A Dual Characterization of Observability for Stochastic Systems

Jin Won Kim and Prashant G. Mehta

Abstract—This paper is concerned with a characterization of the observability for a continuous-time hidden Markov model where the state evolves as a general continuous-time Markov process and the observation process is modeled as nonlinear function of the state corrupted by the Gaussian measurement noise. The main technical tool is based on the recently discovered duality relationship between minimum variance estimation and stochastic optimal control: The observability is defined as a dual of the controllability for a certain backward stochastic differential equation (BSDE). For certain cases, a test for observability is described and comparisons provided with results reported in literature. The proposed duality-based framework allows one to easily relate and compare the linear and the nonlinear cases. A side-by-side summary of this relationship is described in a tabular form.

I. INTRODUCTION

This paper is concerned with the definition of observability for a partially observed pair of continuous-time stochastic processes \((X, Z)\) where the state \(X\) is a Markov process and the observation \(Z\) is a nonlinear function of the state corrupted by the Gaussian measurement noise. The precise mathematical model appears in the main body of the paper.

In deterministic linear settings, observability (more generally detectability) and its dual relationship to the controllability are foundational concepts in systems theory; cf., [1]. It is an important property that a model must satisfy to construct asymptotically stable observers [2]. For a partially observed stochastic linear time-invariant (LTI) system, the detectability property of its deterministic counterpart is necessary to deduce results on asymptotic stability of the optimal (Kalman) filter; cf., [3].

Generalization of these concepts to nonlinear deterministic and stochastic systems has been an area of intense research [4], [5], [6], [7]. In settings more general than this paper, the fundamental definition of observability is due to R. van Handel [8], [9]. The definition is useful to establish results on asymptotic stability of the nonlinear filter [10], [11]. Weak notions of van Handel’s observability definition appear in recent papers [12], [13].

In this paper, we utilize the recently discovered duality relationship between minimum variance estimation and stochastic optimal control (see [14]) to define observability as a dual to the controllability. The latter property is somewhat ‘natural’ because it bears close resemblance to the definition of controllability in linear deterministic settings.

The definition of observability is obtained here through the use of duality. In finite state-space settings, certain Kalman-type rank conditions are derived to test for observability. These conditions are shown to be identical to the ones reported in literature [8] but derived here using alternate means. Given the close similarity of the finite state-space hidden Markov model and the deterministic LTI model, these conditions allow one to compare and contrast the differences between the two. This is important given many attempts and successes over the years to apply methods and tools from linear systems theory to study finite-dimensional Markov and hidden Markov models; cf., [15], [16], [17], [18].

The remainder of this paper is organized as follows: The background on the classical deterministic LTI model appears in Sec. II. The nonlinear model is introduced in Sec. III and its stochastic observability defined and discussed in Sec. IV. The proofs appear in the Appendix.

II. BACKGROUND

In linear algebra, it is an elementary fact that the range space of a matrix is orthogonal to the null space of its transpose. In functional analysis, the closed range theorem provides the necessary generalization of this elementary fact in infinite-dimensional settings. The theorem [19, Theorem 6.5.10] states that

\[
\mathbb{R}(L) = \mathbb{N}(L^\dagger)^{\perp}
\]

where \(\mathbb{R}(L)\) is closure of the range space of a bounded linear operator \(L\) and \(\mathbb{N}(L^\dagger)\) is the null space of its adjoint operator \(L^\dagger\). This dual relationship is of fundamental importance to understand the duality between controllability and observability. The overall procedure is as follows:

1. Define the appropriate function spaces and the associated linear operators; and
2. Express controllability and observability properties in terms of range space and null space of these operators.

We illustrate the procedure first in the classical settings.

A. Background: Deterministic LTI settings

Function spaces: Denote \(\mathcal{U} := L^2([0,T];\mathbb{R}^m)\) to be the Hilbert space of \(\mathbb{R}^m\)-valued (input or output) square-integrable signals on the time interval \([0,T]\). The space is equipped with the inner product \((u,v)_\mathcal{U} = \int_0^T u_t^T v_t \, dt\) for \(u,v \in \mathcal{U}\). Denote \(\mathcal{Y} := \mathbb{R}^d\) to be the Euclidean space equipped with the standard inner product \((y_0,y_0')_\mathcal{Y} := y_0^T y_0\) for \(y_0, y_0' \in \mathcal{Y}\).

Operators: For given matrices \(A \in \mathbb{R}^{d \times d}\) and \(H \in \mathbb{R}^{m \times d}\) define a linear operator \(L: \mathcal{U} \to \mathcal{Y}\) as follows:

\[
Lu = \int_0^T e^{A(t-s)} H u_t \, dt =: y_0
\]
The definition of the adjoint operator from the following calculation:

$$(\mathcal{L}u,x_0)_{\mathcal{Y}} = \int_0^T u_t^\top H e^{At} x_0 dt = (u,\mathcal{L}^\dagger x_0)_{\mathcal{U}} \quad (1)$$

Therefore,

$$(\mathcal{L}^\dagger x_0)(t) = H e^{At} x_0 =: z_t \quad \text{for} \ t \in [0,T]$$

**Controllability and observability:** The operator $\mathcal{L}$ defines the map from a given input signal $u = (u_t; 0 \leq t \leq T)$ to the initial condition $x_0$ for the linear system

$$-\dot{y}_t = A^\top y_t + H^\top u_t, \quad y_T = 0 \quad (2)$$

The range space $R(\mathcal{L})$ is referred to as the controllable subspace. The system (2) is said to be controllable if $R(\mathcal{L}) = \mathcal{Y}$.

