Squeezed condensate of gluons and the mass of the $\eta'$

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Abstract

The relation between the large mass of the $\eta'$ and the structure of the gluon vacuum via the $U_A(1)$ anomaly is discussed. A squeezed gluon vacuum is considered as an alternative to existing models. Considering Witten’s formula for the $\eta_0$ mass we show that the contact term can give a sizable contribution and relate it to the physical gluon condensate. The values of the gluon condensate obtained through this relation are compared with the value by Shifman, Vainshtein and Zakharov and the recent update values by Narison.

PACS number(s): 12.38.Aw, 12.40.Yx, 14.40.Aq, 14.70.Dj

Key words: Squeezed vacuum, gluon condensate, $U_A(1)$ symmetry breaking, $\eta_0$ mass formula

1 Introduction

In the nonet of pseudoscalar mesons the most interesting one for the investigation of the gluon sector of QCD is the relatively heavy $\eta'$, which is related to the so-called $U_A(1)$ problem [1,2]. Since the work of t’Hooft [3] there is little doubt that the large $\eta'$ mass has its origin in the gluon sector of QCD. The singlet $\eta_0$, which is the main component of the $\eta'$ (apart from admixtures of the octet meson $\eta_8$), is expected to couple directly to the gluons via the gluon condensate.
anomaly and gives an additional mass term for the $\eta'$. An important ingredient here is the mixing angle which has been determined rather accurately by experiments via the analyses of the decays of the $\eta$ and $\eta'$ [4,5] during the last years.

There are different approaches to calculate the mass of the $\eta'$. In their pioneering works Witten [6] and Veneziano and Di Vecchia [7] constructed a meson Lagrangian which includes the gluon anomaly. As a byproduct Witten derived a formula which relates the mass of the $\eta_0$ to the topological susceptibility. Witten’s formula has been a key tool for theoretical investigations using rather detailed models of the QCD gluon vacuum such as the instanton model [8,9] or the monopole condensate model [10]. Recently Hutter [11] was able to relate the topological susceptibility to the gluon condensate in the simple picture of the gluon vacuum as an ensemble of uncorrelated instantons and anti-instantons and obtained a good estimate of the mass of the $\eta'$.

Also recently the model of the squeezed gluon vacuum has been considered as an interesting alternative [12]-[16] to the above approaches. In the present paper we discuss the squeezed vacuum in a simple variant and apply it to the $U_A(1)$ problem. We shall directly relate the anomaly term to the gluon condensate. Numerical results for the gluon condensate are compared with other values given by Shifman, Vainshtein and Zakharov [17] and also recently by Narison [18].

The paper is organized as follows: In Section 2 we briefly recall the $U_A(1)$ problem and discuss its connection to the gluon condensate. In Section 3 the effective low energy meson Lagrangian which includes an anomalous gluon term is quoted and in Section 4 the Witten formula is discussed. In Section 5 we outline the squeezed vacuum. In Section 6 the mass of the $\eta'$ is related to the squeezed gluon condensate and numerical results are given. Finally our conclusions are drawn.

2 $U_A(1)$ problem and the gluon condensate

The pseudoscalar mesons have been well understood as irreducible representations of the flavour SU(3). The pions and kaons are members of the octet representation, whereas the $\eta$ and $\eta'$ mesons are related to the octet and singlet pseudoscalar states

\[
\eta_8 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}
\]
\[
\eta_0 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}
\] (1)
via the mixing

\[\begin{align*}
\eta &= \eta_8 \cos \phi - \eta_0 \sin \phi \\
\eta' &= \eta_8 \sin \phi + \eta_0 \cos \phi
\end{align*}\]

with a mixing angle \(\phi\).

Different experimental data have been used to determine the mixing angle \(\phi\). Recent analyses of \(\eta\) and \(\eta'\) decays \([4]\) have obtained a mixing angle

\[\phi = -(18.4 \pm 2)\degree.\]  

The experimental values for the masses of the \(\eta\) and \(\eta'\) are \([19]\)

\[\begin{align*}
m_\eta &= 547.45 \pm 0.19 \text{ MeV} , \\
m_\eta' &= 957.77 \pm 0.14 \text{ MeV} .
\end{align*}\]  

From these experimental values of \(m_\eta, m_\eta'\) and \(\phi\) one obtains the masses \(M_{88}\) and \(M_{00}\) of the \(\eta_8\) and \(\eta_0\)

\[\begin{align*}
M_{88} &= \sqrt{m_\eta^2 \cos^2 \phi + m_{\eta'}^2 \sin^2 \phi} = 591.49^{+9.31}_{-8.58} \text{ MeV} , \\
M_{00} &= \sqrt{m_{\eta'}^2 \cos^2 \phi + m_\eta^2 \sin^2 \phi} = 931.21^{+5.94}_{-5.40} \text{ MeV} .
\end{align*}\]  

