Entropic uncertainty relations from equiangular tight frames and their applications

Alexey E. Rastegin
Department of Theoretical Physics, Irkutsk State University, K. Marx St. 1, Irkutsk 664003, Russia

Finite tight frames are interesting in various topics including questions of quantum information. Each complex tight frame leads to a resolution of the identity in the Hilbert space. Symmetric informationally complete measurements are a special class of equiangular tight frames. Applications of such frames in quantum physics deserve more attention than they have obtained. We derive uncertainty relations for a quantum measurement assigned to an equiangular tight frame. Main results follow from estimation of the corresponding index of coincidence. State-dependent and state-independent formulations are both addressed. Also, we discuss applications of considered measurements to detect entanglement and other correlations.

Keywords: finite frames, generalized entropies, uncertainty relations, entanglement detection

I. INTRODUCTION

Studies of discrete structures in finite-dimensional Hilbert spaces has a long history [1, 2]. Such structures are interesting not only in their own rights but also due to potential application in quantum physics. Emerging technologies of quantum information processing gave a new stimulus to investigate finite sets of states with special properties. Mutually unbiased bases (MUBs) are one of the most known examples of this kind [3, 4]. Another remarkable paradigm of discrete structures is given by symmetric informationally complete measurements [5, 6]. Quantum and unitary designs are considered as a powerful tool of quantum information theory [7–10]. On the other hand, the question of building such structures are often difficult to resolve [11, 12]. For instance, the maximal number of MUBs remains unknown even for $d = 6$, i.e., for the smallest dimension that is not a prime power.

Finite tight frames are a natural generalization of orthonormal bases [13]. Each of such frames can be applied to build a positive operator-valued measure (POVM). POVMs are an indispensable tool in quantum information science [14]. Tight frames have also found use in signal processing and coding [15]. In equiangular tight frames (ETFs), any two frame vectors have the same overlap. In a certain sense, this idea is similar to that is used to define MUBs. Maximal sets of complex equiangular vectors provide symmetric informationally complete POVMs [13]. The existence of such POVMs for arbitrary dimensions is still an open question, though several exact constructions have been found [11, 16, 17]. This gives a reason to study arbitrary ETFs in more detail and use them together with the maximal ones.

In this paper, we consider entropic uncertainty relations for a quantum measurement built of the states of an ETF. Potential applications in quantum information science will be mentioned as well. The paper is organized as follows. In Sec. II the required material on complex tight frames is briefly recalled. Section III is devoted to estimate from above the corresponding index of coincidence, whence many results will be obtained. Fine-grained uncertainty relations are examined in Sec. IV. In Sec. V we formulate entropic uncertainty relations for quantum measurements assigned to ETFs. To quantify the amount of related uncertainties, the Rényi and Tsallis entropies are both used. In Sec. VI the derived relations are applied to entanglement detection, inequalities for conditional von Neumann entropies and quantum coherence. In Sec. VII we conclude the paper with a summary of the results.

II. PRELIMINARIES

In this section, we recall some material concerning ETFs. The authors of [18, 19] discussed the existence of such frames in both the real and complex cases. In the following, all the frames are assumed to be complex. Let $\mathcal{H}_d$ be a $d$-dimensional Hilbert space. A set of $n \geq d$ unit vectors $\mathcal{F} = \{ |\phi_j\rangle \}$ is called a frame if there exist strictly positive numbers $S_0 < S_1 < \infty$ such that

$$S_0 \leq \sum_{j=1}^{n} |\langle \phi_j | \psi \rangle|^2 \leq S_1$$

for all unit $|\psi\rangle \in \mathcal{H}_d$. The numbers $S_0$ and $S_1$ are the minimal and maximal eigenvalues of the frame operator

$$S = \sum_{j=1}^{n} |\phi_j\rangle \langle \phi_j| .$$

The special case $S_0 = S_1 = n/d$ gives a tight frame. Then the frame operator is scalar with the eigenvalue $S = n/d$ of multiplicity $d$. Parseval tight frames obtained with $S = 1$ are equivalent to orthonormal bases. A special kind of
tight frames is known as equiangular ones. The tight frame \( \mathcal{F} \) is called equiangular, when there exists \( c > 0 \) such that
\[
|\langle \phi_i | \phi_j \rangle|^2 = c
\] (2.3)
for each pair \( i \neq j \). By calculations, for an ETF, we have
\[
S = nc + 1 - c = \frac{n}{d}, \quad c = \frac{S - 1}{n - 1} = \frac{n - d}{(n - 1)d}.
\] (2.4)
If there is an ETF with \( n \) elements in dimension \( d \), then \( n \leq d^2 \) and also exists an ETF with \( n \) elements in dimension \( n - d \) \[18, 19\]. The second fact deserves to be considered in more detail. Let us put the \( d \times n \) matrix
\[
\Phi = (|\phi_1 \rangle \cdots |\phi_n \rangle),
\] (2.5)
where the frame states stand as columns. Since \((d/n)\Phi\Phi^\dagger = 1_d\) for an ETF, rescaled rows of \( \Phi \) form an orthonormal set. One can convert \((d/n)\Phi\Phi^\dagger \) into a unitary \( n \times n \) matrix by adding \( n - d \) rows that are mutually orthogonal as well. Collecting these rows into \((n - d) \times n\) matrix and normalizing its columns gives other ETF \[19\]. The latter contains \( n \) vectors in dimension \( n - d \). This way is very close to the method used to build a Naimark extension of rank-one POVM with \( n \) elements. It is important here that the given approach is constructive. In the least case \( n = d^2 \), it is seen from (2.3) and (2.4) that
\[
|\langle \phi_i | \phi_j \rangle|^2 = \frac{1}{d + 1} \quad (i \neq j).
\]
Here, we deal with a symmetric informationally complete measurement (SIC-POVM). As was already mentioned, there is a way to generate new ETFs from the given ones. In particular, a \( d \)-dimensional SIC-POVM induces ETFs in dimensions \( d(d + 1)/2 \) and \( d(d - 1)/2 \) \[12\]. Recently, some generalizations of the above ideas were proposed. The authors of \[20\] introduced the concept of mutually unbiased frames as a general framework for studying unbiasedness. Equioverlapping measurements were considered in \[21\].

The states of an ETF induce the resolution \( \mathcal{E} = \{E_j\} \) of the identity, namely
\[
\frac{d}{n} S = \sum_{j=1}^n E_j = 1_d.
\] (2.6)
Here, the POVM elements are expressed as
\[
E_j = \frac{d}{n} |\phi_j \rangle \langle \phi_j |.
\] (2.7)
When the pre-measurement state is described by density matrix \( \rho \) with \( \text{tr}(\rho) = 1 \), the probability of \( j \)-th outcome is equal to
\[
p_j(\mathcal{E}; \rho) = \frac{d}{n} \langle \phi_j | \rho |\phi_j \rangle.
\] (2.8)

Developing some ideas of \[22\], we aim to derive entropic uncertainty relations for POVMs assigned to ETFs. The composed technique is similar to the method used for mutually unbiased bases \[23\]. Properties of the mentioned discrete structures allow one to impose certain restrictions on generated probabilities. The authors of \[24\] introduced the notion of general symmetric informationally complete measurements. These measurements are not necessarily described by elements of rank one. Similarly, the set of \( d + 1 \) mutually unbiased measurements can be built for arbitrary \( d \), when rank-one projectors are not required \[25\]. Entropic uncertainty relations were obtained for mutually unbiased measurements \[26, 27\] and general SIC-POVMs \[28\]. However, excessive costs may be required to implement measurements with elements that are not of rank one. Thus, ETF-based measurements are still of interest. Due to the results of \[29\], we can often restrict a consideration to POVMs with rank-one elements. For such measurements, there is a Naimark extension with only \( n - d \) extra dimensions (see, e.g., the comments right after (2.5)). In more detail, the question of building a Naimark extension is addressed in section 9-6 of \[30\].

