Higher Equations of Motion in Liouville Field Theory

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Abstract
An infinite set of operator-valued relations in Liouville field theory is established. These relations are enumerated by a pair of positive integers \((m, n)\), the first \((1, 1)\) representative being the usual Liouville equation of motion. The relations are proven in the framework of conformal field theory on the basis of exact structure constants in the Liouville operator product expansions. Possible applications in 2D gravity are discussed.

1. Introduction
To give an idea about the subject of this lecture, let’s start with a simple observation. Consider the classical 2D Liouville equation

\[
\partial \bar{\partial} \varphi = M e^\varphi
\]

Here as usual the complex coordinates \(z = x + i y\) and \(\bar{z} = x - i y\) are implied, so that \(\partial = (\partial_x - i \partial_y) / 2\) and \(\bar{\partial} = (\partial_x + i \partial_y) / 2\). An irrelevant parameter \(M\) is introduced here to make equations below more transparent. It follows directly from (1.1) that the “classical stress tensor” components

\[
T^{(c)} = \frac{1}{4} (\partial \varphi)^2 + \frac{1}{2} \partial^2 \varphi \]
\[
\bar{T}^{(c)} = \frac{1}{4} (\bar{\partial} \varphi)^2 + \frac{1}{2} \bar{\partial}^2 \varphi
\]
are holomorphic and antiholomorphic functions of \( z \) respectively, i.e., \( \bar{\partial}T^{(c)} = 0; \partial \bar{T}^{(c)} = 0 \).

Consider the infinite series of fields \( \Phi_n^{(c)} = e^{(1-n)\varphi/2}, \, n = 1, 2, 3, \ldots \). First representatives of this series, \( e^{-\varphi/2}, e^{-\varphi}, e^{-3\varphi/2}, e^{-2\varphi}, \ldots \) are straightforwardly verified to satisfy the following differential equations

\[
\begin{align*}
\partial \cdot 1 &= 0 \\
(\partial^2 + T^{(c)}) e^{-\varphi/2} &= 0 \\
(\partial^3 + 4T^{(c)} \partial + 2\partial T^{(c)}) e^{-\varphi} &= 0 \\
(\partial^4 + 10T^{(c)} \partial^2 + 10\partial T^{(c)} \partial + (9T^{(c)} + 3\partial^2 T^{(c)}) \partial e^{-3\varphi/2} &= 0 \\
(\partial^5 + 20T^{(c)} \partial^3 + 30\partial T^{(c)} \partial^2 + (64T^{(c)} + 18\partial^2 T^{(c)}) \partial + (64T^{(c)} \partial T^{(c)}) + 4\partial^3 T^{(c)}) \partial e^{-2\varphi} &= 0 \\
\end{align*}
\]

and the same equations with \( \partial \) and \( T^{(c)} \) replaced by \( \bar{\partial} \) and \( \bar{T}^{(c)} \). It is commonly believed that this series of relations goes on ad inf and for each \( \Phi_n^{(c)}, \, n = 0, 1, 2, \ldots \) a unique differential operator exists (albeit its form is unknown in general)

\[
D_n^{(c)} = \partial^n + \ldots + (n-1)\partial^{n-2} T^{(c)}
\]

(1.4)

(the coefficients being graded polynomials in \( T^{(c)} \) and its derivatives) such that

\[
D_n^{(c)} \Phi_n^{(c)} = \bar{D}_n^{(c)} \Phi_n^{(c)} = 0
\]

(1.5)

The “left” operator \( \bar{D}_n^{(c)} = \bar{\partial} + \ldots + (n-1)\bar{\partial}^{n-2} \bar{T}^{(c)} \) differs from \( D_n^{(c)} \) in the same replacement \( \partial \rightarrow \bar{\partial} \) and \( T^{(c)} \rightarrow \bar{T}^{(c)} \).

Let us now take the fields \( \varphi \Phi_n^{(c)}, \, n = 1, 2, 3, \ldots \), i.e., \( \varphi, \, \varphi e^{-\varphi/2}, \, \varphi e^{-\varphi}, \, \varphi e^{-3\varphi/2}, \, \varphi e^{-2\varphi}, \) etc. Then,

\[
\begin{align*}
\bar{D}_1^{(c)} D_1^{(c)} \varphi &= M e^{\varphi} \\
\bar{D}_2^{(c)} D_2^{(c)} (\varphi e^{-\varphi/2}) &= -M^2 e^{3\varphi/2} \\
\bar{D}_3^{(c)} D_3^{(c)} (\varphi e^{-\varphi}) &= 3M^3 e^{2\varphi} \\
\bar{D}_4^{(c)} D_4^{(c)} (\varphi e^{-3\varphi/2}) &= -18M^4 e^{5\varphi/2} \\
\bar{D}_5^{(c)} D_5^{(c)} (\varphi e^{-2\varphi}) &= 180M^5 e^{3\varphi}
\end{align*}
\]

(1.6)

The first line is just the Liouville equation of motion. The next are verified by a straightforward differential calculus with the use of the Liouville equation. It seems natural to surmise in general

\[
D_n^{(c)} D_n^{(c)} (\varphi e^{(1-n)\varphi/2}) = B_n^{(c)} e^{(1+n)\varphi/2}
\]

(1.7)

with

\[
B_n^{(c)} = 2(-)^{n+1} n!(n-1)! \left( \frac{M}{2} \right)^n
\]

