Integrable nonlinear field equations and loop algebra structures

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Abstract

We apply the (direct and inverse) prolongation method to a couple of non-linear Schrödinger equations. These are taken as a laboratory field model for analyzing the existence of a connection between the integrability property and loop algebras. Exploiting a realization of the Kac-Moody type of the incomplete prolongation algebra associated with the system under consideration, we develop a procedure with allows us to generate a new class of integrable nonlinear field equations containing the original ones as a special case.

PACS number: 11.10.L

1 Introduction

The existence of Kac-Moody algebras endowed with a loop structure is one of the most outstanding properties shown by integrable nonlinear field equations (NFE’s). For integrable NFE’s in 1+1 dimensions, algebras of this type emerge in a natural way within the ”direct” Estabrook-Wahlquist prolongation method [1]. Such algebras, which can be regarded as infinite-dimensional realizations of incomplete prolongation Lie algebras (in the sense that not all of the commutators are known), are derived via the introduction of an arbitrary number of prolongation forms having new dependent variables (called pseudopotentials), by assuming the algebraic equivalence between the generators of the prolonged ideals and their exterior differentials. Conversely, in 2+1 dimensions loop algebras were found applying the symmetry reduction procedure (a group-theoretical analysis [2]), rather than the prolongation technique, to
some integrable NFE’s of physical interest, such as the Kadomtsev–Petviashvili (KP) and the Davey-Stewartson (DS) equation, the 3WRI system, and the Ishimori model. Indeed, the prolongation theory in 2+1 independent variables (and, generally, in higher dimensions) is a difficult task and so far only a very few papers have been written on this subject.

The authors of Refs. [3-6] derived essentially the incomplete prolongation Lie algebra associated with certain integrable NFE’s in 2+1 dimensions. In some cases a quotient algebra is used to find the linear eigenvalue problem related to the equations under investigation. Anyway, the techniques devised are not algorithmic and do not seem to be easily implementable. Many problems remain open, such as that of building up a Kac Moody realization with a loop structure of the incomplete prolongation Lie algebras. At this stage we observe that the symmetry algebra (namely, the algebra arising from the symmetry reduction procedure) possessed by an integrable NFE in 2+1 dimensions (KP, DS, ...) is of the Kac-Moody type with a loop structure. Therefore, it seems reasonable to ask whether a link exists between the symmetry approach, which is based on the Lie group theory, and the prolongation method, which is formulated in the language of differential forms. This is an important methodological aspect which deserves a wide investigation. Here we remark only that any approach to this problem should not ignore the contribution by Harrison and Estabrook, where the fundamental concepts of Cartan’s theory of systems of partial differential equations are exploited for obtaining the generators of their invariance groups (isogroups).

Another important aspect inherent to the integrability properties of NFE’s, is the possibility of applying the so-called ”inverse” prolongation method. This consists in starting from a given incomplete Lie algebra to generate the class of NFE’s whose prolongation structure it is. Interesting results in this direction have been obtained in Refs. [10, 11, 12, 13].

On the basis of our present knowledge, an algebraic theory of integrability implies at least the execution of two research programs, i.e. i) the program of founding on a sound ground the prolongation theory in more than 1+1 dimensions and its possible
connection with the symmetry approach; ii) the program of constructing and using (via the inverse prolongation method) Kac-Moody realizations of the loop type of the incomplete prolongation Lie algebras. It is convenient to subordinate the program i) to the second one, ii). This choice is motivated by the fact that, quite recently, within the inverse prolongation scheme a new procedure has been outlined \cite{13} for generating a class of NFE’s starting from a loop algebra realization of a certain incomplete prolongation Lie algebra. This procedure is based on an ansatz which may be generalized, in principle, to the case of more space variables.

From the above considerations it turns out that a close correspondence can be established between loop algebras and integrability. However, in order to achieve a deeper understanding of this correspondence, we have to deal with new case studies, possibly resorting to different approaches. Following this idea, in this Letter we study a couple of nonlinear Schrödinger (NLS) equations in the direct and inverse prolongation framework. This system describes the propagation of waves in nonlinear birefringent optical fibers.

The direct prolongation method provides an incomplete Lie algebra which is used to find the linear eigenvalue problem associated with the system of NLS equations. Furthermore, we devise an inverse prolongation technique which is based on a loop algebra realization of the incomplete prolongation Lie algebra, for obtaining a whole family of NLS pair of equations containing the original ones.

