Screened perturbation theory at four loops

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Abstract

We study the thermodynamics of massless $\phi^4$-theory using screened perturbation theory, which is a way to systematically reorganise the perturbative series. The free energy and pressure are calculated through four loops in a double expansion in powers of $g^2$ and $m/T$, where $m$ is a thermal mass of order $gT$. The result is truncated at order $g^7$. We find that the convergence properties are significantly improved compared to the weak-coupling expansion.

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1. Introduction

Recently, Gynther et al. calculated the pressure of massless $\phi^4$-theory to order $g^6$ in the weak-coupling expansion [1]. The weak-coupling pressure for various orders of $g$ is shown in Fig. 1. Note that it does not seem to converge as higher and higher orders are included. This is a well-known problem, not only in scalar field theory, but also in gauge theories.

Many methods have been devised to improve upon the convergence of this expansion. Among them is screened perturbation theory (SPT), which was first introduced in thermal field theory by Karsch, Patkós and Petreczky [2]. SPT constitutes a reorganisation of the perturbative series so that one resums selected diagrams from all orders of perturbation theory. In the following, we will indeed see that using SPT improves the convergence significantly.

This talk is a brief overview of the calculations and results of Ref. [3].

2. Screened perturbation theory

The Lagrangian density for a massless $\phi^4$-theory is
\[ L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{g^2}{24}\phi^4, \]  

where \( g \) is the coupling constant. The SPT Lagrangian of this theory is defined as

\[ L_{\text{SPT}} = L_{\text{free}} + L_{\text{int}}, \]

where

\[ L_{\text{free}} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \quad \text{and} \quad L_{\text{int}} = \frac{1}{2} m_1^2 \phi^2 - \frac{g^2}{24}\phi^4. \]

If we set \( m_1^2 = m^2 \), it is clear that \( L_{\text{SPT}} = L \). We now take \( m_1^2 \) to be of order \( g^2 \) and expand systematically in powers of \( g^2 \). This defines a reorganisation of the perturbative series, in which the expansion is about an ideal gas of massive particles. The mass \( m \) has the simple interpretation of a thermal mass. A prescription for \( m \) is discussed later; for now we take it formally to be of order \( gT \).

3. Free energy

We calculate the free energy in a double power expansion. That is, first we do a loop expansion in powers of \( g^2 \), and thereafter we expand each diagram in powers of \( m/T \).

The inclusion of the mass term in the interaction yields an additional Feynman rule not present in the original theory, namely \( \int \propto = \frac{1}{2} m_1^2 \). This is called a mass insertion. The free energy can then be written as a series of vacuum diagrams,

\[ \mathcal{F} = \ldots, \]

which we truncate at four loops. Note that mass insertions count as loops, since they are of order \( g^2 \).

As an example of an \( m/T \) expansion, take the one-loop diagram with a single mass insertion:

\[ \mathcal{F}_{1b} = \ldots = -\frac{1}{2} m_1^2 T \sum_{p_0=2\pi nT} \int p \cdot \frac{1}{p^2 + m^2}, \]

There are two momentum scales in this sum-integral; the hard scale, which is of order \( T \), and the soft scale, of order \( gT \). The former arises from the nonzero Matsubara frequencies, whereas the latter comes from the thermal mass \( m \). We isolate the contribution from the zeroth Matsubara mode, as it only contains the soft scale. This yields

\[ \mathcal{F}_{1b} = -\frac{1}{2} m_1^2 T \left[ \int_{p_0 \neq 0} \int p \cdot \frac{1}{p^2 + m^2} + \sum_{p_0=0} \int p \cdot \frac{1}{p^2 + m^2} \right]. \]

Since \( m \ll P \) in the second term, we can expand it in a geometric series:

\[ \mathcal{F}_{1b} = -\frac{1}{2} m_1^2 T \left[ \int_{p_0 \neq 0} \int p \cdot \frac{1}{p^2 + m^2} + \sum_{p_0=0} \int p \cdot \frac{1}{p^2 + m^2} \left( \frac{m^2}{P^2} + \frac{m^4}{P^4} + \cdots \right) \right]. \]
The mass can now be taken outside the sum-integral in each term, and the result is a series of easily-evaluable massless sum-integrals. Finally, the results are truncated at $g^7$.

4. The tadpole mass

The pressure of the original theory is obtained in the limit where the two masses are equal, and is defined as

$$ P = -\mathcal{F}|_{m_1^2 = m^2}. $$

(8)

The parameter $m$ in screened perturbation theory is completely arbitrary, and if we were able to include all loop orders, the result would indeed be independent of $m$. To complete the calculation we must instead find a prescription for $m$ which is physically meaningful.

The simplest choice is the tadpole,

$$ m^2 = \frac{g^2 T}{24} \sum_{p_0} \int \frac{1}{p^2 + m^2}. $$

(9)

In the weak-coupling limit the propagator in the loop is massless, and Eq. (9) reduces to

$$ m^2 = \frac{g^2 T^2}{24}. $$

(10)

Using this value for the mass one obtains the weak-coupling pressure, shown in Fig. 1b. Our result through order $g^6$ agrees with the $N = 1$ result in Ref. [1].

We can generalise this to higher loop orders by taking $m$ to be the tadpole mass,

$$ m^2 = \frac{g^2 T^2}{24} + \cdots = g^2 \frac{\partial \mathcal{F}}{\partial (m^2)} \bigg|_{m_1^2 = m^2}. $$

(11)

With this choice, $m$ is well-defined at all loop orders. Since the propagators in Eq. (11) are massive as well, it means that in calculating the pressure we are doing a selective resummation of diagrams from all orders of perturbation theory.

5. Results

Fig. 1a shows the SPT pressure truncated at various loop orders. The two- and three-loop results are indistinguishable from the exact numerical results found in Ref. [1]. Convergence is rapid—in the two-loop case terms of order $g^5-g^7$ are negligible, while at three loops one can neglect terms of order $g^7$.

There are no exact numerical data available for comparison with our result at four loops, but experience with lower loop orders indicates that this is indeed a good approximation. This can, however, only be confirmed by calculating the pressure through $g^8$. 3
Fig. 1. (a) Pressure normalised to $P_{\text{ideal}}$ through $g^7$ for various loop orders in SPT. (b) Weak-coupling pressure through various orders of $g$.

6. Summary and outlook

We have calculated the pressure of a massless $\phi^4$ theory using screened perturbation theory. As Fig. 1 shows, the successive approximations in SPT seem a lot more stable than in the weak-coupling expansion. The apparent improved convergence seems to be linked to the fact that SPT is basically an expansion about an ideal gas of massive particles, instead of an expansion about an ideal gas of massless particles, which is the case for the weak-coupling expansion.

Note that in Fig. 1b, only terms through order $g^6$ in the weak-coupling expansion have been included. This is because part of the $g^7$-contribution arises from five-loop vacuum diagrams which aren’t considered in Ref. [3]. Evaluation of the free energy to order $g^7$ is work currently in progress [8].

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