BRAIDED AUTOEQUIVALENCES AND QUANTUM COMMUTATIVE 
BI-GALOIS OBJECTS

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Abstract. Let \((H, R)\) be a quasitriangular weak Hopf algebra over a field \(k\). We show that there is a braided monoidal equivalence between the Yetter-Drinfeld module category \(\mathcal{YD}_H\) over \(H\) and the category of comodules over some braided Hopf algebra \(R_H\) in the category \(\mathcal{H}\). Based on this equivalence, we prove that every braided bi-Galois object \(A\) over the braided Hopf algebra \(R_H\) defines a braided autoequivalence of the category \(\mathcal{YD}_H\) if and only if \(A\) is quantum commutative.

In case \(H\) is semisimple over an algebraically closed field, i.e. the fusion case, then every braided autoequivalence of \(\mathcal{YD}_H\) trivializable on \(\mathcal{H}\) is determined by such a quantum commutative Galois object. The quantum commutative Galois objects in \(\mathcal{H}\) form a group measuring the Brauer group of \((H, R)\) as studied in [20] in the Hopf algebra case.

Introduction

Let \(\mathcal{C}\) be a braided fusion category \(\mathcal{C}\), that is, a fusion category equipped with a braiding. Denote by \(\mathbb{Z}(\mathcal{C})\) the Drinfeld center of \(\mathcal{C}\). The braided autoequivalences of \(\mathbb{Z}(\mathcal{C})\) play important roles in the study of braided fusion categories, see [3, 4, 7]. For example, auto-equivalences were used to classify \(G\)-extensions of a given fusion category, see [7]. In order to classify \(G\)-extensions of a given fusion category \(\mathcal{C}\) using the classical homotopy theory, P. Etingof, D. Nikshych and V. Ostrik introduced in [7] a 3-groupoid \(\mathbb{BP}(\mathcal{C})\), called the Brauer-Picard groupoid of \(\mathcal{C}\). This 3-groupoid can be truncated in the usual way into the Brauer-Picard group \(\mathbb{BP}(\mathcal{C})\) of \(\mathcal{C}\), i.e. the group of the equivalence classes of invertible \(\mathcal{C}\)-bimodule categories. It turns out that there is a natural group isomorphism [7, Thm 1.1]:

\[\mathbb{BP}(\mathcal{C}) \cong \text{Aut}^{br}_{\mathcal{Z}(\mathcal{C})}(\mathbb{Z}(\mathcal{C}))\]

where \(\text{Aut}^{br}(\mathbb{Z}(\mathcal{C}))\) is the group of isomorphism classes of braided autoequivalences of \(\mathbb{Z}(\mathcal{C})\). The name "Brauer-Picard group" speaks for itself that the group \(\mathbb{BP}(\mathcal{C})\) has a close relation with the Brauer group \(\text{Br}(\mathcal{C})\) of the category \(\mathcal{C}\) which classifies the Azumaya algebras in \(\mathcal{C}\), see [19]. In fact, every Azumaya algebra in \(\mathcal{C}\) defines an invertible \(\mathcal{C}\)-bimodule category, so that \(\text{Br}(\mathcal{C})\) forms a subgroup of \(\mathbb{BP}(\mathcal{C})\). The characterization of the Brauer group \(\text{Br}(\mathcal{C})\) in the group \(\text{Aut}^{br}(\mathbb{Z}(\mathcal{C}))\) has been done by A. Davydov and D. Nikshych in [3], where the braided autoequivalences corresponding to the Azumaya algebras are those trivializable on the base category \(\mathcal{C}\), that is, \(\text{Br}(\mathcal{C}) \cong \text{Aut}^{br}(\mathbb{Z}(\mathcal{C}), \mathcal{C})\).

Now we look at braided fusion categories from the angle of weak Hopf algebras. Let \(k\) be an algebraically closed field. It is well known that a braided fusion category \(\mathcal{C}\) is equivalent to the category \(\mathcal{H}\) of finite dimensional modules over some finite dimensional quasitriangular semisimple weak
Hopf algebra \((H, R)\) over \(k\), see \([5, 14, 15]\). When the weak Hopf algebra \(H\) happens to be a Hopf algebra, we know that the Brauer group of \(\mathcal{C}\) is the Brauer group \(\text{BM}(H, R)\) of \((H, R)\) consisting of Azumaya \(H\)-module algebras, see \([19]\). In this case, the Brauer group \(\text{BM}(H, R)\) can be characterized by the quantum commutative Galois objects over the braided Hopf algebra \(RH\), the transmutation of the quasitriangular Hopf algebra \((H, R)\), see \([20]\). In fact, we have the following general exact sequence of groups:

\[
1 \longrightarrow \text{Br}(k) \longrightarrow \text{BM}(H, R) \longrightarrow \text{Gal}^q_c(RH),
\]

where \(\text{Gal}^q_c(RH)\) is the group of quantum commutative bi-Galois objects over \(RH\), and \(k\) does not need to be algebraically closed. Now the question is whether the group \(\text{Gal}^q_c(RH)\) is isomorphic to \(\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}), \mathcal{C})\), where \(\mathcal{C} = H.\mathcal{M}\). The answer is positive, see \([5]\). The proof is based on the fact that an autoequivalence of the comodule category over a Hopf algebra \(H\) is defined by a bi-Galois object over \(H\). We don’t know whether this fact still holds for a weak Hopf algebra. However, one direction is always true, that is, a bi-Galois object over a weak Hopf algebra \(H\) defines an autoequivalence of the comodule category over \(H\). In case \(H\) is semisimple over an algebraically closed field, i.e. the braided category \(H.\mathcal{M}_{fd}\) is a fusion category, we can show that both groups \(\text{Gal}^q_c(RH)\) and \(\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}), \mathcal{C})\) are isomorphic to the Brauer group \(\text{BM}(H, R)\), see \([21]\). To obtain the isomorphisms, we first construct a braided Hopf algebra \(RH\) from a quasitriangular weak Hopf algebra \((H, R)\). Unlike the Hopf algebra case, the original algebra \(H\) can not be deformed into a Hopf algebra in the category of \(H\)-modules using Majid’s transmutation theory. Here our braided Hopf algebra \(RH\) is nested on some centralizer subalgebra of \(H\), see \([10]\).

The next step is to use the braided Hopf algebra \(RH\) to describe the Drinfeld center of the category of left \(H\)-modules using the category of left \(RH\)-comodules. Our result is the following (see Theorem 2.5).

**Theorem 1** Let \((H, R)\) be a quasitriangular weak Hopf algebra over a field \(k\). Then the category of Yetter-Drinfeld modules over \(H\) is equivalent to the category of left comodules over the braided Hopf algebra \(RH\) as a braided monoidal category.

Following \([16]\) Thm 5.2 we know that a braided bi-Galois object \(A\) over a braided Hopf algebra \(H\) in a braided monoidal category \(\mathcal{C}\) defines an autoequivalence of the category \(\mathcal{C}^H\) of comodules over \(H\). Now we can apply this result to the braided Hopf algebra \(RH\) in the braided monoidal category \(H.\mathcal{M}\) of a weak quasitriangular Hopf algebra \((H, R)\). Following Theorem 1, we know that the category of left comodules over \(RH\) is braided. Thus a natural question arises: when is the autoequivalence defined by a braided bi-Galois object \(A\) over \(RH\) a braided autoequivalence? Our answer is as follows (see Theorem 3.6):

**Theorem 2** Let \((H, R)\) be a quasi-triangular weak Hopf algebra over a field \(k\). Assume that \(A\) is a braided bi-Galois object. Then the functor \(A \boxtimes -\) defines a braided autoequivalence of the category of Yetter-Drinfeld modules if and only if \(A\) is quantum commutative.

As a consequence, we obtain the following result:

**Theorem 3** Let \(\mathcal{C}\) be a braided fusion category. Then the Drinfeld center of \(\mathcal{C}\) is equivalent to the category of finite dimensional left comodules over some braided Hopf algebra \(RH_{\mathcal{C}}\). If \(A\) is a braided bi-Galois object over \(RH_{\mathcal{C}}\), then the functor \(A \boxtimes -\) defines a braided autoequivalence of the Drinfeld center of \(\mathcal{C}\) trivializable on \(\mathcal{C}\) if and only if \(A\) is quantum commutative.
The paper is organized as follows. In Section 1, we recall some necessary definitions such as a weak Hopf algebra, a Yetter-Drinfeld module and the Drinfeld center of a monoidal category. In Section 2, we show that the category of Yetter-Drinfeld modules over a quasitriangular weak Hopf algebra \((H, R)\) is equivalent to the category of left comodules over the braided Hopf algebra \(R H\). In Section 3, we show that a braided bi-Galois object \(A\) over \(R H\) defines a braided autoequivalence of the category of Yetter-Drinfeld modules if and only if \(A\) is quantum commutative. Such a braided autoequivalence is trivializable on the base category \(H\). In case \((H, R)\) is semisimple and \(k\) is algebraically closed, then every braided auto-equivalence of \(R H\) trivializable on \(H\) is given by a quantum commutative Galois object over \(R H\). The proof will be given in the forthcoming paper [21] as it is a consequence of the exact sequence of the Brauer group. In the last section, we compute the braided Hopf algebras \(R H\) of the face algebras defined by Hayashi in [8] and the quantum commutative Galois objects over \(R H\).

1. Preliminaries

Throughout this paper \(k\) is a fixed field. Unless otherwise stated, unadorned tensor products will be over \(k\). For a coalgebra over \(k\), the coproduct will be denoted by \(\Delta\). We adopt Sweedler’s notation for the comultiplication in [18], e.g., \(\Delta(a) = a_1 \otimes a_2\).

We assume that the reader is familiar with the notions of a (braided) monoidal category, a ribbon or a modular category (see [9]) as well as a braided fusion category in [6]. Moreover, we make free use of the notions of algebras, bialgebras and Hopf algebras in a braided monoidal category, see [12].

