GAP AND RIGIDITY THEOREMS OF $\lambda$-HYPERSURFACES

QIANG GUANG

Abstract. In this note, we study the $\lambda$-hypersurfaces that are critical points of the weighted area functional $\int_\Sigma e^{-\frac{|x|^2}{4}}dA$ for compact variations that preserve weighted volume. First, we prove several gap theorems for complete $\lambda$-hypersurfaces in terms of the norm of the second fundamental form $|A|^2$. Second, we show that in the one dimensional case, the only smooth complete and embedded $\lambda$-hypersurfaces in $\mathbb{R}^2$ with $\lambda \geq 0$ are the lines and round circles. Moreover, we prove a Bernstein type theorem for $\lambda$-hypersurfaces which states that smooth $\lambda$-hypersurfaces that are entire graphs with polynomial volume growth are hyperplanes. All the results can be viewed as generalizations of results for self-shrinkers.

1. Introduction

We follow the notation of [CW14] and call a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ a $\lambda$-hypersurface if it satisfies

$$H - \frac{\langle x, n \rangle}{2} = \lambda,$$

where $\lambda$ is any constant, $H$ is the mean curvature, $n$ is the outward pointing unit normal and $x$ is the position vector.

$\lambda$-hypersurfaces were first introduced by McGonagle and Ross in [MR13], where they study the following isoperimetric type problem on Gaussian weighted Euclidean space.

Let $\mu(\Sigma)$ be the weighted area functional defined by $\mu(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}}dA$ for any hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$. Consider the variational problem of minimizing $\mu(\Sigma)$ among all $\Sigma$ enclosing a fixed Gaussian weighted volume. Note that the variational problem is not to consider $\Sigma$ enclosing a specific fixed weighted volume, but just consider variations that preserve the weighted volume.

They show that the critical points of this variational problem are $\lambda$-hypersurfaces and the only smooth stable ones are hyperplanes. Cheng and Wei [CW14] also introduced the notion of $\lambda$-hypersurfaces by studying the weighted volume-preserving mean curvature flow.

Example 1.1. We give three examples of $\lambda$-hypersurfaces in $\mathbb{R}^3$.

1. The sphere $S^2(r)$ with radius $r = \sqrt{\lambda^2 + 4} - \lambda$.
2. The cylinder $S^1(r) \times \mathbb{R}$, where the $S^1(r)$ has radius $\sqrt{\lambda^2 + 2} - \lambda$.
3. The hyperplane in $\mathbb{R}^3$. 

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Notice that when \( \lambda = 0 \), \( \lambda \)-hypersurfaces are just self-shrinkers and they can be viewed as a generalization of self-shrinkers in some sense. It is well-known that self-shrinkers play a key role in the study of mean curvature flow ("MCF"), since they describe the singularity models of the MCF. First, we state several important results of self-shrinkers. In dimensional one case, smooth complete embedded self-shrinking curves are totally understood and they are just the lines and the round circles by the work of Abresch and Langer [AL86]. In higher dimensions, self-shrinkers are more complicated and there are only few examples, see [Ang92], [KM10], [Møl11] and [Ngu09]. However, there are some classification and rigidity results of self-shrinkers under certain assumptions. Ecker and Huisken [EH89] proved that if a self-shrinker is an entire graph with polynomial volume growth, then it is a hyperplane. Then Wang [Wan11] removed the condition of polynomial volume growth. In [CM12], Colding and Minicozzi proved that the only smooth complete embedded self-shrinkers with polynomial volume growth and \( H \geq 0 \) in \( \mathbb{R}^{n+1} \) are generalized cylinders \( S^k \times \mathbb{R}^{n-k} \) (where the \( S^k \) has radius \( \sqrt{2k} \)) which generalized Huisken’s classification result with \( |A| \) bound.

For self-shrinkers, there exist some gap phenomena in terms of the norm of the second fundamental form \( |A|^2 \). Le and Sesum [LS11] showed that any smooth self-shrinker with polynomial volume growth and satisfying \( |A|^2 < \frac{1}{2} \) is a hyperplane. Cao and Li [CL13] generalized this result to arbitrary codimension and proved that any smooth complete self-shrinker with polynomial volume growth and \( |A|^2 \leq \frac{1}{2} \) is a generalized cylinder.

In this note, we will mainly generalize those gap and rigidity results of self-shrinkers to \( \lambda \)-hypersurfaces. First, we prove the following gap theorem for \( \lambda \)-hypersurfaces.

**Theorem 1.2.** If \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a smooth complete embedded \( \lambda \)-hypersurface satisfying \( H - \langle x, n \rangle^2 = \lambda \) with polynomial volume growth, and satisfies

\[
|A| \leq \frac{\sqrt{\lambda^2 + 2} - |\lambda|}{2},
\]

then, \( \Sigma \) is one of the following:

1. a round sphere \( S^n \),
2. a cylinder \( S^k \times \mathbb{R}^{n-k} \) for \( 1 \leq k \leq n-1 \),
3. a hyperplane in \( \mathbb{R}^{n+1} \).

