Non-axisymmetric oscillations of rapidly rotating relativistic stars by conformal flatness approximation

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We present a new numerical code to compute non-axisymmetric eigenmodes of rapidly rotating relativistic stars by adopting spatially conformally flat approximation of general relativity. The approximation suppresses the radiative degree of freedom of relativistic gravity and the field equations are cast into a set of elliptic equations. The code is tested against the low-order \(f\)- and \(p\)-modes of slowly rotating stars for which a good agreement is observed in frequencies computed by our new code and those computed by the full theory. Entire sequences of the low order counter-rotating \(f\)-modes are computed, which are susceptible to an instability driven by gravitational radiation.

INTRODUCTION

Computing characteristic oscillations of rapidly rotating stars in general relativity (GR) is much harder than in Newtonian theory where it is already a difficult task because of deformations of equilibria from a spherical figure. Relativistic gravity is expressed by a spacetime metric tensor which has its own dynamical degrees of freedom, compared to a single scalar potential in Newtonian gravity. These dynamical degrees of freedom are encoded in the theory in such an intricate way that they may be separated from non-dynamical ones only in cases with a high degree of symmetry. The equilibrium far from a sphere raises another important issue of how to impose boundary conditions for gravitational radiation. Free oscillations of a star in GR are characterized by a condition of gravitational waves that propagate "purely outward" at the infinity. For rapidly rotating stars, there have been so far no prescription to impose this condition.

Up to now the only linear eigenmode problem of rapidly rotating stars solved in full GR is that of the neutral mode when the eigenfrequency vanishes (Stergioulas and Friedman \[1\], hereafter SF98). For rotating stars in post-Newtonian theory Cutler and Lindblom \[2\] performed eigenmode analysis and computed sequences of low order \(f\)-modes. Apart from them the traditional eigenmode problem of oscillations of rapidly rotating stars had been solved only within the Cowling approximation \[3\], where Eulerian perturbations of metric coefficients are neglected.

Remarkable progresses in numerical relativistic hydrodynamics, however, have made direct simulations of stellar oscillations possible. The Thessaloniki-MPA group computed oscillations in the Cowling approximation \[4\] and in the spatially conformally flat approximation \[5\]. The AEI group computed non-linear oscillations of rotating stars in full GR \[6\]. The Tübingen group has developed linear evolution codes specialized for oscillations of rotating stars with the Cowling approximation \[7\]. The Tübingen-LSU-Thessaloniki group computed the low order \(f\)-mode sequences in full GR \[8\].

Despite the remarkable successes in computing oscillations of relativistic rotating stars by direct hydrodynamics simulations, the traditional eigenmode problem still needs to be studied. Firstly extracting eigenmodes by dynamical simulations needs much more computational resources than the eigenmode analysis. Especially the extraction of (the modulus of) an eigenfunction needs multiple numerical runs to obtain a single mode \[4\]. Secondly it is not straightforward to identify eigenmodes by looking at the results of dynamical simulations. As the observational asteroseismology \[9\] the knowledge of the eigenmode analysis is indispensable to interpret what is obtained by a numerical experiment.

In this paper we present a non-axisymmetric eigenmode analysis of rotating relativistic stars beyond the Cowling approximation. For eigenmodes with low order and degree, which may be mainly excited in an astrophysical situation and are of interest in gravitational wave astronomy, the Cowling approximation works poorly with large errors \[8\]. Thus we here take into account gravitational perturbation with suppressing gravitational radiation. For typical compactness of a neutron star (~ 0.2), the damping time of a typical fluid oscillation due to gravitational radiation is three orders of magnitude larger than the oscillation period (e.g., Andersson and Kokkotas \[10\]) and it may be neglected in evaluating the characteristic oscillation frequencies.

