Low Energy Theorems of Hidden Local Symmetries

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We prove to all orders of the loop expansion the low energy theorems of hidden local symmetries in four-dimensional nonlinear sigma models based on the coset space \( G/H \), with \( G \) and \( H \) being arbitrary compact groups. Although the models are non-renormalizable, the proof is done in an analogous manner to the renormalization proof of gauge theories and two-dimensional nonlinear sigma models by restricting ourselves to the operators with two derivatives (counting a hidden gauge boson field as one derivative), i.e., with dimension 2, which are the only operators relevant to the low energy limit. Through loop-wise mathematical induction based on the Ward-Takahashi identity for the BRS symmetry, we solve renormalization equation for the effective action up to dimension-2 terms plus terms with the relevant BRS sources. We then show that all the quantum corrections to the dimension-2 operators, including the finite parts as well as the divergent ones, can be entirely absorbed into a re-definition (renormalization) of the parameters and the fields in the dimension-2 part of the tree-level Lagrangian.

§ 1. Introduction

It is now a popular understanding that the nonlinear sigma model based on a coset space \( G/H \) is equivalent to another model possessing a symmetry \( G_{\text{global}} \times H_{\text{local}} \), \( H_{\text{local}} \) being the hidden local symmetry.\(^1\) If we further add kinetic term for the gauge bosons of the hidden local symmetry, we obtain phenomenological results which are very successful in the particular case of the \( \rho \) meson in the \( [SU(2)_{L} \times SU(2)_{R}]_{\text{global}} \times [SU(2)_{V}]_{\text{local}} \) model.

By choosing a parameter \( a=2 \) in this hidden local symmetry Lagrangian, we have the following tree-level results for the pion and the \( \rho \) meson:\(^3\)

i) universality of the \( \rho \) meson coupling\(^3\)

\[
g_{\text{eff}} = g \quad (g: \text{hidden local gauge coupling}); \quad (1.1)
\]

ii) KSRF relation (version II)\(^4\)

\[
m_{\rho}^2 = 2f_{\pi}^2 g_{\text{eff}}^2; \quad (1.2)
\]

iii) \( \rho \) meson dominance of the electromagnetic form factor of the pion\(^3\)

\[
g_{\text{eff}} = 0. \quad (1.3)
\]

From the tree-level Lagrangian we further obtain an \( a \)-independent relation\(^5\)

\[
\frac{g_{\rho}}{g_{\text{eff}}} = 2f_{\pi}^2 \quad (1.4)
\]

with \( g_{\rho} \) being the strength of \( \rho-\gamma \) mixing. This coincides with another version (version I) of the celebrated KSRF relation.\(^6\)

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The KSRF (I) is a consequence of the symmetry alone and may be regarded as a "low energy theorem" of the hidden local symmetry. Actually, the "low energy theorem" was proved at tree level for any Lagrangian possessing the symmetry and was further anticipated to survive the loop corrections.

Recently, it has in fact been shown at Landau gauge that one-loop effects of the pions and the $\rho$ mesons do not alter at zero momenta the above tree-level relations of the $[SU(2)_L \times SU(2)_R]_{\text{global}} \times [SU(2)_V]_{\text{local}}$ model, particularly the "low energy theorem" mentioned above. These results at zero momentum are actually the relations coming from terms with two derivatives (counting the hidden gauge boson field as one derivative), i.e., dimension-2 operators. The crucial point was that the one-loop renormalization is consistently done within those operators quite independently of the higher dimensional operators.

Since the low energy theorem is the statement on the off-shell amplitudes (at zero momenta), gauge boson identification of the $\rho$ meson in the hidden local symmetry approach is crucial: it has a definite transformation property and hence a definite meaning of the off-shell extrapolation at tree level. However, the gauge symmetry at classical level no longer exists at quantum level due to gauge fixing. Instead, there exists a Becchi-Rouet-Stora (BRS) symmetry at quantum level as a remnant of the classical gauge symmetry. Thus the above one-loop results must be formulated directly as a consequence of the BRS symmetry for the hidden local symmetry, which in turn yields a transparent and powerful method to analyze systematically all orders of the loop corrections beyond one loop.

In this paper we shall prove the above low energy theorem to all orders of the loop expansion. The proof is done in the covariant gauge and for a general case that $G$ and $H$ are arbitrary compact groups not restricted to the chiral case. A particular case of the chiral group is explicitly studied in a separate communication.

For the proof, we follow the techniques developed in the renormalizability proofs of the gauge theories and of the two-dimensional nonlinear sigma model. Our system is a $G_{\text{global}}$ nonlinear sigma model coupled with (propagating) $H_{\text{local}}$ gauge fields. We convert both the $H_{\text{local}}$ gauge transformation and the $G_{\text{global}}$ nonlinear field transformation into a BRS transformation. As usual we assume that there exists a gauge invariant regularization (for example, the dimensional regularization). By using the loop-wise mathematical induction based on the WT identity for the BRS symmetry, we first derive the renormalization equation for the $n$-th loop effective action. Solving the renormalization equation, we show that all the quantum corrections to the dimension-2 operators can be entirely absorbed, including the finite parts as well as the divergent ones, into a re-definition (renormalization) of the parameters and the fields in the dimension-2 part of the original Lagrangian. This implies that all the "low energy theorems" survive loop corrections, since they follow from the form of dimension-2 Lagrangian alone.

One might think that all the loop corrections would respect the symmetry of the tree-level Lagrangian and thus the low energy theorem would trivially follow and needs no "proof". However, the proof of our theorem is by no means trivial. The point is that the present system is the nonlinear sigma model coupled with propagating gauge fields and is crucially different from the separate system of either of them.
Firstly, as a gauge theory, the present system is a novel one of non-renormalizable type. Moreover we have to discuss not only the divergent parts but also finite parts of the loop corrections. Therefore the usual analysis of the renormalization equation restricted to the ordinary-dimension 4 operators does not apply here. Secondly, as a nonlinear sigma model also, the present system has propagating gauge bosons, which generate nonvanishing (in fact, divergent) loop corrections even to the dimension-2 parts in the action, even if the dimensional regularization is used. This is in fact in sharp contrast to the usual pure nonlinear sigma model system containing no propagating gauge bosons. In the latter system, if the dimensional regularization is used, a simple power counting of the decay constant $f_k$ shows that the $L$-loop corrections contribute only to the dimension-2($L+1$) terms in the action,\textsuperscript{13–16} and therefore there appear no loop corrections at all to the dimension-2 part.

The paper is organized as follows. In § 2 we present a brief review of the hidden local symmetry for a general case that $G$ and $H$ are both arbitrary compact groups.\textsuperscript{1,6} This is our model setting. There we also give BRS transformations and the precise statement of our assertion to be proved. Section 3 is the main body of the proof of the assertion based on the WT identity for the BRS symmetry. In § 4 we show that our proof is free from the infrared problem at least in Landau gauge and briefly discuss the extension to the explicit $G_{\text{global}}$ breaking cases. Section 5 is devoted to a summary and discussion. Detailed steps to solve the renormalization equation are given in Appendices A and B.

§ 2. Hidden local symmetry

2.1. $G/H$ algebra

Let $G$ be a compact group with (hermitian) generators $T_i$ satisfying

$$[T_i, T_j] = if_i^k T_k, \quad \text{tr}(T_i T_j) = \frac{1}{2} \delta_{ij}, \quad (2.1)$$

and $H$ be a subgroup of $G$. Then the set of generators $\{T_i\}$ of $G$ is divided into two parts, $\{S_a\}$ of the subgroup $H$ and $\{X_a\}$ of the rest:

$$\{T_i\} = \{S_a \in \mathcal{H}, X_a \in \mathcal{G} - \mathcal{H}\}, \quad (2.2)$$

where $\mathcal{H}$ and $\mathcal{G}$ denote the Lie algebra of $H$ and $G$. It is convenient to choose the generators $\{X_a\}$ of $\mathcal{G} - \mathcal{H}$ to be orthogonal to $\{S_a\}$,

$$\text{tr}(S_a X_a) = 0, \quad \begin{pmatrix} \text{tr}(S_a S_b) = \frac{1}{2} \delta_{ab} \\ \text{tr}(X_a X_b) = \frac{1}{2} \delta_{ab} \end{pmatrix} \quad (2.3)$$

This choice implies $\text{tr}(S_a[S_b, X_a]) = \text{tr}([S_a, S_b]X_a) = 0$ so that $[\mathcal{H}, \mathcal{G} - \mathcal{H}] \subset \mathcal{G} - \mathcal{H}$. Therefore the generators $\{X_a\}$ of $\mathcal{G} - \mathcal{H}$ span a linear representation of $H$ which is generally reducible; namely, $\{X_a\}$ decomposes into a set of irreducible representations $\{X_{a_k}\}$ ($k=1, 2, \cdots, n$) such that
\[ h(\{X_{a_1}\}, \{X_{a_2}\}, \ldots, \{X_{a_n}\}) h^* \]

\[
\frac{\rho^a_{b_1}(h)}{\rho^a_{b_2}(h)} \begin{pmatrix} \rho^a_{b_1}(h) & 0 \\ 0 & \rho^a_{b_2}(h) \end{pmatrix} \]

(2.4)

for \( h \in H \). Let us call the \( k \)-th \( H \)-irreducible space spanned by \( \{X_{a_k}\} \), \( \mathcal{G} - \mathcal{H} \). We treat quite a general case for the subgroup \( H \). \( H \) is generally given by a direct product of factor groups \( H_i \) each of which is simple or \( U(1) \):

\[ H = H_1 \times H_2 \times \cdots \times H_m. \quad (H_1: \text{simple or } U(1)) \]

(2.5)

Thus the generators \( \{S_a\} \) of \( \mathcal{G} \) also decompose into a set of irreducible representations \( \{S_a\} \) \((l = 1, 2, \ldots, m)\) of \( H \); namely, \( \{S_a\} \) is a set of the generators (adjoint representation) of the \( l \)-th factor group \( H_l \), which is trivial under other factor groups \( H_j \) \((j \neq l)\).

2.2. Global X Local model

The nonlinear sigma model \([16]\) based on a coset \( G/H \) gives an effective Lagrangian for the system where a symmetry \( G \) is spontaneously broken down to a subgroup \( H \). Such a \( G/H \) nonlinear sigma model is generally shown \([5]\) to be gauge equivalent to another model possessing \( G_{\text{global}} \times H_{\text{local}} \) symmetry in a certain limit. The \( G_{\text{global}} \times H_{\text{local}} \) model which we discuss in this paper is constructed as follows. \([1, 6]\) The field variable \( \xi(x) \) takes the value of (a unitary matrix representation of) \( G \), which is parameterized as

\[ \xi(x) = \exp[i \phi^i(x) T_i] \in G \quad (\phi^i(x) \in \mathbb{R}) \]

(2.6)

in terms of NG fields \( \phi^i \). The NG fields \( \phi^i \) split into physical ones \( \phi^\ast_i \) corresponding to the "broken" generators \( X_a \in \mathcal{G} - \mathcal{H} \), and unphysical ones \( \phi^\pi_i \) corresponding to the "unbroken" generators \( S_a \in \mathcal{H} \); namely, \( \phi^i T_i = \phi^\ast_i X_a + \phi^\pi_i [S_a] \).

The field variable \( \xi(x) \) transforms under \( G_{\text{global}} \times H_{\text{local}} \) as

\[ \xi(x) \rightarrow \xi'(x) = h(x) \xi(x) g^* \quad g \in G_{\text{global}}, \quad h(x) \in H_{\text{local}}. \]

(2.7)

A basic quantity is a (covariantized) Maurer-Cartan 1-form:

\[ \tilde{\alpha}_\mu(x) = D_\mu \xi(x) \cdot \xi^*(x)/i, \]

(2.8)

where \( D_\mu \) is \( H \)-covariant derivative given by

\[ D_\mu \xi(x) = \partial_\mu \xi(x) - i V_\mu(x) \xi(x) \]

(2.9)

with \( V_\mu \equiv V_\mu^a S_a \) being the hidden gauge boson. The Maurer-Cartan 1-form \( \tilde{\alpha}_\mu \) is Lie-algebra valued \((\in \mathcal{G})\) and transforms under the \( G_{\text{global}} \times H_{\text{local}} \) transformation (2.7) as

\(*\) This parameterization is slightly different from the previously used one \([1]\) \( \xi(x) = \exp[i \sigma^a S_a/\pi] \times \exp[i \pi^a X_a/\pi] \), with \( \sigma \) and \( \pi \) being the unphysical NG fields to be absorbed into the hidden gauge bosons and the physical NG fields living on the coset \( G/H \), respectively.
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\[ \tilde{a}_\mu(x) \rightarrow \tilde{a}_\mu'(x) = h(x) \tilde{a}_\mu(x) h^*(x); \]  

(2.10)

namely, it is \( G_{\text{global}} \)-invariant and transforms homogeneously under \( H_{\text{local}} \). Therefore it splits into two parts, \( \tilde{a}_{\mu \parallel} \) belonging to \( \mathcal{K} \) and \( \tilde{a}_{\mu \perp} \) belonging to \( \mathcal{J} - \mathcal{K} \), and each of them is further decomposed into the above-mentioned \( H \)-irreducible pieces, \( \tilde{a}_{\mu \parallel}^{(l)} \) \((l=1, \cdots, m)\) and \( \tilde{a}_{\mu \perp}^{(k)} \) \((k=1, \cdots, n)\):

\[ \tilde{a}_{\mu \parallel}^{(l)}(x) = \sum_{a_{i} \in \mathcal{K}_{i}} 2 S_{a_{i}} \text{tr}(S_{a_{i}} \tilde{a}_{\mu}(x)), \]

(2.11)

\[ \tilde{a}_{\mu \perp}^{(k)}(x) = \sum_{a_{k} \in \mathcal{J}_{k} - \mathcal{K}_{k}} 2 X_{a_{k}} \text{tr}(X_{a_{k}} \tilde{a}_{\mu}(x)). \]

Thus the most general \( G_{\text{global}} \times H_{\text{local}} \) invariant Lagrangian which contains the least number of derivatives is given by

\[ \mathcal{L} = \sum_{l=1}^{m} a_{\parallel l} \mathcal{L}_{\parallel}^{(l)} + \sum_{k=1}^{n} a_{\perp k} \mathcal{L}_{\perp}^{(k)} + \mathcal{L}_{\text{kin}}(V_{\mu}), \]

(2.12)

\[ \mathcal{L}_{\parallel}^{(l)} = f^{2} \text{tr}[\tilde{a}_{\mu \parallel}^{(l)}]^2, \]

(2.13)

\[ \mathcal{L}_{\perp}^{(k)} = f^{2} \text{tr}[\tilde{a}_{\mu \perp}^{(k)}]^2, \]

(2.14)

\[ \mathcal{L}_{\text{kin}}(V_{\mu}) = \sum_{l=1}^{m} \frac{1}{4g_{l}^{2}} F_{\mu \nu}^{2}(V_{\mu}^{(l)}), \]

(2.15)

where \( a_{\parallel l}, a_{\perp k}, g_{l} \) being arbitrary parameters, and \( F_{\mu \nu}(V_{\mu}^{(l)}) \) is the field strength of the hidden gauge field \( V_{\mu}^{(l)} \) of the \( l \)-th factor group \( H_{l} \). This is the hidden local symmetry Lagrangian.

