Dynamic Set Cover: Improved Algorithms & Lower Bounds

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Abstract

Set cover is a classic problem in combinatorial optimization where a set of \( n \) elements have to be covered by a minimum number of subsets from a given collection of size \( m \). The two traditional lines of inquiry for this problem are via greedy and primal dual algorithms, and respectively yield (tight) approximation factors of \( \ln n \), where \( n \) is the total number of elements, and \( f \), where every element belongs to at most \( f \) sets. Recent research has focused on the dynamic setting, where the set of elements changes over time. Using the same lines of inquiry, this has led to the following results: (a) an \( O(\log n) \)-approximation in \( O(f \log n) \) (amortized) update time (Gupta et al., STOC 2017), and (b) an \( O(f^2) \)-approximation in \( O(f \log (m + n)) \) (amortized) update time (Bhattacharya et al., ICALP 2015). While the former result matches the offline approximation within a constant factor, the latter does not; indeed, the only \( O(f) \)-approximation known in the dynamic setting is by re-solving the problem after every update.

In this paper, we show that it is possible to maintain efficiently a solution (almost) as good as the primal-dual offline one: we give a \((1 + \epsilon) f\)-approximation for set cover in \( O(f^2 \log n/\epsilon) \) (amortized) update time. If we are in a decremental setting, i.e., there are element deletions but no insertions, the update time can be improved to \( O(f^2/\epsilon) \), while still obtaining an \((1 + \epsilon) f\)-approximation. Finally, we study the dependence of the update time on \( f \). Here, we show that a linear dependence on \( f \) is indeed necessary unless we can tolerate much worse approximation factors: using the recent distributed PCP-framework, we show that any dynamic algorithm for set cover that has an amortized update time of \( O(f^{1-\epsilon}) \) must have an approximation factor that is \( \Omega(n^{\delta}) \) for some \( \delta > 0 \) under the Strong Exponential Time Hypothesis. To the best of our knowledge, this is the first application of the growing area of fine-grained complexity theory to show hardness of approximation of a NP-Hard problem.

At the heart of our algorithms is the following observation: we show a simple (deterministic) offline algorithm for the problem achieves a \((1 + \epsilon) f\)-approximation in \( O(f/\epsilon) \) expected (amortized) update time when deletions are randomly ordered. We switch this statement around by transferring the randomness to the algorithm in order to handle an arbitrary sequence of deletions (and insertions in the first result). This recipe of switching the randomness turns out to be very useful, though materializing this insight needs significant new ideas.
1 Introduction

Suppose, we need to solve a combinatorial optimization problem where the input to the problem changes over time. In such a dynamic setting, re-computing the solution from scratch after every update can be prohibitively time-consuming, and it is natural to seek dynamic algorithms that provide faster updates. In the last few decades, efficient dynamic algorithms have been discovered for many combinatorial optimization problems, particularly in graphs such as shortest paths \[4,18,20,29\], connectivity \[5,26,27,38\], maximal independent set and coloring \[7,11,35\]. For many of these problems, maintaining exact solutions is prohibitively expensive under various complexity conjectures \[2,3,24,31\], and thus the best approximation bounds are sought. In their seminal work \[34\], Onak and Rubinfeld proposed an algorithm for matching and vertex cover that maintains an \[O(1)\]-approximate solutions to the maximum matching and minimum vertex cover in the graph. The algorithm runs in \[t \cdot \text{polylog}(n)\] time for any sequence of \(t\) edge insertions and deletions in an \(n\)-vertex graph, i.e., in \(O(\text{polylog}(n))\) time when amortized over all the updates. This has led to a flurry of activity in dynamic algorithms for matching and vertex cover \[8–10,13,14,23,33,37\], and more recently, for the more general set cover problem \[10,12,22\] that we study in this paper.

In the set cover problem, we are given a universe \(X\) of \(n\) elements and a family \(S\) of \(m\) sets on these elements. The goal is to find a minimum-cardinality subfamily of sets \(F \subseteq S\) such that \(F\) covers all the elements of \(X\). The two traditional lines of inquiry for this problem are via greedy and primal dual algorithms, and have respectively led to a \(\ln n\)- and an \(f\)-approximation. Here, \(f\) is the maximum number of sets that an element belongs to in the set system \(S\). Both these results are known to be tight under appropriate complexity-theoretic assumptions \[19,30\]. In the dynamic setting, the set system \(S\) is fixed, but the set of elements that needs to be covered in \(X\) changes over time. In particular, after the insertion of a new element, or the deletion of an existing one, the solution has to be updated to maintain feasibility and the approximation guarantee. The time taken to perform these updates is called the update time of the algorithm, and is often stated amortized over any fixed prefix of updates.

As in the case of the offline problem, dynamic algorithms for set cover have also followed two lines of inquiry. The first is to use greedy-like techniques, which were recently shown to yield an \(O(\log n)\)-approximation in \(O(f \log n)\) update time by Gupta, Kumar, Krishnaswamy, and Panigrahi \[22\]. The second is to use a primal-dual framework, which was employed by Bhattacharya, Henzinger, and Italiano \[12\] to give an \(O(f^2)\)-approximation in \(O(f \log (m+n))\) update time. Gupta et al. \[22\] and Bhattacharya, Chakrabarty, and Henzinger \[10\] also obtained a different but incomparable result using the primal-dual technique, which improves the update time to \(O(f^2)\) thereby removing the dependence on \(n\) and \(m\), but at the cost of a weaker approximation bound of \(O(f^3)\). What stands out in these results is that:

- While dynamic and offline approximation factors match at \(O(\log n)\), there is no \(O(f)\)-approximation known for the dynamic setting. Indeed, the only previous algorithm we are aware of that achieves this bound is one that recomputes the offline \(f\)-approximation after every update.

- The update times of these algorithms depend on \(\log n\) and \(f\). While the dependence on \(\log n\) is not required, at least if we settle for an \(O(f^3)\) approximation \[10,22\], it is not clear if the polynomial dependence on \(f\) is fundamental. For instance, might it be possible to design a dynamic set cover algorithm whose update time only has a logarithmic dependence on \(f^2\)?

\[1\]This set of references is not comprehensive and the reader is referred to papers on dynamic graph algorithms for a more comprehensive list.

\[2\]All update times stated in this paper are amortized, unless stated otherwise.
1.1 Our Results

Our first result closes the gap between offline and dynamic approximation for the set cover problem: for any $\epsilon > 0$, we give a $(1 + \epsilon) f$-approximation algorithm for dynamic set cover with an update time of $O(f^2 \log n / \epsilon)$. Previous algorithms for dynamic set cover heavily rely on deterministically maintaining a greedy-like or primal-dual structure on the set cover solution. Instead, our algorithm is based on the observation that a simple offline algorithm for the set cover problem achieves a $(1 + \epsilon) f$-approximation in $O(f / \epsilon)$ expected update time when the elements are deleted in a random order. We switch this statement around by transferring the randomness to the algorithm in order to handle an arbitrary sequence of deletions (and insertions). As a result, our algorithm is randomized, and our update time bound holds both in expectation and with high probability. (The approximation bound holds deterministically.)

A simplification of the above algorithm leads, in the decremental setting where elements can only be deleted but not inserted, to the same approximation factor of $(1 + \epsilon) f$ in amortized update time $O(f^3 / \epsilon)$. This can be compared with the result of Gupta et al. \[22\] which achieves a (larger) $O(f^3)$-approximation with (roughly) the same update time, but in the fully dynamic case. As far as we know, the approximation bounds of \[10, 22\] do not change when considering the decremental setting, which has been extensively studied in the past \[13, 25, 29\].

Finally, we turn to the problem of determining the dependence of the update time on $f$. Using the recently introduced framework of distributed PCP \[1\] from fine-grained complexity theory, we show that under the Strong Exponential Time Hypothesis (SETH), any dynamic set cover algorithm that has an (amortized) update time of $O(f^{1-\epsilon})$ for any fixed $\epsilon > 0$ must have an approximation factor of $(n / \log f)^{\Omega(1)}$. Since a polynomial dependence on $n$ in the approximation factor is rather weak, this result essentially states that any dynamic set cover algorithm must have a linear dependence on $f$ in the update time. This shows the update time bound of \[22\] to achieve $O(\log n)$ approximation is essentially tight within a $\log n$ factor. Our lower bound holds even if the algorithm is allowed a preprocessing stage with arbitrary polynomial runtime, and it also applies to the set-updates model where the elements are fixed but sets get inserted and deleted. This model is much more popular in the streaming setting \[6, 32\], especially when there are only insertions (see the work by Indyk et al. \[29\] and the many references therein).

Note that a fast algorithm for a dynamic problem usually gives a fast algorithm for its static version. A dynamic $\min\{f, \log n\}$-approximation algorithm that has $o(f)$ update time and $o(fn)$ preprocessing time would lead to a new, faster algorithm for the static case. Such an algorithm would be a breakthrough; however, it is open whether it would be strong enough to refute SETH or any other conjecture from fine-grained complexity. In this paper, we bypass this connection and prove SETH-based lower bounds for dynamic set cover directly. To the best of our knowledge, this is the first application of the growing area of fine-grained complexity theory to show hardness of approximation of a NP-Hard problem.

1.2 Our Techniques

The natural starting point for our work would be to use the deterministic greedy or primal dual techniques for dynamic set cover from \[10, 12, 22\]. A different alternative would be to generalize previous randomized approaches for dynamic vertex cover \[8, 37\]. At a very high level, all these algorithms derive their results from maintaining, either explicitly or implicitly, a very structured dual solution that lower bounds the cost of the algorithm. Indeed, the algorithm of \[22\] for dynamic set cover can be thought of as a derandomization of the dynamic vertex cover algorithm of \[37\]. In order to improve the approximation factor to $O(f)$, these dual solutions must only violate the dual packing constraints by a constant factor (as against an $\Omega(1)$ violation in the previous results), but this requirement is too strict for the analysis framework of these papers that...
Hence, we need a significant new approach to improve the approximation factor to $O(f)$. We start with the following folklore algorithm for offline set cover. Initially, all elements are uncovered and the algorithm has an empty solution. Pick an arbitrary uncovered element and call it a pivot $p$. Then, include all sets containing $p$ in the solution and mark all elements in those sets as covered. Repeat this process until all elements get covered. This algorithm runs in $O(nf)$ time and achieves an $f$-approximation, since no two pivots share a common set and the algorithm picks at most $f$ sets for each pivot. We call this the deterministic covering algorithm.

Now, consider a decremental setting where elements are deleted over time, but in a uniform random order. A small modification to the deterministic covering algorithm gives a $(1 + O(\varepsilon))f$-approximation in $O(f/\varepsilon)$ update time in this setting. Initially, we run deterministic covering to produce a feasible cover. During the deletion phase, the approximation bound may no longer hold because pivots are being deleted. To restore the bound, we re-run the deterministic covering algorithm whenever an $\varepsilon$-fraction of the pivots have been deleted. Since the number of undeleted pivots forms a lower bound on the optimal solution, it follows that this algorithm maintains an $f/(1 - \varepsilon) = (1 + O(\varepsilon))f$-approximation.

Let us now consider the update time. Clearly, deterministic covering takes $O(nf)$ time in every run. So, the question is how frequently do we run it? Because of the random deletion order, we expect to delete an $\varepsilon$-fraction of all elements before an $\varepsilon$-fraction of pivots gets deleted. This suggests an informal amortized update bound of $O(nf/\varepsilon n) = O(f/\varepsilon)$. It turns out that this informal idea can be made formal, but we skip the details here for brevity since we are going to use this only for intuitive purposes.

