Free vibration analysis of rectangular plates with central cutout

Kanak Kalita* and Salil Haldar

Abstract: A nine-node isoparametric plate element in conjunction with first-order shear deformation theory is used for free vibration analysis of rectangular plates with central cutouts. Both thick and thin plate problems are solved for various aspect ratios and boundary conditions. In this article, primary focus is given to the effect of rotary inertia on natural frequencies of perforated rectangular plates. It is found that rotary inertia has significant effect on thick plates, while for thin plates the rotary inertia term can be ignored. It is seen that the numerical convergence is very rapid and based on comparison with experimental and analytical data from literature, it is proposed that the present formulation is capable of yielding highly accurate results. Finally, some new numerical solutions are provided here, which may serve as benchmark for future research on similar problems.

Subjects: Acoustical Engineering; Engineering Mathematics; Mathematics & Statistics for Engineers; Mechanical Engineering

Keywords: finite element method; FSDT; rotary inertia; natural frequency; cutout

1. Introduction
Rectangular plates are widely used across various engineering disciplines and from a technical viewpoint it becomes necessary to know the natural frequencies of such structures. It is well known that natural frequency of the plate depends significantly on its thickness, aspect ratios, and the boundary conditions. Research on free vibration of rectangular plates has a long established history as seen from the excellent comprehensive review articles by Leissa (1978a, 1978b, 1980a, 1980b, 1987a, 1987b), Liew, Xiang, and Kitipornchai (1995), Yamada and Irie (1987) and bibliographical information by Mackerle (1999) among many others. However, more than often it is seen that this research

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PUBLIC INTEREST STATEMENT
Cutouts are often provided in plate structures to meet some functional or esthetic requirement. These cutouts alter the free vibration characteristics of the plates. The focus of the manuscript is on the computation of natural frequencies of such perforated plates with-and-without considering the rotary inertia effect. For the thick plates, it is necessary to consider the first-order shear deformation theory (FSDT) for the displacements with middle-surface shear rotations. Due to the inherent robustness of the formulation and high accuracy of the results, the outcomes from this work are important to engineers and designers.
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is based on the classical Kirchhoff hypothesis, which neglects the effect of shear deformation and rotary inertia resulting in over-estimation of vibration frequencies. This problem is avoided in this article by following Mindlin’s hypothesis. Often these plates contain cutouts or holes which significantly alter the dynamic characteristic. Literature on plates with cutouts having various shapes – rectangular (Liew, Kitipornchai, Leung, & Lim, 2003; Srivastava, Datta, & Sheikh, 2004), circular (Malik & Singru, 2013), etc. can be found.

Although it is difficult to give an exact date for the invention of the finite element (FE) method, the method is born from the need to solve elasticity and structural analysis problems involving complex domain in civil engineering and aeronautics and received her real momentum in the 1960s and 1970s by the developments of Argyris and colleagues at the University of Stuttgart, Clough, and his colleagues at UC Berkeley and Zienkiewicz and colleagues at Swansea university (Zienkiewicz & Taylor, 2005). Since then, several FEs have been developed for the thin/thick plate analysis among which isoparametric elements are most widely used (Batoz, Bathe, & Ho, 1980; Clough & Tocher, 1965; Hrabok & Hrudey, 1984).

Yu (2009) used the Gorman method to calculate the dynamic repose of cantilever plates with attached point mass. Very recently bending and free vibration behavior of laminated soft core skew sandwich plate with stiff laminate face sheets was investigated using a recently developed C₀ FE model based on higher order zigzag theory (Chalak, Chakrabarti, Sheikh, & Iqbal, 2014). A new implementation of the ancient Chinese method called the Max-Min Approach and Homotopy Perturbation Method was presented by Bayat, Pakar, and Bayat (2011) to obtain natural frequency and corresponding displacement of tapered beams. Amabili and Carra (2012) experimentally studied the large-amplitude forced vibrations of a stainless-steel thin rectangular plate carrying different concentrated masses. Dozio (2011) used the trigonometric Ritz method for general vibration analysis of rectangular Kirchhoff plates. Mantari, Oktem, and Guedes Soares (2012) performed bending and free vibration analysis of multilayered plates and shells using higher order shear deformation theory. A number of other numerical methods can be used for the plate vibration problem (Algazin, 2010).

