Research Article

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The (1, 2)-step competition graph of a hypertournament

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Abstract: In 2011, Factor and Merz [Discrete Appl. Math. 159 (2011), 100–103] defined the (1, 2)-step competition graph of a digraph. Given a digraph $D = (V, A)$, the (1, 2)-step competition graph of $D$, denoted $C_{1,2}(D)$, is a graph on $V(D)$, where $xy \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$ such that either $d_{D}(x, z) = 1$ and $d_{D}(y, z) \leq 2$ or $d_{D}(y, z) = 1$ and $d_{D}(x, z) \leq 2$. They also characterized the (1, 2)-step competition graphs of tournaments and extended some results to the $(i, j)$-step competition graphs of tournaments. In this paper, the definition of the (1, 2)-step competition graph of a digraph is generalized to a hypertournament and the (1, 2)-step competition graph of a $k$-hypertournament is characterized. Also, the results are extended to $(i, j)$-step competition graphs of $k$-hypertournaments.

Keywords: $k$-hypertournament, (1, 2)-step competition graph, $(i, j)$-step competition graph

MSC 2020: 05C65, 05C12, 05C20

1 Terminology and introduction

Let $G = (V, E)$ be an undirected graph, or a graph for short. $V(G)$ and $E(G)$ are the vertex set and the edge set of $G$, respectively. The complement $G'$ of a graph $G$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in $G$. Let $G_1$ and $G_2$ be two graphs. The union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Let $D = (V, A)$ be a directed graph, or a digraph for short. $V(D)$ and $A(D)$ are the vertex set and the arc set of $D$, respectively. Let $i \geq 1$, $j \geq 1$. The $(i, j)$-step competition graph of $D$, denoted $C_{i,j}(D)$, is a graph on $V(D)$, where $xy \in E(C_{i,j}(D))$ if and only if there exists a vertex $z \neq x, y$ such that either $d_{D}(x, z) \leq i$ and $d_{D}(y, z) \leq j$ or $d_{D}(y, z) \leq i$ and $d_{D}(x, z) \leq j$. When $(i, j) = (1, 1)$, it is also called the competition graph of $D$.

The notion of competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space. In recent years, many researchers investigated $m$-step competition graphs of some special digraphs and the competition numbers of some graphs etc. (see [2–4]). Particularly, in 1998, Fisher, Lundgren, Merz and Reid [5] studied the domination graphs and competition graphs of a tournament. Recall that a tournament is an orientation of a complete graph. In 2011, Factor and Merz [6] gave the definition of the $(i, j)$-step competition graph of a digraph. They also characterized the $(1, 2)$-step competition graph of a tournament and extended some results to the $(i, j)$-step competition graph of a tournament. They proved the following theorems related to this paper.

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Theorem 1.1. [6] A graph $G$ on $n \geq 5$ vertices is the $(1, 2)$-step competition graph of some strong tournament if and only if $G$ is $K_n$, $K_{n-1} - E(P_2)$, or $K_n - E(P_3)$.

Theorem 1.2. [6] Given a graph on $n$ vertices, is the $(1, 2)$-step competition graph of some tournament if and only if $G$ is one of the following graphs:

1. $K_n$, where $n \neq 2, 3, 4$;
2. $K_{n-1} \cup K_1$, where $n > 1$;
3. $K_n - E(P_3)$, where $n > 2$;
4. $K_n - E(P_2)$, where $n \neq 1, 4$, or
5. $K_n - E(K_3)$, where $n \geq 3$.

Theorem 1.3. [6] If $T$ is a tournament with $n$ vertices, $i \geq 1$ and $j \geq 2$, then $C_{i,j}(T) = C_{j,i}(T)$.

Given two integers $n$ and $k$, $n \geq k > 1$, a $k$-hypertournament $T$ on $n$ vertices is a pair $(V, A)$, where $V$ is a set of vertices, $|V| = n$ and $A$ is a set of $k$-tuples of vertices, called arcs, so that for any $k$-subset $S$ of $V$, $A$ contains exactly one of the $k$! $k$-tuples whose entries belong to $S$. As usual, we use $V(T)$ and $A(T)$ to denote the vertex set and the arc set of $T$, respectively. Clearly, a 2-hypertournament is merely a tournament. When $k = n$, the hypertournament has only one arc and it does not have much significance to study. Thus, in what follows, we consider $3 \leq k \leq n - 1$.

