Tunable signal velocity in the integer quantum Hall effect of tailored graphene

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Topological properties in condensed matter physics are often claimed to be a fruitful resource for technical applications, but so far they only play a minor role in applications. Here we propose to put topological edge states to use in tailored graphene for Fermi velocity engineering. By tuning external control parameters such as gate voltages, the dispersions of the edge states regime are modified in a controllable way. This enables the realizations of devices such as tunable delay lines and interferometers with switchable delays.

I. INTRODUCTION

Non-trivial topological properties of condensed matter systems are believed to represent a valuable resource for various purposes. The key idea is that topological properties are protected so that they are not destroyed by small changes of the system. Hence they are robust against imperfections and unwanted effects. An excellent example is the quantized Chern number in Chern insulators. It is well-established that integer quantum Hall systems represent such Chern insulators and that their Hall conductivity is proportional to the Chern number and thus extremely well quantized\(^1-4\). This has led to the most spectacular application of a topological insulator: the integer quantum Hall effect (IQHE) has become the international gauge standard for resistance measurements, see Ref. \(^5\) and references therein.

Apart, however, from this very important application there has been little application of topological properties so far. We are aware of three-dimensional topological insulators used as thermoelectric elements\(^6,7\). Recently, it has been proposed that the Fermi velocity of electrons in edge states of two-dimensional Chern insulators on lattices can be tuned by the design of the edges and by changing the potential of the outermost sites of a strip of the lattice\(^8,9\). It was conjectured that such systems enable one to tune the signal velocity of a charge signal propagating along the edges by controlling external parameters such as gate voltages. The exponentially localized edge states possess a chiral nature, i.e., the propagation in one direction takes place at one edge while propagation in the opposite direction takes place at the other edge, see Fig. 1 for a generic illustration. Thus tuning of edge-specific properties renders the control of velocities depending on direction possible. The promise is to realize direction-dependent delay lines and interferometers.

The site-specific control of Chern insulators on lattices is a tremendous challenge to experimental realization. Thus, it suggested itself to use the well-established IQHE for the same purpose. Indeed, it is possible to obtain tunable signal velocities in two-dimensional electron gases (2DEG) subjected to a perpendicular magnetic field\(^10\). Still, there are challenges opposing an immediate realization: modifying the edges by periodically aligned bays with the required precision on small length scales of 100 nm represents a tremendous task to sample design. Larger length scales are easier to realize, but the characteristic length \(l_B = h/(e|B|)\) must match the geometric scales so that larger length scales require smaller magnetic fields. At first sight, this seems easy to realize, but the mobilities in the 2DEGs are not high enough to allow for the observation of the IQHE at low magnetic fields.

For this reason, we advocate to explore alternative routes and it is natural to look for other systems displaying an IQHE. Graphene and related compounds are obvious candidates. Graphene is widely known for its special electronic properties\(^11,12\) and its extraordinary structure. It represents an isolated single sheet of graphite\(^13,14\) and as such realizes a two-dimensional (2D) allotrope of carbon. The low-energy band structure of graphene comprises two Dirac cones distinguished by different locations in the Brillouine zone. Thus, electrons near the Fermi level have a linear dispersion relation and therefore behave like massless relativistic particles. Theoretically, low-energy electrons are described by the Dirac equation\(^15,16\) where the speed of light is replaced by the Fermi velocity \(v_F \approx 10^6 \text{ m/s}\).\(^17,18\). Engineering this important parameter has been realized already by varying the substrate\(^17\).
Subjecting graphene to a strong magnetic field at low temperatures leads to the formation of relativistic Landau Levels (LL). As a result, one can observe an unconventional IQHE \(^{11,19}\). The Hall conductivity in graphene appears at half-integer values: 

\[
\sigma_{xy} = \pm \frac{4e^2}{h} \left( |n| + \frac{1}{2} \right).
\]

The four-fold degeneracy given by valley and spin degeneracy yields the prefactor of 4. Comparing \(\sigma_{xy}(B)\) with the IQHE of a non-relativistic 2DEG, the offset \(1/2\) can be attributed to the single LL with energy \(E_0 = 0\) \(^{19}\). This LL is intrinsically half-filled so that one half contributes to the valence band and causes the half-integer conductivity. Due to cleaner samples and more precise measuring instruments the IQHE can be detected down to small external magnetic fields as low as \(\approx 0.1\) T due to a very high electron mobility \(^{20}\). The unique properties of graphene open up many new possibilities for the basic research and technical application, especially in electronics \(^{12,21}\) and spintronics \(^{22,23}\).

