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Bass’ *NK* groups and *cdh*-fibrant Hochschild homology

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Abstract The *K*-theory of a polynomial ring *R*[[*t*]] contains the *K*-theory of *R* as a summand. For *R* commutative and containing \( \mathbb{Q} \), we describe \( K_\ast(R[[t]])/K_\ast(R) \) in terms of Hochschild homology and the cohomology of Kähler differentials for the *cdh* topology.

We use this to address Bass’ question, whether \( K_n(R) = K_n(R[[t]]) \) implies \( K_n(R) = K_n(R[t_1, t_2]) \). The answer to this question is affirmative when *R* is essentially of finite type over the complex numbers, but negative in general.
In 1972, H. Bass posed the following question (see [4], question (VI)):

Does $K_n(R) = K_n(R[t])$ imply that $K_n(R) = K_n(R[t_1, t_2])$?

One can rephrase the question in terms of Bass’ groups $NK_n$, introduced in [3]:

Does $NK_n(R) = 0$ imply that $N^2K_n(R) = 0$?

More generally, for any functor $F$ from rings to an abelian category, Bass defines $NF(R)$ as the kernel of the map $F(R[t]) → F(R)$ induced by evaluation at $t = 0$, and $N^2F = N(NF)$. Bass’ question was inspired by Traverso’s theorem [26], from which it follows that $N \text{Pic}(R) = 0$ implies $N^2 \text{Pic}(R) = 0$.

In this paper, we give a new interpretation of the groups $NK_n(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the $cdh$ topology, for commutative $\mathbb{Q}$-algebras. This allows us to give a counterexample to Bass’ question in the companion paper [8] (see Theorem 0.2 below).

To state our main structural theorem, recall from [30] that each $NK_n(R)$ has the structure of a module over the ring of big Witt vectors $W(R)$. It is convenient to use the countably infinite-dimensional $\mathbb{Q}$-vector spaces $t\mathbb{Q}[t]$ and $\Omega^1_{\mathbb{Q}[t]}$. If $M$ is any $R$-module, then $M \otimes t\mathbb{Q}[t]$ and $M \otimes \Omega^1_{\mathbb{Q}[t]}$ are naturally $W(R)$-modules by [12].

**Theorem 0.1** Let $R$ be a commutative ring containing $\mathbb{Q}$. Then there is a $W(R)$-module isomorphism

$$N^2K_n(R) \cong (NK_n(R) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(R) \otimes \Omega^1_{\mathbb{Q}[t]}).$$

Thus $K_n(R) = K_n(R[t_1, t_2])$ iff $NK_n(R) = NK_{n-1}(R) = 0$ iff $N^2K_n(R) = 0$.

In addition, the following are equivalent for all $p > 0$:

(a) $K_n(R) = K_n(R[t_1, \ldots, t_p])$.
(b) $NK_n(R) = 0$ and $K_{n-1}(R) = K_{n-1}(R[t_1, \ldots, t_{p-1}])$.
(c) $NK_q(R) = 0$ for all $q$ such that $n - p < q \leq n$.

The equivalence of (a), (b) and (c) is immediate by induction, using the formula for $N^2K_n$, and is included for its historical importance; see [27]. Theorem 0.1 also holds for the $K$-theory of schemes of finite type over a field; see Theorem 4.2 below.

Theorem 0.1 allows us to reformulate Bass’ question as follows:

Does $NK_n(R) = 0$ imply that $NK_{n-1}(R) = 0$?
**Theorem 0.2** (a) For any field \( F \) algebraic over \( \mathbb{Q} \), the 2-dimensional normal algebra

\[
R = F[x, y, z]/(z^2 + y^3 + x^{10} + x^7 y)
\]

has \( K_0(R) = K_0(R[t]) \) but \( K_0(R) \neq K_0(R[t_1, t_2]) \).

(b) Suppose \( R \) is essentially of finite type over a field of infinite transcendence degree over \( \mathbb{Q} \). Then \( NK_n(R) = 0 \) implies that \( R \) is \( K_n \)-regular and, in particular, that \( K_n(R) = K_n(R[t_1, t_2]) \).

Part (a) is proven in the companion paper [8], using Theorem 0.1, while part (b) is proven below as Corollary 6.7.

The proof of Theorem 0.1 relies on methods developed in [7] and [9], which allow us to compute the groups \( NK_n \) and \( N^pK_n \) in terms of the Hochschild homology of \( R \), and of the cdh-cohomology of the higher Kähler differentials \( \Omega^p \), both relative to \( \mathbb{Q} \). The groups \( NK_n(R) \) have a natural bigraded structure when \( \mathbb{Q} \subset R \), and it is convenient to take advantage of this bigrading in stating our results. The bigrading comes from the eigenspaces \( NK_n^{(i)}(R) \) of the Adams operations \( \psi^k \) (arising from the \( \lambda \)-filtration) and the eigenspaces of the homothety operations \( [r] \) (i.e. base change for \( t \mapsto rt \)). This bigrading will be explained in Sects. 1 and 5; the general decomposition for Adams weight \( i \) has the form:

\[
NK_n^{(i)}(R) \cong TK_n^{(i)}(R) \otimes_{\mathbb{Q}} t\mathbb{Q}[t].
\]

Here \( TK_n^{(i)} \) denotes the typical piece of \( NK_n^{(i)}(R) \), defined as the simultaneous eigenspace \( \{ x \in NK_n^{(i)}(R) : [r]x = rx, \ r \in R \} \). (See Example 1.6.) We provide a concrete description of the typical pieces in Theorem 5.1, reproduced here:

**Theorem 0.4** If \( R \) is a commutative \( \mathbb{Q} \)-algebra, then \( NK_n^{(i)}(R) \) is determined by its typical pieces \( TK_n^{(i)}(R) \) and (0.3). For \( i \neq n, n + 1 \) we have:

\[
TK_n^{(i)}(R) \cong \begin{cases} 
HH_{n-1}^{(i-1)}(R) & \text{if } i < n, \\
H_{cdh}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n + 2.
\end{cases}
\]

For \( i = n, n + 1 \), we have an exact sequence:

\[
0 \rightarrow TK_n^{(n+1)}(R) \rightarrow \Omega^n_R \rightarrow H^0_{cdh}(R, \Omega^n) \rightarrow TK_n^{(n+1)}(R) \rightarrow 0.
\]
The special case $NK_0 = \bigoplus N K_0^{(i)}$ of Theorem 0.4 is that for $R$ essentially of finite type over a field of characteristic zero, with $d = \dim(R)$,

$$NK_0(R) \cong \left( (R^+/R_{\text{red}}) \oplus \bigoplus_{p=1}^{d-1} H_{\text{cdh}}^p(R, \Omega^p) \right) \otimes_{\mathbb{Q}} \mathbb{Q}[t]. \quad (0.5)$$

Here $R^+$ is the seminormalization of $R_{\text{red}}$; we show in Proposition 2.5 that $R^+ = H^0_{\text{cdh}}(R, \mathcal{O})$. The dimension zero case of Theorem 0.4 is also revealing:

**Example 0.6** If $\dim(R) = 0$ then we get $NK_n(R) \cong HH_{n-1}(R, I) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ for all $n$, where $I$ is the nilradical of $R$. It is illuminating to compare this with Goodwillie’s Theorem [14], which implies that $NK_n(R) \cong NK_n(R, I) \cong NHC_{n-1}(R, I)$. The identification comes from the standard observation (1.2) that the map $HH_* \to HC_*$ induces $NHC_*(R, I) \cong HH_*(R, I) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$.

The calculations of Theorem 0.4 for small $n$ are summarized in Table 1 when $\dim(R) = 2$. We will need the following cases of 0.4 in [8], to prove Theorem 0.2(a).

**Theorem 0.7** Let $R$ be normal domain of dimension 2 which is essentially of finite type over an algebraic extension of $\mathbb{Q}$. Then

(a) $NK_0(R) = NK_0^{(2)}(R) \cong H^1_{\text{cdh}}(R, \Omega^1) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ and

(b) $NK_{-1}(R) = NK_{-1}^{(1)}(R) \cong H^1_{\text{cdh}}(R, \mathcal{O}) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$.

Here is an overview of this paper: Sect. 1 reviews the bigrading on the Hochschild and cyclic homology of $R[t]$ (and $X \times \mathbb{A}^1$), and Sect. 2 reviews the cdh-fibrant analogue. Section 3 describes the sheaf cohomology of the fibers $\mathcal{F}_{HH}(X)$, $\mathcal{F}_{HC}(X)$, etc. of $HH(X) \to \mathbb{H}_{\text{cdh}}(X, HH)$, etc. In Sect. 4 we use these fibers to prove Theorem 0.1, by relating $NK_{n+1}(X)$ to...
We also show that Bass’ question is negative for schemes in Lemma 4.5.

In Sect. 5, we give the detailed computations of the typical pieces \(TK_n^{(i)}(R)\) needed to establish (0.5) and Table 1; these computations employ the main result of [10]. In Sect. 6, we prove Theorem 0.2(b), that the answer to Bass’ question is positive provided we are working over a sufficiently large base field. Finally, Sect. 7 describes how Theorem 0.7 changes if \(R\) is of finite type over an arbitrary field of characteristic 0: the map \(NK_0(R) \to H^1_{cdh}(R, \Omega^1_F) \otimes \mathbb{Q} \otimes [t]\) is onto, and an isomorphism if \(NK_{-1}(R) = 0\).

Notation

All rings considered in this paper should be assumed to be commutative and noetherian, unless otherwise stated. Throughout this paper, \(k\) denotes a field of characteristic 0 and \(F\) is a field containing \(k\) as a subfield. We write \(\text{Sch}/k\) for the category of separated schemes essentially of finite type over \(k\). If \(\mathcal{F}\) is a presheaf on \(\text{Sch}/k\), we write \(\mathcal{F}_{cdh}\) for the associated \(cdh\) sheaf, and often simply write \(H^*_\text{cdh}(X, \mathcal{F})\) in place of the more formal \(H^*_\text{cdh}(X, \mathcal{F}_{cdh})\).

If \(H\) is a functor on \(\text{Sch}/k\) and \(R\) is an algebra essentially of finite type, we occasionally write \(H(R)\) for \(H(\text{Spec} R)\). For example, \(H^*_\text{cdh}(R, \Omega^1)\) is used for \(H^*_\text{cdh}(\text{Spec} R, \Omega^1)\). Note that, because the \(cdh\) site is noetherian (every cover has a finite subcovering) \(H^*_\text{cdh}\) sends inverse limits of schemes over diagrams with affine transition morphisms to direct limits.

