Distributed DP-Helmet: Scalable Differentially Private Non-interactive Averaging of Single Layers

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Abstract

In this work, we propose two differentially private, non-interactive, distributed learning algorithms in a framework called Distributed DP-Helmet. Our framework is based on what we coin blind averaging: each user locally learns and noises a model and all users then jointly compute the mean of their models via a secure summation protocol.

We provide experimental evidence that blind averaging for SVMs and single Softmax-layer (Softmax-SLP) can have a strong utility-privacy tradeoff: we reach an accuracy of 86% on CIFAR-10 for $\varepsilon = 0.4$ and 1,000 users, of 44% on CIFAR-100 for $\varepsilon = 1.2$ and 100 users, and of 39% on federated EMNIST for $\varepsilon = 0.4$ and 3,400 users, all after a SimCLR-based pretraining. As an ablation, we study the resilience of our approach to a strongly non-IID setting.

1 Introduction

Non-interactivity in distributed learning, with just one user message sent, offers minimal communication overhead for privacy-preserving learning, which enables high scalability. Yet, Smith et al.\textsuperscript{[13]} have shown that non-interactive learning has limitations: for any non-interactive learning method, there are problems for which the excess risk increases exponentially with the number of model parameters. By contrast, for a class of convex ERM-based problems, Jayaraman et al.\textsuperscript{[28]} have explored a technique which we coin blind averaging, in which each user first trains locally and perturbs the trained model parameters once and then all users average these models non-interactive. Yet, their convergence bounds prove to be only as good as the local models. Their results give rise to the following open question that we tackle: Can such blind averaging achieve utility-privacy tradeoffs comparable to the central-ized setting, where one user accesses all training data?

As a first step, Jayaraman et al.\textsuperscript{[28]} have shown bounded output sensitivity as a sufficient condition to achieve the same differential privacy (DP) bounds as in a centralized setting: a bounded output sensitivity ensures that exchanging one data point before training has only bounded influence on the resulting model parameters. In contrast to sensitivity bounds only for global optima\textsuperscript{[14, 28]}, the work by Wu et al.\textsuperscript{[49]} accounts for leakage of the optimization algorithm and provides bounds for each training iteration. They have shown that for a variant of stochastic gradient descent (SGD) to have a $s$-bounded output sensitivity it suffices that the objective function has bounded derivatives: it is $\lambda$-strongly convex, $L$-Lipschitz, and $\beta$-smooth, where $s \in O(\sqrt{n})$ for $n$ many data points.

Prior output sensitivity bounds existed only for binary classifiers, e.g. support vector machines (SVM)\textsuperscript{[14]}. For $K$-class classifiers, a one-vs-rest scheme can be used but decreases the privacy bound $\varepsilon$ by a factor of $O(\sqrt{K})$ as we train each of the $K$ SVMs separately.

Contribution. We present Distributed DP-Helmet\textsuperscript{[1]} which realizes non-interactive training for ERMs like SVMs and Softmax-SLP in a scalable manner with a strong utility-privacy tradeoff.

Our framework works in two phases (cf. Fig. 1). In Phase I (Local ERM Learning), each user applies a learning algorithm $T$, e.g. SVM or Softmax-SLP, on its local data and then adds Gaussian noise to the model parameters. In Phase II (Blind Averaging), we compute the average over all noised local parameters via secure summation, a secure computation protocol that outputs the sum of all parameters without leaking partial values. To avoid expensive distributed noise generation protocols, each user locally adds noise which in sum suffices for DP if at least a fraction $t$ (e.g. 50\%) of users are honest. The process of averaging is “blind” in the sense that no further synchronization is needed and it applies equally to any learning method with an output sensitivity bound.

(1) Phase I: Output privacy of Softmax-layer learning does not depend on the number of classes $K$.

We prove the first output sensitivity bound for a truly multi-class Softmax-SLP that is often used for last-layer fine-tuning and show that blind averaging Softmax-SLP works. To get a DP Softmax-SLP, we show that the objective function is smooth, Lipschitz, and strongly convex which suffices for DP\textsuperscript{[19]}. For $K$-class tasks, our proof shows that Softmax-SLP does not suffer the factor...
Distributed DP-Helmet is compatible with heterogeneous data: we evaluate a strongly non-IID scenario where each user solely holds data from one class and the federated EMNIST dataset with real-world users. The accuracy of the SVM variant on CIFAR-10 only deteriorates by 2 percentage points when using strongly non-IID data (cf. Tbl. 2) whereas for federated EMNIST with differing local data sizes, the gap becomes larger (cf. Fig. 6(d)). Moreover, our framework is scalable to millions of users if a highly scalable secure summation protocol is used, e.g. Bell et al. [8]. We extrapolate compelling utility-privacy results for millions of users in a local-DP-like scenario. For 20 million users and $\varepsilon = 10^{-4}$, we reach 87% accuracy (cf. Fig. 7).
Table 1: Comparison to related work for $w$ users with $n$ data points each and $M$ training iterations: utility guarantee, DP noise scale, and number of secure summation (SecSum) invocations. ‘Utility: CC’ denotes whether the local models converge to the centralized setting before noise, i.e., whether blind averaging works. The convergence rate of Jayaraman et al. [25] depends on the dataset size $nw$ whereas we truly converge with the number of iterations (cf. Thm. 6.2). (∗) denotes experimental evidence without formal proof. ∗: In DP-FL, an untrusted aggregator combines the differentially private updates (users add noise and norm-clip those); it does not invoke SecSum but needs a communication round per training iteration.

| SVM Algorithms                       | Utility: CC | Noise | SecSum invocations |
|--------------------------------------|-------------|-------|--------------------|
| Jayaraman et al. [25], gradient perturbation | ✓           | $O(\sqrt{nw})$ | $O(\log(nw))$ |
| Jayaraman et al. [25], output perturbation | (∗)         | $O(1/nw)$   | 1                   |
| **Distributed DP-Helmet: SVM (ours)** | ✓           | $O(1/nw)$   | Thm. 6.2            |
| **Softmax-SLP Algorithms**          |             |       |                    |
| DP federated learning (DP-FL)        | ✓           | $O(\sqrt{nw})$ | *                 |
| **Distributed DP-Helmet: Softmax-SLP (ours)** | (∗)         | $O(1/nw)$   | Thm. 4.1            |
| Baseline: Centralized training       | ✓           | $O(1/nw)$   | 0                   |

2 Related Work

Here and in [Tbl. 1] we discuss the most related work and continue the discussion in Sec. 3.

For interactive ERM via gradient perturbation, Jayaraman et al. [25] show strong utility-privacy tradeoffs. It requires $O(\log(nw))$ invocations of secure summation (SecSum) for $w$ users with $n$ data points each. For non-interactive learning, however, Jayaraman et al. [25] show output perturbation results using results from Chaudhuri et al. [11] for which they only show that the convergence bounds from the local training are preserved, thus leaving a gap of $1/w$ to centralized learning. We prove that SVM learning converges in the limit, closing the $1/w$ gap, and we experimentally show that Softmax-SLP leads to strong results. Moreover, their work does not exclude leakage from the learning algorithm (only from the optimum), while we use bounds on the learning algorithm SGD [19].

In differentially private federated learning (DP-FL) [37, 17], each user protects its local data by only submitting noised local model updates protected via DP guarantees (DP-SGD). A central server then aggregates all incoming updates. However, DP-FL is interactive as its communication rounds increase with the number of training iterations. Moreover, DP-FL achieves a weaker utility-privacy tradeoff than centralized learning, already for hundreds of users. DP-FL’s noise scales with $O(\sqrt{w})$ and has a prohibitively high number of communication rounds.

Other Privacy-preserving Distributed Learning Protocols. The noise overhead of DP-FL can be completely avoided by protocols that rely on cryptographic methods to hide intermediary training updates from a central aggregator. Several secure distributed learning methods protect the contributions during training but do not come with privacy guarantees for the model such as DP: an attacker, e.g., a curious training party, can potentially extract information about the training data from the model. As we focus on differentially private distributed learning methods, we will neglect those methods.

cpSGD [2] is a protocol that utilizes secure multi-party computation (MPC) methods to honestly generate noise and compute DP-SGD. While cpSGD provides the full flexibility of SGD, it does not scale to millions of users as it relies on expensive MPC methods. Trueq et al. [26] relies on a combination of MPC and DP methods which also does not scale to millions of users.

Another line of research aims for the stronger privacy goal of protecting a user’s entire input, called local DP, during distributed learning [3, 24]. Due to the strong privacy goal, federated learning with local DP tends to achieve weaker accuracy. With [Cor. 5.4] evaluated in Fig. 7, we show how Distributed DP-Helmet achieves a comparable guarantee via group privacy: given enough users, any user can protect their entire dataset at once while we still reach good accuracy.

For DP SVM training, other methods besides output perturbation [13, 19] such as objective perturbation [13, 31, 27, 17] and gradient perturbation [34, 13, 22, 17, 23] exist. However, when performed under MPC-based distributed training, both methods would require a significantly higher number of MPC invocations which has a large communication overhead and is thus highly interactive. [Appx. B] discusses those approaches in detail.

Trustworthy Distributed Noise Generation. One core requirement of MPC-based distributed learning is honestly generated and unbreakable noise, otherwise our privacy guarantees would not hold anymore. There is a rich body of work on distributed noise generation [10, 19, 30, 31, 29]. So far, however, no distributed noise generation protocol scales to millions of users. Thus, we use a simple, yet effective technique: we add enough noise if at least a fraction of them (say $t = 50\%$) are not colluding to violate privacy by sharing the noise they generate with each other.

3 Preliminaries

3.1 Differential Privacy

Intuitively, differential privacy (DP) [20] quantifies the protection of any individual’s data within a dataset against an arbitrarily strong attacker observing the output of a computation on the said dataset. Strong protection is achieved by bounding the influence of each individual’s data on the resulting SVMs or Softmax-SLPs.

To ease our analysis, we consider a randomized mechanism $M$ to be a function translating a database to a random variable over possible outputs. Running the mechanism then is reduced to sampling from the random variable. With that in mind, the standard definition of differential privacy looks as follows.

Definition 3.1 ($\varepsilon, \delta$ relation). Let $Obs$ be a set of observations, and $RV(Obs)$ be the set of random variables
over \( \mathcal{O}_s \) and \( \mathcal{D} \) be the set of all databases. A randomized algorithm \( M : \mathcal{D} \rightarrow \text{RV}(\mathcal{O}_s) \) for a pair of datasets \( \mathcal{D}, \mathcal{D}' \), we write \( M(D) \approx_{\varepsilon, \delta} M(D') \) if for all tests \( S \subseteq \mathcal{O}_s \) we have \( \Pr[M(D) \in S] \leq \exp(-\epsilon) \Pr[M(D') \in S] + \delta \).

**Definition 3.2 (Differential Privacy).** Let \( \mathcal{O}_s \) be a set of observations, and \( \text{RV}(\mathcal{O}_s) \) be the set of random variables over \( \mathcal{O}_s \), and \( \mathcal{D} \) be the set of all databases. A randomized algorithm \( M : \mathcal{D} \rightarrow \text{RV}(\mathcal{O}_s) \) for all pairs of databases \( \mathcal{D}, \mathcal{D}' \) that differ in at most 1 element is a \((\varepsilon, \delta)\)-DP mechanism if we have \( M(D) \approx_{\varepsilon, \delta} M(D') \).

In our proofs, we utilize a randomized variant of the sensitivity as proposed by Wu et al. [39] to achieve DP ERMs via SGD Training.

**Definition 3.3 (Randomized Sensitivity).** Let \( q : (D, r) \rightarrow \mathbb{R} \) be a randomized function on dataset \( D \) and randomness \( r \). The sensitivity of \( q \) is defined as \( s = \max_{D, D'} \max_{\varepsilon} \|q(D, r) - q(D', r)\| \), where \( D \approx_{\varepsilon} D' \) denotes that the datasets \( D \) and \( D' \) differ in at most one element. We say that \( q \) is an \( s \)-sensitivity-bounded function.

In the context of machine learning, the randomized algorithm represents the training procedure of a predictor. Our distinguishing element is one data record of the database.

**Computational Differential Privacy.** Note that because of the secure summation, we technically require the computational version of differential privacy [39], where the differential privacy guarantees are defined against computationally bounded attackers; the resulting increase in \( \delta \) is negligible and arguments about computationally bounded attackers are omitted to simplify readability.

**Definition 3.4 (Computational \( \approx_{\varepsilon, \delta} \) Differential Privacy).** Let \( \mathcal{D} \) be the set of all databases and \( \eta \) a security parameter. A randomized algorithm \( M : \mathcal{D} \rightarrow \text{RV}(\mathcal{O}_s) \) for a pair of datasets \( \mathcal{D}, \mathcal{D}' \), we write \( M(D) \approx_{\varepsilon, \delta, \eta} M(D') \) if for any polynomial-time probabilistic attacker \( \Pr[|A(M(D))| = 0] \leq \exp(-\epsilon) \Pr[A(M(D')) = 1] + \delta(\eta) \).

For all pairs of databases \( \mathcal{D}, \mathcal{D}' \) that differ in at most 1 element \( M \) is a computational \((\varepsilon, \delta(\eta))\)-DP mechanism if we have \( M(D) \approx_{\varepsilon, \delta, \eta} M(D') \).

### 3.2 Secure Summation

Hiding intermediary local training results as well as ensuring their integrity is provided by an instance of secure multi-party computation (MPC) called secure summation [11, 8]. It is targeted to comply with distributed summations across a huge number of parties. In fact, Bell et al. [8] has a computational complexity for \( w \) users on an \( l \)-sized input of \( O(\log^2 w + l \log w) \) for the client and \( O(w(\log^2 w + l \log w)) \) for the server as well as a communication complexity of \( O(\log^2 w + l) \) for the client and \( O(w(\log w + l)) \) for the server thus enabling an efficient run-through of roughly \( 10^6 \) users without biasing towards computationally equipped users. Additionally, it offers resilience against client dropouts and colluding adversaries, both of which are substantial features for our distributed setting.

**Definition 3.5 (Secure Summation).** Let \( F(s_1, \ldots, s_n) := \sum_{i=1}^{n} s_i \). We say that \( \pi_{\text{SecSum}} \) is secure summation if there is a probabilistic polynomial-time simulator \( \text{Sim}_F \) such that if a fraction of clients is corrupted \( C \subseteq \{U^{(1)}, \ldots, U^{(w)}\} \), $|C| = \gamma w \), \( \text{Real}_{\text{SecSum}}(s_1, \ldots, s_w) \) is statistically indistinguishable from \( \text{Sim}_F(C, F(s_1, \ldots, s_w)) \), i.e., for an unbounded attacker \( A \) there is a negligible function \( \nu \) such that

\[
\text{Advantage}(A) = |\Pr[(A, \text{Real}_{\text{SecSum}}(s_1, \ldots, s_w)) = 1] - \Pr[(A, \text{Sim}_F(C, F(s_1, \ldots, s_w))) = 1]| \leq \nu(\eta).
\]

Here, \( \text{Sim}_F \) is a potentially interactive simulator that only has access to the sum of all elements and the (sub)-set of corrupted clients. The adversary is unable to distinguish interactions and outputs of the simulator from those of the real protocol. For a detailed definition of the network execution \( \text{Real}_{\omega} \) using the notion of interactive machines we refer to [Appx. A.1].

The following theorem is proven for global network attackers that are passive and statically compromised parties. Formally, the theorem holds for all attackers \( \mathcal{A}' \), \( \mathcal{A}'' \) of the following form. \( \mathcal{A}' \) internally runs \( \mathcal{A}'' \) and ensures that only static compromise is possible and that the attacker remains passive.

**Theorem 3.6 (Secure Aggregation \( \pi_{\text{SecAgg}} \).** Let \( s_1, \ldots, s_n \) be the \( d \)-dimensional inputs of the clients \( U^{(1)}, \ldots, U^{(w)} \). Let \( F \) be the ideal secure summation function: \( F(s_1, \ldots, s_n) := \sum_{i=1}^{n} s_i \). If secure authentication encryption schemes and authenticated key agreement protocol exist, the fraction of dropouts (i.e., clients that abort the protocol) is at most \( \rho \in [0, 1] \), at most \( \gamma \in [0, 1] \) fraction of clients is corrupted \( C \subseteq \{U^{(1)}, \ldots, U^{(w)}\}, |C| = \gamma w \), and the aggregator is honest-but-curious, then there is a secure summation protocol \( \pi_{\text{SecAgg}} \) for a central aggregator and \( w \) users that securely emulates \( F \) as in [Def. 3.3].

### 3.3 Pretraining to boost DP Performance

Recent work [15, 16] has shown that strong feature extractors (such as SimCLR [15, 16]) trained in an unsupervised manner, can be combined with simple learners to achieve strong utility-privacy tradeoffs for high-dimensional data sources like images. As a variation to transfer learning, it delineates a two-step process (cf. Fig. 3) where a simplified representation of the high-dimensional data is learned first before a tight privacy algorithm like DP-SVM-SGD or DP-Softmax-SLP-SGD conducts the prediction process on these simplified representations. For that, two data sources are compulsory: a public data source used for a framework that learns a pertinent simplified representation and our sensitive data source that conducts the prediction process in a differentially private manner. Thereby, the sensitive dataset is protected while strong expressiveness is assured through the feature reduction network. Note that a homogeneous data distribution of the public and the sensitive data is not necessarily required.

