Martingale decomposition of a $L^2$ space with nonlinear stochastic integrals

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Abstract

This paper presents a generalization of the Kunita-Watanabe decomposition of a $L^2$ space with nonlinear stochastic integrals where the integrator is a family of continuous martingales bounded in $L^2$. To get the result, a useful relation between the regularity of the martingale family respect to its parameter and the regularity of the integrand in its martingale decomposition is shown. The decomposition presented in the main result is also the solution of an optimization problem in $L^2$. Finally, an example is given where the optimization problem is solved explicitly.

Keywords: Stochastic calculus, optimization, martingales, orthogonal decomposition.

MSC: 60G44

1 Introduction

Nonlinear stochastic integrals, sometimes called stochastic line integrals, are stochastic integrals where the integrator is a family of semimartingales. Let $\{X(x); x \in E \subset \mathbb{R}\}$ be a family of semimartingales indexed by $x$, where $x$ is often called spatial parameter. Then, $\int X(dt, u_t)$ denotes the nonlinear stochastic integral of the predictable process $u \in E$ respect to the family of semimartingale $\{X(x); x \in E\}$.

This stochastic integral has been defined for relatively general families of semimartingales in [5]. Those results can also be found in [6].
and [7]. More recently, [8] extended the integral to a family of semimartingales which is not necessarily continuous respect to the parameter. Detailed construction of nonlinear stochastic integrals can also be found under different assumptions in [9] and [3].

An interesting application of this type of integrals can be found in mathematical finance, more specifically to represent stock prices in the limit order book. In those models, family of semimartingales are used to model stock prices in the limit order book; the space parameter represents the size of a transaction. The nonlinear stochastic integral is then used to defined the value of the trading portfolio in which the cash flow of a transaction is a nonlinear function of the number of shares traded. One can look at [4] and [2] for examples of financial applications.

In this paper, we look at the problem of approximating a square-integrable random variable in $L^2$ using nonlinear stochastic integrals where the integrator is a family of martingales. Assuming some regularity on the family of martingales respect to the parameter, the integrand minimizing the $L^2$-norm of the difference between the random variable and the stochastic integral is characterized. The result of this characterization is a generalization of the Kunita-Watanabe decomposition.

For classical stochastic integrals, the same problem has been solved by [12] in the case where the integrator is a special semimartingale with the additional assumption that the semimartingale admits a canonical decomposition given by

$$X = X_0 + M + \int \alpha d[M],$$

that is the finite variation process is absolutely continuous respect to the quadratic variation process of the semimartingale. The solution to the problem of approximating random variables in $L^2$ using stochastic integrals led to a new risk management strategy known as quadratic hedging. Quadratic hedging is now viewed as one of the main hedging paradigm in financial mathematics and is supported by a rich literature. The interested reader can look at: [13], [11] and [1] for the theoretical basis and numerical studies of quadratic hedging.

To draw a parallel with this paper, the theory of quadratic-hedging relies on the Follmer-Schweizer, [12], which generalizes the Kunita-Watanabe decomposition, [10], to (linear) stochastic integrals respect to semimartingales. On the other hand, the results of this paper generalize the Kunita-Watanabe decomposition to nonlinear stochastic integrals which are used in some financial models. Accordingly, in addition to their strict mathematical interest, the results presented in here are also of interest for future applications in mathematical finance.
The remainder of this paper is organized in the following way. The next two sections establish the conditions of the probability space, introduce some notations and state the optimizing problem to be solved. Section 4 shows that the regularity of a family of martingales and the integrand in its martingale representation are linked. Section 5 contains the main results and a complete example.

2 Probability space and notations

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space where $\mathbb{F}$ is a right-continuous filtration, complete respect to $P$ and where $\mathbb{F} = \lim_{t \to \infty} \mathcal{F}_t$ is the smallest $\sigma-$algebra containing all $\mathcal{F}_t$. The results of this paper require that all adapted processes are continuous, so we assume that for any stopping times $T$ and $\{T_n\}$ such that $T_n \uparrow T$ then $\bigvee_n \mathcal{F}_{T_n} = \mathcal{F}_T$.

