ASYMPTOTIC BEHAVIORS OF GROUND STATES FOR A MODIFIED GROSS-PITAEVSKII EQUATION

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(Communicated by Thomas Bartsch)

Abstract. In this paper, we consider $L^2$ constrained minimization problem for a modified Gross-Pitaevskii equation with higher order interactions in $\mathbb{R}^2$. By using an auxiliary functional and some detailed energy estimates, the blow-up behavior of ground state for the modified Gross-Pitaevskii equation was obtained under different parameter regimes. Our conclusion extends some results of [3, Theorem 3.4].

1. Introduction. In this paper, we study the following modified Gross-Pitaevskii equation with higher order interactions in two dimensions

$$i\partial_t \Psi = -\Delta \Psi + V(x) \Psi + a|\Psi|^2 \Psi - \delta \Delta (|\Psi|^2) \Psi,$$

where $\Psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential. Equation (1) is used to describe the wave functions of Bose-Einstein condensations with higher order interactions, as well as other nonlinear topics. $a$ and $\delta$ are two dimensionless constants for describing the contact interaction and higher order interaction strengths [4, 17, 18].

Standing waves of (1) are solutions of the form $\Psi(t, x) = e^{-i\mu t} u(x)$, where $\mu \in \mathbb{R}$ is a fixed parameter. Taking it into equation (1), we see that $u(x)$ satisfies

$$-\Delta u + V(x)u - \delta \Delta |u|^2 u = \mu u + a|u|^2 u, \quad x \in \mathbb{R}^N.$$  (2)

To prove the existence of solutions of (2), one approach is to treat equation (2) directly, or transform it into a semilinear elliptic equation by introducing a technique of changing variables, and then apply the mountain pass theorem and other variational methods, see [5, 14, 15] and the references therein. Another approach is to take $\mu \in \mathbb{R}$ as an unknown Lagrange multiplier, then solutions of (2) can be solved by studying the following minimization problem [3, 6, 11, 12, 17, 21]

$$e_a(\delta) = \inf_{u \in A} E_\delta^a(u),$$  (3)

2010 Mathematics Subject Classification. 35J20, 35J60.

Key words and phrases. $L^2$-normalized solutions, modified Gross-Pitaevskii equation, asymptotic behavior.

The first author is supported by NSFC grants 11601173, 11871387 and the Fundamental Research Funds for the Central Universities(WUT: 2017 IVA 076). The second author is supported by NSFC grants 11671394, 11771127 and the Fundamental Research Funds for the Central Universities (WUT: 2018IB014).

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where
\[ E_\delta^a(u) = \frac{\delta}{4} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx - \frac{a}{4} \int_{\mathbb{R}^N} |u|^4 \, dx, \]
and
\[ A = \left\{ \int_{\mathbb{R}^N} |u|^2 \, dx = 1, \; u \in \mathcal{X} \cap \mathcal{H} \right\}. \]

The space \( \mathcal{X} \) and \( \mathcal{H} \) are defined by
\[ \mathcal{X} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx < \infty \right\} \quad \text{and} \quad \mathcal{H} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < \infty \right\}. \]

A minimizer of problem (3) is usually called a ground state of (2). When \( \delta = 0 \), problem (3) was studied in [2, 8, 9, 10, 19, 20] provided \( V(x) \) satisfies the following assumptions
\[ V(x) \in L_\text{loc}^\infty(\mathbb{R}^N; \mathbb{R}^+) \quad \text{inf}_{x \in \mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x| \to \infty} V(x) = 0. \quad (4) \]
Particularly, when \( N = 2 \), problem (3) is mass critical and attracted much attention. It was proved in [2, 8] that there exists \( a^* > 0 \), such that problem (3) has a minimizer if and only if \( a < a^* \). Moreover, the concentration and symmetry breaking of minimizers were investigated in [8, 10] when \( a \not\sim a^* \). Here, \( a^* = |Q|_{L^2}^2 \) with \( Q = Q(|x|) \) being the unique positive solution of the equation [13]
\[ -\Delta u + u - u^3 = 0, \quad u \in H^1(\mathbb{R}^2). \quad (5) \]

When \( \delta \neq 0 \), Colin, Jeanjean and Squassina [6] investigated the existence and stabilities of minimizers of (3) with \( V(x) = 0 \). For \( V(x) \) satisfies the condition (4), Bao, Cai and Ruan proved in [3] that (3) can be attained for all \( a \) when \( N = 1, 2, 3 \). We obtained a similar result results in [21] and proved that (3) has minimizers for more general nonlinear term for any \( \delta > 0 \). In [3], the authors also studied the asymptotic behaviors of minimizers as \( \delta \to 0^+ \) when \( V(x) \) is of the form
\[ V(x) = \sum_{i=1}^{N} \nu_i x_i^2, \quad \text{where} \quad \nu_i > 0 \quad \text{and} \quad x = (x_1, \cdots, x_N). \quad (6) \]
By using the homogeneous of \( V(x) \) and applying some rescaling arguments, they obtained the following results.

