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Markovian properties of passive scalar increments in grid-generated turbulence

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Abstract. Recent research (Renner, Peinke and Friedrich 2001 J. Fluid Mech. 433 383) has shown that the statistics of velocity increments in a turbulent jet exhibit Markovian properties for scales of size greater than the Taylor microscale, \( \lambda \). In addition, it was shown that the probability density functions (PDFs) of the velocity increments, \( v(r) \), were governed by a Fokker–Planck equation. Such properties for passive scalar increments have never been tested.

The present work studies the (velocity and) temperature field in grid-generated wind tunnel turbulence for Taylor-microscale-based Reynolds numbers in the range \( 140 \leq R_\lambda \leq 582 \). Increments of longitudinal velocity were found to (i) exhibit Markovian properties for separations \( r \gtrsim \lambda \) and (ii) be describable by a Fokker–Planck equation because terms in the Kramers–Moyal expansion of order \( >2 \) were small. Although the passive scalar increments, \( \delta(r) \), also exhibited Markovian properties for a similar range of scales as the velocity field, the higher-order terms in the Kramers–Moyal expansion were found to be non-negligible at all Reynolds numbers, thus precluding the PDFs of \( \delta(r) \) from being described by a Fokker–Planck equation. Such a result indicates that the scalar field is less Markovian than the velocity field—an attribute presumably related to the higher level of internal intermittency associated with passive scalars.
1. Introduction

Since Kolmogorov’s pioneering work [1, 2] in 1941 on the ‘turbulent cascade’, the structure and potential universality of small-scale turbulence has received much attention. In the case of a turbulent velocity field, the cascade notion relates to the transport of kinetic energy from large to small scales. Turbulent kinetic energy is injected into a flow at the largest scales (i.e. scales related to the boundary conditions of the flow). The inherent instability of large eddies subsequently transfers kinetic energy to smaller and smaller scales, until the smallest eddies convert their kinetic energy into internal energy by the action of viscosity, thus ending the cascade. The length of the cascade (which can be specified in terms of the ratio of the largest to smallest scales) is related to the (turbulent) Reynolds number. At high Reynolds numbers, it has been hypothesized that the cascade may be long enough for the smallest scales to ‘forget’ the initial, large scales. Consequently, small-scale turbulence may adopt a universal structure. In addition, should the Reynolds number be high enough for there to be a distinct separation of scales between the large and small (dissipative) eddies, a range of scales that is independent of (i) the largest scales and (ii) the smallest scales (and therefore viscosity) may exist. Such a range is called the inertial subrange.

Investigations of the scaling of turbulent fields have focused on increments of turbulent quantities (taken over a given separation) and the statistical moments thereof. Using velocity as an example, the latter—also called structure functions—are defined as

\[ \langle (u_i(r_j))^{n} \rangle = \langle (\Delta u_i(r_j))^{n} \rangle = \langle (u_i(x_j + r_j) - u_i(x_j))^{n} \rangle, \]

where \( u_i \) is a turbulent velocity fluctuation, \( r_j \) a separation (often measured in the direction of the mean flow and calculated using Taylor’s hypothesis), \( v_i(r_j) \) a difference in \( u_i \) measured over \( r_j \) and angular brackets denote an average. Kolmogorov [1, 2] predicted the inertial-range scaling behaviour of the \( n \)th-order structure function to be

\[ \langle (v_i(r))^{n} \rangle = f(\epsilon, r) \propto r^{\kappa_n}, \]

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where, from dimensional considerations,

\[ \zeta_n = n/3. \]  

(3)

Note that \( r = |r_j| \). \( \epsilon \) is the dissipation rate of turbulent kinetic energy (which, in the inertial subrange, is equal to the spectral energy transfer rate) and is equal to \( 2\nu(s_{ij}s_{ij}) \). \( s_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \) is the turbulent strain rate. It has been subsequently shown that equation (3) does not agree with observations of turbulence and that the deviation increases with the structure function order, \( n \) [3]–[8]. The difference is attributed to the large variations in space and time of \( \epsilon \), a phenomenon called internal intermittency. As a result, probability density functions (PDFs) of \( v_j(r) \) cannot be collapsed for different \( r \) and the observed variation of \( \zeta_n \) with \( n \) becomes non-linear. The relationship between \( \zeta_n \) and \( n \) has been the focus of much study of which reviews can be found in [6, 8].

Analogous results have been obtained for passive scalar fields, whose structure functions are given by

\[ ((\delta(r_j))^n) \equiv ((\Delta\theta(r_j))^n) = \langle (\theta(x_j + r_j) - \theta(x_j))^n \rangle, \]  

(4)

where \( \theta \) is a turbulent scalar fluctuation, \( r_j \) a separation and \( \delta(r_j) \) a difference in \( \theta \) measured over \( r_j \). Oboukhov [9] and Corrsin [10] extended the notions of Kolmogorov [1, 2] and predicted the inertial-range scaling behaviour of the \( n \)-th order structure function of a passive scalar to be

\[ ((\delta(r))^n) = f(\epsilon, \epsilon_\theta, r) \propto r^{\xi_n}, \]  

(5)

where, from dimensional considerations,

\[ \xi_n = n/3. \]  

(6)

Here, \( \epsilon_\theta (\equiv 2\alpha \langle (\partial\theta/\partial x_i)^2 \rangle) \) is the rate at which scalar variance is ‘smeared’ by molecular processes and is also known as the scalar dissipation rate. (The latter term is technically inaccurate given that scalar variance is not dissipated, unlike kinetic energy.) Equation (6) does not agree with observations of turbulence [11]–[15], similar to the velocity field. The deviations from equation (6) also increase with order \( n \) and are more significant than those observed for the velocity field. This is attributed to the internal intermittency of the scalar field, which is stronger than that observed for the velocity field. See [14] or [15] for a recent review of turbulent passive scalars.

An alternative to studying moments of increments of turbulent quantities is to study their PDFs (which can subsequently be used to calculate structure functions). Examples of such work include [16, 17] for the velocity field and [18]–[21] for the passive scalar field.

