The super-Eddington nature of supermassive stars

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ABSTRACT
Supermassive stars (SMSs) are massive hydrogen objects that slowly radiate their gravitational binding energy. Such hypothetical primordial objects may have been the seed of the massive black holes (BHs) observed at the centre of galaxies. Under the standard picture, these objects can be approximately described as \( n = 3 \) polytropes, and they are expected to shine extremely close to their Eddington luminosity. Once however one considers the porosity induced by instabilities near the Eddington limit, which give rise to super-Eddington states, the standard picture should be modified. We study the structure, evolution and mass loss of these objects. We find the following. First, the evolution of SMSs is hastened due to their increased energy release. They accelerate continuum-driven winds. If there is no rotational stabilization, these winds are insufficient to evaporate the objects such that they can collapse to form supermassive BHs, however, they do prevent SMSs from emitting a copious amount of ionizing radiation. If the SMSs are rotationally stabilized, the winds evaporate the objects until a normal sub-Eddington star remains having a mass of a few \( \times 100 \, M_\odot \).

Key words: stars: Population III – stars: winds, outflows – dark ages, reionization, first stars.

1 INTRODUCTION
Supermassive stars (hereinafter SMSs) are hydrogen objects with large masses in the range of \( 10^4 - 10^8 \, M_\odot \). Their state is described as a hydrostatic equilibrium between their self-gravity on one hand and radiation pressure on the other hand. Since radiation leaks out as the objects shine, the thermal energy is replenished through the slow release of the gravitational binding energy. This is unlike the less massive stars where the central temperatures and densities are high enough to ignite thermonuclear reactions, the energy of which sustains the hydrostatic equilibrium. Thus, SMSs look like massive radiatively supported stars, but without nuclear reactions (Wagoner 1969; Weinberg 1972; Shapiro & Teukolsky 1983).

Under the standard picture (summarized in Section 1.1), SMSs are expected to shine at nearly the Eddington limit. However, it is now known that objects shining near this luminosity should develop instabilities, reduce the effective opacity in their atmosphere and attain super-Eddington luminosities (Shaviv 2001a,b). SMSs are expected to be no different and should be described with super-Eddington states.

The main qualitative difference introduced by these super-Eddington states is the acceleration of a continuum-driven wind. This high mass-loss rate may affect their evolution. It is also responsible for a larger photosphere, which may affect the amount of ionizing radiation emitted by the SMSs. These effects are studied in the present work.

Two further complications should be considered. First, it was demonstrated by Fowler (1966) that a dynamically insignificant amount of rotation (namely, rotation for which the star is still close to being spherical) can stabilize the SMS against the GR collapse into a massive black hole (BH). In the present analysis we will consider both cases – with and without rotational stabilization.

The second complication was proposed by Begelman (2010) who considered a more realistic model for the formation of SMSs. It was argued that under a natural scenario, SMSs can form with an entropy inversion such that the outer layers which are accreted later have a higher entropy. Such SMSs are not \( n = 3 \) polytropes anymore. Instead, they have a small convective core surrounded by a massive convectively stable envelope. We do not consider this type of model here. Just as the standard model is modified here to consider the super-Eddington states, Begelman’s scenario should be likewise modified. We defer this to subsequent work.

1.1 Supermassive stars: the standard picture
To first approximation SMSs can be modelled as pure polytropes (e.g. Shapiro & Teukolsky 1983). That is, they can be approximated using \( P = K \rho^{(n+1)/n} \). Note that these and other variables used are defined in Table 1. Since radiation pressure dominates the gas pressure in SMSs, the total pressure is approximately given by the radiation pressure. Together with the equation of hydrostatic equilibrium,

\[
\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}
\]

(1)
Table 1. Nomenclature.

| Symbol | Description |
|--------|-------------|
| $A, B$ | Opacity reduction law parameters (equation 20) |
| $a$    | Radiation constant |
| $c$    | Speed of light |
| $\{, \}$ | subscripts -- value at the stellar centre |
| $E$    | Total energy of the star (equation 4) |
| $F$    | Radiative flux |
| $G$    | Gravitational constant |
| $K$    | Polytropic constant |
| $k$    | Boltzmann’s constant |
| $k_1, k_2$ | Constants in the total energy expression (equation 5) |
| $L$    | Total luminosity |
| $L_{\text{conv max}}$ | Maximal convective energy flux (equation 18) |
| $L_{\text{Edd}}$ | Classical Eddington luminosity (equation 3) |
| $m(r)$ | Mass enclosed within radius $r$ |
| $m_\rho$ | Proton mass |
| $m_w$  | Mass loss through continuum-driven wind (equation 12) |
| $m_{\text{limit}}$ | The photon-tiring mass-loss limit (equation 29) |
| $M$    | Stellar mass |
| $n$    | Polytropic index |
| $P, P_r, P_g$ | Total, radiation and gas pressures |
| $r$    | Radial coordinate |
| $R$    | Stellar radius |
| $R_{\text{umic}}$ | Sonic (and critical) point for the wind |
| $s$    | Total specific entropy (gas and radiation) |
| $s_{\text{rad}}$ | Specific entropy of the radiation (equation 6) |
| $T$    | Temperature |
| $t_f$  | Lifetime of the SMS |
| $u$    | Internal energy density |
| $v_s$  | Isothermal speed of sound at the sonic radius |
| $W$    | Dimensionless constant in $m_w(L)$ relation (equation 13) |
| $w$    | Specific enthalpy |
| $\gamma$ | The adiabatic index |
| $\Gamma$ | $L/L_{\text{Edd}}$ |
| $\Gamma_{\text{crit}}$ | $\Gamma$ above which the opacity decreases |
| $\kappa$ | Opacity - absorption per unit mass |
| $\kappa_V$ | Extinction - absorption per unit volume |
| $\kappa_{\text{Th}}$ | Thomson scattering opacity |
| $\kappa_{\text{eff}}$ | The reduced effective opacity (equation 20) |
| $\kappa_{\text{eff}}^{\text{Th}}$ | The reduced effective extinction (equation 11) |
| $\mu$  | Chemical potential |
| $\rho$ | Density |
| $\rho_{\text{sonic}}$ | Density at the sonic of the wind |
| $\tau$ | Optical depth |
| $\Phi$ | Total angular momentum of the star |
| $\Psi$ | The rotational kinetic energy of the star (equation 9) |