In many applications, we often need to couple the system to some external gauge bosons by gauging a part of the \( G_{\text{global}} \) group. For instance, the electromagnetic interaction is introduced by gauging the \( U(1)_{\text{em}} \) part of the chiral group \( [SU(2)_{L} \times SU(2)_{R}]_{\text{global}} \). For the present purpose to discuss generically the renormalization effects in the nonlinear hidden local symmetry Lagrangian, it is convenient to gauge the full \( G_{\text{global}} \) group. So we introduce the external gauge field \( C_{\mu}(x) = C_{\mu}^{(l)}(x) T_{l} \) of \( G_{\text{global}} \) (which is now a local group despite the name), and replace the above covariant derivative (2.9) by

\[ D_{\mu} \xi(x) = \partial_{\mu} \xi(x) - i V_{\mu}(x) \xi(x) + i \xi(x) C_{\nu} V_{\nu}. \]

(2.16)

Then the hidden local symmetry Lagrangian (2.12) is invariant under both local \( G_{\text{global}} \times H_{\text{local}} \) transformation:

\[ \xi(x) \rightarrow \tilde{\xi}(x) = h(x) \xi(x) g^{*}(x). \]

(2.17)

For later reference, we note that, in the Lagrangian (2.12), the hidden gauge field \( V_{\mu}^{(l)} = \sum_{a_{l}} V_{\mu}^{a_{l}} S_{a_{l}} \) of the \( l \)-th factor group \( H_{l} \) appears only in \( \mathcal{L}_{\parallel}^{(l)} \) (aside from \( \mathcal{L}_{\text{kin}}(V_{\mu}) \)) in the form:

\[ \mathcal{L}_{\parallel}^{(l)} = f^{2} \frac{1}{2} \sum_{a_{i} \in \mathcal{K}_{i}} (V_{\mu}^{a_{l}} - J_{\mu}^{a_{l}})^{2}, \]

(2.18)
\[ \mathcal{A}_\mu^a = \frac{2}{i} \text{tr}[S^a(\partial_\mu \xi + i \xi CV_\mu) \xi^\dagger]. \] (2.19)

In the absence of the gauge-field kinetic term \( \mathcal{L}_\text{kin}(V_\mu) \), the hidden local symmetry Lagrangian (2.12) is equivalent to the usual nonlinear sigma model based on \( G/H \); indeed, then \( \mathcal{L}^{(0)} \) in (2.18) vanishes by use of the \( V_\mu^a \) equation of motion, \( V_\mu^a = \mathcal{A}_\mu^a \), and (2.12) reduces to the \( G/H \) Lagrangian \( \sum_{k=1}^n a_{nk} - \mathcal{L}^{(a)} \) [Note that we are using \( f_\pi \) as a unique dimensionful constant and the usual “decay constants” of the true NG fields \( \phi_i^\dagger \) are given by \((a_{nk})^{1/2}f_\pi \).

2.3. BRS transformation

In this paper we do not consider the radiative corrections due to the external gauge field \( CV_\mu \) regarding its coupling as weak. So we do not add the kinetic term nor the gauge-fixing term for \( CV_\mu \). As for the gauge-fixing for the hidden gauge boson \( V_\mu \), we take the covariant gauge, so that the gauge-fixing and Faddeev-Popov (FP) terms are given by

\[ \mathcal{L}_\text{GF} + \mathcal{L}_\text{FP} = B^a \delta^\mu V_\mu^a + \frac{1}{2} a_1 g_1^2 B^a B^a + i C^a \delta^\mu D_\mu C^a, \] (2.20)

where \( a_1 \) is a gauge parameter for the factor group \( H_i \) and we have used a shorthand notation

\[ a_1 g_1^2 B^a B^a \equiv \sum_i a_1 g_1^2 \sum_{a \in \tilde{H}_i} B^{a_i} B^{a_i}. \] (2.21)

The infinitesimal form of the \( G_{\text{global}} \times H_{\text{local}} \) transformation (2.17) is given by

\[ \delta \xi(x) = i \theta(x) \xi(x) - i \xi(x) \theta(x), \]

\[ \theta(x) = \theta^a(x) S_a, \quad \theta(x) = \delta^i(x) T_i. \] (2.22)

This defines the transformation of the field \( \phi^i(x) \) in (2.6), \( \xi(x) = \exp[i \phi^i(x) T_i] \), in the form

\[ \delta \phi^i(x) = \theta^a W^i_a(\phi) + \delta^i \mathcal{W}^i(\phi) \]

\[ = \theta^a W^i_a(\phi) = \theta^a \mathcal{W}^i(\phi) \quad \left( \mathcal{W}^i_a = W^i_a(\phi) \frac{\partial}{\partial \phi^i} \right) \] (2.23)

where \( A = (a, i) \) denotes a set of labels of the \( H_{\text{local}} \) and \( G_{\text{global}} \) generators; \( \theta^a = (\theta^a, \delta^i) \) and \( \mathcal{W}^i_a = (\mathcal{W}^i_a, \mathcal{W}^i_a) \). By this definition, these generators \( \mathcal{W}^i_a \) clearly satisfy the algebra of \( H_{\text{local}} \times G_{\text{global}} \)

\[ [\mathcal{W}^i_a, \mathcal{W}^j_b] = f_{ab}^c \mathcal{W}^i_c, \] (2.24)

i.e.,

\[ [\mathcal{W}^i_a, \mathcal{W}^j_b] = f_{ab}^c \mathcal{W}^k_c, \]

\[ [\mathcal{W}^i_a, \mathcal{W}^j_b] = 0. \]

It is important that we have the same number of \( G_{\text{global}} \) transformation generators \( \mathcal{W}^i_a \) as our field variables \( \phi^i \) and \( \mathcal{W}^i_a \) take the form.
Indeed, as is easily shown we have

\[ \mathcal{Q}_i F(\phi) = 0 \quad \text{for} \quad i \Rightarrow F(\phi) = \text{const (}\phi\text{-independent)}. \] (2.26)

Another point to be noted here is that if we set \( \theta = \theta \) (i.e., \( \theta^a = \theta^a \) and \( \theta^e = 0 \) when writing \( \theta = \theta^a S_a + \theta^e X_e \)), the transformation (2.22) becomes a linear \( H \)-transformation on \( \phi \); therefore we have

\[ (\mathcal{W}_a + \mathcal{Q}_a)\phi^i = \phi^i f_{ja}. \] (2.27)

We call this linear transformation \( H_{\text{diag}} \)-transformation, and the covariance under \( H_{\text{diag}} \) provides us with a useful tool below.

The BRS transformation is given simply by replacing the infinitesimal transformation parameter \( \theta^A = (\theta^a, \theta^i) \) by the FP ghost field \( C^A = (C^a, C^i) \):

\[ \delta_b \phi^i = (C^a \mathcal{W}_a + C^j \mathcal{Q}_j) \phi^i = C^A \mathcal{W}_a \phi^i, \] (2.28)

where \( C^i \) is the FP ghost for the external \( G_{\text{global}} \) gauge field \( C\mathcal{V}_\mu^i \). Note that \( C^i \) is a non-propagating field, since we are not quantizing \( C\mathcal{V}_\mu^i \). The nilpotency requirement \( (\delta_b)^2 = 0 \) on \( \phi^i \) with the algebra (2.24) determines the FP ghost BRS transformation as usual:

\[ \delta_b C^A = -\frac{1}{2} C^a C^c f_{bc} A^A. \] (2.29)

The BRS transformation of the \( H_{\text{local}} \times G_{\text{global}} \) gauge fields \( V^A_\mu = (V^a_\mu, C\mathcal{V}_\mu^i) \) is of course given by

\[ \delta_b V^A_\mu = \partial_\mu C^A + V^a_\mu C^c f_{bc} A^A. \] (2.30)

24. Assertion

For definiteness, let us first define the dimensions of our field as follows:

\[ \text{dim}[\phi^i] = 0, \quad \text{dim}[V^a_\mu] = \text{dim}[C\mathcal{V}_\mu^i] = 1. \] (2.31)

These are canonical dimensions, since we are using the parameterization \( \xi = \exp(i\phi^i T_i) \), and the gauge fields appear in the covariant derivative \( D_\mu \) (of dimension 1). It is also convenient to assign the following dimensions to the FP-ghosts:

\[ \text{dim}[C^a] = \text{dim}[C^i] = 0, \quad \text{dim}[ \tilde{C}_a ] = 2. \] (2.32)

Then the BRS transformation preserves the dimensions.

In this terminology, our hidden local symmetry Lagrangian (2.12) with (2.20) consists of two parts, dimension-2 part \( \sum_n a_n \mathcal{L}_n^{(i)} + \sum_n a_n \mathcal{L}_n^{(k)} \) and dimension-4 part \( \mathcal{L}_{\text{kin}}(V^a_\mu) + \mathcal{L}_{\text{gg}} + \mathcal{L}_{\text{FP}} \). [Here we mean the field plus derivative dimensions.] We consider the quantum corrections to this system at any loop order. What we wish to prove in this paper is the following proposition:

Proposition As far as the dimension-2 operators are concerned, all the quantum
corrections, including the finite parts as well as the divergent parts, can be absorbed into the original dimension-2 Lagrangian $\Sigma a_{|1|} L^{(1)} + \Sigma a_{\perp k} L^{(k)}$ by a suitable redefinition (renormalization) of the parameters $a_{|1|}, a_{\perp k}$, and the fields $\phi^i, V_\mu^a$.

Namely, this implies that the tree-level dimension-2 Lagrangian, with the parameters and fields substituted by the "renormalized" one, already describes the exact action at any loop order, and therefore that all the "low-energy theorems" derived from it receive no quantum corrections at all.

§ 3. Proof of the proposition

3.1. Ward-Takahashi identity

The proof of our proposition goes in quite the same way as the renormalizability proofs for the gauge theories\(^9\)\(^{-10}\) and the nonlinear Lagrangians\(^11\),\(^12\). Actually, our hidden local symmetry Lagrangian is a combined system of the gauge theory and the nonlinear Lagrangian, and hence the proof can be done by use of the techniques for both of them.

Following the usual procedure of introducing external sources for the fields' BRS transforms, we can write down the Ward-Takahashi (WT) identity for the effective action $\Gamma$ both for the gauged-$G_{\text{global}}$ and $H_{\text{local}}$ symmetries. We also make a usual assumption that there exists a gauge invariant regularization. The Nakanishi-Lautrup fields $B^a$ and the FP anti-ghost fields $C^a$ can be eliminated from $\Gamma$ through their equations of motion as usual. After eliminating them, the tree level action $S=\Gamma_{\text{tree}}$ reads

$$S[\Phi, K, a] = S_2[\phi, V] + S_4[\Phi, K], \quad (3\cdot1)$$

$$S_2[\phi, V] = \int d^4x (\Sigma a_{|1|} L^{(1)}(\phi, V) + \Sigma a_{\perp k} L^{(k)}(\phi, V)),$$

$$S_4[\Phi, K] = \int d^4x (\Sigma L_{\text{kin}}(V_\mu) + K_i \delta_B \phi^i + K_A^a \delta_B V_\mu^a + L_A \delta_B C^a) \quad (3\cdot2)$$

with collective notations $a = (a_{|1|}, a_{\perp k}), V_\mu^a = (V_\mu^a, C^a), K^{a\rho} = (K^{a\rho}, K^{\rho i}), L_A = (L_A, L_i), \phi = (\phi^i, V_\mu^a, C^a)$ and $K = (K_i, K_A^a, L_A)$. Here $K_i, K_A^a$ and $L_A$ are the external BRS source fields for the NG field $\phi^i$, the gauge fields $V_\mu^a$ and the ghost fields $C^a$, respectively; i.e., $K_A^a \delta_B V_\mu^a = K_A^a \delta_B V_\mu^a + K^{\rho i} \delta_B C_\rho^i$ and $L_A \delta_B C^a = L_A \delta_B C^a + L_i \delta_B C^i$. Note that according to the field dimension assignment (2·31) and (2·32), the dimension of the BRS source $K$ is given by

$$\dim[K_i] = \dim[L_A] = 4, \quad \dim[K_A^a] = 3. \quad (3\cdot3)$$

Then $S_2$ and $S_4$ in the action (3·1) stand for the parts carrying the (field plus derivative) dimension 2 and 4, respectively. The WT identity for the effective action $\Gamma$ is given by

$$\Gamma \ast \Gamma = 0 \quad (3\cdot4)$$

with the $\ast$ operation defined by
\[ F \star G = (-)^\phi \frac{\delta F}{\delta \Phi} \frac{\delta G}{\delta K} - (-)^\phi \frac{\delta F}{\delta K} \frac{\delta G}{\delta \Phi} \]  

for arbitrary functionals \( F[\Phi, K] \) and \( G[\Phi, K] \). (Here the symbols \( \delta \) and \( \bar{\delta} \) denote the derivatives from the left and right, respectively, and \( (-)^\phi \) denotes +1 or -1 when \( \Phi \) is bosonic or fermionic, respectively.)

