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Existence of guided waves due to a lineic perturbation of a 3D periodic medium

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Abstract

In this note, we exhibit a three dimensional structure that permits to guide waves. This structure is obtained by a geometrical perturbation of a 3D periodic domain that consists of a three dimensional grating of equi-spaced thin pipes oriented along three orthogonal directions. Homogeneous Neumann boundary conditions are imposed on the boundary of the domain. The diameter of the section of the pipes, of order \(\varepsilon > 0\), is supposed to be small. We prove that, for \(\varepsilon\) small enough, shrinking the section of one line of the grating by a factor of \(\sqrt{\mu}\) \((0 < \mu < 1)\) creates guided modes that propagate along the perturbed line. Our result relies on the asymptotic analysis (with respect to \(\varepsilon\)) of the spectrum of the Laplace-Neumann operator in this structure. Indeed, as \(\varepsilon\) tends to 0, the domain tends to a periodic graph, and the spectrum of the associated limit operator can be computed explicitly.

Keywords : guided waves, periodic media, spectral theory.

AMS codes : 78M35, 35J05, 58C40

1 Statement of the problem

Let \(\omega_1, \omega_2\) and \(\omega_3\) be three Lipschitz bounded domains of \(\mathbb{R}^2\) of same area \(|\omega_1| = |\omega_2| = |\omega_3|\) containing the origin \((0,0)\), let \(\varepsilon > 0\) be a parameter (that is going to be small), and let \(a_1, a_2\) and \(a_3\) be three positive real numbers. We denote by \((e_i)_{i \in \{1,2,3\}},\) the standard basis of \(\mathbb{R}^3\). For any \((k,\ell) \in \mathbb{Z}^2\), we consider the three dimensional domain \(D_{k,\ell,3}^\varepsilon\) defined by

\[
D_{k,\ell,3}^\varepsilon = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ such that } (x_1-a_1k)/\varepsilon, (x_2-a_2\ell)/\varepsilon \in \omega_3\},
\]

which is an unbounded cylinder of constant cross section \(\varepsilon \omega_3\). It is infinite along the \(e_3\) direction (invariant with respect to \(x_3\)) and contains the point \((a_1 k, a_2 \ell, 0)\). Similarly, for any \((k,\ell) \in \mathbb{Z}^2\), we define the domains

\[
D_{k,\ell,1}^\varepsilon = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ such that } (x_2-a_2k)/\varepsilon, (x_3-a_3\ell)/\varepsilon \in \omega_1\},
\]

\[
D_{k,\ell,2}^\varepsilon = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ such that } (x_1-a_1k)/\varepsilon, (x_3-a_3\ell)/\varepsilon \in \omega_2\},
\]

and we consider the periodic domain \(\Omega_\varepsilon\) given by

\[
\Omega_\varepsilon = \bigcup_{i \in \{1,2,3\}} \bigcup_{(k,\ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon. \tag{1}
\]

The domain \(\Omega_\varepsilon\) is a three dimensional grating of equi-spaced parallel pipes (of constant cross section) oriented along the three orthogonal directions \(e_1, e_2\) and \(e_3\). It is \(a_j\)-periodic with respect to \(x_j, j = 1,2,3\). Moreover, the points \((ka_1, \ell a_2, ma_3), (k,\ell, m) \in \mathbb{Z}^3\), belong to \(\Omega_\varepsilon\).

In order to create guided modes, we introduce a linear defect (see [1]-[5]-[2]) in the periodic structure by modifying the section size of one pipe of the grating (it is conjectured that guided modes cannot appear in the purely periodic structure, see [4] for the proof in the case of a symmetric medium). More precisely, we assume that the domain \(D_{0,0,3}^\varepsilon\) is replaced with the domain

\[
D_{0,0,3}^{\varepsilon,a} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ such that } (x_1/(\sqrt{\mu}\varepsilon), x_2/(\sqrt{\mu}\varepsilon)) \in \omega_3\},
\]
The graph the restriction of the domain $\Omega$ by $\mu$. We look for guided modes, i.e. solutions to the wave equation $t$ tends to 0. The operator $A_\varepsilon$ is given by
$$A_\varepsilon = \left( \bigcup_{i \in \{1,2\}} \bigcup_{(k,t) \in \mathbb{Z}^2} D^\varepsilon_{k,t,i} \right) \bigcup \left( \bigcup_{(k,t) \in \mathbb{Z}^2 \setminus \{(0,0)\}} D^\varepsilon_{k,t,3} \right) \bigcup D^\varepsilon_{0,0,3}. \quad (2)$$

$\Omega_\varepsilon$ is still $a_3$-periodic with respect to $x_3$. However, the presence of the perturbed pipe $D^\varepsilon_{0,0,3}$ breaks the periodicity with respect to $x_1$ and $x_2$. We emphasize that the domain $\Omega_\varepsilon$ (as well as $\varepsilon$) tends to a 3D periodic graph as $\varepsilon$ tends to 0.