On the other hand the masses \(M_{88}\) and \(M_{00}\) can be obtained from the following Gell-Mann–Okubo formulae \([2]\)

\[\begin{align*}
M_{88}|_{\text{quark}} &= \frac{1}{3} \left( 2m_{K^+}^2 + 2m_{K^0}^2 - 2m_{\pi^+}^2 + m_{\pi^0}^2 \right) , \\
M_{00}|_{\text{quark}} &= \frac{f_\pi^2}{3f_0^2} \left( m_{K^+}^2 + m_{K^0}^2 + m_{\pi^+}^2 \right) ,
\end{align*}\]  

which have been derived using the Gell-Mann–Oakes–Renner relations for the nonet of pseudoscalar Goldstone bosons under the assumption of only small explicit chiral \(SU(3)\) symmetry breaking by the \(u, d\) and \(s\) quark masses. Here \(f_\pi = 93\ \text{MeV}\) is the pion decay constant and \(f_0 \sim f_\pi\) \([5]\) is the singlet decay constant. Taking the kaon and pion masses from experiment, this relation gives \(M_{88}|_{\text{quark}} = 566 \text{ MeV}\), which is in reasonable agreement with (5)\(^3\). For the \(\eta_0\)

\(^3\) A value of \(\phi = -10.1\degree\) would lead to complete agreement \([19]\). This deviation from the experimental value (3) could have its origin in a small deviation from Gell-Mann–Okubo formula which can lead to a relatively large change in the mixing angle.
mass, however, the Gell-Mann–Okubo formula yields only a value $M_{00}|_{\text{quark}} = 413$ MeV for $f_0 = f_\pi$, which is much smaller than the experimental (6). The large squared difference

$$\Delta m_{\eta_0}^2 = M_{00}^2 - M_{00}|_{\text{quark}} = 0.696 \text{ GeV}^2$$

shows that the small explicit chiral symmetry breaking by the current quark masses alone cannot account for the large mass of the $\eta_0$ or the $\eta'$. This constitutes the well-known $U_A(1)$ problem [1].

On the way towards the solution of this puzzle it is important to note that even in the chiral limit of vanishing quark masses the colour singlet axial $U(1)$ quark current $j_5^\mu \equiv i\bar{q}\gamma^\mu \gamma_5 q$ is actually not conserved on the quantum level but afflicted with an anomaly due to the gluon sector of QCD [20]

$$\partial_\mu j_5^\mu = 2i\bar{q}\gamma_5 M_q q + 2N_f Q(x)$$

with

$$Q(x) \equiv \frac{\alpha_s}{8\pi} G^{\mu\nu a}(x) \tilde{G}_a^{\mu\nu}(x), \quad \tilde{G}_a^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma},$$

where $G^a_{\mu\nu}(x)$ is the gluon field strength tensor, $M_q = \text{diag}(m_u, m_d, m_s)$ is the diagonal matrix of the current quark masses, $\alpha_s \equiv g^2/4\pi$ the strong coupling constant and $N_f = 3$ the number of light quark flavours. Although derived only in the one-loop approximation in the presence of classical background gluon fields it is generally accepted that (10) is actually an operator identity (see e.g. the discussion in [20]). The anomalous term $Q(x)$ is known as Pontryagin density. An important condition for a large $\eta'$ mass is that there is a possibility for a nonvanishing $Q(x)$ as will be discussed in the following.

An important concept in this context is the gluon condensate, defined as the expectation value of the local gluonic operator

$$N(x) \equiv \alpha_s G^{a}_{\mu\nu}(x) G^{\mu\nu a}(x)$$

in the nonperturbative QCD vacuum [20]

$$\langle \alpha_s G^2 \rangle \equiv \langle \alpha_s G^a_{\mu\nu}(0) G^{\mu\nu a}(0) \rangle \equiv \langle N(0) \rangle.$$  

Approximate empirical values for the physical gluon condensate are the estimate $\langle \alpha_s G^2 \rangle \simeq 0.04 \text{ GeV}^4$ by Shifman, Vainshtein and Zakharov [17] and the update average value $\langle \alpha_s G^2 \rangle = (0.071 \pm 0.009) \text{ GeV}^4$ obtained by Narison
in a recent analysis of heavy quarkonia mass-splittings in QCD. The value of $\alpha_s$ in the low energy region is not known very well from experiment. The value used by Shifman, Vainshtein and Zakharov [17] is $\alpha_s \approx 1$ and that used by Narison [18] in the low energy region is $\alpha_s(1.3 \text{ GeV}) \approx 0.64^{+0.39}_{-0.18} \pm 0.02$.

