TWO CYCLE-CHORD GRAPHS ARE $e$-POSITIVE

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Abstract. We prove Gebhard and Sagan’s ($e$)-positivity of the line graphs of tadpoles in noncommuting variables. This implies the $e$-positivity of these line graphs. We then extend this ($e$)-positivity result to that of certain cycle-chord graphs, and derive the bivariate generating function of all cycle-chord graphs.

1. Introduction

Stanley [21] introduced the chromatic symmetric function for a simple graph $G$ as

$$X_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where $x = (x_1, x_2, \ldots)$ is a countable set of indeterminates, and the sum is over all proper colorings $\kappa$ of $G$. Chromatic symmetric functions is a generalization of Birkhoff’s chromatic polynomials $\chi_G(k)$ since

$$X_G(1^k, 0, 0, \ldots) = \chi_G(k),$$

see Birkhoff [1].

Chromatic symmetric functions are particular symmetric functions. It is the fundamental theorem of symmetric functions that $\{e_{\lambda}\}$ is a basis of the algebra $\Lambda(x_1, x_2, \ldots)$ of symmetric functions, where

$$e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots \quad \text{and} \quad e_n = \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$  

The algebra $\Lambda(x_1, x_2, \ldots)$ has also some other bases such like the Schur basis $\{s_{\lambda}\}$. For any bases $\{b_{\lambda}\}$, a symmetric function is $b$-positive if its expansion under the basis $b_{\lambda}$ has only nonnegative coefficients, see Macdonald [16] and Stanley [23, Chapter 7]. A graph is said to be $b$-positive if its chromatic symmetric function is $b$-positive. A class of graphs is said to be $b$-positive if every graph in the class is $b$-positive.

This work is originally motivated by Stanley and Stembridge’s 3+1 conjecture, see Stanley and Stembridge [24].

Conjecture 1.1 (Stanley and Stembridge). Any claw-free incomparability graph is $e$-positive.

Only a few methods are known to show the $e$-positivity of a graph, while there are many results on the $e$-positivity of particular graph classes. Wolfgang [25] provided a powerful criterion that every connected $e$-positive graph has a connected partition of every type. Graph classes that are shown

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to be $e$-positive include complete graphs, paths, cycles, generalized bull graphs, $K$-chains, lollipop graphs, triangular ladders, Ferrers graphs; see [2–4, 6, 7, 9, 10, 13, 15, 21]. Graphs that are proved not to be $e$-positive include generalized nets, saltire graphs $SA_{n,n}$, augmented saltire graphs $AS_{n,n}$ and $AS_{n,n+1}$, and triangular tower graphs $TT_{n,n,n}$; see [8, 10]. Dahlberg et al. [8] gave an infinite number of families of non-$e$-positive graphs that are not contractible to the claw; one such family is additionally claw-free, thus establishing that the $e$-positivity is in general not dependent on the existence of an induced claw or of a contraction to a claw.

Note that $e$-positive graphs are Schur positive, since every element $e_\lambda$ is a linear combination of elements $s_\mu$ over Kostka numbers, see Mendes and Remmel [18, Exercise 2.12]. Gasharov [11] obtained the Schur positivity of the graphs in Conjecture 1.1 by smart bijections, see also Shareshian and Wachs [20, Theorem 6.3].

**Theorem 1.2** (Gasharov). Any claw-free incomparability graph is Schur positive.

In view of Theorem 1.2, Stanley [22, Conjecture 1.4] proposed the following concise conjecture, and attributed it to Gasharov.

**Conjecture 1.3** (Stanley). Any claw-free graph is Schur positive.

Stanley [21, Propositions 5.3 and 5.4] demonstrated the $e$-positivity of paths and cycles, which can be considered as the most basic graphs in some sense. Gebhard and Sagan [13] introduced certain ($e$)-positivity of chromatic symmetric functions of graphs in noncommuting variables, which is stronger than the $e$-positivity in the ordinary sense. They established the ($e$)-positivity of paths, cycles, and the so-called $K$-chains. These results can be used to show the ($e$)-positivity of tadpole graphs, see Li, Li, Wang, and Yang [14].

In the same spirit, we manage to show the $e$-positivity of the line graphs of tadpoles, which is a bit less basic. By computing the generating function of the chromatic symmetric functions of these line graphs, one may see that this $e$-positivity is stronger than that of tadpoles, see Corollary 3.5. On the other hand, since $e$-positive graphs are all Schur positive, we obtain the Schur positivity of the line graphs of tadpoles. Recall that Chudnovsky and Seymour [5] discovered a characterization of claw-free graphs, in which line graphs is one of six building blocks. Therefore, the result above confirms a particular case of Conjecture 1.3.

This paper is organized as follows. In Section 2, we give an overview for necessary notion and notation, and known results that will be of use in the sequel. Section 3 is devoted to the ($e$)-positivity of the line graphs of tadpoles. In Section 4, we derive the bivariate generating function of cycle-chord graphs $CC_{a,b}$, which is a slight extension of the line graphs. We also obtain the ($e$)-positivity of $CC_{m,3}$.

## 2. Preliminaries

This section consists of basic properties of chromatic symmetric functions given by Stanley [21], and those for the ($e$)-positivity due to Gebhard and Sagan [13]. They all will be of use in the sequel.

**Proposition 2.1** (Stanley). For any graphs $G$ and $H$,

$$X_{G\sqcup H} = X_G X_H,$$

where $G \sqcup H$ is the disjoint union of $G$ and $H$.

One way of computing a chromatic symmetric function is by using the power symmetric functions $p_\lambda$, which are defined for integer partitions $\lambda = \lambda_1 \lambda_2 \cdots$ by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

and

$$p_n = \sum_{i \geq 1} x_i^n.$$
Proposition 2.2 (Stanley). For any graph $G$,
\[ X_G = \sum_{E' \subseteq E} (-1)^{|E'|} p_{\lambda(E')}, \]
where $\lambda(E')$ is the partition consisting of the component orders of the spanning subgraph $(V, E')$.

The generating function of the power symmetric functions will also be needed (cf. [16, 18]):
\[
\sum_{j \geq 0} p_j (-z)^j = \frac{F(z)}{E(z)},
\]
where
\[ E(z) = \sum_{n \geq 0} e_n z^n \quad \text{and} \quad F(z) = E(z) - z E'(z). \]

The generating functions of paths and cycles are known, see [21, Propositions 5.3 and 5.4].

