Rise of the DIS structure function $F_L$ at small $x$ caused by double-logarithmic contributions

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We present calculation of $F_L$ in the double-logarithmic approximation and demonstrate that the synergic effect of the factor $1/x$ from the $\alpha_s^2$-order and the steep $x$-dependence of the totally resummed double logarithmic contributions of higher orders ensures the power-like rise of $F_L$ at small $x$ and arbitrary $Q^2$.

PACS numbers: 12.38.Cy

I. INTRODUCTION

Theoretical investigation of the DIS structure function $F_L(x, Q^2)$ (and other DIS structure functions) in the context of perturbative QCD began with calculations in the fixed orders in $\alpha_s$. First, there were calculations in the Born approximation, then more involved first-loop and second-loop calculations (see Refs. [1]-[11]) followed by the third-order results[12]. All fixed-order calculations showed that $F_L$ decreases at small $x$. Alternative approach to study $F_L$ was applying all-order resummations. In the first place, $F_L$ was studied with DGLAP[13] and its NLO modifications. In addition, there are approaches where DGLAP is combined with BFKL[14], see e.g. Ref. [15, 16]. Besides, there are calculations in the literature based on the dipole model, see Refs. [18, 19]. This approach was used in the global analysis of experimental data in Ref. [20]. Let us notice that Ref. [21] contains detailed bibliography on this issue.

Applying DGLAP to studying $F_L$ is model-independent. However according to Ref. [21], neither LO DGLAP nor the NLO DGLAP modifications ensure the needed rise of $F_L$ at small $x$ and disagree with experimental data at small $Q^2$, which sounds quite natural because DGLAP by definition is not supposed to be used in the region of small $Q^2$. The modifications of DGLAP in Refs. [15, 16] are based on treating BFKL as a small-$x$ input for the DGLAP equations. The approach of Ref. [17] treats the Pomeron intercept as a parameter fixed from experiment. Implicitly these approaches suggest that the $x$ and $Q^2$-evolutions run separately of each other, which is not obvious at all. Then, the DGLAP equations are suggested for operating with large $x$ whereas BFKL works at small $x$ only. Therefore, any combining BFKL and DGLAP does not look well-grounded from theoretical point of view.

In this paper we present the total resummation of double-logarithmic (DL) contributions to $F_L$. The method we use is self-consistent and does not involve any models. We modify the approach which we used in Ref. [22] to calculate $F_1$ in the Double-Logarithmic Approximation (DLA). This approach has nothing in common with the BFKL equation and its ensuing modifications. Indeed, instead of summing leading logarithms, i.e. the contributions $\sim (1/x)(\alpha_s \ln(1/x))$, which is the BFKL domain, we sum the DL contributions $\sim \alpha_s \ln^2(1/x)$ as well as the DL of $Q^2$. Because of the absence of the factor $1/x$ such contributions were commonly neglected by the HEP community for a long time. However, it has recently been proved in Ref. [22] that the DL contribution to Pomeron is not less important than the BFKL contribution.

We calculate $F_L$ in DLA with constructing and solving Infra-Red Evolution Equations (IREEs). As is well-known, the IREE approach was suggested by L.N. Lipatov[23]. It proved to be a simple and efficient instrument (see e.g. the overviews in Ref. [24]) for calculating many objects in QCD and Standard Model. Constructing and solving IREEs, we obtain general solutions. In order to specify them one has to define the starting point (input) for IREEs. Conventionally in the IREE technology the Born contributions have been chosen as the inputs. However, $F_L = 0$ in the Born approximation, so the input has to be chosen anew. We suggest that the the second-loop expression for $F_L$ can play the role of the input and arrive thereby to explicit expressions for perturbative components of $F_L$. We demonstrate that the total resummation of DL contributions together with the factor $1/x$ appearing in the $\alpha_s^2$-order provide $F_L$ with the rise at small $x$.

We start with considering $F_L$ in the large-$Q^2$ kinematic region

$$Q^2 > \mu^2,$$  \hspace{1cm} (1)
with \( \mu \) being a mass scale. Then we present a generalization of our results to small \( Q^2 \). The scale \( \mu \) is often associated with the factorization scale. The value of \( \mu \) is arbitrary\(^1\) except the requirement \( \mu > \Lambda_{QCD} \) to guarantee applicability of perturbative QCD.