Recent work has shown that for several applications such representation reduction frameworks can be found, such as SimCLR for pictures, FaceNet for face images, UNet for segmentation, or GPT for language data. Without loss of generality, we focus in this work on the unsupervised SimCLR feature reduction network [15, 16]. SimCLR uses contrastive loss and image transformations to align the embeddings of similar images while keeping those of dissimilar images separate [15]. It is based upon a self-supervised training scheme called contrastive loss where no
labeled data is required. Labelless data is especially useful as it exhibits possibilities to include large-scale datasets that would otherwise be unattainable due to the labeling efforts needed.

### 3.4 Dual SVM representation

With the representer theorem, we can completely describe a learned algorithm \( T \) such that there exists a solution that belongs to span\( \{ (x_j, y_j) \}_{j=1}^n \subseteq \mathcal{H} \times \mathcal{Y} \) on a Hilbert space \( \mathcal{H} \) with dim \( \mathcal{H} \) \( \geq 2 \) and label space \( \mathcal{Y} \), and a locally trained model of a learning algorithm \( T \) such that for all \( (x_j, y_j) \in \mathcal{D} \): the parameters \( f, h \) satisfy the definitions after convergence: 

\[
E((f(x_j), y_j)_{j=1}^n) = \frac{1}{n} \sum_{j=1}^n \alpha_i x_j \text{ for some } \alpha_i \in \mathbb{R} \text{ if and only if } \forall j \in \Omega: \mathcal{H} \ni \mathbb{R} \text{ as a non-decreasing function.}
\]

In the case of SVM-SGD and Softmax-SLP-SGD, we have \( \Omega = \| f \|^2 \) which fulfills the requirements of the representer theorem since \( G(z) = z \) is a linear function and the learning algorithm \( T \) follows the definitions after convergence: 

\[
E((f(x_j), y_j)_{j=1}^n) = \frac{1}{n} \sum_{j=1}^n \alpha_i x_j \text{ for some } \alpha_i \in \mathbb{R} \text{ if and only if } \forall j \in \Omega: \mathbb{R} \ni \mathbb{R} \text{ as a non-decreasing function.}
\]

### 3.5 Configuration

**Definition 3.8 (Configuration \( \zeta \)).** A configuration \( \zeta(\mathcal{U}, t, T, \sigma, \xi, \sigma, \mathcal{L}, \mathcal{K}, \mathcal{D}) \) consists of a set of users \( \mathcal{U} \) of which \( t \mathcal{U} \) are honest, an \( s \)-sensitivity-bounded learning algorithm \( T \) on an input distribution \( D^{(i)} \), hyperparameters \( \xi \), a local dataset \( \mathcal{D}(i) \) of user \( i \) with \( n(i) = |\mathcal{D}(i)| \), \( w = |\mathcal{U}| \), and \( \mathcal{D}(i) \), \( n(i) = |\mathcal{D}(i)| \), a number of classes \( K \), and a noise scale \( \sigma^2 \in \mathbb{R}_+ \). The aggregation of \( \mathcal{D} \) is trained with \( T\): 

\[
\text{avg}(T(D^{(i)})) = \frac{1}{p} \sum_{i=1}^p T(D^{(i)}, \xi, K). \quad I \text{ denotes the identity matrix, } K_{\text{comp}}, \text{ the number of compositions depending on } T, \text{ and } p \text{ the dimensionality of each data point in } D^{(i)}. \text{ If unique, we simply write } \zeta.
\]

### 3.6 DP ERM\s via SGD Training

We consider machine learning algorithms that have a strongly convex objective function, like Support Vector Machines (SVMs), logistic regression (LR), or a softmax-activated single-layer perceptron (Softmax-SLP). These algorithms display a unique local minimum and a lower bound on the objective function’s growth, making them ideal for computing tight DP bounds. If we train these algorithms with stochastic gradient descent (SGD) for a bounded learning rate and their objective function is smooth and Lipschitz continuous, then they are uniformly stable [28] which allows a DP bound on the learning algorithm [19]. For that, the sensitivity of the model is bounded in any iteration and does not scale with the number of iterations if we choose a linearly decreasing learning rate schedule. We require a bounded input space which we ensure via norm-clipping as well as an \( R \)-bounded model parameter space which we ensure with projected SGD. The parameters \( c \) and \( R \) influence the Lipschitzness \( L \) and thus the sensitivity for most objective functions.

**Definition 3.9 (Strong convexity).**  For all parameters \( f \) and \( f' \), function \( q \) is \( \lambda \)-strongly convex if 

\[
\sup_{f, f'} q(f) \leq q(f') + \nabla q(f')^T(f - f') + \frac{\lambda}{2} \| f - f' \|^2.
\]

**Definition 3.10 (Lipschitzness).**  For all parameters \( f \) and \( f' \), function \( q \) is \( L \)-Lipschitz continuous if 

\[
\sup_{f, f'} \frac{\| q(f) - q(f') \|}{\| f - f' \|} \leq L.
\]

**Definition 3.11 (Smoothness).**  For all parameters \( f \) and \( f' \), function \( q \) is \( \beta \)-smooth if 

\[
\sup_{f, f'} \frac{\| q(f) - q(f') \|}{\| f - f' \|} \leq \beta.
\]

**Theorem 3.12 (Wu et al. [19] in Lemma 8).** Let \( \zeta \) be a configuration as in Def. 3.8 \( \mathcal{F} \) model parameters, \( c \) an input clustering bound, s.t. \( \| x_i \| \leq c, \forall i, \) and \( R \) a model clipping bound, s.t. \( \| f \| \leq R. \) If a learning algorithm \( T \) is trained with projected SGD on learning rate \( \text{alpha}_m = \min(1/\beta, 1/\lambda_m) \) for iteration \( m \) and has a \( \lambda \)-strongly convex, \( \beta \)-smooth, and \( L \)-Lipschitz objective function \( f \), then the output model \( T(D, \xi) \) has the sensitivity bound \( s = 2L/\lambda_m. \)

Adding Gaussian noise to the result of a sensitivity-bounded function archive DP if the noise is calibrated to the sensitivity. We assume that for each class, one model is trained and noisy. For a binary-class model, set \( K = 1. \)

**Definition 3.13 (Delta-Gauss).**  Given noise scale \( \delta \in \mathbb{R}_+ \), complementary error function erfc, \( \sigma_p := 1/\alpha, \) and \( \mu_p := \sigma_p^2/2, \) \( \delta(\epsilon, K_{\text{comp}}) = 0.5 \cdot \text{erfc}(\frac{\epsilon - \delta(K_{\text{comp}})}{\sqrt{2K_{\text{comp}} \sigma_p}}) \).

**Lemma 3.14 (Gauss mechanism is DP, Theorem 5 in Sommer et al. [43] and Lem. 1.1).** Let \( q_{\mathcal{D}} \) be a sensitivity-bounded functions on dataset \( D \) and \( N \) zero-centered multivariate Gaussian noise. The Gauss mechanism \( D \rightarrow \{ q_{\mathcal{D}}(D) + N(0, \sigma^2 I) \} \) is tightly \( (\epsilon, \delta) \)-DP with \( \delta(\epsilon, K_{\text{comp}}) \) as in Def. 3.13.
Algorithm 1 Our Softmax-SLP-SGD(D, ξ, K) with hyperparameters ξ := (c, Λ, R, M)

Input: dataset D := \{(x_j, y_j)\}_{j=1}^n where x_j is structured as [1, x_{j1}, \ldots, x_{jp}] ; #classes K; input clipping bound: c ∈ R_+; #iterations M; regularization parameter: Λ ∈ R_+; model clipping bound: R ∈ R_+

Result: a model with hyperplane ∈ R^p×K and intercept ∈ R^K; fSR ∈ R^{(p+1)×K}

clipped(x) := c · x/\max(c, ||x||)

J_{\text{loss}}(f, D) = \sum_{(x, y) ∈ D} \frac{1}{n} \sum_{(z, y) ∈ \tilde{D}} \exp(f^T z) - \sum_{k=1}^K y_k \log \frac{\exp(f^T y_k)}{\sum_{j=1}^n \exp(f^T y_j)}

for m to 1, . . . , M do

\[ f_m = \text{SGD}(J_{\text{loss}}(f_{m-1}, D, k), f_{m-1}, c_m), \] with learning rate α_m := \min(\frac{1}{2}, \frac{1}{\Lambda m})

and \[ \beta = \sqrt{(d+1)K\Lambda^2 + 0.5(\Lambda + c^2)^2} \]

f_m := R · f_m/\|f_m\| \quad ▷ projected SGD

Algorithm 2 [Example 3.16] DP ERM with SVM-SGD(D, ξ, K) and hyperparameters ξ := (h, c, Λ, R, M)

Input: dataset D := \{(x_j, y_j)\}_{j=1}^n where x_j is structured as [1, x_{j1}, \ldots, x_{jp}] ; #classes K; input clipping bound: c ∈ R_+; #iterations M; Huber loss relaxation h ∈ R_+; input clipping bound: c ∈ R_+; model clipping bound: R ∈ R_+

Result: models (hyperplanes ∈ R^p with intercepts ∈ R): \{f_m^{(k)}\} \in R^{(p+1)×K}

clipped(x) := c · x/\max(c, ||x||)

J(f, D, k, α_m), with learning rate α_m := \min(\frac{1}{2}, \frac{1}{\Lambda m})

and \[ \beta = \sqrt{(\sqrt{\Lambda\Lambda} + \Lambda m) + p\Lambda^2} \]

for k to 1, . . . , K do

f_m^{(k)} := \text{SGD}(J(f_{m-1}^{(k)}, D, k), f_{m-1}, c_m), \] with learning rate α_m := \min(\frac{1}{2}, \frac{1}{\Lambda m})

and \[ \beta = \sqrt{(\sqrt{\Lambda\Lambda} + \Lambda m) + p\Lambda^2} \]

\[ f_m^{(k)} := R · f_m^{(k)}/\|f_m^{(k)}\| \quad ▷ projected SGD

Corollary 3.15 (Gauss mechanism on T is DP). For configuration ζ as in Def. 3.8 and learning algorithm T of Thm. 3.12 with an output model in R^{p×K}, D → T(D, ξ, K) + N(0, \sqrt{\lambda}I_{p×K}) is tightly (ε, δ)-DP with \( \delta \leq \frac{K}{2^\lambda} \) as in Def. 3.15. \( \text{Kcomp} := \{T_k(D, ξ)\}_{k=1}^K \) and \( \text{Kcomp} := 1 \) on the all-classes-included \[ \text{comp} \] (Softmax-SLP, i.e. T(D, ξ) = T(D, ξ).

In essence, \( \varepsilon \in \mathcal{O}(\sqrt{\text{Kcomp}}) \). Balle et al. 4 have shown a similar tight composition result. By Thm. 1.12 in Appx. 1 we can apply the Gauss mechanism for a deterministic sensitivity (cf. Lem. 3.14) to learning algorithm T of Thm. 3.12 that has a randomized sensitivity.

Example 3.16 (SVM-SGD). Alg. 2 describes the training of an SVM, T = SVM-SGD, in the one-vs-rest (OVR) scheme where we train an SVM for each class against all other classes. The objective function is \( L \)-strongly convex, (\( L \)-C and \( L \text{-Lipschitz} \), and \( (\sqrt{\lambda\lambda} + \lambda^2 + \lambda^2)^{1/2} \)-smooth if we use a Huber loss (cf. Appx. A.3) which is a smoothed hingeloss. Thus, T has a per-class sensitivity of \( s = 2(\sqrt{\lambda} + 1)/\sqrt{\lambda} \).

Example 3.17 (LR-SGD). For a \( L \)-regularized logistic regression T = LR-SGD, we adapt Alg. 2 with the objective function \( J'(f, D) = \frac{1}{2}f^T f - \frac{1}{n} \sum_{(x, y) ∈ D} \ln(1 + \exp(-f^T clipped(x) · y)) \) and \( \beta = ((\sqrt{\lambda\lambda} + \lambda)^{1/2} \)

4 Phase I: Differentially Private Softmax-SLP

For Phase I (cf. Fig. 1), we show that Softmax-layer learning satisfies an output sensitivity bound. The differentially private variant of the Huber-loss SVM (SVM-SGD) 19 in Alg. 2 falls short for multi-class classification. We address this shortcoming by showing differential privacy for a softmax-activated single-layer perceptron (Softmax-SLP-SGD) in Alg. 1.

It has (1) a utility and (2) a privacy boost over SVM-SGD: (1) Since Softmax-SLP-SGD uses the softmax loss, we also optimize the selection of the most dominant class. In contrast, SVM-SGD uses the one-vs-rest (OVR) scheme where each of the K classes is trained against each other, resulting in K many SVMs. Then, the most dominant class is selected by the argmax of the prediction of all SVMs which is static and thus not optimized. (2) The privacy guarantee \( \varepsilon_{\text{softmax}} \) of Softmax-SLP-SGD does not scale with the number of classes K if we keep the hypersurface of trainable parameters \( R \) constant. In contrast, \( \varepsilon_{\text{svm}} \) of SVM-SGD scales with \( \sqrt{K} \) in the same scenario since it does require a K-fold sequential composition. Overall, when scaling with the number of trainable parameters, Softmax-SLP-SGD saves a privacy budget of \( \varepsilon_{\text{softmax}} \propto \frac{\text{AR} + \sqrt{\varepsilon}}{\text{AR} + \sqrt{\varepsilon}} \) with model clipping bound \( R \), input clipping bound \( c \), and regularization parameter \( \Lambda \).

We show differential privacy for Softmax-SLP-SGD by showing that its objective function is \( L \)-strongly convex (cf. Thm. 3.15), \( (\sqrt{d+1}K\Lambda^2 + 0.5(\Lambda + c^2)^2)^{1/2} \)-smooth (cf. Thm. 4.1), and \( (L = AR + 2\varepsilon) \)-Lipschitz (cf. Thm. 3.16). Then, a bounded sensitivity directly follows from Wu et al. 19 Lemma 8).

Theorem 4.1 (Softmax-SLP-SGD sensitivity). For configuration ζ as in Def. 3.8, the learning algorithm T = Softmax-SLP-SGD of Alg. 1 has a sensitivity bound of \( s = 2(\sqrt{\lambda} + 1)/\sqrt{\lambda} \) for the output model.

Corollary 4.2 (Gauss mechanism on Softmax-SLP-SGD is DP). For configuration ζ as in Def. 3.8 and s-sensitivity-bounded T = Softmax-SLP-SGD with output in R^{(p+1)×K} (cf. Thm. 4.1), DP-Softmax-SLP-SGD(D, ξ, K, θ) := Softmax-SLP-SGD(D, ξ, K) + N(0, \sqrt{\lambda}I_{p×K}) is tightly (ε, δ)-DP with \( \delta \leq \frac{K}{2^\lambda} \) as in Def. 3.12. Cor. 4.1 directly follows from a bounded sensitivity (cf. Thm. 4.1) and the Gauss mechanism (cf. Cor. 3.15). Note that, \( \varepsilon \) is in \( \mathcal{O}(s) \) as bounds the sensitivity of all classes thus we do not need to compose any per-class models sequentially.

Lipschitzness. The sensitivity of Softmax-SLP-SGD and thus the privacy budget \( \varepsilon \) is directly proportional to its
Lipschitzness $\Lambda = \Lambda R + \sqrt{2c}$ which we prove in Appx. N to be independent of the number of classes for a fixed $R ([f] \leq R \forall j)$. Our proof bounds the Jacobian of the objective function, i.e. $\sup_{j \in D_f} \| \nabla_j \mathcal{J}_{\text{Softmax}}(f, z) \| \leq \Lambda$.

The first part of the Lipschitz bound $\Lambda R^2$, originates from the 12-regularization term of the objective function $\lambda/2 \| f \|^2$ which is influenced by the size of model $f$ whereas the second part, $\sqrt{2c}$, originates from the softmax loss function for which we use the characteristic of the softmax function that the probabilities of each class add up to 1. In particular, we use the fact that for the $K$ softmax probabilities $s_1, \ldots, s_K$ and class label $y$: $\max_{s_1, \ldots, s_K} (\sum_{k=1}^{K} (s_k - 1/s_k)^2) \leq 2$. The bound $\sqrt{2c}$ directly corresponds to the $\sqrt{2c}$ of the Lipschitz bound $L$.