The adjoint operator $\mathcal{L}^\dagger$ defines the map from a given initial condition $x_0$ to the observation signal $z = (z_t; 0 \leq t \leq T)$ for the linear system

$$\dot{x}_t = A x_t, \quad \text{with init. cond.} \ x_0 \quad (3a)$$
$$z_t = H x_t \quad (3b)$$

The system (3) (henceforth referred to as the linear model $(A,H)$) is said to be observable if $N(\mathcal{L}^\dagger) = \{0\}$.

By the closed-range theorem (or more directly by simply using (1)), $R(\mathcal{L}) = N(\mathcal{L}^\dagger)$. Therefore, the system (3) is observable if and only if the system (2) is controllable. This is useful in the following ways:

1) Definition of observability: as the property of the dual system being controllable.

2) Geometric interpretation of non-observability: If the controllable subspace $R(\mathcal{L}) \not\subseteq \mathbb{R}^d$ then there exists a non-zero vector $\tilde{x}_0 \in N(\mathcal{L}^\dagger)$ such that $y_0^\top \tilde{x}_0 = 0$ for all $y_0 \in R(\mathcal{L})$. The vector $\tilde{x}_0$ has an interpretation of being the “un-observable” direction in the following sense: For any given $x_0 \in \mathbb{R}^d$, $H e^{At} x_0 = H e^{At} (x_0 + \tilde{x}_0)$ for all $t \in [0,T]$. This in turn provides an equivalent definition of observability: The model $(A,H)$ is observable if

$$H e^{At} x_0^{(1)} \equiv H e^{At} x_0^{(2)} \forall t \in [0,T] \text{ implies } x_0^{(1)} = x_0^{(2)} \quad (4)$$

3) Tests for observability: By the use of the Cayley-Hamilton theorem,

$$R(\mathcal{L}) = \text{span} \{H^T, A^T H^T, \ldots, (A^T)^{d-1} H^T \} \quad (5)$$

This provides a straightforward test to verify observability: The model $(A,H)$ is observable if the span on the right-hand side of (5) is $\mathbb{R}^d$.

The aim of this paper is to repeat the above program – viz., (i) the definition of the function spaces $\mathcal{U}$ and $\mathcal{Y}$; (ii) the definition of the linear operator $\mathcal{L}$ and its adjoint $\mathcal{L}^\dagger$; (iii) the mathematical characterization of the controllable subspace $R(\mathcal{L})$; and (iv) its use in definition and geometric interpretation of the observability for a partially observed nonlinear stochastic system.

In the more general settings of this paper, the state evolves as a general continuous-time Markov process. The main restriction is on the observation model which is assumed here to be a nonlinear function of the state corrupted by Gaussian noise. The precise mathematical model is introduced in the following section. However, a summary of the main conclusions and specifically the analogy between the linear-deterministic and the nonlinear-stochastic systems is described as part of the Table I.

### Table I

| Linear-deterministic case (Sec. II-A) | Nonlinear-stochastic case (Sec. IV) |
|------------------------------------|-----------------------------------|
| **Signal space** **$\mathcal{U} = L^2([0,T];\mathbb{R}^m)$** | **$\mathcal{U} = L^2([0,T];\mathbb{R}^m)$** |
| $$(U,V)_{\mathcal{U}} = \int_0^T u_t V_t dt$$ | $$(U,V)_{\mathcal{U}} = \mathbb{E}\left(\int_0^T u_t^T V_t dt\right)$$ |
| **Function space** **$\mathcal{Y} = \mathbb{R}^d$** | **$\mathcal{Y} = \mathcal{C}_x(\mathbb{S})$, $\mathcal{Y}^\dagger = \mathcal{M}(\mathbb{S})$** |
| $$(x,y)_{\mathcal{Y}} = x^\top y$$ | $$(\mu,y)_{\mathcal{Y}} = \mu(y)$$ |
| **Linear operator** **$\mathcal{L} : \mathcal{U} \to \mathcal{Y}$** | **$\mathcal{L} : \mathcal{U} \times \mathcal{R} \to \mathcal{Y}$** |
| $u \mapsto y_0$ by ODE (2) | $$(U,c) \mapsto Y_0$ by BSDE (8)$$ |
| **Adjoint operator** **$\mathcal{L}^\dagger : \mathcal{Y}^\dagger \to \mathcal{U}$** | **$\mathcal{L}^\dagger : \mathcal{Y}^\dagger \to \mathcal{U} \times \mathcal{R}$** |
| $x_0 \mapsto z_t$ by ODE (3) | $\mathbb{R}_0 \mapsto (\pi(h),\mathbb{R}_0(1))$ by Zakai Eq. (10) |
| **Observability** | **Observability** |
| $R(\mathcal{L}) = \mathcal{Y}$ $\iff$ $N(\mathcal{L}^\dagger) = \{0\}$ | $R(\mathcal{L}) = \mathcal{Y}$ $\iff$ $N(\mathcal{L}^\dagger) = \{0\}$ |
| $H e^{At} x_0^{(1)} \equiv H e^{At} x_0^{(2)}$ $\Rightarrow$ $x_0^{(1)} = x_0^{(2)}$ Eq. (4) | $E^\mu(h(X_t)|\mathcal{Z}_t) = E^\nu(h(X_t)|\mathcal{Z}_t) \Rightarrow \mu = \nu$ Eq. (14) |

