Extraction of the $b$-quark shape function parameters using the Belle $B \rightarrow X_s \gamma$ photon energy spectrum

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Abstract

We determine the $b$-quark shape function parameters, $\Lambda_{SF}^b$ and $\lambda_1^{SF}$ using the Belle $B \rightarrow X_s \gamma$ photon energy spectrum. We assume three models for the form of the shape function; Exponential, Gaussian and Roman.
INTRODUCTION

The off-diagonal element $V_{ub}$ in the CKM matrix is extracted from measurements of the $B \to X_u \ell \nu$ process in the limited region of lepton momentum [1], or the hadronic recoil mass $M_X$ [2], or $M_X$ and the leptonic invariant mass squared $q^2$ [3] where the contribution of background from the $B \to X_c \ell \nu$ process is suppressed. In order to determine $|V_{ub}|$ we need to extrapolate measured rates from such limited regions to the whole phase space. This extrapolation factor is evaluated using a theoretical prediction that takes into account the residual motion of the $b$-quark inside the $B$ meson, so called “Fermi motion” [4]. Fermi motion is included in the heavy quark expansion by resumming an infinite set of leading-twist corrections into a shape function of the $b$-quark [5, 6, 7]. Since the shape function is not calculable theoretically, it has to be determined experimentally.

The best way is to make use of the photon energy spectrum for $B \to X_s \gamma$ since both the inclusive decay spectra in $B \to X_u \ell \nu$ and $B \to X_s \gamma$ are expressed by the same shape function up to leading order of $1/m_b$ in the heavy quark expansion [8, 9]. The first results were obtained by CLEO [10], but the errors of the shape function parameters are rather large. Therefore the uncertainty of the shape function dominates the theoretical error of $|V_{ub}|$ at present. Belle has recently provided more precise data than CLEO of the $B \to X_s \gamma$ photon spectrum [11]. We report on the results of determination of the shape function parameters using the Belle $B \to X_s \gamma$ data.

PROCEDURE

We used a method based on that devised by the CLEO collaboration [12]. We fit Monte Carlo (MC) simulated spectra to the raw data photon energy spectrum. “Raw” refers to the spectra that are obtained after the application of the $B \to X_s \gamma$ analysis cuts. The use of “raw” spectra correctly accounts for the Lorentz boost from the $B$ rest frame to the center of mass system, energy resolution effects and avoids unfolding. The method is as follows:

1. Assume a shape function model.
2. Simulate the photon energy spectrum for a certain set of parameters; $(\Lambda_{SF}^{SF}, \lambda_{1}^{SF})$.
3. Perform a $\chi^2$ fit of the simulated spectrum to the data where only the normalization of the simulated spectrum is floated and keep the resultant $\chi^2$ value.
4. Repeat steps 2-3 for different sets of parameters to construct a two dimensional grid with each point having a $\chi^2$.
5. Find the minimum $\chi^2$ on the grid and all points on the grid that are one unit of $\chi^2$ above the minimum.
6. Repeat steps 1-5 for a different shape function model.

Shape function models

Three shape function forms suggested in the literature are employed; Exponential, Gaussian and Roman [8, 9]. These are described in Table 1. The shape function $F$ is a function of
\( k_+(\equiv k_0 - k_3) \), where \( k_\mu \) is the residual momentum of the \( b \)-quark in the \( B \) meson, defined through

\[
p_{b,\mu} = m_b v_\mu + k_\mu, \tag{1}
\]

where \( v_\mu = (1, 0, 0, 0) \) and \( k_3 \) is the \( k \) component along the direction of the \( u \)-quark. The shape function is parameterized by \( \Lambda^\text{SF} \) and \( \lambda^\text{SF}_1 \). These parameters are related to the \( b \) quark mass, \( m_b \), and the average momentum squared of the \( b \) quark, \( \mu^2 \), via the relations,

\[
\Lambda^\text{SF} = M_B - m_b \tag{2}
\]

and

\[
\lambda^\text{SF}_1 = -\mu^2, \tag{3}
\]

where \( M_B \) is the mass of the \( B \) meson. Up to leading order in the non-perturbative dynamics the shape function is universal in describing the \( b \)-quark Fermi motion relevant to \( b \)-to-light quark transitions. The lepton and photon energy spectra in \( B \to X_u l\nu \) and \( B \to X_s \gamma \) decays are given by the convolution of the respective parton-level spectra with the shape function. Example shape function curves are plotted in Figure 1(a).

