Chern-Simons term and Topological Charge on the Lattice

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Abstract

In a somewhat overlooked work by Seiberg, a definition of the topological charge for SU(N) lattice fields was given. Here, it is shown that Seibergs and Lüscher charge definition are related up to the section of the bundle. With the continued interest in baryon number violating processes, Seibergs paper is useful since it allows for a Chern-Simons number also.
1 Introduction

Over the past few years there has been quite an activity in the area of baryon number violating processes. Early on 't Hooft found that both baryon number and lepton number are not conserved in the electroweak theory [1]. While the $B - L$ symmetry remains unbroken due to the anomaly cancellation, $B + L$ is no longer conserved. This so-called baryon number violation is caused by the nontrivial topological winding of the SU(2) gauge fields. The anomaly of the fermionic current relates the winding of the gauge fields and changes the baryon number by an amount

$$B(t_2) - B(t_1) = \frac{N_f}{16\pi^2} \int_{t_1}^{t_2} \int d^3x tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]$$ (1.1)

where $N_f$ is the number of families of quarks and leptons. In the axial gauge $A_0 = 0$ one can relate the change in the baryon number to the change in the Chern-Simons number

$$B(t_2) - B(t_1) = N_f[N_{CS}(t_2) - N_{CS}(t_1)]$$ (1.2)

where the Chern-Simons number $N_{CS}$ is

$$N_{CS} = -\frac{1}{8\pi^2} \int d^3x \epsilon_{ijk} tr[A_i(\partial_j A_k + \frac{2}{3} A_j A_k)].$$ (1.3)

At zero temperature such processes are exponentially suppressed as $\exp(-2\pi/\alpha_W)$, $\alpha \approx 1/30$. This is because any gauge field configuration which changes the winding number has an action at least that of the barrier height $2\pi/\alpha_W$. At high temperatures thermal fluctuations allow the system to tunnel classically, since the only suppression factor is the Boltzmann factor $\exp(-\beta E/T)$ where $E$ is the barrier height and $T$ the temperature. As a consequence any baryon asymmetry generated at the GUT scale will get washed out as the universe approaches the electroweak phase transition from above [2].

Several lattice studies of baryon number violating processes exist in the four dimensional SU(2) Higgs model [3]. The configurations are prepared at high temperature and the system is allowed to change via the classical Hamiltonian equation of motion. Since the axial gauge is used, the Gauss constraint must be implemented in addition. It is then possible to monitor $\Delta N_{CS}$ during the time evolution as a function of the temperature. In practice a naive lattice definition of $F\tilde{F}$ is used to measure $Q$. Only if the fields are very smooth is this method meaningful. Notice,
measuring $Q$ does not say anything about the value of $N_{CS}(t)$. All these calculations are done in the real time formalism, and it is worth while to study the behavior of $N_{CS}$ in Euclidean time. Using a naive lattice transcription of this makes no sense, since $N_{CS}$ must change by an integer under large gauge transformations.

There are now several (all geometric) definitions for $N_{CS}$ in the Euclidean version ref. [4], [5]. For an actual calculation, the work of Seiberg seems to be the most appropriate [6]. In the work of Göckeler et al a continuum field is constructed directly from the lattice field. It is the purpose of this note to explain Seiberg’s lattice Chern-Simons term and topological charge. In particular I will show that Seiberg’s charge is related to Lüscher’s charge up to a section of the Lüscher bundle.

2 Topological charge and the Chern-Simons term in the continuum

I will first define the topological charge and I start with the SU(2) gauge field $A_{\mu}$ and the gauge field tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

(2.1)

Under a a local gauge transformation $g$ the gauge field changes as:

$$\delta A_\mu = g^{-1}[A_\mu + \partial_\mu]g(x),$$

(2.2)

while the gauge field tensor transforms gauge covariantly

$$F_{\mu\nu} \rightarrow g^{-1}F_{\mu\nu}g.$$  

(2.3)

The topological charge $Q$ is gauge invariant and an integer,

$$Q = -\frac{1}{32\pi} \int_M d^4x \epsilon_{\mu\nu\rho\sigma} tr[F_{\mu\nu}F_{\rho\sigma}] \in Z.$$  

(2.4)

The manifold is denoted $M$ and I shall assume that its boundary $\partial M$ is a three sphere $S^3$. The topological charge density $q$ can be written as a perfect derivative

$$q = -\frac{1}{32\pi} \epsilon_{\mu\nu\rho\sigma} tr[F_{\mu\nu}F_{\rho\sigma}] = \partial_\mu K_\mu.$$  

