BOUNDARY OF THE RANGE OF A RANDOM WALK AND THE FÖLNER PROPERTY

GEORGE DELIGIANNIDIS, SÉBASTIEN GOUÉZEL, AND ZEMER KOSLOFF

Abstract. The range process $R_n$ of a random walk is the collection of sites visited by the random walk up to time $n$. In this work we deal with the question of whether the range process of a random walk or the range process of a cocycle over an ergodic transformation is almost surely a Følner sequence and show the following results: (a) The size of the inner boundary $|\partial R_n|$ of the range of recurrent aperiodic random walks on $\mathbb{Z}^2$ with finite variance and aperiodic random walks in $\mathbb{Z}$ in the standard domain of attraction of the Cauchy distribution, divided by $\frac{n}{\log(n)}$, converges to a constant almost surely. (b) We establish a formula for the Følner asymptotic of transient cocycles over an ergodic probability preserving transformation and use it to show that for transient random walk on groups which are not virtually cyclic, for almost every path, the range is not a Følner sequence. (c) For aperiodic random walks in the domain of attraction of symmetric $\alpha$-stable distributions with $1 < \alpha \leq 2$, we prove a sharp polynomial upper bound for the decay at infinity of $|\partial R_n|/|R_n|$. This last result shows that the range process of these random walks is almost surely a Følner sequence.

1. Introduction

Let $G$ be a countable group with identity element $id_G$, $\xi_1, \xi_2, \ldots$ be i.i.d. $G$-valued random variables and define the random walk $(S_n)_n$, where $S_0 := id_G$ and $S_n = \xi_1 \cdot \xi_2 \cdots \xi_n$ for $n \geq 1$. The range of the random walk, denoted

$$R(n) := \{S_1, \ldots, S_n\},$$

is the random subset of $G$ which consists of the sites visited by the random walk up to time $n$. The case where $G = \mathbb{Z}^d$ will serve as a motivating and recurring example in this paper; as this group is abelian we will denote the random walk in this case by $S_n = \sum_{i=1}^{n} \xi_i$.

The range is a natural object to study and understanding its size and shape is of great interest for a variety of models in probability theory such as random walk in random scenery; see for example [Aar12, DK17] where the size of the range is shown to determine the leading term of the asymptotic growth rate of the information arising from the scenery. The size of the range and its fluctuations have been extensively treated in the literature starting with the seminal paper [DE51] where the authors obtained strong laws in the case of the simple random walk on $\mathbb{Z}^2$, see also [ET60] and [Pla76]. A central limit theorem for the range was obtained in [JP72], whereas the case of random walks with stable increments was treated in [LGR91].

More recently, the focus has shifted towards more involved objects, still related to the range. For example [BKYY10] studies the entropy of the range, [ASS16a, ASS16b] its capacity and finally [ASI17, Oka16] study the boundary of the range, henceforth denoted by $\partial R_n$, that is the sites in the range with at least one neighbour not visited by the random walk. Apart from its intrinsic interest, the motivation for studying the range and its relatives often stems from their relevance in more intricate models; the capacity of the range is of interest in random interlacements (see [Szn10]), whereas the range itself is relevant in the study of random walks in random scenery.

The main focus of this paper is on the boundary of the range, and our interest in this particular object is motivated by its relation to the Følner property of the range, which the first and last authors first studied in [DK17]. In the case of transient random walks on $\mathbb{Z}^d$ with $d \geq 1$, Okada in [Oka16] has proved a law of large numbers result for the size of the boundary

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of the range. The starting point in [Oka16] is a result of Spitzer which uses the ergodic theorem to show that for all random walks, almost surely
\[
\lim_{n \to \infty} \frac{|R_n|}{n} = \mathbb{P}(S_n \neq 0, \text{ for all } n \geq 1),
\]
where clearly the limit is positive for transient random walks. In Section 2 we generalise Okada’s result to transient cocycles of ergodic probability measure preserving systems. We first show, see Proposition 1, that Spitzer’s result on the size of the range extends to this setting. Using this we are able to show that for any transient $G$-valued cocycle, almost surely
\[
\lim_{n \to \infty} \frac{|R_n \triangle R_ng|}{|R_n|} = c(g),
\]
where $c(g)$ is given explicitly in terms of return probabilities, see Theorem 3 for a precise statement. The range of the random walk is almost surely a Föllner sequence if $c(g) = 0$ for all $g$. In the last part of Section 2 we focus on random walks and show that if the Green function of the random walk decays at infinity then the range process is almost surely not a Föllner sequence. We show in Appendix 1 that, except for virtually cyclic groups, the Green function of a transient random walk always tends to $0$ at infinity. Therefore, on any non virtually cyclic group, the range of any transient random walk is almost surely not a Föllner sequence. Furthermore, we show that the only transient random walks in $\mathbb{Z}$ satisfying (1) with $c = 0$ are the obvious ones, namely those with positive mean supported on $(-\infty,1]$ and those with negative mean supported on $[-1,\infty)$, thus answering a question of Kaimanovich.

In the case of the simple random walk on $\mathbb{Z}^2$, the first and last authors have shown in [DK17] that the range is almost surely a Föllner sequence. As the range is a connected set, for all $v \in \mathbb{Z}^2$, the random variable $|\partial R_n|$, the cardinality of the boundary, can be used to bound $|R_n \triangle (R_n + v)|$ from above. In Section 3 we show that if $(S_n)_n$ is either a recurrent aperiodic random walk with finite variance on $\mathbb{Z}^2$, or an aperiodic random walk on $\mathbb{Z}$ in the standard domain of attraction of the symmetric Cauchy distribution, there exists $c > 0$ such that
\[
\lim_{n \to \infty} \frac{|\partial R_n|}{n/\log^2(n)} = c, \text{ almost surely.}
\]
We remark that this constant $c$ coincides with that in the asymptotic of $\mathbb{E} |\partial R_n|$ and thus it follows from [Oka16] that for the simple random walk on $\mathbb{Z}^2$, $c \in (\pi^2/2,2\pi^2)$. Since the range in these settings is almost surely of order $dn/\log(n)$ for some $d > 0$, for all $v \in \mathbb{Z}^2$ (or $\mathbb{Z}$ in the Cauchy case), there exists a $C > 0$ such that almost surely for $n$ large enough,
\[
C^{-1} \left( \frac{1}{\log(n)} \right) \leq \frac{|R_n \triangle (R_n + v)|}{|R_n|} \leq C \left( \frac{1}{\log(n)} \right).
\]
This is a rather precise Föllner asymptotic for the range process of these random walks.

Finally in Section 4 we consider the range process of aperiodic, $\mathbb{Z}$-valued random walks in the domain of attraction of the symmetric, $\alpha$-stable distribution with $1 < \alpha < 2$. In these cases the range, scaled appropriately, converges in distribution to a nontrivial random variable, namely the Lebesgue measure of the set $W_{\alpha}([0,1]) := \{W_{\alpha}(t) : 0 \leq t \leq 1\}$, where $W_{\alpha}$ is the symmetric, $\alpha$-stable Lévy process. For the simple random walk on $\mathbb{Z}$, $R_n$ is an interval, and thus there is no scaling to ensure that $|\partial R_n|/ |R_n|$ converges almost surely to a positive constant. Theorem 21 is a quantitative upper bound of these quantities which is optimal in the polynomial term. In particular this shows that for this class of random walks, the range process is almost surely a Föllner sequence. This in turn can be used, for example, to circumvent the use of dyadic partitions in the calculation of the relative complexity of the scenery in [Aar12] as Kieffer’s Shannon-McMillan-Breiman formula applies directly to the sequence of sets $(R_n)_n$.

2. The Range of Cocycles in Countable Groups

Let $T$ be an invertible, ergodic, measure preserving transformation of a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and $(G, \times)$ be a countable discrete group and denote by $\text{id}_G$ and $m_G$ the identity
element and the Haar measure of $G$ respectively. A $G$-valued cocycle is a function $F : \mathbb{Z} \times \Omega \to G$ which satisfies the cocycle identity: For all $m, n \in \mathbb{N}$ and $\omega \in \Omega$,
\[ F(n + m, \omega) = F(m, \omega) \times F(n, T^m \omega). \]

Any function $f : \Omega \to G$ determines a $G$ valued cocycle $F$ via
\[ F(n, \omega) = \begin{cases} f(\omega) \times f \circ T(\omega) \times \cdots \times f \circ T^{n-1}(\omega) & n \in \mathbb{N} \\ \text{id}_G & n = 0 \end{cases} \]

These cocycles appear in the projection to the second coordinate of the skew product map $T_f : \Omega \times G \to \Omega \times G$ defined by $T_f(\omega, g) = (T\omega, gf(\omega))$. Note that $T_f$ preserves the $\sigma$ finite measure $\mathbb{P} \times mg_G$. The cocycle is recurrent if almost surely $N(\omega) := \# \{ n \in \mathbb{N}; F(n, \omega) = \text{id}_G \} = \infty$ and transient if $N(\omega) < \infty$ almost surely. In [Sch77], it is shown that $F$ is recurrent if and only if $T_f$ is conservative, meaning that it satisfies Poincaré recurrence. Denote by
\[ R(n)(\omega) = R_\omega(n) = \{ F(k, \omega) ; 1 \leq k \leq n \} \]

the range (a.k.a. the trace) of the cocycle up to time $N$. When no confusion is possible we will write $R(n)$ for $R_\omega(n)$. For a finite subset $A \subset G$, $|A|$ is the cardinality of $A$.

**Proposition 1.** Let $(\Omega, \mathcal{B}, \mathbb{P})$ be an ergodic probability preserving transformation, $G$ a countable group and $f : \Omega \to G$. Then for $\mathbb{P}$-almost every $\omega$,
\[ \lim_{n \to \infty} \frac{|R_\omega(n)|}{n} = \mathbb{P}(\omega' \in \Omega; \forall n \in \mathbb{N}, F(n, \omega') \neq \text{id}_G) \]

**Proof.** Define
\[ A_n = \{ \omega' \in \Omega : \forall k \in [1, n] \cap \mathbb{N}, F(k, \omega') \neq \text{id}_G \} \]

and $A = A_\infty$. Counting each $z \in R(n)$ according to the last time it has been visited in $[1, n] \cap \mathbb{N}$, it follows that
\[ |R_\omega(n)| = \sum_{k=1}^{n} 1_{\forall k < j \leq n, F(j, \omega') \neq F(k, \omega)} = \sum_{k=1}^{n} 1_{\forall 0 < j \leq n - k, F(j, T^k \omega) \neq \text{id}_G} \]

and for all $n \in \mathbb{N}$ and $N < n$
\[ \sum_{k=1}^{n} 1_A \circ T^k \leq |R(n)| = \sum_{k=1}^{n} 1_{A_{n-k}} \circ T^k \leq N + \sum_{k=1}^{n-N} 1_{A_N} \circ T^k. \]

By the pointwise ergodic theorem, dividing all sides of the inequality by $n$, one has that for all $N \in \mathbb{N}$, for almost every $\omega \in \Omega$,
\[ \mathbb{P}(A) \leq \lim_{n \to \infty} \frac{|R_\omega(n)|}{n} \leq \lim_{n \to \infty} \frac{|R_\omega(n)|}{n} \leq \mathbb{P}(A_N). \]

Noting that $A_N \downarrow A$ as $N \to \infty$ and thus $\mathbb{P}(A_N) \xrightarrow{N \to \infty} \mathbb{P}(A)$ the conclusion follows. $\square$

The former proposition was proved in [Spr76] for random walks in $\mathbb{Z}^d$. That is the case where $T : (\mathbb{Z}^d)^\mathbb{Z} \to (\mathbb{Z}^d)^\mathbb{Z}$ is the Bernoulli shift, $\mathbb{P}$ is a product measure (distribution of an i.i.d. sequence) and $f(\omega) = \omega_0$, where $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)$.

**Corollary 2.** Let $(\Omega, \mathcal{B}, \mathbb{P})$ be an ergodic probability preserving transformation, $G$ a countable group and $f : \Omega \to G$. If $F$ is recurrent then $|R(n)| = o(n)$ almost surely. In the transient case there exists $c > 0$ such that $|R(n)| \sim cn$ almost surely.

**Proof.** It remains to show that if $F$ is transient then $\mathbb{P}(\omega \in \Omega : \forall n \in \mathbb{N}, F(n, \omega) \neq \text{id}_G) =: c > 0$. To see this let $l : \Omega \to \mathbb{N} \cup \{0\}$
\[ l(\omega) := \sup \{ n \in \mathbb{N} \cup \{0\} : F(n, \omega) = \text{id}_G \}. \]
We need to show that \( \mathbb{P}(l = 0) > 0 \). Since \( F \) is transient there exists \( N \in \mathbb{N} \cup \{0\} \) such that \( \mathbb{P}(B) > 0 \) where \( B = \{ \omega \in \Omega; \ l = N \} \). For all \( \omega \in B \) and \( j \in \mathbb{N} \),
\[
\text{id}_G \neq F(N + j, \omega) = F(N, \omega) \times F(j, T^N \omega) = F(j, T^N \omega).
\]
The latter implies that \( T^N B \subset \{ \omega \in \Omega; l = 0 \} \) and since \( T \) preserves \( \mathbb{P} \) we have \( \mathbb{P}(l = 0) > 0 \). \( \square \)

### 2.1. The asymptotic size of the boundary of transient cocycles.

In \([\text{BKYY}10, \text{Oka}16]\), the asymptotic size of the boundary of a random walk in \( \mathbb{Z}^d \), was studied. More specifically let

\[
\partial R(n) = \{ x \in R(n) : \exists y \in \mathbb{Z}^d \setminus R(n), \ |x - y| = 1 \}
\]

and for \( v \in \mathbb{Z}^d \),
\[
\partial_v R(n) = R(n) \setminus (R(n) + v).
\]

One of the main results of \([\text{Oka}16]\) is that if \( X_1, X_2, \cdots \) is a sequence of i.i.d. symmetric random variables in \( \mathbb{Z}^d \), \( d \geq 3 \), then almost surely, writing \( S_k = \sum_{j=1}^k X_j \) and \( R(n) = \{ S_k : 1 \leq k \leq n \} \),
\[
\lim_{n \to \infty} \frac{|\partial R(n)|}{n} = \mathbb{P} \left( \left\{ \forall k \in \mathbb{N}, S_k \neq \text{id}_G, \forall n \in \mathbb{Z}, \ F(n, \omega) \neq g^{-1} \right\} \right),
\]
where we write \( e_1, \ldots, e_d \) for the usual basis vectors of \( \mathbb{Z}^d \). The following proposition is a cocycle version of this result whose proof is almost identical to \([\text{Oka}16]\).