**Remark 1.3.** Note that when \( \lambda = 0 \), this theorem implies the gap theorem of Cao and Li [CL13] in codimension one case.

We also prove that any smooth closed embedded \( \lambda \)-hypersurface \( \Sigma^2 \subset \mathbb{R}^3 \) with \( |A| = \text{constant} \) and \( \lambda \geq 0 \) is a round sphere.

**Theorem 1.4.** Let \( \Sigma^2 \subset \mathbb{R}^3 \) be a smooth closed and embedded \( \lambda \)-hypersurface with polynomial volume growth and \( \lambda \geq 0 \). If the second fundamental form of \( \Sigma^2 \) is of constant length, i.e., \( |A|^2 = \text{constant} \), then \( \Sigma^2 \) is a round sphere.

Next, we show that just as the self-shrinkers in \( \mathbb{R}^2 \), the only smooth complete and embedded \( \lambda \)-hypersurfaces (\( \lambda \)-curves) in \( \mathbb{R}^2 \) with \( \lambda \geq 0 \) are the lines and round circles. The
dynamical picture suggests that there exist some embedded $\lambda$-curves with $\lambda < 0$ which are not round circles. Also it is expected that there exist Abresch-Langer type curves for immersed $\lambda$-curves.

Moreover, we prove a Bernstein type theorem for $\lambda$-hypersurfaces which generalizes Ecker and Huisken’s result.

**Theorem 1.5.** If a $\lambda$-hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is an entire graph with polynomial volume growth satisfying $H - \frac{(x \cdot n)}{2} = \lambda$, then $\Sigma$ is a hyperplane.

**Remark 1.6.** It is expected that one may remove the condition of polynomial volume growth following the method in [Wan11].

Part of the reason we are interested in $\lambda$-hypersurfaces is that closed $\lambda$-hypersurfaces ($\lambda \geq 0$) behave nicely under the rescaled mean curvature flow, see the discussion in the Appendix.

## 2. Background and Preliminaries

In this section, we recall some background and derive several formulas for $\lambda$-hypersurfaces. Throughout this note, we always assume hypersurfaces to be smooth complete embedded, without boundary and with polynomial volume growth.

### 2.1. Notion and conventions.

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface, then $\nabla$, div, and $\Delta$ are the gradient, divergence, and Laplacian, respectively, on $\Sigma$. $n$ is the outward unit normal, $H = \text{div}_\Sigma n$ is the mean curvature, $A$ is the second fundamental form, and $x$ is the position vector. With this convention, the mean curvature $H$ is $n/r$ on the sphere $S^n \subset \mathbb{R}^{n+1}$ of radius $r$. If $e_i$ is an orthonormal frame for $\Sigma$, then the coefficients of the second fundamental form are defined to be $a_{ij} = \langle \nabla e_i e_j, n \rangle$.

**Definition 2.1.** For $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$, the $F$-functional $F_{x_0,t_0}$ of a hypersurface $M \subset \mathbb{R}^{n+1}$ is defined as

$$F_{x_0,t_0}(M) = (4\pi t_0)^{-\frac{n}{2}} \int_M e^{-\frac{|x-x_0|^2}{4t_0}},$$

and the entropy of $M$ is given by

$$\lambda(M) = \sup_{x_0,t_0} F_{x_0,t_0}(M),$$

here the supremum is taking over all $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$.

### 2.2. Simons type identity.

Now we will derive a Simons type identity for $\lambda$-hypersurface $\Sigma$ which plays a key role in our proof of Theorem 1.2. First, recall the operators $\mathcal{L}$ and $L$ from [CM12] defined by

$$\mathcal{L} = \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle,$$
Lemma 2.2. If \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a \( \lambda \)-hypersurface satisfying \( H - \frac{\langle x, n \rangle}{2} = \lambda \), then

\[
(2.5) \quad LA = A - \lambda A^2
\]

\[
(2.6) \quad LH = H + \lambda |A|^2
\]

\[
(2.7) \quad \mathcal{L}|A|^2 = 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2\lambda \langle A^2, A \rangle + 2|\nabla A|^2.
\]

Remark 2.3. More general results of the above formulas were already obtained by Colding and Minicozzi, see proposition 1.2 in [CM13]. For completeness we also include a proof here. Note that when \( \lambda = 0 \), the above formulas are just the Simons’ equation for self-shrinkers in [CM12].

