\textbf{\L^p-IMPROVING AND SPARSE INEQUALITIES FOR DISCRETE SPHERICAL AVERAGES}

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\textbf{Abstract.} Let $\lambda^2 \in \mathbb{N}$, and in dimensions $d \geq 5$, let $A_\lambda f(x)$ denote the average of $f : \mathbb{Z}^d \to \mathbb{R}$ over the lattice points on the sphere of radius $\lambda$ centered at $x$. We prove $\ell^p$ improving properties of $A_\lambda$.

$$\|A_\lambda\|_{p \to p'} \lesssim \lambda^{-d(1 - \frac{1}{p'})}, \quad \frac{d}{d - 2} < p \leq 2.$$  

The inequalities hold for the smaller range of $\frac{d}{d - 1} < p \leq \frac{d}{d - 2}$ under the restriction that $\lambda^2$ has a bounded number of distinct prime factors. We show that these inequalities cannot hold for $p < \frac{d}{d - 1}$. Sparse bounds for discrete lacunary spherical maximal operators are also proved. The proofs use the decomposition of the corresponding multiplier whose properties were established by Magyar-Stein-Wainger, Magyar, and Hunt. Various endpoint estimates are combined with interpolation estimates of different types.

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1. Introduction

The subject of this paper is in discrete harmonic analysis in which continuous objects are studied in the setting of the integer lattice. Relevant norm properties are much more intricate, with novel distinctions with the continuous case arising.

In the continuous setting, $L^p$-improving properties of averages over lower dimensional surfaces are widely recognized as an essential property of such averages [4, 16, 27, 28]. It continues to be very active subject of investigation. In the discrete setting, these questions are largely undeveloped. They are implicit in work on discrete fractional integrals by several authors [22–24, 26], as well as two recent papers [7, 11] on sparse bounds for discrete singular integrals.

Our main results are two fold: First, scale free $\ell^p$ improving estimates for averages over discrete spheres, in dimensions $d \geq 5$, and in dimension $d = 4$, for certain radii. Second, sparse bounds for lacunary spherical maximal functions in the discrete setting. The latter immediately imply novel unweighted, weighted, and vector valued inequalities for this maximal operator.

We recall the continuous case. For dimensions $d \geq 2$, let $d\sigma$ denote Haar measure on the sphere of radius one in $\mathbb{R}^d$, and set $A_1 f = \sigma \ast f$ be convolution with respect to $\sigma$. The classical result of Littman [16] and Strichartz [27] gives the sharp $L^p$ improving property of this average. Here, we are stating the result in a restrictive way, but the full strength is obtained by interpolating with the obvious $L^1 \to L^1$ bound.

**Theorem A.** [16, 27] For dimensions $d \geq 2$, we have $\|A_1\|_{\ell^{d+1} \to \ell^{d+1}}$.

We study the discrete analog of $A_1 f$ in higher dimensions. For $\lambda^2 \in \mathbb{N}$, let $S_\lambda^d := \{n \in \mathbb{Z}^d : |n| = \lambda\}$. For a function $f$ on $\mathbb{Z}^d$, define

$$A_\lambda f(x) = |S_\lambda^d|^{-1} \sum_{n \in S_\lambda^d} f(x - n).$$

The study of the harmonic analytic properties of these averages was initiated by Magyar [18], with Magyar, Stein and Wainger [17] proving a discrete variant of the Stein spherical maximal function theorem [25]. This result holds in dimensions $d \geq 5$, as irregularities in the number of lattice points on spheres presenting obstructions to a positive result in dimensions $d = 2, 3, 4$. In particular, they proved the result below. See Ionescu [10] for an endpoint result, and the work of several others which further explore this topic [1, 5, 8, 19, 21].

**Theorem B.** [Magyar, Stein, Wainger, 17] For $d \geq 5$, there holds

$$\left\| \sup_{\lambda} |A_\lambda f| \right\|_p \leq \|f\|_p, \quad \frac{d}{d-2} < p < \infty.$$

Our first main result is a discrete variant of the result of Littman and Strichartz above. First note that $A_\lambda$ is clearly bounded from $\ell^p$ to $\ell^p$, for all $1 \leq p \leq \infty$. Hence, it trivially
improves any $f \in \ell^p(\mathbb{Z}^d)$ to an $\ell^\infty(\mathbb{Z}^d)$ function. But, proving a scale-free version of the inequality is not at all straightforward.

In dimensions $d = 4$, there is an arithmetical obstruction, namely for certain radii $\lambda$, the number of points on the sphere of radius $\lambda$ can be very small. To address this, let $\Lambda_d := \{0 < \lambda < \infty : \lambda^2 \in \mathbb{N}\}$, for $d \geq 5$, and for $d = 4$,

$$\Lambda_4 := \{0 < \lambda < \infty : \lambda^2 \in \mathbb{N} \setminus 4\mathbb{N}\}$$

Following the work of Magyar [19], we will address the case of dimension $d = 4$ below.

There is a second arithmetical obstruction arising due to nature of the estimate for Kloosterman sums that we will use. A convenient way to address this is through the arithmetic inequality.

Note that in this language, the inequalities in (1.6) depend only on $p$ and $\omega(\lambda^2)$.

\begin{align*}
\langle A_\lambda \rangle_{p \to p'} &\leq C_{p,\omega(\lambda^2)} \lambda^{d(1-\frac{2}{p'})}, & d + 1 < p \leq 2.
\end{align*}

\section*{Theorem 1.2} \textbf{A.} In dimensions $d \geq 5$, the inequality below is universal in $\lambda \in \Lambda_d$.

$$\|A_\lambda\|_{p \to p'} \leq C_p \lambda^{d(1-\frac{2}{p'})}, \quad \frac{d}{2} < p < 2.$$  

\textbf{B.} For dimensions $d \geq 4$, and $\lambda \in \Lambda_d$, the following inequalities depend only on $p$ and $\omega(\lambda^2)$.

\begin{align*}
\langle A_\lambda \rangle_{p \to p'} &\leq C_{p,\omega(\lambda^2)} \lambda^{d(1-\frac{2}{p'})}, & d + 1 < p \leq 2.
\end{align*}

\textbf{C.} Let $d \geq 4$. Suppose that for some $1 < p < 2$, for all $\epsilon > 0$, there holds

$$\|A_\lambda\|_{p \to p'} \leq C_{\epsilon} \lambda^{e+d(1-\frac{2}{p'})}, \quad \lambda \in \Lambda_d.$$  

Then $p \geq \frac{d+1}{d-1}$.

To explain our use of the phrase scale free, we make this definition. For a cube $Q \subset \mathbb{R}^d$ of volume at least one, we set localized and normalized norms to be

$$\langle f \rangle_{Q,p} := \left(|Q|^{-1} \sum_{n \in Q \cap \mathbb{Z}^d} |f(n)|^p \right)^{1/p}, \quad 0 < p \leq \infty.$$  

An equivalent way to phrase parts A and B of the theorem above is the following corollary. Note that in this language, the inequalities in (1.6) are uniform in the choice of $\lambda$.

\begin{align*}
\langle A_\lambda f_1, f_2 \rangle &\leq C_{p_1, p_2, w} \langle f_1 \rangle_{Q,p_1} \langle f_2 \rangle_{Q,p_2} |Q|, \quad \omega(\lambda^2) \leq w, \quad \lambda \in \Lambda_d.
\end{align*}

Above, $Q$ is a cube with side length $\lambda$, and $f_1, f_2$ are supported on $Q$.
Figure 1. The triangle $I_d$ of Theorem 1.5, for the $\ell^p$ improving inequality (1.3), is the dotted triangle with corners $(0,1)$ to $(\frac{d-1}{d+1}, \frac{d-1}{d+1})$ to $(1,0)$.

We prove a second result: Sparse bounds for variants of the lacunary spherical maximal function in the discrete setting. In the continuous case, such results are an easy (and powerful) consequence of the Littman, Strichartz inequality, see [13]. The discrete case, however, requires a separate argument.

Definition 1.7. A collection of cubes $S$ are said to be sparse if for all $Q \in S$ there is a set $E_Q \subset Q$ with $|E_Q| > \frac{1}{2}|Q|$ and $\{E_Q : Q \in S\}$ are pairwise disjoint.

We say that sublinear operator $T$ satisfies a $(p_1, p_2)$ sparse bound if there is a constant $C$ so that for all bounded and compactly supported functions $f_1, f_2$, there holds
\[
|\langle Tf_1, f_2 \rangle| \leq C \sup_{S \text{ sparse}} \sum_{Q \in S} \langle f_1 \rangle_Q, p_1 \langle f_2 \rangle_Q, p_2 |Q|.
\]
The infimum of constants $C$ above is denoted by $\|T\|_{p_1, p_2}$.

In the discrete setting, the lacunary spherical maximal function can be unbounded on $\ell^p$, $1 < p < \frac{d}{d-1}$, due to a clever example of Zienkiewicz [30]. The question of $\ell^p$ bounds, for some choice of $p$ outside the context of the Magyar Stein Wainger theorem is the subject of [8].

Herein, we prove sparse bounds, for particular lacunary sets, namely avoiding the arithmetical obstruction above.

Theorem 1.8. Let $d \geq 4$, and let $S_d$ be the open triangle with vertices $(0, 1), (\frac{d-1}{d+1}, \frac{d-1}{d+1})$, and $(\frac{d-2}{d-1}, \frac{1}{d-1})$. Let $\lambda_0$ be an odd prime, and let $\Lambda_0 = \{\lambda_0^{k/2} : k \in \mathbb{N}\}$. If $(1/p_1, 1/p_2) \in S_d$, the maximal operator below satisfies a $(p_1, p_2)$ sparse bound.
\[
(1.9) \quad A_{\Lambda_0} f := \sup_{\lambda \in \Lambda_0} |A_{\lambda} f|.
\]
Figure 2. The triangle $S_d$ of Theorem 1.5, for the sparse bounds, is the dotted triangle with corners $(0, 1)$ to $(\frac{d-1}{d+1}, \frac{d-1}{d+1})$ to $(\frac{d-2}{d-1}, \frac{1}{d-1})$. The point $a = \frac{d-1}{d}$ is the critical index from the example of Zienkiewicz. The lacunary maximal function need not be bounded on $\ell^p$, for $1 < p < 1/a$. The point $b = \frac{d-2}{d-1}$ is the critical index for the triangle $S_d$. The point $c = \frac{d-2}{d}$ is the critical index for the full supremum, see Theorem B. There are sparse bounds for that operator, which will be addressed in another paper.

