ON HIGHLY-REGULAR GRAPHS

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Abstract. Highly-regular graphs can be regarded as a combinatorial generalization of distance-regular graphs. From this standpoint, we study combinatorial aspects of highly-regular graphs. As a result, we give the following three main results in this paper. Firstly, we give a characterization of a distance-regular graph by using the index and diameter of a highly-regular graph. Secondly, we give two constructions of highly-regular graphs. Finally, we generalize well-known properties of the intersection numbers of a distance-regular graph.

1. Introduction

All graphs which we consider in this paper are finite undirected graphs without loops and multiple edges. For a graph Γ, we denote the vertex set of Γ by \( V(\Gamma) \), the edge set of Γ by \( E(\Gamma) \). For a connected graph Γ and two vertices \( u, v \in V(\Gamma) \), let \( d(u, v) \) be the distance of \( u \) and \( v \), that is, the length of the shortest path from \( u \) to \( v \), and \( \text{diam}(\Gamma) \) be the diameter of a graph Γ, that is, the maximum length of \( d(u, v) \), \( u,v \in V(\Gamma) \). If Γ is not connected, the diameter of Γ is defined as infinity. For a vertex \( u \) and an integer \( i \in \{0, \ldots, \text{diam}(\Gamma)\} \), let \( D_i(u) \) or \( D_i,\Gamma(u) \) be a set of all vertices which are at distance \( i \) from \( u \). Let \( V(\Gamma) = \{v_1, \ldots, v_n\} \) a labeling of \( V(\Gamma) \) and \( A \) be the adjacency matrix of a graph Γ with spectrum \( \text{sp}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \ldots, \lambda_d^{m(\lambda_d)}\} \), where \( \lambda_0 > \cdots > \lambda_d \), and the superscripts \( m(\lambda_i) \) stand for the multiplicities. Let \( E_i, i = 0, \ldots, d \) be the minimal idempotents representing the orthogonal projections on the eigenspaces associated with \( \lambda_i \). The above notation is used throughout this paper.

Classically, many special regular graphs have been widely studied. An important class of regular graphs is the class of strongly-regular graphs. Here, a connected graph Γ is strongly-regular with parameters \( (k, \alpha, \beta) \) if it is \( k \)-regular, for any \( u,v \in V(\Gamma) \) with \( \{u,v\} \in E(\Gamma) \), \( |D_1(u) \cap D_1(v)| = \alpha \) and for any \( u,v \in V(\Gamma) \) with \( \{u,v\} \notin E(\Gamma) \), \( |D_1(u) \cap D_1(v)| = \beta \). The class of strongly-regular graphs is much smaller than the class of regular graphs. There is a wider class which is contained in the class of regular graphs and contains the class of strongly-regular graphs. It is the class of highly-regular graphs. Here, a graph Γ of order \( n \) is highly-regular with collapsed adjacency matrix (CAM, for short) \( C = [c_{i,j}]_{1 \leq i,j \leq m} \) \( (2 \leq m < n) \) if for every vertex \( u \in V(\Gamma) \) there is a partition of \( V(\Gamma) \) into \( V_1(u) = \{u\}, V_2(u), \ldots, V_m(u) \)

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such that each vertex \( y \in V_j(u) \) is adjacent to \( c_{i,j} \) vertices in \( V_i(u) \). In this paper, we allow \( m = n \) only if \( n = 2 \). Namely, we regard the 1-regular graph as a highly-regular graph. The class of highly-regular graphs was introduced as an interesting class of regular graphs by B. Bollobás (cf. [7]). Highly-regular graphs are a combinatorial generalization of strongly-regular graphs (cf. [6]). From this standpoint, strongly-regular graphs are characterized by index of highly-regular graphs (cf. [1]). Here, the index of a highly-regular graph is the least positive integer of the size of CAMs. The index is an important invariant of highly-regular graphs.

In addition to the above two classes, there are other classically well-known classes of regular graphs. One of these classes is the class of distance-transitive graphs. Here, a connected graph \( \Gamma \) is distance-transitive if for every four vertices \( u, v, x, y \in V(\Gamma) \) with \( d(u, v) = d(x, y) \), there exists an automorphism \( \sigma \) of \( \Gamma \) such that \( \sigma(x) = u \) and \( \sigma(y) = v \) (cf. [8]). One of the important and remarkable properties of distance-transitive graphs is that there are only finitely many distance-transitive graphs of fixed valency greater than 2 (cf. [9], [10]). There are many important properties of distance-transitive graphs which are related to the other mathematical theory such as algebraic combinatorics. N. Biggs introduced distance-regular graphs as a combinatorial generalization of distance-transitive graphs by observing that several combinatorial properties of distance-transitive graphs also hold in the class of distance-regular graphs. Here, a connected graph \( \Gamma \) is distance-regular if the integers \( |D_1(v) \cap D_{i-1}(u)| \), \( |D_1(v) \cap D_i(u)| \), and \( |D_1(v) \cap D_{i+1}(u)| \) depend only on \( i = d(u, v) \in \{1, \ldots, \text{diam}(\Gamma)\} \) (cf. [8]). In 1984, E. Bannai and T. Ito conjectured that there are only finitely many distance-regular graphs of fixed valency greater than 2 (cf. [1]). In 2015, S. Bang, A. Dubickas, J. H. Koolen and V. Moulton proved this conjecture conclusively (cf. [3]).

So far, we introduced four classes of regular graphs, that is, strongly-regular graphs, highly-regular graphs, distance-transitive graphs and distance-regular graphs. As we mentioned above, highly-regular graphs and distance-regular graphs were introduced in different contexts. However, there is a naturally connection. Distance-regular graphs are highly-regular graphs, that is, highly-regular graphs can be regarded as a combinatorial generalization of distance-regular graphs. From this standpoint, we study combinatorial aspects of highly-regular graphs. As a result, we give the following three main results in this paper.

Firstly, distance-regular graphs are characterized by index and diameter of highly-regular graphs.

**Theorem 1.1.** A connected highly-regular graph \( \Gamma \) has the index \( \text{diam}(\Gamma) + 1 \) if and only if it is a distance-regular graph.

Secondly, we give a construction of connected highly-regular graphs with diameter 2 which are not distance-regular graphs. Moreover, we also give a construction of highly-regular graphs which do not always have diameter 2 by using a symmetric association scheme.
Theorem 1.2. For a highly-regular graph $\Gamma$ with $3 \leq \text{diam}(\Gamma) < \infty$, the complement of the graph is a highly-regular graph with $\text{diam}(\overline{\Gamma}) = 2$ which is not a distance-regular graph. Moreover, Let $\mathcal{X} = (X, \{R_i \}_{i=0}^d)$ be a symmetric association scheme of class $d$. Then, for each $l \in \{1, \ldots, d\}$, the graph $\Gamma_{R_l} = (X, E_{R_l})$ is a highly-regular graph, where $E_{R_l} = \{\{x,y\} \mid (x,y) \in R_l\}$.

