ON CLUSTER CATEGORIES OF WEIGHTED PROJECTIVE LINES
WITH AT MOST THREE WEIGHTS

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Abstract. Let $X$ be a weighted projective line and $\mathcal{C}_X$ the associated cluster category. It is known that $\mathcal{C}_X$ can be realized as a generalized cluster category of quiver with potential. In this note, under the assumption that $X$ has at most three weights or is of tubular type, we prove that if the generalized cluster category $\mathcal{C}_{(Q,W)}$ of a Jacobi-finite non-degenerate quiver with potential $(Q,W)$ shares a 2-CY tilted algebra with $\mathcal{C}_X$, then $\mathcal{C}_{(Q,W)}$ is triangle equivalent to $\mathcal{C}_X$. As a byproduct, a 2-CY tilted algebra of $\mathcal{C}_X$ is determined by its quiver provided that $X$ has at most three weights. To this end, for any weighted projective line $X$ with at most three weights, we also obtain a realization of $\mathcal{C}_X$ via Buan-Iyama-Reiten-Scott’s construction of 2-CY categories arising from preprojective algebras.

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1. Introduction

Motivated by the theory of cluster algebras [FZ], there has recently been much interest centered around Hom-finite 2-Calabi-Yau (2-CY for short) triangulated categories. Many kinds of Hom-finite 2-CY triangulated categories with cluster-tilting objects have been investigated, for example

- the acyclic cluster category $\mathcal{C}_Q$ for a finite acyclic quiver $Q$ introduced by Buan-Marsh-Reineke-Reiten-Todorov [BMRRT];
- 2-CY triangulated categories arising from preprojective algebras introduced by Geiß-Leclerc-Schröer [GLS06, GLS10] and Buan-Iyama-Reiten-Scott [BIRSc];
- the generalized cluster category $\mathcal{C}_A$ for a finite dimensional algebra $A$ of global dimension $\leq 2$ and the generalized cluster category $\mathcal{C}_{(Q,W)}$ for a quiver with potential $(Q,W)$ constructed by Amiot [A09];
- the cluster category $\mathcal{C}_X$ for a weighted projective line $X$ studied by Barot-Kussin-Lenzing [BKL].

Key words and phrases. generalized cluster category, weighted projective line, 2-Calabi-Yau category, quiver with potential.

Partially supported by the National Natural Science Foundation of China (Grant No. 11971326).
It is known that the acyclic cluster category $C_Q$ can be realized as a 2-CY triangulated category arising from preprojective algebras [GLS10], while 2-CY triangulated categories arising from preprojective algebras and cluster categories of weighted projective lines can be realized as generalized cluster categories of finite dimensional algebras of global dimension $\leq 2$ [ART]. Furthermore, the generalized cluster category $C_A$ can be realized as the generalized cluster category $C_{(Q,W)}$ for a quiver with potential $(Q,W)$ [K11]. However, it is not clear that whether every cluster category of weighted projective line can be realized as a 2-CY triangulated category arising from preprojective algebras. It is conjectured that each algebraic $\Hom$-finite 2-CY category over an algebraically closed field of characteristic zero can be realized as a generalized cluster category for quiver with potential [A08, Y].

Let $K$ be an algebraically closed field and $C$ a $\Hom$-finite algebraic 2-CY triangulated category over $K$. The endomorphism algebra $\End_C(T)$ of a cluster-tilting object $T$ of $C$ is called a $2$-$CY$ tilted algebra. An open question in cluster-tilting theory is that whether the 2-CY tilted algebra $\End_C(T)$ determines the 2-CY triangulated category $C$ up to triangle equivalence? A remarkable progress has been made by Keller-Reiten [KR]. Namely, if $\End_C(T)$ is isomorphic to the path algebra of a finite acyclic quiver $Q$, then $C$ is triangle equivalent to the acyclic cluster category $C_Q$. We also refer to [A09, ART, AIR15] for some other progress on this question. Inspired by Keller-Reiten’s result [KR] and Happel’s classification theorem [H01], it is reasonable to conjecture that if $\End_C(T)$ is isomorphic to a 2-CY tilted algebra of $C_X$ for a weighted projective line $X$, then $C \cong C_X$. The main result of this note provides a partial evidence of a positive answer for the aforementioned question.

**Theorem 1.1** (Theorem 4.4+Theorem 4.7). Let $X$ be a weighted projective line with at most three weights or of tubular type and $C_X$ the cluster category of $X$. Let $(Q,W)$ be a Jacobi-finite quiver with non-degenerate potential and $C_{(Q,W)}$ the associated generalized cluster category. If there is a cluster-tilting object $T$ of $C_{(Q,W)}$ such that $\End_{C_{(Q,W)}}(T)$ is isomorphic to a 2-CY tilted algebra of the cluster category $C_X$, then $C_{(Q,W)}$ is triangle equivalent to $C_X$.

We also consider the relation between cluster categories of weighted projective lines and 2-CY triangulated categories arising from preprojective algebras.

**Theorem 1.2** (Theorem 5.2). Let $X$ be a weighted projective line with at most three weights. The cluster category $C_X$ is triangle equivalent to a 2-CY triangulated category arising from preprojective algebra.

The paper is structured as follows. Section 2 provides the required background from cluster-tilting theory, quivers with potentials and generalized cluster categories. In Section 3, we recollect basic properties for $C_X$. For a given basic cluster-tilting object $T \in C_X$, there is a canonical potential $W_T$ on the Gabriel quiver $Q_T$ of $\End_{C_X}(T)$ via Keller’s construction. We prove that the potential $W_T$ is non-degenerate. Assume that $X$ has at most three weights, we further prove that $W_T$ is the unique non-degenerate potential on $Q_T$ up to right equivalence in Section 4.1. The uniqueness plays a key role in the proof of Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

Throughout this paper, let $K$ be an algebraically closed field. For a differential graded(=dg) $K$-algebra $B$, denote by $D(B)$ the derived category of dg $B$-modules. Let $D^b(B)$ be the full subcategory of $D(B)$ formed by the dg $B$-modules whose homology is of finite total dimension
over $K$. Denote by $\text{per} B$ the perfect derived category of $B$, i.e. the thick subcategory of $\mathcal{D}(B)$ generated by $B$. Let $\mathcal{C}$ be a category and $M \in \mathcal{C}$, denote by $\text{add} M$ the subcategory of $\mathcal{C}$ consisting of objects which are finite direct sum of direct summands of $M$.

2. Preliminaries

2.1. 2-CY triangulated category and cluster-tilting objects. Let $\mathcal{C}$ be a $\text{Hom}$-finite $K$-linear triangulated category with suspension functor $[1]$. The category $\mathcal{C}$ is 2-Calabi-Yau (2-CY for short) if there exist bifunctorial isomorphisms

$$\text{Hom}_{\mathcal{C}}(X, Y[1]) \cong \mathcal{D} \text{Hom}_{\mathcal{C}}(Y, X[1])$$

for arbitrary $X, Y \in \mathcal{C}$, where $\mathcal{D} = \text{Hom}_K(-, K)$ is the duality over $K$.

Let $\mathcal{C}$ be a 2-CY triangulated category. An object $T \in \mathcal{C}$ is a cluster-tilting object of $\mathcal{C}$ if for any object $X \in \mathcal{C}$ with $\text{Hom}_{\mathcal{C}}(T, X[1]) = 0$, we have $X \in \text{add} T$. Let $T = \bigoplus_{i=1}^n T_i$ be a basic cluster-tilting object in $\mathcal{C}$ and $\text{End}_{\mathcal{C}}(T)$ the endomorphism algebra of $T$. Let $Q_T$ be the quiver of $T$, whose vertices correspond to the indecomposable direct summands of $T$ and the number of arrows from $T_i$ to $T_j$ is given by the dimension of the space of irreducible maps from $T_i$ to $T_j$. It is known that the quiver $Q_T$ coincides with the Gabriel quiver of $\text{End}_{\mathcal{C}}(T)$.