Let $T = (V, A)$ be a $k$-hypertournament on $n$ vertices. For an arc $a$ of $T$, $T - a$ denotes a hyperdigraph obtained from $T$ by removing the arc $a$ and $\bar{a}$ denotes the set of vertices contained in $a$. If $v_i, v_j \in \bar{a}$ and $v_i$ precedes $v_j$ in $a$, we say that $v_i$ dominates $v_j$ in $a$. We also say the vertex $v_j$ is an out-neighbour of $v_i$ and use the following notation:

$$N^2_{A}(v_i) = \{v_j \in V \setminus \{v_i\} : v_i \text{ precedes } v_j \text{ in some arc } a \in A(T)\}.$$ 

We will omit the subscript $A$ if the $k$-hypertournament $T$ is known from the context.

A path $P$ in a $k$-hypertournament $T$ is a sequence $v_1a_1v_2a_2v_3 \ldots v_{t-1}a_{t-1}v_t$ of distinct vertices $v_1, v_2, \ldots, v_t$, $t \geq 1$ and distinct arcs $a_1, a_2, \ldots, a_{t-1}$ such that $v_i$ precedes $v_{i+1}$ in $a_i$, $1 \leq i \leq t - 1$. Meanwhile, let the vertex set $V(P) = \{v_1, v_2, \ldots, v_t\}$ and the arc set $A(P) = \{a_1, a_2, \ldots, a_{t-1}\}$. The length of a path $P$ is the number of its arcs, denoted $\ell(P)$. A path from $x$ to $y$ is an $(x, y)$-path. The $k$-hypertournament $T$ is called strong if $T$ has an $(x, y)$-path for every pair $x, y$ of distinct vertices in $T$.

A $k$-hypertournament $T$ is said to be transitive if its vertices are labeled $v_1, v_2, \ldots, v_n$ in such an order so that $i < j$ if and only if $v_i$ precedes $v_j$ in each arc containing $v_i$ and $v_j$.

Now we generalize the $(1, 2)$-step competition graph of a digraph to a $k$-hypertournament.

Definition 1.4. The $(i, j)$-step competition graph of a $k$-hypertournament $T$ with $i \geq 1$ and $j \geq 1$, denoted $C_{i,j}(T)$, is a graph on $V(T)$, where $xy \in E(C_{i,j}(T))$ if and only if there exist a vertex $z \neq x, y$ and an $(x, y)$-path $P$ and a $(y, z)$-path $Q$ satisfying the following:

(a) $y \notin V(P)$, $x \notin V(Q)$;
(b) $\ell(P) \leq i$ and $\ell(Q) \leq j$, or $\ell(Q) \leq i$ and $\ell(P) \leq j$;
(c) $P$ and $Q$ are arc-disjoint.

If $xy \in E(C_{i,j}(T))$, we say that $x$ and $y$ compete if $\ell(P) = 1$ and $\ell(Q) = 1$. $C_{i,j}(T)$ is also called the competition graph of the $k$-hypertournament $T$. Clearly, when $k = 2$, $T$ is a tournament and $C_{1,2}(T)$ is the $(1, 2)$-step competition graph of $T$.

The $k$-hypertournaments form one of the most interesting classes of digraphs. For the class of $k$-hypertournaments, the popular topics are the Hamiltonicity and vertex-pancyclicity (see [7–10]). Besides, some researchers investigated the degree sequences and score sequences of $k$-hypertournaments (see [11,12]). Recently, the $H$-force set of a hypertournament was also studied (see [13]). In this paper, we study the $(1, 2)$-step competition graph of a $k$-hypertournament and extend Theorems 1.1–1.3 to $k$-hypertournaments.

In Sections 2 and 3, useful lemmas are provided to make the proof of the main results easier. In Sections 4 and 5, the $(1, 2)$-step competition graph of a (strong) $k$-hypertournament is characterized. In Section 6, the main results are extended to the $(i, j)$-step competition graph of a $k$-hypertournament.
2 The missing edges of \( C_{1,2}(T) \)

Let \( T = (V, A) \) be a \( k \)-hypertournament. For a pair of distinct vertices \( x \) and \( y \) in \( T \), \( A_T(x, y) \) denotes the set of all arcs of \( T \) in which \( x \) precedes \( y \), \( A_T[x, y] \) denotes the set of all arcs containing \( x, y \) in \( T \) and \( A_T^{xy} \) denotes the set of all arcs containing \( x, y \) in \( T \) and in which neither \( x \) nor \( y \) is the last entry.

**Lemma 2.1.** Let \( T \) be a \( k \)-hypertournament with \( n \) vertices, where \( 3 \leq k \leq n - 1 \). Then \( xy \notin E(C_{1,2}(T)) \) if and only if one of the following holds:

(a) \( N^*(x) = \emptyset \);

(b) \( N^*(y) = \emptyset \);

(c) \( N^*(x) = \{y\} \);

(d) \( N^*(y) = \{x\} \);

(e) \( A_T^{xy} \) contains exactly an arc \( a \), and \( N_{T,a}^{-}(x) \subseteq \{y\} \), \( N_{T,a}^{+}(y) \subseteq \{x\} \).