Poulos showed that finite upper half plane graphs are highly-regular graphs (cf. [2], [15]). The construction of highly-regular graphs by using a symmetric association scheme in Theorem 1.2 is a generalization of Poulos’s result.

Finally, we give a generalization of well-known properties of the intersection numbers of distance-regular graphs.

Theorem 1.3. Let $\Gamma$ be a connected highly-regular graph with $\text{CAM} C = [c_{i,j}]_{i,j \leq m}$ and valency $k$. Here, let a labeling of $C$ be a labeling with respect to distance. For each $i \in \{0, \ldots, \text{diam}(\Gamma)\}$, there exists a nonempty subset $S_i$ of $I = \{1, \ldots, m\}$ such that $D_i(u) = \bigcup_{l \in S_i} V_l(u)$, $u \in V$. For each $i \in \{1, \ldots, \text{diam}(\Gamma)\}$, let the integers $b_{i-1}^{\text{max}}$, $c_i^{\text{max}}$, and $c_i^{\text{min}}$ be the following:

- $b_{i-1}^{\text{max}} = \max \{\sum_{l \in S_i} c_{i,l} \mid l \in S_{i-1}\}$.
- $c_i^{\text{max}} = \max \{\sum_{l \in S_{i-1}} c_{i,l} \mid l \in S_i\}$.
- $c_i^{\text{min}} = \min \{\sum_{l \in S_{i-1}} c_{i,l} \mid l \in S_i\}$.

Then, the following inequalities hold:

1. $k = b_0^{\text{max}} \geq b_1^{\text{max}} \geq \cdots \geq b_{\text{diam}(\Gamma)-1}^{\text{max}} \geq 1$.
2. $1 = c_1^{\text{min}} \leq c_2^{\text{min}} \leq \cdots \leq c_{\text{diam}(\Gamma)}^{\text{min}} \leq k$.

Moreover, we suppose that $\Gamma$ satisfies the following property: \((\ast)\) For any $u \in V(\Gamma)$, $i \in \{0, 1, \ldots, \text{diam}(\Gamma) - 1\}$, $x \in D_i(u)$, the set $D_1(x) \cap D_{i+1}(u)$ is nonempty set. Then, the following inequality holds:

3. If $i \geq 1$ and $i + j \leq \text{diam}(\Gamma)$, then $c_i^{\text{max}} \leq b_j^{\text{max}}$.

It is well-known that distance-regular graphs have interesting combinatorial properties. Moreover, distance-regular graphs have many applications such as coding theory. On the other hand, the class of highly-regular graphs is very wide. For example, vertex-transitive graphs with non-identity stabilizers are highly-regular graphs. For such reasons, it seems that it is difficult to investigate interesting properties of highly-regular graphs. As we described above, however, highly-regular graphs have similar properties of distance-regular graphs. Therefore, we believe that it is an interesting approach to study highly-regular graphs that highly-regular graphs are considered as a generalization of distance-regular graphs.

2. Preliminaries

Let $\Gamma$ be a graph of order $n$. We denote the complement of $\Gamma$ by $\overline{\Gamma}$. If $\Gamma$ is a distance-regular graph, the complement of $\Gamma$ is not always a distance-regular graph. However, if $\Gamma$ is a strongly-regular graph and $\Gamma$ is connected, $\Gamma$ is also a strongly-regular graph. This property is generalized to a highly-regular graph as follows.
Proposition 2.1. (cf. [1, PROPOSITION 1]). A graph $\Gamma$ is a highly-regular graph if and only if the complement of $\Gamma$ is a highly-regular graph.

For two graphs $\Gamma_1$ and $\Gamma_2$, the Cartesian product $\Gamma_1 \square \Gamma_2$ of $\Gamma_1$ and $\Gamma_2$ is a graph such that the vertex set of $\Gamma_1 \square \Gamma_2$ is $V(\Gamma_1) \times V(\Gamma_2)$, and two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent if and only if $u_1 = u_2$ and $\{v_1, v_2\} \in E(\Gamma_2)$, or $v_1 = v_2$ and $\{u_1, u_2\} \in E(\Gamma_1)$. In general, the Cartesian product of two distance-regular graphs is not always a distance-regular graph. However, the Cartesian product of two highly-regular graphs is a highly-regular graph.

Proposition 2.2. (cf. [1, PROPOSITION 6]). If $\Gamma_1$ and $\Gamma_2$ are highly-regular graphs, the Cartesian product $\Gamma_1 \square \Gamma_2$ is a highly-regular graph.

Properties of a partition of a vertex set corresponding to a CAM are important in the class of highly-regular graphs. It is straightforward to see that distance-regular graphs are highly-regular graphs. Therefore, highly-regular graphs are a combinatorial generalization of distance-regular graphs. The following expected properties are satisfied.

Proposition 2.3. (cf. [1, PROPOSITION 3]). Let $\Gamma$ be a connected highly-regular graph with CAM $C = \{c_{ij}\}_{i,j \leq m}$. For $u, v \in V(\Gamma)$, let $V_1(u) = \{u\}, V_2(u), \ldots, V_m(u)$ and $V_1(v) = \{v\}, V_2(v), \ldots, V_m(v)$ be the corresponding partitions of $V(\Gamma)$. Then the following properties are satisfied.

1. For each $i \in \{0, 1, \ldots, \text{diam}(\Gamma)\}$, there exists a nonempty subset $S_i$ of $I = \{1, 2, \ldots, m\}$ such that $D_i(u) = \bigcup_{i \in S_i} V_i(u)$, $D_i(v) = \bigcup_{i \in S_i} V_i(v)$.

2. For $t \in \{1, 2, \ldots, m\}$, $|V_t(u)| = |V_t(v)|$.

3. For $i \in \{0, 1, \ldots, \text{diam}(\Gamma)\}$, the induced subgraph of $D_i(u)$ and the induced subgraph of $D_i(v)$ have the same degree sequence.

In Section 1, we introduced four classes of regular graphs, that is, strongly-regular graphs, highly-regular graphs, distance-transitive graphs and distance-regular graphs. Now, we introduce an invariant of highly-regular graphs (cf. [1]).

Let $\Gamma$ be a highly-regular graph of order $n$. The index of $\Gamma$ is the least positive integer $m$ such that a CAM has the size $m$ ($2 \leq m < n$), and we denote by $i(\Gamma)$. By Proposition 2.1, the index of a highly-regular graph is equal to that of the complement of the graph. By Proposition 2.3, the index $i(\Gamma)$ is greater than $\text{diam}(\Gamma)$ if the graph $\Gamma$ is connected. Moreover, we get a lower bound of $i(\Gamma)$.