For a given basic cluster-tilting object $T = \mathcal{T} \oplus T_k$ in $\mathcal{C}$ with indecomposable direct summand $T_k$, there exists a unique indecomposable object $T^*_k \in \mathcal{C}$ such that $T^*_k \notin T_k$ and $\mu_{T_k}(T) := \mathcal{T} \oplus T^*_k$ is also a cluster-tilting object in $\mathcal{C}$ [FY]. We call $\mu_{T_k}(T)$ the mutation of $T$ at $T_k$. For a vertex $k \in Q_T$, we also denote $\mu_k(T)$ the mutation of $T$ at $T_k$. A cluster-tilting object $T'$ is reachable from $T$, if we can obtain $T'$ from $T$ by a finite sequence of mutations. The cluster-tilting graph $\mathcal{G}_{\text{ct}}(\mathcal{C})$ of $\mathcal{C}$ has as vertices the isomorphism classes of basic tilting objects of $\mathcal{C}$, while two vertices $T$ and $T'$ are connected by an edge if and only if $T'$ is a mutation of $T$. For any basic cluster-tilting objects $T$ and $T'$, it is clear that $T'$ is reachable from $T$ if and only if they belong to the same connected component of $\mathcal{G}_{\text{ct}}(\mathcal{C})$.

2.2. Quivers with potentials. We follow [DWZ, KY]. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. Let $KQ$ be the path algebra of $Q$ over $K$. The complete path algebra $K\langle\langle Q\rangle\rangle$ of $Q$ is the completion of $KQ$ with respect to the ideal generated by the arrows of $Q$. Let $m$ be the ideal of $K\langle\langle Q\rangle\rangle$ generated by arrows of $Q$. In particular, $K\langle\langle Q\rangle\rangle$ is a topological algebra via $m$-adic topology. An element $W \in K\langle\langle Q\rangle\rangle$ is a potential on $Q$ if $W$ is a (possibly infinite) linear combination of cycles in $Q$. Two potentials $W$ and $W'$ are cyclically equivalent if $W - W'$ lies in the closure of the span of all elements of the form $\alpha_s \cdots \alpha_2 \alpha_1 - \alpha_1 \alpha_s \cdots \alpha_2$, where $\alpha_s \cdots \alpha_2 \alpha_1$ is a cyclic path.

Let $W$ be a potential on $Q$ such that $W \in m^2$ and no two cyclically equivalent cyclic paths appear in the decomposition of $W$. Then the pair $(Q, W)$ is called a quiver with potential (QP for short). Two QPs $(Q, W)$ and $(Q', W')$ are right equivalent if $Q$ and $Q'$ have the same set of vertices and there exists an algebra isomorphism $\varphi : K\langle\langle Q\rangle\rangle \rightarrow K\langle\langle Q'\rangle\rangle$ whose restriction on vertices is the identity map and $\varphi(W)$ and $W'$ are cyclically equivalent.

For every arrow $\alpha \in Q_1$, the cyclic derivative $\partial_\alpha$ is the continuous $K$-linear map from the closed subspace of $K\langle\langle Q\rangle\rangle$ generated by all cyclic paths of $Q$ to $K\langle\langle Q\rangle\rangle$ acting on cyclic paths.
by
\[ \partial_\alpha (\alpha_1 \cdots \alpha_m) = \sum_{p, \lambda_\alpha = \alpha} \alpha_{p+1} \cdots \alpha_m \alpha_1 \cdots \alpha_{p-1}. \]

Let \((Q, W)\) be a quiver with potential. Denote by \(J(W)\) the closure of the two-sided ideal of \(K \langle \langle Q \rangle \rangle\) generated by the elements \(\partial_\alpha W\) for all \(\alpha \in Q_1\). The Jacobian algebra of \((Q, W)\) is
\[ J(Q, W) := K \langle \langle Q \rangle \rangle / J(W). \]

A QP \((Q, W)\) is **Jacobi-finite** if the Jacobian algebra \(J(Q, W)\) is finite-dimensional. A QP \((Q, W)\) is **reduced** if \(\partial_\alpha W \in m^2\) for all arrows \(\alpha \in Q_1\). For a reduced Jacobi-finite quiver with potential \((Q, W)\), it is known that the Gabriel quiver of \(J(Q, W)\) is \(Q\).

### 2.3. QP-mutation and non-degenerate potentials

Let \((Q, W)\) be a quiver with potential and \(k\) a vertex of \(Q\). Assume that \((Q, W)\) satisfies the following conditions:

1. \((c_1)\) \(Q\) has no loops;
2. \((c_2)\) \(Q\) has no 2-cycles at \(k\);
3. \((c_3)\) no cyclic path occurring in the expansion of \(W\) starts and ends at vertex \(k\).

Note that under the condition \((c_1)\), any potential is cyclically equivalent to a potential satisfying \((c_3)\). Derksen-Weyman-Zelevinsky [DWZ] defined an operation \(\mu_k\) on \((Q, W)\), called the QP-mutation at vertex \(k\). The resulting \(\mu_k(Q, W)\) is a reduced quiver with potential. We refer to [DWZ, Section 5] and [KY, Section 2.4] for the definition.

**Lemma 2.1.** [DWZ, Thm 5.2] Let \((Q, P)\) be a quiver with potential and \(k\) a vertex of \(Q\) such that \(\mu_k(Q, W)\) is well-defined. Then the right-equivalence class of \(\mu_k(Q, W)\) is determined by the right-equivalence class of \((Q, W)\).

If \(Q\) has no loops nor 2-cycles, then \(\mu_k(Q, W)\) is well-defined for any vertex \(k\) of \(Q\). In this case, the quiver with potential \((Q, W)\) is 2-acyclic. A potential \(W\) is **non-degenerate** on \(Q\), if for any sequence \(i_1, \ldots, i_t\) of vertices, all QPs \(\mu_{i_1}(Q, W), \mu_{i_2}\mu_{i_1}(Q, W), \ldots, \mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(Q, W)\) are 2-acyclic. We also call \((Q, W)\) a non-degenerate QP. A potential \(W\) on \(Q\) is **rigid** if every cycle in \(Q\) is cyclically equivalent to an element of the Jacobian ideal \(J(W)\). According to [DWZ, Cor 8.2], every rigid potential is non-degenerate.

Let \((Q, W)\) be a quiver with potential. Let \(I\) be a subset of \(Q_0\). Let \((Q|_I, W|_I)\) be the restriction of \((Q, W)\) to \(I\).

**Lemma 2.2.** [DWZ, Prop 8.9][LF, Cor 22] If \((Q, W)\) is non-degenerate (resp. rigid), then \((Q|_I, W|_I)\) is non-degenerate (resp. rigid).

### 2.4. QP-mutation and quiver mutation

Let \(Q\) be a finite quiver without loops nor 2-cycles and \(k\) a vertex. The quiver mutation \(\mu_k(Q)\) of \(Q\) at vertex \(k\) is the quiver obtained from \(Q\) as follows:

- for each subquiver \(i \xrightarrow{\beta} k \xleftarrow{\alpha} j\), we add a new arrow \([\alpha\beta] : i \to j\);
- we reverse all arrows with source or target \(k\);
- we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

Quiver mutations have close relation with QP-mutations for non-degenerate QPs.
Lemma 2.3. [DWZ, Prop 7.1] Let \((Q, W)\) be a non-degenerate quiver with potential. Suppose that \((Q', W') = \mu_i((Q, W))\) for some vertex \(i\), then \(Q' = \mu_i(Q)\).

If \(Q'\) can be obtained from \(Q\) by a finite sequence of quiver mutations, then \(Q'\) and \(Q\) are mutation equivalent. The following is well-known (cf. [GLaS, Lem 9.2] for instance).

Lemma 2.4. Assume that \(Q\) is mutation equivalent to an acyclic quiver. Then there exists a unique non-degenerate potential on \(Q\) up to right equivalence.

