BOUNDDEDNESS OF VARIETIES OF FANO TYPE WITH
ALPHA-INVARIANTS AND VOLUMES BOUNDED BELOW

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Abstract. We show that fixed dimensional klt weak Fano pairs with
alpha-invariants and volumes bounded away from 0 and the coefficients
of the boundaries belong to the set of hyperstandard multiplicities \( \Phi(\mathcal{F}) \)
associated to a fixed finite set \( \mathcal{F} \) form a bounded family. We also show
\( \alpha(X, B)^{d-1}\text{vol}(-(K_X + B)) \) are bounded from above for all klt weak
Fano pairs \((X, B)\) of a fixed dimension \(d\).

1. Introduction

Throughout this paper, we work over an uncountable algebraically closed
field of characteristic 0, for instance, the complex number field \(\mathbb{C}\).

In the birational geometry, as the first step of moduli theory, it is inter-

esting to consider whether a certain kind of family of varieties satisfy certain
finiteness. For varieties of Fano type with bounded log discrepancies, Birkar
shows in [Bir16b, Theorem 1.1] that

Theorem 1.1. Fix a positive integer \(d\) and a positive real number \(\epsilon\). The
projective varieties \(X\) satisfying

(1) \(\dim X = d\),
(2) there exists a boundary \(B\) such that \((X, B)\) is \(\epsilon\)-lc,
(3) \(-K_X + B\) is nef and big,

form a bounded family.

Theorem 1.1 was known as the Borisov-Alexeev-Borisov (BAB) Conjec-

ture for decades before Birkar proved it. Equivalently, we can state Theorem
1.1 in the following form of boundedness of varieties of Calabi–Yau type.

Theorem 1.2. Fix a positive integer \(d\) and a positive real number \(\epsilon\). The
projective varieties \(X\) satisfying

(1) \(\dim X = d\),
(2) there exists a boundary \(B\) such that \((X, B)\) is \(\epsilon\)-lc,
(3) \(K_X + B \sim_R 0\) and \(B\) is big,

form a bounded family.

In Theorem 1.1 it is necessary to take \(\epsilon > 0\). In fact, klt Fano threefolds
do not even form a birational family (see Lin03). Nevertheless, Jiang shows
in [Jia17] that if we bound the alpha-invariants and the volumes from the
below, we have

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Theorem 1.3. ([Jia17, Theorem 1.6]) Fix a positive integer $d$ and a positive real number $\theta$. The normal projective klt Fano (i.e. $\mathbb{Q}$-Fano in [Jia17]) varieties $X$ satisfying
\begin{enumerate}
\item $\dim X = d$,
\item $\alpha(X)^d(-K_X)^d > \theta$,
\end{enumerate}
form a bounded family.

Inspired by Theorem 1.3, it is natural to ask if certain boundedness holds for varieties of Fano type or under other more general setting. Thanks to boundedness of complements by Birkar (Theorem 2.4), if the coefficients of the boundaries are well controlled, then the boundedness, which is one of our main theorems, holds as follows.

Theorem 1.4. Fix a positive integer $d$, positive real numbers $\theta$ and $\delta$ and a finite set $\mathcal{R}$ of rational numbers in $[0,1]$. The set of all klt Fano pairs $(X, B)$ satisfying
\begin{enumerate}
\item $\dim X = d$,
\item the coefficients of $B \in \Phi(\mathcal{R})$,
\item $\alpha(X, B)^{d-1+\delta}(-(K_X + B))^d > \theta$,
\end{enumerate}
forms a log bounded family.

Letting $B = 0$ in Theorem 3.7, we have the following corollary, which answers the question asked by Jiang in [Jia17, 1.7].

Corollary 1.5. Fix a positive integer $d$ and two positive real numbers $\delta$ and $\theta$. Then the set of klt Fano varieties $X$ satisfying
\begin{enumerate}
\item $\dim X = d$,
\item $\alpha(X)^d(-K_X)^d > \theta$,
\end{enumerate}
forms a bounded family.

Now we consider $\alpha(X, B)^d(-(K_X + B))^d$ as an invariant for $d$-dimensional klt Fano pairs $(X, B)$. It is well known that this invariant has an upper bound, which can be given by the following lemma.

Lemma 1.6. ([Kol97, Theorem 6.7.1]) Let $(X, B)$ be a klt pair of dimension $d$. Then we have $\text{lct}((X, B), |H|_{\mathbb{Q}})^d H^d \leq d^d$ for any nef and big $\mathbb{Q}$-Cartier divisor $H$ on $X$.

We will show that for $d$-dimensional klt Fano pairs $(X, B)$,
\[\alpha(X, B)^{d-1}(-(K_X + B))^d\]
is also bounded above. In fact, we have the following theorem under a more general setting.

Theorem 1.7. Fix a positive integer $d$. There exists a number $M$, depending only on $d$, such that for any projective normal pair $(X, B)$ and for any big and nef $\mathbb{Q}$-Cartier divisor $H$ on $X$ satisfying
\begin{enumerate}
\item $\dim X = d$,
\item $(X, B)$ is klt,
\end{enumerate}
we have
\[ \text{lct}((X, B), |H|_Q)^{d-1}\tau((X, B), H)H^d \leq M, \]
where \( \tau \) denotes the anti-pseudo-effective threshold (see Definition 2.7).

Remark 1.8. In contrast with the above, for \( d \)-dimensional klt Fano pairs \((X, B)\), \( \alpha(X, B) \) \( d \) \((−(K_X + B))^d \) are not bounded above if \( d' < d - 1 \). To see this, consider the weighted projective spaces \( X_n = \mathbb{P}(1^d, n) \), which are klt Fano varieties of dimension \( d \) with \( (−K_{X_n})^d = \frac{(n+d)^d}{n} \) and \( \alpha(K_{X_n}) = \frac{1}{n+d} \) (cf. [Amb16, 6.3]). Then we have, \( \alpha(X)^{d-1-\delta}(−K_{X_n})^d = \frac{(n+d)^{(1+\delta)}}{n} \), which are not bounded above for any positive real number \( \delta \).

Remark 1.9. We remark that for a klt Fano variety \( X \) of dimension \( d \), a lower bound of \( \alpha(X) \) provides an upper bound of \( (−K_X)^d \) by Lemma 1.6. However, the set of klt Fano varieties with \( (−K_X)^d \) both side bounded does not form a bounded family. As an example, consider the family of weighted projective spaces \( \{X_{p,q,r} = \mathbb{P}(p, q, r)\} \) with \( (p, q, r) \) pairwisely coprime. Then \( \{−K_{X_{p,q,r}}\} = \{(p+q+r)^2\}_{pqr} \) is a dense subset of \( \mathbb{R}_{>0} \). Therefore, for any two positive integers \( a < b \), \( \{X_{p,q,r} | \frac{(p+q+r)^2}{pqr} \in (a, b)\} \) is a family of klt Fano varieties which is not bounded.

