Simulation of POVMs and quantum instruments via state preparation algorithms

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In Ref. [Phys. Rev. A 100, 062317 (2019)], the authors reported an algorithm to implement, in a circuit-based quantum computer, a general quantum measurement (GQM) of a two-level quantum system, a qubit. Even though their algorithm seems right, its application involves the solution of an intricate non-linear system of equations in order to obtain the angles determining the quantum circuit to be implemented for the simulation. In this article, we identify and discuss a simple way to circumvent this issue and implement GQMs on any d-level quantum system through quantum state preparation algorithms. Using some examples for one qubit, one qutrit, and two qubits, we illustrate the easy of application of our protocol. Besides, we show how one can utilize our protocol for simulating quantum instruments, for which we also give an example. All our examples are verified experimentally using IBM’s quantum processors.

Keywords: General quantum measurement; POVM; Quantum instrument; Quantum computer; State preparation algorithm

One of the basic postulates of quantum mechanics says that the measurement of an observable, represented by an hermitian operator $A = \sum_{j=1}^{d_A} a_j \Pi_j^A$, is described by projection operators $\Pi_j^A$, i.e., $\Pi_j^A \Pi_k^A = \delta_{jk} \Pi_j^A$ and $\sum_{j=1}^{d_A} \Pi_j^A = I_A$, with $d_A$ being the dimension of the system Hilbert space $H_A$ and $I_A$ is the identity operator in $H_A$. For a quantum system $A$ prepared in the state $\rho_A$, the measurement outcome corresponding to $\Pi_j^A$ is obtained with probability $Pr(\Pi_j^A | \rho_A) = Tr(\Pi_j^A \rho_A \Pi_j^A)$ and the post-measurement state is $\Pi_j^A \rho_A \Pi_j^A / Tr(\Pi_j^A \rho_A \Pi_j^A)$. Eventually, researchers realized that more general quantum measurements (GQMs) can be defined. These measurements, also named positive operator-value measurements (POVMs), are described by a set of measurement operators $\{M_j\}$ in $H_A$ that satisfy the completeness relation $\sum_j M_j^\dagger M_j = I_A$. In this general setting, for a system prepared in the state $\rho_A$, the probability of obtaining the measurement result corresponding to $M_j$ is $Pr(M_j | \rho_A) = Tr(M_j \rho_A M_j^\dagger)$ and the post-measurement state is $M_j \rho_A M_j^\dagger / Tr(M_j \rho_A M_j^\dagger)$. The fact that $Pr(M_j | \rho_A) = Tr(E_j \rho_A) \geq 0$ with $E_j = M_j^\dagger M_j$ being positive-semidefinite operators motivates the name POVM. The completeness restriction ensures that $\sum_j Pr(M_j | \rho_A) = 1$ [1, 2].

POVMs provide advantages in several applications in Quantum Information Science (QIS), as for example in quantum state estimation [3], shadow quantum state tomography [4], discrimination of quantum states [5], randomness certification [6], acquisition of information from a quantum source [8], quantum key distribution [9], Bell inequalities [10], and device independent quantum information protocols [11]. So, experimentally implementing POVMs is of fundamental importance for QIS and considerable work has been done in this direction recently, as for example in Refs. [12–21].

Of particular interest to us here is Ref. [21], where the authors proposed a deterministic protocol to implement single-qubit POVMs on quantum computers. Even though their protocol seems correct, we realize that for applying it one first has to solve complicated non-linear systems of equations for obtaining the angles determining the quantum circuit to be used in the simulation. Then, motivated by their work, here we identify and discuss a simple way to implement POVMs on any $d_A$-level quantum system through quantum state preparation (QSP) algorithms [23–30]. Using some examples for $d_A = 2$, $d_A = 3$, and $d_A = 4$ we illustrate the simplicity and convenience for application of this new method. In addition to that, we apply our protocol for the simulation of quantum instruments [2].

A POVM with elements $\{M_j\}$ can be implemented coherently through an isometric transformation [2],

$$V_{AB}|k\rangle_A \otimes |0\rangle_B := \sum_j (M_j |k\rangle_A) \otimes |j - 1\rangle_B,$$

followed by a selective projective measurement in the basis $\{j\}_B$ of the auxiliary system $B$, plus discarding of the system $B$. Above $|j\rangle_S$ is the computational basis for the system $S = A, B$. This procedure produces the same
statistics and post-measurement states of the system $A$ as does the POVM $\{M_j\}$, that is to say, $\Pr((j) = \hat{\rho}_{AB}) = \Pr(M_j|\rho_{AB}) = V_{AB}(\rho_{A} \otimes |0\rangle_B \langle 0|) V_{AB}^\dagger$ and $(\mathbb{1}_A \otimes |j\rangle_B \langle j|) \hat{\rho}_{AB}(\mathbb{1}_A \otimes |j\rangle_B \langle j|)/\text{Tr}(\mathbb{1}_A \otimes |j\rangle_B \langle j|) = M_j \rho_{A} M_j^\dagger/\text{Tr}(M_j \rho_{A} M_j^\dagger)$, with $\rho_{AB}$ being the pre-measurement state of system $A$. So, attempts to implement POVMs experimentally usually start from Eq. (1).

