LOCAL MONODROMY OF 1-DIMENSIONAL $p$-DIVISIBLE GROUPS

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ABSTRACT. Let $G$ be a $p$-divisible group over a complete discrete valuation ring $R$ of characteristic $p$. The generic fiber of $G$ determines a Galois representation $\rho$. The image of $\rho$ admits a ramification filtration and a Lie filtration. We relate these filtrations in the case $G$ is one dimensional, giving an equicharacteristic version of Sen’s theorem in this setting. This result generalizes results of Gross and Chai. Additionally, we prove that the representation associated to the étale part of $G$ is irreducible.

1. Introduction

Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$ of characteristic $p$. Let $L$ be a Galois extension of $K$. Then $\text{Gal}(L/K)$ has a ramification filtration which reflects certain arithmetic properties of $L/K$. If $\text{Gal}(L/K)$ is also a $p$-adic Lie group, then $\text{Gal}(L/K)$ also has a Lie filtration which reflects certain analytic properties of $L/K$. One can ask how these two filtrations compare. As the Lie filtration is often more easily understood, a relation between these two filtrations gives an avenue to study certain arithmetic properties of $L/K$.

In the case that $K$ has characteristic 0, Serre conjectured a strong connection between the Lie and ramification filtrations $\text{[Ser67]}$. This conjecture was proved by Wyman in the $\Z_p$ case $\text{[Wym69]}$, and by Sen in the general case $\text{[Sen72]}$.

In the case that $K$ has characteristic $p > 0$, the ramification filtration can exhibit very general behavior which prohibits any meaningful relation with the Lie filtration in general $\text{[GK88]}$ Remark 3]. However, if one restricts to situations ‘coming from geometry’ then some relations between the Lie and ramification filtrations in characteristic $p$ have been discovered. This was first explored by Gross for local monodromy groups of one-dimensional $p$-divisible groups $\text{[Gro79]}$, and more recently by Kramer-Miller for monodromy groups of unit root $F$-isocrystals $\text{[KM18]}$. In this article we further explore the case of local monodromy groups of $p$-divisible groups, following in the footsteps of $\text{[Gro79]}$ and $\text{[Cha00]}$.

Let $G$ be a $p$-divisible group over $R$ whose generic fiber has height $g$ and whose special fiber has height $s$. Set $d = s - g$, the difference between the height of the special fiber and the height of the generic fiber. The action of the absolute Galois group $\mathfrak{G}_K := \text{Gal}(K^{\text{sep}}/K)$ on the (generalized) Tate module associated to the generic fiber of $G$ determines a Galois representation

$$\rho = \prod_{\lambda} \rho^\lambda : \mathfrak{G}_K \to \prod_{\lambda} \text{GL}_{d_{\lambda}}(D_{\lambda})$$

where $D_{\lambda}$ is the $\Q_p$-division algebra with invariant $\lambda$. This representation will be described in more detail in Section 4. The image of $\rho$, which we shall refer to as the local monodromy

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group of $G$, is a closed subgroup of $\prod \text{GL}_{d_{\lambda}}(D_{\lambda})$ and thus inherits the structure of a $p$-adic Lie group.

Suppose $G$ is 1-dimensional. In this case $\rho = \rho^{0/1} \times \rho^{1/g}$. When $g = s - 1$, Gross exhibited a relation between the ramification and Lie filtrations [Gro79]. When $g = 1$ and $s$ arbitrary, Chai showed that there is a similar relation between the ramification and Lie filtrations [Cha00]. In both of these cases an important role is played by an open image theorem. In this setting the classical open image is due to Igusa, whose results imply that when $s = 2$ the image of $\rho^{0/1}$ is open in $\mathbb{Z}_{p}^{\times}$ [Igu68]. Gross showed that when $g = s - 1$, $\rho^{0/1}$ and $\rho^{1/g}$ are both surjective and thus have open image. Chai showed that when $g = 1$ the image of $\rho^{1/1} = \det(\rho^{0/1})$ is open in $\mathbb{Z}_{p}^{\times}$. Beyond these cases, little is known about when $\rho$ has open image, though one can find some other related results at the end of [Cha00] and in [AN10]. We remark that there has, however, been much progress on the surjectivity of the local monodromy representations of universal $p$-divisible groups [Str10, Lau10, Tia09].

The main result (Theorem 7.2.3) of this article extends the results of Gross and Chai to all $g$ with $1 \leq g < d$, under an open image assumption. This gives further support for an equicharacteristic version of Sen’s theorem in this setting. We also prove that for any 1-dimensional $p$-divisible group, the slope zero representation $\rho^{0/1}$ is irreducible, generalizing a result of Chai [Cha00]. Our methods are much the same as those used by Gross and Chai. In particular, we exploit the equivalence of categories between connected $p$-divisible groups and formal groups. This allows us to work very explicitly with formal power series.

1.1. Organization. Sections 2-5 cover background material and our basic setup. In Section 2 we recall basic facts about $p$-adic Lie groups and their Lie filtrations. In Section 3 we review higher ramification theory, covering basic facts about the lower and upper ramification filtrations. We also state an important lemma of Tate which allows one to compute the upper ramification filtration from a Newton polygon in certain circumstances. In Section 4 we consider Galois representations of $p$-divisible groups. We review Gross’s generalized Tate modules and the Galois representations arising from them. We also state an important theorem of Gross concerning Galois characters associated to those representations (Theorem 4.0.3). In Section 5 we review the theory of formal groups and formal modules. We also show how to rephrase the Galois representations of $p$-divisible groups from Section 4 in terms of formal modules.

Sections 6-8 cover results on the local monodromy group of 1-dimensional $p$-divisible groups. In Section 6 we study the $p$-adic Lie towers of local fields arising from $\rho^{0/1}$. Studying properties of these towers we are able to prove that the representation attached in $\rho^{0/1}$ is irreducible (Theorem 6.2.4). In Section 7 we study $p$-adic Lie towers of local fields arising from $\rho^{1/g}$. We prove our main result, an equicharacteristic version of Sen’s theorem for general 1-dimensional $p$-divisible groups (Theorem 7.2.3). In Section 8 we use Theorem 7.2.3 and Theorem 4.0.3 to prove results on the ramification behavior of the Galois characters associated to $\rho^{0/1}$ and $\rho^{1/g}$.

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2. $p$-adic Lie Groups

We cover some basic facts about $p$-adic Lie groups and their Lie filtrations. We refer the reader to [DdSMS99] for details.

Definition 2.0.1 (Uniform pro-$p$ group). A pro-$p$ group $H$ is uniform if it satisfies the following conditions:

1. $H$ is finitely generated.
2. $H^p$, the subgroup generated by $\{h^p : h \in H\}$, is a normal subgroup of $H$ and $H/H^p$ is abelian.
3. For all $i \geq 0$ the $p$-th power map induces an isomorphism

$$H^p^i / H^p^{i+1} \sim H^p^{i+1} / H^p^{i+2}.$$

The following gives an algebraic characterization of $p$-adic Lie groups:

Theorem 2.0.2. [DdSMS99, Theorem 8.32]. A group $G$ is a $p$-adic Lie group if and only if it is a topological group containing an open uniform pro-$p$ subgroup.

Proof. See sections 8.3 and 8.4 of [DdSMS99].

