Abstract

We construct two complex-conjugated rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$. We show that these are the unique surfaces with these invariants and Albanese map of degree 2, apart from the family of product-quotient surfaces given in [Pen11]. This completes the classification of surfaces with $p_g = q = 2$, $K^2 = 8$ and Albanese map of degree 2.

1 Introduction

Despite the work of many authors, surfaces $S$ of general type with the lowest possible value of the holomorphic Euler characteristic, namely such that $\chi(O_S) = 1$, are far from being classified, see e.g. the survey papers [BCP06], [BCP11] and [MLP12] for a detailed bibliography on the subject. These surfaces satisfy the Bogomolov-Miyaoka-Yau inequality $K^2 \leq 9$.

The ones with $K^2 = 9$ are rigid, their universal cover is the unit ball in $\mathbb{C}^2$ and $p_g = q \leq 2$. The fake planes, i.e. surfaces with $p_g = q = 0$, have been classified in [PY07] and [CS10], the two Cartwright-Steger surfaces [CS10] satisfy $q = 1$, whereas no example is known for $p_g = q = 2$.

The next case is $K^2 = 8$. In this situation, Debarre’s inequality for irregular surfaces [Deb82] implies

$$0 \leq p_g = q \leq 4.$$  

The cases $p_g = q = 3$ and $p_g = q = 4$ are nowadays classified ([HP02], [Pir02], [Deb82], Beauville’s appendix), whereas for $p_g = q \leq 2$ some families are known ([Sha78], [BCG05], [Pol06], [Pol08], [CP09], [Pen11]) but there is no complete description yet.

All the examples of surfaces with $\chi = 1$ and $K^2 = 8$ known so far are uniformized by the bidisk $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$ is the Poincaré upper half-plane; so the following question naturally arose:

Is there a smooth minimal surface of general type with invariants $\chi = 1$ and $K^2 = 8$ and whose universal cover is not biholomorphic to $\mathbb{H} \times \mathbb{H}$?

For general facts about surfaces uniformized by the bidisk, we refer the reader to [CF09]. One of the aims of this paper is to give an affirmative answer to the question above. In fact we construct two rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not the bidisk. Moreover, we show that these surfaces are complex-conjugated and that they are the unique surfaces with these invariants and having Albanese map of degree 2, apart from the family of product-quotient surfaces constructed in [Pen11]. This complete the classification of surfaces with $p_g = q = 2$, $K^2 = 8$ and Albanese map of degree 2.

Our results can be summarized as follows, see Proposition 1.3, Theorem 2.5, Theorem 4.8 and Proposition 4.16.

Main Theorem. Let $S$ be a minimal smooth surface of general type with $p_g = q = 2$, $K^2 = 8$ and such that its Albanese map $\alpha : S \to A := \text{Alb}(S)$ is a generically finite double cover. Writing $D_A$ for the branch locus of $\alpha$, there are exactly two possibilities, both of which occur:

2010 Mathematics Subject Classification: 14J29, 14J10

Keywords: Surfaces of general type, Albanese map, double covers, abelian surfaces, rigid surfaces
(I) \( D_A^2 = 32 \) and \( D_A \) is an irreducible curve with one ordinary point of multiplicity 6 and no other singularities. These are the product-quotient surfaces constructed in [Pen11];

(II) \( D_A^2 = 24 \) and \( D_A \) has two ordinary points \( p_1, p_2 \) of multiplicity 4 and no other singularities. More precisely, in this case we can write

\[
D_A = E_1 + E_2 + E_3 + E_4,
\]

where the \( E_i \) are elliptic curves intersecting pairwise transversally at \( p_1, p_2 \) and not elsewhere. Moreover, \( A \) is an \( \hat{\text{etale}} \) double cover of the abelian surface \( A' := E' \times E' \), where \( E' \) denotes the equianharmonic elliptic curve.

Up to isomorphism, there are exactly two such surfaces, which are complex-conjugate. Finally, the universal cover of these surfaces is not biholomorphic to the bidisk \( \mathbb{H} \times \mathbb{H} \).

According to the dichotomy in the Main Theorem, we will use the terminology surfaces of type I and surfaces of type II, respectively. Besides answering the question above about the universal cover, the Main Theorem is also significant because

- it contains a new geometric construction of rigid surfaces, which is usually something hard to do;
- it provides a substantially new piece in the fine classification of minimal surfaces of general type with \( p_g = q = 2 \);
- it shows that surfaces of type II present the so-called Diff ≠ Def phenomenon, meaning that their diffeomorphism type does not determine their deformation class, see Remark 1.19.

Actually, the fact that there is exactly one surface of type II up to complex conjugation is a remarkable feature. The well-known Cartwright-Steger surfaces [CS10] share the same property, however our construction is of a different nature, more geometric and explicit.

The paper is organized as follows.

In Section 1, we provide a general result for minimal surfaces \( S \) with \( p_g = q = 2 \), \( K^2 = 8 \) and Albanese map \( \alpha: S \rightarrow A \) of degree 2, and we classify all the possible branch loci \( D_A \) for \( \alpha \) (Proposition 1.5).

In Section 2, we consider surfaces of type I, showing that they coincide with the family of product-quotient surfaces constructed in [Pen11] (Theorem 2.5).

In Section 3, we start the investigation of surfaces of type II. The technical core of this part is Proposition 3.8, showing that, in this situation, the pair \( (A, D_A) \) can be realized as an \( \hat{\text{etale}} \) double cover of the pair \( (A', D'_A) \), where \( D_A \) is a configuration of four elliptic curves in \( A' = E' \times E' \) intersecting pairwise and transversally only at the origin \( o' \in A' \) (as far as we know, the existence of such a configuration was first remarked in [Hir84]). The most difficult part is to prove that we can choose the double cover \( A \rightarrow A' \) in such a way that the curve \( D_A \) becomes 2-divisible in the Picard group of \( A \) (Proposition 3.15). The rigidity of \( S \) then follows from a characterization of \( A' \) proven in [KH05] (cf. also [Aid]).

In Section 4, we show that there are precisely two surfaces of type II up to isomorphism, and that they are complex-conjugated (Theorem 4.13). In order to do this, we had to study the groups of automorphisms and anti-biholomorphisms of \( A \) that preserve the branch locus \( D_A \), and their permutation action on the set of the sixteen square roots of \( \mathcal{O}_A(D_A) \) in the Picard group of \( A \) (Proposition 4.15).

Finally, we show that the universal cover of the surfaces of type II is not biholomorphic to \( \mathbb{H} \times \mathbb{H} \) (Proposition 4.16), we note that they can be given the structure of an open ball quotient in at least two different ways (Remark 4.18) and we sketch an alternative geometric construction for their Albanese variety \( A \) (Remark 4.20).

Acknowledgments. F. Polizzi was partially supported by GNSAGA-INdAM. C. Rito was supported by FCT (Portugal) under the project PTDC/MAT-GEO/2823/2014, the fellowship SFHR/ BPD/11131/2015 and by CMUP (UID/MAT/00144/2013), which is funded by FCT with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. We thank M. Bolognesi, F. Leprévost, B. Poonen, R. Pardini, C. Ritzenthaler and
M. Stoll for useful conversations and suggestions. We are also indebted to the MathOverflow user Ulrich for his answer in the thread: http://mathoverflow.net/questions/242406

This work started during the workshop Birational Geometry of Surfaces, held at the Department of Mathematics of the University of Roma Tor Vergata from January 11 to January 15, 2016. We warmly thank the organizers for the invitation and the hospitality.

**Notation and conventions.** We work over the field of complex numbers. All varieties are assumed to be projective. For a smooth surface $S$, $K_S$ denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the geometric genus, $q(S) = h^1(S, K_S)$ is the irregularity and $\chi(O_S) = 1 - q(S) + p_g(S)$ is the Euler-Poincaré characteristic.

Linear equivalence of divisors is denoted by $\sim$. If $D_1$ is an effective divisor on $S_1$ and $D_2$ is an effective divisor on $S_2$, we say that the pair $(S_1, D_1)$ is an étale double cover of the pair $(S_2, D_2)$ if there exists an étale double cover $f: S_1 \to S_2$ such that $D_1 = f^* D_2$.

If $A$ is an abelian surface, we denote by $(-1)_A: A \to A$ the involution $x \mapsto -x$. If $a \in A$, we write $t_a: A \to A$ for the translation by $a$, namely $t_a(x) = x + a$. We say that a divisor $D \subset A$ (respectively, a line bundle $L$ on $A$) is symmetric if $(-1)^*_A D = D$ (respectively, if $(-1)^*_A L \simeq L$).

## 1 The structure of the Albanese map

Let us denote by $S$ a minimal surface of general type with $p_g = q = 2$ and maximal Albanese dimension, and by

$$\alpha: S \to A = \text{Alb}(S)$$

its Albanese map. It follows from [Cat13, Section 5] that $\deg \alpha$ is equal to the index of the image of $\wedge^4 H^1(S, \mathbb{Z})$ inside $H^4(S, \mathbb{Z}) = \mathbb{Z}[S]$, hence it is a topological invariant of $S$. So, one can search to classify these surfaces by looking at the pair of invariants $(K_S^2, \deg \alpha)$.

**Lemma 1.1.** Let $S$ be as above and assume that there is a generically finite double cover $\tilde{\alpha}: S \to \tilde{A}$, where $\tilde{A}$ is an abelian surface. Then $\tilde{A}$ can be identified with $A = \text{Alb}(S)$ and there exists an automorphism $\psi: \tilde{A} \to A$ such that $\tilde{\alpha} = \psi \circ \alpha$.

**Proof.** The universal property of the Albanese map ([Bea96, Chapter V]) implies that the morphism $\tilde{\alpha}: S \to \tilde{A}$ factors through a morphism $\psi: A \to \tilde{A}$. But $\tilde{\alpha}$ and $\alpha$ are both generically of degree 2, so $\psi$ must be a birational map between abelian varieties, hence an isomorphism. Thus we can identify $\tilde{A}$ with $A$ and with this identification $\psi$ is an automorphism of $A$. \qed

Throughout the paper, we will assume $\deg \alpha = 2$, namely that $\alpha: S \to A$ is a generically finite double cover. Let us denote by $D_A \subset A$ the branch locus of $\alpha$ and let

$$S \xrightarrow{\epsilon} X \xleftarrow{\alpha} A$$

be its Stein factorization. The map $\alpha_X: X \to A$ is a finite double cover and the fact that $S$ is smooth implies that $X$ is normal, see [BHPVdV04, Chapter I, Theorem 8.2]. In particular $X$ has at most isolated singularities, hence $D_A$ is reduced. Moreover, $D_A$ is 2-divisible in $\text{Pic}(A)$, in other words there exists a divisor $L_A$ on $A$ such that $D_A \simeq 2L_A$.

We have a canonical resolution diagram

$$S \xrightarrow{\beta} X \xleftarrow{\psi} B \xleftarrow{\alpha_X} A$$

see [BHPVdV04, Chapter III, Section 7], [PP13, Section 2] and [Rit10]. Here $\beta: S \to B$ is a finite double cover, $S$ is smooth, but not necessarily minimal, $S$ is the minimal model of $S$ and
\( \varphi: B \rightarrow A \) is composed of a series of blow-ups. Let \( x_1, x_2, \ldots, x_r \) be the centers of these blow-ups and let \( E_i \) be the inverse image of \( x_i \) in \( B \) such that

\[
E_i E_j = -\delta_{ij}, \quad K_B = \varphi^* K_A + \sum_{i=1}^{r} E_i.
\]

Then the branch locus \( D_B \) of \( \beta: \bar{S} \rightarrow B \) is smooth and can be written as

\[
D_B = \varphi^* D_A - \sum_{i=1}^{r} d_i E_i, \quad (3)
\]

where the \( d_i \) are even positive integers, say \( d_i = 2m_i \). Let us introduce the following definitions:

- a **negligible singularity** of \( D_A \) is a point \( x_j \) such that \( d_j = 2 \), and \( d_i \leq 2 \) for any point \( x_i \) infinitely near to \( x_j \);
- a \([2d+1, 2d+1]-\)singularity of \( D_A \) is a pair \((x_i, x_j)\) such that \( x_j \) belongs to the first infinitesimal neighbourhood of \( x_i \) and \( d_i = 2d, d_j = 2d \);
- a \([2d, 2d]-\)singularity of \( D_A \) is a pair \((x_i, x_j)\) such that \( x_i \) belongs to the first infinitesimal neighbourhood of \( x_j \) and \( d_i = 2d, d_j = 2d + 2 \);
- a **minimal singularity** of \( D_A \) is a point \( x_j \) such that its inverse image in \( \bar{S} \) via the canonical resolution contains no \((-1)\)-curves.

