JORDANIAN QUANTUM ALGEBRA $\mathcal{U}_h(\mathfrak{sl}(N))$ VIA CONTRACTION METHOD AND MAPPING

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May 2001

Abstract

Using the contraction procedure introduced by us in Ref. [20], we construct, in the first part of the present letter, the Jordanian quantum Hopf algebra $\mathcal{U}_h(\mathfrak{sl}(3))$ which has a remarkably simple co-algebraic structure and contains the Jordanian Hopf algebra $\mathcal{U}_h(\mathfrak{sl}(2))$, obtained by Ohn, as a subalgebra. A nonlinear map between $\mathcal{U}_h(\mathfrak{sl}(3))$ and the classical $\mathfrak{sl}(3)$ algebra is then established. In the second part, we give the higher dimensional Jordanian algebras $\mathcal{U}_h(\mathfrak{sl}(N))$ for all $N$. The Universal $R$-matrix of $\mathcal{U}_h(\mathfrak{sl}(N))$ is also given.

Keywords: Standard quantization, Nonstandard quantization, contraction procedure, Hopf algebra, universal $R$-matrix, Irreducible representations (irreps.).

1 Introduction

It is well known that the enveloping Lie algebra $\mathcal{U}(\mathfrak{sl}(N))$ has two quantizations: The first one called the Drinfeld-Jimbo deformation or the standard quantum deformation [1, 2] is quasitriangular ($R_{21}R \neq I$), whereas the second one called the Jordanian deformation or the non-standard quantum deformation [3] is triangular ($R_{21}R = I$). A typical example of Jordanian quantum algebras was first introduced by Ohn [4]. In general, nonstandard quantum algebras are obtained by applying Drinfeld twist to the corresponding Lie algebras [3]. The twisting that produces an algebra isomorphic to the Ohn algebra [4] is found in [3, 7].

Recently, the twisting procedure was extensively employed to study a wide variety of Jordanian deformed algebras, such as $\mathcal{U}_h(\mathfrak{sl}(N))$ algebras [8, 9, 10, 11], symplectic algebras [8] and [9, 10].
\( U_h(\text{sp}(N)) \) [12], orthogonal algebras \( U_h(\text{so}(N)) \) [13, 14, 15, 16] and orthosymplectic superalgebra \( U_h(\text{osp}(1|2)) \) [17, 18]. It appears from these studies that:

1. The non-standard quantum algebras have undeformed commutation relations;
2. The Jordanian deformation appear only in the coalgebraic structure;
3. The coproduct and the antipode maps have very complicated forms in comparison with the Drinfeld-Jimbo and the Ohn deformations.

To our knowledge, Jordanian quantum algebra \( U_h(\text{sl}(N)) \) has been written explicitly, with a simple coalgebra, only for \( N = 2 \) [4]. The main object of the present letter is to construct the Jordanian quantum algebra \( U_h(\text{sl}(3)) \) using the contraction procedure developed in [20] and the map studied in Refs. [20, 21]. The \( U_h(\text{sl}(3)) \) algebra presented here has the following properties:

1. The Ohn algebra \( U_h(\text{sl}(2)) \) is included in our structure \( U_h(\text{sl}(3)) \) in a natural way as a Hopf subalgebra and appear here from the longest root generators i.e. from \( e_3, f_3 \) and their corresponding Cartan generator \( h_3 \);
2. Our Jordanian deformed \( U_h(\text{sl}(3)) \) algebra may be regarded as the dual Hopf algebra of the function algebra \( \text{Fun}_h(\text{SL}(3)) \) studied in [22];
3. The present \( U_h(\text{sl}(3)) \) algebra is endowed with a relatively simple coalgebra structure (as compared to previous studies [8, 9, 10, 11]).

Implementing our contraction technique we subsequently obtain higher dimensional Jordanian quantum algebras \( U_h(\text{sl}(N)) \) for arbitrary values of \( N \).

This letter is organized as follows: The Jordanian quantum algebra \( U_h(\text{sl}(3)) \) is introduced via a nonlinear map and proved to be a Hopf algebra in section 2. The irreducible representations (irreps.) of \( U_h(\text{sl}(3)) \) are also given. Higher dimensional algebras \( U_h(\text{sl}(N)) \), \( N \geq 4 \) are presented in the sections 3 and 4.

## 2 \( U_h(\text{sl}(3)) \): Map, Hopf Algebra, Irreps. and \( \mathcal{R}_h \)-matrix

In this letter, \( \hbar \) is an arbitrary complex number. It was proved in [20] that the \( \mathcal{R}_h \)-matrix of the Jordanian quantum algebra \( U_h(\text{sl}(3)) \) can be obtained from the \( \mathcal{R}_q \)-matrix associated to the Drinfeld-Jimbo quantum algebra \( U_q(\text{sl}(3)) \) through a specific contraction which is singular in the \( q \to 1 \) limit. For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. Here we assume the \( U_q(\text{sl}(3)) \) Hopf algebra to be well-known [23].

For brevity and simplicity we limit ourselves to (fundamental irrep.) \( \otimes \) (arbitrary irrep.). Recall that for \( U_q(\text{sl}(3)) \) algebra the \( R_q \)-matrix in the representation (fund.) \( \otimes \) (arb.) reads [23]:

\[
R_q = \left( \pi_{(\text{fund.})} \otimes \pi_{(\text{arb.})} \right) \mathcal{R}_q = \begin{pmatrix}
    q^{\frac{1}{2}(2h_1+h_2)} & q^{\frac{1}{2}(2h_1+h_2)} \Lambda_{12} & q^{\frac{1}{2}(2h_1+h_2)} \Lambda_{13} \\
    0 & q^{-\frac{1}{2}h_2} & q^{-\frac{1}{2}(h_1-h_2)} \Lambda_{23} \\
    0 & 0 & q^{-\frac{1}{2}(h_1+2h_2)}
\end{pmatrix},
\]

