Analytical Collapsing Solutions to Pressureless Navier-Stokes-Poisson Equations with Density-dependent Viscosity $\theta = 1/2$ in $\mathbb{R}^2$

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Abstract

We study the 2-dimensional Navier–Stokes-Poisson equations with density-dependent viscosity $\theta = 1/2$ without pressure of gaseous stars in astrophysics. The analytical solutions with collapsing in radial symmetry, are constructed in this paper.

1 Introduction

The evolution of a self-gravitating fluid can be formulated by the Navier-Stokes-Poisson equations of the following form:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P &= -\rho \nabla \Phi + \text{vis}(\rho, u), \\
\Delta \Phi(t, x) &= \alpha(N) \rho,
\end{align*}
\]

where $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 2$; $\alpha(2) = 2\pi$ and for $N \geq 3$,

\[
\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2 + 1)},
\]

where $V(N)$ is the volume of the unit ball in $\mathbb{R}^N$ and $\Gamma$ is a Gamma function. And as usual, $\rho = \rho(t, x)$ and $u = u(t, x) \in \mathbb{R}^N$ are the density, the velocity respectively. $P = P(\rho)$ is the pressure.

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In the above system, the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ through the Poisson equation.

And $vis(\rho, u)$ is the viscosity function:

$$vis(\rho, u) = \nabla(\mu(\rho) \nabla \cdot u).$$

Here we under a common assumption for:

$$\mu(\rho) = \kappa \rho^\theta$$

and $\kappa$ and $\theta \geq 0$ are the constants. In particular, when $\theta = 0$, it returns the expression for the $u$ dependent only viscosity function:

$$vis(\rho, u) = \kappa \Delta u.$$

And the vector Laplacian in $u(t, r)$ can be expressed:

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} \Delta u.$$  

The equations (1) and (1) $vis(\rho, u) \neq 0$ are the compressible Navier-Stokes equations with forcing term. The equation (1) is the Poisson equation through which the gravitational potential is determined by the density distribution of the density itself. Thus, we call the system (1) the Navier-Stokes-Poisson equations.

Here, if the $vis(\rho, u) = 0$, the system is called the Euler-Poisson equations. In this case, the equations can be viewed as a prefect gas model. For $N = 3$, (1) is a classical (nonrelativistic) description of a galaxy, in astrophysics. See [2], [6] for a detail about the system.

$P = P(\rho)$ is the pressure. The $\gamma$-law can be applied on the pressure $P(\rho)$, i.e.

$$P(\rho) = K \rho^\gamma = \frac{\rho^\gamma}{\gamma},$$

which is a commonly the hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$, $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (1). In particular, the fluid is called isothermal if $\gamma = 1$. With $K = 0$, we call the system is pressureless.

For the 3-dimensional case, we are interested in the hydrostatic equilibrium specified by $u = 0$. According to [2], the ratio between the core density $\rho(0)$ and the mean density $\bar{\rho}$ for $6/5 < \gamma < 2$ is given by

$$\frac{\bar{\rho}}{\rho(0)} = \left(-\frac{3}{z} \hat{y}(z)\right)_{z=z_0}$$

where $y$ is the solution of the Lane-Emden equation with $n = 1/(\gamma - 1)$,

$$\hat{y}(z) + \frac{2}{z} \hat{y}(z) + y(z)^n = 0, \ y(0) = \alpha > 0, \ \hat{y}(0) = 0, \ n = \frac{1}{\gamma - 1}.$$
and \( z_0 \) is the first zero of \( y(z_0) = 0 \). We can solve the Lane-Emden equation analytically for

\[
y_{\text{anal}}(z) = \begin{cases} 
1 - \frac{1}{6}z^2, & n = 0; \\
\sin \frac{z}{z}, & n = 1; \\
\frac{1}{\sqrt{1 + z^2/3}}, & n = 5,
\end{cases}
\]  

(10)

and for the other values, only numerical values can be obtained. It can be shown that for \( n < 5 \), the radius of polytropic models is finite; for \( n \geq 5 \), the radius is infinite.

Gambin [4] and Bezard [1] obtained the existence results about the explicitly stationary solution \((u = 0)\) for \( \gamma = 6/5 \) in Euler-Poisson equations:

\[
\rho = \left( \frac{3KA^2}{2\pi} \right)^{5/4} (1 + A^2r^2)^{-5/2},
\]  

(11)

where \( A \) is constant.