The effective action \( \Gamma \) is calculated in the loop expansion:

\[ \Gamma = S + \hbar \Gamma^{(1)} + \hbar^2 \Gamma^{(2)} + \ldots. \]  

Actually, to calculate the renormalized \( \Gamma \) to \( n \)-th loop order, we need to use the following “bare” action \( (S_0)_n \) which is obtained by substituting the \( n \)-th loop order “bare” fields \( (\Phi_0)_n, (K_0)_n \) and parameters \( (a_0)_n \) into the tree-level action \( S[\Phi, K; a] \) in (3·1):

\[ (S_0)_n = S[(\Phi_0)_n, (K_0)_n; (a_0)_n], \]  

(3·7)

\[ (\Phi_0)_n = \Phi + \hbar \delta \Phi^{(1)} + \cdots + \hbar^n \delta \Phi^{(n)}, \]

(3·8)

\[ (K_0)_n = K + \hbar \delta K^{(1)} + \cdots + \hbar^n \delta K^{(n)}, \]

\[ (a_0)_n = a + \hbar \delta a^{(1)} + \cdots + \hbar^n \delta a^{(n)}. \]

Note that the renormalization of the decay constants \( (a_{\parallel})^{1/2} f_n, (a_{\perp})^{1/2} f_n \) is performed on the parameter \( a \), so that our unique dimensionful parameter \( f_\pi \) is kept fixed.

Therefore, the WT identity (3·4) for \( \Gamma \) calculated based on this “bare” action in fact should read

\[ (-)^\phi \frac{\bar{\delta} \Gamma}{\bar{\delta}(\Phi_0)_n} \frac{\delta \Gamma}{\delta (K_0)_n} - (-)^\phi \bar{\delta} \Gamma \delta (\Phi_0)_n = 0. \]  

(3·9)

However we shall show below that the field renormalization (3·8), \( (\Phi, K) \rightarrow ((\Phi_0)_n, (K_0)_n) \), is a “canonical” transformation for any \( n \) such that the “Poisson bracket” \( F \star G \) remains invariant:

\[ (-)^\phi \bar{\delta} \frac{F}{\delta \Phi} \frac{G}{\delta K} - (-)^\phi \bar{\delta} \frac{F}{\delta K} \frac{G}{\delta \Phi} = (-)^\phi \bar{\delta} \frac{F}{\delta (\Phi_0)_n} \frac{G}{\delta (K_0)_n} - (-)^\phi \bar{\delta} \frac{F}{\delta (K_0)_n} \frac{G}{\delta (\Phi_0)_n}. \]

(3·10)

So we always have \( \Gamma \star \Gamma = 0 \) written in terms of the tree level fields \( \Phi \) and \( K \).

Now the \( \bar{\delta} (\hbar^n) \) term \( \Gamma^{(n)} \), which contains not only the genuine \( n \)-loop terms but also the contributions of lower loop diagrams with counter terms, is further expanded according to the dimensions:

\[ \Gamma^{(n)} = \Gamma^{(n)}_0[\phi] + \Gamma^{(n)}_2[\phi, V] + \Gamma^{(n)}_4[\Phi, K] + \Gamma^{(n)}_6[\Phi, K] + \cdots. \]

(3·11)

Here again we are counting the dimensions of only the fields and derivatives. [The deviation from dimension-4 is compensated by powers of the unique dimensionful parameter \( f_\pi^2 \).] The first dimension-0 term \( \Gamma^{(n)}_0 \) can contain only the dimensionless field \( \phi^4 \) without derivatives. The dimension-2 of the second term \( \Gamma^{(n)}_2 \) is supplied by derivative and/or the gauge field \( V_\mu^A \). The BRS source field \( K \), carrying dimension 4 or 3, can appear only in \( \Gamma^{(n)}_4 \) and beyond: the dimension-4 term \( \Gamma^{(n)}_4 \) is at most linear.
in $\mathbf{K}$, while the dimension-6 term $I_6^{(n)}$ can contain not only linear terms in $\mathbf{K}$ but also a quadratic term in $K_\mu$, the BRS source of the hidden gauge boson $V_\mu^\alpha$.

3.2. Proof of the assertion

Let us now prove the following by mathematical induction with respect to the loop expansion order $n$: for $n=1, 2, \cdots$,

1) $I_0^{(n)}=0$.

2) By choosing suitably the $n$-th order counter terms $\delta\Phi^{(n)}$, $\delta\mathbf{K}^{(n)}$ and $\delta\alpha^{(n)}$ in (3.8), $I_4^{(n)}[\phi, V]$ and the $\mathbf{K}$-linear terms in $I_2^{(n)}[\Phi, \mathbf{K}]$ can be made vanish; $I_4^{(n)}[\phi, V]=I_4^{(n)}[\Phi, \mathbf{K}]|_{\mathbf{K}\text{-linear}}=0$.

3) The field reparameterization (renormalization) $(\Phi, \mathbf{K}) \rightarrow ((\Phi_0)_n, (\mathbf{K}_0)_n)$ is a "canonical" transformation which leaves the $\ast$ operation invariant.

Suppose that these statements hold up to $n-1$, and calculate the $n$-th loop effective action $r_n$, for the moment, using the $(n-1)$-st loop level "bare" action $(S_0)_{n-1}$ [namely, without $n$-th loop level counter terms $\delta\Phi^{(n)}$, $\delta\mathbf{K}^{(n)}$ and $\delta\alpha^{(n)}$. Then we have the WT identity (3.4) thanks to the induction assumption 3), which yields for $\mathbf{K}_n$ terms

$$S \ast r^{(n)} = -\frac{1}{2} \sum_{l=1}^{n-1} I^{(l)} \ast I^{(n-l)}.$$  (3.12)

Substituting the dimensional expansions, $S=S_2+S_4$ [(3.1)] for $S$ and (3.11) for $I^{(l)}$ ($l=1, \cdots, n$), we compare both sides of (3.12) possessing the same dimension. Since $I_4^{(l)}$ and $I_2^{(l)}$ vanish for $1 \leq l \leq n-1$ by the induction assumption, there are no dimension 0 and 2 parts on the RHS of (3.12), so that we have

$$\dim 0: \quad S_4 \ast I_0^{(n)} + S_2 \ast I_2^{(n)} = 0,$$  (3.13)

$$\dim 2: \quad S_2 \ast I_4^{(n)} + S_4 \ast I_4^{(n)} = 0.$$  (3.14)

[Note that the $\ast$ operation lowers the dimension by 4.] For dimension-4 parts, the RHS in (3.12) might seem to have contribution of the form $\sum_{l=1}^{n-1} I_4^{(l)} \ast I_4^{(n-l)}$, but they actually vanish since all the $I_4^{(l)}$ ($1 \leq l \leq n-1$) contain no $\mathbf{K}$ again by the induction assumption. So, we find

$$\dim 4: \quad S_4 \ast I_4^{(n)} + S_2 \ast I_6^{(n)} = 0.$$  (3.15)

These three equations (renormalization equations) (3.13)~(3.15) give enough information for determining the possible forms of $I_0^{(n)}$, $I_2^{(n)}$ and $I_4^{(n)}|_{\mathbf{K}\text{-linear}}$ (the $\mathbf{K}$-linear term in $I_4^{(n)}$) which we are interested in.

Noting that the BRS transformation $\delta_B^\ast$ on the fields $\Phi=(\phi^i, V_{\mu}^\Lambda, C^A)$ can be written in the form

$$\delta_B^\ast = \delta S_4 \delta \Phi \delta \mathbf{K},$$  (3.16)

we see it convenient to define an analogous transformation $\delta_I^\ast$ on the fields $\Phi$ by

$$\delta_I^\ast = \delta I_4^{(n)} \delta \Phi \delta \mathbf{K}.$$  (3.17)
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Then we can write $I_4^{(n)}$ in the form

$$I_4^{(n)} = A_4[\phi, V] + K_i \delta_i \phi^i + K_A \delta_i V_A + L_A \delta_i C^A.$$  \tag{3·18}

In terms of this notation, (3·13)~(3·15) can be rewritten into

$$\delta_B I_0^{(n)} = 0,$$  \tag{3·19}

$$\delta_B I_2^{(n)} + \delta_i S_2 = 0,$$  \tag{3·20}

$$\delta_B I_4^{(n)} + \delta_i S_4 + \frac{\delta I_6^{(n)}}{\delta K} \frac{\delta S_2}{\delta \Phi} = 0,$$  \tag{3·21}

respectively, where use has been made of the fact that $S_2$, $I_0^{(n)}$ and $I_2^{(n)}$ contain no $K$.

From (3·19) it immediately follows that $I_0^{(n)} = 0$. This is because $I_0^{(n)}[\phi]$ is a function of only $\phi^i$ containing no derivatives and the BRS transformation $\delta_B$ on such a function is just a $G_{\text{global}} \times H_{\text{local}}$ transformation, but we know that there is no $G_{\text{global}} \times H_{\text{local}}$-invariant without derivatives. Thus our first statement 1) $I_0^{(n)} = 0$, in the above, has been proved.

To prove the second and the third statement 2) and 3), we need to solve Eqs. (3·20) and (3·21), which is much more non-trivial task. A bit lengthy analysis of (3·20) and (3·21), which is given in Appendices, shows that the general solution is given in the form

$$I_2^{(n)} + I_4^{(n)} \big|_{K\text{-linear}} = A_{261}[\phi, V] - S \ast Y^{(n)}$$  \tag{3·22}

up to irrelevant terms (dimension-6 or $K$-independent dimension-4 terms). Here $A_{261}$ is a dimension-2 gauge-invariant function of $\phi^i$ and $V_A$ and

$$Y^{(n)} = \int d^4x [K_i F^{(n)i}(\phi) + \alpha_l^{(n)} K_{\mu} V_{\mu} + \beta_l^{(n)} L_{a} C^a + \gamma_l^{(n)} f_{abc} K_{\mu} K_{\mu} C^c],$$  \tag{3·23}

where $F^{(n)i}$ are arbitrary functions and $\alpha_l^{(n)}$, $\beta_l^{(n)}$ and $\gamma_l^{(n)}$ are arbitrary constants. In (3·23) we have used shorthand notations like

$$\alpha_l^{(n)} K_{\mu} V_{\mu} = \sum_{a=1}^{m} \alpha_l^{(n)} \left( \sum_{a=1}^{m} K_{\mu} V_{\mu}^a \right);$$  \tag{3·24}

namely, the parameters $\alpha_l^{(n)}$, $\beta_l^{(n)}$ and $\gamma_l^{(n)}$ in (3·23) can take different values for different factor groups $H_i$ in $H = H_1 \times H_2 \times \cdots \times H_m$.

The form (3·22) of the solution already proves our desired statements 2) and 3) in the above, as seen as follows. First recall that the above $I^{(n)}$ was calculated using $(S_0)_{n-1}$ without $n$-th loop level counter terms $\delta \Phi^{(n)}$, $\delta K^{(n)}$ and $\delta a^{(n)}$. If we include those, we have the following additional contributions to $I^{(n)}$:

$$\Delta I^{(n)} = \delta \Phi^{(n)} \frac{\delta S}{\delta \Phi} + \delta K^{(n)} \frac{\delta S}{\delta K} + \delta a^{(n)} \frac{\delta S}{\delta a}$$  \tag{3·25}

with $S[\Phi, K, a]$ being the tree-level action (3·1). So the true $n$-th loop level effective action is $I^{(n)} + \Delta I^{(n)} = I^{(n)}_{\text{total}}$. We now show that these $n$-th loop level counter terms can be chosen such that $I^{(n)}_{\text{total}}$ has no dimension-2 and dimension-4-$K$-linear terms; namely, the quantities in (3·22) can be completely canceled by $\Delta I^{(n)}$. 


First is the $A_{a}^{[\phi, V]}$ term. Since we know that $\mathcal{L}_{i}^{(n)}$ and $\mathcal{L}_{i}^{(k)}$ span a complete set of dimension-2 $G_{\text{global}} \times H_{\text{local}}$ gauge invariants, $A_{a}^{[\phi, V]}$ must be a linear combination of them:

$$
A_{a}^{[\phi, V]} = \int d^{4}x (\sum_{i} b_{i}^{(n)} \mathcal{L}_{i}^{(n)}(\phi, V) + \sum_{k} b_{k}^{(m)} \mathcal{L}_{k}^{(m)}(\phi, V))
$$

(3·26)

with $b_{i}^{(n)} = (b_{i}^{(m)}, b_{i}^{(n)})$ being certain coefficients. But this can be written as $b_{i}^{(n)} \frac{\delta S}{\delta a}$ and can just be canceled by choosing the $a$-parameter counter terms $\delta a^{(n)} = (\delta a_{i}^{(n)}, \delta a_{k}^{(n)})$ as $\delta a^{(n)} = - b_{i}^{(n)}$.

