Effective theory of chiral two-dimensional superfluids

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We construct, to leading orders in the momentum expansion, an effective theory of a chiral \((p_x + ip_y)\) two-dimensional fermionic superfluid at zero temperature that is consistent with nonrelativistic general coordinate invariance. The currents and stress tensor are computed and their linear response to electromagnetic and gravitational sources is calculated. We also consider an isolated vortex in a chiral superfluid and identify the leading chirality effect in the density depletion profile.

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I. INTRODUCTION

Chiral fermionic superfluids enjoy a long-standing and continuous interest in condensed matter physics. This goes back to studies of thin films of \(^3\)He-A which is believed to form a chiral condensate \([1, 2]\). More recently, low-dimensional chiral superfluids attracted some attention due to the presence of Majorana zero-energy edge quasiparticles \([3, 4]\) that might facilitate progress towards a fault-tolerant quantum computer \([5, 6]\).

In this paper we construct and analyze the effective theory of a two-dimensional fermionic superfluid at zero temperature that forms a chiral condensate which in momentum space takes the form

\[
\Delta_p = (p_x \pm ip_y)\hat{\Delta},
\]

where \(\hat{\Delta}\) is a complex function of \(|p|\) and the sign defines the chirality of the condensate. The order parameter is the p-wave eigenstate of the orbital angular momentum. We will assume that (1) is the energetically favored ordering. The usual spontaneous breaking of the particle number \(U(1)_N\) symmetry is accompanied by the breaking of the orbital rotation \(SO(2)_L\) symmetry. The condensate (1) remains invariant under the diagonal combination of \(U(1)_N\) and \(SO(2)_L\) transformations which leads to the symmetry breaking pattern

\[
U(1)_N \times SO(2)_L \to U(1)_V.
\]

This implies the presence of a single gapless Goldstone mode in the spectrum that governs the low-energy and long-wavelength dynamics of the superfluid at zero temperature. The effective theory of this Goldstone field has an infinite number of terms and can be organized in a derivative expansion. In this paper we restrict our attention only to the leading and next-to-leading order contributions.

Our guiding principle is the nonrelativistic version of general coordinate invariance developed in \([7]\). We put the chiral superfluid into a curved space, switch on an electromagnetic source and demand the invariance of the effective theory with respect to nonrelativistic diffeomorphisms and \(U(1)_N\) gauge transformations. Since the general coordinate invariance can be viewed as a local version of Galilean symmetry, the symmetry constraints on the effective theory are (even in flat space) more restrictive than just the ones imposed by the Galilean invariance alone. This approach proved to be useful before and led to new predictions for unitary fermions \([7, 8]\) and quantum Hall physics \([9]\).

While our effective theory might be viewed as a toy model mimicking only some aspects of low-energy dynamics in thin films of \(^3\)He-A and spin-triplet superconductors, our predictions, in principle can be tested experimentally with ultracold spin-polarized fermions confined to two spatial dimensions.

II. EFFECTIVE THEORY OF CHIRAL SUPERFLUIDS

Our starting point is the effective theory of a conventional (s-wave) superfluid constructed in \([7]\). In curved space with a metric \(g_{ij}\) and in the presence of an electromagnetic \(U(1)_N\) source \(A_\mu\), the leading order action of the Goldstone field \(\theta\) was found to be

\[
S[\theta] = \int dtd\mathbf{x} \sqrt{\gamma} P(\mathbf{X}),
\]

\[\text{1 In this paper we follow the notation of [7] except that the mass of the fermion is set to unity.}\]
where \( g \equiv \text{det}g_{ij} \),

\[
X = \partial_i \theta - \frac{g^{ij}}{2} \partial_i \theta \partial_j \theta \quad (4)
\]

and the covariant derivative \( \partial_i \theta = \partial_i \theta - A_i \) with \( \nu = t, x, y \). The superfluid ground state has a finite density, that we characterize by the chemical potential \( \mu \). In the effective action it enters as a background value for the Goldstone field, that is decomposed as

\[
\theta = \mu t - \varphi \quad (5)
\]

with \( \varphi \) standing for a phonon fluctuation around the ground state. This allows to identify the function \( \mu(X) \) in Eq. (3) with the thermodynamic pressure as a function of the chemical potential \( \mu \) at zero temperature. As demonstrated in [7], the expression \( X \) transforms as a scalar under the infinitesimal nonrelativistic diffeomorphism transformation \( x^i \to x^i + \xi^i \) provided

\[
\begin{align*}
\delta \theta &= -\xi^k \partial_k \theta, \\
\delta A_i &= -\xi^k \partial_k A_i - A_k \partial_i \xi^k + g_{ik} \tilde{g}^k, \\
\delta A_k &= -\xi^k \partial_k A_i - A_k \partial_i \xi^k + g_{ik} \tilde{g}^k, \\
\delta g_{ij} &= -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k.
\end{align*} \quad (6)
\]

In addition, \( X \) is invariant under \( U(1)_N \) gauge transformations. These observations ensure that the leading order effective action (3) is invariant under general coordinate transformations.