Proposition 2.3 (Stanley). Denote by $P_n$ the $n$-vertex path and by $C_n$ the $n$-vertex cycle. Then
\[
\sum_{n \geq 0} X_{P_n} z^n = \frac{E(z)}{F(z)} \quad \text{and} \quad \sum_{n \geq 2} X_{C_n} z^n = \frac{z^2 E''(z)}{F(z)}.
\]
As a consequence, paths and cycles are $e$-positive.

A beautiful triple-deletion property can be used to reduce the computation of a chromatic symmetric function, see Orellana and Scott [19, Theorem 3.1, Corollaries 3.2 and 3.3].

Theorem 2.4 (Orellana and Scott). Let $G$ be a graph. Suppose that $G$ contains three vertices $u$, $v$, and $w$ such that no two of them are connected by an edge. Write $e_1 = uv$, $e_2 = vu$, and $e_3 = wu$. For any set $S \subseteq \{1, 2, 3\}$, denote by $G_S$ the subgraph spanned by the edge set $E(G) \cup \{e_j : j \in S\}$. Then
\[ X_{G_{12}} = X_{G_1} + X_{G_{23}} - X_{G_5} \quad \text{and} \quad X_{G_{123}} = X_{G_{12}} + X_{G_{23}} - X_{G_2}. \]

Gebhard and Sagan [13] made a systematic study of the algebra of chromatic symmetric functions in noncommuting variables. Let $G$ be a graph with vertices labeled by $v_1, \ldots, v_d$. They defined the chromatic symmetric function in noncommuting variables of $G$ to be
\[ Y_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}, \]
where the sum runs also over all proper colorings $\kappa$ of $G$. Note that $Y_G$ depends not only on the coloring $\kappa$, but also on the vertex labeling of $G$. Let $\Pi_d$ denote the lattice of partitions of the set
\[ [d] = \{1, \ldots, d\} \]
ordered by refinement. Given $\pi \in \Pi_d$, the elementary symmetric function $e_\pi$ in noncommuting variables is defined by
\[ e_\pi = \sum_{(i_1, \ldots, i_d)} x_{i_1} \cdots x_{i_d}, \]
where the sum runs over all sequences $(i_1, \ldots, i_d)$ of positive integers such that $i_j \neq i_k$ if $j$ and $k$ are in the same block of $\pi$. The type $\lambda(\pi)$ of $\pi$ is the integer partition of $d$ whose parts are the block sizes of $\pi$. It is direct to verify that $e_\pi$ becomes $\lambda(\pi)! e_{\lambda(\pi)}$ if we allow the variables $x_i$ to commute, where the symbol $\lambda!$ stands for the product of factorials of all parts of a partition $\lambda$.

Fix an element $i \in [d]$. Gebhard and Sagan introduced the following congruence relations:
• Two partitions $\pi, \sigma \in \Pi_d$ are said to be congruent modulo $i$, denoted $\pi \equiv_i \sigma$, if

$$\lambda(\pi) = \lambda(\sigma) \quad \text{and} \quad b_{\pi,i} = b_{\sigma,i},$$

where $b_{\pi,i}$ is the size of the block of $\pi$ that contains $i$. Denote by $(\pi)_i$ the congruence class of $\Pi_d$ modulo $i$ that contains $\pi$.

• For any elementary symmetric functions $e_\pi$ and $e_\sigma$, we write $e_\pi \equiv_i e_\sigma$ if and only if $\pi \equiv_i \sigma$. Denote by $e_{(\pi)}$ the congruence class modulo $i$ that contains $e_\pi$.

• For any graph $G$ on $d$ vertices, we write

\begin{equation}
Y_G \equiv_i \sum_{(\pi)_i \subseteq \Pi_d} c(\pi)_i e(\pi),
\end{equation}

where

$$c(\pi)_i = \sum_{\sigma \in (\pi)_i} c_{\sigma} \quad \text{if} \quad Y_G = \sum_{\sigma \in \Pi_d} c_{\sigma} e_\pi.$$ 

They say that $G$ is (e)-positive modulo $i$ if all coefficients $c(\pi)$ are nonnegative. When $i$ equals the order $d$ of the graph $G$, the term “modulo $i$” and the letter $i$ in the notation

$$b_{\pi,i}, \quad \equiv_i, \quad \text{and} \quad (\pi)_i$$

are all ignored. For instance, $G$ is said to be (e)-positive if it is so modulo $d$, and Eq. (2.2) reduces to

\begin{equation}
Y_G \equiv \sum_{(\pi) \subseteq \Pi_d} c(\pi) e_\pi.
\end{equation}

Unlike the $e$-coefficients in $X_G$, it is possible that the $e$-coefficients $c_\pi$ in $Y_G$ is not integral.

Proposition 2.5 builds a bridge between the (e)-positivity and the $e$-positivity of the same graph, which is proved and used in [13]. We state it as a proposition for clarity.

**Proposition 2.5 (Gebhard and Sagan).** Any (e)-positive graph is e-positive.

**Proof.** Suppose that Eq. (2.3) holds and $c(\pi) \geq 0$ for all congruence classes $(\pi)$. Then

$$X_G = \sum_{(\pi) \subseteq \Pi_d} c(\pi) \lambda(\pi)! e_\lambda(\pi)$$

is e-positive. \qed

Gebhard and Sagan defined the induction operation $\uparrow$ on monomials $x_{i_1} \cdots x_{i_{d-1}}$, by

$$\left(x_{i_1} \cdots x_{i_{d-1}}\right) \uparrow = x_{i_1} \cdots x_{i_{d-1}} x_{i_{d-1}}^2$$

and extended it linearly. They discovered the following deletion-contraction property to reduce the computation of the chromatic symmetric functions $Y_G$; see [13, Proposition 3.5].

**Proposition 2.6 (Gebhard and Sagan).** For the edge $e = v_{d-1}v_d$ in a graph $G$ whose vertices are labeled by $v_1, \ldots, v_d$, 

$$Y_G = Y_{G \setminus e} - Y_{G/e} \uparrow,$$

where the contraction of $e$ is labeled by $v_{d-1}$.