Corollary 2.0.3. [DdSMS99, Corollary 8.34(ii)]. If $G$ is a compact $p$-adic Lie group, then $G$ contains an open normal uniform subgroup of finite index.

By the corollary, if $G$ is a $p$-adic Lie group then it admits a normal uniform subgroup $H$. We get a filtration of $G$ by the normal subgroups $G(i) := H^p^i$,

$$G \geq G(0) \geq G(1) \geq \cdots$$

referred to as a Lie filtration.

Theorem 2.0.4 (Closed subgroup theorem). Every closed subgroup $\Gamma$ of a $p$-adic Lie group $G$ is a $p$-adic Lie group. Additionally, if $H$ is an open uniform subgroup of $G$, then $\Gamma$ inherits a Lie filtration

$$\Gamma \geq \Gamma(0) \geq \Gamma(1) \geq \cdots$$

where $\Gamma(i) = H^p^i \cap \Gamma$ are open uniform subgroups of $\Gamma$.

Proof. See [DdSMS99, Theorem 9.6].

Example 2.0.5. The group $G := GL_n(Z_p)$ is a $p$-adic Lie group with open uniform subgroup $H := 1 + p M_n(Z_p)$, where $M_n(Z_p)$ is the set of $n \times n$ matrices with entries in $Z_p$. It admits a Lie filtration

$$GL_n(Z_p) \geq G(0) \geq G(1) \geq \cdots$$

where

$$G(i) := H^p^i = 1 + p^i M_n(Z_p).$$

By the closed subgroup theorem any closed subgroup $\Gamma \subseteq GL_n(Z_p)$, such as the image of a $p$-adic Galois representation, is a $p$-adic Lie group with Lie filtration

$$\Gamma \geq \Gamma(i) \geq \Gamma(i + 1) \geq \cdots$$
where $\Gamma(i) = \Gamma \cap H^i$.

3. Ramification Theory

In this section we review some basic facts from ramification theory. For additional details on ramification theory see [Ser79].

Let $K$ be a field complete with respect to a discrete valuation $v_K$. Let $R_K = \{ a \in K : v_K(a) \geq 0 \}$ denote the valuation ring of $K$. Let $k$ denote the residue field of $K$. Set $p = \text{char}(k)$. Suppose $v_K$ is normalized so that its value group on $K^*$ is $\mathbb{Z}$. Fix an extension of $v_K$ to $\overline{K}$.

Let $L$ be a finite separable extension of $K$ with valuation ring $R_L$. Set $\Gamma_{L/K} = \text{Hom}_K(L, \overline{K})$. Note that if $L/K$ is Galois then $\Gamma_{L/K} = \text{Gal}(L/K)$.

3.1. Lower ramification filtration. For each $x \in \mathbb{R}_{\geq 0}$ define the $x$-th lower ramification group to be

$$\Gamma_x := \left\{ \sigma \in \Gamma_{L/K} : v_K(\sigma(a) - a) \geq \frac{x}{[L : K]} \text{ for all } a \in R_L \right\}.$$ 

This is equivalent to

$$\Gamma_x := \left\{ \sigma \in \Gamma_{L/K} : v_K(\sigma(\alpha) - \alpha) \geq \frac{x}{[L : K]} \right\}$$

where $\alpha$ is such that $R_L = R_K[\alpha]$ as $R_K$ algebras [Ser79 IV §1 Lemma 1]. Note that we have shifted the index of the ramification groups up by 1 compared to most sources.

Proposition 3.1.1. The lower ramification groups form a decreasing chain of subgroups; namely,

1. $\Gamma_0 = \Gamma_{L/K}$.
2. $\Gamma_x \subseteq \Gamma_y$ for all $x \geq y$.
3. $\Gamma_x = \{1\}$ for all sufficiently large $x$.

Definition 3.1.2. We call $x$ a break in the (lower) ramification filtration if $\Gamma_x \neq \Gamma_{x+\epsilon}$ for all $\epsilon > 0$.

When $L/K$ is Galois, the breaks occur only at integers. However, if $L/K$ is not Galois the breaks may occur at other rational numbers. When $L/K$ is totally ramified there is a unique break occurring at $x = 0$.

3.2. Upper ramification filtration. Define the Herbrand transition function:

$$\phi_{L/K} : [0, \infty) \to [0, \infty)$$

$$x \mapsto \int_0^x \frac{\#\Gamma_t}{[L : K]} dt.$$ 

When $M$ is an extension of $K$ containing $L$ we have the identity

(3.1) $$\phi_{M/K} = \phi_{L/K} \circ \phi_{M/L}.$$
Note that the Herbrand transition function is a strictly increasing continuous function, and therefore has an inverse \( \psi_{L/K} \) defined on \([0, \infty)\). Sometimes we will instead write \( \phi_{L/K} \) and \( \psi_{L/K} \).

For each \( x \in \mathbb{R}_{\geq 0} \) we define the \( x \)-th upper ramification group to be

\[
\Gamma^x := \Gamma_{\psi(x)}.
\]

By Proposition 3.1.1 the upper ramification groups form a filtration of \( \Gamma \).

**Definition 3.2.1.** We call \( x \) a break in the (upper) ramification filtration if \( \Gamma^x \neq \Gamma^{x+\epsilon} \) for all \( \epsilon > 0 \).

The upper ramification filtration passes well to quotients in the following sense: For \( M \) a finite Galois extension of \( K \) containing \( L \), set \( G = \text{Gal}(M/K) \) and \( H = \text{Gal}(M/L) \). Then for all \( x \geq 0 \),

\[
\Gamma^x = (G/H)^x = G^xH/H.
\]

From this we may define an upper ramification filtration on infinite Galois extensions \( M/K \):

\[
\text{Gal}(M/K)^x := \{ \sigma \in \text{Gal}(M/K) : \forall \text{ finite extensions } L/K \text{ contained in } M, \sigma \in \Gamma^x_{L/K} \text{Gal}(M/L) \}.
\]

In this case we say \( x \) is a break in the filtration if it occurs as a break in any finite quotient. Suppose now that \( \Gamma = \text{Gal}(L/K) \) is a \( p \)-adic Lie group, so that it has a Lie filtration

\[
\Gamma \supseteq \Gamma(1) \supseteq \Gamma(2) \supseteq \cdots.
\]

We want to relate this Lie filtration to the ramification filtration. From the following proposition we get a weak relation, that will be useful to us later on:

**Proposition 3.2.2.** For any normal subgroup \( H \leq \Gamma \),

\[
\Gamma^x \cap H = H^{\psi_{\Gamma/H}(x)}.
\]

**Proof.** We have

\[
\Gamma^x \cap H = \Gamma_{\psi(x)} \cap H = H_{\psi(x)} = H_{\psi_{\Gamma/H}(x)} = H^{\phi_{\Gamma/H}(x)}
\]

where the final equality follows from (3.1). \( \square \)

Applying the Proposition for \( H = \Gamma(n) \), it follows that for all \( n \geq 1 \),

\[
\Gamma^x \cap \Gamma(n) = \Gamma(n)^{\psi_{\Gamma/n}(x)}.
\]

3.3. **Tate’s Lemma.** We now state a lemma of Tate which allows one to compute the upper breaks in the ramification filtration of \( \Gamma_{L/K} \) when \( L \) is a totally ramified extension of \( K \). First we recall how Newton polygons are defined and state a fundamental result about them.

