A VACUUM-ADAPTED APPROACH TO
QUANTUM FEYNMAN–KAC FORMULAE

ALEXANDER C. R. BELTON, J. MARTIN LINDSAY, AND ADAM G. SKALSKI

Abstract. The vacuum-adapted formulation of quantum stochastic calculus is employed to perturb expectation semigroups via a Feynman–Kac formula, giving an alternative perspective on some recent Feynman-Kac formulae developed by the authors. This generalises earlier work of Lindsay and Sinha, and its extension by Bahn and Park, which use classical stochastic calculus, and also existing quantum-stochastic results, which involve conjugation by unitary processes.

1. Introduction

Let \( \alpha = (\alpha_t)_{t \in \mathbb{R}} \) be an ultraweakly continuous group of normal \(*\)-automorphisms of a von Neumann algebra \( A \) acting faithfully on the Hilbert space \( \mathcal{H} \), and let \( \delta \) be its ultraweak generator. Gaussian subordination may be used to construct an ultraweakly continuous semigroup \( P_0 \) on \( A \) with ultraweak pre-generator \( \frac{1}{2} \delta^2 \) [15, Section 1] in the following manner. If \( B = (B_t)_{t \geq 0} \) is standard Brownian motion on Wiener’s probability space \( \mathbb{W} \), then, by Itô’s formula, the unital \(*\)-homomorphism \( j_t : A \to A \otimes L_\infty(\mathbb{W}; A) \) satisfies the stochastic differential equation

\[
\dot{j}_t(x) = x + \int_0^t j_s(\delta(x)) \, dB_s + \frac{1}{2} \int_0^t j_s(\delta^2(x)) \, ds \quad (x \in \text{Dom } \delta^2)
\]

in the strong sense on \( L_2(\mathbb{W}; \mathcal{H}) \). Thus

\[
P_0^t(a)u := \mathbb{E}_{\mathbb{W}}[j_{t}(a)u] \quad (a \in A, u \in \mathcal{H})
\]

defines an ultraweakly continuous semigroup \( (P_0^t)_{t \geq 0} \) of normal unital completely positive contractions on \( A \) whose ultraweak generator is as desired.

For the case where \( \alpha \) is unitarily implemented, Lindsay and Sinha obtained an ultraweakly continuous semigroup \( P^b \) with Feynman–Kac representation

\[
P_t^b(a)u = \mathbb{E}_{\mathbb{W}}[j_t(a)m^b_t u] \quad (t \geq 0, a \in A, u \in \mathcal{H})
\]

whose ultraweak generator extends \( \frac{1}{2} \delta^2 + \rho_b \delta \), where \( \rho_b : a \mapsto ab \) is the operator on \( A \) of right multiplication by \( b \) [15, Theorem 3.2]. Here \( m^b \) is the exponential

2000 Mathematics Subject Classification. Primary 47D08; Secondary 46L53, 47N50, 81S25.

Key words and phrases. Quantum stochastic cocycle, Markovian cocycle, quantum stochastic flow, Feynman–Kac perturbation, quantum stochastic analysis.
martingale such that
\[ m_t^b = I + \int_0^t j_s(b) m_s^b \, dB_s \quad (t \geq 0), \]
where \( b \in A \) is self adjoint. For the Laplacian on \( \mathbb{R}^{3d} \) and the commutative von Neumann algebra \( L^\infty(\mathbb{R}^{3d}) \), such vector-field perturbations have been studied from this viewpoint in [18]. Other works on quantum Feynman–Kac formulae include [1], [13], [2] and [5], all of which belong to the pre-quantum stochastic era.

The classical Feynman-Kac formula for Schrödinger operators, which is closely related to instances of the Trotter product formula, is well described in the books [19] and [20].

The results of Lindsay and Sinha have been fully generalised in [10]. In that paper a general perturbation theory for quantum stochastic flows is developed, yielding a much wider class of quantum Feynman–Kac formulae.

Here we take our inspiration from [6]. The semigroups defined in (1.2) will not, in general, be positive or even real (i.e., \(*\)-preserving). In this light Bahn and Park investigate a more symmetric form of Feynman–Kac perturbation, using instead an operator process \( n^b_t \) such that
\begin{align}
\frac{n^b_t f}{f} = f + \int_0^t j_s(b) E_W[n^b_s f|B_s] \, dB_s - \frac{1}{2} \int_0^t j_s(b^2) E_W[n^b_s f|B_s] \, ds \tag{1.3}
\end{align}
for all \( f \in L^2(W; \mathfrak{h}) \), where \( (B_t)_{t \geq 0} \) is the canonical filtration of the Brownian motion \( B \).

In this case, letting
\[ Q^b_t u := E_W[(n^b_t)^* j_t(a) n^b_t u] \quad (a \in A, u \in \mathfrak{h}) \]
gives an ultraweakly continuous completely positive semigroup \( (Q^b_t)_{t \geq 0} \) on \( A \), which is contractive if \( n^b \) is and whose generator extends the map
\[ \frac{1}{2} \delta^2 + \lambda b \delta + \rho b \lambda - \frac{1}{2} \lambda b^2 - \frac{1}{2} \rho b^2, \tag{1.4} \]
where \( \lambda_b \) denotes the operator on \( A \) given by left multiplication by \( b \).

In this work we are guided by the form of (1.3); the conditional expectations make it reminiscent of a stochastic differential equation used by Alicki and Fannes for dilating quantum dynamical semigroups [4, Equation (12)]. As observed in [7], this type of equation may be profitably interpreted in the vacuum-adapted form of quantum stochastic calculus. In contrast to [10], where the standard identity-adapted (Hudson–Parthasarathy) theory is used, here the analysis is slightly easier although the algebra becomes a bit more complicated.