**Proposition 3.** Let \((\Omega, \mathcal{B}, \mathbb{P})\) be an ergodic probability preserving transformation, \( G \) a countable group and \( f : \Omega \to G \). Then for all \( g \in G \), for \( \mathbb{P} \)-almost every \( \omega \in \Omega \),
\[
\lim_{n \to \infty} \frac{|R_\omega(n) \setminus R_\omega(n)g|}{n} = \mathbb{P} \left( \left\{ \omega \in \Omega; \ \forall n \in \mathbb{N}, F(n, \omega) \neq \text{id}_G, \forall n \in \mathbb{Z}, \ F(n, \omega) \neq g^{-1} \right\} \right)
\]

**Proof.** Let \( g \in G \) and \( \omega \in \Omega \). By definition, \( z \in R_\omega(n) \setminus R_\omega(n)g \) if and only if there exists \( 1 \leq k \leq n \) such that for all \( 1 \leq j \leq n \),
\[
F(j, \omega)g \neq z = F(k, \omega).
\]

By considering the maximal \( k \leq n \) for which \( F(k, \omega) = z \), it holds that \( |R(n) \setminus R(n)g| = \sum_{k=1}^n 1_{B_n(k)} \) where
\[
B_n(k) = \left\{ \omega \in \Omega; \ \forall j \in [1, n] \cap \mathbb{N}, F(j, \omega) \neq F(k, \omega)g^{-1}, \ \forall k < j \leq n, \ F(j, \omega) \neq F(k, \omega) \right\}
\]

As for all \( k, j \in \mathbb{Z} \), \( F(k, \omega)^{-1} F(j, \omega) = F(j - k, T^k \omega) \), for all \( k \in \{1, \ldots, n\} \),
\[
B_n(k) := T^{-k} \left\{ \omega \in \Omega; \ \forall j \in [-k, n-k] \cap \mathbb{Z}, \ F(j, \omega) \neq g^{-1}, \ \forall j \in [1, n-k] \cap \mathbb{N}, \ F(j, \omega) \neq \text{id}_G \right\}
\]

This implies that for all \( N < n/2 \),
\[
\sum_{k=1}^n 1_{C \circ T^k} \leq |R(n) \setminus R(n)g| \leq 2N + \sum_{k=N+1}^{n-N} 1_{C_N \circ T^k},
\]
where
\[
C_N := \left\{ \omega \in \Omega; \ \forall j \in [-N, N] \cap \mathbb{Z}, \ F(j, \omega) \neq g^{-1}, \ \forall j \in [1, N] \cap \mathbb{N}, \ F(j, \omega) \neq \text{id}_G \right\}
\]
and \( C = C_\infty = \cap N C_N \). The conclusion follows from the ergodic theorem as in the proof of Proposition 1. \( \square \)

**Corollary 4.** Let \((\Omega, \mathcal{B}, \mathbb{P})\) be an ergodic probability preserving transformation, \( G \) a countable group and \( f : \Omega \to G \). If \( F \) is transient then for all \( g \in G \),
\[
\frac{|R(n) \setminus R(n)g|}{|R(n)|} \overset{a.s.}{\underset{n \to \infty}{\longrightarrow}} \mathbb{P} \left( \forall k \in \mathbb{Z}, \ F(k, \cdot) \neq g^{-1} \right) \forall k \in \mathbb{N}, \ F(k, \cdot) \neq \text{id}_G \).
\]

**Proof.** Follows from a combination of the last Proposition, Proposition 1 and Corollary 2. \( \square \)
A sequence \( \{F_n\}_{n \in \mathbb{N}} \) of subsets of \( G \) is a right Følner sequence in \( G \) if for all \( n \in \mathbb{N} \), \( F_n \) is a finite set and for all \( g \in G \),
\[
\frac{|F(n) \triangle F(n)g|}{|F(n)|} \xrightarrow{n \to \infty} 0.
\]
The existence of Følner sequences is equivalent to amenability of the group (existence of a right invariant mean on \( G \)). In [DKT17], it was shown that the range of the symmetric random walk in \( \mathbb{Z}^2 \) is almost surely a Følner sequence. Kaimanovich, in a private communication, suggested to check which cocycles have almost surely Følner ranges. The following is a partial advance on this question for transient cocycles.

**Theorem 5.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be an ergodic probability preserving transformation, \( G \) a countable group and \( f : \Omega \to G \). If \( F \) is transient then for all \( g \in G \), almost surely,
\[
\lim_{n \to \infty} \frac{|R(n) \triangle R(n)g|}{|R(n)|} = \Phi(g) + \Phi\left(g^{-1}\right)
\]
where \( \Phi(g) := \mathbb{P}(\forall n \in \mathbb{Z}, \ F(n, \cdot) \neq g| \forall k \in \mathbb{N}, \ F(k, \cdot) \neq \text{id}_G) \).

**Proof.** It is easy to see, by multiplying by \( g^{-1} \), that for all \( n \in \mathbb{N} \),
\[
|R(n)g \setminus R(n)| = \left|R(n) \setminus R(n)g^{-1}\right|.
\]
Since \( (R(n) \triangle R(n)g) = (R(n) \setminus R(n)g) \cup (R(n)g \setminus R(n)) \), the result follows from Corollary 4. \( \square \)

### 2.2. Følner property and transient random walks

We now turn to an application of Theorem 5 in the context of random walks.

Let \( G \) be a countable group. Given \( p \) a probability measure on \( G \) and \( \xi_1, \xi_2, \ldots \) an i.i.d. sequence with marginals distributed like \( p \), let \( S_n = \xi_1 \xi_2 \cdots \xi_n \) be the corresponding random walk and \( R(n) := \{S_1, \ldots, S_n\} \) be its range process. The ergodic theoretic model of the random walk is the following skew product transformation. Let \( \Omega := G^\mathbb{Z} \), \( \mathbb{P} = p^{\otimes \mathbb{Z}} \) the product measure on \( \Omega \) with marginals distributed like \( p \) and \( T : \Omega \to \Omega \) the full shift defined by \( (T \omega)_n = \omega_{n+1} \). Writing \( m_G \) for the Haar measure of \( G \), and \( f : \Omega \to G \), \( f(\omega) := \omega(0) \), the skew product transformation \( T_f : \Omega \times G \to \Omega \times G \) satisfies
\[
\pi_G(T^\omega_f) = S_n,
\]
where \( \pi_G(\omega, h) = h \) is the projection to the \( G \) coordinate. The advantage of working with the skew product is that the cocycle identity indicates what is the relevant random walk in inverse time. In this case, for write \( n \in \mathbb{Z} \),
\[
S_n^-(\omega) := F(-n, \omega) = \omega(-1)^{-1} \omega(-2)^{-1} \cdots \omega(-n)^{-1}.
\]
Corollary 4 gives the following extension of Okada's result.

**Corollary 6.** Let \( G \) be a discrete countable group and \( p \) a probability measure on \( G \) and \( \xi_1, \xi_2, \ldots \) an i.i.d. sequence with marginal \( p \). Then almost everywhere
\[
\lim_{n \to \infty} \frac{|R(n) \triangle (R(n) \cdot g)|}{|R(n)|} \xrightarrow{\text{a.e.}} \mathbb{P}(\forall n \in \mathbb{N}, \ S_n^- \neq g) \mathbb{P}(\forall n \in \mathbb{N}, \ S_n \neq \text{id}_G)
\]
\[
+ \mathbb{P}(\forall n \in \mathbb{N}, \ S_n^- \neq g^{-1}) \mathbb{P}(\forall n \in \mathbb{N}, \ S_n \neq g^{-1} \ | \forall n \in \mathbb{N}, \ S_n \neq \text{id}_G).
\]

**Proof.** This is a direct consequence of Corollary 4 and the fact that \( \{S_n\}_{n=1}^\infty \) and \( \{S_n^-\}_{n=1}^\infty \) are independent. \( \square \)

**Remark 7.** In the case where \( G \) is Abelian, \( S_n^- \overset{d}{=} (S_n)^{-1} \) and thus for all \( g \in G \),
\[
\mathbb{P}(\forall n \in \mathbb{N}, \ S_n \neq g^{-1}) = \mathbb{P}(\forall n \in \mathbb{N}, \ S_n^- \neq g).
\]
The statement of Corollary 4 can be simplified in this case. Note that for a general group \( (S_n^-)^{-1} \) is a (multiplication from the) left random walk and \( S_n \) is a (multiplication from the) right random walk and their distribution as processes may no longer coincide.
Let us start with the case of $S_n$ a transient random walk on $\mathbb{Z}$. In this setting, if $p$ has finite first moment, by Atkinson’s Theorem \cite{Atkinson} and the law of large numbers, either $\lim_{n \to \infty} S_n = \infty$ almost surely or $\lim_{n \to \infty} S_n = -\infty$ almost surely. If $\lim_{n \to \infty} S_n = \infty$ almost surely and
\[ P(X_1 > 1) = 0, \]
then for almost all $\omega \in \Omega$, $\{R(n)\}_{n=1}^{\infty}$ is an eventually monotone sequence of growing intervals with $|R(n)| \to \infty$ as $n \to \infty$. Consequently the range process is almost surely a F"olner sequence. A similar statement holds for walks with $S_n \to -\infty$ as $n \to \infty$ and $P(X_1 < -1) = 0$. We now show this is the only possibility for $\mathbb{Z}$-valued random walks with almost surely F"olner ranges.

**Theorem 8.** Let $S_n = \sum_{k=0}^{n-1} X_k$ be a $\mathbb{Z}$-valued random walk such that $\lim_{n \to \infty} S_n = \infty$ almost surely and $P(X_1 > 1) > 0$, then almost surely $\{R(n)\}_{n=1}^{\infty}$ is not a F"olner sequence.

**Proof.** As the random walk is transient and $S_n \to \infty$ almost surely, we have
\[ P(\forall n \in \mathbb{N}, S_n > 0) > 0. \]
In addition there exists $\mathbb{Z} \ni j > 1$ such that $P(X_1 = j) > 0$. Therefore,
\[ P(\forall n \in \mathbb{N}, S_n > 1) \geq P(X_0 = j \text{ and } \forall 2 \leq n \in \mathbb{N}, S_n - X_0 > 0) = P(X_0 = j) \cdot P(\forall n \in \mathbb{N}, S_n > 0), \]
by the Markov property of $S_n$.

It follows that $P(\forall n \in \mathbb{N}, S_n > 1) > 0$ and
\[ P(\forall n \in \mathbb{N}, S_n \neq 1|\forall n \in \mathbb{N}, S_n \neq 0) \geq \frac{P(X_1 = j) \cdot P(\forall n \in \mathbb{N}, S_n > 0)}{P(\forall n \in \mathbb{N}, S_n \neq 0)} > 0. \]
As the distributions of $\{S_n^{(-)}\}_{n=1}^{\infty}$ and $\{-S_n\}_{n=1}^{\infty}$ are the same,
\[ P(\forall n \in \mathbb{N}, S_n^{(-)} \neq 1) = P(\forall n \in \mathbb{N}, S_n \neq -1) \geq P(\forall n \in \mathbb{N}, S_n > 0) > 0. \]
We have shown that
\[ P(\forall n \in \mathbb{N}, S_n^{(-)} \neq 1) \cdot P(\forall n \in \mathbb{N}, S_n \neq 1|\forall n \in \mathbb{N}, S_n \neq 0) > 0. \]

By Corollary \cite{corollary}, $\lim_{n \to \infty} |R(n)| / |R(n) + 1| > 0$ almost surely and $R(n)$ is almost surely not a F"olner sequence. □

For other types of groups, it is most often the case that for any transient random walk on $G$, its Green function decays at infinity, see Appendix \cite{appendix}. We next explain what we mean by this and show that in this case the range of a random walk is almost surely not a F"olner sequence. This is applied in Corollary \cite{corollary} to show that for all admissible transient random walks on non-virtually cyclic groups, the range is almost surely not a F"olner sequence.

The Green function of a transient random walk on $G$ is the function $G : G \to [0, \infty]$ defined by
\[ G(g) := \mathbb{E}(L(g)) = \sum_{n=0}^{\infty} P(S_n = g) \quad (2) \]
where $L(g) := |\{n \in \mathbb{N} \cup \{0\}, S_n = g\}| : \Omega \to \mathbb{N} \cup \{0\}$. For $g \in G$ denote by
\[ q(g) := P(\forall n \in \mathbb{N}, S_n \neq g), \]
and write $q := q(id_G)$.

