Abstract

We study Zariski-like topologies on a proper class \( X \subseteq L \) of a complete lattice \( L = (L, \wedge, \vee, 0, 1) \). We consider \( X \) with the so called classical Zariski topology \( (X, \tau_{cl}) \) and study its topological properties (e.g. the separation axioms, the connectedness, the compactness) and provide sufficient conditions for it to be spectral. We say that \( L \) is \( X \)-top iff

\[
\tau := \{X \setminus V(a) \mid a \in L\}, \text{ where } V(a) = \{x \in L \mid a \leq x\}
\]

is a topology. We study the interplay between the algebraic properties of an \( X \)-top complete lattice \( \mathcal{L} \) and the topological properties of \( (X, \tau^L) = (X, \tau) \). Our results are applied to several spectra which are proper classes of \( \mathcal{L} := \text{LAT}(R_M) \) where \( M \) is a left module over an arbitrary associative ring \( R \) (e.g. the spectra of prime, coprime, fully prime submodules) of \( M \) as well as to several spectra of the dual complete lattice \( \mathcal{L}^0 \) (e.g. the spectra of first, second and fully coprime submodules of \( M \)).
Introduction

The spectrum $\text{Spec}(R)$ of prime ideals of a commutative ring $R$ attains the so-called Zariski topology in which the closed sets are the varieties

$$\{V(I) \mid I \in \text{Ideal}(R)\}, \text{ where } V(I) = \{P \in \text{Spec}(R) \mid I \subseteq P\}.$$  

This topology is compact, $T_0$ but almost never $T_2$, and the closed points correspond to the maximal ideals. The Zariski topology proved to be very important in two main aspects: in Algebraic Geometry and in Commutative Algebra. In particular, it provided an efficient tool for studying the algebraic properties of a commutative ring $R$ by investigating the corresponding topological properties of $\text{Spec}(R)$ [9].

Motivated by this, there were many attempts to define Zariski-like topologies on the spectra of prime-like submodules of a given left module $M$ over a (not necessarily commutative) ring $R$. This resulted at the first place in several different notions of prime submodules of $RM$ which reduced to the notion of a prime ideal for the special case $M = R$, a commutative ring (e.g. [21]). The work in this direction was almost limited to studying these prime-like submodules and their duals (the coprime-like submodules) as well as to the families of prime ideals corresponding to them from a purely algebraic point of view. One of the obstacles was that not every module $M$ over a (commutative) ring $R$ has the property that $\text{Spec}(M)$ attains a Zariski-like topology: the proposed closed varieties $\{V(N) \mid N \in \text{LAT}(RM)\}$ are not necessarily closed under finite unions. Modules for which this last condition is satisfied were investigated, among others, by R. L. McCasland and P. F. Smith (e.g. [17], [16]) and called top modules. However, even such modules were studied from a purely algebraic point of view and the associated Zariski-like topologies were not well studied till about a decade ago. In [6], Abuhlail introduced a Zariski-like topology on the spectrum of fully coprime submodules of a given comodule $M$ of a coring $C$ over an associative ring $R$ and studied the interplay between the algebraic properties of $M$ and the topological properties of that Zariski-like topology (see also [5]).

Later, in a series of papers ([3], [4], [2]), Abuhlail introduced and investigated several Zariski-like topologies for a module $M$ over an arbitrary associative ring $R$. These investigations showed that all the (co)prime spectra considered fall in two main classes with several common properties for the spectra in each class. Moreover, these two classes were dual to each other in some sense. This led Abuhlail and Lomp ([7], [1]) to investigate such topologies for a general complete lattice $\mathcal{L} := (L, \wedge, \lor, 0, 1)$ and a proper subset $X \subseteq L\{1\}$. Their main work was on characterizing the so-called X-top lattices (i.e. $\mathcal{L}$ for which the closed varieties $V(a) := \{x \in X \mid a \leq x\}$ are closed under finite unions). In addition to the fact that this approach provided a general framework, it allowed obtaining results on the dual lattice $\mathcal{L}^0 := (L, \land^0, \lor^0, 0^0, 1^0) = (L, \lor, \land, 1, 0)$ and $X \subseteq L\{0\}$ for free.

This paper consists of two sections. After providing some basic definitions and preliminaries, we study in Section 1 Zariski-like topologies for complete lattices using a different approach. Fix a complete lattice $\mathcal{L} = (L, \lor, \land, 1, 0)$, a subset $X \subseteq L\{1\}$ and $\tau := \{X \setminus V(a) \mid a \in L\}$. Inspired by the work of Behboodi and Haddadi [10], [11] on the lattice $\text{LAT}(RM)$ of submodules of a given module $M$ over a ring $R$, and instead of restricting our attention to X-top lattices (i.e. those for which $(X, \tau)$ is a topology), we consider $X$ with the classical Zariski topology $(X, \tau^d)$ which
is constructed on $X$ by considering $\tau$ as a subbase and the finer patch topology $(X, \tau^{fp})$ which has a subbase $\mathcal{B} := \{V(a) \cap X \mid a, b \in L\}$. Indeed, $(X, \tau^{fp}) \leq (X, \tau)$ and $(X, \tau) = (X, \tau^{cl})$ if and only if $\mathcal{L}$ is X-top. In the special case when $\mathcal{L}$ is an X-top lattice, we not only apply the results obtained on $(X, \tau^{cl})$, but obtain also other interesting results especially on the interplay between the algebraic properties of $\mathcal{L}$ and the topological properties of $(X, \tau)$.

In Proposition 1.22, we prove a stronger version of the converse of [1, Proposition 2.7] and conclude in Corollary 1.23 that in case $\mathcal{L}$ is an X-top lattice: $A \subseteq X$ is irreducible if and only if $I(A) := \bigwedge_{x \in A} x$ is (strongly) irreducible in the sublattice $(\mathcal{C}(L), \wedge)$ of radical elements of $\mathcal{L}$. This fact was the key in the proofs of several results including Theorem 2.6. It is worth mentioning that Theorem 2.6 recovers several results of Abuhlail on such 1-1 correspondences for $\mathcal{L} = \text{LAT}(R_{M})$ (e.g. [2], [3], [4]) and Abuhlail/Lomp [1] as special cases (some of these results are recovered under conditions weaker than those assumed in the original results for the different spectra of modules).

In Theorem 1.27, we prove that the class $\text{Max}(X)$ of maximal elements of $X$ coincides with the class of $\text{Max}(\mathcal{C}(\mathcal{L}))$ of maximal radical elements. This yields, assuming that $\mathcal{C}(\mathcal{L})$ satisfies the so called complete max property, that $(X, \tau^{cl})$ is discrete if and only if $(X, \tau^{cl})$ is $T_{1}$. This result generalizes [2, Theorem 5.34], [2, Theorem 4.28] and [3, Theorem 3.46].

A topological space $T$ is said to be spectral [15] iff $T$ is homeomorphic to $\text{Spec}(R)$, the prime spectrum of a commutative ring $R$, with the Zariski topology. Hochster [15] characterized such spaces by giving sufficient and necessary conditions on a topological space to be spectral. We observe in Proposition 1.49 that if the finer patch topology $(X, \tau^{fp})$ is compact, then the classical Zariski topology $(X, \tau^{cl})$ is spectral. Sufficient conditions for $(X, \tau^{fp})$ to be compact were provided in Theorems 1.54 and 1.58. Example 1.64 provides several spectra of modules which are shown to be spectral by Theorem 1.54.

In Section 2, we restrict our investigations to X-top lattices $\mathcal{L} = (L, \vee, \wedge, 1, 0)$ where $X \subseteq L \setminus \{1\}$. We investigate the interplay between the algebraic properties of $\mathcal{L}$ and the topological space $(X, \tau) = (X, \tau^{cl})$. Several types of compactness and connectedness of $(X, \tau)$ are studied in Theorem 2.3. For examples of such an interplay.

The results in Section 1 are applied to the complete lattice $\text{LAT}(R_{M}) := (\mathcal{L}(M), \cap, +, 0, M)$ of submodules of a left module $M$ over an associative ring $R$. In a series of examples 2.9 - 2.14, we apply Theorem 2.6 to a number of spectra $X \subseteq \mathcal{L}(M) \setminus M$ (or $X \subseteq \mathcal{L}(M) \setminus \{0\}$).

1 Zariski-like Topologies for Lattices

Lattices

1.1. ([14]) A lattice $\mathcal{L}$ is a poset $(L, \leq)$ closed under two binary commutative associative and idempotent operations $\wedge$ (meet) and $\vee$ (join), and we write $\mathcal{L} = (L, \wedge, \vee)$. We say that a lattice $(L, \wedge, \vee)$ is a complete lattice iff $\bigwedge_{x \in H} x$ and $\bigvee_{x \in H} x$ exist for any $H \subseteq L$. For two lattices $\mathcal{L} = (L, \wedge, \vee)$ and $\mathcal{L}' = (L', \wedge', \vee')$, a homomorphism of lattices from $\mathcal{L}$ to $\mathcal{L}'$ is a map $\varphi : L \longrightarrow L'$
that preserves finite meets and finite joins, i.e.
\[ \varphi(x \land y) = \varphi(x) \land' \varphi(y) \text{ and } \varphi(x \lor y) = \varphi(x) \lor' \varphi(y) \quad \forall x, y \in L. \]

If \( \mathcal{L} = (L, \land, \lor, 0, 1) \) and \( \mathcal{L}' = (L', \land', \lor', 0', 1') \) are complete lattices, then a morphism of complete lattices from \( \mathcal{L} \) to \( \mathcal{L}' \) is a map \( \varphi : L \rightarrow L' \) that preserves arbitrary meets and arbitrary joins.

1.2. Let \( \mathcal{L} = (L, \land, \lor) \) be a lattice. If \( \mathcal{L} \) has a maximum element \( 1 \) and a minimum element \( 0 \), then \( \mathcal{L} \) is called a bounded lattice and we write \( \mathcal{L} = (L, \land, \lor, 0, 1) \). An element \( x \in L \setminus \{1\} \) is called maximal in \( \mathcal{L} \) iff \( y = x \) or \( y = 1 \) whenever \( x \leq y \); dually, an element \( x \in L \setminus \{0\} \) is called minimal iff \( y = x \) or \( y = 0 \) whenever \( y \leq x \). Notice that every complete lattice is bounded. We make the convention that \( \bigwedge_{x \in \emptyset} x = 1 \) and \( \bigvee_{x \in \emptyset} x = 0 \).

1.3. For every lattice \( \mathcal{L} = (L, \land, \lor) \), there is associated the dual lattice \( \mathcal{L}^0 = (L, \land^0, \lor^0) \) where \( \land^0 = \lor \) and \( \lor^0 = \land \). Indeed, if \( \mathcal{L} = (L, \land, \lor) \) is a complete lattice, then the dual lattice \( \mathcal{L}^0 \) is complete. Moreover, if \( \mathcal{L} = (L, \land, \lor, 0, 1) \) is a bounded lattice, then the dual lattice \( \mathcal{L}^0 = (L, \land^0, \lor^0, 0^0, 1^0) \) is bounded with \( 0^0 = 1 \) and \( 1^0 = 0 \).

Example 1.4. Let \( R \) be a ring.

1. \( S = (\text{Ideal}(R), \cap, +, R, 0) \), where \( \text{Ideal}(R) \) is the set of all (two-sided) ideals of \( R \) is a complete lattice.

2. For any left \( R \)-module \( M \), the set \( \text{LAT}(M) = (\mathcal{L}(M), \cap, +, M, 0) \) is a complete lattice where \( \mathcal{L}(M) \) is the class of all \( R \)-submodules of \( M \).

1.5. Let \( \mathcal{L} = (L, \land, \lor, 0, 1) \) be a complete lattice.

1. An element \( x \in L \setminus \{1\} \) is said to be:
   - irreducible \([7]\) iff for any \( a, b \in L \) with \( a \land b = x \), we have \( a = x \) or \( b = x \);
   - strongly irreducible \([7]\) iff for any \( a, b \in L \) with \( a \land b \leq x \), we have \( a \leq x \) or \( b \leq x \).

   We denote the set of strongly irreducible elements in \( L \) by \( \text{SI}(\mathcal{L}) \).

2. An element \( x \in L \setminus \{0\} \) is said to be:
   - hollow iff whenever for any \( a, b \in L \) with \( x = a \lor b \), we have \( x = a \) or \( x = b \);
   - strongly hollow \([7]\) iff for any \( a, b \in L \) with \( x \leq a \lor b \), we have \( x \leq a \) or \( x \leq b \).

   We denote the set of strongly hollow elements in \( L \) by \( \text{SH}(\mathcal{L}) \).

3. We say that \( \mathcal{L} \) is
   - a hollow lattice iff \( 1 \) is hollow (\( i.e. \) for any two elements \( x, y \in L \setminus \{1\} \) we have \( x \lor y \neq 1 \));
   - a uniform lattice iff \( 0 \) is uniform (\( i.e. \) for any two elements \( x, y \in L \setminus \{0\} \) we have \( x \land y \neq 0 \)).
1.1 X-top Lattices

From now on, we assume that $\mathcal{L} = (L, \land, \lor, 0, 1)$ is a complete lattice.

1.6. Let $X \subseteq L \setminus \{1\}$. For $a \in L$, we define the variety of $a$ as $V(a) := \{ p \in X \mid a \leq p \}$ and set $V(\mathcal{L}) := \{ V(a) \mid a \in L \}$. Indeed, $V(\mathcal{L})$ is closed under arbitrary intersections (in fact, $\bigcap_{a \in A} V(a) = V(\bigvee_{a \in A}(a))$ for any $A \subseteq L$). The lattice $\mathcal{L}$ is called $X$-top (or a topological lattice) iff $V(\mathcal{L})$ is closed under finite unions. The lattice $\mathcal{L}$ is called strongly $X$-top iff $X \subseteq SI(\mathcal{L})$.

