The fate of dynamical phase transitions at finite temperatures and in open systems

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When a quantum system is quenched from its ground state, the time evolution can lead to non-analytic behavior in the return rate at critical times tc. Such dynamical phase transitions (DPT’s) can occur in particular, for quenches between phases with different topological properties in Gaussian models. In this paper we discuss Loschmidt echos generalized to density matrices and obtain results for quenches in closed Gaussian models at finite temperatures as well as for open system dynamics described by a Lindblad master equation. While cusps in the return rate are always smoothed out by finite temperatures we show that dissipative dynamics can be fine-tuned such that DPT’s persist.

I. INTRODUCTION

The macroscopic properties of a quantum system in equilibrium can be understood from the appropriate thermodynamic potential. Studies of Lee-Yang zeros of the grand-canonical potential as a function of a complex fugacity or of Fisher zeros of the canonical potential as a function of complex temperature, in particular, have significantly contributed to our understanding of equilibrium phase transitions.1–4 In recent years, there have been attempts to follow a similar approach to non-equilibrium dynamics. For quench dynamics in closed quantum systems it has been suggested that dynamical phase transitions (DPT’s) can be defined based on the Loschmidt echo5

\[
\mathcal{L}_0(t) = \langle \Psi_0 | e^{-iH_1 t} | \Psi_0 \rangle. \tag{I.1}
\]

Here |Ψ0⟩ is the pure quantum state before the quench and H1 the time-independent Hamiltonian responsible for the unitary time evolution. The Loschmidt echo has the form of a partition function with boundaries fixed by the initial state. In analogy to the Fisher zeros in equilibrium one can thus study the zeros of the Loschmidt echo for complex time t. In Ref. 5 it has been shown that for the specific case of the transverse Ising model these zeros form lines in the complex plane which cross the real axis only for a quench across the equilibrium critical point.

In a many-body system one expects that the overlap between the time-evolved and the initial state is in general exponentially small in system size in analogy to the Anderson orthogonality catastrophe in equilibrium.6 To obtain a non-zero and well-defined quantity in the thermodynamic limit it is thus useful to consider the return rate

\[
l_0(t) = -\lim_{L\to\infty} \frac{1}{L} \ln |\mathcal{L}_0(t)|. \tag{I.2}
\]

where L is the system size. Zeros in L0(t) at critical times tc then correspond to non-analyticities (cusps or divergencies) in l0(t).5,7–11 It is, however, important to stress that in contrast to the particularly simple case of the transverse Ising model there is in general no one-to-one correspondence between dynamical and equilibrium phase transitions.8,12 It is possible to find non-analytical behavior of the return rate without crossing an equilibrium critical point in the quench, and one can cross a critical line without non-analyticities in l0(t) being present. For one-dimensional topological systems it has been shown, in particular, that crossing a topological phase transition in the quench always leads to a DPT but the opposite does not have to be true.13 Thus there are still some issues about the appropriateness of the Loschmidt echo as a useful indicator. Nevertheless the notion of a dynamical phase transition is an exciting concept extending key elements of many-body physics to non-equilibrium.

Lately, DPT’s have also been studied experimentally. In Ref. 14 vortices in a gas of ultracold fermions in an optical lattice were studied and their number interpreted as a dynamical order parameter which changes at a DPT. Even more closely related to the described formalism to classify DPT’s is an experiment where a long-range transverse Ising model was realized with trapped ions. In this case the time-evolved state was projected onto the initial state which allowed access to the Loschmidt echo (I.1) directly.15

While these experiments are an exciting first step to test these far-from-equilibrium theoretical concepts they also lead to a number of new questions. Chief among them is the question how experimental imperfections affect the Loschmidt echo and DPT’s. On the one hand, the initial state is typically not a pure but rather a mixed state at a certain temperature T. This raises the question how the Loschmidt echo can be generalized to thermal states. On the other hand, the dynamics is also typically not purely unitary. Decoherence and particle loss processes do affect the dynamics as well, requiring a generalization of (I.1) to density matrices. Finally dynamical processes and phase transitions can be induced entirely by coupling to reservoirs in which case no pure-state or
In this paper we will address these questions. In Sec. II we discuss various different ways to generalize the Loschmidt echo to finite temperatures. We concentrate, in particular, on projective measurements of time-evolved density matrices relevant for example, for trapped ion experiments, as well as on a proper distance measure between the initial and the time-evolved density matrix following Refs. 18 and 19. We study both of these generalized Loschmidt echos for the case of unitary dynamics of Gaussian fermionic models in Sec. III. As examples, we present results for the transverse Ising and for the Su-Schrieffer-Heeger (SSH) model. In Sec. IV we consider the generalized Loschmidt echo for open-system dynamics of Gaussian fermionic models described by a Lindblad master equation (LME). A short summary and conclusions are presented in Sec. V.

II. THE LOSCHMIDT ECHO

We will first review some properties of the standard Loschmidt echo for unitary dynamics of pure states in Sec. II A before discussing several possible generalizations to mixed states in Sec. II B.