**Definition 3.3.1 (Newton Polygon).** Let \( f(x) \in x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in K[x] \). Then the Newton polygon of \( f \), denoted \( \mathcal{N}(f) \), is the lower convex hull of the set of points \((i, v_K(a_i))\) in \( \mathbb{R}^2 \).

**Proposition 3.3.2 (The theorem of the Newton Polygon).** Suppose that \( f \) is separable over \( K \). For each side \( s_i \) of the Newton polygon of \( f \), let \( \lambda_i \) denote the slope of \( s_i \) and \( \mu_i \) denote the length of the projection of \( s_i \) to the \( x \)-axis. If \( \lambda_i < \infty \), then there are exactly \( \mu_i \) roots of \( f \) with valuation \( -\lambda_i \).
Lemma 3.3.3 (Tate’s Lemma). Let
\[ f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in K[x] \]
be an irreducible separable polynomial with the property that, for each root \( \alpha \) of \( f(x) \), the
extension \( K(\alpha) \) is totally ramified over \( K \). Fix a root \( \alpha \) of \( f(x) \) and set \( L = K(\alpha) \). Let \( \pi_L \)
be a uniformizer of \( L \). Let \( g(x) \) be the ramification polynomial
\[ g(x) := \left(1/\pi_L\right)^n \cdot f(\pi_L x + \alpha) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x \in L[x]. \]
Then the breaks in the upper ramification filtration of \( \Gamma_{L/K} \) occur at the \( y \)-intercepts of the
non-trivial sides of the Newton polygon \( \mathfrak{N}(g) \).

Remarks 3.3.4. Note that the Lemma applies for \( f(x) \) an Eisenstein polynomial, which is the
case due to Tate [Gro79, Lemma 1.5]. The above is actually a mild generalization stated by
Chai [Cha00, Lemma 4.4], which will be crucial to the proof of our main result. For a proof
of Tate’s lemma in terms of Newton copolygons see [Lub13]. The Newton polygon \( \mathfrak{N}(g) \) is
often called the ramification polygon.

Corollary 3.3.5. Suppose \( L \) is a degree \( q = p^r \) totally ramified extensions of \( K \), generated
by a root \( \alpha \) of an irreducible separable polynomial
\[ f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0. \]
with the property that \( K(\sigma(\alpha)) \) is totally ramified for all embeddings \( \sigma : L \to K^{sep} \).
(1) If \( v_K(a_i) \geq v_K(a_1) \) for all \( i \geq 1 \), then both the upper and lower ramification filtrations
of \( \Gamma_{L/K} \) have a unique break at
\[ \frac{q \cdot v_K(a_1)}{q - 1} - 1. \]
(2) If \( q > 2 \) and \( L/K \) is Galois, then \( \text{Gal}(L/K) \cong \mathbb{F}_q^+ \).

4. Galois Representations of \( p \)-Divisible Groups in Characteristic \( p \)

Let \( K \) be a field of characteristic \( p > 0 \) and let \( G = (G_v) \) be a \( p \)-divisible group over \( K \).
Let \( \lambda = r/s \in [0,1] \cap \mathbb{Q} \) be a rational number in lowest terms. Let \( G_\lambda \) denote the \( p \)-divisible
group of slope \( \lambda \), defined to be the unique (up to isomorphism) \( p \)-divisible group over \( \mathbb{F}_p \)
with Dieudonné module
\[ D(G_\lambda) = \mathbb{Z}[F,V]/(F^{s-r} = V^r, FV = VF = p). \]
All endomorphisms of \( G_\lambda \) are defined over \( \mathbb{F}_p^s \) and
\[ \text{End}_{\mathbb{F}_p^s}(G_\lambda) \cong \mathcal{O}_{G_\lambda} \]
where \( \mathcal{O}_{G_\lambda} \) is an order in \( D_\lambda \), the central division algebra over \( \mathbb{Q}_p \) with Hasse invariant \( \lambda \).
By the Dieudonné-Manin classification theorem [Die57, Man63], the \( p \)-divisible group \( G \) is
isogenous over \( \overline{K} \) to
\[ \prod_{\lambda \in \mathbb{Q}} G_\lambda^{d_\lambda} \]
where the \( d_\lambda \) are integers uniquely determined by \( G \), all but finitely many of which are zero.
Definition 4.0.1 (Generalized Tate Module). We define a generalized Tate module

$$T^\lambda(G) := \text{Hom}_K(G_\lambda, G)$$

and corresponding $\mathbb{Q}_p$-vector space

$$V^\lambda(G) := \text{Hom}_K(G_\lambda, G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ 

Observe that $V^\lambda(G)$ is a right $D_\lambda$-module of dimension $d_\lambda$ and a left module of dimension $d_\lambda$ over the opposite algebra $D_\lambda^{op}$. There is a left $G_K$-action,

$$G_K \times V^\lambda(G) \rightarrow V^\lambda(G),$$

$$(\sigma, f) \mapsto \sigma \circ f \circ \sigma^{-1}.$$ 

When $K$ contains $\mathbb{F}_p$, this action is $D_\lambda^{op}$ linear and we thus get a representation

$$\rho^\lambda : G_K \rightarrow \text{Aut}_{D_\lambda^{op}}(V^\lambda(G)) = \text{GL}_{d_\lambda}(D_\lambda).$$

When $K$ contains $\mathbb{F}_p$ we get a representation

$$\rho = \prod_{\lambda \in [0,1] \cap \mathbb{Q}} \rho^\lambda : G_K \rightarrow \prod_{\lambda \in [0,1] \cap \mathbb{Q}} \text{GL}_{d_\lambda}(D_\lambda).$$

Remark 4.0.2. If $\text{char}(K) = 0$ we follow the convention that $G_{0/1} = \mathbb{Q}_p/\mathbb{Z}_p$ and $d_{0/1} = h$.

We define a determinant map on $\prod \text{GL}_{d_\lambda}(D_\lambda)$ as follows:

$$\text{det} = \prod_{\lambda \in [0,1] \cap \mathbb{Q}} \text{Nm}_{\lambda} : \prod_{\lambda \in [0,1] \cap \mathbb{Q}} \text{GL}_{d_\lambda}(D_\lambda) \rightarrow \mathbb{Q}_p^*$$

where $\text{Nm}_{\lambda}$ is the reduced norm in $\text{Mat}_{d_\lambda}(D_\lambda)$ over $\mathbb{Q}_p$. Composing with $\rho$ we get a $p$-adic Galois character $\chi := \text{det} \circ \rho$.

Theorem 4.0.3. [Gro79, Theorem 2.7] If $\text{char}(K) = p$, then $\chi = 1$ in $\text{Hom}(G_K, \mathbb{Q}_p^*)$.

Later we will apply Theorem 4.0.3 in the case that $G$ has dimension 1. In this case $\rho = \rho^{0/1} \oplus \rho^{1/g}$, where $g$ is the dimension of the generic fiber of $G$. Theorem 4.0.3 then says that $\text{Nm}_{1/g} \circ \rho^{1/g}$ is the inverse of $\text{det} \circ \rho^{0/1}$ in $\mathbb{Q}_p^*$. In particular, the Galois characters $\text{det} \circ \rho^{0/1}$ and $\text{Nm}_{1/g} \circ \rho^{1/g}$ determine each other.

