On realizations of the Witt algebra in $\mathbb{R}^3$ *

Renat Zhdanov$^1$ and Qing Huang$^{2,3}$

1. BIO-key International, 55121 Eagan, MN, USA
2. Department of Mathematics, Northwest University, Xi’an 710069, China
3. Center for Nonlinear Studies, Northwest University, Xi’an 710069, China

Abstract

We obtain exhaustive classification of inequivalent realizations of the Witt and Virasoro algebras by Lie vector fields of differential operators in the space $\mathbb{R}^3$. Using this classification we describe all inequivalent realizations of the direct sum of the Witt algebras in $\mathbb{R}^3$. These results enable constructing all possible (1+1)-dimensional classically integrable equations that admit infinite dimensional symmetry algebra isomorphic to the Witt or the direct sum of Witt algebras. In this way the new classically integrable nonlinear PDE in one spatial dimension has been obtained. In addition, we construct a number of new nonlinear (1+1)-dimensional PDEs admitting infinite symmetries.

1 Introduction

Since its introduction in the 19th century, Lie group analysis has become a very popular and powerful tool for solving nonlinear partial differential equations (PDEs). Given a PDE that possesses a nontrivial Lie symmetry, we can utilize symmetry reduction procedure to construct its exact solutions [9, 10].

Not surprisingly, the wider symmetry of an equation under study is, the better off we are when applying the Lie approach to solve it. This is especially the case when its symmetry group is infinite-parameter. If a nonlinear differential equation admits infinite Lie symmetries, then it is often possible either to linearize it or construct its general solution [9].

The classical example is the hyperbolic type Liouville equation

$$u_{tx} = \exp(u), \quad (1)$$

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which admits the infinite-parameter Lie group

\[ t' = t + f(t), \quad x' = x + g(x), \quad u' = u - \dot{f}(t) - \dot{g}(x), \quad (2) \]

where \( f \) and \( g \) are arbitrary smooth functions. The general solution of Eq. (1) can be obtained by the action of transformation group (2) on its particular traveling wave solution of the form \( u(t, x) = \varphi(x + t) \) (see, e.g., [9]). An alternative way to solve the Liouville equation is linearization [9].

Note that the Lie algebra of Lie group (2) is the direct sum of two infinite-dimensional Witt algebras, which are subalgebras of the Virasoro algebra.

Unlike the finite-dimensional algebras, infinite-dimensional ones have not been systematically studied within the context of classical Lie group analysis of nonlinear PDEs. The situation is, however, drastically different in the case of generalized (higher) Lie symmetries which played the critical role in success of the theory of integrable systems in \((1 + 1)\)- and \((1 + 2)\)-dimensions (see, e.g., [10]).

The breakthrough in the analysis of integrable systems has been nicely complemented by development of the theory of infinite-dimensional Lie algebras such as loop [28], Kac-Moody [19] and Virasoro algebras [17].

Virasoro algebra plays an increasingly important role in mathematical physics in general [4, 13] and in the theory of integrable systems in particular. Study of nonlinear evolution equations in \((1+2)\)-dimensions arising in different areas of modern physics shows that many of these equations admit Virasoro algebras as their symmetry algebras. Let us mention among others the Kadomtsev-Petvishilvi (KP) [7, 8, 14], modified KP, cylindrical KP [22], the Davey-Stewartson [6, 15], Nizhnik-Novikov-Veselov, stimulated Raman scattering, \((1+2)\)-dimensional Sine-Gordon [30] and the KP hierarchy [26] equations.

It is a common belief that nonlinear PDEs admitting symmetry algebras of Virasoro type are prime candidates for the roles of integrable systems. Consequently, systematic classification of inequivalent realizations of the Virasoro algebra is a crucial step of symmetry approach to constructing integrable systems (see, e.g., [23, 24]). It should be pointed out that there are a few integrable equations which do not possess Virasoro symmetry algebras, such as the breaking soliton and Zakharov-Strachan equations [30].

Classification of Lie algebras of vector fields of differential operators within the action of local diffeomorphism group has been pioneered by Sophus Lie himself. It remains a very powerful method for group analysis of nonlinear differential equations. Some of the more recent applications of this approach include geometric control theory [18], theory of systems of nonlinear ordinary differential equations possessing superposition principle [31], algebraic approach to molecular dynamics [2, 29] to mention only a few. Still the biggest bulk of results has been obtained in the area of classification of nonlinear PDEs possessing point and higher Lie symmetries (see [3] and references therein). Analysis of realizations of Lie algebras by first-order differential operators is in the core of almost every approach to group classification of PDEs (see, e.g., [1, 5, 10–12, 20, 21]).
In this paper we concentrate on the realizations of the Witt and Virasoro algebras by first-order differential operators in the space \( \mathbb{R}^n \) with \( n \leq 3 \). One of our primary motivations was that with these realizations in hand we can develop a regular way to construct \((1+1)\)-dimensional nonlinear PDEs which are integrable in the sense that they admit infinite symmetries.

The paper is organized as follows. In Section 2 we give a brief account of necessary facts and definitions. In addition, the algorithmic procedure for realizations of the Virasoro algebra is described in detail. We construct all inequivalent realizations of the Witt algebra (a.k.a. centerless Virasoro algebra) in Section 3. Section 4 is devoted to the description of the realizations of the Virasoro algebra. We prove that there are no central extensions of the Witt algebra in the space \( \mathbb{R}^3 \). In Section 5 we construct broad classes of nonlinear PDEs admitting infinite dimensional symmetry algebras, which are realizations of the Witt algebra. Furthermore, all inequivalent realizations of the direct sum of two Witt algebras are obtained in Section 6. This enables us to classify the second-order PDEs whose invariance algebra contains a direct sum of the Witt algebras. We prove that any such PDE is equivalent to one of the four canonical equations (16)–(19). The last section contains a brief summary of the obtained results.

## 2 Notations and definitions

The Virasoro algebra, \( \mathfrak{W} \), is the infinite-dimensional Lie algebra with basis elements \( \{L_n, n \in \mathbb{Z}\} \cup \{C\} \) which satisfy the commutation relations

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m,-n}C, \quad [L_m, C] = 0, \quad m, n \in \mathbb{Z},
\]

where \([Q, P] = QP - PQ\) is the commutator of Lie vector fields \( P \) and \( Q \), and \( \delta_{a,b} \) stands for the Kronecker delta

\[
\delta_{a,b} = \begin{cases} 
1, & a = b, \\
0, & \text{otherwise}.
\end{cases}
\]

The operator \( C \) commuting with all other basis elements is called the central element. If \( C \) equals to zero, the algebra \( \mathfrak{W} \) reduces to the centerless Virasoro algebra or Witt algebra \( \mathfrak{W} \). Consequently, the full Virasoro algebra is the nontrivial one-dimensional central extension of the Witt algebra.

We now consider the Virasoro algebras as the linear subspace of the infinite-dimensional Lie algebra \( \mathfrak{L}_{\infty} \) spanned by the basis elements of the form

\[
Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u
\]

over \( \mathbb{R}^3 \). Applying the transformation

\[
t \rightarrow \tilde{t} = T(t, x, u), \quad x \rightarrow \tilde{x} = X(t, x, u), \quad u \rightarrow \tilde{u} = U(t, x, u),
\]

over \( \mathbb{R}^3 \).
with \( D(T, X, U)/D(t, x, u) \neq 0 \) to (3), we get
\[
\tilde{Q} = (\tau T_t + \xi T_x + \eta T_u)\partial_t + (\tau X_t + \xi X_x + \eta X_u)\partial_x + (\tau U_t + \xi U_x + \eta U_u)\partial_u.
\]

Evidently, \( \tilde{Q} \in \mathcal{L}_\infty \). Hence we see that the set of operators (3) is invariant with respect to the transformation (4).

It is well-known that the correspondence, \( Q \sim \tilde{Q} \), is the equivalence relation and as such it splits the set of operators (3) into some equivalence classes. Any two elements within the same equivalence class are related through a transformation (4), while two elements belonging to different classes cannot be transformed one into another by a transformation of the form (4). Hence to describe all possible realizations of the Virasoro algebra, one needs to construct a representative of each equivalence class. The remaining realizations can be obtained by applying transformations (4) to the representatives in question.