The implied constant is independent of the choice of $\lambda_0$.

Sparse bounds immediately provide a range of $\ell^p$ bounds, vector valued inequalities, and weighted inequalities. This corollary is an example of these consequences, all entirely new in this context.

**Corollary 1.10.** With notation of (1.9), there holds for vector valued functions $\{f_k\}$, and weights $w$

\[
\|A_{\lambda_0} f\|_p \leq \|f\|_p, \quad \frac{d+1}{d-1} < p \leq \frac{d}{d-2},
\]

(1.11)

\[
\left\| A_{\lambda_0} f_k \right\|_{\ell^r(k \in \mathbb{Z})} \leq \left\| f_k \right\|_{\ell^r(k \in \mathbb{Z})}, \quad \frac{d-1}{d-2} < r < 2,
\]

(1.12)

\[
\|A_{\lambda_0} f\|_{\ell^2 (\langle w \rangle)} \leq C \|f\|_{\ell^2 (\langle w \rangle)}.
\]

(1.13)

Above, the inequalities in (1.11) hold in dimension $d \geq 4$, and fall outside the scope of the Magyar Stein Wainger theorem. The inequalities (1.12) are vector valued, and in (1.13) are weighted, and stated in terms of the Muckenhoupt $A_2$ condition. For the vector valued inequalities, see [6], and the weighted inequalities see [2, Prop. 6.4]. There are many more such inequalities that hold.
The proofs of the theorems begin with a subtle decomposition of $A_\lambda$ in terms of its Fourier multiplier. The key elements here were developed by Magyar, Stein and Wainger \[17\], with additional observations of Magyar \[20\] and Hughes \[8\]. We recall this in §2 §3. For the proof of the sufficient direction, namely parts A and B of Theorem 1.2, the estimates of Weil on Kloosterman sums are essential. For the proof of the necessity, estimates on Bourgain on averages of Ramanujan sums are essential. The necessary direction is addressed in §5, which depends upon a certain 'self-improving' property of the $\ell^p$-improving inequalities. The sparse bounds are proved in §6. In contrast to the continuous case, we find that we have to give a proof which is quite similar to the case of $\ell^p$-improving inequalities, but distinct. There is no formal relationship between the two sets of results. A few complements on the main theorems are included in the concluding section 7 of the paper.

The notations set in the paper are listed here.

$\omega(n)$: The number of distinct prime factors of $n$.

$\hat{f}$, $\hat{\phi}$: The Fourier transform, and its inverse, on $\mathbb{Z}^d$, see (2.1).

$e, e_q$: $e(x) = e^{2\pi i x}$ and $e_q(x) = e^{2\pi i x/q}$.

$\mathcal{F}$: The Fourier transform on $\mathbb{R}^d$.

$\mathbb{S}_d$: The discrete sphere $\{n \in \mathbb{Z}^d : |n| = \lambda\}$.

$d\sigma_\lambda$: The uniform measure on the sphere of radius $\lambda$ in $\mathbb{R}^d$.

$A_\lambda$: The normalized discrete spherical average operator $A_\lambda$ in (2.2).

$C_\lambda$: The ‘main term’ given in (2.3).

$R_\lambda$: The ‘residual term’ $R_\lambda = A_\lambda - C_\lambda$.

$c_{\lambda,q}$: The multiplier associated to $C_{\lambda,q}$ in (2.4).

$C_{\lambda,q}^v$, $C_{\lambda,w}^v$: The operators defined in (3.3) and (3.12), respectively

$K(\lambda, q, \ell)$: The Kloosterman sum defined in (2.5).

$\rho(\lambda, q)$: Writing $q = q_12^r$, with $q_1$ odd, $\rho(\lambda, q) = \sqrt{(q_1, \lambda)2^r}$, the term that enters into the Kloosterman refinement (2.7).

$\mathbb{Z}_{q_1}^d$, $\mathbb{Z}_q^d = \mathbb{Z}^d/q\mathbb{Z}^d$.

$\mathbb{Z}_q^\times$: The multiplicative group $\{a \in \mathbb{Z}_q : (a, q) = 1\}$.

$\Phi_q$: A normalized bump function, see text below (2.5).

$c_q(n)$: The Ramanujan sum defined in (2.11).

$I_d$: The region in which the $\ell^p$ improving estimates hold, see Corollary 1.5.
We acknowledge useful conversations with Alex Iosevich and Francesco Di Plinio on the topics of this paper.

2. Decomposition

Throughout \( e(x) = e^{2\pi i x} \). The Fourier transform on \( \mathbb{Z}^d \) is given by

\[
\mathcal{F}_{\mathbb{Z}^d} f(\xi) = \sum_{x \in \mathbb{Z}^d} e(-\xi \cdot x) f(x), \quad x \in \mathbb{T}^d \equiv [0, 1]^d.
\]

We will write \( \mathcal{F}_{\mathbb{Z}^d} f = \hat{f} \), and \( \check{\phi} \) for the inverse Fourier transform, given by

\[
\check{\phi}(n) = \int_{\mathbb{T}^d} e(\xi \cdot n) \phi(\xi) \, d\xi.
\]

The Fourier transform on \( \mathbb{R}^d \) is

\[
\mathcal{F} \phi(\xi) = \int_{\mathbb{R}^d} e(-\xi \cdot x) f(x) \, dx.
\]

We work exclusively with convolution operators \( K : f \mapsto \int_{\mathbb{T}^d} k(\xi) \hat{f}(\xi) \, d\xi \). In this notation, \( k \) is the multiplier, and the convolution is \( k * f \). Lower case letters are frequently, but not exclusively, used for the multipliers, and capital letters for the corresponding convolution operators.

There holds

\[
|\mathbb{S}_n^d| = |\{n \in \mathbb{Z}^d : |n| = \lambda\}| \approx \lambda^{d-2}, \quad \lambda \in \Lambda_d.
\]

Redefine the operator \( A_\lambda f \) to be

\[
A_\lambda f(x) = \lambda^{-d+2} \sum_{n \in \mathbb{Z}^d : |n| = \lambda} f(x - n)
\]

\[
= \int_{\mathbb{T}^d} a_\lambda(\xi) \hat{f}(\xi) \, d\xi,
\]

where \( a_\lambda(\xi) = \lambda^{-d+2} \sum_{n \in \mathbb{Z}^d : |n| = \lambda} e(\xi \cdot n) \).

The decomposition of \( a_\lambda \) into a ‘main’ term \( c_\lambda \) and an ‘residual’ term \( r_\lambda = a_\lambda - c_\lambda \) follows development of Magyar, Stein and Wainger \([17, \S5]\), Magyar \([20, \S4]\) and Hughes \([8, \S4]\). We will be very brief.
Let $\mathbb{Z}^d_q = \mathbb{Z}^d/q\mathbb{Z}^d$ and $\mathbb{Z}^\times_q = \{ a \in \mathbb{Z}_q : (a, q) = 1 \}$ be the multiplicative group. We have

\begin{equation}
 c_\lambda(\xi) = \sum_{1 \leq q \leq \lambda} c_{\lambda, q}(\xi),
\end{equation}

\begin{equation}
 c_{\lambda, q}(\xi) = \sum_{\ell \in \mathbb{Z}^d_q} K(\lambda, q, \ell) \Phi_q(\xi - \ell/q) F d\sigma_\lambda(\xi - \ell/q),
\end{equation}

\begin{equation}
 K(\lambda, q, \ell) = q^{-d} \sum_{a \in \mathbb{Z}^\times_q} \sum_{n \in \mathbb{Z}^d_q} e_q(-a\lambda^2 + |n|^2a + n \cdot \ell).
\end{equation}

Above, $\Phi$ is a smooth non-negative radial bump function, $1_{[-1/8,1/8]^d} \leq \Phi \leq 1_{[-1/4,1/4]^d}$. Further, $\Phi_q(\xi) = \Phi(q\xi)$. Throughout we use $e_q(x) = e(x/q) = e^{2\pi ix/q}$. The term in (2.5) is a Kloosterman sum, a fact that is hidden in the expression above, but becomes clear after exact summation of the quadratic Gauss sums. In addition, $d\sigma_\lambda$ is the continuous unit Haar measure on the sphere of radius $\lambda$ in $\mathbb{R}^d$.

Essential here is the Kloosterman refinement. The estimate below goes back to the work of Kloosterman [12] and Weil [29]. Magyar [20, §4] used it in this kind of setting. (It is essential to the proof of Lemma 2.10.)

**Lemma 2.6.** [20, Proposition 7] For all $\eta > 0$, and all $1 \leq q \leq \lambda$, $\lambda \in \Lambda_d$,

\begin{equation}
 \sup_{\ell} |K(\lambda, q, \ell)| \lesssim q^{-d-1/2+\eta} \rho(q, \lambda),
\end{equation}

where we write $q = q_1 2^r$, with $q_1$ odd, so that $\rho(q, \lambda) = \sqrt{(q_1, \lambda^2)2^r}$, where $(q_1, \lambda^2)$ is the greatest common divisor of $q_1$ and $\lambda^2$.

