Twisted basic Dolbeault cohomology on transverse Kähler foliations

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Abstract
In this paper, we study the twisted basic Dolbeault cohomology and transverse hard Lefschetz theorem on a transverse Kähler foliation. And we give some properties for $\Delta_1\kappa$-harmonic forms and prove the Kodaira–Serre-type duality and Dolbeault isomorphism for the twisted basic Dolbeault cohomology.

Keywords
Riemannian foliation · Transverse Kähler foliation · Basic Dolbeault cohomology · Twisted basic Dolbeault cohomology · Kodaira–Serre duality · Hard Lefschetz theorem

Mathematics Subject Classification 53C12 · 53C21 · 53C55 · 57R30 · 58J50

1 Introduction

Let $(M, \mathcal{F})$ be a smooth manifold with a foliation $\mathcal{F}$. One of the smooth invariants of $\mathcal{F}$ is the basic cohomology. The basic forms of $(M, \mathcal{F})$ are locally forms on the leaf space; that is, forms $\phi$ satisfying $X_\mathcal{F}\phi = X_\mathcal{F}d\phi = 0$ for any vector $X$ tangent to the leaves, where $X_\mathcal{F}$ denotes the interior product with $X$. Basic forms are preserved by the exterior derivative and are used to define basic de Rham cohomology groups $H_B^r(\mathcal{F})$, which is defined by

$$H_B^r(\mathcal{F}) = \frac{\ker d_B}{\text{Im } d_B},$$

where $d_B$ is the restriction of $d$ to the basic forms. In general, the basic de Rham cohomology group does not necessarily satisfy Poincaré duality; in fact, it satisfies the twisted Poincaré duality [17]:

$$H_B^r(\mathcal{F}) \cong H_T^{q-r}(\mathcal{F}),$$

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where $q = \text{codim}(\mathcal{F})$ and $H^r_T(\mathcal{F}) = \frac{\ker d_T}{\text{Im } d_T}$ is the cohomology of $d_T = d_B - \kappa_B \wedge$. Here, $\kappa_B$ is the basic part of the mean curvature form of $\mathcal{F}$.

It is well known [2, 7, 18, 21] that on a compact oriented manifold $M$ with a transversally oriented Riemannian foliation $\mathcal{F}$, $H^r_B(\mathcal{F}) \cong \ker \Delta_B$ is finite dimensional, where $\Delta_B$ is the basic Laplacian. Because of this Hodge theorem, researchers have been able to show relationships between curvature bounds and basic cohomology. In [10], J. Hebda proved that a lower bound on transversal Ricci curvature for a Riemannian foliation of a compact manifold causes the space of leaves to be compact and the first basic cohomology group to be trivial. Another example relating the geometry and the topology came in 1991, when Min-Oo et al. [19] proved that if the transversal curvature operator of $(M, \mathcal{F})$ is positive definite, then the cohomology $H^r_B(\mathcal{F}) = 0$ ($0 < r < q$); that is, any basic harmonic $r$-form is trivial. There are many other examples of known relationships between transversal curvature and basic cohomology.

Recently, Habib and Richardson [8] introduced the twisted basic de Rham cohomology $H^r_\kappa(\mathcal{F}) = \ker d_\kappa \text{Im } d_\kappa$ and proved the Poincaré duality of the twisted basic de Rham cohomology $H^*_{\kappa}(\mathcal{F})$ on foliations, that is, $H^r_{\kappa}(\mathcal{F}) \cong H^{q-r}_{\kappa}(\mathcal{F})$.

Moreover, $H^r_{\kappa}(\mathcal{F}) \cong \ker \Delta_\kappa$, where $\Delta_\kappa = d_\kappa \delta_\kappa + \delta_\kappa d_\kappa$ is the twisted basic Laplacian (cf. Sect. 2.2). From the Weitzenböck formula for the twisted basic Laplacian, if the transversal Ricci curvature $\text{Ric}^Q$ is nonnegative and either $\mathcal{F}$ is nontaut or $\text{Ric}^Q$ is positive at some point, then $H^1_{\kappa}(\mathcal{F}) = \{0\}$. Also, the authors [8] proved that $\mathcal{F}$ is taut if and only if $H^0_{\kappa}(\mathcal{F}) \cong H^q_{\kappa}(\mathcal{F}) \neq \{0\}$ (Tautness theorem).

On a transverse Kähler foliation of codimension $q = 2n$, the basic Dolbeault cohomology group of $\tilde{\partial}_B$ is defined by

$$H^{r,s}_{B}(\mathcal{F}) = \frac{\ker \tilde{\partial}_B}{\text{Im } \tilde{\partial}_B},$$

where $0 \leq r, s \leq n$. The Kodaira–Serre duality for the basic Dolbeault cohomology does not necessarily hold unless $\mathcal{F}$ is taut, but we exhibit a version of Kodaira–Serre duality [14] that actually does hold in all cases. That is,

$$H^{r,s}_{B}(\mathcal{F}) \cong H^{n-r,n-s}_{T}(\mathcal{F}),$$

where $H^{r,s}_{T}(\mathcal{F}) = \frac{\ker \tilde{\partial}_T}{\text{Im } \tilde{\partial}_T}$ is the Dolbeault cohomology group of $\tilde{\partial}_T$ (cf. Sect. 3.2). Recently, Habib and Vezzoni [9] studied the basic Dolbeault cohomology and proved some vanishing theorem for basic Dolbeault cohomology by using the Weitzenböck formula for the twisted basic Laplacian.

In this paper, we define the twisted basic Dolbeault cohomology $H^{r,s}_{\kappa}(\mathcal{F})$ of the twisted operator $\tilde{\partial}_\kappa$ (cf. Sect. 3.2) acting on the basic forms of type $(r, s)$ by

$$H^{r,s}_{\kappa}(\mathcal{F}) = \frac{\ker \tilde{\partial}_\kappa}{\text{Im } \tilde{\partial}_\kappa}.$$
The twisted basic Dolbeault cohomology satisfies the Kodaira–Serre duality (Theorem 3.11), that is,
\[ H^r_{d^*}(F) \cong H^{n-r}_{d^*}(F). \]
Trivially, if \( F \) is taut, then \( H^r_{d^*}(F) \cong H^r_{d^*}(F) \cong H^r_{d^*}(F) \). Also, we prove the Hodge decomposition for the twisted basic Dolbeault cohomology (Theorem 3.9) and give some properties for \( \Delta_\kappa \)-harmonic forms.

**Remark 1.1** The twisted basic de Rham cohomology is the special cohomology of Lichnerowicz basic cohomology on foliations [1]. The Lichnerowicz cohomology is the cohomology of the complex \((\Omega^*(M), d_\theta)\) of differential forms on a smooth manifold \( M \) with the de Rham differential operator \( d_\theta = d + \theta \wedge \) deformed by a closed 1-form \( \theta \). The Lichnerowicz cohomology is a proper tool of locally conformal symplectic geometry [8].

**Remark 1.2** The Lichnerowicz basic cohomology depends only on the basic class of a closed basic 1-form [1, Proposition 3.0.11]. In particular, the twisted basic de Rham cohomology group depends only on the Álvarez class \([ \kappa_B ] \in H^1_B(F)\). That is, for all bundle-like metrics associated with the same transverse structure, the twisted basic de Rham cohomologies are isomorphic [8, Theorem 2.11].

