A new family of $N$ dimensional superintegrable double singular oscillators and quadratic algebra $Q(3) \oplus so(n) \oplus so(N-n)$

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Abstract
We introduce a new family of $N$ dimensional quantum superintegrable models consisting of double singular oscillators of type $(n, N-n)$. The special cases $(2,2)$ and $(4,4)$ have previously been identified as the duals of 3- and 5-dimensional deformed Kepler–Coulomb systems with $u(1)$ and $su(2)$ monopoles, respectively. The models are multiseparable and their wave functions are obtained in $(n, N-n)$ double-hyperspherical coordinates. We obtain the integrals of motion and construct the finitely generated polynomial algebra that is the direct sum of a quadratic algebra $Q(3)$ involving three generators, $so(n)$, $so(N-n)$ (i.e. $Q(3) \oplus so(n) \oplus so(N-n)$). The structure constants of the quadratic algebra itself involve the Casimir operators of the two Lie algebras $so(n)$ and $so(N-n)$. Moreover, we obtain the finite dimensional unitary representations (unirreps) of the quadratic algebra and present an algebraic derivation of the degenerate energy spectrum of the superintegrable model.

Keywords: superintegrable, quadratic algebra, oscillator algebra

1. Introduction
Isotropic and anisotropic harmonic oscillators [1–4, 6–8] are among the most well known maximally superintegrable systems with applications in various areas of physics. However, the aspect of symmetry algebras generated by well-defined integrals of motion is a
complicated issue in quantum mechanics, as recognized in early work by Jaucl and Hill [1]. In the case of isotropic harmonic oscillators in $N$-dimensional Euclidean space one can apply finite dimensional unitary representations (unirreps) and Gel’fand invariants, and their eigenvalues of the Lie algebra $su(n)$ to provide an algebraic derivation of the spectrum and the degeneracies [8]. The 3D case of isotropic harmonic oscillators was discussed using what is now called the Fradkin tensor and connected to $su(3)$ generators [7]. There are, however, various embeddings of the symmetry algebra, some with operators that do not commute with the Hamiltonian [4], in terms of $su(n+1)$, $su(n,1)$ and $sp(n)$. The anisotropic case has been the subject of research for a long time due to its various applications in nuclear physics. An analysis based on well-defined integrals of motion and polynomial algebras of arbitrary order is known only in the 2D case [9].

It has also become known for a long time that harmonic oscillators in 2-dimensional Euclidean space are related to the 2D Kepler–Coulomb system via the so-called Levi-Civita or regularization transformation [10]. This transformation, however, can only be extended to certain specific dimensions. The 3- and the 5-dimensional Kepler–Coulomb systems are related to the harmonic oscillators in 4- and 8-dimensional Euclidean space respectively via the Kustaanheimo–Stiefel and the Hurwitz transformations [11, 12] and these results can be extended to curved spaces [13]. The Kustaanheimo–Stiefel transformations are connected to the so-called Stäckel transformations and were used to classify superintegrable systems in conformally flat spaces [14–16]. These are specific Levi-Civita, Kustaanheimo–Stiefel and Hurwitz transformations. The above-mentioned connections between various models can be reinterpreted in terms of monopole interactions: 4D harmonic oscillators have a duality relation with 3-dimensional Kepler systems with an Abelian $u(1)$ monopole (also referred to as MICZ-Kepler systems) [17–23] and 8D harmonic oscillators are, in fact, dual to 5D Kepler–Coulomb systems with a non-Abelian $su(2)$ monopole (Yang–Coulomb monopole) [24–32].

It has been discovered that some of these duality relations can be extended to deformed MICZ-Kepler systems and 4D singular oscillators that are sums of two 2D singular oscillators. It has been proven that the properties of superintegrability and multiseparability are preserved [33–38]. The dual of a 4D singular oscillator has interesting properties and its classical analog has period motion [37]. It has been recognized this is also the case for the dual of an 8D singular oscillator [40]. Moreover, it has been shown that a quadratic algebra structure exists for these two models with monopole interactions and their duals [39, 40].

In this paper, we introduce a new family of $N$-dimensional superintegrable singular oscillators with an arbitrary partition $(n, N-n)$ of the coordinates. This model is a generalization of the 4D and 8D systems [38–40] obtained from monopole systems via the Hurwitz transformations. However, even in 4 and 8 dimensions, only the symmetric cases $(2, 2)$ and $(4, 4)$ have been studied [34, 39]. We show that quadratic algebra structures exist for all members of the family and can be used to obtain the energy spectrum. Another main objective of this paper is to extend the analysis presented in [41] to more complicated algebraic structures.

The paper is organized as follows: in section 2, we present the integrals of motion of the new family of superintegrable systems. In section 3, we obtain the quadratic algebra and present the Poisson analog and their Casimir operator. We also highlight the structure of higher rank quadratic algebra and decomposition. In section 4, we generalize the realizations as deformed oscillator algebra, construct the Fock space and obtain the finite-dimensional unitary representations (unirreps). Using this analysis, we also provide an algebraic derivation of the energy spectrum. In section 5, we use the method of separation of variables in double
hyperspherical coordinates and compare them with the results obtained algebraically. In the closing section 6, we present some discussion with a few remarks on the physical and mathematical relevance of these algebras.