FORMULATION

In the context of relativistic astrophysics, approximations of GR with the omission gravitational waves are considered by Isenberg (Isenberg \[11\], originally written in 1978). The idea is that the behavior of a gravitating system may be well-approximated without gravitational emission as far as its timescale is longer than the dynamical one. Isenberg proposed several recipes to eliminate the degree of gravitational wave from the
theory. Later Wilson and Mathews [12] rediscovered one of the approximations of Isenberg [11] and applied it to model quasi-equilibria of relativistic binaries (Wilson et al. [13], Flanagan [13], Wilson and Mathews [13]). Since then the conformal flatness approximation (hereafter CFA; also called IWM(Isenberg-Wilson-Mathews) theory or CFC(conformal flatness condition)) has been adopted in numerical relativity [5, 14]. The simplest introduction of CFA is done in (3+1)-decomposition of spacetime (see e.g., Gourgoulhon [14]). A general spacetime metric is written as [15],

$$ds^2 = -\alpha^2 dt^2 + \psi^4 \delta_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt).$$

(C1) CFA assumes \( \hat{g}_{ij} \) to be a flat spatial metric \( f_{ij} \). Assuming the trace of the extrinsic curvature \( K_{ij} \) of the spatial slice to be zero, we obtain the system of elliptic partial differential equations [13, 14]:

$$\nabla^i \nabla_i \psi = -2\pi \psi^5 \left( \rho_{\mu} + \frac{1}{64\pi\alpha^2} \hat{\Lambda}_{ij} \hat{\Lambda}^{ij} \right),$$

(C2)

$$\nabla^i \nabla_i (\alpha \psi) = 2\pi \alpha \psi^5 \left( \rho h (3(\alpha u^i)^2 - 2) + 5p + \frac{7}{64\pi\alpha^2} \hat{\Lambda}_{ij} \hat{\Lambda}^{ij} \right),$$

(C3)

$$\nabla^i \nabla_i \beta^j = 16\pi \alpha \psi^4 S^j - \frac{2}{3} \delta^{ij} (\nabla_k \beta^k) + \hat{\Lambda}^{jk} \hat{\nabla} k \ln \left( \frac{\alpha}{\psi^6} \right).$$

Here \( \nabla_i \) means the covariant derivative with respect to the flat metric \( f_{ij} \). The indices of tensors are raised and lowered by \( f_{ij} \). \( \hat{\Lambda}^{ij} \) is defined as \( \hat{\Lambda}^{ij} = \nabla^i \beta^j + \nabla^j \beta^i - \frac{2}{3} f^{ij} \nabla_k \beta^k \). In the source terms we have \( \rho_{\mu} \equiv n_a n_b T^{ab} = \frac{\xi (\alpha u^i)^2 + \alpha^2 p g^{tt}}{\alpha} \), \( S^j = -\delta^j_1 n_a T^a_c = \frac{\xi \alpha \pi^i \pi^j - \frac{\alpha^2 (\alpha u^i)^2}{\alpha}}{\alpha} \), where \( a \) is the unit normal to the spatial hypersurface of \( t = \text{const}. \) Linearized around the equilibrium state, these equations are elliptic PDEs for Eulerian perturbations of \( \alpha, \psi \) and \( \beta^j \) \( (j = r, \theta, \phi) \). For a given right hand side, these five equations are formally solved for \( \delta \alpha, \delta \psi, \delta \beta^i \), by using appropriate Green’s functions. For scalar variables \( \delta \alpha \) and \( \delta \psi \), we use Green’s function $1/|\vec{r} - \vec{r}'|$, which is expanded as a sum of scalar spherical harmonics. For the vector variable \( \delta \beta^i \), we introduce a vector harmonic expansion of the vector Laplacian operator as in Hill [16].

The assumptions on the equilibrium stars that we perturb are: (1) stationary and axisymmetric, (2) the equation of state (EOS) of the constituent fluid is barotropic. For slowly rotating cases, we follow Yoshida and Kojima [17] to construct the equilibria. In their treatment a stellar structure is computed up to the first order of its rotational frequency, i.e., physical quantities depend on the radial coordinate and they coincide with those of the non-rotating ones except for the \((t, \phi)\)-component of metric coefficient [18]. The EOS adopted in [17] is \( p = K \epsilon^{(1+1)/n} \), where \( \epsilon \) is energy density, \( K \) and \( n \) are constants, and \( p \) is pressure. For the comparison of slowly rotating cases we use the same equilibria as [14]. For rapidly rotating cases we use equilibria computed by COCAL code [19] assuming an EOS of polytropic form as \( p = K \rho^{1+1/n} \), where \( \rho \) is rest mass density. It is known that the metric of a general stationary and axisymmetric spacetime cannot be cast into a conformally flat form on which we perform perturbation. Therefore, for internal consistency we use equilibria constructed by assuming conformal flatness,

$$ds^2 = -\alpha^2 dt^2 + \psi^4 [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\varphi - \omega dt)^2],$$

(C5)

where \( \alpha, \psi \) and \( \omega \) are metric coefficients.

