FREE OBJECTS AND GRÖBNER-SHIRSHOV BASES IN OPERATED CONTEXTS

ZIHAO QI, YUFEI QIN, KAI WANG AND GUODONG ZHOU

ABSTRACT. This paper investigates algebraic objects equipped with an operator, such as operated monoids, operated algebras etc. Various free object functors in these operated contexts are explicitly constructed. For operated algebras whose operator satisfies a set $\Phi$ of relations (usually called operated polynomial identities (aka. OPIs)), Guo defined free objects, called free $\Phi$-algebras, via universal algebra. Free $\Phi$-algebras over algebras are studied in details. A mild sufficient condition is found such that $\Phi$ together with a Gröbner-Shirshov basis of an algebra $A$ form a Gröbner-Shirshov basis of the free $\Phi$-algebra over algebra $A$ in the sense of Guo et al.. Ample examples for which this condition holds are provided, such as all Rota-Baxter type OPIs, a class of differential type OPIs, averaging OPIs and Reynolds OPI.

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6.4. Multiple OPIs

References

Introduction

This paper studies explicit constructions of free objects and Gröbner-Shirshov bases in operated contexts.

In the 1930s, Ritt [47, 48] initiated the algebraic study of differential equations by introducing differential algebras as commutative algebras equipped with a linear operator satisfying the usual Leibniz rule. This mathematical branch has received ample development in the work [37, 41, 46] and has broad applications to other areas such as arithmetic geometry, logic, computer science and mathematical physics [15, 19, 43, 52, 53] etc. In recent years, researchers began to investigate noncommutative differential algebras in order to broaden the scope of the theory to include path algebras, for instance, and to have a more meaningful differential Lie algebra theory [29, 44, 45] and also from an operadic point of view [40]

Another important class of algebras with operators are Rota-Baxter algebras. These algebras (previously known as Baxter algebras) originated with the work of Baxter [7] on probability theory. Baxter’s work was further investigated by, among others, Rota [49] (hence the name “Rota-Baxter algebras”), Cartier [12] and Atkinson [3] etc. Nowadays, Rota-Baxter algebras have numerous applications and connections to many mathematical branches, to name a few, such as combinatorics [25, 50], renormalization in quantum field theory [14], multiple zeta values in number theory [35], operad theory [1, 6], Hopf algebras [14], Yang-Baxter equation [5] etc. For basic theory about Rota-Baxter algebras, we refer the reader to the short introduction [27] and to the comprehensive monograph [28].

Free objects are ubiquitous in mathematics. The general idea is to present an object by realizing it as a quotient of a free object, whence presentations by generators and relations. Explicit construction of free objects are usually very important in many subjects. Earlier construction of commutative free Rota-Baxter algebras were given by Rota [49] and Cartier [12]. In the pioneering work [31, 32], Guo and Keigher gave the third construction in terms of mixable shuffle product algebras, generalizing the well-known construction of shuffle product algebras. Several groups of authors gave explicit constructions of free noncommutative Rota-Baxter algebras on sets, modules or algebras [2, 17, 18, 26, 33].

Guo [26] revisited these constructions from the viewpoint of semigroups, monoids or algebras endowed with operators in the sense of Higgins [36] and Kurosh [38], which Guo renamed as operated semigroups, operated monoids or operated algebras. He constructed the free operated semigroup, the free operated monoid and the free operated algebras from a set in terms of bracketed words, Motzkin paths and rooted trees; see also [16, 9, 54]. When the operator in question satisfies certain relations, Guo, Sit and Zhang [34] introduced the notion of operated polynomial identities (aka. OPIs). By the theory of universal algebra, for a set Φ of OPIs, the free Φ-algebra can be realised as the quotient of the free operated algebras by the operated ideal generated by Φ; see, for instance, [13, Proposition 1.3.6]. This gives a universal construction for free Φ-algebras.

The next step is to develop a theory of rewriting systems and that of Gröbner-Shirshov (aka. GS) bases in this operated context, parallel to and include as a special case, the usual well developed theory for associative algebras [51, 11, 24, 8, 10] and the original theory of rewriting
systems [4]. This has been done in [9, 34, 22, 20] and was applied to various setting, in particular, to two important classes of OPIs: differential type OPIs and Rota-Baxter type OPIs which were carefully studied by Guo et al. [34][22][20]. Recently, there is a need to develop free $\Phi$-algebras over algebras and construct GS bases for these free $\Phi$-algebras as long as a GS basis is known for the given algebra; see [18, 39, 30].

This paper extends and refines some aspects of the above works.

In the first three sections, we present a careful analysis for explicit construction of free objects in various operated contexts.

Denote the category of sets (resp. semigroups, monoids) by $\mathcal{Set}$ (resp. $\mathcal{Sem}$, $\mathcal{Mon}$). We consider categories of structured sets endowed with an operator which is merely a map without any further condition, such as operated sets, operated semigroups and operated monoids; see Definition 1.1. Denote the category of operated sets, that of operated semigroups and that of operated monoids by $\mathcal{OpSet}$, $\mathcal{OpSem}$ and $\mathcal{OpMon}$, respectively. We also investigate vector spaces with structures endowed with a linear operator such as operated vector spaces, operated algebras and operated unital algebras; see Definition 1.2. We use $\mathcal{OpVect}$ (resp. $\mathcal{OpAlg}$, $\mathcal{uOpAlg}$) to denote the category of operated $k$-vector spaces (resp. operated $k$-algebras, operated unital $k$-algebras).

Our goal in the first three sections is to complete the following diagram of free object functors:

Note that these free object functors are left adjoint to the obvious forgetful functors which are not drawn in the diagram.

In the first section, we consider the bottom face and part of the upper face, the second section contains an analysis of the front face and the back face is dealt with in the third section. Except the functors in the bottom face, it seems that any other free object functor has not been appeared in the literature, while some composites were pointed out by Guo [26].

The fourth section recalls the basic theory of operated contexts [26, 34, 22, 20]. In particular, as an application of the previous three sections, we can construct, via universal algebra, the free $\Phi$-algebra over a given algebra for a set of OPIs $\Phi$; see Propositions 4.7 and 4.8.

The first half of the fifth section contains an account of the theory of GS basis in operated contexts [9, 16, 34, 22, 20]. In the second half of Section 5, inspired by [18, 39, 30], we are interested into a question which can be roughly expressed as follows:

**Question 0.1.** Given a unital algebra $A$ with a GS basis $G$ and a set of OPIs $\Phi$, assume that $\Phi$ is GS in the sense of [9, 34, 22, 20]. Let $B$ be the free $\Phi$-algebra over $A$. Under what conditions, $\Phi \cup G$ will be a GS basis for $B$?
We answer this question in the affirmative under a mild condition in Theorem 5.9, which can be considered as the main result of this paper. When this condition is satisfied, \( \Phi \cup G \) is a GS basis for \( B \). As a consequence, we also get a linear basis of \( B \); see Corollary 5.10.

Section 6 contains many applications of Theorem 5.9 and Corollary 5.10. When \( \Phi \) consists of a single OPI of Rota-Baxter type in the sense of [22, 20], this technical condition is fulfilled; see Theorem 6.6. When \( \Phi \) consists of a single OPI of differential type in the sense of [34, 20], the situation is much more involved and we give two particular cases where this technical condition holds; see Proposition 6.11. We also provide a case study for Gröbner-Shirshov \( \Phi \) consisting of several OPIs. For averaging algebras, \( \Phi \) has three elements and for Reynolds algebras, \( \Phi \) will have infinitely many elements; based on [23, 55], we could obtain a GS basis as well as a linear basis for free averaging algebras over algebras and free Reynolds algebras over algebras, respectively.

**Notation:** Throughout this paper, \( k \) denotes a field. All the vector spaces and algebras are over \( k \) and all tensor products are also taking over \( k \).

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1. **Free object functors I**

Firstly, we introduce some relevant definitions and notations.

Recall that \( \mathbb{Set} \) (resp. \( \mathbb{Sem} \), \( \mathbb{Mon} \), \( \mathbb{Vect} \), \( \mathbb{Alg} \), \( \mathbb{uAlg} \)) denotes the category of sets (resp. semigroups, monoids, \( k \)-vector spaces, \( k \)-algebras, unital \( k \)-algebras).

The following two definitions are taken from [26, 22] except the notion of operated sets.

**Definition 1.1.** (a) An operated set is a set \( X \) with a map \( P_X : X \to X \). A morphism of operated sets from \( (X, P_X) \) to \( (Y, P_Y) \) is a map \( f : X \to Y \) such that \( f \circ P_X = P_Y \circ f \). Denote the category of operated sets by \( \mathbb{OpSet} \).

(b) An operated semigroup is a semigroup \( S \) with a map \( P_S : S \to S \) (which is not necessarily a homomorphism of semigroups). Morphisms between operated semigroups can be defined in the obvious way. Denote the category of operated semigroups by \( \mathbb{OpSem} \).

(c) An operated monoid is a monoid \( M \) together with a map \( P_M : M \to M \) (which is not necessarily a homomorphism of monoids). One can define morphisms between operated monoids similarly. Denote the category of operated monoids by \( \mathbb{OpMon} \).

**Definition 1.2.** An operated \( k \)-space (resp. operated \( k \)-algebra, operated unital \( k \)-algebra) is a \( k \)-space (resp. \( k \)-algebra, unital \( k \)-algebra) \( V \) together with a \( k \)-linear map \( P_V : V \to V \). A morphism between operated \( k \)-vector spaces (resp. operated \( k \)-algebras, operated unital \( k \)-algebras) \( (V, P_V) \) and \( (W, P_W) \) is a homomorphism of \( k \)-vector spaces (resp. \( k \)-algebras, unital \( k \)-algebras) \( f : V \to W \) such that \( f \circ P_V = P_W \circ f \).

We use \( \mathbb{OpVect} \) (resp. \( \mathbb{OpAlg} \), \( \mathbb{uOpAlg} \)) to denote the category of operated \( k \)-vector spaces (resp. operated \( k \)-algebras, operated unital \( k \)-algebras).

Now, we are going to complete the bottom face and part of the upper face of the cubic diagram displayed in Introduction.
1.1. The bottom face.

Firstly, let’s consider the following diagram:

\[
\begin{array}{cccc}
\text{Set} & \xrightarrow{\mathcal{F}^\text{Sem}} & \text{Sem} & \xrightarrow{\mathcal{F}^\text{Mon}} \\
\downarrow & & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
\text{Set} & \xrightarrow{\mathcal{U}^\text{Sem}} & \text{Set} & \xrightarrow{\mathcal{U}^\text{Mon}} \\
\end{array}
\]

where, as explained in Introduction, the symbols \( \mathcal{U}^* \) stand for forgetful functors and the free object functors \( \mathcal{F}^* \) are left adjoint to the corresponding forgetful functors.

It is well known that the functor \( \mathcal{F}^\text{Sem}(X) \) assigns to a set \( X \) the free semigroup \( S(X) := \mathcal{F}^\text{Sem}(X) \) generated by it; the functor \( \mathcal{F}^\text{Mon}(X) \) is given by adding a unit element to a semigroup; as a consequence, the composite \( \mathcal{F}^\text{Mon} \circ \mathcal{F}^\text{Sem} \) is just the usual free monoid functor sending a set \( X \) to the free monoid \( M(X) := \mathcal{F}^\text{Mon}(X) \) generated by \( X \).

For any \( \mathbf{k} \)-space \( V \), \( \mathcal{F}^\text{Hilb}(V) \) is the reduced tensor algebra \( \overline{T}(V) = \bigoplus_{n \geq 1} V^\otimes n \); given a non-unital algebra \( A \), the algebra \( \mathcal{F}^\text{Hilb}(A) \) is defined to be the space \( \mathbf{k} \oplus \overline{T}(V) \) endowed with multiplication

\[(\lambda, a)(\mu, b) = (\lambda \mu, \lambda b + \mu a + ab)\]

for \( \lambda, \mu \in \mathbf{k} \) and \( a, b \in A \), which is a unital \( \mathbf{k} \)-algebra with unit \((1_k, 0)\); for any \( \mathbf{k} \)-space \( V \), the free unital \( \mathbf{k} \)-algebra \( \mathcal{F}^\text{Hilb}\mathcal{F}^\text{Rect}(V) = \mathcal{F}^\text{Hilb} \circ \mathcal{F}^\text{Rect}(V) \) generated by \( V \) is just the tensor algebra \( T(V) = \mathbf{k} \oplus \overline{T}(V) = \bigoplus_{n \geq 0} V^\otimes n \).

While the vertical forgetful functors \( \mathcal{U}^* \) in the above diagram is ignoring the \( \mathbf{k} \)-space structure, the vertical free object functors which are left adjoint to the corresponding forgetful functors are exactly given by linearization. For a set (resp. semigroup, monoid) \( X \), \( \mathcal{F}^\text{Sem}(X) \) (resp. \( \mathcal{F}^\text{Mon}(X) \)) is exactly \( \mathbf{k}X \), the \( \mathbf{k} \)-space spanned by elements in \( X \), which is naturally a vector space (resp. algebra, unital algebra). Denote \( \mathcal{F}^\text{Rect} = \mathcal{F}^\text{Hilb} \circ \mathcal{F}^\text{Rect} \). Then for a set \( X \), \( \mathcal{F}^\text{Rect}(X) \) is the free unital algebra generated by \( X \), which is just \( \mathbf{k}M(X) \), the noncommutative polynomial algebra with variables in \( X \).