No literature is found on investigation of variations in calculated natural frequencies of perforated rectangular plates by considering both rotary inertia and without rotary inertia. This work makes a novel attempt to address the issue. The paper is structured in the following way. Section 1 gives a brief idea on the previous research works and motivation for the present work. Section 2 contains the FE formulation (for nine-node isoparametric element) necessary to generate the FEM code. Validation and convergence study is provided in Section 3. Some numerical examples are provided in Section 4 for plate with central cutout followed by summary bulletin of the study under Section 5. The numerical results are compared with existing literature and some novel data are provided.

2. FE formulation

In the current formulation, FEM has been used for free vibration analysis of the plate. The mid plane is assumed to be the reference plane. This was done using the theory of Mindlin plate where it is assumed that the normal to the mid plane of the plate remains straight but not necessarily normal to the deformed mid surface. The first-order shear deformation theory (FSDT) assumes the displacement through the thickness of the plate to be linear. However there is no change in thickness of the plate after deformation. Further, the normal stress throughout the thickness is ignored; a hypothesis which is also called the plane stress state. Though the shear strain is not neglected in this theory, the assumption that it is constant over the entire thickness of the plate is not true. Across the thickness of the plate the shear stress is known to be parabolic. Hence a shear correction factor is applied. The accuracy of solutions of the FSDT is strongly dependent on predicting better estimates for the shear correction factor. In this case, the shear correction factor is assumed to be 5/6. This ensures that the correct amount of internal energy is predicted by the theory. In past, this formulation has been used by the author for analysis of shell (Majumdar, Manna, & Haldar, 2010) and composite plates (Pandit, 2010).
Haldar, & Mukhopadhyay, 2007). A more detailed analysis on vibration of plates can be found in Chakraverty (2008).

A nine-node isoparametric element is used in the current FE formulation. One of the main advantages of the element is that any form of plate can be well managed with an elegant mapping technique that can be defined as

\[ x = \sum_{r=1}^{9} N_r x_r \text{ and } y = \sum_{r=1}^{9} N_r y_r \]  

(1)

Thus by using this simple mapping technique the coordinates at any place within the element \((x, y)\) are expressed as the summation of the product of the Lagrange interpolation function \((N_r)\) and the coordinates of the \(r\)th nodal point \((x_r, y_r)\). Considering the bending rotations as independent field variables (since they are not derivatives of \(w\)), the effect of shear deformation may be incorporated as

\[ \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = \begin{Bmatrix} \theta_x - \frac{\partial w}{\partial x} \\ \theta_y - \frac{\partial w}{\partial y} \end{Bmatrix} \]

Since this is an isoparametric formulation the same interpolation functions used for element geometry have been used to describe the displacement field at any point within the element in terms of nodal variables as

\[ w = \sum_{r=1}^{9} N_r w_r, \theta_x = \sum_{r=1}^{9} N_r \theta_{xr} \text{ and } \theta_y = \sum_{r=1}^{9} N_r \theta_{yr} \]  

(2)

The stresses and strains of any continuous elastic material are connected by a linear relationship that is mathematically similar to Hooke’s law and may be expressed as

\[ \{\sigma\} = |D| \{\epsilon\} \]

(3)

where

\[ \{\sigma\} = \begin{bmatrix} M_x & M_y & M_{xy} & Q_x & Q_y \end{bmatrix} \]

(4)

\[ \{\epsilon\} = \begin{bmatrix} -\partial \theta_x / \partial x \\ -\partial \theta_y / \partial y \\ -\partial \theta_x / \partial y - \partial \theta_y / \partial x \\ \partial w / \partial x - \theta_x \\ \partial w / \partial y - \theta_y \end{bmatrix} \]

(5)

Using Equations (2) and (5),

\[ -\frac{\partial \theta_x}{\partial x} = -\left( \frac{\partial N_r}{\partial x} \right) \theta_{xr} - \frac{\partial \theta_y}{\partial y} = -\left( \frac{\partial N_r}{\partial y} \right) \theta_{yr} \]

\[ \frac{\partial w}{\partial x} - \theta_x = -\left( \frac{\partial N_r}{\partial x} \right) w_r - \left( N_r \right) \theta_{xr} \]

\[ \frac{\partial w}{\partial y} - \theta_y = -\left( \frac{\partial N_r}{\partial y} \right) w_r - \left( N_r \right) \theta_{yr} \]
From Equations (2) and (5), the strain vector may be expressed as

\[
[D] = \begin{bmatrix}
D_{11} & D_{12} & 0 & 0 & 0 \\
D_{21} & D_{22} & 0 & 0 & 0 \\
0 & 0 & D_{33} & 0 & 0 \\
0 & 0 & 0 & D_{44} & D_{45} \\
0 & 0 & 0 & D_{54} & D_{55}
\end{bmatrix}
\]  