To further our understanding of turbulence, the mathematics of Markov processes has been employed in studies of the PDFs of velocity differences for various separations. The applicability of Markov processes to experimental investigations of PDFs of \( v \) was investigated by Friedrich and Peinke [22]. Along with different co-authors, they subsequently continued this work in [23]–[25].

The principal objective of the present work is to study Markov properties of passive scalar increments produced by a mean temperature gradient in grid-generated turbulence. The results
will be compared with previous work treating Markov properties of the velocity field. In addition, the Reynolds number dependence of the Markov properties of the velocity and scalar fields will be examined over the range $140 \leq R_\lambda \leq 582$.

The remainder of this paper is organized as follows. Background theory is presented in section 2 and the apparatus is described in section 3. The results are presented in section 4, where they are divided according to results pertaining to (i) the velocity field and (ii) the scalar field. Conclusions and a discussion are presented in section 5.

2. Background theory

The present section begins with a summary of (some of) the theoretical elements of Markov processes that pertain to the present work. For more information, the reader is referred to [24, 26], from where much of the material is summarized. The section concludes with a brief discussion of the implications of considering turbulence as Markovian to its modelling.

Consider a dynamical system whose state at time $t$ is represented by $X(t)$. If one assumes that (i) the state of the system at an initial time $t_0$ is known and given by $X(t_0) = x_0$, (ii) $X(t)$ for $t > t_0$ may only be predicted probabilistically (that is to say $X(t)$ for $t > t_0$ is a random variable) and (iii) the probability of $X(t)$ at successive instants $t_1, t_2, \ldots, t_n, X(t_1), X(t_2), \ldots, X(t_n)$ (where $t_0 < t_1 < t_2 < \cdots < t_n$) is given by the joint probability density function (JPDF):

$$P_n^{1}(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1 | x_0, t_0) \equiv \text{Prob}\{X(t_i) \in [x_i, x_i + dx_i) \text{ for } i = 1, 2, \ldots, n \}
\text{given that } X(t_0) = x_0 \text{ with } t_0 \leq t_1 \leq \cdots \leq t_n \},$$

then $X(t)$ is a stochastic process. (Using this notation, the superscript represents the number of conditionings and the subscript the number of variables in the JPDF. Note that subordinate JPDFs can exist, e.g. $P_n^{1+i}(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_{j+1}, t_{j+1}; x_j, t_j; \ldots; x_1, t_1; x_0, t_0)$. It is always possible to calculate $P_n^{1+i}$ from $P_n^{1}$ by integrating $P_n^{1}$ over any $x_n$ from $-\infty$ to $\infty$, although the converse is generally not possible.

Should a stochastic process possess the following property:

$$P_n^{1}(x_j, t_j | x_{j-1}, t_{j-1}; \ldots; x_1, t_1; x_0, t_0) = P_n^{1}(x_j, t_j | x_{j-1}, t_{j-1})$$

it is said to be a Markov process. (The last probability can be written without the superscript for the sake of simplicity, i.e. $P_n^{1}(x_j, t_j | x_{j-1}, t_{j-1}) = P(x_j, t_j | x_{j-1}, t_{j-1})$. The physical interpretation of a Markov process is of a process that ‘forgets its past’. In other words, only the most recent conditioning is relevant to the present probability. (The ability to predict the state under consideration will not be enhanced by knowing states further in the past than the most recent conditioning.) Consequently, for a Markov process, a conditional multivariate JPDF can be written in terms of the products of various conditional univariate PDFs. Mathematically,

$$P_n^{1}(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1 | x_0, t_0) = \prod_{i=1}^{n} P(x_i, t_i | x_{i-1}, t_{i-1}),$$
where $t_0 \leq t_1 \leq \cdots \leq t_n$. This result can be derived given equation (8) and the relationship between joint and conditional PDFs:

\[ P(x_i, t_i \mid x_j, t_j, x_k, t_k) = P(x_j, t_j \mid x_i, t_i, x_k, t_k) P(x_i, t_i \mid x_k, t_k). \] (10)

Given a Markov process, it can be shown that the following relation must hold:

\[ P(x_3, t_3 \mid x_1, t_1) = \int_{-\infty}^{\infty} P(x_3, t_3 \mid x_2, t_2) P(x_2, t_2 \mid x_1, t_1) \, dx_2. \] (11)

Equation (11) is called the Chapman–Kolmogorov equation and is readily derived from equation (9). It is a necessary condition for a Markov process, however, not a sufficient one. In addition, equation (11) can be generalized (or ‘compounded’) to an expression for $P(x_n, t_n \mid x_0, t_0)$.

The generalized Chapman–Kolmogorov equation can alternatively be expressed in differential form by means of the Kramers–Moyal expansion. At this point, it is convenient to convert the stochastic process $X(t)$ to a process in terms of spatial (as opposed to temporal) separations, $X(r)$. Doing so, and following the slight modifications proposed in [24, p 391] to give an equation that is physically representative of the turbulent cascade from large to small scales, the Kramers–Moyal expansion is given by

\[-r \frac{\partial}{\partial r} p(x, r \mid x_0, r_0) = \sum_{k=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^k \left[ D_k(x, r) p(x, r \mid x_0, r_0) \right]. \] (12)

The Kramers–Moyal coefficients, $D_k(x, r)$, are given by

\[ D_k(x, r) = \lim_{\Delta r \to 0} M_k(x, r, \Delta r), \] (13)

where $\Delta r = r_0 - r$ and the $k$th-order conditional moments, $M_k$, are given by

\[ M_k(x, r, \Delta r) = \frac{r}{k! \Delta r} \int_{-\infty}^{\infty} (x' - x)^k p(x', r - \Delta r \mid x, r) \, dx'. \] (14)

In general, all Kramers–Moyal coefficients are non-zero. However, it was shown by Pawula [27] that if $D_k$ exists for all $k$, and if $D_k = 0$ for some even $k$, then $D_k = 0$ for all $k \geq 3$. In the words of Pawula [27], ‘it is logically inconsistent to retain more than two terms in the Kramers–Moyal expansion unless all of the terms are retained’. If this is the case, the Kramers–Moyal expansion reduces to the Fokker–Planck (or forward Kolmogorov) equation

\[-r \frac{\partial}{\partial r} p(x, r \mid x_0, r_0) = -\frac{\partial}{\partial x} D_1(x, r) p(x, r \mid x_0, r_0) + \frac{\partial^2}{\partial x^2} D_2(x, r) p(x, r \mid x_0, r_0), \] (15)

where $D_1$ is the drift term and $D_2$ the diffusion term.