2 The incomplete prolongation Lie algebra

Let us consider the pair of NLS equations \cite{14}:

\begin{align}
 i u_x + u_{tt} + \kappa v + (\alpha |u|^2 + \beta |v|^2) u &= 0, \\
 i v_x + v_{tt} + \kappa u + (\alpha |v|^2 + \beta |u|^2) v &= 0,
\end{align}

(2.1a) (2.1b)

where subscripts stand for partial derivatives, $u = u(x,t)$ and $v = v(x,t)$ are the circularly polarized components of the optical field, $x$ and $t$ denote the (normalized) longitudinal coordinate of the fiber and the time variable, respectively, $\kappa$ is the bire-
fringent parameter and the coefficients (\(\alpha\) and \(\beta\) are responsible for the nonlinear properties of the fiber).

First, let us assume that \(\kappa, \alpha\) and \(\beta\) are real parameters, with \(\alpha = \beta\). Then, Eq.(2.1) can be written in the form

\[
i \vec{u}_x + \vec{u}_{tt} + \kappa \sigma \vec{u} + \alpha |\vec{u}|^2 \vec{u} = 0, \tag{2.2a}
\]

\[
-i \vec{u}_x^* + \vec{u}_{tt}^* + \kappa \sigma \vec{u}^* + \alpha |\vec{u}|^2 \vec{u}^* = 0, \tag{2.2b}
\]

where

\[
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \vec{u}^* = \begin{pmatrix} u^* \\ v^* \end{pmatrix}, \quad |\vec{u}|^2 = \vec{u} \cdot \vec{u}^*
\]

(the dot means the scalar product).

The EW prolongation for Eqs. (2.2) can be formulated by introducing the differential ideal defined by the set of (vector) 2-forms:

\[
\vec{\theta}^1 = d\vec{u} \wedge dx - \vec{u}_t \wedge dx, \tag{2.3a}
\]

\[
\vec{\theta}^2 = d\vec{u}^* \wedge dx - \vec{u}^*_t \wedge dx, \tag{2.3b}
\]

\[
\vec{\theta}^3 = -i \, d\vec{u} \wedge dt + d\vec{u}_t \wedge dx + \kappa \sigma \vec{u} dt \wedge dx + \alpha |\vec{u}|^2 \vec{u} dt \wedge dx, \tag{2.3c}
\]

\[
\vec{\theta}^4 = i \, d\vec{u}^* \wedge dt + d\vec{u}^*_t \wedge dx + \kappa \sigma \vec{u}^* dt \wedge dx + \alpha |\vec{u}|^2 \vec{u}^* dt \wedge dx. \tag{2.3d}
\]

One can verify that the ideal (2.3) is closed.

Now we look for the prolongation 1-forms:

\[
\omega^k = dy^k + F^k(\vec{u}, \vec{u}^*, \vec{u}_t, \vec{u}^*_t; y)dx + G^k(\vec{u}, \vec{u}^*; y)dt, \tag{2.4}
\]

where \(y = \{y^m\}\) is the pseudopotential, \((k, m = 1, 2, \ldots, N)\) (\(N\) arbitrary), and \(F^k\) and \(G^k\) are functions to be determined. (In the following we shall drop the index \(k\), for simplicity). The requirement that the forms \(\vec{\theta}^j\) \((j = 1, \ldots, 4)\) and \(\omega^k\) comprise a differential ideal \(\mathcal{I}(\theta^j, \omega^k)\), i.e. \(d\omega^k \in \mathcal{I}(\theta^j, \omega^k)\), implies the set of constraints

\[
i F_{\vec{u}_t} + G_{\vec{u}} = 0, \tag{2.5a}
\]

\[
i F_{\vec{u}^*_t} - G_{\vec{u}^*} = 0, \tag{2.5b}
\]
\[ F_u \cdot \tilde{u}_t + F_{u^*} \cdot \tilde{u}_t^* - \kappa F_{\tilde{u}_t} \cdot \sigma \tilde{u} - \alpha |\tilde{u}|^2 F_{\tilde{u}_t} \cdot \tilde{u} - \kappa F_{\tilde{u}_t^*} \cdot \sigma \tilde{u}^* \]
\[ - \alpha |\tilde{u}|^2 F_{\tilde{u}_t^*} \cdot \tilde{u}^* + [F, G] = 0, \]  
\[ \text{(2.5c)} \]

where

\[ F_u = \text{grad}_u F \equiv (F_u, F_v), \]

the symbol \([F, G]^k = F_j \frac{\partial G^k}{\partial y_j} - G_j \frac{\partial F^k}{\partial y_j},\) and \(F_u = \frac{\partial F}{\partial u};\) and so on.

