Higher $\eta_c(nS)$ and $\eta_b(nS)$ mesons

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Abstract

The hyperfine splittings in heavy quarkonia are studied in a model-independent way using the experimental data on di-electron widths. Relativistic correlations are taken into account together with the smearing of the spin-spin interaction. The radius of smearing is fixed by the known $J/\psi - \eta_c(1S)$ and $\psi(2S) - \eta'_c(2S)$ splittings and appears to be small, $r_{ss} \approx 0.06$ fm. Nevertheless, even with such a small radius an essential suppression of the hyperfine splittings ($\sim 50\%$) is observed in bottomonium. For the $nS$ $b\bar{b}$ states ($n = 1, 2, \ldots, 6$) we predict the values (in MeV) 28, 12, 10, 6, 6, and 3, respectively. For the $3S$ and $4S$ charmonium states the splittings 16(2) MeV and 12(4) MeV are obtained.

1 Introduction

At present two spin-singlet $S$-wave states $\eta_c(1S)$ and $\eta_c(2S)$ have been discovered [1]-[3]. Still, no spin-singlet $\eta_b(nS)$ levels have been seen [4]. Theoretically the masses of the $\eta_b(nS)$ were predicted in many papers [5]-[11].

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however, the calculated hyperfine (HF) splittings,

\[ \Delta_{\text{HF}}(nS) = M(n^3S_1) - M(n^1S_0) \]

(1)
vary in a wide range: from 35 MeV up to 100 MeV for the \( b\bar{b} \) 1S state and for the 2S state between 19 MeV and 44 MeV [11].

However, modern theoretical treatments and current experiments taken together, have produced well-established limits on the factors which determine the spin-spin potential \( V_{\text{HF}}(r) \) in heavy quarkonia. First of all, the wave function (w.f.) at the origin for a given \( n^3S_1(\bar{c}c \text{ or } b\bar{b}) \) state can be extracted from the di-electron width which, as well as some ratios of leptonic widths, are now measured with high accuracy [12, 13]. The quark masses, the pole (current) mass, present in the correct relativistic approach, and the constituent mass, used in the nonrelativistic or in more refined approximations, is also known with good accuracy [14, 15]. Therefore the only uncertainties comes from two sources.

First, in perturbative QCD a strict prescription as to how to choose the renormalization scale \( \mu \) in the strong coupling \( \alpha_{\text{HF}} \), which enters \( V_{\text{HF}}(r) \), is not well established for a bound state, especially for higher excitations.

Secondly, a smearing of the spin-spin interaction is considered to be due to relativistic effects [10] but the true size of the smearing radius \( r_{ss} \) is still not fixed.

In our calculations the radius \( r_{ss} \) is taken to fit the \( J/\psi - \eta_c(1S) \) and \( \psi(2S) - \eta_c(2S) \) splittings. We show that to reach agreement with experiment the smearing radius should be \( r_{ss} \leq 0.06 \text{ fm} \). Our value \( r_{ss} = 0.057 \text{ fm} \) practically coincides with the number used in Ref. [10]. However, in spite of this coincidence the splitting \( \Delta_1 = \Upsilon(1S) - \eta_b(1S) = 28 \text{ MeV} \) found in our calculations appears to be two times smaller than that in Ref. [10], where \( \Delta_1 = 60 \text{ MeV} \) was obtained.

We consider that the use of the w.f. at the origin \( |\tilde{R}_n(0)|^2 \) \text{exp}, extracted from di-electron widths, is the most promising because these w.f.s take implicitly into account the relativistic corrections as well as the influence of open channel(s), in this way drastically simplifying the theoretical analysis. A comparison of these w.f.s with those calculated in different models puts serious restrictions on the static potential used and also on many-channel models.

We also show that the nonperturbative spin-spin interaction gives a contribution of about 9 MeV in the \( J/\psi - \eta_c(1S) \) mass difference.
2 The spin-spin interaction

The HF splitting between the $n^3S_1$ and $n^1S_0$ levels will be considered here for two cases. The first one corresponds to the standard perturbative (P) spin-spin interaction with a $\delta$-function:

$$\hat{V}_{ss}^P(r) = s_1 \cdot s_2 \frac{32\pi}{9\omega^2_q} \alpha_s(\bar{\mu}) \left( 1 + \frac{\alpha_s}{\pi} \rho \right) \delta(r) \equiv s_1 \cdot s_2 V_{HF}(r),$$

(2)

which in one-loop approximation gives the following HF splitting \[6\]:

$$\Delta_{HF}^P(nS) = \frac{8}{9} \frac{\alpha_s(\bar{\mu})}{\omega^2_q} |R_n(0)|^2 \left( 1 + \frac{\alpha_s}{\pi} \rho \right).$$

(3)

where $\rho = \frac{5}{12} \beta_0 - \frac{8}{3} - \frac{3}{4} \ln 2$. This correction is small: $\lesssim 0.5\%$ in bottomonium ($n_f = 5$) and $\lesssim 3\%$ in charmonium ($n_f = 4$).

It is very probable that $\delta(r)$ may be considered as a limiting case and the “physical” spin-spin interaction is smeared with a still unknown “smearing” radius. For example, for the Gaussian smearing function,

$$\delta(r) \to \frac{4\beta^3}{\sqrt{\pi}} \int r^2 dr \exp(-\beta^2 r^2),$$

(4)

the splitting can be rewritten as

$$\Delta_{HF}^P(nS) = \frac{8}{9} \frac{\alpha_s(\bar{\mu})}{\omega^2_q} \xi_n(\beta) |R_n(0)|^2 \left( 1 + \frac{\alpha_s}{\pi} \rho \right),$$

(5)

where by definition “the smearing factor” $\xi_n(\beta)$ is

$$\xi_n(\beta) = \frac{4}{\sqrt{\pi}} \frac{\beta^3}{|R_n(0)|^2} \int |R_n(r)|^2 \exp(-\beta^2 r^2) r^2 dr.$$  

(6)

The general expression Eq. (5) is evidently valid for other smearing prescriptions which may differ from Eq. (4). The factor $\xi_n$ is calculated in Appendix [A].