Our central proposal is to use graphene (or a related system) as IQHE system with accurately tailored edges in order to realize a tunable signal velocity \(v_F\). We point out that tuning \(v_F\) does not influence the DC conductivity often studied in literature. In our view, there are several advantages of graphene as basis material over 2DEGs in semiconductors: (i) the high mobility allows one to reach the IQHE even at low magnetic fields (\(\approx 0.1\) T \(^{20}\)); (ii) the possibilities to modify the edges in a reproducible and accurate way are larger; (iii) the energy separation between the lowest LLs (\(n < 3\)) are much larger so that the IQHE in graphene can be observed for lower magnetic fields and higher temperatures \(^{24}\). So we expect that the tunability of the Fermi velocity \(v_F\) in the edge modes will be feasible in the near future.

The key challenge is to tailor the edges such that periodically arranged bays are weakly coupled to the chiral edge states, see Fig. 2(a). Due to the weak coupling controlled by the width of the opening of the bays the local modes in the bays hybridize with the dispersive edge modes. This hybridization leads to coupled modes with very little dispersion, hence very low Fermi velocity. By applying gate voltages to adjust the chemical potential or the local potential of the bays the Fermi velocity is tuned: choosing the external parameters such that the edge mode is in resonance with the local bay modes reduces the Fermi velocity drastically.

The paper is set up as follows. First, we introduce the model describing the edge states and the LL of graphene. Next, we discuss the geometry of the considered samples. The main part shows the resulting dispersions and the tunability of the Fermi velocity in the edge states controlled by gate voltages as external control parameters. Finally, we conclude by summarizing and discussing possible applications.

\[\text{FIG. 2: Panel (a): Tailored strip of graphene proposed for tunable Fermi velocities. A magnetic field perpendicular to the sheet of graphene induces the IQHE. The sample has periodically aligned bays illustrated here by the zoom with 4 bays. The gate voltage } V_g \text{ is applied only to the bays highlighted in blue. The red area corresponds to the bulk of the strip and the green area to the opening between the bulk and a bay. Panel (b): Schematic sketch of a single unit cell with the notation for the linear dimensions of the bays and the strip. The total number } N_x \text{ of unit cells determines the total length of the sample } L_x = N_x L_{xp}.\]

\[\text{II. MODEL AND METHOD}\]

The electronic properties of graphene in the vicinity of the Fermi energy are well reproduced by a fermionic tight-binding model. Due to the negligible contribution of the spin degree of freedom as well as of interactions we consider spinless fermions in the Hamiltonian

\[
\mathcal{H} = t \sum_{\langle i,j \rangle} c_i^\dagger c_j - V_g \sum_{i \in \text{bay}} c_i^\dagger c_i.
\]  

We focus on zero temperature so that the chemical potential \(\mu\) is identical to the Fermi energy \(E_F\). All states up to \(\mu\) are occupied while the states above \(\mu\) are empty. The relevant Fermi velocity is the derivative of the dis-
persion at the Fermi energy. A pair of nearest neighbors is denoted by \((i, j)\). The tight-binding parameters are the nearest-neighbor hopping \(t = 2.8\) eV and the lattice constant \(a = 0.142\) nm\(^{18}\).

In order to obtain the quantum Hall state in graphene we apply an external perpendicular magnetic field \(\vec{B}\). This leads to the formation of LLs\(^{25,26}\) with energies at

\[
E_n = \text{sgn}(n)\sqrt{2\varepsilon t}\sqrt{B]|n| = \text{sgn}(n)\sqrt{\frac{2t^2}{2B}}|n|. \tag{2}
\]

The second equation stems from \(h\nu_F = 3/2ta\) and the definition of the magnetic length \(l_B = \sqrt{\hbar/(eB)} \approx 25.65564/\sqrt{B} \) nm where the of the magnetic field \(B\) is inserted in units of Tesla. The non-linear spacing between the LL results from the relativistic behavior of the electrons near the Dirac points. The magnetic length plays the same role as in the IQHE in the 2DEG\(^{10,27}\). It sets the scale for the diameter of the circular motion of the electrons due to the Lorentz force. In order that the decoration of the edges by bays has an appreciable effect the geometric dimensions of the bays must be of the order of this magnetic length.

