Low-Energy-Theorem Approach to one-particle singularity in

$\text{QED}_{2+1}$

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Abstract

We evaluate the propagator of scalar and spinor in three dimensional quantum electrodynamics with the use of Ward-Identity for soft-photon emission vertex. We work well in position space to treat infrared divergences in our model. Exponentiation of one-photon matrix element yields a full propagator in position space. It has a simple form as free propagator multiplied by quantum correction. And it shows a new type of mass singularity. But this is not an integrable function so that analysis in momentum space is not easy. Term by term integral converges and they have a logarithmic singularity associated with renormalized mass in perturbation theory. Renormalization constant vanishes for weak coupling, which suggests confinement of charged particle. There exists a critical coupling constant above which the vacuum expectation value of pair condensation is finite.
I. INTRODUCTION

In three dimensional gauge theory infrared divergences is severer than that in four dimension. In 1981, Jackiw first demonstrated it in the massless fermionic $QED_{2+1}$ and after that other authors introduced the infrared counter terms to renormalize the infrared divergences[1,2]. There had been an attempts to solve Dyson-Schwinger equations to examine the dynamical mass generation or chiral symmetry breakings in this model to improve low-energy behaviour[3,4]. Another important feature of this model is that it allows parity violating Chern-Simons term in the Lagrangean without violating gauge invariance[5]. But it has not yet been clear the quantum effects of Chern-Simons term. During the same period infrared behaviour of the propagator in the presence of dynamical mass has been analysed to search the physical cut associated with massless photon[6,7]. To maintain gauge covariance Delbourgo and Jackiw added the vertex correction with Ward-Identities for soft particles in the determination of the infrared behaviour of the propagator with bare mass[9,10]. In their works relation between bare and renormalized masses are given by the spectral function. Their works are fundamental and important to understand the effects of soft-photons in comparison with perturbation theory. Of course their works includes non-perturbative effects. In 1992 Delbourgo applied his method (gauge technique) to $QED_{2+1}$ and shown that massless loop correction to photon soften infrared divergences as T.Appelquist et.al[3,6].
reference[9] infrared behaviour of the scalar propagator in the presence of massless fields (photon, graviton) in four dimension was determined by solving the spectral function. In this work we apply the same method in three dimension to determine the infrared behaviour of the propagator. In section I we analyse scalar QED₃ and evaluate the scalar propagator in position space based on spectral representation. In this case an approximation is made to choose the one-photon, one-meson intermediate state to derive the matrix element \( \langle \Omega | \phi(x) | n \rangle \). Most singular infrared part is assumed to be the contribution of soft photon emitted from external lines. Therefore we introduce photon mass to avoid infrared divergences. Lowest order spectral function contains mass and wave function renormalization up to logarithm in position space. Exponentiation of the lowest-order spectral function yields the full propagator in position space. It shows us a new type of mass singularity. Especially mass renormalization make a drastic change of the propagator. Its has a simple form as free propagator (with renormalized mass) multiplied by quantum correction. However this function is not integrable and it is difficult to make a fourier transform of it. In the perturbative analyses the propagator have linear and logarithmic infrared divergences near on-shell. We mention the gauge transformation property and see our solution satisfies Landau-Kharatonikov transformation. As far as the renormalization constant is concerned we evaluate it by position space propagator. The result is \( Z = 0 \) for weak coupling, which is a signpost of confining phase in our approximation. In section II we study QED₃ with spinor and see the spectral function for fermion is the same in section I. In this case there is a interesting possibility of pair condensation. Evaluating the vacuum expectation value \( \langle \bar{\psi} \psi \rangle \) and we find that there exists a critical coupling constant \( (e^2/(8\pi m) = 1) \) above which \( \langle \bar{\psi} \psi \rangle \) remains finite.

II. SCALAR QED

First we assume the Kallen-Lehmann spectral representation of the propagator for massive boson[1]:
\[ \Delta(p^2) = \int \! d^3 x \exp(ip \cdot x) \langle \Omega|T\phi(x)\phi^+(0)|\Omega \rangle, \]  

(1)

\[ \Delta(p^2) = \int \frac{\sigma(s) ds}{p^2 - s + i\epsilon}, \]

\[ \text{Im} \Delta(p) = \sigma(p^2) = (2\pi)^2 \sum_N \delta(p - p_N) \langle \Omega|\phi|N \rangle \langle N|\phi^+|\Omega \rangle, \]

(2)

\[ |N> = |r; k_1, ... k_n>, r^2 = m^2. \]

(3)

The spectral function is formally written as a sum of multi-photon intermediate states:

\[ \sigma(p^2) = \int \! \frac{d^2 r}{2r_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int \! \frac{d^3 k}{(2\pi)^2} \theta(k_0) \delta^3(k^2) \sum_\epsilon \delta(p - r - \sum_{i=1}^n k_i) \right) \times \langle \Omega|\phi|r; k_1, ... k_n \rangle \langle r; k_1, ... k_n|\phi^+|\Omega \rangle. \]

(4)

Here we use the notation

\[ (f(k))_0 = 1, \]

\[ (f(k))_n = \prod_{i=1}^n f(k_i). \]

