On the best constant of Hardy–Sobolev Inequalities

Adimurthi\textsuperscript{1} & Stathis Filippas\textsuperscript{2,4} & Achilles Tertikas\textsuperscript{3,4}

TIFR center P.O. Box 1234\textsuperscript{1},
Bangalore 560012, India
aditi@math.tifrbng.res.in

Department of Applied Mathematics\textsuperscript{2}
University of Crete, 71409 Heraklion, Greece
filippas@tem.uoc.gr

Department of Mathematics\textsuperscript{3}
University of Crete, 71409 Heraklion, Greece
tertikas@math.uoc.gr

Institute of Applied and Computational Mathematics\textsuperscript{4},
FORTH, 71110 Heraklion, Greece

Abstract

We obtain the sharp constant for the Hardy-Sobolev inequality involving the
distance to the origin. This inequality is equivalent to a limiting Caffarelli–Kohn–Nirenberg inequality. In three dimensions, in certain cases the sharp constant coincides with the best Sobolev constant.

AMS Subject Classification: 35J60, 46E35 (26D10, 35J15)

Keywords: Hardy inequality, Sobolev inequality, critical exponent, best constant, Caffarelli–Kohn–Nirenberg inequality.

1 Introduction

The standard Hardy inequality involving the distance to the origin, asserts that when
\( n \geq 3 \) and \( u \in C_0^\infty(\mathbb{R}^n) \) one has
\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx.
\]
(1.1)

The constant \( \left( \frac{n-2}{2} \right)^2 \) is the best possible and remains the same if we replace \( u \in C_0^\infty(\mathbb{R}^n) \) by \( u \in C_0^\infty(B_1) \), where \( B_1 \subset \mathbb{R}^n \) is the unit ball centered at zero. Brezis and Vázquez [BV] have improved it by establishing that for \( u \in C_0^\infty(B_1) \),
\[
\int_{B_1} |\nabla u|^2 dx \geq \left( \frac{n-2}{2} \right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx + \mu_1 \int_{B_1} u^2 dx,
\]
(1.2)
We then have:

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1 - |x|)^2} \, dx.
\]  

(1.3)

Similarly to (1.2) this has also been improved by Brezis and Marcus in [BM] by proving that

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1 - |x|)^2} \, dx + b_n \int_{B_1} u^2 \, dx,
\]  

(1.4)

for some positive constant \(b_n\). This time the best constant \(b_n\) depends on the space dimension with \(b_n > \mu_1\) when \(n \geq 4\), but in the \(n = 3\) case, one has that \(b_3 = \mu_1\), see [BFT].

On the other hand the classical Sobolev inequality

\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq S_n \left( \int_{\mathbb{R}^n} |u|^\frac{2n}{n-2} \, dx \right) \frac{n-2}{n},
\]  

(1.5)

is valid for any \(u \in C_0^\infty(\mathbb{R}^n)\) where \(S_n = \pi n(n-2) (\Gamma(\frac{n}{2})/\Gamma(n))^{\frac{2}{n}}\) is the best constant, see [A], [T]. Maz’ya [M] combined both the Hardy and Sobolev term in one inequality valid in the upper half space. After a conformal transformation it leads to the following Hardy–Sobolev–Maz’ya inequality

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1 - |x|)^2} \, dx + b_n \left( \int_{B_1} |u|^\frac{2n}{n-2} \, dx \right) \frac{n-2}{n}.
\]  

(1.6)

valid for any \(u \in C_0^\infty(B_1)\). Clearly \(B_n \leq S_n\) and it was shown in [TT] that \(B_n < S_n\) when \(n \geq 4\). Again, the case \(n = 3\) turns out to be special. Benguria Frank and Loss [BFL] have recently established that \(B_3 = S_3 = 3(\pi/2)^{4/3}\) (see also Mancini and Sandeep [MS]).

