Effective local compactness and the hyperspace of located sets

Arno Pauly
Swansea University
Swansea, UK
Arno.M.Pauly@gmail.com

1 Introduction

We revisit the question of how to effectivize the notion of local compactness. Our approach is in the traditions of both synthetic topology [7] (taking the notion of continuous function as a primitive, and employing category-theoretic machinery) and computable topology. Computable topology explores the effective counterparts to definitions and theorems of general topology, which serve in particular as part of the foundations of computable analysis.

There have been previous studies of effective local compactness, albeit restricted to computable Polish spaces. We compare the definitions in Subsection 4.3 and show that they are equivalent. There are some subtleties involved, which could be interpreted as demonstrating that the previous definitions were (even for computable Polish spaces) prima facie too restrictive (as for some examples establishing their effective local compactness would be more work than it should be).

While our definition of effective local compactness (Definition 11) works for arbitrary represented spaces, to make good use the notion we quickly find ourselves desiring additional structure. We thus restrict our attention to countably-based spaces from Section 3 onwards, and obtain the notion of an effectively relatively compact system (ercs) in Definition 5 as the combinatorial underpinning of effective local compactness in countably-based spaces. In, in addition, we demand our spaces to be (computably) Hausdorff, we enter the realm of (computably) metrizable spaces, which we explore in Section 4.

As application of effective local compactness, in Section 5 we study the space \((\mathcal{A} \land V)(X)\) of closed-and-overt subsets of a given space (or of located sets or of closed subsets with full information, depending on the nomenclature). We show that \(X\) admitting an ercs suffices to make \((\mathcal{A} \land V)(X)\) computably compact (Corollary 27) and computably metrizable (Corollary 32). This generalizes a previous result: [12] shows that for a computably compact computable metric space \(X\), the space \((\mathcal{A} \land V)(X) \cong (K \land V)(X)\) is a computably compact computable metric space again. In the restricted setting, the metric on \((\mathcal{A} \land V)(X)\) is the Hausdorff distance. Since [12] thus can work with a known construction of the metric, whereas we employ Schröder’s effective metrization theorem, the proofs are very different.

To establish \((\mathcal{A} \land V)(X)\) as computably compact means that universal quantification over closed-and-overt sets preserves open predicates, following the characterization of computable compactness in [13]. This gives a clear indication of how effective local compactness can be used.
2 Defining effective local compactness

For non-Hausdorff spaces, there are several competing (and non-equivalent) definitions of local compactness, see e.g. the overview provided in the Wikipedia article [1]. We will effective the existence of a compact local neighborhood basis for our purposes:

**Definition 1.** We call a represented space $X$ **effectively locally compact**, if the map

$$\text{CompactBase} : \subseteq X \times \mathcal{O}(X) \Rightarrow \mathcal{O}(X) \times K(X)$$

with $\text{dom}(\text{CompactBase}) = \{(x,U) \mid x \in U\}$ and $(V,K) \in \text{CompactBase}(x,U)$ iff $x \in V \subseteq K \subseteq U$ is computable.

Just as for the classical notion, we see that effective local compactness is preserved by taking open or closed subspaces:

**Proposition 2.** Let $A \in \mathcal{A}(X)$ be computable and $X$ be effectively locally compact. Then the subspace $A$ of $X$ is effectively locally compact.

*Proof.* We are given $x \in A$ and $U \in \mathcal{O}(A)$. We can compute $U \cup A^C \in \mathcal{O}(X)$. We apply $\text{CompactBase}_{X}(x,U \cup A^C)$ to obtain some $(V,K) \in \mathcal{O}(X) \times K(X)$. From that, we can compute $V \cap U \in \mathcal{O}(A)$, and $K \cap A \in K(A)$, and these are a valid output to $\text{CompactBase}_{A}(x,U)$. □

**Proposition 3.** Let $Y \in \mathcal{O}(X)$ be computable and $X$ be effectively locally compact. Then the subspace $Y$ of $X$ is effectively locally compact.

*Proof.* For the computably open subspace $Y$ we have the canonic computable embedding $\text{id} : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. From that, we also get that $\text{id} : \subseteq K(X) \rightarrow K(Y)$ is computable. Together, these yield the claim. □

**Corollary 4.** Let $X$ be effectively locally compact, $A \in \mathcal{A}(X)$ be computable and $Y \in \mathcal{O}(X)$ be computable. Then the subspace $A \cap Y$ of $X$ is effectively locally compact.

*Proof.* Note that $A \cap Y$ is a computably open subset of $A$, and combine Propositions 2, 3. □

Taking more general subspaces does not preserve local compactness. In fact, any locally compact subspace of a Hausdorff space is the intersection of an open and a closed subspace (see e.g. [10]). As such, Corollary 4 already realizes the full extent of what we can hope for.

3 Effective local compactness for countably-based spaces

In countably-based spaces, we can ask for a specific structure that witnesses effective local compactness. Manipulating this structure will be how we prove further results.

