Reflected stochastic differential equations driven by $G$-Brownian motion in non-convex domains

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Abstract

In this paper, we first review the penalization method for solving deterministic Skorokhod problems in non-convex domains and establish estimates for problems with $\alpha$-Hölder continuous functions. With the help of these results obtained previously for deterministic problems, we pathwisely define the reflected $G$-Brownian motion and prove its existence and uniqueness in a Banach space. Finally, multi-dimensional reflected stochastic differential equations driven by $G$-Brownian motion are investigated via a fixed-point argument.

Keywords. Hölder continuity, Skorokhod problem, $G$-Brownian motion, Stochastic differential equations, non-convex reflecting boundaries

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1 Introduction

This paper considers multidimensional reflected stochastic differential equations in a sublinear expectation space. These equations are driven by a new type of Brownian motion associated

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with a sublinear expectation, which are introduced by Peng \cite{18, 19, 20} during the past decade. In the seminal works of Peng, he has established a framework of nonlinear Itô’s calculus and related stochastic analysis which does not rely on a single probability measure. This theory provides basic tools to discuss problems in finance with Knightian uncertainty and in robust statistics. Moreover, the nonlinear Feynman-Kac formula obtained in \cite{20} provides an probabilistic representation of fully nonlinear parabolic PDEs via forward-backward systems.

In \cite{20}, the so-called $G$-Brownian motion is defined as a continuous process with stationary independent increments. Under the associated $G$-expectation, these increments are subject to $G$-normal distribution with volatility uncertainty between two bounds. According to Denis et al. \cite{3}, the $G$-expectation can be regarded as an upper expectation based on a collection of non-dominated martingale measures $\mathcal{P}_G$. Furthermore, a Choquet capacity associated with such a collection can be defined. This leads to a notion of equivalence between two random variables in the sublinear expectation space — “quasi-sure” (q.s.). Instead of “almost-surely” in the classical probability theory, we say a property holds quasi-surely if it holds outside a null set for the referred capacity. In the present paper, we shall examine the following equation in the quasi-sure sense,

\[
\begin{aligned}
X_t &= x_0 + \int_0^t f(s,X_s) \, ds + \int_0^t h(s,X_s) \, dB_s + \int_0^t g(s,X_s) \, dB_s + K_t, \quad 0 \leq t \leq T; \\
K_t &= \int_0^t n_s \, |K|_s; \quad |K|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} \, |K|_s, \quad \text{q.s.}
\end{aligned}
\]

(1.1)

where $B$ is a $d$-dimensional $G$-Brownian motion; $\langle B, B \rangle$ is the covariance matrix of $B$; $X$ is a process reflecting on the boundaries of domain $\overline{D}$ and $K$ is a bounded variations process with variation $|K|$ increasing only when $X \in \partial D$.

In the classical framework, reflected stochastic differential equations driven by Brownian motion have been extensively studied by many authors. Among them, Skorokhod \cite{23, 24} is the first who introduced diffusion processes with reflecting boundaries in the 1960s. Later on, reflecting diffusions in a half-space have been investigated by Watanabe \cite{28}, El Karoui \cite{29}, Yamada \cite{30}, El Karoui and Chleyat-Maurel \cite{3}, El Karoui et al. \cite{2}, etc. The study of multi-dimensional stochastic differential equations on a general domain dates back to Stroock and Varadhan \cite{26}, in which the existence and uniqueness of weak solutions have been proved when the domain is smooth. Afterwards, solutions of such equations has been built on a convex domain by a direct method in Tanaka \cite{27}, whereas Menaldi \cite{19} and Lions et al. \cite{13} have adopted a penalization method to construct them. Concerning the reflecting problem with a non-convex but “admissible” domain, Lions and Szmitman \cite{14} have first solved the deterministic Skorokhod problem and have applied this result to construct pathwisely an iteration sequence in order to approximate the corresponding reflecting diffusion. The results of \cite{14} has been later improved in Saisho \cite{21} and in Saisho and Tanaka \cite{22} by removing the admissibility condition on the domain.

In the context of sublinear expectation, we shall discuss the above mentioned equation (1.1), in which a newly defined Itô stochastic integral with respect to $G$-Brownian motion $\int g dB$ appears in the dynamic. As its counterpart in the classical theory, adapted to Peng’s method, this integral is first defined for simple processes and could subsequently be extended.
to $M^p_G$ due to the $G$-BDG type estimate, where $M^p_G$ is a normed space of processes with slightly additional regularity (see also [9]). Thanks to this extension of the Itô type integral, the notion stochastic differential equations driven by $G$-Brownian motion is brought to this nonlinear stochastic analysis framework. Under the Lipschitz conditions, forward equations are studied by Peng [20] and Gao [6]; backward equation are examined in [7]. Moreover, scalar $G$-diffusion processes with reflection has been considered by Lin in [12] and the multidimensional problem are solved in Lin [11] by the penalization method similar to [16].

The main objective of this paper is to generalize the results of [12, 11] to the multidimensional case when the reflecting boundary is not necessarily convex. We adopt the same assumptions as [21, 22] on the domain and, as the first step, we restrict ourself to the deterministic Skorokhod problems concerning $\alpha$-Hölder continuous paths. Precisely, we revise the estimate for the penalization sequence obtained in [22] by introducing the $\alpha$-Hölder coefficient. Since the $G$-Brownian motion is supported on $C^{0,\alpha}$ ($\alpha < 1/2$), this estimates are accordingly applied to prove that the multidimensional $G$-Brownian motion in this non-convex domain can be approximated in the Banach space $M^p_G$ by a sequence of solutions of Lipschitz equations studied in [6]. Similar arguments also apply for reflected $G$-diffusion with bounded generators. Finally, we pathwisely construct an iteration sequence by using the deterministic result and conduct a fixed-point argument as [14] to prove the wellposedness of (1.1) under the bounded and Lipschitz assumption on coefficients. We remark that the equation (1.1) under consideration can also be examined in a weaker “quasi-sure” sense, which means it holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}_G$. This notion is adopted by Soner et al. for establishing a similar nonlinear stochastic analysis framework – second order backward stochastic differential equations (cf. [25]). We could proceed almost the same procedures of the present paper under each $\mathbb{P} \in \mathcal{P}_G$ and obtain a solution $(X^\mathbb{P}, K^{\mathbb{P}})$ which could be later aggregated to $(X, K)$ by Nutz [17].

The paper is organized as follows. In Section 2, we introduce preliminaries in the framework of $G$-expectation which are necessary for the remainder of this paper. In addition, we revisit the deterministic Skorohod problems in a non-convex domain. In Section 3, we present our main results and Section 4 is devoted to prove the main results.

2 Preliminaires

In this section, we shall briefly introduce the $G$-expectation framework. Moreover, we shall discuss the deterministic Skorohod problem in non-convex domains and the sufficient conditions for its solvability.

2.1 $G$-Brownian motion and $G$-expectation

Adapting to Peng’s framework, we first recall useful notations and results on the $G$-expectation and related $G$-Itô type stochastic calculus. The reader interested in the more detailed description on this topic is referred to Denis et al. [3], Gao [6], Li and Peng [10] and Peng [20].

Let $\Omega$ be the space of all $\mathbb{R}^d$-valued continuous paths with $\omega_0 = 0$, noted by $C_0([0, \infty); \mathbb{R}^d)$,
equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} ((\max_{t \in [0, N]} |\omega^1_t - \omega^2_t|) \wedge 1),$$

and \((B_t)_{t \geq 0}\) be the canonical process, i.e., \(B_t(\omega) := \omega_t\). For each \(t \in [0, \infty)\), we list the following notations:

- \(\Omega_t := \{\omega, \lambda_t : \omega \in \Omega\}; \mathcal{F}_t := \mathcal{B}(\Omega_t)\);
- \(L^0(\Omega)\): the space of all \(\mathcal{B}(\Omega)\)-measurable real functions;
- \(L_{lip}(\Omega_t) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, 0 \leq t_1 < \cdots < t_n \leq t, \varphi \in C_{b, lip}(\mathbb{R}^{d \times n})\};\)
- \(L_{lip}(\Omega) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, 0 \leq t_1 < \cdots < t_n \leq \infty, \varphi \in C_{b, lip}(\mathbb{R}^{d \times n})\};\)

where \(C_{b, lip}(\mathbb{R}^{d \times n})\) is the collection of all bounded Lipschitz functions on \(\mathbb{R}^{d \times n}\).

