Expanding Multi-Market Monopoly and Nonconcavity in the Value of Information

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Abstract

I investigate a Bayesian inverse problem in the specific setting of a price setting monopolist facing a randomly growing demand in multiple possibly interconnected markets. Investigating the Value of Information of a signal to the monopolist in a fully dynamic discrete model employing the Kalman-Bucy-Stratonovich filter, we find that it may be non-monotonic in the variance of the signal. In the classical static settings of the Value of Information literature this relationship may be convex or concave, but is always monotonic. The existence of the non-monotonicity depends critically on the exogenous growth rate of the system.

Keywords: Information Economics, Value of Information, Nonconcavity, Learning, Filtering, Monopoly.

JEL Classification: L12; D42; D80; D83.

1 Introduction

The issue of how the explicit introduction of information impacts traditional economic analysis of, for example, competitive equilibrium or monopolistic behaviour has been investigated fruitfully within the paradigm of asymmetric information. The success of explaining non-perfectly competitive outcomes has however led to a neglect of the issue of how information itself can be considered valuable. A single Value of Information (VoI) literature does not exits and the issue spans over diverse economic fields and even other disciplines that aim to discriminate signal from noise. Of course any investigation of a Value of Information needs a specific reference frame in which such a ”value” may occur.

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In economics, Wilson (1975) investigates a monopoly context in which information may be acquired in a quadratic-normal problem leading to simple objective function containing only the first two modes of the uncertainty. Wilson finds that an optimal choice of the number of signals to be acquired at some fixed cost (while complicated to solve for explicitly) can be shown to lead to corner-solutions, so either no, or an infinite amount of signals are optimally purchased. This creates a problem for the existence of competitive equilibrium. The mirror image of this result implies that the objective function is strictly monotonic (convex) in the precision of the signals.

A literature which has also employed the monopoly setting to investigate the generation of information is the early experimentation literature e.g. Mirman et. al. (1993). The focus in this literature is on the possibility to influence the generation of information itself in order to improve the firm's objective rather than the optimal use of signals as such. Clearly a Value of Information that is not concave in the firm's action may again cause corner-solutions leading to extreme results regarding optimal experimentation.

An influential paper that has argued that the Value of Information may generically exhibit nonconcavities is Radner & Stiglitz (1984), albeit for finite signal realizations only. This assumption precludes many interesting applications that are subject to continuous signals.

Many other academic disciplines such as physics, mathematics, or engineering care about information and have developed sophisticated techniques in areas such as machine learning, c.f. Cover & Thomas (2006). Here however the concept of value often differs from the economists focus and, most frequently, such considerations are simply absent.

Whether one finds non-concavities in the Value of Information clearly also depends on the specification of objective function of the agent(s). Even if results such as Radner & Stiglitz point into the other direction, as noted in Chade & Schlee (2002), p.423 the common intuition remains:

"Why should information exhibit increasing marginal returns over some range? While information as a commodity is admittedly special, it seems that there ought to exist a rich class of problems for which the value of information is concave."

In this paper, as in Behringer (2021) we will adopt the classical microeconomic focus of Mirman et. al. (1993), recently also investigated in Weber and Nguyen (2018) and Weber (2019) in an information acquisition context and concentrate on a single-agent monopoly setting. This framework allows to transfer the issue of information and signals into a value-to-the-firm context smoothly.
We find analytical conditions for which the Value of Information from signals in a fully dynamic setting with a growing market and continuous signals exhibits non-monotonicities in their precision, contrary to the finding of Wilson (1975), thereby adding some scepticism about the above intuition. This implies that there exists a range of parameters in our dynamic setting for which less precise information is beneficial for the agent. Information is updated using the Kalman-Bucy-Stratonovich filter as in Slade’s (1989) Bertrand duopoly setting.

A paper that has previously noticed that less precise information may be beneficial for a firm is Gal-Or (1988) which contrasts with Vives (1984) in competitive settings, which have previously been investigated in Ponssard (1976). However in Gal-Or the mechanism by which less precise information about costs may increase firm’s profits results from the competitive situation only and the effect of information on the incentives for information sharing in oligopoly that will allow for a plethora of outcomes depending on the specification of the uncertainty, see e.g. Raith (1996).

