Kinetics of a network of vortex Loops in He II and a theory of superfluid turbulence

Sergey K. Nemirovskii

Institute of Thermophysics, Lavrentyev ave, 1, 630090, Novosibirsk, Russia,
Novosibirsk State University, Novosibirsk Russia

(Dated: February 6, 2008)
Abstract

A theory is developed to describe the superfluid turbulence on the base of kinetics of the merging and splitting vortex loops. Because of very frequent reconnections the vortex loops (as a whole) do not live long enough to perform any essential evolution due to the deterministic motion. On the contrary, they rapidly merge and split, and these random recombination processes prevail over other slower dynamic processes. To develop quantitative description we take the vortex loops to have a Brownian structure with the only degree of freedom, which is the length $l$ of the loop. We perform investigation on the base of the Boltzmann type ”kinetic equation” for the distribution function $n(l)$ of number of loops with length $l$. This equation describes a slow change of the density of loops (in space of their lengths $l$) due to the deterministic equation of motion and due to fast random change because of the frequent reconnections. By use of the special ansatz in the ”collision” integral we have found the exact power-like solution $n(l) \propto l^{-5/2}$ to ”kinetic equation” in the stationary case. This solution is not (thermodynamically) equilibrium, but on the contrary, it describes the state with two mutual fluxes of the length (or energy) in space of sizes of the vortex loops. The term ”flux” means just redistribution of length (or energy) among the loops of different sizes due to reconnections. Analyzing this solution we drew several results on the structure and dynamics of the vortex tangle in the turbulent superfluid helium. In particular, we obtained that the mean radius of the curvature is of the order of interline space. We also evaluated the full rate of the reconnection events. Assuming, further, that the processes of random collisions are the fastest ones, we studied the evolution of full length of vortex loops per unit volume—the so-called vortex line density $\mathcal{L}(t)$. It is shown this evolution to obey the famous Vinen equation. The properties of the Vinen equation from the point of view of the developed approach had been discussed. Thus, depending on the temperature (and independently on velocity) vortices either develop into the highly chaotic turbulent state (low temperature), or degenerate into few smooth lines (high temperature). This observation can be an alternative explanation for the phenomenon discovered in Helsinki group (Nature 424, 1022–1025 (2003)).

PACS-numbers: 67.25.dk, 47.37.+q, 05.20.-y
I. INTRODUCTION AND SCIENTIFIC BACKGROUND

This paper is the cumulative exposition of a series of preliminary results reported on various scientific meetings and partly published in [1] and in [2]. It concerns an important role of the fusion and breakdown processes of vortex loops in the whole dynamics of a network of vortex filaments in superfluid helium. A network of one-dimensional singularities appears in various physical systems. As examples we would point out vortices in quantum fluids both in turbulent regimes (see, e.g., book [3] and papers [4],[5],[6]) and in a thermodynamically equilibrium state ([7],[8]). Other examples are the flux tubes in superconductors [9], dislocations in solids [10], global cosmic strings [11],[12] and polymer chains [13]. A network of one-dimensional singularities greatly affects many properties of the system where they appear, such as phase transition, thermodynamic and flow properties, structure formation, etc. Therefore, the study of the evolution of these networks is the actual problem.

The evolution of a network of chaotic sets of lines consists of two main ingredients. The first is the motion of the elements of lines, due to equations of the motion, different for each of the cases listed above. For instance, the elements of vortex filaments in superfluid $^4$He move obeying the Biot-Savart law supplemented by the friction force and the external flow/counterflow if any. Cosmic strings move with the speed close to the speed of light $c$ up to some geometrical factor [11]. Velocities of molecules in a polymer chain are determined either by thermal fluctuations of background or by the velocity fluctuations of the surrounding solvent. We will call this ingredient of the evolution as a deterministic motion of lines.

Beside the motion of the each individual loops there is another very important constituent of the whole dynamics, common for all systems, which relates to the collision of loops, or intersection of elements of vortex lines. During intersection of lines the very complicated process, related to arrangement of the vortex core takes place [14]. However this process is relatively short, therefore it is usually accepted that the filaments instantly reconnect whenever they intersect each other. Reconnections of lines result in random fusion and breakdown (recombination) of the loops [15]. The processes of recombination are schematically depicted in Fig. 1. On the left picture we depicted the process of fusion of two loops with lengths $l_1$ and $l_2$ and forming the loop with length $l = l_1 + l_2$. On the right picture we depicted the self-intersection and break down of loop of the length $l$ into two daughter loops with lengths $l_1$ and $l_2$. The rates of these processes are characterized by the rate coefficients
FIG. 1: Schematic sketch of fusion and breaking-down of vortex loops. Rates of these processes characterized by the rate coefficients (number of events per unit time and unit volume) $A(l_1, l_2, l)$ and $B(l, l_1, l_2)$ correspondingly.

It is widely appreciated that the "recombination" processes greatly influence both the structure and dynamics of the vortex tangle. For instance, Feynman in his pioneering paper [4] devoted to superfluid turbulence proposed that the vortex tangle decays due to the cascade-like process of consequent breaking down of vortex loops, and degenerating them into thermal excitations. This scenario is Schematically depicted in Fig. 2. The Feynman’s idea was confirmed in various numerical calculations, where the procedure of artificial elimination of small loops had been used [16]-[19].

To clarify the role of recombination, let us perform the following numerical estimation. The full rate of reconnection $\dot{N}_{\text{rec}}$ (per unit volume) as a function of the vortex line density $L$ had been obtained in papers [1],[20] (see also Subsection IV.3)

$$\dot{N}_{\text{rec}} = C_{\text{rec}} \kappa L^{5/2},$$

where $C_{\text{rec}}$ is the constant of order of unity and $\kappa \approx 10^{-3}$ cm$^2$/s is the quantum of circulation. Let us take, for instance, some typical experiments with superfluid turbulence, with the counterflowing velocity of order of 1 cm/s and with experimental volume of order of 1 cm. Under these conditions the typical value of the vortex line density $L$ is about $L \approx 10^4$ 1/cm$^2$. Interline space $L^{-1/2}$ is of order $10^{-2}$ cm, this quantity coincides with the mean radius of curvature. Then the full rate of reconnection $\dot{N}_{\text{rec}}$ is of the order of $10^7$ collisions per second (per unit volume). Dividing it by $L \approx 10^4$ 1/cm$^2$ we obtain that the rate of reconnection per unit length of the vortex filaments is $C_{\text{rec}} \kappa L^{3/2}$ and is of the order of $10^3$ 1/cm s. Let us take
FIG. 2: Schematic sketch of cascade-like breakdown of vortex loops. (Feynman, 1955, Fig. 10). (a) initial stage, (b) and (c) stages of collapse, (d) reconnection stage, (e) stage of degeneration of vortex rings into thermal excitations.

a loop of length of ten of interline space, \( l \sim 10^{-1} \) cm. This loop undergoes (on average) \( 10^2 \) reconnections per one second, or in other words it exists (on average) \( 10^{-2} \) seconds without reconnection (as a whole). On the other hand, the own vortex filament dynamics (Kelvin waves dynamics) is a much slower process. For instances, if we take again a loop with length \( 10^{-1} \) cm, then any signal on the loop (for instance, degradation of singularity appeared due to the reconnection event) takes time about \( l^2 / \kappa \approx 10 \) seconds. Thus, the characteristic time of the Kelvin waves dynamics exceeds time of existence of the loop by \( 10^3 \) times (!!!). If one takes a smaller loop the situation will be about the same (with other quantitative estimations). In fact up to the smallest loops of the size of interline space, the time of ”life” without reconnection is shorter than time of Kelvin wave running around the loop. Only for the scale of the order of interline space \( \mathcal{L}^{-1/2} \) these times are of the same order. But this means that loops (as a whole) do not live long enough to perform any essential evolution due to the deterministic motion. On the contrary, they frequently merge and split, therefore these recombination processes are the fastest and the basic approach
to study of superfluid turbulence should be grounded on consideration of a set of randomly merging and splitting loops.

It is necessary to do the following two remarks. First, considering superfluid turbulence as evolution of a network of vortex loops we restrict ourselves to the scales of the least loops, which likely coincide with interline space. We are not interested here in what is happening for smaller scales. It is a separate topic connected to evolution of the bending vibrations of vortex lines or the so-called Kelvin waves [22], [23]. This question is of a great interest from point of the vortex tangle decay. Second, we do not consider here the case when the vortex lines are strongly polarized so that the coarse-grained motion induced by the bundles of filaments imitates classical (Kolmogorov) turbulence (See [24], [25], [26]). We consider here the case of the so-called Vinen turbulence, when the vortex loops are highly disordered with zero mean vorticity.

In spite of the recognized importance of the fusion and breakdown processes for the evolution of a network of loops, the numerical results remain the main source of information about this process. The obvious lack of theoretical investigations interferes with the deep insight in the nature of this phenomena (this situation had been recently discussed in [20]). The scarcity of analytic investigations related to the incredible complexity of the problem. Indeed, we have to deal with a set of objects (vortex loops), which do not have a fixed number of elements, they can be born and die. Thus, some analog of the secondary quantization method is required with the difference that the objects (vortex loops) themselves possess an infinite number of degrees of freedom with very involved dynamics. Clearly, this problem can be hardly resolved in the nearest future. Some approach crucially reducing the number of degrees of freedom is required.