Concerning the terms $\rho(q, \lambda)$, we need

**Proposition 2.8.** For $0 < \theta < 1$, and $t > 1$ we have

\begin{equation}
 \sum_{1 \leq q \leq \lambda} q^{-t} \rho(\lambda, q)^\theta \lesssim \begin{cases} P_{t, \omega(\lambda^2)} & t > 1 \\ 1 & t > 1 + \theta/2 \end{cases}.
\end{equation}

That is, we have a universal bound on the sums, for $t > 1 + \theta/2$, and for $1 < t \leq 1 + \theta/2$, the sum is bounded by a constant depending upon $\omega(n)$, the number of distinct prime factors of $\lambda^2$.

**Proof.** Write $q_1 = ds$, where $d = (q_1, \lambda^2)$. Then, $q = ds 2^r$. Since $t > 1$, the left side of (2.9) is at most

\begin{equation}
 \sum_{d : d | \lambda^2} \sum_{s=1}^\infty \sum_{r=1}^\infty \frac{[d 2^r]^\theta}{[ds2^r]^t} \lesssim \sum_{d : d | \lambda^2} d^{\frac{\theta}{2} - t},
\end{equation}
from which our claims follow. If \( t > 1 + \frac{\theta}{2} \), the sum above is clearly finite. Otherwise, it is bounded by a constant depending upon the number of prime factors \( \omega(\lambda^2) \), in particular, the constant is at most
\[
\sum_{j=0}^{\infty} 2^j (\frac{\theta}{2} - t) \sum_{w=1}^{\omega(\lambda^2)} p_w^{\frac{\theta}{2} - t} \leq \sum_{w=1}^{\omega(\lambda^2)} w^{\frac{\theta}{2} - t} = P_{t, \omega(\lambda^2)},
\]
where \( p_1 = 2 < p_2 < \cdots \) are the primes in increasing order. □

The 'main' term is \( C_{\lambda} f \), and the 'residual' term is \( R_{\lambda} = A_{\lambda} - C_{\lambda} \). This is a foundational estimate for us.

**Lemma 2.10.** [20, Lemma 1, page 71] We have, for all \( \epsilon > 0 \), uniformly in \( \lambda \in \Lambda_d \),
\[
\| R_{\lambda} \|_{2 \rightarrow 2} \leq \epsilon \lambda^{-\frac{d+1}{2} + \epsilon}.
\]

In this paper, we will not need to directly examine the residual term. This will save us from a lot of technical difficulties.

We begin the discussion geared towards the endpoint estimates at \( \ell^1 \) and \( \ell^\infty \). Define
\[
(2.11) \quad c_q(n) := \sum_{a \in \mathbb{Z}^\times_q} e_q(\lambda a).
\]

The sum above is a Ramanujan sum, typically written as above. (A sans serif font is used to distinguish it from the main term.) Note that \( c_q(0) = \varphi(q) \), the Euler Totient function. But, Ramanujan sums are distinguished by have remarkable cancellation properties, among other properties.

For a multiplier on \( \mathbb{T}^d \), define a family of related multipliers by
\[
(2.12) \quad m_{\lambda, q} = \sum_{\xi \in \mathbb{T}^d_q} K(\lambda, q, \ell) m(\xi - \ell/q).
\]

These have an explicit inverse Fourier transform, which is well-known. It involves the Ramanujan sums.

**Proposition 2.13.** For a multiplier \( M_{\lambda, q} \) as in (2.12), we have
\[
(2.14) \quad \tilde{m}_{\lambda, q}(n) = \tilde{m}(n) c_q(\lambda^2 - |n|^2), \quad a \in \mathbb{Z}^\times_q.
\]

**Proof.** Rewrite the Kloosterman sum in (2.5) as
\[
(2.15) \quad K(\lambda, q, \ell) = \sum_{a \in \mathbb{Z}^\times_q} e_q(-a\lambda^2) G(a/q, \ell)
\]
\[
(2.16) \quad G(a/q, \ell) := q^{-d} \sum_{n \in \mathbb{Z}^d_q} e_q(|n|^2 a + n \cdot \ell).
\]
The terms $G(a/q, \ell)$ are Gauss sums. Observe that it is a Fourier transform on the group $\mathbb{Z}_q^d$. Namely, if $\phi(\ell) = e(|\ell|^2 a/q)$ is the function on $\mathbb{Z}_q^d$, we have $\hat{\phi}(-\ell) = \hat{\phi}(\ell) = G(a/q, \ell)$. Using the version formula on that group we have

$$
(2.17) \sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) e_q(y \cdot \ell) = e_q(|y|^2 a), \quad y \in \mathbb{Z}^d/q\mathbb{Z}^d.
$$

Define

$$
m^{a/q}(\xi) = e_q(-\lambda^2 a) \sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) m(\xi - \ell/q), \quad a \in \mathbb{Z}_q^*.
$$

By (2.17), we have

$$
\hat{m}^{a/q}(n) = \int_{\mathbb{T}^d} m^{a/q}(\xi) e(-\xi \cdot n) \, d\xi,
$$

$$
= e_q(-\lambda^2 a) \int_{\mathbb{T}^d} \sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) m(\xi - \ell/q) e(-\xi \cdot n) \, d\xi
$$

$$
= e_q((|n|^2 - \lambda^2) a) \tilde{m}(n).
$$

And, this proves our proposition. □

### 3. Decomposition, Part II

We formalize the further decomposition that we will use on the $C_{\lambda,q}$. It is given in terms of multi-frequency Littlewood-Paley operators, which is illustrated in Figure 1. Fix a smooth function $p$ supported on $\{\xi : \frac{1}{2} \leq \xi \leq 2\}$ so that

$$
\sum_{k \in \mathbb{Z}} p(2^k \xi) \equiv 1, \quad \xi \neq 0.
$$

The operators $P_{\lambda,q}^\nu$ that we want are associated to the multipliers

$$
(3.1) \quad P_{\lambda,q}^0(\xi) := \sum_{k \in \mathbb{N} : 2^{k-1} > \lambda} \sum_{\ell \in \mathbb{Z}_q^d} p\left(2^k(\xi - \ell/q)\right),
$$

$$
(3.2) \quad P_{\lambda,q}^\nu(\xi) := \sum_{\ell \in \mathbb{Z}_q^d} p\left(2^k(\xi - \ell/q)\right), \quad \nu > 0, \quad 2\lambda \geq 2^{\nu-k+1} > \lambda.
$$

Recalling the definition of $\Phi$, see below (2.5), we see that

$$
(3.3) \quad C_{\lambda,q} = \sum_{\nu \geq 0 : 2^\nu < 2\lambda/q} C_{\lambda,q}^\nu := \sum_{\nu \geq 0 : 2^\nu < 2\lambda/q} C_{\lambda,q}^\nu.
$$

We recall the stationary phase estimate

$$
(3.4) \quad |F d\sigma(\xi)| \leq |\lambda \xi|^{-\frac{1}{2} + 1}.
$$

From this, we can derive $\ell^2$ estimates for the operators $C_{\lambda,q}^\nu$. 


Lemma 3.5. These estimates hold for all choices of $1 \leq q \leq \lambda$, and integers $v \geq 0$, and $\varepsilon > 0$.

$$\|C_{v,q}\|_{2 \to 2} \lesssim \varepsilon 2^{-\frac{v+1}{4}q^{\varepsilon - \frac{d+1}{2}}\rho(\lambda, q)}.$$ (3.6)

Proof. The estimate is a direct consequence of the Kloosterman refinement (2.7), and the stationary phase estimate (3.4). \qed

The estimates that we need for the associated kernels are as follows.

Lemma 3.7. These estimates hold for all $0 < \varepsilon < 1$

$$\left(\Phi_{q,\lambda} F d\sigma_{\lambda} \sum_{k \in \mathbb{N} : 2^{k-1} > \lambda} p_k\right)\hat{}(n) \lesssim \zeta(\lambda)(n)$$ (3.8)

where $\zeta(\lambda)(n) = \lambda^{-d}[1 + |n|/\lambda]^{-2d/\varepsilon}$. For $v > 0$, there holds

$$\left(\Phi_{q,\lambda} F d\sigma_{\lambda} \cdot p_k\right)\hat{}(n) \lesssim d\sigma_{\lambda} * \zeta(\lambda/2^v)(n), \quad 2^{v-k} \simeq \lambda.$$ (3.9)

In particular, $\|d\sigma_{\lambda} * \zeta(\lambda/2^v)\|_{\infty} \lesssim 2^v\lambda^{-d}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The Littlewood-Paley decomposition of (3.1). The largest circle is centered at a lattice point $\ell/q$, for $\ell \in \mathbb{Z}_q^d$. The innermost circle has radius approximately $1/\lambda$. At the outermost we have $2^v/\lambda \simeq 1/q$, so the decomposition stops.}
\end{figure}
Proof. These are elementary. Convolution with $F^{-1}\Phi_q$ is an average on scale $q \leq \lambda$; convolution with $d\sigma_\lambda$ is spherical average on scale $\lambda$; and convolution with the third term $\sum_{k \in \mathbb{N}} 2k - 1 > \lambda^{-1}$ is an average on scale $\lambda$. Hence the entire convolution is no worse than an average on scale $\lambda$.

The second inequality is seen in a similar way. \qed

The remainder of this section is prelude to the proof of the necessity of $p \geq d+1$ in our main theorem. There is a companion to the Kloosterman refinement, a consequence of this profound theorem of Bourgain [3, (3.44), pg. 126]. It states that $c_q(n)$, is typically of size about one, for $n$ fixed. (We learned of this estimate in Hughes [8].) We will refer to this estimate as the Ramanujan refinement.