If \(H\) is a contravariant functor from \(\text{Sch}/k\) to spectra, (co)chain complexes, or abelian groups that takes filtered inverse limits of schemes over diagrams with affine transition morphisms to colimits (as for example \(K\), \(HH\), \(\mathbb{H}_{cdh}(-, HH)\), and \(\mathcal{F}_{HH}\)), then for any \(k\)-algebra \(R\), we abuse notation and write \(H(R)\) for the direct limit of the \(H(R_\alpha)\) taken over all subrings \(R_\alpha\) of \(R\) of finite type over \(k\). (If \(R\) is essentially of finite type, the two definitions of \(H(R)\) agree up to canonical isomorphism.) In particular, we will use expressions like \(\mathbb{H}_{cdh}(R, HH)\) for general commutative \(\mathbb{Q}\)-algebras even though we do not define the \(cdh\)-topology for arbitrary \(\mathbb{Q}\)-schemes.

We use cohomological indexing for all chain complexes in this paper; for a complex \(C\), \(C[p]^q = C^{p+q}\). For example, the Hochschild, cyclic, periodic, and negative cyclic homology of schemes over a field \(k\) can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [34] and [7, 2.7] for precise definitions. We shall write these presheaves as \(HH(\mathcal{O}/k)\), \(HC(\mathcal{O}/k)\), \(HP(\mathcal{O}/k)\) and \(HN(\mathcal{O}/k)\), respectively, omitting \(k\) from the notation if it is clear from the context.

It is well known (see [33, 10.9.19]) that there is an Eilenberg-MacLane functor \(C \mapsto |C|\) from chain complexes of abelian groups to spectra, and
from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies \( \pi_n(|C|) = H^{-n}(C) \). For example, applying \( \pi_n \) to the Chern character \( K \to |HN| \) yields maps \( K_n(R) \to H^{-n}HN(R) = HN_n(R) \). In this spirit, we will use descent terminology for presheaves of complexes.

1 The bigrading on \( NHH \) and \( NHC \)

Recall that \( k \) denotes a field of characteristic 0. In this section, we consider the Hochschild and cyclic homology of polynomial extensions of commutative \( k \)-algebras. No great originality is claimed. Throughout, we will use the chain level Hodge decompositions \( HH = \prod_{i \geq 0} HH^{(i)} \) and \( HC = \prod_{i \geq 0} HC^{(i)} \).

The Künneth formula for Hochschild homology yields

\[
NHH_n^{(i)}(R) \cong (HH_n^{(i)}(R) \otimes tQ[t]) \oplus (HH_{n-1}^{(i-1)}(R) \otimes \Omega^1_{Q[t]}). \tag{1.1}
\]

From the exact SBI sequence \( 0 \to NHC_{n-1} \xrightarrow{B} NHH_n \xrightarrow{I} NHC_n \to 0 \) (see [33, 9.9.1]), and induction on \( n \), the map \( I \) induces canonical isomorphisms for each \( i \):

\[
NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes tQ[t]. \tag{1.2}
\]

**Remark 1.3** Both (1.1) and (1.2) generalize to non-affine quasi-compact schemes \( X \) over \( k \). Indeed, \( NHH \) and \( NHC \) satisfy Zariski descent because \( HH \) and \( HC \) do and because, for any open cover \( \{U_i \to X\} \), the collection \( \{U_i \times A^1 \to X \times A^1\} \) is also a cover. Thus we have

\[
NHH^{(i)}(X) \cong \mathbb{H}_{Zar}(X, NHH^{(i)})
\]

\[
\cong \mathbb{H}_{Zar}(X, HH^{(i)}) \otimes tQ[t] \oplus \mathbb{H}_{Zar}(X, HH^{(i-1)})[1] \otimes \Omega^1_{Q[t]}
\]

\[
\cong HH^{(i)}(X) \otimes tQ[t] \oplus HH^{(i-1)}(X)[1] \otimes \Omega^1_{Q[t]},
\]

and \( NHC^{(i)}(X) = \mathbb{H}_{Zar}(X, NHC^{(i)}) \cong \mathbb{H}_{Zar}(X, HH^{(i)}) \otimes tQ[t] \cong HH^{(i)}(X) \otimes tQ[t]. \)

It is easy to iterate the construction \( F \mapsto NF \). For example, we see from (1.1) and (1.2) that

\[
N^2HC_n^{(i)}(R) \cong (HH_n^{(i)}(R) \otimes tQ[t] \otimes tQ[t])
\]

\[
\oplus (HH_{n-1}^{(i-1)}(R) \otimes tQ[t] \otimes \Omega^1_{Q[t]}). \tag{1.4}
\]
By induction, we see that $HH_{n-j}^{(i-j)}(R) \otimes (t\mathbb{Q}[t])^{\otimes (p-j)} \otimes (\Omega^1_{\mathbb{Q}[t]})^{\otimes j}$ will occur $\binom{p-1}{j}$ times as a summand of $N^p HC_n^{(i)}(R)$ for all $j \geq 0$. We may write this as the formula:

$$N^p HC_n^{(i)}(R) \cong \bigoplus_{j=0}^{p-1} HH_{n-j}^{(i-j)}(R) \otimes k \wedge^j k^{p-1} \otimes (t\mathbb{Q}[t])^{\otimes (p-j)} \otimes (\Omega^1_{\mathbb{Q}[t]})^{\otimes j}.$$ (1.5)

Cartier operations on $NHH$ and $NHC$

Let $W(R)$ denote the ring of big Witt vectors over $R$; it is well known that in characteristic 0 we have $W(R) \cong \prod_1^\infty R$. (See [30, p. 468] for example.) Cartier showed in [5] that the endomorphism ring $Cart(R)$ of the additive functor underlying $W$ consists of column-finite sums $\sum V_m[r_{mn}]F_n$, using the homotheties $[r]$ (for $r \in R$), and the Verschiebung and Frobenius operators $V_m$ and $F_n$. Restricting the sum to $m \geq m_0$ yields a descending sequence of ideals of $Cart(R)$, making it complete as a topological ring; $W(R)$ is the complete topological subring of all sums $\sum V_m[r_m]F_m$; see [5].

We will be interested in the intermediate (topological) subring $Carf(R)$ of all row and column-finite sums $\sum V_m[r_{mn}]F_n$. As observed in [12, 2.14], there is an equivalence between the category of $R$-modules and the category of continuous $Carf(R)$-modules given by the constructions in the following example. (A left module $M$ is continuous if the annihilator ideal of each element is an open left ideal.)

**Example 1.6** If $M$ is any $R$-module, $N = M \otimes t\mathbb{Q}[t]$ is a continuous $Carf(R)$-module (and hence a $W(R)$-module) via the formulas:

$$[r]t^i = r^i t^i, \quad V_m(t^i) = t^{mi}, \quad F_m(t^i) = \begin{cases} mt^{i/m} & \text{if } m \mid i, \\ 0 & \text{else.} \end{cases}$$

The ring $W(R) = \prod_1^\infty R$ acts on $M \otimes t\mathbb{Q}[t]$ by $(r_1, \ldots, r_n, \ldots) \ast \sum m_it^i = \sum (r_im_i)t^i$. Conversely, every continuous $Carf(R)$-module $N$ has a “typical piece” $M$, defined as the simultaneous eigenspace $\{x \in N : [r]x = rx, \ r \in R\}$, and $N \cong M \otimes t\mathbb{Q}[t]$.

Recall that we can define operators $[r]$ on $NHH_n(R)$ and $NHC_n(R)$, associated to the endomorphisms $t \mapsto rt$ of $R[t]$. There are also operators $V_m$ and $F_m$, defined via the ring inclusions $R[t^m] \subset R[t]$ and their transfers. These operations commute with the Hodge decomposition. The following result follows immediately from [12, 4.11] using the observation that everything commutes with Adams operations.
Proposition 1.7 The operators $[r], V_m$ and $F_m$ make each $NHC_n^{(i)}(R)$ into a continuous $\text{Carf}(R)$-module, and hence a $W(R)$-module. The $R$-module $HH_n^{(i)}(R)$ is its typical piece, and the canonical isomorphism $NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes \mathbb{Q}[t]$ of (1.2) is an isomorphism of $\text{Carf}(R)$-modules, the module structure on the right being given in Example 1.6.

A similar structure theorem holds for $NHH_n(R)$ and its Hodge components, using (1.1). However, it uses a non-standard $R$-module structure on the typical piece $HH_n(R) \oplus HH_{n-1}(R)$; see [12, 3.3] for details.

Remark 1.7.1 The conclusions of Proposition 1.7 still hold for $NHC_n^{(i)}(X)$ and $HH_n^{(i)}(X)$ when $X$ is any scheme, where $W(R)$ and $\text{Carf}(R)$ refer to the ring $R = H^0(X, \mathcal{O})$. That is, $HH_n^{(i)}(X)$ is an $R$-module and $NHC_n^{(i)}(X)$ is a continuous $\text{Carf}(R)$-module, isomorphic to $HH_n^{(i)}(X) \otimes \mathbb{Q}[t]$.

This scheme version of Proposition 1.7 is not stated in [12], which was written before the cyclic homology of schemes was developed in [34]. However, the proof in [12] is easily adapted. Since the operators $V_m, F_m$ and $[r]$ are defined on the underlying chain complexes in [12, 4.1], they extend to operations on the Hochschild and cyclic homology of schemes. The identities required to obtain continuous $\text{Carf}(R)$-module structures all come from the Künneth formula for the shuffle product on the chain complexes (see [12, 4.3]), so they also hold for the homology of schemes.

2 cdh-fibrant $HH$ and $NHC$

Now fix a field $F$ containing $k$; all schemes will lie in the category $\text{Sch}/F$ (essentially of finite type over $F$), in order to use the cdh topology on $\text{Sch}/F$ of [24]. All rings will be commutative $F$-algebras; because they are filtered direct limits of finitely generated $F$-algebras, we can consider their cdh-cohomology.

If $C$ is any (pre-)sheaf of cochain complexes on $\text{Sch}/F$, we can form the cdh-fibrant replacement $X \mapsto \mathbb{H}_{\text{cdh}}^{cdh}(X, C)$ and write $\mathbb{H}_{\text{cdh}}^n(X, C)$ for the $n$th cohomology of this complex. (The fibrant replacement is taken with respect to the local injective model structure, as in [7, 3.3].) For example, the cdh-fibrant replacement of a cdh sheaf $C$ (concentrated in degree zero) is just an injective resolution, and $\mathbb{H}_{\text{cdh}}^n(X, C)$ is the usual cohomology of the cdh sheaf associated to $C$.