To construct all inequivalent realizations of the Virasoro algebra we need to implement the following steps:

- Describe all inequivalent forms of \( L_0, L_1 \) and \( L_{-1} \) such that the commutation relations of the Virasoro subalgebra,

\[
[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0,
\]

hold together with the relations \([L_i, C] = 0, \ (i = 0, 1, -1)\). Note that algebra \( \langle L_0, L_1, L_{-1} \rangle \) is isomorphic to \( sl(2, \mathbb{R}) \).

- Construct all inequivalent realizations of the operators \( L_2 \) and \( L_{-2} \) which commute with \( C \) and satisfy the relations:

\[
[L_0, L_2] = -2L_2, \quad [L_{-1}, L_2] = -3L_1, \quad [L_1, L_{-2}] = 3L_{-1}, \quad [L_0, L_{-2}] = 2L_{-2}, \quad [L_2, L_{-2}] = 4L_0 + \frac{1}{2}C.
\]

- Derive all the remaining basis operators of the Virasoro algebra through the recursion relations

\[
L_{n+1} = (1 - n)^{-1}[L_1, L_n], \quad L_{-n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}]
\]

with

\[
[L_{n+1}, L_{-n-1}] = 2(n + 1)L_0 + \frac{1}{12}n(n + 1)(n + 2)C, \quad [L_i, C] = 0,
\]

where \( i = n + 1, -n - 1 \) and \( n = 2, 3, 4, \ldots \).

In Sections 3 and 4, we will implement the algorithm above to construct all inequivalent realizations of the Witt and Virasoro algebras by operators (3).
3 Realizations of the Witt algebra

Turn now to describing realizations of the Witt algebra \( \mathfrak{W} \). Let us remind that the algebra \( \mathfrak{W} \) is obtained from the Virasoro algebra by putting \( C = 0 \). We begin by letting the vector field \( L_0 \) be of the general form (3), namely,

\[
L_0 = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u.
\]

Transformation (4) maps \( L_0 \) into

\[
\tilde{L}_0 = (\tau T_t + \xi T_x + \eta T_u) \partial_t + (\tau X_t + \xi X_x + \eta X_u) \partial_x + (\tau U_t + \xi U_x + \eta U_u) \partial_u.
\]

We have \( \tau^2 + \xi^2 + \eta^2 \neq 0 \), otherwise \( L_0 \) is trivial. Consequently, we can choose the solutions of equations

\[
\begin{align*}
\tau T_t + \xi T_x + \eta T_u &= 1, \\
\tau X_t + \xi X_x + \eta X_u &= 0, \\
\tau U_t + \xi U_x + \eta U_u &= 0.
\end{align*}
\]

as \( T, X \) and \( U \) and reduce \( L_0 \) to the form \( L_0 = \partial_t \) (hereafter we drop the tildes).

Then \( L_0 \) is equivalent to the canonical operator \( \partial_t \).

With \( L_0 \) in hand we now proceed to constructing \( L_1 \) and \( L_{-1} \) which obey the commutation relations (5). Letting \( L_1 \) be of the general form (3) and inserting it into \([L_0, L_1]\) = \(-L_1\) yield

\[
L_1 = e^{-t} f(x, u) \partial_t + e^{-t} g(x, u) \partial_x + e^{-t} h(x, u) \partial_u,
\]

where \( f, g, h \) are arbitrary smooth functions. To further simplify \( L_1 \) we use an equivalence transformation of the form (4) preserving \( L_0 \). Applying (4) to \( L_0 \) gives

\[
L_0 \rightarrow \tilde{L}_0 = T_t \partial_t + X_t \partial_x + U_t \partial_u = \partial_t.
\]

Hence, transformation

\[
\tilde{t} = t + T(x, u), \quad \tilde{x} = X(x, u), \quad \tilde{u} = U(x, u)
\]

is the most general transformation that does not alter the form of \( L_0 \). It converts Lie vector field \( L_1 \) into

\[
\tilde{L}_1 = e^{-\tilde{t}} (f + g T_x + h T_u) \partial_t + e^{-\tilde{t}} (g X_x + h X_u) \partial_x + e^{-\tilde{t}} (g U_x + h U_u) \partial_u.
\]

To further analyze the realizations of \( L_1 \), we need to consider the inequivalent cases \( g^2 + h^2 = 0 \) and \( g^2 + h^2 \neq 0 \).

**Case 1.** If \( g^2 + h^2 = 0 \), we have \( \tilde{L}_1 = e^{-\tilde{t}} f(x, u) \partial_t \). Choosing \( \tilde{t} = t - \ln |f(x, u)| \) gives \( L_1 = e^{-t} \partial_t \). Let \( L_{-1} \) be of the general form (3). And taking into account (5) we get \( L_{-1} = e^t \partial_t \).

**Case 2.** Provided \( g^2 + h^2 \neq 0 \), we choose \( \tilde{t} = t + T(x, u) \), where \( T(x, u) \) satisfies the relation

\[
e^{-T} = f + g T_x + h T_u,
\]

where \( f, g, h \) are arbitrary smooth functions.
and take $X$ and $U$ to be solutions of the equations

$$ gX_x + hX_u = e^{-T}, \quad gU_x + hU_u = 0. $$

Then $L_1$ is mapped into $e^{-t}(\partial_t + \partial_x)$. Selecting $L_{-1}$ of the form (5) and taking into account commutation relations (5), we arrive at

$$ L_{-1} = e^t(1 - e^{-2x} f_1(u))\partial_t + e^t(-1 - e^{-2x} f_1(u) + e^{-x} g_1(u))\partial_x + e^{t-x} h_1(u)\partial_u, $$

where $f_1, g_1, h_1$ are arbitrary smooth functions.

Acting by the transformation

$$ \tilde{t} = t, \quad \tilde{x} = x + X(u), \quad \tilde{u} = U(u), \quad (7) $$

which keeps $L_0$ and $L_1$ invariant, on $L_{-1}$ gives

$$ \tilde{L}_{-1} = e^t(1 - e^{-2x} f_1(u))\partial_t + e^t(-1 - e^{-2x} f_1(u) + e^{-x} g_1(u))\partial_x + e^{t-x} h_1(u)\tilde{U}\partial_{\tilde{u}}. $$

To complete the analysis, we consider the cases $f_1(u) \neq 0$ and $f_1(u) = 0$ separately.

Assuming that $f_1(u) \neq 0$ we choose

$$ X(u) = -\ln \sqrt{|f_1(u)|}, \quad \phi(u) = (g_1(u) + h_1(u)X(u))/\sqrt{|f_1(u)|}. $$

In addition, we take as $U$ in (7) a solution of $h_1(u)\tilde{U} = 1$ if $h_1 \neq 0$ or an arbitrary non-constant function if $h_1 = 0$. This yields

$$ L_{-1} = e^t(1 + \alpha e^{-2x})\partial_t + e^t(-1 + \alpha e^{-2x} + e^{-x} \phi(u))\partial_x + \beta e^{t-x} \partial_u, $$

where $\alpha = \pm 1$ and $\beta = 0, 1$.

The case $f_1(u) = 0$ gives rise to the realization

$$ \tilde{L}_{-1} = e^t\partial_t + e^t(-1 + e^{-x} g_1(u) + e^{-x} h_1(u)X)\partial_x + e^{t-x} h_1(u)\tilde{U}\partial_{\tilde{u}}. $$

Letting $X = 0$ and $U$ in (7) be a solution of $h_1(u)\tilde{U} = 1$ when $h_1 \neq 0$ or an arbitrary non-constant function otherwise, we get

$$ L_{-1} = e^t\partial_t + e^t(-1 + e^{-x} g_1(u))\partial_x + \beta e^{t-x} \partial_u $$

with $\beta = 0, 1$.

**Lemma 1.** Any triplet of operators $\langle L_0, L_1, L_{-1} \rangle$ obeying the commutation relations of the Witt algebra is equivalent to either

$$ \langle \partial_t, e^{-t}\partial_t, e^t\partial_t \rangle $$

or

$$ \langle \partial_t, e^{-t}(\partial_t + \partial_x), e^t(1 + \alpha e^{-2x})\partial_t + e^t(-1 + \phi(u)e^{-x} + \alpha e^{-2x})\partial_x + \beta e^{t-x} \partial_u \rangle. $$

Here $\alpha = 0, \pm 1$, $\beta = 0, 1$ and $\phi(u)$ is an arbitrary smooth function.
Now to obtain the complete description of all inequivalent Witt algebras we need to extend algebras $\mathcal{W}_2$ and $\mathcal{W}_3$ by the operators $L_2$ and $L_{-2}$ and implement the last two steps of the classification procedure given in Section 2.

