A General Incidence Bound in $\mathbb{R}^d$ and Related Problems

Thao Do∗ Adam Sheffer†

June 13, 2018

Abstract

We derive a general upper bound for the number of incidences with $k$-dimensional varieties in $\mathbb{R}^d$. The leading term of this new bound generalizes to every $k$ previous incidence bounds for the cases of $k = 1, k = d - 1,$ and $k = d/2$. We derive lower bounds showing that this leading term is tight in various cases. Finally, we derive a bound for incidences with planes in $\mathbb{C}^3$, which is also tight in some cases. (In both $\mathbb{R}^d$ and $\mathbb{C}^3$, the bounds are tight up to sub-polynomial factors.)

To prove our incidence bounds, we define the dimension ratio of an incidence problem. This ratio provides an intuitive approach for deriving incidence bounds and isolating the main difficulties in each proof. We rely on the dimension ratio both in $\mathbb{R}^d$ and in $\mathbb{C}^3$, and also in some of our lower bounds.

1 Introduction

Geometric incidence is an important topic in Discrete Geometry. Given a set $P$ of points and a set $V$ of geometric objects (such as circles, or hyperplanes) in $\mathbb{R}^d$, an incidence is a pair $(p, V) \in P \times V$ such that the point $p$ is contained in the object $V$. The number of incidences in $P \times V$ is denoted as $I(P, V)$. In incidence problems, one is usually interested in the maximum number of incidences in $P \times V$, taken over all possible sets $P, V$ of given sizes. Such incidence bounds have many applications in a variety of fields. For a few recent examples, see Guth and Katz’s solution to the Erdős distinct distances problem [9], a number theoretic result by Bombieri and Bourgain [6], and works in Harmonic Analysis such as [7, 10].

When studying an incidence problem between a point set $P$ and a set of objects $V$, we sometimes consider the incidence graph of $P \times V$. This bipartite graph has vertex sets $P$ and $V$, and an edge for every incidence. Deriving an upper bound for the number of incidences is equivalent to finding an upper bound for the number of edges in the incidence graph. When studying incidence problems in dimension $d \geq 3$, one usually assumes that the incidence graph contains no copy of $K_{s,t}$ for some constants $s, t \geq 2$. Such incidence problems can also be thought of as algebraic or geometric variants of the Zarankiewicz problem (for example, see [8]).

In this paper we study incidences with varieties of any dimension in $\mathbb{R}^d$, when the incidence graph contains no copy of $K_{s,t}$ for some constants $s, t \geq 2$. The following theorem

∗Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA. thaodo@mit.edu.
†Department of Mathematics, Baruch College, City University of New York, NY, USA. adamsh@gmail.com. Supported by NSF grant DMS-1710305
describes the main known results that hold for every $d \geq 2$. Throughout this paper, we use the notation $f = O_{a_1, \ldots, a_k}(g)$ to indicate there is some positive constant $c$ that depends on $a_1, \ldots, a_k$, or $c = o(a_1, \ldots, a_k)$ such that $f \leq cg$.

**Theorem 1.1.** Let $P$ be a set of $m$ points and let $V$ be a set of $n$ varieties of degree at most $D$, both in $\mathbb{R}^d$, such that the incidence graph of $P \times V$ contains no copy of $K_{s,t}$.

(a) (Solymosi and Tao [18]) Assume that every variety of $V$ is of dimension at most $d/2$ and that the varieties intersect transversely (that is, whenever two varieties intersect in a point $p$, their tangent spaces at $p$ intersect in a single point). Then for every $\varepsilon > 0$ we have

$$I(P, V) = O_{D, s, t, d, \varepsilon} \left( m^{2s-1\varepsilon+2} n^{2s-2d+1\varepsilon} + m + n \right).$$

(b) (Fox, Pach, Sheffer, Suk, and Zahl [8]) For any $\varepsilon > 0$ we have

$$I(P, V) = O_{D, s, t, d, \varepsilon} \left( m^{k\varepsilon} n^{d-1\varepsilon} + m + n \right).$$

(c) (Sharir, Sheffer, and Solomon [13]) When every variety of $V$ is of dimension at most one, for every $\varepsilon > 0$ there exists a constant $c$ that satisfies the following. For $j = 2, \ldots, d-1$, assume that every $j$-dimensional variety of degree at most $c$ contains at most $q_j$ varieties of $V$, for parameters $q_2 \leq \cdots \leq q_{d-1} \leq q_d = n$. Moreover, for every $2 \leq j < l \leq d$, we have $q_j \geq \left( \frac{q_{j-1}}{q_{l-1}} \right)^{(l-2)} q_{l-1}$. Then for any $\varepsilon > 0$ we have

$$I(P, V) = O_{D, s, t, d, \varepsilon} \left( m^{k\varepsilon} n^{d-1\varepsilon} + \sum_{j=2}^{d-1} m_{j-1, j-1}^{d-1\varepsilon} q_j^{(d-j)(s-1)\varepsilon} + m + n \right).$$

When $2 \leq d \leq 4$, additional results are known. For example, see [4, 16, 19].

The bound of part (a) of Theorem 1.1 is considered “good” only for varieties of dimension exactly $d/2$. For the exact meaning of good, see the discussion below. For now we only state that this bound is known to be tight up to sub-polynomial factors in some cases, but only when the varieties are of dimension exactly $d/2$. The bound of part (b) is considered “good” when the varieties are of dimension $d - 1$ (and is tight up to sub-polynomial factors in some cases for varieties of dimension $d - 1$). The bound of part (c) holds only when the varieties are of dimension one.

All of the bounds in Theorem 1.1 are obtained using the polynomial partitioning technique. When using this technique for $k$-dimensional varieties in $\mathbb{R}^d$, one expects the main term in the incidence bound to be\footnote{To intuitively see how these exponents are obtained, start with $T_{k,d}(m, n) = m^a n^b$. For a constant $r$, we use polynomial partitioning to divide the space into approximately $r^d$ cells, each containing at most $\frac{m}{r^d}$ points and intersecting about $\frac{n}{r^d}$ varieties on average. We inductively apply the incidence bound separately in each cell, intuitively leading to the relation $r^d \cdot T_{k,d}(m, n) \approx T_{k,d}(m, n)$. For the powers of $r$ to cancel out, we require $d + (d - k)\beta = d$. We also require $n = O(m^\alpha n^\beta)$ when $n = O(m^s)$. That is, we intuitively require that $T_{k,d}(m, m^s) \approx m^s$, or $\alpha + s\beta = s$. Solving the two equations yields the asserted exponents.}

$$T_{k,d}(m, n) := m^{\frac{s}{d-k}} n^{\frac{d-k}{d}}.$$
different techniques. For the case of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ with $s > 2$, one can obtain stronger bounds by using the technique of cutting curves into pseudo-segments (see for example [16,19]).

**Our results.** Our main result is a general incidence bound for $k$-dimensional varieties in $\mathbb{R}^d$ with no $K_{s,t}$ in the incidence graph. Our bound has the main term $m^{2k/ds-d+k+\epsilon} n^{ds-d+k}$ for any $d,k,$ and $s$.

The intuition behind our proofs is based on the concept of dimension ratio. When studying incidences between points and $k$-dimensional varieties in $\mathbb{R}^d$, we define the dimension ratio of the problem as $\frac{k}{d}$. As shown in the following lemma, the smaller the dimension ratio is, the smaller the expected leading term is. For a proof of this lemma, see Section 3.

**Lemma 1.2** (Dimension ratio lemma). Consider positive integers $n,m,s,k,d,k',d'$, such that $n = O(m^s)$, $s > 1$, and $\frac{k'}{d'} \leq \frac{k}{d} < 1$. Then $T_{k',d'}(m,n) = O(T_{k,d}(m,n))$.

When handling incidences with $k$-dimensional varieties in $\mathbb{R}^d$, the analysis often involves incidence problems in lower dimensions or with lower-dimensional varieties. When reaching a subproblem with $k'$-dimensional varieties in $\mathbb{R}^{d'}$ such that $k'/d' \leq k/d$, Lemma 1.2 states that the incidence bound of the subproblem is subsumed by the main incidence bound.

As an example for the intuition from the preceding paragraph, consider the problem of studying incidences with three-dimensional varieties in $\mathbb{R}^5$. This problem has a dimension ratio of $3/5$. When part of the analysis leads to incidences with curves in $\mathbb{R}^2$, we expect this case to be easy to handle since $3/5 > 1/2$. On the other hand, we should be worried about incidences with two-dimensional varieties in $\mathbb{R}^3$ and with three-dimensional varieties in $\mathbb{R}^4$ (since $2/3 > 3/5$ and $3/4 > 3/5$). To handle these cases, we can add the assumption that no bounded-degree four-dimensional variety contains many of our three-dimensional varieties, and that no bounded-degree three-dimensional variety has a two-dimensional intersection with many of the varieties. By presenting lower bound constructions, we will demonstrate that such additional restrictions are often necessary.

The above discussion motivates the following definition. For integers $1 \leq k < d$, let

$$R_{k,d} = \left\{(k',d') \in \mathbb{Z}^2 : 1 \leq k' \leq k, \ 2 \leq d' \leq d, \ \frac{k}{d} < \frac{k'}{d'}\right\}.$$ 

That is, $R_{k,d}$ is the set of all “problematic” ratios when studying incidences with $k$-dimensional varieties in $\mathbb{R}^d$.

To demonstrate another difficulty, consider deriving an upper bound for incidences with four-dimensional objects in $\mathbb{R}^7$. Since $\frac{4}{7} < \frac{3}{5} < \frac{3}{3}$, as part of the analysis we need to handle incidences with three-dimensional objects in five-dimensional varieties, and as part of each such sub-problem we need to handle incidences with two-dimensional objects in three-dimensional varieties. Indeed, consider a configuration of points and two-dimensional varieties in $\mathbb{R}^3$ with no $K_{s,t}$ and $T(m,n)$ incidences (for example, see Theorem 1.4 below).

