GLOBAL BIFURCATION SHEET AND DIAGRAMS OF
WAVE-PINNING IN A REACTION-DIFFUSION MODEL
FOR CELL POLARIZATION

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ABSTRACT. We are interested in wave-pinning in a reaction-diffusion model for cell polarization proposed by Y.Mori, A.Jilkine and L.Edelstein-Keshet. They showed interesting bifurcation diagrams and stability results for stationary solutions of a limiting equation by numerical computations. Kuto and Tsujikawa showed several mathematical bifurcation results of stationary solutions of this problem. We show exact expressions of all the solution by using the Jacobi elliptic functions and complete elliptic integrals. Moreover, we construct a bifurcation sheet which gives bifurcation diagram. Furthermore, we show numerical results of the stability of stationary solutions.

1. Introduction. We are interested in wave-pinning in a reaction-diffusion model for cell polarization proposed by Y.Mori, A.Jilkine and L.Edelstein-Keshet[7].

According to [7], the wave-pinning is such a phenomenon that a wave of activation of the species is initiated at one end of the domain, moves into the domain, decelerates, and eventually stops inside the domain, forming a stationary front.

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The model is

\[
\begin{align*}
\varepsilon W_t &= \varepsilon^2 W_{xx} + W(W - 1)(V + 1 - W) \quad \text{in } (0, 1) \times (0, \infty), \\
\varepsilon V_t &= D V_{xx} - W(W - 1)(V + 1 - W) \quad \text{in } (0, 1) \times (0, \infty), \\
W_x(0, t) &= W_x(1, t) = 0, \quad V_x(0, t) = V_x(1, t) = 0 \quad \text{in } (0, \infty), \\
W(x, 0) &= W_0(x), \quad V(x, 0) = V_0(x) \quad \text{in } (0, 1),
\end{align*}
\]

where \( W = W(x, t) \) denotes the density of an active protein, \( V = V(x, t) \) denotes the density of an inactive protein, \( \varepsilon, \ D \) are diffusion coefficients, \( W_0(x) \) denotes the initial density of the active protein, and \( V_0(x) \) denotes initial density of the inactive protein.

It is easy to see that the mass conservation

\[
\int_0^1 (W(x, t) + V(x, t)) dx = \int_0^1 (W_0(x) + V_0(x)) dx = m
\]

holds, where \( m \) is the total mass determined by the mass of the initial densities \( W_0(x) \) and \( V_0(x) \).

Letting \( D \to \infty \) in (TP), we formally obtain the following time dependent limiting equation:

\[
\begin{align*}
\varepsilon W_t &= \varepsilon^2 W_{xx} + W(W - 1)(\bar{V} + 1 - W) \quad \text{in } (0, 1) \times (0, \infty), \\
\frac{d\bar{V}}{dt} &= -\int_0^1 W(W - 1)(\bar{V} + 1 - W) dx \quad \text{in } (0, \infty), \\
W_x(0, t) &= W_x(1, t) = 0 \quad \text{in } (0, \infty), \\
W(x, 0) &= W_0(x) \quad \text{in } (0, 1), \quad \bar{V}(0) = \bar{V}_0,
\end{align*}
\]

where \( W = W(x, t) \), \( \bar{V} = \bar{V}(t) \) is the density depending only on \( t \). \( W_0(x) \) denotes the initial density, and \( \bar{V}_0 \) denotes an initial constant density.

Owing to the mass conservation, the stationary problem of (TP) can be reduced to the following Neumann problem with a nonlocal constraint:

\[
\begin{align*}
\varepsilon^2 W_{xx} + W(W - 1)(V + 1 - W) &= 0 \quad \text{in } (0, 1), \\
D V_{xx} - W(W - 1)(V + 1 - W) &= 0 \quad \text{in } (0, 1), \\
W(x) &> 0, \quad V(x) > 0 \quad \text{in } (0, 1), \\
W_x(0) &= W_x(1) = 0, \quad V_x(0) = V_x(1) = 0, \\
\int_0^1 (W(x) + V(x)) dx &= m,
\end{align*}
\]

where \( W = W(x) \), \( V = V(x) \), and \( m \) is a given initial total mass determined by initial densities.

Straight understanding of a stationary limiting problem for (LP) is

\[
\begin{align*}
\varepsilon^2 W_{xx} + W(W - 1)(\bar{V} + 1 - W) &= 0 \quad \text{in } (0, 1), \\
\int_0^1 W(W - 1)(\bar{V} + 1 - W) dx &= 0, \\
W_x(0) &= W_x(1) = 0, \\
W(x) &> 0 \quad \text{in } (0, 1), \quad \bar{V} > 0, \\
\int_0^1 W(x) dx + \bar{V} &= m.
\end{align*}
\]
The second equation automatically holds from the first and third equation. Hence the above system is equivalent to

\[
\begin{cases}
\varepsilon^2 W_{xx} + W(W - 1)(\tilde{V} + 1 - W) = 0 \quad \text{in } (0, 1), \\
W_x(0) = W_x(1) = 0, \\
W(x) > 0 \text{ in } (0, 1), \quad \tilde{V} > 0, \\
\int_0^1 W(x)dx + \tilde{V} = m.
\end{cases}
\]

For simplicity we concentrate on monotone increasing solutions, since we can obtain other solutions by reflecting this kind of solutions. Thus, we get

\[
\begin{cases}
\varepsilon^2 W_{xx} + W(W - 1)(\tilde{V} + 1 - W) = 0 \quad \text{in } (0, 1), \\
W_x(0) = W_x(1) = 0, \\
W(0) > 0, \quad W_x(x) > 0 \text{ in } (0, 1), \quad \tilde{V} > 0, \\
\int_0^1 W(x)dx + \tilde{V} = m.
\end{cases}
\]

Here it should be noted that we may omit the condition \(W(0) > 0\), since this condition follows from other conditions. Thus we obtain a stationary limiting problem as

\[
\begin{cases}
\varepsilon^2 W_{xx} + W(W - 1)(\tilde{V} + 1 - W) = 0 \quad \text{in } (0, 1), \\
W_x(0) = W_x(1) = 0, \\
W(0) > 0, \quad W_x(x) > 0 \text{ in } (0, 1), \quad \tilde{V} > 0, \\
\int_0^1 W(x)dx + \tilde{V} = m.
\end{cases}
\]



where \(m, \varepsilon\) are given positive constants, \(W = W(x)\) is an unknown function, and \(\tilde{V}\) is an unknown nonnegative constant.