A nonvanishing value of the gluon condensate can lead to a nonvanishing anomalous density $Q(x)$ and therefore to a large mass of the $\eta'$. As discussed by Hutter [11] for a dilute noninteracting gas of statistically independent instantons and anti-instantons, which are (anti-)selfdual field configurations in Euclidean space with $\tilde{G}^a_{\mu\nu} = \pm G^a_{\mu\nu}$, the local gluonic operator $N(x)$ defined in (12) is proportional to the sum of the number densities of instantons and anti-instantons. The Pontryagin density $Q(x)$ given in (11), on the other hand, is equal to the difference of the number densities of instantons and anti-instantons in the dilute instanton gas. In the model of the QCD vacuum as a non-interacting ensemble of instantons and anti-instantons the existence of a gluon condensate is therefore a necessary condition for a nonvanishing value of $Q(x)$ and hence for a large mass of the $\eta'$. Furthermore it has been suggested in [16] and will be discussed in more detail in this paper that also a squeezed condensate in Minkowski space can lead to a large mass of the $\eta'$ through the $U_A(1)$ anomaly.

There are several ways to implement the gluon anomaly in calculations of the meson spectrum in order to obtain the large value of the $\eta'$ mass. In Ref. [3] t’Hooft introduced an effective quark interaction in Minkowski space simulating the anomalous term which breaks $U_A(1)$ but conserves the chiral $SU(3)_L \otimes SU(3)_R$ symmetry. This determinantal interaction has been widely used within effective quark models such as the NJL model [21,22]. Dorokhov and Kochelev [23] model the $\eta'$ as a MIT bag where the nonperturbative vacuum of QCD is allowed to enter the bag which leads to instanton induced quark interactions. They obtain the values $m_\eta = 750 \text{ MeV}$ and $m_{\eta'} = 1150 \text{ MeV}$. The pions obtained in the same scenario, however, turn out to be much too heavy. For the calculation of the mass of the $\eta'$ mass in microscopic models of the gluon vacuum, such as the magnetic monopole condensate, the instanton gas model or the squeezed condensate, another quite general approach is very convenient and will be discussed in the next two sections. It is based on Witten’s formula derived from a low energy meson Lagrangian which contains the axial $U_A(1)$ anomaly at tree level.

3 Effective low energy meson Lagrangian including the chiral anomaly

In a quite general framework, without using the concept of instantons, the authors [24]-[27] start from the low energy effective glueball-meson Lagrangian in a general $\theta$ vacuum.
\[ \mathcal{L}_{\text{meson}}(U, Q) = -\frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^+] + \frac{1}{2} v \text{Tr}[M_q(U^+ + U)] + \frac{i}{2} Q \text{Tr}[\ln U - \ln U^+] + \frac{N_f}{a f_0^2} Q^2 - \theta Q, \]  

(14)

where

\[ U(x) \equiv \exp \left[ i \frac{\sqrt{2}}{f_\pi} \left( \sum_{a=1}^8 B_a(x) \lambda_a + \frac{f_\pi}{\sqrt{3} f_0} \eta_0(x) 1_3 \right) \right], \]  

(15)

with the octet meson fields \( B_a \)

\[
\sum_{a=1}^8 B_a \lambda_a = \begin{bmatrix}
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & \pi^+ & K^+ \\
\pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & K^0 \\
\bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}} \eta_8
\end{bmatrix}
\]  

(16)

and the singlet \( \eta_0 \). The pseudoscalar glueball field \( Q(x) \) plays the role of an auxiliary field included into the meson Lagrangian in order to implement the axial \( U_A(1) \) anomaly on the hadronic level. It can be identified with the vacuum expectation value of the Pontryagin density \( Q(x) \) defined in (11) in terms of gluon degrees of freedom as will be discussed further below. The vacuum angle \( \theta \), the parameters \( a \) and \( v \), as well as the diagonal mass matrix of the current quarks \( M_q = \text{diag}(m_u, m_d, m_s) \) are to be fixed by comparison with experiment. Measurements of the dipole moment of the neutron [28] limit the vacuum angle to \( \theta < 10^{-9} \), so that in practice it can be taken equal to zero.

The kinetic term reads explicitly

\[
-\frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^+] = -\frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 - \partial_\mu \pi^+ \partial^\mu \pi^- - \partial_\mu K^+ \partial^\mu K^- - \partial_\mu K^0 \partial^\mu \bar{K}^- - \frac{1}{2} \partial_\mu \eta_8 \partial^\mu \eta_8 - \frac{1}{2} \partial_\mu \eta_0 \partial^\mu \eta_0
\]  

(17)

in terms of the pion, kaon and eta fields. The explicitly chiral symmetry breaking mass term is

\[
\frac{1}{2} v \text{Tr}[M_q(U^+ + U)] = \frac{1}{2} v \left[ \frac{1}{2} (m_u + m_d)(\pi^0 \pi^0 + 2\pi^+ \pi^-) + (m_u + m_s)K^+ \bar{K}^- + (m_d + m_s)K^0 \bar{K}^0 + \frac{1}{6} (m_u + m_d + 4m_s) \eta_8^2 + \frac{f_\pi^2}{3 f_0^2} (m_u + m_d + m_s) \eta_0^2 \right]
\]
The corresponding meson mass formulae implicit in (18) are the Gell-Mann–Oakes–Renner relations. They can be combined in order to yield the Gell-Mann–Okubo relations (7) and (8).