(6)

where

\[
D_{11} = D_{22} = \frac{E}{(1 - \nu^2)}; D_{12} = D_{21} = \nu D_{11}; D_{33} = \frac{E}{2(1 + \nu)}; D_{44} = D_{55} = \frac{Eh^3}{12(1 - \nu^2)}; D_{45} = D_{54} = \nu D_{44}
\]

From Equations (2) and (5), the strain vector may be expressed as

\[
\{\epsilon\} = \sum_{r=1}^{9} [B]_r \{\delta_r\}_e
\]

(7)

where \([B]\) is the strain displacement matrix containing interpolation functions and their derivatives.

Using the virtual work method the stiffness may be expressed as

\[
[K] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T[D][B]|J|d\xi d\eta
\]

(8)

where \(|J|\) is the determinant of the Jacobian matrix.

Similarly the consistent mass matrix may be expressed as

\[
[M] = \rho h \int_{-1}^{1} \int_{-1}^{1} \left( [N_w]^T[N_w] + \frac{h^2}{12} [N_{wx}]^T[N_{wx}] + \frac{h^2}{12} [N_{wy}]^T[N_{wy}] \right) |J|d\xi d\eta
\]

(9)

The global stiffness matrix \([K_0]\) and global mass matrix \([M_0]\) are calculated by assembling individual stiffness matrix and individual mass matrix of all the elements. Using equation of motion,

\[
[K_0] = \omega^2 [M_0]
\]

(10)

The boundary conditions used are:

Simply supported condition (denoted by S):

\[
w = \theta_x = 0, \text{ at boundary line parallel to } x\text{-axis.}
\]

\[
w = \theta_y = 0, \text{ at boundary line parallel to } y\text{-axis.}
\]

Clamped condition (denoted by C):

\[
w = \theta_x = \theta_y = 0
\]

Free boundary condition (denoted by F):

\[
w \neq 0, \theta_x \neq 0, \theta_y \neq 0
\]

3. Convergence and validation study

Example 1  CSCS rectangular plates with a centrally located cutout
Two opposite edges of the rectangular plate are clamped, while the other two edges are simply supported. The physical dimensions of the rectangular plate (Figure 1) are similar to the experimental results quoted by Aksu and Ali (1976). The aspect ratio ($a/b$) of the plate is $9/8$, Poisson’s ratio 0.3, and thickness ratio ($h/a$) 0.01. The accuracy of the solution is validated by comparing it with the experimental results of Aksu and Ali (1976), the FE solution of Lam, Hung, and Chow (1989), and solutions by Liew, Xiang, and Kitipornchai (1993), they had used Ritz procedure to obtain the results. Excellent convergence in results of rectangular plate with existing literature is seen at 18*18 mesh. Two different mass lumping schemes are introduced in this paper. In the first lumping scheme, the effect of in-plane and transverse movements of mass has been considered. In the second mass lumping scheme, the effect of rotary inertia as well as transverse and in-plane movements of mass has been considered. In this paper, they are called as LSWORI (mass lumping scheme without rotary inertia) and LSWRI (mass lumping scheme with rotary inertia). The difference in natural frequencies with- and without- rotary inertia (for 18*18 mesh) is calculated and shown as percent variation. For example, in Table 1 LSWRI (10*10) means present solution with $10 \times 10$ mesh divisions considering rotary inertia and LSWORI (10*10) present solution with $10 \times 10$ mesh divisions without rotary inertia. The data within the brackets such as (10*10), (18*18), etc. indicate the different mesh sizes.

