EXTREMAL WEIGHT PROJECTORS II

HOEL QUEFFELEC AND PAUL WEDRICH

Abstract. In previous work, we have constructed diagrammatic idempotents in an affine extension of the Temperley–Lieb category, which describe extremal weight projectors for \( sl_2 \), and which categorify Chebyshev polynomials of the first kind. In this paper, we generalize the construction of extremal weight projectors to the case of \( gl_N \) for \( N \geq 2 \), with a view towards categorifying the corresponding torus skein algebras via Khovanov–Rozansky link homology. As by-products, we obtain compatible diagrammatic presentations of the representation categories of \( gl_N \) and its Cartan subalgebra, and a categorification of power-sum symmetric polynomials.

1. Introduction

The topological motivation for this article is the search for an extension of Khovanov–Rozansky link homologies [13, 15] to invariants of links in 3-manifolds other than \( \mathbb{R}^3 \). Since quantum link homologies, in their mode of definition and computation, currently depend on the presentation of links as 2-dimensional projections, the most accessible 3-manifolds in this endeavor are thickened surfaces \( \Sigma \times I \).

Just as Khovanov homology categorifies the Jones polynomial, the surface link homologies should categorify surface skein modules [21, 29, 1]. These admit an algebra structure induced by stacking links, with distinguished bases (conjecturally) satisfying strong integrality and positivity properties [10, 27, 18, 19]. They provide quantizations of surface character varieties that play an important role in quantum Teichmüller theory [5]. Both aspects make such skein algebras prime targets for categorification via link homology technology.

In order to categorify quantum invariants, it is useful to have explicit, combinatorial or diagrammatic descriptions of underlying representation categories. For example, Khovanov homology can be built from a categorification of the Temperley–Lieb category, which describes the representation category of \( U_q(sl_2) \), and all incarnations of Khovanov–Rozansky homology implicitly employ a categorification of the MOY or web calculus for the representation category of \( U_q(gl_N) \) [20, 7].

The main purpose of the present paper is to provide representation-theoretic tools for categorifying skein algebras. The key novelty when working with skein algebras is that their proposed distinguished bases are obtained from links colored not by irreducible representations of the corresponding quantum group, but colored only by their extremal weight spaces. In [26, Section 1.3] we have proposed a strategy for lifting these colorings to the categorified level of toric link homologies, which is inspired by Khovanov’s categorification of the colored Jones polynomial [14]. The main tool necessary in this approach is a diagrammatic presentation of the representation category of a Cartan subalgebra \( U(\mathfrak{h}) \subset U(gl_N) \), which is the first result in this paper.

**Theorem 1** (Corollary [43]). There exists a diagrammatic presentation for the representation category of \( U(\mathfrak{h}) \), given by an affine extension \( N\text{AWeb}^{\text{ess}} \) of the web calculus for \( U(gl_N) \) in which \( \wedge^k(V) \)-labeled essential circles are set to zero for \( 0 < k < N \).

We emphasize that we deal with universal enveloping algebras, rather than their quantizations. Those are intended to control the \( \mathbb{C} \)-linear morphism spaces in our skein module categorifications [26, Section 1.3], whereas the expected quantum parameter \( q \) is promoted to a grading on objects. This is
related to the fact that *annular Khovanov homology* \[11\] has a natural action of \(U(\mathfrak{sl}_2)\), not of \(U_q(\mathfrak{sl}_2)\). An additional quantization seems possible, see Remark \[5\] but will not play a role here. Note that the diagrammatic presentation from Theorem \[1\] is compatible with one that follows from the work of Cautis–Kamnitzer \[6, Section 2.6\].

In \[26\], we have constructed a diagrammatic presentation as in Theorem \[1\] for the case of \(\mathfrak{sl}_2\), and identified idempotent morphisms that encode the projections onto extremal weight spaces in finite-dimensional \(U(\mathfrak{sl}_2)\)-representations. These *extremal weight projectors* are analogous to, but finer than Jones–Wenzl projectors \[12, 30\], and they can also be defined recursively. In this article, we identify and study extremal weight projectors for \(\mathfrak{gl}_N\).

**Theorem 2.** The diagrammatic category \(N\mathsf{AWeb}^\mathrm{ess}\) contains recursively defined idempotents that correspond to projections onto the extremal weight spaces in the \(U(\mathfrak{gl}_N)\)-representations \(\text{Sym}^k(V)\).

In fact we prove a slightly stronger version of this theorem in a central extension of \(N\mathsf{AWeb}^\mathrm{ess}\), which has an additional grading by winding number, that will be important for categorifying skein modules, see Theorem \[32\].

The \(\mathfrak{sl}_2\) extremal weight projectors can be considered as categorifications of Chebyshev polynomials of the first kind by decategorifying their images to elements of the representation ring \(K_0(\text{Rep}(\mathfrak{sl}_2)) \cong \mathbb{Z}[X]\). Analogously, the extremal weight projectors for \(\mathfrak{gl}_N\) categorify power-sum symmetric polynomials in the representation ring of \(\mathfrak{gl}_N\). Such categorifications of classical orthogonal polynomials are of independent interest, see e.g. \[16\]. Motivated by this, we prove a categorified Newton identity.

**Theorem 3 (Theorem \[54\]).** The extremal weight projectors satisfy a categorified version of the Newton identity relating power-sum symmetric and elementary symmetric polynomials.

The main application for extremal weight projectors, however, is in categorifying toric skein modules. In a separate paper \[25\], we will construct a categorification of the \(\mathfrak{gl}_2\) skein module of the thickened torus via a toric \(\mathfrak{gl}_2\) foam category. The category \(2\mathsf{AWeb}^\mathrm{ess}\) describes morphism spaces in this foam category, with affine webs corresponding to rotationally symmetric foams. In particular, the toric foam category contains primitive idempotents given by rotation foams obtained from extremal weight projectors, and it is the target of a toric link homology functor. The use of \(\mathfrak{gl}_2\) foams, as opposed to Bar-Natan cobordisms \[2\], is necessary to guarantee the functoriality of the resulting link homology, see \[4, 8\]. In preparation for this construction, we prove a delooping lemma for \(\mathfrak{gl}_2\) webs and decomposition formulas for tensor products of \(\mathfrak{gl}_2\) extremal weight projectors in Section \[4\].

**Remark 4.** Affine web categories have appeared before in work of the first-named author \[22\] on skein modules, and of Cautis–Kamnitzer \[6\] on a K-theoretic version of the derived geometric Satake correspondence for \(SL_N\). The main differences are that here we work at \(q = 1\), which makes the affine web categories symmetric monoidal, and that we take a quotient by \(\mathbb{L}^k(V)\)-labeled essential circles for \(0 < k < N\). It is unclear to us how to define an analogous quotient for generic \(q\) that would admit extremal weight projectors.

**Remark 5.** Affine web categories at generic \(q\) describe morphism spaces in quantized toric foam categories, which can be defined using a quantized horizontal trace construction. This is analogous to the quantized annular Bar-Natan cobordisms of Beliakova-Putryna-Wehrli \[3\]. However, such a quantization involves a non-canonical choice of a simple closed curve on the torus, that breaks a natural mapping class group action which is desirable for categorified skein modules. In \[25\], we thus proceed with affine webs at \(q = 1\) and unquantized toric foam categories.

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2. AFFINE $\mathfrak{gl}_N$ WEBS AND EXTREMAL WEIGHT PROJECTORS

We start by recalling the diagrammatic calculus of $\mathfrak{gl}_N$ webs, which describes the category of representations of $U_q(\mathfrak{gl}_N)$ that is monoidally generated by exterior powers of the vector representation and their duals.

2.1. The category of $\mathfrak{gl}_N$ webs. The category $N\text{Web}_q$ of $\mathfrak{gl}_N$ webs is the $\mathbb{C}(q)$-linear pivotal tensor category with objects generated by points on the line $\mathbb{R}$ that are labeled with integers in the set $\{1, \ldots, N\}$ and that carry an orientation up or down. We may consider the tensor unit as a 0-labeled point without orientation. The morphisms are generated by trivalent graphs, properly embedded in the strip $\mathbb{R} \times [0, 1]$, with edges oriented and labeled in the same set, with a flow condition at each vertex imposing that the sum of incoming labels equals the sum of the outgoing ones. These graphs are interpreted as mapping from the bottom sequence of boundary points (with labels and orientations) to that at the top. The morphisms are considered modulo isotopy relative to the boundary and subject to the local relations

\[
\begin{align*}
\begin{bmatrix} k + l \\ k \end{bmatrix}_{k+l} &= \begin{bmatrix} N - k \\ l \end{bmatrix}_k, \\
\sum_{t} \begin{bmatrix} k - l + a - b \\ t \end{bmatrix}_{k-t} &= \begin{bmatrix} N - k - l \\ k \end{bmatrix}_k + \begin{bmatrix} N - k - l \\ k \end{bmatrix}_k
\end{align*}
\]

as well as their reflections and orientation reversals. The following relations are useful consequences of the ones above:

\[
\begin{align*}
\begin{bmatrix} N \\ k \end{bmatrix}_k = \begin{bmatrix} N \\ k \end{bmatrix}_k, & \quad \begin{bmatrix} N \\ k \end{bmatrix}_k = \begin{bmatrix} N \\ k \end{bmatrix}_k, & \quad \begin{bmatrix} N \\ k \end{bmatrix}_k = \begin{bmatrix} N \\ k \end{bmatrix}_k
\end{align*}
\]

2.2. Link with $U_q(\mathfrak{gl}_N)$-representation theory. Let $\text{Rep}(U_q(\mathfrak{gl}_N))$ denote the $\mathbb{C}(q)$-linear pivotal tensor category of $U_q(\mathfrak{gl}_N)$-representations that is monoidally generated by exterior powers of the vector representation and their duals. The main purpose of the diagrammatic calculus of $\mathfrak{gl}_N$ webs is to describe this category.

**Theorem 6.** There exists an equivalence of $\mathbb{C}(q)$-linear pivotal tensor categories

\[
\varphi : N\text{Web}_q \to \text{Rep}(U_q(\mathfrak{gl}_N))
\]

that sends $k$-labeled upward points to $k$-fold exterior powers of the vector representation of $U_q(\mathfrak{gl}_N)$.
Essentially, this theorem is due to Cautis–Kamnitzer–Morrison [7], although they state it for $U_q(\mathfrak{sl}_N)$. Versions for $U_q(\mathfrak{g}(N))$ have appeared in [24] [28]. We now describe the functor $\varphi$ explicitly.

Recall that $U_q(\mathfrak{g}(N))$ is the $\mathbb{C}(q)$-algebra generated by $E_i, F_i$ for $1 \leq i \leq N-1$ and $L_j^\pm 1$ for $1 \leq j \leq N$ subject to the following relations:

\begin{align}
(2.3) \quad L_i E_i &= q E_i L_i, \quad L_i F_i = q^{-1} F_i L_i, \quad L_{i+1} E_i = q^{-1} E_i L_{i+1}, \quad L_{i+1} F_i = q F_i L_{i+1} \\
(2.4) \quad [E_i, F_j] &= \delta_{i,j} \frac{L_i L_{j+1} - L_{j+1} L_i}{q - q^{-1}}, \quad [L_i, L_j] = 0.
\end{align}

\begin{align}
(2.5) \quad E_j^2 E_j - [2] E_i E_j E_i + E_j E_i^2 &= 0 \text{ if } |i - j| = 1 \text{ and } [E_i, E_j] = 0 \text{ otherwise; analogously for } F_s.
\end{align}

It is a Hopf algebra with coproduct, antipode and counit as follows:

\begin{align*}
\Delta(E_i) &= E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i, \quad \Delta(L_i^\pm 1) = L_i^\mp 1 \otimes L_i^\pm 1 \\
S(L_i^\pm 1) &= L_i^\mp 1, \quad S(E_i) = -E_i L_i^{-1} L_{i+1}, \quad S(F_i) = -L_i L_{i+1}^{-1} F_i \\
\varepsilon(L_i^\pm 1) &= 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0.
\end{align*}

Let $V = \mathbb{C}(q)\langle v_1, v_2, \ldots, v_N \rangle$ denote the vector representation of $U_q(\mathfrak{g}(N))$ and $V^* = \mathbb{C}(q)\langle v_1^*, v_2^*, \ldots, v_N^* \rangle$ its dual.

To leftward oriented cups and caps, the functor $\varphi$ associates the natural evaluation and co-evaluation maps for duals:

\begin{align*}
\bigcup & \quad \varphi \quad \begin{cases}
\mathbb{C} \to V \otimes V^* \\
1 \mapsto \sum_{k=1}^N v_k \otimes v_k^*
\end{cases} \\
\bigcup & \quad \varphi \quad \begin{cases}
V^* \otimes V \to \mathbb{C} \\
v_k^* \otimes v_l \mapsto \delta_{k,l}
\end{cases}
\end{align*}

Let $\bigwedge^k V$ denote the $k$-th exterior power of $V$. This has a basis indexed by subsets $S = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, N\}$ of size $k$. If $1 \leq i_1 < \cdots < i_k \leq N$, we use the following notation for the corresponding basis vector: $v_S := v_{i_1} \wedge \cdots \wedge v_{i_k}$. The dual $(\bigwedge^k V)^* \cong \bigwedge^k V^*$ then has the dual basis given by vectors $v_S$ and we have corresponding thick $k$ cap and cup morphisms as above.

\begin{align*}
\bigcup & \quad \varphi \quad \begin{cases}
\mathbb{C} \to \bigwedge^k V \otimes \bigwedge^k V^* \\
1 \mapsto \sum_{|S|=k} v_S \otimes v_S^*
\end{cases} \\
\bigcup & \quad \varphi \quad \begin{cases}
\bigwedge^k V^* \otimes \bigwedge^k V \to \mathbb{C} \\
v_S^* \otimes v_T \mapsto \delta_{S,T}
\end{cases}
\end{align*}

The other duality maps, that is, rightward oriented caps and cups, are perturbed by powers of $q$.

\begin{align*}
\bigcup & \quad \varphi \quad \begin{cases}
\mathbb{C} \to \bigwedge^k V^* \otimes \bigwedge^k V \\
1 \mapsto \sum_{|S|=k} q^{-\varepsilon_S} v_S^* \otimes v_S
\end{cases} \\
\bigcup & \quad \varphi \quad \begin{cases}
\bigwedge^k V \otimes \bigwedge^k V^* \to \mathbb{C} \\
v_S \otimes v_T^* \mapsto \delta_{S,T} q^{\varepsilon_S}
\end{cases}
\end{align*}

Here $\varepsilon_S = \sum_{i \in S} (N + 1 - 2i)$.

