A Generalization of the Space-Fractional Poisson Process and its Connection to some Lévy Processes

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Abstract

This paper introduces a generalization of the so-called space-fractional Poisson process by extending the difference operator acting on state space present in the associated difference-differential equations to a much more general form. It turns out that this generalization can be put in relation to a specific subordination of a homogeneous Poisson process by means of a subordinator for which it is possible to express the characterizing Lévy measure explicitly. Moreover, the law of this subordinator solves a one-sided first order differential equation in which a particular convolution-type integral operator called Prabhakar derivative is present. The last section of the paper present a similar model in which the Prabhakar derivative also acts in time. Also in this case, the probability generating function of the corresponding process and the probability distribution are determined.

Keywords: Fractional point processes; Lévy processes; Prabhakar integral; Prabhakar derivative; Time-change; Subordination.

1 Introduction and background

In the last decade, it became apparent that several phenomena that can be modeled in terms of point processes are non-Poissonian in nature (see Barabasi [2010], Jiang et al. [2013] as examples). In parallel, there has been an increased interest in generalizing the Poisson process $N(t)$. The Poisson process is a counting process with many nice properties. It is a Lévy process and, therefore, its increments are time-homogeneous and independent. It is a renewal process, meaning that the sojourn times between points (or events) are independent and identically distributed following the exponential distribution. It is a birth-death Markov process and its counting probability obeys the forward Kolmogorov equation

$$\frac{d}{dt}p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t),$$

(1.1)

where $p_k(t) = \mathbb{P}\{N(t) = k\}$, $k \geq 0$, $t \geq 0$ are the state probabilities of the Poisson process and $\lambda$ is the rate of the Poisson process. There are many possible ways to generalize this process. We are interested in the so-called fractional generalizations of the Poisson process. In these generalizations, either the derivative on the left-hand side or the difference equation on the right-hand side of (1.1) are replaced by suitable fractional operators.
For instance, if one keeps the renewal property and considers sojourn times such that
\[ P[N_\beta(t) = 0] = E_\beta(-t^\beta) \]
for \( 0 < \beta < 1 \), where
\[
E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}
\]
is the one-parameter Mittag-Leffler function, one gets the renewal fractional Poisson process discussed in [Mainardi et al., 2004] leading to the equation for \( p_k(t) = P[N_\beta(t) = k], k \geq 0, t \geq 0 \), [Laskin, 2003]
\[
\frac{d^\beta}{dt^\beta} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t),
\]
where \( d^\beta/dt^\beta \) is the so-called Caputo derivative, a pseudo-differential operator defined as
\[
\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} \frac{df(u)}{du} du.
\]
The renewal fractional Poisson process is not a Lévy process [Politi et al., 2011]. Another possibility is generalizing the Poisson process via the so-called space-fractional Poisson process studied in [Orsingher and Polito, 2012]. Here we aim at a further generalization of this process preserving the Lévy property. This generalized process is constructed by means of a superposition of suitably weighted independent space-fractional Poisson processes. The resulting process is then subordinated by means of a random time process in order to account for the modelization of a possible irregular flow of time.

The motivation at the basis of this study lies in the importance that superposition of, possibly dependent, point processes plays in applied sciences. An important example, amongst all, concerns the modelling of neurons’ incoming signals. It is commonly accepted that each neuron obtains information from the neighbouring neurons in form of spike trains (i.e. signals with powerful bursts), closely resembling realizations of stochastic point processes. Therefore, each neuron, receives a superposition of, possibly rescaled, point processes and its subsequent behaviour depends on the characteristics of this afferent combined input signal. Based on the classical result due to Cox and Smith [1954], Grigelionis [1963], Franken [1963], Cinlar and Agnew [1968] (roughly saying that a superposition of sufficiently sparse independent point processes converges to a Poisson process), the input signal was considered in this applied literature to be well approximated by a Poisson process [see e.g. Hohn and Burkitt, 2001, Shimokawa et al., 1999]. However, it was later shown that the above result does not always apply to superposition of signals from neurons’ activity and that experimental evidence deviates from a Poissonian structure [Lindner, 2006, Câteau and Reyes, 2006, Deger et al., 2012]. Recently, the study of weak convergence of superposition of point processes has regained interest [see e.g. Chen and Xia, 2011, and the references therein] showing that different behaviours are possible. Within this framework, we can consider the model we are going to describe as a weighted finite superposition of independent space-fractional Poisson processes (each of them generalizing the homogeneous Poisson process but also admitting the possibility of jumps of any integer order). Each of these space-fractional Poisson processes can be thought to model different groups of neurons (different areas of the brain) acting together with simultaneous spikes giving rise to the non-unitary jumps. Notice, finally, that the space-fractional Poisson process is a non-renewal process and so is the generalized space-fractional Poisson process. This is a key feature for the combined neurons’ input signal as it is remarked in [Lindner, 2006].

For the sake of clarity and simplicity, we first recall some selected basic mathematical results regarding subordinators which will be useful in the following. Section 2 presents the construction of a random time-change by means of independent subordinators and tempered
subordinators. This is of fundamental importance for the definition of the generalized space-
fractional Poisson process which will be carried out in Section 3. Let us underline that the
obtained process is still tractable both from mathematical and practical point of views.

We start thus by recalling some basic facts on subordinators and tempered stable subor-
dinators. The reader can refer to Bertoin [1996] or Kyprianou [2007] for a more in-depth
explanation. For what concerns our work, we recall that a subordinator is an increasing
Lévy process defined as follows. Consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), where
\(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) is the associated right-continuous filtration. The process \(S_t, t \geq 0\), adapted to \(\mathbb{F}\) and starting from zero is called a subordinator if it has independent and stationary increments or, equivalently, \(\forall s > t \geq 0, S_s - S_t\) is independent of \(\mathcal{F}_t\) and \(S_s - S_t = S_{s-t}\) in distribution. In
this paper we consider only strict subordinators that is those with infinite lifetime and with
the only cemetery state \((\infty)\) located at \(t = \infty\).

For the special class of subordinators a simplified version holds of the well-known Lévy–
Khintchine formula for general Lévy processes. In particular the following theorem gives a
characterization of subordinators in terms of Laplace transforms of their one dimensional law.

**Theorem 1.1.** Any function \(\Phi(\mu), \mu \geq 0\), that can be put in the unique form

\[
\Phi(\mu) = b \mu + \int_0^\infty \left(1 - e^{-\mu x}\right) m(dx),
\]

where \(b \geq 0\) and \(m(dx)\) is a measure concentrated on \((0, \infty)\) such that \(\int_0^\infty (1 + x)m(dx) < \infty\), is the Laplace exponent of a strict subordinator.