Example 2  Square plates with square cutouts

![Figure 1. CSCS rectangular plate with central cutout.](image)

| Cutout size | Source | Mode | 1 | 2 | 3 | 4 |
|-------------|--------|------|---|---|---|---|
| $\frac{a}{3} \times \frac{b}{3}$ | LSWRI(9*9) | 34.1099 | 54.2274 | 64.0188 | 95.4592 |
| | LSWRI(12*12) | 34.1024 | 54.1605 | 63.9469 | 95.4196 |
| | LSWRI(15*15) | 32.5430 | 58.1617 | 65.2617 | 96.8441 |
| | LSWRI(18*18) | 32.5421 | 58.1577 | 65.2573 | 96.8386 |
| | LSWORI(9*9) | 34.1161 | 54.2390 | 64.0333 | 95.4941 |
| | LSWORI(12*12) | 34.1085 | 54.1720 | 63.9614 | 95.4545 |
| | LSWORI(15*15) | 32.5474 | 58.1716 | 65.2744 | 96.8726 |
| | LSWORI(18*18) | 32.5465 | 58.1676 | 65.2700 | 96.8670 |
| | % Variation | 0.0135 | 0.0170 | 0.0195 | 0.0294 |
| Aksu and Ali (1976) | 33.2200 | 53.0100 | 61.9100 | 91.8700 |
| Lam et al. (1989) | 34.0400 | 54.5700 | 65.0500 | 95.3800 |
| Experimental (Aksu & Ali, 1976) | 33.8300 | 53.9900 | 62.4900 | 95.0300 |
| Liew et al. (1993) | 32.4250 | 53.4260 | 62.3530 | 94.8390 |
**Figure 2.** Rectangular plate with central cutout.

**Table 2.** Non-dimensional frequency parameters $\lambda = \omega a^2 \sqrt{h/D}$ of SSSS square plate with central square cutout

| $h/a$ | Source | Mode | Cutout size ($m \times n$) | 0.2$a \times 0.2$a | 0.4$a \times 0.4$a | 0.6$a \times 0.6$a | 0.8$a \times 0.8$a |
|-------|--------|------|-----------------------------|---------------------|---------------------|---------------------|---------------------|
| 0.001 | LSWRI  | 1    | 19.127                      | 20.752              | 28.372              | 57.392              |
|       |        |      | (19.200)                    | (20.807)            | (28.453)            | (57.512)            |
|       |        |      | (20.080))                   | (21.000)            | (29.850)            | (58.070)            |
|       |        | 2    | 47.670                      | 41.051              | 42.628              | 69.634              |
|       |        | 3    | 47.670                      | 41.051              | 42.628              | 69.634              |
|       |        | 4    | 76.427                      | 71.408              | 68.833              | 87.305              |
|       | LSWORI | 1    | 19.127                      | 20.752              | 28.372              | 57.392              |
|       |        | 2    | 47.670                      | 41.051              | 42.628              | 69.634              |
|       |        | 3    | 47.670                      | 41.051              | 42.628              | 69.634              |
|       |        | 4    | 76.427                      | 71.408              | 68.833              | 87.305              |
| 0.1   | LSWRI  | 1    | 18.401                      | 19.816              | 26.297              | 45.825              |
|       |        |      | (18.679)                    | (20.246)            | (27.379)            | (51.465)            |
|       |        |      | (19.070)                    | (21.030)            | (27.540)            | (51.550)            |
|       |        | 2    | 42.754                      | 35.884              | 36.697              | 51.911              |
|       |        | 3    | 42.754                      | 35.884              | 36.697              | 51.912              |
|       |        | 4    | 67.555                      | 63.265              | 53.928              | 59.669              |
|       | LSWORI | 1    | 18.569                      | 20.115              | 27.126              | 51.129              |
|       |        | 2    | 43.319                      | 36.474              | 37.186              | 57.918              |
|       |        | 3    | 43.319                      | 36.474              | 37.186              | 57.918              |
|       |        | 4    | 69.030                      | 64.643              | 55.734              | 66.587              |
| 0.2   | LSWRI  | 1    | 16.942                      | 18.106              | 22.928              | 31.436              |
|       |        |      | (17.452)                    | (19.163)            | (25.688)            | (44.069)            |
|       |        |      | (17.550)                    | (19.270)            | (26.060)            | (44.100)            |
|       |        | 2    | 35.194                      | 29.538              | 29.506              | 33.027              |
|       |        | 3    | 35.194                      | 29.538              | 29.506              | 33.027              |
|       |        | 4    | 45.630                      | 46.570              | 39.418              | 34.772              |
|       | LSWORI | 1    | 17.431                      | 19.013              | 25.371              | 43.647              |
|       |        | 2    | 36.221                      | 30.938              | 32.592              | 46.000              |
|       |        | 3    | 36.221                      | 30.938              | 32.592              | 46.000              |
|       |        | 4    | 45.630                      | 46.570              | 43.552              | 48.653              |

Notes: Data in () indicate fundamental frequencies published in Reddy (1982) and in (()) indicate fundamental frequencies published in Hota and Padhi (2007).
The problem considers fundamental frequency of a square plate (i.e. $b/a = 1$) of isotropic material $\nu = 0.3$ having a central square cutout (i.e. $m = n$) of various sizes for different thickness ratios. Figure 2 is a schematic of the problem. The purpose is to verify the capability of the model to handle problems in the range of thick to thin plates. The first four modes are extracted for each case. The results along with the fundamental frequencies published in Reddy (1982) and Hota and Padhi (2007) are presented in Table 2 and it shows good agreement with the published results.

