Witten-Helffer-Sjöstrand Theory for a
Generalized Morse Function

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March 24, 2022

Abstract
In this paper, we extend the Witten-Helffer-Sjöstrand theory from Morse functions to generalized Morse functions. In this case, the spectrum of the Witten deformed Laplacian $\Delta(t)$, for large $t$, can be seperated into the small eigenvalues (which tend to 0 as $t \to \infty$), large and very large eigenvalues (both of which tend to $\infty$ as $t \to \infty$). The subcomplex $\Omega_0^*(M,t)$ spanned by eigenforms corresponding to the small and large eigenvalues of $\Delta(t)$ is finite dimensional. Under some mild conditions, it is shown that $(\Omega_0^*(M,t), d(t))$ converges to a geometric complex associated to the generalized Morse function as $t \to \infty$.

1 Introduction and Statement of Results
The purpose of this paper is to extend the Witten-Helffer-Sjöstrand theory (cf.[W],[H-S]) for a Morse function on compact manifold to a generalized Morse function. Such a generalized Morse function has all critical points either non-degenerate or of birth-death type, i.e. in some neighbourhood of the critical point and with respect to a convenient coordinate system, the function can be written

$$f(x_1, x_2, \ldots, x_n) = f(0) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2 + ax_n^3$$

for some $a \neq 0$.

The interest of generalized Morse function comes from the following theorem due to H.Chaltin(cf.[I]).

**Theorem:** If $\pi : E \to B$ is a smooth bundle with fibre a compact manifold $M$, then there exists $f : E \to R$ so that for any $t \in B$

$$f_t = f |_{\pi^{-1}(t)} : M_t = \pi^{-1}(t) \to R$$

is a generalized Morse function.
It is easy to see that in general one cannot have such a statement with \( f_t \) a Morse function.

Now, let us state the results of this paper. Let the eigenvalues of the operator

\[-\frac{d^2}{dx^2} + 9x^4 - 6x\]

be

\[0 < e_1 < e_2 \leq \ldots \leq e_l \leq \ldots\]

(See Lemma 2 in §2 for proof.)

Suppose \( M^n \) is a compact orientable Riemannian manifold, \( f \) be a generalized Morse function on \( M \).

Suppose \( x^{j_1}_k, \ldots, x^{j_{k_m}}_k \) are all the non-degenerate critical points of \( f \), of index \( k \), \( y^{j_1}_k, \ldots, y^{j_{k_m}}_k \), are all the critical points of birth-death type, of index \( k \). Also, let \( a_j^{(k)} \in \mathbb{R} \) be associated with \( y^{j}_k \) so that in some neighbourhood of \( y^{j}_k \) and with respect to a suitable oriented co-ordinate system,

\[f(x_1, \ldots, x_n) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a_j^{(k)} x_n^3\]

Suppose for simplicity that

\[|a_1^{(0)}| < |a_2^{(0)}| < \ldots < |a_m^{(0)}| < |a_1^{(1)}| < \ldots < |a_{m_1}^{(1)}| < \ldots < |a_{m_{n-1}}^{(n-1)}| (1)\]

(in fact, the Witten-Helffer-Sjöstrand theory is very similar with minor modifications without assuming (1))

Also, let \( g \) be a Riemannian metric on \( M \) so that in the above co-ordinate system near the critical points \( x_j^k \) or \( y_j^k \), \( g \) is the canonical metric on \( \mathbb{R}^n \).

Consider the Witten deformation of the de Rham complex \((\Omega^*(M), d(t))\) with

\[d(t) = e^{-tf} \text{de}^f : \Omega^*(M) \to \Omega^*(M)\]

Consider the deformed Laplacian

\[\Delta(t) = d(t)d^*(t) + d^*(t)d(t)\]

When the above canonical coordinates near the critical points are used,

\[\Delta(t) = \Delta + t^2|df|^2 + tA\] (2)

where

\[A = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} [dx_i, i_{\partial_j}]\]

and \( i_{\partial_j} \) denotes the contraction along the vector field \( \partial_j \), \( dx_i \) is the exterior multiplication by the form \( dx_i \) and \([dx_i, i_{\partial_j}]\) denotes the commutator \( dx_i i_{\partial_j} - i_{\partial_j} dx_i \).
There are two cases.

Case 1: $x^k_j$ is non-degenerate.

$$f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2$$  \hspace{1cm} (3)

$$|df|^2 = 4(x_1^2 + \ldots + x_k^2)$$

$$A = -2 \sum_{i=1}^{k} [dx_i, i_{\partial_i}] + 2 \sum_{i=k+1}^{n} [dx_i, i_{\partial_i}]$$  \hspace{1cm} (4)

Case 2: $y^k_j$ is of birth-death type.

$$f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a^{(k)}_j x_n^3$$  \hspace{1cm} (5)

$$|df|^2 = 4(x_1^2 + \ldots + x_{n-1}^2) + 9(a^{(k)}_j)^2 x_n^4$$

$$A = -2 \sum_{i=1}^{k} [dx_i, i_{\partial_i}] + 2 \sum_{i=k+1}^{n-1} [dx_i, i_{\partial_i}] + 6a^{(k)}_j x_n [dx_n, i_{\partial_n}]$$  \hspace{1cm} (6)

For each critical point $c$, define the 'localized' operator $\tilde{\Delta}_c(t) : C^\infty(\Lambda^*(R^n)) \to C^\infty(\Lambda^*(R^n))$ which is given by (2) where $A$ is (4) if $c = x^k_j$ is nondegenerate, respectively (6) if $c = y^k_j$ is birth-death. The Laplace operator in (2) then is the canonical Laplace operator corresponding the canonical metric in $R^n$. The operator $\tilde{\Delta}_c(t)$ then extends uniquely to a self-adjoint positive unbounded operator in $L^2(\Lambda^*(R^n))$.

Now suppose $\tilde{\Delta}(t)$ is the 'localized' operator associated to a critical point of birth-death type. Since $L^2(\Lambda^*(R^n)) \cong L^2(\Lambda^*(R^{n-1})) \otimes L^2(\Lambda^*(R))$, $\tilde{\Delta}(t)$ can be written as

$$\tilde{\Delta}(t) = \{ \Delta_{R_{n-1}} + 4t^2(x_1^2 + \ldots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{\partial_i}] \} \otimes id$$

$$+ id \otimes \{ \Delta_R + 9(a^{(k)}_j)^2 t^2 x_n^4 + 6a^{(k)}_j t x_n [dx_n, i_{\partial_n}] \}$$

where

$$\epsilon_i = \begin{cases} -1 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k + 1 \leq i \leq n - 1 \end{cases}$$

and $\Delta_{R_{n-1}}, \Delta_R$ are the Laplace operators on $R^{n-1}$ and $R$ respectively.

By Corollary in §2, $\Delta_R + 9(a^{(k)}_j)^2 t^2 x_n^4 + 6a^{(k)}_j t x_n [dx_n, i_{\partial_n}] : L^2(\Lambda^*(R)) \to L^2(\Lambda^*(R))$ has discrete spectrum with eigenvalues

$$0 < e_1(\vbar{a^{(k)}_j} | t)^{2/3} < e_2(\vbar{a^{(k)}_j} | t)^{2/3} \leq e_3(\vbar{a^{(k)}_j} | t)^{2/3} \leq \ldots$$
Theorem 1 (Quasi-classical limit of eigenvalues)

Each eigenvalue has a multiplicity of 2 with corresponding eigenvectors consisting of a 0-form and a 1-form.

Let $\Delta^k(t) = \Delta(t)|_{L^2(A^k(M))}$.

Let $0 \leq E_1(t) \leq E_2(t) \leq \ldots \leq E_l(t) \leq \ldots$ be all the eigenvalues of $\Delta^k(t)$.

Suppose for simplicity that
\[ e_1 | a_{m_{n-1}}^{(n-1)} |^{2/3} < e_2 | a_1^{(0)} |^{2/3} \]

This together with (1) imply that $e_1 | a_{m_k}^{(k)} |^{2/3} < e_2 | a_1^{(k-1)} |^{2/3}$ for all $k$.

Theorem 1 (Quasi-classical limit of eigenvalues)

\[ \lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_k}(t) = 0 \]

\[ \lim_{t \to \infty} \frac{E_{m_k+1}(t)}{t^{2/3}} = e_1(| a_1^{(k-1)} |^{2/3} < \lim_{t \to \infty} \frac{E_{m_k+2}(t)}{t^{2/3}} = e_1(| a_2^{(k-1)} |^{2/3} < \ldots \]

\[ \ldots < \lim_{t \to \infty} \frac{E_{m_k+m_{n-1}+m_{k}'}(t)}{t^{2/3}} = e_1(| a_1^{(k)} |^{2/3} \]

\[ \left( < \lim_{t \to \infty} \frac{E_{m_k+m_{n-1}+m_{k}'+1}(t)}{t^{2/3}} = e_2(| a_1^{(k-1)} |^{2/3} < \ldots \right) \]

Remarks: 1. In fact, the eigenvectors corresponding to $E_1(t), \ldots, E_{m_k}(t)$ are localized at the non-degenerate critical points of index $k$, while the eigenvectors corresponding to $E_{m_k+1}(t), \ldots, E_{m_k+m_{n-1}+m_{k}'}(t)$ are localized at the birth-death critical points of index $k-1$ and $k$. However, the eigenvectors are not necessarily localized at a single critical point.

