\mathcal{R}(F^L)\eta).
\]

Similarly, one can see that \(\iota_{Z_{F} \eta} = \iota_{(\mathcal{F}L)^{-1}}(\mathcal{R}(F^L)\eta)\). Hence, \(Z_{F^L} = (\mathcal{F}L)^{-1}(Z_{F})\), that is, \((\mathcal{F}L)_* Z_{F^L} = Z_{F}\).

**Definition 27** A simple hybrid time-dependent forced Lagrangian system is described by the tuple \(\mathcal{L}_F = (\mathbb{R} \times TQ, X^L_{t}, S^L_{t}, \Delta^L_{t})\), where \(Q\) is a differentiable manifold, \(X^L_{t}\) is the forced evolution vector field associated with the time-dependent forced Lagrangian system \((L, F^L)\), \(S^L_{t}\) is an embedded submanifold of \(\mathbb{R} \times TQ\) with co-dimension one, the switching surface, and \(\Delta^L_{t} : S^L_{t} \to \mathbb{R} \times TQ\) is a smooth embedding, the **impact map**.

Analogously, one can introduce the notion of simple hybrid time-dependent forced Hamiltonian system \(\mathcal{H}_F = (\mathbb{R} \times T^*Q, X^H_{t}, S^H_{t}, \Delta^H_{t})\), where \(X^H_{t}\) is the forced evolution vector field associated with the time-dependent forced Hamiltonian system \((H, F)\). The relation between both hybrid flows is given by the following result, based on the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case can achieved via the fiber derivative \(\mathcal{F}H\).

**Proposition 28** If \(\chi_{\mathcal{H}_F} = (\Lambda, \beta, \epsilon)\) is a hybrid flow for \(\mathcal{H}_F\), \(S^H_{t}\) is the forced evolution vector field associated with the time-dependent forced Hamiltonian system \((H, F)\), and \(\Delta^H_{t}\) is defined in such a way that \(\mathcal{F}H \circ \Delta^H_{t} = \Delta^L_{t} \circ \mathcal{F}H\), then \(\chi_{\mathcal{L}_F} = (\Lambda, \beta, (\mathcal{F}H)(\epsilon))\) with \((\mathcal{F}H)(\epsilon) = \{(\mathcal{F}H)(\epsilon)\}_{\epsilon \in \Lambda}\).

**Proof.** The proof follows straightforwardly from the proof of Proposition 11.

Let \(\mathcal{L}_F = (\mathbb{R} \times TQ, X^L_{t}, S^L_{t}, \Delta^L_{t})\) be a simple hybrid time-dependent forced Lagrangian system and let \(\psi : G \to Q\) be a free and proper Lie group action with \(\psi^{\mathbb{R} \times TQ}\) denoting its natural lift, namely, \(G\) acts on \(\mathbb{R} \times TQ\) by the identity and on \(TQ\) by \(\psi_{TQ}\). The action \(\psi^{\mathbb{R} \times TQ}\) enjoys the following properties [1], [15]:

- **It is a cosmectic action** meaning that \((\psi^{\mathbb{R} \times TQ})^* \omega = \omega\) and \((\psi^{\mathbb{R} \times TQ})^* \eta = \eta\) for every \(g \in G\)
- **It admits an Ad*-equivariant momentum map** \(\tilde{J} : \mathbb{R} \times T^*Q \to \mathfrak{g}^*\) given by \((\tilde{J}(t, q, p, \xi)) = (p, \xi q)\), for each \(\xi \in \mathfrak{g}\), where \(\xi q(g) = d(\psi_{\exp(t \xi \mathfrak{g})})/dt\) is the infinitesimal generator of \(\xi \in \mathfrak{g}\).

Likewise, \(\psi^{\mathbb{R} \times TQ}\) denotes the natural lift action of \(G\) on \(\mathbb{R} \times TQ\), i.e., \(G\) acts on \(\mathbb{R} \times TQ\) by the identity and on \(TQ\) by \(\psi_{TQ}\).

