Tighter monogamy relations in multi-qubit systems

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Abstract

In this paper, we present some monogamy relations of multiqubit quantum entanglement in terms of the \( \beta \)th power of concurrence, entanglement of formation and convex-roof extended negativity. These monogamy relations are proved to be tighter than the existing ones, together with detailed examples showing the tightness.

Keywords Monogamy relations · Concurrence · Entanglement of formation · Convex-roof extended negativity

1 Introduction

Quantum entanglement is widely used as a very important resource in quantum information processing [1–4]. With the emergence of quantum information theory, quantum entanglement plays a very important role in quantum cryptography, quantum teleportation and measurement-based quantum computing. An important issue related to the entanglement metric is the limited shareability of the two-part entanglement in a multipartite entangled qubit system, that is, the single duality of entanglement [5]. Monogamy of entanglement (MoE) plays a very important role in many quantum information and communication processing tasks, such as security proof of quantum cryptography schemes and security analysis of quantum key distribution [6, 7].

For a tripartite quantum state \( \rho_{ABC} \), MoE can be described as \( E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC}) \), where \( \rho_{AB} = \text{tr}_C(\rho_{ABC}) \), \( \rho_{AC} = \text{tr}_B(\rho_{ABC}) \), \( E(\rho_{A|BC}) \) denotes the entanglement between systems A and BC. A remarkable result was established by Coffman, Kundu and Wootters (CKW) [8] for three qubits that was the simultaneous squares satisfy monogamy inequality. Then, the so-called CKW inequality was generalized to any \( N \)-qubit system [9]. Interestingly, it is further proved that similar
inequalities of polyqubit monogamy can be established for negativity and convex-roof extended negativity (CREN) [10–12], the entanglement of formation (EoF) [13, 14], Rényi-α entanglement [15, 16] and Tsallis-q entanglement [17].

Our paper is organized as follows. In Sect. 2, we present and prove two monogamy inequalities for the $β$th ($β ≥ 2$) power of concurrence in $N$-qubit system. In Sect. 3, we give a tighter monogamy relation for the $β$th ($β ≥ \sqrt{2}$) power of EoF in $2 \otimes 2 \otimes 2^{N-2}$ system. Then, we extend the result to $N$-qubit system. In Sect. 4, the monogamy relation for the $β$th ($β ≥ 2$) power of CREN in $N$-qubit system is discussed. In addition, detailed examples are given to illustrate the tightness. In Sect. 5, we summarize our results.

## 2 Tighter monogamy relations using concurrence

Given a bipartite pure state $|\phi\rangle_{AB}$ on Hilbert space $H_A \otimes H_B$, the concurrence is given by [18–20]

$$C(|\phi\rangle_{AB}) = \sqrt{2(1 - \text{Tr}(\rho_A^2))}, \quad (1)$$

where $\rho_A$ is the reduced density matrix by tracing over the subsystem $B$, $\rho_A = \text{Tr}_B(|\phi\rangle_{AB}\langle \phi|)$. For a bipartite mixed state $\rho_{AB}$, the concurrence is defined by the convex-roof,

$$C(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle_{AB}), \quad (2)$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle\langle \phi_i|$, with $\sum_i p_i = 1$ and $p_i ≥ 0$.

For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$ under bipartite partition $A$ and $B_1\cdots B_{N-1}$ satisfies [21]

$$C^\beta(\rho_{A|B_1\cdots B_{N-1}}) ≥ C^\beta(\rho_{AB_1}) + C^\beta(\rho_{AB_2}) + \cdots + C^\beta(\rho_{AB_{N-1}}), \quad (3)$$

for $β ≥ 2$. Furthermore, for an N-qubit mixed state, if $C_{AB_i} ≥ C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $C_{AB_j} ≤ C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, a generalized monogamy relation for $β ≥ 2$ was presented as [22]:

$$C^\beta(\rho_{A|B_1\cdots B_{N-1}}) ≥ C^\beta(\rho_{AB_1}) + (2^\beta - 1)C^\beta(\rho_{AB_2}) + \cdots + (2^\beta - 1)^{m-1}C^\beta(\rho_{AB_m})$$

$$+ (2^\beta - 1)^{m+1}[C^\beta(\rho_{AB_{m+1}}) + \cdots + C^\beta(\rho_{AB_{N-2}})] + (2^\beta - 1)^m C^\beta(\rho_{AB_{N-1}}), \quad (4)$$

where $1 ≤ m ≤ N - 3$, $N ≥ 4$.

In the following, we will show that these monogamy relations for concurrence can be further tightened under some conditions. Before that, we first introduce two lemmas as follows.
Lemma 1 For any $x \in [0, 1]$ and $t \geq 1$, we have

$$(1 + x)^t \geq 1 + \frac{t}{2} x + \frac{(t-1)^2}{4} x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2} x^{t+1} \geq 1 + \frac{t}{2} x + (2^t - \frac{t}{2} - 1)x^t \geq 1 + (2^t - 1)x^t. \quad (5)$$

Proof Let us consider the function $f(t, x) = \frac{(1+x)^t - 1 - \frac{t}{2} x - \frac{(t-1)^2}{4} x^2 + \frac{(t-1)^2}{2} x^{t+1}}{x^t}$. Thus, Eq. (5) is hold.