We can place this configuration in an arbitrary 3-flat in $\mathbb{R}^5$ and then replace each of the varieties with a three-dimensional variety, while maintaining the same incidence graph. We can then place the new configuration in an arbitrary 5-flat in $\mathbb{R}^7$ and replace each of the varieties with a four-dimensional variety.

The above leads us to make the following definition. A sequence of pairs of positive integers $((k_0,d_0),(k_1,d_1),\ldots,(k_u,d_u))$ is said to be significant if

(i) $k_j < d_j$ for all integers $0 \leq j \leq u$. 


(ii) $k_0 \geq k_1 \geq \cdots \geq k_u \geq 1$.

(iii) $d_0 > d_1 > \cdots > d_u \geq 2$.

(iv) $\frac{k_0}{d_0} < \frac{k_1}{d_1} < \cdots < \frac{k_u}{d_u}$.

Let $S_{k,d}$ be the set of significant sequences $((k_0, d_0), (k_1, d_1), \ldots, (k_u, d_u))$ with $(k_0, d_0) = (k, d)$. Note that $((k, d))$ is also a sequence in $S_{k,d}$. We are finally ready to present our main result. For the proof, see Section 3.

**Theorem 1.3.** Let $k, d, s, t, D$ be positive integers with $s \geq 2$ and $k < d$. For any $\varepsilon > 0$, there exists a constant $c$ such that the following holds. Let $P$ be a set of $m$ points and $V$ be a set of $n$ irreducible varieties of dimension at most $k$ and degree at most $D$, both in $\mathbb{R}^d$. Assume that the incidence graph of $P \times V$ contains no copy of $K_{s,1}$. Moreover, for each pair $(k', d') \in R_{k,d}$, every $d'$-dimensional variety of degree at most $c$ has a $k'$-dimensional intersection with at most $q_{k',d'}$ varieties of $V$. Then

$$I(P, V) = O \left( \sum_{((k_0, d_0), \ldots, (k_u, d_u)) \in S_{k,d}} m^{\frac{k_0}{d_0} - \frac{k_1}{d_1} + \varepsilon} \left( n^{\frac{d_1}{d_2} - \frac{d_2}{d_3} \cdots \frac{d_{u-1}}{d_u}} q_{k_1, d_1} \cdots q_{k_u, d_u} \frac{d_0 - k_0}{s} \frac{d_0 - k_u}{s} \frac{(d_0 - k_0)(s - 1)}{s} \right) + m + n \right).$$

**Remarks.**

(i) The sequence $((k, d)) \in S_{k,d}$ results in (1) containing the leading term $T_{k,d}(m,n)$.

(ii) Both $c$ and the constant hidden in the $O(\cdot)$-notation of (1) depend on $k, d, s, t, D, \varepsilon$.

(iii) If $k = d - 1$ then $R_{k,d}$ is empty and $S_{k,d} = \{(k,d)\}$. This implies that no parameter $q_{k',d'}$ is necessary. Thus, Theorem 1.3 generalizes Theorem 1.1(b).

(iv) When $k = d/2$, Theorem 1.3 has the same leading term as Theorem 1.1(a), but with the additional terms depending on the parameters $q_{k',d'}$. This is because Theorem 1.3 does not make the transversality assumption that appears in Theorem 1.1(a). This extra assumption replaces the assumptions involving $q_{k',d'}$, and allows one to remove the extra terms.

(v) When $k = 1$, Theorem 1.3 has the same leading term as Theorem 1.1(c), but with a different dependency on the parameters $q_{k',d'}$. Theorem 1.1(c) has a simpler dependency in $q_{k',d'}$, but also has additional restriction regarding these parameters. The different dependencies are obtained by relying on properties that are special to curves.

**Lower bounds.** The following theorem shows that the main term in the bound of Theorem 1.1 is tight up to sub-polynomial factors, when $s = 2$ and for specific values of $k$ and $d$. It also provides non-trivial bounds for the case of $s = 3$.

**Theorem 1.4.** For any integer $d \geq 2$ there exists a sufficiently large constant $t$ satisfying the following claims for every $\varepsilon > 0$:

(a) For any $n$ and $m = O(n^d)$, there exists a set $P$ of $m$ points and a set $H$ of $n$ hyperplanes, both in $\mathbb{R}^d$, such that the incidence graph of $P \times H$ contains no $K_{2,t+1}$ and

$$I(P, H) = \Omega \left( m^{\frac{2d}{n^{d-1}}} n^{\frac{d}{n^{d-1}} - \varepsilon} + m + n \right).$$
For any $d \geq 4$, $n$, and $m = O(n^{d-2})$, there exists a set $P$ of $m$ points and a set $H$ of $n$ hyperplanes, both in $\mathbb{R}^d$, such that the incidence graph of $P \times H$ contains no $K_{3,t+1}$ and

$$I(P, H) = \Omega \left( \frac{m^{3d-2+\epsilon}}{n^{3d-1}} n^{\frac{2d}{3d-1} - \epsilon} + m + n \right).$$

(c) Consider integers $a \geq 2$ and $b \geq 1$. Then the bounds of parts (a) and (b) also hold when $H$ is a set of $(a-1)b$-flats in $\mathbb{R}^{ab}$.

**Remarks.**

(i) Theorem 1.4 considers only flats, but it is not difficult to extend it to many other types of varieties. By using the same approach as in [17], one obtains that all three parts of the theorem also hold for spheres, paraboloids, and many other families of varieties of any constant degree.

(ii) While the bound in part (b) does not match the corresponding upper bound $m^{3d-3} n^{\frac{2d}{3d-1}}$, both bounds approach $mn^{2/3}$ as $d$ increases.

(iii) Several cases of part (a) of Theorem 1.4 were previously known. When $d = 2$, this is the standard lower bound of the Szemerédi–Trotter theorem (for example, see [12]). The bound was derived in $\mathbb{R}^3$ by Apfelbaum and Sharir [1]. It was also derived for any $d \geq 4$ in [17], but only when $n = \Theta(m^{3/(d+1)})$ (up to sub-polynomial factors).

The above tempts us to make the following conjecture.

**Conjecture 1.5.** For any $1 \leq k < d$, any positive integers $m, n, t$, and any $\epsilon > 0$, there exists a set $P$ of $m$ points and a set $V$ of $n$ flats of dimension $k$, both in $\mathbb{R}^d$, such that the incidence graph of $P \times V$ contains no copy of $K_{2,t}$ and

$$I(P, V) = \Omega_{t,k,d} \left( \frac{m^{3d-3} n^{\frac{2d}{3d-1}}}{n^{3d-1}} + m + n \right).$$

After considering lower bounds for the main term in the bound of Theorem 1.4, we move to the other terms of that bound. We show that when $s = 2$, this bound must have terms containing many of the parameters $q_{k', d'}$. While the dependency in these parameters cannot be removed, it seems likely that it could be replaced with a better dependency. Such dependency is already known for a few cases involving curves in dimensions $2 \leq d \leq 4$ (for example, see [14, 15]).

**Theorem 1.6.** Consider an incidence problem between points and $k$-flats in $\mathbb{R}^d$ with no $K_{2,t}$ in the incidence graph. For $(k', d') \in R_{k,d}$, let $q_{k', d'}$ be defined as in Theorem 1.3. Moreover, assume that there exist integers $a \geq 2$ and $b \geq 1$ satisfying $(a-1)b < k$, $ab < d$, and $(a-1)/a > k/d$. Then it is impossible to completely remove the dependency in $q_{(a-1)b, ab}$ from the incidence bound of Theorem 1.3.

Proofs for Theorems 1.4 and 1.6 can be found in Section 4.

**Incidences in $\mathbb{C}^3$.** By relying on the dimension ratio approach, we also prove a bound for incidences with planes in $\mathbb{C}^3$. When $s = 2$, Theorem 1.4 implies that this bound is tight up to sub-polynomial factors (the construction in $\mathbb{R}^3$ can be placed in $\mathbb{C}^3$ without any changes). As far as we know, this is the first tight incidence bound in a complex space that is not for lines.
Theorem 1.7. Let $s$ and $t$ be positive integers with $s \geq 2$. Let $\mathcal{P}$ be a set of $m$ points and let $\mathcal{V}$ be a set of $n$ planes, both in $\mathbb{C}^3$. Assume that the incidence graph of $\mathcal{P} \times \mathcal{V}$ contains no copy of $K_{s,t}$. Then for any $\varepsilon > 0$ we have
\[
I(\mathcal{P}, \mathcal{V}) = O\left( m \frac{2s}{3s-1} + \varepsilon n \frac{3s-3}{3s-1} + m + n \right).
\]

A proof of Theorem 1.7 can be found in Section 5.

Acknowledgements. We would like to thank Larry Guth for several helpful discussions.

2 Preliminaries

We will rely on the following variant of the Zarankiewicz problem (for example, see [11, Section 4.5]).

Lemma 2.1. Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^d$ and let $\mathcal{V}$ be a set of $n$ subsets of $\mathbb{R}^d$. If the incidence graph of $\mathcal{P} \times \mathcal{V}$ contains no copy of $K_{s,t}$, then
\[
I(\mathcal{P}, \mathcal{V}) = O_{s,t}(mn^{1-s+t} + n).
\]

Varieties and partitioning. Given polynomials $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_d]$, the variety $V(f_1, \ldots, f_k)$ is defined as
\[
V(f_1, \ldots, f_k) = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : f_j(a_1, \ldots, a_d) = 0 \text{ for all } 1 \leq j \leq k\}.
\]

There are several non-equivalent definitions for the degree of a variety in $\mathbb{R}^d$. For our purposes, we define the degree of a variety $U \subset \mathbb{R}^d$ as
\[
\min_{f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_d]} \max_{1 \leq i \leq k} \deg f_i.
\]

That is, the degree of $U$ is the minimum integer $D$ such that $U$ can be defined with a finite set of polynomials of degree at most $D$.