Before proceeding to the next section, it is worth discussing briefly some of the implications of considering turbulence as Markovian. By definition, PDF methods solve modelled transport equations for the PDF of a given quantity in a turbulent flow [28, 29]. Eulerian PDF methods require the one-point, one-time conditional expectations to be modelled (e.g. the generalized Langevin model, molecular mixing models, etc). Lagrangian PDF methods employ stochastic
models to determine the Lagrangian PDF (from which the Eulerian PDF can subsequently be deduced). Stochastic models have also been used in the study of turbulent dispersion to describe the velocity of a fluid particle or the relative motion of particle pairs in turbulence (see e.g. [30]–[32]).

Most of the stochastic models employed are continuous Markov (or diffusion) processes (i.e. Markov processes whose sample paths are continuous; see [26, 28]). It can be shown that, for a continuous Markov process, the Kramers–Moyal coefficients of order 3 and higher are zero and the Kramers–Moyal expansion reduces to the Fokker–Planck equation (15). The Fokker–Planck equation governs the PDF of a stochastic process that is described by the Langevin equation with Gaussian, δ-correlated forcing [26, 33]. The Langevin equation forms the basis for a large number of stochastic models of turbulence.

In the present work, the stochastic variable $X(t)$ will be a turbulent increment (either $v$ or $\delta$ for the velocity and temperature fields, respectively) evaluated over a spatial separation, $r$. The stochastic properties of the PDFs of $v$ and $\delta$ (measured over different scales) will be used to elucidate the nature of the turbulent cascade. For example, we note that the above Kramers–Moyal expansion for the conditional PDF can be extended to the (unconditional) PDF. Consequently, differential equations for the structure function $\langle x(r)^n \rangle$ can be derived, i.e.

$$- r \frac{\partial}{\partial r} \langle x^n(r) \rangle = \sum_{k=1}^{n} \frac{n!}{(n-k)!} \int_{-\infty}^{\infty} x^{n-k} D_k(x, r) p(x, r) \, dx. \quad (16)$$

Depending on the assumptions for the Kramers–Moyal coefficients, different intermittency models of small-scale turbulence can be derived. For example, when considering velocity increments ($v(r)$), the structure function scaling exponents ($\zeta_n$) corresponding to the log-normal model [3, 4] can be predicted. The expression for $\zeta_n$ can be derived assuming that (i) the Kramers–Moyal coefficients are given by $D_k(v, r) = a_k v^k$, (ii) $D_k = a_k = 0$ for $k \geq 3$, (iii) $\zeta_3 = 1$ (Kolmogorov’s $\frac{4}{5}$ law [2]) and (iv) $\zeta_6 = 2 - \mu$ (from the definition of $\mu$). The result is $\zeta_n = n/3 + \mu n (3 - n)/18$—the same result as obtained with the log-normal model.

3. Apparatus

The measurements were made in the 0.91 m x 0.91 m x 9.1 m low-speed, low-background-turbulence wind tunnel in the Sibley School of Mechanical and Aerospace Engineering at Cornell University. The wind tunnel is of standard open-circuit design with a plenum chamber and contraction that precedes the test sections. The wind tunnel details can be found in [34].

An active grid is used to generate homogeneous, quasi-isotropic, high-Reynolds number turbulence. (Active grids [35, 36] consist of grid bars to which are attached agitator wings. The bars are rotated by stepper motors located outside of the grid. The resulting flow has a significantly higher Reynolds number than ‘passive’ grid turbulence because of its higher turbulence intensity and larger integral scale, $\ell$.) Parallel ribbons located at the entrance to the wind tunnel plenum chamber are differentially heated (see [34, 37]). The result is an imposed mean temperature gradient on the flow in the wind tunnel test section. The action of the turbulence upon the gradient serves to generate the scalar (temperature) fluctuations. The characteristics of the velocity and thermal fields, as well as the details of the apparatus, are documented in [36, 38] and will only be summarized herein. In [38], it is shown that the temperature fluctuations are passive. (The ratio of buoyant production of turbulent kinetic energy to its dissipation, $g\langle v\theta \rangle/(T_0\epsilon)$, is less than 0.004.)
than 2%, the spectra of the transverse velocity fluctuations were no different between the ‘cold’ and ‘heated’ flows, and the velocity field measurements were consistent when the direction of the mean temperature gradient was reversed.)

A TSI 1241 X-wire probe with 3.05 μm diameter tungsten wires was used in conjunction with Dantec 55M01 constant-temperature anemometers to measure the longitudinal and transverse components of the velocity field. The length-to-diameter ratio of each wire was approximately 200 and the inter-wire spacing was 0.5 mm. The temperature field was measured using a TSI 1210 probe and Wollaston wire with a 0.63 μm diameter platinum core. The temperature wire was located 0.5 mm from the X-wire. Its length-to-diameter ratio varied from 500 to 650. The cold-wire constant current thermometry circuits employed a probe current of 250 μA. All wires were calibrated in a laminar calibration jet. The hot-wire signals were compensated for the temperature fluctuations using a modified King’s law with temperature-dependent coefficients [39].

All signals were high- and low-pass filtered and digitized using a 12-bit A/D converter. For the present measurements of PDFs of velocity and temperature increments, time series were required to obtain the desired (longitudinal) separations, which were converted from a temporal to spatial domain by means of Taylor’s hypothesis. The time series were $8 \times 10^5$ samples long and sampled at twice the Kolmogorov frequency.

Measurements are presented for three Reynolds numbers: $R_\lambda = 140, 306$ and 582, where $R_\lambda = \langle u^2 \rangle [15/(\nu \epsilon)]^{1/2}$. The flow parameters for these three cases are summarized in table 1. More details, including profiles, spectra and PDFs of the velocity and temperature fields, can be found in [38].