Smoothness. The linearly-decreasing learning rate schedule of Softmax-MLP-SGD allows a sensitivity independent of the number of training iterations, yet the rate is upper bounded by $1/\beta$. We prove in Appx. O smoothness $\beta = \sqrt{(d+1)K^2A^2 + 0.5(\Lambda + c^2)}$ by bounding the Hessian of the objective: $\sup_{f \in D_f} \| \nabla^2 \mathcal{J}_{\text{Softmax}}(f, z) \| \leq \beta$. The first part of the smoothness bound $\beta$, $(d+1)K^2A^2$, stems from the 12-regularization term of the objective function $\lambda/2 \| f \|^2$ which is influenced by the size of model $f \in \mathbb{R}^{(d+1)\times K}$. In particular, the second derivative of the regularization term is constant in each direction of the derivative thus marking the dependence on the number of model parameters $(d+1)K^2$. The second part of $\beta$, $0.5(\Lambda + c^2)A^2$, stems from the softmax loss function for which we use the characteristic of the softmax function that the probabilities of each class add up to 1. In particular, we also use the fact that $\max_{s_1, \ldots, s_K} \sum_{k=1}^{K} s_k (1 - s_k)(C + s_k) - \sum_{k=1}^{K} s_k = 1 \land \sum_k s_k \geq 0 \forall k \leq 0.25(C+1)^2$ for $C \leq \Lambda/\beta$ which we proof in Lem. O.2 using the KKT conditions. The bound $0.25(C+1)^2$ scaled proportional to $2c^2$ (cf. Thm. O.1) which directly corresponds to the $0.5(\Lambda + c^2)A^2$ of the smoothness bound $\beta$.

Strong convexity. The strong convexity $\Lambda$ stems from the regularization term $\lambda/2 \| f \|^2$ and we show and use in Appx. M that the objective function without the regularization is convex. In particular, the Hessian of the softmax-based cross-entropy loss function $\mathcal{L}_{\text{CE}}$ is convex if it is positive semi-definite: $\nabla^2 \mathcal{L}_{\text{CE}} \succeq 0$.

5 System Design of Distributed DP-Helmet

We present the system design of Distributed DP-Helmet in detail (cf. Alg. 3) and schematically in Fig. 1 including its privacy properties. Each user holds a small dataset while all users jointly learn a model. There are two scenarios: first (differential privacy, see Fig. 6), each person contributes one data point to a user who is a local aggregator; e.g. a hospital; second (local DP, see Fig. 7 for $T = 50$), each user is a person and contributes a small dataset.

Consider a set of users $\mathcal{U}$, each with a local dataset $D$ of size $n^{(i)} = |D^{(i)}|$ that already is in a sufficiently simplified representation by the SimCLR premaining feature extractor [15, 10]. The users collectively train an $(\varepsilon, \delta)$-DP model using a learning algorithm $T$ that is sensitivity-bounded as in Def. 3.3. An example for $T$ is Softmax-MLP-SGD (cf. Alg. 1) with $s = \frac{2(\Lambda R + \sqrt{2c})}{\Lambda n^{(i)}}$ or SVM-SGD (cf. Alg. 2) with $s = \frac{2(\Lambda R + c)}{\Lambda n^{(i)}}$.

Algorithm 3 Distributed DP-Helmet($\zeta$). Softmax-MLP-SGD (Alg. 1) has sensitivity $s = \frac{2(\Lambda R + \sqrt{2c})}{\Lambda M^{(i)}}$ and $\zeta := (c, \Lambda, R, M)$; SVM-SGD (Alg. 2) has $s = \frac{2(\Lambda R + c)}{\Lambda M^{(i)}}$ and $\zeta := (h, c, \Lambda, R, M)$. $\pi_{\text{SecSum}}$ as in Def. 3.5

function Client Distributed DP-Helmet($D, w, K, \tau, t, \xi, \sigma)$

Input: local data $D^{(i)}$ with $n^{(i)} = |D^{(i)}|$; $\#$ users $w$; $\#$ classes $K$; learning algorithm $T$; noise scale $\sigma$; ratio $t$ of honest users; hyperparameters $\xi$;

Result: DP-models $f_{\text{priv}} \in \mathbb{R}^{(p+1) \times K}$ $f_{\text{priv}} \leftarrow T(D, \xi, K) \ldots \Rightarrow T$ is $s$-sensitivity-bounded $f_{\text{priv}} \leftarrow f_{\text{priv}} + \mathcal{N}(0, \delta^2I_{(p+1) \times K})$ with $\delta = \sigma \cdot 1/\sqrt{\tau}$

Run the client code of a secure summation protocol $\pi_{\text{SecSum}}$ on input $n^{(i)}/w \cdot f_{\text{priv}}$

function Server Distributed DP-Helmet($\mathcal{U}$)

Input: users $\mathcal{U}$;

Result: empty string

Run the server protocol of $\pi_{\text{SecSum}}$

Alg. 3 follows the scheme of Jayaraman et al. [28] with an extension to handle differing local data sizes $n^{(i)}$. First, each user separately trains a non-private model $f_{\text{np}}$, using $T$ and the hyperparameters $\xi$, e.g. $\xi := (c, \Lambda, R, M)$ for Softmax-SGD. Next, each user adds to $f_{\text{np}}$, Gaussian noise scaled with $s$ and $1/\sqrt{\tau}$, where $t$ is the number of honest users in the system. Together, the users then run a secure summation protocol $\pi_{\text{SecSum}}$ as in Def. 3.5 where the input of each user is the noised model, which is scaled down by the number of users to yield the average model and scaled up by the local data size $n^{(i)}$. This upsampling keeps all local sensitivities constant and independent of $n^{(i)}$ and thus allows differentially private blind averaging with differing $n^{(i)}$. Utility-wise, the prediction of SVMs and Softmax-MLP are scale-invariant, i.e. the prediction is the same if we scale the model by any constant. If all $n^{(i)}$ are the same, this upsampling corresponds to an averaged model scaled by a constant which has no utility implications due to this scale-invariant nature. Thanks to secure summation, we show centralized-DP guarantees with noise in the order of $O(\nu^{-1})$ after upsampling by $n$ within a threat model akin to that of federated learning with differential privacy. For privacy accounting, we use tight composition bounds like Meiser and Mohammadi [38]. Sommer et al. [41], Balle et al. [4].

Threat model & security goals. For our work, we assume passive, collaborating attackers that follow our protocol. We assume that a fraction of at least $t$ users are honest (say $t = 50\%$). The attacker tries to extract sensitive information about other parties from the interaction and the result. The adversary is assumed to have full knowledge about each user’s dataset, except for one data point of one user. Our privacy goals are $(\varepsilon, \delta)$-differential privacy (protecting single samples) and $(\varepsilon, \delta)$-T-group differential privacy (protecting all samples of a user at once) respectively, depending on whether each user is a local aggregator or a person. Note that even passive adversaries can collude and exchange information about the randomness they used in their local computation. To compensate for untrustworthy users, we adjust the noise added by each user according to the fraction of honest users $t$;
e.g., if \( t = 50\% \), then we double the noise to satisfy our guarantees.

**Non-interactive protocol.** Distributed DP-Helmet is agnostic to the specific secure summation protocol we use. The protocol SecAgg of Bell et al. \cite{Bell2018} requires 4 communication rounds, but the user only shares their local model in the third round. Following Bogetoft et al. \cite{Bogetoft2019}, we introduce \( J \) computation servers that aggregate the model parameters on behalf of the users. Specifically, each user \( i \) sends their model \( f(i) \) in fixed-point arithmetic to server \( j \), where for \( j < J \), each \( f(i) \) is drawn randomly from \( \{1, \ldots, B\} \) for a sufficiently large \( B \) and where \( f(i) = (f(i) - \sum_{j < j} f(j(i))) \mod B \). The computation servers then run SecAgg among each other, yielding the sum of all inputs. All \( f(i) \) cancel out and the sum over all models \( f(i) \) remains. The secret sharing technique is information-theoretically secure if at least one computation server is honest. Security assumptions of SecAgg apply to the computation servers instead of the users. Although secure, this protocol is not robust against active attacks.

### 5.1 Security of Distributed DP-Helmet

First, we derive a tight output sensitivity bound. A naïve approach would be to release each individual predictor, determine the noise scale proportionally to \( \sigma = \sigma(\varepsilon) \) (cf. Cor. \ref{cor:sensitivity}), showing \((\varepsilon, \delta)\)-DP for every user. We can save a factor of \( \sqrt{w_i} \) by leveraging that \( w_i \) is known to the adversary and we have at least \( t = 50\% \). Consequently, local noise of scale \( \tilde{\sigma} = \sigma \cdot \sqrt{\frac{1}{w_i}} \) is sufficient for \((\varepsilon, \delta)\)-DP.

**Lemma 5.1** (Privacy amplification via averaging). For a configuration \( \zeta \) as in Def. \ref{def:config} and a noise scale \( \sigma \) as in Def. \ref{def:noise} of Alg. \ref{alg:DP-Helmet} without noise, \( \text{avg}(n^{(i)}, T(D^{(i)})) \) has a sensitivity of \( \sqrt{w_i} \cdot \sqrt{\frac{1}{w_i}} \) for each model if \( s = s' = 1 \).

The proof is in Appx. II. The sensitivity of the aggregate is bounded to \( n^{(i)} \cdot \sqrt{\frac{1}{w_i}} \) by rescaling the local models by \( n^{(i)} \) which leads to local sensitivities independent of \( n^{(i)} \) and allows blind averaging with varying local data sizes \( n^{(i)} \). The sensitivities of \( T = \text{SVM-SGD} \) (cf. Example \ref{example:sensitivity}) and \( T = \text{Softmax-SLP-SGD} \) (cf. Thm. \ref{thm:softmax}) fulfill the condition in the Lemma as they are proportional to \( n^{-1} \), thus: \( s' = s = n \).

Next, we show that locally adding noise per user \( \sigma \) proportional to \( \sigma \cdot n^{(i)} / \pi \) and taking the mean over the users is equivalent to centrally adding noise \( \tilde{\sigma} \) proportional to \( \sigma \cdot n^{(i)} / \pi \). Adding dishonest noise can be treated as post-processing and does not impact privacy.

**Lemma 5.2**. For a configuration \( \zeta \) as in Def. \ref{def:config} and noise scale \( \tilde{\sigma} = \frac{1}{\pi} \sum_{i=1}^{n} \mathcal{N}(0, (\tilde{\sigma} \cdot \sqrt{\pi})^2) = \mathcal{N}(0, (\tilde{\sigma} \cdot \sqrt{\pi})^2) \).

The proof is in Appx. I. We now prove differential privacy for Distributed DP-Helmet of Alg. \ref{alg:DP-Helmet} with noise scale \( \tilde{\sigma} \) and thus \( \varepsilon \in \mathcal{O}(1 / \sqrt{\pi} \cdot \sqrt{w_i}) \).

**Theorem 5.3** (Main Theorem, simplified). For a configuration \( \zeta \) as in Def. \ref{def:config} of Alg. \ref{alg:DP-Helmet} as in Def. \ref{def:config} and a function \( \nu \) negligible in the security parameter used in \( \pi_{\text{SecSum}} \).

The full statement and proof are in Appx. II. Simplified, the proof follows by applying the sensitivity (cf. Lem. \ref{lem:sensitivity}) to the Gauss mechanism (cf. Lem. \ref{lem:gauss}) where the noise is applied per user (cf. Lem. \ref{lem:noise}). Next, we show how to protect the entire dataset of a single user (e.g., for distributed training via smartphones). The sensitivity-based bound on the Gauss mechanism implies strong \( \nu \)-group privacy results (see Appx. K), leading to guarantees as in local DP.

**Corollary 5.4** (Group-private variant). For a configuration \( \zeta \) as in Def. \ref{def:config} of Alg. \ref{alg:DP-Helmet} in short DP-Helmet\((\zeta) \), satisfies computational \((\nu, \epsilon, \delta)\)-DP with \( \nu \neq \nu \) as in Def. \ref{def:noise} and \( \sigma \) negligible in the security parameter used in \( \pi_{\text{SecSum}} \): for any pair of datasets \( D, D' \) that differ at most \( \nu \) many data points, DP-Helmet\((\zeta, D, D', \epsilon, \delta) \) satisfies \((\nu, \epsilon, \delta)\)-DP.

Theorem \ref{thm:corollary} can be generalized to data oblivious. If the norm of each model is bounded by \( R \), averaging models leads to a sensitivity of \( \sqrt{2R} / \nu \). This method enables other SVM optimizers and can render a tighter sensitivity bound than SVM-SGD for certain settings of \( \nu \) or local data sizes \( n^{(i)} \). In particular, the training procedure of each base learner does not need to satisfy DP. The proof is in Appx. II.

**Corollary 5.5** (User-level sensitivity). Given a learning algorithm \( T \), we say that \( T \) is R-norm bounded if for any \( d \)-dimensional dataset \( D \) with \( n = |D| \), hyperparameter \( \xi \), and class \( k \in \{1, \ldots, K\} \) or \( k = K \), \( |T(D, \xi, k)| \leq R \). Any R-norm bounded learning algorithm \( T \) has a deterministic sensitivity \( s = 2R \). In particular, \( T(D, \xi, K) + N(0, \sigma^2 I_{d \cdot K}) \) satisfies \((\nu, \epsilon, \delta)\)-group differential privacy with \( \nu = 0 \). The proof is in Appx. II.

### 6 Phase II: Non-interactive Blind Average

In Phase II (cf. Fig. 3), the core idea of Distributed DP-Helmet is to locally train models and average their parameters blind, i.e., without further synchronizing or fine-tuning the models: \( \text{avg}(T) \). To show that such a non-interactive training is useful, we provide a utility bound on the blind averaging procedure as follows: we (1) reduce the utility requirement of blindly averaging a regularized empirical risk minimizer (ERM) \( T \) to the coefficients \( \alpha \) of the dual problem of \( T \), (2) leverage the dual problem to show that for a hinge-loss linear SVM trained with SGD, \( T = \text{Hinge-SVM-SGD} \), \( \text{avg}(T) \) gracefully converges in the limit to the best model for the combined local datasets \( D_j \): \( \mathcal{E} [f(\text{avg}(\text{Hinge-SVM-SGD}(D^{(j)})), \text{inf} | \mathcal{F} \{f, \hat{\theta}, \ldots\} \in \mathcal{O}(1/\mu) \) for \( M \) many local train rounds and objective function \( \mathcal{F} \). Thus, convergence holds in a strongly non-IID setting which we illustrate with an example in Fig. 3 where each user has exclusive access to only one class. We now show in Appx. I that if each converged Hinge-SVM \( T(D^{(j)}), \hat{\theta}, \ldots | \mathcal{F} \{f, \hat{\theta}, \ldots\} \in \mathcal{O}(1/\mu) \) on a local dataset \( D^{(j)} \) with the same local data size each, i.e. \( n = n^{(j)} = n^{(i)} \forall j, \hat{\theta}, \ldots \) has support vectors \( V^{(j)} \) then the average of the locally trained SVMs \( \text{avg}(T(D^{(j)})) \) has support vectors \( V = \bigcup_{j=1}^{M} V^{(j)} \). This theorem implies that we can compare and describe the effect of blind averaging by analyzing the selected support vectors, i.e. the non-zero dual coefficients \( \alpha_i \). If only a few \( \alpha_i \) differ between the blind averaged local SVMs and the global SVM, in the worst case the error of blind averaging is significantly smaller as if a lot of the \( \alpha_i \) differ.

**Lemma 6.1** (Support Vectors of averaged SVM). Given a configuration \( \zeta \) as in Def. \ref{def:config} a locally trained model
of hinge-loss based linear SVM $T$, i.e. $T(D^{(i)}, \xi, \_)$ = argmin$_f \frac{1}{n^{(i)}} \sum_{(x,y) \in D^{(i)}} \max(0, 1 - y f(x)) + \lambda \|f\|_1 = f^{(i)}$, has the support vectors $V^{(i)} := \{(x,y) \in D^{(i)} \mid f^{(i)} x y \leq \|f^{(i)}\|_1^{-1}\}$. Then, the average of these locally trained models $\text{avg}(f^{(i)})$ has the support vectors $V = \bigcup_{i=1}^n V^{(i)}$.

If an SVM of the combined local datasets $U$ has the same support vectors as the average of local SVMs then both models converge due to the hinge loss definition (cf. Thm. 6.2 in Appx. I.2). Such a scenario occurs e.g. if the regularization is high and thus the margin is large enough such that all data points are within the margin and thus support vectors.

**Theorem 6.2** (Averaging locally trained SVM converges to a global SVM). Given a configuration $\xi$ as in Def. 3.8 and the same local data sizes $n^{(i)} = n^{(j)} \forall i,j$, there exists a regularization parameter $\lambda$ such that the average of locally trained models $\text{avg}(T(D^{(i)}))$ with a hinge-loss linear SVM as an objective function $T$ trained with projected subgradient descent using weighted averaging (PGDWA), $T = \text{Hinge-SVM-PGDWA}$, converges with the number of local iterations $M$ to the best model for the combined local datasets $U$, i.e. $E[f(\text{avg}(\text{Hinge-SVM-PGDWA}(D^{(i)})), U, \_)] - \inf_f E[f(x, y)] \in O(1/M)$.