When distance is taken from the origin the analogue of (1.6) has been established in [FT] by methods quite different to the ones we use in the present work. To state the result we first define

\[
X_1(a,s) := (a - \ln s)^{-1}, \quad a > 0, \quad 0 < s \leq 1.
\]  

(1.7)

We then have:

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \left( \frac{n-2}{2} \right)^2 \int_{B_1} \frac{u^2}{|x|^2} \, dx + C_n(a) \left( \int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|u|^\frac{2n}{n-2} \, dx \right)^{\frac{n-2}{n}}.
\]  

(1.8)

We note that one cannot remove the logarithm \(X_1\) in (1.8) and actually the exponent \(\frac{2(n-1)}{n-2}\) is optimal. Our main concern in this note is to calculate the best constant \(C_n(a)\) in (1.8). To this end we have:

**Theorem A** Let \(n \geq 3\). The best constant \(C_n(a)\) in (1.8) satisfies

\[
C_n(a) = \begin{cases} 
(n-2)^{\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\
(a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. 
\end{cases}
\]
When restricted to radial functions, the best constant in (1.8) is given by

$$C_{n,\text{radial}}(a) = (n - 2)^{-\frac{2(n-1)}{n}} S_n, \quad \text{for all } a \geq 0.$$ 

In all cases there is no $H^1_0(B_1)$ minimizer.

One easily checks that $C_n(a) < S_n$ when $n \geq 4$. Surprisingly, in the $n = 3$ case one has that $C_3(a) = S_3 = 3(\pi/2)^{1/3} = B_3$, for $a \geq 1$, that is, inequalities (1.5), (1.6) and (1.8) share the same best constant.

Using the change of variables $u(x) = |x|^{-\frac{n-2}{2}} v(x)$ inequality (1.8) is easily seen to be equivalent to

$$\int_{B_1} |x|^{-(n-2)} \nabla v|^2 dx \geq C_n(a) \left( \int_{B_1} |x|^{-n} X_1^{2(n-1)} (a, |x|) |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad v \in C_0^\infty(B_1).$$

For later use we denote by $W^{1,2}_0(B_1; |x|^{-(n-2)})$ the completion of $C_0^\infty(B_1)$ under the norm $\left( \int_{B_1} |x|^{-(n-2)} \nabla v|^2 dx \right)^{1/2}$.

Estimate (1.9) is a limiting case of a Caffarelli–Kohn–Nirenberg inequality. Indeed, for any $-\frac{n-2}{2} < b < \infty$, the following inequality holds:

$$\int_{\mathbb{R}^n} |x|^{2b} \nabla v|^2 dx \geq S(b,n) \left( \int_{\mathbb{R}^n} |x|^{-n} X_1^{2(n-1)} (a, |x|) |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad v \in C_0^\infty(\mathbb{R}^n);$$

see [CKN], Catrina and Wang [CW]. Moreover, for $b = -\frac{n-2}{2}$ estimate (1.10) fails. Clearly, estimate (1.9) is the limiting case of (1.10) for $b = -\frac{n-2}{2}$. Thus we have:

**Theorem A’** Let $n \geq 3$. The best constant $C_n(a)$ in the limiting Caffarelli–Kohn–Nirenberg inequality (1.9) is given

$$C_n(a) = \begin{cases} (n - 2)^{-\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

When restricted to radial functions, the best constant in (1.9) is given by

$$C_{n,\text{radial}}(a) = (n - 2)^{-\frac{2(n-1)}{n}} S_n, \quad \text{for all } a \geq 0.$$ 

In all cases there is no $W^{1,2}_0(B_1; |x|^{-(n-2)})$ minimizer.

We note that the nonexistence of a $W^{1,2}_0(B_1; |x|^{-(n-2)})$ minimizer of Theorem A’ is stronger than the nonexistence of an $H^1_0(B_1)$ minimizer of Theorem A. This is due to the fact that the existence of an $H^1_0(B_1)$ minimizer for (1.8) would imply existence of a $W^{1,2}_0(B_1; |x|^{-(n-2)})$ minimizer for (1.9), see Lemma 2.1 of [FT].