## 2 The basic cohomology

### 2.1 Well-known facts

Let \((M, F, g_Q)\) be a \((p + q)\)-dimensional Riemannian foliation of codimension \( q \) with a holonomy invariant metric \( g_Q \) on the normal bundle \( Q = TM/TF \), meaning that \( L_X g_Q = 0 \) for all \( X \in TF \) on a Riemannian manifold \((M, g_M)\) with a bundle-like metric \( g_M \) adapted to \( g_Q \), where \( TF \) is the tangent bundle of \( F \) and \( L_X \) denotes the Lie derivative. We may also consider the canonical identity \( Q \equiv TF^\perp \). Let \( \nabla \) be the transverse Levi-Civita connection on the normal bundle \( Q \), which is torsion-free and metric with respect to \( g_Q \) [22, 23]. Let \( R^Q \) and \( \text{Ric}^Q \) be the transversal curvature tensor and the transversal Ricci operator of \( F \) with respect to \( \nabla \), respectively. The mean curvature vector field \( \tau \) of \( F \) is given by
\[ \tau = \sum_{i=1}^{p} \pi(\nabla^M_{f_i} f_i), \]
where \( \{ f_i \}_{i=1,...,p} \) is a local orthonormal basis of \( TF \) and \( \pi : TM \to Q \) is a natural projection. Then, the mean curvature form \( \kappa \) of \( F \) is given by
\[ \kappa(X) = g_Q(\tau, \pi(X)) \]
for any tangent vector \( X \in \Gamma TM \). Let \( \Omega^r_B(F) \) be the space of all basic \( r \)-forms, i.e., \( \phi \in \Omega^r_B(F) \) if and only if \( X_\phi = 0 \) and \( L_X \phi = 0 \) for any vector \( X \in \Gamma TF \). The foliation \( F \) is said to be minimal if \( \kappa = 0 \). It is well known [2, 22] that on a compact manifold, \( \kappa_B \) is closed, i.e., \( d\kappa_B = 0 \), where \( \kappa_B \) is the basic part of \( \kappa \). And the mean curvature form satisfies the Rummler’s formula:
\[ d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0, \quad \chi_{\mathcal{F}} \wedge *\varphi_0 = 0, \]
where \( \chi_{\mathcal{F}} \) is the characteristic form of \( F \) and \( * \) is the Hodge star operator associated to \( g_M \).
Now, we recall the transversal star operator \(\tilde{\ast} : \Omega^r_B(F) \to \Omega^{q-r}_B(F)\) given by
\[
\tilde{\ast}\phi = (-1)^p (q-r) \ast (\phi \wedge \chi_F).
\]
Trivially, \(\tilde{\ast}^2 \phi = (-1)^r (q-r) \phi\) for any basic form \(\phi \in \Omega^r_B(F)\). Let \(\nu\) be the transversal volume form, that is, \(\ast \nu = \chi_F\). Then, the pointwise inner product \(\langle \cdot, \cdot \rangle\) on \(\Omega^*_B(F)\) is defined by \(\langle \phi, \psi \rangle \nu = \phi \wedge \tilde{\ast}\psi\).

Let \(d_B = d|_{\Omega^*_B(F)}\) and \(d_T = d_B - \kappa_B \wedge\). Then, the formal adjoint operators \(\delta_B\) and \(\delta_T\) of \(d_B\) and \(d_T\) on basic forms are given by
\[
\delta_B = (-1)^q (r+1) \tilde{\ast} d_T \tilde{\ast} \delta_T + \kappa_B^\sharp \wedge, \quad \delta_T = (-1)^q (r+1) \tilde{\ast} d_B \tilde{\ast},
\]
respectively \([2, 18, 23]\), where \(\kappa_B^\sharp\) be the dual vector of \(\kappa_B\). For a local orthonormal frame \(\{E_a\}_{a=1,\ldots,q}\) of the normal bundle \(Q\), \(\delta_T\) is given \([15]\) by
\[
\delta_T = - \sum_{a=1}^q E_a \llcorner \nabla E_a.
\]

Now, we define two Laplacians \(\Delta_B\) and \(\Delta_T\) acting on \(\Omega^*_B(F)\) by
\[
\Delta_B = d_B \delta_B + \delta_B d_B, \quad \Delta_T = d_T \delta_T + \delta_T d_T,
\]
respectively. The Laplacian \(\Delta_B\) is said to be basic Laplacian. Then,
\[
\Delta_B \tilde{\ast} = \tilde{\ast} \Delta_T.
\]

Also, we have the generalization of the usual de Rham–Hodge decomposition.

**Theorem 2.1** \([18, 23]\) Let \((M, F, g_Q)\) be a transversally oriented Riemannian foliation \(F\) on a compact oriented manifold \(M\) with a bundle-like metric. Then,
\[
\Omega^*_B(F) = \mathcal{H}^*_B(F) \oplus \text{Im} d_B \oplus \text{Im} \delta_B
\]
\[
= \mathcal{H}^r_T(F) \oplus \text{Im} d_T \oplus \text{Im} \delta_T
\]
with finite-dimensional \(\mathcal{H}^*_B(F) = \ker \Delta_B\) and \(\mathcal{H}^r_T(F) = \ker \Delta_T\).

On a compact manifold, \(d_T^2 = 0\) because of \(d \kappa_B = 0\). So the basic de Rham cohomology groups \(H^*_B(F)\) and \(H^r_T(F)\) are defined by
\[
H^*_B(F) = \ker d_B / \text{Im} d_B, \quad H^r_T(F) = \ker d_T / \text{Im} d_T,
\]
respectively. Then, it is well known \([23]\) that
\[
H^*_B(F) \cong \mathcal{H}^*_B(F), \quad H^r_T(F) \cong \mathcal{H}^r_T(F).
\]

From (2.3), we have the twisted duality \([17]\)
\[
H^*_B(F) \cong H^r_T(F) \cong H^{q-r}_T(F).
\]

If the foliation \(F\) is taut (meaning that there exists a Riemannian metric on \(M\) for which all leaves are minimal), then the Poincaré duality holds \([16]\). That is,
\[
H^*_B(F) \cong H^{q-r}_B(F).
\]
Now, we introduce the operator $\nabla^*_{tr} \nabla_{tr}$, which is defined by

$$\nabla^*_{tr} \nabla_{tr} \phi = -\sum_a \nabla E_a \nabla E_a \phi + \nabla_{\kappa^Q} \phi.$$ 

It is well known [11, Proposition 3.1] that the operator $\nabla^*_{tr} \nabla_{tr}$ is nonnegative and formally self-adjoint. Then, we have the following generalized Weitzenböck formula [11]: For any basic form $\phi \in \Omega^1_B(F)$

$$\Delta_B \phi = \nabla^*_{tr} \nabla_{tr} \phi + A_Y(\phi) + F(\phi),$$

(2.5)

where $A_Y(\phi) = L_Y \phi - \nabla Y \phi$ and

$$F(\phi) = \sum_{a,b} \theta^a \wedge E_b \eta R_Q(E_b, E_a) \phi.$$ 

(2.6)

Here, $\theta^a$ is the dual basic 1-form of $E_a$. In particular, if $\phi$ is a basic 1-form, then $F(\phi) \eta = \text{Ric}_Q(\phi \eta)$.

**Remark 2.2** Observe that, from the preceding results, the dimensions of $H^r_B(F)$ and $H^r_T(F)$ are smooth invariants of $F$ and do not depend on the choices of bundle-like metric $g_M$ or on the transversal metric $g_Q$, even though the spaces of harmonic forms do depend on these choices.

**Theorem 2.3** [14] Let $(M, F, g_M)$ be a compact Riemannian manifold with a foliation $F$ of codimension $q$ and a bundle-like metric $g_M$. If the endomorphism $F$ is positive definite, then there are no nonzero basic harmonic forms, that is, $H^r_B(F) = \{0\}$. In particular, if $\text{Ric}_Q$ is positive definite, then $H^1_B(F) = \{0\}$.

### 2.2 The twisted basic cohomology

Now, we recall the twisted differential operators $d_\kappa$ and $\delta_\kappa$ [8], which are given by

$$d_\kappa = d_B - \frac{1}{2} \kappa^Q \wedge, \quad \delta_\kappa = \delta_B - \frac{1}{2} \kappa_{\eta}.$$ 

(2.6)

The operator $\delta_\kappa$ is a formal adjoint operator of $d_\kappa$ with respect to the global inner product. Let $\Delta_\kappa = d_\kappa \delta_\kappa + \delta_\kappa d_\kappa$ be the **twisted basic Laplacian**. Then, we have the following relation: For any basic form $\phi$,

$$\Delta_\kappa \phi = \Delta_B \phi - \frac{1}{2} \left( L_{\kappa^Q} + (L_{\kappa^Q}^*)^* \right) \phi + \frac{1}{4} |\kappa^Q|^2 \phi.$$ 

(2.7)

Also, we have the following relations.

**Lemma 2.4** [8] On $\Omega^r_B(F)$, the following equations hold:

1. $d_\kappa^2 = 0, \quad [d_\kappa, \Delta_\kappa] = [\Delta_\kappa, d_\kappa] = [\Delta_\kappa, \tilde{\phi}] = 0$.
2. $d_\kappa \phi = (-1)^q \tilde{\phi} \delta_\kappa, \quad \tilde{\phi} d_\kappa = (-1)^{q+1} \delta_\kappa \phi$.

Since $d_\kappa^2 = 0$, we can define the **twisted basic de Rham cohomology group** $H^r_\kappa(F)$ by

$$H^r_\kappa(F) = \frac{\ker d_\kappa}{\text{Im} d_\kappa}.$$ 

Then, we have the Hodge decomposition.
Theorem 2.5 [8] Let \((M, \mathcal{F}, g_Q)\) be as in Theorem 2.1. Then,
\[
\Omega_B^r(\mathcal{F}) = \mathcal{H}_k^r(\mathcal{F}) \oplus \text{Im } d_k \oplus \text{Im } \delta_k
\]
with finite-dimensional \(\mathcal{H}_k^r(\mathcal{F}) = \ker \Delta_k\). Moreover, \(\mathcal{H}_k^r(\mathcal{F}) \cong H^r(\mathcal{F})\).

