SINGULAR STRUCTURE OF THE QED EFFECTIVE ACTION

B.A.FAYZULLAEV
Department of Theoretical Physics,
National University of Uzbekistan,
Tashkent 100095, Uzbekistan

March 16, 2022

Abstract

The equations for the QED effective action derived in [3] are considered using singular perturbation theory. The effective action is divided into regular and singular (in coupling constant) parts. It is shown that expression for the regular part coincides with usual Feynman perturbation series over coupling constant, while the remainder has essential singularity at the vanishing coupling constant: \( e \to 0 \). This means that in the frame of quantum field theory it is impossible "to switch off" electromagnetic interaction in general and pass on to "free electron".

Key words: quantum electrodynamics, effective action, singular perturbation expansion, vacuum expectations.
1 Introduction

Investigation of analytic structure of quantum field models usually is reduced to investigation of $S$-matrix as analytic function of its variables such as angular momenta, energy, masses of particles etc. Many results were reached in this way - see, e.g. [1]. But there are no results concerning analyticity of quantum quantities, e.g., vacuum expectations or effective action. Especially, analyticity of vacuum expectations of quantum fields in coupling constants. Expanding amplitude of any process in perturbation series over coupling constant we implicitly suppose analyticity of this amplitude in this coupling constant. But it is well known that perturbation series in quantum field models are asymptotic ones. In this article we want to investigate perturbation series in QED from the following viewpoint: is there any singularity in this series at $e = 0$, where $e$ is electric charge (coupling constant). I.e., can we "switch off" the electromagnetic interaction in Quantum Electrodynamics? In this article we show that this is impossible.

2 Main equations

We will use the following notations: the effective action $W[\eta, \bar{\eta}, J]$ which is generator of all weakly connected Green functions, is connected with the generator of all Green functions $Z[\eta, \bar{\eta}, J]$ (partition function) as follows:

$$Z[\eta, \bar{\eta}, J] = \exp (iW[\eta, \bar{\eta}, J]).$$

Consequently, the vacuum expectations (in the presence of external sources) for quantum fields are

$$\psi_i = \frac{\delta W}{\delta \bar{\eta}_i}, \quad \bar{\psi}_i = -\frac{\delta W}{\delta \eta_i} \quad \text{and} \quad A^\mu_i = \frac{\delta W}{\delta J^\mu_i}.$$ 

Applying to the following classical action for QED (here all the functions are classical fields, not vacuum expectation):

$$S = \frac{i}{2} \overline{\psi} \gamma^\mu \partial_\mu \psi - m \overline{\psi} \psi - e \overline{\psi} A^\mu \psi + \frac{1}{2} A^\mu D^{-1}_{\mu\nu} A^\nu, \quad D^{-1}_{\mu\nu} = g_{\mu\nu} \partial^2 - \left(1 - \frac{1}{\alpha}\right) \partial_\mu \partial_\nu, \quad (1)$$

the DeWitt operator [2]

$$\Lambda = 1 - i h \delta^2 W \frac{\delta^2}{\delta J^\mu_i \delta J^\nu_j} \delta A^\mu_i \delta A^\nu_j + i h \delta^2 W \frac{\delta^2}{\delta J^\mu_i \delta \bar{\eta}_j} \delta A^\mu_i \delta \bar{\psi}_j + i h \delta^2 W \frac{\delta^2}{\delta \eta_i \delta \bar{\eta}_j} \delta A^\mu_i \delta \psi_j + \cdots \quad (2)$$

in accordance with DeWitt equations

$$\Lambda \frac{\delta S}{\delta A^\mu_i} = -J^\mu_i, \quad \Lambda \frac{\delta S}{\delta \bar{\psi}_i} = \bar{\eta}_i, \quad \Lambda \frac{\delta S}{\delta \psi_i} = -\eta_i$$

it may be derived [3] the following set of functional equations for the QED effective action
$W$ (here all the fields are vacuum expectations of quantum fields):

$$
-e\bar{\psi}\gamma^{\mu}\psi + D^{-1\mu\nu}A_{\mu} + i\hbar \text{Tr} \left( \gamma^{\mu} \frac{\delta^2 W}{\delta \eta^{\mu}} \right) = -J^{\mu};
$$

$$
\left[ \bar{\psi}(x) \left( i\partial^{\mu} + eA^{\mu} + m \right) \right]_{\alpha} + i\hbar \left( \frac{\delta^2 W}{\delta J^{\mu}_{\alpha}} \gamma^{\mu} \right) = \eta^{\alpha};
$$

$$
\left[ \left( i\partial^{\mu} - e\hat{A} - m \right) \psi \right]_{\alpha} + i\hbar \left( \frac{\delta^2 W}{\delta \eta^{\mu}} \gamma^{\mu} \right) = -\eta^{\alpha}.
$$