Intuitively, we say that a variety $U \subset \mathbb{R}^d$ has dimension $k$ if there exists a subset of $U$ that is homeomorphic to the open $k$-dimensional cube, but no such subset of $U$ is homeomorphic to the open $(k+1)$-dimensional cube. We refer to a $k$-dimensional variety of degree one (or a $k$-dimensional “plane”) as a $k$-flat. For more information about varieties in $\mathbb{R}^d$ and a more precise definition of dimension, see for example [5].

We will use the following variant of the polynomial partitioning theorem.

Theorem 2.2 ([8]). Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^d$ and let $U \subset \mathbb{R}^d$ be an irreducible variety of degree $k$ and dimension $d'$. Then for every $1 < r < m$ exists $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $O_{d,k}(r)$ such that $U \not\subseteq V(f)$ and every connected component of $U \setminus V(f)$ contains at most $m/r^{d'}$ points of $\mathcal{P}$.

We will also require the following results about irreducible components and connected components of varieties.

Lemma 2.3. Let $U \subset \mathbb{R}^d$ be a variety of degree $k$. Then the number of irreducible components of $U$ is $O_{d,k}(1)$. 


Theorem 2.4 (Barone and Basu [3]). Let $U$ and $W$ be varieties in $\mathbb{R}^d$ such that $W$ is defined by a single polynomial of degree $k_W \geq 2\deg U$. Then the number of connected components of $U \setminus W$ is $O_d\left(k_W^{\dim U} \deg U^{d-\dim U}\right)$.

Singular points, regular points, and tangent flats. The ideal of a variety $U \subseteq \mathbb{R}^d$, denoted $I(U)$, is the set of polynomials in $\mathbb{R}[x_1, \ldots, x_d]$ that vanish on every point of $U$. We say that a set of polynomials $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_d]$ generate $I(U)$ if every element of $I(U)$ can be written as $\sum_{j=1}^\ell f_j g_j$ for some $g_1, \ldots, g_\ell \in \mathbb{R}[x_1, \ldots, x_d]$. We also write $(f_1, \ldots, f_\ell) = I(U)$ to state that $f_1, \ldots, f_\ell$ generate $I(U)$.

The Jacobian matrix of a set of polynomials $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_d]$ is

$$J_{f_1, \ldots, f_k} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_d} \end{pmatrix}$$

Consider a variety $U \subset \mathbb{R}^d$ of dimension $k$, and let $f_1, \ldots, f_\ell \in \mathbb{R}[x_1, \ldots, x_d]$ satisfy $(f_1, \ldots, f_\ell) = I(U)$. We say that $p \in U$ is a singular point of $U$ if $\text{rank}(J(p)) < d - k$. We denote the set of singular points of $U$ as $U_{\text{sing}}$. A point of $U$ that is not singular is said to be a regular point. A $k$-dimensional variety has a unique well-defined tangent $k$-flat at every regular point. We denote the tangent $k$-flat at $p \in U$ as $T_p U$, and think of it as a linear subspace (that is, as incident to the origin). At singular points of a variety, a unique well-defined tangent flat may or may not exist.

Theorem 2.5. Let $U \subset \mathbb{R}^d$ be a variety of degree $D$ and dimension $k$. Then $U_{\text{sing}}$ is a variety of dimension smaller than $k$ and of degree $O_{D,d}(1)$.

The generalizations of the above definitions to varieties of any dimension are more involved. We do not require these definitions for this work, so we do not state them. References for the above claims and additional information can be found, for example, in [3].

3 Upper bounds in $\mathbb{R}^d$

The purpose of this section is to prove Theorem 1.3 — our general incidence bound in $\mathbb{R}^d$. We begin by proving Lemma 1.2 and first repeat the statement of this lemma.

Lemma 1.2 (Dimension ratio lemma). Consider positive integers $n, m, s, k, d, k', d'$, such that $n = O(m^s)$, $s > 1$, and $\frac{d'}{d} \leq \frac{k'}{k} < 1$. Then $T_{k', d'}(m, n) = O\left(T_{k, d}(m, n)\right)$.

Proof. By the assumptions, both $s - 1$ and $d'k - dk'$ are positive. We may thus raise both sides of $n = O(m^s)$ to the power of $(d'k - dk')(s - 1)$, obtaining

$$n^{(d'k - dk')(s - 1)} = O\left(n^{s(d'k - dk')(s - 1)}\right).$$
This implies
\[ m^s k'(ds - d' + k)(ds - d + k) = m^s k'(ds - d + k)(ds - d + k) \]
\[ = O \left( m^s k'(ds - d + k)(ds - d + k) \right) \]
\[ = O \left( m^s k'(ds - d + k)(ds - d + k) \right). \]

Finally, raising both sides to the power of \(1/(ds - d + k)(ds - d + k)\) yields the assertion of the lemma.

Instead of proving Theorem 1.3 we prove the following more general result, where the points are contained in a constant-degree variety. Theorem 1.3 is immediately obtained by setting \( W = \mathbb{R}^d \) and \( d^* = d \) in Theorem 3.1.

**Theorem 3.1.** Let \( k, d, d^*, s, t, D \) be positive integers with \( s \geq 2 \) and \( k < d \). For any \( \varepsilon > 0 \), there exists a constant \( c \) such that the following holds. Let \( \mathcal{P} \) be a set of \( m \) points on an irreducible constant-degree variety \( W \subseteq \mathbb{R}^{d^*} \) and let \( \mathcal{V} \) be a set of \( n \) irreducible varieties of dimension at most \( k \) and degree at most \( D \) in \( \mathbb{R}^{d^*} \). Assume that the incidence graph of \( \mathcal{P} \times \mathcal{V} \) contains no copy of \( K_{s,t} \). Moreover, for each pair \( (k', d') \in R_{k,d} \), every \( d^* \)-dimensional variety of degree at most \( c \) has a \( k' \)-dimensional intersection with at most \( q_{k',d'} \) varieties of \( \mathcal{V} \). Then

\[ I(\mathcal{P}, \mathcal{V}) = O \left( \sum_{\{(k_0,d_0),\ldots,(k_u,d_u)\} \in S_{k,d}} m^{sk_u} \left( \frac{d}{n-\varepsilon} q_{k_1,d_1} \cdots q_{k_u,d_u} \right)^{\frac{d-k}{d-k-1} \cdot \frac{d}{u-1-k_u-1}} \right), \]

Proof. We prove the statement of the theorem by induction on \( d \). In particular, we prove that

\[ I(\mathcal{P}, \mathcal{V}) \leq \alpha_1 \cdot \sum_{\{(k_0,d_0),\ldots,(k_u,d_u)\} \in S_{k,d}} m^{sk_u} \left( \frac{d}{n-\varepsilon} q_{k_1,d_1} \cdots q_{k_u,d_u} \right)^{\frac{d-k}{d-k-1} \cdot \frac{d}{u-1-k_u-1}} + \alpha_2 (m + n), \]

where \( \alpha_1, \alpha_2 \) are sufficiently large constants that depend on \( s, t, c, d, d^* \) and \( \varepsilon \). For the induction basis, we assume that the claim is trivial when \( d = 2 \). Indeed, in this case \( R_{1,2} \) is empty and \( S_{1,2} = \{(k,d)\} \), so the required bound is the one in Theorem 3.1(b).

We prove the induction step using a second induction on \( m + n \). For the base case of this second induction, when \( m \) and \( n \) are sufficiently small the result is obtained by choosing sufficiently large values of \( \alpha_1 \) and \( \alpha_2 \). It remains to handle the induction step of the second induction. The hidden constants in the \( O(\cdot) \)-notations throughout the proof may also depend on \( s, t, D, d, d^* \) and \( \varepsilon \). For brevity we write \( O(\cdot) \) instead of \( O_{s,t,D,d^*,\varepsilon}(\cdot) \).

Since the incidence graph contains no copy of \( K_{s,t} \), Lemma 2.1 implies \( I(\mathcal{P}, \mathcal{V}) = O(mn^{1-1/s} + n) \). When \( m = O(n^{1/s}) \), this implies \( I(\mathcal{P}, \mathcal{V}) = O(n) \), and we may thus assume that

\[ n = O(m^s). \]

8
Partitioning the space. By Theorem \(2.4\) with \(U = W, d' = d\), and a constant \(r\), we obtain a polynomial \(f \in \mathbb{R}[x_1, \ldots, x_{d'}]\) of degree \(O(r)\) such that every connected component of \(W \setminus \mathbf{V}(f)\) contains at most \(m/r^d\) points of \(\mathcal{P}\). The asymptotic relations between the various constants in the proof are

\[
2^{1/\varepsilon} \ll r \ll \alpha_2 \ll \alpha_1.
\]

Denote the cells of the partition as \(C_1, \ldots, C_v\). By Theorem \(2.4\), we have that \(v = O(r^d)\). For each \(1 \leq j \leq v\), denote by \(\mathcal{V}_j\) the set of varieties of \(\mathcal{V}\) that intersect \(C_j\), and set \(\mathcal{P}_j = C_j \cap \mathcal{P}\). We also set \(m_j = |\mathcal{P}_j|, m' = \sum_{j=1}^v m_j, \) and \(n_j = |\mathcal{V}_j|\). Note that \(m_j \leq m/r^d\) for every \(1 \leq j \leq v\). By Theorem \(2.4\), every variety of \(\mathcal{V}\) intersects \(O(r^k)\) cells of \(W \setminus \mathbf{V}(f)\). Therefore, \(\sum_{j=1}^v n_j = O(nr^k)\). For every \((k_u, d_u) \in R_{k,d}\), Hölder’s inequality implies

\[
\sum_{j=1}^v \frac{d(d_u-k_u)(s-1)}{(d-k)(s+d_u-d_u+k_u)} \leq \left( \sum_{j=1}^v \frac{d(d_u-k_u)(s-1)}{(d-k)(s+d_u-d_u+k_u)} \right) \left( \sum_{j=1}^v \frac{d(d_u-k_u)(s-1)}{(d-k)(s+d_u-d_u+k_u)} \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{j=1}^v \left( \frac{d(d_u-k_u)(s-1)}{(d-k)(s+d_u-d_u+k_u)} \right) \right)^{\frac{1}{p}} \leq \left( \sum_{j=1}^v \frac{d(d_u-k_u)(s-1)}{(d-k)(s+d_u-d_u+k_u)} \right)^{\frac{1}{p}}
\]