It is well known [8] that the w.f. at the origin is very sensitive to the form and parameters of the gluon-exchange interaction and also to the value of the quark mass used. Therefore we make the following remarks.
1. To minimize the uncertainties in the w.f. at the origin, $R_n(0)$, we shall use here the w.f.s extracted from the experimental data on the leptonic widths and denote them as $|\tilde{R}_n(0)|_{\exp}^2$. In this way the relativistic corrections to the w.f. and the influence of open channel(s) are implicitly taken into account.

2. In Eqs. (3) and (5) the constituent mass $\omega_q$ enters. This fact can be rigorously deduced from relativistic calculations [14, 15],

$$\omega_q(nS) = \langle \sqrt{p^2 + m^2_{q}} \rangle_{nS},$$  \hspace{1cm} (7)

where under the square-root the pole mass $m_q \equiv m_q(\text{pole})$ is present. This mass is known with good accuracy and we take here $m_b(\text{pole}) = 4.8 \pm 0.1$ GeV and $m_c(\text{pole}) = 1.42 \pm 0.03$ GeV, which correspond to well-established current masses $\tilde{m}_b(\tilde{m}_b) = 4.3(1)$ GeV, $\tilde{m}_c(\tilde{m}_c) = 1.2(1)$ GeV [3]. The important feature of the constituent mass $\omega_q(nS)$ is that it takes into account the relativistic correction and its values depend weakly on the quantum numbers and on the static interaction used. Indeed, they lie in a rather narrow range for all $nS$ states, both in charmonium and bottomonium:

$$\omega_c(1S) = 1.62 \pm 0.03 \text{ GeV}(n = 1), \quad \omega_c(nS) = 1.71 \pm 0.03 \text{ GeV}(n \geq 2), \quad \omega_b(nS) = 5.05 \pm 0.15 \text{ GeV}.$$  \hspace{1cm} (8)

Note that just these mass values are mostly used in nonrelativistic calculations, thus in a hidden way taking into account relativistic corrections.

3. The leptonic width of the $n^3S_1$ state in heavy quarkonia is defined by the Van Royen-Weisskopf formula with QCD correction $\gamma_q$ [16]:

$$\Gamma_{ee}(n^3S_1)_{\exp} = \frac{4e_q^2\alpha^2}{M_n^2}|\tilde{R}_n(0)|_{\exp}^2 \gamma_q.$$  \hspace{1cm} (9)

Here $e_q = \frac{1}{3} \left( \frac{2}{3} \right)$ for a $b(c)$ quark; $\alpha = 1/137$, $M_n \equiv M(n^3S_1)$, and the QCD factor is given by

$$\gamma_q(nS) = 1 - \frac{16}{3\pi} \alpha_s(2m_q).$$  \hspace{1cm} (10)
The renormalization scale $\mu$ in Eq. (10) is supposed to be known, $\mu = 2m_q(\text{pole})$, as in Refs. [9, 10] and in $\eta_b \rightarrow \gamma\gamma$ decay [17]. In some cases $\mu = M_n$ is also taken. With an accuracy of $\lesssim 1\%$ in bottomonium and $\lesssim 10\%$ in charmonium both choices coincide and therefore one can take here $\mu = 2m_q$ ($2m_b = 9.6$ GeV, $2m_c = 2.9$ GeV).

Since for $n_f = 5$ the QCD constant $\Lambda^{(5)}_{\overline{MS}}$ is well known from high energy experiments [3], the factor $\gamma_b$ is supposed to be determined with a good accuracy. For $\Lambda^{(5)}_{\overline{MS}}(3 - \text{loop}) = 210(10)$ MeV, which corresponds to $\alpha_{\overline{MS}}(M_Z) = 0.1185$, one finds

$$\gamma_b = \gamma_{bn} = 0.700(5), \quad \alpha_s(2m_b) = 0.177(3). \tag{11}$$

In charmonium ($n_f = 4$) the strong coupling $\alpha_{\overline{MS}}(2m_c = 2.9$ GeV) is determined with less accuracy and for $\Lambda^{(4)}_{\overline{MS}} = 0.260(10)$ MeV one obtains

$$\alpha_s(2m_c = 2.9 \text{ GeV}) = 0.237(5), \quad \gamma_c = 0.60(2). \tag{12}$$

Here, the theoretical error comes from the uncertainty in the value $\Lambda^{(4)}_{\overline{MS}}$. Then the w.f. at the origin, extracted from the di-electron width in Eq. (5),

$$|\tilde{R}_n(0)|^2_{\text{exp}} = \frac{M_n^2 \Gamma_{ee}(n^3S_1)}{4e_q^2 \alpha^2 \gamma_q}, \tag{13}$$

implicitly takes into account the relativistic corrections as well as the influence of the open channels, which gives rise to smaller values for $|R_n(0)|$ and the HF splitting. An additional decrease of $\Delta_{HF}$ comes from a possible smearing of the $S$-wave spin-spin interaction (for the $P$-wave states this effect is very small).

The extracted values of $|\tilde{R}_n(0)|^2_{\text{exp}}$ in the $b\bar{b}$ and $c\bar{c}$ systems are presented in Tables 1 and 2.

With the use of these w.f.s at the origin and the values $\gamma_b = 0.70$ we also very precisely reproduce the ratios of the leptonic widths measured by the CLEO Collaboration (third Ref. in [12]):

$$\Gamma_{ee}(\Upsilon(2S))/\Gamma_{ee}(\Upsilon(1S)) = 0.46(1), \quad \Gamma_{ee}(\Upsilon(3S))/\Gamma_{ee}(\Upsilon(1S)) = 0.32(1)$$

$$\Gamma_{ee}(\Upsilon(3S))/\Gamma_{ee}(\Upsilon(2S)) = 0.69(2), \tag{14}$$

which confirms our correct choice of equal values of $\gamma_{bn}$ for these states.