In practice, we always firstly select an edge $e$ in Proposition 2.6 according to some reduction strategy, and then we need to relabel the ends of $e$ so that their labels become the largest and second largest labels. The effect of such relabelings is demonstrated by Lemma 2.7, see [13, Lemma 6.6].

**Lemma 2.7 (Gebhard and Sagan).** For any relabeling $\gamma(G)$ of vertices of $G$ that fixes the element $d$, we have $Y_{\gamma(G)} \equiv Y_G$. 

For any block \( B \) that is disjoint with the set \([d]\), they use the symbol \( \pi/B \) to denote the partition that is formed by adding the block \( B \) to \( \pi \), and the symbol \( \pi+(d+1) \) to denote the partition of \([d+1]\) that is formed by inserting the element \( d+1 \) into the block of \( \pi \) that contains \( d \). Proposition 2.8 exhibits the effect of the induction operation acted on \( e_\pi \), see [13, Corollary 6.1].

Proposition 2.8 (Gebhard and Sagan). For any partition \( \pi \in \Pi_d \),

\[ e_\pi \uparrow = \frac{1}{b_\pi} \left( e_{(\pi/d+1)} - e_{(\pi+(d+1))} \right). \]

Since the induction respects the congruence relation of partitions, namely,

\[ e_\pi \equiv e_\sigma \implies e_\pi \uparrow \equiv e_\sigma \uparrow, \]

it is extendable to congruence classes. Precisely speaking, they defined

\[ e_\pi \uparrow \equiv \sum_{(\sigma) \subseteq \Pi_{d+1}} c_{(\sigma)} e_{(\sigma)} \]  

if \( e_\pi \uparrow = \sum_{\sigma \subseteq \Pi_{d+1}} c_\sigma e_\sigma \).

By using the induction operation, Gebhard and Sagan [13, Proposions 6.4 and 6.7] obtained Proposition 2.9, which implies the \((e)\)-positivity of paths \( P_d \) and cycles \( C_d \).

Proposition 2.9 (Gebhard and Sagan). If Eq. (2.3) holds for \( G = P_d \), then

\[ Y_{P_{d+1}} = \sum_{(\pi) \subseteq \Pi_d} \frac{c_\pi}{b_\pi} \left( (b_\pi - 1)e_{\pi/d+1} + e_{\pi+(d+1)} \right) \]  

and  \[ Y_{C_{d+1}} = \sum_{(\pi) \subseteq \Pi_d} c_\pi e_{\pi+(d+1)}. \]

We also need two more results that allow us to construct new \((e)\)-positive graphs from old ones. Proposition 2.10 is a straightforward corollary of [13, Lemma 6.3].

Proposition 2.10. Let \( G \) be a graph on \( d \) vertices. Let \( K_m \) be the complete graph of order \( m \), whose vertices are labeled by \( v_{d+1}, \ldots, v_{d+m} \). If Eq. (2.3) holds, then the disjoint union of \( G \) and \( K_m \) satisfies

\[ Y_{G \cup K_m} = \sum_{(\pi) \subseteq \Pi_d} c_\pi e_{\pi/(d+1, \ldots, d+m)}. \]

Theorem 2.11 is a restatement of [13, Theorem 7.6].

Theorem 2.11 (Gebhard and Sagan). Given \( n, d \geq 1 \), let \( G = (V_1, E_1) \) be a graph with vertex set \( V_1 = \{v_1, \ldots, v_d\} \), and let \( G + K_n = (V_2, E_2) \) be the graph with

\[ V_2 = V_1 \cup \{v_{d+1}, \ldots, v_{d+n-1}\}, \]

\[ E_2 = E_1 \cup \{v_i v_j : i, j \in [d, d+n-1]\}. \]

If \( Y_G \) is \((e)\)-positive, then so is \( Y_{G+K_n} \).

3. The \((e)\)-positivity of the line graphs of tadpoles

The tadpole graph \( T_{p,m,l} \) is obtained by identifying a vertex of the cycle \( C_m \) with an end of the path \( P_l \), see Fig. 3.1. Tadpoles are special squid graphs that is investigated by Martin, Morin, and Wagner [17]. We first give the bivariate generating functions for the chromatic symmetric functions of tadpoles and their line graphs. Recall that the functions \( E(z) \) and \( F(z) \) are defined in Section 2.

Theorem 3.1. The chromatic symmetric functions of tadpole graphs \( T_{p,m,l} \) can be computed by

\[ X_{T_{p,m,l}} = (m-1)X_{P_{m+l}} - \sum_{i=2}^{m-1} X_{P_{m+l-i}} X_{C_i}, \]  

(3.1)
and their bivariate generating function is
\[
\sum_{m \geq 2} \sum_{l \geq 0} X_{T_{p,m,l}} x^m y^l = \frac{x^2}{(x-y)^2} \left[ \frac{x(x-y)E''(x)E(y)}{F(x)F(y)} - y \left( \frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)} \right) \right].
\]

Proof. Let \(m \geq 2\) and \(l \geq 0\). By Proposition 2.1 and Theorem 2.4, we obtain
\[
X_{T_{p,m,l}} = \begin{cases} 
X_{P_{m+l}} + X_{T_{p,m-1,l+1}} - X_{C_{m-1}}X_{P_{l+1}}, & \text{if } m \geq 4, \\
2X_{P_{l+3}} - X_{C_2}X_{P_{l+1}}, & \text{if } m = 3, \\
X_{P_{l+2}}, & \text{if } m = 2.
\end{cases}
\]
If \(m \geq 4\), then we can use the formula above iteratively which gives
\[
X_{T_{p,m,l}} = (m-3)X_{P_{m+l}} + X_{T_{p,3,l}} - \sum_{i=3}^{m-1} X_{C_i}X_{P_{m+i-l}},
\]
which simplifies to Eq. (3.1). It is routine to verify its truth for \(m \leq 3\). Using standard techniques of generating functions, one may translate Eq. (3.1) into the generating function equivalently. \(\square\)

Let \(u\) and \(v\) be two adjacent vertices on the cycle \(C_m\), and let \(w\) be an end of the path \(P_l\). The line graph \(L(T_{p,m,l})\) of the tadpole \(T_{p,m,l}\) can be obtained by adding the edges \(uw\) and \(vw\) to the disjoint union of \(C_m\) and \(P_l\), see Fig. 3.2.