Combining the above with the induction hypothesis implies

\[
\sum_{j=1}^v \int(\mathcal{P}_j, \mathcal{V}_j)
\leq \sum_{j=1}^v \left( \alpha_1 \cdot \sum_{(k_0, d_0), \ldots, (k_u, d_u) \in S_{k,d}} m_{j} \frac{s k_{u}}{s d_{u} - d_{u} + k_{u}} + \varepsilon \left( \frac{d}{d_{u} - k_{u}} d_{d_{1} - k_{1}} \ldots d_{d_{u} - k_{u}} \right) \right)
\]

\[
\leq \alpha_1 \cdot \sum_{(k_0, d_0), \ldots, (k_u, d_u) \in S_{k,d}} m_{j} \frac{s k_{u}}{s d_{u} - d_{u} + k_{u}} + \varepsilon \left( \frac{d}{d_{u} - k_{u}} d_{d_{1} - k_{1}} \ldots d_{d_{u} - k_{u}} \right) \right)
\]

By \(1\) we have \(n = O\left( m_{j} \frac{s k}{s d_{u} - d_{u} + k_{u}} \frac{s d - d}{s d - d + k} \right)\). Thus, when \(\alpha_1\) is sufficiently large with
respect to \( r \) and \( \alpha_2 \), we get
\[
\sum_{j=1}^v I(P_j, V_j) = O\left( \alpha_1 \cdot \sum_{(k_0, d_0), \ldots, (k_u, d_u) \in S_{k, d}}^\frac{m^{k_u}}{d^{d_u-d_k+k_u}} + \varepsilon \left( \sum_{i=1}^{d-d_k} q_{k_i, d_1} \cdots q_{k_u, d_u} \right) \right) + \alpha_2 m'.
\]

When \( r \) is sufficiently large with respect to \( \varepsilon \) and to the constant hidden in the \( O(\cdot) \)-notation, we have
\[
\sum_{j=1}^v I(P_j, V_j) \leq \frac{\alpha_1}{4} \sum_{(k_0, d_0), \ldots, (k_u, d_u) \in S_{k, d}}^\frac{m^{k_u}}{d^{d_u-d_k+k_u}} + \varepsilon \left( \sum_{i=1}^{d-d_k} q_{k_i, d_1} \cdots q_{k_u, d_u} \right) + \alpha_2 m'. \tag{5}
\]

**Incidences on the partition** It remains to bound the number of incidences with points that lie on \( V(f) \). Set \( P_0 = P \cap V(f) \) and \( m_0 = |P_0| = m - m' \). We associate each point \( p \in P_0 \) with an arbitrary component of \( V(f) \cap W \) that \( p \) is incident to. To bound the number of incidences with \( P_0 \), it suffices to separately consider each irreducible component of \( V(f) \cap W \) and the points that are associated with it.

Let \( U \) be an irreducible component of \( V(f) \cap W \). Let \( \overline{P} \) be the set of points of \( P \) associated with \( U \), and set \( \overline{m} = |\overline{P}| \). By the definition of \( f \), we have that \( \dim U < d \). By taking \( c \) to be sufficiently large with respect to \( d^s, s, t, k, \) and \( \varepsilon \), we may assume that the degree of \( U \) is at most \( c \).

Since the incidence graph contains no \( K_{s,t} \), either \( \overline{m} < s \) or at most \( t \) varieties of \( V \) contain \( U \). In the latter case, the varieties of \( V \) that contain \( U \) form \( O(\overline{m}) \) incidences with \( \overline{P} \). In the former case, there are \( O(n) \) such incidences. Moreover, the total contribution to incidences from varieties of \( V \) that have finite intersection with \( U \) is also \( O(n) \). Hence, for each \( h \in V \) we can assume that \( 1 \leq \dim(h \cap U) \leq \min\{k, d-1\} \). For \( 1 \leq k' \leq \min\{d'-1, k\} \), let \( V_{k'} \) denote the set of \( k' \)-dimensional intersections between \( U \) and an element of \( V \). Set \( n_{k'} = |V_{k'}| \) and note that \( \sum_{k'} n_{k'} \leq n \). By definition, for any \( (k', d') \in R_{k,d} \) we have \( n_{k'} \leq q_{k', d'} \).

For some \( 1 \leq k' \leq \min\{d'-1, k\} \), we would like to apply the first induction hypothesis with the set of varieties \( V_{k'}, W = U, d = d' \), and \( k = k' \). Assume that for each \( (k^*, d^*) \in R_{k^*, d^*} \) there are at most \( q_{k^*, d^*} \) varieties of \( V_{k'} \) that has \( k^* \)-dimensional intersection with some \( d^* \) dimensional varieties of degree at most \( c' \) for some sufficiently large \( c' \). If \( (k^*, d^*) \in R_{k,d} \) then \( q_{k^*, d^*} \leq q_{k, d'} \). When \( (k^*, d^*) \notin R_{k,d} \), we will use the trivial bound \( q_{k^*, d^*} \leq n_{k'} \leq n \).

We will now use the dependency of \( \alpha_1 \) and \( \alpha_2 \) in \( d \), and to stress this we change the notation to \( \alpha_{1,d} \) and \( \alpha_{2,d} \). By the first induction hypothesis we have
\[
I(\overline{P}, \overline{V}_{k'}) \leq \alpha_{1,d'} \cdot \sum_{(k_0, d_0), \ldots, (k_u, d_u) \in S_{k', d'}}^\frac{m^{k_u}}{d^{d_u-d_k+k_u}} + \varepsilon \left( \sum_{i=1}^{d-d_k} q_{k_i, d_1} \cdots q_{k_u, d_u} \right) + \alpha_{2,d'} (\overline{m} + n) + O(\overline{m} + n). \tag{6}
\]
Consider a term from (6) of the form
\[
m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} \left( \frac{d'}{q_{k_1,d_1}^{d'} - q_{k_1,d_1}^{d'}} \cdots \frac{d_{u-1}}{q_{k_u,d_u}^{d_{u-1} - q_{k_u,d_u}}} \right)^{\frac{(d_u-k_u)(s-1)}{sd_u-d_u+k_u}}
\]

(7)

If \( k_d \geq \frac{k_u}{d_u} \), we recall that \( \overline{q}_{k_j,d_j} \leq n_k' \leq n \) for every \( 1 \leq j \leq u \). Combining this with Lemma 1.2 implies that (7) is upper bounded by
\[
m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} n^{\frac{d_s-d_u}{sd_u-d_u+k_u}} = T_{d_u,k_u}(m,n,m^\varepsilon) = O(T_{d,k}(m,n) \cdot m^\varepsilon) = O \left( m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} n^{\frac{d_s-d_u}{sd_u-d_u+k_u}} \right).
\]

Next, we consider the case when \( \frac{k}{d} < \frac{k_u}{d_u} \). We set \((k_0,d_0) = (k',d')\) and \( \overline{q}_{k_0,d_0} = n_k' \). Let \( j \) be the smallest non-negative integer that satisfies \( \frac{k}{d} < \frac{k_j}{d_j} \). That is,
\[
\frac{k_0}{d_0} < \cdots < \frac{k_{j-1}}{d_{j-1}} \leq \frac{k}{d} < \frac{k_j}{d_j} < \cdots < \frac{k_u}{d_u}.
\]

Note that \((k,d),(k_j,d_j),\ldots,(k_u,d_u)\) is a significant sequence. We will use the term in (1) corresponding to this significant sequence to upper bound (7). For every \( j' \geq j \) we have \( \overline{q}_{k_j,d_j} \leq q_{k_j,d_j} \). By also recalling that we always have \( \overline{q}_{k_j,d_j} \leq n \), we obtain
\[
\frac{d_j}{\overline{q}_{k_0,d_0}} \frac{d_{j-1}}{\overline{q}_{k_1,d_1}} \cdots \frac{d_{j-2}}{\overline{q}_{k_{j-2},d_{j-2}}} \frac{d_{j-1}}{\overline{q}_{k_{j-1},d_{j-1}}} \leq n^{\frac{d_j}{\overline{q}_{k_j,d_j}}} \frac{d_j}{\overline{q}_{k_{j-1},d_{j-1}}} \frac{d_{j-1}}{\overline{q}_{k_{j-2},d_{j-2}}} \frac{d_{j-2}}{\overline{q}_{k_{j-3},d_{j-3}}} \cdots \frac{d_{j-2}}{\overline{q}_{k_1,d_1}} \frac{d_{j-1}}{\overline{q}_{k_0,d_0}}
\]

Thus, in this case (7) is upper bounded by
\[
m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} \left( \frac{d'}{q_{k_1,d_1}^{d'} - q_{k_1,d_1}^{d'}} \cdots \frac{d_{u-1}}{q_{k_u,d_u}^{d_{u-1} - q_{k_u,d_u}}} \right)^{\frac{(d_u-k_u)(s-1)}{sd_u-d_u+k_u}}
\]

By combining (6) with the two above bounds for (7), we obtain
\[
I(\overline{P},V_{k'}) = O \left( \alpha_{1,d'} \cdot \sum_{((k_0,d_0),\ldots,(k_u,d_u)) \in S_{k,d}} m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} \left( n^{\frac{d}{q_{k_1,d_1}^d - q_{k_1,d_1}^d}} \cdots q_{k_u,d_u}^{d_{u-1} - q_{k_u,d_u}} \right)^{\frac{(d_u-k_u)(s-1)}{sd_u-d_u+k_u}} \right)
\]
\[
+ \alpha_{2,d'} (m + n)
\]