1.1. Weak Hopf algebras. We first recall the notion of a weak Hopf algebra. For more detail on weak Hopf algebras, the reader is referred to [1]. A weak Hopf algebra \(H\) is a \(k\)-algebra \((H, m, \mu)\) and a \(k\)-coalgebra \((H, \Delta, \varepsilon)\) such that the following axioms hold:

\[
\begin{align*}
(i) & \quad \Delta(hk) = \Delta(h)\Delta(k), \\
(ii) & \quad \Delta^2(1) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2, \\
(iii) & \quad \varepsilon(hkl) = \varepsilon(hk_1)\varepsilon(k_2) = \varepsilon(hk_2)\varepsilon(k_1l), \\
(iv) & \quad \text{There exists a } k\text{-linear map } S : H \rightarrow H, \text{ called the antipode, satisfying}
\end{align*}
\]

\[
\begin{align*}
h_1S(h_2) &= \varepsilon(1_1h)1_2, & S(h_1)h_2 &= 1_1\varepsilon(h_12), & S(h) &= S(h_1)h_2S(h_3),
\end{align*}
\]

for all \(h, k, l \in H\). We have two idempotent linear maps \(\varepsilon_t, \varepsilon_s : H \rightarrow H\) defined respectively by

\[
\varepsilon_t(h) = \varepsilon(1_1h)1_2, \quad \varepsilon_s(h) = 1_1\varepsilon(h1_2),
\]

called the target map and the source map respectively. Their images \(H_t\) and \(H_s\) are called the target space and the source space respectively. In fact, \(H_t\) and \(H_s\) are Frobenius-separable subalgebras of \(H\). Moreover, the following equations hold:

\[
\begin{align*}
(1) & \quad h_1 \otimes h_2S(h_3) = 1_1h \otimes 1_2, \\
(2) & \quad S(h_1)h_2 \otimes h_3 = 1_1 \otimes h1_2, \\
(3) & \quad h_1 \otimes S(h_2)h_3 = h1_1 \otimes S(1_2),
\end{align*}
\]
Definition 1.2. Let $H$ be a weak Hopf algebra with a bijective antipode $S$. A quasi-triangular weak Hopf algebra is a pair $(H, R)$, where

$$R = R^1 \otimes R^2 \in \Delta^{\text{cop}}(1)(H \otimes_k H)\Delta(1),$$

satisfies the following conditions:

\begin{align*}
(1) & \quad (id \otimes \Delta)R = R_{13}R_{12}, \\
(2) & \quad (\Delta \otimes id)R = R_{13}R_{23}, \\
(3) & \quad \Delta^{\text{cop}}(h)R = R\Delta(h),
\end{align*}

where $h \in H, R_{12} = R \otimes 1, R_{23} = 1 \otimes R$, etc. Moreover, there exists an element $\overline{R} \in \Delta(1)(H \otimes_k H)\Delta^{\text{cop}}(1)$ such that $\overline{R}R = \Delta^{\text{cop}}(1)$ and $R\overline{R} = \Delta(1)$. Such an element $R$ is often called an $R$-matrix. In particular, $(H, R)$ is called a triangular weak Hopf algebra if $\overline{R} = R^2 \otimes R^1$.

For any $y \in H_s$ and $z \in H_t$, the following equations hold:

\begin{align*}
(11) & \quad (1 \otimes z)R = R(z \otimes 1), \\
(12) & \quad (z \otimes 1)R = (1 \otimes S(z))R, \\
(13) & \quad R(y \otimes 1) = R(1 \otimes S(y)), \\
(14) & \quad (\varepsilon \otimes id)(R) = \Delta(1), \\
(15) & \quad (id \otimes \varepsilon)(R) = (S \otimes id)\Delta^{\text{cop}}(1), \\
(16) & \quad (id \otimes \varepsilon)(R) = (S \otimes id)\Delta(1).
\end{align*}

1.2. Modules over weak Hopf algebras. Let $H$ be a weak Hopf algebra. Denote by $\mathcal{M}_H$ the category of left $H$-modules. Then $\mathcal{M}_H$ forms a monoidal category $(\mathcal{M}_H, \otimes, H_1, a, l, r)$ as follows:

(i) for any two objects $M$ and $N$ in $\mathcal{M}_H$,

$$M \otimes_1 N = \{ \sum m_i \otimes n_i \in M \otimes N | \sum \Delta(1)(m_i \otimes n_i) = \sum m_i \otimes n_i \}.$$
(ii) for any two objects $M$ and $N$ in $\mathbb{H}^\mathbb{M}$, the $H$-module structure on $M \otimes_\mathbb{I} N$ is as follows: $h \cdot (m \otimes_\mathbb{I} n) = h_1 \cdot m \otimes_\mathbb{I} h_2 \cdot n$ for all $h \in H$ and $m \in M$ and $n \in N$;

(iii) $H_t$ is the unit object with $H$-action $h \cdot z = \varepsilon_t(hz)$, where $h \in H$, $z \in H_t$, and the $k$-linear maps $l_M$, $r_M$ and their inverses are given by

\[ l_M(1_1 \cdot z \otimes 1_2 \cdot m) = z \cdot m, \quad l_M^{-1}(m) = 1_1 \cdot 1_H \otimes 1_2 \cdot m \]

\[ r_M(1_1 \cdot m \otimes 1_2 \cdot z) = S(z) \cdot m, \quad r_M^{-1}(m) = 1_1 \cdot m \otimes 1_2, \]

for any $z \in H_t$ and $m \in M$, where $M$ is an object in $\mathbb{H}^\mathbb{M}$.

If $(H, R)$ is a quasi-triangular weak Hopf algebra, then the category $\mathbb{H}^\mathbb{M}$ can be equipped with a braiding $C$ as follows [14 Prop. 5.2]:

\[ C_{M,N}(m \otimes_\mathbb{I} n) = R^2 \cdot n \otimes_\mathbb{I} R^1 \cdot m, \quad \text{for all } m \in M \text{ and } n \in N, \]

where $M$ and $N$ are any two objects in $\mathbb{H}^\mathbb{M}$.

1.3. Yetter-Drinfeld modules and the Drinfeld center.

**Definition 1.3.** Let $H$ be a weak Hopf algebra. A left $H$-module $M$ is called a left *Yetter-Drinfeld module* if $(M, \rho^L)$ is a left $H$-comodule such that the following two conditions:

(i) $\rho^L(m) = m_{[-1]} \otimes m_{[0]} \in H \otimes_\mathbb{I} V$,

(ii) $(h \cdot m)_{[-1]} \otimes (h \cdot m)_{[0]} = h_1 m_{[-1]} S(h_3) \otimes h_2 \cdot m_{[0]}$, are satisfied for all $h \in H$ and $m \in M$. For a Yetter-Drinfeld module $M$, we have the identity:

\[ m_{[-1]} \otimes m_{[0]} = m_{[-1]} S(1_2) \otimes 1_1 \cdot m_{[0]}, \quad \text{for } m \in M. \]

Denote by $\mathbb{YD}_H^H$ the category of left Yetter-Drinfeld modules. A Yetter-Drinfeld morphism is both left $H$-linear and left $H$-colinear. If the antipode $S$ is bijective, then $\mathbb{YD}_H^H$ is a braided monoidal category with the braiding given by

\[ C_{V,W}(v \otimes w) = v_{[-1]} \cdot w \otimes v_{[0]}, \]

where $v \in V \in \mathbb{YD}_H^H$ and $w \in W \in \mathbb{YD}_H^H$. In particular, if $(H, R)$ is a quasi-triangular weak Hopf algebra, then every left $H$-module $M$ is automatically a left Yetter-Drinfeld module with the following left coaction:

\[ \rho^L(m) = R^2 \otimes R^1 \cdot m, \quad \forall m \in M. \]

It is easy to see that the category $\mathbb{H}^\mathbb{M}$ is a braided monoidal subcategory of $\mathbb{YD}_H^H$.

**Definition 1.4.** Let $H$ be a weak Hopf algebra with a bijective antipode $S$. An algebra $A$ in $\mathbb{YD}_H^H$ is called *quantum commutative* if the following equation:

\[ xy = (x_{[-1]} \cdot y)x_{[0]} \]

holds for all $x, y \in A$.

**Definition 1.5.** Let $H$ be a weak Hopf algebra with a bijective antipode. The left *Drinfeld center* $\mathcal{Z}(\mathbb{H}^\mathbb{M})$ of the monoidal category $\mathbb{H}^\mathbb{M}$ is the category, whose objects are pairs $(U, \nu_{U, -})$, where $U$ is an object of $\mathbb{H}^\mathbb{M}$ and $\nu_{U, -}$ is a natural family of isomorphisms, called *half-braidings*:

\[ \nu_{U,V} : U \otimes V \rightarrow V \otimes U, \quad \forall V \in \mathbb{H}^\mathbb{M} \]

satisfying the Hexagon Axiom. Similarly, one can define the right Drinfeld center of $\mathbb{H}^\mathbb{M}$. 

Lemma 1.6. [2] Thm 2.6] Let $H$ be a weak Hopf algebra with bijective antipode. Then $\mathcal{Z}(H, M)$ is equivalent to $\mathcal{H}(H, M)$ as a braided monoidal category.

2. The Drinfeld center of a quasi-triangular weak Hopf algebra

Let $H$ be a quasi-triangular weak Hopf algebra. In this section, we show that there is a braided monoidal equivalence between the Drinfeld center of the category of left $H$-modules and the category of left comodules over some braided Hopf algebra.

Denote by $C_H(H_s)$ the centralizer subalgebra of $H_s$ in $H$. Clearly, $C_H(H_s) = \{1_hS(1)| \forall h \in H\}$. The algebra $C_H(H_s)$ is a left $H$-module algebra with the adjoint action: $h \cdot x = h_1 x S(h_2)$ for all $h \in H$ and $x \in C_H(H_s)$.

Now we need Majid’s transmutation theory in the case of a quasi-triangular weak Hopf algebra. Recall Theorem 3.11 from [10].