**Theorem A (Theorem 3.4 of [3]).** Let \( V(x) \) be given by (6) and \( \phi_0^\delta \) be a nonnegative minimizer of (3). When \( N = 2 \) and \( a > a^* \), let \( \tilde{\phi}(x) = \sqrt{\delta} \phi_0^\delta(\sqrt{\delta} x) \). There exists a subsequence \( \delta_n \to 0^+ \) such that \( \tilde{\phi}_{\delta_n}(x) \to \phi_0 \) in \( H^1(\mathbb{R}^N) \), where \( \phi_0(x) \) is a nonnegative minimizer of the problem (7).

This theorem tells us that the limit behavior of minimizers depends on the value of the parameter \( a \). Its proof mainly involves some technical rescaling arguments. Especially, the proofs rely on the homogeneity of \( V(x) \). In this paper, we intend to extend the results of Theorem A to some general potentials which are not homogeneous, such as the form of (4), and give refined energy estimates for \( c_{\delta}(\delta) \). We also want to study the asymptotic behaviors of minimizers as \( \delta \to 0^+ \) for the case of \( N = 2 \) and \( a = a^* \), which is not involved in [3]. We note that, since the potential \( V(x) \) is not homogeneous, some of the arguments of [3] are no longer applicable for our case. To solve our problem, we need introduce some new ideas and
derive detailed energy estimates of \( e_a(\delta) \). We first introduce the following auxiliary functional

\[
F_a(u) = \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx - \frac{a}{4} \int_{\mathbb{R}^2} |u|^4 \, dx,
\]

and the minimization problem

\[
d_a := \inf_{u \in \mathcal{A}} F_a(u) \quad \text{where} \quad \mathcal{A} = \{ u \in X \text{ and } \int_{\mathbb{R}^2} u^2 \, dx = 1 \}.
\]

**Remark 1.** We note that if \( a > a^* \), then problem (7) has at least one minimizer. Indeed, let \( \bar{u}_\tau = \frac{\tau}{|Q|_{L^2}} Q(\tau x) \), \( \tau > 0 \), one can easily check that \( \bar{u}_\tau \in \mathcal{A} \) and it follows from (21) that

\[
\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = \frac{\tau^4}{|Q|_{L^2}^4} \int_{\mathbb{R}^2} |\nabla Q|^2 \, dx, \quad \int_{\mathbb{R}^2} u^4 \, dx = \frac{2\tau^2}{|Q|_{L^2}^2} \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = \tau^2.
\]

Thus,

\[
d_a \leq F_a(u_\tau) = \frac{1}{4} \frac{\tau^4}{|Q|_{L^2}^4} \int_{\mathbb{R}^2} |\nabla Q|^2 \, dx - \frac{(a - a^*)\tau^2}{2a^*} < 0, \quad \text{if } \tau > 0 \text{ is small enough.}
\]

we then conclude that (7) has at least one minimizer by applying the argument of [6, section 4].

Our following theorem addresses some refined blow-up behaviors of minimizers of (3) as \( \delta \to 0^+ \) for the case of \( N = 2 \).

**Theorem 1.1.** Suppose \( N = 2 \) and \( V(x) \in C^\infty_\text{loc}(\mathbb{R}^2) \) satisfies (4). Let \( u_\delta(x) \) be a nonnegative minimizer of \( e_a(\delta) \) and \( z_\delta \) be one maximum point of \( u_\delta \). Then,

\[
\lim_{\delta \to 0^+} z_\delta = x_0 \text{ with } x_0 \in \mathbb{R}^2 \text{ satisfying } V(x_0) = 0.
\]

Moreover,

(I) if \( a = a^* \), then,

\[
w_\delta = \varepsilon_\delta u_\delta(\varepsilon_\delta x + z_\delta) \to \frac{Q(x)}{|Q|_{L^2}} \text{ in } H^1(\mathbb{R}^2),
\]

where

\[
\varepsilon_\delta := \left( \int_{\mathbb{R}^2} |\nabla u_\delta|^2 \, dx \right)^{-\frac{1}{2}} \to 0^+ \text{ as } \delta \to 0^+.
\]

(II) if \( a > a^* \), then,

\[
c_a(\delta) = \frac{1}{\delta} d_a + o(1) \text{ as } \delta \to 0^+,
\]

and

\[
w_\delta = \delta^{\frac{1}{4}} u_\delta(\delta^{\frac{1}{4}} x + z_\delta) \to w_0 \text{ in } X \text{ as } \delta \to 0^+,
\]

where \( w_0 = w_0(|x|) \) is a nonnegative minimizer of problem (7).

