Linear and nonlinear coherent perfect absorbers on simple layers

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We consider linear and nonlinear coherent perfect absorbers (CPAs) in multidimensional geometries and construct explicitly the respective perfectly absorbed solutions. The multidimensional CPAs have a structure of the so-called simple layers which represent the generalization of the point $\delta$ function potential to higher dimensions. The considered examples include broadband CPAs confined to a straight line (in a two-dimensional setting) and to a plane (in the three-dimensional space); CPAs for topological vortices on an absorbing circle; as well as axially-symmetric CPAs on a sphere. Additionally, it is shown that a paraxial beam propagating along a surface nonlinear CPA embedded in the three-dimensional space can be stable against perturbations. The results are interpreted in applications to optical and acoustic systems.

I. INTRODUCTION

The concept of coherent perfect absorber (CPA) was theoretically introduced in [1], experimentally observed in [2], and received significant attention over the last years [3]. A CPA refers to a complex potential which completely absorbs radiation incident on it without any reflection or emission of outgoing waves. It also can be interpreted as a time-reversed laser [1], i.e., as a complex potential which emits radiation in the absence of incident one. While the simplest CPAs operate in one-dimensional geometry, multidimensional CPAs are also known. In the planar 2D geometry, the perfect absorption can be implemented using composite films of finite width [4] or lossy ultrathin dielectric layers [5]. Broadband perfect absorption of microwaves by ultrathin films was demonstrated experimentally [6]. Plasmonic perfect absorption of TM-polarized modes by a patterned surface was discussed and implemented experimentally [7], too. Further results on multi-dimensional CPAs include the absorption of electromagnetic waves incident on a spherical barrier [8] and excited inside a 2D cavity with absorbing circular boundary [9], perfect absorption of TM-polarized modes by a metallings nanocylinders and nanospheres [10], exploiting coupling to the surface plasmons. Beyond the optics, cylindrical CPAs for acoustic waves have been reported on [11].

Both CPAs and lasers in 1D geometry are characterized by a specific value of the wavelength absorbed and emitted, respectively. Finding such a wavelength requires fine tuning of the parameters. Once the respective wavelength is found and other parameters are fixed, shifting the wavelength (frequency) from the resonant value destroys the CPA or the laser. In 1D, where the theory can be formulated either in terms of the scattering matrix [11] or in terms of the transfer matrix [12], the effect of the “destroying” a CPA or a laser can be easily understood in terms of the zeros of the respective diagonal elements of the transfer matrix, which are displaced from the real axis under the change of the wavelength or of the system parameters. In 2D and 3D cases, there appears an additional degree of freedom, expressed by the angle of incidence of a beam. Thus even for a monochromatic wave one can pose the question of creating CPAs or lasers operating in a finite-band angular spectrum. Physically, this means that such devices totally absorb or emit radiation at different angles of incidence (emission).

Another aspect of the theory, relevant to the present study, is a possibility of finding exact analytical solutions for either CPA (or laser) models. When such solutions are available, they allow for quite complete understanding of the phenomenon. However, even in the 1D setting the problem of spectral singularities (or time-reversal spectral singularities) is not solvable analytically, except for a very few cases, like potentials modeled by the Dirac delta-function [13].

In the present paper, we introduce line (in 2D) and surface (in 3D) absorbing or active potentials which (i) allow for exact analytical solutions, (ii) some of them operate as broadband CPAs (or lasers), (iii) bear Kerr nonlinearity. The corresponding solutions feature totally absorbed incoming radiation without emission (or perfectly emitted radiation without incoming one) in a finite range of the incident angles, although the requirement for the wave to be monochromatic remains.

Absorption of all incoming radiation by linear potentials embedded in nonlinear media was reported previously for effectively 1D [13] and 2D [14] Bose-Einstein condensates. In this work, our interest is in absorbing potentials having either finite or infinite spatial extensions which are either linear or nonlinear and are embedded in a linear medium. In the latter sense, the presented results can be viewed as multidimensional extensions of 1D nonlinear CPAs reported in earlier studies [15]. We show that the multidimensional CPA can be made nonlinear, still preserving the above properties of allowing for exact solutions and operating in a finite range of the wavelengths.

