Semi-Lagrangian Difference Approximations for Different Conservation Laws

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Abstract. The paper demonstrates different ways of using the semi-Lagrangian approximation depending on the fulfillment of conservation laws. A one-dimensional parabolic equation is taken as a simple methodological example. For this equation, the principles of constructing discrete analogues are demonstrated in the context of two different conservation laws. It is important that different conservation laws yield difference problems of different types as well as different ways to justify their stability and convergence.

Introduction

The semi-Lagrangian approach is applied for the numerical solution of the partial differential equations including the transport process. In this approach, the Lagrangian nature of the transport process is used but a fixed computational grid is admissible. Nowadays, a whole group of methods can be attributed to this approach like the modified methods of characteristics, the Eulerian-Lagrangian methods, and the characteristic Galerkin methods. A review of the semi-Lagrangian approach and related methods can be found in [1-5].

Involving the Lagrangian approach in the approximation of the transport operator (with the Eulerian approach for the remaining operators) allows one to improve significantly the properties of the obtained discrete problems. First, in the approximation of the mass conservation law, the Courant-Friedrichs-Levy condition for the time step is eliminated by adapting the stencil of a discrete equation. At high speeds in aerodynamics, the stencil stretches along the transport direction in the combined time-space coordinates. Second, it is not necessary to match space difference grids at the adjacent time layers. This opens up the possibility of flexible condensation of the difference grids independently at each time layer. Finally, the semi-Lagrangian approximation of the transport operator often produces the difference schemes with the inversion of symmetric positive-definite matrices at each time layer unlike other approximations with non-symmetric matrices.

The aim of the paper is also to demonstrate different ways of using the semi-Lagrangian approximation depending on the fulfillment of conservation laws. A one-dimensional parabolic equation is taken as a simple methodological example. For this problem, the principles of constructing discrete analogues are demonstrated for two different conservation laws (of mass and kinetic energy) which are formulated in terms of the discrete norms similar to the $L_1$ and $L_2$ norms. It is important that different conservation laws yield discrete problems of different types as well as different ways to justify their stability.

Stability and Convergence in the Discrete $L_1$ Norm

Consider the equation
\[
\frac{\partial u}{\partial t} - \sigma \frac{\partial^2 u}{\partial x^2} + \partial(au) = f \quad \text{in } \Omega = (0, T) \times (0,1)
\]  
with the initial condition
\[
u(0, x) = u_0(x) \quad \forall x \in (0,1)
\]
and the boundary conditions
\[
u(t, 0) = u_{it}(t), \quad u(t, 1) = u_{it}(t) \quad \forall t \in (0,T).
\]

The coefficient \(\sigma\) is a positive constant (for simplicity); the coefficient \(a(t, x)\) and the right-hand side \(f(t, x)\) are continuous functions defined on \(\Omega\).

From the methodological point of view, this parabolic equation is a simple mathematical model for the transfer with diffusion of a substance density \(u(t, x)\). Such a substance can be agents in economic problems [6] or gas and liquid in theoretical physics [7, 10]. Basing on the meaning of these problems, it is important to ensure the fulfillment of the conservation law in the \(L_1\) - norm
\[
\|\rho\|_1 = \int_0^1 |\rho(x)| \, dx
\]
for transfer integrals from one instant of time to another one. It is a typical situation in the problems of the Kolmogorov or Fokker-Planck type [6].

To discretize the differential statement, introduce uniform grids in time and in space, respectively:
\[
t_k = k\tau, \quad k = 0,\ldots,M, \quad \tau = 1/M;
\]
\[
x_i = ih, \quad i = 0,\ldots,N, \quad h = 1/N;
\]
for integers \(N, M \geq 2\). We take also the midpoints \(x_{i+1/2} = (i+1/2)h, \quad i = 0,\ldots,N-1\). Thus, this function is completely defined by its values \(u^h_{k,j} := u^h(t_k, x_j)\):
\[
u^h(t_k, x) = u^h_{k,i} (x_{i+1} - x)/h + u^h_{k,i+1} (x - x_i)/h \quad \forall x \in \omega_{i+1/2}.
\]