Let $\Gamma$ be a connected highly-regular graph. For any vertex $u \in V(\Gamma)$ and for any $i \in \{1, \ldots, \text{diam}(\Gamma)\}$, we denote the induced subgraph of $D_i(u)$ by $\langle D_i(u) \rangle$. Then, we have the following proposition.

Proposition 2.4. (cf. [1, PROPOSITION 5]). If the cardinality of the degree set of $\langle D_i(u) \rangle$ ($1 \leq i \leq \text{diam}(\Gamma)$) is $k_i$, then

$$i(\Gamma) \geq 1 + \sum_{i=1}^{\text{diam}(\Gamma)} k_i.$$
By the above discussion, we have the following inequalities.

\[ i(\Gamma) \geq 1 + \sum_{i=1}^{\text{diam}(\Gamma)} k_i \geq 1 + \text{diam}(\Gamma). \]

Here, we note that we can easily find highly-regular graphs which do not attain the above equalities.

3. A characterization of distance-regular graphs by using index and diameter of highly-regular graphs

In Section 2, we introduced index of highly-regular graphs. By using the index, we can characterize a strongly-regular graph, that is, a connected highly-regular graph has the index 3 if and only if it is a strongly-regular graph (cf. [1]). In this section, we characterize a distance-regular graph by using the index.

**Theorem 3.1.** A graph \( \Gamma \) is a connected highly-regular graph with the valency \( k \) and CAM of the form (up to a labeling)

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
k & a_1 & c_2 & \vdots & \vdots \\
0 & b_1 & \ddots & \ddots & 0 & \vdots \\
\vdots & 0 & \ddots & \ddots & c_{m-2} & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 & b_{m-2} \\
0 & 0 & \cdots & 0 & b_{m-2} & a_{m-1}
\end{pmatrix}
\]

if and only if it is a distance-regular graph with \( \text{diam}(\Gamma) = m - 1 \).

**Proof.** Let \( \Gamma \) be a connected highly-regular graph which satisfies the above condition. For any \( v \in V(\Gamma) \), there exists a partition of \( V(\Gamma) \) with respect to the above CAM. Let \( V_0(v) = \{v\}, V_1(v), \ldots, V_{m-1}(v) \) be the partition of \( V(\Gamma) \). First, we have \( D_0(v) = \{v\} = V_0(v) \). By the first column of the CAM, we have \( D_1(v) = V_1(v) \). Then, by the second column of the CAM, we have \( D_2(v) = V_2(v) \). By repeating this argument, we have \( D_{\text{diam}(\Gamma)} = V_{m-1}(v) \). Hence, \( m = \text{diam}(\Gamma) + 1 \) and \( \Gamma \) is a distance-regular graph.

Conversely, we suppose a graph \( \Gamma \) is a distance-regular graph with \( \text{diam}(\Gamma) = m - 1 \). By the definition of a distance-regular graph and the index, we have \( 1 + \text{diam}(\Gamma) \geq i(\Gamma) \). By \( i(\Gamma) \geq 1 + \text{diam}(\Gamma) \), we have \( i(\Gamma) = m \). Therefore, we have the desired result.

\[ \square \]

By Theorem 3.1, we conclude the following characterization of a distance-regular graph by using the index.

**Corollary 3.2.** Let \( \Gamma \) be a connected highly-regular graph. Then, the graph \( \Gamma \) has the index \( \text{diam}(\Gamma) + 1 \) if and only if it is a distance-regular graph.
This is very useful to determine whether a highly-regular graph is a distance-regular graph.

4. A CONSTRUCTION OF HIGHLY-REGULAR GRAPHS WITH DIAMETER 2 WHICH ARE NOT DISTANCE-REGULAR GRAPHS

In Section 3, we characterize a distance-regular graph by the index of a highly-regular graph. In this section, we construct highly-regular graphs with diameter 2 which are not distance-regular graphs.

Let $\Gamma$ be a highly-regular graph with $2 \leq \text{diam}(\Gamma) < \infty$.

First, we consider the case where $\Gamma$ is a distance-regular graph. By Corollary 3.2, the graph $\Gamma$ has the index $\text{diam}(\Gamma) + 1$.

If $\text{diam}(\Gamma)$ is equal to 2, the graph $\Gamma$ and the complement $\overline{\Gamma}$ have the index 3. In this case, if $\overline{\Gamma}$ is connected, both $\Gamma$ and $\overline{\Gamma}$ are strongly-regular graphs.

If $\text{diam}(\Gamma)$ is greater than 2, $\text{diam}(\overline{\Gamma})$ is equal to 2. Here, we note that for a connected regular graph with the diameter greater than 2, the diameter of the complement is 2. Hence, we have the following inequality.

$$i(\Gamma) = 1 + \text{diam}(\Gamma) > 1 + \text{diam}(\overline{\Gamma}).$$

By Proposition 2.1, the index of $\Gamma$ is equal to the index of $\overline{\Gamma}$. Therefore, we have the following inequality.

$$i(\overline{\Gamma}) > 1 + \text{diam}(\overline{\Gamma}).$$

By Corollary 3.2, the graph $\overline{\Gamma}$ is a highly-regular graph which is not a distance-regular graph with $\text{diam}(\overline{\Gamma}) = 2$.

Next, we consider the case where the graph $\Gamma$ is not a distance-regular graph. If $\text{diam}(\overline{\Gamma})$ is equal to 2, the index of $\overline{\Gamma}$ is greater than 3. If $\text{diam}(\Gamma)$ is greater than 2, the index of $\overline{\Gamma}$ is greater than 3. Therefore, the graph $\overline{\Gamma}$ is a highly-regular graph which is not a distance-regular graph with $\text{diam}(\overline{\Gamma}) = 2$.

By the above discussion, we conclude the following theorem.

**Theorem 4.1.** For a highly-regular graph $\Gamma$ with $3 \leq \text{diam}(\Gamma) < \infty$, the complement of the graph is a highly-regular graph with $\text{diam}(\overline{\Gamma}) = 2$ which is not a distance-regular graph.

By taking the complement of a distance-regular graph with the diameter greater than 2, we obtain a highly-regular graph with the diameter 2 which is not a distance-regular graph.

5. ANOTHER CONSTRUCTION OF HIGHLY REGULAR GRAPHS BY USING A SYMMETRIC ASSOCIATION SCHEMES

In Section 4, we gave a construction of highly-regular graphs which are not distance-regular graphs. However, this construction can generate only highly-regular graphs with diameter 2 which are not distance-regular graphs. In this section, we
give another construction of highly-regular graphs by using a symmetric association
scheme.

