Resolvent Estimates Related with a Class of Dispersive Equations

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Abstract We present a simple proof of the resolvent estimates of elliptic Fourier multipliers on the Euclidean space, and apply them to the analysis of time-global and spatially-local smoothing estimates of a class of dispersive equations. For this purpose we study in detail the properties of the restriction of Fourier transform on the unit cotangent sphere associated with the symbols of multipliers.

Keywords Resolvent · Dispersive equation · Smoothing effect · Limiting absorption principle

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1 Introduction

This article is concerned with resolvent estimates of elliptic operators on the Euclidean space with constant coefficients. These estimates are equivalent to smoothing properties of solutions to corresponding dispersive evolution equations.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, set $x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n$ and $|\xi| = \sqrt{\xi \cdot \xi}$. Let $a(\xi) \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be a positively homogeneous function of degree one. Suppose that $a(\xi) > 0$ for $\xi \neq 0$. It follows that

$$a'(\xi) = \begin{pmatrix} \frac{\partial a}{\partial \xi_1}(\xi), & \cdots, & \frac{\partial a}{\partial \xi_n}(\xi) \end{pmatrix} \neq 0 \quad \text{for} \quad \xi \neq 0$$
since $a(\xi) = a'(\xi) \cdot \xi$. Set $p(\xi) = a(\xi)^m$ for some fixed number $m > 1$.

Consider the initial value problem of the form

$$D_t u - p(D_x)u = f(t, x) \quad \text{in} \quad \mathbb{R}^{1+n},$$

$$u(0, x) = \phi(x) \quad \text{in} \quad \mathbb{R}^n,$$

where $u(t, x)$ is an unknown function of $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R}^{1+n}$, $f(t, x)$ and $\phi(x)$ are given functions, $i = \sqrt{-1}$, $D_t = -i \partial_t$, $D_x = -i \partial_x$, $\partial = (\partial_1, \ldots, \partial_n)$, and the operator $p(D_x)$ is defined by

$$p(D_x)v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \xi} p(\xi)v(y) dy \, d\xi$$

for an appropriate function $v(x)$. Since $p(\xi)$ is real-valued, the initial value problem (1.1)–(1.2) is $L^2$-well-posed, that is, for any $\phi \in L^2(\mathbb{R}^n)$ and for any $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$, (1.1)–(1.2) possesses a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$. Here $L^q$ and $L^q_{\text{loc}}$ denote a usual Lebesgue space and its local space, respectively, for $q \in [1, \infty]$, and $C(\mathbb{R}; L^2(\mathbb{R}^n))$ is the set of all $L^2(\mathbb{R}^n)$-valued continuous functions on $\mathbb{R}$. Moreover, the unique solution $u$ is explicitly given by

$$u(t, x) = e^{itp(D_x)} \phi(x) + iGf(t, x),$$

$$e^{itp(D_x)} \phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \xi} e^{itp(\xi)}v(y) dy \, d\xi,$n

$$Gf(t, x) = \int_0^t e^{i(t-s)p(D_x)} f(s, x) ds.$$n

A typical example of (1.1) is the Schrödinger evolution equation of a free particle, which is the case $p(\xi) = |\xi|^2$. It is well-known that the solution to the free Schrödinger evolution equation on $\mathbb{R}^n$ gains extra smoothness in comparison with the initial data and the forcing term. This mathematical phenomenon is called local smoothing effect or local smoothing property. In the last two decades, smoothing properties of solutions to more general dispersive partial differential equations and their applications have been vigorously investigated. See, e.g., [1, 2, 4–6, 8, 9, 12, 13, 16, 18, 19] and references therein.

In [2] Doi deeply studied the relationship between the behavior of the geodesic flow and the smoothing effect of the Schrödinger evolution equation on complete Riemannian manifolds. Roughly speaking, he proved that the smoothing effect occurs if and only if all the geodesics go to “infinity”. In other words, if there exists a trapped geodesic, then the smoothing effect breaks down. For more general dispersive equations, the smoothing effect depends on the behavior of the Hamilton flow generated by the principal symbol of the equations. Consider dispersive equations with constant coefficients of the form

$$D_t u - q(D_x)u = f(t, x) \quad \text{in} \quad \mathbb{R}^{1+n},$$

where $q(\xi)$ is a real polynomial of order $m > 1$. Let $q_m(\xi)$ be the principal symbol of $q(D_x)$. $q_m(\xi)$ generates the Hamilton flow $\{(x + tq_m'(\xi), \xi)\}_{t \in \mathbb{R}}$ for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. 

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