For instance, an ordinary double point and an ordinary triple point are both negligible minimal singularities, whereas a \([3, 3]-\)point is neither negligible nor minimal. Every ordinary singularity is minimal, but the converse is not true: a \([4, 4]-\)point is minimal, but not ordinary.

**Lemma 1.2.** In our situation, the following holds:

(a) we have \( S = \bar{S} \) in (2) if and only if all singularities of \( D_A \) are minimal;

(b) if \( S \) contains no rational curves, then \( D_A \) contains no negligible singularities.

**Proof.** (a) If \( D_A \) contains a non-minimal singularity then, by definition, \( \bar{S} \) is not a minimal surface, hence \( \bar{S} \neq S \). Conversely, if all singularities of \( D_A \) are minimal then there are no \((-1)\)-curves on \( \bar{S} \) coming from the resolution of the singularities of \( D_A \). Since the abelian surface \( A \) contains no rational curves, this implies that \( \bar{S} \) contains no \((-1)\)-curves at all, so \( \bar{S} = S \).

(b) Any negligible singularity of \( D_A \) gives rise to some rational double point in \( X \), and hence to some rational curve in \( \bar{S} \) that cannot be contracted by the blow-down morphism \( \bar{S} \rightarrow S \) (again because \( A \) contains no rational curves, so all \((-1)\)-curves in \( \bar{S} \) come from the resolution of singularities of \( X \)). This is impossible because we are assuming that \( S \) contains no rational curves.

By using the formulae in [BHPVdV04, p. 237], we obtain

\[
2 = 2\chi(O_S) = L_A^2 - \sum m_i(m_i - 1), \quad K_S^2 = 2L_A^2 - 2\sum (m_i - 1)^2. \quad (4)
\]

Notice that the sums only involve the non-negligible singularities of \( D_A \simeq 2L_A \). The two equalities in (1) together imply

\[
K_S^2 \geq K_S^2 = 4 + 2\sum (m_i - 1). \quad (5)
\]

We are now ready to analyse in detail the case \( K_S^2 = 8 \).

**Proposition 1.3.** Let \( S \) be a minimal surface with \( p_g = q = 2 \) and \( K_S^2 = 8 \). Then \( S \) contains no rational curves, in particular \( K_S \) is ample. Using the previous notation, if the Albanese map \( \alpha: S \rightarrow A \) is a generically finite double cover then we are in one of the following cases:
(I) \(D_A^2 = 32\) and \(D_A\) has one ordinary singular point of multiplicity 6 and no other singularities;

(II) \(D_A^2 = 24\) and \(D_A\) has two ordinary singular points of multiplicity 4 and no other singularities.

**Proof.** The non-existence of rational curves on \(S\) is a consequence of a general bound for the number of rational curves on a surface of general type, see [Miy84 Proposition 2.1.1].

Since \(K_S^2 = 8\), inequality (5) becomes

\[\sum (m_i - 1) \leq 2.\] (6)

By Lemma 1.2 there are no negligible singularities in \(D_A\), so (6) implies that we have three possibilities:

- \(D_A\) contains precisely one singularity (which is necessarily ordinary) and \(m_1 = 3\), that is \(d_1 = 6\); this is case (I).
- \(D_A\) contains precisely two singularities and \(m_1 = m_2 = 2\), that is \(d_1 = d_2 = 4\). We claim that these two quadruple points cannot be infinitely near. In fact, the canonical resolution of a \([4, 4]\)-point implies that \(S\) contains (two) rational curves and, since a \([4, 4]\)-point is a minimal singularity, this would imply the existence of rational curves on \(S = \tilde{S}\), a contradiction. So we have two ordinary points of multiplicity 4, and we obtain case (II).
- \(D_A\) contains precisely one singularity (which is necessarily ordinary) and \(m_1 = 2\), that is \(d_1 = 4\). An ordinary singularity is minimal, hence we get equality in (5), obtaining \(K_S^2 = 6\) (this situation is considered in [PP13]), which case is excluded.

\[\square\]

**Remark 1.4.** Lemma 1.2 and Proposition 1.3 imply that for any surface \(S\) with \(p_g = q = 2\), \(K_S^2 = 8\) and Albanese map of degree 2, we have \(S = \hat{S}\). Furthermore, referring to diagram (1), the following holds:

- in case (I), the birational morphism \(c: S \to X\) contracts precisely one smooth curve \(Z\), such that \(g(Z) = 2\) and \(Z^2 = -2\). This means that the singular locus of \(X\) consists of one isolated singularity \(x\), whose geometric genus is \(p_g(X, x) = \dim \mathcal{H}^1 c_* \mathcal{O}_S = 2\);
- in case (II), the birational morphism \(c: S \to X\) contracts precisely two disjoint elliptic curves \(Z_1, Z_2\) such that \((Z_1)^2 = (Z_2)^2 = -2\). This means that the singular locus of \(X\) consists of two isolated elliptic singularities \(x_1, x_2\) of type \(E_7\), see [Ish14 Theorem 7.6.4].

**Definition 1.5.** According to the dichotomy in Proposition 1.3 we will use the terminology surfaces of type I and surfaces of type II, respectively.

**Proposition 1.6.** Let us denote as above by \(D_A\) the branch locus of the Albanese map \(\alpha: S \to A\).

Then:

- if \(S\) is of type I, the curve \(D_A\) is irreducible;
- if \(S\) is of type II, the curve \(D_A\) is of the form \(D_A = E_1 + E_2 + E_3 + E_4\), where the \(E_i\) are elliptic curves meeting pairwise transversally at two points \(p_1, p_2\) and not elsewhere. In particular, we have \(E_i E_j = 2\) for \(i \neq j\).

**Proof.** Suppose first that \(S\) is of type I and consider the blow-up \(\varphi: B \to A\) at the singular point \(p \in D_A\). Let \(C_1, \ldots, C_r\) be the irreducible components of the strict transform of \(D_A\) and \(E \subset B\) the exceptional divisor. The curve \(D_A\) only contains the ordinary singularity \(p\), so the \(C_i\) are pairwise disjoint; moreover, the fact that

\[\sum_{i=1}^r C_i = \varphi^* D_A - 6E\]
is 2-divisible in Pic(B) implies that $C_i^2 = C_i(\sum_{i=1}^r C_i)$ is an even integer. Now recall that the abelian surface $A$ contains no rational curves and that every curve of geometric genus 1 on it is smooth and has self-intersection 0; then we infer $g(C_i) \geq 2$ and we can write

$$6 - 4 = \mathcal{E}(\varphi^* D_A - 6\mathcal{E}) + (\varphi^* D_A - 6\mathcal{E})^2$$

$$= K_B \left( \sum_{i=1}^r C_i \right) + \left( \sum_{i=1}^r C_i \right)^2 = \sum_{i=1}^r (2g(C_i) - 2) \geq 2r,$$

that is $r = 1$ and $D_A$ is irreducible.

Assume now that $S$ is of type II, and write $D_A = E_1 + \cdots + E_r$, where each $E_i$ is an irreducible curve. Denote by $m_i$ and $n_i$ the multiplicities of $E_i$ at the two ordinary singular points $p_1$ and $p_2$ of $D_A$, and let $p_0(E_i)$ and $g_i$ be the arithmetic and the geometric genus of $E_i$, respectively. We have $\sum_{i=1}^r m_i = \sum_{i=1}^r n_i = 4$ and

$$E_i^2 = 2p_0(E_i) - 2 = 2g_i - 2 + m_i(m_i - 1) + n_i(n_i - 1).$$

Using this, we can write

$$24 = D_A^2 = \sum_{i=1}^r E_i^2 + 2\sum_{j<k} E_j E_k$$

$$= 2\sum_{i=1}^r g_i - 2r + \sum_{i=1}^r m_i(m_i - 1) + n_i(n_i - 1) + 2\sum_{j<k} (m_jm_k + n_jn_k)$$

$$= 2\sum_{i=1}^r g_i - 2r + \left( \sum_{i=1}^r m_i \right)^2 + \left( \sum_{i=1}^r n_i \right)^2 - \sum_{i=1}^r m_i - \sum_{i=1}^r n_i$$

$$= 2\sum_{i=1}^r g_i - 2r + 24,$$

that is $\sum_{i=1}^r g_i = r$. Since $A$ contains no rational curves we have $g_i \geq 1$, and we conclude that

$$g_1 = \cdots = g_r = 1. \tag{7}$$

But every curve of geometric genus 1 on $A$ is smooth, so (7) implies that $D_A$ is the sum of $r$ elliptic curves $E_i$ passing through the singular points $p_1$ and $p_2$. Therefore $r = 4$, because these points have multiplicity 4 in the branch locus $D_A$.

2 Surfaces of type I

2.1 The product-quotient examples

The following family of examples, whose construction can be found in [Pen11], shows that surfaces of type I do actually exist. Let $C'$ be a curve of genus $g(C') \geq 2$ and let $G$ be a finite group that acts freely on $C' \times C'$. We assume moreover that the action is mixed, namely that there exists an element in $G$ exchanging the two factors; this means that

$$G \subset \text{Aut}(C' \times C') \simeq \text{Aut}(C')^2 \rtimes \mathbb{Z}/2\mathbb{Z},$$

is not contained in $\text{Aut}(C')^2$. Then the quotient $S := (C' \times C')/G$ is a smooth surface with

$$\chi(O_S) = (g - 1)^2|G|, \quad K_S^2 = 8\chi(O_S). \tag{8}$$

The intersection $G^0 := G \cap \text{Aut}(C')^2$ is an index 2 subgroup of $G$, fitting into the non-split extension

$$1 \rightarrow G^0 \rightarrow G \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

and such that the genus of the curve $C := C'/G^0$ equals $g(S)$, see [Fra13] Lemma 2.9.
We have a commutative diagram

\[
\begin{array}{ccc}
C' \times C' & \xrightarrow{t} & C \times C \\
\downarrow & & \downarrow u \\
S & \xrightarrow{\beta} & \text{Sym}^2(C),
\end{array}
\]

where \(t: C' \times C' \to C \times C\) is a \((G^0 \times G^0)\)-cover, \(u: C \times C \to \text{Sym}^2(C)\) is the natural projection onto the second symmetric product and \(\beta: S \to \text{Sym}^2(C)\) is a finite cover of degree \([G^0]\).

Assume now that \(C'\) has genus 3 and that \(G^0 \simeq \mathbb{Z}/2\mathbb{Z}\). Then \(G \simeq \mathbb{Z}/4\mathbb{Z}\) and \(C\) has genus 2. Denoting by \(\Delta \subset C \times C\) the diagonal and by \(\Gamma \subset C \times C\) the graph of the hyperelliptic involution \(\iota: C \to C\), we see that \(\Delta\) and \(\Gamma\) are smooth curves isomorphic to \(C\) and satisfying

\[
\Delta \Gamma = 6, \quad \Delta^2 = \Gamma^2 = -2.
\]

The ramification divisor of \(u\) is precisely \(\Delta\), so \(u(\Delta)^2 = -4\), whereas \(u(\Gamma)\) is a \((-1)\)-curve. The corresponding blow-down morphism \(\varphi: \text{Sym}^2(C) \to A\) is the Abel-Jacobi map, and \(A\) is an abelian surface isomorphic to the Jacobian variety \(J(C)\). The composed map

\[
\alpha = \varphi \circ \beta: S \to A
\]

is a generically finite double cover, that by the universal property coincides, up to automorphisms of \(A\), with the Albanese morphism of \(S\). Such a morphism is branched over \(D_A := (\varphi \circ u)(\Delta)\), which is a curve with \(D_A^2 = 32\) and containing an ordinary sextuple point and no other singularities: in fact, the curves \(u(\Delta)\) and \(u(\Gamma)\) intersect transversally at precisely six points, corresponding to the six Weierstrass points of \(C\).

From this and \([5]\), it follows that \(S\) is a surface with \(p_g = q = 2\), \(K_S^2 = 8\) and of type \(I\). Note that, with the notation of Section \([4]\) we have \(B = \text{Sym}^2(C)\) and \(D_B = u(\Delta)\).

**Remark 2.1.** Here is a different construction of the singular curve \(D_A\) considered in the previous example. Let \(A := J(C)\) be the Jacobian of a smooth genus 2 curve and let us consider a symmetric theta divisor \(\Theta \subset A\). Then the Weierstrass points of \(\Theta\) are six 2-torsion points of \(A\), say \(p_0, \ldots, p_5\), and \(D_A\) arises as the image of \(\Theta\) via the multiplication map \(2_A: A \to A\) given by \(x \mapsto 2x\). Note that \(D_A\) is numerically equivalent to \(4\Theta\).