The Poisson equation (1) can be solved as

\[
\Phi(t, x) = \int_{R^N} G(x - y)\rho(t, y)dy,
\]  

(12)

where \( G \) is the Green’s function for the Poisson equation in the \( N \)-dimensional spaces defined by

\[
G(x) = \begin{cases} 
|x|, & N = 1; \\
\log |x|, & N = 2; \\
-1/|x|^{N-2}, & N \geq 3.
\end{cases}
\]  

(13)

In the following, we always seek solutions in radial symmetry. Thus, the Poisson equation (1) is transformed to

\[
r^{N-1}\Phi_{rr}(t, x) + (N - 1)r^{N-2}\Phi_r = \alpha(N)\rho N^{N-1},
\]  

(14)

\[
\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r \rho(t, s)s^{N-1}ds.
\]

In this paper, we concern the analytical solutions with core collapsing for the 2-dimensional pressureless Navier-Stokes-Poisson equations with the density-dependent viscosity. And our aim is to construct a family of such core collapsing solutions.

Historically in astrophysics, Goldreich and Weber [5] constructed the analytical collapsing (blowup) solution of the 3-dimensional Euler-Poisson equations for \( \gamma = 4/3 \) for the non-rotating gas spheres. After that, Makino [7] obtained the rigorously mathematical proof of the existence of such kind of collapsing solutions. And in [3], the extension of the above collapsing solutions to the higher dimensional cases \((N \geq 4)\). After that, for the 2-dimensional case, Yuen constructed the analytical collapsing solutions for \( \gamma = 1 \), in [8].

For the construction of the analytical solutions to the Navier-Stokes equations in \( R^N \), the Navier-Stokes-Poisson equations in \( R^3 \) without pressure with \( \theta = 1 \) and in \( R^4 \) without pressure with \( \theta = 5/4 \), readers may refer Yuen’s recent results in [9], [10], [11] respectively.
In this article, the analytical collapsing solutions are constructed in the pressureless Navier–Stokes-Poisson equations with density-dependent viscosity in \( R^2 \), with \( \theta = 1/2 \), in radial symmetry:

\[
\begin{cases}
\rho_t + u \rho_r + \rho u_r + \frac{1}{r} \rho u = 0, \\
\rho (u_t + uu_r) = -\frac{2\pi \rho}{r} \int_0^r \rho(t,s)sds + [\kappa \rho^{1/2}]_r u_r + (\kappa \rho^{1/2}) (u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u),
\end{cases}
\]

(15)

in the form of the following theorem.

**Theorem 1** For the 2-dimensional pressureless Navier–Stokes-Poisson equations with \( \theta = 1/2 \), in radial symmetry, (15), there exists a family of solutions,

\[
\begin{cases}
\rho(t,r) = \frac{1}{(T - Ct)^2 y(r)}^2, \\
\rho u = \frac{C}{T - Ct}, \\
\dot{y} + \frac{1}{z} \dot{y} - \frac{2\pi \sqrt{\kappa}}{C y} = 0, \\
\dot{y}(0) = \alpha > 0, \\
\dot{y}(0) = 0,
\end{cases}
\]

(16)

where \( T > 0, \kappa > 0, C \neq 0 \) and \( \alpha \) are constants.

In particular, for \( C > 0 \), the solutions collapse in the finite time \( T/C \).

## 2 Separable Blowup Solutions

Before presenting the proof of Theorem 1, we prepare some lemmas. First, we obtain the solutions for the continuity equation of mass in radial symmetry (15).

**Lemma 2** For the equation of conservation of mass in radial symmetry:

\[
\rho_t + u \rho_r + \rho u_r + \frac{1}{r} \rho u = 0,
\]

(17)

there exist solutions,

\[
\rho(t,r) = \frac{1}{(T - Ct)^2 y(r)}^2, \\
\rho u = \frac{C}{T - Ct},
\]

(18)

with the form \( y \neq 0 \) and \( y \in C^1 \), \( C \) and \( T > 0 \) are constants.

**Proof.** We just plug (18) into (17). Then

\[
\begin{align*}
\rho_t + u \rho_r + \rho u_r + \frac{1}{r} \rho u &= \frac{(-2)(-C)}{(T - Ct)^3 y(r)}^2 + \frac{(-2)(-1)(-C) r \dot{y}(r)}{(T - Ct)^3 y(r)} \\
&+ \frac{(-C) r \dot{y}(r)}{T - Ct (T - Ct) y(r)} + \frac{1}{(T - Ct)^2 y(r)} \frac{(-C)}{T - Ct} \\
&+ \frac{1}{r (T - Ct)^2 y(r)} \frac{(-C)}{T - Ct} = 0.
\end{align*}
\]
The proof is completed. ■

Besides, we need the lemma for stating the property of the function \( y(z) \). The similar lemma was already given in Lemmas 9 and 10, [8], by the fixed point theorem. For the completeness, the proof is also presented here.