More interesting for us is to transition from a random deletion order to an adversarial deletion order. The same update rule gives a $(1 + \varepsilon)f$-approximation, but now, the bound on update time may no longer hold. For instance, if all the pivots are deleted before other elements, the amortized update time is clearly much higher when the first $\varepsilon$-fraction of pivots gets deleted. Our main idea, at this juncture, is to transfer the randomization from the deletion sequence to the algorithm itself. More specifically, instead of picking a pivot arbitrarily from the uncovered elements in each step, let us select it uniformly at random. We call this the random covering algorithm. Our hope is that an (oblivious) adversary deleting a single element will be able to pick a specific pivot with probability no higher than $1/n$. This would ensure that in expectation, an $\varepsilon$-fraction of the elements will have to be deleted before an $\varepsilon$-fraction of pivots is, as in the random deletion scenario.

However, this intuition is not quite correct. While the first pivot is indeed uniformly distributed over all elements, the subsequent pivots are not. To see this, consider the following example: suppose the sets represent edges of a graph containing $(n-2)/f$ cliques on $f$ vertices each, and an isolated edge. For a vertex on the isolated edge to be chosen as the second pivot, it must not be covered by sets containing the first pivot and should be selected as the second pivot; the probability for this event is given by: $(1 - 2/n) \cdot (1/(n-1))$. Clearly, this probability exceeds $1/n$ for $f > 2$. As a consequence, the expected number of element deletions after which we need to run random covering might be smaller than $\varepsilon n$. To overcome this bottleneck, we employ a more fine-grained update procedure: instead of running random covering over the entire undeleted instance, we run it only for a subset of elements. We maintain sufficient structure in the solution to still claim a $(1 + \varepsilon)f$-approximation, while improving the update time to $O(f^2/\varepsilon)$ for the decremental setting, and $O(f^2 \log n/\varepsilon)$ for the fully dynamic setting where elements can be inserted in addition to deletions.

Roadmap. We present the random covering algorithm for the decremental setting in Section 2. In Section 3, we further generalize these ideas to the fully dynamic setting, albeit at a slightly worse update time. Finally, in Section 4, we describe our lower bound results. As mentioned earlier, our update time bounds also hold with high probability–these stronger bounds are given in Appendix H.
2 The Decremental Set Cover Algorithm

In this section, we give a dynamic set cover algorithm for the decremental setting. We denote the initial set system by \((X, S)\), where \(S = \{S_1, S_2, \ldots, S_n\}\) is a collection of subsets of the ground set \(X\) that contains \(n\) elements. The maximum number of subsets that an element belongs to is denoted by \(f\):

\[
f = \max_{x \in X} |\{i : x \in S_i\}|.
\]

The elements are deleted in a fixed sequence, independent of the randomness of the algorithm, that is represented as \(X = \{x_1, x_2, \ldots, x_n\}\).

2.1 The Algorithm

The description of the algorithm comprises two phases: the initial phase where the algorithm selects a feasible solution at the outset, and the update phases where the algorithm changes its solution in response to the deletion of elements from the set system. The feasible solution that the algorithm maintains dynamically is denoted by \(F\). Recall that the goal is to ensure that the cost of \(F\) is at most \((1 + \epsilon)f\) times that of an optimal solution for the set of undeleted elements at all times.

Both the initial and the update phases use a common subroutine that we call the random cover subroutine. We describe this subroutine first.

The Random Cover Subroutine. The random cover subroutine takes as input a set system \((X', S')\) and outputs a feasible set cover solution \(F'\) for this set system. The algorithm is iterative, where each iteration starts with a set of uncovered elements \(Y \subseteq X'\), adds a collection of sets \(F' \subseteq S'\) to the solution \(F'\), and removes all the elements covered by the sets in \(F'\) from the set of the uncovered elements \(Y\) for the next iteration. Initially, all elements in \(X'\) are uncovered, i.e., \(Y = X'\), and the solution \(F'\) is empty, i.e., \(F' = \emptyset\). It only remains to describe an iteration, or more precisely, the sets \(F'\) added to the solution \(F'\) in an iteration. The selection of \(F'\) has three steps. First, the algorithm picks the set in \(S'\) that covers the maximum number of uncovered elements, breaking ties arbitrarily. Let us call this set \(Z\), i.e., \(Z = \arg \max_{S \in S'} |S \cap Y|\). Next, the algorithm chooses an element in \(Z \cap Y\), i.e., an uncovered element in the chosen set, uniformly at random, and calls this element the pivot for the current iteration. Let us call this pivot \(p \in \text{u.a.r. } Z \cap Y\). Finally, all sets in \(S'\) that contain the pivot are added to the solution, i.e., \(F' = \{S' : p \in S\}\). The random cover subroutine ends when all elements in \(X'\) are covered by the solution \(F'\), i.e., \(Y = \emptyset\). This algorithm can be implemented in \(O(f|X'|)\) deterministic time (details in Appendix A).

The above completes the description of the random cover subroutine. However, it will be convenient to create some additional notation for this process that we will use later. Each iteration is characterized by its pivot \(p\). We map the pivot to the set \(S(p) := Z \cap Y\) from which it is chosen. If \(|S(p)| \in \{2^i, 2^{i+1}\}\), we say that pivot \(p\) is a level-\(i\) pivot, and denote \(\ell(p) = i\). Note that by the definition of the random cover subroutine, the pivots chosen in successive iterations have monotonically non-increasing levels, i.e., if pivot \(p\) is chosen in an earlier iteration and pivot \(p'\) in a later iteration, then \(\ell(p) \geq \ell(p')\). Finally, if the sets \(F'\) are added to the solution \(F'\) in an iteration with pivot \(p\), then we denote \(F(p) = F'\). The set of previously uncovered elements that are covered by the sets in \(F'\) is denoted by \(\mathcal{X}(p)\).

Initial Phase. In the initial phase, the random cover subroutine is run on the the entire input set system \((X, S)\). This produces the initial solution \(F\).

In the algorithm, we also maintain sets \(P, D,\) and \(U\) that respectively represent total, deleted, and undeleted pivots. At the end of the initial phase, all the pivots in \(F\) are added to \(P\) and \(U\), and \(D\) is empty. When an element \(e\) is deleted, if \(e\) is in \(P\), then we move \(e\) from \(U\) to \(D\), i.e., change its status from undeleted to deleted. Importantly, we keep this element in \(P\). Changes to \(P\) are done only at the end of an update phase, which we describe below.
**Update Phase.** An update phase is triggered when the number of deleted pivots exceeds an $\epsilon$-fraction of the total number of pivots, i.e., $|D| \geq \epsilon \cdot |P|$. In an update phase, the algorithm first fixes a level $\ell$ using a process that we describe later called the level fixing process. Having fixed this level $\ell$, the algorithm discards all sets $F(p)$ from $F$ that were added by pivots $p$ at levels $q \leq \ell$ or lower, i.e., where $\ell(p) \leq \ell$. Correspondingly, these pivots $p$ are also removed from $P$ and from either $D$ or $U$ depending on whether they are deleted or undeleted. As a result of this change to $F$, some elements become uncovered in $F$; this set is denoted by $X'$. The algorithm now runs the Random Cover subroutine on the instance $(X', S')$ induced by $X'$, where $S' = \{S \cap X' : S \in S, S \cap X' \neq \emptyset\}$. The resulting sets $F'$ are added to the overall solution $F$. Correspondingly, the newly selected pivots are also added to $P$ and $U$. We say that levels $\ell$ and below have been updated in the current update phase. (Note that the newly selected pivots will be at level $\ell$ or below.) Clearly, this restores feasibility of the solution $F$. We already argued that the call to the Random Cover subroutine can be performed in $O(f|X'|)$ deterministic time. The same upper bound holds for the remaining operations related to the construction of the instance $(X', S')$ and to the update of the approximate solution (see Appendix [A] for the details).

**The level fixing process.** We now describe the level fixing process. Let $\{0, 1, \ldots, L = \lfloor \log_2 n \rfloor\}$ be the set of levels. Let $P_j$, $D_j$ and $U_j$ respectively denote the current set of total pivots, deleted pivots, and undeleted pivots at a given level $j$. This process finds a level $\ell$ with the following property: for every level $i \leq \ell$, $\sum_{j=0}^{\ell} |D_j| \geq \epsilon \cdot \sum_{j=0}^{\ell} |P_j|$. In other words, the fraction of deleted pivots in levels $i, i+1, \ldots, \ell$ is at least an $\epsilon$-fraction of the total number of pivots in these levels. We say that level $\ell$ is critical. The next lemma claims that at least one critical level exists whenever the number of deleted pivots is an $\epsilon$-fraction of the total number of pivots.

**Lemma 1.** If $\sum_{j=0}^{L} |D_j| \geq \epsilon \cdot \sum_{j=0}^{L} |P_j|$, then there exists at least one critical level.

**Proof.** Suppose $\sum_{j=0}^{L} |D_j| \geq \epsilon \cdot \sum_{j=0}^{L} |P_j|$. Assume by contradiction that the claim is not true. Hence for each level $\ell$, there exists a level $f(\ell) \leq \ell$ (in case of ties, take, say, the lowest such level) such that the condition does not hold, namely $\sum_{j=f(\ell)}^{\ell} |D_j| < \epsilon \cdot \sum_{j=f(\ell)}^{\ell} |P_j|$. We next define a sequence of levels $\ell_1, \ldots, \ell_q$ as follows. Set $\ell_1 = L$. Given $\ell_i$, halt if $f(\ell_i) = 0$, else set $\ell_{i+1} = f(\ell_i) - 1$ and continue with $\ell_{i+1}$. Observe that the intervals $[f(\ell_i), \ell_i]$ are disjoint and span $[0, L]$. One has

$$
\sum_{j=0}^{L} |D_j| = \sum_{i=1}^{q} \sum_{j=f(\ell_i)}^{\ell_i} |D_j| < \sum_{i=1}^{q} \epsilon \cdot \sum_{j=f(\ell_i)}^{\ell_i} |P_j| = \epsilon \cdot \sum_{j=0}^{L} |P_j|.
$$

This contradicts the assumption. \hfill \square

Next, we give an algorithm to find a critical level. If there are multiple critical levels, this algorithm finds the lowest one, although an algorithm that finds any critical level would suffice for our purpose. To find a critical level, the algorithm maintains the values $|P_j|$ and $|D_j|$. These counters are updated in $O(1)$ time when a new pivot is added to the solution $F$ in either the initial or the update phases, or when a pivot is deleted from the instance. Once a set of levels $\{0, 1, \ldots, \ell\}$ have been updated in an update phase, the number of deleted pivots in these levels is set to 0. Using these counters, we find a critical level $\ell$ in $O(\ell^2)$ time by iterating over possible values of $\ell$ starting with the lowest level, and explicitly checking the condition in constant time for each $i \leq \ell$. More details are given in Appendix [A].

**2.2 Analysis of the Competitive Ratio**

**Lemma 2.** The competitive ratio of the algorithm is at most $f/(1 - \epsilon)$. 


Proof. Consider the data structure right before the $t$-th deletion. Let $P^t$ be the set of pivots at that time, with $U^t$ being the subset of undeleted pivots at that time. We also let $OPT^t$ and $F^t$ be the optimal and approximate solution at that time.