The above results can be easily transformed to the exterior of the unit ball $B_1^c$. For instance we have:

**Corollary** Let $n \geq 3$. For any $u \in C_0^\infty(B_1^c)$, there holds

$$\int_{B_1^c} |\nabla u|^2 dx \geq \left( \frac{n - 2}{2} \right)^2 \int_{B_1^c} \frac{u^2}{|x|^2} dx + C_n(a) \left( \int_{B_1^c} X_1^{2(n-1)} (a, \frac{1}{|x|}) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

(1.11)
where the best constant $C_n(a)$ is the same as in Theorem A.

Our method can also cover the case of a general bounded domain $\Omega$ containing the origin. In particular we have

**Theorem B** Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin. Set $D := \sup_{x \in \Omega} |x|$. For any $u \in C_0^\infty(\Omega)$, there holds

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C_n(a) \left( \int_\Omega X_1^{\frac{2(n-1)}{n-2}} \left( a, \frac{|x|}{D} \right) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (1.12)$$

where the best constant $C_n(a)$ is independent of $\Omega$ and is given by

$$C_n(a) = \begin{cases} 
(n-2) \frac{2(n-1)}{n} S_n, & a \geq \frac{1}{n-2} \\
2 \frac{2(n-1)}{n} S_n, & 0 < a < \frac{1}{n-2}
\end{cases}.$$

It follows easily from Theorem A' that there no minimizers for (1.11) and (1.12) in the appropriate energetic function space.

We next consider the $k$–improved Hardy–Sobolev inequality derived in [FT]. Let $k$ be a fixed positive integer. For $X_1$ as in (1.7) we define for $s \in (0,1)$,

$$X_{i+1}(a, s) = X_1(a, X_i(a, s)), \quad i = 1, 2, \ldots, k. \quad (1.13)$$

Noting that $X_i(a, s)$ is a decreasing function of $a$ we easily check that there exist unique positive constants $0 < a_k < \beta_{n,k} \leq 1$ such that :

(i) The $X_i(a_k, s)$ are well defined for all $i = 1, 2, \ldots, k+1$, and all $s \in (0,1)$ and $X_{k+1}(a_k, 1) = \infty$. In other words, $a_k$ is the minimum value of the constant $a$ so that the $X_i's$, $i = 1, 2, \ldots, k+1$, are all well defined in $(0,1)$.

(ii) $X_1(\beta_{n,k}, 1)X_2(\beta_{n,k}, 1) \cdots X_{k+1}(\beta_{n,k}, 1) = n-2$.

For $n \geq 3$, $k$ a fixed positive integer and $u \in C_0^\infty(B_1)$ there holds:

$$\int_{B_1} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^k \int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) \cdots X_i^{\frac{2(n-1)}{n-2}}(a, |x|) u^2 dx$$

$$+ C_{n,k}(a) \left( \int_{B_1} X_1(a, |x|) \cdots X_{k+1}(a, |x|) \right)^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (1.14)$$

In our next result we calculate the best constant $C_{n,k}(a)$ in (1.14).

**Theorem C** Let $n \geq 3$ and $k = 1, 2, \ldots$ be a fixed positive integer. The best constant $C_{n,k}(a)$ in (1.14) satisfies:

$$C_{n,k}(a) = \begin{cases} 
(n-2) \frac{2(n-1)}{n} S_n, & a \geq \beta_{n,k} \\
\left( \prod_{i=1}^{k+1} X_i(a, 1) \right)^{-\frac{2(n-1)}{n}} S_n, & a_k < a < \beta_{n,k}
\end{cases}.$$

When restricted to radial functions, the best constant of (1.14) is given by

$$C_{n,k,\text{radial}}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \quad \text{for all } a > a_k.$$
Again we notice that $C_{n,k}(a) < S_n$ for $n \geq 4$ but $C_{3,k} = S_3$ for $a \geq \beta_{3,k}$.

