CORRIGENDUM TO: EQUIVARIANT EMBEDDINGS OF RATIONAL HOMOLOGY BALLS

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Abstract. There was an unfortunate oversight in a remark in our paper “Equivariant embeddings of rational homology balls”, Q. J. Math. 69 (2018), no. 3, 1101–1121. Correcting this yields an interesting result which we omitted to observe in that paper: we exhibit smooth embeddings of rational homology balls $B_{p,q}$ in the complex projective plane which do not embed symplectically.

A Markov triple is a positive integer solution $(p_1, p_2, p_3)$ to the Markov equation

$$p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.$$  

Each Markov triple gives rise to an embedding

$$\bigsqcup_{i=1}^3 B_{p_i, q_i} \hookrightarrow \mathbb{CP}^2$$

of a disjoint union of three rational homology balls in the complex projective plane. Here $B_{p,q}$ is the Milnor fibre of the quotient singularity $\frac{1}{p^2}(1, pq-1)$. The embedding in (1) arises by smoothing the three singular points in the weighted projective space $\mathbb{P}(p_1^2, p_2^2, p_3^2)$, and the numbers $q_i$ are given by

$$q_i = \pm 3p_j/p_k \pmod{p_i},$$

where $i, j, k$ is a permutation of 1, 2, 3.

Hacking and Prokhorov proved in [3] that any projective surface with quotient singularities which admits a smoothing to $\mathbb{CP}^2$ is $\mathbb{Q}$-Gorenstein deformation equivalent to some $\mathbb{P}(p_1^2, p_2^2, p_3^2)$ as above. Evans and Smith proved in [2] that any disjoint union $\bigsqcup_{i \in I} B_{p_i, q_i}$ which admits a symplectic embedding in $\mathbb{CP}^2$ arises in this way, with $|I| \leq 3$.

Let $F(2n - 1)$ denote the $n$th odd Fibonacci number, defined by the recursion

$$F(2n+3) = 3F(2n+1) - F(2n-1), \quad F(1) = 1, \quad F(3) = 2.$$  

Then $(1, F(2n-1), F(2n+1))$ is a Markov triple for each $n \in \mathbb{N}$, showing in particular that $B_{F(2n+1), F(2n-3)}$ admits a symplectic embedding in $\mathbb{CP}^2$ for each $n > 1$.

In [4] we mentioned but overlooked the significance of the following result. Here $\Delta_{p,q}$ is a properly embedded surface in the 4-ball whose double branched cover is $B_{p,q}$.
and $P_+$ is the unknotted Möbius band in the 4-ball with normal Euler number 2; see [4] for further details.

**Theorem 1.** For each $n \in \mathbb{N}$, the slice surface $\Delta_{F(2n+1),F(2n-1)}$ admits a simple embedding as a sublevel surface of the unknotted Möbius band $P_+$. Taking double branched covers yields a simple smooth embedding $B_{F(2n+1),F(2n-1)} \hookrightarrow \mathbb{C}P^2$.

These are the first-known smooth embeddings of rational balls $B_{p,q}$ in the complex projective plane that do not arise from symplectic embeddings. This shows that the smooth embedding problem has an as-yet-unknown solution which differs from that to the symplectic problem solved by Evans-Smith.

I am very grateful to Giancarlo Urzúa who reminded me that the Markov triple $(1, F(2n-1), F(2n+1))$ gives rise to a symplectic embedding in $\mathbb{C}P^2$ of $B_{F(2n+1),F(2n-3)}$, and not of $B_{F(2n+1),F(2n-1)}$. I am grateful to Ana Lecuona and Giancarlo Urzúa for helpful comments on an earlier version of this note. Most of the embeddings obtained in [4], but not those given in Theorem 1, have since been reproved and generalised by different methods in [5].

**Proof of Theorem 1.** Induction using (2) yields the Hirzebruch-Jung continued fraction expansion

$$\frac{F(2n+1)}{F(2n-1)} = \left[3^{n-1}, 2\right].$$

Now using [4, Lemma 3.1] we have

$$\frac{F(2n+1)^2}{F(2n+1)F(2n-1) - 1} = \left[3^{n-1}, 5, 3^{n-2}, 2\right].$$

These continued fractions may be used to describe the surface $\Delta_{F(2n+1),F(2n-1)}$, as described in [4].

The proof that $\Delta_{F(2n+1),F(2n-1)}$ is a sublevel surface of $P_+$ is a minor modification of the proof of [4, Theorem 5]. We again refer the reader to that source for details.

Consider the first diagram shown in Figure 1. This represents a surface $\Sigma$ bounded by the unknot, which we claim is $P_+$. Note first that the band move corresponding to the blue band labelled 0 converts the diagram to one of $\Delta_{F(2n+1),F(2n-1)}$, which is the slice disk described by Casson and Harer [1] for the two-bridge knot $S(F(2n+1)^2, F(2n+1)F(2n-1) - 1)$. This shows that $\Delta_{F(2n+1),F(2n-1)}$ is a sublevel surface of the surface $\Sigma$. It remains to see that $\Sigma$ is the unknotted Möbius band $P_+$ whose double branched cover is $\mathbb{C}P^2$.

Figure 2 shows a sequence of isotopies and band slides converting $\Sigma$ to $P_+$ in the first case of interest which is $n = 2$. Taking double branched covers we see that $B_{5,2}$ admits a smooth embedding in $\mathbb{C}P^2$, though we know from [2] that $B_{5,2}$ does not embed symplectically in $\mathbb{C}P^2$. The proof for $n > 2$ follows by an induction argument.
Recall that an embedding of $B_{p,q}$ in a 4-manifold $Z$ is called simple if the resulting rational blow-up of $Z$ is obtainable by a sequence of ordinary blow-ups. The proof that the embeddings described above are simple follows as in [4, Proposition 5.1]. We describe here a slightly shorter version of the proof at the level of double branched covers. The second diagram in Figure 1 represents the surface in the 4-ball pushed in from the black surface of the two-bridge diagram shown, using a chessboard colouring in which the unbounded region is white. The rational blow-up of $\mathbb{C}P^2$ is the double cover $X$ of the 4-ball branched along this black surface, which in turn is the plumbing of disk bundles over $S^2$ corresponding to the linear graph with weights

$$(-3)^{n-1}, -2, -1, (-3)^{n-2}, -2,$$

where $(-3)^m$ denotes $-3$ repeated $m$ times. A sequence of $-1$ blow-downs reduces this to the linear plumbing with weights $-3$ and $0$, which is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. It follows that

$$X \cong \mathbb{C}P^2 \# (2n-1) \overline{\mathbb{C}P^2}.$$
Figure 2. The slice disk $\Delta_{5,2}$ is a sublevel surface of $P_+$. 
Figure 3. **The inductive step.** The band slides are similar to those shown in Figure 2.
References

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