**Definition 5.** Let an effective relatively compact system (ercs) of a represented space be a triple $((U_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}, R)$ where

1. $(U_n \in \mathcal{O}(X))_{n \in \mathbb{N}}$ is a computable sequence of open sets;
2. $(B_n \in K(\mathbb{N}))_{n \in \mathbb{N}}$ is a computable sequence of compact sets;
3. and $R \subseteq \mathbb{N} \times \mathbb{N}$ is a computably enumerable relation such that $(m,n) \in R$ implies $U_m \subseteq B_n$;

...
such that for any open set $U \in \mathcal{O}(X)$ it holds that:

$$U = \bigcup_{\{n \mid U \supseteq B_n\}} \bigcup \{U_m \mid (m,n) \in R\}$$

The idea is that $R$ codes a formal containment relation between the enumerated open and compact sets. We shall write $U_n \ll B_m$ for $(n,m) \in R$.

**Proposition 6.** Let $((U_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}, R)$ be an ercs of $X$. Then $(U_n)_{n \in \mathbb{N}}$ is an effective countable basis of $X$.

**Proof.** We are given some $x \in X$, $U \in \mathcal{O}(X)$ with $x \in U$, and we are searching for some $n \in \mathbb{N}$ such that $x \in U_n \subseteq U$. By assumption, we have that:

$$U = \bigcup_{\{k \mid U \supseteq B_k\}} \bigcup \{U_n \mid (n,k) \in R\}$$

Thus, $x \in U$ implies that there are $k,n$ with $B_k \subseteq U$, $U_n \ll B_k$ and $x \in U_n$. By definition of compact sets, we can effectively enumerate all $k$ such that $B_k \subseteq U$, by definition of $\ll$ we can enumerate all $n$ such that $U_n \ll B_k$, and by definition of open sets we can enumerate all $n$ such that $x \in U_n$. We will thus eventually find $k,n$ as above, and can then output $n$.\qed

Having an ercs is a form of effective local compactness, as witnessed by:

**Proposition 7.** Let $X$ have an ercs. Then the $X$ is effectively locally compact, i.e. the map

$$\text{CompactBase} : \subseteq X \times \mathcal{O}(X) \to \mathcal{O}(X) \times \mathcal{K}(X)$$

with $\text{dom}(\text{CompactBase}) = \{(x,U) \mid x \in U\}$ and $(V,K) \in \text{CompactBase}(x,U)$ iff $x \in V \subseteq K \subseteq U$ is computable.

**Proof.** We proceed as in Proposition 6 but output both $U_n$ as $V$ and $B_k$ as $K$.\qed

Conversely, the existence of an effective countable basis as in Proposition 6 and effective local compactness together are almost enough to imply the existence of an ercs – we only need a mild additional constraint:

**Proposition 8.** Let $X$ admit an effective countable basis $(U_n)_{n \in \mathbb{N}}$, a representation $\delta$ with a computable dense sequence $(p_n)_{n \in \mathbb{N}}$ in $\text{dom}(\delta)$ \footnote{To be precise, we only need that $\text{dom}(\delta)$ is computably overt, and that there is a computable dense sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ for our proof. It is clear that we can construct specific representations fulfilling the more specific criteria, but not the general ones. It is less clear whether this can be extended to an entire equivalence class.} and have the map

$$\text{CompactBase} : \subseteq X \times \mathcal{O}(X) \to \mathcal{O}(X) \times \mathcal{K}(X)$$

be computable. Then we can construct $(B_n \in \mathcal{K}(\mathbb{N}))_{n \in \mathbb{N}}$ and $R \subseteq \mathbb{N} \times \mathbb{N}$ to make an ercs.

**Proof.** First, we construct the $B_k$. For that, set $x_\ell := \delta(p_\ell)$. We go through all $x_\ell$, $U_n$ with $x_\ell \in U_n$ and call $\text{CompactBase}(x_\ell, U_n)$ to obtain some $V_{\ell n} \in \mathcal{O}(X)$ and $B_{\ell n} \in \mathcal{K}(X)$ with $x_\ell \in V_{\ell n} \subseteq B_{\ell n} \subseteq U_n$. Since $(U_n)_{n \in \mathbb{N}}$ is an effective countable basis, we can find some $m$ such
that \( x_\ell \in U_m \subseteq V_{\ell n} \subseteq B_{\ell n} \subseteq U_n \). For any combination of \( n, m \) arising in this way, we pick some \( k(n,m) \) and set \( B_k = B_m \).

Now, we construct \( R \). Given some \( p \in \text{dom}(\delta) \) and some \( n \in \mathbb{N} \) with \( \delta(p) \in U_n \), we can apply first \( \text{CompactBase}(\delta(p), U_n) \) to obtain compact \( K \) and open \( V \) with \( \delta(p) \in V \subseteq K \subseteq U_n \), and then use the fact that \( (U_n)_{n \in \mathbb{N}} \) is an effective countable basis to find some \( m \in \mathbb{N} \) with \( \delta(p) \in U_m \subseteq V \). We only consider the map \( (p, n) \mapsto m \). This has a single-valued computable choice function \( \chi \). We can thus construct the open sets \( O_{nm} = \{ p \in \text{dom}(\delta) \mid \delta(p) \in U_n \land \chi(p, n) = m \} \) of \( \text{dom}(\delta) \). Since \( \text{dom}(\delta) \) contains a computable dense sequence, it is computably overt. Thus, defining \( (m, k(n,m)) \in R \iff O_{nm} \neq \emptyset \) yields a computably enumerable relation, which by construction satisfies that \( U_i \ll B_j \Rightarrow U_i \subseteq B_j \).

