Constant curvature holomorphic solutions of the
supersymmetric grassmannian sigma model: the case of
$G(2, 4)$

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Abstract

We explore the constant curvature holomorphic solutions of the supersymmetric
grassmannian sigma model $G(M, N)$ using in particular the gauge invariance of the
model. Supersymmetric invariant solutions are constructed via generalizing a known
result for $CP^{N-1}$. We show that some other such solutions also exist. Indeed, consid-
ering the simplest case of $G(2, N)$ model, we give necessary and sufficient conditions
for getting the constant curvature holomorphic solutions. Since, all the constant cur-
vature holomorphic solutions of the bosonic $G(2, 4)$ sigma model are known, we treat
this example in detail.

Key words: supersymmetry (susy), grassmannian sigma model, gauge invariance

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1 Introduction

The Weierstrass representation of surfaces in multidimensional spaces [1234], such as
Lie algebras and groups, has generated interest in studying surfaces associated with the
solutions of the grassmannian bosonic $G(M, N)$ sigma model ($\sigma$-model) [567]. Motivated
by the work dealing with $G(2, 4)$ [8] and $G(2, 5)$ [9], a general approach for constructing
holomorphic maps of 2-sphere $S^2$ of constant curvature into $G(M, N)$ have been realized in
two papers [1011].

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Then in [12, 13], most of the above ideas have been generalized to the supersymmetric (susy) \( \mathbb{C}P^{N-1} \) \( \sigma \)-model which is equivalent to the susy \( G(1, N) \). In particular, all the susy invariant solutions with constant curvature of this model have been thoroughly discussed.

The natural question is to extend those results to susy \( G(M, N) \) \( \sigma \)-models for \( M > 1 \). In order to achieve some canonical results we use the full power of the gauge invariance of these susy models. Indeed recently, present authors explored such type of invariance [14]. Although it is well-known [15, 16, 17, 18, 19], to our knowledge, up to now no explicit form of it had been used in an effective way to analyse the solutions of the model. In particular, the gauge invariance of the susy model is much richer than its bosonic counterpart. We will thus use it in the present paper in order to construct constant curvature holomorphic solutions of the susy \( G(2, 4) \) \( \sigma \)-model.

The structure of the article is as follows: In Section 2, we discuss the necessary and sufficient conditions to get the constant curvature holomorphic solutions of the general susy \( G(M, N) \) \( \sigma \)-model. In Section 3 we give a detailed analysis of the susy \( G(2, 4) \) \( \sigma \)-model. Taking into account the susy gauge invariance we present all the holomorphic solutions of this model with constant curvature in a canonical form. In Section 4, a well-known embedding of \( G(2, 4) \) into \( \mathbb{C}P^5 \) is given in order to help to understand some arbitrariness in the susy solutions of \( G(2, 4) \). Finally, we end the article by giving some comments in Section 5.

### 2 Constant curvature holomorphic solutions of the susy \( G(M, N) \) \( \sigma \)-model

For the susy \( G(M, N) \) \( \sigma \)-model [19], a general bosonic superfield \( \Phi : S^2 \mapsto G(M, N) \) has the following expansion

\[
\Phi(x_\pm, \theta_\pm) = \Phi_0(x_\pm) + i \theta_+ \Phi_1(x_\pm) + i \theta_- \Phi_2(x_\pm) - \theta_+ \theta_- \Phi_3(x_\pm),
\]

(2.1)

where \( \Phi_0 \) and \( \Phi_3 \) are \( N \times M \) bosonic complex matrices and \( \Phi_1 \) and \( \Phi_2 \) are \( N \times M \) fermionic complex matrices. Here, \( S^2 \) denotes the superspace \( (x_\pm, \theta_\pm) \) whose bosonic part is compactified to 2-sphere \( S^2 \). This bosonic superfield must satisfy

\[
\Phi^\dagger \Phi = I_M.
\]

(2.2)

As in the bosonic case, holomorphic solutions of the susy \( G(M, N) \) \( \sigma \)-model are trivial solutions of the model [14, 19]. It has been shown that they take the form

\[
\Phi = WL,
\]

(2.3)

where \( W \) is an \( N \times M \) matrix depending only on the coordinates \( (x_+, \theta_+) \) while \( L \) is a non singular \( M \times M \) matrix that depends on the coordinates \( (x_\pm, \theta_\pm) \). It means that the holomorphic superfield \( W \) takes the explicit form

\[
W(x_+, \theta_+) = Z(x_+) + i \theta_+ \eta(x_+) A(x_+)
\]

(2.4)

and the determination of the holomorphic solutions of the susy \( G(M, N) \) \( \sigma \)-model is equivalent [13,19] to the study of these holomorphic superfields.

In the case of the susy \( \mathbb{C}P^{N-1} \) \( \sigma \)-model, the solutions of the susy Euler-Lagrange equations have been shown to be associated with surfaces [20]. The susy Gaussian curvature of the surface corresponding to the susy holomorphic solution \( W \) was given by the formula

\[
\kappa = -\frac{1}{\tilde{g}} \partial_+ \partial_- \ln \tilde{g},
\]

(2.5)

where the supersymmetric expression of the metric was

\[
\tilde{g} = \partial_+ \partial_- \ln (\det(W^\dagger W)).
\]

(2.6)
Clearly the metric and curvature may be functions of \((x_\pm, \theta_\pm)\) even if \(W\) depends only on the coordinates \((x_+, \theta_+).\)

For the case of susy \(G(M, N)\), we assume the same relation between the superfield \(W\), metric \(\tilde{g}\) and curvature \(\tilde{\kappa}\). It means that asking for a constant curvature solution is equivalent to assuming that \(\tilde{\kappa} = \kappa\) where \(\kappa\) is a purely bosonic constant (a strictly positive real number) and must be the curvature associated with the purely bosonic \(G(M, N)\) solution \(Z\) involved in \(W = (2.4).\)

Let us write explicitly the condition \((2.5)\) using the expression of \(W\) in \((2.4)\) and taking into account that \(\tilde{\kappa} = \kappa\). In order to simplify the calculations, we take

\[
T_1 = \theta_+ \eta, \quad T_2 = \theta_- \eta^\dagger. \tag{2.7}
\]

Notice that since \(T_1\) and \(T_2\) are both product of two fermionic functions, we have \(T_1^2 = 0\) and \(T_2^2 = 0\). Moreover, they are bosonic quantities and hence commute with all the other quantities.

