Regularity Criterion to the axially symmetric Navier-Stokes Equations∗†

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Abstract
Smooth solutions to the axially symmetric Navier-Stokes equations obey the following maximum principle:

\[ \|ru_\theta(r, z, t)\|_{L^\infty} \leq \|ru_\theta(r, z, 0)\|_{L^\infty}. \]

We first prove the global regularity of solutions if \( \|ru_\theta(r, z, 0)\|_{L^\infty} \) or \( \|ru_\theta(r, z, t)\|_{L^\infty(r \leq r_0)} \) is small compared with certain dimensionless quantity of the initial data. This result improves the one in Zhen Lei and Qi S. Zhang [10]. As a corollary, we also prove the global regularity under the assumption that \( |ru_\theta(r, z, t)| \leq |\ln r|^{-3/2}, \ \forall 0 < r \leq \delta_0 \in (0, 1/2) \).

Key words axially symmetric, Navier-Stokes Equations, Regularity Criterion.

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1 Introduction

In the cylindrical coordinate system with \((x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z)\), an axially symmetric solution of the Navier-Stokes equations is a solution of the following form

\[ u(x, t) = u_r(r, z, t)e_r + u_\theta(r, z, t)e_\theta + u_z(r, z, t)e_z, \quad p(x, t) = p(r, z, t), \]

where

\[ e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_z = (0, 0, 1). \]

In terms of \((u_r, u_\theta, u_z, p)\), the axially symmetric Navier-Stokes equations are as follows

\[
\begin{align*}
\partial_t u_r + u \cdot \nabla u_r - \Delta u_r + \frac{u_r}{r} - \frac{u_\theta^2}{r} + \partial_r p &= 0, \\
\partial_t u_\theta + u \cdot \nabla u_\theta - \Delta u_\theta + \frac{u_\theta}{r} + \frac{u_r u_\theta}{r} &= 0, \\
\partial_t u_z + u \cdot \nabla u_z - \Delta u_z + \partial_z p &= 0, \\
\partial_r (ru_r) + \partial_z (ru_z) &= 0.
\end{align*}
\]
It is well-known that finite energy smooth solutions of the Navier-Stokes equations satisfy the following energy identity
\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds = \|u_0\|_{L^2}^2 < +\infty. \tag{1.2}
\]

Denote \(\Gamma = ru_\theta\). One can easily check that
\[
\partial_t \Gamma + u \cdot \nabla \Gamma - \Delta \Gamma + \frac{2}{r} \partial_r \Gamma = 0. \tag{1.3}
\]

A significant consequence of (1.3) is that smooth solutions of the axially symmetric Navier-Stokes equations satisfy the following maximum principle (see, for instance, [1] [2])
\[
\|\Gamma\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty}. \tag{1.4}
\]

We can compute the vorticity
\[
\omega = \nabla \times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,
\]
where
\[
\omega_r = -\partial_z(u_\theta), \quad \omega_\theta = \partial_z(u_r) - \partial_r(u_z), \quad \omega_z = \frac{1}{r} \partial_r(r u_\theta).
\]

Denote
\[
\Omega = \frac{\omega_\theta}{r}, \quad J = \frac{\omega_r}{r} = -\frac{\partial_z u_\theta}{r},
\]
then
\[
\begin{aligned}
\partial_t \Omega + u \cdot \nabla \Omega - \left( \Delta + \frac{2}{r} \partial_r \right) \Omega + \frac{2 u_\theta}{r} J &= 0, \\
\partial_t J + u \cdot \nabla J - \left( \Delta + \frac{2}{r} \partial_r \right) J - (\omega_r \partial_r + \omega_z \partial_z) \frac{u_r}{r} &= 0.
\end{aligned} \tag{1.5}
\]

We emphasis that \(J\) was introduced by Chen-Fang-Zhang in [4], while \(\Omega\) appeared much earlier and can be at least tracked back to the book of Majda-Bertozzi in [12]. Both of the two new variables are of great importance in our work.

Our goal is to prove that the smallness of \(\|\Gamma\|_{L^\infty(t \leq r_0)}\) or \(\|\Gamma_0\|_{L^\infty}\) implies the global regularity of the solutions. Here is our result.

**Theorem 1.1.** Let \(r_0 > 0\). Suppose that \(u_0 \in H^2\) such that \(\Gamma_0 \in L^\infty\). Denote
\[
M_1 = (1 + \|\Gamma_0\|_{L^\infty}) \|u_0\|_{L^2} \quad \text{and} \quad M_0 = (\|J_0\|_{L^2} + \|\Omega_0\|_{L^2})M_1^3.
\]

Then there exists an absolute positive constant \(C_0 > 0\) such that if
\[
(a) \quad \|\Gamma\|_{L^\infty(t \leq r_0)} \leq \left( 1 + \ln \left( C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\} + 1 \right) \right)^{-3/2},
\]
then the axially symmetric Navier-Stokes equations are globally well-posed.
Remark 1.1. Choose \( r_0 > 0 \) such that \( M_0^{1/4} \geq r_0^{-1/2} M_1 \) and use (1.4), then we can obtain the following global regularity condition

\[
(b) \quad \|\Gamma_0\|_{L^\infty} \leq (1 + \ln(C_0 M_0^{1/4} + 1))^{-3/2},
\]

this condition depends only on the initial value and is very useful especially when \( \|\Gamma_0\|_{L^\infty} \) is very small, in this sense it improves the result in [10]. On the other hand, if we take \( r_0 \to 0^+ \) in condition (a) we can obtain an important corollary.

