PROGRESSION-FREE SETS IN $\mathbb{Z}_4^n$
ARE EXPONENTIALLY SMALL

ERNIE CROOT, VSEVOLOD F. LEV, AND PÉTER PÁL PACH†

Abstract. We show that for integer $n \geq 1$, any subset $A \subseteq \mathbb{Z}_4^n$ free of three-term arithmetic progressions has size $|A| \leq 4^n$, with an absolute constant $\gamma \approx 0.926$.

1. Background and Motivation

In his influential papers [R52, R53], Roth has shown that if a set $A \subseteq \{1, 2, \ldots, N\}$ does not contain three elements in an arithmetic progression, then $|A| = o(N)$ and indeed, $|A| = O(N/\log \log N)$ as $N$ grows. Since then, estimating the largest possible size of such a set has become one of the central problems in additive combinatorics. Roth’s original results were improved by Heath-Brown [H87], Szemerédi [S90], Bourgain [B99], Sanders [S12, S11], and Bloom [B], the current record due to Bloom being $|A| = O(N(\log \log N)^4/\log N)$.

It is easily seen that Roth’s problem is essentially equivalent to estimating the largest possible size of a subset of the cyclic group $\mathbb{Z}_N$, free of three-term arithmetic progressions. This makes it natural to investigate other finite abelian groups.

We say that a subset $A$ of an (additively written) abelian group $G$ is progression-free if there do not exist pairwise distinct $a, b, c \in A$ with $a + b = 2c$, and we denote by $r_3(G)$ the largest size of a progression-free subset $A \subseteq G$. For abelian groups $G$ of odd order, Brown and Buhler [BB82] and independently Frankl, Graham, and Rödl [FGR87] proved that $r_3(G) = o(|G|)$ as $|G|$ grows. Meshulam [M95], following the general lines of Roth’s argument, has shown that if $G$ is an abelian group of odd order, then $r_3(G) \leq 2|G|/\text{rk}(G)$ (where we use the standard notation $\text{rk}(G)$ for the rank of $G$); in particular, $r_3(\mathbb{Z}_m^n) \leq 2m^n/n$. Despite many efforts, no further progress was made for over 15 years, till Bateman and Katz in their ground-breaking paper [BK12] proved that $r_3(\mathbb{Z}_3^n) = O(3^n/n^{1+\epsilon})$ with an absolute constant $\epsilon > 0$.

Abelian groups of even order were first considered in [L04] where, as a further elaboration on the Roth-Meshulam proof, it is shown that $r_3(G) < 2|G|/\text{rk}(2G)$ for any finite abelian group $G$; here $2G = \{2g : g \in G\}$. For the homocyclic groups of exponent 4 this

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result was improved by Sanders [S11] who proved that $r_3(\mathbb{Z}_n^4) = O(4^n/n(\log n)^\varepsilon)$ with an absolute constant $\varepsilon > 0$. The goal of this paper is to further improve Sanders’s result, as follows.

Let $H$ denote the binary entropy function; that is,

$$H(x) = -x \log_2 x - (1 - x) \log_2(1 - x), \quad x \in (0, 1),$$

where $\log_2 x$ is the base-2 logarithm of $x$. For the rest of the paper, we set

$$\gamma := \max \left\{ \frac{1}{2}(H(0.5 - \varepsilon) + H(2\varepsilon)) : 0 < \varepsilon < 0.25 \right\} \approx 0.926.$$

**Theorem 1.** If $n \geq 1$ and $A \subseteq \mathbb{Z}_n^4$ is progression-free, then $|A| \leq 4^\gamma n$.

The proof of Theorem 1 is presented in the next section.

We note that the exponential reduction in Theorem 1 is the first of its kind for problems of this sort.

Starting from Roth, the standard way to obtain quantitative estimates for $r_3(G)$ involves a combination of the Fourier analysis and the density increment technique; the only exception is [L12] where for the groups $G \cong \mathbb{Z}_q^n$ with a prime power $q$, the above-mentioned Meshulam’s result is recovered using a completely elementary argument. In contrast, in the present paper we use the polynomial method, without resorting to the familiar Fourier analysis – density increment strategy.

