Optimal reinsurance for risk over surplus ratios

Erik Bølviken and Yinzhi Wang
Department of Mathematics
University of Oslo

December 10, 2019
Abstract

Optimal reinsurance when Value at Risk and expected surplus is balanced through their ratio is studied, and it is demonstrated how results for risk-adjusted surplus can be utilized. Simplifications for large portfolios are derived, and this large-portfolio study suggests a new condition on the reinsurance pricing regime which is crucial for the results obtained. One or two layer contracts now become optimal for both risk-adjusted surplus and the risk over expected surplus ratio, but there is no second layer when portfolios are large or when reinsurance prices are below some threshold. Simple approximations of the optimum portfolio is considered, and their degree of degradation compared to the optimum is studied which leads to theoretical degradation rates as the number of policies grow. The theory is supported by numerical experiments which suggest that the shape of the claim severity distributions may not be of primary importance when designing an optimal reinsurance program. It is argued that the approach can be applied to Conditional Value at Risk as well.

Key words and phrases

Asymptotics, degradation rates, large portfolios, one- and two-layer contracts, reinsurance pricing regimes, risk-adjusted surplus.
1 Introduction

Actuarial literature contains countless formulations of what optimal reinsurance should mean, for example, Borch (1960); Arrow (1963); Kaluszka (2001) and Cheung et al. (2014). A criterion with much sense industrially is to balance risk and profit through a ratio where a risk measure is divided on expected surplus. Most of the paper makes use of Value at Risk in this role. That is how the insurance industry is regulated at present, but we shall argue that the perhaps theoretically more appealing Conditional Value at Risk could be handled in the same manner which would yield similar, but not quite identical results; see the companion paper Wang and Bolviken (2019). Ratios of risk and expected surplus are related to risk-adjusted surplus under which many authors have examined how reinsurance could be optimized, see Chi (2012); Asimit et al. (2013); Cheung and Lo (2017) and Chi et al. (2017). Of particular significance for the present paper is Chi et al. (2017) who were able to show that the reinsurance treaties maximizing risk-adjusted surplus are of the multi-layer type. It is not tenable to assume that reinsurance pricing is based on a fixed loading, as noted already by Borch (1960), and Chi et al. (2017) made use of a general formulation of reinsurance pricing that goes back to Bühlmann (1980). Many other researchers, for example, Chi and Tan (2013) and Zhuang et al. (2016) have used this scheme. We follow in their track except that we argue for a modification. The world of reinsurance is above all a market with pricing offers from reinsurers defining a supply curve for such risk, but the variation in time is enormous with big events like the World Trade Center terrorism in 2001 having huge impact. Details are not open to the public in any case, and academic studies must therefore employ ‘premium principles’ as proxies for the real market prices. The question is what conditions should be imposed on them. It is demonstrated in this paper that the original set-up in Bühlmann (1980) would enable the insurer to reinsure everything up to Value at Risk and for large portfolios still obtain profit. No net solvency capital would then be necessary, but is it likely that the insurance market should allow such a situation to exist? We think not and have derived from this viewpoint a new condition that differs from the one in current use. All our results depend on it. One of the consequences is that the multi-layer solution for risk-adjusted surplus in Chi et al. (2017) is reduced to one or two layers with more than one layer only when risk is expensive, and this result is a stepping stone to similar results for the risk over surplus ratio.

One issue that does not seem to have been treated in actuarial literature is what happens when the portfolio size become infinite. This is highly relevant for reinsurance of single risks since many of them are sums of a large number of policies. Such asymptotic studies have in statistics or other branches of applied mathematics often lead to simplification and clarity as indeed they do here. We have limited ourselves to independent risks so that we can lean on the central limit theorem and its Lindeberg extension, but the approach can without doubt be extended to dependent risks influenced by some common random factor; more on that in the concluding section. A part of all asymptotic studies is how well such approximations perform for finite portfolio sizes. This is not in our case a question of error, but rather one of degradation of the criterion in terms of how much it changes compared to its value at the strict optimum. Theoretical degradation studies are developed in Section 4 through large portfolio analysis which leads to approximate rates of decay as the number of policies grows. The numerical side is examined in Section 5 though simulation studies and illustrates how large portfolios must be for the large-portfolio approximations to perform well.
2 Basics and preliminaries

2.1 Notation and formulation

Let $X$ be the total claim losses of a single portfolio of non-life insurance policies over a certain period of time (often one year) and let $R_I = R_I(X)$ be the net risk retained after having received from a reinsurer $I = I(X)$ so that $R_I(X) = X - I(X)$. Natural restrictions on $I(X)$ are

$$0 \leq I(X) \leq X \quad \text{and} \quad 0 \leq I(X_2) - I(X_1) \leq X_2 - X_1 \text{ if } X_1 \leq X_2$$ (2.1)

where the first condition is obvious since the reinsurer will never pay out more than the original claim. The second condition is known as the slow growth property, and it opens for moral hazard if it isn’t satisfied; consult Cheung et al. [2014]. Contracts satisfying (2.1) will be referred to as feasible.

Let $F(x)$ be the distribution function of $X$ which starts at the origin. Risk measures that has attracted much interest are Value at Risk and Conditional Value at Risk with formal mathematical definitions

$$\text{VaR}_\varepsilon(X) = \inf \{x \mid 1 - F(x) \leq \varepsilon\} \quad \text{and} \quad \text{CVaR}_\varepsilon(X) = E\{X \mid X \geq \text{VaR}_\varepsilon(X)\}$$ (2.2)

where $\varepsilon > 0$ is a given level. When these quantities apply to the retained risk $R_I$ we shall be using notation like $\text{VaR}_\varepsilon(R_I)$. The right inequality in (2.1) implies that the retained risk $R_I(X)$ is non-decreasing in $X$ so that if $x_\varepsilon = \text{VaR}_\varepsilon(X)$, then

$$\text{VaR}_\varepsilon(R_I) = R_I(x_\varepsilon) \quad \text{and also} \quad I(x_\varepsilon) = x_\varepsilon - R_I(x_\varepsilon).$$ (2.3)

Premia involved are $\pi$ collected by the cedent from its customers and $\pi(I)$ for the reinsurance. Those are in their simplest form

$$\pi = (1 + \gamma)E(X) \quad \text{and} \quad \pi(I) = (1 + \gamma^*E)E\{I(X)\}$$ (2.4)

with $\gamma > 0$ and $\gamma^* > 0$ given loadings (or coefficients) and with $\gamma^* > \gamma$ in practice. The reinsurance part is inadequate. Prices in that market is likely to increase with risk beyond the fixed coefficient $\gamma^*$ in (2.4) right. In real life that would be captured by offers the cedent company receives from reinsurers, but such information isn’t available and for academic work we must instead supply a so-called premium principle which is dealt with in Section 2.3.

2.2 Expected surplus

We need a mathematical expression for the expected surplus for an insurer under a given reinsurance treaty. Into the account goes the premium $\pi$ collected from clients and out of it their net claims $R_I(X)$ and the reinsurer premium $\pi(I)$. If the cost of holding solvency capital is subtracted too, a simplified summary of the balance sheet becomes

$$A(I) = \pi - R_I(X) - \pi(I) - \beta \text{VaR}_\varepsilon(R_I)$$ (2.5)

with the notation highlighting the dependence on the reinsurance function $I(X)$. The coefficient $\beta \geq 0$ applies per money unit. It seems industrially plausible to attach cost to the entire solvency capital, not
only to the part above the average, as in Chi et al. (2017). Alternatively such cost might be in terms of the Conditional Value at Risk with CVaR \(\epsilon\) replacing VaR \(\epsilon\) on the right in (2.5). Let \(\mathcal{G}(I) = E\{A(I)\}\) and take expectations in (2.5). Inserting (2.3) and (2.4) left yield

\[
\mathcal{G}(I) = \gamma E(X) - \beta x_e - (\pi(I) - E(I(X))) + \beta I(x_e)
\]

(2.6)

with the last two terms depending on the reinsurance contract.

### 2.3 Reinsurance pricing

A standard formulation, used for example in Chi et al. (2017), is to introduce a market factor \(M(Z)\) so that

\[
\pi(I) = E\{I(X)M(Z)\}
\]

(2.7)

where \(Z\) is a positive random variable correlated with \(X\) and \(M(\cdot)\) some function for which an exponential one will be used with the examples in Section 3.2. General models for \((X,Z)\) is constructed through copulas. With \(G(z)\) the distribution function of \(Z\) let

\[
X = F^{-1}(U) \quad \text{and} \quad Z = G^{-1}(V)
\]

(2.8)

with \(F^{-1}(u)\) and \(G^{-1}(v)\) the percentile functions of \(F(x)\) and \(G(z)\) and with \((U,V)\) a dependent pair of uniform variables. This yields an alternative expression for \(\pi_I\). By the rule of double expectation

\[
\pi(I) = E\{I(X)M(Z)\} = E\{E(I(X)M(Z)|U)\} = E(I(X)E(M(Z)|U)),
\]

with the last identity due to \(X = F^{-1}(U)\) being fixed by \(U\). Hence

\[
\pi(I) = E\{I(X)W\{F(X)\}\} \quad \text{where} \quad W(u) = E\{M(Z)|u\}.
\]

(2.9)

The impact of the dependency between \(X\) to \(Z\) is taken care of by \(W(u)\) where the distribution function \(F(x)\) of \(X\) doesn’t enter, and this will prove convenient when \(F(x)\) depends on the underlying portfolio size in Section 4.

It is often assumed that \(E\{M(Z)\} = 1\), but that is hardly an obvious assumption. It is being violated when \(M(Z) = 1 + \gamma^r e\) as in (2.4) right, and there is in the present work no point in restricting the set-up so strongly, more on that later. Note in passing that if \(W(u)\) is a non-decreasing function of \(u\), as is plausible and assumed below, then

\[
\pi(I) = E\{I(X)W\{F(X)\}\} \geq E\{I(X)\}E\{W\{F(X)\}\} = E\{I(X)\}E\{M(Z)\}
\]

since \(U = F(X)\) and \(E\{W(U)\} = E\{M(Z)\}\) by (2.9) right. Hence a non-decreasing \(W(u)\) guarantees the reinsurance premium to be larger than the expected reinsurance pay-out if \(E\{M(Z)\} \geq 1\). Much more general formulations of premium principles can be found in Furman and Zitikis (2009).