**Lemma 9.** If $S_n$ is a $G$-valued transient random walk then for all $g \in G$,
\[ G(g) = \frac{1 - q(g)}{q}. \]

\(^1\)See the proof of Corollary \cite{corollary} where it shown that $P(l = 0) > 0$.\]
Proof. Let \( g \in G \). By the Markov property for the random walk,

\[
\mathbb{P}(L(g) = 0) = q(g)
\]

and thus

\[
\mathbb{E}(L(g)) = q(g) \cdot 0 + (1 - q(g)) \sum_{j=1}^{\infty} j(1-q)^{j-1}q
\]

\[
= (1-q(g)) \mathbb{E}(L(0)) = \frac{1-q(g)}{q}.
\]

Given a random walk \( S_n \) on \( G \), \( G^{(-)} \) denotes the Green function of \( S_n^{(-)} \).

**Theorem 10.** If \( S_n \) is an \( \mathbb{G} \) valued transient random walk and there exists \( \{g_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} \max\left(G\left(g_n\right), G^{(-)}(g_n)\right) = 0 \), then \( \{R(n)\}_{n=1}^{\infty} \) is almost surely not a Föllner sequence.

**Proof.** It follows from Lemma 9 that \( G^{(-)}(g_n) \xrightarrow{n \to \infty} 0 \) implies that \( q(g_n) \xrightarrow{n \to \infty} 1 \). Letting

\[
A_n := \{\forall k \in \mathbb{N}, S_k \neq g_n\}, \quad B := \{\forall k \in \mathbb{N}, S_k \neq \text{id}_G\},
\]

we have that \( \mathbb{P}(A_n) = q(g_n) \to 1 \) and \( \mathbb{P}(B) = q \), whence it easily follows that

\[
\lim_{n \to \infty} \mathbb{P}(\forall k \in \mathbb{N}, S_k \neq g_n|\forall k \in \mathbb{N}, S_k \neq \text{id}_G) = \lim_{n \to \infty} \frac{\mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = 1.
\]

A similar reasoning shows that \( G^{(-)}(g_n) \xrightarrow{n \to \infty} 0 \) implies that

\[
\lim_{n \to \infty} \mathbb{P}(\forall k \in \mathbb{N}, S_k^{(-)} \neq g_n) = 1.
\]

By this

\[
\lim_{n \to \infty} \left[ \mathbb{P}\left(\forall k \in \mathbb{N}, S_k^{(-)} \neq g_n\right) \mathbb{P} \left(\forall k \in \mathbb{N}, S_k \neq g_n|\forall k \in \mathbb{N}, S_k \neq \text{id}_G\right) \right] = 1
\]

and an application of Corollary 6 shows that for all large \( n \in \mathbb{N} \), almost surely

\[
\lim_{k \to \infty} \frac{|R(k) \triangle (R(k) \cdot g_n)|}{|R(k)|} > \frac{1}{2}.
\]

We conclude that almost surely the range is not a Föllner sequence. \( \square \)

Let us say that a random walk driven by a measure \( p \) on the group \( \mathbb{G} \) is *admissible* (or *irreducible*) if the walk starting from the identity can reach any point in the group, i.e., the semigroup generated by the support of \( p \) is the whole group. This is a natural irreducibility assumption. We prove in Appendix B that, except for virtually cyclic groups, the Green function of a transient admissible probability measure tends to 0 at infinity.

**Corollary 11.** Let \( S_n \) be a transient admissible random walk on a group \( \mathbb{G} \) which is not virtually cyclic. Then almost surely \( \{R(n)\}_{n=1}^{\infty} \) is not a Föllner sequence.

In particular, for all \( d \geq 3 \), the range of the simple random walk on \( \mathbb{Z}^d \) is almost surely not a Föllner sequence.

**Proof.** This follows directly from the previous Theorem 10 and Theorem 28 showing that the Green functions \( G \) and \( G^{(-)} \) tend to 0 at infinity. \( \square \)
3. Precise Föllner asymptotics for recurrent planar random walks and Cauchy random walks in $\mathbb{Z}$

Let $\xi_0, \xi_1, \cdots$ be i.i.d. $\mathbb{Z}^d$-valued random variables satisfying one of the following two assumptions

(A1) $d = 2$ and there exists a nonsingular covariance matrix $\Sigma \in M_{2 \times 2}(\mathbb{R})$ such that for all $t \in [-\pi, \pi]^2$,
$$
\phi(t) = \mathbb{E}(\exp(i \langle t, \xi \rangle)) = 1 - \langle \Sigma t, t \rangle + o(|t|^2).
$$

(A2) $d = 1$ and there exists $\gamma > 0$ such that for $t \in [-\pi, \pi]$,
$$
\phi(t) = \mathbb{E}(\exp(it\xi)) = 1 - \gamma|t| + o(|t|).
$$

We will always assume that the random walk is aperiodic, namely that there is no proper subgroup of $\mathbb{Z}^d$ containing the support of $\xi_1$. By [Spi76, Theorem 7.1] this is equivalent to requiring that $\phi(t) = 1$ for $t \in [-\pi, \pi]^d$ if and only if $t = 0$.

We will often need the stronger notion of strong aperiodicity, which is equivalent to $|\phi(t)| < 1$ for all $t \in (-\pi, \pi)^d \setminus \{0\}$. For aperiodic random walks, it amounts to the condition that the greatest common divisor of return times to the origin is 1. For instance, the simple random walk is aperiodic, but not strongly aperiodic.

For $v \in \mathbb{Z}^d$, we define the $v$-boundary of the range as
$$
\partial_v R(n) := R(n) \setminus (R(n) + v).
$$

For $d = 1, 2$, write $E_d := \{\pm e_i : 1 \leq d\}$, where $e_i$ are the usual generators of $\mathbb{Z}^d$. The boundary is defined as
$$
\partial R(n) = \left\{ x \in R(n) : \exists y \in \mathbb{Z}^d \setminus R(n), \ |x - y| = 1 \right\},
$$
which can also be written in the form
$$
\partial R(n) = \bigcup_{v \in E_d} (R(n) \setminus (R(n) + v)) = \bigcup_{v \in E_d} \partial_v R(n). \tag{3}
$$

The following is the main result of this section.

**Theorem 12.** Let $\xi_0, \xi_1, \cdots, \xi_n, \cdots$ be i.i.d. $\mathbb{Z}^d$-valued aperiodic random variables satisfying either Assumption (A1) or Assumption (A2). Then for every $v \in \mathbb{Z}^d$, there exist constants $C_v, C > 0$ such that almost surely,
$$
\lim_{n \to \infty} \frac{\log^2(n)}{n} |\partial_v R(n)| = C_v, \quad \lim_{n \to \infty} \frac{\log^2(n)}{n} |\partial R(n)| = C.
$$

In addition $\mathbb{E}(|\partial_v R(n)|) \sim \frac{C_v n}{\log^2(n)}$ and $\mathbb{E}(|\partial R(n)|) \sim \frac{C n}{\log^2(n)}$ as $n \to \infty$.

The statements in Theorem 12 for $\partial_v R(n)$ and $\partial R(n)$ are proved in a parallel way, the proofs being slightly more involved for $\partial R$ instead of $\partial_v R$ because of the union in (3). We will therefore concentrate on the case of $\partial R$.

In [BKYY10] it was shown that in the case of the symmetric random walk on $\mathbb{Z}^2$, $\mathbb{E} |\partial R(n)|$ is proportional to the entropy of the range (at time $n$) and it is of order constant times $n/\log^2(n)$.

Okada in [Oka16] has shown that $2^{-1} \pi^2 \leq C \leq 2 \pi^2$. The previous Theorem is a law of large numbers type result for a more general class of random walks which includes random walks in the domain of attraction of the (symmetric) Cauchy distribution. It also gives a more precise estimate on the almost sure Föllner property of the range of such random walk, see [DK17] for some consequences of the Föllner property of the range in the model of random walks in random sceneries. The proof goes by first establishing an upper bound for the variance of $|\partial R(n)|$ which gives the convergence in probability. After that we use a method from [Fla76] which improves the asymptotics of the rate of convergence in probability, thus enabling us to use the Borel Cantelli lemma for showing convergence almost surely.
3.1. Auxiliary results. We now state and prove a number of auxiliary results that will be used in the proof of Theorem 12 that we could not find in the literature.

Proposition 13. Under the assumptions of Theorem 12, for all \( j \in \mathbb{Z}^d \setminus \{0\} \) there exist \( c_j, d_j > 0 \) such that

\[
P(\forall 1 \leq k \leq n, \ S_k \notin \{0, j\}) = \frac{c_j}{\log(n)} + o\left(\frac{\log(n)}{n}\right) \quad \text{as} \quad n \to \infty.
\]

and

\[
P(\forall 1 \leq k \leq n, \ S_k \neq j) = \frac{d_j}{\log(n)} + o\left(\frac{\log(n)}{n}\right) \quad \text{as} \quad n \to \infty.
\]

Remark. The previous proposition in the case of symmetric random walks was (essentially) treated in [Oka16].

Proof. The proof is a simple application of [KS63, Theorem 4a]. This theorem states that for any aperiodic random walk in \( \mathbb{Z}^d, W \subset \mathbb{Z}^d \) a finite subset, and any \( x \in \mathbb{Z}^d \) we have

\[
\lim_{n \to \infty} \frac{P^x(S_k \notin W, k = 1, \ldots, n)}{P^0(S_k \neq 0, k = 1, \ldots, n)} = \tilde{g}_W(x),
\]

where for the random walks of interest to us, by [KS63, Eq.(1.16)], and \( x, y \in \mathbb{Z}^d \)

\[
\tilde{g}_W(x, y) := 1_{[x=y]} + \sum_{n=1}^{\infty} P^x(S_n = y, S_k \notin W, 1 \leq k \leq n - 1),
\]

\[
\tilde{g}_W(x) = \lim_{|y| \to \infty} \tilde{g}_W(x, y).
\]

Under assumptions (A1), (A2), it follows from the local limit theorem that

\[
\sum_{j=0}^{n} P(S_j = 0) \sim c \log n.
\]

Using this asymptotic, a standard argument, see for example [DE51], shows that there exists a constant \( \gamma_d > 0 \), depending on the random walk, such that

\[
P^0\{S_k \neq 0, k = 1, \ldots, n\} \sim \frac{\gamma_d}{\log n}.
\]

When \( W = \{0, j\} \), from [KS63, Eq.(5.17) and (5.18)] we have that

\[
\tilde{g}_{\{0,j\}}(j) = \frac{a(j)}{a(j) + a(-j)}, \quad \tilde{g}_{\{0,j\}}(0) = \frac{a(-j)}{a(j) + a(-j)},
\]

where

\[
a(j) = \sum_{n=0}^{\infty} \left[ P^0(S_n = 0) - P^j(S_n = 0) \right] = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1 - \exp(ijt)}{1 - \phi(t)} dt.
\]

Under our assumptions, we have \( a(x) > 0 \) for all \( x \neq 0 \), by [Spi76, Propositions 11.7 and 30.2], whence \( c_j, d_j > 0 \). The result follows from this and \([4] \).

Lemma 14. Under the assumptions of Theorem 12, for any nonempty finite subset \( O \) of \( \mathbb{Z}^d \) and any \( x, y \in \mathbb{Z}^d \), there exists \( C = C(O, x, y) \) such that for all \( n \in \mathbb{N}, \)

\[
P^x[S_n = y, S_j \notin O, 1 \leq j \leq n - 1] \leq \frac{C}{n \log^2(n)}.
\]

Proof. Let

\[
Q^0_{O}(x, y) := P^x[S_n = y, S_j \notin O, 1 \leq j \leq n - 1].
\]

Assume first that (A1) holds, i.e., \( d = 1 \), and that the walk is strongly aperiodic. By [Kes63, Theorems 8 and 9] we have

\[
\lim_{n \to \infty} \frac{Q^0_{O}(x, y)}{\pi \gamma} n \log(n)^2 = 1,
\]

which implies the result.
Under (A2), still with strong aperiodicity, the estimate
\[
\sum_{u,v\in O} Q^0_B(u, v) \sim \frac{2\pi \det(\Sigma)^{1/2}}{n \log(n)^2},
\]
follows from [Kes63, Theorem 9] and [JP72, Theorem 4.1]. Then
\[
Q^0_B(x, y) \sim \frac{2\pi \det(\Sigma)^{1/2}}{n \log(n)^2} \tilde{g}_0(x) \tilde{g}_0(-y),
\]
follows from the above and [Kes63, Theorem 6a]. Indeed, although [Kes63, Theorem 9] is stated only for one-dimensional walks in the domain of attraction of a symmetric stable law, as explained in the proof the result remains true for any recurrent random walk satisfying [Kes63, Equation (11.1)]. This has been established under (A2) for strongly aperiodic random walks in [JP72, Theorem 4.1].