5. Formal Groups and Formal Modules

There is an equivalence of categories between connected $p$-divisible groups of (finite) height $h$ over $R$ and formal groups of height $h$ over $R$ [Tat67, Proposition 1]. In this section we introduce formal groups and formal modules. We will later see how the theory of formal modules can be used to give a more explicit description of the Galois representation $\rho$ introduced in the previous section.

Definition 5.0.1 (Formal Group). A (commutative one-parameter) formal group $\mathcal{F}$ over a ring $R$ is a power series $\mathcal{F}(X, Y) \in R[[r, y]]$ satisfying the following properties:

1. $\mathcal{F}(X, 0) = X$ and $\mathcal{F}(0, Y) = Y$.
2. $\mathcal{F}(X, \mathcal{F}(Y, Z)) = \mathcal{F}(\mathcal{F}(X, Y), Z)$.
3. $\mathcal{F}(X, Y) = \mathcal{F}(Y, X)$. 

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**Definition 5.0.2** (Morphisms of Formal Groups). Let $\mathcal{F}$ and $\mathcal{G}$ be formal groups over $R$. A *homomorphism* from $\mathcal{F}$ to $\mathcal{G}$ is a power series $f \in R[[T]]$ with no constant term satisfying
\[
f(\mathcal{F}(X,Y)) = \mathcal{G}(f(X), f(Y)).
\]
The formal groups $\mathcal{F}$ and $\mathcal{G}$ are isomorphic over $R$ if there are homomorphisms $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{F}$ defined over $R$ satisfying $f(g(T)) = g(f(T)) = T$.

**Definition 5.0.3** (Formal Module). Let $A$ be a ring and assume $R$ is an $A$-algebra. A (one-dimensional one-parameter) formal $A$-module is a (one-parameter) formal group $F$ over $R$ such that for each $a \in A$ there is an endomorphism $[a]_F$ of $F$ satisfying $[a]_F(X) = aX + (\text{higher degree terms})$.

**Definition 5.0.4** (Morphisms of Formal Modules). Let $\mathcal{F}$ and $\mathcal{G}$ be formal $A$-modules over $R$. A homomorphism from $\mathcal{F}$ to $\mathcal{G}$ is a homomorphism of formal groups $f : \mathcal{F} \to \mathcal{G}$ satisfying $f \circ [a]_F = [a]_G \circ f$ for all $a \in A$.

The formal $A$-modules $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* over $R$ if there is an $A$-module homomorphism between them which is an isomorphism of formal groups.

Let $\text{Hom}_R(\mathcal{F}, \mathcal{G})$ denote the set of $A$-module $R$-homomorphisms over from $\mathcal{F}$ to $\mathcal{G}$. Set $\text{End}_R(\mathcal{F}) = \text{Hom}_R(\mathcal{F}, \mathcal{F})$.

### 5.1. Formal $A$-Modules Over Rings of Characteristic $p$.
We consider the case when $A$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ and $R = K$ is a field of characteristic $p$ (e.g. $A = \mathbb{Z}_p$ and $R = \mathbb{F}_p((t))$). Let $\pi$ be a uniformizer for $A$ and set $q = #(A/\pi A)$.

**Definition 5.1.1** (Height). Let $\mathcal{F}$ and $\mathcal{G}$ be formal $A$-modules over $K$ and let $\phi \in \text{Hom}_R(\mathcal{F}, \mathcal{G})$. The height of $\phi$, denoted $\text{ht}(\phi)$, is the largest integer $h$ such that
\[
\phi(T) = f(T^{q^h})
\]
for some power series $f \in K[[T]]$. If there is no such $h$, then set $\text{ht}(\phi) = \infty$.

The *height* of a formal $A$-module $\mathcal{F}$ over $K$, denoted $\text{ht}(\mathcal{F})$, is $\text{ht}([\pi]_\mathcal{F})$.

**Remark 5.1.2.** Though we won’t consider formal group height in this article, we remark that it differs from formal module height by a factor of $[A : \mathbb{Z}_p]$.

**Proposition 5.1.3.** Let $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$ be homomorphisms of formal $A$-modules defined over $R$. Then $\text{ht}(g \circ f) = \text{ht}(f) + \text{ht}(g)$.

**Proof.** See [Sil09, IV.7 Proposition 7.3].

**Proposition 5.1.4.** If $K$ is separably closed then all formal $A$-modules of finite height $h$ are isomorphic. Moreover, any such formal $A$-module has endomorphism ring isomorphic to the ring of integers of the central division algebra over $K$ of Brauer invariant $1/h$.

**Proof.** See [Dri74, Proposition 1.7]
5.2. **A-Typical Formal Modules.** Set $A[v] := A[v_1, v_2, \ldots]$. Let $f(x)$ be the power series with coefficients in $A[v] \otimes \text{Frac}(A) = \text{Frac}(A)[v_1, v_2, \ldots]$ uniquely determined by the functional equation

$$f(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{\pi} f^{(q^i)}(x^{q^i})$$

where $f^{(q^i)}(x)$ denotes the power series obtained from $f(x)$ by replacing each $v_j$ by $v_j^{q^i}$. More explicitly

$$f(x) = \sum_{i=0}^{\infty} b_i x^{q^i}$$

where the coefficients are defined recursively:

$$b_0 = 1$$

$$b_i = \frac{b_0 v_i + b_1 v_i^{q} + b_2 v_i^{q^2} + \cdots + b_{i-1} v_i^{q^{i-1}}}{\pi}$$

Induction on $i$ shows that $\pi^i b_i \in A[v]$ for all $i$.

By Hazewinkel’s functional equation lemma [Haz79, Lemma 4.2], the power series

$$F_V(x, y) := f^{-1}(f(x) + f(y))$$

$$[a]_F(x) := f^{-1}(af(x)) \quad \text{for all } a \in A$$

gives rise to a formal $A$-module, which we shall denote $F_V$. One can show that the multiplication-by-$\pi$ map satisfies the congruence

$$(5.1) \quad [\pi]_F(x) \equiv v_i x^{q^i} \pmod{\pi, v_1, \ldots, v_{i-1}, x^{q^{i+1}}},$$

where we are considering $(\pi, v_1, \ldots, v_{i-1}, x^{q^{i+1}})$ as an ideal in the ring of power series over $A[v]$.

**Definition 5.2.1 (A-Typical Formal $A$-Module).** A formal $A$-module over $R$ is called $A$-typical if it is the specialization of $F_V$ with respect to an $A$-algebra homomorphism $\phi : A[v] \to R$. If $\phi(v)$ denotes the sequence $(\phi(v_1), \phi(v_2), \ldots)$ in $R$, we shall denote the specialization of $F_V$ by $F_{\phi(v)}$. Note that specifying an $A$-algebra homomorphism $A[v] \to R$ is the same as choosing an $r_i \in R$ for each $v_i$.

**Theorem 5.2.2.** [Haz79, Corollary 2.10] Every formal $A$-module over $R$ is isomorphic to an $A$-typical formal $A$-module over $R$.

**Example 5.2.3 (Honda Formal Modules).** Let $F_{1/h}$ denote the $A$-typical formal $A$-module over $\mathbb{F}_q \cong A/\pi A$ with respect to the homomorphism $\phi : A[v] \to \mathbb{F}_q$ which sends $v_h$ to 1 and $v_i$ to 0 for all $i \neq h$. One can show that $F_{1/h}$ is of height $h$ and that

$$[\pi]_{F_{1/h}}(T) = T^{q^h}.$$ 

All endomorphisms of $F_{1/h}$ are defined over $\mathbb{F}_{q^h}$ and

$$\text{End}_{\mathbb{F}_{q^h}}(F_{1/h}) = B_{1/h}.$$
where $B_{1/h}$ is the ring of integers of the central division algebra defined over the field $A \otimes \mathbb{Q}_p$ with Brauer invariant $1/h$.

