Strong Shoda pairs with GAP

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Abstract

We provide algorithms to compute a complete irredundant set of extremely strong Shoda pairs of a finite group $G$ and the set of the primitive central idempotents of the rational group algebra $\mathbb{Q}[G]$ realized by them. These algorithms are also extended to write new algorithms for computing a complete irredundant set of strong Shoda pairs of $G$ and the set of the primitive central idempotents of $\mathbb{Q}[G]$ realized by them. Another algorithm to check whether a finite group $G$ is normally monomial or not is also described.

Keywords : rational group algebra, primitive central idempotents, strong Shoda pairs, extremely strong Shoda pairs, normally monomial groups.

MSC2000 : 16S34, 68W30

1 Introduction

Let $G$ be a finite group and let $\mathbb{Q}[G]$ be the rational group algebra of $G$. A strong Shoda pair of $G$, introduced by Olivieri, del Río and Simón [7], is a pair $(H, K)$ of subgroups of $G$ with the subgroups $H$ and $K$ satisfying some technical conditions. In [1], a strong Shoda pair $(H, K)$ with $H$ normal in $G$ is termed as an extremely strong Shoda pair of $G$. An important property ([7], Proposition 3.3) of the strong Shoda pairs of $G$ is that each such pair $(H, K)$ determines a primitive central idempotent of $\mathbb{Q}[G]$, called the primitive central idempotent of $\mathbb{Q}[G]$ realized by $(H, K)$. Let $E$ be the set of all primitive central idempotents of $\mathbb{Q}[G]$ and $E_{SSP}$ (resp. $E_{ESSP}$) be the set of primitive central idempotents of $\mathbb{Q}[G]$ realized by the strong Shoda pairs (resp. extremely strong Shoda pairs) of $G$. The groups $G$ for which $E = E_{SSP}$ are called strongly monomial groups and are known to constitute

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a large class of monomial groups, including abelian by supersolvable groups \cite{7}. Also, in \cite{1}, it has been proved that $E = E_{SSP} = E_{ESSP}$ if, and only if, $G$ is a normally monomial group i.e., every complex irreducible character of $G$ is induced from a linear character of a normal subgroup of $G$. The GAP package Wedderga \cite{5} features the functions PrimitiveCentralIdempotentsByStrongSP(\text{Q}[G]); that computes the set $E_{SSP}$ for the rational group algebra $\text{Q}[G]$ and the function StrongShodaPairs(G); that determines a subset $X$ of strong Shoda pairs of $G$, which is in bijection with $E_{SSP}$. Such a set $X$ is called a complete irredundant set of strong Shoda pairs of $G$. These functions are based on the search algorithms provided by Olivieri and del Río \cite{6}. Another relevant feature of Wedderga is the function IsStronglyMonomial(G); which checks whether the group $G$ is strongly monomial or not. Using this function, it has been revealed in \cite{4} that all the monomial groups of order less than 1000 are strongly monomial.

In this paper, we provide an algorithm to compute a complete irredundant set of extremely strong Shoda pairs of $G$. This algorithm is based on the work in \cite{1}. We further extend this algorithm by combining it with the search algorithm provided by Olivieri and del Río \cite{6} to obtain a new algorithm that computes a complete irredundant set of strong Shoda pairs of $G$. As a consequence, we obtain algorithms to write the sets $E_{ESSP}$ and $E_{SSP}$ of primitive central idempotents of $\text{Q}[G]$ realized by extremely strong Shoda pairs of $G$ and those realized by strong Shoda pairs of $G$ respectively. Another algorithm to check whether a finite group $G$ is normally monomial or not also follows as a consequence. These algorithms are given in Section 3 and enable us to write following functions in GAP language:

- **ExtStrongShodaPairs(G);** which computes a complete irredundant set of extremely strong Shoda pairs of $G$ i.e., a subset of extremely strong Shoda pairs of $G$, which is in bijection with $E_{ESSP}$.

- **StShodaPairs(G);** which computes a complete irredundant set of strong Shoda pairs of $G$.