**Remark 2.2.** Recently, R. Pignatelli and the first author studied some surfaces with \(p_g = q = 2\) and \(K_S^2 = 7\), originally constructed in \([\text{CF}15]\) and arising as *triple* covers \(S \to A\) branched over \(D_A\), where \((A, D_A)\) is as in the previous example. We refer the reader to \([\text{PP}16]\) for more details.

### 2.2 The classification

The aim of this subsection is to show that every surface of type \(I\) is a product-quotient surface of the type described in Subsection 2.1.

**Lemma 2.3.** Let \(D\) be an irreducible curve contained in an abelian surface \(A\), with \(D^2 = 32\) and having an ordinary point \(p\) of multiplicity 6 and no other singularities. Then, up to translations, we can suppose \(p = 0\) and \(D\) symmetric, namely \((-1)_A^*D = D\).

**Proof.** Since \(D^2 > 0\), the line bundle \(\mathcal{L} = \mathcal{O}_A^3(D)\) is non-degenerate and so, by \([\text{BL}04]\) Chapter 4, \(\mathcal{L}\) is a line bundle \(\mathcal{L}'\) which is algebraically equivalent to \(\mathcal{L}\) and symmetric, i.e. \((-1)_A^*\mathcal{L}' \simeq \mathcal{L}'\). Since algebraically equivalent line bundles differ by a translation, see \([\text{BL}04]\) Section 4.6, up to translations we can assume that \(\mathcal{L}\) itself is symmetric. Then \(D' := (-1)_A^*D\) is an effective divisor linearly equivalent to \(D\), hence the two translates

\[
D_p := t^*pD \quad \text{and} \quad D'_p := t^*pD'
\]

are algebraically equivalent irreducible divisors, both having a sextuple point at 0. If \(D_p\) and \(D'_p\) were distinct, we would have \(D_pD'_p \geq 36\), a contradiction because \(D^2 = 32\); so \(D_p = D'_p\). But \(D'_p = (-1)_A^*D_p\), hence \(D_p\) is a symmetric translate of \(D\) having its sextuple point at 0, as desired. \(\square\)
Proposition 2.4. If $D \subset A$ is as in Lemma 2.3, then there exists a smooth genus 2 curve $C$ such that $A = J(C)$. Furthermore, up to translations, the curve $D$ can be obtained as in Remark 2.1 namely as the image of a symmetric theta divisor $\Theta \subset A$ via the multiplication map $2_A: A \rightarrow A$.

Proof. By Lemma 2.3 we can assume that $D$ is a symmetric divisor and that its sextuple point is the origin $0 \in A$. The geometric genus of $D$ is 2, hence its normalization $C \rightarrow D$ is a smooth genus 2 curve. By the universal property of the Jacobian, the composed map $C \rightarrow D \rightarrow A$ factors through an isogeny

$$\eta: J(C) \rightarrow A,$$

where we can assume, up to translations, that the image $\Theta$ of the embedding $C \rightarrow J(C)$ is a theta divisor containing the origin $0 \in J(C)$. Thus, the abelian surface $A$ is isomorphic to $J(C)/T$, where $T := \ker \eta$ is a torsion subgroup whose order $|T|$ equals the degree $d$ of $\eta$. The group $T$ contains the group generated by the six points

$$0 = p_0, p_1, \ldots, p_5$$

corresponding to the six distinct points of $C$ over $0 \in D$. The restriction of $\eta$ to $C$ is birational, so we have

$$\eta^* D = \Theta_0 + \cdots + \Theta_d,$$

where $\Theta_0 = \Theta$ and the $\Theta_j$ are translates of $\Theta_0$ by the elements of $T$. Since $D^2 = 32$, we obtain $(\eta^* D)^2 = 32 d$. On the other hand, all the curves $\Theta_j$ are algebraically equivalent, hence $\Theta_j \Theta_j = 2$ for all pairs $(i, j)$ and we infer $(\eta^* D)^2 = (\Theta_0 + \cdots + \Theta_d)^2 = 2d^2$. So $32d = 2d^2$, that is $d = 16$.

This shows that the reducible curve $\eta^* D$ has sixteen sextuple points $p_0, \ldots, p_{15}$, such that every curve $\Theta_j$ contains six of them; conversely, since all the $\Theta_j$ are smooth, through any of the $p_k$ pass exactly six curves. We express these facts by saying that the sixteen curves $\Theta_j$ and the sixteen points $p_k$ form a symmetric (166)-configuration. The involution $(-1)_A$ acts on $D$, so the involution $(-1)_C$ acts on $\Theta$, that is $\Theta$ is a symmetric divisor on $J(C)$. Furthermore, the action of $(-1)_A$ induces the multiplication by $-1$ on the tangent space $T_{A,0}$, hence it preserves the six tangent directions of $D$ at $0$; this means that $p_0, \ldots, p_5$ are fixed points for the restriction of $(-1)_D$ to $\Theta$. But a non-trivial involution with six fixed points on a smooth curve of genus 2 must be the hyperelliptic involution, so $p_0, \ldots, p_5$ are the Weierstrass points of $\Theta$. By [Mum84] Chapter 3.2, pp. 28-39, these six points generate the (order 16) subgroup $(J(C)[2])$ of points of order 2 in $J(C)$, thus $T = J(C)[2]$.

Summing up, our symmetric (166)-configuration coincides with the so-called Kummer configuration, see [BL04] Chapter 10); moreover, $A$ is isomorphic to $J(C)$ and the map $\eta$ coincides with the multiplication map $2_A: A \rightarrow A$.

Theorem 2.5. Surfaces of type I are precisely the product-quotient surfaces described in Section 2.1 in particular they form a family of dimension 3. More precisely, denoting by $M_1$ their Gieseker moduli space and by $M_2$ the moduli space of curves of genus 2, there exists a quasi-finite morphism $M_1 \rightarrow M_2$ of degree 15.

Proof. Given any surface $S$ of type I, by Proposition 2.4 there exists a smooth curve $C$ of genus 2 such that $S$ is the canonical desingularization of the double cover $\alpha: X \rightarrow A$, where $A = J(C)$, branched over the singular curve $D_A$ described in the example of Section 2.1 and in Remark 2.1. Equivalently, $S$ arises as a double cover $\beta: S \rightarrow B$, where $B = \text{Sym}^2(C)$, branched over the smooth diagonal divisor $D_B$. There are sixteen distinct covers, corresponding to the sixteen square roots of $D_B$ in Pic$(B)$. One of them is the double cover $u: C \times C \rightarrow B$, whereas the others are fifteen surfaces $S$ with $p_2(S) = q(S) = 2$ and Albanese variety isomorphic to $J(\beta)$; for a general choice of $C$, such surfaces are pairwise non-isomorphic.

On the other hand, once fixed a curve $C$ of genus 2, the product-quotient construction uniquely depends on the choice of the étale double cover $C' \rightarrow C$, that is on the choice of a non-trivial 2-torsion element of $J(C)$. There are precisely fifteen such elements, that necessarily correspond to the fifteen surfaces with $p_2(S) = q(S) = 2$ and $\text{Alb}(S) \simeq J(C)$ found above.

Therefore every surface of type I is a product-quotient example, and the map $M_1 \rightarrow M_2$ defined by $[S] \mapsto [C]$ is a quasi-finite morphism of degree 15.
Remark 2.6. The moduli space of genus 2 curves $C$ with a non-trivial 2 torsion point in $J(C)$ is rational (see [Dol08]). According to the description of $\mathcal{M}_g$ in the proof of Theorem 2.5 we see that $\mathcal{M}_g$ is rational.

Theorem 2.5 in particular implies that the universal cover of $S$ coincides with the universal cover of $C' \times C'$, so we obtain

Corollary 2.7. Let $S$ be a surface of type $I$ and $\tilde{S} \to S$ its universal cover. Then $\tilde{S}$ is biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \}$ is the Poincaré upper half-plane.

3 Surfaces of type II: construction

3.1 Line bundles on abelian varieties and the Appell-Humbert theorem

In this subsection we shortly collect some results on abelian varieties that will be used in the sequel, referring the reader to [BL04] Chapters 1-4 for more details. Let $A = V/\Lambda$ be an abelian variety, where $V$ is a finite-dimensional $\mathbb{C}$-vector space and $\Lambda \subset V$ a lattice. Then the Appell-Humbert Theorem, see [BL04] Theorem 2.2.3, implies that

- the Néron-Severi group $\text{NS}(A)$ can be identified with the group of hermitian forms $h: V \times V \to \mathbb{C}$ whose imaginary part $\text{Im} \, h$ takes integral values on $\Lambda$;
- the Picard group $\text{Pic}(A)$ can be identified with the group of pairs $(h, \chi)$, where $h \in \text{NS}(A)$ and $\chi$ is a semicharacter, namely a map
  \[
  \chi: \Lambda \to U(1), \quad \text{where} \ U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \},
  \]
  such that
  \[
  \chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i \text{Im} \, h(\lambda, \mu)} \quad \text{for all} \ \lambda, \mu \in \Lambda. \tag{9}
  \]
- with these identifications, the first Chern class map $c_1: \text{Pic}(A) \to \text{NS}(A)$ is nothing but the projection to the first component, i.e. $(h, \chi) \mapsto h$.

We will write $\mathcal{L} = \mathcal{L}(h, \chi)$, so that we have $\mathcal{L}(H, \chi) \otimes \mathcal{L}(H', \chi') = \mathcal{L}(H + H', \chi \chi')$. The line bundle $\mathcal{L}(H, \chi)$ is symmetric if and only if the semicharacter $\chi$ has values in $\{ \pm 1 \}$, see [BL04] Corollary 2.3.7. Furthermore, for any $\bar{v} \in \Lambda$ with representative $v \in V$, we have

\[
\ell^*_\lambda \mathcal{L}(h, \chi) = \mathcal{L}(h, \chi e^{2\pi i \text{Im} \, h(v, \cdot)}), \tag{10}
\]

see [BL04] Lemma 2.3.2.

Remark 3.1. Assume that the class of $\mathcal{L} = \mathcal{L}(h, \chi)$ is 2 divisible in $\text{NS}(A)$, that is $h = 2h'$. Then $\text{Im} \, h(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and moreover formula (9) implies that $\chi: \Lambda \to U(1)$ is a character, namely $\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)$. In particular, $\mathcal{L}$ belongs to $\text{Pic}^0(A)$ if and only if there exists a character $\chi$ such that $\mathcal{L} = \mathcal{L}(0, \chi)$.

Proposition 3.2 ([BL04], Lemma 2.3.4). Let $A_1 = V_1/\Lambda_1$ and $A_2 = V_2/\Lambda_2$ be two abelian varieties, and let $f: A_2 \to A_1$ be a homomorphism with analytic representation $F: V_2 \to V_1$ and rational representation $F_\Lambda: \Lambda_2 \to \Lambda_1$. Then for any $\mathcal{L}(h, \chi) \in \text{Pic}(A_1)$ we have

\[
f^* \mathcal{L}(h, \chi) = \mathcal{L}(F^*h, F^*_\Lambda \chi). \tag{11}
\]

Given a point $x \in A$ and a divisor $D \subset A$, let us denote by $m(D, x)$ the multiplicity of $D$ at $x$.

Lemma 3.3 ([BL04], Proposition 4.7.2). Let $\mathcal{L} = \mathcal{L}(h, \chi)$ be a symmetric line bundle on $A$ and $D$ a symmetric effective divisor such that $\mathcal{L} = \mathcal{O}_A(D)$. For every 2 torsion point $x \in A[2]$ with representative $\frac{1}{2} \lambda$, where $\lambda \in \Lambda$, we have

\[
\chi(\lambda) = (-1)^{m(D, 0) + m(D, x)}. \tag{12}
\]
3.2 The equianharmonic product

Let \( \zeta := e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \), so that \( \zeta^2 - \zeta + 1 = 0 \), and consider the equianharmonic elliptic curve

\[
E' := \mathbb{C}/\Gamma_{\zeta}, \quad \Gamma_{\zeta} := \mathbb{Z}\zeta \oplus \mathbb{Z}.
\]

Setting \( V := \mathbb{C}^2 \), we can define

\[
A' := E' \times E' = V/\Lambda_{A'}, \quad \Lambda_{A'} := \Gamma_{\zeta} \times \Gamma_{\zeta}.
\]

Then \( A' \) is a principally polarized abelian surface, that we will call the equianharmonic product.

Denoting by \((z_1, z_2)\) the coordinates of \( V \) and by \( e_1 = (1, 0) \), \( e_2 = (0, 1) \) its standard basis, the four vectors

\[
\lambda_1 := \zeta e_1, \quad \lambda_2 := \zeta e_2, \quad e_1, \quad e_2
\]

form a basis for the lattice \( \Lambda_{A'} \).

