THE HÖRMANDER MULTIPLIER THEOREM III: THE COMPLETE BILINEAR CASE VIA INTERPOLATION

LOUKAS GRAFAKOS AND HANH VAN NGUYEN

ABSTRACT. We prove the optimal version of the Hörmander multiplier theorem concerning bilinear multiplier operators with symbols in the Sobolev space $L^{rs}(\mathbb{R}^2)$, $rs > 2n$, uniformly over all annuli. More precisely, given a smoothness index $s$, we find the largest open set of indices $(1/p_1, 1/p_2)$ for which we have boundedness for the associated bilinear multiplier operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$.

1. Introduction

In the first paper [5] of the present series, we reviewed the Hörmander multiplier theorem concerning a linear operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\sigma(\xi)e^{2\pi i x \cdot \xi}d\xi$$

where $f$ is a Schwartz function on $\mathbb{R}^n$ and $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi}dx$ is its Fourier transform.

Let $\Psi$ be a Schwartz function on $\mathbb{R}^n$ and $\hat{\Psi}(\xi) = \int_{\mathbb{R}^n} f(x)e^{2\pi i x \cdot \xi}dx$ is its Fourier transform. Let $\sigma$ be a Schwartz function whose Fourier transform is supported in the annulus of the form \{$\xi : 1/2 < |\xi| < 2$\} which satisfies $\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. We denote by $\Delta$ the Laplacian and by $(I - \Delta)^{s/2}$ the operator given on the Fourier transform by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$; also for $s > 0$, and we denote by $L^r_s$ the Sobolev space of all functions $h$ on $\mathbb{R}^n$ with norm $\|h\|_{L^r_s} := \|(I - \Delta)^{s/2}h\|_{L^r} < \infty$. Inspired by [1], in [5] we provided a self-contained proof of the following sharp version of the Hörmander multiplier theorem [13] (that extended an earlier result of Mikhlin [14]): if

$$\sup_{k \in \mathbb{Z}} \|\hat{\Psi}\sigma(2^k\cdot)\|_{L^r_s} < \infty$$

and

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n},$$

then $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$. Moreover, condition (2) is optimal in the sense that, if $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself for all multipliers $\sigma$ for which (1) holds, then we must necessarily have $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{s}{n}$.

In this paper we provide bilinear analogues of these results. The study of the Hörmander multiplier theorem in the multilinear setting was initiated by Tomita [19] and was further studied by Fujita and Tomita [2], Tomita [20], Miyachi and Tomita [16, 17], Grafakos and
Si [10], and Grafakos, Miyachi, and Tomita [8]. For a given function $\sigma$ on $\mathbb{R}^{2n}$ we define a bilinear operator

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \sigma(\xi_1, \xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2$$

originally defined on pairs of $\mathcal{C}_0^\infty$ functions $f_1, f_2$ on $\mathbb{R}^n$. We fix a Schwartz function $\Psi$ on $\mathbb{R}^{2n}$ whose Fourier transform is supported in the annulus $1/2 \leq |(\xi_1, \xi_2)| \leq 2$ and satisfies

$$\sum_j \hat{\Psi}(2^{-j} (\xi_1, \xi_2)) = 1, \quad (\xi_1, \xi_2) \neq 0.$$

The following theorem is the main result of this paper:

**Theorem 1.1.** Let $1 < p_1, p_2 < \infty$ and define $1/p = 1/p_1 + 1/p_2$.

(a) Let $n/2 < s \leq n$. Suppose that $rs > 2n$ and that

$$\frac{1}{p_1} < s, \quad \frac{1}{p_2} < s, \quad \frac{1}{n} < \frac{1}{p} < \frac{s}{n} - \frac{1}{2}.$$

Then for all $\mathcal{C}_0^\infty(\mathbb{R}^n)$ functions $f_1, f_2$ we have

$$(4) \quad \|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^r(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}.$$

Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all $\sigma$, then we must necessarily have

$$\frac{1}{p_1} < \frac{s}{n}, \quad \frac{1}{p_2} < \frac{s}{n}, \quad \frac{1}{n} < \frac{1}{p} < \frac{s}{n} - \frac{1}{2}.$$

(b) Let $n < s \leq 3n/2$. Suppose that $rs > 2n$ and that

$$\frac{1}{p} < \frac{s}{n} + \frac{1}{2}.$$

Then (4) holds. Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all $\sigma$, then we must necessarily have

$$\frac{1}{p} \leq \frac{s}{n} + \frac{1}{2}.$$

(c) If $s > \frac{3n}{2}$ and $rs > 2n$ then (4) holds for all $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$.

This theorem uses two main tools: One is the optimal $n/2$-derivative result in the local $L^2$-case contained in [5] and the other is a special type of multilinear interpolation suitable for the purposes of this problem. Figure 1 (Section 4), plotted on a slanted $(1/p_1, 1/p_2)$ plane, shows the regions of boundedness for $T_\sigma$ in the two cases $n/2 < s \leq n$ and $n < s \leq 3n/2$. Note also that in the former case, the condition $1 - \frac{s}{n} < \frac{1}{p_1}$ is only needed when $p > 2$.

Finally, we mention that the necessity of conditions (5) and (7) in Theorem 1.1 are consequences of Theorems 2 and 3 in [6]; these say that if boundedness holds, then we must necessarily have

$$\frac{1}{p_1} \leq \frac{s}{n}, \quad \frac{1}{p_2} \leq \frac{s}{n}, \quad \frac{1}{p} \leq \frac{s}{n} + \frac{1}{2}.$$
Also, if $T_\sigma$ maps $L^{p_1} \times L^{p_2}$ to $L^p$ and $p > 2$, then duality implies that $T_\sigma$ maps $L^{p'} \times L^{p_2}$ to $L^{p_1}$. Now $p'$ plays the role of $p_1$ and so constraint $\frac{1}{p_1} < \frac{\theta}{n}$ becomes $1 - \frac{\theta}{n} \leq \frac{1}{p}$. This proves (5). So the main contribution of this work is the sufficiency of the conditions in (3) and (6).