Conversely, if \(\Phi(\mu)\) is the Laplace exponent of a strict subordinator, there exist a non negative number \(b\) and a unique measure \(m\) with \(\int_0^\infty (1 + x)m(dx) < \infty\) such that (1.5) holds true.

For a detailed proof see Bertoin [1996]. The measure \(m\) is called the Lévy measure and
\(b\) is the drift associated to the strict subordinator \(S_t, t \geq 0\). For the sake of simplicity, in
the following we shall use the term subordinator meaning in fact a strict subordinator.

A particularly simple example of a subordinator is the so-called stable subordinator. This
is characterized by the Laplace exponent \(\Phi(\mu) = \mu^\alpha, \alpha \in (0, 1)\), corresponding to a Lévy mea-
sure \(m(dx) = [\alpha / \Gamma(1 - \alpha)] x^{-1-\alpha} dx\). Note that the case \(\alpha = 1\) is omitted as it is trivial. The
one-parameter \(\alpha\)-stable subordinator is at the basis of the probabilistic theory of anomalous
diffusion based on subordination. In fact it is a building block for the numerous processes con-
ected to fractional evolution equations and their generalizations [Meerschaert and Sikorskii,
2011], D’Ovidio and Polito [2013] and also for other processes connected for example to point
processes [Orsingher and Polito, 2012, Beghin and Macci, 2012]. Moreover, the importance of
stable subordinators stems also from the fact that they are scaling limits of some totally
skewed generalized random walks [Meerschaert and Scheffler, 2004]. Given a subordinator
\(S_t, t \geq 0\), it is possible to define its right-inverse process [Bingham, 1971] as

\[
E_t = \inf\{w > 0 : S_w > t\}, \quad t \geq 0.
\]

In general the inverse process \(E_t\) is non-Markovian with non-stationary and dependent incre-
ments [Veillette and Taqqu, 2010, Lageras, 2005].
Let us now proceed to introducing some basic facts on tempered subordinators, in particular for tempered stable-subordinators. First of all, let us refer the reader to the paper by [Rosiński, 2007] for a complete and detailed account on the general theory of tempered stable processes. However, it is interesting to note that tempered models already appeared in the literature (amongst others, for example, the KoBoL model [Koponen, 1995, Boyarchenko and Levendorskii, 2002]). The class of tempered stable subordinators has been introduced in order to have processes possessing nicer properties than those of standard stable subordinators. In practice the Lévy measure of a stable subordinator is exponentially tempered, thus obtaining
\[
m(dx) = e^{-\xi x} \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} dx, \quad \alpha \in (0,1), \xi > 0. \tag{1.7}
\]
This simple operation produces desirable results as can be ascertained immediately by realizing that the Laplace exponent in this case can be written as
\[
\Phi(\mu) = (\xi + \mu)^\alpha - \xi^\alpha. \tag{1.8}
\]
For more information regarding tempered stable subordinators and associated differential equations, see [Baeumer and Meerschaert, 2010].

In Section 2 we introduce a process which has both the standard stable subordinator and the tempered stable subordinator as special cases and we study its properties. In Section 3 instead, we describe connections of the introduced process with some difference-differential equations involving a generalized fractional difference operator acting in space. An interesting special case related to these equations is that regarding the so-called space-fractional Poisson process [Orsingher and Polito, 2012]. In the last Section 3.3 we present a further generalization leading to a process which is no longer a Lévy process. This generalization is based on the study of similar difference-differential equations but involving the so-called regularized Prabhakar derivative in time that generalizes the Caputo derivative in time.

2 Results on a subordinated combination of independent stable subordinators

Let us consider the filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, \mathbb{P})\) and the process
\[
\eta \mathcal{G}^{\gamma,n}_t = \sum_{r=1}^{n} \binom{n}{r} \eta^r V^r_t = \sum_{r=1}^{n} V^r_t \eta^{n-r} t, \quad t \geq 0, n \in \mathbb{N}, \eta > 0, \forall r \in (0,1) \forall r = 1, \ldots, n, \tag{2.1}
\]
where \(V^r_t, t \geq 0, r = 1, \ldots, n,\) are \(\mathcal{G}\)-adapted independent \(vr\)-stable subordinators. Let us further consider a positive real parameter \(\delta\) such that \([\delta] = n\) and the \(\mathcal{G}\)-adapted tempered \(\delta/n\)-stable subordinator \(\mathcal{G}^{\delta/n}_t, t \geq 0,\) evaluated at \(\eta^n\), i.e. with Laplace exponent
\[
\Phi(\mu) = (\eta^n + \mu)^{\delta/n} - \eta^{\delta}, \quad \delta > 0, \eta > 0. \tag{2.2}
\]

**Proposition 2.1.** The Laplace transform of \(\eta \mathcal{G}^{\gamma,n}_t, t \geq 0,\) reads
\[
\mathbb{E} \exp \left(-\mu \eta \mathcal{G}^{\gamma,n}_t \right) = \exp \left(-t \left[ (\eta^n + \mu)^{\delta/n} - \eta^{\delta} \right] \right), \tag{2.3}
\]
**Proof.** Formula (2.3) can be proven by noticing that
\[
\mathbb{E} \exp \left(-\mu \eta \mathcal{G}^{\gamma,n}_t \right) = \prod_{r=1}^{n} \mathbb{E} \exp \left(-\mu \binom{n}{r} \eta^r V^r_t \right) \tag{2.4}
\]
Theorem 2.1. The \( \eta \mathcal{Q}^{v,\delta}_t \) is a subordinator and derive its associated Lévy measure.

Proposition 2.2. The Laplace transform of the process \( \eta \mathcal{Q}^{v,\delta}_t \), \( t \geq 0 \), can be written as

\[
\mathbb{E} \exp \left( -\mu \eta \mathcal{Q}^{v,\delta}_t \right) = \exp \left( -t \left( \eta + \mu v \right) \delta - \eta^\delta \right), \quad t \geq 0, \mu > 0.
\]

Proof. By recurring to definition (2.7) we can write that

\[
\mathbb{E} \exp \left( -\mu \eta \mathcal{Q}^{v,\delta}_t \right) = \int_0^\infty \mathbb{E} \exp \left( -\mu \eta \gamma^{v,n}_t \right) \mathbb{P} \left( \gamma^{v,\delta/n}_t \in ds \right)
\]

\[
= \int_0^\infty \exp \left( -s \left( \eta + \mu v \right)^n - \eta^n \right) \mathbb{P} \left( \gamma^{v,\delta/n}_t \in ds \right).
\]

Then by using (2.2) we directly arrive at (2.8).