When $h = 1$, the formal group underlying $F_{1/1}$ is the formal multiplicative group $\hat{G}_m$ given by the formal group law

$$X + Y + XY = (1 + X)(1 + Y) - 1.$$ 

5.3. From $p$-Divisible Groups to Formal Modules. Let $F$ be a formal group of height $h$ over $R$. Let $I_v$ denote the ideal of $R[[T]]$ generated by the endomorphism $[p^v]_F(T) \in \text{End}_R(F)$. Then, $G_v := \text{Spec}(R[[T]]/I_v)$ is a finite commutative group scheme of order $p^v$. These $G_v$ combine to determine a connected 1-dimensional $p$-divisible group $G$. One can show that $F$ and $G$ determine each other. See [Tat67, Proposition 1].

Let $F$ be a formal $A$-module over $R$. Let $G_{1/h}$ be the $p$-divisible group corresponding to the formal $A$-module $F_{1/h}$ from example 5.2.3. Suppose the special fiber of $F$ is isomorphic to $F_{1/s}$ over $k$. Let $G_{0/1}$ denote the constant étale $A$-module $(A \otimes \mathbb{Q}_p)/A$. Let $F_K$ denote the generic fiber of $F$. Let $g = \text{ht}(F_K)$. If $\text{char}(K) = 0$ then $G_K$ and $(G_{0/1})^d$ are $\overline{K}$-isomorphic. However, when $\text{char}(K) = p$, $F_K$ is 1-dimensional and therefore there is a $\overline{K}$-isogeny

$$G_K \to G_{1/g} \times (G_{0/1})^d$$

where $d = s - g$.

Define the slope $1/g$-Tate modules

$$T^{1/g}(F) := \text{Hom}_{\overline{K}}(F_{1/g}, F_K),$$

which are of rank 1 over $\text{End}(F_{1/g}) = B_{1/g}$. Define the slope 0-Tate module

$$T^{0/1}(G) := \text{Hom}_{\overline{K}}(G_{0/1}, G_K),$$

which is of rank $d$ over $A$. These give rise to Galois representations

$$\rho^{1/g} : \mathfrak{G}_K \to B_{1/g}^*$$

and

$$\rho^{0/1} : \mathfrak{G}_K \to \text{GL}_d(A)$$

corresponding to the $\rho^\lambda$ in Section 4.

6. Towers Arising From $\rho^{0/1}$

Let $G$ be a 1-dimensional $p$-divisible group over $R$ with special fiber of height $s$, and generic fiber of height $g$ with $1 \leq g < s$. Set $d = s - g$. Let $F$ be the corresponding 1-dimensional formal $A$-module over $R$. In this section we study $p$-adic Lie towers of local fields associated to the Galois representation $\rho^{0/1}$.

6.1. The Torsion Tower and its Subtowers. The representation $\rho^{0/1}$ arises from the action of the absolute Galois group $\mathfrak{G}_K$ on the Tate module $T(G) = \varprojlim G[\pi^n](K^{sep})$. As $T(G)$ is a free $A$-module of rank $d$, we may view $\rho^{0/1}$ as a homomorphism

$$\rho^{0/1} : \mathfrak{G}_K \to \text{GL}_d(A).$$
Let $M$ denote the ring of $d \times d$ matrices in $A$. Note that $M^* = \text{GL}_d(A)$. The Lie filtration on $M^*$,

$$M^* \supseteq 1 + \pi M \supseteq 1 + \pi^2 M \supseteq \cdots$$

gives rise to a tower of fields as follows: Let $\delta_n$ be the reduction map

$$\delta_n : M^* \to \frac{M^*}{1 + \pi^{n+1} M} \cong \left( \frac{M}{\pi^{n+1} M} \right)^*$$

and let $\mathfrak{H}_n \subseteq \mathfrak{G}_K$ be the kernel of $\delta_n \circ \rho^{1/0}$. Letting $K_n \subseteq K^{\text{sep}}$ denote the fixed field of $\mathfrak{H}_n$ we get a $p$-adic Lie tower of local fields:

$$K = K_{-1} \subseteq K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{\infty} \subseteq K^{\text{sep}}$$

where $K_{\infty}$ is the compositum of the $K_n$. We refer to this tower as the \textit{torsion tower}, since the torsion representation $\rho_n$,

$$\rho_n : \mathfrak{G}_K \to \text{Aut}_{A/\pi^{n+1} A}(G[\pi^{n+1}](K^{\text{sep}})) \cong \left( \frac{M}{\pi^{n+1} M} \right)^*$$

is canonically identified with $\delta_n \circ \rho^{0/1}$, compatibly with change in $n$. This representation can be identified with $\delta_n \circ \rho^{0/1}$. Thus the fields $K_n$ are indeed the $\pi^{n+1}$-division fields of $G(K^{\text{sep}})$. In particular the $K_n$ are the $\pi^{n+1}$-division fields of $G(K^{\text{sep}})$.

For each $y = (y_0, y_1, \ldots) \in T(G) = \varprojlim G[\pi^n](K^{\text{sep}})$ we get a tower of local fields

$$K \subseteq K(y_0) \subseteq K(y_1) \subseteq \cdots \subseteq K^{\text{sep}}.$$ 

This tower can be viewed as a \textit{subtower} of the torsion tower in the sense that $K(y_{n-1}) \subseteq K_n$ for all $n$. Additionally, each $K_n$ is the compositum of the $K(y_{n-1})$ as $y$ ranges over all elements of $T(G)$.

\textbf{Remark 6.1.1.} Katz used properties of the \{$(K(y_n))_n$\} towers in his proof of Igusa’s Theorem [Kat73, Theorem 4.3].

\textbf{6.2. Properties of the Subtowers.} We study these towers by considering the multiplication-by-$\pi$ map on $G$. By Example 5.2.3 and the deformation theory of formal modules [Haz78, Theorem 22.4.4], we may replace $\mathcal{F}$ by an isomorphic formal $A$-module, so that the multiplication-by-$\pi$ map, which is a priori a power series, is in-fact a polynomial:

$$[\pi]_F(x) = a_1 x^{q^n} + a_2 x^{q^{n+1}} + \cdots + a_d x^{q^{n+d-1}} + x^{q^d} \in R[x]$$

with each $a_i \in \pi_R R$ and $a_1 \neq 0$. Roots of this polynomial correspond to the $\pi$-torsion points of $G(K)$. If we insist on these $\pi$-torsion points belonging to $G[\pi](K^{\text{sep}})$ then we may instead consider the separable polynomial $V(x) = [\pi]_F(x^{1/q^n})$:

$$V(x) = a_1 x + a_2 x^q + \cdots + a_d x^{q^{d-1}} + x^{q^d}.$$ 

\textbf{Remark 6.2.1.} When $g = 1$, $V$ corresponds to verschiebung for $G$.