From the mathematical point of view, the potentials considered here are known as simple layers [16] which describe the Dirac $\delta$ function “distributed” on a line or on a surface. Simple layers model either absorbing (active)
An image of a paper page is presented, with a segment of text extracted for analysis. The text is about the study of acoustic and electromagnetic wave propagation in the context of layered structures. The page is discussing the Helmholtz equation and the behavior of waves in different geometries, including line potentials in a plane and surface potentials, with a focus on the stability of paraxial beams and absorbing potentials.

The text introduces the Helmholtz equation and discusses its application to different geometries, such as simple layers and spherical layers. The stability of paraxial beams is also considered. The paper concludes with a discussion on the construction of complete absorption surfaces, with references to previous works and the authors' contributions.
of a laser) assuring absorption of incident waves for some angles of incidence in the plane \((x,y)\), we look for an incident-beam solution of Eq. (3) in the form

\[
\psi_\star(\rho) = \int_0^{k_0} \left[ \hat{\psi}_+ (x,q) + \hat{\psi}_- (x,q) \right] e^{-iq\cdot y} dq,
\]

\[
\hat{\psi}_\pm (x,q) = a_\pm(q)e^{\pm \sqrt{k_0^2-q^2}x},
\]

where \(a_\pm(q)\) are, so far, arbitrary complex functions describing amplitudes of the angular-spectrum of the incident beam. Obviously, Eq. (3) solves the 2D Helmholtz equation in the free space, and is continuous at \(y = 0\), i.e. at the points where the line-potential \(\psi_\star\) is located.

Imposing the matching condition for the field derivative at \(y = 0\), one can solve the problem as follows. Given an angular-spectrum distribution of a beam incident from both sides of the surface, i.e., given \(a_\pm(q)\), one can design the dielectric permittivity landscape \(\varepsilon(x)\) which operates as a CPA, i.e., absorbs the incident beams entirely. The solution of this problem is given by

\[
\varepsilon(x) = \frac{2i}{k_0} \int_0^{k_0} \frac{q}{k_0} \left[ \hat{\psi}_+ (x,q) + \hat{\psi}_- (x,q) \right] dq,
\]

and is valid for any beam for which the field is different from zero at \(y = 0\).

A simple example is given by the CPA for an incident plane wave, as well as for a superposition of plane waves having the same wavelength and equal angles of incidence, \(\theta\), the latter defined by \(\cos \theta = q/k_0\). Consider \(\alpha_\pm(q) = \alpha_\pm \delta(q-k_\star)\), where \(k_\star \in (0,k_0)\). Then \(\varepsilon(x) \equiv 2ik_\star/k_0\), i.e., \(\varepsilon_0 \equiv 0\), and the CPA is characterized by a uniform distribution of the absorption: \(\gamma(x) \equiv 2k_\star/k_0\). The corresponding perfectly absorbed solution reads

\[
\psi_\star(\rho) = e^{-ik_\star|y|} (\alpha_+ e^{i\sqrt{k_0^2-k_\star^2}x} + \alpha_- e^{-i\sqrt{k_0^2-k_\star^2}x}).
\]

Here \(\alpha_\pm\) describe amplitudes of the absorbed waves which propagate in opposite \(x\)-directions towards the potential as illustrated in Fig. 1.

### B. Absorption for nondiffracting paraxial beams

The obtained CPA is a straightforward generalization of the 1D result [18]. It also can be viewed as broadband CPA if considered in the context of 3D scattering described by Eq. (2). In this context line \(\psi_\star\) is nothing but the absorbing plane located at \(y = 0\). Let us now take into account that \(z\)-component of the electric field of the form \(\psi(x,y) e^{ik_\star z}\) formally solves the Helmholtz equation with \(\psi(x,y)\) given by Eq. (3) with \(k_0\) replaced by \(\kappa = \sqrt{k_0^2 - k_\star^2}\). Indeed, with the constant dielectric permittivity, the potential in model (2) becomes separable. The absorption depends only on the \(y\)-component of the wavevector. Thus any wave with the same \(k_\star\) and different \(\kappa\), i.e., different \(k_z\), will be coherently absorbed [i.e., solution (3) remains valid for such waves], provided \(k_\star^2 + k_z^2 < k_0^2\).