(2.5)
where the Chern-Simons density $K_\mu$ is

$$K_\mu = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} tr [A_\nu (\partial_\rho A_\sigma + \frac{2}{3} A_\rho A_\sigma)].$$

(2.6)

It is gauge variant and changes under the gauge transformation $g$ by an amount

$$\delta K_\mu = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} tr [g^{-1} \partial_\nu g g^{-1} \partial_\rho g g^{-1} \partial_\sigma g]$$

$$- \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \partial_\sigma tr [\partial_\rho g g^{-1} A_\sigma].$$

(2.7)

I define the Chern-Simons number $N_{CS}$ in the axial gauge as follows:

$$N_{CS} = \int_{\partial M = S^3} d^3 x K_0 \not\in \mathbb{Z}.$$  

(2.8)

While $N_{CS}$ is only an integer for pure gauge configurations, the gauge variation is always an integer (the boundary term vanishes)

$$\delta N_{CS} = -\frac{1}{24\pi^2} \int_{\partial M = S^3} d^3 x \epsilon_{\mu\nu\rho\sigma} tr [g^{-1} \partial_\nu g g^{-1} \partial_\rho g g^{-1} \partial_\sigma g] \in \mathbb{Z}.$$  

(2.9)

This becomes clear from homotopy theory using the mapping $g : S^3 \to SU(2) = S^3$. Such mappings are characterized with the homotopy class $\Pi_3(S^3) \in \mathbb{Z}$.

### 3 Topological charge and the Chern-Simons term on the lattice

I will now consider the lattice version of the topological charge $Q$ and the Chern-Simons number $N_{CS}$. I shall first follow Lüscher’s geometric method to define $Q$. Problems with dislocations will be ignored here. Let the manifold be a four torus $M = T^4$ and assume it is covered with cells (hypercubes) $c(n)$. Consider the gauge potential $A^\mu_n$ defined on $c(n)$ and likewise let $A^{n-\hat{\mu}}_\nu$ be defined on $c(n - \hat{\mu})$. At the faces (cubes) $f(n, \mu) = c(n - \hat{\mu}) \cap c(n)$, one relates the two potentials by a transition function $v_{n,\mu}$

$$A^{n-\hat{\mu}}_\nu = v^{-1}_{n,\hat{\mu}} [A^n_\nu + \partial_\nu] v_{n,\mu}. $$

(3.1)

Lüscher now fixes to a local complete axial gauge in each $c(n)$. For any corner $x$ of $c(n)$ one defines a parallel transporter $w_n(x)$ from $n$ to $x$. This leads to $v_{n,\mu}(x) = w_{n-\hat{\mu}}(x) w_n^{-1}(x)$. It is then possible to interpolate it to the whole cube, thus defining
a bundle. One finds 

\[ v_{n,\mu}(x) = s_{n,\mu}^{-\hat{\mu}}(x)^{-1} v_{n,\mu}(n) \ s_{n,\mu}^\mu(x). \]

Here, the function \( s \) is defined on \( f(n, \mu) \). Let \( p \) be the restriction of \( s \) to \( \partial f(n, \mu) \). The actual expressions are given in ref. [4]. The topological charge is:

\[ Q^L = \sum_n q^L(n) = \frac{1}{2\pi} \sum_{n,\mu} (-1)^\mu (k_{n,\mu} - k_{n+\mu,\mu}), \quad (3.2) \]

where

\[ (-1)^\mu k_{n,\mu} = \frac{1}{12\pi} \epsilon_{\mu\nu\rho\sigma} \int_f d^3x \text{tr} [s \partial_\nu s^{-1} s \partial_\rho s^{-1} s \partial_\sigma s^{-1}] \]

\[ + \quad \frac{1}{4\pi} \epsilon_{\mu\nu\rho\sigma} \int_{\partial f} d^2x \text{tr} [p^{-1} \partial_\rho p s^{-1} \partial_\sigma s]. \quad (3.3) \]

At this point one should note that \( q^L(n) \) is gauge invariant and unrestricted. In Seibergs version no local gauge fixing is performed, but otherwise the same interpolation is performed. Replace \((s, p, k_{n,\mu}) \rightarrow (S, P, K_{n,\mu})\). The difference is that \( S \) and \( P \) only depend on the original gauge fields in the cube. Then

\[ N_{CS} = \frac{1}{2\pi} \sum_{n_s} K_{n_s,\mu} \quad (3.4) \]

is nothing but a Chern-Simons term (the summation is over the spatial lattice only). Like in the continuum it is only an integer for pure gauge field configurations, but under gauge transformations it changes by an integer. The corresponding topological charge is defined as

\[ Q^S = \sum_n \tilde{q}^S(n), \quad -1/2 \leq \tilde{q}^S(n) < 1/2. \quad (3.5) \]