2. New family of superintegrable systems

Let us consider a family of \( N \)-dimensional superintegrable Hamiltonians involving singular terms with any two partitions of the coordinates \((n, N-n)\)

\[
H = \frac{p_i^2}{2} + \frac{\omega_r^2 r^2}{2} + \frac{c_1}{x_1^2 + \ldots + x_n^2} + \frac{c_2}{x_{n+1}^2 + \ldots + x_N^2}, \tag{2.1}
\]

where \( r = (x_1, x_2, \ldots, x_N) \), \( p = (p_1, p_2, \ldots, p_N) \), \( r^2 = \sum_{i=1}^{N} x_i^2 \), \( p_i = -i\hbar \partial_i \) and \( c_1 \) and \( c_2 \) are positive real constants. This family of systems represents the sum of two singular oscillators of dimensions \( n \) and \( N-n \). It is a generalization of the 4D and 8D systems obtained via the Hurwitz transformation for specific cases \((2,2)\) in 4D [38, 39] and \((4,4)\) in 8D [40]. In fact, model (2.1) not only contains cases \((2,2)\) and \((4,4)\), but also cases \((1,3)\) and \((1,7)\), \((2,6)\) and \((3,5)\) for 4D and 8D respectively. An algebraic derivation of the energy spectrum has only previously been obtained for the two symmetric cases \((2,2)\) and \((4,4)\).

The system (2.1) has the following integrals of motion

\[
A = -\frac{\hbar^2}{4} \left\{ \sum_{i,j=1}^{N} x_i^2 \partial_{x_i}^2 - \sum_{i,j=1}^{N} x_i x_j \partial_{x_i} \partial_{x_j} - (N-1) \sum_{i=1}^{N} x_i \partial_{x_i} \right\} + \frac{1}{2} \sum_{i=1}^{N} x_i^2 \left\{ \frac{c_1}{x_1^2 + \ldots + x_n^2} + \frac{c_2}{x_{n+1}^2 + \ldots + x_N^2} \right\}, \tag{2.2}
\]

\[
B = \frac{1}{2} \left\{ \sum_{i=1}^{n} p_i^2 - \sum_{i=n+1}^{N} p_i^2 \right\} + \frac{\omega_r^2}{2} \left\{ \sum_{i=1}^{n} x_i^2 - \sum_{i=n+1}^{N} x_i^2 \right\} + \frac{c_1}{x_1^2 + \ldots + x_n^2} - \frac{c_2}{x_{n+1}^2 + \ldots + x_N^2}, \tag{2.3}
\]

which can be verified to fulfill the commutation relation

\[
[H, A] = 0 = [H, B].
\]

The first order integrals of motion are given by

\[
J_{ij} = x_i p_j - x_j p_i, \quad i, j = 1, 2, \ldots, n, \tag{2.4}
\]

\[
K_{ij} = x_i p_j - x_j p_i, \quad i, j = n+1, \ldots, N. \tag{2.5}
\]

The integrals of motion \( A \) and \( B \) are associated with the separation of variables in double hyper-Eulerian and double hyperspherical coordinates respectively. Let

\[
J_{(2i)} = \sum_{i<j} p_{ij}^2, \quad K_{(2i)} = \sum_{i<j} K_{ij}^2. \tag{2.6}
\]
\(J_{(2)}\) and \(K_{(2)}\) represent the second order Casimir operators and fulfill the commutation relations
\[
[H, J_{(2)}] = [H, K_{(2)}] = [A, J_{(2)}] = [A, K_{(2)}] = [B, J_{(2)}] = [B, K_{(2)}] = [J_{(2)}, K_{(2)}] = 0.
\] (2.7)

These commutation relations can be conveniently represented by the following diagrams

\[
\text{Diagram 1:} \quad H \quad A \quad J_{(2)} \quad \quad \quad \quad K_{(2)} \quad B
\]

\[
\text{Diagram 2:} \quad H \quad K_{(2)} \quad J_{(2)} \quad H
\]

\[
\text{Diagram 3:} \quad K_{(2)} \quad J_{(2)} \quad K_{ij} \quad J_{ij}
\]

The first diagram shows that \(J_{(2)}\) and \(K_{(2)}\) are central elements. The second and third diagrams show that \(J_{(2)}\) and \(K_{(2)}\) are the Casimir operators of \(so(n)\) and \(so(N-n)\) Lie algebras realized by angular momentum \(J_{ij}, i, j = 1, 2, \ldots, n\) and \(K_{ij}, i, j = n+1, \ldots, N\) respectively. The models appear to be minimally superintegrable and the total number of algebraically independent integrals is \(N + 1\). The construction of minimally superintegrable systems is interesting, as research has so far mainly focused on maximally superintegrable systems. In the next section, we construct the quadratic algebra, its Casimir operator, and the finite dimensional unitary representations, which give the energy spectrum of the superintegrable systems.

