EXOTIC COURANT ALGEBROIDS AND T-DUALITY

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Abstract. In this paper, we extend the T-duality isomorphism in [12] from invariant exact Courant algebroids, to exotic exact Courant algebroids such that the momentum and winding numbers are exchanged, filling in a gap in the literature.

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Introduction

In [12], Gualtieri and Cavalcanti showed that T-duality for principal circle bundles in a background H-flux, gives an isomorphism between invariant exact Courant algebroids. The goal in this paper is to extend their isomorphism, to the full Courant algebroid on one principal circle bundle, and the striking answer that we obtain is that the T-dual is an exact exotic Courant algebroid (defined in the text) on the T-dual principal circle bundle, and we conclude that the momentum and winding numbers are exchanged.

Just as the work [12] was inspired by the works [7, 8], where T-duality in a background flux was studied for the first time on invariant differential forms, this paper was inspired by [18, 19] who extended this isomorphism to the full space of differential forms on one principal circle bundle, and the striking thing obtained is that the T-dual is the space of exotic differential forms on the T-dual principal circle bundle.

In more detail, recall that topological T-duality as in [7, 8], asserts that if \( \pi : Z \to M \) denotes a principal circle bundle whose first Chern class is given by \([F] \in H^2(M, \mathbb{Z})\), and let \([H] \in H^3(Z, \mathbb{Z})\) denote a H-flux on \(Z\). Then there exists a T-dual bundle \( \hat{\pi} : \hat{Z} \to M \) whose first Chern class is denoted \([\hat{F}] \in H^2(M, \mathbb{Z})\) and a T-dual H-flux on the T-dual
bundle, \( [\hat{H}] \in H^3(\hat{Z}, \mathbb{Z}) \), satisfying
\[
[\hat{F}] = \pi_*([H]),
\]
\[
[F] = \tilde{\pi}_*([\hat{H}]).
\]
The result of Gualtieri and Cavalcanti [12] asserts that there is an isomorphism of Courant algebroids over \( M \),
\[
\mu_0 : \Gamma(TZ \oplus T^*Z)^{S^1} \longrightarrow \Gamma(T\hat{Z} \oplus T^*\hat{Z})^{S^1}
\]
A natural question is whether it is possible to extend their isomorphism to the (infinite dimensional, graded) Courant algebroid \( \Gamma(TZ \oplus T^*Z) \) over \( M \). In this paper, we achieve this, and prove in Theorem 3.1 that there is a graded T-duality isomorphism of exotic Courant algebroids over \( M \),
\[
\mu : \Gamma(TZ \oplus T^*Z) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \left( \Gamma(T\hat{Z} \oplus T^*\hat{Z}) \otimes \tilde{\pi}^*(L^\otimes n) \right)^{S^1}
\]
such that the restriction of \( \mu \) to \( \Gamma(TZ \oplus T^*Z)^{S^1} \) is equal to \( \mu_0 \). Moreover there is a Clifford action of exotic Courant algebroids on exotic differential forms, and the T-duality isomorphism above is compatible with this action, cf. Theorem 3.2. It enables one to see that the momentum and winding numbers are exchanged as in [19], which is an important aspect of T-duality.

Besides T-duality, some other achievements of this paper is the novel axiomatic definition of exotic Courant algebroids (Definition 2.1), both exact and non-exact, as well as the classification of exact exotic Courant algebroids in Proposition 2.4. These concepts will be studied in further detail in the first author’s Ph.D. thesis [15].

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1. Preliminaries

Throughout this section we recall the notions of a Courant algebroid, along with some standard, but important, examples. There are some excellent articles and reviews on Courant algebras in the literature [2, 3, 4, 11, 16, 17, 24, 29, 30, 21, 22, 20]. We will then introduce topological T-duality and review how the T-duality transformations can be understood in terms of this Courant algebroid framework as in [12], before reviewing the extended T-duality isomorphism for exotic differential forms [19]. All manifolds and maps are assumed to be smooth.

1.1. Courant algebroids.

Given a real manifolds \( M \), let the data \( (E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho) \) consist of a (possibly infinite dimensional) vector bundle \( E \rightarrow M \), a nondegenerate, symmetric bilinear form \( \langle \cdot, \cdot \rangle : \Gamma(E) \otimes \Gamma(E) \rightarrow C^\infty(M) \), a bilinear (Dorfman) bracket \( [\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E) \) defined on the space of sections of \( E \), denoted \( \Gamma(E) \), and a smooth bundle map \( \rho : E \rightarrow TM \) which we term the anchor map.
Definition 1.1. A Courant algebroid over $M$ consists of a structure $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ defined over $M$, which is compatible with the following conditions:

1. $[a, [b, c]] = [[a, b], c] + [b, [a, c]],$
2. $\rho([a, b]) = [\rho(a), \rho(b)],$
3. $[a, hb] = \rho(a) (h) b + h [a, b],$
4. $[a, b] + [b, a] = d(a, b),$
5. $\rho(a)(b, c) = \langle [a, b], c \rangle + \langle b, [a, c] \rangle.$

where $a, b, c \in \Gamma(E), h \in C^\infty(M)$, and $d : C^\infty(M) \to \Gamma(E)$ is the induced differential operator defined by the relation: $\langle dh, a \rangle = \rho(a) h.$

Remark 1.2. It should be noted that, in keeping with the literature, we have included all five conditions (1)-(5). It can, however, be proven that axioms (2) and (3) are simply a consequence of the non-degeneracy of the bracket and axiom (5), and are thus redundant.

Example 1.3. The simplest non-trivial example of a Courant algebroid over $M$ is the standard Courant algebroid. This consists of the vector bundle $TM \oplus T^*M \to M$ given by the direct sum of the tangent and cotangent bundle over $M$, along with the following defining structures:

- $[X + \alpha, Y + \beta]_H = [X, Y] + L_X \beta - \iota_Y d\xi + \iota_X \iota_Y H,$\footnote{This Dorfman bracket can alternatively be defined as the derived bracket of the $H$-twisted differential $d + H$ acting on $\Omega^*(M)$ (i.e., $[\cdot, \cdot]_H = [[d + H, \cdot], \cdot]$), where the sections of $TZ \oplus T^*Z$ act on $\Omega^*(M)$ via the Clifford action given by $(X + \xi) \cdot \omega = \iota_X(\omega) + \xi \wedge \omega.$}
- $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2} (\alpha(Y) + \beta(X)),$
- $\rho(X + \alpha) = X,$

where $X, Y \in \Gamma(TM), \alpha, \beta \in \Omega^1(M),$ and $H \in H^3(M)$. This data defines the standard Courant algebroid $TZ \oplus T^*Z$ twisted by the 3-form $H$.

Now let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ define a Courant algebroid over $M$, and consider the map $\frac{1}{2} \rho^* : T^*M \to E$ defined by the relation: $\langle \frac{1}{2} \rho^*(\xi), a \rangle = \frac{1}{2} \xi(\rho(a)),$ where $\xi \in \Gamma(T^*M), a \in \Gamma(E).$

We can now define the exact Courant algebroids:

Definition 1.4. A Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ over $M$ is said to be exact if its anchor map $\rho$ is surjective, and the kernel of the anchor map coincides with the image of $\frac{1}{2} \rho^*$. That is, the sequence:

$$0 \to T^*M \xrightarrow{\frac{1}{2} \rho^*} E \xrightarrow{\rho} TM \to 0,$$

defines a short exact sequence.