It is of interest to compare the extracted values of $|\tilde{R}_n(0)|^2_{\text{exp}}$ with the theoretically predicted values, which mostly depend on the strong coupling used
Table 1: The w.f. $|\tilde{R}_n(0)|^2_{\text{exp}}$ from Eq. (13) (in GeV$^2$) and the leptonic widths $\Gamma_{ee}(\Upsilon(nS))$ (in keV) for the $\Upsilon(nS)$ states ($\gamma_b = 0.70$) $^a$

| $\Gamma_{ee}(nS)_{\text{exp}}$ | $|\tilde{R}_n(0)|^2_{\text{exp}}$ $^b$ |
|------------------------|------------------|
| 1S                     | 1.314(29)        | 1.336(28)        |
|                        | 7.094(16)        | 7.213(15)        |
| 2S                     | 0.576(24)        | 0.616(19)        |
|                        | 3.49(15)         | 3.73(12)         |
| 3S                     | 0.413(10)        | 2.67(7)          |
| 4S                     | 0.25(3)          | 1.69(20)         |
| 5S                     | 0.31(7)          | 2.21(49)         |
| 6S                     | 0.13(3)          | 0.95(22)         |

$^a$ For the states 1$S$ and 2$S$ the upper entries are taken from PDG $^3$ and the lower ones from the CLEO data $^{12}$. The numbers concerning the 3$S$ state are also taken from the CLEO data $^{12}$. The values for the states 4$S$, 5$S$, and 6$S$ are taken from PDG $^3$.

$^b$ In the third column only experimental errors are given.
Table 2: The w.f. $|\tilde{R}_n(0)|^2_{\exp}$ (in GeV$^3$) and the leptonic widths $\Gamma_{ee}(n\,^3S_1)$ (in keV ) in charmonium ($\gamma_c = 0.60$)$^a$.

|      | 1S         | 2S         | 3S         | 4S         |
|------|------------|------------|------------|------------|
| $\Gamma_{ee}(\exp)$ | 5.40(22)   | 2.12(12)   | 0.75(1)    | 0.47(15)   |
|      | 5.68(24)   | 2.54(14)   | 0.89(8)    | 0.71(10)   |
| $|\tilde{R}_n(0)|^2_{\exp}$ | 0.911(37)  | 0.51(3)    | 0.22(1)    | 0.16(5)    |
|      | 0.959(40)  | 0.61(3)    | 0.26(2)    | 0.24(4)    |

$^a$ The upper entries are taken from Ref. [3] and the lower ones are taken from Ref. [13].

In the gluon-exchange (GE) term. In particular, if the asymptotic freedom (AF) behavior of $\alpha_{\text{static}}(r)$ is neglected, then the theoretical values can be 2–1.5 times larger than $|\tilde{R}_n(0)|^2_{\exp}$, even for those $\Upsilon(nS)(n = 1, 2, 3)$ states, which lie far below the $BB$ threshold $^8$ (see Table 3 in Appendix A).

In our previous analysis of the spectra and fine structure splittings in heavy quarkonia $^7, ^{14}, ^{15}$ we have used the static potential $V_B(r)$ in which the strong coupling in coordinate space $\alpha_B(r)$ is defined as in background perturbation theory (see Eq. (A.3)). In bottomonium (in the single-channel approximation) the potential $V_B(r)$ appears to give values of $|\tilde{R}_n(0)|^2_{\text{th}}$ very close to the numbers $|\tilde{R}_n(0)|^2_{\exp}$. For illustration in Table 3 the ratios

$$S_n = \frac{|\tilde{R}_n(0)|^2_{\exp}}{|\tilde{R}_n(0)|^2_{\text{th}}}$$

are given for all known $nS$ levels in charmonium and bottomonium.

As shown in Table 3 in bottomonium for the potential $V_B(r)$ the influence of open channels appears to be important only for the $4S$ and $6S$ levels, while for other states the single-channel calculations are in good agreement with
Table 3: The factor $S_n$ defined by Eq. (15) for the potential $V_B(r)$ from Eq. (A.2) for the $\Upsilon(nS)$ and $\psi(nS)$ wave functions at the origin$^a$.

|       | 1S  | 2S  | 3S  | 4S  | 5S  | 6S  |
|-------|-----|-----|-----|-----|-----|-----|
| $b\bar{b}$ | 1.08(4) | 1.02(4) | 1.02(4) | 0.72(9) | 1.03(22) | 0.47(10) |
| $c\bar{c}$ | 1.01(4) | 0.82(5) | 0.41(2) | 0.32(10) |     |     |

$^a$ Given numbers correspond to $\Gamma_{ee}$ taken from PDG [3].

experiment. It is not so for many other potentials [8] and this means that any conclusions about the role of open channels cannot be separated from the $q\bar{q}$ interaction used in a given theoretical approach.

In charmonium the effect of open channels is much stronger and reaches $\sim 60\%$ for the $3S$ and the $4S$ states ($S_n \approx 0.4$) and about $20\%$ for the $\psi(2S)$ meson. This number is even smaller (12\%) if the new CLEO data from Ref. [13] are used.

3 The hyperfine splittings in bottomonium

We consider two cases:

A. No smearing effect, i.e. in Eq. (5) the smearing parameter $\xi_{bn} = 1.0$. (for all $n$).

B. The smearing parameter $\xi_{bn}$, Eq. (6), is calculated with the value $\beta = \sqrt{12}$ GeV, corresponding to the smearing radius $r_{ss} = \beta^{-1} = 0.057$ fm.

