Free geometric equations for higher spins

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Abstract

We show how allowing non-local terms in the field equations of symmetric tensors uncovers a neat geometry that naturally generalizes the Maxwell and Einstein cases. The end results can be related to multiple traces of the generalized Riemann curvatures \( R_{\alpha_1 \cdots \alpha_s \beta_1 \cdots \beta_s} \) introduced by de Wit and Freedman, divided by suitable powers of the D’Alembertian operator \( \Box \). The conventional local equations can be recovered by a partial gauge fixing involving the trace of the gauge parameters \( \Lambda_{\alpha_1 \cdots \alpha_{s-1}} \), absent in the Fronsdal formulation. The same geometry underlies the fermionic equations, that, for all spins \( s + 1/2 \), can be linked via the operator \( \gamma \) to those of the spin-\( s \) bosons.

( June, 2002 )

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1. Introduction and summary

String Theory has long been held by several authors to correspond to a broken phase of a higher-spin gauge theory, a viewpoint clearly suggested, for instance, by the BRST formulation of free String Field Theory, that encodes infinitely many higher-spin symmetries in the Stueckelberg mode. However, String Theory presents some clear simplifications with respect to unbroken higher-spin theories, well reflected in the familiar option of associating to large-scale phenomena a low-spin low-energy effective description. This is a general feature of spontaneously broken gauge theories, quite familiar from simpler examples: for instance, differently from Q.C.D., at low energies the electro-weak theory reduces to a low-spin theory with a local Fermi coupling, that for many years has been at the heart of weak interaction phenomenology. On the other hand, it is in Q.C.D. that gauge theory comes to full power, with remarkable infrared phenomena responsible for quark confinement. Even more striking dynamics can thus be expected from these complicated systems, and this is by itself an important motivation to try to gain some familiarity with them.

Free covariant equations for fully symmetric tensors and tensor-spinors were first constructed in the late seventies by Fronsdal and Fang and Fronsdal, starting from the massive equations of Singh and Hagen. These are interesting classes of higher-spin gauge fields, that in four dimensions exhaust all available possibilities, up to dualities, and have the clear advantage of allowing rather simple unified descriptions. Following an important observation of the Göteborg group, that showed how a proper cubic flat-space vertex could be found for higher spins, Fradkin and Vasiliev have led for many years the search for an extension of the free equations to consistent interacting gauge theories of higher spins. Arguments related to the gauge algebra imply that these are bound to involve infinitely many gauge fields of increasing spins, and in the early nineties Vasiliev finally arrived at closed-form dynamical equations for symmetric tensors of arbitrary rank in mutual interaction, but an action principle is still lacking for this complicated system. A crucial input in the constructions of Fradkin and Vasiliev was the inclusion of a cosmological term, that allowed to cancel recursively contributions generated by higher-spin gauge transformations depending on the space-time Weyl tensor, thus bypassing the difficulties met in earlier attempts. Various aspects of the work of Vasiliev and collaborators are reviewed in, while recent, related work is described in.

A peculiar feature of the Fang-Fronsdal equations is the need for unusual constraints, so that, for instance, the bosonic gauge parameters are to be traceless, while the corresponding gauge fields are to be doubly traceless. These constraints manifest themselves as symmetry conditions in the spinor formalism, but appear less natural in the usual component
notation. This letter is thus devoted to showing how one can formulate the dynamics of symmetric tensors and tensor-spinors while foregoing the restrictions implicit in the Fang-Fronsdal equations. One can well work in generic space-time dimensions, with the proviso that for $d > 4$ these fields do not exhaust all available possibilities. The end result is rather amusing, since the free equations contain non-local terms whenever the gauge fields have more than a pair of symmetric Lorentz indices, i.e. in all cases beyond the familiar Maxwell and Einstein examples. However, all non-local terms can be eliminated by a partial gauge fixing using the trace (or, for fermions, the $\gamma$-trace) of the gauge parameter, that reduces the geometric equations to the Fang-Fronsdal form. This analysis will bring us naturally to consider, following de Wit and Freedman [11], higher-spin generalizations of the Christoffel connection, $\Gamma_{\alpha_1 \ldots \alpha_{s-1} ; \beta_1 \ldots \beta_s}$, and of the Riemann curvature, $\mathcal{R}_{\alpha_1 \ldots \alpha_s ; \beta_1 \ldots \beta_s}$, that are totally symmetric under the interchange of any pair of indices within the two sets. In terms of these quantities, the gauge invariant bosonic field equations will be