Hochschild and cyclic homology, as well as differential forms, will be taken relative to $k$. For $C = HH^{(i)}$, it was shown in [9, Theorem 2.4] that

$$\mathbb{H}_{\text{cdh}}(X, HH^{(i)}) \cong \mathbb{H}_{\text{cdh}}(X, \Omega^i)[i]. \quad (2.1)$$

This has the following consequence for $C = NHH^{(i)}$ and $NHC^{(i)}$.  

\[ \text{Springer} \]
Lemma 2.2 Let $H^{(i)}$ denote either $HH^{(i)}$ or $HC^{(i)}$, taken relative to a subfield $k$ of $F$. Then $\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)})$, and:

\[
\begin{align*}
\mathbb{H}_{cdh}(X, NHH^{(i)}) &\cong (\mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t]) \\
&\quad \oplus (\mathbb{H}_{cdh}(X, \Omega^{i-1})[i] \otimes \Omega^1_{\mathbb{Q}[t]});
\end{align*}
\]

\[
\mathbb{H}_{cdh}(X, NHC^{(i)}) \cong \mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t].
\]

Proof The displayed formulas follow from (1.1), (1.2) and (2.1), using the fact that $- \otimes t\mathbb{Q}[t]$ commutes with $\mathbb{H}_{cdh}$. Thus it suffices to verify the first assertion. By resolution of singularities, we may assume that $X$ is smooth.

Recall from [7, 3.2.2] that the restriction of the $cdh$ topology to $\text{Sm}_k$ is called the $scdh$-topology. The product of any $scdh$ cover of $X$ with $\mathbb{A}^1$ is an $scdh$ cover of $X \times \mathbb{A}^1$, and both $HH^{(i)}$ and $HC^{(i)}$ satisfy $scdh$-descent by [9, Theorem 2.4]. Now by Thomason’s Cartan-Leray Theorem [25, 1.56] we have

\[
\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) \cong \mathbb{H}_{cdh}(X, H^{(i)}(- \times \mathbb{A}^1)) \\
\cong \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)}).
\]

This gives the first assertion. Alternatively, we may prove the first assertion by induction on $\dim(X)$, using the definition of $scdh$ descent to see that for smooth $X$ we have $H^{(i)}(X) = \mathbb{H}_{cdh}(X, H^{(i)})$ and

\[
\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = H^{(i)}(X \times \mathbb{A}^1) = H^{(i)}(X) \oplus NH^{(i)}(X).
\]

In particular, the first assertion holds when $\dim(X) = 0$. □

Remark 2.2.1 If $R$ is any commutative $F$-algebra, the formulas of Lemma 2.2 hold for $X = \text{Spec}(R)$ by naturality. This is because we may write $R = \varprojlim R_\alpha$, where $R_\alpha$ ranges over subrings of finite type over $F$, and $\mathbb{H}_{cdh}(X, -) = \varprojlim \mathbb{H}_{cdh}(\text{Spec}(R_\alpha), -)$.

Corollary 2.3 If $X = \text{Spec}(R)$ is in $\text{Sch}/F$, the modules $\mathbb{H}_{cdh}^n(X, HH^{(i)})$ and $\mathbb{H}_{cdh}^n(X, NHC^{(i)})$ are zero unless $0 \leq n + i < \dim(X)$ and $i \geq 0$.

If $n \geq \dim(X)$ and $n > 0$ then $\mathbb{H}_{cdh}^n(X, HH) = 0$.

Proof Because $\mathbb{H}_{cdh}^n(X, \Omega^i)[i] = H_{cdh}^{n+i}(X, \Omega^i)$, this follows from (2.1), Lemma 2.2 and the fact that $H_{cdh}^n(X, \Omega^i) = 0$ for $n \geq \dim(X)$, $n > 0$. This bound is given in [7, 6.1] for $i = 0$, and in [9, 2.6] for general $i$. □

Here is a useful bound on the cohomology groups appearing in Lemma 2.2. Given $X$, let $Q$ denote the total ring of fractions of $X_{\text{red}}$; it is a finite product
of fields $Q_j$, and we let $e$ denote the maximum of the transcendence degrees $\text{tr. deg}(Q_j/k)$.

**Lemma 2.4** Let $X$ be in Sch/$F$. If $i > e$ then $H^n_{\text{cdh}}(X, \Omega^j) = 0$ for all $n$.

**Proof** By [21, 12.24], we may assume $X$ reduced. Since we may write $X$ as an inverse limit of a sequence of affine morphisms of schemes of finite type with the same ring of total fractions $Q$, and $\text{cdh}$-cohomology sends such an inverse limit to a direct limit, we may also assume that $X$ is of finite type over $F$. This implies that $e = \dim(X) + \text{tr. deg}(F/k)$.

The result is clear if $\dim(X) = 0$, since $H^n_{\text{cdh}}(X, -) = H^n_{\text{Zar}}(X, -)$ in that case. Proceeding by induction on $\dim(X)$, choose a resolution of singularities $X' \to X$ and observe that the singular locus $Y$ and $Y \times_X X'$ have smaller dimension. The hypothesis implies that $\Omega^j = 0$ on $X'_\text{Zar}$, so $H^n_{\text{cdh}}(X', \Omega^j) = 0$ by [9, 2.5]. The result now follows by induction from the Mayer-Vietoris sequence of [24, 12.1]. □

If $R$ is a commutative ring, we write $R_{\text{red}}$ and $R^+$ for the associated reduced ring and the seminormalization of $R_{\text{red}}$, respectively. These constructions are natural with respect to localization, so that we may form the seminormalization $X^+$ of $X_{\text{red}}$ for any scheme $X$. Because $X^+ \to X$ is a universal homeomorphism, we have $H^*_\text{cdh}(X, -) \cong H^*_\text{cdh}(X^+, -)$ for every $X$ in Sch/$k$, for any field $k$ of arbitrary characteristic. The case $n = 0$ with coefficients $\mathcal{O}_{\text{cdh}}$ is of special interest; recall our convention that $H^0_{\text{cdh}}(X, \mathcal{O})$ denotes $H^0_{\text{cdh}}(X, \mathcal{O}_{\text{cdh}})$.

**Proposition 2.5** For any algebra $R$, we have $H^0_{\text{cdh}}(\text{Spec } R, \mathcal{O}) = R^+$. Moreover, for every $X$ in Sch/$F$ we have $H^0_{\text{cdh}}(X, \mathcal{O}) = \mathcal{O}(X^+)$.\[\text{Proof} \quad \text{We may assume } R \text{ and } X \text{ are reduced. Writing } R = \lim R_\alpha \text{ as in Remark 2.2.1, we have } R^+ = \lim R^+_\alpha \text{ and } H^0_{\text{cdh}}(R, \mathcal{O}) = \lim H^0_{\text{cdh}}(R_\alpha, \mathcal{O}), \text{ so we may assume that } R \text{ is of finite type. Thus the second assertion implies the first. Since } H^0_{\text{cdh}}(-, \mathcal{O}) \text{ and } \mathcal{O}(-^+) \text{ are Zariski sheaves, it suffices to consider the case when } X \text{ is affine.}

\text{Let } X = \text{Spec } R \text{ be in Sch/$F$, with } R \text{ reduced. There is an injection } R \to Q \text{ with } Q \text{ regular (for example, } Q \text{ could be the total quotient ring of } R). \text{ By [7, 6.3], } H^0_{\text{cdh}}(\text{Spec } Q, \mathcal{O}) = Q, \text{ so } R \text{ injects into } H^0_{\text{cdh}}(\text{Spec } R, \mathcal{O}). \text{ This implies that } \mathcal{O}_{\text{red}} \text{ is a separated presheaf for the } cdh \text{ topology on Sch/$F$. Thus, the ring } H^0_{\text{cdh}}(X, \mathcal{O}) \text{ is the direct limit over all } cdh \text{-covers } p : U \to X \text{ of the } \check{\text{Cech}} H^0. \text{ (See [1, 3.2.3].)}\]

\text{Fix an element } b \in H^0_{\text{cdh}}(\text{Spec } R, \mathcal{O}) \text{ and represent it by } b \in \mathcal{O}(U) \text{ for some } cdh \text{ cover } U \to X. \text{ Now recall from [21, 12.28] or [24, 5.9] that we may}
assume, by refining the cdh cover $U \to X$, that it factors as $U \to X' \to X$ where $X' \to X$ is proper birational cdh cover and $U \to X'$ is a Nisnevich cover. If the images of $b \in \mathcal{O}(U)$ agree in $U \times_X U$, i.e. $b$ is a Čech cycle for $U/X$, then its images agree in $U \times_{X'} U$, i.e. it is a Čech cycle for $U/X'$. But by faithfully flat descent, $b$ descends to an element of $\mathcal{O}(X')$. Thus we can assume that $U$ is proper and birational over $X$.

Next, we can assume that the Nisnevich cover $p : U \to X$ is finite, surjective and birational. Indeed, since $p$ is proper and birational we may consider the Stein factorization $U \xrightarrow{q} Y \xrightarrow{r} X$. By [2, 4.3] or [18, III.11.5 & proof], $q_*(\mathcal{O}_U) = \mathcal{O}_Y$ and $r$ is finite surjective and birational. By [24, 5.8], $r$ is also a cdh cover. Because $q_*(\mathcal{O}_U)(Y) = \mathcal{O}_U(U)$ is an isomorphism. Hence $b$ descends to an element of $\mathcal{O}(Y)$. By Lemma 2.6, $b$ lies in the seminormalization of $R$. □

**Lemma 2.6** Let $A$ be a seminormal ring and $B$ a ring between $A$ and its normalization. Then the Čech complex $A \to B \to B \otimes_A B$ is exact.

**Proof** We use Traverso’s description of the seminormalization (see [26, p. 585]): the seminormalization of a ring $A$ inside a ring $B$ is

$$A^+ = \{ b \in B \mid (\forall \mathfrak{P} \in \text{Spec} A) \ b \in A_{\mathfrak{P}} + \text{rad}(B_{\mathfrak{P}}) \}.$$ 

Let $b \in B$ such that $1 \otimes b = b \otimes 1$. We have to show that $b \in A_{\mathfrak{P}} + \text{rad}(B_{\mathfrak{P}})$, for all primes $\mathfrak{P}$ of $A$. Let $J = \text{rad}(B_{\mathfrak{P}})$; since $B_{\mathfrak{P}}/J$ is faithfully flat over the field $A_{\mathfrak{P}}/P$, the image of $b$ in $B_{\mathfrak{P}}/J$ lies in $A_{\mathfrak{P}}/P$ by flat descent. That is, $b \in A_{\mathfrak{P}} + J$, as required. □

**Remark 2.7** Even if $X$ is affine seminormal, it can happen that $H^i_{\text{cdh}}(X, \mathcal{O}) \neq 0$ for some $i > 0$. For example, if $R$ denotes the subring $F[x, g, yg]$ of $F[x, y]$ for $g = x^3 - y^2$ then it is easy to show that $R$ is seminormal and that $H^1_{\text{cdh}}(\text{Spec}(R), \mathcal{O}) = F$, because the normalization of $R$ is $F[x, y]$ and the conductor ideal is $gF[x, y]$. For another example, the normal ring of Theorem 0.2 has $H^1_{\text{cdh}}(X, \mathcal{O}) \neq 0$, by Theorems 0.1 and 0.7(b).