From (2.5) we obtain

\[ -iF = \bar{a} \cdot \tilde{u}_t + \bar{u}_t^* \cdot M\tilde{u}_t - \bar{b} \cdot \tilde{u}_t^* - \tilde{u}_t^* \cdot M\bar{u} \\
- [\bar{a} \cdot \bar{u}, \bar{b} \cdot \bar{u}^*] - [\bar{a} \cdot \bar{u}, c] + [\bar{b} \cdot \bar{u}^*, c] - \text{id}, \]
\[ \text{(2.6a)} \]

\[ G = \bar{a} \cdot \bar{u} + \bar{b} \cdot \bar{u}^* + \bar{u}^* \cdot M\bar{u} + c, \]
\[ \text{(2.6b)} \]

and the commutation (Lie brackets) relations

\[ [a_j, a_k] = 0, \quad [b_j, b_k] = 0, \quad [a_j, M_{kl}] = 0, \]
\[ [b_j, M_{kl}] = 0, \quad [M_{jk}, M_{lm}] = 0, \quad [c, M_{jk}] = 0, \]
\[ [c, d] = 0, \quad [[a_j, c], a_k] = 0, \quad [[b_j, c], b_k] = 0, \]
\[ i\kappa \sigma \bar{a} + i[[\bar{a}, c], c] - [d, \bar{a}] = 0, \]
\[ i\kappa \sigma \bar{b} + i[[\bar{b}, c], c] + [d, \bar{b}] = 0, \]
\[ [[b_k, a_l], a_j] = \frac{1}{2} \alpha (\delta_{kl} a_j + \delta_{kj} a_l), \]
\[ [[b_k, a_l], b_j] = -\frac{1}{2} \alpha (\delta_{kl} b_j + \delta_{lj} b_k), \]
\[ -i\kappa \sigma (M\sigma - \sigma M)_{jk} + 2i[[b_j, a_k], c] + [d, M_{jk}] = 0. \]
\[ \text{(2.7)} \]

In (2.6) and (2.7), \(M\) is a \(2 \times 2\) matrix whose elements \(M_{jk},\) together with the functions \(c, d\) and the vectors \(\bar{a} \equiv (a_1, a_2), \quad \bar{b} \equiv (b_1, b_2),\) depend arbitrarily on the pseudopotential only. The indices \(j, k, l, m\) take the values 1,2. Furthermore, one has

\[ M^\dagger = M, \quad \bar{u}^* \cdot M\bar{u} = \bar{u} \cdot M^T \bar{u}^* \]
\[ \text{(2.8)} \]

where \(T\) means transposition.

The prolongation algebra (2.7) turns out to be incomplete. However, as we shall see in Sections 3 and 4, some important properties of Eqs. (2.2) arise from an infinite-dimensional realization of the Kac-Moody type with a loop structure.
3 Quotient algebra and spectral problem

For practical purposes, say to find the spectral problem associated with Eqs.(2.2) and, possibly, Bäcklund transformations, the algebra (2.7) can be closed to provide finite-dimensional quotient algebra. This can be done assuming, for example, that

\[ M\sigma = \sigma M, \quad [d, M_{jk}] = 0, \quad (3.1) \]

the quantities \( a_1, a_2, b_1, b_2, [a_1, b_1] \equiv t_{11}, [a_1, b_2] \equiv t_{12}, [a_2, b_1] \equiv t_{21}, [a_2, b_2] \equiv t_{22} \) are independent, and \( c \) and \( d \) are a linear combination of the preceding ones. In doing so, we get (see (2.8) and (3.1))

\[ M = \begin{pmatrix} q & r \\ r & q \end{pmatrix}, \]

\( q \) and \( r \) being two real functions of the pseudopotential;

\[ c = \lambda(t_{11} + t_{22}), \quad (3.3) \]

\[ d = -3i\lambda c - ik \sum_{j,k=1,2} (1 - \delta_{jk})t_{jk}, \quad (3.4) \]

and

\[ [a_j, a_k] = 0, \quad [b_j, b_k] = 0, \quad [a_j, b_k] = t_{jk}, \quad (3.5) \]

\[ [a_k, t_{lm}] = a_k\delta_{lm} + a_l\delta_{km}, \quad [b_m, t_{lk}] = -(b_k\delta_{lm} + b_m\delta_{lk}), \]

\[ [t_{jk}, t_{lm}] = t_{lk}\delta_{jm} - t_{jm}\delta_{kl} \quad (j, k, l, m = 1, 2). \]

