A Metric Lower Bound Estimate for Geodesics in the Space of Kähler Potentials

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Abstract

In this paper we establish a positive lower bound estimate for the second smallest eigenvalue of the complex Hessian of solutions to a degenerate complex Monge-Ampère equation. As a consequence, we find that in the space of Kähler potentials any two points close to each other in $C^2$ norm can be connected by a geodesic along which the associated metrics do not degenerate.

1 Introduction

1.1 Background and Subject

In this paper we will be interested in the solution to degenerate complex Monge-Ampère equations, which play important roles in Kähler geometry.

For a compact Kähler manifold $V$ with a metric $\omega_0$, we define the following space of Kähler potentials,

$$\mathcal{H} = \{ \varphi \in C^\infty(V) | \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \}. \quad (1.1)$$

Then a Riemannian metric can be introduced on this infinite dimensional space, for $\psi_1, \psi_2 \in T_\varphi\mathcal{H}$

$$<\psi_1, \psi_2>_\varphi = \int_V \psi_1 \psi_2 (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n. \quad (1.2)$$

Under this metric (1.2), a function $\Phi : [0,1] \times V \to \mathbb{R}$, with $\Phi(t,*) \in \mathcal{H}$ for each $t \in [0,1]$, is called a geodesic, if it satisfies

$$\Phi_{tt} = \Phi_t g^{\alpha\bar{\beta}} \Phi_{\alpha\bar{\beta}}, \quad \text{in} \quad [0,1] \times V. \quad (1.3)$$

where $g^{\alpha\bar{\beta}} = \omega_{0,\alpha\bar{\beta}} + \Phi_{\alpha\bar{\beta}}$. If we define $S = \{ \zeta \in \mathbb{C} | 0 < \text{Re}(\zeta) < 1 \}$, and consider a function $\Phi$ on $[0,1] \times V$ as a function on $S \times V$, so that $\Phi(\zeta,*)$ does not depend on $\text{Im}(\zeta)$, then (1.3) is equivalent to the following homogenous complex Monge-Ampère equation:

$$\left(\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi\right)^{n+1} = 0, \quad \text{in} \quad S \times V. \quad (1.4)$$

Here if we denote the projection $\mathcal{R} \times V \to V$ by $\pi_V$, then $\Omega_0 = \pi_V^*\omega_0$. This leads to the study of the following problem

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Problem 1.1 (Geodesic Problem in the Space of Kähler Potentials). Given a Kähler manifold $(\mathcal{V}, \omega_0)$ and two functions $\varphi_0, \varphi_1 \in \mathcal{H}$, find a solution to the following Dirichlet boundary value problem on the space $\mathcal{S} \times \mathcal{V}$

\[
(\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi)^{n+1} = 0, \quad \text{in} \quad \mathcal{S} \times \mathcal{V}; \\
\Phi = \varphi_1, \quad \text{on} \quad \{\text{Re}(\zeta) = 1\} \times \mathcal{V}; \\
\Phi = \varphi_0, \quad \text{on} \quad \{\text{Re}(\zeta) = 0\} \times \mathcal{V}; \\
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi \geq 0, \quad \text{in} \quad \mathcal{S} \times \mathcal{V};
\]

and $\Phi$ is independent of $\text{Im}(\zeta)$.

Remark 1.1. Solutions to Problem 1.1 may not satisfy $\Phi(\zeta, \ast) \in \mathcal{H}$, for any $\zeta \in \mathcal{S}$. So a solution is considered as a generalized (or weak) geodesic connecting $\varphi_0$ and $\varphi_1$.

If we replace the infinite strip $\mathcal{S}$ by a compact Riemann surface $\mathcal{R}$, then Problem 1.1 becomes:

Problem 1.2 (Dirichlet Problem for the Homogenous Monge-Ampère Equation on a Product Space). Given a compact Riemann surface with boundary $\mathcal{R}$, a Kähler manifold $(\mathcal{V}, \omega_0)$ and a function $F \in \mathcal{C}^\infty(\partial \mathcal{R} \times \mathcal{V})$, which satisfies

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} F(\tau, \ast) > 0, \quad \text{for any} \quad \tau \in \partial \mathcal{R},
\]

find a solution to the following Dirichlet boundary value problem

\[
(\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi)^{n+1} = 0, \quad \text{in} \quad \mathcal{R} \times \mathcal{V}; \\
\Phi = F, \quad \text{on} \quad \partial \mathcal{R} \times \mathcal{V}; \\
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi \geq 0, \quad \text{in} \quad \mathcal{R} \times \mathcal{V}.
\]

Remark 1.2. Problem 1.1 can be reduced to Problem 1.2 with $\mathcal{R} = \{\tau|1 < |\tau| < 2\}$, because there is a holomorphic covering map from $\mathcal{S}$ to $\mathcal{R} = \{\tau|1 < |\tau| < 2\}$.

Problems 1.1 and 1.2 are introduced in [22][23] and [9]. The global $C^{1,1}$ regularity of the solution is established in [3]. The result is later complemented by Blocki [1], by showing that $\Phi$ is $C^{1,1}$ providing $(\mathcal{V}, \omega_0)$ has non-negative bisectional curvature. In [2] it’s shown that, for any $\tau \in \mathcal{R}$, $\Phi(\tau, \ast)$ has uniform $C^{1,1}$ bound, providing $\omega_0$ is integral. The complete $C^{1,1}$ regularity is established by [7][6]. In [16] and [8], it is shown that $C^{1,1}$ is the optimal regularity for general solutions to Problem 1.1.

Even the optimal regularity for general solutions is known, it’s still interesting to investigate if, in some situations, we can establish higher order regularity results. Theorem 1 of [10] says that when the Riemann surface $\mathcal{R}$ is a disc, the set of smooth functions $F$ for which a smooth solution to Problem 1.2 exists is open in $C^\infty(\mathcal{R} \times \mathcal{V})$. The method cannot be directly applied to other cases, where the Riemann surfaces are not discs. Similar ideas are used in [16] to construct smooth pluri-complex Green functions in a convex domain. In [4], by applying ideas of [10] and [15], we show that for Problem 1.1 if the boundary values have small $C^6$ norm then the solution is $C^4$. In the opposite direction, in [14], we construct a family of analytic functions $\varphi_k$, $k = 1, 2, \ldots$, on $\mathcal{V}$, and $|\varphi_k|_l \to 0$, for any $l > 0$, but none of the $\varphi_k$ can be connected with 0 by a smooth geodesic.
Another question for the degenerate Monge-Ampère equation is whether the Hessian of a solution has maximal rank. In particular, for Problems 1.1 and 1.2 we want to know, if for any \( \tau \in S \) (or \( \mathcal{R} \)),

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast) > 0.
\]

This is shown to be wrong for general solutions. In [21], when the Riemann surface is a disc, a solution \( \Phi \) to Problem 1.2 is constructed, which satisfies

\[
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi = 0,
\]

in an open set in \( \mathcal{R} \times \mathcal{V} \). However in this paper we will show a positive result when the boundary value \( F \) has small \( C^2 \) norm. To present our result, we need to introduce the following approximation to Problem 1.2:

**Problem 1.3** (Dirichlet Problem for the Non-Degenerate Monge-Ampère Equation on a Product Space). Given a bounded domain with smooth boundary \( \mathcal{R} \subset \mathbb{C} \), a Kähler manifold \((\mathcal{V}, \omega_0)\) and a function \( F \in C^\infty(\partial \mathcal{R} \times \mathcal{V}) \), which satisfies for any \( \tau \in \partial \mathcal{R} \)

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast) > 0, \tag{1.13}
\]

find a solution to the following Dirichlet boundary value problem

\[
(\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi)^{n+1} = \epsilon \sqrt{-1} d\tau \wedge \overline{d\tau} \wedge \Omega_0^n, \quad \text{in} \quad \mathcal{R} \times \mathcal{V}; \tag{1.14}
\]

\[
\Phi = F, \quad \text{on} \quad \partial \mathcal{R} \times \mathcal{V}; \tag{1.15}
\]

\[
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi \geq 0, \quad \text{in} \quad \mathcal{R} \times \mathcal{V}, \tag{1.16}
\]

Here \( \epsilon \) is a positive constant and \( \tau \) is the complex coordinate on \( \mathbb{C} \).

The solution to the problem above exists by the non-degenerate Monge-Ampère equation theory, for example [12] [11], and a detailed discussion of this problem can be find in [1]. We know the problem has a smooth solution with

\[
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi > 0.
\]

So for some positive constant \( \sigma(\epsilon) \) depending on \( \epsilon \),

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast) > \sigma(\epsilon) \omega_0, \quad \text{for all} \quad \tau \in \mathcal{R}, \tag{1.17}
\]

But the lower bound estimate \( (1.17) \), established using former elliptic theory, vanishes as \( \epsilon \to 0 \). In this paper, under some conditions, we want to establish a positive lower bound estimate for

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast)
\]

which does not vanish as \( \epsilon \to 0 \). This leads to an estimate in the limiting case of \( \epsilon = 0 \), i.e. Problem 1.3 and Problem 1.1.
1.2 Main Results of the Paper

We will prove

**Theorem 1.1** (Metric Lower Bound Estimates for Monge-Ampère Equations). Given a Kähler manifold \((V, \omega_0)\), a bounded domain with smooth boundary \(\mathcal{R} \subset \mathbb{C}\) and a function \(F \in C^\infty(\partial \mathcal{R} \times V)\), there is a constant \(\hat{\delta}\) depending only on the dimension of \(V\), the curvatures of \(V\) and their covariant derivatives, so that if

\[
0 < \epsilon < \frac{\hat{\delta}}{(\text{diameter of } \mathcal{R})^2},
\]

and

\[
|F(\tau, *)|_{C^2} < \hat{\delta}, \quad \text{for all } \tau \in \partial \mathcal{R},
\]

then the solution \(\Phi\) to Problem 1.3 satisfies

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, *) \geq \frac{1}{2} \omega_0, \quad \text{for all } \tau \in \mathcal{R}.
\]

When measuring the diameter of \(\mathcal{R}\), we use the metric

\[
ds^2 = \frac{1}{2}(d\tau \otimes d\tau + d\tau \otimes d\tau).
\]

Theorem 1.1 implies the following

**Theorem 1.2** (Metric Lower Bound Estimate for Geodesics). Given a Kähler manifold \((V, \omega_0)\), there is a constant \(\delta\) depending on the dimension of \(V\), the curvatures and the covariant derivatives of the curvatures of \(\omega_0\), so that for any \(\varphi_0, \varphi_1 \in \mathcal{H}\), with

\[
|\varphi_0|_{C^2} + |\varphi_1|_{C^2} \leq \delta,
\]

measured by the metric \(\omega_0\), the solution \(\Phi\) to Problem 1.1 satisfies

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, *) \geq \frac{1}{2} \omega_0, \quad \text{for all } \tau \in S.
\]

1.3 Notation

Roman indices, \(i, j, k, \ldots\), are from 0 to \(n\), where \(n\) is the dimension of \(V\). Greek indices, \(\alpha, \beta, \ldots\), are from 1 to \(n\), except \(\tau\) which is the coordinate on \(\mathcal{R}\); and we consider the coordinate \(\tau\) on the domain \(\mathcal{R}\) as the zeroth index.