Twisting the coefficients of $V$ by $q^g$-power Frobenius, we define

$$V^{(q^g)}(x) = a_1^{q^g} x + a_2^{q^g} x^q + \cdots + a_d^{q^g} x^{q^{d-1}} + x^{q^d}.$$
Then specifying an element of the Tate module $T(G) = \varprojlim G[\pi^n](K^{\text{sep}})$ amounts to solving the system of equations

\[
\begin{align*}
V(y_0) &= 0, \\
V^{(q^y)}(y_1) &= y_0, \\
V^{(q^y)}(y_2) &= y_1, \\
V^{(q^y)}(y_3) &= y_2, \\
&\vdots
\end{align*}
\]

If $y_0 \neq 0$ then each $y_n$ will be a point of order $\pi^{n+1}$.

Remark 6.2.2. One can arrive at the polynomial given for $[\pi]_F(x)$ above by more elementary means, avoiding deformation theory of formal modules. To see this choose a model for $F$ such that the multiplication-by-$\pi$ is

$$[\pi]_F(x) = a_1x^{q^0} + a_2x^{2q^0} + \cdots$$

with $a_1 \neq 0$ and $a_i \in \pi R$ for all $i \neq q^d$. Applying the $p$-adic Weierstrass preparation theorem to the power series $[\pi]_F(x)$ gives a polynomial coinciding with the polynomial given for $[\pi]_F(x)$ above. Though this method does not show $[\pi]_F(x)$ is a polynomial, it does give a polynomial whose roots coincide with the roots of $[\pi]_F(x)$ in $\{x \in \overline{K} : v_K(x) > 0\}$. This idea was used in the proof of [Gro79 Lemma 4.14].

The following lemma generalizes [Cha00] Theorem 3.3:

**Proposition 6.2.3.** There exists a positive integer $m$, depending only on the valuations $v_K(a_i)$, such that for each $y = (y_0, y_1, \ldots) \in T(G)$ with $y_0 \neq 0$, $v_K(y_n) = q^{-d} \cdot v_K(y_{n-1})$ for all $n \geq m$. Additionally $K(y_n)/K(y_{n-1})$ is a tamely ramified extension of degree $q^d$ for each $n \geq m$.

**Proof.** We have that $V(y_0) = 0$ and $V^{(q^y)}(y_i) = y_{i-1}$ for all $i > 0$.

Consider the Newton polygon of $V(x)$. Note that $(1, v_K(a_1))$ is the point of highest $y$-value on the (lower) boundary of $\mathfrak{N}(V(x))$. It then follows from Proposition 3.3.2 that the valuation of any of the roots of $V(x)$ is at most $v_K(a_1)$. In particular $v_K(y_0) \leq v_K(a_1)$. Since $v_K(y_{i+1}) \leq v_K(y_i)$ for all $i \geq 0$ it follows that $v_K(y_i) \leq v_K(a_1)$ for all $i$.

Now we consider the Newton polygons of the polynomials $V^{(q^y)}(x) - y_{i-1}$. We have just seen that $v_K(y_i) \leq v_K(a_1)$ for all $i$. On the other hand, for each $j$, $v_K(a_j^{q^y}) = q^jv_K(a_j) \geq q^i$ tends to infinity as $i \to \infty$. From this we see that there exists an $m$, depending only on the valuations of the $a_j$, such that for all $i \geq m$ the boundary of $\mathfrak{N}(V^{(q^y)}(x) - y_{i-1})$ is the line segment between $(0, v_K(y_{i-1}))$ and $(q^d, 0)$. It follows from Proposition 3.3.2 that $v_K(a_i) = q^{-d} \cdot v_K(y_{i-1})$ for all $i \geq m$.

The previous paragraph implies that if $n \geq m$ then the ramification index $e(K(y_n)/K(y_{n-1}))$ is at least $q^d$. On the other hand, since $\deg V^{(q^y)}(x) - y_{i-1}) = q^d$ the index $[K(y_i) : K(y_{i-1})]$ is at most $q^d$. Since $e(K(y_i)/K(y_{i-1}))$ divides $[K(y_i) : K(y_{i-1})]$, it follows that

$$e(K(y_n)/K(y_{n-1})) = [K(y_n) : K(y_{n-1})] = q^d$$

for all $n \geq m$. \qed
The following irreducibility theorem generalizes [Cha00, Theorem 3.5]:

**Theorem 6.2.4.** Every non-zero orbit of the action of the monodromy group \( \rho^{0/1}(\mathfrak{S}_K) \) on the Tate module \( T(G) \) is open. In particular, the representation \( \rho^{0/1} \) is irreducible.

**Proof.** Let \( (y_i) \) be a sequence of elements in \( K^{sep} \) satisfying the conditions in Lemma 6.2.3. By Lemma 6.2.3 the orbit of \( y_n \) under \( \text{Gal}(K^{sep}/K(y_{n-1})) \) has \( q^d \) elements for each \( n \geq N \). As \( y_0 \neq 0 \) and each \( y_{n-1} \) has order \( \pi^i \), the sequence \( (y_i) \) corresponds to an element \( y \) of \( U := T(G) - \pi T(G) \). The coset

\[
(\text{Gal}(K^{sep}/K(y_{n-1}))) \cdot y + \pi n+1 T(G) / \pi n+1 T(G).
\]

is in bijection with the orbit of \( y_n \) under \( \text{Gal}(K^{sep}/K(y_{n-1})) \), and thus contains \( q^d \) elements.

Let \( Z \subset U \) be an orbit under the action of \( \rho^{0/1}(\mathfrak{S}_K) \). We claim that for each \( z = (z_i) \in U \),

\[
((z + \pi n T(G)) \cap Z) + \pi n+1 T(G) = z + \pi n T(G)
\]

for all \( n > N \).

Note that the left hand side, which corresponds to the elements \( z' = (z'_i) \) of \( Z \) for which \( z_i = z'_i \) for all \( i < n \), is contained in the right hand side, which corresponds to all elements \( u = (u_i) \) of \( U \) for which \( z_i = u_i \) for all \( i < n \). Therefore, to prove the equality (6.2), it suffices to show that the two sets have the same cardinality. The left hand side has cardinality \( q^d \), since it follows from (6.1) that there are \( q^d \) possibilities for \( z'_n \) such that \( (z'_0, \ldots, z'_n) = (z_0, \ldots, z_{n-1}, z_n) \). The right hand side also has cardinality \( q^d \), because each of its elements can be written as \( (z_0, \ldots, z_{n-1}, t) \) with \( t \in G[\pi n]^{et} \), and there are precisely \( q^d \) possibilities for \( t \in G[\pi n]^{et} \) since \( T(G) \) is a free \( A \)-module of rank \( d \).