A. Pure states

The Loschmidt echo for unitary dynamics of a pure state is defined by Eq. (1.1). Its absolute value can be used to define a metric in Hilbert space $\phi = \arccos |L_0(t)|$ with $0 \leq |L_0(t)| \leq 1$ which characterizes the distance between the initial state $|\Psi_0\rangle$ and the time-evolved state $|\Psi(t)\rangle = e^{-iH_t t}|\Psi_0\rangle$. From this point of view the Loschmidt echo is a time-dependent version of the fidelity $F = |\langle \Psi_0 | \Psi_1 \rangle|$ which has been widely used to study equilibrium phase transitions. Because of the Anderson orthogonality catastrophe one has to consider a fidelity density $f = -\lim_{L \to \infty} \ln |F|/L$ for a many-body system in the thermodynamic limit $L \to \infty$ in analogy to the Loschmidt return rate defined in Eq. (1.2). If $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are both ground states of a Hamiltonian $H(\lambda)$ for different parameters $\lambda$ then the fidelity susceptibility $\chi_f = (\partial^2 f)/(\partial \lambda)^2|_{\lambda=\lambda_c}$ will typically diverge at an equilibrium phase transition. Similarly, one might expect that a quench can lead to states $|\Psi(t_c)\rangle$ at critical times $t_c$ which are orthogonal to the initial state implying $L_0(t_c) = 0$ and resulting in a non-analyticity in the return rate $\lambda(t_c)$. A peculiarity of the return rate is that its non-analyticity does not only depend on the properties of the initial and final Hamiltonian before and after the quench but also on time. For a quench from $H_0$ to $H_1$, in particular, the critical time $t_c$ will in general depend upon if one starts with the ground state of the initial Hamiltonian or some excited eigenstate.

B. Mixed states

1. Loschmidt echo as a metric

If the Loschmidt echo is primarily seen as defining a metric in Hilbert space, then it is natural to ask if a similar metric can also be defined for density matrices $\rho(t)$. In order for the generalized Loschmidt echo $|L_\rho(\rho(0), \rho(t))|$ to give rise to a proper measure of distance in the space of density matrices we want the following relations to hold

1) $0 \leq |L_\rho(\rho(0), \rho(t))| \leq 1$ and $|L_\rho(\rho(0), \rho(0))| = 1,$

2) $|L_\rho(\rho(0), \rho(t))| = 1$ iff $\rho(0) = \rho(t)$, and

3) $|L_\rho(\rho(0), \rho(t))| = |L_\rho(\rho(t), \rho(0))|.$

Without time dependence, this problem reduces again to the definition of a fidelity for density matrices. A direct generalization of this fidelity leads to

$$L_\rho(t) \equiv L_\rho(\rho(0), \rho(t)) = \text{Tr} \sqrt{\rho(0) \rho(t) \rho(0)}. \quad (II.1)$$

Note that this definition satisfies $\lim_{\beta \to \infty} L_\rho(t) = |L_0(t)|$ if $\rho(0)$ is a thermal density matrix and the time evolution is unitary. $\beta = T^{-1}$ is the inverse temperature with $k_B = 1$. $L_\rho(t)$ is symmetric between $\rho(0)$ and $\rho(t)$ and also satisfies the other conditions above. The induced metric $\phi = \arccos |L(t)|$ also fulfills the triangle inequality. From this point of view, Eq. (II.1) is thus the proper generalization of the Loschmidt echo to density matrices. Despite its relatively complicated appearance, $|L_\rho(p_1, p_2)|$ has a straightforward physical meaning. If we understand $p_1$ and $p_2$ as reduced density matrices obtained by a partial trace over a larger system which is in a pure state $|\phi_{1,2}\rangle$ respectively, then $|L_\rho(p_1, p_2)| = \text{max} |\langle \phi_1 | \phi_2 \rangle |$ where the maximum is taken over all purifications of $p_1$ and $p_2$, respectively. I.e., $L_\rho$ provides the fidelity to the states in the enlarged Hilbert space which are as parallel as possible and consistent with the mixed states of the subsystem.

A seemingly simpler and more straightforward generalization such as

$$|L_\rho(t)| = \sqrt{\frac{\text{Tr} \{ \rho(0) \rho(t) \}}{\text{Tr} \rho^2(0)}} \quad (II.2)$$

does, in general, not fulfill the conditions above. If we start, for example, in a completely mixed state $\rho(0) = \sum_n \frac{1}{N} |\Psi_n\rangle \langle \Psi_n|$ and evolve under dissipative dynamics to a pure state $\rho(t \to \infty) = |\Psi_0\rangle \langle \Psi_0|$ then $|L_\rho(\rho(0))| = |L_\rho(\infty)| = 1$ which clearly is not a desirable property. Using a spectral representation in a basis where $\rho(0) = \sum_n p_n |\Psi_0^n\rangle \langle \Psi_0^n|$ is diagonal, Eq. (II.2) for the special case of unitary time evolution can be represented as

$$|L_\rho(t)|^2 = \sum_{m,n} p_m p_n \langle \Psi_m^n | e^{-iH_t t} | \Psi_0^n \rangle^2 \sum_n p_n^2 , \quad (II.3)$$
where \( p_n \) are weights with \( \sum_n p_n = 1 \).