There are various ways to overcome this problem. For instance, one approach elaborated in context of lambda-transition [27] is to think of the vortex loops as a set of rings of different sizes and to take their radius as the only degree of freedom. Another approach elaborated in the context of cosmic strings (see [11] and references therein) is to imagine the vortex loops as having the Brownian or the random walk structure. This can be motivated by the following consideration. Because of the huge number of random collisions (see [11]) the structure of any loop is determined by numerous previous reconnections. Therefore, any loop consists of small parts, which ”remember” previous collisions. This is depicted schematically in Fig. 3. Points indicate the sites where the previous reconnections occurred. Waves on
FIG. 3: Picture illustrating a Brownian nature of vortex loops. The structure of any loop is determined by numerous previous reconnections (black circles). Therefore, any loop consists of small parts which "remember" previous collision. These parts are uncorrelated since deterministic Kelvin wave signals (wavy parts near black circles) do not have a time to propagate far enough. Thus, the loop has a structure of random walk (like polymer chain).

The main mathematical tool to describe the random walk is the Wiener distribution. We will use it in a form of the so-called generalized Wiener distribution (See [28] and Appendix A of the present paper), which allows to take into account the finite curvature. Here we consider the case of isotropic loops, omitting a possible variant of the polarized loops having their own impulse. In this case the average loop can be imagined as the consisting of many arches with the mean radius of curvature equal \( \xi_0 \), randomly (but smoothly) connected to each other. Quantity \( \xi_0 \) is the important parameter of the approach. It plays a role of the "elementary step" in the theory of polymer. It is low cut-off of the approach developed, the theory does not describe scales smaller then \( \xi_0 \). Having quantity \( \xi_0 \) the only degree of freedom of the random walk is the length of loop \( l \). Resuming the said above we consider the vortex tangle as a collection of vortex loops having various lengths \( l \), and our goal now is
to find distribution function \( n(l) \) of the number of loops in space of their lengths. Knowing quantity \( n(l, t) \) and statistics of each personal loop we are able to evaluate various properties of the real vortex tangle.

The paper is organized as follows. In the next, second Section we derive the "kinetic" Boltzmann type equation for distribution function \( n(l) \) and specify the coefficients, entering it. In Section III, we obtain the power-like stationary solution to the kinetic equation and describe its properties. Section IV is devoted to the structure and dynamics of the real vortex tangle in turbulent He II obtained on the base of the developed approach. In particular, we obtained analytically the equation for evolution of the vortex line density (Vinen equation) and discuss its properties. Two topics are relegated to appendices: the Gaussian model of vortex loops (Appendix A) and detailed calculation of coefficients of the kinetic equation (Appendix B).

II. THE RATE EQUATION

A. Recombination of loops

In introduction we exposed arguments that vortex loops composing the vortex tangle have a random walk structure, which can be described with use of the generalized Wiener distribution. We take parameters of this distribution not to be changed while recombination, so the only degree of freedom of the loop is its length \( l \). This point of view coincides with conception elaborated in paper [11], where similar problem had been studied in context of cosmic strings. Following this work we introduce distribution function \( n(l, t) \), the density of loops in “space” of their lengths. It is defined as the number of loops (per unit of volume) with lengths lying between \( l \) and \( l + dl \). There are two mechanisms for change of \( n(l, t) \). The first one is the mentioned above deterministic process of evolution of elements of the individual loops, during which they move undergo the stretching or shrinking. Other reasons for change of quantity \( n(l, t) \) are the random reconnection processes. We discriminate two types of processes, namely the fusion of two loops into the larger single loop and the breakdown of a single loop into two daughter loops (see Fig. 1) [29]. The kinetics of the vortex tangle are affected by the intensity of the introduced processes, which is number of events per unit volume and unit time. The intensity of the first process is characterized by
the coefficient \( A(l_1, l_2, l) \), which is the rate of collision of two loops with lengths \( l_1 \) and \( l_2 \) and forming the loop of length \( l = l_1 + l_2 \). The intensity of the second process is characterized by the coefficient \( B(l, l_1, l_2) \), which is the rate of self-intersection and breaking down of a loop with length \( l \) into two daughter loops with lengths \( l_1 \) and \( l_2 \). In view of what has been exposed above we can directly write out the master ”kinetic” equation for rate of change of the distribution function \( n(l, t) \)

\[
\frac{\partial n(l, t)}{\partial t} + \frac{\partial n(l, t)}{\partial l} = \ldots \quad (2)
\]

\[
\int \int A(l_1, l_2, l)n(l_1)n(l_2)\delta(l - l_1 - l_2)\,dl_1dl_2 \quad l_1 + l_2 \rightarrow l
\]

\[
- \int \int A(l_1, l_2)\delta(l_2 - l_1 - l)n(l_1)n(l_2)\,dl_1dl_2 \quad l_1 + l \rightarrow l_2
\]

\[
- \int \int A(l_2, l_1, l_1)\delta(l_1 - l_2 - l)n(l_1)n(l_2)\,dl_1dl_2 \quad l_2 + l \rightarrow l_1
\]

\[
- \int \int B(l_1, l_2, l)n(l)\delta(l - l_1 - l_2)\,dl_1dl_2 \quad l \rightarrow l_1 + l_2
\]

\[
+ \int \int B(l_1, l_2, l)\delta(l_1 - l - l_2)n(l_1)\,dl_1dl_2 \quad l_1 \rightarrow l + l_2
\]

\[
+ \int \int B(l_2, l_1, l)\delta(l_2 - l - l_1)n(l_2)\,dl_1dl_2 \quad l_2 \rightarrow l + l_1
\]

All of the processes are depicted at the left of each line. In spite of very cumbersome form, equation (2) is quite transparent. Indeed, let us take for instance the sixth line. It asserts that number of loops of length \( l \) increases whenever a loop with length \( l_1 \) breaks down into two smaller loops and one of the daughter loops has the length \( l \). Rate of growth is proportional to number of larger loops \( n(l_1) \) and to the intensity of breakdown \( B(l_1, l_2, l) \). Then we have to integrate over all sizes \( l_1 \). Delta function \( \delta(l_1 - l - l_2) \) just controls conservation of the total length while recombination. We do not consider here possible small loss of length due to reconnection, this question in context of our approach had been studied in [30]. Kinetic equation (2) has a ”book-keeping” character, moreover, in this form it is applicable for other systems e.g., for network of cosmic strings. Physics of this approach lies in the ”correct” determinations of coefficient \( A(l_1, l_2, l) \) and \( B(l, l_1, l_2) \) of this equation on the base of some more or less plausible model. In next subsections we will outline main idea and derive mathematical identities for the rates coefficient \( A(l_1, l_2, l) \) and \( B(l, l_1, l_2) \) in the case of an arbitrary network of loops. Detailed calculation of these quantities for vortex loops in the turbulent superfluid helium on the Gaussian model is performed in Appendix B.
FIG. 4: Schematic sketch of the self-intersection processes. Elements of line are described as vectors \( s(\xi) \), where the label variable \( \xi \) is taken here as the arc length. We associate the moment of intersection with the vanishing of vector \( S(\xi_1, \xi_1, t) \) connecting points \( s(\xi_2, t) \) and \( s(\xi_1, t) \). Thus, the rate coefficient of break-down is equal to number of zeroes of function \( S_b(\xi_2, \xi_1, t) \) in space of its variables \( \zeta = \{\xi_2, \xi_1, t\} \).

B. Mathematical identities for \( A(l_1, l_2, l) \) and \( B(l, l_1, l_2) \).

In this subsection we formulate mathematical definition for quantities \( A(l_1, l_2, l) \) and \( B(l, l_1, l_2) \). We start with the quantity \( B(l, l_1, l_2) \). By definition its physical meaning is the frequency of events when part of line with total length \( l \) intersects itself and breaks down into two daughter loops with lengths \( l_1 \) and \( l_2 \). As it was already stated we assume that each crossing event leads to the reconnection of lines. Let us consider function

\[
S_b(\xi_2, \xi_1, t) = s(\xi_2, t) - s(\xi_1, t),
\]

which is the vector connecting points \( s(\xi_2, t_2) \) and \( s(\xi_1, t_1) \) (see Fig. 4).