The achievements of this paper are as follows; for simplicity, we restrict here to the case of one-dimensional noise, although the results below are obtained in full generality. The requirement that \( \alpha \) is unitarily implemented is removed; our primary object is a vacuum-adapted quantum stochastic flow. This is an ultraweakly continuous family \( j_t \) of normal \(*\)-homomorphisms which form a vacuum-adapted quantum stochastic cocycle on Boson Fock space over \( L^2(\mathbb{R}_+) \) and which are as unital as vacuum adaptedness permits. The flow \( j_t \) is assumed to satisfy the quantum stochastic differential equation
\begin{align}
dj_t(x) = j_t(\delta_0(x)) \, dA_t^1 + j_t(\pi_0(x)) \, dA_t + j_t(\delta^*_0(x)) \, dA_t + j_t(\tau_0(x)) \, dt \tag{1.5}
\end{align}
for all \( x \) in a subset \( A_0 \) of \( A \), where the structure maps 

\[
\tau_0, \; \delta_0, \; \delta_0^\dagger, \; \pi_0 : A_0 \rightarrow A
\]

must satisfy certain algebraic relations, thanks to the unital and \(*\)-homomorphic properties of \( j \). Equation (1.5) generalises (1.1), which corresponds to the case where \( A_0 = \text{Dom} \delta^2, \pi_0 \) is the inclusion map, 

\[
\delta_0 = \delta_0^\dagger = \delta|_{A_0} \quad \text{and} \quad \tau_0 = \frac{1}{2} \delta^2.
\]

The appearance of the non-zero gauge term \( \tau_0 \) is due to the fact that we are working in the vacuum-adapted set-up: cf. [9, Theorem 7.3]. It follows from (1.5) that the quantum stochastic flow satisfies the equation 

\[
\langle u\Omega, j_t(x)\eta \Omega \rangle = \langle u, v \rangle + \int_0^t \langle u\Omega, j_s(\tau_0(x))\eta \Omega \rangle \; ds \quad (u, v \in \mathfrak{h}, \; t \geq 0, \; x \in A_0),
\]

where \( \Omega \) denotes the Fock vacuum vector. The generator of the vacuum-expectation semigroup \( P^0 := (E \circ j_t)_{t \geq 0} \) therefore extends the map \( \tau_0 \). A natural assumption is that \( \tau_0 \) is a pre-generator of \( P^0 \), although this plays no rôle here.

Starting with Evans and Hudson [12], several authors have used conjugation with a unitary process to perturb quantum stochastic flows. These works focused on the case of bounded structure maps, so that the vacuum-expectation semigroup \( P^0 \) is norm continuous, and considered identity-adapted flows and processes. For \( h = h^* \in A \) and \( l \in A \) there exists a unitary process \( U \) such that 

\[
U_0 = 1, \quad \quad dU_t = j_t(l)U_tdA_t^I + j_t(-l^*)U_t dA_t + j_t(-ih - \frac{1}{2} l^* l)U_t dt,
\]

and the vacuum-expectation semigroup of the perturbed flow \( (a \mapsto U_t^* j_t(a) U_t)_{t \geq 0} \) has generator 

\[
\tau_0 + \lambda_t \delta_0 + \rho_t \delta_0^\dagger + \lambda_t \pi_0 + i[h, \cdot] - \frac{1}{2} \{l^* l, \cdot\},
\]

where \( [\cdot, \cdot] \) and \( \{\cdot, \cdot\} \) denote commutator and anticommutator. The main result obtained here includes this situation as a special case.

For any vacuum-adapted quantum stochastic flow \( j \) and any \( c = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \) in \( A \oplus A \), Theorem 5.3 below gives a process \( M^c \) such that \( M^c - I \) is vacuum adapted and the following quantum stochastic differential equation is satisfied:

\[
d(M^c - I)_t = j_t(c_0) M^c_t dt + j_t(c_1) M^c_t dA_t^I.
\]

Consequently, for any \( d = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \) in \( A \oplus A \), there is an ultraweakly continuous semigroup \( P^{c,d} \) on \( A \) with 

\[
\langle u, P^{c,d}_t(a)v \rangle = \langle u\Omega, (M^c_t)^* j_t(a) M^d_t v\Omega \rangle \quad (u, v \in \mathfrak{h}, \; t \geq 0, \; a \in A).
\]

When \( j \) satisfies (1.5), the ultraweak generator of \( P^{c,d} \) necessarily extends 

\[
\tau_0 + \lambda c_1 \delta_0 + \rho c_1 \delta_0^\dagger + \lambda c_1 \pi_0 + \lambda c_0^* + \rho d_0, \quad (1.6)
\]

This class of semigroups includes both the Lindsay–Sinha and the Bahn–Park examples, as well as those obtained by unitary conjugation; the generators of the latter correspond to the case 

\[
c = d = \begin{bmatrix} -ih - \frac{1}{2} l^* l \\ l \end{bmatrix}, \quad \text{where} \; h = h^*.
\]
1.1. Conventions. Hilbert spaces are complex with inner products linear in their second argument. The linear, Hilbert-space and ultraweak tensor products are denoted by ⊗, ⊗ and ⊗, respectively. For a Hilbert space $H$ we adopt the Dirac-inspired notation $|H|$ for $B(C; H)$ and $\{H\}$ for the topological dual $B(H; C)$, writing $|u\rangle$ for the operator $\lambda \mapsto \lambda u$ and $\langle u|$ for the functional $v \mapsto \langle u, v \rangle$, where $u \in H$. Recall the $E$ notation,

$$E_u := |u\rangle \otimes I \quad \text{and} \quad E_u^* := (E_u)^* = \langle u| \otimes I \quad (u \in H), \quad (1.7)$$

in which $I$ denotes the identity operator on a Hilbert space determined by context. The following commutator and anticommutator notation is also used for elements of an algebra:

$$[a, b] := ab - ba \quad \text{and} \quad \{a, b\} := ab + ba. \quad (1.8)$$

2. Multipliers for quantum stochastic flows

Fix now, and for the rest of the paper, Hilbert spaces $\mathfrak{h}$ and $\mathfrak{k}$, referred to as the initial space and multiplicity space, respectively. Fix also a von Neumann algebra $\mathcal{A}$ acting faithfully on $\mathfrak{h}$. Set $\hat{\mathfrak{k}} := \mathbb{C} \otimes \mathfrak{k}$,

$$\tilde{\epsilon} := \left(\begin{array}{c} 1 \\ \epsilon \end{array}\right) \in \hat{\mathfrak{k}} \quad (\epsilon \in \mathfrak{k}) \quad \text{and} \quad \omega := \hat{0} = \left(\begin{array}{c} 1 \\ 0 \end{array}\right). \quad (2.1)$$

For a subinterval $I$ of $\mathbb{R}_+$, let $\mathcal{F}_I$ denote the Boson Fock space over $L^2(I; \mathfrak{k})$ and let $\mathcal{N}_I := B(\mathcal{F}_I)$. For brevity, set $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+}$, $\mathcal{F}_{[0,t)} := \mathcal{F}_{[0,t]}$ and $\mathcal{F}_{t} := \mathcal{F}_{[t,\infty)}$, with corresponding abbreviations for the noise algebra $\mathcal{N} = B(\mathcal{F})$. The identifications

$$\mathcal{F} = \mathcal{F}_{s} \otimes \mathcal{F}_{[s,t]} = \mathcal{F}_{s} \otimes \mathcal{F}_{[s,t]} \otimes \mathcal{F}_{t} \quad (0 \leq s \leq t < \infty),$$

which arise from the exponential property of Fock space, entail the identifications

$$\mathcal{N} = \mathcal{N}_{s} \otimes \mathcal{N}_{[s,t]} = \mathcal{N}_{s} \otimes \mathcal{N}_{[s,t]} \otimes \mathcal{N}_{t} \quad (0 \leq s \leq t < \infty).$$

The notation $\Omega_{[s,t]}$, $I_{[s,t]}$ and $id_{[s,t]}$ for the vacuum vector in $\mathcal{F}_{[s,t]}$, the identity operator on $\mathcal{F}_{[s,t]}$ and the identity map on $\mathcal{N}_{[s,t]}$, respectively, is also useful, with corresponding abbreviations for other intervals as above.

