Variational inequalities

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Abstract

If $-\infty < \alpha < \beta < \infty$ and $f \in C^3([\alpha, \beta] \times \mathbb{R}^2, \mathbb{R})$ is bounded, while $y \in C^2([\alpha, \beta], \mathbb{R})$ solves the typical one-dimensional problem of the calculus of variations to minimize the function

$$F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) \, dx,$$

then for any $\phi \in C^2([\alpha, \beta], \mathbb{R})$ for which $\phi^{(k)}(\alpha) = \phi^{(k)}(\beta) = 0$ for every $k \in \{0, 1, 2\}$, we prove that

$$\int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y'^2} \phi'^2 - \frac{\partial^3 f}{\partial y'^2 \partial y^2} \phi'^2 \phi'' + \frac{\partial^2 f}{\partial y^2} \phi'^2 \phi'' + \frac{\partial^3 f}{\partial y^2 \partial y'^2} \phi'^2 \phi'' + \frac{\partial^2 f}{\partial y'^2} \phi'^2 \phi'' + \frac{\partial^3 f}{\partial y'^2 \partial y^2} \phi'^2 \phi'' \right) \, dx \geq$$

$$\int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} \phi' \phi'' + \frac{\partial^3 f}{\partial y^2 \partial y'} \phi' \phi'' + \frac{\partial^2 f}{\partial y'^2} \phi' \phi'' + \frac{\partial^3 f}{\partial y'^2 \partial y^2} \phi' \phi'' \right) \, dx,$$

so either the above are variational inequalities of motion or the Lagrangian of motion is not $C^3$.

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1. Definition. If $-\infty < \alpha < \beta < \infty$ and $f : [\alpha, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ is any bounded continuous function, then the typical one-dimensional problem of the calculus of variations is to minimize the function $F$, which is defined by the relation

$$F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) \, dx,$$

where $y : [\alpha, \beta] \to \mathbb{R}$ ranges over a suitably chosen class of functions.

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The following two propositions are well-known. See Appendix D on pages 151-152 of [1].

2. **Proposition.** If $-\infty < \alpha < \beta < \infty$ and $r$ is any positive integer, while $y : [\alpha, \beta] \rightarrow \mathbb{R}$ is any continuous function such that

$$
\int_{\alpha}^{\beta} y(x) \eta(x) dx = 0
$$

for every $\eta \in C^r ([\alpha, \beta], \mathbb{R})$ for which

$$
\eta^{(k)}(\alpha) = \eta^{(k)}(\beta) = 0
$$

for every $k \in \{0, 1, \ldots, r\}$, then

$$
y = 0
$$
on $[\alpha, \beta]$.

3. **Proposition.** If $-\infty < \alpha < \beta < \infty$ and $y : [\alpha, \beta] \rightarrow \mathbb{R}$ is any continuously differentiable function that solves the typical one-dimensional problem of the calculus of variations to minimize the function

$$
F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) dx,
$$

where $f : [\alpha, \beta] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is $C^2$, then

$$
\frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0.
$$

4. **Definition.** Keeping the notation and the assumptions as in the previous proposition, given any $\phi \in C^2 ([\alpha, \beta], \mathbb{R})$ for which

$$
\phi^{(k)}(\alpha) = \phi^{(k)}(\beta) = 0
$$

for every $k \in \{0, 1, 2\}$, it is not difficult to see that the function

$$
I : \mathbb{R} \ni t \mapsto F(y + t\phi) \in \mathbb{R}
$$
is twice differentiable and attains its minimum at $t = 0$, so, by virtue of Proposition 2 of Section 17.1.3 on page 409 of [2], as in Section 10.7 on page 458 of [3] or in Section 1.3 on page 16 of [4], one obtains that

$$
\[ I''(t) = \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) \phi(x)^2 \right. \\
+ \left. \frac{\partial^2 f}{\partial y \partial y'} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) 2\phi(x)\phi'(x) \right. \\
+ \left. \frac{\partial^2 f}{\partial y^2} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) \phi'(x)^2 \right) dx. \]

Our first purpose in this article is to prove the following.

5. Proposition. Keeping the notation as in the previous definition, if \( f \) is \( C^3 \) and \( y \) is \( C^2 \), then

\[ I''(0) = \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y \partial y' \partial y''} 2\phi^3 \right) dx - \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} \phi \phi' \right) dx + \frac{\partial^3 f}{\partial y \partial y'^2} 2\phi^2 \phi' + \frac{\partial^2 f}{\partial y'^2} \phi'' + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \phi'' + \frac{\partial^3 f}{\partial y^2} \phi \phi'' \right) dx. \]

Proof. By virtue of Proposition 2 of Section 17.1.3 on page 409 of [2], one obtains that

\[ I''(0) = \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} \phi^2 + \frac{\partial^2 f}{\partial y \partial y'} 2\phi \phi' + \frac{\partial^2 f}{\partial y^2} \phi'' \right) dx \]

\[ = \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 + \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y^2} \phi'' \right) \phi' dx \]

\[ = \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 + \left[ \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y^2} \phi'' \right) \phi' \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y^2} \phi'' \right) \phi' dx \]

\[ = \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 - \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi + \left( \frac{\partial^2 f}{\partial y \partial y'} \cdot 0 + \frac{\partial^3 f}{\partial y \partial y'^2} \phi + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \right) \phi' \right) dx \]

\[ = \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 - \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi + \left( \frac{\partial^2 f}{\partial y \partial y'} \cdot 0 + \frac{\partial^3 f}{\partial y \partial y'^2} \phi + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \right) \phi' \right) dx \]

\[ = \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^2 f}{\partial y \partial y'} 2\phi' \right) dx - \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y \partial y'} 2\phi' \right. \\
+ \left. \frac{\partial^2 f}{\partial y \partial y'} \phi'' + \frac{\partial^2 f}{\partial y \partial y'} \phi' \phi'' + \frac{\partial^2 f}{\partial y \partial y'} \phi' \phi'' + \frac{\partial^2 f}{\partial y \partial y'} \phi \phi'' \right) dx. \]
Our second purpose in this article is to prove the following.