Clearly, the condition \( S_b(\xi_2, \xi_1, t) = 0 \) implies that the self-crossing event of parts of the line with label-coordinates \( \xi_2, \xi_1 \) occurs at moment of time \( t \). The quantity \( S_b(\xi_2, \xi_1, t) \) is fluctuating 3-component function of three arguments \( \xi_2, \xi_1, t \). We are interested in how often \( S_b(\xi_2, \xi_1, t) \) vanishes in cube of space \( \zeta = \{\xi_2, \xi_1, t\} \). From theory of generalized function it
follows that number of these points (we denote them below as \( \zeta_a \)) can be expressed via \( \delta \)-function of quantity \( S_b(\xi_2, \xi_1, t) \) with the help of the following formula.

\[
\sum_a \delta(\zeta - \zeta_a) = \left[ \frac{\partial (X, Y, Z)}{\partial (\xi_2, \xi_1, t)} \right]_{\zeta = \zeta_a} \delta(S_b(\xi_2, \xi_1, t)). \tag{4}
\]

Here \( X, Y, Z \) are the components of vector \( S_b(\xi_2, \xi_1, t) \). By integration of both parts of (4) over \( d\xi_1 d\xi_2 \) we obtain the full number of intersections (per unit time). The lengths of pieces of the self-intersecting line are however arbitrary. The requirement that pieces should have lengths \( l_1 \) and \( l - l_1 \) can be satisfied by the introducing additional constraint \( \delta(\xi_2 - \xi_1 - l_1) \) into integrand. In addition we have to do averaging over all possible fluctuating configurations. Finally the coefficient \( B(l, l_1, l - l_1) \) with dimension \([s] = s^{-1} cm^{-1}\) is

\[
B(l, l_1, l - l_1) = \int \int d\xi_1 d\xi_2 \delta(\xi_2 - \xi_1 - l_1) \left\langle \left| \frac{\partial (X, Y, Z)}{\partial (\xi_2, \xi_1, t)} \right|_{\zeta = \zeta_a} \delta(S_b(\xi_2, \xi_1, t)) \right\rangle. \tag{5}
\]

To obtain coefficient \( A(l_1, l_2, l) \) we use the similar procedure. Let us consider two loops with length \( l_1 \) and \( l_2 \). Our purpose now to find the rate \( A(l_1, l_2, l) \) of fusion of these two loops into one loop of length \( l = l_1 + l_2 \). Dimension of \( A(l_1, l_2, l) \) is \([A] = cm^3 s^{-1}\). As previously, we describe vortex filaments by positions of radius vectors their elements \( s(\xi_1, t) \) and \( s(\xi_2, t) \). Here we have the two label variables \( \xi_1, \xi_2 \) belonging to different loops and running in the limits \((0 \leq \xi \leq l_1)\) and \((0 \leq \xi \leq l_2)\) respectively. One more important difference with the previous case is that both functions \( s(\xi_1, t) \) and \( s(\xi_2, t) \) should depend on ”initial” positions \( s(\xi_1 = 0, t) = R_1(t) \) and \( s(\xi_2 = 0, t) = R_2(t) \), chosen arbitrary. Of course in previous case of the self-intersection of single loop, quantity \( s(\xi, t) \) also depended on ”initial” positions, but it did not influence the rate of self-intersection. For case of the fusion this dependance is important, since very distant loops have the small probability to collide. Let us introduce the ”fusion” functions

\[
S_f(\xi_2, \xi_1, t) = s(\xi_2, t) - s(\xi_1, t). \tag{6}
\]

Repeating the considerations for case of the single loop we find that the number of reconnection (per unit of time) of points \( \xi_2, \xi_1 \) formally coincides with (4)

\[
\sum \delta(\zeta - \zeta_a) = \left[ \frac{\partial (X, Y, Z)}{\partial (\xi_2, \xi_1, t)} \right]_{\zeta = \zeta_a} \delta(S_f(\xi_2, \xi_1, t)) \tag{7}
\]

with the difference that \( \xi_2, \xi_1 \) belong to different curves. Since intersections of any elements of lines lead to the fusion of loops we have to integrate (7) over \( d\xi_1 d\xi_2 \). The result obtained is
valid for chosen pair of loops. To obtain the total number of events we have to multiply the
result obtained by quantity \( n(l_1)n(l_2)dR_1dR_2 \), which is the full number of loops of chosen
sizes in the whole volume. Comparing with the master kinetic equation \((2)\) we find the final
expression for fusion coefficient \( A(l_1, l_2, l) \)

\[
A(l_1, l_2, l) = \frac{1}{\mathcal{V}} \int \int dR_1dR_2 \int \int d\xi_1d\xi_2 \left\langle \left| \frac{\partial(X, Y, Z)}{\partial(\xi_2, \xi_1, t)} \right| \right|_{\xi = \zeta_a} \delta(S_f(\xi_2, \xi_1, t)) \right\rangle,
\]

where \( \mathcal{V} \) is the total volume of system.

Thus we obtained expressions for the coefficients (5) and (8), which allow to calculate
the rates of the fusion and breakdown of the vortex loops. They are, however, just formal
mathematical identities. Concrete results depend on statistics and dynamics of individual
lines. Therefore to move further we have to ascertain the procedure for averaging. We will do
it with use of the so-called Gaussian model of vortex loops, which is bases on the presentation
them as having a random walk structure. In order not to overcharge the main text we will
expose both ideas of Gaussian model and detailed evaluation of quantities \( B(l_1, l_2, l) \) and
\( A(l_1, l_2, l) \) in Appendices A and B.

III. EXACT SOLUTION OF THE "RATE EQUATION"

A. Zakharov ansatz

In this Section we describe one particular but very important solution to the rate equa-
tion (2). Following results exposed in Appendices we adopt the following expressions for
coefficient \( A \) and \( B \) (see relations (B.5), (B.9) in Appendix B).

\[
A(l_1, l_2, l) = b_m V_l l_1 l_2, \quad B(l_1, l - l_1, l) = b_s \frac{V_l \ell}{(\xi_0 l_1)^{3/2}}. \tag{9}
\]

Quantity \( V_l \) is \((l\)-independent\) characteristic velocity of approaching of elements of line,
\( b_m \) and \( b_s \) are numerical constants. The quantity \( \xi_0 \) associated with the mean radius of the
curvature(see [28] and Appendix A), and it is a low cut-off of the whole approach.

Early the equation similar to (2) with coefficients \( A \) and \( B \) (9) had been studied analyti-
cally in papers [11], and numerically in [12]. Thus, in particular in [11] it was demonstrated
that (2) has the asymptotic solution \( n(l) \propto e^{-\beta l^{-5/2}} \), which describes thermodynamics equi-
librium. It had been obtained in supposition of detailed balance, which implies that each of
the line in the collision integral vanishes. This solution, however, is an approximate solution
of the rate equation \( \text{(2)} \) valid only in case of very small daughter loop.

Here we will search for stationary solution of \( \text{(2)} \) neglecting deterministic terms. As it
had been discussed in the Introduction, processes of recombination (fusion and splitting)
are the fastest, so it is quite natural to suppose that the collision term \( I_{st}\{n\} \) expressed by
the lines 2-7 in \( \text{(2)} \) is the leading one. We assume that the time independent solution of the
equation \( I_{st}\{n\} = 0 \) is the basic equation, and nonstationary processes as well as processes
related to the deterministic motion can be accounted in the frame of the perturbation theory.
Therefore, as a first step we neglect other terms and concentrate on seeking for solution
\( I_{st}\{n\} = 0 \).

Coefficients \( A(l_1, l_2, l) \) and \( B(l_1, l - l_1, l) \) are the power low functions, therefore they are
scale invariant quantities. That implies that for the scales exceeding \( \xi_0 \) there is no character-
istic length in the statement of problem. It points out that equation \( I_{st}\{n\} = 0 \) should have
the scale invariant, or power-like solution of form \( n(l) = C \cdot l^s \). To find power-like solution
we use the Zakharov ansatz, which is the special treatment of the ”collision” integral in
equation \( \text{(2)} \). This trick was elaborated by Zakharov for the wave turbulence (see e.g., [31]),
now we will show how it works in our case. Let us take for instance the first and second
integrals in the ”collision term” of \( \text{(2)} \). Let us further perform in the second integral the
following change of variables.

\[
l = \bar{l}_2 \left( \frac{l}{l_2} \right), \quad l_1 = \bar{l}_1 \left( \frac{l}{l_2} \right), \quad l_2 = l \left( \frac{l}{l_2} \right). \tag{10}
\]

Under this change of variables various factors in the integrand of the second integral trans-
forms as follows

\[
\delta(l_2 - l_1 - l) \rightarrow \left( \frac{l}{l_2} \right)^{-1} \delta(l - \bar{l}_1 - \bar{l}_2),
\]

\[
n(l) \rightarrow n(\bar{l}_2) \left( \frac{l}{l_2} \right)^s, \quad n(l_1) \rightarrow n(\bar{l}_1) \left( \frac{l}{l_2} \right)^s, \tag{11}
\]

\[
A(l_1, l, l_2) \rightarrow \frac{1}{2} V_l \bar{l}_1 \bar{l}_2 \left( \frac{l}{l_2} \right)^2 = A(\bar{l}_1, \bar{l}_2, l) \left( \frac{l}{l_2} \right)^2.
\]

As result the second integral in the ”collision” term takes a form (the additional term 3 in
the power counting appears from the Jacobian of transformation)

\[
\int \int \left( \frac{l}{l_2} \right)^{2+2s-1+3} A(\bar{l}_1, \bar{l}_2, l)n(\bar{l}_1)n(\bar{l}_2)\delta(l - \bar{l}_1 - \bar{l}_2)d\bar{l}_1d\bar{l}_2. \tag{12}
\]
It is easy to see that the transformed second term in the "collision" integral in the right hand side of the master kinetic equation (2) turns into first integral with an additional factor \((l/l_2)^{4+2s}\) in the integrand. Performing the same procedure for all lines we conclude that the "collision integral" of the "rate equation" can be written as

\[
\int \int A(l_1, l_2, l)n(l_1)n(l_2) \left(1 - \left(l/l_1\right)^{4+2s} - \left(l/l_2\right)^{4+2s}\right) \delta(l - l_1 - l_2)dl_1dl_2
\]

\[
-\int \int B(l_1, l_2, l)n(l) \left(1 - \left(l/l_1\right)^{s+3/2} - \left(l/l_2\right)^{s+3/2}\right) \delta(l - l_1 - l_2)dl_1dl_2.
\]

For \(s = -5/2\) both expressions \(1 - \left(l/l_1\right)^{4+2s} - \left(l/l_2\right)^{4+2s}\) and \(1 - \left(l/l_1\right)^{s+3/2} - \left(l/l_2\right)^{s+3/2}\) are equal to \((l - l_1 - l_2)/l\). Thus the integrands of both integrals in (13) include expressions of type \((x)\delta(x)\) and these integrals vanish. This implies in stationary case and neglecting the deterministic terms in (2) the power-like solution \(n = C * l^{-5/2}\) for distribution function \(n(l, t)\) of density of loop in "space" of their lengths takes place.