(1.8)
It is the matter of this lecture to demonstrate that similar, but somewhat extended set of relations holds between the operators in quantum Liouville field theory \cite{1}. The talk is organized as follows. In the next section we describe very briefly the Liouville field theory, mainly to introduce the notations and remind the basic features. In sect. 3 the degenerate primary fields $V_{m,n}$ in Liouville field theory are discussed in some more detail and the basic operators $D_{m,n}$ are defined. Sect.4 introduces the corresponding logarithmic Liouville primaries $V'_{m,n}$, and lines out some of their basic properties. These fields are basically the derivatives of general Liouville exponential $V_\alpha$ at the point $\alpha_{m,n}$. Such fields were considered in the context of 2D Liouville gravity in paper \cite{2}. The main proposition about the effect of $D_{m,n} \bar{D}_{m,n}$ on $V'_{m,n}$ is proven. Then, in sect.5 the quantum higher equations of motion are carried out. Some applications are finally discussed in a pure algebraic problem (sections 6 and 7) as well as in the minimal Liouville gravity (sect.8)

2. Liouville field theory

Liouville field theory (LFT) is defined by the Lagrangian density \cite{1}

$$\mathcal{L} = \frac{1}{4\pi} (\partial_{\alpha} \phi)^2 + \mu e^{2b\phi}$$ (2.1)

where $\phi$ is the Liouville field and $b$ is the dimensionless coupling. Scale parameter $\mu$ is called the cosmological constant. LFT is a particular and rather specific example of conformal field theory (CFT). Conformal invariance follows from the existence of holomorphic and antiholomorphic components of the stress tensor

$$T(z) = -(\partial\phi)^2 + Q\bar{\partial}^2\phi$$
$$\bar{T}(z) = -(\bar{\partial}\phi)^2 + Q\partial^2\phi$$ (2.2)

where the convenient combination

$$Q = b^{-1} + b$$ (2.3)

($Q$ is called traditionally the background charge) is introduced. This stress tensor gives rise to the right and left Virasoro symmetries with central charge

$$c_L = 1 + 6Q^2$$ (2.4)

in the space of states and in the space of local fields.

Action (2.1) describes the local properties of the system. Different problems in LFT are specified by the boundary conditions and correspond to different geometries. E.g., in the so called spherical geometry the field $\phi$ is defined over all two-dimensional plane and has the following behavior at $|x| \to \infty$

$$\phi(x) \sim -2Q \log |x| + O(1) \quad |x| \to \infty$$ (2.5)
Formally Lagrangian (2.1) leads to the following equation of motion for the field $\phi$

$$\partial \bar{\partial} \phi = \pi b \mu e^{2b\phi}$$  \hfill (2.6)

Classical Liouville equation is recovered in the classical limit $b^2 \to 0$ of the quantum construction. In this limit the classical fields and parameters are related to the ones of this section as

$$2b\phi \to \varphi$$
$$T \to b^{-2}T^{(c)}$$
$$2\pi \mu b^2 \to M$$  \hfill (2.7)

In practice it is hard to construct the solution of LFT starting directly from equation of motion (2.6). It turns out more productive to begin with certain set of rather natural premises which permit to construct LFT as a complete and coherent set of CFT-consistent correlation functions of basic primary fields. Then it can be demonstrated that properly defined field $\phi$ (which is not a primary field in the CFT context) satisfies eq.(2.6). This observation makes it natural to interpret the whole construction as the exact solution of LFT.

First, one presumes the existence of the continuous set of basic local primary fields $V_\alpha$ parameterized by (in general complex) parameter $\alpha$. This parameter is usually supposed to satisfy the restriction

$$\text{Re} \alpha < Q/2$$  \hfill (2.8)

called the Seiberg bound. Operators $V_\alpha$ are interpreted as the properly regularized exponentials in the Liouville field $\phi$

$$V_\alpha(x) = e^{2\alpha \phi(x)}$$  \hfill (2.9)

With respect to the conformal symmetry these fields are scalar primaries of dimensions $(\Delta_\alpha, \Delta_\alpha)$

$$\Delta_\alpha = \alpha(Q - \alpha)$$  \hfill (2.10)

It is convenient to normalize these fields through the two-point function $\langle V_\alpha(x)V_\alpha(0) \rangle = S(\alpha) (x\bar{x})^{-2\Delta_\alpha}$ with

$$S(\alpha) = \left(\pi \mu \gamma(b^2)\right)^{(Q-2\alpha)/b} \frac{\gamma(2\alpha b - b^2)}{b^2 \gamma(2 - 2\alpha b^{-1} + b^{-2})}$$  \hfill (2.11)

Here and below the standard notation $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ is used. In this normalization the operator product expansion algebra of these basic primaries is determined by the three-point function in spherical geometry

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{(x_{12}\bar{x}_{12})^{\Delta_1+\Delta_2-\Delta_3}(x_{23}\bar{x}_{23})^{\Delta_3+\Delta_2-\Delta_1}(x_{31}\bar{x}_{31})^{\Delta_3+\Delta_1-\Delta_2}}$$  \hfill (2.12)
where the structure constant \( C(\alpha_1, \alpha_2, \alpha_3) \) has the following explicit form [4, 11]

\[
C(\alpha_1, \alpha_2, \alpha_3) = \left( \pi \mu \gamma(b^2) b^{2-2\mu} \right)^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times \\
\frac{\Upsilon'(0) \Upsilon(2\alpha_1) \Upsilon(2\alpha_2) \Upsilon(2\alpha_3)}{\Upsilon(\sum_{i=1}^3 \alpha_i - Q) \Upsilon(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon(\alpha_3 + \alpha_1 - \alpha_2)}
\] (2.13)

Special function \( \Upsilon(x) \), introduced in [5], is closely related to the Barnes double gamma function [6]. It satisfies the following defining functional relations

\[
\Upsilon(x + b) = \gamma(bx) b^{1-2bx} \Upsilon(x) \quad (2.14)
\]
\[
\Upsilon(x + b^{-1}) = \gamma(b^{-1}x) b^{2b^{-1}x-1} \Upsilon(x)
\]

It is important to note that these basic characteristics are analytic (meromorphic) functions of parameters \( \alpha \). In particular, from the analytic point of view nothing special happens at the Seiberg bound (2.8). This means that the latter is more an interpretational concept than a genuine analytic bound. In fact it is possible to define formally the “exponential” operators (2.9) with \( \text{Re} \alpha \geq Q/2 \) as the analytic continuation from the “physical” region \( \text{Re} \alpha < Q/2 \). Under this continuation the reflection relation between the exponential operators

\[
V_\alpha = S(\alpha) V_{Q-\alpha}
\] (2.15)

holds [11, 15].