3. Quadratic algebra structure

In this section we present the quadratic algebra structure \(Q(3)\) of the superintegrable Hamiltonian systems (2.1). We show how the \(su(N)\) symmetry algebra of isotropic harmonic oscillators is broken down into \(Q(3) \oplus so(n) \oplus so(N-n)\).

3.1. The isotropic harmonic oscillator and \(su(N)\)

In this subsection we review some facts about isotropic harmonic oscillators and \(su(N)\). The Hamiltonian (2.1) in the limit \(c_1 = 0\) and \(c_2 = 0\) reduces to the isotropic harmonic oscillator in the \(N\)-dimension given by
In this limiting case, we define the following operators \([5]\) by

\[
a_i = \frac{1}{\sqrt{2}} (p_i + i \omega x_i), \quad a_i^* = \frac{1}{\sqrt{2}} (p_i - i \omega x_i),
\]

which satisfy the commutation relation \([a_i, a_j^*] = -\omega/\hbar \delta_{ij}\). Furthermore, we can define that the operators

\[
F_{ij}^l = \frac{1}{\sqrt{2}\omega \hbar} \{ a_i, a_j^* \}
\]

are hermitian and satisfy the commutation relations

\[
\left[ F_{ij}^l, F_{kl}^* \right] = \delta_{il} F_{kj}^* - \delta_{jk} F_{li}^*.
\]

Hence \(su(N)\) is the symmetry algebra of the isotropic harmonic oscillator. We briefly present some key elements related to the algebraic derivation of the energy spectrum. The \(su(N)\) symmetry algebra can be embedded in the non-compact algebra \(sp(N)\), and define the operators

\[
F_{ij}^l = -\frac{1}{\sqrt{2}\omega \hbar} \{ a_i^*, a_j^* \}, \quad F_{ij}^0 = \frac{1}{\sqrt{2}\omega \hbar} \{ a_i, a_j \}
\]

satisfying the commutators

\[
\left[ F_{ij}^l, F_{ik}^0 \right] = \delta_{jk} F_{ij}^l + \delta_{jl} F_{ik}^0, \quad \left[ F_{ij}^l, F_{i0}^k \right] = -\delta_{ik} F_{jl}^0 - \delta_{jl} F_{ik}^0.
\]

One can construct the Casimir operator of \(sp(N)\) in the form

\[
Q_2 = B_2 + \frac{H^2}{2} + \frac{1}{\sqrt{2}\omega \hbar} \left\{ F_{ij}^0, F_{ij}^l \right\},
\]

where \(B_2\) is the Casimir operator of \(su(N)\). The eigenvalues of \(Q_2\) and \(B_2\) in the representations of \(sp(N)\) and \(su(N)\) are as

\[
Q_2 = -\frac{N}{2} \left( N + \frac{1}{2} \right), \quad B_2 = -\frac{l(l + N)(N - 1)}{N},
\]

where all representations belonging to the \(H\)-eigenstates are obtained for \(l = 1, 2, \ldots\), and the energy spectrum

\[
E = \omega \hbar \left( l + \frac{N}{2} \right), \quad l = 0, 1, 2, \ldots
\]

### 3.2. Quadratic Poisson algebra

In this subsection we study the Poisson algebra of the classical version of the superintegrable system (2.1) and its Casimir operators. The system has the second order integrals of motion,
\[ A = \frac{1}{4} \left\{ \sum_{i,j=1}^{N} x_i^2 p_j^2 - \sum_{i,j=1}^{N} x_i x_j p_i p_j \right\} + \frac{1}{2} \sum_{i=1}^{N} \left( \frac{c_1}{x_1^2 + \ldots + x_n^2} + \frac{c_2}{x_{n+1}^2 + \ldots + x_N^2} \right). \]  

(3.10)

B and \( J_{(2)}, K_{(2)} \) are given by (2.3) and (2.6) respectively. It can be verified that they satisfy

\[ \{ H, A \}_p = \{ H, B \}_p = 0 = \{ H, J_{(2)} \}_p = \{ H, K_{(2)} \}_p, \]

(3.11)

where \( \{ \cdot, \cdot \}_p \) is the usual Poisson bracket. We can also check these following Poisson brackets

\[ \{ A, J_{(2)} \}_p = \{ A, K_{(2)} \}_p = \{ B, J_{(2)} \}_p = \{ B, K_{(2)} \}_p = 0 = \{ J_{(2)}, K_{(2)} \}_p. \]

The above commutation relations show that \( J_{(2)} \) and \( K_{(2)} \) are second order Casimir operators and central elements. We now construct a new integral of motion from \( A \) and \( B \) as

\[ \{ A, B \}_p = C. \]

(3.12)

The integrals of motion \( C \) is a cubic function of momenta. After a direct but involving computation relying on the properties and identities of the Poisson bracket, we can show that the integrals of motion generate the quadratic Poisson algebra \( QP(3) \),

\[ \{ A, B \}_p = C, \]

(3.13)

\[ \{ A, C \}_p = -4AB + J_{(2)} H - K_{(2)} H + 2(c_1 - c_2) H, \]

(3.14)

\[ \{ B, C \}_p = 2B^2 - 2H^2 + 16\omega^2 A - 4\omega^2 J_{(2)} - 4\omega^2 K_{(2)} - 8\omega^2 (c_1 + c_2). \]

