Growth Model for Vote Distributions in Elections

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There are many factors that can influence the outcome of an election. We here identify two dominant effects that can affect the votes obtained by a candidate, namely, the Majority Effect and the Media Effect. We mimic these two effects in a simple growth model. We put the model on a two-dimensional square lattice and test it against the available data from elections in various countries. By adjusting the two parameters in the model, we are able to fit the vote distributions in all the countries studied.

I. INTRODUCTION

It is now a common practice for a democratic country to hold elections in order to choose an individual to hold formal office. In elections, there are indeed many factors that can influence the outcome of the result. It is noted that the outcome of the distribution of votes for individual candidates has taken interest from researchers in recent years\cite{1-11}. Different countries however, hold regular elections by adopting different electoral systems. In fact, with different electoral systems, one would imagine that the vote distributions of candidates in elections would look very different. In the case of proportional elections held in Brazil\cite{1}, the distribution of votes among candidates for the whole country was shown to follow a power law $N(x) \propto x^{-\alpha}$ with $\alpha = 1.00 \pm 0.02$. On the other hand, in many European countries, the vote distributions turn out to follow a lognormal distribution\cite{2}. Since different countries adopt different electoral systems, one would probably argue that these different distributions are indeed results of different electoral systems.

Figure 1 is an illustration of the vote distributions of elections from different countries\cite{12}. Figures 1(a)-1(f) are vote distributions from Brazil, Finland, USA, Australia, Canada and Taiwan. The bracket next to the name of the country is the electoral system that the country is adopting. Here, $x$ is the number of votes obtained by the candidate divided by the total
number of votes in the election and $P(x)$ is the probability density. It is interesting to note that vote distributions in Fig. 1(c)-(f) display two peak behavior. This is significantly different from either power law or lognormal distributions as in Fig. 1(a) and (b). The apparent differences could probably be thought of related to the different electoral systems used in different countries. However, in the case of Taiwan, the country has changed its electoral systems in 2008 but the essential features of the vote distributions in elections before and in that year is basically unchanged. Therefore it is natural to ask if different vote distributions are in fact results coming from the same mechanism that governs elections instead of different electoral systems.

In this paper, we will show that although different countries adopt different electoral systems, the results of their vote distributions can indeed be explained by two competing effects, viz., the Majority Effect and the Media Effect. We will demonstrate this by mimicking these two effects in a simple growth model and show that one can reproduce the distribution of votes in different countries which adopt different electoral systems. Using models from physics to study topics in political science is common nowadays. For example, one can study political districting problem by using the well-known Potts model in physics[13].

To facilitate our discussion, let us first define what we mean by Majority and Media Effects. By Majority Effect, we here mean that the decision of a person to vote for a particular candidate is mainly influenced by the portion of his neighbors and friends who want to vote for that particular candidate. The larger the portion is, the more likely that the person will vote for this particular candidate. Hence we call it the Majority Effect. It is easy to imagine that most of the less known candidates in fact obtain their votes through this process. On the other hand, a person can also obtain information of a particular candidate through other sources such as the mass media. For example, in the 2008 USA Presidential Election, most Americans obtained their information about Obama and McCain through mass media such as TV, internet, etc, much less from their friends and neighbors. In this way, we can say that people obtain their information about a certain candidate through some long range interactions. We categorize these long range interactions as the Media Effect. We believe that the two effects indeed compete against each other which results in different patterns of vote distributions as observed in different countries.
II. MODEL

We will investigate the two effects one by one and then combine the two effects in our growth model to understand how they compete against each other. Let us first consider only the Majority Effect. To make things simple, we now put all voters on a regular lattice. Let us first put them on a one-dimensional regular lattice (ring if we impose periodic boundary condition). Each of the voters will then take a site on this lattice. In this case, each voter will only be affected by his neighbors whom they will vote for. To begin with, let us assume
that there are \(N\) voters (lattice sites) and \(M\) candidates. The candidates will act as the seeds on the lattice and they are randomly distributed on the lattice. In the first time step, the candidates start to convince his neighbors (neighboring sites) to vote for him. In the next time step, his neighbors (neighboring sites) will convince their neighbors (neighboring sites) to vote for the same candidate. When an undecided voter has decided voters on both sides, his vote will be determined by randomly picking one of his neighbor’s decision. The process will stop when all the voters have decided whom they will vote for. This process is indeed equivalent to a simple one-dimensional growth process with \(M\) seeds to begin with.