Proof of Lemma 2.2. First, note that for a general hypersurface, the second fundamental form \( A \) satisfies

\[
(2.8) \quad \Delta A = -|A|^2 A - HA^2 - Hess_H.
\]

Now we fix a point \( p \in \Sigma \), and choose a local orthonormal frame \( e_i \) for \( \Sigma \) such that \( \nabla e_i e_j (p) = 0 \). Then we have

\[
(2.9) \quad 2Hess_H(e_i, e_j) = \nabla_{e_j} \nabla_{e_i} \langle x, n \rangle = \langle x, -a_{ik} e_k \rangle \rangle_j
\]

\[
= -a_{ikj} \langle x, e_k \rangle - a_{ij} - a_{ik} a_{jk} \langle x, n \rangle
\]

\[
= -\left(\nabla_x T A\right)(e_i, e_j) - A(e_i, e_j) - \langle x, n \rangle A^2(e_i, e_j).
\]

Combining (2.8) with (2.9) gives

\[
(2.10) \quad LA = \Delta A - \frac{1}{2} \nabla_x T A + \left(\frac{1}{2} + |A|^2\right)A = A - (H - \langle x, n \rangle)A^2 = A - \lambda A^2,
\]

This gives (2.5) and taking the trace gives (2.6). For (2.7), we have that

\[
(2.11) \quad \mathcal{L}|A|^2 = 2\langle \mathcal{L} A, A \rangle + 2|\nabla A|^2 = 2\langle LA, A \rangle + 2|\nabla A|^2
\]

\[
= 2|A|^2 - 2\lambda \langle A^2, A \rangle - 2\left(\frac{1}{2} + |A|^2\right)|A|^2 + 2|\nabla A|^2
\]

\[
= 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2\lambda \langle A^2, A \rangle + 2|\nabla A|^2,
\]

This finishes the proof. \( \square \)
2.3. **Weighted integral estimates for** $|A|$. In this subsection, we prove a proposition which will justify our integration when hypersurfaces are not closed and with bounded $|A|$.

**Proposition 2.4.** If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a $\lambda$-hypersurface with polynomial volume growth and satisfies $|A| \leq C_0$, then

$$ \int_{\Sigma} |\nabla A|^2 e^{-\frac{|x|^2}{4}} < \infty. $$

In order to prove this proposition, we need the following two lemmas from [CM12] which show that the linear operator $\mathcal{L}$ is self-adjoint in a weighted $L^2$.

**Lemma 2.5 (CM12).** If $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface, $u$ is a $C^1$ function with compact support, and $v$ is a $C^2$ function, then

$$ \int_{\Sigma} u(\mathcal{L}v) e^{-\frac{|x|^2}{4}} = -\int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-\frac{|x|^2}{4}}. $$

**Lemma 2.6 (CM12).** Suppose that $\Sigma \subset \mathbb{R}^{n+1}$ is a complete hypersurface without boundary. If $u,v$ are $C^2$ functions with

$$ \int_{\Sigma} |u \nabla v| + |\nabla u| |\nabla v| + |u \mathcal{L} v| e^{-\frac{|x|^2}{4}} < \infty, $$

then we get

$$ \int_{\Sigma} u(\mathcal{L}v) e^{-\frac{|x|^2}{4}} = -\int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-\frac{|x|^2}{4}}. $$

**Proof of Proposition 2.4** By Lemma 2.2 and $|A| \leq C_0$, we have

$$ \mathcal{L}|A|^2 = 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2\lambda(A^2, A) + 2|\nabla A|^2 $$

$$ \geq 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2|\lambda||A|^3 + 2|\nabla A|^2 $$

$$ \geq 2|\nabla A|^2 - C, $$

where $C$ is a positive constant depending only on $\lambda$ and $C_0$ and maybe different from line to line. For any smooth function $\phi$ with compact support, we integrate the above inequality against $\frac{1}{2}\phi^2$. By Lemma 2.5 we get

$$ -2 \int_{\Sigma} \phi |A| \langle \nabla \phi, \nabla |A| \rangle e^{-\frac{|x|^2}{4}} \geq \int_{\Sigma} \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}. $$

Using the absorbing inequality $\epsilon a^2 + \frac{b^2}{\epsilon} \geq 2ab$ gives

$$ \int_{\Sigma} (\epsilon \phi^2 |\nabla |A||^2 + \frac{1}{\epsilon} |A|^2 |\nabla \phi|^2) e^{-\frac{|x|^2}{4}} \geq \int_{\Sigma} \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}. $$


Now we choose $|\phi| \leq 1$, $|\nabla \phi| \leq 1$ and $\epsilon = 1/2$, combined with $|\nabla A| \geq |\nabla A|$, then (2.18) gives
\[\int_{\Sigma} (4|A|^2 + C)e^{-\frac{|x|^2}{4}} \geq \int_{\Sigma} \phi^2 |\nabla A|^2 e^{-\frac{|x|^2}{4}}.\]
Now the proposition follows from monotone convergence theorem and $\Sigma$ has polynomial volume growth.

A direct consequence of Proposition 2.4 and Lemma 2.6 is the following corollary.

**Corollary 2.7.** If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a $\lambda$-hypersurface with polynomial volume growth and satisfies $|A| \leq C_0$, then
\[\int_{\Sigma} \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} = 0.\]

3. **Gap Theorems for $\lambda$-Hypersurfaces**

3.1. **Proof of the Theorem 1.2.**

Now we are ready to prove Theorem 1.2.