2. SOME LEMMAS

In this section we state three lemmas needed in our interpolation.

**Lemma 2.1 ([5]).** Let $0 < p_0 < p < p_1 < \infty$ be related as in $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta \in (0,1)$. Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions $h_j^\varepsilon$, $j = 1, \ldots, N_\varepsilon$, supported in cubes on $\mathbb{R}^n$ with pairwise disjoint interiors, and nonzero complex constants $c_j^\varepsilon$ such that the functions

$$f_{\varepsilon} = \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon |\frac{x}{p_0} + \frac{1}{p_1} \cdot z|^\frac{1}{p_1} h_j^\varepsilon$$

satisfy

$$\|f_{\varepsilon} - f\|_{L^2} + \|f_{\varepsilon} - f\|_{L^{p_0}}^{\min(1,p_0)} + \|f_{\varepsilon} - f\|_{L^{p_1}}^{\min(1,p_1)} < \varepsilon$$

and

$$\|f_{\varepsilon} - f\|_{L^{p_0}} \leq \|f\|_{L^p} + \varepsilon', \quad \|f_{\varepsilon} - f\|_{L^{p_1}} \leq \|f\|_{L^p} + \varepsilon',$$

where $\varepsilon'$ depends on $\varepsilon, p_0, p_1, p, \|f\|_{L^p}$ and tends to zero as $\varepsilon \to 0$.

**Lemma 2.2 ([5]).** For $z$ in the strip $a < \Re(z) < b$ and $x \in \mathbb{R}^n$, let $H(z,x)$ be analytic in $z$ and smooth in $x \in \mathbb{R}^n$ that satisfies

$$|H(z,x)| + \left| \frac{dH}{dz} (z,x) \right| \leq H_*(x), \quad \forall a < \Re(z) < b,$$

where $H_*$ is a measurable function on $\mathbb{R}^n$. Let $f$ be a complex-valued smooth function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \max \left\{|f(x)|^a, |f(x)|^b\right\} \left\{1 + |\log(|f(x)|)|\right\} H_*(x) \, dx < \infty.$$

Then the function

$$G(z) = \int_{\mathbb{R}^n} |f(x)|^z e^{i\text{Arg} f(x)} H(z,x) \, dx$$

is analytic on the strip $a < \Re(z) < b$ and continuous up to the boundary.

**Lemma 2.3 ([3, 11]).** Let $F$ be analytic on the open strip $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and continuous on its closure. Assume that for all $0 \leq \tau \leq 1$ there exist functions $A_\tau$ on the real line such that

$$|F(\tau + it)| \leq A_\tau(t) \quad \text{for all } t \in \mathbb{R},$$

and suppose that there exist constants $A > 0$ and $0 < a < \pi$ such that for all $t \in \mathbb{R}$ we have

$$0 < A_\tau(t) \leq \exp \{|Ae^{\delta|t|}|\}.$$

Then for $0 < \theta < 1$ we have

$$|F(\theta)| \leq \exp \left\{ \frac{\sin(\pi \theta)}{2} \int_{-\infty}^{\infty} \frac{\log |A_0(t)|}{\cosh(\pi t) - \cos(\pi \theta)} + \frac{\log |A_1(t)|}{\cosh(\pi t) + \cos(\pi \theta)} \right\}.$$

In calculations it is crucial to note that
\[
\frac{\sin(\pi \theta)}{2} \int_{-\infty}^\infty \frac{dt}{\cosh(\pi t) - \cos(\pi \theta)} = 1 - \theta, \quad \frac{\sin(\pi \theta)}{2} \int_{-\infty}^\infty \frac{dt}{\cosh(\pi t) + \cos(\pi \theta)} = \theta.
\]

3. Multilinear interpolation

In this section we prove the main tool needed in derive Theorem 1.1 by interpolation. We state and prove the main theorem in this section in the \(m\)-linear case, since we anticipate future uses for \(m\)-linear operators. We fix a Schwartz function \(\Psi\) on \(\mathbb{R}^{mn}\) whose Fourier transform is supported in the annulus \(1/2 \leq |\vec{\xi}| \leq 2\) and satisfies
\[
\sum_j \hat{\Psi}(2^{-j} \vec{\xi}) = 1, \quad 0 \neq \vec{\xi} \in \mathbb{R}^{mn}.
\]

**Theorem 3.1.** Let \(0 < p_0^1, \ldots, p_0^m \leq \infty, 0 < p_1^1, \ldots, p_1^m \leq \infty, 0 < q, q_1 \leq \infty, 0 \leq s, s_1 < \infty, 1 < r_0, r_1 < \infty, 0 < \theta < 1,\) and let
\[
\frac{1}{p_l^j} = 1 - \theta + \frac{\theta}{p_0^l}, \quad \frac{1}{q_l^j} = 1 - \theta + \frac{\theta}{q_0}, \quad \frac{1}{r_l} = 1 - \theta + \frac{\theta}{r_0}, \quad s = (1 - \theta)s_0 + \theta s_1
\]
for \(l = 1, \ldots, m.\) Assume \(r_0 s_0 > mn\) and \(r_1 s_1 > mn\) and that
\[
\|T_\sigma(f_1, \ldots, f_m)\|_{L^{p_l^j}(\mathbb{R}^n)} \leq K_1 \sup_{j \in \mathbb{Z}} \|\sigma(2^{j \cdot} \vec{\xi})\|_{L_{p_l^j}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{p_l^j}(\mathbb{R}^n)}
\]
for \(k = 0, 1\) where \(K_0, K_1\) are positive constants. Then we have the intermediate estimate:
\[
\|T_\sigma(f_1, \ldots, f_m)\|_{L^{q_l^j}(\mathbb{R}^n)} \leq C_* K_0^{1 - \theta} K_1^{\theta} \sup_{j \in \mathbb{Z}} \|\sigma(2^{j \cdot} \vec{\xi})\|_{L_{q_l^j}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{q_l^j}(\mathbb{R}^n)}
\]
where \(C_*\) depends on all the indices, \(\theta,\) and the dimension.