**Proof.** First, we show the “if” part. Clearly, if one of (a)–(d) holds, we have \( xy \notin E(C_{1,2}(T)) \). Now we assume that the argument (e) holds. Since \( A_T^{xy} \) contains exactly an arc \( a \), and \( N_{T,a}^{-}(x) \subseteq \{y\} \), \( N_{T,a}^{+}(y) \subseteq \{x\} \), we have to use the unique arc \( a \) to obtain the out-neighbour except \( y \) of \( x \) and the out-neighbour except \( x \) of \( y \). So \( x \) and \( y \) are impossible to (1, 2)-step compete and hence \( xy \notin E(C_{1,2}(T)) \).

Now we show the “only if” part. Assume that \( xy \notin E(C_{1,2}(T)) \). Also, assume that \( x \) and \( y \) do not satisfy (a)–(d). That means \( N^*(x) \cap \{y\} \neq \emptyset , N^*(y) \cap \{x\} \neq \emptyset \). Now we show that \( x \) and \( y \) satisfy (e). Suppose \( A_T^{xy} \) consists of at least two arcs, say \( a_1, a_2 \in A_T^{xy} \). Let \( w_i \) be the last entry of \( a_i \) for \( i = 1, 2 \). If \( w_1 = w_2 \), then \( x \) and \( y \) compete, a contradiction. So assume \( w_1 \neq w_2 \). Note that \( a_1, a_2 \) are distinct from \( x \) and \( y \). Then \( A_T^{xy} \) contains exactly an arc \( a \), and \( N_{T,a}^{-}(x) \subseteq \{y\} \), \( N_{T,a}^{+}(y) \subseteq \{x\} \). Thus \( A_T^{xy} \) contains exactly an arc \( a \), and \( N_{T,a}^{-}(x) \subseteq \{y\} \), \( N_{T,a}^{+}(y) \subseteq \{x\} \).

By the proof of Lemma 2.1, we obtain the following result.

**Corollary 2.2.** Let \( T \) be a strong \( k \)-hypertournament with \( n \) vertices, where \( 3 \leq k \leq n - 1 \). Then \( xy \notin E(C_{1,2}(T)) \) if and only if one of the following holds:

(a) \( N^*(x) = \{y\} \);

(b) \( N^*(y) = \{x\} \);

(c) \( A_T^{xy} \) contains exactly an arc \( a \), and \( N_{T,a}^{-}(x) \subseteq \{y\} \), \( N_{T,a}^{+}(y) \subseteq \{x\} \).

3 The forbidden subgraphs of \( (C_{1,2}(T))^c \)

**Lemma 3.1.** Let \( G \) on \( n \) vertices be the \( (1, 2) \)-step competition graph of some \( k \)-hypertournament \( T \), where \( 3 \leq k \leq n - 1 \). Then the complement \( G^c \) of \( G \) does not contain a pair of disjoint edges.

**Proof.** Suppose the complement \( G^c \) of \( G \) contains a pair of disjoint edges, say \( xy \) and \( zw \). So \( xy, zw \notin E(G) \) and \( x, y, z, w \) are distinct. By Lemma 2.1, we have \( xy \) satisfies one of the cases (a)–(e) and \( zw \) satisfies one of the cases (a)–(e).
Suppose that at least one of $xy$ and $zw$ satisfies one of the cases (a)–(d). W.l.o.g., we assume that $N'(x) \subseteq \{y\}$. Then it must be true that the vertex $z$ dominates $x$ in each arc containing $x, z$ but not containing $w$. Meanwhile, it must be true that the vertex $w$ dominates $x$ in each arc containing $x, w$ but not containing $z$. So $z$ and $w$ compete and hence $zw \in E(G_{1,2}(T)) = E(G)$, a contradiction. Thus, both $xy$ and $zw$ satisfy (e).

However, since $A_T[x, y]$ contains exactly an arc $a$, and $N_{T-a}(x) \subseteq \{y\}, N_{T-a}(y) \subseteq \{x\}$, we have the vertex $x$ must be the last entry in each arc containing $x, z, w$ but not containing $y$ and the vertex $y$ must be the last entry in each arc containing $y, z, w$ but not containing $x$. Thus, $A_T[z, w]$ contains at least two arcs, which contradicts the fact that $zw$ satisfies (e).

The lemma holds.

**Lemma 3.2.** Let $G$ on $n$ vertices be the $(1, 2)$-step competition graph of some $k$-hypertournament $T$, where $3 \leq k \leq n - 1$. Then the complement $G'$ of $G$ does not contain 3-cycle.

**Proof.** Suppose to the contrary that the complement $G'$ of $G$ contains 3-cycle, say $xyz$. So $xy, xz, yz \notin E(G)$ and $x, y, z$ are distinct. By Lemma 2.1, we have $xy, xz$ and $yz$ satisfy one of the cases (a)–(e), respectively.