Corollary 1.1. Let \( \delta_0 \in (0, 1/2) \), \( u \) be the strong solution of the axially symmetric Navier-Stokes equations with initial value \( u_0 \in H^2 \) and \( \|\Gamma_0\|_{L^\infty} < \infty \). If

\[
|\Gamma(r, z, t)| \leq |\ln r|^{-3/2}, \quad \forall \ 0 < r \leq \delta_0,
\]

then \( u \) is regular globally in time.

Denote

\[
K(\varepsilon) = \exp\left(\sqrt{2\varepsilon^{-\frac{4}{3}} - 1} - 1\right), \quad K_0(\varepsilon) = \exp(\varepsilon^{-\frac{4}{3}} - 1)
\]

for \( 0 < \varepsilon \leq 1 \), then one can easily check that

\[
1 + \ln K(\varepsilon) + \frac{1}{2}(\ln K(\varepsilon))^2 = \varepsilon^{-\frac{4}{3}}, \quad \varepsilon^\frac{4}{3} K(\varepsilon) \geq \frac{K_0(\varepsilon)}{C_*} > 0,
\]

for some absolute positive constant \( C_* \) and \( 0 < \varepsilon \leq 1 \). The use of the functions \( K \) and \( K_0 \) is due to a new important observation in Lemma 2.3.

Throughout this paper, we assume \( u \in C([0, T^*); H^2) \) to be the unique strong solution to the Navier-Stokes equations (1.1) with initial value \( u(0) = u_0 \), and the maximal existence time \( T^* > 0 \). We also assume \( u \) to be axially symmetric with \( u_r, u_\theta, u_z, \Gamma, \Omega, J \) defined above, and denote

\[
\Gamma_0 = \Gamma(0), \quad \Omega_0 = \Omega(0), \quad J_0 = J(0), \quad \|\Gamma\|_{L^\infty} = \|\Gamma\|_{L^\infty(R^3 \times (0, T^*))}.
\]

Due to the regularity of solutions to Navier-Stokes equations, \( u \in C((0, T^*); H^4) \), and

\[
\Omega, \quad J \in C([0, T^*); L^2) \cap C((0, T^*); H^2),
\]

\[
\left. \frac{u_\theta}{r}, \frac{u_r}{r} \right|_{r=0} \in C((0, T^*); H^3), \quad \left. \frac{\partial_r u_r}{r} \right|_{r=0} = 0.
\]

Hence, all calculations below are legal for \( t \in (0, T^*) \). (see [10] for more explanation)

Now let us recall some highlights on the study of the axially symmetric Navier-Stokes equations. If the swirl \( u_\theta = 0 \), global regularity result was proved independently by Ukhovskii and Yudovich [15], and Ladyzhenskaya [7], also [11] for a refined proof.

In
the presence of swirl, the global regularity problem is still open. Recently, tremendous efforts and interesting progress have been made on the regularity problem of the axially symmetric Navier-Stokes equations \[1\] \[2\] \[3\] \[4\] \[5\] \[9\] \[10\]. There are many significant results under the sufficient conditions for regularity of axially symmetric solution of type

\[\omega_\theta \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{and} \quad \frac{u_r}{r} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} \leq 2, \quad \frac{3}{2} < q < +\infty\]

in \[1\]. It has been shown in \[4\] that the axially symmetric solution is smooth in \(\mathbb{R}^3 \times (0, T]\) when \(r^d u_\theta \in L^p(0, T; L^q(\mathbb{R}^3))\) with \(d \in [0, 1), \ (p, q) \in \left\{ \left[ \frac{2}{1 - d}, \infty \right] \times \left( \frac{3}{1 - d}, \infty \right), \frac{2}{p} + \frac{3}{q} \leq 1 - d \right\}\), in particular, global regularity is obtained if \(|\Gamma| \leq Cr^a\) for some \(\alpha > 0, \ C > 0\). In \[10\] global regularity is obtained if \(|\Gamma| \leq C |\ln r|^{-2}\) for some \(C > 0\). Clearly, our Corollary 1.1 improves the one in \[10\].

Here global regularity means \(T^* = +\infty\), and we only need to prove \(\Omega \in L^\infty(0, T^*; L^2(\mathbb{R}^3))\), hence we can use the results in \[4\] or \[14\] to obtain global regularity. We can also use Lemma 2.1 in section 2 to obtain \(\nabla \frac{u_r}{r} \in L^\infty(0, T^*; L^2(\mathbb{R}^3))\), and \(\frac{u_r}{r} \in L^\infty(0, T^*; L^6(\mathbb{R}^3))\), then the results in \[1\] or \[6\] imply the global regularity.

This paper is organized as follows, in section 2 we will give some notations and 3 important Lemmas, in section 3 we will first follow the proof in \[10\] then use the Lemmas in section 2 to conclude the proof.

