Heat engines at optimal power: 
Low-dissipation versus endoreversible model

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Low-dissipation model and the endoreversible model of heat engines are two of the most commonly studied models of machines in finite-time thermodynamics. In this paper, we compare the performance characteristics of these two models under optimal power output. We point out a basic equivalence between them, in the linear response regime.

I. INTRODUCTION

The ideal heat cycles discussed in text books, are incapable of modelling realistic thermodynamic machines. For instance, Carnot engine yields maximum work, but its power output (defined as work output per cycle time) is practically zero—due to infeasibly large cycle times. Secondly, the ideal cycle involves no net entropy change in the environment, whereas the operation of real machines, always entails a positive entropy generation. In recent years, research on finite-time models of thermodynamic machines has gained a lot of attention [1–4]. Irreversibilities can be incorporated by assuming a finite rate for heat transfer, internal friction, and heat leakage. Models based on linear irreversible thermodynamics [3–6], the assumption of endoreversibility [7–10], and weak or low dissipation [11–17], are some of the approaches which have been pursued. On the other hand, the finite size of reservoirs, in contrast to infinite reservoirs, also reduces the performance [18–21].

In this paper, we focus on the characteristics—at optimal power output—of two currently studied models in finite-time thermodynamics, viz. low-dissipation model and the endoreversible model. We highlight the similarities and differences between the two models. In particular, we show their equivalence in the linear response regime, i.e. for small difference of bath temperatures.

The plan of the paper is as follows. In Section II, we discuss the optimal features of low-dissipation model. In Section III, we describe optimal features of an endoreversible model assuming a linear irreversible law for heat transfer. In Section IV, we compare the two models.

II. LOW-DISSIPATION MODEL: OPTIMAL OPERATION

Consider a two heat-reservoirs setup, with hot (h) and cold (c) temperatures, \( T_h \) and \( T_c \). A heat engine runs through a four-step cycle by coupling to these reservoirs alternately. The cycle consists of two thermal contacts lasting for time intervals \( \tau_h \) and \( \tau_c \), and two adiabatic steps whose time intervals may be neglected in comparison to the other time scales. Now, the change in entropy of the working medium during heat transfer at the hot/cold contact, can be split as: \( \Delta S_j = \Delta_{\text{rev}} S_j + \Delta_{\text{ir}} S_j \), with \( j = c, h \). Here, the first term accounts for a reversible heat transfer (equal to the amount of heat transferred, divided by the temperature of the reservoir), whereas the second term denotes an irreversible entropy generation during the process. Now, the low-dissipation assumption, which is expected to apply close to the reversibility limit, models the latter term as being inversely proportional to the duration of the time spent (\( \tau_{c, h} \)) on the heat transfer step: \( T_j \Delta_{\text{ir}} S_j = \sigma_j / \tau_j + O(1/\tau_j^2) \), where \( \sigma_j \) is the dissipation constant [12–15]. Thus at the hot and the cold contact, we respectively have

\[
\Delta S_h = \frac{Q_h}{T_h} + \frac{\sigma_h}{T_h \tau_h}, \tag{1}
\]

\[
\Delta S_c = -\frac{Q_c}{T_c} + \frac{\sigma_c}{T_c \tau_c}, \tag{2}
\]

where \( Q_j > 0 \). Given that the other two steps in the heat cycle are adiabatic—with no entropy changes—the cyclic process within the working medium implies \( \Delta S_h + \Delta S_c = 0 \). In other words, \( \Delta S_h = -\Delta S_c = \Delta S > 0 \), where the value \( \Delta S \) is preassigned. Then the amount of heat exchanged with each reservoir can be written as:

\[
Q_h = T_h \Delta S - \frac{\sigma_h}{\tau_h}, \tag{3}
\]

\[
Q_c = T_c \Delta S + \frac{\sigma_c}{\tau_c}. \tag{4}
\]

The work extracted in a cycle with the time period \( \tau \approx \tau_h + \tau_c \) is, \( W = Q_h - Q_c \). So the average power output per cycle is defined as

\[
P = \frac{Q_h - Q_c}{\tau_h + \tau_c}. \tag{5}
\]