and equation of radiative transfer

$$L = -4\pi r^2 \frac{c}{3k\rho} \frac{d}{dr} (aT^4) ,$$

(2)

it is easy to see that the star radiates at its Eddington limit,

$$L = L_{\text{Edd}} = \frac{4\pi c G M}{\kappa_{\text{Th}}},$$

(3)

if the opacity is given by the Thomson scattering opacity, $\kappa_{\text{Th}}$. Once we will allow the opacity to decrease in the super-Eddington states, we will see that the object’s luminosity will necessarily increase.

1.1.1 Stability of non-rotating SMSs

One of the most interesting aspects of non-rotating SMSs is their instability towards gravitational collapse, which arises from corrections due to general relativity. We succinctly summarize the conditions under which this instability arises (for more details see chapter 17 in Shapiro & Teukolsky 1983, hereinafter ST83). These are important as they are required for the description of the evolution of SMSs. The Newtonian energy of a polytrope is given by

$$E = \int_0^R 4\pi r^2 \left( u + \frac{GM\rho}{r} \right) dr ,$$

(4)

where $u$ is the internal energy. This integral is solved for an $n = 3$ polytrope to get

$$E = k_1 K \rho_c^{1/3} M - k_2 G \rho_c^{1/3} M^{5/3},$$

(5)

where $k_1$ and $k_2$ are constants, and $\rho_c$ is the central density of the SMS. For a given mass, the equilibrium configuration occurs when $\partial E/\partial \rho_c = 0$. This condition of equilibrium provides a relation between the total mass and the specific entropy of radiation (e.g. ST83, section 17.2),

$$s_{\text{rad}} = \frac{4m_Ha}{3} \left( \frac{3\pi G}{a} \right)^{3/4} \sqrt{\frac{M}{8\pi}}$$

(6)

Higher order corrections to the energy can be obtained by adding the effects of gas pressure to the internal energy and the post-Newtonian (i.e. GR) corrections to the gravitational energy. The total energy of the star including these corrections is (see section 17.4 in ST83)

$$E = AM \rho_c^{1/3} - BM^{5/3} \rho_c^{1/3} + CM \rho_c^{1/3} \ln \rho_c - DM^{7/3} \rho_c^{-1/3},$$

(7)

where $A, B, C$ and $D$ are constants (the terms proportional to $C$ and $D$ are derived from the additional effects of gas pressure and post-Newtonian corrections). The condition for equilibrium configuration $\partial E/\partial \rho_c = 0$ gives the central density as a function of the mass and the specific entropy. General relativistic instabilities appear when the extremal energy switches from being a minimum to a maximum, i.e. when $\partial^2 E/\partial \rho_c^2 = 0$. Differentiating equation (7) twice yields (see section 17.4 in ST83):

$$0 = \frac{1}{2} CM \rho_c^{-2/3} - \frac{1}{3} DM^{7/3} \rho_c^{-1/3} .$$

(8)

Thus, there is a critical value for the central density above which the star becomes unstable. This analysis will not change once we introduce the existence of super-Eddington states.

1.1.2 Stabilisation by rotation

The general relativistic instability disappears if the SMS is rotating fast enough (Fowler 1966). In this case, the Newtonian term for the rotational energy should be added to the total energy (equation 7). It is given by

$$\Psi = \frac{1}{2} \int r^2 \rho^2 \sin^2(\theta) dV ,$$

(9)

where $\omega$ is the angular velocity, $\theta$ is the polar angle measured from the axis of rotation and the integration is carried out over the whole star. For a constant\(^1\) angular momentum $\Phi$, the critical value of $\Phi$ which stabilizes the SMS against GR instability is given by

$$\frac{\Phi_{\text{crit}}}{M} = 3.6 \times 10^{34} \frac{M}{M_\odot} \text{cm}^2 \text{s}^{-1} .$$

(10)

This value is relatively small even for a SMS with $M = 10^8 M_\odot$. If $\Phi > \Phi_{\text{crit}}$, rotation prevents the gravitational collapse.

\(^1\) For the more general case of non-constant angular momentum, see Fowler (1966).
2 BACKGROUND: SUPER-EDDINGTON STATES

Before proceeding to construct super-Eddington SMSs models, we begin by reviewing the relevant physics pertaining the emergence of super-Eddington states. These include three particular elements. First, the rise of inhomogeneities was shown (Shaviv 1998) to reduce the effective opacity. Secondly, once a super-Eddington state arises, strong continuum-driven winds are accelerated. Finally, if the wind mass loss is too large, wind stagnation and a photon-tired state arise (Owocki & Gayley 1997). In it a layer is formed in which strong shocks mediate a high-energy flux without an excessive mass flux. These components are the necessary building blocks for the super-Eddington states, and we therefore review them below. Two more examples where this theory is applied can be found in Dotan & Shaviv (2011) and Dotan, Rossi & Shaviv (2011).