Next, we will prove that

$$1 + \frac{t-1}{2} x + \frac{(t-1)^2(t-2)}{4t} x^2 + \frac{(t-1)^2}{2t} x^{t+1} \leq (1 + x)^{t-1}, \quad (6)$$

thus \( \frac{df(t,x)}{dx} \leq 0, \) $f(t, x)$ is a decreasing function of $x$, i.e., $f(t, x) \geq f(t, 1) = 2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1$. It follows that $(1+x)^{t-1} \geq 1 + \frac{t}{2} x + \frac{(t-1)^2}{4} x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2} x^{t+1}$.

For the case $1 \leq t \leq 2$, it is obvious that $(1 + x)^{t-1} \geq 1 + (t-1)x + \frac{(t-1)(t-2)}{2} x^2$. Besides, we have

$$\frac{(t-1)(t-2)}{2} x^2 = \frac{t-1}{4t} 2t(t-2)x^2$$

$$= \frac{t-1}{4t} [(t-1)(t-2)x^2 + (2^t + t - 2)x^2 - 2tx^2]$$

$$\geq \frac{t-1}{4t} [(t-1)(t-2)x^2 + 2(t-2)x^{t+1} - 2tx].$$

Thus, Eq. (6) is hold.

For the case $t \geq 2$, it is obvious that $(1 + x)^{t-1} \geq 1 + (t-1)x + \frac{(t-1)(t-2)}{4} x^2$.

Besides, we have

$$\frac{(t-1)(t-2)}{4} x^2 = \frac{t-1}{4t} t(t-2)x^2 = \frac{t-1}{4t} [(t-1)(t-2)x^2 + (2t - 2)x^2 - tx^2]$$

$$\geq \frac{t-1}{4t} [(t-1)(t-2)x^2 + 2(t-1)x^{t+1} - 2tx].$$

Thus, Eq. (6) is hold.

On the other hand, since $x^2 - 2x^{t+1} + x^t \geq 0$ and $\frac{(t-1)^2}{4} \geq 0$, for $t \geq 1$ and $x \in [0, 1]$, we can get $(1 + x)^t \geq 1 + \frac{t}{2} x + \frac{(t-1)^2}{4} x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2} x^{t+1} \geq 1 + \frac{t}{2} x + (2^t - \frac{t}{2} - 1)x^t \geq 1 + (2^t - 1)x^t$. □

Lemma 2 For any mixed state $\rho_{ABC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, suppose that $C_{AB} \geq C_{AC}$, we have

$$C_{A|BC}^\beta \geq C_{AB}^\beta + hC_{AC}^\beta + \frac{\beta}{4} C_{AC}^2 \left( C_{AB}^{\beta-2} - C_{AC}^{\beta-2} \right)$$
for all $\beta \geq 2$, where $h = 2^{\frac{\beta}{2}} - 1$, $C_{A|BC} = C(\rho_{A|BC})$, analogously for $C_{AB}$ and $C_{AC}$. 

**Proof** Since $C_{AB} \geq C_{AC}$, we obtain

$$C_{A|BC}^\beta \geq (C_{AB}^2 + C_{AC}^2)^{\frac{\beta}{2}} = C_{AB}^\beta \left(1 + \frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\beta}{2}}$$

$$\geq C_{AB}^\beta \left[1 + \frac{\beta}{4} C_{AC}^2 \left(\frac{\beta - 2}{16} + C_{AB}^4 \left(\frac{2^\frac{\beta}{2} - \beta - (\beta - 2)^2}{16} - 1\right) \frac{C_{AC}^2}{C_{AB}^2}\right)ight]$$

$$\geq C_{AB}^\beta + hC_{AB}^\beta + \beta \frac{C_{AC}^2}{4} C_{AC}^2 \left(C_{AB}^2 - C_{AC}^2\right)$$

$$+ \frac{(\beta - 2)^2}{16} C_{AC}^4 \left(C_{AB}^2 + C_{AC}^2 - 2C_{AC}^2\right), \quad (8)$$

where the first inequality is due to the fact that $C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2$ for any $2 \otimes 2 \otimes 2^{N-2}$ tripartite state $\rho_{A|BC} [9, 23]$ and the second inequality is due to Lemma 1. 

**Theorem 1** For any $N$-qubit mixed state $\rho_{AB_1\ldots B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\ldots B_{N-1}}$ for $i = 1, 2, \ldots, N - 2$, we have

$$C_{A|B_1\ldots B_{N-1}}^\beta \geq \sum_{i=1}^{N-2} h^{i-1} \left(C_{AB_i}^\beta + P_{AB_i}\right) + h^{N-2} C_{AB_{N-1}}^\beta, \quad (9)$$

for all $N \geq 3$, $\beta \geq 2$, where $h = 2^{\frac{\beta}{2}} - 1$, $P_{AB_i} = \frac{\beta}{4} C_{A|B_{i+1}\ldots B_{N-1}}^2 C_{AB_i}^2 - C_{A|B_{i+1}\ldots B_{N-1}}^2 C_{AB_i}^2$, $C_{A|B_{i+1}\ldots B_{N-1}}^\beta - C_{A|B_{i+1}\ldots B_{N-1}}^\beta - 2C_{A|B_{i+1}\ldots B_{N-1}}^\beta C_{AB_i}$.