Theorem 2.3 implies that \( V(f) \cap W \) has \( O_r(1) \) irreducible components. By summing the above bound over each of these components and over every \( k' \), and assuming \( \alpha_{1,j} \leq \alpha_{1,j+1} \) and \( \alpha_{2,j} \leq \alpha_{2,j+1} \) for every \( j \), we obtain
\[
I(P_0,V) = O \left( \alpha_{1,d-1} \cdot \sum_{((k_0,d_0),\ldots,(k_u,d_u)) \in S_{k,d}} m^{\frac{sk_k}{sd_u-d_u+k_u} + \varepsilon} \left( n^{\frac{d}{q_{k_1,d_1}^{d'} - q_{k_1,d_1}^{d'}}} \cdots q_{k_u,d_u}^{d_{u-1} - q_{k_u,d_u}} \right)^{\frac{(d_u-k_u)(s-1)}{sd_u-d_u+k_u}} \right)
\]
\[
+ \alpha_{2,d-1} (m_0 + n)
\]

11
By recalling that \( n = O \left( m^{\frac{sk}{2d-\varepsilon - k}} n^{\frac{sd-d}{2d-k}} \right) \) and taking \( \alpha_{1,d} \) and \( \alpha_{2,d} \) to be sufficiently large with respect to \( \alpha_{1,d-1}, \alpha_{2,d-1} \), and \( r \), we obtain

\[
I(\mathcal{P}_0, V) \leq \frac{\alpha_{1,d}}{2} \sum_{((k_0,d_0), \ldots, (k_u,d_u)) \in S_{k,d}} \left( n^{\frac{d}{2d-\varepsilon}} q_{k_1,d_1} \cdots q_{k_u,d_u} \right) + \alpha_{2,m_0}.
\]

Combining this bound with \ref{thm:lower-bound-proof} completes the two induction steps and the proof of the theorem. \( \square \)

4 Lower bounds in \( \mathbb{R}^d \).

In this section we prove the lower bounds stated in the introduction. As usual, before each proof we repeat the statement of the theorem.

**Theorem 1.4.** For any integer \( d \geq 2 \) there exists a sufficiently large constant \( t \) satisfying the following claims for every \( \varepsilon > 0 \):

(a) For any \( n \) and \( m = O(n^d) \), there exists a set \( \mathcal{P} \) of \( m \) points and a set \( \mathcal{H} \) of \( n \) hyperplanes, both in \( \mathbb{R}^d \), such that the incidence graph of \( \mathcal{P} \times \mathcal{H} \) contains no \( K_{2,t+1} \) and

\[
I(\mathcal{P}, \mathcal{H}) = \Omega \left( m^{\frac{2d-2}{2d+1}} n^{\frac{d}{2d-\varepsilon}} + m + n \right).
\]

(b) For any \( d \geq 4, n \), and \( m = O(n^{d-2}) \), there exists a set \( \mathcal{P} \) of \( m \) points and a set \( \mathcal{H} \) of \( n \) hyperplanes, both in \( \mathbb{R}^d \), such that the incidence graph of \( \mathcal{P} \times \mathcal{H} \) contains no \( K_{3,t+1} \) and

\[
I(\mathcal{P}, \mathcal{H}) = \Omega \left( m^{\frac{2d-2}{2d+1}} n^{\frac{d}{2d-\varepsilon}} + m + n \right).
\]

(c) Consider integers \( a \geq 2 \) and \( b \geq 1 \). Then the bounds of parts (a) and (b) also hold when \( \mathcal{H} \) is a set of \( (a-1)b \)-flats in \( \mathbb{R}^{ab} \).

**Proof.** (a) To obtain \( m \) incidences, we can place all of the points of \( \mathcal{P} \) on a single hyperplane of \( \mathcal{H} \). To obtain \( n \) incidences, we can set all of the hyperplanes of \( \mathcal{H} \) to be incident to a single point of \( \mathcal{P} \). Thus, we only need to construct a configuration with \( \Omega \left( m^{\frac{2d-2}{2d+1}} n^{\frac{d}{2d-\varepsilon}} \right) \) incidences. For the case of \( d = 2 \), see for example \ref{thm:lower-bound-proof}. We may thus also assume that \( d \geq 3 \).

We consider the point set

\[
\mathcal{P} = \left\{ (x_1, \ldots, x_d) \in \mathbb{Z}^d : 0 \leq x_j \leq m^{1/d} - 1 \right\}.
\]

For a parameter \( N \) that will be set below, let \( \mathcal{L} \) be an \( N \times \cdots \times N \) section of \( \mathbb{Z}^d \) centered at the origin. We say that a nonzero element \( v \in \mathcal{L} \) is a **primitive vector** if there is no integer \( j > 1 \) and \( u \in \mathcal{L} \) such that \( v = j \cdot u \). By \ref{thm:primitive-vector} Corollary 1], there exists a subset \( V \subseteq \mathcal{L} \) of \( \Theta \left( N^{d/(d-1)-\varepsilon} \right) \) primitive vectors such that any hyperplane in \( \mathbb{R}^d \) contains at most \( t \) of these vectors (for a sufficiently large constant \( t \)). Let \( \mathcal{H} \) be the set of hyperplanes in \( \mathbb{R}^d \) that contain at least one point of \( \mathcal{P} \) and whose normal direction is in \( V \) (the sizes of the two vectors may differ). The dot product of a vector from \( V \) and a point of \( \mathcal{P} \) is an integer
of size $O\left(Nm^{1/d}\right)$. Thus, each vector of $V$ corresponds to $O\left(Nm^{1/d}\right)$ hyperplanes of $\mathcal{H}$. This implies that $|\mathcal{H}| = O\left(N^{(2d-1)/(d-1)-\varepsilon'}m^{1/d}\right)$.

By adding generic hyperplanes to $\mathcal{H}$, we can assure that $|\mathcal{H}| = \Theta\left(N^{(2d-1)/(d-1)-\varepsilon'}m^{1/d}\right)$. To have $n \approx |\mathcal{H}|$, we set $N = n^{(d-1)/(2d-1)-(d-1)\varepsilon'}/m^{(d-1)/(2d-1)-(d-1)\varepsilon'}$. The assumption $m = O(n^d)$ implies $N = \Omega(1)$. Every point of $\mathcal{P}$ is incident to exactly one hyperplane of $\mathcal{H}$ with each normal of $V$. By taking $\varepsilon'$ to be sufficiently small with respect to $\varepsilon$, we obtain

$$I(\mathcal{P}, \mathcal{H}) = mN^{\frac{d}{d-1}-\varepsilon'} = \Theta\left(m^{\frac{2d^2-9d+2}{2d-1}} \cdot \left( N^{\frac{d}{d-1}-\varepsilon'} m^{\frac{1}{2d-1}} \right) \right) = \Omega\left(m^{\frac{2d^2-9d+2}{2d-1}} n^{\frac{d}{2d-1}-\varepsilon} \right).$$

Let $\ell \subset \mathbb{R}^d$ be a line. For a hyperplane $h \in \mathcal{H}$ to contain $\ell$, the normal of $h$ must be orthogonal to the direction of $\ell$. That is, the normal of the hyperplane is in a given linear $(d-1)$-dimensional subspace. By the choice of $V$, we obtain that at most $t$ hyperplanes of $\mathcal{H}$ contain any given line. This implies that the incidence graph of $\mathcal{P} \times \mathcal{H}$ contains no copy of $K_{2,t+1}$.

(b) As in part (a) of the proof, it is simple to obtain $\Theta(m + n)$ incidences, so we only need to construct a configuration with $\Theta\left(m^{\frac{2d^2-9d+2}{2d-1}} n^{\frac{d}{2d-1}-\varepsilon} \right)$ incidences.

Let $\mathcal{P}'$ be a $m^{1/(d-2)} \times \cdots \times m^{1/(d-2)}$ section of $\mathbb{Z}^d$ centered at the origin. The distance between a point $(x_1, \ldots, x_d) \in \mathcal{P}'$ and the origin is $\sqrt{x_1^2 + \cdots + x_d^2}$. Every such distance is the square root of an integer between zero and $d \cdot m^{2/(d-2)}$. That is, the points of $\mathcal{P}'$ determine at most $d \cdot m^{2/(d-2)}$ nonzero distinct distances from the origin. By the pigeonhole principle, there exists a distance $\delta$ such that at least $m^{d/(d-2)}/(d \cdot m^{2/(d-2)}) = \Omega(m)$ points are at distance $\delta$ from the origin. In other words, the hypersphere $S_\delta$ centered at the origin and of radius $\delta$ contains $\Omega(m)$ points of $\mathcal{P}'$. Let $\mathcal{P}$ be a set of exactly $m$ of these points (if necessary, we can add extra generic points on $S_\delta$). To recap, $\mathcal{P}$ is a set of $m$ points with integer coordinates on the hypersphere $S_\delta$.

For a parameter $N$ that will be set below, let $\mathcal{L}$ be an $N \times \cdots \times N$ section of $\mathbb{Z}^d$ centered at the origin. By [2, Corollary 1], there exists a subset $V \subset \mathcal{L}$ of $\Theta\left(N^{2d/(d-1)-\varepsilon'}\right)$ primitive vectors such that any $(d-2)$-flat in $\mathbb{R}^d$ contains at most $t$ of these vectors (for a sufficiently large constant $t$). Let $\mathcal{H}$ be the set of hyperplanes in $\mathbb{R}^d$ that contain at least one point of $\mathcal{P}$ and whose normal direction is in $V$ (the sizes of the two vectors may differ).

The dot product of a vector from $V$ and a point of $\mathcal{P}$ is an integer of size $O\left(Nm^{1/(d-2)}\right)$. Thus, the number of hyperplanes in $\mathcal{H}$ is $O\left(N^{(3d-1)/(d-1)-\varepsilon'}m^{1/(d-2)}\right)$. By adding generic hyperplanes to $\mathcal{H}$, we can assure that $|\mathcal{H}| = \Theta\left(N^{(3d-1)/(d-1)-\varepsilon'}m^{1/d}\right)$. To have $n \approx |\mathcal{H}|$, we set $N = n^{(d-1)/(2d-1)-(d-1)\varepsilon'}/m^{(d-1)/(d-2)(3d-1)-(d-1)\varepsilon'}$. The assumption $m = O(n^d)$ implies $N = \Omega(1)$.