III. ON THE INDEX OF COINCIDENCE

The index of coincidence is used in various questions of information theory \[31\]. In cryptography, for example, it gives a measure of the relative frequency of symbols in a ciphertext sample \[32\]. Now, we aim to estimate this
Finally leads to whence the normalization factor of (3.3) follows.

The inverse square root of (3.6) stands right before the sum in the right-hand side of (3.2). Substituting to check the normalization, we note that

$$\langle \varphi\rangle^* = \langle \varphi|\psi\rangle^* = \langle \psi|\varphi\rangle^*.$$ Let us begin with an auxiliary result.

**Proposition III.1** Let \( n \) unit kets \(|\phi_j\rangle\) form an equiangular tight frame in \( \mathcal{H}_d \), and let

\[
|\Psi_0\rangle = \frac{1}{\sqrt{ns}} \sum_{j=1}^{n} |\phi_j\rangle \otimes |\phi_j^*\rangle,
\]

\[
|\Psi_k\rangle = \frac{1}{\sqrt{n - nc}} \sum_{j=1}^{n} \omega_n^{k(j-1)} |\phi_j\rangle \otimes |\phi_j^*\rangle,
\]

where \( k = 1, \ldots, n - 1 \) and \( \omega_n \) is a primitive \( n \)-th root of unity. Then the vectors \|\Psi_0\rangle\) and \|\Psi_k\rangle\) form an orthonormal set in the space \( \mathcal{H}_d \otimes \mathcal{H}_d \).

**Proof.** First, we aim to show that the vectors \(|\Psi_0\rangle\) and \( |\Psi_k\rangle\) are mutually orthogonal. Up to a factor, the inner product \( \langle \Psi_q|\Psi_k\rangle \) with \( k \neq 0 \) is represented as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \omega_n^{k(j-1)} |\langle \phi_i|\phi_j\rangle|^2 = \sum_{j=1}^{n} \omega_n^{k(j-1)} + c \sum_{i,j=1}^{n} \omega_n^{k(j-1)}.
\]

By the definition of \( \omega_n \), one has \( \sum_{j=1}^{n} \omega_n^{k(j-1)} = 0 \) for \( k = 1, \ldots, n - 1 \). We assign this zero sum by the factor \( c \), whence the right-hand side of (3.4) becomes

\[
c \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_n^{k(j-1)} = nc \sum_{j=1}^{n} \omega_n^{k(j-1)} = 0.
\]

Up to a common factor, we express \( \langle \Psi_q|\Psi_k\rangle \) with \( k, q \neq 0 \) as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \omega_n^{-q(i-1)} \omega_n^{k(j-1)} |\langle \phi_i|\phi_j\rangle|^2 = \sum_{j=1}^{n} \omega_n^{(k-q)(j-1)} + c \sum_{i,j=1}^{n} \omega_n^{k(j-1)-q(i-1)}.
\]

For \( q \neq k \), the first sum in the right-hand side of (3.5) is zero. Multiplying it by \( c \), one reduces \( \sum_{j=1}^{n} \omega_n^{k(j-1)} \) to the form

\[
c \sum_{i=1}^{n} \omega_n^{-q(i-1)} \sum_{j=1}^{n} \omega_n^{k(j-1)} = 0.
\]

To check the normalization, we note that

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i|\phi_j\rangle|^2 = n + (n^2 - n)c = ns.
\]

The inverse square root of (3.6) stands right before the sum in the right-hand side of (3.2). Substituting \( q = k \) in (3.5) finally leads to

\[
n + c \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_n^{k(j-1)} = nc = n - nc,
\]

whence the normalization factor of (3.3) follows. \( \square \)
The statement of Proposition III.1 generalizes one of the results of [22]. For a SIC-POVM, we have $n = d^2$, so that the vectors (3.2) and (3.3) form an orthonormal basis and induce a unitary matrix of size $d^2$. It is instructive to consider explicitly an example of such basis. The simplest construction in dimension two contains the kets
\[ |\phi_1\rangle = |0\rangle, \quad \sqrt{3} |\phi_2\rangle = |0\rangle + \sqrt{2} |1\rangle, \quad \sqrt{3} |\phi_3\rangle = |0\rangle + \sqrt{2} \omega_3 |1\rangle, \quad \sqrt{3} |\phi_4\rangle = |0\rangle + \sqrt{2} \omega_3^4 |1\rangle, \] (3.8)
where $\omega_3 = \exp(12\pi/3)$. Calculating the vectors (3.2)–(3.3) in the canonical basis leads to the unitary matrix
\[
\frac{1}{2\sqrt{3}} \begin{pmatrix}
\sqrt{6} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
0 & (1 + \sqrt{3}) \omega_6 & -2 \omega_6 & (1 - \sqrt{3}) \omega_6 \\
0 & (1 - \sqrt{3}) \omega_6^5 & -2 \omega_6^5 & (1 + \sqrt{3}) \omega_6^5 \\
\sqrt{6} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2}
\end{pmatrix}.
\] (3.9)

It is shown in [12] that associated to a SIC in dimension $d$ there exist a projector in dimension $d^2$ and a Hermitian Hadamard matrix of size $d^2$. A square matrix $M$ of size $n$ consisting of unimodular entries is called a Hadamard one, when $M^\dagger M = n I_n$. Its rescaling leads to a unitary matrix all of whose elements have the same absolute value. The matrix built in corollary 2 of [12] is a Hermitian Hadamard one. In contrast, the matrix (3.9) is not proportional to a Hermitian one.

To formulate entropic uncertainty relations for an ETF-based measurement, we proceed to estimating the index of coincidence (3.1) from above. Let us extend one of the derivations presented in [22] for a SIC-POVM.

**Proposition III.2** Let $n$ unit kets $|\phi_i\rangle$ form an ETF in $\mathcal{H}_d$, and let POVM $\mathcal{E}$ be assigned to this frame by (2.7). For the given density matrix $\rho$, it holds that
\[
I(\mathcal{E}; \rho) \leq \frac{S c + (1 - c) \text{tr}(\rho^2)}{S^2},
\] (3.10)
where $S$ and $c$ obey (2.7).