The price on reinsurance can also be expressed through the function

\[
K(u) = \int_u^1 \{W(v) - 1\} dv, \quad 0 \leq u \leq 1
\]

(2.10)
which is sketched in Figure 2.1 below. Consider the reinsurer expected surplus which by (2.9) left is
\[ \pi(I) - E\{I(X)\} = \int_0^\infty (W\{F(x)\} - 1)I(x)dF(x) \]
or since the derivative \( K'(u) = -\{W(u) - 1\} \),
\[ \pi(I) - E\{I(X)\} = -\int_0^\infty K'\{F(x)\}I(x)dF(x). \]
But if \( I_{x>t} = 0 \) if \( x \leq t \) and = 1 otherwise. then
\[ I(x) = \int_0^x I_{x>t}dI(t) \]
so that
\[ \pi(I) - E\{I(X)\} = -\int_0^\infty K'\{F(x)\} \int_0^x I_{x>t}dI(t)dF(x) = -\int_0^\infty \int_t^\infty K'\{F(x)\}dF(x)dI(t), \]
after changing the order of integration. Since \( K\{F(x)\} \to K(1) = 0 \) as \( x \to \infty \), it follows that
\[ \pi(I) - E\{I(X)\} = \int_0^\infty K\{F(t)\}dI(t). \] (2.11)

2.4 Conditions on reinsurance pricing

The function \( K(u) \) will play a key role, and it is possible to extract some useful properties of it if some restrictions are imposed on the reinsurance pricing regime, notably:

**Condition 1.** (i) \( W(u) \) is non-decreasing and (ii) \( E\{M(Z)\} > 1 + \gamma \).

The first assumption assumes a positive type of dependence between the market factor \( M(Z) \) and the risk \( X \), surely reasonable. A sufficient condition for that is \( M(z) \) being non-decreasing in \( z \) and the model for \( (X,Z) \) in (2.8) based on a positive dependent copula for \( (U,V) \) in the sense that \( \Pr(V > v|u) \) is increasing in \( u \) for all \( v \). Most copulas satisfy this, and it is under these circumstances a trivial matter to verify that \( W(u) \) is monotone upwards.

The second assumption which will be needed for the large-portfolio study in Section 4.2, may seem less obvious since many authors assume \( E\{M(Z)\} = 1 \). This condition goes back to B"uhlmann (1980) who derived it through an economic equilibrium argument. Section 4.2 will present alternative reasoning with some resemblance to arbitrage which leads to Assumption (ii). From (2.10)
\[ K(0) = \int_0^1 (W(v) - 1)dv = E\{M(Z)\} - 1 \]
and the assumption is the same as
\[ K(0) > \gamma \] (2.12)
which is the version that will be cited below. To see what it mean suppose there is a fixed loading \( \gamma_r \) in the reinsurance market so that \( W(u) = 1 + \gamma_r \). Then \( K(0) = \gamma_r \), and Assumption (ii) implies \( \gamma_r > \gamma \) with the loading in the reinsurance market the larger one.

Some useful deductions on the form of the function \( K(u) \) can be drawn under Condition 1:

**Lemma 2.1.** Suppose Condition 1 is true. Then \( K(u) \geq 0 \) everywhere, and there is a unique real number \( \delta \) between 0 and 1 so that

\[
K(1 - \delta) = \gamma \quad \text{and} \quad K'(1 - \delta) \leq 0 \tag{2.13}
\]

where \( K'(u) \) is the derivative.

**Proof.** Note that \( K'(u) = -W(u) + 1 \) so that either \( K(u) \) is decreasing everywhere or, as in Figure 2.1, there is an \( u_0 \) between 0 and 1 so that \( W(u_0) = 1 \) which means that \( K(u) \) decreases to the left and increases to the right of \( u_0 \). There is in either case a unique \( \delta \) as in (2.13) with the derivative of \( K(u) \) negative at that point. That \( K(u) \geq 0 \) is immediate when \( u > u_0 \) since the integrand in (2.10) is positive (or zero) everywhere whereas we also have

\[
K(u) = K(0) - \int_0^u (W(v) - 1) dv
\]

which is \( \geq 0 \) when \( u < u_0 \) since the integrand now is negative.

The function \( K(u) \) has been plotted in Figure 2.1 when it has a maximum. Its values at \( 1 - \delta \) where it crosses the \( \gamma \)-line and its value at the Value at Risk level \( 1 - \epsilon \) will in Section 4 play a main role in defining optimal or nearly optimal reinsurance for large portfolios with the \( 1 - \delta \) and \( 1 - \epsilon \) percentiles of the underlying risk variable \( X \) being the lower and upper limit of one-layer contracts. It could happen
that the γ-line crossing in Figure 2.1 takes place to the right of 1 − ϵ. If so, the upper and lower limits coincide, and reinsurance is so expensive that it is optimal for the insurer to carry all risk himself. It will be of some importance that \( K'(1 - \delta) \leq 0 \), and in practice this inequality is sharp which is taken for granted in Proposition 4.2 below.

3 Optimization

3.1 Risk-adjusted surplus

Many contributors to reinsurance optimum theory work with Lagrangian set-ups of the form

\[
\mathcal{L}(I) = G(I) - \lambda \rho(I)
\]

with \( \rho(I) \) a risk measure and \( \lambda > 0 \) a price on risk; see Balbás et al. (2009); Tan et al. (2011); Jiang et al. (2017) and Weng and Zhuang (2017). This risk-adjusted, expected surplus \( \mathcal{L}(I) \) of the insurer is then maximized, and as \( \lambda \) varies the solutions define an efficient frontier tracing out the minimum \( \rho(I) \) obtainable for a given value of \( G(I) \). These solutions are needed when the risk over surplus ratio is studied in Section 3.3.

**Proposition 3.1.** Chi et al. (2017). If \( \rho(I) = \text{VaR}_\epsilon(R_I) \), is Value at Risk, then the optimal reinsurance function in (3.1) is

\[
I_\lambda(x) = \int_0^x I_{\psi_\lambda(y) > 0} dy \quad \text{where} \quad \psi_\lambda(x) = -K\{F(x)\} + (\beta + \lambda)I_{x < x}\epsilon
\]

with \( I_A \) the indicator function of the event \( A \).

**Proof.** It isn’t assumed that \( E\{M(Z)\} = 1 \) as in Chi et al. (2017), but their ingenious argument still works. Insert (2.6) and \( \rho(I) = \text{VaR}_\epsilon(R_I) = x_\epsilon - I(x_\epsilon) \) into (3.1). This yields

\[
\mathcal{L}(I) = \gamma E(X) - (\beta + \lambda) x_\epsilon - (\pi I - E\{I(X)\}) + (\beta + \lambda)I(x_\epsilon)
\]

which can be combined with (2.11) for the next to last term on the right and also

\[
I(x_\epsilon) = \int_0^\infty I_{x < x_\epsilon} dI(x).
\]

for the last one. Hence

\[
\mathcal{L}(I) = \gamma E(X) - (\beta + \lambda) x_\epsilon + \int_0^\infty \psi_\lambda(x) dI(x)
\]

with \( \psi_\lambda(x) \) as in (3.2) right. The restrictions in (2.1) means that \( 0 \leq dI(x)/dx \leq 1 \) so that \( \mathcal{L}(I) \) is maximized by selecting \( dI(x)/dx = 1 \) whenever \( \psi_\lambda(x) > 0 \) and \( dI(x)/dx = 0 \) otherwise, and this yields \( I_\lambda(x) \) in (3.2) as the optimum. \( \square \)
The proposition shows that the optimum is of the multi-layer type. Let

\[ I_a(X) = \max(X - a_1, 0) - \max(X - a_2, 0) \]  

(3.4)

where \( a = (a_1, a_2)^T \) is a vector of coefficients for which \( a_1 \leq a_2 \). This is a single-layer contract which is the solution in Proposition [3.1] when \( \psi_\lambda(x) > 0 \) between \( a_1 \) and \( a_2 \) and \( \leq 0 \) elsewhere. The general solution depends on how many times \( \psi_\lambda(t) \) in (3.2) crosses zero. If there are three or four crossings there is an additional layer \( I_b(x) \) with \( b = (b_1, b_2) \), and the optimum is now \( I_{ab}(x) = I_a(x) + I_b(x) \). In theory we may continue to five or six crossings and a third layer and so on, but arguably there are in practice at most two:

**Corollary 3.1.** If \( \lambda \leq K(0) - \beta \), then the optimum reinsurance function in Proposition 3.1 is under Assumption (i) in Condition 3 a single layer contract \( I_a(x) \) with \( a_2 = x_\epsilon \). In the opposite case where \( \lambda > K(0) - \beta \), there may be an additional layer \( I_b(x) \) with \( b_1 = 0 \) and \( b_2 < a_1 \).

Degenerate situations are covered by this, for example \( a_1 = a_2 \) leading to no reinsurance at all or \( a_1 = 0 \) with the reinsurer covering everything up to \( x_\epsilon \). The (perhaps surprising) \( b \)-layer starting at 0 occurs when \( \lambda \) is large enough; i.e. when the cost attached \( \rho(I) \) weights heavily enough compared to the expected gain \( G(I) \).

**Proof.** Note that if \( x > x_\epsilon \), then \( \psi_\lambda(x) = -K\{F(x)\} \leq 0 \) so that there is under no circumstances reinsurance above \( x_\epsilon \). On the other hand \( \psi_\lambda(0) = -K(0) + \beta + \lambda \) and \( \psi_\lambda(x) \) starts below or at 0 if \( \lambda \leq K(0) - \beta \). Since the derivative \( K'(u) = -\{W(u) - 1\} \), Assumption (i) tells us that either \( K\{F(x)\} \) decreases everywhere or start to increase and then decrease. In either case \( \psi_\lambda(x) \) crosses zero from negative to positive at a single point or not at all. If \( \psi_\lambda(0) > 0 \) there will be a layer starting at zero and a second one if there are two crossings below \( x_\epsilon \). \( \square \)

### 3.2 Numerical illustration

How \( \psi_\lambda(x) \) in (3.2) varies with \( x \) is shown in Figure 3.1 under different combinations of risk parameters. Detailed conditions and assumptions are recorded in Appendix B along with the simulation algorithm used. The portfolio is Poisson/Gamma with 50 claims expected annually and with individual losses on average 10 with standard deviation 15 which allows for huge losses. The model for \( (X, Z) \) is based on the Clayton Copula (consult Appendix B) with a Gamma distribution for \( Z \) with mean 1 and standard deviation 0.3. The market factor is

\[ M(Z) = (1 + \gamma^{re}) \frac{e^{\omega Z}}{E(e^{\omega Z})}, \]

with \( \gamma^{re} \) and \( \omega \) parameters that are varied. Note that \( E\{M(Z)\} = 1 + \gamma^{re} \), and the Buhlman condition corresponds to \( \gamma^{re} = 0 \).

The default set of parameters in Figure 3.1 is \( (\beta, \theta, \gamma^{re}, \omega, \lambda) = (0.06, 10, 0.2, 0.1, 0.1) \), and the cost of capital \( \beta \) is varied (top left), the price on risk \( \lambda \) (top right), \( \gamma^{re} \) (bottom left) whereas finally (bottom right) \( \psi_\lambda(x) \) is shown for several parameter combinations. The optimum is in all but one case a one-layer solution ending at the \( 1 - \epsilon \) percentile \( x_\epsilon \) (\( \psi_\lambda(x) > 0 \) from some lower limit up to \( x_\epsilon \)) or reinsurance everywhere (\( \psi_\lambda(x) > 0 \) for \( x < x_\epsilon \)) or no reinsurance at all (\( \psi_\lambda(x) < 0 \) everywhere). The exception is
the parameter combination with $\gamma_{re} = 0$ so that $E\{M(z)\} = 1$. Now $\psi_\lambda(0) > 0$, and there is a $b$-layer in the beginning and then an $a$-layer ending at $x_\epsilon$. 