1.2. The upper face. Now, let’s clarify the vertical arrows of the following diagram:

\[
\begin{array}{cccc}
\text{OpSet} & \xrightarrow{\mathcal{F}^\text{OpSet}} & \text{Op\mathcal{M}g} & \xrightarrow{\mathcal{U}^\text{Op\mathcal{M}g}} \\
\downarrow & & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
\text{Op\mathcal{M}g} & \xrightarrow{\mathcal{U}^\text{Op\mathcal{M}g}} & \text{Op\mathcal{M}g} & \xrightarrow{\mathcal{U}^\text{Op\mathcal{M}g}} \\
\end{array}
\]

Same as in the previous subsection, the vertical forgetful functors are just forgetting the \( \mathbf{k} \)-space structure and the vertical free object functors are also given by linearization. For an operated set (resp. operated semigroup, operated monoid) \((X, P_X)\), the associated operated vector space \( \mathcal{F}^\text{Op\mathcal{M}g}(X, P_X) \) (resp. operated algebra \( \mathcal{F}^\text{Op\mathcal{M}g}(X, P_X) \), unital operated algebra \( \mathcal{F}^\text{Op\mathcal{M}g}(X, P_X) \)) is
just \((kX, P_{kX})\), where the linear operator \(P_{kX}\) on \(kX\) is just the linear extension of the operator \(P_X\) on \(X\).

2. Free object functors II

In this section, we want to investigate the front face of the diagram displayed in Introduction. More precisely, we will construct all the free object functors in the following diagram of functors:

\[
\begin{array}{cccc}
\mathcal{OpSet} & \mathcal{OpSem} & \mathcal{OpMon} \\
\mathcal{Set} & \mathcal{Sem} & \mathcal{Mon} \\
\mathcal{OpSem} & \mathcal{OpMon} & \mathcal{OpMon} \\
\mathcal{Sem} & \mathcal{Set} & \mathcal{Set} \\
\end{array}
\]

Same as before, the symbols \(\mathcal{U}_s^*\) represent forgetful functors and the free object functors \(\mathcal{F}_s^*\) are left adjoint to the corresponding forgetful functors.

2.1. Free object functor from sets to operated sets.

Let \(X\) be a nonempty set. Denote by \([X]\) the set of the formal elements \([x], x \in X\). Write \([X]^{(0)} = X\), \([X]^{(1)} = X\) and for \(n \geq 1\), put \([X]^{(n)} = [[X]^{(n)}]\). For \(x \in X\), let \([x]^{(n)}\) be the corresponding element in \([X]^{(n)}\). Define \(\Theta\) to be the disjoint union of all \([X]^{(n)}\), \(n \geq 0\) and an operator \(P_X\) on \(\Theta\) mapping \(u \in X\) to \([u]\). Then the pair \(\mathcal{F}_s^* : \mathcal{Set} \to \mathcal{OpSet}\) forms an operated set. When \(X\) is the empty set \(\emptyset\), define \(\mathcal{F}_s^* (\emptyset)\) to be the empty set together with its identity map.

For a map \(f : X \to Y\), introduce \(\mathcal{F}_s^* (f) : (X, P_X) \to (Y, P_Y)\) to be the map sending \([x]^{(n)}\) with \(x \in X\) to \([f(x)]^{(n)}\). It is easy to see that \(\mathcal{F}_s^*\) is a functor from \(\mathcal{Set}\) to \(\mathcal{OpSet}\).

**Proposition 2.1.** The functor \(\mathcal{F}_s^* : \mathcal{Set} \to \mathcal{OpSet}\) is left adjoint to the forgetful functor \(\mathcal{U}_s^* : \mathcal{OpSet} \to \mathcal{Set}\), hence giving the free operated set generated by a set.

**Proof.** Let \(X\) be a set and \((Y, P_Y)\) be an operated set. Given a map \(\theta : X \to Y\), we will show that there is a unique morphism of operated sets \(\tilde{\theta} : \mathcal{F}_s^* (X) \to Y\) which, when restricted to \(X\), recovers \(\theta\).

In fact, it suffices to define the map \(\tilde{\theta}\) from \(X\) to \(Y\) by imposing \(\tilde{\theta}([x]^{(n)}) := P_Y^n (\theta(x))\) for any \(x \in X\). Clearly, \(\tilde{\theta}\) is the unique morphism of operated sets from \((X, P_X)\) to \((Y, P_Y)\) whose restriction on \(X\) is equal to \(\theta\).

Therefore, \(\mathcal{F}_s^*\) is the free object functor from \(\mathcal{Set}\) to \(\mathcal{OpSet}\). \(\square\)

2.2. Free object functor from semigroups to operated semigroups.

In this subsection, we will construct the functor \(\mathcal{F}_s^* : \mathcal{Sem} \to \mathcal{OpSem}\) left adjoint to the forgetful functor \(\mathcal{U}_s^* : \mathcal{OpSem} \to \mathcal{Sem}\).

**Definition 2.2.** Let \(Z\) be a set. Recall for a semigroup \(T\), \(\mathcal{U}_s^* (T)\) is just the underlying set of \(T\). Write \(S(T \sqcup Z) := S(\mathcal{U}_s^* (T) \sqcup Z)\) the free semigroup generated by the disjoint union \(\mathcal{U}_s^* (T) \sqcup Z\). We define a semigroup \(\tilde{S}(T \sqcup Z)\) to be the quotient of \(S(T \sqcup Z)\) by identifying, for all \(t, t' \in T\), their products in \(T\) and in \(S(T \sqcup Z)\).
Notice that if $Z = \emptyset$, $\tilde{S}(T \sqcup Z)$ is exactly $T$ unchanged.

The following lemma will be useful in the sequel, whose easy proof is left to the reader.

**Lemma 2.3.** (a) Let $Z$ be a set and $T$ be a semigroup. Each element in $\tilde{S}(T \sqcup Z)$ has a unique expression

$$a_0 t_1 a_1 t_2 \cdots t_n a_n$$

with $n \geq 0$ and all $t_i \in T$, where for any $0 \leq i \leq n$, $a_i$ lies in $S(Z)$ or is the empty word $\emptyset$, whereas when $n \geq 2$, neither of $a_1, \cdots, a_{n-1}$ can be $\emptyset$ and when $n = 0$, $a_0$ is not empty either.

(b) Let $Z \hookrightarrow Z'$ be an injective map of sets and $T \hookrightarrow T'$ be an injective homomorphism of semigroups. Then the induced homomorphism of semigroups

$$\tilde{S}(T \sqcup Z) \to \tilde{S}(T' \sqcup Z')$$

is injective as well.

Let $T$ be a semigroup. Deduced from Lemma 2.3, the inclusion into the first component $T \hookrightarrow T \sqcup [T]$ induces an injective semigroup homomorphism

$$i_{0,1} : \mathcal{F}_0(T) := T \hookrightarrow \mathcal{F}_1(T) := \tilde{S}(T \sqcup [T]).$$

For $n \geq 2$, assume that we have constructed $\mathcal{F}_{n-2}(T)$ and $\mathcal{F}_{n-1}(T)$ such that $\mathcal{F}_{n-1}(T) = \tilde{S}(T \sqcup [\mathcal{F}_{n-2}(T)])$ endowed with an injective homomorphism of semigroups

$$i_{n-2,n-1} : \mathcal{F}_{n-2}(T) \hookrightarrow \mathcal{F}_{n-1}(T).$$

We define the semigroup

$$\mathcal{F}_n(T) := \tilde{S}(T \sqcup [\mathcal{F}_{n-1}(T)])$$

and again by Lemma 2.3, the natural injection

$$\text{Id}_T \sqcup [i_{n-2,n-1}] : T \sqcup [\mathcal{F}_{n-2}(T)] \hookrightarrow T \sqcup [\mathcal{F}_{n-1}(T)]$$

induces an injective semigroup homomorphism

$$i_{n-1,n} : \mathcal{F}_{n-1}(T) = \tilde{S}(T \sqcup [\mathcal{F}_{n-2}(T)]) \hookrightarrow \mathcal{F}_n(T) = \tilde{S}(T \sqcup [\mathcal{F}_{n-1}(T)]).$$

Define $\mathcal{F}(T) = \varprojlim \mathcal{F}_n(T)$ and the map sending $u \in \mathcal{F}_n(T)$ to $[u] \in \mathcal{F}_{n+1}(T)$ induces an operator $P_{\mathcal{F}(T)}$ on $\mathcal{F}(T)$. By the limit construction, there are injective semigroup homomorphisms $i_n : \mathcal{F}_n(T) \hookrightarrow \mathcal{F}(T), n \geq 0$ and $i : T \hookrightarrow \mathcal{F}(T)$. Thus we have constructed a functor:

$$\mathcal{F}_{\mathsf{OpSem}} : \mathsf{Sem} \to \mathsf{OpSem}$$

$$T \mapsto (\mathcal{F}(T), P_{\mathcal{F}(T)}).$$

**Proposition 2.4.** The functor $\mathcal{F}_{\mathsf{OpSem}} : \mathsf{Sem} \to \mathsf{OpSem}$ is left adjoint to the forgetful functor $\mathcal{U}_{\mathsf{OpSem}} : \mathsf{OpSem} \to \mathsf{Sem}$.

**Proof.** Let $T$ be a semigroup and $(S, P_S)$ be an operated semigroup. Let

$$\theta : T \to \mathcal{U}_{\mathsf{OpSem}}(S, P_S) = S$$

be a homomorphism of semigroups. We will construct a morphism of operated semigroups

$$\tilde{\theta} : \mathcal{F}_{\mathsf{OpSem}}(T) \to (S, P_S)$$

such that $\tilde{\theta} \circ i = \theta$. Moreover, it is unique.
Let $\tilde{\theta}_0 = \theta$. Define a map $\theta_1 : T \sqcup [T] \to S$ by sending any element $t \in T$ to $\theta(t)$ and $[t]$ to $P_S(\theta([t]))$. The universal property of free semigroups gives a homomorphism of semigroups $\tilde{\theta}_1 : S(T \sqcup [T]) \to S$ extending $\theta_1$. For any $t_1, t_2 \in T$, we have

$$\tilde{\theta}_1(t_1 \cdot t_2) = \tilde{\theta}_1(t_1) \cdot \tilde{\theta}_1(t_2) = \theta(t_1) \cdot \theta(t_2) = \theta(t_1 \cdot t_2) = \tilde{\theta}_1(t_1 \cdot t_2),$$

where the first $\cdot$ is the multiplication in $S(T \sqcup [T])$ and the last one in $T$, so there exists an induced semigroup homomorphism

$$\tilde{\theta}_1 : F_1(T) = \bar{S}(T \sqcup [T]) \to S.$$

Assume by induction that we are given homomorphisms of semigroups $\tilde{\theta}_k : F_k(T) \to S$ for $1 \leq k \leq n$ such that $\tilde{\theta}_k \circ i_{k-1,k} = \tilde{\theta}_{k-1}$. Build a map $\theta_{n+1} : T \sqcup [F_n(T)] \to S$ by sending any element $t \in T$ to $\theta(t)$ and $[v]$ to $P_S(\theta_n(v))$ for arbitrary $v \in F_n(T)$. By the universal property of free semigroups, $\theta_{n+1}$ induces a homomorphism of semigroups $\tilde{\theta}_{n+1} : S(T \sqcup [F_n(T)]) \to S$. For any $t_1, t_2 \in T$, we have

$$\tilde{\theta}_{n+1}(t_1 \cdot t_2) = \tilde{\theta}_{n+1}(t_1) \cdot \tilde{\theta}_{n+1}(t_2) = \theta(t_1) \cdot \theta(t_2) = \theta(t_1 \cdot t_2) = \tilde{\theta}_{n+1}(t_1 \cdot t_2),$$

where the first $\cdot$ is the multiplication in $S(T \sqcup [F_n(T)])$ and the last one in $T$, so there exists an induced semigroup homomorphism

$$\tilde{\theta}_{n+1} : F_{n+1}(T) = \bar{S}(T \sqcup [F_n(T)]) \to S.$$

Hence, we have constructed a series of semigroup homomorphisms $\tilde{\theta}_k : F_k(T) \to S, k \geq 0$ which fit into the commutative diagram:

\[
\begin{array}{c}
T \xrightarrow{\theta} S \\
\downarrow \quad | \\
F_0(T) \xleftarrow{i_{0,1}} F_1(T) \xleftarrow{i_{1,2}} F_2(T) \xleftarrow{i_{2,3}} \cdots \xrightarrow{i_{n-1,n}} F_n(T) \xleftarrow{i_{n,n+1}} F_{n+1}(T) \\
\end{array}
\]

The limit construction gives a homomorphism of semigroups $\tilde{\theta}$ from $F(T)$ to $S$, which by construction commutes with the operators, thus $\tilde{\theta}$ is a morphism of operated semigroups.

