Functional form of unitary representations of the quantum “\(az+b\)” group*

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Abstract

The formula for all unitary representations of the quantum “\(az+b\)” group for a real deformation parameter is given. The description involves the quantum exponential function introduced by Woronowicz.

Key words: C*-algebra, quantum group, unitary representation, exponential function.

1 Introduction

In the study of unitary representations of a topological group it is interesting to look for a general structure theorem. The class of abelian locally compact groups provides a good example of a situation in which a general structure theorem is known. It is the SNAG (Stone-Naimark-Ambrose-Godement) theorem ([1, Ch. VI, Thm. 29]) saying that if \(u\) is a strongly continuous unitary representation of a locally compact abelian group \(\Gamma\) on a Hilbert space \(H\) then there exists a spectral measure \(E\) on the dual group \(\hat{\Gamma}\) such that

\[
u_{\gamma} = \int_{\hat{\Gamma}} \chi(\hat{\gamma}, \gamma) dE(\hat{\gamma}),
\]

where \(\chi\) is a bicharacter on \(\Gamma \times \hat{\Gamma}\). The right hand side of (1) is a source of a functional expression for \(u_{\gamma}\). The best known example of such situation is described by Stone’s theorem. In this case \(\Gamma = \hat{\Gamma} = \mathbb{R}\), \(\chi(\hat{\gamma}, \gamma) = \exp(i\gamma \hat{\gamma})\) and (1) leads to a functional expression

\[
u_{\gamma} = \chi(H, \gamma) = e^{i\gamma H},
\]

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where $H = \int_{\mathbb{R}} \hat{\gamma} dE(\hat{\gamma})$ is a selfadjoint operator acting on $H$. No analogy of SNAG theorem is known for nonabelian case, nevertheless in the context of locally compact quantum groups this type of structure theorem was proved e.g. for the quantum $E(2)$ (cf. [8]) and quantum “$az + b$” groups ([5]). Following these ideas, in this report we shall describe the functional form of strongly continuous unitary representations of the quantum “$az + b$” group for real deformation parameter $q \in [0, 1]$.

We shall be concerned with the theory of quantum groups on the $C^*$-algebra level. For the basic notions and notation used in our paper we refer the reader to [6] and [9]. In particular $M(A)$ will denote the multiplier algebra of a $C^*$-algebra $A$, by “$\eta$” we shall denote the affiliation relation in the sense of $C^*$-algebra theory and $\text{Mor} (A, B)$ will denote the set of morphisms from a $C^*$-algebra $A$ to a $C^*$-algebra $B$, i.e. nondegenerate $*$-homomorphisms from $A$ to $M(B)$. Throughout the paper we shall be making extensive use of formulas (2.4)–(2.8) of [9].

### 2 Quantum “$az + b$” group

The quantum “$az + b$” group for real deformation parameter $q \in [0, 1]$ was described in [10, Appendix A]. We shall denote this quantum group by the symbol $G$. Its construction begins with considering the set

$$\Gamma = \{ z \in \mathbb{C} : |z| \in q^\mathbb{Z}\}$$

and its closure in $\mathbb{C}$

$$\overline{\Gamma} = \Gamma \cup \{0\}.$$ 

With the topology and multiplication inherited from $\mathbb{C} \setminus \{0\}$, $\Gamma$ becomes an abelian topological group (isomorphic to $\mathbb{Z} \times S^1$). Clearly its Pontryagin dual is isomorphic to $\Gamma$ and the pairing describing the duality is given by the bicharacter

$$\chi(q^{i\varphi+k}, q^{i\psi+l}) = q^{i(l\varphi + k\psi)}. \quad (2)$$

Considering the natural action by multiplication of $\Gamma$ on $\overline{\Gamma}$ we obtain a $C^*$-dynamical system $(C_\infty(\overline{\Gamma}), \Gamma, \alpha)$, where the action $\alpha$ is

$$(\alpha_{\gamma}f)(\gamma') = f(\gamma \gamma') \quad (3)$$

for all $f \in C_\infty(\overline{\Gamma})$, $\gamma' \in \overline{\Gamma}$ and $\gamma \in \Gamma$. The algebra $C_\infty(\overline{\Gamma})$ is generated by an element $b \eta C_\infty(\overline{\Gamma})$ given by