Under (A1) or (A2), but with strong aperiodicity, we have proved the result of the lemma. We claim that it still holds if one weakens strong aperiodicity to aperiodicity, but a little extra work is needed. Letting \(\{S_j\}_{j\geq 0}\) denote the original random walk, we denote with \(\{\tilde{S}_j\}_{j\geq 0}\) a lazy version of it. In particular, if \(S_j = \xi_1 + \cdots + \xi_j\), we let \(\tilde{S}_j = \tilde{\xi}_1 + \cdots + \tilde{\xi}_j\), where \(\tilde{\xi}_j = B_j \xi_j\) where \(\mathbb{P}\{B_j = 1\} = \rho \in (0, 1)\) and \(\mathbb{P}\{B_j = 0\} = 1 - \rho\). It can be easily checked that \(\{\tilde{S}_j\}_{j\geq 0}\) is then strongly aperiodic and therefore that
\[
\tilde{Q}^n_B(x, y) := \mathbb{P}^x \left[ \tilde{S}_n = y, \tilde{S}_j \notin O, 1 \leq j \leq n - 1 \right] \leq \frac{C}{n \log(n)^2}.
\]
In addition let
\[
T_0 := 0, \quad T_n := \inf\{n > T_{n-1} : \tilde{\xi}_n \neq 0\}, n \geq 1.
\]
It is then clear that for all \(j \geq 1\), \(T_j - T_{j-1}\) are i.i.d. geometrically distributed on the positive integers. With this notation, we can embed a path of \(\{S_j\}_{j\geq 0}\) into a path of \(\{\tilde{S}_j\}_{j\geq 0}\) by letting \(S_j = \tilde{S}_{T_j}\) for \(j \geq 0\). Up to a time-change, the paths of \(\{S_j\}\) and \(\{\tilde{S}_j\}\) coincide and thus
\[
Q^0_B(x, y) = \mathbb{P}^x \left[ S_n = y, S_j \notin O, 1 \leq j \leq n - 1 \right]
= \mathbb{P}^x \left[ \tilde{S}_{T_n} = y, S_j \notin O, 1 \leq j \leq T_n - 1 \right]
= \sum_{k=n}^{\infty} \mathbb{P}^x \left[ \tilde{S}_k = y, S_j \notin O, 1 \leq j \leq k - 1 \right] \mathbb{P}(T_n = k)
\leq C \sum_{k=n}^{\infty} \frac{\mathbb{P}(T_n = k)}{k \log(k)^2} \leq \frac{C}{n \log(n)^2} \sum_{k=n}^{\infty} \mathbb{P}(T_n = k) \leq \frac{C}{n \log(n)^2}.
\]

As \(\mathbb{Z}^d\) is an Abelian group, in this case we consider instead a bi-infinite i.i.d. sequence \(\{X_i\}_{i\in\mathbb{Z}}\) and write for \(n \in \mathbb{N}\),
\[
S_n^{(-)} = - \sum_{k=-n}^{1} X_k.
\]
In what follows we will make use of the fact that for \(k < n\) and \(v \in \mathbb{Z}^d\),
\[
S_k = S_n + v \quad \text{iff} \quad S_n^{(-)} \circ T^n = v \quad \text{iff} \quad S_{n-k} \circ T^k = -v.
\]

**Proposition 15.** Under the assumptions of Theorem [T2], there exists a constant \(C > 0\) such that
\[
\mathbb{E} (|\partial R(n)|) \sim \frac{Cn}{\log^2(n)}, \quad \text{as } n \to \infty.
\]
In addition, there exists \(M > 0\) such that
\[
\text{Var} (|\partial R(n)|) \leq \frac{M n^2 \log \log(n)}{\log^3(n)} \quad \text{as } n \to \infty.
\]
Proof. It follows easily from the equality $\partial R(n) = \bigcup_{v \in E_d} \partial_v R(n)$ in (9) that for any $v \in E_d$ we have

$$|\partial_v R(n)| \leq |\partial R(n)| \leq \sum_{v \in E_d} |\partial_v R(n)|. \quad (10)$$

Using the inclusion-exclusion principle and enumerating the elements of $E_d$ as $v_1, \ldots, v_{2d}$ we have

$$|\partial R(n)| = \sum_{V \subseteq E_d} (-1)^{|V|+1} \left| \bigcap_{v \in V} \partial_v R(n) \right|.$$ 

Given a collection, $V := \{v_1, \ldots, v_l\}$ say, of distinct vectors in $\mathbb{Z}^2$ we have that

$$\bigcap_{v \in V} \partial_v R(n) = R(n) \cap \bigcap_{v \in V} (R(n) + v)^c$$

whence we can write as in the proof of Proposition 3

$$\left| \bigcap_{v \in V} \partial_v R(n) \right| = \sum_{k=1}^n 1_{A_{k,V}(\omega)} 1_{B_{k,V}(n-k)(\omega)}$$

where

$$A_{0,V}(L) = \{\omega \in \Omega : \forall 0 < l < L, S_l^(-) \notin V\}, \quad A_{k,V}(L) = T^{-k} A_{0,V}(L),$$

and

$$B_{0,V}(L) = \{\omega \in \Omega : \forall 0 < l < L, S_l \notin V \cup \{0\}\}, \quad B_{k,V}(L) = T^{-k} B_{0,V}(L).$$

To see why notice that

$$A_k(k) \cap B_k(n-k) = \{\omega \in \Omega : \forall l \in [1, n], S_k \notin S_l + V, \forall m \in [k+1, n], S_m \neq S_k\}.$$ 

Let us first consider the case $V = \{v\}$. Then one has

$$|\partial_v R(n)|(\omega) = \sum_{k=1}^n 1_{A_k(k)(\omega)} 1_{B_k(n-k)(\omega)},$$

where $A_{k,v} := A_{k,\{v\}}, B_{k,v} := B_{k,\{v\}}$. By the Markov property for the random walk, for all $1 \leq k \leq n$, $A_{k,v}$ and $B_{k,v}$ are independent, thus

$$\mathbb{E}(|\partial_v R(n)|) = \sum_{k=1}^n \mathbb{P}(A_k(k)) \mathbb{P}(B_k(n-k))$$

$$= \sum_{k=1}^n \mathbb{P}(\forall 1 \leq l \leq k, S_l \neq v) \mathbb{P}(\forall 1 \leq l \leq n-k, S_l \notin \{0, v\})$$

$$= \sum_{k=1}^n \frac{c_v d_v}{\log(k) \log(n-k)} (1 + o(1)),$$

as $n \to \infty$, where $c_v, d_v > 0$ by Proposition 13. An easy calculation, using the fact that log is slowly varying, shows that

$$\sum_{k=2}^n \frac{1}{\log(k) \log(n-k)} = \sum_{k=\lfloor n/ \log^2 n \rfloor}^{n-\lfloor n/ \log^2 n \rfloor} \frac{1}{\log(k) \log(n-k)} + O\left(\frac{n}{(\log n)^3}\right)$$

$$= \frac{n}{\log(n)^2} (1 + o(1)) + O\left(\frac{n}{(\log n)^3}\right),$$

where we write $\lfloor x \rfloor$ for the integer part of $x$.

Assume now $|V| > 1$. For any set $W$, the proof of Proposition 13 and in particular (4) imply that, as $n \to \infty$

$$\mathbb{P}(\forall 1 \leq k \leq n, S_k \notin W) = \frac{\tilde{g}_W(0)c_d}{\log(n)} + o\left(\log(n)^{-1}\right)$$

as $n \to \infty$. 

with \( \gamma_d \) and \( \hat{g}_V(0) \) as defined in (7) and (5) respectively. Therefore similar arguments show that for any \( V \subset E_d \) we have that

\[
\mathbb{E} \left| \bigcap_{v \in V} \partial_v R(n) \right| \sim \gamma_d^2 \hat{g}_V(0) \hat{g}_{V \cup \{0\}}(0) \frac{n}{\log(n)^2},
\]

where in general it is possible that the constant is zero.

Going back to \( \partial R(n) \), we thus have that

\[
\lim_{n \to \infty} \frac{\log(n)^2}{n} \mathbb{E} |\partial R(n)| = \sum_{V \subseteq E_d} (-1)^{|V|+1} \gamma_d^2 \hat{g}_V(0) \hat{g}_{V \cup \{0\}}(0) \geq c_v d_v > 0
\]

where the last two inequalities follow from (10) and Proposition 13. This proves (8).

For the second part, from (10)

\[
\mathbb{E} \left( |\partial v R(n)|^2 \right) = \mathbb{E} \left( \sum_{v \in E_d} |\partial_v R(n)| \right)^2 \leq C \sum_{v \in E_d} \mathbb{E} \left( |\partial_v R(n)|^2 \right).
\]

Letting \( v \in E_d \) be arbitrary we then have

\[
\mathbb{E} \left( |\partial v R(n)|^2 \right) = \mathbb{E} \sum_{k=1}^n \left( 1_{A_k(k)} 1_{B_k(n-k)} + 2 \right) \sum_{1 \leq k < m \leq n} \mathbb{E} \left( 1_{A_k(k)} 1_{B_k(n-k)} 1_{A_m(m)} 1_{B_m(n-m)} \right).
\]

The first term is equal to \( \mathbb{E} (|\partial_v R(n)|) \). For the second term, notice that for \( 1 \leq k < m \leq n \),

\[
1_{B_k(n-k)} 1_{A_m(m)} \leq 1_{B_k([m-k)/2]} 1_{A_m([m-k)/2]},
\]

since for any \( k, m, B_k(\cdot), A_m(\cdot) \) are decreasing sequences of sets. To keep notation concise, for any integers \( l,k \) we will denote \( \mathbb{P}(A_k(l)) \) by \( \psi(l) \) and \( \mathbb{P}(B_k(l)) \) by \( \theta(l) \), where we can drop the dependence on \( k \) since \( \mathbb{P} \circ T^{-1} = \mathbb{P} \). Since for \( 1 \leq k \leq m \leq n \) the events

\[
A_k(k), B_k \left( \lfloor (m-k)/2 \rfloor \right), A_m \left( \lfloor (m-k)/2 \rfloor \right), B_m(n-m),
\]

are independent, we have

\[
\mathbb{E} \left( 1_{A_k(k)} 1_{B_k(n-k)} 1_{A_m(m)} 1_{B_m(n-m)} \right) \leq \psi(k) \theta \left( \frac{m-k}{2} \right) \psi \left( \frac{m-k}{2} \right) \theta(n-m).
\]

This shows that for \( k < m \),

\[
\iota(k, m) := \mathbb{E} \left( 1_{A_k(k)} 1_{B_k(n-k)} 1_{A_m(m)} 1_{B_m(n-m)} \right) - \mathbb{E} \left( 1_{A_k(k)} 1_{B_k(n-k)} \right) \mathbb{E} \left( 1_{A_m(m)} 1_{B_m(n-m)} \right)
\]

is bounded from above by

\[
\psi(k) \theta(n - m) \left\{ \psi \left( \frac{m-k}{2} \right) \left[ \theta \left( \frac{m-k}{2} \right) - \theta(n-k) \right] + \theta(n-k) \left[ \psi \left( \frac{m-k}{2} \right) \right. \right. \]
\[
\left. \left. - \psi(m) \right] \right\}
\]

Denote by

\[
D(n) := \left\{ (k, m) \in [1, n]^2 : k, n - m > \sqrt{n} \text{ and } m - k \geq \frac{n}{\log^5(n)} \right\}.
\]

Since for \( n \) large enough

\[
\log(n) - 5 \log\log(n) - \log(2) \geq \frac{\log(n)}{2},
\]

it follows that for all \( (k, m) \in D(n) \),

\[
\psi \left( \frac{m-k}{2} \right) \sim \frac{c_v}{\log((m-k)/2)} \leq \frac{c_v}{\log(n) - 5 \log\log(n) - \log(2)} \leq \frac{2c_v}{\log(n)}, \tag{12}
\]

and

\[
\theta \left( \frac{m-k}{2} \right) \sim \frac{d_v}{\log((m-k)/2)} \leq \frac{2d_v}{\log(n)}. \tag{13}
\]
with \( c_v, d_v \) as in Proposition \([13]\). Similarly, for all \((k, m) \in D(n)\) and \(n\) large enough we have that

\[
\theta \left( \frac{m - k}{2} \right) - \theta(n - k) = \sum_{j=\left\lfloor \frac{m-k}{2} \right\rfloor}^{n-k} (\theta(j) - \theta(j + 1)) \\
\leq \sum_{j=\left\lfloor \frac{m-k}{2} \right\rfloor}^{n-k} \mathbb{P}^0 [S_{j+1} \in \{0, v\}, S_l \notin \{0, v\}, \forall 1 \leq l \leq j] \\
= \sum_{j=\left\lfloor \frac{m-k}{2} \right\rfloor}^{n-k} \sum_{w \in \{0, v\}} \mathbb{P}^0 [S_{j+1} = w, S_l \notin \{0, v\}, 1 \leq l \leq j].
\]

By Lemma \([14]\) the term in the sum is bounded by \(C/(j \log^2(j))\). Thus, since \((m - k)/2 \to \infty\) when \(n \to \infty\),

\[
\theta \left( \frac{m - k}{2} \right) - \theta(n - k) \leq C \sum_{j=\left\lfloor \frac{m-k}{2} \right\rfloor}^{n-k} \frac{1}{j \log(j)^2} \\
\leq C \pi \gamma \int_{s=\left\lfloor \frac{m-k}{2} \right\rfloor}^{n-k} \frac{ds}{s \log(s)^2} \\
= C \pi \gamma \left[ \frac{1}{\log((m-k)/2)} - \frac{1}{\log(n-k)} \right] \\
\leq C \pi \gamma \left[ \frac{1}{\log n - 5 \log \log n - \log 2} - \frac{1}{\log n} \right] \\
\leq C \frac{\log \log(n)}{\log^2(n)},
\]

and similarly

\[
\psi \left( \frac{m - k}{2} \right) - \psi(m) \lesssim C \frac{\log \log(n)}{\log^2(n)}. \tag{15}
\]

since for \((k, m) \in D(n)\), it holds that \(n/\log^5(n) \leq m \leq n\).