**Theorem 3.10.** For all $\epsilon > 0$, for all integers $w = 2^w$, there holds for $n \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{w/2 \leq q < w} |c_q(n)| \leq w|\{1 \leq q \leq w : \text{each prime factor of } q \text{ divides } n\}| \lesssim \epsilon w \cdot \min\{w, |n|^\epsilon\}. \quad (3.11)$$

In order to leverage this estimate, we need the following definition. For an integer $w = 2^w$, set

$$(3.12) \quad C_{\lambda, w} := \sum_{q : w/2 \leq q < w} C_{\lambda, q}$$

and similarly for $C_{\lambda, w}^\vee$. This is the detailed estimate we will need to show necessity of $p \geq \frac{d+1}{d-1}$.

**Lemma 3.13.** We have for all $\epsilon > 0$, and $1 \leq 2^\nu \leq 2\lambda/w$,

$$|C_{\lambda, w} f| \lesssim \epsilon \frac{w}{\lambda} A_\lambda f + w^1 \lambda^\epsilon d\sigma_\lambda * \zeta(w) * f, \quad (3.14)$$

$$|C_{\lambda, w}^\vee f| \lesssim \epsilon 2^\nu \frac{w^2}{\lambda^2} A_\lambda f + 2^{\nu w} w^{1+\epsilon} d\sigma_\lambda * \zeta(2^{-\nu} \lambda) * f, \quad (3.15)$$

where $\zeta$ above is as in (3.8).

**Proof.** Our argument is guided by Figure 4. Write the kernel of $C_{\lambda, w}$ as

$$K(n) = \sum_{q : w/2 \leq q < w} c_{\lambda, q}(n) = \sum_{q : w/2 \leq q < w} c_q(\lambda^2 - |n|^2) d\sigma_\lambda * F^{-1}\Phi_q(n).$$

Note that the convolution above is that of a bump function on scale $1 \leq q \leq \lambda$ convolved with $d\sigma_\lambda$. 

\|d\sigma_\lambda * \xi(\lambda^2 - \nu)\|_{\infty} \lesssim 2^\nu \lambda^{-d}

**Figure 4.** A sketch to indicate the estimates (3.14) and (3.15). The convolution \(d\sigma_\lambda * \mathcal{F}^{-1} \Phi_q\) is essentially supported in an annulus around a sphere of radius \(\lambda\) of width about \(q\). We use this with \(q \approx w\). And, \(d\sigma_\lambda * \xi(\lambda^2 - \nu)\) is essentially supported in an annulus around a sphere of radius \(\lambda\) of width \(\lambda^2 - \nu\).

The distinguished case above concerns those \(n \in S_\lambda\). There is no additional cancellation in the Ramanujan sums, so that

\[
|K(n)| \lesssim \sum_{q : w/2 \leq q < w} q \|d\sigma_\lambda * \mathcal{F}^{-1} \Phi_q\|_{\infty}
\]

This is the first term on the right in (3.14).

Otherwise, for \(n \not\in S_\lambda\), we turn to the Ramanujan refinement, namely (3.11), to conclude that

\[
|K(n)| \lesssim w \cdot |\lambda^2 - |n|^2|^{\epsilon/2} d\sigma_\lambda * \xi(\omega)(n)
\]

The definition of \(\xi\) depends upon \(0 < \epsilon < 1\), and passing to the last line, we change \(\epsilon\) to a slightly larger value. This is the second part of (3.14).

For the second conclusion, by (2.14), the kernel of the convolution is

\[
K(n) = \sum_{q : w/2 \leq q < w} c_{\lambda,q}^0(n) = \sum_{q : w/2 \leq q < w} c_q(\lambda^2 - |n|^2) d\sigma_\lambda * \mathcal{F}^{-1} p_k * \mathcal{F}^{-1} \Phi_q(n).
\]
Above, $2^k \simeq \lambda/2^v$. The last convolution is less than $\ll \zeta(\lambda/2^v) \ast d\sigma_\lambda$. Then, one follows the argument for the first inequality, keeping track of the change in the relative values of $\lambda$ and $2^k$. In particular, there is no additional $\lambda^c$ term, in contrast to (3.14). Indeed, it is this feature that motivates the definition of $C_{v,\lambda}^\lambda$.

\[ \square \]

4. \textbf{Proof of the $\ell^p$ improving estimates}

We prove parts \textbf{A} and \textbf{B} of Theorem 1.2. Theorem 1.2 is phrased between dual indices. Those are the estimates that we study herein. Interpolating between dual indices yields bounds for dual indices, preserving the scale free factor $\lambda^{d(1-\frac{2}{p})}$. We are then largely interested in how much the estimates differ from the scale free factor. The principal lemma here is:

\textbf{Lemma 4.1.} For all $\epsilon > 0$, these estimates hold uniformly in $\lambda^2 \in \mathbb{N}$, and $2 \leq q \leq \lambda$.

\begin{align*}
\|C_{\lambda, q}\|_{2 \rightarrow 2} & \lesssim q^{-\frac{d+1}{4} + \epsilon} \rho(\lambda, q), \\
\|C_{\lambda, q}\|_{\frac{d+1}{4} \rightarrow \frac{d+1}{q}} & \lesssim q^{-1 + \epsilon} \lambda^{-\frac{d+1}{2} + \epsilon} \rho(\lambda, q) \frac{q^\epsilon}{\lambda^\epsilon}, \\
\|C_{\lambda, q}\|_{1 \rightarrow \infty} & \lesssim \lambda^{-d-1}. 
\end{align*}

We conclude the proof of the $\ell^p$-improving inequalities.

\textit{Proof of Theorem 1.2, Parts A and B.} We use the expansion of $A_\lambda = C_\lambda + R_\lambda$. Let us address $R_\lambda$ first. Sum (4.4) to see that

\[ \|C_{\lambda, q}\|_{1 \rightarrow \infty} \leq \sum_{1 \leq q < \lambda} \lambda^{-d+1} \lesssim \lambda^{-d+2}. \]

But, we also have $\|A_\lambda\|_{1 \rightarrow \infty} \lesssim \lambda^{-d+2}$, trivially, so that we see that $\|R_\lambda\|_{1 \rightarrow \infty} \lesssim \lambda^{-d+2}$. Note that this bound is larger than we want by a factor of $\lambda^2$. Nevertheless, interpolate with the $\ell^2 \rightarrow \ell^2$ bound for $R_\lambda$ from Lemma 2.10 to complete the norm estimate on $R_\lambda$. Namely, we will have

\[ \|R_\lambda\|_{p \rightarrow p'} \lesssim \lambda^{-d(1-\frac{2}{p})-\delta_p}, \quad \frac{d+1}{d-1} < p \leq 2, \ \delta_p > 0. \]

This is a better outcome than we will have for the term $C_\lambda$, since it is universal in $\lambda$.

Interpolating between (4.2) and (4.3), we see that for all $\frac{d+1}{d-1} < p \leq 2$, there are $\alpha_p, \beta_p > 0$ so that

\[ \|C_{\lambda, q}\|_{p \rightarrow p'} \lesssim q^{-1-\alpha_p} \rho(\lambda, q) \beta_p \lambda^{-d(1-\frac{2}{p})}. \]

By a direct calculation, we have $\alpha_p > \frac{\beta_p}{2}$ if $\frac{d}{d-2} < p < 2$, and $d \geq 5$. Appealing to (2.9), we see that we have

\[ \|C_{\lambda, q}\|_{p \rightarrow p'} \lesssim \lambda^{d(1-\frac{2}{p})} \begin{cases} 
P_{\omega(\lambda^2)} & \frac{d+1}{d-1} < p \leq 2 \\
1 & d \geq 5, \ \frac{d}{d-2} < p < 2 
\end{cases} \]
This completes our proof. □

The first principal estimate (4.2) is a direct consequence of the Kloosterman refinement (2.7). For the proof of the second principal estimate (4.3), we turn to our Littlewood-Paley decomposition (3.1) and (3.2), and show the lemma below. It prove (4.3) by summation over \( \nu \geq 0 \)

**Lemma 4.5.** For all \( \epsilon > 0 \), there holds

\[
\| C_{\lambda, q} \|_{d+1 \to d+1} \lesssim \epsilon 2^{-\nu} q^{1+\epsilon} \lambda^{-d+1+\epsilon} \rho(\lambda, q) \frac{\lambda^2}{\lambda^d},
\]

Proof. We have the \( \ell^2 \to \ell^2 \) estimate (3.6). Combine (3.9) and (2.14) to see that

\[
\| C_{\lambda, q} \|_{1\to\infty} \lesssim q 2^\nu \lambda^{-d}.
\]

Interpolate between these two to get the inequality (4.6). □

**Proof of (4.4).** For the third estimate, the \( \ell^1 \to \ell^\infty \) estimate, we have

\[
\| C_{\lambda, q} \|_{1\to\infty} = \| \hat{c}_{\lambda, q} \|_{\infty} \leq \| \mathcal{F}^{-1} \Phi_q * d\sigma \|_{\infty} \| c_q(\cdot) \|_{\infty} \lesssim \lambda^{-d+1}.
\]

Above, we have appealed to (2.14), and the trivial bound \( q \) on the Ramanujan sum \( |c_q(n)| \leq q \). Note that convolution with \( \mathcal{F}^{-1} \Phi_q \) is an average on scale \( q \leq \lambda \), so the bound above follows. □

5. **Necessity of \( p \geq \frac{d+1}{d-1} \) in the \( \ell^p \) Improving Estimate**

We turn to the proof that the \( \ell^p \) improving inequalities we prove are sharp up to the endpoint, namely we prove part C of Theorem 1.2. Our assumption is (1.4), that the scale-free \( \ell^p \) improving inequality holds with \( \epsilon \) loss in \( \lambda \), and deduce that \( p \geq \frac{d+1}{d-1} \).

Such arguments are typically done by appropriate example. For the classical result of Littman and Strichartz, this example is just the indicator of a thin annulus around the unit circle. In the discrete setting, this would just be the indicator of a single sphere \( S_\lambda \). But, from this, we only see the lower bound of \( \frac{d+2}{d} \), which is smaller than \( \frac{d+1}{d-1} \). Any improvement of this example seems to be very difficult.