**Proof** Due to Eq. (7), we obtain

$$C_{A|B_1\ldots B_{N-1}}^\beta$$

$$\geq C_{AB_1}^\beta + hC_{A|B_2\ldots B_{N-1}}^\beta + \frac{\beta}{4} C_{A|B_2\ldots B_{N-1}}^2 \left(C_{AB_1}^\beta - C_{A|B_2\ldots B_{N-1}}^\beta\right)$$

$$+ \frac{(\beta - 2)^2}{16} C_{A|B_2\ldots B_{N-1}}^4 \left(C_{AB_1}^\beta + C_{A|B_2\ldots B_{N-1}}^\beta - 2C_{A|B_2\ldots B_{N-1}}^\beta C_{AB_1}\right)$$

$$\geq C_{AB_1}^\beta + h \left[C_{AB_2}^\beta + hC_{A|B_3\ldots B_{N-1}}^\beta + \frac{\beta}{4} C_{A|B_3\ldots B_{N-1}}^2 \left(C_{AB_2}^\beta - C_{A|B_3\ldots B_{N-1}}^\beta\right)$$

$$+ \frac{(\beta - 2)^2}{16} C_{A|B_3\ldots B_{N-1}}^4 \left(C_{AB_2}^\beta + C_{A|B_3\ldots B_{N-1}}^\beta - 2C_{A|B_3\ldots B_{N-1}}^\beta C_{AB_2}\right)\right]$$
In addition, since \( h_{\mathcal{A}B} = h_{\mathcal{A}B} + \cdots + h_{N-2}^\mathcal{A}B_{N-1} + h_{N-3}^\mathcal{A}B_{N-1} \), we have

\[
\begin{align*}
C^\beta_{\mathcal{A}B_1 \cdots B_{N-1}} &\geq \sum_{i=1}^{m} h_{i-1}^\mathcal{A}B_i + P_{AB_i} \\
&+ h_{m+1}^\mathcal{A}B_j + P_{AB_j}^1 + h_{m}^\mathcal{A}B_{N-1}^\mathcal{A},
\end{align*}
\]

(11)

for all \( N \geq 4, \beta \geq 2 \), where \( h = 2^\beta - 1 \), \( P_{AB_i} = \frac{\beta}{4} C_{A|B_1 \cdots B_{N-1}}^2 - C_{A|B_1^{\mathcal{A}} \cdots B_{N-1}^{\mathcal{A}}} \),

\[
P_{AB_j}^1 = \frac{\beta}{4} C_{A|B_1 \cdots B_{N-1}}^2 - C_{A|B_1 \cdots B_{N-1}}^2 + \frac{\beta^2}{16} C_{A|B_1 \cdots B_{N-1}}^{2} - C_{A|B_1 \cdots B_{N-1}}^2.
\]

Proof Due to the proof process of Theorem 1, we can get that

\[
C^\beta_{\mathcal{A}B_1 \cdots B_{N-1}} \geq \sum_{i=1}^{m} h_{i-1}^\mathcal{A}B_i + P_{AB_i} + h_{m}^\mathcal{A}B_{m+1} \cdots B_{N-1}.
\]

(12)

In addition, since \( C_{AB_j} \leq C_{A|B_1 \cdots B_{N-1}} \) for \( j = m + 1, \ldots, N - 2 \), hence

\[
C^\beta_{A|B_{m+1} \cdots B_{N-1}}.
\]
\[
\begin{align*}
\geq & \ C_{A|B_{m+2}\cdots B_{N-1}}^{\beta} + hC_{AB_{m+1}}^{\beta} + \frac{\beta^2}{4} C_{AB_{m+1}}^{2} \left( C_{A|B_{m+2}\cdots B_{N-1}}^{\beta-2} - C_{AB_{m+1}}^{\beta-2} \right) \\
& + \frac{(\beta - 2)^2}{16} C_{AB_{m+1}}^{2} \left( C_{A|B_{m+2}\cdots B_{N-1}}^{\beta-4} + C_{AB_{m+1}}^{\beta-4} - 2C_{AB_{m+1}}^{\beta-2} C_{A|B_{m+2}\cdots B_{N-1}}^{\beta-2} \right) \\
\geq & \ \sum_{j=m+1}^{N-2} \left( hC_{AB_j}^{\beta} + P_{AB_j}^{1} \right) + C_{AB_{N-1}}^{\beta}.
\end{align*}
\]

Combining Eqs. (12) and (13), we can get the inequality (11). 

**Example 1** Consider the three-qubit state \(|\psi\rangle_{ABC}\) in generalized Schmidt decomposition form [25, 26]:

\[
|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,
\]

where \(\lambda_i \geq 0\), \(i = 0, 1, 2, 3, 4\), and \(\sum_{i=0}^{4} \lambda_i^2 = 1\). A direct calculation shows that

\[
C_{A|BC} = 2\lambda_0 \sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}, \quad C_{AB} = 2\lambda_0 \lambda_2 \quad \text{and} \quad C_{AC} = 2\lambda_0 \lambda_3.
\]

Set \(\lambda_0 = \frac{\sqrt{7}}{3}, \lambda_1 = 0, \lambda_2 = \frac{\sqrt{2}}{3}, \lambda_3 = \frac{\sqrt{3}}{3}, \lambda_4 = 0\). We have \(C_{A|BC} = \frac{2\sqrt{14}}{9}\), \(C_{AB} = \frac{2\sqrt{10}}{9}\) and \(C_{AC} = \frac{4}{9}\). Then, \(C_{A|BC}^{\beta} = \frac{(2\sqrt{14})\beta}{9} \geq \frac{C_{AB}^{\beta}}{16} + hC_{AC}^{\beta} + \frac{\beta}{4} C_{AC}^{2} (C_{A|BC}^{\beta-2} - C_{AC}^{\beta-2}) + \frac{(\beta - 2)^2}{16} C_{AC}^{4} (C_{A|BC}^{\beta-4} + C_{AC}^{\beta-4} - 2C_{AC}^{\beta-2} C_{A|BC}^{\beta-2}) = \frac{(\sqrt{14})\beta + h(\frac{4}{3})\beta + \frac{\beta}{4} (\frac{4}{3})^2}{16} \left[ \frac{(2\sqrt{14})\beta - 2}{(\frac{4}{3})^2} \right] + \frac{(\beta - 2)^2}{16} \left[ (\frac{\sqrt{14}}{9})^{\beta-2} - (\frac{4}{3})^{\beta-2} \right]
\]

