1 Introduction

This course is concerned with linear groups $\Gamma \leq \text{GL}_n(k)$ where $k$ is some field (usually of characteristic 0). Linearity is one of the most effective and well studied conditions one can put on a general infinite group. The following are two of the most often used consequences of linearity.

(i) a finitely generated linear group $\Gamma$ is residually finite, and

(ii) if in addition $\text{char } k = 0$, then $\Gamma$ is virtually torsion free.

The first of these means that $\Gamma$ has many finite images, and one way to study $\Gamma$ is to investigate these images (equivalently, the profinite completion $\hat{\Gamma}$ of $\Gamma$).

One of the main objectives of this course is the ‘Lubotzky alternative’ for linear groups:

**Theorem 1.1** Let $\Delta \leq \text{GL}_n(k)$ be a finitely generated linear group over a field $k$ of characteristic 0. Then one of the following holds:

(a) the group $\Delta$ is virtually soluble, or

(b) there exist a connected simply connected $\mathbb{Q}$-simple algebraic group $G$, a finite set of primes $S$ such that $\Gamma = G(\mathbb{Z}_S)$ is infinite, and a subgroup $\Delta_1$ of finite index in $\Delta$ such that every congruence image of $\Gamma$ appears as a quotient of $\Delta_1$.

Here $\mathbb{Z}_S = \mathbb{Z}[1/p \mid p \in S]$. In case (b) we can deduce from the *Strong Approximation Theorem* that $\Delta_1$ has many finite images, in particular the groups $\prod_{i=1}^k G(\mathbb{F}_{p_i})$ whenever $p_1, \ldots, p_k$ are distinct primes outside $S$. Now, for all but finitely many primes $p$ the group $G(\mathbb{F}_p)$ is semisimple, in fact it is a perfect central extension of a product of simple groups (of fixed Lie type over $\mathbb{F}_p$). The simple groups of Lie type are very well understood and this enables us to deduce properties of the profinite completion $\hat{\Delta}$ of $\Delta$. 

---

Nikolay Nikolov

Oxford, 10-14 September 2007
For example, if $\Delta$ has polynomial subgroup growth then one can deduce that case (b) of Theorem 1.1 is impossible and hence that $\Delta$ is virtually soluble. Some more applications of Theorem 1.1 are given in section 6 below.

In turn, when $\Delta$ is virtually soluble we have the Lie-Kolchin theorem:

**Theorem 1.2** Suppose that $\Delta \leq \text{GL}_n(K)$ is a virtually soluble linear group over an algebraically closed field $K$. Then $\Delta$ has a triangularizable subgroup $\Delta_1$ of finite index; i.e. $\Delta_1$ is conjugate to a subgroup of the upper triangular matrices in $\text{GL}_n(K)$.

In fact if $\text{char} K = 0$ the index of $\Delta_1$ in $\Delta$ can be bounded by a function of $n$ only (a theorem of Mal’cev and Platonov). This has the corollary:

**Lemma 1.3** Suppose that $\Delta$ is a finitely generated group which is residually in the class of virtually soluble linear groups of degree $n$ in characteristic 0. Then $\Delta$ itself is virtually soluble.

We shall use this Lemma in the proof of Theorem 1.1.

A common feature in the proofs of all these results is to take the Zariski closure $G = \overline{\Delta}$ of $\Delta$ in $\text{GL}_n(K)$. This is a linear algebraic group and we can apply results from algebraic geometry, number theory and the theory of arithmetic groups to study $G$ and its dense subgroup $\Delta$.

The main object of this course is to understand the terminology appearing above and develop the methods by which Theorems 1.1 and 1.2 can be proved. These methods are useful in a variety of other situations involving linear groups.

### 2 Algebraic groups

Throughout, $K$ will denote an algebraically closed field.

#### 2.1 The Zariski topology on $K^n$.

A good reference for the material of this section (with proofs) is the book [1] by Atiyah and Macdonald. For a brief introduction see also the chapter ‘Linear Algebraic Groups’ in [3].

Let $K^n$ be the $n$-dimensional vector space over $K$. Given a subset $S$ of the polynomial ring $R := K[x_1, \ldots, x_n]$

define

$$V(S) = \{x \in K^n \mid f(x) = 0 \, \forall f \in S\}$$

to be set of common zeros of $S$ in $K^n$.

It is easy to that $V(I) = V(S)$ for the ideal $I$ generated by $S$, that

$$V(I) \cup V(J) = V(IJ)$$
for ideals $I$ and $J$ of $R$, and that
\[ \bigcap_{I \in F} V(I) = V(\sum_{I \in F} I) \]
for any family $F$ of ideals of $R$.

The Hilbert Basis Theorem says that each ideal $I$ of $R$ is finitely generated
and so each $V(S)$ can in fact be defined by finitely many polynomial equations.

**Definition 2.1** The Zariski topology of $K^n$ has as its closed sets the sets $V(I)$
for all ideals $I$ of $R$.

The first basic result about the Zariski topology is the following

**Proposition 2.2** (Exercise 3 on p. 30) The space $K^n$ with the Zariski topology
is a compact topological space, in fact it satisfies the descending chain con-
dition on closed subsets.

Note that the closed sets of $K$ coincide with its finite subsets (since a polynomial
in one variable can only have finitely many roots). More generally the
Zariski topology of $K^n$ is never Hausdorff, thus even though the space $K^n$ is
compact one should be careful when applying familiar results from Hausdorff
spaces.

**Example:** Let $V$ be the hyperbola given by the equation $x_1x_2 = 1$ in $K^2$.
Then $V$ is a closed, hence compact subset of $K^2$ but its projection on the $x_1$
axis is $K \setminus \{0\}$ which is not closed. So in the Zariski topology continuous images
of compact sets are not always closed.

The subsets $V(I) \subseteq K^n$ (with the subspace topology induced from the
Zariski topology on $K^n$) are called affine (algebraic) varieties. If $W$ is an affine
variety, the coordinate ring $R(W)$ of $W$ is the algebra $R/J(W)$, where $J(W)$
is the ideal of $R$ consisting of all polynomials vanishing on $W$. The ascending
chain condition on ideals of $R$ (Hilbert’s Basis Theorem) implies the descending
chain condition (minimal condition) for closed sets in $K^n$.

**Theorem 2.3** (Hilbert’s Nullstellensatz) $V(I) = \emptyset$ if and only if $I = R$.

In fact a more general result holds (see [1], Chapter 7): if $W = V(I)$ is an
affine variety then $J(W)/I$ is the radical of $I$, i.e.
\[ J(W) = \{ x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N} \}. \]

The coordinate ring $R(W)$ can be considered as the set of morphisms of $W$
into the one-dimensional variety $K$. In general, a morphism $F$ from $W_1 \subseteq K^{n_1}$
into $W_2 \subseteq K^{n_2}$ is an $n_2$-tuple $(f_1, \ldots, f_{n_2}) \in K[x_1, \ldots, x_{n_1}]^{n_2}$
of polynomial maps such that $F(W_1) \subseteq W_2$. Any such morphism induces a $K$-algebra homo-
morphism $F^* : R(W_2) \rightarrow R(W_1)$ defined by $f \mapsto f \circ F$. Conversely, from the
Nullstellensatz it can be shown that every algebra homomorphism $F^*$ between $R(V_2)$ and $R(V_1)$ arises in this way from a morphism $F : V_1 \to V_2$. In this way the category of affine varieties is anti-equivalent to the category of reduced finitely generated algebras over the algebraically closed field $K$.

**Definition 2.4** A variety is irreducible if it is not the union of two proper closed subsets.

Since $V$ satisfies the minimal condition on closed subsets we can write every variety $W$ as

$$W = W_1 \cup W_2 \cup \cdots \cup W_k,$$

a union of irreducible varieties $W_i$. If we assume that the above decomposition is irredundant, i.e. $W_i \nsubseteq W_j$ whenever $i \neq j$, then it is in fact unique up to reordering of the $W_i$. These are then called the irreducible components of $W$.

For example if $W$ is the variety defined by the single equation

$$x_1x_2(x_1x_2^2 - 1) = 0$$

then its irreducible components are the two lines with equations $x_1 = 0$, $x_2 = 0$ and the curve defined by $x_1x_2^2 = 1$.

It is easy to see that a variety $W$ is irreducible if and only if $J(W)$ is a prime ideal of $W$, i.e. if and only if the coordinate ring $R/J(W)$ is an integral domain.

**Definition 2.5** The dimension, $\dim W$ of an irreducible variety $W$ is the Krull dimension of $R(W)$. This is the transcendence degree of $R(W)$ over $K$, or equivalently the maximal length $d$ of a chain of distinct nonzero prime ideals $0 \subseteq P_1 \subseteq \cdots \subseteq P_d \subseteq R(W)$. The dimension of a general affine variety is the maximal dimension of its irreducible components.

As a consequence, a closed proper subset of an irreducible variety $W$ has strictly smaller dimension than $W$.

### 2.2 Linear algebraic groups as closed subgroups of $\GL_n(K)$.

We identify the $n \times n$ matrix ring $M_n(K)$ with $K^{n^2}$, and use $x_{ij}$ ($i, j = 1, \ldots, n$) as coordinates. Then the subgroup $\SL_n(K)$ of matrices with determinant 1 forms an affine variety, defined by the equation $\det(x_{ij}) = 1$.

**Definition 2.6** A linear algebraic group over $K$ is a Zariski-closed subgroup of $\SL_n(K)$ for some $n$.

**Notes:**

1. The two maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ from $G \times G$ (resp. $G$) to $G$ are morphisms of affine varieties.

$^1$ $R$ is reduced if it contains no non-zero nilpotent elements.
2. There are more general algebraic groups which are not linear. In this course we shall be concerned only with linear algebraic groups and ‘algebraic group’ will always mean ‘linear algebraic group’.

3. The definition we have given is different from the standard one but equivalent to it: one usually defines a linear algebraic group to be an affine variety with maps of group multiplication and inverses which are morphisms of varieties. It can be shown that every such group is in fact isomorphic to a closed subset of some $\text{SL}_n(K)$. See the ‘Linear algebraic groups’ chapter in [3].

A homomorphism of linear algebraic groups $f : G \to H$ is a group homomorphism which is also a morphism between varieties, i.e. $f$ is given by polynomial maps on the realizations of $G \subseteq \text{M}_{n_1}(K)$ and $H \subseteq \text{M}_{n_2}(K)$.

The group $\text{GL}_n(K)$ is isomorphic to a closed subgroup of $\text{SL}_{m+1}(K)$, by the mapping

$$g \mapsto \left( \begin{array}{cc} g & 0 \\ 0 & (\det g)^{-1} \end{array} \right).$$

In this way we consider $\text{GL}_n(K)$ as a linear algebraic group. It is clear that every linear algebraic group is isomorphic to a closed subgroup of $\text{GL}_n(K)$ for some $n$.

2.2.1 Basic examples

For an integer $n \geq 2$ consider the following subgroups of $\text{SL}_n(K)$:

- The group of (upper or lower) unitriangular matrices,
- The upper (upper or lower) triangular matrices,
- The diagonal matrices, or more generally
- The monomial matrices.

It is clear that these are closed subgroups of $\text{SL}_n(K)$ and so are algebraic groups.

Note that when $n = 2$ the first example is isomorphic to the additive group of the field $K$, while the third one is isomorphic to the multiplicative group of $K$. In this way $(K, +)$ and $(K, \times)$ become linear algebraic groups. The first one is denoted by $\mathbb{G}_+$ and the second by $\mathbb{G}_\times$. In can be shown that these are the only connected algebraic groups of dimension 1.

Another family of examples arise from linear groups preserving some form. For example if $(u, v) = u^T P v$ is a bilinear form on the vector space $V = K^n$, then the group $G \leq \text{GL}(V)$ preserving $(-,-)$ can be described as those matrices $X$ in $\text{GL}_n(K)$ such that $X^T PX = P$. This is a collection of $n^2$ polynomial equations on the coefficients of $X = (x_{ij})$ and so $G$ is an algebraic group. Examples are the symplectic group $\text{Sp}_n(K)$ ($n$ even) and the special orthogonal group $\text{SO}_n(K)$. 
2.2.2 Basic properties of Algebraic Groups

Theorem 2.7 (see [5], chapter II) Let \( f : G \to H \) be a homomorphism of algebraic groups. Then

1. \( \text{Im}(f) \) is a closed subgroup of \( H \) and \( \text{ker}(f) \) is a closed subgroup of \( G \).
2. \( \dim G = \dim \text{ker}(f) + \dim \text{Im}(f) \).