For the background metric \(\omega_0\) on the Kähler manifold \(V\), given local coordinates \(\{z^\alpha\}\), we denote

\[
\omega_0 = \sqrt{-1} b_{\alpha\beta} dz^\alpha \wedge \overline{dz^\beta}.
\]

For a solution \(\Phi\) to Problem 1.3 and for each \(\tau \in \overline{\mathcal{R}}\), denote

\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast) = \sqrt{-1} g_{\alpha\beta} dz^\alpha \wedge \overline{dz^\beta}.
\]
We denote the determinant of $g_{\alpha\beta}$ and $b_{\alpha\beta}$ by $g$ and $b$, respectively. Then the equation (1.14) takes another form

$$
\Phi_{\tau\tau} - \Phi_{\tau\beta}g^{\alpha\beta}\Phi_{\alpha\tau} = \frac{eb}{g}.
$$

On $\mathcal{R} \times \mathcal{V}$, denote

$$
\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi = \sqrt{-1}h_{ij}dz^i \wedge d\bar{z}^j.
$$

As a matrix

$$
(h_{ij}) = \begin{pmatrix}
\Phi_{\tau\tau} & \Phi_{\tau\beta} \\
\Phi_{\alpha\tau} & g_{\alpha\beta}
\end{pmatrix}.
$$

When $\epsilon > 0$, $(h_{ij})$ is a positive definite Hermitian matrix, and $\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi$ defines a Kähler metric on the product space $\mathcal{R} \times \mathcal{V}$. The Laplace operator on the product space with this metric is

$$
\tilde{L} = h_{ij}\partial_i\partial_j.
$$

In this paper, it’s more convenient to use a scalar function multiple of this Laplace operator

$$
L = \frac{eb}{g}\tilde{L}.
$$

We write

$$
L = L^{ij}\partial_i\partial_j,
$$

where

$$
(L^{ij}) = \begin{pmatrix}
1 & -g^{\alpha\beta}\Phi_{\alpha\beta} \\
g^{\alpha\beta}\Phi_{\alpha\beta} & -g^{\alpha\beta}\Phi_{\alpha\beta} + \frac{eb}{g}g^{\alpha\beta}
\end{pmatrix}.
$$

Here, $i$ is the column index and $j$ is the row index. It’s also convenient to define $p^{ij}$,

$$
(p^{ij}) = \begin{pmatrix}
1 & -g^{\alpha\beta}\Phi_{\alpha\beta} \\
g^{\alpha\beta}\Phi_{\alpha\beta} & -g^{\alpha\beta}\Phi_{\alpha\beta} + \frac{eb}{g}g^{\alpha\beta}
\end{pmatrix}.
$$

Our curvature notation is

$$
-\nabla_\alpha\nabla_\tau\partial_\theta + \nabla_\tau\nabla_\alpha\partial_\theta = R_{\alpha\tau\beta\gamma}\partial_\mu.
$$

So, for a function $u$ on $\mathcal{V}$,

$$
u_{,\alpha\beta\gamma} = u_{,\alpha\beta\gamma} - R_{\beta\gamma\alpha}^\gamma u_{,\gamma},
$$

$$
u_{,\alpha\beta\gamma\tau} = u_{,\alpha\beta\gamma\tau} - R_{\gamma\tau\beta\alpha}^\mu u_{,\mu\alpha} + R_{\beta\gamma\alpha}^\mu u_{,\mu\beta}.
$$
1.4 Structure of the Paper

To prove Theorem 1.1 we need to control the second order derivatives of \( \Phi \). So in section 3 we construct a quantity

\[
Q = \Phi_{\alpha\beta} \overline{\Phi}_{\gamma\delta} g^\alpha{}^\gamma g^\beta{}^\delta + \overline{\Phi}_{\alpha\beta} \Phi_{\gamma\delta} g^\alpha{}^\gamma g^\beta{}^\delta + \Phi_{\alpha} \overline{\Phi}_{\beta} g^\alpha{}^\beta,
\]

(1.27)

and show that when \( \epsilon \) is small enough for a very small \( \lambda \in \mathbb{R}^+ \),

\[
LQ > -\lambda Q.
\]

The covariant derivatives in (1.27) are taken with respect to the metric \( \omega_0 \). To estimate \( LQ \), we need to compute and estimate \( L(\Phi_{\alpha\beta}) \), \( L(\Phi_{\alpha\bar{\beta}}) \) and \( L(\Phi_{\alpha}) \), these are done in section 2.1, 2.2 and 2.3 respectively. How operator \( L \) acts on tensors is explained in section 2.

In Appendix A we carry out computations in the situation that \( (\mathcal{V}, \omega_0) \) is a 1-dimensional flat torus, in this case we get more complete results. Actually, the main idea of this paper is to generalize Theorem A.2 to general Kähler manifolds.

In Appendix B we study the limiting case of \( \epsilon = 0 \). In this case, there is a leaf structure associated to a solution to the homogeneous complex Monge-Ampère equation, and the computation is simpler.

2 Equations for \( \Phi_{\alpha\beta} \), \( \Phi_{\alpha\bar{\beta}} \) and \( \Phi_{\theta} \)

In this section we compute and estimate \( L(\Phi_{\alpha\beta}) \), \( L(\Phi_{\alpha\bar{\beta}}) \) and \( L(\Phi_{\alpha}) \). Here we consider \( \Phi_{\alpha\beta} \) as a section of the bundle \( \pi^*(T^{1,0}(\mathcal{V}) \otimes T^{0,1}(\mathcal{V})) \), consider \( \Phi_{\alpha\bar{\beta}} \) as a section of the bundle \( \pi^*(T^{1,0}(\mathcal{V}) \otimes T^{1,0}(\mathcal{V})) \) and consider \( \Phi_{\theta} \) as a section of the bundle \( \pi^*(T^{1,0}(\mathcal{V})) \). These bundles have natural metrics induced by the metric \( \omega_0 \):

\[
<dz^\alpha, \overline{dz^\beta}> = b^\alpha{}^\beta,
\]

(2.1)

\[
<dz^\alpha \otimes dz^\beta, \overline{dz^\gamma} \otimes d\overline{z^\delta}> = b^\alpha{}^\gamma b^\beta{}^\delta,
\]

(2.2)

\[
<dz^\alpha \otimes \overline{dz^\beta}, dz^\gamma \otimes d\overline{z^\delta}> = b^\alpha{}^\gamma b^\beta{}^\delta.
\]

(2.3)

We denote the norm of a tensor by \( |\cdot| \). For \( \mathcal{F} = \mathcal{F}_{\alpha\beta} dz^\alpha \otimes \overline{dz^\beta} \) and \( \mathcal{G} = \mathcal{G}_{\alpha\beta} dz^\alpha \otimes d\overline{z^\beta} \),

\[
|\mathcal{F}|^2 = \mathcal{F}_{\alpha\beta} \overline{\mathcal{F}_{\alpha\beta}} b^\alpha{}^\gamma b^\gamma{}^\beta,
\]

(2.4)

\[
|\mathcal{G}|^2 = \mathcal{G}_{\alpha\beta} \overline{\mathcal{G}_{\alpha\beta}} b^\alpha{}^\gamma b^\gamma{}^\beta.
\]

(2.5)

The Chern connection on the bundle \( T^{1,0}(\mathcal{V}) \) naturally induces connections on \( \pi^*(T^{1,0}(\mathcal{V}) \otimes T^{0,1}(\mathcal{V})) \) and \( \pi^*(T^{1,0}(\mathcal{V}) \otimes T^{1,0}(\mathcal{V})) \), and for these connections we have

\[
\nabla^\tau (dz^\alpha \otimes d\overline{z^\beta}) = 0,
\]

(2.6)

\[
\nabla^\theta (dz^\alpha \otimes d\overline{z^\beta}) = (\nabla^\theta dz^\alpha) \otimes d\overline{z^\beta};
\]

(2.7)

\[
\nabla^\tau (dz^\alpha \otimes dz^\beta) = 0,
\]

(2.8)

\[
\nabla^\theta (dz^\alpha \otimes dz^\beta) = (\nabla^\theta dz^\alpha) \otimes dz^\beta + dz^\alpha \otimes (\nabla^\theta dz^\beta);
\]

(2.9)

\[
\nabla^\tau (dz^\alpha \otimes d\overline{z^\beta}) = 0,
\]

(2.10)

\[
\nabla^\theta (dz^\alpha \otimes d\overline{z^\beta}) = 0.
\]
The $\nabla_\tau, \nabla_\tau^{\dagger}$ derivatives are zero, because the metric $\omega_0$ is independent of the $\tau$ variable. The connections are also independent of $\tau$. As a consequence, $\nabla_\tau, \nabla_\tau^{\dagger}$ commute with all the $\nabla_\alpha, \nabla_\beta$, so all components of the curvature tensor, containing $\tau, \tau^{\dagger}$, vanishes, i.e.

$$R_{\tau \tau^{\dagger} \tau^{\dagger} \tau} = R_{\tau \tau^{\dagger} \tau^{\dagger} \tau} = R_{\beta \tau^{\dagger} \tau^{\dagger} \tau} = R_{\beta \tau^{\dagger} \tau^{\dagger} \tau} = 0. \quad (2.10)$$

For norms of curvature tensors $R$, let

$$|R|^2 = \sqrt{g_{\alpha \mu} R_{\alpha \mu \beta \gamma} b^\alpha b^\beta b^\gamma b^{\dagger \gamma}}. \quad (2.11)$$

For covariant derivatives of curvatures, the definition is analogous.

When acting on sections of bundles

$$L = L^i \nabla_\tau \nabla_i.$$

In addition, we denote

$$B_{\alpha \beta} = \Phi_{\alpha \beta}, \quad A_{\alpha \beta} = \Phi_{\alpha \beta}. \quad (2.12)$$

### 2.1 Equations for $\Phi_{\alpha \beta}$

We apply $\partial_\theta$ to (2.11), and get

$$\Phi_{\tau \theta} - \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} = g^{\alpha \beta} \Phi_{\alpha \tau} = 0. \quad (2.13)$$

Here the covariant derivatives are taken with respect to the Chern connection of the metric (2.2).

Then we apply $\partial_\tau$ to (2.12) and get

$$\Phi_{\tau \tau \theta} - \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\tau \theta} g^{\alpha \beta} \Phi_{\alpha \tau} = -g^{\alpha \beta} \Phi_{\alpha \tau}. \quad (2.14)$$

Then we commute indices. In all terms containing $\theta, \tau$, we properly move $\theta, \tau$ leftwards and get

$$\Phi_{\theta \tau \tau} - \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.19)$$

$$\Phi_{\theta \tau \tau} - \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.20)$$

$$\Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\theta \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.21)$$

$$\Phi_{\tau \tau \tau} - \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.22)$$

$$\Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.23)$$

$$\Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} - \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} + \Phi_{\tau \tau \tau} g^{\alpha \beta} \Phi_{\alpha \tau} = (2.24)$$

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We get

\[ - \Phi_{\gamma\eta} g^{\alpha\eta} \Phi_{\phi\gamma} g^{\phi\eta} \Phi_{\tau\alpha} \]  (2.25)

\[ - \Phi_{\gamma\eta} g^{\alpha\eta} \left( R_{\eta\gamma\phi} \Phi_{\phi\tau\eta} + \Phi_{\phi\tau\xi} R_{\xi\gamma\phi} \Phi_{\eta\tau\phi} + R_{\phi\gamma\xi} \Phi_{\xi\tau\gamma} R_{\xi\phi\eta} \Phi_{\eta\tau\xi} \right) g^{\alpha\eta} \Phi_{\tau\alpha} g^{\phi\eta} \]  (2.26)

\[ + \Phi_{\gamma\eta} g^{\alpha\eta} \Phi_{\phi\eta} g^{\phi\eta} \Phi_{\tau\alpha} \]  (2.27)

\[ = - \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\tau\alpha} \right) + \frac{eb}{g} \left( g^{\alpha\eta} \left( R_{\phi\alpha\gamma} \Phi_{\xi\tau\phi} + (R_{\phi\alpha\theta} \Phi_{\phi\theta})_{\eta} \right) \right) \]  (2.28)

\[ + \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\eta} g^{\phi\eta} \Phi_{\tau\alpha} \right) + \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\eta} g^{\phi\eta} \Phi_{\tau\alpha} \right) \]  (2.29)

\[ + \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\eta} g^{\phi\eta} \Phi_{\tau\alpha} \right) - \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\eta} g^{\phi\eta} \Phi_{\tau\alpha} \right). \]  (2.30)

When commuting indices, we used (1.25) (1.26) and (2.10). The terms in (2.30) are fictitious, they add up to be zero.