The uniqueness of $\tilde{\theta}$ is clear.

We are done.

\[\square\]

2.3. **Free object functor from monoids to operated monoids.**

Recall that $\mathcal{M}(X) := F^{\text{Mon}}_{\text{set}}(X)$ is the free monoid generated by set $X$.

**Definition 2.5.** Let $Z$ be a set and $T$ be a semigroup (resp. monoid). Write $\mathcal{M}(T \sqcup Z) := \mathcal{M}(U^{\text{em}}_{\text{set}}(T) \sqcup Z)$ (resp. $\mathcal{M}(U^{\text{Mon}}_{\text{set}}(T) \sqcup Z)$) for the free monoid generated by the disjoint union $U^{\text{em}}_{\text{set}}(T) \sqcup Z$ (resp. $U^{\text{Mon}}_{\text{set}}(T) \sqcup Z$).

When $T$ is a semigroup, we define a monoid $\tilde{\mathcal{M}}(T \sqcup Z)$ to be the quotient of $\mathcal{M}(T \sqcup Z)$ by identifying, for all $t, t' \in T$, their products in $T$ and in $\mathcal{M}(T \sqcup Z)$; when $T$ is a monoid and $e$ is its unit, we need to identify, furthermore, $e$ with the unit of $\mathcal{M}(T \sqcup Z)$.

Notice that if $Z = \emptyset$, then $\tilde{\mathcal{M}}(T \sqcup Z) = F^{\text{Mon}}_{\text{set}}(T)$ (i.e. just adding a unit to $T$) when $T$ is a semigroup and when $T$ is a monoid, it is still $T$ itself.

Similar to Lemma 2.3, the obvious proof of the following lemma is left to the reader.
Lemma 2.6. (1) Let $Z$ be a set and $T$ be a semigroup or a monoid. Any element in $\tilde{M}(T \sqcup Z)$ possesses a unique expression

$$a_0t_1a_1t_2 \cdots t_n$$

with $n \geq 0$, where for $0 \leq i \leq n$, $a_i \in M(Z)$, but for each $1 \leq i \leq n - 1$, $a_i \in S(Z)$; for $1 \leq i \leq n$, $t_i$ belongs to $T$, but neither of them is the unit of $T$ when $T$ is a monoid.

(2) Let $Z \hookrightarrow Z'$ (resp. $T \hookrightarrow T'$) be an injective map of sets (resp. an injective homomorphism of semigroups or monoids). Then the induced homomorphism

$$\tilde{M}(T \sqcup Z) \to \tilde{M}(T' \sqcup Z')$$

is injective, too.

Let $M$ be a monoid. The free operated monoid $F_{\text{opMon}}^M(M)$ generated by $M$ can be built exactly as is done in the preceding subsection except that $\tilde{F}$ can show the functor $F_{\text{opMon}}$ with $\text{Mon} \to \text{Deduced from the natural injection}$

we obtain an injective semigroup homomorphism

is a morphism of operated sets.

Finally, we define

$$\tilde{M}(T \sqcup Z) \to \tilde{M}(T' \sqcup Z')$$

is injective, too.

2.4. Free object functor from operated sets to operated semigroups.

In this subsection, we will construct the functor $F_{\text{opSem}}^M : \text{OpSet} \to \text{OpSem}$ left adjoint to the forgetful functor $U_{\text{opMon}} : \text{OpMon} \to \text{Mon}$. Its construction is a little more involved than as in the preceding subsections.

Let $(X, P_X)$ be an operated set and $F_0(X) := S(X)$. Consider the set $S(X \setminus X)$ obtained by deleting $X$ from $S(X)$ and let $F_1(X) := S(X \sqcup [S(X) \setminus X])$. The natural inclusion $X \hookrightarrow X \sqcup [S(X) \setminus X]$, after applying the free semigroup functor $F_{\text{opSem}}^M$, gives a homomorphism of semigroups

$$i_{0,1} : F_0(X) = S(X) \hookrightarrow F_1(X) = S(X \sqcup [S(X) \setminus X]).$$

For $n \geq 2$, suppose given a sequence of injective semigroup homomorphisms

$$F_0(X) \overset{i_{0,1}}{\hookrightarrow} F_1(X) \overset{i_{1,2}}{\hookrightarrow} \cdots \overset{i_{n-3,n-2}}{\hookrightarrow} F_{n-2}(X) \overset{i_{n-2,n-1}}{\hookrightarrow} F_{n-1}(X)$$

with $F_k(X) = S(X \sqcup [F_{k-1}(X) \setminus X])$ for all $1 \leq k \leq n - 1$, define as above the semigroup

$$F_n(X) := S(X \sqcup [F_{n-1}(X) \setminus X]).$$

Deduced from the natural injection

$$\text{Id}_X \sqcup [i_{n-2,n-1}F_{n-2}(X)] : X \sqcup [F_{n-2}(X) \setminus X] \hookrightarrow X \sqcup [F_{n-1}(X) \setminus X],$$

we obtain an injective semigroup homomorphism

$$i_{n-1,n} : F_{n-1}(X) = S(X \sqcup [F_{n-2}(X) \setminus X]) \hookrightarrow F_n(X) = S(X \sqcup [F_{n-1}(X) \setminus X]).$$

Finally, we define

$$F(X) := \text{lim}_{n \to \infty} F_n(X).$$

Obviously $F(X)$ is a semigroup. By the limit construction, there are injective semigroup homomorphisms $i_n : F_n(X) \hookrightarrow F(X), n \geq 0$ and $i : X \hookrightarrow F(X)$. The operator $P_{F(X)}$ over $F(X)$ can be constructed as follows:

$$P_{F(X)}(u) := \begin{cases} i(P_X(x)) & \text{if } u = i(x), \text{ for some } x \in X, \\ i_{k+1}([u']) & \text{if } u \notin i(X), \text{ but } u = i_k(u'), \text{ for some } u' \in F_k(X). \end{cases}$$

Then $(F(X), P_{F(X)})$ forms an operated semigroup. Now it is obvious that $i : (X, P_X) \to (F(X), P_{F(X)})$ is a morphism of operated sets.
Moreover, it is ready to see that the construction above is functorial.
We have constructed a functor:
\[
\mathcal{F}_{\text{OpSem}} : \text{OpSem} \rightarrow \text{OpSem}
\]
\[
(X, P_X) \mapsto (\mathcal{F}(X), P_{\mathcal{F}(X)}).
\]

Now, we are going to show the main result of this subsection.

**Proposition 2.7.** The functor \(\mathcal{F}_{\text{OpSem}} : \text{OpSem} \rightarrow \text{OpSem}\) is left adjoint to the forgetful functor \(\mathcal{U}_{\text{OpSem}} : \text{OpSem} \rightarrow \text{OpSet}\).

**Proof.** Let \((T, P_T)\) be an operated semigroup. Let
\[
\theta : (X, P_X) \rightarrow \mathcal{U}_{\text{OpSem}}(T, P_T) = (T, P_T)
\]
be a morphism of operated sets. We will construct a morphism of operated semigroups
\[
\tilde{\theta} : \mathcal{F}_{\text{OpSem}}(X, P_X) \rightarrow (T, P_T)
\]
such that \(\tilde{\theta} \circ i = \theta\). Moreover, it is unique.

By the universal property of free semigroup construction, we have a homomorphism of semigroups \(\tilde{\theta}_0 : \mathcal{F}_0(X) = S(X) \rightarrow T\) extending \(\theta\). Now consider the map \(\tilde{\theta}_1 : X \sqcup [S(X)\setminus X] \rightarrow T\) by sending any element \(x \in X\) to \(\theta(x)\) and \([u]\) to \(P_T(\tilde{\theta}_0(u))\) for arbitrary \(u \in S(X)\setminus X\), which induces \(\tilde{\theta}_1 : \mathcal{F}_1(X) = S(X \sqcup [S(X)\setminus X]) \rightarrow T\) such that \(\tilde{\theta}_1 \circ i_{0,1} = \tilde{\theta}_0\).

By iterating this process, we found a series of homomorphisms of semigroups \(\tilde{\theta}_k : \mathcal{F}_k(X) \rightarrow T, k \geq 0\) which fit into the commutative diagram:

\[
\begin{array}{cccccc}
X & \theta & T \\
\mathcal{F}_0(X) \downarrow \mathcal{F}_1(X) \downarrow \mathcal{F}_2(X) \downarrow \cdots \downarrow \mathcal{F}(X) \\
\mathcal{F}_0(X) \downarrow \mathcal{F}_1(X) \downarrow \mathcal{F}_2(X) \downarrow \cdots \downarrow \mathcal{F}(X) \\
\tilde{\theta}_0 & \tilde{\theta}_1 & \tilde{\theta}_2 & \cdots & \tilde{\theta}
\end{array}
\]

It follows that there exists a homomorphism of semigroups \(\tilde{\theta}\) from \(\mathcal{F}(X)\) to \(T\) and by construction \(\tilde{\theta}\) is a morphism of operated semigroups.

The uniqueness of \(\tilde{\theta}\) is clear.

This completes the proof. \(\square\)

### 2.5. Free object functor from operated semigroups to operated monoids.

In this subsection, we will construct the functor \(\mathcal{F}_{\text{OpMon}} : \text{OpSem} \rightarrow \text{OpMon}\) left adjoint to the forgetful functor \(\mathcal{U}_{\text{OpMon}} : \text{OpMon} \rightarrow \text{OpSem}\).

Let \((T, P_T)\) be an operated semigroup, \(\mathcal{F}_0(T) := \tilde{M}(T \sqcup \emptyset) = \tilde{M}(T)\) be the monoid defined before. By Lemma 2.6, the natural inclusion \(T \hookrightarrow T \sqcup [\mathcal{F}_0(T)\setminus T]\) induces an injective monoid homomorphism
\[
i_{0,1} : \mathcal{F}_0(T) = \tilde{M}(T) \hookrightarrow \mathcal{F}_1(T) := \tilde{M}(T \sqcup [\mathcal{F}_0(T)\setminus T]).
\]
For \(n \geq 2\), by induction starting from a series of injective monoid homomorphisms
\[
\mathcal{F}_0(T) \xleftarrow{i_{0,1}} \mathcal{F}_1(T) \xleftarrow{i_{1,2}} \cdots \xleftarrow{i_{n-2,n-1}} \mathcal{F}_{n-2}(T) \xleftarrow{i_{n-2,n-1}} \mathcal{F}_{n-1}(T)
\]
with \(\mathcal{F}_k(T) = \tilde{M}(T \sqcup [\mathcal{F}_{k-1}(T)\setminus T])\) for all \(1 \leq k \leq n - 1\), one imposes
\[
\mathcal{F}_n(T) := \tilde{M}(T \sqcup [\mathcal{F}_{n-1}(T)\setminus T]).
\]
Again by Lemma 2.6, from the injection
\[ \text{Id}_T \sqcup [i_{n-2,n-1}^{\mathcal{F}_{n-2}(T)} : T \sqcup [\mathcal{F}_{n-2}(T)] \hookrightarrow T \sqcup [\mathcal{F}_{n-1}(T)], \]
one can build an injective monoid homomorphism
\[ i_{n-1,n} : \mathcal{F}_{n-1}(T) = \mathcal{M}(T) \sqcup [\mathcal{F}_{n-2}(T)] \hookrightarrow \mathcal{F}_n(T) = \mathcal{M}(T) \sqcup [\mathcal{F}_{n-1}(T)]. \]

Now we impose \( \mathcal{F}(T) := \lim \mathcal{F}_n(T). \) Notice that there are natural induced injective monoid homomorphisms \( i_n : \mathcal{F}_n(T) \hookrightarrow \mathcal{F}(T), \) \( n \geq 0 \) and \( i : T \hookrightarrow \mathcal{F}(T). \) Hence, we obtain an operated monoid
\[
\left( \mathcal{F}(T), P_{\mathcal{F}(T)}(u) := \begin{cases} 
   i(P_T(t)) & \text{if } u = i(t), t \in T \\
   i_{k+1}(u') & \text{if } u \notin i(T), \text{but } u = i_k(u'), \text{ for some } u' \in \mathcal{F}_k(T) \end{cases} \right).
\]

It is ready to see that this forms a functor
\[
\mathcal{F}^{\mathcal{Cp}\mathcal{Em}_{\mathcal{OpMon}}} : \mathcal{Cp}\mathcal{Sem} \rightarrow \mathcal{Cp}\mathcal{Mon} \quad (T, P_T) \mapsto (\mathcal{F}(T), P_{\mathcal{F}(T)}),
\]
and it is left adjoint to the forgetful functor \( \mathcal{U}^{\mathcal{Cp}\mathcal{Mon}}_{\mathcal{Cp}\mathcal{Sem}} : \mathcal{Cp}\mathcal{Mon} \rightarrow \mathcal{Cp}\mathcal{Sem}. \) The details are left to the reader.

