ALL BINOMIAL IDENTITIES ARE ORDERABLE

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Abstract. The main result of this paper is to show that all binomial identities are orderable. This is a natural statement in the combinatorial theory of finite sets, which can also be applied in distributed computing to derive new strong bounds on the round complexity of the weak symmetry breaking task.

Furthermore, we introduce the notion of a fundamental binomial identity and find an infinite family of values, other than the prime powers, for which no fundamental binomial identity can exist.

1. Preliminaries

For any natural number \( n \), we set \([n] := \{1, \ldots, n\}\).

Definition 1.1. A binomial identity is any equality

\[
\binom{n}{a_1} + \cdots + \binom{n}{a_k} = \binom{n}{b_1} + \cdots + \binom{n}{b_m},
\]

where \( n \) is a natural number and \( 0 \leq a_1 < \cdots < a_k \leq n, 0 \leq b_1 < \cdots < b_m \leq n, a_i \neq b_j, \forall i, j.\)

Given a binomial identity (1.1), we associate to it the following data:

- index sets \( A := \{a_1, \ldots, a_k\} \) and \( B := \{b_1, \ldots, b_m\} \);
- the families of subsets \( \mathcal{A} := \{S \subseteq [n] \mid S \in A\} \) and \( \mathcal{B} := \{T \subseteq [n] \mid T \in B\} \).

The binomial identity then simply says that \(|A| = |B|\), i.e., there exists a bijection between \( \mathcal{A} \) and \( \mathcal{B} \).

Definition 1.2. We say that the binomial identity (1.1) is orderable if there exists a bijection \( \Phi : \mathcal{A} \to \mathcal{B} \), such that for each \( S \in \mathcal{A} \) we either have \( S \subseteq \Phi(S) \) or \( S \supseteq \Phi(S) \).

One may view \( \mathcal{A} \) and \( \mathcal{B} \) as subsets of the boolean algebra \( \mathcal{C}^n \), consisting of entire levels indexed by \( A \) and \( B \). The binomial identity is then orderable if and only if there is a perfect matching between elements of \( \mathcal{A} \) and elements of \( \mathcal{B} \), such that we are allowed only to match comparable elements. Our main result says that this can always be done.

Theorem 1.3. All binomial identities are orderable.

We note a special binomial identity \( \binom{n}{k} = \binom{n}{n-k} \), for which Theorem 1.3 is well-known and many explicit bijections have been constructed, e.g., using Catalan factorization of walks. Before we can give our proof of the main theorem, we need to make some constructions and to recall a few facts.

We start with some graph terminology. Given a graph \( G \), we let \( V(G) \) denote its set of vertices, and we let \( E(G) \) denote its set of edges. For a vertex \( v \in V(G) \), we set \( N(v) := \{w \in V(G) \mid (v, w) \in E(G)\} \), the set of all vertices adjacent to \( v \). We extend this notation to

Key words and phrases. boolean algebra, binomial coefficients, shadows, sperner theorem, distributed protocol, weak symmetry breaking.
sets of vertices $S \subseteq V(G)$ by setting $N(S) := \bigcup_{v \in S} N(v)$, so $N(S)$ is the set of all vertices of $G$ adjacent to some vertex of $S$.

A graph $G$ is called bipartite if its set of vertices can be split as a disjoint union $V(G) = U \cup W$, such that every edge of $G$ has one vertex in $U$ and one vertex in $W$. We shall say that $G = (U,W)$ is a bipartite split; note that it may not be unique if the graph is not connected. Note that if $S \subseteq U$, then $N(S) \subseteq W$ and vice versa.

**Definition 1.4.** Assume we are given two disjoint collections of subsets $X, Y \subseteq 2^{[n]}$. We let $\Gamma_{X,Y}$ denote the bipartite graph defined as follows:

- the vertices are the sets in these collections: $V(\Gamma_{X,Y}) = X \cup Y$;
- the sets $S \in X$ and $T \in Y$ are connected by an edge if and only if $S \subset T$ or $T \subset S$.

As said above, a binomial identity is orderable if and only if the bipartite graph $\Gamma_{A,B}$ has a perfect matching. The matching theory is a rich theory, and the following theorem provides a standard criterion for the existence of a perfect matching, see e.g., [Ca, LP].

**Theorem 1.5.** (Hall’s Marriage Theorem). Assume $G = (A,B)$ is a bipartite graph, such that $|A| = |B|$. The graph $G$ has a perfect matching if and only if for every set $Z \subseteq A$ we have

$$\left| N(Z) \right| \geq |Z|.$$  

In addition to the graph terminology, we need some combinatorial notions related to Boolean algebra. For all $0 \leq k \leq n$, we let $C_n^k := \{ S \subseteq [n] \mid |S| = k \}$ denote the $k$-th level in the boolean algebra $C^n$.