2.1 The rise of super-Eddington states

According to common wisdom, spherically symmetric objects cannot shine beyond their classical Eddington limit, $L_{\text{Edd}}$, since no hydrostatic solution exists. In other words, if objects do pass $L_{\text{Edd}}$, they are highly dynamic. They have no steady state, and a huge mass loss should occur since their atmospheres are then gravitationally unbound and they should therefore be expelled. Thus, astrophysical objects according to this picture can pass $L_{\text{Edd}}$ but only for a short duration corresponding to the time it takes them to dynamically stabilize once super-Eddington conditions are forced.

For example, this can be seen in detailed 1D numerical simulations of nova thermonuclear runaways, where novae can be in a super-Eddington state but only for a duration comparable to the dynamical time-scale (e.g. Prialnik & Kovetz 1992). However, once they do stabilize, they are expected and indeed do reach in the simulations a sub-Eddington state. Namely, we naively expect to find no steady-state super-Eddington atmospheres. This however is not the case in nature, where nova eruptions are clearly super-Eddington for durations which are orders of magnitude longer than their dynamical time-scale (Shaviv 2001b). This is exemplified with another clear super-Eddington object — the great eruption of the massive star η-Carinae, which was a few times super-Eddington for over 20 yr (Shaviv 2000).

The existence of a super-Eddington state can be naturally explained once we consider the following.

(i) Atmospheres become unstable as they approach the Eddington limit. In addition to instabilities that operate under various special conditions (e.g. photon bubbles in strong magnetic fields: Arons 1992; Gamme 1998; Begelman 2002 or s-mode instability under special opacity laws: Glatzel 1994; Papaloizou et al. 1997), two instabilities operate in Thomson scattering atmospheres (Shaviv 2001a). It implies that all atmospheres become unstable already below the Eddington limit.

(ii) The effective opacity for calculating the radiative force on an inhomogeneous atmosphere is not necessarily the microscopic opacity. Instead it is given by

\[
\kappa_{\text{eff}} = \frac{\langle F_{\text{k}} \rangle_{V}}{\langle F \rangle_{V}},
\]

where $\langle \cdot \rangle_{V}$ denotes volume averaging and $F$ is the flux. The situation is very similar to the Rosseland versus force opacity means used in non-grey atmospheres, where the inhomogeneities are in frequency space as opposed to real space. The effective opacity is always reduced for the special case of Thomson scattering.

Thus, we find that as atmospheres approach their classical Eddington limit they necessarily become inhomogeneous. These inhomogeneities necessarily reduce the effective opacity such that the effective Eddington limit is not surpassed even though the luminosity can be super-classical-Eddington. This takes place in the outer regions of luminous objects, where the radiation diffusion timescale is shorter than the dynamical time-scale in the atmosphere. Further inside the atmosphere, convection is necessarily excited such that the total energy flux may be super-Eddington, but the radiative part of it is necessarily sub-Eddington with the convective flux carrying the excess (Joss, Salpeter & Ostriker 1973).

2.2 Super-Eddington winds

The atmosphere remains sub-Eddington while being classically super-Eddington only as long as the inhomogeneities comprising the non-linear structure are optically thick. This condition breaks at some radius where the density is low enough. From this radius, the effective opacity returns to its microscopic value and hence the radiative force becomes super-Eddington, thereby accelerating a thick wind. Because it is optically thick, the conditions at the wind affect the structure at the wind’s base.

At the critical point the radiative and gravitational forces balance each other. For a steady-state wind this point has to coincide with the sonic point (where the mass-loss velocity equals the local speed of sound), thus allowing us to obtain the local mass-loss rate. It is given by

\[
m_{\text{w}} = 4\pi R_{\text{sonic}}^{2} \rho_{\text{sonic}} v_{\text{s}}(R_{\text{sonic}}) = \text{const},
\]

where $R_{\text{sonic}}$ is the critical/sonic radius, $\rho_{\text{sonic}}$ is the density at this radius and $v_{s}$ is the local isothermal speed of sound.

Based on the fact that instabilities develop structure with a typical size comparable to the density scale height in the atmosphere, it is possible to estimate the average density at the sonic point (Shaviv 2001b). This is achieved by estimating the density for which the opacity reduction is just sufficient to keep the average radiative force balancing gravity. Below this radius the structure is more opaque, the reduction in the effective opacity is larger and the net radiative force is small. Above this radius the structure becomes optically thin, the effective opacity is closer to its microscopic value and the radiative force is larger than that of gravity. Using the aforementioned density, the mass loss can be estimated to be

\[
m_{\text{w}} = \mathcal{W} \frac{L - L_{\text{Edd}}}{c v_{s}},
\]

where $\mathcal{W}$ is a dimensionless wind function. $\mathcal{W}$ can be calculated from first principles only after the non-linear state of the inhomogeneities is understood. This however is still lacking as it requires elaborate 3D numerical simulations of the non-linear steady state. Nevertheless, it can be done in several phenomenological models which depend on geometrical parameters such as the average size

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2 For example, see the discussion of the spherically symmetric case in section 13.7 of ST83, and the discussion of a non-spherical configuration surpassing the Eddington limit in Paczyński & Wiita (1980).