**Proof.** For arbitrary operator $K$ on $\mathcal{H}_d$ and ket $|\psi\rangle$, one has
\[
\sum_{i=1}^n \sum_{j=1}^n \langle \phi_i | K | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{n^2}{d^2} \text{tr}(K)
\] (3.11)
and
\[
\sum_{j=1}^n |\phi_j\rangle \langle \phi_j | \psi \rangle = \frac{n}{d} |\psi\rangle.
\] (3.12)

They follow by substituting the resolution of the identity in $\text{tr}(\mathbb{1}_d K \mathbb{1}_d) = \text{tr}(K)$ and $\mathbb{1}_d |\psi\rangle = |\psi\rangle$, respectively. Let us represent $\rho \otimes \mathbb{1}_d |\Psi_0\rangle$ in the form
\[
\rho \otimes \mathbb{1}_d |\Psi_0\rangle = \sum_{k=0}^{n-1} a_k |\Psi_k\rangle + |\Theta\rangle,
\] (3.13)
where $\langle \Theta | \Psi_k \rangle = 0$ for all $k = 0, 1, \ldots, n - 1$. It follows from (3.12) and (3.11) that
\[
a_0 = \langle \Psi_0 | \rho \otimes \mathbb{1}_d |\Psi_0\rangle = \frac{1}{n S} \sum_{i=1}^n \sum_{j=1}^n \langle \phi_i | \rho | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{n}{S d^2} = \frac{1}{d}.
\] (3.14)

Using (3.12), for $k \neq 0$, we write the coefficient
\[
a_k = \langle \Psi_k | \rho \otimes \mathbb{1}_d |\Psi_0\rangle = \frac{1}{n \sqrt{S - S c}} \sum_{i=1}^n \sum_{j=1}^n \omega_n^{-k(i-1)} \langle \phi_i | \rho | \phi_j \rangle \langle \phi_j | \phi_i \rangle
\]
\[
= \frac{S^2}{n \sqrt{S - S c}} \sum_{i=1}^n \omega_n^{-k(i-1) p_i},
\] (3.15)
Similarly to (3.14), one also calculates

$$\langle \Psi_0 | (\rho \otimes \mathbb{1}_d)^2 | \Psi_0 \rangle = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i | \rho^2 | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{\text{tr}(\rho^2)}{d^2} \geq \frac{1}{d^2} + \sum_{k=1}^{n-1} a_k^2.$$

Due to (3.15) and \( \sum_{k=1}^{n-1} k(i-j) = n^2 \), we obtain

$$\sum_{k=1}^{n} a_k^2 = \frac{S^3}{n^2(1-c)} \sum_{i=1}^{n-1} \sum_{j=1}^{n} p_i p_j \sum_{k=1}^{n-1} c_k^{i+j} = \frac{n^2 I(\mathcal{E}; \rho) - n}{d^3(1-c)}.$$

Combining (3.16) with (3.17) leads to

$$n^2 I(\mathcal{E}; \rho) \leq n - d(1-c) + d^2(1-c) \text{tr}(\rho^2) = ncd + d^2(1-c) \text{tr}(\rho^2),$$

since \( n - d(1-c) = ncd \) in line with (3.1). Dividing (3.18) by \( n^2 \) gives (3.10) due to \( S = n/d \).

For the maximally mixed state \( \rho_s = \mathbb{1}_d/d \), the inequality (3.10) is saturated. By \( 1-c = S - nc \) and \( Sd = n \), the right-hand side of (3.10) reads as

$$\frac{Scd + 1 - c}{S^2d} = \frac{nc + S - nc}{S^2d} = \frac{1}{n}.$$

It is equal to \( I(\mathcal{E}; \rho_s) = 1/n \) as follows from \( p_j(\mathcal{E}; \rho_s) = 1/n \) for all \( j \). Also, the inequality (3.10) is saturated with any pure state \( | \phi_j \rangle \) taken from the frame kets. Combining this with (3.16) leads us to a conclusion. Namely, the inequality (3.10) is saturated for each density matrix of the form

$$\sigma = \sum_{j=1}^{n} \mu_j | \phi_j \rangle \langle \phi_j |,$$

where non-negative weights \( \mu_j \) sum to 1. Indeed, if two density matrices on \( \mathcal{H}_d \) are such that \( | \Theta \rangle \) in (3.13) is zero, then their convex combination also has this property. Then the sign of inequality in (3.10) is replaced with equality. It is not difficult to check the made conclusion immediately. Let \( \kappa = \sum_{j=1}^{n} \mu_j^2 \), then calculations show

$$Sp_j(\mathcal{E}; \sigma) = \mu_j + c(1-\mu_j),$$

$$S^2I(\mathcal{E}; \sigma) = (1-c)^2 \kappa + 2c(1-c) + nc^2$$

and

$$\text{tr}(\sigma^2) = (1-c) \kappa + c.$$

Substituting (3.22) and (3.23) in (3.10) with the equality sign and eliminating \( \kappa \) gives the formula equivalent to (2.4).

Let us discuss briefly the case of SIC-POVMs. For \( n = d^2 \), \( S = d \) and \( c = (d+1)^{-1} \), the result (3.10) is rewritten as

$$I(\mathcal{E}; \rho) = \frac{1 + \text{tr}(\rho^2)}{d(d+1)}$$

with the probabilities \( p_j(\mathcal{E}; \rho) = d^{-1} \langle \phi_j | \rho | \phi_j \rangle \). The equality holds since the set of kets (3.2) and (3.3) is complete here. As was mentioned, the index of coincidence (3.24) was calculated in [22]. The inequality (3.10) reflects a non-trivial inner structure of ETFs. Hence, various uncertainty relations for the corresponding POVMs immediately follow.

**IV. FINE-GRAINED UNCERTAINTY RELATIONS**

For a pair of observables, uncertainty relations of the Landau–Pollak type are formulated in terms of the two maximal probabilities. Its quantum-mechanical interpretation was mentioned in [34], since the original formulation of Landau & Pollak [35] was focused on signal analysis. Relations of the Landau–Pollak type can also be treated as an example of fine-grained uncertainty relations. The authors of [36] emphasized the role of such relations dealing with a particular combination of the outcomes. When the number of measurement outcomes exceeds dimensionality,
a non-trivial upper bound holds already for a single probability \[22\]. For the given index of coincidence, the maximal probability can be estimated from above as described in \[22\]. Using \[3.10\], we then have

\[
\max_j p_j(\mathcal{E}; \rho) \leq \frac{1}{n} \left( 1 + \sqrt{n - 1} \sqrt{nI(\mathcal{E}; \rho) - 1} \right) \quad \text{(4.1)}
\]

\[
\leq \frac{1}{nS} \left( S + \sqrt{n - 1} \sqrt{nS^2 + n(1 - c) \text{tr}\left(\rho^2\right) - S^2} \right)
= \frac{1}{nS} \left( S + \sqrt{(n - 1)(1 - c)} \sqrt{n \text{tr}\left(\rho^2\right) - S} \right). \quad \text{(4.2)}
\]