Observe that $|F^t| \leq f \cdot |P^t|$ by construction. We claim that $|OPT^t| \geq |U^t|$. This implies the claim since by construction $|P^t| \leq |U^t|/(1 - \varepsilon)$ at any time.

To see that, let us show by a simple induction that, for any two distinct $p, p' \in U^t$, there is no set $S \in F$ covering both $p$ and $p'$. Thus $OPT^t$ needs to include a distinct set for each element of $U^t$. By construction, the Random Cover subroutine applied to $X''$ never selects a pivot $p'$ that is covered by sets selected due to a previous pivot $p$. This implies that the property holds after the initialization step, where $X' = X$. Assume the property holds up to step $t \geq 1$, and consider step $t + 1$. If the pivots are not updated during step $t$, there is nothing to show. Otherwise the algorithm will keep a proper prefix (in order of insertion) $F^t_{pref}$ of the sets $F^t$, and replace the remaining suffix $F^t_{suf}$ with $F^t_{suf+1}$ by means of the Random Cover subroutine. Let $X^t_{suf}$ be the undeleted elements at the beginning of step $t + 1$ covered by $F^t_{suf}$, and $X^t_{suf+1}$ be the remaining undeleted elements at the same time. By the same argument as above, the property holds at step $t + 1$ for any $p, p' \in X^t_{suf}$. It also holds for any $p, p' \in X^t_{pref}$ by inductive hypothesis. If $p \in X^t_{pref}$ and $p' \in X^t_{suf}$, by construction $p$ was selected before any pivot in $X^t_{suf}$. Thus the sets $\cup_{S \in F(p)} S$ and $X^t_{suf}$ (hence $X^t_{suf} \subseteq X^t_{suf}$) are disjoint. \hfill $\square$

2.3 Analysis of Amortized Update Time

Our goal is to show that at any stage of the algorithm, the expected time taken for the update phases till then is $O(f^2)$ times the number of elements deleted till then.

First, we show that following property of the level fixing process.

Lemma 3. Let $\ell$ be a critical level. There exists a $b$-matching between the deleted pivots in levels $\ell$ and below, denoted $D := \cup_{j \leq \ell} D_j$, and all pivots in these levels, denoted $P := \cup_{j \leq \ell} P_j$, such that:

- Each element of $P$ is matched to exactly one element of $D$ and each element of $D$ to at most $b = 1/\varepsilon$ elements of $P$;
- If $d \in D$ is matched to $p \in P$, then $\ell(d) \geq \ell(p)$.

Proof. Let us replace each element $d$ in each $D_j$ with $1/\varepsilon$ copies, and let us call the new set $D'_j$. Each copy inherits the level of the original element. Let us sort $D' := \cup_{j \leq \ell} D'_j$ in non-increasing order of level, and similarly sort $P$. Now, for every element $p \in P$ according to this order, we match $p$ with the first unmatched element $d'$ of $D'$. The $B$-matching is obtained by collapsing the copies of the same node in $D'$. Observe that this way we match all elements of $P$ since, being $\ell$ critical, $|D'| = |D|/\varepsilon \geq |P|$. Note also that obviously we match at most $1/\varepsilon$ copies of each $d \in D$. It remains to show the condition on the levels. Suppose by contradiction that there exists some $d' \in D'$ matched to $p \in P$ with $\ell' = \ell(d') < \ell(p) = \ell''$. By construction this implies that

$$\frac{1}{\varepsilon} \sum_{j = \ell''}^{\ell} |D_j| = \sum_{j = \ell''}^{\ell} |D'_j| \leq \sum_{j = \ell''+1}^{\ell} |D'_j| < \sum_{j = \ell''}^{\ell} |P_j|,$$

which contradicts the assumption that $\ell$ is critical. \hfill $\square$

Let $M$ be the function that maps each $d \in D$ into the corresponding subset of (at most $1/\varepsilon$) elements of $P$ according to the $B$-matching from Lemma 3. We say that a deleted pivot $d \in D$ is responsible for an element $e$ if $e$ is covered by a set that is added to the solution $F$ because of a pivot $p \in M(d)$. To state this in notation, recall that $X(p)$ denotes the set of previously uncovered elements that were covered by the sets
Lemma 4. The total number of elements that a deleted pivot \( d \in D \) at level \( i \) is responsible for is at most \((1/\varepsilon) f 2^i + 1\).

Proof. Consider each \( p \in M(d) \) and recall that \( \ell(p) \leq \ell(d) = i \). By definition of \( \ell(p) \), each set containing \( p \) covers less than \( 2^{\ell(p)+1} \) new elements at the time \( p \) is selected. Hence \( |X(p)| < f \cdot 2^{\ell(p)+1} \leq f \cdot 2^i + 1 \).

The claim follows since \( |M(d)| \leq 1/\varepsilon \).

Recall that for a pivot \( p \), we denote by \( S(p) \) the set of uncovered elements from which \( p \) was randomly selected. We say that a deleted pivot \( p \in D \) is a good pivot if at least half of the elements in \( S(p) \) are deleted before \( p \); otherwise, we say that the deleted pivot is bad.

Lemma 5. Fix any set \( S \). Conditioned on a pivot being selected from \( S \), i.e., \( \exists p \) s.t. \( S(p) = S \), the probability that \( p \) will be a good pivot on its deletion is at least \( 1/2 \). Equivalently, the probability that \( p \) is a good pivot is at least as much as the probability that it is a bad pivot.

Proof. Let \( v_1, \ldots, v_{|S|} \) be the elements of \( S \) in the order of their deletion, and \( p \) be the selected pivot in \( S \). We have that \( \Pr [p = v_i] = \frac{1}{|S|} \). The condition is satisfied iff \( p = v_i \) for \( i \geq \lceil |S|/2 \rceil \), which happens with probability \( \geq 1/2 \).

Consider the first \( t \) deletions. Let \( D_i^t \) denote the set of pivots that have been deleted at level \( i \) up to the \( t \)-th deletion step. By Lemma 4, the total number of elements that have been updated in the update phases of the algorithm is at most \((f/\varepsilon) \cdot \sum_i |D_i^t| \cdot 2^i + 1\). We bound the expectation of this expression next.

Lemma 6. The expected number of elements that have been updated in the update phases of the algorithm is at most \((8f/\varepsilon) \cdot t \) for the first set of \( t \) deletions.

Proof. Recall that the number of updated elements is at most \( O(f/\varepsilon) \cdot \sum_i |D_i^t| \cdot 2^i + 1 \). Let us denote \( \Lambda := \sum_i |D_i^t| \cdot 2^i + 1 \). We have

\[
E[\Lambda] = E \left[ \sum_i |D_i^t| 2^i + 1 \right] = E \left[ \sum_i (G_i + B_i) 2^i + 1 \right],
\]

where \( G_i \) and \( B_i \) denote the number of good and bad pivots in \( D_i^t \), resp. By Lemma 5, \( E[G_i] \geq E[B_i] \) for every level \( i \); hence, \( E[\Lambda] \leq 2 \cdot E \left[ \sum_i G_i 2^i + 1 \right] \). Now, for any good pivot \( p \) in level \( i \), there are at least \( 2^i \) elements in \( S(p) \) and at least half of those elements were deleted. Therefore, the total number of deleted elements \( t \) satisfies: \( t \geq \sum_i G_i 2^i - 1 \). As a consequence,

\[
E[\Lambda] \leq 2 \cdot E \left[ \sum_i G_i 2^i + 1 \right] \leq 8t.
\]

The lemma follows.

Finally, we consider the time spent in the level fixing process.

Lemma 7. The time taken in the level fixing process is at most that in the subsequent update phase.
Proof. Recall that level fixing at level \( \ell \) takes time at most \( c\ell^2 \) for some constant \( c \). Since \( \ell \) is the lowest critical level, \( \ell - 1 \) is not critical, and therefore, there is at least one deleted pivot in level \( \ell \). Thus, the cost of the update phase is at least \( 2^\ell \geq c\ell^2 \) for large enough \( \ell \).

\[\square\]

Lemma 8. The expected amortized update time per deletion is \( O(f^2/\varepsilon) \).

Proof. Consider the first \( t \) deletions and let \( X'_1, \ldots, X'_q \) be sets of elements whose cover is updated during these steps. From the previous discussion and Lemma \[7\] the expected running time of the algorithm is \( O(f \cdot t + \mathbb{E} \left[ \sum_{j=1}^q f \cdot |X'_j| \right]) \). From Lemma\[6\] \( \mathbb{E} \left[ \sum_{j=1}^q |X'_j| \right] \leq (8f/\varepsilon) \cdot t \). The claim follows.

\[\square\]

In Appendix \[3\] we extend this algorithm so that the update time bound holds with high probability.

3 The Fully Dynamic Set Cover Algorithm

In this section, we extend the algorithm for the decremental case to the fully dynamic case. At any time \( t \), let \( A \subseteq X \) denote the elements that are present in the system. In the beginning, \( A = \emptyset \) and \( F = \emptyset \). The elements are then inserted and deleted in a fixed sequence, independent of the randomness of the algorithm. If an element is inserted and then gets deleted and reinserted, we treat the two insertions separately as two copies of the same element.

3.1 The Algorithm

We now describe the update phases where the algorithm changes its solution in response to the insertions and deletions of elements. The update phases are very similar to the decremental algorithm with few changes. To describe the changes, we need to introduce some additional notations. Just like the decremental algorithm, the fully dynamic algorithm maintains a set of pivots \( P \), and at any time, the solution \( F \) can be completely specified by \( P, F = \{ S \mid S \ni p \} \). \( S(p) \) denotes the set of elements from which a pivot \( p \in P \) is chosen and \( X(p) \) denotes the set of elements \( p \) is accounted to cover at any point of time. If \( |S(p)| \in [2^i, 2^{i+1}) \), we say that pivot \( p \) is a level-\( i \) pivot, and denote \( \ell(p) = i \). We call the sets \( \{ S \mid S \ni p, \ell(p) = i \} \) as level-\( i \) sets. In addition, we partition \( X(p) \) into two subsets \( \text{Orig}(p) \) and \( \text{Extra}(p) \), that is \( X(p) = \text{Orig}(p) \cup \text{Extra}(p) \) and \( \text{Orig}(p) \cap \text{Extra}(p) = \emptyset \). An element \( e \in \text{Orig}(p) \) is referred to as an original element and an element \( e \in \text{Extra}(p) \) is referred to as an extra element. \( \text{Orig}(p) \) consists of all elements that \( p \) is accounted to cover at the time when \( p \) was chosen to be a pivot and \( F^+ = \{ S \in S' : p \in S \} \) sets are included in the solution. Thus, \( S(p) \subseteq \text{Orig}(p) \). It is possible that \( p \) is accounted to cover more elements due to later updates. Those elements are added to \( \text{Extra}(p) \). We are now ready to describe the update phases.

Update Phase: Insertion. When a new element \( e \in X \setminus A \) is inserted, if \( \{ S \mid S \ni e \} \cap F \neq \emptyset \), then we select the set \( S \) containing \( e \) at the highest level breaking ties arbitrarily. If \( p \in P \cap S \), then we insert \( e \) in \( \text{Extra}(p) \) and no change to \( F \) is made. Otherwise, we include \( e \) as a level-0 pivot and set \( S(e) = \{ e \} \), \( X(e) = \text{Orig}(e) = \{ e \} \). We update \( F = F \cup \{ S \mid S \ni e \} \) and \( A = A \cup \{ e \} \). This update phase can be implemented in \( O(f) \) time.