Next consider the $-S \ast Y^{(n)}$ term. [This term includes the gauge non-invariant dimension-2 operators

$$
-(F^{(n)}(\phi) \frac{\delta}{\delta \phi} S_{2} + a_{i}^{(n)} V_{\mu} a \frac{\delta}{\delta V_{\mu}} S_{2})
$$

(3·27)

generated by the loop effects* (see (A·58)]. We should note that this term $-S \ast Y^{(n)}$ just represents a “canonical transformation” of $S$ caused by $-Y^{(n)}$ as its generating functional. It is therefore clear that if we choose the $n$-th order field counter terms $\delta \Phi^{(n)}$ and $\delta K^{(n)}$ to be equal to the canonical transformations of $\Phi$ and $K$ generated by $+Y^{(n)}$,

$$
\delta \Phi^{(n)} = \Phi \ast Y^{(n)} = (-)^{n} \frac{\delta Y^{(n)}}{\delta K},
$$

$$
\delta K^{(n)} = K \ast Y^{(n)} = (-)^{n} \frac{\delta Y^{(n)}}{\delta \Phi},
$$

(3·28)

then the additional contributions in (3·25) yield

$$
\delta \Phi^{(n)} \frac{\delta S}{\delta \Phi} + \delta K^{(n)} \frac{\delta S}{\delta K} = S \ast Y^{(n)}
$$

(3·29)

and cancels the $-S \ast Y^{(n)}$ term. Equation (3·28) also shows that the field counter terms $\delta \Phi^{(l)}$ and $\delta K^{(l)}$ ($l = 1, 2, \cdots, n - 1$) at lower loop levels, which are determined at the preceding steps of this induction argument, are also generated by certain generating functional $Y^{(l)}$. Thus the field transformation $(\Phi, K) \rightarrow \left((\Phi_{0}), (K_{0})_{n}\right)$ is an infinitesimal “canonical transformation” generated by $\sum_{l=1}^{n} Y^{(l)}$, so that the “Poisson bracket” $F \ast G$ remains invariant. This completes the proof of our statements 1)~3), and hence our Proposition.

At this point let us comment on the nature of the field renormalization (3·28). The renormalization of the hidden gauge boson $V_{\mu}$ is given by $\delta V_{\mu}^{a} = \delta Y^{(n)}/\delta K_{a}^{*} = a_{i}^{(n)} V_{\mu}^{a}$. Thus $V_{\mu}$ is multiplicatively renormalized** in the covariant gauges even

* It was pointed out that in the nonlinear sigma model (without hidden gauge bosons) such non-invariant terms are generated by the one-loop effects for dimension-4 operators, which are transformed away by the NG-field redefinition involving space-time derivatives. In this case without propagating gauge bosons there exist no loop corrections to the dimension-2 operators (up to quadratic divergences which are absent in the dimensional regularization).

** There is in fact an ambiguity in the expression (3·22); a gauge-invariant term $-2 \epsilon a_{a}^{\text{bulk}} \mathcal{L}_{i}^{(n)}$ in $A_{a}^{[\phi, V]}$ can also be written in the form $S \ast \left[k_{a}^{(n)}(V_{\mu}^{a} - \lambda^{a}{})\right]$. Then the $K_{a}^{*}$ term in $Y^{(n)}$, (3·23), is replaced by $k_{a}^{(n)}(a_{a}^{(n)} V_{\mu}^{a} + \epsilon (V_{\mu}^{a} - \lambda^{a}{})$). This form would imply a mixing of $V_{\mu}^{a} - \lambda^{a}{})$ with $V_{\mu}^{a}$ through the renormalization; $\delta V_{\mu}^{a} = a_{a}^{(n)} V_{\mu}^{a} + \epsilon (V_{\mu}^{a} - \lambda^{a}{})$, i.e., non-multiplicative renormalization of $V_{\mu}^{a}$. However, this mixing can be avoided, since it is obviously equivalent to the renormalization of the parameter $a_{a}^{\text{bulk}}$ with $\delta a_{a}^{\text{bulk}} = -2 \epsilon a_{a}^{\text{bulk}}.$
in this nonlinear system.

On the other hand, the renormalization of the NG fields $\phi^i$ reads $\delta \phi^i = \delta Y^{(n)} / \delta K_i = F^{(n)i}(\phi)$. This implies that the parameterization of the $G$-manifold is successively changed loop by loop through point transformation. It should be emphasized that as explicit one-loop calculation shows, in the presence of the propagating gauge fields the function $F^{(n)i}(\phi)$ does not vanish even if we use the dimensional regularization. Thus there is a nontrivial renormalization on the dimension-2 Lagrangian, in sharp contrast to the nonlinear model without propagating gauge fields (cf. Weinberg and Appelquist-Bernard).

§ 4. Infrared problem and symmetry breaking mass terms

4.1. Infrared divergences

Since we are treating massless Nambu-Goldstone (NG) fields $\phi$, there generally appear infrared divergences which might invalidate the formal discussions presented up to here. But we now show that the dimension-2 effective action $\Gamma_2^{(n)}$, as well as the $K$-linear terms in $\Gamma_4^{(n)}$, which is of our main concern in this paper, is in fact free from the infrared divergences at least in the Landau gauge.

The NG fields $\phi^i$ split into physical ones $\phi^i_\parallel$ corresponding to the “broken” generators $X_a \in \mathcal{G} - \mathcal{H}$, and unphysical ones $\phi^i_\perp$ corresponding to the “unbroken” generators $S_a \in \mathcal{H}$; namely, $\phi^i T_i = \phi^i_\parallel X_a + \phi^i_\perp S_a$. They are in fact further decomposed into the $H$-irreducible pieces $\phi_{\parallel k}^i (k=1, 2, \ldots, n)$ and $\phi_{\perp l}^i (l=1, 2, \ldots, m)$ as explained in § 2. The propagators of the physical NG fields $\phi_{\parallel k}^i$ are determined by the dimension-2 Lagrangian piece $a_{\parallel k} \mathcal{L}^{(n)}$ and given simply as the usual massless ones:

$$\text{F.T.} i\langle 0| T \phi_{\parallel k}^i(x) \phi_{\parallel l}^i(y)|0 \rangle = -\frac{1}{p^2 f_k} \delta_{a_k a_l},$$

with $f_k^2 = a_{\parallel k} f_k^2$ and F.T. denoting the Fourier transformation operation $\int d^4(x-y) e^{i p(x-y)}$. The unphysical NG fields $\phi_{\perp l}^i$, on the other hand, generally mix with the hidden gauge bosons $V_\mu^a$ of the corresponding factor group $H_t$. Their propagators are determined by the following quadratic pieces of the Lagrangian $a_{\parallel l} \mathcal{L}^{(n)} + \mathcal{L}_{\text{kin}}(V_\mu) + \mathcal{L}_{\text{GF}}$:

$$\frac{1}{2} f_t^2 (V_\mu^a - \partial_\mu \phi_{\parallel}^a)^2 - \frac{1}{4 g_\perp^2} \left( \partial_\mu V_\mu^a - \partial_\nu V_\nu^a \right)^2 + B_{\mu}^a \partial^\mu V_\mu^a + \frac{1}{2} a_i g_\perp^2 B_{\mu}^a B_{\nu}^a,$$

with $f_t^2 = a_{\parallel t} f_t^2$. Taking the inverse of the coefficient matrix of this quadratic form, we find the following propagators:

$$\text{F.T.} i\langle 0| T \phi_{\perp}^i(x) \phi_{\perp}^i(y)|0 \rangle = \delta_{a \perp} \left[ -\frac{1}{p^2} g_{\perp}^2 + a_i g_{\perp}^2 f_t^2 \right],$$

$$\text{F.T.} i\langle 0| T V_\mu^a(x) \phi_{\perp}^i(y)|0 \rangle = \delta_{a \perp} \left[ -i p_\mu a_i g_{\perp}^2 f_t^2 \right],$$

$$\text{F.T.} i\langle 0| T V_\mu^a(x) V_\nu^a(y)|0 \rangle = \delta_{a \perp} \left[ -\frac{g_{\perp}^2}{p^2} f_t^2 \left( g_{\mu \nu} - p_\mu p_\nu \left( 1 - a_i g_{\perp}^2 f_t^2 \right) \right) \right].$$
These are the propagators in a general covariant gauge with gauge parameter $\alpha$. In the case of non-Landau gauge $\alpha \neq 0$ there exists a massless dipole $1/p^4$ in these $\phi_\mu \phi_\mu$ and $V_\mu \phi_\mu$ propagators, which are not well-defined as they stand even in the sense of distribution just like the massless boson propagators in two dimensions. [The massless dipole $p_\mu p_\nu/p^4$ in the $V_\mu V_\nu$ propagator, on the other hand, can be well-defined by the presence of $p_\mu p_\nu$ in the numerator.]

So let us consider the Landau gauge case $\alpha = 0$ first. Then there appear no massless dipoles and no $V_\mu \phi_\mu$ transition propagators. We note the following. As far as the infrared behavior is concerned, we can rewrite the vector propagator into

$$
(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \frac{g_1^2}{p^2 - g_1^2 f_1^2} = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \left[ 1 + \frac{p^2}{g_1^2 f_1^2} \right] + \frac{p^2}{g_1^2 f_1^2} \right]^2 + \cdots \right].
$$

This expansion just corresponds to the perturbation expansion in which we treat the dimension-4 Lagrangian $\mathcal{L}_{\text{kin}}(V_\mu)$ as perturbation interactions and use the propagator $-(g_{\mu\nu} - p_\mu p_\nu/p^2)/f_1^2$ determined solely by the dimension-2 Lagrangian $a_{\mu\nu} \mathcal{L}_{\text{kin}}^{(0)}$. This type of perturbation expansion is clearly not valid from the ultraviolet viewpoint, but provides us with completely legitimate one for our present purpose considering the infrared behavior of our Green functions. We mean that we use this expansion only in the vicinity of $p_\mu = 0$ of the loop momentum $p_\mu$ in the relevant Feynman integrands.

With this expansion, all the propagators of $\phi_\perp$, $\phi_\parallel$ and $V_\mu$ particles, (4·1) and (4·3), are now determined by the dimension-2 Lagrangian and hence are proportional to the inverse power of the decay constant, $1/f_1^2$. Moreover, we can rescale the FP anti-ghost field $\bar{\psi}$ into $f_1^2 \bar{\psi}$ so that the FP Lagrangian becomes $\mathcal{L}_{\text{FP}} = i f_1^2 \bar{C} \sigma^\mu D_\mu C^\alpha$, then the FP ghost propagator also becomes proportional to the inverse power $1/f_1^2$. We can now count the power of $f_1^2$ for a general Feynman diagram contributing to $\Gamma$. If we use only the vertices of the dimension-2 Lagrangian or the FP ghost one $\mathcal{L}_{\text{FP}}$, which are all proportional to $f_1^2$, the counting of the inverse power of $f_1^2$ is the same as that of Planck constant $\hbar$, and so we get $(1/f_1^2)^{(L-1)}$ for any $L$-loop diagrams. But if we use the vertices coming from the dimension-4 Lagrangian $\mathcal{L}_{\text{kin}}(V_\mu)$ or the BRS source terms $K_i \delta_\delta \phi^i + \mathcal{K}_a \delta_\delta V_\mu^A + L_A \delta_\delta \mathcal{C}^A$, which have no power of $f_1^2$, we lose a power of $f_1^2$ for each of such vertices. Therefore, a general $L$-loop Feynman diagram possessing $V_4$ vertices of $\mathcal{L}_{\text{kin}}(V_\mu)$ and $K$ BRS source vertices, yields an amplitude proportional to

$$
\left( \frac{1}{f_1^2} \right)^{(L-1+V_4+K)}.
$$

But this implies the following: First the dimension-2 effective action $I_{\text{kin}}^{(2)}$, which is proportional to $f_1^2$, receives no loop corrections at all, since the power $L-1+V_4+K$ is non-negative because $L \geq 1$, $V_4$, $K \geq 0$. Moreover, the $K$-linear terms in the dimension-4 effective action $I_{\text{kin}}^{(4)}$, which is of zero-th power term in $f_1^2$, also have no contributions since the BRS source vertex should be contained once there, $K = 1$, so $L$.

* This counting is similar to that in the nonlinear chiral Lagrangian.13
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-1 + V_4 + K \geq 1. [Note that all these are concerned with only the infrared contributions. There are actually non-zero loop contributions to \( \Gamma_2^{(n)} \) as well as to the K-linear terms in \( \Gamma_4^{(n)} \) coming from the ultraviolet region, for which the above counting of the powers of \( f_\phi^2 \) breaks down.] This finishes the proof of the absence of the infrared divergences in \( \Gamma_2^{(n)} \) and the K-linear terms in \( \Gamma_4^{(n)} \) in the case of Landau gauge.

In the non-Landau gauge case, we should properly define the massless dipole propagator \( 1/\rho^4 \). For instance, it can be defined by introducing a small infrared cutoff for the time-component \( \rho_0 \) as Nakanishi did long time ago in QED. Then, as far as the infrared cutoff is kept finite, there appear no infrared divergences of the dipole propagator origin. So the above argument for the Landau gauge case applies also to this non-Landau gauge case and shows that there appear no infrared divergences in \( \Gamma_2^{(n)} \) and the K-linear terms in \( \Gamma_4^{(n)} \). But there are non-trivial problems whether the theory recovers the Lorentz invariance or not in the limit when the infrared cutoff goes to zero, or whether it remains well-defined in that limit. We here do not pursue these problems any further.

4.2. Symmetry breaking mass terms

Finally, we make a comment on what happens when there exist symmetry breaking mass terms of NG fields \( \phi \). Such mass terms may appear when there are explicit \( G \)-symmetry breaking as in the chiral symmetry in QCD case, or when we want to regularize the infrared divergences as a technical device.