1.7. Let $X \subseteq L \setminus \{1\}$. For any $Y \subseteq X$, we set $I(Y) := \bigwedge_{p \in Y} p$ and $\sqrt{a} := I(V(a))$. We say that $a$ is an $X$-radical element iff $\sqrt{a} = a$. The set of $X$-radical elements of $L$ is

$$\mathcal{C}^X(L) := \{ a \in L \mid \sqrt{a} = a \}.$$

When $X$ is clear from the context, we drop it from the above notation. Notice that $\mathcal{C}(\mathcal{L}) = (\mathcal{C}(L), \land, \lor, \sqrt{0}, 1)$ is a complete lattice, where $\sqrt{Y} := IV(\bigvee(Y))$ for any $Y \subseteq \mathcal{C}(L)$, i.e. $\sqrt{\bigvee_{x \in Y} x} = \bigvee_{x \in Y} \sqrt{x}$. It was proved in [1, Theorem 2.2] that $\mathcal{L}$ is an $X$-top lattice if and only if the map

$$V : (\mathcal{C}(L), \land, \lor, 1, \sqrt{0}) \longrightarrow (\mathcal{P}(X), \cap, \cup, X, \emptyset), \ a \mapsto V(a)$$

is an anti-homomorphism of lattices, that is

$$V(a \land b) = V(a) \cup V(b) \text{ and } V(a \lor b) = V(a) \cap V(b) \text{ for all } a, b \in \mathcal{C}(L).$$

The following lemma appeared in [1] except for (2) which is clear.

**Lemma 1.8.** Let $X \subseteq L \setminus \{1\}$. For any $x, y \in L$ and $A, B \subseteq L$ we have:

1. $A \subseteq B \Rightarrow I(B) \leq I(A)$.
2. $V(x) \subseteq V(y) \Leftrightarrow \sqrt{y} \leq \sqrt{x}$. It follows that $V(x) = V(y) \Leftrightarrow \sqrt{y} = \sqrt{x}$.
3. $V(\sqrt{x}) = V(x)$.
4. $\bigcap_{x \in A} V(x) = V(\bigvee_{x \in A} x)$.
5. $I \circ V \circ I = I$.
6. $V \circ I \circ V = V$.
7. $\mathcal{L}$ is $X$-top $\iff$ $V(x) \cup V(y) = V(x \land y)$ for any $x, y \in \mathcal{C}(L)$.

1.9. Let $X \subseteq L \setminus \{1\}$ and set $\tau := \{ X \setminus V(a) \mid a \in L \}$. We define $\tau^cl$ to be the topology constructed on $X$ by taking $\tau$ as a subbase, that is $\tau^cl$ is the set of all arbitrary unions of finite intersections of elements in $\tau$, and is called the classical Zariski topology on $X$. Moreover, $\mathcal{L}$ is $X$-top (i.e. $\tau$ is closed under finite intersections) if and only if $\tau^cl = \tau$. 

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1.10. Let $Y \subseteq L \setminus \{0\}$. For any $a \in L$, we define the dual variety $V^0(a) := \{q \in Y \mid q \leq a\}$ and set $V^0(L) = \{V^0(a) \mid a \in L\}$. We say that $L$ is dual $Y$-top if the dual lattice $L^0$ is a $Y$-top lattice. For any subset $A \subseteq Y$, we set $H(A) := \bigvee_{q \in A} q$; also we set $\sqrt{a}^0 := H(V^0(a))$, and $\mathcal{H}(L^0) = \mathcal{G}^0(L^0)$. The dual classical Zariski topology $\tau^{dcl}$ on $Y$ is constructed by taking $\tau^0 := \{Y \setminus V^0(a) \mid a \in L\}$ as a subbase for this topology. With this process, one can dualize the results obtained in this section for the (classical) Zariski-topology to results on the dual (classical) Zariski topology.

The following lemma recovers [2, 5.14 and 4.10], [3, 3.23] and [4, 3.21].

**Lemma 1.11.** Let $X \subseteq L \setminus \{1\}$ and assume that $L$ is an $X$-top lattice. The closure of any $Y \subseteq X$ is given by $\overline{Y} = V(I(Y))$.

**Proof.** Let $Y \subseteq X$. Notice that $\overline{Y} = V(a)$ for some $a \in L$, whence $a \leq \bigwedge_{p \in Y} p = I(Y)$ and so $V(I(Y)) \subseteq V(a) = \overline{Y}$. On the other hand, $Y \subseteq V(I(Y))$ and so $\overline{Y} \subseteq V(I(Y))$.

$\blacksquare$

1.12. A non-empty topological space $(T, \tau)$ is said to be:

1. **connected** iff $T$ is not the union of two disjoint non-empty open subsets (equivalently, $T$ is not the union of two disjoint non-empty closed sets).

2. **hyperconnected** (or **irreducible** [12]) iff no two non-empty open sets in $T$ are disjoint (equivalently, $T$ is not the union of two closed subsets).

3. **ultraconnected** [12] iff no two non-empty closed sets in $T$ are disjoint.

1.13. Let $(T, \tau)$ be a topological space. A subset $A \subseteq T$ is called **hyperconnected** [12] (or **irreducible**) iff $A$ is so when considered as a topological space w.r.t. the relative topology induced from $(T, \tau)$ (equivalently, $A$ is non-empty and for any two closed subsets $F_1, F_2$ in $T$ with $A \subseteq F_1 \cup F_2$, we have $A \subseteq F_1$ or $A \subseteq F_2$). The empty set is not considered to be irreducible. A closed subset $F \subseteq T$ is said to have a **generic point** $g \in T$ [12] iff $\{g\} = F$. The topological space $(T, \tau)$ is called **sober** if every closed irreducible subset of $T$ has a unique generic point.

1.14. A subset $A \subseteq T$ is irreducible if and only if the closure $\overline{A}$ is irreducible. An **irreducible component** [12] is an irreducible subset of $X$ which is not a proper subset of any irreducible subset of $T$ (hence an irreducible component of $T$ is indeed a closed subset).

The following result generalizes [10, 3.2 and 3.3].

**Proposition 1.15.** Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^{cl})$.

1. For each $p \in X$, we have $\overline{\{p\}} = V(p)$.

2. $V(p)$ is irreducible $\forall p \in X$. 

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3. If \( Y \subseteq X \) is closed, then \( Y = \bigcup_{p \in Y} V(p) \).

4. If \( \mathcal{L} \) is \( X \)-top, then for any closed subset \( Y \subseteq X \) we have \( Y = \bigcup_{p \in Y} V(p) = V(\bigwedge_{p \in Y} p) \).

**Proof.** Consider \( (X, \tau^l) \).

1. Observe that \( V(p) \) is closed in \( (X, \tau^l) \), whence \( \{p\} \subseteq V(p) \). On the other hand, suppose that \( \{p\} = \bigcap_{i \in I} \left( \bigcup_{j=1}^{n_i} (V(x_{ij})) \right) \), where \( x_{ij} \in L \). Since \( p \in \bigcup_{j=1}^{n_i} (V(x_{ij})) \) \( \forall i \in I \), it follows that \( V(p) \subseteq \bigcup_{j=1}^{n_i} (V(x_{ij})) \) \( \forall i \in I \). Therefore, \( V(p) \subseteq \{p\} \). Clearly, \( \{p\} \subseteq V(p) \), whence \( \{p\} = V(p) \).

2. Notice that \( \{p\} \) is irreducible, whence \( V(p) = \{p\} \) is irreducible.

3. Clear.

4. Let \( Y \subseteq X \) be closed. It follows from (3) that \( Y = \bigcup_{p \in Y} V(p) \subseteq V(\bigwedge_{p \in Y} p) \). Since \( \mathcal{L} \) is assumed to be \( X \)-top, \( Y = V(x) \) for some \( x \in L \) and so \( x \leq \bigwedge_{p \in Y} p \), whence \( V(\bigwedge_{p \in Y} p) \subseteq V(x) = Y \). Consequently, \( Y = \bigcup_{p \in Y} V(p) = V(\bigwedge_{p \in Y} p) \).

\[ \blacksquare \]

**Example 1.16.** Consider the complete ideal lattice \( \mathcal{L} = (\text{Ideal}(\mathbb{Z}), \cap, +, \mathbb{Z}, 0) \). Consider \( X = \text{Spec}^p(\mathbb{Z}) \), the prime spectrum of \( \mathbb{Z} \). It is clear that \( (X, \tau) \) is a topological space (the usual Zariski topology on the spectrum of the commutative ring \( \mathbb{Z} \)). Notice that for \( Y := \text{Spec}^p(\mathbb{Z}) \setminus \{0\} \), we have \( \overline{Y} = V(I(Y)) = V(\bigcap_{p \in Y} P) = V(0) = X \neq \bigcup_{p \in Y} V(p) \). This example shows that [10, Proposition 3.1] fails to hold even for domains. However, the proof of Proposition 1.15 provides a correct proof [10, Corollary 2.3] without using [10, Proposition 3.1].

The following result recovers [10, Proposition 3.8], [3, Proposition 3.24 (1)] and [2, Proposition 5.15 (i)].

**Proposition 1.17.** Let \( X \subseteq L \setminus \{1\} \) and consider \( (X, \tau^l) \).

1. \( X \) is \( T_0 \).

2. Every finite closed irreducible subset of \( X \) has a unique generic point. If \( X \) is finite, then \( X \) is sober.

**Proof.** 1. Let \( p_1, p_2 \in X \) be such that \( \{p_1\} = \{p_2\} \), whence \( V(p_1) = V(p_2) \) and it follows that \( p_1 = p_2 \), which proves that \( X \) is \( T_0 \) (notice that, in general, \( (X, \tau) \) is \( T_0 \) if and only if \( \{p_1\} = \{p_2\} \leftrightarrow p_1 = p_2 \)).
2. In general, if \((X, \tau)\) is \(T_0\), then every finite irreducible closed subset has a unique generic point. To see this, suppose that \(F\) is a closed irreducible finite set that has no generic point. Pick \(x_1 \in F\), whence \(\{x_1\} \neq F\) and so there is \(x_2 \in F \setminus \{x_1\}\). Observe that \(\{x_1\} \cup \{x_2\} \neq F\) as \(F\) is irreducible. So, there is \(x_3 \in F \setminus (\{x_1\} \cup \{x_2\})\). By continuing this process, we conclude that \(F\) is infinite, a contradiction. The uniqueness of the generic point follows directly from the fact that \(T_0\).

\[\]

The following observation generalizes [10, Proposition 2.3].

**Remark 1.18.** Let \(X \subseteq L \setminus \{1\}\). The following are equivalent for \((X, \tau^\ell)\):

1. \(\mathcal{L} = \mathcal{C}(\mathcal{L}')\).

2. For all \(x_1, x_2 \in L\) with \(V(x_1) = V(x_2)\), we have \(x_1 = x_2\).

**Proof.** (1 \(\Rightarrow\) 2) Suppose \(V(x_1) = V(x_2)\) for some \(x_1, x_2 \in L\). It follows that \(x_1 \leq p, \forall p \in V(x_2)\) whence \(x_1 \leq \sqrt{x_2} = x_2\). Similarly, \(x_2 \leq x_1\).

(2 \(\Rightarrow\) 1) \(\forall x \in L\) we have \(V(x) = V(\sqrt{x})\), whence \(x = \sqrt{x}\).

\[\]

**1.19.** Let \(X \subseteq L \setminus \{1\}\) and denote by \(\text{Min}(X)\) the set of minimal elements of \(X\) and by \(\text{Max}(X)\) the set of maximal elements of \(X\). We say that \(X\) is

atomic iff for every \(p \in X\) there is \(q \in \text{Min}(X)\) such that \(q \leq p\);

coatomic iff for every element \(p \in X\) there is \(q \in \text{Max}(X)\) such that \(p \leq q\).

**Remarks 1.20.** Let \(X \subseteq L \setminus \{1\}\) and consider \((X, \tau^\ell)\).

1. If \(X\) satisfies the DCC, then \(X\) is atomic.

2. If \(X\) is atomic, then there is a subset \(A \subseteq X\) such that \(X = \bigcup_{p \in A} V(p)\) with \(V(p)\) and \(V(q)\) are not comparable for any \(p \neq q \in A\) (e.g. take \(A = \text{Min}(X)\)).

3. Let \(X\) be atomic and \(\text{Min}(X)\) finite. Then \(X\) is irreducible if and only if \(\text{Min}(X)\) is a singleton. To see this, observe that \(X = \bigcup_{p \in \text{Min}(X)} V(p)\) with \(p \not\leq q\) for any \(p \neq q\) are in \(\text{Min}(X)\). Clearly, \(X\) is irreducible if and only if \(\text{Min}(X)\) is a singleton.

**Remarks 1.21.** Let \(X \subseteq L \setminus \{1\}\) with \(0 \in X\) and consider \((X, \tau^\ell)\).

1. If \(F \subseteq X\) is closed and \(0 \in F\), then \(F = X\). To prove this, observe that \(X = V(0) = \{0\} \subseteq F\).

2. Every non-empty open subset of \(X\) contains \(0\). To see this, let \(O \subseteq X\) be open. If \(0 \notin O\), then \(0 \in F := X \setminus O\). By (1), \(X \setminus O = X\); i.e. \(O = \emptyset\).

3. \(X\) is irreducible since \(\text{Min}(X) = \{0\}\), a singleton (see Remark 1.20 (3)).
It was proved in [1, Proposition 2.7], that if \( L \) is an X-top lattice and \( A \subseteq X \) is such that \( I(A) \) is irreducible in \((\mathcal{C}(L), \wedge)\), then \( A \) is irreducible in \((X, \tau)\). The following result proves a stronger version of the converse.

**Proposition 1.22.** Let \( X \subseteq L \setminus \{1\} \) and assume that \( L \) is an X-top lattice. If \( A \subseteq X \) is irreducible, then \( I(A) \) is strongly irreducible in \((\mathcal{C}(L), \wedge)\).

**Proof.** Suppose that \( a \wedge b \leq I(A) \) for some \( a, b \in \mathcal{C}(L) \). Now, \( \overline{A} = V(I(A)) \subseteq V(a \wedge b) \) \cite[Theorem 2.2]{1} = \( V(a) \cup V(b) \). Since \( A \) is irreducible, \( \overline{A} \) is also irreducible, whence \( \overline{A} \subseteq V(a) \) or \( \overline{A} \subseteq V(b) \). So, \( a = I(V(a)) \leq I(\overline{A}) = I(V(I(A))) = I(A) \) or \( b = I(V(b)) \leq I(\overline{A}) = I(V(I(A))) = I(A) \).

\[ \Box \]

**Corollary 1.23.** Let \( X \subseteq L \setminus \{1\} \) and assume that \( L \) is an X-top lattice. The following conditions are equivalent for \( A \subseteq X \):

1. \( A \) is irreducible;
2. \( I(A) \) is strongly irreducible in \((\mathcal{C}(L), \wedge)\);
3. \( I(A) \) is irreducible in \((\mathcal{C}(L), \wedge)\).