This inequality is saturated for the maximally mixed state \(\rho_* = \mathbb{1}_d/d\). Indeed, we clearly have \(n \text{tr}(\rho^2) = S\). It is not so obvious that it is saturated for any of the frame states, when the right-hand side of (4.2) reduces to \(d/n\). Of course, the state-independent inequality

\[
\max_j p_j(\mathcal{E}; \rho) \leq \frac{d}{n} \quad \text{(4.3)}
\]

directly follows from (2.7). In the simplest case of orthonormal bases, when \(n = d\) and \(c = 0\), the inequality (4.1) is saturated for all basis states. In general, the inequality (4.2) is not tight for density matrices of the form (3.20), though they saturate (3.10). In fact, the inequality (4.1) becomes equality only if all the probabilities except possibly the maximal one are equal. Let \(p_1\) denote the maximal probability. By convexity of the function \(\xi \mapsto \xi^2\), we have

\[
I(p_1, \ldots, p_n) \geq p_1^2 + \frac{(1 - p_1)^2}{n - 1} = \frac{np_1^2 - 2p_1 + 1}{n - 1},
\]

with equality if and only if \(p_j = (n - 1)^{-1}(1 - p_1)\) for all \(j = 2, \ldots, n\). The latter implies

\[
(n - 1)\left[nI(p_1, \ldots, p_n) - 1\right] = n^2p_1^2 - 2np_1 + n - (n - 1) = (np_1 - 1)^2
\]

and equality in (4.1) due to \(np_1 \geq 1\). In view of (3.21), the inequalities (4.1) and (4.2) are both saturated for (3.20) provided that all the weights except possibly the maximal one are equal.

It is instructive to discuss a contrast case with one half of weights \(\mu_j\) equal to \(2n^{-1}\) and zero others (we assume \(n\) to be even that is not principal for sufficiently large values). Restricting a consideration to \(n = d^2\), we obtain

\[
p_j(\mathcal{E}; \sigma) = \frac{d + 2}{d^2(d + 1)}
\]

for \(j\) with \(\mu_j = 2d^{-2}\) and \(p_j(\mathcal{E}; \sigma) = (d(d + 1))^{-1}\) for \(j\) with \(\mu_j = 0\). In this case, the left-hand side of (4.1) is represented by (4.6). It also holds that

\[
\sqrt{nI(\mathcal{E}; \sigma) - 1} = \frac{1}{d + 1}
\]

and

\[
\frac{1}{n} \left( 1 + \sqrt{n - 1} \sqrt{nI(\mathcal{E}; \sigma) - 1} \right) = \frac{1}{d^2} \left( 1 + \sqrt{\frac{d - 1}{d + 1}} \right). \quad \text{(4.7)}
\]

Comparing (4.6) with (4.7) in our example shows the following. For sufficiently large \(d\), the right-hand side of (4.1) behaves like the doubled maximal probability. On the other hand, this estimation from above approximately equal to \(2d^{-2}\) is essentially better than obvious estimate \(d^{-1}\) based on (4.3).

The structure of an ETF allows us to derive a family of fine-grained uncertainty relations, though in state-independent formulation. Let \(I = \{i_1, \ldots, i_m\}\) be a non-empty subset of the set \(\{1, \ldots, n\}\), and let

\[
\mathbf{Y}_I = (|\phi_{i_1}\rangle \cdots |\phi_{i_m}\rangle).
\]

This is a \(d \times m\) matrix written in terms of \(m\) columns. Recall also that the spectral norm \(\|K\|_{\infty}\) is defined as the square root of the maximum eigenvalue of \(K^\dagger K\). The following statement takes place.

**Proposition IV.1** Let \(n\) unit kets \(|\phi_j\rangle\) form an ETF in \(\mathcal{H}_d\), and let POVM \(\mathcal{E}\) be assigned to this frame by \(2.7\). For the given density matrix \(\rho\) and particular combination \(I = \{i_1, \ldots, i_m\}\) of outcomes, it holds that

\[
\sum_{i \in I} p_i(\mathcal{E}; \rho) \leq \frac{d}{n} \|\mathbf{Y}_I^\dagger \mathbf{Y}_I\|_{\infty},
\]

where \(\mathbf{Y}_I\) is defined by (4.8).
Proof. It is sufficient to prove (4.9) for pure states. For any unit ket $|\psi\rangle$, the sum of probabilities reads as

$$\frac{d}{n} \sum_{i \in I} \langle \psi | \phi_i \rangle \langle \phi_i | \psi \rangle = \frac{d}{n} \langle \psi | \Upsilon^{\dagger}_I \Upsilon^T_I | \psi \rangle \leq \frac{d}{n} \| \Upsilon^T_I \Upsilon^{\dagger}_I \|_\infty = \frac{d}{n} \| \Upsilon^{\dagger}_I \Upsilon_I \|_\infty .$$

(4.10)

The final step uses the fact that $\Upsilon^{\dagger}_I \Upsilon^T_I$ and $\Upsilon^T_I \Upsilon^{\dagger}_I$ have the same non-zero eigenvalues. ■

The above reasons are like the approach proposed in [37] to extend uncertainty relations of the papers [38, 39] to POVMs with rank-one elements. The statement of Proposition IV.1 is a kind of fine-grained uncertainty relations for an ETF-based measurement. Of course, similar reasons hold for arbitrary POVM with rank-one elements. On the other hand, the result can hardly be useful without explicit knowledge of the matrix elements. Nonetheless, the structure of an ETF is such that $m \times m$ matrix $\Upsilon^{\dagger}_I \Upsilon^T_I$ has ones on the main diagonal and off-diagonal entries of the form $\langle \phi_i | \phi_j \rangle = \sqrt{c} \exp(-i \theta_{ij})$. For $m = 2$, we get

$$\max_{i \neq j} \left\{ p_i(\mathcal{E}; \rho) + p_j(\mathcal{E}; \rho) \right\} = \frac{d + \sqrt{c} d}{n} .$$

(4.11)

Indeed, the spectral norm of a positive matrix is equal to the maximum of its eigenvalues. The right-hand side of (4.11) is product of $d/n$ and the maximum of eigenvalues $1 \pm \sqrt{c}$ calculated for

$$\left( \frac{1}{\sqrt{c}} e^{i \theta} \begin{pmatrix} \sqrt{c} e^{-i \theta} \\ 1 \end{pmatrix} \right) .$$

(4.12)

Here, the phase factor corresponds to $\langle \phi_j | \phi_i \rangle = \sqrt{c} e^{i \theta}$. Thus, we obtain a state-independent uncertainty relation with the sum of two probabilities. It is easy to check that the right-hand side of (4.11) is reached with the ket $|\phi_i \rangle + e^{i \theta} |\phi_j \rangle \sqrt{2 + 2 \sqrt{c}}$.

Thus, the state-independent formulation (4.11) cannot be improved under chosen circumstances.

