Fairing of planar curves to log-aesthetic curves

Sebastián Elías Graiff Zurita
Graduate School of Mathematics Kyushu University
744 Motooka, Nishi-ku, Fukuoka 819-0935, Japan

Kenji Kajiwara
Institute of Mathematics for Industry, Kyushu University
744 Motooka, Fukuoka 819-0395, Japan

Kenjiro T. Miura
Graduate School of Science and Technology, Shizuoka University
3-5-1 Johoku, Hamamatsu, Shizuoka, 432-8561, Japan

Abstract

We present an algorithm to fair a given planar curve by a log-aesthetic curve (LAC). We show how a general LAC segment can be uniquely characterized by seven parameters and present a method of parametric approximation based on this fact. This work aims to provide tools to be used in reverse engineering for computer aided geometric design. Finally, we show an example of usage by applying this algorithm to the point data obtained from 3D scanning a car’s roof.

1 Introduction

In 1994, Harada et al., while investigating the mental images that aesthetically pleasing shapes suggest, set up an experiment to quantitatively analyze a curve’s character from the viewpoint of the observer [6]. Their main result may be described as follows: the curves that car designers regard as “aesthetic” have the common property that the frequency histogram of the radius of curvature follows a piecewise linear relation in a log-log scale. An analytic formulation of those curves was provided in [11, 15] defining what will later be called as the log-aesthetic curves (LAC), which promoted theoretical and practical studies of LAC towards their use in computer aided geometric design as indicated by Levien and Séquin [10]. In this regard, several works have been written regarding the implementation and construction of LAC with fixed boundary conditions, see for example [3, 4, 16]. Furthermore, extensions to surfaces have also been considered with an emphasis in providing practical tools for industrial design, see [12]. From a different point of view, LAC have been characterized as curves that are obtained via a variational principle in the framework of similarity geometry, moreover they can also be seen as invariant curves under the integrable flow on plane curves which is governed by the Burgers equation [8]. This fact was also shown to be useful at providing an integrable discretization of the LAC that preserves the underlying geometric structure [9]. All these previous works contributed to constructing methods that generate a desirable shape with given fixed conditions. In this paper, in view reverse engineering applications, we provide a method to characterize a given curve by its closest LAC. For this purpose, we first show how a general LAC segment can be uniquely identify by seven parameters, and we then apply a method of parametric approximation. Our method is based on the one presented in [1] to approximate a given planar curve by an Euler’s elastica, and in its follow-up work [5] for the discrete Euler’s elastica. Finally, we apply our algorithm to the 3D scanner data obtained from an actual car roof and identify their underlying LAC.
2 Log-aesthetic curve

2.1 Basic formulation

First of all, let us review some definitions regarding planar curves. Let $\gamma(s) \in \mathbb{R}^2$ be a planar curve parameterized by arc length $s \in [0, L]$ with total length $L$. We define the tangent and normal vectors as $T(s) = \gamma'(s)$ and $N(s) = R_{\pi/2} T(s)$, respectively, where $R_{\pi/2}$ is a $\pi/2$ rotation matrix and $' = d/ds$. By definition, $\|T(s)\| = 1$, so the tangent vector can be parameterized as

$$T(s) = \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix},$$

where $\theta(s)$ is the turning angle, namely, the angle of the tangent vector measured from the horizontal axis, so that $\kappa(s) = \theta'(s)$ is the curvature and $\rho(s) = 1/\kappa(s)$ the radius of curvature.

As it is shown in [11, 15], by considering the analytic formulation of the work by Harada mentioned in the Introduction, it can be seen that an log-aesthetic curve satisfies

$$\log \left( \frac{ds}{dR} \right) = \alpha R - \log A,$$  \hspace{1cm} (2.2)

for some $\alpha \in \mathbb{R}$ and $A > 0$, where $R = \log \rho$. Assuming that functions are well behaved, we have $\frac{ds}{dR} = \rho / \rho'$, then (2.2) can be rewritten as

$$\rho' \rho^{(\alpha-1)} = A,$$  \hspace{1cm} (2.3)

or equivalently, by taking the derivative of (2.3), as

$$\rho \rho'' + (\alpha - 1)(\rho')^2 = 0.$$  \hspace{1cm} (2.4)

Although (2.2) is presented in [11] as the defining equation of log-aesthetic curves, we consider (2.4) written in terms of the curvature as an alternative definition.

**Definition 2.1** (Log-aesthetic curve) — An arc length parameterized curve $\gamma(s)$ with strictly monotonic radius of curvature is called a log-aesthetic curve (LAC) if its curvature satisfies

$$\kappa \kappa'' - (\alpha + 1) (\kappa')^2 = 0,$$  \hspace{1cm} (2.5)

for some constant $\alpha \in \mathbb{R}$.