Let \(X\) be a vector field on \(Q\). The complete lift of \(X\) at \((t, v_q) \in \mathbb{R} \times T^*Q\) is given by \(X^c_{t,v_q} = (0_t, X^c(v_q))\), where
\( \tilde{X}^c(v_q) \) denotes the complete lift of \( X \) to \( T^*Q \) at \( v_q \). Locally, \( X^c = X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^j}{\partial q^i} \). Analogously, one can define the complete lift of \( X \) to \( \mathbb{R} \times TQ \), locally, \( X^c = X^i \frac{\partial}{\partial q^i} + \xi^i \frac{\partial X^i}{\partial \xi^i} \).

As in the autonomous case, a function \( f \) on \( \mathbb{R} \times T^*Q \) is called a constant of the motion (or a conserved quantity) for \((H,F)\) if it takes a constant value along the trajectories of the system or, in other words, \( X_{H,F} \cdot (f) = 0 \).

For each \( \xi \in \mathfrak{g} \) and \( \alpha_q \in T^*Q \), consider the function \( \tilde{J}^\xi : \mathbb{R} \times T^*Q \to \mathbb{R} \) given by \( \tilde{J}^\xi(t,\alpha_q) = (J(t,\alpha_q), \xi) \).

\[ \Delta^H \circ \psi^g_{R \times T^*Q} |_{S^0_H} = \psi^g_{R \times T^*Q} \circ \Delta^H. \]

From the Cosymplectic Reduction Theorem [2] (see also [17]), we can obtain the non-autonomous analogue of Theorem 20.

**Theorem 29** Let \( \mathcal{H}^F = (\mathbb{R} \times T^*Q, X_{H,F}, S^0_H, \Delta^H) \) be a time-dependent simple hybrid forced Hamiltonian system. Let \( \psi : G \times Q \to Q \) be a hybrid action of a connected Lie group \( G \) on \( Q \). Suppose that \( H \) and \( F \) are \( G \)-invariant and assume that \( J_F \) is a generalized hybrid momentum map. Consider a sequence \( \{ \mu_i \} \) of regular values of \( J_F \), such that

\[ \Delta^H \circ \psi^g_{R \times T^*Q} |_{S^0_H} = \psi^g_{R \times T^*Q} \circ \Delta^H. \]

Let \( (G_F)_{\mu_i} = (G_F)_{\mu_i} \) be the isotropy subgroup in \( \mu_i \) under the co-adjoint action. Then,

(i) \( J^H_F^{-1}(\mu_i) \) is a submanifold of \( \mathbb{R} \times T^*Q \) and \( X_{H,F,*} \) is tangent to it.

(ii) The reduced space \( \mathcal{M}_{\mu_i} := J^H_F^{-1}(\mu_i)/(G_F)_{\mu_i} \) is a cosymplectic manifold, whose cosymplectic structure \((\omega_{\mu_i}, \eta_{\mu_i})\) is uniquely determined by \( \alpha^*_{\mu_i} \omega_{\mu_i} = \eta_{\mu_i} \) and \( \eta_{\mu_i} = \iota_{\mu_i}^* \eta \), where \( \alpha_{\mu_i} : J^H_F^{-1}(\mu_i) \to \mathcal{M}_{\mu_i} \) and \( \iota_{\mu_i} : J^H_F^{-1}(\mu_i) \to \mathbb{R} \times T^*Q \) denote the canonical projection and the canonical inclusion, respectively. In addition, the Reeb vector field \( \mathbb{R} \) projects onto \( \mathbb{R}_{\mu_i} \), the Reeb vector field defined by \( \omega_{\mu_i} \) and \( \eta_{\mu_i} \).

(iii) \( (H,F) \) induces a reduced time-dependent forced Hamiltonian system \((\mathcal{H}_{\mu_i}, F_{\mu_i})\) on \( \mathcal{M}_{\mu_i} \), given by \( H_{\mu_i} \circ \pi_{\mu_i} = H \circ \iota_{\mu_i} \) and \( \pi_{\mu_i} F_{\mu_i} = \iota_{\mu_i}^* F \). Moreover, the forced evolution vector field \( X_{H,F_{\mu_i}} \) projects onto \( X_{H_{\mu_i}, F_{\mu_i}} \).

(iv) \( J^H_F \mu_i \subset S^0_H \) reduces to a submanifold of the reduced space \((S^0_H)_{\mu_i}, J^F_{-1}(\mu_i)/(G_F)_{\mu_i} \).