Combining \([12]\), \([13]\), \([14]\) and \([15]\), we obtain a global constant \(M > 0\) such that for large \(n\), for all \((k, m) \in D(n)\)

\[
\iota(k, m) \lesssim M \frac{\log \log(n)}{\log^2(n)} \frac{1}{\log(n-m)} \frac{1}{\log(k)} \leq 4M \frac{\log \log(n)}{\log^2(n)}. \tag{16}
\]

Since \(\iota(k, m) \leq 1\) for all \(k, m\), we have

\[
\sum_{1 \leq k \leq m \leq n} \iota(k, m) \leq \# \{ (k, l) \in [1, n]^2 \setminus D(n) : k < l \} + \sum_{(k, m) \in D(n)} \iota(k, m) \\
\leq \left( 2n^{3/2} + \frac{n^2}{\log^5(n)} \right) + Mn^2 \frac{\log \log(n)}{\log^4(n)}.
\]

This together with \([8]\) implies \([9]\). \(\square\)

**Corollary 16.** Under the assumptions of Theorem \([12]\), there exists a constant \(C > 0\) such that

\[
\lim_{n \to \infty} \frac{|\partial R(n)|}{n/\log^2(n)} = C, \text{ in probability.}
\]

The gap between the variance of \(|\partial R(n)|\) and the square of its expectation implies via Chebyshev’s inequality that for all \(\epsilon > 0\) there exists \(C > 0\) such that

\[
\mathbb{P} \left( |\partial R(n) - \mathbb{E}(|\partial R(n)|)| > \epsilon \mathbb{E}(|\partial R(n)|) \right) \leq \frac{C \log \log(n)}{\epsilon^2 \log(n)}. \tag{17}
\]
The gap of order $1/\log(n)$ in the decay of these probabilities is enough to guarantee almost sure convergence of $|\partial R_{N_k}|$ for $N_k = \exp(n^a)$ with $a > 1$. By a different method looking at

$$Z_n = \frac{|\partial R(n)|}{\E(|\partial R(n)|)} - 1,$$

one can show almost sure convergence of $Z_{N_k}$ to 0 where $N_k$ is of the form $[\exp(n^a)]$ with $a > 1/2$. This is since for $a > 1/2$, $\sum_{k=1}^{\infty} \E\left(|Z_{N_k}|^2\right) < \infty$.

Unfortunately as $|\partial R(n)|$ is not necessarily monotone, this subsequence is too thin in order to interpolate the almost sure convergence from the subsequence to almost sure convergence along the whole sequence. To that end we use a method from \cite{Fla76}.

**Definition 1.** (i) Given $\delta > 0$, the sequence of random variables $|\partial R(n)|$ satisfies property $A(\delta)$ if for all $\epsilon_0 > 0$ there exists $C = C(\epsilon_0, \delta) > 0$ such that for all $n \geq 2$ and $\epsilon \geq \epsilon_0$,

$$P(|\partial R(n)| > (1 + \epsilon)\E(|\partial R(n)|)) \leq \frac{C}{\epsilon^2 \log^2(n)}.$$

(ii) Given $\delta > 0$, the sequence of random variables $|\partial R(n)|$ satisfies property $D(\delta)$ if for all $\epsilon_0 > 0$ there exists $C = C(\epsilon_0, \delta) > 0$ such that for all $n \geq 2$ and $\epsilon \geq \epsilon_0$,

$$P(|\partial R(n)| < (1 - \epsilon)\E(|\partial R(n)|)) \leq \frac{C}{\epsilon^2 \log^2(n)}.$$

**Theorem 17.** [Theorem 4.2 in \cite{Fla76}] For all $\delta > 0$, if $|\partial R(n)|$ satisfies property $A(\delta)$ (respectively $D(\delta)$) then it satisfies property $A(4\delta/3)$ (respectively $D(4\delta/3)$).

It follows from (17) that for all $0 < \delta < 1$, the sequence $|\partial R(n)|$ satisfies properties $A(\delta)$ and $D(\delta)$.

**Corollary 18.** For all $\delta > 0$, the sequence $|\partial R(n)|$ satisfies property $A(\delta)$ and $D(\delta)$. Consequently, taking $\delta = 5$, for all $\epsilon_0 > 0$ there exists $C = C(\epsilon_0)$ such that for all $\epsilon > \epsilon_0$ and $n \geq 2$,

$$P(|\partial R(n)| - \E(|\partial R(n)|) > \epsilon \E(|\partial R(n)|)) \leq \frac{C}{\epsilon^2 \log^2(n)}.$$  \hfill (18)

The proof of Theorem 17 is similar to the proof of \cite{Fla76} thm. 4.2, hence it is postponed to the appendix.

**Proof of Thm. 12.** By (8) it is enough to show that

$$Z_n := \frac{|\partial R(n)|}{\E(|\partial R(n)|)} - 1 \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

Let $\epsilon > 0$ and write $N_k := \left\lfloor \exp\left(\sqrt[k]{k}\right) \right\rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. By Corollary 18 there exists $C > 0$ such that for all $n \geq 2$,

$$P(Z_{N_k} > \epsilon) \leq \frac{C}{\epsilon^2 \log^2(N_k)} \leq \frac{C}{k^{5/4}}.$$

It follows from the Borel-Cantelli lemma that for almost every $w \in \Omega$,

$$\limsup_{k \to \infty} Z_{N_k} \leq \epsilon$$

As $Z_n \geq 0$ and $\epsilon$ is arbitrary it follows that for almost every $w \in \Omega$,

$$\lim_{n \to \infty} Z_{N_k} = 0.$$

Now for a general $n \in \mathbb{N}$ there exists a unique $m = m(n) \in \mathbb{N}$ such that $N_m \leq n < N_{m+1}$. Since $e^x - 1 \leq 2x$ for all $0 \leq x \leq 1$ and for all $m$ large $\sqrt{m + 1} - \sqrt{m} \leq 1/3m^{3/4}$, it follows that

$$N_{m+1} - N_m = \exp\left(\sqrt[m]{m}\right) \left[\exp\left(\sqrt[m+1]{m} - \sqrt[m]{m}\right) - 1\right] \leq m^{-3/4}N_m.$$  \hfill (19)
In particular, $N_{m+1}/N_m \to 1$. By [3], $\mathbb{E}(|\partial R(n)|) \sim Cn/(\log^2(n))$ is regularly varying. Therefore, since $N_m \leq n \leq N_{m+1}$,
\[
\mathbb{E}(|\partial R(n)|)/\mathbb{E}(|\partial R_{N_m}|) \to 1. \tag{20}
\]

From (19) and the trivial bound
\[
|\partial R(n)| - 2dm \leq |\partial R(n + m)| \leq |\partial R(n)| + 2dm
\]
for all $m$ large enough and $N_m \leq n \leq N_{m+1}$, we have using (19)
\[
\frac{|\partial R(n)| - |\partial R_{N_m}|}{\mathbb{E}(|\partial R_{N_m}|)} \leq \frac{2d(N_{m+1} - N_m)}{\mathbb{E}(|\partial R_{N_m}|)} \leq \frac{2dm^{3/4}N_m}{CN_m/\log^2(N_m)} \leq m^{1/2} \to 0.
\]
Since $|\partial R_{N_m}|/\mathbb{E}(|\partial R_{N_m}|)$ tends almost surely to 1, we obtain that $|\partial R(n)|/\mathbb{E}(|\partial R_{N_m}|)$ also tends to 1. Together with (20), this gives $|\partial R(n)|/\mathbb{E}(|\partial R(n)|) \to 1$ as required. □

4. The Range of Random Walks in the Domain of Attraction of the Symmetric $\alpha$ Stable Distribution for $1 < \alpha \leq 2$

Let $\{X_n\}_{n=-\infty}^{\infty}$ be a sequence of i.i.d., centered, $\mathbb{Z}$-valued random variables, and define the two-sided random walk $\{S_n\}_{n \in \mathbb{Z}}$ as follows, $S_0 = 0$, and for $n \geq 1$ let $S_n = X_0 + \cdots + X_{n-1}$ and $S_n^{(-)} = -X_1 - \cdots - X_{n-1}$. We assume that $\{S_n\}_n$ is aperiodic in the sense of Section 3 and that the random variables $\{X_i\}_i$ belong to the domain of attraction of a nondegenerate, symmetric, $\alpha$-stable distribution with $1 < \alpha \leq 2$. That is, there exists a positive slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that,
\[
Y_n := \frac{S_n}{n^{1/\alpha}L(n)} \xrightarrow{d} Z_\alpha,
\]
where $Z_\alpha$ is a real random variable with characteristic function $\mathbb{E}(e^{itZ_\alpha}) = e^{-|t|^\alpha}$. By Levy’s continuity theorem, writing $\phi(t) := \mathbb{E}(e^{itX_1})$, we see that for all $t > 0$,
\[
\mathbb{E}(e^{itY_n}) = \phi \left( \frac{t}{n^{1/\alpha}L(n)} \right) \xrightarrow{n \to \infty} e^{-|t|^\alpha}.
\]
From this and a Tauberian theorem it follows that, see e.g. [IL75, Theorem 2.6.5],
\[
\phi(t) = 1 - |t|^\alpha L(1/|t|) \left[ 1 + o(1) \right], \quad t \to 0.
\]
From now on $L$ will denote a positive slowly varying function which can change from line to line. If $A \subset \mathbb{Z}$ is a finite subset we define
\[
r_n(x, A) := \mathbb{P}_x(S_k \notin A, 1 \leq k \leq n),
\]
and write $r_n := r_n(0, \{0\})$.

We will need the following result.

**Lemma 19.** Assume that $\{X_i\}_{i \in \mathbb{Z}}$ are aperiodic i.i.d., $\mathbb{Z}$-valued random variables such that $1 - \phi(t)$ is regularly varying of index $\alpha$ as $t \to 0$, for $\alpha \in (1, 2)$. Then $r_n$ is regularly varying of index $1/\alpha - 1$ as $n \to \infty$. In addition, there exists a positive slowly varying function $\mathcal{L}$ such that, for any $x \in \mathbb{Z}$ and any finite nonempty $A \subset \mathbb{Z}$,
\[
\lim_{n \to \infty} n^{1-1/\alpha} L(n) r_n(x, A) = C(x, A), \tag{21}
\]
for some constant $C(x, A) \geq 0$.

**Proof.** When $\lim_{s \to \infty} L(s)$ exists this result is [Kes63, Theorem 8]. So we will consider the case where $L(\cdot)$ does not have a limit at infinity. Similarly to the proof of [Kes63, Theorem 8] we define for $\lambda \in [0, 1)$
\[
U(\lambda) := \sum_{n=0}^{\infty} \lambda^n \mathbb{P}(S_n = 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{1 - \lambda \phi(t)},
\]
\[
R(\lambda) := \sum_{n=0}^{\infty} \lambda^n r_n = (1 - \lambda)^{-1} U(\lambda)^{-1}.
\]
First we will study the asymptotic behaviour of $U(\lambda)$ as $\lambda \to 1$.

Notice that since $|t|^\alpha L(1/t)$ is regularly varying at the origin, by [BGT89] Theorem 1.5.3 there exists a monotone, $\alpha$-regularly varying function $g(t)$, such that $g(t) \sim C|t|^\alpha L(1/t)$ as $t \to 0$, for some $C > 0$. Therefore $\phi(t) = 1 - g(t) [1 + o(1)]$, as $t \to 0$. Next, we use aperiodicity to concentrate on the behaviour around the origin. In particular for any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that $|\phi(t)| < 1 - C(\epsilon)$ for all $|t| > \epsilon$. The main contribution will come from $|t| < \epsilon$. We claim that the function

$$U_\epsilon(\delta) := \int_0^\epsilon \frac{dt}{\delta + g(t)}$$

is slowly varying at 0 with index $1/\alpha - 1$, the asymptotics not depending on $\epsilon$. Let us first see how this claim implies the result. We write

$$U(\lambda) = \frac{U_\epsilon(1 - \lambda)}{\pi} + \frac{1}{2\pi} \left( \int_0^\epsilon \frac{dt}{1 - \lambda \phi(t)} - \tilde{U}_\epsilon(1 - \lambda) \right)$$

$$+ \frac{1}{2\pi} \left( \int_0^\epsilon \frac{dt}{1 - \lambda \phi(t)} - \tilde{U}_\epsilon(1 - \lambda) \right) + \frac{1}{2\pi} \int_{|t| > \epsilon} \frac{dt}{1 - \lambda \phi(t)}.$$

We claim that the term $\tilde{U}_\epsilon(1 - \lambda)$ on the right dominates the others when $\lambda$ tends to 1. First, it dominates the last one as $\tilde{U}_\epsilon(1 - \lambda)$ tends to infinity while the last term remains bounded. The other two terms are similar, let us handle the first one. We write the difference as

$$\int_0^\epsilon \frac{dt}{1 - \lambda \phi(t)} - \tilde{U}_\epsilon(1 - \lambda) = \int_0^\epsilon \left( \frac{1}{1 - \lambda \phi(t)} - \frac{1}{1 - \lambda + g(t)} \right) dt = \int_0^\epsilon \frac{g(t) - \lambda(1 - \phi(t))}{(1 - \lambda \phi(t))(1 - \lambda + g(t))} dt.$$

When $\lambda$ is close enough to 1, the numerator is bounded by $\eta(\epsilon)g(t)$, where $\eta$ tends to 0 with $\epsilon$. Canceling with the first factor of the denominator, it follows that this term is bounded by $\eta(\epsilon)\tilde{U}_\epsilon(1 - \lambda)$, as desired.