Unfortunately, at present there is no precise prescription as to how to choose the renormalization scale in the HF splitting Eq. (3): in $\alpha_{\overline{MS}}(\mu)$ the scale $\mu = m_b$(pole) $\approx 4.80 \pm 0.01$ GeV is often used. With $\Lambda_{\overline{MS}}(n_f = 5) = 210(10)$ MeV (just the same as in our calculations of $\gamma_b$ Eq. (11)), we find

$$\alpha_s(b\bar{b}, \mu) = \alpha_{\overline{MS}}(4.8 \text{ GeV}) = 0.21(1).$$

(16)
Table 4: $\Delta_{HF}^F(nS)$ (in MeV) in bottomonium for $\alpha_{MS}(M_n) = 0.21; \omega_b = 5.10$ GeV and $|R_n(0)|_{\exp}^2$ from Table 1.

| $\xi_b = 1.0$ | $\xi_{bn}$ from Appendix B |
|----------------|----------------------------|
| $\beta = \sqrt{12}$ GeV | $r_{ss} = 0.057$ fm |
| 1S 51 (4) (4) | 28 (2) (3) |
| 2S 25 (3) (2) | 12 (2) (1) |
| 3S 22 (5) (2) | 10 (2) (1) |
| 4S 12 (3) (1) | 5.1 (2) (1) |
| 5S 16 (2) (1) | 6.4 (1) (1) |
| 6S 7 (2) (1) | 2.7 (1) (1) |

With this value of $\alpha_s(\tilde{\mu})$ and $|\tilde{R}_n(0)|_{\exp}^2$ from Table 1, one obtains the HF splittings in bottomonium presented in Table 4 second column.

The numbers in Table 4 contain an experimental error coming from $\Gamma_{ee}(nS)$ [3] (first number), and a theoretical error (second number) $\lesssim 10\%$ ($\lesssim 4\%$ from the $\omega_b$ value and $\lesssim 5\%$ comes from $\alpha_{MS}(m_b)$). For the $\Upsilon(nS)$ states ($n = 1, 2, 3$) the calculated HF splittings ($\xi_n = 1.0$) are very close to the splittings from Refs. [9].

In Table 4 the HF splittings, calculated with the smearing function Eq. (6), are also given (third column). The smearing radius, $r_{ss} = \beta^{-1} = 0.057$ fm ($\beta = \sqrt{12}$ GeV), is taken to fit the experimental values of the $J/\psi - \eta_c(1S)$
and \( \psi(2S) - \eta_c(2S) \) splittings (see the next section). However, even for such a small radius \( \Delta_{\text{HF}}(nS) \) turns out to be 50\% \((n = 1 - 4)\), 60\% \((n = 5, 6)\) smaller compared to the “nonsmearing” case. In particular, the \( \Upsilon(1S) - \eta_b(1S) \) splitting turns out to be 28 MeV instead of 51(4) MeV for \( \xi_{bn} = 1.0 \). For higher excitations very small splittings, \( \Delta_{\text{HF}} \approx 6 \text{ MeV and 3 MeV} \), for the 5S and 6S states are obtained.

Our value of \( r_{ss} = 0.057 \text{ fm} \) is very close to that from Ref. [10] where \( r_{ss} = 0.060 \text{ fm} \) is taken, both in charmonium and in bottomonium. However, in spite of this coincidence our numbers are about two times smaller than in [10], where \( \Upsilon(1S) - \eta_b(1S) = 60 \text{ MeV} \) is obtained. For the 2S state our value of splitting is 12 MeV, still smaller than the value 20 MeV found in Ref. [10].

We can conclude that observation of an \( \eta_b(nS) \) meson could clarify the role of smearing in the spin-spin interaction between a heavy quark and antiquark.

### 4 \( \eta_c(nS) \) masses

The splitting Eq. (15) factually depends on the product \( \alpha_s(\bar{\mu}) \times \xi_n \), therefore it is convenient to discuss an effective HF coupling:

\[
\alpha_{\text{HF}}(nS) = \alpha_s(\bar{\mu}_n) \xi_{cn},
\]

(17)

which is the only unknown factor since \( |\tilde{R}_n(0)|^2 \) can be extracted from the di-electron widths and the constituent masses \( \omega_c(nS) \) are known with \( \lesssim 5\% \) accuracy from relativistic calculations. These masses may be specified for different \( nS \) states, Refs. [14, 15]:

\[
\begin{align*}
\omega_c(1S) &= 1.62(3) \text{ GeV}, & \omega_c(2S) &= 1.71(4) \text{ GeV}, \\
\omega_c(3S) &\approx \omega_c(4S) = 1.73(4) \text{ GeV}.
\end{align*}
\]

(18)

Here the theoretical errors come from a variation of the parameters of the static potential.

As discussed in Ref. [7], the experimental splittings \( J/\psi - \eta_c(1S) \) and \( \psi(2S) - \eta_c(2S) \), can be described if a different \( \alpha_{\text{HF}} \) Eq. (17) for the 1S and 2S states are taken, namely, \( \alpha_{\text{HF}}(1S) \approx 0.36 \) and \( \alpha_{\text{HF}}(2S) \approx 0.30 \). Such a choice implies two possibilities:

A \( \alpha_s(\mu_1) = 0.36, \alpha_s(\mu_2) = 0.30, \alpha_s(\mu_3) = \alpha_s(\mu_4) \leq 0.30, \xi_{cn} = 1.0 \) (for all \( n \)). i.e., the renormalization scale is supposed to grow for larger
excitations. In particular, for the value $\Lambda^{(4)}_{\overline{MS}}(2 - \text{loop}) = 270$ MeV we have $\mu_1 = 1.25$ GeV $\cong \bar{m}_c(\bar{m}_c)$, while the scale $\mu_2 = 1.60$ GeV is essentially larger. For this choice of $\alpha_{HF}$ the perturbative HF splittings are given in Table 5.

B The normalization scale $\mu_n$ (and therefore the coupling constant $\alpha_s(\mu_n)$) is supposed to be equal for all $nS$-states: $\alpha_s(\mu_n) = 0.36$. In this case a smearing of the spin-spin interaction is of principal importance to explain the experimental value for the $\psi(2S) - \eta_c(2S)$ splitting.