The fundamental theorem of planar curves states that an arc length parameterized planar curve is uniquely determined by its curvature up to Euclidean transformations. In addition, the curvature of an LAC determined by (2.5) has two arbitrary parameters. In view of this, here we introduce the basic LAC (Definition 2.2) by fixing this freedom, and show that one can recover arbitrary LAC by applying the similarity transformations and shift of arc length (Proposition 2.3).

Let us see that for a given LAC, its parameter $\alpha$ is invariant under the similarity transformations and the reflections. Noticing that the curvature of a planar curve is invariant under the Euclidean transformations, we only check the invariance under the scale transformations and the reflections over the diagonal \{$(x,x) \in \mathbb{R}^2 | x \in \mathbb{R}$}. For the scale transformations, consider the arc length parameterized LAC $\gamma(s)$ satisfying (2.5), for some $\alpha \in \mathbb{R}$, and define $\tilde{\gamma}(\tilde{s}) := S \gamma(\tilde{s}/S)$, where $S > 0$. Then $\tilde{\kappa}(\tilde{s})$, the curvature of $\tilde{\gamma}(\tilde{s})$, is given by $\tilde{\kappa}(\tilde{s}) = S^{-1} \kappa(\tilde{s}/S)$ and it satisfies (2.5). For the reflections over the diagonal, note that interchanging the $x$- and $y$-component of the curve is equivalent to changing the sign of the curvature, and (2.5) is invariant under that change.
Definition 2.2 (Basic LAC) — Let $\xi^\alpha(s)$ be an LAC defined over an open interval $I \subset \mathbb{R}$, such that $\{0\} \in I$, and satisfying
\[
\begin{cases}
\kappa'(s) = -\kappa(s)^{(\alpha+1)} < 0, & \forall s \in I, \\
\kappa(0) = 1, \\
\theta(0) = 0, \\
\xi^\alpha(0) = 0.
\end{cases}
\] (2.6)

We call $\xi^\alpha(s)$ a basic LAC.

Let us see a more explicit expression for the basic LAC and its related quantities. In what follows, we use the sub-index $\xi^\alpha$, as for example $\kappa_{\xi^\alpha}$, to denote those quantities associated to their respective basic LAC. Taking the initial condition into consideration, the explicit form of the curvature is given by
\[
\kappa_{\xi^\alpha}(s) = \begin{cases}
\exp(-s), & \alpha = 0, \\
(1 + \alpha s)^{-1/\alpha}, & \alpha \neq 0.
\end{cases}
\] (2.7)

Then, the turning angle is obtained from the curvature by $(\theta_{\xi^\alpha})' = \kappa_{\xi^\alpha}$,
\[
\theta_{\xi^\alpha}(s) = \begin{cases}
1 - \exp(-s), & \alpha = 0, \\
\log(s + 1), & \alpha = 1, \\
\frac{(1+\alpha s)^{\frac{\alpha-1}{\alpha} - 1}}{\alpha - 1}, & \alpha \neq 0, 1.
\end{cases}
\] (2.8)

Finally, the position vector can be obtained from the tangent vector as
\[
\xi^\alpha(s) = \int_0^s \begin{pmatrix}
\cos \theta_{\xi^\alpha}(\tilde{s}) \\
\sin \theta_{\xi^\alpha}(\tilde{s})
\end{pmatrix} d\tilde{s}.
\] (2.9)

Although it is not used in this work, we note that the position vector can be expressed in terms of the incomplete gamma function, see for example [16]. For simplicity we consider the case in which $1 + \alpha s > 0$. In this case, the maximal interval $I_{\alpha} \subset \mathbb{R}$ on which the basic LAC can be defined is
\[
I_{\alpha} = \begin{cases}
(-\infty, -1/\alpha), & \alpha < 0, \\
(-\infty, \infty), & \alpha = 0, \\
(-1/\alpha, \infty), & \alpha > 0,
\end{cases}
\] (2.10)

and we assume that all basic LAC are defined over $I_{\alpha}$. Finally, note that the image of $\kappa_{\xi^\alpha}$ is $\kappa_{\xi^\alpha}[I_{\alpha}] = (0, \infty)$.