## 2 Notations and Lemmas

The Laplacian operator \(\triangle\) and the gradient operator \(\nabla\) in the cylindrical coordinate are

\(\triangle = \partial^2_r + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta + \partial^2_z, \quad \nabla = e_r \partial_r + \frac{er}{r} \partial_\theta + e_z \partial_z.\)

We will use \(C\) to denote a generic absolute positive constant whose meaning may change from line to line. If \(|f|^2\) is axially symmetric, we will denote

\[\|f\|_{L^2}^2 = \int |f|^2 r dr dz, \quad dx = r dr dz.\]

The following estimate will be used very often. (see \[5\] \[8\] \[13\])

**Lemma 2.1.** \(\left\| \nabla \frac{u_r}{r} \right\|_{L^2}^2 \leq \|\Omega\|_{L^2}^2, \quad \left\| \nabla^2 \frac{u_r}{r} \right\|_{L^2}^2 \leq \|\partial_z \Omega\|_{L^2}^2.\)
Proof. By virtue of \( \partial_r (ru_r) + \partial_z (ru_z) = 0 \), we can find the stream function \( \psi \) such that

\[
u_r = -\partial_z \psi, \quad u_z = \frac{1}{r} \partial_r (r \psi),
\]

then we can compute

\[- \left( \Delta + \frac{2}{r} \partial_r \right) \frac{\psi}{r} = \Omega, \quad \left( \Delta + \frac{2}{r} \partial_r \right) \frac{u_r}{r} = \partial_z \Omega.\]

Using integration by parts, we have

\[- \int \frac{u_r}{r} \left( \Delta + \frac{2}{r} \partial_r \right) \frac{u_r}{r} \, dx = \left\| \nabla \left( \frac{u_r}{r} \right) \right\|_{L^2}^2 + \int \left| \frac{u_r}{r} (0, z, t) \right|^2 \, dz,\]

therefore,

\[\left\| \nabla \frac{u_r}{r} \right\|_{L^2}^2 \leq - \int \frac{u_r}{r} \left( \Delta + \frac{2}{r} \partial_r \right) \frac{u_r}{r} \, dx\]

\[= - \int \frac{u_r}{r} \partial_z \Omega \, dx = \int \partial_z \frac{u_r}{r} \Omega \, dx \leq \left\| \nabla \frac{u_r}{r} \right\|_{L^2} \left\| \Omega \right\|_{L^2},\]

hence,

\[\left\| \nabla \frac{u_r}{r} \right\|_{L^2} \leq \left\| \Omega \right\|_{L^2}.\]

Since

\[\left\| \partial_z \Omega \right\|_{L^2}^2 = \left\| \left( \Delta + \frac{2}{r} \partial_r \right) \frac{u_r}{r} \right\|_{L^2}^2\]

\[= \left\| \Delta \frac{u_r}{r} \right\|_{L^2}^2 + 4 \left\| \frac{1}{r} \partial_r \frac{u_r}{r} \right\|_{L^2}^2 + 4 \int \Delta \frac{u_r}{r} \left( \frac{1}{r} \partial_r \frac{u_r}{r} \right) \, dr \, dz,\]

and

\[\left\| \Delta \frac{u_r}{r} \right\|_{L^2}^2 = \left\| \nabla^2 \frac{u_r}{r} \right\|_{L^2}^2, \quad \left. \partial_r \frac{u_r}{r} \right|_{r=0} = 0,\]

\[\int \Delta \frac{u_r}{r} \partial_r \frac{u_r}{r} \, dr \, dz = \left. \left( \frac{1}{r} \partial_r \frac{u_r}{r} \right) \right|_{L^2}^2 + \frac{1}{2} \int \left( \left| \partial_z \frac{u_r}{r} \right|^2 - \left| \partial_r \frac{u_r}{r} \right|^2 \right) (0, z, t) \, dz \geq 0,\]

we can obtain \( \left\| \nabla^2 \frac{u_r}{r} \right\|_{L^2}^2 \leq \left\| \partial_z \Omega \right\|_{L^2}^2 \), this completes the proof Lemma 2.1. \( \square \)

Denote

\[v(r, z, t) = \int_0^r |u_\theta (r', z, t)| \, dr', \quad \text{for } r > 0, \quad a(t) = \left\| \frac{v}{r} (t) \right\|_{L^\infty},\]

then we have the following inequality

**Lemma 2.2.** \( a(t)^2 \leq \left\| J(t) \right\|_{L^2} \left\| \frac{u_\theta}{r} (t) \right\|_{L^2}.\)
Proof. For \( r' > 0, \ z' \in \mathbb{R}, \ t > 0 \) as \( v(r', z', t) = \int_{0}^{r'} |u_\theta(r, z', t)| dr, \) by Hölder inequality we have

\[
|v(r', z', t)|^2 \leq \int_{0}^{r'} r dr \int_{0}^{r'} \frac{|u_\theta(r, z', t)|^2}{r} dr = \frac{r^2}{2} \int_{0}^{r'} r dr \int_{z}^{r z'} \frac{1}{r} |u_\theta(r, z, t)|^2 dz = \frac{r^2}{2} \int_{0}^{r'} r dr \int_{z}^{r z'} 2J u_\theta(r, z, t) dz \leq r^2 \int |J| \left| \frac{u_\theta}{r} \right| |(r, z, t) r dr dz \\
\leq r^2 \|J(t)\|_{L^2} \left\| \frac{u_\theta}{r} (t) \right\|_{L^2}.
\]

