On Plücker Equations Characterizing Grassmann Cones

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Abstract
Polynomial solutions to the KP hierarchy are known to be parametrized by a cone over an infinite-dimensional Grassmann variety. Using the notion of Schubert derivations on a Grassmann algebra, we encode the classical Plücker equations of Grassmannians of r-dimensional subspaces in a formula whose limit for \( r \to \infty \) coincides with the KP hierarchy phrased in terms of vertex operators.

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Introduction

The purpose of this article is to advertise the notion of Schubert derivation on an exterior algebra introduced in [10] by showing that it provides another point of view to look at the quadratic equations describing the Plücker embedding of Grassmannians – a very classical and widely studied subject. In particular, it allows i) to “discover” the vertex operators generating the fermionic vertex superalgebra (in the sense of [8, Section 5.3]; ii) to compute their bosonic expressions as in [19, Lecture 5]; iii) to interpret them in terms of Schubert derivations and iv) to provide an almost effortless deduction of the celebrated Hirota bilinear form of the KP hierarchy [19]. If \( B \) denotes the polynomial ring in infinitely many indeterminates \( \mathbb{Q}[x_1, x_2, \ldots] \), recall that an element \( \tau \in B \) is said to be a \( \tau \)-function for the KP-hierarchy (after Kadomtsev and Petviashvili) if

\[
\text{Res}_z \Gamma'(z) \tau \otimes \Gamma(z) \tau = 0,
\]

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where \( z \) is a formal variable. \( \text{Res}_z \) denotes the coefficient of \( z^{-1} \) of a Laurent series and
\[
\Gamma(z) := \exp(\sum_{i \geq 1} x_i z^i) \exp\left(-\sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i}\right) \quad \text{and} \quad \Gamma^*(z) := \exp(\sum_{i \geq 1} x_i z^i) \exp\left(-\sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i}\right).
\]
(2)

It is a fundamental observation, due to Sato \([28, 29]\) and widely developed by his Kyoto school [5, 6, 17], that (1) encodes the Plücker equations of the cone of decomposable elements of a semi-infinite exterior power of an infinite-dimensional vector space. This fact is mentioned and/or explained in a number of different ways, e.g. in [2, 3, 7, 20, 21, 25, 26, 27, 30] and surely in many more references. From our part we shall write the equations characterizing Grassmann cones of decomposable tensors in \( r \)-the exterior powers, for all \( r \geq 1 \), in such a way to recover expression (1) as a limit for \( r \to \infty \).

To this purpose, let \( M_0 \) be a free abelian group of infinite countable rank with a basis \( \mathcal{B}_0 := \{ b_0, b_1, \ldots \} \) and let \( \{ \beta_i \}_{i \geq 1} \) be the basis of the restricted dual \( M_0^\ast \) such that \( \beta_i(b_j) = \delta_{ij} \).

Let \( \wedge^r M_0 \) be the \( r \)-th exterior power of \( M_0 \). The Grassmann cone \( G_r \) of \( \wedge^r M_0 \) is the image of the (non-surjective) multilinear alternating map \( \wedge^r M_0 \to \wedge^r M_0 \), given by \( (m_1, \ldots, m_r) \mapsto m_1 \wedge \cdots \wedge m_r \).

It is well-known, see e.g. \([3, \text{Section 4}]\), that \( m \in \wedge^r M_0 \) belongs to \( G_r \) if and only if
\[
\sum_{i \geq 0} (\beta_{i+1}m) \otimes (b_i \wedge m) = 0, \tag{3}
\]
where \( \beta_{i+1}m \in \wedge^{r-1} M_0 \) denotes the contraction of \( m \) against \( \beta_i \) (Section 1.3). Equation (3) is equivalent to
\[
\text{Res}_z(\sum_{i \geq 0} (\beta_{i+1}z^{i-1}m) \otimes (b_i \wedge m)) = 0 \tag{4}
\]
a trick we learned in [19, \text{Section 7.3}]. We combine (4) with the following observation: there are unique formal power series \( \sigma_{\pm}(z) := \sum_{i \geq 0} \sigma_{\pm}z^i \in \text{End}_\mathbb{C}(\wedge M_0)[[z^{\pm 1}]] \) such that
\[
\begin{align*}
\sigma_{\pm}(z)(m_1 \wedge m_2) &= \sigma_{\pm}(z)(m_1) \wedge \sigma_{\pm}(z)(m_2), \quad \forall m_1, m_2 \in \wedge M_0 \\
\sigma_{\pm}(b_i) &= b_{i \mp 1},
\end{align*}
\]
with the convention that \( b_k = 0 \) if \( k < 0 \). Denote by \( \sigma_{\mp}(z) := \sum_{i \geq 0} \sigma_{\pm}z^i \) the inverse in \( \text{End}_\mathbb{C}(\wedge M_0)[[z^{\pm 1}]] \) of \( \sigma_{\pm}(z) \).

The main result of this paper is:

**0.1 Theorem.** Let \( m \in \wedge^r M_0 \). Then \( m \in G_r \) if and only if the following equality:

\[
\text{Res}_z \left[ \sigma_{+}(z)\sigma_{-}(z)(\beta_0 \sigma_{+}(z)m) \otimes \sigma_{+}(z)\sigma_{-}(z)(b_0 \wedge \sigma_{-}(z)m) \right] = 0
\]

holds in \( \wedge^{r-1} M_0 \otimes \wedge^{r+1} M_0 \), where \( \beta_0.m \) denotes the contraction of \( m \) against \( \beta_0 \) (see 1.3).

Following \([10]\), we have called \textit{Schubert derivations} the maps \( \sigma_{\pm}(z) \) as well as their formal inverses. The reasons are that i) they are derivations of \( \wedge M_0 \) in the sense of Hasse and Schmidt (see \([10, 15]\)) and ii) they satisfy some Pieri and Giambelli formulas (Section 3.1). Therefore the exterior power \( \wedge^r \mathbb{C}^n \) can be regarded as an irreducible module over the cohomology ring \( H^r(\mathbb{C}^n, \mathbb{Z}) \) of the Grassmannian of \( r \)-planes in \( \mathbb{C}^n \) (\([10, 11]\)), which in turn implies the identification \( H^r(\mathbb{C}^n, \mathbb{Z}) \equiv \wedge^r H^*(\mathbb{P}^{n-1}) \). The latter is also known in the literature as \textit{Satake isomorphism} \([9, 16]\). The reason why Theorem 0.1 is interesting is that its proof is cheap and easily returns the equation of the KP hierarchy by letting \( r \) tending to \( \infty \).
To state a relevant consequence of Theorem 0.1 above, we need to introduce a few more pieces of notation. Let \( \mathcal{P}_r : = \{ \lambda : = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r | \lambda_1 \geq \cdots \geq \lambda_r \} \) be the set of all partitions of length at most \( r \) and \( \{ b_\lambda^r \} := b_{\lambda_r} \wedge b_{\lambda_{r-1}} \wedge \cdots \wedge b_{\lambda_1} \). Then \( \bigwedge^r \mathcal{B}_0 := \{ b_\lambda^r | \lambda \in \mathcal{P}_r \} \) is a \( \mathbb{Z} \)-basis of \( \bigwedge^r \mathcal{M}_0 \), i.e. each \( m \in \bigwedge^r \mathcal{M}_0 \) can be uniquely written as a finite linear combination of the form \( \sum_{\lambda \in \mathcal{P}_r} a_\lambda b_\lambda^r \). Moreover, let \( B_r := \mathbb{Z}[e_1, \ldots, e_r] \) be the polynomial ring in the \( r \) indeterminates \( (e_1, \ldots, e_r) \) and \( E_r(z) := 1 - e_1 z + \cdots + (-1)^r e_r z^r \in B_r[z] \). Construct a sequence \( H_r := (h_i)_{i \in \mathbb{Z}} \) of elements of \( B_r \) via the equality
\[
\sum_{i \in \mathbb{Z}} h_i z^i := \sum_{n \geq 0} (1 - \mathfrak{r}(z))^n
\]
understood in the ring of formal Laurent series \( B_r[[z^{-1}, z]] \). By construction, \( h_0 = 0 \) if \( j < 0 \), \( h_0 = 1 \) and, for \( i > 0 \), \( h_i \) is a homogeneous polynomial of degree \( i \) in \( (e_1, \ldots, e_r) \) provided that, for all \( 1 \leq j \leq r \), each \( e_j \) is given weight \( j \). The Schur determinant associated to the sequence \( H_r \) and to \( \lambda \in \mathcal{P}_r \) is by definition
\[
\Delta_\lambda(H_r) := \det(h_{\lambda_1-1}e_1, \ldots, h_{\lambda_r-1}e_r) \in B_r.
\]
Using the well–known fact that \( B_r = \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Z} \cdot \Delta_\lambda(H_r) \), the map \( \mathfrak{r}_r : B_r \to \bigwedge^r \mathcal{M}_0 \) given by \( \Delta_\lambda(H_r) \mapsto [b_\lambda^r] \) defines an isomorphism of abelian groups. It enables to equip \( \bigwedge^r \mathcal{M}_0 \) with a structure of free \( B_r \)-module of rank 1, generated by \( [b_\lambda^r] \), that we shall denote by \( \bigwedge^r \mathcal{M}_r \). We regard \( \sigma_-(z), \tau_-(z) \) as maps \( B_r \to B_r[z^{-1}] \) as well, by defining \( \sigma_-(z) \Delta_\lambda(H_r) = \mathfrak{r}_r^{-1}(\sigma_-(z)[b_\lambda^r]) \) and \( \tau_-(z) \Delta_\lambda(H_r) = \mathfrak{r}_r^{-1}(\tau_-(z)[b_\lambda^r]) \). Then Theorem 0.1 admits the following rephrasing.