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2. Preliminaries

We adopt the standard notation and definitions in [KMM85] and [KM98], and will freely use them.

2.1. Pairs and singularities. A sub-pair \((X, B)\) consists of a normal projective variety \( X \) and an \( \mathbb{R} \)-divisor \( B \) on \( X \) such that \( K_X + B \) is \( \mathbb{R} \)-Cartier. \( B \) is called the sub-boundary of this pair.

A log pair \((X, B)\) is a sub-pair with \( B \geq 0 \). We call \( B \) a boundary in this case.

Let \( f : Y \to X \) be a log resolution of the log pair \((X, B)\), write
\[ K_Y = f^*(K_X + B) + \sum a_iF_i, \]
where \( \{F_i\} \) are distinct prime divisors. For a non-negative real number \( \epsilon \), the log pair \((X, B)\) is called
(a) \( \epsilon \)-kawamata log terminal (\( \epsilon \)-klt, for short) if \( a_i > -1 + \epsilon \) for all \( i \);
(b) \( \epsilon \)-log canonical (\( \epsilon \)-lc, for short) if \( a_i \geq -1 + \epsilon \) for all \( i \);

Usually we write \( X \) instead of \((X, 0)\) in the case when \( B = 0 \). Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense. Also note that \( \epsilon \)-lc singularities only make sense if \( \epsilon \in [0, 1] \), and \( \epsilon \)-klt singularities only make sense if \( \epsilon \in [0, 1] \).

Similarly, sub-\( \epsilon \)-klt and sub-\( \epsilon \)-lc sub-pairs can be defined.

The log discrepancy of the divisor \( F_i \) is defined to be \( a(F_i, X, B) = 1 + a_i \). It does not depend on the choice of the log resolution \( f \).
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F_i is called a non-lc place of \((X, B)\) if \(a_i < -1\). A subvariety \(V \subset X\) is called a non-lc center of \((X, B)\) if it is the image of a non-lc place. The non-klt locus \(\operatorname{Nklt}(X, B)\) is the union of all non-lc centers of \((X, B)\). We recall the Kollár-Shokurov connectedness lemma.

Lemma 2.1. (cf. [Sho93], [Sho94] and [Kol92, 17.4]) Let \((X, B)\) be a log pair, and let \(\pi : X \rightarrow S\) be a proper morphism with connected fibers. Suppose \(- (K_X + B)\) is \(\pi\)-nef and \(\pi\)-big. Then \(\operatorname{Nklt}(X, B) \cap X_s\) is connected for any fiber \(X_s\) of \(\pi\).

2.2. Fano pairs and Calabi–Yau pairs. A normal projective pair \((X, B)\) is a Fano (resp. weak Fano, resp. Calabi–Yau) pair if \(- (K_X + B)\) is ample (resp. \(- (K_X + B)\) is nef and big, resp. \(K_X + B \equiv 0\)). A normal projective variety \(X\) is called Fano if \((X, 0)\) is Fano. It is called \(\mathbb{Q}\)-Fano if it is klt and Fano. It is called of Fano type if \((X, B)\) is klt weak Fano for some boundary \(B\).

2.3. Bounded pairs. A collection of varieties \(D\) is said to be bounded (resp. birationally bounded) if there exists \(h : Z \rightarrow S\) a proper morphism of schemes of finite type such that each \(X \in D\) is isomorphic (resp. birational, or isomorphic in codimension one) to \(Z_s\) for some closed point \(s \in S\).

A couple \((X, D)\) consists of a normal projective variety \(X\) and a reduced divisor \(D\) on \(X\). Note that we do not require \(K_X + D\) to be \(\mathbb{Q}\)-Cartier here.

We say that a collection of couples \(D\) is log birationally bounded (resp. log bounded) if there is a quasi-projective scheme \(Z\), a reduced divisor \(E\) on \(Z\), and a projective morphism \(h : Z \rightarrow S\), where \(S\) is of finite type and \(E\) does not contain any fiber, such that for every \((X, D) \in D\), there is a closed point \(s \in S\) and a birational map \(f : Z_s \dashrightarrow X\) (resp. isomorphic such that \(E_s\) contains the support of \(f_s^{-1}B\) and any \(f\)-exceptional divisor).

A set of log pairs \(P\) is log birationally bounded (resp. log bounded) if the set of the corresponding couples \(\{(X, \operatorname{Supp}B) | (X, B) \in P\}\) is.

2.4. Volumes. Let \(X\) be a \(d\)-dimensional projective variety and \(D\) a Cartier divisor on \(X\). The volume of \(D\) is the real number

\[\operatorname{vol}(X, D) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.\]

For more backgrounds on the volume, see [Laz04, 2.2.C]. By the homogenous property and continuity of the volume, we can extend the definition to \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisors. Moreover, if \(D\) is a nef \(\mathbb{R}\)-divisor, then \(\operatorname{vol}(X, D) = D^d\).

2.5. Complements.

Definition 2.2. Let \((X, B)\) be a pair and \(n\) a positive integer. We write \(B = |B| + \{B\}\). An \(n\)-complement of \(K_X + B\) is a divisor of the form \(K_X + B^+\) such that

1. \((X, B^+)\) is lc,
2. \(n(X, B^+) \sim 0,\)
3. \(nB^+ \leq n|B| + [(n+1)\{B\}].\)
Definition 2.3. For a subset \( R \) of \([0,1]\), we define the set of hyperstandard multiplicities associated to \( R \) to be

\[
\Phi(R) = \{1 - \frac{r}{m} | r \in R, \ m \in \mathbb{N}\}.
\]

Note that the only possible accumulating point of \( \Phi(R) \) is 1 if \( R \) is finite.

Birkar shows the following boundedness of complements.

Theorem 2.4. ([Bir16a, Theorem 1.7]) Fix a positive integer \( d \) and a finite set \( R \) of rational numbers in \([0,1]\). Then there exists a positive integer \( n \) depending only on \( d \) and \( R \), such that if \((X,B)\) is a projective pair with

1. \((X,B)\) is lc dimension \( d \),
2. the coefficients of \( B \in \Phi(R) \),
3. \( X \) is of Fano type,
4. \(-K_X + B\) is nef,

then there is an \( n \)-complement \( K_X + B^+ \) of \( K_X + B \) such that \( B^+ \geq B \).

2.6. \( \alpha \)-invariants, log canonical thresholds and anti-pseudo-effective thresholds.