2 The model

The monopoly game has the following timing: The firm has initial priors over an unknown state variable given by \((\mu_0, \sigma_0^2)\) about the true market potential \(\theta_t\) which has a known evolution over time.

In each period \(t\) the monopolist sets a price \(p_t\) according to its present belief \(\mu_t\) about the market potential which generates an undiscounted noisy per-period profit \(\Pi_t\). The monopolist then observes its own demand \(q(\theta_t, \mu_t, \cdot)\) which acts as a noisy signal about the true \(\theta_t\). This signal can be used to learn about the market potential and find the updated prior moments \((\mu_{t+1}, \sigma_{t+1}^2)\) for the optimal prices in the next period

We thus have to solve for a dynamic pricing strategy given a changing market potential. The state variable for this single market potential changes over time as

\[ \theta_{t+1} = d\theta_t + f e_{t+1} \]

where \(d, f > 0\) are free parameters and the i.i.d. error \(e_t\) is normalized to \(e_t \sim N(0, 1)\) where \(\sigma_e^2 = 1\) is w.l.o.g. as non-unity variances can be embedded in \(f\). To investigate true growth we further assume that \(d > 1\).

At some time \(t\) the monopolist operates in market \(i = 1, \ldots, n\) choosing prices against \(n\) unknown demand functions, each of the linearized form

\[ q_i = \theta_i - p_i + \gamma_i \]

\footnote{We assume that firms are myopic in the sense of Aghion et al. (1988) so that they ignore that by manipulating prices today they can gain information about demand tomorrow but can use past sale data to update their prices which may alternatively be viewed as a sequence of firms. Non-myopic firms are treated in section 4.}
where the noises $\gamma_i \sim N(0,1)$ are independent from $e_t$.

Operating at marginal cost $c$, the monopolist will optimally choose a price

$$p_i = \frac{1}{2} (\theta_i + c + \gamma_i)$$

in each market and obtain a per-period profit of

$$\Pi^* = \frac{(\theta_i - c + \gamma_i)^2}{4}$$

Expectation of these profits in period $t$ absent learning would simply be

$$E\{\Pi^*_i\} = \frac{(\mu_i - c)^2}{4}.$$

However the myopic monopolist knows the parameters of the model and can observe sales in each period in each market.

As we allow for the market potential in one market to also change when the potential in another market changes and to also depend on other markets variations it is more convenient to employ vector notation. The market potentials will then change according to

$$\theta_{t+1} = D\theta_t + Fe_{t+1}$$

(1)

where now $D$ and $F$ are $n \times n$ matrices and $\theta_{t+1}$ and $e_t$ with $e_t \sim N(0, I_n)$ are $n \times 1$ vectors with $I_n$ denoting an $n \times n$ identity matrix. We assume the potential prior to also be normally distributed $\theta_0 \sim N(\mu_0, \Sigma_0)$. All matrices are invertible and transposes are indicated with a prime.

**Lemma 1** The conditional variance of the prior, given previous signals $S_0, ... S_{t-1}$ is

$$Var(\theta_t|S_0, ... S_{t-1}) = D\Sigma_t D' + FF'.$$

**Proof:**

Standard, e.g., Veldkamp (2011). ■

### 2.1 The learning process

The learning process from observing sales allows for conventional Bayesian updating with normal variables which in this dynamic setting becomes the Kalman-Bucy-Stratonovich filter approach.
In a single isolated market in each period $t$ observed market demand depends on the chosen price and therefore the belief about the market potential as

$$q_t = \theta_t - \frac{\mu_t + c}{2} + \gamma_t$$

which can be transformed into an unbiased signal as

$$\tilde{S}_t \mid \theta_t = q_t + \frac{\mu_t + c}{2} = \theta_t + \gamma_t.$$

Allowing for demand interrelations, e.g. competing products, cannibalization, substitutive effects etc. by assuming that demands/signals in one market can depend on other markets demands either systematically or by idiosyncratic (error) realizations of both previous and contemporary periods. This assumption extends the signal system to:

$$S_t = G\theta_t + H\gamma_t$$

where the signal $S_t$ is an $n \times 1$ vector and $G$ and $H$ are $n \times n$ matrices with an $n \times 1$ error vector $\gamma_t \sim N(0, I_n)$.