- **PrimitiveCentralIdempotentsByExtSSP(\text{Q}[G]);** which computes the set of primitive central idempotents of $\text{Q}[G]$ realized by extremely strong Shoda pairs of $G$.

- **PrimitiveCentralIdempotentsByStSP(\text{Q}[G]);** which computes the set of primitive central idempotents of $\text{Q}[G]$ realized by strong Shoda pairs of $G$. 
\* \textbf{Is Normally Monomial}(G); which checks whether the group \(G\) is normally monomial or not.

Using the function \textbf{Is Normally Monomial}(G); we have searched for normally monomial groups among the groups in GAP library of Small Groups. The search indicates that the class of normally monomial groups is a substantial class of monomial groups. It may be also be mentioned that if \(G\) is a normally monomial group, then the output obtained by the functions \textbf{St Shoda Pairs}(G); and \textbf{Primitive Central Idempotents By St SP}(QG); is same as that obtained by \textbf{Ext Strong Shoda Pairs}(G); and \textbf{Primitive Central Idempotents By Ext SSP}(QG); respectively. Furthermore, for a finite group \(G\), the functions \textbf{St Shoda Pairs}(G); and \textbf{Primitive Central Idempotents By St SP}(QG); are alternative to the functions \textbf{Strong Shoda Pairs}(G); and \textbf{Primitive Central Idempotents By Strong SP}(QG); respectively, which are currently available in the GAP package \textit{Wedderga}.

In Section 4, we compare the runtimes of the function \textbf{St Shoda Pairs}(G); with \textbf{Strong Shoda Pairs}(G); for a large sample of groups of order up to 500. Similarly the function \textbf{Primitive Central Idempotents By St SP}(QG); is compared with the function \textbf{Primitive Central Idempotents By Strong SP}(QG); for runtime. It is observed that these new functions show significant improvement in the time taken to compute the same outputs.

\section{Notation and Preliminaries}

Throughout this paper, \(G\) denotes a finite group. By \(H \leq G\) (resp. \(H \unlhd G\)), we mean that \(H\) is a subgroup (resp. normal subgroup) of \(G\). For \(H \leq G\), \([G : H]\) denotes the index of \(H\) in \(G\), \(N_G(H)\) denotes the normalizer of \(H\) in \(G\), \(\text{core}(H) = \bigcap_{x \in G} x H x^{-1}\), and \(\hat{H} = \frac{1}{|H|} \sum_{h \in H} h\), where \(|H|\) is the order of \(H\). For \(K \trianglelefteq H \subseteq G\), write

\[ e(H, K) := \begin{cases} \hat{H}, & \text{if } H = K; \\ \prod (\hat{K} - \hat{L}), & \text{otherwise}, \end{cases} \]

where \(L\) runs over the minimal normal subgroups of \(H\) containing \(K\) properly. Set

\[ e(G, H, K) := \text{the sum of all the distinct } G\text{-conjugates of } e(H, K). \]

Let \(\varphi\) denote the Euler phi function. Denote by \text{Irr}(G), the set of all complex irreducible characters of \(G\). For \(\chi \in \text{Irr}(G)\), \(\mathbb{Q}(\chi)\) denotes the field obtained by adjoining to \(\mathbb{Q}\), all the character values \(\chi(g)\), \(g \in G\), and \(\text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})\) is the Galois group of the extension \(\mathbb{Q}(\chi)\) over \(\mathbb{Q}\).
It is well known that $\chi \mapsto e_Q(\chi) := \frac{\chi(1)}{|G|} \sum_{\sigma \in \text{Gal}(Q(\chi)/Q)} \sum_{g \in G} \sigma(\chi(g^{-1}))g$ defines a map from $\text{Irr}(G)$ to the set of primitive central idempotents of the rational group algebra $Q[G]$. If $\chi$ is the trivial character of $G$, it is easy to see that $e_Q(\chi) = \hat{G}$.