Lemma 2.1. Let $(H, R)$ be a quasi-triangular weak Hopf algebra. Then $C_H(H_s)$ is a Hopf algebra in the braided monoidal category $\mathcal{H}(H, M)$ with the following structures:

(i) the multiplication $\overline{\mu}$ and the unit $\overline{\eta}$ are defined by:
$$\overline{\mu} : C_H(H_s) \otimes C_H(H_s) \rightarrow C_H(H_s), \quad a \otimes b \mapsto (1_1 \cdot a)(1_2 \cdot b),$$
$$\overline{\eta} = \text{Id}_H : H \rightarrow C_H(H_s), \quad x \mapsto x.$$

(ii) The comultiplication $\overline{\Delta}$ and the counit $\overline{\varepsilon}$ are given by:
$$\overline{\Delta} : C_H(H_s) \rightarrow C_H(H_s) \otimes C_H(H_s), \quad x \mapsto x_1 S(R^2) \otimes R^1 \cdot x_2,$$
$$\overline{\varepsilon} = \varepsilon_t : C_H(H_s) \rightarrow H, \quad x \mapsto \varepsilon_t(x).$$

(iii) The antipode is $\overline{S}$ defined by
$$\overline{S} : C_H(H_s) \rightarrow C_H(H_s), \quad x \mapsto R^2 R^2 S(R^1 x S(R^1)) .$$

Moreover, $R H$ is cocommutative cocentral in the sense of [17].

A Hopf algebra in a braided monoidal category is usually called a braided Hopf algebra in case the category does not need to be mentioned. In the sequel, we shall call the Hopf algebra $C_H(H_s)$ in $\mathcal{H}$ a braided Hopf algebra and denote it by $R H$.

Definition 2.2. [12] Let $H$ be a quasitriangular weak Hopf algebra. Let $M$ be a left $H$-module. We call $(M, \rho^l)$ a left $R H$-comodule in the category $\mathcal{H}$ if $(M, \rho^l)$ is a left $R H$-comodule such that $\rho^l$ is left $H$-linear, i.e.,
$$\rho^l(h \cdot m) = h_1 \cdot m(-1) \otimes h_2 \cdot m(0), \forall h \in H, \quad m \in M.$$
Let \((M, \rho')\) and \((N, \rho')\) be two left \(_R H\)-comodules. The tensor product \(M \otimes_N N\) is a left \(_R H\)-comodule with the following comodule structure:

\[
    h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \quad \rho'(m \otimes n) = (\overline{h} \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(\rho \otimes \rho')(m \otimes n),
\]

where \(m \in M\), \(n \in N\), \(h \in H\) and \(C\) is the braiding in \(_H \mathcal{M}\).

Denote by \(\mathcal{H}^H(M, \mathcal{M})\) the category of left \(_R H\)-comodules. Note that a morphism in \(\mathcal{H}^H(M, \mathcal{M})\) is both left \(H\)-linear and left \(_R H\)-colinear. It is easy to see that the category \(\mathcal{H}^H(M, \mathcal{M})\) is a monoidal category with the unit object given by \(H\).

Now we discuss the relation between the category \(\mathcal{H}^H(M, \mathcal{M})\) and the category of left Yetter-Drinfeld \(H\)-modules.

**Lemma 2.3.** Let \(H\) be a quasitriangular weak Hopf algebra. If \((M, \rho')\) is a left \(_R H\)-comodule, then \(M\) is a left Yetter-Drinfeld \(H\)-module with the following \(H\)-comodule structure:

\[
    \rho^L(m) = m_{(-1)} R^2 \otimes R^1 \cdot m_{(0)} \in H \otimes M,
\]

where \(\rho'(m) = m_{(-1)} \otimes m_{(0)}\) for all \(m \in M\).

**Proof.** For any \(m \in M\), we first have

\[
    1_m m_{(-1)} R^2 \otimes 1^2 R^1 \cdot m_{(0)} = m_{(-1)} 1 R^2 \otimes 1^2 R^1 \cdot m_{(0)} = m_{(-1)} R^2 \otimes R^1 \cdot m_{(0)}.
\]

So \(\rho^L(M) \in H \otimes M\). Namely, \(\rho^L\) is well-defined.

Next we verify that \((M, \rho^L)\) is a left \(H\)-comodule. For the coassociativity, we have:

\[
(1 \otimes \rho^L)(\rho^L) = (1 \otimes \rho^L)(m_{(-1)} R^2 \otimes R^1 \cdot m_{(0)}).
\]

The counit axiom holds as well because we have:

\[
(\varepsilon \otimes 1) \rho^L(m) = \varepsilon(m_{(-1)} R^2)(R^1 \cdot m_{(0)}) \stackrel{(5)}{=} \varepsilon(m_{(-1)} \varepsilon(R^2))(R^1 \cdot m_{(0)})
\]

\[
(\varepsilon \cdot 1 \otimes 1) \rho^L(m) = \varepsilon(m_{(-1)} 1_R)(S(1 \otimes 1) \cdot m_{(0)}) = \varepsilon(m_{(-1)} 1_R)(S(1 \otimes 1) \cdot m_{(0)}) = \varepsilon(m_{(-1)} 1_R)(1_R \cdot m_{(0)}) = \varepsilon(1_R m_{(-1)})(1_R \cdot m_{(0)}) = m.
\]
where the last equality follows from the counit of a left \( R H \)-comodule, namely,

\[
l \circ (\varepsilon_1 \otimes 1)(m_{(-1)} \otimes m_{(0)}) = \varepsilon_1(m_{(-1)}) \cdot m_{(0)} = m.
\]

Finally, the compatible condition holds since

\[
h_1(m_{(-1)}R^2) \otimes h_2 \cdot [R^1 \cdot m_{(0)}] = h_11_{[-1]}S(1_2)R^2 \otimes h_2R^1 \cdot m_{(0)}
\]

\[
\overset{(3)}{=} h_11_{[-1]}S(h_2)h_3R^2 \otimes h_4R^1 \cdot m_{(0)}
\]

\[
\overset{(10)}{=} h_11_{[-1]}S(h_2)R^2h_4 \otimes R^1h_3 \cdot m_{(0)}
\]

\[
= (h_1 \cdot m)_{[-1]}R^2h_2 \otimes R^1 \cdot (h_1 \cdot m_{(0)}).
\]

for all \( m \in M \) and \( h \in H \).

The following lemma says that the converse of Lemma 2.3 is also true.

**Lemma 2.4.** Let \( H \) be a quasitriangular weak Hopf algebra with an antipode \( S \). If \( (N, \rho^L) \) is a left Yetter-Drinfeld module, then \( N \) is a left \( R H \)-comodule with the following structure:

\[ \rho^l(n) = n_{[-1]}S(R^2) \otimes R^1 \cdot n_{[0]}, \]

where \( \rho^l(n) = n_{[-1]} \otimes n_{[0]} \) for all \( n \in N \).

**Proof.** First of all, we need to check that \( \rho^l \) is well-defined. For any \( n \in N \),

\[
1_{[-1]}[n_{[-1]}S(R^2)S(1_2) \otimes R^1 \cdot n_{[0]}] = 1_{[-1]}n_{[-1]}S(1_2R^2) \otimes R^1 \cdot n_{[0]}
\]

\[
\overset{(11)}{=} 1_{[-1]}n_{[-1]}S(R^2) \otimes R^11_2 \cdot n_{[0]}
\]

\[
= 1_{[-1]}n_{[-1]}S(R^2) \otimes R^1 \cdot (1_2 \cdot n_{[0]})
\]

\[
= n_{[-1]}S(R^2) \otimes R^1 \cdot n_{[0]};
\]

\[
1 \cdot [n_{[-1]}S(R^2)] \otimes 1_2R^1 \cdot n_{[0]} = [n_{[-1]}S(R^2)]S(1_1) \otimes 1_2R^1 \cdot n_{[0]}
\]

\[
= [n_{[-1]}S(1_1R^2)] \otimes 1_2R^1 \cdot n_{[0]}
\]

\[
= [n_{[-1]}S(R^2)] \otimes R^1 \cdot n_{[0]}.
\]

So \( \rho^l(N) \subset R H \otimes_1 N \). The \( H \)-linearity of the map \( \rho^l \) follows from the equations below:

\[
h_1 \cdot [n_{[-1]}S(R^2)] \otimes h_2R^1 \cdot n_{[0]} = h_1n_{[-1]}S(R^2)S(h_2) \otimes h_3R^1 \cdot n_{[0]}
\]

\[
= h_1n_{[-1]}S(h_2R^2) \otimes h_3R^1 \cdot n_{[0]}
\]

\[
\overset{(10)}{=} h_1n_{[-1]}S(R^2h_3) \otimes R^1h_2 \cdot n_{[0]}
\]

\[
= (h_1n_{[-1]}S(h_3))S(R^2) \otimes R^1 \cdot (h_2 \cdot n_{[0]})
\]

\[
= (h \cdot n)_{[-1]}S(R^2) \otimes R^1 \cdot (h \cdot n_{[0]} = \rho^l(h \cdot n),
\]
for all $h \in H$. Now we show that $(N, \rho^l)$ is a left $R^H$-comodule. For any $n \in N,$

\[
(1 \otimes \rho^l)(n) = n_{(-1)}S(R^2) \otimes (R^1 \cdot n_{(0)}) = n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[11]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[12]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[13]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[14]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[15]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[16]
\]

\[
= n_{(-1)}S(R^2) \otimes R^1 \cdot n_{(0)}[17]
\]

\[
= \sum_{n_{(-1)}S(r^2)} \otimes R^1 \cdot n_{(0)} = (\Delta \otimes 1)\rho^l(n).
\]

Hence the coassociativity holds. Finally, we verify that $\varepsilon_i$ satisfies the counit axiom:

\[
\varepsilon_i(n_{(-1)}S(R^2)) \cdot (R^1 \cdot n_{(0)}) = (\varepsilon_i(n_{(-1)}S(R^2))R^1) \cdot n_{(0)}[18]
\]

\[
= (\varepsilon_i(n_{(-1)}S(R^2))R^1) \cdot n_{(0)}[19]
\]

\[
= (12R^1 \cdot n_{(0)} \varepsilon(1_1) \varepsilon(1_{(-1)}S(R^2))) = (12S(1_1^2)) \cdot n_{(0)} \varepsilon(1_{(-1)}S(1_1^2))
\]

\[
= 12 \cdot n_{(0)} \varepsilon(1_{(-1)}S(1_1)) = n.
\]

Therefore, $(N, \rho^l)$ is a left $R^H$-comodule. 