The average of SVMs has the union of the local dual coefficients as dual coefficients $\alpha$ (cf. Cor. 1.1). This corollary holds not only for hinge-loss linear SVMs but for a broad range of regularized empirical risk minimizers (ERM) for which the representative theorem holds (cf. Thm. 3.7 [3]), including a converged SVM-SGD and Softmax-SLP-SGD. For limitations, we refer to Sec. 8.

### 7 Experimental Results

**Pretraining.** For all experiments, we used a SimCLR pretrained modelootnote{accessible at https://github.com/google-research/simclr} on ImageNet ILSVRC-2012 [42] to get an embedding of the local data (cf. Fig. 9 in Appx. D for an embedding view). It is based on a ResNet152 with selective kernels [35] and a width multiplier of 3. It has been trained on the fine-tuned variant of SimCLR which uses all of the ImageNet labels. Overall, it has 795M parameters and achieves 83.1% classification accuracy (1000 classes) when being applied by a linear prediction head.

**Sensitive datasets.** CIFAR-40, CIFAR-100 [33], and federated EMNIST [17, 12] act as our sensitive datasets. CIFAR is frequently used as a benchmark dataset in differential privacy literature and EMNIST in distributed learning literature. Both CIFAR datasets consist of 60,000 thumbnail-sized, colored images of 10 or 100 classes. Federated EMNIST consists of approximately 750,000 thumbnail-sized, grayscale images of 62 classes and is annotated with 3,400 user-partitions based on the author of the images: users have between 19 and 465 many data points, on average 220 ± 85.

**Evaluation.** We compare Distributed DP-Helmet to DP-SGD-based federated learning (FL) and analyze four research questions:

- (RQ1) If each user has a fixed number of data points, how does performance compare when the number of users increases? Performance improves with an increasing number of users (cf. Fig. 6 (b)). Although DP-Softmax-SLP-SGD training performs subpar to DP-FL for a few users, it takes off after about 100 users on both datasets (cf. Fig. 6 (b)). Our scalability advantage with the number of users becomes especially evident when considering significantly more users (cf. Fig. 7), such as is common in distributed training via smartphones. Here, DP-guarantees...
of $\varepsilon \leq 5 \cdot 10^{-5}$ become plausible with at least 87% prediction performance for a task like CIFAR-10. Alternatively, leveraging [Cor. 5.4] we can consider a local DP scenario (with $\Upsilon = 50$) without a trusted aggregator, yielding an accuracy of 87% for $\varepsilon = 10^{-4}$. Starting from $\Upsilon \geq 2$, a user-level sensitivity (cf. [Cor. 5.5]) is in the evaluated setting mostly tighter than a data point dependent one; hence, the accuracy values are close to the local DP scenario.

**(RQ2)** If we keep the number of overall data points the same, how does distributing them impact performance? If we globally fix the number of data points (cf. Fig. 6 (a,c)) that are distributed over the users, Distributed DP-Helmet’s performance degrades more gracefully than that of DP-FL. Thm. 6.2 supports the more graceful decline; it states that averaging multiple of an SVM similar to SVM-SGD converges for a large enough regularizer to the optimal SVM on all training data. In absolute terms, the accuracy is better for a smaller regularizer which is visible as the noise scales with $\sqrt{\varepsilon}$.

**(RQ3)** How robust is DP-Helmet’s utility if the local user data is non-IID? We observe in the strongly biased non-IID (cf. [Tbl. 2]) where each user exclusively holds data of one class that on CIFAR-10 the utility decline of DP-SVM-SGD is still small whereas DP-Softmax-SLP-SGD needs more users for a similar utility preservation since it is more sensitive to noise. For the real-world federated EMNIST setting with unbalanced local data sizes $n^{(i)}$, the utility gap of blind averaging increases (cf. Fig. 6 (d)) as we favor a tight sensitivity bound at the expense of suboptimally scaling each local SVM and Softmax-SLP. Still, DP-Helmet surpasses the performance of FL.

**(RQ4)** Is there a natural ordering of DP-SVM-SGD, DP-Softmax-SLP-SGD and FL (DP-SGD) in a centralized setting (1 user)? We refer to Appx. C for this ablation study. We observe a natural ordering where DP-SVM-SGD performs worse than DP-Softmax-SLP-SGD while FL outperforms both: for $\varepsilon = 0.6$ on CIFAR-10, DP-SVM-SGD has an accuracy of 87.4% and DP-Softmax-SLP-SGD has 90.2% while FL has 91.6% (cf. Fig. 8). For $\varepsilon = 1.2$ on CIFAR-100, DP-SVM-SGD has an accuracy of 1.5% and DP-Softmax-SLP-SGD has 52.8% while FL has 66.0% (cf. Fig. 6 (c)). The outperformance of DP-Softmax-SLP-SGD above DP-SVM-SGD is supported by our theory where the sensitivity no longer depends on the number of classes for a fixed model size (cf. Thm. 4.1) and its utility boost due to being inherently multi-class. We reckon that the remaining difference between DP-Softmax-SLP-SGD and FL is mostly due to DP-SGD’s noise-correcting property from its iterative-noising.
Figure 7: ($\varepsilon, \Upsilon$)-Heatmap for classification accuracy of Distributed DP-Helmet (cf. Sec. 5) with learning algorithm DP-Softmax-SLP-SGD on CIFAR-10 dataset (top: $\delta = 10^{-12}$; bottom: $\delta = 10^{-12}$) with roughly 200,000 (top) and roughly 20,000,000 (bottom) users. We train 1,000 models on 50 data points each; to emulate having more users we rescale the $\varepsilon$-values ($\varepsilon' := \min_{\Upsilon} \varepsilon / n_{\text{users}}$) to roughly reach the target number of users and report interpolated accuracy values. We extrapolate the privacy guarantees, due to the limited dataset size. Our accuracy values are pessimistic as we keep the accuracy numbers that we got from averaging 1,000 models. Actually taking the mean over roughly 200,000 or even roughly 20,000,000 users should provide better results. $\Upsilon < 50$ group privacy places trust in users as local aggregators whereas $\Upsilon = 50$ is comparable to local DP. Rescaling the $\varepsilon$-values only approximates the $\varepsilon$ guarantee we would get if we actually rescaled the noise scale by the target number of users. For $\Upsilon \geq 2$, a tighter group-privacy bound is possible (cf. Cor. 5.5); hence, the accuracy values are close to $\Upsilon = n \approx 100,000$.

7.1 Experimental Setup

Unless stated differently, we leveraged 5-repeated 6-fold stratified cross-validation for all CIFAR experiments and 10-repeated cross-validation on the pre-defined split for EMNIST ones. We conducted a hyperparameter search across $\Lambda$ and $R$ for each evaluation setting and $\varepsilon$ and reported the highest mean accuracy. Privacy Accounting has been done with the privacy bucket [33, 34] toolbox for Gaussians without subsampling, with Sommer et al. [34] Theorem 5] where both can be extended to multivariate Gaussians (cf. Appx. L). We set DP $\delta = 10^{-5}$ if not stated otherwise which is for CIFAR below $1/nw$ where $nw$ is the size of the combined local data.

Concerning computation resources, our Python implementation of the EMNIST experiments with 3,400 users took under a minute per user on a machine with 2x Intel Xeon Platinum 8168 @24 Cores.

For DP-SVM-SGD-based experiments, we utilized projected SGD (PSGD) as used by Wu et al. [19] and chose a batch size of 20 and the Huber loss with relaxation parameter $h = 0.1$. For CIFAR-10, we chose a hypothesis space radius $R \in \{0.04, 0.05, 0.06, 0.07, 0.08\}$, a regularization parameter $\Lambda \in \{10, 100, 200\}$, and trained for 500 epochs; for the variant where we protect the whole local dataset, we chose $\Lambda \in \{0.5, 1, 2, 5\}$ and $R \in \{0.06, 0.07\}$ instead. For CIFAR-100, we chose $R \in \{0.04, 0.06, 0.08\}$, $\Lambda \in \{3, 30, 100\}$, and trained for 150 epochs. For EMNIST, we chose in the 1 user setting $R = 0.08$ and $\Lambda \in \{3, 10, 100\}$ and in the 3,400 user setting $R = 0.04$ with $\Lambda \in \{30, 100\}$, and trained for 150 epochs.

For DP-Softmax-SLP-SGD-based experiments, we utilized PSGD with a batch size of 20 and trained for 150 epochs. For CIFAR-10, we chose $R \in \{0.1, 0.4, 0.6, 1.0\}$ and $\Lambda \in \{1, 3, 10, 30\}$; for the variant where we protect the whole local dataset, we chose $\Lambda \in \{0.5, 1\}$ and $R \in \{1, 3\}$ instead. For CIFAR-100, we chose the parameter combination $(R, \Lambda) \in \{(0.01, 100), (0.03, 30), (0.03, 100), (0.1, 10), (0.1, 30), (0.3, 3), (0.3, 10), (1, 1), (1, 3)\}$. For EMNIST, we chose in the 1 user setting $R = 6$ and $\Lambda \in \{0.03, 0.1, 0.3\}$ and in the 3,400 user setting $R = 2$ with $\Lambda \in \{0.3, 1, 3\}$ and $R = 6$ with $\Lambda \in \{0.03, 0.1\}$.

For the experiments of [18, 21] we reported the results of the following hyperparameters: (CIFAR10, DP-SVM-SGD, regular & non-iid) $R = 0.06, \Lambda = 100$ for the dataset multiplier 1x and $R = 0.06, \Lambda = 10$ for the dataset multiplier 67x; (CIFAR10, DP-Softmax-SLP-SGD, regular) $R = 1.0, \Lambda = 1$ for both dataset multipliers 1x, 67x; (CIFAR10, DP-Softmax-SLP-SGD, non-iid) $R = 0.6, \Lambda = 3$ for both dataset multipliers 1x, 67x; (CIFAR100, DP-Softmax-SLP-SGD, regular) $R = 1.0, \Lambda = 3$ for dataset multiplier 1x and $R = 1.0, \Lambda = 1$ for both dataset multipliers 67x; (CIFAR100, DP-Softmax-SLP-SGD, non-iid) $R = 1.0, \Lambda = 3$ for both dataset multipliers 1x, 67x.

For the federated learning (FL) experiments, we utilized the opacus PyTorch library [50], which implements DP-SGD [1]. For a fair comparison, we halved the noise scale for the privacy accounting to comply with bounded DP as opacus currently uses unbounded DP: it now uses a noise scale proportional to $2C$ instead of $C$ as the clipping bound. We loosely adapted our hyperparameters to the ones reported by Tramèr and Boneh [45] who evaluated DP-SGD on SimCLR’s embeddings for the CIFAR-10 dataset. In detail, the neural network is a single-layer perceptron with a 6,144 d input and has the following configuration: (CIFAR-10) 61,450 trainable parameters on a 10 d output, (CIFAR-100) 614,500 trainable parameters on a 100 d output, and (EMNIST) 380,990 trainable parameters on a 62 d output. The loss function is the categorical cross-entropy on a softmax activation function and training has been performed with SGD. We set the learning rate to 4, the Poisson sample rate (CIFAR) $q = \frac{1024}{50000}$ (EMNIST) $q = \frac{1024}{671585}$ which in expectation samples a batch size of 1024, trained for 40 epochs, and norm-clipped the gradients with a clipping bound $C = 0.1$.

In the distributed training scenario, instead of running an end-to-end experiment with full SecAgg clients, we evaluate a functionally equivalent abstraction without cryptographic overhead. In our CIFAR experiments, we randomly

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4 accessible at https://github.com/sommerda/privacybuckets

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5 accessible at https://github.com/pytorchi/privacy
split the available data points among the users and emulated scenarios where not all data points were needed by taking the first training data points. The validation size remained constant. For FL, we kept a constant expected batch size: \( q' = \frac{1024}{20000} \) for 20000 and \( q'' = \frac{1024}{5000} \) for 5000 available data points (wan). For FL, we emulated a larger number of users by dividing the noise scale \( \sigma \) by \( \sqrt{n} \) to the benefit of FL. Here, the model performance is not expected to differ as the mean of the gradients of one user is the same as the mean of gradients from different users: SGD computes, just as FL, the mean of the gradients. Yet, the noise will increase by a factor of \( \sqrt{n} \).

8 Limitations & Discussion

For unfavorable like non-IID datasets, blind averaging leads to a reduced signal-to-noise ratio, i.e. model parameters smaller than in the centralized setting. This may explain why blind averaging Softmax-SLPs works better for less noise (cf. \( \text{Tbl. 2} \)). Further, increasing the regularization parameter \( \Lambda \) to help convergence, can lead to poor accuracy of the converged model, e.g. for unbalanced datasets. 

**Distributional shift between the public and sensitive dataset.** For pretraining, we leverage contrastive learning. While very effective generally, it is susceptible to performance loss if the shape of the sensitive data used to train an SVM or Softmax-SLP is significantly different from the shape of the initial public data.

**Input Clipping.** We require bounded input data for a bounded sensitivity. In many pretraining methods like SimCLR, no natural bound exists thus we norm-clip the input data by a constant \( c \). To provide a data-independent \( c \) on CIFAR-10 and EMNIST, \( c \) is based on CIFAR-100 (here: 34.854); its similar data distribution encompasses the output distribution of the pretraining reasonably well. For CIFAR-100, \( c \) is based on CIFAR-10 (here: 34.157).

**Hyperparameter Search.** In SVM-SGD, we have two important hyperparameters that influence the noise scale: the regularization weight \( \lambda \) and the predictor radius \( R \). In the noise scale subterm \( \frac{w}{\lambda + R^2} \), the maximal predictor radius is naturally significantly smaller than \( \frac{1}{\lambda} \) due to the regularization penalty. Thus, a bad-tuned \( R \) often does not have as large of a utility impact as a bad-tuned \( \lambda \). Estimating parameters for a fixed \( \varepsilon \) from public data is called hyperparameter freeness in prior work [27]. For the other \( \varepsilon \) values, we can estimate \( \Lambda \) by fitting a (linear) curve on related public data (proposed by Chaudhuri et al. [14]) or synthetic data (proposed by AMP-NT [27]) as smaller \( \varepsilon \) prefers a higher \( \Lambda \) and vice versa.

**Blind averaging – Signal-to-noise ratio.** For unfavorable yet balanced local datasets, we identify as a main limitation of blind averaging a reduced signal-to-noise ratio: the model is not as large as in the centralized setting w.r.t. the sensitivity analysis. This effect would explain why our experiments show that blind averaging Softmax-SLPs work better with very little noise (cf. \( \text{Tbl. 2} \)).

For SVMs, our formal characterization of the effect of blind averaging enables us to describe its limitations more precisely. In summary, we see two effects that reduce the signal-to-noise ratio. One effect comes from the requirement of the SVM training that the model with the smallest norm shall be found that satisfies the soft-margin constraints of the training data points. The local SVM training has fewer data points and, thus, fewer constraints. Hence, unfavorable local data sets will lead to a smaller model. Another effect comes from the averaging itself. Unfavorable local data sets can lead to local models that point in very different directions. When averaging these models, their norm naturally decreases as for any two vectors \( a, b \in \mathbb{R}^n \) we have \( 0.5|a + b|_2 \leq 0.5(|a|_2 + |b|_2) \), and this discrepancy is larger the smaller the inner product is.

**Blind averaging – Unbalanced data.** Convergence holds if all data points are support vectors (SV) which implies a large margin. Yet a regularly trained SVM chooses roughly equally many SVs per class: by the dual problem, we have the constraint \( y' \alpha = 0 \) for labels \( y_j \in \{-1, 1\} \) and dual coefficients \( \alpha \). If we have an SV inside the margin then \( \alpha_j = \Lambda^{-1} \). Hence, enlarging the margin such that all data points are SVs can lead to poor utility performance. Moreover, unbalanced local data can deteriorate the performance of blind averaging as observed in the EMNIST experiments (cf. \( \text{Fig. 3} \)) as we favor privacy, i.e. a constant sensitivity per user, above utility, i.e. optimal local scaling.

**Active attacks.** Active attackers may deviate from the protocol or send maliciously construed local models. If the used secure summation protocol is resilient against active adversaries and can still guarantee that only the sum of the inputs is leaked, privacy is preserved. This follows from analyzing our algorithm for just the honest users and then leveraging the post-processing property of differential privacy. Secure summation protocols such as Bell et al. [8] leak partial sums under active attacks and will diminish the privacy offered by our work against such adversaries as well.

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A Extended Preliminaries

A.1 Secure Summation

Before formulating the security of the secure summation protocol, we define a network execution against a global network attacker that is active and adaptive. For self-containedness, we briefly present the notion of interactive machines and a sequential activation network execution. More general frameworks for such a setting include, e.g., the universal composability framework [13].

We rely on the notion of interactive machines. For two interactive machines $X, Y$, we write $(X, Y)$ for the interaction between $X$ and $Y$. We write $(X, Y) = b$ to state that the machine $X$ terminates and outputs $b$.

