Abstract

We investigate the application of the residual minimization method (RM) to stabilize the non-stationary Stokes problem. We discretize the trial and test spaces with higher continuity B-spline basis functions from isogeometric analysis (IGA) on a regular patch of elements. We first consider the RM with IGA to stabilize $H^1_0, L^2_0$ formulation of the stationary Stokes problem. We call our method the isoGeometric Residual Minimization (iGRM). Then, we focus on the non-stationary Stokes problem discretized with IGA in space. We employ a time integration scheme that preserves the Kronecker product structure of the matrix and we use RM to stabilize the problem in every time step. We propose a linear computational cost solver utilizing the Kronecker product structure of the iGRM system. We test our method on a manufactured solution problem and the cavity flow problem.

Keywords: isogeometric analysis, residual minimization, Stokes problem, non-stationary Stokes problem, Kronecker product, linear computational cost solver

1. Introduction

In this paper, we focus on the stationary and non-stationary Stokes problem, that requires special stabilization effort, especially for large Reynolds number. There are multiple stabilization methods developed for standard finite element methods [2–6]. We apply a residual minimization method for stabilization [4] and we employ the
B-spline basis functions from isogeometric analysis (IGA) for the discretization in space.

The minimum residual methods aims to find \( u_h \in U_h \) such that

\[
   u_h = \arg\min_{w_h \in U_h} \| b(w_h, \cdot) - \ell(\cdot) \|_{V^*},
\]

where \( U \) and \( V \) are Hilbert spaces, \( b : U \times V \to \mathbb{R} \) is a continuous bilinear (weak) form, \( U_h \subset U \) is a discrete trial space, and \( \ell \in V^* \) is a given right-hand side. Several discretization techniques are particular incarnations of this wide-class of residual minimization methods. These include: the least-squares finite element method \([8]\), the discontinuous Petrov-Galerkin method (DPG) with optimal test functions \([12]\), or the variational stabilization method \([9]\). We approach the residual minimization using its saddle point (mixed) formulation, e.g., as described in \([14]\). In this method, the the residual minimization problem is solved under the constrained enforced by the weak form of our Partial Differential Equation (PDE).

The actual mathematical theory concerning the stability of numerical methods for general weak formulations is based on the famous “Babuška-Brezzi condition” (BBC) developed in years 1971-1974 at the same time by Ivo Babuška, and Franco Brezzi \([7, 10, 11]\). The condition states that the problem is stable when

\[
   \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \geq \gamma \|u\|_U, \quad \forall u \in U. \tag{1}
\]

However, the inf-sup condition in the above form concerns the abstract formulation where we consider all the test functions from \( v \in V \) and look for solution at \( u \in U \) (e.g. \( U = V \)). The above condition is satisfied also if we restrict to the space of trial functions \( u_h \in U_h \)

\[
   \sup_{v \in V_h} \frac{|b(u_h, v)|}{\|v\|_V} \geq \gamma \|u_h\|_{U_h}. \tag{2}
\]

However, if we use test functions from the finite dimensional test space \( V_h = \text{span}\{v_h\} \), where \( V_h \subset V \)

\[
   \sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_{V_h}} \geq \gamma_h \|u_h\|_{U_h}, \tag{3}
\]

we do not have a guarantee that the supremum \([3]\) will be equal to the original supremum \([1]\), since we have restricted \( V \) to \( V_h \). The optimality of the method depends on the quality of the polynomial test functions \( v_h \) defining the test space \( V_h = \text{span}\{v_h\} \) and how far are they from the supremum realized in \([1]\). The residual minimization methods allow finding the best possible approximation in the trial space.
while increasing the test space. Namely, the solution of the residual problem finds the optimal test functions in the given discrete test space, that are utilized for the best possible solution of the weak form in the trial space. The residual minimization method allows stabilizing the numerical simulations of challenging computational problems.

In our formulation with the residual minimization method, we use $V \times P$ as the test spaces for velocity and pressure. Our problem is to find $(u, p) \in V \times P$ such that they minimize $\|Bu - l\|$ in the dual space. We project our problem back using the Ritz operator and we minimize the operator $B(v; p; (u, w)) = a(v; u) - b(p; u) + b(w; v) = (\nabla v, \nabla u) - (p, \text{div } u) + (w, \text{div } v)$ in the norm defined in $V \times P$. Namely, we use the norm induced by the scalar product $((v_1, p_1), (v_2, p_2)) = (\partial_x v_1^1, \partial_x v_2^1)_{L^2} + (\partial_y v_1^2, \partial_y v_2^2)_{L^2} + (p_1, p_2)_{L^2}$.

Later, we employ the splitting scheme from Jean-Luc Guermond and Petar Minev [19] to express the non-stationary Stokes problem as a sequence of pressure predictor, velocity update and pressure corrector. Each of these steps can be solved in a linear computational cost $O(N)$ using the Kronecker product structure of the matrix discretized over the patch of IGA elements.

The method presented in this paper is an extension of the linear computational cost Kronecker product solver that we proposed for the advection-diffusion problem [23]. The algorithm we propose uses the method of lines (discretization in space with time iterations) and delivers a linear computational cost $O(N)$ solver for each time step. In this sense, it is an alternative to the space-time formulation [20], where an iterative solver is employed. The size of the mesh there is for the uniform case equal to $M = N \times k$ where $k$ is the number of time steps. The total cost there is $M \times c = N \times k \times c$ where $c$ us the number of iterations of the iterative solver, and our cost is $N \times k$. However, the space-time formulation allows for adaptation in both space and time. Some parts of the space-time mesh can use “larger” time steps while the others. Thus, in the case of space-time adaptivity, the formulation can be indeed competitive to our method. We also differ from [20] in the sense that we provide automatic residual minimization for Stokes problem, with linear computational cost. It is also worth investigating if the residual minimization can be applied in space-time formulation, in the manner that preserves the Kronecker product structure and the linear computational cost of the solver. In particular, the residual minimization method in space-time may require space-time norms, if we intend to stabilize the problem in the space-time setup.

Our paper is also different from [19] in the sense that they utilize finite difference discretization with linear computational cost direction splitting algorithm, but without the stabilization incorporated. We discretize with B-splines over the patch
of elements, and we use a weak form of the splitting scheme. We also add the residual stabilization for higher Reynolds numbers, in a way that preserves the linear computational cost of the solver. Our method delivers higher continuity of the approximation, and automatic stabilization in every time step.

The residual minimization method we use is very similar to the DPG method developed by [21]. Our spaces, both trial, and test are continuous, while in the DPG method, they are broken. The motivation behind breaking the spaces is to obtain the block-diagonal structure of the matrix. Then, the static condensation practically eliminates the inner product matrix, leaving only the fluxes on the edges between elements. However, breaking the spaces makes the linear cost factorization impossible, since it destroys the Kronecker product structure of the matrix. So does the mesh adaptation. For the DPG method, there are some modern multi-grid solvers developed, allowing for fast factorizations of the system [22].

The structure of the paper is the following. We start in Section 2 from the $H^1_0$, $L^2_0$ stationary Stokes strong and weak problem formulations. Next, in Section 3, we introduce the residual minimization method, and we apply it for the Stokes problem. Later, in Section 4, we describe the Kronecker product structure of the matrices resulting from the stationary problem formulation, and we refer to possible fast solvers solutions. The next Section 5 presents numerical results for the stationary case. Later in Section 6, we present the non-stationary version of the problem, with an alternating direction splitting method. Section 7 presents the residual minimization method applied in every time step of the time-iteration scheme. Finally, Section 8 presents some numerical results for the non-stationary case. We conclude the paper in Section 9.