**Lemma 3** For the ordinary differential equation,

\[
\begin{cases}
\dot{y}(z) + \frac{1}{z} y(z) - \frac{\sigma}{y(z)^2} = 0 \\
y(0) = \alpha > 0, \quad \dot{y}(0) = 0,
\end{cases}
\]

where \( \sigma \) is a positive constant,

has a solution \( y(z) \in C^2 \) and \( \lim_{z \to +\infty} y(z) = \infty \).

**Proof.** By integrating (19), we have,

\[
\dot{y}(z) = \frac{\sigma}{z} \int_0^z \frac{1}{y(s)^2} ds \geq 0.
\]

Thus, for \( 0 < z < 0 \), \( y(x) \) has a uniform lower upper bound

\[ y(z) \geq y(0) = \alpha > 0. \]

As we obtained the local existence in Lemma 3 there are two possibilities:

(1) \( y(z) \) only exists in some finite interval \([0, z_0]\): (1a) \( \lim_{z \to z_0^-} y(z) = \infty \); (1b) \( y(z) \) has an uniformly upper bound, i.e. \( y(z) \leq \alpha_0 \) for some constant \( \alpha_0 \).

(2) \( y(z) \) exists in \([0, +\infty)\): (2a) \( \lim_{z \to +\infty} y(z) = \infty \); (2b) \( y(z) \) has an uniformly upper bound, i.e. \( y(z) \leq \beta \) for some positive constant \( \beta \).

We claim that possibility (1) does not exist. We need to reject (1b) first: If the statement (1b) is true, (20) becomes

\[
\frac{\sigma z}{2 \alpha^2} = \frac{\sigma}{z} \int_0^z \frac{1}{y(s)^2} ds \geq \dot{y}(z).
\]

Thus, \( \dot{y}(z) \) is bounded in \([0, z_0]\). Therefore, we can use the fixed point theorem again to obtain a large domain of existence, such that \([0, z_0 + \delta]\) for some positive number \( \delta \). There is a contradiction. Therefore, (1b) is rejected.

Next, we do not accept (1a) because of the following reason: It is impossible that \( \lim_{z \to z_0^-} y(z) = \infty \), as from (21), \( \dot{y}(z) \) has an upper bound in \([0, z_0]\):

\[
\frac{\sigma z_0}{2 \alpha^2} \geq \dot{y}(z).
\]

Thus, (22) becomes,

\[
y(z_0) = y(0) + \int_0^{z_0} \dot{y}(s) ds \\
\leq \alpha + \int_0^{z_0} \frac{\sigma z_0}{2 \alpha^2} ds \\
= \alpha + \frac{\sigma(z_0)^2}{2 \alpha^2}
\]

Thus, (22) becomes,
Since \( y(z) \) is bounded above in \([0, z_0]\), it contracts the statement (1a), such that \( \lim_{z \to z_0^-} y(z) = \infty \). So, we can exclude the possibility (1).

We claim that the possibility (2b) doesn’t exist. It is because

\[
y(z) = \frac{\sigma}{z} \int_{0}^{z} \frac{s}{y(s)^2} ds \geq \frac{\sigma}{z} \int_{0}^{z} \frac{s}{\beta^2} ds = \frac{\sigma z}{2\beta^2}.
\]

Then, we have,

\[
y(z) \geq \alpha + \frac{\sigma}{4\beta^2} z^2.
\]

By letting \( z \to \infty \), (23) turns out to be,

\[
y(z) = \infty.
\]

Since a contradiction is established, we exclude the possibility (2b). Thus, the equation (19) exists in \([0, +\infty)\) and \( \lim_{z \to +\infty} y(z) = \infty \). This completes the proof. \( \blacksquare \)

Here we are already to give the proof of Theorem 1.

**Proof of Theorem 2.** From Lemma 2 it is clear that (10) satisfy (15). For the momentum equation (16), we get,

\[
\rho u_t + u_r + \frac{2\pi \rho}{r} \int_{0}^{r} \rho(t, s) ds - \mu(\rho) r u_r - \mu(\rho)(u_{rr} + \frac{1}{r^2} u_r - \frac{1}{r^2} u)
\]

\[
= \rho \left[ \frac{(-C)(-1)(C)}{(T-C)t^2} \frac{(T-C)^r}{(T-C)} + \frac{(C)}{T-C} \frac{1}{2} \int_{0}^{r} \frac{1}{(T-C)^2 y}(\frac{s}{T-C})^2 ds \right]
\]

\[
= \frac{(\kappa^1/2)}{T-C} - \mu(\rho) \left( 0 + \frac{1}{r^2} \frac{(-1)}{T-C} - \frac{1}{r^2} \frac{(-1)}{T-C} r \right)
\]