As becomes clear from the symmetry breaking pattern (2), in the case of a chiral superfluid we need a gauge potential for \( SO(2)_L \) orbital rotations. To this end we introduce an orthonormal spatial vielbein \( e^a_i \) with \( a = 1, 2 \). The vielbein satisfies

\[
\begin{align*}
g_{ij} &= e^a_i e^a_j, \\
e^{ab} e^a_i e^b_j &= \varepsilon_{ij}.
\end{align*} \quad (7)
\]

The vielbein is not unique and defined only up to a local \( SO(2)_L \) rotation in the vielbein index \( a \). This allows us to introduce a connection

\[
\begin{align*}
\omega_1 &= \frac{1}{2} \left( e^{ab} e^{aj} \partial_i e^b_j + B \right), \\
\omega_1 &= \frac{1}{2} e^{ab} e^{aj} \nabla_i e^b_j = \frac{1}{2} \left( e^{ab} e^{aj} \partial_i e^b_j - e^{jk} \partial_j g_{ik} \right),
\end{align*} \quad (8)
\]

where \( e^{aj} \equiv e^a_i \delta^{ij} \) and the magnetic field \( B \equiv e^{ij} \partial_i A_j \). Under a local (i.e., time- and position-dependent) infinitesimal \( SO(2)_L \) rotation

\[
e^a_i \to e^a_i + \phi(t, x)e^{ab} e^b_i, \quad (9)
\]

the connection \( \omega_\nu \) transforms as an Abelian gauge field \( \omega_\nu \to \omega_\nu - \partial_\nu \phi \). Using Eq. (8) together with the transformation law of the vielbein one-form

\[
\delta e^a_i = -\xi^k \partial_k e^a_i - e^b_i \partial_i \xi^k, \quad (10)
\]

one can show that \( \omega_\nu \) transforms as a one-form under the nonrelativistic diffeomorphisms, i.e.,

\[
\delta \omega_\nu = -\xi^k \partial_k \omega_\nu - \omega_\nu \partial_i \xi^k. \quad (11)
\]

In hindsight, this simple transformation of \( \omega_\nu \) clarifies the appearance of the magnetic field \( B \) in the definition of \( \omega_\nu \).

We are now in position to construct the leading order effective theory of a chiral superfluid. Quite naturally, the theory is still defined by the action (3) with the covariant derivative now given by

\[
\partial_i \theta = \partial_i \theta - A_i - s \omega_\nu. \quad (12)
\]

For the chiral p-wave superfluid \( s = \pm 1/2 \) with the sign determined by chirality of the ground state. This ensures that the Goldstone boson is coupled to the proper (broken) linear combination of \( U(1)_N \) and \( SO(2)_L \) gauge fields. As will become clear in Sec. III, one must set \( s = \pm n/2 \) for a chiral superfluid with pairing in the \( n \)th partial wave.

It is interesting to note that instead of the usual formalism, where the superfluid is described by a single Goldstone boson \( \theta \), there is an alternative approach in which the Lagrangian depends on the phase of the condensate \( \theta \) and the superfluid velocity \( \nu \). We present this alternative description of a (chiral) superfluid in Appendix A and demonstrate there that by integrating out \( \nu \) one obtains the usual Goldstone effective action.

We point out that Galilean invariance\(^3\) alone is not sufficient to fix the leading order action (3). Under a Galilean boost the magnetic field \( B \) transforms as a scalar, i.e., \( \delta B = -\xi^k \partial_k B \). For this reason the prefactor multiplying the magnetic field \( B \) in the first equation in (8) is not constrained by the Galilei invariance. Thus only the general coordinate invariance fixes uniquely the leading order Lagrangian of a chiral superfluid.

Time reversal and parity act nontrivially as

\[
T : t \to -t, \quad \theta \to -\theta, \quad A_i \to -A_i, \quad \omega_\nu \to -\omega_\nu; \quad P : x_1 \leftrightarrow x_2, \quad A_1 \leftrightarrow A_2, \quad \omega_\nu \to -\omega_\nu, \quad \omega_\nu \leftrightarrow -\omega_\nu. \quad (13)
\]

For a fixed \( s \neq 0 \), the effective theory (3) is not separately invariant under neither time reversal \( T \) nor parity \( P \). On the other hand, \( PT \) is a symmetry of the theory for any value of \( s \).

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\(^2\) Here we introduced the antisymmetric Levi-Civita symbol \( \epsilon^{ij} = \epsilon_{ij}, \epsilon_{ij} \equiv +1 \). The Levi-Civita tensor is then \( \epsilon^{ij} = \frac{1}{\sqrt{g}} \epsilon^{ij} \), \( \epsilon_{ij} = \sqrt{|g|} \epsilon_{ij} \).

\(^3\) The infinitesimal Galilean boost with velocity \( \nu^k \) is realized by a combination of the gauge transformation \( \alpha = \nu_k x^k \) together with the diffeomorphism \( \xi^k = \nu^k t \).
The action (3) is the leading order term in a derivative expansion that follows the counting in [7], where \( \partial_i \theta \sim A_i \sim g' \sim O(1) \) when expanded around the ground state. In addition, in this power-counting a derivative acting on any field \( \phi \) increases the order by one, i.e., \( [\partial_i \phi] = 1 + [\phi] \). Non-linear effects of fields with \( [\phi] = 0 \) are included, since \( [\phi^n] = n[\phi] = 0 \) for any \( n \geq 1 \). For this reason, we conclude from the scaling of the metric that \( [e^s_i] = 0 \) and \( [\omega_{ij}] = 1 \). As a result, \( \partial_i \theta \) turns out to be of a mixed order: it contains terms both of leading and next-to-leading order in the derivative expansion. Since one can show that the only possible next-to-leading order contribution consistent with symmetries is already contained in \( \partial_i \theta \), our theory is complete up to and including next-to-leading order.\(^4\)

By introducing the superfluid density \( \rho \equiv dP/dX \) and the superfluid velocity \( v_j \equiv -\partial_j \theta \) the nonlinear equation of motion for the Goldstone field can be written in the general covariant form

\[
\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} \rho) + \nabla_i (\rho v^i) = 0, \tag{14}
\]

which is the continuity equation in curved space. With respect to nonrelativistic diffeomorphisms, \( \rho \) transforms as a scalar and \( v_i \) transforms as a gauge potential, i.e.,

\[
\begin{align*}
\delta \rho &= -\xi^k \partial_k \rho, \\
\delta v_i &= -\xi^k \partial_k v_i - v_k \partial_i \xi^k + g_{ik} \xi^k.
\end{align*} \tag{15}
\]

By linearizing the equation of motion (14) in the absence of background gauge fields \( (A_i = \omega_i = 0) \) one finds the low-momentum dispersion relation of the Goldstone field to be

\[
\omega^2 = c_s^2 p^2, \tag{16}
\]

where the speed of sound \( c_s \equiv \sqrt{(\partial P/\partial \rho)} \) is evaluated in the ground state.