**Lemma 2** The variance of the unbiased signal is

$$\text{Var}(G^{-1}S_t) = \Sigma_t + G^{-1}HH'(G')^{-1}.$$  

**Proof:**

Premultiplying (2) by $G^{-1}$ we have

$$\text{Var}(G^{-1}S_t) = \text{Var}(\theta_t) + \text{Var}(G^{-1}H\gamma_t)$$

$$= E\{(\theta_t - \mu_t)(\theta_t - \mu_t)\}' + E\{(G^{-1}H\gamma_t)(G^{-1}H\gamma_t)\}'$$

$$= \Sigma_t + G^{-1}HH'(G')^{-1}$$

by independence and $\gamma_t \sim N(0, I_n)$ where $\mu_t$ is the belief about $\theta_t$ conditional on all previous signals, i.e.

$$\mu_t \equiv E\{\theta_t | S_0, ..., S_{t-1}\}.$$

**Lemma 3** Optimal Bayesian beliefs about the unknown market potential can be found from the system

$$\mu_{t+1} = K_t S_t + (D - K_t G)\mu_t$$

where $K_t$ denotes the Kalman-Bucy-Stratonovich Filter (gain) which can be determined as

$$K_t = D\Sigma_t G' [G\Sigma_t G' + HH']^{-1}$$

with the conditional belief variance recursively determined as

$$\Sigma_{t+1} = D\Sigma_t D' + FF' - K_t G\Sigma_t D'.$$
Proof:
Bayesian updating has a linear form due to the normality assumptions. Hence
$$\mu_{t+1} = DK_t' S_t + (D - DK_t' G) \mu_t$$
and let’s define the Kalman gain as $K_t = DK_t'$ so that it includes the known market growth. Now
$$\mu_{t+1} = K_t S_t + (D - K_t G) \mu_t$$
describes an optimal learning process. As $\Sigma_t = E \{(\theta_t - \mu_t)(\theta_t - \mu_t)\}'$ and
$$\theta_{t+1} = D\theta_t + F e_{t+1}$$
we find the conditional variance of the prior in $t+1$ as
$$\Sigma_{t+1} = (D - K_t G) \Sigma_t (D - K_t G)' + FF' + K_t H H' K_t'$$
with the Kalman gain $K_t$ to be determined via the orthogonality principle. Transforming
$$\Sigma_{t+1} = (D - K_t G) \Sigma_t D' - D \Sigma_t G' K_t' + K_t (G \Sigma_t G' + H H') K_t' + FF'$$
Minimizing this variance, noting that the trace of a matrix is equal to the trace of its transpose, thus
$$T[\Sigma_{t+1}] = T[D\Sigma_t D'] - 2T[K_t G \Sigma_t D'] + T[K_t (G \Sigma_t G' + H H') K_t']$$
we differentiate w.r.t. $K_t$ and set to zero to find
$$\frac{dT[\Sigma_{t+1}]}{dK_t} = -2[G \Sigma_t D']' + 2[K_t (G \Sigma_t G' + H H')] = 0$$
or
$$K_t = D \Sigma_t G' [G \Sigma_t G' + H H']^{-1}$$
which leads to the belief updating signal weighting as

\[ \mu_{t+1} = K_t G S_t + (D - K_t G) \mu_t \]

and

\[ \Sigma_{t+1} = D \Sigma_t D' + FF' - D \Sigma_t G' (G \Sigma_t G' + H H')^{-1}) G \Sigma_t D'. \]

**Lemma 4** The normalized Kalman gain \( D^{-1} K_t \) represents the weight to be put on the signal in Bayesian updating.