\[
\frac{1}{\Box^p} \partial \cdot \mathcal{R}^{[p]}_{\alpha_1 \ldots \alpha_{2p+1}} = 0 \quad (1)
\]

for odd spins $s = 2p + 1$, and

\[
\frac{1}{\Box^{p-1}} \mathcal{R}^{[p]}_{\alpha_1 \ldots \alpha_{2p}} = 0 \quad (2)
\]

for even spins $s = 2p$. Here and in the following, a superscript $[p]$ denotes a $p$-fold trace, while $\partial \cdot$ denotes a divergence, but for the sake of brevity low-order traces will be occasionally denoted by “primes”. Moreover, we shall work throughout with a “mostly positive” Minkowski metric.

The analogy with the Maxwell and Einstein cases should be evident, and it is rather pleasing to see a simple pattern extending to all higher-rank tensors. Let us stress that all these equations are manifestly invariant under gauge transformations without any constraints on the gauge fields or on the corresponding gauge parameters and that, after a partial gauge fixing, they can all be reduced to the conventional, local, Fronsdal form.

This geometric form also results in fermionic equations that are closely related to the bosonic ones. In general, the spin-$(s + 1/2)$ fermionic equations can be formally recovered from the spin-$s$ bosonic operators, properly multiplied by $\partial \cdot$, and therefore the geometry underlying the bosonic equations plays a similar, albeit more indirect, role also in the fermionic case. It is amusing to illustrate right away this fact, obvious for the Dirac equation, for a less evident case, the Rarita-Schwinger equation for spin $3/2$, that is quite familiar from supergravity. This is usually written in the form

\[
\gamma^{\mu
u} \partial_\nu \psi_p = 0 \quad , \quad (3)
\]

The double trace condition, however, can be related to the $\text{OSp}(D - 1, 1/2)$ structure of the corresponding system with ghosts. We are grateful to W. Siegel for calling this fact to our attention.
but once combined with its $\gamma$-trace, it becomes

$$\bar{\partial} \psi_\mu - \partial_\mu \psi = 0.$$  \hspace{1cm} (4)

The connection with the Maxwell equation that we are advertising can be exhibited combining it again with its $\gamma$ trace, now multiplied with $\frac{\partial_\mu \partial}{\Box}$, and the end result is indeed

$$\bar{\partial} \left( \Box \psi_\mu - \partial_\mu \partial \cdot \psi \right) = 0.$$  \hspace{1cm} (5)

In section 2 we begin by examining the field equation for spin 3, and we show how to extend the Fronsdal formulation to fully gauge-invariant, albeit non-local, forms, and how to relate the latter to local forms involving Stueckelberg fields with higher-derivative terms. In section 3 we show how one can define via an iterative procedure kinetic operators for all higher spins, derive their Bianchi identities and, making direct use of them, construct corresponding Einstein-like tensors. In section 4 we recover these equations from the geometric notions of connection and curvature for higher-spin gauge fields, originally introduced by de Wit and Freedman [11]. While in [11] the authors linked the local Fang-Fronsdal equations to traces of one and two-derivative connections, the full geometric equations presented here are recovered if one insists on resorting to the connection $\Gamma_{\alpha_1 \cdots \alpha_{s-1} \beta_1 \cdots \beta_s}$ and to the corresponding curvature, that for a spin-$s$ field contain, respectively, $s - 1$ and $s$ derivatives. Whereas unconventional, these are natural ingredients of higher-spin kinetic operators, that in general should contain both the D’Alembertian operator $\Box$ and additional terms with up to $s$ free derivatives. Hence, the non-local structure exposed here is unavoidable in our fully covariant setting. In addition, it anticipates similar properties of the higher-spin interactions. It is conceivable, although by no means clear to the authors at the time of this writing, that corresponding simplifications could take place if the equations of [11] were formulated along these lines. A related observation is that the BRST charge of world-sheet reparametrizations, that lies at the heart of String Field Theory, embodies a massive dynamics of the Fronsdal type, some aspects of which are manifest in the constructions of [12], that therefore bear a direct relationship to the present work, although the field equations are presented there in a local form with compensators that does not exhibit their link with the curvatures.