3. Degenerate Liouville primaries

Among the exponential primaries \( V_\alpha \) there is a discrete subset \( V_{m,n} = V_{\alpha_{m,n}} \) where \( (m, n) \) is a pair of positive integers. They correspond to special values of the parameter \( \alpha = \alpha_{m,n} \)

\[
\alpha_{m,n} = -(m - 1)b^{-1}/2 - (n - 1)b/2
\] (3.1)

These fields are called degenerate, because the corresponding dimensions

\[
\Delta_{m,n} = \frac{Q^2}{4} - \frac{(mb^{-1} + nb)^2}{4}
\] (3.2)

are the Kac dimensions [7] of the degenerate representations of the Virasoro algebra with central charge (2.4). The corresponding (say “right”) null-vectors appear at level \( mn \). They can be represented as the action of certain operators \( D_{m,n} \) on the highest weight vector \( V_{m,n} \). Operator \( D_{m,n} \) is a graded polynomial of level \( mn \) in the right Virasoro algebra generators \( L_n \)

\[
D_{m,n} = L_{-1}^{mn} + d_1^{m,n}(b)L_{-2}L_{-1}^{mn-2} + \ldots
\] (3.3)

Of course there are also “left” null vectors created by the left operators \( \bar{D}_{m,n} \) (with all “right” Virasoro generators \( L_n \) replaced by the “left” ones \( \bar{L}_n \)).
Up to now I ignore the generic form of the coefficients $d_i^{m,n}(b)$ in (3.3). Level by level they can be carried out by a straightforward algebra \[8\]. E.g., at level 2 one finds (below we don’t quote the corresponding “dual” operators since $D_{m,n}$ is related to $D_{n,m}$ simply by the substitution $b \to b^{-1}$)

$$D_{1,2} = L_{-1}^2 + b^2 L_{-2}$$  (3.4)

At level 3

$$D_{1,3} = L_{-1}^3 + 4b^2 L_{-2}L_{-1} + 2b^2 \left(1 + 2b^2\right) L_{-3}$$  (3.5)

and at fourth

$$D_{1,4} = L_{-1}^4 + 10b^2 L_{-2}L_{-1}^2 + 2b^2 \left(5 + 12b^2\right) L_{-3}L_{-1} + 9b^4 L_{-2}^2 + 6b^2 \left(1 + 4b^2 + 6b^4\right) L_{-4}$$

$$D_{2,2} = L_{-1}^2 + 2 \left(b^2 - b^2\right) L_{-2}L_{-1}^2 + 2 \left(b^2 - 3 + b^2\right) L_{-3}L_{-1} + \left(b^4 - 2 + b^4\right) L_{-2}^2 + 3 \left(b^2 + 2 + b^2\right) L_{-4}$$  (3.6)

One of the basic assumptions in the LFT construction \[9\], which allows in fact to reconstruct the whole structure including operator product algebra (2.13) \[10\], is that for each $(m, n)$ the null-vectors created by the action of $D_{m,n}$ or $\bar{D}_{m,n}$ vanish

$$D_{m,n} V_{m,n} = \bar{D}_{m,n} V_{m,n} = 0$$  (3.7)

With standard interpretation of the Virasoro generators in terms of the stress tensor, $L_{-1} \to \partial$ and $(n - 2)!L_{-n} \to \partial^{n-2}T$ at $n > 1$, these equations can be viewed as linear differential equations for $V_{m,n}$, the coefficients being graded polynomials in $T$ and its derivatives. Only the subset $(1, n)$ allows a smooth classical limit. It reduces to (1.3) at $b \to 0$, relation (2.7) between the quantum and classical stress tensor components being taken into account.

### 4. Logarithmic degenerate fields

Define the logarithmic field

$$V'_{\alpha} = \frac{1}{2} \frac{\partial}{\partial \alpha} V_{\alpha} = \phi e^{2\alpha \phi}$$  (4.1)

Corresponding to the degenerate set $V_{m,n}$ we have the following discrete set of logarithmic fields

$$V'_{m,n} = V'_{\alpha} |_{\alpha = \alpha_{m,n}}$$  (4.2)

For example, $V'_{1,1}$ is the Liouville field $\phi$ itself, $V'_{1,2} = \phi e^{-b\phi}$ etc.

**Proposition:**

$$D_{m,n} \bar{D}_{m,n} V'_{m,n}$$  (4.3)

is a primary field.

**Proof:** Consider first $\bar{D}_{m,n} V_{\alpha}$ in the vicinity of $\alpha = \alpha_{m,n}$. From analyticity in $\alpha$, mentioned above, we have $\bar{D}_{m,n} V_{\alpha} = (\alpha - \alpha_{m,n}) A_{m,n} + O((\alpha - \alpha_{m,n})^2)$ where $A_{m,n}$ is an operator of dimension $(\Delta_{m,n} + mn, \Delta_{m,n})$ which is no more left primary but still a right primary. Hence
$D_{m,n}A_{m,n} = 2D_{m,n}D_{m,n}V'_{m,n}$ is also a right primary. In the above consideration one could inverse the roles of $D_{m,n}$ and $\bar{D}_{m,n}$ and therefore $D_{m,n}\bar{D}_{m,n}V'_{m,n}$ is also a left primary. Thus $D_{m,n}\bar{D}_{m,n}V'_{m,n}$ is a primary of dimension $(\Delta_{m,n}, \Delta_{m,n})$ where $\Delta_{m,n} = \Delta_{m,n} + mn$. In the primary field spectrum of the Liouville field theory there is an exponential field $V_\alpha$ with $\alpha = \tilde{\alpha}_{m,n}$, which we denote $\tilde{V}_{m,n} = V_{\tilde{\alpha}_{m,n}}$, of the same dimension $(\Delta_{m,n}, \Delta_{m,n})$. The main goal of this work is to establish the operator-valued relation

$$D_{m,n}\bar{D}_{m,n}V'_{m,n} = B_{m,n}\tilde{V}_{m,n}$$

and calculate explicitly the numerical coefficients $B_{m,n}$.