It was proven by Ševera [28] that there exists a Courant algebroid isomorphism between any such exact Courant algebroid and some standard Courant algebroid that has been twisted by a unique 3-class $H \in H^3(M, \mathbb{R}).$ As a result, every exact Courant algebroid is uniquely determined by an element of the third cohomology class of $M$.

The next example will be that of an invariant Courant algebroid. This is of considerable interest when exploring T-duality and arises as a non-exact Courant algebroid defined

$$[X + \xi, \omega] = \iota_X(\omega) + \xi \wedge \omega.$$
over a principal circle bundle. From here on out we will take \( \pi : Z \to M \) to be a principal circle bundle, let \( v \) denote an invariant period-1 generator of the circle action on \( Z \), and choose an invariant connection form \( A \in \Omega^1(Z) \) normalized such that \( \iota_v A = A(v) = 1 \).

Before we get to the example, we need the following important well-known result for invariant sections of \( TZ \oplus T^*Z \), denoted \( \Gamma(TZ \oplus T^*Z)^{S_1} \), given by

\[
\Gamma(TZ \oplus T^*Z)^{S_1} = \{ x + \alpha \in \Gamma(Z, TZ \oplus T^*Z) \mid \mathcal{L}_v(X + \alpha) = 0 \}
\]

where \( \mathcal{L}_v \) denotes the Lie derivative along \( v \), \( x, \alpha \in \Gamma(TZ) \), and \( \alpha \in \Gamma(T^*Z) \).

**Proposition 1.5.** For any invariant section \( x + \alpha \in \Gamma(TZ \oplus T^*Z)^{S_1} \), there exists a unique \( X \in \Gamma(M, TM) \), \( \xi \in \Omega^1(M) \), and \( f, g \in C^\infty(M) \) such that

\[
(1.1) \quad x + \alpha = h_A(X) + f v + \pi^*(\xi) + gA \in \Gamma(TZ \oplus T^*Z)^{S_1},
\]

where \( h_A \) denotes the horizontal lift with respect to the connection \( A \), and \( v \) the invariant period-1 generator of the circle action on \( Z \).

From this result, it is clear that there is a bijective correspondence between such invariant sections over \( Z \) and arbitrary sections of the bundle \( E := TM \oplus \mathbb{1}_R \oplus T^*M \oplus \mathbb{1}_R \) over \( M \) (where here \( \mathbb{1}_R \) denotes the trivial real line bundle over \( M \)).

Consider the \( [H] \)-twisted standard Courant algebroid over the principal circle bundle \( Z \), and let \( H = H^{(3)} + A \wedge H^{(2)} \) be an invariant representative of \( [H] \), i.e. \( H \in [H] \in H^3(Z) \). We want to restrict all the structures so that they are only defined on the invariant sections, i.e.

\[
\langle \cdot, \cdot \rangle : \Gamma(TZ \oplus T^*Z)^{S_1} \otimes \Gamma(TZ \oplus T^*Z)^{S_1} \to \Gamma(TZ \oplus T^*Z)^{S_1}
\]

where we observe that the invariant sections are closed under the Dorfman bracket.

Now using the bijection \( \phi \) and the fact that the Dorfman bracket is closed under invariant sections, we can transfer the standard Courant algebroid structures over \( Z \) (restricted to the invariant sections) to a Courant algebroid structure over \( M \) with vector bundle \( E = TM \oplus \mathbb{1}_R \oplus T^*M \oplus \mathbb{1}_R \). Doing so, we get that the anchor map \( \rho : E \to TM \) is the composition of the anchor map \( \rho : E \to TZ \) with the pushforward \( \pi_* : TZ \to TM \), whilst the remaining structures defined on the sections \( \Gamma(E) = \Gamma(TM \oplus \mathbb{1}_R \oplus T^*M \oplus \mathbb{1}_R) \) are:

- \( \langle (X, f, \alpha, g), (Y, \tilde{f}, \beta, \tilde{g}) \rangle = \beta(X) + \alpha(Y) + g \tilde{f} + f \tilde{g} \)
- \( [(X, f, \alpha, g), (Y, \tilde{f}, \beta, \tilde{g})]_H = (\phi^{-1}([\phi(X, f, \alpha, g), \phi(Y, \tilde{f}, \beta, \tilde{g})]_H)) = ([X, Y], X(\tilde{f}) - Y(f) + \iota_X \iota_Y F_A, L_X \beta - \iota_Y d\alpha + \tilde{g} df + f \tilde{g} df + g \iota_X F_A - g \iota_Y F_A + \iota_X \iota_Y H^{(3)} + \tilde{f} \iota_X H^{(2)} - \iota_Y H^{(2)}, X(\tilde{g}) - Y(g) + \iota_X \iota_Y H^{(2)}) \).

Then \( E \) defines a non-exact Courant algebroid over \( M \), encoding the invariant data of the standard Courant algebroid \( TZ \oplus T^*Z \). We denote this invariant Courant algebroid over \( M \) by \( (TZ \oplus T^*Z)^{S_1} \).

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\(^2\)Henceforth, in keeping with the literature, instead of writing \( h_A(X) \) will simply write \( X \). Thus, we will take \( X \) to mean both the vector field on \( TM \) as well as its horizontal lift \( h_A(X) \), with which we mean, being clear from the context. Likewise for the basic forms.
Remark 1.7. Notation: we will often express the bracket in terms of invariant sections over \( Z \) to emphasize that the data we are interested in is related to the bundle \( Z \), however the relevant maps will be those with respect to the bundle over \( M \).

1.2. Topological T-duality. We are going to be interested in topological T-duality arising for the case of principal circle bundles with a H-flux. For such a case, one begins with a principal circle bundle \( \pi : Z \to M \) whose first Chern class is given by \([F] \in H^2(M, \mathbb{Z})\), along with a H-flux given by some \([H] \in H^3(Z, \mathbb{Z})\). The aim is to then determine the corresponding data arising after the application of T-duality.

One method for determining the T-dual data is by focusing on the Gysin sequence associated to the bundle \( \pi : Z \to M \), given by:

\[
\cdots \to H^3(M, \mathbb{Z}) \xrightarrow{\pi^*} H^3(Z, \mathbb{Z}) \xrightarrow{\pi_*} H^2(M, \mathbb{Z}) \xrightarrow{[F]^\wedge} H^4(M, \mathbb{Z}) \to \cdots
\]

Letting \([H] \in H^3(Z, \mathbb{Z})\), define \([\hat{F}] = \pi_*([H]) \in H^2(M, \mathbb{Z})\), and make the choice of some principal circle bundle \( \hat{\pi} : \hat{Z} \to M \) with first Chern class \([\hat{F}]\). Having made such a choice, we then consider the Gysin sequence associated to the bundle \( \hat{Z} \) over \( M \),

\[
\cdots \to H^3(M, \mathbb{Z}) \xrightarrow{\hat{\pi}^*} H^3(\hat{Z}, \mathbb{Z}) \xrightarrow{\hat{\pi}_*} H^2(M, \mathbb{Z}) \xrightarrow{[\hat{F}]^\wedge} H^4(M, \mathbb{Z}) \to \cdots
\]

Now using exactness, and the fact that \([F] \wedge [\hat{F}] = [F] \wedge \pi_*([H]) = 0\), there exists an \([\hat{H}] \in H^3(\hat{Z}, \mathbb{Z})\) such that \([\hat{F}] = \hat{\pi}_*([\hat{H}]\)). The following theorem gives a a global, geometric version of the Buscher rules [10].