2. If all the critical points of $f$ are non-degenerate, then the above theorem should be formulated as follows (cf. [S] p219):

Theorem:

\[ \lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_k}(t) = 0 \]

\[ 0 < \lim_{t \to \infty} \frac{E_{m_k+1}(t)}{t} \leq \lim_{t \to \infty} \frac{E_{m_k+2}(t)}{t} \leq \ldots \]

Let us index $a_1^{(0)}, \ldots, a_{m_0}^{(0)}, a_1^{(1)}, \ldots, a_{m_{n-1}}^{(n-1)}$ by $b_1, \ldots, b_N$ where $N = \sum_{k=0}^{n-1} m_k'$.

Also, for $0 \leq l \leq n-1, 1 \leq j \leq m_l'$ let
\[ I_j^{(l)}(\epsilon) = [e_1(| a_{j}^{(l)} |)^{2/3} - \epsilon, e_1(| a_{j}^{(l)} |)^{2/3} + \epsilon] \]

Choose an $\epsilon$ small enough so that the family of intervals
\[ \left\{ [0, \epsilon], I_j^{(l)}(\epsilon), [e_2(| a_{j}^{(0)} |)^{2/3} - \epsilon, \infty) \right\}_{0 \leq l \leq n-1, 1 \leq j \leq m_l'} \]

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is pairwise disjoint. The pairwise disjointness is satisfied if $\epsilon$ is a positive number smaller than
\[
\min_i \left( \frac{e_1}{2} \mid a_1(0) \mid 2/3, \frac{e_1}{2}(\mid b_{i+1} \mid 2/3 - \mid b_i \mid 2/3), \frac{1}{2}(e_2 \mid a_1(0) \mid 2/3 - e_1 \mid a_1^{(n-1)} \mid 2/3) \right)
\]

As a consequence of Theorem 1, for $t$ sufficiently large and $\epsilon$ satisfying the above disjointness condition, we have
\[
\text{Spec}(t^{-2/3} \Delta^k)(t) \subset [0, \epsilon] \cup \left( \bigcup_{j=k-1, k+1 \leq j \leq m_i} I_j(t) \right) \cup \left( e_2(\mid a_1(0) \mid 2/3 - \epsilon, \infty) \right)
\]

and
\[
\begin{cases}
\text{Card} \left( \text{Spec}(t^{-2/3} \Delta^k)(t) \cap [0, \epsilon] \right) = m_k \\
\text{Card} \left( \text{Spec}(t^{-2/3} \Delta^k)(t) \cap I_j(t) \right) = 1 \\
(for l = k - 1, k; 1 \leq j \leq m_i)
\end{cases}
\]

Define
\[
\Omega^k_{\text{small}}(M, t) = \text{Span}\{\psi(t) \in L^2(\Lambda^k(M))| \Delta^k(t)\psi(t) = E(t)\psi(t), t^{-2/3}E(t) \in [0, \epsilon]\}
\]

\[
\Omega^k_{\text{large}, l, j}(M, t) = \text{Span}\{\psi(t) \in L^2(\Lambda^k(M))| \Delta^k(t)\psi(t) = E(t)\psi(t), t^{-2/3}E(t) \in [e_1(\mid a_j(l) \mid 2/3) - \epsilon, e_1(\mid a_j(l) \mid 2/3 + 1)]\}
\]

\[
\Omega^k_{\text{v.large}}(M, t) = \text{Span}\{\psi(t) \in L^2(\Lambda^k(M))| \Delta^k(t)\psi(t) = E(t)\psi(t), t^{-2/3}E(t) \in [e_2 - \epsilon, \infty)\}
\]

\[
(\Omega^*_0(M, t, d(t)) = (\Omega^*_{\text{small}}(M, t, d(t)) \perp \left( \bigcup_{k,j} (\Omega^*_{\text{large}, k,j}(M, t, d(t))\right))
\]

Corollary 1 \((\Omega^*(M), d(t))\) is equal to
\[
(\Omega^*_{\text{small}}(M, t, d(t)) \perp \left( \bigcup_{k,j} (\Omega^*_{\text{large}, k,j}(M, t, d(t))\right)) \perp (\Omega^*_{\text{v.large}}(M, t, d(t))
\]

\((\Omega^*_{\text{small}}(M, t, d(t)), (\Omega^*_0(M, t, d(t))\) are complexes of finite dimensional vector spaces which calculate the de Rham cohomology of \(M\).

Remark: Observe that \((\Omega^*_{\text{large}, k,j}(M, t, d(t))\) has dimension 2. It is spanned by a k-form and a (k+1)-form localized at \(y^k_j\) (the localization of the forms at \(y^k_j\) is due to (1)).

As in the Helffer-Sjöstrand theory for a generic pair \((f, g)\), \((\Omega^*_0(M, t, d(t))\) converges as \(t \to \infty\) to a geometric complex, which can be described as follows.

Let \(f\) be a self-indexing generalized Morse function, i.e.
\[
\begin{cases}
f(x^k_j) = k & \text{if } x^k_j \text{ is a non-degenerate critical point of index } k \\
f(y^k_j) \in (k, k+1) & \text{if } y^k_j \text{ is a birth-death critical point of index } k
\end{cases}
\]
Let $W_{y_j}^−$ be the descending manifold of a non-degenerate critical point $x_j^k$. For a birth-death critical point $y_j^k$, choose an open neighbourhood $U_{y_j^k}$ and a suitable co-ordinate s.t. 

$$f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a_j^{(k)} x_n^3$$

Let 

$$W_{y_j}^{-0} = \{ x \in M | \gamma_x(t) \in U_{y_j^k} \cap R^k \text{ for some } t \in R \}$$

where $\gamma_x$ is the trajectory of Grad f s.t. $\gamma_x(0) = x$ 

$$W_{y_j}^{-,i} = \{ x \in M | \gamma_x(t) \in U_{y_j^k} \cap (R^k \times R_-) \text{ for some } t \in R \}$$

where $R^k \times R_- = \{(x_1, \ldots, x_k, 0, \ldots, 0, x_n) \in R^n | x_n < 0 \}$ 

when $a > 0$ with obvious modifications when $a < 0$. Then $W_{y_j}^{-0}, W_{y_j}^{-,i}$ are manifolds diffeomorphic to $R^k, R^{k+1}$ respectively. Note that $W_{y_j}^{-0} \cap W_{y_j}^{-,1} = \emptyset$ 

Define the descending manifold 

$$W_{y_j}^{-} = W_{y_j}^{-0} \cup W_{y_j}^{-,1}$$

which is then a manifold with boundary diffeomorphic to $R^{k+1}$. The ascending manifold $W_{y_j}^{+}$ is defined similarly.

Suppose the ascending and descending manifolds for any two critical points intersect transversally, then $\{W_{y_j}^{-}, W_{y_j}^{-0}, W_{y_j}^{-,1}\}$ form a CW-complex (see §3 for more details). While the incidence number between $W_{x_j}^{-}$ and $W_{x_{j+1}}^{-}$ is given by the intersection number between the ascending and descending manifolds in $f^{-1}(k+\frac{1}{2})$ i.e. between $W_{x_j}^{+} \cap f^{-1}(k+\frac{1}{2})$ and $W_{x_{j+1}}^{-} \cap f^{-1}(k+\frac{1}{2})$, the incidence number is 1 between $W_{y_j}^{-0}$ and $W_{y_j}^{-,1}$. However, those between $W_{x_j}^{-}$ and $W_{y_j}^{-,i}$ may be non-trivial ($i=0,1$). Let us denote the above described chain complex by $(C_*(M, \delta))$ (with $C_k(M, f) = \text{Span}\{W_{x_j}^{-}, W_{y_j}^{-0}, W_{y_j}^{-,1}\}$), its dual cochain complex by $(C^*(M, f), \delta)$. 

Also, let us rescale the complex $(\Omega^*_0(M, t, d(t)))$ to be

$$(\Omega^*_0(M, t, d(t)) = \left(\Omega^*_{small}(M, t), e^t \sqrt{\frac{n}{2t}} d(t) \right) \perp \left(\Omega^*_{large,j,k}(M, t, d(t))\right)$$

**Theorem 2** There exists $f^*(t) : (\Omega^*_0(M, t, d(t)) \to (C^*(M, f), \delta)$ which is a morphism of co-chain complexes s.t.

$$f^*(t) = I + O(t^{-1})$$

w.r.t. some suitably chosen bases.
Definitions: 1.(i) Suppose $x_j^k, x_j^{k+1}$ are two non-degenerate critical points, $\gamma$ be a generalized trajectory between $x_j^k$ and $x_j^{k+1}$, i.e. $\gamma$ is a piecewise smooth curve with singularities at the birth-death points $y_1, \ldots, y_{n(\gamma)}$ and

$$\gamma = \gamma_{x_j^{k+1} y_1} \cup \{y_1\} \cup \gamma_{y_1 y_2} \cup \{y_2\} \cup \ldots \cup \gamma_{y_{n(\gamma)} x_j^k}$$

where $\gamma_{y_i y_{i+1}}$ is a trajectory between $y_i$ and $y_{i+1}$. Then one can associate $\epsilon = \pm 1$ to the trajectories $\gamma_{x_j^{k+1} y_1}, \gamma_{y_i y_{i+1}}, \gamma_{y_{n(\gamma)} x_j^k}$ as in the Witten-Morse theory. Then define

$$\epsilon_{\gamma_{x_j^{k+1} y_1}} = (-1)^{n(\gamma)} \epsilon_{\gamma_{y_i y_{i+1}}} \prod_{l=1}^{(n(\gamma)-1)} \epsilon_{\gamma_{y_i y_{i+1}}}$$

(ii) Suppose $x_j^k$ is an non-degenerate critical point and $y_i^k$ a birth-death critical point, $\gamma$ be a generalized trajectory between them. With the above notation for $\gamma$ and $y_1 = y_i^k$, define

$$\epsilon_{\gamma_{y_i y_{i+1}}} = (-1)^{n(\gamma)} \prod_{l=1}^{(n(\gamma)-1)} \epsilon_{\gamma_{y_i y_{i+1}}}$$

2. (i) The (generalized) incidence number between two critical points is defined as follows:

$$I(x_j^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_{\gamma_{x_j^{k+1} y_1}}$$

$$I(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_{\gamma_{y_i y_{i+1}}}$$

where $\gamma$ is a generalized trajectory between the initial and end point.