We will simplify the equation above to (2.36). To do this, we need the following notation: \((**)_k\) stands for the \(k\)-th term in \((**), including the sign. For example

\[ (2.28)_1 = - \frac{eb}{g} \left( g^{\alpha\eta} \Phi_{\phi\tau\alpha} \right), \quad (2.19)_2 = - \Phi_{\phi\tau\gamma \alpha} g^{\alpha\gamma} \Phi_{\phi\tau\gamma}. \]  (2.31)

With this notation

\[ (2.29)_1 + (2.30)_2 = F_{\phi\tau}. \]  (2.32)

We get

\[ L^\gamma A_{\phi\gamma,\gamma} = A_{\phi\gamma,\gamma} g^{\gamma\phi} A_{\xi\gamma,\gamma} + B_{\phi_{\mu\xi,\gamma}} g^{\gamma\phi} B_{\xi\gamma,\gamma} + F_{\phi\tau} + U_{\phi\tau}. \]  (2.33)

The expressions for \(F, U\) are

\[ F_{\phi\tau} = \frac{eb}{g} \left( g^{\gamma\phi} \Phi_{\gamma \phi} g^{\gamma\phi} \Phi_{\phi \gamma} \right), \]  (2.34)

and

\[ U_{\phi\tau} = - R_{\phi\tau\alpha} \Phi_{\phi\gamma} g^{\gamma\phi} \Phi_{\tau\alpha} \left( R_{\phi\tau\alpha} \Phi_{\phi\gamma} + (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} + (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \]  (2.35)

\[ - (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \]  (2.36)

\[ + \frac{eb}{g} \left( (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \right) \]  (2.37)

\[ - (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \]  (2.38)

\[ + \frac{eb}{g} \left( (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \right) \]  (2.39)

\[ - (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \]  (2.40)

\[ - (R_{\phi\tau\alpha} \Phi_{\phi\gamma} \right) g^{\gamma\phi} \Phi_{\tau\alpha} \]  (2.41)
\[-(R_{\tau\sigma\rho\mu}^{\tau\sigma\rho\mu})g^{\alpha\beta}\Phi_{\alpha\beta} \Phi_{\rho\sigma}\Phi_{\mu\nu}g^{\mu\nu} \Phi_{\tau\sigma} \Phi_{\rho\mu} \Phi_{\sigma\nu} \Phi_{\tau\nu} \Phi_{\rho\sigma} \Phi_{\mu\nu} \Phi_{\alpha\beta} \Phi_{\nu\sigma} \Phi_{\rho\mu} \Phi_{\mu\nu} (2.42)\]

\[-(R_{\tau\sigma\rho\mu}^{\tau\sigma\rho\mu})g^{\alpha\beta}\Phi_{\alpha\beta} \Phi_{\rho\sigma}\Phi_{\mu\nu}g^{\mu\nu} \Phi_{\tau\sigma} \Phi_{\rho\mu} \Phi_{\sigma\nu} \Phi_{\tau\nu} \Phi_{\rho\sigma} \Phi_{\mu\nu} \Phi_{\alpha\beta} \Phi_{\nu\sigma} \Phi_{\rho\mu} \Phi_{\mu\nu} (2.43)\]

\[+(R_{\tau\sigma\rho\mu}^{\tau\sigma\rho\mu})g^{\alpha\beta}\Phi_{\alpha\beta} \Phi_{\rho\sigma}\Phi_{\mu\nu}g^{\mu\nu} \Phi_{\tau\sigma} \Phi_{\rho\mu} \Phi_{\sigma\nu} \Phi_{\tau\nu} \Phi_{\rho\sigma} \Phi_{\mu\nu} \Phi_{\alpha\beta} \Phi_{\nu\sigma} \Phi_{\rho\mu} \Phi_{\mu\nu} (2.44)\]

**Remark 2.1.** F vanishes when \( \text{dim}(\mathcal{V}) = 1 \), and U vanishes when the metric \( \omega_0 \) is flat (curvature = 0), so when the Kähler manifold \((\mathcal{V},\omega_0)\) is a 1-dimensional flat torus we have more complete results. This is discussed in Appendix A.

In section 3, we will plug (2.36) into the expression of \( LQ \) and try to get \( LQ > -\lambda Q \).

So we need to use non-negative terms to control indefinite terms. The main non-negative terms from \( LQ \) will be

\[E = B_{\theta\gamma,\pi}B_{\xi,\beta}g^{\beta\gamma}g^{\theta\xi} + A_{\theta\gamma,\alpha}A_{\xi,\beta}g^{\beta\gamma}g^{\theta\xi}, \quad (2.45)\]

\[P = B_{\theta\gamma,\pi}B_{\xi,\beta}p^{\beta\gamma}g^{\theta\xi} + A_{\theta\gamma,\alpha}A_{\xi,\beta}p^{\beta\gamma}g^{\theta\xi}, \quad (2.46)\]

and

\[T = \Phi_{\tau\sigma}^{\tau\sigma}g^{\alpha\beta} + \Phi_{\tau\alpha}^{\tau\alpha}g^{\alpha\beta}. \quad (2.47)\]

The \( p^{\beta\gamma} \) in (2.46) is defined by (1.24).

In the previous expression of \( U \), we need to combine (2.41) with (2.42) and combine (2.43) with (2.44), then we can control \( U \) by \( E, T, P \) and \( Q \). By changing some dummy indices we get

\[(2.41) + (2.42) = (\Phi_{\tau\sigma}^{\tau\sigma}g^{\mu\nu})(R_{\alpha\beta\gamma\delta}^{\mu\nu})(\Phi_{\gamma\delta}^{\gamma\delta}g^{\alpha\beta})(\Phi_{\mu\nu}^{\mu\nu}(\Phi_{\rho\sigma}^{\rho\sigma}g^{\gamma\sigma})), \quad (2.48)\]

\[(2.43) + (2.44) = -(\Phi_{\alpha\beta}^{\alpha\beta})(R_{\alpha\beta\gamma\delta}^{\rho\sigma})(\Phi_{\gamma\delta}^{\gamma\delta}g^{\alpha\beta})(\Phi_{\rho\sigma}^{\rho\sigma}g^{\gamma\sigma})), \quad (2.49)\]

Then providing that \( Q < 1 \) and

\[\frac{1}{2}(b_{\alpha\beta}) < (g_{\alpha\beta}) < \frac{3}{2}(b_{\alpha\beta}) \quad (2.50)\]

we can find a constant \( C_A \), which only depends on the dimension of \( \mathcal{V} \), so that

\[|F| \leq C_A \epsilon b g E, \quad (2.51)\]

\[|U| \leq C_A (\text{max} |R| + \text{max} |R|^2 + \text{max} |\nabla R|) \left( T + \sqrt{Q} \sqrt{T} \sqrt{P} + \frac{\epsilon b}{g} \sqrt{Q} \right). \quad (2.52)\]
2.2 Equations for $\Phi_{\mu\beta}$

The computation for $B = \Phi_{\alpha\beta}dz^\alpha \otimes dz^\beta$ is similar to the previous computation for $A$. Apply $\partial_\theta$ and $\nabla_\gamma$ to equation (1.21) gives that

$$L^\gamma B_{\theta\gamma,j} = B_{\theta\mu,j}g^{\mu\nu}A_{\gamma,j,\nu}L^\gamma + A_{\theta\eta,j}g^{\rho\pi}B_{\nu,j,\rho}L^\gamma + H_{\theta\gamma} + V_{\theta\gamma},$$

(2.53)

where

$$H_{\theta\gamma} = \frac{eb}{g} \left( A_{\alpha\beta,\theta} A_{\mu\eta,\gamma} g^{\alpha\pi} g^{\beta\gamma} - A_{\gamma\beta,\alpha} A_{\theta\eta,\gamma} g^{\alpha\pi} g^{\beta\gamma} \right),$$

(2.54)

and

$$V_{\theta\gamma} = -R_{\gamma\mu\beta} \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} \left( A_{\theta,\pi,\eta} - A_{\theta,\eta,\pi} g^{\eta\pi} \Phi_{\pi,\gamma} \right) g^{\alpha\pi} \Phi_{\theta,\tau} - R_{\gamma\mu\beta} \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} \left( A_{\theta,\pi,\eta} - A_{\theta,\eta,\pi} g^{\eta\pi} \Phi_{\pi,\gamma} \right) g^{\alpha\pi} \Phi_{\theta,\tau} + \frac{eb}{g} \Phi_{\lambda} \left( R_{\theta,\pi,\gamma,\eta} + R_{\gamma,\pi,\lambda} \Phi_{\theta,\eta} \right) g^{\alpha\pi} g^{\beta\gamma} + R_{\gamma\mu\beta} \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} \Phi_{\theta,\tau} - R_{\gamma\mu\beta} \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} \Phi_{\theta,\tau} - R_{\gamma\mu\beta} \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} \Phi_{\theta,\tau} + \left( \nabla_\gamma R_{\theta,\pi,\gamma} \right) \Phi_{\gamma,\theta,\alpha} g^{\alpha\beta} + \frac{eb}{g} g^{\alpha\beta} \left( \nabla_\gamma R_{\theta,\pi,\gamma} \right) \Phi_{\mu} + \frac{eb}{g} g^{\alpha\beta} \left( R_{\beta,\gamma,\theta}^{\mu} \Phi_{\mu,\alpha} + R_{\beta,\gamma,\theta}^{\mu} \Phi_{\mu,\theta} + R_{\beta,\gamma,\theta}^{\mu} \Phi_{\mu,\gamma} \right).$$

(2.55 - 2.60)

Remark 2.2. $V$ and $H$ are both symmetric tensors (2.59 is symmetric in $\theta$ and $\gamma$ because of the second Bianchi identity). $H$ vanishes when $\dim(V) = 1$, $V$ vanishes when $\omega_0$ is flat, analogous to $U$ and $F$.