**Proof.** Fix a smooth function \(\hat{\Phi}\) on \(\mathbb{R}^{mn}\) such that \(\text{supp} (\Phi) \subset \{\frac{1}{4} \leq |\vec{\xi}| \leq 4\}\) and \(\hat{\Phi} \equiv 1\) on the support of the function \(\hat{\Psi}.\) Denote \(\varphi_j = (I - \Delta)^{\frac{j}{2}} [\sigma(2^{j \cdot} \vec{\xi})]\) and define
\[
\sigma(z)(\xi) = \sum_{j \in \mathbb{Z}} (I - \Delta)^{-\frac{\varphi_j(z)}{2}} \left[ |\varphi_j|^{\frac{1}{r_0} + \frac{z}{r_1}} e^{i \text{Arg} (\varphi_j)} \right] \right)^{(2^{-j} \vec{\xi})} \hat{\Phi}(2^{-j} \vec{\xi}).
\]
This sum has only finitely many terms and we now estimate its \(L^\infty\) norm.

Fix \(\vec{\xi} \in \mathbb{R}^{mn}\). Then there is a \(j_0\) such that \(|\vec{\xi}| \approx 2^{j_0}\) and there are only two terms in the sum in \([12]\). For these terms we estimate the \(L^\infty\) norm of \((I - \Delta)^{-\frac{\varphi_j(z)}{2}} \left[ |\varphi_j|^{\frac{1}{r_0} + \frac{z}{r_1}} e^{i \text{Arg} (\varphi_j)} \right].\)
For \(z = \tau + it\) with \(0 \leq \tau \leq 1,\) let \(s_\tau = (1 - \tau)s_0 + \tau s_1\) and \(1/r_\tau = (1 - \tau)/r_0 + \tau/r_1.\) By the Sobolev embedding theorem we have
\[
\left\| (I - \Delta)^{-\frac{\varphi_j(z)}{2}} \left[ |\varphi_j|^{\frac{1}{r_0} + \frac{z}{r_1}} e^{i \text{Arg} (\varphi_j)} \right] \right\|_{L^\infty(\mathbb{R}^{mn})} \leq C(r_\tau, s_\tau, mn) \left\| (I - \Delta)^{-\frac{\varphi_j(z)}{2}} \left[ |\varphi_j|^{\frac{1}{r_0} + \frac{z}{r_1}} e^{i \text{Arg} (\varphi_j)} \right] \right\|_{L^r_{\tau}(\mathbb{R}^{mn})} \leq C(r_\tau, s_\tau, n) \left\| (I - \Delta)^{\frac{\varphi_j(z)}{2}} \left[ |\varphi_j|^{\frac{1}{r_0} + \frac{z}{r_1}} e^{i \text{Arg} (\varphi_j)} \right] \right\|_{L^r_{\tau}(\mathbb{R}^{mn})}.
\]
\[ C'(r_\tau, s_\tau, mn)(1 + |s_0 - s_1||t|)^{mn/2 + 1} \left\| \varphi_j \right\|_{L^r(\mathbb{R}^{mn})} |t|^{r/(1 + |t|)} e^{i \text{Arg}(\varphi_j)} \leq C''(r_0, r_1, s_0, s_1, \tau, mn)(1 + |t|)^{mn/2 + 1} \left\| \varphi_j \right\|_{L^r(\mathbb{R}^{mn})} |t|^{r/(1 + |t|)} \]

\[ = C''(r_0, r_1, s_0, s_1, \tau, mn)(1 + |t|)^{mn/2 + 1} \left\| \varphi_j \right\|_{L^r(\mathbb{R}^{mn})} |t|^{r/(1 + |t|)} \]

It follows from this that

\[ \left\| \sigma_{\tau + i \delta} \right\|_{L^\infty(\mathbb{R}^{mn})} \leq C''(r_0, r_1, s_0, s_1, \tau, mn)(1 + |t|)^{mn/2 + 1} \left( \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot \widehat{\psi}) \right\|_{L^s(\mathbb{R}^{mn})} \right)^{r/r_\tau} \]

Let \( T_{\sigma} \) be the family of operators associated to the multipliers \( \sigma_z \). Let \( \varepsilon \) be given.

CASE I: \( \min(q_0, q_1) > 1 \). Fix \( f_t, g \in L^\infty(\mathbb{R}^n) \) and \( 0 < p_0', p_1', q_0, q_1 < \infty \). Given \( \varepsilon > 0 \), by Lemma 2.1 there exist functions \( f_{i,\varepsilon} \) and \( g_{i,\varepsilon} \) of the form (8) such that

\[ \left\| f_{i,\varepsilon} - f_i \right\|_{L^{p_i}} + \left\| g_{i,\varepsilon} - g_i \right\|_{L^{q_i}} < \varepsilon, \quad \left\| g_{i,\varepsilon} - g \right\|_{L^{q_i}} < \varepsilon, \]

and that for all \( l = 1, \ldots, m \) we have

\[ \left\| f_{l,\varepsilon} \right\|_{L^{p_l}} \leq \left( \left\| f_l \right\|_{L^{p_l}} + \varepsilon' \right)^{1/p_l}, \quad \left\| g_{l,\varepsilon} \right\|_{L^{q_l}} \leq \left( \left\| g_l \right\|_{L^{q_l}} + \varepsilon' \right)^{1/q_l} \]