(5)

To evaluate the contribution of the soft-photons, we consider from the beginnings when only the \( n \)th photon is soft. Define the following matrix element

\[ T_n(r; k_1, ..., k_n) = \langle \Omega|\phi|r; k_1, ..., k_n \rangle. \]

(6)

We consider \( T_n \) for \( k_n^2 \neq 0 \), we continue off the photon mass-shell by Lehmann-Symanzik-Zimmermann (LSZ) formula:

\[ T_n = \epsilon^\mu_n T_{n\mu}, \]

(7)

\[ T_{n\mu} = \int \! d^3 x \exp(i k_n \cdot x) \Box x \langle \Omega|T\partial_\mu A_\mu(x)|r; k_1, ..., k_n \rangle \]

\[ = \int \! d^3 x \exp(i k_n \cdot x) \langle \Omega|T\partial_\mu j^\mu(x)|r; k_1, ..., k_{n-1} \rangle, \]

(8)

where the electromagnetic current is

\[ j^\mu(x) = ie\phi^+(x)\overrightarrow{\partial_\mu}\phi(x) - 2e^2 A_\mu\phi^+\phi, \]

(9)
here we assume the second term does not contribute in the infrared behaviour of the propagator and the usual commutation relation for the first term
\[ \delta(x_0 - y_0)[j_0(x), \phi(y)] = e\phi(x). \] (10)
Thus we omit the second term in the present calculus. From the definition (8), \( T_n^\mu \) is seen to satisfy the Ward-Identity
\[ k_n T_n^\mu = eT_{n-1}(r; k_1, .., k_{n-1}). \] (11)
Here we mention the simple proof:
\[
k_n T_n^\mu = \int d^3x k_n^\mu \epsilon_n^\mu \exp(ik_n \cdot x) \langle \Omega | T \phi j^\mu(x) | r, k_1, .., k_{n-1} \rangle
\]
\[
= i \int d^3x \epsilon_n^\mu \exp(ik_n \cdot x) \partial_\mu \langle \Omega | T \phi j^\mu(x) | r, k_1, .., k_{n-1} \rangle
\]
\[
= e\epsilon_n^\mu \langle \Omega | \phi | r, k_1, .., k_{n-1} \rangle,
\] (12)
provided
\[ \partial_\mu T(\phi(y) j^\mu(x)) = \delta(x - y) e\phi(y), \] (13)
where \( e \) is the charge carried by meson. To get the solution of the Ward-Identity we seek the non-singular contribution of photon for \( T_n(r; k_1, .., k_n) \). There is a possible solution which satisfy Ward-Identity[9]
\[ T_n^\mu(r; k_1..k_n) = \frac{e(2r + k_n)^\mu}{2r \cdot k_n + k_n^2} T_{n-1}(r, k_1..k_{n-1}), r^2 = m^2. \] (14)
Here we assume most singular contributions of photons emitted from external lines as in four dimension. These arise from Feynman diagram in which the soft-photon line with momentum \( k_n^\mu \) and the incoming meson line can be separated from the remainder of the diagram by cutting a single meson line. Detailed discussion and evaluation of the explicit form of \( T_n \) are given in ref [9]. Of course for \( n = 1 \)
\[ T_1 = \frac{e(2r + k)^\mu}{2r \cdot k + k^2} \epsilon, \] (15)
and we can determine \( T_n \) recursively with (14). In this case \( T_n \) becomes
\[ T_n = \left( \frac{e(2r + k)^\mu}{2r \cdot k + k^2} \right)_n \epsilon. \] (16)
The one photon matrix element by LSZ is

\[ T_1 = \left\langle in|T(\phi_{in}(x), ie \int d^3 y \phi_{in}^+ \overleftrightarrow{\partial}_\mu \phi_{in}(y) A^\mu_{in}(y))| r; k \; in \right\rangle \]

\[ = ie \int d^3 y d^3 z \Delta_F(x - y) \overleftrightarrow{\partial}_\mu \epsilon^\lambda_\mu(k) \exp(ik \cdot y) \delta^{(3)}(y - z) \exp(ir \cdot z) \]

\[ = \epsilon(2r + k)_\mu \epsilon^\lambda_\mu(k) \left( \frac{1}{(r + k)^2 - m^2} \right) \exp(i(r + k) \cdot x). \quad (17) \]

From this lowest order matrix element the function in (4) is given

\[ F = \sum_{one \ photon} \langle \Omega | \phi(x)| r; k \rangle \langle r; k | \phi^+(0) | \Omega \rangle \]

\[ = \int \frac{d^3 k}{(2\pi)^2} \exp(i k \cdot x) \delta(k^2) \theta(k^0) \left[ \frac{e^2(2r + k)^\mu(2r + k)^\nu \Pi_{\mu \nu}}{(2r \cdot k + k^2)^2} \right], \quad (18) \]

and \( \sigma \) is expressed by

\[ \sigma(p) = \int \frac{d^3 x}{(2\pi)^3} \exp(ip \cdot x) \int \frac{d^2 r}{2r^0} \exp(ir \cdot x) \exp(F). \quad (19) \]