As in Theorem A, one can establish by similar arguments the nonexistence of an $H_0^1(B_1)$ minimizer to (1.14), as well as the analogues of Theorem A', Corollary and Theorem B in the case of the $k$–improved Hardy–Sobolev inequality.

## 2 The proofs

Theorems A follows from Theorem A’, we therefore prove Theorem A’:

### Proof of Theorem A’:

At first we will show that

$$C_n(a) = \left( n - 2 \right) \frac{2(n-1)}{n} S_n, \quad \text{when } a \geq \frac{1}{n-2}. \quad (2.1)$$

We have that

$$C_n(a) = \inf_{v \in C_0^\infty(B_1)} \left( \frac{\int_{B_1} |x|^{-(n-2)} \| \nabla v \|^2 dx}{\left( \int_{B_1} |x|^{-n} X_1(a,|x|) \| v \|^2 dx \right)^{\frac{n-2}{n}}} \right). \quad (2.2)$$

We change variables by $(r = |x|)$

$$v(x) = y(\tau, \theta), \quad \tau = \frac{1}{X_1(a,r)} = a - \ln r, \quad \theta = \frac{x}{|x|}. \quad (2.3)$$

This change of variables maps the unit ball $B_1 = \{ x : |x| < 1 \}$ to the complement of the ball of radius $a$, that is, $B'_c(a) = \{ (\tau, \theta) : a < \tau < +\infty, \ \theta \in S^{n-1} \}$. Noticing that $X_1'(a,r) = \frac{X_1^2(a,r)}{r}$, $d\tau = -X_1'(a,r) = -\frac{dr}{r}$, we also have

$$|\nabla v|^2 = \left( \frac{\partial v}{\partial \tau} \right)^2 + \frac{1}{r^2} |\nabla_\theta v|^2 = e^{2(\tau-a)} (y^2 + |\nabla_\theta y|^2).$$

A straightforward calculation shows that for $y \in C^\infty([a, \infty) \times S^{n-1})$ under Dirichlet boundary condition on $\tau = a$ we have

$$C_n(a) = \inf_{y(a,\theta)=0} \left( \frac{\int_a^\infty \int_{S^{n-1}} (y^2_\tau + |\nabla_\theta y|^2) dSd\tau}{\left( \int_a^\infty \int_{S^{n-1}} \frac{2(n-1)}{n-2} |y|^{\frac{2n}{n-2}} dSd\tau \right)^{\frac{n-2}{n}}} \right). \quad (2.4)$$

In the sequel we will relate $C_n(a)$ with the best Sobolev constant $S_n$. It is well known that for any $R$ with $0 < R \leq \infty$,

$$S_n = \inf_{u \in C_0^\infty(B_R)} \left( \frac{\int_{B_R} |\nabla u|^2 dx}{\left( \int_{B_R} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}} \right). \quad (2.5)$$

We also know that $S_n = S_{n,radial}$ the latter being the infimum when taken over radial functions. Changing variables in (2.5) by

$$u(x) = z(t, \theta), \quad t = |x|^{-(n-2)}, \quad \theta = \frac{x}{|x|}. \quad (2.6)$$
it follows that for any $R \in (0, \infty]$,
\[
(n - 2) \frac{2(n-1)}{n} S_n = \inf_{z(R^{-(n-2)} \theta) = 0} \frac{\int_{R^{-2(n-2)}} \int_{S^{n-1}} (z^2 + (\frac{1}{n-2})^2 |\nabla \theta z|^2) dS dt}{\left( \int_{R^{-2(n-2)}} \int_{S^{n-1}} t^{-\frac{2(n-1)}{n-2}} |z|^\frac{2n}{n-2} dS dt \right)^{\frac{n-2}{n}}}.  \tag{2.7}
\]

