Killing Vector Fields, Maxwell Equations and Lorentzian Spacetimes

Waldyr A. Rodrigues Jr.

Abstract. In this paper we first analyze the structure of Maxwell equations in a Lorentzian spacetime when the potential is \( A = e^K \), i.e., proportional to a 1-form \( K \) physically equivalent Killing vector field. We show that \( A \) necessarily obeys the Lorenz gauge \( \delta A = 0 \). Moreover we determine the form of the current associated with this potential showing that it is of a superconducting type, i.e., proportional to the potential and given by \( 2 R^\beta \alpha A^\beta \), where the \( R^\beta \alpha \) are the Ricci 1-form fields. Finally we study the structure of the spacetime generated by the coupled system consisting of an electromagnetic field \( F = dA \) (with \( A = e^K \)), an ideal charged fluid with dynamics described by an action function \( S \) and the gravitational field. We show that Einstein equations is then equivalent to Maxwell equations with a current given by \( f F A F \) (the product meaning the Clifford product of the corresponding fields), where \( f \) is a scalar function which satisfies a well determined algebraic quadratic equation.

Mathematics Subject Classification (2000). 15A66, 83C05, 83C22.

Keywords. Maxwell equations, Einstein equations.

1. Introduction

In a previous paper [7] we study using the Clifford bundle formalism the effective Lorentzian and teleparallel spacetimes generated by an electromagnetic field moving in Minkowski spacetime.

Here, using the same mathematical apparatus, we study another intriguing connections between gravitation described by Einstein field equations and electromagnetism described by Maxwell equations. In order to do that we first prove in Section 2 a proposition showing that if \( K \) is a Killing vector field on a Lorentzian manifold \((M, g)\) then the form field \( K = g(K_\alpha) \) satisfies \( \delta K = 0 \) and a wave equation given by Eq.(2) in terms of the covariant D’Alembertian. We also show that the Ricci operator (which can be defined only in the Clifford bundle of differential forms) applied to \( K \) it is equal to the covariant D’Alembertian applied to \( K \). Next,
in Section 3 we analyze the structure of Maxwell equations in a Lorentzian spacetime when the potential obeys the Lorenz gauge. Take notice that if a potential is in Lorenz gauge this does not necessarily implies that it is a 1-form physically equivalent to a Killing vector field. Moreover we determine the form of the current associated with this potential showing that it is given by $\frac{1}{2}A^\beta_{,\beta}$, where the $R_{\beta}$ are the Ricci 1-form fields (Eq.(64)). In Section 4 we study the structure of the Lorentzian spacetime representing the gravitational field produced and interacting with an electromagnetic field $F = dA$ (where $A$ is proportional $K = g(K, \cdot)$, with $K$ a Killing vector field) generated by an ideal charged current $J_e$. We show that Einstein equations is in this case represented by Maxwell equations with a current given by $f F A F$ (the product being intended as the Clifford product of the corresponding fields), where $f$ is a scalar function solution of a well determined algebraic quadratic equation (Eq.(36)). In Section 5 we present our conclusions and in the Appendix we recall the main definitions and formulas of the Clifford bundle formalism, proving a result that is need in the proof of Proposition 1.

2. Some Preliminaries

In this paper a spacetime structure is a pentuple $\mathcal{M} = (M, g, D, \tau_g, \uparrow)$ where $(M, g, \tau_g)$ is a Lorentzian manifold, $D$ is the Levi-Civita connection of $g$ and $\uparrow$ is an equivalence relation between timelike vector fields defining the time orientation. Also, $g \in \text{sec} T_0^2M$ denotes the metric of the cotangent bundle, $\bigwedge T^*M$ denotes the bundle of (nonhomogeneous) differential forms and $\mathcal{C}(M, g)$ denotes the Clifford bundle of differential forms. We shall take advantage of the well known fact that $\bigwedge T^*M \hookrightarrow \mathcal{C}(M, g)$ and use in our calculations the powerful Clifford bundle formalism. Let $\{\partial^\mu\}, \partial^\mu := \frac{\partial}{\partial x^\mu}$ be an arbitrary coordinate basis for $TU \subset TM$ and $\{\gamma^\mu = dx^\mu\}$ the corresponding dual basis of $T^*U \subset T^*M = \bigwedge T^*M$. As explained in the Appendix the $\gamma^\mu$ will be though as sections of the Clifford bundle, more precisely, $\gamma^\mu \in \text{sec} T^*U \subset \text{sec} \bigwedge T^*M \hookrightarrow \mathcal{C}(M, g)$. Also, we recall that the set $\{\partial^\mu\}, \partial^\mu = g_{\mu\nu} \frac{\partial}{\partial x^\nu} \in \text{sec} TM$ such that $g(\partial^\mu, \partial^\nu) = \delta^\mu_\nu$ is called the reciprocal basis of $\{\partial^\mu\}$ and the set $\{\gamma^\mu\}$ such that $g(\gamma^\mu, \gamma^\nu) = \delta^\mu_\nu$ is called the reciprocal basis of $\{\gamma^\mu\}$. We denote $g(\gamma^\mu, \gamma^\nu) = \gamma^\mu \cdot \gamma^\nu$, where $\cdot$ denotes the scalar product in $\mathcal{C}(M, g)$. Finally, $\partial = \gamma^\mu D\partial^\mu$ denotes the Dirac operator acting on sections of $\mathcal{C}(M, g)$ and $\square = \nabla \cdot \nabla$ and $\partial \wedge \partial$ denotes respectively the covariant D'Alembertian and the Ricci operators. The operator $\Diamond = \partial^2$ is called Hodge D'Alembertian and the relations between those operators and their main properties are presented in the Appendix.

---

1Our result differs from a factor of 2 from the one presented in [8] and also in [3], where an electromagnetic potential proportional to a Killing vector field is called a Papapetrou field. The important discrepancy is due to the fact that those authors identified the electromagnetic current $J_e$ with $\Box A$ instead of identifying it with $-\delta dA$, as it must be. See the text for details.

2Details may be found, e.g., in [9, 11]