**Proof of the Theorem 1.2.** By Lemma 2.2, we have
\[\frac{1}{2} \mathcal{L}|A|^2 = \left(\frac{1}{2} - |A|^2 - \lambda A^2, A\right) + |\nabla A|^2 \geq (\frac{1}{2} - |A|^2)|A|^2 - |\lambda||A|^3 + |\nabla A|^2 = (\frac{1}{2} - |A|^2 - |\lambda||A||A|^2 + |\nabla A|^2.\]

Corollary 2.7 and Proposition 2.4 give
\[0 = \int_{\Sigma} \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} \geq \int_{\Sigma} \left(\frac{1}{2} - |A|^2 - |\lambda||A||A|^2 e^{-\frac{|x|^2}{4}} + \int_{\Sigma} |\nabla A|^2 e^{-\frac{|x|^2}{4}}.\]

Note that when $|A| \leq \frac{\sqrt{\lambda^2 + 2} - |\lambda|}{2}$, we have
\[\frac{1}{2} - |A|^2 - |\lambda||A| \geq 0,\]
this implies the first term of (3.2) on the right hand side is nonnegative. Therefore, (3.2) implies that all inequalities are equalities, and moreover, we have
\[|\nabla A| = \left(\frac{1}{2} - |A|^2 - |\lambda||A||A|^2 = 0.\]
By Theorem 4 of Laswon [Law69] that every smooth hypersurface with $\nabla A = 0$ splits isometrically as a product of a sphere and a linear space, we finish the proof.

By the above proof of Theorem 1.2, we have the following corollary.
Corollary 3.1. If \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a smooth complete embedded \( \lambda \)-hypersurface satisfying 
\[
H - \frac{(x,n)}{2} = \lambda
\]
with polynomial volume growth, and satisfies 
\[
|A| < \frac{\sqrt{\lambda^2 + 2} - |\lambda|}{2},
\]
then, \( \Sigma \) is a hyperplane in \( \mathbb{R}^{n+1} \).

Remark 3.2. Note that in Theorem 1.2, when \( \Sigma^n \) is a round sphere, this forces \( \lambda = 0 \). We will address this issue and prove a gap theorem for closed \( \lambda \)-hypersurfaces with arbitrary \( \lambda \geq 0 \).

3.2. Gap Theorems for closed \( \lambda \)-hypersurfaces. In this subsection, we consider closed \( \lambda \)-hypersurfaces with \( \lambda \geq 0 \).

Lemma 3.3. Let \( \Sigma^n \subset \mathbb{R}^{n+1} \) be a smooth \( \lambda \)-hypersurface, then 
\[
\mathcal{L}|x|^2 = 2n - |x|^2 - 2\lambda(x,n).
\]

Proof. Note that for any hypersurface, we have \( \Delta x = -Hn \), therefore,
\[
\mathcal{L}|x|^2 = \Delta|x|^2 - \frac{1}{2}(x, \nabla|x|^2)
\]
\[
= 2\Delta(x,x) + 2|\nabla x|^2 - |x|^2
\]
\[
= -2H(x,n) + 2n - |x|^2
\]
\[
= 2n - |x|^2 - 2\lambda(x,n).
\]

Lemma 3.3 has the following immediate consequences.

Corollary 3.4. Let \( \Sigma^n \subset \mathbb{R}^{n+1} \) be a smooth closed \( \lambda \)-hypersurface with \( \lambda \geq 0 \), if 
\[
|x| \leq \sqrt{\lambda^2 + 2n - \lambda},
\]
then \( \Sigma \) is a round sphere.

Next, we prove a gap theorem for closed \( \lambda \)-hypersurface in terms of \( |A| \).

Theorem 3.5. Let \( \Sigma^n \subset \mathbb{R}^{n+1} \) be a smooth closed \( \lambda \)-hypersurface with \( \lambda \geq 0 \), if \( \Sigma \) satisfies 
\[
|A|^2 \leq \frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n},
\]
then \( \Sigma \) is a round sphere with radius \( \sqrt{\lambda^2 + 2n - \lambda} \).

Proof. Since \( \Sigma \) is closed, we consider the point \( p \) where \( |x| \) achieves its maximum. At point \( p \), \( x \) and \( n \) are in the same direction, then \( 2H(p) = 2\lambda + |x|(p) \). By (3.6), we have 
\[
(\lambda + \frac{|x|(p)}{2})^2 = H^2(p) \leq n|A|^2 \leq n\left(\frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n}\right).
\]
This implies 
\[
\max_{\Sigma} |x| \leq |x|(p) \leq \sqrt{\lambda^2 + 2n} - \lambda.
\]
By Corollary 3.4, we conclude that $\Sigma$ is a round sphere.

We also include another weak gap theorem for closed $\lambda$-hypersurface in $\mathbb{R}^3$ in that the proof is interesting and using Gauss-Bonnet Formula, Minkowski Integral Formulas and the Willmore’s inequality.

**Theorem 3.6.** Let $\Sigma^2 \subset \mathbb{R}^3$ be a $\lambda$-hypersurface satisfying $H - \frac{\langle x, n \rangle}{2} = \lambda$, $\lambda \geq 0$ is a constant. If $|A|^2 \leq \frac{1+\lambda^2}{2}$, then $\Sigma$ is a round sphere.

