Control problems for the stationary MHD equations under mixed boundary conditions

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Abstract. The optimal control problems for the stationary magnetohydrodynamic equations under inhomogeneous mixed boundary conditions for a magnetic field are considered. The role of control in control problems under study is played by normal component of the magnetic field on the part of the boundary. The existence of optimal solution is proved and optimality system for considered extremum problem is obtained.

1. Introduction. Statement of the boundary value problem

Control problems for models of magnetic hydrodynamics of viscous electric conducting fluids play an important role in a number of applications [1, 2]. In this paper we study control problems for the stationary model of magnetic hydrodynamics, considered under mixed boundary conditions for the electromagnetic field.

Let $\Omega$ be a bounded domain of the space $\mathbb{R}^3$ with boundary $\Sigma = \partial \Omega$ consisting of two parts $\Sigma_\nu$ and $\Sigma_\tau$. We will consider following system of the stationary magnetohydrodynamic equations of viscous incompressible fluid

$$-\nu \Delta u + (u \cdot \nabla)u + \nabla p - \kappa \text{curl} \mathbf{H} \times \mathbf{H} = f, \quad \text{div} u = 0, \quad (1)$$

$$\nu_1 \text{curl} \mathbf{H} - \rho_0^{-1} \mathbf{E} + \kappa \mathbf{H} \times \mathbf{u} = \nu_1 \mathbf{j}, \quad \text{div} \mathbf{H} = 0, \quad \text{curl} \mathbf{E} = 0, \quad (2)$$

together with the following inhomogeneous boundary conditions:

$$u|_{\Sigma} = 0, \quad \mathbf{H} \cdot \mathbf{n}|_{\Sigma_\nu} = q, \quad \mathbf{H} \times \mathbf{n}|_{\Sigma_\nu} = 0, \quad \mathbf{E} \times \mathbf{n}|_{\Sigma_\tau} = k. \quad (3)$$

Here $u$ is the velocity vector, $\mathbf{H}$ and $\mathbf{E}$ are magnetic and electric fields, respectively, $p = P/\rho_0$, where $P$ is the pressure, $\rho_0 = \text{const}$ is a fluid density, $\kappa = \mu/\rho_0$, $\nu_1 = 1/\rho_0 \sigma = \kappa \nu_m$, $\nu$ and $\nu_m$ are constant kinematic and magnetic viscousity coefficients, $\sigma$ is a constant conductivity, $\mu$ is a constant magnetic permeability, $\mathbf{n}$ is the outer normal to $\Sigma$, $\mathbf{j}$ is the current density. Below we will refer to problem (1)–(3) for given functions $f, \mathbf{j}, \mathbf{k}$ and $q$ as Problem 1. We note that all the quantities in (1)–(3) are dimensional and their physical dimensions are defined in terms of SI units. Physically the boundary conditions for the electromagnetic field in (3) correspond to the situation when the part $\Sigma_\nu$ of the boundary $\Sigma$ is a perfect insulator.

In the particular case when $\Sigma_\nu = \emptyset$ the boundary conditions (3) for the magnetic field take the form $\mathbf{H} \cdot \mathbf{n}|_{\Sigma} = q, \mathbf{E} \times \mathbf{n}|_{\Sigma} = k$. The corresponding boundary value and extremum
problems for model (1), (2) were studied by a number of authors (see, for example [3–13].
Beginning from 2014, the authors published papers [14–16] devoted to the analysis of the
solvability of boundary value problems for model (1), (2) considered under mixed (homogeneous
or inhomogeneous) boundary conditions for the electromagnetic field. Further, authors’ papers
[17, 18] were published, in which the study of control problems for model (1), (2) in some
particular cases was started. Continuing the studies begun in [17, 18], we formulate below the
control problem for model (1), (2) in which the boundary function \( q \in H^s(\Sigma_r) \) where \( s \in [0,1/2] \)
plays the role of control. We study the solvability of the optimal control problem and derive an
optimality system describing the necessary conditions of extremum.