From Eq. (12) the vorticity \( \omega \equiv \frac{1}{2} \epsilon^{ij} \partial_i v_j \) can be expressed as

\[
\omega = \frac{\sqrt{g}}{2} \left( B + \frac{s}{2} R \right), \tag{17}
\]

where we used that the Ricci scalar \( \frac{1}{2} \sqrt{g} R = \epsilon^{ij} \partial_i \omega_j \) in two dimensions. In the absence of a magnetic field in flat space, the superfluid velocity field is irrotational, i.e., \( \omega = 0 \), except for singular quantum vortex defects. For an elementary vortex located at a position \( x_v \), the vorticity is \( \omega(x) = \pi \delta(x - x_v)/2 \). Single-valuedness of the macroscopic condensate wave-function yields quantization of the total magneto-gravitational flux on a compact manifold. For example, for a p-wave superfluid living on a sphere \( S^2 \) with no magnetic field, the total flux is \( \int_{S^2} \omega = \pi \), which is accommodated in two elementary quantum vortices.

### III. CURRENTS AND STRESS TENSOR

In this section, starting from the effective action (3), we construct and analyze the \( U(1)_N \) and \( SO(2)_L \) currents and the stress tensor.

#### A. \( U(1)_N \) current

In curved space the temporal and spatial part of the \( U(1)_N \) three-current are found to be

\[
\begin{align*}
J^t &\equiv -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta A_t} = \rho \\
J^i &\equiv -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta A_i} = \rho g^{ij} v_j + \frac{s}{2} \epsilon^{ij} \partial_j \rho. \tag{18}
\end{align*}
\]

In addition to the usual convective term, we find the parity-odd contribution to the current which is proportional and perpendicular to the gradient of the superfluid density. In a near-homogeneous finite system this part of the current flows along the edge of the sample, where the density changes rapidly. It is important to stress that the edge current is perpendicular to \( \partial_i \rho \), but not to the electric field \( E_j \equiv \partial_i A_j - \partial_j A_i \). Thus there is no static Hall conductivity in the chiral superfluid in agreement with general arguments of [10]. The edge current was found in the study of superfluid 3He-A by Marmin and Muzikar [11].

The current conservation equation is

\[
\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} J^i) + \nabla_i J^i = 0. \tag{19}
\]

Since \( \nabla_i J^i_{\text{edge}} = 0 \), this is consistent with the equation of motion (14).

In a non-homogeneous chiral superfluid the edge current is flowing even in the ground state. This implies that the ground state has a nonvanishing angular momentum \( L_{GS} \). In flat space this is given by

\[
L_{GS} = \int d^2 x \epsilon_{kli} x^k J^l = s \int d^2 x \rho, \tag{20}
\]

where \( N \) denotes the total number of fermions. This result has a simple explanation in the limit of strong interatomic attraction, where the many-body fermionic system can be viewed as a Bose-Einstein condensate of

\(^4\)The term in the Lagrangian of the form \( Q(X)(\partial_i + v^i \partial_i)X \) is consistent with continuous symmetries for an arbitrary function \( Q(X) \) and gives contributions to next-to-leading order. It does not however respect \( PT \) invariance and should not be included.
tightly bound molecules. In the chiral superfluid with pairing in the $n^{th}$ orbital wave every molecule has the intrinsic angular momentum $l = \pm n$ with the sign determined by the chirality of the ground state. In the BEC limit interactions between molecules are weak and thus the total angular momentum is just the sum of internal angular momenta of separate molecules. This explains why one must set $\theta = \pi \left(1 - \frac{1}{n}\right)$. \hspace{1cm} (21)

When $n = 1$ the anyon becomes a boson (scalar) and when $n \to \infty$ it becomes a fermion. Similarly to the chiral superfluid, for any $n > 1$ time reversal and parity invariance are broken. Expanding the action (3) to second order around the ground state we find a term

$$\Delta L = \frac{s}{2} \frac{\rho}{c_s^2} A_t B.$$ \hspace{1cm} (22)

This was identified as responsible for the Landau-Hall effect in the anyon superfluid \cite{12, 13}. The coefficient of this term was determined in \cite{14} to be

$$\Delta L = \frac{c_s^2}{8 \pi} \left(n - \frac{1}{n}\right) A_t B.$$ \hspace{1cm} (23)

For anyons, scale invariance fixes $c_s^2 = \rho/2$ leading to the identification

$$s = \frac{1}{2} \left(n - \frac{1}{n}\right).$$ \hspace{1cm} (24)

Therefore the orbital angular momentum is also fractional, as expected.

\section{SO(2)$_L$ current}

The orbital SO(2)$_L$ current is given by

$$J_{\text{orb}} ^t = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \omega^t_i} = s \rho \epsilon_i^t,$$

$$J_{\text{orb}} ^l = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \omega^l_i} = s \rho \epsilon_i^l.$$ \hspace{1cm} (25)

This current is purely convective and is conserved due to the equation of motion (14).

\section{Stress tensor}

In curved space the invariance of the effective action with respect to an infinitesimal nonrelativistic diffeomorphism $\xi^i$ implies

$$S[\theta + \delta \theta, A_\nu + \delta A_\nu, g_{ij} + \delta g_{ij}, \epsilon^n_i + \delta \epsilon^n_i] = S[\theta, A_\nu, g_{ij}, \epsilon^n_i],$$

and leads to the Euler equation

$$\frac{1}{\sqrt{g}} \partial_t (\sqrt{g} J^t_k) + \nabla_i T^i_k = E_k J^t + \epsilon_{ik} J^t B.$$ \hspace{1cm} (27)

where $T^i_k \equiv T^{ij} g_{jk}$ and the contravariant stress tensor is defined by

$$T^{ij} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}}.$$ \hspace{1cm} (28)

Generically the Euler equation is the conservation equation only if $E_k = 0$ and $B = 0$.