And we have the Poincaré duality for \(d_k\)-cohomology. That is,

Theorem 2.6 [8] (Poincaré duality for \(d_k\)-cohomology) On a compact Riemannian manifold with a Riemannian foliation \(\mathcal{F}\), we have
\[
H^r(\mathcal{F}) \cong H^{q-r}(\mathcal{F}).
\]

Remark 2.7 Theorem 2.6 solves the problem of the failure of Poincaré duality to hold for standard basic de Rham cohomology \(H^r_B(\mathcal{F})\) (cf. (2.4)).

Theorem 2.8 [8] (Tautness theorem) Let \((M, \mathcal{F}, g_Q)\) be as in Theorem 2.1. Then, \(\mathcal{F}\) is taut if and only if \(H^0_\kappa(\mathcal{F}) \cong H^0_\kappa(\mathcal{F}) \neq \{0\}\).

From (2.5) and (2.7), we have the Weitzenböck formula [8] for the twisted basic Laplacian \(\Delta_\kappa\). Namely, for any basic form \(\phi\),
\[
\Delta_\kappa \phi = \nabla^* \nabla \phi + F(\phi) + \frac{1}{4} |\kappa_B|^2 \phi.
\]

Then, we have the following theorem.

Theorem 2.9 [8] Let \((M, \mathcal{F}, g_Q)\) be a Riemannian foliation on a compact, connected manifold \(M\) with a bundle-like metric such that the mean curvature form \(\kappa\) is basic harmonic. Then,

1. If the operator \(F + \frac{1}{4} |\kappa|^2\) is strictly positive, then \(H^r_\kappa(\mathcal{F}) = \{0\}\).
2. If the transversal Ricci curvature \(Ric^Q\) is nonnegative and either \(M\) is nontaut or \(Ric^Q\) is positive at least one point, then \(H^1_\kappa(\mathcal{F}) = \{0\}\).
3. Suppose that the transversal sectional curvatures are nonnegative and positive at least one point. If \(\mathcal{F}\) is nontaut, then \(H^r_\kappa(\mathcal{F}) = \{0\}\) for \(1 < r < q\).

3 The twisted basic Dolbeault cohomology

3.1 The basic Dolbeault cohomology

In this section, we generally use the same notation and cite elementary results from [12, Sect. 3] and [20]. Let \((M, \mathcal{F}, g_Q, J)\) be a transverse Kähler foliation of codimension \(q = 2n\) on a Riemannian manifold \(M\) with a holonomy invariant transverse Hermitian metric \(g_Q\) and an almost complex structure \(J\) on \(Q\) such that \(\nabla J = 0\), with \(\nabla\) being the transversal Levi-Civita connection on \(Q\), extended in the usual way to tensors [20]. In some of what follows, we will merely need the fact that the foliation is transverse Hermitian (all of the above, merely requiring \(J\) is integrable and not that \(\nabla J = 0\)), and other times we will need the full power of the Kähler condition \(\nabla J = 0\). The basic Kähler form \(\omega\) is given by
\[
\omega(X, Y) = g_Q(\pi(X), J\pi(Y))
\]
for all vector fields $X, Y \in TM$. Locally, the basic Kähler form $\omega$ may be expressed by

$$\omega = -\frac{1}{2} \sum_{a=1}^{2n} \theta^a \wedge J\theta^a,$$

where $\{\theta^a\}_{a=1,\ldots,2n}$ is a local orthonormal frame of $Q^\ast$. Here, we extend $J$ to elements of $Q^\ast$ by setting $(J\phi)(X) = -\phi(JX)$ for any $X \in Q_x$ and $\phi \in Q^\ast_x$. When it is convenient, we will also refer to the bundle map $J$ we will also refer to the bundle map $J$.

It is an easy exercise to show that for any complex vector field $X$, $A$ frame fields $\omega$ the form $(0,1)).$ Then, $Q^C = Q_{1,0} \oplus Q_{0,1}$, where

$$Q_{1,0} = \{\theta + i J\theta | \theta \in Q^\ast\} \quad \text{and} \quad Q_{0,1} = \{\theta - i J\theta | \theta \in Q^\ast\}.$$

Let $\Lambda^r_{\mathbb{C}} Q^\ast$ be the subspace of $\Lambda^r Q^\ast$ spanned by $\xi \wedge \eta$, where $\xi \in \Lambda^r Q_{1,0}$ and $\eta \in \Lambda^s Q_{0,1}$. The sections of $\Lambda^r_{\mathbb{C}} Q^\ast$ are said to be forms of type $(r,s)$. Let $\Omega^r_{s,B}(F)$ be the set of the basic forms of type $(r,s)$. Let $\{E_a, J E_a\}_{a=1,\ldots,n}$ be a local orthonormal frame of $Q$ and $\{\theta^a, J\theta^a\}_{a=1,\ldots,n}$ be their dual basic forms on $Q^\ast$. Let $V_a = \frac{1}{\sqrt{2}}(E_a - i J E_a)$ and $\omega^a = \frac{1}{\sqrt{2}}(\theta^a + i J\theta^a)$.

$$\omega^a(V_b) = \bar{\omega}^a(\bar{V}_b) = \delta_{ab}, \quad \omega^a(\bar{V}_b) = \bar{\omega}^a(V_b) = 0.$$

A frame fields $\{V_a\}_{a=1,\ldots,n}$ is a local orthonormal frame of $Q_{1,0}^1$, which is called a normal frame field of type $(1,0)$, and $\{\omega^a\}_{a=1,\ldots,n}$ is a dual frame of $Q_{1,0}^1$.

Now, we extend the connection $\nabla$ on $Q$ in the natural way so that $\nabla_X Y$ is defined for any $X \in \Gamma(TM \otimes \mathbb{C})$ and any $Y \in \Gamma(Q^C)$. We further extend it to differential forms, requiring that $\nabla$ is a Hermitian connection, i.e., for any $V \in Q^C$ and any $\phi, \psi \in \Omega^r_{s,B}(F)$,

$$\nabla(\phi, \psi) = \langle \nabla_V \phi, \psi \rangle + \langle \phi, \nabla_T \psi \rangle.$$

It is an easy exercise to show that for any complex vector field $X$, $\nabla_X$ preserves the $(r,s)$ type of the form or vector field.

Now, the transversal star operator $\check{\ast} : \Omega^r_{s,B}(F) \to \Omega^{r-s,n-r}_{B}(F)$ on $\Omega^r_{s,B}(F)$ is given by

$$\phi \wedge \check{\ast} \psi = \langle \phi, \psi \rangle v$$

for any $\phi, \psi \in \Omega^r_{s,B}(F)$, where $v = \frac{\omega^n}{n!}$ is the transversal volume form. Then, for any $\phi \in \Omega^r_{s,B}(F)$,

$$\check{\ast} \phi = \check{\ast} \check{\ast} \phi, \quad \check{\ast}^2 \phi = (-1)^{r+s} \phi.$$

From (2.1), the adjoint operators $\delta_B$ and $\delta_T$ of $d_B$ and $d_T$ are given by

$$\delta_B = -\check{\ast} d_T \check{\ast}, \quad \delta_T = -\check{\ast} d_B \check{\ast}, \quad (3.1)$$
respectively. Note that $d_B = \partial_B + \bar{\partial}_B$ and $d_T = \partial_T + \bar{\partial}_T$, where

$$
\partial_T \phi = \partial_B \phi - \kappa^{1,0}_B \wedge \phi, \quad \bar{\partial}_T \phi = \bar{\partial}_B \phi - \kappa^{1,0}_B \wedge \phi,
$$

where $\kappa^{1,0}_B = \frac{1}{2} (\kappa_B + i \kappa_B) \in \Omega^{1,0}_B (\mathcal{F})$ and $\kappa^{0,1}_B = \bar{\kappa}_B^{1,0}$ [14].

Let $\partial^*_T, \bar{\partial}^*_T, \partial^*_B$ and $\bar{\partial}^*_B$ be the formal adjoint operators of $\partial_T, \bar{\partial}_T, \partial_B$ and $\bar{\partial}_B$, respectively, on the space of basic forms. Then,

$$
\partial_B = \partial^*_B + \bar{\partial}^*_B, \quad \delta_T = \partial^*_T + \bar{\partial}^*_T,
$$

and from (3.1),

$$
\partial^*_T \phi = -\bar{\phi} \partial_B \phi, \quad \bar{\partial}^*_T \phi = -\bar{\phi} \partial_B \phi,
$$

$$
\partial^*_B \phi = -\bar{\phi} \partial_T \phi, \quad \bar{\partial}^*_B \phi = -\bar{\phi} \partial_T \phi.
$$

Since $\bar{\phi} (\kappa^{0,1}_B \wedge \bar{\phi}) = H^{1,0} \subset [14]$, from (3.2) and (3.5), we have [14]

$$
\partial^*_B \phi = \partial_T^* \phi + H^{1,0} \subset \phi, \quad \bar{\partial}^*_B \phi = \bar{\partial}_T^* \phi + H^{0,1} \subset \phi,
$$

where $H^{1,0} = \frac{1}{2} (\kappa^*_B - i J \kappa^*_B)$ and $H^{0,1} = \bar{H}^{1,0}$. Then, from (2.2) and (3.3)

$$
\partial^*_T \phi = -\sum_{a=1}^n V_a \left( \nabla_{\bar{\partial}_a} \phi \right), \quad \bar{\partial}^*_T \phi = -\sum_{a=1}^n \bar{\nabla}_{\bar{\partial}_a} \phi.
$$

Since $\bar{\partial}_B^2 = 0$, we can define the basic Dolbeault cohomology group $H^{r,s}_B (\mathcal{F})$ by

$$
H^{r,s}_B (\mathcal{F}) = \frac{\ker \bar{\partial}_B}{\text{Im} \bar{\partial}_B}.
$$

Now, let $\square_B = \partial_B \partial^*_B + \partial^*_B \partial_B$ and $\square_B = \bar{\partial}_B \bar{\partial}^*_B + \bar{\partial}^*_B \bar{\partial}_B$. Then, we have the basic Dolbeault decomposition.