Although, the working medium undergoes a cyclic process—with no net entropy change, the whole cycle is irreversible, with a net total entropy production per cycle. This is given by the net change in the entropies of
the two reservoirs:
\[ \Delta_{tot}S = -\frac{Q_h}{T_h} + \frac{Q_c}{T_c}. \]  

(6)

Now, in finite-time thermodynamics, quite often, the desired objective is to maximize the power output. Thus, in order to determine the operating conditions for the maximum power, we set
\[ \left( \frac{\partial P}{\partial \tau_h} \right)_{\tau_c} = 0, \quad \left( \frac{\partial P}{\partial \tau_c} \right)_{\tau_h} = 0, \]  

(7)

where we assume certain given values of the parameters \( \sigma_h, \sigma_c \), and \( \Delta S \). This yields the optimal allocation for the contact times, given by:
\[ \tau_h = \frac{2}{\Delta T \Delta S} \sqrt{\sigma_h(\sqrt{\sigma_h} + \sqrt{\sigma_c})}, \]  

(8)
\[ \tau_c = \frac{2}{\Delta T \Delta S} \sqrt{\sigma_c(\sqrt{\sigma_h} + \sqrt{\sigma_c})}, \]  

(9)

where \( \Delta T = T_h - T_c \). Substituting Eq. (8) in (3), and Eq. (9) in (4), we obtain explicit expressions (at optimal power) for the amounts of heat transferred at each thermal contact:
\[ \dot{Q}_h = T_h \Delta S - \frac{\gamma}{2} \Delta T \Delta S, \]  

(10)
\[ \dot{Q}_c = T_c \Delta S - \frac{(1 + \gamma)}{2} \Delta T \Delta S, \]  

(11)

where \( \gamma = (1 + \sqrt{\sigma_c/\sigma_h})^{-1} \). From Eqs. (10) and (11), the work output per cycle is given by
\[ \dot{W} = \dot{Q}_h - \dot{Q}_c = \frac{1}{2} \Delta T \Delta S. \]  

(12)

The efficiency at optimal power, defined as \( \dot{\eta} = \dot{W} / \dot{Q}_h \), takes the following form:
\[ \dot{\eta} = \frac{\eta_C}{2 - \eta_C}, \]  

(13)

where \( \eta_C = 1 - T_c/T_h \), is the Carnot value. Thus the efficiency at optimal power is bounded as:
\[ \frac{\eta_C}{2} \leq \dot{\eta} \leq \frac{\eta_C}{2 - \eta_C}. \]  

(14)

Then the following extreme cases are of interest. When \( \sigma_c \ll \sigma_h \), or \( \gamma \rightarrow 1 \), it means that the heat transfer at the cold contact approaches the reversible limit, and the efficiency approaches the upper bound. On the other hand, under the condition \( \sigma_h \ll \sigma_c \), or \( \gamma \rightarrow 0 \), the hot contact approaches the reversible limit, and the efficiency approaches its lower bound.

A. Rates of Dissipation

We also note that, under optimal power conditions, the amounts of dissipation at the hot and the cold contacts, defined by \( T_j \Delta_{j}S_j \), is respectively given by:
\[ T_h \Delta_{h}S_h = \frac{\sigma_h}{\tau_h} = \frac{1}{2} \Delta T \Delta S, \]  

(15)
\[ T_c \Delta_{c}S_c = \frac{\sigma_c}{\tau_c} = \frac{1}{2} \Delta T \Delta S. \]  

(16)

Then, at optimal power, the average rates of dissipation at the two thermal contacts, are equal:
\[ \frac{T_h \Delta_{h}S_h}{\tau_h} = \frac{T_c \Delta_{c}S_c}{\tau_c} = \left[ \frac{\Delta T \Delta S}{2(\sqrt{\sigma_h} + \sqrt{\sigma_c})} \right]^2. \]  

(17)

Incidentally, the above rate of dissipation is same as the optimal power output, \( P \).