Combining Lemma 2.3 and Lemma 2.4, we obtain the following theorem.

**Theorem 2.5.** Let $(H, R)$ be a quasitriangular weak Hopf algebra. Then there is a monoidal equivalence $F$ from the category $H^R_H(M)$ of left $R^H$-comodules to the category $H^R_H(D)$ of left Yetter-Drinfeld modules:

\[
F: H^R_H(M) \rightarrow H^R_H(D), \quad (M, \rho^l) \mapsto (M, \rho^r),
\]

where $\rho^r$ is defined in Lemma 2.3. The quasi-inverse of $F$ is

\[
G: H^R_H(D) \rightarrow H^R_H(M), \quad (N, \rho^r) \mapsto (N, \rho^l),
\]

where $\rho^l$ is defined in Lemma 2.4.

**Proof.** We show first that $GF(M) = M$ for any object $M$ in $H^R_H(M)$. It is enough to verify that $\rho^l(m) = m_{(-1)} \otimes m_{(0)}$ for all $m \in M$. Indeed,

\[
\rho^l(m) = m_{(-1)}S(R^2) \otimes R^1 \cdot m_{(0)}[20]
\]

\[
= m_{(-1)}rS(R^2) \otimes R^1 \cdot [r \cdot m_{(0)}] = m_{(-1)}rS(R^2) \otimes (R^1r) \cdot m_{(0)}[21]
\]

\[
= m_{(-1)}S(R^2) \otimes R^1 \cdot m_{(0)}[22]
\]

\[
= m_{(-1)}S(R^2) \otimes 1 \cdot m_{(0)}[23]
\]

\[
= S^{-1}(1_2) \cdot m_{(-1)} \otimes S(1_1) \cdot m_{(0)}[24]
\]

\[
= 1_2 \cdot m_{(-1)} \otimes S(1_1) \cdot m_{(0)}[25]
\]

\[
= m_{(-1)} \otimes m_{(0)}[26]
\]

Thus, $GF(M) = M$. The proof is complete. 

\[\square\]
Next we show that $\mathcal{F} \mathcal{G}(N) = N$ for any object of $\mathcal{H}' \mathcal{D}$. For all $n \in N$,

\[
\rho^L(n) = n_{(-1)} R^2 \otimes R^1 \cdot n_{(0)} = n_{(-1)} S(r^2) R^2 \otimes R^1 \cdot (r^1 \cdot n_{(0)}) \\
= n_{(-1)} S(r^2) R^2 \otimes (R^1 r^1) \cdot n_{(0)} \overset{(\delta)}{=} n_{(-1)} \varepsilon_s R^2 \otimes R^1 \cdot n_{(0)} \\
\overset{(14)}{=} n_{(-1)} 1_{1} \otimes S(1_{2}) \cdot n_{(0)} = n_{(-1)} S(1_{2}) \otimes 1_{1} \cdot n_{(0)} \\
= 1_{1} n_{(-1)} S(1_{2}) \otimes 1_{1} \cdot (1_{2} \cdot n_{(0)}) = 1_{1} n_{(-1)} S(1_{3}) \otimes 1_{2} \cdot n_{(0)} \\
= n_{(-1)} \otimes n_{(0)}.
\]

Finally, we verify that the triple $(\mathcal{G}, \text{Id}, \text{Id})$ is monoidal. It is clear that $\mathcal{G}(H_t) = H_t$. For any two left Yetter-Drinfeld modules $U$ and $V$, the left $RH$-comodule structure on $\mathcal{G}(U) \otimes \mathcal{G}(V)$ is as follows:

\[
(\mu \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(\rho^L \otimes \rho^R)(u \otimes v) = (\mu \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(u_{(-1)} \otimes u_{(0)} \otimes n_{(-1)} \otimes v_{(0)}) = (\mu \otimes 1 \otimes 1)(u_{(-1)} \otimes R^2 \cdot v_{(-1)} \otimes R^1 \cdot u_{(0)} \otimes v_{(0)}) = u_{(-1)}(R^2 \cdot v_{(-1)}) \otimes R^1 \cdot u_{(0)} \otimes v_{(0)},
\]

where $u \in U$ and $v \in V$. Now we have

\[
u_{(-1)}(R^2 \cdot v_{(-1)}) \otimes R^1 \cdot u_{(0)} \otimes v_{(0)} = (u_{(-1)} S(p^2)) R^2(\nu_{(-1)} S(q^2)) S(R^2) \otimes R^1 \cdot (p^1 \cdot u_{(0)} \otimes q^1 \cdot v_{(0)}) \\
\overset{(8)}{=} (u_{(-1)} S(p^2)) r^2(\nu_{(-1)} S(q^2)) S(R^2) \otimes (R^1 r^1 p^1) \cdot u_{(0)} \otimes q^1 \cdot v_{(0)} \\
\overset{(8)}{=} u_{(-1)} \varepsilon_s r^2(\nu_{(-1)} S(q^2)) S(R^2) \otimes (R^1 r^1) \cdot u_{(0)} \otimes q^1 \cdot v_{(0)} \\
\overset{(14)}{=} u_{(-1)} S(1_{2})(\nu_{(-1)} S(q^2)) S(R^2) \otimes (R^1 1_{1}) \cdot u_{(0)} \otimes q^1 \cdot v_{(0)} \\
= u_{(-1)} S(1_{2})(\nu_{(-1)} S(q^2)) S(R^2) \otimes R^1 \cdot (1_{1} \cdot u_{(0)} \otimes q^1 \cdot v_{(0)}) \\
= u_{(-1)} (\nu_{(-1)} S(q^2)) S(R^2) \otimes R^1 \cdot u_{(0)} \otimes q^1 \cdot v_{(0)} \\
= (u_{(-1)} \nu_{(-1)} S(q^2)) S(R^2) \otimes R^1 \cdot u_{(0)} \otimes q^1 \cdot v_{(0)} \\
\overset{(9)}{=} (u_{(-1)} \nu_{(-1)} S(q^2)) S(R^2) \otimes R^1 \cdot (u_{(0)} \otimes v_{(0)}) \\
= [u \otimes v]_{(-1)} S(R^2) \otimes R^1 \cdot (u \otimes v)_{(0)} = \rho^L(u \otimes v).
\]

Hence, $\mathcal{G}(U \otimes V) = \mathcal{G}(U) \otimes \mathcal{G}(V)$. The verification of the other axioms for a monoidal functor are obvious.

Since the category of Yetter-Drinfeld modules is braided, the equivalence $\mathcal{G}$ in Theorem 2.5 induces a braiding in the category of left $RH$-comodules such that the equivalence becomes braided.

**Corollary 2.6.** Let $(H, R)$ be a quasitriangular weak Hopf algebra. Then the category of left $RH$-comodules is a braided monoidal category with a braiding $\tilde{C}$ given by

\[
\tilde{C}(u \otimes v) = u_{(-1)} R^2 \cdot v \otimes R^1 \cdot u_{(0)}, \quad \forall u \in U, \forall v \in V,
\]

where $U$ and $V$ are any two left $RH$-comodules. The inverse of $\tilde{C}$ is given by

\[
\tilde{C}^{-1}(v \otimes u) = R^1 \cdot u_{(0)} \otimes S^{-1}(u_{(-1)} R^2) \cdot v.
\]

Moreover, the functor $\mathcal{G}$ in Theorem 2.5 gives a braided monoidal equivalence.
Proof. Consider the following commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
G(U) \otimes G(V) & \overset{C_{G(U),G(V)}}{\longrightarrow} & G(U \otimes V) \\
G(V) \otimes G(U) & \overset{\cong}{\longrightarrow} & G(V \otimes U),
\end{array}
\]

where the horizontal isomorphisms are given by 
\[Id : G(X) \otimes G(Y) \cong G(Y \otimes X)\]. Thus, the braiding \(\tilde{C}\) is just the composition \(Id^{-1} \circ C_{U,V} \circ Id\). In fact, we have

\[
\tilde{C}_{U,V}(u \otimes v) = Id \circ C_{U,V} \circ Id(u \otimes v) = Id \circ C_{U,V}(u \otimes v) = Id(u_{[-1]} \cdot v \otimes u_{[0]}) = u_{(-1)} R^2 \cdot v \otimes R^1 \cdot u_{(0)}.
\]

Similarly, one can obtain the inverse of \(\tilde{C}\).

By Lemma 1.6 and Corollary 2.6 we obtain the following corollary.

**Corollary 2.7.** Let \((H, R)\) be a quasitriangular weak Hopf algebra. Then the Drinfeld center \(Z_l(HM)\) of left \(H\)-modules is equivalent to the category \(^{RH}(HM)\) of left \(RH\)-comodules as a braided monoidal category.

As a special case, we have the following corollary on a quasitriangular Hopf algebra:

**Corollary 2.8.** Let \((H, R)\) be a quasitriangular Hopf algebra. Then the Drinfeld center of left \(H\)-modules is equivalent to the category of left \(RH\)-comodules as a braided monoidal category.

**Remark 2.9.** (i) when \(H\) is a finite dimensional quasitriangular Hopf algebra, the functor \(G\) was first proved in [20] to have a right adjoint.

(ii) Let \(H\) be a finite dimensional quasitriangular Hopf algebra. Following [11] Prop 4.1] the quantum double \(D(H)\) is isomorphic to a semidirect product \(A \rtimes H\), where \(A = H^*\) is a braided Hopf algebra. By Corollary 2.8 we may choose \(A\) as the dual braided Hopf algebra \((RH)^*\). Thus we have the following equivalences of braided monoidal categories:

\[
(\langle RH \rangle^* \rtimes H, M) \cong D(H, M) \cong ^{H \otimes H}(\mathcal{Y} \mathcal{D}) \cong Z_l(H, M).
\]

In case \(H\) is infinite dimensional, we have neither the usual quantum double \(D(H)\) nor the dual braided Hopf algebra \((RH)^*\). But Corollary 2.8 always holds for any (finite or infinite dimensional) quasitriangular Hopf algebra over any field (or even over a commutative ring). In particular, the Drinfeld center is naturally equivalent to the category of comodules over \(B\mathcal{U}_q(g)\) studied in [11].

