Almost a century ago, Bose [11] and Einstein [2] introduced what is sometimes called the fifth state of matter, the Bose-Einstein Condensate (BEC). It was not until 1995 [3–5] that a BEC was created in a laboratory by using gases of ultra-cold atoms of $^{87}\text{Rb}$ and $^{23}\text{Na}$. The 2001 Nobel prize in physics was awarded to these researchers for the experimental demonstration of a BEC [6, 7]. The development of atomtronics attempts to use a BEC condensate for quantum sensing, quantum computing, and quantum information science [8, 9]. The project reported here is motivated by the recent introduction of a stand-alone device by the company ColdQuanta [10].

The underlying equation for a BEC in a gas has been rigorously shown [12], with reasonable assumptions, to be the Gross-Pitaevskii equation (GPeq) [13, 14]. The GPeq is one type of a non-linear Schrödinger equation [15]. If the gas used for the BEC has $N$ atoms of mass $m$ and a scattering length $a_s$, the nonlinear term in the GPeq is $g = 4\pi\hbar^2a_s/m$. Here $\hbar$ is Plank’s constant divided by $2\pi$. Note that in an atomic BEC $g$ can sometimes be changed experimentally by orders of magnitude and even change its sign [16, 17]. The time-independent GPeq is expressed as [18]

$$\mu\tilde{\psi}(\vec{r}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + \tilde{V}(\vec{r})\right)\tilde{\psi}(\vec{r})$$

with $\mu$ chemical potential. Here $\tilde{\psi}(\vec{r})$ is the wave function of the BEC, and

$$\tilde{V}(\vec{r}) = V(\vec{r}) + gN\left|\psi(\vec{r})\right|^2.$$  

We also detail a prescription to find solutions of the time-independent Schrödinger equation (TISE), namely

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + [V(\vec{r}) - E] \psi(\vec{r}) = 0$$

where for a single mass $m$ particle the wavefunction is $\psi(\vec{r})$ with energy $E$. Note the differences in the physical interpretation between Eq. (1) for $\tilde{\psi}$ and the TISE of Eq. (3) for $\psi$. Nevertheless, mathematically Eq. (1) goes to Eq. (3) with the replacements $\tilde{\psi} \rightarrow \psi$, $\mu \rightarrow E$, and $gN = 0$ so $\tilde{V} \rightarrow V$.

Since the first encounter with the TISE a student learns the traditional way to solve Eq. (3). By the term ‘traditional approach’ we mean: given a $V(\vec{r})$ solve for $\psi(\vec{r})$ and $E$. Then find the pdf using $P(\vec{r}) = |\psi(\vec{r})|^2$. Several interesting applications of this approach are explained in standard textbooks, including [19, 20], for real space as well as momentum space. As opposed to the traditional method, we propose an alternative prescription. Namely, given a pdf $P(\vec{r})$ find which potential $V(\vec{r})$ and energy $E$ in the TISE is needed with $\psi(\vec{r}) = \sqrt{P(\vec{r})}$ to give the desired pdf. Our prescription works only in some cases, requiring certain mathematical properties for $P(\vec{r})$. Nevertheless, our prescription also works in certain cases to obtain solutions of the GPeq for general $gN$, as well as for some other nonlinear Schrödinger equations.

We first outline the prescription for the TISE of Eq. (3). In order to form the 3D potential $V(\vec{r})$ for the given 3D pdf $P(\vec{r})$, we define an analytic function $f(\vec{r})$. We assume we can choose the wavefunction to be real, thereby limiting which problems we can solve. We assume

$$\psi(\vec{r}) = \sqrt{A}\exp\left(-f(\vec{r})/2\right)$$

so $P(\vec{r}) = A\exp(-f(\vec{r}))$, with $A$ for the normalization $\int P(\vec{r})d\vec{r} = 1$. Using first and second-order derivatives of the function $f(\vec{r})$, we can solve both the time independent Schrödinger and Gross-Piteevskii equations.