**Claim 1.** None of $xy, xz$ and $yz$ satisfies the case (a) or (b).

**Proof.** Suppose at least one of $xy, xz$ and $yz$ satisfies the case (a) or (b). W.l.o.g., we assume that $xy$ satisfies (a), i.e. $N'(x) = \emptyset$. Then it must be true that the vertex $y$ dominates $x$ in each arc containing $x, y$ but not containing $z$. Also, the vertex $z$ dominates $x$ in each arc containing $x, z$ but not containing $y$. So $y$ and $z$ compete and $yz \notin E(G_{1,2}(T)) = E(G)$, a contradiction. Thus, none of $xy, xz$ and $yz$ satisfies the case (a) or (b).

**Claim 2.** At most one of $xy, xz$ and $yz$ satisfies the case (c) or (d).

**Proof.** Suppose at least two edges among $xy, xz$ and $yz$ satisfy the case (c) or (d). W.l.o.g., we assume that both $xy$ and $yz$ satisfy (c) or (d). We consider the following four cases.

**Case 1:** Both $xy$ and $yz$ satisfy (c). It means that $N'(x) = \{y\}, N'(y) = \{z\}$. If $xz$ satisfies (c), i.e. $N'(x) = \{z\}$, it contradicts $N'(x) = \{y\}$. If $xz$ satisfies (d), i.e. $N'(x) = \{x\}$, the arcs containing simultaneously $x, y, z$ do not satisfy $N'(x) = \{y\}, N'(y) = \{z\}$ and $N'(z) = \{x\}$, a contradiction. If $xz$ satisfies (e), i.e. $A_T[x, z]$ contains exactly an arc $a$, then there exists a vertex $w$ such that $w \in N'(x)$. Since $N'(x) = \{y\}$, we have $w = y$. So $x$ is the second last entry, $y$ is the last entry and $z$ is any other entry in $a$. Then the vertex $z$ dominates $y$ in $a$. Also, the vertex $x$ dominates $y$ in each arc containing $x, y$ but not containing $z$. So $xz \in E(G_{1,2}(T)) = E(G)$, a contradiction.

**Case 2:** Both $xy$ and $yz$ satisfy (d). It means that $N'(y) = \{x\}, N'(z) = \{y\}$. Similarly to Case 1, we can also get a contradiction.

**Case 3:** $xy$ satisfies (c) and $yz$ satisfies (d). It means that $N'(x) = \{y\}, N'(z) = \{y\}$. Now the arcs containing simultaneously $x, y, z$ do not satisfy $N'(x) = \{y\}, N'(y) = \{z\}$, a contradiction.

**Case 4:** $xy$ satisfies (d) and $yz$ satisfies (c). It means that $N'(y) = \{x\}$ and $N'(y) = \{z\}$. Then $x = z$, a contradiction.

Thus, at most one of $xy, xz$ and $yz$ satisfies the case (c) or (d).

**Claim 3.** At most one of $xy, xz$ and $yz$ satisfies the case (e).

**Proof.** Suppose at least two edges among $xy, xz$ and $yz$ satisfy the case (e). W.l.o.g., we assume that both $xz$ and $yz$ satisfy (e). From the assumption that $xz$ satisfies (e), we get the vertex $y$ dominates $x$ in each arc containing $x, y$ but not containing $z$. From the assumption that $yz$ satisfies (e), we get the vertex $x$ dominates $y$ in each arc containing $x, y$ but not containing $z$. This is a contradiction. Thus, at most one of $xy, xz$ and $yz$ satisfies the case (e).
By Claims 1–3, it is impossible that \( xy, xz, yz \not\in E(G) \) hold simultaneously. Thus, the complement \( G^c \) of \( G \) does not contain 3-cycle. The lemma holds.

**Lemma 3.3.** Let \( G \) on \( n \) vertices be the (1, 2)-step competition graph of some \( k \)-hypertournament \( T \), where \( 3 \leq k \leq n - 1 \). Then the complement \( G^c \) of \( G \) does not contain \( K_{1,3} \), unless \( G = K_{n-1} \cup K_{1} \).

**Proof.** Let \( T \) be a \( k \)-hypertournament on \( n \) vertices, where \( 3 \leq k \leq n - 1 \), and \( G \) the (1, 2)-step competition graph of \( T \). Assume \( G \not= K_{n-1} \cup K_{1} \). Now we show that the complement \( G^c \) of \( G \) does not contain \( K_{1,3} \). Suppose not. Let \( \{x, y, z, w\} \) and \( \{xy, xz, xw\} \) be the vertex set and edge set of the subgraph \( K_{1,3} \), respectively. So \( xy, xz, xw \not\in E(G) \). By Lemma 2.1, we have \( xy, xz \) and \( xw \) satisfy one of the cases (a)–(e).