Proposition 2.3 — Any LAC with $\alpha \neq 1$ and positive and decreasing curvature can be expressed as a basic LAC after applying the similarity transformations and shift of the arc length parameter. In particular, if $\gamma(s), s \in [0, L]$, is an LAC of length $L$, there exists a unique $\gamma_0 \in \mathbb{R}^2$, $\phi \in [0, 2\pi)$, $S \in \mathbb{R} \setminus \{0\}$, and $s_0 \in \mathbb{R}$, such that
\[
\gamma(s) = \gamma_0 + S R_{\phi} \xi^\alpha(s/S + s_0), \quad s \in [0, L],
\] (2.11)

where $\xi^\alpha(s)$ is a basic LAC of length $l := L/S$. 

3
In view of Proposition 2.3, we denote by $\xi^p$ the LAC segment uniquely defined by the parameters

$$p = (\alpha, S, s_0, l, \phi, x_0, y_0),$$

where $y_0 = (x_0, y_0)$. With this notation, note that $\xi^p = \xi^\alpha$ in the case that $p = (\alpha, 1, s_0, l, 0, 0, 0)$ for some $s_0$ and $l$ satisfying that $(s_0, s_0 + l) \subset I_\alpha$.

**Proof of Proposition 2.3.** For a given LAC $\gamma(s)$, $s \in [0, L]$ with $\alpha \neq 1$ and positive and decreasing curvature, from Definition 2.1 its curvature satisfies (2.5), which can be integrated once to obtain

$$\kappa'(s) = -A(\kappa(s))^{(\alpha+1)},$$

for some $A > 0$. Next, we consider the curve $\bar{\gamma}(\bar{s}) := S^{-1}\gamma(\bar{s}S)$, $\bar{s} \in [0, L/S]$, and we set $S = A^{1/(\alpha-1)}$. We see that its curvature satisfies

$$\begin{align*}
\bar{\kappa}'(\bar{s}) &= -\left(\bar{\kappa}(\bar{s})\right)^{(\alpha+1)}, \\
\bar{\kappa}(0) &= A^{1/(\alpha-1)}\kappa(0),
\end{align*}$$

which can be integrated to obtain

$$\bar{\kappa}(\bar{s}) = \begin{cases} 
\exp(-(\bar{s} - \log \bar{\kappa}(0))), & \alpha = 0, \\
1 + \alpha \left(\bar{s} + \frac{\bar{\kappa}(0))^{\alpha-1}}{\kappa(0)}\right)^{-1/\alpha}, & \alpha \neq 0.
\end{cases}$$

By comparing $\bar{\kappa}$ with $\kappa_{\xi^\alpha}$ (2.7) there exists a unique $s_0 \in \mathbb{R}$ such that $\bar{\kappa}(\bar{s}) = \kappa_{\xi^\alpha}(\bar{s} + s_0)$, and from the fundamental theorem of planar curves it follows the curves $\bar{\gamma}$ and $\xi^\alpha$ are congruent up to rigid transformations, i.e.

$$\bar{\gamma}(\bar{s}) = \bar{\gamma}_0 + R_{\phi} \xi^\alpha(\bar{s} + s_0), \quad \bar{s} \in [0, L/S],$$

for some $\bar{\gamma}_0 \in \mathbb{R}^2$ and $\phi \in [0, 2\pi)$. Finally, we use that $\gamma(s) = S\bar{\gamma}(s/S)$ to obtain (2.11) with $\gamma_0 := SL\bar{\gamma}_0$.

**Remark 2.4** — In the proof of Proposition 2.3, the scale transformation is used to alter the value of $A$ in (2.13) without changing the value of $\alpha$. However, in the case $\alpha = 1$ this technique cannot be exploited. This is due to the fact that $\alpha = 1$ corresponds to the logarithmic spiral, which is a self-similar curve. Particularly, $A$ in (2.13) is invariant under the scale transformations, thus the technique used in Proposition 2.3 cannot be used to recover the entire family of LAC with $\alpha = 1$.

**Remark 2.5** — Let $X$ be the reflection of $\mathbb{R}^2$ defined by the map $(x, y) \mapsto (y, x)$. If $\gamma(s)$ with $s \in [0, L]$ is an LAC, then also are

$$\begin{align*}
\gamma^{(1)}(s) &= \gamma(L - s), \\
\gamma^{(2)}(s) &= X\gamma(s), \\
\gamma^{(3)}(s) &= X\gamma(L - s).
\end{align*}$$

Moreover, their respective curvatures satisfy

$$\begin{align*}
\kappa^{(1)}(s) &= -\kappa(L - s), \\
\kappa^{(2)}(s) &= -\kappa(s), \\
\kappa^{(3)}(s) &= \kappa(L - s),
\end{align*}$$

which allow us to use Proposition 2.3 in those cases in which the curvature is not positive and decreasing, by applying one of the transformations (2.17).
2.2 Recovering the parameters of an LAC segment

We focus our attention to the problem of finding the parameters that uniquely identify a given LAC segment. We proceed in three steps, where we solve several linear equations in the least squares sense, with the objective of constructing an algorithm that can be applied to general curves. Given an LAC segment \( \gamma(s), s \in [0, L] \), by possibly applying one of the transformations (2.17), we assume that its curvature is positive and strictly monotonic decreasing. From Proposition 2.3, there exists \( p = (\alpha, S, s_0, l, \phi, x_0, y_0) \) such that \( \gamma(s) = \xi^p(s), s \in [0, L] \).