Interesting bifurcation diagrams are obtained in [7] by numerical computations. Kuto and Tsujikawa [4] obtained several mathematical results for (SLP) with suitable change of variables.

The main purpose of this paper is to show exact expressions of all the solutions for (SLP) by using the Jacob’s elliptic functions and complete elliptic integrals, and construct a global bifurcation sheet in the space \((\tilde{V}, \varepsilon^2, m)\). Furthermore, we show numerical results on the stability of stationary solutions.

Each level curve with the height \(m\) of the sheet corresponds to the bifurcation diagram in the plane \((\tilde{V}, \varepsilon^2)\) for (SLP) with given \(m\). Thus, we can obtain all bifurcation diagrams including all, for instance, even secondary bifurcation branches.

Our method to obtain all the exact solutions essentially based on the method which started in Lou, Ni and Yotsutani [6]. It is developed by Kosugi, Morita and Yotsutani [5] to investigate the Cahn-Hilliard equation treated in Carr, Gurtin and Semrod [1], although we need some extra steps.

This paper is organized as follows. In Section 2 we state main theorems, and show figures of the global bifurcation sheet, bifurcation diagrams and stability results by numerical computations. In Section 3 we give proofs of main theorems by using Propositions 3.1, 3.2, 3.3 and 3.4, where proves of Propositions 3.2 and 3.4 are give there. In Section 4 we give a proof of Proposition 3.1. In Section 5 we give a proof of Proposition 3.3. Finally, in Appendix we give proofs of two inequalities which we use in Section 5.

2. Main results. First we provide the definition for the elliptic functions and the complete elliptic integrals which are used in this paper.
Definition 2.1.

\[ sn(x, k) := F_k^{-1}(x), \quad F_k(\xi) := \int_0^\xi \frac{1}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} dt, \]

\[ sn^2(x, k) + cn^2(x, k) = 1, \]

\[ K(k) := \int_0^1 \frac{1}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} dt, \quad E(k) := \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt, \]

\[ \Pi(\nu, k) := \int_0^1 \frac{1}{(1+\nu t^2)\sqrt{1-k^2t^2}\sqrt{1-t^2}} dt. \]

The following formulas for the complete elliptic integrals are important.

\[
\frac{d}{dk} K(k) = \frac{E(k)}{1-k^2} - \frac{K(k)}{k}, \quad \frac{d}{dk} E(k) = \frac{E(k)}{k} - \frac{K(k)}{k},
\]

\[
\frac{\partial}{\partial \nu} \Pi(\nu, k) = \frac{kE(k)}{(k^2 + \nu)(1-k^2)} - \frac{k\Pi(\nu, k)}{k^2 + \nu},
\]

\[
\frac{\partial}{\partial \nu} \Pi(\nu, k) = \frac{(k^2 - \nu^2)\Pi(\nu, k)}{2(1+\nu)(k^2 + \nu)} - \frac{K(k)}{2(1+\nu)\nu} + \frac{E(k)}{2(1+\nu)(k^2 + \nu)}.
\]

Now, let us introduce an auxiliary problem to investigate (SLP). Let \( \tilde{V} > 0 \) be given, let us consider the problem

\[
\begin{align*}
\varepsilon^2 W_{xx} + W(W-1)(\tilde{V} + 1 - W) &= 0 \quad \text{in } (0, 1), \\
W_x(0) &= W_x(1) = 0, \\
W_x(x) &> 0 \quad \text{in } (0, 1).
\end{align*}
\]

We note that (AP; \( \tilde{V} \)) is equivalent to

\[
\begin{align*}
\varepsilon^2 W_{xx} + W(W-1)(\tilde{V} + 1 - W) &= 0 \quad \text{in } (0, 1), \\
W_x(0) &= W_x(1) = 0, \\
0 < W(0) < \tilde{V} + 1, \quad W_x(x) &> 0 \quad \text{in } (0, 1)
\end{align*}
\]

for given \( \tilde{V} > 0 \), since it is easy to see that a condition \( 0 < W(0) < \tilde{V} + 1 \) holds for any solution of (AP; \( \tilde{V} \)).

The existence and the uniqueness of the solution \( W(x) \) of (AP; \( \tilde{V} \)) is well-known (see, e.g. Smoller and Wasserman [2], and Smoller [3]). However, we need to know more precise information to investigate (SLP). The following theorem gives the representation formula for all solutions of (AP; \( \tilde{V} \)).