Every point of $\mathcal{P}$ is incident to exactly one hyperplane of $\mathcal{H}$ with each normal of $V$. By taking $\varepsilon'$ to be sufficiently small with respect to $\varepsilon$, we obtain

$$I(\mathcal{P}, \mathcal{H}) = mN^{\frac{2d}{d-1}-\varepsilon'} = \Theta\left(m^{\frac{2d^2-9d+2}{2d-1}} \cdot \left( N^{\frac{2d}{d-1}-\varepsilon'} m^{\frac{2d}{2d-1}} \right) \right) = \Omega\left(m^{\frac{2d^2-9d+2}{2d-1}} n^{\frac{2d}{2d-1}-\varepsilon} \right).$$

Let $F \subset \mathbb{R}^d$ be a 2-flat. For a hyperplane $h \in \mathcal{H}$ to contain $F$, the normal of $h$ must be orthogonal to two vectors that span $F$. That is, the normal of the hyperplane is in a given linear $(d-2)$-dimensional subspace. By the choice of $V$, we have that at most $t$ hyperplanes
of $\mathcal{H}$ contain any given 2-flat. Since no line contains three points of $\mathcal{P}$, we conclude that the incidence graph of $\mathcal{P} \times \mathcal{H}$ contains no copy of $K_{3,t+1}$.

(c) We use either part (a) or part (b) to obtain a configuration of hyperplanes in $\mathbb{R}^a$. We take an arbitrary $ab$-flat $F$ in $\mathbb{R}^{ab}$ and place the above copy of $\mathbb{R}^a$ in $F$. Then, we replace each $(a-1)$-flat $h$ in the configuration with a generic $(a-1)b$-flat in $\mathbb{R}^{ab}$ that contains $h$. Since the $(a-1)b$-flats are chosen generically, this process does not yield new incidences. In particular, the incidence graph still contains no copy of either $K_{2,t+1}$ or $K_{3,t+1}$, and the number of incidences remains as in part (a) or (b).

\[ \square \]

**Theorem 1.6.** Consider an incidence problem between points and $k$-flats in $\mathbb{R}^d$ with no $K_{2,t}$ in the incidence graph. For $(k',d') \in R_{k,d}$, let $q_{k',d'}$ be defined as in Theorem 1.3. Moreover, assume that there exist integers $a \geq 2$ and $b \geq 1$ satisfying $(a-1)b < k$, $ab < d$, and $(a-1)/a > k/d$. Then it is impossible to completely remove the dependency in $q_{(a-1)b,ab}$ from the incidence bound of Theorem 1.3.

**Proof.** By Theorem 1.4(c), when $m = O(n^a)$ there exists a set $\mathcal{P}$ of $m$ points and a set $\mathcal{H}$ of $n$ flats of dimension $(a-1)b$, both in $\mathbb{R}^{ab}$, such that the incidence graph of $\mathcal{P} \times \mathcal{H}$ contains no $K_{2,t}$ and

\[
I(\mathcal{P}, \mathcal{H}) = \Theta \left( m^{\frac{2a-2}{2a-1}n^{\frac{a}{2a-1} - \varepsilon}} + m + n \right). 
\]

We take an arbitrary $ab$-flat $F$ in $\mathbb{R}^d$ and place the above copy of $\mathbb{R}^{ab}$ in $F$. Then we replace every $(a-1)b$-flat $h$ in the configuration with a generic $k$-flat that contains $h$. We obtain a configuration of points and $k$-flats in $\mathbb{R}^d$ with no $K_{2,t}$ in the incidence graph and the same number of incidences. Since $(a-1)/a > k/d$, Lemma 1.2 implies that the above number of incidences is larger than the leading term of Theorem 1.3 in this case (when $\varepsilon$ is taken to be sufficiently small). Thus, in this case the leading term in the bound of Theorem 1.3 must be dominated by a term involving $q_{(a-1)b,ab}$. \[ \square \]

5 Planes in $\mathbb{C}^3$

In this section we prove Theorem 1.4. We write any $x \in \mathbb{C}$ as $x_1 + ix_2$, where $x_1, x_2 \in \mathbb{R}$. The polynomial partitioning technique does not work in complex spaces, since removing a variety cannot disconnect a complex space. To overcome this issue, one often thinks of $\mathbb{C}^d$ as $\mathbb{R}^{2d}$. We will rely on this approach, considering the map $\phi: \mathbb{C}^3 \to \mathbb{R}^6$ defined by

\[
\phi(x, y, z) = (x_1, x_2, y_1, y_2, z_1, z_2).
\]

Consider a complex plane $h$ in $\mathbb{C}^3$. By definition, $h$ can be defined using a linear equation $ax + by + cz = d$, where $a, b, c, d \in \mathbb{C}$ and $x, y, z$ are the complex coordinates of $\mathbb{C}^3$. Then, $\phi(h)$ is a variety in $\mathbb{R}^6$ defined by the system

\[
a_1x_1 - a_2x_2 + b_1y_1 - b_2y_2 + c_1z_1 - c_2z_2 = d_1, \\
a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 + c_1z_2 + c_2z_1 = d_2.
\]

It is not difficult to verify that this system defines a 4-flat in $\mathbb{R}^6$, unless $a = b = c = 0$. A point $p \in \mathbb{C}^3$ is incident to $h$ if and only if the point $\phi(p)$ is incident to the flat $\phi(h)$. Thus, we can reduce the problem of incidences with planes in $\mathbb{C}^3$ to incidences with 4-flats in $\mathbb{R}^6$. 

14
In an incidence problem with 4-flats in $\mathbb{R}^6$, our dimension ration is $2/3$. According to Lemma [1.2] we should worry about the dimension ratios $4/5$ and $3/4$; that is, worry about 4-flats contained in five-dimensional components of the partition, and about 4-flats that have a three-dimensional intersection with four-dimensional components of the partition. We will handle the former case using a basic property of 4-flats originating from planes in $\mathbb{C}^3$. We will handle the latter case by using the following lemma.

**Lemma 5.1.** Let $U$ be a four-dimensional variety in $\mathbb{R}^6$, and let $p$ be a regular point of $U$. Let $h_1, h_2, h_3$ be three distinct planes in $\mathbb{C}^3$ such that each of the 4-flats $\phi(h_1), \phi(h_2)$, and $\phi(h_3)$ has a three-dimensional intersection with $U$. Moreover, assume that $p$ is a regular point of $U \cap \phi(h_j)$ for every $1 \leq j \leq 3$. Then $h_1 \cap h_2 \cap h_3$ is a complex line.

**Proof.** For $1 \leq j \leq 3$, let $F_j = T_p(U \cap \phi(h_j))$. Note that each $F_j$ is a 3-flat contained in $T_p U$. Since the 4-flat $T_p U$ contains the 3-flats $F_1, F_2, F_3$ and $p \in F_1 \cap F_2 \cap F_3$, we obtain that $\dim(F_1 \cap F_2 \cap F_3 \cap T_p U) \geq 1$. Since $F_j \subset \phi(h_j)$ for every $1 \leq j \leq 3$, we obtain that $\dim(\phi(h_1) \cap \phi(h_2) \cap \phi(h_3)) \geq 1$. This in turn implies that $\dim(h_1 \cap h_2 \cap h_3) \geq 1$, so the three complex planes intersect in a complex line. \hfill $\Box$

We are now ready to prove Theorem [1.7] and first recall the statement of this theorem.

**Theorem 1.7.** Let $s$ and $t$ be positive integers with $s \geq 2$. Let $\mathcal{P}$ be a set of $m$ points and let $\mathcal{V}$ be a set of $n$ planes, both in $\mathbb{C}^3$. Assume that the incidence graph of $\mathcal{P} \times \mathcal{V}$ contains no copy of $K_{s,t}$. Then for any $\varepsilon > 0$ we have

$$I(\mathcal{P}, \mathcal{V}) = O \left( m^{\frac{2s}{3s-1}+\varepsilon} n^{\frac{3s-3}{3s-1}} + m + n \right).$$

**Proof.** As explained in the beginning of this section, $I(\mathcal{P}, \mathcal{V}) = I(\phi(\mathcal{P}), \phi(\mathcal{V}))$, so it suffices to bound the latter. Note that the incidence graph of $\phi(\mathcal{P}) \times \phi(\mathcal{V})$ also contains no copy of $K_{s,t}$. Abusing the notation, in the rest of the proof we write $\mathcal{P}$ instead of $\phi(\mathcal{P})$ and $\mathcal{V}$ instead of $\phi(\mathcal{V})$.

To prove the theorem, we prove by induction on $m + n$ that

$$I(\mathcal{P}, \mathcal{V}) \leq \alpha_1 m^{\frac{2s}{3s-1}+\varepsilon} n^{\frac{3s-3}{3s-1}} + \alpha_2 (m + n), \quad (8)$$

where $\alpha_1, \alpha_2$ are sufficiently large constants that depend on $s, t, \varepsilon$. For the induction basis, the case where $m$ and $n$ are sufficiently small can be handled by choosing sufficiently large values of $\alpha_1$ and $\alpha_2$. Throughout the proof, the hidden constants in the $O(\cdot)$-notations may also depend on $s, t, \varepsilon$. For brevity we write $O(\cdot)$ instead of $O_{s,t,\varepsilon}(\cdot)$.