Figure 3.1: Plot of $\psi_\lambda(x)$ against $x$ with $\beta$ (top left), $\lambda$ (top right), $\gamma_{re}$ (bottom left) varied and with miscellaneous parameter combinations (bottom right). The default set of parameters and conditions are in the text.
3.3 Risk over expected gain

Another criterion and the main focus in this paper is the ratio of risk over expected surplus; i.e.

\[ C(I) = \frac{\rho(I)}{G(I)} \quad \text{conditioned on} \quad G(I) > 0, \tag{3.5} \]

We search among insurance functions \( I \) for which \( G(I) > 0 \), and \( C(I) \) is to be minimized among such contracts. Inserting (2.3), (2.6) and (2.11) yields the more explicit form

\[ C(I) = \frac{x_e - I(x_e)}{\gamma E(X) - \int_0^\infty K\{F(x)\}dI(x) - \beta\{(x_e - I(x_e)) \}} \tag{3.6} \]

where it is part of the optimization problem to keep the denominator positive. Though not used much in academic literature the criterion reflect industrial thinking very well as a tool to minimize risk per money unit expected gain. Its optimal solutions are still located on an efficient frontier of the Markowitz type and belong to the same class as those maximizing risk-adjusted surplus as the following consequence of Proposition 3.1 shows:

**Proposition 3.2.** Suppose Assumption (i) in Condition 7 is true. Then there exists for any feasible reinsurance function \( \bar{I}(x) \) for which \( G(I) > 0 \) a one or two layer function \( I_{ab}(x) = I_a(x) + I_b(x) \) so that \( b_1 = 0, b_2 \leq a_1, a_2 = x_e, G(I_{ab}) = G(I) \) and \( C(I_{ab}) \leq C(I) \).

**Proof.** Let \( \bar{I}(x) = I(x) \) if \( x \leq x_e \) and \( = I(x_e) \) if \( x > x_e \). Value at Risk is then the same under both \( I \) and \( \bar{I} \) whereas

\[ \int_0^\infty K\{F(x)\}dI(x) \geq \int_0^\infty K\{F(x)\}d\bar{I}(x) \]

since the contribution above \( x_e \) is cut off on the right and \( K(u) \geq 0 \). It follows from (3.6) that

\[ C(\bar{I}) \leq C(I). \]

The idea now is to construct a reinsurance function \( I_\lambda(x) \) satisfying (3.2) for some \( \lambda > 0 \) so that \( G(I_\lambda) = G(\bar{I}) \). Then by Corollary 3.1, \( I_\lambda = I_{ab} \) for some pair of coefficients \( a \) and \( b \), and since this contract maximizes risk-adjusted surplus

\[ G(I_{ab}) - \lambda \rho(I_{ab}) \geq G(\bar{I}) - \lambda \rho(\bar{I}) \quad \text{with} \quad G(I_{ab}) = G(\bar{I}), \]

denoting Value at Risk by \( \rho(I_{ab}) \) and \( \rho(\bar{I}) \). Hence \( \rho(I_{ab}) \leq \rho(\bar{I}) \) which implies

\[ C(I_{ab}) = \frac{\rho(I_{ab})}{G(I_{ab})} \leq \frac{\rho(\bar{I})}{G(\bar{I})} = C(\bar{I}) \leq C(I) \]

which was to be proved. To construct \( I_\lambda(x) \) let \( \lambda_1 = 0 \), and note that \( K(u) \geq 0 \) implies \( \psi_{\lambda_1}(x) < 0 \) for all \( x \) so that Proposition 3.1 implies that the optimal reinsurance function when \( \lambda_1 = 0 \) is cost of risk is \( I_1(x) = 0 \) everywhere. On the other hand if \( \lambda_2 \) is large enough, \( \psi_{\lambda_2}(x) > 0 \) for \( x \leq x_e \) and the optimum contract now is \( I_2(x) = x \) for \( x \leq x_e \) and \( I_2(x) = x_e \) for \( x > x_e \). The construction ensures that

\[ 0 = \int_0^\infty K\{F(x)\}dI_1(x) \leq \int_0^{x_e} K\{F(x)\}dI_1(x) \leq \int_0^{x_e} K\{F(x)\}dx = \int_0^\infty K\{F(x)\}dI_2(x) \]

since \( K(u) \geq 0 \). It follows that \( G(I_1) \geq G(\bar{I}) \geq G(I_2) \). But if we allow \( \lambda \) to grow from \( \lambda_1 = 0 \) to \( \lambda_2 \) there will on continuity be a \( \lambda \) in between so that \( G(I_\lambda) = G(\bar{I}) \) which completes the proof.
4 Large portfolio asymptotics

4.1 Introduction

The search for an optimum reinsurance function was above reduced to the class of two-layer ones \( I_{ab} \) with \( b_1 = 0 \) and \( a_2 = x_e \), and the aim now is to simplify further when portfolios are large. Suppose \( X_J = Y_1 + \cdots + Y_J \) with \( Y_1, \ldots, Y_J \) individual risks and \( J \) large. It might be possible to cover situations where \( Y_1, \ldots, Y_J \) are dependent through some common random factor, but that will not be done, and it is assumed that \( Y_1, \ldots, Y_J \) are independent though not identically distributed. The distribution function \( F(x) \) of \( X_J \) then becomes Gaussian as \( J \to \infty \) by the central limit theorem and its Lindeberg extension which is almost always satisfied. We shall from now on write \( F_J(x) = F(x) \) to emphasize the importance of \( J \), similarly \( x_{\epsilon,J} = x_\epsilon \) for the \( 1 - \epsilon \) percentile, \( G_J(I) = G(I) \), \( \rho_J(I) = \rho(I) \) and \( C_J(I) = C(I) \) and even \( a_{1,J} = a_1 \) and \( a_{2,J} = a_2 \) for the coefficients.

The detailed mathematical calculations are relegated to Appendix A but the crux of the approach is the centered and normalized variable

\[
X_J^0 = \frac{X_J - J\xi}{\sqrt{J} \sigma} \tag{4.1}
\]

with \( \xi \) average mean and \( \sigma^2 \) average variance of the individual risks \( Y_1, \ldots, Y_J \). For the ensuing argument it doesn’t matter that in reality \( \xi = \xi_J \) and \( \sigma = \sigma_J \) depend on \( J \) as long as they converge to fixed values \( \xi \) and \( \sigma \) as \( J \to \infty \). This minor complication is ignored. The distribution function \( F_J^0(x) \) of \( X_J^0 \) with percentiles \( x_{0,J}^0 \) starts at \( x_{0,J}^0 = -\sqrt{J} \xi / \sigma \) (zero below), and we have the elementary relationships

\[
F_J(x) = F_J^0 \left( \frac{x - J\xi}{\sqrt{J} \sigma} \right) \quad \text{and} \quad x_{\epsilon,J}^0 = \frac{x_{\epsilon,J} - J\xi}{\sqrt{J} \sigma}. \tag{4.2}
\]

The reason for introducing \( X_J^0 \) is that \( F_J^0(x) \to \Phi(x) \) and \( x_{\epsilon,J}^0 \to \phi_\epsilon \) as \( J \to \infty \) where \( \Phi(x) \) and \( \phi_\epsilon \) are distribution function and percentile for the standard normal. It is convenient to work with similar versions for the coefficients; i.e.

\[
a_{1,J}^0 = \frac{a_{1,J} - J\xi}{\sqrt{J} \sigma} \quad \text{and} \quad a_{2,J}^0 = \frac{a_{2,J} - J\xi}{\sqrt{J} \sigma} \tag{4.3}
\]

and \( a_{2,J}^0 = x_{a,J}^0 \) when \( a_{2,J} = x_{\epsilon,J} \) is the optimal upper cut-off point. Similar normalized coefficients \( b_{1,J}^0 \) and \( b_{2,J}^0 \) are introduced from \( b_{1,J} \) and \( b_{2,J} \) below.

4.2 A key condition

Consider the one-layer contract \( I_{a,J} \) with limits \( a_{1,J} \) and \( a_{2,J} = x_{\epsilon,J} \). The expected reinsurer surplus (2.11) is

\[
\pi(I_{a,J}) = E_J(I_{a,J}(X_J)) = \int_0^\infty K\{F_J(x)\}dI_{a,J}(x) = \int_{a_{1,J}}^{x_{\epsilon,J}} K\{F_J(x)\}dx
\]

or after changing the integration variable to \( t = (x - J\xi) / (\sqrt{J} \sigma) \) in the last integral

\[
\pi(I_{a,J}) = E_J(I_{a,J}(X_J)) = \sqrt{J} \sigma \int_{a_{1,J}^0}^{x_{\epsilon,J}^0} K\{F_J^0(t)\}dt. \tag{4.4}
\]
Note that $I_{a_J}(x_{e_J}) = (x_{e_J} - a_{1,J})_+$ so that Value at Risk is $x_{e_J} - I_{a_J}(x_{e_J}) = \min(a_{1,J}, x_{e_J})$ which after inserting for $a_{1,J}$ and $x_{e_J}$ becomes

$$x_{e_J} - I_{a_J}(x_{e_J}) = J\xi + \sqrt{J}\sigma \min(a_{1,J}^0, x_{e_J}^0). \quad (4.5)$$

By (2.6)

$$G_J(I_{a_J}) = \gamma E(X_J) - \{\pi(I_{a_J}) - E_J(I_{a_J}(X_J))\} - \beta \{x_{e_J} - I_{a_J}(x_{e_J})\},$$

and after inserting $E(X_J) = J\xi$, (4.4) and (4.5).

$$G_J(I_{a_J}) = J\xi(\gamma - \beta) - \sqrt{J}\sigma \left(\int_{a_{1,J}^0}^{x_{e_J}^0} K\{F_J(t)\} dt + \beta \min(a_{1,J}^0, x_{e_J}^0)\right) \quad (4.6)$$

which will be later needed to prove Proposition 4.1 below.

Suppose $a_{1,J} = 0$ with $a_{2,J} = x_{e_J}$ still. The reinsurer then takes all risk up to the $1 - \epsilon$ percentile so that Value at Risk for the insurer is zero. In (4.6) $a_{1,J}^0 = x_{e_J}^0 = -\sqrt{J}\xi/\sigma$, and when this is inserted, $\beta$ vanishes (as it should) and

$$G_J(I_{a_J}) = J\xi(\gamma - \beta) - \sqrt{J}\sigma \left(\int_{a_{1,J}^0}^{x_{e_J}^0} K\{F_J(t)\} dt\right)$$

or

$$\frac{G_J(I_{a_J})}{J\xi} = \gamma + \frac{1}{x_{e_J}^0} \int_{a_{1,J}^0}^{x_{e_J}^0} K\{F_J(t)\} dt \to \gamma - K(0) \quad \text{as} \quad J \to \infty. \quad (4.7)$$

The limit follows from Lemma A.2 in Appendix A and is also a simple consequence of l’Hôpital’s rule if $K\{F_J(x_{e_J}^0)\} = K(0) > 0$ so that the integral in (4.7) becomes infinite as $J \to \infty$.