**Figure 3.2.** The line graph of a tadpole.

**Theorem 3.2.** The chromatic symmetric functions of the line graphs \(L(T_{p,m,l})\) can be computed by
\[
(3.2) \quad X_{L(T_{p,m,l})} = X_{P_l}X_{C_m} + 2 \sum_{k \geq 1} X_{P_{l-k}}X_{C_{m+k}} - 2lX_{P_{m+l}}, \quad \text{for } m \geq 2 \text{ and } l \geq 0,
\]
or alternatively by
\[
(3.3) \quad X_{L(T_{p,m,l})} = 2X_{T_{p,m,l}} - X_{C_m}X_{P_l}, \quad \text{for } m \geq 2 \text{ and } l \geq 0.
\]
Their bivariate generating function is
\[
\sum_{m \geq 2} \sum_{l \geq 0} X_{L(T_{p,m,l})} x^m y^l = \frac{x^2}{(x-y)^2} \left[ \frac{(x^2-y^2)E''(x)E(y)}{F(x)F(y)} - 2y \left( \frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)} \right) \right].
\]
Proof. By Proposition 2.1 and Theorem 2.4, we obtain Eq. (3.3). Together with Proposition 2.3 and Theorem 3.1, one may deduce Eq. (3.2) and the generating function by routine calculation.

We did not succeed in proving the $e$-positivity of the line graphs $L(T_{p,m,l})$ by analyzing their generating function. Yet powerful enough is Proposition 2.5 to complete this job.

**Theorem 3.3.** Let $m \geq 3$ and $l \geq 1$. Let $C_m$ be the cycle $v_1 \cdots v_m v_1$, and $P_l$ the path $v_{m+1} \cdots v_{m+l}$. Then the line graph $L(T_{p,m,l})$ that is labeled by adding the edges $v_1 v_{m+1}$ and $v_m v_{m+1}$ in the disjoint union $C_m \uplus P_l$ is $(e)$-positive.

**Proof.** We first show it for $l = 1$. Let $G = L(T_{p,m,1})$. By Proposition 2.6,

\[(3.4) \quad Y_G = Y_{T_{p,m,1}} - Y_{C_m} \uparrow,\]

see Fig. 3.1 with $l = 1$ for the labeling of the graph $T_{p,m,1}$. By Lemma 2.7, we can relabel the vertices of $T_{p,m,1}$ so that the resulting graph $T'_{p,m,1}$ is obtained by adding the edge $v_1 v_m$ onto the path $v_1 \cdots v_{m+1}$. Applying Proposition 2.6 to $T'_{p,m,1}$ and by Eq. (3.4), we can infer that

\[(3.5) \quad Y'_G \equiv Y_{T'_{p,m,1}} - Y_{C_m} \uparrow \equiv Y_{C_m \uplus K_1} - 2Y_{C_m} \uparrow.\]

Suppose that

\[(3.6) \quad Y_{P_{m-1}} \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho)e(\rho).\]

By Proposition 2.9,

\[c(\rho) \geq 0 \quad \text{and} \quad Y_{C_m} \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho)e(\rho + m).\]

By Proposition 2.10,

\[(3.7) \quad Y_{C_m \uplus K_1} \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho)e(\rho + m/m + 1).\]

By Proposition 2.8,

\[(3.8) \quad Y_{C_m} \uparrow \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho)e(\rho + m) \uparrow \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left(e(\rho + m/m + m + 1) - e(\rho + m + m + 1)\right).\]

Therefore, substituting Eqs. (3.7) and (3.8) into Eq. (3.5), we obtain

\[Y_G \equiv \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left((b_\rho - 1)e(\rho + m/m + m + 1) + 2e(\rho + m + m + 1)\right).\]

Since $b_\rho \geq 1$ for all $\rho \in \Pi_{m-1}$, we obtain that $Y_G$ is $(e)$-positive.

Applying Theorem 2.11 recursively yields that $Y_{L(T_{p,m,l})}$ is $(e)$-positive.

**Corollary 3.4.** The line graphs $L(T_{p,m,l})$ are $(e)$-positive for any integers $m \geq 3$ and $l \geq 1$.

**Proof.** Immediate by Proposition 2.5.

**Corollary 3.5.** The tadpoles $T_{p,m,l}$ are $(e)$-positive for any integers $m \geq 3$ and $l \geq 1$.

**Proof.** Immediate by Eq. (3.3) and Corollary 3.4.

In fact, tadpoles labeled as in Fig. 3.1 are $(e)$-positive, which is a corollary of Proposition 2.9 and Theorem 2.11, see also [14].
4. The \((e)\)-positivity of the cycle-chord graphs \(CC_{m,3}\)

Regarding the line graphs \(L(T_{p,1})\) as obtained by identifying an edge of the cycles \(C_m\) with an edge of the triangle \(C_3\), we extend this \((e)\)-positive graph class a bit by considering the graphs that are obtained by identifying an edge in \(C_m\) with an edge of the rectangle \(C_4\). In more general, one may consider the cycle-chord graph \(CC_{a,b}\) that is obtained by identifying an edge of \(C_{a+1}\) with an edge of \(C_{b+1}\), where \(a, b \geq 1\). In particular, the graph \(CC_{m,2}\) is \(L(T_{p,1})\).

![Figure 4.1. The cycle-chord graph \(CC_{a,b}\).](image)

In this section, we first present the bivariate generating function for the chromatic symmetric functions of the cycle-chord graphs \(CC_{a,b}\), and then show the \((e)\)-positivity of \(CC_{m,3}\).