| Shape Function | Form |
|----------------|------|
| Exponential | \( F(k_+; a) = N(1 - x)^ae^{(1+a)x} \) |
| Gaussian | \( F(k_+; c) = N(1 - x)^c e^{-b(1-x)^2} \) where \( b = \left( \frac{\Gamma(c+2)}{\Gamma(c+1)} \right)^2 \) |
| Roman | \( F(k_+; \rho) = \frac{N}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4}(\frac{\rho}{\kappa} - \kappa(1-x))^2 \right\} \) where \( \kappa = \frac{\rho}{\sqrt{\pi}} e^{\rho/2} K_1(\rho/2) \) |

where \( x = k_+/\Lambda^\text{SF} \)

\(-m_b \leq k_+ \leq \Lambda^\text{SF}\)

and \( a, c, \rho, N \) are chosen

to satisfy

\( A_0 = 1, A_1 = 0, A_2 = -\lambda^\text{SF}_1/3, \)

where \( A_n = \int k_+^n F(k_+)dk_+ \)

**TABLE I**: The three models used for the shape function forms

**Monte Carlo simulated photon energy spectrum**

We generate \( B \to X_s \gamma \) MC events according to the Kagan and Neubert prescription for each set of the shape function parameter values \(^9\). The generated events are then simulated
FIG. 1: Shape function model curves for Exponential ($\Lambda_{SF}, \lambda_{1SF}^{SF}$) = (0.66, −0.40), Gaussian ($\Lambda_{SF}, \lambda_{2SF}^{SF}$) = (0.63, −0.40), and Roman ($\Lambda_{SF}, \lambda_{3SF}^{SF}$) = (0.66, −0.39), where $\Lambda_{SF}$ and $\lambda_{1SF}^{SF}$ are measured in units of GeV/c$^2$ and GeV$^2$/c$^2$ respectively.

for the detector performance using the Belle detector simulation program. Afterwards $B \rightarrow X_s \gamma$ analysis cuts are applied to the MC events to obtain the raw photon energy spectrum in the $\Upsilon(4S)$ rest frame $^{[1]}$.

Fitting the spectrum

For a given set of shape function parameters, a $\chi^2$ fit of the MC simulated photon spectrum to the raw data spectrum is performed in the interval, $1.5 < E_\gamma^*/\text{GeV} < 2.8^{[15]}$. The normalization parameter is floated in the fit. The raw spectrum is plotted in Figure 2, the errors include both statistical and systematic errors. The latter are dominated by the estimation of the $B\bar{B}$ background and are 100% correlated. Therefore the covariance matrix is constructed as

$$V_{ij} = \sigma_{di}^{\text{stat}} \sigma_{dj}^{\text{stat}} + \sigma_{di}^{\text{sys}} \sigma_{dj}^{\text{sys}}$$

where $i, j = 1, 2, \ldots, 13$ denote the bin number, and $\sigma_d$ is the error in the data. Then the $\chi^2$ used in the fitting is given by

$$\chi^2 = \sum_{ij} (d_i - f_i)(V_{ij})^{-1}(d_j - f_j),$$

where $(V_{ij})^{-1}$ denotes the $ij^{th}$ element of the inverted covariance matrix. The $\chi^2$ value after the fit is used to determine a map of $\chi^2$ as a function of the shape function parameters.

The best fit and $\Delta \chi^2$ contour

The best fit parameters are associated to the minimum chi-squared case, $\chi^2_{\text{min}}$. The $1\sigma$ “ellipse” is defined as the contour which satisfies $\Delta \chi^2 = \chi^2 - \chi^2_{\text{min}} = 1$. The contours are
found to be well approximated by the function

\[ \Delta \chi^2(\Lambda_{\text{SF}}, \lambda_{\text{SF}}^1) = \left( \frac{\lambda_{\text{SF}}^1 + a(\Lambda_{\text{SF}}^2 + b)}{c} \right)^2 + \left( \frac{(\Lambda_{\text{SF}}^2 + d)}{e} \right)^2. \]

(6)

The parameters \( a, b, c, d, \) and \( e \) are determined by fitting the function to the parameter points that lie on the contour.