For the TISE, by substituting Eq. (4) into Eq. (3) we obtain the equation

$$V(\vec{r}) - E = -\frac{\hbar^2}{4m} \nabla^2 f(\vec{r}) + \frac{\hbar^2}{8m} \left[\nabla f(\vec{r}) \cdot \left(\nabla f(\vec{r})\right)\right].$$

(5)
For the time-independent GPeq we also use the ansatz 
\[ \tilde{\psi}(\vec{r}) = \sqrt{A} \exp[-f(\vec{r})/2] \]
and obtain the equation
\[ \tilde{V}(\vec{r}) - \mu = -\frac{\hbar^2}{4m} \nabla^2 f(\vec{r}) + \frac{\hbar^2}{8m} \left[ \left( \nabla \tilde{f} \right) \cdot \left( \nabla \tilde{f} \right) \right]. \] (6)

We expand on the prescription applied to the 1D Gumbel distribution for both Eq. (5) and Eq. (6). The Gumbel pdf is
\[ P_{\text{Gbl}}(x) = \frac{1}{\beta} \exp \left[ -\left( \frac{x-x_0}{\beta} + \exp \left[ -\frac{x-x_0}{\beta} \right] \right) \right] \] (7)
where \( x_0 \) is the mode of the pdf, and \( \beta \) is a scale parameter. From Eq. (7) and in 1D Eq. (4), the function \( f_{\text{Gbl}}(x) \) associated with the distribution \( P_{\text{Gbl}}(x) \)
\[ f_{\text{Gbl}}(x) = \frac{x-x_0}{\beta} + \exp \left[ -\frac{x-x_0}{\beta} \right] + \ln \beta \] (8)
with \( A = 1 \). The potential \( V_{\text{Gbl}}(x) \) for the 1D Gumbel pdf is obtained by substituting the first and second-order derivatives of Eq. (8) into Eq. (5) in 1D. This gives, choosing the zero of energy as the minimum of \( V_{\text{Gbl}}(x) \), as
\[ V_{\text{Gbl}}(x) = \frac{\hbar^2}{4m\beta^2} \left[ \frac{1}{2} \left( 1 - e^{-\left( \frac{x-x_0}{\beta} \right)} \right)^2 - e^{-\left( \frac{x-x_0}{\beta} \right)} + \frac{3}{2} \right] \] (9)
and thus also find the energy associated with this wavefunction to be \( E = 3\hbar^2/(8m\beta^2) \). The potential \( V_{\text{Gbl}} \) is shown in Fig. 1 for select values of \( x_0 \) and \( \beta \). The Gumbel potential \( V_{\text{Gbl}}(x) \) has a minimum at \( x_{\text{min}} = x_0 - \beta \ln(2) \) which has \( V(x_{\text{min}}) = 0 \).

For the 1D Gumbel distribution for the time-independent GPeq of Eq. (1), the same procedure is followed. In fact, after solving for the pdf for the TISE, the 'generalized' potential \( \tilde{V} \) for the GPeq, Eq. (2), is found simply by adding the required nonlinear term to \( V \) of the TISE. Explicitly for the 1D Gumbel pdf from Eq. (2) one obtains
\[ \tilde{V}_{\text{Gbl}}(x) = V_{\text{Gbl}}(x) + gNP_{\text{Gbl}}(x) \] (10)
with \( V_{\text{Gbl}} \) from Eq. (9) and \( P_{\text{Gbl}} \) from Eq. (7). The chemical potential \( \mu \) is the energy difference between \( E = 3\hbar^2/(8m\beta^2) \) and the minimum of the generalized potential \( \tilde{V}(x) \). See Fig. 2 for example plots of \( \tilde{V}_{\text{Gbl}}(x) \).

We implemented our method to obtain solutions of the TISE for some well-known 1D pdfs with various domains. The exact potentials \( V(x) \) and groundstate energy \( E \) are provided in Table I where column 3 contains the potentials for the pdfs in column 2.