\[
= \frac{\partial}{\partial r} \left[ \frac{\kappa}{(T-C)^2 y}(\frac{r}{T-C})^2 \right] ^{1/2} \frac{C}{T-C} + \frac{2\pi \rho}{r} \int_{0}^{r} \frac{1}{(T-C)^2 y}(\frac{s}{T-C})^2 ds
\]

\[
= \frac{1}{2} \left[ \frac{\kappa}{(T-C)^2 y}(\frac{r}{T-C})^2 \right] ^{-1/2} \frac{(-2)}{T-C} y(\frac{r}{T-C}) \frac{C}{T-C} - \frac{\rho}{T-C} \int_{0}^{r} \frac{1}{(T-C)^2 y}(\frac{s}{T-C})^2 ds
\]

\[
= \frac{-C \rho y(\frac{r}{T-C})}{\sqrt{\kappa} (T-C)} + \frac{2\pi \rho}{r} \int_{0}^{r} \frac{1}{(T-C)^2 y}(\frac{s}{T-C})^2 ds
\]

\[
= \frac{-\rho}{T-C} \left[ \frac{C}{\sqrt{\kappa} y}(\frac{r}{T-C}) - \frac{2\pi \rho}{r(T-C)} \int_{0}^{r} \frac{1}{y(\frac{s}{T-C})^2} ds \right]
\]

\[
= \frac{-\rho}{T-C} \left[ \frac{C}{\sqrt{\kappa} y}(\frac{r}{T-C}) - \frac{2\pi \rho}{r(T-C)} \int_{0}^{\frac{r}{T-C}} \frac{1}{y(\tau)^2} d\tau \right]
\]
By letting \( \tau = r/(T - Ct) \), it follows:

\[
\frac{-\rho}{T - Ct} \left[ \frac{C}{\sqrt{\kappa}} \frac{\dot{y}(z)}{T - Ct} - \frac{2\pi}{T - Ct} \int_0^{r/(T - Ct)} \frac{1}{y(\tau)^2} \tau \, d\tau \right] = \frac{-\rho}{T - Ct} Q \left( \frac{r}{T - Ct} \right),
\]

where:

\[
Q \left( \frac{r}{T - Ct} \right) = Q(z) = \frac{C}{\sqrt{\kappa}} \frac{\dot{y}(z)}{z} - \frac{2\pi}{z^2} \int_0^{z} \frac{1}{y(\tau)^2} \tau \, d\tau.
\]

Differentiate \( Q(z) \) with respect to \( z \),

\[
\dot{Q}(z) = \frac{C}{\sqrt{\kappa}} \frac{\dot{y}(z)}{z} - \frac{2\pi}{z^2} \int_0^{z} \frac{1}{y(\tau)^2} \tau \, d\tau
\]

where the above result is due to the fact that we choose the following ordinary differential equation:

\[
\begin{aligned}
\ddot{y}(z) + \frac{1}{z} \dot{y}(z) - \frac{2\pi}{z^2} \frac{1}{y(z)} \frac{1}{\sqrt{\kappa}} = 0 \\
y(0) = \alpha > 0, \quad \dot{y}(0) = 0.
\end{aligned}
\]

With \( Q(0) = 0 \), this implies that \( Q(z) = 0 \). Thus, the momentum equation (15) is satisfied.

With Lemma 3 about \( y(z) \), we are able to show that the family of the solutions collapse in finite time \( T/C \). This completes the proof.

The statement about the blowup rate will be immediately followed:

**Corollary 4** The collapsing rate of the solution (16) is

\[
\lim_{t \to T/C^-} \rho(t,0)(T - Ct)^2 \geq O(1).
\]

**Remark 5** Besides, if we consider the 2-dimensional Navier-Stokes equations with the repulsive force in radial symmetry with \( \theta = 1/2 \),

\[
\begin{aligned}
\rho_t + \rho u_r + \rho u_u + \frac{1}{r} \rho u_r &= 0, \\
\rho (u_t + uu_r) &= +\frac{2\pi \rho}{r} \int_0^r \rho(t,s)ds + [\kappa \rho^{1/2}]u_r + (\kappa \rho^{1/2}) (u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u),
\end{aligned}
\]

the special solutions are:

\[
\begin{aligned}
\rho(t,r) &= \frac{1}{(T - Ct)^2 y \left( \frac{r}{T - Ct} \right)^2}, \quad u(t,r) = \frac{-C}{T - Ct} r \\
y(z) = \frac{1}{z} \dot{y}(z) + \frac{2\pi \sqrt{\kappa}}{C} \frac{1}{y(z)^2} = 0, \quad y(0) = \alpha \neq 0, \quad \dot{y}(0) = 0,
\end{aligned}
\]
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