Control problem for a system of linear loaded differential equations

V R Barseghyan and T V Barseghyan
Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
Yerevan State University, Yerevan, Armenia
E-mail: barseghyan@sci.am

Abstract. The problem of control and optimal control for a system of linear loaded differential equations is considered. Necessary and sufficient conditions for complete controllability and conditions for the existence of a program control and the corresponding motion are formulated. The explicit form of control action for the control problem is constructed and a method for solving the problem of optimal control is proposed.

Introduction

The study of different control processes in various fields of science and technology allows one to conclude that the future course of many control processes turns out to depend not only on the present but also on the prehistory of the process. A mathematical description of these dynamic processes can be carried out by using ordinary differential equations with memory of various types, also called equations with aftereffect or loaded differential equations. In the literature [1–5], the loaded differential equations are usually the equations whose coefficients or the right-hand side contain some functionals (functions) of the solution, in particular, the values of the solution at which the phase state of the process at a point and at a time moment can influence the dynamics of the process as a whole. In practice, such problems arise, for example, when, if it is necessary to make observations of a dynamic process, the phase states are measured at some time moments, and the information is continuously transmitted through the feedback.

The presence of a loaded summand in the dynamics of the system does not always allow one directly to apply the known methods developed in investigations of ordinary (unloaded) dynamic systems. This emphasizes both the theoretical and practical relevance of studying various control problems for loaded differential equations. We note that the presence of a loaded summand in differential equations leads to new problems that have not yet been investigated in the theory of control.

Over the last few years, the loaded differential equations related to various applied problems of mechanics, biology, ecology and chemistry modeled with loaded equations have been studied intensively. The interest of investigators in the control problems of loaded dynamical systems is also stimulated by the capabilities of modern computing and measuring technology, which permits using the most adequate mathematical models of the processes under study.

The loaded ordinary differential equations and boundary-value problems for such equations are considered in [1–5], and conditions for their solvability by various methods are established.
A significant contribution to the development of the theory of loaded equations was made by the article [1] (and in other works by the same author), where the definitions of loaded differential, loaded integro-differential, loaded functional equations, and their numerous applications are given. In the monograph [2], the loaded differential equations are interpreted as perturbations of differential equations. In [5], on the basis of the parametrization method, a linear multi-point boundary-value problem for a system of loaded differential equations is investigated and an algorithm for finding the solution is proposed. Many studies have been devoted to investigation of the existence of solutions of loaded linear differential equations, but relatively little attention has been paid to the problems of their control.

In this work, we consider the problem of control and optimal control of a system of linear loaded differential equations. Necessary and sufficient conditions for complete controllability are formulated. A constructive method for solving the control problem is proposed and conditions for the existence of program control and motion are formulated. The analytic forms of motion and control action for the control problem are constructed, as well as a method for solving the optimal control problem is proposed.

1. Statement of the problem

Let us consider a control process whose dynamics is described by loaded linear differential equations

\[ \dot{x} = A_0(t)x + A_1(t)x(t_1) + A_2(t)x(t_2) + A_3(t)x(t_3) + B(t)u, \]  

(1)

where \( x(t) \in \mathbb{R}^n \) is the phase vector of the system, \( A_k(t) \) and \( B(t) \) \( (k = 0, 1, 2, 3) \) are \( n \times n \) and \( n \times r \) matrices of the system parameters (continuous on \( [t_0, T] \)) and \( u(t) \) is the control action of dimension \( r \times 1 \).

In formula (1), the summands \( A_k(t)x(t_k) \) \( (k = 1, 2, 3) \) influence the system as functions starting from the moment \( t \geq t_k \), since the value of the phase state \( x(t_k) \), as a result of measurement, is determined at the time moment \( t = t_k \) and, from that moment (when \( t \geq t_k \)), continuously affects the system in the form of summands \( A_k(t)x(t_k) \).

