Averaging geometrical objects on a differentiable manifold

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Abstract

We construct a framework within which a mathematically precise, fully covariant, and exact averaging procedure for tensor fields on a manifold can be formulated. In particular, we introduce the Weitzenböck connection for parallel transport and argue that, within the context of averaging, frames and connections are the natural geometrical objects on the manifold.

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# 1 Introduction

In order to address several physical problems in gravitational theories, including the important question of averaging in cosmology and the interpretation of cosmological observations [1], it is necessary to define an averaged or macroscopic theory of gravity. However, due to the non-linear nature of the gravitational field equations in, for example, Einstein’s General Relativity (GR), it is very difficult to define a mathematically precise and covariant averaging procedure for tensor fields. In previous approaches a $3+1$ cosmological space-time splitting has been employed (i.e., this procedure is not generally covariant) where only scalar quantities are averaged [2] and a perturbative approach involving averaging the perturbed Einstein Field Equations (EFE) has been utilized [3].

In the macroscopic gravity (MG) approach to the averaging problem in GR, which is a fully covariant and exact method, a prescription for the correlation functions which emerge in an averaging of the non-linear field equations (without which, the averaging of the EFE simply amount to definitions of the new averaged terms) is given, and a new tensor field with its own set of field equations is introduced [4]. For the cosmological problem, additional assumptions are required: with reasonable cosmological assumptions, the correlation tensor in MG takes the form of a spatial curvature [5]. The formal mathematical issues of averaging tensors on a differential manifold have recently been revisited [6, 7]. In particular, we note that it may be possible to avoid several of the technical problems of averaging by adopting an approach based on scalar curvature invariants [1].

The aim of this paper is to construct a mathematical framework within which a consistent theory of averaged geometric quantities on a differentiable manifold can be formulated. While we develop the formalism with applications to GR in mind (applications to cosmology will be pursued in future work ¹), our primary motive is to set up a smoothing procedure of interest in its own right, similar in spirit to concepts such as Ricci flow [8].

Our intent is to introduce a rigorously defined geometric averaging procedure that captures the intuition of averaging as integrating a quantity over a region $\Sigma$ and dividing by the volume $V_{\Sigma}$. When we average a tensorial object $T$, in order for the averaged object $\overline{T}$ to transform tensorially, $T$ has to be parallel transported from each point $x'$ within $\Sigma$ to a common reference point $x$. Therefore, as in [7], we introduce the Weitzenböck connection for parallel transport and argue that, within the context of averaging, frames and connections are more natural geometrical objects on the manifold. This also suggests alternative theories of gravity (such as, for example, Poincaré Gauge Theories) as being of interest.

¹On small enough scales, our universe is inhomogeneous and it is on these small scales that the theory of GR has been tested. One may therefore adopt the point of view that we should start with GR in the small and average out the inhomogeneities to obtain an effective large-scale theory. The non-linear nature of the EFE means that this large-scale theory is not necessarily GR itself.

²We will sometimes write $\Sigma_x$ when we want to indicate the reference point.
2 Mathematical Framework

We will assume a differentiable $p$-dimensional manifold $\mathcal{M}$, equipped with a local frame co-basis $e'_{a}$ whose associated metric $g_{ab} = e'_{a} e'_{b}$ is of Lorentzian or Riemannian signature and a connection form $\omega'_{Ja}$ taking values in the Lie algebra $\mathfrak{g}$ of the appropriate pseudo-orthogonal structure group $G$, which is $\text{SO}(p-1,1)$ for Lorentzian manifolds and $\text{SO}(p)$ for the Riemannian case. Lower-case Latin indices indicate tensors with respect to diffeomorphisms of $\mathcal{M}$. Upper-case Latin indices transform under $G$. Indices for differential forms will be suppressed when this does not cause confusion.