**1.24.** A maximal element in \( L \) is a maximal element in the poset \((L \setminus \{1\}, \leq)\). An element \( x \in L \) is called minimal in \( L \) iff \( x \) is maximal in \( L^0 \). We denote by \( \text{Max}(L) \) (resp. \( \text{Min}(L) \)) the set of all maximal (resp. minimal) elements in \( L \). The lattice \( L \) is called coatomic iff for every element \( x \in L \setminus \{1\} \), there exists \( y \in \text{Max}(L) \) such that \( x \leq y \). Dually, \( L \) is called atomic iff for every element \( x \in L \setminus \{0\} \), there exists \( y \in \text{Min}(L) \) such that \( y \leq x \).

Let \( A \subseteq L \). The lattice \( L \) is said to have the complete \( A \)-property iff \( \bigwedge_{p \in A \setminus \{q\}} p \nleq q \) for any \( q \in A \). The lattice \( L \) is said to have the complete max property iff \( L \) has the complete \( \text{Max}(L) \)-property.

**Lemma 1.25.** Let \( L \) be an X-top lattice. If \( L \) is coatomic and \( \text{Max}(L) \subseteq X \), then \( \text{Max}(L) = \text{Max}(X) \).

**Proof.** Let \( p \in \text{Max}(X) \). Since \( L \) is coatomic, there is \( y \in \text{Max}(L) \) such that \( p \leq y \) and so \( p = y \) as \( \text{Max}(L) \subseteq X \).

\[ \Box \]

The following result recovers and generalizes [3, Proposition 3.45], [2, Propositions 5.33, 4.27], and [4, Proposition 3.40]. The additional conditions assumed in these results imply that \( \text{Max}(L) = \text{Max}(X) \) (or \( \text{Min}(L) = \text{Min}(X) \) in the dual cases).

**Proposition 1.26.** Let \( X \subseteq L \setminus \{1\} \). The following are equivalent for \((X, \tau^d)\):

1. \( X \) is \( T_1 \);
2. \( \text{Max}(X) = X = \text{Min}(X) \).

**Proof.** \( X \) is \( T_1 \Leftrightarrow \) every singleton is closed \( \Leftrightarrow \{p\} = \overline{\{p\}} = V(p) \forall p \in X \Leftrightarrow \text{Max}(X) = X = \text{Min}(X) \).

\[
\blacksquare
\]

**Theorem 1.27.** Let \( X \subseteq L \setminus \{1\} \) and consider \((X, \tau^\text{cl})\). Then \( \text{Max}(X) = \text{Max}(C(L)) \). Moreover, the following conditions are equivalent:

1. \( X \) is \( T_1 \) and \( C(L) \) satisfies the complete max property;
2. \( X \) is discrete.

**Proof.** Let \( p \in \text{Max}(X) \). Then \( p \in C(L) \) and so \( p \in \text{Max}(C(L)) \); otherwise, there is \( x \in C(L) \setminus \{1\} \) such that \( p \leq x \). Since \( x \neq 1 \), there is \( q \in X \) such that \( x \leq q \) and so \( p \leq q \) (a contradiction). For the reverse inclusion, let \( x \in \text{Max}(C(L)) \). Notice that \( x = \bigwedge_{p \in A} p \) for some \( \emptyset \neq A \subseteq X \).

Since \( A \subseteq C(L) \), it follows by the maximality of \( x \) in \( C(L) \) that \( x = \bigwedge_{p \in A} p = q \) for some \( q \in A \), i.e. \( A \) is singleton and \( x \in X \). Moreover, \( x \in \text{Max}(X) \) as \( X \subseteq C(L) \).

\((1) \Rightarrow (2)\) : Assume that \( C(L) \) satisfies the complete max property. Since \( \text{Max}(X) = \text{Max}(C(L)) \), we have \( \bigwedge_{p \in \text{Max}(X) \setminus \{q\}} p \not\leq q \) for any \( q \in \text{Max}(X) \). Notice that for any \( q \in X \), we have \( X = V(\bigwedge_{p \in \text{Max}(X) \setminus \{q\}} p) \cup \{q\} \) and by our assumption \( q \notin V(\bigwedge_{p \in X \setminus \{q\}} p) \). Hence, every singleton in \( X \) is open, that is \((X, \tau^\text{cl})\) is discrete.

\((2) \Rightarrow (1)\) : Assume that \( X \) is discrete and show that \( C(L) \) satisfies the complete max property. To show this, suppose that \( q \in X \) and let \( Y = X \setminus \{q\} \). Observe that

\[
Y = \overline{Y} = V(I(Y))
\]

as \( \{q\} \) is open. Hence, \( I(Y) \not\leq q \), which completes the proof as \( X = \text{Max}(X) = \text{Max}(C(L)) \).

\[
\blacksquare
\]

The following result generalizes [2, Theorem 5.34], [2, Theorem 4.28] and [3, Theorem 3.46].

**Corollary 1.28.** Let \( X \subseteq L \setminus \{1\} \) and consider \((X, \tau^\text{cl})\). If \( C(L) \) satisfies the complete max property, then the following conditions are equivalent:

1. \( \text{Max}(X) = X = \text{Min}(X) \);
2. \( X \) is \( T_2 \);
3. \( X \) is \( T_1 \);
Corollary 1.29. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^d)$. Assume that $\mathcal{L}$ satisfies the complete max property, $\mathcal{L}$ is coatomic and $\text{Max}(\mathcal{L}) \subseteq \mathcal{C}(L)$. The following are equivalent:

1. $\text{Max}(X) = X = \text{Min}(X)$;
2. $X$ is $T_2$;
3. $X$ is $T_1$;
4. $X$ is discrete.

Proof. Let $p \in \text{Max}(\mathcal{C}(L))$. Since $\mathcal{L}$ is coatomic, there exists $q \in \text{Max}(\mathcal{L})$ such that $p \leq q$. By assumption, $\text{Max}(\mathcal{L}) \subseteq \mathcal{C}(L)$ whence $p = q$. So, $\text{Max}(\mathcal{L}) = \text{Max}(\mathcal{C}(\mathcal{L}))$ and the result follows by Corollary Theorem 3.16.

A topological space is regular [18] iff any non-empty closed set $F$ and any point $x$ that does not belong to $F$ can be separated by disjoint open neighborhoods. A $T_3$ space is one which is both $T_1$ and regular. In general, regular spaces need not be Hausdorff. However, we have a special situation.

Remark 1.30. If $(X, \tau^d)$ is regular, then $(X, \tau^d)$ is $T_3$. To see this, assume that $X$ is regular and let $p \neq q$ be elements in $X$. Assume, without loss of generality, that $p \notin q$ so that $q \notin V(p)$. Since $X$ is regular, there are two disjoint open sets $O_1$ and $O_2$ in $X$ such that $q \in O_1$ and $V(p) \subseteq O_2$. Therefore, $X$ is $T_2$.

A topological space $X$ is normal [18] iff any two disjoint closed sets of $X$ can be separated by disjoint open neighborhoods. The following example shows that the normality of $(X, \tau^d)$ does not guarantee that it is regular.

Example 1.31. Let $R$ be a local ring with $|\text{Spec}(R)| \geq 2$. Then $\text{Spec}(R)$ is normal because it has no disjoint non-empty closed sets. However, $\text{Spec}(R)$ is not $T_1$ whence not regular by Remark 1.30. To see this, notice that the assumption $|\text{Spec}(R)| \geq 2$ implies that there is a prime ideal $p$ of $R$ and a maximal ideal $q$ of $R$ such that $p \subseteq q$. Hence, every open set containing $q$ contains $p$ as well.

1.2 Examples

Through the rest of this section, $R$ is an associative ring, $M$ is a non-zero left $R$-module and $\text{LAT}(M) := (\mathcal{L}(M), \cap, +, M, 0)$ the complete lattice of $R$-submodules of $M$. Moreover, we denote by $\text{Max}(M)$ (resp. $\mathcal{I}(M)$) the possibly empty set of maximal (resp. simple) $R$-submodules of $M$. By $L \leq M$, we mean that $L$ is an $R$-submodule of $M$. With abuse of notation, we mean by $I \leq R$ that $I$ is a (two sided) ideal of $R$. 

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1.32. Let $M$ be a left $R$-module. We call an $R$-submodule $K \subseteq M$: prime \cite{13} iff $K \neq M$ and for any $N \subseteq M$ and $I \subseteq R$, we have

$$IN \subseteq K \Rightarrow N \subseteq K \text{ or } IM \subseteq K.$$ 

first \cite{1} iff $K \neq 0$ and for any $N \subseteq K$ and $I \subseteq R$, we have

$$IN = 0 \Rightarrow N = 0 \text{ or } IK = 0.$$

coprime \cite{2} iff $K \neq M$ and for any $I \subseteq R$, we have

$$IM + K = M \text{ or } IM \subseteq K.$$ 

second \cite{2} iff $K \neq 0$ and for any $I \subseteq R$ we have

$$IK = K \text{ or } IK = 0.$$ 

By $Spec^p(M)$ (resp. $Spec^f(M)$, $Spec^c(M)$, $Spec^s(M)$) we denote the spectrum of prime (resp. first, coprime, second) $R$-submodules of $M$.

1.33. An $R$-submodule $K$ of $M$ is said to be fully invariant \cite{3} (and we write $L \leq^{f.i.} M$) iff $f(L) \subseteq L$ for all $f \in S := End(M)$. If every submodule of $M$ is fully invariant, then $M$ is said to be a duo module \cite{3}. For any $L_1, L_2 \subseteq M$, we define

$$L_1 \ast L_2 = \sum_{f \in \text{Hom}(M, L_2)} f(L_1) \quad \text{and} \quad L_1 \odot_M L_2 = \bigcap_{f \in S, f(L_1) = 0} f^{-1}(L_2);$$

see \cite{3} and \cite{4}. Notice that if $L_1 \leq^{f.i.} M$, then $L_1 \ast L_2 \subseteq L_1 \cap L_2$.

1.34. A fully invariant submodule $K \leq^{f.i.} M$ is: fully prime in $M$ \cite{3} iff $K \neq M$ and for any $L_1, L_2 \leq^{f.i.} M$, we have

$$L_1 \ast L_2 \subseteq K \Rightarrow L_1 \subseteq K \text{ or } L_2 \subseteq K.$$ 

fully coprime in $M$ \cite{4} iff $K \neq 0$ and for any $L_1, L_2 \leq^{f.i.} M$ we have

$$K \subseteq L_1 \odot_M L_2 \Rightarrow K \subseteq L_1 \text{ or } K \subseteq L_2.$$ 

By $Spec^{f.p}(M)$ (resp. $Spec^{f.c.}(M)$) we denote the spectrum of fully prime (resp. fully coprime) $R$-submodules of $M$.

The following example summarizes some facts about some Zariski-like topologies on several spectra of submodules of a given module.

Example 1.35. Consider $X_1 := Spec^p(M), X_2 := Spec^f(M), X_3 := Spec^{f.p}(M), X_4 := Spec^s(M), X_5 := Spec^f(M)$ and $X_6 := Spec^{f.c.}(M)$. Notice that $X_1, X_2, X_3 \subseteq \mathcal{L}(M) \setminus \{M\}$ and so one can construct the classical Zariski topology $\tau_{cl}$ on any of them as we did for general complete lattices $\mathcal{L} = (L, \wedge, \vee, 1, 0)$ and $X \subseteq L \setminus \{1\}$. On the other hand, one can construct dual classical Zariski topologies on $\mathcal{L}$ only any of $X_4, X_5, X_6 \subseteq \mathcal{E}(M) \setminus \{0\}$. Moreover, $M$ is top$^p$-module (resp. a top$^c$-module, a top$^{f.p}$-module) if and only if LAT $(M)$ is $X_1$-top (resp. $X_2$-top, $X_3$-top). On the other hand, $M$ is a top$^s$-module (resp. a top$^f$-module, a top$^{f.c}$-module) iff LAT $(M)$ is dual $X_4$-top (resp. dual $X_5$-top, dual $X_6$-top). The following table summarize some facts about these spaces.
| Type                  | $M \not\in \text{Spec}^{-}(M)$ | $0 \not\in \text{Spec}^{-}(M)$ |
|----------------------|---------------------------------|---------------------------------|
| Subsets of $L$       | $X_1 = \text{Spec}^p(M)$, $X_2 = \text{Spec}^c(M)$, $X_3 = \text{Spec}^{fp}(M)$ | $X_4 = \text{Spec}^f(M)$, $X_5 = \text{Spec}^s(M)$, $X_6 = \text{Spec}^{fc}(M)$ |
| Variety $V^{-}(N)$   | $\{P \in \text{Spec}^{-}(M) \mid N \leq P\}$ | $\{P \in \text{Spec}^{-}(M) \mid P \leq N\}$ |
| Subbase $\tau_-$     | $X \setminus V^{-}(N) \mid N \leq M$ | $X \setminus V^{-}(N) \mid N \leq M$ |
| $\tau^{cl}$ or $\tau^{dcl}$ | $\tau^{cl}$: the topology generated by $\tau_-$ | $\tau^{dcl}$: the topology generated by $\tau_-$ |
| (Dual) X-top         | $\mathcal{L}$ is $X_i$-top $\iff \tau_- = \tau^{cl}$ | $\mathcal{L}^0$ is $X_j$-top $\iff \tau_- = \tau^{dcl}$ |

Table 1: Examples of spectra of submodules of a given module

**Example 1.36.** Let $M$ be a local left module over an arbitrary ring $R$, i.e. $M$ has a unique maximum proper submodule (e.g. the $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, $p$ is any prime and $k$ is any positive integer). Consider $X_1 = \text{Spec}^p(M)$ and $X_2 = \text{Spec}^c(M)$. Then $\mathcal{C}^{X_1}(\text{LAT}(M))$ and $\mathcal{C}^{X_2}(\text{LAT}(M))$ satisfy the complete max property (notice that any maximal submodule is prime and coprime).

**Example 1.37.** Consider $\mathcal{L} := \text{LAT}(M)$, $X_1 = \text{Spec}^p(M)$ and $X_2 = \text{Spec}^c(M)$. Every maximal submodule of $M$ is a prime and a coprime submodule, i.e. $\text{Max}(M) \subseteq X_1$ and $\text{Max}(M) \subseteq X_2$. So, it is enough to assume that $R_M$ is a coatomic module satisfying the complete max property to satisfy the equivalent conditions of Corollary 1.29. Moreover, Corollary 1.28 applies if $\text{RAD}^p(M) := \mathcal{C}^{X_1}(\mathcal{L})$ (resp. $\text{RAD}^c(M) := \mathcal{C}^{X_2}(\mathcal{L})$) satisfies the complete max property as a lattice.