**Proof.** First, by the Gauss-Bonnet Theorem, we have the following identity:

\[
\int_{\Sigma} H^2 = \int_{\Sigma} |A|^2 + 8\pi(1 - g),
\]

here $g$ is the genus of $\Sigma$.

Next, by Minkowski Integral Formulas and Stokes’ theorem, we have

\[
\int_{\Sigma} H \langle x, n \rangle = 2\text{Area}(\Sigma),
\]

\[
\int_{\Sigma} \langle x, n \rangle = 3\text{Volume}(\Omega),
\]

here $\Omega$ is the region enclosed by $\Sigma$.

By (3.10), (3.11) and the $\lambda$-hypersurface equation, we get that

\[
\int_{\Sigma} H \geq \lambda \text{Area}(\Sigma) = \lambda \left( \int_{\Sigma} H^2 - \lambda \int_{\Sigma} H \right),
\]

then,

\[
\int_{\Sigma} H \geq \frac{\lambda}{1 + \lambda^2} \int_{\Sigma} H^2.
\]

Using (3.9), (3.10) and the $|A|$ bound, we have

\[
\int_{\Sigma} H^2 \leq \left( \frac{1+\lambda^2}{2} \right) \text{Area}(\Sigma) + 8\pi(1 - g)
\]

\[
\leq \left( \frac{1+\lambda^2}{2} \right) \left( \int_{\Sigma} H^2 - \lambda \int_{\Sigma} H \right) + 8\pi(1 - g).
\]

Combining this with (3.13) gives

\[
\int_{\Sigma} H^2 \leq 16\pi(1 - g).
\]

Note that for any smooth closed surface $M$ in $\mathbb{R}^3$, the Willmore energy $\int_M H^2$ is greater than or equal to $16\pi$, with the equality holds if and only $M$ is a round sphere. Therefore, by (3.15) we conclude that $\Sigma$ is a round sphere. Actually this case implies that $\lambda = 0$. 

\[
\square
\]
3.3. Closed $\lambda$-hypersurface in $\mathbb{R}^3$ with second fundamental form of constant length. Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth complete embedded self-shrinker, if the norm of the second fundamental form $|A|$ is a constant, then one can show that $\Sigma$ is a generalized cylinder $\mathbb{S}^k \times \mathbb{R}^{2-k}$ for $k \leq 2$, see [Gua14] and [DX11]. One way to prove this is to consider the point where the norm of position vector $|x|$ achieves its minimum. For $\lambda$-hypersurface, we will use a similar method and a result from [HW54] to show that any smooth closed $\lambda$-hypersurface in $\mathbb{R}^3$ with $\lambda \geq 0$ and $|A| = \text{constant}$ is a round sphere, i.e., Theorem 1.4.

First, we recall the following ingredients from [HW54].

**Definition 3.7.** A hypersurface in $\mathbb{R}^3$ is called special Weingarten surface (special $W$-surface) if its Gauss curvature and mean curvature $K$ and $H$, are connected by an identity

$$F(K, H) = 0$$

in which $F$ satisfies the following condition:

- The function $F(K, H)$ is defined and of class $C^2$ on the portion $4K \leq H^2$ of the $(K, H)$-plane and satisfies

$$F_H + HF_K \neq 0 \text{ when } 4K = H^2.$$

In [HW54], Hartman and Wintner proved the following theorem of special $W$-surfaces.

**Theorem 3.8.** [HW54] If a closed orientable surface $S$ of genus 0 is a special $W$-surface of class $C^2$, then $S$ is a round sphere.

One may easily check that closed surface with $|A| = \text{constant}$ is a special $W$-surface, so by Theorem 3.8 we have the following result.

**Corollary 3.9.** Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth closed embedded surface, if $|A| = \text{constant}$, then $\Sigma$ is a round sphere.

Combined with the above result, we are ready to give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** First, by Gauss-Bonnet Formula, Minkowski Integral Formulas and Stokes’ theorem, we have

$$\int_\Sigma H^2 = \int_\Sigma |A|^2 + 8\pi(1 - g),$$  

(3.18)

here $g$ is the genus of $\Sigma$,

$$\int_\Sigma H \langle x, n \rangle = 2\text{Area}(\Sigma),$$  

(3.19)

$$\int_\Sigma \langle x, n \rangle = 3\text{Volume}(\Omega),$$  

(3.20)

here $\Omega$ is the region enclosed by $\Sigma$.

