A NOTE ON THE CONVERGENCE OF THE SOLUTIONS OF THE CAMASSA-HOLM EQUATION TO THE ENTROPY ONES OF A SCALAR CONSERVATION LAW

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ABSTRACT. We consider a shallow water equation of Camassa-Holm type, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solution of the dispersive equation converges to the unique entropy solution of a scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction. The nonlinear evolution equation

$$\partial_t u - \alpha \partial_{xxx}^3 u + \partial_x f(u) = 2\alpha \partial_x u \partial_{xx}^2 u + \alpha u \partial_{xxx}^3 u, \quad f(u) = \frac{3u^2}{2}, \quad (1)$$

is known as the Camassa-Holm equation (see [3]). (1) models the propagation of unidirectional water waves of moderate amplitude over a flat bottom. The unknown $u(t, x)$ represents the fluid velocity at time $t$ in the horizontal direction $x$ (see [3, 21, 33, 34]).

In [24, 25, 26], the authors derived (1) (in a more general form), as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods.

The Camassa-Holm equation goes beyond the Korteweg-de Vries (KdV) and the Benjamin-Bona-Mahony (BBM) ones in the sense that (1) appears as a water-wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves modeled by the incompressible Euler equations, whereas the KdV and BBM equations appear at first order in this asymptotic expansion (see [3, 34]).

Two of the many differences between the KdV and BBM equations and (1) are the following. The KdV and BBM equations admit analytic travelling waves and within the travelling waves for (1) there are the peakons, that present a peak at their crest, being similar to the Stokes waves of greatest height [5, 14, 15, 18, 19]. Moreover,
experiences breaking waves \[5, 20\], namely solutions that remain bounded but whose slope becomes unbounded in finite time, that is not the case for the KdV and BBM equations.

From a mathematical point of view, the Camassa-Holm equation is well studied. Local well-posedness results are proved in \[16, 27, 36, 38\]. It is also known that there exist global solutions for a certain class of initial data and solutions that blow up in finite time for a large class of initial data (see \[13, 16, 17\]). Existence and uniqueness results for global weak solutions of (1) are proven in \[1, 2, 17, 7, 8, 12, 22, 28, 29, 30, 40, 41\]. The convergence of finite difference schemes has been proved in \[9, 10\].

We are interested in the no high frequency limit, i.e., we send \(\alpha \to 0\) in (1). In this way we pass from (1) to the scalar conservation law

\[
\left\{ \begin{array}{l}
\partial_t u + \partial_x f(u) = 0, \\
 u(0, x) = u_0(x),
\end{array} \right. \quad t > 0, x \in \mathbb{R},
\]

where \(u_\varepsilon, \alpha, 0\) is a \(C^\infty\) approximation of \(u_0\) such that

\[
u_\varepsilon, \alpha, 0 \to u_0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}), 1 \leq p < 4, \text{ as } \varepsilon, \alpha \to 0,
\]

\[
\|u_\varepsilon, \alpha, 0\|_{L^2(\mathbb{R})} + \|u_\varepsilon, \alpha, 0\|_{L^4(\mathbb{R})} \leq C_0,
\]

\[
\left(\alpha + \sqrt{\alpha + \varepsilon^2}\right) \|\partial_x u_\varepsilon, \alpha, 0\|_{L^2(\mathbb{R})} + \sqrt{\alpha^2 + \varepsilon^2} \|\partial_x^2 u_\varepsilon, \alpha, 0\|_{L^2(\mathbb{R})} \leq C_0,
\]

\[
\varepsilon \alpha^2 \|\partial_x^2 u_\varepsilon, \alpha, 0\|_{L^2(\mathbb{R})} \leq C_0,
\]

for every \(\varepsilon, \alpha\) and some constant \(C_0\) independent on \(\varepsilon, \alpha\).