Surely this suggests $K(0) \geq \gamma$? Otherwise the insurer by expanding the portfolio and reinsuring everything up to Value at Risk can earn money without having to put up any solvency capital at all. This isn’t quite arbitrage since risk above Value at Risk still rests with the insurer, but it is for large portfolios rather close to it. It seems unlikely that the market should allow reinsurance risk to be priced so cheaply that $K(0) < \gamma$, and $K(0) \geq \gamma$ becomes a fair assumption. The formal condition in (2.12) excludes the case $K(0) = \gamma$. Now Value at Risk is 0 for large portfolios when the insurer reinsures everything below so that its ratio over expected surplus is 0 too (and hence minimized). For large portfolios the situation has become trivial and uninteresting and need not be considered.

### 4.3 Optima for large portfolios

It was shown above that $x_{e_J}$ is the upper cut-off point for the optimal reinsurance function, and it will now turn out that for large portfolios the $1 - \delta$ percentile $x_{\delta,J}$ is the lower one where $\delta$ was defined in Lemma 2.1 consult, in particular (2.13) from which it follows that $\delta$ depends on the reinsurance pricing regime, but on not the actual distribution of risks. Define

$$\hat{a}_{1,J} = \min(x_{\delta,J}, x_{e_J}) \quad \text{and} \quad \hat{a}_{2,J} = x_{e,J}, \quad (4.8)$$
and let \( \hat{a}_J = (\hat{a}_{1,J}, \hat{a}_{2,J}) \). Throughout this section \( \hat{a}_J \) will represent different approximations to the true optimum. For the corresponding one-layer reinsurance function \( I_{\hat{a}_J} \) under (4.8) there is the following result:

**Proposition 4.1.** If Condition \([4]\) is true and \( \gamma > \beta \), the reinsurance function (4.8) satisfies the following: For any \( \eta > 0 \) there exists \( J_\eta \) so that if \( J > J_\eta \), then \( \mathcal{G}_J(I_{\hat{a}_J}) > 0 \) and \( \mathcal{C}_J(I_{\hat{a}_J}) \leq \mathcal{C}_J(I) + \eta \) for any reinsurance function \( I \) for which \( \mathcal{G}_J(I) > 0 \).

In one word \( \mathcal{C}_J(I_{\hat{a}_J}) \) can’t in the limit exceed \( \mathcal{C}_J(I) \) for any other feasible reinsurance function \( I \). The result provides a simple recipe for optimum reinsurance when portfolios are large with the second layer \( I_{B_J} \) not included at all. Note that it could happen that \( \delta < \epsilon \) so that \( x_{\delta,J} > x_{\epsilon,J} \) in (4.8) which implies \( \hat{a}_{1,J} = \hat{a}_{2,J} \), and the optimum for the insurer is now to carry all risk. This happens when reinsurance is expensive. The extra assumption \( \gamma > \beta \) is always satisfied in practice; surely no insurer would operate if the loading \( \gamma \) in the primary insurance market didn’t exceed the cost \( \beta \) per money unit of keeping solvency capital. The proposition is proved in Appendix A.

### 4.4 Degradation asymptotics

The one-layer reinsurance function \( I_{\hat{a}_J} \) with the two percentiles in (4.8) as limits is optimal as \( J \to \infty \), but how much is the solution degraded for finite \( J \)? It may be measured against the true optimum \( I_{\hat{a}_{J^*}} \) based on the coefficients \( \hat{a}_J \) that minimizes \( \mathcal{C}_J(I_{\hat{a}_J}) \). Note that this is the relevant comparison since there is by Proposition 4.1 no second layer when \( J \) is large enough. The degree of degradation in \( I_{\hat{a}_J} \) is therefore

\[
\mathcal{D}_J(I_{\hat{a}_J}) = \mathcal{C}_J(I_{\hat{a}_J}) - \mathcal{C}_J(I_{\hat{a}_{J^*}}),
\]

which is non-negative and \( \to 0 \) by Proposition 4.1 as \( J \to \infty \). The following result provides the rate:

**Proposition 4.2.** If Condition \([4]\) is true and \( \gamma > \beta \), the degradation (4.9) is under (4.8)

\[
\mathcal{D}_J(I_{\hat{a}_J}) = \frac{\zeta_1}{J^{3/2}} + o(1/J^{3/2})
\]

where

\[
\zeta_1 = -\frac{1}{2} \frac{B_1^2}{K'(1-\delta)\Phi'(\phi_\delta)}, \quad B_1 = \sigma^2 \left( \int_{\phi_\delta}^{\phi_\epsilon} K(\Phi(y))dy + \phi_\delta K(1-\delta) \right).
\]

Note that \( \zeta_1 > 0 \) since \( K'(1-\delta) < 0 \) by Lemma 2.1 and the first term on the right in (4.10) is positive as it should. Consult Appendix A.2 for the proof.

Simple, accurate calculation of the coefficients in (4.8) is available through Monte Carlo. If \( X_{J(1)} \leq \cdots \leq X_{J(m)} \) is an ordered sample of simulations of \( X_J \) take \( \hat{a}_{1,J} = \min(X_{J(m_\delta)}, X_{J(m_\epsilon)}) \) and \( \hat{a}_{2,J} = X_{J(m_\epsilon)}^{*} \) where \( m_\delta = (1-\delta)m \) and \( m_\epsilon = (1-\epsilon)m \). But what happens if Gaussian percentiles are used instead so that no Monte Carlo is needed at all? Now instead of (4.8)

\[
\hat{a}_{1,J} = J\xi + \sqrt{J}\sigma\min(\phi_\delta, \phi_\epsilon) \quad \text{and} \quad \hat{a}_{2,J} = J\xi + \sqrt{J}\sigma\phi_\epsilon.
\]
Had the risk $X_J$ been strictly Gaussian, Proposition 4.2 would still apply, but in practice this is only an approximation, and we must suspect a lower degradation rate. The following proposition is proved in Appendix A.3:

**Proposition 4.3.** Suppose Condition 7 is true and that $\gamma > \beta$. If $x_{\epsilon J} \geq \phi_\epsilon$ for large $J$, then the degradation under (4.12) is

$$D_N^J(I_{\hat{a}_J}) = \frac{1}{\sqrt{J}} \zeta_2 (x_{\epsilon J}^0 - \phi_\epsilon) + o(1/J) \quad \text{where} \quad \zeta_2 = \frac{\sigma \{\gamma - K(1 - \epsilon)\}}{\xi (\gamma - \beta)^2}. \quad (4.13)$$

Note that Lemma 2.1 established that $K(1 - \epsilon) < \gamma$ so $\zeta_2 > 0$.

The main contribution to the degradation is thus caused by the discrepancy $x_{\epsilon J}^0 - \phi_\epsilon$ at the upper percentile which may be approximated by the Cornish-Fischer correction

$$x_{\epsilon J} = \phi_\epsilon + \frac{1}{\sqrt{J}} p(\phi_\epsilon) + o(1/\sqrt{J}) \quad \text{where} \quad p(x) = \kappa(x^2 - 1)/6; \quad (4.14)$$

consult (for example) Section 2.5 in Hall (1992). The coefficient $\kappa$ is the average skewness of the individual risk variables underlying the portfolio sum $X_J$, and the usual situation is $\kappa > 0$; consult Chapter 10 in Bolviken (2014), for an expression for $\kappa$. Then $x_{\epsilon J} > \phi_\epsilon$ as assumed in Proposition 4.3, and the degradation now becomes

$$D_N^J(I_{\hat{a}_J}) = \frac{\zeta_2 p(\phi_\epsilon)}{J} + o(1/J). \quad (4.15)$$

Asymptotic results can also be derived when $\kappa \leq 0$ which is so rare that it has little practical interest.

Proposition 4.3 indicates that the accuracy is enhanced when a better approximation of $x_{\epsilon J}^0$ than $\phi_\epsilon$ is used. Suppose in a manner resembling the Normal Power method of reserving in property insurance (4.12) right is replaced by

$$\hat{a}_{1J} = J\xi + \sqrt{J} \sigma \min(\phi_\delta, \phi_\epsilon) + \sigma p(\min(\phi_\delta, \phi_\epsilon)) \quad \text{and} \quad \hat{a}_{2J} = J\xi + \sqrt{J} \sigma \phi_\epsilon + \sigma p(\phi_\epsilon) \quad (4.16)$$

with the Cornish-Fisher correction term added. The error in the approximation of $x_{\epsilon J}^0$ is then of order $o(1/J)$, and it follows from Proposition 4.3 that the degradation $D_{NP}^J(I_{\hat{a}_J})$ now is of order $1/J^{3/2}$.

These results also tell something about the impact of model error. The Poisson distribution, supported by the Poisson point process, is often a reasonable choice for claim numbers, but there is rarely much theory behind the choice of a typical two-parameter family for claim size. Suppose two such families are calibrated so that mean and standard deviation match. The same Gaussian distribution appears in the limit as $J \to \infty$ in either case, and the discrepancies in the optimum value of the criterion are thus of order $O(1/J)$ and not very large for $J$ of some size.
5 Numerical study

5.1 Example and conditions

The Monte Carlo study presented in this section is based on the market factor \( M(Z) = 1 + \gamma^e \) independent of \( Z \) so that in (2.9) right \( W(u) = 1 + \gamma^e \), which yields in (2.10)

\[
K(u) = \gamma^e (1 - u),
\]

and the \( \delta \)-percentile in Lemma 2.1, which is the solution of \( K(1 - \delta) = \gamma \), becomes \( \delta = \gamma^e / \gamma \). Numerical values were \( \gamma = 0.1 \) and \( \gamma^e = 0.2 \) so that \( \delta = 0.5 \) which means that the large-portfolio approximations of the optimal reinsurance function use the 50% percentile of \( X \) as lower limit. The other percentile was \( 1 - \epsilon = 99\% \). Cost of capital was taken as \( \beta = 0 \).

The claim number was Poisson distributed with claim frequency per policy \( \mu = 0.05 \), and the portfolio size varied between \( J = 10^3, 10^4 \) and \( 10^5 \) policies representing small, medium and large portfolios corresponding to \( J\mu = 50, 500 \) and 5000 expected incidents. As model for the individual losses we have taken three classic distributions with strong skewness to the right; i.e Gamma, log-normal and Pareto.

