(S)-convergence and approximation of oscillatory solutions in fluid dynamics

Eduard Feireisl

Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, CZ-115 67 Praha 1, Czech Republic
Institute of Mathematics, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

E-mail: feireisl@math.cas.cz

Received 13 June 2020, revised 3 September 2020
Accepted for publication 1 October 2020
Published 17 February 2021

Abstract
We propose a new concept of (S)-convergence applicable to numerical methods as well as other consistent approximations of the Euler system in gas dynamics. (S)-convergence, based on averaging in the spirit of the strong law of large numbers, reflects the asymptotic properties of a given approximate sequence better than the standard description via Young measures. Similarity with the tools of ergodic theory is discussed.

Keywords: statistical convergence, Euler system, consistent approximation, ergodic theory
Mathematics Subject Classification numbers: 35 A 35, 35 Q 35, 65 J 15.

1. Introduction

As illustrated by numerous recent results, the Euler system describing the motion of an inviscid fluid in the framework of continuum mechanics is (mathematically) ill-posed. This rather pessimistic conclusion applies to models of both incompressible and compressible fluids, see Bressan, Murray [4], Buckmaster, Vicol [5], Chiodaroli [6], DeLellis, Székelyhidi [8], Wiedemann [24], among many others. Accordingly, solutions of the Euler system should be perceived as limits of physically grounded and mathematically well posed approximations, among which various types of zero dissipation limits. A prominent example are approximations by numerical schemes, where the numerical viscosity mimics the physical viscosity present in real fluids.

To study the properties of approximate solutions, we introduce the concept of consistent approximation borrowed from numerics, where the underlying equations are satisfied modulo small consistency errors vanishing in the asymptotic limit. We focus on the case where the
approximate solutions are only bounded, meaning they may exhibit oscillations (called wiggles in numerics) and/or concentrations. In view of the results shown in [3, 12], oscillations and/or concentration appear if, simultaneously,

• the limit Euler system does not admit a regular solution—a consequence of the general weak–strong uniqueness principle stated e.g. in Gwiazda et al [18];
• all accumulation points of the approximate sequence are not smooth, see [3];
• accumulation points of the approximate sequence are not (weak) solutions of the limit system as otherwise the convergence would be strong, see [12].

Anticipating the above scenario we recall the work of DiPerna and Majda [9, 10] on measure-valued solutions, where suitable solutions of the Euler system are identified with the Young measure generated by an approximating sequence. More recently, there has been a series of attempts to compute and visualize the measure-valued solutions to the Euler system, see Fjordholm et al [16, 17] or [15].

The main aim of the present paper is to propose a different strategy based on statistical averaging, inspired by the concept of $K$-convergence, cf Balder [2], Komlós [19]. To illuminate the main idea, consider a simple sequence of functions:

$$U_n = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}.
\end{cases}$$

Of course, more sophisticated and physically relevant examples can be produced. Consider the asymptotic limit of $\{U_n\}_{n=1}^{\infty}$ for $n \to \infty$. A naive but natural solution would be the ‘limit’ is a convex combination of Dirac masses, specifically,

$$U_n \to \frac{1}{2} \delta_1 + \frac{1}{2} \delta_0, \quad (1.1)$$

where $\delta_X$ denotes the Dirac mass at $X$. However, the standard approach based on the theory of Young measures provides a different conclusion: choosing a ‘suitable’ subsequence ($n_k$ odd and $n_l$ even) we get either

$$U_{n_k} \to 1, \quad \text{or, more precisely, } U_{n_k} \to \delta_1, \quad \text{as } k \to \infty$$

or

$$U_{n_l} \to \delta_0 \quad \text{as } l \to \infty.$$

Intuitively, the limit (1.1) reflects better the asymptotic properties of $\{U_n\}_{n=1}^{\infty}$, however, it is never seen if the standard Young measure based theory is applied. The goal of this paper is to introduce the concept of statistical limit that will identify (1.1) as the only eligible option.

We claim that convergence ‘up to a subsequence’, that may be useful in theoretical studies, is of no practical use in numerical experiments, where unconditional convergence is implicitly assumed. What is more, the weak convergence, meaning convergence in the sense of integral averages, is difficult to visualize as well. The same applies to Young measures that are necessarily objects resulting from a weak limit.

Our goal is to introduce a concept of statistical (S)-convergence of approximate solutions, where the limit is identified with a parametrized measure representing a generalized solution of the target problem. The convergence is strong (a.a.) with respect to the physical space and in the Wasserstein metric on the space of probability measures. In particular, all observable quantities like the barycenter, deviation, variance as well as higher order moments of the limit measure can be identified as limits of strongly converging sequences. Moreover, (S)-convergence is robust with respect to statistical perturbations of the approximate sequence.
A sufficient but definitely not necessary condition for a sequence to be (S)-convergent is its asymptotic stationarity. This means, very roughly indeed, that statistical distribution of observable quantities features certain ergodicity. In particular, for strongly (S)-convergent sequences of measurable functions \( \{U_n\}_{n=1}^{\infty} \), the limit of ergodic averages

\[
\frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to \overline{b(U)} \text{ in } L^1 \quad \text{as } N \to \infty, \quad w_N \equiv \sum_{n=1}^{N} w \left( \frac{n}{N} \right),
\]

exists for any bounded continuous function \( b \) and any weight function \( w \in C^1[0,1], w \geq 0, \int_0^1 w(z) \, dz = 1 \). The mapping \( V : b \mapsto \overline{b(U)} \) can be identified with a (parametrized) measure—the (S)-limit of \( \{U_n\}_{n=1}^{\infty} \). Here and hereafter, the symbol \( \overline{b(U)} \) is used to denote a limit associated to the compositions \( b(U_n) \). However, in contrast with the standard notation used in the theory of Young measures, \( \overline{b(U)} \) may not be a weak limit of \( \{b(U_n)\}_{n=1}^{\infty} \).

Our working plan is as follows:

- In section 2, we introduce the concept of (S)-convergence and discuss its basic properties.
- In section 3 we show that (S)-convergence is stable with respect to statistical perturbations. Furthermore, we discuss several examples of (S)-converging sequences, in particular perturbations of stationary approximations in section 4. We also establish several sufficient conditions for a sequence to be (S)-convergent in terms of asymptotic stationarity.
- Finally, in section 5, we apply the abstract theory to approximate sequences of the isentropic Euler system.

## 2. Statistical limit, (S)-convergence

Let \( Q \subset R^M \) denote the physical space. In the context of fluid mechanics, we usually consider

\[
Q = \{(t,x) \mid t \in (0,T), \quad x \in \Omega \},
\]

where \( t \) denotes the time, and \( x \in \Omega \subset R^d, d = 1, 2, 3, \) is the space variable confined to a physical domain \( \Omega \) occupied by the fluid. The state of the system will be denoted by \( U = U(t,x) \). For the Euler system, the vector \( U \) contains the basic state variables, for instance \( U = [\varrho, m, S] \), where \( \varrho \) is the mass density, \( m \) is the momentum, and \( S \) is the total entropy. Accordingly, \( \{U_n\}_{n=1}^{\infty} \) denotes a sequence of approximate solutions.

We introduce the basic concept of (S)-convergence inspired by the theory of stationary stochastic processes, see e.g. Krylov [21, chapter 4], and the approach developed by Das and Yorke [7].

**Definition 2.1 ((S)-convergence).** Let \( U_n : Q \to R^D, n = 1, 2, \ldots, \) be a sequence of measurable functions.