Let \( X \) be a finite set and \( R_i (i = 0, \ldots, d) \) be nonempty subsets of \( X \times X \). A

**symmetric association scheme of class** \( d \) **is a pair** \( \mathfrak{X} = (X, \{R_i\}_{i=0}^d) **satisfying the following conditions:**

- \((SAS-1)\) \( R_0 = \{(x,x) \mid x \in X\}\).
- \((SAS-2)\) \( X \times X = \bigsqcup_{i=0}^d R_i \).
- \((SAS-3)\) \( ^tR_i = R_i \) for any \( i \in \{0, \ldots, d\} \), where \( ^tR_i = \{(y,x) \mid (x,y) \in R_i\}\).
- \((SAS-4)\) for any \( i, j, l \in \{1, \ldots, d\} \), there exists constants \( p_{i,j}^l \) such that for all \( x, y \in X \) with \( (x,y) \in R_l \),
  \( p_{i,j}^l = |\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}| \).

The above constants \( p_{i,j}^l \) are called the **intersection numbers**. For each \( i \in \{0, \ldots, d\} \), we denote the matrix \( [p_{i,j}^l]_{0 \leq j, l \leq d} \) by \( B_i \). The matrix \( B_i \) is called the **i-th intersection matrix** of \( \mathfrak{X} \). For any \( x \in X \) and \( R_i \), we denote a set of elements \( y \in X \) with \( (x,y) \in R_i \) by \( xR_i \).

Let \( \mathfrak{X} = (X, \{R_i\}_{i=0}^d) **be a symmetric association scheme of class** \( d \). Then, we
have the following theorem.

**Theorem 5.1.** For each \( l \in \{0, \ldots, d\} \), the graph \( \Gamma_{R_l} = (X, E_{R_l}) **is a highly-regular graph with CAM** \( B_l \).

**Proof.** We take arbitrary \( x \in X \). By \( (SAS-2) \), we get the decomposition of \( X \) as follows:

\[
X = \bigsqcup_{i=0}^d xR_i.
\]

We take any two partitions \( xR_i, xR_j \) of the above decomposition. By \( (SAS-4) \), for any \( y \in xR_j \), the number of vertices \( z \in xR_i \) such that \( \{y,z\} \in E_{R_l} \) is equal to \( p_{i,j}^l \).

The integer \( p_{i,j}^l \) is independent of choice of an element \( (x,y) \in R_j \). In particular,
the integer \( p_{i,j}^l \) is independent of choice of \( y \in xR_j \). Therefore, the graph \( \Gamma_{R_l} \) is a
highly-regular graph with CAM \( B_l \). \( \square \)

Moreover, let \( G \) be a finite group and \( G \) acts on \( X \) transitively. Let \( S = \{\Delta_0, \ldots, \Delta_d\} \) be the set of \( G \)-orbits of \( X \times X \). We suppose that each \( G \)-orbit of \( X \times X \) is symmetric. Then, \( \mathfrak{X} = (X, S) **is a symmetric association scheme of class** \( d \). This symmetric association scheme is closely related to harmonic analysis
of finite homogeneous spaces (cf. [4], [11]). By Theorem 5.1, we have the following corollary.

**Corollary 5.2.** For each \( l \in \{0, \ldots, d\} \), the graph \( X_{\Delta_l} \) **is a highly-regular graph with CAM** \( B_l \).

By Theorem 5.1 and Corollary 5.2, we get many examples of highly-regular graphs
which are not always distance-regular graphs. For example, Euclidean graphs and
finite upper half plane graphs are highly-regular graphs (cf. [15]). Generalized Euclidean graphs are highly-regular graphs (cf. [5], [13]). Moreover, the other graphs which appear in [13] are also.

In the rest of this section, we construct graphs which are a special case of graphs defined by W. Li in [14]. Let \( p \) be an odd prime, \( r \) be an even number and \( \mathbb{F}_p^r / \mathbb{F}_p \) be a finite field extension of degree \( r \). Let \( N_r \) be the kernel of the norm map of the extension \( \mathbb{F}_p^r / \mathbb{F}_p \). \( N_r \) acts \( \mathbb{F}_p^r \) by multiplication. Then, we consider the semidirect product group \( N_r \rtimes \mathbb{F}_p^r \). The semidirect product group \( N_r \rtimes \mathbb{F}_p^r \) acts \( \mathbb{F}_p^r \) naturally.

Fix \( 0 \in \mathbb{F}_p^r \). The stabilizer of \( 0 \) is \( N_r \rtimes \{0\} \cong N_r \). Then, we have the \( N_r \rtimes \mathbb{F}_p^r \)-orbit decomposition of \( \mathbb{F}_p^r \times \mathbb{F}_p^r \) as follows:

\[
\mathbb{F}_p^r \times \mathbb{F}_p^r = \bigsqcup_{i \in \mathbb{F}_p^r} \Delta_i,
\]

where for each \( i \in \mathbb{F}_p^r \), \( \Delta_i = \{(x, y) \in \mathbb{F}_p^r \times \mathbb{F}_p^r \mid N_{\mathbb{F}_p^r / \mathbb{F}_p}(y - x) = i\} \) and \( \Delta_0 = \{(x, x) \in \mathbb{F}_p^r \times \mathbb{F}_p^r\} \). Since \( p \) is an odd and \( r \) is an even, each \( N_r \rtimes \mathbb{F}_p^r \)-orbit is symmetric. Therefore, the pair \((N_r \rtimes \mathbb{F}_p^r, N_r \rtimes \{0\})\) is a (symmetric) Gelfand pair. By Corollary 5.2, for each \( l \in \{0, \ldots, p-1\} \), the graph \( X_{\Delta_l} \) is a highly-regular graph. We denote \( X_{\Delta_l} \) by \( WL(p, r, l) \) in this paper. Here, we note that we can easily compute the \( N_r \rtimes \mathbb{F}_p^r \)-irreducible decomposition of the \( \ell^2 \)-space \( \ell^2(\mathbb{F}_p^r) \) by using (1).

Therefore, we get Kloosterman sums as the spherical functions and several formulas corresponding to formulas of spherical functions such as convolution property and addition theorem. Moreover, we get a formula of Kloosterman sums by using the fact that they are simultaneous eigenfunctions of the intersection matrices.

**Remark 5.3.** We can apply the above ways to give several formulas of character sums arising as spherical functions including Gauss periods and Kloosterman sums (cf. [5], [13]). Moreover, we note that both eigenvalues and eigenvectors of \( B_l \) are expressed by the same spherical functions. This is an interesting property of highly-regular graphs constructed by using Corollary 5.2.

6. Basic properties of the elements of a CAM

In Section 3, we showed that a connected highly-regular graph has the index \( \text{diam}(\Gamma) + 1 \) if and only if it is a distance-regular graph. Naturally, hence, we can regard the elements of a CAM as generalized constants of the intersection numbers of distance-regular graphs.

Let \( \Gamma \) be a highly-regular graph which satisfies the condition in Theorem 3.1. The following are well-known basic properties of the intersection numbers of a distance-regular graph:

- \( k = b_0 \geq b_1 \geq \cdots \geq b_{m-2} \geq 1 \).
- \( 1 = c_1 \leq c_2 \leq \cdots \leq c_{m-1} \leq k \).