Define

\[ F(z) = \int_{\mathbb{R}^n} T_{\sigma}(f_{1,\varepsilon}, \ldots, f_{m,\varepsilon}) g_{\hat{z},\varepsilon} \, dx \]

\[ = \int_{\mathbb{R}^mn} \sigma_z(\hat{\xi}) \prod_{j=1}^{l-1} f_{i,\varepsilon}(\xi_1) \cdots f_{m,\varepsilon}(\xi_m) g_{\hat{\xi},\varepsilon}(-\xi) \, d\xi \]

\[ = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^mn} \left( I - \Delta \right)^{-s_0(1-z)_{j+1}z} \left[ \left\| \varphi_j \right\|_{L^{1+it\varepsilon}} |t|^{r/(1 + |t|)} e^{i \text{Arg}(\varphi_j)} \right] (\hat{\xi}) \phi(2^{-j} \hat{\xi}) \]

\[ \times \left( \prod_{l=1}^m f_{l,\varepsilon}(\xi_l) \right) g_{\hat{\xi},\varepsilon}(-\xi) \, d\xi \]

\[ = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^mn} \left[ \left\| \varphi_j \right\|_{L^{1+it\varepsilon}} |t|^{r/(1 + |t|)} e^{i \text{Arg}(\varphi_j)} \right] (\hat{\xi}) \phi(2^{-j} \hat{\xi}) \]

\[ \times \left( I - \Delta \right)^{-s_0(1-z)_{j+1}z} \left[ \phi(2^{-j} \hat{\xi}) \left( \prod_{l=1}^m f_{l,\varepsilon}(\xi_l) \right) g_{\hat{\xi},\varepsilon}(-\xi) \right] \, d\xi \]

Notice that

\[ (I - \Delta)^{-s_0(1-z)_{j+1}z} \left[ \phi(2^{-j} \hat{\xi}) \left( \prod_{l=1}^m f_{l,\varepsilon}(\xi_l) \right) g_{\hat{\xi},\varepsilon}(-\xi) \right] \]

is equal to a finite sum (over \( k_1, \ldots, k_m \)) of terms the form

\[ |c_{k_1} f_{k_1} |^{p_1}_x (1-z)_{j+1} + \frac{p_1}{p_1'} \cdots |c_{k_m} f_{k_m} |^{p_m}_x (1-z)_{j+1} + \frac{p_m}{p_m'} \left[ d_{j}^{q_0}(1-z)_{j+1} + q^1 q_{q_1} \left( I - \Delta \right)^{-s_0(1-z)_{j+1}z} \right] \left( \phi(2^{-j} \cdot) ; k_1, \ldots, k_m, l \right) \left( \hat{\xi} \right) \]
which we call $H(\xi, z)$, where $\zeta_{k_1, \ldots, k_m, l}$ are Schwartz functions. Thus $H(\xi, z)$ is an analytic function in $z$.

Lemma 2.2 guarantees that $F(z)$ is analytic on the strip $0 < \Re(z) < 1$ and continuous up to the boundary. Furthermore, by H"older’s inequality,

$$|F(it)| \leq \left\| T_{\sigma_l}(f_1^{it, \epsilon}, \ldots, f_m^{it, \epsilon}) \right\|_{L^p} \left\| g_0^\epsilon \right\|_{L^q},$$

and

$$\left\| T_{\sigma_l}(f_1^{it, \epsilon}, \ldots, f_m^{it, \epsilon}) \right\|_{L^p} \leq K_0 \sup_{k \in \mathbb{Z}} \left\| \sigma_l(2^k \cdot) \hat{\Psi} \right\|_{L^p} \prod_{l=1}^m \left\| f_l^{it, \epsilon} \right\|_{L^p}$$

$$\leq C(m, n, r_0)(1 + |s_1 - s_0| \cdot |t|)^{\frac{m-n}{2} + 1} K_0 \sup_{j \in \mathbb{Z}} \left\| \hat{\varphi}_j \right\|_{L^p} \prod_{l=1}^m \left( \left\| f_l \right\|_{L^{p_l'}} + \epsilon' \right)^{\frac{j}{r_0}}$$

$$= C(m, n, r_0)(1 + |s_1 - s_0| \cdot |t|)^{\frac{m-n}{2} + 1} K_0 \sup_{j \in \mathbb{Z}} \left( I - \Delta \right)^{\frac{j}{2}} [\sigma(2^j \cdot) \hat{\Psi}] \right\|_{L^p} \prod_{l=1}^m \left( \left\| f_l \right\|_{L^{p_l'}} + \epsilon' \right)^{\frac{j}{r_0}}.$$

Thus, for some constant $C = C(m, n, r_0, s_0, s_1)$ we have

$$|F(it)| \leq C(1 + |t|)^{\frac{m-n}{2} + 1} K_0 \sup_{j \in \mathbb{Z}} \left( I - \Delta \right)^{\frac{j}{2}} [\sigma(2^j \cdot) \hat{\Psi}] \right\|_{L^p} \prod_{l=1}^m \left( \left\| f_l \right\|_{L^{p_l'}} + \epsilon' \right)^{\frac{j}{r_0}}.$$

Similarly, we can choose the constant $C = C(m, n, r_1, s_0, s_1)$ above large enough so that

$$|F(1 + it)| \leq C(1 + |t|)^{\frac{m-n}{2} + 1} K_1 \sup_{j \in \mathbb{Z}} \left( I - \Delta \right)^{\frac{j}{2}} [\sigma(2^j \cdot) \hat{\Psi}] \right\|_{L^p} \prod_{l=1}^m \left( \left\| f_l \right\|_{L^{p_l'}} + \epsilon' \right)^{\frac{j}{r_1}}.$$