**Theorem 3.1** [6, 14] Let $(M, \mathcal{F}, g_Q, J)$ be a transverse Kähler foliation on a compact Riemannian manifold $M$ with a bundle-like metric. Then,

$$
\Omega^{r,s}_B (\mathcal{F}) = \mathcal{H}^{r,s}_B (\mathcal{F}) \oplus \text{Im} \bar{\partial}_B \oplus \text{Im} \bar{\partial}^*_B,
$$

where $\mathcal{H}^{r,s}_B (\mathcal{F}) = \ker \square_B$ is finite dimensional. Moreover, $\mathcal{H}^{r,s}_B (\mathcal{F}) \cong H^{r,s}_B (\mathcal{F})$.

Generally, the basic Laplacians do not satisfy the properties which hold on an ordinary Kähler manifold such as $\Delta = 2 \square = 2 \square$. But if $\mathcal{F}$ is taut, then $\Delta_B = 2 \square_B = 2 \square_B$ [14].

### 3.2 The twisted basic Dolbeault cohomology

Let $\partial : \Omega^{r,s}_B (\mathcal{F}) \rightarrow \Omega^{r+1,s}_B (\mathcal{F})$ and $\partial : \Omega^{r,s}_B (\mathcal{F}) \rightarrow \Omega^{r,s+1}_B (\mathcal{F})$ be defined by

$$
\partial \phi = \partial_B \phi - \frac{1}{2} \kappa^{1,0}_B \wedge \phi, \quad \bar{\partial} \phi = \bar{\partial}_B \phi - \frac{1}{2} \kappa^{0,1}_B \wedge \phi,
$$

respectively. From (2.6), it is trivial that $d_B = \partial_B + \bar{\partial}_B$. Let $\partial^*_B$ and $\bar{\partial}^*_B$ be the formal adjoint operators of $\partial_B$ and $\bar{\partial}_B$, respectively. Then, we have the following.
Proposition 3.2 On a transverse Kähler foliation, we have

\[ \partial_k^* \phi = \delta_B^* - \frac{1}{2} H^{1,0} \phi, \quad \bar{\partial}_k^* \psi = \bar{\delta}_B^* - \frac{1}{2} H^{0,1} \psi, \quad \delta_k = \partial_k^* + \bar{\partial}_k^*. \]

Proof From (2.6) and (3.8), the proofs are easy. \(\square\)

Let \( L : \Omega^r_B (F) \rightarrow \Omega^{r+2}_B (F) \) and \( \Lambda : \Omega^r_B (F) \rightarrow \Omega^{r-2}_B (F) \) be given by

\[ L(\phi) = \omega \wedge \phi, \quad \Lambda(\phi) = \omega \cdot \phi, \]

respectively, where \((\xi_1 \wedge \xi_2) \phi = \xi_2^r \xi_1^s \phi\) for any basic 1-forms \(\xi_i\) \((i = 1, 2)\). Trivially, \(\langle L\phi, \psi \rangle = \langle \phi, \Lambda\psi \rangle\) and \(\Lambda = -\hat{\delta} L \hat{\phi}\) [5]. Also, it is well known [12] that

\[ [L, X] = JX^b \wedge, \quad [\Lambda, X^b \wedge] = -JX \wedge, \quad [L, X^b \wedge] = [\Lambda, X] = 0 \] (3.9)

for any vector field \(X \in Q\). From (3.9), we have the following.

Proposition 3.3 [12] On a transverse Kähler foliation, we have

\[ [L, d_B] = [\Lambda, \delta_B] = [L, \partial_B] = [L, \bar{\partial}_B] = [\Lambda, \delta_B^*] = [\Lambda, \bar{\delta}_B^*] = 0, \]

\[ [L, \partial_B^*] = -i \bar{\partial}_r, \quad [L, \bar{\partial}_B^*] = i \partial_r, \quad [\Lambda, \partial_B] = -i \bar{\partial}_r^*, \quad [\Lambda, \bar{\partial}_B] = i \partial_r^*. \]

From (3.9) and Proposition 3.3, we have the following.

Proposition 3.4 On a transverse Kähler foliation, we have

\[ [L, d_k] = [\Lambda, \delta_k] = [L, \partial_k] = [L, \bar{\partial}_k] = [\Lambda, \delta_k^*] = [\Lambda, \bar{\delta}_k^*] = 0, \] (3.10)

\[ [L, \partial_k^*] = -i \bar{\partial}_k, \quad [L, \bar{\partial}_k^*] = i \partial_k, \quad [\Lambda, \partial_k] = -i \bar{\partial}_k^*, \quad [\Lambda, \bar{\partial}_k] = i \partial_k^*. \] (3.11)

Let \(\square_k\) and \(\square_k\) be Laplace operators, which are defined by

\[ \square_k = \partial_k \partial_k^* + \bar{\partial}_k \bar{\partial}_k^* \quad \text{and} \quad \square_k = \bar{\partial}_k \bar{\partial}_k^* + \partial_k \partial_k^*, \]

respectively. Trivially, \(\square_k\) and \(\square_k\) preserve the types of the forms.

Theorem 3.5 On a transverse Kähler foliation, we have

\[ \square_k = \square_k, \quad \Delta_k = 2 \square_k = 2 \square_k. \]

Proof Since \(d_k^2 = 0\), it is trivial that \(\partial_k^2 = \bar{\partial}_k^2 = \partial_k \bar{\partial}_k + \bar{\partial}_k \partial_k = 0\). From (3.11) in Proposition 3.4, we have

\[ i(\partial_k \partial_k^* + \partial_k \partial_k^*) = \partial_k \Lambda \partial_k + \Lambda \partial_k \partial_k - \partial_k \Lambda \partial_k - \bar{\partial}_k \Lambda \partial_k = 0. \]

From (3.9) in Proposition 3.3, we have

\[ i(\bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k \bar{\partial}_k^*) = i(\partial_k \bar{\partial}_k^* + \bar{\partial}_k \partial_k^*) = i(\partial_k \bar{\partial}_k^* + \bar{\partial}_k \partial_k^*) = 0. \]

Hence, \(\square_k = \square_k\) and by a direct calculation,

\[ \Delta_k = \square_k + \square_k = 2 \square_k = 2 \square_k. \] \(\square\)
Remark 3.6 Recall that, if the transverse Kähler foliation is minimal, then a basic form of type \((r, 0)\) is basic harmonic if and only if it is basic holomorphic \([14]\). But, if the foliation is not minimal, then the relation does not hold.

Definition 3.7 On a transverse Kähler foliation, a basic form \(\phi\) is said to be \(\bar{\partial}_k\)-holomorphic if
\[
\bar{\partial}_k \phi = 0.
\]

Theorem 3.8 On a transverse Kähler foliation, a \(\bar{\partial}_k\)-holomorphic form of type \((r, 0)\) is \(\Delta_k\)-harmonic. In addition, if \(M\) is compact, then the converse holds.

Proof Let \(\phi\) be a \(\bar{\partial}_k\)-holomorphic form of type \((r, 0)\). Since \(\bar{\partial}_k \ast \phi = 0\) automatically, \(\bar{\partial}_k \phi = 0\). Conversely, if \(M\) is compact, then \(\Delta_k \phi = 0\) implies that \(\int_M |\bar{\partial}_k \phi|^2 = 0\), i.e., \(\phi\) is \(\bar{\partial}_k\)-holomorphic.

Now, we consider the \(\bar{\partial}_k\)-complex
\[
\cdots \to \Omega^{r,s-1}_B(F) \xrightarrow{\bar{\partial}_k} \Omega^r_B(F) \xrightarrow{\bar{\partial}_k} \Omega^{r,s+1}_B(F) \to \cdots.
\]
Since \(\bar{\partial}_k^2 = 0\), the twisted basic Dolbeault cohomology group is defined by
\[
H^{r,s}_k(F) = \ker \bar{\partial}_k / \text{Im} \bar{\partial}_k.
\]
Then, we have the generalization of the Dolbeault decomposition.