**Example 1.38.** Consider $\mathcal{L} := \text{LAT}(M)$, $X_4 = \text{Spec}^f(M)$ and $X_5 = \text{Spec}^s(M)$. Every simple submodule of $M$ is a second and a first submodule of $M$, i.e. $\mathcal{S}(M) \subseteq X_4$ and $\mathcal{S}(M) \subseteq X_5$. So, it is enough to assume that $R_M$ is an atomic module with the complete min property to satisfy the equivalent conditions of Corollary 1.29 applied to $\mathcal{L}^0$. Moreover, Corollary 1.28 applies if $\mathcal{C}^{X_4}(\mathcal{L}^0)$ (resp. $\mathcal{C}^{X_5}(\mathcal{L}^0)$) satisfies the complete min property as a lattice.

**Remark 1.39.** It was proved in [2], that if $R_M$ is a coatomic $top^c$-module satisfying the complete max property, then

$$\text{Spec}^c(M) = \text{Max}(M) \iff X \text{ is } T_2 \iff X \text{ is } T_1 \iff X \text{ is discrete.}$$

A similar result was proved for $\text{Spec}^{fp}(M)$ assuming that $R_M$ is a self projective coatomic duo module ($S – PCD$). Notice that it was proved in [3, Remark 3.12] that if $R_M$ is self projective and
duo, then every maximal submodule is fully prime. Other conditions were assumed on $M$ in the dual cases to ensure that $\mathcal{S}(M) = \text{Min}(X)$. So, Corollary 1.28 generalizes all the corresponding results in [3] and [2].

1.3 Spectral Spaces

As before, $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a complete lattice.

1.40. A topological space is said to be spectral [15] iff it is homeomorphic to $\text{Spec}(R)$, the prime spectrum of a commutative ring $R$ with the Zariski topology. Hochster [15, Proposition 4] characterized such spaces. A topological space $(X, \tau)$ is spectral if and only if all of the following four conditions are satisfied:

1. $X$ is compact;
2. $X$ has a basis of compact open sets closed under finite intersections;
3. $X$ is sober.

Remark 1.41. Let $X \subseteq L \setminus \{1\}$. If $X$ is finite, then $(X, \tau^c)$ is spectral: By Proposition 1.17, $X$ is $T_0$ and sober. The remaining Hochster’s conditions in 1.40 follow directly from the finiteness of $X$.

Definition 1.42. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^c)$. Set

$$\mathcal{R}(\mathcal{L}) := \{ \sqrt{x} \mid x \in L \text{ and } V(x) \text{ is irreducible} \}. \tag{1}$$

We say that $X$ satisfies the radical condition iff $\mathcal{R}(\mathcal{L}) \subseteq X$.

Lemma 1.43. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^c)$. If $X$ is sober, then $X$ satisfies the radical condition. The converse holds if $\mathcal{L}$ is $X$-top.

Proof. Let $X$ be sober. Let $x \in \mathcal{R}(\mathcal{L})$. Since $X$ is sober, $V(x) = \{p\}$ Proposition 1.15 $V(p)$ for some $p \in X$. It follows by Lemma 1.8 (2) that $\sqrt{x} = p \in X$.

For the converse, assume that $\mathcal{L}$ is $X$-top. Let $F$ be a closed irreducible subset of $X$. Since $\mathcal{L}$ is $X$-top, $F = V(x)$ for some $x \in L$. By our hypothesis, $\sqrt{x} \in X$. By Lemma 1.8 (3), $F = V(x) = V(\sqrt{x})$ and so $\sqrt{x}$ is the unique generic point of $F$ (the uniqueness is obvious). Therefore, $X$ is sober.

Proposition 1.44. Let $X \subseteq L \setminus \{1\}$ and assume that $\mathcal{L}$ is an $X$-top lattice. If $\mathcal{L}$ satisfies the ACC, then every subset of $(X, \tau)$ is compact.
Proof. Let \( A \subseteq X \) and suppose that \( \mathcal{O} = \{X \setminus V(x_i) \mid x_i \in L, i \in I\} \) is an open cover for \( A \). Since \( \mathcal{L} \) satisfies the ACC, \( \bigvee_{i \in I} x_i = \bigvee_{j \in J} x_j \) for some finite subset \( J \subseteq I \). Notice that
\[
A \subseteq \bigcup_{i \in I} (X \setminus V(x_i)) = X \setminus V(\bigvee_{i \in I} x_i) = X \setminus V(\bigvee_{j \in J} x_j) = \bigcup_{i \in J} (X \setminus V(x_j)),
\]
\[
i.e. \ \{X \setminus V(x_j) \mid j \in J\} \text{ is a finite subcover of } \mathcal{O} \text{ for } A.
\]

Proposition 1.45. Let \( X \subseteq L \setminus \{1\} \) and assume that \( \mathcal{L} \) is an \( X \)-top lattice. The following conditions are equivalent:

1. \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC;
2. Every subset of \((X, \tau)\) is compact;
3. Every open set in \((X, \tau)\) is compact.

Proof. \((1 \Rightarrow 2)\) : Consider the complete lattice \((\mathcal{C}(L), \wedge, \vee, \sqrt{0}, 1)\). Since \( V(x) = V(\sqrt{x}) \) for every \( x \in L \), we conclude that \( \mathcal{C} := (\mathcal{C}(L), \wedge, \vee, \sqrt{0}, 1) \) is an \( X \)-top lattice. By our assumption, \( \mathcal{C} \) satisfies the ACC and so every subset of \( X \) is compact by Proposition 1.44.

\((3 \Rightarrow 1)\) : Let \( a_1 \leq a_2 \leq a_3 \leq \cdots \) be an ascending chain in \( \mathcal{C}(\mathcal{L}) \). Notice that \( X \setminus V(a_1) \subseteq X \setminus V(a_2) \subseteq X \setminus V(a_3) \subseteq \cdots \). Setting \( b = \bigvee_{i=1}^{\infty} a_i \), we observe that
\[
X \setminus V(b) = X \setminus V(\bigvee_{i=1}^{\infty} a_i) = X \setminus \bigcap_{i=1}^{\infty} V(a_i) = \bigcup_{i=1}^{\infty} (X \setminus V(a_i)).
\]
By our assumption, the open set \( X \setminus V(b) \) is compact and so \( X \setminus V(b) = \bigcup_{i=1}^{n} X \setminus V(a_i) = X \setminus V(a_n) \) for some \( n \in \mathbb{N} \), i.e. \( b = a_n \) and the ascending chain stabilizes.

Corollary 1.46. Let \( X \subseteq L \setminus \{1\} \) and \( \mathcal{L} \) be an \( X \)-top lattice such that \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC. Then \((X, \tau)\) is spectral \(\iff\) \((X, \tau)\) is sober.

Proof. By Proposition 1.17 \( X \) is \( T_0 \). The result follows now using Proposition 1.45 and Hochster’s characterization for spectral spaces 1.40.

In [11], the so called finer patch topology was used to prove that for any left module \( M \) over an associative ring \( R \), and \( X = \text{Spec}^p(M) \), the classical Zariski topology \((X, \tau^cl)\) is a spectral space provided that the ACC holds for intersections of prime submodules of \( M \).

1.47. Let \( X \subseteq L \setminus \{1\} \). The finer patch topology \( \tau^{fp} \) on \( X \) is the one whose subbase is
\[
\mathcal{B} = \{ V(x) \cap X \setminus V(y) \mid x, y \in L \}.
\]
It is clear that \( \tau^{cl} \subseteq \tau^{fp} \). So, if \( \tau^{fp} \) is compact, then \( \tau^{cl} \) is compact.
Example 1.48. Let \( \mathcal{P} \) be the set of all prime numbers in \( \mathbb{Z} \) and consider the ring \( R = \prod_{p \in \mathcal{P}} \mathbb{Z}_p \). Then the finer patch topology associated with \( \text{Spec}(R) \) is not compact while, trivially, the classical Zariski topology is compact. In general, if \( R \) is a ring with zero dimension and \( \text{Spec}(R) \) is infinite, then the finer patch topology associated with \( \text{Spec}(R) \) is not compact while, trivially, the classical Zariski topology is compact.

Proposition 1.49. Let \( X \subseteq L \setminus \{1\} \). If \( (X, \tau^{fp}) \) is compact, then \( (X, \tau^{cl}) \) is spectral.

Proof. Assume that \( (X, \tau^{fp}) \) is compact. We apply Hochster’s characterizations of spectral spaces to prove that \( (X, \tau^{cl}) \) is spectral. Notice that \( (X, \tau^{cl}) \) is \( T_0 \) by Proposition 1.17 and is compact since \( \tau^{cl} \subseteq \tau^{fp} \).

Claim I: \( (X, \tau^{cl}) \) is sober.

Let \( Y \subseteq X \) be a closed irreducible set in \( (X, \tau^{cl}) \). Then \( Y \) Proposition 1.15 = \( \bigcup_{p \in Y} V(p) \). On the other hand \( Y \) is closed in \( (X, \tau^{fp}) \), whence compact in \( (X, \tau^{fp}) \) (recall that every closed subset of a compact space is compact). Therefore, the open cover \( \mathcal{O} := \{V(p) : p \in Y\} \) of \( Y \) has a finite subcover \( \{V(p_1), \ldots, V(p_n)\} \), i.e. \( Y = \bigcup_{i=1}^{n} p_i \). But \( Y \) is irreducible, whence \( Y = V(p_k) \) for some \( k \in \{1, 2, \ldots, n\} \). Clearly, \( p_k \) is the unique generic point of \( Y \).

Claim II: \( X \) has a basis of compact open sets closed under finite intersections.

We prove this claim in two steps.

Step 1: Every basic open subset of \( (X, \tau^{cl}) \) is compact.

Let \( B \) be a basic open subset of \( (X, \tau^{cl}) \), i.e. \( B = \bigcap_{i=1}^{n} X \setminus V(x_i) \) for some \( \{x_1, \ldots, x_n\} \subseteq L \). Observe that \( X \setminus V(x_i) \) is closed in \( (X, \tau^{fp}) \forall i \in \{1, 2, \ldots, n\} \). So, \( B \) is closed in \( (X, \tau) \), whence compact in \( (X, \tau^{fp}) \). Since \( \tau^{cl} \subseteq \tau^{fp} \), \( B \) is compact in \( (X, \tau^{cl}) \).

Step 2: The collection of open compact subsets of \( (X, \tau^{cl}) \) is closed under arbitrary intersections.

Let \( U \) be an open compact set in \( (X, \tau^{cl}) \). Then we can write \( U = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} X \setminus V(x_{ij}) \) for some subset \( \{x_{ij} | j = 1, 2, \ldots, m_i, i = 1, \ldots, n\} \) (the union is finite because of the compactness of \( U \)). Notice that \( U \) is closed in \( (X, \tau^{fp}) \). So, any intersection of open compact subsets in \( (X, \tau^{cl}) \) is closed in \( (X, \tau^{fp}) \); so it is compact in \( (X, \tau^{fp}) \), whence compact in \( (X, \tau^{cl}) \).

Example 1.50. The ring of integers \( \mathbb{Z} \) is Noetherian and so the finer patch topology on \( \text{Spec}(\mathbb{Z}) \) is compact because the ACC is satisfied on the radical ideals by [11, Theorem 2.2]. This example shows that \( (X, \tau^{fp}) \) can be compact although \( X \) is infinite.

Example 1.51. Let \( L \) be infinite and be such that the elements of \( X := L \setminus \{0, 1\} \) are not comparable (notice that for all \( a \neq b \) in \( X \) we have \( a \land b = 0 \) and \( a \lor b = 1 \)). Notice that \( (X, \tau^{fp}) \) is not compact, whereas \( (X, \tau^{cl}) \) is compact because it is the cofinite topology on \( X \). Notice also that \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC and every element in \( \mathcal{C}(\mathcal{L}) \) can be written as an irredundant meet of elements in \( X \), but this guarantees the compactness for the finer patch topology. Observe that \( \mathcal{L} \) is not \( X \)-top and \( (X, \tau^{cl}) \) is not sober and hence not spectral.
Proposition 1.52. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^L)$. If $V(x)$ is reducible for some $x \in L$, then $V(x) = \bigcup_{i=1}^{n} V(x_i)$ for some elements $x_1, x_2, \cdots, x_n \in L$, where $V(x_i) \nsubseteq V(x)$ for all $i = 1, 2, \cdots, n$.

Proof. Let $V(x)$ be reducible for some $x \in L$, i.e. $V(x) = F_1 \cup F_2$ where both $F_1$ and $F_2$ are closed proper subsets of $V(x)$. Suppose that $F_1 = \bigcap_{i \in I} U_{i=1}^{n_i} V(x_{ij})$ and $F_2 = \bigcap_{i \in I} U_{k=1}^{m_i} V(y_{ik})$ for some $\{x_{ij}\}, \{y_{ik}\} \subseteq L$. Since $F_1$ and $F_2$ are proper subsets of $V(x)$, we have $V(x) \nsubseteq \bigcup_{i=1}^{n_0} V(x_{i0})$ for some $i_0 \in I$ and $V(x) \nsubseteq \bigcup_{k=1}^{m_0} V(y_{0k})$ for some $l_0 \in L$, whence $V(x) \neq \bigcup_{i=1}^{n_0} V(x_{i0}) \cap V(x)$ and $V(x) \neq \bigcup_{k=1}^{m_0} V(y_{0k}) \cap V(x)$. Set $x_r := x_{i0r} \vee x$ for $r = 1, 2, \cdots, n_0$ and $x_{n_0+r} = y_{0r} \vee x$ for $r = 1, 2, \cdots, m_0$ and let $n := n_0 + m_0$. By construction, $V(x) = \bigcup_{r=1}^{n} V(x_r)$, where each $V(x_r)$ is a proper subset of $V(x)$.

As a direct consequence of Proposition 1.52, we obtain the following result which recovers [10, Proposition 2.26] proved for the prime spectrum of a module over a ring.

Corollary 1.53. Let $X \subseteq L \setminus \{1\}$ and assume that $|X| \geq 2$. If $(X, \tau^L)$ is $T_2$, then there exist $x_1, x_2, \cdots, x_n \in L$ such that $X = \bigcup_{i=1}^{n} V(x_i)$, while $X \neq V(x_i)$ for all $i = 1, 2, \cdots, n$.

The radical condition is automatically satisfied by the spectrum of prime submodules of a given left module over an associative ring by [10, Theorem 3.4, Corollary 3.6]. However, we need to check it when dealing with other cases.