(3.15)

The Casimir operator of this quadratic Poisson algebra can be shown to be cubic and explicitly given by

\[ K = C^2 + 4AB^2 - 2\left[ J_{(2)} H - K_{(2)} H + 2(c_1 - c_2) H \right] B + 16\omega^2 A^2 - 2\left[ 8\omega^2 (c_1 + c_2) + 4\omega^2 J_{(2)} + 4\omega^2 K_{(2)} + 2 H^2 \right] A. \]

(3.16)

By means of explicit expressions as functions of the coordinates and the momenta for the generators \( A, B, C \) and the central elements, the Casimir operator becomes

\[ K_1 = -2J_{(2)} H^2 - 2K_{(2)} H^2 - 4(c_1 + c_2) H^2 - \omega^2 J_{(2)}^2 - \omega^2 K_{(2)}^2 + 2\omega^2 J_{(2)} K_{(2)} - 4\omega^2 (c_1 - c_2)^2. \]

The quadratic Poisson algebra and the Casimir operator are useful in deriving the quadratic algebra and Casimir operator: the lowest order terms of \( \hbar \) in the quantum case coincide with the Poisson analog. The first order integrals of motion generate a Poisson algebra isomorphic to the \( so(n) \) Lie algebra

\[ \{ J_{ij}, J_{kl} \}_p = \delta_{ik} J_{jl} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} - \delta_{jk} J_{il}, \]

for \( i, j, k, l = 1, \ldots, n \) and \( so(N-n) \) Lie algebra

\[ \{ K_{ij}, K_{kl} \}_p = \delta_{ik} K_{jl} + \delta_{jl} K_{ik} - \delta_{il} K_{jk} - \delta_{jk} K_{il}, \]

(3.17)

for \( i, j, k, l = n + 1, \ldots, N - n \). So the full symmetry algebra is a direct sum of the quadratic Poisson algebra \( QP(3) \), \( so(n) \) and \( so(N-n) \) Lie algebras.
3.3. Quadratic algebra

We now construct the integral of motion $C$ of the quantum system from (2.2) and (2.3) via commutator

$$[A, B] = C,$$

(3.18)

where $C$ represents a new integral of motion and is a cubic function of momenta. The cubicness of $C$ explains the impossibility of expressing $C$ as a polynomial function of other integrals of motion, which are quadratic functions of momenta. After an involving but direct computation that is based on the properties of commutators and various identities, we obtain the following quadratic algebra $Q(3)$ of the integrals of motion $H$, $A$ and $B$

$$[A, C] = 2\hbar^2 [A, B] - \hbar^2 J_{(2)} H + \hbar^2 K_{(2)} H - \frac{\hbar^2}{4} \left[ 8c_1 - 8c_2 - (N - 4)(N - 2n)\hbar^2 \right] H + \frac{\hbar^4}{4} N (N - 4) B,$$

(3.19)

$$[B, C] = -2\hbar^2 B^2 + 2\hbar^2 H^2 - 16\hbar^2 \omega^2 A + 4\hbar^2 \omega^2 J_{(2)} + 4\hbar^2 \omega^2 K_{(2)}$$

$$+ 8\hbar^2 \omega^2 \left\{ c_1 + c_2 - \frac{\hbar^2}{4} n (N - n) \right\}.$$

(3.20)

It can be demonstrated that the Casimir operator is a cubic expression of the generators and is explicitly given in terms of the generators ($A$, $B$ and $C$) as

$$K = C^2 - 2\hbar^2 [A, B^2] + \frac{\hbar^4}{4} \left[ 16 - N(N - 4) \right] B^2 + 2\hbar^2 \left[ J_{(2)} H - K_{(2)} H \right.$$ 

$$+ \frac{1}{4} \left( 8c_1 - 8c_2 - (N - 4)(N - 2n)\hbar^2 \right) H \right] B - 16\hbar^2 \omega^2 A^2$$

$$+ 2\hbar^2 \left[ 8\omega^2 \left\{ c_1 + c_2 - \frac{\hbar^2}{4} n (N - n) \right\} + 4\omega^2 J_{(2)} + 4\omega^2 K_{(2)} + 2H^2 \right] A.$$ 

(3.21)

Using the realization of the integrals of motion $A$, $B$, $C$, and the central element as differential operators, we can represent the Casimir operator (3.21) only in terms of the central elements $H$, $J_{(2)}$ and $K_{(2)}$.

$$K = 2\hbar^2 J_{(2)} H^2 + 2\hbar^2 K_{(2)} H^2 + \frac{\hbar^2}{4} \left[ 16c_1 + 16c_2 - \left\{ 4(N - 4) \right\}$$

$$- (N - 2n)^2 \hbar^2 H^2 + \hbar^2 \omega^2 J_{(2)} + \hbar^2 \omega^2 K_{(2)} - 2\hbar^2 \omega^2 J_{(2)} K_{(2)}$$

$$+ 4\hbar^2 \omega^2 \left\{ c_1 - c_2 - \frac{1}{4} (N - 4)(N - n)\hbar^2 \right\} J_{(2)} - 4\hbar^2 \omega^2 \left\{ c_1 - c_2 \right\}$$

$$+ \frac{1}{4} n (N - 4)\hbar^2 \left\{ K_{(2)} + 4\hbar^2 \omega^2 \left\{ (c_1 - c_2)^2 - \frac{1}{2} (N - n)(N - 4)\hbar^2 c_1 \right\} \right.$$ 

$$- \frac{1}{2} n (N - 4)\hbar^2 c_2 + \frac{1}{4} n (N - n)(N - 4)\hbar^4 \right].$$