We look for guided modes, i.e. solutions to the wave equation $\partial^2_t u - \Delta u = 0$ in $\Omega_\varepsilon$, satisfying homogeneous Neumann boundary conditions on $\partial \Omega_\varepsilon$ (see [7] for the investigation of the Dirichlet case), that propagate along the defect pipe $D^\varepsilon_{0,0,3}$ (i.e. in the $e_3$ direction) but stay confined in the transversal directions. More precisely, denoting by $B_\varepsilon$ the restriction of the domain $\Omega_\varepsilon$ to the band $|x_3| < a_3/2$,
$$B_\varepsilon = \{(x_1, x_2, x_3) \in \Omega_\varepsilon \text{ such that } |x_3| < a_3/2\}, \quad (3)$$

we search solutions of the form $u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3)e^{i\omega t - \beta x_3}$, where $\beta$ is a real parameter and $v(x_1, x_2, x_3) \in L_2(B_\varepsilon)$ is an $a_3$-periodic function in $x_3$. In fact, it is easily seen that the $\beta$-quasiperiodic function $v(x_1, x_2, x_3)e^{-i\beta x_3}$ is an eigenfunction of the operator
$$A^\varepsilon(\beta) : D(A^\varepsilon(\beta)) \subset L_2(B^\varepsilon) \to L_2(B^\varepsilon), \quad A^\varepsilon(\beta) = -\Delta u \text{ in } B^\varepsilon, \quad (4)$$

with $D(A^\varepsilon(\beta)) = \{u \in H^1_\Delta(B^\varepsilon), \quad u|_{\Sigma^+} = e^{-i\beta} u|_{\Sigma^-}, \quad \partial x_3 u|_{\Sigma^+} = e^{-i\beta} \partial x_3 u|_{\Sigma^-}, \quad \partial u|_{\partial B^\varepsilon \setminus \Sigma^\pm} = 0\}$, where
$$H^1_\Delta(B^\varepsilon) = \{u \in H^1(B^\varepsilon), \text{s.t. } \Delta u \in L_2(B^\varepsilon)\} \quad \text{and} \quad \Sigma^\pm = \{(x_1, x_2, x_3) \in \partial B^\varepsilon, x_3 = \pm a_3/2\}.$$

To study the spectral properties of $A^\varepsilon(\beta)$, we investigate its (formal) limit $A^{\mu}(\beta)$ as $\varepsilon$ tends to 0. The operator $A^{\mu}(\beta)$ is defined on the limit graph $\mathcal{G}$ (see Fig. 1b) and its spectrum can be explicitly computed. In particular, its spectrum has infinitely many gaps (Lemma 2.1), i.e. open intervals $(a, b) \subset \mathbb{R}$ such that the intersection of $[a, b]$ with the spectrum is reduced to $(a, b)$. Moreover, for $\mu < 1$, there is at least one eigenvalue in each gap (Lemma 2.5). Since, in addition, for $\varepsilon > 0$ sufficiently small, the spectrum of $A^\varepsilon(\beta)$ is close to the spectrum of $A^{\mu}(\beta)$, the existence of guided modes is guaranteed (Theorem 3.1).
2 The spectrum of the limit operator $A^\mu(\beta)$

2.1 Definition of the limit operator $A^\mu(\beta)$

The limit operator $A^\mu(\beta)$ is defined on the infinite periodic graph $G = \bigcap_{\varepsilon>0} B^\varepsilon_0$ obtained as the limit of $B^\varepsilon_0$ as $\varepsilon$ tends to 0: $G$ is made of the vertices $\{v_{k,\ell} : (ka_1, la_2, 0), e^{\pm}_{k,\ell} = (ka_1, la_2, \pm a_3/2), (k, \ell) \in \mathbb{Z}^2\}$ connected by the edges

$$\{e_{k+1/2,\ell} = (v_{k,\ell}, v_{k+1,\ell}), e_{k,\ell+1/2} = (v_{k,\ell}, v_{k,\ell+1}), e^{\pm}_{k,\ell} = (v_{k,\ell}, e^{\pm}_{k,\ell}), (k, \ell) \in \mathbb{Z}^2\}.$$ 

It is $a_1$-periodic with respect to $x_1$ and $a_2$-periodic with respect to $x_2$ (see Fig. 1b).