Merges of thick strands act as exterior product:

\begin{align}
(2.6) \quad \begin{cases}
\bigwedge^k V \otimes \bigwedge^l V \to \bigwedge^{k+l} V \\
v_S \otimes v_T \mapsto 0 \quad \text{if } S \cap T = \emptyset \\
v_S \otimes v_T \mapsto (-q)^{\varepsilon_S,T} v_{S \cup T} \otimes v_{S \setminus T} \quad \text{otherwise}
\end{cases}
\end{align}

Here $\varepsilon_{S,T}$ is the number of inversions in the concatenation of the ordered lists of elements of $S$ and $T$. The split vertex acts as follows:

\begin{align}
(2.7) \quad \begin{cases}
\bigwedge^{k+l} V \to \bigwedge^k V \otimes \bigwedge^l V \\
v_S \mapsto (-1)^{|l|} \sum_{T \subset S, |T| = k} (-q)^{-\varepsilon_{S \setminus T,T}} v_{S \setminus T} \otimes v_{S \setminus T}
\end{cases}
\end{align}
Analogous formulas hold for merges and splits of duals, which implies that merges and splits can be slid around caps and cups:

\[
\begin{align*}
\kappa \quad \lambda & \quad \varphi \quad \mapsto \quad \left\{ \begin{array}{ll}
\bigwedge^k V^* \otimes \bigwedge^l V^* & \to \bigwedge^{k+l} V^* \\
v_S^* \otimes v_T^* & \to \begin{cases}
0 & \text{if } S \cap T \neq \emptyset \\
(-1)^{kl}(-q)^{-\varepsilon S \cdot T} v_{S \cup T}^* & \text{otherwise}
\end{cases}
\end{array} \right.
\end{align*}
\]  

(2.8)

and:

\[
\begin{align*}
\kappa \quad \lambda & \quad \varphi \quad \mapsto \quad \left\{ \begin{array}{ll}
\bigwedge^{k+l} V^* & \to \bigwedge^k V^* \otimes \bigwedge^l V^* \\
v_S^* & \mapsto \sum_{T \subset S, |T|=k} (-q)^{-\varepsilon S \cdot T} v_{S \setminus T}^* \otimes v_T^*
\end{array} \right.
\end{align*}
\]  

(2.9)

This concludes the description of the functor \( \varphi \).

The category \( \text{Rep}(U_q(\mathfrak{g}_N)) \) is braided, and by virtue of Theorem 6, so is \( N\text{Web}_q \). The diagrammatic description of the braiding of two fundamental \( U_q(\mathfrak{g}_N) \)-representations in \( N\text{Web}_q \) is given as follows:

\[
(2.10)
\]

In particular, a crossing of two 1-labeled strands is given by:

\[
\begin{align*}
\kappa \quad \lambda & \quad \varphi \quad \mapsto \quad \left\{ \begin{array}{ll}
\bigwedge^k V^* & \to \bigwedge^k V^* \otimes \bigwedge^l V^* \\
v_S^* & \mapsto \sum_{T \subset S, |T|=k} (-q)^{-\varepsilon S \cdot T} v_{S \setminus T}^* \otimes v_T^*
\end{array} \right.
\end{align*}
\]

For negative crossings, one uses the above formulas with \( q \) inverted.

**Lemma 7.** The following analogs of Reidemeister moves hold in \( N\text{Web}_q \), where strands can carry all possible orientations and labels.

\[
q^{-k(N-1)} \quad \varphi \quad \mapsto \quad \left\{ \begin{array}{ll}
\bigwedge^k V^* & \to \bigwedge^k V^* \otimes \bigwedge^l V^* \\
v_S^* & \mapsto \sum_{T \subset S, |T|=k} (-q)^{-\varepsilon S \cdot T} v_{S \setminus T}^* \otimes v_T^*
\end{array} \right.
\]

We will refer to the last relation as a forkslide move.

**Proof.** The Reidemeister II, III and forkslide moves follow from the property of a braiding, and our braiding convention is only a minor rescaling of the one in [7, Corollary 6.2.3], see also [28, Section 2.4]. The Reidemeister I moves can be verified inductively as in [22, Lemma 2.9]. \( \square \)

**Definition 8.** We denote by \( N\text{Web}_q^+ \) the full subcategory of \( N\text{Web}_q \) with objects given by arbitrary sequences with exclusively upward pointing orientations.

In the following we will use the same superscript + to indicate analogous full subcategories of other web categories, consisting of those objects with upward (or outward) pointing orientations. The next lemma is a well-known consequence the proof of Theorem 6 using quantum skew Howe duality, see [7, 28].

**Lemma 9.** The morphism spaces of \( N\text{Web}_q^+ \) are spanned by upward-pointing webs, i.e. webs whose edges have no horizontal tangent vectors.
In the following, we will consider skein modules of isotopy classes of webs embedded in different surfaces, modulo the local relations from (2.1), and we will also vary the ground ring. In the following sections we deal with webs over $\mathbb{C}$, whose defining relations are obtained from (2.1) by specializing $q = 1$. We indicate categories of webs at $q = 1$ by the omission of the $q$-subscript, e.g. $NWeb$ instead of $NWeb_q$. The functor $\varphi$ also specializes to $q = 1$ and then relates $NWeb$ to the symmetric monoidal category of $U(gl_N)$-representations.

Note that setting $q = 1$ identifies the evaluation of positive and negative crossings in terms of webs in (2.10), and so we sometimes omit to display any over- or under-crossing information in graphics. In particular, the braid group action induced by 1-labeled crossings becomes a symmetric group action.

2.3. Affinization at $q = 1$. In [26], we considered an affine extension of the Temperley-Lieb category, and extended an analog of the functor $\varphi$ to this more general category. Just as in this simpler $sl_2$ case, we will consider a more general affine web category that will give us the freedom to extend the diagrammatic presentation of the representation category of $U(gl_N)$ to a Cartan subalgebra.

We define the category $N\text{AWeb}$ to be the $\mathbb{C}$-linear category of webs drawn on an annulus subject to the same local relations as in $NWeb$, i.e. relations (2.1) at $q = 1$. It is easy to see that the morphisms of $N\text{AWeb}$ can be obtained by gluing the strip in which diagrams in $NWeb$ live into an annulus, and adding new wrapping morphisms.

Note that we remember where the ends of the strip have been glued by drawing a dashed line between two base points on the boundary components of the annulus. We denote this segment by $\alpha$. We also stress that in $N\text{AWeb}$ webs can come with any orientation on the boundary.

2.4. Link with representation theory. We will extend the domain of the functor $\varphi$ from $NWeb$ to $N\text{AWeb}$ by sending the wrapping morphisms to maps between $U(gl_N)$-representations, which respect the weight space decomposition but break the $U(gl_N)$-action. This will allow us to build new diagrammatic projectors, and we will now explain how to choose this preferred extension. Note that other extensions via evaluation representations would allow us to preserve the $U(gl_N)$ action, but this would not serve our purpose of finding a diagrammatic presentation of the representation category of a Cartan subalgebra.

We first consider a single counterclockwise wrap morphism $D = D_1$ of a single 1-labeled outward pointing strand.

The requirement that $\varphi(D)$ respects the weight space decomposition of $V$ implies that $\varphi(D)(v_k) = \gamma_k v_k$ for some $\gamma_k \in \mathbb{C}$ and the desired invariance under ambient isotopy forces these scalars to be invertible. In fact, this choice of scalars determines the action of $\varphi(D_k)$, the $k$-labeled version of the wrap: by opening a $k$-blister and sliding one vertex around the wrap $D_k$, the eigenvalues of $\varphi(D_k)$ can be seen to be $k$-fold products of the eigenvalues of $\varphi(D)$: $\varphi(D_k)(v_S) = (\prod_{i \in S} \gamma_i) v_S$. Furthermore, inverse wraps have inverse eigenvalues: $\varphi(D_k^{-1})(v_S) = (\prod_{i \in S} \gamma_i)^{-1} v_S$. Next, we would like to have relations of the form:
To ensure that $\varphi$ respects such isotopy relations for sliding cups and caps around the annulus, we need to have $\varphi(D_k)(v_S) = (\prod_{i \in S} \gamma_i)^{-1} v_S$ and $\varphi(D_k^{-1})(v_S) = (\prod_{i \in S} \gamma_i)v_S$, which determine the maps assigned to inward pointing versions of $D$ and $D^{-1}$.

In order to be able to project onto the 1-dimensional spaces spanned by specific standard basis vectors in $V$, we would like $\varphi(D)$ to have distinct eigenvalues on the $v_k$. Furthermore, we would like to find a set of diagrammatic relations in the annular web category that enforces a choice of $\varphi(D)$ with distinct eigenvalues, or in other words, with a separable characteristic polynomial $\prod_{i=1}^N (X - \gamma_i) = \sum_{k=0}^N X^{N-k}(-1)^{k} e_k(\gamma)$. Here $e_k(\gamma) = e_k(\{\gamma_1, \ldots, \gamma_N\})$ denotes the $k$-th elementary symmetric polynomial evaluated at the complex numbers $\gamma_1, \ldots, \gamma_N$.

**Lemma 10.** The coefficients of the characteristic polynomial of $\varphi(D)$ are determined by the image of $\varphi$ on essential circles in the annulus. More precisely:

$$
\varphi \left( \begin{array}{c} k \\ \cdots \\ 1 \\ \cdots \\ 0 \end{array} \right) = \sum_{|S|=k} (\prod_{i \in S} \gamma_i) \text{id}_C = e_k(\gamma)
$$

**Proof.** The morphism can be written as the composition of a $k$-cup, a $k$-wrap and a $k$-cap. The sum $\sum_{|S|=k}$ comes from the cup and the factors from the action of the wrap on $v_S$. \qed

We now prescribe $\varphi(D)$ to have the separable characteristic polynomial $X^N - 1$, and we may index the roots as $\gamma_k = \zeta^k = e^{2\pi i k/N}$. This choice of relation is homogeneous with respect to a $\mathbb{Z}/N\mathbb{Z}$-grading by winding number, see [20]. We now extend the definition of $\varphi$ to the new generators $D$ in the general case, that is, allowing more than one strand.

**Definition 11.** Let $V \otimes W$ be the image under $\varphi$ of the domain of $D$ and $W \otimes V$ its co-domain. Then we define $\varphi(D)$ to be the linear map determined by $v_k \otimes w \mapsto \zeta^k w \otimes v_k$ for $v_k \in V$ and any $w \in W$. Furthermore we set $\varphi(D^{-1}) = (\varphi(D))^{-1}$ and analogously for the duals.

Let $h$ denote a Cartan subalgebra in $\mathfrak{gl}_N$ and consider $U(h) = (L_1^{\pm 1}, \ldots, L_N^{\pm 1}) \subset U(\mathfrak{gl}_N)$. We denote by $\text{Rep}(h)$ the category of finite-dimensional $U(h)$-representations of integral weights. Note that the inclusion $h \hookrightarrow \mathfrak{gl}_N$ induces a restriction functor $\text{Rep}(\mathfrak{gl}_N) \to \text{Rep}(h)$ and that $\varphi(D)$ and $\varphi(D^{-1})$ are morphisms in $\text{Rep}(h)$.

**Lemma 12.** The functor $\varphi: N\text{AWeb} \to \text{Rep}(h)$ is well-defined.

**Proof.** All morphisms in $N\text{AWeb}$ are compositions of caps or cups between adjacent strands, vertices, as well as the morphisms $D$ and $D^{-1}$. Any relation satisfied by compositions of these generating morphisms is either supported in $N\text{Web} \subset N\text{AWeb}$ (and is thus respected by $\varphi$) or involves some generators $D$ and $D^{-1}$. Since $\varphi$ maps $D$ and $D^{-1}$ to inverse isomorphisms, it suffices to check that $\varphi$ respects the isotopy relations that caps, cups and vertices can be slid along wraps around the annulus. However, the images under $\varphi$ of the wrap morphisms were precisely chosen for these relations to hold. \qed
2.5. The tensor product on annular webs. Let $\otimes: \mathbb{N}A\text{Web} \times \mathbb{N}A\text{Web} \to \mathbb{N}A\text{Web}$ denote the bi-functor given on objects by $(m,n) \mapsto m + n$ and on morphisms by superposing a pair of annular webs $(W_1, W_2)$:

and resolving all crossings via (2.10). This is well-defined thanks to Lemma 7 and it induces a symmetric monoidal structure on $\mathbb{N}A\text{Web}$ such that both the inclusion $\mathbb{N}\text{Web} \to \mathbb{N}A\text{Web}$ and $\varphi: \mathbb{N}A\text{Web} \to \text{Rep}(\mathfrak{h})$ become symmetric monoidal functors. Here we have used that $q = 1$.

Definition 13. For $m \geq 0$ we denote by $\mathbb{N}A\text{Web}(m)$ the endomorphism algebra in $\mathbb{N}A\text{Web}$ of the object consisting of a sequence of $m$ points with label 1 and outward orientation. We denote by $s_i$ for $1 \leq i \leq m$ the element of this endomorphism algebra that is given by the crossing between the strands in positions $i$ and $i + 1$, with positions understood modulo $m$. We also write $u_i = \text{id}_2 - s_i$ for the corresponding dumbbell web.