III. ENDOREVERSIBLE MODEL WITH LINEAR IRREVERSIBLE LAW

In the so-called endoreversible models [4, 7], a specific form of heat-transfer law is assumed between a reservoir and the working medium [8]. Basically, irreversibility arises due to flow of heat—with a finite rate—across a finite heat conductance. In the following, we consider such a model where the heat flux is proportional to the difference of inverse temperatures of the working medium and the reservoir. This particular law is based on the flux-force relation in linear irreversible thermodynamics [22], and is applicable for small temperature gradients. For brevity, we address this model as the linear model. Now, consider \( T_1 \) and \( T_2 \) to be the fixed temperatures of the working medium at hot and cold contacts respectively. Then the heat fluxes are given by
\[ q_h = \alpha_h \left( T_1^{-1} - T_h^{-1} \right), \]  

(18)
\[ q_c = \alpha_c \left( T_2^{-1} - T_c^{-1} \right), \]  

(19)

where \( \alpha_j \), with \( j = c, h \) be the heat conductance. As the fluxes are constant during the times of contact, so the amounts of heat transferred during the times \( t_h \) and \( t_c \), respectively are: \( Q_h = q_h t_h \) and \( Q_c = q_c t_c \). The entropy change in the working medium at hot and cold contacts will be: \( \Delta_S' = Q_h/T_1 \) and \( \Delta_S' = -Q_c/T_2 \), respectively. Again, in the adiabatic steps, the entropy of the working medium stays constant. The cyclicity within the working medium implies, \( \Delta'S_h = -\Delta'S_c = \Delta'S > 0 \), which yields
\[ \frac{Q_h}{T_1} = \frac{Q_c}{T_2} = \Delta'S, \]  

(20)

which is usually known as the endoreversibility condition.

Again, the work extracted per cycle is \( W = Q_h - Q_c \), and the average power per cycle is \( P = (Q_h - Q_c) / (t_h + t_c) \). The efficiency per cycle is
\[ \eta = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_2}{T_1}, \]  

(21)

Now, we optimize the power with respect to variables \( T_1 \) and \( T_2 \). The maximum power is obtained at the following
values:
\[ \tilde{T}_1 = \frac{2(1-\eta_C)}{2 - (1 + \gamma)\eta_C}T_h, \]  
\[ \tilde{T}_2 = \frac{2T_c}{2 - \gamma\eta_C}. \]  

The efficiency at maximum power is
\[ \tilde{\eta} = \frac{\eta_C}{2 - \gamma\eta_C}, \]  
where \( \gamma = (1 + \sqrt{\alpha_c/\alpha_h})^{-1}. \)

Using Eqs. (22) and (23), the heat fluxes at optimal power conditions are given by:
\[ \tilde{q}_h = \frac{1}{2}\alpha_h(1 - \gamma)\left(T_C^{-1} - T_h^{-1}\right), \]  
\[ \tilde{q}_c = \frac{1}{2}\alpha_c\gamma\left(T_c^{-1} - T_h^{-1}\right), \]  
so that \( \tilde{q}_h/\tilde{q}_c = \sqrt{\alpha_h/\alpha_c}. \)

From Eq. (25), we can write for optimal power conditions:
\[ \tilde{Q}_h = \frac{2(1 - \eta_C)}{2 - (1 + \gamma)\eta_C}T_h\Delta'S, \]  
\[ \tilde{Q}_c = \frac{2(1 - \eta_C)}{2 - \gamma\eta_C}T_h\Delta'S, \]  
from which the extracted work per cycle is given by: \( \tilde{W} = \tilde{Q}_h - \tilde{Q}_c. \) From the above expressions, one can easily see that for \( \gamma \to 1, \) the hot contact approaches reversible limit, and the efficiency at maximum power approaches its upper bound of \( \eta_C/(2 - \eta_C). \) Similarly, for \( \gamma \to 0, \) the cold contact becomes reversible and the efficiency approaches the lower bound of \( \eta_C/2. \)

A. Entropy generation

Let us consider the entropy generated at each thermal contact. For the hot contact:
\[ \Delta_{\text{h}}S_h = Q_h (T_1^{-1} - T_h^{-1}). \]  
The average rate of entropy generation at the hot contact is:
\[ \frac{\Delta_{\text{h}}S_h}{t_h} = \frac{Q_h}{t_h} \left(T_1^{-1} - T_h^{-1}\right) \]  
\[ = \frac{\tilde{q}_h^2}{\alpha_h}. \]  

At optimal power, the above rate of entropy generation is given by:
\[ \frac{\Delta_{\text{h}}\tilde{S}_h}{t_h} = \frac{\tilde{q}_h^2}{\alpha_h}. \]  
see Eq. (24). Similarly, we have the corresponding expression of entropy generation, at the cold contact, \( \Delta_{\text{c}}\tilde{S}_c = \tilde{Q}_c (T_c^{-1} - T_2^{-1}) \) with corresponding rate of entropy generation:
\[ \frac{\Delta_{\text{c}}\tilde{S}_c}{t_c} = \frac{\tilde{q}_c^2}{\alpha_c}. \]  

Using the expressions for heat at optimal power, Eqs. (25) and (26), we reach the conclusion that, in the endoreversible model, the rates of entropy production are equal at the hot and the cold contacts, under optimal power.