2. Stokes problem formulation

We consider the Stokes equation over a 2D domain $\Omega = \Omega_x \times \Omega_y \subset \mathbb{R}^2$ with no-slip boundary conditions: Find $v = (v_1, v_2)$ and $p$ such that

\[
\begin{cases}
-\Delta v + \nabla p = f, \\
\nabla \cdot v = 0, \\
v|_{\partial \Omega} = 0.
\end{cases}
\]

(4)

We multiply the first equation of (4) by some test functions $u \in H^1_0(\Omega) = H^1_0(\Omega) \times H^1_0(\Omega)$ and integrating over $\Omega$ yields

\[-(\Delta v, u)_{L^2} + (\nabla p, u)_{L^2} = (f, u)_{L^2}.\]

(5)
Integrating by parts both terms on the left-hand side of (5) gives
\[
(\nabla v, \nabla u)_{L^2} - (p, \nabla \cdot u)_{L^2} = (f, u)_{L^2},
\]
as the boundary terms drop due to homogeneous boundary conditions. Here we have \((\nabla v, \nabla u)_{L^2} = (\nabla v_1, \nabla u_1)_{L^2} + (\nabla v_2, \nabla u_2)_{L^2}\), and likewise \((f, u)_{L^2} = (f_1, u_1)_{L^2} + (f_2, u_2)_{L^2}\). We multiply the second equation of (4) by \(w \in L^2_0(\Omega)\), being \(L^2_0(\Omega)\) the functions in \(L^2(\Omega)\) of zero mean, to get
\[
(\nabla \cdot v, w)_{L^2} = 0.
\]

2.1. Weak formulation

Stokes problem (4) in the weak form can thus be formulated as follows: Find \((v, p) \in H^1_0(\Omega) \times L^2_0(\Omega)\) such that
\[
\left\{ \begin{array}{ll}
(\nabla v, \nabla u)_{L^2} - (p, \nabla \cdot u)_{L^2} = (f, u)_{L^2}, & \forall u \in H^1_0(\Omega), \\
(\nabla \cdot v, w)_{L^2} = 0, & \forall w \in L^2_0(\Omega).
\end{array} \right.
\]

Defining the bilinear forms
\[
a(v, u) := (\nabla v, \nabla u)_{L^2}, \\
b(p, u) := (p, \nabla \cdot u)_{L^2}, \\
B((v, p), (u, w)) := a(v, u) - b(p, u) + b(w, v),
\]
and adding both equations in (8), we can rewrite it as: Find \((v, p) \in H^1_0(\Omega) \times L^2_0(\Omega)\) such that
\[
B((v, p), (u, w)) = (f, u)_{L^2}, \quad \forall (u, w) \in H^1_0(\Omega) \times L^2_0(\Omega).
\]

In our implementation, instead of working directly with \(L^2_0\) space, we force in our system of linear equations that pressure is equal to zero at a single specified point.

As a test space we take \(V = H^1_0(\Omega) \times L^2_0(\Omega)\) with the following scalar product
\[
((u, w), (v, p))_V = (\nabla u, \nabla v)_{L^2} + (w, p)_{L^2}.
\]

2.2. Discrete spaces

Let \(S^{m,k}\) be the space of 2D B-splines of degree \(m\) and global continuity \(k\) and \(S^{m,k}_0 \subset S^{m,k}\) be its subspace without boundary degrees of freedom, i.e.
\[
S^{m,k}_0 = S^{m,k} \cap H^1_0(\Omega).
\]
Let $\tilde{S}^{m,k}$ denote the subspace of $S^{m,k}$ with the $(0,0)$ corner degree of freedom removed.

Let $p$, $q$ denote the trial and test space polynomial orders and $c$, $d$ trial and test space continuities. Discrete trial space $U_h$ and test space $V_h$ are defined as

$$U_h = (S_0^{p,c})^2 \times \tilde{S}^{p,c},$$

$$V_h = (S_0^{q,d})^2 \times \tilde{S}^{q,d}. \quad (13)$$

As the bases of $U_h$ and $V_h$ we take unions of bases of the factors in the above products, i.e. the basis functions are of the form

$$v_1^i = ((N_i,0),0),$$

$$v_2^i = ((0,N_i),0),$$

$$p_i = ((0,0),N_i), \quad (14)$$

where $N_i$ are basis B-spline functions (degree omitted for clarity).

3. Residual minimization method for the global problem

For a general weak problem: Find $u \in U$ such that

$$b(u,v) = l(v), \quad \forall v \in V, \quad (15)$$

we define the operator $B : U \rightarrow V'$ such as

$$\langle Bu, v \rangle = b(u,v), \quad (16)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V$ and $V'$. We can now reformulate problem (15) as

$$Bu - l = 0. \quad (17)$$

We wish to minimize the residual

$$u_h = \arg\min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2. \quad (18)$$

Introducing the Riesz operator being the isometric isomorphism

$$R_V : V \ni v \rightarrow (v,.) \in V', \quad (19)$$
we can project the problem back to $V$ as

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \| R_{V}^{-1}(Bw_h - l) \|_{V}^2,$$

and the minimum is attained at $u_h$ when the Gâteaux derivative is equal to 0 in all directions

$$G(u_h) = \frac{1}{2} \| R_{V}^{-1}(Bu_h - l) \|_{V}^2$$

$$dG(u_h; w_h) = \lim_{h \to 0} \frac{G(u_h + hw_h) - G(u_h)}{h}$$

since $\|a + b\|^2 = \|a\|^2 + 2(a, b) + \|b\|^2$ we have

$$2G(u_h + hw_h) = \| R_{V}^{-1}(Bu_h + hw_h - l) \|_{V}^2 = \| R_{V}^{-1}(Bu_h - l) + h R_{V}^{-1}Bw_h \|_{V}^2 = \| R_{V}^{-1}(Bu_h - l) \|_{V}^2 + 2h \langle R_{V}^{-1}(Bu_h - l), R_{V}^{-1}Bw_h \rangle_{V} + h^2 \| R_{V}^{-1}Bw_h \|_{V}^2$$

so

$$\frac{G(u_h + hw_h) - G(u_h)}{h} = \langle R_{V}^{-1}(Bu_h - l), R_{V}^{-1}Bw_h \rangle_{V} + \frac{1}{2}h \| R_{V}^{-1}Bw_h \|_{V}^2$$

In the limit $h \to 0$ we have

$$dG(u_h; w_h) = \langle R_{V}^{-1}(Bu_h - l), R_{V}^{-1}Bw_h \rangle_{V}$$

If $u_h$ is minimum, then for each $w_h$ we have $G(u_h + hw_h)$ gets minimum for $h = 0$, so the Gâteaux derivative has to be zero

$$\langle R_{V}^{-1}(Bu_h - l), R_{V}^{-1}Bw_h \rangle_{V} = 0 \quad \forall w_h \in U_h$$

We define the residual $r = R_{V}^{-1}(Bu_h - l)$ and we get

$$\langle r, R_{V}^{-1}Bw_h \rangle_{V} = 0 \quad \forall w_h \in U_h$$

From the definition of $R_{V}$ for all functionals $f \in V'$

$$\langle v, R_{V}^{-1}f \rangle_{V} = \langle f, v \rangle = f(v) \text{ from definition of } \langle \cdot, \cdot \rangle$$

so in particular for $f = Bw_h$ and $v = r$ we get

$$\langle Bw_h, r \rangle = 0 \quad \forall w_h \in U_h$$
From the definition of the residual we have
\[(r, v)_V = \langle Bu_h - l, v \rangle, \quad \forall v \in V.\] (28)

Thus, from (27) and (28), our problem reduces to the following semi-infinite problem:
Find \((r, u_h) \in V \times U_h\) such as
\[(r, v)_V - \langle Bu_h - l, v \rangle = 0, \quad \forall v \in V,\]
\[\langle Bw_h, r \rangle = 0, \quad \forall w_h \in U_h.\] (29)

We discretize the test space \(V_h \subset V\) to get the discrete problem: Find \((r_h, u_h) \in V_h \times U_h\) such as
\[(r_h, v_h)_{V_h} - \langle Bu_h - l, v_h \rangle = 0, \quad \forall v_h \in V_h,\]
\[\langle Bw_h, r_h \rangle = 0, \quad \forall w_h \in U_h,\] (30)

where \((, ,)_V\) is an inner product in \(V_h\), \(\langle Bu_h, v_h \rangle = b(u_h, v_h)\), and \(\langle Bw_h, r_h \rangle = b(w_h, r_h)\).

