On Fractional Quantum Hall Solitons in ABJM-like Theory

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Abstract

Using D-brane physics, we study fractional quantum Hall solitons (FQHS) in ABJM-like theory in terms of type IIA dual geometries. In particular, we discuss a class of Chern-Simons (CS) quivers describing FQHS systems at low energy. These CS quivers come from R-R gauge fields interacting with D6-branes wrapped on 4-cycles, which reside within a blown up CP$^3$ projective space. Based on the CS quiver method and mimicking the construction of del Pezzo surfaces in terms of CP$^2$, we first give a model which corresponds to a single layer model of FQHS system, then we propose a multi-layer system generalizing the doubled CS field theory, which is used in the study of topological defect in graphene.

Keywords: Quantum Hall Solitons, ABJM theory, Type IIA superstring, Toric geometry

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1 Introduction

There have been considerable works dealing with connections between string theory and quantum Hall solitons (QHS) in (1+2)-dimensions. The first of them was a construction of Bernevig, Brodie, Susskind and Toumbas describing such models on a 2-sphere using a spherical D2-brane and dissolved $N$ D0-branes moving on it\[^{[1]}\]. The system has been placed in a background of coincident D6-branes extended in the directions perpendicular to the world-volume of the D2-brane on which the quantum Hall effect (QHE) resides. The ten dimensional string picture of the QHE has been extended to the compactification of type IIA superstring theory on the K3 surface with singularities classified by Lie algebras. This extension has been based on the study of quiver gauge theories living on the world-volume of the wrapped D4-branes on intersecting 2-spheres arranged as Dynkin Diagrams\[^{[2, 3]}\].

Recently, some efforts have been devoted to study fractional quantum Hall solitons (FQHS) in relations with an Anti de Sitter/conformal field theory (AdS/CFT) correspondence\[^{[4, 5]}\]. In particular, connections with (2+1)-dimensional Chern-Simons (CS) theory constructed by Aharony-Bergman-Jafferis-Maldacena (ABJM) theory have been discussed in \[^{[6, 7]}\]. The latter is a 3-dimensional $N = 6$ CS quiver with $U(N)_k \times U(N)_{-k}$ gauge symmetry proposed to be dual to M-theory propagating on $AdS_4 \times S^7/Z_k$, with an appropriate amount of fluxes, or type IIA superstring on $AdS_4 \times \mathbb{CP}^3$ for large number of $k, N$ with $k \geq N$ in the weakly interacting regime \[^{[8]}\]. In the decoupling limit, the corresponding CFT$_3$ is generated by the action of multiple M2-branes placed at the orbifold $C^4/Z_k$. In this regard, it has been shown that QHE can be obtained from the world-volume action of the M5-brane filling $AdS_3$ inside $AdS_4$\[^{[6]}\]. The model has been derived from $d = 3$ flavored ABJM theory with the CS levels $(1, -1)$. Alternatively, a FQHE system in $AdS_4$/CFT$_3$ has been realized by adding fractional D-branes (D4-branes wrapping \(\mathbb{CP}^1\)) to the ABJM theory\[^{[7]}\].

Motivated by these investigations, we contribute to these activities by discussing FQHS in $AdS_4$/CFT$_3$ using type IIA dual geometry realized as blown up of $\mathbb{CP}^3$ by four-cycles which are isomorphic to $\mathbb{CP}^2$. In toric geometry, this procedure extends the building of del Pezzo surfaces from $\mathbb{CP}^2$ to $\mathbb{CP}^3$. We will refer to the corresponding CS quiver theory as ABJM-like theory. In this work, we give a class of such quivers describing FQHS from D6-branes interacting with R-R gauge fields in ABJM-like geometry. We propose a stringy hierarchical description in terms of wrapped D6-branes on the blown up four-cycles. We first consider a model corresponding to a single layer of FQHS, then we discuss a multi-layer system which generalizes the doubled CS field theory which is used recently in the topological defect in graphene\[^{[10]}\].