We fix a sublinear continuous and monotone function \(G: \mathbb{S}^d \to \mathbb{R}\). For some bounded and closed subset \(\Gamma \subset \mathbb{R}^d\), this function can be represented by

$$G(A_0) = \sup_{Q \in \Gamma} \left\{ \frac{1}{2} \text{tr} \left[ A_0 Q^T Q \right] \right\}, \quad \text{for } A_0 \in \mathbb{S}^d.$$ 

The related \(G\)-expectation on \((\Omega, L_{lip}(\Omega))\) can be constructed in the following way: for each \(\xi \in L_{lip}(\Omega)\) of the form

$$\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad 0 \leq t_1 < t_2 < \cdots < t_n,$$

define

$$\mathbb{E}_G[\xi] := u_1(0, 0),$$

where \(u_1(0, 0) \in \mathbb{R}\) is obtained by recurrence: for \(k = n, \ldots, 1\), define \(u_k := u_k(t, x; x_1, \ldots, x_{k-1})\), which is a function in \((t, x)\) parameterized by \((x_1, \ldots, x_{k-1}) \in \mathbb{R}^{d \times (k-1)}\), by the solution of the following \(G\)-heat equation defined on \([t_{k-1}, t_k) \times \mathbb{R}^d\):

$$\frac{\partial u_k}{\partial t} - G(D^2 u_k) = 0,$$

with the terminal condition

$$u_k(t_k, x; x_1, \ldots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \ldots, x_{k-1}, x).$$

In particular, \(u_n(t_n, x; x_1, \ldots, x_{n-1}) := \varphi(x_1, \ldots, x_{n-1}, x)\). We say that the canonical process \((B_t)_{t \geq 0}\) is a \(G\)-Brownian motion under this sublinear expectation \(\mathbb{E}_G[\cdot]\), which is with stationary, independent and \(G\)-Gaussian distributed increments (see Definition 1.4 and 1.8 in Chap. I of [20] for the definition of \(G\)-Gaussian distribution and see Definition 3.10 in Chap. I of [20] for the definition of independence under sublinear expectation).
We denote by $L^p_G(\Omega)$ (resp. $L^p_G(\Omega_T)$) the completion of $\text{Lip}(\Omega)$ (resp. $\text{Lip}(\Omega_T)$) with respect to the norm $\| \cdot \|_p := \mathbb{E}_G[| \cdot |^p]^{\frac{1}{p}}$, for $p \geq 1$. We can extend the domain of $G$-expectation $\mathbb{E}_G[\cdot]$ from $\text{Lip}(\Omega)$ to $L^0(\Omega)$ by the procedure introduced in [3], i.e., constructing an upper expectation $\mathbb{E}^{\text{p}}[\cdot]$:

$$
\mathbb{E}^{\text{p}}[X] := \sup_{P \in \mathcal{P}_G} \mathbb{E}^P[X], \quad X \in L^0(\Omega),
$$

where $\mathcal{P}_G$ is a weakly compact family of martingale measures on $(\Omega, \mathcal{B}(\Omega))$. This upper expectation coincides with the $G$-expectation $\mathbb{E}_G[\cdot]$ on $\text{Lip}(\Omega)$ and thus, on its completion $L^1_G(\Omega)$. Naturally, the Choquet capacity related to the upper expectation can be defined by

$$
\bar{C}(A) := \sup_{P \in \mathcal{P}_G} \mathbb{P}(A), \quad A \in \mathcal{B}(\Omega),
$$

and the notation of “quasi-surely” (q.s.) can be introduced as follows:

**Definition 2.1 (Quasi-sure).** A set $A \in \mathcal{B}(\Omega)$ is called polar if $\bar{C}(A) = 0$. A property is said to hold quasi-surely if it holds outside a polar set.

The following Markov’s inequality holds in the context of the upper expectation and the related Choquet capacity (Lemma 13 in [3]).

**Lemma 2.2 (Markov’s inequality).** Let $X \in L^0(\Omega)$ satisfying $\mathbb{E}[|X|^p] < \infty$, for $p > 0$. Then, for each $a > 0$,

$$
\bar{C}(|X| > a) \leq \frac{\mathbb{E}[|X|^p]}{a^p}.
$$

We also have a generalized Fatou’s lemma (cf. e.g. Lemma 2.11 in Bai and Lin [1]) in the $G$-framework.

**Lemma 2.3 (Fatou’s lemma).** Assume that $\{X^n\}_{n \in \mathbb{N}}$ is a sequence in $L^0(\Omega)$ and that for a $Y \in L^0(\Omega)$ satisfying $\mathbb{E}[|Y|] < \infty$ and for all $n \in \mathbb{N}$, $X^n \geq Y$, q.s., then

$$
\mathbb{E}[\liminf_{n \to \infty} X^n] \leq \liminf_{n \to \infty} \mathbb{E}[X^n].
$$

In [20], Peng introduce the Itô type stochastic integral with respect to the $G$-Brownian motion by first considering the simple process space:

$$
M^0([0, T]; \mathbb{R}) = \{ \eta : \eta_t(\omega) = \sum_{i=0}^{n-1} \xi_i(\omega) 1_{[t_i, t_{i+1})}(t),
$$

where $n \in \mathbb{N}^*$, $0 = t_0 < \cdots < t_n = T$, $\xi_i \in \text{Lip}(\Omega_{t_i})$, $i = 0, \cdots, n - 1$.

**Definition 2.4.** For $p \geq 1$, we denote by $M^p_G([0, T]; \mathbb{R})$ the completion of $M^0([0, T]; \mathbb{R})$ under the following norm:

$$
\| \eta \|_p := \left( \mathbb{E} \left[ \frac{1}{T} \int_0^T |\eta_t|^p dt \right] \right)^{1/p}.
$$

Here below is the definition of the $G$-Itô type sintegral. In the sequel, $B^a$ denotes the inner product of $a \in \mathbb{R}^d$ and $B$, which is still a $G$-Brownian motion, and $\sigma_{a} := \mathbb{E}[(a, B_1)^2]$. 
Definition 2.5. For each $\eta \in M^0([0, T]; \mathbb{R})$, we define the Itô type integral

$$I_{[0, T]}(\eta) = \int_0^T \eta_t dB^a_t := \sum_{k=0}^{N-1} \xi_k (B^a_{t_{k+1}} - B^a_{t_k}).$$

Then, thanks to G-Itô’s inequality (cf. Lemma 3.4 in [10]), this linear mapping $I_{[0, T]}$ on $M^0([0, T]; \mathbb{R})$ can be continuously extended to $I_{[0, T]} : M^2_G([0, T]; \mathbb{R}) \rightarrow L^2_G(\Omega_T)$ and for each $\eta \in M^2_G([0, T]; \mathbb{R})$, we define $\int_0^T \eta_t dB^a_t := I_{[0, T]}(\eta)$.

Moreover, we have the following BDG type inequality (cf. Theorem 2.1 in [6]). Define

$$\sigma_{a\bar{a}} := \sup_{Q \in \Gamma} \text{tr}(Q^T Q a \bar{a}).$$

**Lemma 2.6.** Let $p \geq 2$, $a \in \mathbb{R}^d$, $\eta \in M^p_G([0, T]; \mathbb{R})$ and $0 \leq s \leq t \leq T$. Then,

$$E \left[ \sup_{s \leq u \leq t} \left| \int_s^u \eta_u dB^a_t \right|^p \right] \leq C_p \sigma_{a\bar{a}}^{p/2} E \left[ \left( \int_s^t |\eta_u|^2 du \right)^{p/2} \right],$$

where $C_p > 0$ is a constant independent of $a$, $\eta$ and $\Gamma$.

In the $G$-expectation framework the quadratic variation process $\langle B^a \rangle$ is no longer deterministic, which is formulated by

$$\langle B^a \rangle_t := \lim_{\mu(\pi^N_{[0, T]}) \rightarrow 0} \sum_{k=0}^{N-1} (B^a_{t_{k+1}} - B^a_{t_k})^2 = (B^a_t)^2 - 2 \int_0^t B^a_s dB^a_s,$$

where $\pi^N_{[0, T]}$ is a partition of $[0, T]$, i.e., $\pi^N_{[0, T]} = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots < t_N = T$, and $\mu(\pi^N_{[0, T]}) := \max_{1 \leq i \leq N} |t_i^N - t_{i-1}^N|$. For two given vectors $a, \bar{a} \in \mathbb{R}^d$, the mutual variation process of $B^a$ and $B^\bar{a}$ is defined by

$$\langle B^a, B^\bar{a} \rangle_t := \frac{1}{4} (\langle B^{a+\bar{a}} \rangle_t - \langle B^{(a-\bar{a})} \rangle_t).$$

By Corollary 5.7 in Chapter III of Peng [20], for each $0 \leq s \leq t \leq T$,

$$\langle B^a \rangle_t - \langle B^a \rangle_s \leq \sigma_{a\bar{a}}(t - s). \quad (2.1)$$

Let $B^i$ denote the $i$th coordinate of the $G$-Brownian motion $B$ and set $\langle B, B \rangle_t = (\langle B^i, B^j \rangle_t)_{i,j=1,\ldots,d}$. Thus, the path of $\langle B, B \rangle$ quasi-surely has a bounded density and indeed the stochastic integral for $\eta \in M^2_G([0, T]; \mathbb{R})$ with respect to $\langle B^a, B^\bar{a} \rangle$ could be defined pathwisely.