Definition 2.5. Let \((X,B)\) be a projective lc pair and let \( D \) be an effective \( \mathbb{R} \)-Cartier divisor, we define the log canonical threshold of \( D \) with respect of \((X,B)\) to be

\[
\lct((X,B),D) = \sup\{t \in \mathbb{R} \mid (X,B + tD) \text{ is lc}\}.
\]

The log canonical threshold of \( |D| \) with respect of \((X,B)\) is defined to be

\[
\lct((X,B),|D|_\mathbb{Q}) = \inf\{\lct((X,B),M) | M \in |D|_\mathbb{Q}\},
\]

which is equal to

\[
\lct((X,B),|D|_\mathbb{R}) = \inf\{\lct((X,B),M) | M \in |D|_\mathbb{R}\}.
\]

Definition 2.6. Let \((X,B)\) be a projective normal klt weak Fano pair, we define the \( \alpha \)-invariant of \((X,B)\) to be

\[
\alpha(X,B) = \lct((X,B),|-(K_X + B)|_\mathbb{Q}).
\]

In the case when \( B = 0 \), we usually write \( \alpha(X) := \alpha(X,0) \) for convenience.

Definition 2.7. Let \((X,B)\) be a projective normal pair, and \( H \) a big \( \mathbb{R} \)-Cartier divisor. The anti-pseudo-effective threshold of \( H \) respect to \((X,B)\) is defined by

\[
\tau((X,B),H) = \sup\{t \in \mathbb{R} \mid -K_X - B - tH \text{ is pseudo-effective}\} = \sup\{t \in \mathbb{R} \mid K_X + B + tH \text{ is anti-pseudo-effective}\}.
\]

2.7. Potentially birational divisors.

Definition 2.8. (cf. [HMX14, Definition 3.5.3]) Let \( X \) be a projective normal variety, and \( D \) a big \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \). Then, we say that \( D \) is potentially birational if for any two general points \( x \) and \( y \) of \( X \), there is an effective \( \mathbb{Q} \)-divisor \( \Delta \sim_\mathbb{Q} (1 - \epsilon)D \) for some \( 0 < \epsilon < 1 \), such that, after possibly switching \( x \) and \( y \), \((X,\Delta)\) is not lc at \( y \), lc at \( x \) and \( x \) is a non-klt center.
Lemma 2.9. ([HMX13 Lemma 2.3.4]) Let $X$ be a projective normal variety, and $D$ a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. If $D$ is potentially birational, then $|K_X + [D]|$ defines a birational map.

2.8. Exceptional pairs.

Definition 2.10. A projective normal klt pair $(X,B)$ is called exceptional if
\[ \text{lct}((X,B), |-(K_X + B)|_\mathbb{Q}) > 1. \]
In particular, if $(X,B)$ is weak Fano, then it is exceptional if and only if $\alpha(X,B) > 1$.

Theorem 2.11. ([Bir16a, Theorem 1.11]) Fix a positive integer $d$ and a finite set of rational numbers $R$ in $[0,1]$. Then the set of all projective pairs $(X,B)$ satisfying
1. $(X,B)$ is lc dimension $d$,
2. the coefficients of $B \in \Phi(R)$,
3. $X$ is of Fano type,
4. $-(K_X + B)$ is nef,
5. $(X,B)$ is exceptional,
forms a log bounded family.

2.9. Descending chain condition.

Definition 2.12. A set of real numbers $S$ is said to satisfy descending chain condition (DCC for short) if for every non-empty subset $S$ of $S$, there is a minimum element in $S$. $S$ is called a DCC set if it satisfies DCC.

We recall and improve slightly the following proposition of Birkar. It is shown for ample divisors $D$ and $A$ in [Bir16a, 2.31(2)]. We modify it for nef and big divisors $A$ and $D$.

Proposition 2.13. Let $(X,B)$ be a log pair of dimension $d$. Let $D$ and $A$ be big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $X$. Assume that $D^d > (2d)^d$. Then there is a bounded family $\mathcal{P}$ of subvarieties of $X$ such that for each pair $x$, $y \in X$ of general closed points, there is a member $G$ of $\mathcal{P}$ and an effective divisor $\Delta \sim D + (d-1)A$ such that
1. $(X,B + \Delta)$ is lc near $x$ with a unique non-klt place whose centre is $G$,
2. $(X,B + \Delta)$ is not klt at $y$,
3. either $\dim G = 0$ or $A^{d-\dim G} \cdot G \leq d^d$.

Proof. First, by [HMX14, 7.1], there is a bounded family $\mathcal{P}_0$ of subvarieties of $X$ such that for each pair $x$, $y \in X$ of general closed points, there is a member $G_0$ of $\mathcal{P}_0$ and an effective divisor $\Delta_0 \sim D$ such that $(X,B + \Delta_0)$ is lc near $x$ with a unique non-klt place whose centre is $G_0$ and $(X,B + \Delta_0)$ is not klt at $y$.

Now suppose for some $0 \leq i \leq d - 2$, we are given a family $X \mathcal{P}_i$ of subvarieties of $X$ such that for each pair $x$, $y \in X$ of general closed points, there is a member $G_i$ of $\mathcal{P}_i$ and an effective divisor $\Delta_i \sim D + iA$ such that
1. $(X,B + \Delta_i)$ is lc near $x$ with a unique non-klt place whose centre is $G_i$. 

We follow the proof of [Jia17, 3.1]. Taking normalizations and resolutions of $G_i$, we may assume they are smooth. Cutting by general hyperplane sections of $Y$, we may assume the dimension of $G$ to be done independently of $i$. By [Kol97, 6.8.1 and 6.8.1.3] and [Bir16a, 2.32], there are positive rational numbers $\delta \ll 1$ and $c < 1$, such that there is an effective $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} A_i$ such that

$$
\begin{align*}
(1) & \quad (X, B + (1 - \delta)\Delta_i + cH) \text{ is lc near } x \text{ with a unique non-klt place whose centre is } G_i', \\
(2) & \quad (X, B + (1 - \delta)\Delta_i + cH) \text{ is not klt at } y, \\
(3) & \quad \dim G_i' < \dim G_i.
\end{align*}
$$

We set $G_{i+1} = G_i'$ and $\Delta_{i+1} = (1 - \delta)\Delta_i + cH + \delta(D + iA) + (1 - c)A_i + E_i \sim \Delta + A$ in this case. Note that $D, A, A_i$ and $E_i$ are independent of $x$ and $y$.

Set $\mathcal{P}_{i+1} = \{G_{i+1}\}$, then the proposition follows from induction on $i$. Note that either $\dim G_i \leq d - i - 1$ or $A^{d_{-\dim} G_i} \cdot G_i \leq d^d$ implies $\mathcal{P} = \mathcal{P}_{d-1}$ is bounded.

Next, we recall the following lemma by Jiang, which aims to cut down the dimension of $G$ to 0 in the previous proposition. Jiang shows it in [Jia17, 3.1] for $H = -K_X$ being ample. In fact, the proof works under the following setting.