The following result generalizes [11, Theorem 3.2]:

Theorem 1.54. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^L)$. If $X$ satisfies the radical condition, then $\mathcal{C}(\mathcal{L})$ satisfies the ACC if and only if $(X, \tau^\mathcal{L})$ is compact. It follows that, If $\mathcal{C}(\mathcal{L})$ satisfies the ACC and $X$ satisfies the radical condition, then $(X, \tau^L)$ is spectral.

Proof. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the ACC and that $X$ satisfies the radical condition. We need only to prove that $(X, \tau^\mathcal{L})$ is compact since it will follow then, by Proposition 1.49, that $(X, \tau^L)$ is spectral.

Suppose that $(X, \tau^\mathcal{L})$ is not compact, i.e. there is an open cover $\mathcal{A}$ in $\tau^\mathcal{L}$ for $X$ which does not have a finite subcover for $X$.

Let

$$
\mathcal{E} := \{x \in \mathcal{C}(L) \mid V(x) \text{ is not covered by a finite subcover of } \mathcal{A}\}.
$$

Observe that $\sqrt{0} \in \mathcal{E}$, i.e. $\mathcal{E} \neq \emptyset$. Since $\mathcal{C}(\mathcal{L})$ satisfies the ACC, $\mathcal{E}$ has a maximal element $p$. Notice that $V(p) \neq \emptyset$.

Case 1: $p \notin X$. Since $X$ satisfies the radical condition, $V(p)$ is reducible and it follows by Proposition 1.52 that $V(p) = \bigcup_{i=1}^{n} V(x_i)$ for some $x_1, \cdots, x_n \in \mathcal{C}(L)$ (see Lemma 1.8 (3)), where $V(x_i) \nsubseteq V(p)$, whence $p \leq x_i$, for all $i \in \{1, \cdots, n\}$. Since $p$ is maximal in $\mathcal{E}$ and $p \leq x_i$, $V(x_i)$ is covered by a finite subcover of $\mathcal{A}$ for all $i \in \{1, \cdots, n\}$. Hence $V(p)$ is covered by a finite subcover of $\mathcal{A}$, a contradiction.

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Case 2: \( p \in X \). It follows that \( p \in O \) for some \( O \in \mathcal{A} \) and so \( p \in B \), where \( B \) is a basic open subset of \( O \). Assume that

\[
B = \bigcap_{i=1}^{n} (V(x_i) \cap X \setminus V(y_i)) \text{ for some subset } \{x_1, \ldots, x_n, y_1, \ldots, x_n\} \subseteq L.
\]

Observe that \( z_i := y_i \lor p \not\leq p \) as \( y_i \not\leq p \forall i \in \{1,2,\ldots,n\} \).

Claim: \( V(p) \cap \bigcap_{i=1}^{n} X \setminus V(z_i) \subseteq B \). To prove this claim, let \( q \in V(p) \cap \bigcap_{i=1}^{n} X \setminus V(z_i) \) for all \( i \in \{1,2,\ldots,n\} \), whence \( p \leq q \) and \( y_i \lor p \not\leq q \) for all \( i \in \{1,2,\ldots,n\} \). It follows that \( p \leq q \) and \( y_i \not\leq p \forall i \in \{1,2,\ldots,n\} \). But \( x_i \leq p \forall i \in \{1,2,\ldots,n\} \) whence \( x_i \leq q \) and \( y_i \not\leq p \forall i \in \{1,2,\ldots,n\} \), i.e. \( q \in \bigcap_{i=1}^{n} (V(x_i) \cap X \setminus V(y_i)) = B \) as claimed.

Now, notice that for all \( i \in \{1,2,\ldots,n\} \), we have \( p \leq z_i \) and so \( V(z_i) \) is covered by a finite subcover \( \mathcal{A}_i \) of \( \mathcal{A} \). On the other hand, \( V(p) \setminus \bigcup_{i=1}^{n} V(z_i) = V(p) \cap \bigcap_{i=1}^{n} X \setminus V(z_i) \subseteq B \subseteq O \). Hence \( \{O\} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n \) is a finite subcover of \( \mathcal{A} \) for \( V(p) \), which is a contradiction.

Therefore, \((X, \tau^L)\) is compact.

Conversely, assume that \((X, \tau^L)\) is compact. Suppose that \( \mathcal{C}(\mathcal{L}) \) does not satisfy the ACC. Then there is an infinite strictly increasing chain \( a_1 \leq a_2 \leq \ldots \) of elements in \( \mathcal{C}(\mathcal{L}) \). Since \((X, \tau^L)\) is compact, \( V(a_1) \) is compact as it is closed. But one can check that \( \{V(a_i) \cap (X \setminus V(a_{i+1})) \mid i = 1,2,\ldots\} \cup \{V(\bigvee_{i=1}^\infty a_i)\} \) is an open cover for \( V(x_1) \) which does not have a finite subcover, a contradiction.

\[
\blacksquare
\]

Remark 1.55. Let \( X \subseteq L \setminus \{1\} \). The radical condition in Theorem 1.54 is necessary for \((X, \tau^L)\) to be spectral. Recall that this condition is satisfied if \( X \) is sober (see Lemma 1.43).

Definition 1.56. Let \( X \subseteq L \setminus \{1\} \). An element \( p \in X \) is called minimal in \( X \) over \( x \in L \) iff \( p = q \) whenever \( x \leq q \leq p \) for some \( q \in X \).

Corollary 1.57. Let \( X \subseteq L \setminus \{1\} \). Assume that \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC, and that for any \( x \in \mathcal{C}(L \setminus \{1\}) \setminus \mathcal{C}(L \setminus \{1\}) \) with \( V(x) \neq \emptyset \) there is a completely strongly irreducible minimal element in \( X \) over it with respect to \( \mathcal{C}(L, \land) \). Then \((X, \tau^L)\) is compact (and consequently \((X, \tau^L)\) is spectral).

Proof. We claim that \( X \) satisfies the radical condition. Let \( x \in \mathcal{R}(\mathcal{L}) \setminus X \). In particular, \( V(x) \neq \emptyset \).

Let \( p \) be a completely strongly irreducible minimal element in \( X \) over \( x \). Then \( \bigwedge_{q \in V(x) \setminus \{p\}} q \leq p \) (otherwise, \( \bigwedge_{q \in V(x) \setminus \{p\}} q \leq p \) and the complete strong irreducibility of \( p \) would imply that \( q \leq p \) for some \( q \in V(x) \) contradicting the minimality of \( p \) over \( x \)). Therefore, \( V(x) = V(\bigwedge_{q \in V(x) \setminus \{p\}} q) \cup V(p) \) a union of proper closed subsets and so \( V(x) \) is reducible, a contradiction. So, \( X \) satisfies the radical condition. Now, the hypotheses of Theorem 1.54 are satisfied and it follows that \((X, \tau^L)\) is compact and consequently \((X, \tau^L)\) is spectral.
Theorem 1.58. Let $X \subseteq L \setminus \{1\}$ and consider $(X, \tau^I)$. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC and that Min$(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$. Then $(X, \tau^p)$ is compact if and only if $V(p)$ is finite $\forall p \in$ Min$(X)$.

Proof. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC and that Min$(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$. We show that $(X, \tau^p)$ is compact. Notice first of all that $X = \bigcup_{p \in \text{Min}(X)} V(p)$, since $\mathcal{C}(\mathcal{L})$ satisfies the DCC.

Claim: Min$(X)$ is finite. To prove this claim, notice that $\bigwedge_{p \in \text{Min}(X)} p = \bigwedge_{i=1}^{n} p_i$ for some $\{p_1, p_2, \ldots, p_n\} \subseteq$ Min$(X)$ (since $\mathcal{C}(\mathcal{L})$ satisfies the DCC). So, $\bigwedge_{i=1}^{n} p_i \leq p$ for all $p \in$ Min$(X)$. By assumption, Min$(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$, whence $p = p_i$ for some $i \in \{1, 2, \ldots, n\}$. Consequently, Min$(X)$ is finite. If $V(p)$ is finite $\forall p \in$ Min$(X)$, then $X$ is finite, whence $(X, \tau^p)$ is compact.

Conversely, suppose that $(X, \tau^p)$ is compact and that $V(p)$ is infinite for some $p \in$ Min$(X)$.

Case 1: $V(p)$ contains an infinite chain $p = x_1 \leq x_2 \leq \ldots$ which does not stabilize. Consider the open cover $\mathcal{A} := \{V(x_i) \cap (X \setminus V(x_{i+1}) | i = 1, 2, \ldots\} \cup \{\bigvee_{i=1}^{\infty} x_i\}$ for $V(p)$. Clearly $\mathcal{A}$ has no finite subcover for $V(x_1)$, whence $(X, \tau^p)$ is not compact, a contradiction.

Case 2: $V(p)$ does not contain any infinite chain. It follows that there is an infinite subset $A \subseteq V(p)$ of incomparable elements. Since $\mathcal{C}(\mathcal{L})$ satisfies the DCC, it follows that $\bigwedge_{x \in A} x = \bigwedge_{x \in F} x$ for some finite subset $F \subseteq A$. Since $A$ is infinite, there is $q \in A \setminus F$ such that $p \leq q$ for some $p \in F$, a contradiction.

Lemma 1.59. Let $X \subseteq L \setminus \{1\}$ and $\mathcal{L}$ be an X-top lattice. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC. Then $X \subseteq SI(\mathcal{C}(\mathcal{L}))$.

Proof. Since $\mathcal{L}$ is an X-top lattice, we have $\tau = \tau^I$. Notice that for every $p \in X$, the singleton $\{p\}$ is irreducible in $(X, \tau)$, whence $p = I(\{p\})$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$ by Proposition 1.22.

Corollary 1.60. Let $X \subseteq L \setminus \{1\}$ and $\mathcal{L}$ be an X-top lattice. If $\mathcal{C}(\mathcal{L})$ satisfies the DCC, then $(X, \tau^p)$ is compact if and only if $V(p)$ is finite $\forall p \in$ Min$(X)$.

Proof. Follows directly by applying Lemma 1.59 and Theorem 1.58.

Example 1.61. Let $\mathcal{L} = (L, \wedge, \vee, 1, 0)$ be a complete lattice, where $L$ is an infinite ascending chain $x_1 \leq x_2 \leq \cdots$ endowed with a maximum element 1 such that $\bigvee_{i \in I} x_i = 1$ for every infinite subset $I \subseteq \mathbb{N}$. Let $X = L \setminus \{1\}$. Then $\mathcal{C}(\mathcal{L})$ satisfies the DCC, and Min$(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$. Hence, $\tau^p$ is not compact by Theorem 1.58 because $V(x_1)$ is infinite. Moreover, every descending chain of $(X, \tau)$ is a spectral subspace.
In what follows, \( R \) is a ring, \( M \) is a left \( R \)-module and consider \( \mathcal{L} := \text{LAT}(M) \), the complete lattice of left \( R \)-submodules of \( M \).

**Example 1.62.** Let \( X = \text{Spec}^p(M) \), the spectrum of prime \( R \)-submodules of \( M \). By [10, Theorem 3.4 (i)], \( \text{Spec}^p(M) \) satisfies the radical condition. Therefore, Theorem 1.54 recovers [11, Theorem 3.2] as a special case.

**Example 1.63.** Let \( M \) be Noetherian and \( X = \text{SI}(M) \), the spectrum of strongly irreducible \( R \)-submodules of \( M \), whence \( \mathcal{L} \) is \( X \)-top. By [1, Proposition 2.7], \( \text{SI}(M) \) satisfies the radical condition. Therefore (\( \text{SI}(M), \tau^p \)) is compact and (\( \text{SI}(M), \tau \)) is spectral.

**Example 1.64.** Applying Theorem 1.54, we obtain several examples of spectral spaces:

1. If \( R \) is duo and \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC, then \( \text{Spec}^f(M) \) is spectral (notice that \( \text{Spec}^f(M) \) satisfies the radical condition by [3, Proposition 3.30]).

2. If \( R \) is duo and \( \mathcal{H}(L) \) satisfies the DCC, then \( X = \text{Spec}^f(M) \) is spectral (notice that \( X = \text{Spec}^f(M) \) satisfies the radical condition by [4, Proposition 3.25]).

3. If \( R \) is a completely distributive \( \text{top}^c \)-module and \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC, then \( \text{Spec}^c(M) \) is spectral (notice that \( X = \text{Spec}^c(M) \) satisfies the radical condition by [2, Proposition 5.19 (i)]).

4. If \( R \) is a \( \text{top}^c \)-module and \( \mathcal{H}(L) \) satisfies the DCC, then \( \text{Spec}^c(M) \) is spectral (notice that \( X = \text{Spec}^c(M) \) satisfies the radical condition by [2, Proposition 4.14 (i)]).

5. If \( R \) is a \( \text{top}^f \)-module, \( I(A) \) is \( \text{first} \) for every irreducible subset \( A \subseteq \text{Spec}^f(M) \) and \( \mathcal{H}(L) \) satisfies the DCC, then \( \text{Spec}^f(M) \) is spectral (notice that the assumption on the irreducible subsets of \( X = \text{Spec}^f(M) \) is equivalent to \( X \) satisfying the radical condition by [1, Remark 4.25]).

### 2 Algebraic versus Topological Properties

As before, \( \mathcal{L} = (L, \land, \lor, 0, 1) \) is a complete lattice. In this section we study the interplay between the topological properties of \((X, \tau^f)\) where \( X \subseteq L \setminus \{1\} \) (or \((X, \tau^{dcl})\) where \( X \subseteq L \setminus \{0\}\)) and the algebraic properties of \( \mathcal{L} \). Applications will be given to the special case \( \mathcal{L} = \text{LAT}(R,M) \), where \( R \) is a ring and \( M \) is a left \( R \)-module.

#### 2.1. We say that an element \( x \in L \) is **finitely constructed** in \( \mathcal{L} \) iff \( x \) cannot be written as an infinite irredundant join of elements of \( L \), that is, for any collection \( \{x_i\}_{i \in I} \subseteq L \) such that \( \bigvee_{i \in I} x_i = x \), there is a finite sub-collection \( \{x_j\}_{j \in J} \) of \( \{x_i\}_{i \in I} \) with \( \bigvee_{j \in J} x_j = x \). An element \( x \) is called **countably finitely constructed** in \( \mathcal{L} \) iff \( x \) cannot be written as an infinite countable irredundant join of elements of \( L \), i.e. for any countable collection \( \{x_i\}_{i \in I} \subseteq L \) with \( \bigvee_{i \in I} x_i = x \), there is a finite sub-collection \( \{x_j\}_{j \in J} \) of \( \{x_i\}_{i \in I} \) with \( \bigvee_{j \in J} x_j = x \). An element \( x \) is called **countably constructed** in \( \mathcal{L} \) iff \( x \) cannot be written as an uncountable irredundant join of elements of \( L \).
We collect first some remarks:

Remarks 2.2. Let \( \mathcal{L} \) be an \( X \)-top lattice, \( X \subseteq L \setminus \{1\} \) and consider the topological space \( (X, \tau) \).