The magnetic field \(B\) is included in the tight-binding model by the Peierls substitution attributing Aharanov–Bohm phases\(^{28}\) to the hopping processes

\[
t \to t \exp \left(\frac{i e}{\hbar} \int_{r_1}^{r_2} \vec{A} d\vec{r} \right), \tag{3}
\]

where the start and the end site of the hopping process is denoted by \(r_1\) and \(r_2\), respectively. The evolution of the Dirac cones to rather flat LLs has been exhaustively studied and discussed by Delplace and Montambaux\(^{27}\). In order to keep translational invariance in the \(x\)-direction, see Fig. 2(a), we employ the Landau gauge \(\vec{A} = B(-y, 0, 0)\). Thus, the momentum \(k_x\) is a good quantum number in the calculations.

The numerical analysis is facilitated by small system size, i.e., the number of sites in the extended unit cell, see Fig. 2(b), should be rather small so that the dimension of the resulting eigen value problem remains tractable. For the experimental realization, however, it is advantageous to consider rather large extended unit cells. In the following, we briefly discuss these constraints, the employed numerical methods, and justify our choice of parameters.

The tight-binding Hamiltonian (1) comprises only on-site and nearest-neighbor hoppings. Hence, its very large matrix representing the Hamiltonian is mostly populated by zero entries and may therefore be encoded as sparse matrix. The signal transmission is primarily determined by the properties at the Fermi energy \(E_F\). Undoped graphene is a semi-metal with \(E_F = 0\). The states at higher energies hardly influence the low-energy dynamics. So we focus on the low LLs up to the third one, i.e., with \(|n| < 4\). The eigen value solver FEAST\(^{29}\) is used to constrain the considered interval of the energy spectrum. This routine has proved\(^{9}\) to implement a reliable high-performance algorithm to efficiently diagonalize large sparse matrices. It is based on the quantum mechanical density matrix representation and applies counter integration techniques in order to solve the eigen value problem in a fixed interval of the total spectrum. Based on the calculated eigen energies the Fermi velocity as derivative of the dispersion at the Fermi energy is straightforwardly approximated by the ratio of finite differences. We assume periodic boundary conditions in \(x\)-direction with 50 unit cells. This implies a sufficiently fine discretization of the Brillouin zone to display energy crossings and avoided energy crossings.

The experimental constraints consist in the limitations in accurately tailoring the strips of graphene with the desired structure at the edges, i.e., with the periodic structure of bays, while maintaining a high mobility to realize the quantum Hall state. The bay pattern can be customized by electron beam lithography\(^{30,31}\) or by anisotropic etching techniques\(^{32}\). It appears that creating bays of the size of 100 nm can be done without major problems. In our calculations we assume quadratic bays for simplicity. The precise shape of the bays does not matter for the qualitative results although it will have an influence on the quantitative details. The size of the bays and especially the length of the bay opening \(L_0\) to the bulk of the strip, see Fig. 2(b), are crucial for dispersions of the modified edge states. To keep the example neat we aim at a small number of low-lying levels in the bays so that the relevant number of states which may hybridize stays easily tractable. This implies \(L_0 \approx l_B\) and favors small magnetic fields (100 mT with \(l_B \approx 81\) nm). These considerations define the framework for the numerical results in the next section.

