Harnessing Correlations in Distributed Erasure Coded Key-Value Stores

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Abstract

Motivated by applications of distributed storage systems to cloud-based key-value stores, the multi-version coding problem has been recently formulated to efficiently store frequently updated data in asynchronous decentralized storage systems. Inspired by consistency requirements in distributed systems, the main goal in multi-version coding is to ensure that the latest possible version of the data is decodable, even if the data updates have not reached some servers in the system. In this paper, we study the storage cost of ensuring consistency for the case where the data versions are correlated, in contrast to previous work where data versions were treated as being independent. We provide multi-version code constructions that show that the storage cost can be significantly smaller than the previous constructions depending on the degree of correlation between the different versions of the data. Our achievability results are based on Reed-Solomon codes and random binning. Through an information-theoretic converse, we show that our multi-version codes are nearly-optimal in certain regimes.

I. INTRODUCTION

Distributed key-value stores are an important part of modern cloud computing infrastructure. Key-value stores are commonly used by several applications including reservation systems, financial transactions, collaborative text editing, and multi-player gaming. Owing to their utility, there are numerous commercial and open-source cloud-based key-value store implementations such as Amazon Dynamo [2], Google Spanner [3] and Apache Cassandra [4].

Distributed data storage services including key-value stores commonly build fault tolerance and availability into the system by replicating the data across multiple nodes. Unlike archival data storage services, in key-value stores, the data stored is updated frequently, and the time scales of data updates and access are often comparable to the time scales of dispersing the data to the servers (See [2]). In fact, distributed key-value stores are commonly asynchronous, that is, the time scale of data propagation is unpredictable and different nodes can receive the updates at different points of time. In such settings, ensuring that a client gets the latest, most updated,
version of the data requires careful and delicate protocol design as the data updates may not reach all servers. The notion that the latest version of the data must be accessible to the users despite the frequent updates is known as consistency in distributed systems [2], [5], [6].

Modern key-value stores depend on high-speed memory to provide fast read and write operations. High-speed memory is expensive as compared with hard drives, hence the goal of efficiently using memory has motivated significant recent interest in erasure coding based key-value stores. In absence of consistency requirements, reference [7] showed that erasure coding based “in-memory” storage systems have significantly improved latency as compared to replication-based counterparts. Systems research related to erasure coding based consistent data stores also has received significant recent interest (See, e.g., [8] and also [2], which uses erasure coding for Microsoft’s data centers even for non-archival consistent data storage services). Another classical technique used to improve memory efficiency in several data storage systems, where it is desired to store older versions of the data since users may access them, is the delta coding [10] technique. These stores rely on the idea of compressing differences between subsequent versions, which leads to improved storage costs when the data versions are correlated.

The main contribution of our paper is developing an information-theoretic approach that combines the ideas of erasure coding and delta coding to exploit correlations between subsequent updates of the value in key-value stores. Our approach leads to non-trivial code constructions that enable significant memory savings as compared to replication-based schemes, as well as erasure coding based approaches that do not exploit correlations. We show that our codes are approximately optimal for certain regimens through an information-theoretic converse.

We next provide an overview of the salient aspects of consistent data storage algorithms, and then discuss the multi-version coding framework, which is an information-theoretic framework for distributed storage codes tailor-made for consistent key-value stores.

A. Overview of Key-Value Stores

The design principles of modern key-value stores are rooted in the distributed computing-theoretic abstraction known as shared memory emulation [5]. The goal of the read-write shared memory emulation problem is to implement a read-write variable over a distributed system. While there has been much recent interest in archival data storage systems in information theory and coding theory, e.g. [11], [12], the shared memory emulation set up differs from these recent studies in the following aspects:

1) Asynchrony: a new version of the data may not arrive at all servers simultaneously.

2) Decentralized nature: there is no single encoder that is aware of all versions of the data simultaneously, and a server is not aware of which versions are received by other servers.

Shared memory emulation algorithms use quorum-based techniques to deal with the asynchrony. Specifically, in a system of \( n \) servers that tolerates \( f \) server failures, a write operation sends a request to all servers and waits for the acknowledgments from any \( c_W \leq n - f \) servers for the operation to be considered complete. Similarly, a read operation sends a request to all
servers and waits for the response of any \( c_R \leq n - f \) servers to get the data. This strategy ensures that, for every complete version, there are at least \( c_W + c_R - n \) servers that received that version and responded to the read request. Shared memory emulation algorithms require that the \textit{latest} complete version, or a later version, must be recoverable by the read operation. In a replication-based protocol, where servers store an uncoded copy of the latest version that they receive, selecting \( c_W \) and \( c_R \) such that \( c_W + c_R > n \) ensures that the reads can return the value of the latest complete write operation (See [2], [5], [13] for example). The requirement that a read must return the value of the \textit{latest} complete write operation, or a later version, is referred to as consistency\(^2\) in distributed systems literature [5].

Since for every complete write operation, there are \( c := c_W + c_R - n \) servers that store the value of that write operation and respond to a given read operation, it may seem natural to use a maximum distance separable (MDS) code of dimension \( c \) to obtain storage cost savings over replication-based algorithms. However, the use of erasure coding in asynchronous distributed systems where consistency is important leads to interesting algorithmic and coding challenges. This is because, when erasure coding is used, no single server stores the data in its entirety; for instance, if an MDS code of dimension \( c \) is used, each server only stores a fraction of \( 1/c \) of the entire value. Therefore, for a read operation to get some version of the data, at least \( c \) servers must send the codeword symbols corresponding to this version. As a consequence, when a write operation updates the data, a server cannot delete the symbol corresponding to the old version before symbols corresponding to a new version has propagated to a sufficient number of servers. That is, servers cannot simply store the latest version they receive; they have to store older versions at least until a sufficient number of codeword symbols corresponding to the newer version propagate to the other servers (See Fig. 3 in [14]). In fact, this phenomenon is reflected in several erasure coding based algorithms in distributed systems literature (See [15], [16]).

Given that storing multiple versions of the data is inevitable in consistent erasure coding based key-value stores, an important opportunity to improve memory efficiency of such stores is to exploit correlations between the various versions; this opportunity is the main motivation of our paper. We conduct our study through the \textit{multi-version coding} framework [14], [17].

B. Multi-Version Coding

The \textit{multi-version coding} problem abstracts out algorithmic details of shared memory emulation while retaining the essence of consistent storage systems. Specifically, the multi-version coding problem [14] considers a decentralized storage system with \( n \) servers where the objective is storing a message (read-write variable) of length \( K \) bits with \( \nu \) independent versions\(^3\). The

\(^2\)More specifically, our decoding requirement is inspired by the consistency criterion known as \textit{atomicity}, or \textit{linearizability}.

\(^3\)We study the case where each version is \( K \) bits long; although certain applications such as collaborative editing might benefit from dynamic allocation application programming interfaces (APIs), several important applications use a fixed size for various versions of the value. Furthermore, popular key-value stores expose a fixed-sized value to the client and do not expose “malloc”-type dynamic allocation APIs.
versions are totally ordered; messages with higher ordering are referred to as later versions, and lower ordering as earlier versions. Each server receives an arbitrary subset of the $\nu$ versions, and encodes them; note that because of asynchrony, not every server has every version. Because of the decentralized nature, a server is unaware of which versions are available at other servers. Inspired by the structure of quorum-based protocols, we refer to any version that has reached at least $c_W$ servers as a complete version. A decoder connects to any $c_R$ servers, and must recover the latest complete version, or a later version.

Reference [14] showed that there is an inevitable price in terms of storage cost for maintaining consistency in asynchronous decentralized storage systems. In multi-version coding, for any complete version, for any decoder, there are at least $c$ servers that have received that version and have responded to the decoder. In the classical erasure coding model, where $\nu = 1$, the Singleton bound dictates that the storage cost per server is at least $K/c$. However for $\nu > 1$, a server cannot simply store the codeword symbol corresponding to one version, since other servers may not have received it. Reference [14] studied the case where the versions are independent and showed that the storage cost per server is at least $\frac{\nu}{c+\nu-1} K - o(K)$. Since, for $\nu < c$, we have $\frac{\nu}{c+\nu-1} \geq \frac{\nu}{2c}$, and since the per-server cost of storing each version is $K/c$, we may interpret the result as follows: when the versions are independent, to compensate for the asynchrony and still maintain consistency, a server has to store an amount of data that is, from a cost perspective, tantamount to at least $\nu/2$ versions, each stored using an MDS code of dimension $c$.

The study of [14] focuses on coding-theoretic aspects and does not explicitly connect the coding theoretic solutions to the algorithmic aspects. However, the insights obtained from this study have been incorporated into distributed algorithms for the full-fledged shared memory emulation model in references [18], [19]. In particular, reference [18] developed a lower bound on the storage cost of any read-write memory emulation algorithm by creating a worst-case execution mimicking the converse of [14]. Reference [19] developed a distributed read-write memory algorithm based on multi-version coding; we believe that merging the coding-theoretic ideas in our paper and the algorithmic insights of [19] is an interesting area of future work.