In Sec. III we will investigate \( \mathcal{L}_p(t) \) for unitary dynamics in Gaussian models with \( \rho(0) \) being a canonical density matrix at a given finite temperature \( T \). At the same time, we will also briefly discuss the result for \( \tilde{\mathcal{L}}_p(t) \) which—for unitary dynamics—in this specific case does fulfill \( 0 \leq |\tilde{\mathcal{L}}_p(t)| \leq 1 \). This is no longer the case for open system dynamics described by an LME and we will therefore exclusively discuss \( \mathcal{L}_p(t) \) in Sec. IV.

2. Projection onto a pure state

While (II.1) allows to generalize the properties of the Loschmidt echo as a metric to density matrices, \( \mathcal{L}_p(t) \) not necessarily be the quantity measured experimentally. In Ref. 15, for example, DPT’s in the transverse Ising model have been investigated using a system of trapped ions. In this experiment the system is prepared in an initial configuration, the system is then time evolved and the Loschmidt echo measured by a projection. If the system is prepared in a pure state and the projection is onto the same pure state then the Loschmidt echo (I.1) is measured. Here we want to consider the case that the preparation of the system is not ideal—leading to a mixed instead of a pure state—while the projection is still onto the ground state of the initial Hamiltonian. I.e., we consider the case that only one of the states is impure. In this case we can define a generalized Loschmidt echo by replacing \( \rho(0) \rightarrow |\Psi_0^0\rangle\langle\Psi_0^0| \) in Eq. (II.1) leading to

\[
|\mathcal{L}_p(t)|^2 = \frac{\langle\Psi_0^0|\rho(t)|\Psi_0^0\rangle}{\langle\Psi_0^0|\rho(0)|\Psi_0^0\rangle} = \sum_n \frac{p_n}{p_0} \langle\Psi_0^0|e^{-iH_0t}\Psi_n^0\rangle^2. \tag{II.4}
\]

The second line is a spectral representation in the eigenbasis of \( \rho(0) \) and we have introduced a normalization factor such that \( \mathcal{L}_p(0) = 1 \). Note that for a thermal initial density matrix \( \lim_{\beta \rightarrow \infty} |\mathcal{L}_p(t)|^2 = |\mathcal{L}_0(t)|^2 \). In Sec. III we will also investigate this generalization of the Loschmidt echo for unitary dynamics and present results for experimentally relevant cases such as the transverse Ising and the SSH model.

3. Alternative generalizations

The definition of a generalized Loschmidt echo for mixed states is not unique and several other possible generalizations have been discussed previously in the literature. In Ref. 37 and Ref. 38 the quantity

\[
\mathcal{L}_{av} = \text{Tr} \{ \rho(0) U(t) \} = \sum_n p_n \langle\Psi_n^0|e^{-iH_0t}\Psi_n^0\rangle \tag{II.5}
\]

is considered where \( U(t) \) is the time-evolution operator. From the spectral representation for unitary time evolution with a time-independent Hamiltonian shown in the second line of Eq. (II.5) it is clear that this generalization measures an average over pure-state Loschmidt echos rather than the ‘overlap’ between mixed states as defined in Eq. (II.1). Also in contrast to (II.3) only diagonal terms enter; Eq.(II.5) cannot be used to define a measure of distance between two density matrices. For a generic Gibbs ensemble one expects, in general, that \( \mathcal{L}_{av} = 0 \) is only possible if \( p_0 = 1 \), since even if the Loschmidt echos of different states \( |\Psi_n^0\rangle \) will vanish at some time, the corresponding critical times will in general be different. For a Gaussian model in a generalized Gibbs ensemble, where the occupation of each \( k \)-mode is individually conserved, zeros are however also possible at finite temperatures.\(^{38}\)

A similar approach—motivated by the characteristic function of work\(^{39}\)—was also used in Ref. 40 where the specific case of a canonical density matrix as initial condition was considered and a generalized Loschmidt echo defined by

\[
\tilde{\mathcal{L}}_{av} = \frac{1}{Z} \text{Tr} \{ e^{iH_1t} e^{-iH_0t} e^{-\beta H_0} \} = \frac{1}{Z} \sum_n e^{-iH_0t} \langle\Psi_n^0|e^{iH_1t}\Psi_n^0\rangle. \tag{II.6}
\]

The result is a thermal average over the Loschmidt echo of pure states and thus very different from the overlap between density matrices defined in Eq. (II.1).

For all generalized Loschmidt echos discussed here an appropriate return rate (I.2) can be defined. It is the return rate in the thermodynamic limit which we want to study in the following.