Denote by $\Delta$ any of the following projections:

$$P_k \in B(\mathfrak{k}), \quad P_k \otimes 1_{\mathcal{A}} \in B(\mathfrak{k}) \otimes \mathcal{A} \quad \text{and} \quad P_k \otimes 1_{\mathcal{A}} \otimes I_{\mathcal{F}} \in B(\hat{\mathfrak{k}}) \otimes \mathcal{A} \otimes \mathcal{N}, \quad (2.2)$$

where $P_k = \left[\begin{array}{cc} 0 & 0 \\ 0 & f_k \end{array}\right] \in B(\mathfrak{k})$ is the orthogonal projection onto $k$.

The right shift

$$s_t : L^2(\mathbb{R}_+; \mathfrak{k}) \to L^2([t, \infty); \mathfrak{k}); \quad f \mapsto f(\cdot - t) \quad (t \geq 0)$$

has second quantisation

$$S_t : \mathcal{F} \to \mathcal{F}_{t}; \quad \varepsilon(f) \mapsto \varepsilon(s_t f),$$

where $\varepsilon(g)$ denotes the exponential vector corresponding to the vector $g$, and the map

$$\sigma_t : \mathcal{A} \otimes \mathcal{N} \to \mathcal{A} \otimes \mathcal{N}_t; \quad T \mapsto (I_{\mathfrak{h}} \otimes S_t)T(I_{\mathfrak{h}} \otimes S_t)^*$$

is a normal $*$-isomorphism for all $t \geq 0$. 

Definition 2.1. A vacuum-adapted quantum stochastic cocycle $k$ on $A$ is a family of normal completely bounded maps $(k_t : A \to A \otimes N)_{t \geq 0}$ such that, for all $a \in A$ and $s, t \geq 0$,

1. $(\Omega-C \ i)$ $k_0(a) = a \otimes |\Omega \rangle \langle \Omega|$
2. $(\Omega-C \ ii)$ $k_t(a) = k_{t+s}(a) \otimes |\Omega_t \rangle \langle \Omega_t|$, where $k_{t+s}(a) \in A \otimes N_t$,
3. $(C \ iii)$ $k_{s+t} = \hat{k}_s \circ \sigma_s \circ k_t$, where $\hat{k}_s := k_s \otimes \text{id}_s$ and
4. $(C \ iv)$ $r \mapsto k_r(a)$ is ultraweakly continuous.

Such a family is a flow on $A$ if it is also $^*$-homomorphic and each $k_t$ is unital. Following tradition we use the letter $j$ for quantum stochastic flows.

In the standard theory, $(\Omega-C \ i)$ and $(\Omega-C \ ii)$ are replaced by their identity-adapted counterparts,

1. $(I-C \ i)$ $k_0(a) = a \otimes I_F$
2. $(I-C \ ii)$ $k_t(a) = k_{t,j}(a) \otimes I_{[t]}$, where $k_{t,j}(a) \in A \otimes N_t$.

Remark 2.2. The prescription

$$k(t) = (k(t) \cdot \otimes |\Omega_t \rangle \langle \Omega_t|)_{t \geq 0} \mapsto k(I) = (k(t) \cdot \otimes I_{[t]})_{t \geq 0} \ (2.3)$$

gives a bijective correspondence between the class of vacuum-adapted quantum stochastic cocycles and the class of identity-adapted quantum stochastic cocycles. Note that

$$k_{t,j}(a) = E^{\Omega_t} k_t(a) E^{\Omega_t} \quad (t \geq 0, a \in A) \ (2.4)$$

in both cases.

In terms of the orthogonal projection

$$P_t := I_B \otimes I_j \otimes |\Omega_t \rangle \langle \Omega_t|, \quad (2.5)$$

condition $(\Omega-C \ ii)$ becomes

$$k_t(a) = P_t k_t(a) P_t,$$

whereas $(I-C \ ii)$ only implies the weaker commutation relation

$$k_t(a) P_t = P_t k_t(a).$$

Let

$$E := \text{id}_A \otimes \omega_\Omega : A \otimes N \to A$$

denote the vacuum expectation, where $\omega_\Omega$ is the state on $N$ corresponding to the vacuum vector $\Omega$.

Proposition 2.3. Let $k$ be a vacuum-adapted quantum stochastic cocycle on $A$. Then the ultraweakly continuous family of normal completely bounded maps $(E \circ k_t)_{t \geq 0}$ on $A$ forms a semigroup, called the vacuum-expectation semigroup of $k$.

Proof. For all $t \geq 0$, the conditional expectation

$$E_t : A \otimes N \to A \otimes N; \ x \mapsto (\text{id}_A \otimes \omega_{\Omega_t}) (x) \otimes I_{[t]} \ (2.6)$$

has the tower property $E \circ E_t = E$. The claim then follows from the fact that vacuum-adapted (as well as identity-adapted) quantum stochastic cocycles satisfy the identity

$$E_t \circ \hat{k}_t \circ \sigma_t = k_t \circ E \quad (t \geq 0). \ \square$$
Quantum stochastic differential equations of the following form are a basic source of quantum stochastic cocycles.

**Remark 2.4.** Under the correspondence (2.3), \( k(\Omega) \) satisfies a quantum stochastic differential equation of the form

\[
k_0(a) = a \otimes |\Omega\rangle\langle\Omega|, \quad dk_t = \tilde{k}_t(\psi(a)) \, d\Lambda_t
\]

on a subset \( A_0 \) of \( A \), where \( \tilde{k}_t := \text{id}_{B(\hat{k})} \otimes k_t \), if and only if \( k(I) \) satisfies a quantum stochastic differential equation of the form

\[
k_0(a) = a \otimes I_{F}, \quad dk_t = \tilde{k}_t(\phi(a)) \, d\Lambda_t \tag{2.8}
\]

on \( A_0 \), where the maps \( \psi, \phi : A_0 \to B(\hat{k}) \otimes A \) are related by the following identity:

\[
\psi(a) = \phi(a) + \Delta \otimes a \quad (a \in A_0).
\]

This is proved in [9, Theorem 7.3].