### 4. Results

Results treating Markov properties of (longitudinal) velocity increments will be presented in the first subsection and compared with previous work. For consistency with the conventions used in [24], velocity increments are non-dimensionalized by $\sigma_L = \sqrt{2 \langle u^2 \rangle}$—the square root of the large-scale ($r \to \infty$) limit of the second-order velocity structure function. In the second

| $R_\lambda$ | 140 | 306 | 582 |
|-----------|-----|-----|-----|
| $\langle U \rangle$ (m s$^{-1}$) | 3.3 | 3.3 | 7.0 |
| Active grid mode | Synchronous | Random | Random |
| $x/M$ | 62 | 62 | 62 |
| $dT/dy$ (K m$^{-1}$) | 2.5 | 2.7 | 3.6 |
| $\langle u^2 \rangle$ (m$^2$ s$^{-2}$) | 0.0290 | 0.0911 | 0.583 |
| $\langle v^2 \rangle$ (m$^2$ s$^{-2}$) | 0.0209 | 0.0594 | 0.424 |
| $\langle \theta^2 \rangle$ (K$^2$) | 0.176 | 0.800 | 1.07 |
| $\epsilon$ (m$^2$ s$^{-3}$) | 0.0418 | 0.0833 | 0.940 |
| $\epsilon_\theta$ (K$^2$ s$^{-1}$) | 0.277 | 0.799 | 1.74 |
| $\ell$ (m) | 0.11 | 0.30 | 0.43 |
| $\ell_\theta$ (m) | 0.17 | 0.33 | 0.29 |
| $\lambda$ (m) | 0.013 | 0.016 | 0.012 |
| $\lambda_\theta$ (m) | 0.012 | 0.015 | 0.012 |
| $\eta$ (mm) | 0.55 | 0.47 | 0.26 |
subsection, the Markov properties of passive scalar increments will be studied and compared with those of the velocity field. Analogously, scalar increments are non-dimensionalized by \( \sigma_{L,\theta} = \sqrt{2 \langle \theta^2 \rangle} \), the square root of the large-scale limit of the second-order temperature structure function. Most results will be presented for the highest Reynolds number (\( R_\lambda = 582 \)). Unless otherwise specified, results for the lower Reynolds numbers were consistent with those at \( R_\lambda = 582 \).

4.1. Velocity field

4.1.1. Markovian properties of the velocity field. Although not a sufficient condition for the establishment of Markovian properties of a stochastic field, it is nevertheless of interest to test whether the Chapman–Kolmororov equation holds. To this end, equation (11) is tested in figure 1, where \( r_1 = \lambda \), \( r_2 = \ell/2 \) and \( r_3 = \ell \). It shows reasonable agreement between \( P(v_3, r_3 | v_1, r_1) \) and \( \int_{-\infty}^{\infty} P(v_3, r_3 | v_2, r_2) P(v_2, r_2 | v_1, r_1) \, dv_2 \), especially in the inner core of the figures where the statistics are better converged due to larger numbers of samples falling in the central bins of the PDF. To clarify, cross-sections of these contour plots taken at \( v_3 = \pm \sigma_L \) are shown in figure 2. The agreement is good.

The Markovian nature of the statistics of velocity increments is explicitly studied by testing equation (8) in figure 3 (for \( j = 3 \), to render an analysis feasible). The same values of \( r_1, r_2 \)

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3 It is claimed in [22] that non-Markovian processes fulfilling the Chapman–Kolmogorov relation should be rare.
Figure 2. Cross-sections of figure 1 at $v_3 = -\sigma_L$ and $v_3 = +\sigma_L$: ♦, $\int_{-\infty}^{\infty} P(v_1, r_1 | v_2, r_2) P(v_2, r_2 | v_3, r_3) \, dv_2$; ○, $P(v_1, r_1 | v_3, r_3)$.

Figure 3. Verification of the Markovian nature of the velocity increments by a comparison of the conditional PDFs in equation (8). Solid (black) contour lines represent $P(v_1, r_1 | v_2, r_2)$ and dashed (red) contour lines represent $P(v_1, r_1 | v_2, r_2; v_3 = 0, r_3)$. The contour lines are in increments of 0.1 and logarithmically spaced; $R_{\lambda} = 582$. Note that conditioning upon a particular value of $v_3$ is required to plot the PDFs in two dimensions. The value of zero was chosen because it is the location of the peak of the PDF (i.e. the most frequent value of $v_3$) and for consistency with the work
Figure 4. Cross-sections of figure 3 at \( v_2 = -\sigma_L \) and \( v_2 = +\sigma_L \): ♦, \( P(v_1, r_1 | v_2, r_2; v_3 = 0, r_3) \); ○, \( P(v_1, r_1 | v_2, r_2) \).

of [24].

Conditioning of the PDF on other values of \( v_3 \) produced similar results. Like before, cross-sections of figure 3 taken at \( v_3 = \pm \sigma_L \) are shown in figure 4. The agreement is again good. It is worth remarking that fewer samples of \( P(v_1, r_1 | v_2, r_2; v_3 = 0, r_3) \) exist than that of \( P(v_1, r_1 | v_2, r_2) \). This can be seen in that the tails of the former PDF do not extend as far as those of the latter.

In [24], it was shown that the Markovian properties of velocity increment statistics disappeared for differences in scales that were smaller than the Taylor microscale, \( \lambda \). Therefore, it is of interest to verify the Chapman–Kolmogorov equation as well as equation (8) for a set of scales that are closer to each other. Again, analogous to [24], the previous two tests are repeated for \( r_1 = \ell/2 - \lambda/4, r_2 = \ell/2 \) and \( r_3 = \ell/2 + \lambda/4 \), where the separation between the three scales is smaller.

For these new scales, the Chapman–Kolmogorov relation is tested in figure 5. This figure shows that equation (11) is valid for these more closely spaced choice of scales. This is corroborated by cross-sections of figure 5 shown in figure 6. However, this result should be compared with a test of equation (8), which explicitly tests the Markovian properties of the velocity increments. With the present more closely spaced scales, deviations from Markovian behaviour begin to appear in figure 7 and its cross-sections in figure 8.