**Remark 2.** If \( V(x) \in C^\infty_\text{loc}(\mathbb{R}^2) \) satisfies (4), it is well known (see [2] for instance) that the embedding from \( \mathcal{H} \) into \( L^p(\mathbb{R}^2) \) is compact with \( 2 \leq p < \infty \). Hence, from the proof of Theorem 1.4 of [6] or Theorem 2.1 of [3], one can easily to prove that there is a minimizer for problem (3).
When \( a > a^* \), the arguments of [3] only give that \( e_a(\delta) \leq \frac{1}{\delta} d_a + C \) as \( \delta \to 0^+ \), this is not enough to ensure that the blow-up point of \( u_\delta \) must be one minimum point of \( V(x) \). For the case of \( a = a^* \), we see that the blow-up of minimizers also happens as \( \delta \to 0^+ \), but the limit of minimizers is quite different from the case of \( a > a^* \). It also deserves to point out here that, (8) and (10) only indicate that the minimizers blow up around one minimal point of \( V(x) \), however, it does not give any information about the blow-up rate of \( \varepsilon_\delta \). We think the main reason is that the asymptotic expansion of \( V(x) \) near its minimal points is unknown previously. To calculate the precise blow-up rate of the minimizers, motivated by [9, 10] we now assume that \( V(x) \) has exactly \( l \in \mathbb{N}^+ \) different minimal points, i.e.,

\[
\{ x \in \mathbb{R}^2 : V(x) = 0 \} = \{ x_i, \ i = 1, 2, \ldots, l \}
\]

and there exist \( r_0, p_i, \gamma_i > 0 \) such that

\[
V(x) = \gamma_i |x - x_i|^{p_i} + o(|x - x_i|^{p_i}) \text{ for any } x \in B_{r_0}(x_i).
\]

Let

\[
p := \max_{1 \leq i \leq l} p_i, \quad \bar{Z} := \{ x_i : p_i = p, 1 \leq i \leq l \}, \quad \gamma := \min\{ \gamma_i : x_i \in \bar{Z}, 1 \leq i \leq l \}
\]

and denote the flattest minimal point of \( V(x) \) by

\[
Z := \{ x_i : x_i \in \bar{Z} \text{ and } \gamma_i = \gamma, 1 \leq i \leq l \}.
\]

Then, we have the following theorem which gives the blow-up rate and locates the blow-up point of minimizers.

**Theorem 1.2.** Assume \( V(x) \) satisfies (4), (13) and (14). Let \( a = a^* \), \( x_0 \) be the blow-up point given in Theorem 1.1, and \( \varepsilon_\delta \) be given by (10), then \( x_0 \in Z \). Moreover, we have

\[
\lim_{\delta \to 0^+} \varepsilon_\delta / \left( \frac{2 \lambda_1 \delta}{a^* p \gamma \lambda_2} \right)^{\frac{1}{p+4}} = 1,
\]

and

\[
ee_a(\delta) = \frac{(1 + o(1))(p + 4)}{8a^* \gamma \lambda_2} \left( \frac{2 \delta}{a^* p} \lambda_1 \right)^{\frac{1}{p+4}},
\]

where

\[
\lambda_1 = \int_{\mathbb{R}^2} |\nabla Q|^2 dx \text{ and } \lambda_2 = \int_{\mathbb{R}^2} |x|^p Q^2(x) dx.
\]

From the above theorem we see that the minimizers concentrate at one of the flattest minimal points of \( V(x) \) as \( \delta \to 0^+ \). Moreover, (10) and (17) indicate that the blow-up rate of \( \int_{\mathbb{R}^2} |\nabla u_\delta|^2 dx \) is the order of \( \delta^{-\frac{1}{p+4}} \). (18) also gives the precisely energy estimates for \( e_a(\delta) \).

In this paper, we always denote \( |u|_{L^p} \) the \( L^p \)-norm of a function \( u \). \( C > 0 \) denotes some constant which may be different in different place.

**2. Proofs of Theorems 1.1 and 1.2.** We first recall the following Gagliardo-Nirenberg inequality [1]

\[
\int_{\mathbb{R}^2} |u|^4 dx \leq \frac{2}{a^*} \int_{\mathbb{R}^2} |\nabla u|^2 dx \int_{\mathbb{R}^2} |u|^2 dx, \forall u \in H^2(\mathbb{R}^2),
\]

where the “=” holds when \( u = Q(x) \). Moreover, it follows from (5) and (20) that

\[
\int_{\mathbb{R}^2} |\nabla Q|^2 dx = \int_{\mathbb{R}^2} |Q|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |Q|^4 dx.
\]
From Proposition 4.1 of [7], we also have
\[ Q(x), \, |\nabla Q(x)| = O(|x|^{-\frac{\delta}{2}} e^{-|x|}) \quad \text{as} \quad |x| \to \infty. \] (22)

We next estimate the energy of \( e_a(\delta) \) as \( \delta \to 0^+ \) in the following lemma.