More generally, for two objects $\vec{k}$ and $\vec{l}$ we use the shorthand $\mathbb{N}A\text{Web}(\vec{k}, \vec{l}) := \text{Hom}_{\mathbb{N}A\text{Web}}(\vec{k}, \vec{l})$. We will also use these notation conventions in other web categories.

The following lemma will allow us to freely express webs in terms of images of 1-labeled tangles, which will be very useful in a number of proofs.

Lemma 14. Every element of $\mathbb{N}A\text{Web}(m)$ for $m \geq 0$ can be written as a $\mathbb{C}$-linear combination of 1-labeled annular tangles.

As we will see later, we may assume that the closed components are essential circles (possibly carrying higher labels) and the non-closed components are oriented 1-labeled arcs from the inner to the outer boundary circle, that are everywhere outward pointing.

Proof. We include this classical proof for completeness. It is well-known that every closed web can be written as a $\mathbb{C}$-linear combination of webs $W$ with only 1-labeled edges, interacting at most in 2-labeled dumbbells, see e.g. [28, Proof of Lemma 4.1]. Indeed, the argument can be inductively built from the following two operations:

$$
\begin{align*}
\begin{array}{c|c}
\kappa & = \frac{1}{k!} \quad \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \\
\kappa & = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array}
\end{align*}
$$

and

\begin{align*}
\begin{array}{c|c}
\begin{array}{c}
\kappa
\end{array} & = (-1)^k \\
\kappa - 1 & = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} \\
\kappa - 1 & = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{align*}

The remaining 2-labeled dumbbells can now be expanded in terms of crossings and their oriented resolutions, resulting in a linear combination of 1-labeled annular tangles. \hfill \square

An analogous result is true at generic $q$ over $\mathbb{C}(q)$.

Lemma 15. The endomorphism algebra of the empty object in $\mathbb{N}A\text{Web}$ is isomorphic to $\mathbb{C}[c_1, \ldots, c_{N-1}, c_N^{-1}]$, where $c_i$ denotes the counter-clockwise oriented $i$-labeled circle and $c_N^{-1}$ the $N$-labeled clockwise oriented circle.
Proof. The proof proceeds in two steps. First we show that any closed web in the annulus can be written as a $\mathbb{C}$-linear combination of collections of essential circles from the set $\{c_1, c_2, \ldots, c_N, c_N^{-1}\}$. The essential circles commute, which is easily seen by applying Reidemeister II moves, and $c_N$ and $c_N^{-1}$ are indeed mutually inverse. Second, we check that the counter-clockwise oriented circles are algebraically independent.

By Lemma 14 any closed web may be expressed as a linear combination of annular links. Moreover, we may assume that every link component wraps exclusively clockwise or counter-clockwise around the annulus. The annular evaluation algorithm from [23] implies that any such component can be resolved in terms of concentric circles of various labels. Hence, the same is true for the original web. To finish the first part, note that the clockwise-oriented circle of label $i$ is equal to $c_N^{-i}c_N^{-1}$. This follows from the last relation in (2.2).

In order to prove algebraic independence, we use an extension of the functor $\varphi$ from Lemma 12 Let $R = \mathbb{C}[X_{1}^{\pm 1}, \ldots, X_{N}^{\pm 1}]$ be a Laurent polynomial ring in $N$ variables and consider the category $R\text{Rep}(h)$ obtained by tensoring all morphism spaces in $\text{Rep}(h)$ by $R$. Then $\varphi_R : N\text{AWeb} \to R\text{Rep}(h)$ can be defined just as $\varphi$ was above, except that the $X_i$ now play the role of the eigenvalues of the 1-labeled counter-clockwise wrap: $\varphi_R(h)(v_k) := X_kv_k$.

For a closed annular web $W$, it follows that $\varphi(W) \in R$. In particular, counter-clockwise oriented circles $c_i$ evaluate to elementary symmetric polynomials $e_i(X)$ in the variables $X_i$ and their clockwise counterparts evaluate to $e_i(X^{-1})$. As a consequence of the first part of this proof, the evaluation of closed annular webs $\varphi_R(W)$ takes values in the symmetric part $R_{\text{sym}} \cong \mathbb{C}[e_1(X), \ldots, e_{N-1}(X), e_N(X)^{\pm 1}]$ of $R$. Now the algebraic independence of the $c_i$ follows from the algebraic independence of their images $e_i(X)$ under $\varphi_R$.

An analogous version of this result holds for $N\text{AWeb}_q$ over $\mathbb{C}(q)$. The second part in its $q = 1$ version is already sufficient to establish algebraic independence of the counter-clockwise essential circles in $N\text{AWeb}_q$.

Just as in the non-annular case, we denote by $N\text{AWeb}^+$ the full subcategory of $N\text{AWeb}$ with objects given by all upward, or outward, pointing boundary sequences. The following is the analog of Lemma 9.

Lemma 16. The morphism spaces of $N\text{AWeb}^+$ are spanned by webs with all edges outward oriented, but potentially superposed with essential circles.

Note that Lemma 15 allows us to restrict to counter-clockwise essential circles, except for the $N$-labeled ones.

Proof. We will prove the claim for webs $W$ whose source and target objects are only 1-labeled. The general claim follows by a usual merging argument.

Lemma 14 allows us to write $W$ as a linear combination of annular tangles. The closed components of these tangles evaluate to essential circles thanks to Lemma 15 while the non-closed components can be isotoped to be outward oriented arcs. (This shortcut is possible since we work at $q = 1$.) The superposition of such outward arcs is itself a linear combination of outward webs, as desired.

2.6 Equivalences between blocks. The categories $N\text{Web}$, $N\text{AWeb}$ and all of their specializations and quotients considered in the following decompose into blocks (i.e. full subcategories) indexed by $m \in \mathbb{Z}$, which consist of those objects, whose signed sum of labels equals $m$. Here we count upward oriented boundary points positively, and downward pointing ones negatively. We indicate such blocks by the subscript $m$, e.g. $N\text{Web}_m$.

We will use the notation $\lambda$ for the endofunctors of these categories that act on objects by tensoring with a single $N$-labeled upward boundary point and on morphisms by superposing with an upward-oriented $N$-labeled edge. We denote by $\lambda^*$ the analogous operation with downward orientations. We
give an example for $NAWeb$:

Here we display the $N$-labeled strand as crossing over the remaining web for better visibility, even though this has no significance at $q = 1$.

**Lemma 17.** The endofunctors $\lambda$ and $\lambda^*$ restrict to mutually quasi-inverse equivalences between the blocks $NAWeb_m$ and $NAWeb_{m+N}$.

**Proof.** Using isotopy relations, it is easy to see that the following types of webs provide natural isomorphisms between the identity functor on $2AWeb_m$ and the endofunctor given by the composition $\lambda^* \circ \lambda$:

Analogously, there are natural transformations between $\lambda \circ \lambda^*$ and the identity functor on $2AWeb_m$. □

**2.7. The quotients by essential circles.** It is a key observation that the functor $\varphi: NAWeb \to \text{Rep}(\mathfrak{h})$ is not faithful.

**Proposition 18.**

\[
\varphi \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix} = \begin{cases}
(-1)^{N-1}\text{id} & \text{if } k = N \\
0 & \text{otherwise}
\end{cases}.
\]

**Proof.** This follows from (2.12). □

**Definition 19.** We let $NAWeb^{\text{ess}}$ denote the quotient of $NAWeb$ by the ideal generated by the relations for $k < N$ shown in (2.12), with any number of through-strings. We define $NAWeb^{\text{ess}}$ to be quotient of $NAWeb$ by the ideal generated by all relations in (2.12).

Note that the monoidal structure $\otimes$ and the functor $\varphi$ descend to the quotients $NAWeb^{\text{ess}}$ and $NAWeb^{\text{ess}}$. One of the key results of this paper will be to prove that this category $NAWeb^{\text{ess}}$ is equivalent to the full subcategory of $\text{Rep}(\mathfrak{h})$ generated by the objects of $\text{Rep}(\mathfrak{gl}_N)$.

The category $NAWeb^{\text{ess}}$ is a central extension of $NAWeb^{\text{ess}}$ by the invertible element $c_N$ given by $N$-labeled counter-clockwise oriented essential circle, which we denote by $c_N$. Conversely, $NAWeb^{\text{ess}}$ is obtained from $NAWeb^{\text{ess}}$ by setting $c_N = (-1)^{N-1}$.

**Definition 20.** Let $W$ be a web in $NAWeb$. Then the flow winding number $w_f(W)$ of $W$ is given by the algebraic intersection number of the web with the segment $\alpha$ (assuming no trivalent vertex occurs on it), where $k$-labeled edges crossing $\alpha$ count as $\pm k$. 
It is clear that all web relations in $\mathcal{N}A\mathcal{W}eb$ and $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$ preserve the flow winding number. This $\mathbb{Z}$-grading on the morphism spaces of these categories descends to a $\mathbb{Z}/\mathbb{N}\mathbb{Z}$ grading on $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$.

The following corollaries are implied by Lemma 15 and Lemma 16.

**Corollary 21.** The endomorphism algebra of the empty object in $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$ is $\mathbb{C}$ and in $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$ it is isomorphic to the Laurent polynomial ring over $\mathbb{C}$ generated by an essential $\mathcal{N}$-labeled circle.

**Corollary 22.** The morphism spaces of $\mathcal{N}A\mathcal{W}eb^{\text{ess},+}$ are spanned by outward pointing webs. The same is true in $\mathcal{N}A\mathcal{W}eb^{\text{ess},+}$ up to superposition with an integer power of the $\mathcal{N}$-labeled essential circle $c_N$.

**Lemma 23.** In $\mathcal{N}A\mathcal{W}eb^{\text{ess},+}(1)$ we have $D_N = (-1)^{N-1}c_N$. In $\mathcal{N}A\mathcal{W}eb^{\text{ess}}(1)$ this specializes to $D_N = \text{id}_1$.

**Proof.** We compute:

\[
(-1)^{N-1}c_N = (-1)^{N-1} \quad \# \quad = \quad \# \quad \circ D
\]

\[
= (-1)^{N-1} \quad \# \quad \circ D + \quad \# \quad \circ D = \quad \# \quad \circ D^2
\]

\[
= \cdots = - \quad \# \quad \circ D^{N-1} + D^N = D^N
\]

The second equality arises by resolving the crossing while the third is an isotopy. The equalities then alternate between such which hold by expanding a crossing and such that use isotopies and (2.13).

**Remark 24.** Analogous one shows $\sum_{i=0}^{N} D^{N-i}(-1)^i c_i = 0$ in $\mathcal{N}A\mathcal{W}eb$, c.f. [6, Section 8.2].

Consider the algebra $\mathbb{C}[D^{\pm 1}]/(D^N - 1)$. Lemma 23 implies that this surjects onto the subalgebra of $\mathcal{N}A\mathcal{W}eb^{\text{ess}}(1)$ generated by wraps, and the flow winding grading implies that the surjection is an isomorphism. The Chinese remainder theorem implies $\mathbb{C}[D^{\pm 1}]/(D^N - 1) \cong \bigoplus_{j=1}^{N} \mathbb{C}[D]/(D - e^{i2\pi j/N})$ and we denote by $P_k \in \mathbb{C}[D]$ representatives for the idempotents that project onto the direct summands $\mathbb{C}[D]/(D - e^{i2\pi j/N})$. By abuse of notation we also write $P_k$ for the corresponding orthogonal idempotents in $\mathcal{N}A\mathcal{W}eb^{\text{ess}}(1)$. It is a straightforward but crucial observation that $\varphi(P_k(D))$ is the projection $V \twoheadrightarrow \mathbb{C}(v_k) \hookrightarrow V$.

**Theorem 25.** The functor $\varphi : \mathcal{N}A\mathcal{W}eb \rightarrow \text{Rep}(\mathfrak{h})$ is full.

**Proof.** We show that the induced functor on the quotient $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$ is full. For this, let $\vec{k}$ and $\vec{l}$ be objects in $\mathcal{N}A\mathcal{W}eb^{\text{ess}}$ and $\vec{k}$ be obtained from $\vec{k}$ by inverting orientations and the order of the sequence. We consider the isomorphisms of the form $f : \mathcal{N}A\mathcal{W}eb^{\text{ess}}(\vec{k}, \vec{l}) \rightarrow \mathcal{N}A\mathcal{W}eb^{\text{ess}}(\emptyset, \vec{l} \otimes \vec{k})$ and
their inverses, which are given by the following operations on diagrams.

\[
\begin{align*}
  f &= \begin{tikzpicture}[baseline={(0,0)}]
    \draw (-0.5,0) -- (0.5,0);
    \draw (-0.5,0.5) -- (0.5,0.5);
    \draw (-0.5,-0.5) -- (0.5,-0.5);
    \draw (0,0) -- (0,0.5);
  \end{tikzpicture}, &
  f^{-1} &= \begin{tikzpicture}[baseline={(0,0)}]
    \draw (-0.5,0) -- (0.5,0);
    \draw (-0.5,0.5) -- (0.5,0.5);
    \draw (-0.5,-0.5) -- (0.5,-0.5);
    \draw (0,0) -- (0,-0.5);
  \end{tikzpicture},
\end{align*}
\]

Since these operations are given by tensoring with an identity morphism and then pre-composing with cups, or post-composing with caps, there are corresponding isomorphisms \(\varphi(f^{\pm 1})\) between the relevant morphism spaces in the target category \(\text{Rep}(\mathfrak{h})\). To prove the theorem, it suffices to check that \(\varphi\) restricts to a surjective map from the morphism space \(\text{NAWeb}^\text{ess}(\emptyset,\bar{l} \otimes \bar{k})\) to \(\text{Hom}\text{Rep}(\mathfrak{h})(\mathbb{C}, \varphi(\bar{l} \otimes \bar{k}))\), which is isomorphic to the zero weight space in \(\varphi(\bar{l} \otimes \bar{k})\) by evaluating at \(1 \in \mathbb{C}\).