In this section, we give a generalization of the above basic properties of intersection numbers of a distance-regular graph.
Proposition 2.3, for each $i \in \{0, 1, \ldots, \operatorname{diam}(\Gamma)\}$, there exists a nonempty subset $S_i$ of $I = \{1, \ldots, m\}$ such that for any $u \in V$, $D_i(u) = \bigcup_{t \in S_i} V_t(u)$. For each $i \in \{1, \ldots, \operatorname{diam}(\Gamma)\}$, let the integers $b_{i-1}^\text{max}, c_i^\text{max},$ and $c_i^\text{min}$ be the following:

- $b_{i-1}^\text{max} = \max\{\sum_{t \in S_i} c_{t,l} \mid l \in S_{i-1}\}$.
- $c_i^\text{max} = \max\{\sum_{t \in S_i} c_{t,l} \mid l \in S_i\}$.
- $c_i^\text{min} = \min\{\sum_{t \in S_i} c_{t,l} \mid l \in S_i\}$.

Then, we get the following proposition.

**Proposition 6.1.** We have the following inequalities:

1. $k = b_0^\text{max} \geq b_1^\text{max} \geq \cdots \geq b_{\operatorname{diam}(\Gamma)-1}^\text{max} \geq 1$.
2. $1 = c_1^\text{min} \leq c_2^\text{min} \leq \cdots \leq c_{\operatorname{diam}(\Gamma)}^\text{min} \leq k$.

**Proof.** (1) For $i = 1$, it is clear that $b_0^\text{max}$ is equal to $k$. For $i \geq 1$, we take arbitrary $y \in V(\Gamma)$ and $l \in S_i$. Then, there exist elements $z \in D_1(y)$ and $s \in S_{i-1}$ such that $V_l(y) \cap V_s(z) \neq \emptyset$. We take arbitrary $x \in V_l(y) \cap V_s(z)$. First, we show that $D_1(x) \cap D_{i+1}(y) \subset D_1(x) \cap D_i(z)$. We take $w \in D_1(x) \cap D_{i+1}(y)$. The distance $d(z, w)$ is less than or equal to $d(z, x) + d(x, w) = i$. On the other hand, the distance $d(z, w)$ is greater than or equal to $i$ since the distance $d(z, w) + d(z, y)$ is greater than or equal to the distance $d(w, y) = i + 1$. Hence, the element $w$ is in $D_1(x) \cap D_i(z)$. Then, we have $D_1(x) \cap D_{i+1}(y) \subset D_1(x) \cap D_i(z)$. By using this, we have

$$\sum_{t \in S_{i+1}} c_{t,l} \leq \sum_{t \in S_i} c_{t,s}.$$

Then, for any $l \in S_i$, we have

$$\sum_{t \in S_{i+1}} c_{t,l} \leq b_{i-1}^\text{max}.$$

Therefore, we have $b_{i-1}^\text{max} \leq b_{i-1}^\text{max}$.

(2) For $i = 1$, it is clear that $c_i^\text{min}$ is equal to 1. For $i \geq 1$, we take arbitrary $z \in V(\Gamma)$ and $s \in S_{i+1}$. Then, there exist $y \in D_1(z)$ and $l \in S_i$ such that $V_s(z) \cap V_l(y) \neq \emptyset$. We take arbitrary $x \in V_l(y) \cap V_s(z)$. First, we show that $D_1(x) \cap D_{i-1}(y) \subset D_1(x) \cap D_i(z)$. We take $w \in D_1(x) \cap D_{i-1}(y)$. The distance $d(w, z)$ is less than or equal to $i$ since $d(w, z)$ is less than or equal to $d(y, z) + d(y, w)$. On the other hand, the distance $d(w, z)$ is greater than or equal to $i$ since $d(w, z) + d(w, x)$ is greater than or equal to $d(z, x)$. Hence, the element $w$ is in $D_1(x) \cap D_i(z)$ and we have $D_1(x) \cap D_{i-1}(y) \subset D_1(x) \cap D_i(z)$. By using this, we have

$$\sum_{t \in S_{i-1}} c_{t,l} \leq \sum_{t \in S_i} c_{t,s}.$$

By taking any $l \in S_i$, we have

$$\sum_{t \in S_{i-1}} c_{t,l} \leq c_i^\text{min}.$$
Then, for any $k \in S_{i+1}$, we have
\[ c_{i}^{\min} \leq \sum_{t \in S_{i}} c_{t,s}. \]
Therefore, we have $c_{i}^{\min} \leq c_{i+1}^{\min}$. \hfill \Box

**Remark 6.2.** It is clear that the following property holds:
\[ \sum_{j=0}^{m} c_{i,j} = k. \]

**Remark 6.3.** We note that the following statement may not always hold in general:
\((\star)\) For any $u \in V(\Gamma)$, $i \in \{0, 1, \ldots, \text{diam}(\Gamma) - 1\}$, $x \in D_{i}(u)$, the set $D_{i}(x) \cap D_{i+1}(u)$ is nonempty set. In fact, there is a counter-example of this statement such as the graph $WL(7, 2, 1)$ which we defined in Section 5 (cf. Figure 1). The graph $WL(7, 2, 1)$ has the diameter 3 and for each vertex $u$, there exists a vertex $x$ which is at distance 2 from the vertex $u$ such that $D_{1}(x) \cap D_{3}(u) = \emptyset$.

![Figure 1. $WL(7, 2, 1)$](image)

The above phenomenon is a difference between highly-regular graphs and distance-regular graphs. For a highly-regular graph which satisfies the above statement $\,(\star)$, the integers $b_{i-1}^{\max}$ and $c_{i}^{\max}$ are satisfied the following property.

**Proposition 6.4.** We have the following inequality: If $i \geq 1$ and $i + j \leq \text{diam}(\Gamma)$, then $c_{i}^{\max} \leq b_{i}^{\max}$.

**Proof.** We take arbitrary $y \in V(\Gamma)$, $l \in S_{i}$ and $x \in V_{l}(y)$. By the assumption $\,(\star)$, there exists the shortest path from $x$ to some element $z \in D_{i+j}(y)$. Then, there exists $s \in S_{j}$ such that $x \in V_{s}(z)$. First, we show that $D_{1}(x) \cap D_{i-1}(y) \subset D_{1}(x) \cap D_{j+1}(z)$. We take $w \in D_{1}(x) \cap D_{i-1}(y)$. The distance $d(w, z)$ is less than or equal to $1 + j$. The above phenomenon is a difference between highly-regular graphs and distance-regular graphs. For a highly-regular graph which satisfies the above statement $\,(\star)$, the integers $b_{i-1}^{\max}$ and $c_{i}^{\max}$ are satisfied the following property.