Four pdfs (Gaussian, Cauchy, Gumbel, and logistic) have potentials with zero walls since the domain is \( x \in [-\infty, \infty] \). For the Gaussian potential, we can recognize that it is the simple harmonic oscillator (SHO or quadratic) potential with \( V_{\text{min}} \) at \( x = x_0 \) and \( P(x) \) is the groundstate probability distribution. The Lorentzian (or Cauchy) potential has \( V(x \to \pm \infty) \to \hbar^2/2m\gamma^2 \). This potential has a minimum at \( x_{\text{min}} = x_0 \) with \( V(x_{\text{min}}) = 0 \), and it has a maximum located at \( x_{\text{max}} = x_0 \pm \sqrt{2} \) with the value \( V(x_{\text{max}}) = 2\hbar^2/3\gamma^2 \), but we have found one bound state for this potential. The logistic potential becomes zero at it's minimum \( x = x_0 \) and maximum at \( x \to \infty \). We found the groundstate energies (in units where \( \hbar^2/4m = 1 \)) to be \( 1/\sigma^2 \), \( 2/\gamma^2 \), and \( 1/2s^2 \) for the Gaussian, Lorentzian, and logistic distributions, respectively.

We studied the two pdfs, Rayleigh and chi that have a potential with 1 wall since the domain is \( x \in [0, \infty] \).
The Rayleigh distribution is nothing but the chi distribution with two degrees of freedom, $k = 2$. The Rayleigh potential has its maximum at $x \pm \sigma$. We derived the potential which gives the generalization with $x \rightarrow x/\sigma$ of the chi distribution with parameter $k$. Here we limit ourselves to the case where there is a finite minimum for $V(x)$ and a bound state, so the values in Table I are only for $k \geq 3$ which has the ground state energy $E_0 = \left(k - \sqrt{(k-3)(k-1)}\right)/\sigma^2$.

The 1D beta distribution has 2 hard walls since the domain is $x \in [0,1]$. For the beta potential, only certain values of parameters $\alpha$ and $\beta$ give a potential $V(x)$ that does not go to $-\infty$. We outline four different cases for the beta potential considering possible values of the parameters $\alpha$ and $\beta$: 1) $\lim_{x \to 0^+} V(x) \to +\infty$ for $0 < \alpha < 1$ or $3 < \alpha$; 2) $\lim_{x \to 0^-} V(x) \to -\infty$ for $1 < \alpha < 3$; 3) $\lim_{x \to 1^-} V(x) \to +\infty$ for $0 < \beta < 1$ or $3 > \beta$; and finally 4) $\lim_{x \to 1^+} V(x) \to -\infty$ for $1 < \beta < 3$. Thus only in the case where both $0 < \{\alpha, \beta\} < 1$ or $3 < \{\alpha, \beta\}$ is there a finite minimum for $V(x)$.

The prescription outlined here for the TISE works for any 1D pdf, with modest sufficient constraints of being continuous and piece-wise twice continuously differentiable. Although the pdfs in Table I all have a single maximum in $P(x)$, this is not a requirement and very complicated $P(x)$ can thought of that with the prescription gives very complicated $V(x)$. The prescription also works for a pdf that is an excited state, as is easily verified for example for the well-known excited state pdfs for the SHO potential. Solutions for Eq. (3) in higher dimensions are also easily found using this prescription.

In particular, for traditional separable pdfs, written in 2D Cartesian coordinates, $P(x,y) = P_x(x)P_y(y)$ gives $f(x,y) = f_x(x) + f_y(y)$ and hence a potential $V(x,y) = V_x(x) + V_y(y)$. The same result generalizes to higher dimensions and to any other separable set of coordinates. One can argue the reason the prescription works so well is that it is the time-independent version of the nonlinear differential equation which corresponds to quantum mechanics as derived using Fisher information theory [21–23], together with the ansatz $P(\vec{r}) = A \exp(-f(\vec{r}))$. The equations derived using Fisher information are the analogue mathematically of Bohm's formulation of quantum mechanics [24, 25].