0.2 Theorem. The element \( \mathfrak{m} := \sum_{\lambda \in \mathcal{P}_r} a_\lambda [b_\lambda^r] \in \bigwedge^r \mathcal{M}_0 \) belongs to \( \mathfrak{g}_r \) if and only if the equality below holds in \( B_{r-1} \otimes \mathbb{Z} B_{r-1} \)
\[
\text{Res}_z \sum_{\lambda \in \mathcal{P}_r} a_\lambda [b_\lambda^r] E_{r-1}(z) \Delta_\lambda(\sigma_-(z)H_{r-1}) \otimes \frac{1}{E_{r-1}(z)} \tau_-(z) \Delta_\lambda(H_{r-1}) = 0.
\] (5)

For example, for \( r = 2 \), one recovers the Klein’s quadric cutting out the Grassmannian \( G(2, 4) \) in its Plücker embedding (Section 6). Other examples are discussed in the forthcoming [15] (see also [12]). They all indicate that even for detecting the Grassmann cone \( \mathfrak{g}_s \), computations are quite painful, surely not as easy as checking the simpler condition \( \mathfrak{m} \wedge \mathfrak{m} = 0 \). What makes Theorem 0.2 interesting, however, is that, on one hand, the maps \( \Gamma_r(z) : B_r \to B_{r+1}(z) \) and \( \Gamma_r^*(z) : B_r \to B_{r+1}(z) \), defined by
\[
\Gamma_r(z) \Delta_\lambda(H_r) := \frac{1}{E_{r+1}(z)} \tau_-(z) \Delta_\lambda(H_{r+1}) \quad \text{and} \quad \Gamma_r^*(z) \Delta_\lambda(H_r) := E_{r-1}(z) \Delta_\lambda(\sigma_-(z)H_{r-1})
\]
and occurring in formula (5), are precisely truncated versions of the vertex operators displayed in (2) and, on the other hand, the Schubert derivations \( \sigma_-(z) \) and \( \tau_-(z) \) are well defined also for \( r = \infty \).

More precisely, let \( (e_1, e_2, \ldots, e_r) \) be a sequence of infinitely many indeterminates and \( E_\infty(z) := \sum_{i \geq 0} (-1)^i e_i z^i \). For all \( p \in B_\infty \) there is \( r \geq 0 \) such that \( p \in B_r \) for all \( s \geq r \). We say that \( p \) corresponds to an element of the Grassmann cone \( \mathfrak{g}_s \) if \( \Phi_s(p) \in \mathfrak{g}_s \) for all \( s \geq r \). We have:

0.3 Corollary. An element \( p \in B_\infty := \mathbb{Z}[e_1, e_2, \ldots] \) corresponds to an element of the Grassmann cone \( \mathfrak{g}_s \) if and only if
\[
\text{Res}_z \left( E_\infty(z) \sigma_-(z)p \otimes \frac{1}{E_\infty(z)} \tau_-(z)p \right) = 0.
\] (6)
By abuse of notation, let us write \( S_r \otimes \mathbb{Q} \) for the Grassmann cone of decomposable tensors of \( \Lambda^r(M_0 \otimes \mathbb{Q}) \) and let \((x_1, x_2, \ldots)\) be the sequence of indeterminates over \( \mathbb{Q} \), implicitly defined by the equality:

\[
\exp(-\sum_{i \geq 1} x_i z^i) := E_n(z).
\]

An immediate check shows that \( B := B_\infty \otimes \mathbb{Q} = \mathbb{Q}[x_1, x_2, \ldots] \). As before, for each \( p \in B \) there is \( r \geq 0 \) such that \( p \in B_r \otimes \mathbb{Q} \) for all \( s \geq r \). Then we shall say that \( \tau \) corresponds to an element of \( S_\infty \otimes \mathbb{Q} \) if there is \( r \geq 0 \) such that \( \phi_s(\tau) \in \Lambda^s(M_0 \otimes \mathbb{Q}) \) for all \( s \geq r \).

**0.4 Corollary.** A polynomial \( \tau \in B \) corresponds to an element of the Grassmann cone \( S_\infty \otimes \mathbb{Q} \) if and only if (1) holds.

It follows that equation (6) expresses the KP hierarchy (1) over the integers. Notice that it has been obtained by using the indeterminates \( e_i \) and \( h_i \) (that may be interpreted as elementary and complete symmetric polynomials). The latter reveal often more convenient than the variables \((x_i)\), as remarked also in the couple of important and relatively recent articles [13, 18].

The paper is organized as follows. Section 1 sets the preliminaries and notation used throughout the paper. Section 2 recalls a few facts concerning Hasse-Schmidt (HS) derivations on exterior algebras as introduced in [10] and treated in more details in [15]. The section proclaims the most powerful tool of the theory which we call (as in [10]) integration by parts formula. Moreover, that the transpose of an HS-derivation is an HS-derivation as well is also proven, a fact heavily used in Section 4 to prove Theorem 0.1. Schubert derivations are studied in Section 3, where a few technical lemmas leading to the approximation \( B_r \to B_r([z]) \) of the vertex operators are discussed. A pivotal aspect of Section 5 is that the Schubert derivations \( \sigma_\cdot(z) \) and its inverse \( \bar{\sigma}_\cdot(z) \) enjoy a stability property enabling to define them as maps \( B_r \to B_r[z^{-1}] \). Their limit for \( r \to \infty \) gives rise to the ring homomorphisms \( B \to B[z^{-1}] \) which enter in the expression of the vertex operators. The crucial property that \( \sigma_\cdot(z), \bar{\sigma}_\cdot(z) \) commute with taking Schur determinants, proven in Section 5, is obtained by exploiting a powerful determinantal formula due to Laksov and Thorup [22]. Section 6 is entirely devoted to the standard example of decomposable tensors in a second wedge power faced via Theorem 0.2. Eventually, Section 7 is concerned with the limit of formula (5) for \( r \to \infty \), where we (re)show that tau-functions for the KP-hierarchy corresponds to decomposable tensors in an infinite exterior power.

## 1 Preliminaries and Notation

This section is to fix notation and to list the pre-requisites we shall need in the sequel.

**1.1** A partition is a monotonic non increasing sequence \( \lambda := (\lambda_1 \geq \lambda_2 \geq \cdots) \) of non negative integers all zero but finitely many. The length \( |\lambda| \) of a partition is \( \sum_{i \geq 1} \lambda_i \), the number of its non-zero parts; its weight \( \lambda \) := \( \sum_{i \geq 1} \lambda_i \). We shall denote by \( \mathcal{P}_r \) the set of all partitions of length at most \( r \) and by \( \mathcal{P}_r^\lambda := (\lambda := (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_r | |\lambda| \leq r \) the elements of \( \mathcal{P}_r \) bounded by \( r \). The set \( \mathcal{P}_r \) is a monoid with respect to the sum \( \lambda + \mu = (\lambda_1 + \mu_1, \ldots) \) whose neutral element is the null partition \((0)\) with all the parts equal to zero. If \( \lambda, \mu \in \mathcal{P}_r \), we shall write \( \mu \leq \lambda \) if the Young diagram of \( \mu \) is contained in the Young diagram of \( \lambda \).

**1.2** We shall denote by \( M_0 \) the free abelian group \( \mathbb{Z}[X] \) and by \( B_0 := (b_0, b_1, \ldots) \) its standard basis \( (1, X, X^2, \ldots) \). Let \( \bigwedge M_0 := \bigoplus_{r \geq 0} \bigwedge^r M_0 \) be the exterior algebra of \( M_0 \). Then \( \bigwedge^0 M_0 := \mathbb{Z} \) and, for all \( r \geq 1 \), the \( r \)-th exterior power of \( M_0 \) is the \( \mathbb{Z} \)-linear span of \( \bigwedge^r B_0 := \langle \{ b_\lambda \mid \lambda \in \mathcal{P}_r \} \rangle \), where \( b_\lambda := b_{\lambda_1} \wedge b_{\lambda_2} \wedge \cdots \wedge b_{\lambda_r} \). In particular \( b_0^r := b_0 \wedge b_1 \wedge \cdots \wedge b_{r-1} \).

**1.3** Let \( \mathbf{m} \in \bigwedge^r M_0, \ r \geq 1 \). Its contraction, \( \beta \cdot \mathbf{m} \), against \( \beta \in M_0^* \) is the unique element of \( \bigwedge^{r-1} M_0 \) such that

\[
\gamma(\beta \cdot \mathbf{m}) = (\beta \wedge \gamma)(\mathbf{m}), \quad \forall \gamma \in \bigwedge M_0^*.
\]
It turns out that $\beta_D : (\bigwedge M_0, \wedge) \to (\bigwedge M_0, \wedge)$ is the unique derivation of degree $-1$ such that $\beta_D m = \beta(m)$, for all $m \in M_0$.

1.4 Let $B_0 = \mathbb{Z}$ and for $r > 1$ denote by $B_r$ the polynomial ring $\mathbb{Z}[e_1, \ldots, e_r]$. Accordingly, we let $E_0(z) = 1$ and $E_r(z) = 1 - e_1 z + \cdots + (-1)^r e_r z^r \in B_r[z]$ for $r > 1$. The equality

$$\sum_{n \in \mathbb{Z}} h_n z^n = \frac{1}{E_r(z)} = \sum_{i \geq 0} (1 - E_r(z))^i$$

read in the abelian group of formal Laurent series $B_1[[z^{-1}, z]]$ defines the bilateral sequence $H_r := (h_1)_{i \in \mathbb{Z}}$ of elements of $B_r$. By construction, then, $h_j = 0$ if $j < 0$, $h_0 = 1$ and for all $j > 0$, $h_j$ is a polynomial in $e_1, \ldots, e_r$ of weighted degree $j$, after declaring that $e_i$ is given degree $i$. It is well known that (e.g. [24, p. 41])

$$B_r = \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Z} \cdot \Delta_{\lambda}(H_r),$$

where

$$\Delta_{\lambda}(H_r) := \det(h_{\lambda_i - j + i})_{1 \leq i, j \leq r}.$$

1.5 Remark. For the terms of the sequence $H_r$, the more careful notation $h_{r,n}$ should be preferred, in place of just $h_r$, to keep track of their dependence on $r$. To make the notation less heavy we decided however to drop the subscript $r$, hoping for the context being sufficient to avoid confusions.

2 Hasse-Schmidt Derivations on an Exterior Algebra

2.1 Given any module $M$ over a commutative ring $A$ with unit, there is an obvious $A$-module isomorphism

$$\text{End}_A(\bigwedge M)[[z]] \rightarrow \text{Hom}_A(M, M[[z]]),$$

where $\text{End}_A(\bigwedge M)[[z]]$ denotes the formal power series with $\text{End}_A(\bigwedge M)$-coefficients in an indeterminate $z$. If $D(z) \in \text{End}_A(\bigwedge M)[[z]]$, we denote in the same way its image through the map (8).