Our paper is organized as follows: In Sect. II we introduce definitions and notations, then remind how to calculate \( F_L \) through auxiliary invariant functions. Calculations of \( F_L \) in the \( \alpha_s^2 \)-order are considered in Sect. III. We represent them in the way convenient for analysis of contributions from higher loops. Then we explain how to realize our strategy: combining the non-logarithmic results from the \( \alpha_s^2 \)-order with double-logarithmic (DL) contributions from higher-order graphs. Total resummation of DL contributions to \( F_L \) is done in Sect. IV through constructing and solving IREEs. IREEs control both \( x \) and \( Q^2 \)-evolutions of \( F_L \) from the starting point. Specifying the input is done in Sect. V. In Sect. VI we present explicit expressions for leading small-\( x \) contributions to perturbative components of \( F_L \). To make clearly seen the rise of \( F_L \) at small \( x \) we consider the small-\( x \) asymptotics of \( F_L \). We also compare the small-\( x \) behaviour of \( F_L \) and the one of the gluon distribution in the hadrons. Then we consider the generalization of our results on \( F_L \) in region \([1]\) to the small-\( Q^2 \) region. Finally, Sect. VII is for concluding remarks.

II. CALCULATING \( F_L \) THROUGH AUXILIARY AMPLITUDES

The most convenient way to calculate \( F_{1,2} \) and \( F_L \) in Perturbative QCD is the use of auxiliary invariant amplitudes. Below we remind how this approach works. The unpolarized part of the hadronic tensor describing the lepton-hadron DIS is

\[
W_{\mu\nu}(p,q) = \left( -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) F_1 + \frac{1}{pq} \left( p_{\mu} - q_{\mu} \frac{pq}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{pq}{q^2} \right) F_2 \tag{2}
\]

and each of \( F_1, F_2 \) depends on \( Q^2 \) and \( x = Q^2/w \), with \( Q^2 = -q^2 \) and \( w = 2pq \). It is convenient to represent \( F_{1,2}^{(q,g)} \) through auxiliary amplitudes \( A \) and \( B \) which are the convolutions of the tensor \( W_{\mu\nu}^{(q,g)} \) with \( g_{\mu\nu} \) and \( p_{\mu}p_{\nu} \):

\[
-A \equiv g_{\mu\nu}W_{\mu\nu} = 3F_1 + \frac{F_2}{2x} + O(p^2), \tag{3}
\]

\[
B \equiv \frac{p_{\mu}p_{\nu}}{pq}W_{\mu\nu} = -\frac{1}{2x}F_1 + \frac{1}{4x^2}F_2 + O(p^2), \tag{4}
\]

where we use the standard notations \( x = -q^2/w = Q^2/w, w = 2pq \). Neglecting terms \( \sim p^2 \), we express \( F_{1,2} \) through \( A \) and \( B \):

\[
F_1 = \frac{A}{2} + xB, \tag{5}
\]

\[
F_2 = 2xF_1 + 4x^2B,
\]

so that

\[
F_L = F_2 - 2xF_1 = 4x^2B. \tag{6}
\]

Each of \( F_1, F_2 \) includes both perturbative and non-perturbative contributions. According to the QCD factorization concept, these contributions can be separated. In scenario of the single-parton scattering, \( F_1, F_2 \) can be represented in any available form of QCD factorization through the following convolutions (see Fig. 1):

\[
F_1 = F_1^{(q)} \otimes \Phi_{(q)} + F_1^{(g)} \otimes \Phi_{(g)}, \quad F_2 = F_2^{(q)} \otimes \Phi_{(q)} + F_2^{(g)} \otimes \Phi_{(g)}. \tag{7}
\]

\(^1\) for specifying \( \mu \) on basis of Principle of Minimal Sensitivity\([24]\) see Ref. [24]
FIG. 1. QCD factorization for DIS structure functions. Dashed lines denote virtual photons. The upper blobs describe DIS off partons. The straight (waved) vertical lines denote virtual quarks (gluons). The lowest blobs correspond to initial parton distributions in the hadrons. $F^{(q,g)}$ is a generic notation for perturbative components of $F_1^{(q,g)}$, $F_2^{(q,g)}$ and $F_L^{(q,g)}$.