**Remark 1.** We define the discrete test space \(V_h\) in such a way that it is as close as possible to the abstract space \(V\), to ensure stability in a sense that the discrete inf-sup condition is satisfied. In the method, it is possible to gain stability enriching the test space \(V_h\) without changing the trial space \(U_h\).

The iGRM version of the stationary Stokes problem [10] is formulated as follows:
Find \((r, \psi) \in V_h, (v, p) \in U_h\) such that for all \((s, \eta) \in V_h, (u, q) \in U_h\) we have
\[
\begin{aligned}
&((r, \psi), (s, \eta))_V - B((v, p), (s, \eta)) = -(f, s)_{L^2}, \\
&\quad B((u, q), (r, \psi)) = 0,
\end{aligned}
\] (31)

which leads to the following system of equations
\[
\begin{bmatrix}
G & -B \\
B^T & 0
\end{bmatrix}
\begin{bmatrix}
r \\
v
\end{bmatrix}
= 
\begin{bmatrix}
-F \\
0
\end{bmatrix},
\] (32)

where we denote by \(\bar{r} := (r, \psi)\) and \(\bar{v} := (v, p)\).

More precisely, system (32) can be written in the following way
\[
\begin{bmatrix}
G^{xx} & 0 & 0 & -B^{xx} & 0 & -B^{xp} \\
0 & G^{yy} & 0 & 0 & -B^{yy} & -B^{yp} \\
0 & 0 & G^{pp} & -B^{px} & -B^{py} & 0 \\
(B^{xx})^T & 0 & (B^{xx})^T & 0 & 0 & 0 \\
0 & (B^{yy})^T & (B^{yy})^T & 0 & 0 & 0 \\
(B^{xp})^T & (B^{pp})^T & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
r_3 \\
v_1 \\
v_2 \\
p
\end{bmatrix}
= 
\begin{bmatrix}
-f_1 \\
-f_2 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\] (33)
Let $B^x_i$, $B^y_j$ denote one-dimensional B-spline basis functions spanning the trial space, and similarly $\tilde{B}^x_k$, $\tilde{B}^y_l$ for the test space. The entries of the matrices are of the form

$$G^{xx}_{i,j,k,l} = G^{yy}_{i,j,k,l} = \int_{\Omega} \nabla(\tilde{B}^x_i \tilde{B}^y_j) \nabla(\tilde{B}^x_k \tilde{B}^y_l) = \int_{\Omega_x} (\partial_x \tilde{B}^x_i \partial_x \tilde{B}^x_k) \int_{\Omega_y} (\tilde{B}^y_j \tilde{B}^y_l) + \int_{\Omega_x} (\tilde{B}^x_i \partial_y \tilde{B}^y_j) \int_{\Omega_y} (\partial_y \tilde{B}^y_l \partial_y \tilde{B}^y_l),$$

$$G^{yp}_{i,j,k,l} = \int_{\Omega} (\tilde{B}^x_i \tilde{B}^y_j)(\tilde{B}^x_k \tilde{B}^y_l) = \int_{\Omega_x} (\tilde{B}^x_i \tilde{B}^x_k) \int_{\Omega_y} (\tilde{B}^y_j \tilde{B}^y_l),$$

$$B^{xx}_{i,j,k,l} = B^{yy}_{i,j,k,l} = \int_{\Omega} \nabla(B^x_i B^y_j) \nabla(\tilde{B}^x_k \tilde{B}^y_l) = \int_{\Omega_x} (\partial_x B^x_i \partial_x B^x_k) \int_{\Omega_y} (B^y_j \tilde{B}^y_l) + \int_{\Omega_x} (B^x_i \tilde{B}^x_k) \int_{\Omega_y} (\partial_y B^y_j \partial_y \tilde{B}^y_l),$$

$$B^{xp}_{i,j,k,l} = -\int_{\Omega} (B^x_i B^y_j) \partial_x(\tilde{B}^x_k \tilde{B}^y_l) = -\int_{\Omega_x} (B^x_i \partial_x \tilde{B}^x_k) \int_{\Omega_y} (B^y_j \tilde{B}^y_l),$$

$$B^{px}_{i,j,k,l} = \int_{\Omega} \partial_x(B^x_i B^y_j)(\tilde{B}^x_k \tilde{B}^y_l) = \int_{\Omega_x} (\partial_x B^x_i \partial_x B^x_k) \int_{\Omega_y} (B^y_j \tilde{B}^y_l),$$

$$B^{yp}_{i,j,k,l} = -\int_{\Omega} (B^x_i B^y_j) \partial_y(\tilde{B}^x_k \tilde{B}^y_l) = -\int_{\Omega_x} (B^x_i \tilde{B}^x_k) \int_{\Omega_y} (B^y_j \partial_y \tilde{B}^y_l),$$

$$B^{py}_{i,j,k,l} = \int_{\Omega} \partial_y(B^x_i B^y_j)(\tilde{B}^x_k \tilde{B}^y_l) = \int_{\Omega_x} (B^x_i \tilde{B}^x_k) \int_{\Omega_y} (\partial_y B^y_j \tilde{B}^y_l).$$
Therefore, the matrices can be expressed as Kronecker product of 1D matrices as

\[
G_{xx} = G_{yy} = K^x \otimes M^y + M^x \otimes K^y,
\]

\[
G_{pp} = M^x \otimes M^y = \frac{1}{2} M^x \otimes M^y + \frac{1}{2} M^y \otimes M^x,
\]

\[
B_{xx} = B_{yy} = \tilde{K}^x \otimes \tilde{M}^y + M^x \otimes \tilde{K}^y,
\]

\[
B_{xp} = -\tilde{A}^x \otimes \tilde{M}^y,
\]

\[
B_{px} = (\tilde{A}^x)^T \otimes \tilde{M}^y,
\]

\[
B_{yp} = -\tilde{M}^x \otimes \tilde{A}^y,
\]

\[
B_{py} = \tilde{M}^x \otimes (\tilde{A}^y)^T.
\]

(41)

where \(M^x, M^y\) stand for one-dimensional mass matrices, \(K^x, K^y\) for one-dimensional stiffness matrices and \(A_x, A_y\) stand for one-dimensional advection matrices with the dimensions corresponding to the test space, and \(\tilde{M}^x, \tilde{M}^y, \tilde{K}^x, \tilde{K}^y, \tilde{A}^x, \tilde{A}^y\) are their non-symmetric counterparts containing products of trial and test basis functions.