For a finite abelian group $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ with positive integer $m_1 | \cdots | m_k$, denote by $\text{rk}_4(G)$ the number of indices $i \in [1, k]$ with $4 | m_i$. Since, writing $n := \text{rk}_4(G)$, the group $G$ is a union of $4^{-n}|G|$ cosets of a subgroup isomorphic to $\mathbb{Z}_4^n$, as a direct consequence of Theorem 1 we get the following corollary.

**Corollary 1.** If $A$ is a progression-free subset of a finite abelian group $G$ then, writing $n := \text{rk}_4(G)$, we have $|A| \leq 4^{-(1-\gamma)n}|G|$.

**2. Proof of Theorem 1**

We recall that the degree of a multivariate polynomial is the largest sum of the exponents of all of its monomials. The polynomial is **multilinear** if it is linear in every individual variable.

The proof of Theorem 1 is based on the following lemma.

**Lemma 1.** Suppose that $n \geq 1$ and $d \geq 0$ are integers, $P$ is a multilinear polynomial in $n$ variables of total degree at most $d$ over a field $\mathbb{F}$, and $A \subseteq \mathbb{F}^n$ is a set with $|A| > 2 \sum_{0 \leq i \leq d/2} \binom{n}{i}$. If $P(a - b) = 0$ for all $a, b \in A$ with $a \neq b$, then also $P(0) = 0$. 

Proof. Let \( m := \sum_{0 \leq i \leq d/2} \binom{n}{i} \), and let \( K = \{K_1, \ldots, K_m\} \) be the collection of all sets \( K \subseteq [n] \) with \( |K| \leq d/2 \). Writing for brevity

\[
x^I := \prod_{i \in I} x_i, \quad x = (x_1, \ldots, x_n) \in \mathbb{F}^n, \quad I \subseteq [n],
\]

there exist coefficients \( C_{I,J} \in \mathbb{F} \) depending only on the polynomial \( P \), such that for all \( x, y \in \mathbb{F}^n \) we have

\[
P(x - y) = \sum_{I, J \subseteq [n]} C_{I,J} x^I y^J
\]

\[
= \sum_{I \in K} x^I \sum_{J \subseteq [n]\setminus I \atop |J| \leq d - |I|} C_{I,J} y^J + \sum_{J \in K} \left( \sum_{I \subseteq [n]\setminus J \atop |I| \leq d/2 \leq |J| \leq |K|} C_{I,J} x^I \right) y^J.
\]

The right-hand side can be interpreted as the scalar product of the vectors \( u(x), v(y) \in \mathbb{F}^{2m} \) defined by

\[
u_i(x) = x^{K_i}, \quad u_{m+i}(x) = \sum_{I \subseteq [n]\setminus K_i \atop |I| \leq d - |K_i|} C_{I,K_i} x^I
\]

and

\[
u_i(y) = \sum_{J \subseteq [n]\setminus K_i \atop |J| \leq d - |K_i|} C_{K_i,J} y^J, \quad v_{m+i}(y) = y^{K_i}
\]

for all \( 1 \leq i \leq m \). Consequently, if we had \( P(a - b) = 0 \) for all \( a, b \in A \) with \( a \neq b \), while \( P(0) \neq 0 \), this would imply that the vectors \( u(a) \) and \( v(b) \) are orthogonal if and only if \( a \neq b \). As a result, the vectors \( u(a) \) would be linearly independent (an equality of the sort \( \sum_{a \in A} \lambda_a u(a) = 0 \) with the coefficients \( \lambda_a \in \mathbb{F} \) after a scalar multiplication by \( v(b) \) yields \( \lambda_b = 0 \), for any \( b \in A \)). Finally, the linear independence of \( \{u(a) : a \in A\} \subseteq \mathbb{F}^{2m} \) implies \( |A| \leq 2m \), contrary to the assumptions of the lemma. \( \square \)