Since each \( \rho^{0/1}(\mathfrak{S}_K) \)-orbit \( Z \) is a \( p \)-adic analytic subvariety, the only way (6.2) can hold is if \( \dim(Z) = \dim(T(G)) \), i.e. if \( Z \) is open in \( U \). This implies that each orbit in \( T(G) \setminus \{0\} \) is open. \( \square \)

**Corollary 6.2.5.** There are only finitely many \( \rho^{0/1}(\mathfrak{S}_K) \)-orbits in \( T(G) - \pi T(G) \).

**Proof.** This will follow from the compactness of \( T(G) - \pi T(G) \) in \( T(G) \). We know that \( T(G) \) is a free \( A \)-module of rank \( d \), and one can check that the isomorphism of \( A \)-modules \( T(G) \cong A^d \) is in-fact an isomorphism of topological \( A \)-modules. We also have \( T(G) - \pi T(G) \cong A^d - \pi A^d = (A^*)^d \). Note that \( A^* \) is compact in \( A \) (which can be proven in the same way one proves that \( \mathbb{Z}_p^* \) is compact in \( \mathbb{Z}_p \)). Therefore \( (A^*)^d \) is compact in \( A^d \) since it is a product of compact spaces. It follows that \( T(G) - \pi T(G) \) is compact in \( T(G) \). \( \square \)

7. **Towers Arising From \( \rho^{1/g} \)**

In this section we introduce and study certain \( p \)-adic Lie towers of local fields associated to the Galois representation \( \rho^{1/g} \).

7.1. **The \( 1/g \)-Tower.** Recall that the the \( p \)-divisible group \( G \) gives rise to a continuous homomorphism

\[
\rho^{1/g} : \mathfrak{S}_K \to B_{1/g}^*.
\]
where $B_{1/g}$ is the ring of integers of the division algebra over the field $A \otimes \mathbb{Q}_p$ with Hasse invariant $1/g$. Let $\pi_{1/g}$ be a uniformizer for $B_{1/g}$. Then there is a $\pi_{1/g}$-filtration of $B^*_{1/g}$

$$B^*_{1/g} \supset 1 + \pi_{1/g} B_{1/g} \supset 1 + \pi^2_{1/g} B_{1/g} \supset \cdots.$$ 

This filtration gives rise to a $p$-adic Lie tower of local fields as follows: We appropriate our notation from earlier, letting $\delta_n$ be the reduction map

$$\delta_n : B^*_{1/g} \to \frac{B^*_{1/g}}{1 + \pi^{n+1}_{g} B_{1/g}} \cong \left( \frac{B_{1/g}}{\pi^{n+1}_{g} B_{1/g}} \right)^*,$$

and $\mathfrak{H}_n \subseteq \mathfrak{H}_K$ be the kernel of the composition $\rho_n := \delta_n \circ \rho_{1/g}$. Denoting the fixed field of $\mathfrak{H}_n$ by $K_n \subseteq K^{sep}$ we obtain a tower of fields,

$$K = K_{-1} \subseteq K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{\infty} \subseteq K^{sep},$$

where $K_{\infty}$ is the compositum of the $K_n$ and where $\text{Gal}(K_n/K) \cong \mathfrak{H}_K/\mathfrak{H}_n$. We refer to this tower as the $1/g$-tower of $G$.

7.2. **Properties of the $1/g$-Tower.** In analogy to the torsion tower, we consider subtowers which arise from elements of the generalized Tate module $T^{1/g}(F) = \text{Hom}_{K^{sep}}(F_{1/g}, F_k)$. We will focus on invertible elements of $T^{1/g}(F)$, i.e. elements corresponding to $K^{sep}$-isomorphisms. Fix a $K^{sep}$-isomorphism $\varphi : F_{1/g} \to F$. We write the inverse as

$$\varphi^{-1}(T) = c_1 T + c_2 T^2 + \cdots \in K^{sep}[[T]]$$

For $\sigma \in \mathfrak{H}_K$ let $\varphi^\sigma$ denote the morphism of formal $A$-modules obtained by the action of $\sigma$ on the coefficients of the power-series for $\varphi$. Noting that $\mathfrak{H}_K$ acts trivially on the coefficients of $F_{1/g}$, we can write $\rho^{1/g}$ in terms of $\varphi$ as follows:

$$\rho^{1/g} : \mathfrak{H}_K \to \text{Aut}(F_{1/g}) \cong B^*_{1/g}$$

$$\sigma \mapsto \varphi^{-1} \circ \varphi^\sigma.$$ 

Recall that

$$[\pi]_F(x) = a_1 x^{q^0} + a_2 x^{q^1} + \cdots + a_d x^{q^d} + x^{q^1} \in R[x]$$

and

$$[\pi]_{F_{1/g}}(x) = x^{q^0}.$$

Since $\varphi^{-1}$ is an isomorphism of formal modules from $F$ to $F_{1/g}$,

$$\varphi^{-1} \circ [\pi]_F(x) = [\pi]_{F_{1/g}} \circ \varphi^{-1}(x) = c_1^q x^{q^1} + c_2^q x^{2q^1} + \cdots. \quad (7.1)$$

We now study the tower

$$K \subseteq K(c_1) \subseteq K(c_1, c_2) \subseteq K(c_1, c_2, c_3) \subseteq \cdots \subseteq K^{sep}.$$ 

Gross proved the following results about this tower:

**Proposition 7.2.1.** [Gro79 Lemma 4.2(1)] The coefficients $c_i$ of $\varphi$ are integral in $K^{sep}$.

**Proposition 7.2.2.** [Gro79 Lemma 4.2(2)] If $j < q^n$ then $c_j \in K_{n-1}$ and $K_n = K_{n-1}(c_{q^n})$.

We now prove our main result:
**Theorem 7.2.3.** Suppose the image of \( \rho^{1/g} \) is open in \( B^*_1/g \) so that there exists an \( N \) such that \( 1 + \pi_1^{n+1} B \subseteq \rho^{1/g}(\mathfrak{G}_K) \) for all \( n \geq N \). Define

\[
W(n) = \frac{v_N(a_1)q^{n+1}(q^{g+n} - q^g - q^{n-1} + 1)}{(q^g - 1)(q - 1)} - 1.
\]

Then, for \( n > N \), \( W(n) \) are the upper breaks in the upper ramification filtration of \( \rho^{1/g}(\mathfrak{G}_{K_N}) \) and

\[
\rho^{1/g}(\mathfrak{G}_{K_N})^{W(n)} = 1 + \pi_1^n B_{1/g}.
\]

**Remark 7.2.4.** Based on Galois representations in other settings one would expect, or at least hope, that the image of \( \rho^{1/g} \) is always open. This is known in the case that \( d = 1 \) \cite{Gro79} and in the case \( g = 1 \) and \( A = \mathbb{Z}_p \) \cite{Cha00}. Results of Achter and Norman imply that if \( g < s \) and \( g \neq 2 \) then there exists a \( p \)-divisible group with generic fiber of height \( g \) and special fiber of height \( s \) such that the image of \( \rho^{1/g} \) is open \cite{AN10}.