1. The following are equivalent:
   (a) \( (X, \tau) \) is irreducible;
   (b) \( \sqrt{0} \in SI(\mathcal{C}(\mathcal{L})) \);
   (c) If \( X = \bigcup_{i \in I} V(x_i) \), then either \( I \) is infinite or there is \( i_0 \in I \) such that \( x_{i_0} \) is a lower bound for \( X \).

2. \( (X, \tau) \) is \( T_1 \) if and only if \( \text{Max}(X) = X \).

3. \( (X, \tau) \) is Noetherian ⇔ \( \mathcal{C}(\mathcal{L}) \) satisfies the ACC ⇔ each set in \( X \) is compact ⇔ each open set in \( X \) is compact.

4. \( (X, \tau) \) is Artinian ⇔ \( \mathcal{C}(\mathcal{L}) \) satisfies the DCC ⇔ every closed cover for any subset of \( X \) has a finite subcover.

5. \( (X, \tau) \) is (countably) compact if and only if \( 1 \) is (countably) finitely constructed in \( \mathcal{C}(\mathcal{L}) \).

6. If \( SI(\mathcal{C}(\mathcal{L})) \subseteq X \), then \( (X, \tau) \) is sober.

7. If \( X \) satisfies the radical condition, then \( (X, \tau) \) is sober.

8. Assume that \( \mathcal{C}(\mathcal{L}) \) satisfies the complete max property. Then, \( (X, \tau) \) is \( T_1 \) ⇔ \( (X, \tau) \) is discrete.

9. If \( (X, \tau) \) is an atomic, Lindelof (compact) and \( V(p) \) is open \( \forall p \in \text{Min}(X) \), then \( \text{Min}(X) \) is countable (finite).

10. \( V(x) \) is irreducible for every \( x \in X \).

Proof. Let \( \mathcal{L} \) be an \( X \)-top lattice.

1. \( (a \Leftrightarrow b) \) Apply Corollary 1.23 to \( V(0) = X \).
   
   \( (a \Rightarrow c) \) Suppose that \( X = \bigcup_{i \in I} V(x_i) \) with \( I \) finite. Since \( X \) is irreducible, \( V(x_{i_0}) = X \) for some \( i_0 \in I \) whence \( x_{i_0} \) a lower bound for \( X \).
   
   \( (c \Rightarrow a) \) Suppose that \( X = V(x) \cup V(y) \) for some \( x, y \in L \). By our assumption, \( x \) is a lower bound for \( X \) whence \( X = V(x) \) or \( y \) is a lower \( X \) whence \( X = V(y) \). Therefore, \( X \) is irreducible.

2. Apply Proposition 1.26 to \( (X, \tau) = (X, \tau_{cl}) \).
3. It is easy to check that the first two statements are equivalent. The remaining equivalences follow by applying Proposition 1.15 to \((X, \tau) = (X, \tau^c)\).

4. Notice that any open set in \(X\) has the form \(X \setminus V(x)\) where \(x \in C(L)\). The equivalence of the first two statements is straightforward. We claim that they are equivalent to the third statement.

Assume that \(C(L)\) satisfies the DCC. Let \(U \subseteq X\) and \(\{V(x) \mid x \in A\}\) be a closed cover, i.e. \(U \subseteq Y := \bigcup_{x \in A} V(x)\), and assume without loss of generality that \(A \subseteq C(L)\). It follows that \(I(Y) = \bigwedge_{x \in A} x\). Since \(C(L)\) satisfies the DCC, \(I(Y) = \bigwedge_{x \in B} x\) for some finite subset \(B \subseteq I\). It follows that

\[
\forall \text{Lemma } 1.11 \quad V(I(Y)) = V(\bigwedge_{x \in B} x) = \bigcup_{x \in B} V(x).
\]

Therefore, \(U \subseteq \bigcup_{x \in B} V(x)\) for some finite subset \(B \subseteq A\).

Conversely, suppose that \(x_1 \geq x_2 \geq \cdots\) is a descending chain in \(C(L)\) and consider the induced ascending chain \(V(x_1) \subseteq V(x_2) \subseteq \cdots\). Let \(Y = \bigcup_{i=1}^{\infty} V(x_i)\). By assumption, \(Y = \bigcup_{i=1}^{n} V(x_i)\) for some \(n \in \mathbb{N}\), whence \(V(x_n) = V(x_m)\) for all \(m \geq n\) and consequently, \(x_n = x_m\) for all \(m \geq n\) by Lemma 1.8.

5. Assume that \(X\) is (countably) compact and suppose that \(1 = \bigvee_{i \in F} x_i\) where \(x_i \in C(L)\) (and \(F\) is countable). It follows that \(\emptyset = V(\bigvee_{i \in F} x_i) = \bigcap_{i \in F} V(x_i)\), i.e. \(X = \bigcup_{i \in I} (X \setminus V(x_i))\). Since \(X\) is (countably) compact, \(X = \bigcup_{i \in F} (X \setminus V(x_i))\) for some finite subset \(F\) of \(I\) and so \(1 = \bigvee_{j \in F} x_j\). So, \(1\) is (countably) finitely constructed. The converse can be obtained similarly.

6. Let \(F \subseteq X\) be a closed irreducible subset. Then \(F = V(x)\) for some \(x \in L\), whence \(\sqrt{x} \in SL(C(L)) \subseteq X\) by Proposition 1.22. The uniqueness of the generic point is obvious.

7. This follows by Lemma 1.43.

8. This follows by applying Corollary 1.28 to \((X, \tau^c) = (X, \tau)\).

9. Assume that \(X\) is Lindelof (compact). Since \(X\) is atomic, \(X = \bigcup_{p \in Min(X)} V(p)\), whence the open cover \(\{V(p) \mid p \in Min(X)\}\) has a countable (finite) subcover for \(X\), i.e. \(X = \bigcup_{p \in A} V(p)\) for some countable (finite) subset \(A \subseteq Min(X)\). **Claim:** \(Min(X) = A\). Let \(q \in Min(X)\). Since \(X = \bigcup_{p \in A} (V(p))\), we have \(q \in V(p)\) for some \(p \in A\), whence \(q = p\) by the minimality of \(q\). Consequently \(Min(X)\) is countable (finite).

10. This is obtained by applying Proposition 1.15 to \((X, \tau^c) = (X, \tau)\).
Theorem 2.3. Let $X \subseteq L \setminus \{1\}$ and assume that $\mathcal{L}$ is an X-top lattice.

1. The following are equivalent for the sublattice

$$\mathcal{C}'(\mathcal{L}) = \{ x \in \mathcal{C}(L) \mid x \forall y = 1 \text{ and } x \land y = \sqrt{0} \text{ for some } y \in \mathcal{C}(L) \}$$

of $\mathcal{C}(\mathcal{L})$:

(a) $(X, \tau)$ is connected.

(b) If $x \in L$ is such that $\emptyset \neq V(x) \subsetneq X$, then $V(x)$ is not open.

(c) $V(x) \cap V(y) \neq \emptyset$ for all $x \in L$ such that $\sqrt{x} \notin \{\sqrt{0}, 1\}$ and for all $y \in L$ such that $X \setminus V(x) \subseteq V(y)$.

(d) $\mathcal{C}'(\mathcal{L}) = \{\sqrt{0}, 1\}$.

2. Let $(X, \tau)$ be $T_1$. Then $X$ is singleton if and only if $(X, \tau)$ is connected and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property.

3. If $X$ is coatomic and Max$(X)$ is countable (finite), then $(X, \tau)$ is Lindelöf (compact).

4. Let $X$ be coatomic. Then Max$(X)$ is singleton if and only if $(X, \tau)$ is connected and each element in Max$(X)$ is completely strongly irreducible in $(\mathcal{C}(L), \land)$.

5. Let $\mathcal{L}$ be coatomic and Max$(L) \subseteq X$. Then $(X, \tau)$ is ultraconnected if and only if $\mathcal{L}$ is hollow.

6. Let $\emptyset \neq X$ be atomic. Then $(X, \tau)$ is reducible if and only if Min$(X) = I_1 \cup I_2$ such that $\land_{p \in I_2} p \neq q_1$ for some $q_1 \in I_1$ and $\land_{p \in I_1} p \neq q_2$ for some $q_2 \in I_2$.

7. Let $\emptyset \neq X$ be atomic. Then $(X, \tau)$ is connected if and only if for every $\emptyset \neq m \subsetneq \text{Min}(X)$ there exists some $q \in X$ such that $\land_{p \in m} p \lor \land_{p \in \text{Max}(X) \setminus m} p \leq q$.

Proof. Let $X \subseteq L \setminus \{1\}$ and assume that $\mathcal{L}$ is an X-top lattice.

1. Let $x, y \in \mathcal{C}'$. Then there are $x', y' \in \mathcal{C}(L)$ such that $x \lor x' = 1$, $x \land x' = \sqrt{0}$, $y \lor y' = 1$ and $y \land y' = \sqrt{0}$. One can check that $x \land y$ and $x \lor y$ are also in $\mathcal{C}'$ with the corresponding elements $x' \lor y'$ and $x' \land y'$ respectively (recall that if $\mathcal{L}$ is X-top then $\mathcal{C}(\mathcal{L})$ is distributive by [1, Theorem 2.2]).

The equivalence $(a) \Leftrightarrow (b)$ is trivial.

$(a \Rightarrow c)$ Let $x, y \in L$ be such that $\sqrt{x} \notin \{\sqrt{0}, 1\}$ and $X \setminus V(x) \subseteq V(y)$. It follows that $V(x) \cup V(y) = X$, whence $V(x) \cap V(y) \neq \emptyset$ (otherwise, $X$ will be disconnected).
(c \Rightarrow b) Suppose that \( \emptyset \neq V(x) \subseteq X \) is open for some \( x \in L \), so that \( \sqrt{x} \notin \{1, 0, 1\} \). Let \( y \in L \) be such that \( X \setminus V(x) = V(y) \). By our assumption, \( V(x) \cap V(y) \neq \emptyset \) (a contradiction).

(c \Rightarrow d) Let \( x \in \mathcal{C}(L) \). Then there is \( y \in \mathcal{C}(L) \) such that \( x \wedge y = 0 \) and \( x \vee y = 1 \). Clearly, \( x \) and \( y \) satisfy the conditions stated in (c), whence \( V(x \wedge y) = V(x) \cap V(y) \neq \emptyset \), i.e. \( x \vee y \neq 1 \), which is a contradiction.

(d \Rightarrow a) Suppose that \( V(x) \cup V(y) = X \), \( V(x) \cap V(y) = \emptyset \) for some \( x, y \in L \), and assume without loss of generality that \( x, y \in \mathcal{C}(L) \). It is easy to show that \( x, y \in \mathcal{C}(L) \), and it follows by (d) that \( V(x) = X \) or \( V(x) = \emptyset \).

2. Let \((X, \tau)\) be \( T_1 \). If \( \mathcal{C}(L) \) satisfies the complete max property, then applying Corollary 1.28 to \((X, \tau) = (X, \tau^\mathcal{C})\), we conclude that \( X \) is discrete. If \( X \) is moreover connected, then \( X \) is indeed a singleton. The converse is trivial.

3. Let \( X \) be coatomic and \( Max(X) \) be countable (finite). Let \( \mathcal{A} = \{X \setminus V(x) \mid x \in A\} \) be an open cover for \( X \). Then \( \bigcap_{x \in A} V(x) = \emptyset \) and so for any \( p \in Max(X) \), there exists \( x_p \in A \) such that \( p \notin V(x_p) \). **Claim:** \( \bigcap_{p \in Max(X)} V(x_p) = \emptyset \). Suppose that \( q \in \bigcap_{p \in Max(X)} V(x_p) \). Since \( X \) is coatomic, \( q \leq \bar{p} \) for some \( \bar{p} \in Max(X) \) and so \( \bar{p} \in \bigcap_{p \in Max(X)} V(x_p) \), a contradiction.

It follows that \( X = \bigcup_{p \in Max(X)} (X \setminus V(x_p)) \), i.e. \( \{X \setminus V(x_p) \mid p \in Max(X)\} \) is a countable (finite) subcover of \( \mathcal{A} \) for \( X \).

4. Let \( X \) be coatomic.

(\Rightarrow) Assume that \( Max(X) = \{p\} \). For all \( q \in X \), \( q \leq p \) as \( X \) is coatomic and so \( p \) is completely irreducible in the \( (\mathcal{C}, \wedge) \). Also, if \( X = V(x) \cup V(y) \) and \( V(x), V(y) \neq \emptyset \), then \( p \in V(x) \cap V(y) \) and so \( X \) is connected.

(\Leftarrow) Suppose that \( |Max(X)| \geq 2 \) and let \( Max(X) = M' \cup M'' \) with \( M' \cap M'' = \emptyset \) for some \( \emptyset \neq M' \subseteq Max(X) \). Set

\[
A := \{p \in X \mid p \leq q \text{ for some } q \in M' \text{ and } p \notin q \forall q \in M''\},
\]
\[
B := X \setminus A, x := \bigwedge_{p \in A} p \text{ and } y := \bigwedge_{p \in B} p.
\]

**Claim:** \( V(x) \cap V(y) = \emptyset \).

Suppose that \( \bar{p} \in V(x) \cap V(y) \), whence \( y \leq \bar{p} \leq \bar{q} \) for some \( \bar{q} \in Max(X) \). Since \( \bar{q} \) is completely strongly irreducible, \( \bar{q} \in M' \) : otherwise, \( \bar{q} \in M'' \) and \( x = \bigwedge_{p \in A} p \leq \bar{q} \) implies that \( p' \leq \bar{q} \in M'' \) for some \( p' \in A \), a contradiction. Hence, \( y \leq \bar{q} \in M' \). Similarly, since \( \bar{q} \) is completely strongly irreducible, \( q' \leq \bar{q} \) for some \( q' \in B \), which is a contradiction. Therefore \( V(x) \cap V(y) = \emptyset \), and \( V(x) \) and \( V(y) \) are non-empty (\( M' \subseteq V(x) \) and \( M'' \subseteq V(y) \)) with \( V(x) \cup V(y) = X \), whence \( X \) is disconnected.
5. Let $\mathcal{L}$ be coatomic and $\text{Max}(L) \subseteq X$.