**Proof:** Conditional on \( \theta_t \) the signal has the variance of \( G^{-1} H H'(G')^{-1} \). The variance of the prior beliefs about \( \theta \) at time \( t \) is \( \Sigma_t \) so by linear/normal Bayesian updating the agent will give a weight \( \Sigma_t \) to the unbiased signal \( G^{-1} S_t \) (with \( G \) known) given by

\[ \Sigma_t \left[ \Sigma_t + G^{-1} H H'(G')^{-1} \right]^{-1} \]

so the total weighted signal is

\[ G^{-1} y_t \Sigma_t \left[ \Sigma_t + G^{-1} H H'(G')^{-1} \right]^{-1}. \]

The weight can be transformed by adding identities \((G')^{-1} G' = GG^{-1} = I\)

\[ \Sigma_t (G')^{-1} G' \left[ \Sigma_t + G^{-1} H H'(G')^{-1} \right]^{-1} GG^{-1} = \Sigma_t G' [G \Sigma_t G' + H H']^{-1} G \]

so the total weighted signal is

\[ G^{-1} S_t \Sigma_t G' [G \Sigma_t G' + H H']^{-1} G \]

\[ = S_t \Sigma_t G' [G \Sigma_t G' + H H']^{-1} \]

\[ = S_t D^{-1} K_t \]

so that where \( D^{-1} K_t \) represents the weight to be put on a signal in period \( t \).

We next focus on the expected profits of the monopolist:

**Lemma 5** The firm obtains an expected profit in period \( t \) of

\[ E \{ \Pi_t \} = \frac{(D' \mu_t - c I)^2}{4} + \frac{1}{4} K_{t-1} \Sigma_t K_{t-1}' \]

where the second term gives the pecuniary Value of Information of an additional signal

\[ VoI_t = K_{t-1} \Sigma_t K_{t-1}' = D \Sigma_t G' [G \Sigma_t G' + H H']^{-1} \left[ (D - K_t G) \Sigma_t D' + FF' \right] \left[ D \Sigma_t G' [G \Sigma_t G' + H H']^{-1} \right]' \]
Proof:
In each market the monopolist has an expected per-period profit of

\[ E \{ \Pi_t \} = \frac{1}{4} E \{ (D^t \mu_t - cI)^2 \} \]

where again profit is an \( n \times 1 \) vector with \( D^t \) denoting the \( t \)th standard power of the \( D \) matrix, i.e. \( D^t = \Pi_{t=1}^t D_t \). As

\[
\frac{1}{4} E \{ (\mu_t - cI)^2 \} = \frac{1}{4} Var(K_{t-1} GS_t + (D^t - K_{t-1} G)\mu_{t-1}) \\
+ \frac{1}{4} E \{ K_{t-1} GS_t + (D^t - K_{t-1} G)\mu_{t-1} - cI \}^2 \\
= \frac{1}{4} Var(K_{t-1} GS_t) + \frac{1}{4} [D^t \mu_{t-1} - cI]^2 \\
= \frac{1}{4} [D^t \mu_{t-1} - cI]^2 + \frac{1}{4} G K_{t-1} \Sigma_t K'_{t-1} G'.
\]

We normalize \( G = I \) and focus on

\[
\frac{1}{4} [D^t \mu_{t-1} - cI]^2 + \frac{1}{4} K_{t-1} \Sigma_t K'_{t-1}.
\]

Note that the simple Kalman filter setup cannot accommodate a multiplicative error case as for example investigated in Behringer (2021). The problem is that the estimated variance (that goes into the next period) will depend on the particular realization of the signal (via the action of the monopolist) and is no longer an unconditional error covariance as in the white noise case. One approach here would be to consider conditionally Gaussian models, see for example Harvey, (1989) p.155ff, but this lends itself to asymptotic econometric testings only.

In the dynamic setting investigated in this paper however we are still able to derive a non-monotonicity in Value of Information even in the white noise case.

3 \( \text{VoI}_t \) with increasing signal noise

The variance of the prior is recursively given by

\[
\Sigma_{t+1} = D \Sigma_t D' + FF' - D \Sigma_t G' (G \Sigma_t G' + HH')^{-1} G \Sigma_t D'
\]

The system given in Lemma 3 can be guaranteed to possess asymptotically stability, see Harvey, 1989, p.118ff.

In the following we focus on a single market and denote a generic element of a matrix \( X \) as \( X_{ii(t)} \). Alternatively one may assume that \( X \) denotes diagonal
matrices so that one could for example take roots by simply taking the roots of the terms on the major diagonal. Also then the elements of the updated variance \( \Sigma_{t+1} \) are increasing and concave in \( \Sigma_t \) and will be converging asymptotically over time.