First, define

$$\mathcal{M} := \{X : [0, \infty) \times \Omega \to \mathbb{R}; X \text{ is a continuous martingale adapted to } \mathcal{F}\},$$

that is, for $X \in \mathcal{M}$, $t \to X_t$ is continuous a.s., $E[|X_t|] < \infty$ for all $t \geq 0$ and $E[X_t | \mathcal{F}_s] = X_s$ for all $0 \leq s \leq t$. Then, let

$$L^2(P) := \{Y : \Omega \to \mathbb{R}; Y \in \mathcal{F} \text{ and } ||Y||_{L^2} < \infty\}$$

be the space of square-integrable random variables where $||Y||_{L^2} = E[Y^2]$.

In this paper, it is assumed that martingales are continuous and bounded in $L^2(P)$, that is, they belong to the set

$$\mathcal{M}^2 = \{X \in \mathcal{M} : E[\sup_{t \geq 0} (X_t)^2] < \infty\}.$$

However, without loss of generality, many results will be stated for

$$\mathcal{M}_0^2 = \{X \in \mathcal{M}^2 : X_0 = 0\}$$

and the space $L^2_0(P) = \{X \in L^2(P) : E[X] = 0\}$. The general case is recovered by translating processes and random variables. The following convention is also established, since there is an isometry between random variables in $L^2(P)$ (resp. $L^2_0(P)$) and martingales in $\mathcal{M}^2$ (resp. $\mathcal{M}_0^2$), the same notation will be used to define an element of $L^2(P)$ (resp. $L^2_0(P)$) and an element of $\mathcal{M}^2$ (resp. $\mathcal{M}_0^2$). For instance, for $X \in L^2(P)$, $X$ defines the random variable as well as the almost sure limit, $X = \lim_{t \to \infty} X_t$, of the martingale in $\mathcal{M}^2$. Finally, it is assumed that $(\Omega, \mathcal{F})$ is separable, therefore, there exists a countable basis for $L^2(P)$. 

3
3 Optimization problem

In this section, the family of martingales used as an integrator is defined and the optimizing problem is stated.

The family of martingales is defined as

\[ M : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R} \]

\[ (t, x, \omega) \mapsto M(t, x, \omega) \]

such that \( M(x) \) is in \( \mathcal{M}_0^2 \) for each \( x \in \mathbb{R} \). Following the previous convention, \( M(x) = \lim_{t \to \infty} M(t, x) \) a.s.

In the following, the problem that is solved is to approximate a square-integrable random variable \( H \in L_0^2(P) \) with the stochastic integral

\[ \int M(ds, \theta_s) \]

where \( \theta \) is a predictable process in a set \( \mathcal{I}^M \) to be defined. In other words, the goal is to solve

\[ \inf_{\theta \in \mathcal{I}^M} \left\| H - \int M(ds, \theta_s) \right\|_{L^2}. \tag{1} \]

Intuitively, this problem can be seen as minimizing the distance from a point \( H \) to a curve \( \int M(ds, \theta_s) \) in a linear space. To characterize the solution, it is easier, if not necessary, to represent the curve respect to the basis of the linear space and study the properties of its representation. This is the topic of the next section.

4 Regularity conditions

Using the separability assumption of \((\Omega, \mathcal{F})\), there exists \( B \in \mathcal{M}_0^2 \) and a family of predictable processes \( \{\mu(x); x \in \mathbb{R}\} \) such that

\[ M(x) = \int \mu(s, x)dB_s. \]

The existence of \( \mu(x) \) is a simple application of the martingale representation theorem for each \( x \). The next results show that the function \( x \mapsto \mu(s, x) \) inherits some of the regularity properties of the function \( x \mapsto M(x) \). But first, it is shown that \( \mu(x) \) is unique up to an equivalence class of processes. This class of processes is defined by processes which are equal up to a null set respect to the measure \( \beta : \mathcal{B}([0, \infty)) \times \mathcal{F} \to [0, \infty) \) given by \( \beta(A) = \int \int 1_A(s, \omega)d[B]sP(d\omega) \), where \( \mathcal{B}([0, \infty)) \) is the Borel sigma-algebra on \([0, \infty)\) and \( 1_A \) is the indicator function of the set \( A \).