**Lemma 2.14.** Fix positive integers $d > k$. Let $(X, B)$ be a projective normal klt pair of dimension $d$ and $H$ be a nef and big $\mathbb{Q}$-Cartier divisor on $X$. Assume there is a morphism $f : Y \to T$ of projective varieties with a surjective morphism $\phi : Y \to X$ such that a general fiber $F$ of $f$ is of dimension $k$ and $\phi|_F : F \to \phi(F) = G$ is birational, then

$$
H^k \cdot G \geq \frac{\lct((X, B), |H|_{\mathbb{Q}})^{d-k}}{(d)^{(d-k)^{d-k}}H^d}.
$$

**Proof.** We follow the proof of [Jia17, 3.1]. Taking normalizations and resolutions of $Y$ and $T$, we may assume they are smooth. Cutting by general hyperplane sections of $T$, we may assume $\phi$ is generically finite. Therefore, it holds that $\dim Y = d$. Let $\mathcal{I}_{G}^{<m>}$ (resp. $\mathcal{I}_{F}^{<m>}$) be the sheaf of ideal of regular functions vanishing along $G$ (resp. $F$) to order at least $m$. Then $\mathcal{I}_{F}^{<m>} = \mathcal{I}_{F}^{m}$, where $\mathcal{I}_{F}$ denotes the ideal sheaf of $F$. So we have a natural injection

$$
\mathcal{O}_X/\mathcal{I}_{G}^{<m>} \to \phi_*(\mathcal{O}_Y/\mathcal{I}_{F}^{m}).
$$

Now we consider a rational number $l > 0$ and a positive integer $m$ such that $lmH$ is Cartier. By the projection formula, we have

$$
\begin{align*}
h^0(X, \mathcal{O}_X(lmH) \otimes \mathcal{O}_X/\mathcal{I}_{G}^{<m>}) \\
\geq h^0(X, \mathcal{O}_X((lmH) \otimes \phi_*(\mathcal{O}_Y/\mathcal{I}_{F}^{m}))) \\
= h^0(Y, \phi^*\mathcal{O}_X((lmH) \otimes \mathcal{O}_Y/\mathcal{I}_{F}^{m})).
\end{align*}
$$
On the other hand, since $F$ is a general fiber of $f$, the conormal sheaf of $F$ is trivial. That is, we have $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_F^{(d-k)}$. Furthermore, we have

$$\mathcal{I}^{i-1}/\mathcal{I}^i \simeq \mathcal{S}^{i-1}(\mathcal{I}/\mathcal{I}^2) \simeq \mathcal{O}_F^{(d-k+i-2)}$$

for every $i \geq 1$ (see [Har77, II. Theorem 8.24]). Hence,

$$h^0(Y, \phi^* \mathcal{O}_X(lmH) \otimes \mathcal{O}_Y/\mathcal{I}_G^m) \leq \sum_{i=1}^m h^0(Y, \phi^* \mathcal{O}_X(lmH) \otimes \mathcal{I}_F^{-1}/\mathcal{I}_F^i)$$

$$= \sum_{i=1}^m \left(\frac{d-k+i-2}{d-k-1}\right) h^0(Y, \phi^* \mathcal{O}_X(lmH) \otimes \mathcal{O}_F)$$

$$= \left(\frac{d-k+m-1}{d-k}\right) h^0(F, \phi^* \mathcal{O}_X(lmH)|_F).$$

Now we consider the exact sequence

$$0 \to \mathcal{O}_X(lmH) \otimes \mathcal{I}_G^{m>} \to \mathcal{O}_X(lmH) \to \mathcal{O}_X(lmH) \otimes \mathcal{O}_X/\mathcal{I}_G^{m>} \to 0,$$

which implies

$$h^0(X, \mathcal{O}_X(lmH) \otimes \mathcal{I}_G^{m>})$$

$$\geq h^0(X, \mathcal{O}_X(lmH)) - h^0(X, \mathcal{O}_X(lmH) \otimes \mathcal{O}_X/\mathcal{I}_G^{m>})$$

$$\geq h^0(X, \mathcal{O}_X(lmH)) - \left(\frac{d-k+m-1}{d-k}\right) h^0(F, \phi^* \mathcal{O}_X(lmH)|_F).$$

We fix $l$ and consider the asymptotic behavior of the last two terms as $m$ goes to infinity. By the definition of volume,

$$\lim_{m \to \infty} \frac{d!}{m^d} h^0(X, \mathcal{O}_X(lmH)) = \text{vol}(\mathcal{O}_X(lH)) = l^d H^d.$$
Consequently, for \( l > \frac{d-k}{k} \sqrt{\left( \frac{(d)H^k \cdot G}{H^d} \right)} \) and \( m \) sufficiently large, we have

\[
h^0(X, \mathcal{O}_X(lmH)) > \left( \frac{d-k+m-1}{d-k} \right) h^0(F, \phi^* \mathcal{O}_X(lmH)|_F).
\]

Therefore \( h^0(X, \mathcal{O}_X(lmH) \otimes \mathcal{I}_G^{<m>}) > 0 \), so there is an effective \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} H \) such that \( \text{mult}_G D \geq 1 \). Let \( X_{sm} \) be the smooth locus of \( X \). Since \( \phi \) is surjective and \( G \) is the image of a general fiber \( F \) of \( f \), \( G|_{X_{sm}} \) is not empty. By \cite{KM98} Lemma 2.29, \( (X_{sm}, (d-k)D)|_{X_{sm}} \) is not klt along \( G|_{X_{sm}} \). So \( (X, B + (d-k)D) \) is not klt. Hence,

\[
(d-k)l \geq \text{lct}((X, B), \frac{1}{l} D) \geq \text{lct}((X, B), |H|_Q).
\]

Since \( l \) is chosen arbitrarily such that \( l > \frac{d-k}{k} \sqrt{\left( \frac{(d)H^k \cdot G}{H^d} \right)} \), we have

\[
(d-k) \frac{d-k}{k} \sqrt{\left( \frac{(d)H^k \cdot G}{H^d} \right)} \geq \text{lct}((X, B), |H|_Q).
\]

That is,

\[
H^k \cdot G \geq \frac{\text{lct}((X, B), |H|_Q)^{d-k}}{(d-k)^{d-k}} H^d
\]

holds. \( \square \)

**Lemma 2.15.** Under the setting of Lemma 2.14, there is a proper closed subset \( S \) of \( T \), such that

\[
H^k \cdot \phi(F') \geq \frac{\text{lct}((X, B), |H|_Q)^{d-k}}{(d-k)^{d-k}} H^d
\]

for every fiber \( F' \) of \( f \) over \( T - S \), and \( \phi|_{T \setminus (S)} \) does not dominate \( X \).