Since the incidence graph contains no copy of $K_{s,t}$, Lemma 2.1 implies $I(\mathcal{P}, \mathcal{V}) = O(m^{1-1/s} n + n)$. When $m = O(n^{1/s})$, this implies $I(\mathcal{P}, \mathcal{V}) = O(n)$, and we may thus assume that

$$n = O(m^s). \quad (9)$$

**Partitioning the space.** By Theorem 2.2 with $U = \mathbb{R}^6$ and a constant $r$, we obtain a polynomial $f \in \mathbb{R}[x_1, \ldots, x_6]$ of degree $O(r)$ such that every connected component of $\mathbb{R}^6 \setminus \mathcal{V}(f)$ contains at most $m/r^6$ points of $\mathcal{P}$. The asymptotic relations between the various constants in the proof are

$$2^{1/\varepsilon} \ll r \ll \overline{r} \ll \overline{\alpha_2} \ll \overline{\alpha_1} \ll \alpha_2 \ll \alpha_1.$$
Denote the cells of the partition as $C_1, \ldots, C_v$. By Theorem 2.4, we have that $v = O(r^6)$. For each $1 \leq j \leq v$, denote by $V_j$ the set of varieties of $V$ that intersect $C_j$, and set $P_j = C_j \cap P$. We also set $m_j = |P_j|$, $m' = \sum_{j=1}^{v} m_j$, and $n_j = |V_j|$. Note that $m_j \leq m/r^6$ for every $1 \leq j \leq v$. By Theorem 2.4, every variety of $V$ intersects $O(r^4)$ cells of $\mathbb{R}^6 \setminus V(f)$. Therefore, $\sum_{j=1}^{v} n_j = O(n r^4)$. Hölder’s inequality implies

$$\sum_{j=1}^{v} \frac{3v-3}{2n} \leq \left( \sum_{j=1}^{v} \frac{3v-3}{2n} \right) \left( \sum_{j=1}^{v} \frac{2n}{3v} \right) = O\left( \left( n r^4 \right) \frac{3v-3}{2n} \frac{12}{3v-1} \right) = O\left( n \frac{3v-3}{2n} \frac{12}{3v-1} \right).$$

Combining the above with the induction hypothesis implies

$$\sum_{j=1}^{v} I(P_j, V_j) \leq \sum_{j=1}^{v} \left( \alpha_1 m_j \frac{2r^{2\alpha} + \epsilon}{n_j^{\frac{3s-3}{3s-1}}} + \alpha_2 (m_j + n_j) \right) \leq \alpha_1 m \frac{2r^{2\alpha} + \epsilon}{n^{\frac{3s-3}{3s-1}}} + \alpha_2 \left( m' + O(n r^4) \right) \leq \alpha_1 \frac{2r^{2\alpha} + \epsilon}{n^{\frac{3s-3}{3s-1}}} + \alpha_2 \left( m' + O(n r^4) \right).$$

By (9) we have $n = O\left( m \frac{2r^{2\alpha} + \epsilon}{n^{3s-3}} \right)$. Thus, when $\alpha_1$ is sufficiently large with respect to $r$ and $\alpha_2$, we get

$$\sum_{j=1}^{v} I(P_j, V_j) = O\left( \alpha_1 \frac{2r^{2\alpha} + \epsilon}{n^{\frac{3s-3}{3s-1}}} \right) + \alpha_2 m'.$$

When $r$ is sufficiently large with respect to $\epsilon$ and to the constant hidden in the $O(\cdot)$-notation, we have

$$\sum_{j=1}^{v} I(P_j, V_j) \leq \alpha_1 \frac{2r^{2\alpha} + \epsilon}{n^{\frac{3s-3}{3s-1}}} + \alpha_2 m'. \quad (10)$$

**Incidences on the partition** It remains to bound the number of incidences with points that lie on $V(f)$. Set $P_0 = P \cap V(f)$ and $m_0 = |P_0|$. We associate each point $p \in P_0$ with an arbitrary component of $V(f)$ that $p$ is incident to. To bound the number of incidences with $P_0$, it suffices to separately consider each irreducible component of $V(f)$ and the points that are associated with it.

Let $U$ be an irreducible component of $V(f)$, let $P$ denote the set of points that are associated with $U$, and set $m = |P|$. Since $\deg f = O(r)$, the degree of $U$ is also $O(r)$. We divide the analysis of the incidence with $P$ according to $\dim U$.

We first assume that $\dim U \leq 3$. Since the incidence graph contains no copy of $K_{s,t}$, either $U$ is contained in fewer than $t$ flats of $V$ or $m \leq s$. In the former case, flats that contain $U$ have $O(m)$ incidences with $P$, and in the latter they have $O(n)$ incidences. It remains to consider flats of $V$ that intersect $U$ in a variety of dimension at most two. This is an incidence problem between $P$ and at most $n$ varieties of dimension at most two. Let $P_2$ be the set of these varieties.

Let $h$ be a generic 3-flat in $\mathbb{R}^6$, and let $\pi_3(\cdot) : \mathbb{R}^6 \to \mathbb{R}^3$ be the projection on $h$. Since $h$ is a generic 3-flat, we may assume that no two points of $P$ are projected to the same point of
Similarly, we may assume that the projections of $\overline{P}$ and $\overline{V}_2$ on $h$ do not lead to any new incidences. That is, $I(\overline{P}, \overline{V}_2) = I(\pi_3(\overline{P}), \pi_3(\overline{V}_2))$ and the incidence graph of $\pi_3(\overline{P}) \times \pi_3(\overline{V}_2)$ contains no copy of $K_{s,t}$. Theorem 1.1(b) implies

$$I(\overline{P}, \overline{V}_2) = I(\pi_3(\overline{P}), \pi_3(\overline{V}_2)) = O \left( m^{2s-1+\epsilon} n^{4s-3} + m + n \right).$$

We conclude that when $\dim U \leq 3$, we have

$$I(\overline{P}, V) = O_r \left( m^{2s-1+\epsilon} n^{4s-3} + m + n \right). \quad (11)$$

The four-dimensional case. We now assume that $\dim U = 4$. In this case, at most one 4-flat of $V$ can contain $U$, and this flat has $\overline{m}$ incidences with $\overline{P}$. Every other 4-flat of $V$ intersects $U$ in a variety of dimension at most three.

By Theorem 2.5, the singular set $U_\text{sing}$ is a variety of dimension at most three and degree $O_r(1)$. Thus, we can bound the number of incidences with points of $U_\text{sing}$ in the same way we handled the case of $\dim U \leq 3$. In particular, $U_\text{sing}$ consists of $O_r(1)$ components, and each should be handled separately as in the case of $\dim U \leq 3$. Similarly, incidences with 4-flats of $V$ that intersect $U$ in a variety of dimension at most two can be handled using the projection argument involving $\pi_3(\cdot)$. It remains to deal with 4-flats of $V$ that intersect $U$ in a three-dimensional variety.

Let $h$ be a 4-flat of $V$ that has a three-dimensional intersection with $U$. If there are lower-dimensional components in this intersection, we can include these in the projection argument from the previous paragraph. By Theorem 2.5, the singular set of $h \cap U$ is a variety of dimension at most two and degree $O_r(1)$, which can also be included in the projection argument above. It remains to deal with the regular points of the three-dimensional components of $h \cap U$. Let $\overline{V}_3$ be the set of the remaining portions of the intersections between $U$ and $V$. That is, for each intersection we only include the three-dimensional components, and consider only incidences with their regular points.

Points of $\overline{P}$ that are incident to regular points of at most two elements of $\overline{V}_3$ yield at most $2\overline{m}$ incidences, so we may discard these. We remain with points of $\overline{P}$ that are regular points of $U$ and are incident to regular points of more than three elements of $\overline{V}_3$. By Lemma 5.1, for each such point $p$ there exists a complex line $\ell_p \subset \mathbb{C}^3$ that is contained in each of the complex planes whose corresponding elements in $\overline{V}_3$ have $p$ as a regular point. Let $F_p = \phi(\ell_p)$. Let $\mathcal{L}$ be the set of these 2-flats (one 2-flat for each point of $\overline{P}$ that is a regular point of more than two elements of $\overline{V}_3$). To bound the number of incidences between the remaining points and the regular points of the elements of $\overline{V}_3$, it suffices to derive an upper bound for the number of containments between the 2-flats of $\mathcal{L}$ and the 4-flats of $V$.

Let $H$ be a generic 4-flat in $\mathbb{R}^6$. Then $H$ has a two-dimensional intersection with every 4-flat of $V$ and intersects each 2-flat of $\mathcal{L}$ in a point. Inside of $H$, the above containment problem becomes an incidence problem between points and 2-flats. By projecting this incidence problem on a generic 3-flat and applying Theorem 1.1(b), we obtain the bound

$$O \left( m^{2s-1+\epsilon} n^{4s-3} + \overline{m} + n \right).$$

By combining all of the above cases, we conclude that when $\dim U = 4$, the bound (11) holds.

The five-dimensional case. It remains to consider the case where $\dim U = 5$. By Theorem 2.5, the singular set $U_\text{sing}$ is a variety of dimension at most four and degree $O_r(1)$.
We can thus bound the number of incidences with points of $U_{\text{sing}} \cap \overline{P}$ in the same way we handled the cases of $\dim U \leq 4$. In particular, $U_{\text{sing}}$ consists of $O_r(1)$ components, and each should be handled separately as in the cases of $\dim U \leq 4$. Similarly, incidences with 4-flats of $V$ that intersect $U$ in a variety of dimension at most two can be handled using the above projection argument involving $\pi_3(\cdot)$.

Consider two complex planes $h_1, h_2 \subset \mathbb{C}^3$ that intersect at a point $p \in \mathbb{C}^3$. After translating $\mathbb{C}^3$ so that $p$ becomes the origin, we get that the linear subspaces $h_1$ and $h_2$ span $\mathbb{C}^3$. This in turn means that $\phi(h_1)$ and $\phi(h_2)$ span $\mathbb{R}^6$. If $\phi(h_1), \phi(h_2) \in U$, then $\phi(p)$ must be a singular point of $U$. Indeed, if $p$ was a regular point of $U$ then we would have $\phi(h_1), \phi(h_2) \subset T_pU$. This is impossible since at a regular point $p$, the tangent $T_pU$ is a hyperplane while $\phi(h_1), \phi(h_2)$ span $\mathbb{R}^6$. We conclude that for every $p \in \overline{P}$ that is a regular point of $U$, at most one 4-flat of $V$ is contained in $U$ and incident to $p$. There are at most $\overline{m}$ such incidences.