Here \( \Pi_{\mu \nu} \) is a polarization sum we have

\[ \Pi_{\mu \nu} = -(g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2}) - \frac{d k_\mu k_\nu}{k^2}, \quad (20) \]

and the free photon propagator

\[ D_0^{\mu \nu} = \frac{1}{k^2 + i \epsilon} [g^{\mu \nu} - \frac{k_\mu k_\nu}{k^2} + \frac{d k_\mu k_\nu}{k^2}]. \quad (21) \]

we get

\[ F = -e^2 \int \frac{d^3 k}{(2\pi)^2} \exp(i k \cdot x) \theta(k^0) \]

\[ \times \left[ \frac{(2r + k)^2 \delta(k^2)}{(2r \cdot k + k^2)^2} + (d - 1) \frac{\delta(k^2)}{k^2} \right]. \quad (22) \]

It is natural to set \( \delta(k^2)/k^2 \) equals to \( -\delta(k^2) \). We see the reason in calculating the one loop self energy of meson diagram, and take imaginary part. This expression is the same as the one given above with the interpretation \( \delta(k^2)/k^2 = -\delta(k^2) \). We must introduce a small photon mass \( \mu \) as an infrared cut off. Therefore (22) becomes

\[ F = -e^2 \int \frac{d^3 k}{(2\pi)^2} \theta(k^0) \exp(i k \cdot x) [\delta(k^2 - \mu^2)\left( \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)^2} \right) - (d - 1) \frac{\partial}{\partial k^2} \delta(k^2 - \mu^2)]. \quad (23) \]

The evaluation of the integral to \( F \) is described in Appendix. The result is

\[ F = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi r} + \frac{e^2}{8\pi r} \ln(\mu x) - \frac{e^2}{8\pi} x \ln(\mu x) - \frac{e^2}{8\pi} x(d + \gamma - 2). \quad (24) \]
It is helpful to use position space that it shows us easily the short and long-distance behavior. The relation between perturbative spectral function and $F(x)$ is

$$
\sigma^{(2)}(p) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \int \frac{d^2r}{2\pi^2} \exp(ir \cdot x) F(x). \tag{25}
$$

Here we think about the physical meanings of each term.

$$
\sigma(x) = \frac{\exp(-mx)}{4\pi x} \exp(F(x,e,\mu,m)), \tag{26}
$$

$$
\sigma(p) = F.T.(\sigma(x)) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \sigma(x). \tag{27}
$$

The terms proportional to $x$ correspond to mass renormalization and others to wave function renormalization.

$$
\sigma(x) = \exp(A) \frac{\exp(-(m+B)x)}{4\pi x} (\mu x)^{-C x + D}, \tag{28}
$$

$$
A = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi m}, B = \frac{e^2}{8\pi} (d + \gamma - 2), C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}. \tag{29}
$$

From the form of $\sigma(x)$, wave function renormalization effects denoted by constant and $D \ln(\mu x)$ modifies the dimension of the propagator as anomalous dimension in four dimensions. Mass renormalization is proportional to $x$ and $x \ln(\mu x)$. Logarithmic corrections describe the logarithmic infrared divergences and adds a drastic effect as a factor $(\mu x)^{-C x}$. This factor $(\mu x)^{-C x}$ is a new type of mass singularity in three dimension. In perturbative analysis these will be shown clear. By the definition (1), (2), (3) $\sigma(x)$ is a full propagator. Free propagator in three dimension is

$$
\sigma_0(x,m+B) = \frac{\exp(-(m+B)x)}{4\pi x}, \sigma_0(p,m+B) = \frac{1}{(m+B)^2 + p^2}, \tag{30}
$$

and its quantum correction is expressed by $\exp(A)(\mu x)^{-C x + D}$. We find in this expression that $\sigma(x)$ is finite provided

$$
|\sigma(x) = \exp(A) \frac{\exp(-(m+B)x)}{4\pi x} (\mu x)^{-C x + D} | = finite, \tag{31}
$$

$$
0 \leq \int_0^\infty \sigma(x) dx \leq M, \ 0 < D, \tag{32}
$$

and there exists $\sigma(p)$

$$
\sigma(p) = \int_0^\infty \frac{x \sin(px)}{p} \sigma(x) dx. \tag{33}
$$
In this case we have not a simple pole even in the Yennie gauge \( d = 2 \) in which we see the singularity at \( p^2 = (m + e^2\gamma/8\pi)^2 \). Next we treat the momentum dependence of the propagator \( \sigma(p) \) (32). The integral is not analytic and we cannot get the precise expression for \( \sigma(p) \).