(a) We say that \( \{U_n\}_{n=1}^{\infty} \) is weakly (S)-convergent if for any \( b \in C_c(R^D) \) there holds:

- **Weak correlation limit**

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(U_n)b(U_m) \, dy \text{ exists} \quad (2.1)
\]

for any fixed \( m \);
Nonlinearity 34 (2021) 2327

E Feireisl

- Weak correlation disintegration

\[ \lim_{N \to \infty} \sum_{n,m=1}^{N} \frac{1}{N^2} \int_{Q} b(U_n)b(U_m) dy \]

\[ = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \left[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_{Q} b(U_n)b(U_m) dy \right]. \quad (2.2) \]

(b) We say that \( \{U_n\}_{n=1}^{\infty} \) is strongly (S)-convergent if for any \( b \in C_c(\mathbb{R}^d) \) and any \( w \in W = \left\{ w \in C^1[0,1] \mid w \geq 0, \quad \int_0^1 w(z) \, dz = 1 \right\} \) there holds:

- Strong correlation limit

\[ \lim_{N \to \infty} \frac{1}{w_N} \sum_{n=1}^{N} \int_{Q} w \left( \frac{n}{N} \right) b(U_n) b(U_m) \, dy, \quad w_N \equiv \sum_{n=1}^{N} w \left( \frac{n}{N} \right), \quad \text{exists} \]

for any fixed \( m \) and is independent of \( w \);

- Strong correlation disintegration

\[ \lim_{N \to \infty} \sum_{n,m=1}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} b(U_n)b(U_m) \, dy \]

\[ = \lim_{M \to \infty} \frac{1}{w_M} \sum_{m=1}^{M} w \left( \frac{m}{M} \right) \left[ \lim_{N \to \infty} \frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) \int_{Q} b(U_n)b(U_m) \, dy \right]. \quad (2.4) \]

for any \( w \in W \).

Obviously strong \( \Rightarrow \) weak as the choice \( w \equiv 1 \) yields the desired conclusion. Conditions (2.1), (2.2) are rather mild, and, as we shall see below, allow for a large class of approximate sequences (approximate solutions in the context of numerical analysis) that comply with a generalized form of the strong law of large numbers:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b(U_n) \quad \text{exists for any} \quad b \in C_c(\mathbb{R}^d), \]

see theorem 2.6 below. Note that (2.1) and even (2.3) hold as soon as the sequence \( \{U_n\}_{n=1}^{\infty} \) generates a Young measure, specifically,

\[ b(U_n) \to b(U) \quad \text{(weakly)} \quad \text{as} \quad n \to \infty \quad \text{for any} \quad b \in C_c(\mathbb{R}^d). \]

The opposite implication in general fails. Indeed it is easy to construct a sequence of functions (real numbers) that give rise to (infinitely many) different Young measures generated by different subsequences for which (2.1), (2.2) or even (2.3), (2.4) still hold.
2.1. Limit of an (S)-convergent sequence

Our next goal is to associate a limit to an (S)-convergent sequence—a parametrized measure

$$\mathcal{Y} \in L^\infty_{\text{weak}-\ast}(\Omega; \mathcal{M}^+(\mathbb{R}^d))$$

that can be seen as an analogue of the Young measure for weakly converging sequences. To this end, a few preliminary observations are needed.

**Lemma 2.2.** Let $b \in C_c(\mathbb{R}^d)$ and $w \in W$ be given.

The following is equivalent:

- the correlation limit (2.3) exists for any $m$;
- 

$$\frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to b(U) \quad \text{weakly} - \ast \quad \text{in } L^\infty(\Omega) \quad \text{as } N \to \infty. \quad (2.5)$$

Moreover, if the correlation limit is independent of $w$, then so is the weak limit $b(U)$.

**Proof.** Obviously (2.5) $\Rightarrow$ (2.3) for any $m$; whence it is enough to show the opposite implication. Write the Hilbert space $L^2(\Omega)$ as

$$L^2(\Omega) = Y \oplus Y^\perp,$$

where

$$Y \equiv \text{span} \{ b(U_m) \mid m = 1, 2, \ldots \}.$$

Denote

$$B_N \equiv \frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n).$$

Obviously $B_N \in Y$, and, by virtue of (2.3),

$$\lim_{N \to \infty} \int_{\Omega} B_N \phi \, dy \quad \text{exists for any } \phi \in Y.$$

On the other hand,

$$\int_{\Omega} B_N \, v \, dy = 0 \quad \text{whenever } v \in Y^\perp.$$

Writing any $u \in L^2(\Omega)$ as $u = \phi \oplus \phi^\perp$ we may infer that

$$B_N \to b(U) \quad \text{weakly in } L^2(\Omega) \quad \text{as } N \to \infty.$$

As $b(U_n)$ are uniformly bounded, this yields (2.5). \qed

The strong correlation limit postulated in (2.3) may seem difficult to check. It can be simplified as stated in the following assertion.

**Lemma 2.3.** Let $\{U_n\}_{n=1}^\infty$ be a sequence of measurable functions, $U_n : \Omega \to \mathbb{R}^d$. Let $b \in C_c(\mathbb{R}^d)$ be given.
Then the following is equivalent:

- the strong correlation limit (2.3) exists and is independent of \( w \in W \);
- the limit
  \[
  \frac{1}{\beta - \alpha} \lim_{N \to \infty} \frac{1}{N} \sum_{0 < n \leq N} \int_Q b(U_n)b(U_m)dy
  \]
  exists for any fixed \( m \), any \( 0 \leq \alpha < \beta \leq 1 \), and is independent of \( \alpha, \beta \).

**Proof.** Consider a piecewise constant function of \( z \in [0, 1] \),

\[
B_N(z, \phi) = \int_Q b(U_n)\phi \, dy \quad \text{if} \quad z \in \left[ \frac{n-1}{N}, \frac{n}{N} \right), \quad 1 \leq n \leq N.
\]

In accordance with lemma 2.2, the existence of the strong correlation limit (2.5) can be equivalently stated as

\[
\int_0^1 w(z)B_N(z, \phi) \, dz = \sum_{n=1}^N \int_{\frac{n-1}{N}}^{\frac{n}{N}} w(z) \, dz \int_Q b(U_n)\phi \, dy + \sum_{n=1}^N \int_{\frac{n-1}{N}}^{\frac{n}{N}} w \left( \frac{n}{N} \right) \, dz \int_Q b(U_n)\phi \, dy
\]

\[
\quad + \sum_{n=1}^N \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left( w(z) - w \left( \frac{n}{N} \right) \right) dz \int_Q b(U_n)\phi \, dy
\]

\[
\to \int_0^1 w(z) \int_Q b(U)\phi \, dy \, dz \quad \text{for any} \quad \phi \in L^1(Q), \quad w \in W \quad \text{as} \quad N \to \infty.
\]

where we have used that \( w \in C^1[0, 1] \), specifically,

\[
\left| \sum_{n=1}^N \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left( w(z) - w \left( \frac{n}{N} \right) \right) dz \int_Q b(U_n)\phi \, dy \right| \leq c(b, w, \phi) \frac{1}{N}.
\]

As the above convergence holds for any \( w \in W \), we conclude

\[
B_N(z, \phi) \to \int_Q b(U)\phi \, dy \quad \text{weakly} \quad (\ast) \quad \text{in} \quad L^\infty(0, 1),
\]

where the hypothesis that the correlation limit is independent of \( w \) is reflected by the fact that the weak limit \( \int_Q b(U)\phi \, dy \) is independent of \( z \). In fact, relation (2.7) is equivalent with (2.3) which yields the desired conclusion. \( \square \)

We are ready to define a limit for an \((S)\)-convergent sequence. Identifying

\[
L^\infty_{\text{weak-}*}(Q; \mathcal{M}(R^D)) \quad \text{with the dual} \quad \left[ L^1(Q; C_0(R^D)) \right]^*
\]

we define

\[
\langle \mathcal{V}; \varphi b \rangle = \int_Q \varphi \, b(U) \, dy \quad \text{for} \quad \varphi \in L^1(Q), \quad b \in C_0(R^D),
\]

where \( b(U) \) is the (unique) limit identified in (2.5), and \( \langle \cdot; \cdot \rangle \) denotes the duality pairing between the spaces in (2.8). This defines a unique linear functional on \( L^1(Q; C_0(R^D)) \) via standard density argument. Obviously, the limit measure is bounded non-negative,

\[
\mathcal{V} \in L^\infty_{\text{weak-}*}(Q; \mathcal{M}^+(R^D)).
\]
The measure $\mathcal{V}$ will be termed (S)-limit of the sequence $U_{n=1}^\infty$, and we shall write

$$U_n \overset{(S)}{\longrightarrow} \mathcal{V}. \quad (2.9)$$

**Remark 2.4.** The (S)-limit, similarly to the Young measure, can see only oscillations but not concentrations in the approximating sequence. A concentration component could be incorporated in the definition using, for instance, the approach of Alibert–Bouchitté [1]. We will not follow this strategy here for several reasons.