However, the result in [24] is \(C_{A|BC}^{\beta} + hC_{AC}^{\beta} + \frac{\beta^2}{4} C_{AC}^{2} (C_{A|BC}^{\beta-2} - C_{AC}^{\beta-2}) = \frac{(2\sqrt{14})\beta}{9} + h(\frac{4}{3})\beta + \frac{\beta}{4} \left[ \frac{(2\sqrt{14})\beta - 2}{(\frac{4}{3})^2} - (\frac{4}{3})^{\beta-2} \right]
\]

We can see that our results are better than the ones in [24] for \(\beta \geq 2\), see Fig. 1.

### 3 Tighter monogamy relations using EoF

Let \(H_A\) and \(H_B\) be two Hilbert spaces with dimension \(m\) and \(n\) (\(m \leq n\)). The entanglement of formation (EoF) of a pure state \(|\phi\rangle_{AB}\) on Hilbert space \(H_A \otimes H_B\), is defined as [27, 28]

\[
E(|\phi\rangle_{AB}) = S(\rho_A) = -\text{Tr}(\rho_A \log_2 \rho_A),
\]

where \(S(\rho) = -\text{Tr}(\rho \log_2 \rho)\) and \(\rho_A = \text{Tr}_B(|\phi\rangle_{AB} \langle \phi|)\). For a bipartite mixed state \(|\phi\rangle_{AB}\) on Hilbert space \(H_A \otimes H_B\), the EoF is given by

\[
E(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E(|\phi_i\rangle),
\]

where the infimum is taken over all possible pure state decompositions of \(\rho_{AB}\). 

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Fig. 1 Dash dotted line, $C^\beta_{A|BC}$ as a function of $\beta$ ($2 \leq \beta \leq 10$); solid line, the lower bound of $C^\beta_{A|BC}$ as a function of $\beta$ ($2 \leq \beta \leq 10$) in Eq. (11); dash line, the lower bound of $C^\beta_{A|BC}$ as a function of $\beta$ ($2 \leq \beta \leq 10$) in [24].

Let $g(x) = H(\frac{1+\sqrt{1-x}}{2})$ and $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$, it is obvious that $g(x)$ is a monotonically increasing function for $0 \leq x \leq 1$, and satisfies

\[ g^{\sqrt{2}}(x^2 + y^2) \geq g^{\sqrt{2}}(x^2) + g^{\sqrt{2}}(y^2), \]  

where $g^{\sqrt{2}}(x^2 + y^2) = [g(x^2 + y^2)]^{\sqrt{2}}$.

From Eqs. (15) and (16), we have $E(|\phi\rangle) = g(C^2(|\phi\rangle))$ for $2 \otimes d$ ($d \geq 2$) pure state $|\phi\rangle$. And $E(\rho) = g(C^2(\rho))$ for arbitrary two-qubit mixed state $\rho$ [29].

Wootters [8] shows that the EoF does not satisfy the monogamy inequality $E_{AB} + E_{AC} \leq E_{A|BC}$. In [30], the authors shows that EoF is a monotonic function satisfying $E^2(C^2_{A|B_1B_2...B_{N-1}}) \geq E^2 \sum_{i=1}^{N-1} (C^2_{AB_i}).$ For $N$-qubit systems, one has [21]

\[ E^\beta_{A|B_1B_2...B_{N-1}} \geq E^\beta_{AB_1} + E^\beta_{AB_2} + \cdots + E^\beta_{AB_{N-1}}, \]  

for $\beta \geq \sqrt{2}$, where $E_{A|B_1B_2...B_{N-1}}$ is the EoF of $\rho$ under bipartite partition $A|B_1B_2\cdots B_{N-1}, E_{AB_i}$ is the EoF of the mixed state $\rho_{AB_i} = \text{Tr}_{B_1\cdots B_{i-1},B_{i+1}\cdots B_{N-1}}(\rho)$ for $i = 1, 2, \ldots, N - 1$.

Lemma 3 For any mixed state $\rho_{ABC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, $\beta \geq \sqrt{2}$, if $C_{AB} \geq C_{AC}$, then we have

\[ E^\beta_{A|BC} \geq E^\beta_{AB} + hE^\beta_{AC} + \frac{t}{2} E^{\sqrt{2}}_{AC} \left( E^{\beta-\sqrt{2}}_{AB} - E^{\beta-\sqrt{2}}_{AC} \right) \]  

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\[
E_{AB}^{\beta} = \sum_{i=1}^{N-2} h_{i-1} (E_{AB_i}^{\beta} + Q_{AB_i}) + h^{N-2} E_{AB_{N-1}}^{\beta},
\]

where \( t = \frac{\beta}{\sqrt{2}}, h = 2^t - 1 \).