Since

$$-\frac{i}{2} \text{Tr}[\ln U - \ln U^+] = \frac{\sqrt{2N_f}}{f_0} \eta_0,$$  \hspace{1cm} (19)

only the singlet field $\eta_0$ is coupled to $Q(x)$. The Euler-Lagrange equations for $U$ and $U^+$ of the meson Lagrangian (14) include the chiral anomaly

$$\partial_\mu A_\mu^0 = -i v \text{Tr}[M_q(U^+ - U)] + 2N_f Q(x)$$  \hspace{1cm} (20)

with the axial U(1) current $A_\mu^0 = -i f_\pi^2 \text{Tr}[U \partial_\mu U^+ - U^+ \partial_\mu U]$ from Noether's theorem. Eq. (20) is the hadron analogue to (10) with the pseudoscalar glueball field identified as the vacuum expectation value of the Pontryagin density $Q(x)$ defined in (11) in terms of gluon degrees of freedom. The Lagrangian (14) therefore includes the chiral anomaly.

Using $\delta L^\text{meson}/\delta Q = 0$ to eliminate the auxiliary pseudoscalar glueball field $Q(x)$ leads to the reduced Lagrangian

$$L^\text{meson}_{\text{red}}(U) = -\frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^+] + \frac{1}{2} v \text{Tr}[M_q(U^+ + U)]$$

$$-\frac{af_0^2}{4N_f} \left( \theta - \frac{i}{2} \text{Tr}[\ln U - \ln U^+] \right)^2.$$  \hspace{1cm} (21)

Noting (19) we see that the $\eta_0$ field attains an additional contribution

$$\Delta m_{\eta_0}^2 = a$$  \hspace{1cm} (22)

to its mass from the gluon anomaly, which can be chosen in accordance with (9). A different derivation of the reduced Lagrangian (21), which contains the $U_A(1)$ anomaly at tree level, has been given by Witten [6] using large $N_c$ arguments. Note that the kinetic term and the mass term in the Lagrangian (21) are of order $O(1)$ in the number $N_c$ of colours, since $f_0 \sim f_\pi \sim O(N_c^{1/2})$, 

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whereas the third anomalous term like the mass $a$ due to the gluon anomaly itself are of order $O(N_c^{-1})$ and therefore vanish in the large $N_c$ limit.

4 Witten’s formula for the $\eta_0$ mass

From the anomalous low energy meson Lagrangian (21) one can derive the quite general formula by Witten [6], which allows one to calculate the mass shift of the $\eta_0$ due to the gluon anomaly in leading order in $1/N_c$ for microscopic models of the gluon vacuum. Although $\theta$ is practically zero, Witten proposed to use its fluctuations to calculate the quadratic mass shift $\Delta m_{\eta_0}^2$ in the following way. From (21) and (22) one obtains

$$\Delta m_{\eta_0}^2 = \frac{2N_f}{f_\pi^2} \left( \frac{d^2 \varepsilon_0}{d\theta^2} \right)_{\theta=0} \text{no quarks}.$$ (23)

Here $\varepsilon_0(\theta) = E_0(\theta)/V$ is the ground state energy density of the Hamiltonian corresponding to (21) with no quarks, i.e. all meson fields set equal to zero. The singlet decay constant $f_0$ has been replaced by $f_\pi$ which is in agreement with experiment [5] and in accordance with large $N_c$ arguments. As noted by Witten, the ground state energy $E_0(\theta)$ can be alternatively considered in pure gluon QCD with the Lagrangian including an anomalous term as

$$\mathcal{L}^{\text{gluon}} = -\frac{1}{4} G_\mu^a G_\mu^a + \frac{\alpha_s \theta}{8\pi} G_\mu^a \tilde{G}_\mu^a \quad (24)$$

with

$$G_\mu^a G_\mu^a = -2(E_i^a)^2 + 2(B_i^a)^2, \quad G_\mu^a \tilde{G}_\mu^a = -4E_i^a B_i^a \quad (25)$$

$E_i^a$ and $B_i^a$ are the components of the chromoelectric and chromomagnetic field strength. For the quantization in the Hamilton formalism we use the Weyl gauge $A_0 = 0$, such that $E_i^a = A_i^a$. Introducing the canonical momenta

$$\Pi_i^a \equiv \frac{\partial \mathcal{L}_{\text{singlet}}}{\partial A_i^a} = E_i^a + \frac{\alpha_s \theta}{2\pi} B_i^a \quad (26)$$

the Hamiltonian reads

$$H_{\text{gluon}} = \frac{1}{2} \int d^3 x \left[ \left( \Pi_i^a - \frac{\alpha_s \theta}{2\pi} B_i^a \right)^2 + (B_i^a)^2 \right] \quad (27)$$
The Hamiltonian contains a term linear in $\theta$

$$I_1 = -2 \int d^3 \vec{x} \frac{\alpha_s \theta}{4\pi} \Pi_i^0 B_i^a$$

(28)

and a term quadratic in $\theta$

$$I_2 = 2 \int d^3 \vec{x} \left( \frac{\alpha_s \theta}{4\pi} \right)^2 (B_i^a)^2 .$$