2.2 Definition. A Hasse-Schmidt (HS) derivation of $\bigwedge M$ is an algebra homomorphism $D(z) : \bigwedge M \rightarrow \bigwedge M[[z]]$, i.e:

$$D(z)(m_1 \wedge m_2) = D(z)m_1 \wedge D(z)m_2, \quad \forall m_1, m_2 \in \bigwedge M_0.$$  (9)

Define a sequence $D := (D_0, D_1, \ldots)$ of endomorphisms of $\bigwedge M_0$ through the equality:

$$\sum_{j \geq 0} D_j m \cdot z^j := D(z)m.$$  (10)

Then equation (9) holds if and only if the sequence $D$ obeys the higher order Leibniz rules:

$$D_i(m_1 \wedge m_2) = \sum_{j = 0}^i D_j m_1 \wedge D_{i-j} m_2, \quad i \geq 0.$$  (11)

In particular $D_0$ is an algebra homomorphism and $D_1$ is a (usual) derivation of the $\wedge$-algebra $\bigwedge M_0$. If $D_0$ is an automorphism of $\bigwedge M_0$, then the formal power series $D(z)$ is invertible in $\text{End}_A(\bigwedge M)[[z]]$. Denote by $\bar{D}(z)$ its inverse.
2.3 Proposition (see [10]). The set $\text{HS}(\Lambda M)$ of all HS-derivations on $\Lambda M$ is a subalgebra of $\text{End}_\mathbb{Z}(\Lambda M[[z]])$, with respect to the product

$$D(z)E(z) = \sum_{j \geq 0} \sum_{i=0}^{j} (D_1 \circ E_{j-i})z^i,$$

where $\sum_{i \geq 0} D_1 z^i := D(z)$ and $\sum_{i \geq 0} E_1 z^i := E(z)$. In particular if $D_0$ is an automorphism of $\Lambda M$, then the inverse formal power series $\overline{D}(z)$ is an HS-derivation if and only if $D(z)$ is.

2.4 Proposition. Let $f(z) \in \text{End}_\Lambda(M[[z]])$. There exists a unique HS-derivation $D^f(z) : \Lambda M \rightarrow \Lambda M[[z]]$ such that

$$D^f(z)m = f(z)(m)$$

for all $m \in M$.

Proof. Let us first prove the existence. For all $r \geq 1$, consider the unique A-linear extension of the map $D^f(z) : M^r \rightarrow \Lambda^r M[[z]]$ defined by

$$\hat{f}(z)(m_1 \otimes \cdots \otimes m_r) = f(z)m_1 \wedge \cdots \wedge f(z)m_r.$$ 

This map is clearly alternating and thus factorizes through a unique homomorphism $D^f(z) : \Lambda^r M \rightarrow \Lambda^r M[[z]]$ such that $D^f(z)(m_1 \otimes \cdots \otimes m_r) = f(z)m_1 \wedge \cdots \wedge f(z)m_r$. All $m \in \Lambda M$ is a finite sum $m_1 + \cdots + m_i$ of homogeneous elements, i.e. $m_i \in \Lambda^i M$ for some $i \geq 0$ (notice that $D^f(z)m = a$ for all $a \in A$). Define $D^f(z)m$ as $\sum_{i=1}^r D^f(z)m_i$. We want to show that for $m_1, m_2 \in \Lambda M$

$$D^f(z)(m_1 \wedge m_2) = D^f(z)m_1 \wedge D^f(z)m_2.$$ 

Without loss of generality, we may assume they are homogeneous with respect to the graduation of $\Lambda M$, i.e. $m_1 = m_{11} \wedge \cdots \wedge m_{1r}$ and $m_2 = m_{21} \wedge \cdots \wedge m_{2s}$. Thus

$$D^f(z)(m_1 \wedge m_2) = D^f(z)(m_{11} \wedge \cdots \wedge m_{1r} \wedge m_{21} \wedge \cdots \wedge m_{2s}) = f(z)m_{11} \wedge \cdots \wedge f(z)m_{1r} \wedge f(z)m_{21} \wedge \cdots \wedge f(z)m_{2s} = (f(z)m_{11} \wedge \cdots \wedge f(z)m_{1r}) \wedge (f(z)m_{21} \wedge \cdots \wedge f(z)m_{2s}) = D^f(z)m_1 \wedge D^f(z)m_2.$$ 

To prove unicity, let $\overline{D}(z)$ be any HS-derivation on $\Lambda M$ such that $\overline{D}(z)m = f(z)m$ for all $m \in M$. Then for all homogeneous element $m := m_1 \wedge \cdots \wedge m_r \in \Lambda^r M$:

$$\overline{D}(z)m = \overline{D}(z)(m_1 \wedge \cdots \wedge m_r) = \overline{D}(z)m_1 \wedge \cdots \wedge \overline{D}(z)m_r = f(z)m_1 \wedge \cdots \wedge f(z)m_r = D^f(z)(m_1 \wedge \cdots \wedge m_r) = D^f(z)m.$$ 

The main tool of the paper is the following observation for which we omit the totally obvious proof. It is responsible, in our context, of the emergence of the vertex operators.

2.5 Proposition (Integration by Parts). Assume that $D(z) \in \text{HS}(\Lambda M)$ is invertible in the sense of Proposition 2.3. Then the integration by parts formula holds:

$$(D(z)m_1) \wedge m_2 = D(z)(m_1 \wedge \overline{D}(z)m_2), \quad \forall m_1, m_2 \in \Lambda M. \quad (11)$$

2.6 Duality. Let now $M_0$ and $\mathcal{Z}_0$ as in Section 1. Let $\beta_i \in M_0^\vee := \text{Hom}_\mathbb{Z}(M_0, \mathbb{Z})$ such that $\beta_i(b_i) = \delta_{i1}$. The restricted dual of $M_0$ is $M_0^\vee := \bigoplus_{i \geq 0} \mathbb{Z} \cdot \beta_i$. The equality

$$\mu_1 \wedge \cdots \wedge \mu_r(m_1 \wedge \cdots \wedge m_r) := \det(\mu_i(m_j))_{1 \leq i,j \leq r},$$

defines a natural identification of $\Lambda^r M_0^\vee$ with $(\Lambda^r M_0)^\ast$. In particular

$$\{\beta_{\lambda_1} \wedge \beta_{-1 + \lambda_{r-1}} \wedge \cdots \wedge \beta_{r-1 + \lambda_2} \lambda \in \mathcal{P}_r\}$$

is the basis of $\Lambda^r M_0^\vee$ dual of $\Lambda^r \mathcal{Z}_0$, i.e. $\beta_{i_1} \wedge \cdots \wedge \beta_{i_r}(b_{j_1} \wedge \cdots \wedge b_{j_r}) = \delta_{i_1 j_1} \cdots \delta_{i_r j_r}$. 

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2.7 Definition. The transpose of \( \mathcal{D}(z) \in \text{HS}(\wedge M_0) \) is the module homomorphism \( \mathcal{D}(z)^T : \wedge M_0^* \to \wedge M_0^* \), defined by

\[
(\mathcal{D}(z)^T \eta)(m) = \eta(\mathcal{D}(z)m), \quad \forall(\eta, m) \in \wedge M_0^* \times \wedge M_0.
\]

2.8 Proposition. If \( \mathcal{D}(z) \in \text{HS}(\wedge M_0) \), then \( \mathcal{D}(z)^T \) is an HS-derivation of \( \wedge M_0 \).

Proof. By definition, \( \mathcal{D}(z)^T \beta(m) = \beta(\mathcal{D}(z)m) \) for all \( \beta \in M_0^* \). As each \( \eta \in \wedge M_0^* \) is a sum of homogeneous components, without loss of generality we may assume \( \eta \in \wedge^r M_0^n \), i.e. \( \eta := \eta_1 \wedge \cdots \wedge \eta_r \) for some \( \eta_i \in M_0^* \). Thus

\[
\mathcal{D}(z)^T(\eta_1 \wedge \cdots \wedge \eta_r)(m_1 \wedge \cdots \wedge m_r) = \eta_1 \wedge \cdots \wedge \eta_r(\mathcal{D}(z)(m_1 \wedge \cdots \wedge m_r))
\]

\[
= \eta_1 \wedge \cdots \wedge \eta_r(\mathcal{D}(z)m_1 \wedge \cdots \wedge \mathcal{D}(z)m_r)
\]

\[
= \det(\eta_i(\mathcal{D}(z)m_i)) = \det(\mathcal{D}(z)\eta_i(m_i)) =
\]

\[
= \mathcal{D}(z)^T \eta_1 \wedge \cdots \wedge \mathcal{D}(z)^T \eta_r(m_1 \wedge \cdots \wedge m_r).
\]

The unique HS-derivation \( \mathcal{D}(z) \) on \( \bigwedge M_0^* \) such that \( \mathcal{D}(z)\eta = \mathcal{D}(z)^T \eta \) coincides with \( \mathcal{D}(z)^T \) when evaluated on \( \bigwedge^r M_0^n \). Then it must coincide with it and \( \mathcal{D}(z)^T \in \text{HS}(\wedge M_0^*) \).

\[\blacksquare\]

3 Schubert Derivations on \( \mathbb{Z}[X] \)

3.1 With the same notation of 1, Prop. 2.4 guarantees the existence of unique HS-derivations

\[
\sigma_\lambda(z) := \sum_{i \geq 0} \sigma_i z^i : \bigwedge M_0 \to \bigwedge M_0[z], \quad \quad \tau_\lambda(z) := \sum_{i \geq 0} (-1)^i \sigma_i z^i : \bigwedge M_0 \to \bigwedge M_0[[z]]
\]

such that \( \sigma_\lambda(z)b_i = \sum_{j \geq 0} b_{i+j} z^j \) and \( \tau_\lambda(z)b_i = b_i - b_{i+1} \cdot z \). In particular \( \sigma_1 b_i = b_{i+1} \) and \( \tau_1 b_i = 0 \) if \( j \geq 2 \). They are one the inverse of the other:

\[
\sigma_\lambda(z)\tau_\lambda(z) = \sigma_\lambda(z)\tau_\lambda(z) = 1_{\bigwedge M_0}.
\]

We shall call them Schubert derivations, in compliance with the terminology introduced in [10, 11]. The motivation comes from following Pieri-like formula ([10, Theorem 2.4]):

\[
\sigma_i[b]_\lambda^r = \sum_{\mu} [b]_{\mu}^r, \quad (i \geq 0)
\]

(12)

where the sum is taken over all the partitions \( \mu \in \mathcal{P}_r \) such that \( \mu_1 \geq \lambda_1 \geq \cdots \geq \mu_r \geq \lambda_r \) and \( |\mu| = |\lambda| + i \). In addition, a Giambelli-like formula holds ([10, Formula (17)]) or [22, Theorem 0.1]): for all \( \lambda \in \mathcal{P}_r \)

\[
[b]_{\lambda} = \Delta_{\lambda}(\sigma_\lambda)[b]_{\lambda}^r := \det(\sigma_{\lambda_1-i+1} \ldots \sigma_{\lambda_r})[b]_{\lambda}^r
\]

(13)

where by conventions \( \sigma_j = 0 \) if \( j < 0 \). If \( M_{0,n} := \bigoplus_{n=0}^{n-1} \mathbb{Z}b_j \), formula (12) tells us that \( \bigwedge^r M_{0,n} \) is an irreducible representation of the cohomology ring \( H^*(G_{r}(\mathbb{C}^n), \mathbb{Z}) \); the latter is in fact generated as a \( \mathbb{Z} \)-algebra by the special Schubert cycles \( c_i(\Omega_r) \), the \( t \)-th Chern classes of the universal quotient bundle over \( G_r(\mathbb{C}^n) \), traditionally denoted by \( \sigma_t \). So the reason we are using the same notation, more than an abuse, is to emphasize that we are working essentially with the same objects, seeing \( \bigwedge^r M_0 \) as a module over the cohomology (as in [4, p. 303]) of the Grassmannian \( G(r, \mathbb{C}^n) \).