- The concentration require certain structure of non-linearities that is not natural in many applications. For instance, in the context of the Euler system discussed in section 5, enforcing the approach of [1] would require considering rather non-physical state variables as ‘pseudo-velocity’ $\sqrt{\varrho}u$, where $\varrho$ is a mass density and $u$ a velocity of a fluid.
- As we show in section 5, concentrations can be captured by a more elegant way, again in the context of the Euler system, by the concept of dissipative solutions and the associated Reynolds stress defect.

**Remark 2.5.** Similarly to the theory of Young measures, convergence in (2.5) can be extended to a larger class of nonlinearities if suitable integral bounds are available. In particular, if

$$\int_Q F(|U_n|)dy \leq c \quad \text{uniformly for } n \to \infty$$

for some $F \geq 0$, then (2.5) can be extended to the class of functions $b$ satisfying

$$\limsup_{|U| \to \infty} \frac{|b(U)|}{F(|U|)} = 0.$$

### 2.2. Equivalence with convergence of ergodic means

To give a more specific meaning to (2.9), we need the following result.

**Theorem 2.6 (Equivalence principle).**

(a) The following is equivalent:

- $\{U_n\}_{n=1}^\infty$ is weakly (S)-convergent;
- $$\frac{1}{N} \sum_{n=1}^N b(U_n) \to b(U) \quad (\text{strongly in } L^1(Q)) \quad (2.10)$$

for any $b \in C_c(R^d)$.

(b) The following is equivalent:

- $\{U_n\}_{n=1}^\infty$ is strongly (S)-convergent;
- $$\frac{1}{wN} \sum_{n=1}^N w \left( \frac{R}{N} b(U_n) \right) \to b(U) \quad (\text{strongly in } L^1(Q)) \quad (2.11)$$

for any $b \in C_c(R^d)$, and any $w \in W$, where the limit is independent of $w$. 

2333
Proof. It is enough to show the equivalence for strongly (S)-convergent sequences. As (2.11) clearly implies strong (S)-convergence, we focus on the opposite implication. By virtue of lemma 2.2, we have

\[ R_N = \frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to b(U) \quad \text{weakly-* in } L^\infty(Q), \]

where the limit is independent of the weight function \( w \). Accordingly, relation (2.11) can be reformulated as

\[ \int_Q |B_N|^2 \, dy \to \int_Q |b(U)|^2 \, dy \quad \text{as } N \to \infty, \]

which is nothing other than (2.4). \( \square \)

Remark 2.7. As \( b \) is bounded and \( |Q| < \infty \), the convergence in (2.11) can be replaced by

\[ \frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to b(U) \quad \text{in } L^q(Q) \]

for any \( 1 \leq q < \infty \).

Going back, we may rewrite (2.9) in the form

\[ U_n \overset{(S)}{\rightharpoonup} \mathcal{V} \iff \int_Q \left| \frac{1}{N} \sum_{n=1}^{N} \delta_{U_n(y)} ; \mathcal{V} \right| \, dy \to 0 \quad \text{as } N \to \infty, \quad (2.12) \]

where \( \delta \) is the Dirac mass, and \( d_{\text{weak-*}} \) denotes the metric on a bounded ball in \( \mathcal{M}(R^D) \) endowed with the weak-* topology.

Here, the relation \( (S) \rightharpoonup \) is considered for weakly (S)-convergent sequences. If, in addition, the convergence is strong, we may identify the limit measure in (2.12) through

\[ \int_Q d_{\text{weak-*}} \left( \frac{1}{w_N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) \delta_{U_n(y)} ; \mathcal{V} \right) \, dy \to 0 \quad \text{as } N \to \infty \quad (2.13) \]

choosing arbitrary weight function \( w \). As observed by Krengel [20], the rate of convergence for (S)-converging sequences can be rather slow, which may be embarrassing in numerical applications. Das and Yorke [7] therefore proposed to introduce the weights \( w \) and obtained super convergence for certain quasiperiodic sequences. The possibility of choosing suitable \( w \) can possibly improve the rate of convergence of numerical approximations.

Although the analogy between \( \mathcal{V} \) and the Young measure is obvious, we note that an (S)-convergent sequence may not generate any Young measure. Still the following result holds:

Proposition 2.8. Suppose that \( \{U_n\}_{n=1}^\infty \) satisfies

\[ \int_Q F(|U_n|) \, dy \sim 1 \quad \text{uniformly for } n \to \infty, \]

where \( F : [0, \infty) \to [0, \infty) \) is continuous non-decreasing, \( \lim_{r \to \infty} F(r) = \infty \).

Then there is a subsequence \( \{U_{n_k}\}_{k=1}^\infty \) that is weakly (S)-convergent and generates a Young measure that coincides with its (S)-limit.
Proof. Using the celebrated Komlós theorem (see Komlós [19]), Balder [2] showed the existence of a subsequence satisfying (2.10) for any \( b \in C_c(\mathbb{R}^d) \); whence the desired conclusion follows from lemma 2.6. \[ \square \]

2.3. Convergence in Wasserstein distance

Up to now, we did not impose any further hypotheses concerning integrability of the generating sequence. In applications, integrability is usually assumed to ensure finiteness of the first moments, and, in particular, to identify the barycenter of the measure \( V \).

**Proposition 2.9.** Let \( \{U_n\}_{n=1}^\infty \) be a sequence of measurable functions satisfying

\[
\int_Q |U_n|^p \, dy \leq c \quad \text{uniformly for } n = 1, 2, \ldots, \quad p \geq 1.
\]

Let

\[
U_n \overset{(S)}{\to} V.
\]

Then

- \( V \) is a probability measure on \( \mathbb{R}^D \) for a.a. \( y \in Q \), with finite moments of order \( p \).
- If \( p > 1 \), then

\[
\int_Q d_{W_s} \left[ \frac{1}{N} \sum_{n=1}^N \delta_{U_n(y)}; V \right] \, dy \to 0 \quad \text{as } N \to \infty, \quad 1 \leq s < p.
\]

where \( d_{W_s} \) denotes the Wasserstein distance of \( s \)th order (see Villani [23] for the definition of the Wasserstein distance).

In particular, the barycenters converge,

\[
\frac{1}{N} \sum_{n=1}^N U_n \to \langle \mathcal{V}; \tilde{U} \rangle \equiv U \quad \text{as } N \to \infty,
\]

in \( L^s(Q) \), \( 1 \leq s < p \).

**Proof.** The proof is quite standard and can be done in the same way as for the Young measures, see e.g. [14]. \[ \square \]

**Remark 2.10.** If the sequence \( \{U_n\}_{n=1}^\infty \) in proposition 2.9 is strongly (S)-convergent, then

\[
\frac{1}{N} \sum_{n=1}^N \text{ may be replaced by } \frac{1}{w_N} \sum_{n=1}^N w \left( \frac{n}{N} \right)
\]

for any weight \( w \in W \).

