Generalized Huffman Coding for Binary Trees with Choosable Edge Lengths

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Abstract. In this paper we study binary trees with choosable edge lengths, in particular rooted binary trees with the property that the two edges leading from every non-leaf to its two children are assigned integral lengths \(l_1\) and \(l_2\) with \(l_1 + l_2 = k\) for a constant \(k \in \mathbb{N}\). The depth of a leaf is the total length of the edges of the unique root-leaf-path.

We present a generalization of the Huffman Coding that can decide in polynomial time if for given values \(d_1, \ldots, d_n \in \mathbb{N}\), there exists a rooted binary tree with choosable edge lengths having \(n\) leaves of depths at most \(d_1, \ldots, d_n\).

1 Introduction

For a fixed \(k \in \mathbb{N}\) we define \(L'(k) = \{\{i, k - i\} \mid 1 \leq i \leq k - 1, i \in \mathbb{N}\}\). An \(L'(k)\)-tree is a rooted strict binary tree with the property that the two edges leading from every non-leaf to its two children are assigned lengths \(l_1\) and \(l_2\) with \(\{l_1, l_2\} \in L'(k)\). Let \(D = \{d_1, \ldots, d_n\}\) be a multi set of integers such that \(1 \leq d_1 \leq \ldots \leq d_n\). In this paper we show for fixed \(k\) how to decide in polynomial time if there exists an \(L'(k)\)-tree with \(n\) leaves with depths at most \(d_1, \ldots, d_n\) and construct such a tree.

If \(k = 2\), then all edges of the tree have length 1. In this case we have classical binary trees where the depth of a leaf is equal to the number of edges of the unique path from the root to the leaf. By Kraft’s inequality an \(L'(2)\)-tree for \(D\) exists if and only if

\[
\sum_{i \in \{1, \ldots, n\}} 2^{-d_i} \leq 1. \tag{1}
\]

Such a tree can be constructed using the Huffman Coding algorithm [7].

\(L'(k)\)-trees have been first considered by Maßberg and Rautenbach [9]. They are motivated by the so-called repeater tree problem, a problem that occurs in VLSI-design. Repeater trees are required to distribute an electronical signal from a root to several locations on a chip, called sinks, while not exceeding individual time restrictions. By inserting repeaters at a vertex of the tree it is possible to reduce the delay of one of the branches while increasing the delay on the other branch. This effect can be modeled by the \(L'(k)\)-trees: The lengths of the edges correspond to the delays on these edges and the \(d_i\) correspond to the required arrival times for a sink \(i\). Then the task is to build an \(L'(k)\)-tree such that
the signal reaches each sink within time. For more details on the repeater tree problem and its connection to binary trees with choosable edge lengths we refer the reader to [1] and [3]. Beside this application we think that our problem is also of theoretical interest.

In [9] it has been shown how to construct $L'(4)$-trees in polynomial time. For $k > 4$ it was an open problem if there exists a polynomial algorithm that can decide the existence of $L'(k)$-trees for given instances. We show that we can decide the existence with a generalized Huffman coding in time $O(nk^3)$, that is, in polynomial time for constant $k$.

Our problem is related to prefix-free codes with unequal letter costs (see e.g. [2,3,4,5]). Nevertheless, there are significant differences between these problems. First, in our problem we ask for a tree satisfying given depth restrictions for the leaves while for prefix-free codes the task is to find a tree minimizing $\sum_{i \in \{1,\ldots,n\}} \text{depth}(i)w_i$ where $\{w_1, \ldots, w_n\}$ are given numbers and depth($i$) denotes the depth of leaf $i$, $i \in \{1,\ldots,n\}$. Moreover, in our problem we have the freedom to choose the edge lengths from a discrete set of numbers as long as the sum of the lengths of the edges leaving a vertex is $k$. We think that neither a polynomial time algorithm for one of the problems yields an efficient algorithm for the other problem nor vice versa.