For simplicity we assume that perturbed fluid follows the same EOS as in the equilibrium. This suppresses g-modes arising from a stratified chemical composition and a beta-freezing of nucleons in the perturbed fluid [20]. Thus we have

$$\frac{\Delta p}{p} = \left( 1 + \frac{1}{n} \right) \frac{\Delta \epsilon}{\epsilon}$$

for our slowly rotating cases and

$$\frac{\Delta p}{p} = \left( 1 + \frac{1}{N} \right) \frac{\Delta \rho}{\rho}$$

for our rapidly rotating cases. \( \Delta \) here means Lagrangian perturbation of the quantity following it. Relativistic enthalpy \( h \) is defined as \( h = (\epsilon + p)/\rho \). In the present case all the thermodynamic variables \( \epsilon, p, \rho \) are expressed as functions of \( h \).

The perturbation equations for fluid variables are the equation of rest mass conservation and the equation of motion (the conservation of energy-momentum tensor). We introduce perturbed fluid variables \( \delta h, \delta \pi_a, \delta \pi_{a\beta} \), where \( \delta \) means Eulerian perturbation of a quantity that follows it. Momentum density \( \pi_a \) is defined as \( \pi_a = h u_a \), where \( u_a \) is 4-velocity of the fluid. Then the perturbed rest mass conservation is written as

$$\delta \left[ \nabla_a \left( \frac{\rho}{h^{1/2}} \pi^a \right) \right] = 0.$$

(C6)

It should be noticed that \( \nabla_a \) above is the covariant derivative of the four dimensional metric. The perturbed equation of motion is expressed as

$$\delta \left[ \pi^b \partial_b \pi_j - \pi^b \partial_j \pi_b \right] = 0 \quad (j = r, \theta, \phi)$$

(C7)

At a stellar surface where \( p = 0 \), Eq.\((C6)\) is replaced by a boundary condition. We adopt the conventional condition

$$\Delta p = 0,$$

(C8)

i.e., Lagrangian perturbation of pressure vanishes. Lagrangian and Eulerian perturbations are related to each other by using the Lie derivative along Lagrangian displacement vector which is a function of the perturbation.
of 4-velocity [21]. Therefore the boundary condition is cast into a relation between δh and δπj.

The scheme of solutions of eigenmode problem is a modified version of Yoshida and Eriguchi [3]. The equations for fluid variables (Eq. [5], [9], [13]) are cast into a form in which all the fluid variables are on the left hand side and the terms containing perturbed metric coefficients are on the right hand side. If the right hand sides are omitted, the equations in the Cowling approximation used in Yoshida and Eriguchi [3] are recovered. Once the metric perturbation is provided, the right hand sides and the terms containing perturbed metric coefficients are cast into a form in which all the fluid variables are on the left hand side and the terms containing perturbed metric coefficients are on the right hand side. If the right hand sides are omitted, the equations in the Cowling approximation are recovered. Moreover, Eq. (11),

σ = σ0 + mσ′Ω,

where Ω is the rotational frequency of the star. It should be noted that although the metric of a spherical and static background spacetime can be exactly cast into a conformally flat form (by introducing the isotropic coordinate), the perturbed spacetime around it is no longer exactly conformally flat. The frequencies computed in the CFA are slightly different from the quasi-normal ones. The errors in rotational correction (the factor of Ω in the limit of δ → 0, thus σ[δ] = σ0 + σ1δ + σ2δ2 + ⋯. Here σ0 is the value of the numerical eigenfrequency computed by Richardson extrapolation. To evaluate the error size we compare the size of each terms in the Taylor series. For l = m = 2 f-mode of a non-rotating star with M/R = 0.200, we have σ0 = −1.17, σ1 = 2.18, and σ2 = 2.41 × 10−3, when δ is the radial grid spacing normalized by the radius of the equilibrium star. The rotational correction σ′ defined above is developed in the same way and we obtain σ′0 = 6.71 × 10−1, σ′1 = 4.36 × 10−1, σ′2 = −8.20 × 10−2.