III. UNITARY DYNAMICS IN GAUSSIAN MODELS

We consider free fermion models described by the Hamiltonian

\[
H = \sum_{k \geq 0} \Psi_k^\dagger H_k \Psi_k \tag{III.1}
\]

with \( \Psi_k = (c_k, c_{-k}^\dagger)^T \). Here \( c_k \) is an annihilation operator of spinless fermions with momentum \( k \). This Hamiltonian describes models with a single-site unit cell which are biradical in the creation and annihilation operators and can contain pairing terms as in the transverse Ising and Kitaev chains, see Sec. III.B.1. If we identify \( c_k \equiv c_{-k}^\dagger \) then the Hamiltonian (III.1) can also describe models with a two-site unit cell which contain only hopping and no pairing terms such as the SSH and Rice-Mele models, see Sec. III.B.2. The momentum summation in both cases runs over the first Brillouin zone. It is often convenient to write the \( 2 \times 2 \) matrix \( H_k \) as \( H_k = d_k \cdot \sigma \) where \( d_k \) is a three-component parameter vector and \( \sigma \) the vector of Pauli matrices. During the quench the parameter vector \( d_k \) is changed leading to an initial Hamiltonian \( H_0 \) and a
The two Hamiltonians are diagonal and we have

$$H_i = \sum_{k \geq 0} \varepsilon_k^i \left(c_{k}^\dagger c_{k+i} + c_{k-i}^\dagger c_{k} - 1\right)$$  \hspace{1cm} (III.2)

with energies $\varepsilon_k^i > 0$ and $i = 0, 1$. The operators in which the two Hamiltonians are diagonal are related by a Bogoliubov transform

$$c_{k0} = u_k c_{k} + v_k c_{k-1}^\dagger ; \ c_{k1} = u_k c_{k0} - v_k c_{k-1}^\dagger.$$  \hspace{1cm} (III.3)

The Bogoliubov variables can be parametrized by an angle $\theta_k$ as $u_k = \cos \theta_k$ and $v_k = \sin \theta_k$. For each $k$-mode there are 4 basis states. We can either work in the eigenbasis $|\Psi_0^j\rangle$ of $H_0$ or the eigenbasis $|\Psi_1^j\rangle$ of $H_1$ which can be expressed as

$$|\Psi_0^j\rangle = |0\rangle_0 \quad (u_k - v_k c_{k1}^\dagger c_{k-1}^\dagger)|0\rangle_1$$

and

$$|\Psi_1^j\rangle = c_{k1}^\dagger |0\rangle_0$$

or vice versa

$$|\Psi_0^1\rangle = |0\rangle_1 = (u_k + v_k c_{k0}^\dagger c_{k-1}^\dagger)|0\rangle_0$$

and

$$|\Psi_0^2\rangle = c_{k0}^\dagger |0\rangle_0$$

$$|\Psi_0^3\rangle = c_{k0}^\dagger c_{k-1}^\dagger |0\rangle_0 = (v_k + u_k c_{k0}^\dagger c_{k-1}^\dagger)|0\rangle_1$$

Here $|0\rangle_0$ and $|0\rangle_1$ are the ground states of $H_{0,1}$. The Loschmidt echo at zero temperature can be easily calculated using the transformation (III.4) leading to

$$\mathcal{L}_0(t) = \prod_k \left[u_k^2 e^{i\varepsilon_k^1 t} + v_k^2 e^{-i\varepsilon_k^1 t}\right]$$  \hspace{1cm} (III.6)

and $|\mathcal{L}_0(t)|^2 = \prod_k |\mathcal{L}_0^k(t)|^2$ with

$$|\mathcal{L}_0^k(t)|^2 = \left[1 - \sin^2(2\theta_k) \sin^2(\varepsilon_k^1 t)\right].$$  \hspace{1cm} (III.7)

The calculation of (II.1) for the case that $\rho(0)$ is a thermal density matrix is instructive for the dissipative case discussed in Sec. IV so we briefly rederive the known result. For $\rho_p(t)$ here. It is most convenient to perform the calculation in the eigenbasis of the time-evolving Hamiltonian $H_1$ using the transformation (III.5). Because only the states $|\Psi_0^j\rangle$ and $|\Psi_1^j\rangle$ are mixed by the transformation, the initial unnormalized density matrix $\rho_k(0)$ can be rearranged into two $2 \times 2$ block matrices $I_2$ (identity matrix) and $r_k(0)$ with

$$l(t) = -\frac{1}{2\pi} \int \ln |L^k(t)| dk.$$  \hspace{1cm} (III.9)

In the following we will explicitly calculate $l(t)$ for the different generalized Loschmidt echos.