Remark. It is easy to extend the lemma relaxing the multilinearity assumption to the assumption that \( P \) has bounded degree in each individual variable. Specifically, denoting by \( f_\delta(n, d) \) the number of monomials \( x_1^{i_1} \cdots x_n^{i_n} \) with \( 0 \leq i_1, \ldots, i_n \leq \delta \) and \( i_1 + \cdots + i_n \leq d \), if \( P \) has all individual degrees not exceeding \( \delta \), and the total degree not exceeding \( d \), then \( |A| > 2 f_\delta(n, \lfloor d/2 \rfloor) \) along with \( P(a - b) = 0 \) \( (a, b \in A, \ a \neq b) \) imply \( P(0) = 0 \). Moreover, taking \( \delta = d \), or \( \delta = |\mathbb{F}| - 1 \) for \( \mathbb{F} \) finite, one can drop the individual degree assumption altogether.
We will use the estimate
\[
\sum_{0 \leq i \leq z} \binom{n}{i} < 2^{nH(z/n)}
\] (1)
valid for all integer \(n \geq 1\) and real \(0 < z \leq n/2\); see, for instance, [McWS77, Ch. 10, §11, Lemma 8].

Recall, that for integer \(n \geq d \geq 0\), the sum \(\sum_{i=0}^{d} \binom{n}{i}\) is the dimension of the vector space of all multilinear polynomials in \(n\) variables of total degree at most \(d\) over the two-element field \(\mathbb{F}_2\). In particular, the dimension of the vector space of all multilinear polynomials in \(n\) variables over \(\mathbb{F}_2\) is equal to the dimension of the vector space of all \(\mathbb{F}_2\)-valued functions on \(\mathbb{F}_2^n\), and it follows that any non-zero multilinear polynomial represents a non-zero function. These basic facts are used in the proof of Proposition 1 below.

For integer \(n \geq 1\), denote by \(F_n\) the subgroup of the group \(\mathbb{Z}_n^4\) generated by its involutions; thus, \(F_n\) is both the image and the kernel of the doubling endomorphism of \(\mathbb{Z}_n^4\) defined by \(g \mapsto 2g\) \((g \in \mathbb{Z}_n^4))\), and we have \(F_n \cong \mathbb{Z}_2^n\).

Proposition 1. Suppose that \(n \geq 1\) and \(A \subseteq \mathbb{Z}_4^n\) is progression-free. Then for every \(0 < \varepsilon < 0.25\), the number of \(F_n\)-cosets containing at least \(2^{nH(0.5-\varepsilon)+1}\) elements of \(A\) is less than \(2^{nH(2\varepsilon)}\).

Proof. Let \(\mathcal{R}\) be the set of all those \(F_n\)-cosets containing at least \(2^{nH(0.5-\varepsilon)+1}\) elements of \(A\), and for each coset \(R \in \mathcal{R}\) let \(A_R := A \cap R\); thus, \(\cup_{R \in \mathcal{R}} A_R \subseteq A\) (where the union is disjoint), and
\[
|A_R| \geq 2^{nH(0.5-\varepsilon)+1}, \quad R \in \mathcal{R}. \tag{2}
\]

For a subset \(S \subseteq \mathbb{Z}_4^n\), write
\[
2 \cdot S := \{s' + s'': (s', s'') \in S \times S, \ s' \neq s''\} \quad \text{and} \quad 2 * S := \{2s : s \in S\}.
\]
The assumption that \(A\) is progression-free implies that the sets
\[
B := \cup_{R \in \mathcal{R}} (2 \cdot A_R) \subseteq F_n \quad \text{and} \quad C := \cup_{R \in \mathcal{R}} (2 * R) \subseteq F_n
\]
are disjoint: this follows by observing that if \(2r \in 2 \cdot A\) with some \(r \in R\), then for each \(a \in r + F_n\) we have \(2a = 2r \in 2 \cdot A\). Furthermore, the sets \(2 * R\) are in fact pairwise distinct singletons (for \(2r_1 = 2r_2\) is equivalent to \(r_1 - r_2 \in F_n\) and thus to \(r_1 + F_n = r_2 + F_n\)), whence \(|C| = |\mathcal{R}|\).