An example seemingly out of reach, we use a `self-improving' property of our proof in the sufficient direction.

**Lemma 5.1.** Suppose that for \( 1 < p_0 < 2 \), the \( \ell^p \) improving inequality (1.4) holds. Let

\[
\frac{1}{p_2} = \frac{d-3}{d-1} \frac{1}{p_1} + \frac{2}{d-1} \frac{1}{2}, \quad \text{and} \quad \frac{1}{p_1} = \frac{1}{2} + \frac{1}{2p_0}.
\]

Then, these inequalities hold:

\[
\| A_{\lambda} \|_{p \to p'} \leq \epsilon \lambda^{\epsilon + d(1 - \frac{2}{p'})}, \quad p_2 < p < 2.
\]
With the Lemma proved, take $p_{\text{crit}}$ to be the infimum over all such $p_0$. We have from (5.3) that $\frac{1}{p_2} \leq \frac{1}{p_{\text{crit}}}$. Expanding this inequality, note that

$$\frac{1}{p_{\text{crit}}} \geq \frac{d-3}{d-1} \frac{1}{p_1} + \frac{2}{d-1} \frac{1}{2} = \frac{d-3}{d-1} \left( \frac{1}{2} + \frac{1}{2p_{\text{crit}}} \right) + \frac{2}{d-1} \frac{1}{2} = \frac{1}{2} + \frac{d-3}{2p_{\text{crit}(d-1)}}.$$ 

From this, we see that $\frac{1}{p_{\text{crit}}} \geq d - \frac{1}{d-1}$. This is the proof of our necessary condition. The inequalities in (1.3) are sharp, up to the endpoint $p = \frac{d+1}{d-1}$.

We turn to the proof of the Lemma. The core of the proof is the next Lemma.

**Lemma 5.4.** Under the assumptions of Lemma 5.1, these inequalities hold.

(5.5) $\|C_\lambda\|_{p_1 \to p_1'} \leq \varepsilon \lambda^{c + d(1 - \frac{\delta}{p_1}) + 1}$,

(5.6) $\|C_\lambda\|_{p \to p'} \leq \varepsilon \lambda^{c + d(1 - \frac{\delta}{p})}$, $p_2 < p < 2$.

**Proof of Lemma 5.1.** We have the correct inequality for the operator $C_\lambda$, that is (5.6) above. It remains to see that the same inequality holds for the residual term $R_\lambda$. We have already seen that

$$\|A_\lambda\|_{1 \to \infty} \leq \lambda^{-d+2}.$$ 

By definition in (5.2), $\frac{1}{p_1}$ is the midpoint between 1 and $\frac{1}{p_0}$, as defined in (5.2). Hence,

(5.7) $\|A_\lambda\|_{p_1 \to p_1'} \leq \varepsilon \lambda^{c + d(1 - \frac{\delta}{p_1}) + 1}$.

The inequalities (5.5) are the same for $C_\lambda$. Therefore, (5.7) holds for $R_\lambda$ as well.

Now, we also have the crucial fact from Lemma 2.10, $\|R_\lambda\|_{2 \to 2} \leq \varepsilon \lambda^{c - \frac{d+2}{2}}$. Then, the choice of $p_2$ as a convex combination of $\frac{1}{p_1}$ and $\frac{1}{2}$ is chosen so that

$$\|R_\lambda\|_{p \to p'} \leq \varepsilon \lambda^{-d(1 - \frac{\delta}{p})}, \quad p_2 < p < 2.$$ 

Therefore, Lemma 5.1 holds. $\square$

Lemma 5.4 follows from this Lemma and a summation in $w = 2^w \geq 0$. Below, we are using the notation from (3.12).

**Lemma 5.8.** Under the assumptions of Lemma 5.1, these inequalities hold, uniformly in $w$, for a choice of $\delta > 0$.

(5.9) $\|C_{\lambda, w}\|_{p_1 \to p_1'} \leq \varepsilon w^{1 + \varepsilon} \lambda^{c + d(1 - \frac{\delta}{p}) + 1}$,

(5.10) $\|C_{\lambda, w}\|_{p \to p'} \leq \varepsilon w^{-\delta} \lambda^{c + d(1 - \frac{\delta}{p})}$, $p > p_2$, $\delta = \delta_p > 0$.

**Proof.** Since $\frac{1}{p_1} = \frac{1}{2} + \frac{1}{2p_0}$, and $\|A_\lambda\|_{1 \to \infty} \leq \lambda^{-d+2}$, there holds

$$\|A_\lambda\|_{p_1 \to p_1'} \leq \varepsilon \lambda^{1 + \varepsilon + d(1 - \frac{\delta}{p_1})}.$$
Figure 5. The labeled points are $P_1 = \left( \frac{d-1}{d+1}, \frac{d-1}{d+1} \right)$, $P_2 = \left( \frac{1}{d-1}, \frac{d-2}{d-1} \right)$. The two triangles $S_d^1$ and $S_d^2$ make up the region $S_d$ in which the sparse bounds hold.

We turn our attention to $C_{\lambda, w}$, using the pointwise inequality (3.14) so that we can use the inequality above.

There are two terms on the right in (3.14). The first involves only $\frac{w}{\lambda} A_\lambda$, so that it satisfies
$$\left\| \frac{w}{\lambda} A_\lambda \right\|_{p_1 \to p_1'} \leq \lambda^{1+\epsilon + d(1-\frac{2}{p_1})}.$$  

The second term involves $\lambda^\epsilon w^1 d\sigma_\lambda * \zeta(w)$. By convexity, and assumption, it satisfies the estimate (5.9).

We need an $\ell^2$ estimate. From the Kloosterman refinement, we have
$$\|C_{\lambda, w}\|_2 \leq \epsilon \sum_{w/2 \leq q < w} q^{\epsilon - \frac{d+1}{2}} \rho(\lambda, q)$$
$$\leq \epsilon w^{\epsilon - \frac{d-1}{2}}$$

The last inequality is a simple variant of the proof of (2.9).

We have chosen $p_2$ in (5.2) so that the second inequalities (5.10) follow from (5.9) and interpolation with the $\ell^2 \to \ell^2$ just established.  

**6. Proof of the Sparse Bound**

We take up the proof of the sparse bound. Some facts that we will appeal to for this proof are summarized in §6.3.

Throughout this proof, $\Lambda_0 = \{ \lambda_0^{k/2} : k \in \mathbb{N} \}$, where $\lambda_0$ is an odd prime. Estimates will be uniform in $\lambda_0$.  

The interpolation argument for the sparse bounds. Ignoring terms of a logarithmic nature, the norms of operators are either powers of \( w \) or \( \lambda \). These powers are listed at \((\frac{1}{2}, \frac{1}{2}), (0, 1)\) and \((1, 1)\). Interpolate along the dotted lines to the points \( P_1 = (\frac{d-1}{d+1}, \frac{d+1}{d+1}) \) and \( P_2 = (\frac{1}{d-1}, \frac{d-2}{d-1}) \). At these points, the interpolated exponents are zero, ignoring logarithmic terms. In the light gray colored region, we will gain a negative exponent \( w^{-\delta} \), for some \( \delta > 0 \).

We first prove the core estimates for the sparse bound, namely the sparse bound for the 'main term' defined in (2.4), in the region \( S_1^d \), see Figure 5.

**Lemma 6.1.** Let \( S_1^d \) be the interior of the convex hull of \((\frac{1}{2}, \frac{1}{2}), (\frac{1}{d-1}, \frac{d-2}{d-1})\) and \((\frac{d-1}{d+1}, \frac{d+1}{d+1})\). For all \((\frac{1}{p_1}, \frac{1}{p_2}) \in S_1^d\), there holds
\[
\left\| \sup_{\lambda \in \Lambda_0} |C_\lambda^v| \right\|_{(p_1, p_2)} \lesssim 1.
\]
The implied constant depends upon dimension, \((p_1, p_2)\).

The strategy is roughly speaking the same as for the improving bound. But the techniques will be different, as is required to address the supremum. Essential to our argument is the \((1, 1)\) sparse bound in (6.3) below. These are the more technical estimates.

**Lemma 6.2.** There holds
\[
(6.3) \quad \left\| \sup_{\lambda \in \Lambda_0, w \leq 2\lambda} |C_\lambda^v| \right\|_{(1,1)} \lesssim e^{w^2} 2^v
\]
\[
(6.4) \quad \left\| \sup_{\lambda \in \Lambda_0, w < 2\lambda} |C_\lambda^v| \right\|_{1+\epsilon \to 1+\epsilon} \lesssim e^{w^{1+\epsilon}} 2^v.
\]
\[
(6.5) \quad \left\| \sup_{\lambda \in \Lambda_0, w < 2\lambda} |C_\lambda^v| \right\|_{2 \to 2} \lesssim e^{2^{-\frac{d}{2} - \frac{d}{2}}} w^{-\frac{d}{2}}.
\]

These bounds are uniform in \( 1 \leq w \leq 2\lambda \), and \( v \geq 0 \).
Proof of Lemma 6.1. See Figure 6 for a synopsis of the interpolation used in the argument. Using the notation (3.12), we need to show that for all \((\frac{1}{p_1}, \frac{1}{p_2}) \in S^1_d\), there is a \(\delta > 0\) so that

\[
\left\| \sup_{\lambda \in A_0 \atop w < 2\lambda} |C_{\lambda, w}^v| \right\|_{(p_1, p_2)} \lesssim 2^{-\delta(v+w)}.
\]

Interpolating between the \(\ell^2\) estimate (6.5) and the \(\ell^{1+\epsilon}\) bound (6.4) for \(\epsilon = \epsilon(p) > 0\) sufficiently small, we see that

\[
\left\| \sup_{\lambda \in A_0 \atop w < 2\lambda} |C_{\lambda, w}^v| \right\|_{p \to p} \lesssim 2^{-\delta w^{-\delta}}, \quad \frac{d-1}{d-2} < p < 2, \ \delta = \delta_p > 0
\]