Thus, a formal solution of Eq. (2) can be written down as a superposition of four beams \((s_1 = \pm \text{ and } s_2 = \pm)\):

\[
\psi(s_1,s_2)(r) = e^{-ik_\star|y|} \times \int_0^{\sqrt{k_0^2-k_\star^2}} e^{ik_\star z} \alpha_{s_1}(k_z) e^{i|s_1|^2 k_\star^2 - k_z^2 x + is_2 k_z^2} dk_z
\]

where \(\alpha_{s_1}(k_z)\) describe the angular spectra of the beams. Such solution will be coherently absorbed by the surface \(y = 0\) characterized by the absorption

\[
\varepsilon(r) = 2i k_\star/k_0 = i\sigma_\star,
\]

where \(\sigma_\star\) is the surface conductivity. Representing \(k_z = \sqrt{k_0^2 - k_\star^2} \sin \varphi\) and considering \(\varphi\) as an integration variable, we convert to the form

\[
\psi(s_1,s_2)(r) = e^{-ik_\star|y|} \times \int_0^{\pi/2} A_{s_1}(\varphi) e^{i|s_1|^2 k_\star^2 sin \varphi + s_2 k_z^2 cos \varphi} d\varphi
\]

where \(A_{s_1}(\varphi) = \sqrt{k_0^2 - k_\star^2} \cos \varphi \alpha_{s_1}(k_z)\). Using a proper superposition of these four beams, one can readily construct a general nondiffracting beam [19].

\[
\psi(r) = e^{-ik_\star|y|} \int_0^{2\pi} A(\varphi) e^{i|s_1|^2 k_\star^2 (\cos \varphi + z \sin \varphi)} d\varphi.
\]

In [12], \(A(\varphi)\) is a properly defined combinations of amplitudes \(A_{s_\pm}(\varphi)\).

Although, expression (12) formally solves Helmholtz equation with the absorbing plane, being interpreted as a component (say, \(z\)-component) of the electric field, it does not represent the exact solution for the electric field because such field is not a TE wave, and hence the boundary conditions for the field component normal to the surface have to be imposed. Nevertheless, there are at least three cases when formula (12) acquires physical meaning.
First, if z-component of the incident field has much larger amplitude than the other two components: \(|E_{x,y}|-|E_z|\), then expression \([12]\) becomes the leading order approximation for the field. This, in particular, occurs for the paraxial beams, characterized by a sufficiently narrow angular spectrum \(A(\phi)\) with the maximum at \(\phi = 0, \pi\) [notice that the the \(\delta\)-limit \(A(\phi) = \delta(\phi)\) or \(A(\phi) = \delta(\phi - \pi)\) one recovers the plane wave solutions \([9]\)]. In this case, however, the absorbing is not perfect, since it occurs only the z-component of the field in the leading approximation.

Second, solution \([12]\) acquires physical meaning for the pressure field absorbed by a layer (this application is discussed also in Sec. [VI]).

Finally, an interesting situation, when the plane is a CPA for nondiffractive beams, occurs when the beam polarization is in \((x, z)\)-plane, i.e. when it is parallel to the absorbing with the absorption defined by \([10]\). Nondiffractive solutions in vacuum, can be constructed using the expression of \([12]\) type for \(x\)- and \(z\)-components of the field. Indeed, it is straightforward to verify that the field

\[
E_x = e^{-ik_x|y|} \int_0^{2\pi} e^{i\sqrt{k_0^2-k_z^2}(x \cos \varphi + z \sin \varphi)} A(\varphi) \times \left[ \sin(\varphi) \hat{i} - \cos(\varphi) \hat{k} \right] d\varphi, \quad (13)
\]

where \(A(\varphi)\) is a scalar function and \(\hat{i}\) and \(\hat{k}\) are the unitary vectors along \(x\) and \(z\) axes, solves Helmholtz equation \([2]\) with absorbing plane \([and \chi(r) \equiv 0]\), is polarized in the plane of the absorbing layer, and is divergence free: \(\nabla \times E_x = 0\).

The electric field \([13]\) satisfies the continuity boundary conditions at \(y = 0\). However, the respective magnetic field \(B_x = 1/(ik_0)\nabla \times E_x\) in a general case, has continuous only the normal component, while its tangential component \(B_{tx}\) has discontinuity, which is determined by the discontinuous derivatives of the electric field:

\[
B_{tx} = \text{sign}(y) \frac{k_x}{k_0} e^{-ik_x|y|} \int_0^{2\pi} e^{i\sqrt{k_0^2-k_z^2}(x \cos \varphi + z \sin \varphi)} A(\varphi) \times \left[ \cos(\varphi) \hat{i} + \sin(\varphi) \hat{k} \right] d\varphi. \quad (14)
\]