**Theorem 1.8** ([10]). Let \( \pi : Z \to M \) denote a principal circle bundle whose first Chern class is given by \([F] \in H^2(M, \mathbb{Z})\), and let \([H] \in H^3(Z, \mathbb{Z})\) denote a H-flux on \( Z \).

Then there exists a T-dual bundle \( \hat{\pi} : \hat{Z} \to M \) whose first Chern class is denoted \([\hat{F}] \in H^2(M, \mathbb{Z})\) and a T-dual H-flux on the T-dual bundle, \([\hat{H}] \in H^3(\hat{Z}, \mathbb{Z})\), satisfying

\[
[\hat{F}] = \pi_*([H]),
\]

\[
[F] = \hat{\pi}_*([\hat{H}]\)).
\]

Furthermore, letting \( Z \times_M \hat{Z} = \{(a, b) \in Z \times \hat{Z} | \pi(a) = \hat{\pi}(b)\}\), and considering the maps

\[
\begin{align*}
Z \times_M \hat{Z} & \xrightarrow{p} Z \quad \xrightarrow{\pi} M \\
\hat{Z} & \xrightarrow{\hat{\pi}} \hat{Z}
\end{align*}
\]

then if the two H-fluxes \([H]\) and \([\hat{H}]\) satisfy

\[
p^*([H]) = \hat{p}^*([\hat{H}]\),
\]

the T-dual pair is unique up to bundle automorphism, thus defining the T-duality transformation.

**Theorem 1.9** ([10]). Let \( A, \hat{A} \) denote connection forms on \( Z \) and \( \hat{Z} \) respectively, choose some invariant representative \( H \in [H] \) and \( \hat{H} \in [\hat{H}] \), and let \((\Omega^*_S(Z), d + H)\) denote the H-twisted \( \mathbb{Z}_2 \)-graded differential complex of invariant differential forms\(^3\)

\(^3\)Here we have \( \Omega^*_S(Z) := \{ \omega \in \Omega^*(Z)| \mathcal{L}_v(\omega) = 0 \}, \) where \( v \) the invariant period-1 generator of the circle action on \( Z \), and \( \mathcal{L}_v \) the Lie derivative along \( v \).
Then the following map:
\[ T : (\Omega^*_S(Z), d + H) \rightarrow (\Omega^{*+1}_S(\hat{Z}), -(d + \hat{H})) \]
\[ \omega \mapsto \int_{S^1} \omega \wedge e^{A \wedge A}, \]
is a chain map isomorphism between the twisted, \( \mathbb{Z}_2 \)-graded complexes.

Furthermore, this induces an isomorphism on the twisted cohomology:
\[ T : H^{*}_d(Z) \rightarrow H^{*+1}_d(\hat{Z}). \]

In [5], twisted K-theory was proposed as classifying D-branes in a background flux. The proposal was consolidated in [6, 27]. For an account of T-duality in the absence of a background flux, see [23]. In [7], it was also first established that T-duality also gives an isomorphism of twisted K-theory groups,
\[ T : K^*_H(Z) \rightarrow K^{*+1}_H(\hat{Z}). \]

In [11, 26] there are alternate approaches to proving this, via a T-duality classifying space and via noncommutative geometry respectively.

1.3. Exotic differential forms. In [19], the T-duality result in Theorem 1.9 was extended to the full space of the complex-valued differential forms on one principal circle bundle, and the striking thing obtained is that the T-dual is the space of exotic differential forms on the T-dual principal circle bundle. The definition of exotic differential forms was inspired by their previous work, [18].

Let \( L, \hat{L} \) denote the complex line bundles associated to the circle bundles \( Z, \hat{Z} \) and the standard representation of the circle on complex plane respectively. Define the exotic differential forms by
\[ A^\bar{k}(Z)^{S^1} = \bigoplus_{n \in \mathbb{Z}} A^\bar{k}_n(Z)^{S^1} := \bigoplus_{n \in \mathbb{Z}} \Omega^{\bar{k}}(Z, \pi^*(\hat{L}^\otimes n))^{S^1}, \]
\[ A^\bar{k}(\hat{Z})^{S^1} = \bigoplus_{n \in \mathbb{Z}} A^\bar{k}_n(\hat{Z})^{S^1} := \bigoplus_{n \in \mathbb{Z}} \Omega^{\bar{k}}(\hat{Z}, \hat{\pi}^*(L^\otimes n))^{S^1} \]
for \( \bar{k} = k \mod 2 \) for \( \bar{k} = k \mod 2 \).

Define the subspace of weight \(-n\) differential forms on \( Z \),
\begin{align*}
(1.2) \quad \Omega_{-n}^*(Z) &= \{ \omega \in \Omega^*(Z) | \mathcal{L}_n \omega = -n \omega \}.
\end{align*}

It is easy to see that
\[ \Omega^\bar{k}_n(Z) = \Omega_{-n}^{\bar{k}}(Z)^{S^1}, \quad A^\bar{k+1}_n(\hat{Z})^{S^1} = \Omega_{-n}^{\bar{k+1}}(\hat{Z})^{S^1}. \]

Under the above choices of Riemannian metrics and flux forms, their results show that the Fourier-Mukai transform \( T \) can be extended to a sequence of isometries,
\begin{align*}
(1.3) \quad \tau_n : \Omega_{-n}^*(Z) &\rightarrow A^{\bar{k+1}}_n(\hat{Z})^{S^1},
\end{align*}
for \( \bar{k} = k \mod 2 \), and is defined by the exotic Hori formula from \( Z \) to \( \hat{Z} \) given in [19] when \( n = 0 \), \( \tau_0 = T \). The twisted de Rham differential \( d + H \) maps to the differential \(-\hat{\pi}^* \nabla^L \otimes -\iota_{n\theta} + \hat{H})\). One similarly has a sequence of isometries,
\begin{align*}
(1.4) \quad \sigma_n : A^\bar{k}_n(Z)^{S^1} &\rightarrow \Omega^{\bar{k+1}}_{-n}(\hat{Z}),
\end{align*}
for \( \bar{k} = k \mod 2 \), and is defined by the inverse exotic Hori formula form \( Z \) to \( \hat{Z} \) given in equation [19] and the differential \( \pi^* \nabla^L \otimes -\iota_{n\theta} + H \) maps to the twisted de Rham differential \(-(d + \hat{H})\). Note that \( \sigma_0 = T \). Similarly, one can define the sequences of
isometries $\hat{\tau}_n, \hat{\sigma}_n$ on $\hat{Z}$. Although the extension of the Fourier-Mukai transform to all differential forms on $Z$ is slightly asymmetric, one has the crucial identities, verified in [19].

\begin{align}
-\text{Id} = \hat{\sigma}_n \circ \tau_n : \Omega^k_{\tau n}(Z) & \longrightarrow \Omega^k_{\tau n}(Z), \\
-\text{Id} = \hat{\tau}_n \circ \sigma_n : A^k_{\sigma n}(Z)^{S^1} & \longrightarrow A^k_{\sigma n}(Z)^{S^1}.
\end{align}

This is interpreted as saying that T-duality, when applied twice, returns one to minus of the identity. It was verified in the special case when $n = 0$ in [7, 8]. We would like to point out that the minus sign comes from the convention of integration along the fiber.

This shows that for each of either $Z$ or $\hat{Z}$, there are two theories (at degree 0 the two theories coincide), and there are also graded isomorphisms between the two theories of both sides.

Moreover, when $n \neq 0$, the complex $(A^k_{\tau n}(\hat{Z})^{\hat{S}^1}, \hat{\pi}^* \nabla L^n_{\tau n} - \tau n\hat{\nu} + \hat{H})$ has vanishing cohomology. Therefore, when $n \neq 0$, the complex $(\Omega^k_{\tau n}(Z), d + H)$ also has vanishing cohomology. In [19], an explicit homotopy is constructed to show this.