### A. Projection onto a pure state

We want to first investigate the case where only one of the states is impure. A natural generalization is then the Loschmidt echo defined in Eq. (II.4). For the considered Gaussian models (III.1) the Loschmidt echo separates into a product $|\mathcal{L}_p(t)|^2 = \prod_k |\mathcal{L}_p^k(t)|^2$. If we, furthermore, assume that our initial mixed state is described by a canonical ensemble then we obtain

$$|\mathcal{L}_p^k(t)|^2 = \langle \Psi_0^0|\rho_k(t)|\Psi_0^0\rangle / \langle \Psi_0^0|\rho_0(0)|\Psi_0^0\rangle$$  \hspace{1cm} (III.10)

where we have used the spectral representation of the density matrix $\rho_k(t)$ in terms of the eigenstates of $H_0^k$ and $\beta$ is the inverse temperature. The eigenenergies of the 4 eigenstates for each $k$-mode are denoted by $E_{kn}^k = (\varepsilon_k^0, 0, 0, \varepsilon_k^1)$. Using the representation (III.4) of the eigenstates in terms of the operators of the final Hamiltonian $H_1$ one finds

$$|\mathcal{L}_p^k(t)|^2 = \prod_k \left[1 - \left(1 - e^{-2\beta \varepsilon_k^1}\right) \sin^2(2\theta_k) \sin^2(\varepsilon_k^1 t)\right].$$  \hspace{1cm} (III.11)

It is obvious that $\mathcal{L}_p(t)$ is 0 only possible at zero temperature in which case $|\mathcal{L}_p(t)| = |\mathcal{L}_0(t)|$, see Eq. (III.7). If one starts from a mixed state then the DPT’s are washed out even if one projects onto the ground state. With the appropriately chosen ground state and the associated energies $E_{kn}^k$, the result (III.11) also holds for the models with a two-site unit cell such as the SSH and Rice-Mele models.

### B. Thermal density matrices

For any of the generalized Loschmidt echos defined before we can write the return rate as
\[ r_k(0) = \left( \cosh(\beta \varepsilon_k^0) \pm \sinh(\beta \varepsilon_k^0) \cos(2\theta_k) \right) - \sinh(\beta \varepsilon_k^0) \sin(2\theta_k) \cos(\beta \varepsilon_k^0) - \sinh(\beta \varepsilon_k^0) \cos(2\theta_k) \right) . \] 

\( \sqrt{r_k(0)} \) is obtained from (III.12) by replacing \( \beta \to \beta/2 \) and \( r_k(t) \) by replacing \( r_k^{(12)} \to e^{2i\epsilon_k^0 t} r_k^{(12)} \) and \( r_k^{(21)} \to e^{-2i\epsilon_k^0 t} r_k^{(21)} \). The partition function is given by \( Z_k = \text{Tr} \rho_k = \text{Tr}(\mathbf{I}_2) + \text{Tr} r_k(0) = 2 + 2 \cosh(\beta \varepsilon_k^0) \). We can now simplify the generalized Loschmidt echo (II.1) in this case to

\[ \mathcal{L}_p(t) = \prod_k \frac{2 + \lambda_k(1) + \lambda_k(2)}{2 + 2 \cosh(\beta \varepsilon_k^0)} \]  

(III.13)

where \( \lambda_k(1) \) are the eigenvalues of \( \sqrt{r_k(0)} r_k(t) \sqrt{r_k(0)} \) which are given by

\[ \lambda_{k,1,2}(t) = \sqrt{1 + |\mathcal{L}_k(0)|^2 \sinh^2(\beta \varepsilon_k^0) \pm |\mathcal{L}_k(0)| \sinh(\beta \varepsilon_k^0)} , \]

(III.14)

with \( \mathcal{L}_k(0) \) defined in Eq. (III.7). As a final result we thus obtain

\[ \mathcal{L}_p(t) = \prod_k \frac{1 + \sqrt{1 + |\mathcal{L}_k(0)|^2 \sinh^2(\beta \varepsilon_k^0)}}{1 + \cosh(\beta \varepsilon_k^0)} . \]  

(III.15)

For any finite temperature this means that \( \mathcal{L}_p(t) > 0 \) for all times, i.e., there are no DPT’s. For \( \beta \to \infty \) the result reduces to the zero-temperature result, Eq. (III.7).

The result (III.15) also holds for Gaussian models with a two-site unit cell such as the SSH and Rice-Mele models.

We now also briefly discuss the possible generalization \( \tilde{\mathcal{L}}_p(t) \) defined in Eq. (II.2). While this function, in general, does not fulfill the requirements listed in Sec. II.B it turns out that for the case considered here at least \( 0 \leq |\tilde{\mathcal{L}}_p(t)| \leq 1 \) is fulfilled. We start again from a thermal density matrix. The spectral representation using the eigenstates of \( H_1 \) then reads

\[ |\tilde{\mathcal{L}}_p(t)|^2 = \sum_{n,m} e^{i(E_n^k - E_m^k)t} |\langle \Psi_n^1 | e^{-\beta H_0} | \Psi_m^1 \rangle|^2 . \]  

(III.16)

Only the eigenstates \( |\Psi_n^1 \rangle \) and \( |\Psi_0^1 \rangle \) mix and it is easy to check the final result

\[ |\tilde{\mathcal{L}}_p(t)|^2 = \prod_k \left[ \cosh^2(\beta \varepsilon_k^0) + \tanh^2(\beta \varepsilon_k^0) |\mathcal{L}_k(0)|^2 \right] = \prod_k \left[ 1 - \tanh^2(\beta \varepsilon_k^0) \sin^2(2\theta_k) \sin^2(\varepsilon_k^0 t) \right] . \]  

(III.17)

\( \tilde{\mathcal{L}}_p(t) = 0 \) is again only possible if \( T = 0 \).