Therefore, after the reduction procedure, we get a sequence of reduced time-dependent simple hybrid forced Hamiltonian systems \((\mathcal{H}_{\mu_i}^F)\), where \( \mathcal{H}_{\mu_i}^F = (J^H_F^{-1}(\mu_i)/(G_F)_{\mu_i}, \mathcal{M}_{\mu_i}, F_{\mu_i}, S^0_{\mu_i}, \Delta^H_{\mu_i}) \).

Proof: The proof follows straightforwardly from the Proof of Proposition 20. □

By a hybrid action on the simple hybrid system \( \mathcal{L}^F_L = (TQ, X_{L,F_L}, S^0_{L_L}, \Delta^L_L) \) we mean a Lie group action \( \psi : G \times Q \to Q \) such that

- \( L \) is invariant under \( \psi^{R \times TQ} \), i.e., \( L \circ \psi^{R \times TQ} = L \).
- \( \psi^{R \times TQ} \) restricts to an action of \( G \) on \( S^0_{L_L} \).
- \( \Delta^L_L \) is equivariant with respect to the previous action, namely

\[ \Delta^L_L \circ \psi^g_{R \times T^*Q} |_{S^0_L} = \psi^g_{R \times T^*Q} \circ \Delta^L_L. \]

As in the autonomous case, the reduction picture in the Lagrangian side can now be obtained from the Hamiltonian one by adapting the scheme developed in 4 to the cosymplectic setting. Suppose that \( \mathcal{L}^F_L \) is equipped with a hybrid action \( \psi \) and that \( (L,F^L) \) is \( G_{F_L} \)-regular. Since \( L \) is invariant and hyperregular, the Legendre transformation \( F^L \) is a diffeomorphism such that:

- It is equivariant with respect to \( \psi^{R \times TQ} \) and \( \psi^{R \times T^*Q} \).
- It preserves the level sets of the momentum map, that is, \( \mathcal{F} L((J^F_L)^{-1}(\mu_i)) = J^F_L^{-1}(\mu_i) \).
- It relates both cosymplectic structures, that is, \( \mathcal{F} L^* \omega_Q = \omega_L \) and \( \mathcal{F} L^* \eta = \eta \), that is, \( \mathcal{F} L \) is a cosymplectomorphism.

The equivalence of \( \mathcal{F} L \) implies that \( \psi^{R \times TQ} \) admits an Ad*-equivariant momentum map \( J_L : \mathbb{R} \times TQ \to \mathfrak{g}^* \) given by \( J_L = \tilde{J} \circ \mathcal{F} L \).
It follows that the map \( \mathbb{F}L \) reduces to a cosymplectomorphism \( (\mathbb{F}L)_{\text{red}} \) between the reduced spaces. Therefore we get the following commutative diagram

\[
\begin{array}{ccc}
(\mathbb{R} \times TQ, S^1_L, \Delta_L) & \xrightarrow{\mathbb{F}L} & (\mathbb{R} \times T^*Q, \theta, \Delta_{\text{red}}) \\
\text{Red} & & \text{Red} \\
(\mathbb{R} \times TQ, S^1_L, \Delta_L) & \xrightarrow{\mathbb{F}L_{\text{red}}} & (\mathbb{R} \times T^*Q, \theta, \Delta_{\text{red}}) \\
\end{array}
\]

where we have used the notation \( \mathbb{F}L_{\text{red}} := (\tilde{J}_L)^{-1}(\mu_i)/\mu_0 \) and \( \overline{\mathbb{F}L_{\text{red}}} := J^R(\mu_i)/(G_F)^{\mu_0} \).