Recall that a topological space is connected if it cannot be written as the disjoint union of two proper closed (equivalently open) subsets. Clearly an irreducible variety is connected. It turns out that for algebraic groups the converse is also true and so the two concepts coincide. To see this, suppose that \( G \) is a connected algebraic group. Let \( G = V_1 \cup \cdots \cup V_k \) be the decomposition of \( G \) into irreducible components. This decomposition is unique up to the order of the \( V_i \), therefore the action of \( G \) by left multiplication permutes the components \( V_i \). Without loss of generality suppose that \( 1 \in V_1 \). Let \( G_1 = \text{Stab}_G(V_1) := \{ g \in G | gV_1 = V_1 \} \).

Clearly \( G_1 \) is a closed subgroup of finite index \( k \) in \( G \), so it is both open and closed, as are each of its cosets in \( G \). Since \( G \) is connected we must have \( G = G_1 \), so \( k = 1 \) and \( G \) is irreducible.

The above argument easily shows that more generally the connected component of the identity \( G^0 \) of \( G \) is a closed irreducible normal subgroup of finite index in \( G \); it may be characterized as the smallest closed subgroup of finite index in \( G \).

Lemma 2.8 (see [14], 14.15 or [5], §7.5) If \( (H_i)_{i \in I} \) is a family of closed connected subgroups of \( G \) then the subgroup \( \langle H_i | i \in I \rangle \) generated abstractly by the \( H_i \) in \( G \) is closed and connected.

In particular if \( H_1 \) and \( H_2 \) are two closed connected subgroups of \( G \) such that \( H_1H_2 = H_2H_1 \) (e.g., if either of \( H_1 \) or \( H_2 \) is normal in \( G \)) then \( H_1H_2 \) is a closed connected subgroup of \( G \). In general if \( H_1 \), \( H_2 \) are closed subgroups with \( H_1H_2 = H_2H_1 \) and having connected components \( H_1^0 \), \( H_2^0 \) respectively, then \( H_1H_2 \) is a finite union of closed sets \( hH_1^0H_2^0h' \) for some \( h, h' \in G \) and so is a closed subgroup of \( G \).

Theorem 2.9 (Chevalley; see [5], chapter IV) If \( H \) is a closed normal subgroup of \( G \) then the quotient \( G/H \) can be given the structure of a linear algebraic group, so that the quotient map \( G \to G/H \) is a homomorphism of algebraic groups.

2.2.3 Fields of definition and restriction of scalars.

A variety \( V(I) \) is said to be defined over a subfield \( k \subset K \) if the ideal \( I \) is generated (as an ideal of \( R \)) by polynomials with coefficients in \( k \). When the field \( k \) is separable (which is always the case if \( k \) has characteristic 0) there is a useful criterion for \( V \) to be defined over \( k \):
Lemma 2.10 Let $W = V(S)$ be a variety. For $\sigma \in \text{Gal}(K/k)$ define the variety $W^\sigma$ to be $V(S^\sigma)$, i.e., the zero set of the ideal $S^\sigma$ of $R$. Then $W$ is defined over $k$ if and only if $W = W^\sigma$ for all $\sigma \in \text{Gal}(K/k)$.

Similarly, a homomorphism $f : G \to H$ between two algebraic groups is $k$-defined if all the coordinate maps defining $f$ are polynomials with entries in $k$.

Now let $G \leq \text{GL}_n(K)$ be an algebraic group and let $\mathcal{O}$ be a subring of $K$. The group of $\mathcal{O}$-rational points of $G$ is defined to be $\text{GL}_n(\mathcal{O}) \cap G$ and is denoted by $G_\mathcal{O}$.

Suppose that $G$ is defined over some subfield $k$ of $K$ which is a finite extension of $k_0$. In this course we shall study the groups $G_k$ and sometimes we prefer to reduce the situation to a smaller subfield $k_0$ (which will usually be $\mathbb{Q}$).

There is a standard procedure for doing this, called ‘restriction of scalars’. This associates to $G$ another algebraic group $H \leq \text{GL}_n(K)$ where $d = (k : k_0)$; here $H$ is defined over $k_0$ and satisfies $H_{k_0} = G_k$. The algebraic group $H$ is denoted $R_{k/k_0}(G)$. Before presenting the general construction let us study a simple special case which illustrates the idea.

Suppose that $G$ is the multiplicative group of the field $(K, \times)$. This is defined over the integers $\mathbb{Z}$ (i.e. it can be defined by polynomials over $\mathbb{Z}$.) Let $k$ be a number field, i.e. a finite extension field of $\mathbb{Q}$. The group $G_k$ is clearly the multiplicative group $k^\times$ of the field $k$. We want to find a $\mathbb{Q}$-defined algebraic group $H$ such that its group $H_{\mathbb{Q}}$ of $\mathbb{Q}$-rational points is isomorphic (as an abstract group) to $G_k$.

To find $H$ we identify $k$ with the vector space $\mathbb{Q}^d$ by choosing a basis $a_1, \ldots, a_d$ for $k$ over $\mathbb{Q}$, and consider the regular representation of $k$ acting on itself by left multiplication. This gives an algebra monomorphism $\rho : k \to M_d(\mathbb{Q})$ and so $\rho(k)$ is a $d$-dimensional subspace of $M_n(\mathbb{Q})$. This can be defined as the zeroes of some $s = d^2 - d$ linear functionals $F_1, \ldots, F_s : M_n(\mathbb{Q}) \to \mathbb{Q}$. Therefore we can define the algebraic variety $H$ as the set of zeros of $F_1, \ldots, F_s$ in $\text{GL}_d(K)$. Then clearly $H_{\mathbb{Q}} = G_k$ and the only thing that has to be checked is that $H$ is a group, i.e. the variety $H$ is closed under matrix multiplication and inverses. This can be expressed as the vanishing of certain polynomials in the coordinates $x_{ij}$. If one of these polynomials is nontrivial it will be nontrivial for some rational values of its arguments. But we certainly know that $H_{\mathbb{Q}}$ is closed under multiplication and inverses since it is equal to the multiplicative group $k^\times$. So $H$ is indeed an algebraic group.

There is another way to view the algebraic group $H$ just constructed. Let $\sigma_1, \ldots, \sigma_d$ be the $d$ embeddings of $k$ into the algebraically closed field $K$. For an element $h = \sum_{i=1}^d x_i a_i \in k$ with $x_i \in \mathbb{Q}$ consider

$$\lambda(h) = (\lambda_1(h), \ldots, \lambda_d(h)),$$

where

$$\lambda_j(h) = \sum_{i=1}^d x_i \sigma_j(a_i) = \sigma_j(h).$$
The condition \( \det(\rho(h)) \neq 0 \) is equivalent to \( \prod j \lambda_j(h) \neq 0 \). If \( k = \mathbb{Q}(\alpha_1) \) where \( \alpha_1 \) has minimal polynomial \( p(x) = (x-\alpha_1) \cdots (x-\alpha_d) \) over \( \mathbb{Q} \) then \( k \cong \mathbb{Q}[x]/(p(x)) \).

We can extend \( \lambda \) from \( k \) to \( k \otimes \mathbb{Q} \mathbb{C} \) and then

\[
k \otimes \mathbb{Q} \mathbb{C} \cong \frac{\mathbb{C}[x]}{(p(x))} \cong \bigoplus_{i=1}^{d} \mathbb{C}[x]/(x-\alpha_i),
\]

where the second isomorphism comes from the Chinese remainder theorem and coincides with \( \lambda \). Thus \( \lambda \circ \rho^{-1} \) provides a \( K \)-isomorphism of \( H \) with the direct product \( (\mathbb{G}_m)^d \) of \( d \) copies of the multiplicative group \( \mathbb{G}_m \).

In general we are given a \( k \)-defined algebraic group \( G \leq \text{GL}_n(K) \). Consider again an embedding \( \rho : k \rightarrow M_d(k_0) \) given by the regular representation of \( k \) acting on itself. Again the subspace \( \rho(k) \subseteq M_d(k_0) \) is defined by some set of say \( r \) linear equations \( F_i(y_{ab}) \) in the matrix entries \( y_{ab} \) \( (1 \leq a, b \leq d \text{ and } 1 \leq i \leq r) \). Suppose that \( G \) was defined as a variety by the \( l \) polynomials \( P_j(z_{st}) \) in the entries of the matrix \( (z_{st}) \in M_n(K) \) \( (j = 1, \ldots, l, \ 1 \leq s, t \leq n) \).

Now the algebraic group \( H = R_{k/k_0}(G) \) is defined by the following two families equations in the \( (nd)^2 \) variables \( z_{ab}^{st} \):

The first family is

\[
P_j((z_{ab}^{st})_{a,b}) = 0 \in M_d(K), \quad j = 1, 2, \ldots, l,
\]

i.e., we replace each variable \( z_{st} \) in the original polynomial \( P_j \) with a matrix \( (z_{ab})_{a,b} \in M_d(K) \). Note that each \( P_j \) gives \( d^2 \) polynomial equations in \( K \), one for each entry of the matrix in \( M_d(K) \).

The second family is

\[
F_i((z_{ab}^{st})_{a,b}) = 0, \quad i = 1, \ldots, r
\]

for each pair \( (s,t) \) with \( 1 \leq s, t \leq n \).

A typical example is the group

\[
G = \left\{ \begin{pmatrix} a & 2b \\
                             b & a \end{pmatrix} \mid a^2 - 2b^2 \neq 0 \right\}
\]

which is the restriction of scalars \( R_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}} \mathbb{G}_m \). Here, \( G \) is \( K \)-isomorphic to \( \mathbb{G}_m \times \mathbb{G}_m \) via the map \( \begin{pmatrix} a & 2b \\
                             b & a \end{pmatrix} \mapsto (a+ib, a-ib) \), but this isomorphism is not \( \mathbb{Q} \)-defined.

It is easy to see that if we have a \( k \)-defined morphism \( f : G \rightarrow T \) between two \( k \)-defined linear algebraic groups then this induces a \( k_0 \)-defined morphism

\[
R_{k/k_0}(f) : R_{k/k_0}(G) \rightarrow R_{k/k_0}(T).
\]

In this way \( R_{k/k_0} \) becomes a functor between the category of \( k \)-defined groups and morphisms and \( k_0 \)-defined groups and morphisms.
2.2.4 The Lie algebra of $G$

There is a standard way to associate a Lie algebra $L(G)$ to any connected linear algebraic group $G$, so that the map $L : G \mapsto L(G)$ is an equivalence of categories. More precisely the following holds (see III of [5]):

- If $f : G \to H$ is a homomorphism of algebraic groups then there is a uniquely specified homomorphism $L(f) : L(G) \to L(H)$ between their Lie algebras.

- In particular, for any given $g \in G$ the map $x \mapsto g^{-1}xg$ is an automorphism of $G$ and this gives rise to a Lie algebra automorphism denoted $Ad_g : L(G) \to L(G)$. In this way we get a homomorphism of algebraic groups $Ad : G \to \text{Aut} L(G)$, and it is easy to see that $\ker Ad = Z(G)$, the centre of $G$.

- If $H$ is a closed (normal) subgroup of $G$ then $L(H)$ is a Lie subalgebra (resp. an ideal) of $L(G)$.

- If $G$ is defined over a subfield $k$ of $K$ then $L(G)$ is also defined over $k$, i.e., it has a basis such that the structure constants of the lie bracket multiplication are elements of $k$. Moreover if the morphism $f : G \to H$ is $k$-defined then so is the Lie algebra homomorphism $L(f)$.

- The dimension of $L(G)$ (as a vector space over $K$) is equal to the dimension of the algebraic group $G$.

In general if $G$ is not connected we define $L(G)$ to be equal to $L(G^0)$ where $G^0$ is the connected component of the identity in $G$.

Now a linear algebraic group $G$ is an affine subset of $M_n(K)$ so it is defined by an ideal $I \triangleleft \mathbb{R}$ of the polynomial ring $K[X_{11},\ldots,X_{nn}]$. In this setting there is a concrete description of $L(G)$. It is a Lie subalgebra of the Lie algebra $M_n(K)$ with the Lie bracket

$$[A,B] = AB - BA.$$ 

As a vector space $L(G)$ is the tangent space at the identity element $e \in G$. In our situation this is defined as follows.

For a polynomial $P \in R = K[[x_{ij}]]$ and $g = (g_{ij}) \in G \subseteq M_n(K)$ let $\partial P_g$ be the linear functional on $n^2$ variables $X_{ij}$ defined as follows

$$\partial P_g : M_n(K) \to K, \quad \partial P_g((X_{ij})_{i,j}) := \sum_{i,j} \left( \frac{\partial P}{\partial x_{ij}}(g_{ij}) \cdot X_{ij} \right)$$

Then $L(G)$ is the subspace of $M_n(K)$ of common solutions to the equations

$$\partial P_c = 0, \quad \forall P \in I,$$
where \( e = \text{Id}_n \) is the identity matrix in \( G \leq \text{GL}_n(K) \).