2. Function spaces. The preliminary results
As in [15, 16, 17, 18] we begin with describing the main functional spaces. Below we will use
the Sobolev spaces \( H^s(D) \), \( s \in \mathbb{R} \), \( H^0(0) = L^2(D) \), where \( D \) denotes domain \( \Omega \), its boundary
\( \Sigma \) or open nonempty subset \( \Sigma_0 \subset \Sigma \). Corresponding spaces of vector–functions are denoted by
\( H^s(D)^3 \) and \( L^2(D)^3 \). The inner products and norms in the spaces \( H^s(D) \) and \( H^s(D)^3 \) are
denoted by \( (\cdot,\cdot)_{s,D} \) and \( \|\cdot\|_{s,D} \). The inner products and norms in \( L^2(\Omega) \) and \( L^2(\Omega)^3 \) are
denoted by \( (\cdot,\cdot)_{\Sigma} \) and \( \|\cdot\|_{\Omega} \). By \( H^s(\Sigma) \) or \( H^s(\Sigma_0) \) at \( s \geq 0 \) we denote the usual trace spaces of the
space \( H^{s+1/2}(\Omega) \) over \( \Sigma \) or over proper (non empty) open subset \( \Sigma_0 \) of \( \Sigma \). By \( H^{-s}(\Sigma) \) and
\( H^{-s}(\Sigma_0) \) we denote the dual spaces of \( H^s(\Sigma) \) and \( H^s(\Sigma_0) \), respectively, in the case when \( \Sigma_0 \) is
a surface without boundary. By \( \langle \cdot,\cdot \rangle_{\Sigma_0}, \langle \cdot,\cdot \rangle_{\Sigma} \) we denote the duality pairing between \( H^{-s}(\Sigma) \)
and \( H^{-s}(\Sigma_0) \) and \( H^{-s}(\Sigma) \), \( H^{-s}(\Sigma_0) \) respectively.

As in [20, 21] we assume that the following conditions to \( \Omega \) are satisfied:
(i) \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) and the boundary \( \partial \Omega \) is the union of a finite number of
disjoint closed \( C^2 \) surfaces, each surface having finite surface area;
(ii) \( \Sigma_0 \) is nonempty open subset of \( \partial \Omega \) with \( M + 1 \) disjoint nonempty open components
\( \{\sigma_0, \sigma_1, \ldots, \sigma_M\} \) and there is a positive \( d_0 \) such that \( d(\sigma_i, \sigma_j) \geq d_0 > 0 \) when \( i \neq j \) and
\( M \geq 1 \). The boundary of each \( \sigma_i \) is either empty or \( C^{1,1} \) curve. We set \( \Sigma_0 = \partial \Omega \setminus \Sigma_0 \).

Let \( D(\Omega) \) be the space of infinitely differentiable compactly supported functions in \( \Omega \), \( H^0(\Omega) \)
be the closure of \( D(\Omega) \) in \( H^1(\Omega) \). \( V = \{v \in H^0(\Omega)^3 : \text{div } v = 0 \} \), \( H^{-1}(\Omega)^3 = (H_0^1(\Omega)^3)^* \),
\( L^2(\Omega) = \{v \in L^2(\Omega) : (p,1) = 0 \} \), \( H^1(\Omega, \Sigma) = \{v \in H^1(\Omega) : v|_{\Sigma} = 0 \} \), \( C_{\Sigma_0}(\Omega)^3 := \{v \in \mathcal{C}^0(\Omega)^3 : v \cdot n|_{\Sigma_0} = 0, v \times n|_{\Sigma_0} = 0 \} \). In addition to the spaces introduced above we will use the
spaces \( H(\text{div}, \Omega) = \{v \in L^2(\Omega)^3 : \text{div } v \in L^2(\Omega) \}, H(\text{curl}, \Omega) = \{v \in L^2(\Omega)^3 : \text{curl } v \in L^2(\Omega)^3 \},
H(0)(\text{curl}, \Omega) = \{v \in H(\text{curl}, \Omega) : \text{curl } v = 0 \} \) and the space \( H(\text{div}, \Omega) = H(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \),
equipped with the Hilbert norm \( \|u\|_{D(\Omega)}^2 := \|u\|_D^2 + \|\text{div } u\|_0^2 + \|\text{curl } u\|_0^2 \).

Let \( H_{DCs}(\Omega) \) be the closure of \( C_{0,\Omega}(\Omega)^3 \cap H^1(\Omega)^3 \) with respect to the norm \( \|\cdot\|_{DC(\Omega)} \). Let
\( H_{s+1/2}(\Sigma_0) = \{h \in L^2(\Omega)^3 : \text{div } h = 0, \text{curl } h = 0 \} \), \( H_{s+1/2}(\Sigma) = \{h \in L^2(\Omega)^3 : h \times n|_{\Sigma} = 0 \} \),
\( V_{s+1/2}(\Sigma_0) = \{v \in H_{DCs}(\Omega) : \text{div } v = 0 \} \cap H_{s+1/2}(\Sigma_0) \), \( H_{s+1/2}(\Sigma) = H_{s+1/2}(\Omega)^3 \times V_{s+1/2}(\Sigma) \).

A number of properties of the function spaces introduced above has been proved in [20, 21].
We formulate these properties as the following theorem.