By assuming \( G_F \)-regularity we obtain the identification

\[
(\tilde{J}_L)^{-1}(\mu_i)/(G_F)^{\mu_0} \simeq \mathbb{R} \times (T(Q/G) \times Q/G Q)/(G_F)^{\mu_0}.
\]

It is possible to interpret the reduced dynamics on this space as being the Lagrangian dynamics of some regular time-dependent Lagrangian subjected to a time-dependent gyroscopic force (arising from the magnetic term) if one works in the class of magnetic Lagrangians \([32]\), which in the present situation should be extended to include time-dependent Lagrangians and external forces. The Routhian is defined as (the reduction of) the \( (G_F)^{\mu_0} \)-invariant function

\[
\mathcal{R}^{\mu_0}_{\tilde{J}_L} = L - A_{\mu_i}
\]

restricted to \( (\tilde{J}_L)^{-1}(\mu_i) \). Then, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R} \times TQ & \xrightarrow{\mathbb{F}L} & \mathbb{R} \times T^*Q \\
\text{Red} & & \text{Red} \\
\mathbb{R} \times (T(Q/G) \times Q/G Q)/(G_F)^{\mu_0} & \xrightarrow{\mathbb{F}L_{\text{red}}} & \mathbb{R} \times (T^*(Q/G) \times Q/G Q)/(G_F)^{\mu_0} \\
\end{array}
\]

Remark 30 Assume we work with \( Q = S^1 \times M \), where \( M \) is called the shape space and the action is \( (\theta, x) \mapsto (\theta + \alpha, x) \). The forced Lagrangian system has a cyclic coordinate \( \theta \), i.e., \( L \) is a function of the form \( L(t, \theta, x, \dot{x}) \), and \( F \) is of the form \( F(t, \theta, x, \dot{x}) = F_\theta(t, \theta, x, \dot{x})dx \). The conservation of the momentum map \( (\tilde{J}_L)_F = \mu_i \) reads \( \partial L/\partial \theta = \mu_i \), and one can use this relation to express \( \theta \) as a function of the remaining—non-cyclic—coordinates and their velocities, and the prescribed regular value of the momentum map \( \mu_i \), i.e., \( \theta = \theta(t, x, \dot{x}, \mu_i) \). Note that this is the stage at which the \( G_F \)-regularity of \( L \) and \( F^L \) is used: it guarantees that \( \theta \) can be worked out in terms of \( x, \dot{x} \) and \( \mu_i \). If one chooses the canonical flat connection on \( Q \to Q/S^1 = M \), then the Routhian can be computed as

\[
\mathcal{R}^{\mu_0}_{\tilde{J}_L}(t, x, \dot{x}) = \left[ L(t, \theta(t, x, \dot{x}), \dot{x}) - \mu_i \dot{\theta} \right]_{\theta = \theta(t, x, \dot{x}, \mu_i)}
\]

and

\[
F^{\mu_0}_{\tilde{J}_L}(t, x, \dot{x}) = F^L(t, \theta(t, x, \dot{x}), \dot{x}) |_{\theta = \theta(t, x, \dot{x}, \mu_i)}
\]

where the notation means that we have everywhere expressed \( \theta \) as a function of \( (t, x, \dot{x}, \mu_i) \).

As in the non-autonomous case, collisions with the switching surface will, in general, modify the value of the momentum map (non-elastic case). Therefore, if \( \bar{\alpha} := \{ \bar{I}_i \}_{i \in \Lambda} \) is the hybrid interval (see Definition 9), the Routhian has to be defined in each \( \bar{I}_i \) taking into account the value of the momentum \( \mu_i \) after the collision at time \( \tau_k \). Note that this also has an influence in the way the impact map \( \Delta^\mu_{\bar{I}} \) and the switching \( S \) are reduced. Let us denote: (1) \( \mu_i \) the momentum of the system in \( \bar{I}_i = [\tau_k, \tau_{k+1}] \), (2) \( \Delta^\mu_{\bar{I}} \), the reduction of \( \Delta^\mu_L \big|_{\Delta^\mu_{\bar{I}}(\mu_i)} \), and (3) \( (S^\mu_{\bar{I}})^i \), the reduction of \( J^L \big|_{S^\mu_{\bar{I}}}(\mu_i) \). There is a sequence of reduced simple hybrid time-dependent Lagrangian systems (“Coll.” stands for collision and “Red.” stands for reduction):

\[
[\tau_0, \tau_1] \xrightarrow{\text{Red}} (\mathbb{R} \times T(Q/S^1), L_{\mu_0}, (S^\mu_{\bar{I}})^i_{\mu_0}, (\Delta^\mu_{\bar{I}})^i_{\mu_0}) \\
\text{coll.} \downarrow \\
[\tau_1, \tau_2] \xrightarrow{\text{Red}} (\mathbb{R} \times T(Q/S^1), L_{\mu_1}, (S^\mu_{\bar{I}})^i_{\mu_1}, (\Delta^\mu_{\bar{I}})^i_{\mu_1}) \\
\text{coll.} \downarrow \\
(\ldots) \xrightarrow{\text{Red}} (\ldots)
\]