Update Phase: Deletion. When an element \( e \in A \) is deleted, the algorithm checks \( \{ S \mid S \ni e \} \cap F \) and mark \( e \) to be deleted from those sets. It updates \( A = A \setminus \{ e \} \). By doing so, if the number of pivots that are marked as deleted exceed an \( \varepsilon \) fraction of the total number of pivots in \( F \), then the following additional processing is done and we say an update phase has been triggered.

A triggered update phase has two steps, movement step and a covering step. We next describe them. The algorithm first fixes a critical level \( \ell \) using the level fixing process of the decremental algorithm. Having fixed this level \( \ell \), the algorithm discards all sets \( F(p) \) from \( F \) that were added by pivots \( p \) at levels \( \ell \) or lower, i.e., where \( \ell(p) \leq \ell \). As a result, a set of (undeleted) elements become uncovered in \( F \); this set is denoted by \( X' \).
Movement step: The algorithm then checks for each element $e \in X'$, if there exists a set $S \in \mathcal{F}$ containing $e$ at a level $\ell' > \ell$. If yes, the algorithm selects a set $S \ni e$ at the highest level (breaking ties arbitrarily). If $p = P \cap S$, then $e$ is added to $\text{Extra}(p)$ and $X(p)$. We then remove $e$ from $X'$.

Covering step: Let $Y' \subseteq X'$ denote the remaining elements after the movement step. The algorithm now runs the Random Cover subroutine on the instance $(Y', S')$ induced by $Y'$ and adds the resulting sets $\mathcal{F}'$ to the overall solution $\mathcal{F}$. We say that levels $\ell$ and below have been updated in the current update phase. For every newly chosen pivot $p \in Y' \cap P$, if $S(p) \in [2^\ell, 2^{\ell+1})$, then pivot $p$ is a level-$i$ pivot and we include \{ $S \mid S \ni p$ \} at level $i$. Note that it is possible that $i > \ell$ due to new inserted elements. Also note that all the elements of $Y'$ now become original elements after the covering step. We already argued that the call to the Random Cover subroutine can be performed in $O(f|Y'|)$ time. The additional processing related to the construction of the instances corresponding to $X'$ and $Y'$ and to the update of the approximate solution can be done in $O(f|X'|)$ time as well.

3.2 Analysis of the Competitive Ratio

Lemma 9. The competitive ratio of the algorithm is at most $f/(1-\varepsilon)$.

Proof. Consider the data structure right before the $t$-th update. Let $P^t$ be the set of pivots at that time, with $U^t$ being the subset of undeleted pivots at that time. We also let $OPT^t$ and $F^t$ be the optimal and approximate solution at that time.

Observe that $|F^t| \leq f \cdot |P^t|$ by construction. We claim that $|OPT^t| \geq |U^t|$. This implies the claim since by construction $|P^t| \leq |U^t|/(1-\varepsilon)$ at any time.

To see that, let us show by a simple induction that, for any two distinct $p, p' \in U^t$, there is no set $S \in \mathcal{F}$ covering both $p$ and $p'$. Thus $OPT^t$ needs to include a distinct set for each element of $U^t$. Since, we start with an empty set cover, the property holds at the initialization vacuously. Assume the property holds up to step $t \geq 1$, and consider step $t+1$. If an element $e$ is inserted at $t+1$, then $e$ becomes a pivot if and only if $e$ is not covered by any existing set in $\mathcal{F}$. Hence, the result holds. If a non-pivot element $e$ is deleted at $t+1$, or $e$ is a pivot element but its deletion does not trigger an update phase, then since we do not change the set cover, the claim holds by the induction hypothesis. Therefore, assume $e$ is a pivot element and its deletion triggers an update phase with critical level $\ell'$ and $X'$ elements. Note that, we select a set of new pivots from $Y' \subseteq X'$ and $F'$ denotes the new sets that are added after the update phase. Due to the movement step, for every $e \in Y'$, $\{ S \mid S \ni e \} \cap \{ F \setminus F' \} = \emptyset$. Therefore, it is not possible that for $p, p' \in U^{t+1}, p \in P \setminus Y'$ and $p' \in P \cap Y'$, there is a set $S \in \mathcal{F}$ that contains both $p$ and $p'$. By induction hypothesis, for $p, p' \in U^{t+1}, p \in P \setminus Y'$ and $p' \in P \setminus Y'$, there does not exists a set $S \in \mathcal{F}$ containing both $p$ and $p'$ either. Finally, by construction, the Random Cover subroutine applied to $Y'$ never selects a pivot $p'$ that is covered by sets selected due to a previous pivot $p \in Y'$. Thus, it is also not possible that there exists a set $S \in \mathcal{F}$ that contains $p$ and $p'$ where both $p, p' \in U^{t+1}, p \in P \cap Y'$ and $p' \in P \cap Y'$. Therefore, the result holds after the $(t+1)$-th update.

3.3 Analysis of Amortized Update Time

Our goal is to show that at any stage of the algorithm, the expected time taken for the update phases till then is $O(\ell^2 \log \frac{\varepsilon}{\ell})$ times the number of elements deleted or inserted till then.

The property of the level fixing process from Lemma 3 holds for the fully dynamic case as well, since there is no change in the level fixing process.

Let $\ell$ be a critical level. Let $D := \bigcup_{j \leq \ell} D_j$ and $P := \bigcup_{j \leq \ell} P_j$. Then each element of $P$ is matched to exactly one element of $D$ and each element of $D$ to at most $b = 1/\varepsilon$ elements of $P$. Moreover, if $d \in D$ is matched to $p \in P$, then $\ell(d) \geq \ell(p)$. Let $M$ be the function that maps each $d \in D$ into the corresponding
subset of (at most $1/\varepsilon$) elements of $P$ according to the $b$-matching from Lemma \[5\]. We say that a deleted pivot $d \in D$ is responsible for an element $e$ if $e \in \text{Orig}(p)$ where $p \in M(d)$.

Now consider an element $e \in \text{Extra}(p)$ for some $p \in P$ in this phase. Consider the last phase in which $e$ was an original element. If no such phase exists, then $e$ must have been inserted as an extra element and since then it has only taken part in movement steps but never in a covering step. We map $e \in \text{Extra}(p)$ to the insertion of $e$ and that insertion is responsible for $e$.

Otherwise, there exists a last phase such that $e$ was an original element just before the update and became an extra element just after the update. If $e \in \text{Orig}(p')$ and $p' \in M(d')$ during that phase, we map $e$ to $d'$, and say $d'$ is responsible for $e$.

Note that, for every element $v$ in the instance that an update phase operates on, either there is a deleted pivot $d$ involved in the same phase or earlier that is responsible for $v$ or there is an insertion $i$ in the same phase or earlier that is responsible for $v$. Moreover, $d$ is responsible for $v$ if an only if $v \in \text{Orig}(p)$ for some pivot $p$ and $p \in M(d)$.

Consider the first set of $t$ updates. Recall that $L = \lfloor \log_2 n \rfloor$ is the largest level. Let $D^t_i$ denote the set of pivots that have been deleted at level $i$ up to time $t$ and have been updated. Let $I^t$ be the total number of insertions up to time $t$ and $D^t = \cup_{i=0}^t D^t_i$.

**Lemma 10.** The total number of elements, counting multiplicities, that a deleted pivot $d \in D^t_i$ is responsible for is at most $(1/\varepsilon)f2^{i+1}(L+1)$.

**Proof.** Consider each $p \in M(d)$. Note that, $d$ is only responsible for the elements in $\text{Orig}(p)$ and that $\ell(p) \leq \ell(d) = i$. By definition of $\ell(p)$, each set containing $p$ covers less than $2^{\ell(p)+1}$ new elements at the time $p$ is selected. Hence $|\text{Orig}(p)| < f \cdot 2^{\ell(p)+1} \leq f \cdot 2^{i+1}$. Now let us count the number of times $d$ was held responsible for $e \in \text{Orig}(p)$. The element $e$ was an original element just before the update phase that operates on $d$. If $e$ gets processed during that phase by the covering step, then $d$ was responsible for $e$ only once. Otherwise, $e$ gets processed by the movement step during that phase and becomes an extra element. If $e$ is processed $r$ times by the movement step before becoming an original element again (happens immediately when it is processed by the covering step), then $d$ is responsible $r + 1$ times for $e$. However, the level of $e$ strictly increases after each movement step. Therefore, $r \leq L$. Thus, $d$ can be responsible for $e$ at most $L + 1$ times. The claim now follows noting $|M(d)| \leq 1/\varepsilon$. \hfill $\Box$

**Lemma 11.** The total number of elements, counting multiplicities, that an inserted element $e \in I^t$ is responsible for is at most $L + 1$.

**Proof.** An element $e \in I^t$ is only responsible for $e$. If $e$ takes part in $r$ movement steps before becoming an original element for the first time, then $e \in I^t$ is held responsible $r + 1$ times for $e$. Since the level of $e$ strictly increases after each movement step, we have $r \leq L$. Thus, the claim follows. \hfill $\Box$

Recall that for a pivot $p$, we denote by $S(p)$ the set of uncovered elements from which $p$ was selected. We say that a deleted pivot $p \in D$ is a good pivot if at least half of the elements in $S(p)$ are deleted before $p$; otherwise, we say that the deleted pivot is bad. From Lemma \[5\], we have

**Lemma 12.** Fix any set $S$. Conditioned on a pivot being selected from $S$, i.e., $\exists p$ s.t. $S(p) = S$, the probability that $p$ will be a good pivot on its deletion is at least $1/2$. Equivalently, the probability that $p$ is a good pivot is at least as much as the probability that it is a bad pivot.

Consider the first set of $t$ updates. By Lemma \[10\] and Lemma \[11\] the total number of elements that have been updated in the update phases of the algorithm is at most $O(f \log n \sum_i |D_i| \cdot 2^{i+1}) + O(\log n |I|)$. We bound the expectation of this quantity next.
Lemma 13. The expected number of elements that have been updated in the update phases of the algorithm is at most $O\left(\frac{f^2 \log n}{\varepsilon} t\right)$ for the first set of $t$ updates.

Proof. Let $t'$ be the number of deletions. Clearly, $t' \leq t$ and $|I| \leq t$. Recall that the number of updated elements is at most $O\left(\frac{\log n}{\varepsilon} \sum_i |D_i| \cdot 2^{i+1} + \log n |I|\right)$. Let us denote $\Lambda := \sum_i |D_i| \cdot 2^{i+1}$. We have from Lemma 6 $E[|\Lambda|] \leq 8t' \leq 8t$. The lemma follows.

Lemma 14. The expected amortized update time per update is $O\left(f^2 \log n / \varepsilon\right)$.

Proof. Consider the first $t$ updates and let $X'_1, \ldots, X'_q$ be sets of elements whose cover is updated during these steps. From the previous discussion and Lemma 7, the running time of the algorithm is $O(f \cdot t + E\left[\sum_{j=1}^q f \cdot |X'_j|\right])$. From Lemma 13, $E\left[\sum_{j=1}^q |X'_j|\right] \leq O\left(\frac{f^2 \log n}{\varepsilon} \cdot t\right)$. The claim follows.

We summarize the results in the following theorem.