We thus easily get

\[
\det (W^\dagger W) = (\det M_0) \det \left( i_M + iT_1 M_0^{-1} M_1 + iT_2 M_0^{-1} M_2 - T_1 T_2 M_0^{-1} M_3 \right)
\]

with

\[
M_0 = Z^\dagger Z, \quad M_1 = Z^\dagger A, \quad M_2 = A^\dagger Z, \quad M_3 = A^\dagger A. \tag{2.9}
\]

The expressions of \(X_1, X_2\) and \(X_3\) remain to be explicitly computed.

The metric \(\tilde{g}\) = \((2.6)\) takes the form

\[
\tilde{g} = g + \partial_+ \partial_- \ln (1 + iT_1 X_1 + iT_2 X_2 - T_1 T_2 X_3), \tag{2.10}
\]

with

\[
g = \partial_+ \partial_- \ln (\det(Z^\dagger Z)). \tag{2.11}
\]

Using the Taylor expansion of the logarithmic function

\[
\ln (1 + x) = x - \frac{x^2}{2} + \mathcal{O}(x^3), \tag{2.12}
\]

we get

\[
\tilde{g} = g + \partial_+ \partial_- [iT_1 X_1 + iT_2 X_2 - T_1 T_2 (X_3 - X_1 X_2)]. \tag{2.13}
\]

By a similar procedure we can express the quantity \(\partial_+ \partial_- \ln \tilde{g}\) as;

\[
\partial_+ \partial_- \ln \tilde{g} = \partial_+ \partial_- \ln g + iT_1 \partial_+ \partial_- Y_1 + iT_2 \partial_+ \partial_- Y_2
- T_1 T_2 \partial_+ \partial_- (Y_3 - Y_1 Y_2), \tag{2.14}
\]

with

\[
Y_1 \equiv \frac{\kappa}{2} (1 + |x|^2)^2 \partial_+ \partial_- X_1, \quad
Y_2 \equiv \frac{\kappa}{2} (1 + |x|^2)^2 \partial_+ \partial_- X_2, \quad
Y_3 \equiv \frac{\kappa}{2} (1 + |x|^2)^2 \partial_+ \partial_- (X_3 - X_1 X_2). \tag{2.15}
\]

Upon inserting these relations into \((2.5)\) we get the following constraints

\[
\partial_+ \partial_- \ln g + \kappa g = 0 \tag{2.16}
\]
and
\begin{align}
\partial_+ \partial_- (Y_1 + \kappa X_1) &= 0, \\
\partial_+ \partial_- (Y_2 + \kappa X_2) &= 0, \\
\partial_+ \partial_- ((Y_3 - Y_1 Y_2) + \kappa (X_3 - X_1 X_2)) &= 0.
\end{align}

Notice that the two expressions in (2.17) are complex conjugate to each other and hence we have only one independent condition, say the one involving $Y_1$ and $X_1$.

These are necessary and sufficient conditions for the susy holomorphic solutions to have a constant Gaussian curvature and will be the fundamental equations for our analysis.

In the following subsection, we take the special case of susy invariant solutions of $G(M, N)$ and show that it solves our problem.

### 2.1 Susy invariant solutions

We now give a sufficient condition for obtaining a constant curvature solution.

Let us first recall that, in the particular case $M = 1$, we have already shown that the susy holomorphic solutions with constant curvature take the form (up to gauge transformations)\(^{[12]}\)

\[
\omega(x_+, \theta_+) = u(x_+) + i \theta_+ \eta(x_+) \partial_+ u(x_+),
\]

where $u(x_+)^T = (u_1(x_+), \ldots, u_{N-1}(x_+))$ is the Veronese sequence with

\[
u_n(x_+) = \sqrt{N - 1 \choose n} x_+^n, \quad n = 0, 1, 2, \ldots, N - 1.
\]

Such a solution is called susy invariant\(^{[12]}\) since, using Taylor expansion, we have $\omega(x_+, \theta_+) = \omega(y_+)$ with $y_+ = x_+ + i \theta_+ \eta(x_+)$ being a susy translated variable.

In this section, we generalize this result to susy grassmanian $G(M, N) \sigma$-model. Indeed, assuming that the susy holomorphic solution is similarly given by

\[
W(x_+, \theta_+) = Z(x_+) + i \theta_+ \eta(x_+) \partial_+ Z(x_+),
\]

i.e. $A(x_+) = \partial_+ Z(x_+)$ in (2.24) and keeping in mind that the holomorphic solution of the bosonic grassmanian $G(M, N) \sigma$-model is written in the MacFarlane parametrization\(^{[21]}\), we can rewrite (2.21) as

\[
W = \begin{pmatrix} \mathbb{I} & i \theta_+ \eta \partial_+ \end{pmatrix} K.
\]

Here we have taken into account the susy gauge invariance\(^{[14]}\). Since the curvature and metric associated with such a solution are given by (2.5) and (2.6) respectively, we compute first the determinant of the matrix $W^\dagger W$ which could be written as

\[
W^\dagger W = (1 + D) \left( \mathbb{I} + K \right),
\]

where the differential operator $D$ is given by

\[
D = iT_1 \partial_+ + iT_2 \eta \partial_- - T_1 T_2 \partial_+ \partial_-,
\]

using the notation introduced in (2.24).

In order to proceed with the determinant, we use the following lemma which is proven in the Appendix A.

**Lemma 2.1.** Let $D$ be the operator defined in (2.24) and $B(x_+, x_-)$ is an $M \times M$ bosonic matrix. Then we have

\[
det \left[ (1 + D) B(x_+, x_-) \right] = (1 + D) \det \left[ B(x_+, x_-) \right].
\]
Replacing $B(x_+, x_-)$ by $(I_M + K^\dagger K)$ in the above Lemma, we get

$$\det(W^\dagger W) = (1 + \mathcal{D}) \det(I_M + K^\dagger K). \tag{2.26}$$

Now we can give the following theorem.