1. Ising and Kitaev models

The finite-temperature results can be directly applied to concrete models. The Kitaev chain, for example, is defined by

\[ H = \sum_i \left[ \Psi_i^\dagger \left( \Delta \tau^y - J \tau^z \right) \Psi_{i+1} + \text{H.c.} - \Psi_i^\dagger \mu \tau^z \Psi_i \right] \]  

(III.18)

where \( \Psi_i^\dagger = (c_i^\dagger, c_i) \) and \( c_i^{(t)} \) annihilates (creates) a spinless particle at site \( i \). The Kitaev chain is topologically non-trivial when \( \mu < 2 |J| \) and \( \Delta \neq 0 \). Note that \( \Delta = 0 \) is a phase boundary between phases with winding numbers \( \pm 1 \). As a special case the transverse Ising model

\[ H(g) = -\frac{1}{2} \sum_i \sigma_i^x \sigma_{i+1}^x + g \sum_{i=1}^{N} \sigma_i^x \]  

(III.19)

is obtained if one sets \( \mu = -g/2 \) and \( J = 1/4 = -\Delta \) in (III.18). After a Fourier transform, for a chain with periodic boundary conditions, the Hamiltonian (III.18) is of the form of Eq. (III.1) with parameter vector

\[ \mathbf{d}_k = (0, 2 \Delta \sin k, -2J \cos k - \mu) , \]  

(III.20)

and \( \cos(2\theta_k) = \hat{d}_k^0 \cdot \hat{d}_k^1 \). In Fig. 1 we plot the return rate in the thermodynamic limit, Eq. (III.9), for a quench from \( g = 0.5 \) to \( g = 1.5 \). While the cusp in the return rate at the critical time \( t_c \) is only slightly rounded off for temperatures up to \( T = 0.1 \) if we project onto the ground state, Eq. (II.4), signatures of a DPT are already almost lost at this temperature if we use the generalized Loschmidt echo (II.1) which measures the distance between the initial and the time-evolved thermal density matrix.
The SSH model is a special case of the Rice-Mele model
with the identification $d_k \equiv c_{-k}^\dagger$. The parameter vector in this case is given by
$$d_k = (-2 \cos k, 2 \delta \sin k, V) .$$
(III.22)

The SSH model is a special case of the Rice-Mele model obtained by setting the alternating potential $V = 0$.

In Fig. 2 the return rate for a symmetric quench from $\delta = -0.5$ to $\delta = 0.5$ for $V = 0$ is shown.

While the cusp in the return rate at the critical time $t_c$ is washed out in this case as well, a signature of the DPT at zero temperature is more clearly visible also at finite temperatures as compared to the quench in the Ising model shown in Fig. 1.

IV. OPEN SYSTEMS

In systems where the Loschmidt echo has been studied experimentally such as cold atomic gases and trapped
ions\textsuperscript{14,15} interactions with electromagnetic fields are used to control the particles. These systems are therefore intrinsically open systems and decoherence and loss processes are unavoidable. Using the Born-Markov approximation such open systems can be described by a Lindblad master equation
$$\dot{\rho}(t) = -i[H, \rho] + \sum_\mu \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2} \{L_\mu L_\mu^\dagger, \rho\} \right) .$$
(IV.1)

Here $L_\mu$ are the Lindblad operators describing the dissipative, non-unitary dynamics induced by independent reservoirs labelled by $\mu$, and $\{,\}$ is the anti-commutator. In order to have a bilinear LME which can be solved exactly, we continue to consider Hamiltonians as defined in Eq. (III.1) with periodic boundary conditions which can be diagonalized in Fourier space. We consider Lindblad operators that are linear in creation and annihilation operators leading to a linear dynamics
$$L_\mu = \sqrt{\gamma_\mu} c_\mu^\dagger \text{ and } L_\mu = \sqrt{\gamma_\mu} c_\mu$$
(IV.2)
describing particle loss and creation processes with amplitudes $\gamma_\mu > 0$ and $\bar{\gamma}_\mu > 0$, respectively. This form ensures that the dissipative terms in Eq. (IV.1) are also bilinear. More specifically we consider reservoirs that couple each to only one $k$-mode
$$L_k = \sqrt{\bar{\gamma}_k} c_{k}^\dagger \text{ and } L_k = \sqrt{\gamma_k} c_k$$
(IV.3)