For any function $u$ defined on $G$, we denote by $u_{k,\ell}$ (resp. $u^{\pm}_{k,\ell}$) its value at the vertex $v_{k,\ell}$ (resp. $v^{\pm}_{k,\ell}$). The restriction of $u$ to the edge $e_{k+1/2,\ell}$ (resp. $e_{k,\ell+1/2}$ and $e^{\pm}_{k,\ell}$) is denoted by $u_{k+1/2,\ell}(x_1)$ (resp. $u_{k,\ell+1/2}(x_2)$ and $u^{\pm}_{k,\ell}(x_3)$).

The definition of $A^\mu(\beta)$ also requires the introduction of the function spaces $L^2(G)$ and $H^2(G)$ defined as

$$L^2(G) = \left\{ u : \|u\|_{L^2(G)} < +\infty \right\}, \quad H^2(G) = \left\{ u \in C(G) : \|u\|_{H^2(G)} < +\infty \right\},$$

where,

$$\|u\|^2_{L^2(G)} = \sum_{(k, \ell) \in \mathbb{Z}^2} \left( u^{\pm}_{k,\ell} \right)^2 + \|u_{k+1/2,\ell}\|_{L^2_G(e_{k+1/2,\ell}, e_{k,\ell})}^2 + \|u_{k,\ell+1/2}\|_{L^2_G(e_{k,\ell+1/2}, e_{k,\ell})}^2,$$

$$\|u\|^2_{H^2(G)} = \sum_{(k, \ell) \in \mathbb{Z}^2} \left( \sum_{\{u^{\pm}_{k,\ell}\}} \|u^{\pm}_{k,\ell}\|_{H^2_G(e_{k,\ell})}^2 + \|u_{k+1/2,\ell}\|_{H^2_G(e_{k+1/2,\ell})}^2 + \|u_{k,\ell+1/2}\|_{H^2_G(e_{k,\ell+1/2})}^2 \right),$$

and, for any $(k, \ell) \in \mathbb{Z}^2$, $u^{\pm}_{k,\ell}$ is the weight coefficient equal to $\mu$ for $k = \ell = 0$ and 1 otherwise.

The unbounded limit operator in $L^2(G)$ has domain

$$D(\mathcal{A}^\mu(\beta)) = \left\{ u \in H^2(G) : \forall (k, \ell) \in \mathbb{Z}^2, \quad u^{\pm}_{k,\ell} = e^{-i\beta} u^{\pm}_{k,\ell}, \quad (u^{\pm}_{k,\ell})'(a_3/2) = e^{-i\beta} (u^{\pm}_{k,\ell})'(-a_3/2), \right.$$  

$$\left. u'_{k+1/2,\ell}(ka_1) - u'_{k-1/2,\ell}(ka_1) + u'_{k,\ell+1/2}(la_2) - u'_{k,\ell-1/2}(la_2) + u_{k,\ell} \left( (u^{\pm}_{k,\ell})'(0) - (u^{\pm}_{k,\ell})'(0) \right) = 0 \right\},$$

and is defined by

$$\forall u \in D(\mathcal{A}^\mu(\beta)), \quad \mathcal{A}^\mu(\beta) u = -u'' \quad \text{on any edge of the graph } G.$$ 

The functions of $D(\mathcal{A}^\mu(\beta))$ are continuous on $G$ and $\beta$ quasi-periodic. Moreover, they satisfy the Kirchhoff conditions (8) that enforce the weighted sum of the outward derivatives of $u$ to vanish at each vertex $v_{k,\ell}$ $(k, \ell) \in \mathbb{Z}^2$. We point out that the perturbation, which results from a geometrical modification of the domain for the problem (4), is taken into account at the limit by means of the Kirchhoff condition (8) at the vertex $v_{0,0}$ $(u^{\pm}_{0,0} = \mu)$. The formal derivation of the limit model can be found in [6]. It is easily verified that the operator $A^\mu(\beta)$ is self-adjoint (for the weighted scalar product associated with (6)), see also [3]. The objective of the following two sections is to study the spectrum of $A^\mu(\beta)$.