(ii) Here we recall that the (ordinary) incidence number between two critical points(non-degenerate or birth-death) is

$$i(x_j^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_{\gamma_{x_j^{k+1} y_1}}$$

$$i(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_{\gamma_{y_i y_{i+1}}}$$

where $\gamma$ is a trajectory between the two critical points.

Remark: Observe that in general

$$\epsilon_{\gamma_{y_i y_{i+1}}} \neq \epsilon_{\gamma_{x_j^{k+1} y_1}}$$

for a trajectory between two critical points. For example, if $\gamma$ is a trajectory between a birth-death point $y_i^k$ and a non-degenerate critical point $x_j^k$, then $\epsilon_{\gamma_{y_i y_{i+1}}} = -\epsilon_{\gamma_{x_j^{k+1} y_1}}$.
With the above definition, we can reformulate Theorem 2 as follows:

**Theorem 2': (Helffer-Sjöstrand)** There exist orthonormal bases \( \{ E_{y_j}^i(t) \} \) of \( \Omega_{small}^k(M,t) \), \( \{ E_{y_j}^{0i}(t), E_{y_j}^{1i}(t) \} \) of \( \Omega_{large,k,j}^*(M,t) \) s.t.

\[
< E_{x_j}^{k+1}(t), d(t)E_{x_j}^k(t) > = e^{-t} \left( \sqrt{\frac{t}{\pi}} \sum_\gamma e_{\gamma}^{new} + O(t^{-1/2}) \right)
\]

\[
< E_{y_j}^i(t), d(t)E_{y_j}^{0i}(t) > = \sqrt{t}^{1/3} t^{1/3} + O(t^{1/6})
\]

\[
< E_{y_j}^{1i}(t), d(t)E_{y_j}^{2i}(t) > = 0 \text{ if } j_1 \neq j_2 \text{ for } t \text{ sufficiently large}
\]

\[
< E_{x_j}^i(t), d(t)E_{x_j}^j(t) > = < E_{y_j}^i(t), d(t)E_{x_j}^k(t) > = 0 \text{ for } t \text{ sufficiently large}
\]

where \( \sum_\gamma e_{\gamma}^{new} = I(x_j^{k+1}, x_j^k) \) is the incidence number between \( x_j^k \) and \( x_j^{k+1} \) defined above.

Inside the complex \( (C^*(M,f), \delta) \), there is a subcomplex \( (C_{nd}^*(M,f), \delta) \) such that

\[
dim C_{nd}^k(M,f) = m_k
\]

where \( m_k \) is the number of non-degenerate critical points of index \( k \). Note that this subcomplex is not generated by the non-degenerate critical points, since the latter in general do not generate a subcomplex. Instead the subcomplex is obtained by applying Lemma 3 repeatedly as is done in §3. See §3 for details.

**Theorem 2'**: \( f^k(t) |_{\Omega^*_{small}(M,t)}: \left( \Omega^*_{small}(M,t), d(t) \right) \to (C^*(M,f), \delta) \) is an injective homomorphism of co-chain complexes whose image complex converges to \( (C_{nd}^*(M,f), \delta) \) in \( (C^*(M,f), \delta) \) as \( t \to \infty \), more precisely,

\[
f^k(t) \left( E_{x_j}^k(t) \right) = \hat{e}_{x_j}^k + O(t^{-1}) \text{ in } C^*(M,f)
\]

where \( \hat{e}_{x_j}^k = e_{x_j}^k + \sum I(y_j^k, x_j^k) e_{y_j}^{0i} \in C_{nd}^k(M,f) \).

Also, using similar consideration, one can extend the result to any representation \( \rho : \pi_1(M) \to GL(V) \) (cf.[BZ]) or any representation \( \rho : \pi_1(M) \to GL(W_A) \) where \( W_A \) is finite type Hilbert module over a finite von Neuman algebra \( A \) (cf.[BFKM]).

The above question concerning the extension of Witten-Helffer-Sjöstrand theory for generalized Morse functions was raised in Dan Burghelea’s course on \( L_2 \)-topology. I would like to thank him for the problem and help in accomplishing this work. A parametrized version of the above theory will be presented in future work in collaboration with D. Burghelea.
2 Witten Deformation for a Generalized Morse Function

Let \( f \) be a generalized Morse function on \( M^n \), \( y \) be a critical point of birth-death type. Let \((U_y, \varphi)\) be a chart s.t. \( y \in U_y \), and

\[
f(\varphi^{-1}x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + ax_n^3
\]

Let \( g \) be a Riemannian metric on \( M \) s.t. \((\varphi^{-1})^*(g) = \delta_{ij} \).

Define

\[
d(t) = e^{-tf} df
\]

\[
\Delta(t) = d(t)d^*(t) + d^*(t)d(t)
\]

Then in the coordinate system \((U_y, \varphi)\),

\[
\Delta(t) = \Delta + t^2 |df|^2 + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i\partial_i] + 6ax_n t [dx_n, i\partial_n]
\]

where

\[
|df|^2 = 4(x_1^2 + \ldots + x_{n-1}^2) + 9a^2 x_n^4
\]

\[
\epsilon_i = \begin{cases} 
-1 & \text{if } 1 \leq i \leq k \\
1 & \text{if } k+1 \leq i \leq n-1 
\end{cases}
\]

Define \( \overline{\Delta}(t) : L^2(\Lambda^*(R^n)) \to L^2(\Lambda^*(R^n)) \) to be given exactly by the above expression. Recall that

\[
\overline{\Delta}(t) = \{ \Delta + t^2 |df|^2 + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i\partial_i] \} \otimes \text{id}
\]

\[
\text{id} \otimes \{ \Delta_R + 9t^2 a^2 x_n^4 + 6atx [dx, i\partial_n] \}
\]

Observe that the first term is exactly the Witten deformed Laplacian on \( L^2(\Lambda^*(R^{n-1})) \) for the classical Morse theory (cf.\cite{S}\cite{W}) and that the two operators in parenthesis commute with each other. Hence to study the spectrum of \( \overline{\Delta}(t) \), it suffices to find out the spectrum of \( \Delta + 9t^2 a^2 x_n^4 + 6atx [dx, i\partial_n] \) on \( L^2(\Lambda^*(R)) \). Note that

\[
[dx, i\partial_n](f) = -f
\]

\[
[dx, i\partial_n](f dx) = f dx
\]

Therefore

\[
\{ \Delta + 9t^2 a^2 x_n^4 + 6atx [dx, i\partial_n] \} (f) = \left( -\frac{d^2}{dx^2} + 9t^2 a^2 x_n^4 - 6atx \right) (f)
\]
\[
\left\{ \Delta + 9t^2a^2x^4 + 6atx \right\} (f)dx = \left( -\frac{d^2}{dx^2} + 9t^2a^2x^4 + 6atx \right) (f)dx
\]

Define \( R : L^2(R) \to L^2(R) \)

\[
(Rf)(x) = f(-x)
\]

Then

\[
R^{-1} \left( -\frac{d^2}{dx^2} + 9t^2a^2x^4 + 6atx \right) R = -\frac{d^2}{dx^2} + 9t^2a^2x^4 - 6atx
\]

Hence, it is sufficient to consider

\[
P(at) = -\frac{d^2}{dx^2} + 9t^2a^2x^4 - 6atx
\]

where \( P(t) \equiv -\frac{d^2}{dx^2} + 9t^2x^4 - 6tx \) for \( t \in \mathbb{R} \).

Define \( U(\lambda) : L^2(R) \to L^2(R), \lambda > 0 \)

\[
(U(\lambda)f)(x) = \lambda^{1/2} f(\lambda x)
\]

**Lemma 1** For \( t > 0 \),

\[
P(t) = U(t^{1/3}) \left( t^{2/3} P(1) \right) U(t^{-1/3})
\]

**Lemma 2** (i) \( P(1) \) has compact resolvent, hence has discrete spectrum.

\[
0 \leq e_1 \leq e_2 \leq \ldots \leq e_l \leq \ldots
\]

(ii) The smallest eigenvalue of \( P(1) \) is strictly positive and is simple, i.e.

\[
0 < e_1 < e_2 \leq \ldots
\]

(iii) Let \( \Xi_1 \) be a normalized eigenfunction of \( P(1) \) corresponding to the smallest eigenvalue \( e_1 \). Then one can choose \( \Xi_1 \) so that \( \Xi_1(x) > 0 \) for all \( x \in \mathbb{R} \). In particular \( \Xi_1(0)^{-1} \) exists.