We have the following estimates for $H$ and $V$. There is a constant $C_{Bn}$, which only depends on the dimension of the Kähler manifold $V$, so that, when $Q<1$ and (2.50) is satisfied,

$$|H| \leq C_{Bn} \frac{eb}{g} E.$$

(2.61)

$$|V| \leq C_{Bn} (\max |R| + \max |\nabla R|) \left( T + \sqrt{QTP} + \frac{eb}{g} \left( \sqrt{Q} + \sqrt{QE} \right) \right).$$

(2.62)

We also need to compute $L\overline{B}$. By simply commuting indices, we have

$$L^\gamma \overline{B}_{\theta\gamma,j} = L^\gamma \overline{\Phi}_{\theta\gamma,j} = L^\gamma \overline{\Phi}_{\theta\gamma,j} + L^\gamma \left( R_{j\gamma0}^{\alpha} \Phi_{\mu,\alpha} + R_{j\gamma1}^{\alpha} \Phi_{\mu,\beta} \right).$$

(2.63)

$L^\gamma \left( R_{j\gamma0}^{\alpha} \Phi_{\mu,\alpha} + R_{j\gamma1}^{\alpha} \Phi_{\mu,\beta} \right)$ equals $L^\gamma \overline{\Phi}_{\theta\gamma} \left( R_{j\gamma0}^{\alpha} \Phi_{\mu,\alpha} + R_{j\gamma1}^{\alpha} \Phi_{\mu,\beta} \right)$ because of (2.10). We denote

$$W_{\theta\gamma} = V_{\theta\gamma} + L^\gamma \left( R_{j\gamma0}^{\alpha} \Phi_{\mu,\alpha} + R_{j\gamma1}^{\alpha} \Phi_{\mu,\beta} \right).$$

(2.64)
Then

\[ L^\gamma \hat{B}_{\theta\gamma,j} = L^\gamma \hat{B}_{\theta\gamma,i} g^\alpha\beta A_{\alpha\beta,j} + L^\gamma \hat{B}_{\theta\gamma,i} g^\alpha\beta A_{\alpha\beta,j} + H_{\theta\gamma} + W_{\theta\gamma} \]  

(2.65)

The estimate of \( W \) depends on (2.62) and an estimate of \( L^{\beta\gamma} (R_{\beta\pi\iota} \Phi_{\mu\gamma} + R_{\beta\pi\iota} \Phi_{\mu\theta}) \) in the following. By (1.23), we have

\[ (L^{\alpha\beta}) = g^{\alpha\beta} \Phi \tau \phi \mu + \epsilon_{\beta\phi} g^{\alpha\beta} \]  

(2.66)

So for a constant \( C_{\Pi} \), only depending on the dimension of \( \mathcal{V} \), when (2.50) is satisfied, we have

\[ |L^{\beta\gamma} (R_{\beta\pi\iota} \Phi_{\mu\gamma} + R_{\beta\pi\iota} \Phi_{\mu\theta})| \leq C_{\Pi} \cdot \left( T + \frac{eb}{g} \right) \cdot \max |R| \cdot \sqrt{Q} \]  

(2.67)

2.3 Equations for \( \Phi_\theta \)

By commuting indices in (2.13), we get

\[ L^{\gamma} \Phi_{\theta,i,j} = R_{\gamma\phi} \psi g^{\alpha\beta} \Phi_{\phi\gamma} g^{\alpha\beta} \Phi_{\phi\gamma} + \frac{eb}{g} R_{\gamma\phi} \psi \mu \Phi_{\mu} g^{\alpha\beta}. \]  

(2.68)

We denote this by

\[ L^{\gamma} \Phi_{\theta,i,j} = S_{\theta}, \]

and for some constant \( C_{Gn} \), only depending on dimension,

\[ |S| \leq C_{Gn} \cdot \max |R| \cdot \left( T + \frac{eb}{g} \right) \sqrt{Q}, \]  

(2.69)

providing (2.50) is satisfied. For \( \Phi_{\gamma} \), we have

\[ L^{\gamma} \Phi_{\gamma,i,j} = 0 \]  

(2.70)

2.4 Summary

To sum up, we get

\[ L^{\gamma} A_{\theta\gamma,i,j} = A_{\theta\gamma,i} g^{\alpha\beta} A_{\alpha\beta,j} L^{\gamma} + B_{\theta\gamma,i} g^{\alpha\beta} B_{\alpha\beta,j} L^{\gamma} + F_{\theta\gamma} + U_{\theta\gamma}, \]  

(2.71)

\[ L^{\gamma} B_{\theta\gamma,i,j} = B_{\theta\gamma,i} g^{\alpha\beta} A_{\alpha\beta,j} L^{\gamma} + A_{\theta\gamma,i} g^{\alpha\beta} B_{\alpha\beta,j} L^{\gamma} + H_{\theta\gamma} + V_{\theta\gamma}, \]  

(2.72)

and

\[ L^{\gamma} B_{\theta\gamma,i,j} = B_{\theta\gamma,i} g^{\alpha\beta} A_{\alpha\beta,j} L^{\gamma} + B_{\theta\gamma,i} g^{\alpha\beta} A_{\alpha\beta,j} L^{\gamma} + H_{\theta\gamma} + W_{\theta\gamma}, \]  

(2.73)

Providing \( Q < 1 \) and

\[ \frac{1}{2} (b_{\alpha\beta}) < (g_{\alpha\beta}) < \frac{3}{2} (b_{\alpha\beta}), \]  

(2.74)
there is a constant $C_n$ only depending on $\dim(\mathcal{V})$, so that
\[
|F| + |H| + |U| + |V| + |W| + |S| \leq C_n \cdot \left( \max |R| + \max |\nabla R| \right) \frac{eb}{g} \left( \sqrt{Q} + \sqrt{Q_E} \right) \tag{2.75}
\]
\[
+ C_n \left( \max |R| + \max |R|^2 + \max |\nabla R| \right) \left( T + \sqrt{QT} \right) + C_n \frac{eb}{g} E. \tag{2.76}
\]

We denote $C_R = C_n \cdot \left( 1 + \max |R| + \max |R|^2 + \max |\nabla R| \right)$, then
\[
|F| + |H| + |U| + |V| + |W| + |S| \leq C_R \left( T + \sqrt{QT} + \frac{eb}{g} \left( \sqrt{Q} + \sqrt{Q_E} + E \right) \right). \tag{2.77}
\]

3 Estimates of Second Order Derivatives of $\Phi$

In this section, we estimate the second order derivatives of $\Phi$ by estimating the quantity
\[
Q = \Phi_{\alpha \beta \gamma} g^{\alpha \eta} g^{\beta \tau} + \Phi_{\alpha \beta} g^{\alpha \eta} g^{\beta \gamma} + \Phi_{\alpha} g^{\alpha \gamma}, \tag{3.1}
\]
here the covariant derivatives $\Phi_{\alpha \beta \gamma}$ are with respect to the Levi-Civita connection of $\omega_0$. First, with computations and the estimate \eqref{2.77}, we get that when some assumptions are satisfied,
\[
LQ > -\lambda Q, \tag{3.2}
\]
for a small constant $\lambda$. This is the main content of section \ref{3.1} Then with \eqref{3.2}, we prove Proposition \ref{3.1} in section \ref{3.2} which is an apriori estimate showing the value of $Q$ in $\mathcal{R} \times \mathcal{V}$ can be controlled by the value of $Q$ on $\partial \mathcal{R} \times \mathcal{V}$, which only depends on the boundary value $F$. Then with a continuity argument we prove Theorem \ref{1.1}.

3.1 Computation and Estimates for $LQ$

We denote
\[
Q_B = \Phi_{\alpha \beta \gamma} g^{\alpha \eta} g^{\beta \tau}, \tag{3.3}
\]
\[
Q_A = \Phi_{\alpha \beta} g^{\alpha \eta} g^{\beta \gamma}, \tag{3.4}
\]
\[
Q_G = \Phi_{\alpha} g^{\alpha \gamma}, \tag{3.5}
\]
and do the computation separately.

First, for $Q_B$, we have
\[
LQ_B = B_{\alpha \beta \gamma} \overrightarrow{B_{\theta \mu \eta}} g^{\alpha \eta} g^{\beta \tau} L \overrightarrow{L} \tag{3.6}
\]
\[
+ \left( B_{\alpha \beta, \gamma} - B_{\alpha 0} g^{\alpha \eta} A_{\gamma, \xi} - B_{\beta 0} g^{\beta \gamma} A_{\alpha, \xi} \right) \left( \overrightarrow{B_{\theta \mu, \eta}} - A_{\phi, \eta} g^{\phi \tau} B_{\theta \mu} - A_{\phi, \eta} g^{\phi \tau} B_{\mu, \eta} \right) g^{\alpha \eta} g^{\beta \tau} \overrightarrow{L} \overrightarrow{L} \tag{3.7}
\]
\[
- B_{\alpha \beta} g^{\alpha \eta} \overrightarrow{B_{\gamma, \xi}} g^{\beta \gamma} B_{\xi, \gamma} - B_{\alpha \beta} g^{\alpha \eta} B_{\theta \mu} g^{\beta \tau} B_{\phi, \gamma} - B_{\alpha \beta} g^{\alpha \eta} B_{\theta \mu} g^{\beta \tau} B_{\mu, \gamma} \tag{3.8}
\]
\[
+ B_{\alpha \beta} g^{\alpha \eta} B_{\theta \mu} + B_{\alpha \beta} g^{\alpha \eta} g^{\beta \tau} \overrightarrow{W_{\theta \mu}} - H_{\alpha \beta} g^{\alpha \eta} g^{\beta \tau} B_{\theta \mu} - V_{\alpha \beta} g^{\alpha \eta} g^{\beta \tau} \overrightarrow{B_{\theta \mu}} \tag{3.9}
\]
\[
- B_{\alpha \beta} g^{\alpha \eta} F_{\phi, \gamma} g^{\beta \tau} B_{\theta \mu} - B_{\alpha \beta} g^{\alpha \eta} U_{\phi, \gamma} g^{\beta \tau} B_{\theta \mu} \tag{3.10}
\]
\[
- B_{\alpha \beta} g^{\alpha \eta} F_{\phi, \gamma} g^{\beta \tau} g^{\alpha \eta} B_{\theta \mu} - B_{\alpha \beta} g^{\alpha \eta} U_{\phi, \gamma} g^{\beta \tau} g^{\alpha \eta} B_{\theta \mu}. \tag{3.11}
\]
The computation is straightforward, we differentiate \((3.3)\) with Leibniz rule and plug in \((2.71)\), then we note that there are twelve terms, which do not contain \(U, V, W, F, H\), cancel with each other. A simplified computation, when \(U, V, W, F, H\) all vanish, is in Appendix B.

There is a constant \(C_B\), only depending on the dimension, so that

\[
LQ_B \geq (1 - \widetilde{C}_B Q) B_{\alpha\beta\gamma} \overline{D}_{\theta\mu\nu} g^{\alpha\beta} g^{\rho\theta} L^\gamma - \widetilde{C}_B (|W| + |H| + |V| + |F| + |U|) \sqrt{Q},
\]

(3.12)

providing that \(Q < 1\).