Theorem 2.1. We assume that conditions (i), (ii) hold. Then:
1) the spaces \( H_{s}(\Omega) \) and \( H_{s}(\Sigma_0) \) are finite dimensional;
2) \( H_{DCs}(\Omega) \subset H^1(\Omega)^3 \) and the norm \( \|\cdot\|_{DC(\Omega)} \) is equivalent to the norm \( \|\cdot\|_{1,\Omega} \);
3) for all \( h \in V_{s+1/2}(\Omega) \) the coercitivity inequality \( \|\text{curl } h\|_{2,\Omega}^2 \geq \delta_1 \|h\|^2_{s+1/2,\Omega} \) holds where constant \( \delta_1 \)
depends on \( \Omega \) and \( \Sigma_0 \);
4) the orthogonal decomposition \( L^2(\Omega)^3 = \nabla H^1(\Omega, \Sigma_0) \oplus \text{curl } H_{DCs}(\Omega) \oplus H_{s+1/2}(\Omega) \) holds and
curl \( H_{DCs}(\Omega) \equiv \text{curl } V_{s+1/2}(\Sigma_0) \).

Along with spaces \( H_{DCs}(\Omega) \) and \( H^0(\text{curl}, \Omega) \) we will use their subspaces
\( H_{s+1/2}(\Omega) := \{h \in H_{s+1/2}(\Omega) : \text{curl } h \in L^2(\Omega)^3, \text{div } h = 0 \} \cap H_{s+1/2}(\Omega) \).
As a result we arrive at the weak form of Problem 1:

\[ \begin{align*}
\mathcal{H}_{\text{div}}^{s+1/2}(\Omega) := & \{ h \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega) : h \times n|_{\Sigma_\nu} = 0 \}, \\
H_0^0(\text{curl}, \Omega) := & \{ e \in H^0(\text{curl}, \Omega) : e \times n|_{\Sigma_\nu} \in L^2(\Sigma_\nu) \}
\end{align*} \]

equipped with natural norms

\[ \| h \|_{\mathcal{H}_{\text{div}}^{s+1/2}(\Omega)} = \| h \|_{s+1/2, \Omega} + \| \text{curl} h \|_{\Omega}, \quad \| e \|_{H_0^0(\text{curl}, \Omega)} := \| e \|_{\Omega} + \| e \times n \|_{\Sigma_\nu}. \]

The spaces \( \mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\nu) \) and \( H_0^0(\text{curl}, \Omega) \) will be used below for describing properties of the magnetic and electric fields, respectively.

The following technical lemma holds (see for details [6, 20, 21]).

**Lemma 2.1.** Under condition (i) there exist constants \( \delta_i = \delta_i(\Omega) > 0 \) and \( \gamma_i = \gamma_i(\Omega) > 0 \), \( i = 0, 1, \beta > 0 \), depending on \( \Omega \) such that

\[ (\nabla v, \nabla v) \geq \delta_0 \| v \|_{1,\Omega}^2, \quad \forall v \in H_0^1(\Omega)^3, \quad (\text{curl} \Psi, \text{curl} \Psi) \geq \delta_1 \| \Psi \|_{1,\Omega}^2, \quad \forall \Psi \in V_{\Sigma_\nu}(\Omega), \]

\[ |(u \cdot \nabla) v, w| \leq \gamma_0 \| u \|_{1,\Omega} \| v \|_{1,\Omega} \| w \|_{1,\Omega}, \]

\[ |(\text{curl} u \times v, w) - (\text{curl} u \times v, w)| \leq \gamma_1 \| u \|_{1,\Omega} \| v \|_{s+1/2, \Omega} \| w \|_{1,\Omega}, \quad \forall u, w \in H^1(\Omega)^3, \quad v \in H^{s+1/2}(\Omega)^3, \]

\[ \sup_{v \in H^1_0(\Omega)^3, v \neq 0} -\langle \text{div} v, p \rangle / \| v \|_{1,\Omega} \geq \beta \| p \|_{\Omega}, \quad \forall p \in L^2_0(\Omega). \quad (4) \]

Furthermore, the following identities hold true:

\[ (u \cdot \nabla) v, w = -((u \cdot \nabla) v, w) \quad \forall u \in H^1(\Omega)^3 : \text{div} u = 0, \quad (v, w) \in H_0^1(\Omega)^3 \times H^1(\Omega)^3, \]

\[ (\text{rot} \Psi \times H, u) = (H \times u, \text{rot} \Psi) = -(\text{rot} \Psi \times u, H) \quad \forall \Psi, u \in H^1(\Omega)^3, \quad H \in H^{s+1/2}(\Omega)^3. \]

Let the following conditions hold in addition to (i), (ii):

(iii) \( f \in H^{-1}(\Omega)^3, \quad j \in L^2(\Omega)^3, \quad k \in (\gamma_{\tau}|_{\Sigma_\nu}) H_0^0(\text{curl}, \Omega). \)