$(\Rightarrow)$ Assume that $X$ is ultraconnected. Let $x, y \in L \setminus \{1\}$. Since $\mathcal{L}$ is coatomic, there are $p, q \in \text{Max}(\mathcal{L}) \subseteq X$ with $x \leq p$ and $y \leq q$, whence $V(x)$ and $V(y)$ are non-empty. By assumption, $X$ is ultraconnected, whence $V(x \vee y) = V(x) \cap V(y) \neq \emptyset$. Hence $x \vee y \neq 1$. Consequently, $\mathcal{L}$ is hollow.

$(\Leftarrow)$ Assume that $\mathcal{L}$ is hollow. Let $V(x)$ and $V(y)$ be non-empty closed subsets for some $x, y \in L$. Then $x, y \in L \setminus \{1\}$, whence $x \vee y \neq 1$ as $\mathcal{L}$ is hollow. Since $L$ is coatomic, $x \vee y \leq q$ for some $q \in \text{Max}(\mathcal{L}) \subseteq X$. Hence $V(x) \cap V(y) = V(x \vee y) \neq \emptyset$. Therefore, $X$ is ultraconnected.

6. Let $X$ be reducible, i.e. $X = V(x) \cup V(y)$ for some $x, y \in L$ such that $V(x) \subsetneq X$ and $V(y) \subsetneq X$. Set

$$I_1 = \{ p \in \text{Min}(X) \mid x \leq p \} \quad \text{and} \quad I_2 = \{ p \in \text{Min}(X) \mid y \leq p \}.$$ 

Since $X$ is atomic, $\sqrt{x} = \bigwedge_{p \in I_1} p$ and $\sqrt{y} = \bigwedge_{p \in I_2} p$. Indeed, $\sqrt{x} \not\leq q_2$ for some $q_2 \in I_2$; otherwise, $\sqrt{x} \leq p \forall p \in I_2$ and it follows that $V(x) = X$. Similarly, $\sqrt{y} \not\leq q_1$ for some $q_1 \in I_1$. The converse is trivial.

7. Let $\emptyset \neq X$ be atomic.

$(\Rightarrow)$ Assume that $X$ is connected. Let $\emptyset \neq m \subsetneq \text{Min}(X)$, $x := \bigwedge_{p \in m} p$ and $y = \bigwedge_{p \in \text{Max}(X) \setminus m} p$. Since $X$ is atomic, $X = V(x) \cup V(y)$. Since $X$ is connected, $V(x \vee y) = V(x) \cap V(y) \neq \emptyset$, i.e. $\exists q \in X$ such that $x \vee y \leq q$.

$(\Leftarrow)$ Suppose that $X = V(x) \cup V(y)$ for some $x, y \in L$. Set

$$m' := \{ p \in \text{Min}(X) \cap V(x) \} \quad \text{and} \quad m' := \text{Min}(X) \setminus m'.$$

**Case 1:** $m' = \emptyset$. In this case, $X = V(y)$.

**Case 2:** $m' = \text{Min}(X)$. In this case, $X = V(x)$.

**Case 3:** $\emptyset \neq m' \subsetneq \text{Min}(X)$. By our assumption, $\sqrt{x} \vee \sqrt{y} \leq q$ for some $q \in X$ and so

$$V(x) \cap V(y) = V(\sqrt{x}) \cap V(\sqrt{y}) = V(\sqrt{x} \vee \sqrt{y}) \neq \emptyset.$$ 

Consequently, $X$ is connected.

\[\square\]

**Example 2.4.** Let $M$ be a left module over an arbitrary ring $R$. Consider $X_1 = \text{Spec}^p(M)$ and $X_2 = \text{Spec}^c(M)$. Suppose that $\sqrt{0} = 0$ (e.g. the $\mathbb{Z}$-module $\mathbb{Z}[x]$). Then the set $\mathcal{C}'$ which was described in Theorem 2.3 (1) is the set of the prime radical direct summands (resp. the coprime radical direct summands).

**Corollary 2.5.** Let $X \subseteq L \setminus \{1\}$ and assume that $\mathcal{L}$ is an $X$-top lattice.
1. Let $X$ be atomic, coatomic with $|\text{Max}(X)| \leq |\text{Min}(X)|$ and $V(p)$ is open $\forall$ $p \in \text{Min}(X)$, then $(X, \tau)$ is Lindelof (compact) if and only if $\text{Max}(X)$ is countable (finite).

2. Let $X = \text{Max}(\mathcal{L})$. Then $|\text{Max}(\mathcal{L})| = 1$ if and only if $(X, \tau)$ is connected and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property.

Proof. 1. If $(X, \tau)$ is Lindelof, then $\text{Min}(X)$ is countable by Remark 2.2(10). Conversely, assume that $\text{Max}(X)$ is countable (finite). Let $\mathcal{O} = \{X \setminus V(x) \mid x \in A \subseteq L\}$ be an open cover for $X$, i.e. $\bigcap_{x \in A} V(x) = \emptyset$ and assume without loss of generality that $V(x) \neq \emptyset$ for each $x \in A$ (If $V(y) = \emptyset$ for some $y \in A$, then $\{X \setminus V(y)\}$ is a finite subcover of $X$). Pick $x' \in A$ and set $M := \{q \in \text{Max}(X) \mid x' \leq q\}$. Observe that $M$ is non-empty as $V(x') \neq \emptyset$ and $X$ is coatomic. For each $q \in M$, pick $X \setminus V(x_q) \in \mathcal{O}$ that contains $q$.

Claim: $x' \lor \bigvee_{q \in M} x_q \not\leq p$ for each $p \in \text{Max}(X)$.

Case (1): $p \in M$. In this case, $x_p \not\leq p$ and so $x' \lor \bigvee_{q \in M} x_q \not\leq p$.

Case (2): $p \in \text{Max}(X) \setminus M$. In this case, $x' \not\leq p$ and so $x' \lor \bigvee_{q \in M} x_q \not\leq p$.

Therefore, $V(x' \lor \bigvee_{q \in M} x_q) = \emptyset$ and

$$\{X \setminus V(x')\} \cup \{X \setminus V(x_q) \mid q \in M\}$$

is a countable (finite) subcover of $\mathcal{O}$ as $\text{Max}(X)$ is countable (finite).

2. Assume that $\text{Max}(\mathcal{L}) = X$, whence $\text{Max}(X) = X = \text{Max}(\mathcal{L})$. It follows by Corollary 1.28 that $X$ is $T_1$. So, we can use now Theorem 2.3 (2).

\[\square\]

**Theorem 2.6.** Let $X \subseteq L \setminus \{1\}$ and assume the $\mathcal{L}$ is an $X$-top lattice.

1. There is a 1-1 correspondence

$$\mathcal{C}(L) \leftrightarrow \text{closed sets in } (X, \tau).$$

2. If $\text{SI}(\mathcal{C}(\mathcal{L})) \subseteq X$, then there is a 1-1 correspondence

$$X \leftrightarrow \text{Irreducible closed sets in } (X, \tau).$$

3. If $\text{SI}(\mathcal{C}(\mathcal{L})) \subseteq X$, then there is a 1-1 correspondence

$$\text{Min}(X) \leftrightarrow \text{Irreducible components in } (X, \tau).$$
Proof. Since \( L \) is \( X \)-top, the set of closed sets in \( X \) is given by \( \mathcal{V} = \{ V(y) \mid y \in L \} \). Define

\[
\begin{align*}
  f & : \mathcal{C}(L) \longrightarrow \mathcal{V}, \ x \mapsto V(x); \\
  g & : \mathcal{V} \longrightarrow \mathcal{C}(L), \ V(y) \mapsto \sqrt{y}.
\end{align*}
\]

1. For any \( x \in \mathcal{C}(L) \) and \( y \in L \), we have

\[
(g \circ f)(x) = g(V(x)) = \sqrt{x} = x;
\]

\[
(f \circ g)(V(y)) = f(\sqrt{y}) = V(\sqrt{y}) = V(y).
\]

So, \( f \) provides a 1-1 correspondence \( \mathcal{C}(L) \longleftrightarrow \mathcal{V} \) with inverse \( g \).

2. Consider the restrictions of \( f \) to \( X \) and of \( g \) to the class of irreducible closed varieties. For every \( x \in X \), the variety \( V(x) \) is irreducible by Proposition 1.15 (2). On the other hand, if \( V(y) \) is irreducible for some \( y \in L \), then \( \sqrt{y} \) is strongly irreducible in \( \mathcal{C}(L) \) by Proposition 1.22, whence \( \sqrt{y} \in X \) by our assumption.

3. Consider the restrictions of \( f \) to \( \text{Min}(X) \) and of \( g \) to the class of irreducible components in \( (X, \tau) \).

For every \( x \in \text{Min}(X) \). By (2), \( V(x) \) is irreducible. Suppose that \( V(x) \subseteq V(y) \) for some \( y \in L \) with \( V(y) \) irreducible. Since \( \text{SI}(\mathcal{C}(L)) \subseteq X \), it follows by (2) that \( \sqrt{y} \in X \), whence \( \sqrt{y} \leq x \). However, \( x \) in minimal in \( X \), whence \( x = \sqrt{y} \) and \( V(x) = V(y) \).

On the other hand, let \( A \) be an irreducible component in \( (X, \tau) \). Any irreducible component is closed. Moreover, \( I(A) \) is strongly irreducible in \( \mathcal{C}(L) \) as \( A \) is irreducible, hence \( I(A) \in X \). Suppose that \( p \leq I(A) \) for some \( p \in X \). It follows that \( A = \overline{A} = V(I(A)) \subseteq V(p) = \{ p \} \). However, \( V(p) \) is irreducible as it is the closure of a singleton, so \( V(p) = A \) as \( A \) is an irreducible component. So, \( p = I(A) \). Consequently, \( I(A) \in \text{Min}(X) \).

Example 2.7. The first correspondence \( (\mathcal{C}(\mathcal{L}(M)) \longleftrightarrow \text{closed sets in } (X, \tau)) \) of Theorem 2.6 holds for any \( X \subseteq \mathcal{L}(M) \setminus \{ M \} \) such that \( \mathcal{L} \) is \( X \)-top, as well as for any \( X \subseteq \mathcal{L}(M) \setminus \{ 0 \} \) such that \( \mathcal{L} \) is dual \( X \)-top. So, this result recovers [2, 4.12 and 5.16], [3, 3.27] and [4, 3.23] as special cases.

The following table summarizes some of the results we obtained in this section. Some of them generalize results in the literature on Zariski-like topologies for left modules over associative rings, which can be recovered now as special cases. At several occasions, our results were obtained under conditions and assumptions weaker than those in the corresponding results in the literature.
| Assumption & location          | X-top lattice $\mathcal{L}$ | $(X, \tau)$                  | Results recovered                                                                 |
|-------------------------------|------------------------------|------------------------------|----------------------------------------------------------------------------------|
| Proposition 1.26              | $\text{Max}(X) = X$         | $T_1$                        | $[2, 4.27, 5.33], [3, 3.45]$                                                   |
| Proposition 1.45              | $\mathcal{C}(\mathcal{L})$ satisfies the ACC | Each set in $X$ is compact |                                                                                  |
| Remark 2.2 (3)                | $\mathcal{C}(\mathcal{L})$ satisfies the ACC | Noetherian                   | $[2, 4.12, 5.16]$                                                               |
| Proposition 1.45              | $\mathcal{C}(\mathcal{L})$ satisfies the ACC | Each open set in $X$ is compact |                                                                                  |
| Remark 2.2 (4)                | $\mathcal{C}(\mathcal{L})$ satisfies the ACC | Artinian                     | $[2, 4.12, 5.16]$                                                               |
| Remark 2.2 (4)                | $\mathcal{C}(\mathcal{L})$ satisfies the ACC | Every closed cover for any subset of $X$ has a finite subcover |                                                                                  |
| Theorem 1.27                  | $\text{Max}(X) = X$ and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property | Discrete                      | $[2, 4.28, 5.34], [3, 3.46], [1, 4.33]$                                          |
| Theorem 2.3 (1)               | $\mathcal{C} = \{\sqrt{0}, 1\}$ | Connected                    |                                                                                  |
| Corollary 1.23                | $I(A) \in SI(\mathcal{C}(\mathcal{L}))$ | $A \subseteq X$ is irreducible | $[3, 3.30, 3.31]$                                                               |
| Corollary 1.23                | $I(A)$ is irreducible in $\mathcal{C}(\mathcal{L})$ | $A \subseteq X$ is irreducible | $[3, 3.30, 3.31]$                                                               |
| Corollary 1.23                | $\sqrt{0}$ is irreducible in $\mathcal{C}(\mathcal{L})$ | irreducible | $[3, 3.30, 3.31]$                                                               |
| $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ (2.6) | $\sqrt{x} \in X$ | $V(x)$ is irreducible | $[2, 4.17, 5.22], [2, 3.27], [3, 3.33], [10, 3.6], [1, 4.28]$ |
| $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ (2.6) | $\sqrt{x} \in \text{Min}(X)$ | $V(x)$ is irreducible component | $[2, 5.22], [2, 4.17], [3, 3.27], [3, 3.33], [1, 4.28]$ |
| $\text{Max}(\mathcal{L}) = X$ and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property (2.3 (2)) | $|\text{Max}(X)| = 1$ | Connected                      |                                                                                  |
| Remark 2.2 (5)                | $1$ is finitely constructed | Compact                       |                                                                                  |
| Remark 2.2 (5)                | $1$ is countably constructed | Lindelof                      |                                                                                  |

Table 2: Examples on the Interplay between topological properties of $(X, \tau)$ and algebraic properties of the $X-top$ lattice $\mathcal{L}$
Lemma 2.8. Let $R$ be a ring and $M$ a top$^p$-module, that is $\mathcal{L} := \text{LAT}(R, M)$ is $\text{Spec}^p(M)$-top. Then

$$\text{SI}(\mathcal{L}(\text{LAT}(M))) \subseteq \text{Spec}^p(M).$$

Proof. Let $N$ be strongly irreducible in $\mathcal{C}(\mathcal{L})$. Suppose that $IK \subseteq N$ for some ideal $I \subseteq R$ and a submodule $K \subseteq M$. Then $IK \subseteq P$ for any prime submodule $P \in V(N)$, whence $IM \subseteq P$ or $K \subseteq P$ and so $\sqrt{IM} \subseteq P$ or $\sqrt{K} \subseteq P$, whence $\sqrt{IM} \cap \sqrt{K} \subseteq P$ for all $P \in V(N)$. Since $N$ is radical, $\sqrt{IM} \cap \sqrt{K} \subseteq N$. By assumption, $N$ is strongly irreducible in $\mathcal{C}(\mathcal{L})$, whence $IM \subseteq \sqrt{IM} \subseteq N$ or $K \subseteq \sqrt{K} \subseteq N$. Therefore, $N \in \text{Spec}^p(M)$.

