MATHEMATICAL PROBLEMS IN HIGHER ORDER GRAVITY AND COSMOLOGY

S. COTSAKIS

Department of Mathematics
University of the Aegean
Karlovassi 83200, Samos
Greece

We discuss the issue of motivating the analysis of higher order gravity theories and their cosmologies and introduce a rule which states that these theories may be considered as a vehicle for testing whether certain properties may be of relevance to quantum theory.

We discuss the physicality issue arising as a consequence of the conformal transformation theorem, the question of formulating a consistent first order formalism of such theories and also the isotropization problem for a class of generalized cosmologies. We point out that this field may have an important role to play in clarifying issues arising also in general relativity.

1 Motivation.

We consider theories of gravity of the general nonlinear lagrangian type

\[ L_g = f(r), \]  

where \( r = R, \) \( \text{Ric}^2, \) \( \text{Riem}^2, \) called collectively \textit{higher order gravity theories (HOG)}. Doing cosmology with such actions leads to \textit{higher derivative cosmologies (HDCs)}.

The field equations arising from these \( f(r) \)-actions are typically of fourth order. Motivating the study of such systems is non-trivial and indeed there have been mixed feelings in the literature concerning this issue. Among the often quoted virtues of adopting such actions include the fact that they could constitute a first approximation to a non-existent quantum theory of gravity, represent a kind of unifying theory, and also cure ‘problems’ of GR/Cosmology such as providing better singularity behaviour (singularity avoidance) and/or better late-time cosmological behaviour. In fact, there is no rule to force the form of the gravitational action to be necessarily of the usual Einstein–Hilbert type. Adopting this philosophy, one typically ends up with a larger solution space than that of GR. This in turn raises several issues, for instance that of understanding the precise relation between the two solution spaces and also the structure near the singular (conformal–see below) boundaries.

On the other hand, there are several remarks one could make on the negative side of things. Here are some examples: Some authors simply hold that such actions lead to unphysical choises for the gravitational field and as such they cannot be \textit{seriously} considered as a viable approximation to quantum theory. Moreover, it may be thought that the whole issue of assuming such a drastic alteration in the form of the gravitational action is not a ‘burning’ one in view of the fact that, for example, strictly speaking the issue of singularities in GR in not a closed one and
one should clarify this and related matters first. Lastly, it should be pointed out that the initial value problem is not well–posed for some theories.

To proceed, we introduce the following alternative motive for considering such theories which we call the Universal Admittance Rule: If a certain property is valid in GR and in all other physically interesting theories then this may well be an indispensable ingredient of a more fundamental theory (eg, Quantum Gravity).

We believe that under the above rule it becomes meaningful to consider this kind of variants to GR and indeed one may view such alternative theories and their cosmologies as a more friendly, testing ground for discovering which properties may prove to be truly fundamental. Such ‘properties’ may be black hole entropy, inflation, isotropization property, recollapsing property, questions of stability in cosmology, hamiltonian structures/principles, etc.

2 Physicality Issue.

We start with the conformal equivalence theorem cf. 1. This states that any of these higher order systems may be regarded as GR with additional fields in conformal space. For example, taking the starting lagrangian to be an analytic function of the scalar curvature, \( f(R) \), and performing a conformal transformation one obtains

\[
f(R) + T_m \overset{\tilde{g}=\frac{\Omega^2}{g}}{\Rightarrow} \tilde{R} + T_\phi + \tilde{T}_m.
\]

An immediate advantage of this result is that we have cast the original system in the form of a symmetric hyperbolic system which is easier to analyze. However, since there are now two metrics on \( M \) the question naturally arises as to which is the physical metric among \( g, \tilde{g} \). The following result shows that in certain cases only one of these metrics may be the true one 2.

Theorem. \( \tilde{g} \) is always the physical metric for certain manifolds provided

\[
\nabla_a \nabla_b \phi = 0.
\]

The proof consists in constructing the types of admissible manifolds by introducing and exploiting the consequences of a generalized form of a theorem due to Bochner (cf. 3). It also involves an analysis of the behaviour of spacetime metrics with \( \text{Ric} < 0 \) which is, in general, a very delicate, subtle, interesting and open (not completely settled even for Riemannian manifolds–not to mention spacetimes) question.

3 Constrained Variations and Conformal Structure.

Developing a first order formalism for higher order systems of the form discussed above is not a closed issue and in fact these methods (and more generally those of a metric–connection type) may provide us with an alternative to reduction of order. We have recently shown that the field equations obtained from varying the most general, pure–metric, higher order lagrangian with general matter couplings and with an arbitrary symmetric connection \( \nabla \), \( L (g, \nabla g, \nabla^{(m)} g; \psi, \nabla \psi, \ldots, \nabla^{(p)} \psi) \) are equivalent to those found via the Palatini variation of the metric–connection lagrangian \( L' (g, \Gamma, \Lambda, \psi) = L (g, \Gamma, \psi) + L_c (\Lambda, \Gamma) \). A consequence of this result is
the following generalization of the conformal equivalence theorem. Consider the so-called Weyl geometry wherein $\nabla_c g_{ab} = -Q_c g_{ab}$ with $Q_c$ the Weyl covariant vectorfield. Then upon a conformal transformation the $f(R)$ field equations in Weyl geometry reduce to the form

$$\tilde{G}_{ab} = \tilde{M}^Q_{ab} - g_{ab} V(\varphi),$$

(3)

where

$$\tilde{M}^Q_{ab} = -\tilde{\nabla} \left( \tilde{a} \tilde{Q}_b \right) + \tilde{Q}_a \tilde{Q}_b + \tilde{g}_{ab} \left( -\tilde{Q}^2 + \tilde{\nabla}^m \tilde{Q}_m \right).$$

Notice that if the geometry is Riemannian, i.e. $\tilde{Q}_a = 0$ (original Weyl vector is a gradient, $Q_a = \nabla_a \Phi$) this generalised system is reduced to the usual one.

4 Isotropization Theorem for HDCs.

Can the present isotropic state arise from ‘arbitrary’ initial conditions? A first answer to this question is contained in a well-known theorem of Collins and Hawking: The set of spatially homogeneous cosmologies that can approach isotropy at late times is of measure zero in the space of all spatially homogeneous initial data.

The corresponding $f(R)$–isotropization problem may not be obtained directly since in the corresponding Raychaudhuri equation the Ricci term comes from the $f(R)$ field equations. However, if we conformally transform the Raychaudhuri equation we obtain a Raychaudhuri system (generalization of the usual Raychaudhuri equation) and noting that in our case the potential is not necessarily globally convex, we arrive at an isotropization result of a Collins–Hawking type that is, that their theorem I is valid in higher order gravity theories i.e. Bianchi types I, V and VII approach an isotropic state.

5 Future Work.

The field discussed in the present paper certainly contains a host of well-defined problems to consider and these may prove to be fruitful avenues of research in the near future on problems related to the definition of the tendency of cosmological spacetimes to isotropize, recollapse, the role of cosmic no–hair conjecture, their conformal hamiltonian structure, their Cauchy problem and the role of black hole entropy.

References

1. J.D. Barrow and S. Cotsakis, Phys. Lett. B 214, 515 (1988).
2. S. Cotsakis, Phys. Rev. D 47, 1437 (1993); ibid, 49, 6199 (1995).
3. S. Cotsakis, J. Miritzis and L. Querella, (preprint, 1997).
4. C.B. Collins and S.W. Hawking, Ap. J. 180, 317 (1973).
5. S. Cotsakis and J. Miritzis, Phys. Lett. B 383, 377 (1996).