**Definition 1.6.** Assume we are given $S \subseteq C_n^a$ and $0 \leq b \leq n$. We set

$$\text{Sh}_b(S) := \begin{cases} \{ T \in C_n^b \mid T \subseteq S, \text{ for some } S \in S \}, & \text{if } b \leq a; \\ \{ T \in C_n^b \mid T \supseteq S, \text{ for some } S \in S \}, & \text{if } b \geq a. \end{cases}$$

Furthermore, for any set $B \subseteq [0, \ldots, n]$, we set $\text{Sh}_B(S) := \bigcup_{b \in B} \text{Sh}_b(S)$. We call these sets $b$-shadow and $B$-shadow of $S$.

We adopted here the standard terminology from Sperner theory, see [An, Chapter 2], though we do not distinguish between shadows and shades. Clearly, in terms of the bipartite graph $\Gamma_{X,Y}$ above, the shadow operation coincides with the adjacency operation $N(\cdot)$.

## 2. The proof of the main theorem

The crucial fact which we need for our proof is the following result of Sperner.

**Theorem 2.1.** (Local LYM inequality). Assume $n$ is an arbitrary natural number and we are given $S \subseteq C_n^a$, for some $0 \leq a \leq n$. Let $F := |S|/\binom{n}{a}$ denote the fraction of the chosen $a$-subsets, then for any $b$ we have

$$|\text{Sh}_b(S)| \geq F \cdot \binom{n}{b}.$$  

Moreover, we have strict inequality in (2.1), in case $S$ is a proper non-empty subset of $C_n^a$.

Stated colloquially, the local LYM inequality simply says that when viewed proportionally, the shadow of the set family $S$ is at least as large as the family $S$ itself. This is
a standard result in Sperner Theory, which can be found e.g., in [An, Section 2.1]. One says that the Boolean algebra has the normalized matching property.

For the sake of being self-contained we sketch a simple double-counting argument proving Theorem 2.1. Assume for simplicity that \( b \geq a \), the case \( a \geq b \) is completely analogous. Set \( \Lambda := \{ (S, T) \mid S \in \mathcal{S}, T \in \mathcal{C}_B^a, S \subseteq T \} \); this is the set which we want to double-count. On one hand, each \( S \in \mathcal{S} \) is contained in exactly \( \binom{n-a}{b-a} \) subsets of cardinality \( b \), so \( |\Lambda| = |\mathcal{S}| \cdot \binom{n-a}{b-a} \). On the other hand, each set is obtained from \( \Lambda \) by choosing an \( a \)-set inside a \( b \)-set, inside a fixed \( n \)-set, one can either first pick a \( b \)-set inside that \( n \)-set, and then an \( a \)-set inside the chosen \( b \)-set, or first pick an \( a \)-set inside the \( n \)-set, and then complement it to a \( b \)-set, choosing a \( (b-a) \)-set inside the \( (n-a) \)-set.

Note, that if we get equality in (2.1), then we must have \( |\text{Sh}_b(S)| \cdot \binom{b}{a} = |\Lambda| \). In other words, for every set \( T \in \text{Sh}_b(S) \), all of its \( a \)-subsets must be in \( \mathcal{S} \). This means, that if \( A \in \mathcal{S} \), and \( A' \) is obtained from \( A \) by replacing a single element, then \( A' \in \mathcal{S} \) as well. Hence, if \( \mathcal{S} \neq \emptyset \), then \( \mathcal{S} = C_B^a \).

Assume now we are given a set \( B \subseteq \{0, \ldots, n\} \). Then

\[
|\text{Sh}_b(S)| = \sum_{b \in B} |\text{Sh}_b(S)| \geq F \cdot \sum_{b \in B} \binom{n}{b} = F \cdot |B|,
\]

where \( B := \{ T \mid |T| \in B \} \). Again, we get equality in (2.2) if and only if \( \mathcal{S} = \emptyset \) or \( \mathcal{S} = C_B^a \).

We are now ready to prove our main theorem.

**Proof of Theorem 1.3** Assume we have a binomial identity with associated index sets \( A \) and \( B \), and associated collections of sets \( \mathcal{A} \) and \( \mathcal{B} \) as described above. We show that the bipartite graph \( \Gamma_{\mathcal{A}, \mathcal{B}} \) has a perfect matching by checking the condition of the marriage theorem. Let \( \mathcal{Z} \subseteq \mathcal{A}, \mathcal{Z} \neq \emptyset \), and write \( \mathcal{Z} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_k \), where \( \mathcal{Z}_i := \mathcal{Z} \cap C_{a_i}^n \).