**Proof.** We apply the Noetherian induction (see, for example, \cite{Har77} II, exercise 3.16) on Lemma 2.14. Assume that \( T' \) is a closed subset of \( T \). Assume that for any proper closed subset \( U \) of \( T' \) there is a closed subset \( V \) of \( U \) such that

\[
H^k \cdot \phi(F') \geq \frac{\text{lct}((X, B), |H|_Q)^{d-k}}{(d-k)^{d-k}} H^d
\]

for every fiber \( F' \) of \( f \) over \( U - V \), and \( \phi|_{T \setminus (S)} \) does not dominate \( X \). If \( \phi|_{T \setminus (S)} \) does not dominate \( X \), we set \( S' = T' \). On the other hand, if \( \phi|_{T \setminus (S)} \) dominates \( X \), by Lemma 2.14, there is a proper closed subset \( U' \) of \( T' \), such that

\[
H^k \cdot \phi(F') \geq \frac{\text{lct}((X, B), |H|_Q)^{d-k}}{(d-k)^{d-k}} H^d
\]

for every fiber \( F' \) of \( f \) over \( T' - U' \). By the assumption, there is a closed subset \( V' \) of \( U' \) such that

\[
H^k \cdot \phi(F') \geq \frac{\text{lct}((X, B), |H|_Q)^{d-k}}{(d-k)^{d-k}} H^d
\]

for every fiber \( F' \) of \( f \) over \( U' - V' \), and \( \phi|_{T \setminus (S)} \) does not dominate \( X \). We set \( S' = V' \) in this case.
Now we have a closed subset $S'$ of $T'$, such that
\[ H^k \cdot \phi(F') \geq \frac{\lct((X, B), |H|_Q)^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d \]
for every fiber $F'$ of $f$ over $T' - S'$, and $\phi|_{f^{-1}(S')}$ does not dominate $X$, and we are done. Note that $S$ must be a proper subset because $\phi$ is surjective. □

3. Proofs of Theorems

Now we restate and prove the theorems in Section 1.

**Theorem 3.1.** Fix a positive integer $d$ and a positive real number $\theta$. Then there is a number $m$ depending only on $d$ and $\theta$ such that if $X$ is a projective normal variety satisfying

1. $\dim X = d$,
2. there exists a boundary $B$ such that $(X, B)$ is klt,
3. there is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H$ on $X$ with $\lct((X, B), |H|_Q) > \theta$,
4. $H^d > \theta$,

then $|K_X + [B + mH]|$ defines a birational map. Moreover, if $X$ is $\mathbb{Q}$-factorial, then $|K_X + [mH]|$ defines a birational map.

**Proof.** Suppose we are given $X$, $B$ and $H$ as in the assumption.

Set
\[ q = \left\lceil \frac{2d}{\sqrt{\theta}} \right\rceil, \]
\[ p = \max_{1 \leq k \leq d-1} \left\lceil \sqrt[2k]{\binom{d}{k}(d-k)^{d-k}d^d} \frac{\theta^{d-k+1}}{\theta^{d-k+1}} \right\rceil. \]

Then by construction, $(qH)^d > (2d)^d$.

Now apply Proposition 2.13 with $D = qH$ and $A = pH$ and we have a bounded family $\mathcal{P}$ of subvarieties of $X$ such that for each pair $x, y \in X$ of general closed points, there is a member $G_{xy}$ of $\mathcal{P}$ and an effective divisor $\Delta \sim \mathbb{Q} D + (d-1)A$ such that

1. $(X, B + \Delta)$ is lc near $x$ with a unique non-klt place whose centre is $G_{xy}$,
2. $(X, B + \Delta)$ is not klt at $y$,
3. either $\dim G_{xy} = 0$ or $A^{d-\dim G_{xy}} \cdot G_{xy} \leq \theta^d$.

By [Bir16a, Lemma 2.21], this means that there is a finite set $\{ \phi_j : V_j \to T_j \}$ of projective varieties with surjective morphisms $\pi_j : V_j \to X$ such that each member $G \in \mathcal{P}$ is isomorphic through $\pi_j$ to a fiber of $\phi_j$ for some $j$.

Now consider a general fiber $G'$ of $\phi_j$ for each $j$. If $\dim G' > 0$. By Lemma 2.15 there is a closed subset $S_j$ of $T_j$, such that
\[ H^k \cdot G' \geq \frac{\lct((X, B), |H|_Q)^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d > \frac{\theta^{d-k+1}}{\binom{d}{k}(d-k)^{d-k}}. \]
for every image $G$ of $\pi_j$ of a fiber $F'$ of $\phi_j$ over $T_j - S_j$, and $\pi_j|_{\phi_j^{-1}(S_j)}$ is not dominant. On the other hand, we set $S_j$ to be the empty set if $\dim G' = 0$. Now since $x$ and $y$ are general, they are not in $\pi_j(\phi_j^{-1}(S_j))$ for each $j$. So $G_{xy}$ is image of $\pi_j$ of a fiber of $\phi_j$ over $T_j - S_j$ for some $j$. Suppose $\dim G_{xy} > 0$, then by the definition of $p$, $A^k \cdot G' = (pH)^k \cdot G' > d^d$. This contradicts (3) of Proposition 2.13 and thus $\dim G' = 0$ for any general fiber $G'$ of $\phi_j$ for each $j$.

Now $D + (d - 1)A = (q + (d - 1)p)H$. So $B + (q + (d - 1)p + 1)H$ is potentially birational and hence $|K_X + [B + (q + (d - 1)p + 1)H]|$ defines a birational map. Let $m = q + (d - 1)p + 1$ and we are done with the first statement.

If moreover, $X$ is $\mathbb{Q}$-factorial, then $(X, 0)$ is klt and $\lct((X, 0), |H|_Q) \geq \lct((X, B), |H|_Q) > \theta$. Replacing $B$ by 0, we are done.

**Theorem 3.2.** Fix a positive integer $d$. There exists a number $M$, depending only on $d$, such that for any projective normal pair $(X, B)$ and for any big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H$ on $X$ satisfying

1. $\dim X = d$,
2. $(X, B)$ is klt,

we have

$$\lct((X, B), |H|_Q)^{d-1} \varpi((X, B), H)H^d \leq M.$$

**Proof.** By the linearity of $\lct$, $\varpi$ and the volume function with respect to $H$, we may assume that $\varpi((X, B), H) = 2$. So that $-(K_X + B) - H = -(K_X + B) - 2H + H$ is big and then we can write $-(K_X + B) - H = F + E$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $F$ and effective $\mathbb{Q}$-divisor $E$. Assume the theorem fail to hold. Then for any number $M$, there is a pair $(X, B)$ a big and nef $\mathbb{Q}$-Cartier $H$ on $X$ satisfying the assumptions, such that

$$\lct((X, B), |H|_Q)^{d-1}H^d = \lct((X, B), |H|_Q)^{d-1} \varpi((X, B), H) \frac{H^d}{2} \geq \frac{M}{2}.$$ 

By Lemma 1.6 we have

$$\lct((X, B), |H|_Q)^{d-1}H^d \leq \frac{d^d}{\lct((X, B), |H|_Q)}.$$ 

So we may assume $\lct((X, B), |H|) \leq 1$ and $H^d > \frac{M}{2} \geq (4d)^d$. Moreover, for every $0 < k < d$, we may assume

$$M > 2 \binom{d}{k} (d - k)^{d-k} (2(d - 1)d)^d.$$ 

Now apply Proposition 2.13 with $D = \frac{1}{2}H$ and $A = \frac{1}{2(d - 1)}H$. We have a bounded family $\mathcal{P}$ of subvarieties of $X$ such that for each pair $x, y \in X$ of general closed points, there is a member $G_{xy}$ of $\mathcal{P}$ and an effective divisor $\Delta \sim Q D + (d - 1)A = H$ such that