In fact we don’t need to check infinitely many equations. By the Hilbert basis theorem the ideal \( I \) is finitely generated, say by polynomials \( P_1, \ldots, P_t \). Then \( L(G) \) is the common zeroes of the linear functionals \( \partial(P_i)e = 0 \) (\( i = 1, \ldots, t \)).

\[ 2.2.5 \quad \text{Connection with Lie algebras of Lie groups} \]

Let \( G \leq \text{GL}_n \) be a linear algebraic group and suppose that \( k \) is a complete field, for example \( \mathbb{C}, \mathbb{R} \) or the field of \( p \)-adic numbers \( \mathbb{Q}_p \) (see example 2.2 below). We have another topology on \( \text{GL}_n(k) \) by considering it as a subset of the topological space \( M_n(k) = k^{(n^2)} \). In this way the group \( G_k \) of \( k \)-rational points is a topological group, by virtue of being a closed subgroup of \( \text{GL}_n(k) \). In fact \( G_k \) is a complex or real Lie group when \( k = \mathbb{C} \) or \( \mathbb{R} \), and is a \( p \)-adic analytic group when \( k = \mathbb{Q}_p \). In this section we shall use the term analytic group to refer to either a Lie group or a \( p \)-adic analytic group.

There is a standard way to associate a Lie algebra \( L(\mathcal{G}) \) to any (complex or real) Lie group \( \mathcal{G} \) and as explained in \[7\] such a Lie algebra exists for any \( p \)-adic analytic group. One uniform way to define them is as the tangent space at the identity of \( \mathcal{G} \). The Lie bracket is the differential of the commutator map in \( \mathcal{G} \).

The following Proposition is thus almost self evident.

**Proposition 2.11** If the field \( k \) is one from \( \mathbb{C}, \mathbb{R} \) or \( \mathbb{Q}_p \) then the \( k \)-rational points of \( L(G) \), (namely \( L(G)_k = L(G) \cap M_n(k) \)) coincide with the Lie algebra of the analytic group \( \mathcal{G} = G_k \).

For later use we record another basic result. First observe that when we have an analytic group \( \mathcal{G} \leq \text{GL}_n(k) \) with a faithful linear representation in \( \text{GL}_n(k) \) then we can also consider the Zariski topology on \( \mathcal{G} \) as a subset of \( \text{GL}_n \).

**Proposition 2.12** Suppose that the group \( H \) is a Zariski dense subgroup of the analytic group \( G_k \leq \text{GL}_n(k) \) for \( G \) and \( k \) as above. Let \( \text{Ad} \) be the adjoint action of \( G \) on its Lie algebra \( L(G) \). Then \( \text{Ad}(H)_k \) and \( \text{Ad}(G)_k \) have the same span in the vector space \( \text{End}_kL(G)_k \) over \( k \).

Moreover, when \( H \) is an analytic Zariski-dense in \( G \) the Lie algebra \( L(\mathcal{H}) \) of \( \mathcal{H} \) is an ideal of the Lie algebra \( L(G)_k \) of \( G_k \).

**Proof.** The adjoint action of \( G \) on \( L(G) \) is given by a set of polynomials (it coincides with the conjugation action of \( G \) on \( L(G) \) as a subset of \( M_n(K) \) and so the map \( \text{Ad} : G \to \text{End}_K(L(G)) \) is morphism of algebraic varieties, hence a representation of \( G \) as an algebraic group. Since \( H \) is Zariski-dense in \( G_k \) it follows that \( \text{Ad}(H) \) is Zariski-dense in \( \text{Ad}(G_k) \) as subsets of \( \text{End}_kL(G)_k \). Since a vector subspace of \( \text{End}_kL(G)_k \) is Zariski closed the first part of the Proposition follows immediately.

By a standard result of Lie theory \( L(\mathcal{H}) \) is a Lie subalgebra of \( L(G)_k \) which is \( \text{Ad}(\mathcal{H}) \)-invariant. Now the stabilizer \( \text{Stab}(L(\mathcal{H})) \) of \( L(\mathcal{H}) \) in \( \text{End}_kL(G)_k \) is a subspace of \( \text{End}_kL(G)_k \). Since this stabilizer contains \( \text{Ad}(\mathcal{H}) \) it should also
contain Ad($G_k$) by the first part of the Proposition. Therefore $L(\mathcal{H})$ is Ad($G_k$) invariant and so it is an ideal of the Lie algebra $L(G)_k$ of $G_k$.

**Note:** The Lie algebra is a local tool, it was only defined from a neighbourhood of the identity of an analytic group $\mathcal{G}$. So it is the same for any open subgroup of $\mathcal{G}$. In particular any subgroup of finite index in $G_{\mathbb{Z}_p}$ is a compact open subgroup of $G_{\mathbb{Q}_p}$ and hence all of these groups share the same Lie algebra as analytic groups. In fact this property characterises the open subgroups of analytic groups:

**Proposition 2.13** Suppose that the analytic group $\mathcal{H}$ is a closed subgroup of the analytic group $\mathcal{G}$. Then $\mathcal{H}$ is an open subgroup of $\mathcal{G}$ if and only if $\mathcal{H}$ and $\mathcal{G}$ have the same Lie algebra. In particular when $\mathcal{G}$ is compact this happens if and only if $\mathcal{H}$ has finite index in $\mathcal{G}$.

As in the theory of Lie groups the Lie algebra is a very useful tool in the study of algebraic groups. This is best seen in the classification of the simple algebraic groups in next section, but we can already give a nontrivial application.

**Proposition 2.14** A connected linear algebraic group $G$ of dimension less than 3 is soluble. If $\dim G = 1$ then $G$ is abelian.

Indeed $L = L(G)$ is a Lie algebra of dimension at most 2 as a vector space over $K$ and it is easy to see that $L$ must be soluble. If $\dim L = 1$ then $L$ is abelian and then so is $G$.

Note that even at this small dimension we see that two connected groups (for example $G_+\mathbb{R}$ and $G_\mathbb{R}$) may have the same Lie algebra and still be non-isomorphic. However the simply connected semisimple groups are indeed uniquely determined by their Lie algebras as we shall see in the following section.

### 2.3 Semisimple algebraic groups. The classification of simply connected algebraic groups over $K$

**Definition 2.15** A connected algebraic group is called semisimple if it has no nontrivial closed connected soluble normal subgroups.

In general, an algebraic group $G$ has a unique maximal connected soluble normal subgroup. This is called the (soluble) radical and denoted $\text{Rad}(G)$. The group $G/\text{Rad}(G)$ is then semisimple.

**Definition 2.16** A connected algebraic group is simple if it is nonabelian and has no proper nontrivial connected normal subgroups.

This implies that every closed proper normal subgroup of $G$ is central and finite (Exercise: prove this!).

11
Theorem 2.17  A semisimple group $G$ is a central product$$G \cong S_1 \circ S_2 \circ \cdots \circ S_l$$of simple algebraic groups $S_i$. The factors in this product are unique up to reordering.

Recall that a central product $S_1 \circ S_2 \circ \cdots \circ S_l$ is a quotient $P/N$ of the direct product $P = S_1 \times \cdots \times S_l$ by a central subgroup $N$ intersecting each $S_i$ trivially.

So in order to understand semisimple algebraic groups it is sufficient to understand simple algebraic groups and their central extensions.

Analogous definitions apply relative to any field of definition $k$. A connected nonabelian algebraic group defined over $k$ is $k$-simple (resp. $k$-semisimple) if it has no nontrivial closed connected proper (resp. soluble) normal subgroups defined over $k$. Again a $k$-semisimple group is $k$-isomorphic to a central product of $k$-simple groups which are unique up to reordering.

When we speak of simple/semisimple groups without indicating the field the understanding is that it is $K$. In this case $G$ is called absolutely simple (resp. semisimple). **Warning:** a $k$-simple group need not be absolutely simple (though it is semisimple); see Example 2.20 below.

The classification of absolutely simple algebraic groups mirrors entirely the classification of the finite dimensional simple Lie algebras over $K$. Indeed a simple group $G$ has finite centre and $G/Z(G)$ embeds via Ad as a group of automorphisms of its Lie algebra $L = L(G)$. So once the algebra $L(G)$ is known the group $G$ is determined up to an isogeny as a closed subgroup of Aut($L$).

More precisely we have the following classification theorem.

Theorem 2.18  (Chevalley)  For each Lie type $\mathcal{X}$ from the list

$\begin{align*}
A_n \ (n \geq 1), & \quad B_n \ (n \geq 2), \quad C_n \ (n \geq 3), \quad D_n \ (n \geq 4), \quad G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8
\end{align*}$

there are two distinguished simple groups of type $\mathcal{X}$: the so-called simply connected group $G_{sc}$ and the adjoint group $G_{ad} = G_{sc}/Z(G_{sc})$. Every simple group of type $\mathcal{X}$ is an image of $G_{sc}$ modulo a finite central subgroup $T$. Such a quotient map $\pi: G \to G/T$ is called a (central) isogeny; all the groups of the same type $\mathcal{X}$ form one isogeny class.

Every simple algebraic group belongs to exactly one of the isogeny classes described above.

The proof of the uniqueness of the isogeny classes can be found in [5] Chapter XI (see Theorem' in 32.1 there). Their existence is discussed briefly in [5] 33.6 and the construction of the groups of adjoint types is given in [2].

Examples of simply connected groups are $\text{SL}_n(K)$ of type $A_{n-1}$ and $\text{Sp}_{2n}(K)$ (type $C_n$). The group $\text{SO}_n(K)$ is simple of type $B_{(n-1)/2}$ or $D_{n/2}$ (depending on whether $n$ is even or odd) but is not simply connected: its universal cover (i.e. the simply connected group in its isogeny class) is $\text{Spin}_n(K)$, the so-called spinor group.

We extend the definition of ‘simply connected’ to the semisimple groups:
Definition 2.19 A semisimple group is simply connected if it is a direct product of simply connected simple groups.

From Theorem 2.18 it now follows that each semisimple group is the image by a central isogeny of a unique simply connected semisimple group.

In general the $k$-simple algebraic groups are not so easy to describe. In the first place the radical of such a group is defined over $k$ and so it must be trivial. Therefore a $k$-simple (even a $k$-semisimple) group is also absolutely semisimple.

The next example gives a $\mathbb{Q}$-simple group which is not absolutely simple.

Example 2.20 Let $H = R_{\mathbb{Q}(i)/\mathbb{Q}}\text{SL}_2$ be the restriction of scalars of $\text{SL}_2$ (defined over $\mathbb{Q}$) from $\mathbb{Q}(i)$ to $\mathbb{Q}$. Then $H$ is $\mathbb{Q}$-simple.

Indeed, by Exercise 6 on page 23 we see that $H$ is $\mathbb{Q}(i)$-isomorphic to $\text{SL}_2 \times \text{SL}_2$ via an isomorphism $\rho$, say. Composing $\rho$ with complex conjugation $\tau$ has the effect of swapping the two factors $\text{SL}_2$. It follows that none of these two factors is $\mathbb{Q}$-defined as a subgroup of $H$. Now if $H$ had a proper $\mathbb{Q}$-defined normal subgroup $L$ then $L$ must coincide with one of the two factors $\text{SL}_2$ but they are not defined over $\mathbb{Q}$.

Contradiction, hence $H$ is $\mathbb{Q}$-simple.

Suppose now that $G$ is a $k$-simple, connected and simply connected group. This means that over $K$ our group $G$ is isomorphic to a direct product $\prod_i H_i$ of $K$-simple simply connected group $H_i$. It happens that each of $H_i$ is defined over some finite Galois extension $k_1$ of $k$ and we have that $G$ is $k$-isomorphic to the restriction of scalars $R_{k_1/k}H$ where $H = H_1$, say.

The group $H$ is $K$-simple so over $K$ it is isomorphic to one of the (simply connected) groups listed in Theorem 2.18 but we need to classify such groups up to $k_1$-isomorphism.

For example the group

$$\text{SO}_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

is isomorphic to the multiplicative group $\mathbb{C}_\times$ over $K$ but this isomorphism is not defined over the real subfield $\mathbb{R}$.

The $k_1$-isomorphism classes of groups which are $K$-isomorphic to $H$ are called the $k_1$-forms of $H$. These are classified by the non-commutative 1-cohomology set $H^1(\text{Gal}(K/k_1), \text{Aut} H_{ad})$. For example the unitary group $\text{SU}_n$ is isomorphic to $\text{SL}_n$ over $K = \mathbb{C}$ but not over $\mathbb{R}$ and these are the only two real forms of $\text{SL}_n$. Similarly the group $G$ in Exercise 7 on page 31 is a $\mathbb{Q}(i)$-form of $\text{SL}_2$. For more details we about the classification the $\mathbb{Q}$-forms of classical groups we refer to [13], Chapter 2.
2.4 Reductive groups

A class of groups which is more general than semisimple groups but which shares some of their nicer properties is the reductive groups. For example $GL_n(K)$ is not semisimple but still a very important group which we would like to include in our theory.