On the flux \(f\), we assume that it is a \(C^2\) function satisfying

\[
|f'(u)| \leq k_0|u|, \quad |f(u)| \leq k_1u^2, \quad u \in \mathbb{R},
\]

for some constants \(k_0, k_1 > 0\), and the genuinely nonlinear condition

\[
\text{meas}\{u \in \mathbb{R} : f''(u) = 0\} = 0.
\]

In \[11\], under the assumptions (3), (5), (6), (7), (9), (10), and choosing

\[
\alpha = O(\varepsilon^4),
\]

the convergence of the solution of (1) to a distributional solution of (2) is proven. Moreover, following \[23\], under the assumption

\[
\alpha = o(\varepsilon^4),
\]

the dissipation of energy is proven.

In \[31\], under the assumption (12), the convergence of the solution of (1) to the unique entropy solution of (2) is proven. In other to do this, the author used the technique of the kinetic methods, which is introduced in \[32\].
Assume Lemma 2.4. such that

\[ u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R} \cap L^4(\mathbb{R})), \]

such that

\[ u_{\varepsilon_k, \alpha_k} \rightarrow u, \] strongly in \( L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \), for each \( 1 \leq p < 4 \)

\[ u \] is the unique entropy solution of (2).

The paper is organized as follows. In Section 2, we prove several a priori estimates on (4). Those play a key role in the proof of our main result, that is given in Section 3.

2. A priori estimates. This section is devoted to some a priori estimates on \( u_{\varepsilon, \alpha} \). We denote with \( C_0 \) the constants which depend only on the initial data.

**Lemma 2.1.** [11, Lemma 4.1]. For each \( t > 0 \),

\[
\| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \alpha \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \| \partial_x u_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds + 2\alpha \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\]

Moreover,

\[
\| u_{\varepsilon, \alpha} \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \leq C_0 \alpha^{-\frac{1}{2}}.
\]

**Lemma 2.2.** [11, Lemma 4.2]. Assume (10), and (11). Then:

i) the family \( \{ u_{\varepsilon, \alpha} \} \) is bounded in \( L^\infty(\mathbb{R}^+; L^4(\mathbb{R})) \);

ii) the families \( \{ \varepsilon \partial_x u_{\varepsilon, \alpha} \} \), \( \{ \varepsilon \partial_{xx} u_{\varepsilon, \alpha} \} \), \( \{ \varepsilon \partial_{xxx} u_{\varepsilon, \alpha} \} \) are bounded in \( L^\infty(\mathbb{R}^+; L^2(\mathbb{R})) \);

iii) the families \( \{ \varepsilon \partial_x \partial_x u_{\varepsilon, \alpha} \} \), \( \{ \varepsilon \partial_{xx} \partial_x u_{\varepsilon, \alpha} \} \), \( \{ \varepsilon \partial_{xxx} \partial_x u_{\varepsilon, \alpha} \} \) are bounded in \( L^2(\mathbb{R}^+ \times \mathbb{R}) \).

**Lemma 2.3.** We have that

\[
\| \partial_x u_{\varepsilon, \alpha} \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \leq C_0 \alpha^{-\frac{3}{2}}.
\]

**Proof.** Thanks to (15), Lemma 2.2, and the Hölder inequality,

\[
(\partial_x u_{\varepsilon, \alpha}(t, x))^2 = 2 \int_{-\infty}^{x} \partial_x u_{\varepsilon, \alpha}(s, x) \partial_{xx} u_{\varepsilon, \alpha}(s, x) \, dx \leq 2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \alpha}| \partial_{xx} u_{\varepsilon, \alpha} \, dx \leq 2 \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0 \alpha^{-\frac{3}{2}},
\]

Hence,

\[
\| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq C_0 \alpha^{-\frac{3}{2}},
\]

which gives (17).