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3 This is a general property of steady-state transonic winds. Mathematically, it arises from the requirement that both sides of the momentum conservation equation switch sign together. For more details, see e.g. section 3.1 in Lamers & Cassinelli (1999).
3.1.1 Convective core

The convective zone begins at the centre of the star and occupies the bulk of the hydrostatic radius and most of the SMS mass. In this region the first two equations describing the stellar structure (in radial coordinates, using standard notation) are the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2},$$

and the equation of mass conservation:

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

The third is the equation of state, which is obtained as a combination of an ideal ionized gas and blackbody radiation,

$$P = P_e + P_s = \frac{1}{3}aT^4 + \frac{b k T}{\mu m_p},$$

where $\mu$ is the mean molecular weight of the primordial composition gas.

The last equation describes energy transfer inside the star. Under conditions prevailing in SMSs the relevant energy transfer takes place through convection. The temperature gradient is hence given by

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dT}{dr},$$

where $\gamma$ is the adiabatic index. Convection is present if the standard Schwarzschild criterion is satisfied, but although this condition is satisfied all the way out to the stellar surface, at a certain radius the energy transport becomes purely radiative as convection becomes inefficient. For convection to be efficient the convective flux must be smaller than the maximum possible, which is given by (Owocki & Gayley 1997)

$$L_{\text{conv, max}} \approx 4\pi r^2 \rho v_s^3,$$

where $v_s$ is the speed of sound. This is because the maximal convective flux is at most the internal heat content, which is of the order of $\rho v_s^2$, moving at the speed of sound, $v_s$. Convection cannot be more efficient since shocks will inhibit any faster motions.

3.1.2 Atmosphere

If the density is too low convection becomes inefficient, the energy transfer takes place through radiative diffusion, and the radiative luminosity, $L_{\text{rad}}$, becomes super-Eddington. The equation of radiative transfer is modified to take into account the porous nature of the atmosphere:

$$\frac{dT}{dr} = -\frac{3\kappa_{\text{eff}} L_{\text{rad}}}{16\pi a c r^2 T^2},$$

where $\kappa_{\text{eff}}$ is the effective opacity. As described in Section 3.1.1, the atmosphere develops inhomogeneities and the gas becomes porous when the flux approaches a critical value. This reduces the radiative force exerted on the gas.

Shaviv (2001a) found two radiative hydrodynamic instabilities operating in Thomson scattering atmospheres. One of the instabilities is sensitive to the lower boundary conditions and resembles the formation of chimneys but may arise already for $\Gamma = 0.5$. This instability was later found numerically (Turner et al. 2005). A second instability is more general and may be described as a radiative counterpart to the Rayleigh–Taylor instability. It arises for $\Gamma \approx 0.8$. Once the instabilities appear their growth is halted only through the
formation of shocks. For this reason, we can expect the reduction in the opacity to be abrupt (i.e. with a discontinuous derivative).\footnote{Note that when strong magnetic fields are present, we can expect the formation of magnetic 'photon bubbles' (e.g. Turner et al. 2005). However, unlike the radiative hydrodynamic instabilities, the magnetic ones have no threshold, and the associated opacity reduction should be a smooth function of $\Gamma$. Magnetic fields are probably important in super-critical accretion discs, but unlikely in SMSs.}

Thus, we assume that the relation between the effective Eddington factor $\Gamma_{\text{eff}} \equiv L/L_{\text{Edd}}$ and the classical Eddington factor $\Gamma \equiv L/L_{\text{Edd}}$ is empirically given by

$$
\Gamma_{\text{eff}} = 1 - \frac{A}{\Gamma^B} \quad \text{for } \Gamma > \Gamma_{\text{crit}},
$$

$$
\Gamma_{\text{eff}} = \Gamma \quad \text{for } \Gamma < \Gamma_{\text{crit}}.
$$

(20)

$\Gamma_{\text{crit}}$ is the critical $\Gamma$ above which inhomogeneities are excited, so the effective opacity for $\Gamma > \Gamma_{\text{crit}}$ can be written as

$$
\kappa_{\text{eff}} = \kappa_{\text{Th}} \left( 1 - \frac{A}{\Gamma^B} \right) / \Gamma.
$$

(21)

Since we expect a continuous $\Gamma_{\text{eff}}, A$, $B$ and $\Gamma_{\text{crit}}$ satisfy the equation

$$
\Gamma_{\text{crit}} = 1 - A / \Gamma_{\text{crit}}^B.
$$

From theoretical considerations we take $\Gamma_{\text{crit}}$ to be 0.8 (Shaviv 2001a), though we have checked also $\Gamma_{\text{crit}} = 0.5$ as elaborated in Section 5.4. This implies a relation between the normalization constant $A$ and the power law $B$ that is given by

$$
A = (1 - \Gamma_{\text{crit}}) \Gamma_{\text{crit}}^B.
$$

(22)

Since the saturation of the radiative hydrodynamic instabilities has not been thoroughly investigated yet, the exact form of $\Gamma_{\text{eff}}$, and therefore the value of $B$ in the empirical description, has not been determined. Hence, we leave $B$ as a free parameter.

The last equation describing the structure of this region is the equation of hydrostatic equilibrium:

$$
\frac{dP}{dr} = -g \rho.
$$

(23)

where $g = GM/r^2$. To a very good approximation, the mass in the atmosphere is negligible, such that we can safely assume $M$ to be constant.

3.1.3 Continuum-driven winds

As we described in Section 2.2 the average density decreases as we move up the atmosphere. At some point the typical optical depth of the non-linear structure decreases below unity. Once the typical structure becomes transparent, radiation cannot be funnelled around anymore, and the effective opacity returns to its microscopic value. At this point the average radiative force on the gas elements becomes super-Eddington again, and so the gas at this radius experiences a net force outwards. A wind is therefore accelerated from this radius.