Theorem 1. Given an $\varepsilon > 0$, let $\Delta = \frac{f^2 \log n}{\varepsilon}$. There exists a fully-dynamic algorithm for set cover that achieves an $f(1 + \varepsilon)$ approximation and takes $O(\Delta + t)$ total update time on expectation.

Again, the amortized update time bound can be extended to hold with high probability. This requires several new ideas. See Appendix B for details.

4 Conditional Lower Bounds for Dynamic Set Cover

A fast algorithm for a dynamic problem usually gives a fast algorithm for its static version. If we can solve dynamic Set Cover with preprocessing time $P(n, f)$ and update time $T(n, f)$, then we can solve the static Set Cover problem in $P(n, f) + n \cdot T(n, f)$ time: $n$ updates are sufficient in order to create an offline instance. This simple connection immediately leads to some lower bounds for dynamic Set Cover. First, we get that it is NP-Hard to get an $o(\log n)$ approximation with polynomial preprocessing and update times. Second, consider the polynomial time solvable regime of Set Cover where a $\min\{f, \log n\}$ approximation is possible in $O(f n)$ time. A dynamic algorithm with such approximation factors that has $o(f)$ update time and $o(fn)$ preprocessing time would lead to a new, faster algorithm for the static case. Such an algorithm would be a breakthrough; however, it is open whether it would be strong enough to refute SETH or any other conjecture from fine-grained complexity. In this section, we bypass this connection and prove SETH-based lower bounds for dynamic Set Cover directly.

Under SETH, we show that no algorithm can preprocess an instance with $m$ sets and $n$ elements in $\text{poly}(n, m)$ time, and subsequently maintain element (or set) updates in $O(m^{1-\varepsilon})$ time, for any $\varepsilon > 0$, unless the approximation factor is essentially $m^{\delta}$, for some $\delta > 0$. Note that a factor $m$ approximation can be maintained trivially in constant time (pick either zero or all sets). and we show that essentially any $m^{o(1)}$ approximation algorithm requires $\Omega(m^{0.99})$ time.

Theorem 2 (Main Lower Bound). Let $n^{o(1)} < m < 2^{o(n)}$ and $t \geq 2$, such that $t = \left(\frac{n}{\log m}\right)^{o(1)}$. Assuming SETH, for all $\varepsilon > 0$, no dynamic algorithm can preprocess a collection of $m$ sets over a universe $[n]$ in $\text{poly}(n, m)$ time, and then support element (or set) updates in $O(m^{1-\varepsilon})$ amortized time, and answer Set Cover queries in $O(m^{1-\varepsilon})$ amortized time with an approximation factor of $t$.

We can state the following corollary in terms of the frequency bound $f$.

Corollary 1. Assuming SETH, any dynamic algorithm for Set Cover on $n$ elements and frequency bound $f$, where $n^{o(1)} < f < 2^{o(n)}$, that has polynomial time preprocessing and amortized update and query time $O(f^{1-\varepsilon})$, for some $\varepsilon > 0$, must have approximation factor at least $\left(\frac{n}{\log f}\right)^{\Omega(1)}$. 

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Proof. Without assuming anything about the instances in Theorem \ref{thm:approx-factor} we can conclude that \( f \leq m \) while \( m \leq n f < f^2 \). Therefore, any approximation algorithm with factor \( O((n/\log f)^{\delta}) \) also gets an approximation of \( O((n/2 \log f)^{\delta}) = O((n/\log f^2)^{\delta}) \) which is smaller than \( O((n/\log m)^{\delta}) \) and it is enough to refute SETH via Theorem \ref{thm:approx-factor}.

The rest of this section is dedicated to the proof of Theorem \ref{thm:approx-factor}. Our starting point is the following SETH-based hardness of approximation result, which was proven first in \cite{arora2009intractability} with a slightly smaller approximation factor, and was strengthened in \cite{gupta2016approximation} using, in part, the technique of \cite{khot2004linear}. These results use the distributed PCP framework of \cite{arora2009intractability} for hardness of approximation results in \( \text{P} \), and ours is the first application of this framework to dynamic problems.

\textbf{Theorem 3} (\cite{arora2009intractability,gupta2016approximation,khot2004linear}). Let \( n^{o(1)} < m < 2^{o(n)} \) and \( t \geq 2 \), such that \( t = (n/\log m)^{o(1)} \). Given two collections of \( m \) sets \( A, B \) over a universe \([n]\), no algorithm can distinguish the following two cases in \( O(m^{2-\delta}) \) time, for any \( \delta > 0 \), unless SETH is false:

\textbf{YES case} there exist \( A \in A, B \in B \) such that \( B \subseteq A \); and

\textbf{NO case} for every \( A \in A, B \in B \) we have \( |A \cap B| < |B|/t \).

From this theorem and standard manipulations it is easy to conclude the following statement. There are two differences in the statement below: first, the sizes of \( A \) and \( B \) are asymmetric, and second, the approximation is in terms of the number of sets required to cover a single \( b \in B \), rather than the size of the overlap.

\textbf{Lemma 15.} Let \( n^{o(1)} < m < 2^{o(n)} \) and \( t \geq 2 \), such that \( t = (n/\log m)^{o(1)} \), and for all \( 0 < a \leq 1 \). Given two collections of sets \( A, B \) over a universe \([n]\), where \( |B| = m \) and \( |A| = m^a \), no algorithm can distinguish the following two cases in \( O(m^{1+a-\epsilon}) \) time, for any \( \epsilon > 0 \), unless SETH is false:

\textbf{YES case} there exist \( A \in A, B \in B \) such that \( B \subseteq A \); and

\textbf{NO case} there do not exist \( t \) sets \( A_1, \ldots, A_t \in A \), and a set \( B \in B \) such that \( B \subseteq A_1 \cup \cdots \cup A_t \).

\textbf{Proof.} Assume for contradiction that such an algorithm exists. Given an instance \( A, B \) of the problem in Theorem \ref{thm:approx-factor} we show how to solve it in \( O(m^{2-\epsilon}) \) time. Partition \( A \) into \( k = m^{1-a} \) collections \( A_1, \ldots, A_k \) of size \( m^a \) each, and invoke our algorithm on the asymmetric instance \( A_i, B \) for each \( i = 1 \cdots k \). The total time will be \( k \cdot O(m^{1+a-\epsilon}) = O(m^{2-\epsilon}) \). If the original (symmetric) instance was a YES case, then clearly at least one of the \( k \) asymmetric instances is a YES case. On the other hand, if it was a NO case, then any \( A \in A \) cannot cover more than a \( 1/t \) fraction of any set \( B \in B \) and therefore all the asymmetric instances are NO cases.

Next we take this static set-containment problem and reduce it to dynamic Set Cover. We show two distinct reductions, a simpler one for the element updates case, and then a more complicated one with set updates.

\textbf{4.1 Element Updates}

Given an instance \( A, B \) of the problem in Lemma \ref{lem:element-approx} we construct an instance of dynamic Set Cover with approximation factor \( (t - 1) \) as follows. The universe \([n]\) will be the same, and all sets in \( A \) will appear in the instance. However, the sets in \( B \) will not, and they will be implemented implicitly in a dynamic way. Initially, all the universe elements are activated, and the algorithm may preprocess the instance. Note that the number of sets is only \( m^a \).
For each set \( B_i \in B \) we will have a stage. We start the stage by removing from the universe all elements \( e \in B_i \) that belong to \( B_i \). After we do these \( O(n) \) updates, we ask a Set Cover query. If the answer is less than \( t \) then we can stop and answer YES. Otherwise, we finish the stage by adding back all the elements that we removed and move on to the next stage. After we finish all \( m \) stages for all the sets in \( B \), we answer NO.

In total we have \( O(nm) \) updates and queries, and so the final runtime is \( P(n, m^a) + O(nm) \cdot (T(n, m^a) + Q(n, m^a)) \). Assume we have an algorithm with update and query time \( T(n, m^a) + Q(n, m^a) = O(m^a(1−\epsilon)) \) and polynomial preprocessing, \( P(n, m^a) = O(m^{ac}) \) for some \( c \geq 1 \), then we can choose \( a = 1/\epsilon \) and get an algorithm for the problem in Lemma 15 with runtime \( O(m^{1+a−\epsilon}\epsilon) \), contradicting SETH.

Finally, let us show the correctness of the answer. If we are in the YES case, then there is a set \( B \in B \) that is contained in some set in \( A \). When we ask a query at the stage corresponding to this set \( B \), the size of the minimum set cover is 1. To see this note that all active universe elements are the elements of \( B \) and so we can cover all of them with some set in \( A \). Therefore, our \((t−1)\) approximation algorithm must output an answer that is less than \( t \) and we will output YES. On the other hand, if we are in the NO case, then in all stages, the size of the minimum set cover is at least \( t \) since at least \( t \) sets from \( A \) are required to cover any set in \( B \). Thus, the approximation algorithm will always return an answer that is at least \( t \) and we will never output YES.

### 4.2 Set Updates

The previous reduction fails in this case because we are only allowed to update sets, not elements. A natural approach for extending it is to have all sets from \( B \) in our instance and then at each stage we activate one of them. This would work, except that the number of sets grows to \( m \) which would only give us a weaker lower bound. Indeed such a simple reduction can rule out \( O(m^{1−\epsilon}) \) update times if the preprocessing is restricted to take subquadratic time. A different idea is to add \( n \) auxiliary sets, one per element, so that this set only contains that element. Then, if we want to remove an element, we can add this set and somehow ensure that it is a part of the solution so that, effectively, the corresponding element is removed. This is the approach we take. The main challenge, however, is that these auxiliary sets have to be picked in our set cover solution and so they contribute to the size of the optimal solution. That is, we will no longer have a set cover of size 1 in the YES case and the gap between the YES and NO cases changes. To overcome this issue, we introduce another idea where we create many copies of everything and combine them into one instance in a certain way.

Given an instance \( A, B \) of the problem in Lemma 15 we construct an instance of dynamic Set Cover with approximation factor \((t−1)\) as follows.

Our universe will be \( k := n^2 \) times larger, and for each element \( e \in [n] \) in the universe of the original instance we will add \( k \) elements \( e^1, \ldots, e^k \) to our instance. (So, our universe is isomorphic to \([kn]\).)

For each set \( A \in A \) we construct \( t \) sets \( A^1, \ldots, A^k \) in our dynamic instance. All of these sets will remain activated throughout the reduction. The set \( A^i \) contains all elements \( e^j \) such that \( e \in A \). That is, \( A^i \) contains the \( i^{th} \) copy of all the elements that were in \( A \). Note that \( A^i \) does not contain \( e^j \) for any \( i \neq j \).

We also add sets \( S_1, \ldots, S_n \) which will be activated dynamically, and we let \( S_e^i \) contain all copies of the element \( e \in [n] \). That is, \( S_e^i \) contains \( e^1, \ldots, e^k \). These sets will allow us to simulate the deactivation of a set \( B \).

Next we explain the dynamic part of the reduction. For each set \( B \in B \) we have a stage where we effectively deactivate all universe elements that are not in \( B \). To do this, we activate the set \( S_e^i \) for all \( e \notin B \) such that \( e \) is not in \( B \). Note that we have activated up to \( n \) sets \( S_e^i \), and that together they cover all copies of all elements that are in the complement of \( B \). After we perform these \( O(n) \) updates, we ask a Set Cover query. If the answer to the query is at most \((n+k) \cdot (t−1) \) we return YES. Otherwise, we undo the changes we made in this stage and we move on to the next \( B \in B \). After all the stages are done, we return NO.