**Theorem 4.1.** The generating function of cycle-chord graphs \(CC_{a,b}\) is

\[
X_{CC_{a,b}} x^a y^b = \frac{xy}{(x-y)^2} \left[ x^2 E''(x) E(y) + y^2 E''(y) E(x) - 2xy \left( \frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)} \right) \right].
\]

**Proof.** Let \(CC_{a,b} = (V, E)\), where \(V = \{v_0, v_1, \ldots, v_{a+b-1}\}\) and \(E = E_a \cup E_b \cup \{v_0 v_a\}\) with

\[
E_a = \{v_0 v_1, v_1 v_2, \ldots, v_{a-1} v_a\} \quad \text{and} \quad E_b = \{v_a v_{a+1}, v_{a+1} v_{a+2}, \ldots, v_{a+b-1} v_0\}.
\]

Since removing the edge \(v_0 v_a\) from \(CC_{a,b}\) results in the cycle \(C_{a+b}\), by Proposition 2.2,

\[
\sum_{v_0 v_a \notin S \subseteq E} (-1)^{|S|} p_{\lambda(S)} = X_{CC_{a+b}}.
\]

When \(v_0 v_a \in S\), let \(M\) be the component of the graph \((V, S)\) that contains the edge \(v_0 v_a\). Let \(a'\) be the number of edges in \(M \cap E_a\) and \(b'\) the number of edges in \(M \cap E_b\). Then \(0 \leq a' \leq a\) and \(0 \leq b' \leq b\). According to whether \(a' = a\) and \(b' = b\), we obtain

\[
X_{CC_{a,b}} = X_{CC_{a+b}} + \sum_{a'=0}^{a-1} \sum_{b'=0}^{b-1} (-1)^{a'+b'+1} (a' + 1)(b' + 1)p_{a'+b'+2}X_{p_{a-a'-1}}X_{p_{b-b'-1}} + (-1)^{a+b+1}p_{a+b}
\]

\[
+ \sum_{a'=0}^{a-1} (-1)^{a'+b+1}(a' + 1)p_{a'+b+1}X_{p_{a-a'-1}} + \sum_{b'=0}^{b-1} (-1)^{a+b'+1}(b' + 1)p_{a+b'+1}X_{p_{b-b'-1}}
\]

\[
= X_{CC_{a+b}} + \sum_{i=1}^{a} \sum_{j=1}^{b} (-1)^{i+j+1} i \cdot j \cdot p_{i+j}X_{p_{a-i}}X_{p_{b-j}} + (-1)^{a+b+1}p_{a+b}
\]

\[
(4.1) \quad + \sum_{i=1}^{b} (-1)^{b+i} i \cdot p_{b+i}X_{p_{a-i}} + \sum_{j=1}^{b} (-1)^{a+j} j \cdot p_{a+j}X_{p_{b-j}}.
\]

Considering the generating function

\[
H(x, y) = \sum_{a, b \geq 1} X_{CC_{a,b}} x^a y^b,
\]
we compute each term on the right side of Eq. (4.1) by multiplying $x^ax^by^b$ and summing over $a, b \geq 1$. For convenience, we will use the substitutions

$$s = -x, \quad t = -s, \quad \text{and} \quad Q_z = \frac{F(z)}{E(z)}.$$ 

For the first and third term in Eq. (4.1), by Proposition 2.3 and Eq. (2.1),

\begin{equation}
\sum_{a,b \geq 1} X_C a+b x^ax^by^b = \frac{xy}{x-y} \sum_{h \geq 2} X_C h (x^{h-1} - y^{h-1}) = \frac{xy}{x-y} \left( \frac{x E''(x)}{F(x)} - \frac{y E''(y)}{F(y)} \right),
\end{equation}

\begin{equation}
\sum_{a,b \geq 1} (-1)^{a+b+1}p_{a+b}x^ax^by^b = - \sum_{a,b \geq 1} p_{a+b}x^ax^by^b = \frac{xy}{x-y} \sum_{h \geq 2} p_h (s^{h-1} - t^{h-1})
\end{equation}

$$= \frac{xy}{x-y} \left( Q_x - 1 - e_1s \right) \frac{Q_y - 1 - e_1t}{t} \leq \frac{x Q_y - y Q_x}{x-y} - 1.$$

For the fourth term in Eq. (4.1),

$$\sum_{a,b \geq 1} \sum_{i=1}^a (-1)^{i} \cdot i \cdot p_{b+i} X_{P_{a-i}} x^{a} y^{b} = \sum_{b,i \geq 1} \frac{1}{Q_x} \sum_{b,i \geq h \geq 2} p_h \left( s^{h} - s^{h-1} t - s^{h-1} + t^{h-1} \right)$$

$$= \frac{x y}{(x-y)^2 Q_x} \sum_{h \geq 2} p_h \left( h^{s} - h s^{h-1} t - h^{s} + t^{h-1} \right) = \frac{x y \left[(x-y)Q_{x} - (Q_{x} - Q_{y})\right]}{(x-y)^2 Q_{y}}.$$

Exchanging $x$ and $y$ and exchanging $a$ and $b$ in the formula above, we obtain

$$\sum_{a,b \geq 1} \sum_{j=1}^b (-1)^{a+j} \cdot j \cdot p_{a+j} X_{P_{a-j}} x^{a} y^{b} = \frac{x y \left[(y-x)Q_{y} + (Q_{x} - Q_{y})\right]}{(x-y)^2 Q_{y}}.$$ 

For the second term in Eq. (4.1), we compute

$$\sum_{a,b \geq 1} \sum_{i,j \geq 1} (-1)^{i+j+1} \cdot i \cdot j \cdot p_{i+j} X_{P_{a-i,j}} X_{P_{a-j,i}} x^{a} y^{b}$$

$$= - \sum_{i,j \geq 1} i \cdot j \cdot p_{i+j} X_{P_{a-i,j}} X_{P_{a-j,i}} x^{a} y^{b}$$

$$= - \sum_{i,j \geq 1} i \cdot j \cdot p_{i+j} X_{P_{a-i,j}} X_{P_{a-j,i}} x^{a} y^{b}$$

$$= - \frac{1}{Q_x Q_y} \sum_{i,j \geq 1} i \cdot j \cdot p_{i+j} s^{i} t^{j} = - \frac{1}{Q_x Q_y} \sum_{h \geq 2} p_h \sum_{i=1}^{h-1} (h-i) \cdot s^{i} t^{h-i}$$

$$= - \frac{s t}{(s-t)^3 Q_x Q_y} \sum_{h \geq 2} p_h \left[ (h-1)(s^{h+1} - t^{h+1}) - (h+1) s^{h-1} t^{h-1} \right]$$

$$= \frac{x y}{(x-y)^3 Q_x Q_y} \left[(x+y)(Q_{x} - Q_{y}) - (x-y)(x Q_{x} + y Q_{y})\right].$$