Now we wish to compute the stress tensor $T^{ij}$ for the chiral superfluid (3). In this case, the variation of the vielbein is related to the variation of the metric via

$$\epsilon^n_i \to \epsilon^n_i + \frac{1}{2} \epsilon^{mn} \delta g_{ij} + \delta \lambda \epsilon^{ab} e^b_i.$$ \hspace{1cm} (29)

In this way the first relation in Eq. (7) is satisfied up to second order in $\delta g_{ij}$. There is an ambiguity in the transformation of the vielbein parametrized by $\delta \lambda$ which is related to the $SO(2)_L$ gauge freedom of the vielbein. In the following we set $\delta \lambda = 0$. For the contravariant components of the metric, \hspace{1cm} [42]

$$g^{ij} g_{jk} = \delta^i_k \rightarrow \delta g^{ij} = -g^{il} g^{jm} \delta g_{lm},$$ \hspace{1cm} (30)

so one cannot use the metric to raise the indices of the metric variation.

First, consider the s-wave superfluid, i.e., set $s = 0$. Using $\delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{ij} \delta g_{ij}$ we find

$$T^{ij}_{s=0} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}} = P g^{ij} + \rho v^i v^j.$$ \hspace{1cm} (31)

which is the stress tensor of the ideal fluid.

For the chiral superfluid additional variation of the action arise from the variation of the connection (8). Namely, we find

$$\delta \omega^t_i = -\frac{1}{4} \epsilon^{mn} g^{jk} \partial_m g_{nk} g_{ij} - \frac{1}{4} B g^{ij} \delta g_{ij};$$

$$\delta \omega^l_i = -\frac{1}{4} \epsilon^{mn} g^{jk} \partial_m g_{nk} \delta g_{ij} - \frac{1}{2} \epsilon^{jk} \partial_j \delta g_{lk}$$

$$+ \frac{1}{4} \epsilon^{mk} \partial_m g_{lk} g^{ij} \delta g_{ij}.$$ \hspace{1cm} (32)

This leads to

$$\delta S_{\text{ch}} = \int dtdx \sqrt{g} P^\mu \frac{\partial X}{\partial \omega_\mu} \delta \omega_\mu$$ \hspace{1cm} (33)

which in detail is given by
\[ \delta S_{ch} = \frac{s}{4} \int dtdx \sqrt{g} \left[ P' \varepsilon^{jn} g^{jk} \partial_t g_{nk} - P' \mathcal{D}^l \delta \varepsilon^{jn} g^{jk} \partial_t g_{nk} \right] \delta g_{ij} + \frac{s^2}{2} \int dtdx \partial_t (P' \mathcal{D}^i \delta \theta) \delta g_{ij} \]

+ \frac{s}{4} \int dtdx \sqrt{g} P' g^{ij} \left[ B + \varepsilon^{mk} \mathcal{D}^l \partial_m g_{lk} \right] \delta g_{ij}. \quad (34) \]

After a tedious but straightforward calculation we find

\[ \Delta T_{ch}^{ij} = \frac{2}{\sqrt{g}} \frac{\delta S_{ch}}{\delta g_{ij}} \]

\[ = (v^i P_{\text{edge}} + v^i J_{\text{edge}}) + T_{\text{Hall}}^{ij} - \frac{s^2}{4} \rho R g^{ij} \quad (35) \]

with

\[ T_{\text{Hall}}^{ij} = -\eta_H (\varepsilon^{ik} g_{jl} + \varepsilon^{jk} g_{il}) \]

Here the strain rate tensor \( V_{kl} \equiv \frac{1}{2} (\nabla_k v_l + \nabla_l v_k + \partial_t g_{kl}) \) and \( \eta_H = -\frac{1}{2} \rho. \) With respect to general coordinate transformations both \( V_{kl} \) and \( T_{\text{Hall}}^{ij} \) transform as tensors. \( T_{\text{Hall}}^{ij} \) is known as the Hall viscosity part of the stress tensor and was discovered first in [15, 16]. It generically arises in two-dimensional many-body systems that break time reversal and parity [17, 18].

In flat space with the metric \( g_{ij} = \delta_{ij} \) the force density arising from the Hall viscosity is given by

\[ f_{\text{Hall}}^i = -\partial_j T_{\text{Hall}}^{ij} = \eta_H \varepsilon^{ij} \Delta v_j, \quad (37) \]

and thus the net work per unit of time produced by the Hall viscosity in a region \( \mathcal{S} \) surrounded by a boundary \( \partial \mathcal{S} \) is

\[ w = \eta_H \int_{\partial \mathcal{S}} v_i \varepsilon^{ij} \Delta v_j = \eta_H \int_{\partial \mathcal{S}} n^k \varepsilon^{ij} v_i \partial_k v_j, \quad (38) \]

where \( n^k \) is the normal vector to the boundary \( \partial \mathcal{S} \). We conclude that the Hall viscosity is dissipationless in the bulk of the region \( \mathcal{S} \). Alternatively, its contribution to the bulk entropy production is vanishing since \( T_{\text{Hall}}^{ij} V_{ij} = 0. \)

**IV. LINEAR RESPONSE**

In linear response theory the induced current \( \delta \mathcal{J} \) is linear in the source \( \delta \mathcal{A} \), i.e.,

\[ \delta \mathcal{J}^{\mu}(t, \mathbf{x}) = \int dx' \int_{t' < t} dt' \mathcal{K}^{\mu\nu}(t, x; t', x') \delta \mathcal{A}_\nu(t', x'), \quad (39) \]

where we introduced the response kernel \( \mathcal{K}^{\mu\nu}(t, x; t', x') \) with \( \mu, \nu = t, x, y \).