Let the initial

\[ x(t_0) = x_0 \]  

(2)

and final

\[ x(T) = x_T \]  

(3)

states of system (1) be given.

It is assumed that the time moments of loading points and \( 0 \leq t_0 < t_1 < t_2 < t_3 < t_4 = T \) are given. The function \( x(t) \) is continuous on the intervals \( [t_{k-1}, t_k] \) \( (k = 1, 2, 3, 4) \) and has finite left-hand limits \( \lim_{t \to t_k^-} x(t) = x(t_k) \) at the loading points \( t_k \).

We consider the following problems.

**Problem 1.** Determine the conditions under which there exists a program control action \( u = u(t) \) and a program motion transferring the motion of system (1) from the initial state (2) to the final state (3) over the time interval \( [t_0, T] \), as well as construct the control action and motion.

Let a quality criterion \( \varphi[u] \), which can have the meaning of the norm of some normalized space, be given for the selection of the optimal solutions over the time interval \( [t_0, T] \).

The problem of optimal control for system (1) with conditions (2), (3) and the quality criterion \( \varphi[u] \) can be formulated as follows.

**Problem 2.** Determine the optimal control action \( \hat{u}(t), t \in [t_0, T] \), which transfers the motion of system (1) from the initial state (2) to the final state (3) and takes the least possible value of the quality criterion \( \varphi[u] \).
2. Construction of the motion of the loaded system

To construct the motion of system (1), we divide the interval \([t_0, T]\) into parts by loading points: 
\[ t_0, T = \bigcup_{k=1}^{4} [t_{k-1}, t_k). \]
Taking into account the sequence of loading points and the character of the influence of the corresponding loaded summands, we write equation (1) separately on the intervals \([t_{k-1}, t_k)\) \((k = 1, 2, 3, 4)\) in the form

\[
\dot{x} = \begin{cases} 
A_0(t)x + B(t)u & \text{for } t \in [t_0, t_1), \\
A_0(t)x + A_1(t)x(t_1) + B(t)u & \text{for } t \in [t_1, t_2), \\
A_0(t)x + A_1(t)x(t_1) + A_2(t)x(t_2) + B(t)u & \text{for } t \in [t_2, t_3), \\
A_0(t)x + A_1(t)x(t_1) + A_2(t)x(t_2) + A_3(t)x(t_3) + B(t)u & \text{for } t \in [t_3, T], 
\end{cases}
\]

which is the stage by stage changing of differential equations.

We assume that the stage by stage changing of the system of loaded differential equations (1) (or (4)) over the time interval \([t_0, T]\) is completely controllable \([6–8]\).

To solve the above-formulated problems, we construct the motion of stage by stage changing of system (1) (or (4)) over the time interval \([t_0, T]\). For this, we write the solution of system (4) for the time interval \([t_0, t_1]\) in the form

\[
x(t) = X[t, t_0]x(t_0) + \int_{t_0}^{t} H[t, \tau]u(\tau) \, d\tau,
\]

where \(H[t, \tau] = X[t, \tau]B(\tau)\) and \(X[t, \tau]\) denotes the normalized fundamental matrix of the solution of the homogeneous part of \(\dot{x} = A_0(t)x\).

For the time interval \([t_1, t_2]\), the solution of equation (4) is represented as

\[
x(t) = X[t, t_1]X[t_1, t_0]x(t_0) + \int_{t_1}^{t} X[t, \tau](A_1(\tau)x(t_1) + B(\tau)u(\tau)) \, d\tau.
\]

Taking into account that \(\lim_{t \to t_0} x(t) = x(t_0)\), from formula (5), subtracting the value of \(x(t_1)\) and substituting in (6), we obtain the motion of system (4) at the time moment \(t \in [t_1, t_2]\) in the form

\[
x(t) = \bar{X}[t, t_1]X[t_1, t_0]x(t_0) + \bar{X}[t, t_1] \int_{t_0}^{t_1} H[t_1, \tau]u(\tau) \, d\tau + \int_{t_1}^{t} H[t, \tau]u(\tau) \, d\tau.
\]