Hence, we get

\[
a(t)^2 = \sup_{r' > 0, z' \in \mathbb{R}} \left| \frac{v(r', z', t)}{r'} \right|^2 \leq \|J(t)\|_{L^2} \left\| \frac{u_\theta}{r} (t) \right\|_{L^2}.
\]

This completes the proof. \( \square \)

Lemma 2.3. Assume that \( t > 0, \ \|\Gamma\|_{L^\infty(r \leq r_1)} \leq \varepsilon \leq 1, \) and \( 0 < r_1 \leq \frac{\varepsilon K(\varepsilon)}{a(t)}, \) then

\[
\int \frac{|u_\theta(z)|}{r} |f|^2 dx \leq \varepsilon^{-\frac{1}{2}} \int |\partial_r f|^2 dx + C \frac{\|\Gamma\|_{L^\infty} + \varepsilon^{-\frac{1}{2}}}{r_1^2} \int_{r \geq \frac{r_1}{2}} |f|^2 dx, \quad (2.1)
\]

\[
\int |u_\theta(t)|^2 |f|^2 dx \leq \varepsilon^2 \int |\partial_r f|^2 dx + C \frac{\|\Gamma\|_{L^\infty}^2 + \varepsilon^2}{r_1^2} \int_{r \geq \frac{r_1}{2}} |f|^2 dx, \quad (2.2)
\]

for all axially symmetric scalar and vector functions \( f \in H^1. \)

Proof. We first prove that if \( f = 0 \) for \( r \geq r_1, \) then

\[
\int \frac{|u_\theta(t)}{r} |f|^2 r dr dz \leq \varepsilon^{-\frac{1}{2}} \int |\partial_r f|^2 r dr dz. \quad (2.3)
\]

In this case, \( f(r', z) = -\int_{r'}^{r_1} \partial_r f(r, z) dr \) for \( 0 < r' < r_1, \) by Hölder inequality we have

\[
|f(r', z)|^2 \leq \int_{r'}^{r_1} r |\partial_r f(r, z)|^2 dr \int_{r'}^{r_1} \frac{dr}{r}.
\]