Note that $F(z)$ is a combination of finite terms of the form

$$\Lambda_{k_1, \ldots, k_m, l}(z) \int_{\mathbb{R}^m} \sigma(z) h_{j_1}^{\frac{1}{2}}(\xi_1) \cdots h_{j_m}^{\frac{1}{2}}(\xi_m) \hat{g}_j^\epsilon(-((\xi_1 + \cdots + \xi_m)) d\xi,$$

where $\Lambda_{k_1, \ldots, k_m, l}(z) = \left| c_{k_1}^{\frac{1}{2}} z^{1-\epsilon} + c_{k_2}^{\frac{1}{2}} z^{1-\epsilon} \cdots c_{k_m}^{\frac{1}{2}} z^{1-\epsilon} \right|$ and $h_{j_1}^{\frac{1}{2}}, g_j^\epsilon$ are smooth functions with compact supports. Thus for $z = \tau + it$, $t \in \mathbb{R}$ and $0 \leq \tau \leq 1$ it follows from (13) and from the definition of $F(z)$ that

$$|F(z)| \leq C(\tau, \epsilon, f_1, \ldots, f_m, g, r_1, p, q_0, q_1)(1 + |t|)^{\frac{m-n}{2} + 1} \left( \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^p} \right)^{\frac{r}{r_0}} = A_r(t)$$

Since $A_r(t) \leq \exp(Ae^{q|t|})$ it follows that the admissible growth hypothesis of Lemma 2.3 is valid. Applying Lemma 2.3 we obtain

$$|F(\theta)| \leq C K_0^{-\theta} \left( \sup_{j \in \mathbb{Z}} \left( I - \Delta \right)^{\frac{j}{2}} [\sigma(2^j \cdot) \hat{\Psi}] \right) \left( \left\| g \right\|_{L^{p'}} + \epsilon' \right)^{\frac{1}{q'}} \prod_{l=1}^m \left( \left\| f_l \right\|_{L^{p_l'}} + \epsilon' \right)^{\frac{1}{q'}}.$$

But

$$F(\theta) = \int_{\mathbb{R}^m} T_{\sigma_l}(f_1^{\theta, \epsilon}, \ldots, f_m^{\theta, \epsilon}) g_0^{\theta, \epsilon} dx.$$
and then we have
\[
\int_{\mathbb{R}^n} T_\sigma(f_1, \ldots, f_m) g \, dx = F(\theta) + \int_{\mathbb{R}^n} [T_\sigma(f_1, \ldots, f_m) - T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})] g \, dx
\]
\[
+ \int_{\mathbb{R}^n} T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon}) [g - g^{\theta, \varepsilon}] \, dx.
\]
A telescoping identity gives
\[
|T_\sigma(f_1, \ldots, f_m) - T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})| \leq \sum_{l=1}^{m} |T_\sigma(f_1, \ldots, f_{l-1}, f_l - f_1^{\theta, \varepsilon}, f_2^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})|.
\]
In view of the hypothesis that \(T_\sigma\) is bounded from \(L^{p_1} \times \cdots \times L^{p_m}\) to \(L^{q_k}\) and of (14), we obtain
\[
\|T_\sigma(f_1, \ldots, f_m) - T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})\|_{L^{q_k}} \leq C(q_0, q_1) \varepsilon \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}}
\]
and
\[
\|T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})\|_{L^{q_k}} \leq C(q_0, q_1) \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}}
\]
for \(k = 0, 1\). Using the fact that \(\|f\|_{L^q} \leq C\|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{q_1}}^\theta\), we deduce the estimates
\[
\|T_\sigma(f_1, \ldots, f_m) - T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})\|_{L^{q_k}} \leq C\left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{1-\theta} \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{\theta},
\]
\[
\|T_\sigma(f_1^{\theta, \varepsilon}, \ldots, f_m^{\theta, \varepsilon})\|_{L^{q_k}} \leq C\left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{1-\theta} \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{\theta}.
\]
These inequalities together with (14), (15) and Hölder’s inequality yield
\[
\left| \int T_\sigma(f_1, \ldots, f_m) g \, dx \right|
\leq C\varepsilon \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{1-\theta} \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{\theta} \|g\|_{L^{q'}}
\]
\[
+ C\varepsilon \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{1-\theta} \left( \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right)^{\theta}
\]
\[
+ CK_0^{1-\theta} K_1^\theta \sup_{l \in \mathbb{Z}} \left( |(I - \Delta)^{\frac{1}{2}} [\sigma(2^{l-1}\cdot)\hat{\psi}]|_{L^1} \right) \left( \|g\|_{L^{q'}}^{9/2} \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}}^{\theta} + \varepsilon' \right)^{\frac{1}{p_l}} \right).
\]
Now let \(\varepsilon\) go to zero, then obtain
\[
\left| \int T_\sigma(f_1, \ldots, f_m) g \, dx \right| \leq CK_0^{1-\theta} K_1^\theta \sup_{l \in \mathbb{Z}} \left( |(I - \Delta)^{\frac{1}{2}} [\sigma(2^{l-1}\cdot)\hat{\psi}]|_{L^1} \right) \left( \|g\|_{L^{q'}}^{9/2} \prod_{l=1}^{m} \left( \|f_l\|_{L^{p_l}} \right) \right).
\]
Taking supremum over all functions \(g \in L^{q'}\) yields (11).
CASE II: $\min(q_0, q_1) \leq 1$.

Now we need two following lemmas which are proved by Stein and Weiss [18].

Lemma 3.2 ([18]). Let $U : S \to \mathbb{R}$ be an upper semi-continuous function of admissible growth and subharmonic in the strip $S$. Then for $z_0 = x_0 + iy_0 \in S$ we have

$$U(z_0) \leq \int_{-\infty}^{\infty} U(i(y_0 + y))\omega(1 - x_0, y)dy + \int_{-\infty}^{\infty} U(1 + i(y_0 + y))\omega(x_0, y)dy,$$

where

$$\omega(x, y) = \frac{1}{2} \frac{\sin \pi x}{\cos \pi x + \cosh \pi y}.$$ 

Lemma 3.3 ([18]). If $V(x, z)$ is defined on $\mathbb{R}^n \times S$ and analytic in $z \in S$ such that for any $z \in S$ the function $V(\cdot, z)$ lies in $L^1$, then, for $0 < c \leq 1$, $G(z) = \int_{\mathbb{R}^n} |V(x, z)|^c dx$ is continuous and $\log G(z)$ is subharmonic.