**Theorem 2.1.** Let \( \tilde{V} > 0 \). There exists a solution of (AP; \( \tilde{V} \)), if and only if \( (\tilde{V}, \varepsilon^2) \in \mathcal{G} \), where

\[
\mathcal{G} := \left\{ (\tilde{V}, \varepsilon^2) : 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\}.
\]

Moreover, the solution is unique. The solution \( W(x; \tilde{V}, \varepsilon^2) \) has properties

\[
0 < W(x; \tilde{V}, \varepsilon^2) < \tilde{V} + 1, \quad \text{in } (0, 1),
\]

\[
W(x; \tilde{V}, \varepsilon^2) = \tilde{V} + 1 - \tilde{V} \cdot W\left(1-x; \frac{\varepsilon^2}{\tilde{V}^2}\right).
\]
The solution \( W(x; \tilde{V}, \varepsilon^2) \) is represented by

\[
W(x; \tilde{V}, \varepsilon^2) = \tilde{V} + \frac{2}{3} + \frac{1}{\sqrt{3}} \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \frac{\beta(1-hs)sn^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot cn^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)sn^2(K(\sqrt{h})x, \sqrt{h}) + cn^2(K(\sqrt{h})x, \sqrt{h})},
\]

(2.7)

\[
\alpha := \alpha(h, s) = \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3hs^4 - 4(h^2 + h)h^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3}},
\]

(2.8)

\[
\beta := \beta(h, s) = \frac{-hs^2 - 2(1 - h)s + 1}{\sqrt{3hs^4 - 4(h + h)h^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3}}.
\]

(2.9)

where \((h, s) = \left(h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)\right)\) is the unique solution of the following system of transcendental equations

\[
\begin{cases}
\mathcal{E}(h, s) = \sqrt{3} \cdot \frac{\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}}, \\
\mathcal{A}(h, s) = \frac{1}{3\sqrt{3}} \cdot \frac{(1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2)}{\left(\sqrt{\tilde{V}^2 + \tilde{V} + 1}\right)^3},
\end{cases}
\]

(2.10)

(2.11)

(2.12)

where

\[
\mathcal{E}(h, s) := \frac{\sqrt{2s(1-s)(1-sh)/K(\sqrt{h})}}{\sqrt{3hs^4 - 4(h^2 + h)h^3 + (4h^2 + 2h + 4)s^2 - 4(1 + h)s + 3}},
\]

(2.13)

\[
\mathcal{A}(h, s) := \frac{2(hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\left(\sqrt{3hs^4 - 4(h + h)h^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3}\right)^3}.
\]

(2.14)

Here, \(sn(\cdot, \cdot), cn(\cdot, \cdot)\) are Jacobi’s elliptic function. \(K(\cdot)\) is complete elliptic integral of the first kind.

We show the graph of \(\mathcal{A}(h, s)\) and \(\mathcal{E}(h, s)\) in Figures 1 and 2.
Theorem 2.2. Let $W(x; \tilde{V}, \varepsilon^2)$ be the unique solution of $(AP; \tilde{V})$, and
\[ m(\tilde{V}, \varepsilon^2) := \int_0^1 W(x; \tilde{V}, \varepsilon^2)dx + \tilde{V}, \tag{2.15} \]
then
\[ m(\tilde{V}, \varepsilon^2) = 2\tilde{V} + 2 - \tilde{V}m \left( \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2} \right) \quad \text{for any } \tilde{V} > 0, \varepsilon > 0. \tag{2.16} \]
In particular,
\[ m(1, \varepsilon^2) = 2 \quad \text{for any } \varepsilon > 0. \tag{2.17} \]
Moreover, it holds that
\[ m(\tilde{V}, \varepsilon^2) = \frac{4\tilde{V} + 2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \mathcal{M}(h, s), \tag{2.18} \]
\[ \mathcal{M}(h, s) := \frac{-(hs^2 - 2(1 + h)s + 3) + 4(1 - s)(1 - sh)\Pi(-sh, \sqrt{h}/K(\sqrt{h}))}{\sqrt{3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(1 + h)s + 3}}, \tag{2.19} \]
where $h = h(\tilde{V}, \varepsilon^2)$, $s = s(\tilde{V}, \varepsilon^2)$ are given in Theorem 2.1. Here, $K(\cdot)$ is the complete elliptic integral of the first kind, and $\Pi(\cdot, \cdot)$ is the complete elliptic integral of the third kind.

Let us define the global bifurcation sheet $S$ by
\[ S := \left\{ \left( \tilde{V}, \varepsilon^2, \int_0^1 Wdx + \tilde{V} \right) : (\tilde{V}, \varepsilon^2) \in \mathcal{G} \right\}. \]
We obtain exact representation of the global bifurcation sheet $S$ as
\[ S = \left\{ \left( \tilde{V}, \varepsilon^2, m(\tilde{V}, \varepsilon^2) \right) : (\tilde{V}, \varepsilon^2) \in \mathcal{G} \right\} \tag{2.20} \]
by Theorem 2.2. For each $m$, we can obtain the bifurcation diagram by
\[ \left\{ (\tilde{V}, \varepsilon^2) \in \mathcal{G} : m(\tilde{V}, \varepsilon^2) = m \right\} \tag{2.21} \]
directly from the global bifurcation sheet $S$.

We will mathematically investigate precise properties of the global bifurcation sheet and bifurcation diagrams in a forthcoming paper. For instance, we see the following facts:

- For each fixed $\tilde{V} \in (0, \infty)$, $W(x; \tilde{V}, \varepsilon^2) \to 1$ as $\varepsilon^2 \to \tilde{V}/\pi^2$ uniformly on $[0, 1]$.
- For each fixed $\tilde{V} \in (0, \infty)$, $m(\tilde{V}, \varepsilon^2) \to \tilde{V} + 1$ as $\varepsilon^2 \to \tilde{V}/\pi^2$. 

Figure 2. Graph of $\mathcal{E}(h, s)$
For each fixed $\tilde{V} \in (0, 1)$, $W(x; \tilde{V}, \varepsilon^2) \rightarrow \tilde{V} + 1$ as $\varepsilon \rightarrow 0$ in $(0, 1]$.