The regular variation of $\tilde{U}_\epsilon$ then implies that $U(\lambda)$ is regularly varying of index $1/\alpha - 1$. Thus $R(\lambda) \sim C(1 - \lambda)^{-1/\alpha} L(1/(1 - \lambda))$, and since $r_n$ is monotone it follows that $r_n \sim cn^{1/\alpha - 1} L(n)$ for some constants $c, C > 0$. Having established this, [21] follows from [KS63] Theorem 4a.

It remains to prove the claim that $\tilde{U}_\epsilon(\delta)$ is slowly varying at 0 with index $1/\alpha - 1$. Since $g$ is $\alpha$-regularly varying and monotone, its inverse $f(u) := g^{-1}(u)$ will be regularly varying of index $1/\alpha$ and monotone. Next, letting $t = g^{-1}(z)$ we have

$$\tilde{U}_\epsilon(\delta) = \int_{z=0}^{g(\epsilon)} \frac{dg^{-1}(z)}{\delta + z} = \int_{z=0}^{\infty} \frac{dg^{-1}(z)}{\delta + z},$$

interpreting the integral in the Stieltjes sense, and letting $g^{-1}_s := g^{-1}(s)$ for $s < g(\epsilon)$ and $g^{-1}_s := g^{-1}(\epsilon)$ for $s \geq g(\epsilon)$. With this definition it is clear that $g^{-1}_s$ is also monotone and that $g^{-1}_s \sim g^{-1}(s)$ as $s \to 0+$. In particular its behaviour near the origin is independent of $\epsilon$.

The rest is fairly similar to the proof of [BGT89] Theorem 1.7.4. First notice that

$$\tilde{U}_\epsilon(\delta) = \int_{z=0}^{\infty} \frac{dg^{-1}_u}{\delta + z}$$

$$= \int_{z=0}^{\infty} \int_{u=0}^{\infty} e^{-(\delta + z)u} dg^{-1}_u(z) du$$

$$= \int_{u=0}^{\infty} e^{-\delta u} \int_{z=0}^{\infty} e^{-zu} dg^{-1}_u(z) du$$

$$= \int_{u=0}^{\infty} e^{-\delta u} V_\epsilon(u) du,$$

where

$$V_\epsilon(u) := \int_{z=0}^{\infty} e^{-zu} dg^{-1}_u(z).$$

By [BGT89] Theorem 1.7.11 we have that since $g^{-1}_u(z)$ is regularly varying of index $1/\alpha$ at the origin, we have that $V_\epsilon(u)$ is regularly varying of index $-1/\alpha$ as $u \to \infty$. In turn this implies
that

\[ W_\epsilon(u) := \int_0^u V_\epsilon(s) \, ds, \]

is regularly varying of index \( 1 - 1/\alpha \) as \( u \to \infty \) and since

\[ \tilde{U}_\epsilon(\delta) = \int_{u=0}^{\infty} e^{-\delta u} V_\epsilon(u) \, du = \int_{u=0}^{\infty} e^{-\delta u} W_\epsilon(du), \]

by Karamata’s Tauberian theorem \([BGT89] 1.7.1\) we have that \( \tilde{U}_\epsilon(\delta) \) is regularly varying of index \( 1/\alpha - 1 \) as \( \delta \to 0 \).

**Remark 20.** An alternative approach is possible using

\[ |\mathbb{P}(S_n = m) - \mathbb{P}(S_n = 0)| = O \left( \frac{1}{a_n^\alpha} \right), \quad \text{as } n \to \infty \tag{22} \]

where \( m \in \mathbb{Z} \) and \( a_n \) is a \( 1/\alpha \) regularly varying sequence, given in \([GJP84]\).

**Theorem 21.** Let \( S_n \) be an aperiodic random walk satisfying \([21]\) with \( \alpha \in (1, 2] \), then for all \( \epsilon > 0 \), almost surely

\[ \frac{|\partial R_n|}{|R_n|} = o \left( n^{\frac{1}{\alpha} - 1 + \epsilon} \right). \]

**Remark 22.** This theorem, in the case of i.i.d. random variables with \( \mathbb{E}(X_1) = 0 \) and \( \mathbb{E}(X_1^2) = D < \infty \) gives the rate \( o(n^p) \) for every \( p < \frac{1}{2} \). We claim that this is the optimal rate in the polynomial exponent. Indeed, for the simple random walk on \( \mathbb{Z} \), \( |\partial R_n| = 2 \) for all \( n \), and \( \frac{R_n}{\sqrt{n}} \) converges in distribution to \( Y = \text{Leb}(W[0, 1]) \) where \( W \) is the Wiener process and \( W[0, 1] = \{W_s: s \in [0, 1]\} \). As the limiting random variable \( Y \) is unbounded, Theorem \([21]\) is optimal in the polynomial exponent.

**Proposition 23.** Let \( S_n \) be an aperiodic random walk satisfying \([21]\) with \( \alpha \in (1, 2] \), then for all \( \epsilon > 0 \) and \( k \in \mathbb{N} \) there exists \( M > 0 \) such that for all \( x \in \mathbb{Z} \setminus \{0\} \),

\[ \mathbb{E} \left( |R_n \setminus (R_n - x)|^k \right) \leq M n^{k\left(\frac{1}{2} - 1 + \epsilon\right)}. \]

**Proof.** Let \( \epsilon > 0 \) and \( j \in \mathbb{Z} \setminus \{0\} \). In the course of the proof \( C \) will denote a global positive constant whose value can change (increase) from line to line. As before we have the bound

\[ |R_n \setminus (R_n - x)| \leq \sum_{j=1}^n 1_{A_n(j) \cap T^j} \]

where

\[ A_n(m) = \{w \in \Omega : S_j \neq x, \quad \forall j \in [-m, n-m]\} = D_m^-(m) \cap D_m^+(n-m), \]

where for \( y \in \mathbb{Z} \) and \( n \geq 0 \) we define

\[ D_y^-(n) := \{\omega \in \Omega : S_j^{(-)} \neq y, \quad j \in [1, n]\}, \quad D_y^+(n) := \{\omega \in \Omega : S_j \neq y, \quad j \in [1, n]\}. \]

Note that \( D_m^- \) and \( D_m^+ \) are independent, thus

\[ \mathbb{P}(A_n(m)) = \mathbb{P}(D_m^-(m)) \mathbb{P}(D_m^+(n-m)). \]

As \( Z_\alpha \) is symmetric, \( X_1 \) is in the domain of attraction of \( Z_\alpha \) if and only if \( -X_1 \) is in the domain of attraction of \( Z_\alpha \). This together with Lemma \([\ref{lem:critical}]\) implies that there exist positive, slowly varying functions \( L_+, L_- \) such that

\[ \mathbb{P}(D_m^+(n)) \leq \frac{L_+(n)}{n^{1-1/\alpha}}, \]

for \( n \geq 1 \), potentially absorbing any constants in the functions \( L_+(-) \).

We will now show that for all \( k \in \mathbb{N} \) there exists \( C > 0 \) such that,

\[ \mathbb{E} \left( |R_n \setminus (R_n - x)|^k \right) \leq C \left[ n^{k\left(\frac{1}{2} - 1 + \epsilon\right)} + 5k^2 \mathbb{E} \left( |R_n \setminus (R_n - x)|^{k-1} \right) \right]. \]
Then for every \((m_1, m_2, \ldots, m_k) \in \{1, \ldots, n\}^k : \forall 1 < l \leq k, m_l - m_{l-1} \geq 3\),
and \(\Lambda_k(n) := \{(m_1, m_2, \ldots, m_k) \in \{1, \ldots, n\}^k : \exists 1 < l < l' \leq k, |m_l - m_{l'}| \leq 2\},\)
Then
\[
|R_n \setminus (R_n - x)|^k \leq k! \sum_{(m_1, \ldots, m_k) \in \Delta_k(n)} \prod_{l=1}^{k} 1_{A_{m_l}(n)}(k) \circ T^{m_l} + k! \sum_{\Lambda_k(n)} \prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^{m_l}
\]
For every \((m_1, \ldots, m_k) \in \Lambda_k(n)\) there exists a minimal \(1 \leq 1 < k\) such that there exists \(l' > 1\) with \(|m_l - m_{l'}| \leq 2\). Since
\[
\prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^{m_l} \leq \prod_{l \in \{1, \ldots, k\} \setminus \{1\}} 1_{A_{m_l}(n)} \circ T^{m_l},
\]
by a simple counting argument, we can see that
\[
\mathbb{E} \left( \sum_{\Lambda_k(n)} \prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^{m_l} \right) \leq 10 \left( \begin{array}{c} k \\ 2 \end{array} \right) \mathbb{E} \left( |R_n \setminus (R_n - x)|^{k-1} \right).
\]
It remains to bound the other term. For \((m_1, \ldots, m_k) \in \Delta_k(n)\), letting \(q_1 = m_1, q_{k+1} = n - m_k\) and for \(2 \leq j \leq k, q_j = [(m_j - m_{j-1})/2]\), where \([x]\) denotes the integer part of \(x\), we deduce the inequality
\[
\prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^{m_l} \leq 1_{D_x^- (m_1)} \circ T^{m_1} \left( \prod_{l=2}^{k} 1_{D_x^+ (q_l)} \circ T^{m_l-1} 1_{D_x^- (q_l)} \circ T^{m_l} \right) 1_{D_x^+ (q_{k+1})} \circ T^{m_k}
\]
by replacing the restriction that \(S_j + x \neq S_{m_j}\) for \(j \in [1, n]\) by the weaker one that \(S_j + x \neq S_{m_j}\) for \(j \in [m_{l-1} + q_l, m_{l} + q_{l+1}]\); see Figure I where the random walk started at the marked point is constrained to not visit \(x\) for the time indicated by the arrows. The bound above is essentially a product of events that depend on non-overlapping sequences of the random variables \(\{X_j\}_{j \in \mathbb{Z}}\) and therefore by independence we have that
\[
\mathbb{E} \left( \prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^{m_l} \right) \leq \mathbb{P} \left\{ D_x^- (q_1) \right\} \prod_{l=2}^{k} \left[ \mathbb{P} \left\{ D_x^+ (q_l) \right\} \mathbb{P} \left\{ D_x^- (q_l) \right\} \right] \mathbb{P} \left\{ D_x^+ (q_{k+1}) \right\}
\leq C \prod_{l=2}^{k+1} \frac{L_-(q_{l-1})}{q_l^1/\alpha} \frac{L_+(q_l)}{q_l^{1-1/\alpha}}.
\]
In particular since \( L_{\pm}(\cdot) \) are slowly varying, for any \( \epsilon > 0 \) we can find a positive constant \( C > 0 \) such that \( L_{\pm}(n) \leq Cn^{\epsilon} \) and thus

\[
\mathbb{E}\left( \prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^m \right) \leq C \prod_{l=2}^{k+1} \frac{C}{(q_{l-1}q_{l})^{1-1/\alpha - \epsilon}} = C [q_1 q_{k+1}]^{1/\alpha - 1 + \epsilon} \prod_{l=2}^{k} (q_{l})^{2/\alpha - 2 + 2\epsilon}.
\]

Therefore,

\[
\mathbb{E}\left( \sum_{\Delta_k(n)} \prod_{l=1}^{k} 1_{A_{m_l}(n)} \circ T^k \right) \leq C \sum_{q_1, \ldots, q_{k+1}} [q_1 q_{k+1}]^{1/\alpha - 1 + \epsilon} \prod_{l=2}^{k} (q_{l})^{2/\alpha - 2 + 2\epsilon}.
\]

The sum is restricted to the values of \( q_i \) that can be produced by the above process. They satisfy \( q_1 + 2q_2 + \cdots + 2q_k + q_{k+1} \leq n \) and \( q_1 + 2q_2 + \cdots + 2q_k + q_{k+1} \geq n - k \), because of the integer parts in the definition of \( q_i \). The convolution of two sequences which are \( O(n^\alpha) \) and \( O(n^{\alpha^2}) \) with \( \alpha_1, \alpha_2 > 1 \) is \( O(n^{1+\alpha_1+\alpha_2}) \), and the exponent is again \( > -1 \). In the above sum, the exponents are respectively \( 1/\alpha - 1 + \epsilon \) and \( 2/\alpha - 2 + 2\epsilon \). As \( \alpha \in (1, 2] \), they are \( > -1 \). Therefore, one can apply recursively this estimate on convolution of sequences, and deduce that the above sum is \( O(n^\rho) \) for

\[
\rho = k + 2 \cdot \left( \frac{1}{\alpha} - 1 + \epsilon \right) + (k - 1) \cdot \left( \frac{2}{\alpha} - 2 + 2\epsilon \right) = k \cdot \left( \frac{2}{\alpha} - 1 + 2\epsilon \right).
\]

The result follows with \( \epsilon' = 2\epsilon \). \( \square \)