3.2 We similarly define a sort of “mirror” of the Schubert derivation \( \sigma_+(z) \), namely

\[
\sigma_+(z) := \sum_{i \geq 0} \sigma_+ z^{-i} \in \text{HS}(\bigwedge M_0)
\]

(14)
which, by definition, is the unique HS-derivation such that $\sigma_{-j} b_i = b_{i-j}$ if $i \geq j$ and 0 otherwise. Its inverse in $\text{End}_z(\wedge^r M_0[[z^{-1}]],$

$$\overline{\sigma}_-(z) = \sum_{j \geq 0} (-1)^j \sigma_{-j} z^{-j},$$

is the unique HS-derivation such that $\overline{\sigma}_-(z)b_i := b_i - b_{i-1}z^{-1}$ for all $i \geq 0$. In particular, for all $r \geq 0$:

$$\overline{\sigma}_-(z)b^r_0 = [b]_0^r.$$  \hspace{1cm} (15)

We note in passing that for $j > 0$, both $\overline{\sigma}_{-j}$ and $\sigma_{-j}$ are locally nilpotent, i.e. for all $m \in \wedge M_0$ there exists $N \in \mathbb{N}$ such that $(\sigma_{-j})^N m = 0$ (resp. $(\overline{\sigma}_{-j})^N m = 0$).

3.3 Lemma. For all $r \geq 1$, let $(1^r)$ be the partition with $r$ parts equal to one. Then for all $\lambda \in \mathcal{P}_r$ we have

$$\overline{\sigma}_{\pm r}[b]^r_{\lambda} = [b]^r_{\lambda \#(1^r)}.$$  \hspace{1cm} Proof. Indeed $\overline{\sigma}_{\pm r}[b]^r_{\lambda}$ is the coefficient of degree $r$ of the expansion of

$$\overline{\sigma}_{\pm}(z)[b]^r_{\lambda} = \overline{\sigma}_{\pm}(z)(b_{\lambda_r} \wedge \cdots \wedge b_{r-1+\lambda_1})$$

in powers of $z$. Since $\overline{\sigma}_{\pm}(z)$ is a HS-derivation, we have

$$\overline{\sigma}_{\pm}(z)(b_{\lambda_r} \wedge \cdots \wedge b_{r-1+\lambda_1}) = \overline{\sigma}_{\pm}(z)b_{\lambda_r} \wedge \cdots \wedge \overline{\sigma}_{\pm}(z)b_{r-1+\lambda_1}
= (b_{\lambda_r} - b_{\lambda_r+1}z^{-1}) \wedge \cdots \wedge (b_{r-1+\lambda_1} - b_{r-1+\lambda_1+1}z^{-1})$$

and is so apparent that the coefficient of $z^r$ is $[b]^r_{\lambda \#(1^r)}$, as desired. \hfill $\blacksquare$

3.4 Lemma. The following equality holds in $(\wedge^r M_0[[z^{-1}]]$:

$$\overline{\sigma}_+(z)m = (-1)^r z^r \overline{\sigma}_-(z)[\overline{\sigma}_+(z)] m.$$  \hspace{1cm} Proof. Without loss of generality, we may assume $m = [b]^r_{\lambda}$. In this case

$$\overline{\sigma}_+(z)[b]^r_{\lambda} = \overline{\sigma}_+(z)(b_{\lambda_r} \wedge b_{1+\lambda_{r-1}} \wedge \cdots \wedge b_{r-1+\lambda_1})$$

(definition of $[b]^r_{\lambda}$)

$$= \overline{\sigma}_+(z)b_{\lambda_r} \wedge \cdots \wedge \overline{\sigma}_+(z)b_{r-1+\lambda_1}$$

($\overline{\sigma}_+(z) \in \text{HS}(\wedge M_0)$)

$$= (b_{\lambda_r} - b_{\lambda_r+1}z) \wedge \cdots \wedge (b_{r-1+\lambda_1} - b_{r-1+\lambda_1+1}z)$$

(definition of $\overline{\sigma}_+(z)b$)

$$= (-1)^r z^r [(b_{\lambda_r+1} - b_{\lambda_r}z^{-1}) \wedge \cdots \wedge (b_{r-1+\lambda_1} - b_{r-1+\lambda_1+1}z^{-1})]$$

(highlights $(-1)^r z^r$)

$$= (-1)^r z^r [\overline{\sigma}_-(z)(b_{\lambda_r+1} \wedge \cdots \wedge b_{r+\lambda_1})]$$

(definition of $\overline{\sigma}_-(z)$)

$$= (-1)^r z^r [\overline{\sigma}_-(z)(b_{\lambda_r+1} \wedge \cdots \wedge b_{r+\lambda_1})]$$

($\sigma_+(z) \in \text{HS}(\wedge M_0)$)

$$= (-1)^r z^r [\overline{\sigma}_-(z)(b_{\lambda_r})]$$

(3.3 applied to $\overline{\sigma}_+$.)

An analogous of Lemma 3.4 holds for $\overline{\sigma}_-(z)$ as well, up to an additional hypothesis.

3.5 Lemma. Assume that $t(\lambda) = r$ (i.e. that $\lambda_r > 0$). Then

$$\overline{\sigma}_-(z)m = (-1)^r z^{-r} \overline{\sigma}_+(z)[\overline{\sigma}_-(z)] m.$$  \hspace{1cm} (16)
Notice that if \( \ell(\lambda) < r \), the right hand side of (16) is zero and then (16) fails to be true in general.

**Proof.** Taking \( m = |b|^T \), we eventually arrive to the equality
\[
\nabla \sigma(z)|b|^T_\lambda = (b_\lambda - b_{\lambda + 1}z^{-1}) \wedge \cdots \wedge (b_{r-1+\lambda} - b_{r-2+\lambda}z^{-1}),
\]
by just imitating the first few steps of the proof of Lemma 3.4. In turn, the left hand side of (17) can be written as
\[
(-1)^r z^{-r}(b_{\lambda - 1} - b_{\lambda}z) \wedge \cdots \wedge (b_{r-1+\lambda} - b_{r-1+\lambda}z) = (-1)^r z^{-r}\nabla \sigma(z)|b|^T_\lambda = \nabla \sigma(z)|b|^T_\lambda,
\]
i.e., using that \( \nabla \sigma(z) \in HS(\wedge M_0) \):
\[
\nabla \sigma(z)(b_{\lambda - 1} \wedge \cdots \wedge b_{r-1+\lambda}) = (-1)^r z^{-r}\nabla \sigma(z)|b|^T_\lambda,
\]
where the last equality holds because of the hypothesis \( \lambda_t > 0 \).

Recall Definition 2.7. The next easy lemma identifies the transpose \( \sigma^T(z) \) of \( \sigma(z) \) relating it with the Schubert derivations on \( \wedge M_0^* \).

**3.6 Lemma.** Let \( \sum_{i \geq 0} s_i^T z^{-1} := \sigma(z)^T \in \text{End}_t(\wedge M_0^*) \). Then \( \sigma^T(z) b_i = \beta i \).

**Proof.** In fact, for all \( k \geq 0 \):
\[
\sigma^T(z) b_i = \beta_i(\sigma(z) b_k) = \beta_i(b_{k-j}) = \delta_{i,k-j} = \beta_i(\beta_{i+j}(b_k))
\]
which proves the claim.

## 4 Proof of Theorem 0.1

The starting point is the following well-known criterion ([3, Section 4] or, in an infinite-dimensional context, [19, Proposition 7.2]).

**4.1 Proposition.** An element \( m \in \wedge M_0 \) belongs to \( \mathcal{S} \) if and only if the equality
\[
\sum_{i \geq 0} (b_i \wedge m) \otimes [\beta_{i}m] = 0
\]
holds in \( \wedge^{i+1} M_0 \).\wedge^{i+1} M_0.

**Proof.** See [12, Theorem 4.1.4].

Recall from Section 3.1 and Lemma 3.6 that
\[
\sum_{i \geq 0} b_i z^i = \sigma(z)b_0 \quad \text{and} \quad \sum_{i \geq 0} \beta_i z^{-i} = z^{-1}\sigma(z)^T \beta_0.
\]
Equation (18) can be rewritten, imitating [3, 19], in the equivalent form
\[
\Res(z)(\sigma(z)b_0 \wedge m) \otimes \sigma(z)^T \beta_0 \cdot m = 0,
\]
i.e. \( m \in \mathcal{S} \) if and only it satisfies equation (19). We have:

**4.2 Proposition.** The following equality holds in \( \wedge^{r+1} M_0 \):
\[
\sigma(z)b_0 \wedge m = (-1)^r z^r \sigma(z)\nabla \sigma(z)b_0 \wedge \nabla \sigma(z)m.
\]
Proof. First of all
\[ \sigma_+(z) b_0 \wedge m = \sigma_+(z)(b_0 \wedge \mathcal{V}_+(z)m), \]  
(21)
because of integration by parts (11). Lemma 3.4 applied to $\mathcal{V}_+(z)m$ gives, after simplification:
\[ b_0 \wedge \mathcal{V}_+(z)m = (-1)^r z^r \mathcal{V}_+(z)(b_0 \wedge \mathcal{V}_+m). \]
Substituting in the last side of (21) gives (20), as desired. \[ \square \]

4.3 Proposition. The following equality holds in $\mathcal{V}_+^{-1} M_0$:
\[ (z^{-1} \sigma_-(z)^T b_0)_m = (-1)^{r-1} z^{-1} \mathcal{V}_+(z) / \mathcal{V}_+^{-1}(b_0, \sigma_-(z)m). \]
(22)