If we expand \( (\mu x)^{-Cx+D} \)

\[
(\mu x)^{-Cx+D} = 1 + (-Cx + D) \ln(\mu x) + \frac{1}{2}(-Cx + D)^2(\ln(\mu x))^2 + \frac{1}{6}(-Cx + D)^3(\ln(\mu x))^3 + \ldots,
\]

(34)

term by term integral of \( \sigma(p) \) converges. Using the following integrals

\[
I_1 = \int_0^\infty \frac{\sin(px)}{p} \exp(-mx) \ln(\mu x) dx = -\frac{\gamma}{p^2 + m^2} - \frac{\ln((m^2 + p^2)/\mu^2)}{2(p^2 + m^2)} - \frac{\ln((m - \sqrt{-p^2})/(m + \sqrt{-p^2}))}{p^2 + m^2},
\]

(35)

\[
I_2 = \int_0^\infty \frac{\sin(px)}{p} \exp(-mx)x \ln(\mu x) dx = \frac{-m}{(p^2 + m^2)^2} \left[ \ln((m - \sqrt{-p^2})/(m + \sqrt{-p^2})) + \ln((p^2 + m^2)/\mu^2) - 2(1 - \gamma) \right],
\]

(36)

\( \sigma(p) \) up to \( O(e^2) \) is given

\[
\sigma^{(2)}(p) = \left[ \frac{1}{m^2 + p^2} + \frac{A}{p^2 + m^2} - \frac{mB}{2(p^2 + m^2)^2} \right] + DI_1 - CI_2, p = \sqrt{-p^2}.
\]

(37)

In this case the renormalization constant \( Z \) (residue of the pole term) becomes

\[
Z = 1 + A + D(-\gamma + \frac{1}{2} \ln\left(\frac{p^2 + m^2}{\mu^2}\right))_{p^2 \to \infty} \to \infty.
\]

(38)

In the Yennie gauge there remains a renormalization

\[
Z = 1 + \frac{e^2}{16\pi m} \ln((p^2 + m^2)/\mu^2)_{p^2 \to \infty} \to \infty.
\]

(39)

In Minkowski space, \( p^2 \to -p^2 \), with discontinuity

\[
\frac{1}{x - i\epsilon} = P.V. \frac{1}{x} + i\pi \delta(x)
\]

(40)

\[
\frac{1}{(x - i\epsilon)^2} = P.V. \frac{1}{x^2} + i\pi \delta'(x),
\]

(41)

\[
\frac{1}{x - i\epsilon} \ln(x - i\epsilon) = i\pi \delta(x) \ln(x) + P.V. \frac{1}{x} i\pi.
\]

(42)
we can determine the structure near $p^2 = -m^2$ by the imaginary part of $\sigma(p)$. Here we notice the second-order spectral function $\sigma^{(2)}(p)$

$$\sigma^{(2)}(x) = \frac{\exp(-mx)}{4\pi x} (1 + A - Bx + (D - Cx) \ln(\mu x)), \quad (43)$$

$$\sigma^{(2)}(p) = \frac{1}{m^2 + p^2} + \frac{A - \gamma D}{m^2 + p^2} - \frac{2mB}{(m^2 + p^2)^2} - D \ln\left(\frac{\sqrt{(m^2 + p^2)/\mu^2}}{(p^2 + m^2)}\right)$$

$$+ \frac{Cm}{(p^2 + m^2)^2} \ln((p^2 + m^2)/\mu^2) - 2(1 - \gamma)]. \quad (44)$$

Here we look at mass renormalization part

$$Z_m = \frac{m(2B + 2C(1 - \gamma) - C \ln((p^2 + m^2)/\mu^2))}{p^2 + m^2} = \frac{e^2 2(d - 1) - \ln((p^2 + m^2)/\mu^2)}{8\pi} \frac{p^2 + m^2}{p^2 + m^2}. \quad (45)$$

Spectral function $\text{Im} \sigma^{(2)}(-p^2)$ reads

$$\frac{\text{Im} \sigma^{(2)}(-p^2)}{\pi} = (1 + A - \gamma D)\delta(p^2 - m^2) - \frac{e^2}{4\pi} m(d - 1)\delta'(p^2 - m^2)$$

$$- \frac{D\theta(p^2 - m^2)}{m^2 - p^2} + \frac{Cm\theta(p^2 - m^2)}{(m^2 - p^2)^2}. \quad (46)$$

Usually we estimate the second-order spectral function in the Feynman gauge $d = 1$;

$$\frac{\text{Im} \sigma^{(2)}(-p^2)}{\pi} \bigg|_{d=1} = [(1 - \frac{e^2}{8\pi \mu})\delta(p^2 - m^2)$$

$$- \frac{D}{p^2 - m^2} + \frac{Cm}{(m^2 - p^2)^2}] \theta(p^2 - m^2)]. \quad (47)$$

This shows the renormalization constant $Z$ has a linear infrared divergence as $\mu \to 0$. If we add higher order corrections linear infrared divergences cancel each other in the same mechanism in four dimension[15,16,17]. In the Yennie gauge it has cuts

$$\frac{\text{Im} \sigma^{(2)}(-p^2)}{\pi} \bigg|_{d=2} = \delta(p^2 - m^2) - \frac{e^2 m}{4\pi} \delta'(p^2 - m^2)$$

$$\quad + \frac{e^2}{8\pi m} \left( \frac{1}{p^2 - m^2} + \frac{2m^2}{(p^2 - m^2)^2} \right) \theta(p^2 - m^2). \quad (48)$$