**Proposition 24.** Let \( S_n \) be a random walk satisfying \((21)\) with \( \alpha \in (1, 2] \). Then for all \( \epsilon > 0 \), almost surely

\[
\lim_{n \to \infty} \frac{|R_n|}{n^{\alpha - \epsilon}} = \infty.
\]

**Proof.** Let \( \epsilon > 0 \). Le Gall and Rosen have shown in \([LGR91]\) that there exists a \( 1/\alpha \)-regularly varying sequence \( a_n \) such that

\[
\frac{|R_n|}{a_n} \xrightarrow{\text{dist}} \text{Leb}_\mathbb{R} (W_\alpha [0, 1]),
\]

where \( W_\alpha [0, 1] \) is the range of the symmetric \( \alpha \)-stable Levy motion up to time 1. It is well known, see for example \([Boy64]\), that the occupation measure of a one-dimensional \( \alpha \)-stable processes, for \( \alpha > 1 \), defined by

\[
\mu(A) := \int_0^1 1_A \circ W_s ds
\]

is almost surely absolutely continuous with respect to Lebesgue measure. As \( \mu(W_\alpha [0, 1]) = 1 \) this implies that

\[
\mathbb{P}\left\{ \text{Leb}_\mathbb{R} (W_\alpha [0, 1]) > 0 \right\} = 1.
\]

Since \( a_n \) is \( 1/\alpha \)-regularly varying, setting \( t_n := \lfloor n^\epsilon \rfloor \), where \( \kappa > 0 \), we have that \( a_{t_n} \) is \( \kappa/\alpha \)-regularly varying. Let \( \kappa = 1 - \alpha \epsilon/2 \), so that \( \kappa/\alpha > 1/\alpha - \epsilon \). Then for \( n \) large enough we have \( a_{t_n} > n^{1/\alpha - \epsilon} \). Decompose the interval \([0, n] \cap \mathbb{Z}\) into sub intervals \([jt_n, (j + 1)t_n]\), \( j = 0, 1, \ldots, \lfloor n/t_n \rfloor \) of length \( t_n \) plus perhaps a remainder interval which will be ignored. Writing \( R(n, m) = \{S_{n+1}, \ldots, S_m\} \), for \( \delta > 0 \) and all \( n \) large enough we have that

\[
\mathbb{P}\left( |R_n| < \delta n^{\frac{1}{\alpha} - \epsilon} \right) \leq \mathbb{P}\left( \max_{j \in \{1, \ldots, \lfloor n/t_n \rfloor \}} |R(jt_n, (j + 1)t_n)| < \delta a_{t_n} \right) \leq \mathbb{P}\left( |R_{t_n}| < \delta a_{t_n} \right) \leq \mathbb{P}\left( |R_{t_n}| < \delta a_{t_n} \left\lfloor \frac{n}{t_n} \right\rfloor \right).
\]

\footnote{Equivalently almost surely possess a continuous local time \( x \mapsto L_\alpha(1, x) \)}
By (24) we can choose $\delta > 0$ small enough so that $\mathbb{P}(\text{Leb}_{\mathbb{R}}(W_{\alpha}[0,1]) < \delta) < 1$ and by (23) we can choose $N_0$ large enough so that for all $n \geq N_0$ we have
\[
\mathbb{P}(|R_{t_n}| < \delta a_{t_n}) \leq \rho < 1.
\]
Therefore for all $n > N_0$, from the definition of $t_n$ and the above it follows that
\[
\mathbb{P}\left(|R_n| < \delta n^{\frac{1}{2} - \epsilon}\right) \leq \rho^{\frac{n}{2}}.
\]
and since $n/t_n \sim n^\kappa'$ where $\kappa' = \epsilon\alpha/2$, an application of the Borel-Cantelli Lemma shows that
\[
\lim_{n \to \infty} \frac{|R_n|}{n^{\frac{1}{2} - \epsilon}} > \delta.
\]
As $\epsilon > 0$ is arbitrary the conclusion follows. \hfill \Box

**Proof of Theorem 21.** Fix $\delta > 0$. By Proposition 24, it remains to show that
\[
\lim_{n \to \infty} \frac{|\partial R_n|}{n^{\frac{1}{2} - 1 + \delta}} = 0.
\]
First note that
\[
|\partial R_n| = |R_n \setminus (R_n - 1)| + |R_n \setminus (R_n + 1)| = V_n(1) + V_n(2).
\]
Let $i \in \{1, 2\}$. By Proposition 23 for any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$,
\[
\mathbb{E}\left(\left(\frac{V_n(i)}{n^{\frac{1}{2} - 1 + \delta}}\right)^k\right) \leq M n^{k(\epsilon - \delta)}.
\]
Therefore, choosing $\epsilon < \delta$, there exist $k \in \mathbb{N}$ and $M > 0$ such that for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$,
\[
\mathbb{E}\left(\left(\frac{V_n(i)}{n^{\frac{1}{2} - 1 + \delta}}\right)^k\right) \leq M n^{-2}.
\]
A standard use of Markov’s inequality and the Borel Cantelli Lemma shows that for any $\delta > 0$ almost surely
\[
\lim_{n \to \infty} \frac{V_n(1)}{n^{\frac{1}{2} - 1 + \delta}} = \lim_{n \to \infty} \frac{V_n(2)}{n^{\frac{1}{2} - 1 + \delta}} = 0.
\]
Therefore we have that almost surely for all $\delta > 0$ small enough
\[
\lim_{n \to \infty} \frac{|\partial R_n|}{|R_n| n^{\frac{1}{2} - 1 + \delta}} = \lim_{n \to \infty} \frac{|\partial R_n|/n^{\frac{1}{2} - 1 + \delta/2}}{|R_n|/n^{\frac{1}{2} - 1 + \delta/2}} = 0,
\]
which proves the theorem. \hfill \Box

**Appendix A. Proof of Theorem 17 via Flatto’s inequality enhancement procedure**

Assume that $A(\delta)$ holds for $|\partial R(n)|$. Fix $\epsilon_0 > 0$ and denote by $\kappa > 0$ the unique constant, see Proposition 15, so that
\[
\mathbb{E}(|\partial R(n)|) \sim \frac{\kappa n}{\log^2(n)} \quad \text{as } n \to \infty.
\]
For $n \in \mathbb{N}$ we will write $N = N(n) = \lceil \log^{\delta/3}(n) \rceil$. For $1 \leq i \leq N$, write $n_i = |ni/N|$ and divide the range $R(n)$ into $N$-blocks,
\[
X_{n,i} := \{S_{ni-1 + 1}, S_{ni-1 + 2}, \ldots, S_{ni}\}, \quad 1 \leq i \leq N.
\]
As before
\[
\partial_v X_{n,i} := X_{n,i} \setminus \{X_{n,i} + v\}, \quad \partial X_{n,i} = \bigcup_{v \in E_d} \partial_v X_{n,i}.
\]
Clearly
\[ |\partial R(n)| \leq \sum_{i=1}^{N} |\partial X_{n,i}|. \]

Let \( \epsilon \geq \epsilon_0 \) and set
\[ A_i := \left\{ \omega \in \Omega : |\partial X_{n,i}| \geq \left( 1 + \frac{\epsilon}{2} \right) \frac{\kappa n}{N \log^2(n)} \right\} \]
and
\[ B_i := \left\{ \omega \in \Omega : |\partial X_{n,i}| \geq \left( 1 + \frac{\epsilon N}{2} \right) \frac{\kappa n}{N \log^2(n)} \right\}, \]
for \( 1 \leq i \leq N \). Then a simple combinatorial argument (see equation (4.9) in [Fla76]) gives,
\[ \left[ |\partial R(n)| > (1 + \epsilon) \frac{\kappa n}{\log^2(n)} \right] \subset \left( \bigcup_{1 \leq i < j \leq N} A_i \cap A_j \right) \cup \left( \bigcup_{i=1}^{N} B_i \right) \] (27)

First we estimate \( \mathbb{P}(A_i), \mathbb{P}(B_i) \) from above. Writing \( m_i = m_i(n) = n_i - n_{i-1} \) for \( 1 \leq i \leq N \), \( |\partial X_{n,i}| \) is equal in distribution to \( |\partial R(m_i)| \). In addition,
\[ \lim_{n \to \infty} \frac{m_i}{n/(N \log^2(n))} = 1 \]
uniformly in \( 1 \leq i \leq N \). In addition, by [Fla6], since \( m_i \to \infty \) as \( n \to \infty \),
\[ \lim_{n \to \infty} \frac{\mathbb{E}(|\partial R(m_i)|)}{\kappa m_i/\log^2(m_i)} = 1. \]

Consequently there exists \( n = n(\epsilon_0) \), such that for all \( \epsilon > \epsilon_0 \) and \( n > n \),
\[ A_i \subset \left\{ |\partial X_{n,i}| \geq \left( 1 + \frac{\epsilon}{3} \right) \frac{\kappa m_i}{\log^2(m_i)} \right\} \subset \left\{ \mathbb{E}(|\partial X_{n,i}|) \geq \left( 1 + \frac{\epsilon}{4} \right) \mathbb{E}(|\partial R(m_i)|) \right\}. \] (28)

We deduce that for all \( \epsilon > \epsilon_0 \), \( n > n \) and \( 1 \leq i \leq N \),
\[ \mathbb{P}(A_i) \leq \mathbb{P}\left( |\partial X_{n,i}| > \left( 1 + \frac{\epsilon}{4} \right) \mathbb{E}(|\partial R(m_i)|) \right) = \mathbb{P}\left( |\partial R(m_i)| > \left( 1 + \frac{\epsilon}{4} \right) \mathbb{E}(|\partial R(m_i)|) \right), \quad \text{by property A(\( \delta \))} \]
\[ \leq \frac{16C(\epsilon_0/4, \delta)}{\epsilon^2 \log^4(m_i)}. \]

Finally, as \( \log^\delta(m_i) \sim \log^\delta(n) \) as \( n \to \infty \), we can enlarge \( n \) such that for all \( \epsilon > \epsilon_0 \), and \( n > n \),
\[ \mathbb{P}(A_i) \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon^2 \log^4(n)} \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon_0 \epsilon \log^4(n)} \] (29)

To bound \( \mathbb{P}(B_i) \) from above notice that by similar considerations as in [28], for all \( \epsilon > \epsilon_0 \) and \( n > n \),
\[ B_i \subset \left\{ |\partial X_{n,i}| \geq \left( 1 + \frac{\epsilon N}{4} \right) \mathbb{E}(|\partial R(m_i)|) \right\}, \]
consequently for all \( 1 \leq i \leq N \),
\[ \mathbb{P}(B_i) \leq \mathbb{P}\left( |\partial R_{m_i}| \geq \left( 1 + \frac{\epsilon N}{4} \right) \mathbb{E}(|\partial R_{m_i}|) \right) \leq \frac{16C(\epsilon_0/4, \delta)}{\epsilon^2 N^2 \log^4(m_i)}. \]

Now as \( N \sim \log^{\delta/3}(n) \) as \( n \to \infty \), by enlarging \( n \) if needed, we can assume that for all \( n > n \) uniformly on \( 1 \leq i \leq N \) and \( \epsilon > \epsilon_0 \),
\[ \mathbb{P}(B_i) \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon^2 N \log^{\delta/3}(n)}, \] (30)
function $G$ as $n$ walks on groups. The setting is the same as in Paragraph 2.2: we consider a probability $P$ are i.i.d. random variables distributed like $\epsilon$.

Green function $G$

Assume that $p$ defines a transient random walk with finite first moment does not tend to 0 at infinity, as $n \to \infty$. It follows that there exists $C(\epsilon_0, 4\delta/3)$ such that for all $\epsilon \geq \epsilon_0$ and $n \geq 2$

$$
P(\|\partial R(n)\| > (1 + \epsilon)E(\|\partial R(n)\|)) \leq \frac{C(\epsilon_0, 4\delta/3)}{e^2 \log^{4\delta/3}(n)}.
$$

As $\epsilon_0$ is arbitrary this concludes the proof of Theorem 17.

Appendix B. Green functions tend to 0 at infinity

In this appendix, we study the decay at infinity of the Green function of transient random walks on groups. The setting is the same as in Paragraph 2.2 we consider a probability measure $p$ on a group $G$, and let $S_n = \xi_0 \cdots \xi_{n-1}$ be the corresponding random walk, where $\xi_i$ are i.i.d. random variables distributed like $p$. When this random walk is transient, the Green function $G(g)$ of the walk is defined as in (2), by $G(g) = \sum_{k=0}^{\infty} p_n(id_G, g)$, where $p_n(id_G, g) = P(S_n = g)$. More generally, let $G(g, h) = G(g^{-1}h)$. This is the average time that the walk starting from $g$ spends at $h$.

We say that $p$ is admissible if the random walk can reach any point in the group, i.e., the semigroup generated by the support of $p$ is the whole group. This is a natural non-degeneracy assumption.

The main result of this appendix is the following theorem.

Theorem 25. Let $p$ be an admissible probability measure on a finitely generated group $G$. Assume that $p$ defines a transient random walk, and that $G$ is not virtually cyclic. Then the Green function $G(g)$ tends to 0 when $g$ tends to infinity.

The condition that $G$ is not virtually cyclic is necessary for the theorem: in $\mathbb{Z}$, the Green function of a non-centered random walk with finite first moment does not tend to 0 at infinity, by the renewal theorem (and this statement can be extended to virtually cyclic groups).