In a spacetime homogeneous system \( \mathcal{K}^{\mu\nu}(t, x; t', x') = \mathcal{K}^{\mu\nu}(t - t', x - x') \) and it is convenient to transform to Fourier space, where \( \delta \mathcal{J}^{\mu}(\omega, \mathbf{p}) = \mathcal{K}^{\mu\nu}(\omega, \mathbf{p}) \delta \mathcal{A}_\nu(\omega, \mathbf{p}). \)

Following [19–21], one can determine linear response of a superfluid in three steps:

- First, solve the linearized equation of motion of the Goldstone field in the presence of the external source \( \delta \mathcal{A} \).
- Second, substitute the solution \( \theta(\mathcal{A}) \) into the definition of the current \( \mathcal{J} \).
- Third, determine the induced current \( \delta \mathcal{J} \) as a function of the source \( \delta \mathcal{A} \).

The response to electromagnetic sources depends on the electric and magnetic fields, \( E_i \) and \( B \). The response to gravitational sources has similar contributions, that depend on a parity odd “electric” field \( E_{\omega} = \partial_t \omega_i - \partial_i \omega_t \) and the scalar curvature \( R \). In fact, the form of the covariant derivative (12) implies that the full response depends on the combinations

\[ E_i^{\text{tot}} = E_i + s E_{\omega i}, \quad B^{\text{tot}} = B + \frac{s}{2} R. \quad (40) \]

In this section we compute the linear response of the \( U(1)_N \) current and stress tensor to the electromagnetic and gravitational source respectively. The calculation is done in a homogeneous chiral superfluid ground state in flat space with no background \( U(1)_N \) potential \( A_\nu = 0. \)

**A. \( U(1)_N \) current response to electromagnetic source**

In the presence of the electromagnetic source \( A_\nu \), the linearized equation of motion for the phonon field \( \varphi \), defined in Eq. (5), takes the form of the relativistic wave equation

\[ \partial_t^2 \varphi - c_s^2 \Delta \varphi = -\partial_t \left( A_t + \frac{s}{2} B \right) + c_s^2 \partial_i A^i \quad (41) \]

which is solved in momentum space

\[ \varphi(\omega, \mathbf{p}) = \frac{-i \omega (A_t + \frac{s}{2} B) - ic_s^2 p_i A^i}{\omega^2 - c_s^2 \mathbf{p}^2}. \quad (42) \]

By substituting this solution into Eq. (18) we find to linear order in the source

\[ \delta \rho|_{\theta(A)} = \rho|_{\theta(A)} - \rho^{\text{GS}} = \rho^{\text{GS}} \left( \frac{ip_i E^i + \frac{s}{2} p_i B}{\omega^2 - c_s^2 \mathbf{p}^2} \right), \quad (43) \]

\[ J^i|_{\theta(A)} = \sigma(\omega, \mathbf{p}) E^i + i \left( 1 - \frac{s^2}{4 c_s^2} \right) \rho_L(\omega, \mathbf{p}) \varepsilon^{ij} p_j B 
+ \sigma_H(\omega, \mathbf{p}) \varepsilon^{ij} E_j \quad (44) \]
perturbation around the flat background. The linearized equation of motion for the phonon field is given by

\[ \partial_t^2 \varphi - c^2_s \Delta \varphi = -s \partial_t \omega_t + c^2_s \partial_t h/2 + sc^2_s \partial_t \omega^i, \]

(46)

where \( h \equiv \text{Tr} h_{ij} = h_{11} + h_{22} \). This can be solved in the Fourier space with the result

\[ \varphi = \frac{ic^2_s \omega h/2 - is \omega \omega_t - isc^2_s \omega \omega^i}{\omega^2 - c^2_s \mathbf{p}^2}. \]

(47)

First, we find

\[ v_i = ip_i \varphi + s \omega_t \]

\[ = s \sigma(\omega, \mathbf{p}) E_{\omega i} + is \frac{\sigma(\omega, \mathbf{p})}{\rho_{\text{GS}}} \frac{h}{2} + (s \rho_{\text{GS}}(\omega, \mathbf{p}) \mathbf{c}^{ij} R, \]

(48)

\[ \delta P|_{\theta(g)} \equiv P|_{\theta(g)} - P_{\text{GS}} = c^2_s \delta \rho|_{\theta(g)}, \]

(49)

where the parity-odd “electric” field \( E_{\omega i} \equiv -i(\omega \omega_t + p_i \omega_t) \). As explained in Appendix B, \( E_{\omega i} \) is a gradient of the vorticity of the displacement field. The variation of the pressure in Eq. (49) follows from the thermodynamic definition of the speed of sound \( c^2_s = \delta P/\delta \rho \). Now we substitute Eqs. (48), (49) into the stress tensor (35). To linear order in sources the variation of the pure contravariant stress tensor can be written in a matrix form

\[
\delta T|_{\theta(g)} \equiv T|_{\theta(g)} - T_{\text{GS}} = \left( \delta P|_{\theta(g)} - \frac{s^2}{4} \rho_{\text{GS}} R \sigma_0 + P_{\text{GS}} \delta g^{-1} \right. \\
\left. \eta_H \frac{h_{xx} - h_{yy}}{2} - \frac{c^2_s}{2} \frac{\sigma(\omega, \mathbf{p})}{\rho_{\text{GS}}} (p_x^2 - p_y^2) \right) h + i \frac{s}{\rho_{\text{GS}}} \frac{\sigma(\omega, \mathbf{p})}{\rho_{\text{GS}}} (p_x E_{\omega x} - p_y E_{\omega y}) - s \rho_{\text{GS}}(\omega, \mathbf{p}) p_x p_x R \right] \sigma_1 + \\
\eta_H \left[ i \omega h_{xy} + \frac{c^2_s}{2} \frac{\sigma(\omega, \mathbf{p})}{\rho_{\text{GS}}} p_x p_y h - i \frac{s}{\rho_{\text{GS}}} \frac{\sigma(\omega, \mathbf{p})}{\rho_{\text{GS}}} (p_x E_{\omega y} + p_y E_{\omega x}) + s \rho_{\text{GS}}(\omega, \mathbf{p}) (p_y^2 - p_x^2) R \right] \sigma_3,
\]

(50)

where \( \sigma_0 \) is the unity matrix and \( \sigma_i \) are the Pauli matrices. The first two terms in the square brackets of Eq. (50) are linear in \( s \) and break parity, we will discuss them in more detail below. The last two terms are proportional to \( s^2 \) and do not break parity. They are of higher order and might get corrections from next-to-next-to-leading order terms in the Lagrangian and thus are not predicted reliably by our theory.