It is not hard to see that it suffices to prove this in the case where \(\bar{l} \otimes \bar{k}\) consists entirely of entries \(1\), the first \(n\) of which point outward and the last \(n\) inward. Indeed, if all weight zero vectors in this tensor product are hit, then composing with merge morphisms and permutations, one can hit any weight zero vector in a tensor product of fundamentals and their duals.

Actually, it is sufficient to find \(v_k \otimes v_k^*\) in the image of \(\varphi(-)(1)\) applied to \(\text{NAWeb}^\text{ess}(\emptyset, (1, 1^*))\), as then we can take diagrammatic tensor products of such generators, composed with permutations to find any standard basis vector of weight zero. Now we invert the bending process via the isomorphisms \(f^{\pm 1}\) and \(\varphi(f^{\pm 1})\), and the remaining problem becomes equivalent to finding the projection \(V \to \mathbb{C}\langle v_k \rangle \hookrightarrow V\) in the image of \(\varphi\). But this we have already seen; it is given by \(\varphi(P_k)\).

Later, we will prove that \(\varphi\) induces a functor from \(\text{NAWeb}^\text{ess}\) to \(\text{Rep}(\mathfrak{h})\) that is not only full, but also faithful (see Theorem 39). Nevertheless, we will continue to work in the more general framework of \(\text{NAWeb}^\text{ess}\) whenever possible.

2.8. Extremal weight projectors. In [26], we defined the concept of extremal weight projectors in the context of (affine) \(\mathfrak{sl}_2\) skein theory. This involved finding a suitable quotient of the affine Temperley-Lieb category, in which we identified a family of idempotents akin to Jones-Wenzl projector and corresponding, on the representation-theoretic side, to projections onto the direct sum of the top and bottom weight spaces in the tensor powers of the vector representation of \(U(\mathfrak{sl}_2)\). The same question naturally extends beyond the \(\mathfrak{sl}_2\) case, and the definition can be adapted and generalized to the \(\mathfrak{gl}_N\) case as follows.

**Definition 26.** The elements \(T_m \in \text{NAWeb}^\text{ess}(m)\) are recursively defined via:

- \(T_1 = \text{id}_1\),
- \(T_2 = \frac{1}{N} \sum_{k=0}^{N-1} D^{-k} \otimes D^k\),
- \(T_{m+1} = (\text{id}_{m-1} \otimes T_2)(T_m \otimes \text{id}_1)\) for \(m \geq 2\).

**Theorem 27.** The element \(\varphi(T_m)\) is the endomorphism of \(V^\otimes m\) projecting onto the extremal weight space \(\mathbb{C}\langle v_{i_1} \cdots v_i \rangle| i \in \{1, \ldots, N\}\) in \(\text{Sym}^m(V) \subset V^\otimes m\).

**Proof.** For \(m = 1\) this is tautological. For \(m = 2\) we compute \(\varphi(D^{-1} \otimes \text{id}_1)(v_{ij}) = \zeta^{-i}v_{ij}\) and \(\varphi(\text{id}_1 \otimes D)(v_{ij}) = \zeta^jv_{ij}\). Thus we have:

\[
\varphi(T_2)(v_{ij}) = \frac{1}{N} \sum_{k=0}^{N-1} \zeta^{k(j-i)}v_{ij} = \begin{cases} 
  v_{ii} & \text{if } i = j \\
  0 & \text{if } i \neq j 
\end{cases}
\]

For the vanishing, recall that \(0 = (X^N - 1) = (X - 1)(1 + X + \cdots + X^{N-1})\), so if \(X^N = 1\) but \(X \neq 1\), then \(X\) is a zero of the cyclotomic polynomial. This is precisely the case for \(\zeta^{(i-j)}\) for \(i \neq j\).
Lemma 28. In Lemma 29. see Definition 13.

Proof. We compute:

\[ \varphi(e_m \otimes T_2) \varphi(e_m \otimes id_1)(e_m \otimes T_1) = \varphi(e_m \otimes T_2)(e_m \otimes T_1) = \begin{cases} v \varepsilon \varepsilon & \text{if } k = \varepsilon \\ 0 & \text{if } k \neq \varepsilon \end{cases} \]

So \( \varphi(T_{m+1}) \) is the extremal weight projector. \( \square \)

In the following we show that the \( T_m \) are idempotents that satisfy a number of properties analogous to the extremal weight projectors \( \varphi(T_m) \). This will lead to a proof that \( \varphi \) is indeed faithful on \( \overline{\text{AWeb}} \). We start by studying properties of \( T_2 \).

Lemma 29. \( T_2 \) is an idempotent in \( \overline{\text{AWeb}} \) and thus also in the quotient \( \overline{\text{AWeb}} \).

Proof. We compute:

\[ T_2 = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} D^{-k-l} \otimes D^{k+l} = \frac{1}{N} \sum_{l=0}^{N-1} D^{-l} \otimes D^l = T_2 \]

Here we have used that \( D^{-N-i} \otimes D^{N+i} = ( -1 )^{2N-2c_N} c_N^{-1}(D^{-i} \otimes D^i) = D^{-i} \otimes D^i \).

Lemma 30. The idempotent \( T_2 \) absorb the crossing \( s = s_1 \) between its two strands. More precisely \( sT_2 = T_2s = T_2 \) in \( \overline{\text{AWeb}} \) and thus also in \( \overline{\text{AWeb}} \).

The diagrammatic proof is involved and we postpone it until Section 2A10.

Lemma 31. In \( \overline{\text{AWeb}} \) we have \( D^{-1}T_2D = T_2 \).

Proof. For this we rewrite \( T_2 \) in terms of \( D^{-1} \otimes id_1 = D^{-1}s \) and \( id_1 \otimes D = Ds \). Using Lemma 30 and \( (D^{-1}s)^N(Ds)^N = D^{-N} \otimes D^N = 1 \), we compute:

\[ D^{-1}T_2D = D^{-1}sT_2D = \frac{1}{N} \sum_{k=0}^{N-1} D^{-1}s(D^{-1}s)^k(Ds)^kDss = T_2s = T_2 \]

Theorem 32. The elements \( T_m \) of \( \overline{\text{AWeb}} \) satisfy the following properties:

1. \( T_m^2 = T_m \);
2. \( T_m(id_k \otimes T_n \otimes id_{m-n-k}) = (id_k \otimes T_n \otimes id_{m-n-k})T_m = T_m \) for \( 1 \leq n < m \) and \( 0 \leq k \leq m-n \);
3. \( (T_k \otimes id_{m-k})(id_{m-k} \otimes T_l) = (id_{m-k} \otimes T_l)(T_k \otimes id_{m-k}) = T_m \) for \( k + l > m \);
4. \( T_ms_i = s iT_m = T_m \) for \( m \geq 2 \);
5. \( Tムu_i = u_i T_m = 0 \) for \( m \geq 2 \);
6. \( D^{-1}T_mD = T_m \).

Here, \( s_i \) and \( u_i \) again refer to crossings and dumbbell webs between the strands in position \( i \) and \( i+1 \), see Definition 13. \( \square \)
Graphically, we write this as:

Finally the Corollary 33. Projectors.

Lemma 29 implies that these factors commute and so $T_m$ is an idempotent (1) that absorbs smaller $T_n$, i.e. (2). It is also clear that overlapping projectors $T_1$ and $T_k$ combine as in (3). The crossing absorption property of $T_2$ now implies the one for $T_m$ and crossings $s_i$ for $1 \leq i \leq m - 1$.

Using crossing absorption, we obtain the rotation conjugation invariance (6) from the $T_2$ case:

$$D^{-1}T_mD = D^{-1}(T_{m-1} \otimes \text{id}_1)(\text{id}_{m-2} \otimes T_2)(T_{m-1} \otimes \text{id}_1)D$$

$$= (\text{id}_1 \otimes T_{m-1})D^{-1}s_{m-1} \cdots s_2s_1(T_2 \otimes \text{id}_{m-2})s_1s_2 \cdots s_{m-1}D(\text{id}_1 \otimes T_{m-1})$$

$$= (\text{id}_1 \otimes T_{m-1})(D^{-1}T_2D) \otimes \text{id}_{m-2})(\text{id}_1 \otimes T_{m-1})$$

$$= (\text{id}_1 \otimes T_{m-1})(T_2 \otimes \text{id}_{m-2})(\text{id}_1 \otimes T_{m-1}) = T_m$$

This implies the missing crossing absorption relation (4)

$$T_ms_m = T_mD^{-1}s_{m-1}D = D^{-1}T_ms_{m-1}D = D^{-1}T_mD = T_m.$$  

Finally, the $u_i$ annihilation property (5) is equivalent to $s_i$ crossing absorption.

Now we can give an alternative recursion relation for $T_m$ for $m \geq 3$. This is the direct generalization of the defining recursive relation in [26, Definition 15], and it is reminiscent of the Jones-Wenzl projectors.

Corollary 33. The idempotents $T_m$ satisfy the following recursion for $m \geq 3$:

$$T_m = (T_{m-1} \otimes \text{id}_1)s_{m-1}(T_{m-1} \otimes \text{id}_1)$$

Graphically, we write this as:

(2.14)

Proof. We check this identity as follows.

$$(T_{m-1} \otimes \text{id}_1)s_{m-1}(T_{m-1} \otimes \text{id}_1) = (T_{m-1} \otimes \text{id}_1)s_{m-2} \cdots s_2s_1(\text{id}_1 \otimes T_{m-1})s_1s_2 \cdots s_{m-1}$$

$$= (T_{m-1} \otimes \text{id}_1)(\text{id}_1 \otimes T_{m-1})s_1s_2 \cdots s_{m-1} = T_ms_1s_2 \cdots s_{m-1} = T_m$$

The first equation holds by isotopy, the third by item (3) of Theorem 32 and the others follows from crossing absorption.

[Lemma 34. Let $m, n \in \mathbb{N}$ with $m + n \geq 3$, then $(T_m \otimes T_n)s_m(T_m \otimes T_n) = T_{m+n}$. This means, crossing-connected projectors can be combined.]

Proof. We may assume that $m \geq 2$ and compute:

$$(\text{id}_m \otimes T_n)(T_m \otimes \text{id}_n)s_m(T_m \otimes \text{id}_n)(\text{id}_m \otimes T_n) = (\text{id}_m \otimes T_n)(T_{m+1} \otimes \text{id}_{n-1})(\text{id}_m \otimes T_n) = T_{m+n}$$

Here we have used the projector recursion (2.14) and the fact that overlapping projectors can be combined, i.e. (3) in Theorem 32.
Next we consider the images of the idempotents $T_m$ in the quotient category $\text{NAWeb}^{\text{ess}}$. Recall the morphisms $\{P_a\}_{a \in 1, \ldots, N}$ that were introduced just before Theorem 25 as diagrammatic versions of projectors on eigenspaces. We will see in Lemma 35 that they can be combined to give an alternate definition of the extremal weight projectors, which amounts to saying that in the quotient category $\text{NAWeb}^{\text{ess}}$, extremal weight spaces can be broken into individual weight spaces.

**Lemma 35.** In $\text{NAWeb}^{\text{ess}}$ we have $(P_a \otimes P_b) \circ T_2 = T_2 \circ (P_a \otimes P_b) = \delta_{a,b}(P_a \otimes P_b)$.

**Proof.** The $\mathbb{C}$-algebra $R := \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]/(X^{N-1}, Y^{N-1})$ surjects onto the subalgebra of $\text{NAWeb}^{\text{ess}}(2)$ generated by wraps and their inverses via the map $1 \mapsto \text{id}_2$, $X \mapsto D \otimes \text{id}_1$ and $Y \mapsto \text{id}_1 \otimes D = Ds$. We will check the desired equalities in $R$, where $T_2$ is represented by $\sum_{k=0}^{N-1} X^{-k} Y^k$ and $P_a \otimes P_b$ is represented by $P_a(X)P_b(Y)$, which then implies that these equalities also hold in $\text{NAWeb}^{\text{ess}}(2)$.

Note that the idempotents $P_a(X)P_b(Y)$ decompose $R \cong \bigoplus_{a,b} P_a(X)P_b(Y)/R$ into 1-dimensional summands, which precisely consist of simultaneous eigenvectors for multiplication by $X$ and $Y$ with eigenvalues $\zeta_a$ and $\zeta_b$ respectively. Thus we can compute the action of $T_2$ on such an idempotent as:

$$P_a(X)P_b(Y)T_2 = P_a(X)P_b(Y) \frac{1}{N} \sum_{k=0}^{N-1} X^{-k} Y^k = P_a(X)P_b(Y) \frac{1}{N} \sum_{k=0}^{N-1} \zeta^{k(b-a)} = \delta_{a,b} P_a(X)P_b(Y).$$

**Corollary 36.** In $\text{NAWeb}^{\text{ess}}$ we have $T_2 = \sum_{k=1}^{N} P_k \otimes P_k$.

**Corollary 37.** In $\text{NAWeb}^{\text{ess}}$ we have $s(P_k \otimes P_k) = (P_k \otimes P_k)s = P_k \otimes P_k$.

**Proof.** We only consider composing with $s$ on the left:

$$s(P_k \otimes P_k) = sT_2(P_k \otimes P_k) = T_2(P_k \otimes P_k) = P_k \otimes P_k$$

Here we have used Lemma 35 twice and Lemma 30 in between. \hfill \Box

**Lemma 38.** In $\text{NAWeb}^{\text{ess}}$ we have $T_m = \sum_{k=1}^{N} \bigotimes_{m} P_k$.

**Proof.** We have observed this for $m = 2$ in Corollary 36. For $m = 1$ this follows from the decomposition $1 = \sum_{k=1}^{N} P_k$ in $\mathbb{C}[D]/(D^N - 1)$. Also, it is not hard to see that the $P_k$’s slide through crossings, e.g. $s \circ (P_k \otimes \text{id}_1) = (\text{id}_1 \otimes P_k)s$ in $\text{NAWeb}^{\text{ess}}(2)$. Now we proceed inductively for $m \geq 2$:

$$T_{m+1} = (T_m \otimes \text{id}_1)s_m(T_m \otimes \text{id}_1) = \sum_{k=1}^{N} \sum_{l=1}^{N} (P_k \otimes \cdots \otimes P_k \otimes \text{id}_1)s_m(P_l \otimes \cdots \otimes P_l \otimes \text{id}_1)$$

$$= \sum_{k=1}^{N} (\text{id}_{m-1} \otimes P_k \otimes \text{id}_1)s_m(P_k \otimes \cdots \otimes P_k \otimes \text{id}_1)$$

$$= \sum_{k=1}^{N} s_m(P_k \otimes \cdots \otimes P_k \otimes P_k) = \sum_{k=1}^{N} \bigotimes_{m} P_k$$

Here we have used the orthogonality of the idempotents $P_k$ to proceed to the second line and the sliding property to proceed to the third line. The final crossing absorption follows from Corollary 37. \hfill \Box

2.9. **Faithfulness of the diagrammatic presentation.** We will now combine the previous results to prove the following theorem.