IV. THE COMPARISON

Although the two models are based on seemingly different assumptions, there is a remarkable similarity between the expressions for efficiency, \( \eta_C \) and \( \eta_T \), at optimal power. It is apparent that the parameters \( \Delta S \) and \( \Delta T \) also play analogous roles in these models. However, there are points of difference. Thus the expressions for the heat exchanged with the reservoirs, and the work performed per cycle appear to be different. Similarly, as has been shown above, in low-dissipation model, the rates of dissipation at the hot and cold contacts become equal, whereas in endoreversible model, it is the two rates of entropy generation, that are equal at optimal power. Also in a sense, the parameters \( \gamma \) and \( \tilde{\gamma} \) play complementary roles. Thus \( \gamma \to 1 \) corresponds to \( \tilde{\gamma} \to 0, \) which is understandable since if the dissipation constant at the cold contact becomes vanishingly small, it implies approach to reversible limit at that contact. Thus the analogous condition for endoreversible model is that the conductance at the cold contact becomes very large. Similarly, we expect that \( \gamma \to 0 \) corresponds to \( \tilde{\gamma} \to 1. \)

However, as we show below, if we identify the common domain of validity for these models, then the optimal performance of these apparently different models exhibits a basic equivalence.

In fact, the linear irreversible law (Eqs. (18) and (19)), is expected to be applicable for small temperature differences. For the hot contact, it implies that \( 1 - T_1/T_h \ll 1. \) Under conditions of optimal power, this condition is \( 1 - \tilde{T}_1/T_h \ll 1, \) which from Eq. (22) gives the condition \( \eta_C \ll 1. \) Applying a similar argument to the cold contact—at optimal power, the corresponding condition is given by \( \tilde{T}_2/T_c - 1 \ll 1, \) which yields \( \gamma\eta_C \ll 1. \) Since \( \gamma \) lies between zero and unity, so the essential condition is, \( \eta_C \ll 1. \) Thus we see that endoreversible model at optimal power, with the linear law, requires small temperature differences between the reservoirs.

Thus in the linear response regime, which implies small values of \( \Delta T = T_h - T_c, \) the expressions for heat in the endoreversible model, (27) and (28), are simplified as follows:
\[ \tilde{Q}_h = T_h \Delta'T - \frac{1 - \tilde{\gamma}}{2}\Delta T \Delta'T, \]  
\[ \tilde{Q}_c = T_h \Delta'T - \frac{2 - \tilde{\gamma}}{2}\Delta T \Delta'T. \]
The above expressions may be compared to the corresponding expressions (10) and (11) for the low-dissipation model. The extracted work per cycle is: \( \bar{W} = \frac{1}{2} \Delta T \Delta S \). This shows that within linear response (upto first order in \( \Delta T \)), the corresponding expressions for heat and work extracted per cycle, are similar in both models. In particular, the parameter \( \gamma \) is equivalent to \( 1 - \gamma \) in this limit. Accordingly, in this regime, the total entropy generated per cycle in the environment, shows similar behavior within the two models.

The above comparison becomes interesting due to the fact that in the low-dissipation model, there is no intrinsic requirement for the temperature difference \( \Delta T \) to be small \([12, 13]\). However, as the comparison with the linear model shows, the expressions at optimal power which are expected to hold in the linear response regime, also hold for arbitrary temperature differences—according to the low dissipation model. These observations indicate the need to analyze more thoroughly the domain of applicability of the low-dissipation model.

Concluding, we have clearly identified the points of similarity, and difference, between the low-dissipation and endoreversible model, under optimal power conditions. The present study also identifies the equivalence of these two models within the linear response regime, consistent with the principles of linear irreversible thermodynamics.