We will usually take $\omega$ to be the unique torsion-free spin connection associated with the frame field $e$, and in this case, $\omega$ is related to the frame by the Cartan structure equation

\[ \text{d}e^{I} + \omega^{I}_{J} \wedge e^{J} = 0 \]  

(1)

However, it is possible to let $e$ and $\omega$ remain independent. In this case, $\omega$ needs to be determined by other means (see section (6)). The curvature 2-form of $\omega$, $R^{I}_{J} [\omega]$ is defined as

\[ R^{I}_{J} [\omega] = \text{d} \omega^{I}_{J} + \omega^{I}_{K} \wedge \omega^{K}_{J} \]  

(2)

Averaged quantities will generally be denoted with an overline (i.e. $\overline{\omega}$) whereas quantities defined by a compatibility condition with an averaged field will be denoted with tildes (i.e., $\tilde{\omega}$).

3 Details of Averaging Procedure

In order to carry out the parallel transport procedure alluded to in the introduction, we will need to choose a connection and a curve. One possibility would
be to choose the metric-compatible Levi-Civita connection and parallel transport along the geodesic connecting the two points (if one exists). A difficulty in this choice is that one is required to explicitly calculate the equations for the geodesic. Therefore, we will make another choice, one that does not depend on the choice of the curve which is naturally available in theories formulated in terms of a frame field: the Weitzenböck connection $W^{[10]}$. This is a connection with non-zero torsion, zero curvature, and zero non-metricity defined via the frame field by

$$W^b_{ca} = \epsilon^b_{cI} \partial_a e^I$$

The corresponding parallel transport equation for a vector field $v$ is then

$$t^a \partial_a v^b + W^b_{ca} t^a v^c = 0$$

where $t$ is the tangent vector to the curve along which the parallel transport takes place. It can be checked by a direct substitution that

$$v^a(x) = P^a_{a'}(x, x') v^{a'}(x')$$

solves this equation, where $P^a_{a'}(x, x') = e^a_I(x) e^I_{a'}(x')$. The object $P^a_{a'}(x, x')$ is the parallel transport operator that maps a vector at point $x'$ into the parallel transported vector at reference point $x$. In particular, it can be noted that $P^a_{a'}(x, x')$ depends only on the initial and final point and not on the choice of curve connecting the two points; this reflects the flatness of the Weitzenböck connection $W$ (see [7] for additional details).

We now define the average of a tensor field $T^{a_1 \cdots a_n}_{b_1 \cdots b_m}$ as

$$T^{a_1 \cdots a_n}_{b_1 \cdots b_m} = \frac{1}{V_{\Sigma}} \int_{\Sigma} p^{a_1}_{a_1} \cdots p^{a_n}_{a_n} p^{b_1}_{b_1} \cdots p^{b_m}_{b_m} T^{a_1 \cdots a_n}_{b_1 \cdots b_m} e' \, d^p x'$$

(4)

Here, $\Sigma$ is the region over which the averaging is performed, $V_{\Sigma}$ is the volume of that region (as measured by the unaveraged frame field) and primes refer to dependence on the point $x'$ over which the integration is performed, and $e'$ is the determinant of the frame field.

### 4 Averaged Manifold

It can be readily seen from equation (3) that parallel transporting the frame field from point $x'$ to point $x$ will simply give us the frame field at the point $x$. The corresponding statement is also true for the metric, that is, the average of the metric is the metric itself, which is at odds with the intuitive view of averaging as a smoothing process. Therefore, an alternative process is necessary. A suggested course of action is to proceed in the following manner.

One starts the process by averaging the spin connection $\omega$:

$$\overline{\omega}^j_a = \frac{1}{V_{\Sigma}} \int_{\Sigma} e^k_a e^{j}_{K} \omega^j_a e' \, d^p x'$$

(5)
Since \( \omega \) is a Lie algebra-valued one-form and since the averaging procedure ensures that the same is true of \( \overline{\omega} \), one concludes that \( \overline{\omega} \) is truly a connection form.

Given \( \overline{\omega} \), can one find a compatible frame field \( \tilde{e} \) with \( \overline{\omega} \) as its spin connection? The Cartan structure equation, equation (1), has an integrability condition \( R^I_J [\overline{\omega}] \wedge \tilde{e}^J = 0 \) for \( \overline{\omega} \) and \( \tilde{e} \). Under what circumstances is this condition satisfied? It is clear that a necessary condition is that \( \overline{\omega} \) take values in a subalgebra of the pseudo-orthogonal Lie algebra given by the metric signature. In [11], it is shown that this condition is sufficient as well, so the connection determines the metric up to a constant, and therefore the frame up to a constant and a local Lorentz transformation.