$$b(\gamma) = \gamma \quad (4)$$

for all $\gamma \in \overline{\Gamma}$. The algebra of “continuous functions vanishing at infinity on quantum “$az + b$” group” is the $C^*$-algebra crossed product

$$A = C_\infty(\overline{\Gamma}) \rtimes_\alpha \Gamma. \quad (5)$$
Since the canonical injection $C_\infty(\Gamma) \hookrightarrow M(A)$ is a morphism from $C_\infty(\Gamma)$ to $A$, we can regard $b$ as an element affiliated with $A$. By definition of a crossed product there is a strictly continuous family $(\lambda_\gamma)_{\gamma \in \Gamma}$ of unitary elements of $M(A)$ implementing the action $\alpha$:
\[
\lambda_\gamma f \lambda_\gamma^* = \alpha_\gamma(f)
\]
for any $f \in C_\infty(\Gamma)$ considered as an element of $M(A)$. Using the methods developed in [10, Sect. 5] one can show that $\lambda_\gamma$ is of the form
\[
\lambda_\gamma = \chi(a_\gamma, \gamma)
\]
for a normal element $a$ affiliated with $A$, such that $\text{Sp} a \subset \Gamma$. Moreover $a$ is invertible and $a^{-1} \eta A$.

It follows from the definition of the action $\alpha$ (cf. (6) and (7)) that $\chi(a_\gamma, \gamma)b\chi(a_\gamma, \gamma)^* = \gamma b$ for $\gamma \in \Gamma$ and the $C^*$-algebra $A$ is generated by the unbounded affiliated elements $a, a^{-1}$ and $b$.

The basic notion used in our paper is that of a regular $q^2$-pair.

**Definition 2.1** Let $H$ be a Hilbert space and let $(Y, X)$ be a pair of closed densely defined operators on $H$. We shall say that $(Y, X)$ is a regular $q^2$-pair if
\[
\begin{cases}
X \text{ and } Y \text{ are normal}, \\
\text{Sp } X, \text{ Sp } Y \subset \Gamma, \\
\ker X = \{0\} \text{ and} \\
\chi(X, \gamma)Y\chi(X, \gamma)^* = \gamma Y
\end{cases}
\]
for all $\gamma \in \Gamma$.

Let us remark that relations of Definition 2.1 give the precise meaning to the relations of the form
\[
XY = q^2 YX, \quad XY^* = Y^* X
\]
for a pair of normal operators $(Y, X)$.

The pair $(b, a)$ of elements affiliated with the $C^*$-algebra $A$ is a regular $q^2$-pair in the sense that for any nondegenerate representation $\pi$ of $A$ on a Hilbert space $H$, the pair $(\pi(b), \pi(a))$ is a regular $q^2$-pair acting on $H$. We shall also use the notion of a regular $q^2$-pair in the context of any $C^*$-algebra $B$.

Let us note that the $C^*$-algebra $A$ defined by (6) is a universal $C^*$-algebra generated by a regular $q^2$-pair in the following sense (comp. [10, Prop. 4.2]):

**Proposition 2.2** Let $B$ be a $C^*$-algebra and $(b_0, a_0)$ a regular $q^2$-pair of elements affiliated with $B$. Then there exists a unique morphism $\varphi \in \text{Mor} (A, B)$ such that
\[
\varphi(a) = a_0, \quad \varphi(b) = b_0.
\]
It turns out that the operator \( a \otimes b + b \otimes I \) is closeable and (denoting its closure by \( a \otimes b + b \otimes I \) the pair \( (a \otimes b + b \otimes I, a \otimes a) \) is a regular \( q^2 \)-pair (cf. Proposition 3.1) of elements affiliated with \( A \otimes A \). Therefore there exists a unique morphism \( \Delta \in \text{Mor} (A, A \otimes A) \) such that
\[
\Delta(a) = a \otimes a,
\Delta(b) = a \otimes b + b \otimes I.
\] (8)
Moreover \( \Delta \) is coassociative and encodes the group structure of the quantum \( \text{“}az + b\text{”} \) group, briefly \( G = (A, \Delta) \).