The system (2) is an example of a backward ordinary differential equation (ODE) because the terminal condition at time $t = T$ is set (to zero in this case).

The nonlinear model is defined for a pair of continuous-time stochastic processes denoted as $(X,Z)$. The details of the model are as follows:
1) The state \( X = \{X_t : 0 \leq t \leq T\} \) is a Markov process that evolves in the state-space \( S \). The generator of the Markov process \( X \) is denoted as \( A \).

2) The observation process \( Z = \{Z_t : 0 \leq t \leq T\} \) is defined according to the following model:

\[
Z_t = \int_0^t h(X_s) \, ds + W_t \tag{6}
\]

where \( h : S \to \mathbb{R}^m \) is a given observation function and \( \{W_t : t \geq 0\} \) is an \( m \)-dimensional Wiener process (w.p.). It is assumed that \( W \) is independent of \( X \).

3) We refer to the above model as the nonlinear model \((A, H)\).

**Notation:** We denote \( Z_t := \sigma(\{Z_s : s \leq t\}) \) to be the \( \sigma \)-algebra generated by the observations up to time \( t \) and \( Z := \{Z_t : 0 \leq t \leq T\} \) is the entire filtration.

The law of \((X, Z)\) is denoted as \( P \) with the associated expectation operator \( E \). To emphasize the model for initial condition \( X_0 \), we use \( P^\mu \) to denote the law of \((X, Z)\) with initial probability measure \( X_0 \sim \mu \).

For the state-space \( S \), let \( B(S) \) denote the Borel \( \sigma \)-algebra on \( S \); \( \mathcal{M}(S) \) is the vector space of (signed) Radon measures on \( B(S) \); and \( \mathcal{P}(S) \subset \mathcal{M}(S) \) is the set of probability measures. \( C_b(S) \) is used to denote the vector space of continuous and bounded functions on \( S \). Throughout this paper, we will use the notation \( \mu(f) := \int_S f(x) \mu(dx) \) to denote the integral of a measurable function \( f \) with respect to the measure \( \mu \).

**B. Example: Finite state-space**

The guiding example in this paper is when the state space \( S \) is finite. Once the results are understood in this basic setting, the generalization to the more general setting is mainly technical. The following notation is adopted in the finite state-space setting:

1) Without loss of generality, it is convenient to consider the state-space \( S = \{e_1, \ldots, e_d\} \) defined by the canonical basis in \( \mathbb{R}^d \).

2) The measure space \( \mathcal{M}(S) \) and the function space \( C_b(S) \) are both identified with \( \mathbb{R}^d \): any function \( f : S \to \mathbb{R} \) determined by its values at the basis vectors \( \{e_i\} \). We denote these values as a column vector \( f \in \mathbb{R}^d \) and express \( \hat{f}(x) = f^T x, x \in S \). In the remainder of this paper, with a slight abuse of notation, we will drop the tilde to simply write \( f(x) = f^T x \).

3) The set \( \mathcal{P}(S) \) is the probability simplex in \( \mathbb{R}^d \). For a measure \( \mu \in \mathcal{M}(S) \), the integral \( \mu(f) = \sum_i \mu(e_i)f(e_i) = \mu^T f \) is a dot product.

4) The observation function \( h(x) = H^T x, x \in S \), where \( H \in \mathbb{R}^{d \times m} \).

5) The generator \( A \) of the Markov process is identified with a row-stochastic rate matrix \( A \in \mathbb{R}^{d \times d} \) which acts on functions (elements of \( \mathbb{R}^d \)) through right-multiplication: \( A : f \to Af \).

6) Given the identification of \( A \) with matrix \( A \) and the observation function with a matrix \( H \), we refer to the Markov chain as the nonlinear model \((A, H)\). One of the goals of the present work is to compare and contrast observability of this model with the linear model \((A, H)\) for the linear deterministic system [5].

**C. Problem Statement**

The main concern of this paper is to define and understand observability for the nonlinear model \((A, H)\). The fundamental definition of observability for stochastic processes \((X, Z)\) is due to R. van Handel:

**Definition 1:** (R. van Handel [8, Sec. 3]) Suppose \( X \) is a Markov process and \( Z \) is defined according to model (6). Suppose \( P^\mu \) and \( P^\nu \) are two laws of the process \((X, Z)\) with initial measure \( X_0 \sim \mu \) and \( X_0 \sim \nu \), respectively. The model is said to be **observable** if

\[
P^\mu\big|_{Z_T} = P^\nu\big|_{Z_T} \quad \Rightarrow \quad \mu = \nu \tag{O1}
\]

where \( P^\mu\big|_{Z_T} \) denotes the restriction of the probability measure \( P^\mu \) to the \( \sigma \)-algebra \( Z_T \).