Remark 2.6 — In view of Remark 2.4 we omit the case \( \alpha = 1 \), and for simplicity in the formulation of this method we further omit the case \( \alpha = 0 \). Removing these values does not hinder the quality of the algorithm, because they are only single points on the real line.

We start by defining and recalling some useful quantities. From (2.11),
\[
\gamma(s) = \left( \frac{x_0}{y_0} \right) + S R \phi \xi^\alpha(s/S + s_0), \quad s \in [0, L],
\]
implies that
\[
\kappa(s) = S^{-1} \xi^\alpha(s/S + s_0).
\]
Recall that \( R = -\log \kappa \), hence \( R(s) = \log S + R \xi^\alpha(s/S + s_0) \) and using (2.6) we have that \( \log(R' \xi^\alpha) + \alpha R \xi^\alpha = 0 \), thus we obtain
\[
\log(R') + \alpha R = (\alpha - 1) \log S.
\]

Step 1 — Let \( c_1 := \alpha \) and \( c_0 := (\alpha - 1) \log S \). Then, from (2.21), in the least squares sense we have
\[
(c_0, c_1) = \arg \min_{(\tilde{c}_0, \tilde{c}_1)} \left\{ \frac{1}{2} \int_0^L (\log(R') + \tilde{c}_1 R - \tilde{c}_0)^2 \, ds \right\},
\]
which leads to
\[
c_1 = \frac{L \int_0^L R \log(R') \, ds - \int_0^L R \, ds \int_0^L \log(R') \, ds}{(\int_0^L R \, ds)^2 - L \int_0^L R^2 \, ds},
\]
and
\[
c_0 = \frac{1}{L} \int_0^L (\log(R') + c_1 R) \, ds.
\]
Then, \( \alpha = c_1 \) and \( S = \exp(c_0/(c_1 - 1)) \).

Step 2 — From (2.7) and (2.20), we have that \( \kappa(s) = S^{-1}(1 + \alpha(s/S + s_0))^{-1/\alpha} \), which allows us to isolate the parameter \( s_0 \) as
\[
s_0 = \frac{(S \kappa(s))^{-\alpha}}{\alpha} - \frac{1}{\alpha} - \frac{s}{S}.
\]
Then,
\[
s_0 := \arg \min_{\tilde{s}_0} \left\{ \frac{1}{2} \int_0^L \left( \frac{1}{\alpha S \kappa(s)^\alpha} - \frac{1}{\alpha} - \frac{s}{\bar{s}_0} \right)^2 \, ds \right\}
\]
gives
\[
s_0 = \frac{1}{\alpha L S} \int_0^L (\kappa(s))^{-\alpha} \, ds - \frac{1}{\alpha} - \frac{1}{LS} \int_0^L s \, ds.
\]
Similarly, we compute \( s_{\text{end}} := L/S + s_0 \) from \( \kappa^{(3)}(s) = \kappa(L - s) = S^{-1}(1 + \alpha(s_{\text{end}} - s/S))^{-1/\alpha} \). In the least squares sense, we obtain

\[
s_{\text{end}} = \frac{1}{\alpha L S^\alpha} \int_0^L (\kappa(L - s))^{-\alpha} \, ds - \frac{1}{\alpha} + \frac{1}{L S} \int_0^L s \, ds,
\]

(2.28)

Hence,

\[
l = s_{\text{end}} - s_0 = \frac{2}{L S} \int_0^L s \, ds.
\]

(2.29)

**Step 3** — At this point, we are left with finding the rotation and translation parameters. For the former, note that the angle function of \( \gamma \) and \( \xi^\alpha \) differ only by a constant \( \phi \), as

\[
\theta(s) = \phi + \theta_{\xi^\alpha}(s/S + s_0).
\]

(2.30)

Thus, in the least squares sense we obtain

\[
\phi = \frac{1}{L} \int_0^L (\theta(s) - \theta_{\xi^\alpha}(s/S + s_0)) \, ds.
\]

(2.31)