**The network execution $\text{Real}_\pi$.** Next, we define a network execution against a global network attacker that is active and adaptive. Given a protocol $\pi$ with client and server code, we define an interactive machine $\text{Real}_\pi$ that lets each client party run the client code, lets the servers run the server code, and emulates a (sequential-activation-based) network execution, and interacts with another machine, called the attacker $A$. The interaction is written as $(A, \text{Real}_\pi)$. Whenever within this network execution a party $B$ sends a message $m$ over the network to a party $C$, the interactive machine $\text{Real}_\pi$, sends this message $m$ to the attacker, activates the attacker, and waits for a response $m'$ from the attacker. $\text{Real}_\pi$ then lets this response $m'$ be delivered to party $C$, and activates party $C$. Moreover, the attacker $A$ can send a dedicated message $(\text{compromise}, P)$ for compromising a party $P$ within the protocol execution. Whenever the attacker sends the message $(\text{compromise}, P)$ to the network execution $\text{Real}_\pi$, the network execution marks this party $P$ as compromised and sends the internal state of this party to the attacker $A$. For each compromised party $P$, the attacker decides how $P$ acts. Formally, the network execution redirects each message $m$ that is sent to $P$ to the attacker $A$ and awaits a response message $(m', P')$ from the attacker $A$. Upon receiving the response $(m', P')$, the network execution $\text{Real}_\pi$ sends on behalf of $P$ the message $m'$ to the party $P'$.

For convenience, we write that a party $P$ runs the control code of a protocol $\pi$ on input $m$ when the network execution runs for party $P$ the client code of $\pi$ on input $m$.

A.2 DP-SVM-SGD

**Definition A.1.** The Huber loss according to Chaudhuri et al. [14] Equation 7] is with a relaxation parameter $h$ defined as

$$
\ell_{\text{huber}}(h, z) := \begin{cases} 
0 & \text{if } z > 1 + h \\
\frac{1}{2h}(1 + h - z)^2 & \text{if } |1 - z| \leq h \\
1 - z & \text{if } z < 1 - h 
\end{cases}
$$

B Related Work on Differentially Private Empirical Risk Minimization

On differentially private empirical risk minimization for convex loss functions [14] as utilized in this work, the literature discusses three directions: output perturbation, objective perturbation, and gradient perturbation. Output perturbation [14] [29] estimates a sensitivity on the final model without adding noise, and only in the end adds noise that is calibrated to this sensitivity. We rely on output perturbation because it enables us to only have a single invocation of an MPC protocol at the end to merge the models while still achieving the same low sensitivity as if the model was trained at a trustworthy central party that collects all data points, trains a model and adds noise in the end.

Objective perturbation [14] [32] [27] [6] adds noise to the objective function instead of adding it to the final model. In principle, MPC could also be used to emulate the situation that a central party as above trains a model via objective perturbation. Yet, in that case, each party would have to synchronize with every other party far more often, as no party would be allowed to learn how exactly the objective function would be perturbed. That would result in far higher communication requirements.

Concerning gradient perturbation [6] [18] [23] [7] [24], recent work has shown tight privacy bounds. In order to achieve the same low degree of required noise as in a central setting, MPC could be utilized. Yet, for SGD also multiple rounds of communication would be needed as the privacy proof (for convex optimization) does not take into account that intermediary gradients are leaked. Hence, the entire differentially private SGD algorithm for convex optimization would have to be computed in MPC, similar to cpSGD (see above).

C Extended Ablation Study (Centralized Setting)

C.1 Setup of the Ablation Study

For DP-SVM-SMO-based experiments, we used the *liblinear* [21] library via the Scikit-Learn method LinearSVC for classification. *Liblinear* is a fast C++ implementation that uses the SVM-agnostic sequential minimal optimization (SMO) procedure. However, it does not offer a guaranteed and private convergence bound.

More specifically, we used the $L_2$-regularized hinge loss, an SMO convergence tolerance of $\text{tol} := 2 \cdot 10^{-12}$ with a maximum of 10,000 iterations which were seldom reached, and a logarithmically spaced inverse regularization parameter $C \in \{3, 6 \cdot 10^{-8}, 1, 2, 3, 6 \cdot 10^{-7}, 1, 2, 3, 6 \cdot 10^{-6}, 1, 2, 3, 6 \cdot 10^{-5}, 1, 2 \cdot 10^{-4}\}$. To better fit with the *LinearSVC*

[6=https://scikit-learn.org/stable/modules/generated/sklearn.svm.LinearSVC.html] BSD-3-Clause license
implementation, the original loss function is rescaled by \( \frac{1}{\lambda} \) and \( C \) is set to \( \frac{1}{\lambda} \cdot n \) with \( n \) as the number of data points. Furthermore, for distributed DP-SVM-SMO training we extended the range of the hyperparameter \( C \) – whenever appropriate – up to \( 3 \cdot 10^{-3} \) which becomes relevant in a scenario with many users and few data points per user. Similar to DP-SVM-SGD-based experiments, the best-performing regularization parameter \( C \) was selected for each parameter combination.

The non-private reference baseline uses a linear SVM optimized via SMO with the hinge loss and an inverse regularization parameter \( C = 2 \) (best performing of \( C \in \{ \leq 5 \cdot 10^{-5}, 0.5, 1, 2 \} \)).

For the ablation study, we also included the Approximate Minima Perturbation (AMP) algorithm\(^{[27]} \) which resembles an instance of objective perturbation. There, we used a (80-20)-train-test split with 10 repeats and the following hyperparameters: \( L \in \{0.1, 1.0, 34.854\}, \) \( \text{eps} \cdot \text{frac} \in \{9, 95, 98, 99\} \), \( \text{eps} \cdot \text{out} \cdot \text{frac} \in \{0.001, 0.01, 0.1\} \). We selected \((L = 1, \text{eps} \cdot \text{out} \cdot \text{frac} = 0.001, \text{eps} \cdot \text{frac} = 0.99)\) as a good performing parameter combination for AMP. For better performance, we used the GPU-capable \texttt{bayes\_minimize} from the Tensorflow Probability package. To provide better privacy guarantees, we leveraged the results of Kairouz et al.\(^{[29]} \), Murtagh and Vadhan\(^{[41]} \) for tighter composition bounds on arbitrary DP mechanisms.

C.2 Results of the Ablation Study

For the extended ablation study, we considered the centralized setting (only 1 user) and compared different algorithms and different values for the privacy parameter \( \delta \). The results are depicted in Figure 8 and display 5 algorithms: firstly, the differentially private Support Vector Machine with SGD-based training DP-SVM-SGD (cf. Example 3.16), secondly, the differentially private Softmax-activated single-layer perceptron with SGD-based training DP-Softmax-SLP-SGD (cf. Section 5.1), thirdly, a similar differentially private SVM but with SMO-based training which does not offer a guaranteed private convergence bound, fourthly, differentially private Stochastic Gradient descent (DP-SGD)\(^{[1]} \) applied on a 1-layer perceptron with the cross-entropy loss, and fifthly, approximate minima perturbation (AMP)\(^{[27]} \) which is based upon an SVM with objective perturbation. Note that, only DP-SVM-SMO, DP-SVM-SGD, and DP-Softmax-SLP-SGD have an output sensitivity and are thus suited for this efficient Distributed DP-Helmet scheme.

While all algorithms come close to the non-private baseline with rising privacy budgets \( \epsilon \), we observe that although DP-SGD performs best, DP-SVM-SMO and DP-Softmax-SLP-SGD come considerably close. DP-SVM-SGD has a disadvantage above DP-SVM-SMO of about a factor of 2, and AMP a disadvantage of about a factor of 4. We suspect that DP-SGD is able to outperform the variants other than DP-Softmax-SLP-SGD as it directly optimizes for the multi-class objective via the cross-entropy loss while others are only able to simulate it via the one-vs-rest (ovr) SVM training scheme. Additionally, DP-SGD has a noise-correcting property from its iterative noise application. The inherently multi-class DP-Softmax-SLP-SGD performs better than ovr-based DP-SVM-SGD indicating that a joint learning of all classes can boost performance. DP-Softmax-SLP-SGD additionally has a privacy advantage as it does not need to rely on sequential composition as it has an output sensitivity for all classes which is another factor that can lead to the boost of DP-Softmax-SLP-SGD above DP-SVM-SGD. Although DP-SVM-SMO also has an output sensitivity and renders better than DP-SVM-SGD, it does not offer a privacy guarantee when convergence is not reached. In the case of AMP, we have an inherent disadvantage of about a factor of 3 due to an unknown output distribution, and thus bad composition results in the multi-class SVM. Here, the privacy budget of AMP roughly scales linearly with the number of classes.

For DP-SGD, DP-SVM-SGD, DP-Softmax-SLP-SGD, and DP-SVM-SMO, Figure 8 shows that a smaller and considerably more secure privacy parameter \( \delta \ll \frac{1}{\lambda n} \) where \( n \) is the sum of the size of all local datasets is supported although
reflecting on the reported privacy budget \( \varepsilon \).

## D Pretraining Visualisation

Figure 9: 2-d projection of the CIFAR-10 dataset via t-SNE [47] with colored labels. Note that t-SNE is defined on the local neighborhood thus global patterns or structures may be arbitrary.

## E Proof of Lem. 5.1

We recall Lem. 5.1

**Lemma 5.1** (Sensitivity of Distributed DP-Helmet). For a configuration \( \zeta \) as in \[\text{Def. 3.8} \] Distributed DP-Helmet(\( \zeta \)) of Alg. 3 without noise, \( \text{avg}_{i}(n^{(i)} \cdot T(D^{(i)})) \), has a sensitivity of \( s' \cdot 1/w \) for each model if \( s = s' / n \).

**Proof.** Without loss of generality, we consider one arbitrary model which corresponds to one class \( k \). For Softmax-SLP-SGD and all classes \( k: K \) for SVM-SGD we have that \( T \) is an \( s \)-sensitivity bounded algorithm thus

\[
\begin{align*}
\max_{0 \leq i \leq \frac{n}{w} - 1} \max_{0 \leq \eta \leq \frac{n}{w} - 1} \max_{0 \leq r \leq w} \left| T(D^{(i)}_0, \xi, k, r) - T(D^{(i)}_1, \xi, k, r) \right| 
\end{align*}
\]

with \( D^{(i)}_0 \) and \( D^{(i)}_1 \) as 1-neighboring datasets and \( r \) as the randomness of \( T \). For instance, for \( T = \text{SVM-SGD} \) we have \( s = \frac{2(\Lambda R + \varepsilon)}{n \cdot w} \) (cf. Ex. 3.10) and for \( T = \text{Softmax-SLP-SGD} \) we have \( s = \frac{2(\Lambda R + \varepsilon)}{n \cdot w} \) (cf. Thm. 4.1) which fulfill the condition \( s \times \frac{1}{w} = \frac{s'}{n} \).

By Alg. 3 we take the average of multiple local models, i.e. \( \text{avg}_{i}(T(D^{(i)})) = \frac{1}{w} \sum_{i=1}^{w} n^{(i)} \cdot T(D^{(i)}), \xi, K, r) \). The challenge element – i.e. the element that differs between \( D^{(i)}_0 \) and \( D^{(i)}_1 \) – is only contained in one of the \( w \) models. By the application of the parallel composition theorem, we know that the sensitivity reduces to

\[
\begin{align*}
\max_{0 \leq i \leq \frac{n}{w} - 1} \max_{0 \leq \eta \leq \frac{n}{w} - 1} \max_{0 \leq r \leq w} \left| \frac{1}{w} \sum_{i=1}^{w} n^{(i)} \cdot T(D^{(i)}_0, \xi, k, r) - \frac{1}{w} \sum_{i=1}^{w} n^{(i)} \cdot T(D^{(i)}_1, \xi, k, r) \right| 
\end{align*}
\]

Hence, the constant \( n^{(i)}/w \) factor reduces the sensitivity by a factor of \( s'/w \).

## F Proof of Lem. 5.2

We recall Lem. 5.2

**Lemma 5.2.** For a configuration \( \zeta \) as in \[\text{Def. 3.8} \] and noise scale \( \tilde{\sigma} : \frac{1}{w} \sum_{i=1}^{w} \mathcal{N}(0,(\tilde{\sigma} \cdot 1/w)^2) = \mathcal{N}(0,(\tilde{\sigma} \cdot 1/w)^2) \).

**Proof.** We have to show that

\[
\frac{1}{w} \sum_{i=1}^{w} \mathcal{N}(0,(\tilde{\sigma} \cdot 1/w)^2) = \mathcal{N}(0,(\tilde{\sigma} \cdot 1/w)^2).
\]
It can be shown that the sum of normally distributed random variables behaves as follows: Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ two independent normally-distributed random variables, then their sum $Z = X + Y$ equals $Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ in the expectation.

Thus, in this case, we have
\[
\frac{1}{w} \sum_{i=1}^{w} \mathcal{N}(0, (\frac{1}{\sqrt{w}})^2) = \frac{1}{w} \mathcal{N}(0, (\frac{1}{\sqrt{w}})^2) = \frac{1}{w} \mathcal{N}(0, \delta^2).
\]

As the normal distribution belongs to the location-scale family, we get $\mathcal{N}(0, (\frac{1}{\sqrt{w}})^2)$. \qed

\section{Proof of Thm. 5.3}

We state the full version of Thm. 5.3:

\begin{definition}[Main Theorem, full] For a configuration $\zeta$ as in \textit{Def. 3.3} a maximum fraction of dropouts $\rho \in [0, 1]$, and a maximum fraction of corrupted clients $\gamma \in [0, 1]$. Assume that secure summation $\pi_{\text{SecSum}}$ exists as in \textit{Def. 3.5}.

Then Distributed DP-Helmet($\zeta$) (cf. \textit{Alg. 3}) satisfies computational $(\varepsilon, \delta + \nu_1)$-DP with $\delta(\varepsilon, \kappa_{\text{comp}})$ as in \textit{Def. 3.13} for $\nu_1 := (1 + \exp(\varepsilon)) \cdot \nu(\eta)$ and a function $\nu$ negligible in the security parameter $\eta$ used in $\pi_{\text{SecSum}}$.
\end{definition}

\begin{proof}
We first show $(\varepsilon, \delta)$-DP for a variant $M_1$ of Distributed DP-Helmet that uses the ideal summation protocol $F$ instead of $\pi_{\text{SecSum}}$. We conclude that for Distributed DP-Helmet (abbreviated as $M_2$) which uses the real secure summation protocol $\pi_{\text{SecSum}}$ for some negligible function $\nu_1 (\varepsilon, \delta + \nu_1)$-DP holds.

Recall that we assume at least $t \cdot w$ many honest users. As we solely rely on the honest $t \cdot w$ to contribute correctly distributed noise to the learning algorithm $T$, we have for each output model similar to \textit{Lem. 5.2}
\[
\frac{1}{w} \sum_{i=1}^{w} \mathcal{N}(0, (\frac{1}{\sqrt{w}})^2) = \frac{1}{w} \mathcal{N}(0, (\frac{1}{\sqrt{w}})^2) = \mathcal{N}(0, (\frac{1}{\sqrt{w}}^2)) = \mathcal{N}(0, (\frac{1}{\sqrt{w}}^2)).
\]

By \textit{Lem. 5.1} \textit{Lem. 5.2} and \textit{Lem. 3.14} we know that $M_1$ satisfies $(\varepsilon, \delta)$-DP (with the parameters as described above).

Considering an unbounded attacker $A$, we know that for any pair of neighboring data sets $D, D'$ the following holds
\[
\Pr[A(M_1(D)) = 1] \leq \exp(\varepsilon) \Pr[A(M_1(D')) = 1] + \delta
\]

If $\pi_{\text{SecSum}}$ is a secure summation protocol, there is a negligible function $\nu$ such that for any neighboring data sets $D, D'$ (differing in at most one element) the following holds w.l.o.g.: 
\[
\Pr[A(M_2(D)) = 1] - \nu(\eta) \leq \Pr[A(Sim_F(M_1(D))) = 1].
\]

For the attacker $A'$ that first applies $\text{Sim}$ and then $A$, we get:
\[
\Pr[A(M_2(D)) = 1] - \nu(\eta) \leq \exp(\varepsilon) \Pr[A(Sim_F(M_1(D'))) = 1] + \delta
\]
\[
\leq \exp(\varepsilon) \Pr[A(M_2(D')) = 1] + \nu(\eta) + \delta
\]

Thus we have
\[
\Pr[A(M_2(D)) = 1] \leq \exp(\varepsilon) \Pr[A(M_2(D')) = 1] + \delta + (1 + \exp(\varepsilon)) \cdot \nu(\eta).
\]

From a similar argumentation it follows that
\[
\Pr[A(M_2(D')) = 1] \leq \exp(\varepsilon) \Pr[A(M_2(D)) = 1] + \delta + (1 + \exp(\varepsilon)) \cdot \nu(\eta)
\]
holds.