**Corollary**: \( P(at) \) has spectrum

\[
0 < e_1(|at|)^{2/3} < e_2(|at|)^{2/3} \leq e_3(|at|)^{2/3} \leq \ldots \leq e_l(|at|)^{2/3} \leq \ldots
\]

**Proof of Lemma 2**: (i) \( P(1) \) has compact resolvent because \( V(x) = 9x^4 - 6x \to \infty \) as \( |x| \to \infty \) and is bounded from below. (cf. [RS] p249) It is positive because it is the restriction of the deformed Laplacian associated with the function \( x^3 \) on the invariant subspace \( L^2(R) \).
of e chosen s.t. \( \Xi^1_0 \) and is simple. Since \( \omega \) for all \( x \) where \( \xi \)

we have shown that the smallest eigenvalue of \( P \) is \( e_1(\|at\|^2/3) > 0 \) and is simple. Since \(-\frac{d^2}{dx^2} + 9a^2x^2 + 6atx\) is conjugated to \( P \) by an isometry \( R \), the same is true for its smallest eigenvalue. Hence \(-\frac{d^2}{dx^2} + 9a^2x^2 + 6atx[dx,i\frac{\partial}{\partial x}]\) on \( L^2(\Lambda^n(R)) \) has smallest eigenvalue \( e_1(\|at\|^2/3) \) of multiplicity 2, the corresponding eigenvectors are \( \Xi_1(x) \) (\( \in \Omega^0(R) \)) and \( \Xi_1(-x)dx \) (\( \in \Omega^1(R) \)).

Returning to the 'localized' operator,

\[
\overline{\Delta}(t) = \left\{ \Delta_{R^n-1} + 4t^2(x_1^2 + \ldots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i[dx_i, i\partial_x] \right\} \otimes id \\
+ id \otimes \left\{ \Delta_R + 9a^2x^2 + 6atx[dx_n, i\partial_n] \right\}
\]

The smallest eigenvalue is also \( e_1(\|at\|^2/3) \) and of multiplicity 2, whose eigenvectors are spanned by a \( k \)-form and a \( (k+1) \)-form.

\[
\omega_k(t) = t^{(n-1)/4+1/6} \xi_1^{(t^{1/2}x_1)} \ldots \xi_1^{(t^{1/2}x_{n-1})} \Xi_1^{(t^{1/3}x_n)} dx_1 \wedge \ldots \wedge dx_k \\
\omega_{k+1}(t) = t^{(n-1)/4+1/6} \xi_1^{(t^{1/2}x_1)} \ldots \xi_1^{(t^{1/2}x_{n-1})} \Xi_1^{(-t^{1/3}x_n)} dx_1 \wedge \ldots \wedge dx_k \wedge dx_n
\]

where \( \xi_1(x) \) is the groundstate of \(-\frac{d^2}{dx^2} + 4x^2 \) i.e. \( \xi_1(x) = e^{-tx^2} \).

With the above observations, the proof of Theorem 1 follows essentially the arguments in [S](pp 219-222). See Appendix for sketch of proof.

3 Helffer-Sjöstrand Theory for a Generalized Morse Function

**Definition:** A pair \( (f,g) \) is said to satisfy the Morse-Smale condition where \( f \) is a generalized Morse function if for any two critical points \( x \) and \( y \), the ascending manifold \( W^+_x \) and the descending manifold \( W^-_y \), w.r.t. \(-Grad_yf\), intersect transversally.
In the case of a birth-death critical point \( y^k_j \) of index \( k \) such that (5) holds, define

\[
W^+_{y^k_j} = \{ x \in M | \gamma_x(t) \in U^k_j \cap R^{n-k-1} \text{ for some } t \in R \}
\]

where \( R^{n-k-1} = \{ (0,\ldots,0,x_{k+1},\ldots,x_{n-1},0) \in R^n \} \}

\[
W^-_{y^k_j} = \{ x \in M | \gamma_x(t) \in U^k_j \cap (R^{n-k-1} \times R_+ \text{ for some } t \in R \}
\]

where \( R^{n-k-1} \times R_+ = \{ (0,\ldots,0,x_{k+1},\ldots,x_n) \in R^n | x_n > 0 \} \}

while \( W^0_{y^k_j}, W^{-1}_{y^k_j} \) are defined similarly as in §1. Then the ascending and descending manifolds are defined as follows:

\[
W^+_y = W^+_{y^0_j} \cup W^+_{y^1_j}
\]

\[
W^-_y = W^-_{y^0_j} \cup W^-_{y^1_j}
\]

**Proposition 1** For any pair \((f,g)\), there is a \( C^1 \) approximation \( g' \) such that \( g = g' \) in a neighbourhood of the critical points of \( f \) and \((f,g')\) satisfies the Morse-Smale condition.

**Proof**: The proof is the same as in [Sm].

**Definition**: Let \( f \) be a generalized Morse function. \( f \) is said to be self-indexing if

\[
\begin{cases}
  f(x^k_i) = k & \text{if } x^k_i \text{ is a non-degenerate critical point of index } k \\
  f(y^k_j) \in (k,k+1) & \text{if } y^k_j \text{ is birth-death critical point of index } k
\end{cases}
\]

**Proposition 2** For any generalized Morse function \( f \), there exists a self-indexing generalized Morse function \( f' \) such that \( f \) and \( f' \) have the same critical points and corresponding indexes.

**Proof**: The proof is similar as in [M] §4.

**Definition**: A pair \((f,g)\) is called a generalized triangulation if

(i) \( f \) is a self-indexing generalized Morse function on \( M \) and in a neighbourhood \( U_c \) of any critical point \( c \), one can introduce local coordinates s.t. \( g = \delta_{ij} \) and

(a) if \( c \) is non-degenerate

\[
f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2
\]

(b) if \( c \) is of birth-death type

\[
f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + ax_n^3
\]
(ii) \((f,g)\) satisfy the Morse-Smale condition.

Let \(f\) be a generalized Morse function, \(W_{x^j_k}^-\) be the descending manifold of a non-degenerate critical point. For a birth-death critical point \(y^k_j\), recall

\[
W_{y^k_j}^- = \{ x \in M | \gamma_x(t) \in U_{y^k_j}^k \cap R^k \text{ for some } t \in R \}
\]

where \(\gamma_x\) is the trajectory of \(\text{Grad } f\) s.t. \(\gamma_x(0) = x\)

\[
W_{y^k_j}^{-1} = \{ x \in M | \gamma_x(t) \in U_{y^k_j}^k \cap (R^k \times R^-) \text{ for some } t \in R \}
\]

where \(R^k \times R^- = \{(x_1, \ldots, x_k, 0, \ldots, 0, x_n) \in R^n | x_n < 0\}\)

Then we have

**Theorem:** Suppose \((f,g)\) is a generalized triangulation, then

(i) \(\{ W_{x^j_k}^-, W_{y^k_j}^{-0}, W_{y^k_j}^{-1} \}_{0 \leq k \leq n; 1 \leq j \leq m_k} \) or \(m'_k\) is a CW-complex.

(ii) Let \((C_*(M, f), \partial)\) be the cellular chain complex of the above CW-complex (as described in §1), \((C^*(M, f), \delta)\) be its dual co-chain complex.

Then \(\text{Int} : (\Omega^*(M), d) \rightarrow (C^*(M, f), \delta)\)

\[
\omega \mapsto \int_W \omega
\]

is a morphism of co-chain complexes.

**Proof:** A proof of this theorem in the case of a Morse function can be found in [L]. The same argument also works in the case of a generalized Morse function. However, a better argument for this is the following:

(i) One first verify that the partition \(\{ W_{x^j_k}^-, W_{y^k_j}^{-0}, W_{y^k_j}^{-1} \}\) is a stratification in the sense of Whitney (see [V2] for definition). Using the tubular neighbourhood theorem (Proposition 2.6 [V1]) and the fact that each stratum is diffeomorphic with an Euclidean space, one concludes that this partition is a CW-complex.

(ii) The fact that integration is well defined and represents a morphism of cochain complexes follows from Stokes theorem in the framework of integration theory on stratified sets (cf.[F],[V2]). \(\square\)

As a consequence, the composition

\[
(\Omega^*_0(M, f, t), d(t)) \xrightarrow{\partial_*} (\Omega^*(M), d) \xrightarrow{\text{Int}} (C^*(M, f), \delta)
\]

is also a morphism of co-chain complexes.

Let

\[
M_{x^j_k} = M \setminus \left( (\cup_{l \neq j} B(x_l^k, \eta)) \cup (\cup_l B(y_l^k, \eta)) \right)
\]
Let $\Delta_{M_{x_j}}(t)$ be the corresponding Laplace operator on $M_{x_j}$ with Dirichlet boundary condition, $\Psi_{x_j}(t)$ be an eigenvector corresponding to the smallest eigenvalue of $\Delta_{M_{x_j}}(t)$ of norm one.