We split the global operator in (33) into two sub-operators, one with the derivatives with respect to \(x\), and the other one with the derivatives with respect to \(y\)

\[
\begin{pmatrix}
K^x \otimes M^y & 0 & 0 & \tilde{K}^x \otimes \tilde{M}^y & 0 & -\tilde{A}^x \otimes \tilde{M}^y \\
0 & K^x \otimes M^y & 0 & 0 & \tilde{K}^x \otimes \tilde{M}^y & 0 \\
0 & 0 & \frac{1}{2} M^x \otimes M^y & (\tilde{A}^x)^T \otimes \tilde{M}^y & 0 & 0 \\
(\tilde{K}^x)^T \otimes (\tilde{M}^y)^T & 0 & A^x \otimes (\tilde{M}^y)^T & 0 & 0 & 0 \\
0 & (\tilde{K}^x)^T \otimes (\tilde{M}^y)^T & 0 & 0 & 0 & 0 \\
-(\tilde{A}^x)^T \otimes (\tilde{M}^y)^T & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + 
\begin{pmatrix}
M^x \otimes K^y & 0 & 0 & \tilde{M}^x \otimes \tilde{K}^y & 0 & 0 \\
0 & M^x \otimes K^y & 0 & 0 & \tilde{M}^x \otimes \tilde{K}^y & -\tilde{M}^x \otimes \tilde{A}^y \\
0 & 0 & \frac{1}{2} M^x \otimes M^y & 0 & M^x \otimes (\tilde{A}^y)^T & 0 \\
(\tilde{M}^x)^T \otimes (\tilde{K}^y)^T & 0 & 0 & 0 & 0 & 0 \\
0 & (\tilde{M}^x)^T \otimes (\tilde{K}^y)^T & 0 & 0 & 0 & 0 \\
0 & 0 \cdot (\tilde{M}^x)^T \otimes (\tilde{A}^y)^T & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(42)

If we assume that we enrich the test space in the alternating direction manner, e.g., increasing the B-spline continuity during the first sub-step in the \(x\) direction only, and during the second sub-step in the \(y\) direction only, in the first sub-step we have \((\tilde{M}^x)^T = \tilde{M}^x = M^x\) and in the second one \((\tilde{M}^y)^T = \tilde{M}^y = M^y\), and we can use
the following splitting scheme

\[
\begin{bmatrix}
   K^x \otimes M^y & 0 & 0 & -\tilde{K}^x \otimes M^y & 0 & \tilde{A}^x \otimes M^y \\
   0 & K^x \otimes M^y & 0 & 0 & -\tilde{K}^x \otimes M^y & 0 \\
   0 & 0 & \frac{1}{2} M^x \otimes M^y & -\tilde{A}^x \otimes M^y & 0 & 0 \\
   (\tilde{K}^x)^T \otimes M^y & 0 & 0 & 0 & 0 & 0 \\
   - (\tilde{A}^x)^T \otimes M^y & 0 & 0 & 0 & 0 & 0 \\
   M^x \otimes K^y & 0 & 0 & -M^x \otimes \tilde{K}^y & 0 & 0 \\
   0 & M^x \otimes K^y & 0 & 0 & -M^x \otimes \tilde{K}^y & M^x \otimes \tilde{A}^y \\
   0 & 0 & \frac{1}{2} M^x \otimes M^y & 0 & -M^x \otimes (\tilde{A}^y)^T & 0 \\
   M^x \otimes (\tilde{K}^y)^T & 0 & 0 & 0 & 0 & 0 \\
   0 & M^x \otimes (\tilde{K}^y)^T & M^x \otimes \tilde{A}^y & 0 & 0 & 0 \\
   0 & -M^x \otimes (\tilde{A}^y)^T & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[\begin{bmatrix}
   r_1 \\
   r_2 \\
   \psi \\
   v_1 \\
   v_2 \\
   p
\end{bmatrix} = \begin{bmatrix}
   -f_1 \\
   -f_2 \\
   0 \\
   0 \\
   0 \\
   0
\end{bmatrix}, \quad (43)
\]

Now, we decompose the matrices into the Kronecker product form

\[
\begin{bmatrix}
   G_1 \\
   -B_1
\end{bmatrix} \otimes M^y = \begin{bmatrix}
   K^x & 0 & 0 & -\tilde{K}^x & 0 & \tilde{A}^x \\
   0 & K^x & 0 & 0 & -\tilde{K}^x & 0 \\
   0 & 0 & \frac{1}{2} M^x & -\tilde{A}^x & 0 & 0 \\
   (\tilde{K}^x)^T & 0 & 0 & 0 & 0 & 0 \\
   - (\tilde{A}^x)^T & 0 & 0 & 0 & 0 & 0 \\
   M^x \otimes K^y & 0 & 0 & -M^x \otimes \tilde{K}^y & 0 & 0 \\
   0 & M^x \otimes K^y & 0 & 0 & -M^x \otimes \tilde{K}^y & M^x \otimes \tilde{A}^y \\
   0 & 0 & \frac{1}{2} M^x \otimes M^y & 0 & -M^x \otimes (\tilde{A}^y)^T & 0 \\
   M^x \otimes (\tilde{K}^y)^T & 0 & 0 & 0 & 0 & 0 \\
   0 & M^x \otimes (\tilde{K}^y)^T & M^x \otimes \tilde{A}^y & 0 & 0 & 0 \\
   0 & -M^x \otimes (\tilde{A}^y)^T & 0 & 0 & 0 & 0 \\
\end{bmatrix} \otimes M^y, \quad (44)
\]

and

\[
M^x \otimes \begin{bmatrix}
   G_2 \\
   -B_2
\end{bmatrix} = M^x \otimes \begin{bmatrix}
   K^y & 0 & 0 & -\tilde{K}^y & 0 & 0 \\
   0 & K^y & 0 & 0 & -\tilde{K}^y & \tilde{A}^y \\
   0 & 0 & \frac{1}{2} M^y & 0 & -\tilde{A}^y & 0 \\
   (\tilde{K}^y)^T & 0 & 0 & 0 & 0 & 0 \\
   0 & (\tilde{K}^y)^T & \tilde{A}^y & 0 & 0 & 0 \\
   0 & - (\tilde{A}^y)^T & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (45)
\]
where both sub-matrices are now one-dimensional, either diagonal or sparse close to diagonal, and can be factorized fast.

The resulting system of linear equations can be solved using either MUMPS sparse solver or possibly using some iterative solvers dedicated for the Kronecker product structure of the matrices, like e.g., [20]. We can also employ iterative procedure splitting the system into two sub-steps

\[
\begin{bmatrix}
G_1 & -B_1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r^{k+\frac{1}{2}} \\
\tilde{v}^{k+\frac{1}{2}}
\end{bmatrix}
= \begin{bmatrix}
-F^{k+\frac{1}{2}} \\
0
\end{bmatrix}
- \begin{bmatrix}
G_2 & -B_2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r^k \\
\tilde{v}^k
\end{bmatrix},
\] (46)

and

\[
\begin{bmatrix}
G_2 & -B_2 \\
B_2^T & 0
\end{bmatrix}
\begin{bmatrix}
r^{k+1} \\
\tilde{v}^{k+1}
\end{bmatrix}
= \begin{bmatrix}
-F^{k+\frac{1}{2}} \\
0
\end{bmatrix}
- \begin{bmatrix}
G_1 & -B_1 \\
B_1^T & 0
\end{bmatrix}
\begin{bmatrix}
r^{k+\frac{1}{2}} \\
\tilde{v}^{k+\frac{1}{2}}
\end{bmatrix},
\] (47)

where \( F^{k+\frac{1}{2}} \) is the value of the right-hand side in the intermediate time step.

In the remaining parts of the paper, we employ MUMPS solver for the stationary Stokes problems, and linear computational cost Kronecker product solver for the non-stationary case. The possible extensions of the stationary case into a preconditioned iterative solver will be a subject of our future work. We will refer to iterative algorithms for a similar saddle-point structure of the system [30, 31].