3. Quantum commutative Galois objects

In this section we study (braided) Galois objects over the Braided Hopf algebra \(RH\) of a finite dimensional quasitriangular weak Hopf algebra \((H, R)\). We shall construct braided autoequivalences of the Drinfeld center of \(H, M\) from braided bi-Galois objects. For the details about braided Galois objects over a braided Hopf algebra one is referred to [16, 17].
Let \((H, R)\) be a finite dimensional quasitriangular weak Hopf algebra. An object \(X\) in \(H\mathcal{M}\) is flat if tensoring with \(X\) preserves equalizers. A flat object \(X\) is called faithfully flat if tensoring with \(X\) reflects isomorphisms. It is not hard to see that \(RH\) is flat in the category \(H\mathcal{M}\) since \(RH\) is finite and has a dual object.

**Definition 3.1.** \([16]\) An algebra \(A\) in \(H\mathcal{M}\) is called a left \(RH\)-comodule algebra if \(A\) is a left \(RH\)-comodule such that the left comodule map \(\rho^l\) satifies:

\[
\rho^l(ab) = a_{(-1)}(R^2 \cdot b_{(-1)}) \otimes (R^1 \cdot a_{(0)})b_{(0)},
\]

for all \(a, b \in A\), where \(\rho^l(a) = a_{(-1)} \otimes a_{(0)}\). Namely, \(\rho^l\) is an algebra map in \(H\mathcal{M}\).

Similarly, an algebra \(A\) in \(H\mathcal{M}\) is called a right \(RH\)-comodule algebra if \(A\) with a right \(RH\)-coaction \(\rho^r\) is a right \(RH\)-comodule such that

\[
\rho^r(ab) = a_{(0)}(R^2 \cdot b_{(0)}) \otimes (R^1 \cdot a_{(1)})b_{(1)},
\]

where \(a, b \in A\) and \(\rho^r(a) = a_{(0)} \otimes a_{(1)}\). An \(RH\)-bicomodule algebra is both a left and a right \(RH\)-comodule algebra such that the left and the right coactions commute.

Now let \(A\) be a right \(RH\)-comodule algebra. The subalgebra

\[
A_0 = \{a \in A | \rho^r(a) = a \otimes 1 = 1 \cdot a \otimes 1\}
\]

is called the coinvariant subalgebra. Similarly, one can define the coinvariant subalgebra of a left \(RH\)-comodule algebra. An \(RH\)-coinvariant subalgebra \(A_0\) is said to be trivial if \(A_0 = H_r\).

**Definition 3.2.** \([17]\) Defn 2.1] Let \(A\) be a right \(RH\)-comodule algebra. \(A\) is called a right braided \(RH\)-Galois object if \(A\) is faithfully flat and the morphism

\[
\beta : A \otimes_t A \longrightarrow A \otimes_t RH, \quad a \otimes b \longmapsto ab(0) \otimes b_{(1)}
\]

is an isomorphism. Similarly, one can define a left braided \(RH\)-Galois object and a braided bi-Galois object.

The coinvariant subalgebra \(A_0\) of a right \(RH\)-Galois object \(A\) is trivial. So is the coinvariant subalgebra of a left \(RH\)-Galois object \(A\). Moreover, it is not hard to see that \((RH, \tau_{RH}, \ldots)\) is an object in the Drinfeld center \(\mathcal{Z}(H, \mathcal{M})\), where \(\tau_{RH}\) is a half-braiding

\[
\tau_{RH,M} : RH \otimes M \longrightarrow M \otimes RH, \quad h \otimes m \longmapsto r^2 R^1 \cdot m \otimes r^1 hR^2.
\]

Since \(RH\) is cocommutative cocentral, for any left \(RH\)-comodule \((M, \rho^l)\), by \([17]\) there exists a natural right comodule structure induced by the half-braiding \(\tau_{RH,M} : RH \otimes M \longrightarrow M \otimes RH\),

\[
\rho^r = \tau_{RH,M} \circ \rho^l : M \longrightarrow RH \otimes M \longrightarrow M \otimes RH,
\]

so that \((M, \rho^l, \rho^r)\) becomes an \(RH\)-bicomodule. By \([17]\) we call \(M\) cocommutative if the right \(RH\)-comodule is induced by the left \(RH\)-comodule as above.

**Definition 3.3.** A cocommutative braided bi-Galois object \(A\) is called a quantum commutative Galois object if \(A\) is quantum commutative as an algebra in \(H^2 \mathcal{D}\).
By Theorem 2.5 and Corollary 2.6, a left Yetter-Drinfeld module is an $\mathcal{H}$-bicomodule in $\mathcal{H}$-$\mathcal{M}$. Thus we can consider the cotensor product $M \square_{nH} N$, or $M \square N$ for convenience, for two left Yetter-Drinfeld modules $M$ and $N$:

$$M \square N = \{m \otimes_t n \in M \otimes_t N | \rho^a(m) \otimes_t n = m \otimes_t \rho^b(n),$$

or precisely,

$$M \square N = \{m \otimes n \in M \otimes N | r^2 \cdot m_{[0]} \otimes r^1 m_{[-1]} \otimes n = m \otimes n_{[-1]} S(R^2) \otimes R^1 \cdot n_{[0]} \}.$$  \hspace{1cm} (18)

If $A$ is a braided $\mathcal{H}$-bi-Galois object, by [16] we have an isomorphism:

$$\xi : (A \square M) \otimes_t (A \square N) \cong A \square (M \otimes_t N),$$

given by $\xi((a \otimes m) \otimes (b \otimes n)) = a(R^2 \cdot b) \otimes R^1 \cdot m \otimes n$, for all $a, b \in A$, $m \in M$ and $b \in N$. Following [17] the cotensor functor $A \square -$ is a monoidal autoequivalence of $\mathcal{H}(\mathcal{H},\mathcal{M})$.

**Lemma 3.4.** Let $(H, R)$ be a finite dimensional quasitriangular weak Hopf algebra. If $A$ is a quantum commutative Galois object, then the functor $A \square -$ is a braided autoequivalence of $\mathcal{H}(\mathcal{H},\mathcal{M})$.

**Proof.** Let $A$ be a quantum commutative Galois object. By Theorem 2.5 and [19] it suffices to verify that the following diagram is commutative:

$$\begin{array}{ccc}
(A \square M) \otimes_t (A \square N) & \xrightarrow{\cdot \left(\tilde{C}_{ADM, ADN}\right)} & A \square (M \otimes_t N) \\
\downarrow & & \downarrow \left(\ast\right) \\
(A \square N) \otimes_t (A \square M) & \xrightarrow{\cdot \left(\tilde{C}_{M,N}\right)} & A \square (N \otimes_t M)
\end{array}$$

Indeed, on the one hand, for any $a \otimes m \in A \square M$ and $b \otimes n \in A \square N$, we have:

$$\begin{array}{l}
\xi[(a \otimes m)(-1)^2 \cdot (b \otimes n) \otimes r^1 \cdot (a \otimes m)(0)] \\
\xi[a(-1)^2 \cdot (b \otimes n) \otimes r^1 \cdot (a(0) \otimes m)] \\
\xi[a(-1)^2 \cdot b \otimes a(-1) a(0) \otimes r^1 \cdot a(0) \otimes r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes r^1 \cdot m] \\
[a(-1)^2 \cdot b[S(q^2)r^2_1 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes r^1 \cdot m] \\
[a(-1)^2 \cdot b[S(q^2)r^2_1 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m] \\
[a(-1)^2 \cdot b[R^2 r^1 q^1 \cdot a(0)] \otimes R^1 a(-1) a(0) \otimes n \otimes p \cdot r^1 \cdot m]
\end{array}$$

where Corollary 2.6 and Lemma 2.4 were used in the first and fifth equality, respectively. On the other hand, we have:

$$(1 \otimes \tilde{C}) \circ \xi[(a \otimes m) \otimes (b \otimes n)]$$
\[ a(r^2 \cdot b) \otimes \tilde{C}(r^1 \cdot m \otimes n) = a(r^2 \cdot b) \otimes (r^1 \cdot m)(-1)W^2 \cdot n \otimes W^1 \cdot (r^1 \cdot m)(0) = a(r^2 \cdot b) \otimes (r^1 \cdot m(-1))W^2 \cdot n \otimes W^1 r^2_1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(r^2_1)W^2 \cdot n \otimes W^1 r^3_1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(R^2)S(r^2_1)W^2 \cdot n \otimes W^1 r^3_1 R^1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(r^2_1 R^2)W^2 \cdot n \otimes W^1 r^3_1 R^1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(r^3_1)S(R^2)W^2 \cdot n \otimes W^1 r^3_1 R^1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(r^3_1) \cdot n \otimes r^2_1 \cdot m_0 = a(r^2 \cdot b) \otimes r^1_m m(-1)S(12)S(r^3_1) \cdot n \otimes r^2_1 R^1 \cdot m_0 \]

\(^{(14)}\)

A braided autoequivalence. We have the commutative diagram (14). Let \( H, R \) be a finite dimensional quasitriangular weak Hopf algebra. Assume that \( A \) is a braided bi-Galois object. If the functor \( \mathcal{A} \) defines a braided autoequivalence of \( {}^H H \), then \( A \) is quantum commutative.