(1)
where
\[
\begin{align*}
\Lambda_{12} &= q^{-1/2}(q - q^{-1})q^{-h_{12}/2} \hat{f}_1, \\
\Lambda_{13} &= q^{-1/2}(q - q^{-1}) \hat{f}_3 q^{-1/2(h_{13} + h_2)}, \\
\Lambda_{23} &= q^{-1/2}(q - q^{-1})q^{h_{23}/2} \hat{f}_2.
\end{align*}
\]

The elements \(k_i^{\pm} = q^{\pm h_1}, \ k_2^{\pm} = q^{\pm h_2}, \ k_3^{\pm} = q^{\pm (h_1 + h_2)}\), \(\hat{e}_1, \hat{e}_2, \hat{e}_3 = \hat{e}_1 \hat{e}_2 - q^{-1} \hat{e}_2 \hat{e}_1, \hat{f}_1, \hat{f}_2\) and \(\hat{f}_3 = \hat{f}_2 \hat{f}_1 - q \hat{f}_1 \hat{f}_2\) are the \(U_q(sl(3))\) generators. The corresponding classical generators are denoted by \(h_1, h_2, h_3 = h_1 + h_2, e_1, e_2, e_3 = e_1 e_2 - e_2 e_1, f_1, f_2\) and \(f_3 = f_2 f_1 - f_1 f_2\).

We have shown in [20] that the nonstandard \(R_h\)-matrix (in the representation \((\text{fund.}) \otimes \text{(arb.)})\) arise from the \(R_q\)-matrix (in \((\text{fund.}) \otimes \text{(arb.)})\) as follows:

\[
R_h = \lim_{q \to 1} \left[ E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \otimes E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \right]^{-1} R_q \left[ E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \otimes E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \right]
\]

\[
= \lim_{q \to 1} \begin{pmatrix}
E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 & -\frac{\hbar}{q - 1} E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \\
0 & E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 \\
0 & 0 & E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right)
\end{pmatrix} R_q \begin{pmatrix}
E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 & \frac{\hbar}{q - 1} E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \\
0 & E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 \\
0 & 0 & E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
T & 2 \hbar T^{-1/2} e_2 & -\frac{\hbar}{2} (T + T^{-1}) (h_1 + h_2) + \frac{\hbar}{2} (T - T^{-1}) \\
0 & I & -2 \hbar T^{1/2} e_1 \\
0 & 0 & T^{-1}
\end{pmatrix},
\]

where

\[
T = \hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}, \quad T^{-1} = -\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}.
\]

The deformed exponential in (3) is defined by

\[
E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \quad [n]! = [n] \times [n - 1]!,
\]

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [0]! = 1.
\]

The following properties can be pointed out:

1. The corner elements of (3) have exactly the same structure as in the \(R_h\)-matrix of \(U_h(sl(2))\). This implies that the classical generators \(e_3, h_3 = h_1 + h_2\) and \(f_3\) of \(U(sl(3))\) are deformed (for the nonstandard quantization: \(U(sl(3)) \rightarrow U_h(sl(3))\)) as follows [20, 21]:

\[
T = \hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}, \quad T^{-1} = -\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2},
\]

\[
H_3 = \sqrt{1 + \hbar^2 e_3^2} h_3, \quad F_3 = f_3 - \frac{\hbar^2}{4} e_3 (h_3^2 - 1),
\]
and evidently satisfy the commutation relations [4]

\[ T T^{-1} = T^{-1} T = 1, \]
\[ [H_3, T] = T^2 - 1, \quad [H_3, T^{-1}] = T^{-2} - 1, \]
\[ [T, F_3] = \frac{\hbar}{2} (H_3 T + T H_3), \quad [T^{-1}, F_3] = -\frac{\hbar}{2} (H_3 T^{-1} + T^{-1} H_3), \]
\[ [H_3, F_3] = -\frac{1}{2} (T F_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1}). \]  

(7)

With the following definition (see Ref. [4])

\[ E_3 = \hbar^{-1} \ln T = \hbar^{-1} \text{arcsinh } \hbar e_3, \]  

(8)

it follows that the elements \( H_3, E_3 \) and \( F_3 \) satisfy the relations

\[ [H_3, E_3] = 2 \frac{\sinh \hbar E_3}{\hbar}, \]
\[ [H_3, F_3] = -F_3 \left( \cosh \hbar E_3 \right) - \left( \cosh \hbar E_3 \right) F_3, \]
\[ [E_3, F_3] = H_3, \]  

(9)

where it is obvious that as \( \hbar \to 0 \), we have \( (H_3, E_3, F_3) \to (h_3, e_3, f_3) \). It is now evident from (7) that \( \mathcal{U}_h(sl(2)) \subset \mathcal{U}_h(sl(3)) \).

2. The expression (3) of the \( R_h \)-matrix indicates that the simple root generators \( e_1 \) and \( e_2 \) are deformed as follows:

\[ E_1 = \sqrt{\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} e_1 = T^{1/2} e_1, \]
\[ E_2 = \sqrt{\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} e_2 = T^{1/2} e_2. \]  

(10)

To complete our \( \mathcal{U}_h(sl(3)) \) algebra, we introduce the following \( \hbar \)-deformed generators:

\[ F_1 = \sqrt{-\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} f_1 + \frac{\hbar}{2} \sqrt{\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} e_2 h_3 = T^{-1/2} \left( f_1 + \frac{\hbar}{2} e_2 T h_3 \right), \]
\[ F_2 = \sqrt{-\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} f_2 - \frac{\hbar}{2} \sqrt{\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2}} e_1 h_3 = T^{-1/2} \left( f_2 - \frac{\hbar}{2} e_1 T h_3 \right), \]
\[ H_1 = \left( -\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2} \right) \left( \sqrt{1 + \hbar^2 e_3^2} h_1 + \frac{\hbar}{2} e_3 (h_1 - h_2) \right) = h_1 - \frac{\hbar}{2} e_3 T^{-1} h_3, \]
\[ H_2 = \left( -\hbar e_3 + \sqrt{1 + \hbar^2 e_3^2} \right) \left( \sqrt{1 + \hbar^2 e_3^2} h_2 - \frac{\hbar}{2} e_3 (h_1 - h_2) \right) = h_2 - \frac{\hbar}{2} e_3 T^{-1} h_3. \]  