### III. RESULTS

#### A. Dispersions, hybridized edge modes, and localization

In Fig. 3, three representative cases are depicted: uncoupled \((L_o = 0l_B)\), weakly \((L_o = 0.1l_B)\), and moderately \((L_o = 0.5l_B)\) coupled bays. The magnetic field is set to \(B = 0.25\) T which corresponds to a magnetic length \(l_B \approx 51\) nm. The definitions of the various lengths are displayed in Fig. 2(b). They are given by: \(L_{xp} = 3l_B \approx 153\) nm, \(L_y = 6l_B \approx 306\) nm, \(L_b = 2l_B \approx 102\) nm and \(L_o = \{0, 0.1, 0.5\}l_B \approx \{0, 5.26\}\) nm. The width of the strip \(L_y\) is chosen large enough so that the two counter-propagating edge states at the opposite edges do not overlap. This implies that both edges can be modified independent of each other. For clarity, we exploit this simplifying fact and modify only the upper edge while the lower edge remains a bare zigzag edge. The chosen bay size \(L_b\) should be experimentally realizable. The condition \(L_{xp} > L_b\) ensures that the bays are separated from one another. Decreasing the distance between the bays by changing \(L_{xp}\) increases the impact on the dis-
FIG. 3: Dispersions of the lowest positive eigen energies in graphene in the IQHE at $B = 0.25 \, \text{T}$, $L_{x,y} = 3l_B \approx 153 \, \text{nm}$, $L_b = 2l_B \approx 102 \, \text{nm}$, and $L_o = \{0,0.1,0.5\}l_B \approx \{0,5.26\} \, \text{nm}$. Panels (a-c) show the cases of uncoupled, weakly, and moderately coupled bays, respectively. Crossings evolve into avoided crossings due to the hybridization of both modes. The dashed ellipses mark a location where this happens. The insets zoom into the evolution of the avoided crossings at $k_x = \pi/(4a)$, where the two dispersion branches are highlighted in blue and red.

persion of edge states due to the changed fraction of the decorated boundary to the undecorated boundary. The three values for the opening $L_o$ lead to three different degrees of hybridization. As a rule of thumb a wider opening corresponds to a larger hybridization. The effect is discernible in the dispersions in Fig. 3. For the sake of clarity, we display the dispersions up to the first LL so that only two counter-propagating edge states need to be taken into account.

Panel (a) in Fig. 3 shows the uncoupled case where the usual LLs and their related edge states are distinct from the local states. The LLs are essentially flat and turn upwards where their wave functions approach the edges of the strip. In contrast, the eigen states of the modes in the bays are completely local, hence completely flat as function of the wave vector $k_x$. Thus, edge states and local modes show crossings, see inset of Fig. 3(a). Due to the extended unit cell comprising one bay, the edge states are backfolded into the reduced Brillouin zone scheme. Since the dispersion is symmetric with respect to the $k_x$-axis, we only show the positive eigen energies. When the coupling of the bays to the bulk of the strip is switched on, i.e., $L_o \neq 0$, see panels (b) and (c) in Fig. 3, the edge states mix with the local states. We observe that the crossings turn into avoided crossings with the bay modes due to level repulsion, see for instance the encircled regions marked by dashed ellipses. Increasing the opening further and further results in a stronger and stronger level repulsion so that the former local modes become more and more dispersive.

Much to our surprise, in addition to the local states of the bays other almost local states appear upon opening the bays. Such local states have not been observed in the IQHE of the non-relativistic 2DEG. Investigating the probability density of the eigen states at $k_x = 0$ reveals the location of the unexpected modes. In Fig. 4 we display the probability densities of the two states with the energies highlighted by arrows in panel (b) of Fig. 3. Figure 4(a) clearly shows the density of the local state from the bays marked in Fig. 3(b) by the open arrow: it is almost entirely localized within the bay and leaks only weakly into the bulk of the strip. In contrast, Fig. 4(b) clearly shows a strong localization in the opening. This is the density of the additional local state marked in Fig. 3(b) by the filled arrow. Obviously, the opening gives rise to additional localization. Hence, the added sites in the opening, see green area of Fig. 2(b), contribute to the spectra similar to the effect of the bay sites. Independent of the origin of the almost local states, both hybridize with the edge modes at the same en-
FIG. 4: Probability densities $|\psi_{n}(x,y)|^2$ of two almost local eigen states at $k_x = 0$. Clearly, these modes are localized in two different regions. Panel (a) shows the local mode stemming from the bay whereas panel (b) shows the local mode localized in the opening of the bay. The corresponding eigen states are indicated by open (mode from bay) and filled (mode in opening) arrows in panel (b) of Fig. 3. The parameters of field and geometry are $B = 0.25$, $L_{xp} = 3l_B \approx 153$ nm, $L_y = 6l_B \approx 306$ nm, $L_x = 2l_B \approx 102$ nm, and $L_o = 0.1l_B \approx 5$ nm.

e. The additional state hybridizes more strongly as expected since it is localized in the opening very close to the bulk of the strip while the mode from within the bay leaks only weakly into the bulk. The hybridization of both local modes leads to a reduced Fermi velocity.