When there exists such a mass term of \( \phi \), we introduce the following BRS source term corresponding to its BRS transformation:

\[
\mathcal{L}_{\text{mass}} = f_\phi^2 \cdot m^2 f(\phi) + M \delta_0 f(\phi), \tag{4.6}
\]

where the function \( f(\phi) \) is \( \phi^2/2 = f^{(0)}(\phi) \) at the tree level and is generally dimension-0 function containing no derivatives. For simplicity we assume that the mass term preserves the \( H_{\text{diag}} \) symmetry. We assign dimensions 2 and 4 to \( m^2 \) and the BRS source \( M \), respectively. In the presence of this mass term, the WT identity for the effective action becomes

\[
\Gamma * \Gamma = \tilde{m}^2 \frac{\delta \Gamma}{\delta M}, \tag{4.7}
\]

with \( \tilde{m}^2 = m^2 f_\phi^2 \) (which is still dimension 2 in our counting). Then our renormalization equations \( (3.19) \sim (3.21) \) are changed into

\[
\delta_0 \Gamma_0^{(n)} = 0, \tag{4.8}
\]

\[
\delta_0 \Gamma_2^{(n)} + \delta_r S_2 = \tilde{m}^2 \frac{\delta \Gamma_4^{(n)}}{\delta M}, \tag{4.9}
\]

\[
\delta_0 \Gamma_4^{(n)} + \delta_r S_4 + \frac{\delta \Gamma_2^{(n)}}{\delta K} \cdot \frac{\delta S_2}{\delta \Phi} = \tilde{m}^2 \frac{\delta \Gamma_4^{(n)}}{\delta M}, \tag{4.10}
\]

respectively. The first equation is the same as before, so that it still leads to \( \Gamma_0^{(n)} = 0 \). The third equation \( (4.10) \), whose K-linear terms determined the form of \( \delta_r \) previously,
now has a non-vanishing right-hand side. But, fortunately, it does not contribute to the relevant K-linear terms since \( \delta I_4^{(n)} / \delta M \) is of dimension \( 6 - 4 = 2 \) and cannot contain K of dimension 3 or 4. Therefore \( \delta \phi \) is determined in the same form as before as given in Appendix A. [There we need the assumption that the mass terms respect the \( H_{\text{diag}} \) symmetry.] Finally consider the second equation (4·9) with \( \delta \phi \) thus determined. Repeating the same argument as performed before in § A.2 in Appendix A, and noting that the dimension-2 tree action \( S_z \) now contains BRS-(and \( H_{\text{local}} \)-) non-invariant mass term \( m_2 \), we easily find

\[
\delta_\phi S_2 = \delta_\phi \left( \bar{F}^{(n)} S_2 + a_i^{(n)} V_\mu^a \frac{\delta}{\delta V_\mu^a} S_2 \right) - \bar{m}^2 \bar{F}^{(n)}(\delta_\phi f^{(0)}) + \bar{m}^2 \bar{\beta}_i^{(n)} C^a \bar{W}_a f^{(0)}, \tag{4·11}
\]

so that (4·9) turns out to give

\[
\delta_\phi \left[ \Gamma_2^{(n)} + \left( \bar{F}^{(n)} S_2 + a_i^{(n)} V_\mu^a \frac{\delta}{\delta V_\mu^a} S_2 \right) \right] = \bar{m}^2 \left( \bar{F}^{(n)} - \bar{\beta}_i^{(n)} C^a \frac{\delta}{\delta C^a} \right) \delta_\phi f^{(0)} + \bar{m}^2 \frac{\delta \Gamma_4^{(n)}}{\delta M}, \tag{4·12}
\]

where the integral \( \int d^4 x f^{(0)}(\phi) \) is denoted simply by \( f^{(0)} \). The BRS operand \( \Gamma_2^{(n)} + \cdots \) on the left-hand side is dimension-2 quantity and has generally the form

\[
A_2[\phi, V] + \bar{m}^2 A_0[\phi], \tag{4·13}
\]

where \( A_2 \) is a dimension-2 functional with dimensions given by derivatives and/or gauge fields, and \( A_0 \) is a dimension-0 functional of \( \phi \) with no derivatives. [To avoid any confusion, we should note that (4·13) is not a Taylor expansion in \( \bar{m}^2 \). We have written \( \bar{m}^2 \) explicitly in front of the second term since it carries the dimension of that term, but have not written explicitly any \( \bar{m}^2 \)'s in dimensionless form like \( \ln(\bar{m}^2/\mu^2) \), which may still appear both in \( A_2 \) and \( A_0 \).] Let us call the dimension carried by the derivatives and fields alone genuine-dimension. Then the first term \( A_2 \) is a genuine-dimension 2 term and the second \( \bar{m}^2 A_0 \) a genuine-dimension 0 term. Two terms with different genuine-dimensions are of course mutually independent. Note that all the terms on the right-hand side of (4·12) are of genuine-dimension 0, so we see that (4·12) gives the following two equations for the genuine-dimension 2 and 0 parts, respectively:

\[
\delta_\phi A_2 = 0, \tag{4·14}
\]

\[
\delta_\phi A_0 = \left( \bar{F}^{(n)} - \bar{\beta}_i^{(n)} C^a \frac{\delta}{\delta C^a} \right) \delta_\phi f^{(0)} + \bar{m}^2 \frac{\delta \Gamma_4^{(n)}}{\delta M}. \tag{4·15}
\]

The first equation says that \( A_2 \) is given by a gauge-invariant functional \( A_{2\phi}[\phi, V] \) just as in the case of no mass terms. The second equation does not constrain the form of \( A_0 \) but determines the \( M \)-dependence of \( \Gamma_4^{(n)} \); namely, writing the dimension-0 functional \( A_0[\phi] \) as \(-\int d^4 x f^{(n)}(\phi) \) generically, we find that our effective action \( \Gamma^{(n)} \) contains the following additional terms in the presence of mass terms:

\[
\int d^4 x \left[ -\bar{m}^2 f^{(n)}(\phi) - M \delta_\phi f^{(n)}(\phi) \right] - M \left( \bar{F}^{(n)} - \bar{\beta}_i^{(n)} C^a \frac{\delta}{\delta C^a} \right) \delta_\phi f^{(0)}. \tag{4·16}
\]

But, when writing the solution \( \Gamma^{(n)} \) in the form \(-S \ast Y^{(n)}\) as in (3·22), we should note
that \(-S \ast Y^{(n)}\) now contains an additional piece \(- (M \delta_{n} f^{(0)}) \ast Y^{(n)}\) which exactly yields the second term \(-M (\delta_{n} C^{(n)}(\beta C^{(n)}) \delta_{n} f^{(0)})\). Thus the solution of the renormalization equations (4.8)~(4.10) turns out to be given by

\[
\Gamma_{2}^{(n)} + \Gamma_{4}^{(n)}|_{K-\text{linear}} = A_{\delta n}^{[\phi, V]} - S \ast Y^{(n)} + \int d^{4}x \left[ -\tilde{m}^{2} f^{(n)}(\phi) - M \delta_{n} f^{(n)}(\phi) \right]
\]

(4.16)

with the understanding that the "K-linear term" here contains \(M\)-linear terms also. The additional terms which newly appear compared with the previous solution (3.22) are only the last two terms. These two terms can be canceled by renormalizing the mass function \(f(\phi)\) in the mass term (4.6) as

\[
(f(\phi))_{n} = \frac{1}{2} \phi^{2} + f^{(1)}(\phi) + \cdots + f^{(n)}(\phi).
\]

(4.17)

This implies that we can carry out our renormalization procedure even in the presence of \(G\)-symmetry breaking mass terms and, in particular, our main Proposition in § 3 concerning the (genuine-) dimension-2 Lagrangian remains intact.

§ 5. Summary and discussion

We have shown in the covariant gauges that our tree-level dimension-2 action \(S_{2}\), (3.2), if written in terms of renormalized parameters and fields, already gives the exact action \(\Gamma_{2}\) including all the loop effects. The proof was done for \(G_{\text{global}} \times H_{\text{local}}\) model, with \(G\) and \(H\) (\(\subset G\)) being arbitrary compact groups.

Our conclusion in this paper remains unaltered even if the action \(S\) contains other dimension-4 or higher terms, as far as they respect the symmetry. This is because we needed just \((S \ast \Gamma)_{2}\) and \((S \ast \Gamma)|_{K-\text{linear}}\) parts in the WT identity to which only \(S_{2}\) and \(K\)-linear part of \(S_{4}\) can contribute.

Our model includes the chiral model with \(G=U(N)_{L} \times U(N)_{R}\) and \(H=U(N)_{V}\). The case of chiral model was worked out more explicitly in a separate article.\(^{8}\)

When we regard this chiral model as a low energy effective theory of QCD, we must take account of the anomaly and the corresponding Wess-Zumino-Witten term \(\Gamma_{\text{wzw}}\). The WT identity then reads \(\Gamma \ast \Gamma = \text{(anomaly)}\). However, the RHS is saturated already at the tree level in this effective Lagrangian, and hence the WT identity at loop levels, which we need, remains the same as before. The WZW term \(\Gamma_{\text{wzw}}\) or any other intrinsic-parity-odd terms\(^{19}\) in \(S\) are of dimension-4 or higher and hence do not change our conclusion as explained above.

In the chiral model \(S_{2}\) takes a simple form \(\int d^{4}x (\mathcal{L}_{\chi} + a\mathcal{L}_{\nu})\), which (in particular the \(\mathcal{L}_{\nu}\) part) implies that the previously derived relation\(^{5,6}\)

\[
\frac{g_{\nu}(p^{2})}{g_{\text{wzw}}(p^{2}, p_{\pi}^{2} = p_{a_{z}}^{2} = 0)} \bigg|_{p^{2} = 0} = 2f_{\pi}^{2}
\]

(5.1)

is actually an exact low energy theorem valid at any loop order. Of course, this theorem concerns off-shell quantities at \(p^{2} = 0\) of the vector field momentum \(p\), and hence is not physical as it stands. However, as Georgi\(^{20}\) discussed, the consequence
of hidden local symmetry is physical if the vector meson can be regarded as light (e.g., in the "vector limit"). Suppose that the vector mass \( m_v^2 = ag^2 f^2 \) is sufficiently small compared with the characteristic energy scale \( \Lambda^2 \) of the system, which is customarily taken as \( \Lambda^2 \sim 16\pi^2 f^2 \). Then we expect that the on-shell value of \( g_v/g_{\rho\pi\pi} \) at \( p^2 = m_v^2 \) can deviate from the LHS of (5·1) only by a quantity of order \( m_v^2 / \Lambda^2 \sim ag^2 / 16\pi^2 \), since the contributions of the dimension-4 or higher terms in the effective action \( \Gamma \) (again representing all the loop effects) are suppressed by a factor of \( p^2 / \Lambda^2 \) at least. Therefore as far as the vector mass is light, namely, when either \( a \) or \( g^2 / 16\pi^2 \) is small, our theorem is truly a physical one.

In the actual world of QCD, the \( \rho \)-meson mass is not so light \( (ag^2 / 16\pi^2 \sim 1/2) \) so that the situation becomes a bit obscure. Nevertheless, the fact that the KSRF (I) relation \( g_{\rho\pi\pi} = 2f^2 \) holds on the \( \rho \) mass shell with good accuracy strongly suggests that the \( \rho \)-meson is the hidden gauge field and the KSRF (I) relation is a physical manifestation of our low energy theorem.

In this connection we should comment on the gauge choice. In the covariant gauges which we adopted here, the \( G_{\text{global}} \) and \( H_{\text{local}} \) BRS symmetries are separately preserved. Accordingly, the \( V_\mu \) field is multiplicatively renormalized, and the above (off-shell) low energy theorem (5·1) holds. However, if we adopt \( R_\xi \)-gauges (other than Landau gauge), these properties are violated; for instance, \( \delta \phi \) or the external gauge field \( C/V_\mu \) gets mixed in the renormalization of \( V_\mu \), and our off-shell low energy theorem (5·1) is violated. This implies that the \( V_\mu \) field in the \( R_\xi \) gauge generally does not give a smooth off-shell extrapolation; indeed, in \( R_\xi \) gauge with gauge parameter \( \alpha = 1/\xi \), the correction to \( g_{\rho\pi\pi} \) by the extrapolation from \( p^2 = m_v^2 \) to \( p^2 = 0 \) is seen to have a part proportional to \( ag^2 / 16\pi^2 \), which diverges when \( \alpha \) becomes very large. Thus, in particular, the unitary gauge, \(^*\) which corresponds to \( \alpha \to \infty \), gives an ill-defined off-shell field.

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Appendix A

--- Solution to the Renormalization Equations (3·20) and (3·21) ---

In this appendix, we prove that the general solution to (3·20) and (3·21) is given in the form (3·22). We first solve (3·21) which determines the form of the \( K \)-linear term in \( \Gamma^{(n)} \), or equivalently, the form of \( \delta \Gamma \). Then next we solve (3·20) using the obtained \( \delta \Gamma \) and determine the form of \( \Gamma^{(n)}_3 \).