Olivieri et al \cite{7} proved the following:

**Theorem 1** (\cite{7}, Lemma 1.2, Theorem 2.1)

1. If $\chi$ is a non trivial linear character of $G$ with kernel $N$ then

$$e_Q(\chi) = \varepsilon(G, N).$$

2. If $\chi$ is monomial, i.e. $\chi$ is induced from a linear character $\psi$ of a subgroup $H$ of $G$, then $\exists \alpha \in \mathbb{Q}$ such that

$$e_Q(\chi) = \alpha e(G, H, K),$$

where $K$ is the kernel of the character $\psi$. Furthermore $\alpha = 1$, if the distinct $G$-conjugates of $\varepsilon(H, K)$ are mutually orthogonal.

In view of Theorem \cite{7} and Shoda’s irreducibility criteria (\cite{2}, Theorem 45.2), a pair $(H, K)$ of subgroups of $G$ is said to be a *strong Shoda pair* (\cite{7}, Definition 1.4) if the following hold:

(i) $K \trianglelefteq H \trianglelefteq N_G(K)$;

(ii) $H/K$ is cyclic and a maximal abelian subgroup of $N_G(K)/K$;

(iii) the distinct $G$-conjugates of $\varepsilon(H, K)$ are mutually orthogonal.

Further, in addition if $H \trianglelefteq G$, then the strong Shoda pair $(H, K)$ of $G$ is called an *extremely strong Shoda pair* of $G$. It follows from Theorem \cite{7} that if $(H, K)$ is a strong Shoda pair of $G$, then $e(G, H, K)$ is a primitive central idempotent of $Q[G]$, called the primitive central idempotent of $Q[G]$ realized by $(H, K)$. For a strong Shoda pair $(H, K)$ of $G$, we denote by $\text{dim}(H, K)$, the $\mathbb{Q}$-dimension of the simple component $\mathbb{Q}[G]e(G, H, K)$ of $\mathbb{Q}[G]$. In view of (\cite{7}, Proposition 3.4), $\text{dim}(H, K)$ equals $\varphi([H : K])|N_G(K) : H||G : N_G(K)|^2$.

Two strong (resp. extremely strong) Shoda pairs $(H_1, K_1)$ and $(H_2, K_2)$ of $G$ are said to be equivalent if $e(G, H_1, K_1) = e(G, H_2, K_2)$. A complete set of representatives of distinct equivalence classes of strong (resp. extremely strong) Shoda pairs of $G$ is called a *complete irredundant set of strong (resp. extremely strong) Shoda pairs* of $G$. 

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We now recall the method given in [1] to compute a complete irredundant set of extremely strong Shoda pairs of a finite group $G$.

Let $\mathcal{N}$ be the set of all the distinct normal subgroups of $G$. For $N \in \mathcal{N}$, set
- $A_N$: a normal subgroup of $G$ containing $N$ such that $A_N/N$ is an abelian normal subgroup of maximal order in $G/N$.
- $\mathcal{D}_N$: the set of all subgroups $D$ of $A_N$ containing $N$ such that $\text{core}(D) = N$, $A_N/D$ is cyclic and is a maximal abelian subgroup of $N_G(D)/D$.
- $\mathcal{T}_N$: a set of representatives of $\mathcal{D}_N$ under the equivalence relation defined by conjugacy of subgroups in $G$.
- $S_N$: \[\{(A_N, D) \mid D \in \mathcal{T}_N\}\].

**Theorem 2** ([1], Theorem 1) Let $G$ be a finite group. Then,
(i) $\bigcup_{N \in \mathcal{N}} S_N$ is a complete irredundant set of extremely strong Shoda pairs of $G$.
(ii) $\{e(G, A_N, D) \mid (A_N, D) \in S_N, \ N \in \mathcal{N}\}$ is a complete set of primitive central idempotents of $\mathbb{Q}[G]$ if, and only if, $G$ is normally monomial.

**Corollary 1** ([1], Corollary 1) If $G$ is a normally monomial group, then $\bigcup_{N \in \mathcal{N}} S_N$ is a complete irredundant set of strong Shoda pairs of $G$.