Proof. Let $\gamma \in \mathcal{V}_+^{-1} M_0$ be arbitrarily chosen. Then
\[ \gamma(z^{-1} \sigma_-(z)^T b_0, m) = z^{-1}(\sigma_-(z)^T b_0 \wedge \gamma) m \]  
(by definition (7))
\[ = z^{-1} \sigma_-(z)^T(\beta_0 \wedge \mathcal{V}_-(z)^T \gamma) m \]  
(integration by parts (11))
\[ = z^{-1}(\beta_0 \wedge \mathcal{V}_-(z)^T \gamma) \sigma_-(z) m \]  
(definition of $\sigma_-(z)^T$)
\[ = z^{-1} \mathcal{V}_-(z)^T \gamma(\beta_0, \mathcal{V}_-(z)m) \]  
(definition $\mathcal{V}_-(z)^T$)
\[ = z^{-1} \gamma(\mathcal{V}_-(z)^T (\beta_0, \mathcal{V}_-(z)m)) \]  
(definition of $\mathcal{V}_-(z)^T$)
whence the equality
\[ z^{-1} \sigma_-(z)^T \beta_0 \wedge m = z^{-1} \mathcal{V}_-(z)(\beta_0, \sigma_-(z)m). \]

Notice now that $\beta_0, \mathcal{V}_-(z)m$ is a linear combination of elements $[b]_\lambda^{\alpha-1}$ associated to partitions of length exactly $r - 1$. Thus we can apply Lemma 3.5 to get
\[ z^{-1} \sigma_-(z)^T \beta_0 \wedge m = z^{-1} \mathcal{V}_-(z)(\beta_0, \mathcal{V}_-(z)m) = (-1)^{r-1} z^{-1} \mathcal{V}_-(z) / \mathcal{V}_+^{-1}(b_0, \mathcal{V}_-(z)m). \]  
\[ \square \]

4.4 Substitution of expressions (22) and (21) into (19) concludes the proof of Theorem 0.1. \[ \square \]

5 The Grassmann Cone in a Polynomial Ring

5.1 Notation and convention as in Section 1. Let $\mathcal{V}^r M_0$ be the $\mathcal{B}_r$-module structure on $\mathcal{V}^r M_0$ given by:
\[ e_i [b]_\lambda^\alpha := \mathcal{V}_i [b]_\lambda^\alpha, \]
which turns $e_i \in \mathcal{B}_r$ into an eigenvalue of $\mathcal{V}_i$. Accordingly, we have
\[ \mathcal{V}_+(z) [b]_\lambda^\alpha = E_r(z) [b]_\lambda^\alpha \]
and then
\[ \sigma_+(z) [b]_\lambda^\alpha = \sigma_+(z) E_r(z) \sum_{n \geq 0} h_n z^n [b]_\lambda^\alpha = \]
\[ = \sigma_+(z) \mathcal{V}_+(z) \sum_{n \geq 0} h_n z^n [b]_\lambda^\alpha = \sum_{n \geq 0} h_n [b]_\lambda^\alpha z^n. \]

Thus $h_n$ is the eigenvalue of $\sigma_+$ seen as endomorphism of $\mathcal{V}^r M_r$, i.e. $\sigma_+ [b]_\lambda^\alpha = h_n [b]_\lambda^\alpha$. In particular, using (13) the homomorphism of abelian groups $\Phi_r : \mathcal{B}_r \rightarrow \mathcal{V}^r M_0$ given by
\[ \Delta \lambda(H_r) \rightarrow \Delta \lambda(H_r) [b]_0^\alpha = \Delta \lambda(\sigma_+(z)) [b]_0^\alpha = [b]_\lambda^\alpha, \]
(23)
is an isomorphism, as it maps a $\mathcal{Z}$-basis of $\mathcal{B}_r$ to a $\mathcal{Z}$-basis of $\mathcal{V}^r M_0$. \[ \square \]
5.2 Definition. Let \( \sigma_-(z), \sigma_+(z) : B_r \to B_r[z^{-1}] \) be defined as:

\[
(\sigma_-(z) \Delta_1(H_r))|b_0^\sigma = \sigma_-(z) \Delta_1(H_r)|b_0^\sigma = \sigma_-(z)|b \lambda^\sigma,
\]

and

\[
(\sigma_+(z) \Delta_1(H_r))|b_0^\sigma = \sigma_+(z) \Delta_1(H_r)|b_0^\sigma = \sigma_+(z)|b \lambda^\sigma.
\]

The \( \sigma_-(z) \)-image of \( h_n = \Delta_{(n)}(H_r) \) should in principle depend on the integer \( r \). However this is not the case.

5.3 Proposition. For all \( r \geq 1 \), the following equalities hold in the ring \( B_r[z^{-1}] \):

\[
\sigma_-(z)h_n = \sum_{j=0}^n h_{n-i}z^{-i} \quad \text{and} \quad \sigma_+(z)h_n = h_n - h_{n-1}z^{-1}.
\]  

(24)

Proof. We have:

\[
(\sigma_-(z)h_n)|b_0^r = \sigma_-(z)|(h_n|b_0^r) \quad \text{(definition of} \quad \sigma_-(z)h_n) \]

\[
= \sigma_-(z)(|b_0^{r-1} \wedge b_{r-1+n}) \quad \text{(writing} \quad |b_0^{r-1} \wedge b_{r-1+n}) \]

\[
= \sigma_-(z)|b_0^{r-1} \wedge \sigma_-(z)|b_{r-1+n} \quad \text{(since} \quad \sigma_-(z) \in \text{HS}(\wedge M_0)) \]

\[
= \sum_{j=0}^{r-1+n} (|b_0^{r-1} \wedge b_{r-1+n-1}z^{-1}) \quad \text{(apply} \quad (9) \quad \text{and the definition of} \quad \sigma_-(z)) \]

\[
= (\sum_{j=0}^n h_{n-j}z^{-j})|b_0^r,
\]

which proves the first equality in (24). To prove the second equality of (24), we can argue either by observing that \( \sigma_+(z) \) is the inverse of \( \sigma_-(z) \) or again by direct computation:

\[
(\sigma_+(z)h_n)|b_0^r = \sigma_+(z)|(h_n|b_0^r) = \sigma_+(z)|b_0^r \wedge \sigma_+(z)|b_{r-1+n} = |b_0^{r-1} \wedge b_{r-1+n-1}z^{-1}) = (h_n - h_{n-1}z^{-1})|b_0^r.
\]

5.4 Proposition. We have:

\[
b_0 \wedge \sigma_r|b_\lambda^\sigma = |b_\lambda^{r+1} \Delta_0(H_{r+1})|b_0^r
\]

(25)

Proof. Indeed

\[
b_0 \wedge \sigma_r|b_\lambda^\sigma = b_0 \wedge \sigma_r(b_1 + b_1 + \cdots + b_{r+1}) \quad \text{(definition of} \quad |b_\lambda^\sigma) \]

\[
= b_0 \wedge b_1 + b_2 + \cdots + b_{r+1} \quad \text{(definition of} \quad \sigma_r) \]

\[
= |b_\lambda^{r+1} \quad \text{(definition of} \quad |b_\lambda^{r+1})
\]

In the \( B_{r+1} \)-module \( \wedge^{r+1} M_r \) we have then equality (25) (due to \( \phi_{r+1}(\Delta_0(H_{r+1}) = |b_\lambda^{r+1} \) by (23)).

In other words the expression of \( \sigma_-(z)h_n \) and \( \sigma_+(z)h_n \) in the ring \( B_r[z^{-1}] \) does not depend on the integer \( r \).

5.5 To ease notation, let us agree to denote simply by \( \text{Res}(f) \) the coefficient of \( X^{-1} \) of a formal Laurent series \( f \in \mathbb{Z}(X^{-1}) \). To prove Theorem 5.7 below, we need a powerful result due to Laksov and Thorup [22, Theorem 0.1.(2)]. Let us introduce a few new pieces of notation. Let

\[
p_r(X) = X^r E_r \left( \frac{1}{X} \right) = X^r - e_1 X^{r-1} + \cdots + (-1)^r e_r \in B_r[X]
\]
be the generic polynomial of degree \( r \). If \( f := a_0 X^\lambda + a_1 X^{\lambda-1} + \cdots + a_n \in B_r[X] \) is any polynomial of degree \( \leq \lambda \), then for all \( \ell \geq 0 \) an easy computation shows that

\[
\text{Res} \left( \frac{X^{\lambda-1} f(X)}{p_r(X)} \right) := \text{Res} \left( \frac{X^{\lambda-1} f(X)}{X^\lambda} \right) \left( 1 + \frac{h_1}{X} + \frac{h_2}{X^2} + \cdots \right) = \sum_{j=0}^\lambda a_j h_{\ell+r-j}.
\]  

(26)

In agreement with [22], the residue of \( f_0, f_1, \ldots, f_{r-1} \in B_r[X] \) is, by definition:

\[
\text{Res}(f_0, f_1, \ldots, f_{r-1}) = \begin{vmatrix}
\text{Res}(f_0) & \text{Res}(f_1) & \cdots & \text{Res}(f_{r-1}) \\
\text{Res}(X f_0) & \text{Res}(X f_1) & \cdots & \text{Res}(X f_{r-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Res}(X^{r-1} f_0) & \text{Res}(X^{r-1} f_1) & \cdots & \text{Res}(X^{r-1} f_{r-1})
\end{vmatrix}.
\]

(27)

We shall use the following

5.6 **Theorem ([22]).** Let \( f_0, f_1, \ldots, f_{r-1} \in B_r[X] \). Then

\[
\begin{aligned}
f_0(\sigma_1) b_0 \wedge f_1(\sigma_1) b_0 \wedge \cdots \wedge f_{r-1}(\sigma_1) b_0 \\
= \text{Res} \left( f_{r-1}(X) \frac{f_{r-2}(X)}{p_r(X)} \cdots \frac{f_0(X)}{p_r(X)} \right) b_0 \wedge b_1 \wedge \cdots \wedge b_{r-1} \\
= \det \left( \text{Res} \left( \frac{X^{\lambda-1} f(X)}{p_r(X)} \right) \right)_{1 \leq i, j \leq r}.
\end{aligned}
\]

(28)

5.7 **Theorem.** The operators \( \sigma_\cdot(z), \X_\cdot(z) : B_r \rightarrow B_r[z^{-1}] \) commute with taking Schur determinants, i.e.: 

\[
\X_\cdot(z) \Delta_\lambda(H_r) = \Delta_\lambda(\X_\cdot(z) H_r) \quad \text{and} \quad \sigma_\cdot(z) \Delta_\lambda(H_r) = \Delta_\lambda(\sigma_\cdot(z) H_r).
\]

(29)