We now continue the proof of the second case. Let fix functions $f_i$ as in the previous case. Choose an integer $\rho > 1$ such that $\rho \geq \rho \min(q_0, q_1) > q$. Take arbitrarily a positive simple function $g$ with $\|g\|_{L^{c, \rho}} = 1$. Assume that $g = \sum_k c_k \chi_{E_k}$, where $c_k > 0$ and $E_k$ are pairwise disjoint measurable sets. For $z \in \mathbb{C}$, set

$$g^{\rho} = \sum_k c_k^{\lambda(z)} \chi_{E_k},$$

where

$$\lambda(z) = \rho \left[1 - \frac{q}{\rho} \left(1 - \frac{z}{q_0} + \frac{z}{q_1}\right)\right].$$

Now consider

$$G(z) = \int_{\mathbb{R}^n} |T_{\sigma_x}(f_1^{z, \varepsilon}, \ldots, f_m^{z, \varepsilon})(x)|^{\frac{\rho}{q}} |g^{\rho}(x)| dx = \sum_k \int_{E_k} \left|c_k^{\lambda(z)} T_{\sigma_x}(f_1^{z, \varepsilon}, \ldots, f_m^{z, \varepsilon})(x)\right|^{\frac{\rho}{q}} dx.$$ 

It follows from Lemma 3.3 that $G(z)$ is continuous and $\log G(z)$ is subharmonic. Now using Hölder’s inequality with indices $\frac{c\rho}{q}$ and $(\frac{c\rho}{q})^\ast$ and the fact that the $L^{c, \rho}$-norm of $g$ is equal to 1, we have

$$G(it) \leq \left\{ \int_{\mathbb{R}^n} \left|T_{\sigma_x}(f_1^{it, \varepsilon}, \ldots, f_m^{it, \varepsilon})(x)\right|^{q_0} dx \right\}^{\frac{\rho}{q}} \|g^{\rho}\|_{L^{c\rho\rho, \rho}},$$

$$\leq C \left(1 + |t|\right)^{\frac{m+1}{\rho}} \left(K_0 \sup_{j \in Z} \left\|\sigma(2^j \cdot \psi)\right\|_{L^{c\rho, \rho}} \prod_{l=1}^{m} (\|f_l\|_{L^{c\rho, \rho}} + \varepsilon')^{\frac{1}{\rho}} \right)^{\frac{\rho}{q}}.$$ 

Similarly, we can estimate

$$G(1 + it) \leq \left\{ \int_{\mathbb{R}^n} \left|T_{\sigma_x}(f_1^{1+it, \varepsilon}, \ldots, f_m^{1+it, \varepsilon})(x)\right|^{q_1} dx \right\}^{\frac{\rho}{q_1}} \|g^{1+it}\|_{L^{c\rho, \rho}},$$

$$\leq C \left(1 + |t|\right)^{\frac{m+1}{\rho}} \left(K_1 \sup_{j \in Z} \left\|\sigma(2^j \cdot \psi)\right\|_{L^{c\rho, \rho}} \prod_{l=1}^{m} (\|f_l\|_{L^{c\rho, \rho}} + \varepsilon')^{\frac{1}{\rho}} \right)^{\frac{\rho}{q_1}}.$$
By Lemma 3.2 we obtain
\[ G(\theta) \leq C_s \left( K_0^{1-\theta} K_1^{\theta} \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{L^2_j} \prod_{l=1}^m \left( \| f_l \|_{L^{p_l'}} + \varepsilon' \right)^{\frac{1}{p_l'}} \right)^{\frac{1}{\theta}}. \]

Notice that
\[ G(\theta) = \int_{\mathbb{R}^n} \left| T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon})(x) \right|^\frac{2}{\theta} g(x) \, dx, \]

inequality (16) implies that
\[ \left\| T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon}) \right\|_{L^q} = \left\| T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon}) \right\|_{L^\rho}^{\frac{2}{\theta}} = \sup \left\{ \int \left| T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon})(x) \right|^\frac{2}{\theta} \, dx : g \geq 0, g \text{ simple, } \| g \|_{L^\rho'} \leq 1 \right\}^{\frac{1}{\theta}}. \]

Finally, we write
\[ \left\| T_\sigma(f_1, \ldots, f_m) \right\|_{L^q} \leq C(q) \left\| T_\sigma(f_1, \ldots, f_m) - T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon}) \right\|_{L^q} + C(q) \left\| T_\sigma(f_1^{\theta,\varepsilon}, \ldots, f_m^{\theta,\varepsilon}) \right\|_{L^q} \]

and we note that for the second term we use (17), while for the first term we use the approximation argument in case I. Letting \( \varepsilon \to 0 \), we deduce (11).

Note that the proof of Theorem 3.1 is much simpler in the case \( r_0 = r_1 = 2 \), and was proved earlier in [8] Theorem 6.1 Step 1; see also [9] Theorem 2.3. In this case, the domains can be arbitrary Hardy spaces. Here is the statement in this case:

**Theorem 3.4** ([8]). Let \( p_0' \), \( p_1' \), \( p_l' \), \( q_0 \), \( q_1 \), \( q \), \( s_0 \), \( s_1 \), \( s \), and \( \theta \) be positive as in Theorem 3.1 for \( l = 1, \ldots, m \). Assume that \( s_0, s_1 > \frac{mn}{2} \) and that
\[ \left\| T_\sigma(f_1, \ldots, f_m) \right\|_{L^q(\mathbb{R}^n)} \leq K_k \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{L^2_k(\mathbb{R}^{mn})} \prod_{l=1}^m \left\| f_l \right\|_{H^{p_l'}(\mathbb{R}^n)} \]

for \( k = 0, 1 \) where \( K_0, K_1 \) are positive constants. Then we have the intermediate estimate:
\[ \left\| T_\sigma(f_1, \ldots, f_m) \right\|_{L^q(\mathbb{R}^n)} \leq C_s K_0^{1-\theta} K_1^{\theta} \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{L^2_k(\mathbb{R}^{mn})} \prod_{l=1}^m \left\| f_l \right\|_{H^{p_l'}(\mathbb{R}^n)} \]

where \( C_s \) depends on all the indices, \( \theta \), and the dimension.