**Lemma 2.1.** If \( a = a^* \), then
\[ \lim_{\delta \to 0^+} e_{a^*}(\delta) = e_{a^*}(0) = 0. \] (23)

**Proof.** We first recall from [8] that \( e_{a^*}(0) = 0 \), it then yields from (3) that
\[ \lim_{\delta \to 0^+} e_{a^*}(\delta) \geq e_{a^*}(0) = 0. \] (24)

On the other hand, let \( 0 \leq \varphi(x) \in C^1(\mathbb{R}^2) \) be a cut-off function such that
\[ \varphi(x) \equiv 1 \text{ if } |x| < 1 \text{ and } \varphi(x) \equiv 0 \text{ if } |x| > 2. \] (25)

Set
\[ u_\tau := \frac{\tau A_{R\tau}}{|Q|_{L^2}} \varphi(x - \bar{x}_0) Q(\tau |x - \bar{x}_0|) \in \mathcal{A}, \text{ where } \bar{x}_0 \in \mathbb{R}^2 \text{ and } R > 0. \] (26)

Using (21) and (22), direct calculations then give that, as \( \tau \to +\infty, \)
\[ 1 \leq A_{R\tau}^2 \leq 1 + C e^{-R\tau}, \] (27)
\[ \int_{\mathbb{R}^2} |\nabla u_\tau|^2 dx = \tau^2 + O(e^{-R\tau}), \int_{\mathbb{R}^2} |u_\tau|^4 dx = \frac{2\tau^2}{a^*} + O(e^{-R\tau}), \] (28)
\[ \int_{\mathbb{R}^2} V(x) u_\tau^2(x) dx = \frac{A_{R\tau}^2}{a^*} \int_{\mathbb{R}^2} V \left( \frac{x}{\tau} + \bar{x}_0 \right) \varphi \left( \frac{x}{\tau} \right) Q^2(x) dx \to V(\bar{x}_0), \] (29)
and
\[ \int_{\mathbb{R}^2} |\nabla u_\tau|^4 dx = \frac{\tau^4}{a^*^2} \int_{\mathbb{R}^2} |\nabla Q|^2 dx + O(e^{-R\tau}). \] (30)

Choosing \( \bar{x}_0 \) such that \( V(\bar{x}_0) = 0 \) and setting \( \tau = \delta^{-\frac{1}{2}} \), we then have
\[ \lim_{\delta \to 0^+} e_{a^*}(\delta) \leq \lim_{\delta \to 0^+} E_{a^*}(u_\tau) = 0. \]

This together with (24) implies (23). \( \Box \)

**Lemma 2.2.** For any \( \delta > 0 \) and \( a > 0 \), let
\[ \bar{E}_a^\delta(u) = \frac{\delta}{4} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{a}{4} \int_{\mathbb{R}^2} |u|^4 dx, \]
and define
\[ \bar{e}_a(\delta) = \inf_{u \in \mathcal{A}} \bar{E}_a^\delta(u). \]

Then, \( \delta \bar{e}_a(\delta) = d_a. \) Moreover, \( u_\delta \) is a minimizer of \( \bar{e}_a(\delta) \) if and only if \( \delta^{\frac{1}{2}} u_\delta(\delta^{\frac{1}{2}} x) \) is a minimizer of \( d_a. \)

**Proof.** For any \( u \in \mathcal{A}, \) set \( w_\delta = \delta^{\frac{1}{2}} u(\delta^{\frac{1}{2}} x) \). Direct calculations show that \( w_\delta \in \mathcal{A} \) and \( F_a(w_\delta) = \delta \bar{E}_a^\delta(u) \). Hence
\[ \bar{e}_a(\delta) = \inf_{u \in \mathcal{A}} \bar{E}_a^\delta(u) = \inf_{w_\delta \in \mathcal{A}} \frac{1}{\delta} F_a(w_\delta) = \frac{1}{\delta} \inf_{u \in \mathcal{A}} F_a(u) = \frac{1}{\delta} d_a. \]

This also indicates that \( u_\delta \) is a minimizer of \( \bar{e}_a(\delta) \) if and only if and \( \delta^{\frac{1}{2}} u_\delta(\delta^{\frac{1}{2}} x) \) is a minimizer of \( d_a. \) \( \Box \)
Lemma 2.3. If $a > a^*$, then
\[
e_a(\delta) = \bar{e}_a(\delta) + o(1) = \frac{1}{\delta} d_a + o(1) \text{ as } \delta \to 0^+.
\] (31)

Moreover, if $u_\delta(x)$ is a minimizer of $e_a(\delta)$, then there holds that
\[
\int_{\mathbb{R}^2} V(x) u_\delta^2 dx \to 0 \text{ as } \delta \to 0^+.
\] (32)

Proof. Let $w_0(x) = w_0(|x|)$ be a nonnegative radial minimizer of $d_a$. Then there exists $\mu_0 \in \mathbb{R}$ such that
\[
-\Delta (w_0^2) w_0 - \Delta w_0 - a w_0^3 = \mu_0 w_0.
\] (33)

Moreover, it follows from (33) and Remark 1 that
\[
\rho_0 = \int_{\mathbb{R}^2} |\nabla w_\delta|^2 dx + \int_{\mathbb{R}^2} |\nabla w_0|^2 dx - a \int_{\mathbb{R}^2} |w_0|^4 dx
\]
\[
= 4F_a(w_0) - \int_{\mathbb{R}^2} |\nabla w_0|^2 dx = 4d_a - \int_{\mathbb{R}^2} |\nabla w_0|^2 dx < 0.
\]