The probability density functions for Gamma and Pareto were respectively

\[
g(y) = \frac{y^{\alpha-1}e^{-y/\xi}}{(\xi\alpha)^\alpha \Gamma(\alpha)} \quad \text{and} \quad g(y) = \frac{\alpha/(\xi(\alpha - 1))}{(1 + y/\{\xi(\alpha - 1)\})^{\alpha + 1}}
\]

for \( y > 0 \) whereas for the log-normal \( \log(Y) \) was normal with mean \( \alpha \) and variance \( 2(\log(\xi) - \alpha) \). This way of parameterizing means that \( \xi \) is mean loss per event in all three cases whereas \( \alpha \) determines variation. The models were calibrated so that \( \xi = 10 \) and \( \text{sd}(Y) = 15 \) which mean that \( \alpha = 0.44 \) (Gamma), \( \alpha = 1.71 \) (log-normal) and \( \alpha = 3.60 \) (Pareto) with strong skewness in all three cases, respectively 3.00 (Gamma), 7.88 (log-normal) and 5.78 (Pareto). The extreme right tail is heaviest for the Pareto distribution despite its skewness being lower than for the log-normal.

5.2 Results

The optimum Value at Risk over expected surplus had to be optimized numerically as a benchmark against which the approximations could be evaluated. Recall that the upper limit should be the \( 1 - \epsilon \) percentile so the optimization was a simple one-dimensional one to find the lower limit. Monte Carlo was needed to compute the criterion. The number of simulations was \( m = 10^6 \), more than enough to keep Monte Carlo error at a comfortably low level.

Main results are summarized in Table 5.1 for different values of the expected number of incidents \( J\mu \) and the three different loss distributions. All the three approximations (4.8), (4.12), and (4.16) have been evaluated and are recorded as \( D_J(I_{\hat{a}}) \), \( D_J^N(I_{\hat{a}}) \), and \( D_J^{NP}(I_{\hat{a}}) \), and these values in Columns 4 – 6 must be judged against the optimum of the Value at Risk over surplus ratio in Column 3. What counts is the ratios. First note that the criterion itself is strongly dependent on portfolio size with much higher risk over surplus when the expected number of incidents are small. The approximations when \( J\mu = 5 \) are useless, but that changes for larger portfolios with the loss in Column 4 and 6 around 8% when \( J\mu = 50 \), 0.2% when \( J\mu = 500 \) and perhaps 0.0006% when \( J\mu = 5000 \). The normal approximation
Theoretically through large-portfolio studies that lead to degradation rates of order $O$ determined by reinsurance prices. How far this solution is from the true optimum was investigated of the best layer is now defined as fixed percentiles of the underlying risk variable with the lower one that the world of optimum reinsurance is under these circumstances an orderly one. The end points it has for large portfolios been shown that one-layer contracts are close to optimum in any case, and structure as a fixed loading.

It has for large portfolios been shown that one-layer contracts are close to optimum in any case, and that the world of optimum reinsurance is under these circumstances an orderly one. The end points of the best layer is now defined as fixed percentiles of the underlying risk variable with the lower one determined by reinsurance prices. How far this solution is from the true optimum was investigated theoretically through large-portfolio studies that lead to degradation rates of order $O(1/J^{3/2})$ when

| Model   | $J_\mu$ | $C_J(I_{a_j})$ | $D_J(I_{a_j})$ | $D_J^\gamma(I_{a_j})$ | $D_J^{\gamma^r}(I_{a_j})$ |
|---------|---------|----------------|----------------|------------------------|--------------------------|
| Gamma   | 5       | 21.52          | 1.53 $\times$ 10^4 | 2.02 $\times$ 10^4 | 1.63 $\times$ 10^4       |
|         | 50      | 12.46          | 1.00 $\times$ 10^{-1} | 9.82 $\times$ 10^{-1} | 1.00 $\times$ 10^{-1}    |
|         | 500     | 10.68          | 2.33 $\times$ 10^{-3} | 8.54 $\times$ 10^{-2} | 2.24 $\times$ 10^{-3}    |
|         | 5000    | 10.21          | 6.84 $\times$ 10^{-5} | 8.48 $\times$ 10^{-3} | 5.58 $\times$ 10^{-5}    |
| Lognormal | 5       | 20.33          | 8.65 $\times$ 10^0   | 2.00 $\times$ 10^1   | 3.25 $\times$ 10^1       |
|         | 50      | 12.39          | 9.09 $\times$ 10^{-2} | 1.64 $\times$ 10^0   | 1.18 $\times$ 10^{-2}    |
|         | 500     | 10.67          | 2.29 $\times$ 10^{-3} | 1.67 $\times$ 10^{-1} | 3.06 $\times$ 10^{-3}    |
|         | 5000    | 10.21          | 6.66 $\times$ 10^{-5} | 1.57 $\times$ 10^{-2} | 1.08 $\times$ 10^{-4}    |
| Pareto  | 5       | 20.30          | 8.77 $\times$ 10^0   | 1.91 $\times$ 10^1   | 2.17 $\times$ 10^1       |
|         | 50      | 12.37          | 9.02 $\times$ 10^{-2} | 1.56 $\times$ 10^0   | 9.90 $\times$ 10^{-2}    |
|         | 500     | 10.67          | 2.27 $\times$ 10^{-3} | 1.64 $\times$ 10^{-1} | 2.09 $\times$ 10^{-3}    |
|         | 5000    | 10.21          | 6.62 $\times$ 10^{-5} | 1.75 $\times$ 10^{-2} | 5.11 $\times$ 10^{-5}    |

Table 5.1: The degradation based on the three approximations of the optimal $a_{1J}$ and $a_{2J}$ with different loss distributions and conditions as in Section 5.1.

in Column 5 is inferior to the two others as the results in Section 4.3 suggested. Decay rates as $J$ grows match the theoretical ones and are around $1/J$ for the Gaussian approximation in Column 5 and around $1/J^{3/2}$ for the two others with the latter remarkably similar. Discrepancies between the three loss distributions are minor. Since they were calibrated so that mean and standard deviation are equal, the experiment testifies to the lack of importance of the shape of the distributions beyond the first two moments.

6 Concluding remarks

A large-portfolio approach has been introduced which leads to a modification for the market factor $M(Z)$ in the Bühlman pricing regime for reinsurance. Instead of imposing the usual $E\{M(Z)\} = 1$ we have assumed $E\{M(Z)\} > 1 + \gamma$ where $\gamma$ is the loading in the primary market of the insurer. If this condition fails to hold, insurers can in large portfolios earn money with no net solvency capital being needed, arguably an unlikely state of affairs. It was this new condition that reduced the optimum contracts for the Value at Risk adjusted surplus in Chi et al. (2017) to one or two-layer ones, and that applied to the Value at Risk over expected surplus ratio as well. There was only one layer when the price on risk $\lambda$ is below a threshold. If prices in the reinsurance market is of the expected premium principle type with loading $\gamma^*$, the condition boils down to $\lambda < \gamma^* - \beta$ with $\beta$ the cost of solvency capital. Our judgment is that this condition might often be satisfied, but against that view there is the fact that the prices in the reinsurance market are distinctly volatile and nor do they have so simple a structure as a fixed loading.

It has for large portfolios been shown that one-layer contracts are close to optimum in any case, and that the world of optimum reinsurance is under these circumstances an orderly one. The end points of the best layer is now defined as fixed percentiles of the underlying risk variable with the lower one determined by reinsurance prices. How far this solution is from the true optimum was investigated theoretically through large-portfolio studies that lead to degradation rates of order $O(1/J^{3/2})$ when
Monte Carlo approximations of the exact percentiles are used and $O(1/J)$ for Gaussian ones with $J$ the number of policies. There is even a Normal Power modification of the latter that achieves $O(1/J^{3/2})$ too. These results were supported by numerical experiments in Section 5 which suggested considerable robustness with respect to the shape of the underlying claim severity distribution. The important thing for optimal reinsurance seems to be to get mean and variance right.

The studies in this paper can be extended along two lines. We conjecture that similar results are obtained when Value at Risk is replaced by Conditional Value at Risk. The main difference will be that the fixed percentile $1 - \epsilon$ for the upper limit will be replaced by larger one. Then there is the condition of independent risks. They are in many situations some common random factor influencing all of them, for example a random claim frequency. Now the central limit theorem on which the present paper is based no longer holds. Portfolio losses still have a limit distribution, but it is very different from the one in Section 4. It would be of practical interest to develop theory in this situation and investigate how optimal reinsurance is influenced.

References

Arrow, K. J. (1963). Uncertainty and the welfare economics of medical care. *The American Economic Review* 53(5), 941–973.

Asimit, A. V., A. M. Badescu, and A. Tsanakas (2013). Optimal risk transfers in insurance groups. *European Actuarial Journal* 3(1), 159–190.

Balbás, A., B. Balbás, and A. Heras (2009). Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics* 44(3), 374–384.

Bølviken, E. (2014). *Computation and Modelling in Insurance and Finance*. International Series on Actuarial Science. Cambridge University Press.

Borch, K. (1960). An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance. Norges Handelshøyskoles sertrykk-serie. Nr. 35. Uden forlag.

Bühlmann, H. (1980). An economic premium principle. *ASTIN Bulletin: The Journal of the IAA* 11(1), 52–60.

Cheung, K., K. Sung, S. Yam, and S. Yung (2014). Optimal reinsurance under general law-invariant risk measures. *Scandinavian Actuarial Journal* 2014(1), 72–91.

Cheung, K. C. and A. Lo (2017). Characterizations of optimal reinsurance treaties: a cost-benefit approach. *Scandinavian Actuarial Journal* 2017(1), 1–28.

Chi, Y. (2012). Reinsurance arrangements minimizing the risk-adjusted value of an insurer’s liability. *ASTIN Bulletin: The Journal of the IAA* 42(2), 529–557.

Chi, Y., X. S. Lin, and K. S. Tan (2017). Optimal reinsurance under the risk-adjusted value of an insurers liability and an economic reinsurance premium principle. *North American Actuarial Journal* 21(3), 417–432.
Chi, Y. and K. S. Tan (2013). Optimal reinsurance with general premium principles. *Insurance: Mathematics and Economics* 52(2), 180–189.

Furman, E. and R. Zitikis (2009). Weighted pricing functionals with applications to insurance. *North American Actuarial Journal* 13(4), 483–496.

Hall, P. (1992). *The bootstrap and edgeworth expansion*. Springer.

Jiang, W., H. Hong, and J. Ren (2017, dec). On pareto-optimal reinsurance with constraints under distortion risk measures. *European Actuarial Journal* 8(1), 215–243.

Kaluszka, M. (2001). Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics* 28(1), 61–67.

Tan, K. S., C. Weng, and Y. Zhang (2011). Optimality of general reinsurance contracts under cte risk measure. *Insurance: Mathematics and Economics* 49(2), 175–187.

Wang, Y. and E. Bølviken (2019). How much is optimal reinsurance degraded by error? working paper.