We now consider the four 1-dimensional complex subspaces of \( V \) defined as

\[
V_1 := \text{span}(e_1) = \{z_2 = 0\}, \quad V_2 := \text{span}(e_2) = \{z_1 = 0\}, \\
V_3 := \text{span}(e_1 + e_2) = \{z_1 - z_2 = 0\}, \quad V_4 := \text{span}(e_1 + \zeta e_2) = \{\zeta z_1 - z_2 = 0\}.
\]

For each \( k \in \{1, 2, 3, 4\} \), the subspace \( V_k \) contains a rank 1 sublattice \( \Lambda_k \subset \Lambda_{A'} \) isomorphic to \( \Gamma_{\zeta} \), where

\[
\Lambda_1 := \mathbb{Z}\lambda_1 \oplus \mathbb{Z}e_1, \quad \Lambda_2 := \mathbb{Z}\lambda_2 \oplus \mathbb{Z}e_2, \\
\Lambda_3 := \mathbb{Z}(\lambda_1 + \lambda_2) \oplus \mathbb{Z}(e_1 + e_2), \quad \Lambda_4 := \mathbb{Z}(\lambda_1 + \lambda_2 - e_2) \oplus \mathbb{Z}(\lambda_2 + e_1).
\]

Consequently, in \( A' \) there are four elliptic curves isomorphic to \( E' \), namely

\[
E'_k := V_k/\Lambda_k, \quad k \in \{1, 2, 3, 4\}.
\]

**Proposition 3.4** ([Hir84 Section 1]). The four curves \( E'_k \) only intersect (pairwise transversally) at the origin \( o' \in A' \). Consequently, the reducible divisor

\[
D_{A'} := E'_1 + E'_2 + E'_3 + E'_4
\]

has an ordinary quadruple point at \( o' \) and no other singularities.

By the Appell-Humbert Theorem, the Néron-Severi group \( \text{NS}(A') \) of \( A' \) can be identified with the group of hermitian forms \( h \) on \( V \) whose imaginary part takes integral values on \( \Lambda_{A'} \). We will use the symbol \( H \) for the \( 2 \times 2 \) hermitian matrix associated to \( h \) with respect to the standard basis of \( V \) so that, thinking of \( v, w \in V \) as column vectors, we can write \( h(v, w) = v H w^t \). We want now to identify those hermitian matrices \( H_1, \ldots, H_4 \) that correspond to the classes of the curves \( E'_1, \ldots, E'_4 \), respectively.

**Proposition 3.5.** We have

\[
H_1 = \frac{2}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_2 = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
H_3 = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad H_4 = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & -\zeta \\ -\bar{\zeta} & 1 \end{pmatrix},
\]

so that the hermitian matrix representing in \( \text{NS}(A') \) the class of the divisor \( D_{A'} \) is

\[
H := H_1 + H_2 + H_3 + H_4 = \frac{2}{\sqrt{3}} \begin{pmatrix} 3 & -1 - \zeta \\ -1 - \bar{\zeta} & 3 \end{pmatrix}.
\]

Moreover, setting \( \lambda = (a_1 + \zeta a_2, a_3 + \zeta a_4) \in \Lambda_{A'} \), the semicharacter \( \chi_{D_{A'}} \) corresponding to the line bundle \( \mathcal{O}_{A'}(D_{A'}) \) can be written as

\[
\chi_{D_{A'}}(\lambda) = (-1)^{a_1 + a_2 + a_3 + a_4 + a_1(a_2 + a_3 + a_4) + (a_2 + a_3)a_4}.
\]
Proof. The hermitian form $\tilde{h}$ on $\mathbb{C}$ given by $\tilde{h}(z_1, z_2) = \frac{2}{\sqrt{3}}z_1\bar{z}_2$ is positive definite and its imaginary part is integer-valued on $\Gamma$, so it defines a positive class in $\text{NS}(E')$. Moreover, in the ordered basis $\{\zeta, 1\}$ of $\Gamma$, the alternating form $\text{Im} \tilde{h}$ is represented by the skew-symmetric matrix $(-1 0 \, 0 1)$, whose Pfaffian equals 1, so $\tilde{h}$ corresponds to the ample generator of the Néron-Severi group of $E'$, see [BL04 Corollary 3.2.8]. In other words, $\tilde{h}$ is the Chern class of $\mathcal{O}_{E'}(0)$, where 0 is the origin of $E'$. Write $\mathcal{O}_{E'}(0) = \mathcal{L}(h, \nu)$ for a suitable semicharacter $\nu: \Gamma \rightarrow \mathbb{C}$; since $\mathcal{O}_{E'}(0)$ is a symmetric line bundle, the values of $\nu$ at the generators of $\Gamma$ can be computed by using Lemma 3.3 obtaining $\nu(1) = -1$, $\nu(\zeta) = -1$. Consequently, for all $a, b \in \mathbb{Z}$ we get

$$\nu(a + b\zeta) = \nu(a)\nu(b)\exp(2\pi \text{Im} \tilde{h}(a, b\zeta)) = (-1)^a(-1)^b = (-1)^{a+b+ab}. \quad (17)$$

For any $k \in \{1, \ldots, 4\}$ let us define a group homomorphism $F_k: A' \rightarrow E'$ as follows:

$$F_1(z_1, z_2) = z_2, \quad F_2(z_1, z_2) = z_1, \quad F_3(z_1, z_2) = z_1 - z_2, \quad F_4(z_1, z_2) = \zeta z_1 - z_2.$$ 

By (14) we have $E'_k = F_k^*(0)$ and so, setting $\mathcal{O}_{A'}(E'_k) = \mathcal{L}(h_k, \chi_k^e)$, by (11) we deduce

$$h_k = F_k^*\tilde{h}, \quad \chi_k^e = F_k^*\nu. \quad (18)$$

This gives immediately the four matrices $H_1, \ldots, H_4$. Moreover, by using (17) and (18), we can write down the semicharacters $\chi_1, \ldots, \chi_4$; in fact, for any $\lambda = (a_1 + \zeta a_2, a_3 + \zeta a_4) \in \Lambda_{A'}$, we obtain

$$\chi_1^e(\lambda) = (-1)^{a_1+a_4+a_3a_4} \chi_2^e(\lambda) = (-1)^{a_1+a_2+a_3} \chi_3^e(\lambda) = (-1)^{a_1+a_2+a_3+a_4+(a_1+a_3)(a_2+a_4)} \chi_4^e(\lambda) = (-1)^{a_1+a_2+a_3+a_4+(a_1+a_4)(a_2+a_3)}.$$ 

The semicharacter $\chi_{D_{A'}}$ can be now computed by using the formula $\chi_{D_{A'}} = \chi_1^e\chi_2^e\chi_3^e\chi_4^e$. \qed

Remark 3.6. The hermitian matrix

$$H_1 + H_2 = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

represents in $\text{NS}(A')$ the class of the principal polarization of product type $\Theta := E' \times \{0\} + \{0\} \times E'$. 

Remark 3.7. The free abelian group $\text{NS}(A')$ is generated by the classes of the elliptic curves $E'_1$, $E'_2$, $E'_3$, $E'_4$. In fact, since $A' = E' \times E'$ and $E'$ has complex multiplication, it is well-known that $\text{NS}(A')$ has rank 4, see [BL04 Exercise 5.6 (10) p.142], hence we only need to show that the classes of the curves $E'_k$ generate a primitive sublattice of maximal rank in the Néron-Severi group. By Proposition 3.3 the corresponding Gram matrix has determinant

$$\det(E'_k \cdot E'_j) = \det(\delta_{ij}) = -3$$ 

so the claim follows because $-3$ is a non-zero, square-free integer.

3.3 Double covers of the equianharmonic product

In order to construct a surface of type $II$, we must find an abelian surface $A$ and a divisor $D_A$ on it such that

- $D_A$ is 2-divisible in $\text{Pic}(A)$;
- $D_A^2 = 24$ and $D_A$ has precisely two ordinary quadruple points as singularities.
We will construct the pair \((A, D_A)\) as an étale double cover of the pair \((A', D_{A'})\), where \(A' = V/\Lambda_{A'}\) is the equianharmonic product and \(D_{A'} = E'_1 + E'_2 + E'_3 + E'_4\) is the sum of four elliptic curves considered in Proposition 3.4.

By the Appell-Humbert theorem, the sixteen 2-torsion divisors on \(A'\), i.e. the elements of order 2 in \(\text{Pic}^0(A')\), correspond to the sixteen characters

\[
\chi: \Lambda_{A'} \rightarrow \{\pm 1\}.
\]  

Any such character is specified by its values at the elements of the ordered basis \(\{\lambda_1, \lambda_2, e_1, e_2\}\) of \(\Lambda_{A'}\) given in \([13]\), so it can be denoted by

\[
\chi = (\chi(\lambda_1), \chi(\lambda_2), \chi(e_1), \chi(e_2)).
\]

For instance, \(\chi_0 := (1, 1, 1, 1)\) is the trivial character, corresponding to the trivial divisor \(O_{A'}\). We will write

\[
\begin{align*}
\chi_1 & := (-1, -1, 1, -1), & \chi_2 & := (1, -1, -1, 1), & \chi_3 & := (-1, 1, -1, 1), \\
\chi_4 & := (1, 1, -1, 1), & \chi_5 & := (-1, 1, 1, 1), & \chi_6 & := (-1, 1, -1, 1), \\
\chi_7 & := (1, 1, 1, -1), & \chi_8 & := (1, 1, -1, -1), & \chi_9 & := (-1, 1, 1, -1), \\
\chi_{10} & := (1, 1, 1, 1), & \chi_{11} & := (-1, -1, -1, 1), & \chi_{12} & := (1, 1, -1, 1), \\
\chi_{13} & := (1, -1, 1, 1), & \chi_{14} & := (-1, -1, 1, 1), & \chi_{15} & := (-1, -1, 1, -1)
\end{align*}
\]  

(20)

for the fifteen non-trivial characters. To any non-trivial 2-torsion divisor on \(A'\), and so to any non-trivial character \(\chi\) as in \([19]\), it corresponds an isogeny of degree two \(f_\chi: A_\chi \rightarrow A'\); in fact, \(\ker \chi \subset \Lambda_{A'}\) is a sublattice of index 2 and \(A_\chi\) is the abelian surface

\[
A_\chi = V/\ker \chi.
\]  

(21)

Let us set

\[
E_i := f_\chi^*(E'_i), \quad D_{A_\chi} := f_\chi^*(D_{A'}) = E_1 + E_2 + E_3 + E_4
\]

and write \(\Sigma\) for the subgroup of \(\text{Pic}^0(A')\) generated by \(\chi_1\) and \(\chi_2\), namely

\[
\Sigma := \{\chi_0, \chi_1, \chi_2, \chi_3\}.
\]  

(22)

We are now ready to prove the key result of this subsection.

**Proposition 3.8.** The following are equivalent:

(a) the divisor \(D_{A_\chi}\) is 2-divisible in \(\text{Pic}(A_\chi)\);

(a') the divisor \(D_{A_\chi}\) is 2-divisible in \(\text{NS}(A_\chi)\);

(b) every \(E_i\) is an irreducible elliptic curve in \(A_\chi\);

(c) the character \(\chi\) is a non-trivial element of \(\Sigma\).

**Proof.** We first observe that \(\text{Pic}^0(A_\chi) = \text{Pic}(A_\chi)/\text{NS}(A_\chi)\) is a divisible group, so (a) is equivalent to (a').

Next, the curve \(E_i \subset A_\chi\) is irreducible if and only if the 2-torsion divisor corresponding to the character \(\chi: \Lambda_{A'} \rightarrow \{\pm 1\}\) restricts non-trivially to \(E'_i\). This in turn means that \(\chi\) restricts non-trivially to the sublattice \(\Lambda_{A'}\), and so (b) occurs if and only if \(\chi\) restricts non-trivially to all \(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\). By using the generators given in \([15]\), a long but elementary computation (or a quick computer calculation) shows that this happens if and only if (c) holds.