4. Numerical results for the stationary Stokes problem

4.1. Manufactured solution case

We consider the Stokes equations over a 2D domain \( \Omega = (0,1)^2 \) with no-slip boundary conditions: Find \( \mathbf{v} = (v_1, v_2) \) and \( p \) such that

\[
\begin{cases}
-\Delta \mathbf{v} + \nabla p = \mathbf{f}, \\
\nabla \cdot \mathbf{v} = 0, \\
\mathbf{v}|_{\partial \Omega} = 0,
\end{cases}
\] (48)

where \( \mathbf{f} = (f_1, f_2) \) is given by

\[
\begin{align*}
f_1(x, y) &= (12 - 24y)x^4 + (-24 + 48y)x^3 + (-48y + 72y^2 + 12)x^2 \\
&\quad + (-2 + 24y - 72y^2 + 48y^3)x + 1 - 4y + 12y^2 - 8y^3, \\
f_2(x, y) &= (8 - 48y + 48y^2)x^3 + (-12 + 72y - 72y^2)x^2 \\
&\quad + (4 - 24y + 48y^2 - 48y^3 + 24y^4)x - 12y^2 + 24y^3 - 12y^4,
\end{align*}
\] (49)
and the exact solution is
\begin{align}
v_1(x, y) &= x^2(1 - x)^2(2y - 6y^2 + 4y^3), \\
v_2(x, y) &= -y^2(1 - y)^2(2x - 6x^2 + 4x^3), \\
p(x, y) &= x(1 - x).
\end{align}

The resulting \( v_1, v_2, \) and \( p \) scalar fields are presented in Figure 1.

We consider four different constructions of the trial and test spaces:

1. trial space = quadratic B-splines with \( C^1 \) continuity,
   test space = cubic, quartic or quintic B-splines with \( C^1 \) continuity,
2. trial space = B-splines of order \( n \) with \( C^1 \) continuity,
   test space = B-splines of order \( n + 1 \) with \( C^1 \) continuity,
3. trial space = quadratic B-splines with \( C^0 \) continuity,
   test space = cubic, quartic or quintic B-splines with \( C^0 \) continuity,
4. trial space = B-splines of order \( n \) with \( C^0 \) continuity,
   test space = B-splines of order \( n + 1 \) with \( C^0 \) continuity.

We measure the resulting errors (difference between the exact and computed solutions in \( L^2 \) and \( H^1 \) norms) over a mesh of 20 \( \times \) 20 elements and we report them in Tables 1-4. We conclude that the optimal strategy is to increase both trial and test spaces, and both \( C^0 \) or \( C^1 \) continuity spaces gives similar convergence rates.

The graphical comparison of all the points is presented in Figure 2. The horizontal axis denotes the number of floating-point operations performed during the computations, and the vertical axis denotes the numerical error. The points closest to the left top point of the plot are optimal in the Pareto sense of two-criteria optimization. We conclude that the optimal choice is trial (3,1) test (4,1) if we are satisfied with the 0.1 percent error, trial (2,1) test (3,1) if 1 percent error is enough, or trial (4,1) test (5,1) if we want to trade higher cost for higher accuracy. The lower continuity spaces, e.g. trial (4,0) test (5,0) are computationally more expensive.
| Trial | Test  | $L^2 v_1$ | $L^2 v_2$ | $L^2 p$ | $L^2 \nabla \cdot \mathbf{v}$ | $H^1 v_1$ | $H^1 v_2$ | $H^1 p$ | $H^1 \nabla \cdot \mathbf{v}$ |
|-------|-------|-----------|-----------|--------|------------------|-----------|-----------|--------|------------------|
| (2,1) | (3,1) | 0.0179    | 0.0179    | 0.0171 | 0.0127           | 0.31      | 0.31      | 0.31   | 1.95             |
| (2,1) | (4,1) | 0.018     | 0.018     | 0.018  | 0.0127           | 0.31      | 0.31      | 0.31   | 1.96             |
| (2,1) | (5,1) | 0.018     | 0.018     | 0.018  | 0.0127           | 0.31      | 0.31      | 0.31   | 1.96             |

Table 1: Convergence of the numerical errors while increasing the order of the test space with $C^1$ continuity for the Stokes model problem.

| Trial | Test  | $L^2 v_1$ | $L^2 v_2$ | $L^2 p$ | $L^2 \nabla \cdot \mathbf{v}$ | $H^1 v_1$ | $H^1 v_2$ | $H^1 p$ | $L^2 \nabla \cdot \mathbf{v}$ |
|-------|-------|-----------|-----------|--------|------------------|-----------|-----------|--------|------------------|
| (2,1) | (3,1) | 0.0179    | 0.0179    | 0.0171 | 0.0127           | 0.31      | 0.31      | 0.31   | 1.95             |
| (3,1) | (4,1) | 0.00033   | 0.00033   | 0.0018 | 0.000328         | 0.0057    | 0.0057    | 0.0057 | 0.0444           |
| (4,1) | (5,1) | 3.76e-11  | 4.26e-11  | 1.21e-10 | 4.44e-10       | 4.18e-10  | 2.94e-9   | 2.1e-9  |                   |

Table 2: Convergence of the numerical errors while increasing the orders of both trial and test spaces with $C^1$ continuity for the Stokes model problem.

| Trial | Test  | $L^2 v_1$ | $L^2 v_2$ | $L^2 p$ | $L^2 \nabla \cdot \mathbf{v}$ | $H^1 v_1$ | $H^1 v_2$ | $H^1 p$ | $H^1 \nabla \cdot \mathbf{v}$ |
|-------|-------|-----------|-----------|--------|------------------|-----------|-----------|--------|------------------|
| (2,0) | (3,0) | 0.0177    | 0.0177    | 0.069  | 0.0126           | 0.309     | 0.309     | 0.309  | 1.95             |
| (2,0) | (4,0) | 0.0177    | 0.0177    | 0.069  | 0.0126           | 0.309     | 0.309     | 3.21   | 1.95             |
| (2,0) | (5,0) | 0.0177    | 0.0177    | 0.069  | 0.0126           | 0.309     | 0.309     | 3.21   | 1.95             |

Table 3: Convergence of the numerical errors while increasing the order of the test space with $C^0$ continuity for the Stokes model problem.

| Trial | Test  | $L^2 v_1$ | $L^2 v_2$ | $L^2 p$ | $L^2 \nabla \cdot \mathbf{v}$ | $H^1 v_1$ | $H^1 v_2$ | $H^1 p$ | $H^1 \nabla \cdot \mathbf{v}$ |
|-------|-------|-----------|-----------|--------|------------------|-----------|-----------|--------|------------------|
| (2,0) | (3,0) | 0.0177    | 0.0177    | 0.069  | 0.0126           | 0.309     | 0.309     | 0.309  | 1.95             |
| (3,0) | (4,0) | 0.000126  | 0.000126  | 0.00589 | 0.000185         | 0.0032    | 0.0032    | 0.0262 | 0.0478           |
| (4,0) | (5,0) | 2.71e-10  | 5.66e-10  | 1.84e-10 | 1.43e-10       | 3.2e-09   | 5.08e-09  | 4.48e-8| 2.18e-8          |

Table 4: Convergence of the numerical errors while increasing the orders of both trial and test spaces with $C^0$ continuity for the Stokes model problem.
Figure 2: The Pareto front two criteria optimization of the manufactured problem solutions.
4.2. The cavity flow problem

The second numerical example concerns the well-known cavity flow problem, as described in [15].