The proof of any sparse bound is recursive, and so to establish (6.6), we need to establish a corresponding recursive statement. This next statement is a consequence of the \((1, 1)\) sparse bound (6.3). Suppose that \(Q_0\) is a large cube, with side length \(\lambda_0\), and \(f\) is supported on \(Q_0\), there is a collection \(B\) of disjoint cubes \(B \subset Q_0\) with \(\sum_{Q \in B} |Q| < \frac{1}{4} |Q_0|\), which meets this additional condition. Define a stopping time \(\sigma : Q_0 \to (0, \lambda_0)\) by

\[
\sigma(x) = \sum_{Q \in B} \ell(Q)1_Q
\]

and otherwise \(\sigma(x) = 1\). This stopping time satisfies the condition:

\[
\Psi_{\sigma} := \sup_{\lambda \in A_0 \atop \sigma(x) \leq \lambda \leq \lambda_0} |C_{\lambda, w}^v f| \lesssim w^2 \langle f \rangle_{Q_0, 1}.
\]

But, the estimates (6.4)—(6.5) also apply to the function \(\Psi_{\sigma}\). For instance, there holds

\[
\langle \Psi_{\sigma} \rangle_{Q_0, 2} \lesssim 2^{-\frac{d-1}{d-2} w^{-\frac{1}{d-2}}} \langle f \rangle_{Q_0, 2}.
\]

Interpolation between the \(\ell^\infty\) bound (6.7) and all of the inequalities corresponding to (6.4)—(6.5) will complete the proof of (6.6). We omit the easy details.

\[\square\]

Proof of the \((1, 1)\) Sparse Bound (6.3). Recalling the inequality (3.15), we see that the first term on the right dominates in this setting, yielding the estimate

\[
\sup_{\lambda \in A_0 \atop w \leq \lambda} |C_{\lambda, w}^v f| \lesssim w^2 2^v Mf,
\]

where \(M_{HL}\) is the Hardy-Littlewood maximal function. But, the \((1, 1)\) sparse bound for that operator is elementary. So the proof of (6.3) is complete.

\[\square\]
Proof of the $\ell^{1+\epsilon}$ Bound (6.4). Return to the inequality (3.15) to bound $C^v_{\lambda, w} f$. There are two terms on the right in (3.15). The first is easy to control over lacunary $\lambda$.

$$w^{2v} \sum_{\lambda \in \Lambda_0 \atop \lambda < 2\lambda} \lambda^{-2} \|A_\lambda f\|_1 \lesssim 2^v \|f\|_1.$$ 

Forming the supremum over the second term in (3.15), the key fact is Lemma 6.8. □

Lemma 6.8. For $\zeta$ as in (3.8) we have

$$\left\| \sup_{\lambda \in \Lambda_0 \atop \lambda > 2^v} \zeta_{(2^{-v})\lambda} * d\sigma_\lambda * f \right\|_p \leq \|f\|_p, \quad p > 1.$$ 

Proof. This is a transference result, but it does not follow from those that we are aware of in the literature. The key fact is that in the continuous case, the lacunary spherical maximal function

$$\left\| \sup_{\lambda \in \Lambda'} d\sigma_\lambda * f \right\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$ 

Above, $\Lambda'$ is any lacunary set of radii.

The condition $\lambda > 2^v$ implies that $|\nabla \zeta_{(2^{-v})\lambda} * d\sigma_\lambda| \lesssim 1$. So, given $f \in \ell^p(\mathbb{Z}^d)$, we set

$$\varphi = \sum_{n \in \mathbb{Z}^d} f(n) 1_{n+[0,1]^d}.$$ 

It follows that for non-negative $f \in \ell^p(\mathbb{Z}^d)$,

$$\zeta_{(2^{-v})\lambda} * d\sigma_\lambda * f(n) \lesssim \inf_{|x-n| < 1} \zeta_{(2^{-v})\lambda} * d\sigma_\lambda * \varphi(x)$$

where the convolutions on the left are done on $\mathbb{Z}^d$, and on the right on $\mathbb{R}^d$. It is then clear that

$$\left\| \sup_{\lambda \in \Lambda_0 \atop \lambda > 2^v} \zeta_{(2^{-v})\lambda} * d\sigma_\lambda * f \right\|_{L^p(\mathbb{Z}^d)} \leq \left\| \sup_{\lambda \in \Lambda_0 \atop \lambda > 2^v} \zeta_{(2^{-v})\lambda} * d\sigma_\lambda * \varphi \right\|_{L^p(\mathbb{R}^d)} \leq \|\varphi\|_{L^p(\mathbb{R}^d)} = \|f\|_{\ell^p(\mathbb{Z}^d)}.$$ 

□

Proof of $\ell^2 \to \ell^2$ Bound, for $v = 0$. We prove the $\ell^2 \to \ell^2$ maximal inequality (6.5), in the case of $v = 0$, which is of a different nature than $v > 0$. Indeed, this is the core of the matter. The analysis is broken into two cases, the first a restricted supremum, and
the second a supremum over ‘large’ \( \lambda \).

\[
\| \sup_{\lambda \in \Lambda_0} |C^0_{\lambda,w}f| \|_2 \|_2 \leq \epsilon \ w^{-\frac{d}{4}} \|f\|_2,
\]

\[
\sup_{\lambda \in \Lambda_0} |C^0_{\lambda,w}f| \|_2 \leq \epsilon \ w^{-\frac{d}{4}} \|f\|_2.
\]

We treat the cases in order. In the first case of (6.9), we just use the triangle inequality.

\[
\| \sup_{\lambda \in \Lambda_0} |C^0_{\lambda,w}f| \|_2 \leq \sum_{\lambda \in \Lambda_0} \|C^0_{\lambda,w}f\|_2 \leq \epsilon \ \|f\|_2 \sum_{\lambda \in \Lambda_0} \|C^0_{\lambda,w}f\|_2 \leq \epsilon \ \|f\|_2 w^{-\frac{d}{4}}.
\]

Observe that there are only \( \log w \) summands in the sum over \( \lambda \), and that we have appealed to a simple variant of (2.9) above, as well as the fact that \( \lambda^2 \) is the power of a single prime.

In the complementary case \( \lambda \geq 8w^2 \), there is additional orthogonality, in that the Fourier supports of the operators \( C^0_{\lambda,q} \) are pairwise disjoint. (See Figure 3.) This, with a simple square function argument, allows us to replace a sum over \( q \) below by a supremum.

\[
\| \sup_{\lambda \in \Lambda_0} |C^0_{\lambda,w}f| \|_2 \leq 2w \sum_{q : w/2 \leq q < w} \|C^0_{\lambda,q}f\|_2 \leq \|f\|_2 \cdot w \sup_{q : w/2 \leq q < w} \|C^0_{\lambda,q}f\|_2 \|_2 \rightarrow 2.\]

We conclude (6.10) by showing this inequality, using a variant of an argument from Magyar Stein and Wainger [17].

\[
\sup_{q : w \leq 2q < 4w} \| \sup_{\lambda \in \Lambda_0} |C^0_{\lambda,q}f| \|_2 \leq w^{-\frac{d}{4}} \|f\|_2.
\]

A crucial intermediate fact is this multi-frequency inequality. For an integer \( q > 1 \), set

\[
b_{q,\lambda}(\xi) = \sum_{\ell \in \mathbb{Z}^d_q} F\mathcal{d}\sigma_{\lambda}(\xi - \ell/q)\Phi_{q}(\sigma - \ell/q), \quad \lambda \geq q, \ \lambda \in \Lambda_0.
\]

Above, we are using the notation \( \Phi_{q} \) as defined just below (2.4). And, let \( B_{q,\lambda} \) be the corresponding operator.
Lemma 6.12. We have the maximal inequality

\[
\| \sup_{\lambda \in \Lambda_0, \lambda > q} |B_{q, \lambda}| \|_{2 \to 2} \lesssim 1.
\]

This is a simple instance of Magyar, Stein, Wainger [17, Corollary 2.1]. Accordingly, we do not prove this statement here.

We return to the expression defining Kloosterman sums in terms of Gauss sums, as in (2.15) and (2.16). Define

\[
d_{a, q}(\xi) = \sum_{\ell \in \mathbb{Z}} G(a/q, \ell) \Phi_{4q}(\xi - \ell/q).
\]

Denote the corresponding operator by \(D_{a, q}\). A well known estimate for Gauss sums implies that

\[
\|D_{a, q}\|_{2, 2} \lesssim \sup_{\ell} |G(a/q, \ell)| \lesssim q^{-\frac{d}{2}}.
\]

We then estimate from the definition of \(C_{\lambda, q}^0\), see (3.1), by summing over \(q \in \mathbb{Z}^\times\), noting that the Kloosterman sums then get replaced by Gauss sums, which are independent of \(\lambda\).

\[
\| \sup_{\lambda \in \Lambda_0, \lambda \geq 4w^2} |C_{\lambda, q}^0| \|_2 \leq q \sup_{a \in \mathbb{Z}^\times} \| D_{a, q} f \|_2 \lesssim q^{-\frac{d}{2}} \| f \|_2.
\]

Now, observe that for fixed \(a \in \mathbb{Z}^\times\), if we remove the \(P_{\lambda, q}^0\), we can appeal to the maximal inequality (6.13). We have

\[
\| \sup_{\lambda \in \Lambda_0, \lambda \geq 4w^2} |B_{q, \lambda} D_{a, q} f| \|_2 \lesssim \|D_{a, q} f\|_2 \lesssim q^{-\frac{d}{2}} \| f \|_2.
\]

And, using a square function, we see that adding in removing the operator \(P_{\lambda, q}^0\) does not impact the estimate.