The deviation from Markovian behaviour of velocity increments at small scales is further investigated by comparing the PDFs of two conditional velocity increments, denoted \( x \) and \( y \):

\[
\begin{align*}
    x &\equiv v_1(r_1) |_{v_2(r_2)}, \\
y &\equiv v_1(r_1) |_{v_2(r_2), v_3(r_3)}
\end{align*}
\]

as a function of scale. To reduce the number of variables, we let \( r_3 = -r_2 = r_2 - r_1 = \Delta r \) and again condition \( v_3 \) on 0. \( v_2 \) is conditioned on \( |v_2| \leq 0.25\sigma_L \). A comparison of the PDFs of \( x \) and \( y \) is a test of equation (8). It is performed in figure 9 for four values of \( \Delta r \) ranging from 0.1\( \lambda \) to 1.0\( \lambda \). A general trend towards improved Markovian properties is observed with

4 Technically, the PDF is conditioned on \( |v_3| \leq \epsilon\sigma_L \), where \( \epsilon \) is a small parameter. Appropriate values of \( \epsilon \) are a function of the resolution of the measurement equipment, as well as the shape of the PDFs of \( v_3 \), which vary with \( r_3 \) and \( R_\lambda \). The average value of \( \epsilon \) used herein is 0.10.

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Figure 5. Verification of the Chapman–Kolmogorov equation for velocity increments measured with a smaller separation of the scales. Solid (black) contour lines represent \( \int_{-\infty}^{\infty} P(v_1, r_1 \mid v_2, r_2) P(v_2, r_2 \mid v_3, r_3) \, dv_2 \) and dashed (red) lines represent \( P(v_1, r_1 \mid v_3, r_3) \). \( r_1 = \ell/2 - \lambda/4 \), \( r_2 = \ell/2 \) and \( r_3 = \ell/2 + \lambda/4 \). The contour lines are in increments of 0.4 and linearly spaced; \( R_\lambda = 582 \).

Figure 6. Cross-sections of figure 5 at \( v_3 = -\sigma_L \) and \( v_3 = +\sigma_L \): \( \blacklozenge \), \( \int_{-\infty}^{\infty} P(v_1, r_1 \mid v_2, r_2) P(v_2, r_2 \mid v_3, r_3) \, dv_2 \); \( \blacklozenge \), \( P(v_1, r_1 \mid v_3, r_3) \).

increasing \( \Delta r \). The relationship between \( P(x) \) and \( P(y) \) showed no dependence on \( r_2 \), as was also observed in [24].

4.1.2. Kramers–Moyal coefficients of the velocity field. The objective of this subsection is the evaluation of the Kramers–Moyal coefficients, \( D_k \). Knowledge of the coefficients
Figure 7. Verification of the Markovian nature of the velocity increments by a comparison of the conditional PDFs in equation (8) with a smaller separation of scales. Solid (black) contour lines represent $P(v_1, r_1 | v_2, r_2)$ and dashed (red) contour lines represent $P(v_1, r_1 | v_2, r_2; v_3 = 0, r_3)$. $r_1 = \ell/2 - \lambda/4$, $r_2 = \ell/2$ and $r_3 = \ell/2 + \lambda/4$. The contour lines are in increments of 0.4 and linearly spaced; $R_\lambda = 582$.

Figure 8. Cross-sections of figure 7 at $v_2 = 0$ and $v_2 = +\sigma_L$: ♦, $P(v_1, r_1 | v_2, r_2; v_3 = 0, r_3)$; ○, $P(v_1, r_1 | v_2, r_2)$. (Note that the first cross-section is taken at $v_2 = 0$ and not $v_2 = -\sigma_L$ as in the previous figures because there are so few data at $v_2 = -\sigma_L$ that a comparison was not informative.)

characterizes the Kramers–Moyal expansion. It was observed in [24] that the fourth-order term in the Kramers–Moyal expansion was small, from which it can be concluded that $D_4$ is therefore negligible. The Kramers–Moyal expansion therefore only retained its first two

5 It is important to realize that the magnitude of the terms in the Kramers–Moyal expansion should be compared (and not the magnitudes of the coefficients themselves, as has been done by some authors).
Figure 9. Comparison of $P(x)$ (♦) and $P(y)$ (○) for different scale separations: $r_2 = \ell/2$ for all cases: (a) $\Delta r = 0.1\lambda$, (b) $\Delta r = 0.2\lambda$, (c) $\Delta r = 0.5\lambda$, (d) $\Delta r = 1.0\lambda$; $R_\lambda = 582$.

terms and simplified to a Fokker–Planck equation. Whether the same occurs in the present flow will now be tested before answering the same question for the passive scalar field in section 4.2.

To this end, the conditional moments $M_k(v,r,\Delta r)$ are readily estimated from the experimental data. $M_k(v,r,\Delta r)$ (for $k = 1–4$) are shown in figure 10 for $r = \ell/2$ and $\Delta r = \lambda$. As in [24], least-squares polynomials fits are applied to the conditional moments:

\begin{align}
M_1 &= O_1(r, \Delta r) - G_1(r, \Delta r)v + K_1(r, \Delta r)v^2 - E_1(r, \Delta r)v^3, \\
M_2 &= A_2(r, \Delta r) + D_2(r, \Delta r)v + B_2(r, \Delta r)v^2, \\
M_3 &= O_3(r, \Delta r) - G_3(r, \Delta r)v + K_3(r, \Delta r)v^2 - E_3(r, \Delta r)v^3, \\
M_4 &= A_4(r, \Delta r) + D_4(r, \Delta r)v + B_4(r, \Delta r)v^2.
\end{align}
Figure 10. Conditional moments of velocity increments: (a) $M_1(v, r, \Delta r)$, (b) $M_2(v, r, \Delta r)$, (c) $M_3(v, r, \Delta r)$ and (d) $M_4(v, r, \Delta r)$. $r = \ell/2$ and $\Delta r = \lambda$; $R_\lambda = 582$.

$M_1$ is nominally linear, although terms permitting small second- and third-order corrections are retained. $M_2$ and $M_4$ are fit with second-order polynomials, whereas $M_3$ is fit with a third-order polynomial. Such behaviours were observed for all $r$ and $\Delta r$, as was the case in the work of Renner et al [24].