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1In [HW54], they use the average rather than the sum of the principal curvatures.
Combining the above identities, we get
\[(3.21) \quad \int_{\Sigma} H^2 \geq (\lambda^2 + 1) \int_{\Sigma} = (\lambda^2 + 1) \text{Area}(\Sigma).\]

Next, we consider the point \(p \in \Sigma\) where \(|x|\) achieves its minimum. By Lemma 3.3 at point \(p\), we have
\[(3.22) \quad H^2(p) \leq \frac{2 + \lambda^2 + \lambda \sqrt{\lambda^2 + 4}}{2}.\]

At point \(p\), we can choose a local orthonormal frame \(\{e_1, e_2\}\) such that the second fundamental form \(a_{ij} = \lambda_i \delta_{ij}\) for \(i, j = 1, 2\), then
\[(3.23) \quad |\nabla H|^2 = (a_{111} + a_{221})^2 + (a_{112} + a_{222})^2.\]

Since \(|A|^2 = \text{constant}\), we have
\[(3.24) \quad a_{111}a_{111} + a_{22}a_{221} = a_{111}a_{112} + a_{22}a_{222} = 0.\]
Note that at point \(p\), \(|\nabla H| = 0\), this gives
\[a_{111} + a_{221} = a_{112} + a_{222} = 0.\]

Combining this with (3.23), (3.24), we get
\[(3.25) \quad a_{111}(a_{11} - a_{22}) = a_{22} (a_{11} - a_{22}) = 0.\]

If \(a_{11} = a_{22}\), then by (3.22), we have
\[(3.26) \quad |A|^2 = \frac{H^2}{2} \leq \frac{2 + \lambda^2 + \lambda \sqrt{\lambda^2 + 4}}{4}.\]

By Theorem 3.5, this implies \(\Sigma\) is a round sphere.
If \(a_{111} = a_{222} = 0\), then \(|\nabla A|^2 = 0\) and therefore
\[(3.27) \quad (\frac{1}{2} - |A|^2)|A|^2 = \lambda \langle A^2, A \rangle.\]
Then we have
\[(3.28) \quad (|A|^2 - \frac{1}{2})|A|^2 = -\lambda \langle A^2, A \rangle \leq \lambda |A|^3,\]
Therefore,
\[(3.29) \quad |A|^2 \leq \frac{1 + \lambda^2 + \lambda \sqrt{\lambda^2 + 2}}{2}.\]

Combining this with (3.18) and (3.21) gives
\[(3.30) \quad (\lambda^2 + 1) \text{Area}(\Sigma) \leq \int_{\Sigma} H^2 \leq \frac{1 + \lambda^2 + \lambda \sqrt{\lambda^2 + 2}}{2} \text{Area}(\Sigma) + 8\pi (1 - g),\]
Observe that
\[(3.31) \quad \lambda^2 + 1 > \frac{1 + \lambda^2 + \lambda \sqrt{\lambda^2 + 2}}{2},\]
then the genus \(g = 0\). By Corollary \(3.9\), we conclude that \(\Sigma\) is a round sphere. This completes the proof.

\[\square\]

Remark 3.10. Note that our method does not apply to higher dimensional cases. It is desirable that one may remove the conditions of closeness and \(\lambda \geq 0\) to prove that any \(\lambda\)-hypersurface \(\Sigma^2 \subset \mathbb{R}^3\) with \(|A| = \text{constant}\) is a generalized cylinder.

4. Embedded \(\lambda\)-hypersurface in \(\mathbb{R}^2\)

It is well known that smooth complete embedded self-shrinkers in \(\mathbb{R}^2\) are just lines and the unit round circle, this was proven by Abresch and Langer [AL86]. We follow the argument in [Man11] to show that this is also true for \(\lambda\)-hypersurface in \(\mathbb{R}^2\) (\(\lambda\)-curves) with \(\lambda \geq 0\).

**Theorem 4.1.** Any smooth complete embedded \(\lambda\)-hypersurface \(\gamma\) in \(\mathbb{R}^2\) (\(\lambda\)-curve) satisfying \(H - \frac{(x \cdot n)}{2} = \lambda\) with \(\lambda \geq 0\) must either be a line or a round circle.

**Proof.** Suppose \(s\) is an arclength parameter of \(\gamma\). Then the geodesic curvature is \(H = -\langle \nabla_{\gamma'} \gamma', n \rangle\). Note that \(\nabla_{\gamma'} n = H \gamma'\), so we have
\[(4.1) \quad 2H' = \nabla_{\gamma'} \langle x, n \rangle = H \langle x, \gamma' \rangle.\]

If at some point \(H = 0\), then \(H' = 0\), by the uniqueness theorem of ODE, we conclude that \(H \equiv 0\) and then \(\gamma\) is just a line. Therefore, we may assume that \(H\) is always nonzero and possibly reversing the orientation of the curve to make \(H > 0\), i.e., \(\gamma\) is strictly convex.

Differentiating \(|\gamma|^2\) gives
\[(4.2) \quad (|x|^2)' = 2 \langle x, \gamma' \rangle = \frac{4H'}{H}.\]

So \(H = C e^{|x|^2}\) for some constant \(C > 0\).