Besides, we have also calculated the contributions coming from the NP spin-spin interaction. In bottomonium their values are small, $\Delta_{HF}^{NP}(nS) < 1$ MeV and can be neglected. In charmonium, as well as in light mesons, the situation is different. Due to the NP spin-spin interaction in the $1P c\bar{c}$ state a cancellation of the perturbative and NP terms takes place [18]. As a result, the mass difference $M_{cog}(\chi_{cJ}) - M(h_c) = (1 \pm 1)$ MeV turns out to be close to zero or even positive, in accord with experiment [19]. Just the same NP contribution, Eq. (19), provides the correct value of the splitting $M_{cog}(a_J) - M(b_1(1P)) \approx 22$ MeV in light mesons [20].

The values of $\Delta_{HF}^{NP}(nS)$, (in Table 5, fourth column) are calculated in Appendix C. They can be slightly different for different values of the gluonic correlation length $T_g$ which defines the behavior of the vacuum correlation functions (v.c.f.). For $T_g = 0.3$ fm the v.c.f. has an exponential behavior over the whole range, $0 < r < \infty$ [21] and in this case

$$\Delta_{HF}^{NP}(nS) = \frac{\pi^2 G_2}{18 \omega_c^2} 1.20(1 \pm 0.07) \mathcal{F}(nS),$$

where the number $1.20(1\pm0.07)$ follows from lattice data [21]. In Eq. (19) the gluonic condensate $G_2$ is taken to be equal to 0.043(3) GeV$^4$, as in Ref. [18], and the matrix element (m.e.) is defined as

$$\mathcal{F}(nS) = \langle r K_1 (r/T_g) \rangle_{nS}.$$  

Our calculations give

$$\mathcal{F}(nS) = \begin{align*}
0.80 \text{ GeV}^{-1}(1S), & \quad 0.40 \text{ GeV}^{-1}(2S), \\
0.27 \text{ GeV}^{-1}(3S), & \quad 0.20 \text{ GeV}^{-1}(4S).
\end{align*}$$

The corresponding $\Delta_{HF}^{NP}(nS)$ are given in Table 5. One can see that $\Delta_{HF}^{NP}(1S) = 9 \pm 2$ MeV needs to be taken into account in the $J/\psi - \eta_c(1S)$ splitting while
Table 5: The splittings $\Delta_{HF}^{P}(nS)$ and $\Delta_{HF}^{NP}(nS)$ (in MeV) in charmonium$^a$.

|             | $\Delta_{HF}^{P}(nS)$          | $\Delta_{HF}^{P}(nS)$          | $\Delta_{HF}^{NP}(nS)^b$       |
|-------------|-------------------------------|-------------------------------|---------------------------------|
|             | (no smearing: $\xi = 1.0$)    | $r_{ss} = 0.29 \text{ GeV}^{-1}$| $G_2 = 0.043 \text{ GeV}^4$     |
|             | $\alpha_s(\mu_1) = 0.36$     | $\alpha_s(\mu_n) = 0.36$     |                                 |
|             | $\alpha_s(\mu_n) = 0.30$     | $\alpha_s(\mu_n) = 0.36$     |                                 |
|             | $(n = 2, 3, 4)$               | $(n = 1 - 4)$                 |                                 |
| 1S          | 118(5)                        | 100(6)                        | 9 ± 2                           |
|             |                               | 108(7)$^c$                    |                                 |
|             | experiment                    |                               |                                 |
|             | $J/\psi - \eta_c(1S)$        | 117(2)                        | 117(2)                          |
| 2S          | 51(5)                         | 46(3)                         | 3.5 ± 1.5                       |
|             | 61(5)$^c$                     | 55(4)$^c$                     |                                 |
|             | experiment                    |                               |                                 |
|             | $\psi(2S) - \eta_c(2S)$      | 48(4)                         | 48(4)                           |
| 3S          | 21(2)                         | 16(2)                         | 2 ± 1                           |
| 4S          | 15(4)                         | 12(4)                         | 1.5 ± 0.5                       |

$^a$ The w.f. $|\tilde{R}_n(0)|^2_{\text{exp}}$ is taken from Table 2 and corresponds to $\Gamma_{ee}(nS)$ from PDG [3].

$^b$ The NP splittings are calculated in Appendix C.

$^c$ Here $|\tilde{R}_1(0)|^2_{\text{exp}} = 0.959 \text{ GeV}^3$ and $|\tilde{R}_2(0)|^2_{\text{exp}} = 0.61\text{GeV}^3$ are extracted from the CLEO data [13].
for the higher states the values of $\Delta_{NP}^{HF}$ lie within the experimental and theoretical errors. Their values are $\Delta_{NP}^{HF} = 3(2)$ MeV, 2(1) MeV, and 1.5(5) MeV respectively for the 2S, 3S, and 4S states. A smaller value of $T_g (T_g \lesssim 0.2$ fm), which also cannot be excluded, practically does not change the numbers. Adoption of this value would only slightly decrease those splittings. Thus one can conclude that in case A with different renormalization scales $\mu_n (\mu_1 \approx 1.25\text{ GeV})$ is small, $\mu_2 = 1.6$ GeV, for the $J/\psi - \eta_c(1S)$ and $\psi (2S) - \eta_c(2S)$ splittings agreement with experimental values can easily be obtained.

If the renormalization scales $\mu_n$ are supposed to be equal for all $nS$ states: $\alpha_s(\mu_n \approx \bar{m}_c = 1.25 \text{ GeV}) \approx 0.36$, then to explain the relatively small $\psi(2S) - \eta_c(2S)$ splitting a smearing effect needs to be introduced. Then for the potential used the values $\xi_n(c\bar{c}) = 0.85, 0.80, 0.78, 0.76$ for the 1S, 2S, 3S, and 4S states respectively, are calculated in Appendix B and for this case the values of $\Delta_{P}^{HF}(c\bar{c}, nS)$ are also given in Table 3. For the 3S and 4S levels the values we predict are about 21(15) MeV (no smearing) and 16(12) MeV (with smearing), i.e. the difference between the cases A and B is only $\sim 20\%$. Notice that in case B the NP contribution improves the agreement with experiment for the states $J/\psi - \eta_c(1S)$.