For each fixed $\tilde{V} \in (1, \infty)$, $W(x; \tilde{V}, \varepsilon^2) \rightarrow 0$ as $\varepsilon \rightarrow 0$ in $[0, 1)$.  

For each fixed $\tilde{V} \in (0, 1)$, $m(\tilde{V}, \varepsilon^2) \rightarrow 2\tilde{V} + 1$ as $\varepsilon \rightarrow 0$.

For each fixed $\tilde{V} \in (1, \infty)$, $m(\tilde{V}, \varepsilon^2) \rightarrow \tilde{V}$ as $\varepsilon \rightarrow 0$.

For $m \in (0, 1]$, bifurcation diagrams are the empty set.

For $m \in (1, \infty)$, bifurcation diagram given by (2.21) are graphs with $V$ axis (smooth single-valued function in $\tilde{V}$) except $m = 2$ with $\tilde{V} = 1$.

We show figures of bifurcation sheet, bifurcation diagrams, and profiles of $W(x; \tilde{V}, \varepsilon^2)$. Figure 3 shows the global bifurcation sheet $S$ by using the expression (2.18).

![Figure 3. Global bifurcation sheet](image)

Figure 4 shows the bifurcation diagrams for various $m$ with the profiles of solutions of (SLP).

![Figure 4. Bifurcation diagrams for various $m$](image)
Let us explain the stability of solutions of (SLP) observed by numerical computations. Let us see Figure 5. Stationary solutions corresponding to the points on the thick lines are locally stable and those on the broken lines are unstable.

**Figure 5.** Stability of solutions of (SLP)

Figures 6, 7 and 8 show typical results of the behavior of solutions of (TLP) for the three cases $\varepsilon = 0.0576$, $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.03285$, $m = 2$, $\tilde{V}(0) = 0.53846$, and $\varepsilon = 0.07401$, $m = 2.5$, $\tilde{V}(0) = 0.88679$. Here we take initial data $W(x,0)$ as stationary solutions shown by broken lines. Thick lines show stationary solutions which converges as $t \to \infty$.

**Figure 6.** $\varepsilon = 0.0576$, $m = 1.5$, $\tilde{V}(0) = 1.32558$
GLOBAL BIFURCATION SHEET FOR A CELL POLARIZATION MODEL

3. Proofs of Theorems 2.1 and 2.2. We prepare several propositions to prove Theorems 2.1 and 2.2.

Proposition 3.1. Let \( W(x) \) be a solution of \((AP; \tilde{V})\), and

\[
\begin{align*}
    u(x) := & \frac{\sqrt{3}}{\sqrt{\lambda^2 - \lambda + 1}} \left( \lambda W(x) - \left( \frac{1}{3} + \frac{\lambda}{3} \right) \right), \\
    \lambda := & \frac{1}{\tilde{V} + 1}.
\end{align*}
\]

Then \( u(x) \) satisfies

\[
\begin{align*}
    \left( \frac{\sqrt{3} \varepsilon}{\sqrt{V^2 + \tilde{V} + 1}} \right)^2 u_{xx} - u^3 + u \\
    + \frac{1}{3\sqrt{3}} \left( 1 - \tilde{V} \right) \left( 2\tilde{V} + 1 \right) \left( \tilde{V} + 2 \right) \left( \sqrt{V^2 + \tilde{V} + 1} \right)^3 &= 0 \quad \text{in } (0, 1), \\
    u_x(0) &= u_x(1) = 0, \\
    u_x(x) &> 0 \quad \text{in } (0, 1), \\
    \int_0^1 W(x)dx &= \frac{\tilde{V} + 2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{V^2 + \tilde{V} + 1} \int_0^1 u(x)dx.
\end{align*}
\]
Proposition 3.2. Let \( \tilde{V} > 0 \). There exists a solution \( W(x) \) of \( (AP; \tilde{V}) \), if and only if \( (E) \) has a solution \( (h, s) \). For the solution \( (h, s) \) of \( (E) \), \( (AP; \tilde{V}) \) has a solution in the form (2.7) with (2.8) and (2.9).

Proposition 3.3. Let \( \tilde{V} > 0 \). There exists a solution \( (h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)) \) of \( (E) \), if and only if \( (\tilde{V}, \varepsilon^2) \in G \), where \( G \) is defined by (2.4). Moreover, the solution is unique.

Proposition 3.4. Let \( \tilde{V} > 0 \), \( \varepsilon > 0 \), \( (h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)) \) be the unique solution of \( (E) \), \( W(x; \tilde{V}, \varepsilon^2) \) be the unique solution of \( (E) \) in the form (2.7) with (2.8) and (2.9), and \( u(x) \) be defined by (3.1) and (3.2) with \( W(x) = W(x; \tilde{V}, \varepsilon^2) \). Then

\[
\int_0^1 u(x)dx = \mathcal{M}(h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)),
\]

where \( \mathcal{M}(h, s) \) is defined by (2.19).

We use results in Kosugi, Morita and Yotsutani [5] for proofs of Propositions 3.1 - 3.4. We see from Proposition 1.1 and its proof in [5] that the following lemma holds.

Lemma 3.1. Let \( E > 0 \) and \( A \) be constants. Then all the solution of

\[
\begin{align*}
E^2 u_{xx} - u^3 + u - A &= 0 \quad \text{in } (0, 1), \\
u_x(0) &= u_x(1) = 0, \\
u_x(x) &> 0 \quad \text{in } (0, 1)
\end{align*}
\]

are represented by two parameters \((h, s)\) with \( 0 < h < 1 \) and \( 0 < s < 1 \) as follows.

\[
u(x; h, s) = \frac{\beta \cdot (1-hs)sn^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot cn^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)sn^2(K(\sqrt{h})x, \sqrt{h}) + cn^2(K(\sqrt{h})x, \sqrt{h})},
\]

where \( \alpha(h, s) \) and \( \beta(h, s) \) are defined by (2.8) and (2.9), and \((h, s)\) is a solution of the following system of transcendental equations

\[
\begin{align*}
\mathcal{E}(h, s) &= E, \\
\mathcal{A}(h, s) &= A,
\end{align*}
\]

where \( \mathcal{E}(h, s) \) and \( \mathcal{A}(h, s) \) are defined by (2.13) and (2.14) respectively.