2.2 Characterization and properties of the essential spectrum of $A^\mu(\beta)$

By a compact perturbation argument, one can prove that $\sigma_{ess}(A^\mu(\beta)) = \sigma(A(\beta))$, where $\mathcal{A}(\beta) = A^1(\beta)$ is the purely periodic operator corresponding to $A^\mu(\beta)$ for $\mu = 1$. The computation of its spectrum relies on the Floquet-Bloch theory (see [9]). More precisely, we can prove that $\lambda = \omega^2 \in \sigma(\mathcal{A}(\beta))$ if and only if either $\omega = 0$ and $\beta = 0$ or $\omega \neq 0$ and there exists $(k_1, k_2) \in [0, \pi]^2$ such that

$$\sin(\omega a_2) \sin(\omega a_3) (\cos(\omega a_1) - \cos k_1) + \sin(\omega a_3) (\omega a_1) (\cos(\omega a_2) - \cos k_2)$$

$$\quad + \sin(\omega a_1) (\omega a_2) (\cos(\omega a_3) - \cos \beta) = 0.$$

Based on the previous characterization, we prove that the operator $\mathcal{A}(\beta)$ has a countable infinity of gaps that can be separated into three categories (see [10] for the proof):
Lemma 2.1 The following properties hold:
1. \(\sigma_1 \cup \sigma_2 \cup \sigma_3 \subset \sigma(A(\beta))\), where
   \[\sigma_i = \left\{ \left(\frac{pn}{a_i}, n \in \mathbb{Z} \right) \right\} \text{ for } i \in \{1, 2\}, \text{ and } \sigma_3 = \left\{ \left((\pm \beta + 2 \pi n)/a_3, n \in \mathbb{Z} \right) \right\}.\]
2. For any \(\beta \in [0, \pi]\), the operator \(A(\beta)\) has infinitely many gaps whose ends tend to infinity.
3. Let \(W(\beta) = \{pn/a_3, n \in \mathbb{N}^*\}\) if \(\beta \notin \{0, \pi\}\) and \(W(\beta) = \{\beta/a_3 + (2n + 1)\pi/a_3, n \in \mathbb{N}^*\}\) if \(\beta \in \{0, \pi\}\.
   If an interval \((\omega^2, \omega^2)\) is a spectral gap of \(A(\beta)\), then, one of the following possibilities holds:
   (i) \(\omega^2 \notin \sigma_1 \cup \sigma_2\), \(\omega^2 \notin \sigma_1 \cup \sigma_2\), and there is a unique \(\omega_0 \in (\omega_0, \omega_1) \cap W(\beta)\).
   (ii) \(\omega^2 \in \sigma_1 \cup \sigma_2\), \(\omega^2 \notin \sigma_1 \cup \sigma_2\) and \(W(\beta) \cap (\omega_0, \omega_1) = \emptyset\).
   (iii) \(\omega^2 \notin \sigma_1 \cup \sigma_2\), \(\omega^2 \notin \sigma_1 \cup \sigma_2\) and \(W(\beta) \cap (\omega_0, \omega_1) = \emptyset\).

2.3 Computation of the discrete spectrum

Let us now determine the discrete spectrum of \(A^\mu(\beta)\). If \(\lambda = \omega^2\) is an eigenvalue of \(A^\mu(\beta)\), then the corresponding eigenfunction \(u \in D(A^\mu(\beta))\) satisfies the linear differential equation \(u'' + \omega^2 u = 0\) on each edge of the graph \(\mathcal{G}\).