**Example 3  Square plates with rectangular cutouts**

A SSSS isotropic square plate having Poisson's ratio $\nu = 0.3$ with rectangular cutout of different sizes has been analyzed. Table 3 shows the non-dimensional fundamental frequency of the plate having different thickness ratios. The obtained results show a good agreement with the results published by Lee, Lim, and Chow (1990). The problem is solved for various thickness range, ranging from thin to thick plate.

| $h/a$ | Source     | Mode | Cutout size (m × n) | 0.4a × 0.2a | 0.8a × 0.4a | 0.6a × 0.2a | 0.6a × 0.4a |
|-------|------------|------|---------------------|-------------|-------------|-------------|-------------|
| 0.01  | LSWRI      | 1    | 19.036              | 23.501      | 18.964      | 22.572      |
|       |            | 2    | 40.993              | 28.024      | 32.134      | 33.775      |
|       |            | 3    | 46.597              | 55.464      | 47.845      | 46.253      |
|       |            | 4    | 73.718              | 64.747      | 68.616      | 69.294      |
|       | LSWORI     | 1    | 19.038              | 23.506      | 18.966      | 22.577      |
|       |            | 2    | 40.998              | 28.029      | 32.138      | 33.782      |
|       |            | 3    | 46.606              | 55.487      | 47.857      | 46.268      |
|       |            | 4    | 73.740              | 64.768      | 68.635      | 69.316      |
|       | Ref (Lee et al., 1990) | 1    | 19.010              | 23.580      | 18.980      | –           |
|       |            | 2    | 41.430              | 28.260      | 62.530      | –           |
|       |            | 3    | 46.580              | 55.640      | 47.810      | –           |
|       |            | 4    | 74.100              | 65.180      | 69.170      | –           |
| 0.1   | LSWRI      | 1    | 18.281              | 21.899      | 18.109      | 21.358      |
|       |            | 2    | 35.357              | 24.495      | 27.439      | 29.378      |
|       |            | 3    | 42.186              | 49.718      | 43.710      | 41.275      |
|       |            | 4    | 65.045              | 55.716      | 59.407      | 60.687      |
|       | LSWORI     | 1    | 18.480              | 22.313      | 18.326      | 21.761      |
|       |            | 2    | 35.753              | 24.909      | 27.737      | 29.873      |
|       |            | 3    | 42.854              | 51.420      | 44.605      | 42.393      |
|       |            | 4    | 66.379              | 57.097      | 60.530      | 62.136      |
| 0.2   | LSWRI      | 1    | 16.800              | 19.305      | 16.535      | 19.208      |
|       |            | 2    | 28.535              | 20.488      | 22.353      | 24.206      |
|       |            | 3    | 35.176              | 40.523      | 36.783      | 34.065      |
|       |            | 4    | 45.973              | 44.200      | 46.699      | 46.567      |
|       | LSWORI     | 1    | 17.397              | 20.503      | 17.189      | 20.411      |
|       |            | 2    | 29.406              | 21.651      | 23.114      | 25.507      |
|       |            | 3    | 36.523              | 44.295      | 38.680      | 36.812      |
|       |            | 4    | 45.973              | 46.853      | 46.699      | 47.468      |
## Table 4. Non-dimensional frequency parameters $\lambda = \omega a^2 \sqrt{h/D}$ of SSSS rectangular plate with central cutout