The problem models a plane flow of an isothermal fluid in a square lid-driven cavity of size $(0,1)^2$ [16]. We setup the fluid dynamic viscosity $= 1$ and the body force $= 0$. The pressure solution in the problem exhibits two singularities at the corners, as presented in Figure 3. We consider two different setups of the trial and test spaces:

1. trail space = quadratic B-splines with $C^1$ continuity,
   test space = cubic B-splines with with $C^0$ continuity,
2. trail space = cubic B-splines with $C^2$ continuity,
   test space = quartic B-splines with $C^1$ continuity.

We plot the velocity components, the pressure field and their relative $L^2$ errors for increasing mesh sizes in Table 5 for trial (3,1) test (4,1). The order and continuity is selected based on the Pareto front optimization results for the manufactured stationary case. As the fine mesh to compute the relative errors we used the $64 \times 64$ mesh, that is why we do not show the errors there, since they are equal to the numerical zero. The pressure error remains high, since the larger meshes resolves better the jumps at the corners.
| mesh   | velocity $v_1$ | velocity $v_2$ | pressure $p$ |
|--------|----------------|----------------|--------------|
| $L^2$  | 84.6           | 73.0           | 83.5         |
| 2×2    | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) |
| $L^2$  | 45.2           | 46.9           | 71.9         |
| 4×4    | ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) |
| $L^2$  | 22.9           | 26.2           | 55.7         |
| 8×8    | ![Image](image7.png) | ![Image](image8.png) | ![Image](image9.png) |
| $L^2$  | 10.6           | 12.7           | 39.3         |
| 16×16  | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) |
| $L^2$  | 3.73           | 4.62           | 20.4         |
| 32×32  | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) |
| $L^2$  | -              | -              | -            |
| 64×64  | ![Image](image16.png) | ![Image](image17.png) | ![Image](image18.png) |

Table 5: Velocity $(v_1, v_2)$ and pressure $p$ on a series of uniform meshes with trial (3,1) and test (4,1).
5. Time-dependent extension

Let $\Omega = (0,1)^2$ and $I = (0,T] \subset \mathbb{R}$, and we consider the two-dimensional time-dependent Stokes equation

\[
\begin{cases}
\partial_t v - \Delta v + \nabla p = f \quad \text{in } \Omega \times I, \\
\nabla \cdot v = 0 \quad \text{in } \Omega \times I, \\
v = 0 \quad \text{in } \Gamma \times I, \\
v(0) = v_0 \quad \text{in } \Omega,
\end{cases}
\]

(51)

where $v = (v_1,v_2)$ is the velocity and $p$ is the pressure. Here, $\Gamma = \Gamma_x \cup \Gamma_y$ denotes the boundary of the spatial domain $\Omega$, $f$ is a given source and $v_0$ is a given initial condition.

As in [19], we consider the following singular perturbation of problem (51)

\[
\begin{cases}
\partial_t v_\epsilon - \Delta v_\epsilon + \nabla p_\epsilon = f \quad \text{in } \Omega \times I, \\
\epsilon A \phi_\epsilon + \nabla \cdot v_\epsilon = 0 \quad \text{in } \Omega \times I, \\
\epsilon \partial_t p_\epsilon = \phi_\epsilon - \chi \nabla \cdot v_\epsilon \quad \text{in } \Omega \times I, \\
v_\epsilon = 0 \quad \text{in } \Gamma \times I, \\
v_\epsilon(0) = v_0 \quad \text{in } \Omega, \\
p_\epsilon(0) = p_0 \quad \text{in } \Omega,
\end{cases}
\]

(52)

where $A$ is an unbounded operator $A : D(A) \subset L^2_0(\Omega) \rightarrow L^2_0(\Omega)$ and $\phi_\epsilon \in D(A)$. Here, $\epsilon$ is the perturbation parameter and $\chi \in [0,1]$ is a user defined parameter.

5.1. Alternating Direction Implicit (ADI) method

We consider the ADI method presented in [19] with the Peaceman-Rarchford scheme [17, 18] applied to the velocity update. First, we perform an uniform partition of the time interval $\bar{I} = [0,T]$ as

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T,$$

and denote $\tau := t_{n+1} - t_n$, $\forall n = 0,\ldots,N-1$. In (52), we select $\epsilon = \tau$ and $A = (1 - \partial_{xx})(1 - \partial_{yy})$. The scheme reads as follows:

- **Pressure predictor**
  We set $\tilde{p}^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi^{n-\frac{1}{2}}$, $\forall n = 0,\ldots,N-1$ being $p^{-\frac{1}{2}} = p_0$ and $\phi^{-\frac{1}{2}} = 0$. 

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- **Velocity update**
  
  \[ v^{n+\frac{1}{2}} - \frac{\tau}{2} \partial_{xx} v^{n+\frac{1}{2}} = v^n + \frac{\tau}{2} \partial_{yy} v^n - \frac{\tau}{2} \nabla \tilde{p}^{n+\frac{1}{2}} + \frac{\tau}{2} f^{n+\frac{1}{2}}, \quad v^{n+\frac{1}{2}} = 0 \text{ in } \Gamma_x, \]
  
  \[ v^{n+1} - \frac{\tau}{2} \partial_{yy} v^{n+1} = v^{n+\frac{1}{2}} + \frac{\tau}{2} \partial_{xx} v^{n+\frac{1}{2}} - \frac{\tau}{2} \nabla \tilde{p}^{n+\frac{1}{2}} + \frac{\tau}{2} f^{n+\frac{1}{2}}, \quad v^{n+1} = 0 \text{ in } \Gamma_y, \]

  being \( v^0 = v_0 \).

- **Penalty step**

  \[ \psi - \partial_{xx} \psi = -\frac{1}{\tau} \nabla \cdot v^{n+1}, \quad \partial_x \psi = 0 \text{ in } \Gamma_x, \]

  \[ \phi^{n+\frac{1}{2}} - \partial_{yy} \phi^{n+\frac{1}{2}} = \psi, \quad \partial_y \phi^{n+\frac{1}{2}} = 0 \text{ in } \Gamma_y, \]

- **Pressure update**

  \[ p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi^{n+\frac{1}{2}} - \chi \nabla \cdot \left( \frac{1}{2} (v^{n+1} + v^n) \right). \]

5.2. Variational formulation and space discretization

To obtain a variational formulation of the scheme, we multiply (53) and (55) by test functions \( u \in H_0^1(\Omega) \) and \( w \in H_0^1(\Omega) \). Now, denoting by \((\cdot, \cdot)\) both \( L^2(\Omega) \) and \( H^1(\Omega) \) inner products, integrating by parts and applying the corresponding boundary conditions we get

\[
(v^{n+\frac{1}{2}}, u) + \frac{\tau}{2} (\partial_x v^{n+\frac{1}{2}}, \partial_x u) = (v^n, u) - \frac{\tau}{2} (\partial_y v^n, \partial_y u) - \frac{\tau}{2} (\nabla \tilde{p}^{n+\frac{1}{2}}, u) + \frac{\tau}{2} (f^{n+\frac{1}{2}}, u),
\]

\[
(v^{n+1}, u) + \frac{\tau}{2} (\partial_y v^{n+1}, \partial_y u) = (v^{n+\frac{1}{2}}, u) - \frac{\tau}{2} (\partial_x v^{n+\frac{1}{2}}, \partial_x u) - \frac{\tau}{2} (\nabla \tilde{p}^{n+\frac{1}{2}}, u) + \frac{\tau}{2} (f^{n+\frac{1}{2}}, u),
\]

\[
(\psi, w) + (\partial_x \psi, \partial_x w) = -\frac{1}{\tau} (\nabla \cdot v^{n+1}, w),
\]

\[
(\phi^{n+\frac{1}{2}}, w) + (\partial_y \phi^{n+\frac{1}{2}}, \partial_y w) = (\psi, w).
\]