1.4. T-duality of Courant algebroids. T-duality has a natural extension to a duality between certain Courant algebroids defined on principal circle bundles. To see this, again, consider a principal circle bundle $\pi : Z \rightarrow M$ with invariant $H$-flux representative $H \in \Omega^{3}(Z)^{S^1}$, and let $\hat{\pi} : \hat{Z} \rightarrow M$ be the T-dual principal circle bundle over $M$ with T-dual $H$-flux $\hat{H} \in \Omega^{3}(\hat{Z})^{S^1}$.

**Theorem 1.10** ([12]). Consider the invariant Courant algebroids defined over $M$ given by $(T \hat{Z} \oplus T^* \hat{Z})^{S^1}$, $[\cdot, \cdot]_H$ and $(T \hat{Z} \oplus T^* \hat{Z})^{S^1}$, $[\cdot, \cdot]_{\hat{H}}$ (as introduced in example 1.6). Then there exists a Courant algebroid isomorphism between these these objects, given by

\[\chi : (T \hat{Z} \oplus T^* \hat{Z})^{S^1} \rightarrow (T \hat{Z} \oplus T^* \hat{Z})^{S^1}\]

\[\chi(X + f v + \xi + gA) = (X + g \hat{\nu} + \xi + f \hat{A})\]

where $X \in \Gamma(M, TM)$, $\xi \in \Omega^1(M)$, and $f, g \in C^\infty(M)$.

Furthermore, this map defines an isomorphism between Clifford algebras, and so

\[T(a \cdot \omega) = \chi(a) \cdot T(\omega),\]

where $a \in \Gamma(T \hat{Z} \oplus T^* \hat{Z})^{S^1}$ and $\omega \in \Omega^*(Z)^{S^1}$.

Our goal is to generalize this T-duality isomorphism from the invariant Courant algebroid $(T \hat{Z} \oplus T^* \hat{Z})^{S^1}$ over $M$, to an isomorphism from the exact Courant algebroid $(T \hat{Z} \oplus T^* \hat{Z})$ over $Z$, but viewed as an infinite dimensional (graded) Courant algebroid over $M$. We mention that another generalization of Theorem 1.10 to the invariant chiral de Rham complex was established in [25].

2. Exotic Courant algebroids

In this section, we introduce the novel concept of exotic Courant algebroids (see Definition 2.1), which plays a central role in our formulation of T-duality in the last section. We also classify the exact exotic Courant algebroids over a manifold in Proposition 2.4.
2.1. Exotic Courant algebroids. The objects we are going to be interested in are infinite dimensional bundles which are constructed from a (complex) Courant algebroid, $(E, \langle \cdot, \cdot \rangle, [, ], \rho)$, and a (complex) line bundle $L$ over $M$ which has a connection $\nabla : \Gamma(L) \to \Gamma(L \otimes T^*M)$. In particular, we will be interested in the bundle

$$\pi : \bigoplus_{n \in \mathbb{Z}} (E \otimes L^\otimes n) \to M,$$

and we will aim to extend the properties of the Courant algebroid to this infinite dimensional bundle.

On such an object, we will define a bracket, a product and an anchor map. The anchor map for the Courant algebroid is a bundle map $\rho : E \to TM$, where the tangent bundle over $M$, $TM$, is endowed with a Lie algebroid structure (which by definition defines an action on $C^\infty(M)$). Similarly, for the case of the exotic Courant algebroid, we will define our anchor map to be a bundle map $\rho := \bigoplus \rho_n$ where

$$\rho_n : E \oplus L^\otimes n \to TM \otimes L^\otimes n,$$

and we will need to construct some ‘Lie algebroid’-like structure on $TM \otimes L^\otimes n$, which defines an action on section $\Gamma(L^\otimes m)$ for all $m \in \mathbb{Z}$.

To do this, first let the sections $X \otimes s_1 \in \Gamma(TM \otimes L^\otimes n)$ act on $\omega \otimes s_2 \in \Omega^1(M, L^\otimes m)$ by

$$(X \otimes s_1) : \omega \otimes s_2 \mapsto (\iota_X \omega)(s_1 \otimes s_2) \in \Gamma(L^\otimes (n+m)).$$

Now to define a bracket on the infinite dimensional bundle $\bigoplus_n (TM \otimes L^\otimes n)$ over $M$, we let $[\cdot, \cdot] : \Gamma(TM \otimes L^\otimes n) \otimes \Gamma(TM \otimes L^\otimes m) \to \Gamma(TM \otimes L^\otimes (m+n))$, such that

$$[X \otimes s_1, Y \otimes s_2] := [[1 \otimes \nabla^p + d \otimes 1, X \otimes s_1], Y \otimes s_2],$$

where $X \otimes s_1 \in \Gamma(TM \otimes L^\otimes n)$, $Y \otimes s_2 \in \Gamma(TM \otimes L^\otimes m)$, $1$ denotes the identity map, the bracket on the right hand side is the commutator, and the value of $p$ changes depending on the sections it acts on, i.e. $p = n$ when acting on $s_1$ and $p = m$ when acting on $s_2$.

This bracket satisfies the conditions:

1. $[a, b] + [b, a] = 0$,
2. $[a, [b, c]] = [[a, b], c] + [b, [a, c]],$
3. $[a, bh] = a(h)b + h[a, b],$

where $a := X \otimes s \in \Gamma(TM \otimes L^\otimes n)$, $b \in \Gamma(TM \otimes L^\otimes m)$, $c \in \Gamma(TM \otimes L^\otimes p)$ and $h \in \Gamma(L^\otimes p)$, and the action of $a$ on $h$ is given by

$$a(h) := \iota_X (\nabla^p h) \otimes s.$$

We define the bundle $\bigoplus_n (TM \otimes L^\otimes n)$ over $M$ (along with its bracket and action on sections) to be the standard exotic Lie algebroid of the data $(M, L, \nabla)$.

Now we can introduce the concept of an exotic Courant algebroid.

**Definition 2.1.** Let $M$ be a manifold, $E$ a (complex) Courant algebroid over $M$, $L$ a (complex) line bundle over $M$ with connection $\nabla$, and $\mathcal{L} := \bigoplus_{n \in \mathbb{Z}} L^\otimes n$.

An (complex) exotic Courant algebroid over $M$ is given by an infinite-dimensional vector bundle

$$\mathcal{E} := \bigoplus_{n \in \mathbb{Z}} (E \otimes L^\otimes n) \to M,$$

along with a non-degenerate bilinear map $\langle \cdot, \cdot \rangle : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{L})$, a bilinear bracket $[\cdot, \cdot] : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, a bundle map (called the anchor map) $\rho : \mathcal{E} \to \bigoplus_{n \in \mathbb{Z}} (TM \otimes L^\otimes n)$,
and an induced differential operator \( D : \Gamma(L) \to \Gamma(\mathcal{E}) \) defined by the relation
\[
\langle Dh, a \rangle = \rho(a)h,
\]
such that the following properties are satisfied:

1. \([a, [b, c]] = [[a, b], c] + [b, [a, c]]\),
2. \(\rho([a, b]) = [\rho(a), \rho(b)]\),
3. \([a, hb] = \rho(a)(h)b + h[a, b]\),
4. \([a, b] + [b, a] = D(a, b)\), where \(D\) denotes some differential.
5. \(\rho(a)\langle b, c \rangle = \langle [a, b], c \rangle + \langle b, [a, c] \rangle\).

where \(a, b, c \in \Gamma(\mathcal{E})\), \(h \in \Gamma(L)\), and \([, , ]\) denotes the Lie bracket for the standard exotic Lie algebroid of \((M, L, \nabla)\).