6. Proposition. Keeping the notation as in the previous proposition, for any such \( \phi \), we have that

\[
\int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y^2 \partial y} 2\phi^3 \right) dx \geq \int_{\alpha}^{\beta} \left( \frac{\partial^2 f}{\partial y^2 \partial y} 2\phi' \right. \\
+ \left. \frac{\partial^3 f}{\partial y^2 \partial y^2} 2\phi' \phi'' + \frac{\partial^3 f}{\partial y^2 \partial y} \phi' \phi^2 + \frac{\partial^3 f}{\partial y^2 \partial y^2} \phi'' \phi^2 \right) dx ,
\]

so either the above are variational inequalities of motion or the Lagrangian of motion is not \( C^3 \).

Proof. It is enough to notice that

\[ I''(0) \geq 0. \] (19)

7. Remark. Keeping the notation as in the previous proposition, one may take

\[ \phi(x) = \lambda \left( (x - \alpha)(x - \beta) \right)^n , \] (20)

where \( x \in [\alpha, \beta] \), while \( \lambda > 0 \) and \( n \in \mathbb{N} \setminus \{0, 1, 2\} \).

8. Example. If we consider the simple pendulum, where \( g \) is the acceleration of gravity and \( \ell \) is the length of the weightless thread to the one end of which is connected a particle of mass \( m \), while \( \theta \) is the angular displacement as a function of time \( t \), then the Lagrangian of the simple pendulum is

\[ L = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell \cos \theta \] (21)

and it is \( C^\infty \), so

\[ \frac{\partial L}{\partial t} = 0, \] (22)

\[ \frac{\partial L}{\partial \theta} = -mg\ell \sin \theta \] (23)

and

\[ \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta}, \] (24)
which imply that the Euler-Lagrange equation

\[
\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = 0 \quad (25)
\]

in Proposition 3 takes the form

\[
\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0. \quad (26)
\]

So, apart from solving equation (26), \( \theta \) must satisfy the conclusion of Proposition 6 and Remark 7. Since

\[
\frac{\partial^2 L}{\partial \theta^2} = -mg\ell \cos \theta, \quad (27)
\]

\[
\frac{\partial^2 L}{\partial \dot{\theta}^2} = 0 \quad (28)
\]

and

\[
\frac{\partial^2 L}{\partial \theta \partial \dot{\theta}} = m\ell^2, \quad (29)
\]

while

\[
\frac{\partial^3 L}{\partial \theta^2 \partial \dot{\theta}} = 0, \quad (30)
\]

\[
\frac{\partial^3 L}{\partial \theta \partial \dot{\theta} \partial \ddot{\theta}} = 0 \quad (31)
\]

and

\[
\frac{\partial^3 L}{\partial \dot{\theta}^3} = 0, \quad (32)
\]

it follows that if

\[
\phi(t) = \lambda ((t - \alpha)(t - \beta))^n, \quad (33)
\]

where \( t \in [\alpha, \beta] \), while \( \lambda > 0 \) and \( n \in \mathbb{N} \setminus \{0, 1, 2\} \), then

\[
\int_{\alpha}^{\beta} (-mg\ell \cos \theta \cdot \phi^2 - 0 \cdot 2\phi^3) \, dt \\
\geq \int_{\alpha}^{\beta} \left( 0 \cdot 2\phi \dot{\phi} + 0 \cdot 2\phi^2 \ddot{\phi} + m\ell^2 \cdot \phi \dddot{\phi} + 0 \cdot \dddot{\phi} = 0 \cdot \phi \dddot{\phi}^2 \right) dt \quad (34)
\]

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or equivalently
\[-mg\ell \int_\alpha^\beta \cos \theta(t)\phi(t)^2 dt \geq m\ell^2 \int_\alpha^\beta \phi(t)\ddot{\phi}(t) dt \] (35)
or equivalently
\[-g \int_\alpha^\beta \cos \theta(t)\phi(t)^2 dt \geq \ell \int_\alpha^\beta \phi(t)\ddot{\phi}(t) dt \] (36)
or equivalently
\[-g \int_\alpha^\beta \cos \theta(t)\phi(t)^2 dt \geq \ell \left( [\phi(t)\dot{\phi}(t)]_{t=\alpha}^{t=\beta} - \int_\alpha^\beta \dot{\phi}(t)\ddot{\phi}(t) dt \right) \] (37)
or equivalently
\[g \int_\alpha^\beta \cos \theta(t)\phi(t)^2 dt \leq \ell \int_\alpha^\beta \dot{\phi}(t)^2 dt. \] (38)

A formula for \( \theta \), via (26), can be derived from Section 2.1 on pages 69-80 of [5], so if
\[0 < \theta_0 < \pi \] (39)
and
\[\dot{\theta}_0 = 0, \] (40)
then
\[t = \sqrt{\frac{\ell}{g}} \ln \left( \frac{\tan \left( \frac{\pi}{4} - \frac{\theta_0}{4} \right)}{\tan \left( \frac{\pi}{4} - \frac{\theta_0}{4} \right)} \right) \] (41)
and consequently
\[\theta(t) = \pi - 4 \arctan \left( e^{-t\sqrt{\frac{\ell}{g}}} \tan \left( \frac{\pi}{4} - \frac{\theta_0}{4} \right) \right) \] (42)
must satisfy (38) for all \( \phi \) in question.

References

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