**Proof** The proof is similar to the proof of Lemma 2.

In fact, the result can be generalized to \( N \)-qubit mixed state \( \rho_{AB_1 \cdots B_{N-1}} \). The following theorem holds for \( \rho_{AB_1 \cdots B_{N-1}} \).

**Theorem 3** For any \( N \)-qubit mixed state \( \rho_{AB_1 \cdots B_{N-1}} \), if \( C_{AB_i} \geq C_{A|B_{i+1}B_{N-1}} \) for \( i = 1, 2, \ldots, N-2 \), we have

\[
E_{A|B_1 \cdots B_{N-1}}^{\beta} \geq g \left( C_{A|B_1B_2 \cdots B_{N-1}}^2 \right).
\]

In addition, for \( \beta \geq \sqrt{2} \), we have

\[
g^{\beta} (x^2 + y^2) = \left[ g^\sqrt{2} (x^2 + y^2) \right]^{t} \geq \left[ g^\sqrt{2} (x^2) + g^\sqrt{2} (y^2) \right]^{t}
\]

\[
\geq g^{\beta} (x^2) + (2^t - 1) g^{\beta} (y^2) + \frac{t}{2} g^{\sqrt{2}} (y^2)
\]

\[
= g^{\beta - \sqrt{2}} (x^2) + g^{\beta - \sqrt{2}} (y^2) + \frac{(t - 1)^2}{4} g^{\sqrt{2}} (y^2) + g^{\beta - 2 \sqrt{2}} (x^2) + g^{\beta - 2 \sqrt{2}} (y^2) - 2 g^{\beta - 2 \sqrt{2}} (y^2) g^{\sqrt{2}} (x^2)
\]

where the first inequality is due to Eq. (17), and without loss of generality, we can assume \( x^2 \geq y^2 \), then the second inequality is obtained from the monotonicity of \( g(x) \) and Eq. (5).

Thus, combining Eqs. (21) and (22), we obtain

\[
E_{A|B_1B_2 \cdots B_{N-1}}^{\beta} \geq g^{\beta} \left( C_{AB_1}^2 + C_{AB_2}^2 + \cdots + C_{AB_{N-1}}^2 \right)
\]
\[ \geq g^{\beta} \left( C_{AB_1}^2 \right) + h g^{\beta} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) + \frac{t}{2} g^{\sqrt{2}} \left( C_{AB_1}^2 + \cdots + C_{AB_N}^2 \right) \]
\[ \left[ g^{\beta - \sqrt{2}} \left( C_{AB_1}^2 \right) - g^{\beta - \sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) \right] \]
\[ + \frac{(t - 1)^2}{4} g^{2\sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) \]
\[ \left[ g^{\beta - 2\sqrt{2}} \left( C_{AB_1}^2 \right) + g^{\beta - 2\sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) \right] \]
\[ - 2g^{\beta - \sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) g^{-\sqrt{2}} (C_{AB_1}^2) \]
\[ \geq g^{\beta} \left( C_{AB_1}^2 \right) + h g^{\beta} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) \]
\[ + \frac{t}{2} \left[ g^{\sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{\sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] \]
\[ - \frac{(t - 1)^2}{4} \left[ g^{2\sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{2\sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] \]
\[ - 2g^{\beta - \sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) g^{-\sqrt{2}} (C_{AB_1}^2) \]
\[ \geq g^{\beta} \left( C_{AB_1}^2 \right) + h g^{\beta} \left( C_{AB_2}^2 + \cdots + h^{N-2} g^{\beta} \left( C_{AB_N}^2 \right) \right) \]
\[ + h^{N-3} \cdot \frac{t}{2} \cdot g^{\sqrt{2}} \left( C_{AB_N}^2 \right) \left[ g^{\beta - \sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{\beta - \sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] + \cdots \]
\[ + \frac{t}{2} \left[ g^{\sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{\sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] \]
\[ - \frac{(t - 1)^2}{4} \cdot g^{2\sqrt{2}} \left( C_{AB_N}^2 \right) \left[ g^{\beta - 2\sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{\beta - 2\sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] + \cdots \]
\[ + \frac{(t - 1)^2}{4} \left[ g^{2\sqrt{2}} \left( C_{AB_2}^2 + \cdots + g^{2\sqrt{2}} \left( C_{AB_N}^2 \right) \right) \right] \]
\[ - 2g^{\beta - \sqrt{2}} \left( C_{AB_2}^2 + \cdots + C_{AB_N}^2 \right) g^{-\sqrt{2}} (C_{AB_1}^2) \right] , \quad (23) \]

where we have utilized Eq. (3) and the monotonicity of \( g(x) \) to obtain the first inequality, the third and the forth inequalities are due to Eq. (17) and the monotonicity of the function \( g(x) \).