B. Flux of length (energy)

Let us discuss the physical meaning of the solution obtained. First of all we stress that it is not related to detailed balance i.e. it does not describe thermal equilibrium, it rather corresponds to the nonequilibrium state. To clarify the nature of this nonequilibrium state we introduce the length density \(L(t)\) (in space of sizes \(l\)), which the full length accumulated in loops of size \(l\) (per unit of volume) [32]

\[
L(l, t) = n(l, t)l = \frac{\text{the length of all loops with size } l}{\text{unit of volume}\ast\text{interval of length}}.
\]

The total length (per unit volume), or the vortex line density \(\mathcal{L}(t)\) is defined as follows:

\[
\mathcal{L}(t) = \int L(l, t)dl = \int l * n(l, t)dl.
\]

Quantity \(\mathcal{L}(t)\) is obviously conserved during the reconnections events \(d\mathcal{L}(t)/dt = 0\) (see, however, remark in Section II and paper [30]). Conservation of the vortex line density can be expressed in the form of continuity equation for the length density \(L(l, t)\)

\[
\frac{\partial L(l, t)}{\partial t} + \frac{\partial P(l)}{\partial l} = 0.
\]
This form of equation states that the rate of change of length is associated with "flux" of length in space of sizes of the loops. Term "flux" here means just the redistribution of length (or energy, see [32]) among the loops of different sizes due to reconnections. Expression for $P(l)$ is obtained by multiplying the rate equation \ref{eq:rate} by $l$ and by rewriting the "collision" term in the shape of a derivative with respect to $l$. The result is (substitutions $l_1/l = x$ and $l_2/l = y$ have been used below)

$$
P = \left( \frac{l^{5+2s}}{5+2s} \right) \int \frac{1}{2} b_m V_l x y C^2 x^s y^s \left( 1 - \left( \frac{1}{x} \right)^{4+2s} - \left( \frac{1}{y} \right)^{4+2s} \right) \delta(1 - x - y) dxdy$$

$$
- \left( \frac{l^{s+5/2}}{s+5/2} \right) \int \frac{1}{2} b_s V_l \frac{1}{(\xi_0 x)^{3/2}} C \left( 1 - \left( \frac{1}{x} \right)^{s+3/2} - \left( \frac{1}{y} \right)^{s+3/2} \right) \delta(1 - x - y) dxdy.
$$

Both integrals in relation \ref{eq:flux} coincide with integrals in \ref{eq:collision}, therefore they vanish for $s = -5/2$. However they have preintegral factors with the denominators, which also vanish for $s = -5/2$ and we have the indeterminacy $0/0$. Calculating numerically integrals in \ref{eq:flux} as functions of $s$ and taking $s \rightarrow -5/2$ we obtain the final expressions for the "flux" of length in space of sizes of the loops

$$
P_{\text{net}} = P_+ - P_- = \frac{12.555}{2} C^2 b_m V_l - \frac{5.545}{2 \xi_0^{3/2}} C b_s V_l.
$$

The positive sign of the first term corresponds to the flux of length in the direction of large scales. This is justified, since the fusion processes lead to formation of larger and larger loops. The negative sign of the second term corresponds to the flux of length in the direction of small scales. This is justified, since the breaking down processes lead to formation of smaller and smaller loops. Schematically this situation is depicted in Fig. 5. We will also use terms "direct cascade" describing cascade like breakdown of loops and "inverse cascade" responsible for formation of larger and larger loops. As it had been discussed in Introduction the direct cascade of consequent breaking down of vortex loops was predicted by Feynman in his pioneering paper [4]. The Feynman’s idea was confirmed in various numerical calculations, where the procedure of artificial elimination of small loops had been used [16]-[19]. Our analytical calculations confirmed the splendid Feynman’s conjecture and give an exact evaluation for the cascade-like flux (the second term in the right hand side of relation \ref{eq:flux}). In addition we obtained result about inverse cascade responsible for formation of larger and larger loops, so the direction of the net flux is not clear. In more details this fact will be discussed in the next Section.
FIG. 5: Pictures illustrating flux of length (or energy, see [32]) in space of the loop sizes. This flux is just redistribution of total length (energy) among the loops of different sizes due to recombination process. (a) Negative flux, or direct cascade appears due to consequent break-down resulting in formation of smaller and smaller loops. (b) Positive flux, or inverse cascade describes consequent fusion of loops leading to formation of larger and larger loops.

Thus we have found stationary power-solution to the rate equation (2) neglecting deterministic terms. As it was already mentioned this solution connected with recombination of loops describes the fastest processes and can be considered as a first iteration for the whole problem stated by master "rate equation" (2). The approach elaborated above allows to draw several conclusions concerning both the structure and dynamics of the real vortex tangle in the turbulent He II. It will be done in the following Section.
IV. ON THEORY OF SUPERFLUID TURBULENCE.

A. Properties of the vortex tangle with no normal component (zero temperature case).

In this subsection we discuss some properties of the vortex tangle resulting from the solution of equation \( I_{st}(n) = 0 \) obtained and analyzed in the previous Section. As mentioned this solution is a stationary solution of the master kinetic equation (2) neglecting deterministic terms. The latter implies that the interaction with normal component is omitted hence the title of this subsection.

To use formulas derived in previous section we have to specify quantity \( V_l \), which enters into the rates coefficients \( A \) and \( B \) of both the merging and breaking down processes. Basing on the results obtained in Appendix B we estimate the velocity factor \( V_l \) to be of the order of \( \sqrt{2\kappa/\xi_0} \) (\( \kappa \) is the quantum of circulation). Thus the only parameters of the whole theory (at zero temperature) are the quantum of circulation \( \kappa \) and the mean radius of curvature \( \xi_0 \).

1. Vortex Line Density and the mean curvature.

Because of a huge amount of reconnections each of the terms in the right hand side of relation for the net flux (18) are large. The net flux \( P_{net} \), which is the difference between positive \( P_+ \) and negative \( P_- \) constituents is much smaller. Neglecting \( P_{net} \) and equating \( P_+ \) and \( P_- \) we are in position to ascertain constant \( C \)

\[
C = \frac{5.455}{12.555} \frac{b_s}{b_m} \frac{1}{\xi_0^{3/2}} = C_{VLD} \frac{1}{\xi_0^{3/2}}. \tag{19}
\]

New numerical parameter \( C_{VLD} \approx 1.8104 \times 10^{-2} \). Thus the power-like solution \( n(l) \) of the master ”rate equation” (2) is

\[
n(l) = \frac{C_{VLD}}{\xi_0^{3/2}} l^{-5/2}. \tag{20}
\]

The total length \( L \) per unit of volume is evaluated as follows (we recall that quantity \( \xi_0 \) serves as the low cut-off ):

\[
L = \int_{\xi_0}^{\infty} l * n(l) dl = \frac{2C_{VLD}}{\xi_0^2}. \tag{21}
\]
Result (21) is remarkable. It asserts that interline space $\delta = \mathcal{L}^{-1/2}$ is of the order of the mean radius of curvature $\xi_0$, namely

$$\xi_0 = \sqrt{2C_{VLD}\mathcal{L}^{-1/2}}$$

This idea had been launched by Schwarz [16], it is illustrated in Fig. 6. The nature of this phenomenon was not clear. We proved that this relation appears due to kinetics of colliding vortex loops. In more realistic situation of nonzero temperature connection between interline space the mean radius of curvature $\xi_0$ had been obtain numerically by Schwarz. [16]

$$\xi_0 = \frac{1}{c_2(T)\sqrt{2}}\mathcal{L}^{-1/2}.$$  \hspace{1cm} (23)