\(^*\) In the unitary gauge our hidden local symmetry Lagrangian coincides with Weinberg’s old Lagrangian for the \( \rho \) meson.\(^m\)
A.1. **Solving Eq. (3·21)**

To determine the form of $\delta \phi^i$, we in fact use the information only of the $K$-linear terms of (3·21). To the $K$-linear terms of (3·21), only the $K$-linear terms in $\Gamma_1^{(n)}$ and the $K$-quadratic terms in $\Gamma_0^{(n)}$ can contribute. Taking account of the ghost numbers and dimension, we write the general forms of them as

$$\Gamma_1^{(n)}|_{K\text{-linear}} = K_i \delta \phi^i + K_a^a \delta \phi^a + L_a \delta \phi^a$$

(A·1)

with

$$\delta \phi^i = C^A R_A^i(\phi),$$

(A·2)

$$\delta \phi^i V^a = G^a_b(\phi) \partial_{\mu} C^b + [G^a_b(\phi) \partial_{\mu} \phi^i + H_{bc}^a(\phi) V^c + \delta_{bc}^a(\phi) CV^c] C^b,$$

(A·3)

$$\delta \phi^a = -\frac{1}{2} C^b C^c R_{[bc]}^a(\phi),$$

(A·4)

$$\Gamma_0^{(n)}|_{K\text{-quadratic}} = \frac{1}{f^2} F_{[ab][cd]}(\phi) K^{ab} K_{[bc]}^d C^a C^d,$$

(A·5)

where $R_A^i$, $G^a_b$, $H_{bc}^a$, $R_{[bc]}^a$ and $F_{[ab][cd]}$ are dimension-0 functions of $\phi^i$ (without derivatives) carrying the specified group index structures. (The notation $[ab]$ means anti-symmetry under $a \leftrightarrow b$.) They are arbitrary functions at this stage. Note that the BRS source terms $\mathcal{K}_i \delta \phi^i \delta \phi^a$ and $\mathcal{L}_a \delta \phi^a$ do not appear in (A·1). This is because the external $G_{\text{global}} \text{-gauge fields } CV^i$ as well as their ghosts $C^i$ are not quantized but merely $c$-number fields in our system, and therefore their BRS sources $\mathcal{K}_i$ and $\mathcal{L}_a$ appear only in the tree action. So we have $\delta \phi^i \delta CV^i = \delta \phi^a C^a = 0$. Note also that only the $H_{\text{local}}$ ghosts $C^a$ appear in (A·3)~(A·5). This is because the ghost numbers of $H_{\text{local}}$ ghosts $C^a$ and of $G_{\text{global}}$ ghosts $C^i$ are in fact separately conserved in the present system of covariant gauge (2·20), and the BRS sources $K_a^a$ and $L_a$ carry the $H_{\text{local}}$ ghost number $-1$ and $-2$, respectively. [Note also that the $L_a$'s with the index $a$ corresponding to $U(1)$ factor groups $H_i$ are absent since $\delta \phi^a C^a = 0$.] On the contrary, the BRS source $K_i$ for $\phi^i$ cannot be assigned any definite separate ghost number, so that both ghosts $C^a$ and $C^i$ appear in (A·2):

$$C^A R_A^i(\phi) = C^a R_a^i(\phi) + C^i \mathcal{R}_i^a(\phi).$$

(A·6)

Picking up the $K$-linear terms of Eq. (3·21) by inserting (A·1) and (A·5), we find

$$-K_i(\delta_{\phi^i} \delta_{\phi^a}) \phi^i + L_a(\delta_{\phi^i} \delta_{\phi^a}) C^a$$

$$-K_a^a(\{\delta_{\phi^a} \delta_{\phi^a} \phi^a + 2 F_{[ab][cd]}[a_{|il} \{V^b - \delta_{l}^{il} V^b\} C^c C^d] = 0. \text{ (A·7)}$$

In the last term we have used $\delta S_2/\delta V^b = a_{|il} \{V^b - \delta_{l}^{il} V^b\}$ obtained from (2·18), and accordingly the index $l$ of $a_{|il}$ should be understood to refer to the factor group $H_i$ to which the $H_{\text{local}}$-group index $b$ of $V^b$ belongs. Since (A·7) holds as an identity, the terms proportional to $K_i$, $L_a$ and $K_a^a$ have to vanish separately. We examine those in the order i) $L_a$ term, ii) $K_i$ term and iii) $K_a^a$ term.

The $L_a$ term demands $\{\delta_{\phi^i} \delta_{\phi^a} C^a = 0$, the two terms of which are calculated using (2·28), (2·29) and (A·4) to be
\[ \delta_{b}(\delta_{f} C^{a}) = -\frac{1}{2} \delta_{b}(C^{b} C^{c} R_{\{bc\} a}(\phi)) \]
\[ = -\frac{1}{2} \left[ \frac{1}{2} C^{b}(C \times C)^{c} R_{\{bc\} a} - \frac{1}{2}(C \times C)^{b} C^{c} R_{\{bc\} a} + C^{b} C^{c} C^{d}(\tilde{W}_{A} R_{\{bc\} a}) \right] \]  
(A·8)

\[ \delta_{f}(\delta_{b} C^{a}) = -\frac{1}{2} \delta_{f}(C^{b} C^{c} f_{bc} a) \]
\[ = \frac{1}{4} \left[ C^{d} C^{e} R_{\{de\} b} C^{f} C^{g} f_{bc} a - C^{b} C^{d} C^{e} R_{\{de\} f} f_{bc} a \right] \]  
(A·9)

with notation \((C \times C) = C^{b} C^{f} f_{bc} a\). Again (A·8) + (A·9) = 0 is an identity in the field variables. Since the \(G_{\text{global}}\)-ghost fields \(C^{i}\) appear only in the last term in (A·8) in the form \(C^{b} C^{c} C^{i}(\tilde{Q}_{i} R_{\{bc\}})\) [recall that \(C^{A} \tilde{W}_{A} = C^{2} \tilde{W}_{a} + C^{i} \tilde{W}_{i}\)], that term should vanish by itself and so
\[ \tilde{Q}_{i} R_{\{bc\}}(\phi) = 0 . \]  
(A·10)

As explained in (2·26), this holds only when \(R_{\{bc\} a}(\phi)\) is a \(\phi\)-independent constant. But such a constant, which carries the index structure \(R_{\{bc\} a}\) under the (unbroken) \(H_{\text{diag}}\) group and satisfies antisymmetry under \(b+c\), is only the structure constant \(f_{bc} a\), [recall that \(L_{a}\) is absent for \(U(1)\) factor groups and so the index \(a\) here belongs to a certain simple factor group] and so we have
\[ R_{\{bc\} a}(\phi) = \beta_{c} f_{bc} a . \]  
(A·11)

Here the proportionality constant \(\beta_{c}\) may depend on the factor group \(H_{i}\) to which the indices \(a, b, c\) belong. [For later convenience, we take \(\beta_{c} = 0\) for \(U(1)\) factor groups \(H_{i}\) as convention although (A·11) is zero in any case.] Namely, at this stage we find
\[ \delta_{f} C^{a} = -\frac{1}{2} \beta_{c} (C \times C)^{a} = \beta_{c} \delta_{b} C^{a} , \]  
(A·12)

so that \(\{\delta_{b}, \delta_{f}\} C^{a} = 0\) is now clear from the nilpotency of \(\delta_{b}, (\delta_{b})^{2} = 0\).

Next consider the \(K_{i}\) term demanding \(\{\delta_{b}, \delta_{f}\} \phi^{i} = 0\). Using (2·28), (2·29), (A·2) and (A·12), we find
\[ \{\delta_{b}, \delta_{f}\} \phi^{i} = C^{A} C^{B} \left[ \tilde{R}_{A} W_{B}^{i} + \tilde{W}_{A} R_{B}^{i} - \frac{1}{2} f_{AB} C^{c} R_{C}^{i} \right] - \frac{\beta_{c}}{2} C^{a} C^{b} f_{ab} C^{c} W_{c}^{i} \]  
(A·13)

with notation \(\tilde{R}_{A} = R_{A}(\phi)(\partial/\partial \phi^{i})\). Taking account of the anti-commutativity of the ghosts, vanishingness of (A·13) means the relation
\[ [\tilde{W}_{A}, \tilde{R}_{B}] - [\tilde{W}_{B}, \tilde{R}_{A}] = f_{AB} \tilde{C}^{c} R_{C}^{i} + \delta_{A}^{a} \delta_{b}^{b} f_{ab} C^{c} W_{c}^{i} . \]  
(A·14)

This should be solved with respect to \(\tilde{R}_{A}\). The last term, which contributes only when the indices \(A\) and \(B\) are \(H_{\text{local}}\) ones \(a\) and \(b\), can easily be eliminated by shifting the \(\tilde{R}_{A}\) operator as
\[ \tilde{R}_{A} = \tilde{R}_{A} + \delta_{A}^{a} \beta_{c} \tilde{W}_{a} . \]  
(A·15)
Indeed, then using \([\tilde{\mathcal{W}}_A, \tilde{\mathcal{W}}_B]=f_{ab}^{\phi} \tilde{\mathcal{W}}_C\) in (2.24), we find the following homogeneous equation for \(\tilde{R}'_A\):

\[
[\tilde{\mathcal{W}}_A, \tilde{R}_A]-[\tilde{\mathcal{W}}_B, \tilde{R}_A]=f_{ab}^{\phi} \tilde{R}_C.
\]  
(A·16)

Clearly from the algebra \([\tilde{\mathcal{W}}_A, \tilde{\mathcal{W}}_B]=f_{ab}^{\phi} \tilde{\mathcal{W}}_C\) in (2.24), \(\tilde{R}_A\) of the form

\[
\tilde{R}_A=\tilde{\mathcal{W}}_A, \tilde{F} \rightleftharpoons \tilde{F}=F^{i}(\phi) \frac{\partial}{\partial \phi^i},
\]  
(A·17)

with some function \(F^{i}(\phi)\), satisfies (A·16). But it is less trivial whether any solution to (A·16) can be written in the form (A·17). This is proved in Appendix B.

Finally consider the \(K_{\alpha}^\mu\) term. First of all we should note that \(K_{\alpha}^\mu\) with index \(\alpha\) corresponding to any \(U(1)\) factor group in \(H_{\text{local}}\) does not appear in \(\Gamma^{(n)}\). This is because \(K_{\alpha}^\mu\) is contained in the original action \(S\) only in the form \(K_{\alpha}^\mu \partial_{\mu}C^a\) for the \(U(1)\) group index \(\alpha\). Since the \(U(1)\) ghost \(C^\alpha\) is free in this covariant gauge, \(K_{\alpha}^\mu \partial_{\mu}C^a\) itself is like a \(c\)-number source and never appears in \(\Gamma^{(n)}\). So we can assume the group index \(\alpha\) of \(K_{\alpha}^\mu\) belongs to a certain simple factor group \(H_i\) henceforth. Note also that the index \(b\) of \(C^b\) in (A·3) also belongs to the same simple factor group \(H_i\), since the ghost number is in fact conserved separately for each factor group in this covariant gauge.

Let us now analyze the vanishingness condition of the \(K_{\alpha}^\mu\) term in (A·7):

\[
\{\delta_b, \delta^\mu\} V^{a}_{\mu}+2F_{[ab]cd}a_{[l]}(V^{b}_{\mu}-\mathcal{A}^b_{\mu})C^cC^d=0.
\]  
(A·18)

Using (A·3), we calculate

\[
\begin{align*}
\delta_b(\delta^\mu V^{a}_{\mu}) &= C^A(\tilde{\mathcal{W}}_A G^b) \partial_{\mu}C^b \\
&+ [C^A \tilde{\mathcal{W}}_A(G^a_{\mu} \partial_{\mu}\phi^i)+G^{a}_{\mu} \tilde{\mathcal{W}}_A \partial_{\mu}C^A \\
&+ C^A(\tilde{\mathcal{W}}_A H^{a}_{\mu}) V^{a}_{\mu}+H^{a}_{\mu}D_{\mu}C^b \\
&+ C^A(\tilde{\mathcal{W}}_A \mathcal{H}^{a}_{\mu}) V^{a}_{\mu}+\mathcal{H}^{a}_{\mu}D_{\mu}C^b \\
&+ [G_{\mu}^{b} \partial_{\mu} \phi^{i}+H_{\mu}^{b} V^{a}_{\mu}+\mathcal{H}_{\mu}^{b} C^{a}V^{a}_{\mu}](1/2) \mathcal{C} \times \mathcal{C}. 
\end{align*}
\]  
(A·19)

Noting that the \(G_{\text{global}}\) ghosts \(C^i\) do not appear in \(\delta^\mu(\delta_b V^{a}_{\mu})\), we see that the terms containing any \(C^i\) in (A·19) should cancel among them in order for (A·18) to hold. There are various types of such terms, each of which gives the constraint on the coefficient functions in (A·3):

\[
\begin{align*}
\mathcal{C}^i \partial_{\mu}C^b: & \quad \mathcal{W}^i G^b(\phi)=0, \\
\mathcal{C}^i C^b \partial_{\mu} \phi^i: & \quad \mathcal{W}^i(G^a_{\mu}(\phi) \partial_{\mu} \phi^i)=0, \\
\partial_{\mu} \mathcal{C}^i C^b: & \quad G^{a}_{\mu}(\phi) \mathcal{W}^i(\phi)+\mathcal{H}^{a}_{\mu}(\phi)=0, \\
\mathcal{C}^i C^{b} V^{a}_{\mu}: & \quad \mathcal{W}^i H_{\mu}^{a}(\phi)=0, \\
\mathcal{C}^i C^{b} \mathcal{C} V^{a}_{\mu}: & \quad \mathcal{W}^i \mathcal{H}_{\mu}^{a}(\phi)+\mathcal{H}^{a}_{\mu}(\phi)f^{a}_{\mu}=0.
\end{align*}
\]  
(A·20 - A·24)

Equation (A·20) says that \(G_{\mu}^{a}(\phi)\) is \(\phi\)-independent constant by (2·26). By
$H_{\text{diag}}$-covariance and the separate ghost number conservation for each factor group $H_i$, such a constant must be proportional to $\delta_b^a$:

$$G_b^a(\phi) = a_i \delta_b^a,$$  \hfill (A·25)

where the proportionality constant $a_i$ may depend on the simple factor group $H_i$ to which the index $a$ belongs.