1. $(X, B + \Delta)$ is lc near $x$ with a unique non-klt place whose centre is $G_{xy}$,
2. $(X, B + \Delta)$ is not klt at $y$,
3. either $\dim G_{xy} = 0$ or $A^{d-\dim G_{xy}} \cdot G_{xy} \leq d^d.$
By [Bir16a] Lemma 2.21, this means that there is a finite set \( \{ \phi_j : V_j \to T_j \} \) of projective varieties with surjective morphisms \( \pi_j : V_j \to X \) such that each member \( G \in \mathcal{P} \) is isomorph through \( \pi_j \) to a fiber of \( \phi_j \) for some \( j \).

Now consider a general fiber \( G' \) of \( \phi_j \) for each \( j \). If \( \dim G' > 0 \). By Lemma 2.15 there is a closed subset \( S_j \) of \( T_j \), such that

\[
H^k \cdot G \geq \frac{\lct((X, B), |H|_Q)^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d
\]

for every image \( G \) of \( \pi_j \) of a fiber \( F' \) of \( \phi_j \) over \( T_j - S_j \), and \( \pi_j|_{\phi_j^{-1}(S_j)} \) is not dominant. On the other hand, we set \( S_j \) to be the empty set if \( \dim G' = 0 \).

Now since \( x \) and \( y \) are general, they are not in \( \pi_j(\phi_j^{-1}(S_j)) \) for each \( j \). So \( G_{xy} \) is image of \( \pi_j \) of a fiber of \( \phi_j \) over \( T_j - S_j \) for some \( j \). Suppose \( \dim G_{xy} > 0 \), then we have

\[
H^k \cdot G_{xy} \geq \frac{\lct((X, B), |H|_Q)^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d
\]

\[
\geq \frac{\lct((X, B), |H|_Q)^{d-1}}{\binom{d}{k}(d-k)^{d-k}} H^d
\]

\[
> \frac{M}{2^{\binom{d}{k}}(d-k)^{d-k}} > (2(d-1)d)^d.
\]

By construction, \( A^k \cdot G_{xy} = \left( \frac{1}{2^{(d-1)}} \right)^k H^k \cdot G_{xy} > d^d \). This contradicts (3) of the above and thus \( \dim G_{xy} = 0 \).

Now for general \( x \) and \( y \), \( \text{Nklt}(X, B + \Delta) \) contains \( y \) and \( x \) and isolates \( x \). Since \( x \) is general, \( \text{Nklt}(X, B + \Delta + E) \) also isolates \( x \). By Lemma 2.1

\(-K_X - (B + \Delta + E) \sim_{\Q} -(K_X + B) - H - E = F \) is not ample, which is a contradiction.

We recall the following theorem by Hacon and Xu.

**Theorem 3.3.** ([HX13, Theorem 1.3]) Fix a positive integer \( d \) and a DCC set \( \mathcal{I} \) of rational numbers in \([0, 1]\). The set of all projective normal pairs \((X, B)\) satisfying

1. \((X, B)\) is klt log Calabi–Yau of dimension \( d \),
2. \( B \) is big,
3. the coefficients of \( B \in \mathcal{I} \),

forms a bounded family.

Then we show the following log-version of [Bir16a] Lemma 2.26.

**Lemma 3.4.** Fix positive integers \( d \) and \( k \). Let \( \mathcal{P} \) be a set of klt weak Fano pairs of dimension \( d \). Assume that for every element \((Y, B_Y) \in \mathcal{P} \), there is a k-complement \( K_Y + B_Y^+ \) of \( K_Y + B_Y \) such that \((X, B_Y^+) \) is klt and \( B_Y^+ \geq B_Y \). Let \( \mathcal{Q} \) be the set of normal projective pairs \((X, B)\) such that

1. there is \((Y, B_Y) \in \mathcal{P} \) and a birational map \( X \to Y \),
2. there is a common resolution \( \phi : W \to Y \) and \( \psi : W \to X \),
3. \( \phi^*(K_Y + B_Y) \geq \psi^*(K_X + B) \),

Then for every element \((X, B) \in \mathcal{Q} \), there is a k-complement \( K_X + B^+ \) of \( K_X + B \) such that \((X, B^+) \) is klt and \( B^+ \geq B \).
Proof. Let $K_X + B^+$ be the crepant pullback of $K_Y + B_Y^+$ to $X$. Then $(X, B^+)$ is klt. Since $\phi^*(K_Y + B_Y) \geq \psi^*(K_X + B)$, $B^+ - B \geq \psi_* \phi^*(B_Y^+ - B_Y) \geq 0$ is big and $K_X + B^+$ is an $\kappa$-complement of $K_X + B$.

□

Next, we recall the following proposition of Birkar with a small observation.

**Proposition 3.5.** ([Bir16a, Proposition 4.4]) Fix positive integers $d$, $v$ and a positive real number $\epsilon$. Then there exists a bounded set of couples $P$ and a positive real number $c$ depending only on $d$, $v$ and $\epsilon$ satisfies the following.

Assume

- $X$ is a normal projective variety of dimension $d$,
- $B$ is an effective $\mathbb{R}$-divisor with coefficient at least $\epsilon$,
- $M$ is a $\mathbb{Q}$-divisor with $|M|$ defining a birational map,
- $M - (K_X + B)$ is pseudo-effective,
- $\operatorname{vol}(M) < v$, and
- $\mu_D(B + M) \geq 1$ for every component $D$ of $M$.

Then there is a projective log smooth couple $(\overline{W}, \Sigma_W) \in P$, a birational map $\overline{W} \dashrightarrow X$ and a common resolution $X'$ of this map such that

1. $\operatorname{Supp} \Sigma_W$ contains the exceptional divisor of $\overline{W} \dashrightarrow X$ and the birational transform of $\operatorname{Supp}(B + M)$,
2. there is a resolution $\phi : W \to X$ such that $M_W := M|_W \sim A_W + R_W$ where $A_W$ is the movable part of $|M_W|$, $|A_W|$ is base point free, $X' \to X$ factors through $W$ and $A_{X'} := A_W|_{X'} \sim 0/\overline{W}$.

Moreover, if $M$ is nef and $M_W$ is the pushdown of $M_{X'} := M|_{X'}$. Then each coefficient of $M_W$ is at most $c$.

Note that in the original statement of [Bir16a, Proposition 4.4], $M$ is assumed to be nef. We observe from Birkar’s proof that the nefness of $M$ is used only when showing the existence of $c$ and is not necessary when showing (1) and (2) of proposition 3.5.