This set of equations coincides with Schwinger equations [4], of course. Taking into account following relations (all derivatives over the grassmann variables are understand as left ones)

$$
\frac{\delta^2 W}{\delta J^{\mu}_{\alpha}} = -\frac{\delta \bar{\psi}^{i}}{\delta J^{\mu}_{\alpha}} = \frac{\delta A^{\mu}_{i}}{\delta \eta^{\mu}}; \quad \frac{\delta^2 W}{\delta J^{\mu}_{\alpha}} = \frac{\delta \bar{\psi}^{i}}{\delta \eta^{\mu}} = \frac{\delta A^{\mu}_{i}}{\delta \eta^{\mu}};
$$

$$
\frac{\delta^2 W}{\delta \eta^{\alpha}} = -\frac{\delta \bar{\psi}^{i}}{\delta \eta^{\alpha}} = -\frac{\delta \psi^{i}_{\alpha}}{\delta \eta^{\alpha}}. \quad (4)
$$

Eqs. (3) may be bringed to the more convenient for further consideration form:

$$
\text{ie}h \text{Tr} \left( \frac{\delta \bar{\psi}^{i}}{\delta \eta^{\mu}} \gamma^{\mu} \right) = \text{ie}h \text{Tr} \left( \frac{\delta \bar{\psi}^{i}}{\delta \eta^{\mu}} \gamma^{\mu} \right) = J^{\mu}_{i} - e\bar{\psi}\gamma_{\mu}\psi + D^{-1\mu\nu}A^{\nu};
$$

$$
-\text{ie}h \frac{\delta \bar{\psi}^{i}}{\delta J^{\mu}_{\alpha}} \gamma^{\mu}_{\alpha} = \text{ie}h \frac{\delta \hat{A}}{\delta \eta} = \bar{\eta}^{i}_{\alpha} - \left[ \bar{\psi}(x) \left( i\partial^{\mu} + eA^{\mu} + m \right) \right]_{\alpha};
$$

$$
\text{ie}h \gamma_{\mu} \frac{\delta \psi^{i}}{\delta J^{\mu}_{\alpha}} = \text{ie}h \frac{\delta \hat{A}}{\delta \eta} = -\eta^{i}_{\alpha} - \left[ \left( i\partial^{\mu} - e\hat{A} - m \right) \psi \right]_{\alpha}. \quad (5)
$$

Usual way to solve these equations is perturbative expansion over small coupling constant $e$. I.e., at first step we put coupling constant $e = 0$, thereby turning these equations into equations for free particles which can be solved easily. Further, interactions between free particles are taken into account iteratively over small parameter $e$ (really, $e^2/4\pi$) getting perturbative power series:

$$
W = W_0 + eW_1 + e^2W_2 + \cdots. \quad (6)
$$

This is usual way. But there is one circumstance must be taken into account. It is obvious that in front of each derivative term there is coupling constant $e$ (and $\hbar$), this means if we put $e = 0$ then our (variational) differential equations transform to algebraic ones. In the following section we show that in this case a full solution to equations of such type (with small parameter) should contains not only regular but singular (in $e$) part too. In the Sect.5 we will adopt this method to QED vacuum expectations.

3 The method of solution

The formulation of problem under consideration may be explained in the following simple example [5]: find solution to equation

$$
\mu \frac{dx(t)}{dt} = a(t)x(t) + b(t), \quad x(0) = x_0, \quad 0 \leq t < \infty \quad (7)
$$
in the form of perturbative expansion over small $\mu$. It is very easy to find exact solution to this equation obeying given boundary condition:

$$x(t) = x_0 \exp \left( \frac{1}{\mu} \int_0^t a(s) \, ds \right) + \frac{1}{\mu} \int_0^t b(s) \exp \left( -\frac{1}{\mu} \int_s^t a(z) \, dz \right) \, ds. \quad (8)$$

It is obvious that $\mu = 0$ is essential singular point for solution to (7) and, consequently, regular perturbative expansion for small $\mu$ can not exist. The reason for such situation can be seen from Eq.(7) itself: if we put $\mu = 0$ in this equation then it fails to be differential equation and becomes to be algebraic one

$$a(t) \ddot{x}(t) + b(t) = 0. \quad (9)$$

But solution to this (algebraic) equation $\ddot{x}(t) = -b(t)/a(t)$ in general can not obey given boundary condition: $a(0)/b(0) \neq x_0$. It happens loss of boundary condition. This means that solution to Eq.(7) can not be considered even as first approximation to exact solution of Eq.(7). This consideration underlies the reason for singularity at $\mu = 0$.