Moreover,

\[
\int_0^1 u(x)dx = \mathcal{M}(h, s),
\]

where \( \mathcal{M}(h, s) \) is defined (2.19).

Proposition 3.2 immediately follows from the above lemma. Proposition 3.4 follows from the above lemma and Proposition 3.3. We will give proofs of Propositions 3.1 and 3.3 in Sections 4 and 5, respectively.

Now, we give a proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. We see from Proposition 3.2 and Proposition 3.3 that conclusions hold except (2.6).

We see that

\[
\tilde{V} + 1 - \tilde{V} \cdot W \left( 1 - x; \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{V^2} \right)
\]

is a solution of \((AP; \tilde{V})\). Thus, we obtain (2.6) by the uniqueness of solutions of \((AP; \tilde{V})\).
Proof of Theorem 2.2. We obtain conclusions by (2.16), Proposition 3.1, and Proposition 3.4.

4. Proof of Proposition 3.1. In this section we give a proof of Proposition 3.1, which is a simple calculation.

Proof of Proposition 3.1. Let us put

\[ U(x) := \frac{W(x)}{\tilde{V} + 1} \]

We get

\[
\begin{cases}
(\varepsilon \lambda)^2 U_{xx} + U(1 - U)(U - \lambda) = 0 \quad \text{in } (0, 1), \\
U_x(0) = U_x(1) = 0, \\
U_x(x) > 0 \quad \text{in } (0, 1),
\end{cases}
\]

and

\[ \int_0^1 W(x)dx = \frac{1}{\lambda} \int_0^1 U(x)dx, \]

where \( \lambda = 1/(\tilde{V} + 1) \).

We further introduce \( u(x) \) by

\[ u(x) := \left( U(x) - \left( \frac{1}{3} + \frac{\lambda}{3} \right) \right), \quad c := \frac{\sqrt{\lambda^2 - \lambda + 1}}{\sqrt{3}}. \]

We have

\[ U(x) = cu(x) + \frac{1}{3} + \frac{\lambda}{3}, \]

and obtain

\[
\begin{cases}
\left( \frac{\lambda \varepsilon}{c} \right)^2 u_{xx} - u^3 + u + \frac{1}{3\sqrt{3}} \frac{(\lambda - 2)(2\lambda - 1)(\lambda + 1)}{(\lambda^2 - \lambda + 1)^{3/2}} = 0 \quad \text{in } (0, 1), \\
u_x(0) = u_x(1) = 0, \\
u_x(x) > 0 \quad \text{in } (0, 1),
\end{cases}
\]

and

\[ \int_0^1 W(x)dx = \frac{1}{\lambda} \left( c \int_0^1 u(x)dx + \frac{1 + \lambda}{3} \right). \]

Hence, we get

\[
\begin{cases}
\left( \frac{\sqrt{3} \lambda \varepsilon}{\sqrt{\lambda^2 - \lambda + 1}} \right)^2 u_{xx} - u^3 + u \\
\quad \quad + \frac{1}{3\sqrt{3}} \frac{(\lambda - 2)(2\lambda - 1)(\lambda + 1)}{(\lambda^2 - \lambda + 1)^{3/2}} = 0 \quad \text{in } (0, 1), \\
u_x(0) = u_x(1) = 0, \\
u_x(x) > 0 \quad \text{in } (0, 1),
\end{cases}
\]

and

\[ \int_0^1 W(x)dx = \frac{1}{\lambda} \left( \frac{\sqrt{\lambda^2 - \lambda + 1}}{\sqrt{3}} \int_0^1 u(x)dx + \frac{1 + \lambda}{3} \right). \]
Therefore, we obtain
\[
\left\{ \begin{aligned}
  \left( \frac{\sqrt{3} \varepsilon}{\sqrt{\varepsilon^2 + 1}} \right)^2 u_{xx} - u^3 + u \\
  \frac{1}{3\sqrt{3}} \frac{-(1-\hat{V})(2\hat{V}+1)(\hat{V}+2)}{\sqrt{\varepsilon^2 + 1}}
\end{aligned} \right. 
\] in $(0,1)$, 
\[ u_x(0) = u_x(1) = 0, \]
\[ u_x(x) > 0 \text{ in } (0,1), \]

and
\[ \int_0^1 W(x)dx = \frac{\hat{V} + 2}{3} + \frac{1}{\sqrt{3}} \sqrt{\varepsilon^2 + 1} \int_0^1 u(x)dx. \]

5. Proof of Proposition 3.3. First we note that
\[ 3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3 \]
\[ = s^2(3s^2 - 4s + 4)h^2 - 2s(2s^2 - s + 2)h + 4s^2 - 4s + 3 > 0 \]
and $A(h,s)$ and $E(h,s)$ are well-defined in $(h,s) \in (0,1) \times (0,1)$, since
\[ s^2(2s^2 - s + 2)^2 - s^2(3s^2 - 4s + 4 + s^2)(4s^2 - 4s + 3) \]
\[ = -8s^2 s^2 - 3(s - 1)^2 < 0. \]

We prepare several lemmas.

We see from Lemma 3.2 and the proof of Lemma 3.4 of [5] that the following lemma holds.