5. Higher equations of motion

In what follows it will prove convenient to replace sometimes parameters $\alpha$ in the fields $V_\alpha$ by slightly different ones $\lambda$ defined as

$$\alpha = Q/2 - \lambda$$

The reflection $\alpha \to Q - \alpha$ is simply $\lambda \to -\lambda$, dimensions $\Delta_\alpha = Q^2/4 - \lambda^2$ being even functions of $\lambda$. Introducing the notation

$$\lambda_{m,n} = (mb^{-1} + nb)/2$$

let’s define also the following polynomials for every pair $(m, n)$ of positive integers

$$p_{m,n}(x) = \prod_{r,s}(x - \lambda_{r,s})$$

where the pair of integers $(r, s)$ runs over the set

$$r = -m + 1, -m + 3, \ldots, m - 2, m - 1$$

$$s = -n + 1, -n + 3, \ldots, n - 2, n - 1$$

Polynomial $p_{m,n}(x)$ has the order $mn$ in $x$ and the same parity as the product $mn$.

Operator-valued relation (4.5) means that for every multipoint correlation function the equality

$$\langle D_{m,n}\bar{D}_{m,n}V'_{m,n}V_{\alpha_1} \ldots V_{\alpha_N} \rangle = B_{m,n} \langle \tilde{V}_{m,n}V_{\alpha_1} \ldots V_{\alpha_N} \rangle$$

holds. Due to the conformal invariance of the theory it is enough to verify this relation for the structure constant in the operator product expansions, or equivalently, for the three-point function. Hence, we have to compare $C'_{m,n} = \langle D_{m,n}\bar{D}_{m,n}V'_{m,n}(x)V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) \rangle$ and
\[ \tilde{C}_{m,n} = \left\langle \tilde{V}_{m,n}(x)V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) \right\rangle. \]  While the second three point function is obtained from \(^{(2.12)}\) straightforwardly

\[ \tilde{C}_{m,n} = \frac{C(\tilde{\alpha}_{m,n}, \alpha_1, \alpha_2)}{|x_{12}|^{2\Delta_1 + 2\Delta_2 - 2\Delta_{m,n}} |x - x_1|^{2\Delta_{m,n} + 2\Delta_1 - 2\Delta_2} |x - x_2|^{2\Delta_{m,n} + 2\Delta_2 - 2\Delta_1}}. \]  (5.6)

evaluation of the first one involves some subtleties. First notice that \( C(\alpha, \alpha_1, \alpha_2) \) has a first order zero as \( \alpha \rightarrow \alpha_{m,n} \). Thus

\[ \left\langle V'_{m,n}(x)V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) \right\rangle = \left( \frac{\partial C(\alpha, \alpha_1, \alpha_2)}{\partial \alpha} \right)_{\alpha = \alpha_{m,n}} \frac{2 |x_{12}|^{2\Delta_1 + 2\Delta_2 - 2\Delta_{m,n}} |x - x_1|^{2\Delta_{m,n} + 2\Delta_1 - 2\Delta_2} |x - x_2|^{2\Delta_{m,n} + 2\Delta_2 - 2\Delta_1}}{2 |x_{12}|^{2\Delta_1 + 2\Delta_2 - 2\Delta_{m,n}} |x - x_1|^{2\Delta_{m,n} + 2\Delta_1 - 2\Delta_2} |x - x_2|^{2\Delta_{m,n} + 2\Delta_2 - 2\Delta_1}}. \]  (5.7)

Now, the action of \( D_{m,n} \) and \( \bar{D}_{m,n} \) affects only the coordinate dependence of the correlation function, acting on the holomorphic and antiholomorphic factors independently. E.g., \( D_{m,n} \) reduces to a certain differential operator

\[ P_{m,n}(x_{12}) = (\Delta_2 - \Delta_1)^{mn} + \text{lower order terms} \]  (5.9)

where \( P_{m,n} \) is a polynomial in \( \Delta_1 \) and \( \Delta_2 \). Again, it is easy to argue that (consider that each appearance of \( L_n \) with \( n > 1 \) in the explicit expression of \( D_{m,n} \) gives at most a first power of \( \Delta \) while decreasing the order of \( \partial \) in the differential operator \( P_{m,n} \) by at least two) the maximal order (in \( \Delta_1 \) and \( \Delta_2 \)) term in \( P_{m,n}(\Delta_1, \Delta_2) \) is provided by the term \( (\partial/\partial x)^{mn} \) explicitly written down in \((5.8)\). Thus

\[ P_{m,n}(\Delta_1, \Delta_2) = (\Delta_2 - \Delta_1)^{mn} + \text{lower order terms} \]  (5.10)