From above mentioned it follows that any expansion of a solution to Eq.(7) around $\mu = 0$ may be singular one only. Derivation of a singular perturbation series according to [5] consist of the following steps. First, take the second term in (8) and integrate it by parts to get the following series:

$$\frac{1}{\mu} \int_0^t b(s) \exp \left( -\frac{1}{\mu} \int_s^t a(z) \, dz \right) \, ds = - \left[ \frac{b(t)}{a(t)} + \mu \left( \frac{b(t)}{a(t)} \right)' \frac{1}{a(t)} + \cdots \right] +$$

$$+ \left[ \frac{b(0)}{a(0)} + \mu \left( \frac{b(0)}{a(0)} \right)' \frac{1}{a(0)} + \cdots \right] \exp \left( \frac{1}{\mu} \int_0^t a(s) \, ds \right).$$

As a result the following series is obtained:

$$x(t) = - \left[ \frac{b(t)}{a(t)} + \mu \left( \frac{b(t)}{a(t)} \right)' \frac{1}{a(t)} + \cdots \right] +$$

$$+ \left[ x_0 + \frac{b(0)}{a(0)} + \mu \left( \frac{b(0)}{a(0)} \right)' \frac{1}{a(0)} + \cdots \right] \exp \left( \frac{1}{\mu} \int_0^t a(s) \, ds \right).$$

Let’s to make substitution $t = \mu \tau$, $s = \mu \zeta$ in the integrand of the exponent. Then

$$\frac{1}{\mu} \int_0^t a(s) \, ds = \int_0^{\tau} a(\mu \zeta) \, d\zeta = a(0) \tau + \frac{a'(0)}{2} \tau^2 + \mu^2 \frac{a''(0)}{6} \tau^3 + \cdots$$

or,

$$\exp \left( \frac{1}{\mu} \int_0^t a(s) \, ds \right) = e^{a(0) \tau} \left[ 1 + \mu \frac{a'(0)}{2} \tau^2 + \mu^2 \frac{a''(0)}{6} \tau^3 + \mu^2 \frac{\tau^4}{4} a^2(0) + \cdots \right]$$
So it has been derived the following series over $\mu$:

$$x(t, \mu) = \tilde{x}(t) + \Pi x(\tau),$$

where

$$\tilde{x}(t) = \tilde{x}_0(t) + \mu \tilde{x}_1(t) + \cdots = -\frac{b(t)}{a(t)} - \mu \left(\frac{b(t)}{a(t)}\right)' \frac{1}{a(t)} + \cdots$$

(10)

is a regular part of the solution and

$$\Pi x(\tau) = \Pi_0 x(\tau) + \mu \Pi_1 x(\tau) + \mu^2 \Pi_2 x(\tau) + \cdots$$

(11)

is a singular one with

$$\Pi_0 x(\tau) = \left(x_0 + \frac{b(0)}{a(0)}\right) e^{a(0)\tau}; \quad \Pi_1 x(\tau) = \left[x_0 + \frac{b(0)}{a(0)}\right] a'(0) \frac{\tau^2}{2} + \left(\frac{b(0)}{a(0)}\right)' \frac{1}{a(0)} e^{a(0)\tau}$$

(12)

eq\text{etc. The terms } \Pi_k x(\tau) \text{ are called boundary layer terms. It is easy to see that}

$$(\tilde{x}_0(t) + \Pi_0 x(\tau))_{t=0} = x_0; \quad (\tilde{x}_k(t) + \Pi_k x(\tau))_{t=0} = 0, \quad k \geq 1.$$  

(13)

Now we can present the algorithm of singular perturbative solution in the following form. Given an equation with boundary condition (7). Then the solution should be divided into two parts as follows: $x(t) = \tilde{x}(t) + \Pi x(\tau)$ and Eq.(7) can be presented in a form:

$$\mu \frac{d\tilde{x}}{dt} + \frac{d\Pi x(\tau)}{d\tau} = a(t)\tilde{x}(t) + a(\mu \tau)\Pi x(\tau) + b(t), \quad t = \tau / \mu.$$  