There are now two possible identifications for the curvature associated with the averaged manifold. The first is to calculate the corresponding curvature \( R[\overline{\omega}] \) associated with the averaged spin connection. The second is to calculate the curvature \( R[\omega] \) and then average to obtain \( \overline{R[\omega]} \), which is generally distinct from \( R[\overline{\omega}] \).

If we proceed in the latter manner by assuming that \( R[\overline{\omega}] \) is indeed the curvature of some spin connection \( \tilde{\omega} \), can one find the corresponding spin connection \( \tilde{\omega} \)? The integrability condition is \( R^I_J [\tilde{\omega}] \wedge \tilde{\omega}^J - \tilde{\omega}^I_J \wedge \tilde{R}^J_K = d \tilde{R}^I_K \). This integrability condition does not hold for all manifolds, but it is generically satisfied[12].

What if any relation exists between \( R[\omega] \) and \( R[\overline{\omega}] \)? A partial result is possible once we discuss how to obtain neighboring averaging regions. One procedure which yields neighboring averaging regions of similar size and shape to the original averaging region is a process coined “averaging region coordination” first described in [4] and further illustrated in [7, 15]. In this procedure, the given direction vector at \( x, \xi^d(x) \) is mapped (or coordinated) to a direction vector for all other points \( x' \in \Sigma_x \) through parallel transport. The vector

\[
P^d_d(x, x') \xi^d(x)
\]

is the image of the given directional vector \( \xi^d \) at \( x \) to each point \( x' \in \Sigma_x \). Given an averaging region \( \Sigma_x \) at supporting point \( x \) and a direction vector \( \xi^d(x) \), one can now obtain a nearby averaging region \( \Sigma_{x+\Delta \lambda \xi} \) by Lie dragging \( \Sigma_x \) a parametric distance \( \Delta \lambda \) along the integral curves of the parallel transported vector field \( P^d_d(x, x') \xi^d(x) \) for each point \( x' \in \Sigma \). Assuming this “averaging region coordination” one is able to explicitly calculate the difference in the averaged curvature and the curvature of the averaged spin connection. See
[4, 7, 15] for details on the calculation.

\[
R_{Jcd}^I[\omega] = -2\omega_{J[c,d]} + 2\omega_{K[c}\omega^K_{Jd]} \\
= -2 \left[ \omega_{J[c,d]} + \frac{1}{V\Sigma} \int_{\Sigma} G_{[cd]}^{a} \omega_{Ja} P_{a}^{e} d^{p} x' \right] + \frac{2}{V\Sigma} \int_{\Sigma} G_{[cd]}^{a} \omega_{Ja} P_{a}^{e} d^{p} x' \\
= R_{Jcd}^I[\omega] - 2\omega_{K[c}\omega^K_{Jd]} + 2\omega_{K[c}\omega^K_{Jd]} \\
- \frac{2}{V\Sigma} \int_{\Sigma} G_{[cd]}^{a} \omega_{Ja} P_{a}^{e} d^{p} x' - 2 \left[ \omega_{J[c,d]} \frac{1}{e} \right] + 2\omega_{J[c,d]} \frac{1}{e}
\]

where the term containing \( G_{[cd]}^{a} \) is the result of changes in the parallel transporter as one changes position. Explicitly one writes

\[
G_{[cd]}^{a} = P_{a}^{d} (P_{d}^{'b} + P_{d}^{'b} P_{b}^{d} W_{bd}^{'a}).
\]

Using the definitions for both \( P_{a}^{d} \) and \( W_{bd}^{'a} \) one observes that

\[
G_{[cd]}^{a} = W_{bd}^{'a} - P_{a}^{d} P_{b}^{'d} W_{bd}^{'a},
\]

which is just the difference between the Weitzenb"{o}ck connection at \( x \) and the image of the Weitzenb"{o}ck connection at \( x' \) parallel transported to \( x \). The two terms containing the \( e, d \) are the result of changes in the volume as one changes position.