Finally, we recall that Gao [6] proves the $G$-Itô type integral $X. = \int_0^T \eta_s dB^a_s$ has a continuous $C$-modification, for any $\eta \in M^2_G([0, T]; \mathbb{R})$. 

6
2.2 Conditions on the domain

In order to investigate the reflected $G$-Brownian motion in this paper, we shall first recall the results in [22] for the deterministic Skorohod problem in a domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$. In that paper, the following conditions are assumed:

CONDITION (A). For $x \in \partial D$, we denote
\[
\mathcal{N}_{x,r} = \{n \in \mathbb{R}^d : |n| = 1, B(x - rn, r) \cap D = \emptyset\}, \quad r > 0
\]
and $\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r}$, where $B(z, r) := \{y \in \mathbb{R}^d : |y - z| < r\}$, for $z \in \mathbb{R}^d$. We assume that there exists a constant $r_0 > 0$ such that $\mathcal{N}_x \neq \emptyset$, for all $x \in \partial D$.

CONDITION (B). Assume that there exist constants $\delta > 0$ and $\beta \in [1, \infty)$ such that for any $x \in \partial D$, we can find a unit vector $l_x$ such that
\[
\langle l_x, n \rangle \geq \frac{1}{\beta}, \quad \text{for all } n \in \bigcup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y,
\]
where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$.

Throughout this paper, we consider a domain $D \subset \mathbb{R}^d$ satisfying both Condition (A) and (B). For each $x \in \mathbb{R}^d$ such that $\text{dist}(x, \overline{D}) < r_0$, there exists a unique $\overline{x} \in \overline{D}$ with $|x - \overline{x}| = \text{dist}(x, \overline{D})$. If $x \notin \overline{D}$ we have $\overline{x} \in \partial D$ and $\frac{x - \overline{x}}{|x - \overline{x}|} \in \mathcal{N}_{\overline{x}}$ (see e.g. Remark 1.3 in [21]). We keep this notation $\overline{x}$ for the projection of $x$ on $\overline{D}$ in the remainder of this paper.

To consider the solvability of reflected multi-dimensional Skorohod stochastic differential equations on the domain $D$, we assume furthermore

CONDITION (C). There exists a bounded function $\Psi \in C^2_b(\mathbb{R}^d)$ whose first and second derivatives are also bounded, and there exists a $\delta' > 0$, such that
\[
\forall x \in \partial D, \forall y \in \overline{D}, \forall n \in \mathcal{N}_x, \langle y - x, n \rangle + \frac{1}{\delta'} \langle \nabla \Psi(x), n \rangle |y - x|^2 \geq 0.
\]
We note the bound of $\Psi$ and its derivatives by $L_\Psi$. This condition is critical for proving the a priori estimate [11,18], which is similar to its analogue in §3 of [14].

2.3 Deterministic Skorohod problem

We recall here the solvability result of deterministic Skorohod problems in the domain $D$ satisfying Conditions (A) and (B), which could be found in [21]. This result is our starting point of this paper.

Assume that $\phi$ is a continuous function taking values in $\mathbb{R}^d$ and that $\phi$ is of bounded variation over each finite interval. We denote by $|\phi|_t$ the total variation of $\phi$ over $[0, t]$, i.e.,
\[
|\phi|_t := \sup_{0=t_1<t_2<\cdots<t_n=t} \sum_{k=1}^n |\phi_{t_k} - \phi_{t_{k-1}}|.
\]
We also note $|\phi|^s_t := |\phi|_t - |\phi|_s, \ 0 \leq s \leq t$.

For a continuous function $w$ defined on $[0, T], \ T > 0$, taking values in $\mathbb{R}^d$ with $w(0) = 0$ and for $x_0 \in \overline{D}$, we consider the Skorohod equation below:

$$\xi_t = x_0 + w_t + \phi_t, \ t \in [0, T]. \quad (2.2)$$

**Definition 2.7.** We call a couple of functions $(\xi, \phi)$ solution of (2.2), if it satisfies (2.2) and the following conditions:

(i) The function $\xi$ is continuous and takes values in $\overline{D}$;

(ii) The function $\phi$ is continuous and takes values in $\mathbb{R}^d$ with $\phi(0) = 0$. Moreover, it is of bounded variation over $[0, T]$ and for all $t \in [0, T],$

$$\phi_t = \int_0^t n_s |\phi|^s_t;$$

$$|\phi|^t_t = \int_0^t 1_{\{\xi_s \in \partial D\}} d|\phi|^s_s,$$

where $n_s \in N_{\xi_s}$, if $\xi_s \in \partial D$.

**Theorem 2.8.** (Theorem 4.1 in [21]) Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Then there exists a unique solution $(\xi, \phi)$ for the deterministic Skorohod problem (2.2).

3. Main results

In this section, we present our main results on the reflected $G$-Brownian motion and on reflected stochastic differential equations driven by $G$-Brownian motion.

3.1 Reflected $G$-Brownian motion

We replace the deterministic function $w$ in the Skorohod problem (2.2) by the $G$-Brownian motion $B$ and establish the following equation in the “quasi-sure” sense:

$$X_t = x_0 + B_t + K_t, \ x_0 \in \overline{D}, \ 0 \leq t \leq T. \quad (3.1)$$

**Definition 3.1.** We call a couple of processes $(X, K)$ solves the Skorohod problem for the $G$-Brownian motion (3.1), if there exists a polar set $A$, such that

(i) The processes $X$ et $K$ belong to $M^2_G ([0, T]; \mathbb{R}^d)$, and for all $\omega \in A^c$,

$$X_t(\omega) = x_0 + B_t(\omega) + K_t(\omega), \ 0 \leq t \leq T;$$

(ii) For all $\omega \in A^c$, $X(\omega)$ is continuous and takes values in $\overline{D}$;
(iii) For all $\omega \in \bar{A}$, $K(\omega)$ is continuous and takes values in $\mathbb{R}^d$ with $K_0(\omega) = 0$. Moreover, $K(\omega)$ is of bounded variation over $[0,T]$ and for all $t \in [0,T]$,

$$K_t(\omega) = \int_0^t n_s(\omega) \, d|K_s| (\omega);$$

$$|K_t| (\omega) = \int_0^t 1_{\{X_s(\omega) \in \partial D\}} d|K_s| (\omega),$$

where $n_s(\omega) \in \mathcal{N}_{X_s(\omega)}$, if $X_s(\omega) \in \partial D$.

In addition, we call $X$ reflected $G$-Brownian motion on the domain $D$.

We have the following existence and uniqueness theorem for the reflected $G$-Brownian motion. The proof of this theorem is postponed to the next section.

**Theorem 3.2.** Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Then there exists a couple $(X, K) \in (M^2_G([0,T]; \mathbb{R}^d) \times M^2_G([0,T]; \mathbb{R}^d))$ which solves the Skorohod problem (3.1) whenever $x_0 \in \bar{D}$. Moreover, if the problem (3.1) admits two solutions $(X, K)$ and $(X', K')$, then the exists a polar set $\bar{A}$, such that for all $\omega \in \bar{A}$,

$$X(\omega) = X'(\omega) \quad \text{and} \quad K(\omega) = K'(\omega), \quad 0 \leq t \leq T.$$

### 3.2 Reflected stochastic differential equations driven by $G$-Brownian motion

In addition to the reflected $G$-Brownian motion, we shall study reflected stochastic differential equations driven by $G$-Brownian motion, which is formulated as

$$X_t = x_0 + \int_0^t f(s, X_s) \, ds + \int_0^t h^{ij}(s, X_s) \, dB^i_s + \int_0^t g^j(s, X_s) \, dB^j_s + K_t, \quad 0 \leq t \leq T, \quad \text{q.s.} \quad (3.2)$$

Here we adopt the Einstein summation convention. In (3.2), the process $(B, B)$ is the covariation matrix of the $d$-dimensional $G$-Brownian motion $B$. In what follows, we assume that the functions $f$, $h$, $g$ satisfy the following conditions:

**Assumption 3.3.** The functions $f$, $h^{ij}$, $g^j : \Omega \times [0,T] \times \bar{D} \longrightarrow \mathbb{R}^d$, $i, j = 1, 2, \ldots, d$, are functions such that

- *(H1)* For all $x \in \bar{D}$, the processes $f(\cdot, x)$, $h^{ij}(\cdot, x)$, $g^j(\cdot, x)$ belong to $M^2_G([0,T]; \mathbb{R}^d)$;

- *(H2)* The functions $f$, $h^{ij}$, $g^j$ are uniformly bounded by $L_0$ and uniformly $L_0$-Lipschitz, i.e., there exists a constant $L_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0,T]$,

$$\|f(t,x) - f(t,y)\| + \|h^{ij}(t,x) - h^{ij}(t,y)\| + \|g^j(t,x) - g^j(t,y)\| \leq L_0 \|x - y\|, \quad \forall x, y \in \bar{D},$$

where $\| \cdot \|$ denotes the Hilbert-Schmidt norm for matrices.