Weng, C. and S. C. Zhuang (2017). Cdf formulation for solving an optimal reinsurance problem. *Scandinavian Actuarial Journal* 2017(5), 395–418.

Zhuang, S. C., C. Weng, K. S. Tan, and H. Assa (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics* 67, 65–76.

A Proofs of asymptotics

A.1 Proposition 4.1

The following inequality is needed:

**Lemma A.1.** If $K(u)$ and $W(u)$ are as in Section 4 and $F(x)$ is a distribution function, then for all $a$, $b$ and $x$

$$
|\int_a^b K\{F(t)\}dt + aK\{F(b)\}| \leq |x| \left( \int_0^1 |W(u) - 1|du + K\{F(b)\} \right).
$$

(A.1)

**Proof.** Note that

$$
\int_a^b K\{F(t)\}dt + aK\{F(b)\} = \int_a^b \{K\{F(t)\} - K\{F(b)\}\}dt + xK\{F(b)\}
$$

where the integral when inserting (2.10) for $K(u)$ is

$$
\int_a^b \int_{F(b)}^{F(t)} (W(u) - 1)du dt = \int_{F(b)}^{F(x)} (W(u) - 1) \int_{\max(F^{-1}(u),a)}^x dt du
$$

$$
= \int_{F(b)}^{F(x)} (W(u) - 1) \{x - \max(F^{-1}(u),a)\} du
$$

which means that the absolute value is bounded by $|x| \int_0^1 |W(u) - 1|du$, and (A.1) follows. 

□
Lemma A.2. If \( x^0_J = -\sqrt{J}\xi/\sigma \) and \( F^0_J(x) \) is the distribution function of the normalized risk variable \( X^0_J = (X - J\xi)/(\sqrt{J}\sigma) \), then

\[
\frac{1}{x^0_J} \int_{x^0_J}^{x^{0}_J} K\{F^0_J(t)\} dt + K(0) \to 0 \text{ as } J \to \infty. \tag{A.2}
\]

Proof. Insert \( a = x^0_J = -\sqrt{J}\xi/\sigma \), \( x = x^0_{e,J} \), \( b = x^0_J \) in (A.1) and replace \( F(x) \) with \( F^0_J(x) \). Then

\[
| \int_{x^0_J}^{x^{0}_J} K\{F^0_J(t)\} dt + x^0_J K(0) | \leq |x^0_{e,J}| \left( \int_0^1 |W(u) - 1| du + K(0) \right),
\]

and since \( x^0_{e,J} \) tends to the Gaussian percentile as \( J \to \infty \) and \( |W(u) - 1| \) has finite integral, this implies (A.2). \( \square \)

The next lemma shows that one-layer contracts are better than two-layer ones when portfolios are large:

Lemma A.3. There exits for any \( \eta > 0 \) some \( J_{\eta} \) so that if Condition \( \square \) is true,

\[
\sup\{C_J(I_{a,J}) - C_J(I_{a,b,J})\} < \eta \quad \text{when } J > J_{\eta} \tag{A.3}
\]

where the sup is over all sequences of coefficients \( a_J \) and \( b_J \) so that \( 0 = b_{1,J} \leq b_{2,J} \leq a_{1,J} \leq a_{2,J} = x_{e,J} \).

Proof. Consider the reinsurance function \( I_{a,b,J} \) with coefficients as in the lemma for which \( I_{a,b,J}(x_{e,J}) = x_{e,J} - a_{1,J} + b_{2,J} \). Hence Value at Risk becomes

\[
x_{e,J} - I_{a,b,J}(x_{e,J}) = a_{1,J} - b_{2,J} = \sqrt{J}\sigma(a_{1,J} - b_{2,J}^0)
\]

after passing to the normalized coefficients. For the expected surplus of \( I_{a,b,J} \) we need the expected net reinsurance surplus which from (2.11) is

\[
\pi(I_{a,b,J}) - E\{I_{a,b,J}(X)\} = \int_{a_{1,J}}^{x_{e,J}} K\{F_J(x)\} dx + \int_{0}^{b_{2,J}} K\{F_J(x)\} dx
\]

or after substituting \( t = (x - J\xi)/(\sqrt{J}\sigma) \) in the integral

\[
\pi(I_{a,b,J}) - E\{I_{a,b,J}(X)\} = \sqrt{J}\sigma \left( \int_{a_{1,J}^0}^{x_{e,J}^0} K\{F_J^0(t)\} dt + \int_{0}^{b_{2,J}^0} K\{F_J^0(t)\} dt \right)
\]

so that the expected surplus for the insurer becomes

\[
G_J(I_{a,b,J}) = J\xi\gamma - \sqrt{J}\sigma \left( \int_{a_{1,J}^0}^{x_{e,J}^0} K\{F_J^0(t)\} dt + \int_{0}^{b_{2,J}^0} K\{F_J^0(t)\} dt + \beta(a_{1,J}^0 - b_{2,J}^0) \right), \tag{A.4}
\]

and

\[
C_J(I_{a,b,J}) = \frac{\sqrt{J}\sigma(a_{1,J}^0 - b_{2,J}^0)}{G_J(I_{a,b,J})}. \tag{A.5}
\]
Elementary differentiation yields
\[
\frac{\partial C_J(I_{a,J}, b_{2,J})}{\partial b_{2,J}^0} = -\sqrt{J}\sigma - \sqrt{J}\sigma(a_{1,J}^0 - b_{2,J}^0) \frac{\partial G_J(I_{a,J}, b_{2,J})}{\partial b_{2,J}^0}
\]
with
\[
\frac{\partial G_J(I_{a,J}, b_{2,J})}{\partial b_{2,J}^0} = -\sqrt{J}\sigma(K\{F_J^0(b_{2,J})\} - \beta).
\]
After some straightforward calculations
\[
\frac{\partial C_J(I_{a,J}, b_{2,J})}{\partial b_{2,J}^0} = \frac{\sqrt{J}\sigma H_J(a_{1,J}^0, b_{2,J}^0)}{G_J(I_{a,J}, b_{2,J})^2}
\]
where
\[
H_J(a_{1,J}^0, b_{2,J}^0) = -J\xi_\gamma + \sqrt{J}\sigma \left( \int_{a_{1,J}^0}^{x_{J}^0} K\{F_J^0(t)\}dt + \int_{x_{J}^0}^{b_{2,J}^0} K\{F_J^0(t)\}dt + (a_{1,J}^0 - b_{2,J}^0)K\{F_J^0(b_{2,J}^0)\} \right).
\]
Whether $C_J(I_{a,J}, b_{2,J})$ goes up or down with $b_{2,J}^0$ is determined by this function which can be examined through
\[
\frac{H_J(a_{1,J}^0, b_{2,J}^0)}{J\xi} = -\gamma + A_J(a_{1,J}^0, b_{2,J}^0) + B_J(b_{2,J}^0) \tag{A.6}
\]
where since $x_{J}^0 = -\sqrt{J}\xi/\sigma$
\[
A_J(a_{1,J}^0, b_{2,J}^0) = -\frac{1}{x_{J}^0} \left( \int_{a_{1,J}^0}^{x_{J}^0} K\{F_J^0(t)\}dt + a_{1,J}^0K\{F_J^0(b_{2,J}^0)\} \right), \tag{A.7}
\]
\[
B_J(b_{2,J}^0) = -\frac{1}{x_{J}^0} \left( \int_{x_{J}^0}^{b_{2,J}^0} K\{F_J^0(t)\}dt - b_{2,J}^0K\{F_J^0(b_{2,J}^0)\} \right). \tag{A.8}
\]
By Lemma [A.1] with $a = a_{1,J}^0$, $x = x_{J}^0$ and $b = b_{2,J}^0$
\[
\sup |A_J(a_{1,J}^0, b_{2,J}^0)| \leq -\frac{x_{J}^0}{x_{J}^0} \left( \int_0^1 |W(u)| - 1|du + K\{F(b_{2,J}^0)\} \right)
\]
where sup is over $b_{2,J}^0 \leq a_{1,J}^0 \leq x_{J}^0$. Since $x_{J}^0$, $K\{F(b_{2,J}^0)\}$ and the integral on the right are bounded, sup $|A_J(a_{1,J}^0, b_{2,J}^0)| \to 0$. To deal with the other quantity note that
\[
B_J(b_{2,J}^0) - K(0) = -\frac{1}{x_{J}^0} \left( \int_{x_{J}^0}^{b_{2,J}^0} K\{F_J^0(t)\}dt - b_{2,J}^0K\{F_J^0(b_{2,J}^0)\} + x_{J}^0K(0) \right)
\]
\[
= -\frac{1}{x_{J}^0} \left( \int_{x_{J}^0}^{b_{2,J}^0} [K\{F_J^0(t)\} - K(0)]dt + b_{2,J}^0[K(0) - K\{F_J^0(b_{2,J}^0)\}] \right). \]
Moreover, from (2.10)
\[
\int_{x_J^b}^{b_J^0} [K\{F_J^0(t)\} - K(0)]dt = -\int_{x_J^b}^{b_J^0} \int_0^{F_J^0(t)} (W(u) - 1)du dt
\]
which becomes after changing the order of integration
\[
= -\int_0^{F_J^0(b_J^0)} \{b_J^0 - (F_J^0)^{-1}(u)\}(W(u) - 1)du
\]
\[
= -b_J^0[K(0) - K\{F_J^0(b_J^0)\}] + \int_0^{F_J^0(b_J^0)} (F_J^0)^{-1}(u)(W(u) - 1)du
\]
so that
\[
B_J(b_J^0) - K(0) = -\frac{1}{x_J^b} \int_0^{F_J^0(b_J^0)} (F_J^0)^{-1}(u)(W(u) - 1)du.
\]
But it follows from this that
\[
|B_J(b_J^0) - K(0)| \leq \frac{|b_J^0|}{x_J^b} \int_0^{F_J^0(b_J^0)} s_W
\]
where
\[
s_W = \sup_{0 < u < 1 - \epsilon}|W(u) - 1| < \infty.
\]
We have to show that the right hand side of (A.9) \to 0 as \(J \to \infty\) uniformly in \(b_J^0\) when \(-\sqrt{J} \xi/\sigma = x_J^0 \leq b_J^0 \leq x_J^0\). Let \(\eta > 0\) and note that \(F_J^0(x_{1-\eta,J}) = \eta\) and recall that \(x_{1-\eta,J} \to \phi_{1-\eta}\) when \(J \to \infty\). It follows that there exists a \(J_\eta\) so that if \(J > J_\eta\), then \(b_J^0 \leq |x_J^0|\) when \(x_J^0 \leq b_J^0 \leq x_{1-\eta,J}^0\) and under this condition
\[
|B_J(b_J^0) - K(0)| \leq \eta s_W.
\]
In the opposite case when \(x_{1-\eta,J}^0 < b_J^0 \leq x_J^0\), the interval is bounded as \(J \to \infty\), and \(x_J^0\) in the denominator in (A.9) implies that \(|B_J(b_J^0) - K(0)| \to 0\) uniformly in this interval too so that \(\sup |B_J(b_J^0) - K(0)| \to 0\) where the sup is over \(x_J^0 \leq b_J^0 \leq x_J^0\). Finally from (A.6)
\[
\sup |H_J(a_{1,J}^0, b_J^0)/\{J\xi - (K(0) - \gamma)\}| \to 0
\]
where the sup is over all \(a_{1,J}^0\) and \(b_J^0\) for which \(x_J^0 \leq b_J^0 \leq a_{1,J}^0 \leq x_{1,J}\). But since \(K(0) > \gamma\), this uniform bound establishes for sufficiently large \(J\) that \(H_J(a_{1,J}^0, b_J^0) > 0\) for all \(a_{1,J}^0\) and \(b_J^0\) which in turn implies that \(C_J(I_{a_{1,J}, b_J^0})\) for such \(J\) is an increasing function of \(b_J^0\) for all \(a_{1,J}^0\) so that the optimum is to remove the \(b\)-layer completely.