In addition, \(u^h(t_k, x)\) satisfies boundary conditions (3)
\[
u^h_{k,0} = u_{0k} (t_k), \quad u^h_{k,N} = u_{nk} (t_k) \quad \forall k = 0,\ldots,M
\]
and initial condition (2)
\[
u^h_{0,j} = u_0 (x_j) \quad \forall i = 0,\ldots,N.
\]

In [6] we get the following approximation of equation (1):
\[
\left( -\frac{\sigma}{h^2} + \frac{1}{8\tau} \right) u^h_{j,j-1} + \left( \frac{3\sigma}{4h^2} + \frac{1}{4\tau} \right) u^h_{j,j} + \left( -\frac{\sigma}{h^2} + \frac{1}{8\tau} \right) u^h_{j,j+1} = \beta^i_{k-1,j} u^h_{k-1,j} + \beta^i_{k-1,j+1} u^h_{k-1,j+1} + f_{k,j}
\]
with the coefficients

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\[ \beta_{k-1,j}^i = \frac{1}{8} \left( 1 + \frac{2\tau}{h} a_{k-1,j+1/2} \right)^2, \]

\[ \beta_{k-1,j}^i = \frac{1}{8} \left( 3 - \frac{2\tau}{h} a_{k-1,j-1/2} \right) \left( 1 + \frac{2\tau}{h} a_{k-1,j-1/2} \right) + \frac{1}{8} \left( 3 + \frac{2\tau}{h} a_{k-1,j+1/2} \right) \left( 1 - \frac{2\tau}{h} a_{k-1,j+1/2} \right), \quad (8) \]

\[ \beta_{k-1,j+1}^i = \frac{1}{8} \left( 1 - \frac{2\tau}{h} a_{k-1,j+1/2} \right)^2. \]

The approximation order of this expression is \( O(h^2 + \tau) \). Impose the condition

\[ \tau \max_{\Omega} |a| < \frac{h}{2}. \quad (9) \]

Then all three coefficients in (3.9) are non-negative. Now non-positivity of two other off-diagonal coefficients is provided by smallness of \( h^2 \) in comparison with \( \tau \):

\[ h^2 \leq 8\sigma \tau. \quad (10) \]

Combining difference equations (7)-(8) for \( i = 1, \ldots, N-1 \) and \( k = 1, \ldots, M \) with boundary condition (5) and initial condition (6), we get the system of linear algebraic equations for the calculation of \( u^h(t_k, x_i) \) for given \( u^h(t_k-1, x_i) \). The matrix of the system is an M-matrix with diagonal dominance in columns that ensures stability and convergence of \( O(h^2 + \tau) \) in a discrete analogue of the \( L_1 - \) norm [6]:

\[ \|v(x)\|_{1,h} = \sum_{i=1}^{N+1} |v(x_i)| h. \quad (11) \]

**Remark 1.** Note that Courant-like condition (9) is taken here only for simplicity of presentation and is not of fundamental importance in this approach. Here it allows one to use at most 3 neighboring values from the previous time level. If condition (9) is not fulfilled, we can use the integration at the previous time level over other intervals with the same properties of difference scheme [6]. Inequality (10) is more important for this scheme but is less restrictive due to smallness of \( h^2 \).

**Stability and Convergence in the Discrete \( L_2 - \) norm**

Now consider the equation of the form

\[ \frac{\partial u}{\partial t} - \sigma \frac{\partial^2 u}{\partial x^2} + \frac{a}{2} \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial (au)}{\partial x} = f \quad \text{in} \quad \Omega = (0,T) \times (0,1) \]

(12)

with initial condition (2), boundary conditions (3), and the same properties of \( \sigma, a(t,x), f(t,x) \). The transfer operator of this type arises when we try to solve a problem providing stability in the framework of the \( L_2 - \) norm

\[ \|u\|_{2}^2 = \int_0^1 u^2 \, dx. \]