**Theorem 1.** The representation \( M(x) = \int \mu(s, x)dB_s \) is unique except on a set \( A \subset \mathcal{B}([0, \infty)) \times \mathcal{F} \) where \( E[\int_A 1d[B]s] = 0 \).
Proof. Suppose that \( M(x) = \int \mu(s,x)dB_s = \int \nu(s,x)dB_s \).

\[
E \left[ \int (\mu(s,x) - \nu(s,x))^2 d[B]_s \right] = E[(M(x) - M(x))^2] = 0.
\]

Therefore, \( \mu(s,x) = \nu(s,x) \) except maybe on a set \( A \in \mathcal{B}([0,\infty)) \times \mathcal{F} \) such that \( \beta(A) = 0 \).

From the previous theorem, one can say that the integrand in the representation of \( M(x) \) is almost surely unique respect to the sigma finite measure \( \beta \). One gets nonetheless that \( M(x) = \int \mu(s,x)dB_s \) is well defined. For a fixed \( x \in \mathbb{R} \), let \( \hat{\mu}(x) \) be such that \( \mu(t,x,\omega) \neq \hat{\mu}(t,x,\omega) \) on \( A \) where \( \beta(A) = 0 \) and \( \mu(t,x,\omega) = \hat{\mu}(t,x,\omega) \) on \( A^c \). Assume that \( P(\bigcup_{t \geq 0} A(t)) > 0 \). For each \( \omega \), one must have that \( \int_{A(t)} dB_s(\omega) = 0 \), which means that \( A(\omega) \) is a set of times where the martingale \( B \) is constant, hence the value of \( \mu(t,x,\omega) \) for \( t \in A(\omega) \) has no impact on the value \( M(x) \). On the other hand, if \( \int_{A(\omega)} dB_s(\omega) > 0 \), then one must have that \( P(\bigcup_{t \geq 0} A(t)) = 0 \) and \( \hat{\mu}(x) \) is actually a modification of \( \mu(x) \).

To establish our main result, we need the function \( x \mapsto \mu(t,x) \) to be sufficiently smooth. Since we only have its existence, it is not practical to impose conditions on \( \mu(t,x) \). Fortunately, our next results link the continuity of \( x \mapsto \mu(t,x) \) with the continuity of \( x \mapsto M(x) \). In the following theorem, \( C^m \) is the set of \( m \) times continuous differentiable functions.

**Theorem 2.** Let \( \{M(x)\} \) be a family of random variables in \( L_2^0(P) \) such that \( x \mapsto M(x) \) is a.s. in \( C^0 \). Then, for each \( t \geq 0 \), \( x \mapsto M(t,x) \) is a.s. in \( C^0 \) and one can take \( x \mapsto \mu(t,x) \) to be a.s. continuous for all \( t \geq 0 \).

Proof. Given a compact subset \( K \subset \mathbb{R} \), let

\[
A_n = \left\{ \sup_{x,y \in K} |M(x) - M(y)| \leq n \right\}
\]

and define \( M^n(x) = M(x)1_{A_n} \) for all \( x \in \mathbb{R} \). One has that \( M^n(x) \to M(x) \) a.s. and that \( |M^n(x)| \leq |M(x)| \) for all \( n \) and each \( x \in \mathbb{R} \). By the dominated convergence theorem, one has that

\[
E[(M^n(x) - M(x))^2] \to 0.
\]