Remark 2.1. Plainly, when \( \delta = n = 1 \), the process \( \eta \mathcal{Q}^{v,\delta}_t \) coincides with a standard \( v \)-stable subordinator, while for \( v \to 1, \delta \in (0,1) \), it is a tempered \( \delta \)-stable subordinator. As an example, we also note that for \( \delta = 2, v \in (0,1/2) \), \( \eta \mathcal{Q}^{v,\delta}_t \) is the rescaled sum of two independent stable subordinators.

In the following theorem we prove that the \( \mathcal{G} \)-adapted process \( \eta \mathcal{Q}^{v,\delta}_t \), \( t \geq 0 \), is in fact a subordinator and derive its associated Lévy measure.

Theorem 2.1. The \( \mathcal{G} \)-adapted process \( \eta \mathcal{Q}^{v,\delta}_t \), \( t \geq 0 \), is a subordinator. Furthermore, its associated Lévy measure is

\[
m(dx) = \int_0^\infty e^{-\eta t} \mathbb{P} \left( \gamma^{v,n}_t \in dx \right) \frac{\delta}{n} \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma \left(1-\frac{\delta}{n}\right)} dt.
\]

Proof. In order to prove the statement of the theorem we make use of the Lévy–Khintchine formula for subordinators. The Laplace exponent associated to \( \eta \mathcal{Q}^{v,\delta}_t \), \( t \geq 0 \), that is

\[
\Phi(\mu) = \left( \eta + \mu v \right)^\delta - \eta^\delta,
\]

can be retrieved with the following steps.

\[
\int_0^\infty (1-e^{-nx}) \int_0^\infty e^{-\eta t} \mathbb{P} \left( \gamma^{v,n}_t \in dx \right) \frac{\delta}{n} \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma \left(1-\frac{\delta}{n}\right)} dt.
\]
is the Riemann–Liouville fractional derivative
\[ \frac{\delta}{n} \frac{t^{-\left(1+\frac{2}{n}\right)}}{\Gamma\left(1-\frac{2}{n}\right)} \]
Follows.

In order to study the operator associated to the subordinator \( \Psi_t^{\nu,\delta} \) we start from the Laplace transform
\[ E e^{-\mu \Psi_t^{\nu,\delta}} = e^{-\int \left[ (\eta + \mu \nu')^{\delta} - \eta^{\delta} \right] dt}, \quad t \geq 0, \mu > 0. \]
By taking the derivative with respect to time we obtain
\[ \frac{d}{dt} \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} = \left[ (\eta + \mu x)^{\delta} - \eta^{\delta} \right] \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}}. \] (2.17)

From Theorem 2.1 we know that
\[ \left[(\eta + \mu x)^{\delta} - \eta^{\delta}\right] = \int_0^\infty (1 - e^{-x}) m(dx), \] (2.18)
where \( m(dx) \) is given by (2.10). Hence,
\[ \left[(\eta + \mu x)^{\delta} - \eta^{\delta}\right] \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} = \int_0^\infty \left( \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} - e^{-\mu y} \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} \right) m(dy), \] (2.19)
and equation (2.17) becomes
\[ \frac{d}{dt} \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} = - \int_0^\infty \left( \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} - e^{-\mu y} \mathbb{E} e^{-\mu x v_t^{\alpha,\beta}} \right) m(dy). \] (2.20)

Now, a simple inversion of the Laplace transforms leads to
\[ \frac{d}{dt} \mathbb{P} \left( \eta v_t^{\alpha,\beta} \in dx \right) /dx = - \int_0^\infty \left( \mathbb{P} \left( \eta v_t^{\alpha,\beta} \in dx \right) /dx - \mathbb{P} \left( \eta v_t^{\alpha,\beta} \in d(x - y) \right) /d(x - y) \right) m(dy). \] (2.21)

Let us denote now \( v(x, t) = \mathbb{P} \left( \eta v_t^{\alpha,\beta} \in dx \right) /dx \) and with
\[ \eta \Theta_x^{\alpha,\beta} f(x, t) = \int_0^\infty (f(x, t) - f(x - y, t)) m(dy) \] (2.22)
the generating form of the operator. The probability density function \( v(x, t) \) of \( \eta v_t^{\alpha,\beta} \) satisfies
\[ \frac{d}{dt} v(x, t) = - \eta \Theta_x^{\alpha,\beta} v(x, t), \quad x \geq 0, \ t \geq 0. \] (2.23)

**Remark 2.3.** We know [Kilbas et al., 2004, Garra et al., 2014] that the Prabakhar derivative is defined, for a suitable set of functions [see Kilbas et al., 2004 for details], as
\[ D_{\alpha, \eta, \zeta, 0}^{\alpha,\beta} f(t) = D_{0+}^{\eta+\theta} \int_0^t (t - y)^{\alpha-1} E_{\alpha,0}^{-\xi} \left[ \zeta(t - y)^{\beta} \right] f(y) dy, \] (2.24)
with \( \theta, \eta \in \mathbb{C}, \Re(\theta) > 0, \Re(\eta) > 0, \zeta \in \mathbb{C}, \ t \geq 0. \) The fractional derivative appearing in (2.24) is the Riemann–Liouville fractional derivative with respect to time \( t \) and
\[ E_{\kappa,\sigma}^{\gamma}(x) = \sum_{r=0}^{\infty} \frac{x^r(\gamma)}{r! \Gamma(k r + \sigma)}, \quad \kappa, \sigma, \gamma \in \mathbb{C}, \Re(\kappa) > 0, \] (2.25)
is the generalized Mittag–Leffler function (see e.g. Kilbas et al., 2004). The Laplace transform of the Prabakhar derivative is [D'Ovidio and Polito, 2013, Kilbas et al., 2004]
\[ \int_0^\infty e^{-st} D_{\alpha, \eta, \zeta, 0}^{\alpha,\beta} f(t) \ dt = s^{\beta} (1 - \zeta s^{-\alpha})^\xi \hat{f}(s), \quad s > 0. \] (2.26)
Therefore, with our choice of parameters,
\[
\int_0^\infty e^{-\mu t}D^{\delta}_{0^+} v(x,t) \, dt = \mu x (1 + \eta)\delta f(x) = (s^\nu + \eta)\delta f(x), \quad s > 0. \tag{2.27}
\]
This implies that
\[
\eta \Theta^{\nu,\delta}_x = D^{\delta}_{0^+} - \eta \delta
\tag{2.28}
\]
and thus
\[
\int_0^\infty e^{-\mu t} \Theta^{\nu,\delta}_x v(x,t) \, dx = \left[ (\mu^\nu + \eta)^\delta - \eta \delta \right] \tilde{v}(\mu, t). \tag{2.29}
\]