Now we are ready to show the main theorem of this paper. The idea is to follow the strategy of [Bir16a, Proposition 7.13], which is to construct a klt complement with coefficients in a finite set depending only on $d$, $\theta$ and $\mathcal{R}$, and then apply Theorem 3.3.

**Theorem 3.6.** Fix a positive integer $d$, a positive real number $\theta$ and a finite set $\mathcal{R}$ of rational numbers in $[0, 1]$. The set $\mathcal{D}$ of all klt weak Fano pairs $(X, B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \Phi(\mathcal{R})$,
3. $\alpha(X, B) > \theta$,
4. $-(K_X + B))^d > \theta$,

forms a log bounded family. Moreover, there is a finite set of rational numbers $I \subseteq [0, 1]$ depending only on $d$, $\theta$ and $\mathcal{R}$ such that for every element $(X, B) \in \mathcal{D}$, there is a $\mathbb{Q}$-divisor $\Theta \geq B$ with a klt log Calabi–Yau pair $(X, \Theta)$. 

\vspace{1cm}
Proof. By Theorem 3.3, it is enough to show the existence of $I$.

By Lemma 3.4, replacing by a small $\mathbb{Q}$-factorialisation of $X$, we may assume $X$ is $\mathbb{Q}$-factorial.

By Theorem 2.4, there is a positive integer $n$ depending only on $d$ and $\mathcal{R}$, such that there is an $n$-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$. If $(X, B^+)$ is klt, then the theorem follows by Theorem 3.3. So we may assume $(X, B^+)$ is not klt.

Let $m$ be given by Theorem 3.1, such that $|K_X + [m(-K_X - B)]|$ defines a birational map. Replacing $n$ and $m$ by $2mn$, we may assume $n = m > 1$. Since 1 is the only possible accumulating point of $\Phi(\mathcal{R})$, there are only finitely many elements in $\Phi(\mathcal{R}) \cap [0, \frac{m-1}{m}]$. Let $I$ be a positive integer such that $\Phi(\mathcal{R}) \cap [0, \frac{m-1}{m}] \subseteq \frac{1}{I}\mathbb{Z}$. Replacing $m$ by $Im$, we may assume that $I$ divides $m$ and $\text{Supp}(\{m(B^+ - B)\}) \subseteq \text{Supp}(B^+ \cap \text{Supp}B)$. Note that now $K_X + B^+$ is also an $\frac{m}{I}$-complement of $K_X + B$.

Note that this also implies $-B^+ + [m(B^+ - B)] \leq [m(B^+ - B)]$. Since $K_X + B^+$ is an $n$-complement, we have $mK_X \sim mB^+$. Hence, $[m(m(B^+ - B))]$ defines a birational map. Since $m(m(B^+ - B)) \leq [m^2(B^+ - B)]$, replacing $m$ by $m^2$, we may assume $[m(B^+ - B)]$ defines a birational map. On the other hand, by Lemma 1.6, $\text{vol}([m(B^+ - B)]) < v$ for some $v$ depending only on $d$, $m$ and $\theta$.

Let $M$ be a general element of $[m(B^+ - B)]$. By Proposition 3.8, there is a bounded set of couples $\mathcal{P}$ depending only on $d$, $m$ and $\theta$, such that there is a projective log smooth couple $(\overline{W}, \Sigma)$ in $\mathcal{P}$, a birational map $\overline{W} \dashrightarrow X$ and a common resolution $X'$ of this map such that

1. $\text{Supp}\Sigma$ contains the exceptional divisor of $\overline{W} \dashrightarrow X$ and the birational transform of $\text{Supp}(B^+ + M)$,
2. there is a resolution $\phi : \overline{W} \dashrightarrow X$ such that $M|_W := M|_W \sim A_W + R_W$ where $A_W$ is the movable part of $|M_W|$, $|A_W|$ is base point free, $X' \dashrightarrow X$ factors through $W$ and $A_X := A_W|_{X'} \sim 0/\overline{W}$.

Since $M$ is a general element of $[m(B^+ - B)]$, we may assume $M_W = A_W + R_W$ and $A_W$ is general in $|M_W|$. In particular, if $A_{\overline{W}}$ is the pushdown of $A_W|_{X'}$ to $\overline{W}$, then $A_{\overline{W}} \subseteq \Sigma$. Let $M, A, R$ be the pushdowns of $M_W$, $A_W, R_W$ to $X$.

Since $|A_{\overline{W}}|$ defines a birational contraction and $A_{\overline{W}} \subseteq \Sigma$, there exists $l \in \mathbb{N}$ depending only on $\mathcal{P}$ such that $lA_{\overline{W}} \sim G_{\overline{W}}$ for some $G_{\overline{W}} \geq 0$ whose support contains $\Sigma_{\overline{W}}$. Let $K_{\overline{W}} + B_{\overline{W}}^+$ be the crepant pullback of $K_X + B^+$ to $\overline{W}$. Then $(\overline{W}, B_{\overline{W}}^+)$ is sub-lc and $\text{Supp}B_{\overline{W}}^+ \subseteq \text{Supp}\Sigma_{\overline{W}} \subseteq \text{Supp}G_{\overline{W}}$.

Let $G$ be the pushdown of $G_{X'} := G_{\overline{W}}|_{X'}$ to $X$. Since $A_{X'}$ is the pullback of $A_{\overline{W}}$, $lA_{X'} \sim G_{X'}$, and $lA \sim G$. Therefore, $G + lR + l\{m(B^+ - B)\} \sim_{\mathbb{Q}} m(B^+ - B)$.

Take a positive rational number $t \leq (lm)^{-d}\theta$, then

$$(X, B + t(G + lR + l\{m(B^+ - B)\}))$$

is klt. Moreover, we have

$$-K_X - B - t(G + lR + l\{m(B^+ - B)\}) \sim_{\mathbb{Q}} B^+ - B - t(lm(B^+ - B)).$$
By replacing $t$, we may assume $t < \frac{1}{lm}$. Since
\[ B^+ - B - t(lm(B^+ - B)) = (1 - tlm)(B^+ - B) \geq 0, \]
$B^+ - B - t(lm(B^+ - B))$ is nef and big. Therefore
\[ (X, B + t(G + lR + l\{m(B^+ - B)\})) \]
is klt weak Fano.