We perform codes (vector) spherical-harmonic expansion of variables and equations with a truncation of series at some value of the degree of the harmonics. For the l = m f-mode, a scalar quantity as δh is expanded by \{P_m^k(\cos θ)\} (k = 0, 1, 2, ⋯, k_{max}) for a given order m. Thus the convergence in the angular grid is examined through the behavior of the eigenmodes for different k_{max}. For a rapidly rotating star of polytropic index N = 1 and M/R = 0.1 with its surface polar to equatorial axis ratio being 0.65, it is necessary to have k_{max} ≥ 14 to determine an eigenfrequency within 1% of error. A star closest to its mass-shedding limit at which the fluid on its equator rotates at Keplerian velocity, deforms more and the number of harmonic terms necessary to have a good approximation of eigenfunctions increases. For a star with a smaller rotational frequency a smaller number of k_{max} is sufficient.

For rapidly rotating cases, we compute sequences of the low-order counter-rotating f-modes of the order m = 2, 3, 4. These modes correspond to l = m f-modes in the grid spacing, we measure the dependence of the error on it. When the finite grid parameter δ is introduced (say, δ = Δr in the radial grid), a numerically-computed eigenfrequency is developed as a Taylor series around some value of the degree of the harmonics. For the l = m = 2 f-modes in the Cowling approximation, σ′_{CFA}, σ′_{Cw}, σ′_{Cw}', are rotational corrections to the frequencies defined in [17],

σ/CFA = σ0/CFA + σ′/CFA,

σ/Cw = σ0/Cw + σ′/Cw,

σ/Cw' = σ0/Cw' + σ′/Cw',

where CFA and Cw are the frequencies obtained by the Cowling approximation. σ′_{CFA}, σ′_{Cw}, σ′_{Cw}' are the frequencies obtained by the Cowling approximation. σ′_{CFA}, σ′_{Cw}, σ′_{Cw}' are rotational corrections to the frequencies defined in [17],

σ = σ0 + mσ′Ω,

where Ω is the rotational frequency of the star. It should be noted that although the metric of a spherical and static background spacetime can be exactly cast into a conformally flat form (by introducing the isotropic coordinate), the perturbed spacetime around it is no longer exactly conformally flat. The frequencies computed in the CFA are slightly different from the quasi-normal ones with all the GR effects taken into account. As expected, the errors in frequencies computed in the CFA are smaller than the corresponding ones by the Cowling. Although the errors in rotational correction (the factor of Ω in the limit of δ → 0, thus σ[δ] = σ0 + σ1δ + σ2δ2 + ⋯. Here σ0 is the value of the numerical eigenfrequency computed by Richardson extrapolation. To evaluate the error size we compare the size of each terms in the Taylor series. For l = m = 2 f-mode of a non-rotating star with M/R = 0.200, we have σ0 = −1.17, σ1 = 2.18, and σ2 = 2.41 × 10−3, when δ is the radial grid spacing normalized by the radius of the equilibrium star. The rotational correction σ′ defined above is developed in the same way and we obtain σ′0 = 6.71 × 10−1, σ′1 = 4.36 × 10−1, σ′2 = −8.20 × 10−2.

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For rapidly rotating cases, we compute sequences of the low-order counter-rotating f-modes of the order m = 2, 3, 4. These modes correspond to l = m f-modes in the
non-rotating stars. The pattern speed of these modes are retrograde with respect to the stellar rotation. They have been of great interest as because they are the most susceptible modes to an instability driven by gravitational radiation (Chandrasekhar -Friedman-Schutz (CFS) instability ; Friedman 21, Chandrasekhar 23, Friedman and Schutz 24).