Solving explicitly this equation (on each edge), taking into account the quasi-periodicity of \(u\) and the Kirchhoff conditions (8), we can replace the eigenvalue problem \(A^\mu(\beta)u = \lambda u\) with a set of finite differences equations for \((u_{k,\ell}, (k,\ell)) \in \mathbb{Z}^2\):

Lemma 2.2 Assume that \(\omega \notin \{\pi Z/a_1 \cup \pi Z/a_2 \cup \pi Z/a_3\}\) \(u \in D(A^\mu(\beta))\) is an eigenfunction of \(A^\mu(\beta)\) if and only if \((u_{k,\ell}, (k,\ell)) \in \mathbb{Z}^2\) and satisfies

\[
\begin{aligned}
&\frac{u_{k+1,\ell}}{\sin(\omega a_1)} + \frac{u_{k-1,\ell}}{\sin(\omega a_1)} + \frac{u_{k,\ell+1}}{\sin(\omega a_2)} + \frac{u_{k,\ell-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)u_{k,\ell} = 0, \quad \forall (k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\
&\frac{u_{1,0}}{\sin(\omega a_1)} + \frac{u_{-1,0}}{\sin(\omega a_1)} + \frac{u_{0,1}}{\sin(\omega a_2)} + \frac{u_{0,-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)u_{0,0} = 2(\mu - 1)\frac{\cos(\omega a_3) - \cos \beta}{\sin(\omega a_3)}u_{0,0},
\end{aligned}
\]

where we have defined \(g_\beta(\omega) = \frac{1}{\tan(\omega a_1)} + \frac{1}{\tan(\omega a_2)} + \frac{\cos(\omega a_3) - \cos \beta}{\sin(\omega a_3)}\).

As well-known, finite difference schemes may be investigated using the discrete Fourier transform
\[
\mathcal{F} : \mathcal{V} = (v_{k,\ell}, (k,\ell)) \in \mathbb{Z}^2 \mapsto \hat{\mathcal{F}}(\mathcal{V}) = \hat{\mathcal{V}}, \quad \hat{\mathcal{V}}(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}} e^{i(k\xi + \ell\eta)}v_{k,\ell}, \quad (\xi, \eta) \in [0, 2\pi]^2,
\]

where \(\mathcal{F}\) is an isometry between \(\ell_2(\mathbb{Z}^2)\) and \(L_2([0, 2\pi]^2)\). This, together with Lemma 2.2, provides the following characterization for the discrete spectrum of \(A^\mu(\beta)\):

Lemma 2.3 Assume that \(\omega \notin \{\pi Z/a_1 \cup \pi Z/a_2 \cup \pi Z/a_3\}\) \(u \in D(A^\mu(\beta))\) is an eigenfunction of \(A^\mu(\beta)\) if and only if the discrete Fourier transform \(\hat{u}\) of \((u_{k,\ell}, (k,\ell)) \in \mathbb{Z}^2\) belongs to \(L_2([0, 2\pi]^2)\) and satisfies

\[
(f(\xi, \eta, \omega) - \phi_\beta(\omega))\hat{u}(\xi, \eta, \omega) = (\mu - 1)\phi_\beta(\omega)u_{0,0}.
\]

where \(\phi_\beta(\omega) = \frac{\cos(\omega a_3) - \cos \beta}{\sin(\omega a_3)}\) and \(f(\xi, \eta, \omega) = \frac{\cos \xi - \cos(\omega a_1)}{\sin(\omega a_1)} + \frac{\cos \eta - \cos(\omega a_2)}{\sin(\omega a_2)}\).

Under the assumption of Lemma 2.3, (10) can be written as \(f(\xi, \eta, \omega) - \phi_\beta(\omega) = 0\). It follows that \(\lambda = \omega^2\) does not belong to \(\sigma_{ess}(A^\mu(\beta))\) if and only if, for any \((\xi, \eta) \in [0, 2\pi]^2\), \(\phi_\beta(\omega) - f(\xi, \eta, \omega)\) does not vanish. As a consequence, as soon as \(\lambda = \omega^2 \notin \sigma_{ess}(A^\mu(\beta))\), the function \((\xi, \eta) \mapsto \phi_\beta(\omega)/(\phi_\beta(\omega) - f(\xi, \eta, \omega))\) is continuous and bounded. Then, the inverse discrete Fourier transform can be applied to (13) to obtain

\[
u_{k,\ell} = \frac{(1 - \mu)u_{0,0}}{4\pi^2} \int_{[0, 2\pi]^2} \phi_\beta(\omega)/\phi_\beta(\omega) - f(\xi, \eta, \omega) e^{-i(k\xi + \ell\eta)}d\xi d\eta, \quad \forall (k,\ell) \in \mathbb{Z}^2.
\]