**Example 2.2.** The standard exotic Courant algebroid over \(M\) is obtained by taking \(E = TM \oplus T^*M\) and some choice of line bundle \(L \to M\) with connection \(\nabla\).

**Definition 2.3.** An exotic Courant algebroid \(\mathcal{E}\) is said to be exact if there exists a short exact sequence
\[
0 \to \bigoplus_{n \in \mathbb{Z}} (T^*M \otimes L^\otimes n) \xrightarrow{\partial \otimes \phi} \bigoplus_{n \in \mathbb{Z}} (E \otimes L^\otimes n) \xrightarrow{\partial \otimes \phi} \bigoplus_{n \in \mathbb{Z}} (TM \otimes L^\otimes n) \to 0,
\]
for some bundle automorphism \(\phi : L \to L\).

**Proposition 2.4** (Classification of exotic exact Courant algebroids). (1) For a fixed line bundle and fixed connection, \((L, \nabla)\) over \(M\), the exact exotic complex Courant algebroids defined by \((E, L, \nabla)\) are classified by the third cohomology class of \(\bar{H}^3(M, \mathbb{C})\).

(2) The set of all complex, exact, exotic Courant algebroids are classified by the group
\[
\bar{H}^2(M, \mathbb{C}) \oplus \bar{H}^2(M)
\]
where \(\bar{H}^2(M)\) denotes the 2nd differential cohomology group, that classifies line bundles with connection over \(M\).

(Part 1) follows from the classification of (complexified) exact Courant algebroids due to Severa, which can be found in the references on exact Courant algebroids [2, 11, 17].

Part (2) follows from Part (1) and the classification of line bundles with connection by the differential cohomology group \(\bar{H}^2(M)\). We remark that there are several models for differential cohomology groups, such as differential characters [13, 11] where \(\bar{H}^2(M)\) is discussed in Example 19, page 23 [11] and Deligne cohomology \(\bar{H}^1(M, U(1) \xrightarrow{\hat{\sigma}} \Omega^1(M; \mathbb{R}))\),
\[
\hat{\sigma} = \frac{1}{2\pi} d \circ \log
\]

where \(\hat{\sigma}(M, U(1) \xrightarrow{\hat{\sigma}} \Omega^1(M; \mathbb{R}))\) denotes the \(\hat{\sigma}\)Cech hypercohomology group defined in [9].

2.2. **Exotic Courant algebroid examples.** We are now going to introduce two complex, exotic Courant algebroids defined over a principal circle bundle. We motivate the first example from the definition of the standard Courant algebroid over a principal circle bundle, and show that it is isomorphic to an exotic Courant algebroid through the consideration of Fourier expansions about the circle dimension.

**Example 2.5.** Consider the standard Courant algebroid \(\bar{\pi} : TZ \oplus T^*Z \to Z\) where \(\pi : Z \to M\) denotes a \(S^1\)-principal bundle, and let \(\{U_\alpha\}\) denote a good cover on \(M\) such that \(\pi^{-1}(U_\alpha) \cong U_\alpha \times S^1\). Then on the bundle \(Z\) over \(M\), let \(A \in \Omega^1(Z; \mathbb{R})\) denote a non-normalized connection form, which can be expressed locally over \(\pi^{-1}(U_\alpha)\) by
\[
A|_{\alpha} = 2\pi i d\theta + A^*_\alpha,
\]
where \( \theta_\alpha \) denotes the local circle coordinate and \( A_\alpha^* \) is a basic form with respect to the bundle \( \pi \).

Now let \( p: L \to M \) denote the complex line bundle associated to \( Z \), and take the local, nowhere-zero section of \( L \) given by \( \{ s_\alpha \} \in \Gamma(U_\alpha, L|U_\alpha) \) corresponding to the constant map \( U_\alpha \to \{ 1 \} \subset S^1 \).

Now on the infinite dimensional bundle \( \mathcal{E} = \bigoplus_{n \in \mathbb{Z}} E_n := \bigoplus_{n \in \mathbb{Z}} ((TZ \oplus T^* Z) \otimes \pi^*(L^{\otimes n})) \) letting \( n = 0 \), we define the following structures:

- **Differential operator:** Letting \( \nabla^L \) denote the induced connection on \( L \) (induced from the connection \( A \) on \( Z \)), take \( D = \bigoplus_n D_n \) where:
  \[
  D_n = \nabla^{L^{\otimes n}} - nA : \Gamma(L^{\otimes n}) \to \Gamma(E_n).
  \]
  This operator (along with the twisted operator \( D + H \) where \( H \) is an invariant closed 3-form), squares to zero. This is clear when the operator is expanded in local coordinates, say over \( \pi^{-1}(U_\alpha) \), as then: \( (\nabla^{L^{\otimes n}} - nA)^2 = (d - 2\pi i n d\theta_\alpha)^2 = 0 \).

- **Dorfman Bracket:** The bracket on sections of the weighted spaces \( E_n \) is the \( H \)-twisted, derived bracket of the above differential:
  \[
  \{ \cdot, \cdot \}_H : \Gamma(E_n) \otimes \Gamma(E_m) \to \Gamma(E_{n+m})
  \]
  \[
  (a, b_m) \mapsto [a_n, a_m]_H := [[D + H, a_n], a_m].
  \]

- **Bilinear Product:** The \( C^\infty(M) \)-bilinear product on \( E \) which, when restricted to the weight spaces \( E_p \), is given by:
  \[
  \langle \cdot, \cdot \rangle : \Gamma(E_n) \times \Gamma(E_m) \to \Gamma(L^{\otimes (m+n)})
  \]
  \[
  (a \otimes s^n, b \otimes s^m) \mapsto \langle a, b \rangle_{C,A}(s^n \otimes s^m),
  \]
  where \( \langle \cdot, \cdot \rangle_{C,A} \) denotes the inner product on the Courant algebroid \( E = TM \oplus 1_C \oplus T^* M \oplus 1_C \) over \( M \) defined in Example 1.6.

- **Anchor map:** The bundle map \( \rho = \oplus_n \rho_n \), where \( \rho_n : E_n \to TM \otimes L^{\otimes n} \) is the bundle map defined by the relation
  \[
  \langle a_n, Dh \rangle = \rho_n(a_n) h,
  \]
  where \( h \in \Gamma(L^{\otimes n}) \) and \( a_n \in \Gamma(E_n) \).

Such a structure defines an exotic Courant algebroid over \( M \).

We now detail how this exotic Courant algebroid is in fact equivalent to an exact Courant algebroid, where the data defined on one of these objects can be transferred to the other, and vice versa.