(11)

The expressions (6), (10) and (11) constitute a realization of the Jordanian algebra \( \mathcal{U}_h(sl(3)) \) with the classical generators via a nonlinear map. This immediately yields the irreducible representations (irreps.) of \( \mathcal{U}_h(sl(3)) \) in an explicit and simple manner.
Proposition 1 The Jordanian algebra $\mathcal{U}_h(sl(3))$ is an associative algebra over $\mathbb{C}$ generated by $H_1$, $H_2$, $H_3$, $E_1$, $E_2$, $T$, $T^{-1}$, $F_1$, $F_2$ and $F_3$, satisfying, along with (7), the commutation relations

\[
\begin{align*}
[H_1, H_2] &= 0, & [H_1, T^{-1}H_3] &= [H_2, T^{-1}H_3] = 0, \\
[H_1, E_1] &= 2E_1, & [H_2, E_2] &= 2E_2, \\
[H_1, E_2] &= -E_2, & [H_2, E_1] &= -E_1, \\
[T^{-1}H_3, E_1] &= E_1, & [T^{-1}H_3, E_2] &= E_2, \\
[H_1, F_1] &= -2F_1 + \hbar E_2 T^{-1}H_3, & [H_2, F_2] &= -2F_2 - \hbar E_1 T^{-1}H_3, \\
[H_1, F_2] &= F_2 - \hbar E_1 T^{-1}H_3, & [H_2, F_1] &= F_1 + \hbar E_2 T^{-1}H_3, \\
[TH_3, F_1] &= -T^2F_1, & [TH_3, F_2] &= -T^2F_2, \\
[T^{-1}E_1, F_1] &= \frac{1}{2}(T + T^{-1})H_1 + \frac{1}{2}(T - T^{-1})H_2, \\
[T^{-1}E_2, F_2] &= \frac{1}{2}(T + T^{-1})H_2 + \frac{1}{2}(T - T^{-1})H_1, \\
[T^{-1}E_1, F_2] &= 0, & [T^{-1}E_2, F_1] &= 0, \\
[E_1, E_2] &= \frac{1}{2\hbar}(T^2 - 1), \\
[T F_2, T F_1] &= T \left( F_3 - \frac{\hbar}{2} H_3 T H_3 - \frac{\hbar}{8}(T - T^{-1}) \right) \\
[TH_1, T] &= \frac{1}{2}(T^2 - 1), & [TH_1, T^{-1}] &= \frac{1}{2}(T^{-2} - 1), \\
[TH_2, T] &= \frac{1}{2}(T^2 - 1), & [TH_2, T^{-1}] &= \frac{1}{2}(T^{-2} - 1), \\
[H_1, F_3] &= -\frac{T^{-1}}{4} \left( T F_3 + F_3 T + T^{-1}F_3 + F_3 T^{-1} \right) - \frac{\hbar}{4} T^{-1} H_3^2 - \frac{\hbar}{4} H_3 T^{-1} H_3, \\
[H_2, F_3] &= -\frac{T^{-1}}{4} \left( T F_3 + F_3 T + T^{-1}F_3 + F_3 T^{-1} \right) - \frac{\hbar}{4} T^{-1} H_3^2 - \frac{\hbar}{4} H_3 T^{-1} H_3, \\
[E_1, T] &= [E_1, T^{-1}] = [E_2, T] = [E_2, T^{-1}] = 0, \\
[F_1, T] &= \hbar T E_2, & [F_1, T^{-1}] &= -\hbar T^{-1} E_2, \\
[F_2, T] &= -\hbar T E_1, & [F_2, T^{-1}] &= \hbar T^{-1} E_1, \\
[E_1, F_3] &= -\frac{1}{2} \left( T F_2 + F_2 T \right), & [E_2, F_3] &= \frac{1}{2} \left( T F_1 + F_1 T \right), \\
[F_1, F_3] &= \hbar T F_1 - \hbar E_2 F_3 + \frac{\hbar^2}{4} T E_2, \\
[F_2, F_3] &= \hbar T F_2 + \hbar E_1 F_3 - \frac{\hbar^2}{4} T E_1.
\end{align*}
\]

(12)

Here we quoted only the final results. To obtain the realizations of $H_1$ and $H_2$ given in (11), we, in analogy with (6), started with the ansatz $\sqrt{1 + \hbar^2 e_3^2 h_1}$ and $\sqrt{1 + \hbar^2 e_3^2 h_2}$ for these
generators respectively. It is easy to see that

\[
\sqrt{1 + \hbar^2 e_3^2 h_1}, F_3 = -\frac{1}{4} (TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1})
\]
\[
+ \frac{\hbar^2}{4} (e_3(h_1 - h_2)H_3 + H_3e_3(h_1 - h_2)),
\]
\[
\sqrt{1 + \hbar^2 e_3^2 h_2}, F_3 = -\frac{1}{4} (TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1})
\]
\[
- \frac{\hbar^2}{4} (e_3(h_1 - h_2)H_3 + H_3e_3(h_1 - h_2)).
\]

(13)

Then, if we add to \(\sqrt{1 + \hbar^2 e_3^2 h_1}\) and deduct from \(\sqrt{1 + \hbar^2 e_3^2 h_2}\) the term \(\frac{\hbar}{2} e_3(h_1 - h_2)\), we obtain

\[
[(\sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3(h_1 - h_2)), F_3] = -\frac{1}{4} (TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1})
\]
\[
+ \frac{\hbar}{4} T(h_1 - h_2)H_3 + \frac{\hbar}{4} H_3T(h_1 - h_2),
\]
\[
[(\sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3(h_1 - h_2)), F_3] = -\frac{1}{4} (TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1})
\]
\[
- \frac{\hbar}{4} T(h_1 - h_2)H_3 - \frac{\hbar}{4} H_3T(h_1 - h_2).
\]

(14)

These commutation relations suggest the realizations \(H_1 \sim (\sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3(h_1 - h_2))\) and \(H_2 \sim (\sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3(h_1 - h_2))\). Finally, to preserve the Cartan subalgebra, we are obliged to multiply both of these expressions by \(T^{-1}\). The resultant maps for \(H_1\) and \(H_2\) are quoted in (11). The expressions of \(F_1\) and \(F_2\) are obtained in a similar way. The expressions (6), (10) and (11) may be looked now as a particular realization of the \(U_h(sl(3))\) generators. Other maps may also be considered.