Moreover, it is well known since \([8]\), that the left hand side of eq.\((5.9)\) kinematically vanishes if the dimensions \( \Delta_1 \) and \( \Delta_2 \) satisfy the fusion rules of the degenerate operator \( V_{m,n} \). Let me remind that it is this fact that allows to interpret the fusion rules as a consequence of the vanishing of the null-vector \((3.7)\). In terms of \( \lambda_1 \) and \( \lambda_2 \) the fusion relations are conveniently summarized as \([8]\)

\[ \lambda_+ = \lambda_1 + \lambda_2 = \lambda_{r,s} \quad \text{or} \quad \lambda_- = \lambda_2 - \lambda_1 = \lambda_{r,s} \]  (5.11)
with any pair \((r, s)\) from the set \((5.4)\). Therefore the polynomial \(P_{m,n}(\Delta_1, \Delta_2)\) is proportional to \(p_{m,n}(\lambda_+)p_{m,n}(\lambda_-)\). Comparing the maximal order term in the product \((5.3)\) with \((5.10)\) we see that in fact it is equal to this product

\[
P_{m,n}(\Delta_1, \Delta_2) = p_{m,n}(\lambda_1 + \lambda_2)p_{m,n}(\lambda_2 - \lambda_1) \tag{5.12}
\]

The same arguments are applied to the action of \(D_{m,n}\). Thus

\[
C'_{m,n} = \frac{P^2_{m,n}(\Delta_1, \Delta_2)\partial C(\alpha, \alpha_1, \alpha_2)/\partial \alpha|_{\alpha=\alpha_{m,n}}}{2|x_{12}|^{2\Delta_1+2\Delta_2-2\Delta_{m,n}}|x-x_1|^{2\Delta_{m,n}+2\Delta_1-2\Delta_2}|x-x_2|^{2\Delta_{m,n}+2\Delta_1-2\Delta_2}} \tag{5.13}
\]

With explicit expression \((2.13)\) for the structure constant this gives (recall that \(\alpha_{m,n} = Q/2 - \lambda_{m,n}\) and \(\tilde{\alpha}_{m,n} = Q/2 - \lambda_{m,-n}\))

\[
\frac{C'_{m,n}}{C_{m,n}} = \frac{\partial C(\alpha, \alpha_1, \alpha_2)/\partial \alpha|_{\alpha=\alpha_{m,n}}}{2C(\tilde{\alpha}_{m,n}, \alpha_1, \alpha_2)} = \left(\pi \mu \gamma(b^2)|\partial \alpha|_{\alpha=\alpha_{m,n}}\right)^n \frac{\Upsilon'(2\alpha_{m,n})}{\Upsilon'(2\tilde{\alpha}_{m,n})} \times \tag{5.14}
\]

\[
\frac{\Upsilon(Q/2 - \lambda_+ - \tilde{\lambda}_{m,n})\Upsilon(Q/2 + \lambda_+ - \tilde{\lambda}_{m,n})\Upsilon(Q/2 - \lambda_- - \tilde{\lambda}_{m,n})\Upsilon(Q/2 + \lambda_- - \tilde{\lambda}_{m,n})}{\Upsilon(Q/2 - \lambda_+ - \lambda_{m,n})\Upsilon(Q/2 + \lambda_+ - \lambda_{m,n})\Upsilon(Q/2 - \lambda_- - \lambda_{m,n})\Upsilon(Q/2 + \lambda_- - \lambda_{m,n})}
\]

where we’ve introduced \(\tilde{\lambda}_{m,n} = \lambda_{m,-n}\).

The shift relations \((2.14)\) allow to establish the following identity for the \(\Upsilon\)-functions

\[
\frac{\Upsilon(Q/2 + x - \lambda_{m,-n})\Upsilon(Q/2 - x - \lambda_{m,-n})}{\Upsilon(Q/2 + x - \lambda_{m,n})\Upsilon(Q/2 - x - \lambda_{m,n})} = \frac{(-1)^{mn}}{p^2_{m,n}(x)} \tag{5.15}
\]

Thus, all the dependence on the parameters \(\alpha_1\) and \(\alpha_2\) disappears in the ratio of the structure constants leaving a constant, which is interpreted as the constant \(B_{m,n}\) in the operator relation \((4.5)\)

\[
\frac{\langle D_{m,n}D_{m,n}(x)V_{\alpha_1}(x)V_{\alpha_2}(x) \rangle}{\langle \tilde{V}_{m,n}(x)V_{\alpha_1}(x)V_{\alpha_2}(x) \rangle} = B_{m,n} = \left(\pi \mu \gamma(b^2)|\partial \alpha|_{\alpha=\alpha_{m,n}}\right)^n \frac{\Upsilon'(2\alpha_{m,n})}{\Upsilon'(2\tilde{\alpha}_{m,n})} \tag{5.16}
\]

The ratio in the right hand side can be further simplified to

\[
B_{m,n} = \left(\pi \mu \gamma(b^2)\right)^n b^{1+2n-2m}\gamma(m - nb^2) \prod_{\substack{k=1-n\atop l=1-m\atop (k,l)\neq(0,0)}}^{m-1} (lb^{-1} + kb) \tag{5.17}
\]

Let’s consider few properties of this expression.

1. **Classical limit.** Only the \(B_{1,n}\) series of equations admits the classical limit \(b \to 0\). In this case

\[
B_{1,n} = \left(\pi \mu \gamma(b^2)\right)^n \frac{b^{4n-3}(-)^{n-1}((n - 1)!)^2}{\gamma(nb^2)} \tag{5.18}
\]
and in the limit

\[ B_{1,n} \to b \to 0 \left( \pi \mu \gamma b^2 \right)^n \frac{(-)^{n-1}n!(n-1)!}{b} \] (5.19)

This conforms expression (1.8) (remember that \( \varphi \sim 2b\phi \) and \( \pi \mu \gamma (b^2) \to M \) at \( b \to 0 \)).