According to Eq. (21) and the fact that \( g(C^2(\rho)) = E(\rho) \) for arbitrary two-qubit mixed state \( \rho \), we obtain Eq. (20). \( \square \)
Example 2 Let us consider the state in (14) given in Example 1. Set $\lambda_0 = \sqrt{\frac{6}{7}}, \lambda_1 = 0, \lambda_2 = \sqrt{\frac{2}{3}}, \lambda_3 = \frac{1}{3}, \lambda_4 = 0$, we have $E_{A|BC} = 0.91829, E_{AB} = 0.68193, E_{AC} = 0.40416$. Then, $E_{A|BC}^\beta = (0.91829)^\beta \geq E_{AB}^\beta + hE_{AC}^\beta + \frac{\beta}{2\sqrt{2}}E_{AC}^\sqrt{2}(E_{AB}^\sqrt{2} - E_{AC}^\sqrt{2}) + \frac{(\beta - \sqrt{2})^2}{8}E_{AC}^2(E_{AB}^{\sqrt{2} - 2\sqrt{2}} + E_{AC}^{\sqrt{2} - 2\sqrt{2}} - 2E_{AC}^{\sqrt{2} - \sqrt{2}}E_{AB}^{\sqrt{2}}) = (0.68193)^\beta + h(0.40416)^\beta + \frac{\beta}{2\sqrt{2}}(0.40416)^{\sqrt{2}}\left[(0.68193)^{\sqrt{2} - \sqrt{2}} - (0.40416)^{\sqrt{2} - \sqrt{2}}\right] + \frac{(\beta - \sqrt{2})^2}{8}(0.40416)^{2\sqrt{2}}\left[(0.68193)^{\sqrt{2} - 2\sqrt{2}} + (0.40416)^{\sqrt{2} - 2\sqrt{2}} - 2(0.40416)^{\sqrt{2} - \sqrt{2}}\right] (0.68193)^{\sqrt{2}}$. While the result in [24] is $E_{AB}^\beta + hE_{AC}^\beta + \frac{\beta}{2\sqrt{2}}E_{AC}^\sqrt{2}(E_{AB}^{\sqrt{2} - \sqrt{2}} - E_{AC}^{\sqrt{2} - \sqrt{2}}) = (0.68193)^\beta + h(0.40416)^\beta + \frac{\beta}{2\sqrt{2}}(0.40416)^{\sqrt{2}}\left[(0.68193)^{\sqrt{2} - \sqrt{2}} - (0.40416)^{\sqrt{2} - \sqrt{2}}\right]$, we can see that our results are better than the ones in [24], see Fig. 2.

4 Tighter monogamy relations using negativity

The negativity is a well-known quantifier of bipartite entanglement. Given a bipartite state $\rho_{AB}$ in Hilbert space $H_A \otimes H_B$, the negativity is defined as [31]:

$$N(\rho_{AB}) = \frac{\|\rho_{TA_{AB}}\| - 1}{2},$$

(24)
where $\rho_{AB}^{TA}$ is the partial transposed matrix of $\rho_{AB}$ with respect to the subsystem $A$ and $\|X\|$ denotes the trace norm of $X$, i.e., $\|X\| = \text{Tr}\sqrt{XX^T}$. For convenience, we use the definition of negativity as: $\|\rho_{TA}^{AB}\| - 1$ [10].

If a bipartite pure state $|\phi\rangle_{AB}$ with the Schmidt decomposition, $|\phi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |ii\rangle$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, then 

$$N(|\phi\rangle_{AB}) = 2 \sum_{i < j} \sqrt{\lambda_i \lambda_j}.$$  

(25)

From the definition of concurrence (1), we have

$$C(|\phi\rangle_{AB}) = 2 \sqrt{\sum_{i < j} \lambda_i \lambda_j}.$$  

(26)

As a consequence, for any bipartite pure state $|\phi\rangle_{AB}$ with Schmidt rank 2, one has $N(|\phi\rangle_{AB}) = C(|\phi\rangle_{AB})$.

For a mixed state $\rho_{AB}$, the convex-roof extended negativity (CREN) is given by

$$N_c(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i N(|\phi_i\rangle),$$  

(27)

where the minimum is taken over all possible pure state decomposition of $\rho_{AB}$. CREN gives a perfect discrimination between PPT bound entangled states and separable states in any bipartite quantum system [32]. It follows that for any $2 \otimes d$ ($d \geq 2$) mixed state $\rho_{AB}$, we have

$$N_c(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle) = C(\rho_{AB}).$$  

(28)

According to the relation between CREN and concurrence, we have the following results for the lower bound of $N_{cA|B_1\cdots B_{N-1}}^{\beta}$.

**Theorem 4** For any $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $N_{cA|B_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $N_{cA|B_j} \leq N_{cA|B_{j+1}\cdots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, $\forall 1 \leq m \leq N - 3$, then we have

$$N_{cA|B_1\cdots B_{N-1}}^{\beta} \geq \sum_{i=1}^{m} h^{i-1} (N_{cA|B_i}^{\beta} + R_{AB_i})$$

$$+ h^m \sum_{j=m+1}^{N-2} (h N_{cA|B_j}^{\beta} + R_{AB_j}) + h^m N_{cA|B_{N-1}}^{\beta},$$  

(29)

for all $N \geq 4$, $\beta \geq 2$, where $h = 2^\beta - 1$, $R_{AB_i} = \frac{\beta}{4} N_{cA|B_i}^{2} + N_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-2} (N_{cA|B_i}^{\beta-2} - N_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-2}) + \frac{(\beta-2)^2}{16} N_{cA|B_i}^{4} + N_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-4} (N_{cA|B_i}^{\beta-4} + N_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-4} - 2 N_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-2}).$
Example 3 Let us consider the state in (14) given in Example 1. We have $\mathcal{N}_{cA|BC}^\beta$ as a function of $\beta (2 \leq \beta \leq 10)$; solid line, the lower bound of $\mathcal{N}_{cA|BC}^\beta$ as a function of $\beta (2 \leq \beta \leq 10)$ in Eq. (30); dash line, the lower bound of $\mathcal{N}_{cA|BC}^\beta$ as a function of $\beta (2 \leq \beta \leq 10)$ in [24].