Now, let \( \{\mu(x)\} \) be such that \( M(x) = \int \mu(s,x)dB_s \). For \( x,y \in K \), one finds that
\[
E \left[ \int (\mu(s, x) - \mu(s, y))^2 dB|s \right] = E \left[ (M(x) - M(y))^2 \right]
\]
\[
\leq E[(M(x) - M^n(x))^2] + E[(M^n(x) - M^n(y))^2] + E[(M^n(y) - M(y))^2].
\]

Using the dominated convergence theorem and the fact that \(M(x)\) is a.s. in \(C^0\) one has that \(\lim_{n \to y} E[(M^n(x) - M^n(y))^2] = 0\) for all \(n\). Finally, from the convergence of \(M^n(x)\) to \(M(x)\) in \(L^2(P)\), for all \(\epsilon > 0\) one can find \(N_\epsilon\) such that \(E[(M(x) - M^n(x))^2] < \epsilon\) and \(E[(M(x) - M^n(y))^2] < \epsilon\) for all \(n > N_\epsilon\).

We see that for a fixed \(\omega \in \Omega\), one must have that \(\mu(x + h, s) \to \mu(x, s)\) except if \(s \in A(\omega) \in B(\mathbb{R})\) where \(\int_{A(\omega)} d[\mathcal{B}]_s(\omega) = 0\). But since \(A\) is the set where the martingale \(B\) is constant, one can choose \(x \mapsto \mu(x, s)\) to be continuous for \(s \in A\). Finally, for \(s \notin A\) we have that \(\mu(s, x + h) \to \mu(s, x)\) a.s.

To establish conditions for the differentiability of \(\mu(x)\) we first define

\[
C^{m, \delta}(K) = \left\{ f : \mathbb{R} \to \mathbb{R} : \left| \frac{\partial^m}{\partial x^m} f(x) - \frac{\partial^m}{\partial x^m} f(y) \right| \leq K|x - y|^{\delta} \right\}
\]

which is the set of \(m\) times continuously differentiable functions with \(\delta\)-Hölder \(m\)-th derivative. In the following theorem, we need \(M(x)\) to be in \(C^{1, \delta}(K)\) but we can allow \(\delta\) and \(K\) to depend on \(\omega\) which gives a more general result than requiring \(M(x)\) to be uniformly in \(C^{1, \delta}(K)\).

**Theorem 3.** Let \(\{M(x)\}\) be a family of \(L^2(P)\) random variables such that \(M(x)\) is a.s. in \(C^{1, \delta}(K)\), where \(K\) and \(\delta\) are positive \(\mathbb{F}\)-measurable and \(K\) is in \(L^2(P)\). Then, \(\{\frac{\partial}{\partial x} M(x)\}\) is a family of \(L^2(P)\) random variables and there exists a family of predictable processes \(\{\mu(x)\}\) such that \(x \mapsto \mu(x)\) is a.s. in \(C^1\),

\[
M(x) = \int \mu(s, x) dB_s \text{ and } \frac{\partial}{\partial x} M(x) = \int \frac{\partial}{\partial x} \mu(s, x) dB_s.
\]

**Proof.** First, for \(h \in \mathbb{R}\) we find that

\[
\left| \frac{M(x + h) - M(x)}{h} - \frac{d}{dx} M(x) \right| = \left| \frac{d}{dx} M(\alpha(x, h)) - \frac{d}{dx} M(x) \right| < K|h|^{\delta}
\]

where \(\alpha(x, h)\) is a random variable in \((x - h, x + h)\). Moreover, if \(|h| < 1\)
we have that $E[K^2|h^2|] < \infty$. Then,

$$\lim_{h \to 0} E \left[ \frac{(M(x+h) - M(x)}{h} - \frac{d}{dx} M(x) \right]^2$$

$$= \lim_{h \to 0} E \left[ \left( \frac{d}{dx} M(x, h) - \frac{d}{dx} M(x) \right)^2 \right]$$

$$\leq \lim_{h \to 0} E[K^2|h^2|] = 0 \quad (2)$$

We then conclude that $\frac{d}{dx}M(x)$ is in $L^2(P)$ since it is a closed space.