(3.22)

This is a key step in the application of the deformed oscillator algebra approach which relies on both forms of the Casimir operators. The first order integrals of motion, which are simply components of angular momentum, generate an algebra isomorphic to the $so(n)$ Lie
algebra
\[
[ J_{ij}, J_{kl} ] = i \left( \delta_{ik} J_{jl} + \delta_{jk} J_{il} - \delta_{il} J_{jk} - \delta_{jl} J_{ik} \right) \hbar,
\]
for \( i, j, k, l = 1, \ldots, n \) and \( so(N-n) \) Lie algebra
\[
[ K_{ij}, K_{kl} ] = i \left( \delta_{ik} K_{jl} + \delta_{jk} K_{il} - \delta_{il} K_{jk} - \delta_{jl} K_{ik} \right) \hbar,
\]
for \( i, j, k, l = n+1, \ldots, N-n \). So the full symmetry algebra is a direct sum of the quadratic algebra \( Q(3) \), \( so(n) \) and \( so(N-n) \) Lie algebras. Thus the \( su(N) \) Lie algebra generated by the integrals of motion of the \( N \)-dimensional isotropic harmonic oscillators is deformed into higher rank quadratic algebra \( Q(3) \oplus so(n) \oplus so(N-n) \) for the family of superintegrable systems \((2.1)\). The structure constants of the quadratic algebra involve three central elements: the Hamiltonian and the two Casimir operators of the Lie algebras approached in the decomposition. Quadratic algebras involving three generators have been obtained and studied by various authors \([42-45]\). In fact, the one involved in the decomposition would be related to case \( QR(3) \), called quadratic Racah algebra.

4. Energy spectrum and unirreps

We now consider the realizations of the quadratic algebra \((3.18)-(3.20)\) in terms of deformed oscillator algebra \([45, 46]\) \([\mathbb{K}, b^\dagger, b] \) defined by
\[
[\mathbb{K}, b^\dagger] = b^\dagger, \quad [\mathbb{K}, b] = -b, \quad bb^\dagger = \Phi(\mathbb{K} + 1), \quad b^\dagger b = \Phi(\mathbb{K}), \quad (4.1)
\]
where \( \mathbb{K} \) is the number operator and the function \( \Phi(x) \) is a well-behaved real function satisfying the boundary condition
\[
\Phi(0) = 0, \quad \Phi(x) > 0, \quad \forall \ x > 0. \quad (4.2)
\]
The \( \Phi(x) \) is the so-called structure function. The realization of \( Q(3) \) is of the form \( A = A(\mathbb{K}), \ B = b(\mathbb{K}) + b^\dagger \rho(\mathbb{K}) + \rho(\mathbb{K})b \), where \( A(x), b(x) \) and \( \rho(x) \) are functions. Similar to the quadratic algebra for 2D superintegrable systems \([45]\), we have
\[
A(\mathbb{K}) = \hbar^2 \left\{ (\mathbb{K} + u)^2 - \left( \frac{N-2}{2} \right)^2 \right\}, \quad (4.3)
\]
\[
b(\mathbb{K}) = \frac{8c_1 - 8c_2 + 4J_{2\omega} - 4K_{2\omega} + \left( 4N - 8n + 2nN - N^2 \right) \hbar^2}{16 \hbar^2 \left\{ (\mathbb{K} + u)^2 - \frac{1}{2} \right\}}, \quad (4.4)
\]
\[
\rho(\mathbb{K}) = \frac{1}{3.2^{20} \hbar^{16} (\mathbb{K} + u)(1 + \mathbb{K} + u)(1 + 2(\mathbb{K} + u))}, \quad (4.5)
\]
where \( u \) is a constant (in fact a representation dependent constant) to be determined from constraints on the structure function. We construct the structure function \( \Phi(x) \) by using the realization, the quadratic algebra \((3.18), (3.19), (3.20)\) and the two forms of the Casimir operator \((3.21)\) and \((3.22)\).
\( \Phi(x; u, H) = 12288\hbar^{12}\left[ 64c_1^2 + 64c_2^2 - 48h^4 - 32h^2J(2) + 16J(2)^2 - 32\hbar^2K(2) \\
- 32J(2)K(2) + 16K(2)^2 - 64\hbar^2J(2)n + 64\hbar^2K(2)n + 48\hbar^4n^2 \\
+ 32h^4N + 32h^2J(2)N - 32h^2K(2)N - 48h^4nN + 16h^2J(2)nN \\
- 16h^2K(2)nN - 32h^4n^2N + 8h^4N^2 - 8h^2J(2)N^2 + 8h^2K(2)N^2 \\
+ 32h^4n^2N^2 + 4h^4n^2N^2 - 8h^4N^3 - 4h^4nN^3 + h^4N^4 \\
- 16c_1\left[ 4(J(2) - K(2)) + \hbar^2 \left\{ (N - 4)(2n - N) + 4(1 - 2(x + u))^2 \right\} \right] \\
- 16c_1\left[ 8c_2 - 4J(2) + 4K(2) + \hbar^2 \left\{ (N - 4)(N - 2n) \\
+ 4(1 - 2(x + u))^2 \right\} \right] + 32h^2\left[ 4(J(2) + K(2)) + \hbar^2 \left\{ 2n^2 + (N - 2)^2 \\
- 2nN \right\} (x + u) - 32h^2\left[ 4(J(2) + K(2)) + \hbar^2 \left\{ 2(n^2 - 2) \\
- 2(n + 2)N + N^2 \right\} (x + u)^2 - 512h^4(x + u)^2 + 256h^4(x + u)^4 \right\} \right\] \\
\times \left[ H^2 - \hbar^2 \left\{ 1 - 2(x + u)^2 \right\} \omega^2 \right]. \tag{4.6} \end{align*}