2.6. Composites of free object functors.

It is easy to check that the composite \( \mathcal{F}^{\mathcal{Cp}\mathcal{Em}_{\mathcal{OpMon}}} = \mathcal{F}^{\mathcal{Cp}\mathcal{Em}}_{\mathcal{OpMon}} \circ \mathcal{F}^{\mathcal{Em}_{\mathcal{OpMon}}} \) gives exactly the free operated semigroup generated by a set, whose construction firstly appeared in [26]. Although \( \mathcal{F}^{\mathcal{Em}_{\mathcal{OpMon}}} \) is naturally isomorphic to \( \mathcal{F}^{\mathcal{Cp}\mathcal{Em}_{\mathcal{OpMon}}} \circ \mathcal{F}^{\mathcal{Cp}\mathcal{Em}_{\mathcal{OpMon}}} \), the construction of the latter seems to be much more complicated.

Similarly, \( \mathcal{F}^{\mathcal{Cp}\mathcal{Mon}_{\mathcal{OpMon}}} = \mathcal{F}^{\mathcal{Cp}\mathcal{Mon}}_{\mathcal{OpMon}} \circ \mathcal{F}^{\mathcal{Mon}_{\mathcal{OpMon}}} \) is exactly the free operated monoid generated by a set as constructed in [26], whose another realisation \( \mathcal{F}^{\mathcal{Cp}\mathcal{Mon}}_{\mathcal{Cp}\mathcal{Mon}} \circ \mathcal{F}^{\mathcal{Cp}\mathcal{Mon}_{\mathcal{OpMon}}} \) is much more involved, however.

In the sequel, following [26], denote
\[ (\mathcal{Z}(X), P_{\mathcal{Z}(X)}) := \mathcal{F}^{\mathcal{Cp}\mathcal{Em}_{\mathcal{OpMon}}}_{\mathcal{OpMon}}(X) = \mathcal{F}^{\mathcal{Em}_{\mathcal{OpMon}}}_{\mathcal{OpMon}}(S(X)) \]
and
\[ (\mathcal{V}(X), P_{\mathcal{V}(X)}) := \mathcal{F}^{\mathcal{Cp}\mathcal{Mon}_{\mathcal{OpMon}}}_{\mathcal{OpMon}}(X) = \mathcal{F}^{\mathcal{Mon}_{\mathcal{OpMon}}}_{\mathcal{OpMon}}(\mathcal{M}(X)). \]

3. Free object functors III

Parallel with the last section, we will consider the back face of the diagram in Introduction:

![Diagram](image_url)

Our goal in this section is to construct the free object functors \( \mathcal{F}^{\ast}_{\ast} \) which are left adjoint to the corresponding forgetful functors \( \mathcal{U}^{\ast}_{\ast}. \)
Since all the proofs in this section are similar to those in the previous sections, we omit all of them.

3.1. Free object functor from vector spaces to operated vector spaces.

Let $V$ be a $k$-space with a basis $X$. Extend the map $[ ] : X \rightarrow [X]$ to a linear map $[ ] : V \rightarrow [V] := k[X]$, where by abuse of notations, we use the same symbol for the induced linear map. It is easy to see that this definition is independent of the choice of basis. Similarly, we can define $[V]^{(n)}$ and $[v]^{(n)}$ for $n \geq 0$ and $v \in V$. Define $V := \bigoplus_{n \geq 0} [V]^{(n)}$ and an operator $P_V$ on $V$ mapping $[v]^{(n)}$ to $[v]^{(n+1)}$ for all $v \in V$ and $n \geq 0$. Then the pair $F : \text{Vect} \rightarrow \text{OpVect}$ is a functor from vector spaces to operated vector spaces. In fact, it can be seen that $(n = 1)$.

For a linear map $f : V \rightarrow W$, introduce $F_{\text{Vect}}(f) : (V, P_V) \rightarrow (W, P_W)$ to be the linear map sending $[v]^{(n)}$ to $[f(v)]^{(n)}$ for all $n \geq 0$ and $v \in V$. It is easy to see that $F_{\text{Vect}}$ is a functor from $\text{Vect}$ to $\mathcal{P}\text{Vect}$.

It is easy to see that the functor $F_{\text{Vect}} : \mathcal{P}\text{Vect} \rightarrow \mathcal{P}\text{Vect}$ is left adjoint to the forgetful functor $U_{\text{Vect}} : \mathcal{P}\text{Vect} \rightarrow \text{ Vect}$, hence giving the free operated $k$-space generated by a $k$-space.

3.2. Free object functor from algebras to operated algebras.

In this subsection, we will construct the functor $F_{\mathcal{A}_{\text{Alg}}} : \mathcal{A}_{\text{Alg}} \rightarrow \mathcal{P}\mathcal{A}_{\text{Alg}}$ left adjoint to the forgetful functor $U_{\mathcal{A}_{\text{Alg}}} : \mathcal{P}\mathcal{A}_{\text{Alg}} \rightarrow \mathcal{A}_{\text{Alg}}$.

**Definition 3.1.** Let $V$ be a $k$-space, $A$ be a $k$-algebra. We define the following quotient $k$-algebra:

$$\overline{T}(A \oplus V) := \overline{T}(A \oplus V) / \langle a \otimes b - a \cdot b | \forall a, b \in A \rangle,$$

where $\otimes$ and $\cdot$ are respectively the multiplications in $\overline{T}(A \oplus V)$ and $A$. In particular, if $V = 0$, then $\overline{T}(A) = \overline{T}(A) / \langle a \otimes b - a \cdot b | \forall a, b \in A \rangle = A$.

**Lemma 3.2.** Let $V \hookrightarrow V'$ be an injective linear map, $A \hookrightarrow B$ be a monomorphism of $k$-algebras. Then the induced homomorphism of algebras

$$\overline{T}(A \oplus V) \rightarrow \overline{T}(B \oplus V')$$

is injective.

Let $A$ be a $k$-algebra and $F_0(A) := A$. The inclusion into the first component $A \hookrightarrow A \oplus [A] \oplus [A]$ induces an algebra homomorphism

$$i_{0,1} : F_0(A) = A \rightarrow F_1(A) := \overline{T}(A \oplus [A]).$$

For $n \geq 2$, assume that we have a series of monomorphisms of algebras

$$F_0(A) \hookrightarrow F_1(A) \hookrightarrow \cdots \hookrightarrow F_{n-2}(A) \hookrightarrow F_{n-1}(A),$$

with $F_k(A) := \overline{T}(A \oplus [F_{k-1}(A)])$ for $1 \leq k \leq n - 1$. We define the $k$-algebra

$$F_n(A) := \overline{T}(A \oplus [F_{n-1}(A)]).$$

The natural injection

$$\text{Id}_A \oplus [i_{n-2, n-1}] : A \oplus [F_{n-2}(A)] \hookrightarrow A \oplus [F_{n-1}(A)]$$

induces a $k$-algebra injective homomorphism

$$i_{n-1, n} : F_{n-1}(A) = \overline{T}(A \oplus [F_{n-2}(A)]) \rightarrow F_n(A) = \overline{T}(A \oplus [F_{n-1}(A)]).$$
Denote $\mathcal{F}(A) = \lim_{n \to 0} \mathcal{F}_n(A)$ and an operator $P_{\mathcal{F}}$ over $\mathcal{F}$ sending $v \in \mathcal{F}$ to $[v]$. Thus, we have constructed the free object functor:

$$\mathcal{F}^{\mathcal{C}_{\mathcal{pgf}} : \mathcal{U}}_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{C}_{\mathcal{pgf}}$$

$$A \mapsto (\mathcal{F}(A), P_{\mathcal{F}(A)})$$

which is left adjoint to the forgetful functor $\mathcal{U}^{\mathcal{C}_{\mathcal{pgf}} : \mathcal{G}}$.  

3.3. **Free object functor from unital algebras to unital operated algebras.**

**Definition 3.3.** Let $A$ be a unital $k$-algebra and $V$ a $k$-space. We define the following quotient unital $k$-algebra

$$\hat{T}(A \oplus V) := T(A \oplus V) / \langle a \otimes b - a \cdot b, 1_A - 1_{T(A \oplus V)} \mid \forall a, b \in A \rangle.$$  

In particular, if $V = 0$, then $\hat{T}(A) = T(A) / \langle a \otimes b - a \cdot b \mid \forall a, b \in A \rangle = A$.

**Lemma 3.4.** Let $V \hookrightarrow V'$ be an injective linear map, $A \hookrightarrow B$ be a monomorphism of unital $k$-algebras. Then the induced homomorphism

$$\hat{T}(A \oplus V) \to \hat{T}(B \oplus V')$$

is injective.

Let $A$ be a unital $k$-algebra and $\mathcal{F}_0(A) := \hat{T}(A) = A$. By Lemma 3.4, the inclusion $A \hookrightarrow A \oplus |A|$ induces a monomorphism

$$i_{0,1} : \mathcal{F}_0(A) = \hat{T}(A) \hookrightarrow \mathcal{F}_1(A) := \hat{T}(A \oplus |A|)$$

of unital algebras.  

For $n \geq 2$, by induction starting from a series of monomorphisms of unital algebras

$$\mathcal{F}_0(A) \hookrightarrow \mathcal{F}_1(A) \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{n-2}(A) \hookrightarrow \mathcal{F}_{n-1}(A)$$

with $\mathcal{F}_k(A) = \hat{T}(A \oplus |\mathcal{F}_{k-1}(A)|)$ for $1 \leq k \leq n - 1$. Then we define the unital algebra

$$\mathcal{F}_n(A) := \hat{T}(A \oplus |\mathcal{F}_{n-1}(A)|).$$

The natural injection

$$\text{Id}_A \oplus [i_{n-2,n-1}] : A \oplus |\mathcal{F}_{n-2}(A)| \hookrightarrow A \oplus |\mathcal{F}_{n-1}(A)|$$

induces an injective algebra homomorphism

$$i_{n-1,n} : \mathcal{F}_{n-1}(A) = \hat{T}(A \oplus |\mathcal{F}_{n-2}(A)|) \hookrightarrow \mathcal{F}_n(A) = \hat{T}(A \oplus |\mathcal{F}_{n-1}(A)|).$$

Denote $\mathcal{F}(A) = \lim_{n \to \infty} \mathcal{F}_n(A) = \bigcup_{n \geq 0} \mathcal{F}_n(A)$ and define an operator $P_{\mathcal{F}(A)}$ over $\mathcal{F}(A)$ by sending $v \in \mathcal{F}(A)$ to $[v]$. Thus, we have constructed the free object functor:

$$\mathcal{F}^{\mathcal{C}_{\mathcal{pgf}} : \mathcal{G}}_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{C}_{\mathcal{pgf}}$$

$$A \mapsto (\mathcal{F}(A), P_{\mathcal{F}(A)}).$$

left adjoint to the forgetful functor $\mathcal{U}^{\mathcal{C}_{\mathcal{pgf}} : \mathcal{G}}$.  

3.4. Free object functor from operated vector spaces to operated algebras.

In this subsection, we will construct the free object functor $F_{\mathcal{C}^{\text{OpVect}}}$ from $\mathcal{ObVect}$ to $\mathcal{OpAlg}$ which is left adjoint to the forgetful functor $U_{\mathcal{C}^{\text{OpVect}}}$ from $\mathcal{OpAlg}$ to $\mathcal{ObVect}$.

Let $(V, P_V)$ be an operated vector space. As $\bar{T}(V)$ has a canonical splitting $\bar{T}(V) = V \oplus V^\otimes 2 \oplus \cdots$ and we will write $\bar{T}(V)/V = V^\otimes 2 \oplus \cdots$ in the sequel. The reader should understand the notations $\bar{F}_n(V)/V$, $\bar{T}(A)/A$ and $\bar{F}_n(A)/A$ via the canonical splitting.

The inclusion $i_0 : V \hookrightarrow V \oplus \left[ \bar{T}(V)/V \right]$ induces a monomorphism of algebras:

$$i_{0,1} : F_0(V) := \bar{T}(V) \hookrightarrow F_1(V) := \bar{T}(V \oplus \left[ \bar{T}(V)/V \right]).$$

For $n \geq 2$, suppose that a series of monomorphisms of algebras

$$F_0(V) \hookrightarrow F_1(V) \hookrightarrow \cdots \hookrightarrow F_{n-2}(V) \hookrightarrow F_{n-1}(V),$$

has been obtained. Then we define

$$F_n(V) := \bar{T}(V \oplus [F_{n-1}(V)/V]).$$

The injection

$$\text{Id}_V \oplus [i_{n-2,1}]_{F_{n-2}(V)/V} : V \oplus [F_{n-2}(V)/V] \hookrightarrow V \oplus [F_{n-1}(V)/V]$$

induces a $k$-algebra injection

$$i_{n-1, n} : F_{n-1}(V) := \bar{T}(V \oplus [F_{n-2}(V)/V]) \hookrightarrow F_n(V) := \bar{T}(V \oplus [F_{n-1}(V)/V]).$$

Notice that the subspace $V$ has a canonical complement in each $F_i(V)$; for instance, the complement of $V$ in $F_0(V) = \bar{T}(V)$ is exactly $\bigoplus_{n \geq 2} V^\otimes n$. In practice, we always identify $F_k(V)/V$ with this complement.