In what follows we will deal with a weak form of Problem 1. To this end we multiply the first equation in (1) by \( v \in H_0^1(\Omega)^3 \), the first equation in (2) by \( \text{curl} \Psi \) where \( \Psi \in V_{\Sigma_\nu}(\Omega) \), integrate over \( \Omega \), apply Green’s formulas, add the obtained results and make use of the identity [14]

\[ (E, \text{curl} \Psi) = \int_{\Sigma_\nu} (E \times n|_{\Sigma_\nu}) \cdot \Psi_T d\sigma = (k, \Psi_T)|_{\Sigma_\nu} = (E_0, \text{curl} \Psi) \quad \forall \Psi \in V_{\Sigma_\nu}(\Omega). \quad (5) \]

As a result we arrive at the weak form of Problem 1:

\[ \begin{align*}
\nu(\nabla u, \nabla v) + \nu_1 (\text{curl} H, \text{curl} \Psi) + (u \cdot \nabla) u, v + \alpha [(\text{curl} \Psi \times H, u) - (\text{curl} H \times H, v)] - \\
-\langle \text{div} v, p \rangle = (f, v) + (\nu_1 j, \text{curl} \Psi) + \rho_0^{-1}(k, \Psi)|_{\Sigma_\nu} \quad \forall (v, \Psi) \in H_8,
\end{align*} \]

\[ \text{div} u = 0 \quad \text{in} \quad \Omega, \quad H \cdot n = q \quad \text{on} \quad \Sigma_\tau. \quad (6) \]

We will refer below to any triple \( (u, H, p) \in H_0^1(\Omega)^3 \times H_0^{s+1/2}(\Omega, \Sigma_\nu) \times L^2_0(\Omega) \) satisfying (6), (7) as a weak solution to Problem 1.

The identity (6) does not contain electric field \( E \in H_0^0(\text{curl}, \Omega) \) which was eliminated with the help of (5). However, using a condition on a boundary vector \( k \) in (iii) vector \( E \) can be reconstructed uniquely from triple \( (u, H, p) \) satisfying (6) so that the first equation in (2) holds a.e. in \( \Omega \) (see details in [15]).

Let in addition to (i)–(iii) the following condition holds:

(iv) \( \Sigma_\nu \cap \Sigma_\nu' = \emptyset. \)

The following theorem from [15] provides the existence of a weak solution to Problem 1:
Theorem 2.2. Under assumptions (i)–(iv) for any $q \in H^s(\Sigma_r)$, $s \in [0, 1/2]$, there exists a weak solution $(u, H, p)$ to Problem 1 and the following estimates hold:

$$\|u\|_{1, \Omega} \leq M_u, \quad \|H\|_{H^{s+1/2}_{\text{div}}(\Omega)} \leq M_H, \quad \|p\|_{\Omega} \leq M_p,$$

where $M_u, M_H, M_p$ are continuous nondecreasing functions of $\|f\|_{-1, \Omega}, \|j\|_{\Omega}, \|k\|_{\Sigma_r}, \|q\|_{s,\Sigma_r}$.

If, besides, functions $f, j, k, q$ are small (or “viscosities” $\nu, \nu_m$ are large) in the sense

$$\gamma_0 M_u + \gamma_1 (\sqrt{\nu}/2) M_H < \delta_0 \nu, \quad \gamma_1 M_u + \gamma_1 (\sqrt{\nu}/2) M_H < \delta_1 \nu_m,$$

where constants $\delta_0, \delta_1, \gamma_0, \gamma_1$ are introduced in Lemma 2.1, then the weak solution is unique.

The proof of Problem 1′ solvability essentially uses the following Lemma [22]:

Lemma 2.2. Let under assumptions (i), (ii) and (iv) $q \in H^s(\Sigma_r)$ at $s \in [0, 1/2]$. Then there exists a unique function $H_0 \in H^{s+1/2}_{\text{div}}(\Omega, \Sigma_r)$ such that $\text{curl}H_0 = 0$, $\text{div}H_0 = 0$ in $\Omega$, $H_0 \cdot n = q$ on $\Sigma_r$, and $\|H_0\|_{s+1/2, \Omega} \leq C_\Sigma \|q\|_{s, \Sigma_r}$, where constant $C_\Sigma$ does not depend on $q$.

3. Boundary control problem

Let us formulate control problem for Problem 1. For this purpose we divide the set of all data of Problem 1 into two groups: the first contains the functions $f, j$ and $k$ while the second one contains the function $q$. We assume that $q$ can be changed in some subset $K_s$ where

(j) $K_s \subset H^s(\Sigma_r)$, $0 \leq s \leq 1/2$, is nonempty convex closed set.