This is a typical case for the conservation laws for kinetic energy. For example, this problem is a simplest analogue for the Navier-Stokes momentum equations. The main property of the convective part consists in the following conservation law for the “kinetic energy”:

\[ \int_0^1 \left( \frac{\partial u}{\partial t} + \frac{a}{2} \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial (au)}{\partial x} \right) \, dx = \frac{\partial}{\partial t} \int_0^1 \frac{u^2}{2} \, dx + \frac{au^2}{2} \bigg|_{x=0}^{x=1}. \]

This means the conservation of “kinetic energy” on the segment \([0,1]\) taking into account its input and loss through the boundary. It is fruitful to have a discrete analogue of this conservation law.
Introduce the discrete scalar product and the induced norm:

\[ (u, v)_h = \sum_{j=1}^{N} u(x_j)v(x_j)h \quad \text{and} \quad \|u\|_{2,h} = (u, u)_h^{1/2}. \]

To construct the scheme that is stable in this norm, we use the Crank-Nicolson stencil:

\[
\begin{align*}
\left(-\frac{a_{k-1/2,j-1/2}}{4h} - \frac{\sigma}{2h^2}\right) u_{k,j-1}^h + \left(\frac{1}{\tau} + \frac{\sigma}{h^2}\right) u_{k,j}^h + \left(-\frac{a_{k-1/2,j+1/2}}{4h} - \frac{\sigma}{2h^2}\right) u_{k,j+1}^h \\
= \left(-\frac{a_{k-1/2,j-1/2}}{4h} + \frac{\sigma}{2h^2}\right) u_{k-1,j-1}^h + \left(\frac{1}{\tau} - \frac{\sigma}{h^2}\right) u_{k-1,j}^h + \left(-\frac{a_{k-1/2,j+1/2}}{4h} + \frac{\sigma}{2h^2}\right) u_{k-1,j+1}^h + f_{k+1/2,j}. \tag{13}
\end{align*}
\]

Consider the error \( \epsilon_{k,j}^h = u(t_k, x_j) - u_{k,j}^h \) of an approximate solution for which we have the discrete problem with a computational or approximation error \( g_{k,j}^h \):

\[
\begin{align*}
\left(-\frac{a_{k-1/2,j-1/2}}{4h} - \frac{\sigma}{2h^2}\right) \epsilon_{k,j-1}^h + \left(\frac{1}{\tau} + \frac{\sigma}{h^2}\right) \epsilon_{k,j}^h + \left(-\frac{a_{k-1/2,j+1/2}}{4h} - \frac{\sigma}{2h^2}\right) \epsilon_{k,j+1}^h \\
= \left(-\frac{a_{k-1/2,j-1/2}}{4h} + \frac{\sigma}{2h^2}\right) \epsilon_{k-1,j-1}^h + \left(\frac{1}{\tau} - \frac{\sigma}{h^2}\right) \epsilon_{k-1,j}^h + \left(-\frac{a_{k-1/2,j+1/2}}{4h} + \frac{\sigma}{2h^2}\right) \epsilon_{k-1,j+1}^h + g_{k+1/2,j}. \tag{14}
\end{align*}
\]

\( \epsilon_{k,0}^h = \epsilon_{k,N}^h = 0 \quad \forall \ k = 1,...,M; \quad \epsilon_{0,i}^h = 0 \quad \forall \ i = 0,...,N. \tag{15} \)

**Theorem 1.** The solution of problem (14)-(15) satisfies the inequalities

\[
\|\epsilon_{k,i}^h\|_{2,h} \leq \|\epsilon_{k,-1,i}^h\|_{2,h} + \tau \|g_{k,i}^h\|_{2,h}, \quad \forall k = 1,...,M. \tag{16}
\]

\[
\|\epsilon_{k,i}^h\|_{2,h} \leq \tau k \max_{1 \leq k \leq M} \|g_{k,i}^h\|_{2,h}, \quad \forall k = 1,...,M. \tag{17}
\]