| $b/a$ | $h/a$ | Source | 1     | 2     | 3     | 4     | 5     | 6     |
|-------|-------|--------|-------|-------|-------|-------|-------|-------|
|       |       |        |       |       |       |       |       |       |
|       |       |        |       |       |       |       |       |       |
| Cutout size $(\frac{a}{L} \times \frac{a}{b})$ |       |        |       |       |       |       |       |       |
| 1     | 0.01  | LSWRI(15*15) | 19.098 | 47.487 | 47.488 | 76.225 | 95.513 | 103.162 |
|       |       | LSWORI(15*15) | 19.100 | 47.496 | 47.496 | 76.250 | 95.551 | 103.220 |
|       |       | % variation  | 0.010 | 0.018 | 0.018 | 0.032 | 0.039 | 0.056 |
|       |       | Mode Shapes  | ![Image](image1) | ![Image](image2) | ![Image](image3) | ![Image](image4) | ![Image](image5) |
|       | 0.1   | LSWRI(15*15) | 18.402 | 42.742 | 42.741 | 67.510 | 82.323 | 88.856 |
|       |       | LSWORI(15*15) | 18.571 | 43.306 | 43.306 | 68.986 | 84.277 | 92.132 |
|       |       | % variation  | 0.908 | 1.303 | 1.305 | 2.139 | 2.319 | 3.556 |
|       |       | Mode Shapes  | ![Image](image6) | ![Image](image7) | ![Image](image8) | ![Image](image9) | ![Image](image10) |
| 2     | 0.01  | LSWRI(15*15) | 11.427 | 18.941 | 27.632 | 33.664 | 44.926 | 49.538 |
|       |       | LSWORI(15*15) | 11.428 | 18.942 | 27.635 | 33.669 | 44.934 | 49.550 |
|       |       | % variation  | 0.009 | 0.008 | 0.014 | 0.016 | 0.019 | 0.025 |
|       |       | Mode Shapes  | ![Image](image11) | ![Image](image12) | ![Image](image13) | ![Image](image14) | ![Image](image15) |
|       | 0.1   | LSWRI(15*15) | 11.082 | 18.012 | 24.209 | 31.179 | 41.152 | 45.410 |
|       |       | LSWORI(15*15) | 11.176 | 18.143 | 24.517 | 31.606 | 41.772 | 46.342 |
|       |       | % variation  | 0.834 | 0.719 | 1.257 | 1.352 | 1.484 | 2.011 |
|       |       | Mode Shapes  | ![Image](image16) | ![Image](image17) | ![Image](image18) | ![Image](image19) | ![Image](image20) |
| Cutout size $(\frac{a}{L} \times \frac{a}{b})$ |       |        |       |       |       |       |       |       |
| 1     | 0.01  | LSWRI(15*15) | 19.718 | 42.674 | 42.674 | 72.658 | 88.398 | 120.638 |
|       |       | LSWORI(15*15) | 19.720 | 42.681 | 42.681 | 72.680 | 88.426 | 120.699 |
|       |       | % variation  | 0.013 | 0.017 | 0.017 | 0.030 | 0.031 | 0.050 |
|       |       | Mode Shapes  | ![Image](image21) | ![Image](image22) | ![Image](image23) | ![Image](image24) | ![Image](image25) |
|       | 0.1   | LSWRI(15*15) | 18.933 | 37.681 | 37.681 | 64.604 | 73.120 | 99.084 |
|       |       | LSWORI(15*15) | 19.170 | 38.218 | 38.219 | 65.968 | 74.493 | 101.987 |
|       |       | % variation  | 1.237 | 1.406 | 1.406 | 2.068 | 1.842 | 2.847 |
|       |       | Mode Shapes  | ![Image](image26) | ![Image](image27) | ![Image](image28) | ![Image](image29) | ![Image](image30) |
| 2     | 0.01  | LSWRI(15*15) | 11.496 | 19.494 | 32.209 | 36.989 | 47.922 | 48.023 |
|       |       | LSWORI(15*15) | 11.497 | 19.496 | 32.214 | 36.994 | 47.932 | 48.033 |
|       |       | % variation  | 0.006 | 0.008 | 0.013 | 0.013 | 0.020 | 0.020 |
|       |       | Mode Shapes  | ![Image](image31) | ![Image](image32) | ![Image](image33) | ![Image](image34) | ![Image](image35) |
|       | 0.1   | LSWRI(15*15) | 11.176 | 18.712 | 30.424 | 32.622 | 43.806 | 44.220 |
|       |       | LSWORI(15*15) | 11.244 | 18.840 | 30.825 | 32.961 | 44.504 | 44.904 |
|       |       | % variation  | 0.605 | 0.679 | 1.303 | 1.030 | 1.567 | 1.523 |
### Table 5. Non-dimensional frequency parameters $\lambda = \omega a^2 \sqrt{\rho h/D}$ of CCCC rectangular plate with central cutout