Setting $X_s = H^1_0(\Omega) \times H^{s+1/2}_{\text{div}}(\Omega, \Sigma_r) \times L^2_0(\Omega)$ and $Y_s = H^{-1}(\Omega)^3 \times V_{\Sigma_r}(\Omega)^* \times L^2(\Omega) \times H^s(\Sigma_r)$, $x = (u, H, p) \in X_s$ we introduce an operator $F : (X_1, X_2, X_3) : X_s \times K_s \to Y_s$ by

$$\langle F_1(x), (v, \Psi) \rangle = \nu (\nabla u, \nabla v) + \nu_1 (\text{curl} H, \text{curl} \Psi) + \langle (u, \nabla)u, v \rangle + (\nabla^T v, v) + \rho_0^{-1} (k, \Psi_T)_{\Sigma_r}, \forall (v, \Psi) \in H_s,$$

$$\langle F_2(x, r) \rangle = -(\text{div} u, r) \forall r \in L^2(\Omega), \quad F_3(x, q) = H \cdot n - q \in H^s(\Sigma_r)$$

and rewrite the weak form (6), (7) of Problem 1 in the form of the operator equation

$$F(x, q) = F(u, H, p, q) = 0. \quad (10)$$

Let $I : X_s \to \mathbb{R}$ be a weakly lower semicontinuous cost functional. Consider the following optimal control problem:

$$J(x, q) = (\mu_0 / 2) I(x) + (\mu_1 / 2) \|q\|_{s, \Sigma_r}^2 \to \inf, \quad F(x, q) = 0, \quad (x, q) \in X_s \times K_s. \quad (11)$$

Here $\mu_0 > 0$ and $\mu_1 \geq 0$ are nonnegative parameters.

As the cost functional we choose one of the following:

$$I_1(v) = \|v - v_d\|_{Q_s}^2, \quad I_2(H) = \|H - H_d\|_{Q_s}^2, \quad I_3(p) = \|p - p_d\|_{Q_s}^2. \quad (12)$$

Here the function $v_d \in L^2(Q)^3$ denotes some desired velocity field given in a subdomain $Q \subset \Omega$. Functions $H_d \in L^2(Q)^3$ and $p_d \in L^2(Q)$ have similar sense for the magnetic field or pressure. The set of possible pairs for problem (11) is denoted by $Z_{ad} = \{(x, q) \in X_s \times K_s : F(x, q) = 0, J(x, q) < \infty\}$.

Let in addition to (i) the following conditions hold:

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$ and $K_s$ is bounded set, or $\mu_0 > 0$, $\mu_1 > 0$ and $I$ is bounded below.

By analogy with [18], the following theorem concerning with solvability of (11) is proved:

Theorem 3.1. Let under assumptions (i)–(iv) and (jj), (jjj) $I : X_s \to \mathbb{R}$ be a weakly lower semicontinuous functional, where $s \geq 0$, and set $Z_{ad}$ be nonempty. Then problem (11) has at least one solution $(x, q) \in X_s \times K_s$. 


4. Derivation of the optimality system

The following stage of study of problem (11) is a derivation of an optimality system describing necessary conditions of extremum.

Let \( X^*_s = H^{-1}(\Omega)^3 \times H^{s+1/2}(\Omega, \Sigma_\nu)^* \times L^2_0(\Omega) \) and \( Y^*_s = H^0(\Omega)^3 \times V_{\Sigma_\nu}(\Omega) \times L^2_s(\Omega) \times H^s(\Sigma_\nu)^* \) be the duals of spaces \( X_s \) and \( Y_s \) where \( s > 0 \). It is easy to show that the Fréchet partial derivative with respect to \( x \) from operator \( F : X_s \rightarrow Y_s \) at any point \((\hat{x}, \hat{q}) \equiv (\hat{u}, \hat{H}, \hat{p}, \hat{q}) \in X_s \times K_s \) is a linear continuous operator \( F'_x(\hat{x}, \hat{q}) : X_s \rightarrow Y_s \) that maps each element \((w, h, r) \in X_s \) to an element \( F'_x(\hat{x}, \hat{q})(w, h, r) = (\hat{y}_1, \hat{y}_2, \hat{y}_3) \in Y_s \) where the elements \( \hat{y}_1 \in H^s_\nu \), \( \hat{y}_2 \in L^2_0(\Omega) \), \( \hat{y}_3 \in H^s(\Sigma_\nu) \) are defined by triples \((\hat{u}, \hat{H}, \hat{p})\) and \((w, h, r)\) from relations

\[
\langle \hat{y}_1, (v, \Psi) \rangle = \nu(\nabla w, \nabla v) + \nu_1(\text{curl } h, \text{curl } \Psi) + ((\hat{u} \cdot \nabla) w, v) + ((w \cdot \nabla) \hat{u}, v) - (\text{div } v, r) + \\
+\nu_1((\text{curl } \Psi \times h, \hat{u}) + (\text{curl } \Psi \times H, w) - (\text{curl } h \times \hat{H}, v) - (\text{curl } \hat{H} \times h, v)) \forall (v, \Psi) \in H_s, \\
\langle \hat{y}_2, r \rangle = -(\text{div } r, w) \forall r \in L^2_0(\Omega), \quad \hat{y}_3 = h \cdot n |_{\Sigma_\nu}. \tag{13}
\]