As in the symplectic case, the fact that the momentum will, in general, change with the collisions makes the reconstruction procedure more challenging. If one wishes, as usual, to use a reduced solution to reconstruct the original dynamics, one needs to compute the reduced hybrid data after each collision. This means that once the reduced solution has been obtained between two collision events, say at \( t = \tau_n \) and \( t = \tau_{n+1} \), one has to reconstruct this solution to obtain the new momentum after the collision at \( \tau_{n+1} \) and use this new momentum to build a new reduced hybrid system whose solution should be obtained until the next collision event at \( \tau_{n+2} \), and so on (see section 4.1 for details).

Example 31 (Billiard with dissipation and moving walls)

Consider a particle of mass \( m \) in the plane which is free to move inside the surface defined by circle whose radius varies in time according to a given function \( f(t) \), i.e., \( x^2 + y^2 = f(t) \). The surface of the “billiard” is assumed to be rough in such a way that the friction is non-linear on the velocities.

The Lagrangian function \( L : \mathbb{R} \times T\mathbb{R}^2 \to \mathbb{R} \) is given by

\[
L(t, x, y, \dot{x}, \dot{y}) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2)
\]

and \( F^L(t, x, y, \dot{x}, \dot{y}) = F^L_x dx + F^L_y dy \) is an external force given by \( F^L_x = -2ce^{-\frac{t}{t_0}}(\ddot{x} + \dot{y}^2) \), \( F^L_y = 2ce^{-\frac{t}{t_0}}(\ddot{y} + \dot{x}^2) \).
\[ \dot{x}^2, \text{ for a constant } c > 0. \] The equations of motion for the particle off the boundary are then
\[ c\ddot{x} + m\dddot{x} = -2c(\dot{y}\dot{x}^2 - \dot{x}\dot{y}), \quad c\ddot{y} + mj = 2c(\dot{x}\dot{y}^2 - \dot{y}\dot{x}). \]

The guard is the subset of \( \mathbb{R} \times T^2 \simeq \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) given by
\[ S^t_L = (\mathbb{R} \times T^2) \cap \left\{ x^2 + y^2 = f(t), (\dot{x}, \dot{y}) \cdot (x, y) \geq 0 \right\}. \]

This set \( S^t_L \) describes the situation in which the particle hits the moving boundary while heading “outwards” the billiard. For simplicity in the definition of the switching surface, we assume that \( f(t) \) is increasing: this guarantees the particle only hits the boundary when the boundary is also moving outwards. Under the assumption of an elastic collision, the impact map \((t, x, y, \dot{x}^-, \dot{y}^-) \rightarrow (t, x, y, \dot{x}^+, \dot{y}^+)\) is given by:
\[ \dot{x}^+ = \dot{x}^+ + \frac{\dot{f}(t) - 2(x\dot{x}^- + y\dot{y}^-)}{f(t)} x, \]
\[ \dot{y}^+ = \dot{y}^- + \frac{\dot{f}(t) - 2(x\dot{x}^- + y\dot{y}^-)}{f(t)} y. \]

By introducing polar coordinates \( L \) and \( F^L \) become
\[ L(t, \theta, r, \dot{\theta}, \dot{r}) = \frac{m}{2}e^{\mu t}(r^2 + r^2\dot{\theta}^2), \]
\[ F^L(t, \theta, r, \dot{\theta}, \dot{r}) = 2ce^{\mu t}\dot{\theta}dr, \]
respectively. \( L \) is hyperregular and \( L \) and \( F^L \) are independent of \( \theta \). The forced Euler-Lagrange equations (in polar coordinates) are
\[ \ddot{r} = -\frac{2c\dot{r}}{m} \dot{\theta} + c\dot{r} \dot{\theta} - \frac{c}{m} \dot{\theta}. \]

The impact map \( \Delta^t_L \), in polar coordinates, takes the form (observe that \( 2(x\dot{x}^- + y\dot{y}^-) \) is nothing but \( 2r\dot{r}^- \)):
\[ (\dot{r}^+)^2 = (\dot{r}^-)^2 + + \frac{r}{\dot{f}(t)}(\dot{f}(t) - 2r\dot{r}^-) \left( 2\dot{r}^- + \frac{(\dot{f}(t) - 2r\dot{r}^-)r}{\dot{f}(t)} \right), \]
\[ \dot{\theta}^+ = \dot{\theta}^- \].