C. Contributions

In this paper, we extend the scope of the multi-version coding problem to the case where the different versions are correlated. Specifically, we consider a decentralized storage system with $n$ servers storing $\nu$ possibly correlated versions of a message. We assume that each message version is $K$ bits long, and model the correlation between successive versions in terms of the bit-strings that represent them. Given a version, we assume that the subsequent version is uniformly distributed in the Hamming ball of radius $\delta_K K$, centered around that given version and hence this version can be represented using $\log Vol(\delta K, K)$ bits, where $Vol(\delta K, K)$ is the volume of the Hamming Ball of radius $\delta_K K$. We derive three main results for this system:

**Contribution 1:** First we study the case where $\delta_K$ is not known a priori and propose a simple, effective, achievable scheme based on Reed-Solomon codes with a per-server storage cost of
\( \frac{K}{c} + (\nu - 1)\delta_K K (\log K + o(\log K)) \) bits. From a cost viewpoint, this scheme obtains the \( 1/c \) erasure coding gain for the first version and stores every subsequent version via delta coding with a cost of \( \delta_K K (\log K + o(\log K)) \) bits. Note that this scheme is unable to simultaneously obtain the compression gains of both erasure coding and delta coding.

**Contribution 2:** We then study the case where \( \delta_K \) is known a priori and derive a novel achievable scheme based on random binning. Our result significantly improves upon the result of [14] when the correlation is significant. Specifically, our proposed scheme has a per-server storage cost of \( \frac{K}{c} + \frac{\nu - 1}{c} \log Vol(\delta_K K, K) + o(\log K) \) bits. From a cost viewpoint, this scheme is tantamount to storing one version using erasure coding with a cost of \( K/c \) and performing delta coding and erasure coding for the subsequent versions leading to a cost of \( \frac{\log Vol(\delta_K K)}{c} \) bits per version. This scheme outperforms our first contribution in terms of the storage cost because it simultaneously obtains the gains of both erasure coding and delta coding for subsequent versions. We also show the existence of linear codes that obtain this storage cost.

A cost of \( \frac{K}{c} + \frac{\nu - 1}{c} \log Vol(\delta_K K, K) + o(\log K) \) bits is readily achievable in a centralized synchronous setting, where every server receives all the versions, and each server is aware that the other servers have indeed received all the versions. In such a setting, each server can store a fraction of \( 1/c \) of the first version it receives using an MDS code of dimension \( c \). For a new version, each server can calculate the difference between this version and the old version, and then store a fraction of \( 1/c \) of the compressed difference using an erasure code of dimension \( c \). However, this simple scheme would fail in our setting because of the decentralized asynchronous nature. For instance, a server which receives versions 1 and 3 may need to compress version 3 with respect to version 1 and then erasure code it, but a different server that receives only versions 2 and 3 may have to compute the increment of version 3 with respect to version 2 and then encode it; from a decoder’s viewpoint, the erasure coded fractions stored at the two servers would not be compatible. Furthermore, the decentralized nature implies that the server that receives versions 1 and 3 must store some data that would enable successful decoding no matter what versions are received by the other servers. Handling the decentralized and asynchronous nature requires a non-trivial achievability proof and is our main technical contribution.

**Contribution 3:** We extend the converse result of [14] to the case of correlated versions and show our random binning scheme is nearly-optimal for \( \nu < c \).

**D. Related Work**

The idea of exploiting the correlation between the different versions to efficiently update, store or exchange data has a rich history of study in network information theory starting from the classical work of the Slepian-Wolf problem [20] for compressing correlated distributed sources. Linear code constructions that approach the Slepian-Wolf limits have been proposed in [21]–[24] and references therein. The problem of encoding incremental updates efficiently is the motivation of the delta encoding/compression techniques used commonly in data storage. References [25], [26], and references therein refine the notion of delta compression by modeling the data updates.
using the edit distance; in particular, these references develop coding schemes that synchronize a small number of edits between a client and a server efficiently. While we note that the edit distance is relevant to specific key-value store applications such as collaborative text editing, the focus of our paper is on the classical Hamming metric that is used more widely in coding theory for the sake of fundamental understanding and other general applications. The Hamming metric may also have practical applications as the data can be viewed as a vector (or a table, as in Apache Cassandra [4]), and the writes commonly update only a few entries of the vector.

Exploiting correlations between data updates to improve efficiency in distributed storage and caching settings, where multiple servers store codeword symbols corresponding to the data, has been of significant interest [27]–[33]. References [27], [28] devise coding schemes that use as input, the old and the new version of the data, and output a code that can be used to store both versions of the data efficiently in a distributed storage system. References [29], [30] study capacity-achieving update-efficient codes for binary symmetric and erasure channels, where a small change in the message leads to a codeword which is close to the original codeword in Hamming distance. Hence, the constructed codes lead to efficient updates of the data.

Reference [31] studied the communication cost of updating a “stale” server that did not get an updated message, by downloading data from already updated servers in a distributed setting. The reference proposed code constructions and tight bounds for this problem. We note that the problem of [31] has some modeling elements that are common with our approach, specifically, there is a limited degree of asynchrony - a single update does not reach a particular server. A side information problem is presented in [32], where the goal is to send an updated version to a remote entity that has as side information an arbitrary linear transform of an old version. The reference shows that the optimal encoding function is related to a maximally recoverable subcode of the linear transform associated with the side information. The problem of [32] is tangentially related to some of the solutions of our paper, since we aim to use codeword symbols of some versions as side information to store other versions of the data.

Although our problem formulation and solutions have common ingredients with previous works, our setting differs from all previous works because it captures the asynchrony, decentralized nature, and the consistency requirements of the shared memory emulation problem. An important outcome of our study is that correlation can be used to reduce storage costs in distributed systems, despite the asynchrony, decentralized nature, and the consistency requirements.

Organization of the paper

The rest of this paper is organized as follows. Section II presents the multi-version coding problem, background, the results of [14], and the main results of this paper. In Section III we provide our code constructions. Section IV provides a lower bound on the per-server storage cost. Finally, conclusions and future work are discussed in Section V.
II. System Model and Background of Multi-version Codes

We start with some notation. We use boldface for vectors. In the \( n \)-dimensional Euclidean space, the standard basis column vectors are denoted by \( \{e_1, e_2, \cdots, e_n\} \). We denote the Hamming weight of a vector \( \mathbf{x} \) by \( w_H(\mathbf{x}) \) and denote the Hamming distance between any two vectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) by \( d_H(\mathbf{x}_1, \mathbf{x}_2) \). For a positive integer \( i \), we denote by \([i]\) the set \( \{1, 2, \cdots, i\} \). For any set of ordered indices \( S = \{s_1, s_2, \cdots, s_{|S|}\} \subseteq \mathbb{Z} \) where \( s_1 < s_2 < \cdots < s_{|S|} \), and for any ensemble of variables \( \{X_i : i \in S\} \), the tuple \( (X_{s_1}, X_{s_2}, \cdots, X_{s_{|S|}}) \) is denoted by \( X_S \). We use \( \text{BEC}(p) \) to denote a binary erasure channel with erasure probability \( p \). We use \( \log(\cdot) \) to denote the logarithm to the base 2 and \( H(\cdot) \) to denote the binary entropy function. We use the notation \( [2^K] \) to denote the set of \( K \)-length binary strings. A code of length \( n \) and dimension \( k \) over alphabet \( \mathcal{A} \) consists of an injective mapping \( C : \mathcal{A}^k \to \mathcal{A}^n \). When \( \mathcal{A} \) is a finite field and the mapping \( C \) is linear, then the code is referred to as a linear code. We refer to a linear code \( C \) of length \( n \) and dimension \( k \) as an \((n, k)\) code. An \((n, k)\) linear code is called MDS if its minimum distance is \( n - k + 1 \), or equivalently, the mapping projected to any \( k \) co-ordinates is invertible.

A. Multi-version Codes (MVCs)

We now present a variant of the definition of the multi-version code \([14]\), where we model the correlations. We consider a distributed storage system with \( n \) servers storing \( \nu \) possibly correlated versions of the message where \( \mathbf{W}_i \in [2^K] \) is the \( i \)-th version, \( i \in [\nu] \), and \( K \) is the message length in bits. The versions are assumed to be totally ordered, i.e., if \( i > j \), \( \mathbf{W}_i \) is interpreted as a later version with respect to \( \mathbf{W}_j \). We assume that \( \mathbf{W}_1 \to \mathbf{W}_2 \to \cdots \to \mathbf{W}_\nu \) form a Markov chain. \( \mathbf{W}_1 \) is uniformly distributed over the set of \( K \) length binary vectors. Given \( \mathbf{W}_m, \mathbf{W}_{m+1} \) is uniformly distributed in a Hamming ball of radius \( \delta_K K \), \( B(\mathbf{W}_m, \delta_K K) \), where

\[
B(\mathbf{W}_m, \delta_K K) = \{ \mathbf{W} : d_H(\mathbf{W}, \mathbf{W}_m) \leq \delta_K K \}.
\]

We denote the volume of the ball by \( \text{Vol}(\delta_K K, K) = |B(\mathbf{W}_m, \delta_K K)| = \sum_{j=0}^{\delta_K K} \binom{K}{j} \). The \( i \)-th server receives an arbitrary subset of versions \( S(i) \subseteq [\nu] \) that denotes the state of that server. We denote the system state by \( S = \{S(1), S(2), \cdots, S(n)\} \in \mathcal{P}([\nu])^n \), where \( \mathcal{P}([\nu]) \) denotes the power set of \([\nu]\). For the \( i \)-th server with state \( S = S(i) = \{s_1, s_2, \cdots, s_{|S|}\} \), where \( s_1 < s_2 < \cdots < s_{|S|} \), the server stores a codeword symbol generated by the encoding function \( \varphi_S^{(i)} \) that takes an input \( \mathbf{W}_S \) and outputs an element in \([q]\). In state \( S \in \mathcal{P}([\nu])^n \), we denote the set of servers that have received \( \mathbf{W}_u \) by \( \mathcal{A}_S(u) = \{j \in [n] : \mathbf{W}_u \in S(j)\} \). In state \( S \in \mathcal{P}([\nu])^n \), a version \( u \in [\nu] \) is termed complete if \( |\mathcal{A}_S(u)| \geq c_W \). The set of complete versions in state \( S \in [\nu] \) is given by \( \mathcal{C}_S = \{u \in [\nu] : |\mathcal{A}_S(u)| \geq c_W \} \) and the latest among them is denoted by \( L_S = \max \mathcal{C}_S \).