**Definition 3.4.** We call a couple of processes $(X, K)$ solves the Skorohod stochastic differential equation (3.2), if there exists a polar set $A$, such that

- *(i)* The processes $X$ et $K$ belong to $M^2_G([0,T]; \mathbb{R}^d)$ and satisfies (3.2);
(ii) For all $\omega \in A^c$, $X(\omega)$ takes values in $\overline{D}$;

(iii) For all $\omega \in A^c$, $K(\omega)$ takes values in $\mathbb{R}^d$ with $K_0(\omega) = 0$. Moreover, $K(\omega)$ is of bounded variation over $[0, T]$ and for all $t \in [0, T]$,

$$K_t(\omega) = \int_0^t n_s(\omega) d|K|_s(\omega);$$

$$|K_t| = \int_0^t 1_{\{X_s(\omega) \in \partial D\}} d|K|_s(\omega),$$

where $n_s(\omega) \in N_{X_s(\omega)}$, if $X_s(\omega) \in \partial D$.

Using a fixed point type argument, we shall prove in the next section the following existence and uniqueness theorem for the Skorohod stochastic differential equation (3.2).

**Theorem 3.5.** Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Then there exists a unique couple $(X, K) \in (M^2_G([0, T]; \mathbb{R}^d) \times M^2_G([0, T]; \mathbb{R}^d))$ which solves the Skorohod stochastic differential equation (3.2) whenever $x_0 \in \overline{D}$ and the coefficients $f$, $h$, $g$ satisfy Assumptions (H1) and (H2).

4 Proofs

In this section, we shall prove Theorem 3.2 and 3.5. First, we recall the results for deterministic Skorohod problem in Saisho and Tanaka [22] and provide an estimate when the function $w$ is $\alpha$-Hölder continuous, $\alpha \in (0, 1/2)$.

4.1 Estimates for the deterministic Skorohod problem

In (2.2), we assume in addition that $w$ is an $\alpha$-Hölder continuous function on $[0, T]$, where $\alpha \in (0, 1/2)$, i.e.,

$$\|w\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|w_t - w_s|}{|t - s|^\alpha} < \infty.$$

Set

$$\Delta_{s,t}(w) := \sup \{|w_{t_2} - w_{t_1}| : s \leq t_1 < t_2 \leq t\},$$

and $\|w\|_T := \sup\{|w|_t : 0 \leq t \leq T\}$.

We recall the penalization method in [14, 22] and define a sequence of equations: for $m \in \mathbb{N}^*$,

$$\xi^m_t = x_0 + w_t - \frac{m}{2} \int_0^t \nabla U(\xi^m_s) ds,$$  \hspace{1cm} (4.1)
For each $m \in \mathbb{N}^*$, we define
\[
\varepsilon_m^\alpha (w) := \frac{12e^L}{m^\alpha} \|w\|_\alpha.
\] (4.2)

For $\varepsilon > 0$, we note
\[
D_\varepsilon := \left\{ x \in \mathbb{R}^d, \text{ dist}(x, D) < \varepsilon \right\}.
\]
Additionally, set
\[
\tilde{\xi}_m^t := \xi_{m/m}^t,
\]
it is easy to verify that
\[
\tilde{\xi}_m^t = x_0 + \tilde{w}_{t/m} - \frac{1}{2} \int_0^t \nabla U(\tilde{\xi}_s^m)ds, \quad 0 \leq t \leq mT.
\]

Similar to Lemma 4.2 in [22], we have the lemma below.

**Lemma 4.1.** Assume that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon_m^\alpha (w) < r_0/2$ and for all $m' \geq m$, if there exists $u \in (0, m'T)$ such that $\tilde{\xi}_{m'}^u \in \partial D_{\varepsilon_m^\alpha (w)}/2$, then $\{\tilde{\xi}_t^{m'}\}_{t \leq T}$ hits $\partial D_{\varepsilon_m^\alpha (w)}/3$ before hitting $\partial D_{\varepsilon_m^\alpha (w)}$.

**Proof.** This lemma can be proved by slightly modifying the proof of Lemma 4.2 in [22]. Precisely, we could verify that for the given $\varepsilon_m^\alpha$, the number $m$ itself is large enough to ensure Lemma 4.2 in [22] if $\varepsilon_m^\alpha (w)$ is defined as (4.2). For the convenience of the reader, we briefly prove this lemma.

Fix $m' \geq m$ and suppose that $\tilde{\xi}_{m'}^u \in \partial D_{\varepsilon_m^\alpha (w)}/2$. Consider the auxiliary equation:
\[
\eta_t = \tilde{\xi}_{m'}^u - \frac{1}{2} \int_u^t \nabla U(\eta_s)ds, \quad t \geq u.
\]
From Lemma 4.1 in [22], the above equation is solved by
\[
\eta_t = \tilde{\xi}_{m'}^u - \left( \tilde{\xi}_{m'}^u - \tilde{\xi}_u^m \right) \exp\{-u\},
\]
and the function $\eta$ satisfies that for all $t \geq u$,
\[
|\eta_t - \eta_u| = \frac{\varepsilon_m^\alpha (w)}{2} \exp\{-u\}.
\]
Then, we denote
\[
u' := \inf\left\{ t > u : \eta_t \in \partial D_{\varepsilon_m^\alpha (w)/4} \right\},
\]
and it is obvious that $u' = u + \log 2 < u + 1$.

On the other hand, for $u \leq t \leq m'T$, we have
\[
\tilde{\xi}_m^t - \eta_t = w_{t/m'} - w_{u/m'} - \frac{1}{2} \int_u^t \left( \nabla U(\tilde{\xi}_s^{m'}) - \nabla U(\eta_s) \right) ds,
\]
which implies
\[
\left| \tilde{\xi}_m^t - \eta_t \right| \leq \left| w_{t/m'} - w_{u/m'} \right| + L \int_u^t \left| \tilde{\xi}_s^{m'} - \eta_s \right| ds. \tag{4.3}
\]
• If \( u' \leq m'T \), then for all \( u \leq t \leq u' \), we can deduce that
\[
|w_{t/m'} - w_{u/m'}| \leq \frac{\|w\|_\alpha}{m'^\alpha} \leq \frac{\|w\|_\alpha}{m^{\alpha}} = \frac{\varepsilon_m^\alpha(w)}{12e^L}.
\]

We apply Gronwall’s lemma to (4.3) and obtain
\[
\left| \tilde{\varepsilon}_{s_1} - \tilde{\eta}_t \right| \leq \frac{\varepsilon_m^\alpha(w)}{12e^L}e^{L(t-u)} < \frac{\varepsilon_m^\alpha(w)}{12}, \quad u \leq t \leq u'.
\]

Therefore, for \( u \leq t \leq u' \),
\[
\left| \tilde{\varepsilon}_{s_1} - \tilde{\xi}_{s_1} \right| \leq \left| \tilde{\eta}_t - \eta_t \right| + \left| \tilde{\xi}_{s_1} - \tilde{\eta}_t \right| < \frac{\varepsilon_m^\alpha(w)}{2} + \frac{\varepsilon_m^\alpha(w)}{12} < \varepsilon_m^\alpha(w),
\]
wheras
\[
\left| \tilde{\varepsilon}_{s_1} - \tilde{\xi}_{s_1} \right| \leq \left| \eta_t' - \eta_t \right| + \left| \tilde{\xi}_{s_1} - \eta_t \right| < \frac{\varepsilon_m^\alpha(w)}{4} + \frac{\varepsilon_m^\alpha(w)}{12} = \frac{\varepsilon_m^\alpha(w)}{3},
\]
which implies that \( \{ \tilde{\varepsilon}_{s_1} \}_u \leq u' \) hits \( \partial D_{\varepsilon_m^\alpha(w)/3} \) before hitting \( \partial D_{\varepsilon_m^\alpha(w)} \).

• If \( u' > m'T \), then we could repeat the procedure above to prove that for \( u \leq t \leq m'T \),
\[
\left| \tilde{\varepsilon}_{s_1} - \tilde{\xi}_{s_1} \right| < \varepsilon_m^\alpha(w),
\]
which implies that \( \{ \tilde{\xi}_{s_1} \}_u \leq m'T \) never hits \( \partial D_{\varepsilon_m^\alpha(w)} \).