2.5. Generalized cluster category \(C_{(Q,W)}\). Let \((Q, W)\) be a quiver with potential, denote by \(\Gamma(Q, W)\) the complete Ginzburg dg algebra, we refer to [G, Section 4.2] and [KY, Section 2.6] for the precisely construction. It is known that \(\Gamma(Q, W)\) is a non positive dg algebra and the Jacobian algebra \(J(Q, W)\) is the zeroth homology of \(\Gamma(Q, W)\). Keller [K11] proved that the perfect derived category \(\text{per } \Gamma(Q, W)\) contains \(\mathcal{D}^b(\Gamma(Q, W))\). The generalized cluster category \(C_{(Q,W)}\) associated to \((Q, W)\) is defined to be the Verdier quotient

\[
C_{(Q,W)} := \text{per } \Gamma(Q, W)/\mathcal{D}^b(\Gamma(Q, W)).
\]

In general, \(C_{(Q,W)}\) has infinite dimensional \(\text{Hom}\)-space.

Lemma 2.5. [KY, Lem 2.9] Let \((Q, W)\) and \((Q', W')\) be two quivers with potentials. If \((Q, W)\) and \((Q', W')\) are right equivalent, then the complete Ginzburg dg algebras \(\Gamma(Q, W)\) and \(\Gamma(Q', W')\) are isomorphic to each other. In particular, \(C_{(Q,W)}\) is triangle equivalent to \(C_{(Q',W')}\).

Lemma 2.6. [A09, Thm 3.5] Let \((Q, W)\) be a Jacobi-finite quiver with potential. The generalized cluster category \(C_{(Q,W)}\) associated to \((Q, W)\) is \(\text{Hom}\)-finite and \(2\)-CY. Moreover, the image of the free module \(\Gamma(Q, W)\) in \(C_{(Q,W)}\) is a cluster-tilting object. Its endomorphism algebra is isomorphic to the Jacobian algebra \(J(Q, W)\).

Let \((Q, W)\) be a Jacobi-finite quiver with potential. By Lemma 2.6, \(\Gamma(Q, W)\) is a cluster-tilting object in the 2-Calabi-Yau triangulated category \(C_{(Q,W)}\). Therefore, we can form the mutation of \(\Gamma(Q, W)\) at any indecomposable direct summand. In particular, for each vertex \(i\), we have the mutation \(\mu_i(\Gamma(Q, W))\) of \(\Gamma(Q, W)\) at \(e_i\Gamma(Q, W)\), where \(e_i\) is the primitive idempotent associated to \(i\).

The following is a direct consequence of [KY, Thm 3.2] and Lemma 2.6.

Lemma 2.7. Let \((Q, W)\) be a Jacobi-finite quiver with potential and \(i\) a vertex of \(Q\). Assume that \(Q\) has no loops nor 2-cycles. There is a triangle equivalence from \(C_{(Q,W)}\) to \(C_{\mu_i(Q,W)}\) which sends \(\mu_i(\Gamma(Q, W))\) to \(\Gamma(\mu_i(Q, W))\). Consequently, \(\text{End}_{C_{(Q,W)}}(\mu_i(\Gamma(Q, W))) \cong J(\mu_i(Q, W))\).

2.6. Generalized cluster category \(C_A\). Let \(A\) be a finite dimensional \(K\)-algebra with finite global dimension. Denote by \(\mathcal{D}^b(A)\) the bounded derived category of finitely generated right \(A\)-modules and \(\nu_A\) a Serre functor of \(\mathcal{D}^b(A)\). Let \(B\) be the dg algebra \(A \oplus DA[-3]\) with trivial differential and \(p : B \rightarrow A\) the canonical projection. The projection \(p\) induces a triangle functor

\[
p_* : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B).
\]
Let \((A)_B\) be the thick subcategory of \(D^b(B)\) generated by the image of \(p_*\). It is known that \(\text{per} \ B \subset (A)_B\) and we can form the Verdier quotient
\[
C_A := (A)_B / \text{per} \ B.
\]
The category \(C_A\) is called the \textit{generalized cluster category} of \(A\). By [K05, Thm 7.1], the triangulated hull of the orbit category \(D^b(A)/\nu A[-2]\) is \(C_A\). If \(A = KQ\) for a finite acyclic quiver \(Q\), then \(C_{KQ} = D^b(A)/\nu A[-2]\) by [K05, Thm 4]. In this case, \(C_{KQ}\) is the \textit{cluster category} of \(KQ\) introduced by [BMRRT]. In general, \(C_A\) has infinite dimensional \(\text{Hom}\)-space.

**Theorem 2.8.** [A09, Thm 4.10] \(\) Let \(A\) be a finite-dimensional \(K\)-algebra of global dimension \(\leq 2\). If the functor \(\text{Tor}^2_A(?, \mathbb{D}A)\) is nilpotent, then \(C_A\) is a \textit{Hom}-finite 2-CY triangulated category and the object \(A\) is a \textit{cluster tilting object} in \(C_A\).

### 2.7. Quiver with potential for algebra of global dimension \(\leq 2\)

Let \(A = KQ_A/I\) be a finite dimensional algebra of global dimension \(\leq 2\), where \(I\) is an admissible ideal of the path algebra \(KQ_A\). Keller [K11] introduced a quiver with potential \((\tilde{Q}_A, W_A)\) associated to \(A\). Fix a set of representatives of minimal relations of \(I\). For each such representative \(r\) which starts at vertex \(i\) and ends at vertex \(j\), let \(\rho_r\) be a new arrow from \(j\) to \(i\). The quiver \(\tilde{Q}_A\) is obtained from \(Q\) by adding all the arrows \(\rho_r\). The potential \(W_A\) on \(\tilde{Q}_A\) is given by \(W_A = \sum r \rho_r\), where the sum ranges over the set of representatives.

**Lemma 2.9.** [K11, Thm 6.12] \(\) The generalized cluster category \(C_A\) is triangle equivalent to the generalized cluster category \(C_{(Q_A, W_A)}\). Moreover, the endomorphism algebra \(\text{End}_{C_A}(A)\) is isomorphic to the Jacobian algebra \(J(\tilde{Q}_A, W_A)\).

### 3. Cluster categories of weighted projective lines

#### 3.1. Weighted projective lines
Fix a positive integer \(t\). Let \(X = X(p, \lambda)\) be a weighted projective line attached to a weight sequence \(p = (p_1, \ldots, p_t)\) of integers \(p_i \geq 2\) and a parameter sequence \(\lambda = (\lambda_1, \ldots, \lambda_t)\) of pairwise distinct elements of \(\mathbb{P}_1(K)\). Without loss of generality, we may assume that \(\lambda_1 = [1 : 0], \lambda_2 = [0 : 1], \lambda_3 = [1 : 1]\). Denote by \(\text{coh} \ X\) the category of coherent sheaves over \(X\), which is a \textit{Hom}-finite hereditary abelian category with tilting objects.

We refer to [GL] for basic properties of weighted projective lines.

Let \(\mathbb{L}\) be the rank one abelian group generated by \(\bar{x}_1, \ldots, \bar{x}_t\) with the relations
\[
p_1 \bar{x}_1 = p_2 \bar{x}_2 = \cdots = p_t \bar{x}_t =: \bar{c},
\]
where \(\bar{c}\) is called the \textit{canonical element} of \(\mathbb{L}\). For each coherent sheaf \(E\) over \(X\) and \(\bar{x} \in \mathbb{L}\), denote by \(E(\bar{x})\) the grading shift of \(E\) with respect to \(\bar{x}\). Denote by \(\mathcal{O}\) the structure sheaf of \(X\). It is known that each line bundle is given by the grading shift \(\mathcal{O}(\bar{x})\) for a unique element \(\bar{x} \in \mathbb{L}\).

The category \(\text{coh} \ X\) has \(t\) exceptional tubes consisting of sheaves of finite length. In the \(i\)-th exceptional tube of rank \(p_i\), there is a unique simple object \(S_i\) such that \(\text{Hom}_{\text{coh} \ X}(\mathcal{O}, S_i) \neq 0\). Note that \(S_i\) is also the unique simple object in the \(i\)-th exceptional tube of rank \(p_i\) satisfying \(\text{Hom}_{\text{coh} \ X}(\mathcal{O}(\bar{c}), S_i) \neq 0\). For a positive integer \(j\), denote by \(S_i^{[j]}\) the unique indecomposable
object lying in the same tube with \( S_i \) which has length \( j \) and top \( S_i \). The object
\[
T_{sq} = \mathcal{O} \oplus \mathcal{O}([\ell]) \oplus \left( \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{p_i - 1} \mathcal{O}[j] \right)
\]
is a basic tilting object in \( \text{coh} X \), which is called the *squid tilting object* (cf. [BKL, Section 8]).