We first formulate the final results and then present the detailed proof.

**Theorem 1.** There are at most eleven inequivalent realizations of the Witt algebra $\mathcal{W}$ over the space $\mathbb{R}^3$. The representatives $\mathcal{W}_i$, $(i = 1, 2, \ldots, 11)$ of each equivalence class are listed below.

- $\mathcal{W}_1 : \langle e^{-nt} \partial_t \rangle$,
- $\mathcal{W}_2 : \langle e^{-nt} \partial_t + e^{-nt}[n + \frac{1}{2} n(n - 1)a e^{-x}] \partial_x \rangle$,
- $\mathcal{W}_3 : \langle e^{-nt+(n-1)x}[e^{2x} - (n + 1)\gamma e^x + \frac{1}{2} n(n + 1)\gamma^2](e^x - \gamma)^{-n-1}\partial_t + e^{-nt+(n-1)x}[ne^x - \frac{1}{2} n(n + 1)\gamma](e^x - \gamma)^{-n}\partial_x \rangle$,
- $\mathcal{W}_4 : L_0 = \partial_t$, $L_1 = e^{-t} \partial_t + e^{-t} \partial_x$, $L_{-1} = e^t(1 + \gamma e^{-2x}) \partial_t + e^t(-1 + \gamma e^{-2x} + e^{-x} \tilde{\phi}) \partial_x$,
- $L_2 = e^{-2t} f(x, u) \partial_t + e^{-2t} g(x, u) \partial_x$,
- $L_{-2} = e^{2t}[1 + 3\gamma e^{-2x} - \frac{1}{2} e^{-3x}(6\gamma \tilde{\phi} + \tilde{\phi}^3 \pm (4\gamma + \tilde{\phi}^3)^{3/2})] \partial_t + e^{2t}[-2 + 3e^{-x} \tilde{\phi} + 6\gamma e^{-2x} - \frac{1}{2} e^{-3x}(6\gamma \tilde{\phi} + \tilde{\phi}^3 \pm (4\gamma + \tilde{\phi}^3)^{3/2})] \partial_x$,
- $L_{n+1} = (1 - n)^{-1}[L_1, L_n]$, $L_{n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}]$, $n \geq 2$,
- $\mathcal{W}_5 : \langle e^{-nt+(n-1)x}(e^x \pm n)(e^x \pm 1)^{-n} \partial_t + ne^{-nt+(n-1)x}(e^x \pm 1)^{1-n} \partial_x \rangle$,
- $\mathcal{W}_6 : \langle e^{-nt} \partial_t + \gamma e^{-nt}[e^{nx} - (e^x - \gamma)^n](e^x - \gamma)^{1-n} \partial_x \rangle$,
- $\mathcal{W}_7 : L_0 = \partial_t$, $L_1 = e^{-t} \partial_t + e^{-t} \partial_x$, $L_{-1} = e^t(1 + \gamma e^{-2x}) \partial_t + e^t(-1 + \gamma e^{-2x} + e^{-x} \tilde{\phi}) \partial_x$,
- $L_2 = e^{-2t+x} \frac{e^x - \tilde{\phi}}{e^{2x} - e^x \tilde{\phi} - \gamma} \partial_t + e^{-2t+x} \frac{2e^x - \tilde{\phi}}{e^{2x} - e^x \tilde{\phi} - \gamma} \partial_x$,
- $L_{-2} = e^{2t-3x}(3e^{3x} + 3\gamma e^x - \gamma \tilde{\phi}) \partial_t + e^{2t-3x}(2e^x - \tilde{\phi})(-e^{2x} + e^x \tilde{\phi} + \gamma) \partial_x$,
- $L_{n+1} = (1 - n)^{-1}[L_1, L_n]$, $L_{n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}]$, $n \geq 2$. 

\( \mathcal{W}_8 : \langle e^{-nt} \partial_t + e^{-nt}[n - \text{sgn}(n) \gamma \frac{|n| - 1}{2} \sum_{j=1}^{|n| - 1} j(j + 1)e^{-2x}]\partial_x \rangle, \)

\( \mathcal{W}_9 : \langle e^{-nt+(n-1)x} \frac{(-1 + \sum j(j + 1))n + (2n + 1)e^x - (n + 2)e^{2x} + e^{3x}}{(e^x - 1)^{n+2}} + \text{sgn}(n) \frac{|n| - 1}{2} \sum_{j=1}^{|n| - 1} j(j + 1)]\partial_t + \frac{e^{-nt+(n-1)x}}{(e^x - 1)^{n+1}}[(1 - \sum j(j + 1))n \rangle - 2ne^x + ne^{2x} - \text{sgn}(n) \frac{|n| - 1}{2} \sum_{j=1}^{|n| - 1} j(j + 1)]\partial_x), \)

\( \mathcal{W}_{10} : \langle e^{-nt} \partial_t + ne^{-nt}\partial_x + \frac{\text{sgn}(n)}{2} \sum_{j=1}^{|n|} j(j - 1)e^{-nt-2x}\partial_u \rangle, \)

\( \mathcal{W}_{11} : \langle e^{-nt} \partial_t + e^{-nt}[n + \frac{\alpha n(n - 1)}{2}e^{-x}]\partial_x + \frac{n(n - 1)}{2}e^{-nt-x}\partial_u \rangle, \)

where \( n \in \mathbb{Z}, \alpha = 0, \pm 1, \gamma = \pm 1, \text{sgn}(\cdot) \) is the standard sign function, the symbol \( \tilde{\phi}(u) \) stands for either \( u \) or an arbitrary real constant \( c \), and

\[ f(x, u) = e^{x}[4e^{4x} - 10e^{3x}\tilde{\phi} - 36\gamma e^{2x} + 2e^x(31\gamma\tilde{\phi} + 6\tilde{\phi}^3) \pm 6(4\gamma + \tilde{\phi}^2)^{3/2}] - 64\gamma^2 - 54\gamma\tilde{\phi}^2 - 9\tilde{\phi}^4 \pm 9\tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}r^{-1} \]

\[ g(x, u) = e^{x}[8e^{4x} - 16e^{3x}\tilde{\phi} - 2e^{2x}(44\gamma + 5\tilde{\phi})^2 + 2e^x(44\gamma\tilde{\phi} + 9\tilde{\phi}^3) \pm 9(4\gamma + \tilde{\phi}^2)^{3/2}] - 64\gamma^2 - 54\gamma\tilde{\phi}^2 - 9\tilde{\phi}^4 \pm 9\tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}r^{-1}, \]

\[ r = 4e^{5x} - 10e^{4x}\tilde{\phi} - 40\gamma e^{3x} + 10e^{2x}(6\gamma\tilde{\phi} + \tilde{\phi}^3) \pm (4\gamma + \tilde{\phi}^2)^{3/2} - 10e^x(6\gamma^2 + 6\gamma\tilde{\phi}^2) + \tilde{\phi}^4 \pm \tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}] + 30\gamma^2\tilde{\phi} + 20\gamma\tilde{\phi}^3 + 3\tilde{\phi}^5 \pm (2\gamma + 3\tilde{\phi}^2)(4\gamma + \tilde{\phi}^2)^{3/2}. \]

**Proof.** To prove the theorem, it suffices to analyze all possible extensions of the algebras \( \mathcal{W}_8 \) and \( \mathcal{W}_9 \).

**Case 1.** Given the algebra \( \mathcal{W}_8 \) we make use of \( \mathcal{W}_6 \) thus getting

\[ L_2 = e^{-2t}\partial_t, \quad L_{-2} = e^{2t}\partial_t. \]

The remaining basis elements of the corresponding Witt algebra are easily obtained through recursion, which yields \( L_n = e^{-nt}\partial_t, n \in \mathbb{Z} \). We arrive at the realization \( \mathcal{W}_1 \) of Theorem [1].

**Case 2.** Turn now to realization \( \mathcal{W}_9 \). Inserting \( L_0, L_1, L_{-1} \) into the commutation relations \([L_0, L_{-2}] = 2L_{-2} \) and \([L_1, L_{-2}] = 3L_{-1} \) and solving the obtained
PDEs, we have
\[ L_{-2} = e^{2t}(1 + 3\alpha e^{-2x} + \psi_1(u)e^{-3x})\partial_t + e^{2t}(-2 + 3\phi(u)e^{-x} + \psi_2(u)e^{-2x} + \psi_1(u)e^{-3x})\partial_x + e^{2t}(3\beta e^{-x} + \psi_3(u)e^{-2x})\partial_u, \]
where \(\psi_1, \psi_2, \psi_3\) are arbitrary smooth functions of \(u\).