As a matter of fact, the measure \( V \) is a probability measure under the very mild assumption of integrability stated in proposition 2.8. If \( s = 1 \), the celebrated Komlós theorem (see [19]) yields a subsequence \( \{U_{n_k}\}_{k=1}^\infty \) such that

\[
\frac{1}{N} \sum_{k=1}^N U_{n_k} \to U_K \quad \text{a.a. in } Q.
\]
Moreover, the theory of Balder [2] asserts that this subsequence can be chosen in such a way that
\[ U_{n_k} \xrightarrow{(S)} V_K \quad \text{as} \quad k \to \infty, \]
where \( V_K \) is the Young measure generated by \( \{ U_{n_k} \}_{k=1}^{\infty} \), cf proposition 2.8. Note, however, that there may be different subsequences generating different limits; whence, in general
\[ U_K \neq U, \quad V_K \neq V. \]

To conclude, let us point out, that the strength of the concept of (S)-convergence lies in the fact there is no need for subsequence, which is particularly relevant in numerical applications. Moreover, statistical deviations are eliminated and a large class of approximate sequences can be accommodated as we shall show in the next section.

3. Robustness with respect to statistical perturbations

We start by introducing the concept of statistically equivalent sequences, see e.g. León-Saavedra et al [22].

**Definition 3.1 (Statistical equivalence).** We say that two sequences \( \{ U_n \}_{n=1}^{\infty}, \{ V_n \}_{n=1}^{\infty} \) of measurable functions are statistically equivalent, if for any \( \varepsilon > 0 \),
\[
\frac{\# \left\{ k \leq N \left| \int_Q |U_n - V_n| \, dy > \varepsilon \right. \right\}}{N} \to 0 \quad \text{as} \quad N \to \infty. \tag{3.1}
\]
The condition (3.1) allows for large deviations of the two sequences provided they do not occur ‘very often’. Obviously, it is satisfied if the difference \( U_n - V_n \) tends to zero in \( L^1(Q) \). We refer to [22] for more sophisticated criteria of statistical equivalence.

Next we show that statistical perturbations do not influence the (S)-convergence.

**Theorem 3.2 (Statistical perturbation).** Suppose that
\[ \{ U_n \}_{n=1}^{\infty} \overset{(S)}{\approx} \{ V_n \}_{n=1}^{\infty}, \]
and that
\[ U_n \xrightarrow{(S)} \mathcal{V}. \]

Then
\[ V_n \xrightarrow{(S)} \mathcal{V}. \]

**Proof.** In view of lemma 2.6, we have to show that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b(V_n) = \overline{b}(\mathcal{V}) = \overline{b}(U) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b(U_n) \quad \text{for any} \quad b \in C_c(R^D). \tag{3.2}
\]
Suppose \( b \in C_c(\mathbb{R}^d) \) is Lipschitz. Choose \( \varepsilon > 0 \) and define the set
\[
r(\varepsilon) = \{ n \text{ positive integer} \mid \|V_n - U_n\|_{L^1(Q)} \leq \varepsilon \}.
\]
Now, compute
\[
\frac{1}{N} \sum_{n=1}^{N} b(V_n) = \frac{1}{N} \sum_{n \in \mathbb{N}, n \notin r(\varepsilon)} b(V_n) + \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} b(V_n),
\]
where, in view of (3.1),
\[
\frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} b(V_n) \to 0 \text{ in } L^\infty(Q) \text{ as } N \to \infty
\]
as \( b \) is bounded.

Next,
\[
\frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} b(V_n) = \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} b(U_n) + \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} [b(V_n) - b(U_n)],
\]
where, by the same token as above,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} b(U_n) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b(U_n) = b(U) \text{ in } L^1(Q).
\]
Finally, as \( b \) is Lipschitz,
\[
\left\| \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} [b(V_n) - b(U_n)] \right\|_{L^1(Q)} \leq \frac{1}{N} \sum_{n \notin \mathbb{N}, n \notin r(\varepsilon)} \|b(V_n) - b(U_n)\|_{L^1(Q)} \lesssim \varepsilon.
\]
As \( \varepsilon > 0 \) was arbitrary, this shows (3.2) for any Lipschitz \( b \). The rest follows by standard density argument. \( \square \)

**Remark 3.3.** Theorem 3.2 refers to weakly (S)-convergent sequences. However, the same argument can be used to show that if
\[
\{U_n\}_{n=1}^{\infty} \approx_{(S)} \{V_n\}_{n=1}^{\infty},
\]
then
\[
\{U_n\}_{n=1}^{\infty} \text{ strongly (S) - convergent} \iff \{V_n\}_{n=1}^{\infty} \text{ strongly (S) - convergent}.
\]

We immediately get that (S)-convergence accommodates strong convergence as the case may be.

**Corollary 3.4 (Strongly convergent perturbation).**

- **Suppose** that
  \[
  U_n \to U \text{ in } L^1(Q).
  \]
  **Then**
  \[
  \{U_n\}_{n=1}^{\infty} \text{ is strongly (S) - convergent, and } U_n \overset{(S)}{\to} \delta_U.
  \]
Suppose that \( \{U_n\}_{n=1}^{\infty} \) is weakly/strongly \((S)\)-convergent, and

\[
V_n \to 0 \quad \text{in } L^1(Q).
\]

Then \( \{U_n + V_n\}_{n=1}^{\infty} \) is weakly/strongly \((S)\)-convergent, and \((S)\)-converge to the same measure \( \nu \).

4. Asymptotically stationary approximations

There is an important class of \((S)\)-convergent sequences called stationary in stochastic terminology.

**Definition 4.1 (Stationary sequence).** A sequence \( \{U_n\}_{n=1}^{\infty}, \quad U_n : Q \mapsto R^D \), is called stationary if

\[
\int_Q B(U_{k_1}, \ldots, U_{k_j}) dy = \int_Q B(U_{k_1+n}, \ldots, U_{k_j+n}) dy
\]

for any \( B \in C_c(R^D), \) any \( 1 \leq k_1 \leq k_2, \ldots, \leq k_j, \) and any \( n \geq 0. \)

It follows from the celebrated Birkhoff–Khinchin ergodic theorem [21, chapter 4, section 6, theorem 11] that any stationary sequence admits the limit

\[
\frac{1}{N} \sum_{n=1}^{N} b(U_n) \to \mathbb{E}(b(U)) \quad \text{for a.a. } y \in Q
\]

for any bounded Borel function \( b : R^D \to R. \) In particular, we get (2.10); whence any stationary sequence and all its statistical perturbations in the sense of definition 3.1 are weakly \((S)\)-convergent.

4.1. Weak asymptotic stationarity

In practice, we do not expect the approximate sequences to be stationary, however, some kind of asymptotic stationarity can be anticipated.

**Definition 4.2 (Weak asymptotic stationarity).** A sequence \( \{U_n\}_{n=1}^{\infty}, \quad U_n : Q \mapsto R^D \), is called weakly asymptotically stationary if the following holds for any \( b \in C_c(R^D) \):

- **Correlation limit**
  \[
  \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(U_n) b(U_m) dy \text{ exists for any fixed } m; \]

- **Asymptotic correlation stationarity**
  \[
  \left| \int_Q \left[ b(U_{k_1}) b(U_{k_2}) - b(U_{k_1+n}) b(U_{k_2+n}) \right] dy \right| \leq \omega(b, k) \quad (4.3)
  \]

for any \( 1 \leq k \leq k_1 \leq k_2, \) and any \( n \geq 0, \) where

\[
\omega(b, k) \to 0 \quad \text{as } k \to \infty.
\]
Obviously, any stationary sequence is weakly asymptotically stationary. Definition 4.2 is reminiscent of weak stationarity in the stochastic sense, where expectations, second moments, and correlations are required to be stationary. Note however, that we require (4.3) to hold for any \( b \in C_c(\mathbb{R}^d) \).

The next result shows that weakly asymptotically stationary sequences are weakly (S)-convergent.

**Theorem 4.3.** Let \( \{U_n\}_{n=1}^{\infty} \) be a weakly asymptotically stationary sequence in the sense of definition 4.2.