Continuing this procedure, we obtain the formula for representing the motion of system (4) at the time moment \(t \in [t_2, t_3]\) in the form

\[
x(t) = Y[t, t_2]X[t_1, t_0]x(t_0) + Y[t, t_2] \int_{t_0}^{t_1} H[t_1, \tau]u(\tau) \, d\tau
\]

\[+ \bar{X}[t, t_2] \int_{t_1}^{t_2} H[t_2, \tau]u(\tau) \, d\tau + \int_{t_2}^{t} H[t, \tau]u(\tau) \, d\tau,
\]

where

\[
Y[t, t_2] = \bar{X}[t, t_2]\bar{X}[t_2, t_1] + \int_{t_2}^{t} X[t, \tau]A_1(\tau) \, d\tau,
\]
3. Solution of the problems

and at the time moment $t \in [t_3, t_4]$,

$$x(t) = Z[t, t_3]X[t_1, t_0]x(t_0) + Z[t, t_3] \int_0^{t_3} H[t_1, \tau]u(\tau) \, d\tau$$

$$+ \left( \int_{t_3}^t X[t, \tau]A_2(\tau) \, d\tau + \tilde{X}[t, t_3] \tilde{X}[t_3, t_2] \right) \int_{t_3}^t H[t_2, \tau]u(\tau) \, d\tau$$

$$+ \tilde{X}[t, t_3] \int_{t_2}^t H[t_3, \tau]u(\tau) \, d\tau + \int_{t_3}^t H[t, \tau]u(\tau) \, d\tau,$$

where

$$\tilde{X}[t, t_j] = X[t, t_j] + \int_{t_j}^t X[t, \tau]A_j(\tau) \, d\tau \quad (j = 1, 2, 3),$$

$$Z[t, t_3] = \int_{t_3}^t X[t, \tau]A_1(\tau) \, d\tau + \int_{t_3}^t X[t, \tau]A_2(\tau) \, d\tau \tilde{X}[t_2, t_1] + \tilde{X}[t, t_3]Y[t_3, t_2].$$

Thus, having the initial state $x(t_0)$ of system (1) and setting the control action $u(t)$, we use formulas (5) and (7)–(9) to determine the phase state $x(t)$ of system (1) (the solution of the loaded equation (1)) for the respective time intervals $[t_{k-1}, t_k]$ ($k = 2, 3, 4$).

3. Solution of the problems

From formulas (9) for $t = t_4$, we have

$$x(t_4) = Z[t_4, t_3]X[t_1, t_0]x(t_0) + Z[t_4, t_3] \int_0^{t_4} H[t_1, \tau]u(\tau) \, d\tau$$

$$+ \left( \int_{t_3}^{t_4} X[t_4, \tau]A_2(\tau) \, d\tau + \tilde{X}[t_4, t_3] \tilde{X}[t_3, t_2] \right) \int_{t_3}^{t_4} H[t_2, \tau]u(\tau) \, d\tau$$

$$+ \tilde{X}[t_4, t_3] \int_{t_2}^{t_4} H[t_3, \tau]u(\tau) \, d\tau + \int_{t_3}^{t_4} H[t_4, \tau]u(\tau) \, d\tau.$$  \hfill (11)

Now, instead of functions $H[t_k, t]$, we introduce the functions $\bar{H}[t_k, t]$ as

$$\bar{H}[t_1, t] = \begin{cases} H[t_1, t] & \text{for } t_0 \leq t < t_1, \\ 0 & \text{for } t_1 \leq t \leq T, \end{cases} \quad \bar{H}[t_2, t] = \begin{cases} H[t_2, t] & \text{for } t_1 \leq t < t_2, \\ 0 & \text{for } t_2 \leq t \leq T, \end{cases}$$

$$\bar{H}[t_3, t] = \begin{cases} H[t_3, t] & \text{for } t_2 \leq t < t_3, \\ 0 & \text{for } t_3 \leq t \leq T, \end{cases} \quad \bar{H}[t_4, t] = \begin{cases} H[t_4, t] & \text{for } t_3 \leq t \leq T. \end{cases}$$ \hfill (12)