Hence, with $\nu_1 := (1 + \exp(\varepsilon)) \cdot \nu(\eta)$ the mechanism Distributed DP-Helmet mechanism $M_2$ which uses $\pi_{\text{SecSum}}$ is $(\varepsilon, \delta + \nu_1)$-DP. As $\nu$ is negligible and $\varepsilon$ is constant, $\nu_1$ is negligible as well. \qed

\begin{corollary}
Given a configuration $\zeta$, a maximum fraction of dropouts $\rho \in [0, 1]$, and a maximum fraction of corrupted clients $\gamma \in [0, 1]$, if secure authentication encryption schemes and authenticated key agreement protocol exist, then Distributed DP-Helmet($\zeta$) (cf. \textit{Alg. 3}) instantiated with $\pi_{\text{SecAgg}} = \pi_{\text{SecSum}}$ \textit{Def. 3.5} satisfies computational $(\varepsilon, \delta + \nu_1)$-DP with $\delta(\varepsilon, \kappa_{\text{comp}})$ as in \textit{Def. 3.13} for $\nu_1 := (1 + \exp(\varepsilon)) \cdot \nu(\eta)$ and a function $\nu$ negligible in the security parameter $\eta$ used in secure summation.

This follows directly from Thm. 5.3 as by Thm. 3.6 we know that $\pi_{\text{SecAgg}}(s_1, \ldots, s_n)$ securely emulates $F$ (w.r.t. an unbounded attacker).
\end{corollary}
H Proof of Cor. 5.5

We recall Cor. 5.5

**Corollary 5.5** (User-level sensitivity). Given a learning algorithm $T$, we say that $T$ is $R$-norm bounded if for any $d$-dimensional dataset $D$ with $n = |D|$, hyperparameter $\xi$, and class $k \in \{1, \ldots, K\}$ or $k = K$, $\|T(D, \xi, k)\| \leq R$. Any $R$-norm bounded learning algorithm $T$ has a deterministic sensitivity $s = 2R$. In particular, $T(D, \xi, k) + N(0, \sigma^2, I_d, K)$ satisfies $(\Upsilon, \delta)$, $T$-group differential privacy with $\delta(\xi, K_{\text{comp}})$ as in Def. 3.13 and $T = n$.

**Proof.** We know that the deterministic sensitivity of the learning algorithm $T$ is defined as $s = \max_{D \sim \mathcal{D}} \|T(D, \xi, k) - T(D', \xi, k)\|$ for $\mathcal{Y}$-neighboring datasets $D, D'$. Thus, in our case we have $s = 2R$ since any $T(\xi, \xi, k) \in [-R, R]$. As this holds independent on the dataset and by Lem. 3.14 and by Lem. K.1 we can protect any arbitrary number of data points per user, i.e. we have $T$-group DP. \qed

I Non-interactive Blind Averaging

**Corollary I.1** (Averaged Representer theorem). Given a configuration $\zeta$, if on $D^{(i)}$ locally learning algorithm $T$ admits a solution of the form $T(D^{(i)}, \xi, \nu) = \sum_{j=1}^{\nu} \alpha^{(i)}_j x^{(i)}_j$ (cf. Thm. 3.7) then the average of these locally trained models $\text{avg}_i(T(D^{(i)}))$ admits a solution of the form $\text{avg}_i(T(\bar{\xi}, \nu, \zeta)) = \frac{1}{w} \sum_{i=1}^{w} \sum_{j=1}^{n} \alpha^{(i)}_j x^{(i)}_j$.

**Proof.**

\[ \text{avg}_i(T(D^{(i)})) = \frac{1}{w} \sum_{i=1}^{w} T(D^{(i)}) = \frac{1}{w} \sum_{i=1}^{w} \sum_{j=1}^{n} \alpha^{(i)}_j x^{(i)}_j. \]

\qed

I.1 Proof of Lem. 6.1

We recall Lem. 6.1

**Lemma 6.1** (Support Vectors of averaged SVM). Given a configuration $\zeta$ as in Def. 3.8 a locally trained model of hinge-loss linear SVM $T$, i.e. $T(D^{(i)}, \xi, \nu) = \text{argmin} \frac{1}{2} \sum_{(x,y) \in D^{(i)}} \max(0, 1 - yf(x)) + \lambda \|f\|^2 = f^{(i)}$, has the support vectors $V^{(i)} := \{(x, y) \in D^{(i)} \mid f^{(i)} \cdot x \leq \|f^{(i)}\|^{-1}\}$. Then, the average of these locally trained models $\text{avg}_i(f^{(i)})$ has the support vectors $V = \bigcup_{i=1}^{w} V^{(i)}$.

**Proof.** A learning problem that is based on a hinge-loss SVM fulfills the representer theorem requirements due to the L2-regularized ERM objective function. In fact, if a data point $x_j$ is a support vector, i.e. $x_j \in V$, then after successful training its corresponding $\alpha_j$ is restricted by $0 < \alpha_j \leq \Lambda \wedge y_j = 1$ or $0 > \alpha_j \geq -\Lambda \wedge y_j = -1$, or $\alpha_j = 0$ [30] Equation 28-30]. Thus, we denote $V = \{x_j \in D^{(i)} \mid \alpha^{(i)}_j \neq 0\}$. By Cor. I.1 we have that the average of locally trained models $\text{avg}_i(T(D^{(i)})) = \frac{1}{w} \sum_{i=1}^{w} \sum_{j=1}^{n} \alpha^{(i)}_j x^{(i)}_j$. Since the local datasets are disjoint we simplify $\frac{1}{w} \sum_{i=1}^{w} \sum_{j=1}^{n} \alpha^{(i)}_j x^{(i)}_j = \frac{1}{w} \sum_{j=1}^{\sum_{i=1}^{w} n} \alpha_j x_j$ for the combined local dataset $\bar{D} = \bigcup_{i=1}^{w} D^{(i)}$ and a flattened $\alpha = \begin{bmatrix} \alpha^{(1)}_1 & \ldots & \alpha^{(1)}_n \\ \alpha^{(2)}_1 & \ldots & \alpha^{(2)}_n \\ \vdots \\ \alpha^{(w)}_1 & \ldots & \alpha^{(w)}_n \end{bmatrix}$. A model which is represented by $\frac{1}{w} \sum_{j=1}^{\sum_{i=1}^{w} n} \alpha_j x_j$ has the support vectors $V = \{x_j \in \bar{D} \mid \alpha_j \neq 0\} = \bigcup_{i=1}^{w} V^{(i)}$, as the support vector characteristic is uniquely determined by $\alpha$ and each local $\alpha^{(i)}_j$ is element of $\alpha$ and responsible for the same data point. \qed

I.2 Proof of Thm. 6.2

We recall Thm. 6.2

**Theorem 6.2** (Averaged locally trained SVM converges to a global SVM). Given a configuration $\zeta$ as in Def. 3.8 and the same local data sizes $n^{(i)} = n^{(i)} \forall i, j$, there exists a regularization parameter $\Lambda$ such that the average of locally trained models $\text{avg}_i(T(D^{(i)}))$ with a hinge-loss linear SVM as an objective function $J$ trained with projected subgradient descent using weighted averaging (PGDWA), $T = \text{Hinge-SVM-PGDWA}$, converges with the number of local iterations $M$ to the best model for the combined local datasets $\bar{D}$, i.e. $\mathbb{E}[J(\text{avg}_i(\text{Hinge-SVM-PGDWA}(D^{(i)})), \bar{\nu}, \bar{\zeta})] - \inf_{J(f, \bar{\nu}, \bar{\zeta})} \mathbb{E}[J(f, \bar{\nu}, \bar{\zeta})] \in O(1/M)$.

**Proof.** First (1), we show that there exists a regularization parameter $\Lambda$ for which the converged global model equals the average of the converged locally trained models: $T(\bar{D}) = \text{avg}_i(T(D^{(i)}))$. Second (2), we show that both the global and the local models converge with rate $O(1/\sqrt{M})$.

Note that we assume that each data point $x_j$ is structured as $[x, x_j, 1, \ldots, x_{j,p}]$ to include the intercept. We also denote the flattened $\alpha^{(\text{avg}_i, \text{loc})} = \begin{bmatrix} \alpha^{(1)}_1 & \ldots & \alpha^{(n)}_1 \\ \alpha^{(1)}_2 & \ldots & \alpha^{(n)}_2 \\ \vdots \\ \alpha^{(1)}_n & \ldots & \alpha^{(n)}_n \end{bmatrix}$ as the dual coefficients of the averaged local SVM and $\alpha^{(\text{glob})}$ as the dual coefficients of the global SVM.

1. By Lem. 6.1 we know for the combined local datasets $\bar{D} = \bigcup_{i=1}^{w} D^{(i)}$ that

\[ \text{avg}_i(T(D^{(i)})) = \frac{1}{w} \sum_{j=1}^{\sum_{i=1}^{w} n} \alpha^{(\text{avg}_i, \text{loc})}_j x_j = \frac{1}{|\bar{D}|} \sum_{j=1}^{\sum_{i=1}^{w} n} \alpha^{(\text{loc})}_j x_j. \]
Note that we assume a scaled parameter per local SVM: \( T(D^{(i)}) = \frac{1}{n} \sum_{j=1}^{n} \alpha_j x_j \). Without this assumption, we would not average the local SVMs but instead compute their sum.

For the global model, we write by the representer theorem
\[
T(\mathbf{u}) = \frac{1}{|D|} \sum_{j=1}^{|D|} \alpha_j^{(\mathrm{glob})} x_j.
\]

Thus, by parameter comparison we have that \( T(\mathbf{u}) = \operatorname{avg}_r(\hat{T}(D^{(i)})) \) if \( \forall j, \alpha_j^{(\mathrm{glob})} = \alpha_j^{(\mathrm{avg}, \text{ loc})} \). By the characteristic of a hinge-loss linear SVM, we know that any \( \alpha_j \) has the value \( \alpha_j = N y_j \) if a data point is a support vector inside the margin \({\text{Equation 28-30}}\). Hence, \( \forall j, \alpha_j^{(\mathrm{glob})} = \alpha_j^{(\mathrm{avg}, \text{ loc})} \) if the margin is large enough that for both SVMs all data points are inside the margin. Since the margin of a hinge-loss linear SVM is the inverse of the parameter norm, \( \|T(\mathbf{u})\|^{-1} \), and the parameter norm gets smaller with an increased regularization parameter \( \Lambda \) by the definition of the objective function \( \frac{1}{2} \sum_{(x,y) \in D^{(i)}} \max(0, 1 - y f(x) + \Lambda \|f\|^2) \), we derive that there exists a regularization parameter \( \Lambda \) which is large enough s.t. all data points are within the margin.

(2) By Lacoste-Julien et al. \cite{wac}, we know that a hinge-loss linear SVM converges to the optima with rate \( O(M^{-1}) \), if we use projected subgradient descent using weighted averaging (PGDWA) as an optimization algorithm, i.e.
\[
\mathbb{E}[\mathcal{J}(\text{avg}_r(\text{Hinge-SVM-PGDWA}(D^{(i)})), \mathcal{U}_n)] - \inf_f \mathcal{J}(f, \mathcal{U}_n) \in O(1/n).
\]

\section{J Randomized Sensitivity}

\begin{lemma}
Let \( T_{\text{priv}} : (D, r, \kappa) \mapsto U \) be a randomized mechanism on dataset \( D \) with two independent randomnesses \( r \) and \( \kappa \) and universe \( U \). We define \( T_{\text{priv}}^r(D, \kappa) := T_{\text{priv}}(D, r, \kappa), \) i.e., \( T_{\text{priv}}^r : (D, \kappa) \mapsto U \). If \( \forall r, T_{\text{priv}}^r \) is \((\varepsilon, \delta)\)-DP, then \( T_{\text{priv}} \) is \((\varepsilon, \delta)\)-DP.

\begin{proof}
Let \( D, D' \) be neighboring datasets and \( S \subseteq U \) be defined over some universe \( U \) as required. Let \( R \) denote a distribution of randomeness \( r \) which is independent of the data as \( r \) and \( D \) are separate inputs. We show that if \( \forall r, \Pr_{T_{\text{priv}}^r}(D, r, \kappa) \in S \) \( \leq \exp(\varepsilon) \Pr_{T_{\text{priv}}^r}(D', r, \kappa) \in S \) + \( \delta \) then \( \Pr_{T_{\text{priv}}}(D, r, \kappa) \in S \) \( \leq \exp(\varepsilon) \Pr_{T_{\text{priv}}}(D', r, \kappa) \in S \) + \( \delta \). The proof is similar to that of Wu et al. \cite{wu} Lemma 5.

By the law of total probability, we have
\[
\Pr_{T_{\text{priv}}}(D, r, \kappa) \in S = \sum_r \Pr_{T_{\text{priv}}^r}(D, r, \kappa) \in S | R = r] = \sum_r \Pr_{T_{\text{priv}}^r}(D, r, \kappa) \in S] \leq \sum_r \Pr_{T_{\text{priv}}^r}(D, r, \kappa) \in S] + \delta = \exp(\varepsilon) \sum_r \Pr_{T_{\text{priv}}^r}(D', r, \kappa) \in S | R = r] + \sum_r \Pr_{T_{\text{priv}}^r}(D', r, \kappa) \in S] + \delta.
\]
\end{proof}
\end{lemma}

\begin{theorem}
Let \( T_{\text{priv}} : (D, r) \mapsto T(D, r) + \kappa \) be an additive mechanism with a Gaussian randomness \( \kappa \in \text{pdf}_{\mathcal{N}(0, \sigma^2)} \) and noise scale \( \sigma \) where \( T \) is a randomized mechanism with randomness \( r \) and dataset \( D \). \( T \) has a randomized sensitivity \( \max_{D, D'} \max_{\kappa} \|T(D, \kappa) - T(D', \kappa)\| \leq s \) where \( D, D' \) are 1-neighboring datasets. Then \( T_{\text{priv}} \) is \((\varepsilon, \delta)\)-DP.

\begin{proof}
Let \( R \) denote the distribution of randomeness \( r \) which by construction does not depend on data \( D \) or randomness \( \kappa \). We define \( T_{\text{priv}} : T(D, r) \mapsto T_{\text{priv}}(D, r, \kappa) \), i.e., \( T_{\text{priv}} : D \mapsto T_{\text{priv}}(D) \). We make a case distinction over each \( r \in R \).

For each \( r \in R \), we have the mechanism \( T_{\text{priv}}^r : D \mapsto T_{\text{priv}}^r(D) + \kappa \) with a deterministic sensitivity \( \max_{D, D'} \max_{\kappa} \|T(D, r) - T(D', r)\| \leq \max_{D, D'} \|T(D) - T(D')\| \), where \( D, D' \) are 1-neighboring datasets. By construction, \( T_{\text{priv}}^r \) is a Gauss mechanism which is \((\varepsilon, \delta)\)-DP by Lemma 3.14. By Lemma J.1, since \( T_{\text{priv}}^r \) is \((\varepsilon, \delta)\)-DP for all \( r, T_{\text{priv}} \) is \((\varepsilon, \delta)\)-DP.
\end{proof}
\end{theorem}

The same holds if we use the Gauss mechanism in the group privacy extension (cf. Lemma K.1) or in the distributed setting (cf. Lemma 5.3). In each case, we divide the algorithm output by a constant factor \( \text{const} \) which scales both the deterministic and the randomized sensitivity by \( \text{const} \):

\[
\begin{align*}
\max_{D, D'} \max_{\kappa} \frac{\|T(D, r) - T(D', r)\|}{\text{const}} &= \frac{1}{\text{const}} \max_{D, D'} \max_{\kappa} \|T(D, r) - T(D', r)\| = \frac{8_{\text{rand}}}{\text{const}} \\
\max_{D, D'} \max_{\kappa} \frac{\|T(D) - T(D')\|}{\text{const}} &= \frac{1}{\text{const}} \max_{D, D'} \max_{\kappa} \|T(D) - T(D')\| = \frac{8_{\text{det}}}{\text{const}}.
\end{align*}
\]
K  Group Privacy Reduction of Multivariate Gaussian

Lemma K.1. Let pdf_{\mathcal{N}(\lambda, \mu)}[x] denote the probability density function of the multivariate Gauss distribution with location and scale parameters \( \lambda, \mu \) which is evaluated on an atomic event \( x \). For any atomic event \( x \), any covariance matrix \( \Sigma \), any group size \( k \in \mathbb{N} \), and any mean \( \mu \), we get

\[
\frac{\text{pdf}_{\mathcal{N}(0,k^2\Sigma)}[x]}{\text{pdf}_{\mathcal{N}(\mu,k^2\Sigma)}[x]} = \frac{\text{pdf}_{\mathcal{N}(0,\Sigma)}[x/k]}{\text{pdf}_{\mathcal{N}(\mu/k,\Sigma)}[x/k]}
\]  

\text{(7)}

Proof. \[
\frac{\text{pdf}_{\mathcal{N}(0,k^2\Sigma)}[x]}{\text{pdf}_{\mathcal{N}(\mu,k^2\Sigma)}[x]} = \frac{\frac{1}{\sqrt{(2\pi)^d \det(k^2\Sigma)}} \exp\left(-\frac{1}{2} x^T k^2 \Sigma^{-1} x\right)}{\frac{1}{\sqrt{(2\pi)^d \det(k^2\Sigma)}} \exp\left(-\frac{1}{2} (x - \mu)^T k^2 \Sigma^{-1} (x - \mu)\right)} = \exp\left(-\frac{1}{2} k^2 (x - \mu)^T \Sigma^{-1} (x - \mu)\right)
\]

As the Gauss distribution belongs to the location-scale family, Lem. K.1 directly implies that the \((\varepsilon, \delta)\)-DP guarantees of using \(\mathcal{N}(0,k^2\Sigma)\) noise for sensitivity \(k\) and using \(\mathcal{N}(0,\Sigma)\) for sensitivity 1 are the same.