(29)

In order to calculate the topological susceptibility $(d^2 \varepsilon_0 / d\theta^2)_{\theta=0}$ it is only necessary to calculate the vacuum energy $E_0(\theta)$ up to second order in $\theta$. For this purpose one has to do second order perturbation theory in the operator $I_1$ and only first order in $I_2$. This way Witten [6] derived an expression for the topological susceptibility of QCD without quarks:

$$\left( \frac{d^2 \varepsilon_0}{d\theta^2} \right)_{\theta=0}^{\text{no quarks}} = -i \int dt d^3 \vec{x} \langle 0_I | T Q(\vec{x}, t) Q(\vec{0}, 0) | 0_I \rangle_{\text{conn}} + 4 \left( \frac{\alpha_s}{4\pi} \right)^2 \langle 0_I | B_i^a (\vec{0}, 0)^2 | 0_I \rangle .$$

(30)

where $|0_I\rangle$ is the interaction picture gluon vacuum corresponding to $\theta = 0$. The subscript ”conn” denotes the connected part of the Green function and $T$ is the Dyson time ordering operator. Eqs. (23) and (30) together constitute the Witten formula. The first term in (30) is a propagator term whereas the second one is a contact term. The contact term can be incorporated into the propagator term, if instead of the Dyson $T$ ordering the Wick $T^*$ ordering is used [29]. In the following we shall use (30) with the Dyson $T$ ordering. Witten kept only the propagator term in (30) and dropped the contact term. He argued that although the contact term is necessary for the positivity of the result, it should not contribute to the mass of the $\eta'$ since no Goldstone pole can appear in a one point function. Based on Witten’s formulae (23) and (30), dropping the contact term, there were several approaches to calculate the large mass of the $\eta'$. As one of the earliest approaches Novikov et al. [8] used the Euclidean model of the gluon vacuum as an ensemble of noninteracting (anti-)instantons. In this context is valid

$$\langle 0_{ig} | T Q(x) Q(0) | 0_{ig} \rangle_{\text{conn}} = \frac{1}{64\pi^2} \langle 0_{ig} | T N(x) N(0) | 0_{ig} \rangle_{\text{conn}}$$

(31)

with $Q(x)$ and $N(x)$ given in (11) and (12) and $|0_{ig}\rangle$ denoting the dilute instanton gas vacuum. He then related the right hand side of (31) to the
gluon condensate using the Ward identity \[20\]

\[
\int d^4x \frac{1}{8\pi} \langle 0 | T N(x) N(0) | 0 \rangle_{\text{conn}} = \frac{12}{11 N_c} \langle 0 | N(0) | 0 \rangle
\]  

(32)
corresponding to the anomalous breaking of scale invariance. Inserting (31) and (32) into Witten’s formulae (23) and (30) in Euclidean space, he obtained the following relation between the mass of the \(\eta^\prime\) and the physical value of the gluon condensate \(\langle \alpha_s G^2 \rangle\)

\[
\Delta m_{\eta^\prime}^2 = \frac{2 N_f}{8\pi f_\pi^2} \frac{12}{11 N_c} \frac{1}{\gamma} \langle \alpha_s G^2 \rangle .
\]

(33)
The coefficient \(\gamma < 1\), originating from the fermion determinant in the QCD path integral, accounts for the suppression of the gluon condensate value due to the presence of light quarks

\[
\langle \alpha_s G^2 \rangle = \gamma \langle \alpha_s G^2 \rangle_{\text{no quarks}} .
\]

(34)
A discussion of the value for \(\gamma\) has been given by Novikov et al. [8] in the instanton gas scenario. They find values in the range \(\gamma \simeq 1/3 - 1/2\). Recently Hutter [11] showed that formula (33) for the mass of the \(\eta^\prime\) remains valid even for the more general case of the gluon vacuum as an ensemble of noninteracting instantons and anti-instantons. Using the somewhat wider range \(\gamma \simeq 0.4 - 0.7\), Hutter obtains a value \(m_{\eta^\prime} = 884 \pm 116\) MeV neglecting the quark masses.

Another application of the above Witten formula has been the calculation of the mass of the \(\eta^\prime\) in the magnetic monopole condensate scenario by Ezawa and Iwazaki [10]. Using the hypothesis of dominance of the Abelian gauge field components at large distances they find a value \(m_{\eta^\prime} = 550\) MeV neglecting the influence of quarks on the gluon condensate.