3.2. **Classifications of weighted projective lines.** Let \( p \) be the least common multiple of \( p_1, \ldots, p_t \). The genus \( g_X \) of a weighted projective line \( X \) is defined by
\[
g_X = 1 + \frac{1}{2}((t - 2)p - \sum_{i=1}^{t} p/p_i).
\]
A weighted projective line of genus \( g_X < 1 (g_X = 1, \text{resp. } g_X > 1) \) is called of *domestic* (tubular, resp. *wild*) type. It is known that a weighted projective line \( X \) is derived equivalent to a finite dimensional hereditary \( K \)-algebra if and only if \( X \) is of domestic type.

The domestic weight types are, up to permutation, \((q)\) with \( q \geq 1 \), \((q_1, q_2)\) with \( q_1, q_2 \geq 2, (2, 2, n)\) with \( n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5)\), whereas the tubular weight types are, up to permutation, \((2, 2, 2, 2), (3, 3, 3), (2, 4, 4)\) and \((2, 3, 6)\).

3.3. **Cluster category** \( C_X \). Let \( D^b(\text{coh} X) \) be the bounded derived category of \( \text{coh} X \) with suspension functor \([1]\). Let \( \tau : D^b(\text{coh} X) \to D^b(\text{coh} X) \) be the Auslander-Reiten translation functor. The *cluster category* of \( X \) is defined as the orbit category
\[
C_X := D^b(\text{coh} X)/\tau^{-1}[1].
\]
It has been proved by Keller [K05] that \( C_X \) admits a canonical triangle structure such that the projection \( \pi : D^b(\text{coh} X) \to C_X \) is a triangle functor. Moreover, \( C_X \) is a \( \text{Hom} \)-finite 2-CY triangulated category with cluster-tilting objects. By [BKL], the cluster-tilting objects in \( C_X \) are precisely the tilting objects in \( \text{coh} X \) and \( C_X \) has a cluster structure in the sense of [BIRSc]. In particular, for any basic cluster-tilting object \( T \in C_X \), the quiver \( Q_T \) has no loops nor 2-cycles and the mutation of cluster-tilting objects is compatible with the quiver mutation, i.e. \( \mu_i(Q_T) = Q_{\mu_i(T)} \) for each vertex \( i \) of \( Q_T \) (cf. [BKL, Thm 3.1]).

The connectedness of cluster-tilting graph \( G_{\text{ct}}(C_X) \) has been established by [BMRRRT] for domestic type, by [BKL, Thm 8.8] for tubular type and by [FG, Thm 1.2] in full of generality. For a 2-CY triangulated category which shares a 2-CY tilted algebra with \( C_X \), we have

**Lemma 3.1.** Let \( C \) be a \( \text{Hom} \)-finite 2-CY triangulated category over \( K \). If there is a cluster-tilting object \( T \) in \( C \) such that \( \text{End}_C(T) \) is isomorphic to a 2-CY tilted algebra for some cluster-tilting object of \( C_X \), then the cluster-tilting graph \( G_{\text{ct}}(C) \) is connected.

**Proof.** Let \( M \) be a cluster-tilting object of \( C_X \) such that \( \text{End}_{C_X}(M) \cong \text{End}_C(T) =: \Lambda \). According to [AIR14, Thm 4.7], \( \text{Hom}_C(T, -) \) induces a bijection between the set of basic cluster-tilting objects in \( C \) and the set of basic support \( \tau \)-tilting \( \Lambda \)-modules, while \( \text{Hom}_{C_X}(M, -) \) induces a bijection between the set of basic cluster-tilting objects in \( C_X \) and the set of basic support \( \tau \)-tilting \( \Lambda \)-modules. Moreover, the bijections are compatible with mutations. Consequently, \( G_{\text{ct}}(C) \cong G_{\text{ct}}(C_X) \). We conclude that \( G_{\text{ct}}(C) \) is connected by [FG, Thm 1.2].  \( \square \)
3.4. Cluster category $C_\mathcal{X}$ as generalized cluster category. Let $T$ be a basic tilting object of $\text{coh } \mathcal{X}$ and $A = \text{End}_{\text{coh } \mathcal{X}}(T)$ the endomorphism algebra. It is known that $A$ is a finite dimensional algebra of global dimension $\leq 2$ and $\mathcal{D}^b(\text{mod } A) \cong \mathcal{D}^b(\text{coh } \mathcal{X})$. According to [K05, Thm 7.1], we know that $C_A \cong C_\mathcal{X}$ and $\text{End}_{C_\mathcal{X}}(T) \cong \text{End}_{C_A}(A)$. Let $(\tilde{Q}_A,W_A)$ be the quiver with potential associated to $A$ via Keller’s construction (cf. Section 2.7). By Lemma 2.9, we obtain

$$C_\mathcal{X} \cong C_A \cong C_{(\tilde{Q}_A,W_A)} \text{ and } \text{End}_{C_\mathcal{X}}(T) \cong \text{End}_{C_A}(A) \cong J(\tilde{Q}_A,W_A).$$

We remark that the equivalence from $C_\mathcal{X} \cong C_{(\tilde{Q}_A,W_A)}$ sends $T$ to $\Gamma(\tilde{Q}_A,W_A)$.

**Lemma 3.2.** The potential $W_A$ is non-degenerate.

**Proof.** Recall that $Q_T$ is the Gabriel quiver of $\text{End}_{C_\mathcal{X}}(T)$. Since $\text{End}_{C_\mathcal{X}}(T)$ is finite dimensional, it follows that $(\tilde{Q}_A,W_A)$ is Jacobi-finite. Hence we may identify $\tilde{Q}_A$ with $Q_T$. By [BKL, Thm 3.1], $Q_T$ has no loops nor 2-cycles. For each vertex $i$ of $Q_T$, by Lemma 2.7, we have

$$J(\mu_i(Q_T,W_A)) \cong \text{End}_{C_{(\tilde{Q}_A,W_A)}}(\mu_i(\Gamma(\tilde{Q}_A,W_A))) \cong \text{End}_{C_\mathcal{X}}(\mu_i(T)).$$

In particular, $\mu_i(Q_T,W_A)$ is Jacobi-finite and hence the quiver of $\mu_i(Q_T,W_A)$ is isomorphic to $Q_{\mu_i(T)}$. Therefore $\mu_i(Q_T,W_A)$ is 2-acyclic. Continuing this process, we conclude that $(Q_T,W_A)$ is non-degenerate.

The following is a direct consequence of Lemma 3.2.

**Corollary 3.3.** Every 2-CY tilted algebra of $C_\mathcal{X}$ is a Jacobian algebra for a non-degenerate quiver with potential.

4. Cluster category $C_\mathcal{X}$ vs generalized cluster category $C_{(Q,W)}$

4.1. Cluster category $C_\mathcal{X}$ with at most three weights. Let $\mathcal{X}$ be a weighted projective line with weight sequence $(p_1,p_2,p_3)$. Recall that we have a basic tilting object $T_{sq} = \mathcal{O} \oplus \mathcal{O}(\tilde{c}) \oplus (\mathcal{O}^{\oplus 3} \oplus \bigoplus_{j=1}^{p_3-1} \mathcal{O}^{\oplus 1} S_{j}^{[1]}),$ which induces a basic cluster-tilting object in $C_\mathcal{X}$. The Gabriel quiver $Q_{T_{sq}}$ of $\text{End}_{C_\mathcal{X}}(T_{sq})$ is described as follows

$$S_{1}^{[p_1-1]} \quad \cdots \quad S_{1}^{[1]} = S_1$$

$$S_{2}^{[p_2-1]} \quad \cdots \quad S_{2}^{[1]} = S_2$$

$$S_{3}^{[p_3-1]} \quad \cdots \quad S_{3}^{[1]} = S_3.$$

**Lemma 4.1.** Let $\mathcal{X}$ be a weighted projective line with at most three weights. Let $T$ be a basic cluster-tilting object in $C_\mathcal{X}$ and $Q_T$ the quiver of $\text{End}_{C} T$, then there is a unique non-degenerate potential on $Q_T$ up to right equivalence.