2. Duality. Replace the field \( \tilde{V}_{m,n} = V_{m,-n} \) with the “reflected” one \( S(\tilde{\alpha}_{m,n})V_{Q-\tilde{\alpha}_{m,n}} \) from eq. (2.15). Explicitly

\[ S(\tilde{\alpha}_{m,n}) = \left( \pi \mu \gamma (b^2) \right)^{mb - 2 - n} \frac{\gamma(1 - m + nb^2)}{b^2 \gamma(1 + mb^2 - n)} \] (5.20)

so that

\[ B_{n,m} S(\tilde{\alpha}_{n,m}) = \left( \pi \bar{\mu} \gamma (b^2) \right)^{n/b^2} \left( b^{-1} \right)^{1+2n-2m} \gamma(m - nb^2) \prod_{k=1-n \atop l=1-m \atop (k,l) \neq (0,0)} (lb + kb^{-1}) \] (5.21)

This manifests the duality \( \Box \) with respect to \( m \leftrightarrow n \) combined with \( b \to b^{-1} \), \( \mu \to \bar{\mu} \) where

\[ \pi \bar{\mu} \gamma (b^2) = \left( \pi \mu \gamma (b^2) \right)^{1/b^2} \] (5.22)

6. Norms of logarithmic primaries

Take operators \( D_{m,n} \) from eq. (3.3) acting in the space of highest weight vector \( |\alpha\rangle \) representation of Virasoro algebra with central charge (2.4) and dimension \( \Delta = \alpha(Q - \alpha) \). Let the norm be provided by \( \langle \alpha | \alpha \rangle = 1 \) and \( L_n = L_n^\dagger \). In particular, this defines \( D_{mn}^\dagger \). Apparently at \( \alpha \to \alpha_{m,n} \)

\[ \langle \alpha | D_{mn}^\dagger D_{m,n} | \alpha \rangle = (\alpha - \alpha_{mn}) r_{mn} + o ((\alpha - \alpha_{mn})^2) \] (6.1)

where \( r_{m,n} \) are coefficients dependent only on \( b \). Can we say anything about \( r_{m,n} \)? Manipulations with Virasoro algebra give quite suggestive products

\[
\begin{align*}
    r_{12} &= -4b^{-1} (1 - b^2) \left( 1 + b^2 \right) \left( 1 + 2b^2 \right) \\
    r_{13} &= 24b^{-1} (1 - 2b^2) \left( 1 - b^2 \right) \left( 1 + b^2 \right) \left( 1 + 2b^2 \right) \left( 1 + 3b^2 \right) \\
    r_{14} &= -288b^{-1} (1 - 3b^2) \left( 1 - 2b^2 \right) \left( 1 - b^2 \right) \left( 1 + b^2 \right) \left( 1 + 2b^2 \right) \left( 1 + 3b^2 \right) \left( 1 + 4b^2 \right) \\
    r_{22} &= -16b^{-9} (1 - b^2)^2 \left( 1 + b^2 \right)^2 \left( 1 - 2b^2 \right) \left( 1 + 2b^2 \right) \left( 2 - b^2 \right) \left( 2 + b^2 \right) \\
    &\cdots
\end{align*}
\] (6.2)

Is there any general expression?
7. Poincaré disk one-point equation

Being a local operator identities, eqs. (4.5) should hold as well for the Liouville theory in the Poincaré disk geometry [11], in particular for the one-point function. The one-point function on a pseudosphere reads [11]

\[ U_{p,q}(\alpha) = \frac{\sin(\pi b^{-1}Q) \sin(\pi bQ) \sin(\pi pb^{-1}(Q-2\alpha)) \sin(\pi qb(Q-2\alpha))}{\sin(\pi b^{-1}pQ) \sin(\pi bqQ) \sin(\pi b^{-1}(Q-2\alpha)) \sin(\pi b(Q-2\alpha))} U_{1,1}(\alpha) \] (7.1)

where

\[ U_{1,1}(\alpha) = \frac{(\pi \mu \gamma(b^{2}))^{-\alpha/b} \Gamma(bQ)\Gamma(b^{-1}Q)Q}{\Gamma(b(Q-2\alpha))\Gamma(b^{-1}(Q-2\alpha))(Q-2\alpha)} \] (7.2)

We are looking for the following ratio

\[ r_{m,n} = 2 \frac{U_{p,q}(\alpha_m,n)}{U_{p,q}(\alpha_m,n)} B_{m,n} \] (7.3)

It is easy to see that the prefactor in equation (7.1) exactly cancels out in the ratio so that it does not depend on \((p,q)\) and

\[ \frac{U_{1,1}(\alpha_m,n)}{U_{1,1}(\alpha_m,n)} = \frac{(\pi \mu \gamma(b^{2}))^{-n} \Gamma(mb^{-2}) \Gamma(m-nb^{2})}{\Gamma(m-nb^{2})} \prod_{k=-n}^{n} (mb^{-2} + k) \prod_{l=-m+1}^{m-1} (l + nb^{2}) \] (7.4)

Multiplying this by \(2B_{m,n}\) from (5.17) we obtain an expression

\[ r_{m,n} = 2 \prod_{\substack{k=1-n \atop l=1-m \atop (k,l)\neq(0,0)}}^{m-n} (lb^{-1} + kb) \] (7.5)

which generalizes the manual results (6.2). It seems relevant to compare this product with the denominator in eq. (7.10) of ref. [5].