Thus in order for the beam \([13]\) to be coherently absorbed, there must exist a surface current determined by \([14]\). One can ensure that in the case at hand such a current is induced in the conducting surface by the incident field and is given by the Ohm’s law:

\[
J_x = \hat{j} \times [B_{tx}(y = +0) - B_{tx}(y = -0)] = \sigma_x E_x(y = 0), \quad (15)
\]

where \(\sigma_x\) is defined by \([10]\).
the function $\chi(x)$. 1D nonlinear CPA were previously studied in [13]. We again focus on a simple case when the absorber is confined to the straight line $y = 0$. Since in the plane $(x, y)$ outside the line the field solves the homogeneous Helmholtz equation, we consider incidence of a single plane wave from each side of the absorbing line (rather than beams considered above), i.e., we look for solution of (4) in the form

$$\psi_s(\rho) = ae^{i\pi\sqrt{k_0^2 - \rho^2}(x-ik_s y)}, s = \pm 1$$

where $a > 0$ is the wave amplitude, and the freedom of choice $s$ reflects the invariance of the model under the inversion of the $x$ axis. Solution (19) is formally tantamount to that obtained for the linear CPA on a line [Eq. (3)] with $\alpha_+ = 0$ or $\alpha_- = 0$ (Fig. 1). However, in the nonlinear case the matching condition for the $y-$component of the field gradient acquires the form

$$2ik_s - k_0 |\varepsilon(x) + \chi(x)|^2] = 0.$$  

Recalling that the dielectric permittivity of the simple layer is a complex function, and allowing for its real part to vary, $\varepsilon = \varepsilon_0(x) + i\gamma$, we observe that a solution of (20) can be found in the case when the dielectric permittivity and nonlinearity strength are connected as $\varepsilon_0(x) = a^2 \chi(x)$. The perfectly absorbed wave has the form

$$\psi_s(\rho) = a \exp \left(\frac{ik}{2} \sqrt{4 - \gamma^2 k_0 x^2 - i k_0 \gamma |y|}\right)$$

which is valid for $\varepsilon_0(x) \chi(x) > 0$ and sufficiently small losses: $\gamma^2 < 4$.

IV. NONLINEAR CPA FOR VORICTES

All CPAs considered above have infinite extensions. Now we consider an example with different properties. We are interested in a spatially limited CPA which has nonzero curvature, and is embedded in the 2D space. More specifically we describe a CPA in a form of a circle (in 3D this CPA has the cylindrical form) of the radius $\rho_s > 0$ centered at the origin. Concentrating on the simplest case, we assume that the distributions of dielectric permittivity, losses, and nonlinear coefficient are constant on the circle, i.e., $\varepsilon(r) \equiv \varepsilon$ and $\chi(r) \equiv \chi$. In the dimensionless polar coordinates $(\rho, \varphi)$, where $\rho = \sqrt{x^2 + y^2}$ and $\varphi$ is the polar angle, Helmholtz equation (2), for the electric field in the form $E = \psi(\rho) k$, takes the form

$$-\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) - \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} - k_0 \delta(\rho - \rho_s) [\varepsilon - \chi |\psi|^2] \psi = k_0^2 \psi.$$  

(22)

We look for solutions carrying integer vorticity (topological charge) $m$: $\psi = e^{im\varphi} R(\rho)$. Then in the free space we have the well known equation

$$R_{pp} + \rho^{-1}R_p + (k_0^2 - m^2 \rho^{-2})R = 0.$$  

(23)

whose general solution is a linear combination of the Bessel functions $J_m(k_0 \rho)$ and $Y_m(k_0 \rho)$. First, we require the continuity of the field on the circle $S$: $R(\rho_s - 0) = R(\rho_s + 0)$. The second conditions is the discontinuity of the derivative normal to the simple layer:

$$R_p(\rho_s + 0) - R_p(\rho_s - 0) + k_0 [\varepsilon - \chi |R(\rho_s)|^2] R(\rho_s) = 0.$$  

(24)

Note that, generally speaking $R(\rho)$ is a complex valued function, but its amplitude and argument are functions of the radius only.

Using the cross-product

$$P(\rho) = \frac{\pi \rho_s}{2} [J_m(k_0 \rho_s)Y_m(k_0 \rho) - J_m(k_0 \rho)Y_m(k_0 \rho_s)],$$

the sought solution can be written down as

$$R(\rho) = C \left\{ \begin{array}{ll}
J_m(k_0 \rho) & \text{at } \rho < \rho_s, \\
J_m(k_0 \rho) - \varepsilon k_0 J_m(k_0 \rho_s)P(\rho) & \text{at } \rho > \rho_s,
\end{array} \right.$$  

(25)

where $C$ is an arbitrary constant.