Set \( F_i := \left| \mathcal{Z}_i \right| / \binom{n}{a_i} \), this is the fraction of all \( a_i \)-subsets contained in \( \mathcal{Z} \). Choose \( 1 \leq r \leq k \) for which \( F_r = \max F_i \). If \( F_r = 1 \), then \( \text{Sh}_b(\mathcal{Z}) = \mathcal{B} \), so \( |\text{Sh}_b(\mathcal{Z})| = |\mathcal{B}| = |A| > |\mathcal{Z}| \).

Assume now that \( F_r < 1 \), i.e., \( \mathcal{Z}_r \neq C_{a_r}^n \). We have a chain of equalities and inequalities:

\[
|\text{Sh}_b(\mathcal{Z})| \geq \sum_{i=1}^k F_i \cdot \binom{n}{a_i} \geq \sum_{i=1}^k F_i \cdot \binom{n}{a_i} = \sum_{i=1}^k |\mathcal{Z}_i| = |\mathcal{Z}|,
\]

where the second inequality is (2.2), and all the other steps are straightforward. This confirms (1.2), hence the perfect matching exists.

We remark, that we actually proved a slightly stronger condition than required by the Marriage Theorem. Namely, we have shown that

\[
|\text{Sh}_b(\mathcal{Z})| \geq |\mathcal{Z}| + 1,
\]

whenever \( \mathcal{Z} \neq \emptyset, \mathcal{Z} \neq \mathcal{A} \); cf. surplus in [LP].

3. Applications to distributed computing

In order to keep our presentation compact, this section is not made self-contained and we shall use terminology and framework from [HKR, Ko15b]. This section does not contain new mathematical results, and can be skipped by the reader interested in binomial identities.

**Proof** of Theorem 1.3. Assume we have a binomial identity with associated index sets \( A \) and \( B \), and associated collections of sets \( \mathcal{A} \) and \( \mathcal{B} \) as described above. We show that the bipartite graph \( \Gamma_{\mathcal{A}, \mathcal{B}} \) has a perfect matching by checking the condition of the marriage theorem. Let \( \mathcal{Z} \subseteq \mathcal{A}, \mathcal{Z} \neq \emptyset \), and write \( \mathcal{Z} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_k \), where \( \mathcal{Z}_i := \mathcal{Z} \cap C_{a_i}^n \).

Set \( F_i := \left| \mathcal{Z}_i \right| / \binom{n}{a_i} \), this is the fraction of all \( a_i \)-subsets contained in \( \mathcal{Z} \). Choose \( 1 \leq r \leq k \) for which \( F_r = \max F_i \). If \( F_r = 1 \), then \( \text{Sh}_b(\mathcal{Z}) = \mathcal{B} \), so \( |\text{Sh}_b(\mathcal{Z})| = |\mathcal{B}| = |A| > |\mathcal{Z}| \).

Assume now that \( F_r < 1 \), i.e., \( \mathcal{Z}_r \neq C_{a_r}^n \). We have a chain of equalities and inequalities:

\[
|\text{Sh}_b(\mathcal{Z})| \geq \sum_{i=1}^k F_i \cdot \binom{n}{a_i} = F_r \cdot \binom{n}{a_i} = F_r \cdot |\mathcal{A}| = F_r \cdot \sum_{i=1}^k \binom{n}{a_i} \geq \sum_{i=1}^k \binom{n}{a_i} = \sum_{i=1}^k |\mathcal{Z}_i| = |\mathcal{Z}|,
\]

where the second inequality is (2.2), and all the other steps are straightforward. This confirms (1.2), hence the perfect matching exists.

We remark, that we actually proved a slightly stronger condition than required by the Marriage Theorem. Namely, we have shown that

\[
|\text{Sh}_b(\mathcal{Z})| \geq |\mathcal{Z}| + 1,
\]

whenever \( \mathcal{Z} \neq \emptyset, \mathcal{Z} \neq \mathcal{A} \); cf. surplus in [LP].
Assume \( n \) is a natural number, and there exists a binomial identity 

\[ (\begin{array}{c} n \\ 0 \end{array}) + (\begin{array}{c} n \\ a_1 \end{array}) + \cdots + (\begin{array}{c} n \\ a_k \end{array}) = (\begin{array}{c} n \\ 1 \end{array}) + (\begin{array}{c} n \\ b_1 \end{array}) + \cdots + (\begin{array}{c} n \\ b_m \end{array}), \]