**Hint.** Rewrite equation (14):

\[
\begin{align*}
\left(-\frac{a_{k-1/2,j-1/2}}{4h} - \frac{\sigma}{2h^2}\right) \left(\epsilon_{k-1,j-1}^h + \epsilon_{k,j-1}^h\right) + \frac{\sigma}{h^2} \left(\epsilon_{k-1,j}^h + \epsilon_{k,j}^h\right) \\
+ \left(-\frac{a_{k-1/2,j+1/2}}{4h} - \frac{\sigma}{2h^2}\right) \left(\epsilon_{k-1,j+1}^h + \epsilon_{k,j+1}^h\right) + \frac{1}{\tau} \left(\epsilon_{k,j}^h - \epsilon_{k-1,j}^h\right) = g_{k+1/2,j}.
\end{align*}
\]

Multiply it by \( \tau h \left(\epsilon_{k-1,j}^h + \epsilon_{k,j}^h\right) \) and sum up \( \forall i = 1,...,N-1; \)

\[
(\epsilon_{k-1,i}^h, \epsilon_{k,i}^h)_h - (\epsilon_{k-1,i}^h, \epsilon_{k-1,i}^h)_h + \tau \sigma \sum_{j=0}^{N-1} \left(\left(\epsilon_{k-1,j+1}^h + \epsilon_{k,j+1}^h\right) - \left(\epsilon_{k-1,j}^h + \epsilon_{k,j}^h\right)\right)^2/h = \tau \left(g_{k,i}^h, (\epsilon_{k-1,i}^h + \epsilon_{k,i}^h)\right)_h.
\]

Omit the positive sum in the left-hand side and apply the Cauchy-Bunyakovskii inequality:

\[
\|\epsilon_{k,i}^h\|_{2,h} \leq \left(\|g_{k,i}^h, (\epsilon_{k-1,i}^h + \epsilon_{k,i}^h)\|_h\right) \leq \|g_{k,i}^h\|_{2,h} \left(\|\epsilon_{k,i}^h\|_{2,h} + \|\epsilon_{k-1,i}^h\|_{2,h}\right).
\]

After dividing by \( \|\epsilon_{k,i}^h\|_{2,h} + \|\epsilon_{k-1,i}^h\|_{2,h} \) we get (16). Inequality (17) is proved by induction.

From this estimates, we get the convergence of \( O(h^2 + \tau^2). \)

**Two- and Three-dimensional Problems**

There are many ways to generalize the semi-Lagrangian approximations to the two-dimensional continuity equation [7-10]. The most common one is usually applied on rectangular space meshes.
and consists in the separate approximation of one-dimensional parts. For example, the two-dimensional (in space) operator

\[
\frac{\partial u}{\partial t} + \frac{\partial (au)}{\partial x} + \frac{\partial (bu)}{\partial y}
\]

is split into one-dimensional operators

\[
\frac{1}{2} \frac{\partial u}{\partial t} + \frac{\partial (au)}{\partial x}
\quad \text{and} \quad
\frac{1}{2} \frac{\partial u}{\partial t} + \frac{\partial (bu)}{\partial y}
\]

Then each of them is approximated as in the previous sections and thereafter they are combined again.

**Conclusion**

Thus, involving the Lagrangian approach to the approximation of the transport operator (with the Eulerian approach for the remaining part) allows one to improve significantly the properties of the obtained discrete problems. For the approximation with the mass conservation law, the Courant-Friedrichs-Levy condition for the time step is eliminated by the adaptation of the stencil of a discrete equation. Since there is no need to match space difference grids at the neighboring time layers, the flexible condensation of difference grids on each time layer is simplified. Moreover, the semi-Lagrangian approximation in combination with the finite difference or finite element method results in grid problems with inversion of symmetric matrices at each time layer unlike traditional discrete approximations with non-symmetric matrices.

Another aim of the paper is to demonstrate different ways to use the semi-Lagrangian approximation depending on the fulfillment of conservation laws. A one-dimensional parabolic equation is taken as a simple methodological example. For this equation, the principles of constructing discrete analogues are demonstrated in the context of two different conservation laws (or the requirements of stability in two different discrete norms similar to the $L_1$ – and $L_2$ – norms).

It is significant that different conservation laws result in discrete problems of different types as well as in different ways of justification of their stability.

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