**Lemma 5.1.** Let $E(h,s)$ be defined by (2.13). The derivative of $E(h,s)$ with respect to $s$ satisfies
\[
\frac{\partial}{\partial s} E(h,s) \begin{cases} > 0, & s \in (0,\sigma(h)), \quad h \in [0,1), \\ = 0, & s = \sigma(h), \quad h \in [0,1), \\ < 0, & s \in (\sigma(h),1), \quad h \in [0,1), \end{cases} \]
and
\[
E(h,s) < E(h,\sigma(h)) \quad \text{for all } (h,s) \in [0,1] \times [0,1] \setminus \{(h,s) : s = \sigma(h)\}, \quad \sigma(h) := 1/(1 + \sqrt{1-h}).
\]
Moreover,
\[
E(h,\sigma(h)) = \frac{1}{\sqrt{2(2-h)} K(\sqrt{h})},
\]
\[
\frac{d}{dh} E(h,\sigma(h)) < 0 \quad \text{for } h \in [0,1),
\]
and
\[ E(0,\sigma(0)) = \frac{1}{\pi}, \quad E(h,\sigma(h)) \to 0 \text{ as } h \to 1. \]
In addition,
\[ E(0,s) = \frac{2\sqrt{2}s(1-s)}{\pi \sqrt{4s^2 - 4s + 3}}. \]

**Lemma 5.2.** Let
\[ r(v) := \frac{\sqrt{3}}{9} \frac{(1-v)(2v+1)(v+2)}{\sqrt{v^2 + v + 1}}. \]
Then, $r(v)$ is monotone decreasing in $(0,\infty)$ and
\[ r(0) = \frac{2\sqrt{3}}{9}, \quad r(v) \to -\frac{2\sqrt{3}}{9} \text{ as } v \to \infty. \]
Proof. It is obvious from
\[ \frac{dr(v)}{dv} = \frac{3\sqrt{3}}{2} \cdot \frac{v(v+1)}{\sqrt{v^2 + v + 1}}, \tag{5.9} \]

Lemma 5.3. Let \( \mathcal{A}(h, s) \) be defined by (2.14). Then
\[ \mathcal{A}(h, 0) = \frac{2\sqrt{3}}{9}, \quad \mathcal{A}(h, 1) = -\frac{2\sqrt{3}}{9} \quad \text{for all } h \in [0, 1), \tag{5.10} \]
\[ \mathcal{A}_s(h, s) < 0 \quad \text{for all } (h, s) \in (0, 1) \times (0, 1). \tag{5.11} \]
Proof. It is easy to see that (5.10).
We have
\[ \mathcal{A}_s = -32s(1-s)(1-hs) \cdot (s^4h^4 - (s^4 + 2s^3)h^3 + (s^4 - 2s^3 + 6s^2 - 2s + 1)h^2 - (2s + 1)h + 1) \]
\[ / (s^2(3s^2 - 4s + 4)h^2 - 2s(2s^2 - s + 2)h + 4s^2 - 4s + 3)^{5/2}. \tag{5.12} \]
Let us put
\[ f(h, s) = h^4s^4 - (s^4 + 2s^3)h^3 + (s^4 - 2s^3 + 6s^2 - 2s + 1)h^2 - (2s + 1)h + 1. \]
We get
\[ f_h(h, s) = -4h^3s^4 + 3h^2s^4 + 6h^2s^3 - 2hs^4 + 4hs^3 - 12hs^2 + 4hs - 2h + 2s + 1, \]
\[ f_s(h, s) = -2h(2h^3s^3 - 2h^2s^3 - 3h^2s^2 + 2hs^3 - 3hs^2 + 6hs - h - 1). \]
By virtue of a tool of obtaining Groebner basis, we see that a system of algebraic equation
\[ f_h(h, s) = 0, \quad f_s(h, s) = 0, \tag{5.13} \]
is equivalent to the following system of algebraic equation:
\[ h(h - 1) \]
\[ \cdot (32h^9 - 128h^8 + 376h^7 - 845h^6 + 730h^5 + 129h^4 - 227h^3 - 114h^2 + 77h + 2) \]
\[ = 0, \tag{5.14} \]
\[ (h - 1) \]
\[ \cdot (84848288h^9 - 352803136h^8 + 1045170008h^7 - 2378915233h^6 + 2235161600h^5 \]
\[ + 154064557h^4 - 723839784h^3 - 257816310h^2 + 255595505h - 4464738s \]
\[ - 2232369) = 0, \]
\[ 18550598944h^{10} - 93159572768h^9 + 294305940632h^8 - 714679839533h^7 \]
\[ + 929921185907h^6 - 371614736476h^5 - 195277051052h^4 \]
\[ + 69656228620h^3 + 53576856s^3 + 108190730964h^2 - 160730568s^2 \]
\[ - 46074307127h + 160730568s + 127245053 = 0. \]

We can see from Strum’s theorem concerning zeros for a single algebraic equation that (5.2) does not have real zero \( h \) with \( 0 < h < 1 \). Hence the system (5.2) for \( (s, h) \) has no root in \( (0, 1) \times (0, 1) \).
Therefore, \( f(h, s) \) has no critical point in \( (0, 1) \times (0, 1) \). Let us check values of \( f(h, s) \) on the boundary. We have
\[ f(h, 0) = h^2 - h + 1 > 0, \quad f(h, 1) = (h^2 - h + 1)(1 - h)^2 > 0, \]
\[ f_s(h, s) = -2h(2h^3s^3 - 2h^2s^3 - 3h^2s^2 + 2hs^3 - 3hs^2 + 6hs - h - 1). \]

By virtue of a tool of obtaining Groebner basis, we see that a system of algebraic equation
\[ f_h(h, s) = 0, \quad f_s(h, s) = 0, \tag{5.13} \]
is equivalent to the following system of algebraic equation:
\[ f(0, s) = 1 > 0, \quad f(1, s) = (1 - s)^4 > 0, \]
for \(0 < h < 1, \ 0 < s < 1\). Thus we complete the proof.