The Kramers–Moyal coefficients, $D_k$, are determined by taking the limit of $M_k$ as $\Delta r$ tends to zero (see equation (13)). Consequently, they take the following form:

$$D_1(v, r) = \alpha_1(r) - \gamma_1(r)v + \kappa_1(r)v^2 - \epsilon_1(r)v^3,$$

(23)

$$D_2(v, r) = \alpha_2(r) + \delta_2(r)v + \beta_2(r)v^2,$$

(24)

$$D_3(v, r) = \alpha_3(r) - \gamma_3(r)v + \kappa_3(r)v^2 - \epsilon_3(r)v^3,$$

(25)
Figure 11. $A_2$ of the conditional moment $M_2$ (for velocity increments) as a function of $\Delta r$ where $r = \ell/2$: •, correspond to the range of scales where equation (8) is satisfied; ◦, correspond to the range of scales where equation (8) is not satisfied. $\alpha_2$ is calculated by extrapolating the curve fit to the solid circles to $\Delta r = 0$; $R_\lambda = 582$.

\[ D_4(v, r) = \alpha_4(r) + \delta_4(r)v + \beta_4(r) v^2, \tag{26} \]

where the new coefficients correspond to limiting values of the polynomial coefficients of $M_k$ when $\Delta r$ tends to zero, e.g.

\[ a_i(r) = \lim_{\Delta r \to 0} A_i(r, \Delta r). \tag{27} \]

Recalling that the turbulent cascade is not Markovian for small $\Delta r$, the values of the polynomial coefficients cannot simply be evaluated for small $\Delta r$. Instead, they must be appropriately extrapolated from their values when the turbulent cascade indeed exhibits Markovian properties. This is the same method as was used in [24] and is depicted in figure 11. A similar procedure was used for the other coefficients by extrapolating the conditional moments for $\Delta r \geq 0.5 \lambda$ to zero.

Having now determined the Kramers–Moyal coefficients, we follow the method of Renner et al [24] to verify whether the Kramers–Moyal expansion can be accurately represented by its first two terms. This is performed by writing the Kramers–Moyal expansion for a higher-order structure function (see equation (16)) and comparing terms of different orders. Given Pawula’s theorem [27], if $D_4 = 0$ then all $D_k$ (for $k \geq 3$) will be equal to zero (assuming all $D_k$ exist). Consequently, if the ratio of the fourth- to second-order terms in the expansion is small, the effect of the fourth-order term can be neglected (as can all terms of order $\geq 3$) and the Kramers–Moyal expansion simplifies to the Fokker–Planck equation. This procedure is followed for the fourth-order structure function (the highest order for which both the velocity and scalar fields...
were well converged) and the Kramers–Moyal expansion is given by

\[-r \frac{\partial}{\partial r} \langle v(r)^4 \rangle = 4 \int_{-\infty}^{\infty} v^3 D_1(v, r) p(v, r) \, dv + 4 \times 3 \int_{-\infty}^{\infty} v^2 D_2(v, r) p(v, r) \, dv \]

\[+ 4 \times 3 \times 2 \int_{-\infty}^{\infty} v D_3(v, r) p(v, r) \, dv \]

\[+ 4 \times 3 \times 2 \times 1 \int_{-\infty}^{\infty} D_4(v, r) p(v, r) \, dv. \tag{28}\]

The ratio of the fourth- to second-order terms, \( T_4(r)/T_2(r) \), was calculated, as was the upper bound \( \chi(r) \) on this quantity; see appendix B in [24] for a detailed explanation. Here we find that this ratio is of the order of 0.1 for all Reynolds numbers (e.g. \( \chi(r = \ell/2) = 0.16 \) for \( R_\lambda = 140 \), \( \chi(r = \ell/2) = 0.070 \) for \( R_\lambda = 306 \) and \( \chi(r = \ell/2) = 0.10 \) for \( R_\lambda = 582 \)). In [24], \( R_\lambda = 190 \) and a value of \( \chi(r = \ell/2) \approx \chi(r = 5\lambda) = 0.02 \) was observed (when estimating terms in the expansion of the eighth-order structure function). This is smaller than an interpolation of our present results to \( R_\lambda = 190 \). In any case, both the present measured values of \( \chi(r) \) and those in [24] are small, implying that a Fokker–Planck equation can be used to model the PDFs of velocity increments.

4.2. Scalar field

Having analysed the Markovian properties of velocity increments in grid turbulence in section 4.1 (and obtained similar results as in [24] for a turbulent jet), we now proceed to the main objective of the present work—a study of the Markovian properties of scalar increments (in grid turbulence).

4.2.1. Markovian properties of the passive scalar field. Similar to the velocity field, we now test the Chapman–Kolmogorov equation (equation (11)) for passive scalar increments and the Markovian nature (equation (8)) of the passive scalar increments.

The Chapman–Kolmogorov equation (equation (11)) is tested for \( r_1 = \lambda_0 \), \( r_2 = \ell_0/2 \) and \( r_3 = \ell_0 \). Contour plots of conditional PDFs of scalar increments (analogous to figure 1 in the present case) were similar to those for the velocity field and are therefore not reproduced here in the interest of brevity. Cross-sections of the contour plot for fixed \( \delta_3 \) are shown in figure 12. From these figures, the agreement between \( P(\delta_3, r_3 | \delta_1, r_1) \) and \( \int_{\infty}^{\infty} P(\delta_3, r_3 | \delta_2, r_2) P(\delta_2, r_2 | \delta_1, r_1) \, d\delta_2 \) is found to be good.

The Markovian nature of passive scalar increments is tested by means of equation (8) in figure 13. For the scales under consideration, there is a reasonable agreement between \( P(\delta_1, r_1 | \delta_2, r_2) \) and \( P(\delta_1, r_1 | \delta_2, r_2; \delta_3 = 0, r_3) \).
Figure 12. Verification of the Chapman–Kolmogorov equation for temperature increments with $\delta_3 = -\sigma L \theta$ and $\delta_3 = +\sigma L \theta$: $\ast$, $\int_\infty^{-}\infty P(\delta_1, r_1 \mid \delta_2, r_2) P(\delta_2, r_2 \mid \delta_3, r_3) \, d\delta_2$; $\circ$, $P(\delta_1, r_1 \mid \delta_3, r_3)$. $r_1 = \lambda \theta$, $r_2 = \ell \theta / 2$ and $r_3 = \ell \theta$; $R = 582$.