Now, we select tensor product B-splines for spatial discretization. We denote by \( M^{x,y}, K^{x,y} \) and \( R^{x,y} \) the 1D mass, stiffness and advection matrices, respectively. Then, we obtain following system

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\[
\begin{bmatrix}
(M^x + \frac{\tau}{2}K^x) \otimes M^y \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
(M^x + \frac{\tau}{2}K^x) \otimes M^y
\end{bmatrix}
\begin{bmatrix}
v_1^{n+\frac{1}{2}} \\
v_2^{n+\frac{1}{2}}
\end{bmatrix}
= \begin{bmatrix}
M^x \otimes (M^y - \frac{\tau}{2}K^y) \\
0 \\
M^x \otimes (M^y - \frac{\tau}{2}K^y)
\end{bmatrix}
\begin{bmatrix}
v_1^{n} \\
v_2^{n}
\end{bmatrix}
- \frac{\tau}{2}
\begin{bmatrix}
R^x \otimes M^y \\
0 \\
M^x \otimes R^y
\end{bmatrix}
\begin{bmatrix}
\tilde{p}^{n+\frac{1}{2}} \\
\tilde{p}^{n+\frac{1}{2}}
\end{bmatrix}
+ \frac{\tau}{2}
\begin{bmatrix}
F_1^{n+\frac{1}{2}} \\
F_2^{n+\frac{1}{2}}
\end{bmatrix},
\]
\[(58)\]

\[
\begin{bmatrix}
M^x \otimes (M^y + \frac{\tau}{2}K^y) \\
0 \\
M^x \otimes (M^y + \frac{\tau}{2}K^y)
\end{bmatrix}
\begin{bmatrix}
v_1^{n+1} \\
v_2^{n+1}
\end{bmatrix}
= \begin{bmatrix}
(M^x - \frac{\tau}{2}K^x) \otimes M^y \\
0 \\
(M^x - \frac{\tau}{2}K^x) \otimes M^y
\end{bmatrix}
\begin{bmatrix}
v_1^{n+\frac{1}{2}} \\
v_2^{n+\frac{1}{2}}
\end{bmatrix}
- \frac{\tau}{2}
\begin{bmatrix}
R^x \otimes M^y \\
0 \\
M^x \otimes R^y
\end{bmatrix}
\begin{bmatrix}
\tilde{p}^{n+\frac{1}{2}} \\
\tilde{p}^{n+\frac{1}{2}}
\end{bmatrix}
+ \frac{\tau}{2}
\begin{bmatrix}
F_1^{n+\frac{1}{2}} \\
F_2^{n+\frac{1}{2}}
\end{bmatrix},
\]
\[(59)\]

\[
(M^x + K^x) \otimes M^y \psi = -\frac{1}{\tau}(R^x \otimes M^y v_1^{n+1} + M^x \otimes R^y v_2^{n+1}),
\]
\[
M^x \otimes (M^y + K^y) \phi^{n+\frac{1}{2}} = M^x \otimes M^y \psi.
\]

6. Residual minimization for non-stationary Stokes problem

In this section, we apply a residual minimization method in each equation of (56) and (57). We can employ the following inner products for the velocity and pressure

\[
(v, u)_V = (v, u) + (\partial_{x_i} v, \partial_{x_i} u),
\]
\[
(p, w)_V = (p, w) + (\partial_{x_i} p, \partial_{x_i} w),
\]
\[(60)\]

where \(i \in \{1, 2\}\) depending on the first or second sub-step of (56) and (57). Therefore, for each sub-step we have a system of the form

\[
\begin{bmatrix}
G_v & -B_v \\
B^T_v & 0
\end{bmatrix}
\begin{bmatrix}
{r_v} \\
{v}
\end{bmatrix} = \begin{bmatrix}
{-L_v} \\
{0}
\end{bmatrix},
\]
\[(61)\]

\[
\begin{bmatrix}
G_\phi & -B_\phi \\
B^T_\phi & 0
\end{bmatrix}
\begin{bmatrix}
{r_\phi} \\
{\phi}
\end{bmatrix} = \begin{bmatrix}
{-L_\phi} \\
{0}
\end{bmatrix},
\]
\[(62)\]
where we approximate \( \mathbf{v} \) and the test functions for the residual with tensor product B-splines of order \( r \) and we enrich the approximation order to \( s >> r \) in the direction of the splitting for the residual and the test functions for the velocity (similarly for the pressure). Therefore, the matrices coming from the bilinear form in (61) and (62) are

\[
\mathbf{B}_v = \begin{cases}
(M_{rs}^x + \frac{r}{2}K_{rs}^x) \otimes M_y^r & 0 \\
0 & (M_{rs}^x + \frac{r}{2}K_{rs}^x) \otimes M_y^r
\end{cases},
\]

\[
\mathbf{B}_\phi = \begin{cases}
(M_{rs}^x + K_{rs}^x) \otimes M_y^r, \\
M_r^x \otimes (M_{rs}^y + K_{rs}^y)
\end{cases},
\]

where \( M_x^y \) are the 1D mass matrices of order \( r \) and \( M_{rs}^x \) are the non-square mass matrices formed with B-splines of orders \( r \) and \( s \) (similarly for \( K_x^y \)). Employing the inner products defined in (60), the Gramm matrices in (61) and (62) are

\[
\mathbf{G}_v = \begin{cases}
(M_x^x + K_x^x) \otimes M_y^r & 0 \\
0 & (M_x^x + K_x^x) \otimes M_y^r
\end{cases},
\]

\[
\mathbf{G}_\phi = \begin{cases}
(M_x^x + K_x^x) \otimes M_y^r, \\
M_r^x \otimes (M_{s}^y + K_{s}^y)
\end{cases},
\]

Finally, taking into account that \( M_x^y \) are symmetric matrices and the following property of the Kronecker product \((A \otimes B)^T = A^T \otimes B^T\), we can factorize the matrices of systems (61) and (62) as
\[ \begin{bmatrix} G_v & -B_v \\ B_v^T & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} M_s^x + K_s^x & 0 & -M_{rs}^x - \frac{\tau}{2} K_{rs}^x & 0 \\ 0 & M_s^x + K_s^x & 0 & -M_{rs}^x - \frac{\tau}{2} K_{rs}^x \\ (M_{rs}^x + \frac{\tau}{2} K_{rs}^x)^T & 0 & 0 & 0 \\ 0 & (M_{rs}^x + \frac{\tau}{2} K_{rs}^x)^T & 0 & 0 \end{bmatrix} \otimes M_r^y \\ M_r^x \otimes \begin{bmatrix} M_s^y + K_s^y & 0 & -M_{rs}^y - \frac{\tau}{2} K_{rs}^y & 0 \\ 0 & M_s^y + K_s^y & 0 & -M_{rs}^y - \frac{\tau}{2} K_{rs}^y \\ (M_{rs}^y + \frac{\tau}{2} K_{rs}^y)^T & 0 & 0 & 0 \\ 0 & (M_{rs}^y + \frac{\tau}{2} K_{rs}^y)^T & 0 & 0 \end{bmatrix} \end{bmatrix} \]

\[ \begin{bmatrix} G_\phi & -B_\phi \\ B_\phi^T & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} M_s^x + K_s^x & -M_{rs}^x - K_{rs}^x \\ (M_{rs}^x + K_{rs}^x)^T & 0 \end{bmatrix} \otimes M_r^y \\ M_r^x \otimes \begin{bmatrix} M_s^y + K_s^y & -M_{rs}^y - K_{rs}^y \\ (M_{rs}^y + K_{rs}^y)^T & 0 \end{bmatrix} \end{bmatrix} \]

7. Numerical results for time-dependent problem

7.1. Non-stationary manufactured solution

We consider the non-stationary Stokes equation over a 2D spatial domain \( \Omega = (0, 1)^2 \) and \( I = (0, 2] \) with no-slip boundary conditions: Find \( \mathbf{v} = (v_1, v_2) \) and \( p \) such that

\[ \begin{cases} \partial_t \mathbf{v} - \nabla \mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}|_{\partial \Omega} = 0, \\ \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \]  

(63)

with \( \mathbf{f} \) and \( \mathbf{v}_0 \) defined in such a way that the manufactured solution is \( \mathbf{v}(x, y, t) = (\sin(x)\sin(y + t), \cos(x)\cos(y + t)) \) and \( p(x, y) = \cos(x)\sin(y + t) \).