It remains to show the main property of an ercs. It suffices to do so for basic open sets \( U_n \). Moreover, the right-to-left inclusion is trivial. Thus, we start with some \( x \in U_n \). Then there is some name \( p \in \text{dom}(\delta) \) with \( \delta(p) = x \). Note that by construction, we have that \( \bigcup_{m \in \mathbb{N}} O_{nm} = \delta^{-1}(U_n) \), thus there exists some \( m \) such that \( p \in O_{nm} \). But then \( x \in U_m \subseteq B_{k(n,m)} \subseteq U_n \), and \( U_m \ll B_{k(n,m)} \), and thus \( x \) is present on the right hand side of the main property.

**Remark 9.** By dropping any computability requirements from Proposition 8 we can conclude that any countably-based locally compact space will admit an ercs relative to some oracle. In particular, the notion of admitting an ercs passes the fundamental sanity check for being a notion of effective local compactness for countably based spaces.

We briefly explore how admitting an ercs, being compact, and being computably compact are related:

**Proposition 10.** Let \( X \) admit an ercs and be compact. Then \( X \) is computably compact.

**Proof.** By the main property of an ercs, we find that \( X = \bigcup_{k \in \mathbb{N}} \bigcup_{\ell \in \{\ell \mid (\ell, k) \in R \}} U_{\ell} \). This is an open cover of \( X \).Compactness of \( X \) implies that there is a finite subcover, and that in particular there is a finite set \( K \subseteq \mathbb{N} \) with \( X = \bigcup_{k \in K} \bigcup_{\ell \in \{\ell \mid (\ell, k) \in R \}} U_{\ell} \). But this implies \( X = \bigcup_{k \in K} B_k \), and computably compact sets are closed under finite union. \( \square \)

**Example 11.** There is an effectively countably-based computably compact space without an ercs (which is, in fact, not locally compact at all).

**Proof.** Let \( \hat{Q} \) be a one-point compactification of \( Q \) (with the Euclidean topology). This means that the underlying set is \( Q \cup \{\infty\} \), and the topology is generated by open subsets of \( Q \) together with sets of the form \( \{\infty\} \cup (Q \setminus I) \) for finite sets \( I \).

In terms of representation, a name of a point \( x \in \hat{Q} \) starts with listing some rationals \( q_0, q_1, q_2 \), guaranteeing that \( x \neq q_i \). Either this enumeration continues forever and exhausts all rationals, in which case \( x = \infty \). Alternatively, we reach a special stop flag, and after that read a usual \( \mathbb{Q} \)-name of a point different from the rationals enumerated so far.

This space is easily seen to be computably compact (after all, covering \( \infty \) leaves only finitely many points to check). However, the open set \( \mathbb{Q} \subseteq \hat{Q} \) contains only compact sets with empty interior, there cannot be an ercs. \( \square \)
4 Effective local compactness in computable metric spaces

If a space admits an ercs and is computably Hausdorff, it is already computably metrizable. This follows very directly from Schröder’s effective metrization theorem [15, §]. The latter states that computably regular effectively countably-based spaces are computably metrizable. Their formulation of being computably regular actually takes the very same form as the definition of ercs, except that closed sets are used in the place of compact sets. Since being computably Hausdorff suffices to translate from compact sets to closed sets, it follows that a computably Hausdorff space admitting an ercs is already computably regular.

Being computably metrizable is strictly more general than being a computable metric space, although every computably metrizable space embeds into a computable metric space. Still, this shows that restricting to computable metric space is not that restrictive in the context of effective local compactness. We will see that we can say a few more things using the language of metric spaces.

4.1 Finding compact balls

In computable metric spaces, we can be more specific regarding how the sets $B_n$ in ercs look like; namely, we can demand that the compact sets be closed balls:

**Proposition 12.** Let $(X, d)$ be a computable metric space. Then the following are equivalent:

1. $X$ admits an ercs.
2. The map $\text{CompactBall} : X \Rightarrow (\mathbb{N} \times \mathcal{K}(X))$ where $(n, K) \in \text{CompactBall}(x)$ iff $K = \overline{B}(x, 2^{-n})$ is well-defined and computable.