3 Quantum exponential function

In [7] S.L. Woronowicz introduced the quantum exponential function \( F_q \) defined on \( \mathbb{T} \). It is given by the formula
\[
F_q(\gamma) = \prod_{k=0}^{\infty} \frac{1 + q^{2k-1}}{1 + q^{2k-1}}
\]
for \( \gamma \in \mathbb{T} \setminus \{-1, -q^{-2}, -q^{-4}, \ldots\} \) and \( F_q(\gamma) = -1 \) for \( \gamma \in \{-1, -q^{-2}, -q^{-4}, \ldots\} \). Thus defined, \( F_q \) is a continuous function \( \mathbb{T} \to S^1 \). Moreover \( F_q(0) = 1 \).

The most important property of the quantum exponential function is the one contained in the following proposition:

Proposition 3.1 ([7 Thm. 3.1]) Let \( H \) be a Hilbert space and let \( (Y, X) \) be a regular \( q^2 \)-pair acting on \( H \). Then the sum \( X + Y \) is a densely defined closeable operator and its closure \( X + Y \) is a normal operator with \( \text{Sp} \left( X + Y \right) \subset \mathbb{T} \). Moreover
\[
F_q(X + Y) = F_q(Y)F_q(X).
\] (9)

The last statement in Proposition 3.1 justifies the name “quantum exponential function”.

Let us remark that formula [8] holds for more general \( q^2 \)-pairs, without the assumption that \( \ker X = \{0\} \) (cf. [4] Section 2]). Moreover \( F_q \) is the only solution of this type of functional equation in a more general sense. To formulate the corresponding result let \( H \) be a Hilbert space and \( f : \mathbb{T} \ni \gamma \mapsto f(\gamma) \in B(H) \) be a bounded measureable mapping. For a normal operator \( Y \) acting on a Hilbert space \( K \) such that \( \text{Sp} Y \subset \mathbb{T} \) we set
\[
f(Y) = \int_{\mathbb{T}} f(\gamma) \otimes dE_Y(\gamma),
\]
where \( dE_Y(\gamma) \) is the spectral measure of \( Y \). Clearly \( f(Y) \in B(H \otimes K) \).

Theorem 3.2 ([7 Thm. 4.2]) Let \( H \) and \( K \) be Hilbert spaces and let \( (Y, X) \) be a regular \( q^2 \)-pair acting on \( K \). Let \( f : \mathbb{T} \ni \gamma \mapsto f(\gamma) \in B(H) \)
be a borel mapping such that \( f(\gamma) \) is unitary for almost all \( \gamma \in \overline{\Gamma} \). Assume that
\[
 f(X + Y) = f(Y)f(X). \tag{10}
\]
Then there exists a normal operator \( Z \) on \( H \) such that \( \text{Sp} Z \subset \overline{\Gamma} \) and
\[
 f(\gamma) = F_q(Z\gamma)
\]
for almost all \( \gamma \in \overline{\Gamma} \).

Note that any borel solution of the functional equation \((10)\) is in fact a continuous one. This result is crucial for functional description of unitary representations of \( G \) given in the next section.

4 Structure of unitary representations of the quantum “az + b” group

In this section we investigate strongly continuous unitary representations of \( G \).

Definition 4.1 A strongly continuous unitary representation of \( G \) on a Hilbert space \( H \) is a unitary element
\[
 U \in M(\mathcal{K}(H) \otimes A)
\]
such that
\[
 (\text{id} \otimes \Delta) U = U_{12}U_{13} \tag{11}
\]
(where we used the leg numbering notation).

In what follows we shall abbreviate “strongly continuous unitary representations” to “unitary representations”. The main result of the paper is contained in the following theorem:

Theorem 4.2 Let \( U \) be a unitary representation of the quantum “az + b” group on a Hilbert space \( H \). Then there exists a unique regular \( q^2 \)-pair \( (\tilde{b}, \tilde{a}) \) acting on \( H \) such that
\[
 U = F_q(\tilde{b} \otimes b)\chi(\tilde{a} \otimes I, I \otimes a). \tag{12}
\]

Conversely, for any regular \( q^2 \)-pair \( (\tilde{a}, \tilde{b}) \) acting on a Hilbert space \( H \), the operator \( U \) defined by formula \((12)\) is a unitary representation of \( G \).