Now, from Theorem 2 there exists a family of predictable processes $\{\nu(x)\}$ such that $\frac{d}{dx}M(x) = \int \nu(s, x)dB$. Using the same argument as above, we have that

$$\lim_{h \to 0} E \left[ \frac{(M(x+h) - M(x)}{h} - \frac{d}{dx} M(x) \right]^2$$

$$= \lim_{h \to 0} E \left[ \int \left( \frac{\mu(s, x+h)}{h} - \frac{\mu(s, x)}{h} \right)^2 d[B]_s \right] = 0.$$

We conclude that $x \mapsto \mu(s, x)$ is almost surely continuously differentiable almost everywhere respect to $d[B]_s$. As in Theorem 2 one can take $\mu(s, x)$ to be a.s. continuously differentiable for all $s \geq 0$.

### 5 Main results

The next theorem gives the decomposition of the $L^2(P)$ space as an integral respect to $M(x)$ and a martingale orthogonal to an integral respect to $\frac{d}{dx}M(x)$. As we said before, we state the theorem for $L^2_0(P)$, the case for $L^2(P)$ simply requires a translation of the random variables and the martingales.

For this result we need to define the set of integrals respect to $M(x)$ and $\frac{d}{dx}M(x)$ that are in $L^2(P)$. Let $\mathcal{I}^M$ be defined as

$$\mathcal{I}^M = \left\{ \theta : \theta \text{ is predictable and } \int M(ds, \theta_s), \int \frac{\partial}{\partial x} M(ds, \theta_s) \in L^2(P) \right\}$$

and finally

$$\mathcal{H}^M = \left\{ \int M(ds, \theta_s) : \theta \in \mathcal{I}^M \right\}.$$ 

To state the theorem in its full generality we must assume that $\mathcal{H}^M$ is a closed set. This is a strong assumption and, to the best of my knowledge, there is no simple condition on $M(x)$ such that $\mathcal{H}^M$ is closed in $L^2(P)$. However, assuming the closedeness of $\mathcal{H}^M$ is only a
sufficient condition for the existence of the integrand which solves the optimization problem. For an explicit problem, if the characterization in Theorem gives that the minimizing integrand is in , then the closedness of becomes irrelevant.

Another important question about is whether this set of integrands is rich enough. On that aspect, it is easier to find conditions on \( \{ M(x) \} \). For instance, if \( K \) and \( \delta \) from Theorem are such that 

\[
E[K^2|x|^{2\beta}] < \infty \text{ for all } x \in \mathbb{R},
\]

then the set of uniformly bounded predictable processes is a subset of \( I^M \). Also, if \( K \) and \( \delta \) are independent and the distribution of \( \delta \) is not heavy-tailed, then the set of uniformly bounded predictable processes is again a subset of \( I^M \).

**Theorem 4.** Let \( \{ M(x) \} \) be a family of random variables in \( L^2_0(P) \) where \( x \mapsto M(x) \) is a.s. in \( C^1(K) \) where \( \delta \) and \( K \) are positive and \( \mathcal{F} \)-measurable and where \( K \) is in \( L^2(P) \). Assume that \( \mathcal{H}^M \) is closed in \( L^2_0(P) \), then for each \( H \in L^2_0(P) \) there exists \( \theta^H \in I^M \) and \( L^H \in M^2_0 \) such that 

\[
H = \int M(ds, \theta^H_s) + L^H
\]

where \( L^H = H - \int M(ds, \theta^H_s) \) is orthogonal to \( \int \partial_x M(ds, \theta^H_s) \), i.e.

\[
E \left[ \left( H - \int M(ds, \theta^H_s) \right) \int \frac{\partial}{\partial x} M(ds, \theta^H_s) \right] = 0 \tag{3}
\]

Moreover,

\[
\left\| H - \int M(ds, \theta^H_s) \right\|_{L^2} = \inf_{X \in \mathcal{H}^M} \| H - X \|_{L^2}.
\]