**Proof.** 1. $\Rightarrow$ 2. By Proposition [7] given $x \in X$ we can compute some $V \in \mathcal{O}(X)$ and $B \in \mathcal{K}(X)$ with $x \in V \subseteq B$. In a computable metric space, given $x \in X$ and $V \in \mathcal{O}(X)$, we can compute some $n \in \mathbb{N}$ such that $B(x, 2^{-n}) \subseteq V$. Since computable metric spaces are Hausdorff, compact sets are closed, and thus $B(x, 2^{-n}) \subseteq B$ implies $\overline{B}(x, 2^{-n}) \subseteq B$. Moreover, we can compute $\overline{B}(x, 2^{-n}) \in \mathcal{A}(X)$, and $\cap : \mathcal{A}(X) \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is computable for arbitrary represented spaces $X$. Thus, we can obtain $\overline{B}(x, 2^{-n}) = \overline{B}(x, 2^{-n}) \cap B \in \mathcal{K}(X)$.

2. $\Rightarrow$ 1. A computable metric space has a dense sequence $(x_n)_{n \in \mathbb{N}}$. We apply CompactBall to each to obtain $(k_n)_{n \in \mathbb{N}}$ and the sequence $(\overline{B}(x_n, 2^{-k_n}))_{n \in \mathbb{N}}$. Since we can compute any $\overline{B}(x_n, 2^{-i}) \in \mathcal{A}(X)$, and $\cap : \mathcal{A}(X) \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is computable; we obtain the computable double-sequence $(\overline{B}(x_n, 2^{-\max\{k_n,j\}}))_{n,j \in \mathbb{N}}$. As basic open sets we use $(B(x_n, 2^{-i}) \in \mathcal{O}(X))_{n,j \in \mathbb{N}}$, and we set $B(x_n, 2^{-i}) \ll \overline{B}(x_{m}, 2^{-\max\{k_{m},j\}})$ iff $d(x_n, x_m) + 2^{-\max\{k_{m},j\}} < 2^{-i}$. It is a straight-forward calculation to verify that this fulfills the main property of an ercs.

**Corollary 13.** Every computably compact computable metric space admits an ercs.

---

2We need to carefully distinguish between the closed balls $\overline{B}(x, r)$ and the closures of open balls $\text{cl}B(x, r)$ in this context. In general, the former are available as members of $\mathcal{A}(X)$, or, in the compact case, $\mathcal{K}(X)$. The latter are available as members of $\mathcal{V}(X)$.
4.2 Local compactness vs $\sigma$-compactness

A notion somewhat related to local compactness is $\sigma$-compactness. We briefly explore their relationship in the effective setting.

Definition 14. Call $X$ effectively $\sigma$-compact, if there is a computable $(K_n \in K(X))_{n \in \mathbb{N}}$ with $X = \bigcup_{n \in \mathbb{N}} K_n$.

Observation 15. Every space admitting an ercs is effectively $\sigma$-compact, with the sets $(B_n \in K(X))_{n \in \mathbb{N}}$ of the ercs serving as witness.

Being locally compact does not imply being $\sigma$-compact, as is well-known even for metric spaces. A potential counter-example works as follows:

Example 16. Let $S$ have the underlying set $\{\infty\} \cup \{(n,x) \mid n \in \mathbb{N} \land x \in [0,1]\}$ and the metric $d$ satisfying that $d(\infty,(n,x)) = 2^{-n} + x$, $d((n,x),(n,y)) = |x-y|$ and $d((n,x),(m,y)) = x + y + 2^{-n} + 2^{-m}$ for $n \neq m$. This is readily seen to be a computable metric space, and the sets $K_\infty = \{\infty\}$, $K_n = \{n\} \times [0,1]$ form an effective partition into compact sets. Yet any closed ball $\overline{B}(\infty,2^{-j})$ contains the sequence $((\ell,2^{-j-1})_{\ell>j})$ which has no accumulation point. Hence $\overline{B}(\infty,2^{-j})$ is never compact, and $S$ not locally compact.

We can modify this example to yield further separating constructions. In particular, we have:

Example 17. There exists a computable metric space which is effectively $\sigma$-compact and locally compact, but not effectively locally compact.

Proof. Pick some effective enumeration $(\Phi_s)_{s \in \mathbb{N}}$ of some Turing machines. For each $s$, we consider the subspace $S_s$ of $S$ defined by $\infty \in S$ and $(n,x) \in S_s$ iff $x \geq 2^{-j}$ and $\Phi_s$ writes exactly $j$ symbols when run for $n$ steps. We now consider the disjoint union $\bigcup_{s \in \mathbb{N}} S_s$, which is an effectively $\sigma$-compact computable metric space again.