The goal of the multi-version coding problem is to devise encoders such that for every decoder that connects to any arbitrary set of \( c_R \) servers, the latest complete version or a later version is decodable with probability of error that is at most \( \epsilon \). We express this formally next.

**Multi-version coding problem:** An \( \epsilon \)-error \((n, c_W, c_R, \nu, 2^K, q, \delta_K)\) multi-version code (MVC) consists of the following for \( \epsilon > 0 \)
• encoding functions $\varphi^{(i)}_S : [2^K]^{\lvert S \rvert} \rightarrow [q]$, for every $i \in [n]$ and every state $S \subseteq [\nu]$
• decoding functions $\psi^{(T)}_S : [q]^c_R \rightarrow [2^K] \cup \{\text{NULL}\}$

that satisfy the following

$$\Pr \left[ \psi^{(T)}_S \left( \varphi^{(t_1)}_S, \ldots, \varphi^{(t_{c_R})}_S \right) = W_m \text{ for some } m \geq L_S, \text{ if } C_S \neq \emptyset \right] \geq 1 - \epsilon,$$

for every possible system state $S \in \mathcal{P}(\lbrack \nu \rbrack)^n$ and every set of servers $T = \{t_1, t_2, \ldots, t_{c_R}\} \subseteq [n]$, $t_1 < t_2 < \cdots < t_{c_R}$.

The objective of the multi-version coding is to design encoding functions that minimize the per-server storage cost that we define next.

**Definition 1** (Storage Cost of a Multi-version Code). The storage cost of an $\epsilon$-error $(n, c_W, c_R, \nu, 2^K, q, \delta_K)$ MVC is equal to $\log q$ bits.

Reference [14] studied 0-error MVCs with independent versions, that is, the special case of $\epsilon = 0, \delta_K = 1$. We next present an alternative decoding requirement that is shown in [14] to be equivalent to the multi-version coding problem defined above. For any set of servers $T \subseteq [n]$, note that $\max \cap_{i \in T} S(i)$ denotes the latest common version among these servers. The alternate decoding requirement, which we refer to multi-version coding problem with Decoding Requirement A, replaces the two parameters $c_W, c_R$ by one parameter $c$. The decoding requirement requires that the decoder connects to any $c$ servers and decodes the latest common version amongst those $c$ servers, or a later version. We state this formally next.

**Definition 2** ($\epsilon$-error $(n, c, \nu, 2^K, q, \delta_K)$ Multi-version code (MVC) with Decoding Requirement A). An $\epsilon$-error $(n, c, \nu, 2^K, q, \delta_K)$ multi-version code (MVC) consists of the following for $\epsilon > 0$

• encoding functions $\varphi^{(i)}_S : [2^K]^{\lvert S \rvert} \rightarrow [q]$, for every $i \in [n]$ and every state $S \subseteq [\nu]$
• decoding functions $\psi^{(T)}_S : [q]^c \rightarrow [2^K] \cup \{\text{NULL}\}$

that satisfy the following

$$\Pr \left[ \psi^{(T)}_S \left( \varphi^{(t_1)}_S, \ldots, \varphi^{(t_c)}_S \right) = W_m, \text{ for some } m \geq \max \cap_{i \in T} S(i), \text{ if } \cap_{i \in T} S(i) \neq \emptyset \right] \geq 1 - \epsilon,$$

(2)

for every possible system state $S \in \mathcal{P}(\lbrack \nu \rbrack)^n$ and every set of servers $T = \{t_1, t_2, \ldots, t_c\} \subseteq [n]$, $t_1 < t_2 < \cdots < t_c$.

In this paper, we present our achievability results for decoding requirement A. The following lemma establishes the connection between the two decoding requirements.

**Lemma 1.** Consider any three positive integers $n, c_W, c_R, c$ such that $c = c_W + c_R - n$. An $\epsilon$-error $(n, c, \nu, 2^K, q, \delta_K)$ MVC with decoding requirement A exists if and only if an $\epsilon$-error $(n, c_W, c_R, \nu, 2^K, q, \delta_K)$ MVC exists.

The lemma is discussed and proved in [14].
B. Background - Replication and Simple Erasure Coding

Replication and simple MDS codes provide two natural MVC constructions. Suppose that the state of the $i$-th server is $S = \{s_1, s_2, \ldots, s_{|S|}\}$, where $s_1 < s_2 < \ldots < s_{|S|}$.

- **Replication-based MVCs**: In this scheme, each server only stores the latest version it receives. The encoding function is $\varphi_S^{(i)}(W_s) = W_{s_{|S|}}$. Therefore, the storage cost is $K$.

- **Simple MDS codes based MVC (MDS-MVC)**: In this scheme, an $(n, c)$ MDS code is used to encode each version separately. Specifically, suppose that $C: [2^K] \rightarrow [2^{K/c}]$ is an $(n, c)$ MDS code over alphabet $[2^K/c]$, and denote the $i$-th co-ordinate of the output of $C$ by $C^{(i)}: [2^K] \rightarrow [2^{K/c}]$. The MVC is constructed as $\varphi_S^{(i)}(W_s) = (C^{(i)}(W_{s_1}), C^{(i)}(W_{s_2}), \ldots, C^{(i)}(W_{s_{|S|}}))$. That is, each server stores one codeword symbol for each version it receives. In the worst-case where a server receives all versions, the storage cost is $\nu K$. Note that since no server is aware of what versions are present at other servers, a server has to store codeword symbols corresponding to multiple versions as the latest version at the server may not have propagated to a sufficient number of servers.

Reference [14] developed multi-version coding schemes and converse results. An important outcome of the study of [14] is that, when the different versions are independent, i.e., if $\delta_K = 1$, then the storage cost cannot be smaller than $\nu K - c - 1 - o(K)$. In particular, because $\frac{\nu K}{\nu + c - 1} \geq 0.5 \min(\frac{\nu}{c}, 1)$, the best possible MVC scheme is, for large $K$, at most twice as cost-efficient as the better among replication and simple erasure coding. In this paper, we show that replication and simple erasure coding are significantly inefficient if the different versions are correlated, i.e., if $\delta_K$ is smaller than 1. Our schemes resemble simple erasure codes in their construction; however, we exploit the correlation between the versions to store fewer bits per server.

C. Summary of Results

To explain the significance of our results, we begin with a simple motivating scheme. Consider the MDS-MVC scheme of Section II-B. Assume that we use a Reed-Solomon code over a field $\mathbb{F}_p$ of binary characteristic. The generator matrix of a Reed-Solomon code is usually expressed over $\mathbb{F}_p$. However, every element in $\mathbb{F}_p$ is a vector over $\mathbb{F}_2$, and a multiplication over the extension field $\mathbb{F}_p$ is a linear transformation over $\mathbb{F}_2$. Therefore, the generator matrix of the Reed-Solomon code can be equivalently expressed over $\mathbb{F}_2$ as follows

$$G = (G^{(1)}, G^{(2)}, \ldots, G^{(n)}),$$

where $G$ is a $K \times nK/c$ binary generator matrix, and $G^{(i)}$ has dimension $K \times K/c$. Because Reed-Solomon codes can tolerate $n-c$ erasures, every matrix of the form $(G^{(t_1)}, G^{(t_2)}, \ldots, G^{(t_c)})$, where $t_1, t_2, \ldots, t_c$ are distinct elements of $[n]$, has a full rank of $K$ over $\mathbb{F}_2$.

We now describe a simple scheme that extends the MDS-MVC by exploiting the correlation among the versions to reduce storage cost. This scheme requires the knowledge of the parameter $\delta_K$. Suppose that the $i$-th server receives the set of versions $S = \{s_1, s_2, \ldots, s_{|S|}\}$, where
$s_1 < s_2 < \ldots < s_{|S|}$. The server encodes $W_{s_1}$ using the binary code as $W_{s_1}^T G^{(i)}$. For $W_{s_m}$, where $m > 1$, the server finds a difference vector $y^{(i)}_{s_m,s_{m-1}}$ that satisfies the following two conditions:

1) $y^{(i)}_{s_m,s_{m-1}} G^{(i)} = (W_{s_m} - W_{s_{m-1}})^T G^{(i)}$ and

2) $w_H(y^{(i)}_{s_m,s_{m-1}}) \leq (s_m - s_{m-1}) \delta_K K$.