**Theorem 2.2.** Let us assume that $Z : S^2 \to G(M, N)$, is a holomorphic solution of the bosonic Euler-Lagrange equations associated with a constant Gaussian curvature surface. Its susy invariant holomorphic extension \((2.21)\) is also associated with a constant Gaussian curvature surface of the same curvature.

**Proof.** By hypothesis, $Z$ is a holomorphic solution of the bosonic model associated with a constant Gaussian curvature surface. It means that \([10]\) there exists an integer $r$ such that

$$\det(Z^\dagger Z) = \det(I_M + K^\dagger K) = R = (1 + |x|^2)^r, \tag{2.27}$$

for some positive integer $r$ and thus $\kappa = \frac{2}{r}$.

In order to get the expression of the metric \((2.6)\) we first show that

$$\ln [(1 + \mathcal{D}) R] = (1 + \mathcal{D}) \ln R. \tag{2.28}$$

Using the Taylor expansion of the logarithmic function \((2.12)\) and applying it with $x = \frac{1}{R} \mathcal{D} R$, only the first two terms of the expansion contribute because $(\mathcal{D} R)^3 = 0$. We thus get

$$\ln [(1 + \mathcal{D}) R] = \ln R + \ln \left[ 1 + \frac{1}{R} \mathcal{D} R \right] = \ln R + \frac{1}{R} \mathcal{D} R - \frac{1}{2} \left( \frac{1}{R} \mathcal{D} R \right)^2, \tag{2.29}$$

The next step is to show that (see Appendix \([11]\))

$$\tilde{g} = \partial_+ \partial_- \ln [(1 + \mathcal{D}) R] = \partial_+ \partial_- \left( (1 + \mathcal{D}) \ln R \right), \tag{2.30}$$

where

$$\mathcal{D}_\eta = i \theta_+ (\partial_+ \eta) + i \theta_- (\partial_- \eta^\dagger) - \theta_+ \theta_- \left( (\partial_- \eta^\dagger)(\partial_+ \eta) + (\partial_- \eta)(\partial_+ \eta^\dagger) \right). \tag{2.31}$$

Thus the metric becomes

$$\tilde{g} = (1 + \mathcal{D} + \mathcal{D}_\eta) g, \tag{2.32}$$

and (see Appendix \([12]\))

$$\ln \tilde{g} = \ln [(1 + \mathcal{D} + \mathcal{D}_\eta) g] = (1 + \mathcal{D}) \ln g + i \theta_+ (\partial_+ \eta) + i \theta_- (\partial_- \eta^\dagger). \tag{2.33}$$

Taking the mixed derivative, we get

$$\partial_+ \partial_- \ln \tilde{g} = \partial_+ \partial_- \left( (1 + \mathcal{D}) \ln g \right) = (1 + \mathcal{D} + \mathcal{D}_\eta) \left( \partial_+ \partial_- \ln g \right), \tag{2.34}$$

a similar result as in \((2.30)\). Using the expression of the susy Gaussian curvature \((2.5)\) and the fact that $\partial_+ \partial_- \ln g = -\kappa g$, we finally get

$$-\tilde{\kappa} \tilde{g} = \partial_+ \partial_- \ln \tilde{g} = -\kappa \left( 1 + \mathcal{D} + \mathcal{D}_\eta \right) g, \tag{2.35}$$

with $\tilde{g}$ given in \((2.32)\). We conclude that $\tilde{\kappa} = \kappa = \frac{2}{r}$. \hfill \Box
3 Constant curvature holomorphic solutions of the susy G(2, 4) σ-model

In this section we present all the constant curvature holomorphic solutions of the susy G(2,4)-σ-model in a canonical form.

First for the case of $G(2, N)$, the matrices $M_0$, $M_1$, $M_2$, $M_3$ are $2 \times 2$ and the quantities $X_1$, $X_2$, $X_3$ are easily computed from (3.5). We thus get

\begin{align*}
X_1 &= R^{-1} \left[ (M_0)_{11}(M_1)_{22} + (M_0)_{22}(M_1)_{11} - (M_0)_{12}(M_1)_{21} - (M_0)_{21}(M_1)_{12} \right], \\
X_2 &= R^{-1} \left[ (M_0)_{11}(M_2)_{22} + (M_0)_{22}(M_2)_{11} - (M_0)_{12}(M_2)_{21} - (M_0)_{21}(M_2)_{12} \right], \\
X_3 &= R^{-1} \left[ (M_0)_{11}(M_3)_{22} + (M_0)_{22}(M_3)_{11} + (M_1)_{11}(M_2)_{22} + (M_1)_{22}(M_2)_{11} \\
&- (M_0)_{12}(M_3)_{21} - (M_0)_{21}(M_3)_{12} - (M_1)_{12}(M_2)_{21} - (M_1)_{21}(M_2)_{12} \right].
\end{align*}

with $M_0$, $M_1$, $M_2$, $M_3$ given in (3.5) and $R \equiv \det Z^\dagger Z = \det M_0$.

In [8] it has been shown that, up to a $U(4)$ gauge transformation all the constant curvature holomorphic solutions of the purely bosonic case are given by

\begin{align*}
Z_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
Z_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R} \\
Z_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{2}x_+ \sin t \end{pmatrix}, \\
Z_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3}x_+ & 2x_+ \end{pmatrix}.
\end{align*}

(3.4)

Searching for the constant curvature holomorphic solutions of the susy model, we generalize them in the following way

\[ W_r(x_+) = Z_r(x_+) + i\theta_+\eta(x_+)A_r(x_+), \quad r = 1, 2, 3, 4, \]

(3.5)

where the different $Z_r$ are given by (3.4). Our aim is to determine the most general matrices $A_r(x_+)$ that satisfy the conditions of having a constant curvature. Using the gauge invariance of the susy model [14], we take

\[ A_r(x_+) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \beta_{11}(x_+) & \beta_{12}(x_+) \\ \beta_{21}(x_+) & \beta_{22}(x_+) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta(x_+) \end{pmatrix}. \]

(3.6)

Since the solutions $Z_r(x_+)$ are all real functions of $x_+$, we assume that it is also the case for $A_r(x_+)$. For each holomorphic solution $W_r(x_+)$ given in (3.5), the conditions (2.17) and (2.21) have to be satisfied. We investigate each of these cases separately. Interestingly for $W_3$ and $W_4$, the only solutions are the susy invariant ones. However, it is not true for $W_1$ and $W_2$.