It is instructive to illustrate the obtained solution using the quantity $\Pi$ which is proportional to the radial projection of the Poynting vector (the proportionality coefficient depends on the physical units used):

$$\Pi(\rho) = i(\mathbf{R} R^\rho - \mathbf{R}^\rho R).$$  

(26)

One immediately observes that $\Pi(\rho) = 0$ for $\rho < \rho_s$, and hence there is no radiation incident over the absorber in the inner domain. For $\rho > \rho_s$ one computes

$$\Pi(\rho) = -2|\varepsilon k_0 \rho_s^2 J_m^2(k_0 \rho_s).$$  

(27)

In the absorbing regime, $\gamma > 0$, the radial component of the Poynting vector $\Pi(\rho)$ is negative, which is consistent with the energy transfer from the infinity towards the absorbing surface.

The found solution however is, in general, not a CPA solution, since it contains both incident and reflected waves. Indeed, using the well-known asymptotic behavior for Bessel functions:

$$J_m(k_0 \rho) \to \sqrt{\frac{2}{\pi k_0 \rho}} \cos \left(\frac{k_0 \rho - \frac{m \pi}{2} - \frac{\pi}{4}}{2}\right), \quad r \to \infty,$$

$$Y_m(k_0 \rho) \to \sqrt{\frac{2}{\pi k_0 \rho}} \sin \left(\frac{k_0 \rho - \frac{m \pi}{2} - \frac{\pi}{4}}{2}\right),$$

(28)

we observe that at large polar radii solution [25] becomes a linear combination of functions $e^{-ik_0 \rho}/\sqrt{k_0 \rho}$ and $e^{ik_0 \rho}/\sqrt{k_0 \rho}$ i.e., of incoming and outgoing radiation, respectively. The CPA regime is characterized by the absence of any reflected radiation. This means that the general solution [25] corresponds to the CPA only if the contribution with $e^{ik_0 \rho}/\sqrt{k_0 \rho}$ vanishes. Performing straightforward computation, we obtain that this condition is
satisfied if the variation of the dielectric permittivity is chosen as
\[ \varepsilon_* = \chi |C|^2 J_m^2(k_0 \rho_*) - 2Y_m(k_0 \rho_*) - iJ_m(k_0 \rho_*)] / \left( \pi k_0 \rho_* J_m(k_0 \rho_*)^2 \right) \] (29)

The imaginary part of the dielectric permittivity is positive reflecting the fact that in the CPA regime the circle is absorbing. The real part of the dielectric permittivity can be tuned by the nonlinearity coefficient \( \chi \) and by the amplitude of the absorbed solution \( C \). On the other hand, relation (29) can be viewed as the definition of the intensity of the perfectly absorbed nonlinear vortex for a given \( \varepsilon_* \), and suitable choice of the nonlinearity \( \chi \) ensuring the existence of solution of (29) with respect to \( |C|^2 \).

Additionally we notice that if \( k_0 \rho_* \) is a zero of the Bessel function \( J_m(k_0 \rho_*) \), then the expression for \( \varepsilon_* \) diverges, which means that the CPA operating at the given wavelengths cannot be implemented using a surface of the corresponding radius.

Substituting (29) to (25), we finally obtain that in the exterior domain of the circle the CPA solution for perfectly absorbed vortices is given by the expression
\[ R_*(\rho) = C \left( J_m(k_0 \rho_*) H_m^2(k_0 \rho) / H_m^2(k_0 \rho_*) \right), \] (30)

where \( H_m^2 = J_m - iY_m \) is the Hankel function of the second kind. In the interior domain of the circle, \( \rho < \rho_* \), the solution \( R_*(\rho) \) has the form of the Bessel function presented in (25).

In contrast to the CPA solution on the straight line [21], the nonlinear CPA solution (30) features the algebraically decaying field amplitude. Inside the absorbing circle, both the the incident and reflected radiation exist, although they are balanced inside the circle (in fact this property is more general and holds for any, either perfectly or not perfectly, absorbed vortex). Outside the circle only the incident radiation exists, which means that the layer operates as a one-sided CPA, in contrast to the nonlinear CPA on the line constructed above in [21], where the radiation is absorbed perfectly from both sides of the line. This property is physically natural, since perfect absorption of the radiation of the cavity inside the circle would require the existence of a source also located inside the cavity.