For \(Q_A\), by differentiating the expression \((3.4)\) and plugging in \((2.71)\), we get

\[
LQ_A = 2L^\gamma \left( \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} \right)
\]

(3.13)

\[
= 2L^\gamma \left( \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} \right)
\]

(3.14)

\[
+ A_{\alpha\beta\gamma} g^{\rho\sigma} A_{\rho\sigma\tau} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} + B_{\alpha\beta\gamma} g^{\rho\sigma} D_{\rho\sigma\tau} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} \right)
\]

(3.15)

\[
- \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma}
\]

(3.16)

\[
+ \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} + B_{\alpha\beta\gamma} g^{\rho\sigma} D_{\rho\sigma\tau} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma}
\]

(3.17)

\[
+ \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma}
\]

(3.18)

\[
- \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} + A_{\alpha\beta\gamma} g^{\rho\sigma} A_{\rho\sigma\tau} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma}
\]

(3.19)

\[
+ (U_{\alpha\beta} + F_{\alpha\beta}) \Phi_{\rho\sigma\tau} g^{\alpha\beta} g^{\rho\sigma} - \Phi_{\alpha\beta} \Phi_{\rho\sigma\tau} g^{\alpha\beta} \sqrt{Q} (F_{\rho\tau} + U_{\rho\tau}) g^{\rho\sigma} g^{\rho\sigma}
\]

(3.20)

Then under the assumption that \(Q < 1\) and \((2.50)\), there is a constant \(\tilde{C}_A\), so that

\[
LQ_A \geq A_{\alpha\beta\gamma} g^{\alpha\beta} A_{\rho\sigma\tau} g^{\rho\sigma} (2 - \tilde{C}_A \sqrt{Q}) L^\gamma - \tilde{C}_A B_{\alpha\beta\gamma} g^{\alpha\beta} D_{\rho\sigma\tau} g^{\rho\sigma} \sqrt{Q} L^\gamma - \tilde{C}_A (|F| + |U|) \sqrt{Q}
\]

(3.21)

The estimate for \(LQ_G\) is more complicated, because we want \(T\), defined in \((2.47)\), to be the dominating term. The complete expression for \(LQ_G\) is

\[
LQ_G = \frac{1}{2} \Phi_{\alpha\beta\gamma} \Phi_{\mu\nu\rho} g^{\alpha\beta} L^\gamma + \frac{1}{2} \Phi_{\alpha\beta\gamma} \Phi_{\mu\nu\rho}^\mu g^{\alpha\beta} L^\gamma + S_\alpha \Phi_{\mu\nu\rho}
\]

(3.22)

\[
+ \frac{1}{2} \left( \Phi_{\alpha\beta\gamma} - 2 \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) \left( \Phi_{\rho\sigma\tau} - 2 \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) g^{\alpha\beta} L^\gamma
\]

(3.23)

\[
+ \frac{1}{2} \left( \Phi_{\alpha\beta\gamma} - 2 \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) \left( \Phi_{\rho\sigma\tau} - 2 \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) g^{\alpha\beta} L^\gamma
\]

(3.24)

\[
- \left( \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) \left( \Phi_{\rho\sigma\tau} A_{\alpha\beta\gamma} \right) g^{\alpha\beta} L^\gamma - \left( \Phi_{\rho\sigma} A_{\alpha\beta\gamma} \right) \left( \Phi_{\rho\sigma\tau} A_{\alpha\beta\gamma} \right) g^{\alpha\beta} L^\gamma
\]

(3.25)

\[
- \Phi_{\alpha\beta\gamma} \Phi_{\rho\sigma\tau} g^{\alpha\beta} \left( A_{\rho\sigma\tau} g^{\alpha\beta} A_{\alpha\beta\gamma} + B_{\rho\sigma\tau} g^{\alpha\beta} D_{\rho\sigma\tau} + U_{\rho\sigma\tau} + F_{\rho\sigma\tau} \right) L^\gamma.
\]

(3.26)

In the following, we will simplify the expression above into an inequality. First, there is a constant \(\tilde{C}_G\), depending only on the dimension of \(V\), such that

\[
LQ_G \geq \frac{1}{2} \Phi_{\alpha\beta\gamma} \Phi_{\mu\nu\rho} g^{\alpha\beta} L^\gamma + \frac{1}{2} \Phi_{\alpha\beta\gamma} \Phi_{\mu\nu\rho}^\mu g^{\alpha\beta} L^\gamma - \tilde{C}_G (P + \frac{c_b}{g} E) - \tilde{C}_G (|S| + |U| + |F|) \sqrt{Q},
\]

(3.27)
providing that the condition (2.50) is satisfied and $Q < 1$.

In the estimate (3.27), we want to replace
\[
\frac{1}{2} \Phi_{\alpha i} \hat{\Phi}_{\beta j} g^{\alpha \beta} L^{ij} + \frac{1}{2} \Phi_{\alpha j} \hat{\Phi}_{\beta i} g^{\alpha \beta} L^{ij}
\]
by a multiple of
\[
T = \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} + \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta},
\]
which is more convenient to use, as we saw in section 2.

Because $(L^{ij}) \geq (g^{ij})$, we have
\[
\frac{1}{2} \Phi_{\alpha i} \hat{\Phi}_{\beta j} g^{\alpha \beta} L^{ij} + \frac{1}{2} \Phi_{\alpha j} \hat{\Phi}_{\beta i} g^{\alpha \beta} L^{ij} \geq \frac{1}{2} \Phi_{\alpha i} \hat{\Phi}_{\beta j} g^{\alpha \beta} g^{ij} + \frac{1}{2} \Phi_{\alpha j} \hat{\Phi}_{\beta i} g^{\alpha \beta} g^{ij} (3.28)
\]
\[
\geq \frac{1}{2} \left( \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} - \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} g^{\tau \gamma} \Phi^{\gamma \sigma} - \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} g^{\tau \gamma} \Phi^{\gamma \sigma} \Phi^{\eta \tau} \right) (3.30)
\]
\[
+ \frac{1}{2} \left( \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} - \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} g^{\tau \gamma} \Phi^{\gamma \sigma} - \Phi_{\alpha \tau} \hat{\Phi}_{\beta \tau} g^{\alpha \beta} g^{\tau \gamma} \Phi^{\gamma \sigma} \Phi^{\eta \tau} \right) (3.31)
\]
\[
\geq T \left( \frac{1}{2} - \tilde{C}_G \sqrt{Q} \right),
\]
for a constant $\tilde{C}_G$ only depending on the dimension of $\mathcal{V}$. Then plugging the estimate above into (3.27), we get that
\[
LQ \geq T \left( \frac{1}{2} - \tilde{C}_G \sqrt{Q} \right) - \tilde{C}_G (|S| + |U| + |F|) \sqrt{Q},
\]
(3.33)

Put (3.12) (3.21) and (3.33) together, we get that for $\tilde{C} = 2(\tilde{C}_B + \tilde{C}_A + \tilde{C}_G + \tilde{C}_G)$
\[
LQ \geq \left( \frac{1}{2} - \tilde{C} \sqrt{Q} \right) \left( T + P + \frac{eb}{g} E \right) - \tilde{C} (|U| + |V| + |W| + |F| + |H| + |S|) \sqrt{Q},
\]
(3.34)

providing $Q < 1$ and the condition (2.50) is satisfied. Plug in the estimate (2.77), we get, for $\tilde{C} = (\tilde{C} + 1)(C_R + 1)2^{n+2}$
\[
LQ \geq \left( \frac{1}{2} - \tilde{C} \sqrt{Q} \right) (T + P + \frac{eb}{g} E) - \tilde{C} eQ.
\]
(3.35)

So, if $Q \leq \frac{1}{4\tilde{C}^2}$,
\[
LQ \geq -\tilde{C} eQ.
\]
(3.36)

### 3.2 Estimating $Q$

With the inequality (3.36), we can derive the following apriori estimate for $Q$. Here we denote the diameter of $\mathcal{R}$ by $d_{\mathcal{R}}$, and assume that $\mathcal{R}$ is contained in the ball $\{ |\tau| \leq d_{\mathcal{R}} \} \subset \mathbb{C}$. 
Proposition 3.1. There is a constant $\delta$ depending on the dimension, and the curvatures and the covariant derivatives of the curvatures of the manifold $V$, so that for a solution $\Phi$ to Problem 1.3, if the function $Q$, defined by (3.1), satisfies $Q \leq \delta^2$, and $\epsilon < \frac{\delta}{\delta R^2}$, then

$$Q \leq 2 \max_{\partial R \times V} Q, \quad \text{in } R \times V.$$  \hspace{1cm} (3.37)

Proof. First, we require that $\delta < \frac{1}{4}$, then $Q \leq \delta^2$ implies $Q < \frac{1}{10}$. So by (3.1),

$$\Phi_{\alpha\beta} \Phi_{\gamma\delta} g^{\alpha\beta} g^{\gamma\delta} < \frac{1}{10}. \hspace{1cm} (3.38)$$

At any point, we can choose local coordinates, so that

$$\Phi_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}, \quad b_{\alpha\beta} = \delta_{\alpha\beta}. \hspace{1cm} (3.39)$$

Then condition (3.38) implies that

$$|\lambda_\alpha| < \frac{1}{3}, \quad \text{for all } \alpha = 1, ..., n. \hspace{1cm} (3.40)$$

This implies (2.50).

Let

$$u(\tau) = \cos\left(\frac{\pi \text{Re}(\tau)}{4d_R}\right) \cos\left(\frac{\pi \text{Im}(\tau)}{4d_R}\right),$$

then

$$u_{+\tau} = -\frac{\pi^2}{32d_R^2} u. \hspace{1cm} (3.41)$$

We consider $u$ as a function defined on $R \times V$, by letting $u(\tau, z) = u(\tau)$. Then (3.41) becomes

$$u_{+\tau} L^{\bar{\tau}} = -\frac{\pi^2}{32d_R^2} u. \hspace{1cm} (3.42)$$

At an interior maximum point of $\frac{Q}{u}$, we have

$$Q_i = Q_{ui} u, \quad \text{for any } i \in \{0,1,...,n\},$$

and by (3.30) (3.42)

$$0 \geq \left(\frac{Q}{u}\right)_{ij} L^{\bar{\tau}} = \left(\frac{Q_{ij}}{u} - \frac{Q u_{\bar{j}}}{u^2}\right) L^{\bar{\tau}} \geq \left(\frac{\pi^2}{32d_R^2} - \hat{C} \epsilon\right) \frac{Q}{u}. \hspace{1cm} (3.36)$$

If $\epsilon \leq \frac{\pi^2}{32d_R \hat{C}}$, then $\frac{Q}{u}$ cannot achieve interior maximum, except that $Q \equiv 0$. Thus,

$$\frac{Q}{u} \leq \max_{\partial R \times V} \frac{Q}{u}, \quad \text{in } R \times V.$$

Because $1 \geq u \geq \frac{1}{2}$, we have

$$Q \leq 2 \max_{\partial R \times V} Q, \quad \text{in } R \times V.$$

So, we can let $\delta = \min \left\{\frac{\pi^2}{32\hat{C}}, \frac{1}{4}\right\}$. \hfill \Box
Then with the previous proposition we can get an estimate for $Q$. For $\lambda \in [0, 1]$, let $\Phi^\lambda$ be the solution to Problem 1.3 with the boundary value condition being replaced by

$$\Phi^\lambda = \lambda \cdot F, \quad \text{on } \partial R \times V.$$  

Define

$$g^\lambda_{\alpha \beta} = b_{\alpha \beta} + \Phi^\lambda_{\alpha \beta},$$

and denote its inverse by $g^\lambda_{\beta \alpha}$. Then we define

$$Q^\lambda = \Phi^\lambda_{\alpha \beta} \Phi^\lambda_{\gamma \delta} g^\delta_{\lambda \gamma} g^\gamma_{\lambda \delta} + \Phi^\lambda_{\alpha \beta} \Phi^\lambda_{\gamma \delta} g^\delta_{\lambda \gamma} g^\gamma_{\lambda \delta} + \Phi^\lambda_{\alpha \beta} \Phi^\lambda_{\gamma \delta} g^\delta_{\lambda \alpha} g^\alpha_{\lambda \beta}.$$  

(3.43)

For each $(\tau, z) \in R \times V$, $Q^\lambda(\tau, z)$ is a non-decreasing function of $\lambda$, this can be proved by taking $\partial_\lambda$ derivatives of $Q^\lambda$. We assume that

$$\max_{\partial R \times V} Q^\lambda \leq \delta^2,$$  

where the $\delta$ is from Proposition 3.1. Then this implies, by the non-decreasing property, for each $\lambda \in [0, 1],

$$\max_{\partial R \times V} Q^\lambda \leq \delta^2.$$

(3.44)

We assume that $F \in C^\infty(\partial R \times V)$ and $\epsilon > 0$, so all second order derivatives of $\Phi^\lambda$ are continuous with respect to $\lambda$. This can be proved with implicit function theorem. In particular, $\max_{R \times V} Q^\lambda$ is a continuous function of $\lambda$. So, if

$$\max_{R \times V} Q^1 = \max_{R \times V} Q > 2 \max_{\partial R \times V} Q,$$

there is a $\lambda \in (0, 1)$, so that

$$\delta^2 \geq \max_{R \times V} Q^\lambda > 2 \max_{\partial R \times V} Q,$$

(3.46)

because $Q^0 = 0$. But, by Proposition 3.1 $\delta^2 \geq \max_{R \times V} Q^\lambda$ implies

$$\max_{R \times V} Q^\lambda \leq 2 \max_{\partial R \times V} Q^\lambda \leq 2 \max_{\partial R \times V} Q,$$

which is a contradiction.