\[
\sum_{\lambda \in \Lambda_0, \lambda \geq 4w^2} \| (B_{q, \lambda} D_{a, q} - B_{q, \lambda} D_{a, q} P_{\lambda, q}^0) f \|_2 \lesssim \| f \|_2^2.
\]

These estimates complete the proof of (6.11).

\[\square\]

Proof of the \(\ell^2 \to \ell^2\) Estimate (6.5) for \(\nu > 0\). Note that we can estimate

\[
\left\| \sup_{\lambda \in \Lambda_0, \lambda \geq 4w^2} |C_{\lambda, w}^\nu f| \right\|_2^2 \leq w \cdot \sum_{q : w/2 \leq q \leq w} \sum_{\lambda \in \Lambda_0, \lambda \geq 4w^2} \| C_{\lambda, q}^\nu f \|_2^2 \lesssim \varepsilon \| f \|_2^2 w^{2d-2} \sum_{q : w/2 \leq q \leq w} q^{2\varepsilon - (d-1)} \rho(\lambda_w, q)\rho(\lambda_w, q) \lesssim \varepsilon \| f \|_2^2 w^{2d-2} \|| f \|_2^2.\]
The essential point is that the convolution operators $C_{\lambda,q}^{v}$ have Fourier supports that have bounded overlaps. Thus, it is efficient to pass to the square function over $\lambda$ in the top line above. Then, we are free to appeal to (3.6). Note that as $\lambda^2$ is always a power of a fixed prime $\lambda_0$, in the argument of $\rho$ in the second line, we are free to insert $\lambda_w$, the smallest power of $\lambda_0$ that is larger that $w$. Finally, (2.9) completes the proof.

\[\square\]

6.1. The Sparse Bound for the Residual Term. We prove the sparse bounds for the residual term $R_{\lambda}$ in the triangle $S^{1}_{d}$, see Figure 5.

Lemma 6.14. For any lacunary sequence $\Lambda \subset \Lambda_d$, there holds, and and $(\frac{1}{p_1}, \frac{1}{p_2}) \in S^{1}_{d}$, there holds

\[\left\| \sup_{\lambda \in \Lambda} |R_{\lambda}| \right\|_{(p_1,p_2)} \lesssim 1.\]

We have these inequalities at the endpoints for the residual term. For all $\epsilon > 0$, there holds

\[(6.15) \; \left\| R_{\lambda} \right\|_{2 \rightarrow 2} \lesssim \epsilon \lambda^{-\frac{d}{2} + \epsilon},\]

\[(6.16) \; \left\| R_{\lambda} \right\|_{1 \rightarrow 1} \lesssim \epsilon \lambda^{1+\epsilon},\]

\[(6.17) \; \left\| R_{\lambda} \right\|_{1 \rightarrow \infty} \lesssim \epsilon \lambda^{2+\epsilon} \cdot \lambda^{-d}.\]

Indeed, the first estimate (6.15) is the crucial Lemma 2.10. For the other two, write

\[|R_{\lambda}f| \leq |A_{\lambda}f| + |C_{\lambda}f|.\]

By inspection, $A_{\lambda}$ satisfy (6.16) and (6.17). Concerning $C_{\lambda}$, we write $C_{\lambda}$ as a sum over $C_{\lambda,w}$, for $w \leq 2\lambda$. (Recall the definition in (3.12).) Then, appeal to (3.15). We have

\[\left\| C_{\lambda,w} \right\|_{1 \rightarrow 1} \lesssim w^{1+\epsilon},\]

by the second term on the right in (3.15), and

\[\left\| C_{\lambda,w} \right\|_{1 \rightarrow \infty} \lesssim w \cdot \lambda^{1-d}\]

by the first term on the right in (3.15). These are summed over $1 \leq w \leq 2\lambda$ to complete the proofs of (6.16) and (6.17).

The estimate (3.15) also shows that the kernel of $R_{\lambda}$ is very nearly supported in a cube of side length $\lambda$. We have for any $N > d$,

\[\sum_{1 \leq w \leq 2\lambda} |\tilde{c}_{\lambda,w}(n)| \lesssim \lambda^{2-d}(|n|/\lambda)^{-N}, \quad |n| > 2\lambda.\]

We collect sparse bounds.

\[\langle R_{\lambda}f, g \rangle \leq C \epsilon \lambda^{-\frac{d}{2} + \epsilon} \sum_{k=1}^{\infty} \sum_{Q: tQ = 2^k \lambda} 2^{-k} (Q,2\langle g \rangle_Q,2|Q|)\]

\[\langle f \rangle_{Q,2\langle g \rangle_Q,2|Q|} \leq C \epsilon \lambda^{-\frac{d}{2} + \epsilon} \sum_{k=1}^{\infty} \sum_{Q: tQ = 2^k \lambda} 2^{-k} (\langle f \rangle_{Q,2\langle g \rangle_Q,2|Q|})\]
where the sum is over a choice of dyadic cubes of the given side length. There are two more sparse bounds of the same form, but for \((1, 1)\), and \((1, \infty)\) norms on the right hand side and with a loss of \(\lambda^{2+\epsilon}\) and \(\lambda^{1+\epsilon}\) respectively. In particular,

\[
\langle R_\lambda f, g \rangle \leq C_\epsilon \lambda^{2+\epsilon} \sum_{k=1}^{\infty} \sum_{Q: \ell Q = 2^k \lambda} 2^{-k} \langle f \rangle_Q,1 \langle g \rangle_Q,1 |Q|.
\]

These sparse bounds, as the cubes involved are independent of the functions. They can be interpolated, as we know from [14, Lemma 3.5]. See Figure 6 for an illustration of this interpolation.

For \((1/p_1, 1/p_2) \in S_1^d\), we learn that \(\|R_\lambda\|_{(p_1,p_2)} \lesssim \lambda^{-\delta}\), for a choice of \(\delta = \delta(p_1, p_2) > 0\). This estimate is summable over \(\lambda\) in a lacunary collection. This proves Lemma 6.14.

6.2. \textbf{Completing the Proof of the Sparse Bound.} Combining Lemmas 6.1 and 6.14, we have proved the sparse bounds in the triangle \(S_1^d\). It remains to prove the sparse bound in the triangle \(S_2^d\). (See Figure 5.) We use the trivial \(\ell^\infty \to \ell^\infty\) bound to deduce a restricted weak type sparse bound, as defined in (6.22). But such bounds improve to actual sparse bounds, as proved in Lemma 6.23. Thus, it suffices to prove

\textbf{Lemma 6.18.} For \((1/p_1, 1/p_2) \in S_2^d\), the maximal function \(\sup_{\lambda \in \Lambda_0} A_\lambda\) satisfies a restricted weak type sparse bound, in the sense of (6.22).

\textit{Proof.} We know the sparse bound at \((1/p_1, 1/p_2) \in S_1^d\), for the maximal function formed over a lacunary collection of radii \(\Lambda_0\). We will take \((1/p_1, 1/p_2) \in S_1^d\) close to \((d-1/d+1, d-1/d+1)\).

Given a cube \(Q\), set

\[
A_Q^* f(x) = 1_Q \sup_{\lambda \in \Lambda_0 : \lambda \leq \ell(Q)} A_\lambda f.
\]

Note that \(A_Q f = A_Q(1_Q f)\).

Take two non-negative functions \(f_1, f_2\) supported on a cube \(Q_0\). The existence of the sparse bound implies this: There is a collection \(B\) of disjoint subcubes \(B \subset Q_0\), so that the following condition holds. The measure of the cubes is very small.

\[
\sum_{B \in B} |B| \leq 4^{-d} |Q_0|
\]

Associated to the cubes in \(B\) is a ‘stopping time’ \(\sigma : Q_0 \to \Lambda_0\) so that \(\sigma(x) \geq \sum_{B \in B} \ell(Q) 1_B\). We then have

\[
A_{Q_0}^* f_1 = \max \left\{ A_{\sigma(x)} f_1(x), \sum_{B \in B} A_B^* f_1(x) \right\}.
\]

Using this notation, we have

\[
\langle A_{Q_0}^* f_1, f_2 \rangle \leq \langle A_{\sigma(x)} f_1, f_2 \rangle + \sum_{B \in B} \langle A_B^* f_1, f_2 \rangle.
\]

The second terms on the right are of the form to which the sparse bound applies.
Figure 7. The point $Q_1 = (1/p_1, 1/p_2)$ is in $S^1_d$. We deduce the restricted weak type bound at each point on the dotted line between $Q_1$ and $(1,0)$.

We need to control the first term on the right in (6.19). Importantly, we further have the inequalities

$$
\langle A_{\sigma(\cdot)} f_1, f_2 \rangle \lesssim \begin{cases} 
\langle f_1 \rangle_{Q_0, \infty} \langle f_2 \rangle_{Q_0, 1} |Q_0| \\
\langle f_1 \rangle_{Q_0, p_1} \langle f_2 \rangle_{Q_0, p_2} |Q_0| 
\end{cases}.
$$

The top line is trivially true. The bottom line is the additional condition that holds, in view of the sparse bound holding at $(1/p_1, 1/p_2) \in S^1_d$.

But, then, insisting that $f_j = 1_{E_j}$, we see immediately, that for all $0 < \theta < 1$, we have

$$
\langle A_{\sigma(\cdot)} 1_{E_1}, 1_{E_2} \rangle \lesssim \langle 1_{E_1} \rangle_{Q_0} \langle 1_{E_2} \rangle_{Q_0} \frac{1-\theta^{1/p_1} + \theta^{1/p_2}}{\theta} |Q_0|.
$$

Inserting this inequality into (6.19) we can prove the restricted weak type sparse bound for any point $(1/q_1, 1/q_2)$ that is the convex combination of $(1,0)$ and $(1/p_1, 1/p_2)$. See Figure 7. So we have finished the proof of the Lemma. \qed

6.3. Background on Sparse Bounds. We recall some known results about sparse bounds. First, we modify the definitions in Definition 1.7 in the following ways. For $0 < \eta < 1$, we say that a collection of cubes $S$ is $\eta$-sparse if there is a collection of disjoint sets $\{E_Q : Q \in S\}$ with $E_Q \subset Q$ and $|E_Q| \geq \eta$.