**Definition 2.21** An element $g$ of a linear algebraic group $G \leq GL_n(K)$ is called unipotent (resp. semisimple) if $g$ is unipotent (resp. diagonalizable) as a matrix in $GL_n(K)$. This definition is independent of the choice of the linear representation of $G$. The group $G$ is unipotent if it consists of unipotent elements.

For example $G_+$ is unipotent.

Now it can be shown that an algebraic group $G$ has a unique maximal normal unipotent subgroup. This is the unipotent radical of $G$ and is denoted $R_u(G)$. The group $G$ is called reductive if $R_u(G) = 1$. One obvious example of a reductive group is a torus.

**Definition 2.22** An algebraic group $T$ is a torus if it is isomorphic to a direct product $G_m \times \ldots$. The rank of $T$ is the number $m$ of direct factors $G_m$. The torus $T$ is called $k$-split if there is a $k$-defined isomorphism $T \rightarrow G_m^n$

The group of diagonal matrices in $GL_n(K)$ is a torus of rank $n$.

**Theorem 2.23** A connected reductive group $G$ is a product $G = TS$ of a torus $T$ and a semisimple subgroup $S$ such that $[T, S] = 1$ and $T \cap S$ is finite. The subgroups $T$ and $S$ are unique.

For example $GL_n(K)$ is reductive and equal to $Z \cdot SL_n(K)$ where the torus $Z \cong G_x$ is the group of scalar matrices.

2.5 Chevalley groups

**Definition 2.24** Let $G$ be an algebraic group defined over a field $k$. Then $G$ is called $k$-split if it has a maximal $k$-split torus.

There is a unique simple, simply connected and $\mathbb{Q}$-split algebraic group of any given Lie type $\mathcal{X}$ and this is the called the Chevalley group of type $\mathcal{X}$. There is a simple conceptual way to define the adjoint group $G = G/Z(G)$, as described for example in [2], Chapter 1: As we have seen $G$ acts faithfully on the Lie algebra $L = L(G)$ of $G$ and so can be identified with a subgroup of $Aut(L)$. In fact $G$ is defined as the subgroup of $Aut(L)$ generated by the elements

$$\exp(\text{ad}(x)) = 1 + \text{ad}(x) + \frac{\text{ad}(x)^2}{2!} + \frac{\text{ad}(x)^3}{3!} + \cdots$$

where $x$ is an element of a root subgroup of $L$. Note that for such $x$ the linear transformation $\text{ad}(x) : L \rightarrow L$ is nilpotent and so the above series is finite.
Moreover as described in [2] one can find a Lie subring $K$ of $L$ which is a finitely generated torsion free $\mathbb{Z}$-lattice of $L$ and such that $\exp(\text{ad}(x))$ stabilizes $K$ for each $x$ as above. Hence $\overline{G}$ is in fact defined over $\mathbb{Z}$ and one sees that the same is true for the universal cover $G$. Therefore its $R$-rational points $G_R$ are defined for any ring $R$. In particular $G_F$ is defined for any finite field $F$. As we shall see in section 6.1 this is the construction of the untwisted finite simple groups of Lie type.

3 Arithmetic groups and the congruence topology

In this section and below $k$ will denote a number field (a finite extension of $\mathbb{Q}$) and $\mathcal{O}$ its ring of integers. By convention, prime ideals of $\mathcal{O}$ are assumed nonzero. We begin by recalling some information about the ring $\mathcal{O}$.

3.1 Rings of algebraic integers in number fields

$\mathcal{O}$ is the collection of all elements $x \in k$ satisfying a polynomial equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with leading coefficient 1 and each $a_i \in \mathbb{Z}$. This is in fact a subring of $k$. As an additive group it is isomorphic to $\mathbb{Z}^d$, the free abelian group of rank $d$, where $d = (k : \mathbb{Q})$.

The ring $\mathcal{O}$ has Krull dimension 1, i.e. every prime ideal is maximal. Moreover, every nonzero ideal has finite index in $\mathcal{O}$. Each nonzero ideal $I$ can be factorized

$$I = p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m}$$

as a product of prime ideals $p_i$ and this factorization is unique up to reordering of the factors. The Chinese Remainder Theorem says that then

$$\mathcal{O}/I \cong \mathcal{O}/p_1^{e_1} \oplus \mathcal{O}/p_2^{e_2} \oplus \cdots \oplus \mathcal{O}/p_m^{e_m}.$$

Each prime ideal $p$ divides (i.e. contains) a unique rational prime $p \in \mathbb{N}$, and then $p \cap \mathbb{Z} = p\mathbb{Z}$. The quotient $\mathcal{O}/p$ is a finite field of characteristic $p$.

If $p\mathcal{O} = \prod_{i=1}^{g} p_i^{e_i}$ is the factorization of the principal ideal $(p) = p\mathcal{O}$ then

$$d = (k : \mathbb{Q}) = \sum_{i=1}^{g} e_i n_i, \quad \text{where } |\mathcal{O}/p_i| = p^{n_i}. \quad (2)$$

If $k$ is a Galois extension of $\mathbb{Q}$ then $e_1 = \cdots = e_g$ and $n_1 = \cdots = n_g$. Also $e_i \neq 1$ for at most finitely many rational primes $p$ (the so-called ramified ones). The Chebotarev density theorem (see [13], Chapter 1) implies that
for a positive proportion of all rational primes $p$ we have $g = (k : \mathbb{Q})$ and $n_1 = \cdots n_g = 1$, i.e. the ideal $(p)$ splits in $k$. More precisely

$$
\lim_{x \to \infty} \frac{|\{p \leq x | p \text{ a prime which splits in } k\}|}{|\{p \leq x | p \text{ prime }\}|} = \frac{1}{|\text{Gal}(k/\mathbb{Q})|} = \frac{1}{d}.
$$

Here $\text{Gal}(k/\mathbb{Q})$ is the Galois group of $k$ over $\mathbb{Q}$.

Let $S$ be a finite set of prime ideals. An element $a \in k$ is said to be $S$-integral if $Ja \subseteq \mathcal{O}$ where $J$ is some product of prime ideals in $S$. The set of all $S$-integral elements forms a subring $\mathcal{O}_S$ of $k$, containing $\mathcal{O}$, called the ring of $S$-integers of $k$. Of course, when $S$ is empty $\mathcal{O}_S = \mathcal{O}$.

### 3.2 The congruence topology on $\text{GL}_n(k)$ and $\text{GL}_n(\mathcal{O})$

The congruence topology on $k$ has as base of open neighbourhoods of 0 the set of all nonzero ideals of $\mathcal{O}$. The congruence topology on $M_n(k) = k^{n^2}$ is then the product topology, and the congruence topology on $\text{GL}_n(k)$ (and on any closed subgroup) is the one induced by that on $M_n(k)$. This means that a base of neighbourhoods of 1 is the set of subgroups $\text{GL}_d(k) \cap (1_n + M_n(I))$ with $I$ a nonzero ideal of $\mathcal{O}$.

More generally, for any set $S$ of prime ideals, we define the $S$-congruence topology by taking only ideals that are products of prime ideals not in $S$; equivalently, we can take as neighbourhood basis the set of all nonzero ideals of $\mathcal{O}_S$.

It is easy to see that the congruence topology on $k$ and hence on $M_n(k)$ is Hausdorff: if $x \neq y$ are two elements of $k$ then there is an ideal $I$ of $\mathcal{O}$ such that $x - y \not\in I$ and hence $(x + I) \cap (y + I) = \emptyset$. In fact the congruence topology on $M_n(k)$ is finer than the Zariski topology as the following proposition demonstrates.

**Proposition 3.1** Let $W$ be a $k$-defined Zariski closed set of $M_n(K) = K^{n^2}$ defined by an ideal $T$ of polynomials in its $n^2$ coordinates. Then $W_k$ is closed in the congruence topology of $M_n(k)$.

**Proof.** Let $x \in M_n(k)$ be an element of the congruence closure of $W_k$. So for any ideal $I$ of $\mathcal{O}$ we have an element $y \in W_k$ such that $x \equiv y \mod I$. Now let $p$ be a polynomial from $T$ with coefficients in $k$. We may assume that up to a scalar multiple $p$ has coefficients from $\mathcal{O}$. But then $p(x) \equiv p(y) \equiv 0 \mod I$, so $p(x) \in I$ for any ideal $I$ of $\mathcal{O}$. This is possible only if $p(x) = 0$. Since this holds for all polynomials $p$ with coefficients in $k$, and since $W$ is defined over $k$ we deduce that $x \in W$.

**Example:** Let $k = \mathbb{Q}$. Then any finite union or intersection of sets of the form

$$
\{a + m\mathbb{Z}\} \times \{b + n\mathbb{Z}\} \subset \mathbb{Q}^2 \quad a, b, m, n \in \mathbb{Z}
$$

is an open set in the congruence topology of $\mathbb{Q}^2$ but none of them is Zariski open.
It is thus clear that the congruence topology on \(M_n(k)\) has many more closed sets than the Zariski topology. So a Zariski dense subset of matrices may be rather 'sparse' in the congruence topology. From this point of view it is indeed surprising that in the case of simple algebraic groups the property of being Zariski dense has rather strong implication for the congruence closure of a subgroup. This is the main content of Theorem 3.2.1 below.

### 3.2.1 Valuations of \(k\)

For any prime ideal \(p\) of \(\mathcal{O}\) the \(p\)-adic topology is defined in the same way as the congruence topology except that the ideals are only allowed to be positive powers of \(p\). The completion of \(k\) with respect to this topology is denoted \(k_p\) and the closure of \(\mathcal{O}\) in \(k_p\) is denoted \(\mathcal{O}_p\).

The valuation \(v_p\) on \(k_p\) is defined by \(v_p(a) = t\) where \(t \in \mathbb{Z}\) is the largest integer such that \(p^{-t}a \subseteq \mathcal{O}_p\) (if \(a \neq 0\); one sets \(v_p(0) = \infty\)). Thus \(\mathcal{O}_p\) is the valuation ring, consisting of all elements of \(k\) having valuation \(\geq 0\); this implies that \(\mathcal{O}_p\) is a local ring, having \(p\mathcal{O}_p\) as its unique maximal ideal. (One often associates to such a valuation \(v_p\) the corresponding absolute value: \(|a|_p = q^{-v_p(a)}\) where \(q = |\mathcal{O}/p|\), which is multiplicative.)

**Example 3.2 (The \(p\)-adic numbers)** Take \(k = \mathbb{Q}\) with ring of integers \(\mathbb{Z}\). Let \(p\) be a prime. The \(p\)-adic valuation \(v_p(x)\) is the usual one where \(v_p(x) = t\) is the largest integer such that \(x = p^t a/b\) with integers \(a\) and \(b\) coprime to \(p\). A base for neighbourhoods of 0 in the \(p\)-adic topology on \(\mathbb{Q}\) is the family of subgroups \(\{ \mathbb{Z}^l_p a/b \mid a, b \in \mathbb{Z}, (p, b) = 1\}, l = 1, 2, \ldots\). The completion of \(\mathbb{Q}\) with respect to this topology is the field \(\mathbb{Q}_p\) of \(p\)-adic numbers. Inside \(\mathbb{Q}_p\) we have the closure \(\mathbb{Z}_p\) of \(\mathbb{Z}\), which is the ring of \(p\)-adic integers. We can view \(\mathbb{Z}_p\) as a ring of infinite power series in \(p\):

\[
a_0 + a_1p + \cdots + a_k p^k + \cdots, \quad a_i \in \{0, 1, \ldots, p-1\}
\]

with the obvious addition and multiplication. The finite such sums comprise the subring \(\mathbb{Z}\). The unique maximal ideal is \(p\mathbb{Z}_p\) and every element \(x \in \mathbb{Q}_p\) can be written uniquely as \(x = p^t y\) for some \(y \in \mathbb{Z}_p \setminus p\mathbb{Z}_p\) and \(t \in \mathbb{Z}\).

In general if \(p_i\) is a prime ideal of \(\mathcal{O}\) dividing \(p\) then \(k_{p_i}\) is a vector space over \(\mathbb{Q}_{p_i}\) of dimension \(e_in_i\), the number in (2) above.