**Proof.** According to Lemma 3.2, it remains to prove the uniqueness.

If $\mathcal{X}$ has at most two weights, then $\mathcal{X}$ is of domestic type. In this case, $\text{coh } \mathcal{X}$ is derived equivalent to the path algebra $KQ$ of an acyclic quiver $Q$ of affine type. Consequently, $Q_T$ is
mutation equivalent to \( Q \). By Lemma 2.4, we conclude that \( Q_T \) has a unique non-degenerate potential up to right equivalence.

Suppose that \( X \) has weight sequence \((p_1, p_2, p_3)\). Let \( Q \) be the quiver as follows

```
1 ---- 4 ---- 3
|       |       |
|       |       |
|       |       |
|       |       |
|       |       |
5 ---- 2
```

It is straightforward to check that \( \mu_2\mu_3\mu_4\mu_5(Q) \) is an acyclic quiver. Therefore \( Q \) has a unique non-degenerate potential \( S \) up to right equivalence. Note that \( Q \) is a full subquiver of \( QT_{sq} \) and every cycle of \( QT_{sq} \) lies in \( Q \). If \( W \) is a non-degenerate potential on \( QT_{sq} \), then \( W \) is a non-degenerate potential on \( Q \) by Lemma 2.2. Hence, \((Q, W)\) is right equivalent to \((Q, S)\). Clearly, the right equivalence between \((Q, W)\) and \((Q, S)\) lifts to a right equivalence between \((QT_{sq}, W)\) and \((QT_{sq}, S)\). Therefore \( S \) is the unique non-degenerate potential on \( QT_{sq} \) up to right equivalence.

By [FG, Thm 1.2], \( T \) is reachable from \( T_{sq} \). Since the mutation of cluster-tilting objects in \( C_X \) is compatible with quiver mutation, it follows that there is a sequence of vertices \( i_1, \ldots, i_k \) of \( Q_T \) such that \( QT_{sq} = \mu_{i_k} \cdots \mu_{i_1}(Q_T) \). By Lemma 2.1 and Lemma 2.3, the uniqueness of non-degenerate potential on \( Q_T \) follows from the uniqueness of non-degenerate potential on \( QT_{sq} \).

**Corollary 4.2.** Let \( X \) be a weighted projective line with at most three weights. Let \( Q \) be the Gabriel quiver of a 2-CY tilted algebra of \( C_X \). If \( W \) is a non-degenerate potential on \( Q \), then \( C_{(Q,W)} \cong C_X \).

**Proof.** Let \( T \) be a basic cluster-tilting object of \( C_X \) such that \( Q_T = Q \). By Lemma 3.2, there is a non-degenerate potential \( W_T \) on \( Q_T \) such that \( C_X \cong C_{(Q_T, W_T)} \). Now the result follows from Lemma 4.1 and Lemma 2.5. \( \square \)

Let \( T \) be a basic cluster-tilting object of the cluster category \( C_X \) of a weighted projective line \( X \). The 2-CY tilted algebra \( \text{End}_{C_X}(T) \) is determined by its quiver if for any basic cluster-tilting object \( T' \) of the cluster category \( C_{X'} \) of some weighted projective line \( X' \) such that \( Q_T \cong Q_{T'} \), then \( \text{End}_{C_X}(T) \cong \text{End}_{C_{X'}}(T') \).

**Corollary 4.3.** Let \( X \) be a weighted projective line with at most three weights. Then 2-CY tilted algebras arising from \( C_X \) are uniquely determined by their quivers.

**Proof.** Let \( T \) be a basic cluster-tilting object of \( C_X \) and \( T' \) a cluster-tilting object of \( C_{X'} \) for some weighted projective line \( X' \) such that \( Q_T \cong Q_{T'} \).

By Lemma 3.2, there exist non-degenerate quivers with potentials \((Q_T, W_T)\) and \((Q_{T'}, W_{T'})\) such that \( C_X \cong C_{(Q_T, W_T)} \) and \( C_{X'} \cong C_{(Q_{T'}, W_{T'})} \). Moreover, \( \text{End}_{C_X}(T) \cong J(Q_T, W_T) \) and \( \text{End}_{C_{X'}}(T') \cong J(Q_{T'}, W_{T'}) \). By Lemma 4.1, \((Q_T, W_T)\) and \((Q_{T'}, W_{T'})\) are right equivalent. Hence, \( C_{(Q_T, W_T)} \cong C_{(Q_{T'}, W_{T'})} \) and \( J(Q_T, W_T) \cong J(Q_{T'}, W_{T'}) \) by Lemma 2.5. \( \square \)
Theorem 4.4. Let \((Q, W)\) be a Jacobi-finite quiver with non-degenerate potential. If there is a cluster-tilting object \(T\) in \(\mathcal{C}_{(Q, W)}\) such that \(\text{End}_{\mathcal{C}_{(Q, W)}}(T)\) is isomorphic to a 2-CY tilted algebra arising from some weighted projective line \(\mathcal{X}\) with at most three weights, then \(\mathcal{C}_{(Q, W)}\) is triangle equivalent to \(\mathcal{C}_\mathcal{X}\).

Proof. By Lemma 2.6, \(\Gamma(Q, W)\) is a cluster-tilting object in \(\mathcal{C}_{(Q, W)}\). Consequently, \(\Gamma\) is reachable from \(\mathcal{T}\) by Lemma 3.1. In particular, there exists a sequence of vertices \(i_1, \ldots, i_t\) of \(Q\) such that \(\mathcal{T} = \mu_{i_t} \cdots \mu_{i_2} \mu_{i_1}(\Gamma(Q, W))\). Denote by \((Q', W') = \mu_{i_t} \cdots \mu_{i_2} \mu_{i_1}(Q, W)\), which is a non-degenerate quiver with potential. Applying Lemma 2.7, we have \(\mathcal{C}_{(Q', W')} \cong \mathcal{C}_{(Q, W)}\) and \(\text{End}_{\mathcal{C}_{(Q, W)}}(T) \cong J(Q', W')\). Note that \((Q, W)\) is Jacobi-finite, which implies that \((Q', W')\) is Jacobi-finite. Hence \(Q' = Q_T\). It follows from Corollary 4.2 that \(\mathcal{C}_{(Q, W)} \cong \mathcal{C}_{(Q', W')} \cong \mathcal{C}_\mathcal{X}\).

\[
\begin{array}{c}
\text{1} \quad \text{2} \quad \text{3} \\
\text{4} \quad \text{5} \quad \text{6} \quad \text{7}
\end{array}
\]

4.2. Cluster category \(\mathcal{C}_\mathcal{X}\) of tubular type. Let \(\mathcal{X}\) be a weighted projective line with weight sequence \((2, 2, 2, 2)\) and parameter sequence \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\). Recall that \(\lambda_1 = [1 : 0], \lambda_2 = [0 : 1], \lambda_3 = [1 : 1]\). The point \(\lambda_4\) is uniquely determined by \(\lambda \in K \setminus \{0, 1\}\) by setting \(\lambda_4 = [\lambda : 1]\). Hence a weighted projective line with weight type \((2, 2, 2, 2)\) can by denote by \(\mathcal{X}(2, 2, 2, 2; \lambda)\) for \(\lambda \neq 0, 1\).