**Remark 2.4.** Note that an alternative representation for the operator \(\eta \Theta^{\nu,\delta}_x\) can be given in terms of Riemann–Liouville derivatives. We have
\[
\eta \Theta^{\nu,\delta}_x = (\eta + D^{\nu}_{0^+})^\delta - \eta \delta. \tag{2.30}
\]
This can be proved by simply computing the Laplace transform, as follows.
\[
\int_0^\infty e^{-\mu x} \left[ (\eta + D^{\nu}_{0^+})^\delta - \eta \delta \right] f(x) \, dx. \tag{2.31}
\]

We can perform a formal expansion by means of Newton’s theorem obtaining
\[
\int_0^\infty e^{-\mu x} \sum_{r=1}^\infty \left( \frac{\delta}{r} \right)^r D^{\nu-r}_{0^+} f(x) \, dx = \left[ (\mu^\nu + \eta)^\delta - \eta \delta \right] \tilde{f}(\mu). \tag{2.32}
\]
Clearly, we considered the set of functions for which the semigroup property for the Riemann–Liouville derivative holds, and also that \(D^{\nu}_{0^+}\) is the identity function. Note finally that (2.32) coincides with (2.29).

### 3 A generalization of the space-fractional Poisson process

In this section we present a study of possible generalizations of the so-called space-fractional Poisson process [Orsingher and Polito, 2012] (see also Orsingher and Toaldo [2013] for other more generalized results and special cases). Furthermore, in the following we will see how the process introduced in the preceding section arises rather naturally in this framework as a time-change of an independent homogeneous Poisson process. Section 3.3 shows the effect of replacing the integer-order time derivative in the governing difference-differential equations with a non-local integro-differential operator with a three-parameter Mittag–Leffler function in the kernel, i.e. the so-called regularized Prabhakar derivative.

#### 3.1 Classical model

In order to make the paper as self-contained as possible, we summarize here the basic construction of the space-fractional Poisson process as it was carried out in [Orsingher and Polito, 2012]. The original aim was to generalize a homogeneous Poisson process in a fractional sense by introducing a fractional difference operator in the governing equations acting on the state space. The chosen operator is \((1-B)^\alpha, \alpha \in (0,1)\) (where \(B\) is the backward-shift operator) which appears in the study of long memory time series. Space-fractional models
with continuous state-space are object of intense study (see e.g. Meerschaert and Sikorskii [2011], Baeumer and Meerschaert [2010] and references therein). Similar models for point processes should be developed and studied and put in relation to their time-fractional counterparts. In particular, it is important to note that the introduction of the fractional difference operator \((1 - B)^{\alpha}\) implies a dependence of the probability of attaining a particular state to those of all the states below. The difference-differential equations for the state probabilities of the space-fractional Poisson process read

\[
\begin{align*}
\frac{d}{dt} p_k(t) &= -\lambda^\alpha (1 - B)^{\alpha} p_k(t), \quad \alpha \in (0, 1], \\
p_k(0) &= \delta_{k,0}.
\end{align*}
\] (3.1)

where \(p_k(t) = \Pr\{N^\alpha(t) = k\}, k \geq 0, t \geq 0\), are the state probabilities of the space-fractional homogeneous Poisson process \(N^\alpha(t), t \geq 0\). If \(\alpha = 1\), the process \(N^\alpha(t)\) reduces to the homogeneous Poisson process of rate \(\lambda\). The space-fractional Poisson process possesses independent and stationary increments and \(\mathbb{E}(N^\alpha(t))^h = \infty, h = 1, 2, \ldots\). The Cauchy problem (3.1) is easily solved by recurring to the probability generating function \(G(u, t)\) and by an application of the Laplace transform. It turns out that the probability generating function can be written as

\[
G(u, t) = e^{-\lambda^{\alpha} t (1 - u)^{\alpha}}, \quad |u| \leq 1,
\] (3.2)

and by means of a simple Taylor expansion the state probability distribution is recognized as a discrete stable distribution and reads

\[
p_k(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^{\alpha} t)^r}{r!} \frac{\Gamma(ar + 1)}{\Gamma(ar + 1 - k)} , \quad k \geq 0.
\] (3.3)

The behaviour of the process is made apparent by the subordination representation

\[
N^\alpha(V^\gamma t) \overset{d}{=} N^\alpha(t), \quad \alpha, \gamma \in (0, 1).
\] (3.4)

For \(\alpha = 1\), the time-changed process \(N^1(V^\gamma t)\) increases super-linearly with jumps of any integer size. Following the same lines, in the next section we will construct a generalized model by suitably adapting the operator acting on the state-space.

### 3.2 Generalized model

In order to make the construction clear and easy to follow for the reader, we shall proceed step-by-step starting from a simple modification of the Cauchy problem (3.1). Consider the following difference-differential equations.

\[
\begin{align*}
\frac{d}{dt} p_k(t) &= -\lambda^\alpha (1 - B)^{\alpha} p_k(t) - \lambda^\beta (1 - B)^{\beta} p_k(t), \quad p_k(0) = \delta_{k,0}, \quad k \geq 0, t \geq 0,
\end{align*}
\] (3.5)

where \(\alpha, \beta \in (0, 1)\). Note that these generalize the difference-differential equations governing the state probabilities of a space-fractional Poisson process [Orsingher and Polito, 2012] i.e.

\[
\begin{align*}
\frac{d}{dt} p_k(t) &= -\lambda^\alpha (1 - B)^{\alpha} p_k(t), \quad p_k(0) = \delta_{k,0}, \quad k \geq 0, t \geq 0,
\end{align*}
\] (3.6)

and of course that of the classical homogeneous process for \(\alpha \to 1\) and \(\beta \to 0\).
Theorem 3.1. The subordinated Poisson process

\[ \mathcal{N}(t) = N(V_t^\alpha + V_t^\beta), \quad t \geq 0, \]  

is such that \( \mathbb{P}[\mathcal{N}(t) = k] \) is the solution to the Cauchy problem \((3.5)\). In \((3.7)\), \( N(t), t \geq 0, \)

is a homogeneous Poisson process of parameter \( \lambda > 0 \) and \( V_t^\alpha, V_t^\beta \), are two independent stable subordinators also independent of \( N(t) \).