The valuations \(v_p\) for all prime ideals \(p\) of \(\mathcal{O}\) comprise the non-archimedean valuations of \(k\). Now, suppose that the number field \(k\) has \(s\) embeddings \(v_i : k \to \mathbb{R}(i = 1, \ldots, s)\) and \(2t\) non-real embeddings \(v_j, \overline{v}_j : k \to \mathbb{C}(j = s+1, \ldots, s+t)\). Composing these with the ordinary real or complex absolute value gives the set \(V_\infty\) of \(s + t\) archimedean absolute values on \(k\). For \(v \in V_\infty\) we put \(k_v = \mathbb{R}\) or \(k_v = \mathbb{C}\) according as the corresponding embedding of \(k\) is real or non-real.

The ring of \(S\)-integers has a more natural definition in terms of valuations:

\[
\mathcal{O}_S = k \cap \bigcap_{p \notin S} \mathcal{O}_p.
\]
A word of warning: the notation $O_S$ can be a bit confusing: if $S = \{q\}$ consists of a single prime then $O_{\{q\}}$ is the ring of $\{q\}$-integers, while $O_q$ is the completion of $O$ at $q$. For example $\mathbb{Z}(p) = \mathbb{Z}[1/p] \subset \mathbb{Q}$ while $\mathbb{Z}_p$ is the ring of $p$-adic integers.

3.3 Arithmetic groups

Suppose we are given a linear algebraic group $G$ defined over $k$ with a faithful representation $G \hookrightarrow \text{GL}_n(K)$, also defined over $k$.

Definition 3.3 A subgroup $\Gamma$ of $G_k$ is called arithmetic if it is commensurable with the group of $O$-integral points $G_O$ (in other words $\Gamma \cap G_O$ has finite index in both $\Gamma$ and $G_O$).

It turns out that this definition is independent of the choice of $k$-defined linear representation of $G$.

More generally we can define the $S$-arithmetic subgroups of $G(k)$ as those commensurable with $G_{O_S}$. When the set $S$ has not been specified we shall always assume that it is empty.

The simplest examples of arithmetic groups are $(O,+)$ and $(O^*,\times)$ the additive and multiplicative groups of the ring of integers of $k$. We thus see that the study of arithmetic groups is a generalization of classical algebraic number theory.

One of the most general results about arithmetic groups is the following

Theorem 3.4 ([13], chapter 4) Let $\Gamma$ be an arithmetic subgroup of a $k$-defined linear algebraic group $G$ as above. Then $\Gamma$ is finitely presented and has only finitely many conjugacy classes of finite subgroups.

For $S$-arithmetic groups the above statement is still true, provided that $G$ is reductive.

Now an $(S)$-arithmetic group $\Gamma$ has its own $(S)$-congruence topology induced from the $(S)$-congruence topology of $\text{GL}_n(k)$. We call a subgroup $\Delta \leq \Gamma$ an $(S)$-congruence subgroup if is is open in this topology, i.e. if $\Delta$ contains a principal congruence subgroup $\Gamma \cap (1_n + M_n(I))$ for some nonzero ideal $I$ of $(\text{coprime to } S)$. The congruence images $\Gamma/N$ of $\Gamma$ are those with kernel a congruence subgroup $N \lhd \Gamma$.

Clearly a congruence subgroup of $\Gamma$ has finite index, but the converse is not true in general. When it does hold, that is if every subgroup of finite index is a congruence subgroup, $\Gamma$ is said to have the congruence subgroup property (CSP).

There is a neat way to state CSP in term of profinite groups. If $X$ is an intersection-closed family of normal subgroups of finite index in $\Gamma$, one defines
the $\mathcal{X}$-completion of $\Gamma$ to be the inverse limit

$$\widehat{\Gamma}_X = \lim_{\leftarrow N \in \mathcal{X}} \Gamma \slash N$$

$$= \{(\gamma_N)_{N \in \mathcal{X}} \mid p_{NM}(\gamma_N) = \gamma_M \ \forall N \leq M \in \mathcal{X}\} \leq \prod_{N \in \mathcal{X}} \Gamma \slash N,$$

where $p_{NM} : \Gamma \slash N \to \Gamma \slash M$ denotes the natural quotient map for each $N \leq M$. (With the topology induced from the product topology on the Cartesian product, $\widehat{\Gamma}_X$ becomes a compact topological group, a profinite group).

A natural example of inverse limits is the valuation ring $O_p$ of $\mathcal{O}$ for a prime ideal $p$ of $\mathcal{O}$. The inverse limit

$$\lim_{\leftarrow n \in \mathbb{N}} O \slash p^n O$$

is isomorphic as a ring to the completion $O_p$ of $\mathcal{O}$ with respect to the $p$-adic topology defined by the powers of the ideal $p$. This also shows that $O_p \slash p^n O_p$ is isomorphic to $O \slash p^n O$.

We are interested in two special choices for $\mathcal{X}$. When $\mathcal{X}$ consists of all normal subgroups of finite index, $\widehat{\Gamma}_X = \widehat{\Gamma}$ is the profinite completion of $\Gamma$. When $\mathcal{X}$ consists of all the normal congruence subgroups, $\widehat{\Gamma}_X = \widehat{\Gamma}$ is the congruence completion of $\Gamma$. There is an obvious natural projection $\pi : \widehat{\Gamma} \to \widehat{\Gamma}$, which is clearly surjective.

Now we can reformulate the congruence subgroup property as saying that the map $\pi$ is bijective. For many purposes the following generalization of CSP is more relevant: the arithmetic group $\Gamma$ is said to have the generalized congruence subgroup property (GCSP for short) if the kernel of $\pi : \widehat{\Gamma} \to \widehat{\Gamma}$ is finite. Group theoretically this says that any subgroup of finite index in $\Gamma$ is commensurable ‘with bounded index’ with a congruence subgroup. There is a famous conjecture by Serre which characterizes the $S$-arithmetic groups (in semisimple algebraic groups) with GCSP as those having $S$-rank at least 2: see section 4.1.

### 4 The Strong Approximation Theorem

The congruence images of the $S$-arithmetic group $\Gamma = G_{\mathcal{O}_S}$ are easier to understand when $G$ has the strong approximation property. In order to explain this we need several more definitions.

Recall that $k_p$ and $O_p$ are the completions of $k$ and $\mathcal{O}$ with respect to the $p$-adic topology defined by powers of the prime ideal $p \triangleleft O$. As usual we set $G_{k_p} = G \cap M_n(k_p)$ and $G_{\mathcal{O}_p} = G \cap M_n(O_p)$. The first of these is a locally compact totally disconnected topological group and the second is a compact subgroup. In fact $G_{\mathcal{O}_p}$ is an example of a $p$-adic analytic group. We refer to $G_{k_p}$ as the completion of $G$ at $p$.

Similarly, if $v$ is an archimedean real (resp. complex) absolute value of $k$ associated to an embedding $\nu_i$, then we write $G_v$ for $G^{\nu_i}_{\mathbb{R}}$ (resp. $G^{\nu_i}_{\mathbb{C}}$) where
$G^\nu \leq \text{GL}_n(\mathbb{C})$ is the group obtained by applying $\nu$ to the defining equations of the affine variety $G$.

The profinite groups $G_{O_S}$ are in close relationship with the congruence images of $G_{O_S}$:

Recall that the algebraic group $G$ is $k$-defined. It is easy to see that for almost all prime ideals $\mathfrak{p}$ the coefficients of the equations defining $G$ in $\text{GL}_n$ are not divisible by $\mathfrak{p}$. Therefore we can consider these equations modulo $\mathfrak{p}^n$ for any $n \in \mathbb{N}$. Denote the set of their solutions in $O/\mathfrak{p}^n$ by $G_{O/\mathfrak{p}^n}$: this is a finite subgroup of $\text{GL}_n(O/\mathfrak{p}^n)$ and is called the reduction of $G$ modulo $\mathfrak{p}^n$. See [13] p. 142-146 for more details about reductions of affine algebraic varieties and groups.

Now consider the quotient mapping

$$O_p \to O_p/\mathfrak{p}^n O_p \simeq O/\mathfrak{p}^n O.$$

This induces a homomorphism

$$\pi_{\mathfrak{p}^n} : G_{O_p} \to G_{O/\mathfrak{p}^n O}.$$

The following is the content of Proposition 3.20 of [13].

**Proposition 4.1** The maps $\pi_{\mathfrak{p}^n}$ are surjective for all but finitely many primes $\mathfrak{p}$ (and all integers $n$).

We say that $G$ has *good reduction* for such primes $\mathfrak{p}$.

Assume from now on that $\mathfrak{p}$ is not in the finite set $S$. The restriction of $\pi_{\mathfrak{p}^n}$ to its dense subgroup $G_{O_S} \leq G_{O_p}$ is the homomorphism

$$G_{O_S} \to G_{O_S}/\mathfrak{p}^n O_S$$

obtained by reducing all entries of $\Gamma = G_{O_S} \leq \text{GL}_n(O_S)$ modulo $\mathfrak{p}^n$. So the images of $\pi_{\mathfrak{p}^n}$ are all congruence images of $\Gamma$. What is not clear at this point is how to combine these to describe the congruence images of $\Gamma$ at composite ideals. This is the content of the strong approximation theorem below.

Define

$$G_S := \prod_{v \in V_\infty} G_v \times \prod_{\mathfrak{p} \in S} G_{k_p}.$$  

This is a locally compact group and the image of $\Gamma$ in $G_S$ under the diagonal embedding in each factor is a *lattice* in $G_S$, i.e., a discrete subgroup of finite co-volume. As a consequence the arithmetic subgroup $\Gamma = G_{O_S}$ is infinite if and only if the group $G_S$ is non-compact.

Let

$$G_{\hat{O}_S} = \prod_{\mathfrak{p} \not\in S} G_{O_p}.$$  

Again there is an obvious diagonal embedding $i : \Gamma \to G_{\hat{O}_S}$ and the congruence topology of $\Gamma$ coincides with the topology induced in $i(\Gamma)$ as a subgroup
of the profinite group \( G_{\mathcal{O}_S} \). Hence the congruence completion \( \tilde{\Gamma} \) is isomorphic to the closure \( \overline{i(G)} \) of \( i(G) \) in \( G_{\mathcal{O}_S} \). The strong approximation theorem states that under certain conditions \( i(G) \) is dense in \( G_{\mathcal{O}_S} \), and therefore \( \tilde{\Gamma} \simeq G_{\mathcal{O}_S} \).

**Theorem 4.2 (Strong approximation for arithmetic groups)** ([13], Theorem 7.12). Let \( G \) be a connected simple simply connected algebraic group defined over a number field \( k \) and let the groups \( \Gamma = G_{\mathcal{O}_S}, GS, Gb \mathcal{O}_S \) and the embedding \( i : \Gamma \rightarrow G_{\mathcal{O}_S} \) be as above. Assume that \( \Gamma \) is infinite (which is equivalent to \( GS \) being non-compact). Then \( i(\Gamma) \) is dense in \( G_{\mathcal{O}_S} \) and hence \( \tilde{\Gamma} \simeq G_{\mathcal{O}_S} \).

When the conclusion holds we say that \( G_{\mathcal{O}_S} \) has the **strong approximation property**, or that \( G \) has the strong approximation property w.r.t. \( S \).

Note: Usually the strong approximation theorem is formulated for the group of \( k \)-rational points \( G_k \) and says that \( G_k \) is dense in the adelic group \( G_A \), the statement we have given above is equivalent to this (and more transparent for arithmetic groups); see [13], Chapter 7.

More generally, a connected algebraic group \( G \) has the strong approximation property if its maximal reductive quotient \( H = G/R_u(G) \) is a direct product of simple simply connected groups, and \( HS \) is non-compact.

The strong approximation theorem can be viewed as a generalization of the Chinese remainder theorem, which in this setting says that the diagonally embedded image of \( \mathbb{Z} \) is dense in \( \prod_{p \text{ prime}} \mathbb{Z}_p \). In the general situation the theorem says that the finite images of the product \( G_{\mathcal{O}_S} \) coincide with the congruence images of \( \Gamma \).

Note: The condition that \( G \) be simply connected is indeed necessary (Exercise 10).

Set \( F_{q(p)} \) where \( q(p) = |\mathcal{O}/p| \). Then theorem 4.2 and Proposition 4.1 give

**Corollary 4.3** Under the hypotheses of Theorem 4.2 we have \( \pi_p(\Gamma) = G_{F_{q(p)}} \) for all but finitely many primes \( p \notin S \).

In turn the groups \( G_{F_{q(p)}} \) are easy to describe when \( G \) is semisimple, see Proposition 6.1 below.

For the moment we shall note case the relationship between \( G \) and \( H \) when \( G = R_{k/Q} H \) is a restiction of scalars of \( H \).