In Fig.1 the pattern speed $\omega_p$ of counter-rotating f-modes with $m = 2, 3, 4$ are plotted for the stellar equilibrium sequence of $N = 1$ with a fixed rest mass. It is defined as $\omega_p = \sigma/m$. The ratio of the rotational energy to the gravitational energy $T/W$ is adopted as a parameter characterizing the degree of stellar rotation 25. The non-rotating limit of the equilibrium sequence have a compactness $C$ (i.e., the ratio of gravitational mass to the circumferential radius) value of 0.2. When the gravitational radiation is taken into account, an inviscid star rotating faster than the zeroes of these modes becomes unstable due to the CFS instability. As is expected the difference between a mode sequence computed in the CFA and that computed in the Cowling approximation becomes smaller for larger $m$. Introducing gravitational perturbations, the pattern speed tends to be larger than in the cases with the Cowling approximation. Thus the Cowling approximation underestimate the CFS instability 3. It should be also remarked that the slope of the pattern-speed curve around $T/W = 0$ is larger than that around the actual neutral point. Therefore an estimate of neutral points by using the slow rotation approximation would rather overestimate the instability.

Eigenfunctions of perturbed fluid and metric variables are plotted in Fig.2 and Fig.3 respectively (although metric variables are computed also in the vacuum domain, they are displayed inside the star). Each curve corresponds to the value of a function on a $\theta = \text{const.}$ direction. Notice that the ratio of the polar axis to the equatorial axis of

**FIG. 1.** Dimensionless pattern speed $\omega_p$ (defined as $\sigma/m$ where $\sigma$ is the eigenfrequency and $m$ is the azimuthal quantum number of the eigenmode) of counter-rotating f-modes as a function of a stellar rotational parameter $T/W$. These modes become $l = m$ f-modes in the non-rotating limit of the star. For a slowly rotating star their pattern rotates against the stellar spin. These modes becomes unstable to the CFS instability at their zeroes when viscosity is neglected. The rightmost points of the mode sequence correspond to the mass-shedding limit of the star. The black dots are the neutral points of the instability computed by SF98. From right to left they correspond to $m = 2, 3, 4$ f-modes.

**FIG. 2.** Perturbed fluid variables of the counter-rotating bar mode near its neutral point of CFS instability ($T/W = 0.0843$, the ratio of the polar to the equatorial radius is 0.650). The value of functions along each $\theta = \text{const.}$ direction is plotted as a function of $r$. Each panel corresponds to the following functions ; top-left: $\delta h$, top-right: $\delta \pi$, bottom-left: $\delta \beta_{\theta}/r$, bottom-right: $\delta \pi_{\phi}/r \sin \theta$.

**FIG. 3.** Metric perturbations for the same model as Fig.2 are plotted inside the star. Each panel corresponds to the following functions ; top-left: $\delta \alpha$, top-right: $\delta \psi$, centre-left: $\delta \beta_{\theta}$, centre-right $\delta \beta_{\phi}/r$ and bottom: $\delta \beta_{\phi}/r \sin \theta$. 

the star is 0.65 and the rightmost point of each curve corresponds to an equilibrium surface point. These are plots for the counter-rotating bar mode at $T/W = 0.0843$ where it is neutrally stable against the bar-mode CFS instability. Eigenfunctions are normalized as $\delta h = 1$ at the closest surface point to the equator. As a star spins up, the eigenfunctions in enthalpy and in the momentum change their profiles in such a way that their amplitudes are peaked more toward the equator [3].

CONCLUSION

We presented a new code that solves the eigenmode problem for rapidly rotating stars in GR. The code is intended to “go beyond the Cowling approximation” on which the precedent eigenmode solvers in GR have been mainly developed. Our code adopts the CFA to take into account the non-radiative part of relativistic gravity. For slowly rotating stars we see that the mode frequency computed by the CFA rather improves those obtained by Cowling approximation.

A particularly interesting issue in rapidly rotating relativistic stars is their instability. The zeroes of counter-rotating modes marks the onset of the CFS instability and the slow rotation approximation is rather erroneous in evaluating them. Thus the application of the current code on this issue is promising. We compared the critical $T/W$ code on this issue is promising. We compared the critical numbers there and compared them with our results. For slowly rotating stars we see that the mode frequency might be coincidental. These results are consistent with the analysis done by non-linear simulations [8].

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