**RESULTS**

The best fit parameters are given in Table II. The parameter values are found to be consistent across all three shape function forms. The minimum \( \chi^2 \) fit for each shape function model is displayed in Figure 3. The fits to the contour with \( \Delta \chi^2 = 1 \) points are shown in Figure 4. The imposed shape function form acts to correlate \( \Lambda_{\text{SF}} \) and \( \lambda_{\text{SF}}^1 \).

| Shape   | \( \chi^2_{\text{min}} \) | \( \Lambda_{\text{SF}} \) (GeV/\( c^2 \)) | \( \lambda_{\text{SF}}^1 \) (GeV/\( c^2 \)) |
|---------|-----------------|-----------------|-----------------|
| Exponential | 4.883 | 0.66 | -0.40 |
| Gaussian | 4.272 | 0.63 | -0.33 |
| Roman | 5.020 | 0.66 | -0.39 |

**TABLE II:** The best fit shape function parameter values.
**Strong Coupling $\alpha_s$**

The strong coupling constant, $\alpha_s$, is an input into the parton-level calculations for both $B \to X_s\gamma$ and $B \to X_u\ell\nu$ spectra. By default $\alpha_s(\mu)$ is evaluated at the mass scale $\mu = m_b$. To investigate the systematic effect of this choice the analysis is redone for $\mu = m_b/2$ and $\mu = 2m_b$ in the case of the exponential shape function model. The $\Lambda_{SF}$ and $\lambda_1^{SF}$ parameter values corresponding to $\chi^2_{\text{min}}$ are given in Table III.

**COMPARISON WITH CLEO**

The CLEO collaboration has provided points which lie on their equivalent $\Delta\chi^2 = 1$ contour for the case of an exponential shape function model[14]. The data points are slightly different from those given in the Gibbons’ report[10] since the present data now includes the
TABLE III: The best fit parameters for various $\alpha_s$ using the exponential shape function model

| $\mu$ | $\alpha_s(\mu)$ | $\Lambda_{SF}$ (GeV/$c^2$) | $\lambda_{SF}^1$ (GeV$^2$/$c^2$) |
|-------|-------------------|-----------------------------|---------------------------------|
| $m_b$  | 0.210             | 0.66                        | -0.40                           |
| $m_b/2$| 0.257             | 0.65                        | -0.41                           |
| $2m_b$ | 0.177             | 0.68                        | -0.43                           |

uncertainty in the $B\bar{B}$ background Monte Carlo normalization[14].

We fit the functional form given in equation 6 to their contour data and find excellent agreement ($a = 2.378$, $b = -0.347$, $c = 0.178$, $d = -0.426$, $e = 0.256$). The minimum $\chi^2$ point for the CLEO data corresponds to $(\Lambda_{SF}, \lambda_{SF}^1)_{Exp} = (0.545, -0.342)$. We compare the CLEO and Belle contours in Figure 5. The regions bounded by the contours marginally overlap. The uncertainty in the Belle result is much reduced with respect to that of CLEO.

Unfortunately we can not produce a combined $\Delta \chi^2 = 1$ contour of the two experiments since a precise map of $\Delta \chi^2$ as a function of $\Lambda_{SF}$ and $\lambda_{SF}^1$ is not currently available for CLEO.

FIG. 5: The fitted $\Delta \chi^2 = 1$ contours for CLEO (blue) and Belle (red) assuming an exponential shape function form.

SUMMARY

We have determined the $b$-quark shape function parameters, $\Lambda_{SF}$ and $\lambda_{SF}^1$, from fits of Monte Carlo simulated spectra to the raw Belle measured $B \rightarrow X_s \gamma$ photon energy spectrum. Raw refers to the spectrum as measured after the application of analysis cuts. We used three models for the form of the shape function; Exponential, Gaussian and Roman. We found
the best fit parameters; $(\Lambda_{\text{SF}}, \lambda_{\text{SF}}^1)_{\text{Exp}} = (0.66, -0.40)$, $(\Lambda_{\text{SF}}, \lambda_{\text{SF}}^1)_{\text{Gauss}} = (0.63, -0.33)$, and $(\Lambda_{\text{SF}}, \lambda_{\text{SF}}^1)_{\text{Roman}} = (0.66, -0.39)$, where $\Lambda_{\text{SF}}$ and $\lambda_{\text{SF}}^1$ are measured in units of GeV/$c^2$ and GeV$^2$/c$^2$ respectively. We also determined the $\Delta \chi^2 = 1$ contours in the $(\Lambda_{\text{SF}}, \lambda_{\text{SF}}^1)$ parameter space for each of the assumed models.

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