This long-run (steady-state) variance in any market can be calculated as

\[
\Sigma^*_{ii} = \frac{1}{2G_{ii}^2} \left( \sqrt{H_{ii}^2 \left( H_{ii}^2 (D_{ii} - I_{ii})^2 (D_{ii} + I_{ii})^2 + 2F_{ii}^2 G_{ii}^2 (D_{ii}^2 + I_{ii}) \right) + F_{ii}^4 G_{ii}^4} + H_{ii}^2 (D_{ii} - I_{ii}) (D_{ii} + I_{ii}) + F_{ii}^2 G_{ii}^2 \right)
\]

which is approximately quadratic in the signal noise. In the "no-growth" case \( D = I \) with diagonal element \( I_{ii} = 1 \) we find

\[
\Sigma^*_{ii} \mid_{D=I} = \frac{F_{ii}}{2G_{ii}} \sqrt{F_{ii}^2 G_{ii}^2 + 4H_{ii}^2} + F_{ii}^2 G_{ii}^2
\]

to be approximately linear in the signal noise.

3.1 Pro-rata case \( Voi_t \left[ K'_{t-1} \right]^{-1} \)

As noted in Radner & Stiglitz (1984) in linear prediction settings monotone transformations of the information structure (e.g. by a square) may alter findings. This motivates our investigation of a time \( t \) pro-rata time \( Voi_t \), where the Kalman gain matrix is not multiplied with itself. While a good economic interpretation of this concept may be hard to come by we include it for its important didactic purposes. We find that:

**Lemma 6** \( Voi_t \left[ K'_{t-1} \right]^{-1} = K_{ii,t-1} \Sigma^*_{ii} \) has a global minimum at the lowest signal noise \( H_{ii} = 0 \).

**Proof:**

The result is not obvious as \( K_{ii,t-1} \), has a global maximum at \( H_{ii} = 0 \) and \( \Sigma^*_{ii} \), is convex in the signal noise with a global minimum at \( H_{ii} = 0 \) so their product needs to be explicitly determined. The first order condition is multiplicative in \( H_{ii} \) and thus shows the existence of an extreme value. Calculating its second order condition at this extreme value we find that

\[
\frac{\partial^2 (K_{ii,t-1} \Sigma^*_{ii})}{\partial H_{ii}^2} \bigg|_{H_{ii}=0} = (F_{ii} G_{ii})^2 D_{ii} (2G_{ii}^2 + I_{ii})^4 (D_{ii} - I_{ii}) (D_{ii} + I_{ii}) > 0
\]

which holds for any matrices as \( D > I \) holds element-wise.

This finding implies that the long-run conditional prior variance always increases faster in the signal noise than the Kalman gain decreases. We next show that this is no longer the case once we look at the full stage \( Voi_t \) with a squared Kalman gain as is implied in the chosen monopoly setting.

9
3.2 Full case $\text{VoI}_t$

The effect on the full stage Value of Information is fundamentally different once the square of the Kalman gain is taken into account. It turns out that in this case the increasing variance of the prior term only dominates the total effect for large values of signal noise, not small ones. This leads to a non-monotonicity in the monopoly $\text{VoI}_t$, implying that changes in the signal noise have an ambiguous effect on expected profits the exact direction of which will depend on the growth rate of the market. This is formally shown next:

**Proposition 7** \[ \text{VoI}_t = K_{ii,t-1} \Sigma_{ii} K'_{ii,t-1} \] has an (local) extreme value at the lowest signal noise $H_{ii} = 0$ and there exists a range of market growths $D_{ii}$ such that this extreme value is a maximum implying a non-monotonicity of the $\text{VoI}_t$ with respect to the signal precision.

**Proof:**

The first order condition $\partial (K_{ii,t-1} \Sigma_{ii} K'_{ii,t-1}) / \partial H_{ii}$ is again multiplicative in $H_{ii}$. The second derivative of the $\text{VoI}_t$ at the extreme value can be found as

$$\frac{\partial^2 (K_{ii,t-1} \Sigma_{ii} K'_{ii,t-1})}{\partial H_{ii}^2} \bigg|_{H_{ii}=0} = \frac{D_{ii}^2}{G_{ii}^4} (2G_{ii}^2 (D_{ii} - I_{ii})(D_{ii} + I_{ii}) + D_{ii}^2 - 3)$$

so that the extreme value is a maximum if the growth satisfies

$$I_{ii} \leq D_{ii} \leq \sqrt{\frac{8G_{ii}^2 + 4G_{ii}^4 + 3 [2G_{ii}^2 + I_{ii}]}{2G_{ii}^2}}$$

As $\text{VoI}_t$ is unbounded above with $D > I$ again holding element-wise, this implies a non-monotonicity in signal noise which completes the proof. Note that this result does not depend on the prior noise $F_{ii}$.