**Remark 2.5 ([16, Section 6]).** Let the map \( \phi : A \to B(\hat{k}) \otimes A \) have the block-matrix form

\[
\phi(a) = \begin{bmatrix}
[ih, a] - \frac{1}{2} \{r^*r, a\} + r^*\pi(a)r & ar^* - r^*\pi(a) \\
ra - \pi(a)r & \pi(a) - I_k \otimes a
\end{bmatrix} \quad (a \in A), \tag{2.9}
\]

where \( h \in A \) is self adjoint, \( r \in |k\rangle \otimes A \) and \( \pi : A \to B(k) \otimes A \) is a normal unital \(*\)-homomorphism. Then the quantum stochastic differential equation (2.8) has a unique solution and this is an identity-adapted quantum stochastic flow. Conversely, if an identity-adapted quantum stochastic flow satisfies (2.8) for some normal bounded map \( \phi : A \to B(\hat{k}) \otimes A \) then \( \phi \) has the form (2.9).

**Definition 2.6.** Let \( j \) be a vacuum-adapted quantum stochastic flow on \( A \). A family of operators \( M = (M_t)_{t \geq 0} \) in \( A \otimes N \) is a multiplier for \( j \) if, for all \( s, t \geq 0 \),

\begin{align*}
(M \text{ i}) & \quad M_0 = I_{A \otimes F}, \\
(M \text{ ii}) & \quad M_t P_s = P_s M_t, \\
(M \text{ iii}) & \quad M_{s+t} = J_s(M_t)M_s, \quad \text{where } J_s := \hat{j}_s \circ \sigma_s \\
(M \text{ iv}) & \quad r \mapsto M_t \text{ is strongly continuous}.
\end{align*}

The Banach–Steinhaus Theorem and condition (M iv) imply that \( M \) is locally bounded.

**Theorem 2.7 (Cf. [6, Theorem 2.1]).** Let \( M \) and \( N \) be multipliers for the vacuum-adapted quantum stochastic flow \( j \). The ultraweakly continuous normal completely bounded family

\[
\mathcal{P} := (a \mapsto E[M^*_t j_t(a)N_t])_{t \geq 0}
\]

forms a semigroup, which is completely contractive if \( M \) and \( N \) are contractive and is completely positive if \( M = N \).
To prove the semigroup property, let $a \in A$ and $s, t \geq 0$. By the tower property for the conditional expectation $E_s$ defined in (2.6), it follows that

$$
P_{s+t}(a) = \mathbb{E}[E_s[M^*_s J_s(M^*_t J_s(j_t(a))J_t(N_t)N_s)]
$$

by (C iii) and (M iii)

$$
= \mathbb{E}[M^*_s E_s[J_s(M^*_t j_t(a)N_t)]N_s]
$$

by (M ii)

$$
= \mathbb{E}[M^*_s j_s(E[M^*_t j_t(a)N_t])N_s]
$$

by (M ii) and the tower property

$$
= P_s (P_t(a)).
$$

For the equality (2.10), note that if $a \in A$ and $x \in \mathbb{N}$ then

$$
\mathbb{E}_s[J_s(a \otimes x)] = \langle \Omega, x\Omega \rangle j_s(a) = j_s(\mathbb{E}[a \otimes x]);
$$

thus $\mathbb{E}_s \circ J_s = j_s \circ \mathbb{E}$, by linearity and ultraweak continuity. \hfill $\square$

**Remark 2.8.** Many of the ideas in this section go back to work of Accardi [1, Sections 2 and 4]; see also [3, Section 2.3].

### 3. A vacuum-adapted quantum stochastic differential equation

Let $(u_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group in $A$, let $B = (B_t)_{t \geq 0}$ be the canonical Brownian motion on Wiener’s probability space $\mathbb{W}$ and identify $L^2(\mathbb{W})$ with $\mathcal{F}$, where $k = C$, via the Wiener–Itô–Segal isomorphism. If the unitary operator $U_t \in A \otimes \mathbb{N}$ is such that

$$
U_t \xi : \omega \mapsto u_{B_t(\omega)} \xi(\omega) = u_{\omega(t)} \xi(\omega) \quad (\xi \in L^2(\mathbb{W}; h))
$$

then the family of maps $(j_t^B : a \mapsto U_t(a \otimes I_{\mathcal{F}})U_t^*)_{t \geq 0}$ is an identity-adapted quantum stochastic flow on $A$ (cf. [14, Section 5]).

Bahn and Park considered the operator stochastic differential equation

$$
M^a_t = I_{h \otimes \mathcal{F}}, \quad dM^a_t = j_t^B(a)P_tM^a_t dB_t - \frac{1}{2}j_t^B(a^2)P_tM^a_t dt,
$$

where $a \in A$, and obtained a solution pointwise in $L^2(\mathbb{W}; h)$ [6, Proposition 3.2]. They showed that the collection of operators $(M^a_t)_{t \geq 0}$ forms a multiplier for the quantum stochastic flow $j^B$ [6, Proposition 3.3].

Fix $a \in A$ and set $N_t := M^a_t - I_{h \otimes \mathcal{F}}$ for all $t \geq 0$, so that

$$
N_t \xi = \int_0^t Q_s \xi dB_s - \frac{1}{2} \int_0^t R_s \xi ds + \int_0^t Q_s N_s \xi dB_s - \frac{1}{2} \int_0^t R_s N_s \xi ds
$$

for all $\xi \in L^2(\mathbb{W}; h)$, where

$$
Q_t := j_t^B(a)P_t \quad \text{and} \quad R_t := j_t^B(a^2)P_t.
$$

As $(j_t^B(b))_{t \geq 0}$ is identity adapted for all $b \in A$, the processes $Q$ and $R$ are vacuum adapted. Hence the quantum stochastic differential equation

$$
N_0 = 0, \quad dN_t = Q_t dA^I_t - \frac{1}{2} R_t dt + Q_t N_t dA^I_t - \frac{1}{2} R_t N_t dt
$$

has a unique vacuum-adapted solution.
To see that (3.2) is the correct quantum stochastic generalisation of (3.1), for simplicity take \( \mathfrak{h} = \mathbb{C} \) and let \( \hat{z}(f) \) denote the Brownian exponential corresponding to \( f \in L^2(\mathbb{R}_+) \), i.e., the unique element of \( L^2(\mathbb{W}) \) such that