**Proposition 2** In terms of the Chevalley generators (simple roots) \(\{E_1, E_2, F_1, F_2, H_1, H_2\}\), the algebra \(U_h(sl(3))\) is defined as follows:

\[
T = \left(1 + 2\hbar[E_1, E_2]\right)^{1/2},
\]
\[
T^{-1} = \left(1 + 2\hbar[E_1, E_2]\right)^{-1/2},
\]
\[
[H_1, H_2] = 0,
\]
\[
[H_1, E_1] = 2E_1,
\]
\[
[H_1, E_2] = -E_2,
\]
\[
[H_1, F_1] = -2F_1 + \hbar E_2(H_1 + H_2),
\]
\[
[H_1, F_2] = F_2 - \hbar E_1(H_1 + H_2),
\]
\[
[T^{-1}E_1, F_1] = \frac{1}{2}(T + T^{-1})H_1 + \frac{1}{2}(T - T^{-1})H_2,
\]
\[ [T^{-1}E_2, F_2] = \frac{1}{2}(T + T^{-1})H_2 + \frac{1}{2}(T - T^{-1})H_1, \]
\[ [T^{-1}E_1, F_2] = [T^{-1}E_2, F_1] = 0, \]
\[ E_1^2E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0, \]
\[ E_2^2E_1 - 2E_2E_1E_2 + E_1E_2^2 = 0, \]
\[ (TF_1)^2TF_2 - 2TF_1TF_2TF_1 + TF_2(TF_1)^2 = 0, \]
\[ (TF_2)^2TF_1 - 2TF_2TF_1TF_2 + TF_1(TF_2)^2 = 0, \]
\[ (15) \]

or, briefly
\[ [H_i, H_j] = 0, \]
\[ [H_i, E_j] = a_{ij}E_j, \]
\[ [H_i, F_j] = -a_{ij}F_j + T^{-1}[F_j, T](H_1 + H_2), \]
\[ [T^{-1}E_i, F_j] = \delta_{ij} \left( T^{-1}H_i + \frac{1}{2}(T - T^{-1})(H_1 + H_2) \right), \]
\[ (ad E_i)^{1-a_{ij}}(E_j) = 0, \quad i \neq j, \]
\[ (ad TF_i)^{1-a_{ij}}(TF_j) = 0, \quad i \neq j, \]
\[ (16) \]
where \( (a_{ij})_{i,j=1,2} \) is the Cartan matrix of \( \text{sl}(3) \), i.e. \( a_{11} = a_{22} = 2 \) and \( a_{12} = a_{21} = -1 \).

3. We now turn to the coalgebraic structure:

**Proposition 3** The Jordanian quantum algebra \( \mathcal{U}_h(\text{sl}(3)) \) admits a Hopf structure with coproducts, antipodes and counits determined by

\[ \Delta(E_1) = E_1 \otimes 1 + T \otimes E_1, \]
\[ \Delta(E_2) = E_2 \otimes 1 + T \otimes E_2, \]
\[ \Delta(T) = T \otimes T, \]
\[ \Delta(F_1) = F_1 \otimes 1 + T^{-1} \otimes F_1 + hH_3 \otimes E_2 \]
\[ = F_1 \otimes 1 + T^{-1} \otimes F_1 + T(H_1 + H_2) \otimes T^{-1}[F_1, T], \]
\[ \Delta(F_2) = F_2 \otimes 1 + T^{-1} \otimes F_2 - hH_3 \otimes E_1 \]
\[ = F_2 \otimes 1 + T^{-1} \otimes F_2 + T(H_1 + H_2) \otimes T^{-1}[F_2, T], \]
\[ \Delta(F_3) = F_3 \otimes T + T^{-1} \otimes F_3, \]
\[ \Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - \frac{1}{2}(1 - T^{-2}) \otimes T^{-1}H_3 \]
\[ = H_1 \otimes 1 + 1 \otimes H_1 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2), \]
\[ \Delta(H_2) = H_2 \otimes 1 + 1 \otimes H_2 - \frac{1}{2}(1 - T^{-2}) \otimes T^{-1}H_3 \]
\[ = H_2 \otimes 1 + 1 \otimes H_2 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2), \]
\[ \Delta(H_3) = H_3 \otimes T + T^{-1} \otimes H_3, \]
\[
S(E_1) = -T^{-1}E_1, \quad S(E_2) = -T^{-1}E_2,
\]
\[
S(T) = T^{-1}, \quad S(T^{-1}) = T,
\]
\[
S(F_1) = -TF_1 + hTH_3T^{-1}E_2 = -TF_1 + T^2(H_1 + H_2)T^{-2}[F_1, T],
\]
\[
S(F_2) = -TF_2 - hTH_3T^{-1}E_1 = -TF_2 + T^2(H_1 + H_2)T^{-2}[F_2, T],
\]
\[
S(F_3) = -TF_3T^{-1},
\]
\[
S(H_1) = -H_1 - \frac{1}{2}(T - T^{-1})H_3 = -H_1 - \frac{1}{2}(T^2 - 1)(H_1 + H_2),
\]
\[
S(H_2) = -H_2 - \frac{1}{2}(T - T^{-1})H_3 = -H_2 - \frac{1}{2}(T^2 - 1)(H_1 + H_2),
\]
\[
S(H_3) = -TH_3T^{-1},
\]
\[
\epsilon(a) = 0, \quad \forall a \in \{H_1, H_2, H_3, E_1, E_2, F_1, F_2, F_3\},
\]
\[
\epsilon(T) = \epsilon(T^{-1}) = 1. \quad (17)
\]

All the Hopf algebra axioms can be verified by direct calculations. Let us remark that our coproducts have simpler forms as compared to Refs. [8, 9, 10, 11].