For the purposes of this section, we define a sparse form

$$
L_{S, p_1, p_2}(f_1, f_2) := \sum_{Q \in S} \langle f_1 \rangle_{3Q, p_1} |f_2|_{3Q, p_2} |Q|.
$$
We will use this notation for a sparse collection of cubes $S$. Because we impose the local norm on the triple of the cubes above, we can further restrict $S$ to consist of dyadic cubes.

We then define the $(p_1, p_2)$ sparse norm of a sublinear operator $T$ to be the smallest constant $C$ so that for all finitely supported functions $f, g$ we have

$$|\langle Tf, g \rangle| \leq C \sup_{S \text{ is } \eta\text{-sparse}} L_{S, p_1, p_2}(f, g), \quad C' > C.$$ 

We denote the best constant $C$ above by $\|T\|_{(p_1, p_2, \eta)}$.

All of these norms are equivalent. There holds $\|T\|_{(p_1, p_2, \eta)} \simeq \|T\|_{(p_1, p_2, 1/10)}$, for all $0 < \eta < 1/2$. This is a convenient fact, as from time to time we can change the implied value of $\eta$ to insure that certain arguments are valid. Accordingly, we will suppress $\eta$ in the notation below.

The sparse norm is (quasi)-sublinear, as this ‘One Form’ Lemma implies.

**Lemma 6.20.** [15, Lemma 4.7] For all compactly supported $f, g$, there is a choice of sparse collection $S_0$ so that

$$\sup_{S \text{ sparse}} L_{S, p_1, p_2}(f_1, f_2) \lesssim L_{S_0}(f_1, f_2).$$

This allows the following corollary.

**Corollary 6.21.** Suppose that for a sublinear operator $T$ there are non-negative constants $\{c_j\}$ so that for all $f, g$ there are sparse forms $\{L_{j, p_1, p_2}\}$ so that

$$\langle Tf, g \rangle \leq \sum_j c_j L_{j, p_1, p_2}(f, g).$$

Then, $\|T\|_{(p_1, p_2)} \lesssim \sum_j c_j$.

We define a restricted weak-type sparse bound as follows. Given a sublinear operator $T$, and $1 \leq p_1, p_2 \leq \infty$ we say that $T$ satisfies a $(p_1, p_2)$ restricted weak-type sparse bound if for all finitely supported functions $|f| \leq 1_F$ and $|g| \leq 1_G$, we have

$$|\langle Tf, g \rangle| \leq \sup_{S \text{ sparse}} L_{S, p_1, p_2}(1_F, 1_G).$$

The advantage of restricted bounds is that they self-improve to sparse bounds, for slightly weaker bounds.

**Lemma 6.23.** Suppose that sublinear operator $T$ satisfies a $(p_1, p_2)$ restricted weak-type sparse bound. Then, it also satisfies a $(q_1, q_2)$ sparse bound for all $q_j > p_j, j = 1, 2$.

**Proof.** This is a standard level set argument, although the argument is new in this setting. In the argument below, we will improve the assumed restricted weak type inequality to the desired conclusion in one of the functions. For compactly supported $|f| \leq 1_F$, and finitely supported $g$, there holds

$$|\langle Tf, g \rangle| \leq \sup_{S \text{ sparse}} L_{S, p_1, p_2}(1_F, g), \quad q > p_2.$$
An iteration of the argument will prove the Lemma as stated.

We can assume that $g$ is a finitely supported function bounded by one. Write $g = \sum_{v \in \mathbb{N}} 2^{-v} g_v$, where $\frac{1}{2} 1_{G_k} \leq |g_k| \leq 1_{G_k}$. Fix a points $(1/p_1, 1/p_2) \in U$.

Apply the restricted weak type sparse bound to $(f, g_k)$. There is a sparse collection $S_k$ so that

$$|\langle Tf, g_k \rangle| \lesssim L_{S_k,p_1,p_2}(f, g_k)$$

Further, let $\mathcal{R}$ be a collection of dyadic stopping cubes for the averages $\langle f \rangle_{p_1,3Q}$. That is, for all dyadic cubes $Q$, we have $\langle f \rangle_{p_1,Q} \lesssim \langle f \rangle_{p_1,Q'}$, where $Q'$ is the least element of $\mathcal{R}$ that contains $Q$. We then estimate

$$|\langle Tf, g \rangle| \leq \sum_{k=1}^{\infty} 2^{-k} |\langle Tf, g_k \rangle|$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k} L_{S_k,p_1,p_2}(1_f, 1_{G_k})$$

$$\lesssim \sum_{R \in \mathcal{R}} \sum_{k=1}^{\infty} \sum_{Q \in S_k} \langle f \rangle_{p_1,3Q} 2^{-k} \langle 1_{G_k} \rangle_{p_2,3Q} |Q|$$

$$\lesssim \sum_{R \in \mathcal{R}} \langle 1_f \rangle_{p_1,3R} \sum_{k=1}^{\infty} \sum_{Q \in S_k} 2^{-k} \langle 1_{G_k} \rangle_{p_2,3Q} |Q|.$$  

We claim that uniformly over $R \in \mathcal{R}$, we have

$$\sum_{k=1}^{\infty} \sum_{Q \in S_k \atop Q' = R} 2^{-k} \langle 1_{G_k} \rangle_{p_2,3Q} |Q| \lesssim \langle g \rangle_{3R,q}^q |R|, \quad q > p_2.$$  

This completes the proof of our Lemma.

To see $(6.24)$, fix $k \in \mathbb{N}$ and $q : p_2 < \bar{q} < q$. We appeal to the Carleson embedding inequality below:

$$\sum_{Q \in S_k} \langle 1_{G_k} \rangle_{p_2,3Q} |Q| \leq \left[ \sum_{Q \in S_k} \langle 1_{G_k} \rangle_{1,3Q} \bar{q} \right]^{1/\bar{q}} \left[ \sum_{Q \in S_k} |Q| \right]^{1/q}$$

$$\lesssim |G_k \cap 3R|^{1/\bar{q}} |R|^{1/q}.$$  

Then, summing over $k \in \mathbb{N}$ gives us

$$\sum_{k \in \mathbb{N}} 2^{-k} |G_k \cap 3R|^{1/\bar{q}} \lesssim \|g 1_{3R}\|_{q}^{1/q} |R|^{1/q} - \frac{1}{q}.$$  

We are finished. \qed
7. Complements

(1) The results of Hughes [9] were obtained independently of us. Fixed radius inequalities are proved, although none of them are scale free. More general surfaces than spheres are considered, and no sparse bounds are proved. Similarly, [8] considers lacunary spherical maximal functions, although for very particular examples of lacunary radii.

(2) Apply the $\ell^p$ improving inequality to $f = 1_{S_\lambda}$ to see that for $0 < c, \epsilon < 1$,
\[
|\{A_{\lambda}1_{S_\lambda} > c/\lambda\}| \leq C_{\epsilon, \omega(\lambda^2)} \lambda^{d - \frac{d-1}{2} + \epsilon}, \quad \lambda^2 \in \mathbb{N}.
\]

The set on the left are those points in $n \in \mathbb{N}^d$ at which the sphere $n + S_{\lambda}$ intersects $S_{\lambda}$ in about the expected number of $\approx \lambda^{d-3}$ points. And, the estimate above shows that these points can only be a small proportion of the ball of radius $\lambda$. The estimate (7.1) is better than what follows from the $\ell^1 \to \ell^1$ bound for $A_{\lambda}$ in dimensions $d \geq 6$. Estimates of this set seem to be of some interest, but we could not find any literature on this subject.

(3) We don’t know the example that shows that $p = \frac{d+1}{d-1}$ is the sharp index at which the $\ell^p$-improving inequality can hold, with some arithmetical constants. It would be interesting to find such an example.

(4) The intricate ‘self-improving’ property of §5 does not have a variant in the case of the lacunary radii maximal function. The question of the sharp bound for the boundedness of such maximal functions is interesting. We certainly have no idea about how to improve our bound.

(5) The first author will establish results for the full Stein type maximal function, both in the $\ell^p$-improving and sparse setting. The $\ell^p$ improving inequality is
\[
\langle \sup_{q/2 \leq \lambda \leq q} A_{\lambda}f \rangle_{Q,p'} \leq \langle f \rangle_{Q,p}, \quad \frac{d}{d-2} < p \leq 2,
\]
where $Q$ is a cube of side length about $q$, and $f$ is supported on $Q$. Note the appearance of the Magyar Stein Wainger index $\frac{d}{d-2}$ above.

(6) The sparse bounds for Stein maximal function in (1.1) will hold for $(p_1, p_2)$ provided $(1/p_1, 1/p_2)$ are in the open triangle with vertices $(0, 1)$, $(d-2, \frac{d-2}{d})$, and $(\frac{d-2}{d}, 2)$. See Figure 8. This is of course a more restrictive set of conditions. We see that our sparse bounds hold, by accommodation of additional arithmetical features of the operator, in a larger range of indices.

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Figure 8. Above, $A = \left( \frac{d-1}{d+1}, \frac{d-1}{d+1} \right)$, $B = \left( \frac{d-1}{d+1}, \frac{2}{d+1} \right)$, $C = \left( \frac{d-2}{d}, \frac{2}{d} \right)$, $D = \left( \frac{d-2}{d}, \frac{d-2}{d} \right)$. The triangle $F_d$ is the region in which sparse bounds hold, in dimension $d$, for the Stein maximal operator in (1.1). Note that $F_4$ is empty. The gray polygonal area is the region in which sparse bounds hold for the lacunary radii maximal function, assuming an arithmetic restriction on the radii.

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