3.1 The case of $Z_1$

This is the simplest solution of the bosonic $G(2, 4)$ model with $\det Z_1^\dagger Z_1 = (1 + |x|^2)$, i.e.; $r = 1$ or $\kappa = 2$. It is easy to see that the condition (2.17) is trivially satisfied for $W_1$ given in (3.5). Hence we are left with the condition (2.18). It reads as

\[ |x_+ (\partial^2_{x_+} \beta_{22}) + 2(\partial_+ \beta_{22})|^2 + |\partial^2_{x_+} \beta_{12}|^2 + |\partial^2_+ \beta_{21}|^2 = 0. \]

(3.7)
Since $\beta_{11}$ does not appear in this equation, it will remain arbitrary. Equation (3.7) implies that
\[
\rho_0^2 \beta_0 = 0, \quad \rho_0^2 \beta_0 = 0, \quad x_+ (\rho_0^2 \beta_2) + 2(\rho_0^2 \beta_2) = 0, \quad (3.8)
\]
which further fix the matrix $A_1$. We thus get constant curvature susy holomorphic solutions of the form
\[
W_1 = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
x_+ & 0 \\
0 & 0
\end{pmatrix} + i \theta \eta \begin{pmatrix}
0 & 0 \\
0 & 0 \\
\beta_{11}(x_+) & b_1 x_+ + b_0 \\
c_1 x_+ + c_0 & d_0
\end{pmatrix}, \quad (3.9)
\]
where $b_1, b_0, c_1, c_0$ and $d_0$ are arbitrary constants. Notice that when $b_0 = b_1 = c_0 = c_1 = d_0 = 0$, we get in particular the susy invariant solution. It is clear that we have more solutions than the susy invariant one in this case.

### 3.2 The case of $Z_2$

We have a family of bosonic solutions, labeled by the parameter $t$:
\[
Z_2(x_+, t) = \begin{pmatrix}
I_2 \\
K_2(t)
\end{pmatrix}, \quad K_2(x_+, t) = \begin{pmatrix}
x_+^2 \cos 2 t \\
\sqrt{2} x_+ \cos t \\
\sqrt{2} x_+ \sin t \\
0
\end{pmatrix}. \quad (3.10)
\]
Since
\[
\det Z_2^1 Z_2 = \det \left( I_2 + K_2^1 K_2 \right) = (1 + |x|^2)^2, \quad (3.11)
\]
the associated curvature is $\kappa = 1$.

In [8], the parameter $t$ can take any real values but due to the properties of the trigonometric functions, using a residual gauge invariance, we have been able to show (see Appendix C) that $t \in [0, \pi]$.

Considering now the corresponding susy holomorphic solution
\[
W_2(x_+, \theta, t) = Z_2(x_+, t) + i \theta \eta(x_+) A_2(x_+, t), \quad (3.12)
\]
where $A_2(x_+, t)$ takes the form (3.10), the conditions (2.17) and (2.18) have to be satisfied in order to get a family of constant curvature solutions.

Introducing $W_2$ given in (3.12) into (2.17), we get two different cases:

1. The first case corresponds to $\cos 2t \neq 0$. Condition (2.17) implies
\[
\beta_{11}(x_+, t) = x_+ \left( \sqrt{2} \cos \beta_{12}(x_+, t) - \sqrt{2} \sin \beta_{21}(x_+, t) + x_+ \sin 2t \beta_{22}(x_+, t) \right). \quad (3.13)
\]
So we have only one condition (2.18) to resolve three unknown functions. Interestingly, starting with a polynomial form in $x_+$ of the unknown functions we get a pattern. Indeed, we find that
\[
\beta_{12}(x_+, t) = c_0 + c_1 x_+ + F(x_+), \quad (3.14)
\]
\[
\beta_{21}(x_+, t) = (c_0 + F(x_+)) \tan t + a_1 x_+, \quad (3.15)
\]
\[
\beta_{22}(x_+, t) = \frac{\cos t}{\sqrt{2}} \left( a_1 - c_1 \tan t \right), \quad (3.16)
\]
where $a_1, c_0$ and $c_1$ are constants, solve our problem. Thus the matrix $\beta(x_+, t)$ takes the form
\[
\beta(x_+, t) = \begin{pmatrix}
(c_0 + F(x_+)) \\
\frac{2 x_+ \cos 2 t}{\sqrt{2} \sin t} \\
\frac{\sqrt{2} \cos t}{\sqrt{2} \sin t} \\
0
\end{pmatrix} + a_1 \begin{pmatrix}
-\sqrt{2} x_+^3 \sin^3 t \\
x_+ \\
\frac{1}{\sqrt{2}} \cos t \\
0
\end{pmatrix} + c_1 \begin{pmatrix}
\sqrt{2} x_+ \cos^3 t \\
0 \\
-\frac{x_+}{\sqrt{2} \sin t} \\
0
\end{pmatrix}. \quad (3.17)
\]
The susy invariant solution is obtained when $a_1 = c_1 = 0$. Again the case $W_2$ gives other solutions to our problem than the susy invariant ones.
2. The second case corresponds to \( \cos 2t = 0 \) or \( t = \frac{\pi}{4} \) (the case \( t = \frac{3\pi}{4} \) is gauge equivalent) so that

\[
K_2(x_+, \frac{\pi}{4}) = \begin{pmatrix} 0 & x_+ \\ x_+ & 0 \end{pmatrix}.
\]  \tag{3.18}

Since \( K_2(x_+, \frac{\pi}{4}) \) is symmetric, we assume that the matrix \( \beta(x_+) \) is also symmetric, i.e.

\[
\beta_{21}(x_+) = \beta_{12}(x_+). \tag{3.19}
\]

These quantities will remain arbitrary since the condition \( \beta_{21} \) depends only on \( \beta_{11} \) and \( \beta_{22} \) and the susy invariant solutions will be obtained when \( \beta_{11} = \beta_{22} = 0 \).

