Gamow Shell Model description of Li isotopes and their mirror partners

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Background: Weakly bound and unbound nuclei close to particle drip lines are laboratories of new nuclear structure physics at the extremes of neutron/proton excess. The comprehensive description of these systems requires an open quantum system framework that is capable of treating resonant and nonresonant many-body states on equal footing.

Purpose: In this work, we construct the minimal complex-energy configuration interaction approach to describe binding energies and spectra of selected $5 \leq A \leq 11$ nuclei.

Method: We employ the complex-energy Gamow shell model (GSM) assuming a rigid $^4$He core. The effective Hamiltonian, consisting of a core-nucleon Woods-Saxon potential and a simplified version of the Furutani-Horiuchi-Tamagaki interaction with the mass-dependent scaling, is optimized in the $sp$ space. To diagonalize the Hamiltonian matrix, we employ the Davidson method and the Density Matrix Renormalization Group technique.

Results: Our optimized GSM Hamiltonian offers a good reproduction of binding energies and spectra with the root-mean-square (rms) deviation from experiment of 160 keV. Since the model performs well when used to predict known excitations that have not been included in the fit, it can serve as a reliable tool to describe poorly known states. A case in point is our prediction for the pair of unbound mirror nuclei $^{10}$Li-$^{10}$N in which a huge Thomas-Ehrman shift dramatically alters the pattern of low-energy excitations.

Conclusion: The new model will enable comprehensive studies of structure and reactions aspects of light drip-line nuclei.

I. INTRODUCTION

With progress in radioactive beam experimentation and the development of microscopic nuclear theory, light nuclei provide an excellent ground for testing both nuclear interactions and many-body approaches. Of particular interest are weakly bound and unbound nuclear systems with extreme neutron-to-proton imbalance, whose structure is profoundly affected by the coupling to the continuum of decay and reaction channels [1-5]. Impressive progress has been achieved in describing such nuclei using $A$-body methods rooted in realistic inter-nucleon interactions [6-10]. Examples include microscopic computations of $^{11}$Be [11, 12], $^7$He [13], and $^9$He [14].

On a more phenomenological level, configuration integration techniques, based on the concept of valence nucleons coupled to an inert core have reached a high level of sophistication. Approaches such as the real-energy continuum shell model [15, 16] and shell model embedded in the continuum [17, 18] have been applied to systems near particle-emission threshold with one/two nucleons allowed in the continuum space. Another powerful tool is the complex-energy GSM [21, 22], an open quantum system extension of the interacting shell model. GSM has been successfully used to describe structural and reaction properties of exotic nuclei (see Refs. [23-28] for recent representative applications).

This study can be viewed as a continuation of previous work on quantitative GSM description of light nuclei using a $^4$He-nucleon potential and finite-range valence interaction. In the first paper [28], carried out in the $spdf$ model space, the core-nucleon potential was optimized to $^4$He phase shifts and separation energies of $^3$He and $^3$Li. By means of the principal-component analysis, it was concluded that a very reasonable description of energies of $6 \leq A \leq 9$ nuclei (rms deviation from experiment of 250 keV) could be achieved with only four interaction parameters. In the follow-up paper [29], carried out in the $sp$ space, experimental energies and widths of $5-8^4$He could be reproduced within tens of keV precision by adjusting only one parameter (the strength of spin-singlet central neutron-neutron term). In this work, we construct the minimal GSM model to describe binding energies and spectra of $5 \leq A \leq 11$ nuclei in the $sp$ space by carrying out simultaneous optimization of the core-nucleon potential and the valence two-body interaction with the mass-dependent interaction scaling to effectively account for the missing three-body forces.

This paper is organized as follows. The theoretical model is outlined in Sec. [1] which contains a short overview of GSM, description of the GSM Hamiltonian, and the optimization protocol. Results are presented in Sec. [II] with the optimization results discussed first, followed by predictions for lithium isotopes and their mirror partners. Finally, Sec. [IV] presents conclusions and perspectives for future studies.
II. THEORETICAL MODEL

A. Gamow Shell Model

Here we briefly recall the GSM formalism. In this work, we describe the lithium isotopes and their mirror partners in terms of valence nucleons coupled to the $^4$He core. This picture is justified by the fact that the $^4$He nucleus is a tightly bound system with the first excited state located 20.21 MeV above the ground state (g.s.) [30].