The protocol given in Ref. [21] follows this path. The authors said that the quantum circuit shown in Fig. 1 prepares the state

$$|\Psi\rangle = \sum_{j=1}^{n-1} (M_j|\psi_0\rangle) \otimes |o_1^{(j)}\rangle + (M_n|\psi_0\rangle) \otimes |o_2^{(n-1)}\rangle,$$

with $|\psi_0\rangle$ being the system $A$ pre-measurement state and $\{o_1^{(j)}\}, \{o_2^{(n-1)}\}$ are orthonormal states of the auxiliary system $B$, thus implementing a one-qubit POVM with an arbitrary number $n$ of elements:

$$M_j = \begin{cases} V_1^{(1)} D_1^{(1)} U, \text{ para } j = 1, \\ V_1^{(j)} D_1^{(j)} \prod_{k=1}^{j-1} V_2^{(k)} D_2^{(k)} U, \text{ para } 1 < j < n, \\ \prod_{k=1}^{j-1} V_2^{(k)} D_2^{(k)} U, \text{ para } j = n. \end{cases}$$

In these equations, $U$ and $V_j$ are general one-qubit unitaries and $D_1^{(k)} = \cos \theta_1^{(k)} |0\rangle_0 \langle 0| + \cos \theta_2^{(k)} |1\rangle_0 \langle 1|$ and $D_2^{(k)} = \sin \theta_1^{(k)} |0\rangle_0 \langle 0| + \sin \theta_2^{(k)} |1\rangle_0 \langle 1|$ are positive operators if $\theta_1^{(k)}, \theta_2^{(k)} \in [0, \pi/2]$.

As a general one-qubit unitary transformation can be recast in terms of four angles as [1]

$$\begin{bmatrix} e^{i(\alpha - \beta)/2} \cos \frac{\delta}{2} \\ -e^{i(\alpha + \beta)/2} \sin \frac{\delta}{2}
\end{bmatrix}, \begin{bmatrix} e^{-i(\alpha - \beta)/2} \cos \frac{\delta}{2} \\ e^{i(\alpha + \beta)/2} \sin \frac{\delta}{2} \end{bmatrix},$$

we see that, given the matrices for the POVM elements in the left hand side of Eq. (2), the implementation of the algorithm of Ref. [21] involves the solution of an intricate system of nonlinear equations for the angles appearing on the right hand side of Eq. (3). Perhaps this complication is related to the wrong examples presented in Ref. [21], for which the measurement operators do not satisfy the completeness restriction. It is worthwhile mentioning also that the protocol of Ref. [21], if implemented exactly as in the quantum circuit of Fig. 1, requires $O(n)$ auxiliary qubits. For diminishing this number to $O(\log_2 n)$, one has to make some modifications/additions to this quantum circuit, as exemplified in Fig. 2 for $n = 4$.

Quantum state preparation (QSP) algorithms are used as subroutines for performing many tasks [23–30], as for example for implementing the general quantum Fourier transform [23]. Motivated by the issues just discussed about the POVM simulation algorithm of Ref. [21], here we present a simple protocol that implements POVMs on any discrete quantum system $A$ associated with a Hilbert space $\mathcal{H}_A$ through QSP algorithms. For the dimension $d_A$ of the system $A$ on which the POVM is to be implemented and any number of elements of the POVM, if the measurement operators $M_j$ are known, in principle it is possible to calculate the right hand side of Eq. (1):

$$|\Psi\rangle_{AB} = \sum_j (M_j|k\rangle_A) \otimes |j - 1\rangle_B.$$  

Once obtained this vector, we can use QSP algorithms to prepare it. Afterwards, a projective measurement on the basis $\{|j\rangle_B\}$ is done. Running several times this procedure, we can extract the probabilities. It is worth mentioning that, since the projective measurements are done on the system $B$, by applying post-selection we can use quantum state tomography to obtain the post-measurement state of the system $A$. Therefore, our protocol can be summarized as follows:

1. Given $\{M_j\}$, obtain $|\Psi\rangle_{AB}$ of Eq. (4).

2. Implement $|\Psi\rangle_{AB}$ using algorithms for quantum state preparation.

3. Make projective measurements on system $B$ and extract the measurement statistics.

4. If needed, implement quantum state tomography to obtain the system $A$ post-measurement state, with post-selection of the measurement results on system $B$.