The consideration is more complicated for $m = 3$. By $|\phi_i\rangle$, $|\phi_j\rangle$ and $|\phi_k\rangle$, we denote three columns of the matrix (4.8). The eigenvalues of $\Upsilon^{\dagger}_I \Upsilon^T_I$ obey characteristic equation

$$\lambda^3 - 3\lambda^2 + (3 - 3c)\lambda - (1 - 3c + 2e^{3/2} \cos \gamma) = 0 , \quad \gamma = \theta_{ij} + \theta_{jk} + \theta_{ki} .$$

(4.13)

By the standard approach, we rewrite (4.13) as $(\lambda - 1)^3 - 3c(\lambda - 1) - 2e^{3/2} \cos \gamma = 0$ and substitute $u + v = \lambda - 1$, $uv = c$. It then follows that both the terms $u^3$ and $v^3$ satisfy quadratic equation

$$\xi^2 - 2\xi e^{3/2} \cos \gamma + c^3 = 0$$

with the roots $\xi_{\pm} = e^{3/2} \exp(\pm i \gamma)$. Doing usual algebra finally gives the set of eigenvalues

$$\left\{ 1 + 2 \sqrt{c} \cos \frac{\gamma}{3} , 1 + 2 \sqrt{c} \cos \left( \frac{\gamma}{3} \pm \frac{2\pi}{3} \right) \right\} .$$

None of these eigenvalues exceeds $1 + 2 \sqrt{c}$, whence

$$\max_{i \neq j \neq k \neq i} \left\{ p_i(\mathcal{E}; \rho) + p_j(\mathcal{E}; \rho) + p_k(\mathcal{E}; \rho) \right\} \leq \frac{d + 2 \sqrt{c} d}{n} .$$

(4.14)

In general, this inequality is not tight, since we replaced exact values of cosines with 1. On the other hand, the exact phases are known in each of concrete cases. Using these phases allows one to improve (4.11). We refrain from exemplifying this possibility here. Being state-independent, the inequalities (4.11) and (4.14) can be applied to detect non-classical correlations.

V. ENTROPIC UNCERTAINTY RELATIONS

Entropic uncertainty relations are a prosperous alternative to more traditional approach [40, 41]. For a general discussion of entropic uncertainty relations and their role, see the review [42] and references therein. To express
uncertainty relations, we will use the Rényi and Tsallis entropies. For $\alpha > 0 \neq 1$, the Rényi $\alpha$-entropy is expressed as

$$\R_{\alpha}(p) = \frac{1}{1-\alpha} \ln \left(\sum_j p_j^\alpha\right).$$

This quantity reduces to the Shannon entropy $H_1(p) = -\sum_j p_j \ln p_j$ in the limit $\alpha \to 1$. For $\alpha = 0$ we have logarithm of the number of non-zero probabilities, whereas the limit $\alpha \to \infty$ gives the min-entropy

$$\R_\infty(p) = -\ln(\max p_j).$$

For $\alpha > 0 \neq 1$, the Tsallis $\alpha$-entropy is defined by

$$H_\alpha(p) = \frac{1}{1-\alpha} \left(\sum_j p_j^\alpha - 1\right) = -\sum_j p_j^\alpha \ln(p_j).$$

Here, the $\alpha$-logarithm of positive $\xi$ is expressed as

$$\ln_\alpha(\xi) = \begin{cases} \frac{\xi^{1-\alpha} - 1}{1-\alpha}, & \text{for } 0 < \alpha \neq 1, \\ \ln \xi, & \text{for } \alpha = 1. \end{cases}$$

By $\R_{\alpha}(E; \rho)$ and $H_\alpha(E; \rho)$, we respectively mean the entropies (5.1) and (5.3) computed with the probabilities (2.8). By construction, the Tsallis is concave with respect to probability distributions. The Rényi $\alpha$-entropy is certainly concave for $0 < \alpha \leq 1$ [45]. For $\alpha > 1$, the answer depends on dimensionality of probability distributions [46].

Entropic uncertainty relations for a POVM with elements (2.7) follow from the inequality (5.10). Let us begin with uncertainty relations in terms of Rényi entropies. Combining (4.2) with (5.2) leads to the min-entropy uncertainty relation,

$$\R_\infty(E; \rho) \geq \ln(nS) - \ln\left(S + \sqrt{(n-1)(1-c)} \sqrt{n \text{tr}(\rho^2) - S}\right),$$

where $S$ and $c$ are defined by (2.4). In the case $n = d^2$ with a SIC-POVM, the inequality (5.3) reduces to one of the results of [22]. Substituting $\alpha = 2$ in (5.1) gives the collision entropy $R_2(E; \rho) = -\ln I(E; \rho)$. Combining the latter with (5.10), we obtain

$$R_2(E; \rho) \geq 2 \ln S - \ln\left(Sc + (1-c) \text{tr}(\rho^2)\right)$$

$$\geq \ln\left(\frac{n^2 - n}{d^2 - 2d + n}\right).$$

Similarly to (5.10), the inequality (5.5) is saturated for states of the form (3.20). If all the weights except possibly the maximal one are equal then such states also saturate (5.4). As was shown in [22], the Rényi $\alpha$-entropy of order $\alpha \in [2, \infty)$ satisfies

$$R_{\alpha}(E; \rho) \geq \frac{\alpha - 2}{\alpha - 1} \R_{\infty}(E; \rho) + \frac{1}{\alpha - 1} R_2(E; \rho).$$

Combining the above inequalities, we have arrived at a conclusion.

**Proposition V.1** Let $n$ unit kets $|\phi_j\rangle$ form an ETF in $\mathcal{H}_d$, and let POVM $E$ be assigned to this frame by (2.7). For the given density matrix $\rho$ and $\alpha \in [2, \infty]$, it holds that

$$R_{\alpha}(E; \rho) \geq \frac{\alpha \ln S + (\alpha - 2) \ln n - \ln(Sc + (1-c) \text{tr}(\rho^2))}{\alpha - 1}$$

$$- \frac{\alpha - 2}{\alpha - 1} \ln\left(S + \sqrt{(n-1)(1-c)} \sqrt{n \text{tr}(\rho^2) - S}\right)$$

$$\geq \ln n - \frac{(\alpha - 2) \ln d}{\alpha - 1} + \frac{1}{\alpha - 1} \ln\left(\frac{n - 1}{d^2 - 2d + n}\right).$$

The statement of Proposition V.1 gives Rényi-entropy uncertainty relations for the POVM assigned to an ETF. These relations reflect features of ETFs with the use of (3.10) in combination with some properties of Rényi entropies. It is well known that the entropy (5.1) cannot increase with growth of $\alpha$. Hence, the inequality (5.5) remains valid with the Rényi $\alpha$-entropy of order $\alpha \in [0, 2]$. It would be interesting to obtain for this interval more precise inequalities. Apparently, other ways of derivation should be considered here.

To derive uncertainty relations in terms of Tsallis entropies, we apply the corresponding methods of [22]. The following statement takes place.
The probability \( p \) where the binary entropy \( H(p) \) satisfies

\[
H_\alpha(\mathcal{E}; \rho) \geq \ln_\alpha \left( \frac{S^2}{Sc + (1 - c) \text{tr}(\rho^2)} \right).
\]

The state-independent inequality is posed as

\[
H_\alpha(\mathcal{E}; \rho) \geq \ln_\alpha \left( \frac{n^2 - n}{d^2 - 2d + n} \right).
\]

**Proof.** For \( 0 < \alpha \leq 2 \), one writes

\[
H_\alpha(p) \geq \ln_\alpha \left( \frac{1}{I(p)} \right).
\]

The latter holds by applying Jensen’s inequality to the function \( \xi \mapsto \ln_\alpha(1/\xi) \). Since this function decreases, combining (5.10) with (5.12) implies (5.10). The result (5.11) immediately follows due to \( \text{tr}(\rho^2) \leq 1 \). ■