Now we are ready to extract the dynamic Hall viscosity from the linear response calculation. Indeed, as argued in [22], the gravitational wave

\[ h_{ij}(t) = h_{ij} \exp(-i \omega t) \]

(51)
induces the following perturbation of the off-diagonal component of the contravariant stress tensor
\[
\delta T^{xy} = -P^{GS} h_{xy} + i \omega \eta(\omega) h_{xy} - i \omega \eta_H(\omega) (h_{xx} - h_{yy}),
\]
where \( \eta(\omega) \equiv \eta(\omega, p = 0) \) and \( \eta_H(\omega) \equiv \eta_H(\omega, p = 0) \). Direct comparison with Eq. (50) gives us
\[
\eta(\omega) = 0,
\]
i.e., there is no shear viscosity because at zero temperature superfluid does not dissipates energy. In addition we find
\[
\eta_H(\omega) = \eta_H = -\frac{s}{2} \rho^{GS}.
\]
It is natural that in the leading-order theory (3) the Hall viscosity does not depend on frequency.

C. Relation between Hall conductivity and viscosity

It was demonstrated in [9] that for Galilean-invariant quantum Hall states the static electromagnetic Hall response at small momenta \( p \) receives a contribution from the Hall viscosity. Subsequently, the relation between the Hall conductivity and stress response was generalized to other Galilean-invariant parity-violating systems [23]. In addition, it was shown in [23] that the relation can be extended to all frequencies. In particular, for a chiral superfluid one finds
\[
\eta_H(\omega) = \frac{\omega^2}{2} \frac{\partial^2}{\partial p_x^2} \sigma_H(\omega, p) \bigg|_{p=0}.
\]
Using Eqs. (45) and (54), we checked that this relation is satisfied within our leading-order effective theory.

D. Parity breaking terms and Kubo formula

The parity breaking contributions to the stress tensor due to gravitational sources can be grouped in the odd viscosity tensor, defined through the linear response formula
\[
T_{odd}^{ij} = -\eta_{odd}^{ijkl} \partial_i h_{kl}.
\]

The odd viscosity can be obtained from the Kubo formula involving the parity-odd part of the retarded two-point function of the stress tensor [23]. We define the frequency and momentum-dependent odd viscosity
\[
\eta_{odd}^{ijkl}(\omega, p) \equiv \frac{1}{i \omega^+} G_R^{ijkl}(\omega^+, p)_{odd},
\]
where \( \omega^+ \equiv \omega + i \epsilon \) with \( \epsilon \to 0 \). In fact, the total viscosity tensor contains an additional contact term inversely proportional to the compressibility [23], but it is parity-even and thus does not affect our discussion of the odd viscosity.

The retarded two-point function is
\[
G_R^{ijkl}(\omega, p) = i \int_0^\infty dt \int d^2 x e^{i \omega t - i p \cdot x} \langle [\tau^{ij}(t, x), \tau^{kl}(0, 0)] \rangle,
\]
where \( \tau^{ij} \) is the stress tensor in the absence of gravitational sources.

To leading order in the phonon fluctuation, the parity even and odd contributions to the stress tensor are
\[
\tau_{even}^{ij} = P^{GS} \delta^{ij} - \rho^{GS} \delta^{ij} \partial_t \varphi,
\]
\[
\tau_{odd}^{ij} = -\eta_H (\epsilon^k \delta^{il} + \epsilon^l \delta^{ik}) \partial_k \partial_l \varphi.
\]
The parity-odd contributions to the two-point function come from cross terms of even and odd contributions. Therefore, to leading order
\[
\langle \tau_{even}^{ij} \rangle = \rho^{GS} \eta_H \delta^{ij} \left[ \sigma^1 (\partial_x^2 - \partial_y^2) + 2 \sigma^3 \partial_y \partial_x \right] \partial_t \langle \varphi \varphi \rangle,
\]
\[
\langle \tau_{odd}^{ij} \rangle = \rho^{GS} \eta_H \sigma^0 (\epsilon^k \delta^{il} + \epsilon^l \delta^{ik}) \partial_k \partial_l \langle \varphi \varphi \rangle,
\]
where we are using the same matrix notation as before for the first entry of the stress tensor in the two-point function. In addition, we used \( \langle \varphi \rangle = 0 \).

Using the Fourier transform of the two-point function of the phonon field
\[
G^{\nu \nu}(\omega, p) = \frac{1}{\rho^{GS}} \frac{1}{\omega^2} \frac{1}{\omega^2 - p^2},
\]
one can recover the results we have derived before for the linear response of the stress tensor to gravitational sources. Specifically, the odd-even contribution produce the terms proportional to the trace of the metric perturbation in the brackets in Eq. (50). These terms are especially interesting because they are related to the fact that the superfluid is compressible. The metric perturbation changes the volume form by a term proportional to its trace \( \delta \sqrt{g} = \hbar/2 \). This excites phonons that produce the stress we have computed above. The even-odd terms give contributions to the variation of the pressure.