Proof. Let
\[
 a_0(\gamma) = \gamma, \\
 b_0(\gamma) = 0
\]
for all $\gamma \in \Gamma$. Then $a_0, b_0 \eta C_\infty(\Gamma)$ and if we represent $C_\infty(\Gamma)$ on $L^2(\Gamma)$ by multiplication operators, $(b_0, a_0)$ becomes a regular $q^2$-pair on $L^2(\Gamma)$. By the universal property of $A$ (Proposition 2.2) there exists a $\varphi \in \operatorname{Mor}(A, C_\infty(\Gamma)) = C(\Gamma, \operatorname{Mor}(A, \mathbb{C}))$ such that

$$
\varphi(a) = a_0,
\varphi(b) = b_0.
$$

Thus $\varphi$ is a continuous family $\varphi = (\varphi_\gamma)_{\gamma \in \Gamma}$ where the $\varphi_\gamma$ are multiplicative functionals on $A$:

$$
\varphi_\gamma(x) = (\varphi(x))(\gamma).
$$

Moreover the map $\Gamma \ni \gamma \mapsto \varphi_\gamma \in A^*$ is a homomorphism in the sense that

$$
\varphi_{\gamma_1} \ast \varphi_{\gamma_2} = \varphi_{\gamma_1 \gamma_2},
$$

where the convolution

$$
\varphi_{\gamma_1} \ast \varphi_{\gamma_2} = (\varphi_{\gamma_1} \otimes \varphi_{\gamma_2}) \circ \Delta.
$$

Let

$$
\phi(x) = (\text{id} \otimes \varphi) \Delta(x)
$$

for $x \in A$. Then $\phi \in \operatorname{Mor}(A, A \otimes C_\infty(\Gamma)) = C(\Gamma, \operatorname{Mor}(A, A))$. Again we identify $\phi$ with a continuous family $\phi = (\phi_\gamma)_{\gamma \in \Gamma}$ and it is easy to see that for all $\gamma \in \Gamma$ the maps $\phi_\gamma$ are automorphisms of $A$:

$$
\phi_\gamma \in \operatorname{Aut}(A).
$$

In other words

$$
\phi \in C(\Gamma, \operatorname{Aut}(A)).
$$

It follows from (13) that $(\phi_\gamma)_{\gamma \in \Gamma}$ is a continuous group of automorphisms of $A$. It is also easy to check that

$$
\phi_\gamma(a) = \gamma a,
\phi_\gamma(b) = b
$$

for all $\gamma \in \Gamma$. In other words the action $\hat{\Gamma} = \Gamma \ni \gamma \mapsto \phi_\gamma \in \operatorname{Aut}(A)$ is the dual action to the action $\alpha$ of $\Gamma$ on $C_\infty(\Gamma)$ (cf. (3) and (5)).

Define

$$
u = (\text{id} \otimes \varphi) U \in M(K(H) \otimes C_\infty(\Gamma)) = C_{\text{bounded}}(\Gamma, B(H)).$$

We can thus view $u$ as a continuous family $(u_\gamma)_{\gamma \in \Gamma}$ of unitary elements in $B(H)$.

Since by (13) and (13)

$$
u_{\gamma_1} \nu_{\gamma_2} = ((\text{id} \otimes \varphi_{\gamma_1}) U)(\text{id} \otimes \varphi_{\gamma_2}) U
= (\text{id} \otimes \varphi_{\gamma_1} \otimes \varphi_{\gamma_2}) U_{12} U_{13}
= (\text{id} \otimes \varphi_{\gamma_1} \ast \varphi_{\gamma_2}) U = u_{\gamma_1 \gamma_2},$$

the map $\nu \in (\text{id} \otimes \varphi)^{-1}(M(\mathbb{K}) \otimes C_\infty(\Gamma))$ is the dual to $\varphi$.
$u$ is a strongly continuous representation of $\Gamma$ in the Hilbert space $H$. By SNAG theorem (cf. [1, Ch. VI, Thm. 29] and Section 1)
\[ u_\gamma = \chi(\tilde{a}, \gamma) \]
where $\tilde{a}$ is a normal operator on $H$ such that $\text{Sp} \tilde{a} \subset \overline{\Gamma}$ and $\ker \tilde{a} = \{0\}$.