If the $\Phi_s$ are such that any $\Phi_s$ will write only finitely many symbols at all, then $\bigcup_{s \in \mathbb{N}} S_s$ is locally compact, since it is a discrete union of countably many singletons and copies of the unit interval. For it to be effectively locally compact, we need to be able to compute the map CompactBall by Proposition 12 which in particular would mean that given $s \in \mathbb{N}$ we can compute some $d \in \mathbb{N}$ such that $\overline{B}(\infty_s,2^{-d})$ is compact, where $\infty_s$ is the $s$-th copy of $\infty$. The reasoning in Example 16 shows that that means that $\Phi_s$ will never write more than $d$ symbols. We can easily chose a family $(\Phi_s)_{s \in \mathbb{N}}$ such that each $\Phi_s$ writes only finitely many times, but such that we cannot compute an upper bound on how often from $s$.\[\square\]

4.3 Comparison to notions in the literature

[17] Definition 3] defines a computable metric space $X$ to be effectively locally compact, if there is a computable positive function $\gamma : X \to \mathbb{R}$ such that each $\overline{B}(x,\gamma(x))$ is compact. Proposition 12 shows that our definition implies theirs. Any compact metric space trivially satisfies their condition, but by combination of Proposition 10 and Corollary 13 a compact computable metric space admits an ercs iff it is computably compact. Thus, a compact but not computably compact computable metric space separates the two notions.

In [16] (the journal version of the conference paper [17]), the requirement is that given $x \in X$ and $0 < \delta \in \mathbb{R}$, one can compute some $\overline{B}(x,\rho) \in (K \land V)(X)$ with $\overline{B}(x,\rho) \subseteq B(x,\delta)$. We observe the following:
Proposition 18. The map $\text{Radius} : \subseteq X \times (K \land V)(X) \Rightarrow \mathbb{R}$ mapping $x, K$ to some $\rho \in \mathbb{R}$ such that $K = B(x, \rho)$ is computable.

Proof. We can compute $\rho_- \in \mathbb{R}_+$ such that $\rho_- < r$ iff $K \subseteq B(x, r)$ since we have $K$ as a compact set. We can also compute $\rho_+ \in \mathbb{R}_+$ such that $r < \rho_+$ iff $B(x, r) \cap K \neq \emptyset$ since we have $K$ as an overt set. Now any $\rho$ with $\rho_1 > \rho \leq \rho_+$ is a valid answer. We can start with the assumption that $\rho_1 = \rho_+$, and start computing $\mathbb{R} \ni \rho = \rho_- = \rho_+$ from that. If $\rho_- < \rho_+$, we will eventually notice, and can then output some rational in that interval. □

Since the \(\Rightarrow\) 1.-direction of Proposition [12] does not make use of the radius having the special form $2^{-n}$, we see that the definition of effective local compactness from [16] implies the existence of an ercs. The difference lies in [16] requiring that the closed balls are produced as compact and overt sets. In general, we can compute $clB(x, r)$ as an overt set given $x \in X$ and $r \in \mathbb{R}$, but not $\overline{B}(x, r)$. Of course, for the cases that $clB(x, r) = \overline{B}(x, r)$, we can obtained the ball as both closed (respectively compact) and overt set. This is, in a way, the typical case:

Lemma 19 (Banakh [21]). Let $(X, d)$ be a separable metric space and $x \in X$. Then $\{r \in \mathbb{R} | clB(x, r) \neq \overline{B}(x, r)\}$ is countable.

Proof. Fix a countable basis $(U_n)_{n \in \mathbb{N}}$. Let $R = \{r \in \mathbb{R} | clB(x, r) \neq \overline{B}(x, r)\}$. For each $r \in R$, pick some $y(r) \in \overline{B}(x, r) \setminus clB(x, r)$. Then there must be some $n(r) \in \mathbb{N}$ with $y(r) \in U_{n(r)}$ and $U_{n(r)} \cap B(x, r) = \emptyset$. If $|R| > |\mathbb{N}|$, there must be $r_1, r_2 \in R$ with $r_1 < r_2$ but $n(r_1) = n(r_2)$. Now we have that $y(r_1) \in U_{n(r_1)}$, $d(x, y(r_1)) = r_1$ and $U_{n(r_1)} \cap B(x, r_2) = \emptyset$, contradiction. So $R$ is countable. □

Proposition 20. Let $X$ be computable Polish space. The map $\text{NiceRadius} : \subseteq X \times \mathbb{R}^\geq_0 \times K(X) \Rightarrow \mathbb{R}$ with $(x, r, K) \in \text{dom}(\text{NiceRadius})$ iff $K = \overline{B}(x, r)$ and $r' \in \text{NiceRadius}$ iff $0 < r' < r$ and $clB(x, r') = \overline{B}(x, r')$ is computable.

Proof. For each $d \in \mathbb{R}$ with $0 < d < r$ we can compute $\{y \in X | d(x, y) = d\} \in K(X)$, as this is trivially computable as a closed set, and we then take the intersection with the provided compact set $\overline{B}(x, r)$. Now given $n \in \mathbb{N}$ and $\{y \in X | d(x, y) = d\} \in K(X)$, we can semidecide if $\forall y \in X (d(x, y) = d \Rightarrow B(x, d) \cap B(y, 2^{-n}) \neq \emptyset)$, since this is a universal quantification over a compact set and an open predicate. From this, we see that for each $n \in \mathbb{N}$, we can obtain the open set:

$$U_n = \{d \in \mathbb{R} | 0 < d < r \land \forall y \in X (d(x, y) = d \Rightarrow B(x, d) \cap B(y, 2^{-n}) \neq \emptyset)\}$$

Note that $clB(x, d) = \overline{B}(x, d) \Leftrightarrow \forall n \in \mathbb{N} d \in U_n$. The set $D := \bigcap_{n \in \mathbb{N}} U_n$ is available to us as a $\Pi^0_2$-set, and by Lemma [19] it is co-countable and hence co-meager. We can thus apply the Computable Baire Category theorem [5] to compute some $r' \in D$. □

Corollary 21. A computable Polish space is effectively locally compact in the sense of [16] iff it admits an ercs.