A further interesting possibility to explain the large mass of the \(\eta^\prime\), which has been suggested recently [16] is the model of a squeezed gluon vacuum to be discussed in the following two sections.

5 The model of the squeezed gluon vacuum

The squeezed condensate of gluons has been investigated recently [12]-[16] in order to construct a Lorentz and gauge invariant stable QCD vacuum in Minkowski space. Different alternative approaches have not solved this problem. For instance the simple perturbative vacuum is unstable [30], and there is
no stable (gauge invariant) coherent vacuum in Minkowski space \[31\]. From the physical point of view, the squeezed state differs from the coherent one by the condensation of colour singlet gluon pairs rather than of single gluons. In analogy to the Bogoliubov model \[32\] we consider the case of a homogeneous condensate, but in a squeezed instead of a coherent state.

Let \(|n⟩\) denote the eigenstates of the pure gluon Hamiltonian (27) for vanishing vacuum angle \(\theta = 0\). In order to introduce the squeezed states let us consider the gluon system to be enclosed in a large finite volume \(V\). The squeezed states \(|n_{sq}[ξ]⟩\) as candidates for the gluon eigenstates \(|n⟩\), in particular the squeezed vacuum \(|0_{sq}[ξ]⟩\) as a candidate for a homogeneous colourless gluon vacuum \(|0⟩\), are constructed from the nonperturbative states \(|n^{(0)}⟩ \equiv |n_{sq}[ξ]⟩|_{ξ=0}\), further specified below, according to

\[
|n_{sq}[ξ]⟩ = U_{sq}^{-1}[ξ]|n^{(0)}⟩ .
\] (35)

The squeezing operator

\[
U_{sq}[ξ] = \exp \left[ iξ \frac{V}{2} (A_{i}^{a} E_{i}^{a} + E_{i}^{a} A_{i}^{a}) \right]
\] (36)

with the zero momentum components \(A_{i}^{a}\) and \(E_{i}^{a}\) of the fields and their canonical momenta contains the parameter \(ξ\) given below. This special transformation for the homogeneous condensate does not violate Lorentz invariance, since the gauge fields are massless \[33\]. The question of gauge invariance of such a procedure is a difficult open problem and first steps towards a clarification are under current investigation \[34\]. As in Ref. \[35\] we suppose here the gauge invariance of \(A_{i}^{a} E_{i}^{a}\) and hence of the squeezing operator as a colourless functional of the spatial zero momentum components of the gauge fields. The multiplicative transformations of fields corresponding to (35) and (36) are

\[
U_{sq}[ξ] A_{i}^{a} U_{sq}^{-1}[ξ] = e^{ξ} A_{i}^{a} ,
\]

\[
U_{sq}[ξ] E_{i}^{a} U_{sq}^{-1}[ξ] = e^{-ξ} E_{i}^{a} .
\] (37)

From this canonical transformation it follows that the expectation values in the squeezed state basis as functions of the squeezing parameter \(ξ\) behave like

\[
\langle n_{sq}[ξ]| (B_{i}^{a})^{2} |n_{sq}'[ξ]⟩ = e^{4ξ} \langle n^{(0)}| (B_{i}^{a})^{2} |n^{(0)}⟩ ,
\] (38)

\[
\langle n_{sq}[ξ]| (E_{i}^{a})^{2} |n_{sq}'[ξ]⟩ = e^{-2ξ} \langle n^{(0)}| (E_{i}^{a})^{2} |n^{(0)}⟩ ,
\] (39)
\[ \langle \text{n}_\text{sq}[\xi]|E_i^a B_i^a|\text{n}'_\text{sq}[\xi]\rangle = e^\xi \langle \text{n}^{(0)}|E_i^a B_i^a|\text{n}'^{(0)}\rangle, \quad (40) \]

with \(B_i^a \equiv f^{abc} \epsilon_{ijk} A_j^b A_k^c\). Let the reference states \(|\text{n}^{(0)}\rangle\) be such that the expectation values \(\langle \text{n}^{(0)}|(B_i^a)^2|\text{n}'^{(0)}\rangle\), \(\langle \text{n}^{(0)}|E_i^a E_i^b|\text{n}'^{(0)}\rangle\) and \(\langle \text{n}^{(0)}|E_i^a B_i^a|\text{n}'^{(0)}\rangle\) behave in the large volume limit \((V \to \infty)\) like \(V^{-4/3}\) in accordance with dimensional analysis. The parameter of the squeezing transformation \(\xi\) can be chosen so that the magnetic condensate density \((38)\) remains finite in the large volume limit \((e^{4\xi} \sim V^{4/3})\)

\[ \lim_{V \to \infty} \langle \text{n}_\text{sq}|(B_i^a)^2|\text{n}'_\text{sq}\rangle = \mathcal{O}[1]. \quad (41) \]