\[^{1}\] see also \[^{[9]}\].
2 U(1)$_k$ Chern-Simons theory and ABJM theory

To start, recall that fractional quantum Hall states were proposed first by Laughlin and they are characterized by the filling factor $\nu_L = \frac{1}{k}$ where $k$ is an even integer for a boson and an odd integer for a fermion[11]. At low energy, this model can be described by a 3-dimensional $U(1)_k$ Chern-Simons theory. The corresponding effective action reads as

$$S_{CS} = -\frac{k}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge dA + \frac{q}{2\pi} \tilde{A} \wedge dA,$$

(2.1)

where $A$ is the dynamical gauge field, $\tilde{A}$ is an external electromagnetic field, and $q$ is the charge of the electron [12, 13]. In the present work, we will see that $U(1)_k$ CS theory, describing FQH, can be derived from ABJM theory with $U(N)_k \times U(N)_{-k}$ gauge symmetry. Roughly speaking, $U(1)_k$ CS gauge theory can be obtained from D6-branes wrapped on four-cycles which are embedded in type IIA geometry. Indeed, let us take a stack of $M$ D6-branes in AdS$_4 \times \mathbb{CP}^3$ type IIA geometry. A priori, D6-branes can wrap various cycles. However, here, we consider a stack of $M$ D6-branes wrapping a particular complex hypersurface class $[C]$ in $H^4(\mathbb{CP}^3, \mathbb{Z})$ which is one dimension. On the gauge theory side, the gauge symmetry $U(N)_k \times U(N)_{-k}$ becomes $U(N + M)_k \times U(N)_{-k}$. To get the action (2.1), we take just the $U(1)$ part of $U(M)_k$ corresponding to a single D6-brane wrapping the four-cycle $C$ and consider other gauge factors as spectators as made in [6, 7]. Indeed, on a seven dimensional world-volume of the D6-brane lives an $U(1)$ gauge symmetry with the following action

$$S_{D6} = S_{DBI} + S_{WZ}$$

(2.2)

where

$$S_{DBI} \sim T_6 \int_{\mathbb{R}^{1,6}} e^{-\phi} \sqrt{-\det(G + 2\pi F)}$$

(2.3)

and where the action $S_{WZ}$ depends on the R-R gauge fields of type IIA superstring. To obtain the first term of Chern-Simons action, we ignore the $S_{DBI}$ action and take $S_{WZ}$ as

$$S_{WZ} \sim T_6 \int_{\mathbb{R}^{1,6}} F \wedge F \wedge A_3,$$

(2.4)

where $T_6$ is the D6-brane tension and where $A_3$ is the R-R 3-form coupled to the D2-brane of type IIA superstring. Performing a simple integration by parts and integrating the result over $C_4$, we get the first term of the action (2.1), namely

$$-\frac{k}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge F$$

(2.5)

where $k = \frac{1}{2\pi} \int_{C_4} (dA_3)$ is produced now by $k$ D4-flux. To couple the system to an external gauge field, we need to turn on the RR 5-form $A_5$ which is coupled to the D4-brane. This gauge field should be decomposed as follows

$$A_5 \rightarrow \tilde{A} \wedge \omega$$

(2.6)
where $\omega$ is a harmonic 4-form dual to the four-cycle $C_4$. In this way, the WZ term $\int_{\mathbb{R}^1} A_5 \wedge F$ on a D6-brane gives the second term of the action (2.1), namely

$$\mathcal{A} = \int_{\mathbb{R}^{1,2}} \mathcal{A} \wedge F$$

(2.7)

where $q = \int_C w$ and $\mathcal{A}$ is an U(1) gauge field serves as the external gauge field that couples to gauge fields living on the D6-brane world-volume. Integrating out the gauge field $A$ and using the electric current expression, the above effective action reproduces now the following filling factor

$$\nu = \frac{q^2}{k}. \tag{2.8}$$

From this equation, it follows that the filling factor depends on the D4-branes and the harmonic 4-forms defined on $\mathbb{C}P^3$. It turns out that known values could be reached by taking particular choices of such parameters.