The GSM Hamiltonian can be written as

$$H = \sum_{i} \left( \frac{p_i^2}{2\mu_i} + U_c(i) \right) + \sum_{i=1,j>i}^{N_{\text{val}}} V_{i,j} + \frac{P_i P_j}{M_c},$$

(1)

where $N_{\text{val}}$ denotes the number of valence nucleons, $\mu_i$ and $M_c$ are the reduced mass of the nucleon and the mass of the core, respectively, $U_c$ is the core-nucleon potential, and $V_{i,j}$ is the interaction between valence nucleons. The Hamiltonian (1) is written in the cluster orbital shell model coordinates [31] defined with respect to the center of mass of the core.

The GSM Hamiltonian is diagonalized in the Berggren basis [32], which allows to consistently treat bound, resonance, and scattering states. In the complex-momentum space, the Berggren basis obeys the unitarity relation for each partial wave ($\ell, j$):

$$\sum_{n=b,d} |\tilde{u}_{n}\rangle \langle u_{n}| + \int_{\mathcal{L}^+} |\tilde{u}(k)\rangle \langle u(k)| \, dk = 1,$$

(2)

where $b$ and $d$ stand for the discrete bound states and decaying resonances respectively, and the contour $\mathcal{L}^+$ representing the non-resonant scattering states is located in the fourth quadrant of the complex $k$-plane. The specific shape of $\mathcal{L}^+$ is not important as long as it lies below the resonances included in the model space. In practical applications, the contour is discretized for each ($\ell, j$), which results in a finite number of single-particle (s.p.) states. From this discretized set of shells one constructs Slater determinants, which form a many-body basis within which $H$ is diagonalized. Due to inclusion of resonances and complex-momentum scattering states, the representation of $H$ in the Berggren basis is complex symmetric [23].

As in any configuration interaction models, the dimension of the Hamiltonian matrix grows quickly with the number of active particles. In the context of the GSM, it increases more quickly than in the conventional shell model due to the presence of discretized scattering states. To this end, we truncate the model space by working with naturals orbitals which provide an optimized set of s.p. states [28, 33, 34].

The natural orbitals are first computed in a truncated space where few valence particles are allowed to occupy continuum shells. A truncation is then performed on the s.p. basis by keeping only natural orbitals for which the modulus of the occupation number is greater than a certain (small) value. Finally, a new set of Slater determinants is constructed, for which also a truncation on the number of particles in the continuum is enforced, and the numerical diagonalization is performed using the Davidson method [35].

To check the accuracy of this truncation procedure in the case of the largest systems, the supplementary computation was also performed using the Density Matrix Renormalization Group (DMRG) [36, 37] method. The DMRG allows performing calculations without the s.p. particle basis truncation and without restrictions on the number of particles in the continuum. In this approach, the many-body Schrödinger equation is solved iteratively in tractable truncated spaces, which are gradually increased until the numerical convergence is reached. We have checked that, in all cases discussed in this work, the GSM results are in good agreement with those of DMRG.

B. Gamow Shell Model Hamiltonian

The core-nucleon potential is taken as a Woods-Saxon (WS) field, with a central and spin-orbit terms. The WS radius and diffuseness parameters were taken from Ref. [28]: $R_0(n) = 2.15$ fm, $R_0(p) = 2.06$ fm, $a(n) = 0.63$ fm, and $a(p) = 0.64$ fm. The Coulomb potential is generated by a spherical Gaussian charge distribution with the radius $R_{ch} = 1.681$ fm [38].

The effective two-body interaction is constructed based on the finite-range Furutani-Horiuchi-Tamagaki (FHT) force [28, 39, 40], which has been used successfully to describe structure and reactions involving light nuclei [25, 29, 41, 42]. The FHT interaction contains central ($c$), spin-orbit ($LS$), and tensor ($T$) terms. For each term, the radial form factor is represented by a sum of three Gaussians with different widths representing the short, intermediate and long ranges of the nucleon-nucleon interaction [28]. There are seven interaction strengths $V_{\eta}^{ST}$ ($\eta = c, LS, T$) in different spin-isospin channels, which are adjusted to experimental data. In Ref. [28], the FHT interaction was used in the GSM description of bound and unbound nuclei with $A \leq 9$. While a good energy reproduction was achieved, the systematic statistical study of the parameters carried out in Ref. [28] demonstrated that some of the terms in the FHT interaction were sloppy, i.e., not well constrained.