L  Representing Multivariate Gaussians as Univariate Gaussians

For completeness, we rephrase a proof that we first saw in Abadi et al. [1] that argues that sometimes the multivariate Gauss mechanism can be reduced to the univariate Gauss mechanism.

Lemma L.1. Let \text{pdf}_{\mathcal{N}(0,\text{diag}(x^2)))} denote the probability density function of a multivariate \((p \geq 1)\) spherical Gauss distribution with location and scale parameters \( \mu \in \mathbb{R}^p, \sigma \in \mathbb{R}^p \). Let \( M_{\text{gauss},p,q} \) be the \( p \) dimensional Gauss mechanism \( D \rightarrow q(D) + \mathcal{N}(0, \Sigma^2 \cdot I_p) \) for \( \Sigma^2 > 0 \) of a function \( q : \mathcal{D} \rightarrow \mathbb{R}^p \), where \( \mathcal{D} \) is the set of datasets. Then, for any \( p \geq 1, \) if \( q \) is \( s\)-sensitivity-bounded, then for every \( p \geq 1, \) there is another \( s\)-sensitivity-bounded function \( q' : \mathcal{D} \rightarrow \mathbb{R}^p \) such that the following holds: for all \( \varepsilon \geq 0, \delta \in [0,1] \) if \( M_{\text{gauss},1,q'} \) satisfies \((\varepsilon, \delta)\)-ADP, then \( M_{\text{gauss},p,q} \) satisfies \((\varepsilon, \delta)\)-ADP.

Proof. First observe that for any \( s\)-sensitivity-bounded function \( q'' \), two adjacent inputs \( D, D' \) (differing in one element) with \( \|q''(D) - q''(D')\|_2 = s \) are worst-case inputs. As a spherical Gauss distribution (covariance matrix \( \Sigma = \sigma^2 \cdot I_p \)) is rotation invariant, there is a rotation such that the difference only occurs in one dimension and has length \( s \). Hence, it suffices to analyze a univariate Gauss distribution with sensitivity \( s \). Hence, the privacy loss distribution of both mechanisms (for the worst-case inputs) is the same. As a result, for all \( \varepsilon \geq 0, \delta \in [0,1] \) (i.e. the privacy profile is the same) if \((\varepsilon, \delta)\)-ADP holds for the univariate Gauss mechanism it also holds for the multivariate Gauss mechanism.

M  Strong Convexity of CE Loss

Theorem M.1. Let \( J \) denote the objective function \( J(f, D) := \frac{1}{2} \sum_{k=1}^K (f^T f)_k + \frac{1}{n} \sum_{(x, y) \in D} \mathcal{L}_{\text{CE}}(y, f^T x) \) with the cross-entropy loss \( \mathcal{L}_{\text{CE}}(y, f^T x) := -\sum_{k=1}^K y_k \log \sum_{x=1}^n \exp(x_k) \) and parameters \( f \in \mathbb{R}^{d+1} \), dataset \( D \) where \( (x, y) \in D \) with data points \( x \in \mathbb{R}^{d+1} \) structured as \( [1 \ x_1 \ldots \ x_d] \) and labels \( y \in \{0,1\}^K \), number of classes \( K \), and regularization parameter \( \lambda \). \( J \) is \( \lambda \)-strongly convex.
Proof. \( \mathcal{J} \) is \( \mu \)-strongly convex if \( \mathcal{J} - \frac{\mu}{2} \| f \| \) is convex. In our case, with \( \mu = \Lambda \), it remains to be shown that the cross-entropy loss \( \mathcal{L}_{CE}(y, z) \) is convex since a linear layer like \( f^T x \) represents an affine map which preserves convexity [9].

It is known that the cross-entropy loss is convex by a simple argumentation: If the Hessian is positive semi-definite \( \nabla^2 \mathcal{L}_{CE}(y, z) \geq 0 \) then \( \mathcal{L}_{CE} \) is convex. By the Gershgorin circle theorem, a symmetric diagonally dominant matrix is positive semi-definite if the diagonals are non-negative.

Since the second derivative of the cross-entropy loss is \( \frac{\partial^2}{\partial y^2} \mathcal{L}_{CE} = s_p(1 - s_q) \) for the softmax probabilities \( s_p = \frac{\exp(x_p)}{\sum_{j=1}^K \exp(x_j)} \), we conclude that the diagonals are non-negative since \( s_p(1 - s_p) \) for \( 0 \leq s_p \leq 1 \) is always non-negative. The Hessian is diagonally dominant if for every row \( p \) the absolute value of the diagonal entry is larger or equal to the sum of the absolute values of all other row entries. In our case, we have

\[
\forall_p \ |s_p(1 - s_q)| \geq \sum_{q=1, q \neq p}^K |s_p - s_q| \iff \forall_p (1 - s_q) \geq \sum_{q=1, q \neq p}^K s_q \iff \forall_p (1 - s_q) \geq (1 - s_p).
\]

\( \square \)

\( \text{N Lipschitzness of CE Loss} \)

**Theorem N.1.** Let \( \mathcal{J} \) denote the objective function \( \mathcal{J}(f, D) := \frac{1}{2} \sum_{k=1}^K (f^T f)_k + \frac{1}{2} \sum_{(x, y) \in D} L_{CE}(y, f^T x) \) with the cross-entropy loss \( \mathcal{L}_{CE}(y, z) := - \sum_{k=1}^K y_k \log \frac{\exp(x_k)}{\sum_{j=1}^K \exp(x_j)} \) and parameters \( f \in \mathbb{R}^{d+1 \times K} \), dataset \( D \) where \((x, y) \in D \) with data points \( x \in \mathbb{R}^{d+1} \) structured as \( [1 \ x_1 \ \cdots \ x_d] \) and labels \( y \in \{0, 1\}^K \), number of classes \( K \), and regularization parameter \( \Lambda \). \( \mathcal{J} \) is \( L \)-Lipschitz with \( L = \Lambda R + \sqrt{2}\epsilon \) where \( \|x\| \leq c \) and \( \|f\| \leq R \).

**Proof.** In the following, we abbreviate \( d' := d + 1 \), flatten \( f \in \mathbb{R}^{d'K} \) and note \( z := (x, y) \).

The Lipschitz continuity is defined as:

\[
\sup_{z \in D, f, f'} \frac{\| \mathcal{J}((f, z)) - \mathcal{J}((f', z)) \|}{\| f - f' \|} \leq L.
\]

We first (1) show

\[
\sup_{z \in D, f, f'} \frac{\| \mathcal{J}((f, z)) - \mathcal{J}((f', z)) \|}{\| f - f' \|} \leq \sup_{z \in D, f} \| \nabla_f \mathcal{J}(f, z) \|
\]

using the mean value theorem and subsequently (2) bound \( \sup_{z \in D, f} \| \nabla_f \mathcal{J}(f, z) \| \leq L \).

(1) Recall that the multivariate mean value theorem states that for some function \( g : G \rightarrow \mathbb{R} \) on an open subset \( G \subseteq \mathbb{R}^n \), some \( x, y \in G \) and some \( c \in [0, 1] \), we have

\[
g(y) - g(x) = (\nabla g((1-c)x + cy), y - x).
\]

In our case, we write

\[
\sup_{z \in D, f, f'} \frac{\| \mathcal{J}((f, z)) - \mathcal{J}((f', z)) \|}{\| f - f' \|} \leq \sup_{z \in D, f} \| \nabla_f \mathcal{J}(f, z) \|
\]

by the multivariate mean value theorem for some \( c \in [0, 1] \)

\[
= \sup_{z \in D, f, f'} (\| \nabla_f ((1-c)f' - cf, z) \|)
\]

for \( f'' := (1-c)f' - cf \) and by the Cauchy-Schwarz inequality \( \| \nabla_f \mathcal{J}(f'', z) \| \cdot \| f' - f'' \| \leq \sup_{z \in D, f, f'} \| \nabla_f \mathcal{J}(f'', z) \| \).

(2) We know that for \( 1 \leq j \leq d', 1 \leq p \leq K \) the partial derivative of \( \mathcal{J} \) is \( \frac{\partial}{\partial y^{(j, p)}} \mathcal{J}(f, (x, y)) = \Lambda f_{j, p} + x_l (s_p - 1_{y_{(p, l)}}) \) with \( s_p := \frac{\exp(x_p)}{\sum_{j=1}^K \exp(f_{j, p})} \). Thus, we have

\[
\| \nabla_f \mathcal{J}(f, z) \| = \sqrt{\sum_{l=1}^{d'} \sum_{p=1}^{K} \left( \Lambda f_{j, p} + x_l (s_p - 1_{y_{(p, l)}}) \right)^2}
\]

\[
= \sqrt{\sum_{l=1}^{d'} \sum_{p=1}^{K} \left( \Lambda^2 f_{j, p}^2 + 2 \Lambda f_{j, p} x_l (s_p - 1_{y_{(p, l)}}) + x_l^2 (s_p - 1_{y_{(p, l)}})^2 \right)}
\]

\[
= \sqrt{\Lambda^2 \| f \|^2 + 2 \Lambda \sum_{l=1}^{d'} x_l \sum_{p=1}^{K} f_{j, p} (s_p - 1_{y_{(p, l)}}) + \sum_{l=1}^{d'} x_l^2 \sum_{p=1}^{K} (s_p - 1_{y_{(p, l)}})^2}
\]
due to the Cauchy-Schwarz inequality, we have $\sum_{p=1}^{K} f_{lp}(s_p - 1_{y=p}) \leq \sqrt{\sum_{p=1}^{K} f_{lp}^2} \sqrt{\sum_{p=1}^{K} (s_p - 1_{y=p})^2}$ and

$$\sum_{l=1}^{d} x_l \sqrt{\sum_{p=1}^{K} f_{lp}^2} \leq \sqrt{\sum_{l=1}^{d'} x_l^2} \sqrt{\sum_{l=1}^{d'} f_{lp}^2} = \|x\| \|f\|$$

$$\leq \sqrt{\Lambda^2 \|f\|^2 + 2 \Lambda \|x\| \|f\|} \left(\sum_{p=1}^{K} (s_p - 1_{y=p})^2 + \left(\sum_{p=1}^{d'} x_p^2\right) \left(\sum_{p=1}^{d'} (s_p - 1_{y=p})^2\right)\right)$$

since $\max_{s_1, \ldots, s_K} \left\{ (s_p - 1)^2 + \sum_{q=1, q \neq p}^{K} s_q^2 \right\} = 1 \wedge s_k \geq 0 \forall k$, $s_q \neq 0$ for $q \neq p$

$$\leq \sqrt{\Lambda^2 \|f\|^2 + 2 \sqrt{2} \Lambda \|x\| \|f\| + 2 \|x\|^2} = \Lambda \|f\| + \sqrt{2} \|x\|$$

Thus, with $\|x\| \leq c$, $\|f\| \leq R$ we conclude that

$$\sup_{z \in D, f, f'} \frac{\|J(f, z) - J(f', z)\|}{\|f-f'\|} \leq \sup_{z \in D, f} \|\nabla J(f, z)\| \leq LR + \sqrt{2c} = L$$

\[\square\]

O Smoothness of CE Loss

**Theorem O.1.** Let $J$ denote the objective function $J(f, D) := \frac{1}{n} \sum_{y_k} \log \frac{\exp(x_k)}{\sum_{y \in \{0, 1\}^K} \exp(y_k)}$ and parameters $f \in \mathbb{R}^{d+1 \times K}$, dataset $D$ where $(x, y) \in D$ with data points $x \in \mathbb{R}^{d+1}$ structured as $[1 \ x_1 \ x_2 \ \cdots \ x_d]$ and labels $y \in \{0, 1\}^K$, number of classes $K$, and regularization parameter $\Lambda$. $J$ is smooth with $\beta = \sqrt{(d + 1)K^2 \Lambda^2 + 0.5(A^2 + c^2)}$ where $\|x\| \leq c$.

**Proof.** In the following, we abbreviate $d' := d + 1$, flatten $f \in \mathbb{R}^{d' \times K}$ and notate $z := (x, y)$.

\[\beta\]-Smoothness is defined as:

$$\sup_{z \in D, f, f'} \frac{\|\nabla f J(f, z) - \nabla f' J(f', z)\|}{\|f-f'\|} \leq \beta.$$  

We first (1) show

$$\sup_{z \in D, f, f'} \frac{\|\nabla f J(f, z) - \nabla f' J(f', z)\|}{\|f-f'\|} \leq \sup_{z \in D, f} \|\nabla f J(f, z)\|$$

using the mean value theorem and subsequently (2) bound $\sup_{z \in D, f} \|\nabla f J(f, z)\| \leq \beta$.

(1) Recall that the multivariate mean value theorem states that for some function $g : G \to \mathbb{R}$ on an open subset $G \subset \mathbb{R}^n$, some $x, y \in G$ and some $c \in [0, 1]$, we have

$$g(y) - g(x) = \langle \nabla g((1 - c)x + cy), y - x \rangle.$$

In our case, write

$$\sup_{z \in D, f, f'} \frac{\|\nabla f J(f, z) - \nabla f' J(f', z)\|}{\|f-f'\|} = \sup_{z \in D, f, f'} \frac{\left(\sum_{i=1}^{d'} \sum_{j=0}^{K} \left(\nabla g_{i,j}((1-c)f - cf, s)\right)^2\right)^{\frac{1}{2}}}{\|f-f'\|}$$

by the multivariate mean value theorem for some $c \in [0, 1]$ and $g_{i,j}(f, z) := \nabla f_{i,j} J(f, z)$

$$= \sup_{z \in D, f, f'} \frac{\left(\sum_{i=1}^{d'} \sum_{j=0}^{K} \left(\nabla g_{i,j}((1-c)f - cf, s)\right)^2\right)^{\frac{1}{2}}}{\|f-f'\|}$$

for $f'' := (1 - c)f - cf$ and by the Cauchy-Schwarz inequality $\|\langle \nabla g_i, f, s \rangle - f'' \| \leq \|\nabla g_i, f, s \| \|f''\| \leq \sup_{z \in D, f''} \|\nabla f'' J(f'', z)\| \leq \sup_{z \in D, f''} \|\nabla f'' J(f'', z)\|$.

(2) We know that with $1 \leq l \leq d', 1 \leq p \leq K$ the first-order partial derivative of $J$ is $\frac{\partial}{\partial f_{lp}} J(f, (x,y)) = \Lambda f_{lp} + x_l \cdot \exp(f_{lp} x_l) / \sum_{i=1}^{K} \exp(f_{i,p} x_i)$. 

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With $1 \leq j \leq d', 1 \leq q \leq K$ we know that the second-order partial derivative of $J$ is \( \frac{\partial^2}{\partial x_{pq} \partial x_{pq}} J(f, (x, y)) = 1_{[p=q]} \cdot \Lambda + x_i \cdot x_j \cdot s_p(1_{[p=q]} - s_q) \). Thus, we have
\[
\|H_f(J(f, z))\| = \sqrt{\sum_{l=1}^{d'} \sum_{q=1}^{K} \left( 1_{[p=jq]} \cdot \Lambda + x_i \cdot x_j \cdot s_p(1_{[p=q]} - s_q) \right)^2}
\]
\[
= d'K \sum_{l=1}^{d'} \left( \Lambda + x_i^2 s_p(1 - s_p) \right)^2 + \sum_{j=1}^{d'} x_i^2 x_j^2 s_p^2 (1 - s_p)^2 + \sum_{k=1}^{K} x_k^2 s_p^2 \sum_{j=1}^{d'} \alpha_j
\]

Since we have $\max_{s_1, \ldots, s_K} \left\{ \sum_{q=1}^{K} s_q^2 \right\}$ and $\sum_{q=1}^{K} s_q = 1 - s_p \land s_i \geq 0 \forall i$ we conclude
\[
= d'K \Lambda^2 + \sum_{j=1}^{d'} x_j^2 s_p(1 - s_p) \left( 2 \Lambda + x_j^2 s_p(1 - s_p) + 2 s_p(1 - s_p) \sum_{j=1}^{d'} x_j \right)
\]

Thus, with $\|x\| \leq c$ we conclude that
\[
\sup_{z \in D, f'} \frac{\|\nabla_f J(f, z) - \nabla_f J(f', z)\|}{\|f - f'\|} \leq \frac{\|H_f(J(f, z))\|}{\sqrt{d'K}} \leq \sqrt{d'K} \Lambda^2 + 0.5(\Lambda + c)^2 = \beta.
\]

Lemma O.2. Let \( s_p \}_{p=1}^K \) denote probabilities such that $\sum_{p=1}^K s_p = 1$, and $C \in \mathbb{R}_+$ a constant, then we have
\[
\max_{s_p} \left\{ \sum_{p=1}^K s_p (1 - s_p) (C + s_p) \mid \sum_{p=1}^K s_p = 1 \land s_i \geq 0 \forall i \right\} \leq 0.25(C + 1)^2
\]

with $(s_p = 1/2 \land s_q = 0) \forall p \in \{1, \ldots, K\}, s_p \in \{0, 1\}$, i.e. for some arbitrary but fixed dimensions $k : 1 \leq k \leq K$, the solution has $k$-times $s_p = 1/2$ and $(K - k)$-times $s_p = 0$.