Theorem 3.9 Let \((M, F, g_Q, J)\) be a transverse Kähler foliation on a compact manifold \(M\) with a bundle-like metric. Then,
\[
\Omega^{r,s}_B(F) = \mathcal{H}^{r,s}_k(F) \oplus \text{Im} \bar{\partial}_k \oplus \text{Im} \tilde{\bar{\partial}}^*_k,
\]
where \(\mathcal{H}^{r,s}_k(F) = \ker \square_k\) is finite dimensional.

Proof The proof is similar to the one in Theorem 2.2. See \([18]\) precisely.

As a corollary of Theorem 3.9, we have the Dolbeault isomorphism.

Corollary 3.10 (Dolbeault isomorphism) Let \((M, F, g_Q, J)\) be as in Theorem 3.9. Then,
\[
\mathcal{H}^{r,s}_k(F) \cong H^{r,s}_k(F).
\]

Proof The proof is similar to the proof of the Hodge isomorphism.

Then, we have the Kodaira–Serre duality.

Theorem 3.11 (Kodaira–Serre duality) Let \((M, F, g_Q, J)\) be as in Theorem 3.9. Then,
\[
H^{r,s}_k(F) \cong H^{n-r,n-s}_k(F).
\]

Proof We define the operator \(\sharp : \Omega^{r,s}_B(F) \to \Omega^{n-r,n-s}_B(F)\) by
\[
\sharp \phi := \tilde{\bar{\partial}} \phi,
\]
which is an isomorphism. Since \(\tilde{\bar{\partial}}(k_B^{1.0} \wedge \phi) = H^{0,1} \phi\) \([14]\), we have that for \(\phi \in \Omega^{r,s}_B(F)\),
\[
\tilde{\bar{\partial}}(k_B^{0,1} \wedge \phi) = (-1)^{r+s} H^{1,0} \phi
\]
(3.12)
and from (3.4), we have
\[ \bar{*} \bar{\partial}_B \phi = (-1)^{r+s+1} \partial_T^{*} \bar{*} \phi. \] (3.13)

From (3.12) and (3.13), we get that on \( \Omega^{r,s}_B(F) \),
\[ \bar{*} \bar{\partial}_k = (-1)^{r+s+1} \partial_T^{*} \bar{*}. \] (3.14)

Also, from (3.5), we get that on \( \Omega^{r,s}_B(F) \),
\[ \bar{*} \bar{\partial}_k^{*} = (-1)^{r+s} \partial_T^{*} \bar{*}. \] (3.15)

From (3.14) and (3.15), we have that on \( \Omega^{r,s}_B(F) \),
\[ \bar{*} \bar{\partial}_k \bar{*} \partial k^{*} = - \partial_T^{*} \bar{*} \bar{\partial}_k = - \partial_T^{*} \bar{*} \bar{\partial}_k, \]
which implies
\[ \bar{*} \bar{\square}_k = \bar{\square}_k \bar{*}. \]

Hence, for any basic form \( \phi \in \Omega^{r,s}_B(F) \), we get
\[ \# \bar{\square}_k \phi = \bar{*} \square_k \phi = \bar{\square}_k \bar{*} \phi = \bar{\square}_k \bar{\phi}. \]

That is, \( \# \) preserves \( \mathcal{H}^{r,s}_k(F) = \ker \bar{\square}_k \). From the Dolbeault isomorphism (Corollary 3.10), the proof follows. \( \square \)

**Remark 3.12** In general, the Kodaira–Serre duality does not hold for \( \bar{\partial}_B \)-cohomology. In fact, \( H^{r,s}_B(F) \approx H^{n-r,n-s}_T(F) \), where \( H^{r,s}_T(F) = \ker \bar{\partial}_T / \im \partial_T \) is the \( \bar{\partial}_T \)-cohomology. \( H^{r,s}_T(F) \) is a type of Lichnerowicz basic cohomology. The interested reader may consult [3, Sect. 3] and [24, Sect. 3, called “adapted cohomology” here] for information about ordinary Lichnerowicz cohomology and [1] for the basic case. Theorem 3.11 resolves the problem of the failure of Kodaira–Serre duality to hold for \( \bar{\partial}_B \)-cohomology.

From Theorem 3.5, we have the following Hodge decomposition for the twisted basic Dolbeault cohomology.

**Proposition 3.13** Let \( (M, F, g_Q, J) \) be as in Theorem 3.9. Then,
\[ H^{l}_{k}(F) = \bigoplus_{r+s=l} H^{r,s}_k(F) \] (3.16)
for \( 0 \leq l \leq 2n \) and
\[ \dim_{\mathbb{C}} H^{r,s}_k(F) = \dim_{\mathbb{C}} H^{s,r}_k(F). \] (3.17)

**Proof** Since \( \Delta_k = 2 \square_k \) and \( \square_k \) preserves the space \( \Omega^{r,s}_B(F) \), the proof of (3.16) follows by Hodge isomorphism (Theorem 3.11). The proof of (3.17) follows from that the map \( \mathcal{H}^{r,s}_k(F) \to \mathcal{H}^{s,r}_k(F) \) is a conjugate linear isomorphism. \( \square \)

**Remark 3.14** The Hodge decomposition for \( \bar{\partial}_B \)-cohomology does not hold unless the mean curvature vector \( \kappa^{\mu}_B \) of \( F \) is transversally automorphic, that is, \( L_{\kappa_B^{\mu}} J = 0 \) [15, Corollary 6.7].

Now let \( \Omega^{r,s}_{B,P}(F) \) be the set of all *primitive* basic \( r \)-forms \( \phi \), that is, \( \Lambda \phi = 0 \). Then, by \( \mathfrak{sl}_2(\mathbb{C}) \) representation theory, we have the following proposition.
Proposition 3.15  [15] Let \((M, \mathcal{F}, g_Q, J)\) be as in Theorem 3.9. Then, we have the following.

1. \(\Omega^r_{B,p}(\mathcal{F}) = 0\) if \(r > n\).
2. If \(\phi \in \Omega^r_{B,p}(\mathcal{F})\), then \(L^s \phi \neq 0\) for \(0 \leq s \leq n - r\) and \(L^s \phi = 0\) for \(s > n - r\).
3. The map \(L^s : \Omega^r_{B}(\mathcal{F}) \to \Omega^{r+2s}_{B}(\mathcal{F})\) is injective for \(0 \leq s \leq n - r\).
4. The map \(L^s : \Omega^r_{B}(\mathcal{F}) \to \Omega^{r+2s}_{B}(\mathcal{F})\) is surjective for \(s \geq n - r\).
5. \(\Omega^r_{B}(\mathcal{F}) = \oplus_{s \geq 0} L^s \Omega^{r-2s}_{B}(\mathcal{F})\).

Theorem 3.16 (Hard Lefschetz theorem) Let \((M, \mathcal{F}, g_Q, J)\) be a transverse Kähler foliation on a compact manifold \(M\) with a bundle-like metric. Then, the hard Lefschetz theorem holds for twisted basic cohomology. That is, the map

\[
L^s : H^r_\varphi(\mathcal{F}) \to H^{r+2s}_\varphi(\mathcal{F})
\]

is injective for \(0 \leq s \leq n - r\) and surjective for \(s \geq n - r\), \(s \geq 0\). Moreover,

\[
H^r_\varphi(\mathcal{F}) = \oplus_{s \geq 0} L^s H^{r-2s}_\varphi(\mathcal{F}),
\]

\[
H^{r,s}_\varphi(\mathcal{F}) = \oplus_{t \geq 0} L^t H^{r-t,s-t}_\varphi(\mathcal{F}),
\]

where \(H^r_\varphi(\mathcal{F}) \cong \Omega^r_{B,p}(\mathcal{F}) \cap \ker \Delta_\varphi\) and \(H^{r,s}_\varphi(\mathcal{F}) \cong \Omega^r_{B,p}(\mathcal{F}) \cap \ker \Delta_\varphi\).

Proof Since \(\partial_k \partial_k + \partial_k \partial_k = 0, [L, \square_k] = [L, \square_k] = 0\) by Proposition 3.4, and so \([L, \Delta_\varphi] = 0\). Hence, by Proposition 3.15 and Hodge isomorphism (Theorem 2.5), the proofs of (3.18) and (3.19) follow. The proof of (3.20) follows from the Dolbeault isomorphism (Corollary 3.10).

Remark 3.17 Generally, hard Lefschetz theorem for basic cohomology does not hold unless \([\partial_B \kappa^{0,1} B] = 0\) is trivial. (cf. [15, Theorem 5.11]).