It remains to prove that (a') and (c) are equivalent. The isogeny \(f_\chi: A_\chi \rightarrow A\) lifts to the identity \(1_V: V \rightarrow V\) so, if \(h: V \times V \rightarrow \mathbb{C}\) is the hermitian form that represents the class of \(D_{A'}\) in \(\text{NS}(A')\), then the same form also represents the class of \(D_{A_\chi}\) in \(\text{NS}(A_\chi)\). By the Appell-Humbert theorem the group \(\text{NS}(A_\chi)\) can be identified with the group of hermitian forms on \(V\) whose imaginary part takes integral values on the lattice \(\ker \chi\), so (21) implies that condition (a') is equivalent to

\[
\text{Im } h(\ker \chi, \ker \chi) \subseteq 2\mathbb{Z}.
\]  

(23)
The non-zero values assumed by the alternating form $\text{Im} \, h$ on the generators $\lambda_1$, $\lambda_2$, $e_1$, $e_2$ of $\Lambda_A^\ast$ can be computed by using the hermitian matrix $H$ given in Proposition 3.5 obtaining Table 1 below:

| $(\cdot, \cdot)$ | $(\lambda_1, \lambda_2)$ | $(\lambda_1, e_1)$ | $(\lambda_1, e_2)$ | $(\lambda_2, e_1)$ | $(\lambda_2, e_2)$ | $(e_1, e_2)$ |
|------------------|---------------------|------------------|------------------|------------------|------------------|------------------|
| $\text{Im} \, h(\cdot, \cdot)$ | $-1$ | $3$ | $-2$ | $-1$ | $3$ | $-1$ |

Table 1: Non-zero values of $\text{Im} \, h$ at the generators of $\Lambda_A^\ast$.

Now we show that \[ (23) \] holds if and only if $\chi$ is a non-trivial element of $\Sigma$. In fact we have seen that, if $\chi \notin \{ \chi_1, \chi_2, \chi_3 \}$, then one of the effective divisors $E_i = f_{\chi}^\ast(E_i')$ is a disjoint union of two elliptic curves, say $E_i = E_{i1} + E_{i2}$. But then, using the projection formula, we find

$$D_A \cdot E_{i1} = f_{\chi}^\ast(D_{A'}) \cdot E_{i1} = D_{A'} \cdot f_{\chi}^\ast(E_{i1}) = D_{A'} \cdot E_{i}' = 3$$

which is not an even integer, so $D_{A_{\chi}}$ is not 2-divisible in this situation.

Let us consider now the case $\chi \in \{ \chi_1, \chi_2, \chi_3 \}$. We can easily see that the integral bases of $\ker \chi_1$, $\ker \chi_2$, $\ker \chi_3$ are given by

$$\begin{align*}
\mathcal{B}_1 & := \{ e_1, \lambda_1 + e_2, \lambda_2 + e_2, 2e_2 \}, \\
\mathcal{B}_2 & := \{ \lambda_2 + e_1, \lambda_1, e_2, 2e_1 \}, \\
\mathcal{B}_3 & := \{ \lambda_1 + e_2, \lambda_2, 2e_2, e_1 + e_2 \},
\end{align*}$$

respectively. Then, by using Table 1, it is straightforward to check that $\text{Im} \, h(b_1, b_2) \in \mathbb{Z}$ for all $b_1, b_2 \in \mathcal{B}_1$; for instance, we have

$$\text{Im} \, h(\lambda_1 + e_2, \lambda_2 + e_2) = \text{Im} \, h(\lambda_1, \lambda_2) + \text{Im} \, h(\lambda_1, e_2) + \text{Im} \, h(e_2, \lambda_2) + \text{Im} \, h(e_2, e_2)
$$

$$= -1 - 2 - 3 + 0 = -6 \in \mathbb{Z}.$$

This shows that the inclusion \[ (23) \] holds for $\chi_1$. The proof that it also holds for $\chi_2$ and $\chi_3$ is analogous.

\textbf{Remark 3.9.} Writing the details in the proof of Proposition 3.8 one sees that every non-trivial character $\chi$ in \[ (20) \] restricts trivially to at most one curve $E_i'$. Identifying $A'$ with $\text{Pic}^0(A')$ via the principal polarization $\Theta$ described in Remark 3.6, this corresponds to the fact that every 2-torsion point of $A'$ is contained in at most one of the $E_i'$. More precisely, every $E_i'$ contains exactly three non-zero 2-torsion points of $A'$, so it remain $16 - (4 \times 3 + 1) = 3$ of them that are not contained in any of the $E_i'$. Via the identification above, they clearly correspond to the three 2-torsion divisors restricting non-trivially to all the $E_i'$, namely, to the three non-trivial characters in the group $\Sigma$.

Summing up, we have the following existence result for surfaces of type II.

\textbf{Proposition 3.10.} Let $\chi$ be any non-trivial element in the group $\Sigma$, and write $f: A \to A'$ instead of $f_{\chi}: A_{\chi} \to A'$. Then there exists a double cover $\alpha_X: X \to A$ branched precisely over the 2-divisible effective divisor $D_A \subset A$. The smooth minimal resolution $S$ of $X$ is a surface with $p_g = q = 2$, $K^2 = 8$ and Albanese map of degree 2, belonging to type II.

\textbf{Proof.} It only remains to compute the invariants of $S$. From the double cover formulas (see \cite[Chapter V.22]{BHPVdV04}) we see that, if we impose an ordinary quadruple point to the branch locus, then $\chi$ decreases by 1 and $K^2$ decreases by 2, hence we get

$$\chi(O_S) = \frac{1}{8} D_A^3 - 2 - 1 \quad \text{and} \quad K_S^2 = \frac{1}{2} D_A^3 - 4 = 8.$$

Since $q(S) \geq q(A) = 2$, we have $p_g(S) = q(S) \geq 2$. Assume that $p_g(S) = q(S) \geq 3$. By \cite[Beauville appendix]{HP02}, \cite{Pir02} and \cite{Deb82}, we have two possibilities:

- $p_g(S) = q(S) = 4$ and $S$ is the product of two curves of genus 2;
• \( p_g(S) = q(S) = 3 \) and \( S = (C_2 \times C_3)/\mathbb{Z}_2 \), where \( C_2 \) is a smooth curve of genus 2 with an elliptic involution \( \tau_2 \), \( C_3 \) is a smooth curve of genus 3 with a free involution \( \tau_3 \), and the cyclic group \( \mathbb{Z}_2 \) acts freely on the product \( C_2 \times C_3 \) via the involution \( \tau_2 \times \tau_3 \).

In both cases above, \( S \) contains no elliptic curves. On the other hand, all our surfaces of type II contain four elliptic curves, coming from the strict transform of \( D_A \). Therefore the only possibility is \( p_g(S) = q(S) = 2 \).

4 Surfaces of type II: classification

4.1 Holomorphic and anti-holomorphic diffeomorphisms of cyclic covers

Let \( n \geq 2 \) be an integer and let \( D \) be an effective divisor on a smooth projective variety \( Y \), such that

\[ \mathcal{O}_Y(D) = \mathcal{L}^\otimes n = \mathcal{L}_2^\otimes n \]

for some line bundles \( \mathcal{L}_1, \mathcal{L}_2 \in \text{Pic}(Y) \). Canonically associated to such data, there exists two simple \( n \)-cyclic covers

\[ \pi_1: X_1 \rightarrow Y \text{ and } \pi_2: X_2 \rightarrow Y, \]

both branched over \( D \). We want to provide conditions ensuring that the two compact complex manifolds underlying \( X_1 \) and \( X_2 \) are biholomorphic or anti-biholomorphic.

Following \[ \text{KK02 Section3} \], let us denote by \( \text{Kl}(Y) \) the group of holomorphic and anti-holomorphic diffeomorphisms of \( Y \). There is a short exact sequence

\[ 1 \rightarrow \text{Aut}(Y) \rightarrow \text{Kl}(Y) \rightarrow H \rightarrow 1, \]

where \( H = \mathbb{Z}/2\mathbb{Z} \) or \( H = 0 \). To any anti-holomorphic element \( \sigma \in \text{Kl}(Y) \) we can associate a \( \mathbb{C} \)-antilinear map

\[ \sigma^* : \mathbb{C}(Y) \rightarrow \mathbb{C}(Y) \]

on the function field \( \mathbb{C}(Y) \) by defining

\[ (\sigma^* f)(x) := \overline{f(\sigma(x))} \]

for all \( f \in \mathbb{C}(Y) \). That action extends the usual action of \( \text{Aut}(Y) \) on \( \mathbb{C}(Y) \) in a natural way (note that in \[ \text{KK02} \] the notation \( \sigma^* \) is used only for holomorphic maps, whereas for anti-holomorphic maps the corresponding notation is \( \sigma^* \)). We have \( \sigma^{-1}(\text{div}(f)) = \text{div}(\sigma^* f) \), hence the action \( \sigma^* : \text{Div}(Y) \rightarrow \text{Div}(Y) \) induces an action \( \sigma^* : \text{Pic}(Y) \rightarrow \text{Pic}(Y) \), such that \( \sigma^* K_Y = K_Y \). Moreover, the intersection numbers are also preserved by the action of any \( \sigma \in \text{Kl}(Y) \).

Example 4.1. Let \( A_1 = V_1/\Lambda_1 \) and \( A_2 = V_2/\Lambda_2 \) be two abelian varieties, and let \( \sigma : A_2 \rightarrow A_1 \) be an anti-holomorphic homomorphism with analytic representation \( \mathcal{G} : V_2 \rightarrow V_1 \) and rational representation \( \mathcal{G}_A : A_2 \rightarrow A_1 \) (note that \( \mathcal{G} \) is a \( \mathbb{C} \)-antilinear map). Then, for any \( \mathcal{L}(h, \chi) \in \text{Pic}(A_1) \), we have the following analog of \( (\Pi) \):

\[ \sigma^* \mathcal{L}(h, \chi) = \mathcal{L}(\overline{\mathcal{G}^* h}, \overline{\mathcal{G}^*_A \chi}), \tag{25} \]

In fact, looking at the transition function of the anti-holomorphic line bundle \( \mathcal{L}(\mathcal{G}^* h, \overline{\mathcal{G}^*_A \chi}) \) we see that, in order to obtain a holomorphic one, we must take the conjugated hermitian form \( \mathcal{G}^* h \) and the conjugated semicharacter \( \overline{\mathcal{G}^*_A \chi} \).

Let us now denote by \( \text{Kl}(Y, D) \) and \( \text{Aut}(Y, D) \) the subgroups of \( \text{Kl}(Y) \) and \( \text{Aut}(Y) \) given by diffeomorphisms such that \( \sigma^* D = D \). Again, \( \text{Aut}(Y, D) \) is a normal subgroup of \( \text{Kl}(Y, D) \), of index 1 or 2.

Proposition 4.2.

(i) Let \( \sigma \in \text{Kl}(Y, D) \) be such that \( \sigma^* \mathcal{L}_2 \simeq \mathcal{L}_1 \). Then there exists a diffeomorphism \( \tilde{\sigma} : X_1 \rightarrow X_2 \) such that \( \sigma \circ \pi_1 = \pi_2 \circ \tilde{\sigma} \). Moreover, \( \tilde{\sigma} \) is holomorphic (respectively, anti-holomorphic) if and only if \( \sigma \) is so.
(ii) A diffeomorphism \( \sigma \in \text{Kl}(Y) \) lifts to \( X_i \) if and only if \( \sigma \in \text{Kl}(Y, D) \) and \( \sigma^*L_i \simeq L_1 \).

Moreover, if \( \sigma \) lifts than it lifts in \( n \) different ways.

Proof. Let us prove (i). Let \( L_1, L_2 \) be the total spaces of \( L_1, L_2 \) and let \( p_1 : L_1 \rightarrow Y, p_2 : L_2 \rightarrow Y \) be the corresponding projections. Let \( s \in H^0(Y, \mathcal{O}_Y(D)) \) be a section vanishing exactly along \( D \) (if \( D = 0 \), we take for \( s \) the constant function 1). If \( t_i \in H^0(L_i, p_i^*L_1) \) denotes the tautological section, by \([BHPvdV04] 1.17\) it follows that global equations for \( X_1 \) and \( X_2 \), as analytic subvarieties of \( L_1 \) and \( L_2 \), are provided by

\[
\begin{align*}
&\quad t_1^2 - p_1^*s = 0 \quad \text{and} \quad t_2^2 - p_2^*s = 0,
\end{align*}
\]

the covering maps \( \pi_1 \) and \( \pi_2 \) being induced by the restrictions of \( p_1 \) and \( p_2 \), respectively. Since \( \sigma \in \text{Kl}(Y, D) \), we have \( \sigma^*s = \lambda s \) with \( \lambda \in \mathbb{C}^* \). Moreover, \( \sigma^*L_2 \simeq L_1 \) implies that there exists \( \tilde{\sigma} : L_1 \rightarrow L_2 \) such that \( p_2 \circ \tilde{\sigma} = \sigma \circ p_1 \), hence

\[
\tilde{\sigma}^*(p_2^*s) = p_1^*(\sigma^*s) = \lambda p_1^*s.
\]

Moreover, we have \( \tilde{\sigma}^*t_2 = \mu t_1 \), with \( \mu \in \mathbb{C}^* \). Up to rescaling \( t_2 \) by a constant factor we can assume \( \mu = \sqrt{\lambda} \), so that

\[
\tilde{\sigma}^*(t_2^2 - p_2^*s) = \lambda(t_1^2 - p_1^*s).
\]

This means that \( \tilde{\sigma} : L_1 \rightarrow L_2 \) restricts to a diffeomorphism \( \tilde{\sigma} : X_1 \rightarrow X_2 \), which is compatible with the two covering maps \( \pi_1 \) and \( \pi_2 \). By construction, such a diffeomorphism is holomorphic (respectively, anti-holomorphic) if and only if \( \sigma \) is so.