such that \( 2 < a_1 < \cdots < a_k < n, 1 < b_1 < \cdots < b_m < n, \) and \( a_i \neq b_j, \) for all \( i, j. \) Set \( A := \{0, a_1, \ldots, a_k\}, B := \{1, b_1, \ldots, b_m\}, \) \( \mathcal{A} := \{S \subseteq [n] \mid |S| \in A\} \) and \( \mathcal{B} := \{T \subseteq [n] \mid |T| \in B\}. \) Assume furthermore that there exists a bijection \( \Phi : \mathcal{A} \rightarrow \mathcal{B}, \) such that

- \( \Phi(\emptyset) = [n], \)
- for all \( S \in \mathcal{A} \) we have either \( \Phi(S) \subseteq S \) or \( \Phi(S) \supseteq S. \)

Then there exists a 3-round IIS protocol solving WSB for \( n \) processes.

**Proof.** We can construct the desired bijection \( \Phi \) as follows. To start with, set \( \Phi(\emptyset) := [n]. \) Let \( \Gamma \) be obtained from \( \Gamma_{A,B} \) by deleting \( \emptyset \) and \([n].\) In the proof of Theorem 3.3, we have actually showed that \( \Gamma_{A,B} \) satisfies the stronger condition (2.3). This means that the graph \( \Gamma \) satisfies the Marriage theorem condition and hence has a perfect matching. Thus we get a bijection \( \Phi \) satisfying all the necessary conditions.

The binomial identities of the type (3.1) exist for all \( n = 6t, \) where \( t \) is an arbitrary natural number: \( \sum_{k=0}^{t-1} \binom{n}{3k} = \sum_{k=0}^{t-1} \binom{n}{3k+1}, \) see also [Ko15b]. This shows that there are infinitely many values of \( n \) for which such identity exists. On the other hand, when \( n \) is a prime power \( p, \) the left hand side of (3.1) is equal to 1 modulo \( p, \) whereas the right hand side is divisible by \( p, \) so no such identity can exist.

Here are examples of binomial identities for \( n = 15, 20, 21, \) satisfying conditions in Theorem 3.2

\[
\begin{align*}
(\begin{array}{c} 15 \\ 0 \end{array}) + (\begin{array}{c} 15 \\ 4 \end{array}) + (\begin{array}{c} 15 \\ 6 \end{array}) + (\begin{array}{c} 15 \\ 13 \end{array}) = (\begin{array}{c} 15 \\ 1 \end{array}) + (\begin{array}{c} 15 \\ 3 \end{array}) + (\begin{array}{c} 15 \\ 5 \end{array}) + (\begin{array}{c} 15 \\ 10 \end{array}), \\
(\begin{array}{c} 20 \\ 0 \end{array}) + (\begin{array}{c} 20 \\ 5 \end{array}) + (\begin{array}{c} 20 \\ 6 \end{array}) + (\begin{array}{c} 20 \\ 13 \end{array}) = (\begin{array}{c} 20 \\ 1 \end{array}) + (\begin{array}{c} 20 \\ 3 \end{array}) + (\begin{array}{c} 20 \\ 4 \end{array}) + (\begin{array}{c} 20 \\ 8 \end{array}), \\
(\begin{array}{c} 21 \\ 0 \end{array}) + (\begin{array}{c} 21 \\ 4 \end{array}) + (\begin{array}{c} 21 \\ 5 \end{array}) + (\begin{array}{c} 21 \\ 7 \end{array}) + (\begin{array}{c} 21 \\ 14 \end{array}) + (\begin{array}{c} 21 \\ 19 \end{array}) = (\begin{array}{c} 21 \\ 1 \end{array}) + (\begin{array}{c} 21 \\ 3 \end{array}) + (\begin{array}{c} 21 \\ 6 \end{array}) + (\begin{array}{c} 21 \\ 8 \end{array}).
\end{align*}
\]
The Theorem 3.2 implies then that Weak Symmetry Breaking can be solved in 3 rounds for \( n = 15, 20, \) and 21.

4. Fundamental binomial identities

We now fix a certain family of binomial identities whose existence appears to be an interesting but difficult question, which is also important in various contexts.

**Definition 4.1.** Let \( n \) be an arbitrary natural number. A binomial identity of the form

\[
\binom{n}{0} + \binom{n}{a_1} + \cdots + \binom{n}{a_k} = \binom{n}{b_1} + \cdots + \binom{n}{b_m},
\]

such that \( 1 \leq a_1 < \cdots < a_k < n \), \( 1 \leq b_1 < \cdots < b_m < n \), is called a fundamental binomial identity associated to \( n \).