**Corollary 2** ([1], Corollary 2) A finite group $G$ is normally monomial if, and only if
\[\sum_{N \in \mathcal{N}} \sum_{(A_N, D) \in S_N} \dim(A_N, D) = |G|\.

3 Algorithms

We shall use the notation developed in the previous section.

3.1 Extremely Strong Shoda Pairs

We provide Algorithm 1, which computes the set $\text{ESSP}$, which is a complete irredundant set of extremely strong Shoda pairs of a given finite group $G$. This algorithm is based on Theorem 2. It mainly requires the set $\mathcal{N}$ of normal subgroups of $G$ and the computation of $S_N$ for each $N \in \mathcal{N}$. In the algorithm, the process of computation of $S_N$ for certain normal subgroups $N$ of $G$ can be cut down with the following:

**Lemma 1** For a normal subgroup $N$ of $G$, the following hold:
(i) If $G/N$ is abelian, then

$$S_N = \begin{cases} \{(G, N)\}, & \text{if } G/N \text{ is cyclic;} \\ \emptyset, & \text{otherwise.} \end{cases}$$

(ii) If $G/N$ is non abelian and $A_N/N$ is cyclic, then

$$S_N = \begin{cases} \{(A_N, N)\}, & \text{if } A_N/N \text{ is a maximal abelian subgroup of } G/N; \\ \emptyset, & \text{otherwise.} \end{cases}$$

**Proof.** Follows immediately from the definition of $S_N$. □

**Lemma 2** If $M \subseteq \mathcal{N}$ is such that

$$\sum_{N \in M} \sum_{(A_N, D) \in S_N} \dim(A_N, D) = |G|,$$

then $S_N = \emptyset \forall N \in \mathcal{N} \setminus M$.

**Proof.** The primitive central idempotents $e(G, A_N, D)$ for $(A_N, D) \in S_N$, $N \in \mathcal{N}$, are distinct, as $\bigcup_{N \in \mathcal{N}} S_N$ is a complete irredundant set of extremely strong Shoda pairs of $G$. Therefore, $\bigoplus_{N \in \mathcal{N}} \bigoplus_{(A_N, D) \in S_N} \mathbb{Q}[G] e(G, A_N, D)$ is a direct summand of $\mathbb{Q}[G]$, and hence its $\mathbb{Q}$-dimension is at most $|G|$. Consequently,

$$|G| \geq \sum_{N \in \mathcal{N}} \sum_{(A_N, D) \in S_N} \dim(A_N, D) \geq \sum_{N \in M} \sum_{(A_N, D) \in S_N} \dim(A_N, D) \quad (\because M \subseteq \mathcal{N})$$

$$= |G|.$$ 

This yields that $S_N = \emptyset \forall N \notin M$ and completes the proof. □

**Lemma 3** If $(H, K)$ is a strong Shoda pair of $G$ with $N = \text{core}(K)$, then the centre of $G/N$ must be cyclic.

In particular, if $N \in \mathcal{N}$ is such that the centre of $G/N$ is not cyclic then $S_N = \emptyset$.

**Proof.** Let $aK$ be a generator of $H/K$ and let $\zeta$ be a primitive $m^{th}$ root of unity, where $m = [H : K]$. Consider the linear representation $\rho : H \to \mathbb{C}$ given by $x \mapsto \zeta^i$, if $xK = a^iK$, for $x \in H$. Since $(H, K)$ is a strong Shoda pair, $\rho^G$ is an irreducible representation of $G$. Now, as $\ker(\rho^G) = \bigcap_{x \in G} x(\ker \rho)x^{-1} = \bigcap_{x \in G} xKx^{-1} = N$, the result follows from ([3], Lemma 2.27). □
Data: A finite group $G$.