**B. Fermi velocities for signal transmission**

In order to tune the Fermi velocity of the edge state at the upper edge of the strip of graphene we have to change the derivative of its dispersion at the Fermi level. To do so two ways suggest themselves in particular. The most transparent one is to change the Fermi energy, i.e., the chemical potential, such that the Fermi level lies in a rather flat region of the edge state dispersion. This can be achieved by a gate close to the total strip of graphene. Alternatively, one may conceive a control of the energy level of the bays alone. This can be achieved by a voltage applied to an appropriate gate close to the edge of the strip, see Fig. 2(a). Undoubtedly, other ways of tuning can be devised as well. Below we present results for both approaches to show that tuning of the Fermi velocity is possible.

Figure 5 depicts the Fermi velocity as function of the chemical potential. Panel (a) and (b) correspond to the case of weakly and moderately coupled bays, respectively, of which the dispersions are displayed in Fig. 3. The Fermi velocity in the weakly coupled case shows three deep dips. The two extremely narrow and steep dips can be attributed to the hybridization with two local levels from the isolated bays. The energetic position of these local levels is indicated by vertical red lines in the two panels. The closeness of the steep dips to these lines underlines their physical origin. The slight shifts of the dips relative to the red lines result from the energy shift due to the hybridization. This is supported by the fact that the shift is larger for panel (b) which refers to bays with a wider opening and hence stronger hybridization. The broader dips (one in panel (a) and two in panel (b)) are related to the additional local states localized in the openings, see Fig. 4(b). The fact that these dips are broader is explained by the vicinity of these states to the bulk of the strips implying a stronger hybridization with the edge mode than for the local states from within the bays.

By tuning $\mu$ into these dips the Fermi velocity can be reduced by orders of magnitude. In particular for narrow bay openings a strong reduction can be achieved. For instance, $v_F$ in panel (a) of Fig. 5 at $\mu \approx 2$ meV is reduced by a factor of $\approx 65000$ from its value without tailored edges, i.e., without bays, see green curve in Fig. 5. Increasing the bay opening leads to a broadening and a shift of the dips, see Fig. 5(b). The shift of the local modes results from the hybridization pushing the local modes downwards in energy due to level repulsion. Furthermore, a wider opening possesses more low-lying energy modes so that the number of dips is increased. If the chemical potential is out-of-resonance with local modes, the reduction of the Fermi velocity is insignificant. This is particularly true for narrow opening where there is no significant difference between the green and the blue line out-of-resonance.

Tuning of the bay potential $V_g$ leads to similar results even though changing this gate voltage does not simply shift the local modes relative to the bulk modes. Still, significant changes of the Fermi velocity can be realized as illustrated in Fig. 6 for which we assume a generic Fermi energy $E_F = 0.0005t = 1.4$ meV in the regime of doped graphene. The finite value of the Fermi energy is necessary to be in the dispersive regime of the edge states. In order to realize a substantial reduction of the Fermi velocity, the local states must be brought into resonance with the edge states. This tuning of the bay potential results in similar dips of the Fermi velocity as the tuning of the chemical potential $\mu$. Negative values of $V_g$ lift the bay spectrum up in energy so that they come into resonance with the edge state resulting in a steep dip.