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**Proof.** We take arbitrary $y \in V(\Gamma)$, $l \in S_{i}$ and $x \in V_{l}(y)$. By the assumption $\,(\star)$, there exists the shortest path from $x$ to some element $z \in D_{i+j}(y)$. Then, there exists $s \in S_{j}$ such that $x \in V_{s}(z)$. First, we show that $D_{1}(x) \cap D_{i-1}(y) \subset D_{1}(x) \cap D_{j+1}(z)$. We take $w \in D_{1}(x) \cap D_{i-1}(y)$. The distance $d(w, z)$ is less than or equal to $1 + j$. The above phenomenon is a difference between highly-regular graphs and distance-regular graphs. For a highly-regular graph which satisfies the above statement $\,(\star)$, the integers $b_{i-1}^{\max}$ and $c_{i}^{\max}$ are satisfied the following property.

**Proposition 6.4.** We have the following inequality: If $i \geq 1$ and $i + j \leq \text{diam}(\Gamma)$, then $c_{i}^{\max} \leq b_{i}^{\max}$.

**Proof.** We take arbitrary $y \in V(\Gamma)$, $l \in S_{i}$ and $x \in V_{l}(y)$. By the assumption $\,(\star)$, there exists the shortest path from $x$ to some element $z \in D_{i+j}(y)$. Then, there exists $s \in S_{j}$ such that $x \in V_{s}(z)$. First, we show that $D_{1}(x) \cap D_{i-1}(y) \subset D_{1}(x) \cap D_{j+1}(z)$. We take $w \in D_{1}(x) \cap D_{i-1}(y)$. The distance $d(w, z)$ is less than or equal to $1 + j$.
since \( d(w, x) + d(x, z) \) is greater than or equal to \( d(w, z) \). On the other hand, the distance \( d(w, z) \) is greater than or equal to \( 1 + j \) since \( d(w, z) + d(w, y) \) is greater than or equal to \( d(z, y) \). Hence, the element \( w \) is in \( D_1(x) \cap D_{j+1}(z) \). Then, we have \( D_1(x) \cap D_{i-1}(y) \subset D_1(x) \cap D_{j+1}(z) \). By using this, we have
\[
\sum_{t \in S_{i-1}} c_{t,l} \leq \sum_{t \in S_{j+1}} c_{t,s}.
\]
Then, for any \( l \in S_i \), we have
\[
\sum_{t \in S_{i-1}} c_{t,l} \leq b_{j}^{\max}.
\]
Therefore, we have \( c_{i}^{\max} \leq b_{j}^{\max} \). □

By Propositions 6.1, 6.4, we conclude the following theorem.

**Theorem 6.5.** For a connected highly-regular graph, we have the following inequalities:

1. \( k = b_0^{\max} \geq b_1^{\max} \geq \cdots \geq b_{m-1}^{\max} \geq 1 \).
2. \( 1 = c_1^{\min} \leq c_2^{\min} \leq \cdots \leq c_m^{\min} \leq k \).

Moreover, we suppose that \( \Gamma \) satisfies the following property: \((\star)\) For any \( u \in V(\Gamma) \), \( i \in \{0, 1, \ldots, \operatorname{diam}(\Gamma) - 1\} \), \( x \in D_i(u) \), the set \( D_1(x) \cap D_{i+1}(u) \) is nonempty set. Then, the following inequality holds:

3. If \( i \geq 1 \) and \( i + j \leq \operatorname{diam}(\Gamma) \), then \( c_i^{\max} \leq b_j^{\max} \).

We give an example of Theorem 6.5. Let \( C \) be the following matrix:

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
4 & 0 & 2 & 1 & 0 \\
0 & 2 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

We consider the graph \( C_5 \square C_5 \) (cf. Figure 2). Here, \( C_5 \) is the cycle graph of order 5. Let the labeling of vertices be the same as in Figure 2. It is easy to check that this graph is a highly-regular graph with CAM \( C \) which satisfies the condition \((\star)\). For the vertex \( 25 \in V(C_5 \square C_5) \), we can take the partition of \( V(C_5 \square C_5) \) with respect to \( 25 \in V(C_5 \square C_5) \) as follows:

\[
\begin{align*}
V_0(25) &= \{25\}, \\
V_1(25) &= \{5, 20, 21, 24\}, \\
V_2(25) &= \{1, 4, 16, 19\}, V_2(25) &= \{10, 15, 22, 23\}, \\
V_3(25) &= \{2, 3, 6, 9, 11, 14, 17, 18\}, \\
V_4(25) &= \{7, 8, 12, 13\}.
\end{align*}
\]

The entries of the matrix \( C \) satisfy the inequalities in Theorem 6.5.
APPENDIX A. INFINITE FAMILIES OF CONNECTED HIGHLY-REGULAR GRAPHS WITH FIXED VALENCY WHICH ARE NOT DISTANCE-REGULAR GRAPHS

In this section, we give infinite families of connected highly-regular graphs with fixed valency which are not distance-regular graphs explicitly. More precisely, we discuss the following question:

Question 1. Are there only finitely many connected highly-regular graphs of fixed valency greater than 2 which are not distance-regular graphs?

First, we construct highly-regular graphs of the valency 3 and 4 which are not distance-regular graphs explicitly.

Let $n, m$ be positive integers greater than 1. We denote the graph $C_n \Box C_m$ by $T_{n,m}$, where $C_n$ and $C_m$ are cycle graphs of order $n$ and $m$ respectively. Here, we note that $C_2$ is in Figure 3. Also, we note that the graph $T_{n,m}$ is vertex-transitive. Without loss of generality, We may assume $n \leq m$.

By Proposition 2.2, the graph $T_{n,m}$ is a connected highly-regular graph since a cycle graph is a distance-regular graph. Then, we have the following theorem.

Proposition A.1. The graph $T_{n,m}$ is a connected highly-regular graph which is not a distance-regular graph except for the cases $(n, m) = (2, 2), (2, 4), (3, 3), (4, 4)$.

Proof. In the case $n = 2$, $T_{2,m}$ has the valency 3. If $m = 2$, $T_{2,2}$ is the cycle graph of order 4. Hence, $T_{2,2}$ is a distance-regular graph. If $m = 3$, we take $v \in V(T_{2,3})$. 
Then, the cardinality of the degree set of $\langle D_1(\mathbf{v}) \rangle$ is 2. By Proposition 2.4, $i(T_{2,3})$ is greater than $1 + \text{diam}(T_{2,3})$. By Corollary 3.2, $T_{2,3}$ is not a distance-regular graph. If $m = 4$, $T_{2,4}$ is the Hamming graph of order 8 (cube). Hence, $T_{2,4}$ is a distance-regular graph. In the case $m > 4$, we take arbitrary $\mathbf{v} \in V(T_{2,m})$. If $D_1(\mathbf{v})$ is divided, the graph $T_{2,m}$ is not a distance-regular graph. If $D_1(\mathbf{v})$ is not divided, at least $D_2(\mathbf{v})$ is divided into the following as a partition in a highly-regular graph:

$$D_2(\mathbf{v}) = \{ \mathbf{w} \in D_2(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 2 \} \cup \{ \mathbf{w} \in D_2(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 1 \}.$$  

By Corollary 3.2, the graph $T_{2,m}$ is not a distance-regular graph.