2. The spin-3 case

Let us begin by describing the spin-3 Fronsdal equation [3], that for the sake of brevity we shall write in the form

$$\mathcal{F}_{123} = 0,$$  \hspace{1cm} (6)
where

\[ F_{123} \equiv \Box \phi_{123} - (\partial_1 \partial \cdot \phi_{23} + \partial_2 \partial \cdot \phi_{13} + \partial_3 \partial \cdot \phi_{12}) + \partial_1 \partial_2 \phi_3' + \partial_1 \partial_3 \phi_2' + \partial_2 \partial_3 \phi_1' , \]  

exposing only the subscripts of the three Lorentz indices involved. A gauge transformation of the spin-3 field \( \phi_{123} \),

\[ \delta \phi_{123} = \partial_1 \Lambda_{23} + \partial_2 \Lambda_{31} + \partial_3 \Lambda_{12} , \]

transforms \( F \) according to

\[ \delta F_{123} = 3 \partial_1 \partial_2 \partial_3 \Lambda' , \]

and therefore, as is well known, \( F \) is gauge invariant only if the parameter is subject to the constraint

\[ \Lambda' = 0 . \]

An additional subtlety, already met in the spin-2 case, is that (6) does not follow directly from a Lagrangian. In order to proceed, one must therefore introduce an analogue of the linearized Einstein tensor,

\[ G_{123} = F_{123} - \frac{1}{2} ( \eta_{12} F_{3}' + \eta_{23} F_{1}' + \eta_{31} F_{2}' ) , \]

where \( \eta \) denotes the Minkowski metric. The Bianchi identity

\[ \partial \cdot F_{23} = \frac{1}{2} ( \partial_2 F_3' + \partial_3 F_2' ) , \]

then implies that

\[ \partial \cdot G_{23} = - \frac{1}{2} \eta_{23} \partial \cdot F' , \]

and together with eq. (10) this result is instrumental in deriving a gauge-invariant Lagrangian for this system, since

\[ \partial \cdot F' = 3 \Box \partial \cdot \phi' - 2 \partial \cdot \partial \cdot \partial \cdot \phi \]

does not vanish identically. Integrating

\[ \delta L = \delta \phi^{123}_2 \cdot F_{123} \]

one can finally recover the Fronsdal action

\[ L = - \frac{1}{2} ( \partial_\mu \phi_{123} )^2 + \frac{3}{2} ( \partial \cdot \phi_{12} )^2 + \frac{3}{2} ( \partial_\mu \phi_1' )^2 + \frac{3}{4} ( \partial \cdot \phi' )^2 + 3 \phi_1' \partial \cdot \partial \cdot \phi' . \]
Our aim is now to extend the gauge symmetry, modifying the kinetic operator \( F_{123} \), by itself a sort of connection for the trace of the original gauge parameter. This case is simple enough to arrive quickly at a fully gauge invariant equation, for instance

\[
F_{123} - \frac{1}{3} \Box \left( \partial_1 \partial_2 F_3' + \partial_2 \partial_3 F_1' + \partial_3 \partial_1 F_2' \right) = 0 .
\]  

(17)

As in the Fronsdal case, one can then define an Einstein-like tensor \( G_{123} \) and arrive at

\[
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \phi_{123})^2 + \frac{3}{2} (\partial \cdot \phi_{12})^2 + \phi_1' \partial \cdot \partial \cdot \phi_1 - \frac{3}{2} (\partial \cdot \phi')^2 \\
+ \frac{1}{2} (\partial_{\mu} \phi_1')^2 + \partial \cdot \partial \cdot \phi_1 \frac{1}{3} \Box \partial \cdot \partial \cdot \phi + \partial \cdot \partial \cdot \partial \cdot \phi \frac{1}{3} \partial \cdot \phi' .
\]  

(18)

It is also possible to recast the Lagrangian in a local form, introducing a Stueckelberg field \( \varphi \), such that

\[ \delta \varphi = \Lambda', \]  

(19)

but, as we shall see, the non-local forms will turn out to underlie an interesting structure. At any rate, this compensator allows one to construct the two gauge invariant expressions

\[
\partial_{\mu} \varphi - \phi_{\mu}' + \frac{1}{3} \Box \partial \cdot \partial \cdot \varphi_{\mu} - \frac{1}{3} \partial_{\mu} \partial \cdot \partial \cdot \phi ,
\]  

(20)

\[
\Box \varphi + \frac{2}{3} \partial \cdot \partial \cdot \phi - \partial \cdot \phi' ,
\]  