**Proposition 4.4** Let \( k \) be a finite Galois extension of \( \mathbb{Q} \) and \( G = R_{k/Q} H \) be the restriction of scalars of some \( \mathbb{Q} \)-defined algebraic group \( H \). Then for almost all primes \( p \)

\[
G_{F_p} = \prod_{p \mid q} H_{F_{q(p)}},
\]

where the product on the right is over all prime ideals \( p \) of \( k \) dividing \( p \).
Indeed all but finitely many primes $p$ are unramified in $k$ and therefore

$$\mathcal{O}/p\mathcal{O} = \prod_{p|\mathcal{O}} \mathcal{O}/p = \prod_{p|\mathcal{O}} \mathbb{F}_q(p),$$

where $\mathcal{O}$ is the ring of integers of $k$. The proposition follows immediately from $G_\mathbb{Z} = H_\mathcal{O}$ and Proposition 4.1.

### 4.1 An Aside: Serre’s Conjecture

We now have most of the definitions to state Serre’s conjecture.

**Definition 4.5** For a valuation $v$ of $k$ the $k_v$-rank of the topological group $G_v$ is the largest integer $n$ such that $G_v$ contains the direct product $(k_v^*)^n$. The $S$-rank of an algebraic group $G$ is

$$\sum_{v \in V_\infty \cup S} k_v\text{-rank of } G_v$$

where $V_\infty$ is the set of all archimedean valuations of $k$.

**Conjecture 4.6** (J-P. Serre) A connected simply connected simple algebraic group $G$ has the generalized $S$-congruence subgroup property if and only if the $S$-rank of $G$ is at least 2.

For example the group $\text{SL}_n(\mathbb{Z})$ has CSP if $n > 2$ but not if $n = 2$.

Currently Serre’s conjecture is open for some groups of $S$-rank 1 and also when $G$ is a totally anisotropic form of $A_n$, see [13], §9.5.

### 5 The Nori-Weisfeiler theorem and Lubotzky’s alternative

It will be too much to expect that the Strong Approximation Theorem holds for linear groups in general, indeed it doesn’t hold for algebraic tori. Nevertheless there is something that can be said when the group is non-solvable.

**Theorem 5.1** (Nori [12], Weisfeiler [13]) Let $\Delta$ be a Zariski-dense subgroup of a $\mathbb{Q}$-simple simply connected linear algebraic group $G \leq \text{GL}_n(\mathbb{C})$ and suppose that $\Delta \leq G_{\mathbb{Z}}$ for some finite set of primes $S$. Let $i : \Delta \to G_{\mathbb{Z}}$ be the diagonal embedding.

Then the closure $\overline{i(\Delta)}$ of $i(\Delta)$ in $G_{\mathbb{Z}}$ is an open subgroup of $G_{\mathbb{Z}}$.

It follows that for all but finitely many primes $p$, all the groups $G_{\mathbb{Z}}/(p^m \mathbb{Z})$ appear as congruence images of $\Delta$.

There are several different proofs of this theorem. We shall sketch one of them in section 7. For the moment, let us assume this result and deduce Theorem 1.1. We restate it here:
Theorem 1 Let $\Delta \leq \text{GL}_n(k)$ be a finitely generated linear group over a field $k$ of characteristic 0. Then one of the following holds:

(a) the group $\Delta$ is virtually soluble, or

(b) there exist a connected simply connected $\mathbb{Q}$-simple algebraic group $G$, a finite set of primes $S$ such that $\Gamma = G_{2S}$ is infinite, and a subgroup $\Delta_1$ of finite index in $\Delta$ such that every congruence image of $\Gamma$ appears as a quotient of $\Delta_1$.

Proof of Theorem 1: Suppose that we have a finitely generated linear group $\Delta \leq \text{GL}_n(\mathbb{C})$. Then in fact $\Delta \leq \text{GL}_n(J)$ for some finitely generated subring $J$ of $\mathbb{C}$.

Now the Jacobson radical (the intersection of the maximal ideals of $J$) is trivial and so $J$ is residually a number field. Indeed if $m$ is a maximal ideal of $J$ then $J/m$ is a finitely generated algebra which is a field. By Corollary 7.10 in [1] (The weak Nullstellensatz), $J/m$ is a finite extension of $\mathbb{Q}$, i.e. a number field.

Hence $\Delta$ is residually in $\text{GL}_n(k_i)$ for some number fields $k_i$. Suppose that $\Delta$ is not virtually soluble. By Lemma 1.3 it follows that there is $i \in I$ such that the image of $\Delta$ in $\text{GL}_n(k_i)$ is not virtually soluble. Replacing $\Delta$ with this image we may assume that $\Delta \leq \text{GL}_n(k)$ for some number field $k$.

Consider $\text{GL}_n(k)$ as a subgroup of $\text{GL}_n(d)$ where $d = (k : \mathbb{Q})$. Let $G$ be the Zariski-closure of $\Delta$ in $\text{GL}_n(d)$. This is a $\mathbb{Q}$-defined linear algebraic group and we take its connected component $G_0$ at the identity.

Let $\Delta_1 = G_0 \cap \Delta$. This has finite index in $\Delta$ and is Zariski-dense in $G_0$. Since $\Delta$ is not virtually soluble the connected algebraic group $G_0$ is not soluble. By Exercise 12 we see that there exists a $\mathbb{Q}$-simple connected algebraic group $G$ and a $\mathbb{Q}$-defined epimorphism $f : G_0 \to G$. Now $f(\Delta_1)$ is dense in $G$ and we may replace $\Delta$ by $f(\Delta_1)$ and $G_0$ by $G$ to reduce the situation to where we have a finitely generated Zariski-dense subgroup $\Delta \leq G_{\mathbb{Q}}$ of a $\mathbb{Q}$-simple connected linear algebraic group $G$. The main difference with the setup of Theorem 5.1 is that $G$ may not be simply connected. However $G$ is isogenous to its simply connected cover $\tilde{G}$, i.e., there is a $\mathbb{Q}$-defined surjection $\pi : \tilde{G} \to G$, where $\ker \pi = Z$ is a finite central subgroup of $\tilde{G}$.

It is not in general true that $\pi(\tilde{G}_\mathbb{Q}) = G_\mathbb{Q}$ but at least we have the following

Proposition 5.2 The group $G_{\mathbb{Q}}/\pi(\tilde{G}_\mathbb{Q})$ is abelian of finite exponent dividing $|Z|$.

Proof: Let $A$ be the Galois group of $K/\mathbb{Q}$ where $K$ is the algebraic closure of $\mathbb{Q}$. Then $\tilde{G}_\mathbb{Q}$ consists of all those $g \in \tilde{G}_K$ such that $g^\alpha = g$ for all $\alpha \in A$. On the other hand $\pi^{-1}(G_{\mathbb{Q}})$ consists of those $g \in \tilde{G}_K$ such that $g^\alpha = g \mod Z$ for all $\alpha \in A$. Suppose that $g, h \in \pi^{-1}(G_{\mathbb{Q}})$, thus $g^\alpha = g$ and $h^\alpha = h \mod Z$ for all $\alpha \in A$. Now using that $Z$ is central in $\tilde{G}$ we see that $[g, h]^\alpha = [g^\alpha, h^\alpha] = [g, h]$ and hence that $[g, h] \in \tilde{G}_{\mathbb{Q}}$. Let $m = \exp Z$. In the same way also we see that if $g^m \equiv g \mod Z$ then $(g^m)^\alpha = g^m$ and therefore $g^m \in \tilde{G}_\mathbb{Q}$. So $\pi^{-1}(G_{\mathbb{Q}})/\tilde{G}_\mathbb{Q}$ is abelian of exponent dividing $|Z|$ and this implies the Proposition. $\square$
Now take $\Delta_0 = \Delta \cap \pi(\tilde{G}_Q)$, this is a subgroup of finite index in $\Delta$ because $\Delta/\Delta_0$ is a finitely generated abelian group of finite exponent. Let $U_0 = \pi^{-1}(\Delta_0) \cap \tilde{G}_Q$, then $U_0/(U_0 \cap Z) \simeq \Delta_0$; $U_0$ is a finitely generated linear group it is residually finite. So we can find a subgroup $U$ of finite index in $U_0$ such that $U \cap Z = \{1\}$. Then $U$ is isomorphic to $\pi(U)$ which is a subgroup of finite index in $\Delta_0$ and hence in $\Delta$.

Now take $\Delta_1 = \pi(U) \simeq U$. Observe that $U$ is Zariski dense in the $\mathbb{Q}$-simple, connected and simply connected algebraic group $\tilde{G}$. In addition $U$ is finitely generated and inside $\tilde{G}_Q$. It follows that there is a finite set $S$ of rational primes such that $U \leq \tilde{G}^S$. All the conditions of Theorem 5.1 are now satisfied with $U$ and $\tilde{G}$ in place of $\Delta$ and $G$. Hence we deduce that the congruence completion of $U$ is an open subgroup of

$$G_S = \prod_{p \notin S} G_{\mathbb{Z}_p}.$$  

This open subgroup projects onto all but finitely many of the factors in the product $G_S$. So by enlarging $S$ to some finite set $S_1$ we may ensure that the congruence completion of $U$ maps onto $\prod_{p \notin S_1} G_{\mathbb{Z}_p}$. Since $U$ is isomorphic to $\Delta_1$ Theorem 1.1 follows.

6 Some applications to Lubotzky’s alternative

As noted in the introduction, Theorem 1.1 puts a substantial restriction on the finite images of a linear group in characteristic 0. First we need to introduce

6.1 The finite simple groups of Lie type.

For a detailed account of the material of this section we refer to Carter’s book [2].

The untwisted simple groups of Lie type are the groups $L = G_{\mathbb{F}_q}/Z$ where $G$ is a simply connected Chevalley group defined over $\mathbb{Z}$ and $Z$ is the centre of the group of rational points $G_{\mathbb{F}_q}$ over the finite field $\mathbb{F}_q$. The type of $L$ is just the Lie type $X$ of $G$.

The twisted simple groups arise as the fixed points $L^\sigma$ of a specific automorphism $\sigma$ (of order 2 or 3) of some untwisted simple group $L$. Such twisted Lie type simple groups are for example $\text{PSU}_n(q)$. The (untwisted) type of $L^\sigma$ is just the Lie type of $L$. For example the untwisted Lie type of $\text{PSU}_n(q)$ is $A_{n-1}$.

A finite group $L$ is quasisimple if $L = [L, L]$ and $L/Z(L)$ is simple. Similarly to the isogenies described in Theorem 2.18, the quasisimple finite groups break up into families with the same simple quotient. The members of each family have the same simple quotient, say $S$ and there is a largest member of the family $L$, called the universal cover of $S$. All the other members of the family are the quotients $L/A$ where $A \leq Z(S)$. The type (twisted or not) of a quasisimple group is the same as that of its simple quotient.
6.2 Refinements

Let us return to Corollary 4.3. Recall that $G$ was a simple simply connected linear algebraic group defined over an algebraic number field $k$ and $\Gamma = G_{\mathcal{O}_S}$ for a ring of algebraic $S$-integers $\mathcal{O}_S$ of $k$. The group $\Gamma$ then maps onto $G_{\mathbb{F}_q(p)}$ for almost all $p \not\in S$.

**Proposition 6.1** Assume in the above situation that $G$ is absolutely simple. Then for almost all prime ideals $p$ outside $S$ the reduction $G_{\mathbb{F}_q(p)}$ of $G$ modulo $p$ is a quasisimple finite group.

Now from the description of the $k$-forms of $G$ it follows that $G$ splits over $\mathbb{F}_{q(p)}$ if and only if some specific polynomials in $k[x]$ (depending only on $G$) splits completely in linear factors in the finite field $\mathbb{F}_{q(p)}$. The Chebotarev density Theorem now implies that $G_{\mathbb{F}_q(p)}$ is an untwisted quasisimple group for a positive proportion of the primes $p$ of $k$.