The temperature dependent parameter $c_2(T)$ is one of the structure constant of the vortextangle introduced by Schwarz. It is responsible for kinking of the vortex filaments. Function $c_2(T)$ decreases when the temperature grows, this imply that the vortex tangle becomes more kinked at low temperatures. This fact had been reported in numerical works (see e.g., [16],[19]). Comparing (23) and (22) we conclude that parameter $C_{VLD}$, obtained in our approach is the zero-temperature limit of quantay $1/4c_2^2(T)$.For the minimal temperature $T = 1.07$ K covered in the Schwarz’s simulations the latter quantity is approximately 2.04 × 10$^{-2}$. Note that this value is very close to $C_{VLD} \approx 1.8104 \times 10^{-2}$, obtained for the zero temperature case.
In previous subsection we had ascertained the constant $C$ in relation $n(l) = C \ast l^s$ (see (20)) and found the connection between the vortex line density $\mathcal{L}$ with mean curvature $\xi_0$ (see (21)). This enables us to express the net flux $P_{\text{net}}$ (18) via quantity $\mathcal{L}$. Substituting (21) into relation for the net flux (18) we arrive at conclusion that both the positive constituent $P_+$ and the negative one $P_-$ are proportional to the squared vortex line density $\mathcal{L}^2$. In unsteady case at finite temperature quantities $P_+$ and $P_-$ do not compensate each other, so the net flux $P_{\text{net}}$ does not vanish and it is also proportional to the squared vortex line density $\mathcal{L}^2$. That means that the rate of decay of quantity $\mathcal{L}(t)$ due to fluxes carrying away the length from the system can be written as

$$\frac{d\mathcal{L}(t)}{dt} \propto -\mathcal{L}^2.$$  

Relation (24) is the particular case of the so-called Vinen equation discussed in detail in the next Subsection. It is remarkable that relation (24) appears due to the reconnection processes. The own dynamics of filament specific for various systems is absorbed by the quantity $\xi_0$, which has dropped out of the Vinen equation at all. Thus the (24) has the universal character and can be applied for other systems. It reflects growth $\delta = \mathcal{L}^{-1/2} \propto \sqrt{t}$ of the interdefects space, which is general behavior for nonconserved order parameter (see [33]). The result obtained requires one comment. We used stationary solution of the rate equation to describe unsteady situation. This can be justified only when change of $\mathcal{L}(t)$ is slow and structure of loops (namely quantity $\xi_0$) has a time to adjust its equilibrium value expressed by (21). This was confirmed in numerical simulations in [34]. Resuming result of this Subsubsection we would like to stress that our calculations confirmed the Feynman’s conjecture on the formation of cascade-like breakdown of vortex loops, leading to decay of the vortex tangle.

3. Full rate of reconnections.

The full rate of reconnection $\dot{N}_{\text{rec}}$ can be evaluated directly from collision term in the master ”rate equation” (2). Indeed, this term describes change of $n(l)$ due to reconnection events. It takes into account signs of events, depending on whether the loop of size $l$ appears or dies in result of reconnection. Therefore, if we take all terms in collision integral with
the plus sign and use for estimation our solution for \( n(l) \) we obtain the total number of reconnections. The according calculations lead to this result

\[
\dot{N}_{rec} = \frac{1}{3} \kappa \left( b_s C_{VD} + b_m^2 C_{VPD} \right) \xi_0^5 = C_{rec} \kappa L^{5/2},
\]

where \( C_{rec} \) one more constant of the order \( 0.1 - 0.5 \). This result agrees with the recent numerical investigation [20].

**B. Vinen equation**

The aim of this subsection is to study one of the key questions of the theory of superfluid turbulence, namely the evolution of the vortex line density defined in relation (15). Unlike the previous subsection we do not omit deterministic terms in (2), which implies that the interaction with the normal component is taken into consideration. Let us multiply the kinetic equation (2) by \( l \) and integrate over all sizes.

\[
\frac{dL(t)}{dt} = \int \frac{\partial n(l,t)}{\partial t} dl - \int \frac{\partial n(l,t)}{\partial l} \frac{\partial l}{\partial t} dl - |P_{net}|.
\]

(25)

The first term in the right hand side of (25) describes a change of vortex line density \( L(t) \) due to the deterministic motion, in fact due to the mutual friction. Quantity \( P_{net} \) is the net ”flux” of the length (or energy, see subsection II. B) in \( l \)-space. We use the absolute value of \( |P_{net}| \) because the net flux \( P_{net} \) always carries away the vortex line density \( L \) from the system, and different signs refer to direction of the cascade.

Let us treat the deterministic term in equation (25). We calculate the rate of a change in the length of each loop on the base of the motion equation of the line in the so-called local approximation (see e.g., [16]). In this approach the velocity of the line element \( v_l(\xi) \) is

\[
v_l = \beta s' \times s'' + \alpha s' \times (V_{ns} - \beta s' \times s'') + \alpha' s' \times s' \times (V_{ns} - \beta s' \times s'').
\]

(26)

Here \( s' \) and \( s'' \) are the first and second derivatives from position of line \( s(\xi) \) with respect to label variable \( \xi \), which coincides here with the arc length. Quantaties \( \alpha \) and \( \alpha' \) are the temperature dependent friction coefficients. To calculate \( \partial l/\partial t \), we use the relation for the rate of a change in the length \( \partial \delta l/\partial t \) for some arbitrary element with length \( \delta l \). Assuming for a while that the label variable \( \xi \) is not exactly the arc length, we have \( \delta l = |s'| \delta \xi \). Then
the following chain of relations takes place
\[
\frac{\partial \delta l}{\partial t} = \frac{\partial |s'| \delta \xi}{\partial t} = \frac{|s'| \partial |s'| \delta \xi}{\partial t} = \frac{s' \partial s' \delta \xi}{|s'| \partial t} = s' v'_l \delta \xi. \tag{27}
\]

On the last stage we return to condition $|s'| = 1$. Differentiating (26) and multiplying by $s'$ we have after little algebra
\[
\frac{\partial \delta l}{\partial t} = (\alpha (s' \times s'') V_{ns} - \alpha \beta (s' \times s'')^2) \delta \xi. \tag{28}
\]

Terms with $\alpha'$ vanish due to symmetry. The next step is to average expression (27) over all possible configurations of the vortex loops. We do it with use of the Gaussian model of the vortex tangle\[35\]. In accordance with this model
\[
\langle s' \times s'' \rangle = \frac{I_l}{\sqrt{2c_2 \xi_0 |V_{ns}|}}, \quad \langle (s' \times s'')^2 \rangle = \langle (s'')^2 \rangle = \frac{1}{2c_2^2}. \tag{29}
\]

Quantity $I_l$ is another (together with $c_2$) structure constant introduced by Schwarz \[16\]. Substituting (29) into (averaged equation (27)) and then into (25) and integrating by part we get the contribution into $dL(t)/dt$ from the deterministic term
\[
\left(\alpha \frac{I_l |V_{ns}|}{\sqrt{2c_2 \xi_0}} - \alpha \beta \frac{1}{2c_2^2} \right) \int \frac{\partial n(l, t)}{\partial l} l^2 dl = -\alpha \frac{2I_l |V_{ns}|}{\sqrt{2c_2 \xi_0}} L + \frac{\alpha \beta}{\xi_0} L. \tag{30}
\]

Now we have to treat the "flux" term in equation (25). We consider consequently the collision and reconnection events to put the system into the equilibrium (with respect to solution (20)) state much faster than the slow deterministic processes. This implies that the parameters $\xi_0$, $c_2$ and $I_l$ have a time to adjust to their equilibrium values. This assumption is widely adopted and it was confirmed in numerical simulations\[34\]. By use of expression for the net flux\[18\], ridding of the constant $C$ with the help of the normalization condition $L(t) = \int n(l) dl$ and using definition of the Schwarz number $c_2(T)$ (see Subsussection IV.A.1, relations (21),(23)) we rewrite the expression for flux (18) in form $P_{net} = C_F \kappa L^2$, where the temperature constant $C_F$ is
\[
C_F \approx (2.22b_m - 3.926c_2^2b_s). \tag{31}
\]

We named this constant in honor of Feynman who was the first person to discuss evolution of vortex line density due to the reconnection processes. We would like to recall that Feynman supposed the decay of a vortex tangle due to the cascade-like breakdown of vortex loops with further disappearance of them on very small scales. The approach elaborated
here quantitatively confirmed Feynman’s splendid conjecture. Relation (31) shows, however, that there is also possible the inverse cascade, which corresponds to the cascade-like fusion of vortex loops. Unfortunately our approach has too approximate character to do any strong quantitative conclusion. It is clear, however, that for low temperatures, where the vortex tangle is more kinked, correspondingly $c_2$ is large, the quantity $C_F$ is negative. This corresponds to the direct cascade in region of very small loops. On the contrary for high temperature lines are smoother, $c_2$ is small, and $C_F$ is positive, which implies that there is inverse cascade with formation of large loops. If we for instance adopt values for $b_m$ and $b_s$ and use for $c_2$ values offered by Schwarz (see [16]), then we get $C_F \approx -0.252$ for the temperature $1.07$ K and $C_F \approx 0.4$ K for the temperature $2.01$ K.

Collecting contribution into $dL(t)/dt$ from both the deterministic and collision processes (25), (30) and (31), and taking into account that $P_{net} = |C_F| \kappa L^2$ we finally have

$$\frac{dL(t)}{dt} = \frac{5}{2} \alpha I_t |V_{ns}| L^2 - \frac{5}{2} \alpha \beta c_2^2 L^2 - |C_F| \kappa L^2. \tag{32}$$

Thus, starting with kinetics of a network of vortex loops, we get the famous Vinen equation [5].

Let us discuss the meaning of various terms entering this equation. The first, generating term in the right hand side of the Vinen equation describes the growth of the vortex tangle due to the mutual friction. The second term is also connected to the mutual friction, however this term is responsible for a decrease of the vortex line density. This point of view coincides with ideas by Schwarz [16] who obtained the deterministic contribution into $dL(t)/dt$ using a bit different approach. The third term in the right hand side of (32) is related to the random collisions of vortex loops. It describes a decrease of the vortex line density due to the flux of length carrying away the length from the system. Depending on an interplay between coefficients $b_m$ and $b_s$ and the Schwarz parameters $c_2(T)$ the flux can be either positive or negative. We stress again that independently on the sign of the net flux, this third term should result in a decrease of the vortex line density. The negative flux appears when the breakdown of loops prevails and the cascade-like process of generation of smaller and smaller loops forms. There exists a number of mechanisms of disappearance of the vortex energy on very small scales. It can be e.g., the acoustic radiation, collapse of lines, Kelvin waves etc. These dissipative mechanisms balance the growth of the line length due to the mutual friction. As a result, fully developed turbulence with the flux of energy in
direction of small scales is formed, what implies highly chaotic picture of the vortex tangle (see Fig. 7).