To do this we will use the fact that every element \( a \in \Gamma(Z, TZ \oplus T^* Z) \) can be expressed locally over \( \pi^{-1}(U_\alpha) \cong U_\alpha \times S^1 \) as a sum over its weight space \( \Gamma_n(TZ \oplus T^* Z) := \{ a \in \Gamma_n(TZ \oplus T^* Z) | L_a a = na \} \), using the family Fourier expansion:

\[
 a_{|U_\alpha} = \sum_{n \in \mathbb{Z}} e^{-2\pi in\theta_\alpha} (X_{n,a} f_{n,a} v + \xi_{n,a} + g_{n,a} A) \in \bigoplus_{n \in \mathbb{Z}} \Gamma_n \left( T^*Z \oplus T^* Z |_{\pi^{-1}(U_\alpha)} \right),
\]

\[\text{Observe that we incorrectly expressed the map as } D_n = \nabla^{\otimes n} - nA, \text{ which maps on the sections over } Z \text{ not over } M. \text{ It should be understood that the map of interest defined over } M \text{ is } \psi \circ D_n \circ \psi^{-1}. \text{ This will be implicit from here on out} \]
where $X_{n,\alpha} \in \Gamma(TU_a)$ is a horizontal vector field, $\xi_{n,\alpha} \in \Gamma(T^*U_a)$ is a basic form, and $f_{n,\alpha}, g_{n,\alpha} \in C^\infty(U_a).$

**Proposition 2.6.** The exotic Courant algebroid over $M$ defined in Example (2.5) is equivalent to the $H$-twisted standard Courant algebroid over $Z$ detailed in Example (1.3).

**Proof.** Consider the map given by
\[
\phi = \oplus_{n \in \mathbb{Z}} \phi_n : \Gamma(Z, TZ \oplus T^*Z) \to \Gamma((TZ \oplus T^*Z) \otimes \pi^*(L^{\otimes n}))^S_1
\]
that, when decomposed into its weight spaces, defines the maps
\[
\phi_n : \Gamma_n(Z, TZ \oplus T^*Z) \to \Gamma((TZ \oplus T^*Z) \otimes \pi^*(L^{\otimes n}))^S_1.
\]
which can locally be expressed over $\pi^{-1}(U_a)$ by the invertible map
\[
\phi_n(e^{-2\pi i \theta_n}(X_{n,\alpha} + f_{n,\alpha} \eta + \xi_{n,\alpha} + g_{n,\alpha} A))|_{\pi^{-1}(U_a)} = (X_{n,\alpha} + f_{n,\alpha} \eta + \xi_{n,\alpha} + g_{n,\alpha} A) \otimes \pi^*(s^n_\alpha).
\]

Next, observe that there is a bijective correspondence between the invariant sections over $Z$ and sections over $M$ given by the map:
\[
\psi : \Gamma((TM \oplus 1_C \oplus T^*Z \oplus 1_C) \otimes L^{\otimes n}),
\]
which can be expressed locally over on each weight space by
\[
\psi((X_{n,\alpha} + f_{n,\alpha} \eta + \xi_{n,\alpha} + g_{n,\alpha} A) \otimes \pi^*(s^n_\alpha))|_{\pi^{-1}(U_a)} = (X_{n,\alpha} + f_{n,\alpha} \eta + \xi_{n,\alpha} + g_{n,\alpha} A) \otimes s^n_\alpha.
\]
Let $\bar{\phi} = \psi \circ \phi$ which defines a bijection by construction. Then $\bar{\phi}$ satisfies the following conditions:
\[
\begin{align*}
[\bar{\phi}(a), \bar{\phi}(b)]_H &= \bar{\phi}([a, b]_{H,C,A}), \\
\langle \bar{\phi}(a), \bar{\phi}(b) \rangle &= \bar{\phi}(\langle a, b \rangle_{C,A}), \\
\rho(\bar{\phi}(a)) &= \bar{\phi} \circ \rho(a),
\end{align*}
\]
where $a, b \in \Gamma(TZ \oplus T^*Z)$, and we use $\bar{\phi} : C^\infty(Z) \xrightarrow{\approx} \oplus_n \Gamma(L^{\otimes n})$, given locally by
\[
\bar{\phi}_n(e^{-2\pi i \theta_n}) = s^n_\alpha.
\]
By decomposing the Courant algebroid $TZ \oplus T^*Z$ into its weight spaces we can transfer all its data to that of the exotic Courant algebroid. Given this map is invertible, the result is proven. \hfill \Box

**Remark 2.7.** We will often refer to $TZ \oplus T^*Z$ as having an exotic Courant algebroid structure, where by this, we mean the exotic Courant algebroid $\mathcal{E}$ of which it is equivalent to.

Keeping all notation the same, and once again letting $\hat{\pi} : \hat{Z} \to M$ denote the T-dual bundle to $Z$ with invariant T-dual H-flux representative $\hat{H} \in \Omega^1(\hat{Z})$ we come to our second example:

**Example 2.8.** Consider the infinite-dimensional bundle over $M$ given by the space
\[
\hat{\mathcal{E}} = \bigoplus_{n \in \mathbb{Z}} \hat{E}_n := \bigoplus_{n \in \mathbb{Z}} ((T\hat{Z} \oplus T^*\hat{Z}) \otimes \hat{\pi}^*(L^{\otimes n}))^S_1 = \bigoplus_{n \in \mathbb{Z}} ((TM \oplus 1_C \oplus T^*M \oplus 1_C) \otimes L^{\otimes n}),
\]
and consider the following structures defined on it:

---

5Recall that $X_{n,\alpha}$ should be written as $h^{-1}_A(X_{n,\alpha})$ where $h_A : M \to Z$ denotes the horizontal lift associated to the connection $\nabla$, and $\xi_{n,\alpha}$ should be written as $\pi^*(\xi_{n,\alpha})$. Keeping with the literature, however, we will take $X_{n,\alpha}$ to mean both the vector field on $TU_a$ as well as its horizontal lift defined on $TZ|_{\pi^{-1}(U_a)}$, with which being clear from the context. Likewise for the basic forms.
• **Differential operator:** The differential operator given by 
\[ \hat{D} := \bigoplus_n \hat{D}_n \]
where 
\[ \hat{D}_n = - (\mathring{\nabla}^\otimes n - i_{n\nu}) : \Gamma(L^\otimes n) \to \Gamma(M, \hat{E}_n). \]
The operator 
\[- (\mathring{\nabla}^\otimes n - i_{n\nu} + \hat{H}) \]
squares to zero for some invariant closed 3-form \( \hat{H} \), as detailed in Theorem 2.1 of \[19\].

• **Dorfman Bracket:** The \( \hat{H} \)-twisted derived bracket of the above differential, defined on the weight spaces by:
\[
[\cdot, \cdot]_{\hat{H}} : \Gamma(\hat{E}_n) \otimes \Gamma(\hat{E}_m) \to \Gamma(\hat{E}_{n+m}) \\
(a_n, b_m) \mapsto [a_n, a_m]_{\hat{H}} := [\hat{D} + \hat{H}, a_n], a_m].
\]

• **Bilinear product:** The \( \mathcal{C}^\infty(M) \)-bilinear product which, when restricted to the weight spaces \( \hat{E}_n, \hat{E}_m \), is given by:
\[
\langle \cdot, \rangle : \Gamma(\hat{E}_n) \times \Gamma(\hat{E}_m) \to \Gamma(L^\otimes (m+n)) \\
(a \otimes s_1, b \otimes s_2) \mapsto \langle a, b \rangle_{\mathcal{C},A}(s_1 \otimes s_2),
\]
where \( \langle \cdot, \cdot \rangle_{\mathcal{C},A} \) defines the inner product on sections of the Courant algebroid \( TM \oplus \mathbb{R}C \oplus T^*M \oplus \mathbb{R}C \) over \( M \) defined in Example 1.6.

• **Anchor map:** The bundle map \( \hat{\rho} = \bigoplus_n \hat{\rho}_n \), where 
\[ \hat{\rho}_n : \hat{E}_n \to (TM \oplus \mathbb{R}C) \otimes L^\otimes n, \]
is the bundle map defined by the relation
\[ \langle a_n, \hat{D}h \rangle = \hat{\rho}_n(a_n)h, \]
where \( h \in \bigoplus_n \Gamma(L^\otimes n) \) and \( a_n \in \Gamma(\hat{E}_n) \).