**Theorem 39.** The functor $\varphi : \text{NAWeb}^{\text{ess}} \rightarrow \text{Rep}(\mathfrak{h})$ is faithful.

We partition the proof of the theorem into three parts.

**Proposition 40.** $\varphi$ is injective when restricted to the endomorphism algebra $\text{NAWeb}^{\text{ess}}(n)$. 

Proof. To see this result, we will exhibit a spanning set in $\text{NAWeb}^{\text{ess}}(n)$ that is sent under $\varphi$ to a linear basis. Consider $\varepsilon, \varepsilon' \in \{1, \cdots, N\}^n$ so that $\{|i, \varepsilon_i = k\} = \{|i, \varepsilon'_i = k\}$ for all $k = 1, \cdots, N$. Choose $\sigma'_\varepsilon \in \mathcal{S}_n$ to be the smallest in length so that $\varepsilon'_{\sigma(i)} = \varepsilon_i$. For example, it can be inductively defined by assigning to 1 the smallest $r$ so that $\varepsilon'_r = \varepsilon_1$, etc. Recall the notation $P_k$ for the polynomial such that $P_k(D) \in \text{NAWeb}^{\text{ess}}(1)$ is the projector onto the $\chi_k$ eigenspace of $D$. In $\text{NAWeb}^{\text{ess}}(n)$, denote $w_i = \text{id}_{i-1} \otimes D \otimes \text{id}_{n-i}$ the complete wrap of the $i$-th strand. We define:

$$
\varphi^\varepsilon_{\varepsilon'} := \sigma P_{s_n}(w_n) \cdots P_{s_1}(w_1).
$$

It is easy to see that the set $\{\varphi^\varepsilon_{\varepsilon'}\}$ is linearly independent, because it is so under $\varphi$.

On the other hand, we can deduce from Lemma~\ref{lem:injectivity} and its proof that this set spans $\text{NAWeb}^{\text{ess}}(n)$. Indeed, given the essential circle relations, we first deduce that any element $W \in \text{NAWeb}^{\text{ess}}(n)$ is made of a composition of elements from $\mathcal{S}_n \to \text{NAWeb}^{\text{ess}}(n)$ and the wraps $w_i$. From there, using far-commutation and the formulas:

$$
\begin{align*}
\bar{w}_i s_{i-1} &= s_{i-1} w_{i-1}, & \bar{w}_i s_i &= s_i w_{i+1},
\end{align*}
$$

one can see that this gives an algebra epimorphism $(w_i) \times \mathcal{S}_n \to \text{NAWeb}^{\text{ess}}(n)$.

Now, by construction, the polynomials $\{P_k(w)\}_{k \in \{0, \cdots, N-1\}}$ form a basis of $\mathbb{C}[w]/(w^N - 1)$, and since the $w_i$’s commute, it follows that the elements $P_{s_n}(w_n) \cdots P_{s_1}(w_1)$ span $\langle w_i \rangle_{i \in \{1, \cdots, n-1\}} \subset \text{NAWeb}^{\text{ess}}(n)$. Thus, any element in $\text{NAWeb}^{\text{ess}}(n)$ can be written as a linear combination of terms of the kind $\bar{\sigma} \varphi^\varepsilon_{\varepsilon'}$ with $\bar{\sigma}$ in the image of $\mathcal{S}_n$. It remains to see that $\bar{\sigma}$ can be assumed to be of minimal length, which is equivalent to saying that no two strands corresponding to the same value in $\varepsilon$ cross. Via isotopies, this reduces to the identities proven in Corollary~\ref{cor:centralizer}.

This proves that the set $\{\varphi^\varepsilon_{\varepsilon'}\}$ spans $\text{NAWeb}^{\text{ess}}(n)$ and concludes the proof. \hfill $\square$

**Proposition 41.** $\varphi$ is injective when restricted to morphism spaces in $\text{NAWeb}^{\text{ess}, +}$.

**Proof.** Suppose a linear combination $W$ of webs in such a morphism space are sent to zero under $\varphi$. Then for each web in this linear combination we pre-compose with a merge web $M$ and post-compose with a splitter web $S$ in order to obtain an endomorphism $SWM$ in $\text{NAWeb}^{\text{ess}}([k])$. Then we have $\varphi(SWM) = \varphi(S)\varphi(W)\varphi(M) = 0$ and Proposition~\ref{lem:injectivity} implies that $SWM = 0$. This implies $W = c M S W M S = 0$ where $c \neq 0$ is a scalar resulting from opening bigons. \hfill $\square$

**Proof of Theorem 29.** Let $W$ be a linear combination of webs in some morphism space of $\text{NAWeb}^{\text{ess}}$, which is sent to zero under $\varphi$. Then there exists a composition of invertible bending and braiding operations similar to those used in the proof of Theorem~\ref{thm:twist} that transforms $W$ into a linear combination $W'$ of webs in a morphism space as in Proposition~\ref{prop:twist}, which is also sent to zero under $\varphi$. The proposition then implies $W' = 0$ and, by invertibility of the operations, $W = 0$. \hfill $\square$

The final result of this section is best expressed in terms of Karoubi envelopes, the definition of which we recall now.

**Definition 42.** The Karoubi completion of a category $\mathcal{C}$ is the category with objects given by pairs $(X, e)$, where $X$ is an object of $\mathcal{C}$ and $e \in \text{Hom}_C(X, X)$ an idempotent. Morphisms between $(X, e)$ and $(Y, f)$ are of the form $f \circ g \circ e$ with $g \in \text{Hom}_C(X, Y)$.

If $\mathcal{C}$ is additive or monoidal, then the Karoubi completion $\text{Kar}(\mathcal{C})$ inherits these structures. If $\mathcal{C}$ is not yet additive but linear, then we pass to the additive closure before taking the idempotent completion as above. The resulting category $\text{Kar}(\mathcal{C})$ will be additive and linear. If the morphism spaces of $\mathcal{C}$ furthermore admit a $\mathbb{Z}$-grading, i.e. $\mathcal{C}$ is $\mathbb{Z}$-pre-graded, then so will be Karoubi completion, which we denote by $\text{Kar}(\mathcal{C})^*$. In this case, we reserve the notation $\text{Kar}(\mathcal{C})$ for the $\mathbb{Z}$-graded, additive, linear category whose objects are generated by formal grading shifts $w^k(X, e)$ of the objects $(X, e)$ in
Kar(\mathcal{C})^* and morphism required to be of degree zero. I.e. \( g: w^k(X,e) \to w^l(Y,f) \) is required to satisfy \( \deg(g) = l - k \).

**Corollary 43.** The functor \( \varphi \) induces an equivalence of additive, \( \mathbb{C} \)-linear pivotal categories:

\[
\text{Kar}(\mathcal{N}A\text{Web}^{\text{ess}})^* \simeq \text{Rep}(\mathfrak{h}).
\]

**Proof.** This follows from Theorems 25 and 39 since \( \text{Rep}(\mathfrak{h}) \) is already idempotent complete and any of its object can be written as the direct sum of \( \varphi \)-images of idempotents in \( \mathcal{N}A\text{Web}^{\text{ess}} \). \( \square \)

Let \( \text{Rep}^+(\mathfrak{h}) \) denote the full subcategory of \( \text{Rep}(\mathfrak{h}) \) containing only integral \( \mathfrak{h} \)-representations whose weights have non-negative entries.

**Remark 44.** The functor \( \varphi \) restricts to a fully faithful functor \( \mathcal{N}A\text{Web}^{\text{ess},+} \to \text{Rep}^+(\mathfrak{h}) \) that induces an equivalence of \( \mathbb{C} \)-linear monoidal categories \( \text{Kar}(\mathcal{N}A\text{Web}^{\text{ess},+})^* \simeq \text{Rep}^+(\mathfrak{h}) \). \( \square \)

2.10. **Proof of Lemma 30.** This section contains a proof of the fact that the \( T_2 \) projector absorbs crossings. It can be safely skipped on a first read-through.

In order to prove Lemma 30 we study the endomorphism algebra of the object 2 in \( \mathcal{N}A\text{Web}^{\text{ess}} \). For \( k \geq 1 \) we introduce the following notation:

\[
E_k := \quad , \quad B_k := \quad , \quad A_k := \quad , \quad D_2 := \quad.
\]

Here we set \( A_1 = 0, B_2 = D_2 \) and \( A_2 = \text{id} \), and doubled edges stand for 2-labeled edges. Note that in the definition of \( B_k \), we haven’t depicted the orientation of one of the strands: this is because it depends on \( k \). More precisely, we have:

\[
B_1 := , \quad B_2 = D_2, \quad B_k := \quad \text{if } k \geq 3.
\]

Note also that \( A_k = B_k = E_k = 0 \) for \( k > N \).

**Lemma 45.** The following statements hold in the endomorphism algebra of the 2-labeled upward point in \( \mathcal{N}A\text{Web}^{\text{ess}} \):

1. \( E_k = \delta_{k,N}c_N \).
2. \( D_2 \) is invertible and central,
3. \( B_k = A_kD_2 \) for \( k \geq 2 \)
4. \( B_1A_k = -E_{k-1} + A_{k-1}D_2 + A_{k+1} \) for \( k \geq 2 \)
5. \( B_N = E_N = c_N \) and thus \( A_N = c_ND_2^{-1} \).

Note that only (1) and (5) depend on the value of \( N \).
Proof. (1) holds by definition of \(\text{NAWeb}^{\infty}\), (2) and (3) follow from isotopies. For (4) we resolve the crossing in \(E_{k-1}\) to obtain:

\[
\begin{align*}
\ast & = \ast - \ast + \ast \\
\ast & = \ast - \ast + \ast \\
\ast & = B_1A_k
\end{align*}
\]

Here the first and third summands are \(B_{k-1} = A_{k-1}D_2\) and \(A_{k+1}\) respectively. The web in the second summand simplifies as follows:

\[
\begin{align*}
\ast & = \ast - \ast + \ast \\
\ast & = \ast - \ast + \ast \\
\ast & = B_1A_k
\end{align*}
\]

As a corollary, we get that the elements \(A_k\) can be written in terms of powers of \(B_1\) and \(D_2\):

**Corollary 46.** The elements \(A_k\) for \(k < N + 2\) satisfy the recursion \(A_k := B_1A_{k-1} - A_{k-2}D_2\) for \(k \geq 5\) with initial conditions \(A_3 = B_1\) and \(A_4 = B_1^2 - D_2\).

**Proof.** We induct on \(k\). For \(k = 1 < N\) we use Lemma 45 to obtain \(B_1 = B_1A_2 = -E_1 + A_1D_2 + A_3 = A_3\). Similarly, for \(k = 2 < N\), we get from Lemma 45 that \(B_1^2 - D_2 = B_1A_3 - A_2D_2 = -E_2 + A_4 = A_4\). For the recurrence relation we compute:

\[
B_1A_{k-1} - A_{k-2}D_2 = -E_{k-2} + A_k = A_k \quad \Box
\]

**Remark 47.** We will now find expressions for \(T_2\) which are more convenient in the following proof. First we introduce the notation \(t = D^{-1}sD\) for the rotation conjugate of the crossing. Now we use the facts that \(D^2\) is central in \(\text{NAWeb}^{\infty}\) (2) and that \((sD)^{-1}\) and \((Ds)\) commute to write \((sD)^{-k}(Ds)^k = (ts)^k\), which gives:

\[
T_2 = \frac{1}{N} \sum_{k=0}^{N-1} (ts)^k
\]

Further, we resolve the crossings as \(s = id_2 - u\) and \(t = id_2 - v\) where \(u = u_1\) is the 2-labeled dumbbell and \(v\) its conjugate. So we have:

\[
T_2 = \frac{1}{N} \sum_{k=0}^{N-1} ((id_2 - v)(id_2 - u))^k
\]

If we write \(M\) and \(S\) for the merge and split vertices on two strands, such that \(SM = u\), we get the following equality for \(k \geq 2\):

\[
\underbrace{\ldots vuvu}_{k \text{ factors}} = D^{1-k}SB_1^{k-1}M
\]

We will now attempt to rewrite the expressions \(X_n := D^{1-n}S\underbrace{A_n}_nM\) in terms of powers of \(B_1\). We can then use the relation \(X_N = 0\) to deduce a relation between compositions of the webs \(u\) and \(v\). To this end we define \(R_{2k-1} := \underbrace{(id_2 - v)(id_2 - u)\cdots(id_2 - v)}_{2k-1 \text{ factors}}\) and \(S_x := \sum_{k=1}^{x} R_{2k-1}\).
Lemma 48. For $2 \leq n \leq N$ we have:

$$X_n = \begin{cases} u - R_{n-1}u - uS_{n/2-1}u & \text{even } n \\ u - uS_{(n-1)/2}u & \text{odd } n \end{cases}$$