Using the relations \([L_0, L_2] = -2L_2\) and \([L_{-1}, L_2] = -3L_1\) in a similar fashion, we derive that
\[ L_2 = e^{-2t}f(x, u)\partial_t + e^{-2t}g(x, u)\partial_x + e^{-2t}h(x, u)\partial_u, \]
f, g, h satisfying the following system of PDEs
\[
\begin{align*}
-3(\alpha e^{-2x} + 1)f + 2\alpha e^{-2x}g + (\phi e^{-x} + \alpha e^{-2x} - 1)f_x + \beta e^{-x}f_u + 3 &= 0, \\
(1 - \phi e^{-x} - \alpha e^{-2x})f + (\phi e^{-x} - 2)g - \phi_u e^{-x}h + (\phi e^{-x} + \alpha e^{-2x})g_x &= 0, \\
+ \beta e^{-x}g_u + 3 &= 0,
\end{align*}
\]
\[ \beta e^{-x}f - \beta e^{-x}g + 2(1 + \alpha e^{-2x})h - (\phi e^{-x} + \alpha e^{-2x} - 1)h_x - \beta e^{-x}h_u = 0. \] (10c)

Inserting the expressions for the basis elements \(L_2\) and \(L_{-2}\) into the commutation relation \([L_2, L_{-2}] = 4L_0\) yields three more PDEs
\[
\begin{align*}
4(\psi_1 e^{-3x} + 3\alpha e^{-2x} + 1)f - 3e^{-2x}(\psi_1 e^{-x} + 2\alpha)g + e^{-3x}\dot{\psi}_1 h &= 0, \\
- (\psi_1 e^{-3x} + \psi_2 e^{-2x} + 3\phi e^{-x} - 2)f_x - e^{-x}(\psi_3 e^{-x} + 3\beta)f_u - 4 &= 0, \\
2(\psi_1 e^{-3x} + \psi_2 e^{-2x} + 3\phi e^{-x} - 2)f - (\psi_1 e^{-3x} - 2(3\alpha - \psi_2)e^{-2x} + 3\phi e^{-x} - 2)g_x &= 0, \\
+ e^{-x}(\psi_1 e^{-2x} + \psi_2 e^{-x} + 3\phi)h - (\psi_1 e^{-3x} + \psi_2 e^{-2x} + 3\phi e^{-x} - 2)g_x &= 0, \\
- e^{-x}(\psi_3 e^{-x} + 3\beta)g_u &= 0,
\end{align*}
\]
\[ 2e^{-x}(\psi_3 e^{-x} + 3\beta)h - e^{-x}(2\psi_3 e^{-x} + 3\beta)g + (2\psi_1 e^{-3x} + (6\alpha + \dot{\psi}_3)e^{-2x} + 2)h_x - e^{-x}(\psi_3 e^{-x} + 3\beta)h_u = 0. \] (11)

To determine the forms of \(L_2\) and \(L_{-2}\), we have to solve Eqs. (10) and (11). It is straightforward to verify that the relation
\[ \Delta = e^{-t-4x}[\beta e^{3x} + \psi_3 e^{2x} + (\beta \psi_2 - \phi \psi_3 - 3\alpha \beta)e^x + \beta \psi_1 - \alpha \psi_3] \neq 0 \]
is the necessary and sufficient condition for the system of equations (10) and (11) to have the unique solution in terms of \(f_x, f_u, g_x, g_u, h_x\) and \(h_u\). By this reason, we need to differentiate between the cases \(\Delta = 0\) and \(\Delta \neq 0\).

**Case 2.1.** Let \(\Delta = 0\) or, equivalently, \(\beta = \psi_3 = 0\). Eqs. (10) and (11) do not contain derivatives of the functions \(f, g, h\) with respect to \(u\). That is why the
derivatives $f_x, g_x, h_x$ can be expressed in two different ways using (10) and (11). Equating the right-hand sides of the two expressions for $h_x$ yields
\begin{equation}
he^{x}e^{4x} - 2\phi e^{3x} - \psi_2 e^{2x} - 2\psi_1 e^x + 3\alpha^2 + \phi \psi_1 - \alpha \psi_2 = 0.
\end{equation}
Hence $h = 0$. Similarly, the compatibility conditions for the derivatives $f_x$ and $h_x$ give two more linear equations for the functions $f$ and $g$. The determinant of the obtained system of three linear equations does not vanish. Thus the system in question has the unique solution for $f$ and $g$. Computing the derivatives of the so obtained $f$ and $g$ with respect to $x$ and comparing the results with the previously obtained expressions for $f_x$ and $g_x$, we arrive at the equations
\begin{equation}
(\psi_2 - 6\alpha)(\phi^3 + \phi \psi_2 + 2\psi_1)e^{11x} + F_{10}[x, u] = 0,
\end{equation}
and
\begin{equation}
(10\phi^3\psi_1 - 3\alpha^2(3\psi_2 - 8\alpha) + 3\phi \psi_1(2\alpha + 3\psi_2) + 2(5\psi_1^2
-4\alpha(2\alpha^2 - 3\alpha + \psi_2^2)))e^{10x} + F_9[x, u] = 0.
\end{equation}
Hereafter $F_n[x, u]$ ($n \in \mathbb{N}$) denotes a polynomial in $\exp(x)$ of the power less than or equal to $n$. To find $f$ and $g$ we need to construct the most general $\phi$ and $\psi_i$ ($i = 1, 2, 3$) satisfying Eqs. (12) and (13). If (12) holds, then at least one of the following equations $\psi_2 = 6\alpha$ and $\psi_1 = -(\phi^3 + \phi \psi_2)/2$ should be satisfied.

Case 2.1.1. When $\psi_2 = 6\alpha$, Eqs. (12) and (13) hold if and only if
\begin{equation}
16\alpha^3 + 3\alpha^2 \phi^2 - 6\alpha \phi \psi_1 - \phi^3 \psi_1 - \psi_1^2 = 0,
\end{equation}
whence $\psi_1 = (6\alpha \phi - \phi^3 \pm (4\alpha + \phi^2)\sqrt{2})/2$.

Case 2.1.1.1. Suppose now that $\psi_1 = (6\alpha \phi - \phi^3 - (4\alpha + \phi^2)\sqrt{2})/2$. Provided $\alpha = 0$, we have either $\psi_1 = 0$ or $\psi_1 = -\phi^3$. The case $\alpha = \psi_1 = 0$ leads to $L_{-1} = e^t \partial_t + e^t(-1 + e^{-x}) \partial_x$. Making the equivalence transformation $\tilde{x} = x + X(u)$, we can reduce $\phi$ to one of the forms $a = 0, \pm 1$. Thus
\begin{equation}
f = 1, \quad g = 2 + ae^{-x}.
\end{equation}
Making use of the recurrence relations of the Witt algebra, we arrive at the realization $\mathfrak{W}_2$.

Provided $\alpha = 0$ and $\psi_1 = -\phi^3$, we can reduce the function $\phi$ to the form $b = 0, \pm 1$ with the equivalence transformation $\tilde{x} = x + X(u)$. The case $b \neq 0$ gives rise to the following $f$ and $g$:
\begin{equation}
f = \frac{e^x(e^{2x} - 3be^x + 3b^2)}{(e^x - b)^3}, \quad g = \frac{e^x(2e^x - 3b)}{(e^x - b)^2}.
\end{equation}
Hence the realization $\mathfrak{W}_3$ is obtained. Note that the case $b = 0$ leads to the particular case of $\mathfrak{W}_2$. 

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Assuming $\alpha = \pm 1$, we have $\psi_1 = (-6\alpha \phi - \phi^3 - (4\alpha + \phi_2^2)) / 2$ which yields $W_4$.

**Case 2.1.1.2.** Let $\psi_1 = (-6\alpha \phi - \phi^3 + (4\alpha + \phi_2^2)) / 2$. If $\alpha = 0$, then we have either $\psi_1 = 0$ or $\psi_1 = -\phi^3$. This case has already been considered when we analyzed the Case 2.1.1. When $\alpha = \pm 1$, we get the realization $W_4$.