Note that it is not necessary that $y^{(i)}_{s_m,s_{m-1}} = W_{s_m} - W_{s_{m-1}}$; however, the fact that $W_{s_m} - W_{s_{m-1}}$ satisfies the two conditions above implies that the encoder can find at least one vector $y^{(i)}_{s_m,s_{m-1}}$ satisfying these two conditions. Since $w_H(y^{(i)}_{s_m,s_{m-1}}) \leq (s_m - s_{m-1}) \delta_K K$, there are at most $Vol((s_m - s_{m-1}) \delta_K K, K)$ possible choices for $y^{(i)}_{s_m,s_{m-1}}$, and therefore, an encoder aware of $\delta_K$ can represent $y^{(i)}_{s_m,s_{m-1}}$ by $\log Vol((s_m - s_{m-1}) \delta_K K, K)$ bits. The server simply stores a corresponding representation of $y^{(i)}_{s_m,s_{m-1}}$. Property 1) implies that a decoder that connects to the $i$-th server can obtain $W_{s_m}^T G^{(i)}$, for any $m \in [|S|]$, by applying $W_{s_1}^T G^{(i)} + \sum_{t=2}^{m} y^{(i)}_{s_t,s_{t-1}} G^{(i)} = W_{s_m}^T G^{(i)}$. Therefore, from any subset $\{t_1, t_2, \ldots, t_c\}$ of $c$ servers, for any common version $s_m$ among these servers, a decoder can recover $W_{s_m}^T G^{(t_1)}, W_{s_m}^T G^{(t_2)}, \ldots, W_{s_m}^T G^{(t_c)}$ from these servers and can therefore recover $W_{s_m}$. Importantly, the decoder can recover the latest common version, and thus, the above scheme is a valid multi-version code.

The worst-case storage cost of this scheme is obtained when each server receives all the $\nu$ versions, which results in a storage cost of $\frac{K}{c} + (\nu - 1) \log Vol(\delta_K K, K)$. Intuitively, the above scheme stores the first version using erasure coding - $K/c$ bits - and the remaining $(\nu - 1)$ versions using delta coding, which adds a storage cost of $\log Vol(\delta_K K, K)$ bits per version.

The scheme we described above motivates the following two questions.

Q1: Can we obtain a MVC construction that is oblivious to the parameter $\delta_K$ with a storage cost of $\frac{K}{c} + (\nu - 1) \log Vol(\delta_K K, K)$?

Q2: Can we use erasure coding based ideas to construct a MVC with a storage cost of $\frac{K}{c} + \frac{\nu - 1}{c} \log Vol(\delta_K K, K)$?

Our result of Theorem 2, which we prove in Section III-A, provides a MVC construction that addresses question Q1.

**Theorem 2.** [Reed-Solomon Update-Efficient MVC] There exists a 0-error $(n,c,\nu,2K,q,\delta_K)$ multi-version code whose worst-case storage is given by

$$\frac{K}{c} + (\nu - 1) \min(\delta_K K \log \left( \frac{Kn_p}{c \log n_p} \right), K/c),$$

where $n_p = 2^{\lceil \log_2 n \rceil}$.

The code construction for the above scheme does not require knowledge of $\delta_K$. Intuitively speaking, this scheme is able to obtain the erasure coding savings factor of $1/c$ for the first version available at a server, and stores the subsequent versions via delta coding, which results in a storage cost of $(\nu - 1) \delta_K K \log \left( \frac{Kn_p}{c \log n_p} \right)$ for the other versions.

We next discuss question Q2. The answer to this question is non-trivial because of the nature of the multi-version coding problem. For instance, for $c = 3$, one server may have versions $W_1$ and $W_3$. The second server may not receive $W_1$, but instead receive $W_2$ and $W_3$. The third
server may only have $W_3$. The first server has to encode the difference between $W_1$ and $W_3$, whereas the second server has to encode the difference between $W_2$ and $W_3$ and the third server has to encode $W_3$ in a manner that $W_3$ is decodable from these three servers. The development of constructions that satisfy this decoding constraint, and still obtain a $1/c$ erasure coding gain factor for the difference is non-trivial in such scenarios.

In Section III-B, we develop MVCs that give a positive answer to question Q2, by obtaining an storage-efficient scheme whose cost described in the following theorem.

**Theorem 3.** [Random Binning MVC] There exists an $\epsilon$-error $(n, c, \nu, 2^K, q, \delta_K)$ multi-version code whose worst-case storage cost is given by

$$\frac{K}{c} + \frac{(\nu - 1) \log Vol(\delta_K K, K)}{c} + \frac{\nu(\nu - 1)/2 - \nu \log 2^{-\epsilon n}}{c}. \quad (4)$$

Intuitively speaking, this obtains the erasure coding factor of $1/c$, not only for the first version, but also for the subsequent increments. Moreover, the scheme is able to harness the delta compression gain to compress the increment to $\log Vol(\delta_K K, K)$.

The construction associated with Section III-B is inspired by Cover’s random binning proof of the Slepian-Wolf source coding problem [34]; the more familiar expressions of the Slepian-Wolf theorem which involve entropy can be obtained when $\delta_K$ is equal to a constant $\delta_0$ using Stirling’s inequality [34]. Motivated by the fact that linear codes have lower complexity, in the Appendix, we show that linear codes exist that achieve the storage cost of Theorem 3. Our proof in the appendix is inspired by reference [21].

We next state our converse results in the following theorem.

**Theorem 4.** [Storage Cost Lower Bound] An $\epsilon$-error $(n, c, \nu, 2^K, q, \delta_K)$ multi-version code with correlated versions such that $W_1 \rightarrow W_2 \rightarrow \ldots \rightarrow W_\nu$ form a Markov chain, $W_m \in [2^K]$ and given $W_m$, $W_{m+1}$ is in a Hamming ball of radius $\delta_K K$ centered around $W_m$ must satisfy

$$\log q \geq \frac{K + (\nu - 1) \log Vol(\delta_K K, K)}{c + \nu - 1} + \frac{\log(1 - \epsilon 2^{\nu n}) - \log \left(\frac{c + \nu - 1}{\nu!}\right)}{c + \nu - 1}. \quad (5)$$

For $\nu < c$, the achievable scheme of Theorem 3 is at most twice the lower bound of Theorem 4. Thus, our achievable scheme is approximately optimal for $\nu < c$. The main results of our paper are tabulated in Table I.

III. CODE CONSTRUCTIONS (PROOFS OF THEOREM 2 AND THEOREM 3)

In this section, we provide our code constructions. We start with the case where $\delta_K$ is not known a priori and present update-efficient MVC for this case that is based on Reed-Solomon code. Later on in this section, we study the case where $\delta_K$ is known a priori and propose a random binning code argument for this case.
| Scheme                                      | Worst-case Storage Cost | Comments.                                                                 |
|--------------------------------------------|-------------------------|--------------------------------------------------------------------------|
| Replication                                | $K,$                    | oblivious to $\delta_K.$                                               |
| Simple erasure codes                        | $\nu \frac{K}{c},$     | outperforms replication if $\nu < c;$ oblivious to $\delta_K.$         |
| Reed-Solomon update-efficient code          | $\frac{K}{c} + (\nu - 1)\delta_K (\log K + o(\log K)),$ | asymptotically outperforms the above schemes for $\nu < c$ and $\delta_K = o(1/\log K);$ oblivious to $\delta_K.$ |
| Motivating scheme of Section II-C           | $\frac{K}{c} + (\nu - 1)\log Vol(\delta_K K, K);$ | not oblivious to $\delta_K.$                                           |
| Random Binning                              | $\frac{K}{c} + \nu - 1 - \frac{1}{c} - 1 \log Vol(K, \delta_K K) + O(1);$ | asymptotically outperforms all the above schemes for $\nu < c;$ not oblivious to $\delta_K.$ |
| Lower Bound                                 | $\frac{K}{c + \nu - 1} + \frac{1}{c - 1} \log Vol(K, \delta_K K) + O(1);$ | applicable for all $\delta_K.$                                       |

**TABLE I: Storage cost.**

**A. Update-efficient Multi-version Codes**

We develop simple multi-version coding scheme that exploits the correlation between the different versions and have smaller storage cost as compared with [14]. In this scheme, the servers do not know the correlation degree $\delta_K$ in advance. We begin by recalling the definition of the update efficiency of a code from [29].

**Definition 3 (Update efficiency).** For a code $C$ of length $N$ and dimension $K$ with encoder $C : \mathbb{F}^K \rightarrow \mathbb{F}^N$, the update efficiency of the code is the maximum number of codeword symbols that must be updated when a single message symbol is changed. Formally, the update efficiency is expressed as follows

$$t = \max_{W, W' \in \mathbb{F}^K : d_H(C(W), C(W')) = 1} d_H(C(W), C(W')).$$

We observe that the update efficiency of a linear code is the maximum row weight of the generator matrix of this code.

**Definition 4 (Update efficiency of a server).** Suppose that $C^{(i)} : \mathbb{F}^K \rightarrow \mathbb{F}^{N/n}$ denotes the $i$-th co-ordinate of the output of $C$ stored by the $i$-th server. The update efficiency of the $i$-th server is the maximum number of codeword symbols that must be updated in this server when a single message symbol is changed. Formally, the update efficiency of the $i$-th server is given by

$$t^{(i)} = \max_{W, W' \in \mathbb{F}^K : d_H(C^{(i)}(W), C^{(i)}(W')) = 1} d_H(C^{(i)}(W), C^{(i)}(W')).$$

Suppose that $G = (G^{(1)}, G^{(2)}, \ldots, G^{(n)})$ is the generator matrix of a linear code $C$, where $G^{(i)}$ is of dimension $K \times N/n$ and corresponds to the $i$-th server. The update efficiency of the $i$-th server is the maximum row weight of $G^{(i)}$. 
Definition 5 (Maximum update efficiency per server). The maximum update efficiency per server is the maximum number of codeword symbols that must be updated in any server when a single message symbol is changed. Formally, the maximum update efficiency per serve is given by

\[ t_s = \max_{i \in [n]} t^{(i)}. \] (8)

An \((N, K)\) code \(C\) is referred to as update-efficient code if it has an update efficiency of \(o(N)\).