Consequently,

\[
\int |u_\theta(r', z, t)|^2 |f(r', z)|^2 dr' \leq \int r |\partial_r f(r, z)|^2 dr \int_{0}^{r_1} |u_\theta(r', z, t)| \int_{r'}^{r_1} \frac{dr}{r} dr', \quad (2.4)
\]
by the definition of $v$ we have
\[
\int_0^{r_1} |u_\theta(r', z, t)| \int_{r'}^{r_1} \frac{dr'}{r} = \int_0^{r_1} v(r, z, t) \frac{dr}{r}, \tag{2.5}
\]
by the definition of $a(t)$ we have $v(r, z, t) \leq ra(t)$. On the other hand, $|u_\theta| = \frac{|\Gamma|}{r} \leq \varepsilon$ for $0 < r < r_1$. Hence, if $\frac{\varepsilon}{a(t)} \leq r \leq r_1$, then
\[
v(r, z, t) = v\left(\frac{\varepsilon}{a(t)}, z, t\right) + \int_{\frac{\varepsilon}{a(t)}}^{r} |u_\theta(r', z, t)| dr' \leq \frac{\varepsilon}{a(t)} a(t) + \int_{\frac{\varepsilon}{a(t)}}^{r} \varepsilon dr' = \varepsilon \left(1 + \ln \frac{ra(t)}{\varepsilon}\right).
\]
The above estimates of $v$ implies
\[
\int_0^{r_1} \sqrt{\frac{v(r, z, t)}{r}} dr \leq \int_0^{\frac{\varepsilon}{a(t)}} \frac{ra(t)}{r} dr + \int_{\frac{\varepsilon}{a(t)}}^{r} \frac{\varepsilon}{r} \left(1 + \ln \frac{ra(t)}{\varepsilon}\right) dr
\]
\[
= \varepsilon + \int_1^{K(\varepsilon)} \varepsilon (1 + \ln r) \frac{dr}{r} = \varepsilon \left(1 + \ln K(\varepsilon) + \frac{1}{2} (\ln K(\varepsilon))^2\right) = \varepsilon^{-\frac{1}{2}},
\tag{2.6}
\]
here we used $0 < r_1 \leq \frac{\varepsilon K(\varepsilon)}{a(t)}$. By (2.4), (2.5), (2.6), we have
\[
\int |u_\theta(r', z, t)||f(r', z)|^2 dr' \leq \varepsilon^{-\frac{1}{2}} \int r|\partial_r f|^2 dr,
\]
integrate in $z$, we obtain (2.3). Now we discuss general $f$. Take a smooth cut-off function of $r$ such that (i) $\phi' \leq 0$, (ii) $\phi \equiv 1$ if $0 \leq r \leq \frac{1}{2}$, (iii) $\phi \equiv 0$ if $r \geq 1$. Then we have
\[
\int \frac{|u_\theta(t)|}{r} |f|^2 r dr dz = \int |u_\theta(t)| \left| \phi \left(\frac{r}{r_1}\right) f \right|^2 r dr dz + \int |u_\theta(t)| \left(1 - \phi \left(\frac{r}{r_1}\right)^2 \right) |f|^2 r dr dz
\]
\[
\leq \varepsilon^{-\frac{1}{2}} \int \left| \partial_r \left[ \phi \left(\frac{r}{r_1}\right) f \right] \right|^2 r dr dz + \int \frac{|\Gamma|}{r} |f|^2 r dr dz
\]
\[
\leq \varepsilon^{-\frac{1}{2}} \left( \int |\partial_r f|^2 r dr dz + C \frac{1}{r_1^2} \int_{r \geq \frac{1}{2}} |f|^2 r dr dz \right) + \frac{4 \|\Gamma\|_{L^\infty}}{r_1^2} \int_{r \geq \frac{1}{2}} |f|^2 r dr dz
\]
\[
\leq \varepsilon^{-\frac{1}{2}} \left( \int |\partial_r f|^2 r dr dz + C \frac{||\Gamma||_{L^\infty} + \varepsilon^{-\frac{1}{2}}}{r_1^2} \int_{r \geq \frac{1}{2}} |f|^2 r dr dz, \right.
\]
\[
\leq \varepsilon^{-\frac{1}{2}} \left( \int |\partial_r f|^2 r dr dz + C \frac{||\Gamma||_{L^\infty} + \varepsilon^{-\frac{1}{2}}}{r_1^2} \int_{r \geq \frac{1}{2}} |f|^2 r dr dz, \right.
\]
here we used (2.3) and the fact that
\[ \int |r \left[ \phi \left( \frac{r}{r_1} \right) f \right]|^2 r dr dz = \int \left[ \phi \left( \frac{r}{r_1} \right)^2 |\partial_r f|^2 + |f|^2 |\partial_r \phi \left( \frac{r}{r_1} \right)|^2 + \frac{\partial_r |f|^2 \phi \left( \frac{r}{r_1} \right) \partial_r \phi \left( \frac{r}{r_1} \right)}{r} \right] r dr dz \]
\[ = \int \left[ \phi \left( \frac{r}{r_1} \right)^2 |\partial_r f|^2 + |f|^2 |\partial_r \phi \left( \frac{r}{r_1} \right)|^2 \right] r dr dz - \int |f|^2 \partial_r \left[ \phi \left( \frac{r}{r_1} \right) \partial_r \phi \left( \frac{r}{r_1} \right) \right] r dr dz \leq \int |\partial_r f|^2 + C \int_{r \geq \frac{3}{4}} |f|^2 r dr dz. \]
Similarly we have
\[ \int |u_\theta(t)|^2 |f|^2 r dr dz = \int |u_\theta(t)|^2 \left| \phi \left( \frac{r}{r_1} \right) \right|^2 r dr dz + \int |u_\theta(t)|^2 \left( 1 - \phi \left( \frac{r}{r_1} \right)^2 \right) |f|^2 r dr dz \leq \|\Gamma\|_{L^\infty(r \leq r_1)} \int |u_\theta(t)| \left| \phi \left( \frac{r}{r_1} \right) \right|^2 r dr dz + \int \frac{|\Gamma|^2}{r^2} |f|^2 r dr dz \leq \varepsilon^{-\frac{3}{4}} \left( \int |\partial_r f|^2 r dr dz + \frac{C}{r_1^2} \int_{r \geq \frac{3}{4}} |f|^2 r dr dz \right) + \frac{4\|\Gamma\|^2_{L^\infty}}{r_1^4} \int_{r \geq \frac{3}{4}} |f|^2 r dr dz \leq \varepsilon\frac{3}{4} \int |\partial_r f|^2 r dr dz + C \frac{\|\Gamma\|^2_{L^\infty} + \varepsilon^{\frac{3}{4}}}{r_1^2} \int_{r \geq \frac{3}{4}} |f|^2 r dr dz. \]
This completes the proof. \[ \square \]

3 Proof of the results

Proof of Theorem 1.1. By applying standard energy estimate to \( J \) equation, we have
\[ \frac{1}{2} \frac{d}{dt} \|J\|^2_{L^2} + \int J(u \cdot \nabla) J dx - \int J \left( \Delta + \frac{2}{r} \partial_r \right) J dx - \int J(\omega, \partial_r + \omega_2 \partial_z) \frac{u_r}{r} dx = 0. \]
Using \( \nabla \cdot u = 0 \), one has
\[ \int J(u \cdot \nabla) J dx = \frac{1}{2} \int J^2 (\nabla \cdot u) dx = 0. \]
On the other hand, by direct calculations, one has
\[ - \int J \left( \Delta + \frac{2}{r} \partial_r \right) J dx = \|\nabla J\|^2_{L^2} + \int |J(0, z, t)|^2 dz. \]
Consequently, we have
\[ \frac{1}{2} \frac{d}{dt} \|J\|^2_{L^2} + \|\nabla J\|^2_{L^2} + \int |J(0, z, t)|^2 dz = \int J(\omega, \partial_r + \omega_2 \partial_z) \frac{u_r}{r} dx. \quad (3.1) \]
Similarly, by applying the energy estimate to the equation of $\Omega$, one obtains that

$$
\frac{1}{2} \frac{d}{dt} \|\Omega\|^2_{L^2} + \|\nabla \Omega\|^2_{L^2} + \int \Omega(0, z, t)|^2 dz = -2 \int \left[ \frac{u_\theta}{r} J \Omega \right] dx. \quad (3.2)
$$