Proof. We will use the notation $Y_n$ for the entries on the right-hand side of the equation in the statement of the lemma. The proof of $X_n = Y_n$ proceeds by induction on $n$. For $n = 2$ we have $X_2 = vu = u - (id_2 - v)u = Y_2$. Similarly, for $n = 3$, we have

$$X_3 = D^{-2}SA_1M = D^{-2}S(B_1^2 - D_2)M = uvu - u = u - u(1 - v)u = Y_3$$

since $u^2 = 2u$ by the bigon relation. We prove the remaining cases recursively. For this, note that the elements $X_n = D^{1-n}SA_{n+1}M$ inherit a recurrence relation from the elements $A_{n+1}$:

$$(2.16) \quad X_n = D^{1-n}SA_{n+1}M = D^{1-n}SB_1A_nM - D^{1-n}SA_{n-1}D_2M$$

$$= D^{1-n}SB_1A_nM - D^{3-n}SA_{n-1}M = \begin{cases} uX_{n-1} - X_{n-2} & \text{even } n \\ uX_{n-1} - X_{n-2} & \text{odd } n \end{cases}$$

Here we have used $D^{1-n}SB_1 = vD^{2-n}S$ for even $n$ and $D^{1-n}SB_1 = uD^{2-n}S$ for odd $n$. Now it remains to check that the $Y_n$ also satisfy this recurrence (2.16). Indeed, for odd $N > 4$ we can verify:

$$Y_n - uY_{n-1} + Y_{n-2} = (u - uS_{(n-1)/2}u) - u(u - R_{n-2}u - uS_{(n-3)/2}u) + (u - uS_{(n-3)/2}u)$$

$$= u(-S_{(n-1)/2} + R_{n-2} + S_{(n-3)/2})u = 0$$

Here we have used that $S_x - S_{x-1} = R_{2x-1}$. In order to check the recurrence for $Y_n$ in the case of even $n$ we need an auxiliary computation. For odd $x \geq 1$ we have

$$S_x = (id_2 - v) + (id_2 - v)(id_2 - u)S_{x-1}$$

$$= (id_2 - v) + (id_2 - v)S_{x-1} - uS_{x-1} + vuS_{x-1}$$

$$= 2id_2 - v + (id_2 - u)S_{x-2} - uS_{x-1} + vuS_{x-1}$$

which implies:

$$vuS_{n/2-1}u = -2u + vu + S_{n/2}u - (id_2 - u)S_{n/2-2}u + uS_{n/2-1}u$$

$$= -2u + vu + R_{n-1}u + R_{n-3}u + uS_{n/2-2}u + uS_{n/2-1}u$$

Here we have used $(id_2 - v)S_x = id_2 + (id_2 - u)S_{x-1}$. Now we check the recurrence for even $n > 3$:

$$Y_n - vY_{n-1} + Y_{n-2} = (u - R_{n-1}u - uS_{n/2-1}u) - v(u - uS_{n/2-1}u) + (u - R_{n-3}u - uS_{n/2-2}u)$$

$$= 2u - vu - R_{n-1}u - R_{n-3}u - uS_{n/2-2}u - uS_{n/2-1}u + vuS_{n/2-1}u = 0$$

This completes the proof of the Lemma. $\square$

Proof of Lemma 40. We only prove $T_{2s} = T_2$, which is equivalent to $NT_2u = 0$ by expanding the crossing and multiplying by $N$. Using (2.15) we compute:

$$NT_2u = \sum_{k=0}^{N-1} \left( (id_2 - v)(id_2 - u) \right)^k u = u - \sum_{k=1}^{N-1} \left( (id_2 - v)(id_2 - u) \right)^{k-1} (id_2 - v)u = u - \sum_{k=1}^{N-1} R_{2k-1}u$$

Now note that $id_2 = (sD)^{-N}(Ds)^N = ((id_2 - v)(id_2 - u))^N$ implies that

$$(2.17) \quad R_{2k-1} = ((id_2 - u)R_{2N-2k-1}(id_2 - u))^{-1} = (id_2 - u)R_{2N-2k-1}(id_2 - u).$$
Now we distinguish two cases. For even $N$ we expand:

$$NT_2 u = u - \sum_{k=1}^{N/2} R_{2k-1} u - \sum_{k=N/2+1}^{N-1} R_{2k-1} u$$

$$= u - \sum_{k=1}^{N/2} R_{2k-1} u - (\text{id}_2 - u) \sum_{l=1}^{N/2-1} R_{2l-1}(\text{id}_2 - u)$$

$$= u - \sum_{k=1}^{N/2} R_{2k-1} u + (\text{id}_2 - u) \sum_{l=1}^{N/2-1} R_{2l-1} u = u - R_{N-1} u - uS_{N/2-1} u = X_N$$

Here we have used (2.17) for the second equality and Lemma 48 for the last equality. For odd $N$ we expand analogously:

$$NT_2 u = u - \sum_{k=1}^{(N-1)/2} R_{2k-1} u - \sum_{k=(N+1)/2+1}^{N-1} R_{2k-1} u$$

$$= u - \sum_{k=1}^{(N-1)/2} R_{2k-1} u + (\text{id}_2 - u) \sum_{l=1}^{(N-1)/2} R_{2k-1} u = u - uS_{(N-1)/2} u = X_N$$

We conclude the proof by noting that $X_N = D^{-N}S_{N+1} M = 0$ in $\overline{\mathbf{N} \text{Web}}^\infty$ since $A_{N+1} = 0$. □

**Remark 49.** The expression $X_N$ in the rewritten form in Lemma 48 expresses the longest Kazhdan–Lusztig basis element $H_{\text{sts...}} = H_{\text{sts...}}$ in the type $I_2(N)$ Hecke algebra in terms of products of $H_v := u$ and $H_v := v$, see [9] Section 2.3. In particular, the relation $X^N = 0$ suggests that $\overline{\mathbf{N} \text{Web}}^\infty(2)$ is related to a quotient of the Hecke algebra, by the 2-cell containing the basis element associated to the longest word.

### 3. Categorification of power-sum symmetric polynomials

Before turning to the topological applications of our work, we will in this section focus on identifying more precisely the structures that are categorified by the categories defined before. The main result of this section consists in a categorification of the Newton’s identities for power-sum and elementary symmetric polynomials (see Theorem 54).

Let $\mathbb{X} = \{X_1, \ldots, X_N\}$ be an alphabet of $N$ variables and denote by $\text{Sym}(\mathbb{X}) := \mathbb{C}[\mathbb{X}]$ the ring of symmetric polynomials in $\mathbb{X}$. Recall that $\text{Sym}(\mathbb{X}) \cong \mathbb{C}[e_1(\mathbb{X}), \ldots, e_N(\mathbb{X})]$, where $e_j(\mathbb{X})$ denotes the $j^{th}$ elementary symmetric polynomial in $\mathbb{X}$. We use the notation $h_j(\mathbb{X})$ for the $j^{th}$ complete symmetric polynomial.

**Definition 50.** The split Grothendieck group of an additive category $\mathcal{C}$ is the abelian group $K_0(\mathcal{C})$ defined as the quotient of the free abelian group spanned by the isomorphism classes $[X]$ of objects $X$ of $\mathcal{C}$, modulo the ideal generated by relations of the form $[A \oplus B] = [A] + [B]$ for objects $A, B$ of $\mathcal{C}$.

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ inherits a unital ring structure with multiplication $[A] \cdot [B] := [A \otimes B]$.

The following lemma is classical.

**Lemma 51.** There is an isomorphism

$$K_0(\text{Rep}^+(\mathfrak{sl}_N)) \otimes \mathbb{C} \cong K_0(\text{Kar}(\text{Rep}^+(\mathfrak{sl}_N))) \otimes \mathbb{C} \cong \text{Sym}(\mathbb{X}) \cong \mathbb{C}[e_1(\mathbb{X}), \ldots, e_N(\mathbb{X})]$$

sending the classes of the fundamental representations $[\wedge^k V]$ to the elementary symmetric polynomials $e_k(\mathbb{X})$. The class of the simple representation indexed by the partition $\lambda$ is then given by the Schur
polynomial \( \pi(\mathfrak{X}) \). If one includes duals, one obtains

\[ K_0(\text{Rep}(\mathfrak{gl}_N)) \otimes \mathbb{C} \cong K_0(\text{Kar}(\text{Rep}(\mathfrak{gl}_N))) \otimes \mathbb{C} \cong \mathbb{C}[\mathbb{X}^{\pm 1}]^{\mathbb{S}_N} \cong \mathbb{C}[e_1(\mathbb{X}), \ldots, e_{N-1}(\mathbb{X}), e_N^{\pm 1}(\mathbb{X})]. \]

For example, the classes of the symmetric and anti-symmetric power representations are related as follows.

\[ h_{m+1}(\mathbb{X}) = \sum_{i=1}^{m+1} (-1)^i h_{m+1-i}(\mathbb{X}) e_i(\mathbb{X}) \]

This can also be seen in the Grothendieck group of \( N \text{Web} \), at the cost of passing to the Karoubi envelope. For this, we recall the symmetric clasps [17], which are higher-rank analogs of Jones–Wenzl projectors, and their anti-symmetric counterparts.

**Definition 52.** The symmetric and anti-symmetric clasps \( P_m \in N \text{Web} \) and \( V_m \in N \text{Web} \) are defined by

\[ P_{m+1} = P_m - \frac{m}{m+1} P_m P_m, \quad V_{m+1} = \frac{1}{(m+1)!} V_m \]

Note that the clasps are related by:

\[ P_{m+1} = P_m - \frac{2m}{m+1} P_m V_m, \quad V_{m+1} = V_m - \frac{2m}{m+1} V_m P_m \]

It is well-known that \( \varphi \) sends \( P_m \) and \( V_m \) to the projections onto simple representations in \( V \otimes m \) given by the \( m \)-fold symmetric and anti-symmetric powers of the vector representation respectively.

**Theorem 53.** In \( \text{Kar}(N \text{Web}) \) there is an isomorphism

\[ \bigoplus_{i=0}^{\left\lfloor \frac{m+1}{2} \right\rfloor} (k, V_{k-2i} \otimes P_{2i+1}) \simeq \bigoplus_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \bigoplus_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (k, V_{k-2j} \otimes T_{2i}) \]

which categorifies (3.1).

The proof of this is similar to but easier than the proof of Theorem 54 below, and thus omitted.

### 3.1. Categorified Newton’s identities.

We now explicitly show that the projectors \( T_m \) categorify the power-sum symmetric polynomials \( p_m(\mathbb{X}) = X_1^m + \cdots + X_N^m \) in the same sense as the clasps \( P_m \) categorify the complete symmetric polynomials. To this end, we prove that the projectors \( T_m \) satisfy categorified versions of the classical Newton identities:

\[ p_k(\mathbb{X}) = (-1)^{k-1} k e_k(\mathbb{X}) - \sum_{j=1}^{k-1} (-1)^{k-j} e_{k-j}(\mathbb{X}) p_j(\mathbb{X}) \quad \text{for } 1 \leq k \]

**Theorem 54.** In \( \text{Kar}(N A \text{Web}^{\text{new}})^* \), there is an isomorphism:

\[ \bigoplus_{i=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (k, V_{k-2i} \otimes T_{2i+1}) \simeq \bigoplus_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \bigoplus_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (k, V_{k-2j} \otimes T_{2i}) \]

**Proof.** The desired isomorphism takes the shape of a “zig-zag”, i.e. each direct summand maps non-trivially to at most two direct summands on the other side. A typical segment of the zig-zag looks as
Above, gray-colored boxes stand for the anti-symmetric clasps $V_m$, as pictured in Definition 52, while the other rectangles are our extremal weights projectors $T_m$. The isomorphism takes this form locally only when $l \geq 2$, and we will deal with the $l = 1$ term at the end.

We check that the composite of the maps to the right with the maps to the left induce the identity on the components of the left. For this, we compute

\[
(k - l + 1) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l-1 & i & \ldots \\
  k-l & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
\begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l & i & \ldots \\
  k-l-1 & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
= -(k - l - 1) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l & i & \ldots \\
  k-l-1 & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
+ (k - l) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l & i & \ldots \\
  k-l-1 & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
\]

by expanding the middle projectors on the left-hand side using the recursions in (2.14) and 52. Adding both equations, we obtain the desired equality:

\[
(k - l + 1) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l-1 & i & \ldots \\
  k-l & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
+ (k - l) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l & i & \ldots \\
  k-l-1 & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
= \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l & i & \ldots \\
  k-l-1 & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
\]

We also check that all other components of this endomorphism of the left-hand side are zero:

\[
(k - l) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l+1 & i & \ldots \\
  k-l & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
= 0 = (k - l + 1) \begin{array}{ccc}
  \ldots & t & \ldots \\
  k-l+1 & i & \ldots \\
  k-l & i & \ldots \\
  \ldots &  & \ldots \\
\end{array}
\]

These expressions are zero because the middle projectors can be absorbed into the top and bottom projectors respectively. The results have anti-symmetric clasps and extremal weight projectors that share two strands, which forces them to equal zero.

To finish the proof, we need to look at the top and bottom ends of the zig-zag. The top end, which contains $(k, T_k)$ is treated precisely as in the generic case. The bottom end is more interesting, as it involves $k$ copies of the object $(k, V_k)$. 
In order to obtain the desired isomorphism, one needs to express the following term as a sum of \( k \) orthogonal projections onto \( k \) copies of the exterior power:

\[
\sum_{A \subset \{0, \ldots, N-1\}} |A| = k, \quad s \in \mathcal{S}_k
\]

and more generally in the case of \( k \) parallel strands, for a tuple \( I = (i_1, \ldots, i_k) \) we use the diagrams

\[
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from the first \(k - 1\) ones, or a repeat. Now, we make use of the following equality:

\[
\begin{array}{ccc}
2 & 2 & 2 \\
\hline
2 & 2 & 2 \\
\end{array}
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
\hline
\otimes & \otimes & \otimes \\
\end{array} = (-1)^2 
\begin{array}{ccc}
2 & 2 & 2 \\
\hline
2 & 2 & 2 \\
\end{array}
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
\hline
\otimes & \otimes & \otimes \\
\end{array}
\]

to rewrite the two summands in (3.3). The first sum is partitioned into \(k\) terms, the \(j\)-th of which contains those projectors \(sA\) that project to the \(j\)-th largest element of \(A\) on the right-most strand. For a fixed \(A\) and \(j\), there are \((k - 1)!\) permutations \(s\) such that \(sA\) is of this type, each of which produces an identical summand by the previous equation.