2 Generalized Huffman Coding

Let $D = \{d_1, \ldots, d_n\}$ be a multi set of integers and $T$ be a rooted binary tree with $n$ leaves $v_1, \ldots, v_n$. Define $d : \{v_1, \ldots, v_n\} \rightarrow \mathbb{Z}$, $d(v_i) = d_i$ for $i \in \{1, \ldots, n\}$. We want to decide, if we can assign lengths to the edges of $T$ such that $T$ is an $L'(k)$-tree with $v_i$ at depth at most $d(v_i)$ for $i \in \{1, \ldots, n\}$.

First we note that we can assume the $d_i$‘s not to be too big.

Remark 1. As $T$ has $n$ leaves, every root-leaf-path contains at most $n-1$ edges. Moreover, every edge has length at most $k-1$. Thus every leaf has depth at most $(k-1)(n-1)$ and we can assume w.l.o.g. that $d_n \leq (k-1)(n-1)$.

For two integers $a_1, a_2 \in \mathbb{Z}$ we define

$$\omega_k(a_1, a_2) := \min\{a_1, a_2\} - \max\left\{1, \left\lfloor \frac{k - |a_1 - a_2|}{2} \right\rfloor \right\}. \quad (2)$$

We extend the function $d$ to the internal vertices: Let $u$ be an internal vertex and $v$ and $w$ be the two children of $u$ in $T$ such that $d(v)$ and $d(w)$ have already been defined. Then we set $d(u) = \omega_k(d(v), d(w))$. Note that $d(u)$ is the maximum possible depth for $u$ such that we can assign edge lengths $l_1, l_2$ to the edges $uv$ and $uw$ with $\{l_1, l_2\} \in L'(k)$ and $v$ and $w$ are at depth at most $d(v)$ and $d(w)$, respectively.

This implies that $T$ can be transformed into a feasible $L'(k)$-tree such that $v_i$ is at depth at most $d(v_i)$ if and only if $d(r) \geq 0$ for the root $r$ of $T$.

Let $D = \{d_1, \ldots, d_n\}$ be an instance of our decision problem. The previous observation gives us a simple (but inefficient) method to decide if an $L'(k)$-tree
exists for \(D\): For every pair \(d_i, d_j \in D, d_i \leq d_j\) construct a new instance of size \(n - 1\) by replacing \(d_i\) and \(d_j\) in \(D\) by \(d' = \omega_k(d_i, d_j)\). We will call \((d_i, d_j, d')\) a replacement triple of \(D\). Denote by \(M_{n-1}\) the set of all instances we get from \(D\) by applying this replacement step. An \(\mathcal{L}'(k)\)-tree exists for \(D\) if and only if an \(\mathcal{L}'(k)\)-tree exists for at least one of the instances \(A \in M_{n-1}\). Recursively, we can construct the sets \(M_{n-2}, \ldots, M_1\). By induction we conclude that an \(\mathcal{L}'(k)\)-tree exists for \(D\) if and only if \(\{i\} \in M_1\) for some \(i \geq 0\).

The main challenge of this generalization of Huffman Coding is that the sets \(M_i\) can get exponentially large. A key idea of our algorithm is to traverse the binary trees from the leaves to the root in a reasonable way. With such an appropriate traversing strategy we can show that the sets \(M_i\) have a size polynomially bounded in \(n\). Let \(D = \{d_1, \ldots, d_n\}\) be an instance and assume we have a rooted binary tree \(T\) associated with this instance. Replacing two numbers \(d_i, d_j \in D, i \leq j\), by a new number \(d'\) corresponds to removing two leaves \(v, w \in V(T)\) with a common parent \(u\) and \(d(v) = d_i\) and \(d(w) = d_j\) such that \(u\) becomes a new leaf with \(d(u) = d'\). When choosing \(v\) and \(w\) we will use the following strategy: Always remove leaves \(v, w\) with a common parent \(u\) such that \(d(u)\) is maximum. Using this strategy repeatedly the tree finally reduces to the root \(r\) and \(d(r)\).