Let \(d = n - [2\varepsilon n]\) so that, in view of (2) and (1),
\[
2 \sum_{0 \leq i \leq d/2} \binom{n}{i} < 2^{nH(0.5-\varepsilon)+1} \leq |A_R|, \quad R \in \mathcal{R}. \tag{3}
\]
Denoting by $\mathcal{C}$ the complement of $C$ in $F_n$, and assuming, contrary to what we want to prove, that $|R| \geq 2^{nH(2\varepsilon)}$, from (1) we get
\[
\sum_{i=0}^d \binom{n}{i} = 2^n - \sum_{i=0}^{[2\varepsilon n] - 1} \binom{n}{i} > 2^n - 2^{nH(2\varepsilon)} \geq 2^n - |R| = 2^n - |C| = |\mathcal{C}|.
\]
(This is the computation where the assumption $\varepsilon < 0.25$ is used.) Consequently, identifying $F_n$ with the additive group of the vector space $\mathbb{F}_2^n$, and accordingly considering $B$ and $C$ as subsets of $\mathbb{F}_2^n$, we conclude that the dimension of the vector space of all multilinear $n$-variate polynomials over the field $\mathbb{F}_2$ exceeds the dimension of the vector space of all $\mathbb{F}_2$-valued functions on $\mathcal{C}$. Thus, the evaluation map, associating with every polynomial the corresponding function, is degenerate. As a result, there exists a non-zero multilinear polynomial $P \in \mathbb{F}_2[x_1, \ldots, x_n]$ of total degree $\deg P \leq d$ such that $P$ vanishes on $\mathcal{C}$. In particular, $P$ vanishes on $B \subseteq \mathcal{C}$, and therefore on each set $2 \cdot A_R$, for all $R \in \mathcal{R}$.

Fixing arbitrarily an element $r \in R$, the polynomial $P(2r + x)$ thus vanishes whenever $x \in 2 \cdot (A_R - r)$. Hence, also $P(2r) = 0$ by Lemma 1 (which is applicable in view of (3)); that is, $P$ also vanishes on each singleton set $2 \ast A_R$, for all $R \in \mathcal{R}$. It follows that $P$ vanishes on $C$. However, $P$ was chosen to vanish on $\mathcal{C}$. Therefore, $P$ vanishes on all of $\mathbb{F}_2^n$, and it follows that $P$ is the zero polynomial. This is a contradiction showing that $|\mathcal{R}| < 2^{nH(2\varepsilon)}$, and thus completing the proof.

**Proof of Theorem 1.** For $x \geq 0$, let $N(x)$ denote the number of $F_n$-cosets containing at least $x$ elements of $A$; thus $N(x) = 0$ for $x > 2^n$, and we can write
\[
|A| = \int_0^{2^{n+1}} N(x) \, dx.
\]
(4)

Trivially, we have $N(x) \leq 2^n$ for all $x \geq 0$, so that
\[
\int_0^{2^{nH(1/4)+1}} N(x) \, dx \leq 2^{(H(1/4)+1)n+1} < 2 \cdot 4^n.
\]
(5)

On the other hand, the substitution $x = 2^{nH(0.5-\varepsilon)+1}$ gives
\[
\int_{2^{nH(1/4)+1}}^{2^{n+1}} N(x) \, dx = n \int_0^{1/4} 2^{nH(0.5-\varepsilon)+1} N(2^{nH(0.5-\varepsilon)+1}) \log \frac{0.5 + \varepsilon}{0.5 - \varepsilon} \, d\varepsilon,
\]
(6)

and applying Proposition 1, the integral in the right-hand side can be estimated as
\[
2n \int_0^{1/4} 2^{nH(0.5-\varepsilon)+H(2\varepsilon)} \log \frac{0.5 + \varepsilon}{0.5 - \varepsilon} \, d\varepsilon < 3n \int_0^{1/4} 2^{nH(0.5-\varepsilon)+H(2\varepsilon)} \, d\varepsilon < n \cdot 4^n.
\]
(7)

From (4)–(7) we get $|A| \lesssim (n + 2) \cdot 4^n$, and to conclude the proof we use the tensor power trick: for integer $k \geq 1$, the set $A \times \cdots \times A \subseteq \mathbb{Z}_4^{kn}$ is progression-free and therefore
\[
|A|^k \lesssim (kn + 2) \cdot 4^{kn}.
\]
by what we have just shown. This readily implies the result. □

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E-mail address: ecroot@math.gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

E-mail address: seva@math.haifa.ac.il

Department of Mathematics, The University of Haifa at Oranim, Tivon 36006, Israel

E-mail address: ppp@cs.bme.hu

Department of Computer Science and Information Theory, Budapest University of Technology and Economics, 1117 Budapest, Magyar tudósok körútja 2, Hungary