### 3 The fibers $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$

If $C$ is a presheaf of complexes on $\text{Sch}/F$, we write $\mathcal{F}_C$ for the shifted mapping cone of $C \to \mathbb{H}_{\text{cdh}}(-, C)$, so that we have a distinguished triangle:

$$\mathbb{H}_{\text{cdh}}(X, C)[-1] \to \mathcal{F}_C(X) \to C(X) \to \mathbb{H}_{\text{cdh}}(X, C). \quad (3.1)$$

**Example 3.1.1** When $C$ is concentrated in degree 0 we have $H^n \mathcal{F}_C = 0$ for all $n < 0$. For $C = \mathcal{O}$ and $X = \text{Spec}(R)$, we see from Proposition 2.5 that

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$H^0_{\mathcal{F}_\mathcal{O}}(X) = \text{nil}(R)$, $H^1_{\mathcal{F}_\mathcal{O}}(X) = R^+/R$, and $H^n_{\mathcal{F}_\mathcal{O}}(X) = H^n_{\text{cdh}}(X, \mathcal{O})$ for $n \geq 2$. Note that, if $X = \text{Spec } R \in \text{Sch}/F$, then $H^n_{\mathcal{F}_\mathcal{O}}(X) = 0$ for $n > \text{dim}(X)$ by [7, 6.1].

We now consider the Hochschild and cyclic homology complexes, taken relative to a subfield $k$ of $F$. For legibility, we write $F_{HH}^{(i)}$ for $F_{HH}(i)$, etc. By the usual homological yoga, $F_{HH}^{(i)}$ is the direct sum of the $F_{HH}^{(i)}$, $i \geq 0$, and similarly for $F_{HC}^{(i)}$.

**Example 3.1.2** If $X$ is smooth over $F$ then $F_{HH}(X) \simeq 0$ by [9, 2.4].

**Lemma 2.2** and Remarks 2.2.1 and 1.3 imply the following analogue for $N_{\mathcal{F}}$.

**Lemma 3.2** If $X$ is in $\text{Sch}/F$, or if $X = \text{Spec } R$ for an $F$-algebra $R$, we have quasi-isomorphisms:

$$N_{\mathcal{F}}_{HH}^{(i)}(X) \cong (\mathcal{F}_{HH}^{(i)}(X) \otimes \mathbb{Q}[t]) \oplus (\mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes \Omega^1_{\mathbb{Q}[t]})$$

$$N_{\mathcal{F}}_{HC}^{(i)}(X) \cong \mathcal{F}_{HH}^{(i)}(X) \otimes \mathbb{Q}[t].$$

Mimicking the argument that establishes (1.4) and (1.5) yields:

**Corollary 3.3** If $X$ is in $\text{Sch}/F$, or if $X = \text{Spec } R$ for an $F$-algebra $R$,

$$N^2_{\mathcal{F}}_{HC}^{(i)}(X) \cong (\mathcal{F}_{HH}^{(i)}(X) \otimes \mathbb{Q}[t] \otimes \mathbb{Q}[t]) \oplus (\mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes \mathbb{Q}[t] \otimes \Omega^1_{\mathbb{Q}[t]})$$

and

$$N^p_{\mathcal{F}}_{HC}^{(i)}(X) \cong \bigoplus_{j=0}^{p-1} \mathcal{F}_{HC}^{(i-j)}(X)[j] \otimes_k \wedge^j k^{p-1} \otimes \mathbb{Q}[t]^\otimes (p-j) \otimes (\Omega^1_{\mathbb{Q}[t]})^\otimes j.$$

The cohomology of the typical pieces $\mathcal{F}_{HH}^{(i)}(R)$ is given as follows.

**Lemma 3.4** If $R$ is an $F$-algebra and $i \geq 0$, then there is an exact sequence:

$$0 \to H^{-i}_{\mathcal{F}}_{HH}^{(i)}(R) \to \Omega^i_R \to H^0_{\text{cdh}}(R, \Omega^i_R) \to H^{1-i}_{\mathcal{F}}_{HH}^{(i)}(R) \to 0.$$
Proof. As in Remark 2.2.1, we may assume \( R \) is of finite type. Since \( HH_i^1(R) = \Omega^1_R \) for all \( i \geq 0 \), and \( HH_i(R) = 0 \) when \( i > n \) (see [33, 9.4.15] or [19, 4.5.10]), it suffices to use (2.1) and to observe that \( \mathbb{H}^{i-n}_{\text{cdh}}(R, HH_i) = H_{\text{cdh}}^{i-n}(R, \Omega^1) \) vanishes when \( n > i \).

Example 3.5 Let \( X = \text{Spec}(R) \) be in \( \text{Sch}/F \). Since \( HH(0) = \mathcal{O} \), \( \mathcal{F}_{HH}^0(R) \) is described in Example 3.1. Applying Corollary 2.3 and Lemma 3.4 for \( i > 0 \), and using [9, 2.6] to bound the terms, we see that if \( d = \dim(R) \) then
\[
H^n \mathcal{F}_{HH}(X) = 0 \quad \text{for} \quad n > d.
\]
If \( d = 1 \), then the only nonzero positive cohomology of \( \mathcal{F}_{HH} \) is
\[
H^1 \mathcal{F}_{HH}(R) \cong \mathcal{F}_{HH}^1(R) = \mathcal{O}.
\]
If \( d > 1 \), we have:
\[
H^1 \mathcal{F}_{HH}(R) \cong \bigoplus \mathbb{H}^1_{\text{cdh}}(X, \mathcal{O}) \oplus \cdots \oplus \mathbb{H}^{d-1}_{\text{cdh}}(X, \Omega^{d-1}).
\]

Example 3.6 When \( R \) is essentially of finite type over \( F \) and \( \text{tr.deg}(F/k) < \infty \), \( H^n \mathcal{F}_{HH}(R) \) is Hochschild homology for large negative \( m \). To see this, observe that \( e = \text{tr.deg}(R/k) \), the maximum transcendence degree of the residue fields of \( R \) at its minimal primes, is finite. Using Lemmas 2.4 and 3.4, we get \( H^n \mathcal{F}_{HH}^i(R) = 0 \) and \( H^n \mathcal{F}_{HH}(R) = \Omega^n_R \) for all \( n > e \).

If \( R = k \oplus R_1 \oplus R_2 \oplus \cdots \) is graded, and \( \widetilde{HC}_*(R) = HC_*^1(R)/HC_*(k) \), it is well known that the map \( \widetilde{HC}_*(R) \rightarrow \widetilde{HC}_{*-2}(R) \) is zero. (See [33, 9.9.1] for example.) In Lemma 3.8 below, we prove a similar property for \( \mathcal{F}_{HH} \) and \( \mathcal{F}_{HC} \), which we derive from Lemma 3.2 using the following trick.

Standard Trick 3.7 If \( R \) is a non-negatively graded algebra, there is an algebra map \( \nu : R \rightarrow R[t] \) sending \( r \in R_n \) to \( rt^n \). The composition of \( \nu \) with evaluation at \( t = 0 \) factors as \( R \rightarrow R_0 \rightarrow R \), and so if \( H \) is a functor on algebras taking values in abelian groups, then the composition \( H(R) \rightarrow H(R[t]) \rightarrow H(R) \) is zero on the kernel \( \tilde{H}(R) \) of \( H(R) \rightarrow H(R_0) \). Similarly, the composition of \( \nu \) with evaluation at \( t = 1 \) is the identity. That is, \( \nu \) maps \( \tilde{H}(R) \) isomorphically onto a summand of \( NH(R) \), and \( \tilde{H}(R) \) is in the image of \( (t = 1) : NH(R) \rightarrow H(R) \).

Lemma 3.8 If \( R = k \oplus R_1 \oplus \cdots \) is a graded algebra, then for each \( m \) the map \( \pi_m \mathcal{F}_{HC}(R) \rightarrow \pi_{m-2} \mathcal{F}_{HC}(R) \) is zero and there is a split short exact
sequence:

\[ 0 \to \pi_{m-1} \mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{I} \pi_{m+1} \mathcal{F}_{HC}(R) \to 0. \]

Similarly, there are split short exact sequences:

\[ 0 \to \widetilde{H}_{cdh}^{m+1}(R, HC) \xrightarrow{B} \widetilde{H}_{cdh}^m(R, HH) \xrightarrow{I} \widetilde{H}_{cdh}^m(R, HC) \to 0 \]

and

\[ 0 \to \widetilde{H}_{cdh}^{m-1}(R, \Omega^{<i}) \xrightarrow{B} \widetilde{H}_{cdh}^{m-i}(R, \Omega^i) \xrightarrow{I} \widetilde{H}_{cdh}^m(R, \Omega^{\leq i}) \to 0. \]

**Proof** It suffices to show that \( I \) is onto and split. By [9, 2.4], \( \mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0 \), so \( \mathcal{F}_{HH} = \mathcal{F}_{HH} \) and \( \mathcal{F}_{HC} = \mathcal{F}_{HC} \). By the Standard Trick 3.7, it suffices to show that the maps \( N\pi_m \mathcal{F}_{HH}(R) \to N\pi_m \mathcal{F}_{HC}(R) \) and \( N\mathbb{H}_{cdh}^m(R, HH) \to N\mathbb{H}_{cdh}^m(R, HC) \) are split surjections. But this is evident from the decompositions of \( N\mathcal{F}_{HC}^{(i)}(R) \) and \( \mathbb{H}_{cdh}(R, NH^{HC}(i)) \) in Lemmas 3.2 and 2.2.

The third sequence is obtained from the second one by taking the \( i \)th component in the Hodge decomposition, described in Lemma 2.2.