Let \(\mathcal{X} = \mathcal{X}(2, 2, 2, 2; \lambda)\) and \(\mathcal{X}' = \mathcal{X}(2, 2, 2, 2; \lambda')\) be weighted projective lines. It is known that \(\text{coh } \mathcal{X}\) and \(\text{coh } \mathcal{X}'\) are equivalent abelian categories if \((\mathcal{D}^b(\text{coh } \mathcal{X}))\) and \((\mathcal{D}^b(\text{coh } \mathcal{X}'))\) are triangle equivalent if \(\lambda' \in O(\lambda) = \{\lambda, \lambda^{-1}, 1 - \lambda, 1 - \lambda^{-1}, (1 - \lambda)^{-1}, (1 - \lambda)^{-1}, \frac{\lambda}{1 - \lambda}\}\). Therefore, \(\mathcal{C}_\mathcal{X}\) and \(\mathcal{C}_{\mathcal{X}'}\) are triangle equivalent if \(\lambda' \in O(\lambda) = \{\lambda, \lambda^{-1}, 1 - \lambda, 1 - \lambda^{-1}, (1 - \lambda)^{-1}, (1 - \lambda)^{-1}, \frac{\lambda}{1 - \lambda}\}\).

Let \(Q^{(2,2,2,2)}\) be the following quiver

\[
\begin{array}{c}
\text{1} \quad \text{2} \quad \text{3} \quad \text{4} \\
\text{a} \quad \text{b} \quad \text{c} \quad \text{d} \quad \text{e} \quad \text{f} \quad \text{g} \quad \text{h} \quad \text{i} \quad \text{j} \quad \text{k} \quad \text{l}
\end{array}
\]

For each \(\lambda \in K \setminus \{0, 1\}\), set
\[
W_\lambda = \lambda abc - dge + dki - afi + jgh - ebf + efl - jkl.
\]

Let \(A = kQ_A/I_A\), where \(Q_A\) is the following quiver

\[
\begin{array}{c}
\text{1} \quad \text{2} \quad \text{3} \\
\text{4} \quad \text{5} \quad \text{6}
\end{array}
\]

and \(I = \langle dg - \lambda ab, dk - af, jg - eb, ef - jk \rangle, \lambda \notin \{0, 1\}\). It is straightforward to check that \(A\) is a tubular algebra of type \((2, 2, 2, 2; \lambda)\). In particular, \(\mathcal{D}^b(\text{mod } A) \cong \mathcal{D}^b(\text{coh } \mathcal{X})\) for a weighted projective line \(\mathcal{X}\) with weight sequence \((2, 2, 2, 2; \lambda)\). Consequently, \(\mathcal{C}_A \cong \mathcal{C}_\mathcal{X}\). Applying Keller's construction to \(A\), we obtain a quiver with potential \((\tilde{Q}_A, W_A) = (Q^{(2,2,2,2)}, W_\lambda)\). Hence, by Lemma 2.9, we have \(\mathcal{C}_\mathcal{X} \cong \mathcal{C}_A \cong \mathcal{C}_{(Q^{(2,2,2,2)}, W_\lambda)}\).

Lemma 4.5. [GLaS, Prop 9.15]
(1) Each non-degenerate potential on $Q^{(2,2,2)}$ is right equivalent to one of the potentials $W_\lambda$ with $\lambda \notin \{0, 1\}$.

(2) The Jacobian algebras $J(Q^{(2,2,2)}, W_\lambda)$ and $J(Q^{(2,2,2)}, W_{\lambda'})$ are isomorphic if and only if $\lambda' \in \{\lambda, \lambda^{-1}\}$.

Let $X$ be a weighted projective line of type $(2, 2, 2; \lambda)$. It has been observed by [GG, Prop 2.5.2 (b)] that every 2-CY tilted algebra of $C_X$ satisfies the vanishing condition of [BIRSm, Thm 5.2]. Consequently, we obtain

**Lemma 4.6.** Let $X$ be a weighted projective line of type $(2, 2, 2; \lambda)$. Let $C$ be a 2-CY triangulated category over $K$ and $M \in C$ a basic cluster-tilting object such that $\text{End}_C(M)$ is a 2-CY tilted algebra of $C_X$. If there is a potential $W$ on $Q_M$ such that $J(Q_M, W) \cong \text{End}_C(M)$, then for any vertex $k$ of $Q_M$, $\text{End}_C(\mu_k(M)) \cong J(\mu_k(Q_M, W))$.

**Theorem 4.7.** Let $(Q, W)$ be a Jacobi-finite quiver with non-degenerate potential. If there is a cluster-tilting object $T$ in $\mathcal{C}_{(Q,W)}$ such that $\text{End}_C(T)$ is isomorphic to a 2-CY tilted algebra arising from some weighted projective line $X$ of tubular type, then $C$ is triangle equivalent to $C_X$.

**Proof.** According to Section 3.2, $X$ is of tubular type if and only if $X$ has weight sequences $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ or $(2, 2, 2)$. By Theorem 4.4, it remains to consider the case $(2, 2, 2)$.

Let $X = X(2, 2, 2; \lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Note that we have $C_X \cong C_{(Q^{(2,2,2)}, W_\lambda)}$. In particular, there is a basic cluster-tilting object in $C_X$ whose quiver is $Q^{(2,2,2)}$. Recall that mutation of cluster-tilting objects of $C_X$ is compatible with quiver mutations and the cluster-tilting graph of $C_X$ is connected. Since $\text{End}_{C_{(Q,W)}}(T)$ is a 2-CY tilted algebra for $C_X$, it follows that $Q_T$ is mutation equivalent to $Q^{(2,2,2)}$. In particular, there is a sequence of vertices $j_1, j_2, \ldots, j_s$ of $Q_T$ such that $Q^{(2,2,2)} = \mu_{j_s} \cdots \mu_{j_1}(Q_T)$.

Let $M$ be a basic cluster-tilting object of $C_X$ such that $\text{End}_{C_{(Q,W)}}(T) \cong \text{End}_{C_X}(M)$. Clearly, we may identify quiver $Q_M$ with $Q_T$. By Lemma 3.2, there is a non-degenerate potential $W_M$ on $Q_M$ such that $\text{End}_{C_{(Q,W)}}(T) \cong \text{End}_{C_X}(M) \cong J(Q_M, W_M)$ and $C_{(Q_M,W_M)} = C_X$. Denote by $M' := \mu_{j_s} \cdots \mu_{j_1}(M) \in \mathcal{C}_X$ and $(Q', W') = \mu_{j_s} \cdots \mu_{j_1}(Q_M, W_M)$. It is clear that $(Q', W')$ is a non-degenerate quiver with potential. According to Lemma 2.3, we have $Q' = Q^{(2,2,2)}$.

By Lemma 4.5, any non-degenerate potential on $Q^{(2,2,2)}$ has the form $W_t$ for some $t \notin \{0, 1\}$ up to right equivalence. Without loss of generality, we may assume $W' = W_{t_0}$ for some $t_0 \in K \setminus \{0, 1\}$. By Lemma 4.6, we have $\text{End}_{C_X}(M') \cong J(Q^{(2,2,2)}, W_{t_0})$.

Since $(Q, W)$ is Jacobi-finite, $\Gamma(Q, W)$ is a cluster-tilting object of $C_{(Q,W)}$. Note that $\text{End}_{C_{(Q,W)}}(T)$ is a 2-CY tilted algebra for $C_X$, it follows that $T$ is reachable from $\Gamma(Q, W)$ by Lemma 3.1. Consequently, there is a sequence of vertices $i_1, \ldots, i_t$ of $Q$ such that

$$T = \mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(\Gamma(Q, W)) \quad \text{and} \quad \text{End}_{C_{(Q,W)}}(T) \cong J(\mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(Q, W))$$

by Lemma 2.7. Clearly, the quiver of $\mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(Q, W)$ is $Q_T$ and we may write $(Q_T, W'') = \mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(Q, W)$. Consequently, $\text{End}_{C_X}(M') \cong J((Q_T, W'')).$ By Lemma 4.6, we conclude that

$$\text{End}_{C_X}(M') \cong J(\mu_{j_s} \cdots \mu_{j_2}\mu_{j_1}(Q_T, W')).$$
Let us write \( \mu_{j_1} \cdots \mu_{j_2} \mu_{j_1} (Q_T, W') = (Q^{(2,2,2,2)}, W_{t_1}) \) for some \( t_1 \in K\setminus\{0,1\} \), which is a non-degenerate quiver with potential. Hence \( J(Q^{(2,2,2,2)}, W_{t_0}) \cong J(Q^{(2,2,2,2)}, W_{t_1}) \). By Lemma 4.5 (2), we have either \( t_0 = t_1 \) or \( t_0 = t_1^{-1} \). In any case, \( C_{(Q^{(2,2,2,2)}, W_{t_0})} \cong C_{(Q^{(2,2,2,2)}, W_{t_1})} \). By Lemma 2.5 and Lemma 2.7, we conclude that

\[
C(Q, W) \cong C(Q, W') \cong C_{(Q^{(2,2,2,2)}, W_{t_0})} \cong C_{(Q^{(2,2,2,2)}, W_{t_1})} \cong C_{(Q,M, W,M)} \cong C_X. \]