Proof. On the one hand we calculate the probability generating function \( \mathbb{E}u^{\mathcal{N}(t)}, |u| \leq 1 \).

\[ \mathbb{E}u^{\mathcal{N}(t)} = \int_0^\infty \mathbb{E}N(t)q_{a,\beta}(s,t) \, ds, \]

where \( q_{a,\beta}(s,t) = \mathbb{P}[V_t^\alpha + V_t^\beta \in ds]/ds = q_a(s,t) * q_\beta(s,t). \) Then

\[ \mathbb{E}u^{\mathcal{N}(t)} = \int_0^\infty e^{-\lambda(1-u)}q_{a,\beta}(s,t) \, ds = \int_0^\infty e^{-\lambda(1-u)}q_a(s,t) \, ds \int_0^\infty e^{-\lambda(1-u)}q_\beta(s,t) \, ds \]

\[ = e^{-\lambda(1-u)}e^{-t\lambda(1-u)^{\alpha}} = e^{-t[\lambda^\alpha(1-u)^{\alpha}]} \]

On the other hand we can determine the probability generating function directly from the equations \((3.5)\), thus obtaining

\[ \frac{\partial}{\partial t} G_{a,\beta}(u,t) = -\lambda^a \sum_{r=0}^\infty \sum_{k=0}^\infty u^k \Gamma(\alpha + 1)(-1)^r \frac{1}{r!\Gamma(\alpha + 1 - r)}p_{k-r}(t) - \lambda^\beta \sum_{r=0}^\infty \sum_{k=0}^\infty u^k \Gamma(\beta + 1)(-1)^r \frac{1}{r!\Gamma(\beta + 1 - r)}p_{k-r}(t) \]

\[ = -\lambda^a G_{a,\beta}(u,t)(1-u)^a - \lambda^\beta G_{a,\beta}(u,t)^\beta \]

\[ = -G_{a,\beta}(u,t) \left[ \lambda^a(1-u)^a + \lambda^\beta(1-u)^\beta \right] \]

with the initial condition \( G_{a,\beta}(u,0) = 1 \). The solution to \((3.10)\) reads

\[ G_{a,\beta}(u,t) = e^{-t[\lambda^a(1-u)^a + \lambda^\beta(1-u)^\beta]}, \quad |u| \leq 1, \]

which coincides with \((3.9)\). \( \square \)

Remark 3.1. Notice that \((3.10)\) is the probability generating function of a superposition of two independent and unweighted space-fractional Poisson processes \( \mathcal{N}_1(t) \) and \( \mathcal{N}_2(t) \) of parameters \((\lambda, \alpha)\) and \((\lambda, \beta)\), respectively, such that \( \mathcal{N}(t) = \mathcal{N}_1(t) + \mathcal{N}_2(t), \)

This remark should be kept in mind for the subsequent generalizations.

In order to arrive to a different fractional generalization of the classical Poisson process we now construct equations of the same type of \((3.5)\) but this time involving \( n \) fractional difference operators.

\[ \frac{d}{dt} p_k(t) = -k_1(1-B)^{\alpha_1} p_k(t) - k_2(1-B)^{\alpha_2} p_k(t) - \cdots - k_n(1-B)^{\alpha_n} p_k(t), \quad n \in \mathbb{N}. \]

\[ (3.12) \]

Specializing properly the coefficients \( k_j, j = 1, \ldots, n \), and the parameters \( \alpha_j, j = 1, \ldots, n \), we have

\[ \frac{d}{dt} p_k(t) = -[(1 + \lambda^\alpha(1-B)^\alpha)^n - 1] p_k(t), \quad \forall n \in (0,1), n \in \mathbb{N}, \]

\[ (3.13) \]
with the initial condition \( p_k(0) = \delta_{k,0} \). Expanding the right hand side of the above equation we obtain

\[
\begin{aligned}
\frac{d}{dt} p_k(t) &= - \sum_{r=1}^{n} \binom{n}{r} \lambda^r (1 - B)^r p_k(t), \quad \nu r \in (0, 1), \quad r = 1, \ldots, n, \\
p_k(0) &= \delta_{k,0}.
\end{aligned}
\] (3.14)

The different conditions on the exponents will be clear in the following. The probability generating function in this case can be determined as before.

\[
\frac{\partial}{\partial t} G(u, t) = - \sum_{r=1}^{n} \binom{n}{r} \lambda^r \sum_{m=0}^{\infty} \binom{\nu r}{m} (-1)^n \sum_{k=m}^{\infty} u^k p_{k-m}(t) \\
= - \sum_{r=1}^{n} \binom{n}{r} \lambda^r \sum_{m=0}^{\infty} \binom{\nu r}{m} (-1)^n \sum_{k=0}^{\infty} u^{k+m} p_k(t) \\
= - \sum_{r=1}^{n} \binom{n}{r} \lambda^r \sum_{m=0}^{\infty} \binom{\nu r}{m} (-u)^m G(u, t) \\
= -G(u, t) \sum_{r=1}^{n} \binom{n}{r} \lambda^r (1-u)^{\nu r} \\
= -G(u, t) \left[ (1 + \lambda^r (1-u)^{\nu r} - 1 \right], \quad G(u, 0) = 1.
\] (3.15)

The solution is

\[
G(u, t) = e^{-t[(1+\lambda^r(1-u)^{\nu r})^{-1}], \quad |u| \leq 1.}
\] (3.16)

**Theorem 3.2.** The subordinated process \( N_1(Y_i^{\nu,n}), t \geq 0 \), where \( N(t), t \geq 0 \), is a homogeneous Poisson process of rate \( \lambda \) independent of \( Y_i^{\nu,n} \), has state probability distribution \( \mathbb{P}\{N_1(Y_i^{\nu,n}) = k\}, k \geq 0 \), which is the solution to (3.13).