**Case 2.1.2.** If $\psi_1 = -(\phi^3 + \phi \psi_2) / 2$, then Eq. (12) takes the form

$$(4\alpha + \phi^2)(\psi_2 - (4\alpha - 5\phi^2) / 4)(\psi_2 - (2\alpha - \phi^2)) e^{10x} + F_9[x, u] = 0.$$ 

To solve the above equation, we need to consider the following three subcases.

**Case 2.1.2.1.** Given $\psi_2 = (4\alpha - 5\phi^2) / 4$, Eqs. (12) and (13) hold if and only if $4\alpha + \phi^2 = 0$.

Consequently $\alpha \leq 0$ and $\phi = 2b(-\alpha) \phi$ with $b = \pm 1$.

If $\alpha = -1$, we have $\phi = 2b, \psi_1 = 2b, \psi_2 = -6$ and furthermore

$$f = \frac{e^x(e^x - 2b)}{(e^x - b)^2}, \quad g = \frac{2e^x}{e^x - b},$$

which leads to $W_5$.

In the case when $\alpha = 0$ and $\phi = \psi_1 = \psi_2 = 0$, we arrive at the realization $W_2$ with $\alpha = 0$.

**Case 2.1.2.2.** Let $\psi_2 = 2\alpha - \phi^2$ and suppose that Eqs. (12) and (13) hold.

Provided $\alpha = 0$ we can transform $\phi$ to $b = \pm 1$ (note that the case $b = 0$ has already been considered). Consequently,

$$f = 1, \quad g = \frac{2e^x - b}{e^x - b}$$

and the realization $W_6$ is obtained.

Given $\alpha = \pm 1$, we have

$$f = \frac{e^x(e^x - \phi)}{e^{2x} - e^x \phi - b}, \quad g = \frac{e^x(2e^x - \phi)}{e^{2x} - e^x \phi - b},$$

where $b = \pm 1$. Since $\phi$ can be reduced to the form $\tilde{u}$ by the equivalence transformation $\tilde{u} = \phi$ with $\tilde{u} \neq 0$, we get the realization $W_7$.

**Case 2.1.2.3.** If $4\alpha + \phi^2 = 0$ and Eqs. (12) and (13) holds, we get $\alpha \leq 0$, whence $\alpha = 0, -1$.

Given the relation $\alpha = 0$, we can reduce $\phi$ to the form $a = 0, \pm 1$. With this we obtain $f = 1$ and $g = 2 - ae^{-x}$, thus getting $W_8$.

In the case when $\alpha = -1$, we have

$$f = \frac{e^x(e^{3x} - 4e^{2x} + 5e^x + 4 + \psi_2)}{(e^x - 1)^4}, \quad g = \frac{e^x(2e^{2x} - 4e^x - 4 - \psi_2)}{(e^x - 1)^3}.$$
And what is more the function $\psi_2$ is reduced to the form $\tilde{u}$ by the equivalence transformation $\tilde{u} = \psi_2$, provided $\psi_2$ is a nonconstant function. As a result, we get $\mathfrak{M}_9$.

Summing up we conclude that the case $\Delta = 0$ leads to the realizations $\mathfrak{M}_i$, $i = 2, 3, \cdots, 9$.

**Case 2.2.** If $\Delta \neq 0$, or equivalently, $\beta^2 + \psi_3^2 \neq 0$, then we can solve Eqs. (10) and (11) for $f_x, f_u, g_x, g_u, h_x$ and $h_u$. The compatibility conditions

$$f_{xu} - f_{ux} = 0, \quad g_{xu} - g_{ux} = 0, \quad h_{xu} - h_{ux} = 0$$

can be rewritten as the following system of three linear equations for the functions $f, g, h$

$$a_1 f + a_2 g + a_3 h + d_1 = 0,$$

$$b_1 f + b_2 g + b_3 h + d_2 = 0,$$

$$c_1 f + c_2 g + c_3 h + d_3 = 0.$$  

Here $a_i, b_i, c_i, d_i, (i = 1, 2, 3)$ are functions of $t, x, \phi, \psi_1, \psi_2, \psi_3$.

It is straightforward to verify that the above system has the unique solution $f, g, h$ when $\beta^2 + \psi_3^2 \neq 0$. We do not present here the explicit formulae for these functions as they are very cumbersome. Inserting these $f, g, h$ into Eq. (10a) yields

$$\alpha \beta^6 e^{42x} + F_{41}[x, u] = 0.$$ 

Consequently, we have either $\alpha = 0$ or $\beta = 0$.

**Case 2.2.1.** If $\beta = 0$, then Eq. (10a) takes the form

$$\alpha \psi_3^6 e^{36x} + F_{35}[x, u] = 0,$$

which gives $\alpha = 0$ and $\psi_3 \neq 0$ (since $\Delta = 0$ otherwise). In view of these relations we can rewrite Eq. (11) as follows

$$\psi_1 \psi_3^6 e^{36x} + F_{35}[x, u] = 0,$$

$$\psi_1 \psi_3^6 e^{36x} + F_{35}[x, u] = 0,$$

$$\psi_1 \psi_3^6 e^{36x} + F_{35}[x, u] = 0.$$  

Hence we conclude that $\phi = \psi_1 = \psi_2 = 0$. Inserting these formulas into the initial Eqs. (10) and (11) and solving the obtained system yield

$$f = 1, \quad g = 2, \quad h = -e^{-2x} \psi_3.$$ 

The function $\psi_3$ can be reduced to the form $-1$ by the transformation $\tilde{u} = U(u)$, where $\dot{U} = -1/\psi_3$. As a result, we have $\mathfrak{M}_{10}$.
Case 2.2.2. Provided \( \alpha = 0 \), Eq. (10c) turns into

\[
\beta^5 \left( 4\beta \psi_3 - 6\psi_3^2 + \beta^2 \dot{\psi}_3 \right) e^{41x} + 30\beta^5 \phi \psi_3^2 e^{40x} + F_{39} [x, u] = 0.
\]

Note that the case \( \alpha = \beta = 0 \) has already been analyzed in Case 2.2.1. Consequently, without any loss of generality we can restrict our considerations to the two cases \( \psi_3 = 0, \beta = 1 \) and \( \phi = 0, \beta = 1 \).

If \( \psi_3 = 0 \), then it follows from (11) and (10c) that \( \psi_1 = \psi_2 = 0 \). In view of these relations, we get from (10) and (11) that

\[
f = 1, \quad g = 2 + e^{-x} \phi, \quad h = e^{-x}.
\]

What is more, the function \( \phi \) can be reduced to one the forms \( 0, \pm 1 \) by the equivalence transformations \( \tilde{x} = x + X(u) \) and \( \tilde{u} = U(u) \). Hence we get the realization \( \mathcal{W}_{11} \).

In the case \( \phi = 0 \), Eqs. (10) and (11) are incompatible.

We check by direct computation that the realizations \( \mathcal{W}_i \ (i = 1, 2, \ldots, 11) \) cannot be mapped one into another by a transformation of the form (4). Consequently, they are inequivalent. This completes the proof of the theorem.

While proving Theorem 1, we have also obtained the exhaustive description of the Witt algebras in the spaces \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \), as a by-product.

**Theorem 2.** There is only one inequivalent realization, \( \mathcal{W}_1 \), of the Witt algebra in the space \( \mathbb{R} \).

**Theorem 3.** The realizations \( \mathcal{W}_1 - \mathcal{W}_9 \) with \( \tilde{\phi} = c \in \mathbb{R} \) exhaust the list of inequivalent realizations of Witt algebra in the space \( \mathbb{R}^2 \).

## 4 Realizations of the Virasoro algebra

To construct all inequivalent realizations of the Virasoro algebra \( \mathfrak{V} \), we need to extend inequivalent Witt algebras in Theorem 1 by all possible nonzero central elements \( C \). In this section, we will prove that there are no realizations of the Virasoro algebra with nonzero central element in the space \( \mathbb{R}^3 \).

Let us begin by constructing all possible central extensions of the subalgebra \( \langle L_0, L_1, L_{-1} \rangle \). According to Lemma 1, it suffices to consider the algebras (8) and (9).