We next present an update-efficient MVC construction that is based on Reed-Solomon code and has an update efficiency of \(n\). The maximum update-efficiency per server of this construction is 1. We provide the construction and prove Theorem 2 next.

Proof of Theorem 2 We start by describing the code construction.

Construction 1 (Reed-Solomon Update-Efficient MVC). Suppose that the \(i\)-th server receives the versions \(S = \{s_1, s_2, \ldots, s_{|S|}\}\), where \(s_1 < s_2 < \cdots < s_{|S|}\). We divide a version \(W_{sj}, j \in [|S]|\), into \(\frac{K}{c \log n_p}\) blocks, each block is of length \(c \log n_p\), where \(n_p = 2^{|\log_d n|}\). In each block, we represent every consecutive string of \(\log n_p\) bits by a symbol in \(F_{n_p}\); thus each block can be viewed as a length \(c\) vector over \(F_{n_p}\). We denote the representation of \(W_{sj}\) over \(F_{n_p}\) by \(W_{sj}\).

Each block is encoded by a \((n, c)\) Reed-Solomon code with a generator matrix \(G\) given by

\[
\tilde{G} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{c-1} & \lambda_2^{c-1} & \cdots & \lambda_n^{c-1}
\end{bmatrix},
\] (9)

where \(\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset F_{n_p}\) is a set of distinct elements. For \(W_{s_1}\), the \(i\)-th server stores \(\tilde{W}_{s_1}^T G^{(i)}\), where \(G^{(i)}\) is a \(\frac{K}{\log n_p} \times \frac{K}{c \log n_p}\) matrix that is given by

\[
G^{(i)} = \begin{bmatrix}
\tilde{G} e_i & 0 & \cdots & 0 & 0 \\
0 & \tilde{G} e_i & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tilde{G} e_i
\end{bmatrix}
\] (10)

and \(e_i\) is \(i\)-th standard basis vector. For \(W_{s_m}\), where \(m > 1\), the server may only store the updated symbols from the old version \(\tilde{W}_{s_{m-1}}\) or store all symbols. Storing the index of an updated symbol requires \(\log(\frac{K}{c \log n_p})\) bits and storing the value requires \(\log n_p\), thus the server stores at most \(\min((s_m - s_{m-1})\delta_K K (\log(\frac{K}{c \log n_p}) + \log n_p), K/c)\) bits of \(W_{s_m}\).

The worst-case storage cost corresponds to the case where a server receives all versions. Therefore, the storage cost per server is at most

\[
\frac{K}{c} + \sum_{i=1}^{\nu-1} \min(d_H(W_{i+1}, W_i)(\log(\frac{Kn_p}{c \log n_p}), K/c).
\]
Hence, the worst-case storage cost is at most
\[ \frac{K}{c} + (\nu - 1) \min(\delta_K K \log \left( \frac{Kn_p}{c \log n_p} \right), K/c). \]

B. Slepian-Wolf Inspired Multi-version Codes

We next introduce a random binning argument showing the existence of a multi-version code satisfying the result of Theorem 3. Recall that Slepian-Wolf coding [20], [35] is a distributed data compression technique for correlated sources that are drawn in independent and identical manner according to a given distribution. In the Slepian-Wolf setting, the decoder is interested in decoding the data of all sources. In the multi-version coding problem, the decoder is interested in decoding the latest common version, or a later version, among any set of \( c \) servers.

Our model differs from the standard Slepian-Wolf setting as we do not aspire to decode all the versions and we are interested in a broader correlation model, where the bits of subsequent versions are not drawn in an independent and identical manner given the previous versions. Moreover, in our model the correlation degree \( \delta_K \) may depend on \( K \).

The lossless source coding problem with a helper [36], [37] may seem to be related to our approach, since the side information of the older versions may be interpreted as helpers. In an optimal strategy for the helper setting, the helper side information is encoded via a joint typicality encoding scheme, whereas the random binning is used for the message. However, in the multi-version coding setting, a version that may be a side information for one state may be required to be decoded in another state. For this reason, a random binning scheme for all versions leads to schemes with a near-optimal storage cost.

Before proving Theorem 3, we introduce the following useful definitions.

**Definition 6** (\( \delta_K \)-possible Set of Tuples). The set \( A_{\delta_K} \) of \( \delta_K \)-possible set of tuples \( (w_{u_1}, w_{u_2}, \ldots, w_{u_L}) \) is defined as follows

\[
A_{\delta_K} (W_{u_1}, W_{u_2}, \ldots, W_{u_L}) = \{ (w_{u_1}, w_{u_2}, \ldots, w_{u_L}) : w_{u_1} \in [2^K], w_{u_2} \in B(w_{u_1}, \delta_K (u_2 - u_1)K), w_{u_3} \in B(w_{u_2}, \delta_K (u_3 - u_2)K), \ldots, w_{u_L} \in B(w_{u_{L-1}}, \delta_K (u_L - u_{L-1})K) \},
\]

where \( u_1 < u_2 < \cdots < u_L \).

We omit the dependency on the messages and simply write \( A_{\delta_K} \), when it is clear from the context. Similarly, we can also define the set of possible tuples \( w_{F_1} \) given a particular tuple \( w_{F_2} \), \( A_{\delta_K} (W_{F_1} | W_{F_2}) \), where \( F_1, F_2 \) are two subsets of \( \{u_1, u_2, \ldots, u_L\} \).

We next prove Theorem 3.

**Proof of Theorem 3** We first describe the random binning construction.

**Construction 2** (Random binning multi-version code). Suppose that the \( i \)-th server receives the versions \( S = \{s_1, s_2, \ldots, s_{|S|}\} \subseteq [\nu] \), where \( s_1 < s_2 < \cdots < s_{|S|} \).
• **Random code generation:** At the i-th server, for a version \( s_j \) the encoder assigns an index at random from \( \{1, 2, \ldots, 2^{R_{s_j}^i/c}\} \) uniformly and independently to each vector of length \( K \) bits, where \( R_{s_j}^i/c \) is the rate assigned by the i-th server to version \( s_j \).

• **Encoding:** The server stores the corresponding index to each version that it receives and the decoder is also aware of this mapping. The encoding function of the i-th server is given by

\[
\varphi_{s_j}^{(i)} = (\varphi_{s_j}^{(i)}, \varphi_{s_j}^{(i)}, \ldots, \varphi_{s_j}^{(i)}),
\]

where \( \varphi_{s_j}^{(i)} : [2^K] \to \{1, 2, \ldots, 2^{KR_{s_j}^i/c}\} \), for \( j \in [|S|] \) and we choose the rates as follows

\[
KR_{s_j}^i = K + (s_1 - 1) \log \text{Vol}(\delta K, K) + (s_1 - 1) - \log 2^{-\nu n},
\]

\[
KR_{s_j}^i = (s_j - s_j - 1) \log \text{Vol}(\delta K, K) + (s_j - 1) - \log 2^{-\nu n}, \quad j \in \{2, 3, \ldots, |S|\}.
\]

Suppose that a version \( s_j, j \in [|S|] \), is received by a set of servers \( \{i_1, i_2, \ldots, i_r\} \subseteq T \), then the bin index corresponding to this version is given by

\[
\varphi_{s_j} = (\varphi_{s_j}^{(i_1)}, \varphi_{s_j}^{(i_2)}, \ldots, \varphi_{s_j}^{(i_r)}).
\]

In this case, the rate of version \( s_j \) is given by

\[
R_{s_j} = \frac{1}{c} \sum_{i \in \{i_1, i_2, \ldots, i_r\}} R_{s_j}^i.
\]

• **Decoding:** Consider a state \( S \in \mathcal{P}([\nu])^n \) and assume that the decoder connects to the servers \( T = \{t_1, t_2, \ldots, t_c\} \subseteq [n] \). The decoder employs the possible set decoding strategy that we explain next. Assume that \( \mathbf{W}_{u_L} \) is the latest common version in state \( S \in \mathcal{P}([\nu])^n \) and that the versions \( \mathbf{W}_{u_1}, \mathbf{W}_{u_2}, \ldots, \mathbf{W}_{u_{L-1}} \) are older versions such that each of them is received by at least one server out of those \( c \) servers. We denote this set of versions by \( S_T \) and define it formally as follows

\[
S_T = \{u_1, u_2, \ldots, u_L\} = \left( \bigcup_{t \in T} S(t) \right) \setminus \{u_L + 1, u_L + 2, \ldots, \nu\},
\]

where \( u_1 < u_2 < \cdots < u_L \). Given the bin indices \( (b_{u_1}, b_{u_2}, \ldots, b_{u_L}) \), the decoder finds all tuples \( (\mathbf{w}_{u_1}, \mathbf{w}_{u_2}, \ldots, \mathbf{w}_{u_L}) \in A_{\delta K} \) such that \( (\varphi_{u_1}(\mathbf{w}_{u_1}) = b_{u_1}, \varphi_{u_2}(\mathbf{w}_{u_2}) = b_{u_2}, \ldots, \varphi_{uL}(\mathbf{w}_{uL}) = b_{uL}) \). If all of these tuples have the same latest common version \( \hat{\mathbf{W}}_{u_L} \), the decoder declares \( \hat{\mathbf{W}}_{u_L} \) to be the estimate of the latest common version \( \hat{\mathbf{W}}_{u_L} \). Otherwise, it declares an error.