In the second summand, we similarly reorder the \(x\) in \(sB\) to the right-most strand, which collects together \(k - 1\) identical terms.

\[
\begin{align*}
(3.4) \quad & \quad \begin{array}{ccc}
2 & 2 & 2 \\
\hline
2 & 2 & 2 \\
\end{array}
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
\hline
\otimes & \otimes & \otimes \\
\end{array} = \left( \sum_{j=1}^{k} \sum_{A \subseteq \{0,\ldots,N-1\} \mid |A| = k, \ x \ j\text{-th largest entry of } A} (k - 1)! \right) + (k - 1) \left( \sum_{x \in \{0,\ldots,N-1\} \mid |A| = k, \ |C| = k-2 \sum_{s \in \mathfrak{S}_{k-2}} \right)
\end{align*}
\]

The remaining summation in the second term is over \((k - 2)\)-element subsets \(C\) and their permutations. Faithfulness of \(\varphi\) implies that the anti-symmetric clasps absorb this sum of projectors and so we get:

\[
(k - 1) \left( \sum_{x \in \{0,\ldots,N-1\} \mid |A| = k, \ |C| = k-2 \sum_{s \in \mathfrak{S}_{k-2}} \right) = (k - 1) \left( \sum_{x \in \{0,\ldots,N-1\} \mid |A| = k, \ |C| = k-2 \sum_{s \in \mathfrak{S}_{k-2}} \right) = (k - 1)
\]

The \(k\) terms in the first summand in (3.4) are clearly orthogonal to each other and to the second summand, from which it follows that they are idempotents. It remains to argue that they are the are isomorphic to anti-symmetric clasps in the Karoubi envelope. To this end, we fix a \(j\) and argue that

\[
\sum_{A \subseteq \{0,\ldots,N-1\} \mid |A| = k, \ x \ j\text{-th largest entry of } A} (k - 1)! \quad \text{and} \quad \sum_{A \subseteq \{0,\ldots,N-1\} \mid |A| = k, \ x \ j\text{-th largest entry of } A} k!
\]

give the desired inverse isomorphisms. So we simplify both composites:

\[
\sum_{A,B \subseteq \{0,\ldots,N-1\} \mid |A| = |B| = k, \ x \ j\text{-th largest entry of } A \ y \ j\text{-th largest entry of } B} k!(k - 1)! = \sum_{A \subseteq \{0,\ldots,N-1\} \mid |A| = k} k! = \sum_{A \subseteq \{0,\ldots,N-1\} \mid |A| = k} k!
\]

Summands in the composite of the left-hand side are zero unless \(x = y\) and \(A \setminus x = B \setminus y\) and the anti-symmetric clasp is absorbed at the cost of dividing by \((k - 1)!\). The second equality is just a
reordering, while the third uses that the anti-symmetric projectors absorb the sum of all $A$-projectors at the cost of dividing by $k!$. For the other composite we get:

$$\sum_{A,B \subseteq \{0,\ldots,N-1\}} k!(k-1)! \quad \sum_{A \subseteq \{0,\ldots,N-1\}} (k-1)!$$

Here we have used the same absorption property as in the first step in the computation of the other composite. □

**Remark 55.** In the case $k = N$ we can give an alternative characterization of the more mysterious part of the isomorphism in Theorem 54, which involves the $N$-fold direct sum of objects $(N,V_N)$. Indeed, after (3.4) and the following displayed equation, the goal was to decompose

$$\sum_{A \subseteq \{0,\ldots,N-1\}} (k-1)! \quad \sum_{s \in S_N}$$

into a sum of $N$ orthogonal idempotents, which are individually isomorphic to $(N,V_N)$. It is easy to check that the above is equal to

$$\sum_{x=1}^{N} (\text{id}_{N-1} \otimes D^{-x})V_N(\text{id}_{N-1} \otimes D^x),$$

which is manifestly a sum of idempotents, which are orthogonal since $V_N(\text{id}_{N-1} \otimes D^{k-1})V_N = \delta_{k,l}V_N$.

**Question 56.** Can the extremal weight projectors and symmetric clasps be used to give categorifications of the following identities?

(3.5)  

$$kh_k(\mathbb{X}) = \sum_{j=1}^{k} h_{k-j}(\mathbb{X})p_j(\mathbb{X}) \quad \text{for } 1 \leq k \leq N$$

An isomorphism categorifying this identity for $k = 2$ is easy to construct. For $k \geq 3$ such an isomorphism cannot be of zig-zag shape as for (3.2).

### 3.2. Categorification of the symmetric polynomial ring

An easy consequence of Corollary 43 is the following.

**Lemma 57.** There are isomorphisms  

$$K_0(\text{Kar}(N\text{AWeb}^{\text{ess},+})^\ast) \otimes \mathbb{C} \cong K_0(\text{Rep}^+(\mathfrak{h})) \otimes \mathbb{C} \cong \mathbb{C}[\mathbb{X}]$$

sending the object $(1, P_i)$ to $[\mathbb{C}(v_i)]$ and further to $X_i$.

We have seen that the extremal weight projectors in $N\text{AWeb}^{\text{ess},+}$ categorify the power sum symmetric polynomials. However, by Lemma 57 the Grothendieck group of the Karoubi envelope of $N\text{AWeb}^{\text{ess},+}$ is larger than the symmetric polynomial ring $\text{Sym}(\mathbb{X}) \cong K_0(\text{Kar}(N\text{Web}^+)) \cong K_0(\text{Rep}^+(\mathfrak{gl}_N))$. To see this, recall that the objects in $\text{Rep}^+(\mathfrak{h})$ are direct sums of non-negative integral $\mathfrak{gl}_N$ weight spaces. However, in the Grothendieck group, such direct sums can be written as formal differences of $\mathfrak{gl}_N$-representations only if they are orbits of the action for the Weyl group $\mathfrak{S}_N$. In this section, we identify a sub-category of $N\text{AWeb}^{\text{ess},+}$ that is $\mathfrak{S}_N$-equivariant, that contains the extremal weight projectors and has $\text{Sym}(\mathbb{X})$ as Grothendieck group.
Definition 58. We let \( \text{Rep}(\mathfrak{g})^{\otimes N} \) denote the subcategory of \( \text{Rep}(\mathfrak{g}) \) with objects that are invariant under \( \mathfrak{S}_N \) and morphisms that are \( \mathfrak{S}_N \)-equivariant.

Lemma 59. The category \( \text{Rep}(\mathfrak{g})^{\otimes N} \) is semi-simple and the homomorphism

\[
\text{Sym}(X) \cong K_0(\text{Rep}^+(\mathfrak{g})) \otimes \mathbb{C} \to K_0(\text{Rep}(\mathfrak{g})^{\otimes N}) \otimes \mathbb{C}
\]

induced by the inclusion is an isomorphism.

Proof. The indecomposable objects in \( \text{Rep}(\mathfrak{g})^{\otimes N} \) are orbits of the form \( \mathbb{C}\langle v_{s(\epsilon_1),\ldots,s(\epsilon_N)}|s \in \mathfrak{S}_N \rangle \).
Through the braiding, such an object is isomorphic to an \( \mathfrak{S}_N \)-orbit of a vector \( v_{0,0,1,\ldots,1,N-1} \) with multiplicities \( n_i \) of the weights \( i \) determined by a partition \( \lambda : n_0 \geq n_1 \geq \cdots \geq n_{N-1} \) of \( n \). There are no morphisms between distinct indecomposables and their endomorphism algebras are 1-dimensional over \( \mathbb{C} \). This shows that \( \text{Rep}(\mathfrak{g})^{\otimes N} \) is semi-simple. The isomorphism follows since the classes of these indecomposables can be expressed as linear combinations of the classes of tensor products of fundamental representations in the same way as monomial symmetric polynomials can be expressed as polynomials in elementary symmetric polynomials. \( \square \)

We aim to describe the subcategory \( \text{Rep}(\mathfrak{g})^{\otimes N} \) of \( \text{Rep}(\mathfrak{g}) \) by a subcategory of \( \text{NAWeb}_s^{+,\text{ess}} \).

Definition 60. Let \( \text{NAWeb}_s^{+,\text{ess}} \) denote the symmetric monoidal \( \mathbb{C} \)-linear subcategory of \( \text{NAWeb}_s^{+,\text{ess}} \) with the same objects, but with morphisms spaces generated (under tensor product and composition) by morphisms in \( \text{NWeb}^+ \) and the extremal weight projectors \( T_m \) for \( m \geq 1 \).

Note that the restriction of \( \varphi \) to the subcategory \( \text{NAWeb}_s^{+,\text{ess}} \) has image contained in \( \text{Rep}(\mathfrak{g})^{\otimes N} \).

Proposition 61. The functor \( \varphi : \text{NAWeb}_s^{+,\text{ess}} \to \text{Rep}(\mathfrak{g})^{\otimes N} \) is fully faithful and induces an equivalence of \( \mathbb{C} \)-linear monoidal categories \( \text{Kar}(\text{NAWeb}_s^{+,\text{ess}}) \cong \text{Rep}(\mathfrak{g})^{\otimes N} \).

Proof. Faithfulness is inherited from Theorem [20]. We shall prove fullness by showing that the image of \( \varphi \) contains the projections onto the simple objects in \( \text{Rep}(\mathfrak{g})^{\otimes N} \) as identified in the proof of Lemma [59]. Indeed, if \( \lambda : n_0 \geq n_1 \geq \cdots \geq n_{N-1} \) of \( n \), then we will construct an idempotent morphism in \( \text{NAWeb}_s^{+,\text{ess}} \) that projects onto the \( \mathfrak{S}_N \)-orbit of the vector \( v_{0,0,1,\ldots,1,N-1} \) with weights \( i \) appearing with multiplicities \( n_i \).

To this end, we first define an auxiliary projector \( O_n \) in \( \text{NAWeb}_s^{+,\text{ess}} \) for the case where \( n_i \in \{0,1\} \) for \( 1 \leq i \leq N \). We set \( O_1 = \text{id}_1 \) and \( O_2 = \text{id}_2 - T_2 \). For \( n \geq 2 \) we inductively define:

\[
O_{n+1} := s_1(\text{id}_1 \otimes O_n)s_1(\text{id}_1 \otimes O_n)(O_n \otimes \text{id}_1)
\]

It is easy to check that the image of \( O_n \) under \( \varphi \) is the desired projection, and so the \( O_n \) are the desired diagrammatic idempotents by faithfulness of \( \varphi \). It is also clear that the \( O_n \) are contained in \( \text{NAWeb}_s^{+,\text{ess}} \).

Now let \( \lambda : n_0 \geq n_1 \geq \cdots \geq n_k \) be a partition of \( n \) with \( k \) non-zero parts \( n_i \). Then consider the projector built as the composite of \( T_{n_k} \otimes \cdots \otimes T_{n_k} \), the permutation given as the product of transpositions that interchange the strands in position \( n_i \) and \( n - i \) for \( 1 \leq i \leq k \), the projector \( \text{id}_{n-k} \otimes O_k \), the inverse permutation and again \( T_{n_k} \otimes \cdots \otimes T_{n_k} \).

The image of this element under \( \varphi \) is the idempotent projecting onto the \( \mathfrak{S}_N \)-orbit of the vector \( v_{0,0,1,\ldots,1,N-1} \) with weights \( i \) of multiplicities \( n_i \), and by faithfulness of \( \varphi \) it is itself an idempotent in \( \text{NAWeb}_s^{+,\text{ess}} \).

\( \square \)

Corollary 62. \( \text{Kar}(\text{NAWeb}_s^{+,\text{ess}}) \) categorifies the symmetric polynomial ring \( \text{Sym}(\mathbb{C}) \) and its objects \( T_m \) categorify the power-sum symmetric polynomials.
Remark 63. The isomorphisms of Theorem 54, which categorify the Newton identities, holds in $\text{N AWeb}_+^{\text{ess}}$, although the direct sum decomposition $\bigoplus_k (k, V_k)$ on the right-hand side is not $\mathfrak{S}_N$-equivariant.

4. Special properties of the $\mathfrak{gl}_2$ case

We now review some of the special properties of the extremal weight projectors in the $N = 2$ case. In this context, we encode 2-labeled edges in webs as double edges and henceforth omit the labels. For convenience, we list the $\mathfrak{gl}_2$ web relations for generic $q$ separately.

\[
\begin{align*}
\bigcirc & = (q + q^{-1}) \emptyset = \bigcirc, & \bigcirc & = \emptyset = \bigcirc \\
\bigcirc & = (q + q^{-1}) \bigcirc, & \bigcirc & = \bigcirc = \bigcirc \\
\bigcirc & = \bigcirc = \bigcirc = \bigcirc = \bigcirc \\
\end{align*}
\]

In fact, $\mathfrak{gl}_2$ webs satisfy generalizations of the 1-labeled circle relation in (4.1) that we now describe.

Lemma 64 (The delooping lemma). Let $W$ be a $\mathfrak{gl}_2$-web (in a disc or some other surface) and denote by $c(W)$ the unoriented multi-curve obtained by erasing all 2-labeled edges. Suppose that $c(W)$ contains a circle $c$ which bounds a disc $D$ in the complement of $c(W)$. Then $W = (q + q^{-1}) V$, where $V$ is a web that agrees with $W$ outside a neighborhood of the disc $D$ and with underlying curve $c(V)$ obtained by removing the circle in question from $c(W)$.