The forces acting on the material in this region are the pressure gradient outwards, and gravity inwards. Thus, the equation of motion of the gas (assuming a steady-state wind, namely, that $\partial / \partial t \rightarrow 0$) is given by

$$
\frac{dv}{dr} = -\frac{GM}{r^2} - \frac{1}{\rho} \frac{dP}{dr},
$$

(24)

where $v$ is the wind velocity. Note that the pressure is the total pressure. The continuity equation implies

$$
\dot{m} = 4\pi r^2 \rho v = \text{const}.,
$$

(25)

where $\dot{m}$ is the mass-loss rate. The equation of energy conservation is

$$
L(r) = \dot{E} - \dot{m} \left[ \frac{v^2}{2} + w(r) - \frac{GM}{r} \right],
$$

(26)

where $w = 5kT / 2\mu m_\rho + 4P_{\text{rad}} / \rho$ is the specific enthalpy and $\dot{E}$ is the total energy loss rate from the sonic point, given by

$$
\dot{E} = L_{\text{sonic}} + \dot{m} \left[ \frac{1}{2} v_{\text{sonic}}^2 + w(R_{\text{sonic}}) - \frac{GM}{R_{\text{sonic}}} \right].
$$

(27)

Because the wind is optically thick, the transfer of energy can be described by radiative diffusion. Note that although conditions for convection are still satisfied, we can neglect this energy transport since it is necessarily unimportant in the supersonic flow. Thus, the temperature gradient in the wind is

$$
\frac{dT}{dr} = \frac{3kL\rho}{16\piacr^2T^2}.
$$

(28)

3.1.4 Photon-tired winds

The mass loss is predicted according to conditions at the sonic point. If the mass loss is too high and the potential well is too deep, the luminosity may be insufficient to push the wind to infinity. This gives rise to so called photon-tired winds (Owocki, Gayley & Shaviv 2004). The mass loss for which the wind becomes photon tired is

$$
\dot{m}_{\text{tiring}} = \frac{L_{\text{sonic}}R_{\text{sonic}}}{MG}.
$$

(29)

The behaviour of photon-tired winds was numerically studied by van Marle, Owocki & Shaviv (2009). It was found that shocks form between in-falling material and outflowing wind. This forms a layer of shocks in which there is a large kinetic flux but without the associated mass flux. When photon-tired winds arise, the mass loss from the top of the layer of shocks is reduced to less than the photon-tiring limit. Following the aforementioned simulations, we take it to be

$$
\dot{m} = 0.9\dot{m}_{\text{tiring}}.
$$

(30)

In other words, most of the luminosity is used to accelerate almost as much mass as possible. The rest of the mass stagnates and falls back to the star.

It is interesting to note that although superficially photon-tired winds resemble convection, because both have gas motion advecting energy outwards, the two mechanisms are quite different for several reasons. First, convection advects energy primarily as heat content. For this reason, it is quenched when radiative diffusion is important. The shock structure of photon-tired winds advects energy primarily as kinetic energy of the gas. Secondly, convection is intrinsically a multidimensional phenomenon, but the shock structure can exist in 1D. Finally, the conditions which give rise to convection depend on the local thermal gradient. The shock structure inherently depends on global conditions, i.e. the fact that the luminosity is insufficient to drive the gas to infinity.

3.1.5 Boundary conditions

The structure of the star is determined by the specific entropy at the centre (which almost uniquely determines the mass) and by the central density (see Section 1.1.1). The only free variable in a given structure is therefore the luminosity. The value chosen for the
luminosity should agree with the surface conditions on the radiation field, which should satisfy the blackbody radiation law:

\[ L_{\text{ph}} = 4\pi r^2 \rho \sigma T_{\text{eff}}^4, \quad (31) \]

where the \( T_{\text{eff}} \) is the temperature at optical depth \( \tau = 2/3 \). The optical depth in the wind is given by

\[ \tau = \int_r^\infty \kappa \rho \, dr. \quad (32) \]

However, if the wind is sufficiently thick, the gas temperature can decrease enough for recombination of the ionized hydrogen to occur before the photosphere is reached (i.e. for \( \tau > 2/3 \)). This implies that the opacity decreases from its Thomson value to effectively zero, beyond which the gas is transparent. Thus, the photosphere in this case is going to be located at the recombination front. Since it is relatively sharp, we can take the classical Eddington result that \( \tau = 2/3 \) at this radius and require equation (31) to be valid there. The actual temperature of recombination depends on the density, but since the structure is not sensitive to the actual recombination temperature, we take a nominal \( T_{\text{rec}} = 4500 \) K. As we will show in the results, the states of very low effective temperature for which \( T_{\text{eff}} \lesssim 4500 \) K are very common in SMSs.

### 3.2 Evolution of non-rotating SMSs

Assuming that the central temperature never becomes high enough for nuclear burning to be important, and because in the mass range of \( 4 \times 10^4 \)–\( 10^9 \) M\(_\odot\) the thermal evolution time-scale is longer than the hydrostatic time-scale, the evolution of the SMS is described by a sequence of hydrostatic equilibrium states. The star begins as a large spherical cloud with little binding energy. As the star radiates its binding energy, it becomes denser and smaller. In addition, the wind from the star changes the total mass as the star evolves and the central entropy is no longer constant.\(^5\) For the evolution of the SMS we need to know its total energy. The internal component is given by

\[ E_{\text{int}} = \int_0^R 4\pi r^2 u \, dr, \quad (33) \]

where \( u \) is the internal energy density. The gravitational Newtonian energy is given by

\[ E_{\text{grav}} = -\int_0^R 4\pi r G m \rho \, dr. \quad (34) \]

#### 3.2.1 End stage of non-rotating SMSs

The star ends its hydrostatic equilibrium life when it reaches the critical radius at which general relativistic effects ultimately destabilize the star. This critical radius is given by (see ST83 for details)

\[ \frac{G M}{R_{\text{crit}} c^2} = 0.6295 \left( \frac{M}{M_\odot} \right)^{-1/2}. \quad (35) \]

Specifically, the criterion for instability is given in form of a critical density (equation 8) above which the destabilizing GR corrections overcome the stabilizing effects of the gas pressure. When this density is reached, the evolution of the star ends and a BH is formed.