8. Application in “minimal” gravity

What such equations can be good for in physical context? Very likely they might be useful in Liouville gravity (LG) (i.e., the field theory approach to 2D gravity based on the Liouville field theory). To illustrate the idea consider the following simple example. Let \(\langle U_1 \ldots U_n \rangle_{\text{LG}}\) be the LG correlation function of any number of matter primary fields \(\Phi_i\) “dressed” by appropriate Liouville exponentials \(V_{\alpha_i}\) so that \(U_i = \Phi_i V_{\alpha_i}\). From (2.1) we have

\[ \frac{\partial}{\partial \mu} \langle U_1 \ldots U_n \rangle_{\text{LG}} = \int \langle U_1 \ldots U_n e^{2b\phi(x)} \rangle_{\text{LG}} d^2x \] (8.1)
Substituting in the r.h.s $\exp(2b\phi)$ from the basic equation of motion (2.6) one reduces (8.1) to

$$\frac{\partial}{\partial \mu} \langle U_1 \ldots U_n \rangle_{\text{LG}} = \frac{1}{\pi \mu b} \int \langle U_1 \ldots U_n \partial \bar{\partial} \phi(x) \rangle_{\text{LG}} \, d^2 x$$

which can be integrated by parts, i.e., turned to contour integrals. Considering the operator product expansions

$$\phi(x)V_\alpha(x') = \alpha \log |x - x'| V_\alpha(x') + \ldots$$

and the asymptotic

$$\phi(x) \sim Q \log |x|$$

as $|x| \to \infty$ one summarizes the boundary terms as

$$\frac{\partial}{\partial \mu} \langle U_1 \ldots U_n \rangle_{\text{LG}} = \left( \sum_{i=1}^{n} \alpha_i - Q \right) \frac{\mu}{b} \langle U_1 \ldots U_n \rangle_{\text{LG}}$$

The basic Liouville equation of motion allows to carry out explicitly every (integrated) insertion of the particular operator $U_I = \exp(2b\phi)$ reducing it to the correlation function without insertions.

Of course, for more general dressed insertion this trick is not supposed to work any more. Something exceptional happens in the case of “minimal” gravity. The term minimal gravity (MG) stands here for the Liouville gravity induced by a single minimal CFT model $\mathcal{M}_{p,q}$ (+ ghosts) and then possibly perturbed by all primary fields of $\mathcal{M}_{p,q}$. Unlike general Liouville gravity, MG is expected to be exactly solvable. This is because the matrix model approach to 2D gravity (for a review see e.g., [12]) provides explicit expressions for many observables in what is believed to be equivalent to MG.

In Liouville gravity it’s reasonable to start with unperturbed (conformal) matter. Conformal matter doesn’t interact with the background except for through the conformal anomaly. The Liouville gravity is decoupled to pure matter CFT, Liouville and ghosts. In minimal $p,q$ gravity the matter central charge is

$$c_M = 1 - 6(b^{-1} - b)^2$$

where $b = \sqrt{p/q}$. We keep the same notation $b$ for this parameter as for the Liouville parameter in the previous sections because it’s in fact the one. For, the Liouville central charge correctly adds up with (8.6)

$$c_M + c_L = 26$$

In principle $b^2$ is a rational number but for what follows it doesn’t really matter, it can arbitrary.

Possible perturbations are the fields $\Phi_{m,n}$ from the spectrum of $\mathcal{M}_{p,q}$. They have dimensions

$$\Delta_{m,n}^{(M)} = \frac{\left( b^{-1}m - bn \right)^2 - (b^{-1} - b)^2}{4} = 1 - \Delta_{m,n}$$

$$12$$
with $\tilde{\Delta}_{m,n} = \tilde{\alpha}_{m,n} (Q - \tilde{\alpha}_{m,n})$ and $\tilde{\alpha}_{m,n}$ from eq.\ref{4.4}. Thus the Liouville field $\tilde{V}_{m,n}$ of sect.4 can be used as the “gravitational dressing” of $\Phi_{m,n}$ and the perturbing operator of dimension $(1,1)$ is

$$U_{m,n}(x) = \Phi_{m,n} \tilde{V}_{m,n}(x) \quad (8.9)$$

It is one of the peculiarities of MG: all matter fields are “dressed” by Liouville exponentials entering the higher equations of motion \ref{4.3}.

Our first goal is to learn to integrate such insertions

$$\int \langle U_{m,n}(x) \ldots \rangle_{\text{MG}} d^2 x \quad (8.10)$$

where $\langle \ldots \rangle_{\text{MG}}$ stands for the joint correlation function of matter, Liouville and ghosts, and $\ldots$ is for any observable. Higher equations of motion \ref{4.5} allow to substitute the Liouville part of this insertion as

$$\int \langle U_{m,n}(x) \ldots \rangle_{\text{MG}} d^2 x = \frac{1}{B_{m,n}} \int \langle \Phi_{m,n} \bar{D}_{m,n} D_{m,n} V_{m,n}(x) \ldots \rangle_{\text{MG}} d^2 x \quad (8.11)$$

What is also specific for MG is that the matter fields $\Phi_{m,n}$ are all degenerate \footnote{D\textsuperscript{(M)} is the matter version of the Liouville operator $D_{m,n}$. In fact, $D_{m,n}$ is explicitly obtained from $D_{m,n}$ by replacing all $L_n$ with the matter conformal generators $L_n^{(M)}$ and substituting $b^2 \to -b^2$. This allows to define the joint operators

$$\mathcal{D}_{m,n} = D_{m,n} - D_{m,n}^{(M)} \quad (8.13)$$

and write

$$\int \langle U_{m,n}(x) \ldots \rangle_{\text{MG}} d^2 x = \frac{1}{B_{m,n}} \int \langle \mathcal{D}_{m,n} D_{m,n} \Theta'_{m,n}(x) \ldots \rangle_{\text{MG}} d^2 x \quad (8.14)$$

where $\Theta'_{m,n} = \Phi_{m,n} V'_{m,n}$.}

A very likely statement is that if $\ldots$ in this correlation function is BRST closed, the integrand in the right hand side of \ref{8.11} is a derivative in $x$ and $\bar{x}$

$$\mathcal{D}_{m,n} D_{m,n} \left( \Theta'_{m,n} \right) = \partial \bar{\partial} \left( \hat{H}_{m,n} H_{m,n} \Theta'_{m,n} \right) + \text{BRST exact}$$

and thus can be reduced to boundary terms. Here $H_{m,n}$ are operators of level $mn$ and ghost number 0 constructed from $L_n$, $L_n^{(M)}$ and ghosts. Few such operators were explicitly derived in refs.\cite{13, 14, 15}.