As a particular example, let us address in more detail the case when the nonlinearity-assisted CPA is implemented using an absorbing layer with the real part of the dielectric permittivity fixed to be equal to zero, i.e., \( \varepsilon_{*,0} = 0 \). From (29), the corresponding requirement reads
\[ \chi |C|^2 = \left( \frac{2Y_m(k_0 \rho_*)}{\pi k_0 \rho_* J_m^2(k_0 \rho_*)} \right) \] (31)

If the nonlinearity is defocusing \( (\chi > 0) \) then the nonlinear CPA on such a layer is possible only if \( Y_m(k_0 \rho_*) / J_m(k_0 \rho_*) > 0 \). Examples of perfectly absorbed states with topological charges \( m = 0, 1, 2 \) are presented in Fig. 2. In this figure, the nonlinearity coefficient is fixed \( (\chi = 0.1) \) and the amplitude of a vortex perfectly absorbed by a purely imaginary layer is computed from (31).

The constructed CPA solution can be easily converted to the solution on a lasing circle. To this end, it is sufficient to take the complex conjugate of \( R_*(\rho) \) and of \( \varepsilon_* \) (that is to invert sign of imaginary part of the dielectric permittivity replacing the absorbing surface with a lasing one).
V. PARAXIAL APPROXIMATION AND STABILITY

So far we considered the field distributions in the whole space. Now we turn to the propagation of a paraxial beam along $z$-axis in the half-space $z > 0$ in the presence of an absorbing surface defined by a line in the plane $(x, y)$ and homogeneous along $z$. Making the standard ansatz $E_z(r) = \psi(x, y, z)e^{ik_0z}$ with $\psi$ being a slow function of $z$ $|\partial \psi / \partial z| \ll k_0 z$, we obtain the equation of paraxial approximation [recall that $\rho = (x, y)$]:

$$2ik_0\psi_z = -\nabla^2 \rho \psi - k_0^2 \delta_S(\rho / \ell) [\varepsilon(\rho) - \chi(\rho)] |\psi|^2 \psi. \quad (32)$$

On the whole $z$-axis, for the plane wave solution corresponding to $z$-independent amplitude $\psi$ we recover the model $\psi = 0$.

Now a perfectly absorbed solution, i.e. a beam having no radiation outgoing from the absorbing simple layer, has the form $\psi(z, \rho) = \psi_s(\rho) e^{- i k z}$, and the question of its stability (at $z > 0$) arises. To check the stability of such beam we consider propagation of a perturbed solution in the form $\psi = [\psi_s + \eta(z, \rho)] e^{- i k z}$, where $\xi$ is small perturbation whose behavior can be described in terms of the linearized version of equation (32):

$$ik_0 \eta_z = -\nabla^2 \rho \eta - k_0^2 \delta_S(\rho / \ell) [\varepsilon(\rho) \eta - \chi(\rho) (|\psi_s|^2 \eta + \psi_s^2 \eta^*)]. \quad (33)$$

Since the waveguiding medium is linear and the CPA-surface bears defocusing nonlinearity, one can expect that the eventual instability can result only from spatially localized perturbations. We therefore limit our analysis to perturbations carrying finite power $X(z) = \int_{R^2} |\eta|^2 d\rho$.

Direct integration over the whole space $\mathbb{R}^2$ gives

$$\frac{d X(z)}{d z} = -2 \int_S \gamma(\rho) |\eta|^2 dS + 2 \text{Im} \int_S \chi(\rho) \psi_s^2(\eta^*)^2 dS. \quad (34)$$

If

$$\chi(\rho) |\psi(\rho)|^2 \leq \gamma(\rho) \quad \text{for all} \quad \rho \in S, \quad (35)$$

then the r.h.s. of (34) is negative, i.e., the power of perturbations monotonously decays.

In particular, the stability condition (35) for the CPA solution on the line (21) takes the form $0 < \varepsilon(x) \leq \gamma$. For CPA on the nonlinear ring in Eq. (25) stability condition takes the form $\chi |C|^2 J_m^2 (k_0 \rho_s) \leq \gamma$.