**Case 1.** Given the realization (8), we have

\[
L_0 = \partial_t, \quad L_1 = e^{-t} \partial_t, \quad L_{-1} = e^t \partial_t.
\]

Letting the basis element \( C \) be of the general form (3) and inserting it into the commutation relations \([L_i, C] = 0, \ (i = 0, 1, -1) \) yield

\[
C = \xi(x, u) \partial_x + \eta(x, u) \partial_u, \quad \xi^2 + \eta^2 \neq 0.
\]
Applying the transformation
\[
\hat{t} = t, \quad \hat{x} = X(x, u), \quad \hat{u} = U(x, u),
\]
which preserves \(L_0, L_1\) and \(L_{-1}\), to the central element \(C\), we get
\[
C \to \tilde{C} = (\xi X_x + \eta X_u)\partial_x + (\xi U_x + \eta U_u)\partial_u.
\]
We choose solutions of the equations
\[
\xi X_x + \eta X_u = 0, \quad \xi U_x + \eta U_u = 1
\]
as \(X\) and \(U\), and get \(C = \partial_u\).

Proceed now to constructing \(L_2\). Making use of the commutation relations
\[
[L_0, L_2] = -2L_2, \quad [L_{-1}, L_2] = -3L_1 \text{ and } [L_2, C] = 0,
\]
yields \(L_2 = e^{-2t}\partial_t\). Next, let \(L_{-2}\) be of the form (5). With this \(L_{-2}\), the commutation relations (6) involving \(L_{-2}\) are equivalent to an over-determined system of PDEs for the unknown functions \(\tau\), \(\xi\) and \(\eta\). This system turns out to be incompatible. Hence realization (8) cannot be extended up to a realization of the Virasoro algebra with nonzero central element.

**Case 2.** Consider now the algebra (9). Since \(C\) should commute with \(L_0\) and \(L_1\), we have
\[
C = f(u)e^{-x}\partial_t + (g(u) + f(u)e^{-x})\partial_x + h(u)\partial_u,
\]
where \(f\), \(g\) and \(h\) are arbitrary smooth functions. Acting by transformation (7), that does not alter \(L_0\) and \(L_1\), on \(C\) gives
\[
\tilde{C} = f(u)e^{-x}\partial_t + (g(u) + f(u)e^{-x} + h(u)\dot{X}(u))\partial_x + h(u)\dot{U}(u)\partial_u.
\]
To further simplify \(\tilde{C}\), we analyze the cases \(f(u) \neq 0\) and \(f(u) = 0\) separately.

If \(f(u) \neq 0\), then choosing \(X(u) = -\ln|f(u)|\) we have \(\tilde{C} = e^{-x}\partial_t + (e^{-x} + \beta g + h\dot{X}))\partial_x + \beta h\dot{U}\partial_u\), where \(\beta = \pm 1\). Provided \(h = 0\) and \(\dot{g} \neq 0\), we can make the transformation \(\hat{u} = g(u)\) and thus get \(C_1 = e^{-x}\partial_t + (e^{-x} + u)\partial_x\). The case \(h = \dot{g} = 0\) leads to \(C_2 = e^{-x}\partial_t + (e^{-x} + \lambda)\partial_x\), where \(\lambda\) is an arbitrary constant. Next, if \(h \neq 0\) then we choose solutions of the equations \(g + h\dot{X} = 0\) and \(h\dot{U} = 1/\beta\) as \(X\) and \(U\) and thus \(C_3 = e^{-x}\partial_t + e^{-x}\partial_x + \partial_u\) is obtained.

Provided \(f(u) = 0\), we have \(\tilde{C} = (g + \dot{X})\partial_x + h\dot{U}\partial_u\). If \(h \neq 0\), we can reduce \(C_4\) to the form \(\partial_u\) by a suitable choice of \(X\) and \(U\).

Given the condition \(h = 0\), we have \(\tilde{C} = g\partial_x\). If \(g\) is not a constant, then selecting \(U = g(u)\) yields \(C_5 = u\partial_x\). The case of constant \(g\) leads to \(C_6 = \partial_x\).

Summing up, we conclude that there exist six inequivalent nonzero central element \(C\) for the case when \(L_0 = \partial_t\) and \(L_1 = e^{-t}\partial_t + e^{-t}\partial_x\). Now we need to extend the realizations \(\langle L_0, L_1, C_i \rangle, \ (i = 1, 2, \cdots, 6)\) up to realizations of the full Virasoro algebra. Here we present the calculation details for the case \(i = 1\) only. The remaining five cases are handled in a similar fashion.
To extend \( \langle L_0, L_1, C_1 \rangle \) up to a realization of the full Virasoro algebra, we need to construct all possible realizations of \( L_{-1} \). Taking into account (5) we have

\[
L_{-1} = e^{t-2x}(ue^x - 1) \partial_t - \frac{e^{t-2x}(ue^x + 1)^2}{u^2} \partial_x.
\]

With \( L_{-1} \) in hand, we proceed to constructing \( L_2 \). Using the commutation relations (6) yields

\[
L_2 = \frac{ue^x(ue^x + 2)}{e^{2t}(ue^x + 1)^2} \partial_t + \frac{2ue^x}{e^{2t}(ue^x + 1)} \partial_x.
\]

While constructing \( L_{-2} \), we arrive at the incompatible system of PDEs for its coefficients. Hence, the algebra \( \langle L_0, L_1, C_1 \rangle \) cannot be extended to a realization of the full Virasoro algebra. The same result holds for the remaining realizations \( C_2, C_3, \ldots, C_6 \).

**Theorem 4.** There are no realizations of the Virasoro algebra with nonzero central element \( C \) in the space \( \mathbb{R}^n \), \((n = 1, 2, 3)\).

### 5 PDEs invariant under the Witt algebras

In this section we construct a number of new classes of second-order evolution equations in \( \mathbb{R}^2 \) that admit the Witt algebra. Given a realization of the Witt algebra, we can apply the Lie infinitesimal approach to construct the corresponding invariant equation [25, 27]. Differential equation

\[
F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0
\]

is invariant with respect to the Witt algebra \( \langle L_n \rangle \) if and only if the condition

\[\text{pr}^{(2)} L_n(F)|_{F=0} = 0\]

holds for any \( n \in \mathbb{N} \), where \( \text{pr}^{(2)} L_n \) is the second-order prolongation of the vector field \( L_n \), that is

\[\text{pr}^{(2)} L_n = L_n + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \eta^{tt} \partial_{u_{tt}} + \eta^{tx} \partial_{u_{tx}} + \eta^{xx} \partial_{u_{xx}}\]

with

\[
\begin{align*}
\eta^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\eta^{tt} &= D_t(\eta^t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\eta^{tx} &= D_x(\eta^t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi).
\end{align*}
\]
Here the symbols $D_t$ and $D_x$ stand for the total differentiation operators with respect to $t$ and $x$, correspondingly,
\[
D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{xt} \partial_{u_x} + \cdots ,
\]
\[
D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \cdots .
\]

As an example, we present the procedure of constructing $\mathfrak{W}_1$ invariant equations in detail. Utilizing the formulas above, we obtain
\[
\text{pr}^{(2)} L_n = e^{-nt} \partial_t + ne^{-nt} u_t \partial_u + (2ne^{-nt} u_{tt} - n^2 e^{-nt} u_t) \partial_{u_{tt}} + ne^{-nt} u_{tx} \partial_{u_{tx}}. \quad (14)
\]

The next step is computing the full set of functionally-independent second-order differential invariants, $I_m(t, x, u, u_t, u_x, u_{tx}, u_{xx})$ ($m = 1, 2, \ldots, 7$), associated with $L_n$. To get $I_m$, we need to solve the corresponding characteristic equations
\[
\frac{dt}{e^{-nt}} = \frac{dx}{0} = \frac{du}{ne^{-nt} u_t} = \frac{du_t}{0} = \frac{du_x}{2ne^{-nt} u_{tt} - n^2 e^{-nt} u_t} = \frac{du_{tx}}{ne^{-nt} u_{tx}} = \frac{du_{xx}}{0}.
\]

Integration of the above equations yields
\[
I_1 = x, \quad I_2 = u, \quad I_3 = u_x, \quad I_4 = u_{xx}, \quad I_5 = \frac{u_{tx}}{u_t}, \quad I_6 = e^{-nt} u_t, \quad I_7 = e^{-2nt} u_{tt} - ne^{-2nt} u_t.
\]

Hence the most general $L_n$-invariant equation is of the form
\[
F(I_1, I_2, \ldots, I_7) = 0.
\]

Since this equation should be invariant under every basis element of the Witt algebra $\mathfrak{W}_1$, it must be independent of $n$. To meet this requirement, function $F$ has to be independent of $I_6$ and $I_7$. Thus the most general second-order PDE invariant under $\mathfrak{W}_1$ has the form
\[
F(I_1, I_2, I_3, I_4, I_5) = 0,
\]
or, equivalently,
\[
F \left( x, u, u_x, u_{xx}, \frac{u_{tx}}{u_t} \right) = 0.
\]

What is more, we have succeeded in constructing the general forms of PDEs invariant under $\mathfrak{W}_2$, $\mathfrak{W}_6$, $\mathfrak{W}_8$ and $\mathfrak{W}_{10}$. We list the corresponding invariant equations in Table 1, where $F$ is an arbitrary smooth real-valued function.