**Proof.** Let us prove the first of equalities (29). We have

\[
(\X_\cdot(z) \Delta_\lambda(H_r)) [b]\bigg|^r_0 = \X_\cdot(z) [b]_\lambda^r = \X_\cdot(z) b_\lambda \wedge \cdots \wedge \X_\cdot(z) b_{r-1}\lambda_1
\]

\[
= f_0(\sigma_1) b_0 \wedge f_1(\sigma_1) b_0 \wedge \cdots \wedge f_{r-1}(\sigma_1) b_0,
\]

where \( f_1(\sigma_1) \) stands for \( f_1(X) = X^{\lambda r-1} - X^{\lambda r-1} z^{-1} \) evaluated at \( X = \sigma_1 \). By Theorem 5.6 and formula (26):

\[
f_0(\sigma_1) b_0 \wedge f_1(\sigma_1) b_0 \wedge \cdots \wedge f_{r-1}(\sigma_1) b_0 = \text{Res} \left( f_{r-1}(X) \frac{f_{r-2}(X)}{p_r(X)} \cdots \frac{f_0(X)}{p_r(X)} \right) b_0 \wedge b_1 \wedge \cdots \wedge b_{r-1} \\
= \det(h_{\lambda_{j-1}} - h_{1+r-z^{-1}})_{1 \leq i, j \leq r} = \Delta_\lambda(\sigma_\cdot(z) H_r).
\]

To prove the second equality of (29) we exploit the fact that \( \sigma_\cdot(z) \) and \( \X_\cdot(z) \) are one inverse of the other. Then

\[
\sigma_\cdot(z) \Delta_\lambda(H_r) = \sigma_\cdot(z) \Delta_\lambda(\X_\cdot(z) H_r) = \sigma_\cdot(z) \Delta_\lambda(\sigma_\cdot(z) H_r) = \Delta_\lambda(\sigma_\cdot(z) H_r).
\]

\[
\square
\]

5.8 **Lemma.** For all \( \lambda \in P_r \) the following equality holds in \( \bigwedge^{r-1} M_0 \)

\[
\X_{r+1}(\beta_0, \lambda) = \Delta_\lambda(H_{r-1}) [b]^r_0.
\]

(30)

**Proof.** If \( \ell(\lambda) = r \) then \( \lambda_r > 0 \) and so \( \beta_0, \lambda [b]^r_\lambda = 0 \). So, both sides of (30) vanish. If \( \ell(\lambda) < r \) instead, one may write

\[
[b]^r_\lambda = b_0 \wedge [b]^{r-1}_{\lambda + (1^r-1)},
\]

where \( \lambda + (1^r-1) = (\lambda_1 + 1, \ldots, \lambda_r - 1 + 1) \). Thus

\[
\X_{r+1}(\beta_0, \lambda) = \X_{r+1}(b_0 \wedge [b]^{r-1}_{\lambda + (1^r-1)}) = \X_{r+1}(b_0 \wedge [b]^{r-1}_{\lambda + (1^r-1)}) = [b]^{r-1}_\lambda = \Delta_\lambda(H_{r-1}) [b]^r_0
\]

as claimed.

\[
\square
\]
5.9 Proposition. The equality below holds in $\bigwedge^{r-1} M_0$:

$$\vartheta_{r+1}(\beta_0, \sigma_-(z)|b^\lambda_\alpha) = \Delta_\lambda(\sigma_-(z)H_{r-1})|b^r_{0-1}$$

Proof. If one defines $\Delta_{\lambda_\alpha}(\sigma_-(z)H_0)$ to be $z^{-n}$, the equality holds for $r = 1$. To check the formula in the remaining cases let us preliminary observe that $\Delta_\lambda(\sigma_-(z)H_r)$ is a linear combination $\sum_{\mu \in \Lambda} a_\mu(z^{-1})\Delta_\mu(H_r)$, whose coefficients $a_\mu(z^{-1}) \in \mathbb{Z}[z^{-1}]$ do not depend on the chosen $r > 1$. Thus

$$\vartheta_{r+1}(\beta_0, \sigma_-(z)|b^\lambda_\alpha) = \vartheta_{r+1}(\beta_0, \sigma_-(z)(\Delta_\lambda(H_r)|b^\mu_0))$$

$$= \vartheta_{r+1}(\beta_0, \sigma_-(z)\Delta_\lambda(H_r)|b^\mu_0)$$

$$= \vartheta_{r+1}\beta_0, \sum_{\mu \in \Lambda} a_\mu(z^{-1})\Delta_\mu(H_r)|b^\mu_0$$

$$= \sum_{\mu \in \Lambda} a_\mu(z^{-1})\vartheta_{r+1}(\beta_0, \sigma_-(z)\Delta_\mu(H_r)|b^\mu_0)$$

$$= \sum_{\mu \in \Lambda} a_\mu(z^{-1})\Delta_\lambda(H_{r-1})|b^r_{0-1} = \Delta_\lambda(\sigma_-(z)H_{r-1})|b^r_{0-1}. \quad \blacksquare$$

5.10 Let $\Gamma_r(z): B_r \to B_{r+1}((z))$ and $\Gamma_r^*(z): B_r \to B_{r-1}((z))$ defined by:

$$\Gamma_r(z)\Delta_\lambda(H_r) := \frac{1}{E_{r+1}(z)}\vartheta_r(z)\Delta_\lambda(H_{r+1}), \quad (31)$$

and

$$\Gamma_r^*(z)\Delta_\lambda(H_r) := E_{r-1}(z)\Delta_\lambda(\sigma_-(z)H_{r-1}). \quad (32)$$

Expression (31) can be also written in the form

$$\Gamma_r(z)\Delta_\lambda(H_r) = \frac{1}{E_{r+1}(z)}\Delta_\lambda(\vartheta_r(z)H_{r+1}),$$

by virtue of Theorem 5.7, according which $\vartheta_r(z)\Delta_\lambda(H_{r+1}) = \Delta_\lambda(\vartheta_r(z)H_{r+1})$ for all $\lambda \in \mathcal{P}_{r+1}$. Similarly, the equality $\Delta_\lambda(\sigma_-(z)H_{r-1}) = \sigma_-(z)\Delta_\lambda(H_{r-1})$ surely holds for all $\lambda \in \mathcal{P}_{r-1}$. However, if $f(\lambda) = r$, in general $\Delta_\lambda(\sigma_-(z)H_{r-1}) \neq \sigma_-(z)\Delta_\lambda(H_{r-1})$, because $\Delta_\lambda(\sigma_-(z)H_{r-1}) \neq 0$ in spite of the vanishing of $\Delta_\lambda(H_{r-1})$. For example, if $r = 1$, $\sum_{n \geq 0} h_n z^n = (1 - e_1 z)^{-1} = \sum_{n \geq 0} e_1^n z^n$, i.e. $h_n = h^n$. This implies

$$\Delta_{1,1}(H_1) = \begin{vmatrix} h_1 & 1 \\ h_2 & h_1 \end{vmatrix} = h_1^2 - h_2 = h_2 = h_1^2 - h_1^2 = 0.$$  

On the other hand

$$\Delta_{1,1}(\sigma_-(z)H_1) = \begin{vmatrix} h_1 + \frac{1}{z} & 1 \\ \frac{h_1}{z} + \frac{1}{z} & h_1 + \frac{1}{z} \end{vmatrix} = \left(h_1 + \frac{1}{z}\right)^2 - h_2 - \frac{h_1}{2} = \frac{h_1}{z} \neq 0.$$  

5.11 Proof of Theorem 0.2. According to Theorem 0.1, it follows that $\sum_{\lambda \in \mathcal{P}_{r,n}} a_\lambda|b^r_\lambda \in \mathbb{S}_r$ if and only if

$$\text{Res}_{\lambda \mu \in \mathcal{P}_{r}} \sum_{\lambda \mu \in \mathcal{P}_{r}} a_\lambda a_\mu \vartheta_r(z)\vartheta_{r+1}(\beta_0, \sigma_-(z)|b^\lambda_{\alpha}) \otimes \mathbb{S}_r(z)\vartheta_\nu(z)|b_0 \otimes \vartheta_r|b^r_{\mu-1}) \quad (33)$$
vanishes in \( (\wedge^r M_{r-1} \otimes \wedge^i M_{r+1}) (z) \). Since \( \varpi_{r+1} (\beta_0 . \sigma_+ (z) [b]_0^{r+1} ) \in \wedge^{r-1} M_{r-1} \), it is an eigenvector of \( \varpi_r (z) \) corresponding to the eigenvalue \( E_{r-1} (z) \). Similarly, \( \varpi_r (z) [b]_0 \otimes \varpi_r (b) [b]_0^{r+1} \) belongs to \( \wedge^{r+1} M_0 \), which is an eigenmodule of \( \sigma_+ (z) \) corresponding to the eigenvalue \( 1/E_{r+1} (z) \). Thus, expression (33) can be rewritten as

\[
0 = \left( \text{Res}_{\lambda, \mu} \sum_{\lambda, \mu \in \mathcal{P}_r} a_\lambda a_\mu E_{r-1} (z) (\varpi_{r+1} (\beta_0 . \sigma_+ (z) [b]_0^{r+1} )) \otimes \frac{1}{E_{r+1} (z) (\varpi_r (z) (b_0 \otimes \varpi_r (b)) [b]_0^{r+1} )} =
\]

\[
= \left( \text{Res}_{\lambda, \mu} \sum_{\lambda, \mu \in \mathcal{P}_r} E_{r-1} (z) \Delta (\sigma_+ (z) H_{r-1}) [b]_0^{r-1} \otimes \frac{1}{E_{r+1} (z) (\varpi_r (z) (b_0 \otimes \varpi_r (b)) [b]_0^{r+1} )} =
\]

\[
= \left( \text{Res}_{\lambda, \mu} \sum_{\lambda, \mu \in \mathcal{P}_r} E_{r-1} (z) \Delta (\sigma_+ (z) H_{r-1}) [b]_0^{r-1} \otimes \frac{1}{E_{r+1} (z) (\varpi_r (z) (b_0 \otimes \varpi_r (b)) [b]_0^{r+1} )} \right) [b]_0^{r-1} \otimes [b]_0^{r+1} ,
\]

where in last equality we used Propositions 5.4 and 5.9. Let us now identify the tensor product \( B_{r-1} \otimes B_{r+1} \) with the polynomial ring in \( 2r \) indeterminates:

\[
B_{r-1} \otimes B_{r+1} = \mathbb{Z} [e_1', \ldots, e_{r-1}', e_{r}''', \ldots, e_{r+1}'].
\]

Since \( [b]_0^{r-1} \otimes [b]_0^{r+1} \) is a basis of \( \wedge^{r-1} M_{r-1} \otimes \wedge^{r+1} M_{r+1} \) over \( B_{r-1} \otimes B_{r+1} \) and using the definitions (31) and (32) of \( E_r (z) \) and \( E_r' (z) \), we obtain (5).