By \( F'_x(\hat{x}, \hat{q})^* : Y^*_s \rightarrow X^*_s \) we denote operator adjoint to \( F'_x(\hat{x}, \hat{q}) \). Following to the general theory of smooth-convex extremum problems [23] we introduce an element \( y^* = ((\xi, \eta), \sigma, \zeta) \in Y^*_s \) to which we will refer as adjoint state and define the Lagrangian \( L : X_s \times K_s \times R \times Y^*_s \rightarrow R \) by formula \( L(x, q, \lambda_0, y^*) = \lambda_0 J(x, q) + \langle y^*, F(x, q) \rangle_{Y_s \times X_s} \equiv \lambda_0 J(x, q) + \langle F_1(x, q), (\xi, \eta) \rangle_{H^s_\nu \times H_s} + (F_2(x, q), \sigma) + \langle \zeta, F_3(x, q) \rangle_{s, \Sigma_\nu} \).

The following theorem holds.

**Theorem 4.1.** Let under assumptions (i)-(iv) and (j), (jj) at \( s \in [0, 1/2] \), the element \((\hat{x}, \hat{q}) \equiv (\hat{u}, \hat{H}, \hat{p}, \hat{q}) \in X_s \times K_s \) be a local minimizer in problem (11) and let the cost functional \( I \) be continuously Fréchet differentiable with respect to state \( x \) in point \( \hat{x} \). Then there exists a nonzero Lagrange multiplier \((\lambda_0, y^*) = (\lambda_0, \xi, \eta, \sigma, \zeta) \in R^+ \times Y^*_s \) such that the Euler–Lagrange equation takes place \( F'_x(\hat{x}, \hat{q})^* y^* = -\lambda_0 J'_x(\hat{x}, \hat{q}) \) in \( X^*_s \), which is equivalent to relations

\[
\nu(\nabla w, \nabla \xi) + \nu_1(\text{curl } h, \text{curl } \eta) + ((\hat{u} \cdot \nabla) w, \xi) + ((w \cdot \nabla) \hat{u}, \xi) - (\text{div } w, \sigma) + \\
+\nu_1((\text{curl } \Psi \times h, \hat{u}) + (\text{curl } \Psi \times H, w) - (\text{curl } h \times \hat{H}, v) - (\text{curl } \hat{H} \times h, v)) \forall (v, \Psi) \in H_s, \\
(\text{div } r, w) = -\lambda_0(\mu_0/2)(I'_u(\hat{x}), w) + (I'_H(\hat{x}), h) \forall (w, h) \in H^2_0(\Omega)^3 \times H^{s+1/2}(\Omega, \Sigma_\nu), \tag{14}
\]

and minimum principle \( L(\hat{x}, \hat{q}, \lambda_0, y^*) \leq L(x, q, \lambda_0, y^*) \forall q \in K_s \) holds, which is equivalent to

\[
\lambda_0 \mu_1(\hat{q} - \hat{\bar{q}})_{s, \Sigma_\nu} - \langle \zeta, \sigma - \bar{\sigma} \rangle_{s, \Sigma_\nu} \geq 0 \forall \hat{q} \in K_s. \tag{16}
\]

If, besides, (9) holds for all \( q \in K_s \), then any nonzero Lagrange multiplier \((\lambda_0, y^*)\), satisfying (14)–(16) is regular, i.e. it has the form (1, \( y^* \)) and is determined uniquely for given pair \((\hat{x}, \hat{q})\).

**Proof.** According to [23, p. 79], to prove Theorem 4.1, it suffices to show that \( F'_x(\hat{x}, \hat{q}) : X_s \rightarrow Y_s \) is a Fredholm operator. By virtue of (13), the operator \( F'_x(\hat{x}, \hat{q}) : X_s \rightarrow Y_s \) can be presented as \( F'_x = \Phi + \Phi \equiv (\Phi_1, \Phi_2, \Phi_3) + (\Phi_4, 0, 0) \) where \( \Phi_2(x) = \text{div } w, \Phi_3(x) = h \cdot n |_{\Sigma_\nu} \) and operators \( \Phi_1, \Phi_1 : X_s \rightarrow H^s_\nu := H^{-1}(\Omega)^3 \times V_{\Sigma_\nu}(\Omega)^* \) act by formulæ