In this study, we use a simplified version of the FHT interaction where we consider the central $V_{c}^{10}, V_{c}^{01},$ and tensor $V_{T}^{10}$ terms. This choice is not only informed by the previous statistical work [28] but also justified by the Effective Field Theory (EFT) arguments [43, 47]. Indeed, in the EFT expansion of the bare nucleon-nucleon interaction, these three terms appear at leading order, whereas the other terms present in the original FHT interaction correspond to higher orders of EFT. However, we have observed that adding the central term $V_{c}^{10}$ improves the overall description of the nuclei considered in
this work and hence we have also included it in $V_{i,j}$. We want to mention here that a similar approach was employed in Ref. [29] to construct an effective neutron-neutron interaction for the description of the helium isotopic chain in the Berggren basis. In that case, using only the central term $V^{01}_{ij}$, a good reproduction of weakly-bound and unbound states in helium nuclei was achieved.

As it is customary in shell model studies [48–49], a mass-dependent interaction-scaling factor of the form $(6/A)^{\alpha}$ is introduced to effectively account for the missing three-body forces [50–51]. We found that the value $\alpha = 1/3$ gives a very reasonable description of experimental energies. Finally, the Coulomb interaction between valence protons is treated by incorporating its long-range part into the basis potential and expanding the short-range two-body component in a truncated basis of HO states [52–53].

C. Interaction Optimization Protocol

To optimize the GSM interaction, we minimize the $\chi^2$ penalty function:

$$\chi^2(p) = \sum_{i=1}^{N_d} \left( \frac{O_i(p) - O_i^{\text{exp}}}{\delta O_i} \right)^2$$

(3)

where $p$ is the vector of parameters used, $N_d$ is the number of observables, $O_i(p)$ are the calculated observables, $O_i^{\text{exp}}$ are experimental values, and $\delta O_i$ are adopted model errors (experimental uncertainties are much smaller compared to the model uncertainties and can safely be ignored). Since in this work we only consider energies in the optimization, the adopted errors are arbitrary. The penalty function is usually normalized at the minimum $\chi^2(p_0)$ to the number of degrees of freedom $N_{\text{dof}} = N_d - N_p$ [54]. The adopted error becomes $\delta O_i \sqrt{\chi^2(p_0)/N_{\text{dof}}}$. Such renormalization procedure guarantees that the calculated statistical uncertainties are meaningful [55].

The minimization of $\chi^2$ is done using the Gauss-Newton method. Since the GSM Hamiltonian is linear in strength parameters, the Jacobian matrix

$$J_{\alpha\beta} = \frac{1}{\delta O_i} \left. \frac{\partial O_i}{\partial p_\alpha} \right|_{p_0}$$

(4)

can be calculated exactly using Hellmann-Feynman theorem [55]. The covariance matrix $C$ can be expressed in terms of $J$:

$$C \simeq (J^T J)^{-1}$$

(5)

The uncertainty for each parameter is the corresponding diagonal elements in the covariance matrix $C$, and the correlation between parameters $p_\alpha$ and $p_\beta$ is given by

$$c_{\alpha\beta} = \frac{C_{\alpha\beta}}{\sqrt{C_{\alpha\alpha} C_{\beta\beta}}}$$

(6)

In the situation where the Jacobian matrix is non-invertible or has a very small determinant, the Gauss-Newton method becomes unstable as the Jacobian matrix must be inverted therein. This typically happens when a parameter is sloppy, i.e., not well constrained by observables. In order to stabilize the calculation, the matrix inversion is replaced by its pseudo-inverse, derived from the singular value decomposition (SVD) of the Jacobian matrix [28].