Thus, in the $\psi(nS) - \eta_c(nS)$ splittings the smearing effect appears to be less prominent that in bottomonium.

In Appendix B we also show that the relativistic correlations as well as the constituent masses stop to grow when the many-channel description is effectively used. This fact is very important for the study of higher excitations in charmonium.

5 Conclusions

In our paper we have shown that

1. In bottomonium $\Delta_{P}^{HF}(nS)$ appears to be very sensitive to smearing of the spin-spin interaction. Due to this effect the splitting decreases from 51 MeV to 28 MeV for the 1S state and from 25 MeV to 12 MeV for the 2S state; very small values are obtained for higher states.

2. In charmonium there are two possibilities to describe $\Delta_{HF}(1S)$ and $\Delta_{HF}(2S)$, which are known from experiment. The first one refers to a different choice of the renormalization scale: $\mu_1 = 1.25$ GeV and $\mu_2 \approx$
1.60 GeV for the $1S$ and $2S$ states, if the smearing effect is absent. The second possibility implies equal renormalization scales $\mu_n(n = 1 - 4)$ for all $nS$ states. Then to explain the $\psi(2S) - \eta_c(2S)$ splitting the smearing of the spin-spin interaction needs to be taken into account. We also expect that for the $1S$ level a small contribution ($\sim 9$ MeV) comes from the NP spin-spin interaction.

3. The $\psi(3S) - \eta_c(3S)$ splitting is predicted to be around 16(2) MeV, without and 12(4) MeV with smearing effect.

To understand the true role of the smearing effect in the spin-spin interaction the observation of an $\eta_b(nS)$ is crucially important.

A The wave function at the origin

Two factors strongly affect the w.f. at the origin:

(i) The AF behavior of the coupling in the GE term: $V_0(r) = \sigma r - \frac{4}{3} \alpha_{GE}(r)$,

(ii) The influence of open channels.

For illustrations of these statements in Table 6.1 the w.f.s $|R_n(0)|^2$ are given for several cases.

1. For the Cornell potential, $V_c(r) = \sigma r - \frac{4}{3} \alpha_c$ in single-channel approximation ($\alpha_c = 0.39, \sigma = (2.34)^{-2}$ GeV$^2, \omega_c = 1.84$ GeV).

2. For the same Cornell potential while open channels are taken into account with the use of the nonrelativistic Cornell coupled-channel (CCC) model [22]. Then the w.f. in the coupled-channel system, $\psi_n(r) \equiv \psi(n^3S_1)$, is presented as the composition of different contributions, e.g.

$$
\begin{align*}
\psi_1(1^3S_1) &= 0.983|1S > -0.050|2S > -0.009|3S > -0.003|4S > , \\
\psi_2(2^3S_1) &= 0.103|1S > +0.010|2S > -0.085|3S > \\
&-0.017|4S > -0.007|5S > +0.040|1D > -0.008|2D > , \\
\psi_3(3^3S_1) &= 0.02e^{-i0.05\pi}|1S > +0.19e^{-i0.30\pi}|2S > +0.67|3S > \\
&+0.07e^{i0.54\pi}|4S > +0.04e^{i0.59\pi}|1D > +0.04e^{i0.59\pi}|2D > .
\end{align*}
$$

(A.1)
In single-channel approximation the w.f. $|R_{nS}^C(0)|^2$ for the Cornell potential and the derivatives $|R''_{1D}(0)|^2$ were calculated in [8]: $|R_{1S}^C(0)|^2 = 1.454$ GeV$^3$, $|R_{2S}^C(0)|^2 = 0.927$ GeV$^3$, $|R_{3S}^C(0)|^2 = 0.791$ GeV$^3$, $|R_{4S}^C(0)|^2 = 0.725$ GeV$^3$; $|R''_{1D}(0)|^2 = 0.030$ GeV$^7$; $|R''_{2D}(0)|^2 = 0.0655$ GeV$^7$. From Eq. (A.1) it follows that in the CCC model the $2^3S_1 − 1^3D_1$ and the $3^3S_1 − 2^3D_1$ mixings appear to be small compared to the analysis in [23], where the admixture of the $1^3D_1$ state is $\sim 22\%$.

3. For the potential

$$V_B(r) = \sigma r - \frac{4}{3} \frac{\alpha_B(r)}{r}, \quad (A.2)$$

the coupling is defined as in Refs. [14, 15]:

$$\alpha_B(r) = \frac{8}{\beta_0} \int dq \frac{\sin qr}{q} \left[ 1 - \frac{\beta_1 \ln t_B}{\beta_0^2} \right], \quad (A.3)$$

with

$$t_B(q) = \ln \left( \frac{q^2 + M_B^2}{\Lambda_B^2(n_f)} \right).$$

Here $M_B = 0.95$ GeV, is the background mass, $\Lambda_B(n_f)$ is expressed through the $\Lambda_{\overline{MS}}(n_f)$ and in 2-loop approximation $\Lambda_B(n_f = 4) = 0.360(10)$ MeV [14]. The string tension $\sigma = 0.18$ GeV$^2$ and the same values of $\omega_c = 1.65$ GeV are taken here for simplicity for all $nS$ states.