We use the functions introduced in (12) to write the integral relations (11) as

$$x(t_4) = Z[t_4, t_3]X[t_1, t_0]x(t_0) + \int_0^T \left( \sum_{i=1}^4 H_i[t] \right) u(t) \, dt,$$ \hfill (13)

where

$$H_1[t] = Z[t_4, t_3]\bar{H}[t_1, t], \quad H_2[t] = \left( \int_{t_3}^{t_4} X[t_4, \tau]A_2(\tau) \, d\tau + \tilde{X}[t_4, t_3] \tilde{X}[t_3, t_2] \right) \bar{H}[t_2, t],$$

$$H_3[t] = \tilde{X}[t_4, t_3] \bar{H}[t_3, t], \quad H_4[t] = \bar{H}[t_4, t].$$
The integral relations (13) become
\[ \int_{t_0}^{T} \left( \sum_{i=1}^{4} H_i[t] \right) u(t) \, dt = x(t_4) - Z[t_4, t_3] X[t_1, t_0] x(t_0) = \eta(t_0, \ldots, T). \] (14)

Thus, we see that system (1) over the time interval \([t_0, T]\) is completely controllable if and only if, for any vector \(\eta(t_0, \ldots, T) = x(T) - Z[T, t_3] X[t_1, t_0] x(t_0)\) in the space \(R^n\), a control \(u = u(t, \eta)\) satisfying condition (14) can be indicated.

The question of solvability is fundamental for any control problem and reduces to analyzing the controllability of the system.

We can formulate the assertion for the complete controllability of system (1) with conditions (2) and (3) as follows.

For the loaded system (1) (or the stage by stage changing of system (4)) to be completely controllable over the interval \([t_0, T]\), it is necessary and sufficient that the column vector of the matrix \(\left( \sum_{i=1}^{4} H_i[t] \right)^T\) be linearly independent over this interval. Hereinafter “\(^T\)” denotes the operation of transposition.

Following [6, 7, 9], we seek the function \(u(t), t \in [t_0, T]\), satisfying the integral relation (14) in the form
\[ u(t) = \left( \sum_{i=1}^{4} H_i[t] \right)^T C + v(t), \] (15)
where \(C\) is a constant vector to be determined and \(v(t)\) is a vector function satisfying the orthogonality condition
\[ \int_{t_0}^{T} \left( \sum_{i=1}^{4} H_i[t] \right) v(t) \, dt = 0. \] (16)

Substituting (15) into (14), we obtain
\[ Q(t_0, \ldots, T) C = \eta(t_0, \ldots, T), \] (17)
where
\[ Q(t_0, \ldots, T) = \int_{t_0}^{T} \left( \sum_{i=1}^{4} H_i[t] \right) \left( \sum_{i=1}^{4} H_i[t] \right)^T \, dt. \] (18)

Equation (17) present a system of \(n\) algebraic equations in the unknowns \(C_j\) \((j = 1, \ldots, n)\). Equation (17) has a solution if \(\text{det} Q \neq 0\) or the rank of the matrix \(Q\) coincides with the rank of the extended matrix \(\{Q, \eta\}\).

If \(\text{det} Q \neq 0\), then the solution of equation (17) is \(C = Q^{-1} \eta\), and consequently, it follows from (15) that
\[ u(t) = \left( \sum_{i=1}^{4} H_i[t] \right)^T Q^{-1} \left( x(T) - Z[T, t_3] X[t_1, t_0] x(t_0) \right) + v(t). \] (19)

Therefore, the solution of Problem 1 can be formulated as a theorem similar to that proved in [6, 9].