**Lemma 2.4.** Assume (7), (8), (9), and (11). For each \( t > 0 \),

\[
\varepsilon^2 \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} + \varepsilon^2 \alpha \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^3}{2} \int_0^t \| \partial_{xx} u_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds + 2\varepsilon^3 \alpha \int_0^t \| \partial_{xxx} u_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\]
In particular,

\[ \varepsilon^3 \int_0^t \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^4(\mathbb{R})}^4 \, ds \leq C_0. \]  

(19)

Proof. Let \( t > 0 \). Multiplying (4) by \(-2\varepsilon^2 \partial_x^2 u_{\varepsilon,\alpha}\), integrating on \( \mathbb{R} \), we have that

\[
\frac{d}{dt} \left( \varepsilon^2 \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \alpha \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 \right)
= -2\varepsilon^2 \int_{\mathbb{R}} f'(u_{\varepsilon,\alpha}) \partial_x u_{\varepsilon,\alpha} \partial_x^2 u_{\varepsilon,\alpha} \, dx - 4\varepsilon^2 \alpha \int_{\mathbb{R}} \partial_x u_{\varepsilon,\alpha} (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx
- 2\varepsilon^2 \alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x^2 u_{\varepsilon,\alpha} \partial_x^3 u_{\varepsilon,\alpha} \, dx - 2\varepsilon^3 \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2
- 2\varepsilon^3 \alpha \| \partial_x^3 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2.
\]  

(20)

Since

\[
-4\varepsilon^2 \alpha \int_{\mathbb{R}} \partial_x u_{\varepsilon,\alpha} (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx - 2\varepsilon^2 \alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x^2 u_{\varepsilon,\alpha} \partial_x^3 u_{\varepsilon,\alpha} \, dx
= -3\varepsilon^2 \alpha \int_{\mathbb{R}} \partial_x u_{\varepsilon,\alpha} (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx,
\]

from (9), and (20), we have that

\[
\frac{d}{dt} \left( \varepsilon^2 \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \alpha \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 \right)
+ 2\varepsilon^3 \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon^3 \alpha \| \partial_x^3 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2
\]

\[
= -2\varepsilon^2 \int_{\mathbb{R}} f'(u_{\varepsilon,\alpha}) \partial_x u_{\varepsilon,\alpha} \partial_x^2 u_{\varepsilon,\alpha} \, dx - 3\varepsilon^2 \alpha \int_{\mathbb{R}} \partial_x u_{\varepsilon,\alpha} (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx
\]

\[
\leq 2\varepsilon^2 \int_{\mathbb{R}} |f(u_{\varepsilon,\alpha})| |\partial_x u_{\varepsilon,\alpha}| |\partial_x^2 u_{\varepsilon,\alpha}| \, dx + 3\varepsilon^2 \alpha \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\alpha}| (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx
\]

\[
\leq 2k_0 \varepsilon^2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\alpha}| (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx + 3\varepsilon^2 \alpha \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\alpha}| (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx.
\]  

(21)

Due to the Young inequality,

\[
2k_0 \varepsilon^2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\alpha}| (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx = 2 \int_{\mathbb{R}} \varepsilon^2 u_{\varepsilon,\alpha} |\partial_x u_{\varepsilon,\alpha}| \left| \varepsilon^2 \partial_x^2 u_{\varepsilon,\alpha} \right| \, dx
\leq k_0^2 \varepsilon \| u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 + \varepsilon^3 \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2.
\]  

(22)

From (11),

\[ \alpha \leq D \varepsilon^4, \]

where \( D \) is a positive constant that will be specified later. Due to (17), and (22),

\[
3\varepsilon^2 \alpha \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\alpha}| (\partial_x^2 u_{\varepsilon,\alpha})^2 \, dx \leq 3\varepsilon^2 \alpha \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^\infty(\mathbb{R})} \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}
\leq C_0 \varepsilon \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}
\leq C_0 D \varepsilon^3 \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}.
\]

Therefore, from (21), we gain

\[
\frac{d}{dt} \left( \varepsilon^2 \| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \alpha \| \partial_x^2 u_{\varepsilon,\alpha}(t,\cdot) \|_{L^2(\mathbb{R})}^2 \right)
\]
An integration on $(0, t)$, from (7), (8), and Lemma 2.2, we have (18).