(21)

and adding suitable combinations of these to \( \mathcal{L} \) finally yields the local Lagrangian

\[
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \phi_{123})^2 + \frac{3}{2} (\partial \cdot \phi_{12})^2 + 3 \phi_1' \partial \cdot \partial \cdot \phi_1 + \frac{3}{4} (\partial \cdot \phi')^2 \\
+ \frac{3}{2} (\partial_{\mu} \phi_1')^2 - \frac{9}{2} \varphi \Box \partial \cdot \phi' + 3 \varphi \partial \cdot \partial \cdot \phi + \frac{9}{4} \varphi \Box^2 \varphi ,
\]  

(22)

that, differently from (16), is invariant under gauge transformations with an \textit{unconstrained} parameter.

It is interesting to notice, however, that this fully gauge invariant equation is not unique, another possibility being

\[
F_{123} - \frac{\partial_1 \partial_2 \partial_3}{\Box^2} \partial \cdot F' = 0 ,
\]  

(23)

that can actually be obtained combining eq. (17) with its trace. The corresponding non-local Lagrangian

\[
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \phi_{123})^2 + \frac{3}{2} (\partial \cdot \phi_{12})^2 + 3 \phi_1' \partial \cdot \partial \cdot \phi_1 - \partial \cdot \partial \cdot \partial \cdot \phi \frac{1}{3} \Box \partial \cdot \partial \cdot \phi \\
+ 3 \partial \cdot \partial \cdot \partial \cdot \phi \frac{1}{3} \partial \cdot \phi' + \frac{3}{2} (\partial_{\mu} \phi_1')^2 - \frac{3}{2} (\partial \cdot \phi')^2
\]  

(24)
can also be brought to a local form, making use again of the compensator $\varphi$. The end result, obtained adding to (24) the square of the gauge-invariant expression (21), is again, not surprisingly, the local Lagrangian (22). Notice that, under a gauge transformation with generic parameter $\Lambda_{123}$,

$$\delta \left( \frac{\partial \cdot F'}{\Box^2} \right)_{\mu} = 3 \Lambda'_{\mu},$$

(25)

and therefore the form (23) of the geometric equation makes it rather transparent that the trace of the gauge parameter suffices to bring it to the local Fronsdal form.

3. **Kinetic operators for spin-$s$ bosons**

It is possible to extend the results of the previous section to symmetric tensors of arbitrary spin. To this end, it is quite convenient to introduce a shorthand notation that eliminates the need for explicit indices. A generic spin-$s$ tensor will be denoted simply by $\phi$, while derivatives, divergences and traces will be denoted by $\partial \phi$, $\partial \cdot \phi$ and $\phi'$ (or, more generally, $\phi^{[p]}$), respectively, with the understanding that in all cases the implicit indices are totally symmetrized. With this proviso, one can see that the somewhat unconventional rules

$$\left(\partial^p \phi\right)' = \Box \partial^{p-2} \phi + 2 \partial^{p-1} \partial \cdot \phi + \partial^p \phi'$$

(26)

$$\partial^p \partial^q = \begin{pmatrix} p + q \\ p \end{pmatrix} \partial^{p+q}$$

(27)

$$\partial \cdot (\partial^p \phi) = \Box \partial^{p-1} \phi + \partial^p \partial \cdot \phi$$

(28)

$$\partial \cdot \eta^k = \partial \eta^{k-1},$$

(29)

hold. For instance, a special case of (29) is

$$\partial \cdot \eta^2 \equiv \partial_1 (\eta_{12} \eta_{34} + \eta_{13} \eta_{24} + \eta_{14} \eta_{23}) = (\partial_2 \eta_{34} + \partial_3 \eta_{24} + \partial_4 \eta_{23}) \equiv \partial \eta,$$

(30)

and the advantages of the compact notation should be evident.