Example 3.18 We consider the Carrière example from [4]. Also, see [8, Sect. 7.1] and [15, Example 9.1]. Let \(A\) be a matrix in \(\text{SL}_2(\mathbb{Z})\) of trace strictly greater than 2. We denote, respectively, by \(w_1\) and \(w_2\) unit eigenvectors associated with the eigenvalues \(\lambda \) and \(\frac{1}{\lambda}\) of \(A\) with \(\lambda > 1\) irrational. Let the hyperbolic torus \(T^3\) be the quotient of \(\mathbb{T}^2 \times \mathbb{R}\) by the equivalence relation which identifies \((m, t)\) to \((A(m), t + 1)\), where \(\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2\) is the torus. The flow generated by the vector field \(W_2\) is a Riemannian foliation with bundle-like metric (letting \((x, s, t)\) denote local coordinates in the \(w_2\) direction, \(w_1\) direction, and \(\mathbb{R}\) direction, respectively)

\[
g = \lambda^{-2t} dx^2 + \lambda^{2s} ds^2 + dt^2.
\]

Since \(A\) preserves the integral lattice \(\mathbb{Z}^2\), it induces a diffeomorphism \(A_0\) of the torus \(\mathbb{T}^2\). So the flow generated by \(W_2\) is invariant under the diffeomorphism \(A_0\) of \(\mathbb{T}^2\). Note that the mean curvature of the flow is \(\kappa = \kappa_B = \log(\lambda) dt\), since \(\chi_{\mathcal{F}} = \lambda^{-t} dx\) is the characteristic form and \(dx \wedge \chi_{\mathcal{F}} = -\log(\lambda) \lambda^{-t} dx \wedge dx = -\kappa \wedge \chi_{\mathcal{F}}\). It is well known that all twisted de Rham cohomology groups vanish, that is, \(H^r_\varphi(\mathcal{F}) = \{0\}\) for all \(r = 0, 1, 2\) in [8].

Now, we will show that all twisted basic Dolbeault cohomology groups satisfy Kodaira–Serre duality. First, we note that an orthonormal frame field for this manifold is \(\{X = \lambda^t \partial_t, S = \lambda^{-t} \partial_s, T = \partial_t\}\) corresponding to the orthonormal coframe \(\{X^* = \chi_{\mathcal{F}}, S^* = \lambda^t dx, T^* = dt\}\). Then, letting \(J\) be defined by \(J(S) = T, J(T) = -S,\) the Nijenhuis tensor

\[
N_J(S, T) = [S, T] + J([J(S, T) + [S, JT]]) - [J(S, JT)]
\]
clearly vanishes, so that \( J \) is integrable. The corresponding transverse Kähler form is seen to be \( \omega = T^* \wedge S^* = \lambda^i dt \wedge ds = d(\frac{1}{\log \lambda} S^*) \), an exact form in basic cohomology. From the above,

\[
\kappa_B = -i \left( \log \lambda \right) Z^* + i \left( \log \lambda \right) \bar{Z}^*,
\]

where \( Z^* = \frac{1}{2} (S^* + i T^*) \in \Omega^{1,0}_B (\mathcal{F}) \). Then,

\[
\kappa_B^{1,0} = -i \log (\lambda) Z^* = -\frac{i}{2} (\log \lambda) (\lambda^i ds + i dt)
\]

\[
\bar{\partial} B \kappa_B^{1,0} = d \kappa_B^{1,0} = (\log \lambda)^2 \bar{Z}^* \wedge Z^*
\]

\[
\bar{\partial}_k \kappa_B^{1,0} = \bar{\partial} B \kappa_B^{1,0} - \frac{i}{2} \kappa_B^{1,0} \wedge \kappa_B^{1,0} = \frac{9}{8} \bar{\partial} B \kappa_B^{1,0}.
\]

It is impossible to change the metric so that \( \bar{\partial}_B \kappa_B^{1,0} = 0 \). Hence, \( \mathcal{F} \) is nontaut [15, Corollary 5.12].

The basic Dolbeault cohomology groups are given by \( H_B^{0,0} = \mathbb{R}, \ H_B^{1,0} = \{0\}, \ H_B^{0,1} = \mathbb{R} \) and \( H_B^{1,1} = \{0\} \). Then, observe that the ordinary basic cohomology Betti numbers for this foliation are \( h_B^0 = h_B^1 = 1, \ h_B^2 = 0 \), we see that the basic Dolbeault Betti numbers satisfy

\[
h_B^{0,0} = h_B^{0,1} = 1, \ h_B^{1,0} = h_B^{1,1} = 0.
\]

So even though it is true that

\[
h_B^j = \sum_{r+s=j} h_B^{r,s},
\]

and the foliation is transversely Kähler, we also have (with \( n = 1 \))

\[
h_B^{r,s} \neq h_B^{s,r}, \ h_B^{r,s} \neq h_B^{n-r,n-s}.
\]

Thus, for a nontaut, transverse Kähler foliation, it is not necessarily true that the odd basic Betti numbers are even, and the basic Dolbeault numbers do not have the same kinds of symmetries as Dolbeault cohomology on Kähler manifolds.

Now, we compute the twisted basic Dolbeault cohomology groups \( H_k^{*,*}(\mathcal{F}) \). Let \( f \in H_k^{0,0}(\mathcal{F}) \), that is, \( f \) is a periodic function alone \( t \) and \( \bar{\partial} f = 0 \). Equivalently,

\[
\bar{\partial} B f = \frac{1}{2} f \kappa_B^{0,1}.
\] (3.21)

On the other hand, since \( \bar{\partial} B f \in \Omega_{0,1}^1(\mathcal{F}) \), by a direct calculation, \( \bar{\partial} B f = if'(t) \overline{Z}^* \). Hence, from (3.34), \( f'(t) \overline{Z}^* = \frac{1}{2} f(\log \lambda) \overline{Z}^* \). That is, \( f' = \frac{1}{2} (\log \lambda) f \). Then, \( f = c \lambda^{-t} \) for some constant. Since \( f \) is periodic, \( f(t) = 0 \). Hence,

\[
H_k^{0,0}(\mathcal{F}) = \{0\}.
\]

Let \( \varphi \in H_k^{1,0} \). That is, \( \varphi = f(t) Z^* \in \Omega^{1,0}_B(\mathcal{F}) \) and \( \bar{\partial} \varphi = 0 \) for a periodic function \( f \). Hence,

\[
\bar{\partial} B \varphi = \frac{1}{2} \kappa_B^{0,1} \wedge \varphi.
\] (3.22)
By a direct calculation, \( \bar{\partial}_B \phi = i^2 (\log \lambda) f \mathcal{Z}^* \wedge \mathcal{Z}^* \). So from (3.35)

\[
f' = -\frac{3}{2} (\log \lambda) f
\]

and so \( f(t) = c \lambda^{-\frac{3}{4}} \) for some \( c \in \mathbb{R} \). Since \( f \) is periodic, it is zero. Thus, \( \phi = 0 \). That is, \( H_{1,0}^k(\mathcal{F}) = \{0\} \).

By complex conjugation, we get \( H_{0,1}^k(\mathcal{F}) = \{0\} \).

Let \( \phi \in H_{1,1}^k(\mathcal{F}) \). Then, \( \phi \in \Omega_{B}^{1}(\mathcal{F}) \) is of the form \( \phi = f(t) \mathcal{Z}^* \wedge \mathcal{Z}^* \), where \( f \) is a periodic function. Trivially, \( \bar{\partial}_k \phi = 0 \). Now, let \( \bar{\partial}_k^* \phi = 0 \). Since

\[
\delta_k \phi = \delta_T \phi + \frac{1}{2} \kappa^\nu \phi
\]

we have \( \bar{\partial}_k^* \phi = -\frac{i}{2} (f' - \frac{3}{4} (\log \lambda) f) (\mathcal{Z}^* + \overline{\mathcal{Z}}^*) \),

we have \( \bar{\partial}_k^* \phi = -\frac{1}{2} (f' - \frac{3}{4} (\log \lambda) f) \mathcal{Z}^* \). Thus, the solution of \( f' - \frac{3}{4} (\log \lambda) f = 0 \) reduced to zero for periodic functions \( f \). Hence, \( H_{1,1}^k(\mathcal{F}) = \{0\} \).

This shows that the Kodaira–Serre duality and Hodge isomorphism are satisfied for the twisted basic Dolbeault cohomology.