**Proof.** If \( \hat{q}_i(s, t) = \mathbb{P}\{N^{\nu,n}(s) = ds\} / ds \), we can write that

\[
G(u, t) = \int_0^\infty \mathbb{E} \mathbb{E}[N^{\nu,n}(s,t) ds = \mathbb{E} \exp \left(-(1-u)\lambda \sum_{r=1}^{n} \binom{n}{r} 1^{\nu r} V^{\nu r}_t \right) \\
= \prod_{r=1}^{n} \exp \left(-\lambda(1-u) \binom{n}{r} 1^{\nu r} V^{\nu r}_t \right) = \prod_{r=1}^{n} \exp \left(-t \binom{n}{r} \lambda^{\nu r}(1-u)^{\nu r} \right) \\
= \exp \left[-t \sum_{r=1}^{n} \binom{n}{r} \lambda^{\nu r}(1-u)^{\nu r} \right] = e^{-t[(1+\lambda^r(1-u)^{\nu r})^{-1}],
\] which coincides with (3.16). \( \square \)

The above construction can be generalized in a fractional sense simply by replacing the parameter \( n \) with a more general parameter \( \delta \in \mathbb{R}^+ \). The obtained Cauchy problem reads

\[
\begin{aligned}
\frac{\partial}{\partial t} p_k(t) &= - \left[ (1 + \lambda^r (1-B)^r)^{\delta} - 1 \right] p_k(t), \quad \nu r \in (0, 1), \quad \delta \in \mathbb{R}^+, \quad n = [\delta], \\
p_k(0) &= \delta_{k,0}.
\end{aligned}
\] (3.18)

Analogously as before we have

\[
G(u, t) = \sum_{k=0}^{\infty} u^k p_k(t) = e^{-t[(1+\lambda^r(1-u)^{\nu r})^{\delta} - 1]}, \quad |u| \leq 1. \] (3.19)
Theorem 3.3. The subordinated process $N(\mathcal{Z}^a, t), t \geq 0$, where $N(t), t \geq 0$, is a homogeneous Poisson process of rate $\lambda$ independent of $\mathcal{Z}^a$, is such that $\mathbb{P}(N(\mathcal{Z}^{a}, t) = k), k \geq 0$, is the solution to (3.18).

Proof. It suffices to note that

$$G(u, t) = e^{-t \left[ \left( 1 + \gamma (1-u) \right)^\delta - 1 \right]} = \mathbb{E} e^{-t \left[ \left( 1 + \gamma (1-u) \right)^\delta - 1 \right]} q_{\delta/n}(t) ds,$$

where $q_{\delta/n}(t) ds = \mathbb{P}(\delta^{\delta/n} \in ds)$.

Let us generalize even more. Let $\xi \in \mathbb{R}^+$, $m = \lfloor \xi \rfloor$, and consider the Cauchy problem

$$\begin{cases}
\frac{d}{dt}p_k(t) = - \sum_{j=1}^{m} \binom{\gamma}{j} \left[ 1 + \lambda^\gamma (1-u) \right]^\delta j - 1 p_k(t), & \forall mn \in (0, 1), \delta, m \in \mathbb{N}, \\
p_k(0) = \delta_{k,0},
\end{cases}$$

(3.21)

which is equivalent to

$$\begin{cases}
\frac{d}{dt}p_k(t) = - \left[ 1 + \lambda^\gamma (1-u) \right] p_k(t), & \forall mn \in (0, 1), \delta \in \mathbb{R}^+, m \in \mathbb{N}, \\
p_k(0) = \delta_{k,0},
\end{cases}$$

(3.22)

and by letting $m = \lfloor \xi \rfloor$ and $\gamma = \delta \xi$ we arrive at

$$\begin{cases}
\frac{d}{dt}p_k(t) = - \left[ 1 + \lambda^\gamma (1-u) \right]^\delta p_k(t), & \forall \gamma \in \mathbb{R}^+, \gamma \in \mathbb{R}^+, \\
p_k(0) = \delta_{k,0}.
\end{cases}$$

(3.23)

The Cauchy problems (3.23) and (3.18) are equivalent and therefore also in this case we have that $\mathbb{P}(N(\mathcal{Z}^{a}, t) = k), k \geq 0, t \geq 0$, is the associated solution.

The most general situation we can deal with is the following. Consider the equations

$$\begin{cases}
\frac{d}{dt}p_k(t) = - \left\{ 1 + \lambda^\gamma (1-u) \right\}^\delta p_k(t), & \forall \gamma n \in (0, 1), \delta \in \mathbb{R}^+, \eta \in \mathbb{R}^+, \\
p_k(0) = \delta_{k,0},
\end{cases}$$

(3.24)

Proceeding as before, we obtain immediately that $G(u, t) = e^{-t \left[ (\eta + \lambda^\gamma (1-u))^\delta - \eta^\delta \right]}$.

Theorem 3.4. We have that $\mathbb{P}(N(\mathcal{Z}^{a}, t) = k), k \geq 0, t \geq 0$, where $N(t), t \geq 0$, is an independent homogeneous Poisson process of rate $\lambda > 0$, is the solution to (3.24).

Proof. First recall that

$$e^{-t \left[ (\eta + \mu)^\delta - \eta^\delta \right]} = \int_0^\infty e^{-t \left[ (\eta + \mu)^\delta - \eta^\delta \right]} h_{\delta/n}(s, t) ds.$$

(3.25)

Furthermore recall also that

$$e^{-t \left[ (\eta + \mu)^\delta - \eta^\delta \right]} = e^{-t \left[ (\eta + \mu)^\delta - \eta^\delta \right]} = e^{-t \sum_{r=1}^{\infty} \binom{\gamma}{r} \eta^{(r-1)} \mu^r} = \prod_{r=1}^{\infty} e^{-t \binom{\gamma}{r} \eta^{(r-1)} \mu^r} = \prod_{r=1}^{\infty} e^{-t \eta^{(r-1)} \mu^r} = \prod_{r=1}^{\infty} e^{-t \eta^{(r-1)} \mu^r}.$$

(3.26)

Clearly, if $\eta \bar{q}(s, t) ds = \mathbb{P}(\eta \gamma^{\infty, n} \in ds)$

$$\int_0^\infty \mathbb{E} \eta^{(s)} \eta \bar{q}(s, t) ds = \mathbb{E} e^{-t \left[ (\eta + \lambda^\gamma (1-u))^\delta - \eta^\delta \right]} = e^{-t \left[ (\eta + \lambda^\gamma (1-u))^\delta - \eta^\delta \right]}.$$

(3.27)

and then by using (3.25) we get the claimed result. □
The form of the probability generating function shows that the mean value of the associated process is infinite unless the subordinating process reduces to the tempered stable subordinator, i.e. if $\nu = 1$, $\delta \in (0, 1)$. From the probability generating function $G(u, t)$, it is possible, by a simple Taylor expansion, to obtain the state probability distribution $p_k(t) = \mathbb{P}[N(\eta, \nu, \delta)_t = k]$, $k \geq 0$, $t \geq 0$. We have the following theorem.