The gauge transformation of the spin-$s$ field then reads

$$\delta \phi = \partial \Lambda,$$

(31)

while the generic spin-$s$ Fronsdal equation becomes

$$\mathcal{F} = \Box \phi - \partial \partial \cdot \phi + \partial^2 \phi',$$

(32)
whose gauge variation is
\[ \delta F = 3 \partial^3 \Lambda'. \] (33)

The spin-\(s\) Fronsdal operator satisfies in general the “anomalous” Bianchi identity
\[ \partial \cdot F - \frac{1}{2} \partial F' = -\frac{3}{2} \partial^3 \phi'', \] (34)
where the difference with respect to the spin-3 case should be noted, and as a result one can define the Einstein-like tensor
\[ G = F - \frac{1}{2} \eta F', \] (35)
such that
\[ \partial \cdot G = -\frac{3}{2} \partial^3 \phi'' - \frac{1}{2} \eta \partial \cdot F'. \] (36)
This relation is at the heart of the usual restrictions, present in the Fronsdal formulation, to traceless gauge parameters and doubly traceless fields, needed to ensure that
\[ \delta \mathcal{L} = \delta \phi \, G \] (37)
vanish if \(\delta \phi\) is given by eq. (31).

One can now define recursively a sequence of kinetic operators, as
\[ F^{(n+1)} = F^{(n)} + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\Box} F^{(n)} - \frac{1}{n+1} \frac{\partial}{\Box} \partial \cdot F^{(n)}, \] (38)
where \(F^{(1)} = F\), and an inductive argument then shows that
\[ \delta F^{(n)} = (2n+1) \frac{\partial^{2n+1}}{\Box^{n+1}} \Lambda^{[n]}, \] (39)
where, as anticipated, \(\Lambda^{[n]}\) denotes the \(n\)-fold trace of the gauge parameter \(\Lambda\). Notice that this is only available for spin \(s > 2n + 1\), and therefore this procedure yields a gauge-invariant kinetic operator after a certain number of iterations.

If, as in [13], the gauge field \(\phi_{1\ldots s}\) is contracted with a vector \(\xi\), it is simple to convince oneself that traces and divergences of the resulting expression
\[ \hat{\Phi}(x, \xi) = \frac{1}{s!} \xi^1 \cdots \xi^s \phi_{1\ldots s} \] (40)
can be recovered applying to it the differential operators \(\partial \xi \cdot \partial\) and \(\partial \xi \cdot \partial\), where \(\partial \xi\) denotes a derivative with respect to \(\xi\). The least singular non-local field equations obtained from the Fronsdal term
\[ \hat{\mathcal{F}}(\hat{\Phi}) = \left[ \Box - \xi \cdot \partial \partial \cdot \partial \xi + (\xi \cdot \partial)^2 \partial \xi \cdot \partial \xi \right] \hat{\Phi}, \] (41)
by successive iterations can then be written in the compact form
\[ \prod_{k=0}^{n-1} \left[ 1 + \frac{1}{(k+1)(2k+1)} \frac{(\xi \cdot \partial)^2}{\Box} \partial_\xi \cdot \partial_\xi - \frac{1}{k+1} \frac{\xi \cdot \partial}{\Box} \partial_\xi \cdot \partial \right] \hat{\mathcal{F}}(\hat{\Phi}) = 0 , \quad (42) \]

where for spin \( s \) a fully gauge invariant operator is first obtained after \( \left\lceil \frac{s+1}{2} \right\rceil \) iterations. Expanding this expression and combining it with its trace it is possible to show that the field equations can all be reduced to the form
\[ \mathcal{F} = \partial^3 \mathcal{H} , \quad (43) \]

where under a gauge transformation \( \delta \mathcal{H} = 3 \Lambda' \), and therefore the local form (41) can always be recovered from (42) making use of the trace of the gauge parameter \( \Lambda \).

These kinetic operators satisfy the “anomalous” Bianchi identities
\[ \partial \cdot \mathcal{F}^{(n)} - \frac{1}{2n} \partial \mathcal{F}^{(n)} = - \left( 1 + \frac{1}{2n} \right) \frac{\partial^{2n+1}}{\Box^{n-1}} \phi^{(n+1)} , \quad (44) \]
that generalize eq. (12). This result can be also justified by an inductive argument, and implies similar relations for successive traces of the \( \mathcal{F}^{(n)} \),
\[ \partial \cdot \mathcal{F}^{(n)}[k] - \frac{1}{2(n-k)} \partial \mathcal{F}^{(n)}[k+1] = 0 , \quad (k \leq n-1) \quad (45) \]
here written for \( n \) large enough so that the “anomaly” on the r.h.s. of (44) vanishes identically. Notice that for odd spin \( s = 2n - 1 \) the second term vanishes for the last trace, so that
\[ \partial \cdot \mathcal{F}^{(n)}[n-1] = 0 . \quad (46) \]