Before presenting the main result, it is useful to review some concepts from the theory of nonlinear filtering [20, Ch. 5]:

**Change of measure:** Given \( P \), define a new measure \( \hat{P} \) according to the Radon-Nikodym derivative

\[
\frac{d\hat{P}}{dP}(\omega) := \exp\left(-\int_0^T h^T(X_s) \, dz_s + \frac{1}{2} \int_0^T |h(X_s)|^2 \, ds\right)
\]

By the Girsanov theorem, \( Z \) is a \( \hat{P} \) Wiener process. For a given function \( f \), the un-normalized filter is defined by

\[
\sigma_t(f) := \hat{E}(D_t f(X_t) | Z_t)
\]

where \( \hat{E}(\cdot) \) denotes the expectation operator with respect to the new measure \( \hat{P} \) and

\[
D_t = \exp\left(\int_0^t h^T(X_s) \, dz_s - \frac{1}{2} \int_0^t |h(X_s)|^2 \, ds\right)
\]

The un-normalized filter \( \sigma_t(f) \) solves the Zakai equation of nonlinear filtering. The nonlinear filter is given by

\[
E(f(X_t) | Z_t) = \frac{\sigma_t(f)}{\sigma_t(1)} \tag{7}
\]

where \( 1(x) \equiv 1 \forall x \in S \) is the constant function.

**IV. STOCHASTIC OBSERVABILITY**

**A. Function spaces**

In nonlinear settings, the signal space \( \mathcal{U} = L^2_S([0, T]; \mathbb{R}^m) \) is the Hilbert space of \( \mathbb{R}^m \)-valued stochastic processes on \([0, T]\). The subscript \( Z \) denotes the fact that the signals are (forward) adapted to the filtration \( Z \). The space is equipped with the inner product

\[
\langle U, V \rangle_{\mathcal{U}} := \hat{E}\left(\int_0^T U_s^T V_s \, ds\right)
\]

It is noted that the expectation \( \hat{E} \) is with respect to the measure \( \hat{P} \). For the proof that \( \mathcal{U} \) is a Hilbert space with respect to this inner product, cf., [21, p. 99].
The space $\mathcal{Y} = C_b(\mathcal{S})$ and its dual $\mathcal{Y}^* = \mathcal{M}(\mathcal{S})$. For a function $y \in C_b(\mathcal{S})$ and a measure $\mu \in \mathcal{M}(\mathcal{S})$, the dual pairing [22, Ch. 21] is as follows:

$$\langle \mu, f \rangle_\mathcal{Y} = \langle \mu, f \rangle = \mathcal{S} f(x)(\mu(dx))$$

In the finite state-space case, $\mathcal{Y} = \mathcal{Y}^* = \mathbb{R}^d$ and $\langle \mu, y \rangle_\mathcal{Y} = \mu^T y$ is the standard dot product.

### B. Controllability

The goal is to define the controllable subspace as the range space of a certain bounded linear operator. For this purpose, we introduce the backward stochastic differential equation (BSDE):

$$-dY_t(x) = \left( AY_t(x) + h^T(x)(U_t + V_t(x)) \right)dt - V_t^T(x)dz_t, \quad Y_T(x) = c1(x) \quad \forall x \in \mathcal{S}$$

where $c \in \mathbb{R}$ and the input signal $U \in \mathcal{U}$. The solution $(Y, V) := \{(Y_t(x), V_t(x)) : t \in [0, T], x \in \mathcal{S}\}$ of the BSDE is adapted to the filtration $\mathcal{Z}$. For the purposes of this paper, well-posedness (existence, uniqueness and regularity) of the solution $(Y, V) \in L^2_{\mathcal{Z}}([0, T]; C_b(\mathcal{S})) \times L^2_{\mathcal{Z}}([0, T]; C_b(\mathcal{S}; \mathbb{R}^m))$ is assumed; cf., [23]. The justification for considering the BSDE appears in Appendix A where our prior work [14] on the topic of duality is briefly reviewed.

The linear operator $L : \mathcal{U} \times \mathbb{R} \rightarrow \mathcal{Y}$ is defined through the solution of the BSDE as follows:

$$L(U, c) = Y_0$$

and its range space $R(L) = \{Y_0 \in \mathcal{Y} : U \in \mathcal{U}, c \in \mathbb{R}\}$ is referred to as the controllable space. The BSDE (8) is said to be controllable if $R(L)$ is dense in $\mathcal{Y}$. It is noted that in finite state-space settings $R(L)$ is a subspace of $\mathbb{R}^d$. Therefore, in this setting, the system is controllable if $R(L) = \mathbb{R}^d$. Duality is used to propose an indirect definition of observability:

**Definition 2:** The nonlinear model $(A, h)$ is said to be observable if

$$R(L) \text{ is dense in } \mathcal{Y} \quad (O2)$$

A more direct definition is obtained by considering the dual operator.