| $b/a$ | $h/a$ | Source | Modes |
|-------|-------|--------|-------|
|       |       |        | 1    | 2    | 3    | 4    | 5    | 6    |
|       |       |        | 0.01 | LSWRI($15\times15$) | 36.612 | 69.424 | 69.424 | 103.715 | 126.563 | 140.943 |
|       |       |        |      | LSWOR($15\times15$) | 36.617 | 69.438 | 69.438 | 103.752 | 126.617 | 141.015 |
|       |       |        |      | % variation | 0.014 | 0.020 | 0.020 | 0.036 | 0.043 | 0.065 |
|       |       |        |      | Mode shapes | | | | | | |
|       |       |        | 0.1  | LSWRI($15\times15$) | 33.164 | 57.583 | 57.584 | 83.949 | 98.699 | 108.815 |
|       |       |        |      | LSWOR($15\times15$) | 33.505 | 58.258 | 58.258 | 85.592 | 100.670 | 112.466 |
|       |       |        |      | % variation | 1.019 | 1.157 | 1.157 | 1.920 | 1.958 | 3.246 |
|       |       |        | 0.1  | LSWRI($15\times15$) | 24.828 | 31.437 | 45.704 | 53.760 | 61.532 | 68.849 |
|       |       |        |      | LSWOR($15\times15$) | 24.830 | 31.440 | 45.712 | 53.768 | 61.546 | 68.865 |
|       |       |        |      | % variation | 0.009 | 0.009 | 0.018 | 0.015 | 0.023 | 0.023 |
|       |       |        |      | Mode shapes | | | | | | |
|       |       |        | 0.1  | LSWRI($15\times15$) | 23.093 | 28.614 | 40.848 | 44.566 | 53.409 | 58.666 |
|       |       |        |      | LSWOR($15\times15$) | 23.254 | 28.808 | 41.407 | 45.042 | 54.265 | 59.501 |
|       |       |        |      | % variation | 0.694 | 0.673 | 1.349 | 1.056 | 1.576 | 1.404 |

### Cutout size ($\frac{1}{4} \times \frac{1}{4}$)

| $b/a$ | $h/a$ | Source | Modes |
|-------|-------|--------|-------|
|       |       |        | 1    | 2    | 3    | 4    | 5    | 6    |
|       |       |        | 0.01 | LSWRI($15\times15$) | 42.900 | 64.218 | 64.219 | 99.137 | 113.253 | 153.247 |
|       |       |        |      | LSWOR($15\times15$) | 42.909 | 64.233 | 64.233 | 99.172 | 113.291 | 153.330 |
|       |       |        |      | % variation | 0.019 | 0.022 | 0.022 | 0.035 | 0.033 | 0.054 |
|       |       |        |      | Mode shapes | | | | | | |
|       |       |        | 0.1  | LSWRI($15\times15$) | 38.441 | 53.350 | 53.349 | 80.678 | 85.836 | 114.779 |
|       |       |        |      | LSWOR($15\times15$) | 38.994 | 54.143 | 54.142 | 82.240 | 87.314 | 117.697 |
|       |       |        |      | % variation | 1.417 | 1.465 | 1.465 | 1.900 | 1.693 | 2.479 |
|       |       |        | 0.1  | LSWRI($15\times15$) | 28.626 | 31.744 | 46.198 | 49.912 | 64.666 | 66.359 |
|       |       |        |      | LSWOR($15\times15$) | 28.629 | 31.747 | 46.207 | 49.923 | 64.681 | 66.379 |
|       |       |        |      | % variation | 0.011 | 0.010 | 0.020 | 0.021 | 0.023 | 0.030 |
|       |       |        |      | Mode Shapes | | | | | | |
|       |       |        | 0.1  | LSWRI($15\times15$) | 26.470 | 28.672 | 39.825 | 43.527 | 55.121 | 57.161 |
|       |       |        |      | LSWOR($15\times15$) | 26.696 | 28.890 | 40.411 | 44.194 | 55.942 | 58.367 |
|       |       |        |      | % variation | 0.847 | 0.755 | 1.450 | 1.510 | 1.467 | 2.067 |
4. Results and discussion