The four strengths of the WS potential and four parameters of the two-body interaction are simultaneously optimized to reproduce 15 energy levels in lithium isotopes and their mirror partners given in Table 1. The radius and diffuseness parameters for the WS potentials are taken from [28].

| Nucleus | State | $E_{\text{exp}}$ (MeV) | $E_{\text{GSM}}$ (MeV) |
|---------|-------|---------------------|---------------------|
| $^6\text{Li}$ | $1^+$ | −3.70 | −3.72 |
| | $0^+$ | −0.14 | −0.10 |
| $^7\text{Li}$ | $3/2^-$ | −10.95 | −11.02 |
| | $1/2^-$ | −10.47 | −10.14 |
| $^8\text{Li}$ | $2^+$ | −12.98 | −13.14 |
| | $1^+$ | −12.00 | −11.93 |
| $^9\text{Li}$ | $3/2^-$ | −17.05 | −16.90 |
| | $1/2^-$ | −14.35 | −14.50 |
| $^{11}\text{Li}$ | $3/2^-$ | −17.41 | −17.48 |
| $^{7}\text{Be}$ | $3/2^-$ | −9.30 | −9.36 |
| | $1/2^-$ | −8.88 | −8.53 |
| $^8\text{B}$ | $2^+$ | −9.44 | −9.60 |
| | $1^+$ | −8.67 | −8.50 |
| $^9\text{C}$ | $3/2^-$ | −10.74 | −10.85 |
| | $1/2^-$ | −8.52 | −8.59 |

The calculations are performed in a model space which includes $s_{1/2}$, $p_{3/2}$ and $p_{1/2}$ partial waves for both protons and neutrons. Since the optimization involves energies only, for the sake of speeding-up the optimization and for better stability, we used a deeper WS potential to generate the basis, in which the $0p_{3/2}$ and $0p_{1/2}$ poles are bound. A real contour was then used to describe the non-resonant continuum space. The contour $L^+$ was divided into 3 segments with $k_{\text{peak}} = 0.25 \text{ fm}^{-1}$, $k_{\text{mid}} = 0.5 \text{ fm}^{-1}$, and the cutoff momentum $k_{\text{max}} = 4 \text{ fm}^{-1}$. Discretizing each segment with 10 points using the Gauss-Legendre quadrature guarantees the convergence of results.

To calculate resonance’s width, one has to generate a basis based on a more shallow basis-generating WS potential, in which the $0p_{3/2}$ and $0p_{1/2}$ poles are decaying resonances. In this case, a complex contour defined by a complex value of $k_{\text{peak}}$ is employed. It is to be noted that calculation of the width is more demanding than that of energy. A higher discretization of (20,20,20) is needed.
for each segment of the contour. Due to the Coulomb repulsion, the mean field used to generate the s.p. basis for proton rich nuclei varies with proton number. The contour is adjusted separately for each system to assure that the Berggren completeness relation is met. To ensure the numerical stability, the chosen contour should neither lie too close to the Gamow poles nor lie too far from the real-k axis. In this work, \( k_{\text{peak}} \) is chosen to lie slightly (0.05 fm\(^{-1}\)) below the position of the \( 0p_{3/2} \) and \( 0p_{1/2} \) poles, but with the imaginary part greater than \(-0.2 \text{ fm}^{-1}\). The calculations were repeated with several slightly different values of \( k_{\text{peak}} \) to assure the full convergence of results.

In this study, we used a newly developed GSM code that is based on the two-dimensional partitioning of the Hamiltonian matrix [57]. First, we computed natural orbitals with occupations greater than \( 10^{-6}\). The GSM problem was then solved in a model space with at most three particles in the continuum shells. We checked the accuracy of this truncation by performing the full DMRG calculations for the heaviest systems. (There is no s.p. basis truncation nor restrictions on the number of particles in the continuum in DMRG.) In all cases considered, the results obtained with the GSM turned out to agree well with DMRG.

### III. RESULTS

#### A. Optimized Interaction

As one can see in Table II a very good consistency between theoretical and experimental energies has been achieved. The root-mean-square deviation from experimental values is 160 keV. The largest discrepancy is obtained for the 1/2\(^{−}\) states of \(^{7}\)Li and \(^{7}\)Be, where the deviation from the data is \( \sim 340 \) keV.

| Nucleus  | \( E_{\text{GSM}} \) (MeV) | \( E_{\text{exp}} \) (MeV) | \( \Gamma_{\text{GSM}} \) (MeV) | \( \Gamma_{\text{exp}} \) (MeV) |
|----------|----------------|----------------|----------------|----------------|
| \(^{3}\)He | 0.74 | 0.798 | 640 | 648 |
| \(^{5}\)Li | 1.6 | 1.69 | 1300 | 1230 |

### TABLE II. Central and spin-orbit strengths of the core-nucleon WS potential optimized in this work. The statistical uncertainties are given in parentheses.