5.12 Let \( E_r' (z) = 1 - e_1' z + \cdots + (-1)^{r+1} e_{r-1}' z^r \in B_{r-1} [z] \) and \( E_r'' (z) = 1 - e_1'' z + \cdots + (-1)^{r+1} e_{r+1}'' z^r \in B_{r+1} [z] \). Similarly, let

\[
H_r' (z) = \sum_{n \geq 0} h_n' z^n \quad \text{and} \quad H_r'' (z) = \sum_{n \geq 0} h_n'' z^n
\]

be the inverses of \( E_r' (z) \) and \( E_r'' (z) \) in \( B_{r-1} [z] \) and \( B_{r+1} [z] \) respectively. Then formula (5) reads

\[
\text{Res}_{\lambda, \mu} \frac{E_r' (z)}{E_r'' (z)} \sum_{\lambda, \mu} a_\lambda a_\mu \Delta (\sigma_+ (z) H_r' (z)) \cdot \varpi_r (z) (b_0 \otimes \varpi_r (b)) \Delta (H_r'' (z)) = 0.
\]

6 An Example

To show how formula (34) works, in this section we proceed to determine all the polynomials

\[
p(H) = a_0 + a_1 h_1 + a_2 h_2 + a_1 \Delta (H_2) + a_2 \Delta (H_2) + a_2 \Delta (H_2) + a_2 \Delta (H_2) \in B_2.
\]

such that \( p(H) [b]_0^2 \) belongs to the Grassmann cone \( \mathcal{G}_2 \) of \( \wedge^2 M_0 \). Useless to say, we expect to find the same expression of the Klein quadric hypersurface of \( \mathbb{P}^5 \). To this purpose we need a few preliminaries to speed up computations, which by the way could be carried out directly.

6.1 Consider the two sub-modules \( \sigma_n M_0 = \wedge (b_n \otimes \wedge M_0) \) and \( M_{0,n} := \wedge (b_n \otimes \wedge M_0) \) of \( \wedge M_0 \).

They fit in the exact sequence

\[
0 \longrightarrow \sigma_n M_0 \longrightarrow M_0 \longrightarrow M_{0,n} \longrightarrow 0,
\]

where \( t_n \) is the truncation which maps \( b_n \) to itself if \( 0 \leq i \leq n-1 \) and to zero otherwise. Hence \( \wedge M_0 \) is the submodule \( \wedge (b_n \otimes \wedge M_0) \) of \( \wedge M_0 \). It can be seen itself as the epimorphic image of the truncation \( t_n : \wedge M_0 \rightarrow \wedge M_0 \), which maps \( [b]_n \) to itself if \( \lambda \in \mathcal{P}_r \) and to zero otherwise. Again, we have an exact sequence

\[
0 \longrightarrow \sigma_n M_0 \longrightarrow \wedge M_0 \longrightarrow \wedge M_0 \longrightarrow 0,
\]
where $\sigma_i M_0 \cap \wedge^{r-1} M_0$ is the submodule of $\wedge^r M_0$ generated by all $b_{i_1} \wedge \ldots \wedge b_{i_r} \in \wedge^r M_0$ such that $1 \leq i_j \leq n$ for at least one $1 \leq j \leq r$.

Let $I_{r,n}$ be the kernel of the composition $t'_r \circ \varphi_r : B_r \to \wedge^r M_{0,n}$, where $\varphi_r : B_r \to \wedge^r M_0$ is the isomorphism (23). It is a simple exercise to check that $h_{n-r+1} \in I_{r,n}$ for all $j \geq 0$. Indeed, a simple argument (see [15, Chapter 5] or [11] for details) shows that $I_{r,n} = (h_{n-r+1}, \ldots, h_n)$.

Let

$$\tau_n : B_r \to B_{r,n} := \frac{Z[e_1, \ldots, e_r]}{(h_{n-r+1}, \ldots, h_n)}$$

be the canonical projection. Then $B_{r,n} = \bigoplus_{\lambda \in P_{r,n}} Z \Delta_\lambda (H_{r,n})$, where by $H_{r,n}$ we have denoted the sequence $(h_{1+r,n})_{j \geq 0}$ of $B_{r,n}$ and then $\Delta_\lambda (H_{r,n}) = \Delta_\lambda (H_r) + I_{r,n}$. Clearly, $\Delta_\lambda (H_{r,n}) | b_0^r = \Delta_\lambda (H_r) | b_0^r$ if $\lambda \in P_{r,n}$. Define $\sigma_j (h_{1+r,n}) = \sigma_j h_n + I_{r,n}$ (resp. $\sigma_j (h_{1+r,n}) = \sigma_j h_n + I_{r,n}$).

Then Theorem 0.2 has the following

**6.2 Corollary.** A tensor $\sum_{\lambda \in P_{r,n}} a_\lambda | b_\lambda^r$ is decomposable if and only if

$$\text{Res}_2 \sum_{\lambda \in P_{r,n}} a_\lambda \alpha \Delta_\lambda (\sigma_j (z) H_{r-1,n}) \otimes \Delta_\lambda (\sigma_j (z) | H_{r+1,n}) = 0$$

in the tensor product $(B_{r-1,n} \otimes Z B_{r+1,n})(\{z\})$.

**Proof.** In fact by Theorem 0.2, $\sum_{\lambda \in P_{r,n}} a_\lambda | b_\lambda^r \in \mathcal{G}_r$ if and only if

$$0 = \text{Res}_2 \sum_{\lambda \in P_{r,n}} E_{r-1} \Delta_\lambda (\sigma_j (z) H_{r-1,n}) | b_0^r \otimes \frac{1}{E_{r+1} (z)} \Delta_\lambda (\sigma_j (z) | H_{r+1,n}) | b_0^{r+1} = \text{Res}_2 \sum_{\lambda \in P_{r,n}} E_{r-1} \Delta_\lambda (\sigma_j (z) H_{r-1,n}) | b_0^r \otimes \frac{1}{E_{r+1} (z)} \Delta_\lambda (\sigma_j (z) | H_{r+1,n}) | b_0^{r+1}.$$ 

Moreover formula (34) and Corollary 6.2 easily imply:

**6.3 Corollary.** The tensor $\sum_{\lambda \in P_{r,n}} a_\lambda | b_\lambda^r \in \wedge^r M_0$ is decomposable if and only if

$$\text{Res}_2 \sum_{\lambda \in P_{r,n}} a_\lambda \alpha \Delta_\lambda (\sigma_j (z) H_{r-1,n}) \cdot \Delta_\lambda (\sigma_j (z) | H_{r+1,n}) = 0.$$ 

**6.4** Going back to our original purpose of finding all $p(H_2) \in B_2$ such that $p(H_2) | b_0^r \in \mathcal{G}_2$, where $p(H_2)$ is like in (35). We use the $B_2$-module structure of $\wedge^2 M_2 := B_2 \otimes \wedge^2 M_0$. Since $p(H_2) | b_0^r \in \wedge^2 M_2$, the $B_2$-module structure of $\wedge^2 M_2$, factors through that of $B_{2,4}$, i.e. $p(H_2) | b_0^r = p(H_{2,4}) | b_0^2$. Then we have

$$B_{1,4} := \frac{B_1}{(h_4)} \cong \frac{Z[x]}{(x^4)}$$

and

$$B_{3,4} := \frac{B_3}{(h_2, h_3, h_4)} \cong \frac{Z[y]}{(y^4)},$$

where $x = e_1 + (h_4)$ and $y = e_1 + (h_2, h_3, h_4)$. In particular, $h_1 + (h_4) = x^4$ and $h_3 + (h_2, h_3, h_4) = y$. Thus

$$p(\sigma_j (z) H_{1,4}) = a_0 + a_1 \sigma_j (z) h_1 + a_2 \sigma_j (z) h_2 + a_{11} \Delta_{11} (\sigma_j (z) H_1)$$

$$+ a_{21} \Delta_{21} (\sigma_j (z) H_1) + \Delta_{22} (\sigma_j (z) H_1) + I_{1,4}$$

$$= a_0 + a_1 (x + z^4) + a_2 (x^2 + x z^3 + z^{-2})$$

from which

$$\gamma^*_r (z) p(H_{2,4}) = E_1 (z) \sum_{\lambda \in P_{2,4}} a_\lambda \Delta_\lambda (\sigma_j (z) H_{1,4}) + I_{1,4}$$

$$= \frac{1 - x z}{z^2} \left( a_0 + a_1 \left( \frac{x + 1}{z} \right) + a_2 \left( x^2 + x + \frac{1}{z^2} \right) \right) +$$

$$+ a_{11} \frac{x}{z} + a_{21} \left( \frac{x^2}{z^2} + \frac{x}{z} \right) + a_{22} \frac{x^2}{z^2}.$$

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On the other hand
\[
p(\varpi(z)H_{3,4}) = \sum_{\lambda \in \mathcal{P}_{2,4}} a_\lambda \Delta_\lambda(\varpi(z)H_{3,4})
\]
\[
= \sum_{\lambda \in \mathcal{P}_{2,4}} a_{(\lambda_1, \lambda_2)} \det(h_{\lambda_1-l+1} - h_{\lambda_1+j+1}z^{-1}) + I_{3,4}
\]
and so the equality
\[
\Gamma_2(z)p(H_{2,4}) = \frac{z^2}{E_3(z)} \sum_{\lambda \in \mathcal{P}_{2,4}} a_{(\lambda_1, \lambda_2)} \det(h_{\lambda_1-l+1} - h_{\lambda_1+j+1}z^{-1}) + I_{3,4}
\]
holds in $B_{3,4}((z))$. Since
\[
\frac{1}{E_3(z)} + I_{3,4} = 1 + h_1z + I_{3,4} = 1 + yz,
\]
formula (36) becomes:
\[
\Gamma_2(z)p(H_{2,4}) = z^2(1 + yz) \left[ a_0 + a_1 \left( y - \frac{1}{z} \right) - a_2 \frac{y}{z} + a_{11} \left( y^2 - \frac{y}{z} + \frac{1}{z^2} \right) \right] + a_{21} \left( \frac{y}{z^2} - \frac{y^2}{z} \right) + a_{22} \frac{y^2}{z^2}.
\]
A few computations, carried out by means of the CoCoA software [1], eventually tell
\[
\text{Res}_z(\Gamma_2(z)p(H_{2,4})) \cdot (\Gamma_2^* z)p(H_{2,4})) = (-a_{11}a_2 + a_1a_{21} - a_0a_{22})x^3 + (a_{11}a_2 - a_1a_{21} + a_0a_{22})x^2y
\]
\[
- (a_{11}a_2 - a_1a_{21} + a_0a_{22})xy^2 + (a_{11}a_2 - a_1a_{21} + a_0a_{22})y^3
\]
\[
= (a_{11}a_2 - a_1a_{21} + a_0a_{22})(y^3 - y^2x + x^2y - x^3).
\]
So, the latter expression is identically zero if and only if the following Plücker equation holds:
\[
a_{11}a_2 - a_1a_{21} + a_0a_{22} = 0. \tag{37}
\]