In Table 6 for comparison the values of $|\tilde{R}_n(0)|^2_{\exp}$ extracted from the di-electron widths are also given. Then one can see that in single-channel approximation very different $|R_n(0)|^2_{\exp}$ are obtained for $V_c(r)$ and $V_B(r)$. For the potential $V_B(r)$ the w.f. $|R_1(0)|^2$ appears to be very close to the “experimental” number: $|\tilde{R}_1(0)|^2 = 0.91$ for $J/\psi$, while for the Cornell potential $|R_n(0)|^2_{\exp}$ is about two times larger than $|\tilde{R}_n(0)|^2_{\exp}$ (single-channel approximation). Moreover, even in the CCC model $|R_n(0)|^2_{\exp}$ is still too large, being $\sim 15\% (30\%)$ larger for the $2S(1S)$ states compared to $|\tilde{R}_n(0)|^2_{\exp}$. Correspondingly, for such multi-channel w.f.s the leptonic widths will be 15% and 30% larger than the experimental value, respectively.

At this point we would like to stress that one cannot fit the leptonic widths $\Gamma_{ee}(nS)$ taking a much smaller value for the factor $\gamma_c$, Eq. (11) (or larger $\alpha_s(\mu)$ in $\gamma_c$). Such a change in the renormalization scale would come in contradiction with the description of other “annihilation” decays like $\eta_c(nS) \rightarrow \gamma \gamma$, ...
Table 6: Comparison of $|\tilde{R}_n(0)|^2_{\text{exp}}$ (in GeV$^3$) from Eq. (13) with the w.f. $|R_n(0)|^2_{\text{th}}$ for the Cornell potential (single-channel approximation and also in the coupled-channel model) and for $V_B(r)$ Eq. (A.2).

|                           | 1S   | 2S   | 3S   |
|---------------------------|------|------|------|
| Cornell potential         |      |      |      |
| single-channel            | 1.454| 0.928| 0.790|
| approximation             |      |      |      |
| coupled-channel           | 1.268| 0.703| 0.520|
| model [21]                |      |      |      |
| $V_B(r)$,                 |      |      |      |
| single-channel approx.    | 0.900| 0.616| 0.534|
| $|\tilde{R}_n(0)|^2_{\text{exp}}$ from Table 2$^a$ | 0.91(4)$^\text{exp}(5)_{\text{th}}$ | 0.51(3)$^\text{exp}(2)_{\text{th}}$ | 0.22(3)$^\text{exp}(1)_{\text{th}}$ |
|                           | 0.96(4)$^\text{exp}(5)_{\text{th}}$ | 0.61(3)$^\text{exp}(2)_{\text{th}}$ | 0.26(3)$^\text{exp}(1)_{\text{th}}$ |

$^a$ See the footnote to Table 2 for the origin of these numbers.
where just the value $\alpha_s(\mu = 2m_c) = 0.24(1)$, as in our analysis is taken. Also we would like to notice that in the CCC model the $2^3S_1 - 1^3D_1$ and $3^3S_1 - 2^3D_1$ mixings are small, as seen from Eqs. (A.1), and instead large admixtures to the $2S(3S)$ states come from the neighboring $1S(2S)$ states.

For the $V_B(r)$ potential Eq. (A.2) we have obtained good agreement with the “experimental” $|\tilde{R}_1(0)|_{\text{exp}}^2$ (for the $J/\psi \sim 5\%$ accuracy) and $20\%$ discrepancy for $\psi(2S)$. For higher states open channels give rise to a suppression of $|\tilde{R}_n(0)|_{\text{th}}^2$ (single-channel approximation) by $\sim 20\%$ for $\psi(2S)$ and about $60\%$ for the $\psi(4040)$ and $\psi(4415)$ mesons. It is precisely the effect of the open channels that appears to be responsible for the screening of the GE interaction which was discussed in Ref. [24], where open channels have been considered effectively through flattening of the confining potential and switching off of the GE interaction.

\section*{B The smearing factor $\xi_n$}

For the smearing function Eq. (4) the factor

$$
\xi_n \equiv \xi(nS) = \frac{4\beta^3}{\sqrt{\pi}} \frac{J_n}{|\tilde{R}_n(0)|^2},
$$

(B.1)

is calculated here for the potential $V_B(r)$ Eq. (A.2). This factor appears to be weakly dependent on a variation of the mass $m_q$. In Table 4 its values are given for the parameter $\beta = \sqrt{12}$ GeV which corresponds to the smearing radius $r_{ss} = \beta^{-1} = 0.057$ GeV. For smaller $\beta$ (larger $r_{ss}$) the smearing effect is becoming even larger (the $\xi_n$ are smaller).

From Table 4 one can conclude that in bottomonium (due to smearing) the HF splittings decrease by a factor of two for all $nS$ levels ($n = 1 - 6$), while in charmonium the smearing effect is weaker and mainly important for the $3S$ and $4S$ states (see Tables 4 and 5).

We would also like to make several remarks about relativistic corrections. Partly this correction is taken into account through the use of the constituent mass $\omega_q$ – the average over the quark kinetic energy Eq. (7). Actually, in the Hamiltonian, or the spinless Salpeter equation (SSE):

$$(T + V)\psi^{R}_{nL}(r_1) = M_0(nL)\psi^{R}_{nL}(r)$$

(B.2)

we use the expansion of the square root near the point $p^2 + m_q^2 - \omega_q^2$. Then
Table 7: The smearing factor $\xi_n$ Eq. (6) in bottomonium and charmonium for the potential $V_B(r)$ Eq. (A.2). $\sigma = 0.1826 \text{ GeV}^2$, $\alpha_B(r)$ is defined by Eq. (A.3).

|       | $\xi_n(\bar{b}b)^{a)}$ | $\xi_n(\bar{c}c)^{b)}$ | $\xi_n(\bar{c}c)^{b)}$ |
|-------|--------------------------|--------------------------|--------------------------|
| 1S    | 0.61                     | 0.85                     | 0.69                     |
| 2S    | 0.55                     | 0.80                     | 0.62                     |
| 3S    | 0.51                     | 0.78                     | 0.57                     |
| 4S    | 0.49                     | 0.76                     | 0.53                     |
| 5S    | 0.47                     | 0.74                     | 0.50                     |