**Lemma 5.4.** Let \( \hat{V} > 0 \) be fixed. There exists a unique curve
\[ s(h; \hat{V}) \in C^\infty[0, 1] \] (5.15)
such that
\[ A(h, s(h; \hat{V})) = \frac{1}{3\sqrt{3}} \frac{(1 - \hat{V})(2\hat{V} + 1)(\hat{V} + 2)}{\sqrt{\hat{V}^2 + \hat{V} + 1}}, \quad 0 < s(h; \hat{V}) < 1. \] (5.16)
Moreover
\[ s(0; \hat{V}) = \frac{1}{2} - \frac{1 - \hat{V}}{\sqrt{2}\sqrt{(\hat{V} + 2)(2\hat{V} + 1)}}. \] (5.17)
\[ E(0, s(0; \hat{V})) = \frac{\sqrt{3}\sqrt{\hat{V}}}{\pi \sqrt{\hat{V}^2 + \hat{V} + 1}}. \] (5.18)
and
\[ E(h, s(h; \hat{V})) \to 0 \text{ as } h \to 1. \] (5.19)

**Proof.** We obtain the existence and uniqueness of \( s(h; \hat{V}) \) by Lemmas 5.2 and 5.3. The assertion (5.15) is obtained by employing standard implicit function theorems.
We obtain (5.19) by Lemma 5.1.
We will show (5.17) and (5.18). By the construction of \( s(0; \hat{V}) \), it is the unique solution of
\[ \frac{2(-2s + 1)}{(4s^2 - 4s + 3)^{3/2}} = \frac{1}{3\sqrt{3}} \frac{(1 - \hat{V})(2\hat{V} + 1)(\hat{V} + 2)}{\left(\sqrt{\hat{V}^2 + \hat{V} + 1}\right)^3}. \]
We can obtain the solution exactly, and get
\[ s(0; \hat{V}) = \frac{1}{2} - \frac{1 - \hat{V}}{\sqrt{2}\sqrt{(\hat{V} + 2)(2\hat{V} + 1)}}. \]
Thus, we obtain (5.18) by
\[ E(0, s) = \frac{\sqrt{2}\sqrt{s(1 - s)}}{\pi\sqrt{4s^2 - 4s + 3}}. \]

Let us show that \( E(h, s(h; \hat{V})) \) is decreasing with respect to \( h \).

**Lemma 5.5.** Let \( E(h, s) \) be defined by (2.13), and \( s(h; \hat{V}) \) defined in Lemma 5.4, then for each fixed \( \hat{V} > 0 \)
\[ \frac{dE(h, s(h; \hat{V}))}{dh} < 0 \text{ in } (0, 1). \] (5.20)

**Proof.** Let us denote \( s(h; \hat{V}) \) by \( s(h) \) or \( s \), since \( \hat{V} \) is given and fixed.
It holds that
\[ \frac{dE(h, s(h))}{dh} = E_h + E_s \cdot \frac{ds(h)}{dh}. \]
and
\[ A_h + A_s \cdot \frac{ds(h)}{dh} = 0. \]
Hence, we get
\[
\frac{dE(h, s(h))}{dh} = \frac{A_sE_h - E_sA_h}{A_s}.
\] (5.21)

We have (5.12),
\[
A_h = -16s^2(1 - s)^2(1 - hs)\{s^3h^2 + (-2s^3 + 3s^2 - 3s + 2)h - 1\}
/ \left((3s^4 - 4s^3 + 4s^2)h^2 + (-4s^3 + 2s^2 - 4s)h + 4s^2 - 4s + 3\right)^{5/2},
\]
\[
E_s = 3\left(hs^2 - 2hs + 1\right)\left(hs^2 - 2s + 1\right)(1 - hs^2)
/ \left(\sqrt{2s(1 - s)(1 - hs)}K(\sqrt{h})\right) 
\cdot \left((3s^4 - 4s^3 + 4s^2)h^2 + (-4s^3 + 2s^2 - 4s)h + 4s^2 - 4s + 3\right)^{3/2},
\]
\[
E_h = \left(s(1 - s)(1 - hs)\right)
\cdot \left((3s^4 - 4s^3 + 4s^2)h^2 + (-4s^3 + 2s^2 - 4s)h + 4s^2 - 4s + 3\right)E(\sqrt{h})
- s(1 - s)(1 - h)
\cdot \left((s^4 + 2s^3)h^2 + (-8s^3 + 8s^2 - 6s)h + 4s^2 - 4s + 3\right)K(\sqrt{h})
/ \left(-h(1 - h)\sqrt{2s(1 - s)(1 - hs)}K(\sqrt{h})^2\right) 
\cdot \left((3s^4 - 4s^3 + 4s^2)h^2 + (-4s^3 + 2s^2 - 4s)h + 4s^2 - 4s + 3\right)^{3/2}.
\]