### C. Observability

In the Prop. 1 (stated below), it is shown that the adjoint to the BSDE (8) is the Zakai equation:

$$\tilde{\pi}_t(f) = \tilde{\pi}_0(f) + \int_0^t \tilde{\pi}_s(Af)ds + \int_0^t \tilde{\pi}_s(h^Tf)dz_s \quad \forall f \in \mathcal{Y}$$

where the initial condition $\tilde{\pi}_0 \in \mathcal{M}(\mathcal{S})$ is given. For a given function $f \in \mathcal{Y}$, the solution of the Zakai equation (10) is denoted as $\tilde{\pi}(f) := \{\tilde{\pi}_t(f) : 0 \leq t \leq T\}$. It is noted that in finite state-space settings, the Zakai equation is simply a linear SDE on $\mathbb{R}^d$ with initial measure $\tilde{\pi}_0 \in \mathbb{R}^d$.

The following proposition is proved in Appendix B

**Proposition 1:** Consider the linear operator (9). Its adjoint $L^T : \mathcal{Y}^* \rightarrow \mathcal{U} \times \mathbb{R}$ is given by

$$L^T \tilde{\pi}_0 = (\tilde{\pi}(h), \tilde{\pi}_0(1))$$

where $\tilde{\pi}(h) = \{\tilde{\pi}_t(h) : 0 \leq t \leq T\}$ is the solution of the Zakai equation (10) with $f = h$ and the initial measure $\tilde{\pi}_0 \in \mathcal{Y}^*$.

For the purposes of defining observability, the adjoint’s null space $N(L^T) = \{\tilde{\pi}_0 \in \mathcal{Y}^* : \tilde{\pi}(h) = 0, \tilde{\pi}_0(1) = 0\}$ is of interest. It is noted that in the finite state-space settings $N(L^T)$ is a subspace of $\mathbb{R}^d$. This suggests a following direct definition of observability:

**Definition 3:** The nonlinear model $(A, h)$ is said to be observable if

$$N(L^T) = \{0\}$$

The two definitions (O2) and (O3) are equivalent: By the closed range theorem, $R(L) = N(L^T)^\perp$. If the controllable subspace $R(L) \subseteq \mathcal{Y}$ then there exists a non-zero measure $\tilde{\pi}_0 \in N(L^T)$ such that $\tilde{\pi}_0(Y_0) = 0$ for all $Y_0 \in R(L)$. The measure $\tilde{\pi}_0$ has an interpretation of being the “un-observable” measure in the following sense: For any given $v \in \mathcal{P}(\mathcal{S})$, suppose $\varepsilon \in \mathbb{R}$ is chosen such that $\mu = \varepsilon + \tilde{\pi}_0 \in \mathcal{P}(\mathcal{S})$ then $E^\mu(h(X)_t)Z_t = E^\varepsilon(h(X)_t)Z_t, \forall t \in [0, T]$. This leads to the third equivalent definition of observability:

**Definition 4:** The nonlinear model $(A, h)$ is said to be observable if

$$E^\mu(h(X)_t)Z_t = E^\varepsilon(h(X)_t)Z_t, \forall t \in [0, T] \Rightarrow \mu = \varepsilon \quad (O4)$$

It is noted that (O4) is the stochastic analog of (4). The proof of the equivalence of the Definitions 2, 3 and 4 appears in Appendix C.

### D. Test for observability

The following theorem provides an explicit characterization of $R(L)$. Its proof appears in the Appendix D.

**Theorem 1:** Consider the linear operator (9). Its range space $R(L)$ is the smallest such subspace $C \subset \mathcal{Y}$ that satisfies the following two properties:

1. The constant function $1 \in C$;
2. If $g \in C$ then $Ag \in C$ and $g \cdot h \in C$. $(g \cdot h)$ is the Hadamard (element-wise) product of functions $g$ and $h$.

As an example, consider the finite state-space model introduced in Sec. 2.3-B with the generator $A$ given by the (row-stochastic) rate matrix $A$ and the function $h(x) = H^T x$. The BSDE (8) is expressed as follows:

$$-dY_t = \left( AY_t + HU_t + \text{diag}^T(HV^T) \right)dt - V_t dz_t, \quad Y_T = c1$$

where $1$ is a vector of ones in $\mathbb{R}^d$ and $\text{diag}^T(HV^T)$ is the vector of the diagonal elements of the matrix $HV^T$. The solution pair is $(Y, V) \in L^2_{\mathcal{Z}}([0, T]; \mathbb{R}^d) \times L^2_{\mathcal{Z}}([0, T]; \mathbb{R}^{d \times m})$.

3. We assume the linear operators are bounded based on the well-posedness of the solution of the BSDE (8).

4. For a vector-valued function $h(x) = [h_1(x), \ldots, h_m(x)]$, $g \cdot h \in \mathcal{C}$ means $g \cdot h_i \in \mathcal{C}$ for each $i = 1, \ldots, m$. The Hadamard product is simply the product of functions, i.e., $(g \cdot h_i)(x) = g(x)h_i(x)$ for all $x \in \mathcal{S}$. 