This theorem is rather weak, in the sense that it is not quantitative. Much stronger results are available on specific classes of groups, see [Woe00]. For instance, when $G$ is non-amenable, then the probabilities $p_n(id_G, x)$ tend to zero exponentially fast, uniformly in $x$, from which the result follows readily. On amenable groups, for symmetric walks with finite support or more generally a second moment, one can sometimes use isoperimetric techniques to obtain much stronger results, specifying the speed of decay of $p_n(id_G, id_G)$ and of the Green function at infinity. However, in general, there is no hope to get a quantitative version of Theorem 25 as we have made no moment assumption (think of the case where $p$ is chosen so that $\xi_1$ is at distance $2^{2n}$ of id$_G$ with probability $1/n^2$ – then the Green function decays at most like $1/\log \log d(id_G, x)$). Moreover, there are surprisingly few tools that apply in all classes of groups, regardless of their geometry.

Most groups only carry transient random walks. Indeed, the only groups on which there are recurrent random walks are the groups which are virtually cyclic or virtually $\mathbb{Z}^2$, see [Woe00, Theorem 3.24]. In the proof of the theorem, we will have to separate the case where $G$ is virtually $\mathbb{Z}^2$. Let us start with this case.

Lemma 26. Assume that $G$ is virtually $\mathbb{Z}^2$, and that the admissible probability measure $p$ on $G$ defines a transient random walk. Then its Green function tends to 0 at infinity.
Proof. Assume first that $G = \mathbb{Z}^2$. Then the convergence to 0 at infinity of the Green function is [Spi76, 24.P.5], as $p$ is admissible and therefore aperiodic.

Assume now that $G$ has a finite index subgroup $H$ which is isomorphic to $\mathbb{Z}^2$. Replacing $H$ with the intersection of its (finitely many) conjugates, one can even assume that $H$ is normal in $G$. The measure $p$ induces a measure on the group $G/H$, which defines a recurrent walk as $G/H$ is finite. In particular, almost every trajectory of the random walk returns to $H$. The distribution of this first return is an admissible probability measure $p_H$ on $H$, to which one can apply the previous result: its Green function tends to 0 at infinity. Moreover, the Green functions of $p$ and $p_H$ coincide on $H$ as the trajectories of the random walk associated to $p_H$ can be obtained from the trajectories of the random walk for $p$ by restricting to the times where the walk is in $H$. It follows that $G(g)$ tends to 0 when $g$ tends to infinity along $H$.

By Harnack’s inequality [Woe00, 25.1], there exists a constant $C$ such that, for all $g_1, g_2 \in G$, one has $G(g_1) \leq C \delta(g_1,g_2) G(g_2)$. As $H$ has finite index in $G$, every point of $G$ is within uniformly bounded distance of $H$. Therefore, the convergence to 0 of the Green function along $H$ extends to the whole group.

To prove Theorem 25, we can therefore restrict to the case where the simple random walk is transient. We will use its symmetry to obtain the decay at infinity of its Green function. Then, we will compare a general random walk to the simple random walk.

Lemma 27. Assume that a symmetric random walk on a finitely generated group $G$ is transient. Then its Green function tends to 0 at infinity.

Proof. Let $p_n(g, h)$ denote the probability that the walk starting at $g$ is at position $h$ at time $n$. Then it is a standard fact that $\sum_{n} p_n(g) = 1$ for all $g \in G$. This is proved using Cauchy-Schwarz inequality as follows:

$$p_{2n}(id_G, g) = \sum_{h} p_n(id_G, h)p_n(h, g) \leq \left( \sum_{h} p_n(id_G, h)^2 \right)^{1/2} \left( \sum_{h} p_n(h, g)^2 \right)^{1/2}
= \left( \sum_{h} p_n(id_G, h)p_n(h, id_G) \right)^{1/2} \left( \sum_{h} p_n(g, h)p_n(h, g) \right)^{1/2}
= p_{2n}(id_G, id_G)^{1/2}p_{2n}(g, g)^{1/2} = p_{2n}(id_G, id_G).$$

Conditioning on the position of the walk at time 1, one also gets $p_{2n+1}(id_G, g) \leq p_{2n}(id_G, id_G)$. Therefore, for any $g$ and any $N$,

$$\sum_{n=2N}^{\infty} p_n(id_G, g) \leq 2 \sum_{n=2N}^{\infty} p_n(id_G, id_G).$$

The right hand side is the tail of the converging series $\sum_{n} p_n(id_G, id_G) = G(id_G)$. If $N$ is large enough, it is bounded by an arbitrarily small constant $\epsilon$. For each $n < 2N$, the measure $p_n$ is a probability measure on $G$. Hence, there are only finitely many points $g$ for which $p_n(id_G, g) > \epsilon/(2N)$. Summing over $n < 2N$, it follows that for all but finitely many points one has $\sum_{n<2N} p_n(id_G, g) \leq \epsilon$, and therefore $G(g) = \sum_{n} p_n(id_G, g) \leq 2\epsilon$.

The next lemma is the main step of the proof of Theorem 25.

Lemma 28. Consider a finitely generated group $G$, with a finite generating set $S$. Assume that the simple random walk on $G$ is transient. Let $c > 0$. Consider a probability measure $p$ on $G$ with $p(s) \geq c$ for all $s \in S$. Then its Green function tends to 0 at infinity.

Proof. The proof will rely on a classical comparison lemma, making it possible to relate general random walks to symmetric ones. Denote by $G$ the Green function associated to $p$, and by $G_S$ the Green function associated to the simple random walk. Under the assumptions of the lemma, [Var83, Proposition, Page 251] ensures that for any nonnegative square-integrable function $f$ on $G$,

$$\sum_{g,h \in G} f(g)G(g,h)f(h) \leq c^{-1} \sum_{g,h \in G} f(g)G_S(g,h)f(h).$$

(31)
Let $\Omega = \mathbb{G}^n$ be the space of all possible trajectories, with the (Markov) probability $\mathbb{P}$ corresponding to the distribution of the random walk given by $p$ starting from $\text{id}_\mathbb{G}$. A cylinder set is a set of the form
\[ [g_0, \ldots, g_n] = \{ \omega \in \Omega \mid \omega_0 = g_0, \ldots, \omega_n = g_n \} \subseteq \Omega. \]

Assume by contradiction that $G(g)$ does not tend to 0 at infinity. Then one can find $\epsilon > 0$ and an infinite set $I \subseteq \mathbb{G}$ on which $G(g) > \epsilon G(\text{id}_\mathbb{G})$. Let $M > 0$ be large enough (how large will be specified at the end of the argument). We define a sequence $h_n$ of elements of $I$ as follows.

- First, take $h_0 = \text{id}_\mathbb{G}$. Let also $T_0 = [\text{id}_\mathbb{G}] \subseteq \Omega$ be the set of all trajectories starting from $\text{id}_\mathbb{G}$, and $R_0 = \{ \text{id}_\mathbb{G} \}$.
- Then, take $h_1 \in I$ at distance at least $M$ of $h_0$. As $G(h_1)/G(\text{id}_\mathbb{G}) > \epsilon$ is the probability to reach $h_1$ starting from $\text{id}_\mathbb{G}$, one can find (by throwing away very long trajectories or very unlikely trajectories in finite time) a finite number of cylinder sets starting at $\text{id}_\mathbb{G}$ and ending at $h_1$ with total probability $> \epsilon$. Denote this set of trajectories by $T_1$, with $\mathbb{P}(T_1) > \epsilon$. Let $R_1$ be the set of points that trajectories in $T_1$ reach before $h_1$. As $T_1$ is a finite union of cylinders, $R_1$ is finite.
- Then, take $h_2 \in I$ at distance at least $M$ of $h_0$ and $h_1$. We can also require that it does not belong to $R_0 \cup R_1$, as this set is finite while $I$ is infinite. As above, we can then define a set $T_2$ which is a finite union of cylinders ending at $h_2$ with $\mathbb{P}(T_2) > \epsilon$, and $R_2$ the finite set of points reached by these trajectories before $h_2$.
- The construction goes on inductively to define $h_n$.

Let us denote by $F(g,h) = G(g,h)/G(\text{id}_\mathbb{G})$ the probability to reach $h$ starting from $g$. The point of the previous construction is that for $i \leq j$ we have the inequality
\[ \mathbb{P}(T_i \cap T_j) \leq F(h_i, h_j). \] (32)

Indeed, this is clear for $i = j$. For $i < j$, note that trajectories in $T_i \cap T_j$ reach $h_i$ and then $h_j$, in this order as $h_j \notin R_i$. Therefore,
\[ \mathbb{P}(T_i \cap T_j) \leq \mathbb{P}(\exists n < m, S_n = h_i \text{ and } S_m = h_j) = F(\text{id}_\mathbb{G}, h_i) \cdot F(h_i, h_j) \leq F(h_i, h_j), \]
where the central equality follows from the Markov property. This proves (32).

Let us now take $N$ large, and apply the inequality (31) to the characteristic function of $\{h_0, \cdots, h_{N-1}\}$. We obtain
\[ \sum_{i,j < N} G(h_i, h_j) \leq c^{-1} \sum_{i,j < N} G_S(h_i, h_j). \] (33)

We will bound the left hand side from below and the right hand side from above to get a contradiction. Thanks to (32), we have
\[ \sum_{i,j < N} G(h_i, h_j) \geq \sum_{i,j < N} G(h_i, h_j) = G(\text{id}_\mathbb{G}) \sum_{i,j < N} F(h_i, h_j) \geq G(\text{id}_\mathbb{G}) \sum_{i,j < N} \mathbb{P}(T_i \cap T_j) \]
\[ \geq \frac{G(\text{id}_\mathbb{G})}{2} \sum_{i,j < N} \mathbb{P}(T_i \cap T_j) = \frac{G(\text{id}_\mathbb{G})}{2} \int \left( \sum_{i < N} 1_{T_i} \right)^2 \mathbb{P} \geq \frac{G(\text{id}_\mathbb{G})}{2} \left( \int \sum_{i < N} 1_{T_i} \mathbb{P} \right)^2 \]
\[ = \frac{G(\text{id}_\mathbb{G})}{2} \left( \sum_{i < N} \mathbb{P}(T_i) \right)^2 \geq \frac{G(\text{id}_\mathbb{G})}{2} \epsilon^2 N^2. \]

The Green function $G_S$ tends to 0 at infinity, by Lemma 27. As the distance between $h_i$ and $h_j$ is at least $M$ for $i \neq j$ by construction, it follows that $G_S(h_i, h_j) \leq \eta(M)$ where $\eta$ tends to 0 with $M$. We obtain
\[ \sum_{i,j < N} G_S(h_i, h_j) \leq \sum_{i=j} G_S(h_i, h_j) + \sum_{i \neq j} G_S(h_i, h_j) \leq NG_S(\text{id}_\mathbb{G}) + N^2 \eta(M). \]
Combining these two estimates with (33) yields
\[ \frac{G(\text{id}_\mathbb{G})}{2} \epsilon^2 N^2 \leq c^{-1} NG_S(\text{id}_\mathbb{G}) + c^{-1} N^2 \eta(M). \]
We obtain a contradiction by taking $M$ large enough so that $c^{-1} \eta(M) < G(\text{id}_G) \epsilon^2/2$, and then letting $N$ tend to infinity.

\begin{proof}[Proof of Theorem 23] The result follows from Lemma 26 if $G$ is virtually $\mathbb{Z}^2$. Hence, we can assume that this is not the case, and therefore that the simple random walk on $G$ is transient.

The Green functions for the probability measures $p$ and $(p+\delta_{\text{id}_G})/2$ are related by the identity

\[ G((p+\delta_{\text{id}_G})/2)(g) = 2G_p(g), \]

by [Voe00, Lemma 9.2]. Without loss of generality, we can therefore replace $p$ with $(p+\delta_{\text{id}_G})/2$ and assume $p(\text{id}_G) > 0$. As $p$ is admissible, it follows that there exists $N$ with $p_N(\text{id}_G, s) > 0$ for all $s$ in the generating set $S$. By Lemma 28 the Green function associated to $p_N$, denoted by $G_N$, tends to 0 at infinity.

To compute $G(g)$, split the arrival times at $g$ according to their values modulo $N$. For times of the form $i+kN$, such arrivals can be realized by following $p$ for $i$ steps, and then $p_N$ for $k$ steps. It follows that

\[ G(g) = \sum_{h \in G} \sum_{i<N} p_i(\text{id}_G, h)G_N(h^{-1}g). \]

Let $\epsilon > 0$. Take a finite set $F \subset G$ such that

\[ \sum_{h \notin F} \sum_{i<N} p_i(\text{id}_G, h) < \epsilon. \]

Then

\[ G(g) \leq \sum_{h \in F} \sum_{i<N} p_i(\text{id}_G, h)G_N(h^{-1}g) + \epsilon\|G_N\|_{L^\infty}. \]

When $g$ tends to infinity, the first term tends to 0 as this is a finite sum and $G_N$ tends to 0 at infinity. For large enough $g$, we get $G(g) \leq 2\epsilon\|G_N\|_{L^\infty}$.
\end{proof}

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Department of Statistics, University of Oxford, Oxford OX1 3LB, UK

and The Alan Turing Institute, 96 Euston Road, London NW1 2DB, UK

E-mail address: deligian@stats.ox.ac.uk

Laboratoire Jean Leray, CNRS UMR 6629, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes, France

E-mail address: sebastien.gouezel@univ-nantes.fr

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Edmond J. Safra Campus, Jerusalem 91904, Israel

E-mail address: zemer.kosloff@mail.huji.ac.il