Figure 13. Verification of the Markovian nature of the temperature increments by a comparison of the conditional PDFs in equation (8) with $\delta_2 = -\sigma L \theta$ and $\delta_2 = +\sigma L \theta$: $\ast$, $P(\delta_1, r_1 \mid \delta_2, r_2; \delta_3 = 0, r_3)$; $\circ$, $P(\delta_1, r_1 \mid \delta_2, r_2)$. $r_1 = \lambda \theta$, $r_2 = \ell \theta / 2$ and $r_3 = \ell \theta$; $R = 582$.

For smaller separations (i.e. $r_1 = \ell \theta / 2 - \lambda \theta / 4$, $r_2 = \ell \theta / 2$ and $r_3 = \ell \theta / 2 + \lambda \theta / 4$) the same two tests are performed. The Chapman–Kolmogorov equation (equation (11)) is tested in figure 14 and the Markovian nature of passive scalar increments tested by means of equation (8) in figure 15. Similar to the velocity field, at these smaller scales, we observe that the Chapman–Kolmogorov equation continues to hold, although the passive scalar increments lose the Markovian property (equation (8)).

As can be deduced from the previous figures, the passive scalar field behaves similar to that of the velocity field. For large scale separations, the passive scalar increments exhibit Markovian properties. For smaller separations, these properties disappear. This can alternatively be observed in figure 16, where PDFs of conditional scalar increments, denoted by $x_\theta$ and $y_\theta$ (and defined analogously to the conditional velocity increments in equations (17) and (18)), are compared for different separations $\Delta r$. 

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We also note that equation (8) was tested employing scales divisions by a factor of 2 (i.e. $r_3 = \ell$, $r_2 = \ell/2$, $r_1 = \ell/4$ and $r_3 = 4\lambda$, $r_2 = 2\lambda$, $r_1 = \lambda$). The results reaffirmed that the Markov approximation (for both the velocity and passive scalar fields) is dependent on the difference in scales (i.e. $\Delta r$) and not their absolute magnitude (e.g. $r_2$).

Lastly, it is worth remarking on the different shapes of the passive scalar increment PDFs. Compared with the PDFs of velocity increments, the passive scalar increment PDFs are more pointy and possess broader tails. This is a characteristic of the higher level of internal intermittency of a turbulent passive scalar field when compared with the velocity field [14].
4.2.2. Kramers–Moyal coefficients of the passive scalar field. It is now of interest to calculate the Kramers–Moyal coefficients for the passive scalar increments and evaluate their relative magnitudes to determine the accuracy of modelling the PDF of the scalar increments with a Fokker–Planck equation. As was done for the velocity field, the conditional moments of the passive scalar increments, $M_k(\delta, r, \Delta r)$, are calculated. Typical results for $r = \ell_\theta/2$ and $\Delta r = \lambda_\theta$ are shown in figure 17. Polynomial fits of the following form are applied to first four conditional moments:

\[
M_1 = O_1(r, \Delta r) - G_1(r, \Delta r)\delta + K_1(r, \Delta r)\delta^2 - E_1(r, \Delta r)\delta^3, \quad (29)
\]

\[
M_2 = A_2(r, \Delta r) + X_2(r, \Delta r)\delta + B_2(r, \Delta r)\delta^2 + Z_2(r, \Delta r)\delta^3 + W_2(r, \Delta r)\delta^4, \quad (30)
\]

\[
M_3 = O_3(r, \Delta r) - G_3(r, \Delta r)\delta + K_3(r, \Delta r)\delta^2 - E_3(r, \Delta r)\delta^3, \quad (31)
\]

\[
M_4 = A_4(r, \Delta r) + X_4(r, \Delta r)\delta + B_4(r, \Delta r)\delta^2 + Z_4(r, \Delta r)\delta^3 + W_4(r, \Delta r)\delta^4. \quad (32)
\]

For first and third orders, the form of the curve fits are identical to those used for the velocity field. However, for second and fourth order, the order of the polynomials has been
increased from 2 to 4 given the results in figure 17, which suggest that a polynomial of order >2 is required.

The Kramers–Moyal coefficients $D_k(\delta, r)$ are calculated as before—by extrapolating the values of $M_k(\delta, r, \Delta r)$ to $\Delta r = 0$, using only data for values of $\Delta r$ for which the passive scalar field exhibits Markovian properties (i.e. $\Delta r \geq 0.5\lambda$). They take the following form:

\begin{align*}
D_1(\delta, r) &= o_1(r) - \gamma_1(r)\delta + \kappa_1(r)\delta^2 - \epsilon_1(r)\delta^3, \\
D_2(\delta, r) &= \alpha_2(r) + \xi_2(r)\delta + \beta_2(r)\delta^2 + \zeta_2(r)\delta^3 + \omega_2(r)\delta^4, \\
D_3(\delta, r) &= o_3(r) - \gamma_3(r)\delta + \kappa_3(r)\delta^2 - \epsilon_3(r)\delta^3, \\
D_4(\delta, r) &= \alpha_4(r) + \xi_4(r)\delta + \beta_4(r)\delta^2 + \zeta_4(r)\delta^3 + \omega_4(r)\delta^4.
\end{align*}

To estimate the relative magnitude of the fourth Kramers–Moyal coefficient $D_4(\delta, r)$, the same procedure employed for the velocity field is used. The ratio of the fourth-order term to the second order term, $T_4(r)/T_2(r)$, in the Kramers–Moyal expansion for the fourth-order passive
5. Conclusions and discussion

In the previous section, it was shown that turbulent velocity increments in grid-generated wind tunnel turbulence over the range $140 \leq R_\lambda \leq 582$ behaved similar to those measured in [24] for a turbulent jet at $R_\lambda = 190$. Equation (8) was observed to hold for scales $r \gtrsim \lambda$ and terms in the Kramers–Moyal expansion of order $>2$ were small, thus allowing the PDFs of velocity increments (for large enough scales) to be well approximated by a Fokker–Planck equation. In addition, the present work showed that the Chapman–Kolmogorov equation was satisfied for all cases studied—even those for which the velocity increments no longer satisfied equation (8). This result clearly indicates how satisfaction of the Chapman–Kolmogorov equation is only a necessary (and not sufficient) condition for Markov processes.