The distribution of \(M\) random seeds (candidates) should follow an exponential distribution \(e^{-(x/x_0)}\) where \(x\) is the distance between two neighboring seeds and \(x_0 = 1/M\), the average lattice distance between two neighboring seeds. We here normalize \(x\) to be from 0 to 1. At the end of the process, there will be a distribution of votes from all the seeds (candidates). The form of the distribution of votes in this simple case can be obtained analytically. The argument is as follows. In order for a candidate to get a portion of votes \(x\), he has to have two nearest neighboring candidates to be of a distance \(2x\) apart. The probability to have two neighboring candidates of distance \(2x\) apart is \(e^{-(2x/x_0)}\). The seed can be anywhere between these two neighboring candidates. Integrating out all possibilities will give a factor proportional to \(x\). The vote distribution in this simple case is therefore equal to \(axe^{-(2x/x_0)}\), where \(a\) is the normalization factor. \(a\) can be fixed by integrating over \(x\) from 0 to 1/2, since the largest separation between two sites with a seed in between should be less than 1. Notice that there are no free parameters in this simple one-dimensional case.

One can easily extend this analysis to higher dimensional cases. For a certain number of seeds randomly distributed on a two-dimensional surface, their distribution takes the form \(re^{-\rho \pi r^2}\), where \(r\) is the distance measured from a specified seed and \(\rho\) is the density of seeds in the two-dimensional plane. In a similar fashion, the vote distribution on the two-dimensional square lattice would take a form \(ax^b e^{-cx}\), where \(a, b, c\) are constants and \(x\) is the portion of votes that a certain seed will get. Figure 2 is an illustration of simulations of both the one- and two-dimensional lattices with only Majority Effect and with \(N = 1,000,000\) and \(M = 1,000\). The curves from theoretical arguments are also included for comparison. In the two-dimensional case, the theoretical fit gives \(a = 6.19 \times 10^6\), \(b = 0.012\) and \(c = 2.48\) respectively.

The distribution of votes for the Majority Effect changes significantly if one introduces
FIG. 2: Simulations of both the one- and two-dimensional lattices with only Majority Effect. Here, \( N = 1,000,000 \) and \( M = 1,000 \).

an extra factor which takes the form

\[
\left\{ \frac{N_i}{\sum_{i=1}^{M} N_i} \right\}^\beta, \tag{1}
\]

where \( i \) refers to the \( i^{\text{th}} \) candidate, \( N_i \) is the total number of voters who have decided to vote for the this candidate and \( \sum_{i=1}^{M} N_i \) is the total number of voters who have already decided to vote for one of the \( M \) candidates. \( \beta \) here is a free parameter. The introduction of the factor in Eq.(1) has the following meaning. A decided voter would want to convince his neighbors and friends to vote for that particular candidate. However, the probability that he could succeed would also depend on how well the candidate has been doing so far. One can conceive that if the candidate had been doing very poorly, the probability that the decided voter could successfully convince his neighbors and friends to vote for the candidate would be low. On the other hand, if the candidate had been doing very well, it could have been much easier for the decided voter to convince his neighbors and friends to vote for the candidate. The case of \( \beta = 0 \) is the extreme case which corresponds to what we have discussed above. This is the case when a decided voter can always convince his neighbors to vote for the candidate that he decided to vote for, regardless of how the candidate has
been doing so far. Figure 3 is an illustration of how the distribution of votes will change as we change $\beta$. As $\beta$ increases, the tails on both ends of the distribution will start to move up until it displays a power law like distribution when $\beta \approx 0.7$.

As mentioned above, a few of the candidates can obtain a large portion of votes largely by Media Effect in an election. We investigate this effect again on a lattice. We here again assume that there are $M$ candidates, each candidate can put $L$ random seeds on the lattice with a total number of $N$ sites. Since the $M \times L$ seeds are randomly put on the lattice, one would then expect the distribution to follow a Gaussian distribution. Figure 4 shows the result of the distribution of votes with $M = 4$ candidates on a two dimensional square lattice with a total of $N = 500 \times 500$ sites. We have tried two different situations here. In Figure 4(a), $L = 512$ and the random seeds are all put on the lattice in the very beginning. We let the system evolve similar to the case of the Majority Effect as described above. One can see that the curve can be fitted well by a Gaussian distribution with its peak around 0.25 as expected. In Figure 4(b), we start with only one seed for each of the $M$ candidates. In each time step, we give a probability $\alpha$ that one of the undecided sites will become the seed of one of the $M$ candidates. Again, the curve can be described by a Gaussian distribution.
FIG. 4: Vote distributions with only Media Effect. The number of candidates $M = 4$ and each has $L = 512$ random seeds on a square lattice with 250,000 sites. (a) All the random seeds are put on the lattice at the beginning and, (b) the number of seeds grow as time evolves.