Similar to the proof of Lemma 5.10 in [16] (see also [3, Theorem 2.2]), one can deduce from (33) that
\[
w_0(x) \leq C e^{-\beta |x|} \text{ for some } \beta > 0, \text{ if } |x| > 0 \text{ is large enough.}
\] (34)

Let $\varphi(x)$ be the cut-off function given by (25), and still denote $\tilde{u}_\delta := \delta^{-\frac{1}{2}} u_\delta(\delta^{-\frac{1}{2}} x)$ a nonnegative minimizer of $\bar{e}_a(\delta)$ by Lemma 2.2. For any $x_0 \in \mathbb{R}$, we set
\[
\tilde{u}_\delta(x) = A_\delta \varphi(x - x_0) \delta^{-\frac{1}{2}} w_0(\delta^{-\frac{1}{2}} (x - x_0)) = A_\delta \varphi(x - x_0) \tilde{u}_\delta(x - x_0),
\]
where $A_\delta \geq 1$ such that $\int_{\mathbb{R}^2} \tilde{u}_\delta^2 dx \equiv 1$ and $A_\delta \to 1$ as $\delta \to 0^+$. Using the exponential decay of $w_0$ in (34), we have
\[
0 \leq A_\delta^2 - 1 = \frac{\int_{|\delta^{\frac{1}{2}} x| \geq 1} |1 - \varphi^2(\delta^{\frac{1}{2}} x)| w_\delta^2(x) dx}{\int_{\mathbb{R}^2} \varphi^2(\delta^{\frac{1}{2}} x) w_\delta^2(x) dx} \leq C e^{-2\beta \delta^{-\frac{1}{2}}} \text{ as } \delta \to 0^+,
\]
\[
\int_{\mathbb{R}^2} V(x) \tilde{u}_\delta^2(x) dx = A_\delta^2 \int_{\mathbb{R}^2} V(\delta^{\frac{1}{2}} x + x_0) \varphi^2(\delta^{\frac{1}{2}} x) w_\delta^2 dx \to V(x_0) \text{ as } \delta \to 0^+,
\]
and
\[
\int_{\mathbb{R}^2} |\tilde{u}_\delta|^4 dx = \delta^{-1} A_\delta^4 \int_{\mathbb{R}^2} \varphi^4(\delta^{\frac{1}{2}} x) |w_0|^4 dx = \delta^{-1} \int_{\mathbb{R}^2} |w_0|^4 dx + O(e^{-4\beta \delta^{-\frac{1}{2}}})
\]
\[
= \int_{\mathbb{R}^2} |\tilde{u}_\delta|^4 dx + O(e^{-4\beta \delta^{-\frac{1}{2}}}) \text{ as } \delta \to 0^+.
\]

Similarly, one can also prove that
\[
\int_{\mathbb{R}^2} |\nabla \tilde{u}_\delta|^2 dx = \int_{\mathbb{R}^2} |\nabla \tilde{u}_\delta|^2 dx + O(e^{-4\beta \delta^{-\frac{1}{2}}}) \text{ as } \delta \to 0^+
\]
and
\[
\int_{\mathbb{R}^2} |\nabla \tilde{u}_\delta|^2 dx = \int_{\mathbb{R}^2} |\nabla \tilde{u}_\delta|^2 dx + O(e^{-2\beta \delta^{-\frac{1}{2}}}) \text{ as } \delta \to 0^+.
\]
Therefore, choosing \( x_0 \in \mathbb{R}^2 \) such that \( V(x_0) = 0 \), we then deduce from the above estimates that

\[
0 \leq e_a(\delta) - \bar{e}_\delta = E_a^\delta(\bar{u}_\delta(x)) - \bar{E}_a(\bar{u}_\delta(x)) \\
= E_a^\delta(\bar{u}_\delta(x)) - \bar{E}_a(\bar{u}_\delta(x)) + \frac{1}{2} \int_{\mathbb{R}^2} V(x) \bar{u}_\delta^2(x) dx \\
= \frac{1}{2} V(x_0) + O(e^{-2\beta \delta^{-\frac{1}{2}}}) + o(1) \to 0 \quad \text{as} \quad \delta \to 0^+.
\]

This implies (31) by applying Lemma 2.2. Meanwhile, if \( u_\delta \) is a nonnegative minimizer of \( e_a(\delta) \), then

\[
\int_{\mathbb{R}^2} V(x)u_\delta^2 dx = e_a(\delta) - \bar{E}_a(\bar{u}_\delta) \leq e_a(\delta) - \bar{e}_\delta \to 0 \quad \text{as} \quad \delta \to 0^+.
\]

We therefore obtain (32) and the proof of the lemma is finished.