**Theorem 3.5.** The state probability distribution $p_k(t) = \mathbb{P}[N(\eta, \nu, \delta)_t = k]$, $k \geq 0$, $t \geq 0$, reads

$$p_k(t) = e^{\eta t} \frac{(-1)^k}{k!} \sum_{m=0}^{\infty} \frac{\Gamma(vm + 1)}{m! \Gamma(vm - k + 1)} \left( \frac{\lambda^v}{\eta} \right)^m \sum_{r=0}^{\infty} \frac{(-t \eta^v)^r \Gamma(r \delta + 1)}{r! \Gamma(r \delta - m + 1)} \right),$$

(3.28)

where $\eta \psi_1(\delta)$ is the generalized Wright function (see Kilbas et al. [2006], page 56, formula (1.11.14)).

**Proof.** Starting from the probability generating function $G(u, t)$, $t \geq 0$, $|u| \leq 1$, we have

$$G(u, t) = e^{\nu t} e^{-t(\eta + \lambda^v(1-u)^{\nu})} = e^{\nu t} \sum_{r=0}^{\infty} \frac{[-t(\eta + \lambda^v(1-u)^{\nu})]^r}{r!}$$

= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{m=0}^{\infty} \frac{\Gamma(r \delta m \lambda^v(1-u)^v)}{m! \Gamma(r \delta - m + 1)} \sum_{h=0}^{\infty} \frac{(-u)^h}{h!}$$

= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} \sum_{r=0}^{\infty} \frac{\Gamma(r \delta + 1)}{r!} \sum_{m=0}^{\infty} \frac{\Gamma(vm + 1)}{m! \Gamma(vm - h + 1)} \frac{\eta^v \delta^m \lambda^{vm}}{\Gamma(vm - h + 1)}$$

= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} \sum_{r=0}^{\infty} \frac{\Gamma(r \delta + 1)}{r!} \sum_{m=0}^{\infty} \frac{\Gamma(vm + 1)}{m! \Gamma(vm - h + 1)} \frac{\eta^v \delta^m \lambda^{vm}}{\Gamma(vm - h + 1)}$$

and thus the claimed formula (3.28).

**Remark 3.2.** For $\delta = 1$ we obtain the discrete stable distribution (2.15) of Orsingher and Polito [2012] characterizing the behaviour of the space-fractional Poisson process. Indeed it suffices to compute

$$\eta^{-m} e^{\eta t} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r + 1)}{r! \Gamma(r - m + 1)} = \eta^{-m} e^{\eta t} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r + 1)}{r! \Gamma(r - m + 1)} = \eta^{-m} e^{\eta t} (-\eta)^m e^{-\eta t} = (-t)^m.$$  

(3.30)

See also Orsingher and Tolino [2013], Section 4.1. For general information on discrete stable random variables, the reader can refer to Steutel and van Harn [1977] or Devroye [1993]. For $\nu = 1$, $\delta \in (0, 1)$, a time-change given by a tempered $v$-stable subordinator is retrieved (see Orsingher and Tolino [2013], Section 4.2).

**Remark 3.3.** Since $N(t)$ is a Lévy process, the subordinated process $N(\eta, \nu, \delta)_t$ is a Lévy process with Laplace exponent

$$\Phi(\mu) = (\eta + \lambda^v(1-e^{-\mu})^\nu) \delta - \eta^\delta,$$

(3.31)

and thus it possesses stationary and independent increments. Moreover $N(\eta, \nu, \delta)_t$ is not in general a renewal process. This comes simply from Kingman [1963] (see also Grandell [1976]).
and from the fact that, since \( \eta^w_t^\gamma,\delta \) is a properly rescaled and time-changed linear combination of independent stable subordinators, its inverse process \( \mathcal{E}_t = \inf\{w > 0 : \eta^w_t^\gamma,\delta > t\} \) has in general non-stationary and dependent increments.

**Remark 3.4.** The reader can compare the explicit form of the state probabilities (3.28) with the results obtained by Orsingher and Toaldo [2013], Section 2, with a suitably specialized Bernstein function \( f \).

### 3.3 A further time-fractional generalization involving regularized Prabhakar derivatives

We saw from the above analysis (see Remark 2.3 and the subordination result) that the difference operator (3.24) is in practice connected with the Prabhakar derivative. This suggests that a further generalization which can be still treatable could involve a regularized Prabhakar derivative acting in time. This is what was considered in Garra et al. [2014] in the case of the classical difference operator and in D’Ovidio and Polito [2013] for the generator of a Lévy process.

Let us first recall the definition of the regularized Prabhakar derivative.

**Definition 3.1.** Let \( \beta, \omega, \gamma, \alpha \in \mathbb{C}, \Re(\beta), \Re(\alpha) > 0, n = [\Re(\beta)] \) and \( f \in \mathcal{AC}^n[0, b], \) \( 0 < x < b \leq \infty, \) where

\[
\mathcal{AC}^n[a, b] = \left\{ f : [a, b] \to \mathbb{R} : \frac{d^{n-1}}{dx^{n-1}} f(x) \text{ is absolutely continuous in } [a, b] \right\}.
\]

The regularized Prabhakar derivative is defined as

\[
C^\gamma D^\gamma_{a,\beta,-\omega,0^+} f(x) = D^\gamma_{a,\beta,-\omega,0^+} \left( f(x) - \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0^+) \right). \tag{3.32}
\]

For further details the reader can consult Garra et al. [2014].

By means of the regularized Prabhakar derivative we can construct the following Cauchy problem.

\[
\begin{aligned}
\left\{ C^\gamma D^\gamma_{a,\beta,-\omega,0^+} p_k(t) = -\left\{ [\eta + \lambda^\gamma(1-B)^\gamma]^{\delta} - \eta^{\delta} \right\} p_k(t), \quad &\forall t \in (0, 1), \delta \in \mathbb{R}^+, \eta \in \mathbb{R}^+, \\
p_k(0) = \delta_k,0, \quad &\end{aligned} \tag{3.33}
\]

with the constraints \( \omega > 0, \gamma \geq 0, 0 < \alpha \leq 1, 0 < \beta \leq 1. \) We also have \( 0 < \beta [\gamma] / \gamma - r \alpha < 1, \forall r = 0, \ldots, \lfloor \gamma \rfloor, \) if \( \gamma \neq 0. \) With similar calculations of (3.15) we obtain for \( G(u, t) = \sum_k p_k(t) \)

\[
\begin{aligned}
\left\{ C^\gamma D^\gamma_{a,\beta,-\omega,0^+} G(u, t) = -\left\{ [\eta + \lambda^\gamma(1-u)^\gamma]^{\delta} - \eta^{\delta} \right\} G(u, t), \quad &\forall t \in (0, 1), \\
G(u, 0) = 1, &\end{aligned} \tag{3.34}
\]