Proof. Assume that the functor \( \mathcal{A} \) defines a braided autoequivalence. We have the commutative diagram (A). Let \( M \) and \( N \) be two left \( RH \)-comodules. Following the proof of Lemma 3.3, we obtain
the following equation:
\[ a_{(0)}(r^2 \cdot b) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \cdot n \otimes r_2^1 p^1 \cdot m \]
(19)  
\[ = [a_{(-1)} r_1^1 b][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^2 \cdot n \otimes r_2^1 \cdot m, \]
for all \( a \otimes m \in A \square M \) and \( b \otimes n \in A \square N \). Now let \( M = RH \). Since \( a_{(0)} \otimes a_{(1)}, b_{(0)} \otimes b_{(1)} \in A \square RH \), we may substitute them for the elements \( a \otimes m \) and \( b \otimes n \) in the above equation and obtain the following equation:
\[ a_{(0)}(r^2 \cdot b_{(0)}) \otimes (r^1 \cdot a_{(1)})[-1] \cdot b_{(1)} \otimes (r^1 \cdot a_{(1)})[0] \]
\[ = [a_{(-1)} r_1^1 b_{(0)}][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^2 \cdot b_{(1)} \otimes r_2^1 \cdot a_{(1)}. \]
Now we apply the map \( 1 \otimes \varepsilon_t \otimes \varepsilon_t \) to the foregoing equality and obtain the following:
\[ [a_{(-1)} r_1^1 b_{(0)}][R^2 r_1^1 \cdot a_{(0)}] \otimes \varepsilon_t(R^1 a_{(-1)} r_2^2 \cdot b_{(1)}) \varepsilon_t(r_2^1 \cdot a_{(1)}) \]
\[ = a_{(0)}(r^2 \cdot b_{(0)}) \otimes \varepsilon_t[(r^1 \cdot a_{(1)})[-1] \cdot b_{(1)}] \varepsilon_t[(r^1 \cdot a_{(1)})[0]]. \]
Since \( \varepsilon_t \) is an algebra map in the category \( H \square \mathcal{M} \) and \( A \) is a right \( RH \)-comodule algebra, we have
\[ [a_{(-1)} r^2 \cdot b][r^1 \cdot a_{(0)}] = ab, \]
which is equivalent to
\[ ab = (a_{[-1]} \cdot b)a_{[0]}. \]
Thus \( A \) is quantum commutative.

Now we show that \( A \) is cocommutative. Namely, we need to verify that the right coaction \( \rho^L \) on \( A \) is induced by its left coaction \( \rho^L \) and the half-braiding. Note that the regular left \( H \)-module \( H \) has an induced Yetter-Drinfeld module structure, where the comodule structure is given by
\[ \rho^L(h) = R^2 \otimes R^1 h := h_{[-1]} \otimes h_{[0]} \]
By Lemma 2.4 we have a left \( RH \)-comodule structure on \( H \), where \( \rho^L(h) = 1 \otimes_k h \) for any \( h \in H \). Namely, \((H, \rho^L)\) is a trivial left \( RH \)-comodule. Now consider \( A \square RH \) and \( A \square H \). Note that \( 1_A \otimes_1 H \in A \square H \) and \( a_{(0)} \otimes a_{(1)} \in A \square RH \). Using Equation (19) we easily get:
\[ a_{(0)}(r^2 \cdot 1_A) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \]
\[ = [a_{(-1)} r_1^1 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^2 \otimes r_2^1 \cdot a_{(1)}. \]
Now on the one hand, we have:
\[ a_{(0)}(r^2 \cdot 1_A) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \]
\[ = a_{(0)}(\varepsilon_t(r^2) \cdot 1_A) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \]
(15) \[ = a_{(0)}(1^L_2 \cdot 1_A) \otimes S(I^L_1) 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \]
\[ = 1^L_1 \cdot a_{(0)} \otimes 1_2^L 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \]
\[ = a_{(0)} \otimes 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \]
\[ = a_{(0)} \otimes a_{(1)} p^2 S(1_3) \otimes 1_1 p^1 \cdot a_{(2)} \]
(11) \[ = a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)}. \]
On the other hand, we have:
\[ [a_{(-1)} r_1^1 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^2 \otimes r_2^1 \cdot a_{(1)} \]
\[ = a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)} \]
(11)
= \varepsilon_s(a_{(-1)} r_1) \cdot 1_A [R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
= [\varepsilon_s(a_{(-1)} 1) \cdot 1_A [R^2 r_1^1 \cdot a_{(0)}] \otimes R^2 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
\overset{\text{(4)}}{=} [\varepsilon_s(a_{(-1)} S(1)) \cdot 1_A [R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
= [\varepsilon_s(a_{(-1)} 1) \cdot 1_A [R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
= [1_1 \cdot 1_A [R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
= S(1) R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \\
\overset{\text{(11)}}{=} R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)}. \\

Thus, the following equation holds:

\[ a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)} = R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)} \in A \otimes H \otimes H. \]

Applying the map \((1 \otimes 1 \otimes \varepsilon)\) to right side of the above equation, we obtain:

\[ (1 \otimes 1 \otimes \varepsilon)(R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r_2^1 \otimes r_2^1 \cdot a_{(1)}) \]
\[ \overset{\text{(5)}}{=} R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(r_2^1) a_{(1)} S(r_2^1)] \]
\[ \overset{\text{(2)}}{=} R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S(r_2^1))] \]
\[ = R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S(r_2^1))] \]
\[ = R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S(r_2^1))] \]
\[ \overset{\text{(3)}}{=} R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1 \varepsilon_s(a_{(1)} S^2(1))] \]
\[ = R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1] \]
\[ = R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1] \]
\[ = R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1], \]

where the counit of a right \(RH\)-comodule \(A\) was used in the last equality. Now we have

\[ R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r_2^1] = (1 \otimes 1 \otimes \varepsilon)(a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)}) \]
\[ = a_{(0)} \otimes a_{(1)} p^2 \varepsilon(p^1 \cdot a_{(2)}) \]
\[ = a_{(0)} \otimes a_{(1)} p^2 \varepsilon_s(p^1 S(p^1)) \]
\[ = a_{(0)} \otimes a_{(1)} p^2 \varepsilon_s(a_{(2)} S(p^1)) \]
\[ = a_{(0)} \otimes a_{(1)} p^2 \varepsilon_s(a_{(2)} S(p^1)) \]
\[ \overset{\text{(14)}}{=} a_{(0)} \otimes a_{(1)} 1_2 \varepsilon_s(a_{(2)} S(1)) \]
\[ = a_{(0)} \otimes a_{(1)} 1_2 \varepsilon_s(1 a_{(2)}) \]
\[ = a_{(0)} \otimes a_{(1)} \varepsilon_s(a_{(2)}) \]
\[ = a_{(0)} \otimes a_{(1)}, \]

where the counit on \(RH\) was used in the last equality. This means that a right \(RH\)-comodule structure on \(A\) is indeed induced by its left \(RH\)-coaction. Therefore, \(A\) is a quantum commutative Galois object. \(\square\)
Summarizing the foregoing arguments, we obtain the main result of this section:

**Theorem 3.6.** Let $(H, R)$ be a finite dimensional quasitriangular weak Hopf algebra. Assume that $A$ is a braided bi-Galois object. Then the functor $A \Box -$ defines a braided autoequivalence of the category $H_H \mathcal{Y} \mathcal{D}$ of Yetter-Drinfeld modules if and only if $A$ is quantum commutative.

**Proof.** Assume that $A$ is a braided bi-Galois object. By Lemma 3.4 and Lemma 3.5 the functor $A \Box -$ defines a braided autoequivalence of $H^H(\mathcal{M}, \mathcal{D})$ if and only if $A$ is quantum commutative. Since $H^H(\mathcal{M}, \mathcal{D}) \cong H_H \mathcal{Y} \mathcal{D}$ as braided monoidal categories, the functor $A \Box -$ induces a braided autoequivalence of $H_H \mathcal{Y} \mathcal{D}$ if and only if $A$ is quantum commutative. \qed

Recall that the Drinfeld center $Z(H, \mathcal{D})$ is tensor equivalent to the Yetter-Drinfeld module category $H_H \mathcal{Y} \mathcal{D}$. Thus the functor $A \Box -$ defines a braided autoequivalence of the Drinfeld center if and only if $A$ is quantum commutative. This holds as well for any quasitriangular Hopf algebra.

In order to deal with the case of a braided fusion category, we need to restrict ourself to the category of finite dimensional representations. Denote by $H_0 \mathcal{M}^{\text{f.d}}$ and $H_H \mathcal{Y} \mathcal{D}^{\text{f.d}}$ the category of finite dimensional left $H$-modules and the category of finite dimensional left Yetter-Drinfeld modules respectively. Then $Z(H_0 \mathcal{M}^{\text{f.d}}, \mathcal{D}) \cong H_H \mathcal{Y} \mathcal{D}^{\text{f.d}}$. Thus, Theorem 3.6 applies to $H_0 \mathcal{M}^{\text{f.d}}$.

**Corollary 3.7.** Let $C$ be a braided fusion category. Then the Drinfeld center of $C$ is equivalent to the category of finite dimensional left comodules over some braided Hopf algebra $RH_C$. Moreover, if $A$ is a braided bi-Galois object over $RH_C$, then the cotensor functor $A \Box -$ defines a braided autoequivalence of the Drinfeld center of $C$ if and only if $A$ is quantum commutative.

**Proof.** Suppose that $C$ is a braided fusion category. By [15] there exists a semisimple connected weak Hopf algebra $H_C$ such that $C$ is (tensor) equivalent to the category $H_C \mathcal{M}^{\text{f.d}}$ of finite dimensional left $H_C$-modules. Similar to the proof of Corollary 2.6, one can endow the category $H_C \mathcal{M}^{\text{f.d}}$ with a braiding $\Phi$ such that the equivalence between the two categories preserves the braidings. Following [14] Prop 5.2] one can define a quasitriangular structure $R$ on $H_C$ so that the braiding $\Phi$ of $H_C \mathcal{M}^{\text{f.d}}$ is induced by the quasi-triangular structure $R$ of $H_C$. \qed

To end this section, we show that the quantum commutative Galois objects over $RH$ form a subgroup of the group of braided bi-Galois objects (see [16]). In the Hopf algebra case, this subgroup was defined in [20]. In what follows, we fix a finite dimensional quasitriangular weak Hopf algebra $(H, R)$. A Galois object means a braided bi-Galois object over the braided Hopf algebra $RH$ in the category $H \mathcal{D}$. It is easy to see that $RH \boxtimes -$ defines the identity functor of $H^H(\mathcal{M}, \mathcal{D})$. So $RH$ is a quantum commutative Galois object.