Then \( \{U_n\}_{n=1}^{\infty} \) is weakly (S)-convergent, in particular,

\[
\frac{1}{N} \sum_{n=1}^{N} b(U_n) \rightarrow b(U) \quad \text{as } N \rightarrow \infty \quad \text{in } L^1(Q)
\]

for any \( b \in C_c(\mathbb{R}^d) \).

**Proof.** It is enough to show that \( \{U_n\}_{n=1}^{\infty} \) enjoys the property (2.2). We know from lemma 2.2 that

\[
\frac{1}{N} \sum_{n=1}^{N} b(U_n) \rightarrow b(U) \quad \text{weakly} \quad \text{in } L^\infty(Q); \quad (4.4)
\]

hence (2.2) reduces to showing

\[
\lim_{N \to \infty} \sqrt{N} \sum_{n=1}^{N} b(U_n) \rightarrow b(U) \quad \text{as } N \rightarrow \infty \quad \text{in } L^1(Q).
\]

Fixing \( k > 0 \) we first observe that

\[
\lim_{N \to \infty} \sqrt{N} \sum_{n=1}^{N} b(U_n) \rightarrow b(U) \quad \text{as } N \rightarrow \infty \quad \text{weakly} \quad \text{in } L^\infty(Q); \quad (4.5)
\]

Next, regrouping terms, we get

\[
\sum_{n,m=1}^{N} b(U_n) b(U_m) = \sum_{n=0}^{N} \left( \sum_{m=0}^{N} b(U_{k+n}) b(U_{k+m}) \right)
\]

\[
= \sum_{n=0}^{N} b(U_k) \sum_{m=0}^{N} b(U_{k+m}) + \sum_{n=0}^{N} \sum_{m=0}^{N} \left( b(U_{k+n}) b(U_{k+m}) - b(U_k) b(U_{k+m}) \right).
\]

Furthermore,

\[
\sum_{m=0}^{N} b(U_{k+n}) b(U_{k+m}) = \sum_{m \leq n} b(U_{k+n}) b(U_{k+m}) + \sum_{m=n+1}^{N} b(U_{k+n}) b(U_{k+m})
\]

\[
= \sum_{m \leq n} \left[ b(U_{k+n}) b(U_{k+m}) - b(U_k) b(U_{k+n-m}) \right] + \sum_{m=0}^{N} b(U_k) b(U_{k+m}).
\]

2339
Going back to (4.7) we obtain
\[
\sum_{m=n+1}^{N} \left[ b(U_{k+n})b(U_{k+m}) - b(U_k)b(U_{k+m-n}) \right] + \sum_{m=n+1}^{N} b(U_k)b(U_{k+m}) \]
\[
= \sum_{m \leq n} \left[ b(U_{k+n})b(U_{k+m}) - b(U_k)b(U_{k+n-m}) \right] \]
\[
+ \sum_{m=n+1}^{N} \left[ b(U_{k+n})b(U_{k+m}) - b(U_k)b(U_{k+m-n}) \right] + \sum_{m=0}^{N} b(U_k)b(U_{k+m}).
\]

As \( \{U_n\}_{n=1}^{\infty} \) is asymptotically stationary, it follows from (4.3) that
\[
\left| \int_{Q} \left[ b(U_{k+n})b(U_{k+m}) - b(U_k)b(U_{k+m-n}) \right] d\mu \right| \leq \omega(k), \quad n \geq m \geq 0
\]
\[
\left| \int_{Q} \left[ b(U_{k+n})b(U_{k+m}) - b(U_k)b(U_{k+m-n}) \right] d\mu \right| \leq \omega(k), \quad m \geq n \geq 0.
\]
Thus performing the limit in (4.6) we obtain
\[
\int_{Q} b(U_k)b(U) d\mu = \omega(k) \leq \liminf_{N \to \infty} \sum_{n=1}^{N} \frac{1}{N^2} \int_{Q} b(U_n)b(U) d\mu
\]
\[
\leq \limsup_{N \to \infty} \sum_{n=1}^{N} \frac{1}{N^2} \int_{Q} b(U_n)b(U) d\mu \leq \int_{Q} b(U_k)b(U) d\mu + \omega(k),
\]
(4.9)

where we have used the weak convergence stated in (4.8).

Finally, summing (4.9) with respect to \( k \), we get
\[
\frac{1}{M} \sum_{k=1}^{M} \int_{Q} b(U_k)b(U) d\mu - \frac{1}{M} \sum_{k=1}^{M} \omega(k) \leq \liminf_{N \to \infty} \sum_{n=1}^{N} \frac{1}{N^2} \int_{Q} b(U_n)b(U) d\mu
\]
\[
\leq \limsup_{N \to \infty} \sum_{n=1}^{N} \frac{1}{N^2} \int_{Q} b(U_n)b(U) d\mu \leq \frac{1}{M} \sum_{k=1}^{M} \int_{Q} b(U_k)b(U) d\mu + \frac{1}{M} \sum_{k=1}^{M} \omega(k).
\]
Thus letting \( M \to \infty \) and using (4.4) once more, we get (2.2). \( \square \)
Remark 4.4. As revealed in the proof of proposition 4.3, the hypothesis (4.3) can be replaced by a weaker stipulation:

$$\limsup_{N \to \infty} \frac{1}{N^2} \sum_{n,m=0}^{N} \left| \int_{Q} b(U_{k+n})b(U_{k+m}) - b(U_{k})b(U_{k+[n-m]}) \, dy \right| \leq \omega(b, k), \quad (4.10)$$

$$\omega(b, k) \to 0 \text{ as } k \to \infty.$$

4.2. Strong asymptotic stationarity

Our final goal in this section is to show a sufficient condition for strong (S)-convergence.

Definition 4.5 (Strong asymptotic stationarity). A sequence \( \{U_n\}_{n=1}^{\infty}, U_n : Q \to \mathbb{R}^d, \) is called strongly asymptotically stationary if the following holds for any \( b \in C_c(\mathbb{R}^d) \):

- **Strong correlation limit**

  $$\lim_{n \to \infty} \int_{Q} b(U_{n})b(U_{m}) \, dy \text{ exists} \quad (4.11)$$

  for any fixed \( m; \)

- **Asymptotic correlation stationarity**

  $$\left| \int_{Q} \left[ b(U_{k_1})b(U_{k_2}) - b(U_{k_1+n})b(U_{k_2+n}) \right] \, dy \right| \leq \omega(b, k) \quad (4.12)$$

  for any \( 1 \leq k \leq k_1 \leq k_2, \) and any \( n \geq 0, \) where

  $$\omega(b, k) \to 0 \quad \text{as } k \to \infty.$$  

The condition (4.11) is actually stronger than its counterpart (2.5). It can be shown that (4.11) implies weak convergence of \( \{b(U_n)\}_{n=1}^{\infty}, \) whence (4.11) \( \Rightarrow \) (2.5).

We claim the following analogue of theorem 4.3.