The case with inverse is less clear. The inverse cascade implies the cascade-like process of generation of larger and larger loops. Unlike the previous case of the direct cascade, there is no an apparent mechanism for disappearance of very large loops. The probable scenario is that the parts of large loops are pinned to the walls. Finally, a state with few lines stretching from wall to wall with poor dynamics and rare events is realized, this is a degenerated state of the vortex tangle (See Fig. 8). Some of numerical investigators [16], [18], report on this situation. This observation can be an alternative explanation for a phenomenon discovered in Helsinki group [36], who observed transition to superfluid turbulence governed by the temperature.

V. SUMMARY AND CONCLUSION

The evolution of a network of vortex loops in He II, which merge and break down due to reconnections has been considered. It was discussed that because of very frequent reconnections of the vortex loops these processes of recombination is the leading mechanism in the whole dynamics of the vortex tangle. To develop the quantitative description we take the
vortex loops to have a Brownian structure with the only degree of freedom, which is length $l$ of the loop. We perform investigation on the basis of the Boltzmann type "kinetic equation" for distribution function $n(l)$ of number of loops with lengths $l$. By the use of special substitution of variables in the collision integral (Zakharov ansatz) we had found the power-like stationary solution to this equation. This is a non-equilibrium solution characterized by two mutual fluxes of length (energy) in the space of loop sizes. The negative flux (direct cascade) corresponds to the cascade-like breaking down the vortex loops with consequent dissipation of energy on a very small scale. This situation fully coincides with the scenario of superfluid turbulence proposed by Feynman [4]. The positive flux (inverse cascade) corresponds to the cascade-like formation of larger and larger loops. Analyzing this solution we drew several results on the structure and dynamics of the vortex tangle in the superfluid turbulent helium. In particular, we obtained that the mean radius of the curvature is of the order of the interline space. We also evaluated the full rate of reconnection events. Assuming, further, that processes of random colliding are the fastest we studied evolution of the vortex line density $L(t)$ in a presence of mutual friction (for finite temperatures). This evolution was shown to obey the famous Vinen equation. In conclusion we discuss the properties of the Vinen equation from the point of view of the developed approach. Thus, depending on the

FIG. 8: The high temperature numerical simulation of the vortex tangle[18]. It is seen that vortex tangle is generated into the state with very few lines.
temperature (and independently on velocity) vortices either develop into a highly chaotic picture (turbulence), or degenerate into few smooth lines.

ACKNOWLEDGMENTS

This work was partially supported by grants N 05-08-01375 and 07-02-01124 from the RFBR and grant of President Federation on the state support of leading scientific schools RF NSH-6749.2006.8. I am grateful to participants of the workshop ”Superfluidity under Rotation” (Jerusalem, 2007) for useful discussion of the results exposed above.

APPENDIX A: GAUSSIAN MODEL

To evaluate quantities $B(l_1, l_2, l)$ and $A(l_1, l_2, l)$ written in form (5) and (8) one needs to know statistics of individual loops. In general, this statistics should be extracted from an investigation of the full dynamical problem. The according statement of such problem includes equation of the motion (Biot-Savart law for quantum vortices) and dissipative effects (interaction with normal component). This problem is very involved, and at this stage we choose another way, namely, we use the Gaussian model of the vortex tangle elaborated by author [28]. Gaussian model uses the supposition that vortex loops have a random walk (or Brownian) structure. Main mathematical tool to describe the random walk structure is the Wiener distribution (see e.g., [9],[13]). The pure Wiener distribution has some deficiencies to describe real vortex filament. The most apparent one is that Wiener distribution does not have finite average $\langle s'(\xi, t)s'(\xi, t) \rangle$, which is a squared tangent vector. Moreover it does not have the squared second derivative $\langle s''(\xi, t)s''(\xi, t) \rangle$, which is a squared curvature vector.. In classical form it also does not describe possible anisotropy and polarization of the loops. To overcome these difficulties the so-called generalized Wiener distribution had been offered in paper [28]. The generalized Wiener distribution allows to take into account the possible anisotropy and finite curvature. Namely, the probability $\mathcal{P}(\{s(\xi, t)\})$ to find some particular configuration $\{s(\xi, t)\}$ is expressed by the probability distribution functional (see for details the paper by author [28])

$$\mathcal{P}(\{s(\xi, t)\}) = \mathcal{N} \exp \left( - \int_0^t \int_0^t s'^\alpha(\xi_1, t) \Lambda^{\alpha\beta}(\xi_1 - \xi_2)s'^\beta(\xi_2, t) d\xi_1 d\xi_2 \right). \quad (A.1)$$
FIG. 9: Snapshot of the ”average” vortex loop obtained from analysis of the statistical properties. Close ($\Delta \xi \ll \xi_0$) parts of the line are separated in 3D space by distance $\Delta \xi$. The distant parts ($\xi_0 \ll \Delta \xi$) are separated in 3D space by the distance $\sqrt{\xi_0 \Delta \xi}$, i.e. the vortex loop has the typical random walk structure. The scale $\xi_0$ is depicted here in the left upper corner.

Here $\mathcal{N}$ is normalizing factor, $l$ is the length of curve. Parameters of this generalized Wiener distribution (elements of matrix $\Lambda^\alpha\beta(\xi_1 - \xi_2)$) were taken so that some quantities (e.g., mean curvature, coefficients of anisotropy, etc.) evaluated on the basis of (A.1) give the values known from both experimental studies and numerical simulations. Typical form of function $\Lambda^\alpha\beta(\xi, \xi')$ is a smoothed $\delta$ function of a Mexican hat shape with the width equal $\xi_0$. According to this model the ”average” vortex loop has a typical structure shown in Fig. 9. The average loop can be imagined as consisting of many arches with the mean radius of curvature equal $\xi_0$ randomly (but smoothly) connected to each other. The close parts of the loop separated (along line) by distance $\xi_2 - \xi_1$ smaller then the mean radius of curvature $\xi_0$ are strongly correlated, $\langle s'(\xi_1, t)s'(\xi_2, t) \rangle \to 1$, ($s'$ is the tangent vector) and line is smooth. Remote parts of the line $(\xi_2 - \xi_1 \gg \xi_0)$ are not correlated at all, $\langle s'(\xi_1, t)s'(\xi_2, t) \rangle \to 0$. Thus for large separations the vortex loop has a typical ”random walk” structure. This ”semifractal” behavior satisfies to the generalized Wiener distribution.

Being the Gaussian, the Wiener distribution allows to calculate readily any average functional $A(\{s(\xi, t)\})$ depending on configuration $\{s(\xi, t)\}$. It can be done evaluating the following path integral

$$\langle A(\{s(\xi, t)\}) \rangle = \int DsA(\{s(\xi, t)\}) \mathcal{P}(\{s(\xi, t)\}).$$
In practice it is more convenient to deal with the characteristic functional 
\[ W(\{ P(\xi, t) \}) \]
defined as
\[
W(\{ P(\xi, t) \}) = \left\langle \exp \left( i \int_0^l P(\xi, t) s'(\xi, t) d\xi \right) \right\rangle.
\] (A.2)

The characteristic functional enables us calculate any averages depending on vortex lines configuration \( \{ s(\xi, t) \} \) by a simple functional differentiation. For instance, the average tangent vector \( \langle s'_\alpha(\xi_1) \rangle \), or the correlation function between orientation of the different elements of the vortex filaments \( \langle s'_\alpha(\xi_1) s'_\beta(\xi_2) \rangle \) are readily expressed via the characteristic functional accordingly to the following rules:

\[
\langle s'_\alpha(\xi_1) \rangle = \frac{\delta W}{i \delta P^\alpha(\xi_1)} \bigg|_{P=0}, \quad \langle s'_\alpha(\xi_1) s'_\beta(\xi_2) \rangle = \frac{\delta^2 W}{i \delta P^\alpha(\xi_1) i \delta P^\beta(\xi_2)} \bigg|_{P=0}.
\] (A.3)

Calculation of the characteristic functional \( W(\{ P(\xi, t) \}) \) on the base of the probability functional (A.1) is reduced to the functional integration, which, in turn, reduces to the ”full square procedure”. The result is

\[
W(\{ P(\xi, t) \}) = \exp \left\{ - \int_0^l \int_0^l P^\alpha(\xi_1) N^{\alpha\beta}(\xi_1 - \xi_2) P^\beta(\xi_2) d\xi_1 d\xi_2 \right\}.
\] (A.4)

At this stage, to avoid lengthy calculations we simplify the model expressed by the probability distribution functional (A.1), namely we omit both the anisotropy and polarization. We also disregard the closure condition of lines, which is not significant for the rate coefficients. In this case the matrix \( N^{\alpha\beta}(\xi_1 - \xi_2) \) used in (28) is proportional to unit matrix and has to be can be taken as

\[
N^{\alpha\beta}(\xi_1 - \xi_2) = \frac{\delta_{\alpha\beta}}{6} \exp \left[ -\frac{(\xi_1 - \xi_2)^2}{4\xi_0^2} \right].
\] (A.5)

For small separation \( (\xi_2 - \xi_1 \ll \xi_0) \) the sum \( \sum_\alpha N^{\alpha\alpha}(\xi_1 - \xi_2) \to 1/2 \), that guarantees that \( \langle s'(\xi, t)s'(\xi, t) \rangle = 1 \), as it should be for smooth lines. For large separation \( (\xi_2 - \xi_1 \gg \xi_0) \) the exponents in (A.5) tends to \( \delta_{\alpha\beta}(2\sqrt{\pi}\xi_0/6)\delta(\xi_1 - \xi_2) \), and correlation between tangent vectors weakens, \( \langle s'(\xi, t)s'(\xi, t) \rangle \to 0 \), which implies the random walk behavior. Thus the characteristic functional with function \( N \) satisfies to necessary ”semifractal” behavior of closed line and will be used further for evaluation of the rate reconnection coefficients.
Resuming, we exposed main ideas and relations of the gaussian model, which will be used to calculate intensities of fusion and breakdown of vortex loop. It will be done in the Appendix B.