\(^5\) Note that for a homogeneous \( n = 3 \) polytropic star the specific entropy depends only on the mass and chemical potential, not on the radius (e.g. see ST83). Thus without mass loss or chemical evolution, the specific entropy remains constant.

### 3.3 Evolution of rotating SMSs

While the initial stages of rotating SMSs are the same as the non-rotating SMSs, the following steps change drastically. As mentioned in Section 1.1.2, SMSs having enough angular momentum do not collapse when they reach the critical density. Instead they keep evolving through a sequence of hydrostatic states, while radiating their binding energy and evaporating some of their mass. While contracting the central temperature increases, and nuclear burning becomes important. At some point the energy released by nuclear burning becomes large enough to support the SMSs from further contraction.

We divide the evolution of rotating SMSs into three stages.

(i) **Stage Ia: super-Eddington contraction.** The evolution is governed by the SMS contraction and the radiation of its binding energy. For non-rotating SMSs this stage ends with GR collapse. This sets the duration of this stage. For rotating SMSs the stage ends when the central temperature is high enough for hydrogen to be ignited.

(ii) **Stage Ib: super-Eddington CNO burning.** Although the pp-chain and triple-\( \alpha \) burning cannot supply a sufficient amount of energy to halt the contraction, the temperature can be high enough for these reactions to build up\(^{12}\) C. This build-up sets the duration of this stage.

(iii) **Stage II: normal massive star.** At this stage the luminosity is sub-Eddington, and the SMS becomes a normal massive star. Here the evolutionary time-scale is governed by nuclear burning. This final stage is the longest. Since the SMSs at this stage are normal stars, they have already been extensively described in the literature (e.g. Bond, Arnett & Carr 1984) and therefore will not be described here.

### 4 NUMERICAL METHODS

The problem we are required to solve can be divided into two parts. First, we need to solve the hydrostatic model of the SMS with a given mass and energy. Then we need to evolve these models to describe the evolution of the SMSs from their formation to their collapse. Here we describe the main methods we used.

#### 4.1 Solving a stellar model

The stellar structure is determined by solving equations (14)–(17) for the convective region, equations (19) and (23) for the atmosphere, and equations (24)–(26) and (28) for the wind. These equations are solved by first guessing the total luminosity. The solution is then iterated using the shooting method until the outer boundary condition given in equation (31) is fulfilled.

#### 4.2 Evolving the stellar models

After we obtain a single snapshot of a SMS we can proceed to find how it evolves with time. Each model solution predicts a luminosity \( L \) and a mass-loss rate \( \dot{m} \). The stellar mass and total energy are then
evolved through conservation of mass and energy:

\[ \frac{dM}{dt} = -\dot{m} \]  
\[ \frac{dE}{dt} = L_{\text{nuc}} - L, \]  

where \( L_{\text{nuc}} \) is the total nuclear energy generation rate, while \( L \) is the total luminosity. \( E \) is the sum of the internal and gravitational energies. Since \( \rho_c \) and \( s \) are the real variables describing the models and not \( M \) and \( E \), the model evolution proceeds by choosing a small central density increment \( \Delta \rho_c = \epsilon \rho_c \) (specifically with \( \epsilon = 0.001 \)), and then solving for the new model.

5 RESULTS

We now proceed to describe the results of our simulations. We present the structure and evolution of the SMSs from their early phase to the time they reach the critical radius and collapse if they are non-rotating, or until they become normal massive stars if they do rotate.

We also perform a parameter study. Since the exact porous opacity law required is not adequately known, we evolve solutions for the same mass but different effective opacity laws. This allows us to study the influence of the atmospheric effective opacity on the stellar structure and evolution.

5.1 Quasi-static configuration

The early phase (\( R_{\text{sonic}} \approx 2000R_{\text{crit}} \)) of a SMS with \( M_0 \approx 5.5 \times 10^4 M_\odot \) is depicted in Figs 2 and 3. In Fig. 2 we present the gas density and the temperature in both the convective zone and the atmosphere. In the latter the density decreases by three orders of magnitude, enabling a much lighter wind. The wind velocity and temperature are described in Fig. 3, from the sonic point to the photosphere. In the case depicted here the wind is optically and geometrically thick (\( R_{\text{ph}} \sim 1.5R_{\text{sonic}} \)).

5.2 Stage Ia: Temporal evolution before collapse or ignition of nuclear burning

Figs 4 and 5 plot the temporal evolution of the sonic and photospheric radii, the luminosity and the mass of a star with an initial mass of \( M = 5.5 \times 10^4 M_\odot \). This star loses about 13 per cent of its initial mass during this stage (until ignition of nuclear burning). As can be seen in Fig. 4 the wind is optically thick, causing the effective
temperature to remain low. Thus, a negligible amount of ionizing radiation is emitted during this stage of evolution, unlike the standard lore for SMSs. The results for non-rotating SMSs (which only have this stage of evolution) are summarized in Table 2, for different initial masses.