### 3 FQH systems with $U(1) \times U(1)$ gauge symmetry

Once given a system with a single U(1) gauge symmetry, we will discuss a possible generalization described by an effective Chern-Simons gauge theory with a series of U(1) gauge fields. The general form of the abelian part of these effective theories is given by

$$S \sim \frac{1}{4 \pi} \int_{\mathbb{R}^{1,2}} K_{ij} A^i \wedge dA^j + 2q_i \mathcal{A} \wedge dA^i, \tag{3.1}$$

where $K_{ij}$ is a real, symmetric and invertible matrix ($\det K \neq 0$) with $q_i$ a vector of charges. The external gauge field $\mathcal{A}$ couples now to each current $\star dA^i$ with charge strengths $e q_i$. The $K_{ij}$ matrix and the $q_i$ charge vector in this effective field action are suggestive of some physical concepts. Following the Wen-Zee model [13], $K_{ij}$ and $q_i$ are interpreted as order parameters and classify the various QHS states. Integrating over the all gauge fields $A^i$, one gets the formulae for the filling factor

$$\nu = q_i K_{ij}^{-1} q_j. \tag{3.2}$$

In the following, we will see that the corresponding models can be also obtained from D6-branes wrapping individually more than one four-cycles in type IIA geometry. In ABJM backgrounds, one way to achieve this extended geometry is to use the blown up $\mathbb{C}P^3$ at singular toric vertices. In particular, we will explore the same analysis for constructing del Pezzo surfaces from the projective space $\mathbb{C}P^2$ using toric geometry language[14, 15]. Indeed, the meaning of a blown up $\mathbb{C}P^2$ is to replace a point on it by a $\mathbb{C}P^1$. In toric geometry framework, the realization of $\mathbb{C}P^2$ involves a triangle over each point of which there is a 2-torus $T^2$. Blowing up $\mathbb{C}P^2$ consists of replacing a vertex of the triangle by a $\mathbb{C}P^1$ and that is realized in toric geometry by an interval. This geometric operation can be extended to $\mathbb{C}P^n$ which is identified with $T^n$ fibered over a
For $n=3$, the toric realization is given by a tetrahedron. Applying the blowing up procedure to $\mathbb{CP}^3$, we can replace one vertex of the tetrahedron by an exceptional four-cycle $E$ which is isomorphic to $\mathbb{CP}^2$. In toric geometry, this is realized by a triangle. The resulting blown up space $\tilde{\mathbb{CP}}^3$ has now two Kahler parameters and the 4-cycle homology group $H^4(\tilde{\mathbb{CP}}^3, \mathbb{Z})$ is two dimension and is generated by $\{C_4, E\}$. Roughly speaking, consider now type IIA superstring on $\text{AdS}_4 \times \tilde{\mathbb{CP}}^3$ in the presence of a new stack of $M'$ D6-branes wrapping on the blown up $E$ space. Inspired by the study of quiver gauge theories in connection with AdS/CFT conjecture, this blowing up can be understood by the introduction of a new gauge factor. In particular, the appearance of the new factor is partly motivated by the result of the geometry of $\text{AdS}_5 \times M_5$, where $M_5 = S^5/\Gamma$ with $\Gamma$ being a discrete subgroup of $\text{SO}(6)$, dual to D3-branes at ADE singularities of Calabi-Yau threefolds generalizing the conifold singularity [16] [17]. It can be also supported also by results on $N=1$ four-dimensional quiver theories arising on the world-volume of D3-branes transverse to cones over complex two-dimensional toric varieties, e.g. del Pezzo surfaces [18]. Based on such quivers built on $\mathbb{CP}^2$ and del Pezzo surfaces, we propose that new gauge factors can be enhanced due to the blowing up procedure. In this way of thinking, the gauge symmetry of ABJM $U(N)_k \times U(N)_{-k}$ changes into $U(N+M)_k \times U(N)_{-k} \times U(M')_{k'}$. The corresponding CS theory is represented by a quiver with three vertices which is represented by the following triangle

$$
\begin{array}{c}
\text{U}(N+M)_k \\
\text{U}(N)_{-k} \\
\text{U}(M')_{k'}
\end{array}
$$