\[
\hat{z}(f)_t := \mathbb{E}_W[\hat{z}(f)|\mathcal{B}_t] = 1 + \int_0^t f(s)\mathbb{E}_W[\hat{z}(f)|\mathcal{B}_s] \, dB_s \quad (t \geq 0),
\]

where \( (\mathcal{B}_t)_{t \geq 0} \) is the canonical filtration generated by the Brownian motion \( B \). (Recall that \( \hat{z}(f) \) corresponds to \( \varepsilon(f) \) and \( \mathbb{E}_W[\cdot|\mathcal{B}_t] \) to \( P_t \).) If \( (X_t)_{t \geq 0} \) is a process of bounded operators on \( \mathcal{F} \) with locally bounded norm and such that \( X_t P_t = P_t X_t \) for all \( t \geq 0 \) then, by the Itô product formula,

\[
\mathbb{E}_W[\hat{z}(f) \int_0^t X_s P_s \hat{z}(g) \, dB_s] = \mathbb{E}_W\left[ \int_0^t \hat{z}(f) X_s \hat{z}(g) \, ds \right] 
= \left\langle \varepsilon(f), \int_0^t X_s P_s \, dA_s \varepsilon(g) \right\rangle \quad (f, g \in L^2(\mathbb{R}_+)).
\]

**Definition 3.1.** For a Hilbert space \( \mathcal{H} \), a bounded process in \( B(\mathcal{H}) \otimes \mathcal{A} \) is a family of operators \( Z = (Z_t)_{t \geq 0} \) in \( B(\mathcal{H}) \otimes \mathcal{A} \otimes \mathcal{N} \) such that

\[
t \mapsto \langle \zeta', Z_t \zeta \rangle
\]

is measurable \((\zeta, \zeta' \in \mathcal{H} \otimes \mathfrak{h} \otimes \mathcal{F})\); such a process is **vacuum adapted** if

\[
Z_t = (I_{\mathcal{H}} \otimes P_t)Z_t(I_{\mathcal{H}} \otimes P_t) \quad (t \geq 0)
\]
or, equivalently,

\[
Z_t = Z_{t|} \otimes |\Omega_t\rangle\langle \Omega_t|, \quad Z_{t|} \in B(\mathcal{H}) \otimes \mathcal{A} \otimes \mathcal{N}|_{0, t} \quad (t \geq 0).
\]

A vacuum-adapted bounded process \( G \) in \( B(\mathfrak{k}) \otimes \mathcal{A} \) is an **integrand** process if its block-matrix form \( \begin{bmatrix} k & m \\ l & n \end{bmatrix} \) is such that

\[
\| G \|_t := \| k \|_{1, t} + \| l \|_{2, t} + \| m \|_{2, t} + \| n \|_{\infty, t} < \infty \quad (t \geq 0),
\]

where \( \| f \|_{p, t} \) denotes the \( L^p \) norm of the function \( 1_{[0, t]} f \).

The following result is the coordinate-independent version of [8, Proposition 37], with non-trivial initial space. Recall the \( E \) notations (1.7).

**Proposition 3.2.** Let \( G \) be an integrand process. There is a unique bounded vacuum-adapted process \( \int G \, d\Lambda = \left( \int_0^t G_s \, d\Lambda_s \right)_{t \geq 0} \) in \( \mathcal{A} \) such that

\[
\langle u \varepsilon(f), \int_0^t G_s \, d\Lambda_s | v \varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), E^{\varepsilon(s)} G_s E^{-\varepsilon(s)} v \varepsilon(g) \rangle \, ds \quad (t \geq 0)
\]

for all \( u, v \in \mathfrak{h} \) and \( f, g \in L^2(\mathbb{R}_+; \mathfrak{k}) \). Moreover, the following inequality holds:

\[
\| \int_0^t G_s \, d\Lambda_s \| \leq \| G \|_t \quad (t \geq 0).
\]

We shall need to pass suitably adapted operators inside quantum stochastic integrals. The following result takes care of this.
**Lemma 3.3.** Let $G$ be an integrand process such that $G\Delta \equiv 0$ and let $X$ be a bounded vacuum-adapted process in $\mathfrak{A}$. Then

$$\int_s^t G_r \, d\Lambda_r X_s = \int_s^t G_r (I_k \otimes X_s) \, d\Lambda_r \quad (0 \leq s \leq t). \quad (3.3)$$

**Proof.** Let $u, v \in \mathfrak{h}$ and $f, g \in L^2(\mathbb{R}_+; k)$; note that

$$\langle u\varepsilon(f), \int_s^t G_r \, d\Lambda_r \, v\varepsilon(g) \rangle = \int_0^t \langle u\varepsilon(f), E_{(r)} G_r \, v\varepsilon(g) \rangle \, dr,$$

since $\Delta \hat{E}_\xi = E_\xi$ for all $\xi \in k$. If $A \in B(\mathfrak{h} \otimes \mathcal{F}_s)$ and $\xi \in \mathfrak{h} \otimes \mathcal{F}$ then, setting $P_s := |\Omega_s\rangle\langle\Omega_s|$ for brevity, it follows that

$$\langle u\varepsilon(f), \int_s^t G_r \, d\Lambda_r (A \otimes P_s)\xi \rangle = \int_s^t \langle u\varepsilon(f), E_{(r)} G_r (A \otimes P_s)\xi \rangle \, dr$$

$$\quad = \int_s^t \langle u\varepsilon(f), E_{(r)} G_r (I_k \otimes A \otimes P_s)E_\xi \rangle \, dr$$

$$\quad = \langle u\varepsilon(f), \int_s^t G_s (I_k \otimes A \otimes P_s) \, d\Lambda_r \xi \rangle. \quad \square$$

The following existence and uniqueness theorem is sufficiently general for present purposes.

**Theorem 3.4.** Let $G$ and $X$ be as in Lemma 3.3, with $X$ locally bounded in norm. Then there is a unique vacuum-adapted process $Z$ in $\mathfrak{A}$ such that

$$Z_t = X_t + \int_0^t G_s (I_k \otimes Z_s) \, d\Lambda_s \quad (t \geq 0). \quad (3.4)$$

Furthermore,

$$\|Z\|_{\infty,t} \leq \sqrt{2} \|X\|_{\infty,t} \exp(2\|t\|_{2,t}^2 + 2\|k\|_{1,t}^2) \quad (t \geq 0),$$

where $\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$ is the block-matrix form of $G$, and $Z$ is norm continuous if and only if $X$ is.