We note that a function $u$ is radial in $x$ and only if the function $z$ is a function of $t$ only. Looking at (2.4) and (2.7) we have that

\[
C_n(a) \leq C_{n, \text{radial}}(a) = (n - 2)\frac{2(n-1)}{n} S_{n, \text{radial}} = (n - 2)\frac{2(n-1)}{n} S_n.  \tag{2.8}
\]

On the other hand let us take $R = a^{-\frac{1}{n-2}}$ (so that $a = R^{-(n-2)}$) and assume that $a \geq \frac{1}{n-2}$. Then $\left(\frac{1}{n-2}\right)^{\frac{n}{2}} t \leq 1$ since $t \geq a \geq \frac{1}{n-2}$, and therefore

\[
C_n(a) \geq \left(\frac{1}{n-2}\right)^{\frac{2(n-1)}{n}} S_n.
\]

Combining this with (2.8) we conclude our claim (2.1).

Our next step is to prove the following: For any $a > 0$ we have that

\[
C_n(a) \leq a^{\frac{2(n-1)}{n}} S_n.  \tag{2.9}
\]

To this end let $0 \neq x_0 \in B_1$ and consider the sequence of functions

\[
U_\varepsilon(x) = (\varepsilon + |x - x_0|^2)^{-\frac{n-2}{2}} \phi_{\delta}(|x - x_0|), \tag{2.10}
\]

where $\phi_{\delta}(t)$ is a $C^\infty_0$ cutoff function which is zero for $t > \delta$ and equal to one for $t < \delta/2$; $\delta$ is small enough so that $|x_0| + \delta < 1$ and therefore $U_\varepsilon \in C^\infty_0(B_\delta(x_0)) \subset C^\infty_0(B_1)$.

Then, it is well known, cf [BN], that

\[
S_n = \lim_{\varepsilon \to 0} \frac{\int_{B_1} |\nabla U_\varepsilon|^2 dx}{\left( \int_{B_1} |U_\varepsilon|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}. \tag{2.11}
\]

From (2.2) we have that for any $\varepsilon > 0$ small enough,

\[
C_n(a) = \inf_{v \in C^\infty_0(B_1)} \frac{\int_{B_1} |x|^{-n-2} |\nabla v|^2 dx}{\left( \int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}} (a, |x|)|v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}} \leq \frac{\int_{B_1} (x_0) |x|^{-n-2} |\nabla U_\varepsilon|^2 dx}{\left( \int_{B_1(x_0)} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}} (a, |x|)|U_\varepsilon|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}} \leq (|x_0| + \delta)^{-n-2} \frac{1}{X_1^{\frac{2(n-1)}{n-2}} (a, |x_0| - \delta)} \frac{\int_{B_1(x_0)} |\nabla U_\varepsilon|^2 dx}{\left( \int_{B_1(x_0)} |U_\varepsilon|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}.  \tag{2.12}
\]
where we used the fact that \( X_1(a, s) \) is an increasing function of \( s \). Taking the limit \( \varepsilon \to 0 \) we conclude:

\[
C_n(a) \leq \left( \frac{|x_0| + \delta}{|x_0| - \delta} \right)^{n-2} \frac{S_n}{X_1^{\frac{2(n-1)}{n}}(a, |x_0| - \delta)}.
\]

This is true for any \( \delta > 0 \) small enough, therefore

\[
C_n(a) \leq X_1^{\frac{2(n-1)}{n}}(a, |x_0|) S_n.
\]

Since \( |x_0| < 1 \) is arbitrary and \( X_1(a, s) \) is an increasing function of \( s \), we end up with

\[
C_n(a) \leq X_1^{-2}(n-1) S_n = a^{2(n-1)} S_n,
\]

and this proves our claim (2.12).