The statement of Proposition V.2 gives Tsallis-entropy uncertainty relations for the measurement assigned to an ETF. There are especially interesting values of \( \alpha \). For \( \alpha = 2 \), the Tsallis entropy is equal to 1 minus the index of coincidence. Here, the inequality (5.10) is saturated for all states of the form (5.20). Moreover, for a SIC-POVM we always have

\[
H_2(\mathcal{E}; \rho) = 1 - \frac{1 + \text{tr}(\rho^2)}{d(d + 1)}.
\]

In the case \( \alpha = 1 \), the inequality (5.10) can be improved due to the results of the paper [31]. Its authors considered information diagrams that represent the Shannon entropy as a function of the index of coincidence. It is seen from (5.12) that \( H_1(p) \geq -\ln I(p) \). Here, the smooth line \( \xi \mapsto -\ln \xi \) can be replaced with the polygonal line connecting the points \((1/q, \ln q)\) with integer \( q = 1, \ldots, n \). Each of them corresponds to a uniform distribution with the corresponding number of non-zero probabilities. Thus, the final inequality reads as

\[
H_1(\mathcal{E}; \rho) \geq \max_{1 \leq q \leq n-1} \left\{ \ln(q + 1) + q \ln(1 + q^{-1}) - q(q + 1) \ln(1 + q^{-1}) \frac{Sc + (1 - c) \text{tr}(\rho^2)}{S^2} \right\}.
\]

Extending the methods of the paper [31] to generalized entropies is an open question. Entropic uncertainty relations are a tool for deriving several kinds of criteria used in quantum information processing [42]. Such criteria are of interest, since ETFs are easier to construct than SIC-POVMs. In more detail, applications of the derived relations will be discussed in the next section.

Another reason to use Tsallis-entropy uncertainty relations deals with the case of detection inefficiencies. To the given efficiency \( \eta \in [0; 1] \) and probability distribution \( \{p_j\} \), we assign a distorted distribution

\[
p_{(\eta)}^{\eta} = \eta p_j, \quad p_{(\eta)}^{\eta} = 1 - \eta.
\]

The probability \( p_{(\eta)}^{\eta} \) corresponds to the no-click event. It is easy to check that

\[
H_\alpha(p^{(\eta)}) = \eta^\alpha H_\alpha(p) + h_\alpha(\eta),
\]

where the binary entropy \( h_\alpha(\eta) = (1 - \alpha)^{-1} [\eta^\alpha + (1 - \eta)^\alpha - 1] \). For \( \alpha \in (0, 2] \), the Tsallis \( \alpha \)-entropy calculated with

\[
H_\alpha^{(\eta)}(\mathcal{E}; \rho) \geq \eta^\alpha \ln_\alpha \left( \frac{S^2}{Sc + (1 - c) \text{tr}(\rho^2)} \right) + h_\alpha(\eta).
\]

The latter immediately follows from (5.10) and (5.16). For the Shannon entropy, we also have

\[
H_1^{(\eta)}(\mathcal{E}; \rho) \geq \max_{1 \leq q \leq n-1} \left\{ \ln(q + 1) + q \ln(1 + q^{-1}) - q(q + 1) \ln(1 + q^{-1}) \frac{Sc + (1 - c) \text{tr}(\rho^2)}{S^2} \right\} + h_1(\eta).
\]
In other words, entropies of the actual probability distribution reflect not only quantum uncertainties, but also an additional uncertainty inserted by detectors. The right-hand sides of (5.17) and (5.18) also allow us to estimate a required amount of detector efficiency. In both the formulae, the first term should be sufficiently large in comparison with the corresponding binary entropy.

A utility of relations with parametric dependence has been exemplified in [34]. Varying entropic parameters is important in asking for possible values of unknown characteristics at the given ones. Let us discuss this for relations of the form (5.10) in application to a SIC-POVM. Then the index of coincidence is connected with the state purity by (3.24). If \( n - 2 \) probabilities are given, then the remaining two are determined from the normalization condition and the value of \( H_2(E; \rho) \) calculated in line with (5.13). When \( \alpha \) deviates from 2, ambiguity in determining unknown probabilities increases. A special role of the value \( \alpha = 2 \) was already emphasized in [48]. In the case of detection inefficiencies, the picture becomes more complicated, but a possibility to vary entropic parameters is still important due to relations of the form (5.17).

VI. SOME APPLICATIONS

MUBs give an important tool to detect entanglement [49]. The use of SIC-POVMs in entanglement detection was briefly discussed in [22]. This task can also be approached with measurements assigned to ETFs. By \( A \) and \( B \), we denote subsystems of a bipartite system with Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). A bipartite mixed state is called separable, when its density matrix can be represented as a convex combination of product states [50, 51]. To use ETFs for entanglement detection, we extend the method considered in [22]. Let us consider a bipartite system of two \( d \)-dimensional subsystems. To the given ETF, we assign the POVM \( \mathcal{N}_{AB} \) with elements

\[
\frac{d^2}{n^2} |\phi_j \rangle \langle \phi_j| \otimes |\phi_k \rangle \langle \phi_k|.
\]

For a density matrix \( \rho_{AB} \) of the bipartite system, the \((j,k)\)-probability reads as

\[
\frac{d^2}{n^2} \langle \phi_j \phi_k^* | \rho_{AB} | \phi_j \phi_k^* \rangle.
\]

Summing these probabilities over all \( j = k \), we obtain the correlation measure

\[
G(\mathcal{N}_{AB}; \rho_{AB}) = \frac{d^2}{n^2} \sum_{j=1}^{n} \langle \phi_j \phi_j^* | \rho_{AB} | \phi_j \phi_j^* \rangle.
\]

For the case of SIC-POVMs, the quantity (6.2) proposed in [22]. It is also similar to the mutual predictability used in [49]. For each product state \( \rho_A \otimes \rho_B \), the probability (6.1) is a product of two local probabilities. Using (5.10) and the Cauchy–Schwarz inequality, we then obtain

\[
G(\mathcal{N}_{AB}; \rho_A \otimes \rho_B) \leq \frac{1}{n^2} \sqrt{Sc + (1 - c) \text{tr}(\rho_A^2)} \sqrt{Sc + (1 - c) \text{tr}(\rho_B^2)} \leq \frac{d^2 - 2d + n}{n^2 - n}.
\]

This inequality takes place for all separable states. Its violation implies that the tested state is entangled. Using some orthonormal basis \( \{|b_i\rangle \} \) in \( \mathcal{H}_d \), we write a maximally entangled state

\[
|\Phi_+ \rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |b_i \rangle \otimes |b_i \rangle.
\]

It is immediate to get \( \langle \phi_j \phi_j^* |\Phi_+ \rangle = 1/\sqrt{d} \), whence the \((j,j)\)-probability is equal to \( d/n^2 \) and

\[
G(\mathcal{N}_{AB}; |\Phi_+ \rangle \langle \Phi_+|) = \frac{d}{n}.
\]

To estimate a region of detectability, we divide the right-hand side of (6.3) by (6.4). The ratio is calculated as

\[
\frac{d - 2 + n/d}{n - 1} < 1,
\]

\[
\frac{d - 2 + n/d}{n - 1} < 1,
\]
whenever \( n > d \). In this way, the most efficient scheme takes place for \( n = d^2 \), when the used ETF is maximal with reaching a SIC-POVM. Isotropic states are typically used to test entanglement criteria. For \( \nu \in [0,1] \), one considers the density matrix

\[
\rho_{\text{iso}} = \nu |\Phi_+\rangle \langle \Phi_+| + \frac{1 - \nu}{d^2} \mathbb{1}_d \otimes \mathbb{1}_d. \tag{6.5}
\]