### 5.3 Ignition and evolution of rotating SMSs

Figs 6 and 7 show stages Ib and II of a rotating SMS with an initial mass $M = 5 \times 10^4 \, M_\odot$. Specifically, Fig. 6 depicts the mass, luminosity (as $\Gamma_s$, the Eddington parameter at the base of the wind), central temperature ($T_c$) and effective temperature ($T_{ph}$) of this SMS. Fig. 7 describes the sonic and photospheric radii as a function of time. The two stages Ib and II discussed in Section 3.3 are easily distinguished. Stages Ib lasts about $10^3$ yr. Here the SMS radiates its binding energy, while contracting (as can be seen as a fast decrease of the sonic radius). The central temperature increases from about $10^6$ to greater than $10^7$ and the luminosity is very high (greater than $10L_{\text{Edd}}$). Stage II lasts of the order of $10^7$ yr. Here the evolution is ruled by the wind mass loss from the atmosphere. The total mass and luminosity decrease, but the central temperature and the sonic radius remain almost constant. Stage III (not shown) is the last and longest. At this stage the luminosity is sub-Eddington and the evolution is ruled by the nuclear reactions. If a wind is present, it is a weak-line-driven wind due to the small amount of CNO that was produced.

### 5.4 The effects of the atmospheric opacity law

One of the uncertainties in the model is the opacity law behaviour of the porous atmosphere. The reasons it is not known well are because it depends on the non-linear radiative hydrodynamic con-

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**Table 2. Summary of the pre-collapse stage of non-rotating SMSs.** Tabulated are the various SMS characteristics as a function of the initial mass. The values $M(t_f)$, $T_c(t_f)$ and $\rho_s(t_f)$ are the mass, central temperature and central density of the star at the time collapse, $t_f$. A few points to note are the following. (A) The two lower masses represented ($5.5 \times 10^4$ and $1.0 \times 10^5 \, M_\odot$) will not collapse even if there is no rotation, because at some point of their evolution the thermonuclear energy generation becomes important, supporting the star against additional contraction. Thus, these objects are not SMSs, but ‘Very Massive Stars’ instead (e.g. see Bond et al. 1984) (B) The SMSs always have a large enough photosphere and therefore a low effective temperature (determined by recombination in the wind). As a consequence, SMSs do not contribute a significant amount of ionizing radiation. (C) The mass shed by the objects is small relative to the total mass of the star. Thus, as long as thermonuclear reactions are not ignited, the total wind mass loss does not significantly affect the evolution. (D) Masses larger than the above range may collapse to become a black hole, but they do not form an intermediate hydrostatic object because of the short gravitational collapse time-scale.

| $M_{\text{initial}}$ ($M_\odot$) | $t_f$(lifetime) (yr) | $M(t_f)$/$M_{\text{initial}}$ | $T_c(t_f)$ (K) | $\rho_s(t_f)$ (g cm$^{-3}$) |
|--------------------------|-----------------|--------------------------|-----------------|------------------|
| $5.5 \times 10^4$        | –               | 0.87                     | 2.1 $\times 10^8$ | 6                |
| $1.0 \times 10^5$        | –               | 0.86                     | 2.2 $\times 10^8$ | 4.9              |
| $2.5 \times 10^5$        | 8.6 $\times 10^2$ | 0.94                     | 1.0 $\times 10^8$ | 0.3              |
| $5 \times 10^5$          | 3.2 $\times 10^2$ | 0.97                     | 5.2 $\times 10^7$ | 2.7 $\times 10^{-2}$ |
| $1.5 \times 10^6$        | 60              | 0.98                     | 1.7 $\times 10^7$ | 5 $\times 10^{-4}$ |
| $5.0 \times 10^6$        | 10              | >0.99                    | 5.0 $\times 10^6$ | 8 $\times 10^{-6}$ |
| $1.0 \times 10^7$        | <10             | 1.00                     | 2.5 $\times 10^6$ | 7.5 $\times 10^{-7}$ |
We obtained numerical solutions for rotating and non-rotating SMSs in the mass range between $5 \times 10^2$ and $1 \times 10^7 \, M_\odot$. Within this range, the hydrodynamical time-scale is shorter than the thermal time-scale such that the star is always in hydrostatic equilibrium during its evolution. Above this range, there is no equilibrium phase and the star collapses on a dynamical time-scale. For SMSs with $M > 1 \times 10^7 \, M_\odot$, the thermonuclear energy generation of the non-rotating SMSs is negligible; thus in this range, the SMSs are hydrostatic objects which slowly release their binding energy, and shrink until they end their life through a collapse into a supermassive BH. On the other hand, rotating SMSs do not collapse but instead ignite nuclear burning and lose mass through a wind until a sub-Eddington star is left. The mass of this star depends on the Eddington parameter $\Gamma_{\text{crit}}$ for which the system becomes sub-Eddington (see Section 5.4).

Our analysis included two main parts.

(i) We constructed hydrostatic models of SMSs. These included the full calculation of the SMS structure while considering that luminous atmospheres are unstable and become porous as the Eddington luminosity is approached, and that continuum-driven winds are accelerated by these objects. When relevant it also included nuclear reactions.

(ii) We then evolved the models for SMSs using the hydrostatic configurations as basic building blocks. The evolution allows us to calculate various general properties such as the total mass loss, the energy radiated in ionizing photons or the total lifetime.