[1] B. Andresen, P. Salamon, and R. S. Berry, “Thermodynamics in finite time,” Phys. Today 37, 62 (1984).
[2] A. Bejan, “Entropy generation minimization: The new thermodynamics of finite-size devices and finite-time processes,” J. Appl. Phys. 79, 1191–1218 (1996).
[3] P. Salamon, J.D. Nulton, G. Siragusa, T.R. Andersen, and A. Limon, “Principles of control thermodynamics,” Energy 26, 307–319 (2001).
[4] Bjarne Andresen, “Current trends in finite-time thermodynamics,” Angewandte Chemie International Edition 50, 2690–2704 (2016).
[5] C. Van den Broeck, “Thermodynamic efficiency at maximum power,” Phys. Rev. Lett. 95, 190602 (2005).
[6] Yang Wang and Z. C. Tu, “Bounds of efficiency at maximum power for linear, superlinear and sublinear irreversible Carnot-like heat engines,” EPL (Europhysics Letters) 98, 40001 (2012).
[7] P. L. Curzon and B. Ahlborn, “Efficiency of a Carnot engine at maximum power output,” Am. J. Phys. 43, 22–24 (1975).
[8] L. Chen and Z. Yan, “The effect of heat transfer law on performance of a twoheatsource endoreversible cycle,” J. Chem. Phys. 90, 3740 (1989).
[9] Y. Apertet, H. Ouerdane, A. Michot, C. Goupil, and Ph. Lecoeur, “On the efficiency at maximum cooling power,” EPL (Europhysics Letters) 103, 40001 (2013).
[10] Luis A. Correa, José P. Palao, Gerardo Adesso, and Daniel Alonso, “Optimal performance of endoreversible quantum refrigerators,” Phys. Rev. E 90, 062124 (2014).
[11] F. Schmied and U. Seifert, “Efficiency at maximum power: An analytically solvable model for stochastic heat engines,” Europhys. Lett. 81, 20003 (2008).
[12] Massimiliano Esposito, Ryoichi Kuwai, Katja Lindenberg, and Christian Van den Broeck, “Efficiency at maximum power of low-dissipation Carnot engines,” Phys. Rev. Lett. 105, 150603 (2010).
[13] Yang Wang, Mingxing Li, Z. C. Tu, A. Calvo Hernández, and J. M. M. Roco, “Coefficient of performance at maximum figure of merit and its bounds for low-dissipation Carnot-like refrigerators,” Phys. Rev. E 86, 011127 (2012).
[14] C. de Tomás, A. Calvo Hernández, and J. M. M. Roco, “Optimal low symmetric dissipation Carnot engines and refrigerators,” Phys. Rev. E 85, 010104 (2012).
[15] Van den Broeck, C., “Efficiency at maximum power in the low-dissipation limit,” EPL 101, 10006 (2013).
[16] Viktor Holubec and Artem Ryabov, “Efficiency at and near maximum power of low-dissipation heat engines,” Phys. Rev. E 92, 052125 (2015); Erratum-ibid. 93, 059904 (2016).
[17] Julian Gonzalez-Ayala, A. Calvo Hernández, and J. M. M. Roco, “From maximum power to a trade-off optimization of low-dissipation heat engines: Influence of control parameters and the role of entropy generation,” Phys. Rev. E 95, 022131 (2017).
[18] M. J. Ondrechen, B. Andresen, M. Mozurchewich, and R. S. Berry, “Maximum work from a finite reservoir by sequential Carnot cycles,” Am. J. Phys. 49, 681 (1981).
[19] Harvey S. Leff, “Available work from a finite source and sink: How effective is a maxwells demon?” American Journal of Physics 55, 701–705 (1987).
[20] Y. Izumida and K. Okuda, “Work output and efficiency at maximum power of linear irreversible heat engines operating with a finite-sized heat source,” Phys. Rev. Lett. 112, 180603 (2014).
[21] R. S. Johal and R. Rai, “Near-equilibrium universality and bounds on efficiency in quasi-static regime with finite source and sink,” EPL (Europhys. Lett.) 113, 10006 (2016).
[22] S. R. de Groot and P. Mazur, Non-equilibrium Thermodynamics (North-Holland Publishing Company, Amsterdam-London, 1969).