**Proof.** Define a sequence of processes $(X^{(n)})_{n \geq 0}$ inductively by letting $X^{(0)} := X$ and

$$X^{(n+1)}_t := \int_0^t G_s (I_k \otimes X^{(n)}_s) \, d\Lambda_s \quad (t \geq 0).$$

This process is well defined and such that

$$\|X^{(n+1)}_t\|_{\infty,t} \leq \|k\|_{1,t} \|X^{(n)}_t\|_{1,t} + \|t\|_{2,t} \|X^{(n)}_t\|_{2,t} \quad (t \geq 0),$$

so

$$\|X^{(n+1)}_t\|_{\infty,t}^2 \leq 2\|k\|_{1,t}^2 \|X^{(n)}_t\|_{1,t}^2 + 2\|t\|_{2,t}^2 \|X^{(n)}_t\|_{2,t}^2 \leq \int_0^t c(s) \|X^{(n)}_t\|_{\infty,s}^2 \, ds,$$

where

$$c(s) := 4\|k_s\| \int_0^s \|k_r\| \, dr + 2\|t_s\|^2.$$
It follows that
\[
\|X^{(n+1)}\|_{\infty,t}^2 \leq \frac{1}{n!} \left( \int_0^t c(s) \, ds \right)^n \|X\|_{\infty,t}^2 \quad (n \geq 0, \ t \geq 0),
\]
so \( Z_t := \sum_{n=0}^{\infty} X_t^{(n)} \) exists for all \( t \geq 0 \), the series being convergent in norm. A dominated-convergence argument shows that \( Z \) satisfies (3.4) and, since
\[
\|Z_t\|^2 \leq 2\|X_t\|^2 + 2 \int_0^t c(s)\|Z_s\|^2 \, ds \quad (t \geq 0),
\]
the inequality and so uniqueness follow from Gronwall’s lemma. The final claim is immediate. \( \square \)

4. Multipliers via quantum stochastic differential equations

Fix a vacuum-adapted quantum stochastic flow \( j \) on \( A \) and let
\[
J_t := \hat{j}_t \circ \sigma_t : A \otimes N \to A \otimes N_{t} \otimes N|t = A \otimes N
\]
and \( \hat{J}_t := \text{id}_{\hat{B} \otimes} \otimes J_t \), for all \( t \geq 0 \). The ultraweakly continuous family of normal unital \(*\)-homomorphisms \((J_t)_{t \geq 0}\) form a semigroup (cf. [17, Proposition 4.3]).

The following result is a vacuum-adapted version of [10, Lemma 5.1] which suffices for the present paper.

**Lemma 4.1.** If the integrand process \( G \) is norm continuous then the family of operators \( \{1_{[s,\infty]}(r)J_s(G_{r-s})\}_{r \geq 0} \) defines an integrand process such that
\[
J_s \left( \int_0^t G_r \, d\Lambda_r \right) = \int_s^{s+t} \hat{J}_s(G_{r-s}) \, d\Lambda_r \quad (t \geq 0).
\]

**Sketch proof.** Apply the ampliation of the vector functional \( A \mapsto \langle \epsilon(f), A\epsilon(g) \rangle \) to the left-hand side, then consider suitable Riemann sums. \( \square \)

With this technical lemma we can construct multipliers of \( j \) by solving quantum stochastic differential equations with coefficients driven by \( j \).

**Lemma 4.2.** For all \( c \in [\hat{k}] \otimes A \) there is a unique process \( M^c = (M^c_t)_{t \geq 0} \) in \( A \) such that \( M^c - I = (M^c_t - I_{0 \otimes x})_{t \geq 0} \) is vacuum adapted and
\[
M^c_t = I_{0 \otimes x} + \int_0^t \hat{j}_s(cE^\omega)(I_{\hat{k}} \otimes M^c_s) \, d\Lambda_s,
\]
where \( \hat{j}_s := \text{id}_{\hat{B} \otimes} \otimes j_s \) and \( \omega := \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \in \hat{k} \), i.e.,
\[
\langle u\epsilon(f), (M^c - I)u\epsilon(g) \rangle = \int_0^t \langle u\epsilon(f), j_s(E^\omega(s)c)M^c_s \epsilon(g) \rangle \, ds \quad (t \geq 0)
\]
for all \( u, v \in \mathfrak{h} \) and \( f, g \in L^2(\mathbb{R}_+; \mathfrak{k}) \). The process \( M^c \) is norm continuous.

**Proof.** Define an integrand process \( G \) by setting \( G_t := \hat{j}_t(cE^\omega) \) for all \( t \geq 0 \). In view of the identity \( \hat{j}(\cdot)\Delta = \hat{j}(\cdot, \Delta) \), which exploits the abuse of notation (2.2),
and the fact that $E^\omega \Delta = 0$, Theorem 3.4 gives a vacuum-adapted process $N$ in $A$ which is norm continuous and such that

$$N_t = \int_0^t G_s \, d\Lambda_s + \int_0^t G_s (I_k \otimes N_s) \, d\Lambda_s \quad (t \geq 0).$$

Hence $M^c_t := I_h \otimes F + N_t$ is a norm-continuous process as required; uniqueness holds because the solution of (4.1) is unique. □

**Theorem 4.3.** For all $c \in [\hat{k} \otimes A$ the process $M^c$ given by Lemma 4.2 is a multiplier for $j$.

**Proof.** It suffices to verify that condition (M iii) of Definition 2.6 holds. Fix $s \geq 0$ and let

$$M_t := \begin{cases} M^c_t & \text{if } t \in [0, s), \\ J_s(M^c_{t-s})M^c_s & \text{if } t \in [s, \infty). \end{cases}$$

Now $\tilde{J}_r \circ \tilde{J}_{r-s} = \tilde{J}_r$ for all $r \geq s$, by (C iii) of Definition 2.1, so Lemma 3.3 and Proposition 4.1 imply that

$$M_{t+s} = J_s(I_h \otimes F + \int_0^s \tilde{J}_r (cE^{\omega})(I_k \otimes M^c_r) \, d\Lambda_r)M^c_s + \int_0^{t+s} \tilde{J}_r (cE^{\omega})(I_k \otimes M^c_r) \, d\Lambda_r.$$

By Lemma 4.2, $M \equiv M^c$ and $M^c_{t+s} = M_{t+s} = J_s(M^c_t)M^c_s$, as required. □

5. Semigroup perturbation

For vacuum-adapted integrands the quantum Itô product formula takes the following form [8, Section 5.4].

**Lemma 5.1.** Let $Z := \int G \, d\Lambda$ and $Z' := \int G' \, d\Lambda$ for integrand processes $G$ and $G'$. Then

$$H := (I_k \otimes Z)\Delta^\perp G' + G\Delta^\perp (I_k \otimes Z') + G\Delta G'$$

defines an integrand process such that $ZZ' = \int H \, d\Lambda$.