$N := \text{Normal subgroups of } G \text{ (in decreasing order);}$

$ESSP := \emptyset$;

$\text{SumDim} := 0$;

if $G$ is abelian then

while $\text{SumDim} \neq |G|$ do

for $N$ in $\mathcal{N}$ do

if $G/N$ is cyclic then

Add $[G,N]$ to the list $ESSP$;

$\text{dim} := \text{dim}(G,N)$; $\text{SumDim} := \text{SumDim} + \text{dim}$;

end

end

else

List1 := $\emptyset$; List2 := $\emptyset$; List3 := $\emptyset$;

for $N$ in $\mathcal{N}$ do

if $A = G$ then

Add $[A,N]$ to the list List1;

else if $A/N$ is cyclic then

Add $[A,N]$ to the list List2

else

Add $[A,N]$ to the list List3

end

end

while $\text{SumDim} \neq |G|$ do

for $p = [p[1],p[2]]$ in List1 do

if $p[1]/p[2]$ is cyclic then

Add $p$ to the list $ESSP$;

$\text{dim} := \text{dim}(p[1],p[2])$; $\text{SumDim} := \text{SumDim} + \text{dim}$;

end

end

for $p = [p[1],p[2]]$ in List2 do

if $p[1]/p[2]$ is maximal abelian subgroup of $G/p[2]$ then

Add $p$ to the list $ESSP$;

$\text{dim} := \text{dim}(p[1],p[2])$; $\text{SumDim} := \text{SumDim} + \text{dim}$;

end

end

List4 := $\emptyset$;

for $p = [p[1],p[2]]$ in List3 do

if Centre$(G/p[2])$ is cyclic then

Add $p$ to the list List4;

end

end

LIST := List4;

end

while LIST is non empty do

A := LIST[1][1];

$NA := \text{Normal subgroups } D \text{ of } A \text{ such that } A/D \text{ is cyclic;}$

LIST0 := pairs $p = [p[1],p[2]]$ in LIST such that $p[1] = A$;

for $q = [q[1],q[2]]$ in LIST0 do

$D := \text{Subgroups } D \in NA \text{ such that } \text{core}(D) = q[2]$;

$T := \text{Distinct conjugate representatives of } D;$

for $T$ in $T$ do

if $A/T$ is maximal abelian subgroup of $N_G(T)/T$ then

Add $[A,T]$ to the list $ESSP$;

$\text{dim} := \text{dim}(A,T)$; $\text{SumDim} := \text{SumDim} + \text{dim}$;

end

end

end

LIST := LIST \ LIST0;

end

Result: $ESSP$

Algorithm 1: Extremely Strong Shoda Pairs of $G$
We now describe Algorithm 1. The first step of the algorithm is to list all the normal subgroups of $G$ in decreasing order. If $G$ is abelian, then for every normal subgroup $N$, the corresponding set $S_N$ is computed using Lemma 1(i). If $G$ is non abelian, then for every normal subgroup $N$ of $G$, the set $S_N$ is computed with the help of Lemmas 1 and 3, in addition the procedure described in Section 2. In either of the two cases, if $S_N \neq \emptyset$, then the elements of $S_N$ are added to the list $ESSP$, which is initially an empty list. Also, the sum of $\mathbb{Q}$-dimensions of simple components of $\mathbb{Q}[G]$ corresponding to the extremely strong Shoda pairs of $G$ in $S_N$ is simultaneously added to $SumDim$, which is initially set to be 0. Thus, at any stage of the computation, the list $ESSP$ contains the extremely strong Shoda pairs of $G$ found by then, and $SumDim$ denotes the sum of $\mathbb{Q}$-dimensions of the simple components of $\mathbb{Q}[G]$ corresponding to the elements in $ESSP$. In view of Lemma 2, the process stops when either $SumDim = |G|$ or when all the normal subgroups of $G$ are exhausted. The normal subgroups $N$ of $G$ are selected in decreasing order i.e., if the normal subgroup $N_1$ is chosen before the normal subgroup $N_2$, then $|N_1| \geq |N_2|$. This has been done keeping in view the ease of computation of $S_N$, if $G/N$ has small order. This algorithm enables us to write the function $\text{ExtStrongShodaPairs}(G)$; in GAP language.