Adding the five results above up, we obtain

$$H(x, y) = \frac{x y}{x-y} \left( \frac{x E''(x)}{F(x)} - \frac{y E''(y)}{F(y)} \right) + \frac{x Q_y - y Q_x}{x-y} - 1$$

$$+ \frac{x y \left[(x-y)Q_{x} - (Q_{x} - Q_{y})\right]}{(x-y)^2 Q_x} + \frac{x y \left[(y-x)Q_{y} + (Q_{x} - Q_{y})\right]}{(x-y)^2 Q_y}$$

$$+ \frac{x y}{(x-y)^3 Q_x Q_y} \left[(x+y)(Q_{x} - Q_{y}) - (x-y)(x Q_{x} + y Q_{y})\right],$$

which simplifies to the desired formula.
In the procedure of showing the \((e)\)-positivity of \(Y_{CC_{m,3}}\), we will use Proposition 2.6 and Lemma 2.7 frequently in which we need to relabel the largest vertex. We write \((d, i)\) to denote the transposition of the elements \(d\) and \(i\). For any partition \(\pi \in \Pi_d\) and any graph \(G\) on \(d\) vertices, we write

- \(\pi \circ (d, i)\) to denote the partition of \([d]\) that interchanges the elements \(d\) and \(i\),
- \(G \circ (d, i)\) to denote the relabeling of \(G\) that interchanges the vertex labels \(v_d\) and \(v_i\), and
- \(Y_G \circ (d, i)\) to denote the chromatic symmetric function \(Y_{G \circ (d, i)}\).

It follows that

\[
Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi \implies Y_G \circ (d, i) = \sum_{\pi \in \Pi_d} c_\pi e_{\pi \circ (d, i)}.
\]

Another basic result of the transposition is Proposition 4.2.

**Proposition 4.2.** For any partition \(\pi \in \Pi_d\) and any element \(i \in [d]\),

\[
e_{\pi \circ (d, i)} \uparrow \equiv \frac{1}{b_{\pi \circ (d, i)}} \left( e_{(\pi/d+1)} - e_{(\pi+(d+1))} \right) \circ (d, i),
\]

where \(\pi + (d + 1)\) is the partition of \([d + 1]\) formed by inserting the element \(d + 1\) into the block of \(\pi\) that contains \(i\).

**Proof.** By Proposition 2.8, we have

\[
e_{\pi \circ (d, i)} \uparrow \equiv \frac{1}{b_{\pi \circ (d, i)}} \left( e_{(\pi/d+1)} - e_{(\pi+(d+1))} \right),
\]

which simplifies to the desired congruence relation. \(\square\)

Note that Proposition 4.2 reduces to Proposition 2.8 when \(i = d\).

Here comes the main result of Section 4.

**Theorem 4.3.** For any integer \(m \geq 3\), the cycle-chord graph \(CC_{m,3}\) that is labeled by adding the edge \(v_2v_{m+1}\) to the cycle \(v_1 \cdots v_{m+2}v_1\) is \((e)\)-positive.

**Proof.** By Proposition 2.6,

\[
Y_{CC_{m,3}} = Y_{T_{P_{m,2}}} - Y_{L(T_{P_{m,1}})} \uparrow,
\]

see the left part of Fig. 4.2 for the vertex labeling of \(CC_{m,3}\).

**Figure 4.2.** The cycle-chord graph \(CC_{m,3}\) and the line graph \(Y_{L(T_{P_{m,1}})}\).

We relabel the vertices of the tadpole \(T_{P_{m,2}}\) so that it is obtained by adding the edge \(v_1v_m\) in the path \(v_1 \cdots v_{m+2}\). Write the resulting graph as \(T'_{P_{m,2}}\). We also relabel the vertices of the line graph \(L(T_{P_{m,1}})\) so that it is obtained by adding the edge \(v_{m-1}v_{m+1}\) in the cycle \(v_1 \cdots v_{m+1}v_1\). Write
the resulting graph as $L(T_{p,m})'$, see the right part of Fig. 4.2. By Lemma 2.7 and by applying Proposition 2.6 another twice, we can deduce that

\[
\begin{align*}
Y_{C_{m-3}} &= m+2 Y_{T_{p,m,2}} - Y_{L(T_{p,m,1})}'
\equiv m+2 \left( Y_{T_{p,m,1}} \uparrow_{K_1} - Y_{T_{p,m,1}} \uparrow \right) - Y_{L(T_{p,m,1})}'
\equiv m+2 Y_{T_{p,m,1}} \uparrow_{K_1} - (Y_{C_{m-1}} \uparrow_{K_1} - C_{m-1} \uparrow_{K_1}) - Y_{L(T_{p,m,1})}'
\equiv m+2 Y_{T_{p,m,1}} \uparrow_{K_1} + Y_{C_{m-1}} \uparrow_{K_1} - Y_{C_{m-1}} \uparrow_{K_1} - Y_{L(T_{p,m,1})}' .
\end{align*}
\]

(4.3)

We proceed by computing the four terms in Eq. (4.3) separately. In order to compute the last term, we pause here and give a lemma for computing $Y_{L(T_{p,m,1})}'$.

**Lemma 4.4.** If Eq. (3.6) holds, then

\[
Y_{L(T_{p,m,1})}' = \sum_{(\rho) \leq \Pi_{m-1}} c(\rho) \left( \frac{2}{b_\rho + 1} e_{(\rho/m+m+1)} - \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} e_{(\rho/m+m+1)} + \frac{b_\rho - 1}{b_\rho} e_{(\rho')}, \right)
\]

where $\rho' \in \Pi_{m+1}$ is the partition $\rho \setminus \{m-1\} + m + (m+1)/m - 1$.