The product of three integrals gives the following.
Corollary 5.2. Let $G$, $G'$ and $G''$ be integrand processes and let $Z := \int G \, d\Lambda$, $Z' := \int G' \, d\Lambda$ and $Z'' := \int G'' \, d\Lambda$. Then

$$H := (I_k \otimes ZZ')\Delta^\perp G'' + (I_k \otimes Z)\Delta^\perp G'\Delta^\perp (I_k \otimes Z') + G\Delta^\perp (I_k \otimes Z' Z'')$$

is an integrand process such that $ZZ'Z'' = \int H \, d\Lambda$.

Theorem 5.3. Let $\psi : A_0 \to A \overline{\otimes} B(\hat{k})$, where $A_0 \subseteq A$, and suppose $j$ satisfies the vacuum-adapted quantum stochastic differential equation

$$j_0(x) = x \otimes |\Omega\rangle \langle \Omega|, \quad dj_t(x) = \tilde{j}_t(\psi(x)) \, d\Lambda_t \quad (x \in A_0).$$

Let $\tau$ be the ultraweak generator of the semigroup $P := \{\mathbb{E}(M^\tau_i) \, j_t(\cdot) \, M^\tau_i) \}_{t \geq 0}$, where $c, d \in [\hat{k}] \overline{\otimes} A$. Then $\text{Dom} \, \tau \supseteq A_0$ and, for all $x \in A_0$,

$$\tau(x) = E^{\psi}(x)E_\omega + c^* \Delta \psi(x)E_\omega + E^{\psi}(x)\Delta d + c^* \Delta \psi(x)\Delta d + c^* E_\omega x + x E^\omega d. \quad (5.1)$$

Proof. Let $x \in A_0$ and $t \geq 0$; note that $(M^\tau_i \, j_t(x)M^\tau_i - j_t(x))$ equals

$$(M^\tau_i - I)^*_t (j_t - j_0)(x)(M^\tau_i - I)_t + (M^\tau_i - I)^*_t(j_t - j_0)(x)$$

$$+ (j_t - j_0)(x)(M^\tau_i - I)_t + (M^\tau_i - I)^*_t j_0(x)(M^\tau_i - I)_t$$

$$+ (M^\tau_i - I)^*_t j_0(x) + j_0(x)(M^\tau_i - I)_t. \quad \quad \quad (5.2)$$

If $u, v \in \mathfrak{h}$ and $f, g \in L^2(\mathbb{R}_+; k)$ then, writing $P_\Omega$ for $|\Omega\rangle \langle \Omega| \in \mathbb{N},$

$$\langle u \varepsilon(f), j_0(x)(M^\tau_i - I)_t v \varepsilon(g) \rangle = \langle (x^* u) \Omega, \int_0^t \tilde{j}_s(dE^\omega)(I_k \otimes M^\tau_i) \, d\Lambda_s \, v \varepsilon(g) \rangle$$

$$= \int_0^t \langle (x^* u) \Omega, E^{\varepsilon}_s \tilde{j}_s(dE^\omega)(I_k \otimes M^\tau_i) E^{-\varepsilon}_s(v \varepsilon(g)) \rangle \, ds$$

$$= \int_0^t \langle u \varepsilon(f), (x \otimes P_\Omega) j_s(E^\omega d) M^\tau_i v \varepsilon(g) \rangle \, ds,$$

therefore

$$j_0(x)(M^\tau_i - I)_t = \int_0^t (x \otimes P_\Omega) j_s(E^\omega d) M^\tau_i ds,$$

$$(M^\tau_i - I)^*_t j_0(x) = \int_0^t (M^\tau_i)^* j_s(c^* E_\omega)(x \otimes P_\Omega) ds$$

and

$$(M^\tau_i - I)^*_t j_0(x)(M^\tau_i - I)_t = \int_0^t (M^\tau_i - I)^*_t(x \otimes P_\Omega) j_s(E^\omega d) M^\tau_i ds$$

$$+ \int_0^t (M^\tau_i)^* j_s(c^* E_\omega)(x \otimes P_\Omega)(M^\tau_i - I)_s ds.$$
This implies that the sum of the last three terms in (5.2) equals
\[
\int_0^t (M_s^e)^*((x \otimes P_\Omega)j_s(E^\omega d) + j_s(c^*E_\omega)(x \otimes P_\Omega))M_s^d \, ds
\]
\[
= \int_0^t (\tilde{M}_s^e)^*((I_k \otimes x \otimes P_\Omega)\tilde{j}_s(\Delta^\omega dE^\omega) + \tilde{j}_s(E_\omega c^*\Delta^\omega)(I_k \otimes x \otimes P_\Omega))\tilde{M}_s^d \, d\Lambda_s,
\]
where $\tilde{M}_s^e := I_k \otimes M_s^e$ for $e = c, d$.

After some working, with the aid of Lemma 5.1 and Corollary 5.2, it follows that $(M_t^c)^*j_t(x)M_t^d - j_0(x)$ equals
\[
\int_0^t (\tilde{j}_s(A_1) + \tilde{j}_s(A_2)\tilde{M}_s^d + (\tilde{M}_s^e)^*\tilde{j}_s(A_3) + (\tilde{M}_s^e)^*\tilde{j}_s(A_4)\tilde{M}_s^d) \, d\Lambda_s,
\]
where
\[
A_1 := \Delta \psi(x)\Delta,
\]
\[
A_2 := \Delta \psi(x)\Delta^\omega + \Delta \psi(x)\Delta dE^\omega,
\]
\[
A_3 := \Delta^\omega \psi(x)\Delta + E_\omega c^*\Delta \psi(x)\Delta
\]
and
\[
A_4 := \Delta^\omega \psi(x)\Delta^\omega + E_\omega c^*\Delta \psi(x)\Delta^\omega + \Delta^\omega \psi(x)\Delta dE^\omega + E_\omega c^*\Delta^\omega \Delta dE^\omega + E_\omega c^*\Delta^\omega (I_k \otimes x) + (I_k \otimes x)\Delta^\omega dE^\omega.
\]
Hence
\[
\langle u, (\mathcal{P}_t(x) - x)v \rangle = \langle u, (M_t^c)^*j_t(x)M_t^d - j_0(x)\rangle v\Omega
\]
\[
= \int_0^t \langle u, \left( j_s(E^\omega A_1 E_\omega) + j_s(E^\omega A_2 E_\omega)M_s^d
\right.
\]
\[
+ (M_s^e)^*j_s(E^\omega A_3 E_\omega) + (M_s^e)^*j_s(E^\omega A_4 E_\omega)M_s^d) \rangle v\Omega \rangle \, ds
\]
\[
= \int_0^t \langle u, \left( M_t^c)^*j_t(E^\omega A_4 E_\omega)M_t^d \rangle v\Omega \rangle \, ds
\]
\[
= \int_0^t \langle u, \mathcal{P}_s(y)v \rangle \, ds,
\]
where
\[
y = E^\omega \psi(x)E_\omega + c^*\Delta \psi(x)E_\omega + E^\omega \psi(x)\Delta d + c^*\Delta \psi(x)\Delta d + c^*E_\omega x + xE^\omega d,
\]
as required. \qed