The regular Hall viscosity term on the other hand does not involve the phonon propagator and therefore it is a kinematic response that will appear in the correlation function as a contact term. Its origin is similar to the diamagnetic current, which is obtained from a term in the action quadratic in sources.

V. VORTEX SOLUTION

Here we consider a vortex in a chiral superfluid in flat space placed at the origin. We treat this problem in
polar coordinates \((r, \phi)\). Far away from the core, due to the single valuedness of the condensate wave function

\[
v_r = 0, \quad v_\phi = \frac{n}{2} n \in \mathbb{Z}.
\] (62)

Here \(v_r\) and \(v_\phi\) are the coefficients in the decomposition \(v = v_r \mathbf{e}_r + v_\phi \mathbf{e}_\phi\) with \(\mathbf{e}_r\) and \(\mathbf{e}_\phi\) denoting the unit vectors in the radial and angular directions.

We will determine the asymptotic behavior of the superfluid density \(\rho(r)\) as \(r \to \infty\). From Eq. (27) the static Euler equation reads

\[
\rho v^i \partial_j v^i = -\partial^2 p + \eta_H \epsilon^{ij} \Delta v_j - \partial_j (v^i J^j_{\text{edge}} + v^i J^j_{\text{edge}})
\] (63)

which after the projection onto the radial direction becomes

\[
\frac{\rho}{r} v^2 = \epsilon_s \partial_r \rho - \eta_H \left[ \frac{1}{r} \partial_r v_\phi - \frac{1}{r^2} v_\phi \right] - f_{\text{edge}},
\] (64)

where \(f_{\text{edge}} \equiv -\delta_{ik} \epsilon_k \partial_j (v^i J^j_{\text{edge}} + v^i J^j_{\text{edge}}) = -\frac{sn}{2r^2} \partial_r \rho\). For the velocity given by Eq. (62) we have \(\nabla \cdot \mathbf{v} = 0\) and thus \(\Delta \mathbf{v} = 0\). This simplifies the previous equation to the form

\[
\rho \frac{n^2}{4r^3} = \left[ \epsilon_s + \frac{sn}{2} \right] \partial_r \rho.
\] (65)

To leading order in the large-distance expansion

\[
\left[ \epsilon_s + \frac{sn}{2} \right] \rightarrow \epsilon_s \equiv \left. \frac{\partial P}{\partial \rho} \right|_{r=\infty}
\] (66)

and the differential equation (65) simplifies to

\[
\frac{d \rho}{\rho} = \frac{n^2 dr}{4\epsilon_s \rho^3}.
\] (67)

It is easily integrated giving the density profile

\[
\rho(r) = \rho_\infty \left[ 1 - \frac{n^2}{8\epsilon_s \rho^2} + O(r^{-4}) \right],
\] (68)

where \(\rho_\infty\) is the asymptotic value of the superfluid density away from the vortex core. Since the chirality parameter \(s\) does not appear in Eq. (67), the leading order tail of the density profile is invariant under \(n \to -n\).

Chirality effects arise first at next-to-leading order in the large-distance expansion. By using

\[
c_2^2 = \left. \frac{\partial P}{\partial \rho} \right|_{r=\infty} = \epsilon_s + \frac{sn}{2} \left( r \rightarrow \infty \right) \frac{1}{2} \partial_r \rho
\] (69)

together with Eq. (68), we find that to next-to-leading order we can replace

\[
\left[ \epsilon_s + \frac{sn}{2} \right] \rightarrow \epsilon_s + \left( \frac{sn}{2} - \frac{3n^2}{8\epsilon_s} \right) \left( r \rightarrow \infty \right) \frac{1}{2} \partial_r \rho
\] (70)

in Eq. (65). The solution of Eq. (65) now gives the superfluid density profile up to next-to-leading order

\[
\rho(r) = \rho_\infty \left[ 1 - \frac{n^2}{8\epsilon_s \rho^2} + \frac{n^4}{128\epsilon_s^3 \rho^4} \left( r \rightarrow \infty \right) \right] + \frac{sn^3}{32\epsilon_s^3 \rho^4} + O(r^{-6})
\] (71)

Since in the chiral superfluid time reversal and parity are spontaneously broken, the density profile is not invariant under \(n \to -n\). The leading parity-violating correction appears first in the \(1/r^4\) tail. The difference between the densities of a vortex (with \(n = 1\)) and an antivortex (with \(n = -1\)) is asymptotically given by

\[
\Delta \rho = \frac{8sn}{16\epsilon_s^3 \rho^4} + O(r^{-6}).
\] (72)

VI. CONCLUSION

In this paper we constructed the leading-order low-energy and long-wavelength effective hydrodynamic theory of the simplest chiral fermionic superfluid in two spatial dimensions. Due to chirality, the effective theory breaks time reversal and parity and thus naturally gives rise to the edge particle current and Hall viscosity. In agreement with [23], we found a relation between the Hall conductivity and Hall viscosity response functions. As an application of the formalism, we constructed a quantum vortex solution and discovered that the leading chirality effect appears first in the \(1/r^4\) tail of the density depletion. Our predictions might be tested in experiments with spin-polarized two-dimensional ultracold fermions.

In the future it would be interesting to extend our theory to higher orders in the derivative expansion, where the nonrelativistic general coordinate invariance might put many more additional constraints compared to those imposed by considering Galilean invariance alone. Generalization of the theory to chiral superfluids with spin degrees of freedom might prove useful for better understanding of thin films of \(^3\)He.