The argument above can be illustrated by Figure 1. By Proposition 3.1, for no $\lambda \in (0, 1)$, $(\lambda, \max_{R \times V} Q^\lambda)$ can stay in the shadowed area. This is because Proposition 3.1 says if $Q^\lambda < \delta^2$ in $R \times V$, then $\max_{R \times V} Q^\lambda \leq 2 \max_{\partial R \times V} Q^\lambda$. Therefore, since $\max_{R \times V} Q^\lambda$ is a continuous function of $\lambda$, with $\max_{R \times V} Q^0 = 0$, the curve

$$\{(\lambda, \max_{R \times V} Q^\lambda) | \lambda \in [0, 1]\}$$

(3.47)

has to stay below the shadowed area. So, for all $\lambda \in [0, 1],

$$\max_{R \times V} Q^\lambda \leq 2 \max_{\partial R \times V} Q^\lambda \leq 2 \max_{\partial R \times V} Q^1 \leq \delta^2.$$
As a conclusion, if for any $\tau \in \partial \mathcal{R}$ the second order derivatives of $F(\tau, \ast)$ are small enough, so that

$$\max_{\partial \mathcal{R} \times \mathcal{V}} Q \leq \frac{\delta^2}{2},$$

(3.48)

and $\epsilon < \frac{\delta}{10d_{\mathcal{R}^2}}$, we have $Q < \delta^2 \leq \frac{1}{10}$ and so

$$\frac{1}{2} (b_{\alpha \beta}) < (g_{\alpha \beta}) < \frac{3}{2} (b_{\alpha \beta}).$$

(3.49)

Theorem 1.1 is proved.

**Remark 3.1.** The $\hat{\delta}$ of Theorem 1.1 can be chosen as

$$\hat{\delta} = \min \left\{ \frac{1}{C \cdot (1 + \max |\mathcal{R}| + \max |\mathcal{R}|^2 + \max |\nabla \mathcal{R}|)^{1/16} \right\},$$

(3.50)

for some constant $C$ only depending on the dimension of the manifold $\mathcal{V}$.

Now given two functions $\varphi_1, \varphi_0 \in \mathcal{H}$, with small $C^2$ norm, and $\epsilon \in (0, \frac{\hat{\delta}}{10d_{\mathcal{R}^2}})$, we can use the theory of non-degenerate Monge-Ampère equation, and get a solution $\Phi^\epsilon$ to Problem 1.3 with $\mathcal{R} = \{\tau | 1 < |\tau| < \epsilon\}$, and

$$\Phi^\epsilon = \varphi_0, \text{ on } \{|\tau| = 1\} \times \mathcal{V};$$

(3.51)

$$\Phi^\epsilon = \varphi_1, \text{ on } \{|\tau| = \epsilon\} \times \mathcal{V}.$$  

(3.52)

With Theorem 1.1 we know

$$\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi^\epsilon(\tau, \ast) > \frac{1}{2} \omega_0, \text{ for any } \tau \in \mathcal{R}.$$ 

(3.53)

When $\epsilon \to 0$, the sequence of $\Omega_0$–plurisubharmonic functions $\Phi^\epsilon$ monotone increase, and converge to a $\Omega_0$–plurisubharmonic function $\Phi$, in $C^1$ norm. Then $\Phi(\epsilon^\ast, \ast)$ is a solution to Problem 1.1 in the weak sense and it satisfies

$$\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi^\epsilon(\epsilon^\ast, \ast) > \frac{1}{2} \omega_0, \text{ for any } \zeta \in \mathcal{S},$$

(3.54)

weakly. Theorem 1.2 is proved.
 Estimates with 1-Dimensional Flat Torus

In this appendix, we prove some estimates for solutions to Problem 1.3, in the situation that the Kähler manifold \((V, \omega_0)\) is a flat torus. The torus under consideration is

\[ T = \mathbb{C}/\sim, \quad z \sim z + 1, \quad z \sim z + \lambda, \]

for \(\lambda \in \mathbb{C} \) and \(\lambda \neq 1\), the metric on the torus is \(\omega_0 = \sqrt{-1} dz \wedge d\bar{z}\). On such a flat torus, we define

**Definition A.1.** For a function \(\varphi \in C^2(T)\), we say \(\varphi\) is \(\omega_0\)-convex if

\[
\frac{|\varphi_{zz}|}{1 + \varphi_{z\bar{z}}} < 1, \quad \text{and} \quad 1 + \varphi_{z\bar{z}} > 0. \tag{A.1}
\]

**Remark A.1.** Condition (A.1) is equivalent to

\[
D^2 \varphi + dz \otimes d\bar{z} + d\bar{z} \otimes dz > 0, \tag{A.2}
\]

and it coincides with the concept of \(g\)-convexity defined in [13], if we let the \((V, \nabla, g)\) in [13] be \((T, \partial, dz \otimes d\bar{z} + d\bar{z} \otimes dz)\).

The results of this appendix are: Theorem A.1, which says that if, for any \(\tau \in \partial \mathcal{R}\), \(F(\tau, \ast)\) is \(\omega_0\)-convex, then for the solution \(\Phi\) to Problem 1.3, \(\Phi(\tau, \ast)\) is \(\omega_0\)-convex, for any \(\tau \in \mathcal{R}\); Theorem A.2, if for all \(\tau \in \partial \mathcal{R}\), \(F(\tau, \ast)\) is \(\omega_0\)-convex, then for all \(\tau \in \mathcal{R}\), the metric

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \ast)
\]

has a lower bound, independent of \(\epsilon\); and Theorem A.3, for the solution \(\Phi\) to Problem 1.3, \(|\Phi_{zz}|\) and \(\Phi_{z\bar{z}}\) has a very precise upper bound, independent of \(\epsilon\). These results will be generalized to higher dimensional case in an upcoming paper.

We start with

**Theorem A.1.** Given \(F \in C^\infty(\partial \mathcal{R} \times T)\), if for all \(\tau \in \partial \mathcal{R}\), \(F(\tau, \ast)\) is \(\omega_0\)-convex, then for the solution \(\Phi\) to Problem 1.3, \(\Phi(\tau, \ast)\) is \(\omega_0\)-convex, for all \(\tau \in \mathcal{R}\).

**Proof.** In the following, we denote

\[
b = \Phi_{zz}, \quad a = \Phi_{z\bar{z}}. \tag{A.3}
\]

With \((V, \omega_0)\) being a flat torus, the Monge-Ampère equation becomes

\[
(1 + \Phi_{z\bar{z}}) \Phi_{z\bar{z}} - \Phi_{z\bar{z}} \Phi_{z\bar{z}} = \epsilon. \tag{A.4}
\]

Computations in section 2 give equations for \(a, b\):

\[
h^{i\bar{j}} a_{i\bar{j}} = \left( \frac{a_i a_{\bar{j}}}{1 + a} + \frac{b_i \delta_{i\bar{j}}}{1 + a} \right) h^{i\bar{j}}, \tag{A.5}
\]

\[
h^{i\bar{j}} b_{i\bar{j}} = h^{i\bar{j}} \frac{2 a_i b_{i\bar{j}}}{1 + a}. \tag{A.6}
\]
Remark A.2. Comparing with section 2, the computation here is much simpler. \( H, F, U, V, W \) all vanish, because the dimension of \( V \) is 1, and \( \omega_0 \) is flat.

In this appendix, we define

\[ Q = \frac{b \bar{b}}{(1 + a)^2}, \tag{A.7} \]

and show that for a very large constant \( K \),

\[ h^T (e^{KQ}) h \geq 0. \tag{A.8} \]

Here,

\[ (h) = \begin{pmatrix} \Phi_{\tau \tau} & \Phi_{\tau \tau} \\ \Phi_{\tau \tau} & 1 + \Phi_{\tau \tau} \end{pmatrix}, \tag{A.9} \]

and

\[ (h^2) = \frac{1}{\epsilon} \begin{pmatrix} 1 + \Phi_{\tau \tau} & -\Phi_{\tau \tau} \\ -\Phi_{\tau \tau} & \Phi_{\tau \tau} \end{pmatrix}, \tag{A.10} \]

where \( i \) is the row index and \( j \) is the column index. Direct computation gives,

\[ h^T (e^{KQ}) h = Ke^{KQ} h \cdot h^T \tag{A.11} \]

\[ = Ke^{KQ} \left[ \frac{b_i}{a + 1} - \frac{2a_i b}{(1 + a)^2} \right] \left( \frac{b_j}{a + 1} - \frac{2a_j b}{(1 + a)^2} \right) + \frac{b_i \bar{b}_i}{(1 + a)^2} (1 - 2Q) \]

\[ + K \left[ \frac{b_i \bar{b}_i}{(1 + a)^2} + \frac{b_i \bar{b}_i}{(1 + a)^2} - \frac{2Qa_i}{1 + a} \left( \frac{b_i \bar{b}_i}{(1 + a)^2} + \frac{b_i \bar{b}_i}{(1 + a)^2} - \frac{2Qa_j}{1 + a} \right) \right]. \tag{A.12} \]

We denote

\[ b_i - \frac{2a_i b}{1 + a} = \beta_i, \tag{A.14} \]

and

\[ M = \begin{pmatrix} 1 + KQ & K \frac{\beta^2}{(1 + a)^2} \\ K \frac{\beta^2}{(1 + a)^2} & 1 - 2Q + KQ \end{pmatrix}, \tag{A.15} \]

then

\[ h^T (e^{KQ}) h = Ke^{KQ} h \cdot \beta_i M (\beta_j, \bar{b})^T. \tag{A.16} \]