We have thus shown that even in a setup without a multiplicative signal error as investigated in Behringer (2021), in a dynamic model with a growing market we can find that the VoI to a monopoly firm may be non-monotonic in the signal precision. In the former model without growth the conditional prior variance is concave in the signal noise. In contrast in the present dynamic setting with growth, the long run conditional prior variance is convex in the signal noise.

In static settings, e.g. Wilson (1975) the weight to be put on the signal will be strictly decreasing in the signal noise while the variance of the prior is fixed and given. Hence the VoI in the static case in unambiguously decreasing in this noise. In a dynamic setting with growth the priors will be updated and the VoI at each stage will be subject to these updates. The VoI is composed of the updated prior variance and the optimal weights. As in the static setting the optimal weights will be decreasing in the signal noise but now, contrary to static models, the updated belief prior will increase in the signal noise.
4 Non-myopic case

Let the random linear(ized) demand now be

\[ q = \theta - B\beta p + H\gamma \]

where \( B \) and \( H \) are known matrices, additive noise is \( \gamma \sim N(0, I) \) and multiplicative noise is \( \beta \sim N(I, I) \) as in the two-stage game of Behringer (2021) where the interplay of the two noises results in a VoI non-monotonicity. Note that this implies a continuum of possible demand curves with varying slopes and intercepts. The demand intercept is:

\[ \theta_{t+1} = D\theta_t + F e_{t+1} \]

For this \( AR(1) \) process \( D \) and \( F \) are known matrices and \( \theta_t \) and \( e_t \) with \( e_t \sim N(0, I) \) are vectors and \( \theta_0 \sim N(\mu_0, \Sigma_0) \).

\[ \mu_t = E_t \{ \theta_t \} \text{ and } \Sigma_t = E_t \{ (\theta_t - \mu_t)^2 \} \]

Firms thus observe a signal system:

\[ S_{t+1} = \theta_t - B\beta_t p_t + H\gamma_t \]

where learning is more complicated as the random effects cannot be disentangled. The conditional signal variance is then:

\[ \text{Var}(S|\theta, p) = B\beta_t p_t' B + HH' \]

Optimization for per-period profit without learning yields the necessary condition for expected optimal myopic prices as

\[ \frac{dE_t \{ \Pi_t \}}{dp_t} = E_t \{ d\Pi_t \} = \mu_t + Bc - 2Bp_t = 0. \]

**Lemma 8** Optimal Bayesian beliefs about the unknown market potential can be found from the system

\[ \mu_{t+1} = D[K_t S_{t+1} + (I - K_t)\mu_t] \]

where \( K_t \) denotes the (now growth independently defined) Kalman-Bucy-Stratonovich gain

\[ K_t = \Sigma_t [\Sigma_t + Bp_t p_t' B + HH']^{-1} \]

with the conditional belief variance recursively determined as

\[ \Sigma_{t+1} = D(I - K_t)\Sigma_t D' + FF' = D\Sigma_t D' + FF' - DK_t \Sigma_t D'. \]
Full dynamic optimization via the Bellman equation with prior estimate \( \mu_t \) and variance \( \Sigma_t \) as state variables:

\[
V(\mu_t, \Sigma_t) = \max_{p_t} \left[ \Pi_t + \delta V(\mu_{t+1}, \Sigma_{t+1}) \right]
\]

and constant discount factor \( \delta \in (0, 1) \). Thus \( V(\mu_t, \Sigma_t) = \)

\[
\max_{p_t} \left\{ \Pi_t + \delta V(D(K_tS_{t+1} + (I-K_t)\mu_t), D(I-K_t)\Sigma_tD'+FF') \right\}
\]

where now \( K_t \) (and thus also \( \Sigma_{t+1} \)) depends on the choice of \( p_t \).