So let \(A = \{a_1, \ldots, a_z\}\) be an instance where we replace two values \(a_i\) and \(a_j\) by a new number \(w\) and let \(B = \{b_1, \ldots, b_{z-1}\}\) denote the resulting set. By our strategy we can assume that in the prospective replacing steps we will never create a new value greater than \(w\), that is, all internal vertices in an \(\mathcal{L}'(k)\)-tree \(T\) for \(B\) will have depth at most \(w\). As all edges have length at most \(k - 1\), the depth of all leaves of an \(\mathcal{L}'(k)\)-tree for \(B\) is at most \(w + k - 1\). Thus we can replace all values \(b_i \in B\) with \(b_i > w + k - 1\) by \(b_i = w + k - 1\).

An additional strategy in order to improve the running time is to eliminate sets that are “dominated” by other sets:

**Remark 2.** Let \(A = \{a_1, \ldots, a_n\}\) and \(B = \{b_1, \ldots, b_n\}\) be multi sets such that \(a_1 \leq \cdots \leq a_n\) and \(b_1 \leq \cdots \leq b_n\). If \(a_i \leq b_i\) for all \(i \in \{1, \ldots, n\}\) and if there exists a \(\mathcal{L}'(k)\)-tree for \(A\), then there also exists an \(\mathcal{L}'(k)\)-tree for \(B\). In this case, \(A\) is dominated by \(B\) and we write \(A \preceq B\).

Note that “\(\preceq\)” defines a partial order on multi sets of integers of the same cardinality. By Remark 2 we can restrict ourselves to maximal elements with respect to this partial order.

Consider a multi set \(A = \{a_1, \ldots, a_z\}, a_1 \leq \cdots \leq a_z\). We denote by \(P(A) = \{(a_i, a_j, w) : 1 \leq i < j \leq z, w = \omega_k(a_i, a_j)\}\) the set of replacement triples for \(A\). Note that \(|P(A)| \in O(z^2)\). Using the idea of dominated sets we can reduce the number of elements in \(P(A)\) without removing relevant elements. Assume there are two triples \((a, a^1, w), (a, a^2, w) \in P(A)\) with \(a^1 < a^2\), that is, they only differ in the second entry. For \(i \in \{1, 2\}\) denote by \(B_i\) the multi set we obtain by replacing \(\{a, a^i\}\) in \(A\) by \(w\). Then \(B_1 = (B_2 \setminus \{a^1\}) \cup \{a^2\}\) and thus \(B_2 \preceq B_1\). In this case \((a, a^1, w)\) dominates \((a, a^2, w)\). We conclude that it is sufficient to consider only replacement triples in \(P(A)\) that are not dominated
by other replacement triples. Denote by $P'(A)$ the set of dominating replacement triples for $A$.

**Proposition 1.** For a multi set $A = \{a_1, \ldots, a_z\}, a_1 \leq \ldots \leq a_z$ we have $|P'(A)| \leq k z$.

**Proof.** Let $(a_i, a_j, w) \in P'(A)$. As $w = \omega_k(a_i, a_j) = a_i - \max\left\{1, \left\lceil \frac{k-a_j}{2} \right\rceil \right\}$ and $a_i \leq a_j$ we conclude $a_i - \left\lceil \frac{k}{2} \right\rceil \leq w \leq a_i - 1$. For given $a_i$ and $w$ the second value $a_j$ either is unique or no triple $(a_i, a_j, w) \in P'(A)$ exists. Thus we have at most $z - 1$ possibilities for the first value and at most $k$ possibilities for the third value of a replacement triple in $P'(A)$. This completes the proof. □

Note, that the set $P'(A)$ can be computed in time $O(k z)$.

Our observations lead us to the following algorithm.