The values of the parameters for the WS potentials and the two-body interaction are displayed, along with their statistical uncertainties, in Tables II and IV, respectively. As one can judge from small parameter uncertainties in Tables II and IV the GSM Hamiltonian fit is well constrained. As expected [28], the central term \( V_{c}^{00} \) has the largest uncertainty of \( \sim 12\%\).

It is to be noted that the core-nucleon potential developed in the present study, optimized simultaneously with the two-body interaction, is slightly more shallow than the WS field optimized in Ref. [28] to the experimental \( s \) and \( p \) nucleon-\(^{4}\)He scattering phase shifts. To assess the quality of the WS potential obtained in this work, Table III shows the predicted energies and widths of the 3/2\(^{−}\) g.s. of \(^{5}\)He and \(^{5}\)Li. These values are indeed very close to predictions of Ref. [28] for \(^{5}\)He and \(^{5}\)Li.

### TABLE III. Ground-state energies (in MeV) and widths (in keV) of \(^{5}\)He and \(^{5}\)Li obtained from the optimized core-nucleon potential and compared to experiment [58, 59].

| Nucleus  | \( E_{\text{GSM}} \) (MeV) | \( E_{\text{exp}} \) (MeV) | \( \Gamma_{\text{GSM}} \) (MeV) | \( \Gamma_{\text{exp}} \) (MeV) |
|----------|----------------|----------------|----------------|----------------|
| \(^{5}\)He | 0.74 | 0.798 | 640 | 648 |
| \(^{5}\)Li | 1.6 | 1.69 | 1300 | 1230 |

### TABLE IV. Strengths \( V_{\ell}^{\text{ST}} \) of the two-body interaction optimized in this work. The statistical uncertainties are given in parentheses.

| \( V_{\ell}^{\text{ST}} \) (MeV) | \( V_{\ell}^{\text{ST}} \) (MeV fm\(^{−2}\)) |
|----------------|----------------|
| \( V_{0}^{01} \) | \(-9.425 \) (70) |
| \( V_{1}^{01} \) | \(-8.309 \) (90) |
| \( V_{0}^{10} \) | \(-8.995 \) (1130) |
| \( V_{2}^{10} \) (MeV fm\(^{−2}\)) | \(-22.418 \) (970) |

Figure II shows the energies calculated in GSM for the ground states and selected excited states in lithium isotopes. As one can see, the optimized interaction allows for a good reproduction of experimental energies. It is to be noted that the results for higher-excited states not included in the fit are very satisfactory. For instance, the calculated 3\(^{+}\) state in \(^{6}\)Li at \(-1.57 \text{ MeV}\) is only 60 keV below the experimental energy. The experimental widths for the second 5/2\(^{−}\) state in \(^{7}\)Li (89 keV) and 5/2\(^{−}\) state in \(^{7}\)Li (88 keV) are very reasonable: the GSM values are, respectively, \(22 \text{ keV}\) and \(62 \text{ keV}\). In general, we do not expect the same quality of data reproduction for all excited states due to the fact that the higher partial waves with \( \ell \geq 2 \), which may contribute to the wave functions of these states, are not included in the model space.

#### B. Structure of \(^{10}\)Li

Several experiments [60, 63] and theoretical studies [22, 64] have indicated that the structure of the ground state in \(^{10}\)Li may correspond to a valence neutron in a virtual s-state. In a recent experiment [65], the presence of an appreciable low-energy \( \ell = 0 \) strength has not been confirmed. Their conclusion was, however, challenged in theoretical studies [66, 67].

The optimized WS potential given in Table II does not support the virtual s.p. s-state. We recall here that in the GSM computation of unbound states, one needs to have an approximation of the targeted state in order to correctly identify it in the continuum spectrum obtained from the Hamiltonian diagonalization. In the present
context, the lack of a s.p. virtual state makes the identification of a hypothetical many-body virtual state an extremely difficult task. For that reason, we limited our calculations to resonant states in $^\text{10}\text{Li}$.

The computed ground-state $2^+$ and the first excited state $1^+$ are predicted, respectively, at 0.35 MeV and 0.68 MeV above the $n+^9\text{Li}$ threshold. As seen in Fig. 1, the practically degenerate $1^-$ and $2^-$ states are calculated at 1.05 MeV. A comment is in order here. To achieve the numerical stability, the calculation of the resonances in $^{10}\text{Li}$ had to be performed by employing a basis-generating WS potential that is deeper than the optimized core-nucleon potential. We have checked that in this way we could obtain very stable results for the energies, with accuracy below 1 keV. On the other hand, the computed widths, of the order of few hundreds keV, are not stable. For that reason, we do not show them in Fig. 1.