Denoting \( E \) as the event that there is an error, we can express it as follows

\[
E = \{ \exists (\mathbf{w}_{u_1}', \mathbf{w}_{u_2}', \ldots, \mathbf{w}_{u_L}') \in A_{\delta K} : \mathbf{w}_{u_L}' \neq \mathbf{W}_{u_L} \text{ and } \varphi_u(\mathbf{w}_u') = \varphi_u(\mathbf{W}_u), \forall u \in S_T \}.
\]

The error event in decoding can be equivalently expressed as follows

\[
E = \bigcup_{I \subseteq S_T: u_L \in I} E_I
\]
where
\[ E_I \equiv \{ \exists w'_{u} \neq W_u, \forall u \in I : \varphi_u(w'_u) = \varphi_u(W_u), \forall u \in I \text{ and } (w'_I, W_{S_T \setminus I}) \in A_{K,K} \}, \]  
(19)
for \( I \subseteq S_T \) such that \( u_L \in I \). By the union bound, the probability of error can be upper-bounded as follows
\[ P_e(S, T) \equiv P(E) = P \left( \bigcup_{I \subseteq S_T : u_L \in I} E_I \right) \leq \sum_{I \subseteq S_T : u_L \in I} P(E_I), \]  
(20)
and we require that \( P_e(S, T) < \epsilon 2^{-n} \). Thus, for every \( I \subseteq S_T \) such that \( u_L \in I \), it suffices to show that \( P(E_I) < \epsilon 2^{-(L-1)}2^{-n} \).

We now proceed in a case by case manner. We first consider the case where \( u_{L-1} \notin I \), later we consider the case where \( u_{L-1} \in I \). For the case where \( u_{L-1} \notin I \), we have
\[ E_I \subseteq \tilde{E}_{u_{L-1}} \equiv \{ \exists w'_{uL} \neq W_u : \varphi_{uL}(w'_{uL}) = \varphi_{uL}(W_u) \text{ and } (W_{u_{L-1}}, w'_{uL}) \in A_{K,K} \}. \]  
(21)
Consequently, we have \( P(E_I) < P(\tilde{E}_{u_{L-1}}) \), and we can upper-bound \( P(\tilde{E}_{u_{L-1}}) \) as follows
\[
P(\tilde{E}_{u_{L-1}}) = \sum_{(w_{u_{L-1}}, w_{uL})} P(w_{u_{L-1}}, w_{uL}) 
\]
\[
\leq \sum_{(w_{u_{L-1}}, w_{uL})} P(w_{u_{L-1}}, w_{uL}) \sum_{w'_{uL} \neq w_{uL}} \sum_{(w_{u_{L-1}}, w'_{uL}) \in A_{K,K}} P(\varphi_{uL}(w'_{uL}) = \varphi_{uL}(w_{uL})) 
\]
\[
= \sum_{(w_{u_{L-1}}, w_{uL})} P(w_{u_{L-1}}, w_{uL}) \sum_{w'_{uL} \neq w_{uL}} \prod_{i=1}^{c} P(\varphi_{uL}(w'_{uL}) = \varphi_{uL}(w_{uL})) 
\]
\[
= \sum_{(w_{u_{L-1}}, w_{uL})} P(w_{u_{L-1}}, w_{uL}) |A_{K,K}(W_{uL}|W_{u_{L-1}})| \prod_{i=1}^{c} 2^{-KR_{u_{L-1}}/c} 
\]
\[
= \sum_{(w_{u_{L-1}}, w_{uL})} P(w_{u_{L-1}}, w_{uL}) 2^{-(K R_{uL} - \log Vol((u_L - u_{L-1}) \delta_K K, K))} 
\]
\[
= 2^{-(K R_{uL} - \log Vol((u_L - u_{L-1}) \delta_K K, K))}, \]  
(22)
where (a) follows since each server assigns an index independently from the other servers and (b) follows from (15).

Choosing \( R_{uL} \) to satisfy \( KR_{uL} \geq \log Vol((u_L - u_{L-1}) \delta_K K, K) + (L - 1) - \log \epsilon 2^{-n} \) ensures that \( P(E_I) < \epsilon 2^{-(L-1)}2^{-n} \).
Now, we consider the case where \( u_{L-1} \in \mathcal{I} \). In this case, we consider the following two cases. First, we consider the case where \( u_{L-2} \notin \mathcal{I} \), later we consider the case where \( u_{L-2} \in \mathcal{I} \). For the case where \( u_{L-2} \notin \mathcal{I} \), we have

\[
E_I \subseteq \tilde{E}_{u_{L-2}} \coloneqq \{ \exists w'_{u_{L-1}} \neq W_{u_{L-1}}, w'_L \neq W_u : \varphi_{u_{L-1}}(w'_{u_{L-1}}) = \varphi_{u_{L-1}}(W_{u_{L-1}}), \varphi_{u_L}(w'_{u_L}) = \varphi_{u_L}(W_u) \land (W_{u_{L-2}}, w'_{u_{L-1}}, w'_L) \notin A_{\delta K} \}.
\]

Therefore, we have \( P(E_I) < P(\tilde{E}_{u_{L-2}}) \), and we can upper-bound \( P(\tilde{E}_{u_{L-2}}) \) as follows

\[
P(\tilde{E}_{u_{L-2}}) < \sum_{(w_{u_{L-2}}, w_{u_{L-1}}, w_u)} p(w_{u_{L-2}}, w_{u_{L-1}}, w_u) P(\varphi(w'_{u_{L-1}}) = \varphi(w_{u_{L-1}})) P(\varphi(w'_L) = \varphi(w_L))
\]

\[
\leq \sum_{(w_{u_{L-2}}, w_{u_{L-1}}, w_u)} p(w_{u_{L-2}}, w_{u_{L-1}}, w_u) 2^{-K(R_{u_{L-1}} + R_u) |A_{\delta K}(W_{u_{L-1}}, W_u | w_{u_{L-2}})}
\]

\[
= \sum_{(w_{u_{L-2}}, w_{u_{L-1}}, w_u)} p(w_{u_{L-2}}, w_{u_{L-1}}, w_u) 2^{-K(R_{u_{L-1}} + R_u)}
\]

\[
\geq 2\log Vol((u_{L-1} - u_{L-2}) \delta K, K) + \log Vol((u_{L-2} - u_{L-3}) \delta K, K).
\]

(23)

In this case, we choose the rates as follows

\[
K(R_{u_{L-1}} + R_u) \geq \sum_{j=L-1}^L \log Vol((u_j - u_{j-1}) \delta K, K) + (L - 1) - \log \epsilon 2^{-\nu n}.
\]

We next consider the other case where \( u_{L-2} \in \mathcal{I} \). In this case, we also have two cases based on whether \( u_{L-3} \) is in \( \mathcal{I} \) or not.

By applying the above argument repeatedly, we obtain the following conditions for the overall probability of error to be upper bounded by \( \epsilon \).

\[
K \sum_{j=1}^L R_{u_j} \geq \sum_{j=1}^L \sum_{j=i}^L \log Vol((u_j - u_{j-1}) \delta K, K) + (L - 1) - \log \epsilon 2^{-\nu n}, \forall i \in \{2, 3, \ldots, L\},
\]

\[
K \sum_{j=1}^L R_{u_j} \geq K + \sum_{j=2}^L \log Vol((u_j - u_{j-1}) \delta K, K) + (L - 1) - \log \epsilon 2^{-\nu n}.
\]

(24)

(25)

We notice that \( \log Vol(m \delta K, K) \leq m \log Vol(\delta K, K), \forall m \in \mathbb{Z}^+ \). Therefore, it suffices if the rates satisfy the following

\[
K \sum_{j=1}^L R_{u_j} \geq \sum_{j=1}^L (u_j - u_{j-1}) \log Vol(\delta K, K) + (L - 1) - \log \epsilon 2^{-\nu n}, \forall i \in \{2, 3, \ldots, L\},
\]

\[
K \sum_{j=1}^L R_{u_j} \geq K + \sum_{j=2}^L (u_j - u_{j-1}) \log Vol(\delta K, K) + (L - 1) - \log \epsilon 2^{-\nu n}.
\]

(26)
The rates chosen in (12), (13) satisfy the above inequalities, therefore, our construction has a probability of error bounded by $\epsilon 2^{-\nu m}$. Moreover, as in [38, Chapter 7], it follows that there exists a deterministic code that has a probability of error that is bounded by $\epsilon$.