Notice that

$$
\int J(\omega_r \partial_r + \omega_\theta \partial_\theta) \frac{u_r}{r} dx = \int \left[ \nabla \times (u_\theta e_\theta) \right] \cdot \left( J \nabla \frac{u_\theta}{r} \right) dx
$$

and take $f = \frac{u_r}{r}$, let

$$
\|\nabla \frac{u_\theta}{r}\|^2_{L^2},
$$

and by (3.1), we have

$$
\frac{d}{dt} \|J\|^2_{L^2} + \|\nabla J\|^2_{L^2} + 2 \int |J(0, z, t)|^2 dz \leq \|u_\theta \nabla \frac{u_r}{r}\|^2_{L^2}. \quad (3.3)
$$

Now we estimate the right hand side of (3.2) and (3.3), under the assumption of condition (b), let

$$
\varepsilon = \left( 1 + \ln \left( C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\} + 1 \right) \right)^{-3/2}, \quad (3.4)
$$

then $\|\Gamma\|_{L^\infty(r \leq r_0)} \leq \varepsilon \leq 1$, and we can apply Lemma 2.3 with

$$
r_1 = r(t) = \min \left\{ \frac{\varepsilon K(\varepsilon)}{\alpha(t)}, r_0 \right\},
$$

and take $f = J$, $\Omega$ in (2.1), we have

$$
-2 \int \left[ \frac{u_\theta}{r} J \Omega \right] dx \leq \varepsilon \frac{1}{2} \int \left[ \frac{u_\theta}{r} \right] |\Omega|^2 dx + \varepsilon \frac{1}{2} \int \left[ \frac{u_\theta}{r} \right] |J|^2 dx
$$

$$
\leq \int \left[ \partial_r \Omega \right]^2 dx + \frac{C(1 + \varepsilon \frac{1}{2} \|\Gamma\|_{L^\infty})}{r(t)^2} \int_{r \geq \varepsilon M_0} \left[ \Omega \right]^2 dx
$$

$$
+ \varepsilon \frac{3}{2} \int \left[ \partial_r J \right]^2 dx + \frac{C \varepsilon \frac{1}{2} (1 + \varepsilon \frac{1}{2} \|\Gamma\|_{L^\infty})}{r(t)^2} \int_{r \geq \varepsilon M_0} \left| J \right|^2 dx. \quad (3.5)
$$

Choosing $f = \partial_r \frac{u_r}{r}$, $\partial_\theta \frac{u_r}{r}$ in (2.2), we have

$$
\left\| u_\theta \nabla \frac{u_r}{r} \right\|^2_{L^2} \leq \varepsilon \frac{3}{2} \int \left[ \partial_r \nabla \frac{u_r}{r} \right]^2 dx + \frac{C \varepsilon \frac{1}{2} (1 + \varepsilon \frac{1}{2} \|\Gamma\|_{L^\infty})}{r(t)^2} \int_{r \geq \varepsilon M_0} \left| \nabla \frac{u_r}{r} \right|^2 dx. \quad (3.6)
$$

Denote

$$
M_2 = 1 + \varepsilon \frac{1}{2} \|\Gamma\|_{L^\infty} + \varepsilon \frac{3}{2} \|\Gamma\|_{L^\infty}^2,
$$

by Lemma 2.1 we have

$$
\int \left| \partial_r \nabla \frac{u_r}{r} \right|^2 dx \leq \left\| \nabla^2 \frac{u_r}{r} \right\|^2_{L^2} \leq \left\| \partial_r \Omega \right\|^2_{L^2}.
$$
Inserting (3.5), (3.6) into (3.2), (3.3), we have
\[ \frac{d}{dt} \left( \| J(t) \|_{L^2}^2 + \frac{\varepsilon^2 t}{2} \| \Omega(t) \|_{L^2}^2 \right) \leq \frac{CM_2}{r(t)^2} \int_{r \geq \varepsilon K(t)} \left[ \frac{\varepsilon^2}{r} \left( \left\| \nabla \frac{u_r}{r} \right\|^2 + \| \Omega \|^2 \right) + \| J \|^2 \right] \, dx. \tag{3.7} \]
Denote
\[ A(t) = \| J(t) \|_{L^2}^2 + \frac{\varepsilon^2 t}{2} \| \Omega(t) \|_{L^2}^2, \]
then
\[ A(t) \in C[0, T^*) \cap C^1(0, T^*). \]