### 4.1 Change of averages under change of base point

While a full investigation is beyond the scope of this paper, we will here show that when the base point \( x \) of \( \Sigma_x \) is shifted, on average the average of a scalar quantity \( f \) changes more slowly than the corresponding unaveraged quantity.

Let us start with the definition of averaging for a scalar function \( f \) on \( M \):

\[
\bar{f} = \frac{1}{V} \int_{\Sigma_x} f dV
\]

where \( dV \) is the volume element associated with the frame field \( e \) and \( V = \int_{\Sigma_x} dV \) is the volume of \( \Sigma_x \). Let us now shift the region of integration \( \Sigma_x \) by an amount \( \delta x \) to a new region \( \Sigma_{x'} \) with a base point at \( x' \). We assume \( \delta x \) to be infinitesimal. In addition, we will need to assume that the map \( x'^{a} = x^{a} + \delta x^{a} \) is volume-preserving.

A calculation leads to the induced shift \( \delta \bar{f} \) of the average value \( \bar{f} \) being given by

\[
\delta \bar{f} = \frac{1}{V} \int_{\Sigma_x} \partial_{a} (f(x) \delta x^{a}) n_{a} dS
\]
By comparison, the corresponding change $\delta f$ in the unaveraged quantity $f$ is given by

$$\delta f = \delta x^a \partial_a f$$

It then follows from the triangle inequality and the assumptions stated above that

$$|\delta \bar{f}| \leq |\delta f|$$

where the average on the right-hand side is taken over $\Sigma_x$. This equation can be interpreted as stating that on average, the averaged $f$ changes more slowly than the unaveraged $f$.

## 5 An example

Let us as an example explicitly perform the averaging for the manifold in volume preserving coordinates to eliminate the effects of changes in the volume. Consider the manifold $(\mathbb{R}^2, ds^2)$, with $ds^2 = f(x)^{-2}dx^2 + f(x)^2dy^2$. The line element yields the orthonormal frame field $e^1 = f(x)^{-1}dx$, $e^2 = f(x)dy$. The corresponding nontrivial components of the Weitzenböck connection are

$$W^x_{xx} = -\frac{f'(x)}{f(x)} \quad W^y_{yx} = \frac{f'(x)}{f(x)}$$

From the Cartan structure equation (1), we obtain

$$\omega_2^1 = -f(x)f'(x)dy = -\frac{1}{2}([f(x)]^2)'dy$$
as the only non-zero component. If we pick \( \Sigma \) to be a square with side coordinate length \( 2\delta \) centred at \((x, y)\), equation (5) now tells us that

\[
\omega^1_2 = \left[ \frac{1}{4\delta^2} \int_{\Sigma} e_y e_y e^1 \omega_{20} e' d^2 x' \right] dy
\]

\[
= \left[ \frac{1}{4\delta^2} \int_{y+\delta}^{y} \int_{x-\delta}^{x+\delta} f(x) \frac{f(x')}{f(x')} (-f'(x')f(x')) dx' dy' \right] dy
\]

\[
= \left[ -\left( \frac{f(x + \delta) - f(x - \delta)}{2\delta} \right) f(x) \right] dy
\]

Equation (6)

It is a simple exercise in taking limits to show that as \( \delta \to 0 \), \( \omega^1_2 \to -f'(x)f(x)dy \) as it should, since in that limit \( \omega \) should reduce to the unaveraged connection \( \omega \).