We will show how the finite dimensional unirreps can be obtained using an appropriate Fock space. In 2D we need \( n \in [n, E] \) such that \( \mathcal{H}(n, E) = n(n, E) \). However, in our case one needs to define the quantum numbers associated with certain subalgebra chains. We use two subalgebra chains \( \mathcal{H}(n, E) = n(n, E) \) and related chains of quadratic Casimir operators \([47]\) \( J^{(\alpha)}_2 \) and \( K^{(\alpha)}_2 \) can be written as

\[
J^{(\alpha)}_2 = \sum_{\alpha} J_{\alpha}, \quad \alpha = 2, \ldots, n, \tag{4.7}
\]

\[
K^{(\alpha)}_2 = \sum_{\alpha} K_{\alpha}, \quad \alpha = n + 2, \ldots, N. \tag{4.8}
\]

In fact, \( \mathcal{H}(n, E) = n(n, E) \), and \( H \) in \( \Phi(x, u, H) \) replaced by \( E \). Hence, by using the eigenvalues of \( J_{(2)}, K_{(2)} \) and \( H \), the structure function becomes the following factorized form:

\[
\Phi(x) = -12582912\hbar^{18}\omega^2 \left[ x + u - \frac{1}{4}(2 + m_1 + m_2) \right] \left[ x + u - \frac{1}{4}(2 - m_1 - m_2) \right] \\
\times \left[ x + u - \frac{1}{4}(2 + m_1 - m_2) \right] \left[ x + u - \frac{1}{4}(2 - m_1 - m_2) \right] \\
\times \left( x + u - \frac{H + \hbar\omega}{2\hbar\omega} \right) \left( x + u - \frac{H - \hbar\omega}{2\hbar\omega} \right). \tag{4.9}
\]

where \( \hbar^2m_1^2 = 8c_2 + 4J(2)_2 + \hbar^2(n - 2)^2 \) and \( \hbar^2m_2^2 = 8c_2 + 4K(2)_2 + \hbar^2(N - n - 2)^2 \). For unirreps to be finite-dimensional, we impose the following constrains on the structure function:

\[
\Phi(p + 1; u, E) = 0; \quad \Phi(0; u, E) = 0; \quad \Phi(x) > 0, \quad \forall x > 0, \tag{4.10}
\]

where \( p \) is a positive integer and \( u \) is an arbitrary constant. We then obtain finite \( (p + 1) \)-dimensional unirreps. The solution of the constrains \((4.10)\) gives the energy \( E \) and the constant \( u \). Thus, we obtain the following possible structure functions and energy spectra for

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\( \epsilon_1 = \pm 1, \epsilon_2 = \pm 1, \eta = 24576h^{18} \omega^2 \). Set-1:

\[
u = \frac{-E + \hbar \omega}{2\hbar \omega}, \quad E = 2\hbar \omega \left( p + 1 + \frac{\epsilon_1 m_1 + \epsilon_2 m_2}{4} \right),
\]

\[
\Phi(x) = \eta x \left[ 4 + 4p - 4x - (1 - \epsilon_1)m_1 + (1 + \epsilon_2)m_2 \right]
\times \left[ 4 + 4p - 4x + (1 + \epsilon_1)m_1 - (1 - \epsilon_2)m_2 \right] [4 + 4p - 4x - (1 - \epsilon_1)m_1 - (1 - \epsilon_2)m_2] [4 + 4p - 4x + (1 + \epsilon_1)m_1 + (1 + \epsilon_2)m_2].
\]