Note that the particle bounces on the boundary after the collision, so the “minus” square root is considered in \( \dot{r}^+ \).

Note that the momentum map \( (\bar{J}_L) F \) for \( L \)
\[ (\bar{J}_L) F(t, \dot{r}, \dot{\theta}) = m\mu e^{\mu t}r\dot{\theta} \]
is preserved. Hence, by considering \( \mu = m\mu e^{\mu t}r^2\dot{\theta} \) (i.e., \( \dot{\theta} = \frac{\mu}{m\mu} e^{\mu t} \)) the time-dependent Routhian and the reduced time-dependent force takes the form
\[ R^t_F(t, r, \dot{r}) = \frac{m}{2}e^{\mu t}r^2 - \frac{\mu^2}{2mr^2}e^{\mu t}, \quad F^t_L = 2cr\mu \frac{dr}{m}, \]
so, the time-dependent forced reduced Euler-Lagrange equations for the Routhian \( R^t_F \) are given by
\[ \ddot{r} = \frac{\mu^2}{m^2r^3}e^{\mu t} - \frac{2cr\mu}{m^2}e^{\mu t} - \frac{\mu\dot{c}}{m}. \]

As a matter of fact, \((\bar{J}_L)F\) is a hybrid momentum map. The reduced impact map is given by (7) for \( \dot{r}^+ \) (note that the expression drops to the quotient since it only involves \( r, \dot{r} \) and \( f(t) \)). The reduced impact surface is \((S^t_L)\mu = \{r^2 = f(t), \dot{r} > 0\} \). Hence, we have the following simple hybrid time-dependent forced Lagrangian system \( \mathcal{L}^t_F = (Q_{\text{red}}, R^t_F, F^t_L, (S^t_L)\mu, (\Delta^t_L)\mu) \), with \( Q_{\text{red}} \simeq \mathbb{R}^+ \) parametrized by the coordinate \( r \).

![Figure 1. Simulation for \( c = 0.005 \). The figure in the left corresponds with the reduced trajectory while the figure to the right corresponds with the reconstructed solution](image)

Figures 1 and 2 show numerical results using PYTHON for two different values of the dissipation parameter \( c \). The remaining parameters are the same for both simulations: \( m = 1, r(0) = 0.5590, \dot{r}(0) = 2.8621, \theta(0) = 1.1071 \) (rad), \( \dot{\theta}(0) = -3.0400 \) (rad/s), with \( f(t) \) given by
\[ f(t) = 2 - \exp(t/10). \]

The reduced dynamics corresponding to \( R^t_F \) is solved numerically (dashed black line) and used to integrate (numerically) the reconstruction equation
\[ \dot{\theta} = \exp\left(-\frac{c}{m}\right) \mu \frac{\dot{c}}{mr^2}, \]
with \( \mu \) determined from the initial conditions.

**Example 32 (Rolling disk with dissipation hitting a moving wall)** Consider a homogeneous circular disk of radius \( R \) and mass \( m \) moving in the vertical plane \( xOy \) (see Example 25).

Suppose that there are two rough walls at the axis \( y = 0 \) and at \( y = f(t) \), where \( f(t) \geq h = \alpha R \) for some constant...
In polar coordinates, we have

\[
\left( \dot{x}^-, \dot{y}^-, \dot{\vartheta}^- \right) \mapsto \left( \frac{R^2 \dot{x}^- + k^2 \dot{\vartheta}^-}{k^2 + R^2}, -\dot{y}^- \frac{R \dot{x}^- + k^2 \dot{\vartheta}^-}{k^2 + R^2} \right),
\]

where the switching surface is given by

\[
S = \{(x, y, \vartheta, \dot{x}, \dot{y})| y = R \ or \ y = f(t)-R, \ and \ \dot{x} = R \dot{\vartheta}\}.
\]

It is worth noting that \( S \) is time-dependent, even though in this case \( \Delta \) is time-independent.