Subsequently studied was the behaviour of turbulent passive scalar increments produced by the imposition of a mean temperature gradient upon grid-generated wind tunnel turbulence. Similar to the velocity field, the passive scalar field satisfied the Chapman–Kolmogorov equation for all scales studied herein and satisfied equation (8) for scales larger than the Taylor microscale. However, in contrast with the velocity field, it was found that the fourth-order term in the Kramers–Moyal expansion governing passive scalar increments was notably

![Figure 18.](http://www.njp.org/)
more significant. Consequently, all terms in the expansion must be retained and the PDFs of passive scalar increments cannot be modelled by a Fokker–Planck equation.

Given that a continuous Markov process must have a Kramers–Moyal expansion that reduces to its first two terms (and therefore simplifies to the Fokker–Planck equation), the results for the scalar field may initially appear contradictory. However, it should be remarked that neither the velocity field nor the scalar field is purely Markovian. Hosokawa has noted this in the past, indicating that the Fokker–Planck equation obtained by Renner et al [25] leads to negative scaling exponents of velocity structure functions \((\zeta_n)\) for \(n > 5\), which is clearly implausible (see [40, 41]). He remarks ‘that a Fokker–Planck equation looks too simple and restrictive to govern the real PDF of velocity increments in turbulence, and terms with higher-order derivatives in the Kramers–Moyal expansion seem hardly completely negligible, however small the coefficients may be.’ Given that the Fokker–Planck equation governs the PDF of a stochastic process described by the Langevin equation, and a fundamental limitation of the Langevin equation is that the process it models is not differentiable whereas turbulent quantities are [28], it might be expected that the approximation of turbulent quantities will not be exact over all scales, especially when considering smaller scales and higher-order statistics. The inability of a process described by the Langevin equation to be differentiated derives from the Gaussian, \(\delta\)-correlated forcing and means that all related models will be qualitatively incorrect when examined on a small time scale [28]. Should the forcing no longer be \(\delta\)-correlated, the Markov property will be destroyed [33]. Since equation (8) becomes invalid for \(r \ll \lambda\) (for both the velocity and passive scalar statistics), the Taylor microscale can potentially be viewed as a scale below which the turbulence is no longer effectively \(\delta\)-correlated [24]. Although the assumption of a Markov process will never truly be accurate, the present results imply that—at least in the case of modelling the turbulent cascade—the use of Markovian tools will be more appropriate in describing the hydrodynamic cascade than that of the passive scalar.

Physically, we interpret the fact that PDFs of passive scalar increments (approximately) satisfy equation (8) yet have non-vanishing higher-order terms of their Kramers–Moyal expansion as a characteristic of the strong connection between large and small scales of a turbulent passive scalar—a trait presumably at odds with the random, \(\delta\)-correlated forcing of a Langevin equation. This connection results in the ramp-cliff morphology ubiquitous to turbulent passive scalars and to which is attributed the persistent scalar anisotropies observed in turbulent flows with mean scalar gradients [14]. It is also related to the higher level of internal intermittency present in passive scalar fields.

Given that internal intermittency arises from large, rare fluctuations in the dissipation rate that can be ‘found’ in the tails of PDFs of increments of turbulent quantities, one might expect its effects to be contained in the higher-order terms of the Kramers–Moyal expansion. As stated in [24], ‘the influence of higher-order Kramers–Moyal coefficients on \(\langle v(r)^n\rangle\) increases with order \(n\)’. Therefore, when internal intermittency is strong, the tails of the PDFs of turbulent increments become more important and the role of the higher-order Kramers–Moyal coefficients increases correspondingly. This may explain the present results. In the velocity field where internal intermittency is weaker,\(^6\) the higher-order terms in the Kramers–Moyal expansion

\(^6\) See figure 28 in [38] for a quantification of the internal intermittency of the velocity and passive scalar fields in the present flow. Therein, it was shown that the inertial-range velocity field internal intermittency is effectively non-existent for \(R_\lambda \lesssim 100\), after which it begins to slowly increase with Reynolds number. This behaviour was quite different from the observed internal intermittency of the passive scalar field, which assumed levels very close to its high-Reynolds-number limit for Reynolds numbers as low as \(R_\lambda = 85\).
do not play significant roles. However, internal intermittency in the scalar field is strong for all Reynolds numbers and therefore the higher-order terms in the Kramers–Moyal expansion should be important at all Reynolds numbers—a notion consistent with figure 18(b). We remark that a Fokker–Planck equation is capable of capturing the effects of internal intermittency in the velocity field (see e.g. [24]). The question is whether, using only two terms of the Kramers–Moyal expansion, it is capable of capturing all its effects, as was addressed by Hosokawa [40, 41].

We point out that the velocity field has been studied in high-Reynolds-number flows (see e.g. [25]). However, it is not yet clear whether the truncation of the Kramers–Moyal expansion after two terms has also been validated at the higher Reynolds numbers or whether it was simply assumed to be valid given the previous lower-Reynolds-number velocity field work. Should the higher-order terms in the Kramers–Moyal expansion for the velocity field be negligible at all Reynolds numbers (as has been theoretically predicted in [42]), the fact that such terms need to be retained for scalar increments could be indicative of the differences in hydrodynamic and scalar turbulence. However, should the higher-order terms increase in importance as the Reynolds number (and therefore the internal intermittency) increases, the present results for the scalar field could be interpreted as highlighting a close relationship between internal intermittency and the higher-order terms in the Kramers–Moyal expansion. Given that the scalar field possesses a high level of internal intermittency at low Reynolds numbers, it could be viewed as foreshadowing the results for the velocity field at high Reynolds number, when the latter’s internal intermittency has also become intense. To resolve this question, an unambiguous test of the importance of higher-order terms in the Kramers–Moyal expansion for velocity increments at very large Reynolds numbers (i.e. $R_\lambda > 10^3$) is required and would be of significant insight and benefit.

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