(14)

Further one should to expand each term in both sides of this equation in series over $\mu$. Equating coefficients in front of the same powers of $\mu$, separately for terms depending on $t$ and terms depending on $\tau$, one obtains equations for determination of terms of the expansions (10) and (11). It is easy to check out that in this way the series (10) and (12) will be obtained. And it is not so hard to see, that this algorithm is equivalent to consider Eq.(14) as divided in to two parts - first part includes terms depending on $t$, and second part includes terms depending on $\tau$. Solutions of these equations connect each other through boundary conditions (13).

4 The regular part of the effective action

Let’s to introduce new scaled variables $\rho = \eta/e, \bar{\rho} = \bar{\eta}/e$ and $j_\mu = J_\mu/e$. And then present each field in Eq.(5) in the form, divided into regular and singular parts:

$$\psi = \psi^R(J, \eta, \bar{\eta}; e) + \Pi \psi(ej, e\rho, e\bar{\rho}; e), \quad \bar{\psi} = \bar{\psi}^R(J, \eta, \bar{\eta}; e) + \Pi \bar{\psi}(ej, e\rho, e\bar{\rho}; e),$$

$$A_\mu = A^R_\mu(J, \eta, \bar{\eta}; e) + \Pi A_\mu(ej, e\rho, e\bar{\rho}; e).$$

Further acting in accordance with above mentioned (in the end of preceding section) method one should to divide Eqs.(5) into part depending on $J, \eta, \bar{\eta}$ and part, depending on $j, \rho, \bar{\rho}$. Let’s for simplification of equations denote the sources as follows: $J, \eta, \bar{\eta} \Leftrightarrow s$ and scaled sources as follows: $j, \rho, \bar{\rho} \Leftrightarrow \sigma$. These settings allows one to write down equations in more shorter form because now it is possible to denote: $\psi^R(J, \eta, \bar{\eta}; e) = \psi^R(s; e)$ for regular part.
and \( \Pi\psi(J, \eta, \bar{\eta}; e) = \Pi\psi(e\sigma; e) \) for singular part of the field \( \psi \). The same notations will be used for other fields too. Then equations for regular parts will be (for sake of simplicity in the below equations for regular parts we will omit the superscript \( R \)):

\[
 i\hbar \text{Tr} \left( \frac{\delta \psi^i(s; e)}{\delta \eta^i} \gamma_\mu \right) = i\hbar \text{Tr} \left( \frac{\delta \psi^i(s; e)}{\delta \eta^i} \gamma_\mu \right) = J^i_\mu - e\bar{\psi}(s; e)\gamma_\mu \psi(s; e) + D^{-1}_{\mu\nu} A^\nu(s; e);
\]

\[
 -i\hbar \frac{\delta \bar{\psi}^i(s; e)}{\delta J^i_\mu} \gamma_\mu = i\hbar \frac{\delta \bar{A}(s; e)}{\delta \eta} = \eta^i - \left[ \bar{\psi}(s; e) \left( i\hat{\partial} + e\bar{A}(s; e) + m \right) \right]_\alpha;
\]

\[
 i\hbar \gamma_\mu \frac{\delta \psi^i(s; e)}{\delta J^i_\mu} = i\hbar \frac{\delta \bar{A}(s; e)}{\delta \eta} = -\eta^i - \left[ (i\hat{\partial} - e\bar{A}(s; e) - m) \psi(s; e) \right]_\alpha.
\]

Solutions to these equations will be searched in the regular perturbation form:

\[
 \psi(s, e) = \psi_0(s) + \psi_1(s) + e^2 \psi_2(s) + \cdots; \quad \bar{\psi}(s, e) = \bar{\psi}_0(s) + e\bar{\psi}_1(s) + e^2 \psi_2(s) + \cdots;
\]

\[
 A_\mu(s, e) = A_{0\mu}(s) + eA_{1\mu}(s) + e^2A_{2\mu}(s) + \cdots.
\]

It is very simple to find these series by iterations. Equations for zeroth order terms:

\[
 J^i_\mu + D^{-1}_{\mu\nu} A^\nu_0(s) = 0; \quad \bar{\eta} - \bar{\psi}_0(s) \left( i\hat{\partial} + m \right) = 0; \quad -\eta^i_\alpha - \left( i\hat{\partial} - m \right) \psi_0(s) = 0.
\]