The part (ii) follows from part (i), setting \( L_1 = L_2 \), so that \( X_1 = X_2 \). The existence of \( n \) different choices for the lifting of \( \sigma \) is a consequence of the fact that there are \( n \) different choices for \( \sqrt{\lambda} \).

In the case of double covers induced by the Albanese map, we have the following converse of Proposition 4.2 (i).

**Proposition 4.3.** Set \( n = 2 \), let \( Y = A \) be an abelian variety and assume that the double cover \( \pi_i : X_i \rightarrow A \) is the Albanese map of \( X_i \), for \( i = 1, 2 \). If there is a holomorphic (respectively, anti-holomorphic) diffeomorphism \( \tilde{\sigma} : X_1 \rightarrow X_2 \), then there exists a holomorphic (respectively, anti-holomorphic) diffeomorphism \( \sigma \in \text{Kl}(A, D) \) such that \( \sigma^*L_2 \simeq L_1 \).

Proof. We first assume that \( \tilde{\sigma} \) is holomorphic. By the universal property of the Albanese map the morphism \( \pi_2 \circ \tilde{\sigma} : X_1 \rightarrow A \) factors through \( \pi_1 \), in other words there exists \( \sigma : A \rightarrow A \) such that \( \sigma \circ \pi_1 = \pi_2 \circ \tilde{\sigma} \). The map \( \sigma \) is an isomorphism because \( \tilde{\sigma} \) is an isomorphism, then it sends the branch locus of \( \pi_1 \) to the branch locus of \( \pi_2 \), or equivalently \( \sigma^*D = D \). Finally, looking at the direct image of the structural sheaf \( \mathcal{O}_{X_1} \), we get

\[
\begin{align*}
&(\sigma \circ \pi_1)_*\mathcal{O}_{X_1} = (\pi_2 \circ \tilde{\sigma})_*\mathcal{O}_{X_1}, \quad \text{that is} \\
&\sigma_*(-L_{-1}) = \pi_2^*(\tilde{\sigma}^*\mathcal{O}_{X_1}), \quad \text{that is} \\
&\mathcal{O}_A \oplus (\sigma_*L_{-1}) = \mathcal{O}_A \oplus L_{-1}.
\end{align*}
\]

By \([Ati56]\) a direct sum decomposition of a vector bundle into irreducible subbundles is unique up to isomorphisms, so we obtain \( \sigma_*L_{-1} \simeq L_{-1} \). By using the projection formula and dualizing we infer \( L_1 = \sigma^*L_2 \), as desired. If \( \tilde{\sigma} \) is anti-holomorphic, it suffices to apply the same proof to the holomorphic diffeomorphism which is complex-conjugated to it.

Summing up, Propositions 4.2 and 4.3 together imply

**Corollary 4.4.** With the same assumption as in Proposition 4.3, there exists a holomorphic (respectively, anti-holomorphic) diffeomorphism \( \tilde{\sigma} : X_1 \rightarrow X_2 \) if and only if there exists a holomorphic (respectively, anti-holomorphic) element \( \sigma \in \text{Kl}(A, D) \) such that \( \sigma^*L_2 = L_1 \).
4.2 The uniqueness of the abelian surface $A$

We follow the notation of Section 3.3 If $\chi_i$ is any non-trivial element of the group $\Sigma = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ for the sake of brevity we will write $A_1, D_{A_1}$ and $f_j: A_1 \rightarrow A'$ instead of $A_{\chi_i}, D_{\chi_i}$, and $f_{\chi_i}: A_{\chi_i} \rightarrow A'$, respectively. We will also denote by $L_i \in \text{Pic}^0(A')$ the 2-torsion line bundle corresponding to $\chi_i$, so that $f_{\chi_i}^*O_{A_1} = O_{A'} \oplus L_i^{-1}$.

**Proposition 4.5.** The abelian surfaces $A_1, A_2, A_3$ are pairwise isomorphic. More precisely, for all $i, j \in \{1, 2, 3\}$ there exists an isomorphism $\tilde{\gamma}_{ij}: A_j \rightarrow A_i$ such that $\tilde{\gamma}_{ij}^*D_{A_i} = D_{A_j}$.

**Proof.** By Proposition 4.2 it suffices to prove that there exists an automorphism corresponding to $f$. Let us first recall the notions of deformation equivalence and global rigidity, (4.3 A rigidity result for surfaces of type II).

**Definition 4.6.**

- Two complex surfaces $S_1, S_2$ are said to be direct deformation equivalent if there is a proper holomorphic submersion with connected fibres $f: Y \rightarrow D$, where $Y$ is a complex manifold and $D \subset \mathbb{C}$ is the unit disk, and moreover there are two fibres of $f$ biholomorphic to $S_1$ and $S_2$, respectively;

- two complex surfaces $S_1, S_2$ are said to be deformation equivalent if they belong to the same deformation equivalence class, where by deformation equivalence we mean the equivalence relation generated by direct deformation equivalence;

- a complex surface $S$ is called globally rigid if its deformation equivalence class consists of $S$ only, i.e. if every surface which is deformation equivalent to $S$ is actually isomorphic to $S$.

The following result is a characterization of the equianharmonic product, that can be found in [KH05 Proposition 5].

**Proposition 4.7.** Let $A'$ be an abelian surface containing four elliptic curves, that intersect pairwise at the origin $0$ and not elsewhere. Then $A'$ is isomorphic to the equianharmonic product $E' \times E'$ and, up to the action of $\text{Aut}(A')$, the four curves are $E'_1, E'_2, E'_3, E'_4$.

A more conceptual proof of Proposition 4.7, exploiting some results of Shioda and Mitani on abelian surfaces with maximal Picard number, can be found in [Aid]. Using Proposition 4.7 we obtain:

**Theorem 4.8.** Let $S$ be a surface with $p_g(S) = q(S) = 2$, $K_S^2 = 8$ and Albanese map $\alpha: S \rightarrow A$ of degree 2. If $S$ belongs to type $\mathbb{I}$, then the pair $(A, D_A)$ is isomorphic to an étale double cover of the pair $(A', D_{A'})$, where $A'$ is the equianharmonic product and $D_{A'} = E'_1 + E'_2 + E'_3 + E'_4$. In particular, all surfaces of type $\mathbb{II}$ arise as in Proposition 3.10. Finally, all surfaces of type $\mathbb{II}$ are globally rigid.
Proof. Let us consider the Stein factorization $\alpha_X : X \rightarrow A$ of the Albanese map $\alpha : S \rightarrow A$; then $\alpha_X$ is a finite double cover branched over $D_A$.

By Proposition 4.10 we have $D_A = E_1 + E_2 + E_3 + E_4$, where the $E_i$ are four elliptic curves intersecting pairwise transversally at two points $p_1$, $p_2$ and not elsewhere. Up to a translation, we may assume that $p_1$ coincides with the origin of $o \in A$. Then $p_2 = a$, where $a$ is a non-zero, 2-torsion point of $A$ (in fact the $E_i$ are subgroups of $A$, so the same is true for their intersection $\{o, a\}$).

If we consider the abelian surface $A' := A/(a)$, then the projection $f : A \rightarrow A'$ is an isogeny of degree 2. Moreover, setting $E'_i := f(E_i)$, we see that $E'_1, \ldots, E'_4$ are four elliptic curves intersecting pairwise transversally at the origin $o' \in A'$ and not elsewhere. Then the claim about $A'$ and $D_A'$ follows from Proposition 4.7.

Since there are finitely many possibilities for both the double covers $f : A \rightarrow A'$ and $a_X : X \rightarrow A$, it follows that $S$, being the minimal desingularization of $X$, belongs to only finitely many isomorphism classes. This implies that $S$ is globally rigid, because by [Gie77] the moduli space of surfaces of general type is separated. 

Remark 4.9. It is straightforward to check that the class of the point $\zeta e_1 = (\zeta, 0)$ in $A = V/\Lambda_A$ is contained in all the curves $E_1, \ldots, E_4$, so we obtain $a = \zeta e_1 + \Lambda_A$.

4.4 The groups $\text{Aut}(A, D_A)$ and $\text{Kl}(A, D_A)$

In the sequel we will write $A := V/\Lambda_A$ in order to denote any of the pairwise isomorphic abelian surfaces $A_1$, $A_2$, $A_3$, see Proposition 4.5. Choosing for instance $\Lambda_A = \ker \chi_1$, by (21) we have

$$\Lambda_A = \Z e_1 \oplus \Z e_2 \oplus \Z e_3 \oplus \Z e_4,$$

where

$$e_1 := e_1, \quad e_2 := \lambda_1 + e_2, \quad e_3 := \lambda_2 + e_2, \quad e_4 := 2e_2.$$

Note that $e_1 = (1, 0)$ and $e_2 = (\zeta, 1)$ form a basis for $V$. Let $\Gamma_\zeta$ and $E'$ be as in (22) and set

$$E'' := C/\Gamma_{2\zeta}, \quad \Gamma_{2\zeta} := \Z [2\zeta].$$

The next result implies that $A$ is actually isomorphic to the product $E'' \times E'$.

Lemma 4.10. We have $\Lambda_A = \Gamma_{2\zeta} e_1 \oplus \Gamma_{\zeta} e_2$.

Proof. We check that the base-change matrix between the $\Q$-bases $e_1, e_2, e_3, e_4$ and $e_1, e_2, 2\zeta e_1, \zeta e_2$ of $H_1(A, \Q)$ is in $\text{GL}(4, \Z)$.

We will use Lemma 4.10 in order to describe the groups $\text{Aut}(A, D_A)$ and $\text{Kl}(A, D_A)$. In what follows, we will identify an automorphism $A \rightarrow A$ with the matrix of its analytic representation $V \rightarrow V$ with respect to the standard basis $\{e_1, e_2\}$. Moreover, we will write $\tau = \tau_a : A \rightarrow A$ for the translation by the 2-torsion point $a = \zeta e_1 + \Lambda_A$ defined in the proof of Theorem 4.8 see also Remark 4.9.

Proposition 4.11. The following holds.

(a) We have

$$\text{Aut}(A, D_A) = \text{Aut}_0(A, D_A) \times \Z/2\Z,$$

where $\Z/2\Z$ is generated by the translation $\tau$, whereas $\text{Aut}_0(A, D_A)$ is the subgroup of group automorphisms of $A$ generated by the elements

$$g_2 = \begin{pmatrix} \zeta & -1 \\ \zeta & -\zeta \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & \zeta & 1 \\ 1 - \zeta & \zeta - 1 \end{pmatrix}.$$

As an abstract group, $\text{Aut}_0(A, D_A)$ is isomorphic to $\text{SL}(2, \F_3)$; it has order 24.
(b) The group $\text{Kl}(A, D_A)$ is generated by $\text{Aut}(A, D_A)$ together with the anti-holomorphic involution $\sigma: A \rightarrow A$ induced by the $\mathbb{C}$-antilinear involution of $V$ given by

$$(z_1, z_2) \mapsto ((\zeta - 1)z_2, (\zeta - 1)\bar{z}_1).$$

Furthermore, the two involutions $\tau$ and $\sigma$ commute, so that we can write

$$\text{Kl}(A, D_A) = \text{Kl}_0(A, D_A) \times \mathbb{Z}/2\mathbb{Z},$$

where $\text{Kl}_0(A, D_A)$ contains $\text{Aut}_0(A, D_A)$ as a subgroup of index 2.