**Claim 1.** None of \( xy, xz \) and \( xw \) satisfies the case (a) or (b).

**Proof.** Suppose at least one of \( xy, xz \) and \( xw \) satisfies the case (a) or (b). W.l.o.g., we assume that \( xy \) satisfies (a), i.e. \( N^*(x) = \emptyset \). Let \( V(T) = \{v_1, v_2, \ldots, v_n\} \) and \( x = v_n \). By Lemma 2.1, for all \( 1 \leq i \leq n - 1 \), we have \( v_iv_n \not\in E(C_{i,2}(T)) \). Indeed,

- For \( 1 \leq i < j \leq n - (k - 1) \), the vertex \( v_i \) dominates \( v_n \) by the arc consisting of \( v_i, v_{1,2,3}, v_{n-1}, v_n \), and the vertex \( v_j \) dominates \( v_n \) by the arc consisting of \( v_j, v_{1,2,3,4}, v_{n-1}, v_n \). Then \( v_i \) and \( v_j \) compete and \( v_iv_j \in E(C_{i,2}(T)) \).
- For \( n - (k - 2) \leq i < j \leq n - 1 \), the vertex \( v_i \) dominates \( v_n \) by the arc consisting of \( v_{n-1}, v_{n-3}, v_{n-1}, v_n \) for \( n - (k - 2) \leq i \leq n - 3 \) and by the arc consisting of \( v_{i,2,3}, v_{i,2,3}, v_{n-3}, v_{n-1}, v_n \) for \( i = n - 2 \) and the vertex \( v_j \) dominates \( v_n \) by the arc consisting of \( v_{n-1}, v_{n-3}, v_{n-1}, v_n \). Then \( v_i \) and \( v_j \) compete and \( v_iv_j \in E(C_{i,2}(T)) \).
- For \( 1 \leq i < n - k < n - (k - 2) \leq j \leq n - 1 \), the vertex \( v_i \) dominates \( v_n \) by the arc consisting of \( v_{k-3}, v_{k-3}, v_{n-1}, v_n \), and the vertex \( v_j \) dominates \( v_n \) by the arc consisting of \( v_{n-1}, v_{n-3}, v_{n-1}, v_n \). Then \( v_i \) and \( v_j \) compete and \( v_iv_j \in E(C_{i,2}(T)) \).
- For \( i = n - (k - 1) \) and \( n - (k - 2) \leq j \leq n - 1 \), the vertex \( v_{n-1} \) dominates \( v_n \) by the arc consisting of \( v_{3}, v_{2,3}, v_{n-2}, v_{n-1}, v_n \) for \( k = 3 \) and by the arc consisting of \( v_{n-k}, v_{n-1}, v_{n-3}, v_{n-1}, v_n \) for \( 4 \leq k \leq n - 1 \) and the vertex \( v_{j} \) dominates \( v_n \) by the arc consisting of \( v_{n-1}, v_{n-3}, v_{n-1}, v_n \). Then \( v_j \) and \( v_{n-1} \) compete and \( v_jv_{n-1} \in E(C_{i,2}(T)) \).

Then \( C_{i,2}(T) = K_{n-1} \cup K_1 \), a contradiction. Thus, none of \( xy, xz \) and \( xw \) satisfies the case (a) or (b).

**Claim 2.** At most one of \( xy, xz \) and \( xw \) satisfies the case (c) or (d).

**Proof.** Suppose at least two edges among \( xy, xz \) and \( xw \) satisfy the case (c) or (d). W.l.o.g., we assume that both \( xy \) and \( xz \) satisfy the case (c) or (d). We consider the following four cases.

**Case 1:** Both \( xy \) and \( xz \) satisfy (c). It means that \( N^*(x) = \{y\} \) and \( N^*(z) = \{x\} \). Then \( y = z \), a contradiction.

**Case 2:** Both \( xy \) and \( xz \) satisfy (d). It means that \( N^*(y) = \{x\} \) and \( N^*(z) = \{x\} \). Now the arcs containing simultaneously \( x, y, z \) do not satisfy \( N^*(y) = \{x\} \), \( N^*(z) = \{x\} \), a contradiction.