Next consider the situation of Theorem 1.1. There we have a $\mathbb{Q}$-simple algebraic group $G$ such that all congruence images of $G_{\mathbb{Z}_S}$ occur as quotients of $\Delta_1$. Now $G$ may not be absolutely simple but in any case there is a finite Galois extension $k$ of $\mathbb{Q}$ and an absolutely simple $k$-defined group $H$ such that $G = R_{k/\mathbb{Q}}H$. Proposition 4.4 gives that for almost all rational primes $p$ outside $S$

$$G_{\mathbb{F}_p} = \prod_{p | p} H_{\mathbb{F}_q(p)}.$$ 

where as before the product on the right is over all prime ideals $p$ of $k$ dividing the (unramified) prime $p$. Note that the degree of $\mathbb{F}_{q(p)}$ over $\mathbb{F}_p$ is bounded by $(k : \mathbb{Q})$

Therefore Theorem 1.1 in combination with Corollary 4.3 gives

**Corollary 6.2** Suppose that $\Gamma < \text{GL}_n(K)$ is a finitely generated linear group in characteristic 0 which is not virtually soluble. Then there is

- a positive integer $d$,
- a Lie type $X$,
- for every prime $p$ a finite field $\mathbb{F}_{p^f}$ of degree $f \leq d$ over $\mathbb{F}_p$ and a finite simple group $L(p^f)$ of Lie type over $\mathbb{F}_{p^f}$ whose untwisted type is $X$ (e.g. if $X = A_{n-1}$ then $L(p^f)$ is either $\text{PSL}_n(p^f)$ or $\text{PSU}_n(p^f))$, and
- a subgroup of finite index $\Gamma_0$ in $\Gamma$,

such that $\Gamma_0$ maps onto $L(p^f)$ for almost all primes $p$. Moreover, for a positive proportion of these primes one has $f = 1$ and the group $L(p)$ is untwisted.

One consequence of this is that $\Gamma$ cannot have polynomial subgroup growth because the Cartesian product $\prod_{p \text{ prime}} L(p)$ doesn’t have polynomial subgroup growth, see 14 Chapter 5.2 for details.
The untwisted type $\mathcal{X}$ of the simple groups $L(p)$ is not completely arbitrary: Let $G$ be the simple algebraic group of type $\mathcal{X}$ as stated in Theorem 2.18. Then $G$ is an image of the connected component of the Zariski closure of $\Gamma$ in $G_n(K)$.

There is one particular case when the group $G$ is explicitly determined: when $\Gamma$ is a subgroup of $\text{GL}_2(\mathbb{C})$. Then the dimension of $G$ is at most 4. On the other hand, from the classification in Theorem 2.18 it follows that the only simple algebraic group of dimension less than 8 is $\text{SL}_2$. Therefore we obtain the following

**Proposition 6.3** A finitely generated subgroup $\Gamma$ of $\text{GL}_2(\mathbb{C})$ which is not virtually soluble has a subgroup of finite index $\Gamma_0$ which maps onto $\text{PSL}_2(p)$ for infinitely many, in fact for a positive proportion of all primes $p$.

This result is used in [8] where the authors prove that any lattice $\Lambda$ in $\text{PSL}_2(\mathbb{C})$ has a collection $\{N_i\}_i$ of subgroups of finite index such that $\bigcap_i N_i = \{1\}$ and $\Lambda$ has property $\tau$ with respect to $\{N_i\}_i$. As a corollary, the authors obtain that any hyperbolic 3-manifold has a co-final sequence of finite covers with positive infimal Heegaard gradient.

### 6.3 Normal subgroups of linear groups

A normal subgroup $N$ of a finitely generated group does not need to be finitely generated. So it comes as no surprise that when this happens in linear groups we can put further restriction on the finite images of $N$.

**Proposition 6.4** Let $\Gamma$ be a finitely generated linear group with a finitely generated normal subgroup $\Delta$. Assume that $\Delta$ is not virtually soluble. Then there exist a number $C > 0$ and a Lie type $\mathcal{X}$ with the following property: For infinitely many primes $p \in \mathbb{N}$ the group $\Delta$ has a normal $\Gamma$-invariant subgroup $N$ with $\Delta/N$ isomorphic to a direct product of at most $k$ copies of the untwisted finite simple group $L(p)$ over $\mathbb{F}_p$.

**Sketch of Proof:** Using similar arguments to those in the proof of Theorem 5.1, we can reduce to the case when $\Gamma \leq \text{GL}_d(\mathbb{Q})$ for some integer $d$ and $\Delta$ is Zariski dense in some absolutely semisimple simply connected algebraic group $G \leq \text{GL}_d$ defined over $\mathbb{Q}$ with isomorphic simple factors. Moreover, we have $\Gamma \leq \text{GL}_d(\mathbb{Z}_S)$ for some finite set of rational primes $S$.

Let $t$ be the number of simple factors of $G$.

As before for a rational prime $p \not\in S$ let $\pi_p$ be the homomorphism $\text{GL}_d(\mathbb{Z}_S) \to \text{GL}_d(\mathbb{F}_p)$ obtained by reducing $\mathbb{Z}_S$ mod $p$.

From Theorem 5.1, we deduce that for all but finitely many primes $p$ outside $S$ one has $\pi_p(\Delta) = G_{F_p} = \pi_p(G_{Z_S})$. Let $M_p = \ker \pi_p$ and $N_p = \Delta \cap M_p$. Then $\Delta/N_p \cong G_{F_p}$ is a central product of at most $t$ quasisimple groups of the same Lie type as the factors of $G$. Also for infinitely many primes $p$ these factors are untwisted quasisimple groups.

Now the only thing remaining is to observe that $N_p$ is normal in $\text{GL}_d(\mathbb{Z}_S)$ and therefore $N_p = \Delta \cap M_p$ is invariant under $\Gamma$. Hence $G_{F_p}/Z(G_{F_p})$ is the required $\Gamma$-invariant quotient of $\Delta$. 

26
As suggested by Lubotzky Proposition 6.4 may be relevant in the following open problem:

**Conjecture 6.5** Let \( n > 2 \) and consider \( \text{Aut}(F_n) \), the automorphism of the free group on \( n \) free generators. If \( \rho \) is a complex linear representation of \( \text{Aut}(F_n) \) then \( \rho(\text{Inn}(F_n)) \) is virtually soluble, where \( \text{Inn}(F_n) \) is the subgroup of inner automorphism of \( F_n \).

7 **Theorem 5.1**

Our sketch of the proof of Theorem 5.1 follows the argument in [10], Window 9.

Suppose that \( \Gamma \leq G_{\mathbb{Z}_p} \) is Zariski dense in the simply connected \( \mathbb{Q} \)-simple algebraic group \( G \). Now \( G \) may not be absolutely simple, but in any case there is a number field \( k \) and an absolutely simple group \( H \) defined over \( k \) such that \( G = \mathcal{R}_{k/\mathbb{Q}}(H) \). We have \( G_{\mathbb{Q}} = H_k \), and for each prime \( p \)

\[
G_{\mathbb{Z}_p} = \prod_j H_{\mathcal{O}_p/j}
\]

where \( p\mathcal{O} = \prod_j p_j^e_j \) is the factorization of the principal ideal \( (p) \) in \( \mathcal{O} \). This means that \( k \otimes \mathbb{Q}_p = \prod_j k_{p_j} \).

From now on assume that the prime \( p \) is unramified in \( k \), i.e. all \( e_j = 1 \). In addition assume that \( G \) has good reduction mod \( p \). This holds for all but finitely many rational primes \( p \) (see Proposition 4.1).

Since \( L(G) \) is \( \mathbb{Q} \)-defined we have that \( L(G)_{\mathbb{Q}_p} = L(G) \otimes \mathbb{Q}_p \). Therefore \( L(G)_{\mathbb{Q}_p} = \prod_j L(H)_{k_{p_j}} \). Similarly since \( p \) is unramified

\[
L(G)_{\mathbb{F}_p} = \prod_j L(H)_{\mathcal{O}_p/j} \quad \text{and} \quad (3)
\]

\[
G_{\mathbb{F}_p} = \prod_j H_{\mathcal{O}_p/j}.
\]

The group \( H \) is absolutely simple so for almost all primes \( p \) the Lie algebras \( L(H)_{\mathcal{O}_p/j} \) are simple and the groups \( H_{\mathcal{O}_p/j} \) are quasisimple.

**Step 1:** Let \( D_p \) be the closure of \( \Delta \) in the \( p \)-adic analytic group \( G_{\mathbb{Q}_p} \). Since \( \Delta \) is Zariski-dense in \( G \) then by Proposition 2.12 the Lie algebra of \( D \) is an ideal of the Lie algebra \( L(G)_{\mathbb{Q}_p} \) of \( G_{\mathbb{Q}_p} \). But \( \Delta \leq G_{\mathbb{Q}} \), so the Lie algebra \( L(D_p) \) is defined over \( \mathbb{Q} \). Hence the projections of \( L(D_p) \) in each of the factors \( L(H)_{k_{p_j}} \) of \( L(G)_{\mathbb{Q}_p} \) are isomorphic. So for almost all primes \( p \) we have \( L(D_p) = L(G)_{\mathbb{Q}_p} \) which means that \( D_p \) is an open subgroup of \( G_{\mathbb{Q}_p} \) for almost every \( p \) (see Proposition 2.13). In fact, since we are assuming \( p \notin S \), we have \( \Delta \subset G_{\mathbb{Z}_p} \), and so \( D_p \) is an open subgroup of the compact open subgroup \( G_{\mathbb{Z}_p} \).
Next we want to prove that for almost all primes $p$ our group $\Delta$ is dense in $G_{\mathbb{Z}_p}$.

**Step 2:** For almost all primes the Frattini subgroup of $G_{\mathbb{F}_p}$ is contained in the kernel of $G_{\mathbb{Z}_p} \rightarrow G_{\mathbb{F}_p}$. In follows that a subgroup $\Delta$ is dense in $G_{\mathbb{Z}_p}$ if and only if $\Delta$ maps onto $G_{\mathbb{F}_p}$. This is proved in [10], Window 9, Proposition 7 using the structure of the finite images of the $p$-adic analytic group $G_{\mathbb{Z}_p}$.

**Step 3:** We shall prove that $D_p = G_{\mathbb{Z}_p}$ for almost all primes $p$. By Step 2 it is enough to show that $\Delta$ maps onto $G_{\mathbb{F}_p}$ for almost all primes $p$.

Let $\pi_p$ be the projection of $G_{\mathbb{Z}_p}$ onto $G_{\mathbb{F}_p}$ and further let $\pi_{j,p}$ and $\tau_{j,p}$ be the projections of $G_{\mathbb{Z}_p}$ and $L(G)_{\mathbb{Z}_p}$ onto their direct factors $H_{\mathcal{O}/p}$ and $L(H)_{\mathcal{O}/p}$, respectively.

At this stage we need the following

**Proposition 7.1** Let $\Gamma$ be a subgroup of $G_{\mathbb{F}_p}$ such that

(a) For all $j$ the image $\pi_j(X)$ of $\Gamma$ in $H_{\mathcal{O}_{\mathbb{F}_p}}$ has order divisible by $p$, and

(b) Every subspace of $L(G)_{\mathbb{F}_p}$ invariant under $\Gamma$ is an ideal.

Then provided $p$ is sufficiently large compared to $\dim G$ we have $\Gamma = G_{\mathbb{F}_p}$.

Let us check that the conditions (a) and (b) above are satisfied for the group $\pi_p(\Delta) \leq G_{\mathbb{F}_p}$, for almost all primes $p$.

Suppose that (a) fails for a set $A$ of infinitely many primes. Then there is $j = j_p$ such that $\pi_{j,p}(\Delta)$ has order coprime to $p$ and so is a completely reducible subgroup of $\text{GL}_n(\mathbb{F}_p)$, where $n$ depends only on $G$ and not on $p$. A variation of Jordan’s Theorem [8] then says that there is a number $f = f(n)$ such that $\pi_{j,p}(\Gamma)$ has an abelian subgroup of index at most $f$.

Since the set $A$ of rational primes is infinite we have

$$G_{\mathbb{Z}_S} \cap \bigcap_{p \in A} \ker \pi_{j,p} = \{1\}$$

This implies that $\Delta$ itself is virtually abelian (it is finitely generated so it has only finitely many subgroups of index at most $f(n)$). But $\Delta$ is Zariski-dense in the $\mathbb{Q}$-simple algebraic group $G$: contradiction.

So condition (a) of Proposition 7.1 holds for almost all primes.

Condition (b) is immediate: $H$ is absolutely simple and so for almost all primes each of the $L(H)_{\mathcal{O}/p}$, is a simple module for $H_{\mathcal{O}/p}$. Since $\Delta$ is Zariski-dense in $H_k$ the group $\text{Ad}(\Delta)$ spans $\text{End}_k L(H)_k$ so for almost all primes $\text{Ad}(\pi_{j}(\Delta))$ spans $\text{End}_{\mathcal{O}/p} L(H)_{\mathcal{O}/p}$. This means that each summand $L(H)_{\mathcal{O}/p}$ of $L(H)_{\mathbb{F}_p}$ is a simple module for $\pi_p(\Delta)$. So the decomposition of $L(G)_{\mathbb{F}_p}$ into minimal Lie ideals is also a decomposition into irreducible $\mathbb{F}_p\pi_p(\Delta)$-modules. So every irreducible module for $\pi_p(\Delta)$ in $L(G)_{\mathbb{F}_p}$ is an ideal, proving that (b) holds.