**Lemma 3.8.** If $A$ and $B$ are two quantum commutative Galois objects, so is $A \Box B$.

**Proof.** Assume that $A$ and $B$ are quantum commutative Galois objects. Then $A \Box -$ and $B \Box -$ are braided autoequivalences. So is the composition $(A \Box B) \Box -$. Thus by Proposition 3.3 $A \Box B$ is quantum commutative. \qed

Let $A$ a bi-Galois object $A$. One can define a braided bi-Galois object $A^{-1} := (RH \otimes A)^{\text{cop}} H \subset RH \otimes A^{\text{op}}$ such that $A \Box A^{-1} \cong RH$ and $A^{-1} \Box A \cong RH$. For more detail on $A^{-1}$, one may refer to [16].
Lemma 3.9. If A is a quantum commutative Galois object, so is \( A^{-1} \).

Proof. Suppose that A is a quantum commutative Galois object. The functor \( A \boxtimes - \) is a braided autoequivalence functor. It is easy to see that \( A^{-1} \boxtimes - \) gives the inverse of the functor \( A \boxtimes - \). By Lemma 3.5 the Galois object \( A^{-1} \) is quantum commutative. □

Denote by \( \text{Gal}^{qc}(R\mathcal{H}) \) the set of isomorphism classes of the quantum commutative Galois objects. Let \([A]\) denote the isomorphism class of a quantum commutative Galois object A. By Lemma 3.8 and Lemma 3.9 we obtain the following.

Theorem 3.10. The set \( \text{Gal}^{qc}(R\mathcal{H}) \) forms a group. The multiplication is induced by the cotensor product \( \boxtimes \) over \( R\mathcal{H} \), the identity is given by \([R\mathcal{H}]\) and the inverse of an element \([A]\) is represented by \( A^{-1} \).

It is well-known that the category \( \mathcal{M} \) is braided subcategory of the Yetter-Drinfeld module category \( \mathcal{D} \). If \( M \) is a left \( H \)-module. Then \( M \) possesses a left \( H \)-comodule structure:

\[
\rho^L(m) = R^2 \otimes R^1 \cdot m := m_{[-1]} \otimes m_{[0]},
\]

so that \((M, \rho^L)\) is a left Yetter-Drinfeld module. It follows from Lemma 2.4 that the induced left \( RH \)-comodule structure on \( M \) is trivial, namely, \( \rho^L(m) = 1 \otimes m \) for all \( m \in M \). If \( A \) is a braided bi-Galois object, then \( A \boxtimes M \cong M \). Thus the functor \( A \boxtimes - \) restricts to the identity functor on the category of left \( H \)-modules.

Now we consider the image of the group \( \text{Gal}^{qc}(R\mathcal{H}) \) in the group \( \text{Aut}^br(\mathcal{D})_{\mathcal{M}} \) of braided autoequivalences of the Yetter-Drinfeld module category.

Definition 3.11. \([3, \text{Defn 2.1}]\) A braided autoequivalence \( F \) of \( \mathcal{D} \) is called trivalizable on \( \mathcal{M} \) if the restriction \( F|_{\mathcal{M}} \) is isomorphic to the identity functor as a braided tensor functor.

Denote by \( \text{Aut}^br(\mathcal{D}, \mathcal{M}) \) the group of isomorphism classes of braided autoequivalences of \( \mathcal{D} \) trivalizable on \( \mathcal{M} \).

Corollary 3.12. The group \( \text{Gal}^{qc}(R\mathcal{H}) \) is a subgroup of the group \( \text{Aut}^br(\mathcal{D}, \mathcal{M}) \).

We expect that the two groups are isomorphic for any finite dimensional quasitriangular weak Hopf algebras \((H, R)\). This is the case when \( H \) is a Hopf algebra, see \([3]\). In case \( H \) is semisimple over an algebraically closed field, i.e. the fusion case, the two groups are indeed isomorphic (to the Brauer group of the braided fusion category), see \([21]\) or \([22]\).

Example 3.13. Let \( k \) be a field with \( \text{ch}(k) \not= 2 \). Let \( H_4 \) be the Sweedler 4-dimensional Hopf algebra over \( k \). Namely, \( H_4 \) is generated by two elements \( g \) and \( h \) satisfying

\[
g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.
\]

The comultiplication, the counit and the antipode are given as follows:

\[
\Delta(g) = g \otimes g, \quad \Delta(h) = 1 \otimes h + h \otimes g \\
\varepsilon(g) = 1, \quad S(g) = g, \quad \varepsilon(h) = 0, \quad S(h) = gh.
\]
It is known that $H_4$ has a quasitriangular structure $R_0$. All quantum commutative Galois objects were computed in $[20]$. Moreover, the group $Gal^{qc}(R_0, H)$ is isomorphic to $\Gamma \rtimes Z_2$, where $\Gamma \cong k^\times \times K^*/K^*2$.

4. Face algebras

In this section we compute the groups of quantum commutative Galois objects of a class of weak Hopf algebras, namely, the face algebras introduced by Hayashi in $[8]$.

Let $N \geq 2$ be an integer and $\mathbb{Z}_N$ the cyclic group $\mathbb{Z}/N\mathbb{Z}$. Let $\omega \in \mathbb{C}$ be a primitive $N$th root of unity. Let $H$ be the $\mathbb{C}$-linear span of $\{X_i^j(s) | i, j, s \in \mathbb{Z}_N\}$. $H$ is a quasitriangular weak Hopf algebra equipped with the following structures:

$$\Delta(X_i^j(s)) = \sum_{p+q=s} X_i^j(p) \otimes X_{i+p}^j(q), \quad \varepsilon(X_i^j(s)) = \delta_{s,0},$$

$$X_i^j(p)X_i^k(q) = \delta_{j,k}\delta_{p,q}X_i^j(p), \quad 1 = \sum_{i,p} X_i^i(p),$$

$$S(X_i^j(p)) = X_{i+p}^j(-p),$$

$$R_1 \otimes R_2 = \sum_{i,j,p} X_j^i(p) \otimes X_{j+p}^1(i-j)\omega^{-p(i-j)},$$

$$R_1' \otimes R_2' = \sum_{i,j,p} X_{i+p}^j(-p) \otimes X_{j+p}^i(i-j)\omega^{-p(i-j)},$$

where the target subalgebra $H_t$ of $H$ is the $\mathbb{C}$-linear span of $\{\sum_p X_i^j(p) | i \in \mathbb{Z}_N\}$. Denote by $1^i$ the sum $\sum_p X_i^j(p)$ for all $i \in \mathbb{Z}_N$. Then $H_t$ is commutative and is equal to the direct sum $\bigoplus_{i \in \mathbb{Z}_N} \mathbb{C}1^i$.

Now we compute the braided Hopf algebra $RH$.

**Lemma 4.1.** The braided Hopf algebra $RH$ is equal to the $\mathbb{C}$-linear span of $\{X_i^j(p) | i, p \in \mathbb{Z}_N\}$ equipped with the following structures:

$$\Delta'(X_k^i(s)) = \sum_{w+q=s} X_k^i(w) \otimes X_k^i(q), \quad \varepsilon(X_k^i(s)) = \delta_{s,0} \sum_p X_i^i(p),$$

$$X_i^j(p)X_k^j(q) = \delta_{i,k}\delta_{p,q}X_i^j(p), \quad 1 = \sum_{i,p} X_i^i(p),$$

$$S(X_k^i(s)) = X_k^i(-s).$$

**Proof.** Note that $\Delta(1_R) = \Delta(\sum_{i,s} X_i^i(s)) = \sum_{i,s} \sum_{p+q=s} X_i^i(p) \otimes X_{i+p}^{i+p}(q)$. We have

$$1_R X_n^m(r)S(1_R) = \sum_{i,s} \sum_{p+q=s} X_i^i(p)X_n^m(r)S(X_{i+p}^{i+p}(q))$$

$$= \sum_{i,s} \sum_{p+q=s} X_i^i(p)X_n^m(r)X_{i+p+q}^{i+p+q}(-q)$$

$$= \sum_{i,s} \sum_{p+q=s} \delta_{i,n}\delta_{n,i+p+q}\delta_{p,r}\delta_{q,r}X_{i+p+q}^s$$

$$= \sum_{i} \delta_{i,m}\delta_{n,i}X_i^r = \delta_{m,n}X_n^m(r),$$

for all $m, n, r \in \mathbb{Z}_N$. So $RH$ is the $\mathbb{C}$-linear span of $\{X_i^j(p) | i, p \in V\}$. 
Using the expression $\Delta(R^1) \otimes R^2 = \sum_{i,j,p} \sum_{u+v=p} X^i_j(u) \otimes X^{i+u}_j(v) \otimes X^j_{i+p}(i-j)\omega^{-p(i-j)}$, we compute the deformed comultiplication as follows:

$$\Delta'(X^k_h(s)) = \sum_{u+q=s} X^k_h(w)S(R^2) \otimes R^1 \cdot X^{k+w}_{k+w}(q)$$

$$= \sum_{u+q=s} X^k_h(w)S(R^2) \otimes R^1 X^{k+w}_{k+w}(q)S(R^2)$$

$$= \sum_{u+q=s} \sum_{i,j,p} X^k_h(w)S(X^j_{j+p}(i-j)) \otimes X^j_{j+p}(u)X^{k+w}_{k+w}(q)S(X^{i+u}_j + u(v))\omega^{-p(i-j)}$$

$$= \sum_{u+q=s} \sum_{i,j,p} \sum_{u+v=p} \sum_{i,j,p} X^k_h(w)X^{i+p}(j-i) \otimes X^j_{j+p}(u)X^{k+w}_{k+w}(q)X^{i+u+v}_j(-v)\omega^{-p(i-j)}$$

$$= \sum_{u+q=s} \sum_{i,j,p} \delta_{w,j-i} \delta_{k,i} \delta_{k,j} X^k_h(w) \otimes X^i_j(q)$$