**Case 3:** \( xy \) satisfies (c) and \( xz \) satisfies (d). It means that \( N^*(x) = \{y\} \) and \( N^*(z) = \{x\} \). If \( xw \) satisfies (c), then \( N^*(x) = \{w\} \), contradicting \( N^*(x) = \{y\} \). If \( xw \) satisfies (d), then \( N^*(w) = \{x\} \). Now the arcs containing simultaneously \( x, z, w \) do not satisfy \( N^*(z) = \{x\} \), \( N^*(w) = \{x\} \), a contradiction. If \( xw \) satisfies (e). Let \( b \) be an arc containing \( x, z, w \). \( N^*(x) = \{y\} \) yields \( z \) dominates \( x \) in \( b \). \( N^*(z) = \{x\} \) yields \( z \) is the second last entry, \( x \) is the last entry and \( w \) is any other entry of \( b \). Clearly, \( b \not\in A^*_{T}[x,w] \), i.e. \( z \in N^*_{T}[w] \), which contradicts the fact that \( N^*_{T}[x,w] \subseteq \{x\} \).

**Case 4:** \( xy \) satisfies (d) and \( xz \) satisfies (c). Similarly to Case 3, we can also get a contradiction. Thus, at most one of \( xy, xz \) and \( xw \) satisfies the case (c) or (d).

**Claim 3.** At most one of \( xy, xz \) and \( xw \) satisfies the case (e).

**Proof.** Suppose at least two edges among \( xy, xz \) and \( xw \) satisfy the case (e). Assume that \( xy \) and \( xz \) satisfy (e). From the assumption that \( xy \) satisfies (e), we get the vertex \( z \) dominates \( y \) in each arc containing \( y, z \) but
not containing $x$. From the assumption that $xz$ satisfies (e), we get the vertex $y$ dominates $z$ in each arc containing $y, z$ but not containing $x$. This is a contradiction. Thus, at most one of $xy, xz$ and $xw$ satisfies the case (e).

By Claims 1–3, it is impossible that $xy, xz, xw \notin E(G)$ hold simultaneously. Thus, the complement $G^c$ of $G$ does not contain $K_{3, 3}$ unless $G = K_{n-1} \cup K_1$. The lemma holds.

By Corollary 2.2 and the proof of Lemma 3.3, we obtain the following result.

**Corollary 3.4.** Let $G$ on $n$ vertices be the $(1, 2)$-step competition graph of some strong $k$-hypertournament $T$, where $3 \leq k \leq n - 1$. Then the complement $G^c$ of $G$ does not contain $K_{3, 3}$. 

### 4 Strong $k$-hypertournaments

**Theorem 4.1.** A graph $G$ on $n$ vertices is the $(1, 2)$-step competition graph of some strong $k$-hypertournament $T$ with $3 \leq k \leq n - 1$ if and only if $G$ is $K_n, K_n - E(P_2)$, or $K_n - E(P_3)$.

**Proof.** We first show the “if” part. Let $T$ be a transitive $k$-hypertournament with the vertices $v_1, v_2, \ldots, v_n$. Let $T_i$ be a $k$-hypertournament obtained from $T$ by replacing the arc $(v_1, v_2, v_{n-(k-3)}, \ldots, v_n)$ with $(v_1, v_{n-3}, v_2, v_n)$. It is easy to check that $T_i$ is strong. Now we show that $C_i(T_i) = K_n - E(P_2)$. For convenience, let $a = (v_1, v_2, v_{n-(k-3)}, v_2, v_1)$. We claim that $v_n - v_1 \notin E(C_i(T_i))$. Indeed, for $k = 3$, the vertex $v_{n-1}$ has a unique out-neighbour $v_n$ and Corollary 2.2(a) implies $v_{n-1}v_n \notin E(C_i(T_i))$. For $4 \leq k \leq n - 1$, $A^+_i[v_{n-1}, v_n]$ contains exactly an arc $a$, and $N_{T_i-o}(v_{n-1}) = [v_1]$, $N_{T_i-o}(v_2) = \emptyset$. Corollary 2.2(c) implies $v_{n-1}v_1 \notin E(C_i(T_i))$. We also claim $v_i v_j \in E(C_i(T_i))$ for all $i, j \neq [n-1, n]$. W.l.o.g., we assume $i < j$.