\( \square \)

We now give a proposition which is a straightforward corollary of Lemma 4.1. The proof of this proposition is omitted and we refer the reader to Proposition 4.1 in [22].

**Proposition 4.2.** Assume that the domain \( D \subset \mathbb{R}^d \) is open and satisfies Conditions (A) and (B). For each \( m \in \mathbb{N}^* \) such that \( 0 < \varepsilon_m^\alpha(w) < r_0/2 \) and for all \( m' \geq m \),
\[
\varepsilon_m^{m'}(w) \in D_{\varepsilon_m^\alpha(w)}, \quad 0 \leq t \leq T.
\]

In particular, the results in the remainder of this subsection is based on the fact that if \( m \in \mathbb{N}^* \) is large enough such that \( 0 < \varepsilon_m^\alpha(w) < r_0/2 \) then
\[
\varepsilon_m^\alpha(w) \in D_{\varepsilon_m^\alpha(w)}, \quad 0 \leq t \leq T.
\]

For \( m, n \in \mathbb{N}^* \), set
\[
T_{m,0} = \inf \{ t \geq 0 : \tilde{\varepsilon}_{s_1} \in \partial D \} ;
\]
\[
t_{m,n} = \inf \{ t > T_{m,n-1} : \tilde{\varepsilon}_{s_1} - \tilde{\varepsilon}_{T_{m,n-1}} \geq \delta/2 \} ;
\]
\[
T_{m,n} = \inf \{ t \geq t_{m,n} : \tilde{\varepsilon}_{s_1} \in \partial D \} ,
\]
where the constant \( \delta \) is from Condition (B). Moreover, denote by
\[
\tilde{\phi}_t^m := \frac{-m}{2} \int_0^t \nabla U(\varepsilon_s^m)ds, \quad 0 \leq t \leq T.
\]

In the remainder of this subsection, we shall provide an estimate of \( |\tilde{\phi}_t^m|_T^0 \). First, we prove that for sufficient large \( m \), there is a lower bounded for \( T_{m,n} - T_{m,n-1} \).

For simplicity, note \( \gamma := \frac{2e^2(r_0/2)\delta}{r_0} \) and \( \lambda(w) := \exp \{ \gamma (\|w\|_T + \delta) \} \), where \( \kappa(r_0/2) \) is a constant from Lemma 2.1 in [22] such that for all \( |x - \overline{x}| < r_0/2 \) and \( |y - \overline{y}| < r_0/2, \ |x - y| \leq \kappa(r_0/2)|x - y| \). Obviously, \( \lambda > 1 \).
Lemma 4.3. Assume that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon^\alpha_m (w) < \frac{\delta}{180\beta \lambda (w)} \wedge r_0/2$ and for each $n \geq 1$ such that $T_{m,n} < \infty$,

$$|T_{m,n} - T_{m,n-1}| \geq h := \left( \frac{\delta}{36\beta \lambda (w) \|w\|_\alpha} \right)^{1/\alpha},$$

Proof. From Lemma 5.3 in [22], we have

$$\Delta_{s,t} (\xi^m) \leq (8\beta \lambda (w) + 1) (\Delta_{s,t} (w) + \varepsilon^\alpha_m (w)). \quad (4.4)$$

Then, from Proposition 4.2 and the definitions of $t_{m,n}$ and $T_{m,n}$, we have

$$\frac{\delta}{2} - 2\varepsilon^\alpha_m (w) = \left| \xi^m_{t_{m,n}} - \xi^m_{T_{m,n-1}} \right| - 2\varepsilon^\alpha_m (w)
\leq \left| \xi^m_{t_{m,n}} - \xi^m_{T_{m,n-1}} \right|
\leq 9\beta \lambda (w) \left( \Delta_{T_{m,n},T_{m,n-1}} (w) + \varepsilon^\alpha_m (w) \right),$$

which implies

$$\Delta_{T_{m,n},T_{m,n-1}} (w) \geq \frac{\delta - 4\varepsilon^\alpha_m}{18\beta \lambda (w)} - \varepsilon^\alpha_m \geq \frac{\delta}{18\beta \lambda (w)} - 5\varepsilon^\alpha_m \geq \frac{\delta}{36\beta \lambda (w)}.$$}

Furthermore, we deduce the desired result by the definition of $\|w\|_\alpha$. 

Proposition 4.4. Assume that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon^\alpha_m (w) < \frac{\delta}{180\beta \lambda (w)} \wedge r_0/2$, we have

$$|\phi^m|^S |_{T_{m,n-1}} \leq C_0 \left( \|w\|_\alpha^{1+1/\alpha} + \|w\|_\alpha \right) \exp \left\{ \gamma (1 + 1/\alpha) \|w\|_T \right\}, \quad (4.5)$$

where $C_0$ depends only on $\alpha, \beta, L, \delta$ and $T$.

Proof. For any $s, t$, such that $T_{m,n-1} \leq s \leq t \leq T_{m,n}$, we know from Lemma 5.1 in [22] that

$$|\phi^m|^S |_{t} \leq \beta \left( \Delta_{s,t} (\xi^m) + \Delta_{s,t} (w) \right),$$

We combine this inequality with (4.4) and deduce

$$|\phi^m|^S |_{t} \leq 10\beta^2 \exp \left\{ \gamma (\|w\|_T + \delta) \right\} (\Delta_{s,t} (w) + \varepsilon^\alpha_m (w)).$$

Thus,

$$|\phi^m|^S |_{T_{m,n-1}} \leq 10 \left( \frac{T}{h} + 1 \right) \beta^2 \exp \left\{ \gamma (\|w\|_T + \delta) \right\} (\Delta_{0,T} (w) + \varepsilon^\alpha_m (w))
\leq 10 \left( \frac{T}{h} + 1 \right) \beta^2 \exp \left\{ \gamma (\|w\|_T + \delta) \right\} \left( T^\alpha \|w\|_\alpha + \frac{12eL}{m^\alpha} \|w\|_\alpha \right).$$

By definition, $\exp \{\gamma \|w\|_T \} \geq 1$, then we can complete the proof by recalling the the definition of $h$. □
4.2 The existence and uniqueness for the reflected G-Brownian motion

By Theorem 2.8 for each \( \omega \in \Omega \), there exists a pair \((X(\omega), K(\omega))\) that solves the deterministic Skorohod problem for \( B(\omega) \), i.e.,

\[
X(\omega) = x_0 + B(\omega) + K(\omega), \quad x_0 \in \mathcal{D}.
\]  

(4.6)

It is easy to see that the pair of processes \((X,K)\) satisfy (i), (ii) and (iii) in Definition 3.1. Thus, to prove Theorem 3.2, it suffices to show that both \( X \) and \( K \) belong to \( M^{p}_{\mathcal{G}}([0,T];\mathbb{R}^d) \).

**Lemma 4.5.** Assume that the domain \( D \subset \mathbb{R}^d \) is open and satisfies Conditions (A) and (B). For each \( \omega \in \Omega \), we define the pair of processes \((X,K)\) by the unique solution of the deterministic Skorohod problem

\[
X(\omega) = x_0 + B(\omega) + K(\omega), \quad x_0 \in \mathcal{D},
\]

where \( B \) is a G-Brownian motion. Then, \( X \) and \( K \) belong to \( M^{p}_{\mathcal{G}}([0,T];\mathbb{R}^d) \).

The main idea to prove this lemma is to construct a sequence of \( \{(X^m,K^m)\}_{m \in \mathbb{N}^*} \) formed by elements from \( (M^p_{\mathcal{G}}([0,T];\mathbb{R}^d) \times M^p_{\mathcal{G}}([0,T];\mathbb{R}^d)) \) and then to show that the following convergences hold

\[
X^m \to X, \quad K^m \to K, \quad \text{in} \quad M^p_{\mathcal{G}}([0,T];\mathbb{R}^d), \quad \text{for} \quad p \geq 2.
\]

Similarly to the previous subsection, we define for each \( m \in \mathbb{N}^* \),

\[
X^m_t = x_0 + B_t - \frac{m}{2} \int_0^t \nabla U(X^m_s)ds, \quad 0 \leq t \leq T,
\]

which is a stochastic differential equations driven by G-Brownian motion with bounded Lipschitz coefficients. For each \( m \in \mathbb{N}^* \), the above equation admits a unique solution in \( M^p_{\mathcal{G}}([0,T];\mathbb{R}^d) \), \( p \geq 2 \) (cf. Theorem 4.2 in [6]). Moreover, one can find a version of \( X^m \), denoted still by \( X^m \), such that there exists a polar set \( A^m \), for all \( \omega \in (A^m)^c \), \( X^m \) is continuous and

\[
X^m_t(\omega) = x_0 + B_t(\omega) - \frac{m}{2} \int_0^t \nabla U(X^m_s(\omega))ds, \quad 0 \leq t \leq T.
\]

We note \( A := \bigcup_{m \in \mathbb{N}^*} A^m \), which is still a polar set, and we note

\[
K^m_t(\omega) := - \frac{m}{2} \int_0^t \nabla U(X^m_s(\omega))ds.
\]

In what follows, we shall find a bound uniform in \( m \) for \( \{\mathbb{E}\left[ \sup_{0 \leq t \leq T} |X^m_t|^{p} \right] \}_{m \in \mathbb{N}^*} \) and \( \{\mathbb{E}\left[ \left( |K^m_t|^{p} \right)^{\frac{p}{2}} \right] \}_{m \in \mathbb{N}^*} \).