Equation (A·21) does not say that $G_{bi}^a \partial_\mu \phi^i$ is constant since it contains derivative $\partial_\mu \phi^i$, but says it is $G_{\text{global}}$-invariant. We know that the only $G_{\text{global}}$-invariant containing the first order derivative $\partial_\mu \phi^i$ is given by $a_i^a(\phi) = (2/i) \text{tr}(T^i \partial_\mu \xi^i)$, or, separating the $H$ and $G/H$ generator parts, by $a_i^a(\phi) = (2/i) \text{tr}(S^a \partial_\mu \xi^i)$ and $a_i^a(\phi) = (2/i) \text{tr}(X^a \partial_\mu \xi^i)$. Therefore the $G_{\text{global}}$-invariant $G_{bi}^a(\phi) \partial_\mu \phi^i$ must be a linear combination of them:

$$G_{bi}^a(\phi) \partial_\mu \phi^i = g_{ba}^c a_i^a(\phi) + g_{ba}^c a_{l,1}(\phi).$$  \hfill (A·26)

Since the indices $a$ and $b$ belong to the same simple factor group $H_i$ as noted in the above, the $H_{\text{diag}}$-covariance demands the coefficient $g_{ba}^c$ corresponding to the $H$ group index $c$ to take the form

$$g_{ba}^c = \gamma f_{bc}^d + \widetilde{\gamma} d_{bc}^d + \tilde{\gamma}_{u_1}^d \delta_b^a \delta_c^u,$$  \hfill (A·27)

where $d_{abc} = 2 \text{tr}(S_a S_b S_c)$ for $S_a, S_b, S_c$ belonging to the same simple factor group $H_i$ and the index $u_1$ runs over the $U(1)$ factor groups in $H$. The second coefficient $g_{ba}^{l, a}$ corresponding to the $G/H$ index $a$ also takes a similar form:

$$g_{ba}^{l, a} = \gamma f_{a1}^e \delta_a^e + \gamma_{a1}^e d_{ba}^e \delta_c^e + \tilde{\gamma}_{a1}^e \delta_b^a \delta_c^e,$$  \hfill (A·28)

where $\gamma f_{a1}$ and $\gamma_{a1}$ can be non-vanishing only when the $H$-irreducible representation $[a]$ to which the $G/H$ generator $X_a$ belongs happens to be the same representation as the $H_i$ generators, namely, adjoint under $H_i$ and singlet under the other factor groups $H_k$ ($k \neq i$). The index $u_1$ in the last term in (A·28) runs over all the $H$-singlet indices among the $G/H$ generators $X_a$.

Equation (A·22) simply gives $\mathcal{M}_{bi}^a(\phi)$ in terms of $G_{bi}^a(\phi)$:

$$\mathcal{M}_{bi}^a(\phi) = - G_{bi}^a(\phi) W_i(\phi).$$  \hfill (A·29)

This form implies an interesting fact: in (A·3), the terms $G_{bi}^a \partial_\mu \phi^i$ and $\mathcal{M}_{bi}^a c \nu_{\mu}^i$ are combined to yield

$$G_{bi}^a(\phi) \partial_\mu \phi^i + \mathcal{M}_{bi}^a(\phi) c \nu_{\mu}^i = G_{bi}^a(\partial_\mu \phi^i - c \nu_{\mu}^i W_j(\phi)).$$  \hfill (A·30)

This is nothing but a $G_{\text{global}}$-covariant derivative since $\delta \phi^i = W_j^i$ is just an infinitesimal $G_{\text{global}}$ transformation by $T_j$. In view of (A·26), we therefore find that (A·30) must equal

$$(A·30) = g_{bc}^a \frac{2}{i} \text{tr}[S^c \partial_\mu \xi^c + i \xi^c c \nu_{\mu}^i \cdot \xi^i] + g_{ba}^c \frac{2}{i} \text{tr}[X^a \partial_\mu \xi^a + i \xi^a c \nu_{\mu}^i \cdot \xi^i]$$

\[= g_{bc}^a A_{\mu}^c(\phi) + g_{ba}^c A_{\mu,1}^c(\phi), \]

with $A_{\mu}^a(\phi)$ defined in (2·19) and $A_{\mu,1}^a(\phi)$ defined similarly.
Equation (A·23) again says about the constantness of $H_{bc}^a(\phi)$ so that we have from $H_{\text{diag}}$-covariance

$$H_{bc}^a(\phi) = h_{ij}^{bc} + \tilde{h}_{ij}^{bc} + \tilde{h}_{ij}^{bc} \delta_b \delta_c^{u_1} \quad (\equiv h_{bc}^a) \quad (\text{A·32})$$

The final equation (A·24) is easily seen to be satisfied automatically using (A·21) and (A·29).

At this stage, the $\delta V_{p}^a$ in (A·3) is already rather simplified:

$$\delta_{\gamma} V_{p}^a = a' \partial_{p} C^a + g_{bc}^a \mathcal{A}_{p}^c(\phi) + g_{bc}^a \mathcal{A}_{p}^{\mu}(\phi) + h_{bc}^a V_{p}^c$$

$$= a' \partial_{p} C^a + U_{p}^a(\phi) C^a + f_{bc}^a B_{p}^c(\phi) C^b + d_{bc}^a \tilde{B}_{p}^c(\phi) C^b \quad (\text{A·33})$$

with notations

$$U_{p}^a(\phi) = \tilde{h}_{ij}^{u_1} \mathcal{A}_{p}^{u_1}(\phi) + \tilde{h}_{ij}^{u_1} \mathcal{A}_{p}^{\mu}(\phi) + \tilde{h}_{ij}^{u_1} V_{p}^{u_1},$$

$$\mathcal{B}_{p}^c(\phi) = \gamma_{ij} \mathcal{A}_{p}^c(\phi) + \gamma_{ij} \mathcal{A}_{p}^{\mu}(\phi) + h_{p} V_{p}^c,$$

$$\mathcal{B}_{p}^c(\phi) = \tilde{\gamma}_{ij} \mathcal{A}_{p}^c(\phi) + \tilde{\gamma}_{ij} \mathcal{A}_{p}^{\mu}(\phi) + \tilde{h}_{p} V_{p}^c. \quad (\text{A·34})$$

Since now the indices $a$, $b$ and $c$ in the second expression in (A·33) for $\delta V_{p}^a$ are all those belonging to the same simple factor group $H_i$, it is much more convenient to switch to matrix notation and rewrite (A·33) with (A·34) into

$$\delta_{\gamma} V_{p} = a' \partial_{p} C + U_{p} C + i[B_{p}, C] + \{B_{p}, C\}, \quad (\text{A·35})$$

$$B_{p} = \gamma_{ij} \mathcal{A}_{p}^{\mu} + \gamma_{ij} \mathcal{A}_{p}^{\mu}(\phi) + h_{p} V_{p}^c, \quad \mathcal{B}_{p} = \tilde{\gamma}_{ij} \mathcal{A}_{p}^{\mu} + \tilde{\gamma}_{ij} \mathcal{A}_{p}^{\mu}(\phi) + \tilde{h}_{p} V_{p}^c, \quad (\text{A·36})$$

where the matrices mean, for instance,

$$A' C = \sum_{a} \sum_{a \in H_i} C^a S_a, \quad h V_{p} = \sum_{a} h_{i} \sum_{a \in H_i} V_{p}^a S_a$$

with summation taken only over simple factor groups $H_i$ in $H$. Noting that $V_{p}^a - \mathcal{A}_{p}^a(\phi)$ and $\mathcal{A}_{p}^{\mu}(\phi)$ are $G_{\text{global}}$-invariant and $H_{\text{local}}$-covariant, we find

$$\delta_{bc} A_{p} = \partial_{p} C - i[C_{p}, \mathcal{A}_{p}], \quad \delta_{bc} A_{p}^{\mu} = -i[C_{p}, \mathcal{A}_{p}^{\mu}], \quad (\text{A·37})$$

$$\delta_{bc} U_{p} = (\tilde{h}_{ij}^{u_1} + \tilde{h}_{ij}^{u_1}) \partial_{p} C^{u_1} = \partial_{p} C^{u_1(1)}, \quad (\text{A·38})$$

and hence

$$\delta_{bc} B_{p} = (\gamma + h) \partial_{p} C - i[B_{p}, C], \quad \delta_{bc} [B_{p}, C] = (\gamma + h) \partial_{p} (C^2), \quad (\text{A·39})$$

and so on. Using these we calculate

$$\delta_{b} (\delta_{\gamma} V_{p}) = (a' + \gamma + h) \partial_{p} (i C^2) + (\partial_{p} C^{u_1(1)}) C + i U_{p} C^2$$

$$+(\tilde{\gamma} + \tilde{h})[\partial_{p} C, C] + 2i C \tilde{B}_{p} C, \quad (\text{A·40})$$

$$\delta_{b} (\delta_{\gamma} V_{p}) = (\beta - \alpha') \partial_{p} (i C^2) - 2i U_{p} C^2 + [B_{p}, C]^2$$

$$-i[B_{p}, C^2] - 2i C \tilde{B}_{p} C. \quad (\text{A·41})$$

These (A·40) and (A·41) should add up to cancel the last $F_{[ab][cd]}$ term in (A·18). Since the latter term comes from $F_{[ab][cd]} K_{a}^{a} K_{b}^{a} C_{c} C_{d}$ term in $\Gamma_{\phi}$, the indices $a$ and...
b must be of simple factor groups, so that the last term contains neither $\mathcal{A}_{\mu}^{\mu}$, $\mathcal{A}_{\mu}^{\nu}$ nor $V_\mu^{\nu}$ of $U(1)$ factor group or $H$-singlet indices. Therefore the $U_\mu C_\mu$ terms should cancel already in $(A\cdot 40)+(A\cdot 41)$, but it demands $U_\mu$ itself vanishes (i.e., $\tilde{\varphi} = \tilde{\varphi} = \tilde{h} = 0$) and so $C_{\mu t t} = 0$. Moreover, the $F_{[a b] t c d}$ term in $(A\cdot 18)$ contains only $V_\mu b - \mathcal{A}_\mu b$ of the simple factor groups $H_t$, but no $\mathcal{A}_\mu a$ of the $G/H$ generators $X_\alpha$, so that the terms containing $\mathcal{A}_\mu a$ in $(A\cdot 40)+(A\cdot 41)$ should also vanish; that is, $\gamma^a = \tilde{\gamma}^a = 0$. Further, since the $F_{[a b] t c d}$ term in $(A\cdot 18)$ does not contain $\partial_\alpha C$ either, the terms containing $\partial_\alpha C$ should also vanish in $(A\cdot 40)+(A\cdot 41)$. This yields

$$\beta_t + \gamma_t + h_t = 0,$$  \hspace{1cm} (A\cdot 42)

so that $(A\cdot 40)+(A\cdot 41)$ becomes, at this stage,

$$\{\delta_b, \delta_t\} V_\mu = - \gamma [V_\mu - \mathcal{A}_\mu, C^a] + i \tilde{\gamma} [V_\mu - \mathcal{A}_\mu, C^a],$$  \hspace{1cm} (A\cdot 43)

or, in terms of the original notation,

$$\{\delta_b, \delta_t\} V_\mu = - \frac{1}{2} \gamma (V_\mu - \mathcal{A}_\mu) b (C \times C) + \frac{1}{2} \tilde{\gamma} (V_\mu - \mathcal{A}_\mu) b (C \times C).$$  \hspace{1cm} (A\cdot 44)

The second term proportional to $d_{bc} a$ is symmetric under $a \leftrightarrow b$, and cannot be canceled with the $a \leftrightarrow b$ anti-symmetric $F_{[a b] t c d}$ term in $(A\cdot 18)$. So

$$\tilde{\gamma} = 0,$$  \hspace{1cm} (A\cdot 45)

and then $(A\cdot 18)$ turns out to give

$$F_{[a b] t c d}(\phi) = - \frac{1}{4} \frac{\gamma_t}{a_{[t]} f_{a b c d}}.$$  \hspace{1cm} (A\cdot 46)

Putting $(A\cdot 42), (A\cdot 45), \gamma^a = \tilde{\gamma}^a = 0$ and $U_{\mu t} = 0$ together into $(A\cdot 33)$, we finally obtain

$$\delta_t V_\mu = \alpha_t \partial_\mu C^a + \beta_t (V_\mu \times C) + \gamma_t (V_\mu - \mathcal{A}_\mu) \times C^a = \alpha_t \partial_\mu C^a + \beta_t \delta_b V_\mu + \gamma_t \delta_b (V_\mu - \mathcal{A}_\mu)$$  \hspace{1cm} (A\cdot 47)

with $\alpha_t = \alpha_t - \beta_t$.

We thus have finished solving the renormalization equation (3·21) and found the following collecting the results obtained above:

$$I_4^{(n)}_{K \text{-linear}} = K [C^a (\bar{W}_a F^i (\phi) - F \bar{W}^a_i (\phi)) + C^a W_\mu^i (\phi)]$$

$$+ K^a [\alpha_t \partial_\mu C^a + \beta_t \delta_b V_\mu + \gamma_t \delta_b (V_\mu - \mathcal{A}_\mu) (\phi)]$$

$$+ L_\alpha (\beta_t \delta_b C^a),$$  \hspace{1cm} (A\cdot 48)

$$I_6^{(n)}_{K \text{-quadratic}} = - \frac{1}{2 f_{\pi}^2} \alpha_{[t]} (K_\mu K_\nu) a (C \times C)^a,$$  \hspace{1cm} (A\cdot 49)

where $F^i (\phi)$ is an arbitrary function and $\alpha_t, \beta_t, \gamma_t$ are arbitrary parameters dependent on the simple factor group $H_t$.