It remains to consider 4-flats of $V$ that have a three-dimensional intersection with $U$. Let $\nabla_3$ denote the set of such three-dimensional intersections. We clearly have $|\nabla_3| \leq n$. We derive an upper bound for $I(\overline{P}, \nabla_3)$ using a second induction. In particular, we prove by induction on $\overline{m} + n$ that

$$I(\overline{P}, \nabla_3) \leq \overline{m}^{\frac{2s}{3s-1} + \epsilon} n^{\frac{3s-3}{3s-1}} + \overline{a}_2(\overline{m} + n),$$

(12)

where $\overline{a}_1, \overline{a}_2$ are sufficiently large constants that depend on $s, t, r, \text{ and } \epsilon$. For the induction basis, the case where $\overline{m}$ and $n$ are sufficiently small can be handled by choosing sufficiently large values of $\overline{a}_1$ and $\overline{a}_2$.

For the induction step, we will use a second polynomial partitioning. By Theorem 2.2 with $U$ and a sufficiently large constant $\overline{m}$, there exists a polynomial $f \in \mathbb{R}[x_1, \ldots, x_6]$ of degree $O(\overline{m})$ that does not vanish identically on $U$, such that every connected component of $U \setminus V(f)$ contains at most $\overline{m}/\overline{m}^5$ points of $\overline{P}$.

Denote the cells of the partition as $\overline{C}_1, \ldots, \overline{C}_\overline{m}$. By Theorem 2.4 we have that $\overline{m} = O(\overline{m}^5)$. For each $1 \leq j \leq \overline{m}$, denote by $\overline{V}_j$ the set of varieties of $\overline{V}_3$ that intersect $\overline{C}_j$, and set $\overline{P}_j = \overline{C}_j \cap \overline{P}$. We also set $\overline{m}_j = |\overline{P}_j|, \overline{m}' = \sum_{j=1}^{\overline{m}} \overline{m}_j$, and $\overline{m}_j = |\overline{V}_j|$. Note that $\overline{m}_j \leq \overline{m}/\overline{m}^5$ for every $1 \leq j \leq \overline{m}$. By Theorem 2.4 every variety of $\overline{V}_3$ intersects $O(\overline{m}^5)$ cells of $U \setminus V(f)$. Therefore, $\sum_{j=1}^{\overline{m}} \overline{m}_j = O(n\overline{m}^3)$. Hölder’s inequality implies

$$\sum_{j=1}^{\overline{m}} \overline{m}_j^{\frac{3s-3}{3s-1}} \leq \left( \sum_{j=1}^{\overline{m}} \overline{m}_j \right)^{\frac{3s-3}{3s-1}} = O \left( \left( \sum_{j=1}^{\overline{m}} \overline{m}_j \right) \left( \sum_{j=1}^{\overline{m}} 1 \right)^{\frac{3s-3}{3s-1}} \right) = O \left( \left( \sum_{j=1}^{\overline{m}} \overline{m}_j \right)^{\frac{3s-3}{3s-1}} \right) = O \left( \left( \sum_{j=1}^{\overline{m}} \overline{m}_j \right)^{\frac{3s-3}{3s-1}} \right).$$

Combining the above with the induction hypothesis implies

$$\sum_{j=1}^{\overline{m}} I(\overline{P}_j, \nabla_j) \leq \sum_{j=1}^{\overline{m}} \left( \overline{a}_1 \overline{m}_j^{\frac{2s}{3s-1} + \epsilon} \overline{m}_j^{\frac{3s-3}{3s-1}} + \overline{a}_2(\overline{m}_j + \overline{m}_j) \right) \leq \overline{a}_1 \overline{m}^{\frac{2s}{3s-1} + \epsilon} \sum_{j=1}^{\overline{m}} \overline{m}_j^{\frac{3s-3}{3s-1}} + \overline{a}_2 \left( \overline{m} + O(n\overline{m}^3) \right) \leq \overline{a}_1 \overline{m}^{\frac{2s}{3s-1} + \epsilon} \overline{m}^{\frac{3s-3}{3s-1}} + \overline{a}_2 \left( \overline{m} + O(n\overline{m}^3) \right).$$
We recall that \( n = O \left( m^{\frac{2s}{3-\varepsilon} n^{\frac{3s-3}{5s-1}}} \right) \). When \( \alpha_1 \) is sufficiently large with respect to \( r \) and \( \alpha_2 \), we get
\[
\sum_{j=1}^{\overline{r}} I(\mathcal{P}_j, V_j) = O \left( \frac{m^{\frac{2s}{3-\varepsilon}} + \varepsilon}{\alpha_1} n^{\frac{3s-3}{5s-1}} \right) + \alpha_2 m. \]

When \( \bar{r} \) is sufficiently large with respect to \( \varepsilon \) and to the constant hidden in the \( O(\cdot) \)-notation, we have
\[
\sum_{j=1}^{\bar{r}} I(\mathcal{P}_j, V_j) \leq \frac{\alpha_1}{2} m^{\frac{2s}{3-\varepsilon}} + \varepsilon n^{\frac{3s-3}{5s-1}} + \alpha_2 (m - m' + n). \tag{13}
\]

It remains to bound the number of incidences with points of \( \mathbf{V}(f) \cap \mathcal{P} \). Since \( \mathbf{V}(f) \) is a variety of dimension at most four and degree \( O(r'(1)) \), we can simply repeat the analysis of the cases of \( \text{dim } U \leq 4 \). We get the upper bound in (11) for this number of incidences (with \( r \) replaced by the larger constant \( r' \)). By taking \( \alpha_1 \) and \( \alpha_2 \) to be sufficiently large with respect to \( r' \), we obtain
\[
I(\mathcal{P} \cap \mathbf{V}(f), V_3) \leq \frac{\alpha_1}{2} m^{\frac{2s}{3-\varepsilon}} + \varepsilon n^{\frac{3s-3}{5s-1}} + \alpha_2 (m - m' + n).
\]

Combining this with (13) completes the proof of the second induction step. We conclude that when \( \text{dim } U = 5 \), the bound (11) also holds.

**Completing the proof.** By going over the various cases above, we note that we always have the bound (11). By Lemma 2.3, the partition \( \mathbf{V}(f) \) consists of \( O(r'(1)) \) irreducible components. Summing up (11) over every irreducible component of \( \mathbf{V}(f) \) leads to
\[
I(\mathcal{P}_0, \mathcal{V}) = O_{r, r'} \left( m^{\frac{2s}{3s-1} + \varepsilon} n^{\frac{3s-3}{5s-1}} + m_0 + n \right)
\]

Taking \( \alpha_1 \) and \( \alpha_2 \) to be sufficiently large, we obtain
\[
I(\mathcal{P}_0, \mathcal{V}) \leq \frac{\alpha_1}{2} m^{\frac{2s}{3s-1} + \varepsilon} n^{\frac{3s-3}{5s-1}} + \alpha_2 m_0.
\]

Combining this bound with (10) completes the first induction step and the proof of the theorem. \( \square \)

**References**

[1] R. Apfelbaum and M. Sharir, Large complete bipartite subgraphs in incidence graphs of points and hyperplanes, *SIAM Journal on Discrete Mathematics* **21** (2007), 707–725.

[2] M. Balko, J. Cibulka, and P. Valtr, Covering lattice points by subspaces and counting point-hyperplane incidences, [arXiv:1703.04767](http://arxiv.org/abs/1703.04767).

[3] S. Barone, and S. Basu, Refined bounds on the number of connected components of sign conditions on a variety, *Discrete Comput. Geom.* **47** (2012), 577–597.

[4] S. Basu and M. Sombra, Polynomial partitioning on varieties of codimension two and point-hypersurface incidences in four dimensions, *Discrete Comput. Geom.* **55** (2016), 158–184.
[5] J. Bochnak, M. Coste, and M. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin, 1998.

[6] E. Bombieri and J. Bourgain, A problem on sums of two squares, *International Mathematics Research Notices* **2015.11** (2015), 3343–3407.

[7] J. Bourgain and C. Demeter, New Bounds for the Discrete Fourier Restriction to the Sphere in 4D and 5D, *International Mathematics Research Notices* **2015.11** (2015), 3150–3184.

[8] J. Fox, J. Pach, A. Sheffer, A. Suk, and J. Zahl, A semi-algebraic version of Zarankiewicz’s problem, *J. of the European Mathematical Society*, **19** (2017), 1785–1810.

[9] L. Guth and N.H. Katz, On the Erdős distinct distances problem in the plane, *Annals Math.* **181** (2015), 155–190.

[10] N. H. Katz and J. Zahl, An improved bound on the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^3$, arXiv:1704.07210.

[11] J. Matoušek, *Lectures on Discrete Geometry*, Springer Verlag, Heidelberg, 2002.

[12] J. Pach and M. Sharir, Geometric incidences, in *Towards a Theory of Geometric Graphs* (J. Pach, ed.), Amer. Math. Soc., Providence, RI, 2004, 185–223.

[13] M. Sharir, A. Sheffer, and N. Solomon, Incidences with curves in $\mathbb{R}^d$, *Electronic J. Combinat.* **23** (2016), P4.16.

[14] M. Sharir and N. Solomon, Incidences between points and surfaces and points and curves, and distinct and repeated distances in three dimensions, *Proc. 28th ACM-SIAM Symp. on Discrete Algorithms* (2017), 2456–2475.

[15] M. Sharir and N. Solomon, Incidences Between Points and Lines in $\mathbb{R}^4$, *Discrete and Computational Geometry* **57** (2017): 702–756.

[16] M. Sharir and J. Zahl, Cutting algebraic curves into pseudo-segments and applications, *J. Combin. Theory Ser. A* **150** (2017), 1–35.

[17] A. Sheffer, Lower bounds for incidences with hypersurfaces, Discrete Analysis 2016:16.

[18] J. Solymosi and T. Tao, An incidence theorem in higher dimensions, *Discrete Comput. Geom.* **48** (2012), 255–280.

[19] J. Zahl, Breaking the 3/2 barrier for unit distances in three dimensions, *IMRN*, to appear.