**Proof of Lemma 4.4.** By using Proposition 2.6, we obtain

\[
Y_{C_{m}} = Y_{J_{m}} - Y_{C_{m-1}} \uparrow_{K_1} .
\]

It follows that

\[
Y_{C_{m-1}} \uparrow_{K_1} = Y_{J_{m}} - Y_{C_{m}} \uparrow_{K_1} .
\]

(4.5)

Let $i \in [m-1]$. It will be carefully set to facilitate the computation in the sequel. By using Proposition 2.6 thrice and using Eq. (4.5), we can deduce

\[
\begin{align*}
Y_{L(T_{p,m,1})}' &= Y_{T_{p,m,1}} \circ (m,m-1) - Y_{C_{m}} \uparrow_{K_1}
\equiv Y_{T_{p,m-1}} \uparrow_{K_1} Y_{T_{p,m-1}} \uparrow_{K_1} - Y_{C_{m}} \uparrow_{K_1}
\equiv Y_{T_{p,m-1}} \uparrow_{K_1} - (Y_{C_{m-1}} \circ (m,i) \uparrow_{K_1} - Y_{C_{m-1}} \uparrow_{K_1}) + Y_{C_{m}} \uparrow_{K_1}
\equiv Y_{T_{p,m-1}} \uparrow_{K_1} - Y_{C_{m-1}} \circ (m,i) \uparrow_{K_1} + Y_{C_{m}} \uparrow_{K_1} - 2Y_{C_{m}} \uparrow_{K_1} ,
\end{align*}
\]

(4.6)

where the graph $C_{m-1} \uparrow_{K_1}$ is labeled by $C_{m-1} = v_1 \cdots v_{m-1} v_1$ and $V(K_1) = \{v_m\}$. We will calculate the four terms in Eq. (4.6) independently. Suppose that

\[
\begin{align*}
Y_{T_{p,m-2}} &= \sum_{(\tau) \leq \Pi_{m-2}} h(\tau) c(\tau),
Y_{T_{p,m-1}} &= \sum_{(\rho) \leq \Pi_{m-1}} c(\rho) c(\rho), 	ext{ and}
Y_{C_{m-1}} &= \sum_{(\rho) \leq \Pi_{m-1}} c'(\rho). \end{align*}
\]

Then by Proposition 2.9,

\[
c(\rho) = \begin{cases} 
\frac{h(\tau)}{b_\rho} (b_\rho - 1), & \text{if } \rho = \tau/(m-1), \\
\frac{h(\tau)}{b_\rho - 1}, & \text{if } \rho = \tau + (m-1), \\
0, & \text{otherwise},
\end{cases}
\]

\[
c'(\rho) = \begin{cases} 
h(\tau), & \text{if } \rho = \tau + (m-1), \\
0, & \text{otherwise}.
\end{cases}
\]
It follows that \( c'_\rho = h(\tau) = (b_\rho - 1)c_\rho \), if \( \rho = \tau + (m - 1) \). By using Propositions 2.10 and 4.2, we obtain

\[
Y_{C_{m-1} \sqcup K_1 \circ (m, i)} \uparrow = \sum_{(\rho) \leq \Pi_{m-1}} c'_\rho c_{(\rho/m)} \circ (m, i) \uparrow
\]

\[
= \sum_{(\rho) \leq \Pi_{m-1}} \frac{c'_\rho}{b_\rho} \left( e_{(\rho/m, m+1)} - e_{\rho/m, (m+1)} \right) \circ (m, i)
\]

\[
= \sum_{(\rho) \leq \Pi_{m-1}} \frac{(b_\rho - 1)c_\rho}{b_\rho} \left( e_{(\rho/m, m+1)} - e_{\rho/m, (m+1)} \right) \circ (m, i).
\]

Taking \( i = m - 1 \), we obtain

\[(4.7)\quad Y_{C_{m-1} \sqcup K_1 \circ (m, i)} \uparrow = \sum_{(\rho) \leq \Pi_{m-1}} \frac{(b_\rho - 1)c_\rho}{b_\rho} \left( e_{(\rho/m, m+1)} - e_{\rho'/m} \right).\]

On the other hand, by Proposition 2.9, we obtain

\[(4.8)\quad Y_{P_m} \equiv \sum_{(\rho) \leq \Pi_{m-1}} c_\rho \left( (b_\rho - 1)e_{(\rho/m)} + e_{(\rho/m)} \right) \quad \text{and} \quad
\]

\[(4.9)\quad Y_{C_m} \equiv \sum_{(\rho) \leq \Pi_{m-1}} c_\rho e_{(\rho/m)}.
\]

By Proposition 2.9, we obtain

\[
Y_{P_{m+1}} \equiv \sum_{(\rho) \leq \Pi_{m-1}} \frac{(b_\rho - 1)c_\rho}{b_\rho (b_\rho/m)} \left( (b_\rho/m - 1)e_{(\rho/m, m+1)} + e_{(\rho/m, (m+1))} \right)
\]

\[
+ \sum_{(\rho) \leq \Pi_{m-1}} \frac{c_\rho}{b_\rho} \left( (b_\rho/m - 1)e_{\rho/m, m+1} + e_{\rho/m, (m+1)} \right).
\]

\[(4.10)\quad \equiv \sum_{(\rho) \leq \Pi_{m-1}} c_\rho \left( \frac{b_\rho - 1}{b_\rho} e_{(\rho/m, m+1)} + \frac{1}{b_\rho} e_{(\rho/m, m+1)} + \frac{1}{b_\rho (b_\rho + 1)} e_{(\rho/m, m+1)} \right).
\]

Applying the induction to Eqs. (4.8) and (4.9) respectively, and by Proposition 2.8, we obtain

\[
Y_{P_m} \uparrow \equiv \sum_{(\rho) \leq \Pi_{m-1}} \frac{c_\rho}{b_\rho} \left( (b_\rho - 1)e_{(\rho/m)} \uparrow + e_{(\rho/m)} \uparrow \right)
\]

\[
= \sum_{(\rho) \leq \Pi_{m-1}} \frac{c_\rho}{b_\rho} (b_\rho - 1) \left( e_{(\rho/m, m+1)} - e_{\rho/m, (m+1)} \right)
\]

\[
+ \sum_{(\rho) \leq \Pi_{m-1}} \frac{c_\rho}{b_\rho (b_\rho + 1)} \left( e_{(\rho/m, m+1)} - e_{(\rho+m, (m+1))} \right), \quad \text{and}
\]

\[(4.11)\quad Y_{C_m} \uparrow = \sum_{(\rho) \leq \Pi_{m-1}} c_\rho e_{(\rho/m)} \uparrow = \sum_{(\rho) \leq \Pi_{m-1}} \frac{c_\rho}{b_\rho + 1} \left( e_{(\rho/m, m+1)} - e_{(\rho+m, (m+1))} \right).
\]