4. THE PROOF OF THE MAIN RESULT VIA INTERPOLATION

We now turn to the proof of Theorem 1.1.

**Proof.** (a) Assume \( n/2 < s \leq n \) and pick \( 0 < \varepsilon < \frac{s}{n} - \frac{1}{2} \leq \frac{1}{2} \). Let \( \Gamma \) be the convex set of all points \( \left( \frac{1}{p_1'}, \frac{1}{p_2'} \right) \in (1, \infty) \times (1, \infty) \) that satisfy the constraints in (3) and let \( \Gamma_\varepsilon \) be the set of all points \( \left( \frac{1}{p_1'}, \frac{1}{p_2'} \right) \in \Gamma \) that satisfy the constraints in (5) with \( s \) being replaced by \( \frac{n}{2} + \varepsilon n \).
Note that $\Gamma_\varepsilon$ is the hexagon $NDKLGH$; see Figure 1(A). Since $\Gamma = \cup_{0<\varepsilon<\frac{n}{2}} \Gamma_\varepsilon$, we just need to prove that
\begin{equation}
    \|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^\infty(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}
\end{equation}
for every $(\frac{1}{s_1}, \frac{1}{s_2}) \in \Gamma_\varepsilon$ and all $0 < \varepsilon < \frac{n}{2} - \frac{1}{2}$. By interpolation, it suffices to verify (18) at the vertices $N(\frac{1}{2} - \varepsilon, 0)$, $H(0, \frac{1}{2} - \varepsilon)$, $D(\frac{1}{2} + \varepsilon, 0)$, $G(0, \frac{1}{2} + \varepsilon)$, $K(\frac{1}{2} + \varepsilon, \frac{1}{2})$, and $L(\frac{1}{2}, \frac{1}{2} + \varepsilon)$ of $\Gamma_\varepsilon$.

When $1 \leq p < \infty$, by duality, if $T_\sigma$ maps $L^{p_1} \times L^{p_2} \to L^p$, then it also maps $L^{p'} \times L^{p_2} \to L^{p'}$, since the Hörmander condition $\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^\infty(\mathbb{R}^{2n})}$ is invariant under the duality; see (10), (11). Therefore, if $T_\sigma$ is bounded at $D$, then $T_\sigma$ is also bounded at $N$ by duality. Also by symmetry, it remains to prove (18) at $D$ and $K$.

Recall the boundedness of $T_\sigma$ at the point $B(\frac{1}{2}, \frac{1}{2})$, i.e., mapping from $L^2 \times L^2$ to $L^1$, which is established in (6) with smoothness $s > \frac{n}{2}$. Therefore the duality deduces the boundedness of this operator also at the dual points $A(\frac{1}{2}, 0)$ and $C(0, \frac{1}{2})$ of $B$ with the same smoothness. By Theorem 3.1 we obtain the boundedness of $T_\sigma$ on the entire local triangle $ABC$. Note that the constraints for this triangle are $0 \leq \frac{1}{p_1} \leq \frac{1}{2}$, $0 \leq \frac{1}{p_2} \leq \frac{1}{2}$, and $0 \leq \frac{1}{p'} \leq \frac{1}{2}$, from which the last inequality gives $\frac{1}{2} \leq \frac{1}{p} \leq 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Boundedness holds in the shaded regions and unboundedness in the white regions. The local $L^2$ region is shaded in a lighter color.}
\end{figure}

Now we establish the boundedness of $T_\sigma$ at $D$. To do so, we need to use Theorem 3.1 to interpolate between $A(\frac{1}{2}, 0)$ with smoothness $s_0 > \frac{n}{2}$ and a point near $I(1, 0)$ with smoothness $s_1 > n$. The same argument can be applied to obtain the boundedness of $T_\sigma$ at $K$. More precisely, we have the following estimate (see (6)):
\begin{equation}
    \|T_\sigma(f_1, f_2)\|_{L^1(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^\infty(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}, \quad s > \frac{n}{2}.
\end{equation}

By duality, (19) yields the estimate below for $s > \frac{n}{2}$
\begin{equation}
    \|T_\sigma(f_1, f_2)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^\infty(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \|f_2\|_{L^{\infty}(\mathbb{R}^n)}.
\end{equation}
Note that, the smoothness $s > n$ is required to obtain the boundedness of $T_\sigma$ at any point near $I(1,0)$, $J(0,1)$, $P(\frac{3}{2},1)$, and $Q(1,\frac{3}{2})$. In [10] and [7] it was shown that

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\langle I - \Delta_{\xi_j}\rangle^{\frac{\gamma}{2}} (I - \Delta_{\xi_j})^{\frac{\gamma}{2}} [\sigma(2^j \cdot \xi) \hat{\psi}] \|_{L^2(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}$$

for $\gamma > \frac{n}{2}$ and $1 < p_1, p_2 \leq \infty$, $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2}$, and $p = \frac{1}{p_1} + \frac{1}{p_2}$. Since

$$\sup_{j \in \mathbb{Z}} \|\langle I - \Delta_{\xi_j}\rangle^{\frac{\gamma}{2}} (I - \Delta_{\xi_j})^{\frac{\gamma}{2}} [\sigma(2^j \cdot \xi) \hat{\psi}] \|_{L^2(\mathbb{R}^{2n})} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \xi) \hat{\psi}\|_{L^2(\mathbb{R}^{2n})}$$

for $\gamma = \frac{s}{2} > \frac{n}{2}$, we obtain

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \xi) \hat{\psi}\|_{L^2(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}$$

for all $1 < p_1, p_2 \leq \infty$, $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2}$, and $p = \frac{1}{p_1} + \frac{1}{p_2}$.