One can check that \((\tilde{J}_L)_F (r, \dot{r}, \theta, \dot{\theta}, \dot{\vartheta}) = (mv^2 \hat{\vartheta}, mk^2 \hat{\vartheta})\) is a generalized hybrid momentum map but not a hybrid momentum map, i.e., \((\tilde{J}_L)_F (q_1, \dot{q}_1^-) = (\tilde{J}_L)_F (q_2, \dot{q}_2^-)\) implies that \((\tilde{J}_L)_F (q_1, \dot{q}_1^+) = (\tilde{J}_L)_F (q_2, \dot{q}_2^+)\) but \((\tilde{J}_L)_F (q_1, \dot{q}_1^+) \neq (\tilde{J}_L)_F (q_1, \dot{q}_1^-)\).

In polar coordinates, we have

\[
\dot{\vartheta}^+ = -\dot{\vartheta}^-,
\]

\[
\dot{r}^+ = (2 \cos^2 \theta - 1) \dot{r}^- - 2r \sin \theta \cos \theta \dot{\vartheta}^-, \quad \text{and}
\]

\[
\dot{\theta}^- = \frac{R \left( \cos \vartheta \dot{r}^- - \sin \vartheta \dot{\vartheta}^- \right) + k^2 \dot{\vartheta}^-}{k^2 + R^2}.
\]

The switching surface can be written as

\[
S = \left\{ (r, \theta, \vartheta, \dot{r}, \dot{\theta}) | r \sin \theta = R \ or \ r \sin \theta = f(t) - R, \right.
\]

\[
\left. \quad \text{and} \ r \cos \theta - r \dot{\vartheta} \sin \theta = R \dot{\vartheta} \right\}.
\]

Let \((\mu_1^-, \mu_2^-)\) and \((\mu_1^+, \mu_2^+)\) be the value of the momentum map before and after the impact, respectively. We can write

\[
\dot{\theta}^\pm = \mu_1^\pm / mr^2 \quad \text{and} \quad \dot{\vartheta}^\pm = \mu_2^\pm / mk^2,
\]

so the reduced switching map is

\[
\dot{r}^- \mapsto (2 \cos^2 \theta - 1) \dot{r}^- - 2r \sin \theta \cos \theta \frac{\mu_1^-}{mr^2},
\]

with the relations

\[
\mu_1^+ = -\mu_1^- \quad \text{and} \quad \mu_2^+ = \mu_2^-.
\]

The reduced switching surface can be written as

\[
S_{(\mu_1, \mu_2)} = \{(r, \dot{r}) | r \sin \varphi = R \ or \ r \sin \varphi = f(t) - R, \right.
\]

\[
\left. \quad \text{and} \ r \cos \varphi - r \frac{\mu_1}{mr^2} \sin \varphi = R \frac{\mu_2}{mk^2}, \quad \right\} \text{for some } \varphi \in [0, 2\pi).
\]

6 Conclusions and future work

The celebrated symplectic reduction of (conservative) mechanical systems with symmetries, due to Marsden and Weinstein \([1,40,42]\), was recently extended for forced autonomous Lagrangian \([15,37]\) as well as Hamiltonian \([16]\) systems. In this paper we have gone a step further and considered simple hybrid forced mechanical systems, both autonomous and non-autonomous, with a generalized hybrid momentum map. The main difference in the reduction and reconstruction of these systems with respect to continuous systems, or hybrid systems with a hybrid momentum map, is that, since nonelastic collisions with the switching surface will modify the value of the momentum map (see Examples 25 and 32, respectively), we have a sequence of reduced simple hybrid forced reduced Hamiltonian systems. In particular, we have considered \( S^1 \)-invariant hybrid forced autonomous and non-autonomous mechanical systems.