Proof. (a) Let us work using the basis $\{e_1, e_2\}$ of $V$ defined in [25]. With respect to this basis, using (14) we see that the four elliptic curves $E_1, \ldots, E_4$ have tangent spaces

$$V_1 = \text{span}(e_1), \quad V_2 = \text{span}(-\zeta e_1 + e_2), \quad V_3 = \text{span}((1 - \zeta)e_1 + e_2), \quad V_4 = \text{span}((1 - 2\zeta)e_1 + e_2).$$

Then, up to the translation $\tau$, we are looking at the subgroup $\text{Aut}_0(A, D_A)$ of the group automorphisms of $A$ whose elements have matrix representation preserving the set of four points $D = \{P_1, P_2, P_3, P_4\} \subset \mathbb{P}^1$, where

$$P_1 = [1 : 0], \quad P_2 = [-\zeta : 1], \quad P_3 = [1 - \zeta : 1], \quad P_4 = [1 - 2\zeta : 1].$$

The cross ratio $(P_1, P_2, P_3, P_4)$ equals $\zeta^{-1}$, hence $D$ is an equianharmonic quadruple and so the group $\text{PGL}(2, \mathbb{C})$ acts on it as the alternating group $A_4$, see [Mai07, p. 7]. Such a group can be presented as

$$A_4 = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = (\alpha\beta)^3 = 1 \rangle,$$

where $\alpha = (12)(34)$ and $\beta = (123)$, so we need to find matrices $\tilde{g}_2, \tilde{g}_3 \in \text{GL}(2, \mathbb{C})$, acting as an isomorphism on the lattice $\Lambda_A$ and inducing the permutations $(P_1 P_2)\langle P_3 P_4 \rangle$ and $(P_1 P_2)\langle P_3 P_4 \rangle$ on $D$, respectively. Using Lemma 1.10 we see that $\tilde{g} \in \text{GL}(2, \mathbb{C})$ preserves $\Lambda_A$ if and only if it has the form

$$\tilde{g} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with $a_{11} \in \Gamma_\zeta$, $a_{12} \in 2\Gamma_\zeta$, $a_{21}, a_{22} \in \Gamma_\zeta$, and its determinant belongs to the group of units of $\Gamma_\zeta$, namely $\{ \pm 1, \pm \zeta, \pm \zeta^2 \}$.

Now an elementary computation yields the matrices $\tilde{g}_2, \tilde{g}_3$ of our automorphisms in the basis $\{e_1, e_2\}$ of $V$ by taking

$$g_1 = N\tilde{g}_iN^{-1}, \quad \text{with } N = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix}. $$

This gives (30). Setting $h = -I_2$ and lifting the presentation (33) we get the presentation

$$\text{Aut}_0(A, D_A) = \langle g_2, g_3, h \mid h^2 = 1, \quad g_2^2 = g_3^3 = (g_2g_3)^3 = h \rangle,$$  

(34)

showing the isomorphism $\text{Aut}_0(A, D_A) \cong \text{SL}(2, F_3)$.

Finally, a standard computation shows that $\tau$ commutes with both $g_2$ and $g_3$. Since $\text{Aut}(A)$ is the semidirect product of the translation group of $A$ by the group automorphisms, it follows that $\text{Aut}(A, D_A)$ is the direct product of $\langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$ by $\text{Aut}_0(A, D_A)$, hence we obtain (29).

(b) Let us consider the $\mathbb{C}$-antilinear map $V \rightarrow V$

$$(z_1, z_2) \mapsto (\bar{z}_1, (\zeta - 1)\bar{z}_1 + \zeta z_2),$$

expressed with respect to the basis $\{e_1, e_2\}$. It preserves the lattice $\Lambda_A$ and so it defines an anti-holomorphic involution $\sigma: A \rightarrow A$, inducing the transposition $(P_1 P_2)$ on the set $D = \{P_1, P_2, P_3, P_4\}$. Since $\text{Aut}(A, D)$ has index at most 2 in $\text{Kl}(A, D_A)$, it follows that $\text{Aut}(A, D_A)$ and $\sigma$ generate $\text{Kl}(A, D_A)$. Moreover, a change of coordinates allows us to come back to the basis $\{e_1, e_2\}$ and to obtain the expression of $\sigma$ given in (31). The subgroup $\text{Kl}_0(A, D_A)$ generated by $g_2, g_3$ and the involution $\sigma$ contains $\text{Aut}_0(A, D_A)$ as a subgroup of index 2; a straightforward computation now shows that $[\tau, \sigma] = 0$, so (32) follows from (29) and the proof is complete. \qed
Remark 4.12. The proof of Proposition 4.11 also shows that there are central extensions
\[ 1 \longrightarrow (-I_2) \longrightarrow \text{Aut}_0(A, D_A) \longrightarrow A_4 \longrightarrow 0, \]
\[ 1 \longrightarrow (-I_2) \longrightarrow \text{Kl}_0(A, D_A) \longrightarrow S_4 \longrightarrow 0 \]
such that \(-I_2 = [g_2, g_3]^2\). In fact, \(\text{Kl}_0(A, D_A)\) is isomorphic as an abstract group to \(\text{GL}(2, \mathbb{F}_3)\), see the proof of Theorem 4.13.

4.5 The action of \(\text{Kl}(A, D_A)\) on the square roots of \(\mathcal{O}_A(D_A)\)

The main result of this subsection is the following

Theorem 4.13. Up to isomorphism, there exist exactly two surfaces of type II. These surfaces \(S_1, S_2\) have conjugated complex structures, in other words there exists an anti-holomorphic diffeomorphism \(S_1 \rightarrow S_2\).

In order to prove this result, we must study the action of the groups \(\text{Aut}(A, D_A)\) and \(\text{Kl}(A, D_A)\) on the sixteen square roots \(\mathcal{L}_1, \ldots, \mathcal{L}_{16}\) of the line bundle \(\mathcal{O}_A(D_A) \subset \text{Pic}(A)\). The Appell-Humbert data of such square roots are described in the following

Proposition 4.14. For \(k \in \{1, \ldots, 16\}\), we have \(\mathcal{L}_k = \mathcal{L}\left(\frac{1}{2}h_A, \psi_k\right)\) where

- \(h_A : V \times V \rightarrow \mathbb{C}\) is the hermitian form on \(V\) whose associated alternating form \(\text{Im} h_A\) assumes the following values at the generators \(e_1, \ldots, e_4\) of \(\Lambda_A\):

| \((\cdot, \cdot)\) | \((e_1, e_2)\) | \((e_1, e_3)\) | \((e_1, e_4)\) | \((e_2, e_3)\) | \((e_2, e_4)\) | \((e_3, e_4)\) |
|------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(\text{Im} h_A(\cdot, \cdot)\) | -4 | 0 | -2 | -6 | -4 | 6 |

Table 2: The values of \(\text{Im} h_A\) at the generators of \(\Lambda_A\)

- Using the notation \(\psi_k := (\psi_k(e_1), \psi_k(e_2), \psi_k(e_3), \psi_k(e_4))\), the semicharacters \(\psi_k : \Lambda_A \rightarrow \mathbb{C}^*\) are as follows:

\[
\begin{align*}
\psi_1 &:= (i, 1, i, 1), \\
\psi_2 &:= (-i, -1, i, -1), \\
\psi_3 &:= (i, -1, -i, 1), \\
\psi_4 &:= (-i, 1, i, -1), \\
\psi_5 &:= (i, 1, -i, 1), \\
\psi_6 &:= (-i, 1, i, 1), \\
\psi_7 &:= (-i, 1, -i, 1), \\
\psi_8 &:= (i, 1, i, -1), \\
\psi_9 &:= (-i, 1, i, -1), \\
\psi_{10} &:= (-i, 1, -i, 1), \\
\psi_{11} &:= (i, 1, i, -1), \\
\psi_{12} &:= (-i, 1, -i, 1), \\
\psi_{13} &:= (i, -1, 1, 1), \\
\psi_{14} &:= (-i, -1, -i, 1), \\
\psi_{15} &:= (-i, 1, -i, 1), \\
\psi_{16} &:= (i, 1, 1, -i).
\end{align*}
\]

(35)

Proof. Let us consider the double cover \(f : A \rightarrow A'\). If the hermitian form \(h : V \times V \rightarrow \mathbb{C}\) and the semicharacter \(\chi_{D_A} : \Lambda_A \rightarrow \{\pm 1\}\) are as in Proposition 4.5 and Table 1 then we have \(\mathcal{O}_A(D_A) = \mathcal{L}(h_A, \chi_{D_A})\), where \(h_A = f^* h\). \(\chi_{D_A} = f^* \chi_{D_A}\). From this, using (28) we can compute the values of the alternating form \(\text{Im} h_A\) and of the semicharacter \(\chi_{D_A}\) at \(e_1, \ldots, e_4\), obtaining Table 2 and

\[ \chi_{D_A} = (\chi_{D_A}(e_1), \chi_{D_A}(e_2), \chi_{D_A}(e_3), \chi_{D_A}(e_4)) = (-1, 1, -1, 1). \]

Then, setting \(\mathcal{L}_k = \mathcal{L}(h_k, \psi_k)\), the equality \(\mathcal{L}_k^{\otimes 2} = \mathcal{O}_A(D_A)\) implies

\[ 2h_k = h_A, \quad \psi_k^2 = \chi_{D_A}, \]

hence \(h_k = \frac{1}{2}h_A\) for all \(k\). Moreover we can set \(\psi_1 = (i, 1, i, 1)\), whereas the remaining 15 semicharacters \(\psi_k\) are obtained by multiplying \(\psi_1\) by the 15 non-trivial characters \(\Lambda_A \rightarrow \{\pm 1\}\). □
The hermitian form $h_A$ is $\text{Kl}(A, D_A)$-invariant (according with the fact that the divisor $D_A$ is so), hence the action of $\text{Kl}(A, D_A)$ on the set $\{L_1, \ldots, L_{16}\}$ is completely determined by its permutation action on the set $\{\psi_1, \ldots, \psi_{16}\}$, namely

$$\varrho: \text{Kl}(A, D_A) \rightarrow \text{Perm}(\psi_1, \ldots, \psi_{16}), \quad \varrho(g)(\psi_k) := g^* \psi_k.$$ 

After identifying the group $\text{Perm}(\psi_1, \ldots, \psi_{16})$ with the symmetric group $S_{16}$ on the symbols $\{1, \ldots, 16\}$, we get the following

**Proposition 4.15.** With the notation of Proposition 4.11 we have

$$\varrho(g_2) = (1 \ 3 \ 7 \ 12)(2 \ 9 \ 14 \ 16)(3 \ 5 \ 15 \ 6)(4 \ 11 \ 8 \ 10),$$

$$\varrho(g_3) = (1 \ 13 \ 5 \ 7 \ 12 \ 6)(2 \ 4 \ 11 \ 14 \ 8 \ 10)(3 \ 15)(9 \ 16),$$

$$\varrho(-I) = \varrho(g_2^2) = \varrho(g_3^3) = \varrho(\sigma) = (1 \ 7)(2 \ 14)(3 \ 15)(4 \ 8)(5 \ 6)(9 \ 16)(10 \ 11)(12 \ 13),$$

$$\varrho(\sigma) = (1 \ 14)(2 \ 7)(3 \ 16)(4 \ 5)(6 \ 8)(9 \ 15)(10 \ 12)(11 \ 13).$$

**Proof.** Using the explicit expressions given in Proposition 4.11 by a standard computation we can check that $g_2$, $g_3$, $\tau$ and $\sigma$ send the ordered basis $\{e_1, e_2, e_3, e_4\}$ of $\Lambda_A$ to the bases

$$\{e_2 + e_3 - e_4, -2e_1 + e_2 - e_4, -e_1 - e_2 - 2e_3 + 2e_4, -2e_1 - 2e_3 + e_4\},$$

$$\{-e_3 + e_4, -e_1 + e_2 + e_3 - e_4, -2e_1 + e_2 + e_3 - 2e_4, -2e_1 + 2e_2 + 2e_3 - 3e_4\},$$

$$\{-e_3 - e_2, -e_1, e_4\},$$

$$\{e_3 - e_4, -e_1 + e_2 + e_3 - e_4, -e_1 + 2e_2 - e_4, -e_1 + 2e_2 - e_4\},$$

respectively. For any $g \in \text{Aut}(A, D_A)$, calling $G_A: \Lambda_A \rightarrow \Lambda_A$ the corresponding rational representation we have $\varrho(g)(\psi_k) = \psi_k \circ G_A$, whereas $\varrho(\sigma)(\psi_k) = \psi_k \circ \overline{G_A}$ (see (25)). Then another long but straightforward calculation using (35) and (36) concludes the proof.