**Theorem.** For the existence of the program control (15) and its corresponding solution of system (1) satisfying conditions (14), (16), it is necessary and sufficient that the matrix (18) be nonsingular or that the ranks of the matrix \(Q\) and \(\{Q, \eta\}\) coincide.
With (12) taken into account, under the assumption that \(\det Q \neq 0\) and \(v(t) = 0\), it follows from (19) that the control action \(u(t)\) is can be presented in the form

\[
u(t) = \begin{cases} 
\left( Z[t_4, t_3]H[t_1, t]\right)^T Q^{-1}(x(T) - Z[T, t_3]X[t_1, t_0]x(t_0)) & \text{for } t \in [t_0, t_1), \\
\left( \int_{t_3}^{t_4} X[t_4, \tau]A_2(\tau) d\tau + \tilde{X}[t_4, t_3]X[t_3, t_2]H[t_2, t]\right)^T & \text{for } t \in [t_1, t_2), \\
\left( \tilde{X}[t_4, t_3]H[t_3, t]\right)^T Q^{-1}(x(T) - Z[T, t_3]X[t_1, t_0]x(t_0)) & \text{for } t \in [t_2, t_3), \\
\left( H[t_4, t]\right)^T Q^{-1}(x(T) - Z[T, t_3]X[t_1, t_0]x(t_0)) & \text{for } t \in [t_3, T].
\end{cases}
\] (20)

By substituting the control action from (20) into (5), (7)–(9), respectively, we obtain the program motion of system (1) over the time interval \([t_0, T]\) and satisfy conditions (2), (3).

To solve Problem 2, we note that the left-hand side of condition (14) is a linear operation generated by the function \(u(t)\) over the time interval \([t_0, T]\). Consequently, if the functional \(\varphi[u]\) is the norm of some linear normalized space, then the solution of Problem 2 should be constructed by the algorithm for solving the corresponding problem of moments [6, 8]. Then the solution of Problem 2 is given by the optimal control action \(u^0(t), t \in [t_0, T]\), minimizing the functional \(\varphi[u]\).

**Conclusions**

A constructive approach to studying the problem of control for a system of linear loaded differential equations is proposed. A formula for determining the phase state of such dynamic systems at any time moment in a given initial state is obtained. Necessary and sufficient conditions for complete controllability and conditions for the existence of the program control and motion are formulated. The solution of the control problem for linear loaded differential equations is constructed, and a method for solving the problem of optimal control is proposed.

**References**

[1] Nakhushev A M 2012 *Loaded Equations and Their Applications* (Moscow: Nauka) p 232 [in Russian]
[2] Dzhenaliev M T and Ramazanov M I 2010 *Loaded Equations as Perturbations of Differential Equations* (Almaty: Fylym) p 336 [in Russian]
[3] Dzhenaliev M T 1989 Optimal control of linear loaded parabolic equations *Diff. Uravn.* **25** (4) 641–51
[4] Kozhanov A I 2004 Nonlinear Loaded Equations and Inverse Problems *Comput. Math. Math. Phys.* **44** (4) 657–75
[5] Bakirova E A and Kadirbayeva Zh M 2016 On a solvability of linear multipoint boundary-value problem for the loaded differential equations *Izv. Nats. Akad. Nauk Resp. Kazakhstan. Ser. Fiz.-Mat.* **5** (309) 168–75
[6] Barseghyan V R 2016 *Control of Compound Dynamic Systems and of Systems with Multipoint Intermediate Conditions* (Moscow: Nauka) p 230 [in Russian]
[7] Barseghyan V R 2012 Control of stage by stage changing linear dynamic systems *Yugoslav J. Operat. Res.* **22** (1) 31–9
[8] Krasovskii N N 1968 *The Theory of Motion Control* (Moscow: Nauka) p 476 [in Russian]
[9] Zubov V I 1975 *Lectures on the Theory of Control* (Moscow: Nauka) p 496 [in Russian]