Proof. We show this Lemma as follows: First, we use the Karush–Kuhn–Tucker (KKT) conditions to find the $s_p$’s which maximize the maximization term. Thereby, we obtain a set of four solution candidates where we encode all $s_p$’s in closed form and introduce two new variables $k, j$ which serve as a solution counter. Second, we insert the solution candidates into the maximization term and show that the result is always bounded by $0.25(C + 1)^2$ by calculating the optimal front across all possible values of the solution counters $k, j$.

Let $f(s) := \sum_{p=1}^K s_p (1 - s_p) (C + s_p)$ denote the function to maximize, $h(s) := \sum_{p=1}^K s_p - 1$ the equality constraint, and $g_p(s) := -s_p, \forall p$ the inequality constraints. To find the constrained maximum, we maximize the Lagrangian function $L_{\text{Lagrangian}}(s) = f(s) + \mu g_p(s) + \lambda h(s)$ with $\mu, \lambda$ as slack variables. This suffices since $s_p$ does not have unbounded border cases: the only valid configuration of all $s_p$’s is on a hyperplane (\( \sum_{p=1}^K s_p = 1 \)) bounded in all dimensions ($s_p \geq 0$). Using the slack variable $\mu$, we already cover whether its corresponding $s_p$ is on the border ($\mu > 0$) or not ($\mu = 0$). Following the KKT conditions, the following conditions have to hold for the maximum:

1. Stationarity: $\nabla_s L_{\text{Lagrangian}}(s) = C + 2s_p - 2C s_p - 3s_p^2 + \mu - \lambda = 0, \forall p$

2. Primal feasibility: $h(s) = 0$ and $g_p(s) \leq 0, \forall p$
(3) Dual feasibility: $\mu_p \geq 0, \forall p$

(4) Complementary slackness: $\mu_p g_p(s) = 0, \forall p$

Informally, it suffices for the solution of the KKT conditions to analyze the cases where $s_p > 0, \forall 1 \leq p \leq k$ for all fixed number of dimensions $k : 1 \leq k \leq K$ since if $s_p = 0$ then we have already proved the same result for one less dimension.

Formally and without loss of generality, we show for all fixed numbers of dimensions $k : 1 \leq k \leq K$ that for the solution of the KKT conditions it suffices to analyze the cases where $s_p > 0, \forall 1 \leq p \leq k$. For the induction base case ($k = 1$ dimensional), we have $s_1 > 0$ and thus by condition (4) $\mu_1 = 0$. If and only if $s_1 = 1$, we satisfy conditions (2) and (1) with $\lambda = -C-1$. With $s_1 = 0$ we would not be able to satisfy the equality constraint of condition (2), i.e. $\lambda = 1$.

For the $k \mapsto k+1$ induction case, we know that $s_p > 0, \forall 1 \leq p \leq k$. If $s_{p+1} > 0$, by the induction hypothesis we know that $\forall 1 \leq p \leq k+1$, $s_p = 0$. If $s_{p+1} = 0$ then by conditions (3) and (4) we have $\mu_{p+1} > 0$ and thus by condition (1), $\mu_{p+1} = \lambda - C$. Inserting $s_{p+1} = 0$, $\mu_{p+1} = \lambda - C$ into conditions (1) to (4), we obtain the same set of equations and inequalities as for the $k$-dimensional case which already holds by the induction hypothesis.

We solve the KKT conditions (1) to (4) as follows: First, we solve the system of equations of condition (1) for $s_p$ via the quadratic formula:

$$s_p^\pm = \frac{-2(2C) \pm \sqrt{(2(2C))^2 - 4(-3)(C-C_j)}}{2(C-C_j)} = 1/3 \cdot (\pm \sqrt{C^2 + 3C - 1} + C + 1).$$ (16)

Second, we plug $s_p^\pm$ into the equality constraint ‘$h(s) = 0$’ of condition (2) and solve for $\lambda$ which gives us for some solution counter $j \in \mathbb{N}, 0 \leq j \leq k$ with $2j \neq k$:

$$h(s^\pm) = 0 \iff (\sum_{i=1}^{j} s_i^+) + (\sum_{i=j+1}^{k} s_i^-) = 1 \iff j \cdot (C^2 + C - 3\lambda + 1 + 1 \cdot (k-j) - (C^2 - 3\lambda + 1 + 1) = 3 \iff (2j-k) \cdot (C^2 - 3\lambda + 1 = Ck - k + 3 \Rightarrow C^2 + C - 3\lambda + 1 = \frac{(Ck-k^2)^2}{(2j-k)^2} \iff \lambda = \frac{(2j-k)^2(C^2 - Cj + 1 - (Ck-k^2)^2)}{2(C-j)^2}.$$

The solution counter $j$ quantifies how often we plug the ‘positive’ variant of $s_p^\pm$ into $h(s^\pm)$:

$$s^\pm := [s_1^+ \ldots s_j^+ s_{j+1}^- \ldots s_k^-]$$

or any permutation of the dimensions of $s^\pm$.

Note that at $2j = k$, we have a special case and by the equality constraint ‘$h(s) = 0$’ of condition (2)

$$h(s^\pm) = 0 \land 2j = k \iff (\sum_{p=1}^{j} s_p^+) + (\sum_{p=j+1}^{k} s_p^-) = 1 \iff k(1-C) = 3 \iff C = \frac{-3}{k}.$$

Thus, at $2j = k$, $C = \frac{-3}{k}$ we simplify the solution in Eq. (16) to

$$s_k^\pm \cdot C = -3/3 = 1/3 \cdot (\pm \sqrt{1-3k} + 1 = \mp Q + \frac{1}{k}.$$ If we now insert $s_k^\pm, C = -3/3$ into $f(\cdot)$ and maximize for all remaining variables, we find the maximum at

$$\max_{k,j,\lambda} \{ f(s_k^\pm, C = -3/3) \mid 2j = k \land C = \frac{-3}{k} \land s_k^\pm, C = -3/3 \geq 0 \}$$

$$\leq \max_{k,j,\lambda} \{ \sum_{p=1}^{j} (\frac{1}{k} + Q)(1 - (\frac{1}{k} + Q)) (C + (\frac{1}{k} + Q)) + \sum_{p=j+1}^{k} (\frac{1}{k} - Q)(1 - (\frac{1}{k} - Q)) (C + (\frac{1}{k} - Q)) \mid 2j = k \land C = \frac{-3}{k}\}$$

$$= \max_{j} \{ \frac{1}{k} (1 - (\frac{1}{k} + 1)) \} = \max_{j} \{ \frac{1}{k} - \frac{1}{k+1} \} = \max_{j} \{ \frac{1}{k} - \frac{1}{k+1} \} = \max_{j} \{ \frac{1}{k} - \frac{1}{k+1} \} \leq 0.25(\frac{1}{j+1} + 1) = 1 - \frac{1}{j+1} + \frac{1}{2k^2}.$$

Thus, at $2j = k$, $\mathcal{L}_{\text{agrange}}$ is maximal at $C = \frac{-3}{k}$ which is always strictly below the maximum we will show in this lemma if $C = \frac{-3}{k}$. In the following, we continue the proof for $2j \neq k$.

Third, by plugging $\lambda$ into Eq. (16) which is derived from the system of equations in condition (1) and solving for $s_p$ we obtain the following two solution candidates for $2j \neq k$

$$s_p^{(\pm)} = \frac{1}{3} \left( \pm \sqrt{C^2 + C - 3 \cdot (\frac{2(j-k)^2(C^2 - Cj + 1 - (Ck-k^2)^2)}{2(j-k)^2} + 1 - C + 1) \right)$$

$$= \frac{1}{3} \left( 1 - C \pm \frac{C(k-k^2)}{2(j-k)^2} \right) = \frac{2(j-k)(1-C) \pm (Ck-k^2)}{(2(j-k)^2)} = \frac{2Cj + 2j - 3 \pm (Ck-k^2)}{2(j-k)^2} = \frac{2(j-k) - 3 \pm (Ck-k^2)}{2(j-k)^2}.$$

The same argumentation holds for situations where the dimensions are permuted.
Observe that if we replace $\tilde{j} := k - j$ in $s_p^{(i)}$ we get $s_p^{(-)}$ with $\tilde{j}$ instead of $j$. To abbreviate, we write
\[ s_p^{(j')} = \frac{-2k'c + 2j' - 3}{6j' - 3k} \]
for $j' \in \{j, k - j\}$. Because of the similar structure of $s_p^{(j')}$ and $s_p^{(k-j)}$, restricting $j$ by $0 \leq 2j < k$ suffices since we would otherwise count the same maximum twice. With $s_p^{(j')}$ as our solution candidate, the equality constraint ‘$h(s) = 0$’ in condition (2) holds when we have $(k - j)$ times $s_p^{(j')}$ and $j$ times $s_p^{(k-j)}$:
\[ s_{sol} := \begin{bmatrix} s_1^{(k-j)} & \ldots & s_{j}^{(k-j)} & s_{j+1}^{(j')} & \ldots & s_k^{(j')} \end{bmatrix} \]
or any permutations of the dimensions of $s_{sol}$. This goes by construction of $s^\perp$ where the solution counter $j$ quantifies how often we plug in $s_p^{(j)}$ into $h(s^{(i)}, \ldots)$.

We next compute the second partial derivative test to determine for which parameters the solution candidate $s_{sol}$ is a local maximum or minimum: We have a maximum if the Hessian of $\mathcal{L}_{\text{agrange}}$ is positive definite and a minimum if the Hessian of $\mathcal{L}_{\text{agrange}}$ is negative definite. In our case, the second partial derivatives of $\mathcal{L}_{\text{agrange}}$ are $\nabla^2 s_p \mathcal{L}_{\text{agrange}}(s) = 2 - 2k + 6s_p$ and $\nabla^2 s_q \mathcal{L}_{\text{agrange}}(s) = 0$ with $p \neq q$. Thus, we have a diagonal Hessian matrix. Hence, if $2 - 2k - 6s_{sol} < 0$ we have a maximum and if $2 - 2k - 6s_{sol} > 0$ we have a minimum. Because of the second partial derivative test, we also know that if the Hessian has both positive and negative eigenvalues then we have a saddle point. This holds in our case when we have both positive and negative values on the diagonals of the Hessian, i.e. for some $p$ we have $2 - 2k - 6s_p < 0$ and for some $q$ we have $2 - 2k - 6s_q > 0$. Furthermore, if we have a zero eigenvalue this test is indecisive.

We rearrange the maximum condition for any entry of $s_{sol}$ (here: $s_p^{(j')}$) as follows:
\[
2 - 2k - 6s_p^{(k-j)} < 0 \iff \begin{cases} kC - k + 3 > 0 & \text{if } 0 \leq 2j' < k \\ kC - k + 3 < 0 & \text{if } 2j' > k \\
C > \frac{k}{3} - \frac{2}{3} & \text{if } 0 \leq 2j' < k \\ C < \frac{k}{3} - \frac{2}{3} & \text{if } 2j' > k 
\end{cases}
\]
Similarly, we rearrange the minimum condition, such that
\[
2 - 2k - 6s_p^{(k-j)} > 0 \iff \begin{cases} C < \frac{k}{3} - \frac{2}{3} & \text{if } 0 \leq 2j' < k \\ C > \frac{k}{3} - \frac{2}{3} & \text{if } 2j' > k 
\end{cases}
\]
Recall that at this point we only consider $2j \neq k$. We now distinguish three cases for the second partial derivative test for the vector $s_{sol}$: $C < \frac{k}{3}, C > \frac{k}{3}$, $C = \frac{k}{3}$.

At $C < \frac{k}{3}$, we write
\[
\begin{align*}
2 - 2k - 6s_1^{(k-j)} &< 0 \\
\ldots \\
2 - 2k - 6s_{j}^{(k-j)} &< 0 \\
2 - 2k - 6s_{j+1}^{(j')} &> 0 \\
\ldots \\
2 - 2k - 6s_k^{(j')} &> 0
\end{align*}
\]
and at $C > \frac{k}{3}$, we write similarly
\[
\begin{align*}
2 - 2k - 6s_1^{(k-j)} &> 0 \\
\ldots \\
2 - 2k - 6s_{j}^{(k-j)} &> 0 \\
2 - 2k - 6s_{j+1}^{(j')} &< 0 \\
\ldots \\
2 - 2k - 6s_k^{(j')} &< 0
\end{align*}
\]
Recall the saddle point criteria as $\exists_k \exists_\delta \{2 - 2k - 6s_{sol} < 0 \wedge 2 - 2k - 6s_{sol} > 0\}$ and the maximum criteria as $2 - 2k - 6s_{sol} < 0$. By the above test criteria, for $C \neq \frac{k}{3}$, we have a saddle point for all $j \in [1, k - 1]$ as well as a maximum for $j = k \wedge C < \frac{k}{3}$ and for $j = 0 \wedge C > \frac{k}{3}$ at
\[
s_{max} := \begin{bmatrix} s_1^{(k-k)} & \ldots & s_k^{(k-k)} \\ s_1^{(0)} & \ldots & s_k^{(0)} \end{bmatrix} = \begin{bmatrix} 1/k & \ldots & 1/k \end{bmatrix} \wedge C \neq \frac{k}{3}
\]
since only at $j \in \{0, k\}$ do we have the case that either $s_p^{(j)}$ or $s_p^{(k-j)}$ is present in the solution $s_{sol}$.

At $C = \frac{k}{3}$, we have for any entry of $s_{sol}$ (here: $s_p^{(j')}$)
\[
s_p^{(j', C= k-j/3)} = \frac{-2(k - 3)j'/k + 2j' - 3}{6j' - 3k} = \frac{6j'/k - 3}{6j' - 3k} = \frac{1}{k}
\]
Thus, although the second partial derivative test is indecisive since $2 - 2k - 6j/k = 0$, we have at $C = \frac{k}{3}$ always the same solution as in $s_{max}$. This renders $s_{max}$ for all $C$ as the maximal solution.
Next, we plug the solution $s^{\text{max}}$ into $f(s)$ and calculate the optimal front with the inequality constraint ‘$g_p(s) \leq 0$’ of condition (2) and across all number of dimensions $k$ and range of the solution counter $j \in \{0, k\}$:

$$\max_{s,j} \left\{ f(s^{\text{max}}) \mid s_p^{(j)} \geq 0 \wedge s_p^{(k-j)} \geq 0 \wedge j \in \{0, k\} \right\}$$

$$= \max_k \left\{ \sum_{p=1}^{k} s_p^{(0)}(1 - s_p^{(0)})(C + s_p^{(0)}) \mid s_p^{(0)} \geq 0 \right\}$$

$$= \max_k \left\{ \sum_{p=1}^{k} \frac{1}{k}(1 - \frac{1}{k})(C + \frac{1}{k}) \mid \frac{1}{k} \geq 0 \right\}$$

$$= \max_k \left\{ C + \frac{1-C}{k} - \frac{1}{k} \right\}$$

(for $k = \frac{2}{1-C}$, the term $C + \frac{1-C}{k} - \frac{1}{k}$ is maximal for which we need the derivative to be zero: $\frac{d}{dk}(C + \frac{1-C}{k} - \frac{1}{k}) = \frac{C - 1}{k^2} + \frac{1}{k} = 0$)

$$= C + \frac{1}{k}(1-C)^2 - \frac{1}{k}(1-C)^2$$

$$= \frac{C^2}{k^2} + \frac{C}{k} + \frac{1}{k} = 0.25(C + 1)^2$$

Thus, we conclude that $f(s^{\text{max}})$ is equal to or below the convex hull $0.25(C + 1)^2$ for any solution counter $j$ and any number of classes $k$.

Note: In this proof, we assumed $k \in \mathbb{R}_+$, however, we can restrict the number of classes $k$ even further: $k \in \mathbb{N}$ and $k \leq K$. Yet, this restriction does not have much impact on the bound on $f$ for a reasonable $C, K$. Now, we only have $K$ possible maxima ($s_p^{\text{max}} = \{1/2, \ldots, 1/k\}$) where for a given $C$ only one of these maxima are dominant. This also means that our $0.25(C + 1)^2$-bound is a convex hull and only matches the maxima in a few selected points. However, already for little $K$ does the maximum come considerably close to the hull as shown in Fig. 10.

Figure 10: Precise maximum of $f(s^{\text{max}})$ per constant $C$ and restricted, discretized number of classes $k \leq K, k \in \mathbb{N}$ versus convex hull of the maximum of $f(s^{\text{max}})$ across all number of classes $k \in \mathbb{R}_+$. 27