Based on the above lemmas, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** (I) Let \( u_\delta \) be a nonnegative minimizer of \( e_a(\delta) \), then there exists \( u_\delta \in \mathbb{R} \) such that \( u_\delta \) satisfies

\[
-\triangle u + V(x)u - \delta \triangle |u|^2 u = \mu u + a^*|u|^2 u, \quad x \in \mathbb{R}^2.
\]

From (20) and (23) we have

\[
0 \leq E_a^0(u_\delta) \leq E_a^\delta(u_\delta) = e_a(\delta) \to e_a^*(0) = 0 \quad \text{as} \quad \delta \to 0^+
\]

and

\[
\int_{\mathbb{R}^2} V(x)|u_\delta|^2 dx \to 0 \quad \text{as} \quad \delta \to 0^+.
\]

We next prove that

\[
\lim_{\delta \to 0^+} \int_{\mathbb{R}^2} |\nabla u_\delta|^2 dx = +\infty.
\]

On the contrary, if \( \{|\nabla u_\delta|^2\}_{\delta, x} \) is bounded, it then follows from (37) that \( \{u_\delta\} \) is bounded in \( \mathcal{H} \). Applying Lemma 2.1 of [9], there exists \( u_0 \in \mathcal{H} \) such that

\[
u_\delta \to u_0 \quad \text{in} \quad L^p(\mathbb{R}^2) \quad \text{for any} \quad p \in [2, +\infty) \quad \text{as} \quad \delta \to 0^+.
\]

Thus, \( \int_{\mathbb{R}^2} |u_0|^2 dx = 1 \), and it follows from (36) that

\[
0 \leq E_a^0(u_0) \leq \lim_{\delta \to 0^+} E_a^\delta(u_\delta) = 0 = e_a^*(0),
\]

this implies that \( u_0 \) is a minimizer of \( e_a^*(0) \), which is however contradicts [8, theorem1] since it was proved that \( e_a^*(0) \) cannot be attained therein.

Let \( \varepsilon_\delta > 0 \) be given by (10), we then deduce from (38) that \( \lim_{\delta \to 0^+} \varepsilon_\delta = 0 \), and it follows from (35) that

\[
\mu \varepsilon_\delta^2 = 4\varepsilon_\delta^2 e_a(\delta) - \varepsilon_\delta^2 \int_{\mathbb{R}^2} (|\nabla u_\delta|^2 + V(x)u_\delta^2) dx \to -1 \quad \text{as} \quad \delta \to 0^+.
\]

Moreover, set

\[
\bar{w}_\delta(x) := \varepsilon_\delta u_\delta(\varepsilon_\delta x),
\]

then,

\[
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \bar{w}_\delta|^2 + V(\varepsilon_\delta x)\bar{w}_\delta^2 dx + \frac{\delta \varepsilon_\delta^2}{4} \int_{\mathbb{R}^2} |\nabla \bar{w}_\delta|^2 dx - \frac{a^*}{4} \int_{\mathbb{R}^2} |\bar{w}_\delta|^4 dx = \varepsilon_\delta^2 E_a^\delta(u_\delta).
\]
From (36) we thus have
\[
\lim_{\delta \to 0^+} \left( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \bar{w}_\delta|^2 \, dx - \frac{\alpha^*}{4} \int_{\mathbb{R}^2} |\bar{w}_\delta|^4 \, dx \right) = 0, \tag{40}
\]
\[
\lim_{\delta \to 0^+} \frac{\delta \varepsilon_\delta^2}{4} \int_{\mathbb{R}^2} |\nabla \bar{w}_\delta|^2 \, dx = 0, \tag{41}
\]
and
\[
\int_{\mathbb{R}^2} |\nabla \bar{w}_\delta|^2 \, dx = 1, \lim_{\delta \to 0^+} |\bar{w}_\delta|^4 \, dx = \frac{2}{a^*}. \tag{42}
\]
Applying the concentration compactness principle as the proof of [21, Lemma 2.5], one can deduce from (40) and (42) that there exists \( \bar{z}_\delta \in \mathbb{R}^2 \) and \( y_0 \in \mathbb{R}^2 \) such that
\[
\bar{w}_\delta(x) := \bar{w}_\delta(x + \bar{z}_\delta) \to \frac{Q((x - y_0))}{|Q|_{L^2}} \text{ in } H^1(\mathbb{R}^2) \text{ as } \delta \to 0^+. \tag{43}
\]
Moreover, (37) indicates that
\[
\int_{\mathbb{R}^2} V(\varepsilon \delta x + \bar{z}_\delta) w_\delta^2 \, dx \to 0 \text{ as } \delta \to 0^+.
\]
This together with (43) yields that, up to a subsequence, there exists \( x_0 \in \mathbb{R}^2 \) satisfying \( V(x_0) = 0 \) such that
\[
\lim_{\delta \to 0^+} \bar{z}_\delta = x_0. \tag{44}
\]
From (41) and (43), we also have
\[
\lim_{\delta \to 0^+} \delta \cdot \varepsilon_\delta^2 = 0. \tag{45}
\]
(35) and (43) imply that \( \bar{w}_\delta \) satisfies
\[
- \Delta \bar{w}_\delta + \varepsilon_\delta^2 V(\varepsilon \delta x + \bar{z}_\delta) \bar{w}_\delta - \delta \varepsilon_\delta^2 \Delta |\bar{w}_\delta|^2 \bar{w}_\delta = \mu \varepsilon_\delta^2 \bar{w}_\delta + a^* |\bar{w}_\delta|^2 \bar{w}_\delta, \quad x \in \mathbb{R}^2. \tag{46}
\]
Employing a similar argument to the proof of Lemma 5.10 in [16] (see also [3, Theorem 2.2]), one can deduce from (39), (43) and (45) that
\[
\bar{w}_\delta(x) \leq C e^{-\beta|x|} \text{ for some } \beta > 0, \text{ if } |x| > 0 \text{ is large enough.} \tag{47}
\]
Let \( z_\delta \) be any global maximum point of \( u_\delta(x) \), and note that \( \bar{w}_\delta(x) \) attains its maximum at \( x = \frac{z_\delta - \bar{z}_\delta}{\varepsilon_\delta} \). As a consequence of (43) and (47), we then have
\[
\limsup_{\delta \to 0^+} \frac{|z_\delta - \bar{z}_\delta|}{\varepsilon_\delta} < \infty. \tag{48}
\]
Set
\[
w_\delta(x) := \varepsilon_\delta u_\delta(\varepsilon_\delta x + z_\delta) = \bar{w}(x + \frac{z_\delta - \bar{z}_\delta}{\varepsilon_\delta}).
\]
Since \( x = 0 \) is the unique maximum point of \( Q(x) \), and each \( w_\delta(x) \) obtains its maximum point at \( x = 0 \), it then yields from (43) that
\[
\lim_{\delta \to 0^+} w_\delta(x) = \frac{Q(x)}{|Q|_{L^2}} \text{ in } H^1(\mathbb{R}^2) \text{ as } \delta \to 0^+.
\]
This gives (9). (8) follows from (44) and (48).