The condition \( \beta_{21} \) may be written as follows, taking in particular \( x_+ = x_- = x \):

\[
(1 + x^2)^2 \left( 4(x^2 - 1)(\beta'_{11})^2 + (\beta''_{22})^2 \right) + (1 + x^2)^2 \left( (\beta'_{11})^2 + (\beta''_{22})^2 \right) 
- 8x(1 + x^2)(x^2 - 2)(\beta_{11}\beta'_{22} + \beta_{22}\beta'_{11}) + 4x^2(1 + x^2)(\beta_{11}\beta''_{22} + \beta_{22}\beta''_{11}) 
+ 4(1 - 4x^2 + x^4)(\beta'_{11})^2 - 4x(1 + x^2)^3 (\beta'_{11}\beta''_{22} + \beta_{22}\beta''_{11}) = 0. \tag{3.20}
\]

Let us first mention the invariance of this equation with respect to the exchange \( \beta_{11} \leftrightarrow \beta_{22} \).

After some trials we first get a solution choosing

\[
\beta_{22}(x) = x\beta_{11}(x). \tag{3.21}
\]

Condition \( \beta_{21} \) thus becomes very simple

\[
(1 + x^2)^5 (\beta''_{11}(x))^2 = 0, \tag{3.22}
\]

which implies that

\[
\beta_{11}(x) = a_0 + d_2 x, \quad \beta_{22}(x) = x(a_0 + d_2 x). \tag{3.23}
\]

Using this observation, we assume that \( \beta_{11}(x) \) and \( \beta_{22}(x) \) are real polynomial in \( x \). We can easily show that they must be at most of degree 2. If we take

\[
\beta_{11}(x) = a_2 x^2 + a_1 x + a_0, \quad \beta_{22}(x) = d_2 x^2 + d_1 x + d_0, \tag{3.24}
\]

and identify the coefficients of different powers of \( x \) in \( \beta_{21} \), we get three independent equations for the parameters \( a_i \) and \( d_i \),

\[
a_0^2 - a_1^2 + a_2^2 - d_1^2 + d_2^2 = 0, \quad a_0 a_2 + d_0 d_2 = 0, \quad a_0 a_1 - a_1 a_2 + d_1 (d_0 - d_2) = 0. \tag{3.25}
\]

Let us first assume that \( a_0 \neq 0 \), we then get

\[
\beta_{11}(x) = a_0 + (d_2 - d_0)x - \frac{d_0 d_2}{a_0} x^2, \\
\beta_{22}(x) = d_0 + (a_0 + \frac{d_0 d_2}{a_0}) x + d_2 x^2, \tag{3.26}
\]

where \( a_0, d_0 \) and \( d_2 \) remain arbitrary real parameters. Clearly the solution \( \beta_{21} \) is obtained when \( d_0 = 0 \).

Now, consider \( a_0 = 0 \). We then get different subcases (\( \epsilon = \pm1 \))

- \( d_2 \neq 0, d_0 = 0 \implies \begin{cases} 
\beta_{11}(x) = a_2 x^2 + \epsilon d_2 x, \\
\beta_{22}(x) = d_2 x^2 - \epsilon a_2 x,
\end{cases} \]
- \( d_0 \neq 0, d_2 = 0 \implies \begin{cases} 
\beta_{11}(x) = a_2 x^2 + \epsilon d_0 x, \\
\beta_{22}(x) = \epsilon a_2 x + d_0.
\end{cases} \]
3.3 The case of $Z_3$

In this case we have $\det Z_3 Z_3 = (1 + |x|^2)^3$, i.e., $r = 3$ and $\kappa = \frac{2}{3}$. With the solution $W_3$ as in (3.35), the condition (2.17) becomes a third degree polynomial in $x_-$. Equating the coefficients of different powers of $x_-$ to zero we obtain the following equations:

\[
2x_+^3 \left( \sqrt{2} \beta_{12}'' + 5 \beta_{22}'' \right) - x_+^3 \left( 3 \beta_{11}'' + 8 \sqrt{2} \beta_{12}' + 6 \sqrt{2} \beta_{21}' + 40 \beta_{22}' \right) + 6x_+ \left( 3 \beta_{11}' + 2 \sqrt{2} \beta_{12}' + 6 \sqrt{2} \beta_{21}' + 10 \beta_{22}' \right) - 36 \beta_{11} - 72 \sqrt{2} \beta_{21} = 0, \quad (3.27)
\]
\[
x_+^2 \left( -\beta_{11}'' + 8 \sqrt{2} \beta_{12}' - 4 \sqrt{2} \beta_{21}' + 6x_+ \beta_{22}' + 4 \beta_{22}' \right) - x_+ \left( 4 \beta_{11}' + 24 \sqrt{2} \beta_{12}' - 8 \sqrt{2} \beta_{21}' + 48 \beta_{22}' \right) + 28 \beta_{11} + 16 \sqrt{2} \beta_{21} = 0, \quad (3.28)
\]
\[
x_+ \left( \beta_{11}' - 2 \sqrt{2} x_+ \beta_{12}' + 8 \sqrt{2} \beta_{21}' - 2 \sqrt{2} \beta_{22}' + 2x_+ \beta_{22}' + 16 \beta_{22}' \right) - 10 \beta_{11}' + 12 \sqrt{2} \beta_{12}' - 4 \sqrt{2} \beta_{21}' + 12 \beta_{22} = 0, \quad (3.29)
\]
\[
-3 \beta_{11}'' + 4 \sqrt{2} x_+ \beta_{12}'' + 8 \sqrt{2} \beta_{21}'' + 2x_+ \beta_{22}'' + 4 \beta_{22}'' = 0. \quad (3.30)
\]