For a small number $0 < \delta < \frac{1}{2} - \varepsilon$ to be chosen later, interpolating between (20) with $p_1^0 = 2$, $p_2^0 = \infty$, and $s_0 = \frac{n}{2} + \delta n$ and (22) with $p_1^1 = \frac{1}{1-\theta}$, $p_2^1 = \infty$, and $s_1 = n + \delta n$ yields the boundedness of $T_\sigma$ at $\delta$ with smoothness

$$s = (1 - \theta) s_0 + \theta s_1 = \frac{n}{2} + \delta n + \frac{\theta n}{2}.$$

Note that

$$\frac{1}{p_1} = \frac{1}{2} + \varepsilon = \frac{1 - \theta}{2} + \theta(1 - \delta) = \frac{1}{2} + \theta\left(\frac{1}{2} - \delta\right).$$

Therefore

$$\frac{s}{n} - \frac{1}{2} = \delta + \frac{\theta}{2} = \delta + \frac{\varepsilon}{1 - 2\varepsilon}.$$

Since $0 < \varepsilon < \frac{s}{n} - \frac{1}{2} \leq \frac{1}{2}$, we can choose $0 < \delta < \frac{1}{2} - \varepsilon$ so that the above equality holds.

Likewise, we obtain the boundedness of $T_\sigma$ at $K$, interpolating between (19) at $B$ and (22) at the point $(1 - \delta, \frac{1}{2})$ on the segment $BP$ and near $P$. This completes the proof of the part (a) of the theorem.

(b) Assume $n < s \leq \frac{3n}{2}$. Let $\Gamma_e$ be the set of all points $(\frac{1}{p_1}, \frac{1}{p_2}) \in (1, \infty) \times (1, \infty)$ that satisfy $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2} + \varepsilon$, where $0 < \varepsilon < \frac{s}{n} - 1 \leq \frac{1}{2}$. In this case $\Gamma_e$ is the pentagon $OIRSJ$ with $S(\frac{1}{2} + \varepsilon, 1)$ and $R(1, \frac{1}{2} + \varepsilon)$; see Figure 1(B). Since $s > n$, (22) yields (18) for all points inside the trapezoid $PRSQ$. It remains to show the boundedness of $T_\sigma$ at every point inside the trapezoid $PRSQ$. For $s_0 > n$ and $s_1 > \frac{3n}{2}$, using (21) and a result in [10] or [7], we have

$$\|T_\sigma(f_1, f_2)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \xi) \hat{\psi}\|_{L^2_{s_0}(\mathbb{R}^{2n})} \|f_1\|_{H^s(\mathbb{R}^n)} \|f_2\|_{L^2(\mathbb{R}^n)},$$

and

$$\|T_\sigma(f_1, f_2)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \xi) \hat{\psi}\|_{L^2_{s_1}(\mathbb{R}^{2n})} \|f_1\|_{H^s(\mathbb{R}^n)} \|f_2\|_{H^1(\mathbb{R}^n)}.$$
The last case when \( s > \frac{3n}{2} \) is a consequence of the case \( s = 3 \) \( n/2 \) noting that
\[
\sup_{\mathbb{Z}} \| \sigma(2^j \cdot \hat{\Psi}) \|_{L_p^n(\mathbb{R}^n)} \leq \sup_{\mathbb{Z}} \| \sigma(2^j \cdot \hat{\Psi}) \|_{L_p^n(\mathbb{R}^n)}
\]
and that condition (7) reduces to \( p \geq 1/2 \) when \( s = 3 \) \( n/2 \). The proof of Theorem 1.1 is now complete. \( \Box \)

5. AN APPLICATION

We consider the following multiplier on \( \mathbb{R}^n \): \([a, b]((\xi_1, \xi_2)) = \psi((\xi_1, \xi_2))(\xi_1, \xi_2) \cdot \frac{1}{\xi_1, \xi_2} \frac{1}{2} \) where \( a > 0, a \neq 1, b > 0, \) and \( \psi \) is a smooth function on \( \mathbb{R}^n \) which vanishes in a neighborhood of the origin and is equal to 1 in a neighborhood of infinity. One can verify that \([a, b] \) satisfies (1) on \( \mathbb{R}^n \) with \( s = b/a \) and \( r > 2n/s \).

The range of \( p \)'s for which \([a, b] \) is a bounded multiplier on \( L^p(\mathbb{R}^n) \) can be completely described by the equation \( \frac{1}{p} - \frac{1}{2} \leq \frac{b/a}{2n} \) (see Hirschman \[12], comments after Theorem 3c], Wainger \[21], Part II], and Miyachi \[15], Theorem 3]); similar examples are contained in Miyachi and Tomita \[16], Section 7].

In the bilinear setting, as a consequence of Theorem 1.1 we obtain that the bilinear multiplier operator associated with \([a, b] \) is bounded from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) in the following cases:

(i) when \( n \geq b/a > n/2 \) and
\[
\frac{1}{p_1} < \frac{b}{an}, \quad \frac{1}{p_2} < \frac{b}{an}, \quad 1 - \frac{b}{an} < \frac{1}{p} < \frac{b}{an} + \frac{1}{2}.
\]

(ii) when \( 3n/2 \geq b/a > n \) and
\[
\frac{1}{p} < \frac{b}{an} + \frac{1}{2};
\]

(iii) when \( b/a > 3n/2 \) in the entire range of exponents \( 1 < p_1, p_2 \leq \infty, \frac{1}{2} < p < \infty \).

As of this writing, it is unknown to us, whether this specific bilinear multiplier is unbounded outside the aforementioned range of indices.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: grafakosl@missouri.edu

Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487, USA

E-mail address: hvnguyen@ua.edu