We shall denote the corresponding squeezed states simply by \(|\text{n}_\text{sq}\rangle\). This entails that the electric component \((39)\) and the mixed component \((40)\) of the gluon condensate vanish in the large volume limit

\[ \lim_{V \to \infty} \langle \text{n}_\text{sq}|(E_i^a)^2|\text{n}'_\text{sq}\rangle = \mathcal{O}[1/V^2], \quad (42) \]

\[ \lim_{V \to \infty} \langle \text{n}_\text{sq}|E_i^a B_i^a|\text{n}'_\text{sq}\rangle = \mathcal{O}[1/V]. \quad (43) \]

Hence we conclude that in the squeezed vacuum \((35)\) the gluon condensate is equal to its magnetic part,

\[ \langle \alpha_s G^2 \rangle_{\text{no quarks}} = \langle 0_{\text{sq}}|\alpha_s G_{\mu \nu}^a(0) G_{\mu \nu}^a(0)|0_{\text{sq}}\rangle = 2 \langle 0_{\text{sq}}|\alpha_s (B_i^a)^2|0_{\text{sq}}\rangle. \quad (44) \]

Note that this model is consistent with the picture of the QCD vacuum as a homogenous magnetic medium. As discussed e.g. in [36], the vacuum energy \(\varepsilon_{\text{vac}}\) of full QCD can be related to the gluon condensate \(\langle \alpha_s G^2 \rangle\) via the one-loop result for the scale anomaly in the dilatation current

\[ \varepsilon_{\text{vac}} = \frac{1}{4} \langle \theta_\mu^\mu \rangle \approx -\frac{b}{32\pi} \langle \alpha_s G^2 \rangle. \quad (45) \]

Here \(\langle \theta_\mu^\mu \rangle\) is the vacuum expectation value of the trace of the energy momentum tensor and \(b = \frac{11}{3} N_c - \frac{2}{3} N_f\) the QCD \(\beta\)-function coefficient. Since the vacuum energy \(\varepsilon_{\text{vac}}\) is negative and \(b\) positive, the gluon condensate \(\langle \alpha_s G^2 \rangle\) is expected to be positive and hence

\[ \langle \alpha_s B^2 \rangle > \langle \alpha_s E^2 \rangle, \quad (46) \]
which is a Lorentz invariant statement and corresponds to a magnetic vacuum. Quark condensate terms have been neglected in this estimation and are expected to soften the inequality (46) but not to turn it to the reverse. The squeezed vacuum has vanishing electric field $E$ and thus is in accordance with (46).

6 The $\eta'$ mass in the model of the squeezed gluon vacuum

For the calculation of the $\eta'$ mass in the squeezed vacuum it is useful to rewrite the Witten formula (30) in the Schrödinger picture as [29]

$$\left( \frac{d^2 \varepsilon_0}{d \theta^2} \right)_{\theta=0}^{\text{no quarks}} = -2 \sum_{n \neq 0} \frac{|\langle n | Q(\vec{0}) | 0 \rangle|^2}{\varepsilon_n - \varepsilon_0} + 4 \left( \frac{\alpha_s}{4\pi} \right)^2 \langle 0 | (B_1^a(\vec{0}))^2 | 0 \rangle .$$

(47)

It contains the exact eigenstates $|n\rangle$ and eigenvalues $\varepsilon_n$ of the pure gluon QCD Hamiltonian. We approximate (47) by replacing the exact eigenstates and eigenvalues by the nonperturbative squeezed states and the corresponding energy expectation values. In order to see whether the propagator and the contact term give finite contributions we inspect their volume dependence. Since in the squeezed vacuum

$$\langle 0_{sq} | (B_i^a(\vec{0}))^2 | 0_{sq} \rangle = \langle 0_{sq} | (B_i^a)^2 | 0_{sq} \rangle ,$$

(48)

the contact term gives a finite contribution to the topological susceptibility according to (41).

In addition a further, negativ contribution might arise from the propagator term. Despite the fact that the matrix elements are suppressed in the large volume limit the denominator can simultaneously become very small due to states arbitrarily close to the vacuum. Whereas in the instanton model of the gluon vacuum only the propagator term is considered, as discussed in Section 4, we shall here not further investigate the propagator term but consider only the finite contribution from the contact term.

The contact term by itself gives the following contribution to the $\eta_0$ mass via (23)

$$\Delta m_{\eta_0}^2 \bigg|_{\text{contact}} = \frac{3\alpha_s}{2\pi^2 f_\pi^2} \langle 0_{sq} | \alpha_s (B_i^a)^2 | 0_{sq} \rangle .$$

(49)

Using the expression (44) for the squeezed gluon condensate and relation (34)
to account for the suppression of the physical gluon condensate due to the presence of light quarks by a factor $\gamma < 1$ we obtain

$$\langle \alpha_s G^2 \rangle = \frac{4 \gamma^2 f^2}{3 \alpha_s} \Delta m^2_{\eta_0}|_{\text{contact}} .$$

This formula is the main result of our investigation. It relates the gluon condensate to the $U_A(1)$ breaking contact contribution to the mass shift of the $\eta_0$.