First, by Kolmogorov’\textprime s Criterion (cf. Theorem 36 in Denis et al. [3]), for any \( \alpha \in (0,1/2) \), there exists a polar set \( A' \) such that for all \( \omega \in (A')^c \), the path of G-Brownian motion \( B(\omega) \) is \( \alpha \)-Hölder continuous. Moreover, for any \( p > 0, \alpha \in (0,1/2) \),

\[
\mathbb{E}\left[ \left( \sup_{0 \leq s \leq t \leq T} \frac{|B_s - B_t|}{|t-s|^\alpha} \right)^p \right] < \infty.
\]  

(4.7)
For each $m \in \mathbb{N}^*$, define
\[
\mathcal{A}^m := \left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) < \frac{\delta}{180\beta\lambda(B(\omega))} \wedge r_0 \right\} \cap A^c \cap (A')^c.
\]

The following lemma gives an estimate for $c(\mathcal{A}^m)^c$.

**Lemma 4.6.** Fix $\alpha \in (0,1/2)$. For $m \in \mathbb{N}^*$,
\[
c(\mathcal{A}^m) \leq \frac{C_{\alpha,p}}{m^p}, \quad p \geq 1,
\]
where $C_{\alpha,p}$ depends on $T$, $\Gamma$, $\alpha$, $r_0$, $\delta$, $L$ and $\beta$.

**Proof.** It is clear that
\[
c(\mathcal{A}^m) \leq c\left(\left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) \geq \frac{r_0}{2} \right\}\right) + c\left(\left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) \geq \frac{\delta}{180\beta\lambda(B(\omega))} \right\}\right).
\]
We calculate by Markov’s inequality, for any $p \geq 1$,
\[
c\left(\left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) \geq \frac{r_0}{2} \right\}\right) = c\left(\left\{ \omega \in \Omega : \|B(\omega)\|_\alpha \geq \frac{r_0e^{-L}}{24}m^\alpha \right\}\right) \leq \frac{C_1\mathbb{E}\left[\|B\|_{\alpha}^{p/\alpha}\right]}{m^p},
\]
where $C_1 > 0$ depends on $r_0$ and $L$. On the other hand,
\[
c\left(\left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) \geq \frac{\delta}{180\beta\lambda(B(\omega))} \right\}\right) 
\leq \frac{C_2\mathbb{E}\left[\|B\|_{\alpha}^{p/\alpha}\exp\left(\frac{2\alpha}{\alpha}\|B\|_T\right)\right]}{m^p} 
\leq \frac{C_2\mathbb{E}\left[\|B\|_{\alpha}^{2p/\alpha}\right]^{1/2}\mathbb{E}\left[\exp\left(\frac{2\alpha}{\alpha}\|B\|_T\right)\right]^{1/2}}{m^p},
\]
where $C_2 > 0$ depends on $r_0$, $\delta$, $L$ and $\beta$. By Theorem 3.3 in Luo and Wang [15], we have
\[
\mathbb{E}\left[\exp\left(\frac{2\gamma p}{\alpha}\|B\|_T\right)\right] \leq C',
\]
where $C'$ depends on $T$, $\Gamma$, $p$, $\alpha$ and $\gamma$. We combine this with [4.7] to conclude the desired result.

**Proposition 4.7.** Assume that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Fix $\alpha \in (0, \frac{1}{2})$, then we have
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^p\right] + \mathbb{E}\left[\left(|K_t^m|^0\right)^p\right] \leq C_{\alpha,p}',
\]
where $C_{\alpha,p}'$ depends on $T$, $\Gamma$, $\alpha$, $p$, $r_0$, $\delta$, $L$ and $\beta$.
Proof. We denote by $\Lambda$ an upper bound of $|\nabla U|$, then

$$
\mathbb{E}\left[\left(\frac{|K^m|}{T}\right)^p 1_{\mathcal{A}^m}\right] \leq \left(\frac{mT^2}{2}\right)^p \mathcal{C}(\mathcal{A}^m) = C_{\alpha,p} \left(\frac{T^2}{2}\right)^p.
$$

(4.8)

From (4.5), we have, for each $\omega \in (\mathcal{A}^m)^c$,

$$
|K^m|_T^0(\omega) \leq C_0 \left(\|B(\omega)\|_T^{1+1/\alpha} + \|B(\omega)\|_T\right) \exp\left\{\gamma \left(1 + \frac{1}{\alpha}\right)\|B(\omega)\|_T\right\},
$$

(4.9)

where $C_0$ is the constant from (4.5). Thus,

$$
\mathbb{E}\left[\left(\frac{|K^m|}{T}\right)^p 1_{(A^m)^c}\right] \leq C_0 \left(\mathbb{E}\left[\|B\|_T^{2p(1+\alpha)/\alpha}\right]^{1/2} + \mathbb{E}\left[\|B\|_T^2\right]^{1/2}\right) \mathbb{E}\left[\exp\left\{\frac{2\gamma p(1+\alpha)}{\alpha}\|B\|_T\right\}\right]^{1/2}.
$$

Recall that for some $C_p > 0$, which depends only on $p$,

$$
|X_t^m| \leq C_p \left(\|x_0\| + \|B_t\| + |K_t^m|\right),
$$

we can deduce that

$$
\sup_{0 \leq t \leq T} |X_t^m| \leq C_p \left(\|x_0\| + \sup_{0 \leq t \leq T} |B_t| + \left(\frac{|K^m|_T^0\right)^p\right).
$$

We take the $G$-expectation on both sides and apply the BDG type inequality, (4.8) and (4.9) to conclude the desired result. \hfill \Box

Now we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. From (6.4) in [22], we have for $\omega \in (A \cup A')^c$,

$$
X_t^m(\omega) \rightarrow X_t(\omega), \text{ uniform on } [0, T],
$$

and for each $\omega \in (\mathcal{A}^m)^c$,

$$
\sup_{0 \leq s \leq t} |X_s^m(\omega) - X_s(\omega)|^2 \leq 4\varepsilon_m(\omega) \left(|K^m(\omega)|_t^0 + |K(\omega)|_t^0\right)
$$

$$
+ \frac{\gamma}{2\beta} \int_0^t \sup_{0 \leq u \leq s} (|X_u^m(\omega) - X_u(\omega)|^2) d\left(|K^m(\omega)|_s^0 + |K(\omega)|_s^0\right),
$$

which implies

$$
\sup_{0 \leq t \leq T} |X_t^m(\omega) - X_t(\omega)|^2 \leq 4\varepsilon_m(\omega) \left(|K^m(\omega)|_T^0 + |K(\omega)|_T^0\right) \exp\left\{\frac{\gamma}{2\beta} \left(|K^m(\omega)|_T^0 + |K(\omega)|_T^0\right)\right\}.
$$

Now we shall prove that

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^m(\omega) - X_t(\omega)|^2\right] \rightarrow 0, \text{ as } m \rightarrow \infty.
$$