**Example 3.9** Splicing the final sequences of Lemma 3.8 together, we see that the de Rham complexes are exact:

\[ 0 \to k \to R \xrightarrow{d} \tilde{H}^0_{cdh}(R, \Omega^1) \xrightarrow{d} \tilde{H}^0_{cdh}(R, \Omega^2) \to \cdots \quad (3.9a) \]

\[ 0 \to H^n_{cdh}(R, \mathcal{O}) \xrightarrow{d} H^n_{cdh}(R, \Omega^1) \xrightarrow{d} H^n_{cdh}(R, \Omega^2) \to \cdots, \quad n > 0. \quad (3.9b) \]

An analogous exact sequence

\[ \cdots \to \pi_{m-1} \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1} \mathcal{F}_{HH}(R) \to \cdots \]

is obtained by splicing the other sequences in Lemma 3.8. Using the interpretation of their Hodge components, described in Lemma 3.4, produces two more exact sequences:

\[ 0 \to \text{nil}(R) \to \text{tors} \Omega^1_R \to \text{tors} \Omega^2_R \to \text{tors} \Omega^3_R \to \cdots \quad (3.9c) \]

\[ 0 \to (R^+ / R) \to \Omega^1_{cdh}(R) / \Omega^1_R \to \Omega^2_{cdh}(R) / \Omega^2_R \to \cdots. \quad (3.9d) \]

Here we have written \( \Omega_{cdh}^i(R) \) for \( H^0_{cdh}(R, \Omega^i) \), and \( \text{tors} \Omega^i_R \) is defined as the kernel of \( \Omega^i_R \to \Omega^i_{cdh}(R) \); the notation reflects the fact that if \( R \) is reduced then \( \text{tors} \Omega^i_R \) is the torsion submodule of \( \Omega^i_R \) (see Remark 5.3.1 below).
4 Bass’ groups $NK_\ast(X)$

In this section, we relate algebraic $K$-theory to our Hochschild and cyclic homology calculations relative to the ground field $k = \mathbb{Q}$. Consider the trace map

$$NK_{n+1}(X) \to NHC_n(X) = NHC_n(X/\mathbb{Q})$$

induced by the Chern character. In the affine case, it is defined in [29]; for schemes it is defined using Zariski descent. As explained in [29], it arises from the Chern character from the spectrum $NK(X)$ to the Eilenberg-Mac Lane spectrum $|NHC(X)[1]|$ associated to the cochain complex $NHC(X)[1]$. Note that our indexing conventions are such that $\pi_{n+1}|NHC(X)[1]| = H^{-n}NHC(X) = NHC_n(X)$.

**Proposition 4.1** Suppose that $R = \Gamma(X, \mathcal{O})$ for $X$ in $\text{Sch}/F$, or that $X = \text{Spec}(R)$ for an $F$-algebra $R$. Then for all $n$, the Chern character induces a natural isomorphism

$$NK_{n+1}(X) \cong H^{-n}F_{HH}(X) \otimes t\mathbb{Q}[t].$$

This is an isomorphism of graded $R$-modules, and even $\text{Carf}(R)$-modules, identifying the operations $[r]$, $V_m$ and $F_m$ on $NK_\ast(X)$ with the operations on the right side described in Example 1.6.

**Proof** By Remark 2.2.1, we may suppose $X \in \text{Sch}/F$. By [9, 1.6], the Chern character $K \to HN$ induces weak equivalences $F_K(X) \simeq |F_{HC}(X)[1]|$ and $F_K(X \times \mathbb{A}^1) \simeq |F_{HC}(X \times \mathbb{A}^1)[1]|$. Since for any presheaf of spectra $E$ we have a natural objectwise equivalence $E(- \times \mathbb{A}^1) \simeq E \times NE$, we obtain a natural weak equivalence from $NK(X)$ to $|NF_{HC}(X)[1]|$. Now take homotopy groups and apply Lemma 3.2.

As observed in [12, 4.12], the Chern character also commutes with the ring maps used to define the operators $[r]$, $V_m$, and with the transfer for $R[t^n] \to R[t]$ defining $F_m$. That is, it is a homomorphism of $\text{Carf}(R)$-modules. Since the transfer is defined via the ring map $R[t] \to M_n(R[t^n])$, followed by Morita invariance, there is no trouble in passing to schemes. \(\square\)

We now come to one of our main results, which implies Corollary 0.1.

**Theorem 4.2** For all $n$, $N^2K_n(X) \cong (NK_n(X) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(X) \otimes \Omega^1_{\mathbb{Q}[t]})$, and

$$N^{p+1}K_n(X) \cong \bigoplus_{j=0}^{p} NK_{n-j}(X) \otimes \wedge^j \mathbb{Q}^p \otimes (t\mathbb{Q}[t])^{\otimes (p-j)} \otimes (\Omega^1_{\mathbb{Q}[t]})^{\otimes j}.$$
This holds for every \( X \) in \( \text{Sch}/F \), as well as for \( \text{Spec}(R) \) where \( R \) is an arbitrary commutative \( F \)-algebra.

**Proof** As in Proposition 4.1 it follows that the Chern character induces a natural weak equivalence \( N^2 K(X) \cong |N^2 F_{HC}(X)[1]| \). Now take homotopy groups and apply Corollary 3.3. \( \square \)

**Remark 4.2.1** Jim Davis has pointed out (see [11]) that a computation equivalent to 4.2 can also be derived—for arbitrary rings \( R \)—from the Farrell-Jones conjecture for the groups \( \mathbb{Z}^r \). This particular case is covered by F. Quinn’s proof of hyperelementary assembly for virtually abelian groups; see [22].

As an immediate consequence of 4.2 and [3, XII(7.3)], we deduce:

**Corollary 4.3** Suppose that \( X \) is in \( \text{Sch}/F \), or that \( X = \text{Spec}(R) \) for an \( F \)-algebra \( R \). Then:

(a) If \( NK_n(X) = NK_{n-1}(X) = 0 \) then \( N^2 K_n(X) = 0 \).
(b) If \( NK_n(X) = 0 \) and \( K_{n-1}(X) = K_{n-1}(X \times \mathbb{A}^p) \) then \( K_n(X) = K_n(X \times \mathbb{A}^{p+1}) \).
(c) \( K_n(X) = K_n(X \times \mathbb{A}^p) \) if and only if \( NK_q(X) = 0 \) for all \( q \) such that \( n - p < q \leq n \).

Recall that \( X \) is called \( K_n \)-regular if \( K_n(X) = K_n(X \times \mathbb{A}^p) \) for all \( p \).

**Corollary 4.4** Suppose that \( X \) is in \( \text{Sch}/F \), or that \( X = \text{Spec}(R) \) for an \( F \)-algebra \( R \). Then the following conditions are equivalent:

(a) \( X \) is \( K_n \)-regular.
(b) \( NK_n(X) = 0 \) and \( X \) is \( K_{n-1} \)-regular.
(c) \( NK_q(X) = 0 \) for all \( q \leq n \).

**Remark 4.4.1** This gives another proof of Vorst’s Theorem [27, 2.1] (in characteristic 0) that \( K_n \)-regularity implies \( K_{n-1} \)-regularity, and extends it to schemes.

The assumption that the scheme be affine is essential in Bass’ question—here is a non-affine example where the answer is negative.

**Negative answer to Bass’ question for non-affine curves**

Let \( X \) be a smooth projective elliptic curve over a number field \( k \) and let \( L \) be a nontrivial degree zero line bundle with \( L^3 \) trivial. For example, if \( X \) is the Fermat cubic \( x^3 + y^3 = z^3 \), we may take the line bundle associated to the divisor \( P - Q \), where \( P = (1 : 0 : 1) \) and \( Q = (0 : 1 : 1) \).
Lemma 4.5 Write $Y$ for the nonreduced scheme with the same underlying space as $X$ but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L = \text{Sym}(L)/(L^2)$, that is, $L$ is regarded as a square-zero ideal.

Then $NK_7(Y) = 0$ but $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega^1_{\mathbb{Q}[t]}$ is nonzero.

Proof In this setting, the relative Hochschild homology presheaf $HH_n(Y, L)$ is the kernel of $HH_n(Y) \to HH_n(X)$; sheafifying, $\mathcal{H}HH_n(Y, L)$ is the kernel of $\mathcal{H}HH_n(Y) \to \mathcal{H}HH_n(X)$. Since $\Omega_X^1 \cong \mathcal{O}_X$ we see from Lemma 5.3 of [9] that $\mathcal{H}HH_n(Y, L)$ is: $L \otimes L^3 \oplus L \otimes L^5$ if $n = 4$; $L \otimes L^3 \oplus L \otimes L^5$ if $n = 5$; and $L \otimes L^3 \oplus L \otimes L^7$ if $n = 6$. By Serre duality, $H^*(X, L \otimes i) = 0$ if $3 \nmid i$ (cf. [9, 5.1]). By Zariski descent, this implies that $HH_5(Y, L) \cong H^1(X, \mathcal{H}\mathcal{H}_4) \cong H^1(X, L \otimes L^3) \cong k$ and $HH_6(Y, L) = 0$. Since $\mathcal{F}_{HH}(Y) \cong HH(Y, L)$, it follows from 4.1 and 4.2 that $NK_7(Y) = 0$ but $NK_6(Y) \cong t\mathbb{Q}[t]$ and $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega^1_{\mathbb{Q}[t]} \cong t\mathbb{Q}[t] \otimes \Omega^1_{\mathbb{Q}[t]}$.

We conclude this section by refining Proposition 4.1 and Corollary 4.3 to take account of the Adams/Hodge/$\lambda$-decompositions on $K$-theory and Hochschild homology, and by establishing the triviality of $K_n^{(i)}(X)$ for $i \leq 0$.

Recall that by definition, $K_n^{(i)}(X) = \{ x \in K_n(X) \otimes \mathbb{Q} : \psi^k(x) = k^i x \}$. For $n < 0$, the Adams operations cannot be defined integrally. However, it is possible to define the operations $\psi^k$ on $K_n(X) \otimes \mathbb{Q}$ for $n < 0$ using descending induction on $n$ and the formula $\psi^k(x, t) = k \{ \psi^k(x), t \}$ in $K_{n+1}(X \times (\mathbb{A}^1 - 0))$ for $x \in K_n(X)$ and $\mathcal{O}(\mathbb{A}^1 - 0) = F[t, 1/t]$. This definition was pointed out in [32, 8.4].

By [13, 2.3] or [10, 7.2], the Chern character $NK_{n+1}(X) \to NHC_n(X)$ commutes with the Adams operations $\psi^k$ in the sense that it sends $NK_{n+1}^{(i+1)}(X)$ to $NHC_n^{(i)}(X)$ for all $i \leq n$ (and to 0 if $i > n$). Here is the $\lambda$-decomposition in Proposition 4.1:

Proposition 4.6 Suppose that $X \in \text{Sch}/F$, or that $X = \text{Spec}(R)$ for an $F$-algebra $R$. Then for all $n$ and $i$, the Chern character induces a natural isomorphism:

$$NK_{n+1}^{(i)}(X) \cong H^{-n}\mathcal{F}_{HH}^{(i-1)}(X) \otimes t\mathbb{Q}[t].$$

In particular, if $i \leq 0$ then $NK_{n}^{(i)}(X) = 0$ for all $n$.