To solve the Lindblad equation we will use the superoperator formalism.\textsuperscript{41} The $n \times n$ density matrix $\rho$ is recast into an $n^2$-dimensional vector $|\rho\rangle$ and the Hamiltonian and Lindblad operators become superoperators acting on this vector. The LME (IV.1) and its solution can then be written as
$$|\dot{\rho}\rangle = \mathcal{L} |\rho\rangle \text{ ; } |\rho\rangle(t) = \exp(\mathcal{L}t) |\rho(0)\rangle .$$
(IV.4)

For the purely unitary time evolution considered in the previous section the Lindbladian $\mathcal{L}$ takes the form
$$\mathcal{L} = -i (H \otimes \mathbf{I}_n + \mathbf{I}_n \otimes H^\dagger)$$
(IV.5)
where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Similarly, the individual Lindblad operators (IV.3) can be written as superoperators acting on $|\rho\rangle$. The solution vector $|\rho\rangle(t)$ can then be recast into a matrix allowing one to calculate the generalized Loschmidt echos also for open systems.

A. Particle loss

We consider again free fermionic models of the type (III.1) with the 4 basis states (III.4) for each $k$-mode.

As a first example, we investigate a simple mixed initial state $\rho_k(0) = \frac{1}{4} (|\Psi_1^0\rangle |\Psi_1^0\rangle + |\Psi_2^0\rangle |\Psi_2^0\rangle)$ and a time evolution under the Lindblad operators $L_{1k} = \sqrt{\gamma_k} c_k$ and $L_{2k} = \sqrt{\bar{\gamma}_k} c_{-k}$. In this case it is straightforward to show that the density matrix takes the form

![Graph showing return rate $l(t)$ for the SSH chain in the thermodynamic limit for a quench from $\delta = -0.5$ to $\delta = 0.5$ at different temperatures $T$.](image-url)
\[ \rho_k(t) = \frac{1}{2} \text{diag}(2 - e^{-\gamma_k t} - e^{-\gamma_{-k} t}, e^{-\gamma_k t}, e^{-\gamma_{-k} t}, 0). \]

The non-equilibrium steady state (NESS) is thus the completely empty state for \( \gamma \neq 0 \). Since both \( \rho(0) \) and \( \rho(t) \) are diagonal it follows immediately that the general
ized Loschmidt echo is given by

\[ L_\rho(t) = \frac{1}{2} \prod_k \left( e^{-\gamma_k t/2} + e^{-\gamma_{-k} t/2} \right), \quad (IV.6) \]

As one might have expected, \( L_\rho(t) \) shows an exponential decay in this case. If \( \gamma_k = \gamma_{-k} = \gamma = \text{const} \) then the return rate in the thermodynamic limit (III.9) increases linearly, \( l(t) = \gamma t/2 \), and thus diverges only at infinite time.

B. Quench in Kitaev-type models with particle loss

Next, we want to consider a quench for a Kitaev-type model with Hamiltonian (III.2) with the basis states (III.5). As in Sec. III.B we start with a thermal density matrix \( \rho(0) \) but now also allow for particle loss processes as in the example above. Crucially, the matrix \( \rho_k(t) \) still can be decomposed into two \( 2 \times 2 \) block matrices. We can therefore write

\[ L_\rho^k(t) = \text{Tr} \sqrt{M_1} + \text{Tr} \sqrt{M_2} \]

with \( M_i = \sqrt{\rho_i^k(0) \rho_k^i(0)} \) and \( \rho_i^{1,2} \) being the two block matrices. With \( \text{Tr} \sqrt{M_i} = \sqrt{\lambda_1^i + \lambda_2^i} > 0 \) we can write

\[ (\text{Tr} \sqrt{M_i})^2 = \lambda_1^i + \lambda_2^i + 2 \sqrt{\lambda_1^i \lambda_2^i} = \text{Tr} M_i + 2 \sqrt{\text{det} M_i}. \]

For the Loschmidt echo we therefore find

\[ L_\rho(t) = \prod_k \sum_{i=1,2} \sqrt{\text{Tr} M_i + 2 \sqrt{\text{det} M_i}}. \quad (IV.7) \]

Using this formula it is straightforward to obtain an explicit result for \( L_\rho(t) \) which, however, is quite lengthy for finite temperatures. We therefore limit ourselves here to presenting the result for \( T = 0 \) only. In this case one of the block matrices is zero and we obtain the following closed-form expression

\[ L_\rho^k(t) = \prod_k e^{-\Gamma_k^\pm t} \left[ \cos 2\theta_k \sinh(\Gamma_k^- t) - \sin^2 2\theta_k \sin^2(\xi_k^0 t) \right. \]

\[ + \left. \frac{1}{2} \sin^2 2\theta_k \left( 1 - \cosh(\Gamma_k^- t) \right) + \cosh(\Gamma_k^+ t) \right]. \quad (IV.8) \]

Here we have defined \( \Gamma_k^\pm = (\gamma_k \pm \gamma_{-k})/2 \). It is easy to see that this result reduces to Eq. (III.7) for \( \gamma_k = \gamma_{-k} = 0 \). Furthermore, there are no DPT’s for finite loss rates.