**Step 4** We now know that the closure $\overline{\Delta}$ of $\Delta$ in $G_{\mathbb{Z}_S} = \prod_{p \in S} G_{\mathbb{Z}_p}$ projects onto all but finitely many of the factors $G_{\mathbb{Z}_p}$. Now it is easy to show (see Exercise 16) that in this case $\overline{\Delta}$ contains their Cartesian product. Combined with Step 1 (which says that $\overline{\Delta}$ projects onto an open subgroup in each of the remaining factors) we easily see that $\overline{\Delta}$ is open in $G_{\mathbb{Z}_S}$.
7.1 Proposition 7.1

There are now at least three different proofs of Proposition 7.1. One is by Matthews, Vaserstein and Weisfeiler [11], it uses the Classification of the Finite simple groups to deduce properties of a proper subgroup of $G_{F_p} \leq \text{GL}_n(F_p)$ which are incompatible with (a) and (b).

There is also a proof using logic by Hrushovkii and Pillay [4].

We shall focus on the original proof by Nori [12]. It studies unipotently generated algebraic groups and their Lie algebras in large finite characteristic $p$. This is motivated by the construction of the Chevalley groups described on section 2.5. Recall that the adjoint chevalley group $G$ is generated by certain automorphisms $\exp(\text{ad}(x))$ for certain ad-nilpotent elements $x$ of the Lie algebra of $G$. If we fix such an element $x$ then the set

$$\{\exp(\text{ad}(tx)) \mid t \in K\}$$

is a unipotent subgroup of $G$ and is isomorphic to $G_+$. Nori generalizes this situation in two directions: He proves an analogue of this not just for algebraic groups but for Zariski-dense subgroups of $\text{GL}_n(F_p)$ and secondly, he does this not just in the algebraic closure $F_p$ of $F_p$ but in the finite field $F_p$ (provided $p$ is large enough compared to $n$).

The details are as follows:

For a group $\Gamma \leq \text{GL}_n(F_p)$ let $\Gamma^+$ be the subgroup generated by its unipotent elements. When $p \geq n$ these are just the elements of order $p$ in $\Gamma$. Similarly for an algebraic group $G \leq \text{GL}_n(K)$ let $G^+$ be the subgroup generated by its unipotent elements.

Now for an element $g \in \text{GL}_n(F_p)$ of order $p$ let $X_g$ be the unipotent 1-dimensional algebraic group over $F_p$ generated by $g$. In other words define

$$X_g = \left\{g^t \defeq \sum_{i=0}^{p} t^i \left( g - 1 \right)^i \mid t \in F_p \right\},$$

where $F_p$ is the algebraic closure of $F_p$. Note that $X_g$ is defined over $F_p$ and is isomorphic to the additive group of the field $F_p$.

Now, given $\Gamma \leq \text{GL}_n(F_p)$ define the algebraic group $T = T(\Gamma)$ as

$$T = \langle X_g \mid \forall g \in \Gamma, g^p = 1 \rangle \leq \text{GL}_n(F_p).$$

Recall that the subgroup generated by a collection of closed connected subgroups is closed and connected, so $T$ is indeed a connected algebraic group. Observe that since $X_g$ is the smallest connected algebraic group containing $g$ and $g \in G_{F_p}$, it follows that $X_g \leq G$ and hence $T \leq G$.

Nori’s main result is that in the above setting we have

$$\Gamma^+ = (T_{F_p})^+$$

provided $p$ is large enough compared to $n$. 

29
Now, it is known that for large primes \( p \) one has

\[
(T_{\mathbb{F}_p})^+ = (T_{\mathbb{F}_p}).
\]

So \( \Gamma^+ \) is the group of \( \mathbb{F}_p \)-rational points of the connected algebraic group \( T \).

Now, suppose that condition (b) of Proposition 7.1 holds. Clearly \( \Gamma \) normalizes the algebraic group \( T \leq G \) since \( (X_g)^\gamma = X_{g^\gamma} \) for any \( \gamma, g \in \Gamma \) with \( g^p = 1 \). Therefore the Lie algebra \( L(T) \leq L(G) \) of \( T \) is normalized by \( \Gamma \).

It follows that the subspace \( L(T)_{\mathbb{F}_p} \) of \( L(G)_{\mathbb{F}_p} \) is invariant under \( \Gamma \) and so it is an ideal of \( L(G)_{\mathbb{F}_p} \). Not only that, \( L(T) \) is defined over \( \mathbb{F}_p \) and so its projections on the direct factors of \( L(G) \) are isomorphic. In the same way as in Step 1 above we deduce that \( L(T) = L(G) \) and since both \( G \) and \( T \leq G \) are connected we have \( T = G \). So

\[
\Gamma \geq \Gamma^+ = T_{\mathbb{F}_p} = G_{\mathbb{F}_p} \geq \Gamma
\]
giving that \( \Gamma = G_{\mathbb{F}_p} \), as required.

### 8 Exercises

1. Show that every open set in \( K^n \) can be regarded as closed affine set in some \( K^m, m \geq n \).

2. Prove that \( \dim V \) for an irreducible affine variety \( V \) is the largest \( d \) such that we can find a chain \( \emptyset \neq V_1 \subset V_2 \subset \cdots V_d \subset V \) of distinct irreducible closed subvarieties \( V_i \) in \( V \). You may use any of the equivalent definitions of \( \dim V \) in §2.1.

3. (Proposition 2.2) Show that each affine variety is a compact topological space and that in fact it satisfies the descending chain condition on closed subsets.

A subset \( X \subset V \) of an affine variety \( V \) is constructible if it can be obtained from the open or closed subsets of \( V \) by a finite process of forming unions and intersections. A theorem of Chevalley says that an image of a constructible set under a morphism of varieties is constructible.

4. ([14], Lemma 14.10.) Prove that a constructible (abstract) subgroup \( H \) of a linear algebraic group \( G \) is in fact closed and so is algebraic. Deduce with Chevalley’s theorem that an image of an algebraic group under a homomorphism is an algebraic group.

5. ([14], Lemma 14.14) Let \( G \) be a linear algebraic group and \( (X_i)_{i \in I} \) be a family of constructible irreducible subsets of \( G \) each containing the identity. Show that \( X_i \) together generate a closed irreducible subgroup of \( G \). Hence deduce that if \( G \) is connected, then the derived subgroup \( G' = \langle [x, y] | x, y \in G \rangle \) is both closed and connected.
6. Suppose that $k/k_0$ is a finite extension of fields and $H = \mathcal{R}_{k/k_0}(G)$. Show that $H$ is $K$-isomorphic to

$$G^{\sigma_1} \times G^{\sigma_2} \times \cdots \times G^{\sigma_d}$$

where $\sigma_i$ are all the embeddings of $k$ in $K$ which fix the elements of $k_0$ and $G^{\sigma_i}$ is the algebraic group defined by the ideal $I^{\sigma_i}$ where the ideal $I$ defines $G = V(I)$ as a variety in $M_n(K)$. Hint: use the map $\lambda$ on page 3 and the isomorphism (1).

7. Let $G$ be the multiplicative group of norm one quaternions defined over $\mathbb{Q}$. For example we can take $G$ in its left regular representation

$$G = \left\{ \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \mid a^2 + b^2 + c^2 + d^2 = 1 \right\}$$

Show that $G$ is $\mathbb{Q}(i)$-isomorphic to $\text{SL}_2$ but it is not $\mathbb{Q}$-isomorphic to it. Hint: Send the $4 \times 4$ matrix with first column $a, b, c, d$ as above to

$$\begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix}.$$

8. Show that if $G = \text{SL}_n(K)$ then $L(G) = \text{sl}_n(K)$, the Lie algebra of matrices of trace 0 in $M_n(K)$.

9. Show that $\Gamma = \text{SL}_2(\mathbb{Z})$ does not have the generalized congruence subgroup property. You may use that $\Gamma$ has a nonabelian free subgroup of finite index.

10. Show that $\text{SL}_n(\mathbb{Z})$ has the strong approximation property. (Hint: use the fact that for a finite ring $R$ the group $\text{SL}_n(R)$ is generated by elementary matrices.)

11. Show that $\text{PGL}_2(\mathbb{Z})$ fails to have the strong approximation property (as an arithmetic subgroup of $G = \text{PGL}_2$).

12. Show that if a connected linear algebraic group $G$ is not soluble then it maps onto a simple algebraic group. (Hint: Let $M = \text{Rad} G$ be the soluble radical of $G$. Then $G/M$ is semisimple.)

13. Suppose that $\Gamma$ is a Zariski-dense subgroup of a connected algebraic group $G$ and that $\Delta$ is a subgroup of finite index in $\Gamma$. Show that $\Delta$ is also Zariski-dense in $G$.

14. Suppose that $G \leq \text{GL}_n(K)$ is a connected algebraic group which has a normal subgroup $N$ which preserves a one-dimensional subspace $(v)$. Show that either $N$ acts as scalars or else $G$ stabilizes a nontrivial subspace of $K^n$.

15. Show that a connected soluble algebraic group $G \leq \text{GL}_n(K)$ has a common eigenvector. Deduce that $G$ is triangularizable and hence prove Theorem 1.2. (Hint: use Exercise 14 with $G'$ in place of $N$.)
16. Suppose that $L$ is a closed subgroup of $K = \prod_{p \in A} G_{\mathbb{Z}_p}$ for some set $A$ of primes, where $G$ is a $\mathbb{Q}$-simple connected and simply connected algebraic group.

(a) Show that if $p$ is sufficiently large then if $L$ maps onto the direct factor $G_{\mathbb{Z}_p}$ of $K$ then in fact it contains it.

(b) On the other hand if $A$ is finite set of primes and $L$ maps onto an open subgroup of each factor $G_{\mathbb{Z}_p}$ of $K$ show that then $L$ is an open subgroup of $K$.

17. Show that for any algebraic group $G$ in characteristic 0 the group $G^+$ generated by its unipotent elements is connected. (Hint: use exercise 5)

18. Using Theorems 2.23 and 2.18 show that if a connected algebraic group consists of semisimple elements then it is a torus. (Hint: a nontrivial semisimple group contains a copy of $SL_2$ or $PSL_2$.)

19. (9) Let $n > 1$ be an integer. Show using Strong Approximation that there is a finite set $A$ of rational primes with the following property: If $S \subseteq SL_n(\mathbb{Z})$ is a subset whose image generates $SL_n(\mathbb{F}_p)$ for some prime $p \notin A$, then for almost all primes $q$, the image of $S$ in $SL_n(\mathbb{F}_q)$ generates $SL_n(\mathbb{F}_q)$. Generalize this to any absolutely simple, connected, simply connected group $G$ defined over $\mathbb{Z}$.

References

[1] M.F. Atiyah, I.G. MacDonald, Introduction to commutative algebra, Addison-Wesley series in Mathematics, 1969.

[2] R. W. Carter, Simple groups of Lie type, London - New York, Wiley, 1972.

[3] R. W. Carter, I. G. MacDonald and G. B. Segal, Lectures on Lie groups and Lie algebras, LMS Student Texts 32, CUP, Cambridge, 1995.

[4] E. Hrushovskii, E. Pillay, Definable subgroups of algebraic groups over finite fields. J. eine angew. 462 (1995), 69-91.

[5] J. Humphreys, Linear algebraic groups, Graduate Texts in Mathematics No. 21, Springer-Verlag, 1975.

[6] Jordan’s theorem

[7] B. Klopsch, Five lectures on analytic pro-$p$ groups: a meeting-ground between finite $p$-groups and Lie theory. this book.

[8] D.D. Long, A. Lubotzky, A.W. Reid, Heegaard gradient and Property $\tau$ for hyperbolic 3-manifolds. http://arxiv.org/abs/0709.0101

[9] A. Lubotzky, One for almost all: Generation of $SL(n, p)$ by subsets of $SL(n, \mathbb{Z})$, in: Algebra, K-. Theory, Groups and Education, Contemp. Math., 243, Amer. Math. Soc., Providence, RI, 1999. 125-128.

[10] A. Lubotzky, D. Segal, Subgroup growth, Birkhäuser, Basel, 2003.
[11] C.R. Matthews, L.N. Vaserstein, B. Weisfeiler, Congruence properties of Zariski-dense subgroups, *Proc. London Math. Soc.* 48 (1984), 514-532.

[12] M. Nori, On subgroups of $\text{GL}_n(\mathbb{F}_p)$, *Invent. Math.* 88 (1987), 257-275.

[13] V. Platonov, A. Rapinchuk, *Algebraic groups and number theory* Academic Press, 1994.

[14] B. Wehrfritz, *Infinite linear groups*, Springer-Verlag, 1973.

[15] B. Weisfeiler, Strong approximation for Zariski-dense subgroups of semisimple algebraic groups, *Annals of Math.* 120, (1984),