Set-2:

\[
u = \frac{E + \hbar \omega}{2\hbar \omega}, \quad E = 2\hbar \omega \left( p + 1 + \frac{\epsilon_1 m_1 + \epsilon_2 m_2}{4} \right),
\]

\[
-\eta x \left[ 4 + 4p + 4x - (1 - \epsilon_1)m_1 + (1 + \epsilon_2)m_2 \right]
\times \left[ 4 + 4p + 4x + (1 + \epsilon_1)m_1 - (1 - \epsilon_2)m_2 \right] [4 + 4p + 4x - (1 - \epsilon_1)m_1 - (1 - \epsilon_2)m_2] [4 + 4p + 4x + (1 + \epsilon_1)m_1 + (1 + \epsilon_2)m_2].
\]

Set-3:

\[
u = \frac{1}{4} \left( 2 + \epsilon_1 m_1 + \epsilon_2 m_2 \right), \quad E = 2\hbar \omega \left( p + 1 + \frac{\epsilon_1 m_1 + \epsilon_2 m_2}{4} \right),
\]

\[
\Phi(x) = \eta (p + 1 - x) [4x - (1 - \epsilon_1)m_1 - (1 - \epsilon_2)m_2]
\times \left[ 4x + (1 + \epsilon_1)m_1 - (1 - \epsilon_2)m_2 \right] [4x - (1 - \epsilon_1)m_1 + (1 + \epsilon_2)m_2]
\times \left[ 4x + (1 + \epsilon_1)m_1 + (1 + \epsilon_2)m_2 \right] [2 + 2p + 2x + \epsilon_1 m_1 + \epsilon_2 m_2].
\]

The structure functions \( \Phi(x) > 0 \) for \( \epsilon_1 = 1, \epsilon_2 = 1 \) and \( m_1, m_2 > 0 \). In the limit \( \epsilon_1 = 0 \) and \( \epsilon_2 = 0 \), these results coincide with \( N \)-dimensional harmonic oscillators with the following relation among the quantum numbers \( l = \frac{2}{n} + \frac{1}{N-n} \) and the algebraic derivation using the \( su(N) \) and \( sp(N) \) Lie algebra and their Casimir operators and eigenvalues. Let us mention that the value of \( \eta \) does not play a role—only the sign needs to be taken into account for the constraint to obtain the finite-dimensional unirreps.

### 5. Separation of variables

Each member of the family of superintegrable Hamiltonian systems (2.1) is multiseparable and allows for the Schrödinger equation the separation of variables in double hyper-Eulerian and double hyperspherical coordinates. We can rewrite (2.1) into the sum of two singular oscillators of dimensions \( n \) and \( N-n \) as

\[ H = H_1 + H_2, \quad \text{(5.1)} \]

where

\[ H_1 = \frac{1}{2} \left( p_1^2 + \ldots + p_n^2 \right) + \frac{\omega^2 r^2}{2} + \frac{\epsilon_1}{n}, \quad \text{(5.2)} \]
and the position vectors \( r_1 = x_1 + ... + x_{N-1} \), \( \mathbf{r}_2 = x_{N+1} + ... + x_N \). The Schrödinger equation \( H\psi(r, \Omega) = E\psi(r, \Omega) \) can also be written as

\[
H_1\psi(r_1, \Omega) = E_1\psi(r_1, \Omega), \quad H_2\psi(r_2, \Omega) = E_2\psi(r_2, \Omega),
\]

where \( E_1 \) and \( E_2 \) are the eigenvalues of \( H_1 \) and \( H_2 \) respectively. The \( N \)-dimensional hyperspherical coordinates are given by

\[
x_1 = r \sin(\Phi_{N-1}) \sin(\Phi_{N-2}) ... \sin(\Phi_1),
\]

\[
x_2 = r \sin(\Phi_{N-1}) \cos(\Phi_{N-2}) ..., \cos(\Phi_1),
\]

\[
\vdots \quad x_{N-1} = r \sin(\Phi_{N-1}) \cos(\Phi_{N-2}),
\]

\[
x_N = r \cos(\Phi_{N-1}),
\]

(5.5)

where the examples of \( N \times \) are Cartesian coordinates in the hyperspherical coordinates, \( \{\Phi_1, ..., \Phi_{N-1}\} \) are the hyperspherical angles and \( r \) is the hyperradius. We can also introduce a double type of hyperspherical coordinates system by considering two copies and setting \( N = n \) and \( N = N - n \) for the \( H_1 \) component and the \( H_2 \) component respectively. The Schrödinger equation of \( H_1 \) in \( n \)-dimensional hyperspherical coordinates

\[
\frac{\partial^2}{\partial r_1^2} + \frac{n - 1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_1^2} \mathcal{L}^2(n) - \omega^2 r_1^2 - \frac{2c'_1}{r_1^2} + 2E'_1 \psi(r_1, \Omega) = 0,
\]

(5.6)

where \( c'_1 = \frac{c_1}{r_1^2}, \quad \omega' = \frac{\omega}{r_1}, \quad E'_1 = \frac{E_1}{r_1^2} \) and the grand angular momentum operator \( \mathcal{L}^2(n) \), which satisfies the recursive formula

\[
-\mathcal{L}^2(n) = \frac{\partial^2}{\partial \Phi_{n-1}^2} - (n - 2) \cot(\Phi_{n-1}) \frac{\partial}{\partial \Phi_{n-1}} - \frac{\mathcal{L}^2(n - 1)}{\sin^2(\Phi_{n-1})},
\]