It is not clear whether the requirements of having a surrounding compact ball, or of the space being Polish, are actually needed. We thus raise the following question:

Open Question 22. Is the map $\text{OvertBall} : X \times \mathbb{R}^\geq_0 \Rightarrow \mathcal{V}(X) \times \mathbb{R}^\geq_0$ defined by $(K, r') \in \text{OVERTBALL}(x, r)$ iff $r' < r$ and $\overline{K} = \overline{B}(x, r')$, computable for all computable metric spaces? Or at least for all computable Polish spaces?
is considering countably-based Hausdorff spaces, and is defining effective local compactness in terms of an enumeration of a basis \((O_n)_{n \in \mathbb{N}}\) making \(O_n \subseteq O_m\) and \(\text{cl} O_n \subseteq O_m\) decidable in \(n\) and \(m\); and moreover there is a cover \(X = \bigcup_{i \in \mathbb{N}} X_i\) by compact subspaces, such that \(X_i \setminus O_n \subseteq O_{m_1} \cup \ldots \cup O_{m_k}\) is decidable in the indices. It is immediate that this requirement implies ours, but asking for actual containment to be decidable rather than employing formal containment makes the framework of [6] much more restrictive.

The definition of effectively local compactness for computable Polish spaces in [11] is asking for a cover \(X = \bigcup_{i \in \mathbb{N}} X_i\), where \((X_i \in (V \land K)(X))_{i \in \mathbb{N}}\) is a computable sequence, and from \(x \in X\) we can compute some \(0 < \delta \in \mathbb{R}\) and \(i \in \mathbb{N}\) such that \(B(x, \delta) \subseteq X_i\). From Proposition [12] we conclude that their definition implies ours. What is missing for the converse is that we only get a cover by compact sets, not by compact and overt sets. We can use Proposition 20 to circumvent this, and conclude that for computable Polish spaces the definitions of effective local compactness from [16], from [11] and from this paper all agree.

5 The hyperspace \((A \land V)(X)\)

We move on to our application of the machinery of effective local compactness. We study the hyperspace \((A \land V)(X)\) of sets given as both closed and overt. In the language of Weyhrauch [7], this is the full information representation of the closed sets. In constructive mathematics, the computable elements of \((A \land V)(X)\) are often called located.

Our main result is that whenever \(X\) admits an ercs, then \((A \land V)(X)\) is computably compact and computably metrizable. This generalizes a result from [12] for computably compact computable metric spaces.

5.1 Compactness of \((A \land V)(X)\)

We fix a space \(X\) with an ercs \(((U_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}, R)\).

**Definition 23.** Call \(p \in \{0, 1\}^\mathbb{N}\) consistent, if whenever \(p(n) = 1\), \(U_n \ll B_k\) and \(B_k \subseteq U_{n_0} \cup \ldots \cup U_{n_\ell}\), then \(p(n_i) = 1\) for some \(i \leq \ell\).

**Theorem 24.**

1. The set of consistent \(p\) is \(\Pi^0_1\).
2. If for some set \(A\) we have that \(p(n) = 1 \iff A \cap U_n \neq \emptyset\), then \(p\) is consistent.
3. If \(p\) is consistent, then \(\{U \in \mathcal{O}(X) \mid \exists n, \ell \in \mathbb{N}, p(n) = 1 \land U_n \ll B_\ell \land B_\ell \subseteq U\}\) uniformly defines an overt set \(A \in \mathcal{V}(X)\).
4. Moreover, we have that \(X \setminus A = \bigcup_{\{n \mid p(n) = 0\}} U_n\).

**Proof.**

1. We can enumerate all conditions that need to be fulfilled, and decide for any enumerated condition whether it is indeed fulfilled.
2. If \(p(n) = 1\), then there exists some \(x \in A \cap U_n\). If then \(U_n \ll B_k\) and \(B_k \subseteq U_{n_0} \cup \ldots \cup U_{n_\ell}\), then in particular \(U_n \subseteq U_{n_0} \cup \ldots \cup U_{n_\ell}\), hence there is some \(i \leq \ell\) with \(x \in U_{n_i}\). By assumption, it follows that \(p(n_i) = 1\), i.e. \(p\) is consistent.
3. Given some consistent \( p \), consider the set \( A = \{ x \in X \mid \forall n \in \mathbb{N} \ x \in U_n \rightarrow p(n) = 1 \} \). We claim that for any open set \( U \), we have that \( \exists n, \ell \in \mathbb{N} \ p(n) = 1 \wedge U_n \ll B_\ell \wedge B_\ell \subseteq U \iff U \cap A \neq \emptyset \).