Since the curve is strictly convex, we introduce a new variable \(\theta\) defined by \(\theta = \arccos \langle e_1, n \rangle\). Differentiating with respect to the arclength parameter gives
\[(4.3) \quad \partial_s \theta = -H,\]
\[(4.4) \quad H_\theta = -\frac{H'}{H} = -\frac{\langle x, \gamma' \rangle}{2},\]
\[(4.5) \quad H_{\theta \theta} = \frac{\partial_s H_\theta}{-H} = \frac{1 - 2H(H - \lambda)}{2H} = \frac{1}{2H} - H + \lambda.\]

Multiplying both sides of the above equation by \(2H_\theta\), then we get
\[(4.6) \quad \partial_\theta (H_\theta^2 + H^2 - \log H - 2\lambda H) = 0,\]
so the quantity $E = H_\theta^2 + H^2 - \log H - 2\lambda H$ is a constant.

Consider the function $f(t) = t^2 - \log t - 2at$, $t > 0$. It’s easy to verify that $f(t) \geq f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2})$, therefore $E \geq f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2})$. If $E = f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2})$, then we can conclude that $H$ is constant and $\gamma$ must be a round circle.

Now we assume that $E > f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2})$. Note that $H = Ce^{\frac{|x|^2}{2H}}$ and $H \leq |x/2| + |\lambda|$, then $H$ has an upper bound and $|x|$ is bounded. By the embeddedness and completeness of $\gamma$, we conclude that $\gamma$ must be closed, simple and strictly convex.

If $\gamma$ is not a round circle, then we consider the critical points of the curvature $H$. By our assumption that $E > f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2})$, when $H_\theta = 0$, then $H_{\theta\theta} = \frac{1}{2H} - H + \lambda \neq 0$. So the critical points are not degenerate, hence, by the compactness of the curve, they are finite and isolated.

Without loss of generality, we may assume $H_\theta = \frac{1}{2H} - H + \lambda$, then $(H^2)_{\theta\theta} + 4(H^2)_\theta - 6\lambda H_\theta$. Then we compute

$$2 \int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = \int_0^{\frac{T}{2}} \sin 2\theta((H^2)_{\theta\theta} + 4(H^2)_\theta - 6\lambda H_\theta) d\theta$$

$$= \sin 2\theta(H^2)_{\theta\theta}|_{0}^{\frac{T}{2}} - 2 \int_0^{\frac{T}{2}} \cos 2\theta(H^2)_\theta d\theta + 4 \int_0^{\frac{T}{2}} \sin 2\theta(H^2)_\theta d\theta$$

$$- 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta$$

$$= 2 \sin T[H^2_{\theta}(\frac{T}{2}) + H(\frac{T}{2})H_{\theta\theta}(\frac{T}{2})] - 2 \cos 2\theta(H^2)_\theta|_{0}^{\frac{T}{2}}$$

$$- 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta$$

$$= 2 \sin TH(\frac{T}{2})H_{\theta\theta}(\frac{T}{2}) - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta.$$

By (4.5) and $H_\theta(\frac{T}{2}) = 0$, we get

$$2 \int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = 2 \sin T[1 - H^2(\frac{T}{2}) + \lambda H(\frac{T}{2})] - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta.$$
Now look at the integral (4.8), since $H$ is decreasing from 0 to $\frac{T}{2}$ and $\sin 2\theta$ is nonnegative, then the left-hand side is nonpositive. For the right-hand side, notice that $H\left(\frac{T}{2}\right)$ is a minimum, then the first term is nonnegative, and $\lambda \geq 0$ implies the second term is nonpositive. So the right-hand side of (4.8) is nonnegative, and this gives a contradiction and we conclude that $\gamma$ is a round circle.

\[\square\]

**Remark 4.2.** For noncompact case, we do not need the condition $\lambda \geq 0$ to prove it is a line, and we do need $\lambda \geq 0$ for closed case. When $\lambda < 0$, there may exist some embedded $\lambda$-curves which are not round circles.

5. A Bernstein type theorem for $\lambda$-hypersurfaces

In this section, we prove Theorem 1.5 which generalizes Ecker and Huisken’s result [EH89]. The key ingredient is that for a $\lambda$-hypersurface $\Sigma$, the function $\langle v, n \rangle$ is an eigenfunction of the operator $L$ with eigenvalue $1/2$, here $v \in \mathbb{R}^{n+1}$ is any constant vector. Notice that the result is also true for self-shrinkers. This eigenvalue result was also obtained by McGonagle and Ross [MR13].

**Lemma 5.1.** If $\Sigma \subset \mathbb{R}^{n+1}$ is a $\lambda$-hypersurface, then for any constant vector $v \in \mathbb{R}^{n+1}$, we have

$$L\langle v, n \rangle = \frac{1}{2}\langle v, n \rangle.$$

**Proof.** Set $f = \langle v, n \rangle$. Working at a fixed point $p$ and choose $e_i$ to be a local orthonormal frame. Then,

$$\nabla_{e_i} f = \langle v, \nabla_{e_i} n \rangle = -a_{ij} \langle v, e_j \rangle.$$

Differentiating again and using Codazzi equation gives that

$$\nabla_{e_k} \nabla_{e_i} = -a_{ijk} \langle v, e_j \rangle - a_{ij} a_{jk} \langle v, n \rangle.$$