We want to choose a large enough \( K \) so that \( M \geq 0 \), which implies \( A.16 \geq 0 \). First, we need \( K > 2 \), which implies that the diagonal elements of \( M \) are positive. Then we make \( K \) larger, so that

\[ \det(M) = 2K \left[ -Q^2 + \left( 1 - \frac{1}{K} \right)Q + \frac{1}{2K} \right]. \]
is also positive. As \( K \to \infty \), two roots of

\[
-Q^2 + (1 - \frac{1}{K})Q + \frac{1}{2K}
\]  
(A.17)

converge to 0 and 1. We need to choose \( K \) large enough, so that the larger one of two roots

\[
\sigma_2 = \frac{1 - \frac{1}{K} + \sqrt{1 + \frac{1}{K^2}}}{2} > \max_{\partial R \times T} Q.
\]  
(A.18)

Because \( \sigma_2 < 1 \), for any \( K > 0 \), so (A.18) can only be achieved under the condition that

\[
\max_{\partial R \times T} Q < 1.
\]  
(A.19)

Since the smaller one of two roots of (A.18)

\[
\sigma_1 = \frac{1 - \frac{1}{K} - \sqrt{1 + \frac{1}{K^2}}}{2}
\]  
(A.20)

is always negative, we have showed that if \( 0 \leq Q < \sigma_2 \), in \( R \times V \), then \( M \geq 0 \) and

\[
h^\partial (e^{KQ})_Q \geq 0.
\]  
(A.21)

Because \( e^{KQ} \) is a strictly monotone increasing function of \( Q \), for positive \( K \), the maximum principle for Laplace equation tells

\[
Q \leq \max_{\partial R \times T} Q, \quad \text{in} \ R \times T.
\]  
(A.22)

The estimate above is an analog of Proposition 3.1. Form here, we can use a continuity argument to show that if

\[
Q < 1, \quad \text{on} \ \partial R \times T,
\]  
(A.23)

then

\[
Q < 1, \quad \text{in} \ \partial R \times T.
\]

The argument is in the following. Similar to the proof in section 3.2, for \( \lambda \in [0,1] \), let \( \Phi^\lambda \) be the solution to Problem 1.3, with the boundary condition (1.15) being replaced by

\[
\Phi^\lambda = \lambda F, \quad \text{on} \ \partial R \times T.
\]  
(A.24)

Define

\[
Q^\lambda = \frac{|\Phi^\lambda_{zz}|^2}{(1 + \Phi^\lambda_{zz})^2}.
\]

By taking \( \partial_\lambda \) derivatives, we can show that for any fixed \((\tau, z) \in \partial R \times T\), \( Q^\lambda \) is a non-decreasing function of \( \lambda \). This implies, \( \max_{\partial R \times T} Q^\lambda \) is a non-decreasing function of \( \lambda \), so, if (A.23) is satisfied, then for any \( \lambda \in [0,1] \),

\[
\max_{\partial R \times T} Q^\lambda < 1.
\]  
(A.25)
Figure 2: Continuity Method with $\mathcal{V}$ being a Torus

We choose $K$ large enough so that

$$\max_{\partial \mathcal{R} \times \mathcal{T}} Q < \sigma_2 < 1.$$  \hfill (A.26)

Then, by estimate (A.22), for any $\lambda \in (0, 1)$,

$$\max_{\mathcal{R} \times \mathcal{T}} Q^\lambda \not\in (\max_{\partial \mathcal{R} \times \mathcal{T}} Q^\lambda, \sigma_2),$$  \hfill (A.27)

i.e. as illustrated by Figure 2, $(\lambda, \max_{\mathcal{R} \times \mathcal{T}} Q^\lambda)$ cannot stay in the dotted area. Since $\max_{\mathcal{R} \times \mathcal{T}} Q^\lambda$ is a continuous function of $\lambda$, and $\max_{\mathcal{R} \times \mathcal{T}} Q^0 = 0$, we know

$$\max_{\mathcal{R} \times \mathcal{T}} Q = \max_{\mathcal{R} \times \mathcal{T}} Q^1 \leq \sigma_2 < 1.$$  \hfill (A.28)

In the same way, we have estimates

$$\frac{|\Phi_{zz} - \gamma|^2}{(1 + \Phi_{xz})^2} < 1,$$  \hfill (A.29)

with $\gamma \in \mathbb{C}$ being small enough. By using these estimates, we can get the following metric lower bound estimate independent of $\epsilon$.

**Theorem A.2.** Given $F \in C^\infty(\partial \mathcal{R} \times \mathcal{T})$, if for all $\tau \in \partial \mathcal{R}$, $F(\tau, \ast)$ is $\omega_0$-convex, then there is a constant $\delta > 0$, so that the solution $\Phi$ to Problem 1.3 with boundary value $F$ satisfies

$$\partial_{xx} \Phi(\tau, \ast) + 1 > \delta, \quad \text{for all } \tau \in \mathcal{R}.$$  \hfill (A.30)

$\delta$ only depends on the second order derivatives of $F(\tau, \ast)$, for all $\tau$ in $\mathcal{R}$.

**Proof.** For all $\gamma \in \mathbb{C}$, define

$$Q_\gamma = \frac{|\Phi_{zz} - \gamma|^2}{(1 + \Phi_{xz})^2}.$$

Then computations show that

$$h^\mathcal{G} (e^{KQ_\gamma})_{\mathcal{G}} \geq 0.$$
providing that $K$ is big enough and $Q_\gamma < 1$. The computations are the same as that of Theorem A.1, we only need to replace $b$ by $\Phi_{zz} - \gamma$. This is because equations (A.5) and (A.6) only contain derivatives of $b$, so adding a constant to $b$ does not affect the equations.

Because for each $\tau \in \partial R$, $F(\tau, \ast)$ is $\omega_0-$convex, we have

$$\max_{\partial R \times T} \frac{|F_{zz}|^2}{(1 + F_{zz})^2} < 1.$$  

Since $\partial R \times T$ is a compact set, we can find $\delta > 0$, only depending on

$$\max_{\partial R \times T} \frac{|F_{zz}|^2}{(1 + F_{zz})^2} \quad \text{and} \quad \min_{\partial R \times T} |1 + F_{zz}|,$$

so that for all $\gamma \in \mathbb{C}$, with $|\gamma| \leq \delta$,

$$\max_{\partial R \times T} \frac{|F_{zz} - \gamma|^2}{(1 + F_{zz})^2} < 1.$$  

Then, using a continuity method argument, similar to that of section 3.2 or the proof of Theorem A.1 we know

$$\max_{\mathbb{R} \times T} \frac{|\Phi_{zz} - \gamma|^2}{(1 + \Phi_{zz})^2} < 1, \quad \text{for all } \gamma, \text{ with } |\gamma| \leq \delta.$$  

In particular, we have

\begin{align*}
|\Phi_{zz} - \delta| &\leq 1 + \Phi_{zz}, \quad \text{in } \mathbb{R} \times T, \quad \text{(A.31)} \\
|\Phi_{zz} + \delta| &\leq 1 + \Phi_{zz}, \quad \text{in } \mathbb{R} \times T. \quad \text{(A.32)}
\end{align*}

Adding (A.31) and (A.32) together, we get

$$1 + \Phi_{zz} > \delta, \quad \text{in } \mathbb{R} \times T.$$

The results and ideas in Theorem A.1 and Theorem A.2 can be illustrated by Figure 3. First, we define

$$\mathcal{L} = \{(b, a) | b \in \mathbb{C}, a \in (-1, \infty)\}. \quad \text{(A.33)}$$

For $\tau \in \mathbb{R}$, we define a jet map, $J_\tau : T \to \mathcal{L}$,

$$J_\tau(z) = (\Phi_{zz}(\tau, z), \Phi_{zz}(\tau, z)),$$

and $J_\tau$ can also be considered as a map from $\mathbb{R} \times T$ to $\mathcal{L}$,

$$J(\tau, z) = J_\tau(z).$$

For each $\gamma \in \mathbb{C}$, define a cone

$$C_\gamma = \{(b, a) | b \in \mathbb{C}, a > -1, |b - \gamma| < a + 1\}.$$
In particular, $C_0 = \{(b,a) | b \in \mathbb{C}, a > -1, |b| < a + 1\}$. Theorem A.1 says that if
\[
\text{Image}(J_\tau) \subset C_0, \quad \text{for any } \tau \in \partial R,
\]
then
\[
\text{Image}(J_\tau) \subset C_0, \quad \text{for any } \tau \in R.
\]
Theorem A.2 says that since
\[
J(\partial R \times V) = \bigcup_{\tau \in \partial R} \text{Image}(J_\tau) \subset C_0,
\]
there will be a positive distance in between $J(\partial R \times V)$ and $\partial C_0$ then, for $\gamma$ with $|\gamma| \leq \delta$,
\[
\text{Image}(J_\tau) \subset C_\gamma, \quad \text{for any } \tau \in \partial R
\]
and so
\[
\text{Image}(J_\tau) \subset C_\gamma, \quad \text{for any } \tau \in R.
\]
This is equivalent to
\[
J(R \times V) = \bigcup_{\tau \in R} \text{Image}(J_\tau) \subset \bigcap_{\gamma, |\gamma| \leq \delta} C_\gamma,
\]
which gives the metric lower bound, because
\[
\bigcap_{\gamma, |\gamma| \leq \delta} C_\gamma \subset \{a + 1 > \delta\}.
\]

![Figure 3: Jet Maps](image)

**Remark A.3.** The construction of the function $Q$ is inspired by two facts. First, a quadratic polynomial
\[
P(z) = (1 + a)z \bar{z} + \frac{1}{2} \left( b \bar{z}^2 + \overline{b z^2} \right)
\]
is convex if and only if $|b| < 1 + a$. Second, the space $\mathcal{L}$ can be given a Lorentzian metric

$$ds^2 = \frac{-da^2 + db\bar{b}}{(1+a)^2}.$$  \hfill (A.34)

With this metric, the cone $C_0$ is a convex set which means that every geodesic with ends in $C_0$, this is explained in [20] and the appendix of [14]. It’s also an interesting fact that with the metric (A.34), the complement of $C_0$, which we denote by $C_0^c$, is also convex, in the sense that if two ends of a geodesic segment stay in $C_0^c$, then this geodesic stays in $(C_0)^c$. This fact leads to the following theorem.