3.2 Strong Shoda Pairs

We now describe Algorithm 2 to compute the set $StSP$, which is a complete irredundant set of strong Shoda pairs of a given finite group $G$.

```plaintext
Data: A finite group $G$.

$StSP:= A$ complete irredundant set of extremely strong Shoda Pairs of $G$;

$SumDim:= the$ sum of $\mathbb{Q}$-dimensions of simple components of $\mathbb{Q}[G]$ corresponding to the primitive central idempotents realized by the extremely strong Shoda pairs of $G$;

if $SumDim = |G|$ then
  return $StSP$;
else
  $PCIs:= Primitive$ central idempotents of $\mathbb{Q}[G]$ realized by strong Shoda pairs in $StSP$;
  $C:= Conjugacy$ classes of the subgroups of $G$, which are not normal;

  while $SumDim \neq |G|$ do
    for $c$ in $C$ do
      $K:= Representative(c)$; $N:= core(K)$;
      if $Centre(G/N)$ is cyclic then
        $H:= a$ subgroup of $G$ such that $(H, K)$ is a strong Shoda pair and $H \nmid G$;
        $e:= e(G, H, K)$;
        if $e$ is not in the list $PCIs$ then
          Add $e$ to the list $PCIs$;
          Add $[H, K]$ to the list $StSP$;
          $dim:= \dim(H, K)$; $SumDim:= SumDim + dim$;
        end
      end
    end
  end

Result: $StSP$
```

Algorithm 2: $\text{StShodaPairs}(G)$;
Initially, $StSP$ is the list $ESSP$ of extremely strong Shoda pairs of $G$ obtained using Algorithm 1. Also, $SumDim$ is set to be the the sum of $Q$-dimensions of simple components of $Q[G]$ corresponding to the primitive central idempotents realized by extremely strong Shoda pairs of $G$. In case $SumDim = |G|$, by Corollaries 1 and 2, $StSP$ is a complete irredundant set of strong Shoda pairs of $G$ and the algorithm terminates. Otherwise, to find the remaining strong Shoda pairs of $G$, we make use of the algorithm provided by Olivieri and del Río [6] with desired modifications. For a strong Shoda pair $(H,K)$ of $G$, we use the fact that $G/\text{core}(K)$ must be cyclic (Lemma 3). Moreover, if $(H,K)$ realizes a primitive central idempotent of $Q[G]$ different from the one realized by an extremely strong Shoda pair of $G$, then none of $H$ or $K$ is normal in $G$. This algorithm allows us to write the function $StShodaPairs(G)$; in GAP language.

### 3.3 Primitive Central Idempotents

The algorithm to compute the primitive central idempotents of $Q[G]$ realized by extremely strong Shoda pairs of $G$ is similar to Algorithm 1. The only difference is that at any stage of the computation, instead of collecting the elements of $S_N$, one collects the primitive central idempotents realized by them. Using this algorithm, we write the function $PrimitiveCentralIdempotentsByExtSSP(QG)$; in GAP language which computes the set of primitive central idempotents realized by extremely strong Shoda pairs of $G$. To compute the primitive central idempotent $e(G,H,K)$ of $Q[G]$ realized by the strong Shoda pair $(H,K)$ of $G$, we use the function $Idempotent_eGsum(QG,H,K)$; currently available in Wedderga.

Similarly, the algorithm to compute the primitive central idempotents of $Q[G]$ realized by strong Shoda pairs of $G$ is obtained from Algorithm 2 and the corresponding function $PrimitiveCentralIdempotentsByStSP(QG)$; is also obtained.

### 3.4 Normally Monomial Groups

The algorithm to check whether a finite group $G$ is normally monomial or not is obtained by replacing the result $ESSP$ of Algorithm 1 with $SumDim$. In view of Corollary 2, $G$ is normally monomial if and only if $SumDim=|G|$. This algorithm enables us to write the function $IsNormallyMonomial(G)$; in GAP language.