These generalized Bianchi identities suffice to define for all spin-\( s \) fields fully gauge invariant analogues of the Einstein tensor,
\[ \mathcal{G}^{(n)} = \sum_{p \leq n} \frac{(-1)^p}{2^p \ p! \ \binom{n}{p}} \eta^p \mathcal{F}^{(n)}[p] \quad (47) \]
that, for \( n \) large enough, have vanishing divergence like their spin-2 counterpart. This is attained directly by the subtractions for all even spins, while for odd spins the last term vanishes on account of (16). From \( \mathcal{G}^{(n)} \), integrating eq. (37) one can then construct generalized Lagrangians that are fully gauge invariant without any restrictions on the gauge fields or on the gauge parameters.
4. Geometric forms of the spin-s field equations

Following [11], one can define generalized connections of various orders in the derivatives for all spin-s gauge fields. This can be done by an iterative procedure, so that, in the compact notation of the previous section, for any field of spin s after m iterations one can define

\[ \Gamma^{(m)} = \frac{1}{m + 1} \sum_{k=0}^{m} \left( \frac{(-1)^k}{(m-k)} \right) \partial^{m-k} \nabla^{k} \phi , \]

(48)

where we are now using two types of derivatives for two sets of symmetrized indices, \( \partial \) for the s symmetric indices \((\beta_1 \cdots \beta_s)\) and \( \nabla \) for the other \( m \) symmetric ones \((\alpha_1 \cdots \alpha_m)\). It is simple to show, by an inductive argument, that the gauge transformation of \( \Gamma^{(m)} \) is

\[ \delta \Gamma^{(m)} = \partial^{m+1} \Lambda , \]

(49)

where all \( m \) indices of the first set are within the gauge parameter. Hence,

\[ \Gamma^{(s-1)} = \frac{1}{s} \sum_{k=0}^{s-1} \left( \frac{(-1)^k}{(s-k)} \right) \partial^{s-k-1} \nabla^{k} \phi , \]

(50)

is the proper analogue of the Christoffel connection for a spin-s gauge field, since its gauge transformation contains a single term. That these objects can be defined in general can be also recognized noticing that the spin-s gauge variation of eq. (31) and the rules of symmetric calculus of the previous section imply that

\[ \delta \left( \partial^{s-1} \phi \right) = s \partial^{s} \Lambda , \]

(51)

and therefore one can in principle retrieve a composite connection \( \Gamma_{\alpha_1 \cdots \alpha_{s-1};\beta_1 \cdots \beta_s} \) such that

\[ \delta \Gamma_{\alpha_1 \cdots \alpha_{s-1};\beta_1 \cdots \beta_s} = \partial_{\beta_1} \cdots \partial_{\beta_s} \Lambda^{\alpha_1 \cdots \alpha_{s-1}} \]

(52)

inverting the linear system

\[ \partial^{s-1} \phi = \Gamma_{\{s-1\};\{s\}} , \]

(53)

with \( \binom{2s-1}{s} \) unknowns, a higher-derivative analogue of the linearized metric postulate for Einstein gravity. Moreover, all \( \Gamma \)'s with \( m > s \) are gauge invariant, and in particular

\[ \Gamma^{(s)} = \frac{1}{s + 1} \sum_{k=0}^{s} \left( \frac{(-1)^k}{(s)} \right) \partial^{s-k} \nabla^{k} \phi , \]

(54)

is the proper analogue of the Riemann curvature tensor. This generalized curvature \( R_{\alpha_1 \cdots \alpha_s;\beta_1 \cdots \beta_s} \) is totally symmetric under the interchange of any two indices within the two sets. In addition, as shown in [11],

\[ R_{\alpha_1 \cdots \alpha_s;\beta_1 \cdots \beta_s} = (-1)^s R_{\beta_1 \cdots \beta_s;\alpha_1 \cdots \alpha_s} , \]

(55)
and a generalized cyclic identity holds. These concepts can also be related to an interesting generalization of the exterior differential, whereby the familiar condition $d^2 = 0$ is replaced by $d^{s+1} = 0$\footnote{12}.