Compare Eq. (6) with Eq. (5), together with the definition of $V(\vec{r})$ of Eq. (3). Thus in any case where the prescription works for the TISE there is a corresponding solution of the GPeq. One illustration of this is in Fig. 2 for the 1D Gumbel distribution. Of course one must keep in mind that at constant temperature and pressure for the Gibbs free energy $G$ that the chemical potential is $\mu = \left(\frac{\partial G}{\partial N}\right)_{T,p}$.

To provide another concrete 2D example, we use the 2D pdf for the hydrogen atom in momentum space. The H atom in momentum space has a long history starting in 1928 [26, 27], but we use the notation from the 2020 article [28]. From [29] we use their two variables for $p_r$ and $\theta_p$ and wavefunctions from their Eq. [36], while performing the integral over their variable $\phi_p$. The unit for momentum we use is $p_0 = 2\pi \hbar/a_0$ with $a_0$ the Bohr radius of the hydrogen atom. These atomic physics pdfs $P_{n,\ell,m}(p_r, \theta_p)$ for the quantum numbers $n, \ell$, and $m$ for the hydrogen atom in momentum space can be viewed as just given examples of a pdf. We choose to plot both the pdf and the potential in Cartesian coordinates with $x' = p_r$ and $y' = \theta_p$. Consequently, we ask which $V_{n,\ell,m}(x', y') = V_{n,\ell,m}(p_r, \theta_p)$ gives the desired pdf $P_{n,\ell,m}(x', y')$. An example is shown in Fig. 3.

![Image](image-url)

**FIG. 3.** (a) The 2D pdf $P_{2,1,0}(p_r, \theta_p)$ for the hydrogen atom in momentum space, plotted in Cartesian coordinates $x' = p_r$ and $y' = \theta_p$. The units used are $p_0 = 1$. (b) The 2D potential $V_{2,1,0}(p_r, \theta_p)$ that gives the 2D pdf $P_{2,1,0}(p_r, \theta_p)$. 3
TABLE I. Seven 1D pdfs, $P(x)$, that satisfy the TISE (so $gN = 0$) together with their potentials $V(x)$. The mode of the top four pdfs is $x_0$. With the minimum of $V(x)$ set to the zero of energy, the energy of the groundstate is $E_0$. For the chi distribution the results are only for $k \geq 3$. The beta distribution only has a minimum in $x < \infty$. For the logistic distribution only has a minimum in $x < \infty$. For other cases the GPeq parameter gives a wide class of new solutions for bound states of the TISE, and works in any dimension. The GPeq solutions to be taught to students the first time they are introduced to this differential equation. The introduced prescription (GPeq) and the time-independent Schrödinger equation (TISE). The method of solving the TISE is simple enough to be taught to students the first time they are introduced to this differential equation. The introduced prescription gives a wide class of new solutions for bound states of the TISE, and works in any dimension. The GPeq solutions obtained using the prescription are also relatively easy, and can be used for any value (and sign) of $g$. For an atomic BEC the GPeq parameter $g$ can be of either sign as well as be large or small, and still the prescription will work. With experimentally obtained ‘painted’ potentials [10, 11] for an atomic BEC the prescription may have additional potential technological benefits. The prescription will also work in other cases where the GPeq may be valid, for example in the case of $\psi$-dark-matter [30]. Future work will be to implement a related study for the time-dependent Schrödinger equation and Gross-Pitaevskii equation for one or many Bose-Einstein condensates [31].