Table V lists the squared amplitudes of the dominant neutron configurations for the four low-lying states of $^{10}\text{Li}$. The positive parity states are primarily made from the $0p_{3/2}$ and $0p_{1/2}$ resonant shells. The negative parity
states contain one neutron in the 1s1/2 shell. The contribution from the non-resonant continuum space to the low-lying states is very small.

**TABLE V.** Squared amplitudes of dominant configuration of valence neutrons and protons for low-lying levels of 10Li and 11N, respectively. The odd proton in 10Li and the odd neutron in 11N occupy the 0p3/2 Gamow state. The tilde sign labels non-resonant continuum components.

| configuration                        | \(^{10}\text{Li}\) | \(^{11}\text{N}\) |
|--------------------------------------|----------------|----------------|
| \((0p_{3/2})^4(0p_{1/2})\)^1        | 2\(^+\)         | 2\(^+\)         |
| \((0p_{3/2})^3(0p_{1/2})^2\)^1      | 0.84            | 0.81            |
| \((0p_{3/2})^4(1s_{1/2})\)^1        | 1\(^-\)         | 1\(^-\)         |
| \((0p_{3/2})^4(\tilde{s}_{1/2})\)^1 | 0.72            | 0.73            |
| \((0p_{3/2})^3(0p_{1/2})^2(1s_{1/2})\)| 0.14           | 0.14            |
| \((0p_{3/2})^2(0p_{1/2})^2(\tilde{s}_{1/2})\)| 0.06           | 0.06            |
| \((0p_{3/2})^2(0p_{1/2})^2(\tilde{s}_{1/2})\)| 0.07           | 0.07            |
| \((0p_{3/2})^2(0p_{1/2})^2(\tilde{s}_{1/2})\)| 0.03           | 0.03            |

In Ref. [70] they observed two positive-parity states at 0.24 MeV and 0.53 MeV above the \(n^+\)Li threshold. The \(J^\pi = 1^+\) assignment for the lower state was questioned in Ref. [71] who suggested a \(J^\pi = 2^+\) assignment. Considering large widths of the predicted positive-parity doublet, both assignments are consistent with the GSM results. The computed position of the negative-parity \(1^-,2^-\) doublet is consistent with the observation of a negative-parity state at \(\sim 1.5\) MeV [65].

### C. Mirror partners of lithium isotopes

The level schemes for the mirror partners of lithium isotopes are shown in Fig. 2. As in the Li case, we obtain a very reasonable agreement with experiment. The 5/2\(^-\) and 7/2\(^-\) excited states in \(^7\)Be are slightly (\(< 300\) keV) above the corresponding experimentally values, whereas the position of the resonant 3\(^+\) states in \(^8\)B and 5/2\(^-\) state in \(^8\)C are well reproduced, as well as the weakly-bound g.s. of \(^8\)B and \(^8\)C.

In the following we focus on the unbound nuclei \(^{10}\)N and \(^{11}\)O. Due to the presence of the Coulomb barrier, the 1s1/2 single-proton state is a resonance rather than a virtual state [69,72]. To capture this state, a complex contour is used with a \(k_{\text{peak}} = (0.25, -0.05)\) fm\(^{-1}\).

The spectrum of \(^{10}\)N is not experimentally known with certainty. In Fig. 2, we show the tentative level assignments used in Ref. [30]. According to Refs. [73,74], the ground state of \(^{10}\)N is most likely a 1\(^-\) state with \(E_P\) in the range from 1.81 to 1.94 MeV. In a more recent work [72], they observed two low-lying negative-parity states but they were not able to assign \(J^\pi\) values.

Our calculations for \(^{10}\)N predict the ground state to be a 1\(^-\) state with \((E,\Gamma) = (-8.93, 0.9)\) MeV that lies 1.92 MeV above the 1p threshold. The first excited state is predicted to be a 2\(^-\) state with \(\Gamma = 0.3\) MeV slightly below the value quoted in Ref. [73]. This result is consistent with the recent Gamow coupled-channel analysis of Ref. [72]. We also predict an excited 1\(^+\) state with \(\Gamma = 0.3\) MeV, lying 2.9 MeV above the \(^9\)C+p threshold, as well as a second positive-parity 2\(^+\) state with a width of 0.36 MeV.