By restricting the charge to this interval one has a gauge invariant charge definition. After some algebra the Chern-Simons term is found to have the correct naive continuum limit, that is after writing \( U_{n,\mu} = \exp(a A_{n,\mu}) \) and letting \( a \rightarrow 0 \):

\[ (-1)^\mu K_{n,\mu} = \frac{a^3}{4\pi} \epsilon_{\mu\nu\rho\sigma} [A_{n,\nu}(\partial_\rho A_{n,\sigma} + \frac{2}{3} A_{n,\rho} A_{n,\sigma})]. \quad (3.6) \]

As an interesting corollary I find that the two charge definitions are related. I first introduce the section of the Lüscher bundle, \( w(x), x \in \partial c(n) \), which relates \( s \) and \( S \): \( s(x) = w(0) S(x) w^{-1}(x) \) [7]. The following identity is useful

\[ \epsilon_{\mu\nu\rho\sigma} \text{tr} [(sw) \partial_\nu (sw)^{-1} (sw) \partial_\rho (sw)^{-1} (sw) \partial_\sigma (sw)^{-1}] \]

\[ = \quad \epsilon_{\mu\nu\rho\sigma} \text{tr} [s \partial_\nu s^{-1} s \partial_\rho s^{-1} s \partial_\sigma s^{-1}] \]

\[ - \quad \epsilon_{\mu\nu\rho\sigma} \text{tr} [w^{-1} \partial_\nu w w^{-1} \partial_\rho w w^{-1} \partial_\sigma w] \]

\[ + \quad 3 \epsilon_{\mu\nu\rho\sigma} \text{tr} [w \partial_\rho w^{-1} s^{-1} \partial_\sigma s]. \quad (3.7) \]
This gives

\[ (-1)^\mu K_{n,\mu} = (-1)^\mu k_{n,\mu} - \frac{1}{12\pi} \varepsilon_{\mu\nu\rho\sigma} \int_f d^3 x \text{tr} [w^{-1} \partial_\nu w w^{-1} \partial_\rho w w^{-1} \partial_\sigma w] \]

\[ + \frac{1}{4\pi} \varepsilon_{\mu\nu\rho\sigma} \int_{\partial f} d^2 x \text{tr} [P^{-1} \partial_\rho P w^{-1} \partial_\sigma w]. \]  

(3.8)

Using the identity

\[ \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu)} d^2 x \text{tr} [P^{-1} \partial_\rho P w^{-1} \partial_\sigma w] = 0, \]  

(3.9)

and others I arrive at

\[ q^S(n) = q^L(n) - q^w(n) \]

\[ = q^L(n) - \frac{1}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int_{\partial \Omega(n)} d^3 x \text{tr} [w^{-1} \partial_\nu w w^{-1} \partial_\rho w w^{-1} \partial_\sigma w], \]  

(3.10)

where \( q^w(n) \) is the topological charge (integer) of the section. Since \( q^w(n) \) is not gauge invariant, the same is true for \( q^S(n) \). It can change by an integer under a gauge transformation. This is why one must use \( q^S(n) \) instead of \( q^S(n) \). Notice also that \( Q_S = \sum_n q^S(n) = 0 \) since \( Q^L = \sum_n q^L(n) = \sum_n q^w(n) \). Therefore \( q^S(n) = q^L(n) \) up to integers. For smooth fields like instantons they always agree, while for realistic configurations this is true for almost every hypercube. It can happen in a few hypercubes that \( |q^L(n)| > 1/2 \) and this often make the charges different. Since Seiberg interpolates the original gauge fields, he has a much rougher function to integrate. Therefore it is necessary to perform a global Landau or axial gauge fixing, to make the integrals converge. For the topological charge this is fine since it is gauge invariant. The same is true for the restricted Chern-Simons term \( \tilde{N}_{CS} \).

As a check I have taken an instanton configuration. For the evaluation of the integrals see ref. [6]. The size of the lattice is \( 8^3 \times 12 \) and the core size is \( \rho = 2 \). For both charge definitions I find \( Q = -1 \). In Fig. 1 I have plotted the Chern-Simons number as a function of the time slice. The complete axial gauge was chosen to resemble a tunneling event. Notice that \( N_{CS} = 0 \) in the first time slice and \( N_{CS} = 1 \) in the next to last time slice. Of course, at the last time slice the Chern-Simons number must return to its initial value due to periodicity.

**Acknowledgement**

I appreciate valuable discussions with F. Karsch and B. Plache.
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Figure captions

Figure 1. Profile of $N_{CS}$ through an instanton configuration. The lattice is $8^3 \times 12$ and the core size $\rho = 2$. 