The main results obtained are the following:

(i) The solution describes objects which radiate above their Eddington limit. This stands in contrast to the standard solution where the luminosity is just below the Eddington limit. The luminosities obtained vary between roughly $2L_{\text{Edd}}$ and $50L_{\text{Edd}}$ at the base of the wind (the larger the mass the higher the luminosity), but the actual luminosities emitted from the photosphere vary between roughly $10^{-2}L_{\text{Edd}}$ and $2L_{\text{Edd}}$ (the larger the mass, the lower is the photospheric luminosity). The rest of the energy is used to drive a wind.

(ii) Because the binding energy at the onset of the GR collapse into a BH does not depend on the outer layers of the star, higher luminosities imply shorter lifetimes in non-rotating stars.

(iii) The mass loss driven by the wind in the pre-collapse phase is typically between 5 and 20 per cent of the total mass, where less massive SMSs have a larger mass loss. This immediately implies that mass loss is crucial to stars less massive than the least massive SMSs. These stars, which have prolonged lifetimes (due to nuclear reactions), will lose significant amounts of mass, making mass loss the dominant factor in their evolution.

(iv) The effective temperature of the SMSs remains low during their evolution (i.e. the photosphere is determined by recombination at the wind, taking place at roughly 4500 K) due to the large photospheric radius. Hence, the amount of ionizing radiation emitted by these objects is negligible.

(v) Rotating SMSs lose most of their mass in the later stage (II, see Section 3.3) through their winds. The mass loss continues until a sub-Eddington massive star is left.

### 6.1 Differences from the standard solution

As mentioned before, the solution for SMSs obtained here is notably different from the standard lore of SMSs. It is thus worthwhile to emphasize these differences and their causes.

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**Figure 7.** Sonic radius $R_{\text{sonic}}$ (solid), and radius of the photosphere (dashed) of the same SMS as Fig. 6, during stages Ib and II of the evolution. In stage III (not shown), the luminosity is sub-Eddington, so there is no continuum-driven wind and the photosphere is located at the top of the atmosphere.

1.0 and 1.25. We find that there is no significant difference in the appearance of SMSs under different opacity laws. However, there are some differences with respect to SMS evolution.

Because the luminosity depends on the opacity, the lifetime somewhat depends on the opacity law. For example, the lifetime of stages Ia and Ib of a $5.5 \times 10^4 \, M_\odot$ SMS (i.e. the lifetime of the gravitational collapse until ignition of CNO nuclear burning) ranges between 2800 yr for $B = 0.75$ and 7000 yr for $B = 1.25$.

However, there is one major characteristic which is sensitive to the opacity law, and it is the mass of the sub-Eddington star that will end the super-Eddington phase. The opacity law can be described by a $\Gamma_{\text{crit}}$, which is the Eddington parameter below which the system becomes sub-Eddington. Using the Eddington quartic relation and the approximation that the SMS can be described by an $n = 3$ polytrope, it is possible to estimate the critical mass for which the SMS will become sub-Eddington (see Owocki & Shaviv 2012):

$$M_{\text{crit}} = \frac{\mu^{-\frac{1}{2}} \Gamma_{\text{crit}}^{1/2}}{(1 - \Gamma_{\text{crit}})^2} 18.3 \, M_\odot,$$

where $\mu$ is the molecular weight.

Since there is little depletion of hydrogen during the SMS evolution, we can take the primordial value of 0.75 for the hydrogen mass fraction. Moreover, Shaviv (2001a) found two instabilities: one with $\Gamma_{\text{crit}}$ of about 0.8, which is more general, and one for which $\Gamma_{\text{crit}} = 0.5$, but which depends on the boundary conditions. Other instabilities, when magnetic fields are present could operate at even lower values of $\Gamma_{\text{crit}}$ (see e.g. the numerical analysis of instabilities in luminous accretion discs by Turner et al. 2005). For the above value of $X$ and range of $\Gamma_{\text{crit}}$, we find $M_{\text{crit}}$ ranging between 150 and 1200 $M_\odot$. Namely, the normal sub-Eddington star left once the super-Eddington phase is over is very sensitive to the critical Eddington parameter above which atmospheres become porous.

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In SMSs in general, the radiation to gas pressure ratio is very large. If the opacity is not modified and instead remains at its microscopic value of Thomson scattering, it is straightforward to show that the luminosity should be that of Eddington minus a small relative correction, due to the small gas to total pressure ratio. Hence, the luminosity in the classical SMS solutions is taken to be the Eddington luminosity.

However, this does not consider the rise of instabilities when the radiation pressure is dominant. Once taken into consideration, the effective opacity is reduced and the effective Eddington luminosity is increased. This gives rise to solutions with \( L > L_{\text{Edd}} \), in which the outer regions of the hydrostatic part of the star is porous.

Since the structure of the convective zone does not depend on the luminosity, this inner region (which comprises most of the stellar mass) does not change in the super-Eddington solution. Nevertheless, this region does not fit exactly the polytropic solution with a polytropic index of \( n = 3 \). This deviation from the polytropic solution has two causes: the finite pressure contributed by the gas and the limited convective efficiency at the upper boundary. Neither effect is due to the super-Eddington nature of the solution.

Another intrinsic characteristic of the present solutions is the high mass loss rate. This has several effects. First, the high mass loss rates imply that large amounts of energy are required to push the gas out of the gravitational potential well. As a result, the luminosity left above the wind can be significantly reduced, and even make the intrinsically super-Eddington SMSs appear to be sub-Eddington.

Secondly, it implies that the photosphere resides at large radii, inside the wind. As a consequence the photospheric temperature is lower by an order of magnitude relative to the standard SMS solution. Thus, only a negligible amount of ionizing radiation is emitted. This is in marked contrast with the standard SMS models, and it implies that it is unlikely that SMSs have played a major role in recognizing the early Universe.

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