This protocol works for any dimension of the system $A$. The minimum dimension of the auxiliary system $B$ is equal to the number of POVM elements, independently of the dimension of the system $A$, i.e., our protocol uses $O(\log_2 n)$ auxiliary qubits. Of course, for implementing our protocol on quantum computers based on qubits, it is necessary to choose how to codify the qudit states in terms of qubit states.

In the sequence we present examples of applications of our protocol. Let us start by considering a one-qubit POVM with two elements:

$$M_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} |0\rangle \langle 0| + \sqrt{3}|1\rangle \langle 0| + 2|1\rangle \langle 1| \end{bmatrix},$$

$$M_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} |0\rangle \langle 0| - \sqrt{3}|1\rangle \langle 0| + 2|1\rangle \langle 1| \end{bmatrix}.$$  

We set the pre-measurement state of system $A$ to $|0\rangle_A$. So the POVM probabilities are given by

$$\Pr(M_1|0\rangle = (0|M_1^\dagger M_1|0\rangle = 1/2,$$

$$\Pr(M_2|0\rangle = (0|M_2^\dagger M_2|0\rangle = 1/2.$$ 

In this case we use a qubit as the auxiliary system $B$. For implementing our protocol, we have to prepare the state

$$|\Psi_{AB}\rangle = 2^{-3/2} \left(|00\rangle + \sqrt{3}|10\rangle + |01\rangle - \sqrt{3}|11\rangle \right).$$

Here we use the algorithm of Ref. [24] for QSP. This algorithm was already implemented in Qiskit [32]. After state
Figure 1: Adapted from the quantum circuit reported in Ref. [21] to simulate general one-qubit POVMs. $U$ and $V^j$ are general one-qubit gates and $\theta^j$ represents the $R_y(\theta^j)$ gate. We also used $N = \log_2 n$.

Figure 2: Adaptation of the quantum circuit of Ref. [21], shown in Fig. 1, for simulating a four-elements one-qubit POVM.

preparation, a projective measurement is performed in the basis $\{|0\rangle_B, |1\rangle_B\}$. For performing the experiments, we used the IBMQ [31] quantum chip *ibmq_belem*. The simulation and experimental results for this first example are shown in Fig. 3.

As a second example, let us consider a one-qubit three-element POVM with measurement elements associated with the sequence of states in the $xz$ plane of the Bloch sphere separated by $2\pi/3$ radians:

\[ M_1 = \sqrt{\frac{2}{3}} |0\rangle \langle 0|, \]

\[ M_2 = \sqrt{\frac{2}{3}} |\psi(2\pi/3, 0)\rangle \langle \psi(2\pi/3, 0)|, \]

\[ M_3 = \sqrt{\frac{2}{3}} |\psi(4\pi/3, 0)\rangle \langle \psi(4\pi/3, 0)|, \]

where we used the qubits $b$ and $c$ to encode the states of the qutrit $B$. The probabilities, given that the qubit is prepared in the $|0\rangle_A$, are

\[ P_r(M_1|0) = \langle 0|M_1^\dagger M_1|0\rangle = 2/3, \]

\[ P_r(M_2|0) = \langle 0|M_2^\dagger M_2|0\rangle = 1/6. \]

\[ P_r(M_3|0) = \langle 0|M_3^\dagger M_3|0\rangle = 1/6. \]

After state preparation, a projective measurement is performed on the basis $\{|00\rangle_{bc}, |01\rangle_{bc}, |10\rangle_{bc}, |11\rangle_{bc}\}$ of the auxiliary system. The obtained probabilities are shown in Fig. 4.

As a third example, let us consider a one-qutrit three-element POVM given by

\[ M_1 = \frac{1}{2} (|0\rangle + |2\rangle)(\langle 0\rangle + \langle 2|), \]

\[ M_2 = \frac{1}{2} (|0\rangle - |2\rangle)(\langle 0\rangle - \langle 2|), \]

\[ M_3 = |1\rangle \langle 1|. \]

We set the pre-measurement state of system $A$ to $|\psi_0\rangle_A = \frac{1}{\sqrt{3}} (|0\rangle + e^{2i\pi/3}|1\rangle + e^{4i\pi/3}|2\rangle)$. In this case the global state

Figure 3: Simulation and experimental statistics for the one-qubit two-element POVM of Eq. (6) implemented using our protocol, for the qubit prepared in the state $|0\rangle_A$.