Theorem 4.6. Let \( \{U_n\}_{n=1}^{\infty} \) be strongly asymptotically stationary in the sense of definition 4.5. Then \( \{U_n\}_{n=1}^{\infty} \) is strongly (S)-convergent, in particular,

$$\frac{1}{wN} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to b(U) \quad \text{in } L^1(Q),$$

for any \( b \in C_c(\mathbb{R}^d), w \in W. \)

Proof. In view of lemma 2.6, it is enough to show that \( \{U_n\}_{n=1}^{\infty} \) satisfies (2.11). To begin, following the arguments of the proof of lemma 2.2 we observe that (4.11) implies

$$b(U_n) \to b(U) \quad \text{weakly - (*) in } L^\infty(Q), \quad (4.13)$$

which yields

$$\frac{1}{wN} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) b(U_n) \to b(U) \quad \text{as } N \to \infty \quad \text{weakly - (*) in } L^\infty(Q). \quad (4.14)$$
Consequently, it remains to show
\[
\lim_{N \to \infty} \sum_{n,m=1}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} b(U_n)b(U_m)dy = \int_{Q} |\overline{b(U)}|^2 dy. \quad (4.15)
\]

First,
\[
\sum_{n,m=1}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} b(U_n)b(U_m)dy - \int_{Q} |\overline{b(U)}|^2 dy
\]
\[
= \sum_{n,m=1}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy
\]
\[
= \sum_{n,m=1}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy
\]
\[
+ \sum_{n,m=k}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy. \quad (4.16)
\]

Now observe that
\[
\frac{w_N}{N} = \frac{1}{N} \sum_{n=1}^{N} w \left( \frac{n}{N} \right) \to \int_{0}^{1} w(z)dz = 1 \quad \text{as } N \to \infty, \quad (4.17)
\]
and, consequently,
\[
\sum_{n=1}^{k-1} \left( \sum_{m=1}^{N} + \sum_{m=k}^{N} \right) \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy \to 0 \quad \text{as } N \to \infty
\]
for any fixed \( k \).

Going back to (4.16), we write the last integral as
\[
\sum_{n,m=k}^{N} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy
\]
\[
= \sum_{n,m=k,|n-m|<l} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy
\]
\[
+ \sum_{n,m=k,|n-m| \geq l} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right)
\]
\[
\times \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy.
\]

Using (4.17) once more, we get
\[
\left| \sum_{n,m=k,|n-m|<l} \frac{1}{w_N} w \left( \frac{n}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_n)b(U_m) - |\overline{b(U)}|^2] dy \right| \lesssim \frac{ln}{w_N} \to 0 \quad \text{as } N \to \infty
\]
for any fixed \( l \).

Finally,

\[
\sum_{n,m=|k|,n-m| \geq |l|}^{N} \frac{1}{w_{N}^{n}} w \left( \frac{m}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_{n})b(U_{m}) - \left| b(U) \right|^{2}] \, dy
\]

\[
= 2 \sum_{n,m=|k|,n-m| \geq |l|}^{N} \frac{1}{w_{N}^{n}} w \left( \frac{m}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_{n})b(U_{m}) - b(U_{k})b(U_{m-n+k})] \, dy
\]

\[
= 2 \sum_{n,m=|k|,n-m| \geq |l|}^{N} \frac{1}{w_{N}^{n}} w \left( \frac{m}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_{n})b(U_{m}) - b(U_{k})b(U_{m-n+k})] \, dy.
\]

In view of hypothesis (4.12),

\[
\left| \sum_{n,m=|k|,n-m| \geq |l|}^{N} \frac{1}{w_{N}^{n}} w \left( \frac{m}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_{n})b(U_{m}) - b(U_{k})b(U_{m-n+k})] \, dy \right| \leq \varepsilon(b, k),
\]

where \( \varepsilon(b, k) \to 0 \) as \( k \to \infty \), uniformly for any \( l \). Now, in view of (4.13), given \( k \), we can fix \( l = l(\varepsilon(b, k), k) \) such that

\[
\left| \sum_{n,m=|k|,n-m| \geq |l|}^{N} \frac{1}{w_{N}^{n}} w \left( \frac{m}{N} \right) w \left( \frac{m}{N} \right) \int_{Q} [b(U_{n})b(U_{m}) - b(U_{k})b(U_{m-n+k})] \, dy \right| < \varepsilon(b, k).
\]

We strongly believe that asymptotic stationarity is satisfied by consistent approximations of the Euler system. For relevant numerical evidence, see [14].

5. Applications, convergence of consistent approximations for the compressible Euler system

Our ultimate goal is to apply the abstract theory to the isentropic Euler system:

\[
\partial_{t} \varrho + \text{div}_{x} \mathbf{m} = 0, \quad \varrho(0, \cdot) = \varrho_{0},
\]

\[
\partial_{t} \mathbf{m} + \text{div}_{x} \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_{x} p(\varrho) = 0, \quad \mathbf{m}(0, \cdot) = \mathbf{m}_{0},
\]

with the isentropic equation of state \( p(\varrho) = a\varrho^{\gamma} \), \( \gamma > 1 \). Here \( \varrho = \varrho(t, x) \) is the mass density and \( \mathbf{m} = \mathbf{m}(t, x) \) the linear momentum of a compressible gas in the isentropic regime. For the sake of simplicity, we consider the periodic boundary conditions \( \Omega = \mathbb{T}^{d} \). We recall the associated energy inequality in the integrated form

\[
\int_{\mathbb{T}^{d}} E(\varrho, \mathbf{m})(t, \cdot) \, dx \leq \int_{\mathbb{T}^{d}} E(\varrho_{0}, \mathbf{m}_{0}) \, dx,
\]

2343
where

\[ E(\rho, m) = \begin{cases} \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0, \ m = 0, \\ \infty & \text{otherwise} \end{cases} \]

5.1. Dissipative solutions

We recall the definition of dissipative solution of the Euler system, see [3]:

- **Equation of continuity**
  \[ \partial_t \rho + \text{div}_x m = 0 \quad \text{in } D'(0, T) \times T^d; \]

- **Momentum equation**
  \[ \partial_t m + \text{div}_x \left( \frac{m_\rho}{\rho} \right) + \nabla_x p(\rho) + \text{div}_x \mathfrak{R} = 0, \quad \text{in } D'(0, T) \times T^d; R^d, \]
  with the Reynolds stress \( \mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(T^d; R^d_{sym})) \);

- **Energy balance**
  \[ \int_{T^d} E(\rho, m)(t, \cdot) \, dx + \min \left\{ \frac{1}{2}, \frac{1}{\gamma} \right\} \int T^d \, dtr[\mathfrak{R}](t, \cdot) \leq \int_{T^d} E(\rho_0, m_0) \, dx \]
  for a.a. \( t \in (0, T) \).

It follows that

\( \rho \in C_{\text{weak}}([0, T]; L^\gamma(T^d)), \quad m \in C_{\text{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(T^d; R^d)); \)

hence the initial data are well defined.

If \( \mathfrak{R} \equiv 0 \), the above definition yields the standard (admissible) weak solution. Note that the Reynolds stress \( \mathfrak{R} \) is a positively definite matrix valued measure that accommodates possible oscillations/concentration inherited from the approximation process. Although definitely more general, the dissipative solutions share many important properties with the weak solutions, among which the weak–strong uniqueness principle. The reader may consult [3] or [11] for other interesting properties of dissipative solutions.

5.2. Consistent approximation

The class of dissipative solutions is large enough to accommodate limits of various approximate schemes that are consistent with the Euler system, cf [12] for the vanishing viscosity limit, and [13] for a finite volume numerical scheme.