We are now ready to give the

**Proof of Theorem 4.13**. Since any surface $S$ of type II is a double cover $f: S \rightarrow A$, branched over $D_A$, by Proposition 4.3 it follows that the number of surfaces of type II up to isomorphisms (respectively, up to holomorphic and anti-holomorphic diffeomorphisms) equals the number of orbits for the permutation action of $\text{Aut}(A, D_A)$ (respectively, of $\text{Kl}(A, D_A)$) on the set $\{L_1, \ldots, L_{16}\}$. We have seen that such an action is determined by the permutation action on the set of sixteen semicharacters $\{\psi_1, \ldots, \psi_{16}\}$, so we only have to compute the number of orbits for the subgroup of $S_{16}$ whose generators are described in Proposition 4.15. This can be done by hand, but it is easier to write a short script using the Computer Algebra System GAP4 (**GAP16**):

```bash
g2:=(1, 13, 7, 12)(2, 9, 14, 16)(3, 5, 15, 6)(4, 11, 8, 10);;
g3:=(1, 13, 5, 7, 12, 6)(2, 4, 11, 14, 8, 10)(3, 15)(9, 16);;
sigma:=(1, 14)(2, 7)(3, 16)(4, 5)(6, 8)(9, 15)(10, 12)(11, 13);;
Aut:=Group(g2, g3);;
Kl:=Group(g2, g3, sigma);;
StructureDescription(Aut) = "SL(2,3)";
OrbitsPerms(Aut, [ 1 .. 16 ] ) =
[ [ 1, 7, 12, 13, 3, 15, 5, 6 ], [ 2, 14, 16, 9, 10, 11, 4, 8 ] ];
StructureDescription(Kl) = "GL(2,3)";
OrbitsPerms(Kl, [ 1 .. 16 ] ) =
[ [ 1, 7, 12, 13, 15, 3, 6, 5, 14, 2, 10, 11, 9, 16, 8, 4 ] ];
```

The output shows that $\varrho$ induces an embedding of $\text{Aut}_0(A, D_A)$ in $S_{16}$, and that the corresponding permutation subgroup has precisely two orbits. Therefore there are exactly two surfaces $S_1$, $S_2$ of type II, up to isomorphisms. Furthermore, $\varrho$ induces an embedding of $\text{Kl}_0(A, D_A)$ in $S_{16}$, and the corresponding permutation subgroup has only one orbit. This means that there exists an anti-holomorphic diffeomorphism $S_1 \rightarrow S_2$, hence these surfaces are not isomorphic, but they have conjugated complex structure. 

\hfill \Box
Let us finally show that surfaces of type $II$ are not uniformized by the bidisk (unlike surfaces of type $I$, see Corollary 2.7).

**Proposition 4.16.** Let $S$ be a surface of type $II$ and $\tilde{S} \rightarrow S$ its universal cover. Then $\tilde{S}$ is not biholomorphic to $\mathbb{H} \times \mathbb{H}$.

**Proof.** Looking at diagram (2) in Section 1, we see that, in case $II$, the map $\varphi: B \rightarrow A$ is the blow-up of $A$ at the two quadruple points $p_1, p_2$ of the curve $D_A$ and that $\tilde{S} = S$. Moreover, considering $\beta: S \rightarrow B$ we have

$$\beta^* D_B = C_1 + C_2 + C_3 + C_4,$$

where the $C_i$ are (pairwise disjoint) elliptic curves with $C_i^2 = -1$. The embedding $C_i \rightarrow S$, composed with the universal cover $\mathbb{C} \rightarrow C_i$, gives a non-constant holomorphic map $\mathbb{C} \rightarrow S$, that in turn lifts to a non-constant holomorphic map $\mathbb{C} \rightarrow \tilde{S}$. If $\tilde{S}$ were isomorphic to $\mathbb{H} \times \mathbb{H}$, projecting onto one of the two factors we would obtain a non-constant holomorphic map $\mathbb{C} \rightarrow H$, whose existence would contradict Liouville’s theorem because $\mathbb{H}$ is biholomorphic to the bounded domain $D = \{ z \in \mathbb{C} : |z| < 1 \}$. $\square$

### 4.6 Concluding remarks

**Remark 4.17.** In the argument in the proof of Proposition 4.16, we could have used one of the elliptic curves $Z_1, Z_2$ instead of the $C_i$ (see Remark 1.4).

**Remark 4.18.** Denoting by $\chi_{\text{top}}$ the topological Euler number, we have

$$\left( K_S + \sum_{i=1}^{4} C_i \right)^2 = 12 = 3 \chi_{\text{top}} \left( S - \sum_{i=1}^{4} C_i \right)$$

and

$$\left( K_S + \sum_{i=1}^{2} Z_i \right)^2 = 12 = 3 \chi_{\text{top}} \left( S - \sum_{i=1}^{2} Z_i \right).$$

This implies that the open surfaces $S - \sum_{i=1}^{4} C_i$ and $S - \sum_{i=1}^{2} Z_i$ both have the structure of a complex ball-quotient, see [Rou14] for references and further details.

**Remark 4.19.** The two non-isomorphic surfaces of type $II$ exhibit a new occurrence of the so-called $\text{Diff} \Rightarrow \text{Def}$ phenomenon, meaning that their diffeomorphism type does not determine their deformation class. In fact, they are (anti-holomorphically) diffeomorphic, but not deformation equivalent since they are rigid. See [Man01], [KK02], [Cat03], [CW07] for further examples of this situation.

**Remark 4.20.** It is possible to give a different geometric construction of the abelian surfaces $A'$, $A$ and of the divisor $D_A$ as follows. Unfortunately, at present we do not know how to recover the $2$-divisibility of the curve $D_A$ in $\text{Pic}(A)$ by using this alternative approach.

Let $F_1, F_2, F_3$ and $G_1, G_2, G_3$ be general fibres of the two rulings $f, g: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, respectively; then the two reducible divisors $F_1 + F_2 + F_3$ and $G_1 + G_2 + G_3$ meet at nine distinct points. Consider three of these points, say $p_1, p_2, p_3$, with the property that each $F_i$ and each $G_i$ contain exactly one of them. Then there exists precisely one (smooth) curve $C_1$ of bidegree $(1, 1)$ passing through $p_1, p_2, p_3$. Similarly, if we choose three other points $q_1, q_2, q_3 \notin \{ p_1, p_2, p_3 \}$ with the same property, there exist a unique curve $C_2$ of bidegree $(1, 1)$ passing through $q_1, q_2, q_3$. The curves $C_1$ and $C_2$ meet at two points, say $r_1, r_2$, different from the points $p_i$ and $q_i$.

Let us call $F_i$ and $G_i$ the fibres of $f$ and $g$ passing through one of these two points, say $r_1$. Then the reducible curve $B$ of bidegree $(4, 4)$ defined as

$$B = F_1 + \cdots + F_4 + G_1 + \cdots + G_4$$

has sixteen ordinary double points as only singularities, and the double cover $\phi: Q' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ branched over $B$ gives a singular Kummer surface $Q'$; let us write $A'$ for the associate abelian surface. We can easily show that

$$\phi^* C_1 = C_{11} + C_{12}, \quad \phi^* C_2 = C_{21} + C_{22},$$

21
where all the $C_{ij}$ are smooth and irreducible. Moreover we see that $C_{11}$ and $C_{22}$ intersect at exactly one point, which is a node of $Q'$. Writing
\[ \phi^* F_4 = 2 \widehat{F}_4, \quad \phi^* G_4 = 2 \widehat{G}_4, \]
we see that the rational curves $C_{11}, C_{22}, \widehat{F}_4, \widehat{G}_4$ meet at one node of $Q'$ and that each of them contains precisely four nodes of $Q'$. Hence the pullback of these curves via the double cover $A' \to Q'$ yields four elliptic curves in $A'$ intersecting pairwise and transversally at a single point.

Let us choose now $i, j, h, k \in \{1, 2, 3, 4\}$, with $i \neq j$ and $h \neq k$, and consider the eight nodes of $B$ different from the nodes of the curve $H = F_i + F_j + G_h + G_k$. The 2-divisibility of $H$ in $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ implies that the corresponding set $\Xi$ of eight nodes in the Kummer surface $Q'$ is 2-divisible, so we can consider the double cover $Q \to Q'$ branched over $\Xi$. The surface $Q$ is again a singular Kummer surface and, calling $A$ the abelian surface associate with $Q$, we obtain a degree 2 isogeny $A \to A'$. We can choose (in three different ways) $i, j, h, k$ so that each of the four curves $C_{11}, C_{22}, \widehat{F}_4$ and $\widehat{G}_4$ contains exactly two nodes of $\Xi$. Therefore we obtain four rational curves in $Q$, all passing through two of the nodes of $Q$ and containing four nodes each. This in turn gives four elliptic curves in $A$ meeting at two common points and not elsewhere, and the union of these curves is the desired divisor $D_A$.

References

[Ａid] V. Aide. Uniqueness of the elliptic curve configurations of Hirzebruch and Holzapfel. Submitted. (Cited on pages 2, 16.)

[Ati56] M. Atiyah. On the Krull-Schmidt theorem with application to sheaves. Bull. Soc. Math. France, 84:307–317, 1956. (Cited on page 15.)

[BCG05] I. Bauer, F. Catanese, and F. Grunewald. Beauville surfaces without real structures. In Geometric methods in algebra and number theory, volume 235 of Progr. Math., pages 1–42. Birkhäuser Boston, Boston, MA, 2005. (Cited on page 1.)

[BCP06] I. Bauer, F. Catanese, and R. Pignatelli. Complex surfaces of general type: some recent progress. In Global aspects of complex geometry, pages 1–58. Springer, Berlin, 2006. (Cited on page 1.)

[BCP11] I. Bauer, F. Catanese, and R. Pignatelli. Surfaces of general type with geometric genus zero: a survey. In Complex and differential geometry, volume 8 of Springer Proc. Math., pages 1–48. Springer, Heidelberg, 2011. (Cited on page 1.)

[Bea96] A. Beauville. Complex algebraic surfaces, volume 34 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. (Cited on page 2.)

[BHPVdV04] W. Barth, K. Hulek, C. Peters, and A. Van de Ven. Compact complex surfaces, volume 4 of Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, second edition, 2004. (Cited on pages 3, 4, 13, 15.)

[BL04] C. Birkenhake and H. Lange. Complex abelian varieties, volume 302 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2004. (Cited on pages 7, 8, 9, 11.)

[Cat03] F. Catanese. Moduli spaces of surfaces and real structures. Ann. of Math. (2), 158(2):577–592, 2003. (Cited on page 21.)

[Cat13] F. Catanese. A superficial working guide to deformations and moduli. In Handbook of moduli. Vol. I, volume 24 of Adv. Lect. Math. (ALM), pages 161–215. Int. Press, Somerville, MA, 2013. (Cited on pages 3, 16.)
D. Mumford. Tata lectures on theta. II. Jacobian theta functions and differential equations. 43:xiv+272, 1984. With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura. (Cited on page 8.)

M. Penegini. The classification of isotrivially fibred surfaces with $p_g = q = 2$. Collect. Math., 62(3):239–274, 2011. With an appendix by Sönke Rollenske. (Cited on pages 1, 2, 6.)

G. P. Pirola. Surfaces with $p_g = q = 3$. Manuscripta Math., 108(2):163–170, 2002. (Cited on pages 1, 13.)

F. Polizzi. Surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2. Trans. Amer. Math. Soc., 358(2):759–798, 2006. (Cited on page 4.)

F. Polizzi. On surfaces of general type with $p_g = q = 1$ isogenous to a product of curves. Comm. Algebra, 36(6):2023–2053, 2008. (Cited on page 1.)

M. Penegini and F. Polizzi. Surfaces with $p_g = q = 2$, $K^2 = 6$, and Albanese map of degree 2. Canad. J. Math., 65(1):195–221, 2013. (Cited on pages 3, 5.)

R. Pignatelli and F. Polizzi. A family of surfaces with $p_g = q = 2, k^2 = 7$ and Albanese map of degree 3. Mathematische Nachrichten (2016), DOI: 10.1002/mana.201600202, 2016. (Cited on page 7.)

G. Prasad and S.-K. Yeung. Fake projective planes. Invent. Math., 168(2):321–370, 2007. (Cited on page 1.)

C. Rito. Involutions on surfaces with $p_g = q = 1$. Collect. Math., 61(1):81–106, 2010. (Cited on page 3.)

X. Roulleau. Bounded negativity, Miyaoka-Sakai inequality and elliptic curves configuration. arXiv:1411.6996 [math.AG], 2014. To appear on IMRN. (Cited on page 21.)

I. H. Shavel. A class of algebraic surfaces of general type constructed from quaternion algebras. Pacific J. Math., 76(1):221–245, 1978. (Cited on page 4.)

Francesco Polizzi
Dipartimento di Matematica e Informatica, Università della Calabria
Cubo 30B, 87036 Arcavacata di Rende (Cosenza), Italy
polizzi@mat.unical.it

Carlos Rito
Permanent address:
Universidade de Trás-os-Montes e Alto Douro, UTAD
Quinta de Prados
5000-801 Vila Real, Portugal
www.utad.pt, crito@utad.pt
Temporary address:
Departamento de Matemática
Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre 687
4169-007 Porto, Portugal
www.fc.up.pt, crito@fc.up.pt

Xavier Roulleau
Aix-Marseille Université, CNRS, Centrale Marseille,
I2M UMR 7373,
13453 Marseille, France
Xavier.Roulleau@univ-amu.fr