In the case $n = 3$, $T_{3,m}$ has the valency 4. If $m = 3$, $T_{3,3}$ is the graph as in Figure 4, and $T_{3,3}$ is a distance-regular graph. If $m$ is greater than 3, we take arbitrary $\mathbf{v} \in V(T_{3,m})$. Then, at least $D_1(\mathbf{v})$ is divided into the following as a partition of a vertex set in a highly-regular graph:

$$D_1(\mathbf{v}) = \{ \mathbf{w} \in D_1(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 1 \} \cup \{ \mathbf{w} \in D_1(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 0 \}.$$  

By Corollary 3.2, the graph $T_{3,m}$ is not a distance-regular graph.

In the case $n = 4$, $T_{4,m}$ has the valency 4. If $m = 4$, $T_{4,4}$ is the Hamming graph of order 16. Hence, $T_{4,4}$ is a distance-regular graph. If $m$ is greater than 4, we take arbitrary $\mathbf{v} \in V(T_{4,m})$. If $D_1(\mathbf{v})$ is divided, the graph $T_{4,m}$ is not a distance-regular graph. If $D_1(\mathbf{v})$ is not divided, at least $D_2(\mathbf{v})$ is divided into the following as a partition in a highly-regular graph:

$$D_2(\mathbf{v}) = \{ \mathbf{w} \in D_2(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 1 \} \cup \{ \mathbf{w} \in D_2(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 2 \}.$$  

By Corollary 3.2, the graph $T_{4,m}$ is not a distance-regular graph.

In the case $n > 4$, we take arbitrary $\mathbf{v} \in V(T_{n,m})$. If $D_1(\mathbf{v})$ is divided, the graph $T_{n,m}$ is not a distance-regular graph. If $D_1(\mathbf{v})$ is not divided, at least $D_2(\mathbf{v})$ is divided into the following as a partition in a highly-regular graph:

$$D_2(\mathbf{v}) = \{ \mathbf{w} \in D_2(\mathbf{v}) \mid |D_1(\mathbf{w}) \cap D_1(\mathbf{v})| = 2 \}.$$  

By Corollary 3.2, the graph $T_{n,m}$ is not a distance-regular graph.

Remark A.2. As we mentioned above, $T_{2,m}$ $(m > 2)$ has the valency 3, and $T_{n,m}$ $(n \geq 3)$ has the valency 4.

Remark A.3. The graph $T_{n,m}$ has the diameter $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor$. Here, the symbol $\lfloor x \rfloor$ denotes the largest integer less than $x$. Therefore, $T_{n,m}$ has the diameter 2 if and only if $(n, m) = (2, 2), (2, 3), (3, 3)$. Then, we can easily observe the following:

- $\overline{T_{2,2}}$ is not connected.
- $\overline{T_{2,3}}$ is the cycle graph of order 6.
- $\overline{T_{3,3}} \simeq T_{3,3}$ is a distance-regular graph (cf. Figure 4).

Next, we construct some infinite families of connected highly-regular graphs of fixed valency greater than 4 which are not distance-regular graphs.
Proposition A.4. Let $\Gamma_1$ be a connected highly-regular graph which is not a distance-regular graph and $\Gamma_2$ be a connected highly-regular graph. Then, the Cartesian product $\Gamma_1 \square \Gamma_2$ is a connected highly-regular graph which is not a distance-regular graph.

Proof. First, we note that $\text{diam}(\Gamma_1 \square \Gamma_2) = \text{diam}(\Gamma_1) + \text{diam}(\Gamma_2)$. By Proposition 2.2, the Cartesian product $\Gamma_1 \square \Gamma_2$ is a highly-regular graph. We take arbitrary $v = (v_1, v_2) \in V(\Gamma_1) \times V(\Gamma_2)$. For each $j \in \{0, 1, \ldots, \text{diam}(\Gamma_1) + \text{diam}(\Gamma_2)\},$

$$D_{j,\Gamma_1 \square \Gamma_2}(v) = \bigcup_{k,l} \{(v, w) \in V(\Gamma_1) \times V(\Gamma_2) \mid v \in D_{k,\Gamma_1}(v_1) \text{ and } w \in D_{l,\Gamma_2}(v_2)\},$$

where $k$ and $l$ run through $0 \leq k \leq \text{diam}(\Gamma_1)$, $0 \leq l \leq \text{diam}(\Gamma_2)$ such that $k + l = j$. Since $\Gamma_1$ is not a distance-regular graph, there exist $i \in \{0, \ldots, \text{diam}(\Gamma_1)\}$, $j \in \{i - 1, i, i + 1\}$, $u_1, u_2 \in D_{i,\Gamma_1}(v_1)$ such that

$$|D_{1,\Gamma_1}(u_1) \cap D_{j,\Gamma_1}(v_1)| \neq |D_{1,\Gamma_1}(u_2) \cap D_{j,\Gamma_1}(v_1)|. \tag{2}$$

We consider the vertices $(u_1, v_2), (u_2, v_2) \in D_{i,\Gamma_1 \square \Gamma_2}(v)$. Then, $D_{j,\Gamma_1}(v_1) \cap D_{1,\Gamma_1}(u_1)$ and $D_{j,\Gamma_1}(v_1) \cap D_{1,\Gamma_1}(u_2)$ are divided into the following:

$$D_{1,\Gamma_1 \square \Gamma_2}((u_1, v_2)) \cap D_{j,\Gamma_1 \square \Gamma_2}(v) = \{(u_1, w) \in V(\Gamma_1) \times V(\Gamma_2) \mid \{w, v_2\} \in E(\Gamma_2) \text{ and } d(w, v_2) = j - i\}$$

$$\cup \{(v, v_2) \in V(\Gamma_1) \times V(\Gamma_2) \mid v \in D_{1,\Gamma_1}(u_1) \cap D_{j,\Gamma_1}(v_1)\},$$

$$D_{1,\Gamma_1 \square \Gamma_2}((u_2, v_2)) \cap D_{j,\Gamma_1 \square \Gamma_2}(v) = \{(u_2, w) \in V(\Gamma_1) \times V(\Gamma_2) \mid \{w, v_2\} \in E(\Gamma_2) \text{ and } d(w, v_2) = j - i\}$$

$$\cup \{(v, v_2) \in V(\Gamma_1) \times V(\Gamma_2) \mid v \in D_{1,\Gamma_1}(u_2) \cap D_{j,\Gamma_1}(v_1)\}.$$

By (2), we have

$$|D_{1,\Gamma_1 \square \Gamma_2}((u_1, v_2)) \cap D_{j,\Gamma_1 \square \Gamma_2}(v)| \neq |D_{1,\Gamma_1 \square \Gamma_2}((u_2, v_2)) \cap D_{j,\Gamma_1 \square \Gamma_2}(v)|.$$

Therefore, $\Gamma_1 \square \Gamma_2$ is not a distance-regular graph. \qed
Let $\mathcal{P}_1$ be the infinite family of connected highly-regular graphs of fixed valency 3 and $\mathcal{P}_2$ be the infinite family of connected highly-regular graphs of fixed valency 4 which we construct explicitly in Proposition A.1.