We plan to extend our results for more general settings. We would like to obtain a reduction procedure for dissipative hybrid systems in the framework of contact geometry. Moreover, we could consider the reduction of systems with both continuous and discrete time dynamics which are not simple hybrid systems. For instance, we could consider \( D \) of co-dimension different from 1, or a system having several domains and switching surfaces that separate them \([13, 24, 23, 8]\). Furthermore, we intend to develop a Hamilton-Jacobi theory for hybrid systems with dissipation arising from impulsive constraints.

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Let now $(H, F)$ be a forced Hamiltonian system on $T^*Q$. A function $f$ on $T^*Q$ is called a constant of the motion (or a conserved quantity) for $(H, F)$ if it takes a constant value along the trajectories of the system or, in other words, $X_{H,F}(f) = 0$. A vector field $\hat{X}$ on $T^*Q$ is called a symmetry of the forced Hamiltonian if $\hat{X}(H) + F(\hat{X}) = 0$ and $\mathcal{L}_{\hat{X}}\theta_L = df$ for some function $f$ on $T^*Q$. If $\hat{X}$ is a symmetry of the forced Hamiltonian, then $f - \theta_Q(\hat{X})$ is a constant of the motion.

In addition, the Hamiltonian and Lagrangian symmetries are related as follows. Suppose that $(L, F^L)$ is the Lagrangian counterpart of $(H, F)$, namely, $H = H_L + F^L - F$. Let $\check{X}$ be a vector field on $TQ$ and let $\hat{X}$ be a $\mathcal{F}F$-related vector field on $T^*Q$, i.e., $\mathcal{F}F \circ \hat{X} = \check{X} \circ \mathcal{F}F$. Then, the following relations hold:

(i) $[\hat{X}, X_{H,F}] = 0$ if and only if $\hat{X}$ is a dynamical symmetry of $(L, F^L)$.

(ii) $\mathcal{L}_{\hat{X}}\theta_L = df$ if and only if $\mathcal{L}_{\hat{X}}\theta_L = (f \circ \mathcal{F}F)$.

(iii) $\mathcal{L}_{\hat{X}}\theta_Q = df$ if and only if $\mathcal{L}_{\hat{X}}\theta_L = df$ for some function $f$ on $TQ$.

Similarly, let $\check{X}$ be a vector field on $TQ$. Then, $\check{X}$ is called a dynamical symmetry if $[\check{X}, X_{H,F}] = 0$ and a Noether symmetry if $\mathcal{L}_{\check{X}}\theta_L = df$ for some function $f$ on $TQ$.

Moreover, the following relations between symmetries and constants of the motion hold:

(i) $X$ is a symmetry of the forced Lagrangian if and only if $X^\nu(L)$ is a constant of the motion.

(ii) If $X$ satisfies that $\mathcal{L}_X \theta_L = df$, then $X$ is a Noether symmetry if and only if $f - X^\nu(L)$ is a constant of the motion.

(iii) If $X$ is a Noether symmetry, it is also a Lie symmetry if and only if $t_X \cdot d\beta = 0$.

(iv) $X$ is a Lie symmetry if and only if $X^c$ is a dynamical symmetry.

(v) $X$ is a Noether symmetry if and only if $X^c$ is a Cartan symmetry.

(vi) If $\hat{X}$ satisfies that $\mathcal{L}_{\hat{X}}\theta_L = df$, then $\hat{X}$ is a Cartan symmetry if and only if $f - (S\check{X})(L)$ is a constant of the motion. Here $S$ is the vertical endomorphism.

(vii) If $\check{X}$ is a Cartan symmetry, it is also dynamical if and only if $t_{\check{X}}\cdot d\beta = 0$.

When the set of transformations that leave invariant a forced Lagrangian (or Hamiltonian) system form a Lie group (so the vector fields generating the infinitesimal symmetries close a Lie subalgebra), we can introduce a momentum map, which associates an independent constant of the motion to each of the generators of the Lie algebra (see Section 4). This, when the group action “behaves well”, allows to project the dynamics of the system to a reduced space of less dimensions. As a matter of fact, if we know that an (unforced) Hamiltonian system $H$ can be reduced by the action of a Lie group $G$, in order to reduce the forced Hamiltonian system $(H, F)$ it suffices to consider the Lie subgroup $G_F$ whose infinitesimal generators act as symmetries of the forced Hamiltonian (see Remark 17).