Their solutions:

\[
 A_{0\mu}(s) = -D_{\mu\nu} J^\nu; \quad \bar{\psi}_0(s) = \frac{1}{i\hat{\partial} + m}; \quad \psi_0(s) = -\frac{1}{i\hat{\partial} - m} \eta.
\]

Regular part of zeroth order effective action:

\[
 W_0 = -\bar{\eta} \frac{1}{i\hat{\partial} - m} \eta - \frac{1}{2} J^\mu D_{\mu\nu} J^\nu.
\]

Equations for first order terms:

\[
 \bar{\psi}_1(s) \left( i\hat{\partial} + m \right) + \bar{\psi}_0(s) \bar{A}_0(s) = 0; \quad \left( i\hat{\partial} - m \right) \psi_1(s) - \bar{A}_0(s) \psi_0(s) = 0;
\]

\[
 i\hbar \text{Tr} \left( \frac{1}{i\hat{\partial} - m} \gamma_\mu \right) + \bar{\psi}_0 \gamma_\mu \psi_0(s) = D^{-1}_{\mu\nu} A^\nu_1(s).
\]

Solutions to them:

\[
 \psi_1(s) = -\frac{1}{i\hat{\partial} - m} \gamma_\mu \frac{\eta}{i\hat{\partial} - m} D_{\mu\nu} J^\nu; \quad \bar{\psi}_1 = -\frac{\bar{\psi}}{i\hat{\partial} + m} \bar{A}_0 \left( i\hat{\partial} + m \right)^{-1};
\]

\[
 A_{1\mu}(s) = -\bar{\eta} \left( i\hat{\partial} + m \right)^{-1} \gamma_\mu \frac{1}{i\hat{\partial} - m} \eta + i\hbar D_{\mu\nu} \text{Tr} \left( \frac{1}{i\hat{\partial} - m} \gamma_\nu \right).
\]
Regular part of first order effective action:

\[ W_1 = -\bar{\psi} \left( \frac{1}{i\partial - m} \Gamma_\mu \right) \Gamma_\mu \psi + \frac{1}{i\partial - m} \eta J^\mu \psi + i\hbar J^\mu D_\mu \psi \left( \frac{1}{i\partial - m} \gamma^\nu \right). \]

Eq. (19) reproduces free propagators for electron-positron and photon fields. Eq. (21) reproduces first order Feynman diagrams, including one-loop tadpole diagram. Acting this way one can to reproduce the Feynman diagrams of all order in coupling constant (electric charge) \( e \). This is why we have called this series as regular perturbation ones. But the existence of the coupling constant (electric charge) \( e \) in front of terms with derivatives in Eqs. \( \delta \) set one thinking about possible singularity at \( e = 0 \).

## 5 Singular parts of vacuum expectations in QED

Equations for singular (boundary layer) parts are more complicated:

\[ -i\hbar \frac{\delta \bar{\psi}(e\sigma; e)}{\delta j_\mu} \gamma_\mu = i\hbar \frac{\delta \bar{\psi}(e\sigma; e)}{\delta j_\mu} \gamma_\mu = -\Pi \varphi(e\sigma; e) \left( i\partial + m \right) - e\Pi \bar{\psi}(e\sigma; e) \tilde{A}(e\sigma; e) - e\Pi \bar{\psi}(e\sigma; e) \Pi \tilde{A}(e\sigma; e); \]
\[ i\hbar \frac{\delta \Pi \varphi(e\sigma; e)}{\delta j_\mu} \gamma_\mu = -\Pi \varphi(e\sigma; e) \left( i\partial + m \right) + e\Pi \bar{\psi}(e\sigma; e) \bar{\psi}(e\sigma; e) + e\Pi \tilde{A}(e\sigma; e) \Pi \varphi(e\sigma; e); \]

Recall that fields \( \psi(e\sigma; e) \), \( \bar{\psi}(e\sigma; e) \) and \( A_\mu(e\sigma; e) \) are regular parts of corresponding fields, but they are functions not of \( s = (J_\mu, \eta, \bar{\eta}) \) but of \( e\sigma = (eJ_\mu, e\rho, e\bar{\rho}) \).