Finally, for the translation \((x_0, y_0)\) we solve \((2.11)\) in the least squares sense,

\[
\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{L} \int_0^L (\gamma(s) - S R_{\phi} \xi^\alpha(s/S + s_0)) \, ds.
\]

(2.32)

### 3 Approximation of planar curves

#### 3.1 Methodology

Now we consider the case where a general curve segment is given and we want to find an LAC segment that is the closest in a \( L^2 \)-distance sense. For the applications that we have in mind, the input is always a discrete curve and thus we must consider a discretization of the LAC. From previous section, we know that any LAC segment, after a possible change of parameterization or reflection, can be expressed as

\[
\xi^P(s) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + S R_{\phi} \xi^\alpha(s/S + s_0), \quad s \in [0, L].
\]

(3.1)

Using the equations for the basic LAC \((2.7), (2.8)\) and \((2.9)\), we write

\[
\xi^P(s) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^s \left( \frac{\cos(\theta_{\xi^\alpha}(t/S + s_0) + \phi)}{\sin(\theta_{\xi^\alpha}(t/S + s_0) + \phi)} \right) \, dt, \quad s \in (0, L).
\]

(3.2)

Let \( N \in \mathbb{N} \), and define

\[
\begin{cases}
h := \frac{L}{N-1}, \\
z := \frac{L}{S(N-1)},
\end{cases}
\]

(3.3)
and the discrete curve $\xi_n^{\Theta} \in \mathbb{R}^2$, $n = 0, \ldots, N - 1$, such that

$$
\begin{align*}
\xi_n^{\Theta} &= \xi_{n-1}^{\Theta} + h \begin{pmatrix}
\cos(\theta_{\xi_n} (zn + s_0) + \phi) \\
\sin(\theta_{\xi_n} (zn + s_0) + \phi)
\end{pmatrix}, \quad n = 1, \ldots, N - 1, \\
\xi_0^{\Theta} &= \begin{pmatrix}
x_0 \\
y_0
\end{pmatrix},
\end{align*}
\tag{3.4}
$$

where we use introduce the notation $\xi^{\Theta}$ for the discrete curve that depends on the parameters $\Theta = (x_0, y_0, h, \phi, z, s_0, \alpha)$.

Note that, the recursive expression (3.4) gives an approximation of $\xi^p(s)$ of second order, in the sense that it satisfies

$$
\xi_n^{\Theta} = \xi_{n+1}^{\Theta} + O(h^2), \quad n = 0, \ldots, N - 1.
\tag{3.6}
$$

The input curve is a list of $N$ equidistant $(x, y)$-points, which we express as the discrete curve

$$
\gamma_n = \begin{pmatrix}
x_n \\
y_n
\end{pmatrix}, \quad n = 0, \ldots, N - 1.
\tag{3.7}
$$

Given $\gamma_n$ we look for the closest $\xi_n^{\Theta}$ in the $L^2$-distance sense, i.e., we seek to find a set of parameters $\Theta^*$ such that

$$
\Theta^* = \arg \min_{\Theta \in U} \left\{ \frac{1}{2} \sum_{n=0}^{N-1} \| \xi_n^{\Theta} - \gamma_n \|^2 \right\},
\tag{3.8}
$$

with the admissible set $U$ given by

$$
U = \left\{ (x_0, y_0, \phi, h, z, s_0, \alpha) \in \mathbb{R}^7 \mid (x_0, y_0) \in \mathbb{R}^2, h > 0, \phi \in [0, 2\pi), \ z > 0, s_0 \in I_\alpha \text{ and } \alpha \in \mathbb{R} \backslash \{0, 1\} \right\}.
\tag{3.9}
$$

The optimization problem is solved using the Interior Point Optimizer (IPOPT) package, which for our purpose can be seen as a gradient descent-like method for nonlinear optimizations (see [14]), so we need to compute the gradient of the objective function in (3.8),

$$
L(\Theta) := \frac{1}{2} \sum_{n=0}^{N-1} \| \xi_n^{\Theta} - \gamma_n \|^2.
\tag{3.10}
$$