**Proof.** First, take a sequence \( \{ X^n \} \subset \mathcal{H}^M \) such that 

\[
\| H - X^n \|_{L^2} \to \inf_{X \in \mathcal{H}^M} \| H - X \|_{L^2}.
\]

Thanks to the parallelogram law, one has that \( X^n \to X \) in \( L^2(P) \). From the assumption that \( \mathcal{H}^M \) is closed, there exists \( \theta^n, \theta^H \in \mathcal{I}^M \) such that \( \int M(ds, \theta^n) = X^n \) and \( \int M(ds, \theta^H) = X \).

Let \( F(\epsilon) = \| H - \int M(ds, \theta^H + \epsilon) \|_{L^2}^2 \), then we know that \( \frac{d}{d\epsilon} F(\epsilon)|_{\epsilon=0} = 0 \). We now need to show that one gets the same result by differentiating inside the norm. Let \( \{ \mu(x) \} \) be the family of predictable processes given by Theorem and define the sequence of stopping times

\[
\tau_n = \inf \left\{ t > 0; \sup_{\{ h \}, \| h \| < 1} \left| \int_0^t \frac{\partial}{\partial x} M(ds, \theta^H_s + h_s) - \int_0^t \frac{\partial}{\partial x} M(ds, \theta^H_s) \right| \geq n \right\}
\]

Orthogonality is defined respect to \( L^2(P) \) as opposed to strong orthogonality of martingales.
where the supremum is taken over predictable processes \( \{h\} \) with \(|h_s| < 1 \) for all \( s \geq 0 \). For each \( n \),
\[
\lim_{\epsilon \to 0} E \left[ \int_0^{t \wedge \tau_n} \left( \frac{\mu(s, \theta_s^H - \epsilon) - \mu(s, \theta_s^H)}{\epsilon} - \frac{\partial}{\partial x} \mu(s, \theta_s^H) \right)^2 d[B]_s \right]
\]
\[
= \lim_{\epsilon \to 0} E \left[ \int_0^{t \wedge \tau_n} \left( \frac{\partial}{\partial x} \mu(s, \theta_s^H + h_s(\epsilon)) - \frac{\partial}{\partial x} \mu(s, \theta_s^H) \right)^2 d[B]_s \right]
\]
\[
E \left[ \lim_{\epsilon \to 0} \int_0^{t \wedge \tau_n} \left( \frac{\partial}{\partial x} M(ds, \theta_s^H + h_s(\epsilon)) \right)^2 \right] = 0,
\]
where the process \( h(\epsilon) \) is predictable and \( h_s(\epsilon) \in (\theta_s^H - \epsilon, \theta_s^H + \epsilon) \). We see that for each \( n \), \( \int_0^{t \wedge \tau_n} M(ds, \theta_s^H + \epsilon) \) is a.s. differentiable respect to \( \epsilon \) for \(|\epsilon| < 1 \).

Now, from the Kunita-Watanabe decomposition, there exists a predictable process \( h \) and a \( L^2(P) \)-martingale \( \lambda^H \) such that \( H = \int h_s dB_s + \lambda^H \). Since \( \lambda^B \) is strongly orthogonal to \( B \) it is also strongly orthogonal to \( \int M(ds, \theta_s) \) and \( \int \frac{\partial}{\partial x} M(ds, \theta_s) \) for any \( \theta \in \mathcal{I}_M \), therefore we can set \( \lambda^M \equiv 0 \) without loss of generality.