Remark 5.4. In terms of the direct-sum decomposition $\hat{k} = C \oplus k$, if
\[
\psi = \begin{bmatrix} \tau_0 & \delta_0^T \\ \delta_0 & \pi_0 \end{bmatrix}, \quad c = \begin{bmatrix} k_1 \\ l_1 \end{bmatrix} \text{ and } d = \begin{bmatrix} k_2 \\ l_2 \end{bmatrix}
\]
then (5.1) becomes
\[
\tau(x) = \tau_0(x) + l_1^T \delta_0(x) + \delta_0^T(x)l_2 + l_1^T \pi_0(x)l_2 + k_1^T x + x k_2 \quad (x \in A_0).
\]
When \( \psi \) is bounded and everywhere defined, \( \delta_0 \) is a bounded \( \pi_0 \)-derivation. Since \( \delta_0(A_0) \subseteq A \otimes |k\rangle \), it follows that \( \delta_0 \) is implemented ([11], see [14, Chapter 6]) and so

\[
\tau(x) = i[h, x] - \frac{1}{2} \{r^*r, x\} + r^*\pi_0(x)r \\
+ (xr^* - r\pi_0(x))l_2 + l_1^*(rx - \pi_0(x)r) + l_1^*\pi_0(x)l_2 + k_1^*x + xk_2
\]

for some \( h = h^* \in A \) and \( r \in |k\rangle \otimes A \). Equivalently,

\[
\tau(x) = d_1^*\pi_0(x)d_2 + e_1^*x + xe_2,
\]

where \( d_i = l_i - r \) and \( e_i = k_i + r^*l_i - \frac{1}{2}r^*r - ih \) for \( i = 1, 2 \).

Acknowledgments. We thank Kalyan Sinha for constructive remarks at an early stage of this work. Support from the UK-India Education and Research Initiative grant Quantum Probability, Noncommutative Geometry and Quantum Information is gratefully acknowledged.

References

1. Accardi, L.: On the quantum Feynman–Kac formula, Rend. Sem. Mat. Fis. Milano 48 (1978) 135–180.
2. Accardi, L.; Frigerio, A.: Markovian cocycles, Proc. Roy. Irish Acad. Sect. A 83 (1983) no. 2, 251–263.
3. Accardi, L.; Frigerio, A.; Lewis, J. T.: Quantum stochastic processes, Publ. Res. Inst. Math. Sci. 18 (1982) no. 1, 97–133.
4. Allicki, R.; Fannes, M.: Dilations of quantum dynamical semigroups with classical Brownian motion, Comm. Math. Phys. 108 (1987) no. 3, 353–361.
5. Arveson, W.: Ten lectures on operator algebras, CBMS Regional Conference Series in Mathematics 55, American Mathematical Society, Providence, 1984.
6. Bahn, C.; Park, Y. M.: Feynman–Kac representation and Markov property of semigroups generated by noncommutative elliptic operators, Infinite Dim. Anal. Quantum Probab. 6 (2003) no. 1, 103–121.
7. Belton, A. C. R.: Quantum \( \Omega \)-semimartingales and stochastic evolutions, J. Funct. Anal. 187 (2001) no. 1, 94–109.
8. Belton, A. C. R.: An isomorphism of quantum semimartingale algebras, Q. J. Math. 55 (2004) no. 2, 135–165.
9. Belton, A. C. R.: Random-walk approximation to vacuum cocycles, J. London Math. Soc. (2) 81 (2010) no. 2, 412–434.
10. Belton, A. C. R.; Lindsay, J. M.; Skalski, A. G.: Quantum Feynman–Kac perturbations, preprint, 2011.
11. Christensen, E.; Evans, D. E.: Cohomology of operator algebras and quantum dynamical semigroups, J. London Math. Soc. (2) 20 (1979) no. 2, 358–368.
12. Evans, M. P.; Hudson, R. L.: Perturbations of quantum diffusions, J. London Math. Soc. (2) 41 (1990) no. 2, 373–384.
13. Hudson, R. L.; Ion, P. D. F.; Parthasarathy, K. R.: Time-orthogonal unitary dilations and noncommutative Feynman–Kac formulae, Comm. Math. Phys. 83 (1982) no. 2, 261–280; Time-orthogonal unitary dilations and noncommutative Feynman–Kac formulae II, Publ. Res. Inst. Math. Sci. 20 (1984) no. 3, 607–633.
14. Lindsay, J. M.: Quantum stochastic analysis — an introduction, in: Quantum Independent Increment Processes I (2005) 181–271, Lecture Notes in Mathematics 1865, Springer, Berlin.
15. Lindsay, J. M.; Sinha, K. B.: Feynman–Kac representation of some noncommutative elliptic operators, J. Funct. Anal. 147 (1997) no. 2, 400–419.
16. Lindsay, J. M.; Wills, S. J.: Existence, positivity and contractivity for quantum stochastic flows with infinite dimensional noise, Probab. Theory Related Fields 116 (2000), 505–543.
17. Lindsay, J. M.; Wills, S. J.: Markovian cocycles on operator algebras adapted to a Fock filtration, J. Funct. Anal. 178 (2000) no. 2, 269–305.
18. Parthasarathy, K. R.; Sinha, K. B.: A stochastic Dyson series expansion, in: Theory and Application of Random Fields (Bangalore, 1982) (1983) 227–232, Lecture Notes in Control and Information Science 49, Springer, Berlin.
19. Reed, M; Simon, B.: Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.
20. Simon, B.: Functional Integration and Quantum Physics, Academic Press, New York, 1979.

ACRB: DEPARTMENT OF MATHEMATICS AND STATISTICS, LANCASTER UNIVERSITY, LANCASTER LA1 4YF, UNITED KINGDOM
E-mail address: a.belton@lancaster.ac.uk

JML: DEPARTMENT OF MATHEMATICS AND STATISTICS, LANCASTER UNIVERSITY, LANCASTER LA1 4YF, UNITED KINGDOM
E-mail address: j.m.lindsay@lancaster.ac.uk

AGS: MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8,
P.O. BOX 21, 00-956 WARSAWA, POLAND
E-mail address: a.skalski@impan.pl