**Definition 5.1 (Consistent approximation).** We say that a sequence \( \{\rho_n, m_n\}_{n=1}^\infty \) is a consistent approximation of the isentropic Euler system if the following holds:

- **Approximate equation of continuity**
  \[ \int_0^T \int_{T^d} \left[ \partial_t \varphi + m_n \cdot \nabla_x \varphi \right] \, dx \, dt = - \int_{T^d} \rho_0 \varphi(0, \cdot) \, dx + e_n^i[\varphi] \]
  for any \( \varphi \in C_0^\infty([0, T) \times T^d); \)
• Approximate momentum equation

\[
\int_0^T \int_{\mathbb{T}^d} \left[ m_n \cdot \partial_t \varphi + \left( 1_{\varrho_n > 0} \frac{m_n \otimes m_n}{\varrho_n} \right) : \nabla \varphi + p(\varrho_n) \text{div} x \varphi + \mathcal{R}_n : \text{div} x \varphi \right] \, dx \, dt \\
= \int_{\mathbb{T}^d} \varphi_0 \varphi(0, \cdot) \, dx + \epsilon_1^n[\varphi]
\]

for any \( \varphi \in C^\infty_0([0, T) \times T^d; \mathbb{R}^d) \);

• Approximate energy balance

\[
\int_{\mathbb{T}^d} E(\varrho_n, m_n)(t, \cdot) \, dx + \min \left\{ \frac{1}{2}, \frac{1}{\gamma} \right\} \int_{\mathbb{T}^d} [\mathcal{R}_n](t, \cdot) \leq E_n
\]

• Consistency

\[
e_1^n[\varphi] \to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad \varphi \in C^\infty_0([0, T) \times T^d),
\]

\[
e_2^n[\varphi] \to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad \varphi \in C^\infty_0([0, T) \times T^d; \mathbb{R}^d),
\]

\( \varrho_{0,n} \to \varrho_0 \) weakly in \( L^1(Q) \),

\( m_{0,n} \to m_0 \) weakly in \( L^1(Q; \mathbb{R}^d) \),

\[
\limsup_{n \to \infty} E_n \leq \int_{\mathbb{T}^d} E(\varrho_0, m_0) \, dx.
\]

Note that the above definition differs from [12, 13] as it allows for ‘approximate Reynolds stress’ \( \mathcal{R}_n \) that is set to be zero in [12, 13]. Accordingly, the present definition accommodates a larger class of consistent approximations than [12, 13]. There are two crucial observations:

(a) Any weak limit of a sequence of consistent approximations \( \{ \varrho_n, m_n \}_{n=1}^\infty \) is a dissipative solution with the initial data \( (\varrho_0, m_0) \). This follows from the general compactness results proved in [3, section 3, proposition 3.1].

(b) If \( \{ \varrho_n, m_n \}_{n=1}^\infty \) is a consistent approximation, then

\[
\varrho_N = \frac{1}{N} \sum_{n=1}^N \varrho_n, \quad m_N = \frac{1}{N} \sum_{n=1}^N m_n, \quad N = 1, 2, \ldots
\]

is another consistent approximation (of the same problem, with the same data). This reflects a general principle that a convex combination of dissipative solutions is a dissipative solution. It remains to observe that the error terms satisfy

\[
\frac{1}{N} \sum_{n=1}^N e_1^n[\varphi], \quad \frac{1}{N} \sum_{n=1}^N e_2^n[\varphi] \to 0 \quad \text{as} \quad N \to \infty.
\]

**Corollary 5.2 (Convergence of (S)-convergent consistent approximations).** Suppose that a sequence \( \{ \varrho_n, m_n \}_{n=1}^\infty \) is a consistent approximation of the Euler system that is weakly (S)-convergent in the sense of definition 2.1.
Then

\[
\frac{1}{N} \sum_{n=1}^{N} q_n \to q \quad \text{in } L^q((0,T) \times \mathbb{T}^d), \quad 1 \leq q < \gamma,
\]
\[
\frac{1}{N} \sum_{n=1}^{N} m_n \to m \quad \text{in } L^q((0,T) \times \mathbb{T}^d), \quad 1 \leq q < \frac{2\gamma}{\gamma + 1},
\]

where \((\varrho, m)\) is a dissipative solution of the Euler system;

- there exists a (unique) parametrized measure \((\mathcal{V}_{t,x})_{(t,x)\in(0,T)\times\mathbb{T}^d}\), \(\mathcal{V}_{t,x} \in \mathcal{P}(\mathbb{R}^{d+1})\), such that

\[
(\varrho_n, m_n) \overset{(S)}{\rightharpoonup} \mathcal{V},
\]

and

\[
\int_0^T \int_{\mathbb{T}^d} dW_{n} \left[ \frac{1}{N} \sum_{n=1}^{N} \delta_{(\varrho_n(t,x), m_n(t,x))}; \mathcal{V}_{t,x} \right]^{s} \ dx \ dt \to 0 \quad \text{for any } 1 \leq s < \frac{2\gamma}{\gamma + 1};
\]

- If, in addition, the sequence \(\varrho_n, m_n\) is strongly (S)-convergent, the

\[
\frac{1}{N} \sum_{n=1}^{N} \ 	ext{may be replaced by } \frac{1}{wN} \sum_{n=1}^{N} w \left( \frac{n}{N} \right)
\]

for arbitrary weight \(w \in \mathbb{W}\).

The parametrized measure \(\mathcal{V}\) can be seen as a generalized solution of the Euler system. It shares all fundamental properties with the Young measures generated by (sub)sequences of consistent approximations. In particular,

\[
\mathcal{V}_{(t,x)} = \delta_{(\varrho(t,x), m(t,x))} \quad \text{for a.a. } (t,x) \in (0,T) \times \mathbb{T}^d
\]

if either the limit Euler system admits a smooth solution, or the barycenter of \(\mathcal{V}—[\varrho, m]—\) is of class \(C^1\).

### 5.3. (S)-convergence to weak solutions

We know that weakly (S)-convergent approximate sequences to the isentropic Euler system generate the parametrized measure \(\mathcal{V}\). Our ultimate goal is to discuss validity of the following statement:

\[\varrho = \langle \mathcal{V}; \varrho \rangle,\]
\[m = \langle \mathcal{V}; m \rangle \quad \text{is a weak solution to the Euler system}\]
\[\Rightarrow \mathcal{V}_{t,x} = \delta_{(\varrho(t,x), m(t,x))} \quad \text{for a.a. } (t,x) \in (0,T) \times \mathbb{T}^d.
\]

Here \(\langle \mathcal{V}; b(\varrho, m) \rangle\) denotes the application of the measure \(\mathcal{V}\) to a (Borel) function \(b\) of the variables \((\varrho, m)\). In particular, \(\langle \mathcal{V}; \varrho \rangle; \langle \mathcal{V}; m \rangle\) is the barycenter of \(\mathcal{V}\).

First observe that if \((\varrho, m)\) is a weak (distributional) solution of the Euler system, then the Reynolds defect tensor satisfies

\[
\text{div}_1 \mathfrak{R} = 0 \quad \text{in } D'(T; \mathbb{T}^d; \mathbb{R}^d)
\]
from which we easily deduce
\[ \text{div}_x \mathcal{R}(t, \cdot) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^d), \quad \mathcal{R} \in \mathcal{M}^+(\mathbb{T}^d, R_{0}^{d \times d}) \quad \text{for a.a.} \quad t \in (0, T), \] (5.3)
where \( \mathcal{M}^+(\mathbb{T}^d, R_{0}^{d \times d}) \) is the cone of parametrized non-negative matrix-valued measures.

Moreover, the limit Reynolds stress can be written as
\[ \mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2, \]
where
\[ \mathcal{R}_1 = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \mathcal{R}_n \quad \text{weak} - (*) \quad \text{in} \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^d, R_{0}^{d \times d})), \quad \mathcal{R}_1(t, \cdot) \in \mathcal{M}^+(\mathbb{T}^d, R_{0}^{d \times d}), \]
\[ \mathcal{R}_2 = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \left[ \left( 1_{\rho_n > 0} \left( \frac{m_n \otimes m_n}{\rho_n} \right) + p(\rho_n)I \right) \right] - \left( 1_{\rho > 0} \left( \frac{m \otimes m}{\rho} \right) + p(\rho)I \right) \quad \text{weak} - (*) \quad \text{in} \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^d, R_{0}^{d \times d})), \]
where
\[ \mathcal{R}_2(t, \cdot) \in \mathcal{M}^+(\mathbb{T}^d, R_{0}^{d \times d}) \quad \text{for a.a.} \quad t \in (0, T). \]

Thanks to the convexity argument, specified in [12, section 3], implication (5.2) follows as soon as we can show that \( \mathcal{R} = 0 \). Moreover, as \( \mathcal{R} \) is positively semi-definite and satisfies (5.3), it is enough to show that \( \mathcal{R} \) vanishes in a neighborhood of the boundary of \( \Omega = \mathbb{T}^d \), see [12, section 4, proposition 4.3]. As \( \mathcal{R}_1 \) usually vanishes for consistent approximation of the Euler system, we have to make sure that \( \mathcal{R}_2 \) vanishes in a neighborhood of \( \partial \Omega \). Note that here we identify \( \mathbb{T}^d \) with a bounded subset of \( \mathbb{R}^d \). Following step by step the arguments of [12, section 4], we can show the following result.