4.1. Uniform thickness rectangular plates with central cutout

Isotropic rectangular plates with central cutout (Figure 2) of different sizes are analyzed by considering the two different mass lumping schemes. The analysis is carried out for two different thickness ratios ($h/a = 0.01$ and 0.1) and two aspect ratios ($b/a = 1$ and 2). Poisson’s ratio is taken as 0.3. Since this study attempts to study the frequency variations in isotropic plate with respect to boundary conditions, aspect ratio, and thickness ratio, the elastic moduli is so adjusted that it depends on thickness and the resulting frequency is obtained in non-dimensional form. The frequency parameters for SSSS and CCCC plate investigated with and without rotary inertia are presented here in non-dimensional form in Tables 4 and 5. It is observed that with increase in cutout size the frequency parameters increase, due to reduction in mass of the plate. For the sake of brevity modes shapes are shown for simply supported and clamped plates for thickness $h/a = 0.01$ only. It is interesting to note that for square plates with square cutouts the modes shapes at $h/a = 0.01$ are similar for both simply supported and clamped plates.

The aspect ratios, thickness ratios, and boundary conditions have significant influence on the modes of vibration of rectangular plates. It is seen that as the thickness of the plate increases the frequency parameter decreases clearly showing the effect of rotary inertia and shear deformation on free vibration of plates. If rotary inertia are omitted there is an overestimation of natural frequencies. Also the obtained results clearly show that frequency parameters increase if more constraints are included. For example, SSSS have lower frequency than CCCC due to clamping in all four sides in the later. This means that as the constraints on the edges increases the flexural rigidity of the plate increases and hence there is an increase in the frequency. It is observed that the frequency decreases as the aspect ratio increases for all types of boundary conditions. It is also seen that as the thickness of the plates increases the percentage variation between LSWRI and LSWORI results increases.

5. Conclusions

In this paper, the dynamic characteristics of thick and thin, rectangular plates with central cutouts have been investigated by incorporating the FSDT in the FE method. The non-dimensional frequency parameters for plates of various aspect ratio, boundary conditions, and thickness ratios have been computed. The convergence of the eigen solutions was first checked and the results were then validated by comparing with known experimental and numerical data. It is evident that the results are within reasonable agreement of the published literature. A maximum deviation of 3% was seen in the worst case and in most cases the present results were within 1% of established results. This concludes beyond doubt the accuracy and rigor of the present formulation. Thus, the formulation presented here is theoretically sound, clear, and simple and does not require any complicated mathematical knowledge. The following conclusions are drawn based on the study:

- For thick plates, rotary inertia is very significant; for thin plates and shells, rotary inertia has no effect.
- With increase in thickness ratio the percentage variation in frequency parameters calculated with and without rotary inertia increases.
- The above-said effect decreases with increase in the aspect ratio for the same boundary condition.
- With increase in thickness ratio the frequency decreases.
- The increase in the cutout area causes fundamental frequency to increase.
- Natural frequency is lowest when an edge is kept free, followed by a simply supported edge and maximum for clamped edge i.e. natural frequencies increase if constraints at the boundary increase.
Due to the inherent features of the current analytical solution, the present findings will be useful as benchmark solutions for evaluating other analytical and numerical methods. Future works will show the results for plates with cutouts in plates with linearly and parabolically varying thickness.

Symbols

\[ [B] \] strain displacement matrix
\[ [D] \] rigidity matrix
\[ [K] \] global stiffness matrix
\[ [N] \] shape function
\[ [N_r] \] null matrix
\[ [M] \] consistent mass matrix
\[ [J] \] Jacobian matrix
\[ [N_r] \] interpolation function of the \( r \)th point
\[ [K_0] \] overall stiffness matrix
\[ [M_0] \] overall mass matrix
\[ w \] transverse displacement
\[ \theta_x, \theta_y \] total rotations in bending
\[ E \] modulus of elasticity
\[ G \] modulus of rigidity
\[ \nu \] Poisson's ratio
\[ h \] thickness of plate
\[ a, b \] plate dimensions
\[ D \] flexural rigidity
\[ \omega \] natural frequency
\[ \phi_x, \phi_y \] average shear rotation
\[ \theta_x, \theta_y \] total rotation in bending
\[ \{\sigma\} \] stress vector
\[ \{\epsilon\} \] strain vector
\[ M_x, M_y \] bending moments in x and y direction
\[ M_{xy} \] twisting moment
\[ Q_x, Q_y \] transverse shear forces
\[ \xi, \eta \] natural coordinates
\[ \rho \] density

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