A.2. Solving Eq. (3·20)

Now that we have determined the form of $\delta_t$, we can solve the renormalization
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We calculate $\delta S_2$ using (A·48) and (A·49) as follows:

$$\delta S_2 = (\delta \phi^i) \frac{\delta S_2}{\delta \phi^i} + (\delta V_\mu^a) \frac{\delta S_2}{\delta V_\mu^a}$$  
(A·50)

$$= C[A[\Phi, F]] S_2$$  
(A·51)

$$+ \beta \left( C W_a^i(\phi) \frac{\delta}{\delta \phi^i} + D_\mu C^a \frac{\delta}{\delta V_\mu^a} \right) S_2$$  
(A·52)

$$+ \left[ a_0 \delta \mu C^a + i_l \left( (V_\mu - A_\mu) \times C \right)^a \right] \frac{\delta S_2}{\delta V_\mu^a}.$$  
(A·53)

First note that

$$[\delta_F, F] = \left[ C^A W_a^i(\phi) \frac{\delta}{\delta \phi^i} + D_\mu C^a \frac{\delta}{\delta V_\mu^a} - \frac{1}{2} (C \times C)^A \frac{\delta}{\delta C^A}, F^j(\phi) \frac{\delta}{\delta \phi^j} \right]$$  

$$= [C^A W_a^i, F],$$  
(A·54)

which together with $\delta \mu S_2 = 0$ leads to

$$C[A[\Phi, F]] S_2 = \delta \mu (F S_2).$$  
(A·55)

Next we note that $\beta (C W_a + D_\mu C^a \delta / \delta V_\mu^a)$ appearing in (A·52) is a $H_\mu$-local gauge transformation with angle $\theta^a = \beta C^a$, and so (A·52) vanishes because of the $H_\mu$-gauge invariance of $S_2$. Thirdly, using the fact that $\delta S_2 / \delta V_\mu^a = a_1 f_\mu^a (V_\mu - A_\mu)^a$ by (2·18), we see

$$\delta \mu S_2 = (a_0 \delta \mu C^a) \cdot a_1 f_\mu^a (V_\mu - A_\mu)^a$$  

$$= \delta \mu (a_1 V_\mu^a \cdot a_1 f_\mu^a (V_\mu - A_\mu)^a)$$  

$$= \delta \mu \left( a_1 V_\mu^a \frac{\delta}{\delta V_\mu^a} S_2 \right).$$  
(A·56)

Thus we find that $\delta \mu S_2$ can be written in the form $\delta \mu (\star)$, and the renormalization equation (3·20) becomes

$$\delta \mu I_2^{(n)} + \delta \mu \left( F S_2 + a_1 V_\mu^a \frac{\delta}{\delta V_\mu^a} S_2 \right) = 0.$$  
(A·57)

The general solution to this is clearly given by

$$I_2^{(n)} = A_{2\mu}[\phi, V] - \left( F S_2 + a_1 V_\mu^a \frac{\delta}{\delta V_\mu^a} S_2 \right)$$  

(A·58)

with arbitrary gauge-invariant function $A_{2\mu}$ of dimension 2.

We have finished solving (3·20). It is now a trivial matter to check that our solutions (A·58) plus (A·48) and (A·49) are combined into a simple form

$$I_2^{(n)} + I_4^{(n)} \left|_{\text{linear}} + I_6^{(n)} \right|_{\text{quadratic}} = A_{2\mu}[\phi, V] - S * Y,$$  
(A·59)

$$Y = \int d^4 x \left[ K_0 F^i(\phi) + a_1 K_\mu^a V_\mu^a + \beta_1 L_a C^a + \frac{\gamma_1}{2 a_1 f_\mu^a} f_{abc} K_\mu^a K_\mu^b C^c \right].$$  
(A·60)
aside from irrelevant terms which we are not discussing. This just agrees with (3·22) with (3·23) which we wanted to prove.

**Appendix B**

--- The Proof of (A·17) ---

We first note that (A·16) splits into the following three types according to whether the group indices $A$ and $B$ refer to the $G_{\text{global}}$ or $H_{\text{local}}$ ones: recalling $W_A = (\bar{W}_i, \bar{W}_a)$ and $\bar{R}_A = (\bar{R}_i, \bar{R}_a)$ we have

\[ [\bar{W}_i, \bar{R}_i] - [\bar{W}_j, \bar{R}_j] = f_{ij}^k \bar{R}_k, \quad (B·1) \]

\[ [\bar{W}_i, \bar{R}_a] - [\bar{W}_a, \bar{R}_i] = 0, \quad (B·2) \]

\[ [\bar{W}_a, \bar{R}_a] - [\bar{W}_b, \bar{R}_a] = f_{ab}^c \bar{R}_c. \quad (B·3) \]

We shall show below that the first set of equations, (B·1), already gives enough information to determine the form of the general solution $\bar{R}_i$ as

\[ \bar{R}_i = [\bar{W}_i, \bar{F}]. \quad (B·4) \]

[Recall that there are as many $G_{\text{global}}$ generators $\bar{W}_i = -\partial/\partial \phi^i + \cdots$ as the variables $\phi^i$.] Assuming that (B·4) is proved for $\bar{R}_i$, we first prove that $\bar{R}_a$ is given by $[\bar{W}_a, \bar{F}]$ in terms of the same $\bar{F}$. Equations (B·4), (B·2) and the Jacobi identity with the algebra $[\bar{W}_i, \bar{W}_a] = 0$ lead to

\[ [\bar{W}_i, (\bar{R}_a - [\bar{W}_a, \bar{F}])] = 0. \quad (B·5) \]

This does not immediately imply the desired equation $\bar{R}_a = [\bar{W}_a, \bar{F}]$, since there are operators which commute with the $\bar{W}_i$'s. But, note that the $\bar{W}_i$'s span a complete set of Lie algebra generators of the group $G$ corresponding to the right-multiplication. Therefore, as is well-known, the complete set of first order differential operators which are commutative with all the $\bar{W}_i$'s, is given by the Lie algebra generators of $G$ corresponding to the left-multiplication which is denoted by $\bar{W}_i$. Our $H_{\text{local}}$ generators $\bar{W}_a$ are just a subset of $\bar{W}_i$'s: $\bar{W}_{iwa} = \bar{W}_a$. Thus, (B·5) generally says that the difference $\bar{R}_a - [\bar{W}_a, \bar{F}]$ is given by a linear combination of the left-multiplication generators:

\[ \bar{R}_a - [\bar{W}_a, \bar{F}] = z_a^i \bar{W}_i = \bar{Z}_a \quad (B·6) \]

with certain coefficients $z_a^i$. We now use (B·3). Since $\bar{R}_a$ and clearly $[\bar{W}_a, \bar{F}]$ also satisfy (B·3), so does $z_a^i \bar{W}_i = \bar{Z}_a$:

\[ [\bar{W}_a, \bar{Z}_a] - [\bar{W}_b, \bar{Z}_a] = f_{ab}^c \bar{Z}_c. \quad (B·7) \]

On the other hand, the $H_{\text{diag}}$-covariance implies that both sides of (B·6) have the same transformation law under $\bar{W}_a + \bar{W}_a$ [see (2·27)], so that we have

\[ [\bar{W}_a, \bar{Z}_a] = [\bar{W}_a + \bar{W}_a, \bar{Z}_a] = f_{ab}^c \bar{Z}_c. \quad (B·8) \]

Substituting this into (B·7), we find

\[ f_{ab}^c \bar{Z}_c = 0, \quad (B·9) \]
which implies $Z_a = 0$ for the indices $a$ belonging to any simple factor group $H_t$. For the indices $a$ corresponding to $U(1)$-factor groups (if any), however, $Z_a = z_a \tilde{W}_t$ may still not vanish. But, again by the covariance under $H_{\text{diag}}$, the coefficients $z_a$ can be nonvanishing only for $H_{\text{diag}}$-singlet indices $i$. Recall that we are discussing the $K_t$-term appearing in $\Gamma_t^{(n)}$, which now takes the form, by (A·15), (B·4) and (B·6)

$$K_t [C^j R_t^j(\phi) + C^a R_t^a(\phi)]$$

$$= K_t [C^j [\bar{W}_j, \bar{F}] \phi^i + C^a (\beta_t \bar{W}_a + [\bar{W}_a, \bar{F}] + z_a \tilde{W}_j) \phi^i]. \quad (B·10)$$

Noting $z_a \tilde{W}_j \phi^i = z_a + O(\phi)$ (since $\tilde{W}_j = \partial / \partial \phi^j + \cdots$), we see that the last term contributes only when the index $i$ is of $H_{\text{diag}}$-singlet and the index $a$ is of $U(1)$ factor group. But such a term proportional to $K_t C^a$ with $H_{\text{loc}}$-singlet $i$ and $U(1)$ index $a$ has a particular property in the original action $S$. In $S$, $K_t$ is contained in the form $K_t (C^j \bar{W}_j(\phi) + C^a \bar{W}_a(\phi))$. Since the relation $W_a(\phi) = - \bar{W}_a(\phi)$ holds for $H_{\text{diag}}$-singlet index $i$ and the $U(1)$ ghosts $C^a$ are free, the external (non-quantized) ghosts $C = C^a$ and the $U(1)$ ghosts $C^a$ are contained in the $H_{\text{diag}}$-singlet $K_t$ term on the same footing in the form $K_t (C^a - C) W^a(\phi)$. Therefore the $n$-th loop level effective action $\Gamma_t^{(n)}$ has to contain the $K_t C^a$ term with the same coefficient function of $\phi$ as the $-K_t C^a$ term. (Recall that $\beta_t = 0$ for the $U(1)$ index $a$.) In view of the RHS of (B·10), this implies $z_a = 0$. Thus we have proved $\tilde{R}_t = \tilde{W}_a + \bar{F}$ for any $a$.

Remaining is the proof of (B·4). We prove it by mathematical induction with respect to the powers in $\phi$ of the solution $\bar{R}_t^i$ to (B·1). Both $\bar{W}_j(\phi)$ and $\bar{R}_t^i(\phi)$ have terms of powers $\phi^n$ with $n = 0, 1, 2, \cdots$. The $n$-th power terms are denoted by $\bar{W}_j^{(n)}$ and $\bar{R}_t^{(n)}$, and the corresponding operators $\bar{W}_j^{(n)}(\partial / \partial \phi^j)$ and $\bar{R}_t^{(n)}(\partial / \partial \phi^i)$ by $\bar{W}_j^{(n)}$ and $\bar{R}_t^{(n)}$. Then our claim is the following: For any solution $\bar{R}_t$ to (B·1), there exists an $(n+1)$-th order polynomial $F_{n+1}(\phi)$ in $\phi$ starting from a linear term

$$F_{n+1}^i(\phi) = F_1^i \phi^i + F_2^i \phi^j \phi^j + \cdots + F_{n+1}^i \phi^j \phi^j \phi^{j+n+1}$$

$$= F_1^i(\phi) + F_2^i(\phi) + \cdots + F_{n+1}^i(\phi) \quad (B·11)$$

for which the commutator $[\bar{W}_i, \bar{R}_t^{(n+1)}] = F_{n+1}(\phi) \partial / \partial \phi^i$ gives the solution $\bar{R}_t^i$ correctly up to the $n$-th power terms; namely,

$$\bar{R}_t^i = [\bar{W}_i, \bar{R}_t^{(n+1)}] + O(\phi^{n+1}) \times \partial / \partial \phi^i. \quad (B·12)$$

Proof We can start the induction from $n = -1$. Then $F_{n+1}^i(\phi) = F_0^i(\phi)$ of (B·11) is zero by definition, but $\bar{R}_t^i$ itself is $O(\phi^0) \times \partial / \partial \phi$ and so (B·12) is trivially true. If (B·12) holds for a certain $n$, then the difference $\tilde{R}_t^i = \bar{R}_t^i - [\bar{W}_i, \bar{R}_t^{(n+1)}]$ starts from an $(n+1)$-th power term in $\phi$. Since both $\bar{R}_t^i$ and $[\bar{W}_i, \bar{R}_t^{(n+1)}]$ satisfy (B·1), the difference $\tilde{R}_t^i$ also satisfies it:

$$[\bar{W}_i, \tilde{R}_t^i] = [\bar{W}_j, \tilde{R}_t^i] = f_{ijk} \tilde{R}_t^k. \quad (B·13)$$

Consider the $n$-th power terms on both sides of this equation. Such terms exist only on the left-hand side and come from the lowest power terms of both $\bar{W}_i$ and $\tilde{R}_t^i$. So we have

$$[\bar{W}_i^{(0)}, \tilde{R}_t^{(n+1)}] - [\bar{W}_j^{(0)}, \tilde{R}_t^{(n+1)}] = 0. \quad (B·14)$$
Recalling $\mathcal{W}_i^{(0)} = -\partial/\partial \phi^i$ \((2\cdot25)\) and writing $r_i^{(n+1)} = r_i^{(n+1)}(\phi)\partial/\partial \phi^i$, this simply gives
\[
\frac{\partial}{\partial \phi^i} r_j^{(n+1)k}(\phi) - \frac{\partial}{\partial \phi^j} r_i^{(n+1)k}(\phi) = 0. \tag{B·15}
\]
But this is just an integrability condition and guarantees that there exists an \((n+2)\)-th power homogeneous function $F_{(n+2)}^i(\phi)$ such that
\[
r_j^{(n+1)k}(\phi) = -\frac{\partial}{\partial \phi^j} F_{(n+2)}^i(\phi) = [\mathcal{W}_j^{(0)}, F_{(n+2)}^i] \phi^i.
\]
This implies that if we define \((n+2)\)-th order polynomial $F_{n+2}^i(\phi)$ by
\[
F_{n+2}^i(\phi) = F_{n+1}^i(\phi) + F_{(n+2)}^i(\phi),
\]
it satisfies
\[
\mathcal{R}_i = [\mathcal{W}_i, F_{n+2}] + \mathcal{O}(\phi^{n+2}) \times \partial/\partial \phi,
\]
namely, \((B·12)\) with \(n\) raised by 1. This finishes the proof.

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