Let
\[ V = \chi(\tilde{a} \otimes I, I \otimes a) \in M(\mathcal{K}(H) \otimes A). \]

We have
\[ (\text{id} \otimes \varphi)V = u. \]

Using (11) we obtain
\[ (\text{id} \otimes \phi_\gamma)U = (\text{id} \otimes \text{id} \otimes \varphi_\gamma)\Delta(U) \]
\[ = U((\text{id} \otimes \varphi_\gamma)U) \]
\[ = U(u_\gamma \otimes I) = U(\chi(\tilde{a}, \gamma) \otimes I) \quad (14) \]

and by (8) and the definition of $\varphi_\gamma$ we have
\[ (\text{id} \otimes \phi_\gamma)V = (\text{id} \otimes \text{id} \otimes \varphi_\gamma)(\text{id} \otimes \Delta)\chi(\tilde{a} \otimes I, I \otimes a) \]
\[ = \chi(\tilde{a} \otimes I, I \otimes \gamma a) = V(\chi(\tilde{a}, \gamma) \otimes I). \]

Define
\[ W = UV^*. \]

It follows that for all $\gamma \in \Gamma$
\[ (\text{id} \otimes \phi_\gamma)W = W. \quad (15) \]

At this point one expects that
\[ W \in M(\mathcal{K}(H) \otimes C_\infty(\overline{\Gamma})). \quad (16) \]

It is known that $M(\mathcal{K}(H) \otimes C_\infty(\overline{\Gamma})) = C_{\text{bounded}}(\overline{\Gamma}, B(H))$, therefore $W = f(b)$ where $f \in C_{\text{bounded}}(\overline{\Gamma}, B(H))$ and $f(z)$ is unitary for any $z \in \overline{\Gamma}$. Unfortunately in the context of $C^*$-algebra crossed products the invariance condition (15) is not sufficient to support (16). In addition to (15) one needs to know that $\Gamma \ni \gamma \mapsto (I \otimes \lambda_\gamma)W(I \otimes \lambda_\gamma)^*X$ is norm continuous for any $X \in \mathcal{K}(H) \otimes C_\infty(\overline{\Gamma}) \subset M(\mathcal{K}(H) \otimes A)$ (cf. [3, Proposition 7.8.9]). However we have no argument to justify this. On the other hand for $W^*$-dynamical systems the additional condition is not relevant. Then we only have borel measureability of the corresponding function $f$. Nevertheless this fact combined with a functional equation for $f$ (cf. (10)) will imply continuity of $f$. Therefore we are able to justify (16) only a posteriori.

Let us now extend the $C^*$-dynamical system $\left(\mathcal{K}(H) \otimes C_\infty(\overline{\Gamma}), \Gamma, \text{id} \otimes \alpha\right)$ to a $W^*$-dynamical system $\left(B(H) \otimes L^\infty(\overline{\Gamma}), \Gamma, \text{id} \otimes \alpha\right)$. We may assume that $C_\infty(\overline{\Gamma})$ is faithfully represented in a Hilbert space. Then
\[ W \in M(\mathcal{K}(H) \otimes A) \subset B(H) \otimes A'' = (B(H) \otimes L^\infty(\overline{\Gamma})) \rtimes_{\text{id} \otimes \alpha} \Gamma \]
and $W$ is fixed under the action $\gamma \mapsto (\text{id} \otimes \phi_\gamma)$ which clearly is dual to the action $\gamma \mapsto (\text{id} \otimes \alpha_\gamma)$. By [3, Theorem 7.10.4] the element $W$ belongs to the von Neuman algebra

$$B(H) \otimes L^\infty(\Gamma)$$

which means that $W = f(b)$ where $f : \Gamma \to B(H)$ is a unitary operator-valued borel function. Now we shall show that $f$ satisfies the functional equation (10).