By the limit construction, there are injective homomorphisms $i_n : F_n(V) \hookrightarrow F(V), n \geq 0$ and $i : V \hookrightarrow F(V)$. Then, we define an operated $k$-algebra

$$\left( F(V) := \lim_{\longrightarrow} F_n(V), P_{F(V)}(u) := \begin{cases} i(P_V(v)) & \text{if } u = i(v) \text{ for some } v \in V \\ i_{k+1}([u']) & \text{if } u \notin i(V), \text{ but } u = i_k(u') \text{ for some } u' \in F_k(V) \end{cases} \right).$$

Thus we get a functor

$$F_{\mathcal{C}^{\text{OpVect}}} : \mathcal{ObVect} \longrightarrow \mathcal{OpAlg}

(V, P_V) \longmapsto (F(V), P_{F(V)}).$$

We obtain the following result:

**Proposition 3.5.** The functor $F_{\mathcal{C}^{\text{OpVect}}}$ is the left adjoint of the forgetful functor $U_{\mathcal{C}^{\text{OpVect}}}$ : $\mathcal{OpAlg} \rightarrow \mathcal{ObVect}$.

3.5. Free object functor from operated algebras to unital operated algebras.

In this subsection, we will construct the free object functor $F_{\mathcal{C}^{\text{OpAlg}}}$ from $\mathcal{OpAlg}$ to $\mathcal{uOpAlg}$ which is left adjoint to the forgetful functor $U_{\mathcal{C}^{\text{OpAlg}}}$ from $\mathcal{uOpAlg}$ to $\mathcal{OpAlg}$.

**Definition 3.6.** Let $V$ be a $k$-space, $A$ be a $k$-algebra. We define the following quotient unital $k$-algebra:

$$\bar{T}(A \oplus V) := \bar{T}(A \oplus V)/ \langle a \otimes b - a \cdot b \mid \forall a, b \in A \rangle,$$

where $\otimes$ and $\cdot$ are respectively the multiplications in $\bar{T}(A \oplus V)$ and $A$. In particular, if $V = 0$ then $\bar{T}(A \oplus V) = \bar{T}(A)/ \langle a \otimes b - a \cdot b \mid \forall a, b \in A \rangle = k \oplus A$.
Lemma 3.7. Let $V \hookrightarrow V'$ be an injective linear map, $A \hookrightarrow B$ be an injective homomorphism of \(k\)-algebras. Then the induced homomorphism of unital algebras
\[
\hat{T}(A \oplus V) \rightarrow \hat{T}(B \oplus V')
\]
is injective.

Let $(A, P_A)$ be an operated \(k\)-algebra and $F_0(A) := \hat{T}(A)$. Deduced from Lemma 3.7, the inclusion $i_0 : A \hookrightarrow A \oplus \left[\hat{T}(A)/A\right]$ induces an injective homomorphism of unital algebras
\[
i_{0,1} : F_0(A) = \hat{T}(A) \hookrightarrow F_1(A) := \hat{T}(A \oplus \left[\hat{T}(A)/A\right]).
\]
For $n \geq 2$, assume that a series of injective homomorphisms of unital algebras
\[
F_0(A) \xhookrightarrow{i_{0,1}} F_1(A) \xhookrightarrow{i_{1,2}} \cdots \xhookrightarrow{i_{n-2,n-1}} F_{n-2}(A) \xhookrightarrow{i_{n-1,n}} F_{n-1}(A),
\]
with $F_k(A) = \hat{T}(A \oplus \left[\hat{T}_{k-1}(A)/A\right])$ for all $1 \leq k \leq n - 1$, has been constructed. Then we define
\[
F_n(A) := \hat{T}(A \oplus \left[\hat{T}_{n-1}(A)/A\right]).
\]

By Lemma 3.7, the injection
\[
Id_A \oplus \left[i_{n-2,n-1}\right]_{F_{n-2}(A)/A} : A \oplus \left[\hat{T}_{n-2}(A)/A\right] \hookrightarrow A \oplus \left[\hat{T}_{n-1}(A)/A\right]
\]
induces an injective homomorphism of unital \(k\)-algebras:
\[
i_{n-1,n} : F_{n-1}(A) = \hat{T}(A \oplus \left[\hat{T}_{n-2}(A)/A\right]) \hookrightarrow F_n(A) = \hat{T}(A \oplus \left[\hat{T}_{n-1}(A)/A\right]).
\]

By the limit construction, there are injective homomorphisms $i_n : F_n(A) \hookrightarrow F(A), n \geq 0$ and $i : A \hookrightarrow F(A)$. Finally, we define an operated unital \(k\)-algebra
\[
\left(F(A) := \lim_{\rightarrow} F_n(A), P_{F(A)}(u) := \begin{cases} i(P_A(a)) & \text{if } u = i(a) \text{ for some } a \in A \\ i_{k+1}(u') & \text{if } u \notin i(A), \text{but } u = i_k(u') \text{ for some } u' \in F_k(A) \end{cases}\right).
\]

Thus we obtain a functor
\[
F^{-\text{unC}^\text{filq}} : \text{OpFilq} \rightarrow \text{uOpFilq}
\]
\[
(A, P_A) \mapsto (F(A), P_{F(A)})
\]
and we can prove the following result:

Proposition 3.8. The functor $F^{-\text{unC}^\text{filq}}$ is the left adjoint of the forgetful functor $U^{\text{unC}^\text{filq}} : \text{uOpFilq} \rightarrow \text{OpFilq}$, thus it is the free object functor from $\text{OpFilq}$ to $\text{uOpFilq}$.

3.6. Compositions of free object functors.

It is easy to check that the composite $F^{-\text{C}^\text{filq}} \circ F^{-\text{unC}^\text{filq}} = F^{-\text{C}^\text{filq}} \circ F^{-\text{filq}}$ provides exactly the free unital operated \(k\)-algebra generated by a set, whose construction firstly appeared in [26]; for a set X, $F^{-\text{C}^\text{filq}} \circ F^{-\text{unC}^\text{filq}}(X) = \text{kFil}(X)$.

Although $F^{-\text{C}^\text{filq}}$ has many other constructions, as indicated by the cubic diagram in Introduction, these different constructions seem to be much more complex.

Remark 3.9. From the analysis of the first three sections, it is interesting to notice that except the four squares in the front face and the back face, all other squares of free object functors are actually commutative.

To conclude this section, we include a result which considers the action of $F^{-\text{unC}^\text{filq}}$ on a unital algebra $A = \hat{T}(V)/I_A$. 

Proposition 3.10. Let $A = \mathcal{T}(V)/\langle I \rangle$ be a unital algebra. Then we have:

$$\mathcal{F}_{\text{uAlg}}(A) = \mathcal{F}_{\text{Vect}}(\mathcal{T}(V))/\langle \langle I \rangle \rangle,$$

where $I$ is considered as a subset in $\mathcal{F}_{\text{uAlg}}(V)$ via the canonical inclusion

$$j : \mathcal{T}(V) = \mathcal{F}_{\text{uAlg}}(\mathcal{T}(V)) \hookrightarrow \lim_{\rightarrow} \mathcal{F}_{\text{uAlg}}(\mathcal{T}(V)) = \mathcal{F}_{\text{uAlg}}(V).$$

Proof. Write canonical surjections $\pi : \mathcal{T}(V) \to A$ and $p : \mathcal{F}_{\text{uAlg}}(V) \to \mathcal{F}_{\text{uAlg}}(V)/\langle \langle I \rangle \rangle$. Since $I \subseteq \mathcal{T}(V) \cap \langle \langle I \rangle \rangle$, the inclusion $j$ induces a morphism of unital algebras

$$i : A \to \mathcal{F}_{\text{uAlg}}(V)/\langle \langle I \rangle \rangle.$$

We have obviously $p \circ j = i \circ \pi$.

Let $(B, P)$ be a unital operated algebra and $f : A \to B$ be a homomorphism of unital algebras. We will show that there exists a unique morphism of unital operated algebras

$$\tilde{f} : \mathcal{F}_{\text{uAlg}}(V)/\langle \langle I \rangle \rangle \to (B, P)$$

such that $\tilde{f} \circ i = f$.

Consider the following diagram:

The composition $f \circ \pi$ is a homomorphism of unital algebras and sends $I$ to zero. By the universal property of $\mathcal{F}_{\text{uAlg}}(\mathcal{T}(V)) = \mathcal{F}_{\text{uAlg}}(V)$, there is a unique morphism $\hat{f}$ of unital operated algebras from $\mathcal{F}_{\text{uAlg}}(V)$ to $(B, P)$ such that $\hat{f} \circ j = f \circ \pi$. As $j$ is injective, we also have $I \subseteq \ker(\hat{f})$, thus the operated ideal $\langle I \rangle$ generated by $I$ is also contained in $\ker(\hat{f})$. So $\hat{f}$ factors through $\mathcal{F}_{\text{uAlg}}(V)/\langle I \rangle$, which induces a morphism $\tilde{f}$ of unital operated algebras from $\mathcal{F}_{\text{uAlg}}(V)/\langle I \rangle$ to $(B, P)$ so that $\tilde{f} \circ p = \hat{f}$. Since $i \circ \pi = p \circ j$, we have

$$\tilde{f} \circ i \circ \pi = \tilde{f} \circ p \circ j = \hat{f} \circ j = f \circ \pi.$$

Hence, $\tilde{f} \circ i = f$, by the surjectivity of $\pi$.

For the uniqueness, assume that there is another morphism of unital operated algebras

$$\tilde{g} : \mathcal{F}_{\text{uAlg}}(V)/\langle \langle I \rangle \rangle \to (B, P)$$

such that $\tilde{g} \circ i = f$. So $\tilde{g} := \tilde{g} \circ p$ satisfies

$$\tilde{g} \circ j = \tilde{g} \circ p \circ j = \tilde{g} \circ i \circ \pi = f \circ \pi.$$

By the uniqueness of $\tilde{f}$, we get $\tilde{g} = \tilde{f}$ and thus $\tilde{g} = \tilde{f}$ as $p$ is surjective. □
4. Free $\Phi$-algebras via universal algebra

Free objects in operated contexts have been constructed in the previous sections. In this section, we will consider an operated algebra whose operator satisfies extra relations. It will be seen that free operated algebras with prescribed relations on the operator are given by quotients of free operated algebras. Our method is in the realm of universal algebra following [26].

Throughout this section, let $X$ be a set. Recall that $k\mathfrak{M}(X)$ denotes the free unital operated algebra generated by $X$.

**Definition 4.1** ([34, 22, 20]). Let $\phi(x_1, \ldots, x_n) \in k\mathfrak{M}(X)$ with $n \geq 1, x_1, \ldots, x_n \in X$. We call $\phi(x_1, \ldots, x_n) = 0$ (or just $\phi(x_1, \ldots, x_n)$) an operated polynomial identity (aka OPI).

Let $\phi = \phi(x_1, \ldots, x_n)$ be an OPI in $k\mathfrak{M}(X)$ and $(A, P)$ be a unital operated algebra. Let $\theta : X \to A$ be a map with $r_i = \theta(x_i), 1 \leq i \leq n$. By the universal property of $k\mathfrak{M}(X)$, there is a unique morphism of unital operated algebras $\bar{\theta} : k\mathfrak{M}(X) \to A$. We denote:

$$\phi(r_1, \ldots, r_n) := \bar{\theta}(\phi(x_1, \ldots, x_n)).$$

**Definition 4.2** ([34, 22, 20]). Let $\phi(x_1, \ldots, x_n)$ be an OPI. A unital operated algebra $(A, P)$ is said to satisfy the OPI $\phi(x_1, \ldots, x_n)$ if

$$\phi(r_1, \ldots, r_n) = 0 \text{ for all } r_1, \ldots, r_n \in A.$$

In this case, $(A, P)$ is called a $\phi$-algebra and $P$ is called a $\phi$-operator.

Generally, for a family of OPIs $\Phi \subset k\mathfrak{M}(X)$, we call a unital operated algebra $(A, P)$ a $\Phi$-algebra if it is a $\phi$-algebra for any $\phi \in \Phi$. Denote the full subcategory of $\Phi$-algebras in $u\mathfrak{Alg}$ by $\Phi\mathfrak{Alg}$.

**Definition 4.3** ([34, 22, 20]). Let $(A, P)$ be a unital operated algebra. An ideal $I$ of $A$ is called an operated ideal if it is closed under the action of the operator $P$. The operated ideal generated by a subset $S \subset A$ is denoted by $\langle S \rangle_{u\mathfrak{Alg}}$.

It is easy to see that the quotient of a unital operated algebra by an operated ideal is naturally a unital operated algebra.

Let $\Phi$ be a set of OPIs in $k\mathfrak{M}(X)$. For a unital algebra $A$, define $R_\Phi(A) \subseteq \mathcal{F}_{u\mathfrak{Alg}}(A)$ to be the following set

$$R_\Phi(A) := \{\phi(u_1, \ldots, u_n) | u_1, \ldots, u_n \in \mathcal{F}_{u\mathfrak{Alg}}(A), \phi(x_1, \ldots, x_n) \in \Phi\}.$$

In particular, when $A = k\mathfrak{M}(Z)$ is the free unital algebra generated by a set $Z$, we also use the notation $R_\Phi(Z) := R_\Phi(k\mathfrak{M}(Z)) \subseteq k\mathfrak{M}(Z)$ for short. We also define another set $S_\Phi(Z) \subseteq \mathfrak{M}(Z)$ by

$$S_\Phi(Z) := \{\phi(u_1, \ldots, u_k) | u_1, \ldots, u_k \in \mathfrak{M}(Z), \phi(x_1, \ldots, x_k) \in \Phi\}.$$
(b) The multilinear property on OPIs is not strong in the sense that one can always use a polarization so long as the OPIs in question are multilinear, at least when the base field is of characteristic zero; see [21, Corollary 2.18]. We will assume from now on that all OPIs are multilinear.