In (3.3), (3.4) and (3.5) we have performed the changes \( \vec{a} \rightarrow \frac{1}{\alpha}\vec{a}, \quad \lambda \rightarrow \frac{2}{\alpha}\lambda \), where \( \lambda \) is a free parameter. The commutation relations (3.5) define an sl(3,\( C \)) quotient algebra of the incomplete prolongation algebra (2.7). We notice that exploiting (3.5), (2.4) and (2.6), the spectral problem related to Eqs.(2.2) is furnished.

Another interesting realization of (2.7) is given by an infinite-dimensional Lie algebra of the Kac-Moody type. This realization can be built up by putting \( M = 0 \) in (2.7) and resorting to the correspondence

\[ a_m \rightarrow T^{(n)}_{0m}, \quad b_m \rightarrow T^{(-n)}_{m0}, \]

\[ ... \]
\[ t_{lm} \rightarrow i T_{lm}^{[n(\delta_{lm} - \delta_{mm})]}, \quad c \rightarrow \lambda \left( T_{11}^{(l)} + T_{22}^{(l)} \right), \]
\[ d \rightarrow -3i \lambda^2 \left( T_{11}^{(2l)} + T_{22}^{(2l)} \right) - i \kappa \left( T_{12}^{(0)} + T_{21}^{(0)} \right), \]  
(3.6)

where \( n, l \in \mathbb{Z} \).

The vector fields \( T_{lm}^{(n)} \) satisfy the commutation relations
\[
\left[ T_{lm}^{(n)}, T_{l'm'}^{(n')} \right] = i \epsilon_{lml'm'k} T_{k}^{(n+n')} \]  
(3.7)

where the structure constants are expressed by
\[
\epsilon_{lml'm'k} = \delta_{lm} \delta_{l'0} \delta_{k0} \delta_{m'0} \delta_{j0} \delta_{l'k} + \delta_{l'm'} \delta_{l0} \delta_{m0} \delta_{j0} \delta_{l'k} \delta_{l0} + \delta_{m'l'} \delta_{k0} \delta_{m0} \delta_{j0} \delta_{l0} \delta_{k0} + \delta_{m'0} \delta_{l0} \delta_{j0} \delta_{l0} \delta_{k0} + \delta_{m0} \delta_{l0} \delta_{j0} \delta_{l0} \delta_{k0} + \delta_{l0} \delta_{j0} \delta_{l0} \delta_{k0} \delta_{m0} \delta_{l0} \delta_{j0} \delta_{l0} \delta_{k0} \]  
(3.8)

\((m, l, m', l', j, k = 0, 1, 2)\). We observe that \( T_{00}^{(n)} = 0, \forall n \in \mathbb{Z} \).

If, in particular, \( l, m, l', m' \neq 0, \) Eq.(3.7) becomes
\[
\left[ T_{lm}^{(n)}, T_{l'm'}^{(n')} \right] = i \delta_{m'l'} T_{l'm}^{(n+n')} - i \delta_{m'l} T_{lm}^{(n+n')} . \]  
(3.9)

It is noteworthy that (3.7) admits a representation in terms of the prolongation variables, namely
\[
T_{lm}^{(r)} = -i \sum_{k=-\infty}^{+\infty} y_{l}^{(r+k)} \frac{\partial}{\partial y_{m}^{(k)}} + i \delta_{lm} \sum_{k=-\infty}^{+\infty} y_{0}^{(r+k)} \frac{\partial}{\partial y_{0}^{(k)}}, \]  
(3.10)

\( r \in \mathbb{Z}, \) where the pseudopotential \( y \) is expressed through the independent infinite-dimensional vectors \( y_0, y_1 \) and \( y_2 \). \( (y_0^{(k)}, y_1^{(k)}, y_2^{(k)}) \) denote the \( k \)-component of \( y_0, y_1 \) and \( y_2 \), respectively.