- For $1 \leq i < j < n - (k - 1)$, $v_i$ dominates $v_j$ by the arc $(v_i, v_{n-(k-2)}, v_1, v_n)$ and $v_j$ dominates $v_i$ by the arc $(v_j, v_{n-(k-2)}, v_1, v_n)$. Then $v_i$ and $v_j$ compete and $v_i v_j \in E(C_i(T_i))$.
- For $n - (k - 2) \leq j < n - 1$, $v_i$ dominates $v_n$ by the arc $(v_i, v_{n-(k-2)}, v_1, v_n)$ and $v_j$ dominates $v_j$ by the arc $(v_j, v_{n-(k-2)}, v_1, v_n)$. Then $v_i$ and $v_j$ compete and $v_i v_j \in E(C_i(T_i))$.
- For $i = 1$ and $n - (k - 2) \leq j \leq n - 1$, $v_1$ dominates $v_2$ by the arc $(v_1, v_2, v_{n-(k-2)}, v_1, v_n)$ and $v_j$ dominates $v_i$ by the arc $(v_j, v_1, v_2, v_{n-(k-2)}, v_1, v_n)$. Then $v_i$ and $v_j$ compete and $v_i v_j \in E(C_i(T_i))$.
- For $2 \leq i \leq n - (k - 1)$ and $n - (k - 2) \leq j \leq n - 1$, $v_i$ dominates $v_n$ by the arc $(v_i, v_{n-(k-2)}, v_1, v_n)$ and $v_j$ dominates $v_i$ by the arc $(v_j, v_1, v_2, v_{n-(k-2)}, v_1, v_n)$. Then $v_i$ and $v_j$ compete and $v_i v_j \in E(C_i(T_i))$.
- For $i = 1$ and $j = n$, $v_1$ dominates $v_j$ by the arc $(v_1, v_2, v_{n-(k-2)}, v_1, v_n)$ and $v_j$ dominates $v_1$ by the arc $(v_1, v_{n-(k-2)}, v_1, v_n)$, which is a contradiction. Then $v_1$ and $v_n (1, 2)$-step compete and $v_1 v_n \in E(C_i(T_i))$.

Thus, $C_i(T_i) = K_n - E(P_2)$.

Let $T_i$ be a $k$-hypertournament obtained from $T_i$ above by replacing the arc $(v_1, v_2, v_{n-(k-2)}, v_n)$ with $(v_{n-1}, v_{n-(k-2)}, v_2, v_1)$. It is easy to check that $T_i$ is strong. Now we show that $C_i(T_i) = K_n$.

- For $(i, j) \neq [n-1, n]$, similarly to the proof of $T_i$, we have $v_i v_j \in E(C_i(T_i))$.
- For $i = n - 1$ and $j = n$, $v_{n-1}$ dominates $v_1$ by the arc $(v_{n-1}, \ldots, v_1, v_2, v_1)$ and $v_n$ dominates $v_1$ by the arc $(v_1, v_{n-(k-3)}, v_2, v_1)$. Then $v_{n-1}$ and $v_1$ compete and $v_{n-1} v_1 \in E(C_i(T_i))$.

Thus, $C_i(T_i) = K_n$.

Let $T_j$ be a $k$-hypertournament with the vertices $v_1, v_2, \ldots, v_n$ satisfying the following:

1. Each arc excluding $v_1, v_2$ satisfies $i < j$ if and only if $v_i$ precedes $v_j$;
(2) Each arc including $v_i, v_j$ satisfies that $v_i$ is the second last entry, $v_j$ is the last entry and the remaining $k - 2$ entries satisfy $i < j$ if and only if $v_i$ precedes $v_j$.
(3) Each arc including $v_i$ but excluding $v_j$ satisfies that $v_i$ is the last entry and the remaining $k - 1$ entries satisfy $i < j$ if and only if $v_i$ precedes $v_j$.
(4) Each arc including $v_j$ but excluding $v_i$, $v_j$ satisfies that $v_2$ is the last entry and the remaining $k - 1$ entries satisfy $i < j$ if and only if $v_i$ precedes $v_j$.
(5) Each arc including $v_2, v_3$ but excluding $v_i$ satisfies that $v_2$ is the second last entry, $v_3$ is the last entry and the remaining $k - 2$ entries satisfy $i < j$ if and only if $v_i$ precedes $v_j$.

It is easy to check that $T_1$ is strong. Now we show that $G_{1,2}(T_1) = K_n - E(P_3)$. Note that $N'(v_1) = \{v_2, v_3\}$ and $N'(v_2) = \{v_3\}$. By Corollary 2.2(a), we have $v_1v_2, v_2v_3 \notin E(G_{1,2}(T_1))$. Now we consider the arc $v_1v_j$ for $[i, j] \neq \{1, 2\}$ and $[i, j] \neq \{2, 3\}$. W.l.o.g., we assume $i < j$.

- For $3 \leq i < j \leq n - (k - 3)$, $v_i$ dominates $v_j$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_i, v_j)$ and $v_j$ dominates $v_i$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_i, v_j)$. Then $v_i$ and $v_j$ compete and $v_1v_j \in E(G_{1,2}(T_1))$.
- For $n - (k - 4) \leq i < j \leq n$, $v_i$ dominates $v_j$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_i, v_j)$ and $v_j$ dominates $v_i$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_i, v_j)$. Then $v_i$ and $v_j$ compete and $v_1v_j \in E(G_{1,2}(T_1))$.
- For $3 \leq i \leq n - (k - 3)$ and $n - (k - 4) \leq j \leq n$, $v_i$ dominates $v_j$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_i, v_j)$ for $3 \leq i \leq n - (k - 2)$ and by the arc $(v_i, \ldots, v_{i+(k-3)}, v_{i-1}, v_i, v_j)$ for $i = n - (k - 3)$ and $v_j$ dominates $v_i$ by the arc $(v_i, \ldots, v_{i+(k-3)}, v_{i-1}, v_i, v_j)$. Then $v_j$ and $v_1v_i$ compete and $v_1v_j \in E(G_{1,2}(T_1))$.