Similar to Proposition 4.4, we can construct the free $\Phi$-algebra generated by a given algebra as well.

**Proposition 4.7.** Let $\Phi \subseteq k\mathcal{M}(X)$ be a set of OPIs. Given a unital $k$-algebra $A$, the quotient operated algebra $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A) := \mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)/(R_{\Phi}(A))_{\mathcal{U}_{\mathcal{V}}} \Phi$ is the free $\Phi$-algebra generated by the unital algebra $A$.

**Proof.** Let $B$ be a $\Phi$-algebra. Given a homomorphism of unital algebras from $A$ to $\mathcal{U}_{\mathcal{V}}(B)$, by the universal property of $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)$, we get a morphism of unital operated algebras from $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)$ to $B$. Since $B$ is a $\Phi$-algebra, any morphism of unital operated algebras from $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)$ to $B$ will factor through the $\Phi$-algebra $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)/(R_{\Phi}(A))_{\mathcal{U}_{\mathcal{V}}} \Phi$, thus we obtain a morphism of $\Phi$-algebras from $\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)/(R_{\Phi}(A))_{\mathcal{U}_{\mathcal{V}}} \Phi$ to $B$ fulfilling the required universal property. \hfill $\square$

**Proposition 4.8.** Let $X$ be a set and $\Phi \subseteq k\mathcal{M}(X)$ a system of OPIs. Let $A = k\mathcal{M}(Z)/I_{A}$ be a unital algebra with generating set $Z$. Then we have:

$$\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A) = k\mathcal{M}(Z)/\langle S_{\Phi}(Z) \cup I_{A}\rangle_{\mathcal{U}_{\mathcal{V}}} \Phi.$$ 

**Proof.** By Proposition 3.10 and Remark 4.6, we have

$$\mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A) = \mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(A)/(R_{\Phi}(A))_{\mathcal{U}_{\mathcal{V}}} \Phi = \mathcal{F}_{\mathcal{U}_{\mathcal{V}}}(kZ)/(R_{\Phi}(Z) \cup I_{A})_{\mathcal{U}_{\mathcal{V}}} \Phi = k\mathcal{M}(Z)/\langle S_{\Phi}(Z) \cup I_{A}\rangle_{\mathcal{U}_{\mathcal{V}}} \Phi.$$ 

\hfill $\square$

5. **Gröbner-Shirshov bases for free $\Phi$-algebras over algebras**

In this section, as a generalisation of the well developed theory of Gröbner-Shirshov bases (aka GS bases) for associative algebras [51, 11, 24, 8, 10], the GS basis theory for operated algebras is recalled following [9, 34, 22, 20]. Inspired by [39, 30], we will also consider the general problem of GS bases for free $\Phi$-algebra generated by a given algebra.

**Definition 5.1 ([9, 34, 22, 20]).** Let $Z$ be a set, $\star$ a symbol not in $Z$.

(a) Define $\mathcal{M}^{\star}(Z)$ to be the subset of $\mathcal{M}(Z \cup \star)$ consisting of elements with $\star$ occurring only once.

(b) For $q \in \mathcal{M}^{\star}(Z)$ and $u \in \mathcal{M}(Z)$, we define $q|_{u} \in \mathcal{M}(Z)$ obtained by replacing the symbol $\star$ in $q$ by $u$.

(c) For $q \in \mathcal{M}^{\star}(Z)$ and $s = \sum_{i}c_{i}u_{i} \in \mathcal{M}(Z)$ with $c_{i} \in k$ and $u_{i} \in \mathcal{M}(Z)$, we define

$$q|_{s} := \sum_{i}c_{i}|_{u_{i}}.$$ 

(d) An element $u \in \mathcal{M}(Z)$ is a subword of another element $w \in \mathcal{M}(Z)$ if $w = q|_{u}$ for some $q \in \mathcal{M}^{\star}(Z)$. 
For $f = \sum c_i q_i \in \mathcal{M}(Z)$ with $q_i \in \mathcal{M}^*(Z)$, $c_i \in k$ and $u \in \mathcal{M}(Z)$, we define its substitution of $u$ by $s \in \mathcal{M}(Z)$ to be

$$f|_{u\rightarrow s} := \sum_i c_i q_i s_i.$$ 

**Definition 5.2 ([9, 34, 22, 20]).** Let $Z$ be a set, $\leq$ a linear order on $\mathcal{M}(Z)$ and $f \in k\mathcal{M}(Z)$.

(a) Let $f \not\in k$. The leading monomial of $f$, denoted by $\bar{f}$, is the largest monomial appearing in $f$. The leading coefficient of $f$, denoted by $c_f$, is the coefficient of $\bar{f}$ in $f$. We call $f$ monic with respect to $\leq$ if $c_f = 1$.

(b) Let $f \in k$ (including the case $f = 0$). We define the leading monomial of $f$ to be $1$ and the leading coefficient of $f$ to be $c_f = f$.

(c) A subset $S \subseteq k\mathcal{M}(Z)$ is called monicized with respect to $\leq$, if each nonzero element of $S$ has leading coefficient $1$. Obviously, each subset $S \subseteq \mathcal{M}(Z)$ can be made monicized if we divide each nonzero element by its leading coefficient.

**Definition 5.3 ([9, 34, 22, 20]).** Let $Z$ be a set. We denote $u < v$ if $u \leq v$ but $u \neq v$ for an order $\leq$.

(a) A monomial order on $\mathcal{M}(Z)$ is a well-order $\leq$ on $\mathcal{M}(Z)$ such that

$$u < v \Rightarrow wuz < wvz \text{ for any } u, v, w, z \in \mathcal{M}(Z)$$

(e.g., the deg-lex order, see Definition 6.5).

(b) A monomial order on $\mathcal{M}(Z)$ is a well-order $\leq$ on $\mathcal{M}(Z)$ such that

$$u < v \Rightarrow q|u < q|v \text{ for all } u, v \in \mathcal{M}(Z) \text{ and } q \in \mathcal{M}^*(Z).$$

We need another notation. Let $Z$ be a set. For $u \in \mathcal{M}(Z)$ with $u \neq 1$, as $u$ can be uniquely written as a product $u_1 \cdots u_n$ with $u_i \in Z \cup \{\mathcal{M}(Z)\}$ for $1 \leq i \leq n$, call $n$ the breadth of $u$, denoted by $|u|$; for $u = 1$, we define $|u| = 0$.

**Definition 5.4 ([9, 34, 22, 20]).** Let $\leq$ be a monomial order on $\mathcal{M}(Z)$ and $f, g \in k\mathcal{M}(Z)$ be monic.

(a) If there are $w, u, v \in \mathcal{M}(Z)$ such that $w = \bar{f}u = v\bar{g}$ with $\max\{|\bar{f}|, |\bar{g}|\} < |w| < |\bar{f}| + |\bar{g}|$, we call

$$(f, g)_{w}^{u,v} := fu - vg$$

the intersection composition of $f$ and $g$ with respect to $w$.

(b) If there are $w \in \mathcal{M}(Z)$ and $q \in \mathcal{M}^*(Z)$ such that $w = \bar{f} = q|\bar{g}$, we call

$$(f, g)_{w}^{q} := f - q\bar{g}$$

the inclusion composition of $f$ and $g$ with respect to $w$.

**Definition 5.5 ([9, 34, 22, 20]).** Let $Z$ be a set and $\leq$ a monomial order on $\mathcal{M}(Z)$. Let $\mathcal{G} \subseteq k\mathcal{M}(Z)$.

(a) An element $f \in k\mathcal{M}(Z)$ is called trivial modulo $(\mathcal{G}, w)$ for $w \in \mathcal{M}(Z)$ if

$$f = \sum_i c_i q_i |s_i \text{ with } q_i |s_i < w, \text{ where } c_i \in k, \ q_i \in \mathcal{M}^*(Z) \text{ and } s_i \in \mathcal{G}.$$ 

(b) The subset $\mathcal{G} \subseteq k\mathcal{M}(Z)$ is called a GS basis in $k\mathcal{M}(Z)$ with respect to $\leq$ if, for all pairs $f, g \in \mathcal{G}$ monicized with respect to $\leq$, every intersection composition of the form $(f, g)_{w}^{u,v}$ is trivial modulo $(\mathcal{G}, w)$, and every inclusion composition of the form $(f, g)_{w}^{q}$ is trivial modulo $(\mathcal{G}, w)$.

To distinguish from usual GS bases for associative algebras, from now on, we shall rename GS bases in operated contexts by operated GS bases.
Theorem 5.6 ([9, 34, 22, 20]). (Composition-Diamond Lemma) Let $Z$ be a set, $\leq$ a monomial order on $\mathcal{M}(Z)$ and $\mathcal{G} \subseteq \mathcal{M}(Z)$. Then the following conditions are equivalent:

(a) $\mathcal{G}$ is an operated GS basis in $\mathcal{M}(Z)$.

(b) Let $\eta : \mathcal{M}(Z) \rightarrow \mathcal{M}(Z)/\langle \mathcal{G} \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}$ be the quotient morphism. Denote $\text{Irr}(\mathcal{G}) := \mathcal{M}(Z) \setminus \{q_{\mathcal{G}} | s \in \mathcal{G}, \ q \in \mathcal{M}^*(Z)\}$.

As a $\mathbf{k}$-space, $\mathcal{M}(Z) = \mathcal{M}(Z)/\langle \mathcal{G} \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}$ and $\eta(\text{Irr}(\mathcal{G}))$ is a $\mathbf{k}$-basis of $\mathcal{M}(Z)/\langle \mathcal{G} \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}$.

Proposition 5.7. Let $Z$ be a set and $\leq$ a monomial order on $\mathcal{M}(Z)$. Clearly when restricted to $\mathcal{M}(Z)$, it is still a monomial order. Then a GS basis $G \subseteq \mathcal{M}(Z)$ with respect to the restriction of $\leq$ to $\mathcal{M}(Z)$ is also an operated GS basis in $\mathcal{M}(Z)$ with respect to $\leq$.

Proof. For any intersection composition $(f, g)_{w}^{u,v}$ in $\mathcal{M}(Z)$ with $f, g \in G \subseteq \mathcal{M}(Z)$, we have $w, u, v \in S(Z)$ and $(f, g)_{w}^{u,v}$ is trivial modulo $(G, w)$ in $\mathcal{M}(Z)$; the case of inclusion compositions is similar. It follows that $G$ is a GS basis in $\mathcal{M}(Z)$. \qed

Now, let’s consider operated GS bases for free $\Phi$-algebras over unital algebras. Recall that we always assume that all OPIs are multilinear.

Definition 5.8 ([20]). Let $X$ be a set and $\Phi \subseteq \mathcal{M}(X)$ a system of OPIs. Let $Z$ be a set and $\leq$ a monomial order on $\mathcal{M}(Z)$. We call $\Phi$ operated GS on $Z$ with respect to $\leq$ if $S_{\Phi}(Z)$ is a GS basis in $\mathcal{M}(Z)$ with respect to $\leq$. We call $\Phi$ operated GS if, for each set $Z$, there is a monomial order $\leq$ on $\mathcal{M}(Z)$ such that $\Phi$ is GS on $Z$ with respect to $\leq$.

Theorem 5.9. Let $X$ be a set and $\Phi \subseteq \mathcal{M}(X)$ a system of OPIs. Let $A = \mathcal{M}(Z)/I_{A}$ be a unital algebra with generating set $Z$. Assume that $\Phi$ is operated GS on $Z$ with respect to a monomial order $\leq$ in $\mathcal{M}(Z)$ and that $G$ is a GS basis of $I_{A}$ in $\mathcal{M}(Z)$ with respect to the restriction of $\leq$ to $\mathcal{M}(Z)$.

Suppose that the leading monomial of any OPI $\phi(x_1, \ldots, x_n) \in \Phi$ has no subword in $S(X) \setminus X$, and that for all $u_1, \ldots, u_n \in \mathcal{M}(Z)$, $\phi(u_1, \ldots, u_n)$ vanishes or its leading monomial is still $\overline{\phi}(u_1, \ldots, u_n)$. Then $S_{\Phi}(Z) \cup G$ is an operated GS basis of $S_{\Phi}(Z) \cup I_{A}$ in $\mathcal{M}(Z)$ with respect to $\leq$.

Proof. By Proposition 4.8, we have $\mathcal{F}_{\Phi, \mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}(A) = \mathcal{M}(Z)/\langle S_{\Phi}(Z) \cup I_{A} \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}$.