Now we are ready to derive the spectral problem for Eqs.(2.2) using the Kac-Moody loop algebra (3.7). To this aim, from (2.4) and (2.6) (with \( M = 0 \)) we obtain
\[
\tilde{\Psi}_{x}(\lambda) = L_{1} \tilde{\Psi}(\lambda), \]  
(3.11a)
\[
\tilde{\Psi}_{t}(\lambda) = L_{2} \tilde{\Psi}(\lambda), \]  
(3.11b)

where \( \tilde{\Psi} \) is a 3-component vector such that
\[
\tilde{\Psi}(\lambda) \equiv \tilde{\Psi}(x, t; \lambda) = \sum_{k=-\infty}^{+\infty} \lambda^{k} \tilde{y}^{(k)}, \]  
(3.12)
\( \vec{y}^{(k)} = (y_0^{(k)}, y_1^{(k)}, y_2^{(k)})^T \), and \( L_1 \) and \( L_2 \) are the \( 3 \times 3 \) traceless matrices

\[
L_1 = \begin{pmatrix}
-i\left(\frac{1}{2}\alpha |u|^2 + \frac{1}{2}\alpha |v|^2 + 6\lambda^2\right) & u_t^* + 3\lambda u^* & v_t^* + 3\lambda v^* \\
\frac{1}{2}\alpha (-u_t + 3\lambda u) & i\left(\frac{1}{2}\alpha |u|^2 + 3\lambda^2\right) & i\left(\frac{1}{2}\alpha v^* u + \kappa\right) \\
\frac{1}{2}\alpha (-v_t + 3\lambda v) & i\left(\frac{1}{2}\alpha v^* + \kappa\right) & i\left(\frac{1}{2}\alpha |v|^2 + 3\lambda^2\right)
\end{pmatrix},
\]

\[ L_2 = \begin{pmatrix}
2\lambda & iu^* & iv^* \\
i\frac{1}{2}\alpha u & -\lambda & 0 \\
i\frac{1}{2}\alpha v & 0 & -\lambda
\end{pmatrix}. \tag{3.13}
\]

The compatibility condition for Eqs.(3.11), i.e. \( L_1 t - L_2 x + [L_1, L_2] = 0 \), reproduces just the equations (2.2).

4 Incomplete Lie algebras vs. nonlinear field equations

The prolongation of a nonlinear field equation can be interpreted as a Cartan-Ehresmann connection. In this way an incomplete Lie algebra of vector fields can be related to a differential ideal \([10]\). Here we start from the incomplete Lie algebra (2.7) to yield the differential ideal associated with Eqs.(2.2). This can be carried out specifying the form of the connection. In doing so, let us assume that the connection

\[
\omega^k = dy^k + A_j^k \eta^j \tag{4.1}
\]

exists such that

\[
d\omega^k = A_j^k d\eta^j - \frac{1}{2} [A_i, A_j]^k \eta^i \wedge \eta^j = 0 \mod \omega^k, \tag{4.2}
\]

where \( \eta^j \) are 1-forms, \( A_j \) (j=1,2,..., 18) are defined by

\[
A_1 = a_1, \ A_2 = a_2, \ A_3 = b_1, \ A_4 = b_2, \ A_5 = c \\
A_6 = d, \ A_7 = t_{11}, \ A_8 = t_{12}, \ A_9 = [a_1, c], \ A_{10} = t_{21}, \\
A_{11} = t_{22}, \ A_{12} = [a_2, c], \ A_{13} = [b_1, c], \ A_{14} = [b_2, c], \ A_{15} = M_{12} \\
A_{16} = M_{21}, \ A_{17} = M_{11}, \ A_{18} = M_{22},
\]
involved in the incomplete Lie algebra (2.7), and \( \omega^k \) means that all the exterior products between \( \omega^k \) and 1-forms of the Grassmann algebra have not been considered.

By expliciting (4.2) and taking into account (2.7), we get the set of exterior differential equations