| pdf        | $P(x)$                                      | $V(x)$ with $\frac{\hbar^2}{4m} = 1$ | domain   | $E_0$ with $\frac{\hbar^2}{4m} = 1$ |
|------------|--------------------------------------------|-------------------------------------|----------|--------------------------------------|
| Gaussian   | $\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-x_0}{\sigma} \right)^2}$ | $\frac{(x-x_0)^2}{2\sigma^4}$     | $-\infty < x < \infty$ | $\frac{1}{\sigma^2}$ |
| Lorentzian (Cauchy) | $\frac{1}{\pi \gamma} \left[ \frac{x^2}{(x-x_0)^2 + \gamma^2} \right]$ | $\frac{6(x-x_0)^2}{[x-x_0]^2 + \gamma^2}] - \frac{2}{(x-x_0)^2 + \gamma^2} + \frac{2}{\gamma^2}$ | $-\infty < x < \infty$ | $\frac{2}{\pi^2}$ |
| Gumbel     | $\frac{1}{\beta} \exp \left[ - \left( \frac{x-x_0}{\beta} \right) \right] + e^{\left( \frac{x-x_0}{\beta} \right)} \right]$ | $\frac{1}{\beta^2} \left( 1 - e^{-\left( \frac{x-x_0}{\beta} \right)^2} \right)$ | $-\infty < x < \infty$ | $\frac{3}{2\beta^2}$ |
| Logistic   | $\frac{1}{2s} \text{sech}^2 \left( \frac{x-x_0}{2s} \right)$ | $\frac{1}{s^2} \tanh^2 \left( \frac{x-x_0}{2s} \right)$ | $-\infty < x < \infty$ | $\frac{1}{2s^2}$ |
| Rayleigh   | $\frac{x}{\sigma^2} \exp \left( - \frac{x^2}{2\sigma^2} \right)$ | $\frac{1}{7} \left( \frac{x^2}{\sigma^2} - \frac{1}{\sigma^2} \right)$ | $0 \leq x \leq \infty$ | $\frac{2}{\sigma^2}$ |
| Chi        | $\frac{1}{\sigma^2 \frac{\pi}{2} \Gamma \left( \frac{k-1}{2} \right)} \left( \frac{x}{\sigma} \right)^{k-1-1} \times \exp \left( - \frac{x^2}{2\sigma^2} \right)$ | $\frac{x^2}{2\sigma^2} + \frac{(k-1)(k-3)}{2x^2}$ | $0 \leq x \leq \infty$ | $\frac{k-\sqrt{(k-1)(k-3)}}{\sigma^2}$ |
| Beta       | $\frac{\Gamma \left( \alpha + \beta \right)}{\Gamma \left( \alpha \right) \Gamma \left( \beta \right)} x^{\alpha-1} \times (1-x)^{\beta-1}$ | $\frac{(\alpha+\beta-2)(\alpha+\beta-4)x^2}{2x^2(1-x)^2} - \frac{2x(\alpha-1)(\alpha+\beta-4)}{2x^2(1-x)^2} + \frac{(\alpha^2-4\alpha+3)}{2x^2(1-x)^2}$ | $0 \leq x \leq 1$ | see text |

In summary, we have introduced a new methodology to solve both the time-independent Gross-Pitaevskii equation (GPeq) and the time-independent Schrödinger equation (TISE). The method of solving the TISE is simple enough to be taught to students the first time they are introduced to this differential equation. The introduced prescription gives a wide class of new solutions for bound states of the TISE, and works in any dimension. The GPeq solutions obtained using the prescription are also relatively easy, and can be used for any value (and sign) of $g$. For an atomic BEC the GPeq parameter $g$ can be of either sign as well as be large or small, and still the prescription will work. With experimentally obtained ‘painted’ potentials [10, 11] for an atomic BEC the prescription may have additional potential technological benefits. The prescription will also work in other cases where the GPeq may be valid, for example in the case of $\psi$-dark-matter [30]. Future work will be to implement a related study for the time-dependent Schrödinger equation and Gross-Pitaevskii equation for one or many Bose-Einstein condensates [31].

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