**Input:** An instance $D = \{d_1, \ldots, d_n\}$ and $k \in \mathbb{N}, k \geq 2$.

**Output:** Returns true iff there exists an $L'(k)$-tree for $D$.

1. $M_0 \leftarrow \{D\}$;
2. $l(D) \leftarrow \infty$;
3. for $z = n - 1$ to 1 do
   4. $M_z \leftarrow \emptyset$;
   5. foreach $A = \{a_1, \ldots, a_{z+1}\} \in M_{z+1}$ do
      6. Compute the set $P'(A) \subseteq \{(a_i, a_j, \omega_k(a_i, a_j)) : 1 \leq i < j \leq z+1\}$ of dominating replacement triples for $A$;
      7. foreach $(c, d, w) \in P'(A)$ do
         8. $B = \{b_1, \ldots, b_z\} \leftarrow (A \setminus \{c, d\}) \cup \{w\}$;
         9. $m \leftarrow \min\{w + k - 1, (z - 1)(k - 1)\}$;
         10. for $b_i \in B : b_i > m$ do
              11. $b_i \leftarrow m$;
         12. end
      13. end
   14. $p(B) \leftarrow A$;
   15. $l(B) \leftarrow \min\{l(A), w\}$;
   16. $M_z \leftarrow M_z \cup \{B\}$;
   17. end
   18. (optional: Remove sets $A \in M_z$ that are dominated by other sets $B \in M_z$);
19. end
20. if $\{i\} \in M_1$ for some $i \geq 0$ then
21.    return true;
22. else
23.    return false;
24. end

**Algorithm 1:** $L'(k)$-tree decision algorithm.

Let $D = \{d_1, \ldots, d_n\}$ be the input of the algorithm. In order to be able to reproduce the replacement steps we introduce a function $p(\cdot)$ that assigns
a multi set $B$ to the set that it replaces. For a multi set $B$ we will denote by $l(B)$ the minimum value of an element that has been added to $B$. This implies that we have only removed and changed elements $b$ with $b > l(B)$. Thus if $B = \{b_1, \ldots, b_z\}$ with $b_1 \leq \ldots \leq b_z$ then

$$b_i = d_i \text{ for all } i \leq \max \{j \mid b_j < l(B)\} = \max \{j \mid d_j < l(B)\}.$$  

(3)

Note that the Lines 9 and 11 are due to the observations that no leaf in an $L'(k)$-tree with $z$ leaves has depth greater than $(z - 1)(k - 1)$ by Remark 1 and that no leaf has depth greater than $w + k - 1$ by our traversing strategy.

Recall that $A$ and $B$ are multi sets. Thus in Line 8 we remove an element $a_i = c$ and an element $a_j = d$, $i \neq j$ from $A$ and insert a new element $w$ to $B$. The set $\mathcal{M}_1$ contains exactly one element, as otherwise one element dominates another. For simplicity of notation we set $m(A) = \max_{a \in A} a$. We will prove now, that the size of the sets $\mathcal{M}_z$ is polynomially bounded in $n$.

**Proposition 2.** If $z \in \{1, \ldots, n - 1\}$, then all sets $B \in \mathcal{M}_z$ satisfy

$$m(B) \leq l(B) + k - 1.$$  

(4)

**Proof.** For contradiction we assume that $B$ is a set of maximum cardinality computed by the algorithm that contradicts (4).

Set $A = p(B)$. Obviously, $l(B) \leq l(A)$ and $m(B) \leq m(A)$. If $l(B) = l(A)$ then $m(A) \geq m(B) > l(B) + k - 1 = l(A) + k - 1$, i.e. $A$ also does not satisfy (4), contradicting the maximality of $B$. Thus $l(B) < l(A)$. But in this case $l(B) = w$ and by the setting in Line 11 of the algorithm we obtain $m(B) \leq m \leq w + k - 1 = l(B) + k - 1$, which is a contradiction and completes the proof. \[\square\]

**Lemma 1.** If $z \in \{1, \ldots, n - 1\}$, then

$$|\mathcal{M}_z| \leq z^k.$$  

(5)

**Proof.** Define $\mathcal{M}_z := \{A \in \mathcal{M}_z : l(A) = i\}$ for $i \in \mathbb{N}$. By Remark 1 we know $\mathcal{M}_i = \emptyset$ for $i > (z - 1)(k - 1)$.