**Proposition 4** The universal \(\mathcal{R}_h\)-matrix has the following form:

\[
\mathcal{R}_h = F^{-1} \mathcal{F}, \quad (18)
\]

where

\[
\mathcal{F} = \exp(hTH_3 \otimes E_3) \exp(2hTE_1 \otimes T^{-2}E_2). \quad (19)
\]

The \(\mathcal{R}\)-matrix properties are verified using MAPLE. The element (18) coincides with the universal \(\mathcal{R}\)-matrix of the Borel subalgebra and gives exactly the expression (3) in the representation (fund.) \(\otimes\) (arb.).

4. Following Drinfeld’s arguments [4], it is possible to construct a twist operator \(G \in \mathcal{U}(sl(3))^\otimes[[h]]\) relating the Jordanian coalgebraic structure given by (17) with the corresponding classical coalgebraic structure. For an invertible map \(m : \mathcal{U}_h(sl(3)) \to \mathcal{U}(sl(3)), m^{-1} : \mathcal{U}(sl(3)) \to \mathcal{U}_h(sl(3))\), the following relations hold:

\[
(m \otimes m) \circ \Delta \circ m^{-1}(\mathcal{X}) = G\Delta_0(\mathcal{X})G^{-1}, \quad m \circ S \circ m^{-1}(\mathcal{X}) = gS_0(\mathcal{X})g^{-1}, \quad (20)
\]

where \(\mathcal{X} \in \mathcal{U}(sl(3))[[h]]\) and \((\Delta_0, \epsilon_0, S_0)\) are the coproduct, counit and the antipode maps of the classical \(\mathcal{U}(sl(3))\) algebra. The transforming operator \(g(\in \mathcal{U}(sl(3))[[h]])\) and its inverse may be expressed as

\[
g = \mu \circ (\text{id} \otimes S_0)G, \quad g^{-1} = \mu \circ (S_0 \otimes \text{id})G^{-1}, \quad (21)
\]

where \(\mu\) is the multiplication map.
For the map presented here in (6), (10) and (11), we have the construction

\[ G = 1 \otimes 1 - \frac{1}{2} h \hat{r} + \frac{1}{8} h^2 \left[ \hat{r}^2 + 2(e_3 \otimes e_3) \Delta_0(h_3) \right] - \frac{1}{48} h^3 \left[ \hat{r}^3 + 6(e_3 \otimes e_3) \Delta_0(h_3) \hat{r} - 4(\Delta_0(e_3))^2 \hat{r} \right] + \frac{1}{384} h^4 \left[ \hat{r}^4 - 16(\Delta_0(e_3))^2 \hat{r}^2 + 12(e_3 \otimes e_3) \Delta_0(h_3) \hat{r}^2 + 12((e_3 \otimes e_3) \Delta_0(h_3))^2 + 6(e_3^2 \otimes 1 - 1 \otimes e_3^2)^2 \Delta_0(h_3) + 12(\Delta_0(e_3))^2 (e_3^2 \otimes 1 + 1 \otimes e_3^2) \Delta_0(h_3) - 8 \Delta_0(e_3)(e_3^3 \otimes 1 + 1 \otimes e_3^3) \Delta_0(h_3) - 10(\Delta_0(e_3))^4 \Delta_0(h_3) \right] + O(h^5), \]

(22)

where \( \hat{r} = h_3 \otimes e_3 - e_3 \otimes h_3 \). The above twist operators, while obeying the requirement (20) for the full \( \mathcal{U}(sl(3))[[h]] \) algebra, are, however, generated only by the elements \((e_3, h_3)\), related to the longest root. This property accounts for the embedding of the \( \mathcal{U}_h(sl(2)) \) algebra in the higher dimensional \( \mathcal{U}_h(sl(3)) \) algebra. The transforming operator \( g \) is obtained in (22) in a closed form. The series expansion of the twist operator \( G \) may be developed up to an arbitrary order in \( h \). The expansion (22) of the twist operator \( G \) in powers of \( h \) satisfies the cocycle condition

\[ (1 \otimes G)(id \otimes \Delta_0)G = (\Delta_0 \otimes 1)(1 \otimes G)(id \otimes \Delta_0) \]

(23)

upto the desired order. The present discussion of the twist operator relating to the \( \mathcal{U}_h(sl(3)) \) algebra may be easily extended to higher dimensional Jordanian algebras. (A systematic study of twists for \( \mathcal{U}_h(sl(2)) \) can be found in [21]).