**Corollary 7.2.5.** Suppose the image of \( \rho^{1/g} \) is open in \( B^*_1/g \) and \( N \) is as in the above theorem. Set \( e_N = [K_N : K] \) and

\[
u_N = \inf\{x : \text{Gal}(K_N/K)^x = (1)\}
\]
\[
l_N = \inf\{x : \text{Gal}(K_N/K)_x = (1)\}.
\]

Then for all \( n > N \) such that

\[(7.2)\]
\[W(n) > l_N\]

we have

\[
\rho^{1/g}(\mathfrak{G}_K)^{W(n)-l_N+u_N} = 1 + \pi_1^n B_{1/g}.
\]

**Proof.** By Proposition 3.2.2 if \( (7.2) \) holds then

\[
\rho^{1/g}(\mathfrak{G}_K)^{W(n)-l_N+u_N} = \rho^{1/g}(\mathfrak{G}_{K_n})^{W(n)}.
\]

The result then follows from Theorem 7.2.3. \( \square \)

Theorem 7.2.3 will follow as an immediate consequence of the following proposition and its corollary:

**Proposition 7.2.6.** Suppose the image of \( \rho^{1/g} \) is open in \( B^*_1/g \) so that there exists an \( N \) such that \( 1 + \pi_1^{n+1} B \subseteq \rho^{1/g}(\mathfrak{G}_K) \) for all \( n \geq N \). Then for all \( n > N \), the ramification filtration of \( \text{Gal}(K_n/K_{n-1}) \cong (B/\pi_1^n B)^* \) has a unique upper and lower break at

\[
B(n) := \frac{q^{g+n}v_{n-1}(a_1)}{q^g - 1} - 1.
\]

**Proof.** Let \( \varphi \in T^{1/g}(\mathcal{F}) \) be invertible with inverse

\[
\varphi^{-1}(T) = c_1 T + c_2 T^2 + \cdots \in K^{sep}[T].
\]

\[15\]
By Proposition 7.2.2, \( K_n = K_{n-1}(c_{q^n}) \) for all \( n \). As \( \rho^{1/g}(\mathfrak{g}_K) \) is open in \( B^*_1 \), there exists an \( N \) such that \( 1 + \pi_g^{n+1} \subseteq \rho^{1/g}(\mathfrak{g}_K) \) for all \( n > N \). This implies that
\[
\text{Gal}(K_n/K_{n-1}) \cong \frac{1 + \pi_1^{1/g} B}{1 + \pi_1^{n+1} B} \cong \left( \frac{B}{\pi^{1/g} B} \right)^*.
\]

In particular, \( K_n \) is a totally ramified cyclic extension of degree \( q^a \) over \( K_{n-1} \). Comparing coefficients of \( x^{q^a} \) in identity (7.1), we have that
\[
a_1^{q^n} c_{q^n} + b = c_{q^n}
\]
where \( b \in k[a_1, \ldots, a_d, c_1, \ldots, c_{q^n-1}] \). Thus \( z^{q^a} - a_1^{q^n} z - b \) is the minimal polynomial for \( c_{q^n} \) over \( K_{n-1} \). Note that for each root \( \alpha \) of this minimal polynomial, \( K_{n-1}(\alpha) = K_n \) is totally ramified over \( K_{n-1} \). Therefore we may apply Chai’s extension of Tate’s Lemma (Corollary 3.3.5) to deduce that the ramification filtration of \( \text{Gal}(K_n/K_{n-1}) \) has a unique upper and lower break at
\[
\frac{q^{g^a} v_{n-1}(a_1^{q^n})}{q^g - 1} - 1 = \frac{q^{g^a} v_{n-1}(a_1^{q^n})}{q^g - 1} - 1.
\]

From the ramification breaks of \( \text{Gal}(K_n/K_{n-1}) \) the upper and lower ramification filtrations of \( \text{Gal}(K_n/K_N) \) can be computed inductively using the identity of Herbrand transition functions (3.1):

**Corollary 7.2.7.** Let \( B(n) \) and \( W(n) \) be as in Proposition 7.2.6 and Theorem 7.2.3 respectfully. The lower ramification filtration of \( \Gamma = \text{Gal}(K_n/K_N) \) is given by
\[
\begin{align*}
\Gamma_x &= \text{Gal}(K_n/K_N) & \text{for } 0 \leq x \leq B(N + 1) \\
\Gamma_x &= \text{Gal}(K_n/K_{N+1}) & \text{for } B(N + 1) < x \leq B(N + 2) \\
& \vdots \\
\Gamma_x &= \text{Gal}(K_n/K_{n-1}) & \text{for } B(n - 1) < x \leq B(n) \\
\Gamma_x &= \text{Gal}(K_n/K_n) = (1) & \text{for } B(N + n) < x.
\end{align*}
\]

The upper ramification filtration of \( \Gamma = \text{Gal}(K_n/K_N) \) is given by
\[
\begin{align*}
\Gamma^x &= \text{Gal}(K_n/K_N) & \text{for } 0 < x \leq W(N + 1) \\
\Gamma^x &= \text{Gal}(K_n/K_{N+1}) & \text{for } W(N + 1) < x \leq W(N + 2) \\
& \vdots \\
\Gamma^x &= \text{Gal}(K_n/K_{n-1}) & \text{for } W(n - 1) < x \leq W(n) \\
\Gamma^x &= \text{Gal}(K_n/K_n) = (1) & \text{for } W(n) < x.
\end{align*}
\]

8. **Ramification Breaks of Galois Characters**

For this section set \( A = \mathbb{Z}_p \). We will be studying the ramification breaks of \( p \)-adic Lie towers associated to the Galois characters
\[
\chi_{0/1} := \det \circ \rho_{0/1} : \mathfrak{g}_K \to \mathbb{Z}_p^*.
\]
\( \chi_{1/g} = \Nm_{1/g} \circ \rho^{1/g} : G_K \to \mathbb{Z}_p^* \).

By Theorem 4.0.3, \( \Nm_{1/g} \circ \rho^{1/g} \) is the inverse of \( \det \circ \rho^{0/1} \) in \( \mathbb{Z}_p \). Our strategy will be to use our result on ramification breaks of towers associated to \( \rho^{1/g} \) to study the ramification breaks of \( \det \circ \rho^{0/1} \).

**Remark 8.0.1.** One may hope to determine when the image of \( \chi_{0/1} (G_K) \), or equivalently \( \chi_{1/g} (G_K) \), is open in \( \mathbb{Z}_p^* \). Since the determinant and reduced norm are open maps, this is technically easier than proving that either \( \rho^{1/g} \) or \( \rho^{0/1} \) have open image. However, it appears that the only cases when \( \chi_{0/1} \) is known to be open are in the cases when \( \rho^{0/1} \) or \( \rho^{1/g} \) are open; namely when \( d = 1 \), in which case \( \chi_{0/1} = \rho^{0/1} \) is surjective [Gro79], and when \( g = 1 \), in which case \( \chi_{1/g} = \rho^{1/g} \) and it can be shown that the image is infinite and hence open (since all infinite closed subgroups of \( \mathbb{Z}_p \) are open) [Cha00].

### 8.1. Ramification Breaks of \( \chi_{0/1} \) and \( \chi_{1/g} \)

The following is a corollary of Theorem 7.2.3.

**Corollary 8.1.1.** Suppose the image of \( \rho^{1/g} \) is open in \( B_{1/g}^* \) and \( N \) is as in Theorem 7.2.3. Then
\[
\chi^{0/1}(G_K)^{W(n)} = 1 + p^n \mathbb{Z}_p
\]
for all \( n > N \).

**Proof.** By [Rie70] Lemma 5:
\[
\Nm_{1/g}(1 + \pi_{1/g}^{n} B_{1/g}) = 1 + p^{[n/g]} \mathbb{Z}_p.
\]

The corollary then follows from Theorem 7.2.3 and Theorem 4.0.3.

Combining the above Corollary with Corollary 7.2.5.

**Corollary 8.1.2.** Suppose the image of \( \rho^{1/g} \) is open in \( B_{1/g}^* \) and \( N \) is as in Theorem 7.2.3. Then for all sufficiently large \( n \),
\[
\chi_{0/1}(G_K)^{aW(n) + b} = 1 + p^n \mathbb{Z}_p
\]
for some fixed constants \( a, b \).

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