In contrast, a positive gate voltage $V_g$ decreases the energy of the additional local model in the opening. The ensuing resonance leads to a broader dip due to the larger hybridization of the opening mode to the edge mode. Note that larger absolute values $V_g$ are needed to reach the resonance compared to what can be estimated from
FIG. 5: Fermi velocity $v_F$ of the right-moving edge states localized at the upper edge as function of the chemical potential $\mu$ equivalent to the Fermi energy. Panels (a) and (b) show the weakly and moderately coupled cases $L_o = \{0.1, 0.5\}l_B \approx \{5, 26\}$ nm. The other parameters are $B = 0.25$ T, $L_{xp} = 3l_B \approx 153$ nm, $L_y = 6l_B \approx 306$ nm, $L_o = 2l_B \approx 102$ nm.

FIG. 6: Fermi velocity $v_F$ of the right-moving edge states as function of the gate voltage $V_g$ changing only the energy offset of the bays, see Fig. 2(a). The parameters are $B = 0.25$ T, $L_{xp} = 3l_B \approx 153$ nm, $L_y = 6l_B \approx 306$ nm, $L_b = 2l_B \approx 102$ nm, and $L_o = 0.1l_B \approx 5$ nm. The less smooth dip at about $V_g = -1.7$ meV results from the discretization of the Brillouin zone with 50 points. A finer mesh would lead to a smoother curve, but is computationally demanding.

IV. SUMMARY AND DISCUSSION

The above results clearly show that substantial tuning of the Fermi velocity in graphene with tailored edges is possible. This can be used to construct tunable delay lines for charge signals: by controlling the external parameters such as gate voltages the temporal delay can be tuned at will by choosing an appropriate velocity of signal transmission through the sample. This may help to control and to switch properties of nanoscale devices including devices for quantum information processes. The delayed signal itself does not need to have quantum character, but the tuned delay can help to deliver control precisely at required time instants.

Another promising idea in the same spirit is to construct interferometers in which two signals are superposed which have propagated along different pathways. One of these pathways contains the graphene sheet with tunable signal velocity. By adjusting the Fermi velocity the delay in this path can be altered such that destructive or constructive interference takes place. If the signal along the other pathway is propagating through a sample of an unknown compound or an unknown device its transmission properties can be investigated in this way. These two suggestions are meant to exemplify promising applications of tunable signal velocities. We emphasize that the tuning can be done very fast on the time scales on which the gate voltages can be changed.

To obtain an idea of the order of magnitude of the delays we assume the conditions used for Fig. 5(b), but with only $N_x = 20$. For simplicity, rounded values are used. The sample length is $L_x \approx 3 \mu$m and we consider the broader dip because its less susceptible to imperfections. For $\mu = 8$ meV the undecorated strip of graphene displays a velocity $v_F = 12 000$ m/s. It will be reduced by a factor of 20 down to $v_F = 600$ m/s in the minimum of the dip. The time required by a signal to propagate along

the dispersion. This fact can be explained by the level repulsion from the the bulk states which are mixed in. Thus, both dips are shifted relative to the energy differences in the uncoupled dispersion, see panel (a) in Fig. 3. We conclude that tuning the bay potential is also a suitable knob to control the Fermi velocity. In summary, tuning both gate voltages leads to resonances with local modes so that substantial reductions of the velocity of signal transmission are achieved.
the edge is delayed from 0.25 ns to 5 ns. This change of transmission time is readily detectable.

We are aware that the calculations presented in this article assume idealized conditions, for instance zero temperature and small samples without imperfections or disorder. So further research is called for to address these points as we did already for the non-relativistic IQHE. Nevertheless, the origin of the predicted effect is clearly elucidated and does not rely on subtleties of the model, for instance the shape of the bays does not matter as long as the bays host local modes. With state-of-the-art techniques the IQHE can be detected in graphene samples which are not smaller than $3 \times 3 \mu m^2$. For smaller samples, imperfections and disorder spoil the Hall states. As long as the IQHE can be observed and the properties of at least one of the edges can be externally controlled tuning of the signal velocity will be within reach.

The take-home message is that graphene represents a promising material to realize the IQHE with externally tunable dispersions of the edge states on the basis of today's technology. Tailoring of the edges is an exacting prerequisite. We hope that our results trigger further studies, in particular experimental ones, paving the way towards tunable signal velocities. As an outlook we point out that further work on the influence of imperfections, disorder, and the presence of the spin degree of freedom are in order. The latter opens up particular applications in spintronics.

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