\[
\mathcal{N}_{cA|BC}^{\beta - 2}, \quad R_{AB_j}^1 = \frac{\beta}{4} \mathcal{N}_{cAB}^\beta (\mathcal{N}_{cA|B_j+1...B_{N-1}}^{\beta - 2} - \mathcal{N}_{cAB}^{\beta - 2}) + \frac{(\beta - 2)^2}{16} \mathcal{N}_{cAB}^{\beta - 4} (\mathcal{N}_{cA|B_j+1...B_{N-1}}^{\beta - 4} + \mathcal{N}_{cA|B_j+1...B_{N-1}}^{\beta - 4} - 2 \mathcal{N}_{cA|B_j+1...B_{N-1}}^{\beta - 2} - \mathcal{N}_{cAB}^{\beta - 2}).
\]

Theorem 5 For any $N$-qubit mixed state $\rho_{AB_1...B_{N-1}}$, if $\mathcal{N}_{cA|B_1...B_{N-1}}^\beta \geq \mathcal{N}_{cA|B_{i+1}...B_{N-1}}^\beta$ for $i = 1, 2, \ldots, N - 2$, then we can obtain

\[
\mathcal{N}_{cA|B_1...B_{N-1}}^\beta \geq \sum_{i=1}^{N-2} h^{i-1} \left( \mathcal{N}_{cA|B_i}^\beta + R_{AB_i} \right) + h^{N-2} \mathcal{N}_{cA|B_{N-1}}^\beta,
\]

for all $N \geq 3$, $\beta \geq 2$, where $h = 2^\beta - 1$, $R_{AB_i} = \frac{\beta}{4} \mathcal{N}_{cA|B_{i+1}...B_{N-1}}^\beta (\mathcal{N}_{cA|B_{i+1}...B_{N-1}}^{\beta - 2} - \mathcal{N}_{cA|B_{i+1}...B_{N-1}}^{\beta - 2} - \mathcal{N}_{cAB}^{\beta - 2}).$

Example 3 Let us consider the state in (14) given in Example 1. We have $\mathcal{N}_{cA|BC} = 2\lambda_0 \sqrt{\lambda_2 + \lambda_3 + \lambda_4, \mathcal{N}_{cAB} = 2\lambda_0 \lambda_2}$ and $\mathcal{N}_{cAC} = 2\lambda_0 \lambda_3$. Set $\lambda_0 = \frac{\sqrt{5}}{5}, \lambda_1 = 0, \lambda_2 = \frac{\sqrt{5}}{5}, \lambda_3 = \frac{\sqrt{2}}{5}, \lambda_4 = 0$. We have $\mathcal{N}_{cA|BC}^\beta \geq \mathcal{N}_{cAB}^\beta + h \mathcal{N}_{cAC}^\beta + \frac{\beta}{4} \mathcal{N}_{cAB}^\beta (\mathcal{N}_{cA|BC}^{\beta - 2} + \mathcal{N}_{cAC}^{\beta - 2}) + \frac{(\beta - 2)^2}{16} \mathcal{N}_{cAB}^{\beta - 4} (\mathcal{N}_{cA|BC}^{\beta - 4} + \mathcal{N}_{cAC}^{\beta - 4} - 2 \mathcal{N}_{cA|BC}^{\beta - 2} - \mathcal{N}_{cAC}^{\beta - 2}) = \left( \frac{2\sqrt{10}}{9} \right)^\beta + h \left( \frac{4}{9} \right)^\beta + \frac{\beta}{4} \left( \frac{2\sqrt{10}}{9} \right)^\beta - \left( \frac{4}{9} \right)^\beta - 2 \left( \frac{4}{9} \right)^\beta - 2 \left( \frac{2\sqrt{10}}{9} \right)^{-2}.

While the result in [24] is $\mathcal{N}_{cAB}^\beta + h \mathcal{N}_{cAC}^\beta + \frac{\beta}{4} \mathcal{N}_{cAC}^\beta (\mathcal{N}_{cA|BC}^{\beta - 2} - \mathcal{N}_{cAC}^{\beta - 2}) = \left( \frac{2\sqrt{10}}{9} \right)^\beta + \left( \frac{4}{9} \right)^\beta - 2 \left( \frac{4}{9} \right)^\beta - 2 \left( \frac{2\sqrt{10}}{9} \right)^{-2}.$
\[ h\left(\frac{4}{9}\right)^\beta + \beta \left(\frac{4}{9}\right)^2\left[\left(\frac{2\sqrt{10}}{9}\right)^\beta - 2 - \left(\frac{4}{9}\right)^\beta\right]. \]

We can see that our result is better than the one in [24] for \(\beta \geq 2\), see Fig. 3.

5 Conclusion

Entanglement monogamy relations are fundamental properties of multipartite entangled states. In this paper, we have provided the multipartite entanglement based on the monogamy relations for \(\beta\)th power of concurrence \(C_{A|B_1\cdots B_{N-1}}^\beta (\beta \geq 2)\), entanglement of formation \(E_{A|B_1\cdots B_{N-1}}^\beta (\beta \geq \sqrt{2})\) and convex-roof extended negativity \(N_{cA|B_1\cdots B_{N-1}}^\beta (\beta \geq 2)\). Our monogamy relations have larger lower bounds and are tighter than the existing results [24]. These tighter monogamy inequalities can also provide a finer description of the entanglement distribution. In multi-qubit system, our research results provide a rich reference for future research on multi-party quantum entanglement. Our method can also be applied to the study of other properties of monogamy related to quantum correlations.