**Question.** For which values of \( n \) does a fundamental identity associated to \( n \) exist?

A fundamental binomial identity certainly does not exist for the degenerate case \( n = 1 \). Furthermore, when \( n = p^r \) is a prime power, we see that \( p \) divides the right hand side of (4.1), while it does divide the left hand side. So again, no identity exists, and \( n \) must be at least 6.

Clearly, any binomial identity of the type (3.1) is fundamental, so from the previous section we know that such an identity exists for \( n = 6 \). Furthermore, the nonexistence of fundamental identities will also imply the nonexistence of identities of type (3.1). The first value for which no fundamental binomial identity exists is \( n = 10 \). It is easy to prove the following more general proposition.

**Proposition 4.2.** Assume that \( n = p^k \cdot q^m \), where \( p \) and \( q \) are different prime numbers, \( k \geq 1 \), and \( m \geq 0 \). If we have

\[
p \geq 2^m,
\]

then there does not exist any binomial identity of the type (3.1).

**Proof.** The case \( m = 0 \) has already been settled, so assume \( m \geq 1 \). Note that \( p \) divides \( \binom{n}{t} \), unless \( t = \alpha p^k \). Furthermore, \( \binom{n}{\alpha p^k} \equiv \binom{q^m}{q^m} \mod p \), for all \( \alpha = 0, \ldots, q^m \). So a binomial identity of the type (3.1) would imply that we have

\[
\epsilon_1 \binom{q^m}{1} + \cdots + \epsilon_{q^m-1} \binom{q^m}{q^m-1} \equiv 1 \mod p,
\]

for some \( \epsilon_1, \ldots, \epsilon_{q^m-1} \in \{-1, 0, 1\} \). On the other hand, we have

\[
\left| \epsilon_1 \binom{q^m}{1} + \cdots + \epsilon_{q^m-1} \binom{q^m}{q^m-1} \right| \leq 2^{q^m} - 2 \leq p - 2,
\]

where the last inequality uses our assumption (4.2). Combining (4.3) with (4.4) we obtain

\[
\epsilon_1 \binom{q^m}{1} + \cdots + \epsilon_{q^m-1} \binom{q^m}{q^m-1} = 1,
\]

which is impossible, since the left hand side is divisible by \( q \). \( \square \)

As mentioned above, when \( m = 0 \) in Proposition 4.2, we recover the case when \( n \) is a prime power. When \( m = 1, q = 2 \), we get the case \( n = 2p^k \), for \( p \geq 5 \), covering the special cases \( n = 10, 14, 22, \ldots \). Further special values of \( q \) and \( m \) will yield the cases \( n = 3p^k \), for \( p \geq 11, n = 4p^k \), for \( p \geq 17, n = 5p^k \), for \( p \geq 37, n = 7p^k \), for \( p \geq 131 \), etc.

1At the time writing we are not aware of any \( n \) for which a fundamental binomial identity exists, but an identity of the type (3.1) does not exist.
5. Final remarks

Curtis Greene, \[Gr\], has suggested that the following stronger version of \[Ko15b, Conjecture 11.5\] might be true. Our argument above yields this more general result as well.

**Corollary 5.1.** Assume that we have an inequality
\[
\left( \binom{n}{a_1} + \cdots + \binom{n}{a_k} \right) \leq \left( \binom{n}{b_1} + \cdots + \binom{n}{b_m} \right).
\]
Set \( \Sigma := \{S \mid |S| \in \{a_1, \ldots, a_k\} \} \), \( \Lambda := \{T \mid |T| \in \{b_1, \ldots, b_m\} \} \). Then there exists an injection \( \Phi : \Sigma \to \Lambda \) with \( S \subseteq \Phi(S) \) or \( S \supseteq \Phi(S) \), for all \( S \in \Sigma \).

**Proof.** Follows immediately from our argument together with a stronger version of the Hall’s Marriage Theorem, e.g., see \[LP, Theorem 1.3.1\], since the corresponding bipartite graph here has deficiency 0. \( \Box \)

As a final note, we would like to remark that Hall’s Marriage Theorem has many constructive proofs, see, e.g., \[LP\]. This means that once we have an identity (5.1), there is a way to *construct* the corresponding perfect matching. That in turn, combined with the direct path construction in \[Ko15b\], yields an explicit 3-round distributed protocol solving the Weak Symmetry Breaking for \( n \) processes.

**Acknowledgments.** We thank Curtis Greene for engaged discussions.

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