We need still another lemma which utilizes that \(F_J^0(x) \to \Phi(x)\) uniformly in \(x\) as \(J \to \infty\) where \(\Phi(x)\) is the standard Gaussian distribution function; consult Hall (1992) (for example) for this result.
Lemma A.4. Let \( x^0_{\delta J} \) be the \( 1 - \delta \) percentile of \( X^0_J \) which satisfies \( K\{F^0_J(x^0_{\delta J})\} = \gamma \) and let \( z^0_J \) be a sequence so that \( K\{F^0_J(z^0_{\delta J})\} \to \gamma \) as \( J \to \infty \). Then \( z^0_J - x^0_{\delta J} \to 0 \).

Proof. Under Condition 1 there is a unique \( \delta \) between 0 and 1 so that \( K(1 - \delta) = \gamma \) which implies that \( F^0_J(z^0_J) \to 1 - \delta = F^0_J(x^0_{\delta J}) \) so that \( F^0_J(z^0_J) - F^0_J(x^0_{\delta J}) \to 0 \). But since \( F^0_J(x) \to \Phi(x) \) uniformly this cannot occur unless \( z^0_J - x^0_{\delta J} \to 0 \). \( \square \)

Finalizing the argument Proposition 3.2 established that the search for the optimal reinsurance function can be carried out within the two-layer class \( I_{a_Jb_J} \) with \( b_{1,J} = 0 \) and \( a_{2,J} = x_{\epsilon J} \), and for large portfolios Lemma A.3 further reduced the candidates to the one-layer sub-class \( I_{a_J} \), with \( a_{2,J} = x_{\epsilon J} \). The coefficient \( a_{1,J} \) with its normalized version \( a^0_{1,J} \) is then the only remaining unknown to optimize over, and it is convenient to simplify notation so that \( C_J(a^0_{1,J}) = C_J(I_{a_J}) \) and \( \mathcal{G}_J(a^0_{1,J}) = \mathcal{G}_J(I_{a_J}) \) (the same convention is used everywhere below).

Value at Risk is now simply \( a_{1,J} = J\xi + \sqrt{J}\sigma a^0_{1,J} \) so that the risk over surplus ratio becomes

\[
C_J(a^0_{1,J}) = J\xi + \sqrt{J}\sigma a^0_{1,J} \quad \mathcal{G}_J(a^0_{1,J})
\]

where after removing the \( b \)-layer in (A.4)

\[
\mathcal{G}_J(a^0_{1,J}) = J\xi - \sqrt{J}\sigma \int_{a^0_{1,J}}^{x^0_J} K\{F^0_J(t)\} dt - \beta(J\xi + \sqrt{J}\sigma a^0_{1,J}) \quad \text{(A.10)}
\]

Differentiation yields

\[
\frac{\partial C_J(a^0_{1,J})}{\partial a^0_{1,J}} = \sqrt{J}\sigma \frac{J\xi + \sqrt{J}\sigma a^0_{1,J}}{\mathcal{G}_J(a^0_{1,J})^2} - \sqrt{J}\sigma \frac{K\{F^0_J(a^0_{1,J})\} - \beta}{\mathcal{G}_J(a^0_{1,J})^2},
\]

or after some straightforward calculations

\[
\frac{\partial C_J(a^0_{1,J})}{\partial a^0_{1,J}} = \sqrt{J}\sigma \frac{J\xi[\gamma - K\{F^0_J(a^0_{1,J})\}] - \sqrt{J}\sigma H_J(a^0_{1,J})}{\mathcal{G}_J(a^0_{1,J})^2}
\]

where

\[
H_J(a^0_{1,J}) = \int_{a^0_{1,J}}^{x^0_J} K\{F^0_J(t)\} dt + a^0_{1,J} K\{F^0_J(a^0_{1,J})\} \quad \text{(A.11)}
\]

Hence

\[
\frac{\partial C_J(a^0_{1,J})}{\partial a^0_{1,J}} = J^{3/2} \sigma \frac{J\xi[\gamma - K\{F^0_J(a^0_{1,J})\}] - \sigma H_J(a^0_{1,J})/\sqrt{J}}{\mathcal{G}_J(a^0_{1,J})^2} \quad \text{(A.12)}
\]

By Lemma A.1 with \( x = x^0_{\epsilon J}, a = a^0_{1,J} = b, \)

\[
|H_J(a^0_{1,J})| \leq |x^0_{\epsilon J}| \left( \int_0^1 |W(u) - 1| du + K\{F^0_J(a^0_{1,J})\} \right)
\]

23
and $\sup |H_J(a_{1,j}^0)|/\sqrt{J} \to 0$ with the sup taken over all $a_{1,j}^0 \leq x_{e,j}^0$. Let $\tilde{a}_{1,j}^0$ be the value minimizing $C_J(a_{1,j}^0)$ which from (A.12) must satisfy that

$$K\{F_J^0(\tilde{a}_{1,j}^0)\} + (\sigma/\xi)H_J(\tilde{a}_{1,j}^0)/\sqrt{J} = \gamma,$$

and since $H_J(a_{1,j}^0)/\sqrt{J} \to 0$ uniformly in $\tilde{a}_{1,j}^0$, it follows that $K\{F_J^0(\tilde{a}_{1,j}^0)\} \to \gamma$. But $K\{F_J^0(x_{\delta,j}^0)\} = \gamma$, and by Lemma A.4 this can not occur unless $\tilde{a}_{1,j}^0 - x_{\delta,j}^0 \to 0$. Hence $\tilde{a}_{1,j}^0 = x_{\delta,j}^0$ so that $C_J(\tilde{a}_{1,j}^0) = C_J(x_{\delta,j}^0) \to 0$ as well, and there exists for any $\eta > 0$ some $J_\eta$ so that for any $a_{1,j}^0$

$$C_J(a_{1,j}^0) \geq C_J(\tilde{a}_{1,j}^0) \geq C_J(x_{\delta,j}^0) - \eta$$

which completes the proof of Proposition 4.1.

### A.2 Proposition 4.2

**Part 1** We need asymptotic expressions for the first and second derivative of $C_J(a_{1,j}^0)$ at $a_{1,j}^0 = x_{\delta,j}^0$. In (A.12) the first term in the numerator then vanishes since $K\{F_J^0(x_{\delta,j}^0)\} = \gamma$ so that

$$\frac{\partial C_J(x_{\delta,j}^0)}{\partial a_{1,j}^0} = -J\sigma^2H_J(x_{\delta,j}^0)\frac{x_{\delta,j}^0}{G_J(x_{\delta,j}^0)^2}, \tag{A.13}$$

whereas from (A.10)

$$G_J(x_{\delta,j}^0) = J\xi(\gamma - \beta) + o(J),$$

and from (A.11) since $F_J^0(x_{\delta,j}) = 1 - \delta$

$$H_J(x_{\delta,j}^0) = \int_{x_{\delta,j}^0}^{x_{\delta,j}^0} K\{F_J^0(t)\}dt + x_{\delta,j}^0K(1 - \delta).$$

But $F_J^0(x) \to \Phi(x)$, $x_{\delta,j}^0 \to \phi_\delta$ as $J \to \infty$ and $x_{e,j}^0 \to \phi_e$ so that

$$H_J(x_{\delta,j}^0) = \int_{\phi_\delta}^{\phi_e} K\{\Phi(t)\}dt + \phi_\delta K(1 - \delta) + o(1),$$

and when the expressions for $G_J(x_{\delta,j}^0)$ and $H_J(x_{\delta,j}^0)$ are inserted in (A.13), it emerges that

$$\frac{\partial C_J(x_{\delta,j}^0)}{\partial a_{1,j}^0} = -\frac{B_1}{J\xi^2(\gamma - \beta)^2} + o(1/J) \quad \text{with} \quad B_1 = \sigma^2\left(\int_{\phi_\delta}^{\phi_e} K\{\Phi(t)\}dt + \phi_\delta K(1 - \delta)\right) \tag{A.14}$$

The second derivative can be calculated from (A.12) and has a complicated expression. However, only the leading term is required, and this boils down to

$$\frac{\partial^2 C_J(x_{\delta,j}^0)}{\partial (a_{1,j}^0)^2} = -\frac{K\{F_J^0(x_{\delta,j})\}f_J^0(x_{\delta,j})}{\xi(\gamma - \beta)^2} \frac{1}{\sqrt{J}} + o(1/\sqrt{J})$$

24
where $f_i^o(x) = dF_i^o(x)/dx$. But the central limit theorem on density form yields $f_i^o(x) \rightarrow \Phi'(x)$ and as above

$$
\frac{\partial^2 C_J(x_{\delta J}^0)}{\partial (a_{1 J}^0)^2} = - \frac{K'(1 - \delta)\Phi'(\phi_0)}{\xi(\gamma - \beta)^2} \frac{1}{\sqrt{J}} + o(1/\sqrt{J}). \tag{A.15}
$$