**Theorem A.3.** Given $F \in C^\infty(\partial R \times T)$, let

$$S = \max_{\partial R \times T} (F_{zz} + 1 + |F_{zz}|)$$ \hfill (A.35)

then for the solution $\Phi$ to Problem 1.3, we have

$$|\Phi_{zz}| \leq S - 1 - \Phi_{zz}, \quad \text{in } R \times T.$$

**Remark A.4.** This implies $|\Phi_{zz}| \leq S$ and $\Phi_{zz} \leq S - 1$.

**Proof.** By the definition of $S$ (A.35), for any $\eta \in \mathbb{C}$, with $|\eta| > S$,

$$\frac{|F_{zz} - \eta|}{1 + F_{zz}} > 1.$$

Computations analogous to those for Theorem 1.1 give us:

$$\sigma_i = (e^{-\nu Q_{\eta}})_{ij} = Pe^{-\nu Q_{\eta}}h_i^j(\beta_i, b_i)N(\beta_j, b_j)\left(1 + \frac{1}{(1+\eta)^2}\right),$$ \hfill (A.36)

where we denote

$$\beta_i = b_i - \frac{2a_i(b - \eta)}{1 + a},$$

and

$$N = \begin{pmatrix} PQ_{\eta} - 1 & P \frac{(b - \eta)^2}{(1+a)^2} \\ P \frac{(b - \eta)^2}{(1+a)^2} & (P + 2)Q_{\eta} - 1 \end{pmatrix}. \hfill (A.37)$$

If we choose $P > 1$, then since $Q_{\eta} > 1$, we get the diagonal elements of $N$ are positive. We also need to make $\det N > 0$, which then implies $N > 0$.

$$\det N = 2P \left(Q_{\eta}^2 - (1 + \frac{1}{P})Q_{\eta} + \frac{1}{2P}\right).$$

We choose $P$ large, so that the bigger one of two roots of $Q_{\eta}^2 - (1 + \frac{1}{P})Q_{\eta} + \frac{1}{2P}$

$$\sigma_2 = \frac{1}{2}(1 + \frac{1}{P} + \sqrt{1 + \frac{1}{P^2}}) < \min_{\partial R \times T} \frac{|F_{zz} - \eta|^2}{(1 + F_{zz})^2}.$$
Then, for $Q \eta > \sigma_2$, we have

$$h^\mathcal{T} (e^{-PQ}) \geq 0.$$  

Using an analogous continuity method argument in section 3.2, we get

$$\frac{|\Phi_{zz} - \eta|^2}{(1 + \Phi_{z\tau})^2} = Q \eta \geq \min_{\partial R \times T} Q \eta = \min_{\partial R \times T} \frac{|F_{zz} - \eta|^2}{(1 + F_{z\tau})^2} > 1.$$  

This says that for any $\eta$, with $|\eta| > S$,

$$J(R \times T) \subset C^c_{\eta}.$$  

So

$$J(R \times T) \subset \bigcap_{\eta, |\eta| > S} C^c_{\eta} = \{(b, a)|a + 1 > 0, |b| < S - a - 1\}.$$  

Now we proved

$$|\Phi_{zz}| \leq S - \Phi_{z\tau} - 1.$$  

\[\square\]

### B Computation Along the Leaf

In this appendix, we show how the computations in section 2 and 3 work in the limiting case of Problem 1.3 when $\epsilon = 0$, or Problem 1.2. The manifold $V$ is of arbitrary dimension.

To carry out the computation, it’s necessary to make the following assumptions:

1. $\Phi \in C^4(R \times V)$,
2. $\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, *) > 0$.

However, these assumptions may not be satisfied in general as shown by [10][8][21] and [14]. So the computations in this section provide a guideline for Section 2 and 3.

According to [10], under assumptions 1 and 2, kernels of $\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi$ form a foliation in $R \times V$, and at a point $(\tau_0, p_0) \in R \times V$, the leaf is locally the graph of a holomorphic map, $f : \mathcal{U} \rightarrow V$, where $\mathcal{U}$ is an open set in $R$; $\mathcal{U} \ni \tau_0$ and $f(\tau_0) = p_0$.

In the case that $R$ is a disc, each leaf is also a disc, and $f$ can be defined on $R$. However, when $R$ is not simply connected, a leaf is a covering of $R$, of finite or infinite index. More information on the foliation structure can be found in [10] and [5].

In the following we will do computations locally on an open set of a leaf:

$$L \triangleq \{(\tau, f(\tau))|\tau \in \mathcal{U}\}.$$  

Denote the projection $R \times V \rightarrow R$ by $\pi_R$, then $\pi_R^{-1} \tau \triangleq X$ is a complex coordinate on $L$. Actually $X = \pi_R$, if we consider $\pi_R$ as a map to $\mathbb{C}$ $(\mathbb{C} \supset R)$. Because the tangent vector of the leaf lies in the kernel of $\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi$, we have

$$\partial_X = \partial_\tau - \Phi_{z\tau} g^{\alpha\overline{\beta}} \partial_\alpha.$$  

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Consider \( A = \Phi_{\alpha\beta} dz^\alpha \otimes dz^\beta \) and \( B = \Phi_{\alpha\beta} dz^\alpha \otimes dz^\beta \) as sections of bundles \( i^*\pi_V^*(T^{1,0}(V) \otimes T^{*0,1}(V)) \) and \( i^*\pi_V^*(T^{1,0}(V) \otimes T^{*1,0}(V)) \) over \( L \). Here \( i \) is the inclusion map \( L \hookrightarrow \mathcal{R} \times \mathcal{X} \).

With these notations, equations (2.71), (2.72) and (2.73) become

\[
\begin{align*}
A_{\alpha\beta,\mathcal{X}} &= A_{\alpha\beta,\mathcal{X}} \mu^\alpha B_{\alpha\beta,\mathcal{X}} + B_{\alpha\beta,\mathcal{X}} \mu^\beta B_{\alpha\beta,\mathcal{X}} + U_{\alpha\beta}; \\
B_{\alpha\beta,\mathcal{X}} &= A_{\alpha\beta,\mathcal{X}} \mu^\alpha B_{\alpha\beta,\mathcal{X}} + B_{\alpha\beta,\mathcal{X}} \mu^\beta B_{\alpha\beta,\mathcal{X}} + V_{\alpha\beta}; \\
\mathcal{B}_{\alpha\beta,\mathcal{X}} &= A_{\alpha\beta,\mathcal{X}} \mu^\alpha B_{\alpha\beta,\mathcal{X}} + B_{\alpha\beta,\mathcal{X}} \mu^\beta B_{\alpha\beta,\mathcal{X}} + W_{\alpha\beta}.
\end{align*}
\]

Terms \( F, H \) in (2.71), (2.72) and (2.73) vanish because \( \epsilon = 0 \). See the expression of \( F, H \) (2.37) and (2.54). It’s convenient to consider \( A, B, U, V, W \) as matrices, then we can use matrix multiplication to simplify the equations to:

\[
\begin{align*}
A_{\alpha,\mathcal{X}} &= A_{\alpha,\mathcal{X}} G^{-1} A_{\alpha,\mathcal{X}} + B_{\alpha,\mathcal{X}} G^{-1} B_{\alpha,\mathcal{X}} + U, \\
B_{\beta,\mathcal{X}} &= B_{\beta,\mathcal{X}} G^{-1} B_{\beta,\mathcal{X}} + B_{\beta,\mathcal{X}} G^{-1} B_{\beta,\mathcal{X}} + V, \\
\mathcal{B}_{\alpha,\mathcal{X}} &= \mathcal{B}_{\alpha,\mathcal{X}} G^{-1} A_{\alpha,\mathcal{X}} + \mathcal{B}_{\alpha,\mathcal{X}} G^{-1} B_{\alpha,\mathcal{X}} + W.
\end{align*}
\]

In the equations above, \( G = (g_{\alpha\beta}) \) and \( G^{-1} = (g^{\alpha\beta}) \), where \( \alpha \) is the row index and \( \beta \) is the column index. With matrix multiplications, the quantity \( Q_B \), introduced in (3.3), is

\[
Q_B = \text{tr} \left(B G^{-1} B G^{-1}\right).
\]

In the case that curvatures of \((\mathcal{V}, \omega_0)\) are 0, terms \( U, V, W \) all vanish and computations will be largely simplified. We choose a coordinate at the point of computation, so that \( g_{\alpha\beta} = \delta_{\alpha\beta} \) and the Christoffel symbols are zero. Then apply \( X \) and \( \mathcal{X} \) to (B.7). By Leibniz rule and equations (B.4)–(B.6), we get

\[
(Q_B)_{X\mathcal{X}} = \text{tr} \left( B A X B + A X B X B - B X A X B - B X B X A + B X A X B \right)
\]

There are 12 terms canceling each other:

\[
(B.9)_2 + (B.9)_6 = 0, \quad (B.9)_1 + (B.11)_6 = 0, \quad (B.11)_2 + (B.11)_4 = 0, \\
(B.11)_3 + (B.11)_5 = 0, \quad (B.10)_1 + (B.10)_2 = 0, \quad (B.12)_4 + (B.12)_5 = 0.
\]

Here we are using a notation mentioned at (2.31), term \((\ast \ast)_k\) stands for the \( k \)-th term in \((\ast \ast)\), for example \((B.9)_6 = -\text{tr}(B X A X B)\). For the first two equalities of (B.13), we used that transposing a matrix does not change its trace and for matrices \( A_1, A_2, \ldots, A_k \),

\[
\text{tr}(A_1 A_2 \cdots A_k) = \text{tr}(A_k A_1 A_2 \cdots A_{k-1}).
\]
In addition, as introduced in Section 1.3, \( \omega \) is the norm of a tensor, with respect to the metrics induced by \( \omega_0 \). Similarly, for \( Q_A \) and \( Q_G \), we have that

\[
(Q_A)_{\mathcal{X}} \geq (2 - C\sqrt{Q})\text{tr}(A_{\mathcal{X}} A_{\mathcal{X}}) - C|\mathcal{X}|\sqrt{Q} - C\sqrt{Q} \text{ tr}(\mathcal{B}_{\mathcal{X}} A_{\mathcal{X}}),
\]

and

\[
(Q_G)_{\mathcal{X}} \geq \frac{1}{2} \left( \Phi_{\theta,\mathcal{X}} \Phi_{\mathcal{X},\mathcal{X}} + \Phi_{\mathcal{X},\mathcal{X}} \Phi_{\theta,\mathcal{X}} \right) - C(|\mathcal{X}| + |\mathcal{X}|)\sqrt{Q}.
\]

To control \(|S|, |U|, |V|, |W|\), we define

\[
P = B_{\alpha\beta,\mathcal{X}} B_{\beta,\gamma,\mathcal{X}} g_{\alpha\gamma}^\theta \text{g}^\rho \text{g}^\tau + A_{\alpha,\mathcal{X}} A_{\beta,\gamma,\mathcal{X}} g_{\alpha\gamma}^\theta \text{g}^\rho \text{g}^\tau + \Phi_{\theta,\mathcal{X}} \Phi_{\mathcal{X},\mathcal{X}} g_{\theta\tau} + \Phi_{\theta,\mathcal{X}} \Phi_{\mathcal{X},\mathcal{X}} g_{\theta\tau}.
\]

At the point of computation, where \( g_{ij} = \delta_{ij} \),

\[
P = \text{tr}(\mathcal{B}_{\mathcal{X}} A_{\mathcal{X}}) + \Phi_{\theta,\mathcal{X}} \Phi_{\mathcal{X},\mathcal{X}} + \Phi_{\theta,\mathcal{X}} \Phi_{\mathcal{X},\mathcal{X}}.
\]
$\mathcal{P}$ defined here is roughly a combination of $P$ and $T$, defined in (2.46) and (2.47). It’s easy to see that

$$\mathcal{P} \geq P + (1 - C \sqrt{Q})T.$$  \hspace{1cm} (B.27)

So, when $Q$ is small enough, we can simplify (2.77) and get the following estimates for $|U|, |V|, |W|, |S|$

$$|U| + |V| + |W| + |S| \leq C_R \mathcal{P}.$$ \hspace{1cm} (B.28)

Here $C_R$ is a constant depending on curvatures of $V$ and their covariant derivatives. Then combining (B.22) (B.23) and (B.24) together, we find, when $Q$ is smaller than a dimensional constant,

$$Q_{\overline{X}X} \geq \frac{1}{2} \mathcal{P} - C_R \sqrt{Q} \mathcal{P}.$$  

When $Q$ is smaller than a curvature related constant, we have $Q_{\overline{X}X} \geq 0$. This leads to an upper bound for $Q$, and so an upper bound for $|D^2\Phi(\tau,*)|_{\omega_0}, \tau \in \mathcal{R}$.

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