The worst-case storage cost is when a server receives all versions. Therefore, the worst-case storage cost is given by

$$K - \log \epsilon 2^{-\nu m} - \frac{(\nu - 1)(\log Vol(\delta K, K) - \log \epsilon 2^{-\nu m} + \nu/2)}{c}.$$ 

**Remark 1.** From a technical standpoint, the proof of the theorem above uses ideas that resemble simultaneous non-unique decoding [39], which is previously used in the several multi-user scenarios including the broadcast and interference channels, to decode the latest common version. In particular, with our non-unique decoding approach to decode $W_{u_L}$, the decoder picks the unique $w_{u_L}$ such that $(w_{u_1}, w_{u_2}, \ldots, w_{u_L}) \in A_{\delta K}$ for some $w_{u_1}, w_{u_2}, \ldots, w_{u_{L-1}}$, which is consistent with the bin index. We use this strategy since unlike the Slepian-Wolf problem where all the messages are to be decoded, we are only required to decode one of the messages (the latest common version), using the older versions as side information. In contrast, the unique decoding approach employed by Slepian-Wolf coding would require the decoder to obtain for some subset $S \subseteq \{u_1, u_2, \ldots, u_L\}$ such that $u_L \in S$, the unique $w_S$ in the possible set that is consistent with the bin-indices; unique decoding, for instance, would not allow for correct decoding if there are multiple possible tuples even if they happen to have the same latest common version $w_{u_L}$.

The discussion in [40], which examined the necessity of non-unique decoding, motivates the following question: Can we use the decoding ideas of Slepian-Wolf - where all the messages are decoded - however, for an appropriately chosen subset of messages to recover the same rates? In other words, if we take the union of the unique decoding rate regions over all possible subsets of \{W_{u_1}, W_{u_2}, \ldots, W_{u_L}\}, does the rate allocation of (12), (13), lie in this region? We provide in a technical report associated with this paper [41] (See Remark 4 in Section 5), a counter-example that answers this question in the negative and shows that non-unique decoding provides better rates than unique decoding for our problem.

**IV. LOWER BOUND ON THE STORAGE COST (PROOF OF THEOREM 4)**

In this section, we extend the lower bound on the per-server worst-case storage cost of [14] for the case where we have correlated versions, and we require successful decoding with probability that is at least $1 - \epsilon$. We begin with the proof for the case where $\nu = 2$.

**Proof of Theorem 4 for $\nu = 2$.** Consider any $\epsilon$-error $(n, c, 2, 2^K, q, \delta K)$ multi-version code, and consider the first $c$ servers, $T = [c]$, for decoding. Suppose we have two versions $W[2] = (W_1, W_2)$. We partition the set of possible tuples $A_{\delta K}$ into disjoint sets as $A_{\delta K} = A_{\delta K,1} \cup A_{\delta K,2}$, where $A_{\delta K,1}$ is the set of tuples $(w_1, w_2) \in A_{\delta K}$ for which we can decode the latest common
version or a later version successfully for all \( S \in \mathcal{P}([\nu])^n \). \( A_{\delta K,2} \) is the set of tuples where we cannot decode successfully at least for one state \( S \in \mathcal{P}([\nu])^n \), which can be expressed as follows

\[
A_{\delta K,2} = \bigcup_{S \in \mathcal{P}([\nu])^n} A^{(S)}_{\delta K,2},
\]

where \( A^{(S)}_{\delta K,2} \) is the set of tuples for which we cannot decode successfully given a particular state \( S \in \mathcal{P}([\nu])^n \). Consequently, we have \( |A_{\delta K,2}| \leq \sum_{S \in \mathcal{P}([\nu])^n} |A^{(S)}_{\delta K,2}| \). For any state \( S \in \mathcal{P}([\nu])^n \), we require the probability of error, \( P_e \), to be at most \( \epsilon \). Since all tuples in the set \( A_{\delta K} \) are equiprobable, we have

\[
P_e = \frac{|A^{(S)}_{\delta K,2}|}{|A_{\delta K}|}. \tag{28}
\]

Therefore, we have

\[
|A_{\delta K,1}| = |A_{\delta K}| - |A_{\delta K,2}| \geq |A_{\delta K}| - \sum_{S \in \mathcal{P}([\nu])^n} |A^{(S)}_{\delta K,2}| > |A_{\delta K}| - \sum_{S \in \mathcal{P}([\nu])^n} \epsilon |A_{\delta K}|
\]

\[
> |A_{\delta K}|(1 - \epsilon 2^{\nu n}). \tag{29}
\]

Suppose that \((W_1, W_2) \in A_{\delta K,1}\). Because of the decoding requirements, if \( W_1 \) is available at all servers, the decoder must be able to obtain \( W_1 \), and if \( W_2 \) is available at all servers, then the decoder must return \( W_2 \). Therefore, as shown in [14], there must be two states \( S_1, S_2 \in \mathcal{P}([\nu])^n \) such that (a) \( S_1 \) and \( S_2 \) differ only in the state of one server indexed by \( B \in [c] \), and (b) \( W_1 \) can be recovered from the first \( c \) servers in state \( S_1 \) and \( W_2 \) can be recovered from the first \( c \) servers in \( S_2 \). Therefore both \( W_1 \) and \( W_2 \) are decodable from the \( c \) codeword symbols of the first \( c \) servers in state \( S_1 \), and the codeword symbol of the \( B \)-th server in state \( S_2 \). Thus, we require the following

\[
c \log q^{c+1} \geq |A_{\delta K,1}| > |A_{\delta K}|(1 - \epsilon 2^{\nu n}). \tag{30}
\]

Because \( W_1 \) is uniformly distributed in the set of all \( K \) length binary vectors and given \( W_1 \), \( W_2 \) is uniformly distributed in a Hamming ball of radius \( \delta_K K \) centered around \( W_1 \), we have

\[
|A_{\delta K}| = 2^K Vol(\delta_K K, K).
\]

In this case, the storage cost can be lower-bounded as follows

\[
\log q \geq \frac{K + \log Vol(\delta_K K, K)}{c + 1} + \frac{\log(1 - \epsilon 2^{\nu n}) - \log c}{c + 1}. \tag{31}
\]

The proof for the case where \( \nu \geq 3 \) extends the proof of [14] by allowing probabilistic decoding and considering correlated versions. We now provide a proof sketch for this case.

**Proof sketch of Theorem 4 for \( \nu \geq 3 \).** Consider any \( \epsilon \)-error \((n, c, \nu, 2^K, q, \delta_K)\) multi-version code, and consider the first \( c \leq n \) servers, \( T = [c] \), for decoding. Suppose we have \( \nu \) versions
$W_{[\nu]} = (W_1, W_2, \cdots, W_{\nu})$. Similar to the case where $\nu = 2$, we partition the set of possible tuples $A_{\delta_K}$ into two disjoint sets $A_{\delta_K,1}$ and $A_{\delta_K,2}$, where $A_{\delta_K,1}$ is the set of tuples $(w_1, w_2, \cdots, w_{\nu}) \in A_{\delta_K}$ for which we can decode the latest common version or a later version successfully for all $S \in \mathcal{P}(\nu)^n$.

Suppose that $W_{[\nu]} \in A_{\delta_K,1}$. We construct auxiliary variables $Y_{[c-1]}$, $Z_{[\nu]}$, $B_{[\nu]}$, where $Y_i, Z_j \in [q], i \in [c-1], j \in [\nu], 1 \leq B_1 \leq \cdots \leq B_{\nu} \leq c$ and a permutation $\Pi : [\nu] \rightarrow [\nu]$, such that there is a bijection mapping from these variables to $A_{\delta_K,1}$. In order to construct these auxiliary variables, we use the algorithm of reference [14]. Therefore, we have

$$q^{c+\nu-1} = (c + \nu - 1)_\nu! \geq |A_{\delta_K,1}| > |A_{\delta_K}|(1 - e^{2^\nu n})$$ (32)

where the first inequality follows since $Y_i, Z_j \in [q]$, there are at most $(c+\nu-1)_\nu$ possibilities of $B_{[\nu]}$ and at most $\nu!$ possible permutations. The second inequality follows as in [29]. Finally, we have $|A_{\delta_K}| = 2^K Vol(\delta_K K, K)^{(\nu-1)}$. Therefore, the storage cost is lower-bounded as follows

$$\log q \geq \frac{K + (\nu - 1) \log Vol(\delta_K K, K)}{c + \nu - 1} + \frac{\log(1 - e^{2^\nu n}) - \log((c+\nu-1)_\nu\nu!)}{c + \nu - 1}.$$ (33)

\[\square\]

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V. CONCLUSION

In this paper, we have proposed multi-version codes to efficiently store correlated updates of data in a decentralized asynchronous storage system. These constructions are based on Reed-Solomon codes and random binning. An outcome of our results is that the correlation between versions can be used to reduce storage costs in asynchronous decentralized systems, even if there is no single server or client node who is aware of all data versions, in applications where consistency is important. In addition, our converse result shows that these constructions are nearly-optimal in certain regimes. The development of practical coding schemes for the case where $\delta_K$ is known a priori is an open research question, which would require non-trivial generalizations of previous code constructions for the Slepian-Wolf problem [22], [23].

APPENDIX

In this appendix, we show that there exist linear codes that achieve the rate of Theorem 3. The proof uses linear binning instead of random binning, but mirrors the random binning proof in other respects, for the sake of brevity, we focus on the key differences here.
Lemma 5. Let $G$ be an $N \times M$ matrix whose entries are chosen according to Bernoulli($p$) independently of each other. Let $u$ be any non-zero $N \times 1$ vector. We have

$$\mathbb{P}(u^T G = 0) = ((1 + (1 - 2p)^{w_H(u)})/2)^M. $$

Proof. Consider $k$ Bernoulli trials where the probability of success of each trial is $p$. It can be shown that an even number of successes among the $k$ trials occurs with probability $(1 + (1 - 2p)^k)/2$. Therefore, we have $\mathbb{P}(u^T G = 0) = ((1 + (1 - 2p)^{w_H(u)})/2)^M.$

Proof of Theorem 3 using linear binning. We first describe the construction.