To complete the calculation of \( C_n(a) \) we will finally show that

\[
C_n(a) \geq a^{2(n-1)} S_n, \quad \text{when } 0 < a < \frac{1}{n-2}.
\]

To prove this we will relate the infimum \( C_n(a) \) to a Caffarelli–Kohn–Nirenberg inequality. We will need the following result:

**Proposition 2.1** Let \( b > 0 \) and

\[
S_n(b) := \inf_{v \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |x|^{2b} |
\n\nA straightforward calculation shows that for any \( R' \),

\[
(n - 2 + 2b)^{-\frac{2(n-1)}{n}} S_n \leq \inf_{z(R', \theta) = 0} \frac{\int_{R'}^{\infty} \int_{S^{n-1}} \left( z^2 + \frac{1}{(n-2+2b)^2} t^2 |\n\nTaking \( R' = a \) and comparing (2.16) with (2.4) we have that if

\[
1 \geq \frac{1}{(n - 2 + 2b)^2} t^2, \quad \text{for } t \geq a,
\]

then

\[
C_n(a) \geq (n - 2 + 2b)^{-\frac{2(n-1)}{n}} S_n.
\]
Condition (2.17) is satisfied if we choose $b \in (0, +\infty)$ such that
\[
\frac{1}{n-2} > a = (n - 2 + 2b)^{-1} > 0.
\] (2.19)
For such a $b$ it follows from (2.18) that
\[
C_n(a) \geq a \frac{2^{2(n-1)}}{n} S_n,
\]
and this proves our claim (2.13).

We finally establish the nonexistence of an energetic minimizer. We will argue by contradiction. Suppose that $\bar{v} \in W^{1,2}_0(B_1; |x|^{-(n-2)})$ is a minimizer of (2.2). Through the change of variables (2.3), the quotient in (2.4) admits also a minimizer $\bar{y}$.

Consider first the case when $a \geq \frac{1}{n-2}$. Comparing (2.4) and (2.7) with $R = a^{-\frac{1}{n-2}}$, we conclude that $\bar{y}$ is a radial minimizer of (2.7) as well. It then follows that (2.5) admits a radial $H_0^1(B_R)$ minimizer $\bar{u}(r) = \bar{y}(t)$, $t = r^{-(n-2)}$, which contradicts the fact that the Sobolev inequality (2.5) has no $H_1^0$ minimizers.

In the case when $0 < a < \frac{1}{n-2}$, we use a similar argument comparing (2.4) and (2.16) to conclude the existence of a radial minimizer to (2.16) with $b$ as in (2.19). This contradicts the nonexistence of minimizer for (2.14). The proof of Theorem A’ is now complete.

Proof of Corollary: One can argue in a similar way as in the previous proof, or apply Kelvin transform to the inequality of Theorem A.

Proof of Theorem B: The lower bound on the best constant follows from Theorem A, the fact that if $u \in C^\infty_0(\Omega)$ then $u \in C^\infty_0(B_D)$ (since $\Omega \subset B_D$) and a simple scaling argument.

To establish the upper bound in the case where $0 < a < \frac{1}{n-2}$ we argue exactly as in the proof of (2.9) using the test functions (2.10) that concentrate near a point of the boundary of $\Omega$, that realizes the $\max_{x \in \Omega} |x|$. Let us now consider the case where $a \geq \frac{1}{n-2}$. For $a > 0$ and $0 < \rho < 1$, we set
\[
\bar{C}_n(a, \rho) := \inf_{u \in C^\infty_0(B_\rho)} \frac{\int_{B_\rho} |\nabla u|^2 \, dx - \left(\frac{n-2}{2}\right)^2 \int_{B_\rho} \frac{u^2}{|x|^{n-2}} \, dx}{\left(\int_{B_\rho} X_1^{2(n-1)}(a, |x|) |u|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}}}.
\]
A simple scaling argument and Theorem A shows that:
\[
\bar{C}_n(a, \rho) = C_n(a - \ln \rho).
\]
Thus, for $\rho$ small enough we have that
\[
\bar{C}_n(a, \rho) = (n - 2)^{-\frac{2(n-1)}{n}} S_n.
\]
Since for $\rho$ small, $B_\rho \subset \Omega$ the upper bound follows easily in this case as well.