We first solve (3.30) for $\beta_{11}$ and get

\[
\beta_{11}(x_+) = \frac{4 \sqrt{2} x_+}{3} \beta_{12}(x_+) + \frac{2x_+}{3} \beta_{22}(x_+) + c_1 x_+ + c_2, \quad (3.31)
\]

where $c_1$ and $c_2$ are integration constants. Upon introducing (3.31) into some linear combinations of (3.27), (3.28) and (3.29) we obtain

\[
\beta_{22}(x_+) = \frac{\sqrt{2}}{4} \beta_{12}(x_+) + \frac{3 \sqrt{2}}{4x_+} \beta_{21}(x_+) + \frac{3}{4} c_1 + \frac{3}{4x_+} c_2. \quad (3.32)
\]

In order to satisfy all the equations (3.27), (3.29), the integration constants $c_1$ and $c_2$ must vanish. Hence, we can give the final form of $\beta_{11}$ and $\beta_{22}$ as

\[
\beta_{11}(x_+) = \frac{3x_+}{\sqrt{2}} \beta_{12}(x_+) + \frac{1}{\sqrt{2}} \beta_{21}(x_+), \quad (3.33)
\]
\[
\beta_{22}(x_+) = \frac{\sqrt{2}}{4} \beta_{12}(x_+) + \frac{3 \sqrt{2}}{4x_+} \beta_{21}(x_+). \quad (3.34)
\]

Introducing (3.33) and (3.34) into the condition (2.18) we obtain

\[
\partial_+ \partial_- \left( \frac{(1 + 3|x|^2) \beta_{21} - x_+ (1 + |x|^2) \partial_+ \beta_{21}}{|x|^4 (1 + |x|^2)^2} \right)^2 = 0, \quad (3.35)
\]

or equivalently

\[
\left( \frac{(1 + 3|x|^2) \beta_{21} - x_+ (1 + |x|^2) \partial_+ \beta_{21}}{|x|^4 (1 + |x|^2)^2} \right) = f(x_+) + g(x_-), \quad (3.36)
\]

for arbitrary functions $f$ and $g$ of given variables. Requiring it to be satisfied when $x_+ = 0$ and $x_- = 0$ separately we obtain

\[
\beta_{21}(x_+) = \gamma_1 x_+, \quad (3.37)
\]

where $\gamma_1$ is an arbitrary constant. Upon introducing it into (3.36) we get

\[
f(x_+) + g(x_-) = \frac{4|x|^2 \gamma_1^2}{(1 + |x|^2)^2}, \quad (3.38)
\]
which immediately implies that \( \gamma_1 = 0 \) and hence \( \beta_{21} = 0 \).

The necessary and sufficient conditions \([2.17]\) and \([2.18]\) are thus satisfied and finally the constant curvature holomorphic solution \( W_3 \) is given by the form

\[
W_3 = Z_3 + i\theta + \sqrt{3}\eta(x_+)\beta_{22}(x_+)^2 Z_3. \tag{3.39}
\]

Hence in this case we have obtained the susy invariant solution as the unique constant curvature holomorphic solution.

### 3.4 The case of \( Z_4 \)

In this case we have \( \det Z_4 = (1 + |x|^2)^4 \), i.e., \( r = 4 \) and \( \kappa = \frac{1}{2} \).

Again the condition \([2.17]\) becomes a third degree polynomial in \( x_- \) after introducing the solution \( W_4 \) given in \( 3.39 \). Similarly as what we did with \( W_3 \), we equate the coefficients of different powers of \( x_- \) to zero and now get

\[
\beta_{11}(x_+) = 3x_-^2 \beta_{22}(x_+), \tag{3.40}
\]

\[
\beta_{21}(x_+) = -\beta_{12}(x_+) + 2\sqrt{x_-} \beta_{22}(x_+). \tag{3.41}
\]

We are left with the last condition \([2.18]\). Introducing \([3.40]\) and \([3.41]\) into this last condition we find that \( \beta_{21}(x_+) = \beta_{12}(x_+) \). Finally, the constant curvature holomorphic solution \( W_4 \) is given as

\[
W_4 = Z_4 + i\theta + \frac{1}{2} \eta(x_+)\beta_{22}(x_+)^2 Z_4. \tag{3.42}
\]

Again, we have obtained in this case the susy invariant solution as the unique constant curvature holomorphic solution.

### 4 About the Plücker embedding of \( G(2, 4) \) into \( \mathbb{C}P^5 \)

It is well-known \([22, 23, 24]\) for the bosonic model that Plücker embedding of \( G(2, N) \) into \( \mathbb{C}P^{N(N-1)-1} \) is obtained by introducing the map \( \Phi_N : G(2, N) \to \mathbb{C}P^{N(N-1)-1} \).

In our case, we get explicitly the map \( \Phi_4 : G(2, 4) \to \mathbb{C}P^5 \), on the form

\[
\Phi_4(Z) = (1, -z_{31}, z_{32}, -z_{41}, z_{42}, z_{31}z_{42} - z_{32}z_{41})^T, \tag{4.1}
\]

when

\[
Z = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
z_{31} & z_{32} \\
z_{41} & z_{42}
\end{pmatrix}. \tag{4.2}
\]

It is thus easy to show, up to gauge equivalence \( \hat{\Phi}_4(Z_+) = V_r \Phi_4(Z_+) (V_r \in U(6)) \) is a constant matrix, explicitly given in Appendix D for \( r = 1, 2, 3, 4 \) that we get

\[
\hat{\Phi}_4(Z_1) = (1, x_+, 0, 0, 0, 0)^T, \tag{4.3}
\]

\[
\hat{\Phi}_4(Z_2) = (1, \sqrt{2}x_+, x_+^2, 0, 0, 0)^T, \tag{4.4}
\]

\[
\hat{\Phi}_4(Z_3) = (1, \sqrt{3}x_+, \sqrt{3}x_+^2, x_+^3, 0, 0)^T, \tag{4.5}
\]

\[
\hat{\Phi}_4(Z_4) = (1, 2x_+, \sqrt{6}x_+^2, 2x_+^3, x_+^4, 0)^T. \tag{4.6}
\]

For the bosonic case, such a correspondence has helped \([10]\) constructing the holomorphic solutions with constant curvature of \( G(2, N) \) from the Veronese curves embedded in \( \mathbb{C}P^{N(N-1)-1} \).
The Veronese curve in $\mathbb{CP}^5$

\[
(1, \sqrt{5}x_+, \sqrt{10}x_+^2, \sqrt{10}x_+^3, \sqrt{5}x_+^4, x_+^5)^T
\]  

(4.7)

does not give rise to a solution of $G(2, 4)$. Indeed, the constraint on the Plücker coordinates is not satisfied [10].