The runtime analysis is similar to before since the only difference is in the universe size which increased...
from $n$ to $kn = n^3$ but it is still $m^{o(1)}$. We have $O(nm)$ updates and queries, and so the final runtime is $P(nk, m^a) + O(nm) \cdot (T(nk, m^a) + Q(nk, m^a))$. Assume we have an algorithm with update and query time $T(nk, m^a) + Q(nk, m^a) = O(m^{a(1-\varepsilon)})$ and polynomial preprocessing, $P(nk, m^a) = O(m^{a-c})$ for some $c \geq 1$, then we can choose $a = 1/c$ and get an algorithm for the problem in Lemma [15] with runtime $O(m^{1+a-c})$, contradicting SETH.

Finally, let us show the correctness of the answer. If we are in the YES case, then there is a set $B \in \mathcal{B}$ that is contained in some set in $\mathcal{A}$. When we ask a query at the stage corresponding to this set $B$, the size of the minimum set cover is at most $n + k$. This is because of the following set cover: Choose all sets $S_e$ that are active in this stage; this cover all copies of all universe elements that are not in $B$. Then choose all copies $A^i$ of the set $A \in \mathcal{A}$ that contains our $B$; this covers all copies of all universe elements that are in $B$. Therefore, our $(t-1)$ approximation algorithm must output an answer that is at most $(n + k)(t - 1)$ and we will output YES. On the other hand, if we are in the NO case, then in all stages, the size of the minimum set cover is at least $k \cdot t$. This is because at least $t$ sets from $\mathcal{A}$ are required to cover any set in $\mathcal{B}$, and in a stage of some set $B$ the only way we can cover copies of elements that belong to $B$ is by choosing copies of sets $A$ that contain them. There are $k$ copies of the universe elements, and for each such copy we have to choose at least $t$ sets from $\mathcal{A}$ to cover the elements of that copy, and these sets do not contain any elements from any other copy of the universe. Thus, the approximation algorithm will always return an answer that is at least $kt$, which is larger than $(n + k)(t - 1)$ since $k = n^2$ and $t = n^{o(1)}$, and we will never output YES.

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Each time we remove an element $e$ of cardinality $i$, sets that are added to the solution because of $P$ initialized in linear time have $|S|$ $\leq f \cdot n'$. We have a vector $SET$ indexed by elements, where $SET[e]$ is the list of sets $S'(e)$ containing $e$. Observe that $SET[e]$ contains at most $f$ elements. We assume that sets are described by a vector $ELEM$ indexed by sets, where $ELEM[S]$ is a list of elements contained in set $S$. We keep a pointer from each $e \in ELEM[S]$ and the corresponding entry $S$ in $SET[e]$ and vice versa.

In order to implement deletions, we proceed as follows. We maintain a Boolean vector $DEL$ indexed by $e \in X'$, which is initialized to false. When element $e$ is deleted, we set $DEL[e] = true$. Furthermore, we scan $SET[e]$, and for each $S \in SET[e]$ we remove $e$ from $ELEM[S]$. Note that this can be done in $O(1)$ time for each set $S$ using the mentioned pointers, hence in time $O(f)$ per element $e$. This also implies that deleting all the elements one by one costs $O(f |X'|)$ in total.

The Random Cover Subroutine. We next describe in more detail the behavior of the Random Cover subroutine. We assume to receive in input a set cover instance $(X', S')$, as described above. In particular we have $|S'| \leq f |X'|$ and each element $e \in X'$ is contained in at most $f$ sets $S'(e)$ of $S'$. The procedure will compute a sequence of pivots $P'$. Furthermore, for each pivot $p \in P'$, it will compute a collection $F(p)$ of sets that are added to the solution because of $p$, and the corresponding set $\mathcal{X}(p)$ of newly covered elements. This can be done in $O(f + |\mathcal{X}(p)|)$ time.

We assume to have a vector $DEL$ as described before, and denote by $(X'', S'')$ the set cover instance induced by undeleted elements. At each step we extract $S \in S''$ of maximum cardinality. Then we sample a pivot $p \in S$ uniformly at random, and add it to $P'$. We let $S(p)$ denote the set from which $p$ is sampled, and $\ell(p)$ be the corresponding level. We add the at most $f$ sets in $S''$ covering $p$ to $F(p)$, and all their elements in $X''$ to $\mathcal{X}(p)$. Then we invoke the deletion procedure on each $v \in \mathcal{X}(p)$, hence updating $DEL$ and $(X'', S'')$ consequently.

It remains to specify how we efficiently extract a set $S$ of maximum cardinality at each step. We maintain a list $SORT$ whose entries are pairs $(i, L_i)$, where $i$ is the cardinality of a set and $L_i$ is the list of sets of cardinality $i$. We store all such entries with $L_i$ not empty, in decreasing order of $i$. This list can be initialized in linear time $O(f |X'|)$ (say, using radix sort). We also maintain pointers from each set $S$ to the corresponding entry in the list $L_i[S]$.

The first element of the first list $L_i$ is the selected set $S$ at each step. Then we update $SORT$ as follows. Each time we remove an element $e$ from some set $S'$ of cardinality $i$, we remove $S'$ from $L_i$ and add it to $L_{i-1}$. Note that this might involve creating a new entry $(i-1, L_{i-1})$ in $SORT$ (if $S$ becomes the only set of cardinality $i - 1$), or deleting the entry $(i, L_i)$ from $SORT$ (if $S$ was the only set of cardinality $i$).
case this operations can be performed in $O(1)$ time. It follows that the entire procedure can be implemented in time $O(f \cdot |X'|)$.

**Approximate solution representation.** We store and maintain the approximate solution as follows. We maintain the set cover instance under deletions as described before. Furthermore, we maintain vectors $F$ and $\mathcal{X}$. For a pivot $p$, $F(p)$ is the corresponding list of selected sets because of $p$, and $\mathcal{X}(p)$ is the associated list of newly covered elements due to the latter sets. These two lists are simply empty if $p$ is not a pivot.

**Level selection.** We maintain counters $D$ and $P$ labelled by levels $i = 0, \ldots, \lceil \log_2 n \rceil$, where $D[i]$ (resp., $P[i]$) is the number of deleted pivots (resp., pivots) of level $i$. When we delete a pivot at level $i$, we increment $D[i]$. When we update at critical level $\ell$, we set $D[i] = P[i] = 0$ for all $i \leq \ell$. Furthermore we increment $P[i]$ for each newly computed pivot of level $i$. Clearly these operations have amortized cost $O(1)$ per update.

We similarly maintain the total number $\tilde{D}$ and $\tilde{P}$ of deleted pivots and pivots, respectively. By comparing $\tilde{D}$ and $\tilde{P}$ at each deletion we can check whether the condition for the update of a suffix is satisfied. In that case, by means of $D$ and $P$ it is easy to compute in $O(\ell^2)$ time the lowest critical level $\ell$.

**Update procedure.** We next describe how, given a critical level $\ell$, we update the approximate solution. We keep a list $\text{GREEDY}$ whose entries are pairs $(j, P_j)$. Here $j$ is a level and $P_j$ is the list of pivots of that level. We keep such entries only for non-empty $P_j$, in increasing order of $j$.

Given a critical level $\ell$, we scan the list $\text{GREEDY}$, and compute $P' := \cup_{j \leq \ell} P_j$ together with $X' := \cup_{p \in P'} \mathcal{X}(p)$ (represented as lists). We remove all the corresponding entries from $\text{GREEDY}$, and reset the corresponding values of $\ell(p)$, $S(p)$, $F(p)$ and $\mathcal{X}(p)$.

Let us show how to build the data structures for the subinstance $(X', S')$, with $S' = \{S \cap X' : S \in S \text{ and } S \cap X' \neq \emptyset\}$, in time $O(|f| |X'|)$. Let $n' = |X'|$ and $m' = |F'| \leq fn'$. By scanning the entries of $\text{SET}$ corresponding to $X'$, we build the list of indexes $S'$. We now map $X'$ into a set of new indexes in $[1, O(n')]$ by means of a perfect hash function, and similarly map $F'$ into a set of new indexes in $[1, O(m')]$. These perfect hash functions can be build in linear expected time using well-known constructions (say, 2-level hashing [21]). Observe that some indexes might not be used: we interpret those indexes as dummy elements and sets. Given this, we can easily build in linear time the data structures $\text{SET}'$ and $\text{ELEM}'$ for the new instance.

However, the use of random hash functions can be avoided by assuming along with the set instances we are given access to two arrays $\text{MAP}_{\text{elem}}$ and $\text{MAP}_{\text{set}}$ respectively of size $n$ and $m \leq nf$ initialized to all zeros. We now map $X'$ to $[1, n']$ and $S'$ to $[1, m']$ as follows. We create vectors $\text{MAP}_{\text{elem}}^{-1}$ and $\text{MAP}_{\text{set}}^{-1}$ of size $n'$ and $fn'$, resp., which are initialized to zero. We iterate over $X'$: when considering the $j$-th element of $X$ of global id $k$ (that is it is the actual $k$-th element in $[1, n]$), then we set $\text{MAP}_{\text{elem}}[k] = j$ and $\text{MAP}_{\text{elem}}^{-1}[j] = k$. Similarly, we iterate over $S'$ and update $\text{MAP}_{\text{set}}$ and $\text{MAP}_{\text{set}}^{-1}$ analogously. In order to handle possible duplicates in $S'$ (and possibly in $X'$), we simply do not perform the update when we find an entry of $\text{MAP}_{\text{set}}$ (or $\text{MAP}_{\text{elem}}$) which is already non-zero. Now $X'$ has been mapped to $[1, n']$ and $S'$ has been mapped to $[1, m']$. Now we can build vectors $\text{ELEM}'$ and $\text{SET}'$ in the same way as above. Once the update phase has ended, we reset to $0$ $\text{MAP}_{\text{elem}}$ and $\text{MAP}_{\text{set}}$ and $\text{MAP}_{\text{set}}^{-1}$ we can map back the indexes of the corresponding elements and sets into the original indexes.

We feed the vectors $\text{SET}'$ and $\text{ELEM}'$ to the Random Cover subroutine, that will output a list $P'$ of pivots, plus the associated values $\ell(p)$, $S(p)$, $F(p)$, and $\mathcal{X}(p)$ for each $p \in P'$. Using $\text{MAP}_{\text{elem}}^{-1}$ and $\text{MAP}_{\text{set}}^{-1}$ we can map back the indexes of the corresponding elements and sets into the original indexes.

We remark that by construction we will have $\ell(p) \leq \ell$. Now we build a list $\text{GREEDY}'$ of the same type as $\text{GREEDY}$, however restricted to pivots in $P'$ and to the respective levels. We finally set $\text{GREEDY} \leftarrow \text{GREEDY}' \circ \text{GREEDY}$, where $\circ$ indicated the concatenation of two lists.
A.1 Fully-Dynamic Case

The only difference in the fully-dynamic case happens in the update phase, where before calling the random cover subroutine on \((X', S')\) from a critical level \(\ell\), we check for each entry in \(e \in X'\), whether there exists a set \(S\) in the current solution containing \(e\) at a level \(\ell' > \ell\). We can do this check in \(O(f|X'|)\) time by maintaining the level of each set in the solution and having its level as \(-1\) if it is not in the solution.