To see the singularity structure near $p^2 = -m^{*2}$ it is better to expand around $p^2 = -m^{*2}$. We get

$$\sigma^{(2)}(p) = \left[ \frac{1}{m^{*2} + p^2} + (\exp(A) - 1) \times (-CI_2(m \to m^{*}) + DI_1(m \to m^{*}) \right]. \quad (49)$$
In this case the renormalization constant $Z$:

$$Z = 1 + (\exp(A) - 1) + D(-\gamma + \frac{1}{2} \ln(\frac{m^*^2 + p^2}{\mu^2}))$$  \hspace{1cm} (50)$$

is divergent at $p^2 = -m^*^2$. There is an interesting contribution from $\exp(F_2)$

$$\int_0^\infty \frac{\sin(px)}{p} \exp(-mx)(\mu x)^a dx$$

$$= -\Gamma(a + 1) \frac{\cos(\pi a)}{2} (p^2 + m^2)^{1-a/2} \mu^a$$

$$\times \frac{1}{\sqrt{-p^2}} [(\sqrt{-p^2} + m)(\sqrt{-p^2} - m)^{-a/2} + (\sqrt{-p^2} - m)(\sqrt{-p^2} + m)^{a/2}]$$

$$\sim (\sqrt{-p^2} - m)^{-1-a} \text{ near } p^2 = -m^2, a = \frac{e^2}{8\pi m}.$$  \hspace{1cm} (51)$$

Here we notice that this type of singularity also appears in four dimensions where $a = -\alpha(d - 3)/(2\pi)$ but in this case $a$ is gauge invariant. Formally fourier transform of $\sigma(p)$ can be written by double fourier transform

$$\sigma(p) = \int \frac{d^3q}{(2\pi)^3} F.T.\frac{\exp(-mx)(\mu x)^D(q)}{x} \times F.T.(\mu x)^{-Cx}(p - q),$$

but we do not discuss details of it in this paper. As we see in the analysis in terms of $e^2$, the contribution of the function $(\mu x)^{-Cx}$ leads to a new type of mass singularity: divergent series at $p^2 = -m^*^2$ in $\sigma (2)(p)$. We have seen that the fourier transformation of the propagator is very difficult. It is interesting to study the phase structure of the model. First we estimate the renormalization constant by the following sum rule[10,11],

$$Z^{-1} = \int \sigma(\omega)d\omega.$$  \hspace{1cm} (53)$$

$$Z^{-1} = \lim_{p^2 \to \infty} p^2 \int \sigma(\omega)d\omega \sin(px)\exp(-\omega x)dx,$$  \hspace{1cm} (54)$$

$$\lim_{p \to \infty} F(p) = \lim_{p \to \infty} \int_0^{\pi x} \frac{\pi x \sin(px)}{p} F(x)dx = \lim_{p \to \infty} \frac{\pi x}{p^2} (xF(x))(0).$$  \hspace{1cm} (55)$$

we can evaluate this quantity by direct substitution of $\sigma(x)$ into the above equations and take the limit

$$Z^{-1} = \exp(A) \lim_{x \to 0^+} \exp(-mx)(\mu x)^{-Cx+D} = 0.$$  \hspace{1cm} (56)$$
Next we consider the gauge dependence of the propagator\cite{12,13,14}. When we write the photon propagator as
\[ D_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + k_\mu k_\nu M(k), \]
we seek the change of the propagator. Under the gauge transformation defined:
\[ \phi(x) \rightarrow \exp(i e \chi(x)), \phi^+(x) \rightarrow \phi^+(x) \exp(-i e \chi(x)), \]
\[ A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \chi, \delta(\phi^+ \partial_\mu \phi) = -\partial_\mu \chi(\phi^+ \partial_\mu \phi - 2 i e \phi^+ \phi A_\mu), \]
propagator changes
\[ D_{\mu\nu} \rightarrow i \left\langle T [A_\mu(x) - \partial_\mu \chi(x)][A_\nu(y) - \partial_\nu \chi(y)] \right\rangle = D_{\mu\nu} + i \left\langle T [\partial_\mu \chi(x) \partial_\nu \chi(y)] \right\rangle, \]
\[ \delta D_{\mu\nu} = \partial_\mu \partial_\nu M(x - y), M(x - y) = i \left\langle T \chi(x) \chi(y) \right\rangle \]
\[ \Delta \rightarrow -i \left\langle T \phi(x) \exp(i e \chi(x)) \exp(-i e \chi(y)\phi^+(y)) \right\rangle = \Delta \left\langle T \exp(i e \chi(x)) \exp(-i e \chi(y)) \right\rangle. \]
Here we expand in $\delta \chi$
\[ \left\langle T \exp(i e \delta \chi(x)) \exp(-i e \delta \chi(y)) \right\rangle = -\frac{1}{2} e^2 \left\langle \delta \chi^2(x) + \delta \chi^2(y) - 2 T \delta \chi(x) \delta \chi(y) \right\rangle, \]
\[ \delta \Delta = i e^2 \Delta^{(0)}(x - y)[M(0) - M(x - y)], \]
\[ i[M(0) - M(x - y)] = -\left\langle T \delta \chi(x) \delta \chi(y) \right\rangle. \]
Usually $M$ is identified as the gauge fixing term
\[ M(k) = -d/k^4, \]
\[ i e^2 M(x) = -i e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\exp(ik \cdot x) - 1}{k^4} = -\frac{e^2 d}{8\pi} x, \]
with subtraction of infrared divergence at $k = 0$. We get the propagator\cite{59} in covariant $d$ gauge for three dimension
\[ \Delta(d, x) = \exp(-\frac{e^2 d}{8\pi} x) \Delta(0, x) \]
\[ = \exp(-\frac{e^2 d}{8\pi} x) \sigma(0, x), \]
Since $\sigma(x)$ is a full propagator it obeys the Landau-Kharatonikov transformation in our approximation except for linear infrared divergent factor $A$