Since $U = WV = f(b)\chi(\tilde{a} \otimes I, I \otimes a)$ we have by (8)

$$(\text{id} \otimes \Delta)U = f(a \otimes b + b \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a).$$

(17)

Therefore

$$(\text{id} \otimes \phi_\gamma \otimes \text{id})U_{12}U_{13} = (\text{id} \otimes \varphi_\gamma \otimes \text{id})(\text{id} \otimes \Delta)U$$

$$= f(\gamma b)\chi(\tilde{a} \otimes I, I \otimes \gamma a)$$

$$= f(\gamma b)(\chi(\tilde{a}, \gamma) \otimes I)\chi(\tilde{a} \otimes I, I \otimes a).$$

On the other hand (cf. (14))

$$(\text{id} \otimes \varphi_\gamma \otimes \text{id})U_{12}U_{13} = (\chi(\tilde{a}, \gamma) \otimes I)U$$

and thus

$$U = (\chi(\tilde{a}, \gamma) \otimes I)^*f(\gamma b)(\chi(\tilde{a}, \gamma) \otimes I)\chi(\tilde{a} \otimes I, I \otimes a)$$

$$= \chi(\tilde{a} \otimes I, \gamma I \otimes I)^*f(\gamma b)\chi(\tilde{a} \otimes I, \gamma I \otimes I)\chi(\tilde{a} \otimes I, I \otimes a).$$

(18)

for all $\gamma \in \Gamma$. But $U$ does not depend on $\gamma$, so integrating both sides of (18) over $\gamma$ with respect to the spectral measure of $a$ we obtain

$$U_{13} = \chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)^*f(a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes I \otimes a)$$

$$= \chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)^*f(a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a).$$

Now since

$$U_{12} = f(b \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)$$

we obtain

$$U_{12}U_{13} = f(b \otimes I)f(a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a).$$

(19)

Comparing (19) with (17) we obtain a functional expression

$$f(a \otimes b + b \otimes I) = f(b \otimes I)f(a \otimes b).$$

(20)

Denote $X = b \otimes I$, $Y = a \otimes b$. It is easy to check that $(Y, X)$ is a regular $q^2$-pair. Therefore by (20) and Theorem 3.2 there exists a normal operator $\tilde{b}$ acting on $H$ with $\text{Sp} \tilde{b} \subset \Gamma$ such that

$$f(\gamma) = F_q(\tilde{b}\gamma).$$

Consequently

$$f(b) = F_q(\tilde{b} \otimes b)$$
and
\[ U = F_q(\tilde{b} \otimes b)\chi(\tilde{a} \otimes I, I \otimes a). \] (21)

So far we know that \( \tilde{a} \) and \( \tilde{b} \) are normal operators on \( H \) such that \( \text{Sp} \tilde{a}, \text{Sp} \tilde{b} \subset \Gamma \) and \( \ker \tilde{a} = \{0\} \). To end the proof of the existence part of Theorem 4.2 cf. Definition 2.1) we need to show that
\[ \chi(\tilde{a}, \gamma)\tilde{b}\chi(\tilde{a}, \gamma)^* = \gamma\tilde{b} \] (22)
for all \( \gamma \in \Gamma \).

To that end observe that inserting the information about \( U \) given by (21) into the identity (18) we obtain
\[ (\chi(\tilde{a}, \gamma) \otimes I)F_q(\tilde{b} \otimes b)(\chi(\tilde{a}, \gamma) \otimes I)^* = F_q(\gamma\tilde{b} \otimes b) \]
which by unitarity of \( (\chi(\tilde{a}, \gamma) \otimes I) \) means that
\[ F_q \left( (\chi(\tilde{a}, \gamma)\tilde{b}\chi(\tilde{a}, \gamma)^*) \otimes b \right) = F_q(\gamma\tilde{b} \otimes b). \] (23)

Setting \( T_1 = \chi(\tilde{a}, \gamma)\tilde{b}\chi(\tilde{a}, \gamma)^* \) and \( T_2 = \gamma\tilde{b} \) we can rewrite (23) as
\[ F_q(T_1 \otimes b) = F_q(T_2 \otimes b). \] (24)

For \( z \in \Gamma \) let \( \omega_z \in \text{Mor} \left( C_\infty(\Gamma), \mathbb{C} \right) \) be given by
\[ \omega_z(b) = z. \]

Applying \( (\text{id} \otimes \omega_z) \) to both sides of (21) we get
\[ F_q(zT_1) = F_q(zT_2) \]
and this equality holds for any \( z \in \Gamma \). Now the equality of \( T_1 \) and \( T_2 \) follows from the following result:

**Lemma 4.3** Let \( T_1 \) and \( T_2 \) be normal operators acting on a Hilbert space \( K \) such that \( \text{Sp} T_1, \text{Sp} T_2 \subset \Gamma \). Then
\[ \left( F_q(zT_1) = F_q(zT_2) \right) \iff \left( T_1 = T_2 \right). \]

(We omit the proof since it is analogous to the proof of [1, Lemma 3.5].)