(II) Let \( \tilde{w}_\delta(x) = \delta^{-\frac{1}{2}} u_\delta(\delta^{\frac{1}{2}} x) \), it then follows from Lemmas 2.2 and 2.3 that
\[
\delta^{-1} d_a = \tilde{e}_a(\delta) \leq e_a(\delta) = \bar{E}_a(\delta) + \frac{1}{2} \int_{\mathbb{R}^2} V(x) u_\delta^2 \, dx
\]
\[
= \delta^{-1} F_a(\bar{w}_\delta) + o(1) \leq \tilde{e}_a(\delta) + o(1).
\]
Thus,
\[ d_a \leq F_a(\tilde{\omega}_\delta) + o(1)\delta \leq d_a + o(1). \]
This indicates that \{\tilde{\omega}_\delta\} is a minimizing sequence of \(d_a\). Therefore, it follows from [6, section 4] that there exists \(x_\delta \in \mathbb{R}^2\) such that, up to a subsequence,
\[ w_\delta(x) := \tilde{\omega}_\delta(x + x_\delta) = \delta^{\frac{1}{2}} u_\delta(\delta^{\frac{1}{2}} x + \delta^{\frac{1}{2}} x_\delta) \rightarrow w_0 \text{ in } \mathcal{X}, \quad (49) \]
where \(w_0\) is a minimizer of \(d_a\). Moreover, it follows from (32) that
\[ \int_{\mathbb{R}^2} V(x)u_\delta^2 dx = \int_{\mathbb{R}^2} V(\delta^{\frac{1}{2}} x + \delta^{\frac{1}{2}} x_\delta)w_\delta^2 dx \rightarrow 0 \text{ as } \delta \rightarrow 0^+. \]
This together with (49) indicates that \(z_\delta := \delta^{\frac{1}{2}} x_\delta \rightarrow x_0\) as \(\delta \rightarrow 0^+\), where \(x_0 \in \mathbb{R}^2\) satisfying \(V(x_0) = 0\). Repeating the arguments of Part (I), one can further prove that (8) and (12).

Based on Theorem 1.1 and the assumptions (13) and (14), we now give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first prove that
\[ 0 < e_a(\delta) \leq \frac{(1 + o(1))p + 4}{8a^*} \left( \gamma \lambda_2 \right)^{\frac{1}{p+1}} \left( \frac{2\delta}{a^*p} \lambda_1 \right)^{\frac{1}{p+1}}, \quad (50) \]
where \(\lambda_1\) and \(\lambda_2\) are given by (19). Let \(u_\tau\) be the test function given by (27). Taking \(\bar{x}_0 = x_i\) with some \(x_i \in Z\) and \(R = \frac{c_0}{\tau}\), it then follows from (14) and (29) that
\[ \int_{\mathbb{R}^2} V(x)u_\tau^2 dx = \frac{A_{Rz}}{a^*} \int_{\mathbb{R}^2} V(\bar{x}_\tau + \bar{x}_0)\varphi(\bar{x}_\tau)Q^2(x) dx \leq \frac{\gamma(1 + o(1))}{a^* p^*} \int_{\mathbb{R}^2} |x|^p Q^2(x) dx. \]
Combining with (28) and (30), we thus have
\[ E_a^*(u_\tau) \leq \frac{\delta \tau^4}{4a^*} \int_{\mathbb{R}^2} \frac{1 + o(1)\gamma}{2a^* p^*} \int_{\mathbb{R}^2} |x|^p Q^2(x) dx + O(e^{R\tau}). \]
Taking \(\tau = \left( \frac{a^* p^* \gamma \lambda_2}{2 \lambda_1 \delta} \right)^{\frac{1}{p+1}}\), we then deduce from above that (50) holds.