$$\sigma(d, x) = \frac{\exp(-(m^* + de^2/8\pi)x)}{x} \exp(A)\exp(Cx + D). \quad (65)$$

Therefore we notice that the Landau-Karatonikov transformation is regularization dependent. We keep the Landau gauge and not take the covariant $d$ gauge to avoid longitudinal photon. Here we mention the radiative correction of mass to understand the mass shift. In the Landau gauge our approximation changes the mass $m$ to $m^* = m + e^2(\gamma - 2)/(8\pi)$. The constant $\gamma$ appeared after the expansion of $\text{Ei}(1, \mu x)$ in $\mu$. In ref.[5], gauge dependent self-energy is estimated at $O(e^2)$

$$\Sigma(m) = \frac{de^2}{4\pi}, \quad (66)$$

and the physical mass for fermion defined in the Landau gauge

$$m_{phy} = m + \Sigma(m). \quad (67)$$

If we estimate one-loop self-energy in the Landau gauge

$$\Sigma(m) = \frac{e^2}{2\pi}, \quad (68)$$

which is two-times larger than $e^2/(4\pi)$ in $m^*$ since we used $D_\pm(x)$ instead of $D(x)$. We see the same effects occurred for scalar with vertex correction. We call $m^* = m + e^2(\gamma - 2)/8\pi$ the renormalized mass in our approximation. At first sight, mass shift seems to be peculiar but infrared behaviour is governed by one-particle intermediate state.

### III. SPINOR QED

In this section we study the spectral function for massive fermion. Similar to scalar case propagator and the spectral function are defined

$$S(p) = \int d^3x \exp(i p \cdot x) \langle \Omega | T(\bar{\psi}(x)\psi(0)) | \Omega \rangle$$

$$= \left( \int_m^\infty + \int_{-\infty}^{-m} \right) \frac{\rho(\omega)d\omega}{p \cdot \gamma - \omega + i\epsilon}, \quad (69)$$

$$\rho(p^2) = (2\pi)^2 \sum_N \delta^{(3)}(p - p_N) \langle \Omega | \psi | N \rangle \langle N | \overline{\psi} | \Omega \rangle. \quad (70)$$
Matrix element is

\[ T_n = \langle \Omega | \psi | r; k_1, \ldots, k_n \rangle, \tag{71} \]

\[ T_n^\mu = - \int d^3 x \exp(ik_n \cdot x) \langle \Omega | T \psi j^\mu | r; k_1, \ldots, k_{n-1} \rangle, \tag{72} \]

provided

\[ \Box_x T \psi A_\mu(x) = T \psi \Box_x A_\mu(x) = T \psi (-j_\mu(x) + \partial^\mu \partial \cdot A(x)). \tag{73} \]

In the similar way to the scalar case \( T_n \) satisfies Ward-Identity:

\[ \partial^\mu T(\psi j_\mu(x)) = -e \psi(x), \tag{74} \]
\[ \partial^\mu T(\overline{\psi} j_\mu(x)) = e \overline{\psi}(x), \tag{75} \]
\[ k_{\mu \nu} T_n^\mu(r, k_1, k_2, \ldots, k_n) = e T_{n-1}(r, k_1, k_2, \ldots, k_{n-1}), r^2 = m^2. \tag{76} \]

One photon matrix element by LSZ is

\[
T_1 = \left\langle in | T(\psi_{in}(x), ie \int d^3 y \psi_{in}(y) \gamma_\mu \psi_{in}(y) A_\mu^{in}(y)) | r; k in \right\rangle \\
= ie \int d^3 y d^3 z S_F(x - y) \gamma_\mu \delta^{(3)}(y - z) \exp(i(k \cdot y + r \cdot z)) e^{\mu}(k, \lambda) U(r, s) \\
= -ie \frac{(r + k) \cdot \gamma + m}{(r + k)^2 - m^2} \gamma_\mu e^{\mu}(k, \lambda) \exp(i((k + r) \cdot x)) U(r, s), \tag{77} \]

where \( U(r, s) \) is a two-component free particle spinor with positive energy:

\[ \sum_s U(r, s) \overline{U}(r, s) = \frac{\gamma \cdot r + m}{2m}. \tag{78} \]