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References

1. Jafarpour, M., Kazemi Hasanvand, F., Afshar, D.: Dynamics of entanglement and measurement-induced disturbance for a hybrid qubit-qutrit system interacting with a spin-chain environment: a mean field approach. Commun. Theor. Phys. 67, 27 (2017)
2. Deng, F.G., Ren, B.C., Li, X.H.: Quantum hyperentanglement and its applications in quantum information processing. Sci. Bull. 62, 46 (2017)
3. Huang, H.L., Goswami, A.K., Bao, W.S., Panigrahi, P.K.: Demonstration of essentiality of entanglement in a Deutsch-like quantum algorithm. Sci. China Phys. Mech. Astron. 61, 060311 (2018)
4. Wang, M.Y., Xu, J.Z., Yan, F.L., Gao, T.: Entanglement concentration for polarization-spatial-time-bin hyperentangled Bell states. Europhys. Lett. 123, 60002 (2018)
5. Terhal, B.: Is entanglement monogamous? IBM J. Res. Dev. 48, 71 (2004)
6. Bennett, C.H.: Quantum cryptography using any two nonorthogonal states. Phys. Rev. Lett. 68, 3121 (1992)
7. Pawlowski, M.: Security proof for cryptographic protocols based only on the monogamy of Bell’s inequality violations. Phys. Rev. A 82, 032313 (2010)
8. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61, 052306 (2000)
9. Osborne, T.J., Verstraete, F.: General monogamy inequality for bipartite qubit entanglement. Phys. Rev. Lett. 96, 220503 (2006)
10. Kim, I.S., Das, A., Sanders, B.C.: Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extend negativity. Phys. Rev. A 79, 012329 (2009)
11. Ou, Y.: Violation of monogamy inequality for higher-dimensional objects. Phys. Rev. A 75, 034305 (2007)
12. Yany, Y., Chen, W., Li, G., Zheng, Z.J.: Generalized monogamy inequalities and upper bounds of negativity for multiqubit systems. Phys. Rev. A 97, 012336 (2018)
13. Rungta, P., Caves, C.M.: Concurrence-based entanglement measures for isotropic states. Phys. Rev. A 67, 012307 (2003)
14. Ren, X.J., Jiang, W.: Entanglement monogamy inequality in a 2–2–4 system. Phys. Rev. A 81, 024305 (2010)
15. Kim, J.S., Sanders, B.C.: Monogamy of multi-qubit entanglement using Rényi entropy. J. Phys. A: Math. Theor. 43, 445305 (2010)
16. Wang, Y.X., Mu, L.Z., Vedral, V., Fan, H.: Entanglement Rényi α-entropy. Phys. Rev. A 93, 022324 (2016)
17. Luo, Yu., Li, Y.M.: Hierarchical polygamy inequality for entanglement of Tsallis q-entropy. Commun. Theor. Phys. 69, 532 (2018)
18. Uhlmann, A.: Fidelity and concurrence of conjugated states. Phys. Rev. A 62, 032307 (2000)
19. Albeverio, S., Fei, S.M.: A note on invariants and entanglements. J. Opt. B Quantum Semiclass Opt. 3, 223 (2001)
20. Rungta, P., Buzek, V., Caves, C.M., Hillery, M., Milburn, G.J.: Universal state inversion and concurrence in arbitrary dimensions. Phys. Rev. A 64, 042315 (2001)
21. Zhu, X.N., Fei, S.M.: Entanglement monogamy relations of qubit systems. Phys. Rev. A 90, 024304 (2014)
22. Jin, Z.X., Li, J., Li, T., Fei, S.M.: Tighter monogamy relations in multiqubit systems. Phys. Rev. A 97, 032336 (2018)
23. Ren, X.J., Jiang, W.: Entanglement monogamy inequality in a 2 ⊗ 2 ⊗ 4 system. Phys. Rev. A 81, 024305 (2010)
24. Zhang, J.B., Jin, Z.X., Fei, S.M., Wang, Z.X.: Enhanced monogamy relations in multiqubit systems. Int. J. Theor. Phys. 59, 3449–3463 (2020)
25. Acín, A., Andrianov, A., Costa, L., Jané, E., Latorre, J.I., Tarrach, R.: Generalized Schmidt decomposition and classification of three-quantum-bit states. Phys. Rev. Lett. 85, 1560 (2000)
26. Gao, X.H., Fei, S.M.: Estimation of concurrence for multipartite mixed states. Eur. Phys. J. Spec. Top. 159, 71 (2008)
27. Bennett, C.H., Bernstein, H.J., Popescu, S., Schumacher, B.: Concentrating partial entanglement by local operations. Phys. Rev. A 53, 2046 (1996)
28. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A 54, 3824 (1996)
29. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245 (1998)
30. Bai, Y.K., Zhang, N., Ye, M.Y., Wang, Z.D.: Exploring multipartite quantum correlations with the square of quantum discord. Phys. Rev. A 88, 012123 (2013)
31. Vidal, G., Werner, R.F.: Computable measure of entanglement. Phys. Rev. A 65, 032314 (2002)
32. Lee, S., Chi, D.P., Oh, S.D., Kim, J.S.: Convex-roof extended negativity as an entanglement measure for bipartite quantum systems. Phys. Rev. A 68, 062304 (2003)

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