### 3.3 The \( d_k^c d_k^c \) Lemma

Let \( (M, \mathcal{F}, g_Q, J) \) be a transverse Kähler foliation of codimension \( 2n \) on a compact Riemannian manifold \( M \) with bundle-like metric. First, we recall the operator \( C : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F}) \) defined by [15]

\[
C = \sum_{0 \leq r, s \leq n} (\sqrt{-1})^{r - s} P_{r,s},
\]

where \( P_{r,s} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r,s}(\mathcal{F}) \) is the projection. Then,

\[
C^* = C^{-1} = \sum_{0 \leq r, s \leq n} (\sqrt{-1})^{s - r} P_{r,s}.
\]

Now, we define \( d_k^c : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F}) \) by

\[
d_k^c = C^* d_k C = C^{-1} d_k C.
\]

Then, \( d_k^c = \sqrt{-1} (\bar{\partial}_k - \partial_k) \) and

\[
d_k d_k^c \big|_{\Omega_B^*(\mathcal{F})} = -d_k^c d_k \big|_{\Omega_B^*(\mathcal{F})}.
\]

Let \( \delta_k^c \) be the adjoint operator of \( d_k^c \), which is given by

\[
\delta_k^c = C^* \delta_k C = C^{-1} \delta_k C.
\]
Let $\Delta^c_\kappa = d^c_\kappa \delta^c_\kappa + \delta^c_\kappa d^c_\kappa$. Since $\Delta_\kappa$ preserves the type of differential form, we have

$$\Delta^c_\kappa = C^{-1} \Delta_\kappa C = \Delta_\kappa.$$ 

**Lemma 3.19** (d$^c_\kappa$ Lemma) Let $(M, \mathcal{F}, g_Q, J)$ be a transverse Kähler foliation on a compact manifold $M$ with a bundle-like metric. Then, on $\Omega^*_B(\mathcal{F})$,

$$\ker d_\kappa \cap \text{Im} d^c_\kappa = \text{Im} d_\kappa d^c_\kappa.$$ 

**Proof** Let $\alpha \in \ker d_\kappa \cap \text{Im} d^c_\kappa \cap \Omega^*_B(\mathcal{F})$. That is, for some basic $(r-1)$-form $\beta$, $\alpha = d^c_\kappa \beta$ and $\beta = \gamma + d_\kappa \gamma_1 + \delta^c_\kappa \gamma_2$ with $\gamma \in \mathcal{H}^{r-1}_\kappa(\mathcal{F})$ by the Hodge decomposition (Theorem 2.5). Since $M$ is compact, by Theorem 3.5, $\Delta_\kappa \gamma = 0$ implies $\square_\kappa \gamma = \square_\kappa \gamma = 0$. Therefore, $\partial_\kappa \gamma = \bar{\partial}_\kappa \gamma = 0$ and so $d^c_\kappa \gamma = 0$. Hence,

$$\alpha = d^c_\kappa \beta = d^c_\kappa \gamma + d^c_\kappa d_\kappa \gamma_1 + d^c_\kappa \delta^c_\kappa \gamma_2 = d^c_\kappa d_\kappa \gamma_1 + d^c_\kappa \delta^c_\kappa \gamma_2 = d_\kappa d^c_\kappa (-\gamma_1) + d^c_\kappa \delta^c_\kappa \gamma_2. \quad (3.23)$$

Moreover, since $d_\kappa \alpha = 0$, by the equation above,

$$0 = d_\kappa d^c_\kappa \beta = d_\kappa d^c_\kappa \delta^c_\kappa \gamma_2 = -d_\kappa \delta^c_\kappa d^c_\kappa \gamma_2.$$ 

The last equality follows from $d^c_\kappa \delta^c_\kappa + \delta^c_\kappa d^c_\kappa = 0$ (Theorem 3.5). By integrating,

$$0 = \int_M (d_\kappa \delta^c_\kappa d^c_\kappa \gamma_2, d^c_\kappa \gamma_2) = \int_M \| \delta^c_\kappa d^c_\kappa \gamma_2 \|^2 = \int_M \| d^c_\kappa \delta^c_\kappa \gamma_2 \|^2.$$ 

That is, $d^c_\kappa \delta^c_\kappa \gamma_2 = 0$. So from (3.23),

$$\alpha = d_\kappa d^c_\kappa (-\gamma_1),$$

which implies that $\alpha \in \text{Im} d_\kappa d^c_\kappa$. \qed

**Remark 3.20** If $\mathcal{F}$ is taut, then the $d^c_\kappa$ Lemma implies that $dd^c_\kappa$ Lemma [13, Lemma 7.3].

### 3.4 $\Delta_\kappa$-harmonic forms

Let $(M, \mathcal{F}, g_Q, J)$ be a transverse Kähler foliation on a compact Riemannian manifold $M$ with a bundle-like metric. We define two operators

$$\nabla_T^* \nabla_T \phi = -\sum_a \nabla_{V_a} \nabla_{\bar{V}_a} \phi + \nabla_{H^{01}} \phi,$$

$$\bar{\nabla}_T^* \bar{\nabla}_T \phi = -\sum_a \nabla_{\bar{V}_a} \nabla_{V_a} \phi + \nabla_{H^{10}} \phi.$$

Then, by a direct calculation, we have

$$\nabla_T^* \nabla_T \phi = \bar{\nabla}_T^* \bar{\nabla}_T \phi + \nabla_{H^{01} - H^{10}} \phi - \sum_a R^Q(V_a, \bar{V}_a) \phi \quad (3.24)$$

for any basic form $\phi$. Then, the operators $\nabla_T^* \nabla_T$ and $\bar{\nabla}_T^* \bar{\nabla}_T$ are formally self-adjoint and positive definite [14].
Proposition 3.21 [14] Let \((M, \mathcal{F}, g_Q, J)\) be a transverse Kähler foliation on a Riemannian manifold \(M\) with a bundle-like metric. Then, for all \(\phi \in \Omega^r_B(\mathcal{F})\),

\[
\Box_B \phi = \nabla^*_T \nabla_T \phi + \sum_{a,b} \tilde{\omega}^{ab} \wedge \tilde{V}_b \circ R^Q(V_b, \tilde{V}_a) \phi + \sum_a \tilde{\omega}^a \wedge (\nabla_{\tilde{V}_a} H^{0,1})_\phi, \tag{3.25}
\]

From (3.24), we have the following.

Proposition 3.22 [14] Let \((M, \mathcal{F}, g_Q, J)\) be as in Proposition 3.21.

1. If \(\phi\) is a basic form of type \((r, 0)\), then

\[
\Box_B \phi = \nabla^*_T \nabla_T \phi. \tag{3.26}
\]

2. If \(\phi\) is a basic form of type \((r, n)\), then

\[
\Box_B \phi = \nabla^*_T \nabla_T \phi + \sum_a R^Q(V_a, \tilde{V}_a) \phi + \text{div}_\gamma(H^{0,1}) \phi \tag{3.28}
\]

On the other hand, by a direct calculation, we have the following.

Proposition 3.23 On a transverse Kähler foliation, we have

\[
\Box_{\kappa} = \Box_B - \frac{1}{2} \left( \epsilon(\kappa_B^{0,1}) \tilde{\partial}_B^* + \tilde{\partial}_B^* \epsilon(\kappa_B^{0,1}) \right) - \frac{1}{2} \left( \tilde{\partial}_B H^{0,1} + H^{0,1} \tilde{\partial}_B \right) + \frac{1}{2} \kappa_B^{0,1} |^2. \tag{3.29}
\]

From Propositions 3.21 and 3.23, we have the following.

Proposition 3.24 On a transverse Kähler foliation, we have

\[
\Box_{\kappa} \phi = \nabla^*_T \nabla_T \phi + \sum_{a,b} \tilde{\omega}^{ab} \wedge \tilde{V}_b \circ R^Q(V_b, \tilde{V}_a) \phi + \sum_a \tilde{\omega}^a \wedge (\nabla_{\tilde{V}_a} H^{0,1})_\phi
\]

\[
- \frac{1}{2} \left( \epsilon(\kappa_B^{0,1}) \tilde{\partial}_B^* + \tilde{\partial}_B^* \epsilon(\kappa_B^{0,1}) \right) \phi - \frac{1}{2} \left( \tilde{\partial}_B H^{0,1} + H^{0,1} \tilde{\partial}_B \right) \phi + \frac{1}{2} \kappa_B^{0,1} |^2 \phi. \tag{3.30}
\]

Remark 3.25 Proposition 3.24 is also shown in [9, Theorem 3.1], but the expression is little bit different. The authors in [9] proved the vanishing theorem of transversally holomorphic basic \((r, 0)\)-form by using the Weitzenböck formula for the twisted basic Laplacian \(\Box_{\kappa}\) [9, Theorem 3.4]. In this research, we deal with the vanishing theorem of \(\tilde{\partial}_{\kappa}\)-holomorphic \((r, 0)\)-form (Corollary 3.31).

Proposition 3.26 [15] Let \((M, \mathcal{F}, g_Q, J)\) be a transverse Kähler foliation on a closed Riemannian manifold \(M\). Then, there exists a bundle-like metric compatible with the Kähler structure such that \(\tilde{\partial}_B^* \kappa_B^{1,0} = 0\), or \(\tilde{\partial}_B^* \kappa_B^{0,1} = 0\).