Finally, we prove (19). [11, Lemma 4] says that

$$
\int_\mathbb{R} (\partial_x u_{\varepsilon, \alpha})^4 \, dx \leq c_1 \int_\mathbb{R} u_{\varepsilon, \alpha}^2 \, dx \int_\mathbb{R} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx, \quad (23)
$$

for some constant $c_1 > 0$. Therefore,

$$
\varepsilon^3 \int_\mathbb{R} (\partial_x u_{\varepsilon, \alpha})^4 \, dx \leq c_1 \varepsilon^3 \int_\mathbb{R} u_{\varepsilon, \alpha}^2 \, dx \int_\mathbb{R} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx. \quad (24)
$$

It follows from (15) and (24) that

$$
\varepsilon^3 \|\partial_x u_{\varepsilon, \alpha}(t, \cdot)\|_{L^4(\mathbb{R})} \leq C_0 \varepsilon^3 \|\partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot)\|_{L^2(\mathbb{R})}. \quad (25)
$$

An integration on $(0, t)$, and (18) give (19).

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1. In other to do this, the following technical lemma is needed [37].

**Lemma 3.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n \in \mathbb{N}}$ of distributions is bounded in $W^{-1, \infty}(\Omega)$. Suppose also that

$$\mathcal{L}_n = \mathcal{L}_{1, n} + \mathcal{L}_{2, n},$$

where $\{\mathcal{L}_{1, n}\}_{n \in \mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$ and $\{\mathcal{L}_{2, n}\}_{n \in \mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{\text{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n \in \mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$.

Moreover, we consider the following definition.

**Definition 3.2.** A pair of functions $(\eta, q)$ is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function and $q : \mathbb{R} \to \mathbb{R}$ is defined by

$$q(u) = \int_0^u f'(\xi)\eta'(\xi) \, d\xi.$$  

An entropy–entropy flux pair $(\eta, q)$ is called convex/compactly supported if, in addition, $\eta$ is convex/compactly supported.

**Proof of Theorem 1.1.**

$$\partial_t \eta(u_{\varepsilon, \alpha}) + \partial_x q(u_{\varepsilon, \alpha})
= \varepsilon^3 \eta'(u_{\varepsilon, \alpha}) \partial_{xx}^2 u_{\varepsilon, \alpha} - \alpha \eta'(u_{\varepsilon, \alpha}) \partial_{xxxx} u_{\varepsilon, \alpha} + \alpha \eta'(u_{\varepsilon, \alpha}) \partial_{xx}^4 u_{\varepsilon, \alpha}
+ 2 \alpha \eta'(u_{\varepsilon, \alpha}) \partial_{xx}^2 u_{\varepsilon, \alpha} + \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_{xx}^3 u_{\varepsilon, \alpha}
= I_{1, \varepsilon, \alpha} + I_{2, \varepsilon, \alpha} + I_{3, \varepsilon, \alpha} + I_{4, \varepsilon, \alpha} + I_{5, \varepsilon, \alpha} + I_{6, \varepsilon, \alpha}.$$
where
\begin{align*}
I_{1, \varepsilon, \alpha} &= \partial_x(\varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha}), \\
I_{2, \varepsilon, \alpha} &= -\varepsilon \eta''(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2, \\
I_{3, \varepsilon, \alpha} &= -\partial_x(\varepsilon \alpha \eta'(u_{\varepsilon, \alpha}) \partial_{xxx} u_{\varepsilon, \alpha}), \\
I_{4, \varepsilon, \alpha} &= \varepsilon \alpha \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{xxx}^2 u_{\varepsilon, \alpha}, \\
I_{5, \varepsilon, \alpha} &= \partial_x(\varepsilon \alpha \eta'(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2), \\
I_{6, \varepsilon, \alpha} &= -\varepsilon \alpha \eta''(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^3, \\
I_{7, \varepsilon, \alpha} &= \partial_x(\varepsilon \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}), \\
I_{10, \varepsilon, \alpha} &= -\varepsilon \alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha}, \\
I_{11, \varepsilon, \alpha} &= \alpha \eta'(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}.
\end{align*}