In this case the function \( F \) becomes

\[
F = e^2 \int \frac{d^3 k}{(2\pi)^3} \exp(ik \cdot x) \theta(k^0) \delta(k^2) tr[ \frac{(r + k) \cdot \gamma + m}{((r + k)^2 - m^2)} \gamma_\mu \frac{\gamma \cdot r + m}{2m} \frac{(r + k) \cdot \gamma + m}{((r + k)^2 - m^2)} \Pi^{\mu \nu} ] \\
= -e^2 \int \frac{d^3 k}{(2\pi)^3} \exp(ik \cdot x) \theta(k^0) [\delta(k^2 - \mu^2)(\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)^2}) + (d - 1) \frac{\partial \delta(k^2 - \mu^2)}{\partial k^2}] . \tag{79} \]

\[ F = \frac{e^2 m^2}{8\pi r^2} \left( -\frac{1}{\mu} + x(1 - \gamma) - x \ln(\mu x) \right) + \frac{e^2}{8\pi r} \left( \ln(\mu x) + \gamma \right) + \frac{e^2}{8\pi} (d - 1) \left( \frac{1}{\mu} - x \right). \tag{80} \]

Thus the lowest order spectral function is exactly the same with that in the scalar case except for the normalization factor in the phase space integral:
\[
\rho(p^2) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \rho(x) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \int d^2r \frac{m}{r_0} \exp(ir \cdot x) \exp(F). \tag{81}
\]

After angular integral we get
\[
\rho(p) = \frac{m}{4\pi p} \exp(A) \int_0^\infty dx \sin(px) \frac{\exp(-Bx)}{x} (\mu x)^{-C_D} x. \tag{82}
\]

Therefore the structure is the same with scalar case as we mentioned in the last section. Consequently there is no infrared divergences at \(p^2 = m^2\).

For the renormalization constants we apply the same argument as for the scalar, full propagator is expressed in the dispersion integral
\[
S_F(x) = -\int \rho(\omega) d\omega (i\gamma \cdot \partial + \omega) \frac{\exp(-\omega x)}{4\pi x} = -(i\gamma \cdot \partial \sigma(x) + \sigma(x)),
\]

Here we use the Fourier transform of \(S_F\) to determine \(Z^{-1}\). For free case the propagator in position space is
\[
S_F^{(0)}(x) = (i\gamma \cdot \partial) \frac{1}{4\pi \sqrt{-x^2}} = \frac{i\gamma \cdot x}{4\pi (-x^2)^{3/2}} = \frac{i\gamma \cdot x}{\sqrt{-x^2}} \frac{1}{4\pi x^2}. \tag{83}
\]

We see the dimension of the \(S_F^{(0)}(x)\) is equal to \(1/x^2\). In momentum space we obtain
\[
\int_0^\infty \frac{x \sin(px)}{p} \frac{1}{x^2} dx = \frac{\pi}{2p}, \tag{84}
\]

and this shows the ordinary expression in momentum space
\[
S_F^{(0)}(p) = \frac{\gamma \cdot p}{p^2} = \frac{\gamma \cdot p}{p^2}. \tag{85}
\]

\[
Z^{-1} = \lim_{x \to 0^+} \left[(i\gamma \cdot \partial \sigma(\sqrt{-x^2})) \frac{i\gamma \cdot x}{\sqrt{-x^2}}\right] = \lim_{x \to 0^+} \left(\frac{d\sigma(\sqrt{-x^2})}{d\sqrt{-x^2}}\right) \tag{86}
\]

\[
= \pi \exp(A) \lim_{x \to 0^+} \frac{d}{dr} \left[\frac{\exp(-mr)}{r} (\mu r)^{-C_D}\right] = \begin{bmatrix} 0 & (1 < D) \\ \infty & (1 \geq D) \end{bmatrix},
\tag{87}
\]

\[
\frac{d}{dr} \left[\frac{\exp(-mr)}{r} (\mu r)^{-C_D}\right] = \exp(-mr)(\mu r)^{-C_D} \left[-\frac{m+C}{r} + \frac{D-1}{r^2} - \frac{C \ln(\mu r)}{r}\right],
\tag{88}
\]

\[
m_0 Z^{-1} = \int \omega \rho(\omega) d\omega = \exp(A) \lim_{x \to 0^+} (\mu x)^{-C_D} = 0. \tag{89}
\]
Therefore there is a confining phase; $Z_2 = 0 (Z^{-1}_2 = \infty)$ of charged particle for weak coupling constant. In this case bare mass vanishes for all coupling. Order parameter for the vacuum expectation value of pair condensate is given

$$\langle \bar{\psi} \psi \rangle = -itrS_F(x) = -2m^2 \exp(A) \lim_{x \to 0^+} (\exp(-mx)(x)^{-C_\lambda+D-1}))$$

$$= -2m^2 \begin{cases} 0 & (1 < D) \\ finite & (1 = D) \\ \infty & (1 > D) \end{cases} .$$

(90)

Here we notice that there is a critical coupling constant $D_{cr} = 1 (e^2/(8\pi m) = 1)$.