Let $k$ be an integer greater than 4. Then, there exist $r_1, r_2, r_3 \in \mathbb{Z}_{\geq 0}$ with $(r_2, r_3) \neq (0, 0)$ such that $k = r_1 \cdot 1 + r_2 \cdot 3 + r_3 \cdot 4$. Let $\mathcal{P}(r_2)$ and $\mathcal{P}(r_3)$ be the following:

- $\mathcal{P}(r_2) = \{\square_{j=1}^{r_2} \Gamma_{1,j} \mid \Gamma_{1,j} \in \mathcal{P}_1, 1 \leq j \leq r_2\}$.
- $\mathcal{P}(r_3) = \{\square_{l=1}^{r_3} \Gamma_{2,l} \mid \Gamma_{2,l} \in \mathcal{P}_2, 1 \leq l \leq r_3\}$.

Moreover, let $\mathcal{P}(r_1, r_2, r_3)$ be the following:

$$\mathcal{P}(r_1, r_2, r_3) = \{(\square_{r_2}^{C_2}) \square \Gamma_1 \square \Gamma_2 \mid \Gamma_1 \in \mathcal{P}(r_2), \Gamma_2 \in \mathcal{P}(r_3)\}.$$  

By Proposition A.4, we have the following proposition.

**Proposition A.5.** The families $\mathcal{P}(r_1, r_2, r_3)$ with $r_1 \cdot 1 + r_2 \cdot 3 + r_3 \cdot 4 = k$, $(r_2, r_3) \neq (0, 0)$ are infinite families of connected highly-regular graphs of fixed valency $k$ which are not distance-regular graphs.

By Proposition A.4, A.5 we conclude the following theorem.

**Theorem A.6.** There are infinitely many connected highly-regular graphs of fixed valency greater than 2 which are not distance-regular graphs.

**Appendix B. Local spectral properties of highly-regular graphs**

In this section, we give the following local spectral properties of highly-regular graphs. For vertices $u, v \in V$ and an eigenvalue $\lambda_i$, $uv$-crossed multiplicity $m_{uv}(\lambda_i)$ is $uv$-entry of $E_i$. Let $\Gamma$ be a connected highly-regular graph with CAM $C = [c_{i,j}]_{1 \leq i,j \leq m}$. Here, let a labeling of $C$ with respect to a distance. For each $i \in \{0, 1, \ldots, \text{diam}(\Gamma)\}$, there exists a nonempty subset $S_i$ of $I = \{1, \ldots, m\}$ such that $D_i(u) = \bigsqcup_{i \in S_i} V_i(u)$, $u \in V$. Then, we have the following theorem.

**Theorem B.1.** A connected highly-regular graph $\Gamma$ is a spectrally-regular graph. Moreover, for each $u \in V$ and for two vertices $v, w \in V_s(u_0)$, $s \in S_j$, we have $m_{uv}(\lambda_i) = m_{uw}(\lambda_i)$, for any $\lambda_i$.

**Proof.** We fix a vertex $u \in V$. We define the matrix $P_u \in M_m(\mathbb{R})$ whose entries are given by

$$(P_u)_{t,w} = \begin{cases} 1 & \text{if } w \in V_i(u), \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that this matrix intertwines the adjacency matrix $A$ and collapsed adjacency matrix $C$, that is, $P_u A = CP_u$. For each eigenvalue $\lambda_i$, there exists unique polynomial such that $E_i = Z_i(A)$ by using Lagrange interpolation. This implies that for each $l = 0, \ldots, d$, we have $P_u Z_l(A) = Z_l(C) P_u$. For $u \in V$, $t \in S_i$, $s \in S_j$, $w \in V_s(u)$, we have

$$\sum_{z \in V_t(u)} Z_t(A)_{z,w} = Z_t(C)_{t,s}. \quad (3)$$
Putting $s = t = 1 \in S_0$, $w \in V_1(u)$, we have

$$m_u(\lambda_l) = Z_l(A)_{u,u} = Z_l(C)_{1,1}.$$ 

Therefore, $\Gamma$ is a spectrally-regular graph. Moreover, putting $t = 1$ in $[3]$, we have

$$Z_l(A)_{u,w} = Z_l(C)_{1,s}.$$ 

This implies that for $v, w \in V_s(u)$, $m_{u,v}(\lambda_l) = Z_l(C)_{1,s} = m_{u,w}(\lambda_l).$ □

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References

[1] Y. Alavi, G. Chartrand, D. R. Lick and H. C. Swart, Highly regular graphs, Ann. New York Acad. Sci. 576 (1989) 20–29.
[2] J. Angel, Finite upper half planes over finite fields, Finite Fields Appl. 2 (1996) 62–86.
[3] S. Bang, A. Dubickas, J. H. Koolen and V. Moulton, There are only finitely many distance-regular graphs of fixed valency greater than two, Adv. Math. 269 (2015) 1–55.
[4] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1984.
[5] E. Bannai, O. Shimabukuro and H. Tanaka, Finite Euclidean graphs and Ramanujan graphs, Discrete Math. 309 (2009) 6126–6134.
[6] B. Bollobás, Graph theory: An Introductory Course, Springer-Verlag, Berlin, 1979.
[7] B. Bollobás, Modern graph theory, Springer-Verlag, Berlin, 2002.
[8] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-regular graphs, volume 18 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1989.
[9] P. J. Cameron, There are only finitely many finite distance-transitive graphs of given valency greater than two, Combinatorica 2 (1) (1982) 9–13.
[10] P. J. Cameron, C. E. Praeger, J. Saxl and G. M. Seitz, On the Sims conjecture and distance transitive graphs, Bull. Lond. Math. Soc. 15 (5) (1983) 499–506.
[11] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Representation Theory and Harmonic Analysis of wreath products of finite groups, Cambridge University Press (2014).
[12] M. A. Fiol, E. Garriga and J. L. A. Yebra, Locally pseudo-distance-regular graphs, J. Combin. Theory Ser. B 68 (1996) 179–205.
[13] W. M. Kwok, Character tables of association schemes of affine type, European J. Combin. 13 (1992) 167–185.
[14] W. Li, Character sums and abelian Ramanujan graphs, J. Number Theory 41 (1992) 199–217.
[15] A. Terras, Fourier analysis of finite groups and applications, London Mathematical Society Student Texts, 43, Cambridge University Press, Cambridge, 1999.

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