\[
\begin{align*}
\text{d}\eta^1 - \kappa \eta^5 \wedge \eta^{12} - \alpha \eta^1 \wedge \eta^7 - \frac{1}{2} \alpha (\eta^1 \wedge \eta^{11} + \eta^2 \wedge \eta^8) &= 0, \quad (4.3a) \\
\text{d}\eta^2 - \kappa \eta^5 \wedge \eta^9 - \alpha \eta^2 \wedge \eta^{11} - \frac{1}{2} \alpha (\eta^1 \wedge \eta^{10} + \eta^2 \wedge \eta^7) &= 0, \quad (4.3b) \\
\text{d}\eta^3 - \kappa \eta^5 \wedge \eta^{14} + \alpha \eta^3 \wedge \eta^7 + \frac{1}{2} \alpha (\eta^3 \wedge \eta^{11} + \eta^4 \wedge \eta^{10}) &= 0, \quad (4.3c) \\
\text{d}\eta^4 - \kappa \eta^5 \wedge \eta^{13} + \alpha \eta^4 \wedge \eta^{11} + \frac{1}{2} \alpha (\eta^3 \wedge \eta^8 + \eta^4 \wedge \eta^7) &= 0, \quad (4.3d)
\end{align*}
\]

and another set of constraints, which we omit for brevity, whose solutions can be written as

\[
\begin{align*}
\eta^j &= -\alpha^j dx + i\alpha^j dt \quad (j = 1, 2), \\
\eta^k &= -\alpha^k dx - i\alpha^k dt \quad (k = 3, 4), \\
\eta^5 &= dt, \quad \eta^6 = dx, \\
\eta^7 &= -i\alpha^1 \alpha^3 dx, \quad \eta^8 = -i\alpha^1 \alpha^4 dx, \quad \eta^9 = \alpha^1 dx, \\
\eta^{10} &= -i\alpha^2 \alpha^3 dx, \eta^{11} = -i\alpha^2 \alpha^4 dx, \eta^{12} = \alpha^2 dx, \\
\eta^{13} &= \alpha^3 dx, \quad \eta^{14} = \alpha^4 dx,
\end{align*}
\]

where \( \alpha^l \) (\( l = 1, \ldots, 4 \)) are arbitrary 0-forms. At this stage, by choosing \( \alpha^1 = u, \quad \alpha^2 = v, \quad \alpha^3 = u^*, \quad \alpha^4 = v^* \), by virtue of (4.4), Eqs. (4.3) give exactly the original system (2.2).

In the following we solve the inverse prolongation problem for Eqs. (2.2) by adopting a procedure based on the Kac-Moody realization (3.10). Precisely, let us look for the class of NFE’s whose prolongation structure is assumed to be given by the 1-forms

\[
\omega = dy + F(\overline{u}, \overline{u}^*, \overline{u}_t, \overline{u}_t^*; y) dx + G(\overline{u}, \overline{u}^*; y) dt,
\]

where
where \( F \) and \( G \) are defined by

\[
F = -i \sum_{k=1}^{2} \left[ \alpha_k T_{0k}^{(n)} + \beta_k T_{k0}^{(-n)} + \gamma_k T_{0k}^{(n+l)} + \phi_k T_{k0}^{(-n+l)} + \psi_k T_{1k}^{(0)} + \chi_k T_{2k}^{(0)} + \rho T_{kk}^{(2l)} \right], \tag{4.6a}
\]

\[
G = -i \sum_{k=1}^{2} \left[ p_k u_k T_{0k}^{(n)} + q_k u_k^* T_{k0}^{(-n)} + \sigma T_{kk}^{(l)} \right], \tag{4.6b}
\]

\( \alpha_k, \beta_k, \gamma_k, \phi_k, \chi_k \), are functions of \( u = u_1, v = u_2, u^* = u_1^*, v^* = u_2^* \) to be determined in such a way that the operators \( T(\cdot) \) satisfy the Kac-Moody algebra \( (3.7) \) and \( p_k, q_k, \rho, \) and \( \sigma \) are constants.

At this point, by setting \( \omega = 0 \) into Eqs. \( (4.5) \) from the compatibility condition \( \vec{y}^{(i)}_{xt} = \vec{y}^{(i)}_{tx} \) we obtain the constraints

\[
\beta_{1t} - (\psi_1 + \chi_2) q_1 u_1^* = \sum_{j=1}^{2} q_j \psi_j u_j^* + q_1 u_{1x}, \tag{4.7}
\]

\[
\beta_{2t} - (\psi_1 + \chi_2) q_2 u_2^* = \sum_{j=1}^{2} q_j \chi_j u_j^* + q_2 u_{2x}, \tag{4.8}
\]

\[
\alpha_{k,t} + p_k u_k (\psi_1 + \chi_2) + p_1 u_1 \psi_k + p_2 u_2 \chi_k = p_k u_{k,x} \tag{4.9}
\]