Therefore

$$\Delta f = \langle v, \nabla H \rangle - |A|^2 f. \quad (5.1)$$

Using the equation of $\lambda$-hypersurface, we have

$$\langle v, \nabla H \rangle = \langle v, -\frac{1}{2}a_{ij} \langle x, e_j \rangle e_i \rangle = \frac{1}{2} \langle x, \nabla f \rangle. \quad (5.2)$$

Combining (5.1) and (5.2), we obtain that

$$L f = \Delta f - \frac{1}{2} \langle x, \nabla f \rangle + \left(\frac{1}{2} + |A|^2\right) f = \frac{1}{2} f. \quad (5.3)$$

Now we prove the Theorem 1.5.
Proof of Theorem 1.5. Since \( \Sigma \) is an entire graph, we can find a constant vector \( v \) such that 
\[
  f = \langle v, n \rangle > 0.
\]
Let \( u = 1/f \), then we have
\[
  \nabla u = -\frac{\nabla f}{f^2}, \quad \Delta u = -\frac{\Delta f}{f^2} + \frac{2|\nabla f|^2}{f^3}.
\]
By Lemma 5.1 we can easily get
\[
  \mathcal{L}u = |A|^2 u + \frac{2|\nabla u|^2}{u}.
\]
Since \( \Sigma \) has polynomial volume growth, we get
\[
  \int_{\Sigma} (|A|^2 u + \frac{2|\nabla u|^2}{u}) e^{-\frac{|x|^2}{4}} = 0.
\]
Therefore \( |A| = 0 \), and \( \Sigma \) is a hyperplane in \( \mathbb{R}^{n+1} \).

6. Appendix

Following the notation of [CIMW13], we call the quantity
\[
  H - \frac{\langle x, n \rangle}{2}
\]
rescaled mean curvature. Instead of considering \( \lambda \)-hypersurfaces, we consider more general hypersurfaces that have nonnegative rescaled mean curvature, i.e.,
\[
  H - \frac{\langle x, n \rangle}{2} \geq 0.
\]
Notice that \( \lambda \)-hypersurfaces (\( \lambda \geq 0 \)) are just special cases of (6.2). Closed hypersurfaces satisfying (6.2) are closely related to a conjecture in [CIMW13] and they behave nicely under the rescaled mean curvature flow. Recall that a one-parameter family of hypersurfaces \( M_t \subset \mathbb{R}^{n+1} \) flows by rescaled mean curvature if
\[
  \partial_t x = -\left( H - \frac{\langle x, n \rangle}{2} \right) n.
\]
Rescaled mean curvature flow is the negative gradient flow for the \( F \)-functional and self-shrinkers are the stationary points for this flow.

Under the rescaled MCF, closed hypersurfaces satisfying (6.2) have several nice properties. Let \( M_0 \subset \mathbb{R}^{n+1} \) be a smooth closed hypersurface satisfying (6.2), then
- Nonnegative rescaled mean curvature, i.e., (6.2) is preserved under rescaled MCF, just like mean convexity is preserved under MCF.
- As long as \( (H - \frac{\langle x, n \rangle}{2}) > 0 \) holds at least at one point of \( M_0 \), then the rescaled MCF \( M_t \) will develop a singularity in finite time.
- If \( M_0 \) satisfies an entropy condition, \( \lambda(M_0) < 3/2 \), then at a singularity, there is a multiplicity one tangent flow of the form \( S^k \times \mathbb{R}^{n-k} \) with \( k > 0 \).
Basically, these three properties give a classification of singularities for rescaled MCF starting from a closed hypersurface satisfying (6.2) and with low entropy.

Using corresponding Simons type identity and the parabolic maximum principle, the first two properties are not hard to prove. The last property is highly technical, and the proof involves analysis of the singular part of a weak solution of the self-shrinker equation (an integral rectifiable varifold) using theories of stationary tangent cones.

Combining the above properties and a perturbation result, the following theorem was obtained in [CIMW13].

**Theorem 6.1** ([CIMW13]). Given \( n \), there exists \( \epsilon = \epsilon(n) > 0 \) so that if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a closed self-shrinker not equal to the round sphere, then \( \lambda(\Sigma) \geq \lambda(S^n) + \epsilon \). Moreover, if \( \lambda(\Sigma) \leq \min\{\lambda(S^n), \frac{3}{2}\} \), then \( \Sigma \) is diffeomorphic to \( S^n \).

Actually, the proof of this theorem implies that the round sphere minimizes entropy among not only all closed self-shrinkers, but also all closed hypersurfaces satisfying (6.2).

By Huisken’s monotonicity formula, the entropy is monotone non-increasing under the MCF and therefore the entropy of the initial hypersurface gives a bound of the entropy at all future singularities. So the study of entropy will help us to better understand the singularities of the MCF. A related conjecture is the following:

**Conjecture 6.2** ([CIMW13]). Theorem 6.1 holds with \( \epsilon = 0 \) for any closed hypersurface \( M^n \) with \( n \leq 6 \).

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E-mail address: qguang@math.mit.edu