\[
\langle \Phi_1(w, h, r), (v, \Psi) \rangle = \nu(\nabla w, \nabla v) + \nu_1(\text{curl } h, \text{curl } \Psi) + ((\hat{u} \cdot \nabla) w, v) + \\
+\nu_1((\text{curl } \Psi \times h, \hat{u}) + (\text{curl } \Psi \times H, w) - (\text{curl } h \times \hat{H}, v) - (\text{curl } \hat{H} \times h, v)), \\
\langle \Phi_4(w, h, r), (v, \Psi) \rangle = ((w \cdot \nabla) \hat{u}, v) + \nu_1((\text{curl } \Psi \times w, \hat{u}) + (\text{curl } \Psi \times H, v)).
\]
Let us show that the operator $\Phi : X_s \to Y_s$ is an isomorphism. For this purpose it is enough to show that for any triple $(F, \theta, q) \in H^s_x \times \times L^2(\Omega) \times H^s(\Sigma_t)$ there exists a unique solution $(w, h, p) \in X_s$ of problem

$$
\nu(\nabla w, \nabla v) + \nu_1(\text{curl} \, h, \text{curl} \Psi) + \langle (\hat{u} \cdot \nabla)w, v \rangle - \langle \text{div} \, v, p \rangle \nonumber$$

$$+ \alpha[(\text{curl} \Psi \times \hat{H}, w) - (\text{curl} \, h \times \hat{H}, v)] = \langle F, (v, \Psi) \rangle \quad \forall (v, \Psi) \in H^s,
$$

(17)
div $w = \theta$ in $\Omega$, $h \cdot n = q$ on $\Sigma_r$.  
(18)

that continuously depends on $(F, q, \theta)$. The existence of the solution of (17), (18) and its continuous dependence on $(F, q, \theta)$ is proved using the scheme proposed in [6]. In order to prove the uniqueness let us assume that there exists two solutions $(w_i, h_i, p_i) \in X_s$, $i = 1, 2$, of problem (17), (18). Then the difference $w = w_1 - w_2$, $h = h_1 - h_2$ and $p = p_1 - p_2$ belongs to $V \times V_{\Sigma}^*(\Omega) \times L^2(\Omega)$ and satisfies

$$
\nu(\nabla w, \nabla v) + \nu_1(\text{curl} \, h, \text{curl} \Psi) + \langle (\hat{u} \cdot \nabla)w, v \rangle + 
$$

$$+ \alpha[(\text{curl} \Psi \times \hat{H}, w) - (\text{curl} \, h \times \hat{H}, v)] - \langle \text{div} \, v, p \rangle = 0 \quad \forall (v, \Psi) \in H^s, 
$$

(19)
div $w = 0$ in $\Omega$, $h \cdot n = 0$ on $\Sigma_r$.  
(20)

Setting $v = w$, $\Psi = h$ in (19), we arrive at the relation $\nu(\nabla w, \nabla w) + \nu_1(\text{curl} \, h, \text{curl} \Psi) = 0$. By virtue of Lemma 2.1 it implies that $w = 0$ and $h = 0$ or $w_1 = w_2$ and $h_1 = h_2$ in $\Omega$. Then, from (19), taking into account (4), we derive that $p_1 = p_2$ in $\Omega$. Thus, we proved that the operator $\Phi : X_s \to Y_s$ is surjective and invertible. Then, by Banach theorem, operator $\Phi$ is an isomorphism. Finally, from compactness of embeddings $H^s_{\text{div}}(\Omega) \subset L^3(\Omega)^3$ at $s > 0$, and $H^1(\Omega)^3 \subset L^4(\Omega)^3$ and estimates of Lemma 2.1 follows that continuity and compactness of the operator $\Phi$.

Let us prove the regularity of the multiplier $(\lambda_0, y^*)$, i.e. that $\lambda_0 \neq 0$. To this end we denote by $y^* \equiv (\xi, \eta, \sigma, \zeta)$ an arbitrary solution of system (14)–(16) at $\lambda_0 = 0$. Setting $w = \xi$, $h = \eta$ and $r = \sigma$ in this system we come to relation

$$
\nu(\nabla \xi, \nabla \xi) + \nu_1(\text{curl} \, \eta, \text{curl} \, \eta) + \langle (\xi \cdot \nabla)\hat{u}, \xi \rangle + \alpha[(\text{curl} \eta \times \eta, \hat{u}) - (\text{curl} \hat{H} \times \eta, \xi)] = 0. 
$$

(21)

Arguing as in [7] one can easily prove using (21) that $\xi = 0$ and $\eta = 0$ in $\Omega$ under condition (9). In this case, from (14), (15) follows that

$$
-(\text{div} \, w, \sigma) + \langle \zeta, \text{h} \cdot \text{n} \rangle_{\Sigma_r} = 0 \quad \forall w \in H^s_0(\Omega)^3, \text{h} \in H^{1/2+s}_{\text{div}}(\Omega, \Sigma_r). 
$$

(22)

Choosing as $\text{h}$ an arbitrary function from $V^*_{\Sigma} \subset H^{1/2+s}_{\text{div}}(\Omega, \Sigma_r)$ we derive from (22) that $(\text{div} \, w, \sigma) = 0$ for all $w \in H^s_0(\Omega)^3$. By (4) this identity means that $\sigma = 0$ a.e. in $\Omega$. Then (22) transforms to $\langle \zeta, \text{h} \cdot \text{n} \rangle_{\Sigma_r} = 0$ for all $\text{h} \in H^{1/2+s}_{\text{div}}(\Omega, \Sigma_r)$. This means that $\zeta = 0$ in $H^s(\Sigma_t)^*$ and therefore $y^* = 0$. Uniqueness of the regular Lagrange multiplier $(1, y^*)$ under conditions (9) follows from Fredholm property of the operator $F_\lambda'(\hat{x}, \hat{q})$.  