Similarly, let

$$M_{x_j} = M \setminus \left[(\cup_i B(x^{i}_k, \eta)) \cup (\cup_{\not\in j} B(y^{j}_k, \eta))\right]$$

With $\Delta_{M_{y_j}}(t)$ similarly defined, let $\Psi_{y_j}^0(t)$, respectively $\Psi_{y_j}^1(t)$ be the smallest eigenvector of $\Delta_{M_{y_j}}(t), \Delta_{M_{y_j}}^{k+1}(t)$ of norm one.

Define $J_k(t) : C^k(M, f) \rightarrow \Omega^k(M)$ by

$$J_k(t) \left(e_{x_j}^i(t)\right) = \Psi_{x_j}^i(t)$$
$$J_{k+1}(t) \left(e_{y_j}^i(t)\right) = \Psi_{y_j}^i(t), \ i = 0, 1$$

where $\{e_{x_j}^i, e_{y_j}^i\}$ is the dual basis of $\{W_{x_j}^i, W_{y_j}^i\}$.

Let $Q_{k,small}(t), Q_{k+i,k,j}(t)$ be the orthogonal projection onto $\Omega_{small}^k(M, t)$ and $\Omega_{large,k,j}^{k+i}(M, t)$ respectively.

Define $Q_k(t) : J_k(t) \left(C^k(M, f)\right) \rightarrow \Omega^k_0(M, t)$ by

$$Q_k(t) \left(\Psi_{x_j}^i(t)\right) = Q_{k,small}(t) \left(\Psi_{x_j}^i(t)\right)$$
$$Q_{k+1}(t) \left(\Psi_{y_j}^i(t)\right) = Q_{k+i,k,j}(t) \left(\Psi_{y_j}^i(t)\right)$$

Let

$$H_k(t) = (Q_k(t)J_k(t))^* \left(Q_k(t)J_k(t)\right)$$
$$\tilde{J}_k(t) = Q_k(t)J_k(t)(H_k(t))^{-1/2}$$

Then $\tilde{J}_k(t) : C^k(M, f) \rightarrow \Omega^k_0(M, t)$ is an isometry.

Define $E_{x_j}^i(t) = \tilde{J}_k(t) \left(e_{x_j}^i\right), E_{y_j}^i(t) = \tilde{J}_k(t) \left(e_{y_j}^i\right)$.

Note that $E_{x_j}^i(t) \in \Omega_{small}^k(M, t), E_{y_j}^i(t) \in \Omega_{large,k,j}^{k+i}(M, t)$.

**Proposition 3**

$$E_{x_j}^i(t) = \left(\frac{2t}{\pi}\right)^{n/4} e^{-t(x_1^2 + \ldots + x_k^2)} \left(dx_1 \wedge \ldots \wedge dx_k + O(t^{-1})\right)$$

$$E_{y_j}^i(t) = \left(\frac{2t}{\pi}\right)^{n/4} e^{-t(x_1^2 + \ldots + x_{n-1}^2)} \left(d\mid t\left(\mid(\cdot)^{1/3} x_n\right) \left(dx_1 \wedge \ldots \wedge dx_k + O(t^{-1})\right)$$

on $U_{x_j}^*$ and $U_{y_j}^*$ respectively.
Proposition 4

Remark: Note that \(\left(\frac{2t}{\pi}\right)^{n-1}\) and \(|at|^{1/6}\) are the normalization constants for \(e^{-t(x_1^2+\ldots+x_{n-1}^2)}\) and \(\Xi_1((at)^{1/3}x_n)\) respectively, i.e.

\[
\|\left(\frac{2t}{\pi}\right)^{n-1}e^{-t(x_1^2+\ldots+x_{n-1}^2)}\| = \| |at|^{1/6}\Xi_1((at)^{1/3}x)\| = 1
\]

Proof: One can follow the argument in [HS] or [BZ]. Note that the term \(|at|^{-1/6}\Xi_1((at)^{1/3}x)\) is the ground state of \(-\frac{d^2}{dx^2} + 9\gamma^2t^2x^4 - 6atx\). So it is also the first term of the asymptotic expansion. \(\square\)

Recall that we have defined \(e_{n,w}^k\) for a generalized trajectory between two critical points and the incidence number \(I(x,y)\) between two critical points. See \(\S 1\) for definitions. Also define \(Int_k = Int |\Omega^v(M)|\). With these definitions, we have

Proposition 4 (i)

\[Int_k e^{tf} \left( E_{x_j}^k(t) \right) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{2}k} e^{tk} \left( e_{x_j} + \sum_l I(y_l^k, x_j^k)e_{y_l^k}^0 + O(t^{-1}) \right)\]

(ii)

\[Int_k e^{tf} \left( E_{y_j^k}^0(t) \right) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}k} e^{tf(x_j^k)} \Xi_1(0) |a_j^{(k)}t|^{1/6} \left( e_{y_j^k}^0 + \sum_{l \neq j} \beta_l(t)e_{y_l^k}^0 + O(t^{-1}) \right)\]

(iii)

\[Int_{k+1} e^{tf} \left( E_{y_j^k}^l(t) \right) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}k} e^{tf(x_j^k)} \Xi_1(0) |a_j^{(k)}t|^{1/3} \delta \left( e_{y_j^k}^0 + \sum_{l \neq j} \beta_l(t)e_{y_l^k}^0 \right) + O(t^{-1})\]

Proof: We introduce the following notations. Let \(y_1^k, \ldots, y_{m_k}^k\) be all the birth-death critical points of index \(k\). Let

\[f(y_1^k) = \ldots = f(y_{r_1}^k) < f(y_{r_1+1}^k) = \ldots = f(y_{r_1+r_2}^k) < f(y_{r_1+r_2+1}^k) < \ldots < f(y_{r_1+\ldots+r_k-1}^k) = \ldots = f(y_{r_1+\ldots+r_k}^k)\]

where \(m_k = r_1 + \ldots + r_k\).

Also, for \(1 \leq q \leq l_k\) let

\[L_q^{(k)} = \{y_l^k | r_1 + \ldots + r_{q-1} + 1 \leq l \leq r_1 + \ldots + r_{q-1} + r_q\}\]

(i) It is clear from [HS],[BZ] that

\[\int_{W_{x_j}^k} e^{tf} E_{x_j}^k(t) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{2}k} e^{tk} \left( 1 + O(t^{-1}) \right)\]
Suppose \( y^k_i \in L_1^{(k)} \), let

\[
\partial W_{y^k_i}^{−1} = W_{y^k_i}^{−0} + i(y^k_i, x^k_j)W_{x^k_j}^{−} + \left( \sum_{i \neq j} i(y^k_i, x^k_j)W_{x^k_j}^{−} + \sum_i i(y^k_i, y_i^{k−1})W_{y_i^{k−1}}^{−1} \right)
\]

where \( i(y^k_i, x^k_j) \) is the (ordinary) incidence number between \( y^k_i \) and \( x^k_j \) defined in §1. Denote the expression inside the parenthesis by \( R \), the remainder term.

Therefore,

\[
\int_{\partial W_{y^k_i}^{−1}} e^{tf} E_{x^k_j}(t) = \int_{W_{y^k_i}^{−0}} e^{tf} E_{x^k_j}(t) + i(y^k_i, x^k_j) \int_{W_{x^k_j}^{−}} e^{tf} E_{x^k_j}(t) + \int_R e^{tf} E_{x^k_j}(t)
\]

But by Stoke’s Theorem,

\[
\int_{\partial W_{y^k_i}^{−1}} e^{tf} E_{x^k_j}(t) = \int_{W_{y^k_i}^{−0}} e^{tf} \left( d(t)E_{x^k_j}(t) \right) = \int_{W_{y^k_i}^{−0}} e^{tf} \left( \sum_i \lambda_i(t)E_{x^k_{j+i}}(t) \right)
\]

for some exponentially decaying functions \( \lambda_i(t) \). Since \( |E_{x^k_{j+i}}(t)(x)| \) decreases as \( e^{-t|f(x)−f(x^{k+j})|} \), \( \int_{\partial W_{y^k_i}^{−1}} e^{tf} E_{x^k_j}(t) \) is of smaller order compared with \( \int_{W_{y^k_i}^{−0}} e^{tf} E_{x^k_j}(t) \). The same is true for \( \int_R e^{tf} E_{x^k_j}(t) \).

Hence,

\[
\int_{W_{y^k_i}^{−0}} e^{tf} E_{x^k_j}(t) = -i(y^k_i, x^k_j) \int_{W_{x^k_j}^{−}} e^{tf} E_{x^k_j}(t) + O(t^{-1})
\]

\[
= I(y^k_i, x^k_j)(\frac{2t}{\pi})^{\frac{n−2k}{2}} 0 \frac{2t}{\pi} e^{tk} + O(t^{-1})
\]

One can show by using finite induction on \( q \) that for any \( y^k_i \in L_q^{(k)} \)

\[
\int_{W_{y^k_i}^{−0}} e^{tf} E_{x^k_j}(t) = I(y^k_i, x^k_j)(\frac{2t}{\pi})^{\frac{n−2k}{2}} e^{tk} + O(t^{-1})
\]

This proves (i).