Fix $T > 0$. Arguing as [11, Lemma 5.1], we have that
\begin{align*}
I_{1, \varepsilon, \alpha} &\to 0 \quad \text{in } H^{-1}((0, T) \times \mathbb{R}), T > 0, \\
\{I_{2, \varepsilon, \alpha}\}_{\varepsilon, \alpha > 0} &\text{ is bounded in } L^1((0, T) \times \mathbb{R}), \\
I_{3, \varepsilon, \alpha} &\to 0 \quad \text{in } H^{-1}((0, T) \times \mathbb{R}), T > 0, \\
I_{5, \varepsilon, \alpha} &\to 0 \quad \text{in } H^{-1}((0, T) \times \mathbb{R}), T > 0, \\
I_{7, \varepsilon, \alpha} &\to 0 \quad \text{in } H^{-1}((0, T) \times \mathbb{R}), T > 0, \\
I_{9, \varepsilon, \alpha} &\to 0 \quad \text{in } H^{-1}((0, T) \times \mathbb{R}), T > 0.
\end{align*}

We claim that
\begin{align*}
I_{4, \varepsilon, \alpha} &\to 0 \quad \text{in } L^1((0, T) \times \mathbb{R}), T > 0, \text{ as } \varepsilon \to 0.
\end{align*}

By (11), Lemmas 2.1, 2.4, and the Hölder inequality
\begin{align*}
\|\varepsilon \alpha \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{xxx}^2 u_{\varepsilon, \alpha}\|_{L^1((0,T) \times \mathbb{R})} \\
&\leq \varepsilon \alpha \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon, \alpha} \partial_{xxx}^2 u_{\varepsilon, \alpha}| dt \, dx \\
&\leq \varepsilon \alpha \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon, \alpha}| \|\partial_{xxx}^2 u_{\varepsilon, \alpha}\|_{L^2((0,T) \times \mathbb{R})} dt \, dx \\
&\leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\varepsilon^2}{\varepsilon} \leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \varepsilon \to 0.
\end{align*}

We have that
\begin{align*}
I_{6, \varepsilon, \alpha} &\to 0 \quad \text{in } L^1((0, T) \times \mathbb{R}), T > 0, \text{ as } \varepsilon \to 0.
\end{align*}

By (11), Lemmas 2.1, 2.2, and the Hölder inequality,
\begin{align*}
\|\alpha \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{tx}^2 u_{\varepsilon, \alpha}\|_{L^1((0,T) \times \mathbb{R})} \\
&\leq \frac{\varepsilon \alpha}{\varepsilon} \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon, \alpha} \partial_{tx}^2 u_{\varepsilon, \alpha}| dt \, dx \\
&\leq \frac{\varepsilon \alpha}{\varepsilon} \|\eta''\|_{L^\infty(\mathbb{R})} \|\partial_x u_{\varepsilon, \alpha}\|_{L^2((0,T) \times \mathbb{R})} \|\partial_{tx}^2 u_{\varepsilon, \alpha}\|_{L^2((0,T) \times \mathbb{R})}
\end{align*}
We show that
\[ I_{8, \varepsilon, \alpha} \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
By (11), (15), (19), and the Young inequality,
\[
\| \alpha \eta''(u_{\varepsilon, \alpha})(\partial_x u_{\varepsilon, \alpha})^3 \|_{L^1((0, T) \times \mathbb{R})} \\
\leq \alpha \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon, \alpha}|^3 dt dx \\
\leq \frac{C_0 \alpha}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})}^2 + \frac{C_0 \alpha}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})}^4 \\
\leq \frac{C_0 \varepsilon^4}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})}^2 + C_0 \varepsilon^4 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})}^4 \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon^3 \to 0.
\]
We obtain that
\[ I_{10, \varepsilon, \alpha} \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
By (11), Lemmas 2.1, 2.2, and the Hölder inequality,
\[
\| \alpha \eta''(u_{\varepsilon, \alpha})(\partial_x u_{\varepsilon, \alpha})^2 \partial_{xx} u_{\varepsilon, \alpha} \|_{L^1((0, T) \times \mathbb{R})} \\
\leq \frac{\varepsilon \alpha}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |u_{\varepsilon, \alpha}|^2 |\partial_{xx} u_{\varepsilon, \alpha}| |\partial_x u_{\varepsilon, \alpha}| dt dx \\
\leq \frac{C_0 \varepsilon}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_{xx} u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \frac{\alpha}{\varepsilon} \to 0.
\]
We have that
\[ I_{11, \varepsilon, \alpha} \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
By (11), Lemmas 2.1, 2.4, and the Hölder inequality,
\[
\| \alpha \eta''(u_{\varepsilon, \alpha})(\partial_x u_{\varepsilon, \alpha})^2 \partial_{xx} u_{\varepsilon, \alpha} \|_{L^1((0, T) \times \mathbb{R})} \\
\leq \alpha \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon, \alpha}|^2 |\partial_{xx} u_{\varepsilon, \alpha}| dt dx \\
\leq \alpha \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_{xx} u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \\
\leq \varepsilon^2 C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_{xx} u_{\varepsilon, \alpha} \|_{L^2((0, T) \times \mathbb{R})} \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \frac{1}{\varepsilon} \to 0.
\]
Therefore, (13) follows from Lemmas 2.1, 2.2 and the \(L^p\) compensated compactness of [39].