Assume that \( x \in U \cap A \). With the argument in Proposition 7, given \( x \in U \) we can find \( n, \ell \in \mathbb{N} \) such that \( x \in U_n, U_n \ll B_\ell \) and \( B_\ell \subseteq U \). Since \( x \in A \), from \( x \in U_n \) we can conclude \( p(n) = 1 \), and conclude this direction of the argument.

For the other direction, we assume that \( \exists n, \ell \in \mathbb{N} \ p(n) = 1 \wedge U_n \ll B_\ell \wedge B_\ell \subseteq U \) yet \( U \cap A = \emptyset \), and seek to derive a contradiction. It immediately follows that \( B_\ell \cap A = \emptyset \). By definition of \( A \), that means that for each \( x \in B_\ell \) there exists some \( U_n(x) \) with \( x \in U_n(x) \) and \( p(n(x)) = 0 \). Since \( B_\ell \subseteq \bigcup_{x \in B_\ell} U_n(x) \) and \( B_\ell \) is compact, there are finitely many \( n_0, \ldots, n_j \) chosen amongst the \( n(x) \) such that \( B_\ell \subseteq U_{n_0} \cup \ldots \cup U_{n_j} \). Together with \( U_n \ll B_\ell, p(n) = 1 \) and consistency of \( p \) we would get that \( p(n_i) = 1 \) for some \( i \leq j \), contradicting the choice of the \( n_i \).

4. If \( x \in U_n \) and \( p(n) = 0 \), then it follows by construction that \( x \notin A \). Conversely, if \( x \notin A \), then there must be some \( n \in \mathbb{N} \) with \( x \in U_n \) and \( p(n) = 0 \).

\[ \square \]

**Corollary 25.** If \( X \) admits an ercs, then there is a \( \Pi^0_1 \)-subset \( A \) of \( \{0,1\}^\mathbb{N} \) such that \( V(X) \) is computably isomorphic to \( EC^{-1}(A) \subseteq \mathcal{O}(\mathbb{N}) \), where \( EC : \mathcal{O}(\mathbb{N}) \rightarrow \{0,1\}^\mathbb{N} \) maps an enumeration of a set to its characteristic function.

**Corollary 26.** If \( X \) admits an ercs, then there is a \( \Pi^0_1 \)-subset \( A \) of \( \{0,1\}^\mathbb{N} \) and a computable surjection \( \psi : A \rightarrow (A \wedge V)(X) \).

**Corollary 27.** If \( X \) admits an ercs, then \( (A \wedge V)(X) \) is computably compact.

### 5.2 Metrizability \( (A \wedge V)(X) \)

**Proposition 28.** \( \not\subseteq : V(X) \times A(X) \rightarrow S \) is computable.

**Proof.** Note that \( A \not\subseteq B \) is equivalent to \( A \cap B^C \neq \emptyset \), and the latter is semidecidable by definition of \( V \) and \( A \) in terms of \( \mathcal{O} \). \( \square \)

**Corollary 29.** For arbitrary \( X \), \( (A \wedge V)(X) \) is computably Hausdorff.

**Proposition 30.** Let \( X \) admit an ercs. Then \( (A \wedge V)(X) \) embeds into \( T^\omega \).

**Proof.** Fix an \( (U_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}, R \). We define the computable embedding \( \phi : (A \wedge V)(X) \rightarrow T^\omega \) as follows: If \( U_n \cap A \neq \emptyset \), then \( \phi(A)(n) = 1 \) – we can recognize this having access to \( A \in V(X) \). If there exists some \( \ell \in \mathbb{N} \) with \( B_\ell \cap A = \emptyset \) (which we can recognize by knowing \( A \in A(X) \)) and \( U_n \ll B_\ell \), then \( \phi(A)(n) = 0 \). Else \( \phi(A)(n) = \bot \).

To compute the inverse of \( \phi \), first note that we can recover \( A \in A(X) \) by noting that \( A^C = \bigcup_{n|\phi(A)(n)=0} U_n \). Then, note that by the main property of an ercs, we have that \( U \cap A \neq \emptyset \) for open \( U \) iff \( \exists n, \ell \in \mathbb{N} \ \text{such that} \ U \supseteq B_\ell, U_n \ll B_\ell \) and \( U_n \cap A \neq \emptyset \). Using this twice, we can recover \( A \in V(X) \) from \( \phi(A) \in T^\omega \) via \( U \cap A \neq \emptyset \) iff \( \exists n, \ell \in \mathbb{N} \ \phi(A)(n) = 1 \wedge U_n \ll B_\ell \wedge B_\ell \subseteq U \). \( \square \)

**Corollary 31.** Let \( X \) admit an ercs. Then \( (A \wedge V)(X) \) admits an effective countable basis.
By combining Corollaries 27, 29 and 31 we see that whenever $X$ admits an ercs, then $(A \land V)(X)$ is a computably compact computably Hausdorff effectively countably-based space. It was shown in [14] (using Schröder’s effective metrization theorem [15, 8]) that these conditions together imply computable metrizability. We thus get:

**Corollary 32.** Let $X$ admit an ercs. Then $(A \land V)(X)$ is computably metrizable.

### 5.3 Related work in classical topology

[9] shows that for a Hausdorff space $X$, the Fell topology is normal iff $X$ is Lindelöf and locally compact; and [1] shows that for Hausdorff $X$, the Fell topology is Hausdorff iff $X$ is locally compact. Compactness of the Fell topology, however, holds for arbitrary Hausdorff spaces [3]. We thus see the precise opposite for the behaviour of the Fell topology and of the space $(A \land V)(X)$. This perplexing feature shall serve as a reminder that the hyperspace constructions in our work are exploiting the cartesian-closure of our ambient category, and thus do not apply to $\text{TOP}$. 

### Acknowledgements

This research was partially supported by the Royal Society International Exchange Grant 170051 “Continuous Team Semantics: On dependence and independence in a continuous world”.

The author is grateful to Matthew de Brecht for enlightening discussion.

### References

[1] Locally Compact Space. Wikipedia. Available at [https://en.wikipedia.org/wiki/Locally_compact_space](https://en.wikipedia.org/wiki/Locally_compact_space).

[2] Taras Banakh. Closed balls vs closure of open balls. MathOverflow. Available at [https://mathoverflow.net/q/303836](https://mathoverflow.net/q/303836). URL:https://mathoverflow.net/q/303836 (version: 2018-06-28).

[3] G. Beer (1993): Topologies on Closed and Closed Convex Sets. Kluwer Academic, Dordrecht.

[4] G. Beer & R. Tamaki (1994): The infimal value functional and the uniformization of hit-and-miss hyperspace topologies. Proc. Amer. Math. Soc. 122, pp. 601–611. Available at [https://www.jstor.org/stable/2161055](https://www.jstor.org/stable/2161055).

[5] Vasco Brattka (2001): Computable versions of Baire’s Category Theorem. In: Mathematical Foundations of Computer Science 2001, LNCS 2136, Springer, pp. 224–235.

[6] Abbas Edalat (2009): A computable approach to measure and integration theory. Information and Computation 207(5), pp. 642 – 659. Available at [http://www.sciencedirect.com/science/article/pii/S0890540109000200](http://www.sciencedirect.com/science/article/pii/S0890540109000200) From Type Theory to Morphological Complexity: Special Issue dedicated to the 60th Birthday Anniversary of Giuseppe Longo.

[7] Martín Escardó (2004): Synthetic topology of datatypes and classical spaces. Electronic Notes in Theoretical Computer Science 87.

[8] Tanja Grubba, Matthias Schröder & Klaus Weihrauch (2007): Computable Metrization. Mathematical Logic Quarterly 53(4-5), pp. 381–395.

[9] L. Holá, S. Levi & J. Pelant (1999): Normality and paracompactness of the Fell topology. Proc. Amer. Math. Soc. 127, pp. 2193–2197.
user642796 (https://math.stackexchange.com/users/8348/user642796). *Locally compact subspace is an intersection of an open and closed set.* Mathematics Stack Exchange. Available at https://math.stackexchange.com/q/644089.

Hiroyasu Kamo (2000): *Effective Contraction Theorem and its Applications.* In: Peter Hertling Jens Blanck, Vasco Brattka, editor: *Computability and Complexity in Analysis: 4th International Workshop, CCA 2000, Swansea, UK, September 17-19, 2000. Selected Papers*, Springer.

Chansu Park, Jiwon Park, Sewon Park, Dongseong Seon & Martin Ziegler (2017). *Computable Operations on Compact Subsets of Metric Spaces with Applications to Fréchet Distance and Shape Optimization.* arXiv 1701.08402. Available at http://arxiv.org/abs/1701.08402.

Arno Pauly (2016): *On the topological aspects of the theory of represented spaces.* Computability 5(2), pp. 159–180. Available at http://arxiv.org/abs/1204.3763.

Arno Pauly & Hideki Tsuiki (2016). $T^\omega$-representations of compact sets. arXiv:1604.00258.

Matthias Schröder (1998): *Effective metrization of regular spaces.* In: K.-I. Ko, A. Nerode, M. B. Pour-El, K. Weihrauch & J. Wiedermann, editors: *Computability and Complexity in Analysis, Informatik Berichte* 235, FernUniversität Hagen.

Klaus Weihrauch & Zheng Xizhong (1999): *Effectiveness of the global modulus of continuity on metric spaces.* Theoretical Computer Science 219(1), pp. 439 – 450. Available at http://www.sciencedirect.com/science/article/pii/S0304397598002990.

Klaus Weihrauch & Xizhong Zheng (1997): *Effectiveness of the global modulus of continuity on metric spaces.* In: Eugenio Moggi & Giuseppe Rosolini, editors: *Category Theory and Computer Science (CTCS 97)*, Springer.