(ii) A direct computation shows that

\[
\int_{W_{y^k_i}^{−0}} e^{tf} E_{y^k_i}(t) = (\frac{2t}{\pi})^{\frac{n−1−2k}{2}} e^{tf(y^k_i)x}(0) a^{(k)}_j t^{1/6} (1 + O(t^{-1}))
\]

Next note that by choosing a coordinate system \( (x^{(1)}_1, \ldots, x^{(j)}_k) \) on \( W_{y^k_i}^{−0} \) and extending it to a neighbourhood of \( W_{y^k_i}^{+} = W_{y^k_i}^{−0} \cup W_{y^k_i}^{+1} \), one can show as in [HS] p 276-8 that if \( x \neq y^k_i \),

\[
E_{y^k_i}(t) = (\frac{2t}{\pi})^{\frac{n−1}{2}} | at |^{1/6} e^{-td(x,y^k_i)} \left( dx^{(j)}_1 \wedge \ldots \wedge dx^{(j)}_k + O(t^{-1}) \right)
\]

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where $d(x, y_i^k)$ is the Agmon distance between $x$ and $y_i^k$, i.e. w.r.t. the metric $|df|^2 dg$, but it is not necessarily true for $x = y_i^k$. So we let

$$E_{y_j^k}(t)(y_i^k) = \left( \frac{2t}{\pi} \right)^{\frac{n-1}{2}} |at|^{1/6} e^{-td(y_i^k, y_j^k)} c_{ij}(t)$$

where $c_{ij}(t) \in \Lambda^k(T_{y_i^k}(M))$. By [HS],

$$|c_{ij}(t)| = O(e^{\epsilon t}) \text{ for any } \epsilon > 0$$

For $x \in U_{y_j^k} \cap W_{y_j^k}^0$,

$$f(x) = f(y_j^k) - (x_1^{(l)})^2 - \cdots - (x_k^{(l)})^2$$

Therefore,

$$\int_{W_{y_j^k}^0} e^{tf} E_{y_j^k}(t) = e^{tf(y_j^k)} \int_{U_{y_j^k} \cap W_{y_j^k}^0} e^{-t((x_1^{(l)})^2 + \cdots + (x_k^{(l)})^2)} E_{y_j^k}(t) + \text{ lower order terms}$$

By the stationary phase approximation formula ([D] p23-4), we have

$$\int_{W_{y_j^k}^0} e^{tf} E_{y_j^k}(t) = e^{tf(y_j^k)} \left( \frac{2 \pi}{e} \right)^{\frac{1}{2}} \left| at \right|^{1/6} \left( \prod \partial \right) (x_1, \ldots, x_k) + \text{ lower order terms}$$

since $d(y_j^k, y_i^k) = f(y_j^k) - f(y_i^k)$ and for some $\beta_{ij}(t)$. Hence (ii) is proved.

Note that (cf.[HS] p265, [HS1] p138)

$$|\beta_{ij}(t)| = O(e^{\epsilon t}) \text{ for any } \epsilon > 0$$

(iii) Since $d(t)E_{y_j^k}(t) = \lambda(t)E_{y_j^k}(t)$ for some $\lambda(t) \neq 0$ when $t$ is sufficiently large (this follows from (7) below) and

$$\text{Int}_{k+1} \left( e^{tf} d(t)E_{y_j^k}(t) \right) = \delta \left( \text{Int}_k e^{tf} E_{y_j^k}(t) \right)$$

by (ii), we have

$$\text{Int}_{k+1} e^{tf} \left( E_{y_j^k} \right) = \left( \frac{2 \pi}{e} \right)^{\frac{1}{2}} \left( \prod \partial \right) (x_1, \ldots, x_k) \int_{at} \left( \frac{\lambda(t)}{at} \right)^{1/6} \left( \delta e_{y_j^k} + \sum_{l \neq j} \beta_{ij}(t)e_{y_l^k} \right) + O(t^{-1})$$

(iii) is proved by noting that

$$\lambda(t) = \sqrt{e_1} \left| at \right|^{1/2} + \text{ lower order terms}$$
Using Proposition 4(i), we can prove

**Theorem 2’**: (Helffer-Sjöstrand) There exist orthonormal bases \( \{E_{x,\gamma}^k(t)\} \) of \( \Omega_{small}^k(M, t) \), \( \{E_{y,\gamma}^0(t), E_{y,\gamma}^1(t)\} \) of \( \Omega_{large,k,j}^*(M, t) \) s.t.

\[
< E_{x,\gamma}^{k+1}(t), d(t) E_{x,\gamma}^k(t) > = e^{-t} \left( \frac{|t|}{\pi} \sum_{\gamma} \epsilon_{\gamma}^{new} + O(t^{-1/2}) \right)
\]

\[
< E_{y,\gamma}^1(t), d(t) E_{y,\gamma}^0(t) > = \sqrt{e_1(a_{j1}^{(k)})^{1/3}} + O(t^{1/6})
\]

\[
< E_{y,\gamma}^{j_1}(t), d(t) E_{y,\gamma}^{j_2}(t) > = 0 \text{ if } j_1 \neq j_2 \text{ for } t \text{ sufficiently large}
\]

\[
< E_{x,\gamma}^k(t), d(t) E_{x,\gamma}^l(t) >=< E_{y,\gamma}^l(t), d(t) E_{x,\gamma}^k(t) >= 0 \text{ for } t \text{ sufficiently large}
\]

where \( \sum_{\gamma} \epsilon_{\gamma}^{new} = I(x_i^{k+1}, x_j^k) \) is the incidence number between \( x_i^{k+1} \) and \( x_j^k \) defined in §1.

**Proof**: We prove the first and second equalities, the others are obvious.

Let \( d(t) E_{x,\gamma}^k(t) = \sum_l \lambda_{ij}(t) E_{x_{i,j}^{k+1}}(t) \) for some \( \lambda_{ij}(t) \).

Since

\[
\text{Int}_{k+1}e^{tf} \left( d(t) E_{x,\gamma}^k(t) \right) = \delta \text{Int}_k e^{tf} \left( E_{x,\gamma}^k(t) \right)
\]

By Proposition 4(i), we have

\[
\sum_l \lambda_{ij}(t) \left( \frac{2l}{\pi} \right)^{\frac{n-2k-2}{2}} e^{t(k+1)} \left( e_{x_{i,j}^{k+1}} + \sum_l I(y_i^{k+1}, x_i^{k+1}) e_{y_i^{k+1}} + O(t^{-1}) \right)
\]

\[
= \left( \frac{2l}{\pi} \right)^{\frac{n-2k-2}{2}} \left( \delta e_{x_{i,j}^k} + \sum_l I(y_i^k, x_j^k) e_{y_i^k} + O(t^{-1}) \right)
\]

By comparing the coefficients of \( e_{x_{i,j}^{k+1}} \),

\[
\lambda_{ij}(t) \left( \frac{2l}{\pi} \right)^{-1/2} e^t = i(x_i^{k+1}, x_j^k) + \sum_l I(y_i^k, x_j^k) i(x_i^{k+1}, y_i^k) + O(t^{-1})
\]

But by definition of \( I(x, y), \)

\[
I(x_i^{k+1}, x_j^k) = i(x_i^{k+1}, x_j^k) + \sum_l I(y_i^k, x_j^k) i(x_i^{k+1}, y_i^k) = \sum_{\gamma} \epsilon_{\gamma}^{new}
\]

Hence the first equality is proved. For the second equality, note that \( E_{y,\gamma}^l(t) \) are normalized eigenforms in \( \Omega_{large,k,j}^*(M, t) \) and

\[
\|d(t) E_{y,\gamma}^l(t)\|^2 =< E_{y,\gamma}^l(t), \Delta^{k}(t) E_{y,\gamma}^l(t) > = c_1 |a_{j1}^{(k)}|^2 + \text{lower order terms}
\]

(7)

So,
< e_{y_j}^1(t), d(t) E_{y_j}^0(t) >= \pm \sqrt{\epsilon_1(| a_j^{(k)} | t)^{1/3}} + \text{lower order terms} \quad \Box

In view of Proposition 4, define \( f^k(t) : \Omega^*_0(M,t) \to C^k(M,f) \) s.t.

\[
\begin{align*}
  f^k(t) \left( E_{x_j}^k(t) \right) &= (\pi/2^k) e^{-tk} \text{Int}_k e^{t f} \left( E_{x_j}^k(t) \right) \\
  f^k(t) \left( E_{y_j}^0(t) \right) &= \text{Int}_k e^{t f} \left( E_{y_j}^0(t) \right) \\
  f^{k+1}(t) \left( E_{y_j}^1(t) \right) &= \text{Int}_{k+1} e^{t f} \left( E_{y_j}^1(t) \right)
\end{align*}
\]

Let

\[
\begin{align*}
  \left( \Omega^*_0(M,t), d(t) \right) &= \left( \Omega^*_{\text{small}}(M,t), e^{t(\pi/2t)^{1/2}} d(t) \right) \\
  &\downarrow \left( \text{Int}_{k,j} \left( \Omega^*_{\text{large},k,j}(M,t), d(t) \right) \right)
\end{align*}
\]

Also define

\[
\begin{align*}
  \tilde{e}_{x_j}^k &= e_{x_j}^k + \sum_l I(l_j^k, x_j^k) e_{y_l}^0 \\
  \tilde{e}_{y_j}^0 &= e_{y_j}^0 \\
  \tilde{e}_{y_j}^1 &= \delta(e_{y_j}^0)
\end{align*}
\]

Then we have

**Proposition 5** \( f^*(t) : (\Omega^*_0(M,t), d(t)) \to (C^*(M,f), \delta) \) is a morphism of co-chain complexes s.t.

\[
f^k(t) \left( E_{x_j}^k(t) \right) = \tilde{e}_{x_j}^k + O(t^{-1})
\]

with \( f^k(t) \left( E_{y_j}^0(t) \right), f^{k+1}(t) \left( E_{y_j}^1(t) \right) \) given by (ii) and (iii) in Proposition 4.