Arguing as [11, Lemma 5.1], we have that \(u\) is a distributional solution of (2). We conclude by proving that \(u\) is the entropy solution of (2). Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\), and \(\phi \in C^2_c((0, \infty) \times \mathbb{R})\) a non–negative test function. Fix \(T > 0\). We have to prove that
\[
\int_0^T \int_\mathbb{R} \left( \partial_t \eta(u) + \partial_x q(u) \right) \phi dt dx \leq 0. \quad (27)
\]
We define

\[ u_k := u_{\varepsilon_k, \alpha_k} \]  

and have

\[
\int_0^\infty \int_\mathbb{R} \left( \partial_t \eta(u_k) + \partial_x q(u_k) \right) \phi dt dx
\]

\[
= \varepsilon_k \int_0^\infty \int_\mathbb{R} \partial_x \left( \eta'(u_k) \partial_x u_k \right) \phi dt dx + \varepsilon_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \left( \partial_x u_k \right)^2 \phi dt dx
\]

\[
+ \alpha_k \int_0^\infty \int_\mathbb{R} \partial_x \left( \eta''(u_k) \partial_x^2 u_k \right) \phi dt dx - \alpha_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \left( \partial_x u_k \right)^3 \phi dt dx
\]

\[
+ \alpha_k \int_0^\infty \int_\mathbb{R} \partial_x \left( \eta'(u_k) \partial_x^2 u_k \right) \phi dt dx - \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \left( \partial_x u_k \right)^2 \phi dt dx
\]

\[
- \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_x u_k \partial_x^2 u_k \phi dt dx
\]

\[
\leq \varepsilon_k \int_0^\infty \int_\mathbb{R} \left( \eta'(u_k) \partial_x u_k \partial_x \phi + \varepsilon_k \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_x^2 u_k \phi dt dx \right)
\]

\[
+ \varepsilon_k \alpha_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \partial_x u_k \partial_x^3 u_k \phi dt dx - \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \left( \partial_x u_k \right)^2 \phi dt dx
\]

\[
- \alpha_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \left( \partial_x u_k \right)^3 \phi dt dx - \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_x u_k \partial_x^2 u_k \phi dt dx
\]

\[
- \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_x u_k \partial_x^2 u_k \phi dt dx - \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_x u_k \partial_x^2 u_k \phi dt dx
\]
(27) follows from (11), (17), Lemmas 2.1, 2.2, 2.4, and the Young inequality.

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