Proof. We only consider $W$ in a neighborhood of the disc $D$ bounded by $c$. We will find a sequence of web relations which reduce the interaction of 2-labeled edges with $c$ until $c$ can be removed via a the first relation in (4.1). There are three types of interaction of $c$ with 2-labeled edges to consider in sequence:

1. Any 2-labeled circle contained in $D$ can be removed using one of the relations in (4.1), starting with an innermost one.
2. Suppose there exists a 2-labeled edge in the interior of $D$ with boundary on $c$. We take an innermost such edge, i.e. one which encloses a region in the disc with no other 2-labeled edges in the interior. Such an intersection edge can be removed via the bigon relations in (4.2), provided there are no 2-labeled edges hitting the boundary of $D$ from the outside in the relevant region. Otherwise, jump to (3) to remove external edges first. Note that they always come in pairs for orientation reasons.
3. There is a pair of 2-labeled edges, hitting $c$ from the outside $D$, which are adjacent in the sense that an arc along $c$ connects them without hitting other 2-labeled edges. Then one application of the saddle relations in (4.3) creates a 2-labeled edge connecting two points on $c$ from the outside (see the right side of Figure 4), which can be removed as in (2).

This algorithm relates $W$ to a web that contains $c$ as an oriented 1-labeled circle that can be removed via (4.1).

In the following, we again work in the $q = 1$ specialization.

Lemma 65. The morphism spaces in $\text{AWeb}^{\text{ess}}_+^+$ are spanned by outward pointing webs, except for the endomorphism space of the empty object, which is isomorphic to $\mathbb{C}[c_2^{\pm 1}]$.

Proof. This follows from Corollaries 21 and 22.
4.1. **Decomposing the tensor product of extremal weight projectors.** We have seen that the tensor product of extremal weight projectors $T_m \otimes T_n$ contains a copy of $T_{m+n}$. For $\mathfrak{gl}_2$ we will explicitly describe the difference $T_m \otimes T_n - T_{m+n}$ in terms of the projector $T_{|m-n|}$. The situation here is very similar to the $\mathfrak{sl}_2$-case investigated in [26].

Let $p\text{Tr}_1$ denote the linear maps on the morphism spaces of $\mathcal{2AWeb}^+$ that act on a web $W$ by first tensoring with $\text{id}_1$ and then pre- and post-composing the result with splitter and merge webs between the new strand and the two rightmost 1-labeled bottom and top boundary strands if they exist—otherwise we declare the result to be zero. We use the shorthand $p\text{Tr}_n := (p\text{Tr}_1)^n$. The following is an example of $p\text{Tr}_2$ applied to a web $W$:

We can decompose $p\text{Tr}_n(W) = M_n(W \otimes \text{id}_n)S_n$ where $S_n$ is a splitter web and $M_n$ is a merge web:

(4.4) \[ S_n = \quad \text{and} \quad M_n = \]

Recall that $\lambda$ denotes the endofunctor of $\mathcal{2AWeb}^{\text{ess}}$ given on morphisms by tensoring with a 2-labeled strand on the right as shown in the following:

**Lemma 66.** The extremal weight projectors in $\mathcal{2AWeb}^{\text{ess}}$ satisfy $p\text{Tr}_n(T_m) = \lambda^n(T_{m-n})$ for $1 \leq n < m$ and also for $n = m$ if we set $T_0 = 2$. 
Proposition 71. The idempotent $e_{m,m}$ is isomorphic to $\lambda^m(\emptyset) \oplus w\lambda^m(\emptyset)$ in $\text{Kar}(\text{2AWeb}^{\text{ess,ss}^+})$. 

Proof. We use Lemma 68 to write $e_{m,m} = (T_m \otimes T_n)(T_{m-n} \otimes S_n)(T_{m-n} \otimes M_n)(T_n \otimes T_n)$. Then it is immediate from Lemma 69 that the maps $(T_m \otimes T_n)(T_{m-n} \otimes S_n)$ and $(T_{m-n} \otimes M_n)(T_n \otimes T_n)$ are inverse isomorphisms between the elements of the Karoubi element represented by the idempotents $e_{m,n}$ and $\lambda^m(T_{m-n})$. The proof for $e_{m,n}$ is similar. □
Proof. For $m = 1$ we have $e_{1,1} = u_1/2 + D^{-1}u_1D/2$. The two summands are orthogonal idempotents. The first is isomorphic to $\lambda(\emptyset)$ in $\text{Kar}(\mathcal{A}_{s.s}^{\text{ess}^*})$, while the conjugation by $D$ in the second summand makes it isomorphic to $w\lambda(\emptyset)$. For $m > 1$ we rewrite
\[
e_{m,m} = (T_m \otimes T_m)\varphi_1 \psi_1 + \varphi_2 \psi_2 = (T_m \otimes T_m)S_m M_m(T_m \otimes T_m)/2 + (T_m \otimes T_m)D^{-1}u_{2m-1}D(id_1 \otimes (S_m - M_m) \otimes id_1)T_m \otimes T_m)/2
\]
The equality in the second line can be verified by inserting $D^{-1}(S_m - M_m \otimes T_2)D$ between two factors of $T_m \otimes T_m$ and realizing that the result is zero. To prove the proposition it remains to verify that $\psi_1 \varphi_2 = \delta_{i,j}\lambda^m(\emptyset)$. We give one example for orthogonality:
\[
\psi_1 \varphi_2 = M_m(T_m \otimes T_m)D^{-1}(S_m - S_1)/2 = M_m(T_m \otimes id_m)D^{-1}(S_m - S_1)/2 = \lambda^{m-1}(\emptyset) \otimes (M_1D^{-1}S_1) = 0
\]
Here we have used the proof of Lemma 69, an isotopy and the essential torus relation. The proof of $\psi_2 \varphi_1 = 0$ is analogous. $\psi_1 \varphi_1 = \lambda^m(\emptyset)$ follows from Lemma 60. It remains to check
\[
\psi_2 \varphi_2 = (M_m - M_1)D(T_m \otimes T_m)D^{-1}(S_m - S_1)/2 = \lambda^m(\emptyset).
\]
For $m = 2$ this follows by expanding the left copy of $T_2$ and seeing that all terms except the identity term die. The result is evaluated using Lemma 60. For $m \geq 3$, we use the recursion on the left copy of $T_m$, absorb the resulting copies of $T_{m-1}$ as the proof of Lemma 69 and then simplify via Lemma 60. The result is a equal to $\lambda^{m-2}(\emptyset)$ superposed with the $m = 2$ case, which we have already checked.

4.2. Skeleta. Recall that we put basepoints on the boundary components of the annulus and fix a connecting arc $\alpha$ between them, which cuts the annulus into a square – this is drawn as a dashed line above.

Lemma 72. The endomorphism algebra of $T_m$ in $\text{Kar}(\mathcal{A}_{s.s}^{\text{ess}^*})$ is isomorphic to $\mathbb{C}[D^{\pm 1}]$ if $m \geq 1$ and isomorphic to $\mathbb{C}[c_2^{\pm 1}]$ if $m = 0$. In $\text{Kar}(\mathcal{A}_{s.s})$ both are given by $\mathbb{C}$.

Proof. Any endomorphism of $T_m$ is represented by a linear combination of outward pointing webs with $m$ 1-labeled input and output strands. Such webs factor into dumbbells $u_i$ and wraps $D^{\pm 1}$. Since $T_m$ kills all $u_i$, the endomorphism algebra is generated by the isomorphisms $D^{\pm 1}$. Since all web relations in $\mathcal{A}_{s.s}$ preserve the flow winding number of the web around the annulus, it is clear that the elements $D^m$ for $m \in \mathbb{Z}$ are linearly independent.

Lemma 73. The endomorphism algebra of $\lambda^k(T_m)$ in $\text{Kar}(\mathcal{A}_{s.s}^{\text{ess}^*})^*$ is isomorphic to $\mathbb{C}[\lambda^k(D^{\pm 1})]$ if $m \geq 1$ and isomorphic to $\mathbb{C}[D_2^{\pm 1}]$ if $m = 0$. In $\text{Kar}(\mathcal{A}_{s.s})$ both types of endomorphism algebras are isomorphic to $\mathbb{C}$. Furthermore, in both versions of the Karoubi envelope, there are no non-zero morphisms between $\lambda^k(T_m)$ and $\lambda^l(T_n)$ (or their $w$-shifts) unless $m = n$ and $k = l$.

Proof. The first part is an immediate corollary of Lemma 72 due to the fact that $\lambda$ is an auto-equivalence, which guarantees that it induces isomorphisms on endomorphism algebras. For the second part, it is clear that we need $m + 2k = n + 2l$ to have both objects in the same block. Now suppose, without loss of generality, that $k < l$. Then an application of the essential inverse $(\lambda^l)^k$ provides an isomorphism between the morphism space in question and $\text{Hom}(T_m, \lambda^{l-k}(T_n))$. However, any web representing a morphism in that space necessarily contains a merge vertex, which is killed by $T_m$. Thus $\text{Hom}(T_m, \lambda^{l-k}(T_n)) = 0$. □
Lemma 74. For $m \geq 1$ we have $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess}})(w^a \lambda^k(T_m), w^b \lambda^k(T_m)) \cong \mathbb{C}(\lambda^k(D^{b-a}))$ for any $a$, $b$ and $k$. On the other hand:

$$\text{Kar}(\overline{\text{AWWeb}}^{\text{ess}})(w^a \lambda^k(\emptyset), w^b \lambda^k(\emptyset)) \cong \begin{cases} \mathbb{C}(\mathbb{D}_{2}^{(b-a)/2}) & \text{if } a - b \text{ is even and } k \geq 1 \\ \mathbb{C}(\mathbb{C}_{2}^{(b-a)/2}) & \text{if } a - b \text{ is even and } k = 0 \\ 0 & \text{if } a - b \text{ is odd} \end{cases}$$

Clearly, all such non-zero morphisms are isomorphisms.

In particular, this implies, that all objects of the form $w^a \lambda^k(T_m)$ are actually isomorphic to unshifted objects $\lambda^k(T_m)$ if $m \geq 1$. The objects $w^a \lambda^k(\emptyset)$, on the other hand, are isomorphic to their versions with $a = 1$ or $a = 0$.

Lemma 75. Any object in $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess},+})$ is isomorphic to a direct sum of objects $\lambda^k(T_m)$ for $m > 1$, $k \geq 0$ or $\lambda^k(\emptyset)$ or $w\lambda^k(\emptyset)$ for $k \geq 0$.

Proof. It suffices to decompose the objects in $\overline{\text{AWWeb}}^{\text{ess},+}$ into a formal direct sum of objects of the above type. Moreover, since 2-labeled objects are isomorphic to idempotents on 1-labeled objects in the Karoubi envelope, we only need to decompose $\text{id}_m$. Then any idempotent endomorphism of $\text{id}_m$ will give rise to an idempotent endomorphism of the decomposition, which is necessarily block-diagonal (there are no morphisms between distinct objects of the form $\lambda^k(T_{m-2k})$ or $w\lambda^{m/2}(\emptyset)$) and has entries in $\mathbb{C}$. Such idempotent matrices can be diagonalized, and thus decompose into objects of type $\lambda^k(T_{m-2k})$ or $w\lambda^{m/2}(\emptyset)$.

The decomposition for $\text{id}_m$ follows inductively from the parallel product formulas in Propositions 70 and 71. More precisely, if we already know that $\text{id}_m$ is isomorphic to a direct sum of terms $\lambda^{k_1}(T_{m-2k_1})$ and possibly $w\lambda^{m/2}(\emptyset)$ if $m$ is even, then $\text{id}_{m+1}$ can be decomposed into summands

$$\lambda^{k_1}(T_{m-2k_1}) \otimes \text{id}_1 \cong \lambda^{k_1}(T_{m-2k_1} \otimes T_1) \cong \begin{cases} \lambda^{k_1}(T_{m-2k_1+1}) \oplus \lambda^{k_1+1}(T_{m-2k_1-1}) & m - 2k_1 > 1 \\ \lambda^{k_1}(T_2) \oplus \lambda^{k_1+1}(\emptyset) & m - 2k_1 = 1 \end{cases}$$

as well as $\lambda^{m/2}(\emptyset) \otimes \text{id}_1 \cong w\lambda^{m/2}(\emptyset) \otimes \text{id}_1 \cong \lambda^{m/2}(T_1)$.

Since orientation of the boundary can be reversed by means of the auto-equivalences $\lambda$ and $\lambda^*$, one easily extends the previous result to the whole category:

Corollary 76. Any object in $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess}})$ is isomorphic to a direct sum of objects $\lambda^k(T_m)$ for $m > 1$, $k \in \mathbb{Z}$ or $\lambda^k(\emptyset)$ or $w\lambda^k(\emptyset)$ for $k \in \mathbb{Z}$. Here we identify $\lambda^{-1} = \lambda^*$.

Definition 77. A subcategory $D$ of a category $C$ is a skeleton of $C$ if the inclusion of $D$ into $C$ is an equivalence of categories and additionally no two distinct objects of $D$ are isomorphic.

Lemma 78. The full subcategory of $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess}})$ containing (lexicographically ordered direct sums of) the objects $\lambda^k(T_m)$ and $s\lambda^k(\emptyset)$ for $k, m \geq 0$ is a skeleton of $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess},+})$. Moreover, this skeleton is semisimple.

Proof. The inclusion of this full subcategory is essentially surjective by Lemma 75. Moreover, by Lemma 73 the decomposition of an object of $\text{Kar}(\overline{\text{AWWeb}}^{\text{ess},+})$ into a lexicographically ordered direct sum of such simples is essentially unique. In particular, there are also no isomorphisms between distinct direct sums of simples. We have also seen that the endomorphism algebras of simples are isomorphic to $\mathbb{C}$, which implies that the skeleton is semisimple.
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