By applying the Laplace transform with respect to time \( t \) we have that

\[
\hat{G}(u, s) = \frac{s^{\beta-1}(1 + \omega s^{-\alpha})^\gamma}{s^{\beta}(1 + \omega s^{-\alpha})^\gamma + (\eta + \lambda^\gamma(1-u)^\gamma)^\delta - \eta^{\delta}}. \tag{3.35}
\]

which can be written, for \( [[(\eta + \lambda^\gamma(1-u)^\gamma)\delta - \eta^{\delta}]/[s^{\beta}(1 + \omega s^{-\alpha})^\gamma]] < 1, \)

\[
\hat{G}(u, s) = \sum_{n=0}^{\infty} \left\{ [((\eta + \lambda^\gamma(1-u)^\gamma)\delta - \eta^{\delta})^n] s^{\beta n-1}(1 + \omega s^{-\alpha})^{-\eta n}. \right\} \tag{3.36}
\]
We can now invert term by term the Laplace transform by using result (2.19) by Kilbas et al. [2004] and Theorem 30.1 by Doetsch [1974]. The probability generating function can be written as

\[
G(u, t) = \sum_{n=0}^{\infty} \left\{ \frac{(-\eta)^n}{n!} E_{\alpha, \beta n+1}^\gamma(-\omega t^\alpha) \right\} \frac{\gamma}{\Gamma(m+1)} \frac{\Gamma(m-k+1)}{(\gamma-\delta)^m}. (3.37)
\]

Note that formula (2.20) of Orsingher and Tolaldo [2013] and equation (3.37) only coincide when \( \gamma = 0 \) in (3.37) and, at the same time, in (2.20) of Orsingher and Tolaldo [2013], \( f(.) \) is specialized to \( (\eta + \lambda(\cdot))^{\delta} - \eta^{\delta} \).

By expanding the probability generating function (3.37) we can derive the state probability distribution for this generalized model. Indeed we have

\[
G(u, t) = \sum_{n=0}^{\infty} (-t^{\beta})^n E_{\alpha, \beta n+1}^\gamma(-\omega t^\alpha) \sum_{r=0}^{\infty} \left\{ \eta^{\delta - r} \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{(\gamma-\delta)^m} \sum_{h=0}^{\infty} \frac{(vm)!}{(vm - h)!} (-u^h) \right\}. (3.38)
\]

Rearranging, we get

\[
\nu_k(t) = \frac{(-1)^k}{k!} \sum_{n=0}^{\infty} (-t^{\beta})^n E_{\alpha, \beta n+1}^\gamma(-\omega t^\alpha) \sum_{r=0}^{\infty} (-\eta^r)^n \frac{\Gamma(m+1)}{(\gamma-\delta)^m} \frac{\Gamma(vm-k+1)}{(\gamma-\delta)^m}. (3.39)
\]

**Remark 3.5.** Note that for \( \gamma = 0, \delta = 1 \), the generating function (3.37) reduces to that of the space-time fractional Poisson process [Orsingher and Tolaldo 2013], while for \( \gamma = 0, \nu = 1, \delta \in (0, 1) \), we obtain that of the tempered space-time fractional Poisson process.

We now show that the probabilities \( p_k(t) \) of (3.39) are in fact state probabilities of a suitably time-changed homogeneous Poisson process.

Consider the stochastic process, given as a sum of subordinated independent stable subordinators

\[
\mathcal{B}_t = \sum_{r=0}^{n} \nu_{\Phi(t)}^{\beta} V_{\nu_{\Phi(t)}}^{\alpha}, \quad t \geq 0. (4.0)
\]

The random time change is defined as

\[
\Phi(t) = \frac{[\gamma \mathcal{B}_t]}{r} V_{\nu_{\Phi(t)}}^{\alpha}, \quad t \geq 0, (4.1)
\]

where \( V_{\nu_{\Phi(t)}}^{\alpha} \) is a stable subordinator independent of all the others and where 0 < \( \beta[\gamma]/\gamma - r/\alpha < 1 \) holds for each \( r = 0, 1, \ldots, [\gamma] \). The hitting time process can be defined in turn as

\[
\mathcal{U}_t = \inf\{ s \geq 0 : \mathcal{B}_s > t \}, \quad t \geq 0. (4.2)
\]

We are now ready to state the following theorem.

**Theorem 3.6.** Let \( \mathcal{U}_t, t \geq 0, \) be the hitting-time process presented in formula (4.2). Furthermore let \( \mathcal{U}_t^{\gamma, \nu, \delta} \) be a homogeneous Poisson process of parameter \( \lambda > 0, \) subordinated by the process \( \mathcal{U}_t^{\gamma, \nu, \delta}, \) defined in (2.7) and independent of \( \mathcal{U}_t. \) The time-changed process

\[
\mathcal{N}(t) = \mathcal{U}_t^{\gamma, \nu, \delta}, \quad t \geq 0, (4.3)
\]

has state probabilities (3.39).
Proof. The claimed result can be proved by writing the probability generating function related to the time-changed process \( N(t) \) as

\[
\sum_{k=0}^{\infty} u^k \mathbb{P}(N(t) = k) = \int_0^\infty e^{-y[(\eta + \lambda \gamma(1-u)^\nu)/\nu - \eta^\delta]} \mathbb{P}(U_t \in dy).
\] (3.44)

Therefore, by taking the Laplace transform with respect to time and taking into consideration Theorem 2.2 of D'Ovidio and Polito [2013] we have

\[
\int_0^\infty \int_0^\infty e^{-y[(\eta + \lambda \gamma(1-u)^\nu)/\nu - \eta^\delta] - s t} \mathbb{P}(U_t \in dy) dt = \frac{s\beta - 1 - \alpha}{s\beta - 1 - \alpha + (\eta + \lambda \gamma(1-u)^\nu)} \frac{\eta^\delta}{\gamma s\beta - 1 - \alpha},
\] (3.45)

which coincides with (3.35).

As a byproduct of our analysis we obtain

\[
\mathbb{E} e^{-\mu \eta^\delta t} = \sum_{k=0}^{\infty} \left( -\left[ (\gamma + \mu)^\delta - \eta^\delta \right] t^\beta \right)^k \mathbb{E}^{\mu t}_{a,\beta+1}(-\omega t^\alpha), \quad t \geq 0, \ s > 0,
\] (3.46)

that generalizes formula (3.37) of Beghin and D'Ovidio [2014] with the Laplace exponent suitably specialized.

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