Let \(u_\delta\) be the nonnegative minimizer of \(e_a(\delta)\) and \(w_\delta(x)\) be given by (9). Since \(V(x_0) = 0\), without loss of generality, we assume \(x_0 = x_{i_0}\) for some \(1 \leq i_0 \leq l\). From (9) we see that
\[ \lim_{\delta \rightarrow 0^+} \frac{\lambda_1 \delta}{4a^*} \int_{\mathbb{R}^2} |\nabla w_\delta|^2 dx = \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}^2} |\nabla w_\delta|^2 dx \geq \frac{1}{a^*} \int_{\mathbb{R}^2} |\nabla Q|^2 dx. \]
We claim that
\[ p_{i_0} = p \quad \text{and} \quad \frac{|z_\delta - x_{i_0}|}{\varepsilon_\delta} \text{ is bounded}. \quad (52) \]
On the contrary, if \(p_{i_0} < p\) or \(\lim_{\delta \rightarrow 0^+} \frac{|z_\delta - x_{i_0}|}{\varepsilon_\delta} = +\infty\), it then follows from (9) and (14) that for any \(M > 0\) large enough,
\[ \lim_{\delta \rightarrow 0^+} \frac{1}{p_{i_0}^*} \int_{\mathbb{R}^2} V(x)u_\delta^2 dx = \lim_{\delta \rightarrow 0^+} \frac{1}{p_{i_0}^*} \int_{\mathbb{R}^2} V(z_\delta x + z_\delta)w_\delta^2 dx \]
\[ = \lim_{\delta \rightarrow 0^+} \frac{1}{p_{i_0}^*} \int_{\mathbb{R}^2} \frac{V(z_\delta x + z_\delta)}{|z_\delta x + z_\delta - x_{i_0}|^{p_{i_0}}}|x + \frac{z_\delta - x_{i_0}}{\varepsilon_\delta}|^{p_{i_0}}w_\delta^2 dx \geq M > 0. \]
As a consequence, we then deduce from (51) that
\[ e_{a^*}(\delta) \geq \frac{\lambda_1 \delta}{4a^*} + \frac{M \varepsilon_\delta^p}{2} \geq CM \tau^\frac{1}{p} \delta \tau^\frac{1}{p^*}, \]
this however contradicts (50). Claim (52) is proved. □
From (52), we may assume that, up to a subsequence, there exists \( z_0 \in \mathbb{R}^2 \) such that \( \lim_{\delta \to 0^+} \frac{z_i - x_{i0}}{\varepsilon \delta} = z_0 \). Applying (9) and (14) again, we have

\[
\lim_{\delta \to 0^+} \frac{1}{\varepsilon \delta} \int_{\mathbb{R}^2} V(x) u_{\delta}^2 dx = \lim_{\delta \to 0^+} \frac{1}{\varepsilon \delta} \int_{\mathbb{R}^2} V(\varepsilon \delta x + z_0) w_{\delta}^2 dx
\]

\[
= \lim_{\delta \to 0^+} \int_{\mathbb{R}^2} \frac{V(\varepsilon \delta x + z_0)}{|x + z_0|^{p}} |x + z_0 - x_{i0}|^p \frac{1}{\varepsilon \delta^2} w_{\delta}^2 dx
\]

\[
\geq \frac{\gamma}{a^*} \int_{\mathbb{R}^2} |x + z_0|^p Q^2(x - y_0) dx = \frac{\gamma}{a^*} \int_{\mathbb{R}^2} |x + z_0 + y_0|^p Q^2(x) dx
\]

\[
\geq \frac{\gamma}{a^*} \int_{\mathbb{R}^2} |x|^p Q^2(x) dx,
\]

where the “=” in the last inequality holds if and only if \( \gamma_{i0} = \gamma \) (i.e., \( x_{i0} \in Z \)) and \( z_0 = -y_0 \). It then follows from (51) and (53) that

\[
e_{\alpha^*}(\delta) \geq \frac{(1 + o(1))\lambda_1 \delta}{4a^* \varepsilon \delta^4} + \frac{(1 + o(1))\lambda_2 \varepsilon^p}{2a^*} \geq \frac{(1 + o(1))(p + 4)}{8a^*} \left( \frac{25}{a^* p} \right)^{\frac{2}{p+2}} \lambda_1^{\frac{p}{p+2}},
\]

where the “=” in the last inequality holds if and only if (17) holds.

From (50) and (54) we see that (18) holds. Therefore, all (53) and (54) are indeed equalities. This further implies that \( x_{i0} \in Z \) and (17) holds.

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Received September 2018; revised March 2019.

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