Proof By [10], the Chern character $K \to HN$ sends $K^{(i)}(X)$ to $HN^{(i)}(X)$. The proof in [10] shows that the lift $\mathcal{F}_K(X) \to \mathcal{F}_{HN}(X)$, shown to be a weak equivalence in [9, 1.6], may be taken to send $\mathcal{F}_K^{(i)}(X)$ to $\mathcal{F}_{HN}^{(i)}(X)$. Since $HC \to HN$ sends $HC^{(i-1)}$ to $HN^{(i)}$, the weak equivalence $\mathcal{F}_{HC}[1] \simeq \mathcal{F}_{HN}$ identifies $\mathcal{F}_{HC}^{(i-1)}[1]$ and $\mathcal{F}_{HN}^{(i)}$. Finally $\mathcal{F}_{HH}^{(i-1)} = 0$ for $i \leq 0$. $\square$
Corollary 4.7 Suppose that \( R \) is essentially of finite type over \( F \) and has dimension \( d \). If \( n < 0 \) then \( NK_n^{(i)}(R) = 0 \) unless \( 1 \leq i \leq d + n \), in which case
\[
NK_n^{(i)}(R) = H_{cdh}^{i-n-1}(R, \Omega_{i-1}) \otimes tQ[t].
\]
In particular, \( NK_n(R) = 0 \) for all \( n \leq -d \).

If \( d \geq 2 \) then:
\[
NK_0(R) \cong [(R^+ / R) \oplus H_{cdh}^1(R, \Omega^1) \oplus \cdots \oplus H_{cdh}^{d-1}(R, \Omega^{d-1})] \otimes tQ[t],
\]
\[
NK_{-1}(R) \cong [H_{cdh}^1(R, \mathcal{O}) \oplus H_{cdh}^2(R, \Omega^1) \oplus \cdots \oplus H_{cdh}^{d-1}(R, \Omega^{d-2})] \otimes tQ[t],
\]
\[
:\vdots
\]
\[
NK_{-d}(R) \cong H_{cdh}^{d-1}(R, \mathcal{O}) \otimes tQ[t].
\]
If \( d = 1 \) then \( NK_0(R) = (R^+ / R) \otimes tQ[t] \) and \( NK_n(R) = 0 \) for \( n < 0 \).

Proof This is straightforward from Proposition 4.6 and Lemma 3.4. \(\square\)

Remark 4.7.1 The \( d = 1 \) part of Corollary 4.7 holds for any 1-dimensional noetherian ring by [28, 2.8].

Corollary 4.8 \( K_n^{(i)}(X) \cong K_n^{(i)}(X \times \mathbb{A}^p) \) if and only if \( NK_n^{(i-j)}(X) = 0 \) for all \( j = 0, \ldots, p - 1 \).

Theorem 4.9 For \( X \) in \( \text{Sch}/F \) or \( X = \text{Spec}(R) \), and all integers \( n \), we have:

1. For \( i < 0 \), \( K_n^{(i)}(X) = 0 \).
2. For \( i = 0 \), \( K_n^{(0)}(X) \cong KH_n^{(0)}(X) \cong H_{cdh}^{-n}(X, \mathbb{Q}) \).

Here \( KH \) denotes the homotopy \( K \)-theory of [31]. Theorem 4.9 answers Question 8.2 of [32].

Proof We first show that \( K_n^{(i)}(X) \cong KH_n^{(i)}(X) \) when \( i \leq 0 \). Covering \( X \) with affine opens and using the Mayer-Vietoris sequences of [31, 5.1], it suffices to consider the case \( X = \text{Spec}(R) \).

Since \( K(R)_{\mathbb{Q}} \) is the product of the eigen-components, the descent spectral sequence \( E_{p,q}^{1} = N_{p,q}K_{p,q}(R)_{\mathbb{Q}} \Rightarrow KH_{p+q}^{(i)}(R)_{\mathbb{Q}} \) (see [31, 1.3]) breaks up into one for each eigen-component. If \( i \leq 0 \), the spectral sequence collapses by Proposition 4.6 to yield \( K_n^{(i)}(R) \cong KH_n^{(i)}(R) \) for all \( n \).

To determine the groups \( KH_n^{(i)}(R) \) when \( i \leq 0 \), we use the \( cdh \) descent spectral sequence of [17, 1.1]. If \( i < 0 \), then the \( cdh \) sheaf \( K_{cdh}^{(i)} \) is trivial as

\(\square\) Springer
X is locally smooth, so we have $KH_n^{(i)}(R) = 0$ for all $n$. If $i = 0$ then the cdh sheaf $K^{(0)}_{cdh}$ is the sheaf $\mathbb{Q}_{cdh}$; see [23, 2.8]. Hence we have $K_n^{(0)}(R) = KH_n^{(0)}(R) = H_{cdh}^{-n}(X, \mathbb{Q})$.

5 The typical pieces $TK_n^{(i)}(R)$

In this section, $R$ will be a commutative $F$-algebra. The default ground field $k$ for Kähler differentials and Hochschild homology will be $\mathbb{Q}$.

As stated in (0.3), the Adams summands $NK_n^{(i)}(R)$ of $NK_n(R)$ decompose as $NK_n^{(i)}(R) = TK_n^{(i)}(R) \otimes t\mathbb{Q}[t]$ for each $n$ and $i$; the decomposition is obtained from an action of finite Cartier operators precisely as the corresponding one for $NHC$ and $NHH$, explained in Sect. 1. The typical pieces $TK_n^{(i)}(R)$ are described by the following formulas.

**Theorem 5.1** Let $R$ be a commutative $F$-algebra. For $i \neq n, n + 1$ we have:

$$TK_n^{(i)}(R) \cong \begin{cases} H^{i-1}_n(R), & \text{if } i < n, \\ H^{n-i}_n(R, \Omega^{i-1}_1) & \text{if } i \geq n + 2. \end{cases}$$

For $i = n, n + 1$, the typical piece $TK_n^{(i)}(R)$ is given by the exact sequence:

$$0 \to TK_{n+1}^{(n+1)}(R) \to \Omega^n_R \to H^0_{cdh}(R, \Omega^n) \to TK_{n+1}^{(n+1)}(R) \to 0.$$

**Proof** By Proposition 4.6, $TK_n^{(i)} = H^{1-n}F^{(i-1)}_{HH}$. The rest is a restatement of Lemma 3.4. \qed

**Remark 5.1.1** If $R$ is essentially of finite type over a field $F$ whose transcendence degree is finite over $\mathbb{Q}$, then the $TK_n^{(i)}(R)$ are finitely generated $R$-modules. This fails if $\text{tr.deg}(F/\mathbb{Q}) = \infty$ because then $\Omega^1_{F/\mathbb{Q}}$ is infinite dimensional. For instance, Example 0.6 implies that, for $R = F[x]/(x^2)$, we have $TK_2^{(2)}(R) = HH_1(R, x) = F \oplus \Omega^1_{F/\mathbb{Q}}$.

**Remark 5.1.2** Observe that Corollaries 4.7 and 4.4 imply that $R$ is $K_{-d}$-regular. This recovers the affine case of one of the main results in [7].

Here is a special case of the calculations in Theorem 5.1, which proves Theorem 0.7. We will use it to construct the counterexample to Bass’ question in the companion paper [8].

**Theorem 5.2** Let $F$ be a field of characteristic 0 and $R$ a normal domain of dimension 2, essentially of finite type over $F$. Then
(a) $H^1\mathcal{F}_{\mathcal{H}}(R/F) \cong H^1_{\text{cdh}}(R, \Omega^1_F)$,
(b) $H^2\mathcal{F}_{\mathcal{H}}(R/F) \cong H^1_{\text{cdh}}(R, \mathcal{O})$,
(c) $NK_0(R) \cong H^1_{\text{cdh}}(R, \Omega^1) \otimes t\mathbb{Q}[t], and$
(d) $NK_{-1}(R) \cong H^1_{\text{cdh}}(R, \mathcal{O}) \otimes t\mathbb{Q}[t]$.

Proof Parts (a) and (b) are immediate from Example 3.5 and the fact that $R$ is reduced and seminormal. Parts (c) and (d) follow from (a) and (b) using Proposition 4.1; cf. Corollary 4.7. □

In order to compare the torsion submodules $\text{tors} \Omega^*_R$ with the typical pieces of $NK^*_R$, we need the affine case of the following lemma. Following tradition, we write $F(X)$ for the total ring of fractions of $X_{\text{red}}$. That is, $F(X)$ is the product of the function fields of the irreducible components of $X_{\text{red}}$. When $X = \text{Spec}(R)$ is affine, we write $Q$ instead of $F(X)$.

Lemma 5.3 Let $X \in \text{Sch}/F$; for $F(X)$ as above, the map $\Omega^i_{\text{cdh}}(X) \to \Omega^i_{F(X)}$ is an injection.

Proof We may assume $X$ reduced, and proceed by induction on $d = \dim(X)$, the case $d = 0$ being trivial. Choose a resolution of singularities $X' \to X$ and let $Y$ be the singular locus of $X$, with $Y' = Y \times_X X'$. By [24, 12.1], there is a Mayer-Vietoris exact sequence

$$0 \to \Omega^i_{\text{cdh}}(X) \to \Omega^i_{\text{cdh}}(X') \oplus \Omega^i_{\text{cdh}}(Y) \to \Omega^i_{\text{cdh}}(Y') \overset{\partial}{\to} H^1_{\text{cdh}}(X, \Omega^i) \to \cdots.$$ 

Since $F(Y) \subseteq F(Y')$, $\Omega^i_{F(Y)} \subseteq \Omega^i_{F(Y')}$. Because $\dim(Y') < d$, the inductive hypothesis implies that $\Omega^i_{\text{cdh}}(Y) \to \Omega^i_{\text{cdh}}(Y')$ is an injection. Hence $\Omega^i_{\text{cdh}}(X) \to \Omega^i_{\text{cdh}}(X')$ is an injection. But $X'$ is smooth, so by $\text{scdh}$ descent for $\Omega^i$ (see [9, 2.5]) we have $\Omega^i_{\text{cdh}}(X') \cong \Omega^i(F(X)) \subseteq \Omega^i_{F(X)} = \Omega^i_{F(X)}$. □

Remark 5.3.1 Lemma 5.3 remains true if, instead of $\Omega^i$, we use $\Omega^i_k$ for $k \subseteq F$. In particular, if $X = \text{Spec}(R)$ is reduced affine, then $\Omega^i_{\text{cdh}}(R/k) = H^0_{\text{cdh}}(R, \Omega^i_{k/F})$ injects into $\Omega^i_{Q/k}$. Thus $\text{tors}(\Omega^i_{R/k})$, defined as the kernel of $\Omega^i_{R/k} \to \Omega^i_{\text{cdh}}(R/k)$ in (3.9c), is the torsion submodule of $\Omega^i_{R/k}$.