In nonlinear filtering, the Zakai equation is considered with initial measure $\tilde{\pi}_0 \in \mathcal{P}(\mathcal{S})$. In this paper, the initial measure is allowed to be a signed measure.
Consider now the simple case with scalar-valued observation so \( m = 1 \) and \( H \in \mathbb{R}^{d \times 1} \) is a column vector. In this case (using the dot to denote the element-wise product),

\[
R(\mathcal{L}) = \text{span}\{1, H, AH, A^2H, A^3H, \ldots, \\
H \cdot H, A(H \cdot H), H \cdot (AH), A^2(H \cdot H), \ldots, \\
H \cdot (H \cdot H), (AH) \cdot (H \cdot H), H \cdot A(H \cdot H), \ldots\}
\]

This provides a test to verify observability: The nonlinear model \((A, H)\) is observable if the vectors in the righthand-side of (11) span \(\mathbb{R}^n\). Compare this with the test \(\mathcal{S}\) for observability in the linear model \((A, H)\). It is clear that if the linear model \((A, H)\) is observable (in the sense of \(\mathcal{S}\)) then the nonlinear model is also observable. However, the latter property is in general much weaker than the linear observability. This is shown in the following proposition whose proof appears in Appendix E.

**Proposition 2 (A sufficient condition for observability):** Consider the nonlinear model \((A, H)\) for the finite state-space. Then \((A, H)\) is observable if \(h(x) = H^T x\) is an injective map from \(\mathcal{S}\) into \(\mathbb{R}^n\). (The map is injective if and only if \(H_i + H_j\) for all \(i \neq j\) where \(H_i\) is the \(i\)th row of the \(d \times m\) matrix \(H\)). If \(A = 0\) then the injective property of the function \(h\) is also necessary for observability.

**Remark 1:** The test (11) for observability appears in the work of van Handel [8, Lemma 9]. The test is obtained by explicitly calculating the probability of the event \(\{h(X_{t_i}) = n_1, h(X_{t_2}) = n_2, \ldots, h(X_{t_k}) = n_k\}\) and applying (11) [8, Lemma 8]. For a general class of linear BSDE-s, the controllable subspace is identically defined in [24, Lemma 3.2]. However, its use in the study of observability appears to be new.

**E. Relationship to van Handel’s (Definition 1 of) observability**

R. van Handel’s definition of observability applies to a more general class of stochastic processes \((X, Z)\) whereby the independent increment of the measurement noise may not be Gaussian (as is assumed here). For the finite state space example, the two definitions of observability yield the same test for observability (see Remark 1). This suggests that the two definitions are possibly equivalent for the class of models considered in this paper. A partial result in this direction appears in the following proposition whose proof appears in Appendix E.

**Proposition 3:** If the model \((A, h)\) is observable (according to one of the equivalent Definitions 2, 3 or 4) then it is also observable according to Definition 1.

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**APPENDIX**

**A. Duality between estimation and control**

This section includes a brief review of the duality between nonlinear filtering and stochastic optimal control introduced in our recent paper [14].

**Dual optimal control problem:**

\[
\begin{align*}
\text{Min}_{U \in \mathcal{U}} & \quad J(U) = E \left[ \frac{1}{2} |Y_0(X_0) - \bar{\nu}(X_0)|^2 + \int_0^T \left( \ell(Y_t, V_t, U_t; X_t) \right) \, dt \right] \\
\text{Subj.} & \quad - dY_t(x) = \left( A_V Y_t(x) + h^T(x)(U_t + V_t(x)) \right) \, dt - V_t^T(x) \, dZ_t \\
& \quad Y_T(x) = f(x) \quad \forall x \in \mathcal{S}
\end{align*}
\]
where the set of admissible control $U := L^2_\mathbb{F}([0,T]; \mathbb{R}^m)$ and the cost function
\[
l(y,v,u;x) = \frac{1}{2}Q(y;x) + \frac{1}{2}(u + v(x))^\top R(u + v(x))
\]
where $Q(y;x)$ is a certain non-negative function. Explicit formulæ of $Q$ for particular examples (finite state-space and Itô-diffusions) of Markov processes appear in [14, Sec. 2]. It is noted that the constraint is a backward stochastic differential equation (BSDE) with solution $(Y,V) := (Y_t,V_t): t \in [0,T] \in L^2_\mathbb{F}([0,T]; C_b(S)) \times L^2_\mathbb{F}([0,T]; C_b(S; \mathbb{R}^m))$. The terminal condition $f \in C_b(S)$ is prescribed.

Consider the following linear structure of the estimator:
\[
S_T = \pi_0(Y_0) - \int_0^T U_t^\top dZ_t
\]
where $U \in U$ is an admissible control and $Y_0$ is obtained by (12). The precise duality relationship is as follows:

**Proposition 4 (Prop. 1 in [14]):** Consider the observation model (9), together with the dual optimal control problem (12). Then for any choice of admissible control $U \in U$:
\[
J(U) = \frac{1}{2}E(S_T - f(X_T))^2
\]

The significance of the duality relationship is as follows: The problem of obtaining the minimum variance estimate $S_T$ of $f(X_T)$ (minimizer of the right-hand side of the equality) is converted into the problem of finding the optimal control $U$ (minimizer of the left-hand side of the identity). Additional details including the use of the dual optimal control problem (12) to derive the nonlinear filter can be found in [14].