5. Let us mention that there is a \( \mathbb{C} \)-algebra automorphism \( \phi \) of \( \mathcal{U}_h(sl(3)) \) such that

\[
\begin{align*}
\phi(T^{\pm 1}) &= T^{\pm 1}, \\
\phi(F_3) &= F_3, \\
\phi(H_3) &= H_3, \\
\phi(E_1) &= E_2, \\
\phi(F_1) &= F_2, \\
\phi(H_1) &= H_2, \\
\phi(E_2) &= -E_1, \\
\phi(F_2) &= -F_1, \\
\phi(H_2) &= H_1.
\end{align*}
\]

(24)

(For \( h = 0 \), this automorphism reduces to the classical one \((h_1, e_1, f_1, h_2, e_2, f_2) \rightarrow (h_2, e_2, f_2, h_1, -e_1, -f_1)\)). Also there is a second \( \mathbb{C} \)-algebra automorphism \( \varphi \) of \( \mathcal{U}_h(sl(3)) \) defined as:

\[
\begin{align*}
\varphi(T^{\pm 1}) &= -T^{\pm 1}, \\
\varphi(F_3) &= -F_3, \\
\varphi(H_3) &= -H_3, \\
\varphi(E_1) &= E_1, \\
\varphi(F_1) &= F_1, \\
\varphi(H_1) &= H_1, \\
\varphi(E_2) &= E_2, \\
\varphi(F_2) &= F_2, \\
\varphi(H_2) &= H_2.
\end{align*}
\]

(25)

6. The expressions (6), (10) and (11) permit immediate explicit construction of the finite-dimensional irreducible representations of \( \mathcal{U}_h(sl(3)) \). For example, the three-dimensional irreducible representations are spanned by

\[
H_1 = \begin{pmatrix} 1 & 0 & \frac{h}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{h}{2} \\ 0 & 0 & 0 \end{pmatrix},
\]

9
Here from the longest roots, i.e. from $e$, $h$.

The three-irrep. (27) is simply obtained from the irrep. (26) using the automorphism $\varphi$. The irrep. (27) has evidently no classical $(h = 0)$ limit.

3 $U_h(sl(4))$: Map and $R_h$-matrix

The major interest of our approach is that it can be generalized for obtaining Jordanian quantum algebras $U_h(sl(N))$ of higher dimensions. Here we illustrate our method using $U(sl(4))$ as an example. Let $h_1 = e_{11} - e_{22} \equiv h_{12}$, $h_2 = e_{22} - e_{33} \equiv h_{23}$, $h_3 = e_{33} - e_{44} \equiv h_{34}$, $e_1 \equiv e_{12}$, $e_2 \equiv e_{23}$, $e_3 \equiv e_{34}$, $f_1 \equiv e_{21}$, $f_2 \equiv e_{32}$ and $f_3 \equiv e_{43}$ be the standard Chevalley generators (simple roots) of $U(sl(4))$. The others roots obtained by action of the Weyl group are denoted by $e_{13} = [e_{12}, e_{23}]$, $e_{14} = [e_{13}, e_{34}]$, $e_{24} = [e_{23}, e_{34}]$, $e_{31} = [e_{32}, e_{21}]$, $e_{41} = [e_{43}, e_{31}]$, $e_{42} = [e_{43}, e_{32}]$, $h_{13} = h_{12} + h_{23}$, $h_{14} = h_{12} + h_{23} + h_{34}$ and $h_{24} = h_{23} + h_{34}$. As for $U_h(sl(3))$, the Jordanian deformation arises here from the longest roots, i.e. from $e_{14}$, $e_{41}$ and $h_{14}$. These generators are deformed as follows:

$$
T = he_{14} + \sqrt{1 + h^2 e_{14}^2},
$$

$$
T^{-1} = -he_{14} + \sqrt{1 + h^2 e_{14}^2},
$$

$$
E_{41} = e_{41} - \frac{h^2}{4} e_{14}(h_{14}^2 - 1),
$$

$$
H_{14} = \sqrt{1 + h^2 e_{14}^2} h_{14},
$$

with the well-known coproducts

$$
\Delta(T) = T \otimes T,
\Delta(T^{-1}) = T^{-1} \otimes T^{-1},
\Delta(E_{41}) = E_{41} \otimes T + T^{-1} \otimes E_{41},
\Delta(H_{14}) = H_{14} \otimes T + T^{-1} \otimes H_{14}.
$$

(28)
It is now easy to verify that $H_{23} + H_{34} = H_{24}$, $[E_{12}, E_{23}] = E_{13}$, $[E_{32}, E_{21}] = E_{31}$, $H_{12} + H_{23} = H_{13}$, $[E_{23}, E_{34}] = E_{24}$, $[E_{43}, E_{32}] = E_{42}$. (33)

*Each subsets forms a $\mathcal{U}(sl(3))$ subalgebra in $\mathcal{U}(sl(4))$. 
Proposition 5  The generating elements \( H_1 \equiv H_{12}, \; H_2 \equiv H_{23}, \; H_3 \equiv H_{34}, \; E_1 \equiv E_{12}, \; E_2 \equiv E_{23}, \; E_3 \equiv E_{34}, \; F_1 \equiv E_{21}, \; F_2 \equiv E_{32}, \; F_3 \equiv E_{43} \) of the Jordanian quantum algebra \( \mathcal{U}_h(sl(4)) \) obey the following commutation rules:

\[
T = \left(1 + 2h[E_1, [E_2, E_3]]\right)^{1/2}, \; \quad T^{-1} = \left(1 + 2h[E_1, [E_2, E_3]]\right)^{-1/2},
\]

\[
[H_1, H_2] = [H_1, H_3] = [H_2, H_3] = 0, \; \quad [H_1, E_1] = 2E_1, \; \quad [H_1, E_2] = -E_2, \; \quad [H_1, E_3] = 0, \; \quad [H_2, E_1] = -E_1, \; \quad [H_2, E_2] = 2E_2, \; \quad [H_2, E_3] = -E_3, \; \quad [H_3, E_1] = 0, \; \quad [H_3, E_2] = -E_2, \; \quad [H_3, E_3] = 2E_3,
\]

\[
[H_1, F_1] = -2F_1 + T^{-1}[F_1, T](H_1 + H_2 + H_3), \; \quad [H_1, F_2] = F_2, \; \quad [H_1, F_3] = T^{-1}[F_3, T](H_1 + H_2 + H_3),
\]

\[
[H_2, F_1] = F_1, \; \quad [H_2, F_2] = -2F_2, \; \quad [H_2, F_3] = F_3, \; \quad [H_3, F_1] = T^{-1}[F_1, T](H_1 + H_2 + H_3), \; \quad [H_3, F_2] = F_2,
\]