By Lemma 2.1 the antipode is given by $\Xi(x) = R^2 R^2 S^2(R^1)S(R^1x)$. For convenience, we first compute $R^2 S^2(R^1)$. Indeed,

$$R^2 S^2(R^1) = \sum_{i,j,p} X^j_{j+p}(i-j)S^2(X^j_j(p))\omega^{-p(i-j)}$$

$$= \sum_{i,j,p} X^j_{j+p}(i-j)X^i_j(p)\omega^{-p(i-j)}$$

$$= \sum_{i,j,p} \delta_{j+p,i} X^j_{j+p}(i-j)\omega^{-p(i-j)}.$$
Lemma 4.3. For all equipped with the following structures:

\[
\sum_{i} \sum_{j} \delta_{i-k,j} \delta_{j,-s} \delta_{k+s,j} X_{i+s}^{k}(-s) \omega^{-(s(k)+s(j))} = \sum_{i} \sum_{j} \delta_{i-k,j} \delta_{j,-s} \delta_{k+s,j} X_{i+s}^{k}(-s) \omega^{-(s(k)+s(j))}\]

Thus, the proof is completed. \qed

Take \( i \in \mathbb{Z}_{N} \). Define \( H^{i} \) to be the \( \mathbb{C} \)-linear span of \( \{X_{i}(p)|p \in \mathbb{Z}_{N}\} \). It is obvious that \( H^{i} \) is a subalgebra of \( R H \) with unity \( 1^{i} \). Moreover, \( R H \) is the direct sum of all these \( H^{i} \), i.e., \( R H = \bigoplus_{i \in \mathbb{Z}_{N}} H^{i} \). We will show that every \( H^{i} \) is also an ordinary Hopf algebra and so \( R H \) is actually the direct sum of all these Hopf algebras. In order to verify that every \( H^{i} \) can be equipped with a coalgebra structure, we need to decompose the vector space \( R H \otimes_{\mathbb{C}} R H \).

Lemma 4.2. \( R H \otimes_{\mathbb{C}} R H = \bigoplus_{i \in \mathbb{Z}_{N}} (H^{i} \otimes H^{i}) \).

Proof. It is equivalent to show that

\[
1_{1} \cdot X_{u}^{a}(b) \otimes 1_{2} \cdot X_{u}^{a}(w) = \delta_{u,a} X_{u}^{a}(b) \otimes X_{u}^{a}(w),
\]

for all \( a, b, u, w \in \mathbb{Z}_{N} \). Indeed, we have

\[
1_{1} \cdot X_{u}^{a}(b) \otimes 1_{2} \cdot X_{u}^{a}(w) = \sum_{i, s} \sum_{p+q=s} X_{i}^{a}(p) \cdot X_{u}^{a}(b) \otimes X_{i+p}^{a}(q) \cdot X_{u}^{a}(w) = \sum_{i, s} \sum_{p+q=s} \delta_{i, u} \delta_{p, 0} X_{i}^{a}(b) \otimes \delta_{i+p, u} \delta_{q, 0} X_{i+p}^{a}(w) = \sum_{i} \delta_{i, u} X_{i}^{a}(b) \otimes \delta_{i, u} X_{i}^{a}(w) = \delta_{u,a} X_{u}^{a}(b) \otimes X_{u}^{a}(w),
\]

for all \( a, b, u, w \in \mathbb{Z}_{N} \). \qed

Lemma 4.3. For all \( i \in \mathbb{Z}_{N} \), \( H^{i} \) is a coalgebra over \( \mathbb{C} 1^{i} \) with the following structures:

\[
\Delta'(X_{i}^{a}(s)) = \sum_{w+q=s} X_{i}^{a}(w) \otimes X_{i}^{a}(q),
\]

\[
\varepsilon_{t}(X_{i}^{a}(s)) = \delta_{s, b} \sum_{p} X_{i}^{a}(p).
\]

Proof. Follows from Lemma 4.1 and Lemma 4.2. \qed

Proposition 4.4. For all \( i \in \mathbb{Z}_{N} \), \( H^{i} \) is a commutative and cocommutative Hopf algebra over \( \mathbb{C} 1^{i} \) equipped with the following structures:

\[
X_{i}^{a}(p) X_{i}^{a}(q) = \delta_{p, q} X_{i}^{a}(p), \quad 1_{H^{i}} = 1^{i},
\]

\[
\Delta'(X_{i}^{a}(s)) = \sum_{w+q=s} X_{i}^{a}(w) \otimes X_{i}^{a}(q),
\]

\[
\varepsilon_{t}(X_{i}^{a}(s)) = \delta_{s, 0} \sum_{p} X_{i}^{a}(p), \quad S(X_{i}^{a}(s)) = X_{i}^{a}(-s).
\]
Proof. Since we know already that $H^1$ is both an algebra and a coalgebra, it remains to be proved that $\Delta'$ and $\varepsilon_t$ are multiplicative, and that the axioms of the antipode $S$ hold. We first check that $\Delta'$ is multiplicative. Indeed,

$$\Delta'(X^i_s(t))\Delta''(X^i_t(s)) = \sum_{p+q=s} X^i_p \otimes X^i_q \left[ \sum_{p'+q'=t} X^i_{p'} \otimes X^i_{q'} \right]$$

$$= \sum_{p+q=s} \sum_{p'+q'=t} [X^i_p X^i_{p'} \otimes X^i_q X^i_{q'}]$$

$$= \sum_{p+q=s} \sum_{p'+q'=t} \delta_{p,p'} \delta_{q,q'} [X^i_p \otimes X^i_q]$$

$$= \delta_{s,t} \sum_{p+q=s} X^i_p \otimes X^i_q$$

$$= \Delta'(X^i_s(t) X^i_t(s)),$$

for all $i, s, u, t \in \mathbb{Z}_N$.

Note that $\Delta'(1) = 1 \otimes 1$. It follows from Lemma 4.2 that $\Delta'(1^i) = 1^i \otimes 1^i$.

Next we verify that $\varepsilon_t$ is an algebra map. For all $s, t \in \mathbb{Z}_N$, we have

$$\varepsilon_t(X^i_s(t))\varepsilon_t(X^i_t(s)) = \delta_{s,0} \delta_{t,0} \left( \sum_p X^i_p \right) \left( \sum_q X^i_q \right)$$

$$= \delta_{s,0} \delta_{t,0} \left( \sum_p X^i_p \right) = \delta_{s,t} \delta_{s,0} \varepsilon_t(X^i_s(t))$$

$$= \varepsilon_t(X^i_s(t)) \varepsilon_t(X^i_t(s)).$$

Finally, we prove that the antipode axioms hold. Indeed,

$$m(1 \otimes S) \Delta''(X^i_s(s)) = \sum_{p+q=s} X^i_p S(X^i_q) = \sum_{p+q=s} X^i_p X^i_{-q}$$

$$= \delta_{p,-q} \sum_{p+q=s} X^i_p = \delta_{s,0} \sum_{p \in \mathbb{Z}_N} X^i_p = \varepsilon_t(X^i_s(s)).$$

for any $s \in \mathbb{Z}_N$. Similarly, we also have

$$\sum_{w+q=s} S(X^i_w) X^i_q = \sum_{w+q=s} X^i_{-w} X^i_q = \sum_{w+q=s} \delta_{-w,q} X^i_q$$

$$= \sum_q \delta_{s,0} X^i_q = \varepsilon_t(X^i_s(s)).$$

Hence, $H^1$ is an ordinary Hopf algebra over $\mathbb{C}1^i$. \hfill \Box

In fact, $H^1$ is isomorphic to the dual Hopf algebra of the group Hopf algebra $k\mathbb{Z}_N$.

Corollary 4.5. The braided Hopf algebra $\mathcal{R}H$ has a decomposition:

$$\mathcal{R}H = \bigoplus_{i \in \mathbb{Z}_N} H^i,$$

where $H^1$ is a Hopf algebra over $\mathbb{C}1^i$ with unity $1^i$. Moreover, there exists a Hopf algebra isomorphism from $H^i$ to $H^j$ defined by

$$\iota^j_i : X^i_p \rightarrow X^j_p.$$
for all $i,j,p \in \mathbb{Z}_N$.

**Proof.** Follows from Proposition 4.4. □

Corollary 4.5 indicates that braided bi-Galois objects over $R H$ can be obtained from bi-Galois objects over a Hopf algebra $H^i$.

Let the notations be as above. Let $A$ be a quantum commutative Galois object over $R H$. Corollary 4.5 implies that there is a decomposition: $A = \bigoplus_{i \in \mathbb{Z}_N} A^i$, where $\rho'(A^i) \in A^i \otimes H^i$. Furthermore, every $A^i$ is just a Galois object over $H^i$ (automatically a bi-Galois object as $H^i$ is cocommutative).

Conversely, given a Galois object $A'$ over Hopf algebra $H^i$ for some $i \in \mathbb{Z}_N$, we can get a quantum commutative Galois object over $R H$ as the direct sum $\bigoplus_{i \in \mathbb{Z}_N} A'^i$, where every algebra $A'^i$ is a copy of $A'$. Now we state the relation between quantum commutative Galois object over $R H$ and Galois object over $H^i$ as follows:

**Proposition 4.6.** Let $A$ be a $C$-algebra with unity. Then $A$ is a quantum commutative Galois object over $R H$ if and only if $A$ is the direct sum $\bigoplus_{i \in \mathbb{Z}_N} A^i$, where every $A^i$ is an $H^i$-Galois object. Moreover, there exists a group isomorphism

$$\Omega : \text{Gal}^{qc}(R H) \rightarrow \text{Gal}(H^i), \quad A \mapsto A^i,$$

for any fixed $i \in \mathbb{Z}_N$. The inverse of $\Omega$ is given as follows:

$$\Omega' : \text{Gal}(H^i) \rightarrow \text{Gal}^{qc}(R H), \quad A' \mapsto \bigoplus_{i \in \mathbb{Z}_N} A'^i.$$

The detailed proof of the statement above is given in [22] following a tedious and long computation.

So the group $\text{Gal}^{qc}(R H)$ can be obtained by computing the group $\text{Gal}(H^i)$ of Galois objects over $H^i$. Since the Hopf algebra $H^i$ is commutative and cocommutative isomorphic to $k\mathbb{Z}_N$, we know that the group $\text{Gal}(H^i)$ is actually given by the second Galois cohomology group $H^2(\mathbb{Z}_N, k)$.

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