Corollary 5.4 For all $n \geq 1$, $TK^{(n)}_n(R) \cong \ker(\Omega^{n-1}_R \to \Omega^{n-1}_Q)$. In particular if $R$ is reduced, then $TK^{(n)}_n(R)$ is the torsion submodule of $\Omega^{n-1}_R$.

Proof By Theorem 5.1, $TK^{(n)}_n(R)$ is the kernel of $\Omega^{n-1}_R \to \Omega^{n-1}_{\text{cdh}}(R)$, so Lemma 5.3 applies. □
We introduce some notation to make the statement of the next theorem more readable. The letter \( e \) denotes the maximum transcendence degree of the component fields in the total ring of fractions \( Q \) of \( R_{\text{red}} \). For simplicity, we write \( \Omega^i_{\text{cdh}}(X) \) for \( H^0_{\text{cdh}}(X, \Omega^i) \), and we have written \( \Omega^i_{\text{cdh}}(R)/\Omega^i R \) for the cokernel of \( \Omega^i R \to \Omega^i_{\text{cdh}}(R) \).

**Definition 5.5** For any commutative ring \( R \) containing \( Q \), we define:

\[
E_n(R) = \Omega^n_{\text{cdh}}(R)/\Omega^n_R \oplus \bigoplus_{p=1}^{\infty} H^p_{\text{cdh}}(R, \Omega^{n+p});
\]

\[
\widetilde{HH}_n(R) = \ker(HH_n(R) \to \Omega^n_Q) = \ker\left(\Omega^n_R \to \Omega^n_Q\right) \oplus \bigoplus_{i=1}^{n-1} HH_n^{(i)}(R).
\]

**Theorem 5.6** Let \( R \) be a commutative ring containing \( Q \). Then for all \( n \):

\[
NK_n(R) \cong \left[ \widetilde{HH}_{n-1}(R) \oplus E_n(R) \right] \otimes \mathbb{Q}[t].
\]

If furthermore \( R \) is essentially of finite type over a field, and \( n \geq e + 2 \), then \( NK_n(R) \cong HH_{n-1}(R) \otimes \mathbb{Q}[t] \).

**Proof** Assembling the descriptions of the \( TK_n^{(i)}(R) \) in Theorem 5.1 yields the first assertion. The second part is immediate from this and Example 3.6. \( \square \)

**Remark 5.6.1** The Chern character \( NK_n(R) \to NH^n_{-1}(R) \cong HH_{n-1}(R) \otimes \mathbb{Q}[t] \) is an isomorphism for \( n \geq e + 2 \). If \( n \leq e + 1 \), neither it nor the map \( H^{1-n}FHH_n(R) \to HH_{n-1}(R) \) of Proposition 4.1 need be a surjection.

The typical pieces of \( NK_1^{(2)}(R) \) and \( NK_2^{(2)}(R) \) of Theorem 5.1 and Corollary 5.4 may be described as follows.

**Proposition 5.7** For all reduced \( F \)-algebras \( R \), the typical pieces \( TK_1^{(2)}(R) = \Omega^1_{\text{cdh}}(R)/\Omega^1_R \) and \( TK_2^{(2)}(R) = \text{tors}(\Omega^1_R) \) fit into an exact sequence:

\[
0 \to \text{tors}(\Omega^1_R) \to \text{tors}(\Omega^1_{R/F}) \to \Omega^1_F \otimes (R^+/R) \to \frac{\Omega^1_{\text{cdh}}(R)}{\Omega^1_R} \to 0.
\]

**Proof** We may assume \( \text{Spec } R \in \text{Sch}/F \). Recall from [9, 4.2] that there is a bounded second quadrant homological spectral sequence for all \( p \) (0 ≤ \( i < p \),
When $p = 1$, this spectral sequence degenerates to yield exactness of the bottom row in the following commutative diagram; the top row is the First Fundamental Exact Sequence for $\Omega^1$ [33, 9.2.6].

\[
\begin{array}{cccc}
\Omega^1_F \otimes R & \longrightarrow & \Omega^1_R & \longrightarrow & \Omega^1_{R/F} & \longrightarrow & 0 \\
& & & & & & \\
0 & \longrightarrow & \Omega^1_F \otimes R^+ & \longrightarrow & \Omega^1_{cdh}(R) & \longrightarrow & \Omega^1_{cdh}(R/F) & \longrightarrow & 0.
\end{array}
\]

The upper left horizontal map is an injection because the left vertical map is an injection. Now apply the snake lemma, using Remark 5.3.1.

\section{Bass’ question for algebras over large fields}

We will now show that the answer to Bass’ question is positive for algebras $R$ essentially of finite type over a field $F$ of infinite transcendence degree over $\mathbb{Q}$.

Recall from Proposition 4.1 that $NK_{n+1}(R) \cong H^{-n}_{HH}(R/\mathbb{Q}) \otimes t\mathbb{Q}[t]$. In light of this identification, the version of Bass’ question stated before Theorem 0.2 becomes the case $k = \mathbb{Q}$ of the following question:

Does $H^m_{HH}(R/k) = 0$ imply that $H^{m+1}_{HH}(R/k) = 0$? \hspace{1cm} (6.1)

In Theorem 6.6, we show that the answer to question (6.1) is positive provided $R$ is of finite type over a field $F$ that has infinite transcendence degree over $k$. The proof is essentially a formal consequence of the Künneth formula in Lemma 6.3.

\textbf{Lemma 6.2} \hspace{1cm} \textit{Let $R$ be a commutative $F$-algebra, and suppose $k$ is a subfield of $F$. Then $H^{-*}_{HH}(R/k)$ and $\mathbb{H}^{-*}_{cdh}(R, HH(\cdot/k))$ are graded modules over the graded ring $\Omega^*_{F/k}$.}

\textbf{Proof} \hspace{1cm} As in Remark 2.2.1, we may suppose that $R$ is of finite type over $F$. Consider the functor on $F$-algebras that associates to an $F$-algebra $A$ the Hochschild complex $HH(A/k)$. The shuffle product makes this into a functor to $dg-HH(F/k)$-modules. Since the $cdh$-site has a set of points (corresponding to valuations by [15, 2.1]), we can use a Godement resolution.
to find a model for the cdh-hypercohomology $\mathbb{H}_{cdh}(-, HH(/k))$ which is also a functor to $dg-HH(F/k)$-modules. It follows that there is a model for $\mathcal{F}_{HH}(R/k)$ that is a $dg-HH(F/k)$-module, functorially in $R$. This implies the assertion, since $\Omega^\bullet_{F/k} = H^{-\bullet}HH(F/k)$.

**Lemma 6.3 (Künneth formula)** Suppose that $Q \subseteq k \subseteq F_0 \subseteq F$ are fields. Let $R_0$ be an $F_0$-algebra, and set $R = F \otimes F_0 R_0$.

(i) Let $T = \{t_i\}$ be transcendence basis of $F/F_0$; writing $F[FdT]$ for the exterior algebra on the set $\{dt_i\}$, we have $\Omega^\bullet_{F/k} = F[FdT] \otimes F_0 \Omega^\bullet_{F_0/k}$

In particular, the graded algebra homomorphism $\Omega^\bullet_{F_0/k} \rightarrow \Omega^\bullet_{F/k}$ is flat.

(ii) $HH_*(R/k) \cong \Omega^\bullet_{F/k} \otimes \Omega^\bullet_{F_0/k}$. $HH_*(R_0/k) \cong F[dT] \otimes F_0 HH_*(R_0/k)$.

**Proof** It is classical that $F[dT] = \Omega^\bullet_{F_0/k}$. The tensor product decomposition of part (i) follows from the fact that the fundamental sequence

$$0 \rightarrow F \otimes F_0 \Omega^1_{F_0/k} \rightarrow \Omega^1_F \rightarrow \Omega^1_{F/F_0} \rightarrow 0$$

is split exact. This proves (i). To prove (ii), choose a free chain $dg-F_0$-algebra $\Lambda$ and a surjective quasi-isomorphism of $dg$-algebras $\Lambda \rightarrow R_0$. Then $\Lambda' = F \otimes F_0 \Lambda \rightarrow F \otimes F_0 R_0 = R$ is a free chain model of $R$ as a $k$-algebra. Write $\Omega^\bullet_{\Lambda/k}$ for differential forms; consider $\Omega^\bullet_{\Lambda/k}$ as a chain $dg$-algebra with the differential $\delta$ induced by that of $\Lambda$. Note $\Lambda$ and $\Lambda'$ are homologically regular in the sense of [6], so that Theorem 2.6 of [6] applies. Combining this with part (i), we obtain

$$HH_*(R/k) = HH_*(\Lambda'/k) = H_*(\Omega^\bullet_{\Lambda'/k})$$

$$= H_*(\Omega^\bullet_{F/k} \otimes \Omega^\bullet_{F_0/k} \Omega^\bullet_{\Lambda/k}) = \Omega^\bullet_{F/F_0} \otimes \Omega^\bullet_{F_0/k} HH_*(\Omega^\bullet_{\Lambda/k})$$

$$= \Omega^\bullet_{F/k} \otimes \Omega^\bullet_{F_0/k} HH_*(R_0/k).$$

□

Here is an easy consequence of Lemmas 6.2 and 6.3.

**Proposition 6.4** Suppose $Q \subseteq k \subseteq F_0 \subseteq F$ are field extensions, that $R_0$ is an $F_0$-algebra and $R = F \otimes F_0 R_0$. Then there is an isomorphism of graded $\Omega^\bullet_{F/k}$-modules

$$F[dT] \otimes F_0 H^{-*}\mathcal{F}_{HH}(R_0/k) \cong H^{-*}(\mathcal{F}_{HH}(R/k)).$$

We also need the following lemma to prove the main result of this section.
Lemma 6.5 Let $R$ be essentially of finite type over $F \supset \mathbb{Q}$, and let $H_n(R)$ denote either $HH_n(R)$ or $H^{-n}\mathcal{F}_{HH}(R)$. Assume that $H_n(R) = 0$ for some finite set $\{n_1, \ldots, n_r\}$ of positive integers. Then there exist an $F$-algebra of finite type $R'$, and a multiplicatively closed set $S$ such that $R \cong S^{-1}R'$ and $H_n(R') = 0$ for $1 \leq i \leq r$.

Proof Because $R$ is essentially of finite type, it is the localization $R = S^{-1}R''$ of some finite type $F$-algebra $R''$. It is well known that $HH_n(S^{-1}R'') \cong S^{-1}HH_n(R'')$ (see [33, 9.1.8]), and $H^{-n}\mathcal{F}_{HH}(S^{-1}R'') \cong S^{-1}H^{-n}\mathcal{F}_{HH}(R'')$ by [9, 2.8–9].