For the susy case, we see that the arbitrariness in the possible choices of the fermionic contribution $A(x_+)$ may thus come from some arbitrariness in the corresponding solutions of $\mathbb{CP}^5$. A detailed discussion of such a correspondence is out of the scope of this paper.

5 Conclusions and final comments

In this article we give some criteria for having constant curvature holomorphic solutions of the susy grassmanian $G(M, N)$ $\sigma$-model. With the help of the susy gauge invariance of the model we first show that the susy holomorphic solution given in [22] (i.e.; generalisation of bosonic holomorphic solution) leads to a constant curvature surface. This kind of a solution is called a susy invariant one, in analogy with the discussion given in [12].

Then we restrict ourselves to the susy $G(2, N)$ $\sigma$-model and give the necessary and sufficient conditions to get such solutions. The case of $G(2, 4)$ is studied in detail taking into account the classification of bosonic solutions [8].

The existence of an embedding of $G(2, 4)$ into $\mathbb{CP}^5$ shows a connection between the corresponding solutions. It will be relevant when we consider higher dimensional models. Indeed the case of the susy $G(2, 5)$ $\sigma$-model could be treated taking into account the results for the bosonic case [9] and its relation to $\mathbb{CP}^9$. Some results for the bosonic $G(2, N)$ [25] will thus be used to study similar solutions of the susy $G(2, N)$ $\sigma$-model.

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A Proof of Lemma 2.1

Here we will prove Lemma 2.1.

Proof. Let us first show that the Lemma 2.1 holds for $M = 2$ and then generalize it to general $M$. For $M = 2$, we have

\[
(1 + D)B = \begin{pmatrix}
(1 + D) b_{11} & (1 + D)b_{12} \\
(1 + D)b_{21} & (1 + D)b_{22}
\end{pmatrix},
\]  

(A.1)

whose determinant is

\[
\det [(1 + D) B] = (1 + D)b_{11}(1 + D)b_{22} - (1 + D)b_{12}(1 + D)b_{21}. 
\]  

(A.2)

It is enough to show that (A.2) can be expressed as $(1 + D) (b_{11}b_{22} - b_{12}b_{21})$. Let us consider the following two equations which has to be verified:

\[
(1 + D) b_{11}(1 + D)b_{22} = (1 + D)(b_{11}b_{22}),
\]  

(A.3)

\[
(1 + D)b_{12}(1 + D)b_{21} = (1 + D)(b_{12}b_{21}).
\]  

(A.4)
We concentrate our calculations on (A.3). The result for (A.4) will then follow immediately. In order to simplify the calculations we separate the differential operator $D$ into first- and second-order parts as

$$D_1 = i\theta_+ \eta \partial_+ + i\theta_- \eta^\dagger \partial_-, \quad D_2 = -\theta_+ \theta_- |\eta|^2 \partial_+ \partial_-.$$  \hspace{1cm} \text{(A.5)}

Since $D_1$ is a first-order operator and $D_2$ is a second-order operator, it is not difficult to verify (A.3) by considering the properties of grassmann variables and the following identity

$$D_2(b_{11} b_{22}) = b_{11}(D_2 b_{22}) + (D_2 b_{11})b_{22} + (D_1 b_{11})(D_1 b_{22}).$$  \hspace{1cm} \text{(A.6)}

Hence the Lemma (A.3) is shown to be true for $M = 2$. For general $M > 2$, $(1 + D)B$ will be an $M \times M$ matrix whose determinant can be expressed as

$$\det [(1 + D)B] = \sum (-1)^{\sigma(\nu_i)}(1 + D)b_{1\nu_1}(1 + D)b_{2\nu_2}...(1 + D)b_{N-1\nu_{N-1}}, \quad \text{(A.7)}$$

where the sum is over the permutations $\nu_i$. By using (A.3) it is clear that the terms of this sum can be rewritten as

$$(1 + D)b_{1\nu_1}(1 + D)b_{2\nu_2}(1 + D)b_{3\nu_3}...(1 + D)b_{N-1\nu_{N-1}} = (1 + D)(b_{1\nu_1} b_{2\nu_2})(1 + D)b_{3\nu_3}...(1 + D)b_{N-1\nu_{N-1}}, \quad \text{(A.8)}$$

and following the same strategy, they are equal to

$$(1 + D)(b_{1\nu_1} b_{2\nu_2})(1 + D)b_{3\nu_3}...(1 + D)b_{N-1\nu_{N-1}} = (1 + D)(b_{1\nu_1} b_{2\nu_2} b_{3\nu_3}...b_{N-1\nu_{N-1}}). \quad \text{(A.9)}$$

By iteration, one obtains

$$(1 + D)b_{1\nu_1}(1 + D)b_{2\nu_2}...(1 + D)b_{N-1\nu_{N-1}} = (1 + D)(b_{1\nu_1} b_{2\nu_2} b_{3\nu_3}...b_{N-1\nu_{N-1}}). \quad \text{(A.10)}$$

By applying this argument to all of the terms in the sum (A.7), it is clear that for any value of $M$, we have

$$\det [(1 + D)B(x_+, x_-)] = (1 + D) \det [B(x_+, x_-)]. \quad \text{(A.11)}$$

**B Equations (2.30) and (2.33)**

In this appendix we give some explicit calculations, which are needed in the proof of the theorem (2.23). In particular, we derive (2.30) and (2.33). In (2.30) we have

$$\tilde{g} = \partial_+ \partial_-(\ln R) = \partial_+ \partial_-(D \ln R) + \partial_+ \partial_-(D \ln R) . \quad \text{(B.1)}$$