At the end of the random cover subroutine, we check the size of \(S(p)\) for each newly selected pivot and based on that compute its level as well as the level of all sets that contain it. Again, this extra check can be done in \(O(f|X'|)\) time.

B High Probability Bounds

In Sections 2 and 3 our approximation guarantees are achieved in the worst case, and we provide bound on the expected amortized update time. In this section, we show how the amortized update time bound can be maintained with probability \(1 - \frac{1}{\text{poly}(n)}\). Obtaining a high-probability bound requires several new ideas.

To start with, we provide a relatively simple bound. Suppose \(\Delta\) denotes the expected amortized update time bound of our algorithm. Therefore \(\Delta = O\left(\frac{\log n}{t}\right)\) for the decremental algorithm, and \(\Delta = O(\frac{\log n}{e})\) for the fully dynamic case. Then with probability \(1 - \frac{1}{\text{poly}(n)}\) after \(t\) updates, the total running time is \(O(t \cdot \Delta + n \log n \cdot \Delta)\). Thus, if \(t \geq n \log n\), we get the required amortized update time bound with high probability.

Of course, such a bound is not directly useful for the decremental case since there \(t \leq n\), and we will handle it separately.

B.1 Fully Dynamic: High Probability Bound for \(t \geq n \log n\)

We follow the same algorithm as described in Section 3. The only changes happen in the analysis. We point out to the relevant changes.

Recall that \(D^i_t\) is the number of deleted pivots from level \(i\) that the update phases have operated on up to \(t\) updates. \(D^i_t = \cup_{i=0}^{\log n} D^i_t\). For simplicity of notations, we drop the superscript \(t\) in this section when the context is clear. We have the following lemmas.

**Lemma (10).** The total number of elements, counting multiplicities, that a deleted pivot \(d \in D_i\) is responsible for is at most \((1/e) f 2^{i+1} \lfloor \log n \rfloor\).

**Lemma (11).** The total number of elements, counting multiplicities, that an inserted element \(e \in I\) is responsible for is at most \(\lfloor \log n \rfloor + 1\).

Recall that for a pivot \(p\), we denote by \(S(p)\) the set of uncovered elements from which \(p\) was selected. We say that a deleted pivot \(p \in D\) is a good pivot if at least half of the elements in \(S(p)\) are deleted before \(p\); otherwise, we say that the deleted pivot is bad. From Lemma 5 we have the following.

**Lemma (12).** Fix any set \(S\). Conditioned on a pivot being selected from \(S\), i.e., \(\exists p\) s.t. \(S(p) = S\), the probability that \(p\) will be a good pivot on its deletion is at least \(1/2\). Equivalently, the probability that \(p\) is a good pivot is at least as much as the probability that it is a bad pivot.

Consider the first set of \(t\) updates. By Lemma 10 and Lemma 11 the total number of elements that have been updated in the update phases of the algorithm is at most \(O(\frac{\log n}{t} \sum_i |D_i| \cdot 2^{i+1}) + O(\log n |I|) = O(\frac{\log n}{t} (n \log n + \sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1}) + O(\log n |I|)).\) We now show that \(\sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1} = O(t)\) with probability \(1 - \frac{1}{\text{poly}(n)}\). This will achieve the desired high-probability bound.
Lemma 16. \( \sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1} = O(t) \) for the first set of \( t \) updates with probability \( 1 - \frac{\log n}{n^2} \).

Proof. Let \( t_i \) denote the number of deletions at level \( i \). Then \( \sum_{i:|D_i| \geq 32 \ln n} t_i \leq t \). Let us denote \( \Lambda := \sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1} \). We have

\[
E[\Lambda] = E \left[ \sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1} \right] = E \left[ \sum_{i:|D_i| \geq 32 \ln n} (G_i + B_i)2^{i+1} \right],
\]

where \( G_i \) denotes the number of good pivots among the deleted pivots in \( D_i \) in level \( i \), and \( B_i \) the number of the remaining bad pivots in \( D_i \).

Fix two sets \( S_1 \) and \( S_2 \) from which two pivots \( p \) and \( p' \) in \( \cup_{i:|D_i| \geq 32 \ln n} \) were selected with \( p \neq p' \). Then \( S_1 \neq S_2 \). Now we show the events \( p \) is a good pivot conditioned on being selected from \( S_1 \) and \( p' \) is a good pivot conditioned on being selected from \( S_2 \) are independent. If \( S_1 \cap S_2 = \emptyset \), then clearly, the events are independent. Otherwise, say \( S_1 \cap S_2 \neq \emptyset \), this can only happen if \( p \) (similarly for \( p' \)) is selected from \( S_1 \), then gets deleted and during a later update phase, \( p' \) is selected from \( S_2 \). However, in that case, all the elements in \( S_1 \) that get deleted before \( p \) are not in \( S_2 \), and \( p \) being a good/bad pivot does not reveal any ordering among the elements in \( S_2 \).

Therefore, by the standard Chernoff bound argument and noting \( E[|G_i|] \geq |D_i|/2 \geq 16 \ln n \), we have

\[
Prob(|G_i| \leq \frac{1}{2} E[|G_i|]) \leq e^{-\frac{1}{n^2}}.
\]

Thus, with probability at least \( 1 - \frac{\log n}{n^2} \),

\[
\Lambda = \sum_{i:|D_i| \geq 32 \ln n} |D_i| \cdot 2^{i+1} \\
\leq \sum_{i:|D_i| \geq 32 \ln n} 4|G_i| \cdot 2^{i+1} \\
\leq 16 \sum_{i:|D_i| \geq 32 \ln n} t_i \leq 16 \cdot t.
\]

This gives the desired high probability bound for \( t \geq n \log n \).

Theorem 4. Given an \( \varepsilon > 0 \), let \( \Delta = \frac{f^2 \log n}{\varepsilon} \). There exists a fully-dynamic algorithm for set cover that achieves an \( f(1 + \varepsilon) \) approximation and takes \( O(\Delta(t + n \log n)) \) total update time over \( t \) updates with probability at least \( 1 - \frac{1}{n} \).

B.2 Fully-Dynamic: High Probability Bound for \( t < n \log n \)

We now show for all \( t < n \log n \), it is possible to design an algorithm that has the same approximation bound of \( f(1 + \varepsilon) \) but has a total running time that is bounded by \( O(\Delta + \lceil \frac{f \log n}{\varepsilon} \rceil O(f) \log (f \log n)) \) with probability \( 1 - \frac{1}{n} \).
B.2.1 Algorithm.

This is achieved in two steps. We maintain two data structures simultaneously as follows.

**First Data Structure.** The first one is the same data structure that we had for the fully-dynamic algorithm Section 3. When updating this data structure, we use a slightly different level fixing criteria (described later). The update on insertion is exactly the same as before. After a deletion, if an $\epsilon$ fraction of the pivots are deleted and the total number of pivots $P$ (undeleted and deleted) is $\geq \frac{100 \log^2 n}{\epsilon}$, we trigger the update phase. Otherwise, we simply mark the deleted element. When an update phase could not be triggered because $|P| < \frac{100 \log^2 n}{\epsilon}$, we make the status of the data structure passive. In the passive state, we continue following the update routines for insertion and deletion except that we never trigger an update phase. After being passive, if the total number of pivots $P$ again becomes $> \frac{100 \log^2 n}{\epsilon}$ due to insertions, then we return the state of the data structure to active. Therefore, an update phase can only be triggered in the active state. Also, it is possible that an update phase is triggered immediately after returning to the active state depending upon the number of deleted pivots.

**Second Data Structure.** We maintain a small sample of elements such that whenever the optimum set cover size of the entire instance is $O(\ell^* \log^2 n)$, any set cover solution of the sample is guaranteed to provide a cover of the entire instance. Chitnis et al. showed how such a sample can be maintained in $O(f)$ update time 17. The total sample size is $O(\ell^*(\log^2 n))^f$. We next describe their sample process.

**Sampling process of 17.** We have a pallet of $k = \frac{100 + \log^2 n}{\epsilon}$ colors. Each set in the set system receives a color from this pallet, assigned to be it by using a pair-wise independent hash function. Colors of the sets containing an element defines the color of the element. Therefore, there are at most $C = k^f$ possible colors for the elements. Sample $O(\log n)$ elements from each color class uniformly at random to obtain a sample $\hat{E}$. This sample can be maintained in $O(f)$ time upon element insertion and deletion as follows. When an element is deleted, if the element is not present in the sample, simply remove it from the set of active elements without affecting the sample. Otherwise, delete the element from the sample and insert a newly chosen element from the same color class uniformly at random among those which were not present in the sample (assuming not all elements of that color class are already present in the sample in which case, do nothing). When an element is inserted, we either make no change to the sample $\hat{E}$ or delete an element from the same color class and insert this new element based on whether this new element is chosen by reservoir sampling, a standard technique to maintain uniform random sampling under insertions.

We maintain our original fully-dynamic data structure on this sample over insertions and deletions. An update phase is triggered in this sample if and only if the first data structure is in a passive state and at least epsilon fraction of pivots are deleted in the second data structure.

At any time $t$, if the first data structure is in an active state, then the set cover solution maintained by it is the intended solution. Otherwise, the set cover solution maintained by the second data structure is the intended solution.

B.2.2 Analysis of the Competitive Ratio

We use the following theorem from [17] restated to fit our context. Let $OPT$ denote the set cover size of the entire instance (all active elements).
Theorem (17). If $OPT \leq \frac{100f + \log^2 n}{\varepsilon}$, then any set cover of $\hat{E}$ is also a set cover of the entire instance and an optimum set cover of $\hat{E}$ provides an optimum set cover of the entire instance with probability $1 - \frac{1}{n^c}$ for $c \geq 3$.

Lemma 17. The competitive ratio of the algorithm is at most $f/(1 - \varepsilon)$ with probability $1 - \frac{1}{n}$.

Proof. If the first data structure is in an active state, then by Lemma 9 the competitive ratio is at most $f/(1 - \varepsilon)$. Otherwise, the first data structure is in a passive state which means the total number of pivots $P < \frac{100f + \log^2 n}{\varepsilon}$. Therefore, there exists a cover of the current active elements that contain at most $\frac{100f + \log^2 n}{\varepsilon}$ sets. Thus, $OPT \leq \frac{100f + \log^2 n}{\varepsilon}$. Thus from the theorem above, the optimum set cover of $\hat{E}$ is also an optimum set cover of the entire instance with probability $1 - \frac{1}{n^c}$. By union bound over $t$ updates, the optimum set cover of $\hat{E}$ is also an optimum set cover of the entire instance whenever $OPT \leq \frac{100f + \log^2 n}{\varepsilon}$ holds with probability at least $1 - \frac{\log n}{n^c}$. Since the set cover solution maintained by our algorithm in the second data structure is $f/(1 - \varepsilon)$-competitive for $\hat{E}$, so it is for all active elements. \hfill \square

B.2.3 Analysis of the Update Time

We first show the update time bound on the second data structure.

Lemma 18. Over $t$ updates, $t \leq n \log n$, the total running time of the second data structure is $O(t \cdot f + \Delta(t + T))$ where $T = O(\frac{L}{\varepsilon} \log^2 n)^f \log (f \log n)$ with probability $1 - \frac{1}{n}$.