7 The KP Hierarchy

7.1 Proposition. Let $r \geq 1$ be fixed. Then for all $(i_1, \ldots, i_r) \in \mathbb{N}^r$, the equalities below holds in $B_r$:
\[
\sigma_-(z)(h_{i_1} \cdots h_{i_r}) = \sigma_-(z)h_{i_1} \cdots \sigma_-(z)h_{i_r} \quad \text{and} \quad \varpi_-(z)(h_{i_1} \cdots h_{i_r}) = \varpi_-(z)h_{i_1} \cdots \varpi_-(z)h_{i_r}. \tag{38}
\]

Proof. Let us begin by checking the first of (38). Since $(\Delta_\lambda(H_r) \mid \lambda \in \mathcal{P}_r)$ is a basis of $B_r$, each product of the form $h_{i_1} \cdots h_{i_r}$ is a linear combination $\sum_{\lambda \in \mathcal{P}_r} a_\lambda \Delta_\lambda(H_r)$. Therefore
\[
\sigma_-(z)(h_{i_1} \cdots h_{i_r}) = \sigma_-(z)\left( \sum_{\lambda \in \mathcal{P}_r} a_\lambda \Delta_\lambda(H_r) \right) = \sum_{\lambda \in \mathcal{P}_r} a_\lambda \sigma_-(z)\Delta_\lambda(H_r) = \sum_{\lambda \in \mathcal{P}_r} a_\lambda \Delta_\lambda(\sigma_-(z)H_r),
\]
where last equality is due to Theorem 5.7. In other words:
\[
\sigma_-(z)(h_{i_1} \cdots h_{i_r}) = \sum_{\lambda \in \mathcal{P}_r} a_\lambda \det(\sigma_-(z)h_{\lambda_1-j+1}) = \sigma_-(z)h_{i_1} \cdots \sigma_-(z)h_{i_r}.
\]
The proof for $\varpi_-(z)$ is evidently analogous. \hfill \blacksquare
7.2 Let $B_\infty$ be the polynomial ring $\mathbb{Z}[e_1, e_2, \ldots]$ in infinitely many indeterminates $(e_1, e_2, \ldots)$. It is the projective limit of the rings $B_r$ in the category of graded $\mathbb{Z}$-algebras, in the following sense. For all $s > r$ there are obvious projection maps $(B_s)_w \to (B_r)_w$, defined by $\Delta_\lambda(H_r) \mapsto \Delta_\lambda(H_r)$ if $\lambda \in \mathcal{P}_r$ and to 0 otherwise, for all $\lambda \in \mathcal{P}_s$ such that $|\lambda| = w$. Let $(B_\infty)_w := \lim (B_r)_w$.

Clearly, for all $w \in \mathbb{N}$ there exists $r > 0$ such that $(B_\infty)_w = (B_r)_w$. The ring $B_\infty$ is by definition the direct sum $\bigoplus_{w \geq 0} (B_r)_w$. Let $E_\infty(z) = 1 - e_1 z + e_2 z^2 + \cdots$ and $\sum_{n \geq 0} h_n z^n = (E_\infty(z))^{-1} \in B_\infty[[z]]$. In this case $(h_1, h_2, \ldots)$ are algebraically independent and so

$$B_\infty := \mathbb{Z}[h_1, h_2, \ldots]$$

as well.

7.3 **Corollary.** The maps $\sigma_-(z), \varpi_-(z) : B_\infty \to B_\infty[z^{-1}]$ are ring homomorphisms and are one inverse of the other, when regarded as elements of $[\text{End}_\mathbb{Z}(B_\infty)][z^{-1}]$.

**Proof.** Consider an arbitrary product in $h_{i_1} \cdots h_{i_s} \in B_\infty$ and let $w = i_1 + \cdots + i_s$. There exists a sufficiently large $r > \max(w, s)$ such that $(B_\infty)_w = (B_r)_w$. For such a choice of $r$, we have

$$\sigma_-(z)(h_{i_1} \cdots h_{i_s}) = \sigma_-(z)h_{i_1} \cdots \sigma_-(z)h_{i_s} \quad \text{and} \quad \varpi_-(z)(h_{i_1} \cdots h_{i_s}) = \varpi_-(z)h_{i_1} \cdots \varpi_-(z)h_{i_s}$$

by virtue of Proposition 7.1.

7.4 **Proof of Corollary 0.3.** Each $p \in B_{\infty}$ is a finite linear combination $\sum_{\lambda \in \mathcal{P}} a_{\lambda} \Delta_\lambda(H_\infty)$, where $\mathcal{P} := \cup_{r \geq 1} \mathcal{P}_r$ is the set of all the partitions. Then, by Theorem 0.2, $\Phi_\infty(p)$ corresponds to a decomposable tensor if and only if

$$\text{Res}_z \sum_{\lambda, \mu \in \mathcal{P}} a_{\lambda} a_{\mu} E_\infty(z) \Delta_\lambda(H_\infty) \otimes \frac{1}{E_\infty(z)} \varpi_-(z) \Delta_\mu(H_\infty) = 0. \quad (39)$$

Corollary 7.3 implies the commutation $\Delta_\lambda(\sigma_-(z)H_\infty) = \sigma_-(z)\Delta_\lambda(H_\infty)$, for all $\lambda \in \mathcal{P}$. Thus (39) can be rewritten as

$$0 = \text{Res}_z \sum_{\lambda \in \mathcal{P}} E_\infty(z) a_{\lambda} \sigma_-(z) \Delta_\lambda(H_\infty) \otimes \frac{1}{E_\infty(z)} \sum_{\mu \in \mathcal{P}} a_{\mu} \varpi_-(z) \Delta_\mu(H_\infty)$$

$$= \text{Res}_z E_\infty(z) \sigma_-(z)p \otimes \frac{1}{E_\infty(z)} \varpi_-(z)p. \quad \square$$

7.5 Let $B := B_\infty \otimes \mathbb{Q}$ and define the sequence $X := (x_1, x_2, \ldots)$ through the equality

$$\sum_{n \geq 0} h_n x^n = \exp(\sum_{j \geq 1} x_j z^j),$$

holding in $B[[z]]$, in such a way that each $h_n$ can be regarded as a function of $(x_1, x_2, \ldots)$. Standard calculations show that $h_n$ is a polynomial expression of $(x_1, \ldots, x_n)$, homogeneous of degree $n$ with respect to the weight graduation of $B_\infty$ (for which $h_n$ and $x_n$ have degree $n$). We have, easily:

7.6 **Lemma.** The following equalities hold in the ring $B$ for all $j \geq 1$ and $n \geq 0$:

$$\frac{\partial^j h_n}{\partial x_1^j} = \frac{\partial h_n}{\partial x_j} = h_{n-j}. \quad (40)$$

**Proof.** For all $j \geq 1$:

$$\frac{\partial}{\partial x_j} \sum_{n \geq 0} h_n z^n = \sum_{n \geq 0} \frac{\partial h_n}{\partial x_j} z^n = z^j \exp(\sum_{j \geq 1} x_j z^j) = \sum_{n \geq 0} h_n z^{n+j}.$$
Comparing the coefficient of \( z^n \) in the first and last side gives
\[
\frac{\partial h_n}{\partial x_j} = h_{n-1}.
\]
In particular \( \partial h_n/\partial x_1 = h_{n-1} \). Iterating \( j \)-times \( \partial/\partial x_1 \) gives (40), as desired. 

Let
\[
\Gamma_\infty(z) \Delta\lambda(H_\infty) := \frac{1}{E_\infty(z)} \varpi_\infty(z) \Delta\lambda(H_\infty) \in B_\infty((z))
\]
and
\[
\Gamma_\infty^*(z) \Delta\lambda(H_\infty) := E_\infty(z) \varpi_\infty(z) \Delta\lambda(H_\infty) \in B_\infty((z)).
\]

Define \( \Gamma(z), \Gamma^*(z) \) to be, respectively, \( \Gamma_\infty(z) \odot 1_Q : B \to B((z)) \) and \( \Gamma_\infty^*(z) \odot 1_Q : B \to B((z)) \). The proof of Corollary 0.4 is an immediate consequence of the following:

7.7 Theorem. We have:
\[
\Gamma(z) = \exp \left( \sum_{i \geq 1} x_i z^i \right) \exp \left( - \sum_{i \geq 1} \frac{1}{i z} \frac{\partial}{\partial x_i} \right)
\]
\[(41)\]
and
\[
\Gamma^*(z) = \exp \left( - \sum_{i \geq 1} x_i z^i \right) \exp \left( \sum_{i \geq 1} \frac{1}{i z} \frac{\partial}{\partial x_i} \right).
\]
\[(42)\]

Proof. Since
\[
\frac{1}{E_\infty(z)} = \sum_{n \geq 0} h_n z^n = \exp(\sum_{i \geq 1} x_i z^i)
\]
it follows that \( E_\infty(z) = \exp(-\sum_{i \geq 1} x_i z^i) \) and then the first factors involved on the left hand side of (41) and (42) are explained. Let us now observe that:
\[
\varpi_\infty(z) h_n = h_n - \frac{h_{n-1}}{z} = \left( 1 - \frac{1}{z} \frac{\partial}{\partial x_1} \right) h_n.
\]
Evaluating the well-known identity \( 1 - t = \exp \left( - \sum_{n \geq 1} t^n \right) \) at \( t = z^{-1} \frac{\partial}{\partial x_1} \), and using (40), we have
\[
\varpi_\infty(z) h_n = \exp \left( - \sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i} \right) h_n = \exp \left( - \sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i} \right) h_n.
\]\[(43)\]
Now, we observe that
\[
\exp \left( - \sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i} \right) : B \to B[z^{-1}]
\]
is a ring homomorphism, because it is the exponential of the first order differential operator
\[
- \sum_{i \geq 1} \frac{1}{i z} \frac{\partial}{\partial x_i}.
\]
Thus
\[
\varpi_\infty(z) = \exp \left( - \sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i} \right),
\]
\[
\text{18}
\]
because both sides are ring homomorphisms and they coincide on $h_n$, for all $n \geq 0$, which generate $B$ as a $\mathbb{Q}$-algebra. The proof that

$$\sigma^{-1}(z) = \exp\left(\sum_{i \geq 1} \frac{1}{i} \frac{\partial}{\partial x_i}\right)$$

is similar, but arguing that $\sigma^{-1}(z)$ is the inverse of $\overline{\sigma}(z) \in \text{End}_\mathbb{Q}(B)[z^{-1}]$ turns it easier.

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