**Theorem 5.3.** Let \( \{ \rho_n, m_n \}_{n=1}^{\infty} \) be a consistent approximation of the isentropic Euler system in the sense of definition 5.1, with \( \mathcal{R}_n \equiv 0 \). Moreover, suppose that
\[ (\rho_n, m_n) \xrightarrow{(S)} \mathcal{V}, \]
and
\[ \frac{1}{N} \sum_{n=1}^{N} \int_{0}^{T} \int_{\mathcal{U}} E(\rho_n, m_n) \, dx \, dt \to \int_{0}^{T} \int_{\mathcal{U}} E(\rho, m) \, dx \, dt, \quad \rho = \langle V \rangle, \quad m = \langle \mathcal{V} \rangle, \] (5.4)
where \( \mathcal{U} \) is an open neighborhood of \( \partial \Omega \), \( \Omega = \mathbb{T}^d \). Finally, suppose that \( (\rho, m) \) is a weak solution of the Euler system.

Then
\[ \mathcal{V}(t, x) = \delta_{(\rho_0, m_0, t, x)} \quad \text{for a.a.} \quad (t, x) \in (0, T) \times \mathbb{T}^d. \]

We finish this part by a short discussion when hypothesis (5.4) can be anticipated. Suppose that
\[ \rho_0 = \overline{\rho} > 0, \quad m_0 = \overline{m} \quad \text{for} \quad |x| \geq R. \]
Supposing the finite-speed of propagation for the Euler system, we may infer that
\[ E(\varrho, m) = E(\varrho', m') \quad \text{for all} \quad t \in (0, T), \quad |x| \leq R + ct. \]

Consistently, we assume that
\[ E(\varrho_n, m_n) \to E(\varrho, m) \quad \text{in} \quad L^1 \{ t \in (0, T), \quad |x| \leq R + ct \}. \quad (5.5) \]

In numerical approximations, (5.5) is usually guaranteed by imposing a (CFL) condition.

Finally, we claim that hypothesis (5.4) can be dropped in the case of complete Euler system, provided the reference variables are the density \( \varrho \), the momentum \( m \), and the total entropy \( S \).

The interested reader may elaborate the details following [12, section 2]. With a bit of extrapolation, we may conclude that if the barycenter of the (S)-limit \( V \) is a weak solution of the Euler system, then \( V \) is a (parametrized) Dirac mass.

6. Conclusion

The (S)-convergence provides a tool to study the limits of approximate sequences even in the case when the weak limit does not exist. This is in particular convenient for numerical schemes, where the procedure of picking up a suitable subsequence is practically not applicable. The notion is stable under very general statistical perturbations that may 'pollute' the approximation procedure. The limit measure \( V \) is attained in the strong topology of the underlying physical space and in the space of probability measures endowed with suitable Wasserstein distance. In particular, deviation, variance, barycenter and other parameters of the limit measure can be effectively computed. This should be seen as the main advantage of this approach as the standard Young measures, if non-trivial, are necessarily products of weak convergence that is difficult to 'visualize' numerically.

Given a sequence of consistent approximations, it is a hard problem to determine whether or not it is (S)-convergent. Note that it is possible to construct examples of consistent approximations that are not (S)-convergent at least for certain class of initial data. Given a consistent approximation resulting from the vanishing viscosity process or as a limit of a specific numerical scheme, a rigorous verification of validity of any form (weak or strong) of (S)-convergence remains an outstanding open problem. There is, however, a piece of (experimental) evidence that certain numerical schemes produce (S)-convergent approximate sequences, see [14].

References

[1] Alibert J J and Bouchitté G 1997 Non-uniform integrability and generalized Young measures J. Convex Anal. 4 129–47
[2] Balder E J 2000 Lectures on Young measure theory and its applications in economics Rend. Istit. Mat. Univ. Trieste 31 1–69 workshop on measure theory and real analysis (Italian) (Grado, 1997)
[3] Breit D, Feireisl E and Hofmanová M 2020 Solution semiflow to the isentropic Euler system Arch. Ration. Mech. Anal. 235 167–94
[4] Bressan A and Murray R 2020 On self–similar solutions to the incompressible Euler equations J. Differ. Equ. 269 5142–203
[5] Buckmaster T and Vicol V 2019 Convex integration and phenomenologies in turbulence EMS Surv. Math. Sci. 6 173–263
[6] Chiodaroli E 2014 A counterexample to well-posedness of entropy solutions to the compressible Euler system J. Hyperbolic Differ. Equ. 11 493–519
[7] Das S and Yorke J A 2018 Super convergence of ergodic averages for quasiperiodic orbits
Nonlinearity 31 491–501
[8] De Lellis C and Székelyhidi L Jr 2010 On admissibility criteria for weak solutions of the Euler
equations Arch. Ration. Mech. Anal. 195 225–60
[9] DiPerna R J and Majda A J 1987 Oscillations and concentrations in weak solutions of the
incompressible fluid equations Commun. Math. Phys. 108 667–89
[10] DiPerna R J 1985 Measure-valued solutions to conservation laws Arch. Ration. Mech. Anal. 88
223–70
[11] Feireisl E 2020 A note on the long-time behavior of dissipative solutions to the Euler system preprint
series IM AS, Praha, preprint No. IM-2020-4
[12] Feireisl E and Hofmanová M 2020 On convergence of approximate solutions to the compressible
Euler system Ann. PDE 6 11
[13] Feireisl E, Lukáčová-Medvid’ová M and Mizerová H 2019 $K$–convergence as a new tool in
numerical analysis (arXiv:1904.00297)
[14] Feireisl E, Lukáčová-Medvid’ová M, Mizerová H, She B and Wang Y 2019 Computing oscillatory
solutions to the Euler system via $K$–convergence (arXiv:1910.03161)
[15] Feireisl E and Lukáčová-Medvid’ová M 2018 Convergence of a mixed finite element-finite volume
scheme for the isentropic Navier–Stokes system via dissipative measure-valued solutions Found.
Comput. Math. 18 703–30
[16] Fjordholm U K, Käppeli R, Mishra S and Tadmor E 2017 Construction of approximate entropy
measure valued solutions for hyperbolic systems of conservation laws Found. Comput. Math. 17
763–827
[17] Fjordholm U S, Mishra S and Tadmor E 2016 On the computation of measure-valued solutions Acta
Numer. 25 567–679
[18] Gwiazda P, Świerczewska-Gwiazda A and Wiedemann E 2015 Weak-strong uniqueness for
measure-valued solutions of some compressible fluid models Nonlinearity 28 3873–90
[19] Komlós J 1967 A generalization of a problem of Steinhaus Acta Math. Acad. Sci. Hungar. 18 217–29
[20] Krenkel U 1978 On the speed of convergence in the ergodic theorem Monatsh. Math. 86 3–6
[21] Krylov N V 2002 Introduction to the Theory of Random Processes (Graduate Studies in Mathematics
vol 43) (Providence, RI: American Mathematical Society)
[22] León-Saavedra F, del Carmen Listán-García M, Pérez Fernández F J and Romero de la Rosa M P
2019 On statistical convergence and strong Cesáro convergence by moduli J. Inequ. Appl. 12 298
[23] Villani C 2002 Limites hydrodynamiques de l’equation de Boltzmann Astérisque 282 365–405
[24] Wiedemann E 2011 Existence of weak solutions for the incompressible Euler equations Ann. Inst.
Henri Poincare 28 727–30

2349