Proof. For each original insertion or deletion, we perform at most two insertions or deletions on $\hat{E}$. Now the lemma follows from Theorem 4 noting that here the total number of elements is $O(\frac{L}{\varepsilon} \log^2 n)^f$. \hfill \square

We now want to show the update time bound on the first data structure. We start by showing that a critical level according to the new criteria exists whenever $P \geq \frac{100f + \log^2 n}{\varepsilon}$, that is the first data structure is in an active state. Note that an update phase is triggered in the first data structure if and only if it is in an active state and an $\varepsilon$ fraction of pivots have been deleted.

Lemma 19. If $\sum_{j=0}^{L} |D_j| \geq \varepsilon \cdot \sum_{j=0}^{L} |P_j|$ and $\sum_{j=0}^{L} |P_j| > \frac{100f + \log^2 n}{\varepsilon}$ then there exists at least one critical level.

Proof. Call a level $i$ good if it satisfies the following condition.

$$\sum_{k=j}^{i} |D_k| \geq \frac{\varepsilon}{2} \sum_{k=j}^{i} |P_k| \quad \forall j \leq i.$$ (Cond. 1)

From Lemma 19 such a good level exists.

Starting with $j = L$, find the highest level $j$ such $|D_j| > \frac{\varepsilon}{2} |P_j|$. If $j$ is a good level, stop. Else, there must exist a level $j_1 < j$ such that $\sum_{k=j_1}^{j} |D_k| \leq \frac{\varepsilon}{2} \sum_{k=j_1}^{j} |P_k|$. Set $j = j_1 - 1$, and repeat this search process. If this search process does not find a good level, then $\sum_{j=0}^{L} |D_j| < \frac{\varepsilon}{2} \sum_{j=0}^{L} |P_j|$ and we get a contradiction. Thus the search process ends and suppose, it returns a level $i$.

We have from the search process

$$\sum_{j=i+1}^{L} |D_j| \leq \frac{\varepsilon}{2} \sum_{j=i+1}^{L} |P_j|$$
Therefore,

\[
\sum_{j=0}^{L} |D_j| - \sum_{j=i+1}^{L} |D_j| \geq \varepsilon \sum_{j=0}^{L} |P_j| - \frac{\varepsilon}{2} \sum_{j=i+1}^{L} |P_j|
\]

or,

\[
\sum_{j=0}^{i} |D_j| \geq \frac{\varepsilon}{2} \sum_{j=0}^{L} |P_j|
\]

Consider the first level \(i_1 \leq i\) such that \(|D_{i_1}| > 50 * \log n\). If no such level exists, then \(\sum_{j=0}^{i} |D_j| \leq 50 * i * \log n\). But, \(\sum_{j=0}^{i} |D_j| \geq \frac{\varepsilon}{2} \sum_{j=0}^{L} |P_j|\). Hence, \(P \leq \frac{100}{\varepsilon} * i * \log n \leq \frac{100 \log^2 n}{\varepsilon}\).

Else, such a level \(i_1 \leq i\) such that \(D_{i_1} > 50 * \ln n\) exists. If \(i_1\) satisfies Cond. 1, then \(i_1\) is our final level. Else \(i_1\) does not satisfy Cond. 1, then there exists a level \(g_1\) such that \(\sum_{j=g_1}^{i_1} |D_j| \leq \frac{\varepsilon}{2} \sum_{j=g_1}^{i_1} |P_j|\).

Now search for \(i_2 < g_1\) such that \(|D_{i_2}| > 50 * \log n\). If no such level exists, then we have

\[
\varepsilon \sum_{j=0}^{L} |P_j| \leq \sum_{j=0}^{L} |D_j| = \sum_{j=0}^{g_1-1} |D_j| + \sum_{j=g_1}^{i_1} |D_j| + \sum_{j=i_1+1}^{L} |D_j| \\
\leq 50 * (g_1 + (i - i_1)) \log n + \frac{\varepsilon}{2} \sum_{j=g_1}^{i_1} |P_j| + \sum_{j=i_1+1}^{L} |D_j| \\
\leq 50 * (g_1 + (i - i_1)) \log n + \frac{\varepsilon}{2} \sum_{j=0}^{L} |P_j|
\]

Therefore,

\[
P \leq \frac{100}{\varepsilon} * (g_1 + (i - i_1)) \log n \leq \frac{100 \log^2 n}{\varepsilon}
\]

Otherwise, such a level \(i_2\) exists. If \(i_2\) satisfies Cond. 1, then \(i_2\) will be our final level. Otherwise, there exists a level \(g_2\) such that \(\sum_{j=g_2}^{i_2} |D_j| \leq \frac{\varepsilon}{2} \sum_{j=g_2}^{i_2} |P_j|\). Now search for \(i_2 < g_1\) such that \(|D_{i_2}| > 50 * \log n\). If no such level exists, then we again come to the contradiction that

\[
P \leq \frac{100}{\varepsilon} * (g_1 + (i - i_1)) \log n \leq \frac{100 \log^2 n}{\varepsilon}
\]

Thus, continuing, if \(P > \frac{100 \log^2 n}{\varepsilon}\), then there must exist a critical level. \(\square\)

Let us recall the following property of the level fixing process from Section 2. Since the new level fixing process satisfies the old condition as well, the following property holds here.

**Lemma 20.** Let \(\ell\) be a critical level. There exists a \(b\)-matching between \(D := \cup_{j \leq \ell} D_j\) and \(P := \cup_{j \leq \ell} P_j\) such that:

- Each element of \(P\) is matched to exactly one element of \(D\) and each element of \(D\) to at most \(b = \frac{1}{\varepsilon}\) elements of \(P\);
- If \(d \in D\) is matched to \(p \in P\), then \(\ell(d) \geq \ell(p)\).
Let $\hat{M}_1$ be the function that maps each $d \in \hat{D}$ into the corresponding subset of $T$ according to the $B$-matching from Lemma[20].

Let $\ell = \ell_0$ be a critical level. Let $\ell_1 < \ell$ be the first level such that $|D_{\ell_1}| > |D_{\ell_0}|$. Similarly, let $\ell_2 < \ell_1$ be the first level below $\ell_1$ such that $|D_{\ell_2}| > |D_{\ell_1}|$. Continuing in this fashion, we get $\ell_0 > \ell_1 > \ell_2 > \ldots > \ell_s$ such that $|D_{\ell_i}| > |D_{\ell_{i-1}}|$ for $i = 0, 1, \ldots, s$ and $|D_{\ell_i}| \leq |D_{\ell_s}|$ for all $i \in [s-1, \ldots, 0]$. We now construct a $B$-matching between $\bigcup_{j=0}^{s} D_{\ell_j}$ to $\bigcup_{j \leq \ell_0 \in \mathbb{R}} D_j$ with the following property.

**Lemma 21.** Let $R = \{\ell_0, \ell_1, \ldots, \ell_s\}$. There exists a mapping between $T_1 = \bigcup_{j=0}^{s} D_{\ell_j}$ to $T_2 = \bigcup_{j \leq \ell_0, j \not\in R} D_j$ such that:

- Each element of $T_2$ is mapped to exactly one element of $T_1$ and each element of $T_1$ to at most one element of $T_2$ from each level.

**Proof.** We construct such a mapping explicitly. By construction, we know for all $k \in [\ell_j - 1, \ell_{j+1} + 1]$, $|D_k| \leq |D_{\ell_j}|$. Therefore, we can map each $d \in D_{\ell_j}$ to at most one $d \in D_k$ for $k \in [\ell_j - 1, \ell_{j+1} + 1]$ to cover all of $T_2$.

Let $\hat{M}_2$ be the function that maps each $d \in D_{\ell_j}$ to at most $(\ell_j - \ell_{j+1})$ elements in $T_2$ according to the above mapping for $j = 0, 1, \ldots, s$ and to itself. We say a deleted pivot $d \in D_{\ell_j}$ is responsible for another deleted pivot $d'$ if $\hat{M}_2(d) = d'$.

We recall the following two lemmas.

**Lemma (10).** The total number of elements, counting multiplicities, that a deleted pivot $d \in D_i$ responsible for is at most $(1/e)f^{2i+1}|\log n|$.

**Lemma (11).** The total number of elements, counting multiplicities, that an inserted element $e \in I$ is responsible for is at most $|\log n| + 1$.

From the above two lemmas we have, the total number of elements that have been updated in the update phases of the algorithm is at most $O\left(\frac{\log n}{e} \sum_{i} |D_i| \cdot 2^{i+1} + O(|\log n| I)\right)$. We next bound this quantity in terms of $\sum_{j=0}^{s} |D_{\ell_j}|$.

**Lemma 22.** The total number of elements that have been updated in the update phases of the algorithm is $O\left(\frac{\log n}{e} \sum_{j} |D_{\ell_j}| \cdot 2^{j+2} + O(|\log n| I)\right)$

**Proof.** Using the mapping $\hat{M}_2$, we have $\sum_{i} |D_i| \cdot 2^{i+1} \leq \sum_{j} |D_{\ell_j}| \cdot (2^{\ell_j+1} + 2^{\ell_j} + \ldots + 2^{\ell_{j+1}+2}) \leq \sum_{j} |D_{\ell_j}| \cdot 2^{\ell_j+2}$ and hence the lemma now follows from Lemma 10 and Lemma 11. \qed

Since each $|D_{\ell_j}| \geq 50 \log n$, we have from Lemma 16.

**Lemma 23.** $\sum_{j} |D_{\ell_j}| \cdot 2^{j+2} = O(t)$ for the first set of $t$ updates with probability $1 - \frac{\log n}{n^2}$.

**Lemma 24.** Over $t$ updates, $t \leq n \log n$, the total running time of the first data structure is $O(t \ast f + \Delta \ast t)$ with probability $1 - \frac{\log n}{n^2}$ where $\Delta = \frac{\log n}{\log \log n}$. Let $T = O\left(\frac{\log n}{\log \log n}\right) \log (f \log n)$. Therefore, from Lemma 18 and Lemma 24 we get

**Lemma 25.** Over $t$ updates, $t \leq n \log n$, the total running time is $O(t \ast f + \Delta \ast (t + T))$ with probability $1 - \frac{1}{n}$.

**Theorem 5.** Given an $\varepsilon > 0$, let $\Delta = \frac{\varepsilon^2 \log n}{\log \log n}$. There exists a fully-dynamic algorithm for set cover that achieves an $f(1 + \varepsilon)$ approximation and takes $O(\Delta(t + T))$ total update time over $t \leq n \log n$ updates with probability at least $1 - \frac{1}{n}$.\text{24}
Corollary 2. Given an $\varepsilon > 0$, let $\Delta = \frac{f^2}{\varepsilon}$. There exists a decremental algorithm for set cover that achieves an $f(1+\varepsilon)$ approximation and takes $O(\Delta(t+T))$ total update time over $t \leq n \log n$ updates with probability at least $1 - \frac{1}{n}$. In particular, if $f$ is a constant, then the update time is $O(1/\varepsilon)$ with high probability for all $t \geq \text{poly} \log n$.

Combining Theorem 4 and Theorem 2, we get our final theorem for the high probability bound.

Theorem 6. Given an $\varepsilon > 0$, let $\Delta = \frac{f^2 \log n}{\varepsilon}$. There exists a fully-dynamic algorithm for set cover that achieves an $f(1 + \varepsilon)$ approximation and takes $O(\Delta(t + \min[n \log n, T]))$ total update time over $t$ updates with probability at least $1 - \frac{1}{n}$.