**B. Proof of Prop. 7**

By linearity, $L(U;c) = L(U;0) + c1$ for $U \in U$ and $c \in \mathbb{R}$. Therefore, for $\pi_0 \in Y^\top$,
\[
(\pi_0, L(U;c))_Y = (\pi_0, L(U;0))_Y + c \pi_0(1)
\]
Thus, the main calculation is to transform $(\pi_0, L(U;0))_Y$. For this purpose, consider (8) with $c = 0$ and express $(\pi_0, L(U;0))_Y = \pi_0(Y_0)$.

Using the Itô-Wentzell formula for measures [25, Theorem 1.1],
\[
d(\pi_t(Y_t)) = \left(\pi_t(A Y_t) dt + \pi_t(h^\top Y_t) dZ_t \right) + \pi_t(h^\top V_t) dt + \pi_t(-A Y_t - h^\top U_t - h^\top V_t) dt + \pi_t(V_t) dZ_t
\]
\[= -U_t^\top \pi_t(h) dt + \pi_t(h^\top Y_t + V_t^\top) dZ_t
\]
Integrating both sides,
\[
\pi_t(Y_t) - \pi_0(Y_0) = \int_0^t U_s^\top \pi_s(h) ds + \int_0^t \pi_s(h^\top Y_s + V_s^\top) dZ_s
\]
Under the probability measure $\tilde{P}$, $Z$ is a Wiener process. Hence,
\[
\pi_0(Y_0) = E\left(\int_0^T U_t^\top \pi_t(h) ds\right) = (\pi(h), U)_U
\]
Therefore,
\[
(\pi_0, L(U;c))_Y = (\pi(h), U)_U + c \pi_0(1)
\]

**C. Proof of equivalency of observability definitions**

(13) and (14) are equivalent by the closed range theorem. The proof of (14) $\iff$ (13) is presented next.

**Necessity:** We first show $\text{(O3)} \implies \text{(O4)}$. For a given $\pi_0 \in N(L^\top)$, there exist $\nu, \nu \in P(S)$ such that $\epsilon \pi_0 = \mu - \nu$ for some constant $\epsilon = 0$ [22, Theorem 21.2]. By linearity of the Zakai equation (10),
\[
\epsilon \pi_0(h) = \sigma^\mu(h) - \sigma^\nu(h)
\]
Since $\pi_0 \in N(L^\top)$ implies $\pi_t(h) \equiv 0$,
\[
\sigma^\mu(h) = \sigma^\nu(h) \quad \forall t \in [0,T]
\]
Using the Zakai Eq. (10) with $f = 1$ (the constant function),
\[
\sigma^\mu(1) = 1 + \int_0^T \sigma^\mu(h^\top) dZ_s
\]
Therefore (13) implies that the normalization constant $\sigma^\mu(1) = \sigma^\nu(1)$ for all $t \in [0,T]$. Thus, using (7),
\[
E^\mu(h(X_t)|Z_t) = E^\nu(h(X_t)|Z_t) \quad \forall t \in [0,T]
\]

**Sufficiency:** Assume (13) is not true: There exists $\mu \neq \nu \in P(S)$ such that $E^\mu(h(X_t)|Z_t) = E^\nu(h(X_t)|Z_t)$ for all $t \in [0,T]$. Now (14) and (13) are combined into:
\[
\sigma^\mu(1) = 1 + \int_0^T \sigma^\mu(h^\top) dZ_s
\]
This shows $E^\mu(h(X_t)|Z_t) = E^\nu(h(X_t)|Z_t)$ implies $\sigma^\mu(1) = \sigma^\nu(1)$ for all $t \in [0,T]$, and therefore $\pi_t(h) \equiv \pi^\nu(h)$ by (7). Since $\mu(1) - \nu(1) = 0$, $\mu - \nu \in N(L^\top)$.

**D. Proof of Theorem 7**

For notational ease, we assume $m = 1$. The objective is to show $C = R(L)$. The proof below is adapted from [24].

The definition of $N(L^\top)$ is:
\[
\pi_0 \in N(L^\top) \iff \pi_0(1) = 0 \text{ and } \pi_t(h) \equiv 0 \quad \forall t \in [0,T]
\]
Since $N(L^\top)$ is the annihilator of $R(L)$, we have $1, h \in R(L)$. Consider next the Zakai equation (10) with the initial condition $\pi_0 \in N(L^\top)$ and $f = h$:
\[
\pi_t(h) = \tilde{\pi}_0(h) + \int_0^t \tilde{\pi}_s(A h) ds + \int_0^t \tilde{\pi}_s(h^2) dZ_s
\]
Since $t$ is arbitrary, the left-hand side is identically zero for all $t \in [0,T]$ if and only if
\[
\tilde{\pi}_0(h) = 0, \quad \tilde{\pi}_t(A h) \equiv 0, \quad \tilde{\pi}_t(h^2) \equiv 0 \quad \forall t \in [0,T]
\]
and in particular, this implies $A h, h^2 \in R(L)$.