Figure 4: As a third example, let us consider a one-qutrit three-element POVM given by

\[ M_1 = \frac{1}{2} (|0\rangle + |2\rangle)(\langle 0\rangle + \langle 2|), \]

\[ M_2 = \frac{1}{2} (|0\rangle - |2\rangle)(\langle 0\rangle - \langle 2|), \]

\[ M_3 = |1\rangle \langle 1|. \]

We set the pre-measurement state of system $A$ to $|\psi_0\rangle_A = \frac{1}{\sqrt{3}} (|0\rangle + e^{2i\pi/3}|1\rangle + e^{4i\pi/3}|2\rangle)$. In this case the global state...
with $\alpha = \sqrt{\frac{3}{12}}$, $\beta = \sqrt{\frac{3}{4}}$ and $\gamma = \sqrt{\frac{3}{12}}$. Using qubits $a$ and $b$ to encode the states of target qutrit $A$ and qubits $c$ and $d$ to encode the states of qutrit $B$, the state vector above can be represented by

$$|\Psi\rangle_{abcd} = \alpha (|00\rangle + |10\rangle)_{ab} \otimes |00\rangle_{cd} - \gamma (|01\rangle + |11\rangle)_{ab} \otimes |01\rangle_{cd} + \beta (|00\rangle - |10\rangle)_{ab} \otimes |01\rangle_{cd}. \quad (21)$$

In this case, given the pre-measurement state $|\psi\rangle_A$ above, the probabilities are

$$Pr (M_1|\psi\rangle) = \langle\psi|M_1^\dag M_1|\psi\rangle = 1/6, \quad (22)$$
$$Pr (M_2|\psi\rangle) = \langle\psi|M_2^\dag M_2|\psi\rangle = 1/2, \quad (23)$$
$$Pr (M_3|\psi\rangle) = \langle\psi|M_3^\dag M_3|\psi\rangle = 1/3. \quad (24)$$

As in the previous example, after state preparation, a projective measurement is performed on the basis $\{ |00\rangle_{cd}, |01\rangle_{cd}, |10\rangle_{cd}, |11\rangle_{cd} \}$ of the auxiliary system, and the obtained probabilities are shown in Fig. 5.

As a last example, let us consider a two-qubit four element POVM, with measurement operators given as follows

$$M_1 = \sqrt{\frac{2}{3}}|\Phi_+\rangle\langle\Phi_+|, \quad (25)$$
$$M_2 = \sqrt{\frac{2}{3}}|\Psi (2\pi/3)\rangle\langle\Psi (2\pi/3)|, \quad (26)$$
$$M_3 = \sqrt{\frac{2}{3}}|\Psi (4\pi/3)\rangle\langle\Psi (4\pi/3)|, \quad (27)$$
$$M_4 = |\Phi_-\rangle\langle\Phi_-| + |\Psi_-\rangle\langle\Psi_-|. \quad (28)$$

with $|\Psi (\theta)\rangle = \cos \left(\frac{\theta}{2}\right)|\Phi_+\rangle + \sin \left(\frac{\theta}{2}\right)|\Phi_+\rangle$. The pre-measurement state of the system $AB$ is given by $|\psi\rangle_{AB} = |00\rangle$, so the global state to be prepared is

$$|\Phi_{abcd}\rangle = \frac{1}{6} (|00\rangle_{ab} \otimes |00\rangle_{cd} + |11\rangle_{ab} \otimes |00\rangle_{cd})$$
$$+ \frac{1}{4\sqrt{6}} (|00\rangle_{ab} \otimes |01\rangle_{cd} + |01\rangle_{ab} \otimes |01\rangle_{cd})$$
$$+ \frac{1}{4\sqrt{2}} (|01\rangle_{ab} \otimes |01\rangle_{cd} + |10\rangle_{ab} \otimes |01\rangle_{cd})$$
$$+ \frac{1}{4\sqrt{6}} (|00\rangle_{ab} \otimes |10\rangle_{cd} + |11\rangle_{ab} \otimes |10\rangle_{cd})$$
$$- \frac{1}{4\sqrt{2}} (|01\rangle_{ab} \otimes |10\rangle_{cd} + |10\rangle_{ab} \otimes |10\rangle_{cd})$$
$$+ \frac{1}{2} (|00\rangle_{ab} \otimes |11\rangle_{cd} - |11\rangle_{ab} \otimes |11\rangle_{cd}), \quad (29)$$

where we use the qubits $b$ and $c$ to encode the states of the auxiliary quutrit system. For the pre-measurement state $|\psi\rangle_{AB}$ above, we have the following probabilities