VI. SPHERICAL CPA FOR ACOUSTIC WAVES

Equation (2) also describes interaction of acoustic waves with an absorbing layer, the latter being modeled by the simple layer. Taking into account the boundary conditions employed in this paper, i.e., the continuity of the field and the discontinuity of the normal derivative, the physical meaning of $\psi(\rho)$, in such a statement, is pressure variation field, which is considered dimensionless. Now $k_0 = \omega / c$, where $c$ is the sound velocity in the homogeneous medium, and the sound velocity in the layer is considered complex and defined by: $c_\delta^2 = c^2 (1 + \delta_S (r / \ell) \varepsilon(r))$. The problem for CPA of acoustic wave can be posed in 3D space for the spherical simple layer (in 2D setting corresponding to cylindrical absorber a similar problem was considered in [11]).

Specifically, we consider an acoustic CPA on a sphere of a radius $r_*$ and focus on the simplest axially symmetric solutions which depend only on the spherical radius $r = |r|$, i.e., on solutions of the form $\psi(r) = R(\rho)$. If the sphere is characterized by the uniform distribution of $\varepsilon(r) = \varepsilon_0 + i \gamma = \text{const}$, then the Helmholtz equation (2) reduces to

$$-\frac{1}{r^2} \frac{d}{dr} \left( r^2 dR / dr \right) - k_0 \varepsilon_0 \delta(r - r_*) R = k_0^2 R. \quad (36)$$

In the free space the general solution is a linear combination of functions $\cos(k_0 r)/r$ and $\sin(k_0 r)/r$. Imposing the matching conditions at $r = r_*$, one finds

$$R(r) = \frac{\sin(k_0 r)}{k_0 r} \quad \text{at} \quad 0 < r < r_*, \quad (37)$$

and

$$R(r) = \frac{\sin(k_0 r)}{k_0 r} - \frac{\varepsilon}{k_0 r} \sin(k_0 (r - r_*)) \sin(k_0 r_*) \quad \text{at} \quad r > r_* \quad (38)$$

Like in the case of circular absorbing layer, the obtained solution is characterized by the freedom of choice of the coefficients $\varepsilon_0$ and $\gamma$. The radial component of the energy flow can be estimated using the expression $\Pi(r) = i (RR^* - R^* R)$ Thus $\Pi(r)$ vanishes in the cavity inside the absorbing sphere, while for $r > r_*$ one computes

$$\Pi(r) = - \frac{2 \gamma \sin^2(k_0 r_*)}{k_0 r^2}, \quad (39)$$

which is always negative for $\gamma > 0$, thus corresponding to the energy flow from the infinity towards the absorber.

Since the constructed solution (38) is a linear combination of incident ($e^{-ik_0 r}/r$) and reflected ($e^{ik_0 r}/r$) waves, it is, in general, not the perfectly absorbed one. In order to obtain the parameters when the sphere operates as a CPA, we require the outgoing radiation to vanish in the exterior region of the absorber. This requirements leads to the following expression for the variation of the dielectric permittivity:

$$\varepsilon_* = \cot(k_0 r_*) + i. \quad (40)$$

Substituting (40) to the general solution (38), we obtain that for $r > r_*$ the perfectly absorbed solution reads

$$R_*(r) = \frac{\sin(k_0 r_*)}{k_0 r} e^{-ik_0 (r - r_*)}. \quad (41)$$

Inside the CPA sphere, $r < r_*$, the solution $R_*(r)$ is given by expression (37).
VII. CONCLUSION

In this work, we have constructed explicit solutions for several examples of coherent perfect absorbers (CPAs) confined in straight lines or surfaces embedded in two-dimensional and three-dimensional spaces. From the mathematical point of view, the considered surfaces represent simple layers, which are multidimensional generalizations of the Dirac delta function. We explored different geometries, including line, surface, cylindric and spheric surfaces. Both linear and nonlinear CPAs are presented. Some of the surface CPAs appear to be broadband with respect to the angular spectrum of incidence. The closed lines or surfaces perfectly absorbing vortices do not require incident energy flows from both sides of the surface, and in this sense can be viewed as unidirectional.

CPAs. Nevertheless, the field is nonzero in the interior domain of the CPA-surface and perfectly absorbed vortices are characterized by the energy flows tangential to the surface either in the interior and exterior domains of the respective CPA. Stability of the resulting CPA solutions on nonlinear surfaces has been demonstrated, provided that the intensity of the solution is below a certain threshold value.

Finally, the presented coherent perfect absorbers can be straightforwardly transformed in multidimensional simple-layer lasers by changing absorption by gain.

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