Writing the previous relation for \(k = \ell = 0\) yields the following criterion of existence of an eigenvalue:
Lemma 2.4 Assume that $\omega \notin \{\pi \mathbb{Z}/a_1\} \cup \{\pi \mathbb{Z}/a_2\} \cup \{\pi \mathbb{Z}/a_3\}$ and that $\lambda = \omega^2 \notin \sigma_{ess}(\mathcal{A}^\mu(\beta))$. Then, $\lambda$ is an eigenvalue of $\mathcal{A}^\mu(\beta)$ if and only if

$$
\mu = 1 - F_\beta(\omega) \quad \text{where} \quad F_\beta(\omega) = \left( \frac{1}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} \, d\xi d\eta \right)^{-1}.
$$

(14)

The study of the behavior of the function $F_\beta$ leads to the existence of at least one eigenvalue in each gap of $\mathcal{A}^\mu(\beta)$ as soon as $\mu < 1$, the minimal number of eigenvalues in each gap depending on the type of gaps (cf. Lemma 2.1-3 for the classification):

Lemma 2.5 For $\mu > 1$, the operator $\mathcal{A}^\mu(\beta)$ has no eigenvalue. For $0 < \mu < 1$, let $(\omega^2_b, \omega^2_t)$ be a spectral gap of the operator $\mathcal{A}^\mu(\beta)$:

(a) If $(\omega^2_b, \omega^2_t)$ is a gap of type (i), then $\mathcal{A}^\mu(\beta)$ has at least two eigenvalues $\lambda_1 = \omega^2_t$ and $\lambda_2 = \omega^2_1$ that satisfy $\omega_b < \omega_1 < \omega_0 < \omega_2 < \omega_t$ (see Lemma 2.1-3 for the definition of $\omega_0$).

(b) If $(\omega^2_b, \omega^2_t)$ is a gap of type (ii) or (iii), then $\mathcal{A}^\mu(\beta)$ has at least one eigenvalue $\lambda_1 = \omega^2_t$ such that $\omega_b < \omega_1 < \omega_t$.

The sketch of the proof of the previous lemma is the following, a complete proof being available in [10] (Theorem 5.2.1): First, one can verify that $F_\beta(\omega) \geq 0$ in any gap, which, together with (14) proves that $\mathcal{A}^\mu(\beta)$ has no eigenvalue for $\mu > 1$. Then, if $(\omega^2_b, \omega^2_t)$ is a gap of type (i), one can show that

$$
\lim_{\omega \to \omega^2_b} (1 - F_\beta(\omega)) = \lim_{\omega \to \omega^2_t} (1 - F_\beta(\omega)) = 1 \quad \text{and} \quad \lim_{\omega \to \omega_0} (1 - F_\beta(\omega)) = 0.
$$

By continuity of $F_\beta$ inside the gap, (a) directly results from the intermediate value theorem and (14). If $(\omega^2_b, \omega^2_t)$ is a gap of type (ii), the intermediate value theorem also permits us to conclude since

$$
\lim_{\omega \to \omega^2_b} (1 - F_\beta(\omega)) \leq 0 \quad \text{and} \quad \lim_{\omega \to \omega^2_t} (1 - F_\beta(\omega)) = 1.
$$

A similar argument works for a gap of type (iii).

3 Guided modes for the operator $\mathcal{A}^\mu_\varepsilon(\beta)$: an asymptotic result

Finally, thanks to the general result [8] (Theorem 2.13 convergence of the spectrum of $\mathcal{A}^\mu_\varepsilon(\beta)$ toward the spectrum of $\mathcal{A}^\mu(\beta)$), we can prove the following result of existence of eigenvalue for the operator $\mathcal{A}^\mu_\varepsilon(\beta)$:

Theorem 3.1 Let $\mu \in (0,1)$, $(\lambda_b, \lambda_t)$ be a spectral gap of the operator $\mathcal{A}^\mu(\beta)$ and $\lambda_0 \in (\lambda_b, \lambda_t)$ be a (simple) eigenvalue of this operator. Then, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $\mathcal{A}^\mu_\varepsilon(\beta)$ has an eigenvalue $\lambda_\varepsilon$ inside a spectral gap $(\lambda^\varepsilon_b, \lambda^\varepsilon_t)$. Moreover, $\lambda_\varepsilon = \lambda_0 + O(\sqrt{\varepsilon})$.

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