Using the function $IsNormallyMonomial(G)$; we have found by a computer search that 98.84% of the monomial groups of order up to 500 are normally monomial. Also, 97.88% of all the finite groups of order up to 500 are normally monomial. An exhaustive computer search also yields that among the groups of odd
order up to 2000, the only groups which are not normally monomial are:

- SmallGroup(375,2); SmallGroup(1029,12); SmallGroup(1053,51);
- SmallGroup(1125,3); SmallGroup(1125,7); SmallGroup(1215,68);
- SmallGroup(1875,18); SmallGroup(1875,19);

It may be pointed out that all the groups in the above list, except the first two, are non monomial. Moreover, in the first two groups, only the second group is strongly monomial.

4 Runtime Comparison

We now present an experimental runtime comparison between the following two sets of functions:

1. StrongShodaPairs(G); with StShodaPairs(G);

2. PrimitiveCentralIdempotentsByStrongSP(QG); with PrimitiveCentralIdempotentsByStSP(QG);

This experiment has been performed on the computer with Intel Core i7-4770 CPU @ 3.40GHz, 4GB RAM, for a sample \( S \) of 6358 groups of order up to 500. We first describe the sample \( S \). For \( 1 \leq n \leq 500 \), if the number of non isomorphic groups of order \( n \) is less than 200, then \( S \) contains all the groups of order \( n \). Otherwise, we include in the sample \( S \), at least 100 groups of order \( n \), which are evenly spread in the GAP library of small groups.

For \( 1 \leq n \leq 500 \), let \( t(n) \) be the average of the runtimes, taken in milliseconds, for the groups in \( S \) of order \( n \). Define \( T(n) = \sum_{i=1}^{n} t(i) \), \( n \geq 1 \). The graph of \( n \) versus \( T(n) \) for the comparison of the functions StrongShodaPairs(G); and StShodaPairs(G); is presented in Fig.1. Similarly, Fig.2 presents the runtime comparison of the function PrimitiveCentralIdempotentsByStrongSP(QG); with the function PrimitiveCentralIdempotentsByStSP(QG);

In Fig.1, SSP and StSP are the curves for the functions StrongShodaPairs(G); and StShodaPairs(G); respectively. Similarly, in Fig.2, PCIsBySSP and PCIsByStSP are the curves for the functions PrimitiveCentralIdempotentsByStrongSP(QG); and PrimitiveCentralIdempotentsByStSP(QG); respectively.
It is observed that the running time of the computation of strong Shoda pairs of $G$ using the function $\text{StShodaPairs}(G)$; is significantly better than that of $\text{StrongShodaPairs}(G)$;. For the computation of primitive central idempotents of $\mathbb{Q}[G]$ realized by strong Shoda pairs of $G$, the running time of the function $\text{PrimitiveCentralIdempotentsByStSP}(\mathbb{Q}G)$; also shows noticeable improvement over $\text{PrimitiveCentralIdempotentsByStrongSP}(\mathbb{Q}G)$; which is currently available in GAP package Wedderga [5].
References

[1] G.K. Bakshi and S. Maheshwary, *The rational group algebra of a normally monomial group*, J. Pure Appl. Algebra 218 (2014), no. 9, 1583–1593.

[2] W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[3] I. M. Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976, Pure and Applied Mathematics, No. 69.

[4] Eric Jespers, Gabriela Olteanu, and Ángel del Río, *Rational group algebras of finite groups: from idempotents to units of integral group rings*, Algebr. Represent. Theory 15 (2012), no. 2, 359–377.

[5] A. Konovalov A. Olivieri G. Olteanu ´A. del R´ıo O. Broche Cristo, A. Herman and I. van Geldar, *wedderga — wedderburn decomposition of group algebras*, Version 4.7.1; (2014), (http://www.cs.st-andrews.ac.uk/~alexk/wedderga).

[6] A. Olivieri and ´Angel del Río, *An algorithm to compute the primitive central idempotents and the Wedderburn decomposition of a rational group algebra*, J. Symbolic Comput. 35 (2003), no. 6, 673–687.

[7] Aurora Olivieri, Ángel del Río, and Juan Jacobo Simón, *On monomial characters and central idempotents of rational group algebras*, Comm. Algebra 32 (2004), no. 4, 1531–1550.