\[
[H_3, F_3] = -2F_3 + T^{-1}[F_3, T](H_1 + H_2 + H_3), \; \quad [T^{-1}E_1, F_1] = T^{-1}H_1 + \frac{1}{2}(T - T^{-1})(H_1 + H_2 + H_3),
\]

\[
[E_2, F_2] = H_2, \; \quad [T^{-1}E_3, F_3] = T^{-1}H_3 + \frac{1}{2}(T - T^{-1})(H_1 + H_2 + H_3), \; \quad [T^{-1}E_1, F_2] = 0,
\]

\[
[E_2, F_1] = [E_2, F_3] = 0, \; \quad [T^{-1}E_3, F_1] = [T^{-1}E_3, F_2] = 0,
\]

\[
[E_1, E_3] = [T F_1, TF_3] = 0, \; \quad E_1 E_2 - 2E_1 E_2 E_1 + E_2 E_1^2 = 0, \; \quad E_1 E_2^2 - 2E_2 E_1 E_2 + E_2^2 E_1 = 0,
\]

\[
E_2^2 E_3 - 2E_2 E_3 E_2 + E_3 E_2^2 = 0, \; \quad E_2 E_3^2 - 2E_3 E_2 E_3 + E_3^2 E_2 = 0,
\]

\[
(TF_1)^2F_2 - 2TF_1F_2TF_1 + F_2(TF_1)^2 = 0, \; \quad TF_1F_2^2 - 2F_2TF_1F_2 + F_2^2TF_1 = 0,
\]

\[
(TF_3)^2F_2 - 2TF_3F_2TF_3 + F_2(TF_3)^2 = 0, \; \quad F_2^2TF_3 - 2F_2TF_3F_2 + TF_3F_2^2 = 0 \quad (34)
\]

or, briefly,

\[
[H_i, H_j] = 0, \; \quad [H_i, E_j] = a_{ij}E_j, \; \quad [H_i, F_j] = -a_{ij}F_j + (\delta_{i1} + \delta_{i3})T^{-1}[F_j, T](H_1 + H_2 + H_3),
\]

\[
[T^{-\delta_{i1}+\delta_{i3}}E_i, F_j] = \delta_{ij}\left(T^{-\delta_{i1}+\delta_{i3}}H_i + \frac{(\delta_{i1} + \delta_{i3})(T - T^{-1})(H_1 + H_2 + H_3)}{2}\right),
\]

\[
[E_i, E_j] = [T^{\delta_{i1}+\delta_{i3}}F_i, T^{\delta_{i1}+\delta_{i3}}F_j] = 0, \; \quad AD_{E_i}^{1-a_{ij}}(E_j) = 0, \; \quad (i \neq j), \; \quad AD(T^{\delta_{i1}+\delta_{i3}}F_i)^{1-a_{ij}}(T^{\delta_{i1}+\delta_{i3}}F_j) = 0, \; \quad (i \neq j), \quad (35)
\]
where \((a_{ij})_{i,j=1,2,3}\) is the Cartan matrix of \(sl(4)\).

**Proposition 6** The non-cocommutative coproduct structure of \(U_h(sl(4))\) reads:

\[
\begin{align*}
\Delta(E_1) &= E_1 \otimes 1 + T \otimes E_1, \\
\Delta(E_2) &= E_2 \otimes 1 + 1 \otimes E_2, \\
\Delta(E_3) &= E_3 \otimes 1 + T \otimes E_3, \\
\Delta(F_1) &= F_1 \otimes 1 + T^{-1} \otimes F_1 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_1, T], \\
\Delta(F_2) &= F_2 \otimes 1 + T^{-1} \otimes F_2, \\
\Delta(F_3) &= F_3 \otimes 1 + T^{-1} \otimes F_3 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_3, T], \\
\Delta(H_1) &= H_1 \otimes 1 + 1 \otimes H_1 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3), \\
\Delta(H_2) &= H_2 \otimes 1 + 1 \otimes H_2, \\
\Delta(H_3) &= H_3 \otimes 1 + 1 \otimes H_3 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3). 
\end{align*}
\]

(36)

In the (fund.) \(\otimes\) (arb.) representation, the \(R_h = (\pi_{\text{fund.}} \otimes \pi_{\text{arb.}}) R_h\) take the following simple form:

\[
R_h = \begin{pmatrix} 
T & 2hT^{-1/2}e_{24} & 2hT^{-1/2}e_{34} & -\frac{h}{2}(T + T^{-1})(h_1 + h_2 + h_3) + \frac{h}{2}(T - T^{-1}) \\
0 & I & 0 & -2hT^{1/2}e_{12} \\
0 & 0 & I & -2hT^{1/2}e_{13} \\
0 & 0 & 0 & T^{-1} \end{pmatrix}.
\]

(37)

**Proposition 7** The universal \(R_h\)-matrix for \(U_h(sl(4))\) may be cast in the form:

\[
R_h = F^{-1}_{21} F, 
\]

(38)

where

\[
F = \exp \left( hTH_{14} \otimes E_{14} \right) \exp \left( 2hTE_{34} \otimes T^{-2}E_{13} + 2hTE_{24} \otimes T^{-2}E_{12} \right),
\]

(39)

\[
E_{14} = h^{-1} \ln T = h^{-1} \text{arcsinh} \, h e_{14}.
\]

(40)

The \(R_h\)-matrix (38) coincides with the universal \(R\)-matrix of the Borel subalgebra. Let us just note that the tensor elements \(TE_{34} \otimes T^{-2}E_{13}\) and \(TE_{24} \otimes T^{-2}E_{12}\) commute.