Since $\eta$ and $\eta^\dagger$ in the definition of operator $D$ (2.24) are functions of $x_+$ and $x_-$, respectively, we can develop the last term in (B.1) as

$$\partial_+ \partial_-(D \ln R) = \partial_+ \partial_-(i\theta_+ \eta \partial_+ \ln R + i\theta_- \eta^\dagger \partial_- \ln R \cdots) = \partial_+ \partial_-(i\theta_+ \eta \partial_+ \ln R + i\theta_- \eta^\dagger \partial_- \ln R \cdots) . \quad \text{(B.2)}$$
Deriving again with respect to $\partial_+$, we finally get
\[
\partial_+ \partial_- (D \ln R) = \left( D + i \theta_+ (\partial_+ \eta) + i \theta_- (\partial_- \eta^\dagger) \right.
\]
\[
- \theta_+ \theta_- \left( (\partial_- \eta^\dagger) (\partial_+ \eta) + (\partial_- \eta^\dagger \eta \partial_+ + \eta^\dagger (\partial_+ \eta) \partial_- + \eta^\dagger \eta \partial_+ \partial_-) \right) \partial_+ \partial_- \ln R ,
\]
\[
= \left( D + D_\eta \right) \partial_+ \partial_- \ln R ,
\]
\[
(B.3)
\]
where we define $D_\eta$ as in (2.31) and hence (2.30) holds.

Let us now show that (2.33) holds. Using the Taylor expansion of the logarithmic function (2.12), we have
\[
\ln \tilde{g} = \ln \left(1 + \frac{1}{g} (D + D_\eta) g \right) ,
\]
\[
= \ln g + \frac{1}{g} (D + D_\eta) g - \frac{1}{2} \left( \frac{1}{g} (D + D_\eta) g \right)^2 .
\]
\[
(B.4)
\]
Expressions $(D + D_\eta) g$ and $(D + D_\eta) g^2$ in (B.3) can be expressed as
\[
(D + D_\eta) g = \left( i \theta_+ ((\partial_+ \eta) + \eta \partial_+) + i \theta_- ((\partial_- \eta^\dagger) + \eta^\dagger \partial_-) \right.
\]
\[
- \theta_+ \theta_- \left( (\partial_- \eta^\dagger) (\partial_+ \eta) + (\partial_- \eta^\dagger \eta \partial_+ + \eta^\dagger (\partial_+ \eta) \partial_- + \eta^\dagger \eta \partial_+ \partial_-) \right) \right) g ,
\]
\[
(B.5)
\]
and
\[
(D + D_\eta) g^2 = -2 \theta_+ \theta_- \left( (\partial_- \eta^\dagger) (\partial_+ \eta) g^2 + \eta^\dagger (\partial_+ \eta) (\partial_- g) g + (\partial_- \eta^\dagger \eta g (\partial_+ g) + \eta^\dagger \eta (\partial_+ g) (\partial_- g) \right). \]
\[
(B.6)
\]

Upon introducing (B.5) and (B.6) into (B.4) and making necessary cancellations we arrive at (2.33).

C Residual gauge invariance for $Z_2$

Here we want to prove that the bosonic holomorphic solutions $Z_2(x_+, t)$ may be considered in the interval $t \in [0, \pi]$. For obtaining the admissible gauge transformations $V$, we refer [14]. In the case of $Z_2$ we have
\[
K = K_2 = \left( \begin{array}{cc}
x_+^2 \cos 2t & \sqrt{2} x_+ \cos t \\
\sqrt{2} x_+ \sin t & 0 \end{array} \right).
\]
\[
(C.1)
\]
First notice that since $V$ is a constant matrix and $\lim_{x_+ \to 0} K_2 = 0$, we have
\[
\lim_{x_+ \to 0} K_{2G} = 0 .
\]
\[
(C.2)
\]
Thus $V_{21} = 0$ and by unitarity $V_{12} = 0$. Finally, $K_{2G}$ and $K_2$ are related as
\[
K_{2G} = V_{22} K_2 V_{11}^\dagger.
\]
\[
(C.3)
\]
Moreover, since the entries of $K_2$ are real functions of $x_+$, the matrices $V_{11}$ and $V_{22}$ are real and orthogonal. In particular, this means that
\[
\det K_{2G} = \pm \det K_2 = \pm x_+^2 \sin 2t .
\]
\[
(C.4)
\]
Since $K_{2G}$ has the same form as $K_2$, it must have the same determinant as $K_2$ up to a sign. Assuming that $t_0$ is fixed in $K_2$, the only admissible values of $t$ in $K_{2G}$ are
\[ t = \pm t_0 + k \frac{\pi}{2}. \quad (C.5) \]
It can easily be shown that $V_{11}$ and $V_{22}$ can be fixed, such that
\[ K_2(\pm t_0 + \pi) \simeq K_2(t_0), \quad (C.6) \]
and
\[ K_2(\pm t_0 + \frac{3\pi}{2}) \simeq K_2(t_0 + \frac{\pi}{2}). \quad (C.7) \]

$K_2(t_0)$ and $K_2(t_0 + \frac{\pi}{2})$ are not gauge equivalent and we have reduced the interval of values of the parameter $t$ between $[0, \pi]$.

D The transformation matrices $V_r$ of Section 4

Here, we give the explicit form of the transformation matrices $V_r$ that we used in Section 4

\[
V_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
V_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 &\cos t & -\sin t & 0 & 0 & 0 \\
0 &-\cos 2t & 0 & 0 & 0 & -\sin 2t \\
0 & 0 &\sin t & \cos t & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 &-\sin 2t & 0 & 0 & 0 & \cos 2t
\end{pmatrix},
\]
\[
V_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} & 0
\end{pmatrix},
V_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0
\end{pmatrix}.
\]

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