Let $i \in \{0, \ldots, (n - 1)(k - 1)\}$ and $A = \{a_1, \ldots, a_z\} \in \mathcal{M}_z$ such that $a_1 \leq \ldots \leq a_z$. Recall that $D = \{d_1, \ldots, d_n\}$, $d_1 \leq \ldots \leq d_n$, is the input of the algorithm. Set $t = \max \{j : d_j < l(A)\}$. By the definition of $l(A)$ and (3), we conclude

$$a_j = d_j$$  

(6)

for $j \in \{1, \ldots, t\}$. On the other hand, by Prop. 2

$$a_j \in \{l(A), \ldots, l(A) + k - 1\}$$  

(7)

for $j \in \{t + 1, \ldots, z\}$.

This implies that the sets $A \in \mathcal{M}_z$ only differ in the largest $(z - t)$ elements and each of these elements can only take one of $k$ different values. Thus the number of different sets $A$ (without removing dominated ones) is at most the
number of integral partitions of \( z - t \) into the sum of \( k \) non-negative integers. This number equals \( \left( \frac{(z-t)k}{k-1} \right)^{z-1} \) and is bounded by \( \frac{z}{z-1} \).

Altogether we have at most \( 1 + (z - 1)(k - 1) \) sets \( M_z \) that are non-empty and each of these sets has at most \( z^{k-1} \) elements. Hence

\[
|M_z| = \sum_{i \in \mathbb{N}} |M^i_z| \leq (1 + (z - 1)(k - 1)) \frac{z^{k-1}}{k-1} \leq z^k. \tag{8}
\]

This finishes the proof. \( \Box \)

**Theorem 1.** For a given multi set of integers \( D = \{d_1, \ldots, d_n\} \) it can be decided in time \( O(n^{k+3}) \) if there exists an \( L'(k) \)-tree with \( n \) leaves at depths at most \( d_1, \ldots, d_n \). Moreover, such a tree can be constructed in the same running time.

**Proof.** The running time is dominated by the number of iterations of the **foreach** loop in Line 7. By Proposition 1 this loop is executed for every \( z \in \{1, \ldots, n-1\} \) and every element in \( M_z \) at most \( kz \) times. Thus the total number of iterations is bounded by

\[
\sum_{z \in \{1, \ldots, n-1\}} k |M_z| \leq \sum_{z \in \{1, \ldots, n-1\}} k z^{k+1} \in O(n^{k+2}). \tag{9}
\]

In each of these iterations we consider a set containing at most \( n \) elements. Thus the total running time of the algorithm is bounded by \( O(n^{k+2}) \).

As we have seen before, there exists an \( L'(k) \)-tree as required if and only if \( M_1 = \{\{i\}\} \) for an \( i \geq 0 \). This tree can be reconstructed using the predecessor function \( p(\cdot) \). \( \Box \)

**Remark 3.** Note that we use the concept of dominated sets in order to reduce the number of replacement triples (see Line 6 of Alg. 1). It is also possible to remove dominated sets after each iteration of the main loop (see Line 18 of Alg. 1). In practice this reduces the cardinalities of the sets \( M_z \) and the running time of the algorithm significantly. Nevertheless, it seems that the theoretical worst case running times do not decrease in the general.

**Remark 4.** If \( k = 2 \) we are in the case of ordinary binary trees with all edges having length 1. In this case it is easy to show that the set \( B \) constructed for the replacement triple \( (a_z, a_{z+1}, a_z - 1) \in P'(A) \) always dominates all sets \( B' \) constructed for \( (c, d, w) \in P'(A) \), \( (c, d, w) \neq (a_z, a_{z+1}, a_z - 1) \). Thus \( |M_z| = 1 \) for all \( z \in \{1, \ldots, n\} \) and the algorithm is equivalent to the Huffman coding.

### 3 Conclusion and Future Work

In this paper we have proven that a generalized version of the Huffman Coding can be used in order to build rooted binary trees with choosable edge lengths in polynomial time. It is still open if there is an algorithm for the \( L'(k) \)-tree with a significantly better running time, for example an algorithm with a running time that is polynomially bounded not only in \( n \) but also in \( k \).
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