We have,

$$
\frac{\partial}{\partial \Theta_i} L(\Theta) = \sum_{n=0}^{N-1} \left( \xi_n^{\Theta} - \gamma_n, \frac{\partial}{\partial \Theta_i} \xi_n^{\Theta} \right), \quad \Theta_i = x_0, y_0, \phi, h, z, s_0, \alpha,
\tag{3.11}
$$

which is computed recursively from equation (3.4), using that

$$
\frac{\partial}{\partial \Theta_i} \xi_n^{\Theta} = \frac{\partial}{\partial \Theta_i} \xi_{n-1}^{\Theta} + \begin{cases} 
0 & \Theta_i = x_0, y_0, \\
T_n & \Theta_i = h, \\
h \frac{\partial}{\partial s_i} T_n & \text{otherwise,}
\end{cases}
\tag{3.12}
$$
with
\[
\frac{\partial}{\partial \Theta_i} \xi_0^\Theta = \begin{cases} \frac{1}{2} & \Theta_i = x_0, \\
\frac{1}{2} & \Theta_i = y_0, \\
0 & \text{otherwise,}
\end{cases}
\] (3.13)

where we denoted \(T_n\) as
\[
T_n := \begin{pmatrix} \cos(\theta_{\xi^a}(zn + s_0) + \phi) \\
\sin(\theta_{\xi^a}(zn + s_0) + \phi) \end{pmatrix}.
\] (3.14)

Then, using that \((\theta_{\xi^a})' = \kappa_{\xi^a}\), we obtain the gradient of \(T_n\) as
\[
\frac{\partial}{\partial \Theta_i} T_n = R_{\pi/2} T_n \times \begin{cases} 1, & \Theta_i = \phi, \\
\kappa_{\xi^a}(zn + s_0), & \Theta_i = s_0, \\
n\kappa_{\xi^a}(zn + s_0), & \Theta_i = z, \\
\frac{\partial}{\partial \alpha} \theta_{\xi^a}(zn + s_0), & \Theta_i = \alpha,
\end{cases}
\] (3.15)

where
\[
\frac{\partial}{\partial \alpha} \theta_{\xi^a}(s) = \frac{(1 + s\alpha)^{-\frac{1}{\alpha}} \left((\alpha - 1)(1 + s\alpha) \log(1 + s\alpha) - \alpha(\alpha + 2s\alpha - s)\right)}{\alpha^2(\alpha - 1)^2} + \frac{1}{(\alpha - 1)^2},
\] (3.16)

which is obtained by direct computation from (2.8). We compute the initial guess \(\tilde{\Theta}\) by using a discrete analogue of the three steps described in Section 2.2. We proceed as follows; we first approximate the curvature at each point \(\gamma_{n}, n = 1, \ldots, N - 2\), by
\[
\kappa_n = \frac{2 \det(T_{n-1}, T_n)}{h (1 + \langle T_{n-1}, T_n \rangle)}, \quad n = 1, \ldots, N - 2,
\] (3.17)

which is obtained from the inscribe circle tangent at the middle point of two consecutive segments [7], where \(T_n = (\gamma_{n+1} - \gamma_{n})/h\). Then, let us define the logarithm of the radius of curvature by
\[
R_n = -\log \kappa_n, \quad n = 1, \ldots, N - 2,
\] (3.18)

and its discrete derivative by
\[
\Delta R_n = -\frac{\kappa_{n+1} - \kappa_n}{\kappa_n}, \quad n = 1, \ldots, N - 3.
\] (3.19)

From Step 1, we obtain
\[
\tilde{\alpha} = \frac{(N - 3) \sum_{n=1}^{N-3} R_n \log \Delta R_n - \sum_{n=1}^{N-3} R_n \sum_{n=1}^{N-3} \log \Delta R_n}{\left(\sum_{n=1}^{N-3} R_n\right)^2 - (N - 3) \sum_{n=1}^{N-3} R_n^2},
\] (3.20)

and
\[
\tilde{S} = h^{1-\tilde{\alpha}} \exp\left(\frac{1}{(\tilde{\alpha} - 1)(N - 3)} \sum_{n=1}^{N-3} (\log \Delta R_n + \tilde{\alpha} R_n)\right).
\] (3.21)
From Step 2, and using that \( \bar{z} = \bar{I}/(N - 1) \), we obtain

\[
\bar{s}_0 = \frac{1}{\bar{a}(N - 2)S^2} \sum_{n=1}^{N-2} (\kappa_n)^{-\bar{a}} - \frac{1}{\bar{a}} \frac{(N - 1)h}{2S},
\]

and

\[
\bar{z} = \frac{(N - 1)h}{\bar{S}}.
\]  

Finally, from Step 3, we obtain

\[
\bar{\phi} = \frac{1}{N - 1} \sum_{n=0}^{N-2} (\theta_n - \theta_{\xi\bar{a}}(\bar{z}n + \bar{s}_0)),
\]

and

\[
\begin{pmatrix} \bar{x}_0 \\ \bar{y}_0 \end{pmatrix} = \frac{1}{N} \sum_{n=0}^{N-1} \left( \gamma_n - \xi_{\bar{\phi}} \right).
\]

As a first test of this algorithm we use synthetic data: discrete curves with constant step size based on Bézier curves. These curves were split in segments with sign preserving monotonic curvature. Then, we found the initial guess parameter, plot the resulting curve, and finally applied the \( L^2 \)-distance minimization algorithm. Some examples are shown in Figure 1. In the next section we apply this method to real data.