We will not study the general case. Instead, we will discuss a particular model motivated by the study of topological defect in graphene. This model which is called doubled level-$k$ CS field theory $U(1) \times U(1)$ is dealt with in [10]. The action of this model takes the form

$$
S_{CS} = -\frac{1}{4\pi} \int_{\mathbb{R}^{1,2}} kA^1 \wedge dA^1 - kA^2 \wedge dA^2.
$$

To make contact with IIA superstring on $\text{AdS}_4 \times \tilde{\mathbb{CP}}^3$, one requires the following condition on CS levels and gauge group ranks

$$
k(M+N) - kN + k'M' = 0.
$$
This is inspired from the study of CS quivers with $\prod_i U(N_i)_{k_i}$ gauge symmetry\cite{19}, where the constraint on CS levels $k_i$ is given by

$$\sum_i k_i N_i = 0.$$  \hspace{1cm} (3.5)

Taking into account this observation, the link we are after force us to consider a particular solution given by

$$M = M' \quad k' = -k.$$  \hspace{1cm} (3.6)

As before, thinking that ABJM symmetry as spectators, the CS quiver theory describing FQH system will be in the $U(M)_k \times U(M)_{-k}$ part of $U(N + M)_k \times U(N)_{-k} \times U(M)_{-k}$. Extracting an $U(1) \times U(1)$ abelian part of $U(M)_k \times U(M)_{-k}$, we can obtain the doubled CS field theory model from two D6-branes wrapping individually the two four-cycles $C_4$ and $E$. Using quiver gauge theory living on their world-volumes and proceeding now similarly as in the preceding model, we can get a model with the following $K_{ij}$ matrix

$$K_{ij} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$  \hspace{1cm} (3.7)

This matrix, which describes a system without interactions between gauge fields, produces the effective action given in (3.3). Indeed, the first term of this action can be obtained from a D6-brane wrapping $C_4$ as we did before. Similarly, the second term can be derived from D6-branes wrapping $E$ in the presence of the R-R gauge field sourced by the anti-D4-flux, which leads to $-k = \frac{1}{2\pi} \int_E (dA_3)$. To couple the system to an external gauge field, one needs an extra D4-brane coupled to a RR 5-form $A_5$. This gauge field can be decomposed in terms of a harmonic 4-form which is dual to a generic 4-cycle described by a linear combination of $\{C_4, E\}$ as follows

$$C = q_1 C_4 + q_2 E.$$  \hspace{1cm} (3.8)

Evaluating (3.2) for the charges $q_i = (q_1, q_2)$ yields

$$\nu = \frac{q_1^2 - q_2^2}{k}.$$  \hspace{1cm} (3.9)

Let us give a concrete example to illustrate how the formulae works. Taking $q_1 = 2$, $q_2 = 1$ and specializing the general expression to the value $k = 2$, we obtain $\nu = \frac{3}{2}$. The corresponding model has been studied extensively theoretically and experimentally. The computation for $q_1 = n + 1$, $q_2 = n$ and $k = n + 1$, gives a remarkable sequence $\nu = \frac{2n+1}{n+1}$. In this way, we can generate the following values

$$\frac{3}{2} \rightarrow \frac{5}{3} \rightarrow \frac{7}{4} \rightarrow \frac{9}{5} \rightarrow \ldots$$  \hspace{1cm} (3.10)
It is worth noting at this point that this sequence can be viewed as the inverse of Fary series used in the hierarchy structure and scaling theory of FQHE [20]. It follows that, the vanishing filling factor behavior has to do with the line defined by $q^2 = 0$ in the $(k, q^2)$ parameter space, where $q^2 = q_1^2 - q_2^2$. This can happened if the condition $q_1 = q_2$ is satisfied, which means that we can fix the world-volume flux by the number of D4-branes. The Hall conductivity is quantized in terms of the CS level being identified with the D4-flux and the quadratic charge $q^2$ given in terms of the integral of harmonic 4-forms over dual four-cycles used in the the blowing up of CP^3.