At first step we should to extract zeroth order (in \( e \)) equations from above mentioned ones:

\[ i\hbar \frac{\delta \Pi_0 \varphi(\sigma)}{\delta \rho} \gamma_\mu = i\hbar \frac{\delta \Pi_0 \bar{\psi}(\sigma)}{\delta \rho} \gamma_\mu = D_{\mu\nu}^{-1} \Pi_0 A^\nu(\sigma); \]
\[ -i\hbar \frac{\delta \Pi_0 \bar{\psi}(\sigma)}{\delta j_\mu} \gamma_\mu = i\hbar \frac{\delta \Pi_0 \bar{\psi}(\sigma)}{\delta j_\mu} \gamma_\mu = -\Pi_0 \bar{\psi}(\sigma) \left( i\partial + m \right); \]
\[ i\hbar \frac{\delta \Pi_0 \psi(\sigma)}{\delta j_\mu} = i\hbar \frac{\delta \Pi_0 \bar{\psi}(\sigma)}{\delta j_\mu} \gamma_\mu = - \left( i\partial + m \right) \Pi_0 \psi(\sigma). \]

It is easy to find a general form of solutions to equations for \( \Pi_0 \psi(\sigma) \) and \( \Pi_0 \bar{\psi}(\sigma) \):

\[ \Pi_0 \psi(\sigma) = \exp \left[ i\hbar \int \left( \frac{\tau}{m} \right)(i\partial - m) \right] f + c\psi_D; \]
\[ \Pi_0 \bar{\psi}(\sigma) = \bar{f} \exp \left[ -i\hbar \int \left( \frac{\tau}{m} \right)(i\partial + m) \right] \bar{\psi} + c\bar{\psi}_D, \]
where $\psi_D$ and $\bar{\psi}_D$ are solutions to free Dirac equations, $c$ - an arbitrary constant, $f$ - some spinor field.

Now it is the time to apply equation for boundary condition (13). From Eq.(18) follows that regular parts of fields under consideration vanish at $J_\mu = \eta = \bar{\eta} = 0$. Further, from Lorentz invariance it follows that vacuum expectation for spinor field in external source-free case vanishes:

$$\psi \Big|_{J_\mu = \eta = \bar{\eta} = 0} = \bar{\psi} \Big|_{J_\mu = \eta = \bar{\eta} = 0} = 0.$$ 

These conditions give us that $f \big|_{J_\mu = \eta = \bar{\eta} = 0} = -c\psi_D$. In general let’s present the function $f$ as follows:

$$f = -c\psi_D + \sum_{n=0}^{\infty} c_n (\bar{\rho} \rho)^n s \rho,$$

where $s$ should be found from corresponding indicial equation (after defining of corresponding differential equation for $f$). After substitution of Eq.(29) into (28) we have following expression for singular part of the spinor field:

$$\Pi_0 \psi(\sigma) = c \left[ 1 - \exp \left( \frac{i}{4\hbar} \hat{j}(i\hat{\partial} - m) \right) \right] \psi_D + \exp \left( \frac{i}{4\hbar} \hat{j}(i\hat{\partial} - m) \right) \sum_{n=0}^{\infty} c_n (\bar{\rho} \rho)^n + s \rho =$$

$$\exp \left( \frac{i}{4\hbar} \hat{j}(i\hat{\partial} - m) \right) \sum_{n=0}^{\infty} c_n (\bar{\rho} \rho)^n + s \rho = \exp \left( \frac{i}{4\hbar} \hat{j}(i\hat{\partial} - m) \right) \sum_{n=0}^{\infty} c_n (\bar{\rho} \rho)^n + s \rho$$

and conjugate expression for $\Pi_0 \bar{\psi}(\sigma)$. These expressions has essential singularity at zero coupling limit $e \to 0$. Conclusion about essential singularity at zero coupling limit $e \to 0$ can be referred to $\Pi_0 A^\nu$ too.

So, any vacuum expectation in QED has essential singularity at zero coupling limit $e \to 0$.

### 6 Conclusion

The singularity at $e = 0$ is very interesting - its existence means that in general we can’t ”switch off” electromagnetic interaction and go to ”free electron”. It is the time to remember Dyson’s proof \[6\] that perturbation series in QED is asymptotic one. Our consideration shows that QED effective action can’t be an analytic function in the neighborhood of $e = 0$, consequently, any series in this region can’t be convergent one. In the light of this singularity the notion of ”free electron” should be revised - because it is impossible to ”switch off” the electromagnetic interaction the existence of free, noninteracting electrically charged particle is questionable. But this point is very hard one and more accurate studies required to be conclusively established.

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