and

\[
\alpha_k = \frac{\rho}{3\sigma^2} p_k u_{k,t}, \quad \beta_k = -\frac{\rho}{3\sigma^2} q_k u_{k,t}^*,
\]

\[
\gamma_k = \frac{\rho}{\sigma} p_k u_k, \quad \phi_k = \frac{\rho}{\sigma} q_k u_k^*,
\]

\[
\psi_k = -\frac{\rho}{3\sigma^2} q_1 p_k u_1^* u_k + \eta_k,
\]

\[
\chi_k = -\frac{\rho}{3\sigma^2} q_2 p_k u_2^* u_k + \mu_k, \tag{4.10}
\]

where \( \eta_k \) and \( \mu_k \) are arbitrary constants and \( k = 1, 2 \). Since \( \frac{3\sigma^2}{\rho} \) is an imaginary number, \( p_k q_k \) is a real constant, and \( \eta_2 = -\mu_1^*, \mu_2 = -\mu_2^*, \eta_1 = -\eta_1^* \), without lost of generality we can put \( \rho = 3i\sigma^2, p_1 = p_2 = i, q_1 = q_2 = \frac{i}{\sigma} \epsilon \), where \( \epsilon \) is a real quantity. Then, with the help of \( (4.10) \), Eqs.(4.7)-(4.9) yield

\[
i \ddot{u}_x + \dddot{u}_t + A \ddot{u} + \epsilon |\ddot{u}|^2 \dddot{u} = 0, \tag{4.11a}
\]
\[-i \vec{u}_x + \vec{u}_{tt} + A^T \vec{u} + \epsilon |\vec{u}|^2 \vec{u} = 0, \quad (4.11b)\]

where

\[A = \begin{pmatrix} \Delta_1 & \kappa \\ \kappa^* & \Delta_2 \end{pmatrix}, \quad (4.12)\]

\[\Delta_1 = -i (2\eta_1 + \mu_2) \in \mathcal{R}, \Delta_2 = -i (\eta_1 + 2\mu_2) \in \mathcal{R}, \kappa = -i \mu_1 \in \mathcal{C}, \text{ and } A^T = A^*.\]

We remark that the nonlinear field equations obtained by our method, i.e. Eqs.(4.11), are more general than Eqs.(2.2). These can be found for \(\Delta_1 = \Delta_2 = 0\) and \(\kappa = \kappa^*\). Furthermore, we point out that our inverse prolongation technique, based on the ansatz (4.5) which exploits the Kac-Moody realization (3.10), is a powerful tool for generating new integrable nonlinear field equations of physical significance. In fact, in the present case, by choosing \(\Delta_1 = -\Delta_2 = \Delta \neq 0\) and \(\kappa = \kappa^*\), Eqs.(4.11) become the coupled equations related to twisted birefringent optical fibers [15].

5 Conclusions

The results achieved in this Letter show that the Estabrook-Whalquist prolongation method presents several advantages in the study of nonlinear field equations. The method works out both in the "direct" and in the "inverse" direction. Our investigation confirms the existence of a deep connection between prolongation Lie algebras and the integrability property of nonlinear field equations. In particular, concerning the nonlinear Schrödinger equations (2.2), we point out that the related incomplete Lie algebra (2.7) admits an infinite-dimensional realization of the Kac-Moody type which yields the linear spectral problem associated with the system under consideration. On the other hand, it is noteworthy that the same Kac-Moody algebra is involved in the inverse prolongation procedure. This, which is based on the ansatz (4.5), produces the new class of linearizable nonlinear field equations (4.11), containing the starting ones, i.e. Eqs. (2.2), as a particular case. In this context, we have that for \(\Delta_1 = -\Delta_2 \equiv \Delta \neq 0\) and \(\kappa = \kappa^*\), Eqs.(4.11) coincide with the system describing twisted birefringent optical fibers [15]. This result is interesting mostly in the sense that so far the integrability property of such equations was not known.
Let us conclude with a few comments. First, the prolongation method possesses an intrinsic predictive character. In other words, in the specific application carried out here, the prolongation algebra related to Eqs.(2.1) allows a nontrivial closure if and only if $\alpha \equiv \beta \in \mathcal{R}$. Second, although some examples of prolongation calculations exist in the direct framework in more than 1 + 1 dimensions [7], it seems that, on the contrary, the inverse method has not been still explored. This is an open problem, together with the attempt of building up an algebraic foundation of integrability of nonlinear field equations. The prolongation strategy might accomplish this plan.

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