*a) $m_b = 5.1 \text{ GeV}$

*b) $\omega_c = 1.70 \text{ GeV}$
Table 8: The ratio \( \eta = \frac{|R_{nS}^R(0)|^2}{|R_{nS}^A(0)|^2} \) in charmonium for the linear potential \( \sigma_0 r \) and for the flattening potential \( \sigma(r) \cdot r \) \( (\sigma_0 = 0.18 \text{ GeV}^2) \). The mass \( m_c(\text{pole}) = 1.45 \text{ GeV} \). The parameters of \( \sigma(r) = \sigma_0[1 - \gamma(r)] \) are taken from Ref. [24].

|       | 1S  | 2S  | 3S  | 4S  |
|-------|-----|-----|-----|-----|
| \( \sigma_0 r \) potential | 1.108 | 1.196 | 1.258 | 1.310 |
| \( \sigma(r) \cdot r \) potential | 1.093 | 1.153 | 1.147 | 1.045 |

The kinetic term is

\[
T_R = 2 \sqrt{p^2 + m_q^2} \equiv 2\omega \sqrt{1 + \frac{p^2 + m_q^2 - \omega_q^2}{\omega_q^2}} \approx \omega_q + \frac{m_q^2}{\omega_q} + \frac{p^2}{\omega_q} = T_{EA}, \quad (B.3)
\]

which is different from the standard nonrelativistic case. This expansion, or so-called Einbein approximation (EA), appears to be a good approximation to the SSE [26]. While relativistic corrections are important, as in charmonium, it provides much better accuracy than the NR approximation, where the difference between the constituent mass \( \omega_q(nL) \) and \( m_q = \text{const.} \) is neglected. Still, for the w.f. at the origin a small difference between the exact solution \( R_{nL}^R(r) \) of the SSE and an approximate solution for the EA equation occurs

\[
(T_{EA} + V)\psi_{EA}(r) = M_{EA}(nL)\psi_{EA}(r). \quad (B.4)
\]

As an example, in Table 8 the ratio \( \eta = \frac{|R_{nS}^R(0)|^2}{|R_{nS}^A(0)|^2} \) are given for two confining potentials. The first one refers to the single-channel case with the string tension \( \sigma_0 = \text{const.} \). In this case the relativistic correction (the factor \( \eta \)) is growing for higher levels and reaches about 30% for the 4S state.

However, if there are open channels, the linear potential is flattening, and such a flattening potential \( \sigma(r) \cdot r \) effectively takes into account the open channels. This feature provides an essential shift down for high excitations...
Due to the flattening, already for the $3S$ state the factors $\eta$ as well as $\omega_c(3S)$, stop to grow, i.e., an open channel affects the w.f. at the origin much more strongly than the relativistic correction. Since at present there is no relativistic multichannel theory and one cannot distinguish between both effects, we expect that rather small relativistic corrections ($\leq 15\%$) will take place for all $nS$ states in charmonium. Note that this relativistic correction does not affect the smearing factor $\xi_n(\bar{c}c)$.

C Nonperturbative spin-spin interaction

The NP contribution to the HF splitting $M_{\text{cog}}(\chi_{cJ}) - M(h_c)$ has been discussed in detail in Refs. \[18, 20\], where the following general expression has been obtained

$$\Delta_{\text{HF}}^{\text{NP}}(1P) = \frac{\pi^2 g_2^2}{18 \omega_c^2} 1.20(1 \pm 0.07)e^{-\frac{r_0}{T_g}}F(nL). \quad (C.1)$$

Here the matrix element is given by

$$F(nL) = \langle \theta(r-r_0) r K_1(r/T_g) \rangle_{nL}, \quad (C.2)$$

with $T_g$ the gluonic correlation length, which is still not rigorously fixed in a lattice measurement (see the discussion in Ref. \[20\]). The parameter $r_0$ in Eq. (C.1) characterizes the size of the region near $r = 0$ where the vacuum correlation functions are not well defined. In lattice data for large $T_g = 0.3$ fm ($n_f = 4$) it is supposed that $r_0 = 0$ and then the expressions (C.1) and (C.2) go over into Eqs. \[19, 20\]. If $r_0 \neq 0$, (e.g. for $T_g = 0.2$ fm when the value $r_0 \approx T_g = 0.2$ fm) an additional term needs to be added to $\Delta_{\text{HF}}^{\text{NP}}$ (C.1).

At small $r$, $(r \to 0)$, the NP spin-spin potential was defined in Ref. \[18\],

$$\hat{V}_{\text{HF}}^{\text{NP}} = s_1 \cdot s_2 V_{\text{HF}}^{\text{NP}}, \quad V_{\text{HF}}^{\text{NP}}(r \to 0) = \frac{\pi^2}{9\omega_c^2} G_2(r_0 + T_g). \quad (C.3)$$

Then assuming that the v.c.f. has a kind of plateau near the origin, one obtains the following contribution from the region $r \leq r_0$ (Eq. \[18\]) in Ref. \[18\]:

$$V_{\text{HF}}^{\text{NP}}(r \to 0) = \frac{\pi^2}{9\omega_c^2} G_2(r_0 + T_g) \sqrt{r_0^2 - r^2} \theta(r_0 - r). \quad (C.4)$$
Therefore, in the most general case the total NP contribution is
\[
\Delta_{\text{HF}}^{\text{NP}}(nL) = \frac{\pi^2 G_2}{18 \omega_c^2} \left\{ 1.20(1 \pm 0.07) e^{\tau g} \mathcal{F}(nL) \theta(r-r_0) \\
+ 2(r_0 + T_g) \langle \sqrt{r_0^2 - r^2} \rangle_{nL} \theta(r_0-r) \right\}. \tag{C.5}
\]

Note that for \( T_g = r_0 = 0.2 \) fm the second term in Eq. (C.5) is important only for the \( S \)-wave states, being small for the \( P \)-wave states, i.e. for the splitting \( M_{\text{cog}}(\chi_{cJ}) - M(h_c) \) considered in Ref. [18].

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