These states are known to be entangled if and only if \( \nu > (d+1)^{-1} \). It can be checked that the \((j,j)\)-probability is equal to \( n^{-2}(\nu d + 1 - \nu) \), whence

\[
G(N_{AB}; \rho_{\text{iso}}) = \frac{\nu d + 1 - \nu}{n}. \tag{6.6}
\]

Combining this with (6.3) shows that the violation occurs for

\[
\nu > \frac{d - 1}{n - 1}. \tag{6.6}
\]

With a SIC-POVM, the latter gives the maximally wide interval, i.e., \( \nu > (d+1)^{-1} \). At the same time, ETFs with sufficiently large number of elements are also of interest to detect entanglement, at least for isotropic states. No possibility for entanglement detection we see only for a single orthonormal basis, when \( n = d \). In addition, the use of several ETFs provides a collection of separability conditions. There is another way to use entropic uncertainty relations of this kind for POVMs. The approach of [53] uses the isometry \( U_E : \mathcal{H}_A \to \mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_A \) defined by

\[
U_E = \sqrt{\frac{d}{n}} \sum_{j=1}^{n} |x_j\rangle \otimes |y_j\rangle \otimes |\phi_j\rangle \langle \phi_j|. \tag{6.7}
\]

By \(|x_j\rangle\) and \(|y_j\rangle\), one means kets of the orthonormal bases in the \(n\)-dimensional spaces \( \mathcal{H}_X \) and \( \mathcal{H}_Y \), respectively. To each tripartite state \( \rho_{ABC} \), we further assign the conditional von Neumann entropies

\[
H_1(X|B) = H_1(\rho_{XB}) - H_1(\rho_B), \tag{6.8}
\]

\[
H_1(X|C) = H_1(\rho_{XC}) - H_1(\rho_C). \tag{6.9}
\]

Here, the von Neumann entropy of each \( \rho \) is defined as \( H_1(\rho) = -\text{tr}(\rho \ln \rho) \), and the reduced states are obtained as the corresponding partial traces of the matrix

\[
(U_E \otimes \mathbb{1}_{BC})\rho_{ABC}(U_E^\dagger \otimes \mathbb{1}_{BC}).
\]

In the considered case, theorem 5 of [54] reads as

\[
H_1(X|B) + H_1(X|C) \geq -\sum_{j=1}^{n} p_j(E; \rho_A) \ln \|Q_j\|_{\infty}, \tag{6.10}
\]

where \( \rho_A = \text{tr}_{BC}(\rho_{ABC}) \) and

\[
Q_j = \frac{d^3}{n^3} \sum_{k=1}^{n} |\phi_k\rangle \langle \phi_k| \langle \phi_j| \langle \phi_j| = \frac{d^3}{n^3} \left( (1 - c)|\phi_j\rangle \langle \phi_j| + cS \right).
\]

As the frame operator reads as \( S = S\mathbb{1}_A \), we finally have

\[
Q_j = S^{-3}((1 - c)|\phi_j\rangle \langle \phi_j| + Sc \mathbb{1}_A). \tag{6.11}
\]

The eigenvalues of this positive operator are equal to

\[
\frac{1 - c + Sc}{S^3} = \frac{d^3 - 2d^2 + nd}{n^3 - n^2}. \tag{6.12}
\]
with multiplicity 1 and $S^{-2}c$ with multiplicity $d - 1$. The spectral norm of (6.11) is equal to (6.12), whence

$$H_1(X|B) + H_1(X|C) \geq \ln \left( \frac{n^3 - n^2}{d^3 - 2d^2 + nd} \right).$$

(6.13)

For an orthonormal basis, the right-hand side of (6.13) is equal to zero as expected. Otherwise, we have a non-trivial bound for a single ETF-based POVM. It is instructive to compare this bound with the right-hand side of (6.11) for $\alpha = 1$. The difference between these right-hand sides is equal to $\ln S$. When the system $A$ is not entangled with others, the result following from (6.13) is weaker than (6.11). In general, the two uncertainty relations are independent. It would be interesting to seek a way to improve (6.13). There is a hope to address this question in a future work.

It is known that entropic uncertainty relations immediately lead to steering inequalities. Steering is a phenomenon for bipartite quantum systems that is related to entanglement but is not precisely the same [55]. For a general discussion of this important topic, see the review [56] and references therein. Steering inequalities in terms of the standard entropic functions were considered in [48]. Using quantum designs, the papers [59, 60] derived corresponding uncertainty relations and their application to steering criteria. Applying the results of the previous section to quantum steering and comparing them with similar inequalities taken from the literature is rather the subject of a separate investigation.

In last years, quantum coherence is intensively studied as an informational and computational resource. A theoretical framework for quantifying coherence was developed in [61]. The main idea is to characterize quantitatively a distinction of the given density matrix from matrices of the form

$$\tau = \sum_{i=1}^{d} t_i |b_i\rangle\langle b_i|, \quad \sum_{i=1}^{d} t_i = 1.$$  
(6.14)

Such states are completely incoherent with respect to the orthonormal basis $B = \{|b_i\rangle\}$. There are several ways to choose a distinguishability measure that should be minimized for the given orthonormal basis and state of interest. Various candidates to quantify coherence and their properties are reviewed in [62]. The quantum relative entropy leads to the quantity [61]

$$C_1(B; \rho) = H_1(B; \rho) - H_1(\rho),$$

(6.15)

which is referred to as the relative entropy of coherence. Since different computational bases can be preferred depending on the theoretical or experimental setup, coherence quantifiers are applied to a variety of bases, including non-orthogonal ones. POVM-based coherence measures and corresponding incoherent operations were examined in [63]. For a rank-one POVM, the question is naturally resolved with Naimark’s extension [64]. In this case, we merely replace (6.15) with

$$C_1(\mathcal{E}; \rho) = H_1(\mathcal{E}; \rho) - H_1(\rho).$$

(6.16)

Combining this with (5.14), we estimate from below the relative entropy of coherence with respect to the chosen ETF. In contrast, the use of other entropic functions does not allow simple expressions similarly to (6.13) and (6.16).