Construction 3 (Random Linear binning multi-version code). Suppose that the $i$-th server receives the versions $S = \{s_1, s_2, \cdots, s_N\} \subseteq [\nu]$, where $s_1 < s_2 < \cdots < s_N$.

- **Random code generation:** At the $i$-th server, for version $s_j$ the encoder creates a $K \times K/c$ random binary matrix $G_{s_j}^{(i)}$, where each entry is chosen as Bernoulli$(1/2)$ independently of all the other entries in the matrix, and all other matrices. We denote by $G_{s_j,m}^{(i)}$ the first $m$ columns of $G_{s_j}^{(i)}$.

- **Encoding:** The server stores $W_{s_j}^T G_{s_j,K_R}^{(i)}/c$, for version $s_j$, where $R_{s_j}^{(i)}/c$ is the rate assigned by the $i$-th server to version $s_j$. The decoder is also aware of the matrix $G_{s_j}^{(i)}$ a priori. The encoding function of the $i$-th server is defined as follows

$$\varphi_{s_j}^{(i)} = (W_{s_1}^T G_{s_1,K_R}^{(i)}/c, W_{s_2}^T G_{s_2,K_R}^{(i)}/c, \cdots, W_{s_N}^T G_{s_N,K_R}^{(i)}/c, W_{s|S|} G_{s|S|, K_R}^{(i)}/c),$$

(35) where we choose the rates as given by (12), (13). Suppose that a version $s_j, j \in [|S|]$, is received by a set of servers $\{i_1, i_2, \cdots, i_r\} \subseteq T$, then the bin index corresponding to this version is given by

$$\varphi_{s_j} = (W_{s_j}^T G_{s_j,K_R}^{(i_1)}/c, W_{s_j}^T G_{s_j,K_R}^{(i_2)}/c, \cdots, W_{s_j}^T G_{s_j,K_R}^{(i_r)}/c),$$

(36) In this case, the rate of version $s_j$ is given by

$$R_{s_j} = \frac{1}{c} \sum_{i \in \{i_1, i_2, \cdots, i_r\}} R_{s_j}^{(i)}.$$

(37)

- **Decoding:** Consider a state $S \in \mathcal{P}([\nu])^n$ and assume that the decoder connects to the servers $T = \{t_1, t_2, \cdots, t_e\} \subseteq [n]$. Let $W_{u_L}$ be the latest common version among these servers and that the versions $W_{u_1}, W_{u_2}, \cdots, W_{u_L}$ are the older versions such that each is received by at least one server out of those $c$ servers. This set of versions is denoted by $S_T$ and defined in (16). Given the bin indices $(b_{u_1}, b_{u_2}, \cdots, b_{u_L})$, the decoder finds all tuples $(w_{u_1}, w_{u_2}, \cdots, w_{u_L}) \in A_R$ such that $(\varphi_{u_1}(w_{u_1}) = b_{u_1}, \varphi_{u_2}(w_{u_2}) = b_{u_2}, \cdots, \varphi_{u_L}(w_{u_L}) = b_{u_L})$. If all of these tuples have the same latest common version $w_{u_L}$, the decoder declares $w_{u_L}$ to be the estimate of the latest common version $\hat{W}_{u_L}$. Otherwise, the decoder declares an error.
As in the proof of Theorem 3, the probability of error in decoding the latest common version among the $c$ servers is upper-bounded as follows

$$P_e(S, T) = P(E) = P \left( \bigcup_{I \subseteq S^T : u_L \in I} E_I \right) \leq \sum_{I \subseteq S^T : u_L \in I} P(E_I),$$

and we require that $P_e(S, T) < \epsilon 2^{-\nu n}$. We proceed in a case by case manner as in the proof of Theorem 3. We first consider the case where $u_{L-1} \notin \mathcal{I}$, later we consider the case where $u_{L-1} \in \mathcal{I}$. For the case where $u_{L-1} \notin \mathcal{I}$, we have the following

$$E_I \subset \tilde{E}_{u_{L-1}} := \{ \exists w_{u_L}^{'} \neq w_{u_L} : w_{u_L}^{T} G^{(t_i)}_{u_L, K R_{u_L}^{(t_i)}} = w_{u_L}^{T} G^{(t_i)}_{u_L, K R_{u_L}^{(t_i)}} / c, \forall i \in [c] \}
\text{and} \ (w_{u_L-1}, w_{u_L}^{'}) \in A_{\delta_K}. \ \ (38)$$

Consequently, we have $P(E_I) < P(\tilde{E}_{u_{L-1}})$, and we can upper-bound $P(\tilde{E}_{u_{L-1}})$ as follows

$$P(\tilde{E}_{u_{L-1}}) = \sum_{(w_{u_{L-1}}, w_{u_L})} p(w_{u_{L-1}}, w_{u_L}) \ \ (39)$$

$$\leq \sum_{(w_{u_{L-1}}, w_{u_L})} p(w_{u_{L-1}}, w_{u_L}) \sum_{(w_{u_{L-1}}, w_{u_L})} \prod_{i=1}^{c} P \left( (w_{u_{L-1}} + w_{u_L})^T G^{(t_i)}_{u_L, K R_{u_L}^{(t_i)}} / c = 0 \right)$$

$$= \sum_{(w_{u_{L-1}}, w_{u_L})} p(w_{u_{L-1}}, w_{u_L}) Vol((u_{L} - u_{L-1}) \delta_K K, K) \prod_{i=1}^{c} 2^{-KR_{u_L}^{(t_i)} / c}$$

$$= 2^{-(K R_{u_{L}} - \log Vol((u_{L} - u_{L-1}) \delta_K K, K)}.$$ \ (40)

where (a) follows since the matrices $G^{(t_1)}, G^{(t_2)}, \ldots, G^{(t_c)}$ are chosen independently and (b) follows from Lemma 5. Choosing $R_{u_{L}}$ to satisfy $K R_{u_{L}} \geq \log Vol((u_{L} - u_{L-1}) \delta_K K, K) + (L - 1) - \log \epsilon 2^{-\nu n}$ ensures that $P(E_I) < \epsilon 2^{-(L-1)2^{-\nu n}}$.

Now, we consider the case where $u_{L-1} \in \mathcal{I}$. In this case, we consider the following two cases. First, we consider the case where $u_{L-2} \notin \mathcal{I}$, later we consider the case where $u_{L-2} \in \mathcal{I}$. For the case where $u_{L-2} \notin \mathcal{I}$, we have

$$E_I \subset \tilde{E}_{u_{L-2}} := \{ \exists w_{u_{L-1}}^{'} \neq w_{u_{L-1}}, w_{u_L}^{'} \neq w_{u_L} : w_{u_{L-1}}^{T} G^{(t_i)}_{u_{L-1}, K R_{u_{L-1}}^{(t_i)}} = w_{u_{L-1}}^{T} G^{(t_i)}_{u_{L-1}, K R_{u_{L-1}}^{(t_i)}} / c,$$

$$w_{u_L}^{T} G^{(t_i)}_{u_L, K R_{u_L}^{(t_i)}} = w_{u_L}^{T} G^{(t_i)}_{u_L, K R_{u_L}^{(t_i)}} / c, \text{and} \ (w_{u_{L-2}}, w_{u_{L-1}}^{'}, w_{u_{L}}^{'}) \in A_{\delta_K}. \ \ (41)$$

Therefore, we have $P(E_I) < P(\tilde{E}_{u_{L-2}})$, and we can upper-bound $P(\tilde{E}_{u_{L-2}})$ as follows

$$P(\tilde{E}_{u_{L-2}}) < \sum_{(w_{u_{L-2}}, w_{u_{L-1}}, w_{u_L})} p(w_{u_{L-2}}, w_{u_{L-1}}, w_{u_L})$$

\(\text{if} \ u_{L-2} \notin \mathcal{I}\). \(\text{if} \ u_{L-2} \in \mathcal{I}\).
\[
\sum_{\substack{\mathbf{w}_{u_{L-1}}^t, \mathbf{w}_{u_{L-1}}' \\ \mathbf{w}_{u_{L-2}}^t, \mathbf{w}_{u_{L-1}}', \mathbf{w}_{u_{L}}' \\ \mathbf{w}_{u_{L}}^t}} P \left( \left( \mathbf{w}_{u_{L-1}}^t + \mathbf{w}_{u_{L-1}}' \right)^T G_{u_{L-1}K'u_{L-1}}^{t} / c = 0, \left( \mathbf{w}_{u_{L}}^t + \mathbf{w}_{u_{L}}' \right)^T G_{u_{L}K'u_{L}}^{t} / c = 0 \right)
\]

In this case, we choose the rates to satisfy

\[
k(u_{L-1} + R_{u_{L}}) + \sum_{j=L-1}^{L} \log \text{Vol}((u_{j} - u_{j-1})\delta_K K, K) + (L-1) - \log 2^{-\mu}. \]

By applying the above argument repeatedly, we obtain the rate region given by (25). \qed

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