Proof of Theorem C: To simplify the presentation we will write $X_i(|x|)$ instead of $X_i(a, |x|)$. Let $k$ be a fixed positive integer. We first consider the case $a \geq \beta_{k,n}$. We change variables in (1.14) by
\[
u(x) = |x|^{-\frac{n-2}{2}} X_1^{-1/2}(|x|) X_2^{-1/2}(|x|) \ldots X_k^{-1/2}(|x|) v(x),
\]
to obtain
\[
\int_{B_1} |x|^{-(n-2)} X_1^{-1}(|x|) \ldots X_k^{-1}(|x|)|\nabla v|^2 \, dx \geq C_{n,k}(a) \left( \int_{B_1} |x|^{-n} X_1(|x|) \ldots X_k(|x|) X_{k+1}^{2(n-1)} \frac{2a}{n} \right)^{\frac{n-2}{n}}, \quad v \in C_0^\infty(B_1). \tag{2.20}
\]

We further change variables by
\[
v(x) = y(\tau, \theta), \quad \tau = \frac{1}{X_{k+1}(r)}, \quad \theta = \frac{x}{|x|} \quad (r = |x|).
\]

This change of variables maps the unit ball \( B_1 = \{ x : |x| < 1 \} \) to the complement of the ball of radius \( r_a := X_{k+1}^{-1}(1) \), that is, \( B_{r_a}^c = \{ (\tau, \theta) : X_{k+1}^{-1}(1) < \tau < +\infty, \quad \theta \in S^{n-1} \} \).

Note that
\[
d\tau = -\frac{X_{k+1}'(r)}{X_{k+1}^2(r)} dr = -\frac{X_1(r) \ldots X_k(r)}{r} dr.
\]

Let us denote by \( f_i(t) \) the inverse function of \( X_i(t) \). We also set \( f_{i+1}(t) = f_i(f_i(t)), \quad i = 1, 2, \ldots, k \). Consequently, \( r = f_{k+1}(\tau^{-1}) \). Also, \( X_1(r) = f_k(\tau^{-1}), \quad X_2(r) = f_{k-1}(\tau^{-1}), \ldots, X_k(r) = f_1(\tau^{-1}) \).

We then find
\[
C_{n,k}(a) = \inf_{y(r_a, \theta) = 0} \left( \int_{r_a}^\infty \int_{S^{n-1}} (\tau^{2} + (f_1(\tau^{-1}) \ldots f_k(\tau^{-1}))^{-2} |\nabla_\theta y|^2) \, dSd\tau \right)^{\frac{n-2}{n}}.
\tag{2.21}
\]

Again, we will relate this with the best Sobolev constant \( S_n \). From (2.7) we have that
\[
(n-2)^{-\frac{2(n-1)}{n}} S_n = \inf_{z(r_a, \theta) = 0} \left( \int_{r_a}^\infty \int_{S^{n-1}} \tau^{\frac{2(n-1)}{n-2}} |\nabla_\theta z|^2 \, dSd\tau \right)^{\frac{n-2}{n}}.
\tag{2.22}
\]

Comparing this with (2.21) we have that
\[
C_{n,k}(a) \leq C_{n,k,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_{n,radial} = (n-2)^{-\frac{2(n-1)}{n}} S_n. \tag{2.23}
\]