This bundle along with the above structures defines an exotic Courant algebroid over \( M \). Observe that although the bundle of this example is the same as that of Example 2.5, the remaining data is distinct, and thus the resulting exotic Courant algebroids are also distinct.

### 3. T-duality for exotic Courant algebroids

In this section, we prove our main T-duality isomorphism for exotic Courant algebroids in Theorem 3.1 that generalizes the T-duality isomorphism in \[12\] on the invariant Courant algebroid \( (TZ \oplus T^*Z)^{S^1} \) over \( M \), to an isomorphism from the exact Courant algebroid \( (TZ \oplus T^*Z) \) over \( Z \), but viewed as an infinite dimensional (graded) Courant algebroid over \( M \). We also define a Clifford action of exotic Courant algebroids on exotic differential forms in Theorem 3.2 that is compatible with T-duality. Subsection 3.1 illustrates this isomorphism in the special case of trivial circle bundles with trivial flux.

Let \( \tau = \bigoplus_n \tau_n \) where \( \tau_n : \mathcal{O}_{\mathcal{L}}^n(Z) \to A_n^{k+1}(Z)^{S^1} \) denote the exotic Hori formula as defined in \[19\], which extends the standard T-duality isomorphisms on invariant differential forms.

Recall the exotic Courant algebroid from Example 2.2 and Example 2.5 defined over \( Z \) and \( M \) respectively, given by:
\[
\mathcal{E}_Z := \bigoplus_n \mathcal{E}_{Z,n} = \bigoplus_n ((TZ \oplus T^*Z) \otimes \pi^*(L^\otimes n)),
\]
\[
\mathcal{E}_M := \bigoplus_n \mathcal{E}_{M,n} = \bigoplus_n ((TM \oplus \mathbb{R}C \oplus T^*M \oplus \mathbb{R}C) \otimes L^\otimes n).
\]
Similarly, the exotic Courant algebroid from Example 2.2 and Example 2.8 defined over \( Z \) and \( M \) respectively, given by:

\[
\hat{E}_Z := \bigoplus_n \hat{E}_{Z,n} = \bigoplus_n \left( (T\hat{Z} \oplus T^*\hat{Z}) \otimes \hat{\pi}^*(L^{\otimes n}) \right),
\]

\[
\hat{E}_M := \bigoplus_n \hat{E}_{M,n} = \bigoplus_n \left( (TM \oplus \mathbb{C} \oplus T^*M \oplus \mathbb{C}) \otimes L^{\otimes n} \right),
\]

and we recall that there is a bijection, \( \psi : \Gamma(E_Z)^{s1} \rightarrow \Gamma(E_M) \) and \( \hat{\psi} : \Gamma(\hat{E}_Z)^{s1} \rightarrow \Gamma(\hat{E}_M) \).

Defining the map \( \varphi = \oplus_n \varphi_n \) where \( \varphi_n : \Gamma(E_{M,n}) \rightarrow \Gamma(\hat{E}_{M,n}) \) is given by

\[
\varphi_n((X,f,\xi,g) \otimes s^n) = -(X,g,\xi,f) \otimes s^n,
\]

and using the map \( \hat{\phi} \) from (2.10)

\[
\hat{\phi} : \Gamma(Z,TZ \oplus T^*Z) \rightarrow \Gamma(E_M),
\]

we want to consider the composition:

(3.1)

\[
\mu := \varphi \circ \hat{\phi} : \Gamma(TZ \oplus T^*Z) \rightarrow \Gamma(\hat{E}_M),
\]

where we can decompose this map such that \( \mu = \bigoplus_n \mu_n \) and

\[
\mu_n : \Gamma(-n)(TZ \oplus T^*Z) \rightarrow \Gamma(\hat{E}_{M,n}).
\]

**Theorem 3.1** (T-duality of exotic Courant algebroids). The map defined in equation (3.1) above:

\[
\mu := \Gamma(TZ \oplus T^*Z) \rightarrow \Gamma(\hat{E}_M),
\]

is an isomorphism between the exotic Courant algebroid structures of each space. Furthermore, it is equal to the complexified T-duality isomorphism of Cavalcanti-Gualtieri [12] when the domain is restricted to the invariant sections of the exact Courant algebroid.

**Proof.** The map \( \hat{\phi} \) which can be expressed locally over \( \pi^{-1}(U_\alpha) \) by:

\[
\hat{\phi}(e^{-2\pi in_\alpha}(X_{n,\alpha} + f_{n,\alpha}v + \xi_{n,\alpha} + g_{n,\alpha}) \mid_{\pi^{-1}(U_\alpha)}) = -(X_{n,\alpha}, g_{n,\alpha}, \xi_{n,\alpha}, f_{n,\alpha}) \otimes s^n,
\]

defines an equivalence between the Courant algebroid structure on \( TZ \oplus T^*Z \) and the exotic Courant algebroid structure on \( \hat{E}_M \) as a result of (2.3). So what we need to show is that \( \varphi \) defines an exotic Courant algebroid isomorphism from \( \hat{E}_M \) to \( \hat{E}_M \). First, it is clear that \( \varphi \) is invertible. Simply take the map

\[
\varphi_n^{-1} : \Gamma(M,\hat{E}_{M,n}) \rightarrow \Gamma(M,E_{M,n})
\]

\[
(X_{n,\alpha}, f_{n,\alpha}, \xi_{n,\alpha}, g_{n,\alpha}) \otimes s^n \mapsto -(X_{n,\alpha}, g_{n,\alpha}, \xi_{n,\alpha}, f_{n,\alpha}) \otimes s^n,
\]

and compose this with the map \( \hat{\phi}^{-1} \) to get

\[
\mu_n^{-1} : \Gamma(M,\hat{E}_{M,n}) \rightarrow \Gamma(Z,TZ \oplus T^*Z)
\]

\[
(X_{n,\alpha}, f_{n,\alpha}, \xi_{n,\alpha}, g_{n,\alpha}) \otimes s^n = e^{-2\pi in_\alpha}(X_{n,\alpha} + f_{n,\alpha}v + \xi_{n,\alpha} + g_{n,\alpha}).
\]

Then \( \varphi \) transfers the Courant algebroid structure bijectively, as can be proven explicitly by simply showing:

\[
[\varphi(a), \varphi(b)]_H = \varphi([a,b]_H),
\]

\[
\langle \varphi(a), \varphi(b) \rangle = \hat{\phi}(\langle a, b \rangle),
\]

\[
\hat{\rho}(\varphi(a)) = \varphi \circ \rho(a).
\]
Furthermore, if we restrict to the case of when $n = 0$, then we get that all the sections are trivial, $s_α = 1$ for all $α$, and thus:

$$\varphi_0 : \Gamma(TZ \oplus T^* Z)^{S^1} \rightarrow \Gamma(T \hat{Z} \oplus T^* \hat{Z})^{\hat{S}^1}$$

$$(X, f, \xi, g) \mapsto -(X, g, \xi, f),$$

which is the T-duality map of Cavalcanti-Gaultieri from Theorem [12] (the negative sign arising due to considering the Courant algebroid with derived bracket of the differential operator $-(d + \hat{H})$ instead of $d + \hat{H}$).