By Lemma 2.1 and the fact that
\[ \nabla \frac{u_r}{r} = \nabla u_r - \frac{u_r}{r^2} e_r, \]
we obtain
\[ \int_{r \geq \varepsilon K(t)} \left( \left\| \nabla \frac{u_r}{r} \right\|^2 + \| \Omega \|^2 \right) \, dx \leq \left\| \nabla \frac{u_r}{r} \right\|_{L^2}^2 + \| \Omega \|_{L^2}^2 \leq 2\| \Omega \|_{L^2}^2, \]
\[ \int_{r \geq \varepsilon K(t)} \left( \left\| \nabla \frac{u_r}{r} \right\|^2 + \| \Omega \|^2 + \| J \|^2 \right) \, dx \leq \frac{C}{r(t)^2} \int \| \nabla u \|^2 \, dx, \]
hence we have
\[ \frac{d}{dt} A(t) \leq \frac{CM_2}{r(t)^2} \int_{r \geq \varepsilon K(t)} \left[ \frac{\varepsilon^2}{r} \left( \left\| \nabla \frac{u_r}{r} \right\|^2 + \| \Omega \|^2 \right) + \| J \|^2 \right] \, dx \]
\[ \leq \frac{CM_2}{r(t)^2} \min \left\{ 2\varepsilon \| \Omega \|_{L^2}^2 + \| J \|_{L^2}^2, \frac{C}{r(t)^2} \int \| \nabla u \|^2 \, dx \right\} \tag{3.8} \]
\[ \leq \frac{CM_2}{r(t)^2} \min \left\{ A(t), \frac{\| \nabla u(t) \|_{L^2}^2}{r(t)^2} \right\}. \]

Fix \( t \in (0, T^*) \), if \( r(t) = \frac{\varepsilon K(t)}{a(t)} \leq r_0 \), by Lemma 2.2, we have
\[ a(t)^2 \leq A(t)^{\frac{1}{2}} \left\| \nabla u(t) \right\|_{L^2}, \]
\[ \frac{1}{r(t)^2} = \frac{a(t)^2}{(\varepsilon K(t))^2} \leq \frac{A(t)^{\frac{1}{2}} \left\| \nabla u(t) \right\|_{L^2}}{(\varepsilon K(t))^2}. \]

And (3.8) implies
\[ \frac{d}{dt} A(t) \leq \frac{CM_2 A(t)^{\frac{1}{2}} \left\| \nabla u(t) \right\|_{L^2}}{(\varepsilon K(t))^2} \min \left\{ A(t), \frac{A(t)^{\frac{1}{2}} \left\| \nabla u(t) \right\|_{L^2}^2}{(\varepsilon K(t))^2} \right\} \]
\[ = \frac{CM_2 A(t) \left\| \nabla u(t) \right\|_{L^2}^2}{(\varepsilon K(t))^2} \min \left\{ A(t)^{\frac{1}{2}}, \frac{\left\| \nabla u(t) \right\|_{L^2}^2}{(\varepsilon K(t))^2} \right\} \]
\[ \leq \frac{CM_2 A(t)^{\frac{1}{2}} \left\| \nabla u(t) \right\|_{L^2}^2}{(\varepsilon K(t))^2}, \]
otherwise, we have \( r(t) = r_0 \) and
\[
\frac{d}{dt} A(t) \leq \frac{C M_2 \|
abla u(t)\|_{L^2}^2}{r_0^4}.
\]
Combining the above two cases we have
\[
\frac{d}{dt} A(t) \leq C M_2 \|
abla u(t)\|_{L^2}^2 \max \left\{ \frac{A(t)^{\frac{4}{3}}}{(\varepsilon K(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4} \right\}.
\]
Denote \( F(y) = \int_y^{+\infty} \left[ \max \left\{ \frac{y^{\frac{4}{3}}}{(\varepsilon K(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4} \right\} \right]^{-1} dy \), then
\[
\frac{d}{dt} F(A(t)) \geq C M_2 \|
abla u(t)\|_{L^2}^2,
\]
and we can use the energy identity to obtain
\[
F(A(0)) - F(A(t)) \leq \int_0^t C M_2 \|
abla u(s)\|_{L^2}^2 ds \leq C_1 M_2 \|u_0\|_{L^2}^2.
\]
Therefore, if the condition \((a)’\) : \( F(A(0)) > C_1 M_2 \|u_0\|_{L^2}^2 \) is satisfied, then
\[
\inf_{0 < t < T^*} F(A(t)) > 0, \sup_{0 < t < T^*} A(t) < +\infty, \sup_{0 < t < T^*} \|\Omega(t)\|_{L^2}^2 < +\infty,
\]
and these imply the global regularity.