Now we argue that the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ are in a set of hyperstandard multiplicities associated to a finite set. Write
\[ B + tl\{m(B^+ - B)\} = (1 - tlm)B + tlmb^+ - tl\{m(B^+ - B)\}. \]

Denote $B^{\leq 1 - \frac{1}{m}}$ to be the sum of irreducible components of $B$ with coefficients at most $1 - \frac{1}{m}$ and $B^{> 1 - \frac{1}{m}} = B - B^{\leq 1 - \frac{1}{m}}$. Since 1 is the only possible accumulating point of $\Phi(\mathcal{R})$, $\mathcal{R} := \Phi(\mathcal{R}) \cap [0, \frac{m}{lm}]$ is a finite set. Therefore, the coefficients of $B^{\leq 1 - \frac{1}{m}} \in \mathcal{R}$. On the other hand, if $D$ is a component of $B^{> 1 - \frac{1}{m}}$, then by assumption, $\mu_D(B) = 1 - \frac{b}{m'} > 1 - \frac{1}{m}$ for some $b \in \mathcal{R}$ and $m' \in \mathbb{N}$. Since $B^+$ is an $m$-complement, $\mu_D(B^+) \in \{1\} \cup \mathbb{N}$. So $\mu_D(B^+) = 1$ and $\mu_D([m(B^+ - B)]) = 0$. Then we have
\[ \mu_D((1 - tlm)B^{> 1 - \frac{1}{m}} + tlmB^+ - tl\{m(B^+ - B)\}) \]
\[ = (1 - tlm)(1 - \frac{b}{m'}) + tlm = 1 - \frac{(1 - tlm)b}{m'}. \]

Therefore, the coefficients of $(1 - tlm)B^{> 1 - \frac{1}{m}} + tlmB^+ - tl\{m(B^+ - B)\} \in \Phi((1 - tlm)\mathcal{R} \cup tl\mathbb{N})$ since $B^+$ and $\{m(B^+ - B)\}$ are integral. Since $B + tl\{m(B^+ - B)\} = (1 - tlm)B + tlmB^+ - tl\{m(B^+ - B)\} = (1 - tlm)B^{\leq 1 - \frac{1}{m}} + (1 - tlm)B^{> 1 - \frac{1}{m}} + tlmB^+ - tl\{m(B^+ - B)\}$, the coefficients of $B + tl\{m(B^+ - B)\} \in ((1 - tlm)\mathcal{R} \cup \{0\} + tl\mathbb{N}) \cup \Phi((1 - tlm)\mathcal{R})$. Consequently, since $G$ and $R$ are integral, the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ belongs to
\[ ((1 - tlm)\mathcal{R} \cup \Phi((1 - tlm)\mathcal{R} \cup \{0\}) + t(\mathbb{N} \cup \{0\}). \]

Moreover, since $(X, B + t(G + lR + l\{m(B^+ - B)\}))$ is klt, the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ is less than 1. Since $(1 - tlm)\mathcal{R}$ and $(1 - tlm)\mathcal{R}$ are both finite, the only possible accumulating point of $\mathcal{I} := ((1 - tlm)\mathcal{R} \cup \Phi((1 - tlm)\mathcal{R} \cup \{0\})$ is 1. Thus, $(\mathcal{I} + t(\mathbb{N} \cup \{0\})) \cap [0, 1] = \mathcal{I} \cup ((\mathcal{R} \cap [0, 1 - t]) + t(\mathbb{N} \cup \{0\})) \cap [0, 1] = \mathcal{I} \cup \mathcal{I} \cup \mathcal{I} \subseteq \mathcal{I} \cup \mathcal{I} \cup \mathcal{I} \subseteq \Phi((1 - tlm)\mathcal{R} \cup \mathcal{R})$, where $\mathcal{I} := \{1\} - ((1 - tlm)\mathcal{R} \cup \{0\} \cup \mathcal{I})$ is a finite set.

By Theorem 2.4, there is a positive integer $n'$ depending only on $d, \mathcal{R} m, l$ and $t$ such that there is an $n'$-complement $K_X + \Omega$ of $K_X + B + t(G + lR + l\{m(B^+ - B)\})$, such that
\[ \Omega \geq B + t(G + lR + l\{m(B^+ - B)\}). \]

On the other hand, let
\[ \Delta_W := B^+_W + \frac{t}{m} A_W - \frac{t}{lm} G_W. \]
Then $(\overline{W}, \Delta_{\overline{W}})$ is sub-$\epsilon$-klt for some $\epsilon > 0$ depending only on $P$, $t$, $l$ and $m$ since $\text{Supp}B_{\overline{W}} \subseteq \text{Supp} \Sigma_{\overline{W}} \subseteq \text{Supp} G_{\overline{W}}$, $(\overline{W}, \Sigma_{\overline{W}})$ is log smooth, $(\overline{W}, B_{\overline{W}})$ is sub-lc, and $A_{\overline{W}}$ is not a component of $[B_{\overline{W}}]$. Moreover, $K_{\overline{W}} + \Delta_{\overline{W}} \sim_\mathbb{Q} 0$.

Let

$$\Delta := B^+ + \frac{t}{m} A - \frac{t}{lm} G.$$ 

Then $K_X + \Delta \sim_\mathbb{Q} 0$. $(X, \Delta)$ is sub-klt since $K_X + \Delta$ is the crepant pullback of $K_{\overline{W}} + \Delta_{\overline{W}}$.

Let $\Theta = \frac{1}{2} \Delta + \frac{1}{2} \Omega$. Then

$$\Theta = \frac{1}{2} B^+ + \frac{t}{2m} A - \frac{t}{2lm} G + \frac{1}{2} \Omega \geq \frac{1}{2} B^+ + \frac{t}{2m} A - \frac{t}{2lm} G + \frac{1}{2} \Omega \geq \frac{1}{2} B^+ = \frac{1}{2} \left( B + \frac{t}{2} (G + lR) \right) \geq \frac{t}{2} \frac{G + lR}{2} \geq 0.$$ 

Since $(X, \Delta)$ is sub-klt, $K_X + \Delta \sim_\mathbb{Q} 0$ and $(X, \Omega)$ is lc log Calabi–Yau, $(X, \Theta)$ is klt log Calabi–Yau. The coefficients of $\Theta$ belong to a fixed finite set $I$ depending only on $t$, $l$, $m$ and $n'$. Moreover, $\text{Supp} B \subseteq \text{Supp} \Omega \subseteq \text{Supp} \Theta$. By Theorem 3.3 we are done. \hfill \Box

**Theorem 3.7.** Fix a positive integer $d$, positive real numbers $\theta$ and $\delta$ and a finite set $\mathcal{R}$ of rational numbers in $[0, 1]$. The set of all klt Fano pairs $(X, B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \Phi(\mathcal{R})$,
3. $\alpha(X, B)^{d-1+\delta}(-K_X + B)^d > \theta$,

forms a log bounded family.

**Proof.** This follows from Theorem 2.11, Theorem 3.2 and Theorem 3.6. \hfill \Box

**Corollary 3.8.** Fix a positive integer $d$ and two positive real numbers $\delta$ and $\theta$. Then the set of klt-Fano varieties $X$ satisfying

1. $\dim X = d$,
2. $\alpha(X)^{d-1+\delta}(-K_X)^d > \theta$,

forms a bounded family.

**Proof.** This follows either from the previous theorem or from Theorem 2.11, Theorem 3.2 and Theorem 1.3. \hfill \Box

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