The next step is to find a frame \( \tilde{e} \) compatible with \( \omega \). Again, we use equation (1) and this time we find the system of equations

\[
\begin{align*}
\partial \tilde{e}^1 + \omega^1_2 \wedge \tilde{e}^2 &= 0 \\
\partial \tilde{e}^2 + \omega^2_1 \wedge \tilde{e}^1 &= 0
\end{align*}
\]

Equation (7)

Assume that \( \tilde{e}^1 = F(x)dx \) for some as yet undetermined function \( F(x) \). This leads to

\[
-\frac{f(x)}{2\delta} \left( f(x + \delta) - f(x - \delta) \right) dy \wedge \tilde{e}^2 = 0
\]

Equation (8)

which implies that \( \tilde{e}^2 = h(x,y)dy \). Inserting this into the second equation in (7) gives

\[
\frac{\partial h}{\partial x} (dx \wedge dy + \frac{f(x)}{2\delta} \left( f(x + \delta) - f(x - \delta) \right) F(x)dx \wedge dy = 0
\]

Equation (9)

which in turn shows that

\[
h(x,y) = h_0(y) + \frac{1}{2\delta} \int f(x) \left( f(x + \delta) - f(x - \delta) \right) F(x)dx
\]

Equation (10)

for some function \( h_0(y) \). By requiring that \( h(x,y) \to f(x) \) as \( \delta \to 0 \), a small calculation shows that \( h_0(y) = 0 \) and \( F(x) = f(x)^{-1} \) resulting in a frame field of the form

\[
\tilde{e}^1 = f(x)^{-1}dx \quad \tilde{e}^2 = \left[ \frac{1}{2\delta} \int (f(x + \delta) - f(x - \delta))dx \right] dy
\]

Equation (11)

As for the curvatures, we start by calculating \( R^j_i[\omega] \)

\[
R^j_i[\omega] = -[f(x)f'(x)]' dx \wedge dy = -\frac{1}{2}([f(x)]^2)'' dx \wedge dy
\]

Equation (12)
with all the other components of $R_I^J[\omega]$ equal to zero. Averaging, we find

$$R_{12}^2(\omega) = -\frac{1}{2\delta}\left[\frac{f(x+\delta)f'(x+\delta) - f(x-\delta)f'(x-\delta)}{2}\right] dx \wedge dy \tag{13}$$

We can also compute the curvature of the averaged connection $\omega$; this leads to

$$R_{12}^2(\omega) = -\frac{1}{2\delta}\left[\frac{f(x)f'(x+\delta) - f(x)f'(x-\delta) + f(x+\delta)f'(x) - f(x-\delta)f'(x)}{2}\right] dx \wedge dy \tag{14}$$

One can show that the two curvatures both approach $-\frac{1}{2}\left[\frac{f(x)f'(x)}{2}\right] dx \wedge dy = R_{12}^2[\omega]$ in the limit $\delta \to 0$.

Two-dimensional spaces of constant curvature are prescribed if $f(x) = \sqrt{1-|k|x^2}$ for $k = -1, 0, 1$. Substitution of this expression into equations (12) and (13) shows that $R_{12}^2(\omega) = R_{21}^2(\omega) = k dx \wedge dy$, as might be expected for spaces of constant curvature. In contrast, $R_{12}^2(\omega) \neq R_{21}^2(\omega)$, which suggests that the appropriate primary object to consider when averaging or smoothing manifolds is the curvature and not the connection.

6 Averaging and gravitational theories

In physical applications, $\epsilon$ and $\omega$ will ultimately be given by the field equations of some geometric gravitational theory, along with some boundary conditions. If $\omega$ is assumed to be the spin connection compatible with $\epsilon$, we may for instance choose the GR action in Palatini form, since the compatibility of $\epsilon$ and $\omega$ then falls out naturally as a field equation. On the other hand, if $\epsilon$ and $\omega$ are a priori assumed to be independent, it is natural to consider other theories of gravity, such as Einstein-Cartan theory and its Poincaré Gauge Theory cousins [13]. For an overview of actions for geometric gravitational theories, see [14]. For instance, for general relativity the action would take the form

$$S = \int e^a_I e^b_J R^{I J}_{ab}[\omega] e d^p x$$

Varying this action with respect to the frame $e_{aI}$ gives the usual vacuum Einstein equation $R_{ab} = 0$. If we consider $\epsilon$ and $\omega$ to be independent, varying with respect to the connection $\omega$ will give the compatibility condition (1).

No matter which gravitational theory is considered, the field equations for the unaveraged $\epsilon$ and $\omega$ will determine equations and integrability conditions for the corresponding averaged quantities. The form of these equations will be studied in a future paper.
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