We select the trial \((2, 1)\) and test \((3, 1)\) spaces over two meshes of 20 \times 20 and 40 \times 40 elements. We expect to obtain the error of the order of 1, as concluded from the analysis performed for the stationary case. We choose the time step \( \tau = 10^{-1} \) or \( \tau = 10^{-2} \) and we employ the linear computational cost solver \( \mathcal{O}(N) \) employing the Kronecker product structure of each sub-step of the time iteration algorithm.

The resulting \((v_1, v_2)\), and \( p \) scalar fields are presented in Tables 6-8. The order of the error is proportional to the one obtained when we analyzed the stationary Stokes problem.
| $t$ [s] | Velocity ($v_1, v_2$) | Pressure $p$ |
|--------|----------------------|-------------|
| $L^2$  | 10.23, 4.79          | 0.42        |
| $t=0.0$|                      |             |
| $L^2$  | 7.36, 12.04          | 0.39        |
| $t=0.5$|                      |             |
| $L^2$  | 3.40, 29.20          | 0.41        |
| $t=1.0$|                      |             |
| $L^2$  | 4.28, 17.14          | 0.46        |
| $t=1.5$|                      |             |
| $L^2$  | 9.51, 9.00           | 0.50        |
| $t=2.0$|                      |             |

Table 6: Velocity ($v_1, v_2$) and pressure $p$ at different time steps for $20 \times 20$ mesh with trial (2,1) test (3,1), $\tau = 10^{-1}$. 

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| t [s] | Velocity \((v_1, v_2)\) | Pressure \(p\) |
|-------|-----------------|---------|
| \(L^2\) | 10.23, 4.79 | 0.42 |
| t=0.0 | ![Image](image1) | ![Image](image2) |
| \(L^2\) | 7.33, 12.03 | 0.39 |
| t=0.5 | ![Image](image3) | ![Image](image4) |
| \(L^2\) | 3.39, 29.20 | 0.41 |
| t=1.0 | ![Image](image5) | ![Image](image6) |
| \(L^2\) | 4.26, 17.15 | 0.46 |
| t=1.5 | ![Image](image7) | ![Image](image8) |
| \(L^2\) | 9.48, 9.01 | 0.50 |
| t=2.0 | ![Image](image9) | ![Image](image10) |

Table 7: Velocity \((v_1, v_2)\) and pressure \(p\) at different time steps for 40 × 40 mesh with trial (2,1) test (3,1), \(\tau = 10^{-1}\).
Table 8: Velocity \((v_1, v_2)\) and pressure \(p\) at different time steps for 40 × 40 mesh with trial (2,1) test (3,1), \(\tau = 10^{-2}\).
7.2. Non-stationary cavity flow

We consider the non-stationary cavity flow problem over a 2D domain $\Omega = (0, 1)^2$

\[
\begin{aligned}
\partial_t \mathbf{v} - \frac{1}{Re} \nabla^2 \mathbf{v} + \nabla p & = 0 \\
\nabla \cdot \mathbf{v} & = 0 \\
\mathbf{v}_1(1, y) & = 1 \text{ for } y \in (0, 1) \\
\mathbf{v}_1(0, y) & = 0 \text{ for } y \in (0, 1) \\
\mathbf{v}_2(x, 0) & = 0 \text{ for } x \in (0, 1) \\
\mathbf{v}_1(x, 1) & = 0 \text{ for } x \in (0, 1) \\
\mathbf{v}_2(x, y) & = 0 \text{ for } (x, y) \in \partial \Omega
\end{aligned}
\]

We select trial space (3,1) and test space (4,1) as concluded from the analysis performed for the stationary case. We choose the time step $\tau = 10^{-2}$ and we employ the linear computational cost solver $O(N)$ employing the Kronecker product structure of each sub-step of the time iteration algorithm. We run our simulation on $80 \times 80$ mesh. We compare with Galerkin method where we use the splitting scheme from [19] but without the stabilization. For $Re = 1, 10, 100, \text{ and } 1000$ the Galerkin and iGRM methods gives identical results without oscillation. For $Re = 10000$, the Galerkin method produces polluted oscillating results, as presented in Figure 9, and the iGRM method removes the oscillations and preserves the stability, see Figure 11.

From the point of view of the $L^2$ and $H^1$ errors, the velocity and pressure norms converge to the stable state, see Table 11.

8. Conclusions

In this paper, we applied the residual minimization (RM) to the Stokes problem in a simple $H^1_0, L^2_0$ formulation. We applied the continuous B-splines basis functions from isogeometric analysis (IGA) for both trial and test spaces. We experiment with different order and continuity of both trial and test spaces. Then, we focused on the non-stationary Stokes problem. We employed the splitting scheme originally proposed in [19] for the finite-difference simulations. Next, we augment the problem with the residual minimization (RM) applied in every time step for stabilization. We obtain a linear computational cost solver for non-stationary Stokes problem with IGA and RM. We tested our method on the manufactured solution problems and on the cavity flow problems. The RM method allowed to increase the Reynolds number for which the scheme of the non-stationary problem is stable. Future work may include the development of an iterative algorithm based on ideas presented in [30, 31]. We will also target parallelization of the method [27], possibly using the decomposition of the solver algorithm into basic undividable tasks [28, 29]. Our future work will also extend this method to Maxwell problems [24, 26].
Table 9: Galerkin method for $Re = 10000$ for mesh $80 \times 80$. Components $v_1, v_2$ of the velocity and $p$ pressure scalar field at time steps 20, 50, 80, 100 of the solution of the non-stationary cavity flow problem.
Table 10: iGRM method with trial (3,1), test (4,1) for mesh $80 \times 80$ with for $Re = 10000$. Components $v_1, v_2$ of the velocity and $p$ pressure scalar field at time steps 20, 50, 80, 100 of the solution of the non-stationary cavity flow problem.

| time [s] | $L^2 v_1$ | $L^2 v_2$ | $H^1 v_1$ | $H^1 v_2$ | $L^2 p$ |
|----------|------------|------------|------------|------------|----------|
| 0.2      | 0.0538284  | 0.124031   | 0.124031   | 0.0538284  | 0.00473062|
| 0.5      | 0.0671889  | 0.392994   | 0.0671889  | 0.392994   | 0.00402624|
| 0.8      | 0.0749734  | 0.587544   | 0.0749734  | 0.587544   | 0.00360512|
| 1.0      | 0.0789135  | 0.699199   | 0.0789135  | 0.699199   | 0.00347476|
| 1.5      | 0.0865951  | 0.845704   | 0.0865951  | 0.845704   | 0.00347966|
| 2.0      | 0.0913917  | 0.899765   | 0.0913917  | 0.899765   | 0.00346295|

Table 11: $L^2$ and $H^1$ norms of the solution for non-stationary cavity flow Stokes problem for trial (3,1) test (4,1), mesh $80 \times 80$ time step $\tau = 10^{-2}$.
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