Thus, we have
\[
A_sE_h - E_sA_h = 
\left(8(-1 + hs)^2\sqrt{2s^2(s - 1)^2}\right)\left(F_1(h, s)E(\sqrt{h}) - F_2(h, s)K(\sqrt{h})\right)
/ \left((1 - h)h((3s^4 - 4s^3 + 4s^2)h^2 + (-4s^3 + 2s^2 - 4s)h + 4s^2 - 4s + 3)\right)
\cdot \left(K(\sqrt{h})^2\sqrt{s(1 - s)(1 - hs)}\right),
\] (5.22)

where
\[
F_1(h, s) := 2s^4h^4 - (2s^4 - 4s^3)h^3 + (2s^4 - 4s^3 + 12s^2 - 4s + 2)h^2 - (4s + 2)h + 2,
\]
\[
F_2(h, s) := s^4h^4 + (-3s^4 + 4s^3 - 6s^2)h^3 + (2s^4 - 4s^3 + 6s^2 + 4s + 1)h^2 + (-4s - 3)h + 2.
\]

We note that
\[
E(\sqrt{h}) - K(\sqrt{h})\sqrt{1 - h} > 0 \quad \text{for } h \in (0, 1),
\]
which is easy to prove by differentiation. On the other hand, we show that
\[
F_1(h, s) > 0 \quad \text{for } (h, s) \in (0, 1) \times (0, 1)
\] (5.23)
in Appendix.

Thus we may show
\[
F := F_1(h, s)\sqrt{1 - h} - F_2(h, s) > 0 \quad \text{in } (0, 1) \times (0, 1).
\]
Now let us put \( H := \sqrt{1 - h} \), that is, \( h := 1 - H^2 \). We get
\[
F_1(1 - H^2, s) = -2s^2H^8 + (6s^4 - 4s^3)H^6 + (-8s^4 + 16s^3 - 12s^2 + 4s - 2)H^4
+ (6s^4 - 20s^3 + 24s^2 - 12s + 2)H^2 - 2s^4 + 8s^3 - 12s^2 + 8s - 2,
\]
\[
F_2(1 - H^2, s) = s^4H^8 + (-s^4 - 4s^3 + 6s^2)H^6
+ (-s^4 + 8s^3 - 12s^2 + 4s + 1)H^4 + (s^4 - 4s^3 + 6s^2 - 4s + 1)H^2.
\]
Therefore, we obtain
\[
F = H(1 - H)^2F_3(H, s),
\]
where
\[
F_3(H, s) := 2s^3H^6 + 3s^4H^5 + (-2s^4 + 4s^3)H^4 + (-6s^4 + 12s^3 - 6s^2)H^3
+ (-2s^4 + 4s^3 - 4s + 2)H^2 + (3s^4 - 12s^3 + 18s^2 - 12s + 3)H
+ 2s^4 - 8s^3 + 12s^2 - 8s + 2.
\]
We show that
\[
F_3(H, s) > 0 \quad \text{in} \ (0, 1) \times (0, 1) \quad (5.24)
\]
in Appendix. Thus we complete the proof.

**Proof of Proposition 3.3** First, we note that
\[
0 < \frac{\sqrt{3\varepsilon}}{\sqrt{V^2 + V} + 1} < \mathcal{E}(0, s(0; \tilde{V}))
\]
is equivalent to
\[
0 < \frac{\sqrt{3\varepsilon}}{\sqrt{V^2 + V} + 1} < \frac{\sqrt{3\sqrt{V}}}{\pi \sqrt{V^2 + V} + 1},
\]
that is,
\[
0 < \varepsilon < \frac{\sqrt{V}}{\pi}.
\]
Thus we complete the proof by Lemmas 5.4 and 5.5.

6. **Appendix.** In this section we give proofs of inequalities (5.23) and (5.24).

**Proof of (5.23).** We use exactly same argument which is showed in Lemma 5.3. We can show that there exists no critical point in \((0, 1) \times (0, 1)\). We omit details here. We may check values of the boundary. We have
\[
F_1(h, 0) = 2h^2 - 2h + 2 > 0, \quad F_1(h, 1) = 2(h^2 - h + 1)(h - 1)^2 > 0,
\]
\[
F_1(0, s) = 2 > 0, \quad F_1(1, s) = 2(s - 1)^4 > 0
\]
for \(0 < h < 1, \ 0 < s < 1\). Thus we complete the proof.

**Proof of (5.24).** We use exactly same argument which is showed in Lemma 5.3. We can show that there exists no critical point in \((0, 1) \times (0, 1)\). We omit details here. We may check values of the boundary. We have
\[
F_3(H, 0) = 2H^2 + 3H + 2 > 0, \quad F_3(H, 1) = H^4(2H^2 + 3H + 2) > 0,
\]
\[
F_3(0, s) = 2(s - 1)^4 > 0, \quad F_3(1, s) = 24s^2 - 24s + 7 > 0
\]
for \(0 < H < 1, \ 0 < s < 1\). Thus we complete the proof.

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REFERENCES

[1] J. Carr, M. E. Gurtin, and M. Semrod, Structured phase transitions on a finite interval, *Arch. Rational Mech. Anal.*, 86 (1984), 317–351.

[2] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations*, 39 (1981), 269–290.

[3] J. Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer, 1994.

[4] K. Kuto and T. Tsujikawa, Bifurcation structure of steady-states for bistable equations with nonlocal constraint, *Discrete and Continuous Dynamical Systems Supplement*, 2013C467–476D.

[5] S. Kosugi, Y. Morita, and S. Yotsutani, Stationary solutions to the one-dimensional Cahn-Hilliard equation: Proof by the complete elliptic integrals, *Discrete and Continuous Dynamical Systems*, 19 (2007), 609–629D.

[6] Y. Lou, W-M. Ni and S. Yotsutani, On a limiting system in the Lotka-Voltera competition with cross diffusion, *Discrete Contin. Dyn. Syst.*, 10 (2004), 435–458.

[7] Y. Mori, A. Jilkine and L. Edelstein-Keshet, Asymptotic and bifurcation analysis of wave-pinning in a reaction-diffusion model for cell polarization *SIAM J. Appl. Math.*, 71 (2011), C1401–1427D.

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