Because $R''$ is of finite type over $F$, we may write $R'' = F \otimes_{F_0} R_0$ for some finitely generated field extension $F_0$ of $\mathbb{Q}$ and some finite type $F_0$-algebra $R_0$. Note $R_0$ is essentially of finite type over $\mathbb{Q}$, whence $H_p(R_0)$ is a finitely generated $R_0$-module ($p \geq 0$). By Lemma 6.3 and/or Proposition 6.4, $H_p(R'')$ is isomorphic, as an $R''$-module, to a direct sum of copies of $R'' \otimes_{R_0} H_q(R_0)$ with $q \leq p$. In particular, $M = \bigoplus_{i=1}^{r} H_n_i(R'')$ is a finite sum of $R''$-modules, each of which is a—possibly infinite—direct sum of copies of one finitely generated module.

Given that $M$ has this form, the hypothesis that $S^{-1}M = 0$ implies that there exists a nonzero element $s \in \text{Ann}(M) \cap S$. Consider the finite type $F$-algebra $R' = R''[1/s]$. Then $R \cong S^{-1}R'$ and we have $\bigoplus_i H_n_i(R') = M[1/s] = 0$. \hfill \Box

Theorem 6.6 Suppose $k \subset F$ is an extension with $\text{tr.deg}(F/k) = \infty$, and $R$ is essentially of finite type over $F$. If $H^n(\mathcal{F}_{HH}(R/k)) = 0$, then $H^m(\mathcal{F}_{HH}(R/k)) = 0$ for all $m \geq n$.

Proof By Lemma 6.5, we may assume that $R$ is of finite type over $F$. There is a finitely generated field extension $F_0 \subset F$ of $k$ and a finite type $F_0$-algebra $R_0$ such that $R = R_0 \otimes_{F_0} F$. Note that $\text{tr.deg}(F/F_0) = \infty$. By Lemma 6.3 and Proposition 6.4, $\Omega^i_{F/F_0} \otimes_{F_0} H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ is a direct summand of $H^n(\mathcal{F}_{HH}(R/k))$ for each $i \geq 0$. Since $\Omega^i_{F/F_0} \neq 0$ for all $i$, all the $H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ vanish as well. Similarly, $H^m(\mathcal{F}_{HH}(R/k))$ is a direct sum of copies of the groups $\Omega^j_{F/F_0} \otimes_{F_0} H^{m+j}(\mathcal{F}_{HH}(R_0/k))$ for $j \geq 0$, all of which vanish when $m \geq n$, as we just observed. \hfill \Box

Corollary 6.7 Let $\mathbb{Q} \subset F$ be a field extension of infinite transcendence degree, and suppose $R$ is essentially of finite type over $F$. Then $NK_n(R) = 0$ implies that $R$ is $K_n$-regular.

Proof Combine Theorem 6.6 with Proposition 4.1 and Corollary 4.4. \hfill \Box
Here is another proof of Corollary 6.7, which is essentially due to Murthy and Pedrini and given in their 1972 paper [20]; they stated the result only for \( n \leq 1 \) because transfer maps for higher \( K \)-theory and the \( W(R) \)-module structure had not yet been discovered. We are grateful to Joseph Gubeladze [16] for pointing this out to the authors.

**Lemma 6.8** If \( R \) is an algebra over a field \( k \) of characteristic 0, \( N^p K_n(R[t]) \rightarrow N^p K_n(R \otimes_k k(t)) \) is injective.

*Proof* The proof in [20, 1.3–1.6] goes through, taking into account that the norm map and localization sequences used there for \( K_0, K_1 \) are now known for all \( K_n \). \( \square \)

**Lemma 6.9** Suppose that \( k \) is an algebraically closed field of infinite transcendence degree over \( \mathbb{Q} \), and that \( R \) is a finitely generated \( k \)-algebra. If \( NK_n(R) \) is zero, then \( K_n(R) \rightarrow K_n(R[x_1, \ldots, x_p]) \) for all \( p > 0 \).

*Proof* Muthy and Pedrini prove this in [20, 2.1.]; although their result is only stated for \( i \leq 1 \), their proof works in general. Note that since \( NK_n(R) \) has the form \( \mathbb{T} K_n(R) \otimes \mathbb{Q}[t] \) by (0.3) (a result which was not known in 1972), \( NK_n(R) \) is torsionfree, and has finite rank if and only if it is zero. \( \square \)

*Proof of Corollary 6.7* Let \( \Phi \) denote the functor \( N^p K_n \). If \( k \subset k_1 \) is a finite algebraic field extension and \( R \) is a \( k \)-algebra, then \( \Phi(R) \rightarrow \Phi(R \otimes_k k_1) \) is an injection because its composition with the transfer \( \Phi(R \otimes_k k_1) \rightarrow \Phi(R) \) is multiplication by \( [k_1 : k] \), and \( \Phi(R) \) is a torsionfree group. Since \( \Phi \) commutes with filtered colimits of rings, \( \Phi(R) \rightarrow \Phi(R \otimes_k \mathbb{K}) \) is an injection. Thus Lemma 6.9 suffices to prove Corollary 6.7 when \( R \) is of finite type. \( \square \)

7. \( NK_0 \) of surfaces

We conclude with a general description for affine surfaces of the canonical map \( \Omega^1_F \otimes_F NK_{-1} \rightarrow NK_0 \). This sheds light on the difference between the cases of small and large base fields, and also explains some results of [35].

If \( R \) is a 2-dimensional noetherian ring then \( NK_0(R) \) is the direct sum of \( NK_0^{(1)}(R) = N \text{Pic}(R) \) and \( NK_0^{(2)}(R) \).

**Theorem 7.1** Let \( R \) be a 2-dimensional normal domain of finite type over a field \( F \) of characteristic 0. There is an exact sequence:

\[
0 \rightarrow NK_1^{(2)}(R) \rightarrow (H^0(R, \Omega^1_{r/F})/\Omega^1_{r/F}) \otimes \mathbb{Q}[t] \rightarrow \Omega^1_F \otimes_F NK_{-1}(R) \rightarrow NK_0(R) \rightarrow H_{cdh}^1(R, \Omega^1_{r/F}) \otimes \mathbb{Q}[t] \rightarrow 0.
\]
Proof  Consider the following short exact sequence of sheaves in \((\text{Sch}/F)_{\text{cdh}}\):

\[ 0 \to \Omega^1_F \otimes_F \mathcal{O} \to \Omega^1 \to \Omega^1_F \to 0. \]

Applying \(H_{\text{cdh}}\) yields

\[ 0 \to \Omega^1_F \otimes_F R \xrightarrow{\iota} H^0(R, \Omega^1) \to H^0(R, \Omega^1_F) \]

\[ \to \Omega^1_F \otimes_F H^1_{\text{cdh}}(R, \mathcal{O}) \to H^1_{\text{cdh}}(R, \Omega^1) \to H^1_{\text{cdh}}(R, \Omega^1_F) \to 0. \]

Note that, because \(\Omega^1_R \to \Omega^1_{R/F}\) is onto, the map \(\partial\) kills the image of \(\Omega^1_{R/F}\). Similarly, the image of \(\iota\) is contained in that of \(\Omega^1_R\). Thus we obtain

\[ 0 \to H^0(R, \Omega^1)/\Omega^1_R \to H^0(R, \Omega^1_{R/F})/\Omega^1_{R/F} \]

\[ \to \Omega^1_F \otimes_F H^1_{\text{cdh}}(R, \mathcal{O}) \to H^1_{\text{cdh}}(R, \Omega^1) \to H^1_{\text{cdh}}(R, \Omega^1_{R/F}) \to 0. \]

Now apply \(\otimes t \mathbb{Q}[t]\) and use Theorem 5.1 and parts (c) and (d) of Theorem 5.2. \(\square\)

Corollary 7.2  Let \(R\) be a 2-dimensional normal domain of finite type over a field \(F\) of characteristic 0. If \(NK_{-1}(R) = 0\) then \(NK_0(R) \cong H^1_{\text{cdh}}(R, \Omega^1_F) \otimes t \mathbb{Q}[t]\).

Example 7.3  Let \(R\) be a 2-dimensional normal domain of finite type over \(\mathbb{Q}\), and put \(RF = R \otimes F\). By Propositions 4.1 and 6.4,

\[ NK_*(RF) \cong NK_*(R) \otimes \Omega^*_F/\mathbb{Q}. \quad (7.4) \]

Keeping track of the \(\lambda\)-decomposition, as in Theorem 5.1, we see from Theorem 0.7 that

\[ TK_1^{(2)}(RF) \cong TK_1^{(2)}(R) \otimes F \cong H^0(R, \Omega^1) \otimes F/\Omega^1_R \otimes F \]

\[ \cong H^0(R_F, \Omega^1_{R/F})/\Omega^1_{R/F}. \]

From Theorem 7.1 we get an exact sequence

\[ 0 \to \Omega^1_{F/\mathbb{Q}} \otimes_F NK_{-1}(RF) \to NK_0(RF) \to H^1_{\text{cdh}}(RF, \Omega^1_{R/F}) \otimes t \mathbb{Q}[t] \to 0. \quad (7.5) \]

Using (7.4) and Theorem 0.7 again, we see that the sequence (7.5) is isomorphic to the sum

\[ \otimes \text{Springer} \]
\[
(0 \to \Omega^1_{F/Q} \otimes \mathrm{H}^1_{\text{cdh}}(R, \mathcal{O}) \otimes \mathbb{Q}[t] \\
\overset{\sim}{\longrightarrow} \Omega^1_{F/Q} \otimes \mathrm{H}^1_{\text{cdh}}(R, \mathcal{O}) \otimes \mathbb{Q}[t] \to 0 \to 0)
\]
\[
(0 \to 0 \to F \otimes \mathrm{H}^1_{\text{cdh}}(R, \Omega^1) \otimes t \mathbb{Q}[t] \overset{\sim}{\longrightarrow} F \otimes \mathrm{H}^1_{\text{cdh}}(R, \Omega^1) \otimes t \mathbb{Q}[t] \to 0).
\]

For example, for \( R_F := F[x, y, z]/(z^2 + y^3 + x^{10} + x^7y) \) the results of [8] show that:

\[
\begin{align*}
\NK_{-1}(R_F) &= F \otimes \mathbb{Q}[t], \\
\NK_0(R_F) &= \Omega^1_{F/Q} \otimes t \mathbb{Q}[t] \cong \bigoplus_{p=1}^{\text{tr. deg}(F)} F \otimes t \mathbb{Q}[t].
\end{align*}
\]

In other words, both typical pieces \( TK_{-1}(R_F) \) and \( TK_0(R_F) \) are \( F \)-vectorspaces, but while \( \dim_F TK_{-1}(R_F) = 1 \) for all \( F \), any cardinal number \( \kappa \) can be realized as \( \dim_F TK_0(R_F) \) for an appropriate \( F \).

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