□

Now we want to show that there is an action of the exotic Courant algebroid on the exotic differential forms such that the extended T-duality isomorphism acts as a module isomorphism. That is, given any $A \in \Gamma(TZ \oplus T^* Z)$, $ω \in Ω^*(Z)$,

$$\tau(A \cdot ω) = μ(A) \cdot τ(ω).$$

We know that there is a group action of $Γ(\hat{E}_Z)^{S^1}$ (and thus $Γ(\hat{E}_M)$) on the exotic differential forms $A^*(Z)$, given by

$$((\hat{X} + \hat{α}) \otimes s_1) \cdot (ω \otimes s_2) := (τ_X \hat{ω} + \hat{α} \wedge \hat{ω}) \otimes s_1 \otimes s_2.$$

Therefore, we get that

$$((\dot{X} + \dot{α}) \otimes s_1)^2 \cdot (\dot{ω} \otimes s_2) := ((\dot{X} + \dot{α}) \otimes s_1) \cdot ((\dot{X} + \dot{α}) \otimes s_1) \cdot (\dot{ω} \otimes s_2)$$

$$= (X + \dot{α}, X + \dot{α}) \hat{ω} \otimes s_1^2 \otimes s_2.$$

As a result, take the algebra of interest to be generated by the set $Γ(\hat{E}_M)$ satisfying the relation

$$((\hat{X} + \hat{α}) \otimes s)^2 = (\hat{X} + \hat{α}, \hat{X} + \hat{α}) \hat{ω} \otimes s^2.$$

**Theorem 3.2** (Clifford action of exotic Courant algebroids on exotic differential forms). The algebra of sections $Γ(\hat{E}_M) \cong Γ(\hat{E}_Z)^{S^1}$ defines an action on the set $A^{k+1}(\hat{Z})^{\hat{S}^1}$, such that the extended T-duality isomorphism mapping to the exotic differential forms,

$$\tau : Ω^k(Z) → A^{k+1}(\hat{Z})^{\hat{S}^1}$$

defines a module isomorphism.

**Proof.** Let $a \in Γ_{-n}(TZ \oplus T^* Z)$ and $ω \in Ω_{-m}^k(Z)$, such that locally, over $π^{-1}(U_α)$,

$$a = e^{-2πi m θ_a} (X_{-n, α} + f_{-n, α} v + ξ_{-n, α} + g_{-n, α} A)$$

$$ω = e^{-2πi m θ_a} (ω_{-m, α, 1} + ω_{-m, α, 0} A).$$

Then acting $a$ on $ω$, we get

$$a \cdot ω|_π^{-1}(U_α) = e^{-2πi(m+n)θ_a} (t_{X_{-n}ω_{-m, 1}} + (-1)^{k-1} f_{-n}ω_{-m, 0} + ξ_{-n} \wedge ω_{-m, 1}

+ (t_{X_{-n}ω_{-m, 0}} + ξ_{-n} \wedge ω_{-m, 0} + (-1)^k g_{-n}ω_{-m, 1}) \wedge A)$$

where we have removed the $α$ subscript in the last line to simplify notation.

Now under the extended T-duality isomorphism of [19], $a \cdot ω$ gets mapped locally to

$$\tau(a \cdot ω) = \int^{(a_{-n} \cdot ω_{-m})} e^{A \wedge A}|_{π^{-1}(U_α)}$$

$$= (-1)^k (t_{X_{-n}ω_{-m, 0}} + ξ_{-n} \wedge ω_{-m, 0} + (-1)^k g_{-n}ω_{-m, 1} +

+ (t_{X_{-n}ω_{-m, 1}} + (-1)^{k-1} f_{-n}ω_{-m, 0} + ξ_{-n} \wedge ω_{-m, 1}) \wedge \hat{A}) \otimes θ^*(s^{n+m}).$$

□
Furthermore, under T-duality,
\[ \tau(\omega) = (-1)^{k-1}(\omega_{-m,0} + \omega_{-m,1} \wedge \hat{A}) \otimes \pi^*(s^m) \]
\[ \mu(a) = -(X_{-n,0} + g_{-n,0} + \xi_{-n,0} + f_{-n,0}) \otimes \pi^*(s^n) \]
Therefore, acting \( \mu(a) \) on \( \tau(\omega) \), we find
\[ \mu(a) \cdot \tau(\omega) = \tau(a \cdot \omega) \]
Furthermore, taking the inverse extended T-duality mapping \( \sigma = -\tau^{-1} \) defined in [19], we get that \( \tilde{\mu}^{-1} \) defines the relevant module isomorphism for this inverse.

3.1. The case of trivial circle bundles. Consider now T-duality in the case of trivial circle bundles. That is,
\[ Z = M \times S^1, \quad \hat{Z} = M \times \hat{S}^1 \]
and \( H, \hat{H} \) are equal to 0 and the connections are all the trivial ones.

Let \( \theta \) be the coordinate function of the circle \( S^1 \) and \( \hat{\theta} \) be the coordinate function of the T-dual circle \( \hat{S}^1 \). The connections on each bundle will thus be \( \partial \theta \) and \( \partial \hat{\theta} \) respectively.

Take \( (x + \alpha)_{-n} \in \Gamma_{-n}(TZ \oplus T^*Z) \). Then globally, \( (x + \alpha)_{-n} \) is of the form
\[ e^{-2\pi i n \theta} (X_{-n} + f_{-n}v + \xi_{-n} + g_{-n,0}d\theta) \]
where \( X_{-n} \in \Gamma(TM) \) is a vector field on \( M \), \( \xi_{-n} \in \Gamma(T^*M) \) is a 1-form on \( M \), and \( f_{-n}, g_{-n} \in C^\infty(M) \) and \( v = \frac{\partial}{\partial \theta} \).

Then given \( \hat{v} = \frac{\partial}{\partial \hat{\theta}} \), under the T-duality mapping,
\[ \mu_n((x + \alpha)_{-n}) = -e^{2\pi i n \theta} (X_{-n,0} + g_{-n,0}v + \xi_{-n} + f_{-n}d\theta) \]
So we see that
\[ \mu_n(x_{-n} + \alpha_{-n}) \in \Gamma_n(T\hat{Z} \oplus T^*\hat{Z}) \]
On the other hand, if \( (\hat{y} + \hat{\beta})_{-n} \in \Gamma_{-n}(T\hat{Z} \oplus T^*\hat{Z}) \), then \( (\hat{y} + \hat{\beta})_{-n} \) is globally equal to
\[ e^{-2\pi i n \hat{\theta}} (\hat{Y}_{-n} + \hat{f}_{-n}v + \hat{\xi}_{-n} + \hat{g}_{-n,d\hat{\theta}}) \]
where \( \hat{Y}_{-n} \in \Gamma(TM) \) is a vector field on \( M \), \( \hat{\xi}_{-n} \in \Gamma(T^*M) \) is a 1-form on \( M \), and \( \hat{f}_{-n}, \hat{g}_{-n} \in C^\infty(M) \).

Now consider the map \( \mu^{-1} : \Gamma(T\hat{Z} \oplus T^*\hat{Z}) \to \Gamma(TM \oplus T^*Z) \) such that \( \hat{\sigma}_n((\hat{y} + \hat{\beta})_{-n}) \) is equal to
\[ -e^{2\pi i \hat{\theta}} (\hat{Y}_{-n} + \hat{g}_{-n}v + \hat{\xi}_{-n} + \hat{f}_{-n}d\theta) \]
And so evidently
\[ \hat{\mu}^{-1}((\hat{y} + \hat{\beta})_{-n}) \in \Gamma_n(TZ \oplus T^*Z) \]

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