\qed

**Remark 4.8.** If \( X \) is of type \((2,2,2,2; \lambda)\), then the condition ‘non-degenerate’ in Theorem 4.7 is superfluous. One can prove that the quiver with potential \((Q, W)\) is non-degenerate. Indeed, by Lemma 3.2, there is a non-degenerate potential \( W_T \) on \( Q_T \) such that \( C_{(Q_T, W_T)} \cong C_X \) and \( J(Q_T, W_T) \cong \text{End}_{C(Q, W)}(T) \). Applying Lemma 4.6, we know that \( \text{End}_{C(Q,W)}(\mu_k(T)) \cong J(\mu_k(Q_T, W_T)) \) for each vertex \( k \) of \( Q_T \). Consequently, \( \text{End}_{C(Q,W)}(\mu_k(T)) \) is a 2-CY tilted algebra of \( C_X \). Continuing this process, we conclude that the endomorphism algebra \( \text{End}_{C(Q,W)}(M) \) of a basic cluter-tilting \( M \in C(Q,W) \) reachable from \( T \) is a 2-CY tilted algebra of \( C_X \). It follows from Lemma 3.1 that \( \Gamma(Q, W) \) is reachable from \( T \). Hence \( J(Q, W) \cong \text{End}_{C(Q,W)}(\Gamma(Q, W)) \) is a 2-CY tilted algebra of \( C_X \). Let \( M \in C_X \) be the basic cluster-tilting object such that \( J(Q, W) \cong \text{End}_{C_X}(M) \). We may identify \( Q \) with \( Q_M \). By Lemma 4.6, for each vertex \( k \) of \( Q \), we have \( J(\mu_k(Q,W)) \cong \text{End}_{C_X}(\mu_k(M)) \). In particular, \( \mu_k(Q,W) \) is 2-acyclic. Continuing this process, we conclude that \((Q,W)\) is non-degenerate.

5. Cluster category \( C_X \) as stable category arising from preprojective algebra

5.1. 2-CY categories associated with elements in Coxeter groups. We follow [BIRSc]. Let \( Q \) be an acyclic quiver with vertices \( 1, 2, \cdots, n \). Denote by \( \Lambda \) the preprojective algebra associated \( Q \). For each vertex \( i \), denote by \( e_i \) the associated primitive idempotent and \( I_i = \Lambda(1 - e_i)\Lambda \) the ideal of \( \Lambda \). Let \( W_Q \) be the Coxeter group associated to \( Q \). For a reduced expression \( w = s_{i_1} \cdots s_{i_t} \) of \( w \in W_Q \), denote by \( I_w = I_{i_1} \cdots I_{i_t} \) and \( \Lambda_w = \Lambda/I_w \). It is known that \( I_w \) and \( \Lambda_w \) are independent of the choice of reduced expressions of \( w \), and \( \Lambda_w \) is a finite dimensional \( K \)-algebra. Denote by \( \text{Sub} \Lambda_w \) the full subcategory of \( \text{mod} \Lambda_w \) whose objects are the submodules of finitely generated projective \( \Lambda_w \)-modules. By [BIRSc], \( \text{Sub} \Lambda_w \) is a Frobenius category and the stable category \( \text{Sub} \Lambda_w \) is a Hom-finite 2-CY triangulated category. Moreover, \( T(i_1, \ldots, i_t) := \Lambda/I_{i_1} \oplus \Lambda/I_{i_1} I_{i_2} \oplus \cdots \oplus \Lambda/I_{i_1} \cdots I_{i_t} \) is a basic cluster-tilting object in \( \text{Sub} \Lambda_w \), which induces a basic cluster-tilting object \( \overline{T}(i_1, \ldots, i_t) \) in \( \text{Sub} \Lambda_w \). By abuse of notations, for a fixed reduced expression \( w = s_{i_1} \cdots s_{i_t} \), we also denote by \( T_w := T(i_1, \ldots, i_t) \) and \( \overline{T}_w := \overline{T}(i_1, \ldots, i_t) \).

The Gabriel quiver of \( \text{End}_{\text{Sub} \Lambda_w}(\overline{T}_w) \) can be constructed from the reduced expression \( w = s_{i_1} \cdots s_{i_t} \) directly. Namely, we associate with the reduced expression \( w = s_{i_1} \cdots s_{i_t} \) a quiver \( \overline{Q}(w) \) as follows, where the vertices correspond to the \( s_{i_k} \).

- For two consecutive \( i(i \in \{1, \cdots, n\}) \), draw an arrow from the second one to the first one.
- For each edge \( i \rightarrow d_{ij} \rightarrow j \) of the underlying diagram of \( Q \), pick out the expression consisting of the \( i_k \) which are \( i \) or \( j \), so that we have \( \cdots ii \cdots iij \cdots jii \cdots i \). Draw \( d_{ij} \).
arrows from the last \( i \) in a connected set of \( i \) to the last \( j \) in the next set of \( j \), and do the same from \( j \) to \( i \).

Denote by \( Q(w) \) the quiver obtained from \( \hat{Q}(w) \) by removing the last \( i \) for each vertex \( i \) of \( Q \). It was shown by [BIRS, Thm III.4.1] that the Gabriel quiver of \( \text{End}_{\text{Sub}}(Q(w)) \) is \( Q(w) \). Moreover, there is a rigid potential \( W_w \) on \( Q(w) \) such that \( \text{End}_{\text{Sub}}(Q(w)) \cong J(Q(w), W_w) \) by [BIRS, Thm 6.4].

**Lemma 5.1.** If the quiver \( Q(w) \) of \( \text{End}_{\text{Sub}}(Q(w)) \) is isomorphic to the quiver of a 2-CY tilted algebra arising from a weighted projective line \( \mathbb{X} \) with at most three weights, then \( \text{Sub}_{\text{A}} \) is triangle equivalent to \( C_{\mathbb{X}} \).

**Proof.** By [ART, Theorem 4.4] and Lemma 2.9, there is a potential \( W_w' \) on \( Q(w) \) such that \( \text{Sub}_{\text{A}} \cong C_{\text{w}(Q(w), W_w')} \). Moreover, \( W_w' \) is cyclically equivalent to the potential \( W_w \) according to the proof of [ART, Proposition 3.12]. Consequently, \( W_w' \) is non-degenerate by [DWZ, Corollary 8.2].

Let \( T \) be a basic cluster-tilting object in \( C_{\mathbb{X}} \) such that \( QT = Q(w) \). It follows from Lemma 3.2 that there is a non-degenerate potential \( W_T \) on \( Q_T \) such that \( \text{End}_{C_{\mathbb{X}}}(T) \cong J(Q_T, W_T) \) and \( C_{Q_T, W_T} \cong C_{\mathbb{X}} \). Since \( \mathbb{X} \) has at most three weights, there is a unique non-degenerate potential on \( Q_T \) up to right equivalence by Lemma 4.1. Hence \( W_T \) and \( W_w' \) are right equivalent and

\[
\text{Sub}_{\text{A}} \cong C_{Q(w), W_w'} \cong C_{Q_T, W_T} \cong C_{\mathbb{X}}
\]

by Lemma 2.5.