APPENDIX A: APPENDIX B: EVALUATION OF $A(L_1, L_2, L)$ AND $B(L, L_1, L_2)$

1. Evaluation of $B(l, l_1, l_2)$

We start with the self-intersection processes. Our goal to evaluate $B(l, l_1, l_2)$ in accordance with relation (5). Positions of the line elements $s(\xi_2, t), s(\xi_1, t)$ and the relative vector $S_b(\xi_2, \xi_1, t)$ are the strongly fluctuating quantities having the Gaussian statistics. Due to the Wick theorem the average in integrand of (5) can be taken as a sum of all possible pairs of quantity $S_b(\xi_2, \xi_1, t)$ and its derivatives. Because of uniformity in $\xi$ space, quantity $S_b(\xi_2, \xi_1, t)$ depends on $|\xi_2 - \xi_1|$, for this reason all averages of structure $\langle (\partial X/\partial \xi_1)\delta(X(\xi_2, \xi_1, t)) \rangle$ vanish, therefore only the pairs separately from $S_b(\xi_2, \xi_1, t)$ and from its derivatives survive. As a result the average of production is equal to production of averages and each of the factors can be evaluated separately

$$\left\langle \frac{\partial(X, Y, Z)}{\partial(\xi_2, \xi_1, t)} \delta(S_b(\xi_2, \xi_1, t)) \right\rangle_{\xi_2=\xi_1} = \left\langle \frac{\partial(X, Y, Z)}{\partial(\xi_2, \xi_1, t)} \right\rangle_{\xi_2=\xi_1} \langle \delta(S_b(\xi_2, \xi_1, t)) \rangle. \quad (B.1)$$

As mentioned, the use of the characteristics functional (A.2), (A.4) allows to calculate any averaged functional of configurations $\{s(\xi, t)\}$. Let us show how to evaluate $\langle \delta S_s(\xi_2, \xi_1, t) \rangle$. With use of the standard integral representation for $\delta$-function

$$\delta(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{ixy} dy,$$

we rewrite $\langle \delta(S_b(\xi_2, \xi_1, t)) \rangle$ as

$$\langle \delta(S_b(\xi_2, \xi_1, t)) \rangle = \frac{1}{(2\pi)^3} \int \langle \exp [iy(s(\xi_2, t) - s(\xi_1, t))] \rangle d^3y =$$

$$\frac{1}{(2\pi)^3} \int \left\langle \exp \left( i \int_{\xi_1}^{\xi_2} y s'(\xi, t) d\xi \right) \right\rangle d^3y. \quad (B.2)$$

Comparing (A.2) and (B.2) we conclude that the integrand in the last term of (B.2) is just the characteristic functional $W(\{P(\xi, t)\})$, taken at value of $P(\xi, t)$

$$P(\xi) = -y \theta(\xi - \xi_1) \theta(\xi_2 - \xi). \quad (B.3)$$
Here $\theta(\xi)$ is the unit step-wise function. Relation (B.3) implies that we choose in integrand of the characteristic functional only points lying in interval from $\xi_1$ to $\xi_2$ on the curve. Calculation the average of $\delta(S(\xi_2, \xi_1, t))$ we can use the model of pure random walk with elementary step equal to $\xi_0$. Practically we can change function $N^{\alpha\beta}(\xi_1 - \xi_2)$ by $\delta_{\alpha\beta}(2\sqrt{\pi}\xi_0/6)\delta(\xi_1 - \xi_2)$. According calculation lead to the following result

$$\langle \delta(S_b(\xi_2, \xi_1, t)) \rangle = \frac{1}{(2\pi)^3} \int \exp \left[ -y^2 \frac{2\sqrt{\pi} \xi_0}{6}(\xi_2 - \xi_1) \right] d^3y$$

$$= \left( \frac{3\sqrt{3}}{8\pi^{9/4}} \right) \left( \frac{1}{\xi_0 (\xi_2 - \xi_1)} \right)^{3/2}.$$

Evaluation of absolute value of Jacobian in (B.1) we perform by use of relation $|J| = \sqrt{J^2}$ (See [33]). The Jacobian consists of production of derivatives with respect to time and the label variable $\xi$. The latter can be calculated directly from (A.5). Averages including $V_l = ds/dt$ also can be calculated in explicit form, expressing velocity via the vortex filament configuration $\{s(\xi)\}$. However it would be convenient for the sake of generalization to use the velocity factor $V_l = ds/dt$, (see comments at the end of this Section). Calculation of $J^2$ can be fulfilled writing Jacobian in explicit form and subsequent applying of the Wick theorem. Simple but tedious calculations lead to result that

$$J^2 = \langle V^2_{lx} \rangle \langle (\partial s_y / \partial \xi_1)^2 \rangle \langle (\partial s_z / \partial \xi_2)^2 \rangle + \langle V^2_{lx} \rangle \langle (\partial s_y / \partial \xi_2)^2 \rangle \langle (\partial s_z / \partial \xi_1)^2 \rangle + \text{p.p.,} \quad (B.4)$$

where p.p. all permutations with respect to $x, y, z$. Taking $\langle (\partial s_y / \partial \xi_1)^2 \rangle$ and similar terms to be equal to $1/3$, we obtain that $|J| = \sqrt{12/27} |V_l|$. After use of the integration $\int \int d\xi_1 d\xi_2 \delta(\xi_2 - \xi_1 - l_1)$ (see [3]) we finally obtain

$$B(l, l_1, l - l_1) = b_s V_l \frac{l}{(\xi_0 l_1)^{3/2}}, \quad (B.5)$$

where constant $b_s = \sqrt{3/64\pi^{-9/4}} \approx 1.6477 \times 10^{-2}$. We introduced in the coefficient $B$ the additional factor $1/2$ to avoid the over-counting of the reconnection events, since decays $l \rightarrow l_1 + l_2$ and $l \rightarrow l_2 + l_1$ describe the same process, though the both enter into equations.

2. **Evaluation of $A(l_1, l_2, l)$**

Let us now evaluate quantity $A(l_1, l_2, l)$ defined by relation (8). We again (as for the previous case) evaluate the averages from Jacobian and $\delta$-function separately. Contribution
from Jacobian coincides with the previous result $|J| = \sqrt{2}|V_l|/3$. The rest $\delta$-function part can be evaluated with the help of the CF obtained above. Unlike the previous case we have to know two-loop distribution function. Since we omit interaction of loops (until the reconnection event occurs) the CF for two loops with lengths $l_1$ and $l_2$ is just the production of the expressions of type (A.4)

$$ W(\{P_1(\xi)\}, \{P_2(\xi)\}) = \exp \left\{ - \int_0^{l_1} \int_0^{l_2} P^\alpha(\xi_1) N_1^{\alpha\beta}(\xi_1 - \xi_2) P^\beta(\xi_2) d\xi_1 d\xi_2 \right\} \times \quad (B.6) $$

$$ \exp \left\{ - \int_0^{l_1} \int_0^{l_2} P^\alpha(\xi_1) N_2^{\alpha\beta}(\xi_1 - \xi_2) P^\beta(\xi_2) d\xi_1 d\xi_2 \right\}. $$

Quantities $N_1^{\alpha\beta}(\xi_1 - \xi_2)$ and $N_2^{\alpha\beta}(\xi_1 - \xi_2)$ differ from each other only by lengths of loops $l_1$ and $l_2$, entering expressions for $N_1^{\alpha\beta}$. Further, by use of the standard integral representation for $\delta$-function we have

$$ \langle \delta(S_j(\xi_2, \xi_1, t)) \rangle = \frac{1}{(2\pi)^3} \int \exp [i y(s_2(\xi_2, t) - s_1(\xi_1, t))] d^3 y. \quad (B.7) $$

We stress again that the label variables $\xi_2$ and $\xi_1$ belongs two different loops. Let us introduce initial points $s_1(0)$ and $s_2(0)$ and rewrite (B.7) in the following form:

$$ \frac{1}{(2\pi)^3} \int \exp [-i y(s_2(0) - s_1(0))] \langle \exp [i y(s_2(\xi_2) - s_2(0))] \exp [-i y(s_1(\xi_2) - s_1(0))] \rangle d^3 y. $$

Identifying further the ”initial” positions $s_1(0), s_2(0)$ with quantities $R_1, R_2$ in formula (8) we rewrite it as

$$ A(l_1, l_2, l) = \frac{1}{V} \frac{\sqrt{2}}{3} |V_l| \int dR_1 dR_2 \int d\xi_1 d\xi_2 \quad (B.8) $$

$$ \frac{1}{(2\pi)^3} \int \exp [-i y(R_2 - R_1)] \langle \exp [i y(s_2(\xi_2) - s_2(0))] \exp [-i y(s_1(\xi_2) - s_1(0))] \rangle d^3 y. $$

Let us introduce variables $R_1 - R_2, (R_1 + R_2)/2$. Integration over $R_2 - R_1$ gives $\delta(y)$, integration over $(R_1 + R_2)/2$ gives the total volume of system. Further, integration over $y$ gives unity, and integration over $\xi_1, \xi_2$ gives the production $l_1 l_2$. Thus we obtain the remarkable result, that for noninteracting loops the rate coefficient $A(l_1, l_2, l)$ responsible for merging of loops does not depend on statistics of the individual loop at all and is equal to

$$ A(l_1, l_2, l) = b_m V_l l_1 l_2. \quad (B.9) $$
Here $b_m = 1/\sqrt{18} \approx 0.2357$. As earlier we introduced additional factor $1/2$ to avoid the over-counting of the reconnection events.