Let \( F_n(t, \epsilon) = \left\| \int_0^{t \wedge \tau_n} (h_s - \mu(ds, \theta_s^H + \epsilon)) dB_s \right\|^2 \). Then, one finds that
\[
\frac{\partial}{\partial \epsilon} F_n(t, \epsilon)|_{\epsilon=0} = (-2)E \left[ \int_0^{t \wedge \tau_n} \left( h_s - \mu(ds, \theta_s^H) \right) \frac{\partial}{\partial x} \mu(s, \theta_s^H) dB_s \right] = 0.
\]

Finally, from the Cauchy-Schwartz inequality and the fact that \( \theta^H \in \mathcal{I}_M \),
\[
E \left[ \left( H - \int M(ds, \theta_s^H) \right) \int \frac{\partial}{\partial x} M(ds, \theta_s^H) \right] \leq \left\| H - \int M(ds, \theta_s^H) \right\|^2 \left\| \int \frac{\partial}{\partial x} M(ds, \theta_s^H) \right\|^2 < \infty.
\]
This inequality allows us to take the limit respect to \( n \) and \( t \) in (4) to show that
\[
E \left[ \left( H - \int M(ds, \theta_s^H) \right) \int \frac{\partial}{\partial x} M(ds, \theta_s^H) \right] = 0
\]
which concludes the proof.

As opposed to the Kunita-Watanabe decomposition, the characterization of \( \theta^H \) in Theorem 3 does not guarantee that a process satisfying Equation (3) is the minimizing integrand. Since the convexity of \( \mathcal{H}^M \)
is not assumed, a process satisfying Equation (3) could give a local minimum. For the same reason, the minimizing integrand for Problem (1) might not be unique. Alike the condition of closedness, it is difficult to find practical conditions to guarantee the convexity of \( \mathcal{H}^M \) and, at the same time, providing the uniqueness of the solution to Problem (1). Once again, for an explicit problem, it is possible to determine if the process satisfying Equation (3) is unique or not and, in some case, to determine whether a minimizer is global or not. Example 5 below shows a situation where the minimizer is not unique but can be identified explicitly without the need of verifying if \( \mathcal{H}^M \) is closed or convex.

In the case where the Kunita-Watanabe decomposition of \( H \) is known, the next corollary shows that the characterization of \( \theta^H \) becomes an almost sure characterization.

**Corollary 1.** Assume the conditions of Theorem 4 are satisfied and that \( \mathcal{I}^B \subset \mathcal{I}^M \) where \( \mathcal{I}^B = \{ \theta : \theta \text{ is predictable and uniformly bounded} \} \). Let \( H \) has the following Kunita-Watanabe decomposition, \( H = \int h_s dB_s + \lambda^H \), then

\[
(h_s - \mu(s, \theta^H_s)) \frac{\partial}{\partial x} \mu(s, \theta^H_s) \equiv 0
\]

for all \( s \geq 0 \).

**Proof.** It is clear that

\[
\mathcal{H}' = \left\{ \int \theta_s \frac{\partial}{\partial x} M(ds, \theta^H_s); \theta \in \mathcal{I}^B \right\} \subset L^2(P).
\]

By using the definition of the orthogonal projection on linear space, one has that

\[
|| \int h_s dB_s - (\int M(ds, \theta^H_s) + \int \alpha_s \frac{\partial}{\partial x} M(ds, \theta^H_s)) ||^2_{L^2} = 0
\]

for all \( \alpha \in \mathcal{I}^B \). This equality is equivalent to

\[
E \left[ \int (h_s - \mu(s, \theta^H_s)) \frac{\partial}{\partial x} \mu(s, \theta^H_s) d[B]_s \right] = 0
\]

for all \( \alpha \in \mathcal{I}^B \). Since, the set \( \{ \int \alpha_s dB_s; \alpha \in \mathcal{I}^B \} \) is dense in

\[
\left\{ X \in L^2(P); X = \int \theta_s dB_s \right\}
\]

respect to the \( L^2(P) \) norm, hence

\[
E \left[ \int (h_s - \mu(s, \theta^H_s)) \frac{\partial}{\partial x} \mu(s, \theta^H_s) dB_s \right] = 0.
\]