### 3.2 Application

In order to test the fairing algorithm shown in the previous section, we characterize some simple profile lines of a car’s roof (Toyota Prius). A 3D model was obtained by measuring a scale model car with a 3D laser scanner (Hexagon 8330-7), see Figure 2. This 3D model, was stored in STL (Standard Triangle/Tessellation Language) file, which encodes the geometry of the object in a triangular mesh. Using Rhinoceros 6, a computer-aided design software, we intercepted the 3D model with vertical planes, see Figure 3. Finally, we projected the curves into the plane and processed the discrete point to obtain a planar discrete curve with constant step size. The length of each curve is approximately 1500 mm, and the separation between curve is 100 mm. We observed that the curvature of the keylines is highly irregular, as a product of the measuring technique employed. To reduce the noise, we proceeded as follows. Let the curve \( \gamma_n \in \mathbb{R}^2, n = 0, \ldots, N - 1 \) with a constant step size \( \|\gamma_{n+1} - \gamma_n\| = h > 0 \), be the raw data; then:

(0) — Let \( \bar{N} = 3 \).

(1) — For \( \bar{N} < N \), apply the Ramer–Douglas–Peucker algorithm (see [2, 13]) to \( \gamma_n \), to obtain a new curve \( \bar{\gamma}_n, n = 0, \ldots, \bar{N} - 1 \), such that \( \bar{\gamma}_0 = \gamma \) and \( \bar{\gamma}_{\bar{N}-1} = \gamma_{N-1} \).

(2) — Construct a cubic spline curve \( \gamma_{cs}(t), t \in [0, L] \) using \( \bar{\gamma}_n, n = 0, \ldots, \bar{N} - 1 \), as the control points; hence, \( \gamma_{cs}(0) = \gamma_0 \) and \( \gamma_{cs}(L) = \gamma_{N-1} \).

(3) — Construct a discrete curve \( \tilde{\gamma}_n, n = 0, \ldots, N - 1 \) with step size \( h \), by sampling the cubic spline \( \gamma_{cs}(t_n) \) in a appropriate manner such that \( \|\gamma_{cs}(t_{n+1}) - \gamma_{cs}(t_n)\| = h \) and \( \tilde{\gamma}_n = \gamma_{cs}(t_n) \).
Figure 1: Examples of discrete curves approximated by LAC. Black curves are the input curves, blue curves are the first guesses obtained from our algorithm and the red curves are the final outputs of the IPOPT program.

(4) — If the residual $R$ is greater than a prescribed error $\eta$,

$$R = \frac{1}{L^2} \sum_{n=0}^{N-1} \| \tilde{\gamma}_n - \gamma_n \| h > \eta,$$

(3.26)

repeat from step (1), with a greater value of $\tilde{N}$. Otherwise, the process ends.

In our case, we used $\eta = 10^{-3}$, and we observed that this produces a smoother plot for the curvature, see Figure 4. Hence, we use $\tilde{\gamma}_n$, instead of $\gamma_n$, as the input curve for the approximation algorithm. However, we noticed that the admissible set $U$ for the parameters $\Theta$, as defined in (3.9), was too broad and unexpected jumps in the value of the parameters produced unrealistic outputs. To keep the optimization relatively close to the initial guess, we decided to constrain the admissible set to $U[\tilde{\Theta}]$, defined by

$$\bar{U}_{0.1} = U \cap \{ \Theta \in \mathbb{R}^7 : |\Theta_i - \bar{\Theta}_i| < 0.1 \bar{\Theta}_i \},$$

(3.27)

where $\bar{\Theta}$ is the initial guess for the IPOPT method. In this way, $\bar{U}_{0.1}$ constrain the final result to be in a $10\%$ range of the initial guess. Final results are presented in Figure 5. We note that the
Figure 2: Left: Hexagon 8330-7 (7 axis arm type 3D laser scanner with measurement accuracy of 0.078 mm). Right: 3D model (STL data).