Part 2 We seek the asymptotic degradation when using the normalizing coefficients $a_{1 J}^0 = x_{\delta J}$ instead of the optimal $\bar{a}_{1 J}^0$ with in both cases $a_{2 J}^0 = x_{e J}^0$ as upper limit. An elementary one-variable Taylor argument yields

$$
C_J(x_{\delta J}^0) = C_J(\bar{a}_{1 J}^0) + \frac{1}{2} \frac{\partial^2 C_J(v_{1 J})}{\partial (a_{1 J}^0)^2} (x_{\delta J}^0 - \bar{a}_{1 J}^0)^2
$$

with $v_{1 J}$ between $\bar{a}_{1 J}^0$ and $x_{\delta J}^0$. Note that the linear term has vanished since $\bar{a}_{1 J}^0$ is minimizing. The degradation $D_J(x_{\delta J}^0) = C_J(x_{\delta J}^0) - C_J(\bar{a}_{1 J}^0)$ in using $a_{1 J}^0 = x_{\delta J}$ instead of $a_{1 J}^0 = \bar{a}_{1 J}^0$ is then

$$
D_J(x_{\delta J}^0) = \frac{1}{2} \frac{\partial^2 C_J(v_{1 J})}{\partial (a_{1 J}^0)^2} (x_{\delta J}^0 - \bar{a}_{1 J}^0)^2 \tag{A.16}
$$

where an assessment of $x_{\delta J}^0 - \bar{a}_{1 J}^0$ is needed. The mean value theorem implies that

$$
\frac{\partial C_J(x_{\delta J}^0)}{\partial a_{1 J}^0} = \frac{\partial C_J(\bar{a}_{1 J}^0)}{\partial a_{1 J}^0} + \frac{\partial^2 C_J(v_{2 J})}{\partial (a_{1 J}^0)^2} (x_{\delta J}^0 - \bar{a}_{1 J}^0) = \frac{\partial^2 C_J(v_{2 J})}{\partial (a_{1 J}^0)^2} (x_{\delta J}^0 - \bar{a}_{1 J}^0) \tag{A.17}
$$

with $v_{2 J}$ between $\bar{a}_{1 J}^0$ and $x_{\delta J}^0$. But the second order derivatives in (A.16) and (A.17) both tends to $\frac{\partial^2 C_J(x_{\delta J}^0)}{\partial (a_{1 J}^0)^2}$ since $x_{\delta J}^0 - \bar{a}_{1 J}^0 \rightarrow 0$ and both $v_{1 J}$ and $v_{2 J}$ are squeezed in between. By combining (A.16) and (A.17) it follows that

$$
D_J(x_{\delta J}^0) = \frac{1}{2} \left( \frac{\partial C_J(x_{\delta J}^0)}{\partial a_{1 J}^0} \right)^2 \frac{\partial^2 C_J(x_{\delta J}^0)}{\partial (a_{1 J}^0)^2}^{-1} + o(1/J^{3/2}) \tag{A.18}
$$

where the error term is a consequence of those in (A.14) and (A.15), and inserting those leads after a straightforward calculation to

$$
D_J(x_{\delta J}^0) = - \frac{1}{J^{3/2}} \frac{B_i^2/\xi^3(\gamma - \beta)^2}{K'(1 - \delta)\Phi'(\phi_0)} + o(1/J^{3/2})
$$

as claimed in Proposition 4.2.

A.3 Proposition 4.3

The normalized coefficients when the reinsurance function are using the Gaussian percentiles in (4.12) are $a_{1 J}^0 = \phi_0$ and $a_{2 J}^0 = \phi_e$, and the upper limit deviates from the exact one $x_{e J}^0$ which changes things considerably. Value at Risk is now

$$
a_{1 J} + (x_{e J} - a_{2 J})_+ = J\xi + \sqrt{J}\sigma(\phi_0 + (x_{e J}^0 - \phi_0)_+).$$
after inserting $a_{1,J} = J\xi + \sqrt{J}\sigma \phi_\delta$, $a_{2,J} = J\xi + \sqrt{J}\sigma \phi_\epsilon$ and $x_{\epsilon,J} = J\xi + \sqrt{J}\sigma x_0^{\epsilon,J}$. Recall that we are assuming $x_0^{\epsilon,J} > \phi_\epsilon$, and the degradation in using $a_{1,J} = \phi_\delta$ and $a_{2,J} = \phi_\epsilon$ instead of the optimal $a_{1,J} = \tilde{a}_1^{0,J}$ and $a_{2,J} = x_0^{\epsilon,J}$ then becomes

$$D_J(\phi_\delta, \phi_\epsilon) = \frac{J\xi + \sqrt{J}\sigma(\phi_\delta + x_0^{\epsilon,J} - \phi_\epsilon)}{\mathcal{G}_J(\phi_\delta, \phi_\epsilon)} - \frac{J\xi + \sqrt{J}\sigma \tilde{a}_1^{0,J}}{\mathcal{G}_J(\tilde{a}_1^{0,J}, x_0^{\epsilon,J})}$$

with $\mathcal{G}_J(\phi_\delta, \phi_\epsilon)$ and $\mathcal{G}_J(\tilde{a}_1^{0,J}, x_0^{\epsilon,J})$ the expected surplus terms. This may be rewritten

$$D_J(\phi_\delta, \phi_\epsilon) = A_{1,J} + A_{2,J} + A_{3,J}$$

where

$$A_{1,J} = \sqrt{J}\sigma \frac{x_0^{\epsilon,J}}{\mathcal{G}_J(\phi_\delta, \phi_\epsilon)},$$

$$A_{2,J} = \frac{J\xi + \sqrt{J}\sigma \phi_\delta}{\mathcal{G}_J(\phi_\delta, x_0^{\epsilon,J})} - \frac{J\xi + \sqrt{J}\sigma \tilde{a}_1^{0,J}}{\mathcal{G}_J(\tilde{a}_1^{0,J}, x_0^{\epsilon,J})},$$

$$A_{3,J} = \frac{J\xi + \sqrt{J}\sigma \phi_\delta}{\mathcal{G}_J(\phi_\delta, \phi_\epsilon)} - \frac{J\xi + \sqrt{J}\sigma \phi_\delta}{\mathcal{G}_J(\phi_\delta, x_0^{\epsilon,J})}.$$  

The second of these terms represents degradation due to the difference between $\tilde{a}_1^{0,J}$ and $\phi_\delta$ with the upper limit the optimal $x_0^{\epsilon,J}$ and is on the argument that lead to Proposition 4.2 of order $o(1/J)$ whereas $A_{3,J}$ must be examined further. Note that

$$A_{3,J} = \frac{J\xi + \sqrt{J}\sigma \phi_\delta}{\mathcal{G}_J(\phi_\delta, \phi_\epsilon)} \{ \mathcal{G}_J(\phi_\delta, x_0^{\epsilon,J}) - \mathcal{G}_J(\phi_\delta, \phi_\epsilon) \}.$$  

where

$$\mathcal{G}_J(\phi_\delta, \phi_\epsilon) = J\xi - \sqrt{J}\sigma \int_{\phi_\delta}^{\phi_{\epsilon,J}} K\{ F_0^J(t) \} dt - \beta \{ J\xi + \sqrt{J}\sigma (\phi_\delta + x_0^{\epsilon,J} - \phi_\epsilon) \},$$

$$\mathcal{G}_J(\phi_\delta, x_0^{\epsilon,J}) = J\xi - \sqrt{J}\sigma \int_{\phi_\delta}^{x_0^{\epsilon,J}} K\{ F_0^J(t) \} dt - \beta \{ J\xi + \sqrt{J}\sigma \phi_\delta \}. $$

Hence

$$\mathcal{G}_J(\phi_\delta, x_0^{\epsilon,J}) - \mathcal{G}_J(\phi_\delta, \phi_\epsilon) = \sqrt{J}\sigma \left( -\int_{\phi_\delta}^{x_0^{\epsilon,J}} K\{ F_0^J(t) \} dt + \int_{\phi_\delta}^{\phi_{\epsilon,J}} K\{ F_0^J(t) \} dt + \beta (x_0^{\epsilon,J} - \phi_\epsilon) \right)$$

$$= \sqrt{J}\sigma (-K\{ F_0^J(\phi_\epsilon) \} + \beta)(x_0^{\epsilon,J} - \phi_\epsilon) + o(1)$$

after Taylor’s formula has been applied to the difference between the integrals. But $F_0^J(\phi_\epsilon) \to 1 - \epsilon$ as $J \to \infty$ so that (A.23) after some straightforward calculations becomes

$$A_{3,J} = \frac{1}{\sqrt{J}} \frac{\sigma(-K(1-\epsilon) + \beta)}{\xi(\gamma - \beta)^2} (x_0^{\epsilon,J} - \phi_\epsilon) + o(1/J).$$
This is a quantity of the same order of magnitude as $A_{1J}$ which by (A.20) can be rewritten

$$A_{1J} = \frac{1}{\sqrt{J}} \frac{\sigma}{\xi(\gamma - \beta)} (x_{\epsilon, J}^0 - \phi_\epsilon) + o(1/J).$$

These two must be added whereas $A_{2J}$ is of smaller order and can be ignored so that (A.19) is

$$D_J(\phi_\delta, \phi_\epsilon) = A_{1J} + A_{2J} + o(1/J)$$

or after a little calculation

$$D_J(\phi_\delta, \phi_\epsilon) = \frac{1}{\sqrt{J}} \zeta_2 (x_{\epsilon, J}^0 - \phi_\epsilon) + o(1/J)$$

where

$$\zeta_2 = \frac{\sigma(\gamma - K(1 - \epsilon))}{\xi(\gamma - \beta)^2}$$

as claimed in Proposition 4.3.

**B The simulation experiment in Section 3**

Section 3.2 required the calculations of $K\{F(x)\}$ which is the main part of $\psi_\lambda(x)$ in (3.2). This requires a joint model for the uniform pair $(U, V)$ underlying $(X, Z)$. We have used the Clayton Copula

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad 0 < u, v < 1$$

with $\theta = 10$, and $U$ and $V$ are then passed on through $X = F^{-1}(U)$ and $Z = G^{-1}(V)$ where $F(x)$ and $G(z)$ are the distribution functions of $X$ and $Z$. From the definition of $K(u)$ in (2.10) and $W(u)$ in (2.9) right it follows that

$$K\{F(x)\} = E\{M(Z) - 1 \mid I_{U > F(x)}\}$$

which can be approximated by Monte Carlo through the following steps:

1. Generate simulations $X_1^*, \ldots, X_m^*$ of $X$.
2. Approximate $F(x)$ through the kernel density estimate

$$F^*(x) = \frac{1}{m} \sum_{i=1}^m \Phi\{(x - X_i^*)/(s^* h)\}$$

with $\Phi(x)$ the Gaussian integral, $s^*$ the standard deviation of $X_1^*, \ldots, X_m^*$ and $h = 0.2$.
3. Calculate $U_i^* = F^*(X_i^*)$, $i = 1, \ldots, m$.
4. Generate $Y_i^* \sim$ uniform, $i = 1, \ldots, m$.
5. Calculate $V_i^* = \{1 + (U_i^*)^{-\theta} (Y_i^{-\theta/(1 + \theta)} - 1)^{-1/\theta}\}$, $i = 1, \ldots, m$.
6. Calculate $Z_i^* = G^{-1}(V_i^*)$, $i = 1, \ldots, m$.

The approximations of $K\{F(x)\}$ then becomes

$$K^*\{F^*(x)\} = \frac{1}{m} \sum_{i=1}^m M(Z_i^*) I_{U_i^* > F^*(x)}.$$

Note that all Monte Carlo simulations have been *-marked. The first step is carried out by an ordinary program for simulating portfolio losses whereas steps 4 and 5 is one of the ways the Clayton copula can be simulated; consult p.208 in Bølviken (2014). The final step 6 makes use of the percentile function $G^{-1}(z)$ which is available for all standard distributions.