The proof is completed by taking the norm,

\[
|| \int (h_s - \mu(s, \theta^H_s)) \frac{\partial}{\partial x} \mu(s, \theta^H_s) dB_s ||^2_{L^2} = E \left[ \int (h_s - \mu(s, \theta^H_s)) \frac{\partial}{\partial x} \mu(s, \theta^H_s) \right]^2 d[B]_s = 0.
\]
Theorem 4 and Corollary 1 give a generalization of the Kunita-Watanabe decomposition. If one set \( M(x) = xB \), then \( \frac{\partial}{\partial x} M(x) = B \) and defining \( L^H = H - \int \theta^H_s dB_s \) one has that
\[
E \left[ L^H \int_0^T \alpha_s B_s \right] = 0
\]
for all bounded predictable processes \( \alpha \). Consequently, with Corollary 1, the processes \( L^H \) and \( \int \frac{\partial}{\partial x} M(ds, \theta^H_s) \) in Theorem 4 are strongly orthogonal.

5.1 Example

The following example shows that for some specific cases, Theorem 4 and Corollary 1 allow to explicitly identify the minimizing integrand of Problem (1).

Let \( F_t = \{W_s^{(1)}, W_s^{(2)} ; 0 \leq s \leq t \} \) where \( W^{(i)}, i = 1, 2 \) are two independent Brownian motions, \( F = F_T \) for some \( T > 0 \) and define \( W_t = \rho W_t^{(1)} + \sqrt{1 - \rho^2} W_t^{(2)} \), \( \forall t \geq 0 \) with \( \rho \in (0, 1) \). Then, define the family of martingales \( \{M(x)\} \subset M^2_0 \) by \( M(t, x) = e^{x W_t - \frac{1}{2} x^2} - 1 \) for \( 0 \leq t \leq T \). Finally, let \( H_T = (W_T^{(1)})^2 - T = 2 \int_0^T W_s^{(1)} dW_s^{(1)} \). In the following we find the predictable process \( \theta_H \) which solves
\[
\inf_{\theta \in \mathcal{L}^M} E \left[ \left( H_T - \int_0^T M(ds, \theta_s) \right)^2 \right].
\] (5)

Using Itô’s formula one finds that \( M(t, x) = \int_0^t M(s, x) x dW_s \) and consequently,
\[
\int_0^t M(ds, \theta_s) = \int_0^t M(s, \theta_s) \theta_s dW_s
\]
for any predictable processes \( \theta \) which is square-integrable respect to \( \{M(x)\} \). Differentiating respect to \( x \) one finds that \( \frac{\partial}{\partial x} M(t, x) = M(t, x)(W_t - x) \) and, using Theorem 3 one has that \( \frac{\partial}{\partial x} M(t, x) = \int_0^t M(s, x)(xW_s - x^2 s + 1) dW_s \), which leads to
\[
\int_0^T \frac{\partial}{\partial x} M(ds, \theta_s) = \int_0^T M(s, \theta_s)(\theta_s W_s - (\theta_s)^2 s + 1) dW_s.
\]

At this point, we find the Kunita-Watanabe decomposition for \( H_T \) respect to \( W \), which can be done by performing a simple orthogonal projection on the linear space \( \{\int_0^T \theta dW_s\} \subset L^2(P) \). One finds that
\( H_T = \int_0^T 2\rho W_s^{(1)} dW_s + \lambda_T^H \). With the Kunita-Watanbe decomposition of \( H_T \) known, we can use Corollary 1 to find that

\[
\left( 2\rho W_s^{(1)} - \theta^H_s M(s, \theta^H_s) \right) \left( \theta^H_s W_s - (\theta^H_s)^2 s + 1 \right) M(s, \theta^H_s) \equiv 0.
\]

Defining \( \theta^H \) by \( \theta^H_s M(s, \theta^H_s) = 2\rho W_s^{(1)} \) for all \( s \) then

\[
E \left[ \left( H_T - \int_0^T M(ds, \theta^H_s) \right)^2 \right] = E[(\lambda_T^H)^2],
\]

which shows that we have found the minimizer. One should note that this solution is not unique in general.

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