There is another, perhaps more obvious way, to generate a gauge invariant quantity from a connection $\Gamma^{(s-1)}$ that transforms as in (52), taking a curl with respect to any of its $\beta$ indices. However, the choice of $\Gamma^{(s-1)}$ has the virtue of simplicity, since it results automatically in a tensor with two totally symmetric sets of indices. If we now restrict our attention to the $\Gamma$’s with $m$ even, and for the sake of clarity let $m = 2n$, eq. (33) implies that the total trace of $\Gamma^{(2n)}$ over pairs of $\beta$ indices, $\Gamma^{(2n)[n]}$, is in general a totally symmetric spin-$s$ tensor such that

$$\delta \left( \frac{1}{\Box^{n-1}} \Gamma^{(2n)[n]} \right) = \partial^{2n+1} \Lambda^{[n]} .$$

(56)

Up to an overall proportionality constant, this is exactly the gauge transformation of our $\mathcal{F}^{(n)}$, the corrected kinetic operators for spin-$s$ gauge fields, and in particular if $s = 2n$ $\Gamma^{(2n)[n]}$ is gauge invariant and proportional to the spin-$s$ analogue of the Riemann tensor defined above. This therefore means that the iterative procedure of the previous section is actually providing a rôle for the higher-spin connections of \cite{14}, so that the geometric gauge-invariant equations for even spin $s = 2n$ can be written in the form

$$\frac{1}{\Box^{n-1}} \mathcal{R}^{[n]:\mu_1 \cdots \mu_{2n}} = 0 ,$$

(57)

a natural generalization of the Einstein equation.

The odd-spin case $s = 2n+1$ presents a further minor subtlety, in that the corresponding curvatures $\Gamma^{(2n+1)}$ have an odd number of $\beta$ indices. The simplest option is in this case to take a trace over $n$ pairs of $\beta$ indices in $\Gamma^{(2n+1)}$ and a divergence over the remaining one. The end result for spin $s = 2n + 1$ is then

$$\frac{1}{\Box^{n}} \partial \cdot \mathcal{R}^{[n]:\mu_1 \cdots \mu_{2n+1}} = 0 ,$$

(58)

in complete analogy with the Maxwell case. Notice that the Maxwell and Einstein cases are the only ones when these geometric equations are local, while the Fronsdal operators provide local, albeit partly gauge fixed, forms for them. As anticipated in the previous sections, these are the least singular fully gauge invariant kinetic operators, while more singular forms can be obtained combining eqs. (57) and (58) with their traces, as we saw in section 2.
5. Fermionic equations

One can also arrive at similar non-local geometric equations for fermion fields. In this case the local equations of \[ S \equiv i (\not\partial \psi - \partial \not\psi) = 0 \] (59) are gauge invariant under \[ \delta \psi = \partial \epsilon \] (60) only if the gauge parameter is subject to the constraint \[ \not\epsilon = 0 \] (61).

In addition, \( S \) satisfies the “anomalous” Bianchi identity \[ \partial \cdot S - \frac{1}{2} \partial S' - \frac{1}{2} \not\partial S = i \partial^2 \psi' , \] (62) and therefore the gauge variation of the generic Lagrangian \[ \delta \mathcal{L} = \delta \bar{\psi} \left[ S - \frac{1}{2} \left( \eta S' + \gamma S \right) \right] \] (63) vanishes only if \[ \psi'' = 0 \] (64), the fermionic analogue of the double trace condition for boson fields.

It is convenient to notice that the fermionic operators for spin \( s + 1/2 \) are related to the corresponding bosonic operators for spin \( s \) according to \[ S_{s+1/2} - \frac{1}{2} \not\square \not\partial S_{s+1/2} = i \not\partial \mathcal{F}_s(\psi) . \] (65)

This amusing link generalizes the obvious one between the Dirac and Klein-Gordon operators, and actually extends to their non-local counterparts. Hence, it allows one to relate corrected fermionic kinetic operators \( S^{(n)} \), defined recursively as \[ S^{(n+1)} = S^{(n)} + \frac{1}{n(2n+1)} \not\square \partial^2 S^{(n)}' - \frac{2}{2n+1} \not\square \partial \cdot S^{(n)} \] (66) and such that \[ \delta S^{(n)} = - 2 i n \not\square \partial^{2n} \not\epsilon^{[n-1]} \] (67) to the corresponding corrected bosonic operators of section 3, according to \[ S_{s+1/2}^{(n)} - \frac{1}{2n} \not\square \not\partial S_{s+1/2}^{(n)} = i \not\partial \mathcal{F}_s^{(n)}(\psi) . \] (68)
This relation also determines the “anomalous” Bianchi identities of the $S^{(n)}$,

$$\partial \cdot S^{(n)} - \frac{1}{2n} \partial S^{(n)} - \frac{1}{2n} \vartheta S^{(n)} = i \frac{\partial^{2n}}{\square_{n-1}} \psi^{[n]},$$

and therefore the corrected Einstein-like operators

$$G^{(n)} = S^{(n)} + \sum_{0 < p \leq n} \frac{(-1)^p}{2^p p!} \left( \binom{n}{p} \eta^{p-1} \left[ \eta S^{(n)} [p] + \gamma S^{(n)} [p-1] \right] \right).$$