Relations (6), (7) for the main state $(\text{u}, \text{H}, p)$ together with identities (14), (15) for the adjoint state $(\xi, \eta, \sigma, \zeta)$ and variational inequality (16) for control $\hat{q}$ form an optimality system for problem (11). We emphasize that the optimality system plays an important role in the study of the optimal solutions’ properties. In particular, based on the analysis of the optimality system one can derive stability estimates of the optimal solutions (see, e.g., [8]) The authors propose to do this in future paper for all three cost functionals defined in (12).
References

[1] Meir A J and Schmidt P G 2001 On electromagnetically and thermally driven liquid-metal flows Nonlinear Anal. Theor. 47 3281–94
[2] Sterl A 1990 Numerical simulation of liquid metal MHD flows in rectangular ducts J. Fluid Mech. 216 161–91
[3] Hou L and Meir A 1995 Boundary optimal control of MHD flows Appl. Math. Optim. 32 143–62
[4] Alekseev G V 2003 Control problems for the steady-state equations of magnetohydrodynamics of a viscous incompressible fluid J. Appl. Mech. Tech. Phys. 44 890–99
[5] Alekseev G V 2004 Control problems for stationary equations of magnetic hydrodynamics Dokl. Math. 69 310–13
[6] Alekseev G 2004 Solvability of control problems for stationary equations of magnetohydrodynamics of a viscous fluid Siberian Math. J. 45 197–213
[7] Alekseev G V and Brizitskii R V 2005 Control problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid under mixed boundary conditions Comp. Math. Math. Phys. 45 2049–65
[8] Alekseev G and Brizitskii R 2012 Stability estimates of solutions of control problems for the stationary equations of magnetic hydrodynamics Diff. Eq. 48 397–409
[9] Griess R and Kunish K 2006 Optimal control for stationary MHD system in velocity–current formulation SIAM J. Control Optim. 45 1822–45
[10] Gunzburger M and Trenchea C 2007 Analysis of an optimal control problem for the three–dimensional coupled modified Navier–Stokes and Maxwell equations J. Math. Anal. Appl. 333 295–310
[11] Brizitskii R and Tereshko D 2007 On the solvability of boundary value problems for the stationary magnetohydrodynamic equations with inhomogeneous mixed boundary conditions Diff. Eq. 43 246–58
[12] Ravindran S 2008 On the dynamics of controlled magnetohydrodynamic systems Nonlinear Anal. Modell. 13 351–77
[13] Bermudez A, Munoz-Sola R and Vazquez R 2010 Analysis of two stationary magnetohydrodynamics systems of equations including Joule heating J. Math. Anal. Appl. 368 444–68
[14] Alekseev G V and Brizitskii R V 2014 Solvability of the boundary value problem for stationary magnetohydrodynamic equations under mixed boundary conditions for the magnetic field Appl. Math. Lett. 32 13–8
[15] Alekseev G 2016 Mixed boundary value problems for steady-state magnetohydrodynamic equations of viscous incompressible fluid Comp. Math. Math. Phys. 56 1426–39
[16] Alekseev G 2016 Mixed Boundary Value Problems for Stationary Magnetohydrodynamic Equations of a Viscous Heat-Conducting Fluid J. Math. Fluid Mech. 18 591–607
[17] Alekseev G V 2017 Problems on control for a steady-state magnetic-hyrdodynamic model of a viscous heat-conducting fluid under mixed boundary conditions Dokl. Phys. 62 128–32
[18] Brizitskii R V 2017 Study of control problems for the stationary MHD equations J. Phys.: Conf. Ser. Physics 894 012015
[19] Fernandes P and Gilardi G 1997 Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions Math. Models Meth. Appl. Sci. 7 957–91
[20] Auchmuty G 2004 The main inequality of vector field theory Math. Models Meth. Appl. Sci. 14 79–103
[21] Auchmuty G and Alexander J S 2006 Finite energy solutions of mixed 3d div-curl systems Quart. Appl. Math. 64 335–57
[22] Alonso A and Valli A 1996 Some remarks on the characterization of the space of tangential traces of H(rot; Ω) and the construction of the extension operator Manuscr. Math. 89 159–78
[23] Ioffe A D and Tikhomirov V M 1978 Theory of extremal problems (Amsterdam: Elsevier)