Let the matrix associated to the linear map \( f^k(t) \) w.r.t. the bases

\[
\begin{align*}
  \left\{ E_{x_j}^k(t), E_{y_j}^0(t), E_{y_j}^1(t) \right\} \quad \text{and} \quad \left\{ \tilde{e}_{x_j}^k, e_{y_j}^0, e_{y_j}^1 \right\}
\end{align*}
\]

be

\[
F^k(t) = \begin{pmatrix}
  I & O(e^{kt}) & N_k^k(t) \\
  O(t^{-1}) & M^k(t) & N_k^k(t) \\
  O(t^{-1}) & O(e^{kt}) & N_k^k(t)
\end{pmatrix}
\]

where \( O(t^{-1}) \) in a certain entry of the matrix means that the corresponding entry is of the order \( O(t^{-1}) \). Here \( M^k(t) \) is \( \left( \frac{\pi}{2^k} \right)^{n-2k} \Xi_1(0) \) times the following
Observe that localized at the corresponding birth-death points.

\[
\begin{pmatrix}
| a_1^{(k)} t | \hat{e}^{t f(y_1^k)} & | a_2^{(k)} t | \hat{e}^{t (f(y_1^k) - c_0)} \beta_{12}(t) & \cdots & | a_{m_k}^{(k)} t | \hat{e}^{t (f(y_1^k) - c_0)} \beta_{1m_k}(t) \\
| a_1^{(k)} t | \hat{e}^{t f(y_2^k)} \beta_{21}(t) & | a_2^{(k)} t | \hat{e}^{t f(y_2^k)} & \cdots & | a_{m_k}^{(k)} t | \hat{e}^{t f(y_2^k)} \beta_{2m_k}(t) \\
\vdots & \vdots & \ddots & \vdots \\
| a_1^{(k)} t | \hat{e}^{t f(y_{m_k}^k)} \beta_{m_k+1}(t) & | a_2^{(k)} t | \hat{e}^{t f(y_{m_k}^k)} \beta_{m_k+2}(t) & \cdots & | a_{m_k}^{(k)} t | \hat{e}^{t f(y_{m_k}^k)} \\
\end{pmatrix}
\]

Note that we have used the fact that the birth-death points are indexed such that

\[f(y_1^k) \leq f(y_2^k) \leq \cdots \leq f(y_{m_k}^k)\]

so that the above matrix is approximately lower triangular. Hence, \(M^k(t)\) is invertible for sufficiently large \(t\).

Also, let

\[A^k(t) = \text{diag}\left( (\sqrt{\epsilon_1} | a_1^{(k)} t |^{1/3})^{-1}, \ldots, (\sqrt{\epsilon_1} | a_{m_k}^{(k)} t |^{1/3})^{-1} \right)\]

Define for \(1 \leq j \leq m_k'\),

\[\hat{E}_{y_j^k}^0(t) = (M^k(t))^{-1} \left( E_{y_j^k}^0(t) \right)\]

\[\hat{E}_{y_j^k}^1(t) = (M^k(t)A^k(t))^{-1} E_{y_j^k}^1(t)\]

Observe that \(\{\hat{E}_{y_j^k}^j(t)\}\) is approximately orthogonal whose elements are still localized at the corresponding birth-death points.

Let

\[B^k(t) = \begin{pmatrix} I & 0 & 0 \\ 0 & (M^k(t))^{-1} & 0 \\ 0 & 0 & (M^{k-1}(t)A^{k-1}(t))^{-1} \end{pmatrix}\]

Then the matrix associated to \(f^k(t)\) w.r.t. the new bases

\[\{E_{x_j^k}(t), \hat{E}_{y_j^k}^0(t), \hat{E}_{y_j^k}^1(t)\}\] and \(\{\hat{e}_{x_j^k}, e_{y_j^k}^0, e_{y_j^k}^1\}\)

is

\[F^k(t)B^k(t) = \begin{pmatrix} I & O(e^{kt}) & N_1^k(t) \\ O(t^{-1}) & M^k(t) & N_2^k(t) \\ O(t^{-1}) & O(e^{kt}) & N_3^k(t) \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & (M^k(t))^{-1} & 0 \\ 0 & 0 & (M^{k-1}(t)A^{k-1}(t))^{-1} \end{pmatrix}\]
Suppose that
\[
\delta^{(k-1)}(k-1) = \begin{pmatrix}
\delta_{11}^{(k-1)} & \delta_{12}^{(k-1)} & \delta_{13}^{(k-1)} \\
\delta_{21}^{(k-1)} & \delta_{22}^{(k-1)} & \delta_{23}^{(k-1)} \\
\delta_{31}^{(k-1)} & \delta_{32}^{(k-1)} & \delta_{33}^{(k-1)}
\end{pmatrix}
\]
w.r.t. the bases \(\{\hat{e}_x^k, e_0^k, e_1^k\}\).

Then,
\[
F^k(t)B^k(t) = \begin{pmatrix}
I & O(t^{-1}) & \delta_{13}^{(k-1)} + O(t^{-1}) \\
O(t^{-1}) & I & \delta_{23}^{(k-1)} + O(t^{-1}) \\
O(t^{-1}) & O(t^{-1}) & \delta_{33}^{(k-1)} + O(t^{-1})
\end{pmatrix}
\]

To see this, it suffices to show
\[
\text{Int}_{k \in [k]} (E^1_{yj} - 1)(t) = \delta(e_0^{y_1} + O(t^{-1})
\]

But by Proposition 4(iii),
\[
\text{Int}_{k \in [k]}(E^1_{yj} - 1)(t) = \delta \left( \sum_i (M^{k-1}(t)A^{k-1}(t))_{ij} e_0^{y_i} + O(t^{-1}) \right)
\]

Using the definition of \(E^1_{yj} - 1(t)\), (8) follows.

Hence, with the definition of \(\hat{e}^i_{y_j} - 1\) on p18, we finally have

**Theorem 2:** \(f^*(t) : (\Omega^*_0(M, t), d(t)) \rightarrow (C^*_0(M, f), \delta)\) is a morphism of cochain complexes s.t.

\[
f^*(t) = I + O(t^{-1})
\]
w.r.t. the bases \(\{E^i_{y_j} - 1(t), \hat{E}^i_{y_i} - 1(t)\}\) and \(\{\hat{e}^i_{y_j}, e^i_{y_i}\}\).

Inside \((C^*_0(M, f), \delta)\), there is a subcomplex \((C^*_{nd}(M, f), \delta)\) such that

\[
\dim C^{k}_{nd}(M, f) = m_k
\]

where \(m_k\) is the number of non-degenerate critical points of index \(k\). This subcomplex can be obtained by application of the following Lemma.

**Lemma 3** Suppose \((C^*, \delta)\) is a cochain complex such that

\[
C^* = \begin{cases}
\tilde{C}^* & \text{if } * \neq k, k + 1 \\
\tilde{C}^* \oplus R & \text{if } * = k \text{ or } k + 1
\end{cases}
\]