As an example for the broadening of the cusps in the return rate (III.9) we consider the same quench in the transverse Ising model as before. Fig. 3 shows that small loss rates already lead to a significant broadening of the first cusp at \( t = t_c \) and completely wash out the cusps at longer times. Furthermore, the NESS for a non-zero loss rate is always the empty state so that the return rate at infinite times becomes independent of the loss rate and is given by

\[ l(t \to \infty) = -\frac{1}{2\pi} \int_0^\infty \ln \left( \frac{1 + d_k^0 \cdot d_k^0}{2} \right) dk. \quad (IV.9) \]

FIG. 3. (Color online) The return rate \( l(t) \) for the Ising chain in the thermodynamic limit for a quench from \( g = 0.5 \) to \( g = 1.5 \) at \( T = 0 \) for different particle loss rates \( \gamma = \gamma_k = \gamma_{-k} \). Inset: Broadening of the first cusp at \( t = t_c \).

C. Quench in Kitaev-type models with particle creation and loss

So far we have seen that both finite temperatures and particle loss processes destroy DPT’s. One can then ask if it is possible to engineer dissipative processes in an open quantum system in such a way that DPT’s persist. By constructing a concrete example we will show that this is indeed possible.

We consider the case that particles with momentum \( k \) are annihilated with rate \( \gamma_k \) while particles with momentum \(-k\) are created with rate \( \tilde{\gamma}_{-k} \). As in the case with particle loss considered in Sec. IV.B the density matrix \( \rho_k(t) \) still has block structure and a calculation along the same lines is possible. At \( T = 0 \) we obtain a result which is very similar to Eq. (IV.8) and reads

\[ L_\rho^k(t) = \prod_k e^{-\tilde{\Gamma}_k^\pm t} \left[ \cos 2\theta_k \sinh(\tilde{\Gamma}_k^- t) - \sin^2 2\theta_k \sin^2(\xi_k^0 t) \right. \]

\[ + \left. \frac{1}{2} \sin^2 2\theta_k \left( 1 - \cosh(\tilde{\Gamma}_k^- t) \right) + \cosh(\tilde{\Gamma}_k^+ t) \right]. \quad (IV.10) \]

The rates are now defined as \( \tilde{\Gamma}_k^\pm = (\gamma_k \pm \tilde{\gamma}_{-k})/2 \). The essential difference when comparing Eq. (IV.10) with the previous result (IV.8) is that inside the bracket only the rate \( \tilde{\Gamma}_k^- \) is present. For \( \tilde{\Gamma}_k^- = 0 \), i.e. \( \gamma_k = \gamma_{-k} \), the Loschmidt echo becomes \( L_\rho^k(t) = \prod_k \exp(-\tilde{\Gamma}_k^+ t)|L_\rho^k(0)|^2 \) which is the zero-temperature result (III.7) with an additional exponential decay. DPT’s are thus still present for this particular case at the same critical times \( t_c \) despite the dissipative processes.

As an example, we consider again the quench in the transverse Ising chain. In Fig. 4 we show results for the fine-tuned point \( \gamma = \gamma_k = \tilde{\gamma}_{-k} \). The cusps remain clearly visible for finite dissipation rates. For a \( k \) independent
transitions which are based on thermal averages over the Loschmidt echos of pure states and are only applicable to unitary dynamics.

For bilinear one-dimensional fermionic lattice models with periodic boundary conditions we have shown that finite temperatures always wash out the non-analyticities in the return rate of the generalized Loschmidt echo. Dynamical phase transitions only exist at zero temperature.

For open quantum systems described by a Lindblad master equation we similarly find that particle loss processes smooth out cusps in the return rate so that signatures of the dynamical phase transition are hard to detect even if the loss rates are very small.

Finally, we showed that it is possible to fine-tune particle loss and creation processes in such a way that dynamical phase transitions can be observed despite the dissipative dynamics.

The generalized Loschmidt considered in this paper can be understood as a tool to measure distances between density matrices. As such it might be helpful in engineering and controlling specific states using dissipative dynamics. Zeros of the Loschmidt echo signal, in particular, that a mixed state has been reached such that all purifications to states in an enlarged Hilbert space are orthogonal to purifications of the initial state.

Shortly after submitting this paper Ref. 42 became available, which is on a related topic.

V. CONCLUSIONS

We have studied a generalization of the Loschmidt echo to density matrices which is applicable both to finite temperatures and to open systems. It is based on a direct generalization of the fidelity for mixed states to dynamical problems and provides a measure of the distance between the initial and the time-evolved density matrix. As such it is very different from previous generalizations studied in the context of dynamical phase transitions which are hard to detect.

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