### 4 \(U_h(sl(N))\): Generalization

The \(U_h(sl(5))\) algebra is derived in a similar way: The elements \(E_2, E_3, F_2, F_3, H_2, H_3\) are not affected by the nonstandard quantization. From these above studies, It is easy to see that:
Proposition 8  The Jordanian quantization deform $\mathcal{U}_h(sl(N))$’s Chevalley generators as follows:

$$
T = h[e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]] + \sqrt{1 + h^2([e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]])^2},
$$

$$
T^{-1} = -h[e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]] + \sqrt{1 + h^2([e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]])^2},
$$

$$
E_i = T^{(\delta_{i1} + \delta_{i,N-1})/2} e_i,
$$

$$
F_i = T^{-(\delta_{i1} + \delta_{i,N-1})/2} \left( f_i + \frac{h}{2} T[f_i, [e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]](h_1 + \cdots + h_{N-1}) \right)
$$

$$
H_i = h_i - \frac{(\delta_{i1} + \delta_{i,N-1})h}{2}[e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}] \cdots]]T^{-1}(h_1 + \cdots + h_{N-1}) \quad (i = 1, \cdots, N-1)
$$

and they satisfy the commutation relations

$$
[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j,
$$

$$
[H_i, F_j] = -a_{ij} F_j + (\delta_{i1} + \delta_{i,N-1})T^{-1}[F_j, T](H_1 + \cdots + H_{N-1}),
$$

$$
[T^{-(\delta_{i1} + \delta_{i,N-1})} E_i, F_j] = \delta_{ij} \left( T^{-(\delta_{i1} + \delta_{i,N-1})} H_i + \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (T - T^{-1})(H_1 + \cdots + H_{N-1}) \right),
$$

$$
[E_i, E_j] = 0, \quad |i - j| > 1,
$$

$$
[T^{(\delta_{i1} + \delta_{i,N-1})} F_i, T^{(\delta_{j1} + \delta_{j,N-1})} F_j] = 0, \quad |i - j| > 1,
$$

$$
(ad E_i)^{1-a_{ij}}(E_j) = 0, \quad (i \neq j),
$$

$$
(ad T^{(\delta_{i1} + \delta_{i,N-1})} F_i)^{1-a_{ij}}(T^{(\delta_{j1} + \delta_{j,N-1})} F_j) = 0, \quad (i \neq j),
$$

where $(a_{ij})_{i,j=1,\cdots,N}$ is the Cartan matrix of $sl(N)$, i.e. $a_{ii} = 2$, $a_{i,i\pm 1} = -1$ and $a_{ij} = 0$ for $|i - j| > 1$.

The algebra (42) is called the Jordanian quantum algebra $\mathcal{U}_h(sl(N))$. The expressions (41) may be regarded as a particular nonlinear realization of the $\mathcal{U}_h(sl(N))$ generators.

Proposition 9  The Jordanian algebra $\mathcal{U}_h(sl(N))$ (42) admits the following coalgebra structure:

$$
\Delta(E_i) = E_i \otimes 1 + T^{(\delta_{i1} + \delta_{i,N-1})} \otimes E_i,
$$

$$
\Delta(F_i) = F_i \otimes 1 + T^{-(\delta_{i1} + \delta_{i,N-1})} \otimes F_i + T(H_1 + \cdots + H_{N-1}) \otimes T^{-1}[F_i, T],
$$

$$
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i - \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (1 - T^{-2}) \otimes (H_1 + \cdots + H_{N-1}),
$$

$$
S(E_i) = -T^{-(\delta_{i1} + \delta_{i,N-1})} E_i,
$$

$$
S(F_i) = -T^{(\delta_{i1} + \delta_{i,N-1})} F_i + T^2(H_1 + \cdots + H_{N-1})T^{-2}[F_i, T],
$$

$$
S(H_i) = -H_i + \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (1 - T^2)(H_1 + \cdots + H_{N-1}),
$$

$$
\epsilon(E_i) = \epsilon(F_i) = \epsilon(H_i) = 0. \quad (43)
$$
 Proposition 10 The $R_{\hbar}$-matrix of $U_{\hbar}(\mathfrak{sl}(N))$ has the following general form:

$$R_{\hbar} = F_{21}^{-1} F,$$

(44)

where

$$F = \exp\left(\hbar TH_{1N} \otimes E_{1N}\right) \exp\left(\sum_{k=2}^{N-1} 2\hbar TE_{kN} \otimes T^{-2}E_{1k}\right),$$

(45)

$$H_{1N} = T(H_1 + \cdots + H_{N-1}),$$

(46)

$$E_{1N} = \hbar^{-1} \ln T = \hbar^{-1} \arcsinh \hbar e_{1N},$$

(47)

$$E_{kN} = [E_k, \cdots, [E_{N-2}, E_{N-1}]], \quad k = 2, \cdots, N - 2,$$

(48)

$$E_{N-1,N} = E_{N-1},$$

(49)

$$E_{12} = E_1,$$

(50)

$$E_{1k} = [E_1, \cdots, [E_{k-2}, E_{k-1}]], \quad k = 3, \cdots, N - 1$$

(51)

and may be obtained from the $R_q$-matrix associated to $U_q(\mathfrak{sl}(N))$ via the contraction procedure discussed above, i.e.

$$R_{\hbar} = \lim_{q \to 1} \left[ E_q\left(\frac{\hbar e_{1N}}{q - 1}\right) \otimes E_q\left(\frac{\hbar e_{1N}}{q - 1}\right)\right]^{-1} \mathcal{R}_q \left[ E_q\left(\frac{\hbar e_{1N}}{q - 1}\right) \otimes E_q\left(\frac{\hbar e_{1N}}{q - 1}\right)\right].$$

(52)

It is interesting to note that, via the nonlinear map (41), the $\hbar$-deformed generators $(E_i, F_i, H_i)$ may be also equipped with an induced co-commutative coproduct. Similarly, the undeformed generators $(e_i, f_i, h_i)$, via the inverse map, may be viewed as elements of the $U_{\hbar}(\mathfrak{sl}(N))$ algebra; and, thus, may be endowed with an induced noncommutative coproduct.

Acknowledgments: One of us (BA) wants to thank Professor Peter Forgacs for a kind invitation to the University of Tours, where parts of this work was done. He is also grateful to the members of the group for their kind hospitality.

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