The subspace $C$ is obtained by continuing to repeat the steps ad infinitum: If at the conclusion of the $k^{th}$ step, we find a function $g \in C$ such that $\tilde{\pi}_t(g) \equiv 0$ for all $t \in [0,T]$. Then through the use of the Zakai equation,
\[
\tilde{\pi}_0(g) = 0, \quad \tilde{\pi}_t(A g) \equiv 0, \quad \tilde{\pi}_t(h g) \equiv 0 \quad \forall t \in [0,T]
\]
s $Ag, hg \in C$. By construction, because $\tilde{\pi}_0 \in N(L^\top)$, $C = R(L)$. 


E. Proof of Prop. 2

Step 1: We first provide the proof for the case when \( m = 1 \). In this case, \( H \) is a column vector and \( H_i \) denotes its \( i \)th element. We claim that:

\[
\text{span}\{1, H, H \cdot H, \ldots, H \cdot H \cdot \cdots \cdot H\} = \mathbb{R}^d \tag{15}
\]

where (as before) the dot denotes the element-wise product. Assuming that the claim is true, the result easily follows because the vectors on left-hand side are contained in \( \mathbb{R}(L) \) (see (11)). It remains to prove the claim. For this purpose, express the left-hand side of (15) as the column space of the following matrix:

\[
\begin{pmatrix}
1 & H_1 & H_1^2 & \ldots & H_1^{d-1} \\
1 & H_2 & H_2^2 & \ldots & H_2^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & H_d & H_d^2 & \ldots & H_d^{d-1}
\end{pmatrix}
\]

This matrix is easily seen to be full rank by using the Gaussian elimination:

\[
\begin{pmatrix}
1 & H_1 & H_1^2 & \ldots & H_1^{d-1} \\
0 & H_2 - H_1 & H_2^2 - H_1^2 & \ldots & H_2^{d-1} - H_1^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \prod_{i=1}^{d-1} (H_d - H_i)
\end{pmatrix}
\]

The diagonal elements are non-zero because \( H_i \neq H_j \).

Step 2: In the general case, \( H \) is a \( d \times m \) matrix and \( H_i \) denotes its \( i \)th row. We claim that if \( H_i \neq H_j \) for all \( i \neq j \) then there exists a vector \( \tilde{H} \) in the column span of \( H \) such that \( \tilde{H}_i \neq \tilde{H}_j \) for all \( i \neq j \). Assuming that the claim is true, the result follows from the \( m = 1 \) case by considering (15) with \( \tilde{H} \). It remains to prove the claim. Let \( \{e_1, \ldots, e_d\} \) denote the canonical basis in \( \mathbb{R}^d \). The assumption means \( (e_i - e_j)^\top H \) is a non-zero row-vector in \( \mathbb{R}^m \) for all \( i \neq j \). Therefore, the null-space of \( (e_i - e_j)^\top H \) is a \( (m-1) \)-dimensional hyperplane in \( \mathbb{R}^m \). Since there are only \( \binom{m}{2} \) such hyperplanes, there must exist a vector \( a \in \mathbb{R}^m \) such that \( (e_i - e_j)^\top H a \neq 0 \) for all \( i \neq j \). Pick such an \( a \) and define \( \tilde{H} := Ha \).

Step 3: To show the necessity of the injective property when \( A = 0 \), assume \( H_i = H_j \) for some \( i \neq j \). Then the corresponding row is identical, so it cannot be rank \( d \).

F. Proof of Prop. 3

We show the following:

\[
P^\mu|_{\mathcal{H}_T} = P^\nu|_{\mathcal{H}_T} \Rightarrow E^\mu(h(X_t)|Z_t) = E^\nu(h(X_t)|Z_t) \quad \forall t \in [0, T]
\]

Let \( \mathcal{H}_T = \sigma(\{h(X_t) : t \in [0, T]\}) \). By [8, Sec. 5.1], if \( P^\mu|_{\mathcal{H}_T} = P^\nu|_{\mathcal{H}_T} \), then \( P^\mu|_{\mathcal{H}_T} = P^\nu|_{\mathcal{H}_T} \). This is because of the additive Gaussian nature of the measurement noise.

Recall the definition of conditional expectation:

\[
\int_B h(X_t) dP^\mu = \int_B E^\mu(h(X_t)|Z_t) dP^\mu, \quad \forall B \in \mathcal{Z}_t
\]

Since \( E^\mu(h(X_t)|Z_t) \) is a \( \mathcal{Z}_T \)-measurable random variable,

\[
\int_B E^\mu(h(X_t)|Z_t) dP^\mu = \int_B E^\nu(h(X_t)|Z_t) dP^\nu
\]

Now because \( P^\mu|_{\mathcal{H}_T} = P^\nu|_{\mathcal{H}_T} \),

\[
\int_B h(X_t) dP^\mu = \int_B h(X_t) dP^\nu
\]

By combining these three equations, we conclude

\[
\int_B (h(X_t) - \nu_t) dP^\mu = \int_B (h(X_t) - \nu_t) dP^\nu, \quad \forall B \in \mathcal{Z}_t
\]

Therefore,

\[
E^\mu(h(X_t)|Z_t) = E^\nu(h(X_t)|Z_t) \quad \forall t \in [0, T]
\]

The result follows.