(4.10)
For $\epsilon > 0$, by Markov’s inequality, we could first fix a constant $M_0 > 0$, such that
\[
(2C'_{\alpha, A})^{\frac{1}{2}} \mathbb{E} \left[ 1_{\|B\|_\alpha \geq M_0} \right] \leq \frac{(2C'_{\alpha, A})^{\frac{1}{2}} \mathbb{E} [|B|_\alpha]}{M_0} \leq \frac{\epsilon}{2},
\] (4.11)
where $C'_{\alpha, A}$ is the constant from Proposition 4.7. Then, we choose $m_0 \in \mathbb{N}^*$ sufficiently large such that
\[
\frac{12e^L}{m_0^3} M_0 \leq \frac{r_0}{2} \quad \text{and} \quad \frac{12e^L}{m_0^3} M_0 \leq \frac{\delta}{180\beta \exp \{\gamma (\delta + M_0 T^\alpha)\}};
\] (4.12)
\[
\frac{48e^L}{m_0^3} M_0 \left( 4C_0 M_0^{1+1/\alpha} \exp \left\{ \gamma \left( 1 + \frac{1}{\alpha} \right) M_0 T^\alpha \right\} \right) \exp \left\{ \frac{2\gamma C_0}{\beta} M_0^{1+1/\alpha} \exp \left\{ \gamma \left( 1 + \frac{1}{\alpha} \right) M_0 T^\alpha \right\} \right\} \leq \frac{\epsilon}{2},
\] (4.13)
where $C_0$ is the constant from Proposition 4.4. From (4.12), we know for $\omega \in \{\|B\|_\alpha < M_0\} \cap A^c \cap A'\cap A^c$ and $m \geq m_0$,
\[
ed_\alpha (B, \omega) < \frac{\delta}{180\beta \lambda (B, \omega)} \wedge \frac{r_0}{2}.
\]
It follows that for $m \geq m_0$,
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^n_m(t) - X^n(t)|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^n_m(t) - X^n(t)|^2 1_{\{\|B\|_\alpha < M_0\} \cap A^c \cap A^c} \right]
\+ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^n_m(t) - X^n(t)|^2 1_{\{\|B\|_\alpha \geq M_0\} \cap A^c \cap A^c} \right]
\leq \frac{\epsilon}{2} + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^n_m(t) - X^n(t)|^4 \right]^{\frac{1}{2}} \mathbb{E} \left[ 1_{\{\|B\|_\alpha \geq M_0\}} \right] \leq \epsilon,
\]
where the last inequality is deduced from (4.11) and (4.13). Therefore, (4.10) holds true. From (1.10), it is obvious that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |K^n_m(\omega) - K^n(\omega)|^2 \right] \rightarrow 0, \quad \text{as } m \rightarrow \infty.
\]
We end the proof. $\square$

Proof of Theorem 3.2. We define pathwisely a couple $(X, K)$ by the solution of the deterministic problem (4.6). Then, we apply Lemma 4.5 to prove that $X$ and $K$ belong to $M^2_G([0, T]; \mathbb{R}^d)$. Therefore, $(X, K)$ is a couple satisfying Definition 3.1. The uniqueness of the solution is inherited from the pathwise uniqueness. $\square$

Instead of the $G$-Brownian motion, if we consider a $G$-Itô process as
\[
Y_t = \int_0^t \alpha_s ds + \int_0^t \eta_s^i d(B^i_t, B^j_t)_s + \int_0^t \beta_s^i dB^j_s, \quad 0 \leq t \leq T,
\] (4.14)
where $\alpha, \eta^{ij}, \beta^j : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, $i, j = 1, 2, \ldots, d$, are bounded functions in $M^2_G([0, T]; \mathbb{R}^d)$, then a similar result holds due to the fact that for any $p \geq 2$,

$$E \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] \leq C_p,$$

which can be easily obtained by the BDG type inequality.

**Corollary 4.8.** Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Then there exists a couple $(X, K) \in (M^2_G([0, T]; \mathbb{R}^d) \times M^2_G([0, T]; \mathbb{R}^d))$ which solves the Skorohod problem

$$X_t = x_0 + Y_t + K_t, \quad 0 \leq t \leq T,$$

whenever $x_0 \in \overline{D}$ and $Y$ is defined by (4.14). Moreover, if the above problem admits two solutions $(X, K)$ and $(X', K')$, then there exists a polar set $A$, such that for all $\omega \in A^c$,

$$X(\omega) = X'(\omega) \quad \text{and} \quad K(\omega) = K'(\omega), \quad 0 \leq t \leq T.$$

**Remark 4.9.** Indeed, thanks to Proposition 4.7, we could have a stronger convergence instead of (4.10), that is, for any $p \geq 2$,

$$E \left[ \sup_{0 \leq t \leq T} |X_t^m(\omega) - X_t(\omega)|^p \right] \rightarrow 0, \quad \text{as} \ m \rightarrow \infty.$$

Thus, the couple of solution $(X, K)$ in both Theorem 3.2 and Corollary 4.8 could be found in $(M^p_G([0, T]; \mathbb{R}^d) \times M^p_G([0, T]; \mathbb{R}^d))$.

### 4.3 The existence and uniqueness for the RGSDE

Without loss of generality, we consider in this subsection the following equation instead of (3.2),

$$X_t = x_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB_s + K_t, \quad 0 \leq t \leq T, \quad \text{q.s.}$$

(4.16)

However, all result here holds for the more general case (3.2) due to the boundedness of the density of the process $\langle B, B \rangle$ (see §III-4 in [20]).

If the coefficients $f$ and $g$ satisfy Assumptions (H1) and (H2) and (4.16) admits a solution couple $(X, K)$, then $(X, K)$ can be regarded as the solution couple of the Skorohod problem (4.15) in the domain $D$ for

$$Y_t = \int_0^t f(s, X_s) \, ds + \int_0^t g^{ij}(s, X_s) \, dB_s^i, \quad 0 \leq t \leq T.$$

Then, it is straightforward that for any $p \geq 2$ there exists a constant $C_p' > 0$ such that

$$E \left[ \sup_{0 \leq t \leq T} |X_t|^p \right] + E \left[ \left( |K_T^0| \right)^p \right] \leq C_p'.$
Proposition 4.10. Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A), (B) and (C). For $i = 1, 2$, the couple $(\tilde{X}^i, K^i)$ are solutions of the following Skorohod problems

$$\tilde{X}^i_t = x_0 + \int_0^t f^i (s, \tilde{X}^i_s) \, ds + \int_0^t g^i (s, \tilde{X}^i_s) \, dB_s + K^i_t, \quad i = 1, 2,$$

$$|K^i_t| = \int_0^t 1_{\{\tilde{X}^i_t \notin \partial D\}} \, d|K^i_s| \quad \text{and} \quad K^i_t = \int_0^t n^i_s \, d|K^i_s| \quad \text{with} \quad n^i_s \in N_{\tilde{X}^i_t},$$

where the coefficients $f^i$ and $g^i$ satisfy Assumptions (H1) and (H2). Then, there exists a constant $C > 0$ that depends on $\Gamma$, $d$, $\delta'$, $L_\Psi$ and $L_0$,

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\tilde{X}^1_s - \tilde{X}^2_s|^4 \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq t} |K^1_s - K^2_s|^4 \right] \leq C \int_0^t \left( \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X^1_u - X^2_u|^4 \right] + \mathbb{E} \left[ \sup_{0 \leq u \leq s} |\tilde{f}_u|^4 \right] + \mathbb{E} \left[ \sup_{0 \leq u \leq s} |\tilde{g}_u|^4 \right] \right) \, ds,$$

(4.17)

(4.18)

where $\tilde{f}_s := f^1 (s, X^2_s) - f^2 (s, X^2_s)$ and $\tilde{g}_s := g^1 (s, X^2_s) - g^2 (s, X^2_s)$.

Proof. The proof is similar to the one of Lemma 3.1 in [14], so we only display the key steps for the convenience of the readers. First, we have

$$|\tilde{X}^1_t - \tilde{X}^2_t|^2 = 2 \int_0^t \langle \tilde{X}^1_s - \tilde{X}^2_s, f^1 (s, X^1_s) - f^1 (s, X^2_s) + \tilde{f}_s \rangle \, ds$$

$$+ 2 \int_0^t \langle \tilde{X}^1_s - \tilde{X}^2_s, (g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s) \rangle \, dB_s$$

$$+ 2 \int_0^t \langle \tilde{X}^1_s - \tilde{X}^2_s, n^1_s \rangle \, d|K^1_s| - 2 \int_0^t \langle \tilde{X}^1_s - \tilde{X}^2_s, n^2_s \rangle \, d|K^2_s|$$

$$+ \int_0^t \text{tr} \left[ (g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s) \, d \langle B, B \rangle_s \right] (g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s) \right],$$

and

$$\Psi (\tilde{X}^i_s) = \Psi (x_0) + \int_0^t \langle \nabla \Psi (\tilde{X}^i_s), f^i (s, X^i_s) \rangle \, ds + \int_0^t \langle \nabla \Psi (\tilde{X}^i_s), g^i (s, X^i_s) \rangle \, dB_s$$

$$+ \int_0^t \langle \nabla \Psi (\tilde{X}^i_s), n^i_s \rangle \, d|K^i_s| + \frac{1}{2} \int_0^t \text{tr} \left[ H(\Psi (\tilde{X}^i_s)) g^i (s, X^i_s) \, d \langle B, B \rangle_s \right] (g^i (s, X^i_s)) \right].$$
Then,
\[
\exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_t \right) + \Psi \left( \tilde{X}^2_t \right) \right) \right\} \times |\tilde{X}^1_t - \tilde{X}^2_t|^2
\]