By Proposition 5.7, $G$ is an operated GS basis in $\mathcal{M}(Z)$ with respect to $\leq$ and by assumption, so is $S_{\Phi}(Z)$. To show that $S_{\Phi}(Z) \cup G$ is an operated GS basis of $\langle S_{\Phi}(Z) \cup I_{A} \rangle_{\mathcal{U} \mathcal{C} \mathcal{P} \mathcal{U} \mathcal{L} \mathcal{O}}$ in $\mathcal{M}(Z)$, we need to check the triviality of any inclusion composition $(f, g)_{w_1}^{u_1}$ modulo $(S_{\Phi}(Z) \cup G, w_1)$, and also that of any intersection composition $(f, g)_{w_2}^{u_2}$ modulo $(S_{\Phi}(Z) \cup G, w_2)$, for only $(f \in S_{\Phi}(Z), g \in G)$ or $(f \in G, g \in S_{\Phi}(Z))$.

For an inclusion composition $(f, g)_{w_1}^{u_1}$, it is easy to see that we only need to consider the case $f \in S_{\Phi}(Z), g \in G$. Since for $w_1 \in \mathcal{M}(Z)$ and $q \in \mathcal{M}^*(Z)$, we have $w_1 = \overline{f} = q|_{\overline{g}}$, we can write $f = \phi(u_1, \ldots, u_k) \in S_{\Phi}(Z)$ with $\phi(x_1, \ldots, x_k) \in \Phi$ and $u_1, \ldots, u_k \in \mathcal{M}(Z)$. By assumption, the leading monomial of any OPI in $\Phi$ has no subword in $S(X) \setminus X$, so there exists $u_i \in \mathcal{M}(Z)$ such that $u_i = q'|_{\overline{g}}$. One gets $f = \sum_j c_j q_j|_{\overline{g}}$ with $q_j \in \mathcal{M}^*(Z)$. By multilinearity of $\phi$,

$$(f, g)_{w_1}^{u_1} := f - q|_{\overline{g}} = f - f|_{\overline{g} \rightarrow g} + f|_{\overline{g} \rightarrow g} - \overline{f}|_{\overline{g} \rightarrow g} = f|_{\overline{g} \rightarrow g} + (f - \overline{f})|_{\overline{g} \rightarrow g}.$$

Observe three facts: $f|_{\overline{g} \rightarrow g} \in S_{\Phi}(Z), f|_{\overline{g} \rightarrow g} < w_1$ and $(f - \overline{f})|_{\overline{g} \rightarrow g}$ is trivial modulo $(G, w_1)$. Consequently, we obtain that $(f, g)_{w_1}^{u_1}$ is trivial modulo $(S_{\Phi}(Z) \cup G, w_1)$.

For an intersection composition $(f, g)_{w_2}^{u_2}$, we only consider the case $f \in S_{\Phi}(Z), g \in G$, the other case being similar. Our method is to reduce this case to the case of inclusion compositions.
Write $f = \phi(u_1, \ldots, u_k) \in S_\Phi(Z)$ with $\phi(x_1, \ldots, x_k) \in \Phi$ and $u_1, \ldots, u_k \in \mathcal{M}(Z)$. For the intersection composition $(f, g)_{w_2}^{u,v}$, we have $w_2 = f \bar{u} = v\bar{g}$ with $\max(|f|, |\bar{g}|) < |w_2| < |f| + |\bar{g}|$. So $\bar{f}$ must have a right factor in $S(Z)$ and thus $\bar{\phi}$ has a right factor in $S(X)$. Since the leading monomial of any OPI in $\Phi$ has no subword in $S(X) \setminus X$, without loss of generality, we can write $\bar{\phi}(x_1, \ldots, x_k)$ as $\psi(x_1, \ldots, x_{k-1})x_k$, for some $\psi(x_1, \ldots, x_{k-1}) \in \mathcal{M}(X)$ with no right factor in $S(X)$. So we have

$$v\bar{g} = w_2 = f \bar{u} = \bar{\phi}(u_1, \ldots, u_k)u = \psi(u_1, \ldots, u_{k-1})u_1u = \bar{\phi}(u_1, \ldots, u_k),$$

and $u_k = s\bar{g}$ for some $s \in \mathcal{M}(Z)$. Let $q = \bar{\phi}(u_1, \ldots, u_k, s\star) \in \mathcal{M}^*(Z)$, then

$$w_2 = \bar{\phi}(u_1, \ldots, u_k) = \bar{\phi}(u_1, \ldots, u_k, s\bar{g}) = q|_{\bar{\phi}(u_1, \ldots, u_k)}.$$

Notice that we obtain an inclusion composition:

$$(\phi(u_1, \ldots, u_k), g)^q_{w_2} = \phi(u_1, \ldots, u_k)u - \bar{\phi}(u_1, \ldots, u_k, s\bar{g}) = \phi(u_1, \ldots, u_k) - v\bar{g},$$

which is trivial modulo $(S_\Phi(Z) \cup G, w_2)$ by the case of inclusion compositions already dealt with above.

Consider two elements $\phi(u_1, \ldots, u_k)u$ and $\phi(u_1, \ldots, u_k)$ in $\mathbb{k}\mathcal{M}(Z)$. Let $q' = \star u \in \mathcal{M}^*(Z)$, then we have

$$w_2 = \bar{\phi}(u_1, \ldots, u_k) = \bar{\phi}(u_1, \ldots, u_k)u = q'|_{\bar{\phi}(u_1, \ldots, u_k)}.$$ 

One gets another inclusion composition:

$$(\phi(u_1, \ldots, u_k), \phi(u_1, \ldots, u_k))^q_{w_2} = \phi(u_1, \ldots, u_k)u - \bar{\phi}(u_1, \ldots, u_k)u,$$

which is trivial modulo $(S_\Phi(Z), w_2)$ by the assumption that $S_\Phi(Z)$ is a GS basis in $\mathbb{k}\mathcal{M}(Z)$. Now it is obvious that $(\phi(u_1, \ldots, u_k), \phi(u_1, \ldots, u_k))^q_{w_2}$ is also trivial modulo $(S_\Phi(Z) \cup G, w_2)$.

Return to our intersection composition and we obtain that

$$(f, g)_{w_2}^{u,v} = \phi(u_1, \ldots, u_k)u - v\bar{g} = \phi(u_1, \ldots, u_k)u - \bar{\phi}(u_1, \ldots, u_k)u + \phi(u_1, \ldots, u_k)u - v\bar{g} = -\bar{\phi}(u_1, \ldots, u_k)u, \phi(u_1, \ldots, u_k))^q_{w_2} + (\phi(u_1, \ldots, u_k), g)^q_{w_2}$$

is trivial modulo $(S_\Phi(Z) \cup G, w_2)$, by the triviality of the two inclusion compositions proved before. 

By Theorem 5.6, we have the following result.

**Corollary 5.10.** With the same assumption as Theorem 5.9, denote by

$$\eta : \mathbb{k}\mathcal{M}(Z) \to \mathcal{T}_{\mathbb{k}\mathcal{M}(Z), \mathcal{M}(Z)}(A) = \mathbb{k}\mathcal{M}(Z)/\langle S_\Phi(Z) \cup I_A \rangle_{\mathbb{k}\mathcal{M}(Z), \mathcal{M}(Z)}$$

the natural quotient map. Then $\eta(\text{Irr}(S_\Phi(Z) \cup G))$ is a $\mathbb{k}$-basis of $\mathcal{T}_{\mathbb{k}\mathcal{M}(Z), \mathcal{M}(Z)}(A)$.

**Remark 5.11.** In Theorem 5.9, it is necessary to require that the leading monomial of any OPI in $\Phi$ has no subword in $S(X) \setminus X$.

For example, let $A := \mathbb{k}\mathcal{M}([z_1, z_2])/(z_1z_2 - 1)$. It is obvious that $\{z_1z_2 - 1\}$ is a GS basis in $\mathbb{k}\mathcal{M}([z_1, z_2])$. Consider the OPI

$$\phi(x, y) = [xy] - x[y] - [x]y,$$

which defines differential algebras of weight zero. We know that $S_\phi([z_1, z_2])$ is an operated GS basis in $\mathbb{k}\mathcal{M}([z_1, z_2])$ with respect to a certain monomial order; see [34].

However, let $w = [z_1z_2]$, and $q = [\star] \in \mathcal{M}^*([z_1, z_2])$, then $w = \bar{\phi}(z_1, z_2) = q|_{z_1z_2}$. There is an inclusion composition

$$(\phi(z_1, z_2), z_1z_2 - 1)^q_w = \phi(z_1, z_2) - [z_1z_2 - 1] = -z_1[z_2] - [z_1]z_2,$$
which it is not trivial modulo \( S_\phi((z_1, z_2)) \cup \{z_1z_2 - 1\}\). So the set \( S_\phi((z_1, z_2)) \cup \{z_1z_2 - 1\}\) is not an operated GS basis in \( k\mathfrak{m}(\{z_1, z_2\})\) with respect to the given monomial order.

6. Examples of operated GS bases for free \( \Phi \)-algebras over algebras

In this section, we present many examples for which Theorem 5.9 will provide operated GS bases. These include all OPIs of Rota-Baxter type \([22, 20]\), a class of OPIs of differential type \([34, 20]\) and two other examples, say, averaging algebras and Reynolds algebras.

6.1. Preliminaries about rewriting systems. We need some basic notions about rewriting systems in order to introduce OPIs of Rota-Baxter type and of differential type. The basic references about rewriting systems are, for instance, [4] and the recent lecture notes [42].

Definition 6.1. Let \( V \) be a \( k \)-space with a \( k \)-basis \( Z \).

(a) For \( f = \sum_{w \in Z} c_w w \in V \) with \( c_w \in k \), the support \( \text{Supp}(f) \) of \( f \) is the set \( \{w \in Z \mid c_w \neq 0\} \).

(b) Let \( f, g \in V \). We use \( f + g \) to indicate the property that \( \text{Supp}(f) \cap \text{Supp}(g) = \emptyset \). If this is the case, we say \( f + g \) is a direct sum of \( f \) and \( g \), and use \( f + g \) also for the sum \( f + g \).

(c) For \( f \in V \) and \( w \in \text{Supp}(f) \) with the coefficient \( c_w \), write \( R_w(f) := c_w w - f \in V \). So \( f = c_w w + (-R_w(f)) \).

Definition 6.2. Let \( V \) be a \( k \)-space with a \( k \)-basis \( Z \).

(a) A term-rewriting system \( \Pi \) on \( V \) with respect to \( Z \) is a binary relation \( \Pi \subseteq Z \times V \). An element \((t, v) \in \Pi \) is called a (term-)rewriting rule of \( \Pi \), denoted by \( t \rightarrow v \).

(b) The term-rewriting system \( \Pi \) is called simple with respect to \( Z \) if \( t \rightarrow v \) for all \( t \rightarrow v \in \Pi \).

(c) If \( f = c_i t + (-R_i(f)) \in V \), using the rewriting rule \( t \rightarrow v \), we get a new element \( g := c_i v - R_i(f) \in V \), called a one-step rewriting of \( f \) and denoted by \( f \rightarrow \Pi g \).

(d) The reflexive-transitive closure of \( \rightarrow \Pi \) (as a binary relation on \( V \)) is denoted by \( \rightarrow^* \Pi \) and, if \( f \rightarrow^* \Pi g \), we say \( f \) rewrites to \( g \) with respect to \( \Pi \).

(e) Two elements \( f, g \in V \) are joinable if there exists \( h \in V \) such that \( f \rightarrow^* \Pi h \) and \( g \rightarrow^* \Pi h \); we denote this by \( f \downarrow^* \Pi g \).

(f) A term-rewriting system \( \Pi \) on \( V \) is called terminating if there is no infinite chain of one-step rewriting \( f_0 \rightarrow^* \Pi f_1 \rightarrow^* \Pi f_2 \cdots \).

(g) A term-rewriting system \( \Pi \) is called compatible with a linear order \( \geq \) on \( Z \), if \( t \geq v \) for each \( t \rightarrow v \in \Pi \).

6.2. Single OPI of Rota-Baxter type.

Let us introduce OPIs of Rota-Baxter type following \([22, 20]\).

Definition 6.3 (\([22, 20]\)). An OPI \( \phi \) is said to be of Rota-Baxter type if \( \phi \) is of the form \([x][y] - [B(x, y)]\), where \( B(x, y) \) satisfies the following conditions:

(a) \( B(x, y) \) is linear in \( x \) and \( y \);

(b) no monomial of \( B(x, y) \) contains any subword of the form \([u][v]\) for any \( u, v \in \mathfrak{m}(\{x, y\}) \backslash \{1\} \);

(c) for every set \( Z \), the rewriting system \( \Pi_\phi(Z) := \{[u][v] \rightarrow [B(u, v)] \mid u, v \in \mathfrak{m}(Z)\} \) is terminating;

(d) for any set \( Z \) and \( u, v, w \in \mathfrak{m}(Z) \backslash \{1\} \),

\[
B(B(u, v), w) - B(u, B(v, w)) \rightarrow^* \Pi_\phi(Z) 0.
\]
Gao, Guo, Sit and Zheng gave a list of OPIs of Rota-Baxter type and they conjectured that these are all possible OPIs of Rota-Baxter type.