$$Pr (M_1|00\rangle) = \langle 00|M_1^\dag M_1|00\rangle = 1/3, \quad (30)$$
$$Pr (M_2|00\rangle) = \langle 00|M_2^\dag M_2|00\rangle = 1/2, \quad (31)$$
$$Pr (M_3|00\rangle) = \langle 00|M_3^\dag M_3|00\rangle = 1/2, \quad (32)$$
$$Pr (M_4|00\rangle) = \langle 00|M_4^\dag M_4|00\rangle = 1/2. \quad (33)$$
As in the previous examples, after state preparation a projective measurement is performed on the computational basis of the auxiliary system \( \{|00\rangle_{cd}, |01\rangle_{cd}, |10\rangle_{cd}, |11\rangle_{cd}\} \), allowing the extraction of the probabilities presented in Fig. 6.

Now, let us show how our protocol can be used for implementing quantum instruments (QI), that are quantum operations having as input a quantum state and as output a quantum state and a classical variable [2]:

\[
\Gamma(|\psi_0\rangle_A) = \sum_j \varepsilon_j(|\psi_0\rangle_A \otimes |j\rangle_J \langle j|, \tag{34}
\]

in which \( \{|j\rangle_J\} \) is an orthonormal basis for the system \( J \) and

\[
\varepsilon_j(|\psi_0\rangle_A) = \sum_k M_{j,k} |\psi_0\rangle_A \langle \psi_0 | M_{j,k}^\dagger \tag{35}
\]

is a trace non-increasing quantum operation, i.e., \( \text{Tr}(\varepsilon_j(|\psi_0\rangle_A)) \leq 1 \). Above, \( M_{j,k} \) are the elements of a POVM, i.e., \( \sum_{j,k} M_{j,k}^\dagger M_{j,k} = I \). One can verify that the quantum instrument in Eq. (34) can be obtained from the purification

\[
|\Psi_{AJE,J}\rangle = \sum_{j,k} M_{j,k} |\psi_0\rangle_A \otimes |j\rangle_J \otimes |k\rangle_E \otimes |j\rangle_E_J. \tag{36}
\]

That is to say, \( \Gamma(|\psi_0\rangle_A) = \text{Tr}_E,J_E \{ |\Psi_{AJE,J}\rangle \langle \Psi_{AJE,J} | \} \). So, given the QI, that is to say, given the completely positive maps \( \varepsilon_j \) in terms of the set of measurement operators \( \{M_{j,k}\}_k \), we can simulate this QI by preparing the state \( |\Psi_{AJE,J}\rangle \) and by taking the partial trace over the auxiliary systems \( E_J, E_J \). By measuring the system \( J \) in the basis \( \{|j\rangle_J\}_j \) and post-selecting the results, we can also reconstruct the action of the operators \( \varepsilon_j(|\psi_0\rangle_A) \). In what follows, we exemplify the application of this simulation protocol. Let us consider a one-qubit QI defined by the following set of trace non-increasing quantum operations:

\[
\varepsilon_0 \equiv \left\{ M_{00} = \frac{1}{\sqrt{2}} |0\rangle_A \langle 0|, M_{01} = \frac{1}{\sqrt{2}} |+\rangle_A \langle +| \right\}, \tag{37}
\]

\[
\varepsilon_1 \equiv \left\{ M_{10} = \frac{1}{\sqrt{2}} |1\rangle_A \langle 1|, M_{11} = \frac{1}{\sqrt{2}} |\rangle_A \langle -| \right\}. \tag{38}
\]

We set the pre-measurement state of system \( A \) to \( |\psi_0\rangle_A = |0\rangle_A \). The global state to be prepared for the simulation of this QI is

\[
\sqrt{2}|\Psi_{AJE,J}\rangle = |0000\rangle_{AJE,J} + \frac{1}{2} |0010\rangle_{AJE,J} + \frac{1}{2} |1010\rangle_{AJE,J} + \frac{1}{2} |0111\rangle_{AJE,J} - \frac{1}{2} |1111\rangle_{AJE,J}. \tag{39}
\]

From this quantum state, we obtain the quantum instru-
level states (one qubit) to $d$-level states, avoiding also the complications of their algorithm regarding the implementation of general unitary transformations on a multiqubit system. We exemplified the application of our protocol for one qubit, one qudit and two qubit POVMs and for simulating quantum instruments. We implemented these examples experimentally using IBM quantum computers. The simulation results matched the theoretical predictions. The experimental results are in fairly good agreement with theory, but can be further improved if this protocol is executed in lesser noise quantum devices. So, we believe that the simplicity and easy to use of this protocol will foster further research involving POVMs.

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