Let’s present here an explicit calculation for the simplest example $(m,n) = (1,2)$. In this case

$$\mathcal{D}_{1,2} = \partial_L^2 - \partial_M^2 + b^2 L_{-2} \quad (8.15)$$
where $\mathcal{L}_n = L_n + L_n^{(M)}$. It is verified straightforwardly that

$$\mathcal{D}_{1,2} \mathcal{D}_{1,2} O'_{12} - \bar{\partial} \bar{H}_{12} H_{12} \Theta'_{12} = b^4 \{ Q, \{ Q, BB\Theta'_{12} \} \} - b^2 \{ Q, \bar{\partial} (\bar{H}_{12} B \Theta'_{12}) \} - b^2 \{ Q, \bar{\partial} (\bar{H}_{12} B \Theta'_{12}) \} \quad (8.16)$$

Here $B$ and $C$ are ghost fields (we use here the unusual upper case letters for the usual ghosts for not to mix $B$ with the parameter $b$ of minimal gravity), $Q$ is the BRST charge

$$Q = \oint (C(T_L + T_M) + C\partial CB) \frac{dz}{2\pi i}$$

and operator $H_{12}$ reads explicitly [14, 15]

$$H_{12} = \partial M - \partial L + b^2 CB \quad (8.18)$$

Notice, that in the usual ground ring constructions [13, 16] (see also [17] for more recent developments) $H_{m,n}$ is applied to $\Theta_{m,n} = \Phi_{m,n} V_{m,n}$ to obtain the ground ring element

$$O_{m,n} = H_{m,n} \bar{H}_{m,n} \Theta_{m,n} \quad (8.19)$$

Now it is applied to the product $\Phi_{12} V'_{12}$ involving the logarithmic degenerate field of Liouville theory.

If we introduce also the operator $\hat{Q}$ acting on fields as

$$\hat{Q} f = \{ Q, f \} \quad (8.20)$$

eq.(8.16) can be rewritten in quite a compact and suggestive form

$$\mathcal{D}_{1,2} \mathcal{D}_{1,2} O'_{12} = (\hat{Q} B - b^2 \partial H_{12}) \left( \hat{Q} \bar{B} - b^2 \bar{\partial} \bar{H}_{12} \right) O'_{12} \quad (8.21)$$

where $\bar{\partial}$ and $\bar{\partial}$ inside the brackets act on everything to the right of them.

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**References**

[1] J.Teschner. Liouville theory revisited. Class.Quant.Grav., 18 (2001) R153-R222, hep-th/0104158

[2] J.Polchinski. Ward identities in two dimensional gravity. Nucl.Phys., B357 (1991) 241–270.
[3] N. Seiberg. Notes on quantum Liouville theory and quantum gravity, in “Random Surfaces and Quantum Gravity”, ed. O. Alvarez, E. Martinec and P. Windey, Plenum Press, 1990.

[4] H. Dorn and H.-J. Otto. On correlation functions for non-critical strings with $c < 1$ but $d > 1$. Phys. Lett., B291 (1992) 39, hep-th/9206053
Two and three point functions in Liouville theory. Nucl. Phys., B429 (1994) 375, hep-th/9403141

[5] A. Zamolodchikov and Al. Zamolodchikov. Structure constants and conformal bootstrap in Liouville field theory. Nucl. Phys., B477 (1996) 577.

[6] E. W. Barnes. The genesis of the double gamma function. Proc. London Math. Soc., 31 (1899) 358;
The theory of the double gamma function. Phil. Trans. Roy. Soc., A196 (1901) 265.

[7] V. G. Kac. Infinite-dimensional Lie algebras. Prog. Math., Vol. 44, Birkhäuser, Boston, 1984.

[8] A. Belavin, A. Polyakov and A. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys., B241 (1984) 333–380.

[9] J.-L. Gervais and A. Neveu. Novel triangle relation and absence of tachyons in Liouville string field theory. Nucl. Phys., B238 (1982) 125.

[10] J. Teschner. On the Liouville three-point function. Phys. Lett., B363 (1995) 63, hep-th/9507109

[11] A. Zamolodchikov and Al. Zamolodchikov. Liouville field theory on a pseudosphere. hep-th/0101152

[12] P. Ginsparg and G. Moore. Lectures on 2D gravity and 2D string theory. TASI 1992, hep-th/9304011
P. Di Francesco, P. Ginsparg and J. Zinn-Justin. 2D gravity and random matrices. Phys. Rep., 254 (1995) 1–113, hep-th/9306153

[13] E. Witten. Ground ring of two dimensional string theory. Nucl. Phys., B373 (1992) 187, hep-th/9108004

[14] P. Bouwknegt, J. McCarthy and K. Pilch. BRST analysis of physical states for 2D gravity coupled to $c \leq 1$ matter. Commun. Math. Phys., 145 (1992) 541.

[15] C. Imbimbo, S. Mahapatra and S. Mukhi. Construction of physical states of non-trivial ghost number in $c < 1$ string theory. Nucl. Phys., B375 (1992) 399.

[16] D. Kutasov, E. Martinec and N. Seiberg. Ground rings and their modules in 2-D gravity with $c \leq 1$ matter. Phys. Lett., B276 (1992) 437, hep-th/9111048
[17] M. Douglas, I. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg. A new hat for the $c = 1$ matrix model. [hep-th/0307195]
N. Seiberg and D. Shih. Branes, rings and matrix models in minimal (super)string theory. [hep-th/0312170]