**List 6.4 ([22, Conjecture 2.37]).** For any $c, \lambda \in k$, the OPI $\phi := [x][y] - [B(x, y)]$, where $B(x, y)$ is taken from the list below, is of Rota-Baxter type.

1. $x[y]$ (average operator),
2. $[x]y$ (inverse average operator),
3. $x[y] + y[x],$
4. $[x]y + [y]x$, 
5. $x[y] + [x]y - [xy]$ (Nijenhuis operator),
6. $x[y] + [x]y + \lambda xy$ (Rota-Baxter operator of weight $\lambda$), 
7. $x[y] - x[1]y + \lambda xy$, 
8. $[x]y - x[1]y + \lambda xy$, 
9. $x[y] + [x]y - x[1]y + \lambda xy$ (generalized Leroux TD operator with weight $\lambda$), 
10. $x[y] + [x]y - xy[1] - x[1]y + \lambda xy$, 
11. $x[y] + [x]y - x[1]y - [xy] + \lambda xy$, 
12. $x[y] + [x]y - x[1]y - [1]xy + \lambda xy$, 
13. $cx[1]y + \lambda xy$ (generalized endomorphisms), 
14. $cy[1]x + \lambda xy$ (generalized antimorphisms).

It is showed in [22, Theorem 4.9] that for an OPI $\phi$ of Rota-Baxter type, $S_{\phi}(Z)$ is a GS basis of $\langle S_{\phi}(Z) \rangle_{u \in \mathbb{N}_{\phi \in \mathbb{N}}}$ with respect to some monomial order, denoted by $\leq_{db}$ (see [22, Lemma 5.5]); the leading monomial of $\phi$ under $\leq_{db}$ is $[x][y]$, and the restriction of this order on $P(V)$ is the degree lexicographical order $\leq_{dlex}$ defined as follows.

**Definition 6.5.** Let $Z$ be a set endowed with a well order $\leq_{Z}$. By convention, define $\deg_{Z}(1) = 0$; for $u = u_{1} \cdots u_{r} \in M(Z) \setminus \{1\}$ with $u_{1}, \ldots, u_{r} \in Z$, define $\deg_{Z}(u) = r$.

Define the degree lexicographical order $\leq_{dlex}$ on $M(Z)$ by taking, for any $u, v \in M(Z), u <_{dlex} v$ if

(a) either $\deg_{Z}(u) < \deg_{Z}(v)$, or 
(b) $\deg_{Z}(u) = \deg_{Z}(v)$, and $u = mu_{n}, v = mv_{n}n'$ for some $m, n, n' \in M(Z)$ and $u_{i}, v_{i} \in Z$ with $u_{i} <_{Z} v_{i}$.

It is surprising to see that Theorem 5.9 could deal with all OPIs of Rota-Baxter type.

**Theorem 6.6.** Let $Z$ be a set, $A = kM(Z)/I_{A}$ a unital $k$-algebra and $\phi$ an OPI of Rota-Baxter type. Then we have:

$$\mathcal{F}_{\mathbb{N}_{\phi \in \mathbb{N}}}^{\phi}(A) = kM(Z)/\langle S_{\phi}(Z) \cup I_{A} \rangle_{u \in \mathbb{N}_{\phi \in \mathbb{N}}}.$$

Moreover, assume $I_{A}$ has a GS basis $G$ with respect to the degree lexicographical order $\leq_{dlex}$. Then $S_{\phi}(Z) \cup G$ is an operated GS basis of $\langle S_{\phi}(Z) \cup I_{A} \rangle_{u \in \mathbb{N}_{\phi \in \mathbb{N}}}$ in $kM(Z)$ with respect to $\leq_{db}$.

**Proof.** Since $[x][y]$ has no subword in $S(X) \setminus X$, the result is immediately obtained by our main result Theorem 5.9.

By Corollary 5.10, one could determine a linear basis of the free $\phi$-algebra of Rota-Baxter type over a unital algebra.
Theorem 6.7. Let $Z$ be a set, $A = kM(Z)/I_A$ a unital algebra with a GS basis $G$ with respect to $\leq_{dlex}$. Let $\phi$ be an OPI of Rota-Baxter type. Then
\[
\text{Irr}(S_\phi(Z) \cup G) = \mathcal{M}(Z) \setminus \{a|b, q|a|v| : s \in G, q \in \mathcal{M}^*(Z), u, v \in \mathcal{M}(Z)\}
\]
is a basis of the free $\phi$-algebra $\mathcal{F}_{\phi, \mathcal{M}(Z)}(A)$ over $A$.

Example 6.8. Consider the OPI $\phi_{Nij}$ of type (5) in Example 6.4:
\[
\phi_{Nij}(x, y) = [x][y] - [[x][y]] - [x][y] + [xy].
\]
A $\phi_{Nij}$-algebra is called a Nijenhuis algebra.

In [39], given a linear basis of a $k$-algebra $A$, Lei and Guo gave a $k$-basis of the free Nijenhuis algebra $\mathcal{F}_{\phi_{Nij}, \mathcal{M}(Z)}(A)$ over $A$. When the $k$-basis of $A$ can be deduced from a GS basis of $A$, it can be seen that their result is a consequence of Theorem 6.7.

6.3. Single OPI of differential type.

Definition 6.9 ([34, 20]). An OPI $\phi$ is said to be of differential type if $\phi$ is of the form $[xy] - N(x, y)$, where $N(x, y)$ satisfies the following conditions:

(a) $N(x, y)$ is linear in $x$ and $y$;
(b) no monomial of $N(x, y)$ contains any subword of the form $[uv]$ for any $u, v \in \mathcal{M}((x, y))\{1\}$;
(c) for any set $Z$ and $u, v, w \in \mathcal{M}(Z)\{1\}$,
\[
N(uv, w) - N(u, vw) \to_{\Pi_\phi(Z)} 0,
\]
where $\Pi_\phi(Z) := \{[uv] \to N(u, v) : u, v \in \mathcal{M}(Z)\}$.

Guo, Sit and Zhang gave a list of OPIs of differential type and they conjectured that these are all possible OPIs of differential type.

List 6.10 ([34, Conjecture 4.7]). For any $a, b, c, \lambda_{ij} \in k$, $i, j \geq 0$, the OPI $\phi := [xy] - N(x, y)$, where $N(x, y)$ is taken from the list below, is of differential type.

1. $a[x][y] + [x][y] + b[x][y] + cxy$ where $a^2 = a + bc$,
2. $ab^2xy + bxy + a[y][x] - ab(y)[x] + [y][x]$,
3. $\sum_{i,j \geq 0} \lambda_{ij}[1]^i[y][x][1]^j$ with the convention that $[1]^0 = 1$,
4. $[x][y] + a[x][y] + bxy$,
5. $[xy] + a[x][y] - xy[1]$, $[xy] + a[x][y] - [1]xy$.

It is showed in [34, Theorem 5.7] that for an OPI $\phi$ of differential type, $S_\phi(Z)$ is an operated GS basis of $\left\langle S_\phi(Z) \right\rangle_{\alpha \in \mathcal{M}(Z)}$ with respect to some monomial order, under which the leading monomial of $\phi$ is $[xy]$. Moreover, the restriction of this order on $\mathcal{T}(V)$ is $\leq_{dlex}$.

Since the leading term $[uv]$ contains a subword $uv \in S(X) \setminus X$, Theorem 5.9 can not be applied to differential type directly. However, in some special cases, we can still compute the operated GS basis by rewriting the OPI. For case (1) with $a = b = 0$ Example 6.10, we substitute $xy$ by $z$ and get the following two types of OPIs both with leading monomial $[z]$,

1. $[z] - cz$ for any $c$,

They define the same $\phi$-algebras as case (1) (with $a = b = 0$) in Example 6.10. By Theorem 5.9 and Corollary 5.10, we have the following result.
Proposition 6.11. Let \( Z \) be a set, \( A = kM(Z)/I_\Lambda \) a unital algebra with a GS basis \( G \) with respect to \( \leq_{\text{dlex}} \). Let \( \phi \) be an OPI of type (1'). Then \( S_{\phi}(Z) \cup G \) is an operated GS basis of \( \left\{ S_{\phi}(Z) \cup I_\Lambda \right\}_{u \in M(Z)} \) in \( kM(Z) \) with respect to the monomial order on \( M(Z) \) given in [34], and
\[
\text{Irr}(S_{\phi}(Z) \cup G) = M(Z) \setminus \{ q \mid s \in G, q \in M^*(Z), u \in M(Z) \} = M(Z) \setminus \{ q \mid s \in G \}
\]
is a linear basis of the free \( \phi \)-algebra \( F_{\text{Diff}}(A) \) over \( A \).

Remark 6.12. Consider the OPI \( \phi_{\text{Diff}} \in kM(\{x, y\}) \) of type (1) in Example 6.10, by taking \( a = 1, b = \lambda \) and \( c = 0 \):
\[
\phi_{\text{Diff}}(x, y) = [xy] - x[y] - [x]y - \lambda [x][y].
\]
A \( \phi_{\text{Diff}} \)-algebra is called a differential algebra of weight \( \lambda \).

In [30], Guo and Li showed that the free differential algebra \( F_{\text{Diff}}(A) \) over a unital \( k \)-algebra \( A = kM(X)/I_\Lambda \) is the same as the free differential algebra over the set \( X \) modulo the differential ideal generated by the ideal \( I_\Lambda \), which can be deduced from Proposition 4.8 as well.

Guo and Li also gave a \( k \)-basis of \( F_{\text{Diff}}(A) \) by using the theory of differential GS bases. Their method is completely different from ours.

Moreover, our result Theorem 5.9 could not apply directly to differential algebras under the monomial order introduced in [34]; see Remark 5.11. However, it may be possible to modify the monomial order so that we can apply Theorem 5.9 and this task will be taken in a subsequent paper.

6.4. Multiple OPIs.

Example 6.13 (Averaging algebra). Take \( \Phi_{Av} \) to be the following set of operated polynomial identities:
\[
\{[x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2] \}.
\]
Then a \( \Phi_{Av} \)-algebra is called an averaging algebra.

Let
\[
\Phi_{Av} = \{[x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2], [x_1][x_2] - [x_1][x_2] \}.
\]
Note that \( \Phi_{Av} \) has three elements. Consider the free averaging algebra \( F_{\Phi_{Av}}(Z) \) over a given set \( Z \). It is shown in [23, Theorem 3.10] that the set \( S_{\Phi_{Av}}(Z) \) is an operated GS basis of \( F_{\Phi_{Av}}(Z) \) with respect to the order defined in [34] and the leading monomials of the elements in \( \Phi_{Av} \) are \([x_1][x_2], [x_1][x_2], [x_1][x_2] \) and \([x_1][x_2], [x_1][x_2] \) respectively.

By Theorem 5.9 and Corollary 5.10, given a unital algebra \( A \) with a GS basis \( G \) with respect to \( \leq_{\text{dlex}} \), the set \( S_{\Phi_{Av}}(Z) \cup G \) is an operated GS basis of the free averaging algebra \( F_{\Phi_{Av}}(A) \) over \( A \) and the set
\[
\text{Irr}(S_{\Phi_{Av}}(Z) \cup G) = M(Z) \setminus \{ q \mid s \in G, q \in M^*(Z), u, v \in M(Z) \}
\]
is a linear basis of \( F_{\Phi_{Av}}(A) \).

Example 6.14 (Reynolds algebra). Let \( \phi_{\text{Rey}} \) be the following operated polynomial identity:
\[
[x_1][x_2] - [x_1][x_2] - [x_1][x_2] + [x_1][x_2].
\]
Then a \( \phi_{\text{Rey}} \)-algebra is called a Reynolds algebra.
Consider the free Reynolds algebra $F_{\Rey}$ over a given set $Z$. By Theorem 5.6 and [55, Proposition 2.13 and Theorem 3.9], the set $S_{\Rey}(Z)$ is an operated GS basis in $F_{\Rey}(Z)$ with respect to the order defined in [34], where

$$\Phi_{\Rey} = \{[[x_1] \cdots [x_n]] - \sum_{i=1}^{n} [[[x_1] \cdots [x_{i-1}]x_i[x_{i+1}] \cdots [x_n]] + [x_1] \cdots [x_n] | n \geq 2],$$

and the leading monomials of the elements in $\Phi_{\Rey}$ are of the form $[[x_1][x_2] \cdots [x_n]]$.

By Theorem 5.9 and Corollary 5.10, given a unital algebra $A$ with a GS basis $G$ with respect to $\leq_{dlex}$, the set $S_{\Rey}(Z) \cup G$ is an operated GS basis of the free Reynolds algebra $F_{\Rey}(A)$ over $A$ and the set

$$\operatorname{Irr}(S_{\Rey}(Z) \cup G) = \mathcal{M}(Z) \setminus \{q|s, q|[u_1, \ldots, u_n] | s \in G, q \in \mathcal{M}(Z), u_1, \ldots, u_n \in \mathcal{M}(Z), n \geq 2\}$$

is a linear basis of $F_{\Rey}(A)$.

Note that in this case, $\Phi_{\Rey}$ has infinitely many elements.

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