**Lemma 9** The Bellman equation can be rewritten as

\[
E_t \left\{ \frac{d\Pi_t}{dp_t} \right\} + \frac{\partial(K_t\sqrt{\Sigma_t + HH'} + Bp_t)'}{\partial \Sigma_t} \frac{\partial V}{\partial \Sigma_t} - E_t \left\{ \frac{\partial V}{\partial \Sigma_t} \right\} \times \delta D \left( \frac{\partial(K_t\sqrt{\Sigma_t + HH'} + Bp_t)'}{\partial \Sigma_t} \right) \left( I - \frac{\partial(K_t\Sigma_t)}{\partial \Sigma_t} + \frac{\partial(K_t\Sigma_t)}{\partial p_t} \right) D' = 0.
\]

**Proof:**

As the new signal \( S_{t+1} \) is normal with mean \( \mu_t \) and variance \( \Sigma_t + HH' + Bp_t'B' \) we have that

\[
\varepsilon_{t+1} = \frac{S_{t+1} - \mu_t}{\sqrt{\Sigma_t + HH' + Bp_t'B'}}
\]

is standard normally distributed. Replacing \( S_{t+1} \) in the Bellman equation yields

\[
V(\mu_t, \Sigma_t) = \max_{p_t} \left\{ \Pi_t + \delta V(D(\mu_t + K_t(\sqrt{\Sigma_t + HH'} + Bp_t'B')\varepsilon_{t+1}), \frac{D(I-K_t)\Sigma_tD'+FF'}{D(I-K_t)\Sigma_tD'+FF'}) \right\}.
\]

The Euler condition for this equation (with variables at \( t \) known) is:

\[
E_t \left\{ \frac{d\Pi_t}{dp_t} \right\} + \delta D \frac{\partial(K_t(\sqrt{\Sigma_t + HH'} + Bp_t'B'))}{\partial p_t} E_t \left\{ \frac{\partial V}{\partial \mu_{t+1}} \right\} - \delta \frac{\partial(DK_t\Sigma_tD')}{\partial p_t} E_t \left\{ \frac{\partial V}{\partial \Sigma_{t+1}} \right\} = 0.
\]

An application of the envelope theorem to the Bellman equation

\[
\max_{p_t} \left\{ \Pi_t + \delta V(D(\mu_t + K_t(\sqrt{\Sigma_t + HH'} + Bp_t'B')\varepsilon_{t+1}), \frac{D(I-K_t)\Sigma_tD'+FF'}{D(I-K_t)\Sigma_tD'+FF'}) \right\}
\]
yields

\[
\frac{\partial V}{\partial \Sigma_t} = \delta D \frac{\partial (K_t \sqrt{\Sigma_t + HH' + Bp_t p_t' B' t})}{\partial \Sigma_t} E_t \left\{ \frac{\partial V}{\partial \mu_{t+1} \varepsilon_{t+1}} \right\} + \\
\delta D \left( I - \frac{\partial (K_t \Sigma_t)}{\partial \Sigma_t} \right) D'E_t \left\{ \frac{\partial V}{\partial \Sigma_{t+1}} \right\}.
\]

Rearranging and substituting this into the Bellman equation we find the above expression. ■

**Lemma 10** The Euler equation can be simplified to

\[
E_t \left\{ \frac{d\Pi_t}{dp_t} \right\} - \frac{2Bp_t B' \Sigma_t}{\Sigma_t + 2Bp_t p_t' B' + 2HH'} \left( \frac{\partial V}{\partial \Sigma_t} - \delta D'E_t \left\{ \frac{\partial V}{\partial \Sigma_{t+1}} \right\} \right) = 0.
\]