**Table 1.** Second-order PDEs admitting Witt algebra

| Symmetry algebra | Invariant equation |
|------------------|--------------------|
| $\mathfrak{W}_1$ | $F(x, u, u_x, u_{xx}, \frac{u_{tx}}{u_t}) = 0$ |
| $\mathfrak{W}_2$ | $F(u, u_x, u_{xx}, \frac{u_{tx}-u_{tx}u_x}{e^u u_t}) = 0$, $\alpha = 0$ |
| $\mathfrak{W}_6$ | $F(u, u_{xx} - u_x u_{xx}, e^u u_t + 2u u_{xx} + u_x^2 - 2\alpha u_x) = 0$, $\alpha = \pm 1$ |
| $\mathfrak{W}_8$ | $F(u, \frac{u_{xx} - 2u u_x}{u_t^2}) = 0$ |
| $\mathfrak{W}_{10}$ | $F(u_x + 2u, u_{xx} - 4u) = 0$ |
6 The direct sums of the Witt algebras

This section is devoted to classification of realizations of the direct sum of the Witt algebras in $\mathbb{R}^3$. We obtain the complete description of inequivalent realizations of the direct sums of two Witt algebras.

According to Theorem 1, it suffices to consider realizations of the form

$$\mathcal{M}_i \oplus \langle \tilde{L}_n, n \in \mathbb{Z} \rangle, \quad i = 1, 2, \cdots, 11,$$

where $\mathcal{M}_i$ are given in Theorem 1 and $\tilde{L}_n, n \in \mathbb{Z}$ are basis elements of the Witt algebra commuting with the corresponding realization $\mathcal{M}_i$.

We begin by considering the realization $\mathcal{M}_i \oplus \langle \tilde{L}_n \rangle$. Let us choose $\tilde{L}_n$ in the general form (13). As $\tilde{L}_n$ should commute with $\mathcal{M}_i$, we have

$$\tilde{L}_n = f_n(x,u)\partial_x + g_n(x,u)\partial_u. \quad (15)$$

Here $f_n$ and $g_n$ are arbitrary smooth functions. We have established in Section 3 that the realizations $\langle \tilde{L}_n \rangle$ with basis operators (15) exhaust the list of inequivalent realizations of the Witt algebra in the space $\mathbb{R}^3$ of the variables $t$ and $x$. Consequently, we can replace $t, x$ with $x, u$ respectively in $\mathcal{M}_i, (i = 1, \cdots, 9)$ presented in Theorem 3, thus getting all possible inequivalent realizations of $\mathcal{M}_i \oplus \langle \tilde{L}_n \rangle$.

The realizations $\mathcal{M}_i, (i = 2, \cdots, 11)$ are handled in the same way. We skip rather tedious and cumbersome computations and present the final results in the assertion below.

**Theorem 5.** Any realization of the direct sum of two Witt algebras in $\mathbb{R}^3$ is equivalent to one of the realizations, $\{\mathcal{D}_i, i = 1, 2, \cdots, 10\}$, below

\[
\begin{align*}
\mathcal{D}_1 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx} \partial_x \rangle, \\
\mathcal{D}_2 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx} \partial_x + ne^{-nx} \partial_u \rangle, \\
\mathcal{D}_3 & : \langle e^{-mt} \partial_t + me^{-mt} \partial_x \rangle \oplus \langle ne^{-nu} \partial_x + e^{-nu} \partial_u \rangle, \\
\mathcal{D}_4 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx} \partial_x + \gamma e^{-nx} [e^{nu} \pm (e^u - \gamma)^n] (e^u - \gamma)^{1-n} \partial_u \rangle, \\
\mathcal{D}_5 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx} \partial_x + e^{-nx} [n - \text{sgn}(n) \frac{\gamma}{2} \sum_{j=1}^{[n]-1} j(j+1)e^{-2u}] \partial_u \rangle, \\
\mathcal{D}_6 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx+(n-1)u} (e^u \pm n)(e^u \pm 1)^{-n} \partial_x \\
& \quad + ne^{-nx+(n-1)u} (e^u \pm 1)^{1-n} \partial_u \rangle, \\
\mathcal{D}_7 & : \langle e^{-mt} \partial_t \rangle \oplus \langle e^{-nx+(n-1)u} [e^{2u} - (n+1)\gamma e^u + \frac{1}{2} n(n+1)] (e^u - \gamma)^{-n} \partial_x \\
& \quad + e^{-nx+(n-1)u} [ne^u - \frac{1}{2} n(n+1)\gamma] (e^u - \gamma)^{-n} \partial_u \rangle,
\end{align*}
\]
\[ D_8 : \langle e^{-mt} \partial_t \rangle \oplus \langle J_1 \partial_x + J_2 \partial_u \rangle, \]
\[ D_9 : \langle e^{-mt} \partial_t \rangle \oplus \tilde{W}_4, \]
\[ D_{10} : \langle e^{-mt} \partial_t \rangle \oplus \tilde{W}_7. \]

Here
\[ J_1 = \frac{e^{-nx+(n-1)u}}{(e^u-1)^{n+2}} \left[ (-1 + \sum_{j=1}^{n-1} (2j+1))n + (2n+1)e^u - (n+2)e^{2u} + e^{3u} \right. \]
\[ + \sgn(n) \frac{c}{2} \sum_{j=1}^{n-1} j(j+1) \right], \]
\[ J_2 = \frac{e^{-nx+(n-1)u}}{(e^u-1)^{n+1}} \left[ (1 - \sum_{j=1}^{n-1} (2j+1))n - 2ne^u + ne^{2u} - \sgn(n) \frac{c}{2} \sum_{j=1}^{n-1} j(j+1) \right], \]

\[ n \in \mathbb{Z}, \ m \in \mathbb{Z}, \ c \in \mathbb{R} \text{ and the symbols } \tilde{W}_4 \text{ and } \tilde{W}_7 \text{ stand for the realizations obtained from } W_4 \text{ and } W_7 \text{ listed in Theorem 3 by replacing } (t,x) \text{ with } (x,u). \]

Analysis of second-order differential equations invariant under the direct sum of the Witt algebras yields that there are no equations that admit realizations \( D_4 \), \( D_5 \) and \( D_7 \)–\( D_{10} \). The remaining realizations of the direct sum of the Witt algebras gives rise to the following invariant nonlinear PDEs:

\[ D_1 : \quad F \left( u, \frac{u_{tx}}{u_t u_x} \right) = 0, \quad (16) \]
\[ D_2 : \quad F \left( \frac{u_{tx} e^{-u}}{u_t} \right) = 0, \quad (17) \]
\[ D_3 : \quad F \left( \frac{u_t u_{xx} - u_x u_{tx}}{u_x^3 e^{-x}} \right) = 0, \quad (18) \]
\[ D_6 : \quad F \left( \frac{u_{tx}(1 - u_x \pm e^u) + u_t (u_{xx} - u_x^2 + u_x)}{u_t (e^{2u} + (u_x - 1)(u_x - 1 \mp 2e^u))} \right) = 0. \quad (19) \]

Here \( F \) is an arbitrary smooth real-valued function.

Let us reiterate, any second-order PDE, in two independent variables, which is invariant under the direct sum of the Witt algebras, is equivalent to one of the equations, (16)–(19).

PDEs (16)–(19) are classically integrable in the sense that they admit infinite symmetry groups involving two arbitrary functions of one variable.

Eq. (16) can be rewritten in the equivalent form
\[ u_{tx} = f(u) u_t u_x. \]
Making the change of variables $u \to \tilde{u} = U(u)$ with appropriately chosen $U(u)$ reduces the above PDE to the linear wave equation $\tilde{u}_{tx} = 0$.