Table V shows the squared amplitudes of the dominant proton configurations for the four low-lying states of \(^{10}\)N. Similar to \(^{10}\)Li, the positive parity states are primarily made from the 0p3/2 and 0p1/2 resonant shells. The dominant configurations of negative parity states contain one \(\ell = 0\) proton, which can either be in the 1s1/2 shell or in a non-resonant continuum state.

The unbound \(^{11}\)O is the mirror partner of the 2n-halo nucleus \(^{11}\)Li. The first observation of \(^{11}\)O was achieved recently [69]. A broad peak with a width of 3.4 MeV was observed. Our GSM calculations predict a 3/2\(^-\) g.s. with a width of 0.13 MeV and the first excited 5/2\(^+\) state with \(\Gamma \approx 1\) MeV, see Fig. 2. These predictions are consistent with the Gamow coupled-channel calculations of Ref. [72].

![Fig. 3.](image_url) Level schemes of Li isotopes with (a) \(A = 7\), (b) \(A = 8\), (c) \(A = 9\), (d) \(A = 10\), and their mirror partner predicted in our GSM calculations. The energies are plotted with respect to the g.s. energy (at zero). The one-nucleon emission thresholds are marked.

To study the effect of particle continuum due to different positions of particle thresholds in mirror partners, or Thomas-Ehrman effect [76,77], in Fig. 3 we compare the level schemes of Li isotopes and their mirror partners. (For the early GSM study of the Thomas-Ehrman shifts...
in light nuclei, see Ref. [25].) As expected, the proton-unbound states in proton-rich mirror nuclei are shifted down in energy as compared to the states in neutron-rich partners, which lie below, or slightly above the $1n$ threshold.

The $^{10}\text{Li}-^{10}\text{N}$ mirror pair is the most interesting one as both nuclei lie above the particle-emission thresholds. As seen in Table V, the effect of the very low $^9\text{C}+p$ threshold in $^{10}\text{N}$ on the negative-parity states $1^-$ and $2^-$ containing the $s$-wave proton is huge: it results in a rather dramatic shift of both negative parity states when going from $^{10}\text{Li}$ to $^{10}\text{N}$ that gives rise to a different structure of low-lying resonances in these nuclei.

IV. CONCLUSION

In this work, we studied level schemes of $^{6-11}\text{Li}$ and their mirror partners in the framework of the complex-energy Gamow shell model assuming the rigid $^4\text{He}$ core. The effective interaction between valence nucleons is constructed based on a simplified version of the FHT potential.

By fitting four FHT coupling constants and four parameters of the core-nucleon potential, to the experimental energies of fifteen states in $^{6-9,11}\text{Li}$, $^7\text{Be}$, $^8\text{B}$ and $^9\text{C}$, we managed to construct a well constrained interaction. A rms deviation from experiment of 160 keV was reached, with the statistical errors of the GSM Hamiltonian parameters not exceeding 10%.

We assessed the predictive power of the optimized Hamiltonian by making predictions for excited states not included in the fit. In general, very good agreement with testing data was obtained.

Predictions were also made for the particle-unstable nuclei $^{10}\text{Li}$, $^{10}\text{N}$, and $^{11}\text{O}$. The computed $3/2^-$ ground state of $^{11}\text{O}$ is consistent with the recent Gamow coupled-channel calculations [69, 72]. The ground state of $^{10}\text{Li}$ is predicted to be a $2^-$ state about 0.35 MeV above the neutron-emission threshold, in accordance with Ref. [71] while the lowest negative-parity state $1^-$ is expected to lie $\sim 1.0\text{MeV}$ higher, in agreement with Ref. [65]. Due to a spectacularly strong Thomas-Ehrman effect, for $^{10}\text{N}$ we predict the $1^-$ ground state and $2^-$ first excited state.

By successfully reproducing the structure of lithium isotopes and their mirror partners with an optimized interaction, we demonstrated that the quantified GSM is capable of quality predictions for exotic light nuclei with several valence protons and neutrons. Our future efforts will focus on Be and B isotopes, which exhibit complex structure due to the intricate effects of continuum coupling and clustering [78, 79].

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