Finally, we recall the Brukner–Zeilinger approach to quantify an amount of quantum information [65, 66]. Brukner and Zeilinger proposed an operationally invariant measure of information in quantum measurements. It was reasoned in [67] that the difference between two indices of coincidence is a useful measure. Combining this approach with (3.10) gives

$$I(\mathcal{E}; \rho) - I(\mathcal{E}; \rho_\alpha) \leq \frac{d - 1}{n^2 - n} \left[ d \operatorname{tr}(\rho^2) - 1 \right].$$

(6.17)

For a SIC-POVM, the inequality (6.17) is saturated so that both the sides are equal to [67]

$$\frac{\operatorname{tr}(\rho^2) - \operatorname{tr}(\rho_\alpha^2)}{d(d+1)}.$$  
(6.18)

The numerator of (6.18) is the total information gained in measurements with the use of $d + 1$ MUBs. But the existence of $d + 1$ MUBs is proved only when $d$ is a prime power [68]. There are reasons to believe that SIC-POVMs exist for all $d$, and this conjecture is due to Zauner [5]. Meantime, ETF-based measurements allow us to estimate the Brukner–Zeilinger measure of quantum information.
VII. CONCLUSIONS

We have derived uncertainty relations for quantum measurements assigned to ETFs. Such frames are interesting in their own rights as well as due to applications in many disciplines. Complex ETFs include a class of SIC-POVMs used in quantum information theory. It is natural to assume that the use of ETFs expands available tools in building protocols of information processing. The presented results support this conclusion. Many of them were obtained by a development of the reasons known for SICs. Thus, ETFs deserve more quantum applications than they have obtained.

Each of ETFs leads to a POVM-measurement. The inner structure of ETFs imposes certain restrictions on generated probabilities. In particular, the index of coincidence and the maximal probability are bounded from above. Hence, fine-grained and entropic uncertainty relations with their corollaries were formulated. The elaborated framework is an extension of the facts found earlier for SICs. It is shown that ETF-based measurements allow us to test quantum correlations. The above results are presented with the aim to stimulate further use of ETFs in quantum information theory.

[1] van Lint JH, Seidel JJ. 1966 Equilateral point sets in elliptic geometry. *Indag. Math.** 28, 335–348.
[2] Lemmens PWH, Seidel JJ. 1973 Equiangular lines. *J. Algebra** 24, 494–512.
[3] Ivanović ID. 1981 Geometrical description of quatum state determination. *J. Phys. A: Math. Gen.** 14, 3241–3145.
[4] Wootters WK, Fields BD. 1989 Optimal state-determination by mutually unbiased measurements. *Ann. Phys.* **191**, 363–381.
[5] Zauner G. 2011 Quantum designs: Foundations of a noncommutative design theory. *Int. J. Quantum Inf.* **9**, 445–507.
[6] Renes JM, Blume-Kohout R, Scott AJ, Caves CM. 2004 Symmetric informationally complete quantum measurements. *J. Math. Phys.* **45**, 2171–2180.
[7] Gross D, Audenaert K, Eisert J. 2007 Evenly distributed unitaries: on the structure of unitary designs. *J. Math. Phys.* **48**, 052104.
[8] Vermerden B, Elben A, Dalmonte M, Cirac JJ, Zoller P. 2018 Unitary n-designs via random quenches in atomic Hubbard and spin models: Application to the measurement of Rényi entropies. *Phys. Rev. A** 97, 023604.
[9] Dankert C, Cleve R, Emerson J, Livine E. 2009 Exact and approximate unitary 2-designs and their application to fidelity estimation. *Phys. Rev. A** 80, 012304.
[10] Czartowski J, Goyeneche D, Grassl M, Życzkowski K. 2020 Isoentangled mutually unbiased bases, symmetric quantum measurements, and mixed-state designs. *Phys. Rev. Lett.* **124**, 090503.
[11] Fuchs CA, Hoang MC, Stacey BC. 2017 The SIC question: history and state of play. *Axioms*** 6, 21.
[12] Appleby M, Bengtsson I, Flammia S, Goyeneche D. 2019 Tight frames, Hadamard matrices and Zauner’s conjecture. *J. Phys. A: Math. Theor.* **52**, 295301.
[13] Waldron SFD. 2018 *An Introduction to Finite Tight Frames*. New York, USA: Birkhäuser.
[14] Nielsen MA, Chuang IL. 2010 *Quantum Computation and Quantum Information*. Cambridge, UK: Cambridge University Press.
[15] Casazza PG, Kutyniok G, eds. 2013 *Finite Frames: Theory and Applications*. Boston, USA: Birkhäuser.
[16] Appleby M, Bengtsson I. 2019 Simplified exact SICs. *J. Math. Phys.* **60**, 062203.
[17] Appleby M, Flammia S, McConnell G, Yard J. 2020 Generating ray class fields of real quadratic fields via complex equiangular lines. *Acta Arith.* **192**, 211–233.
[18] Sustik MA, Tropp JA, Dhillon IS, Heath RW. 2007 On the existence of equiangular tight frames. *Linear Algebra Appl.* **426**, 619–635.
[19] Fickus M, Mixon DG. 2015 Tables of the existence of equiangular tight frames. E-print [arXiv:1504.00253 [math.FA]].
[20] Pérez FC, Avella VG, Goyeneche D. 2022 Mutually unbiased frames. *Quantum*** 6, 851.
[21] Feng L, Luo S. 2022 Equioverlapping measurements. *Phys. Lett. A*** 445, 128243.
[22] Rastegin AE. 2013 Uncertainty relations for MUBs and SIC-POVMs in terms of generalized entropies. *Eur. Phys. J. D*** 67, 269.
[23] Wu S, Yu S, Molmer K. 2009 Entropic uncertainty relation for mutually unbiased bases. *Phys. Rev. A*** 79, 022104.
[24] Gour G, Flammia S, Goyeneche D. 2014 Construction of all general symmetric informationally complete measurements. *J. Phys. A: Math. Theor.* **47**, 335302.
[25] Kameyama A, Gour G. 2014 Mutually unbiased measurements in finite dimensions. *New J. Phys.* **16**, 053038.
[26] Rastegin AE. 2015 On uncertainty relations and entanglement detection with mutually unbiased measurements. *Open Syst. Inf. Dyn.* **22**, 1550005.
[27] Chen B, Fei S-M. 2015 Uncertainty relations based on mutually unbiased measurements. *Quantum Inf. Process.* **14**, 2227–2238.
[28] Rastegin AE. 2014 Notes on general SIC-POVMs. *Phys. Scr.* **89**, 085101.
[29] Davies EB. 1978 Information and quantum measurement. *IEEE Trans. Inf. Theory* **24**, 596–599.
[30] Peres A. 2002 *Quantum Theory: Concept and Methods*. Dordrecht, the Netherlands: Springer.
[31] Harremoës P, Topsøe F. 2001 Inequalities between entropy and index of coincidence derived from information diagrams.
Durt T, Englert B-G, Bengtsson I, Życzkowski K. 2010 On mutually unbiased bases.

Rastegin AE. 2015 On the Brukner–Zeilinger approach to information in quantum measurements.

Rastegin AE. 2018 Coherence quantifiers from the viewpoint of their decreases in the measurement process.

Streltsov A, Adesso G, Plenio MB. 2017 Quantum coherence as a resource.

Bischof F, Kampermann H, Bruß D. 2019 Resource theory of coherence based on positive-operator-valued measures.

Ketterer A, Gühne O. 2020 Entropic uncertainty relations from quantum designs.

Brukner Č, Zeilinger A. 2002 Young’s experiment and the finiteness of information. Phil. Trans. R. Soc. Lond. A 360, 1061–1069.

Rastegin AE. 2015 On the Brukner–Zeilinger approach to information in quantum measurements. Proc. R. Soc. A 471, 20150435.

Durt T, Englert B-G, Bengtsson I, Życzkowski K. 2010 On mutually unbiased bases. Int. J. Quantum Inf. 8, 535–640.