Results (B.5) and (B.9) (with not well determined factors $b_s$ and $b_m$) were also obtained in papers [11], [12]. Authors used some qualitative picture of moving and colliding elements of lines. This fact confirms the validity of approach made in our work and allows us to use it for more complicated (in comparison with the Brownian loops) cases.

Let us discuss the velocity factor $V_l$ introduced in relations (B.5), (B.9). In accordance with formulas (5), (8), calculation of the coefficients $A(l_1, l_2, l)$ and $B(l, l_1, l_2)$ includes calculations of the time derivatives for the line elements (velocity factors) and derivatives with respect to the label variable $\xi$ (structure factors). Various systems such as polymer chains, cosmic strings, vortex loops have a similar structure, namely, the random walk or semi-random walk, therefore the structure factors for these systems are evaluated in a similar manner. The situation with velocity factor is rather different, which reflects essentially different dynamics for the different systems. Even for quantized vortices, studied in the present paper, the velocity of elements can be determined differently depending on whether the self-induced motion or an external flow prevails in dynamics of the line (cf. with [20]). In view of the said above and for the sake of generality, we prefer to use the general velocity factor $V_l$ instead of calculation in an explicit form. It allows us to extend our formalism for other systems. Nevertheless we could avoid introductions of the velocity factor $V_l$ and act as follows. First we have to express the velocity of the line elements $ds/dt$ via functional depending on configuration $\{s(\xi, t)\}$. This should be done be use of the motion equation (see e.g., [3]) with the full Biot-Savart law for the self-induced motion. Then we have to substitute the components of vector $ds/dt$ into determinant entering (5), (8), and calculate the averaged of absolute value of this determinant with the use of probability functional (A.1). An extremely long chain of calculations brings the result that the velocity factor $V_l$ can be estimated as $V_l = C_v \kappa/\xi_0$, where $\kappa$ is the quantum of circulation and $C_{v1}$ is a constant about unity. There is a much simpler (but cruder) way to estimate constant $C_v$. Velocity of line element with curvature $\xi_0$ in local approximation and neglecting both the mutual friction and the external flow is

$$v_l = \frac{\kappa}{\xi_0} n.$$  

Here $n$ is the unit vector directed along the binormal [37]. The relative velocity of elements
in points $\xi_1$ and $\xi_2$ is
\[
V_l = \left( \frac{\kappa}{\xi_0} n(\xi_1) \right) - \frac{\kappa}{\xi_0} n(\xi_2))
\] (B.10)

In accordance with (B.4) we have to square and to average (B.10). Performing it and taking \( \langle n(\xi_1) n(\xi_2) \rangle = 0 \) we obtain that velocity factor $V_l$ is equal $\sqrt{2\kappa/\xi_0}$, so constant $C_v$ for real loop is close to $\sqrt{2}$.

Results of Appendixes A and B are that we calculate $A(l_1, l_2, l)$ and $B(l, l_1, l_2)$ on the base of Gaussian model. Relations (B.5) and (B.9) are the key results of these Sections. They make the general ”rate equation” (2) to be the closed problem.

---

[1] Sergey K.Nemirovskii, Phys. Rev. Lett., 96, 015301, (2006).
[2] Sergey K.Nemirovskii, J. Low Temperature Physics, Vol. 142, Nos.5/6, (2006).
[3] R.J.Donnelly, Quantized Vortices in HeliumII, (Cambridge University Press, 1991).
[4] R.P.Feynman, in Progress in Low Temperature Physics, edited by C.J.Gorter (North-Holland, Amsterdam, 1955), Vol.I, p.17.
[5] W.F.Vinen, Proc. R. Soc. London A 242, 493 (1957).
[6] S.K.Nemirovskii and W.Fiszdon, Rev. Mod. Phys. 67, 37 (1995).
[7] W. H. Zurek, Nature (London) 317, 505 (1985); Acta Phys. Pol. B 24, 1301 (1993); Phys. Rep., 276,177, (1996).
[8] S.K.Nemirovskii, Theoretical and Mathematical Physics, 141, 141, (2004).
[9] H. Kleinert, Gauge Fields in Condenced Matter Physics, World Scientific Publishing, Singapore,1990.
[10] F.R.N. Nabarro, Theory of Crystal Dislocation, (Oxford Univ. Press. Oxford, 1967).
[11] E.J.Copeland, T.W.B.Kibble and D.A.Steer, Phys.Rev. D, textbf{58}, 043508, (1998).
[12] Joao Magueijo, Havard Sandvik, and Dani’ele A. Steer, http://lanl.arxiv.org/abs/astro-ph/9905363
[13] F.W.Wiegel, Introduction to Path-Integral Methods in Physics and Polymer Sciences, World Scientific Publishing Co Pte Ltd, 1986.
[14] J.Koplik and H.Levine, Phys. Rev. Lett., 71, 1375(1993).
[15] In present work we restrict ourselves by the case, when network of vortex lines consist of only closed loops. We do not consider very long lines stretching from wall to wall. Thus we deal with
the so-called "uniform" superfluid turbulence. We also do not consider turbulence in rotating containers.

[16] K.W.Schwarz, Phys. Rev. B, 38, 2398(1988).

[17] C.F.Barenghi, D.C.Samuels, G.H.Bauer and R.J.Donnelly, Phys. Fluids \{bf 9\}, 2631(1997).

[18] R.G.K.M.Aarts and A.T.A.M.de Waele, Phys. Rev. B 50, 10069, (1994).

[19] M. Tsubota, T. Araki and S. K. Nemirovskii, Phys. Rev. B 62, 11751 (2000).

[20] C.F.Barenghi and D.C.Samuels, Journal of Low Temperature Physics, 136, Nos. 5/6, September 2004.

[21] As it is seen from this paper there are about 20000 loops of length of 1 mm and higher. They accumulate about 70% of total length. The 3D size of such loop is of the order 0.3 mm.

[22] S. K. Nemirovskii, J. Pakleza and W. Poppe, Russian J. Eng. Thermophysics, 3, 369 (1993).

[23] B. V. Svistunov, Phys. Rev. B 52, 3646 (1995).

[24] W.F. Vinen, Phys. Rev. B 61, 1410 (2000).

[25] JETP Letters 78, 553, (2003).

[26] L. Skrbek, Pis’ma v ZhETP, 80, 541, 2004.

[27] G.A. Williams, J. Low Temp. Phys., 101, 421, (1993).

[28] S.K. Nemirovskii, Phys. Rev B57, 5792 (1997).

[29] We omit the rare events when two loops cross each other in two (or more) different points at the same time, or the single loop undergoes two (or more) simultaneous self-intersection. These processes may be important only for the very dense tangle, but the estimations for the vortex line density requires information about details (e.g., duration) of the reconnection process.

[30] Kuzmin P.A. Pis’ma v JETP (Russian), 84, 238, 2006.

[31] V.E. Zakharov, V.S. L’vov, G. Falkovich, Kolmogorov Spectra of Turbulence I, Springer-Verlag, 1992

[32] In the local induced approximation the energy of line is proportional to its length, therefore, instead of the "length" conservation, we can equally to say about conservation of the energy $E(l, t)$ accumulated in loops of size $l$, instead of the flux of "length", to say about the flux of energy and so on.

[33] R.J. Rivers, arXiv:cond-mat/0105171 v1, May 2001.

[34] K.W.Schwarz and Rozen , Phys. Rev. B, 44, 7563 (1991).
The presence of counterflow velocity violates an assumptions of the isotropic Wiener distribution used in previous Sections. This, however, concerns only relative velocity $\mathbf{V}_l$, which is determined only in order of magnitude (see, Appendix B).

A. P. Finne, T. Araki, R. Blaauwgeers, V. B. Eltsov, N. B. Kopnin, M. Krusius, L. Skrbek, M. Tsubota, and G. E. Volovik, Nature, 424, 1022 (2003).

In fact in theis there should present factor of the order $(1/4\pi) \ln(\xi_0/a_0)$ ($a_0$ is the core size of vortex filament). In reality this factor is very close to unity and we omit it.