\[
= 2 \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\}
\times \left\{ \left\langle \tilde{X}^1_s - \tilde{X}^2_s, f^1 (s, X^1_s) - f^1 (s, X^2_s) + \tilde{f}_s \right\rangle ds
\right.
\]

\[
+ \left( \tilde{X}^1_s - \tilde{X}^2_s \right) \left( g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s \right) dB_s
\]

\[
+ \left\langle \tilde{X}^1_s - \tilde{X}^2_s, n^1_s \right\rangle d |K^1|_s - \left\langle \tilde{X}^1_s - \tilde{X}^2_s, n^2_s \right\rangle d |K^2|_s
\right\}
\]

\[
+ \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\}
\times \text{tr} \left[ \left( g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s \right) dB_s, \right.
\]

\[
-\frac{1}{\delta^t} \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\} \times |\tilde{X}^1_s - \tilde{X}^2_s|^2
\]

\[
\times \left\{ \left\langle \nabla \Psi \left( \tilde{X}^1_s \right), f^1 (s, X^1_s) \right\rangle + \left\langle \nabla \Psi \left( \tilde{X}^2_s \right), f^2 (s, X^2_s) \right\rangle \right\} ds
\]

\[
+ \left( \nabla \Psi \left( \tilde{X}^1_s \right) \right) g^1 (s, X^1_s) + \left( \nabla \Psi \left( \tilde{X}^2_s \right) \right) g^2 (s, X^2_s) dB_s
\]

\[
+ \left\langle \nabla \Psi \left( \tilde{X}^1_s \right), n^1_s \right\rangle d |K^1|_s + \left\langle \nabla \Psi \left( \tilde{X}^2_s \right), n^2_s \right\rangle d |K^2|_s
\right\}
\]

\[
-\frac{1}{2\delta^t} \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\} \times |\tilde{X}^1_s - \tilde{X}^2_s|^2
\]

\[
\times \text{tr} \left[ \mathbf{H} \left( \Psi \left( \tilde{X}^1_s \right) \right) g^1 (s, X^1_s) dB_s, \right.
\]

\[
+ \mathbf{H} \left( \Psi \left( \tilde{X}^2_s \right) \right) g^2 (s, X^2_s) dB_s \right\]
\]

\[
+ \frac{1}{2} \left( \frac{1}{\delta^t} \right)^2 \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\} \times |\tilde{X}^1_s - \tilde{X}^2_s|^2
\]

\[
\times \left( \nabla \Psi \left( \tilde{X}^1_s \right) \right) g^1 (s, X^1_s) + \left( \nabla \Psi \left( \tilde{X}^2_s \right) \right) g^2 (s, X^2_s)
\]

\[
\times dB_s \right( \nabla \Psi \left( \tilde{X}^1_s \right) \right)^t (g^1 (s, X^1_s)) + \nabla \Psi \left( \tilde{X}^2_s \right)^t (g^2 (s, X^2_s)) \right) \]

\[
-\frac{2}{\delta^t} \int_0^t \exp \left\{ -\frac{1}{\delta^t} \left( \Psi \left( \tilde{X}^1_s \right) + \Psi \left( \tilde{X}^2_s \right) \right) \right\}
\times \left( \tilde{X}^1_s - \tilde{X}^2_s \right) \left( g^1 (s, X^1_s) - g^1 (s, X^2_s) + \tilde{g}_s \right)
\]

\[
\times dB_s \right( \nabla \Psi \left( \tilde{X}^1_s \right)^t (g^1 (s, X^1_s)) + \nabla \Psi \left( \tilde{X}^2_s \right)^t (g^2 (s, X^2_s)) \right) .
\]

Thanks to Condition (C), we know that the integrals with respect to $d|K|$ are negative. Since the set $\Gamma$, the function $\Psi$ and its derivatives, the functions $f^i$ and $g^i$, $i = 1, 2$, are bounded,
we have
\begin{align*}
&\exp\left\{-\frac{2M}{\delta'}\right\} \times \left|X^1_s - \tilde{X}^2_s\right|^2 \\
&\leq 2 \int_0^t \exp\left\{-\frac{1}{\delta'} \left(\Psi(X^1_s) + \Psi(\tilde{X}^2_s)\right)\right\} \left(\tilde{X}^1_s - \tilde{X}^2_s\right) \left(g^1(s, X^1_s) - g^1(s, X^2_s) + \hat{g}_s\right) dB_s \\
&\quad - \frac{1}{\delta'} \int_0^t \exp\left\{-\frac{1}{\delta'} \left(\Psi(X^1_s) + \Psi(\tilde{X}^2_s)\right)\right\} \times \left|\tilde{X}^1_s - \tilde{X}^2_s\right|^2 \\
&\quad \times \left(t \left(\nabla \Psi(\tilde{X}^1_s)\right) g^1(s, X^1_s) + t \left(\nabla \Psi(\tilde{X}^2_s)\right) g^2(s, X^2_s)\right) dB_s \\
&\quad + C \int_0^t \left(|\tilde{X}^1_s - \tilde{X}^2_s|^2 + |X^1_s - X^2_s|^2 + |\hat{f}_s|^2 + |\hat{g}_s|^2\right) ds,
\end{align*}
where $C > 0$ is a constant that depends on $\Gamma, d, \delta', L_q$ and $L_0$, which may vary from line to line. We square both sides and apply the BDG type inequality to obtain
\begin{align*}
&E\left[\sup_{0 \leq s \leq t} \left|X^1_s - \tilde{X}^2_s\right|^4\right] \\
&\leq C \int_0^t \left(E\left[\sup_{0 \leq u \leq s} \left|X^1_u - \tilde{X}^2_u\right|^4\right]\right) + E\left[\sup_{0 \leq u \leq s} |X^1_u - X^2_u|^4\right] + E\left[\sup_{0 \leq u \leq s} |\hat{f}_u|^4\right] + E\left[\sup_{0 \leq u \leq s} |\hat{g}_u|^4\right] ds.
\end{align*}

The desired result follows from the Gronwall inequality. \hfill \Box

**Proof of Theorem 3.5.** The uniqueness of solution is straightforward by Proposition 4.10. We now turn to prove the existence. Indeed, by Corollary 4.8 one can construct a sequence $\{(X^m, K^m)\}_{m \in \mathbb{N}^*}$ by the Picard type iteration starting with $X^0 \equiv K^0 = 0$,

\begin{align*}
Y^m_t + 1 = \int_0^t f(s, X^m_s) ds + \int_0^t g(s, X^m_s) dB_s, \quad X^m_t = x_0 + Y^m_t + K^m_t,
\end{align*}

\begin{align*}
|K^m|_t = \int_0^t 1_{\{X^m \in \partial D\}} |\partial^m|_t \quad \text{and} \quad K^m_t = \int_0^t \mathbf{n}^m dB^m \quad \text{such that} \quad \mathbf{n}^m \in \mathcal{N}_{X^m}.
\end{align*}

Thanks to the a priori estimate (4.18), we can proceed a similar argument as the proof of Theorem 2.1 and 4.1 in [5] to find a couple of processes $(X, K)$ such that

\begin{align*}
&E\left[\sup_{0 \leq t \leq T} |X^m_t - X_t|^2\right] + E\left[\sup_{0 \leq t \leq T} |K^m_t - K_t|^2\right] \to 0,
\end{align*}

\begin{align*}
&E\left[\sup_{0 \leq t \leq T} |Y^m_t - X_t|^2\right] \to 0, \quad \text{as} \quad m \to \infty,
\end{align*}

and such that there exists a polar set $A$ and subsequence $\{(X^{m_k}, K^{m_k})\}_{k \in \mathbb{N}^*}$, such that for each $\omega \in A^c$, $(X^{m_k}(\omega), K^{m_k}(\omega))$ is the solution couple for the deterministic Skorohod problem with $(Y^{m_k}(\omega), D)$, and

\begin{align*}
&\sup_{0 \leq t \leq T} |X^{m_k}(\omega) - X_t(\omega)| + \sup_{0 \leq t \leq T} |K^{m_k}(\omega) - K_t(\omega)| \to 0,
\end{align*}

\begin{align*}
&\sup_{0 \leq t \leq T} Y^{m_k}(\omega) - \left(\int_0^t f(s, X_s) ds - \int_0^t g(s, X_s) dB_s\right)(\omega) \to 0, \quad \text{as} \quad k \to \infty.
\end{align*}
It is clear that \((X, K) \in \left( M^2_G([0,T]; \mathbb{R}^d) \times M^2_G([0,T]; \mathbb{R}^d) \right)\). Besides, for each \(\omega \in A^c\), \((X(\omega), K(\omega))\) verified (i) (ii) (iii) of Definition 3.1 which can be proved by the last step of the proof to Theorem 4.1 in [21]. We complete the proof.

\[\square\]

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