(5.7)

valid for \( n > 0 \) and \( \mathcal{L}^2(1) = 0 \). The separation of the radial and angular parts of (5.6) can be performed by setting \( \psi(r_1, \Omega) = R_1(r_1) y(\Omega_{n-1}) \). Thus, we obtain

\[
\frac{\partial^2 R_1(r_1)}{\partial r_1^2} + \frac{n - 1}{r_1} \frac{\partial R_1(r_1)}{\partial r_1} + \left( 2E'_1 - \omega^2 r_1^2 - \frac{2c'_1}{r_1^2} - \frac{l_a(l_a + n - 2)}{r_1^2} \right) R_1(r_1) = 0,
\]

(5.8)

\[
\mathcal{L}^2(n)y(\Omega_{n-1}) = l_a(l_a + n - 2)y(\Omega_{n-1}),
\]

(5.9)

where \( l_a(l_a + n - 2) \) is the general form of the separation constant. Now the radial equation (5.8) can be converted by setting \( u = e^{\alpha r_1}, \quad R_1(u) = e^{-\alpha l_a} f(u) \) and \( f(u) = e^{\delta_1 f(u)} \), to

\[
u f''_1(u) + \left( 2\left( \delta_1 + \frac{l_a + n}{4} \right) - u \right) f'_1(u) + \left( \frac{E'_1}{2\omega'} - \left( \delta_1 + \frac{l_a + n}{4} \right) \right) f_1(u) = 0,
\]

(5.10)
where

$$\delta_1 = \left\{ \frac{1}{2} l_n + \frac{n - 2}{4} \right\}^2 + \frac{1}{2} c_1' - \frac{n - 2}{4} - \frac{1}{2} l_n. \tag{5.11}$$

Set

$$N_1 = \frac{E_1}{2\omega} - \left( \delta_1 + \frac{1}{2} l_n + \frac{n}{4} \right). \tag{5.12}$$

Then the equation (5.10) represents the confluent hypergeometric equation and it gives the solution in terms of special function

$$\psi_{N_1}(u) = \sqrt{2} \frac{\left( N_1 + 2 \left( \delta_1 + \frac{1}{2} l_n + \frac{n}{4} \right) \right)}{N_1!} \frac{ae^{- \frac{u}{2} (\delta_1 + l_n / 2)}}{\sqrt{2} \left( \delta_1 + \frac{1}{2} l_n + \frac{n}{4} \right)} \times {}_2F_1 \left( -N_1; 2 \left\{ \delta_1 + \frac{1}{2} l_n + \frac{n}{4} \right\}; u \right) \tag{5.13}$$

In order for the wave functions to be square integrable, the parameter $N_1$ needs to be positive. Hence we obtain the discrete energy $E_1$ of $H_1$ from (5.12) as

$$E_1 = \frac{2\hbar \omega}{p + 1} \left( N_1 + \frac{\alpha_1}{2} + \frac{1}{2} \right), \tag{5.14}$$

where $\alpha_1 = 2\delta_1 + l_n + \frac{n-2}{2}$. Let us point out that the wave functions can also be written in terms of Laguerre polynomials. The wave equation of $H_2$ in $(N-n)$-dimensional hyperspherical coordinates provides a similar solution $\psi_{N_1,n}$, replacing $r_1, l_n, c_1', \delta_1$ by $r_2, N-n, l_{N-n}, c_2', \delta_2$ respectively in (5.13), and the energy $E_2$, replacing $N_1$, $\alpha_1$ by $N_2$, $\alpha_2$ respectively in (5.14). Hence the energy spectrum of the $N$-dimensional double singular oscillators

$$E = \frac{2\hbar \omega}{p + 1} \left( N_1 + N_2 \right), \tag{5.15}$$

where $p = N_1 + N_2$, which coincides with the energy expression obtained algebraically. In fact, these multiseparability properties are also shared by its classical analog, as the Hamilton–Jacobi can also be separated in double type coordinates systems.

### 6. Conclusion

In this paper, we have extended the symmetric double singular oscillators in 4D and 8D to arbitrary dimensions with any partition $(n, N-n)$ of the coordinates. This provides a new family of superintegrable systems. As main results, we construct and obtain their realization as deformed oscillator algebras. We also construct finite dimensional unitary representations which enable the algebraic derivation of the energy spectrum. These are compared with the results from the separation of variables method. Moreover, the new family may also include duals of deformed higher dimensional Kepler–Coulomb systems involving non-Abelian monopoles [50, 51].

Our systems are generalisations of one of the four 2D Smorodinsky–Winternitz models [48]. Further generalisations are possible. For example,
\[ H = \frac{p^2}{2} + \frac{\omega^2 r^2}{2} + \sum \frac{c_i}{x_i^2} + \ldots + \frac{c_n}{x_n^2} \]

generalizes the \( N \)-dimensional Smorodinsky–Winternitz model [49] to the one with any partition \((n_1, n_2, \ldots, n_k)\) such that \(n_1 + n_2 + \ldots + n_k = N\). This model would also be superintegrable, but with a more complicated quadratic algebra and embedded structure seen in [52, 53].

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