Therefore, substituting Eqs. (4.7) and (4.10) to (4.12) into Eq. (4.6) with \( i = m - 1 \), we obtain

\[
Y_{L(T_{p_{m-1}})} \equiv Y_{P_{m+1}} - Y_{C_{m-1} \sqcup K_1 \circ (m, m - 1)} \uparrow + Y_{P_m} \uparrow - 2Y_{C_m} \uparrow
\]

\[
= \sum_{(\rho) \leq \Pi_{m-1}} \frac{(b_\rho - 1)c_\rho}{b_\rho} \left( e_{(\rho/m, m+1)} + \frac{2}{b_\rho + 1} e_{(\rho/m, m+1)} - \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} e_{(\rho/m, m+1)} \right)
\]

\[
- \sum_{(\rho) \leq \Pi_{m-1}} \frac{(b_\rho - 1)c_\rho}{b_\rho} \left( e_{(\rho/m, m+1)} - e_{(\rho')} \right),
\]

which simplifies to Eq. (4.4). \qed
Now we are in a position to complete the proof of Theorem 4.3. Applying the induction to Eq. (4.4), and by Proposition 2.8, we can compute
\[
Y_{L(T_{p,m},r)} \uparrow = \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) \left( \frac{2}{b_\rho + 1} e^{(\rho+m+(m+1))} - \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} e^{(\rho+m/m+1)} \uparrow + \frac{b_\rho - 1}{b_\rho} e^{(\rho')} \uparrow \right)
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) \left( \frac{2}{(b_\rho + 1)(b_\rho + 2)} \left( e^{(\rho+m+(m+1)/m+2)} - e^{(\rho+m+(m+1)+(m+2))} \right) - \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} \left( e^{(\rho/m+1/m+2)} - e^{(\rho/m+(m+1)+(m+2))} \right) \right.
+ \left. \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} \left( e^{(\rho'/m+2)} - e^{(\rho'/(m+2))} \right) \right)
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) \left( \frac{2}{(b_\rho + 1)(b_\rho + 2)} \left( e^{(\rho+m+(m+1)/m+2)} - e^{(\rho+m+(m+1)+(m+2))} \right) + \frac{b_\rho - 1}{b_\rho (b_\rho + 1)} \left( e^{(\rho+m/(m+1)+(m+2))} - e^{(\rho'(m+2))} \right) \right).
\]

Suppose that Eq. (3.6) holds. By Proposition 2.10 and Eq. (4.9), we obtain
\[
(4.13) \quad Y_{C_m \cup K_1} = \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) e^{(\rho+m/m+1)}.
\]

Applying Proposition 2.6 to the graph $T_{p,m,1}$, and by using Eqs. (4.12) and (4.13), we can deduce that
\[
Y_{T_{p,m,1}} = Y_{C_m \cup K_1} - Y_{C_m} \uparrow
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) e^{(\rho+m/m+1)} - \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left( e^{(\rho+m/m+1)} - e^{(\rho+m/(m+1))} \right)
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left( b_\rho e^{(\rho+m/m+1)} + e^{(\rho+m/(m+1))} \right).
\]
It follows by Proposition 2.10 that
\[
Y_{T_{p,m,1} \cup K_1} = \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left( b_\rho e^{(\rho+m/m+1)/m+2} + e^{(\rho+m/(m+1)/m+2)} \right).
\]

Applying the induction to Eqs. (4.12) and (4.13) respectively, and by Proposition 2.8, we can infer that
\[
Y_{C_m} \uparrow \uparrow = \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left( e^{(\rho+m/m+1)} \uparrow - e^{(\rho+m/(m+1))} \uparrow \right)
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} \left( e^{(\rho+m/m+1/m+2)} - e^{(\rho/m+(m+1)+(m+2))} \right)
- \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{(b_\rho + 1)(b_\rho + 2)} \left( e^{(\rho+m/(m+1)/m+2)} - e^{(\rho+m/(m+1)+(m+2))} \right)
\]
and
\[
Y_{C_m \cup K_1} \uparrow = \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) e^{(\rho+m/m+1)} \uparrow
\]
\[
= \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) \left( e^{(\rho+m/m+1)+m+2)} - e^{(\rho+m/(m+1)+(m+2))} \right).
\]
Now we can continue calculating $Y_{CC_m,3}$ by Eq. (4.3) as follows.

\[
Y_{CC_m,3} = Y_{T_{P_m,1} \uparrow \uparrow K_1} + Y_{C_m \uparrow \uparrow} - Y_{C_m \uparrow \uparrow K_1} - Y_{L(T_{P_m,1})} \uparrow \\
\equiv \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{\rho^2 + 1} \left( b_\rho \epsilon(\rho + m/m + 1/m + 2) + c(\rho + m + 1/m + 2) - c(\rho + m + (m + 1)/m + (m + 2)) \right) \\
\quad + \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{\rho^2 + 1} \left( \epsilon(\rho + m/m + 1/m + 2) - c(\rho + m + m + 1/m + 2) \right) \\
\quad - \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} (b_\rho + 2) \left( \epsilon(\rho + m + (m + 1)/m + 2) - c(\rho + m + (m + 1)/m + (m + 2)) \right) \\
\quad - \sum_{(\rho) \subseteq \Pi_{m-1}} c(\rho) \left( \epsilon(\rho + m + (m + 1)/m + 2) - c(\rho + m + (m + 1)/m + (m + 2)) \right) \\
\quad + \frac{b_\rho - 1}{b_\rho + 1} \left( \epsilon(\rho + m + (m + 1)/m + (m + 2)) - c(\rho + (m + 2)) \right) \right) \\
\equiv \sum_{(\rho) \subseteq \Pi_{m-1}} \frac{c(\rho)}{b_\rho + 1} \left( \frac{b_\rho - 1}{b_\rho + 2} \epsilon(\rho + m + (m + 1)/m + 2) + \frac{b_\rho^2 - b_\rho + 1}{b_\rho} c(\rho + m + (m + 1)/m + (m + 2)) \right) \\
\quad + \frac{3}{b_\rho + 2} \epsilon(\rho + m + (m + 1)/m + (m + 2)) + \frac{b_\rho - 1}{b_\rho} c(\rho + (m + 2)) \right).
\]

Since $b_\rho \geq 1$ for all $\rho \in \Pi_{m-1}$, we obtain that $Y_{CC_m,3}$ is $(\epsilon)$-positive. \qed

**Corollary 4.5.** The cycle-chord graphs $CC_m,3$ are $\epsilon$-positive for any integer $m \geq 3$.

**Proof.** Immediate by Proposition 2.5 and Theorem 4.3. \qed

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