IV. SUMMARY

We have seen how the Ward-Identity for soft photon may be applied to three-dimensional electro-dynamics to extract full propagator in position space and the infrared behaviour of the propagator in momentum space. Neglecting unconventional terms in the electromagnetic current for scalar, spectral functions for scalar and spinor coincide each other. Exponentiation of the lowest order spectral function corresponds to the infinite ladder approximation. Since the theory is super renormalizable the mass are corrected to add some finite radiative correction. We found a new type of mass singularity in three dimensional QED in position space as $(\mu x)^{-C_\lambda x}$. In momentum space it shows us logarithmic mass renormalization and yields a singular infrared structure of the propagator. Wave function renormalization $D$ plays the role as anomalous dimension. These are the consequences of logarithmic infrared divergences. Renormalization constant vanishes for spinor case and there is a confining phase for weak coupling. This picture is consistent with perturbative logarithmic infrared divergences at $p^2 = m^2$. If the coupling constant is smaller than $D_{cr}$ ($D = e^2/8\pi m \leq 1$), vacuum expectation value $\langle \bar{\psi} \psi \rangle$ becomes finite. In our lowest order spectral function there remains linear infrared divergences that was regularized by photon mass. $O(e^2)$ correction to the external line and the cancellation of infrared divergences are now in progress[17].
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VI. APPENDIX

To evaluate $F$ we use the function $D_+(x)$ in three dimension:

$$D_+(x) = \frac{1}{(2\pi)^2 i} \int \exp(ik \cdot x) \theta(k^0) \delta(k^2 - \mu^2) d^3 k$$

$$= \frac{1}{(2\pi)^2 i} \int_0^\infty J_0(kx) \frac{\pi k dk}{2\sqrt{k^2 + \mu^2}} = \frac{\exp(-\mu x)}{8\pi i x}, \quad (91)$$

with the following parameter trick as in ref[9]

$$\lim_{\epsilon \to 0^+} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = \frac{\exp(ik \cdot x)}{k \cdot r}, \quad (92)$$

$$\lim_{\epsilon \to 0^+} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = \frac{\exp(ik \cdot x)}{(k \cdot r)^2}. \quad (93)$$

The function $F$ is written in terms of the parameter integrals

$$F = ie^2 m^2 \int_0^\infty d\alpha D_+(x + \alpha r, \mu) - e^2 \int_0^\infty d\alpha D_+(x + \alpha r, \mu)$$

$$- ie^2 (d - 1) \frac{\partial}{\partial \mu^2} D_+(x, \mu)$$

$$= \frac{e^2 m^2}{8\pi r^2} \left( - \frac{\exp(-\mu x)}{\mu} + x \text{Ei}(1, \mu x) \right) - \frac{e^2}{8\pi r} \text{Ei}(1, \mu x) + (d - 1) \frac{e^2}{8\pi \mu} \exp(-\mu x), \quad (94)$$

where the function $\text{Ei}(n, x)$ is defined

$$\text{Ei}(n, x) = \int_1^\infty \frac{\exp(-xt)}{t^n} dt. \quad (95)$$

It is understood that all terms which vanish with $\mu \to 0$ are ignored.

$$\text{Ei}(1, \mu x) = -\gamma - \ln(\mu x) + O(\mu x), \quad (96)$$
\[ F_1 = \frac{e^2 m^2}{8\pi r^2} \left( \frac{1}{\mu} + x(1 - \ln(\mu x) - \gamma) \right) + O(\mu), \]  
(97)

\[ F_2 = \frac{e^2}{8\pi r} (\ln(\mu x) + \gamma) + O(\mu), \]  
(98)

\[ F_g = \frac{e^2}{8\pi} \left( \frac{1}{\mu} - x \right)(d - 1) + O(\mu). \]  
(99)

Here \( \gamma \) is Euler’s constant. Using the integrals

\[ \int d^3 x \exp(ip \cdot x) \int d^3 r \delta(r^2 - m^2) \exp(ir \cdot x) f(r) = f(m), \]  
(100)

\[ \int d^3 x \exp(ip \cdot x) \int \frac{d^3 r}{(2\pi)^3} \delta(r^2 - m^2) = \frac{1}{m^2 + p^2}. \]  
(101)

we get \( F \) in position space

\[ F = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi r} + \frac{e^2}{8\pi r} \ln(\mu x) - \frac{e^2}{8\pi} x \ln(\mu x) - \frac{e^2}{8\pi} x(d + \gamma - 2). \]  
(102)

After integration over \( r \)

\[ \int_0^\pi \exp(ir \cdot x \cos(\theta)) d\theta = \pi J_0(rx), \]  
(103)

\[ \int_0^\infty dr \frac{\pi r J_0(rx)}{\sqrt{r^2 + m^2}} = \frac{\exp(-mx)}{x}, \]  
(104)

we obtain the full propagator

\[ \sigma(x) = \exp(A) \frac{\exp(-(m + B)x)}{4\pi x} (\mu x)^{-C + D}, \]  
(105)

\[ A = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi m}, \quad B = \frac{e^2}{8\pi} (d + \gamma - 2), \quad C = \frac{e^2}{8\pi}, \quad D = \frac{e^2}{8\pi m}. \]  
(106)

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