On the other hand for \( a \geq \beta_{k,n} \) and \( \tau \geq r_a \) we have that
\[
\left( \tau^{-1} f_1(\tau^{-1}) \ldots f_k(\tau^{-1}) \right)^{-2} \geq \left( r_a^{-1} f_1(r_a^{-1}) \ldots f_k(r_a^{-1}) \right)^{-2} = (X_1(a, 1) \ldots X_k(a, 1) X_{k+1}(a, 1))^{-2} \geq \frac{1}{(n-2)^2},
\]

therefore
\[
\left( f_1(\tau^{-1}) \ldots f_k(\tau^{-1}) \right)^{-2} \geq \frac{1}{(n-2)^2 \tau^2}, \quad \tau \geq r_a,
\]

and consequently,
\[
C_{n,k}(a) \geq (n-2)^{-\frac{2(n-1)}{n}} S_n.
\]
From this and (2.23) it follows that

$$C_{n,k}(a) = (n-2) \frac{2(n-1)}{n} S_n, \quad \text{when } a \geq \beta_{k,n}.$$ 

The case where $a_k < a < \beta_{k,n}$ is quite similar to the case $0 < a < \frac{1}{n-2}$ in the proof of Theorem A'. That is, testing in (2.20) the sequence $U_\varepsilon$ as defined in (2.10), we first prove that

$$C_{n,k}(a) \leq \left( \prod_{i=1}^{k+1} X_i(a,1) \right)^{-\frac{2(n-1)}{n}} S_n,$$

by an argument quite similar to the one leading to (2.12). Finally, in the case $a_k < a < \beta_{k,n}$, we obtain the opposite inequality by comparing the infimum in (2.21) with the infimum in (2.16). This time we take $R' = r_a$ and $b > 0$ is chosen so that

$$\prod_{i=1}^{k+1} X_i(a,1) = n - 2 + 2b.$$

We omit further details.

Acknowledgments Adimurthi is thanking the Departments of Mathematics and Applied Mathematics of University of Crete for the invitation as well as the warm hospitality. The authors thank the referee for raising the question of existence or nonexistence of minimizers.

References

[A] Aubin, T., Problème isopérimétrique et espace de Sobolev, *J. Differential Geometry*, 11, (1976), 573–598.

[BFL] Benguria R. D., Frank R. L. and Loss M., The sharp constant in the Hardy–Sobolev–Maz’ya inequality in the three dimensional upper half space, *Math. Res. Lett.*, 15, (2008), 613–622.

[BFT] Barbatis G., Filippas S. and Tertikas A., Refined geometric $L^p$ Hardy inequalities, *Comm. Cont. Math.*, 5, (2003), 869–881.

[BM] Brezis H. and Marcus M., Hardy’s inequality revisited, *Ann. Sc. Norm. Pisa*, 25, (1997), 217-237.

[BN] Brezis H. and Nirenberg L., Positive solutions of nonlinear elliptic problems involving critical exponents, *Comm. Pure Appl. Math.*, 36, (1983), 437–477.

[BV] Brezis H. and Vázquez J.L., Blow–up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Comp. Madrid*, 10, (1997), 443–469.

[CKN] Caffarelli L., Kohn, R. and Nirenberg, L. First order interpolation inequalities with weights. *Compositio Math.*, 53 (1984), no. 3, 259–275.

[CW] Catrina F. and Wang Z.–Q., On the Caffarelli–Kohn–Nirenberg inequalities: Sharp constants, existence (and nonexistence) and symmetry of extremal functions. *Comm. Pure Appl. Math.*, LIV, (2001), 229–258.
[FT] Filippas S. and Tertikas A., Optimizing Improved Hardy inequalities. *J. Funct. Anal.*, 192, (2002), 186–233; Corrigendum, *J. Funct. Anal.* to appear (2008).

[MS] Mancini G. and Sandeep K., On a semilinear elliptic equation in $\mathbb{H}^n$, preprint.

[M] V. G. Maz’ya, Sobolev spaces, Springer-Verlag, 1985.

[T] Talenti, G., Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, (4), 110, (1976), 353–372.

[TT] Tertikas A. and Tintarev K., On existence of minimizers for the Hardy–Sobolev–Maz’ya inequality. *Ann. Mat. Pura Appl.*, 186(4) (2007), 645–662.