Figure 3: Position of sampled curves, obtained from the interception of the 3D model and vertical planes (Software: Rhino 6). The black lines represent the intercepting planes, and the red lines, in the highlighted area, represent the resulting curves. From the center to the bottom, curves are labelled as ps_1, ps_2, ps_3, ps_4 and ps_5.

parameters that we obtained provide a good fit of the input curve. We can observe that, despite the curves being similar in shape to each other, the values of the parameter $\alpha$ have big variations. However, the final result is still a good fit, after finding the remaining parameters. It is interesting to note that, the algorithm that we provided allow us to input the parameter $\alpha$ by hand (or by replacing (3.20) by an alternative expression or algorithm) and then we can continue with the next steps without any further change. We conclude that the method proposed has a good performance, however further analysis on the recovery of the parameter $\alpha$ is required.

4 Concluding remarks

In the present work we briefly reviewed the notion of log-aesthetic curve, and showed how the family of LAC with a given parameter $\alpha$ can be described as segments of a basic LAC after applying the similarity transformations and shift of the arc length parameter (Proposition 2.3). This constitutes
the main result of this work. In a second instance, we used this characterization and provided an algorithmic way to recover the seven parameters $p = (\alpha, S, s_0, l, \phi, x_0, y_0)$ that uniquely identify a given LAC segment. Finally, this algorithm was later used to obtain the initial input for the gradient descent-like method used to approximate a general planar curve segment by LAC (Equation (3.8)). Regarding the applications, we developed this method aiming to be used as a reverse engineering tool that characterizes existing objects. In view of this, we mention that further improvements in the methodology employed to find the parameter $\alpha$ might be necessary, in particular, if we are interested in characterize a given design by a concrete value of $\alpha$. Further analysis regarding this will be considered in further works.

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Figure 5: Discrete curves approximated by LAC. Each curve represent a different section of half of the car roof, taken at 100 mm apart. Black lines: input curves. Red lines: LAC output.

References

[1] D. Brander, J. Gravesen, T.B. Nørjerg, Approximation by planer elastic curves, Adv. Comput. Math. 43 (2017) 25–43.

[2] D. Douglas and T. Peucker, Algorithms for the reduction of the number of points required to represent a digitized line or its caricature, The Canadian Cartographer 10 2 (1973) 112–122.

[3] R. U. Gobithaasan and K. T. Miura, Aesthetic spiral for design, Sains Malaysiana 40 (2011) 1301–1305.

[4] R. U. Gobithaasan, Y. Siew Wei, K. T. Miura, Log-aesthetic curves for shape completion problem, J. Appl. Math. 2014 (2014) 960302.

[5] S. E. Graiff Zurita, K. Kajiwara and T. Suzuki, Fairing of discrete planar curves by integrable discrete analogue of Euler’s elasticae, preprint, arXiv:2111.00804v1 (2021).

[6] T. Harada, N. Mori, K. Sugiyama, Study of quantitative analysis of curve’s character, Bull. JSSD 40 (1994) 9–16. (in Japanese)

[7] T. Hoffmann, Discrete differential geometry of curves and surfaces, MI Lecture Notes 18, Kyushu University, Fukuoka (2009).
[8] J. Inoguchi, K. Kajiwara, K. T. Miura and M. Sato, W. K. Schief and Y. Shimizu, Log-aesthetic curves as similarity geometric analogue of euler’s elasticae, Comput. Aided Geom. Des. 61(2018) 1–5.

[9] J. Inoguchi, Y. Jikumaru, K. Kajiwara, K. T. Miura and W. K. Schief, Log-aesthetic curves: similarity geometry, integrable discretization and variational principles, preprint, arXiv:1808.03104v2 (2021)

[10] R. Levien and C.H. Séquin, Interpolating splines: which is the fairest of them all? Comput. Aided Des. Appl. 6(2009) 91–102.

[11] K. T. Miura, J. Sone, A. Yamashita, T. Kaneko, Derivation of a general formula of aesthetic curves, Proceedings of the Eighth International Conference on Humans and Computers (HC2005) (2005) 166–171.

[12] K. T. Miura and R. U. Gobithaasan, Aesthetic design with log-aesthetic curves and surfaces, in: Mathematical Progress in Expressive Image Synthesis III, eds. by Y. Dobashi and H. Ochiai, Mathematics for Industry 24 (Springer, Singapore, 2015) 107–120.

[13] U. Ramer, An iterative procedure for the polygonal approximation of plane curves, Comput. Graph. Image Process. 1 (1972) 244–256.

[14] A. Wächter and L.T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, Math. Program. Ser. A 106 (2006) 25–57.

[15] N. Yoshida and T. Saito, Interactive aesthetic curve segments, Visual Comput 22 (2006) 896—905.

[16] R. Ziatdinov, N. Yoshida, T. Kim, Analytic parametric equations of log-aesthetic curves in terms of incomplete gamma functions. Comput. Aided Geom. Des. 29 2 (2012) 129–140.