Now we claim that, if \( C_0 > C_* \max \left\{ 1, \sqrt{C_1/3} \right\} \), then (3.4) implies condition \((a)’\). Here \( C_1, C_* \) are absolute positive constants. Notice that
\[
F(A(0)) \geq F \left( \max \left\{ A(0), \frac{(\varepsilon K(\varepsilon))^{\frac{2}{3}}}{r_0^4} \right\} \right)
= 3 \max \left\{ A(0), \frac{(\varepsilon K(\varepsilon))^{\frac{2}{3}}}{r_0^4} \right\}^{-\frac{4}{3}} (\varepsilon K(\varepsilon))^{\frac{8}{3}},
\]
\[
A(0)^{\frac{4}{3}} = \left( \|J(0)\|_{L^2}^2 + \frac{\varepsilon^{\frac{2}{3}}}{2} \|\Omega(0)\|_{L^2}^2 \right)^{\frac{1}{2}} \leq \|J_0\|_{L^2} + \|\Omega_0\|_{L^2},
\]
from the definition of \( M_2, M_1, M_0 \) and (1.4) we have
\[
M_2 < \varepsilon^{-\frac{4}{3}} (1 + \|\Gamma\|_{L^\infty})^2, \quad M_2 \|u_0\|_{L^2}^2 \leq \varepsilon^{-\frac{4}{3}} M_1^2, \quad A(0)^{\frac{4}{3}} M_1^3 \leq M_0.
\]
And (3.3) implies
\[
K_0(\varepsilon) > C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\},
\]
and (3.4) also implies
\[
\inf_{0 < t < T^*} A(t) > 0, \sup_{0 < t < T^*} \|\Omega(t)\|_{L^2}^2 < +\infty,
\]
from the definition of \( M_2, M_1, M_0 \) and (1.4) we have
\[
M_2 < \varepsilon^{-\frac{4}{3}} (1 + \|\Gamma\|_{L^\infty})^2, \quad M_2 \|u_0\|_{L^2}^2 \leq \varepsilon^{-\frac{4}{3}} M_1^2, \quad A(0)^{\frac{4}{3}} M_1^3 \leq M_0.
\]
And (3.3) implies
\[
K_0(\varepsilon) > C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\},
\]
and (3.4) also implies
\[
\inf_{0 < t < T^*} A(t) > 0, \sup_{0 < t < T^*} \|\Omega(t)\|_{L^2}^2 < +\infty,
\]
from the definition of \( M_2, M_1, M_0 \) and (1.4) we have
\[
M_2 < \varepsilon^{-\frac{4}{3}} (1 + \|\Gamma\|_{L^\infty})^2, \quad M_2 \|u_0\|_{L^2}^2 \leq \varepsilon^{-\frac{4}{3}} M_1^2, \quad A(0)^{\frac{4}{3}} M_1^3 \leq M_0.
\]
And (3.3) implies
\[
K_0(\varepsilon) > C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\},
\]
and (3.4) also implies
\[
\inf_{0 < t < T^*} A(t) > 0, \sup_{0 < t < T^*} \|\Omega(t)\|_{L^2}^2 < +\infty,
\]
from the definition of \( M_2, M_1, M_0 \) and (1.4) we have
\[
M_2 < \varepsilon^{-\frac{4}{3}} (1 + \|\Gamma\|_{L^\infty})^2, \quad M_2 \|u_0\|_{L^2}^2 \leq \varepsilon^{-\frac{4}{3}} M_1^2, \quad A(0)^{\frac{4}{3}} M_1^3 \leq M_0.
\]
And (3.3) implies
\[
K_0(\varepsilon) > C_0 \max \left\{ M_0^{1/4}, r_0^{-1/2} M_1 \right\},
\]
hence we obtain

\[
\left( \frac{M_2 \|u_0\|_{L^2}^2}{F(A(0))} \right)^{\frac{1}{2}} \leq \frac{\varepsilon^{-1} M_1^3 \max \left\{ A(0)^2, \varepsilon K(\varepsilon)/r_0^{3/2} \right\}}{3^{3/2}(\varepsilon K(\varepsilon))^4} \\
\leq \frac{\max \left\{ M_0, \varepsilon K(\varepsilon) M_1^3/r_0^{3/2} \right\}}{3^{3/2}\varepsilon (\varepsilon K(\varepsilon))^4} \\
\leq 3^{-\frac{3}{2}} \max \left\{ \frac{M_0 C_4^4}{K_0(\varepsilon)^4}, \frac{C_0 C_4^3 M_1^3}{K_0(\varepsilon)^3 r_0^{3/2}} \right\} \\
< 3^{-\frac{3}{2}} \max \left\{ \frac{C_4^4}{C_0^4}, \frac{C_4^3}{C_0^3} \right\} = 3^{-\frac{3}{2}} \left( \frac{C_4}{C_0} \right)^3 \\
\leq 3^{-\frac{3}{2}} \left( \sqrt{\frac{3}{C_1}} \right) = C_1^{-\frac{3}{2}},
\]

and \( F(A(0)) > C_1 M_2 \|u_0\|_{L^2}^2. \)

Therefore the claim is true, this completes the proof of Theorem 1.1. \(\square\)

**Proof of Corollary 1.1.** First, we can take \( r_0 \in (0, \delta_0) \) such that

\[
r_0^{-1/2} M_1 \geq M_1^{1/4}, \quad C_0 M_1 r_0^{-1/2} + 1 < e^{-1} r_0^{-1},
\]

by (1.6), we have \( \ln r_0 \leq \| \Gamma \|_{L^\infty(r \leq r_0)}, \) using the property of this \( r_0 \) we have

\[
\left( 1 + \ln \left( C_0 \max \left\{ M_1^{1/4}, r_0^{-1/2} M_1 \right\} + 1 \right) \right)^{-3/2} = \left( 1 + \ln \left( C_0 r_0^{-1/2} M_1 + 1 \right) \right)^{-3/2} > (1 + \ln (e^{-1} r_0^{-1}))^{-3/2} = |\ln r_0|^{-3/2},
\]

Therefore condition (a) in Theorem 1.1 is satisfied, and we can use Theorem 1.1 to get the global regularity, this completes the proof of Corollary 1.1. \(\square\)

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