with its peak around 0.25. The results in both cases agree well with our expectation. From the above analysis, we therefore propose that the function that describes the various effects on vote distribution in elections would take the general form

$$a x^b \exp\left(-\left(\frac{x - c}{\sigma}\right)^\gamma\right),$$

where $a, b, c, \gamma$ and $\sigma$ are all constants and with the vote distributions of the two effects studied in the above as particular cases.
III. RESULTS

We now include both effects in our growth model and investigate how they compete against each other. We again use the two-dimensional square lattice as an example. To begin with, we have a number of candidates $M$ and a lattice with $N$ lattice sites. Of the $M$ candidates, $M_1$ of them acquire Media Effect while the rest can only obtain their votes through Majority Effect. The rules that mimic the two effects have been introduced above so we will not repeat here. Figure 5 are the distributions of votes with both effects included in the model. Figures 5(a) - (f) are the fits using both our simulations and theoretical fit of the vote distributions corresponding to Figure 1(a)-(f). In each of the cases, we fit the real data using both simulations from our growth model and the theoretical function of Eq. (2). For example, Figure 5(a) is a fit of the 2006 election in Brazil. The solid triangles are from real data while the circles are the simulated result of our growth model with $M = 300$ and $N = 1,000,000$. We use $\alpha = 0, \beta = 0.65$ in the simulation. The solid line is a theoretical fit using our function above with $a = 9.94 \times 10^{19}, b = 1.57, c = 0, \sigma = 2.38 \times 10^{-20}$ and $\gamma = 0.089$.

Table 1 lists the values of $\alpha$ and $\beta$ that we use in order to fit the vote distributions in different countries. We should also mention here that in the simulation, we use the same number of voters (sites) corresponding to the number of voters in the election held by that country. Therefore, even with the same set of $\alpha$ and $\beta$, the result of the simulation could be different. An example is the simulations of Australia and Taiwan. Even with the same values of $\alpha$ and $\beta$, the simulated results look somewhat different. Theoretical fits for other countries can be done in a similar way but a sum of two functions similar to Eq. (2) is needed for Figure 5(c)-(f). The peaks on the right of the distributions in Figure 5(c)-(f) refer to the group of candidates who obtain their votes through Media Effect. One could therefore study the relative importance of the two effects in different electoral systems within our simple growth model which would have real applications in elections.

| Brazil | Finland | USA | Australia | Canada | Taiwan |
|--------|---------|-----|-----------|--------|---------|
| $\alpha$ | 0.0 | 0.0 | 1.0E-6 | 1.0E-5 | 1.0E-5 | 1.0E-5 |
| $\beta$  | 0.65 | 0.6 | 0.35 | 0.6 | 0.55 | 0.6 |

TABLE I: Parameters of Media Effect ($\alpha$) and Majority Effect ($\beta$) in Figure 5.
FIG. 5: Vote distributions from (a) Brazil, (b) Finland, (c) USA, (d) Australia, (e) Canada and, (f) Taiwan.

IV. DISCUSSION

The fact that there are two different groups of candidates in an election might be understood in the language of networks. The candidates who only obtain votes through Majority Effect can be conceived as people who communicate with others through a regular network with nearest neighbor interactions and could only affect others through some kind of diffusion process which is rather slow. On the other hand, candidates who obtain votes through Media Effect are more likely connected to networks similar to small world or scale free networks in which case they can affect voters from distance away.

In summary, we have identified two dominant effects in elections and mimic them in a
simple growth model. We put the growth model on a two-dimensional square lattice and test it against the available data from elections in various countries. By adjusting the two parameters $\alpha$ and $\beta$ in the model, we are able to fit the vote distributions of elections in all the countries that we have studied. We also give a theoretical argument of the general form of the function it can take and fit it with data. Other effects as well as lattices with higher dimensions and/or different structures can be used and their effects on vote distributions will be investigated in the future.

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