4 On extended models

In this section, we would like to note that the above analysis can be extended in many ways. One of them is to use the blown up CP^3 at $\ell$ points. In this way, the 4-cycle homology group $H^4(\tilde{CP}^3, Z)$ is now $\ell + 1$ dimension and is generated by $\{C_4, E_\ell\}$. In type IIA superstring theory, the corresponding model appears as a CS quiver model on the world-volume of $\ell + 1$ D6-branes wrapped separately on 4-cycles generating $H^4(\tilde{CP}^3, Z)$. On the world-volume of these wrapped D6-branes lives an U$(1)^{\ell+1}$ gauge theory in 3-dimensional space-time on which FQHS system will reside. The general study is beyond the scope of the present work, though we will consider a simple brane system extending the above doubled CS field theory model. This can be obtained by requiring that the matrix CS coupling to be

$$K_{ij} = k\eta_{ij}$$

(4.1)

where $\eta_{ij} = diag(\ell, -1, \ldots, -1)$. This matrix is obtained by solving a similar equation as the one appearing in (3.4). As in the previous discussed model, the external gauge field arises from the decomposition of the RR 5-form $A_5$ in terms of a combination of harmonic four-forms where the dual four-cycle takes the following form

$$C = q_1C_4 + q_2E_1 + \ldots + q_{\ell+1}E_\ell.$$

(4.2)

For the vector charge $q_i = (q_1, \ldots, q_{\ell+1})$, the filling factor can be simplified as

$$\nu = \frac{q^2}{\ell k}$$

(4.3)

where $q^2 = q_1^2 - \ell \sum_{i=1}^{\ell+1} q_i^2$. For $\ell = 1$, we recover the model associated with doubled CS field theory. As before, the vanishing filling factor behavior has to do with the line defined by $q^2 = 0$ in the $(k, q^2, \ell)$ parameter space. This can happened if the condition $q_1^2 = \ell \sum_{i=1}^{\ell+1} q_i^2$ is satisfied which means that we can fix the world-volume flux by the number of D4-branes and the number of the blown up points.
5 Discussion

In this paper, we have discussed FQHS in ABJM-like theory. The corresponding geometries are realized as a blown up $\mathbb{CP}^3$ projective space. To reach such systems, we have considered abelian parts and we have assumed the remaining part as spectators. Using toric geometry, we have built CS quivers describing FQHS in terms of D6-branes and R-R gauge fields. For the blown up $\mathbb{CP}^3$ at $\ell$ points, we have given a model where the filling factor depends on the extra parameter $\ell$. It is worth noting that this new parameter $\ell$ gives more freedom to recover some known values of the filling factor. It follows also that as we blow up the type IIA geometry, the FQHS system undergoes a scaling transformation from $\nu = \frac{q^2}{k}$ to $\nu = \frac{q^2}{\ell k}$. It should be interesting to understand, in more detail, this scaling of the CS couplings in terms of Kahler moduli space of type IIA geometry.

This work comes up with many open questions. First, we have analyzed a blown up $\mathbb{CP}^3$, it would be important to consider 3-dimensional toric varieties extending $\mathbb{CP}^3$ using technics of F-theory compactifications. On the other hand, it has been given a nice string realization of graphene [21] and its relation to CFT$_3$, it should be interesting to make contact with such a 3-dimensional model. We shall address these open questions in the future.

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