Example 2.9. Let $R$ be a ring and $M$ a top$^p$-module. By Lemma 2.8, we have $\text{SI}(\mathcal{C}(\text{LAT}(R, M))) \subseteq \text{Spec}^p(M)$. So, all the 1-1 correspondences in Theorem 2.6 hold for this special case. Behboodi and Haddadi proved the second correspondence in [10, Corollary 3.6].

Example 2.10. Let $R$ be a ring and $R, M$ a left top$^c$-module (i.e. $\mathcal{L} = \text{LAT}(R, M)$ is $X$-top, where $X = \text{Spec}^c(M)$). If $R, M$ is completely distributive, then $\text{SI}(\mathcal{C}(\mathcal{L})) \subseteq X$ by [2, Proposition 5.19] and the 1-1 correspondences of Theorem 2.6 hold. In [2, Proposition 5.22], these correspondences were proved under the additional condition that every coprime submodule of $M$ is strongly irreducible.

Example 2.11. Let $R$ be a ring and $R, M$ a left top$^s$-module (i.e. $\mathcal{L} = \text{LAT}(R, M)$ is dual $X$-top, where $X = \text{Spec}^s(M)$). By [2, Proposition 4.14], $\text{SH}(\mathcal{H}(\mathcal{L})) \subseteq X$ and so the 1-1 correspondences of Theorem 2.6 hold. These were proved in this special case in [2, Proposition 4.17] under the additional condition that every second submodule of $M$ is strongly hollow.

Example 2.12. Let $R$ be a ring and $R, M$ a left top$^p$-module (i.e. $\mathcal{L} = \text{LAT}(R, M)$ is $X$-top, where $X = \text{Spec}^p(M)$). If $R, M$ is duo, then $\text{SI}(\mathcal{C}(\mathcal{L})) \subseteq X$ by [3, 3.30] and the 1-1 correspondences of Theorem 2.6 hold. These were also obtained under the same condition in [3, Proposition 3.33].

Example 2.13. Let $R$ be a ring and $R, M$ a left top$^f$-module (i.e. $\mathcal{L} = \text{LAT}(R, M)$ is dual $X$-top, where $X = \text{Spec}^f(M)$). If $R, M$ is duo, then $\text{SH}(\mathcal{H}(\mathcal{L})) \subseteq X$ by [4, Proposition 3.35] and Proposition 1.22 and the 1-1 correspondences of Theorem 2.6 hold. These were also obtained under the same condition for this special case in [4, Proposition 3.28].

Example 2.14. Let $R$ be a ring and $R, M$ a left top$^f$-module (i.e. $\mathcal{L} = \text{LAT}(R, M)$ is dual $X$-top, where $X = \text{Spec}^f(M)$). If $R, M$ has the property that $H(A)$ is first whenever $A$ is irreducible, then $\text{SH}(\mathcal{H}(\mathcal{L})) \subseteq X$ and so the 1-1 correspondences of Theorem 2.6 hold. This was proved under the same condition in [1].

Example 2.15. Let $R$ be a PID with an infinite number of non-zero prime ideals (e.g. $R = \mathbb{Z}$), $\mathcal{L} := \text{Ideal}(R)$, $X = \text{Max}(R)$ and consider the topological space $(X, \tau)$.

1. $X = V(0)$ is irreducible since $0$ is strongly irreducible. However, $0 = \sqrt{0} \notin X$ and so $X$ is not sober by Remark 2.2 (7), whence not spectral.
2. $X$ is $T_1$ as $\text{Max}(X) = X$.

3. $X$ is cofinite: consider a closed set $\emptyset \neq V(I) \subseteq X$, where $I = (a)$ for some $a \in R \setminus \{0\}$. Since $R$ is a PID, the unique prime factorization of $a$ implies that $I$ is contained in a finite number of primes, i.e. $V(I)$ is finite.

4. $X$ is not regular, not $T_2$, and not normal. Observe that $X$ is infinite and cofinite, so it does not have disjoint non-empty open sets, although it has disjoint non-empty closed sets.

**Example 2.16.** Let $R$ be a ring, $M$ a left $R$-module, $X \subseteq \text{LAT}(R M) \setminus \{M\}$ (resp. $X \subseteq \text{LAT}(R M) \setminus \{0\}$) and assume that $\mathcal{L} := \text{LAT}(R M)$ is $X$-top (resp. dual $X$-top). If $\mathcal{C}(\mathcal{L})$ is uniform (resp. $\mathcal{H}(\mathcal{L})$ is hollow), then $(X, \tau)$ (resp. $(X, \tau^0)$) is connected by Theorem 2.3 (1).

**Example 2.17.** Let $R$ be a commutative domain, $\mathcal{L} := \text{Ideal}(R)$, $X \subseteq \text{Ideal}(R) \setminus \{R\}$ (resp. $X \subseteq \text{Ideal}(R) \setminus \{0\}$), and assume that $\mathcal{L}$ is $X$-top (resp. $\mathcal{L}$ is dual $X$-top). If $\sqrt{0} = 0$ (resp. $\sum_{p \in X} p = R$).

Then $(X, \tau)$ (resp. $(X, \tau^0)$) is connected.

**Example 2.18.** Let $R$ be a UFR with zero devisors. Consider $\mathcal{L} := \text{Ideal}(R)$, $X := \text{Spec}(R)$ (the prime spectrum of $R$) and assume that $\text{Min}(X)$ is infinite (e.g. $R = \mathbb{Z}[x]$ with $x$ not prime). Notice that $\sqrt{0} = 0$ (since $\text{Min}(X)$ is infinite, if $0 \neq x \in \sqrt{0}$ then $x \in \bigcap_{p \in \text{Min}(X)} p$, but this is impossible as $R$ is a UFR).

- $(X, \tau)$ is connected by Theorem 2.3 (7).

**Claim:** the intersection of any infinite collection of minimal elements of $X$ is zero. Suppose that $0 \neq I := \bigcap_{q \in m'} q$ for some infinite subcollection $m'$ of $\text{Min}(R)$. For any $x \in I \setminus \{0\}$, we have $x = p_1 \cdots p_n$ where $p_1, \ldots, p_n$ are prime elements of $R$. Notice that $p_1, \ldots, p_n \in I$. For every $q \in m'$, we have $q = (p_i)$ for some $i \in \{1, 2, \ldots, n\}$, whence $m'$ is finite (a contradiction).

- $(X, \tau)$ is reducible by Remark 2.2 (1). To prove this, suppose that $(X, \tau)$ is irreducible and that $I \cap J = 0$ for some ideals $I, J \leq R$. Then $V(0) = V(I \cap J) = V(I) \cup V(J) = V(\sqrt{I}) \cup V(\sqrt{J}) = V(\sqrt{I} \cap \sqrt{J})$, whence $\sqrt{I} \cap \sqrt{J} = \sqrt{0} = 0$. Since $\sqrt{0} \in SI(\mathcal{C}(\mathcal{L}))$ (by Remark 2.2 (1)), it follows that $I = \sqrt{I} = 0$ or $J = \sqrt{J} = 0$, whence $R$ is a domain, a contradiction.

**Example 2.19.** Let $(G, +)$ be a group and set

$$L := \{H \mid H \leq G \text{ is a normal subgroup of } G\},$$
$$X := \{H \mid H \leq G \text{ is a finite normal subgroup of } G\} \setminus \{G\}.$$

Notice that $\mathcal{L} = (L, \cap, +, G, 0)$ is a complete lattice endowed with $\bigvee_{i \in I} N_i := \sum_{i \in I} N_i$ and $\bigwedge_{i \in I} N_i := \bigcap_{i \in I} N_i$. 

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1. \( \mathcal{C}(L) = X \cup \{G\} \) as the intersection of any non-empty family of finite normal subgroups is a finite normal subgroup.

2. \( SI(\mathcal{C}(L)) \subseteq X \) and so all the 1-1 correspondences of Theorem 2.6 hold.

3. \( 0 = \sqrt{0} \in X \) and so \( (X, \tau^l) \) is irreducible and connected (observe that \( \{0\} = X \) and \( \{0\} \) is irreducible).

4. \( \mathcal{C}(L) \) satisfies the DCC but need not satisfy the ACC (e.g. a \( p \)-quasicyclic group [19]).

5. \( SI(\mathcal{C}(L)) = X \) if and only if \( \mathcal{L} \) is an \( X \)-top lattice.

6. If \( \mathcal{L} \) is \( X \)-top, then the intersection of any nonzero elements in \( X \) is nonzero.

7. By Corollary 1.28: \( (X, \tau^l) \) is \( T_1 \) if \( T_2 \) is a singleton \( (X, \tau^l) \) is \( T_2 \) if \( (X, \tau^l) \) is discrete.

8. Suppose that \( \mathcal{L} \) is an \( X \)-top lattice and \( (X, \tau^l) \) is compact with each element in \( G \) having a finite order. Then \( G \) is a finite \( p \)-group for some prime \( p \). Indeed, since \( X \) is compact, by Theorem 2.3 (5), \( G \) is finitely constructed. But \( G \) is the union of all proper cyclic subgroups, say \( G = \sum_{i \in I} H_i \). Then \( G = \sum_{j \in F} H_j \) where \( F \) is a finite subset of \( I \). Hence \( G \) is finite. Consequently, the Prüfer group is not \( X \)-top (\( X \) is the set of all proper subgroups) as it is infinite.

Example 2.20. Let \((G, +)\) be a group, \( Z(G) \) the center of \( G \) and set
\[
L := \{H \mid H \trianglelefteq G \text{ is a normal subgroup of } G\}, \\
X := \{H \mid H \leq Z(G)\} \setminus \{G\}.
\]
Notice that \( \mathcal{L} = (L, \cap, +, G, 0) \) is a complete lattice with \( \bigvee_{i \in I} N_i := \sum_{i \in I} N_i \) and \( \bigwedge_{i \in I} N_i := \bigcap_{i \in I} N_i \).

1. \( \mathcal{C}(L) = X \cup \{G\} \) as the intersection of any non-empty family of subgroups of the center is again in the center.

2. \( SI(\mathcal{C}(L)) \subseteq X \) and so all correspondences of Theorem 2.6 hold.

3. \( 0 = \sqrt{0} \in X \) and so \( (X, \tau^l) \) is irreducible and connected.

4. By Corollary 1.28: \( (X, \tau^l) \) is \( T_1 \leftrightarrow (X, \tau^l) \) is singleton \( (X, \tau^l) \) is \( T_2 \leftrightarrow (X, \tau^l) \) is discrete.

5. \( SI(\mathcal{C}(L)) = X \leftrightarrow \mathcal{L} \) is \( X \)-top. Hence, if \( \mathcal{L} \) is an \( X \)-top lattice, then the intersection of any distinct nonzero subgroups in \( X \) is nonzero.

6. If \( G \) is finite, then \( (X, \tau^l) \) is spectral by Remark 1.41.
7. Suppose that $\mathcal{L}$ is an $X$-top lattice and $(X, \tau^{cl})$ is compact with each element in $G$ having a finite order. Then $G$ is a finite $p$-group for some prime $p$.

8. $X$ is coatomic and $Z(G)$ is the unique maximal element of $X$.

9. If $\mathcal{L}$ is $X$-top, then $X$ is compact as $X$ is coatomic and $\text{Max}(X)$ is finite (by Theorem 2.3 (3)).

**Example 2.21.** Let $(G, +)$ be a group, $Z(G)$ the center of $G$ and set

$$L := \{H \mid H \trianglelefteq G \text{ is a normal subgroup of } G\},$$

$$X := \{H \mid H \leq Z(G) \text{ is finite } \} \setminus \{G\}.$$ 

Notice that $\mathcal{L} = (L, \cap, +, G, 0)$ is a complete lattice with $\bigvee_{i \in I} N_i := \bigoplus_{i \in I} N_i$ and $\bigwedge_{i \in I} N_i := \bigcap_{i \in I} N_i$.

1. $\mathcal{C}(L) = X \cup \{G\}$ as the intersection of any non-empty family of finite subgroups of the center is again finite and in the center.

2. $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ and so all correspondences of Theorem 2.6 hold.

3. $0 = \sqrt{0} \in X$ and so $(X, \tau^{cl})$ is irreducible and connected.

4. By Corollary 1.28: $(X, \tau^{cl})$ is $T_1$ if $(X, \tau^{cl})$ is singleton if $(X, \tau^{cl})$ is $T_2$ if $(X, \tau^{cl})$ is discrete.

5. $SI(\mathcal{C}(\mathcal{L})) = X \iff \mathcal{L}$ is $X$-top. Hence, if $\mathcal{L}$ is an $X$-top lattice, then the intersection of any distinct nonzero subgroups in $X$ is nonzero and so $X$ can only be $\{0\}$ or a collection of $p$-subgroups for some fixed prime $p$. Otherwise, $H \in X$ has order $p^n q^m l$ with primes $p$ and $q$ not dividing $l$ and so by the Sylow Theorem [19, Theorem 5.2] there is a Sylow $p$-subgroup $K_1$ of order $p^n$ and a Sylow $q$-subgroup $K_2$ of order $q^m$. By Lagrange’s Theorem [19, Theorem 1.26], the order of their intersection must divide $p^n$ and $q^m$ and so the intersection must be zero, whereas $K_1$ and $K_2$ are nonzero elements of $X$. The uniqueness of $p$ is clear also by Lagrange’s Theorem.

6. If $G$ is finite, then $(X, \tau^{cl})$ is spectral by Remark 1.41.

7. Suppose that $\mathcal{L}$ is an $X$-top lattice and $(X, \tau^{cl})$ is compact with each element in $G$ having a finite order. Then $G$ is a finite $p$-group for some prime $p$.

8. $X$ is coatomic and $Z(G)$ is the unique maximal element of $X$.

9. If $\mathcal{L}$ is $X$-top, then $X$ is compact as $X$ is coatomic and $\text{Max}(X)$ is finite (by Theorem 2.3 (3)).
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