Lemma 3.27 For any \(\phi \in \Omega^r_B(\mathcal{F})\), we get

\[
\tilde{\partial}_B^* \epsilon(\kappa_B^{0,1}) \phi = - \nabla_{H^{1,0}} \phi. \tag{3.27}
\]
Proof From (3.12) and (3.13), for any \( \phi \in \Omega^r_B(\mathcal{F}) \), since \( \bar{V}_a \phi = 0 \), we have
\[
\bar{\partial}_B^*e(\kappa_B^{0,1})\phi = -\sum_{a} \bar{V}_a \bar{\nabla}_a \kappa_B^{0,1} \wedge \phi - \sum_{a} \kappa_B^{0,1}(\bar{V}_a) \bar{\nabla}_a \phi + H^{0,1}_B e(\kappa_B^{0,1})\phi
\]= \(\bar{\partial}_B^*\kappa_B^{0,1}\) \wedge \phi - \nabla_{H^{0,1}_B} \phi + |\kappa_B^{0,1}|^2 \phi.
\]
By Proposition 3.26, if we choose the bundle-like metric such that \(\bar{\partial}_B^*\kappa_B^{0,1} = 0\), the proof follows. \(\square\)

From Proposition 3.26 and Lemma 3.27, we get

Theorem 3.28 On a transverse Kähler foliation, the following hold:

1. If \( \phi \in \Omega^r_B(\mathcal{F}) \), then
\[
\Box_\kappa \phi = \nabla_T^* \nabla_T \phi - \frac{1}{2} \bar{\partial}_B^* e(\kappa_B^{0,1}) \phi - \frac{1}{2} H^{0,1}_B \bar{\partial}_B \phi + \frac{1}{2} |\kappa_B^{0,1}|^2 \phi
\]
(3.30)
\[
= \nabla_T^* \nabla_T \phi + \sum_{a} R^Q(\bar{V}_a, \bar{V}_a) \phi + \nabla_{H^{0,1}_B} \phi + \frac{1}{2} \bar{\partial}_B^* e(\kappa_B^{0,1}) \phi - \frac{1}{2} H^{0,1}_B \bar{\partial}_B \phi + \frac{1}{2} |\kappa_B^{0,1}|^2 \phi.
\]
(3.31)

2. If \( \phi \in \Omega^{r,n}_B(\mathcal{F}) \), then
\[
\Box_\kappa \phi = \nabla_T^* \nabla_T \phi - \nabla_{H^{1,0}} \phi + \frac{1}{2} \nabla_{H^{0,1}} \phi + \frac{1}{2} \text{div}_V(H^{0,1}) \phi - \frac{1}{2} \kappa_B^{0,1} \wedge \bar{\partial}_B \phi + \frac{1}{2} |\kappa_B^{0,1}|^2 \phi
\]
(3.32)
\[
= \nabla_T^* \nabla_T \phi + \sum_{a} R^Q(\bar{V}_a, \bar{V}_a) \phi - \frac{1}{2} \nabla_{H^{1,0}} \phi - \frac{1}{2} \kappa_B^{0,1} \wedge \bar{\partial}_B \phi + \frac{1}{2} \text{div}_V(H^{0,1}) \phi + \frac{1}{2} |\kappa_B^{0,1}|^2 \phi.
\]
(3.33)

Proof Let \( \phi \) be a basic form of type \((r, 0)\). Since \( Z \phi = 0 \) for any \( Z \in Q^{0,1} \), from Proposition 3.24, (3.30) is proved. Now, we choose the bundle-like metric such that the mean curvature form is basic harmonic, that is, \(\bar{\partial}_B^*\kappa_B^{0,1} = 0\). From Lemma 3.27 and (3.24), the proof of (3.31) follows. From Proposition 3.21(2) and Proposition 3.22, the proofs of (3.32) and (3.33) follow. \(\square\)

Proposition 3.29 Let \((M, \mathcal{F}, g, \mathcal{J})\) be a transverse Kähler foliation on a compact manifold \(M\) with a bundle-like metric. If \( \phi \in \Omega^r_B(\mathcal{F}) \) is a \(\Delta_\kappa\)-harmonic form, then
\[
\nabla_V \phi = 0
\]
for any \( V \in \Gamma Q^{0,1} \).

Proof Since \(\Delta_\kappa \phi = 0\), by Theorem 3.5, \(\Box_\kappa \phi = 0\) and so \(\bar{\partial}_B \phi = 0\). That is, \(\bar{\partial}_B \phi = \frac{1}{2} \kappa_B^{0,1} \wedge \phi\).

Since \( \phi \) is type of \((r, 0)\), we have
\[
H^{0,1}_B \bar{\partial}_B \phi = \frac{1}{2} H^{0,1}_B (\kappa_B^{0,1} \wedge \phi) = \frac{1}{2} |\kappa_B^{0,1}|^2 \phi.
\]
From (3.30), we get
\[
\nabla_T^* \nabla_T \phi - \frac{1}{2} \bar{\partial}_B^* e(\kappa_B^{0,1}) \phi + \frac{1}{4} |\kappa_B^{0,1}|^2 \phi = 0.
\]
(3.34)

By integrating (3.34), we get
\[
0 = \int_M \|\nabla_T \phi\|^2 - \frac{1}{2} \int_M (\kappa_B^{0,1} \wedge \phi, \bar{\partial}_B \phi) + \frac{1}{4} \int_M |\kappa_B^{0,1}|^2 \phi^2
\]

\(\square\)
\[ \int_M \| \nabla_T \phi \|^2 - \frac{1}{4} \int_M \| \kappa_B^{0,1} \wedge \phi \|^2 + \frac{1}{4} \int_M | \kappa_B^{0,1} | \| \phi \|^2. \]  

(3.35)

Since \( H^{0,1} \phi = 0 \) for any \( \phi \in \Omega_B^{r,0}(\mathcal{F}) \), we get
\[ \langle \kappa_B^{0,1} \wedge \phi, \kappa_B^{0,1} \wedge \phi \rangle = \langle \phi, H^{0,1} (\kappa_B^{0,1} \wedge \phi) \rangle = | \kappa_B^{0,1} | \| \phi \|^2. \]

From (3.35), we get
\[ \int_M \| \nabla_T \phi \|^2 = 0, \]
which implies \( \nabla_T \phi = 0 \), that is, for any \( V \in \Gamma Q^{0,1}, \nabla_V \phi = 0 \).

**Theorem 3.30** Let \((M, \mathcal{F}, g_Q, J)\) be a transverse Kähler foliation on a compact manifold \(M\) with a bundle-like metric.

(1) If the transverse Ricci curvature is nonnegative and positive at some point, then \( \mathcal{H}^{r,0}_{\kappa} (\mathcal{F}) = \{0\} \) for all \( r > 0 \).

(2) If \( \mathcal{F} \) is transversely Ricci-flat and nontaut, then \( \mathcal{H}^{r,0}_{\kappa} (\mathcal{F}) = \{0\} \) for all \( r > 0 \).

**Proof** Let \( \phi \) be a \( \Delta_{\kappa} \)-harmonic form of type \((r,0)\). If we choose the bundle-like metric such that \( \bar{\partial}^* \mathcal{B} \kappa_B^{0,1} = 0 \), then from (3.24), Lemma 3.27 and Proposition 3.29,
\[ \bar{\nabla}_T^* \bar{\nabla}_T \phi + \bar{\partial}^* \mathcal{B} \epsilon(\kappa_B^{0,1}) \phi - \sum_a R^Q (V_a, \bar{V}_a) \phi = 0. \]  

(3.36)

Note that \( \sum_a R^Q (\bar{V}_a, V_a) \omega^b = Ric^Q (E_b, E_b) \omega^b \) [14, Remark 4.7]. By integrating (3.36), we get
\[ \int_M \| \bar{\nabla}_T \phi \|^2 + \int_M \| \kappa_B^{0,1} \wedge \phi \|^2 + \sum_{i=1}^r \int_M R^Q (E_{ai}, E_{ai}) \| \phi \|^2 = 0. \]

If \( Ric^Q \) is nonnegative and positive at some point, then \( \phi = 0 \). So the proof of (1) is proved.

If \( Ric^Q = 0 \), then
\[ | \kappa_B^{0,1} | \| \phi \| = 0. \]

So if \( \mathcal{F} \) is nontaut, then \( \phi = 0 \), that is, the proof of (2) is finished.

**Corollary 3.31** Let \((M, \mathcal{F}, g_Q, J)\) be as in Theorem 3.30 with a transversely Ricci-flat foliation. If \( \mathcal{F} \) is nontaut, then
\[ H^{r,0}_{\kappa} (\mathcal{F}) \cong H^{0,r}_{\kappa} (\mathcal{F}) \cong H^{n,\ast}_{\kappa} (\mathcal{F}) \cong H^{s,0}_{\kappa} (\mathcal{F}) = \{0\} \]
for \( r, s > 0 \).

**Proof** The proofs follow from complex conjugation, Kodaira–Serra duality and Dolbeault isomorphism.

The vanishing theorems for the basic harmonic space were proved in [9] and [14], respectively.

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