Thus, $G_{1,2}(T_1) = K_n - E(P_3)$.

Now we show the “only if” part. Let $T$ be a strong $k$-hypertournament and $G$ the $(1, 2)$-step competition graph of $T$. We show that $G^c$ contains at most two edges. Suppose to the contrary that $G^c$ contains at least three edges, say $e_1, e_2, e_3 \in E(G^c)$. Let $e_l = x_l y_l$ for $l = 1, 2, 3$. By Lemma 3.1, $e_1$ and $e_2$ have a common end-point. W.l.o.g., we assume that $y_1 = x_2$. By Lemma 3.1, $e_3$ and $e_1$ have a common end-point, and $e_3$ and $e_2$ have also a common end-point. So either $e_3 = x_3 y_2$ or $e_2$ is an end-point of $e_2$. However, this implies $G^c$ contains 3-cycle or $K_{3,3}$, which contradicts Lemma 3.2 and Corollary 3.4. So $G^c$ contains at most two edges. Thus, if $G^c$ contains two edges, Lemma 3.1 implies $G = K_n - E(P_3)$; if $G^c$ contains one edge, then $G = K_n - E(P_3)$; if $G^c$ contains no edge, then $G = K_n$.

Therefore, the theorem holds.

5 Remaining $k$-hypertournaments

**Theorem 5.1.** A graph $G$ on $n$ vertices is the $(1, 2)$-step competition graph of some $k$-hypertournament $T$ with $3 \leq k \leq n - 1$ if and only if $G$ is $T$, $K_n - E(P_3)$, or $K_{n-1} \cup K_i$.

**Proof.** The “if” part follows from the proof of Lemma 3.3 and Theorem 4.1. Now we show the “only if” part. Let $T$ be a $k$-hypertournament and $G$ the $(1, 2)$-step competition graph of $T$. We show that $G$ is $K_n, K_n - E(P_3)$,
Similarly to the proof of “only if” of Theorem 4.1, we get $G^c$ contains at most two edges unless $G = K_{n-1} \cup K_1$. Thus, if $G^c$ contains two edges, Lemma 3.1 implies $G = K_n - E(P_2)$; if $G^c$ contains one edge, then $G = K_n - E(P_2)$; if $G^c$ contains no edge, then $G = K_n$.

Therefore, the theorem holds. $\square$

6 The $(i, j)$-step competition graph of a $k$-hypertournament

We generalize the $(1, 2)$-step competition graph to the $(i, j)$-step competition graph as follows. By the definition of the $(i, j)$-step competition graph for a $k$-hypertournament $T$, we obtain that if $i \geq 1$, $j \geq 2$, then $E(C_{i,j}(T)) \subseteq E(C_{i,k}(T))$. It is easy to see that the proof of Lemma 2.1 implies the following corollary.

**Corollary 6.1.** Let $T$ be a $k$-hypertournament with $n$ vertices satisfying $3 \leq k \leq n-1$ and $i \geq 1$, $j \geq 2$ integers. Then $xy \notin E(C_{i,j}(T))$ if and only if one of the following holds:

(a) $N^+(x) = \emptyset$;

(b) $N^+(y) = \emptyset$;

(c) $N^+(x) = \{y\}$;

(d) $N^+(y) = \{x\}$;

(e) $A^+_T(x, y)$ contains exactly an arc $a$, and $N^-_{i-j}(x) \subseteq \{y\}$, $N^-_{i-j}(y) \subseteq \{x\}$.

**Theorem 6.2.** Let $T$ be a $k$-hypertournament with $n$ vertices satisfying $3 \leq k \leq n-1$ and $i \geq 1$, $j \geq 2$ integers. Then $C_{i,j}(T) = C_{i,2}(T)$.

**Proof.** Clearly, $V(C_{i,j}(T)) = V(C_{i,2}(T)) = V(T)$. Since $E(C_{i,j}(T)) \subseteq E(C_{i,k}(T))$, it suffices to show that $E(C_{i,j}(T)) \subseteq E(C_{i,2}(T))$. Let $xy \in E(C_{i,j}(T))$. Suppose $xy \notin E(C_{i,2}(T))$. By Lemma 2.1, $x$ and $y$ must satisfy one of the cases (a)–(e). This contradicts Corollary 6.1. Thus, $xy \in E(C_{i,2}(T))$ and $E(C_{i,j}(T)) \subseteq E(C_{i,2}(T))$. $\square$

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