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Appendix A: Alternative description of superfluid

We postulate that $v^\mu = (1, v^i)$ transforms like a vector under spatial coordinate transformations, i.e.,

$$\delta v^\mu = -\xi^k \partial_k v^\mu + v^\lambda \partial_\lambda \xi^\mu, \quad (A1)$$

where $\xi^i = 0$. This means

$$\delta v^i = -\xi^k \partial_k v^i + v^k \partial_k \xi^i + \xi^i,$n

$$\delta v_i \equiv g_{ij} v^j = -\xi^k \partial_k v_i - v_k \partial_i \xi^k + g_{ik} \xi^k. \quad (A2)$$

If such $v^i$ exists, then one can define the improved gauge potentials

$$\tilde{A}_i \equiv A_0 + \frac{1}{2} g_{ij} v^j, \quad (A3)$$

$$\tilde{A}_i \equiv A_i - g_{ij} v^j, \quad (A4)$$

so that $\tilde{A}_\mu$ transforms like a one-form

$$\delta \tilde{A}_\mu = -\xi^k \partial_k \tilde{A}_\mu - \tilde{A}_\nu \partial_\nu \xi^k. \quad (A5)$$

In this formalism the action of a conventional s-wave superfluid is

$$S[\theta, v^i] = \int dt dx \sqrt{g} \left[ \rho^\mu \left( \partial_\mu \tilde{A}_\nu - \tilde{A}_\nu \right) - \epsilon(\rho) \right], \quad (A6)$$

where $\epsilon(\rho)$ is the density of the internal energy that is not associated with the macroscopic motion of the superfluid. After expanding $\tilde{A}_\mu$, this action becomes

$$S = \int dt dx \sqrt{g} \left[ \rho \partial_t \theta + \rho v^i \partial_i \theta + \frac{1}{2} \rho g_{ij} v^i v^j - \epsilon(\rho) \right]. \quad (A7)$$

By integrating out $v^i$, we find that

$$v^i = -g^{ij} \partial_j \theta, \quad (A8)$$

i.e., $v^i$ is the superfluid velocity. Moreover, by integrating the theory $(A6)$ over $v^i$ and $\rho$ we reproduce the effective theory $(3)$. To describe chiral superfluids in this formalism we need to use the improved connection

$$\omega_i \equiv \frac{1}{2} \left( \epsilon^{ab} \epsilon^{ij} \partial_0 v^b + \epsilon^{ij} \partial_j v^i \right), \quad (A9)$$

$$\omega_i \equiv \frac{1}{2} \epsilon^{ab} \epsilon^{ij} \nabla^a e^{ij}. \quad (A10)$$

It can be checked that $\omega_\mu$ transforms like a one-form. The action of a chiral superfluid becomes

$$S[\theta, v^i] = \int dt dx \sqrt{g} \left[ \rho v^\mu \left( \partial_\mu \theta - \tilde{A}_\mu - s \omega_\mu - \epsilon(\rho) \right) \right]. \quad (A11)$$

Note that in this formalism the $U(1)_N$ current is pure convective

$$J^\mu \equiv -\frac{1}{\sqrt{g}} \delta S \delta \tilde{A}_\mu = \rho v^\mu. \quad (A12)$$

Appendix B: Interpretation of $E_{\omega,i}$

In the linearized approximation around flat background $e^a_i = \frac{1}{2} h_i^a$, and one can check that the components of the connection $(8)$ are

$$\omega_i \sim O(h^2), \quad \omega_i = -\frac{1}{2} \epsilon^{jk} \partial_j h_{ki}. \quad (B1)$$

The spatial metric can be interpreted as the stress source produced by a deformation $x^i \rightarrow x^i + \xi^i$, where $\xi^i$ is the displacement vector. In this case $h_{ij} = -\partial_i \xi_j - \partial_j \xi_i$, which is fixed by the transformation law of the metric under nonrelativistic diffeomorphisms. The spatial part of the connection $(8)$ now becomes

$$\omega_i = \frac{1}{2} \partial_i (\epsilon^{jk} \partial_j \xi_k). \quad (B2)$$

This can be viewed as the gradient of the torsion of the displacement field. To linear order, the parity-odd “electric” field is then

$$E_{\omega,i} \simeq \partial_i \omega_i = \partial_i \left( \frac{1}{2} \epsilon^{jk} \partial_j \partial_k \xi_k \right) \equiv \partial_i \Omega, \quad (B3)$$

where $\Omega$ is the vorticity of the displacement field.

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[1] D. Vollhardt and P. Wölfle, The Superfluid Phases of Helium 3, CRC Press (1990).
[2] G. E. Volovik, The Universe in a Helium Droplet, OUP Oxford (2009).
[3] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[4] J. Alicea, Reports on Progress in Physics 75, 076501 (2012).
[5] A. Kitaev, Ann. Phys. 303, 2 (2003).
[6] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[7] D. Son and M. Wingate, Ann. Phys. 321, 197 (2006).
[8] D. T. Son, Phys. Rev. Lett. 98, 020604 (2007).
[9] C. Hoyos and D. T. Son, Phys. Rev. Lett. 108, 066805 (2012).
[10] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[11] N. D. Mermin and P. Muzikar, Phys. Rev. B 21, 980 (1980).
[12] Y.-H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, Int. J. Mod. Phys. B 3, 1001 (1989).
[13] M. Greiter, F. Wilczek, and E. Witten, Mod. Phys. Lett. B 3, 903 (1989).
[14] B. I. Halperin, J. March-Russell, and F. Wilczek, Phys. Rev. B 40, 8726 (1989).
[15] J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. 75, 697 (1995).
[16] J. E. Avron, arXiv:physics/9712050.
[17] N. Read, Phys. Rev. B 79, 045308 (2009).
[18] N. Read and E. H. Rezayi, Phys. Rev. B 84, 085316 (2011).
[19] V. Ambegaokar and L. Kadanoff, Il Nuovo Cim. 22, 914 (1961).
[20] P. I. Arseev, S. O. Loiko, and N. K. Fedorov, Phys. Usp. 49, 1 (2006).
[21] R. Roy and C. Kallin, Phys. Rev. B 77, 174513 (2008).
[22] O. Saremi and D. Son, JHEP 2012, 1 (2012).
[23] B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B 86, 245309 (2012).