Without any loss of generality, we can rewrite (17) in the form

$$u_{tx} = \lambda u_t e^u, \quad \lambda \in \mathbb{R}.$$ Integrating it above with respect to $t$ yields

$$u_x = \lambda e^u + \frac{g''(x)}{g'(x)},$$

where $g(x)$ is an arbitrary smooth function satisfying $g' \neq 0$. The obtained equation can be represented in the equivalent form

$$(u - \ln g'(x))_x = \lambda e^{(u - \ln g'(x))} e^{\ln g'(x)}.$$ It is straightforward to integrate the equation above and thus get the general solution of the initial nonlinear PDE (17)

$$u(t, x) = \ln \frac{g'(x)}{h(t) - \lambda g(x)},$$

where $g, h$ are arbitrary smooth real-valued functions with $g' \neq 0$.

Eq. (18) is equivalent to the following PDE:

$$u_t u_{xx} - u_x u_{tx} = \lambda e^{x^2} u_x^3, \quad \lambda \in \mathbb{R}.$$ The hodograph transformation $x \to u, u \to x$ and re-scaling $t \to \lambda t$ reduce it to the Liouville equation (11), which is known to be integrable.

To the best of our knowledge, Eq. (19) is the new classically integrable nonlinear PDE.

## 7 Concluding Remarks

In this paper, we perform the exhaustive classification of the realizations of the Witt and Virasoro algebras by Lie vector fields in the space $\mathbb{R}^n$ with $n = 1, 2, 3$. The complete lists of inequivalent realizations are given in Theorems 1–5.

The main classification results can be briefly summarized as follows:

- There exists only one inequivalent realization of the Witt algebra in $\mathbb{R}$.
- There are nine inequivalent realizations of the Witt algebra in $\mathbb{R}^2$.
- There exist eleven inequivalent realizations of the Witt algebra in $\mathbb{R}^3$ space.
- There are no realizations of the Virasoro algebra with nonzero central element in the space $\mathbb{R}^n$ with $n \leq 3$. 

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• There exist ten inequivalent realizations of the direct sum of the Witt algebras in $\mathbb{R}^3$.

As an application, we construct a number of new nonlinear PDEs which are invariant under various realizations of the Witt algebra.

What is more, we completely classify the nonlinear second-order PDEs in two independent variables admitting direct sums of the Witt algebras and obtain four canonical invariant equations (16)–(19) which possess infinite-dimensional algebras involving two arbitrary functions. As we have mentioned before, the well-known massless wave and Liouville equations are typical examples of such PDEs. Among them, Eqs. (16)-(18) are well-known, while the nonlinear PDE (19) is seemingly new.

Furthermore, since Virasoro algebra is a subalgebra of the Kac-Moody-Virasoro algebra, the results obtained here can be directly applied to classify the integrable KP type equations in $(1 + 2)$ dimensions. The starting point would be describing inequivalent realizations of the Kac-Moody-Virasoro algebras by differential operators in $\mathbb{R}^4$.

This problem is under study now and will be reported in our future publications.

References

[1] M. Ackerman and R. Hermann, Sophus Lie’s 1880 Transformation Group Paper, Math. Sci. Press: Brookline, 1975.
[2] Y. Alhassid, J. Engel and J. Wu, Algebraic approach to the scattering matrix, Phys. Rev. Lett. 53(1) (1984) 17-20.
[3] P. Basarab-Horwath, V. Lahno and R. Zhdanov, The structure of Lie algebras and the classification problem for partial differential equations, Acta Appl. Math. 69 (2001) 43-94.
[4] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241(2) (1984) 333-380.
[5] J. Campbell, Introductory Treatise on Lie’s Theory of Finite Continuous Transformation Groups, Chelsea: New York, 1966.
[6] B. Champagne, P. Winternitz, On the infinite dimensional symmetry group of the Davey-Stewartson equation, J. Math. Phys. 29 (1988) 1-8.
[7] D. David, N. Kamran, D. Levi and P. Winternitz, Subalgebras of Loop algebras and symmetries of the Kadomtsev-Petviashvili equation, Phys. Rev. Lett. 55 (1985) 2111-2113.
[8] D. David, N. Kamran, D. Levi, P. Winternitz, Symmetry reduction for the Kadomtsev-Petviashvili equation using a loop algebra, J. Math. Phys. 27(5) (1986) 1225-1237.
[9] W. I. Fushchych, W. M. Shtelen and N. I. Serov, Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer: Dordrecht, 1993.
[10] W. Fushchych and R. Zhdanov, Symmetries and Exact Solutions of Nonlinear Dirac Equations, Mathematical Ukraina Publishing: Kyiv, 1997.
[11] A. González-López, N. Kamran and P. Olver, Lie algebras of the vector fields in the real plane, Proc. London Math. Soc. 64(3) (1992) 339-368.
[12] A. González-López, N. Kamran and P. Olver, Lie algebras of differential operators in two complex variables, Amer. J. Math. 114(6) (1992) 1163-1185.
[13] O. Gray, On the complete classification of unitary $N = 2$ minimal superconformal field theories, Comm. Math. Phys. 312(3) (2012) 611-654.
[14] F. Gürsoy and P. Winternitz, Generalized Kadomtsev-Petviashvili equation with an infinite-dimensional symmetry algebra, J. Math. Anal. Appl. 276 (2002) 314-328.
[15] F. Gürsoy and Ö. Aykanat, The generalized Davey-Stewartson equations, its Kac-Moody-Virasoro symmetry algebra and relation to Davey-Stewartson equations, J. Math. Phys. 47 (2006) 013510.
[16] N. H. Ibragimov, Transformation Groups Applied to Mathematical Physics, Reidel: Dordrecht, 1985.
[17] K. Iohara and Y. Koga, Representation Theory of The Virasoro Algebra, Springer-Verlag: London, 2011.
[18] V. Jurdjevic, Geometric Control Theory, Cambridge University Press: Cambridge, 1997.
[19] V. G. Kac, Infinite Dimensional Lie Algebras, Cambridge University Press: Cambridge, 1994.
[20] N. Kamran and P. J. Olver, Equivalence of differential operators, SIAM J. Math. Anal. 20 (1989) 1172-1185.
[21] D. Kosloff and R. Kosloff, A Fourier method solution for the time dependent Schrödinger equation as a tool in molecular dynamics, J. Comp. Phys. 52(1) (1983) 35-53.
[22] D. Levi, P. Winternitz, The cylindrical Kadomtsev-Petviashvili equation, its Kac-Moody-Virasoro algebra and relation to the KP equation, Phys. Lett. A 129 (1988) 165-167.
[23] S. Y. Lou, J. Yu and J. Lin, (2+1)-dimensional models with Virasoro-type symmetry algebra, J. Phys. A: Math. Gen. 28 (1995) L191-L196.
[24] S. Y. Lou and X. Y. Tang, Equations of arbitrary order invariant under the Kadomtsev-Petviashvili symmetry group, J. Math. Phys. 45(3) (2004) 1020-1030.
[25] P. Olver, Applications of Lie Groups to Differential Equations, Springer: New York, 1986.
[26] A. Yu. Orlov, P. Winternitz, Algebra of pseudodifferential operators and symmetries of equations in the Kadomtsev-Petviashvili hierarchy, J. Math. Phys. 38 (1997) 4644-4674.
[27] L. V. Ovsyannikov, Group Analysis of Differential Equations, Academic Press: New York, 1982.
[28] A. Pressley and G. Segal, Loop groups, Clarendon Press: Oxford, 1986.
[29] M. Salazar-Ramírez, D. Martínez, R. D. Mota and V. D. Granados, An $su(1,1)$ algebraic approach for the relativistic Kepler-Coulomb problem, J. Phys. A: Math. Theor. 43(44) (2010) 445203.
[30] M. Senthilvelan and M. Lakshmanan, Lie symmetries, Kac-Moody-Virasoro algebras and integrability of certain (2+1)-dimensional nonlinear evolution equations, J. Nonl. Math. Phys. 5(2) (1998) 190-211.
[31] S. Shnider and P. Winternitz, Nonlinear equations with superposition principles and the theory of transitive primitive Lie algebras, Lett. Math. Phys. 8(1) (1984) 69-78.
[32] R. Z. Zhdanov, V. I. Laino and W. I. Fushchych, On covariant realizations of the Euclid group, Comm. Math. Phys. 212(3) (2000) 535-556.