---

### 5.2. Cluster category \( C_{\mathbb{X}} \) via Buan-Iyama-Reiten-Scott’s construction.

Let \( \mathbb{X} \) be a weighted projective line with weight sequence \((p_1, p_2, p_3)\) with \( p_i \geq 2 \) for each \( i \). Recall that \( S_i \) is the unique simple object in the \( i \)-th exceptional tube of rank \( p_i \) satisfying \( \text{Hom}_{\text{coh}}(\mathbb{X}, S_i) \neq 0 \). For a positive integer \( j \), \( S_i^{[j]} \) is the unique indecomposable object in the same tube with \( S_i \) such that \( S_i^{[j]} \) has length \( j \) and top \( S_i \).

Denote by

\[
T^{(p_1, p_2, p_3)} = \mathcal{O} \oplus \mathcal{O}(\bar{c}) \oplus \mathcal{O}(\bar{x}_1) \oplus \mathcal{O}(\bar{x}_2) \oplus \mathcal{O}(\bar{x}_3) \oplus \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{p_i-2} S_i^{[j]}.
\]

It is straightforward to check that \( T^{(p_1, p_2, p_3)} \) is a tilting object in \( \text{coh} \mathbb{X} \), hence a cluster-tilting object in \( C_{\mathbb{X}} \). The endomorphism algebra \( B = \text{End}_{\text{coh}}(T^{(p_1, p_2, p_3)}) \) is given by the following quiver with relations:

- \( \mathcal{O}(\bar{x}_1) \longrightarrow S_1^{[p_1-2]} \longrightarrow S_1^{[p_1-3]} \longrightarrow \cdots \longrightarrow S_1^{[1]} \)
- \( \mathcal{O}(\bar{x}_2) \longrightarrow S_2^{[p_2-2]} \longrightarrow S_2^{[p_2-3]} \longrightarrow \cdots \longrightarrow S_2^{[1]} \)
- \( \mathcal{O}(\bar{x}_3) \longrightarrow S_3^{[p_3-2]} \longrightarrow S_3^{[p_3-3]} \longrightarrow \cdots \longrightarrow S_3^{[1]} \).
According to Section 2.7 and 3.4, the Gabriel quiver $Q_{\Delta(p_1,p_2,p_3)}$ of $\text{End}_{\mathcal{C}_X}(\Delta(p_1,p_2,p_3))$ is as follows:

$$Q_{\Delta(p_1,p_2,p_3)} :$$

- $\mathcal{O}(\vec{x}_1) \leftarrow S_{p_1}^{[p_1-2]} \rightarrow S_{p_1}^{[p_1-3]} \rightarrow \cdots \rightarrow S_{p_1}^{[1]}$
- $\mathcal{O}(\vec{x}_2) \rightarrow S_{p_2}^{[p_2-2]} \leftarrow S_{p_2}^{[p_2-3]} \rightarrow \cdots \rightarrow S_{p_2}^{[1]}$
- $\mathcal{O}(\vec{c}) \rightarrow \mathcal{O}(\vec{c})$
- $\mathcal{O}(\vec{x}_3) \leftarrow S_{p_3}^{[p_3-2]} \rightarrow S_{p_3}^{[p_3-3]} \rightarrow \cdots \rightarrow S_{p_3}^{[1]}$.

**Theorem 5.2.** Let $\mathcal{X}$ be a weighted projective line with at most three weights and $\mathcal{C}_X$ the associated cluster category. There is an acyclic quiver $Q$ and a reduced word $w \in W_Q$ such that $\mathcal{C}_X$ is triangle equivalent to $\text{Sub}\Lambda_w$.

**Proof.** Let us assume that $\mathcal{X}$ has at most two weights or $\mathcal{X}$ has three weights but at least two of them are 2. In particular, $\mathcal{X}$ is of domestic type. In this case, $\text{coh}\mathcal{X}$ is derived equivalent to the module category of $KQ$ for some acyclic quiver $Q$ of affine type. By [BIRSc, Thm III.3.4], each cluster category $\mathcal{C}_{KQ}$ can be realized as the stable category $\text{Sub}\Lambda_w$ for some reduced word $w \in W_Q$. Therefore $\mathcal{C}_X \cong \mathcal{C}_{KQ} \cong \text{Sub}\Lambda_w$.

Now assume that $\mathcal{X}$ has three weights $(p_1,p_2,p_3)$ and at most one of them is 2. Up to permutation, we may assume that $p_1 \leq p_2 \leq p_3$.

Suppose that $p_1 = 2$. By assumption, $p_2 > 2$ and $p_3 > 2$. Let $Q$ be the following quiver

$$b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_{p_2-2}$$

$$a_1 \rightarrow o \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_{p_3-2}.$$

Let $w = s_0s_{b_1}s_{c_1}s_{a_1}s_0s_{b_2} \cdots s_{c_{p_3-2}}s_{a_2}s_0s_{b_2} \cdots s_{c_{p_3-2}}$ be a word in $W_Q$.

It follows from [ORT, Thm 2.3] that $w$ is a reduced expression. Applying the construction in Section 5.1, we obtain the quiver $Q(w)$ of $\text{End}_{\text{Sub}\Lambda_w}(\mathcal{T}_w)$ as follows

$$Q(w) :$$

- $a_1 \rightarrow o \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_{p_2-2}$
- $c_1 \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_{p_3-2}.$

In particular, $Q(w) = Q_{\Delta(2,p_2,p_3)}$. Consequently, $\mathcal{C}_X \cong \text{Sub}\Lambda_w$ by Lemma 5.1.

Now suppose that $p_1 \geq 3$. Let $Q$ be the following quiver

$$a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_{p_1-2}$$

$$b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_{p_2-2}$$

$$c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_{p_3-2}.$$
Let $W_Q$ be the Coxeter group of $Q$, define
\[ w = s_0s_{a_1}s_{b_1}s_{c_1}s_0s_{a_1} \cdots s_{p_1} \cdots s_{p_2-1}s_{b_1} \cdots s_{c_{p_3-2}}s_0s_{a_1} \cdots s_{p_1} \cdots s_{p_2-1}s_{b_1} \cdots s_{c_{p_3-2}} \in W_Q. \]
Again by [ORT, Thm 2.3], $w$ is a reduced word. Similarly, we obtain the quiver $Q(w)$ of $\text{End}_{\text{Sub}_w}(T_w)$ as follows
\[
Q(w):
\begin{align*}
  & a_1 \quad a_2 \quad \cdots \quad a_{p_1-2} \\
  b_1 \quad & b_2 \quad \cdots \quad b_{p_2-2} \\
  c_1 \quad & c_2 \quad \cdots \quad c_{p_3-2}.
\end{align*}
\]
In particular, $Q(w) = Q_{T(p_1,p_2,p_3)}$. By Lemma 5.1, $\mathcal{C}_X \cong \text{Sub}_w$. □

Corollary 5.3. Let $X$ be a weighted projective line with at most three weights. Then each 2-CY tilted algebra arising from $\mathcal{C}_X$ is a Jacobian algebra of a quiver with rigid potential.

Proof. By Theorem 5.2, there is an acyclic quiver $Q$ and a reduced word $w \in W_Q$ such that $\mathcal{C}_X$ is triangle equivalent to $\text{Sub}_w$. Moreover, there is a basic cluster-tilting object $T$ in $\mathcal{C}_X$ such that $Q_T = Q(w)$ and $\text{End}_{\mathcal{C}_X}(T) \cong \text{End}_{\text{Sub}_w}(T_w)$. By Lemma 3.2, $\text{End}_{\mathcal{C}_X}(T) \cong J(Q_T,W_T)$ for a non-degenerate potential $W_T$ on $Q_T$. On the other hand, there is rigid potential $W_w$ on $Q(w)$ such that $\text{End}_{\text{Sub}_w}(T_w) \cong J(Q(w),W_w)$ by [BIRSc, Thm 6.4].

Since $X$ has at most three weights, by Lemma 4.1, there is a unique non-degenerate potential on $Q_T$ up to right equivalence. Consequently, $W_T$ is right equivalent to $W_w$. It is known that the mutation of a rigid potential is rigid. For any vertex $i$ of $Q_T$, we obtain $\text{End}_{\mathcal{C}_X}(\mu_i(T)) \cong J(\mu_i(Q_T,W_w))$ by Lemma 2.7. Now the result follows from [FG, Thm 1.2] that each basic cluster-tilting object is reachable from $T$.

□

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