The geometry underlying the bosonic case thus bears a close, if less direct, relation to the fermionic operators $S^{(n)}$, that can also be retrieved from the iterated bosonic terms $\mathcal{F}^{(n)}$.

**Acknowledgments**

It is a pleasure to thank M. Vasiliev for stimulating conversations. This work was supported in part by I.N.F.N., by the EC contract HPRN-CT-2000-00122, by the EC contract HPRN-CT-2000-00148, by the INTAS contract 99-1-590 and by the MURST-COFIN contract 2001-025492. D.F. was supported by an I.N.F.N. Pre-Doctoral Fellowship. This work was presented at the Third International Sakharov Conference, Moscow, June 24-29, 2002, held at the Lebedev Institute, that the authors would like to thank for the kind hospitality.

**References**

[1] W. Siegel and B. Zwiebach, Nucl. Phys. B263 (1986) 105; T. Banks and M. E. Peskin, Nucl. Phys. B264 (1986) 513; E. Witten, Nucl. Phys. B268 (1986) 253. For a review see, for instance, W. Siegel, [arXiv:hep-th/0107094].

[2] C. Fronsdal, Phys. Rev. D18 (1978) 3624; T. Curtright, Phys. Lett. B85 (1979) 219.

[3] J. Fang and C. Fronsdal, Phys. Rev. D18 (1978) 3630.

[4] L. P. Singh and C. R. Hagen, Phys. Rev. D9 (1974) 898, Phys. Rev. D9 (1974) 910.

[5] A. K. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B227 (1983) 31.

[6] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. B291 (1987) 141, Annals Phys. 177 (1987) 63, Phys. Lett. B189 (1987) 89, JETP Lett. 44 (1986) 622 [Pisma Zh. Eksp. Teor. Fiz. 44 (1986) 484], Int. J. Mod. Phys. A3 (1988) 2983.
[7] M. A. Vasiliev, Phys. Lett. B243 (1990) 378, Class. Quant. Grav. 8 (1991) 1387, Phys. Lett. B257 (1991) 111, Phys. Lett. B285 (1992) 225.

[8] C. Aragone and S. Deser, Phys. Lett. B86 (1979) 161; F. A. Berends, J. W. van Holten, B. de Wit and P. van Nieuwenhuizen, in C79-02-25.5 J. Phys. A13 (1980) 1643.

[9] For recent reviews see: M. A. Vasiliev, Int. J. Mod. Phys. D5 (1996) 763 [arXiv:hep-th/9611024], Nucl. Phys. Proc. Suppl. 56B (1997) 241, [arXiv:hep-th/9910096], [arXiv:hep-th/0104246].

[10] E. Sezgin and P. Sundell, [arXiv:hep-th/0205131], [arXiv:hep-th/0205132], [arXiv:hep-th/0112100], supergravity,” JHEP 0109 (2001) 025 [arXiv:hep-th/0107186].

[11] B. de Wit and D. Z. Freedman, Phys. Rev. D21 (1980) 358; T. Damour and S. Deser, Annales Poincare Phys. Theor. 47 (1987) 277.

[12] V. E. Lopatin and M. A. Vasiliev, Mod. Phys. Lett. A 3 (1988) 257; M. A. Vasiliev, Nucl. Phys. B 301 (1988) 26; S. Deser and A. Waldron, Nucl. Phys. B 607 (2001) 577 [arXiv:hep-th/0103198]; I. L. Buchbinder, A. Pashnev and M. Tsulaia, Phys. Lett. B 523 (2001) 338 [arXiv:hep-th/0109067], and references therein.

[13] V. Bargmann and I. T. Todorov, J. Math. Phys. 18 (1977) 1141.

[14] M. Dubois-Violette and M. Henneaux, Lett. Math. Phys. 49 (1999) 245 [arXiv:math.qa/9907133], [arXiv:math.qa/0110088].