We point out that Nielsen et al. [37] have derived the same formula for the shift of the $\eta'$ mass as (49) (except for the factor $\gamma$) using the Cheshire cat principle instead of the Witten formula.

A main source of uncertainty is the reduction factor of the gluon condensate due to the presence of light quarks. We shall use here the estimate $\gamma \approx 1/3 - 1/2$ obtained by Novikov et al. [8] in the instanton gas scenario.

The value of $\alpha_s$ in the low energy region is not known very well from experiment. The value used by Shifman, Vainshtein and Zakharov [17] is $\alpha_s \approx 1$ and that used by Narison [18] in the low energy region is $\alpha_s(1.3 \text{ GeV}) \approx 0.64^{+0.36}_{-0.18} \pm 0.02$. In order to check whether relation (50) is in agreement with empirical data we have plotted in Fig. 1 the gluon condensate $\langle \alpha_s G^2 \rangle$ against $\alpha_s$ for two limiting values $1/3$ and $1/2$ of $\gamma$. We estimate the contact contribution by the full empirical mass shift $\Delta m^2_{\eta_0} = 0.696 \pm 0.02 \text{ GeV}^2$ obtained in (9) for $\phi = -18.4^\circ$. The curves in Fig. 1 give our result for these two limiting values $\gamma = 1/3$ (solid line) and $\gamma = 1/2$ (dashed line) and the mixing angle $\phi = -18.4^\circ$. The thin lines indicate the error $\pm 2^\circ$ in the value of the mixing angle which is negligibly small. Also shown are the gluon condensate value $\langle \alpha_s G^2 \rangle \approx 0.04 \text{ GeV}^4$ by Shifman, Vainshtein and Zakharov [17] (filled square) and the update average value $\langle \alpha_s G^2 \rangle = (0.071 \pm 0.009) \text{ GeV}^4$ for the gluon condensate obtained by Narison [18] (filled triangle) in a recent analysis of heavy quarkonia mass-splittings in QCD. The gluon condensate values described by our result (50) are in good agreement with both the Shifman, Vainshtein and Zakharov and the Narison value for the respective values of $\alpha_s$ in the range of Narison’s update average $\alpha_s$ (filled circle). Compared with our previous result [16] we have included in this work a rather detailed discussion of the error due to the experimental uncertainty in the mixing angle and due to the influence of the quarks on the gluon condensate.
Fig. 1. The gluon condensate $\langle \alpha_s G^2 \rangle$ vs. QCD coupling $\alpha_s$ according to (50) and (9) for the two limiting values $1/3$ (solid line) and $1/2$ (dashed line) of the suppression factor $\gamma$ for the mixing angle $\phi = -18.4^\circ$. The thin lines indicate the error $\pm 2^\circ$ in the value of the mixing angle. Also shown are the gluon condensate values obtained by Narison [18] (filled triangle) and by Shifman, Vainshtein and Zakharov [17] (filled square) which both are compatible with the $\alpha_s(1.3 \text{ GeV})$ value (filled circle) of Ref. [18] according to relation (50).

7 Conclusions

In the present work we have pointed out the possibility to resolve the $U_A(1)$ problem via the model of a homogeneous squeezed gluon condensate as an interesting alternative to existing models of the QCD gluon vacuum such as the instanton gas model. In particular we have discussed that in the squeezed vacuum the contact term in Witten’s formula can give a sizable contribution to the gluonic part of the $\eta_0$ mass. In the framework of a homogeneous squeezed vacuum we have obtained a relation between the value of the gluon condensate and the mass shift of the $\eta_0$ as a function of the strong coupling constant. An interesting aspect to consider the contact term is the fact that H.B. Nielsen et al. [37] have derived exactly the same relation using the Cheshire cat principle.
instead of Witten’s formula. The gluon condensate values found in our estimate are in quite good agreement with both the “standard” value $0.04 \text{ GeV}^4$ by Shifman, Vainsthein and Zakharov and the update average value $0.071 \text{ GeV}^4$ by Narison for reasonable values of the strong coupling in the low energy region.

In our simple model of the squeezed vacuum only the zero momentum mode of the gluon field operators has been squeezed. This is a Lorentz invariant operation since the gauge field is massless. The question of the gauge invariance of the procedure is still an open problem and under current investigation [34] as well as the possibility of a Lorentz and gauge invariant extension to the squeezing of nonzero momentum modes.

Acknowledgement

We thank D. Ebert, I.I. Kogan, E.A. Kuraev and C.D. Roberts for fruitful discussions on the subject. H.-P.P. is grateful to the Deutsche Forschungsgemeinschaft for support under contract No. RO 905/11-1. M.K.V. acknowledges financial support provided by INTAS under Grant No. W 94-2915 and by the Max-Planck Gesellschaft as well as the hospitality of the Fachbereich Physik at the University of Rostock.

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