**Proof:**

As optimal learning implies that

\[K_t = \Sigma_t \left[ \Sigma_t + Bp_t p_t' B' + HH' \right]^{-1}\]

calculating the derivatives we find that:

\[
\frac{\partial (K_t \sqrt{\Sigma_t + HH' + Bp_t p_t' B' t})}{\partial \Sigma_t} = \\
\left( I - \frac{\partial (K_t \Sigma_t)}{\partial \Sigma_t} \right) + \frac{\partial (K_t \Sigma_t)}{\partial p_t} = \\
\frac{\partial (K_t \sqrt{\Sigma_t + HH' + Bp_t p_t' B' t})}{\partial \Sigma_t} = \\
\frac{\partial (K_t \sqrt{\Sigma_t + HH' + Bp_t p_t' B' t})}{\partial p_t} = -2Bp_t B' \Sigma_t \\
\Sigma_t + 2Bp_t p_t' B' + 2HH'
\]

so that the Euler equations simplifies. ■

The equation represents the trade-off of present (with myopically optimal price) and future gains from optimal non-myopic pricing. As \(\frac{\partial V}{\partial \Sigma_t}\) and \(\frac{\partial V}{\partial \Sigma_{t+1}}\) still depend on it, an explicit calculation of the optimal non-myopic price is prevented.

We next proceed to show that an interplay between the multiplicative and the additive demand error remains if the latter is large, as in the non-monotonicity finding of Behringer (2021).

**Proposition 11** With large signal noise \(H_{it}\) the optimal non-myopic price will still depend on the degree of multiplicative demand variance indicating a non-monotonicity in the VoI.
Proof:
Again the Ricatti recursion for \( \Sigma_{t+1} \) converges to a long-run (steady-state) belief variance it can be solved directly as

\[
\Sigma^*_ii = \frac{1}{2} \left( \sqrt{(H^2_{ii} + B^2_{ii} p^2_{ii})^2 (D_{ii} - I_{ii})^2 (D_{ii} + I_{ii})^2 + 2F^2_{ii} (H^2_{ii} + B^2_{ii} p^2_{ii}) (D^2_{ii} + I_{ii}) + F^4_{ii}} \right).
\]

The critical term in the Euler equation attains a limit for large signal noise:

\[
\lim_{H_{ii} \to \infty} \left( -\frac{2B^2_{ii} p_t \Sigma^*_ii}{\Sigma^*_ii + 2B^2_{ii} p^2_{ii} + 2H^2_{ii}} \right) = -\frac{2B^2_{ii} p_t (D_{ii} - I_{ii}) (D_{ii} + I_{ii})}{D^2_{ii} + I_{ii}}.
\]

so that while falling and convex in the growth rate for \( D_{ii} > 0 \), even for a very unreliable signal the term still depends on the multiplicative noise in the demand function implying an effect on any optimal non-myopic price as \( E_t \{ d\Pi_t / dp_t \} \) is affine.

5 Conclusion

The two-stage analysis with multiplicative error investigated in Behringer (2021) has found a non-monotonicity in the Value of information for the monopolist resulting from the interplay of the variance of the multiplicative error and the level of the prior for the market potential demand and cost. A higher variance of the total signal noise (additive and multiplicative) will eventually decrease the total Value of Information in the two stage setup, in which there is only one signal and the conditional variance is only updated once, to zero. This results from the signal weight in the Bayesian process being strictly decreasing in the signal noise and the posterior conditional variance in the second stage being increasing but bounded.

In a fully dynamic framework where the moments of the estimates are state variables, a non-monotonicity of the Value of information can still be found when allowing for myopic agents and additive errors only. The long run updated conditional variance of the prior resulting from a Bayesian learning process will be increasing in the signal noise if market demand increases over time. Despite the fact that the signal weight in the Bayesian process resulting from the Kalman-Bucy-Stratonovich filter is strictly decreasing in the signal noise the pro-rata stage VoI will strictly increase. However, when squaring the weight in the full VoI analysis the decreasing weight will dominate for low values of market growth and signal noise leading to a non-monotonicity of the stage VoI in the signal noise, contrary to the findings in static models such as Wilson (1975).
With large signal noise an interplay of multiplicative and additive error (which leads to a non-monotone VoI in Behringer 2021) even if uncorrelated, prevails also in a fully dynamic, non-myopic setting. These results imply that the concerns expressed in Radner & Stiglitz (1984) are robust to the introduction of continuous signals and true dynamics. The overall intuition raised in Chade & Schlee (2002) or Arrow (1985) prevails but will have to be tested for even the classical quadratic settings (here motivated by monopoly) which allow for a monetization of the information concept. This may in some cases (including the present one) allow for analytical solutions and guidance.

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