Therefore \( (\tilde{b}, \tilde{a}) \) is a regular \( q^2 \)-pair.

To prove uniqueness of \( (\tilde{b}, \tilde{a}) \) let us apply \( (\text{id} \otimes \varphi_\gamma) \) to \( U \):
\[ (\text{id} \otimes \varphi_\gamma)U = \chi(\tilde{a}, \gamma). \]
It shows that the operator $\tilde{a}$ is determined uniquely. Indeed: we have (cf. (2))
\[
\chi(\gamma, q) = \text{Phase} \gamma,
\]
\[
\chi(\gamma, q^\dagger) = |\gamma|^{|\dagger}
\]
for all $\gamma \in \Gamma$. Therefore, by functional calculus for normal operators,
\[
\text{Phase} \tilde{a} = (\id \otimes \varphi_q)U, \\
|\tilde{a}|^{|\dagger} = (\id \otimes \varphi_{q^\dagger})U
\]
which determines $\tilde{a}$ completely. Now the operator $\tilde{b}$ is also determined uniquely. In fact if
\[
U = F_q(\tilde{b} \otimes b)\chi(\tilde{a} \otimes I, I \otimes a)
\]
then
\[
F_q(\tilde{b}' \otimes b) = F_q(\tilde{b} \otimes b)
\]
and the reasoning presented after (24) shows that $\tilde{b}' = \tilde{b}$. This ends the proof of the first part of our theorem.

For the proof of the second part let $(\tilde{b}, \tilde{a})$ be a regular $q^2$-pair acting on a Hilbert space $H$ and
\[
U = F_q(\tilde{b} \otimes b)\chi(\tilde{a} \otimes I, I \otimes a).
\]
Elements $\tilde{b} \otimes b$, $\tilde{a} \otimes I$ and $I \otimes a$ are affiliated with $\mathcal{K}(H) \otimes A$. Therefore $U \in M(\mathcal{K}(H) \otimes A)$ and $U$ is unitary. Now by (3)
\[
(\id \otimes \Delta)U = F_q(\tilde{b} \otimes a \otimes b + \tilde{b} \otimes b \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a).
\]
Since $(\tilde{b} \otimes b \otimes I, \tilde{b} \otimes a \otimes I)$ is a $q^2$-pair, we have (cf. remark after Proposition 3.1)
\[
F_q(\tilde{b} \otimes a \otimes b + \tilde{b} \otimes b \otimes I) = F_q(\tilde{b} \otimes b \otimes I)F_q(\tilde{b} \otimes a \otimes b).
\]
Moreover by the character property of $\chi$
\[
\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a) = \chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes I \otimes a)
\]
and $\gamma \tilde{b} \chi(\tilde{a}, \gamma) = \chi(\tilde{a}, \gamma) \tilde{b}$ by (22). Therefore
\[
(\tilde{b} \otimes a)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a) = (\chi(\tilde{a} \otimes I, I \otimes a))\tilde{b} \otimes I)
\]
and
\[
(\tilde{b} \otimes a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I) = \chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)\tilde{b} \otimes I \otimes b).
\]
Now
\[
F_q(\tilde{b} \otimes b \otimes I)F_q(\tilde{b} \otimes a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes a)
\]
\[
= F_q(\tilde{b} \otimes b \otimes I)F_q(\tilde{b} \otimes a \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes I \otimes a)
\]
\[
= F_q(\tilde{b} \otimes b \otimes I)\chi(\tilde{a} \otimes I \otimes I, I \otimes a \otimes I)F_q(\tilde{b} \otimes I \otimes b)\chi(\tilde{a} \otimes I \otimes I, I \otimes I \otimes a)
\]
\[
= U_{12}U_{13}.
\]
This combined with (25) and (26) shows that $U$ is a unitary representation of $G$ (cf. Definition 4.1). Q.E.D.
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