Let \(\dim(\tilde{C}^q) = n_q\), for \(0 \leq q \leq n\),

\[
\{e^q_{x_1}, \ldots, e^q_{x_n}\}
\] be a basis of \(\tilde{C}^q\)
so that
\[ \left\{ e_{x_1^k}, \ldots, e_{x_{n_k}^k}, e_y^0 \right\} \] is a basis of \( \tilde{C}^k \oplus R \)
and
\[ \left\{ e_{x_{n_k+1}^k}, \ldots, e_{x_{n_{k+1}^k}}, e_y^1 \right\} \] is a basis of \( \tilde{C}^{k+1} \oplus R \)
and w.r.t. the above bases,
\[
\delta^{(k)} = \begin{pmatrix}
i(x_{1}^{k+1}, x_{1}^k) & \cdots & i(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & i(x_{1}^{k+1}, y) \\
\vdots & \ddots & \vdots & \vdots \\
i(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & \cdots & i(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & i(x_{n_{k+1}}^{k+1}, y)
\end{pmatrix}
\]
Then with the following change of bases in \( C^k \) and \( C^{k+1} \),
\[
\left\{ e_{x_i^k}, e_y^0 \right\}_{1 \leq i \leq n_k} \rightarrow \left\{ e_{x_i^k} - i(y, x_i^k)e_y^0, e_y^0 \right\}_{1 \leq i \leq n_k}
\]
\[
\left\{ e_{x_i^{k+1}}, e_{y}^1 \right\}_{1 \leq i \leq n_{k+1}} \rightarrow \left\{ e_{x_i^{k+1}}, \delta e_y^0 = e_y^1 + \sum_{j=1}^{n_{k+1}} i(x_{j}^{k+1}, y)e_{x_j^{k+1}} \right\}_{1 \leq i \leq n_{k+1}}
\]
we have
\[
(i) \quad \delta^{(k)} = \begin{pmatrix}
i'(x_{1}^{k+1}, x_1^k) & \cdots & i'(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
i'(x_{n_{k+1}}^{k+1}, x_1^k) & \cdots & i'(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & 0
\end{pmatrix}
\]
where \( i'(x_{i}^{k+1}, x_j^k) = i(x_{i}^{k+1}, x_j^k) - i(x_{i}^{k+1}, y)i(y, x_j^k) \).
\[
(ii) \quad \delta^{(k-1)} = \begin{pmatrix}
i(x_{1}^{k}, x_{1}^{k-1}) & \cdots & i(x_{n_k}^{k}, x_{n_k-1}^{k-1}) \\
\vdots & \ddots & \vdots \\
i(x_{n_k}^{k}, x_{1}^{k-1}) & \cdots & i(x_{n_k}^{k}, x_{n_k-1}^{k-1}) \\
0 & \cdots & 0
\end{pmatrix}
\]
\[
\delta^{(k+1)} = \begin{pmatrix}
i(x_{1}^{k+2}, x_{1}^{k+1}) & \cdots & i(x_{n_{k+2}}^{k+2}, x_{n_{k+1}}^{k+1}) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
i(x_{n_{k+2}}^{k+2}, x_{1}^{k+1}) & \cdots & i(x_{n_{k+2}}^{k+2}, x_{n_{k+1}}^{k+1}) & 0
\end{pmatrix}
\]
Corollary: Let
\[ (C')^* = \begin{cases} 
C^n & \text{if } * \neq k \\
\text{Span}\{e_{x_i^k} - i(y, x_i^k)e_y^0\}_{1 \leq i \leq n_k} & \text{if } * = k
\end{cases} \]
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\[
(C')^* = \begin{cases}
0 & \text{if } * \neq k, k + 1 \\
e_0^0 & \text{if } * = k \\
\delta e_0^0 & \text{if } * = k + 1
\end{cases}
\]

Then
\[
(C^*, \delta) = \left( (C')^*, \delta \right) \oplus \left( (C'')^*, \delta \right)
\]

In particular, \( (C')^* \) is a subcomplex of \( (C^*, \delta) \). Both of them calculate the same cohomology.

**Remark:** For the application of Lemma 3, it is clear that \( \tilde{C}^* \) need not be generated only by cells corresponding to non-degenerate critical points, but it can also be generated by cells corresponding to birth-death points. Let

\[k < f(y_1^k) \leq \ldots \leq f(y_{m_k}^k) < k + 1\]

We 'eliminate' \( y_1^k \) first by applying Lemma 3 and obtain a subcomplex \( ((C(1))^*, \delta) \). Note that

\[
\begin{align*}
e_0^0 &= e_0^0 - i(y^k_1, y^k_2)e_0^0 \\
e_1^1 &\in (C(1))^{k+1} \\
\delta e_0^0 &= e_1^1 + \ldots
\end{align*}
\]

Therefore, the assumptions in Lemma 3 are satisfied and we can apply Lemma 3 to 'eliminate' \( y_2^k \) and obtain \( ((C(2))^*, \delta) \). Hence by applying Lemma 3 repeatedly, \( (C_{nd}^*(M, f), \delta) \) is obtained.

**Proof of Lemma 3:** The Lemma can be proved by direct calculation.

**Theorem 2':** \( f^*(t) \mid_{\Omega_{\text{small}}^*(M, t)}: \left( \Omega^*_\text{small}(M, t), \tilde{d}(t) \right) \longrightarrow (C^*(M, f), \delta) \) is an injective homomorphism of cochain complexes whose image complex converges to \( (C_{nd}^*(M, f), \delta) \) in \( (C^*(M, f), \delta) \) as \( t \to \infty \), more precisely,

\[f^k(f) \left( E_{x_j^k}(t) \right) = \dot{e}_{x_j^k} + O(t^{-1}) \text{ in } C^*(M, f)\]

**Remark:** One can show by induction that

\[\dot{e}_{x_j^k} = e_{x_j^k} + \sum_l I(y^k_l, x_j^k)e_0^0 \in C_{nd}^k(M, f)\]

**Appendix**

**Sketch of Proof (of Theorem 1):**

Let \( C_{bd} \) be the set of birth-death critical points of \( f \),

\( C_{nd} \) be the set of non-degenerate critical points of \( f \),

\[0 = e^k_0 = \ldots = e^k_{m_k} \text{ be the smallest eigenvalues of } \bigoplus_{j \in C_{nd}} \Delta_j^k(1)\]

\[0 \leq e^k_{m_k+1} \leq e^k_{m_k+2} \leq \ldots \text{ be the eigenvalues of } \bigoplus_{j \in C_{nd}} \Delta_j^k(1)\]

Let \( \{ \Psi^k_l(1) \}_{l}^{\infty} \) be the eigenvectors corresponding to \( e^k_l \) of \( \bigoplus_{j \in \text{Crit}(f)} \Delta_j^k(1) \).
More generally, let \( \{ \Psi^j(t) \}_{t}^{\infty} \) be the eigenvectors corresponding to \( e^j t^{2/3} \) of \( \bigoplus_{j \in \text{Crit}(f)} \Delta^j(t) \).

Note that

\[
\Psi^j(t) = \begin{cases} 
U(t^{1/2}) \Psi^j(1) & \text{if } 1 \leq t \leq m_k \\
U(t^{1/3}) \Psi^j(1) & \text{if } m_k + 1 \leq t
\end{cases}
\]

Then Theorem 1 is essentially equivalent to

\[
\lim_{t \to \infty} \frac{E_i(t)}{t^{2/3}} = e^j_i
\]

The proof is divided into 2 steps.

(i) \( \lim_{t \to \infty} \frac{E_i(t)}{t^{2/3}} \leq e^j_i \)

This follows by similar arguments as in [S]. Let \( \{ J_j \}_{j \in \text{Crit}(f) \cup \{ 0 \}} \) be a partition of unity on \( M \) (cf. [S] p 27). Let \( \Psi^j(t) \) be an eigenvector of \( \Delta^j(t) \), define

\[
\varphi^j(t) = J_j(t) \Psi^j(t).
\]

Then \( \{ \varphi^j(t) \} \) form a set of approximate eigenvectors in \( L^2(\Lambda^k(M)) \) with

\[
< \varphi_m(t), \Delta^k(t) \varphi_n(t) > = e^j_i t^{2/3} \delta_{mn} + o(t^{2/3})
\]

by using the minimax principle, (i) follows.

(ii) \( \lim_{t \to \infty} \frac{E_i(t)}{t^{2/3}} \geq e^j_i \)

To prove (ii), one has to modify slightly the arguments in [S]. It suffices to construct, for \( e \in (e^k_i, e^k_{i+1}) \), a symmetric operator \( R(t) \) of rank \( t \) s.t.

\[
\Delta^k(t) \geq t^{2/3} e + R(t) + o(t^{2/3}) \ldots (*)
\]

To construct \( R(t) \) define \( \Delta^j_k(t) : C^\infty(\Lambda^k(M)) \to C^\infty(\Lambda^k(M)) \)

\[
\Delta^j_k(t) = \Delta^k + t^2 f_j + tA
\]

\[
f_j(x) = \begin{cases} 
| df(x) |^2 & \text{if } x \in U_j \\
0 & \text{if } x \notin U_j
\end{cases}
\]

Let \( 0 \leq E_1^{(j)}(t) \leq E_2^{(j)}(t) \leq \ldots \leq E_j^{(j)}(t) \leq \ldots \) be the eigenvalues of \( \Delta^j_k(t) \),

\( \Psi_1^{(j)}(t), \Psi_2^{(j)}(t), \ldots, \Psi_j^{(j)}(t), \ldots \) be the corresponding eigenvectors of \( \Delta^j_k(t) \).

For \( j \in \text{Crit}(f) \), let \( 0 \leq e_1^{(j)} \leq e_2^{(j)} \leq \ldots \leq e_j^{(j)} \leq \ldots \) be the eigenvalues of \( \Delta^j_k(1) \), then one can show that

\[
\lim_{t \to \infty} \frac{E_t^{(j)}}{t^{2/3}} = e_j^{(j)}
\]

Define \( n_j \) s.t. \( e_{n_j}^{(j)} < e < e_{n_j+1}^{(j)} \)

\[
P_j(t) = \begin{cases} 
\text{orthogonal projection onto } \text{span} \{ \Psi_l^{(j)} \}_{l \leq n_j} & \text{if } j \in \text{Crit}(f) \\
\text{orthogonal projection onto smallest eigenvector of } \Delta^j_k(t) & \text{if } j \in \text{Crit}(f)
\end{cases}
\]

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\[ R_j(t) = (\Delta^k_j(t) - t^{2/3}e) P_j(t) \]

Define

\[ R(t) = \sum_{j \in \text{Crit}(f)} J_j R_j(t) J_j \]

which is a symmetric operator of rank \( l \).

To verify (*), observe that by IMS localization formula (cf. [S] p28)

\[ \Delta^k(t) \geq t^{2/3}e \Delta J + \sum_{j \in \text{Crit}(f)} J_j \Delta^k_j(t) J_j + O(1) \]

Then (*) follows from the definition of \( R(t) \).

One finishes the proof by showing that

\[ \lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_k}(t) = 0 \]

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