where $\Phi_1^{(q,g)}$ and $\Phi_2^{(q,g)}$ stand for initial parton distributions whereas $F_1^{(q,g)}$, $F_2^{(q,g)}$ are perturbative components of the structure functions $F$ and $F_2$ respectively. The superscripts $q$ and $g$ in Eq. (7) or $F_1^{(q,g)}$ and $F_2^{(q,g)}$ mean that the initial partons in the perturbative Compton scattering are quarks (gluons). The DIS off the partons is parameterized by the same way as Eq. (2):

\begin{equation}
W_{\mu\nu}^{(q,g)}(p,q) = \left(-g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2}\right) F_1^{(q,g)} + \frac{1}{p q} \left( p_{\mu} - q_{\mu} \frac{pq}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{pq}{q^2} \right) F_2^{(q,g)},
\end{equation}

with $p$ denoting the initial parton momentum. Throughout the paper we will neglect virtualities $p^2$, presuming the initial partons to be nearly on-shell. Introducing the auxiliary amplitudes $A^{(q,g)}$ and $B^{(q,g)}$ similarly to Eqs. (3,4), one can express $F_1^{(q,g)}$ and $F_2^{(q,g)}$ in terms of $A^{(q,g)}$ and $B^{(q,g)}$ so that

\begin{equation}
F_L^{(q,g)} = F_2^{(q,g)} - 2x F_1^{(q,g)} = 4x^2 B^{(q,g)},
\end{equation}

with

\begin{equation}
B^{(q,g)} = \frac{p_{\mu} p_{\nu}}{pq} W_{\mu\nu}^{(q,g)}.
\end{equation}

Applying (9,10) to $W_{\mu\nu}^{(q,g)}$ in the Born and the first-loop approximation yields [1]-[11] that $F_L^{(q)} = F_L^{(g)} = 0$ in the Born approximation whereas the first-loop results are:

\begin{equation}
\left( F_L^{(q)} \right)_1 = \frac{\alpha_s C_F}{\pi} x^2, \quad \left( F_L^{(g)} \right)_1 = 0.
\end{equation}

Eq. (11) suggests that $F_L$ should decrease $\sim x^2$ at $x \rightarrow 0$. However, the second-order results bring a slower decrease.

III. LEADING CONTRIBUTIONS TO $B$ IN THE SECOND-LOOP APPROXIMATION

The second loop brings a radical change to the small-$x$ behaviour of $B$ compared to the first-loop result. Namely, there appear contributions $\sim 1/x$ in contrast to logarithmic dependence of $B$ in the first loop. Such contributions were calculated in Ref. [10]. Nevertheless, we prefer to repeat these calculations in order to represent the results in the way convenient for applying to the total resummation of higher loops in DLA. Doing so, we account for the leading contributions only. Throughout the paper we use the Feynman gauge for virtual gluons.
In the first place we consider ladder graphs contributing to $B$. The ladder graphs contributing to $W_{\mu\nu}$ in the $\alpha_s^2$-order are depicted in Fig. 2. Graphs (a) and (b) correspond to DIS off quarks whereas graphs (c) and (d) are for DIS off gluons. Calculations in the small-$x$ kinematics are simpler when the Sudakov variables\cite{26} are used. In terms of them, momenta $k_i$ of virtual partons are parameterized as follows:

$$k_i = \alpha_i q' + \beta_i p' + k_{i\perp},$$  \hspace{1cm} \text{(12)}

where $q'$ and $p'$ are the massless (light-cone) momenta made of momenta $p$ and $q$:

$$p' = p - q(p^2/w) \approx p, \quad q' = q - p(q^2/w) = q + xp.$$ \hspace{1cm} \text{(13)}

In Eq. (13) $q$ denotes the virtual photon momentum while $p$ is momentum of the initial parton. We remind that we presume that $p^2$ is small, so we will neglect it throughout the paper. Invariants involving $k_i$ look as follows in terms of the Sudakov invariants:

$$k_i^2 = w(\alpha_i\beta_i - z_i), \quad 2p k_i = w\alpha_i, \quad 2q k_i = w(\beta_i - x\alpha_i),$$

$$2k_i k_j = w((\alpha_i - \alpha_j)(\beta_i - \beta_j) - k_{i\perp}^2 - k_{j\perp}^2 = w((\alpha_i - \alpha_j)(\beta_i - \beta_j) - z_i - z_j).$$ \hspace{1cm} \text{(14)}

We have introduced in Eq. (14) dimensionless variables $z_{i,j}$ defined as follows:

$$z_i = k_{i\perp}^2/w.$$ \hspace{1cm} \text{(15)}

### A. Contributions to $B$ for DIS off quarks

We start with calculating the second-loop contribution $B_{q}^{(2)}$ of the two-loop ladder graph (a) in Fig. 2 to $B$ for DIS off quarks. It is given by the following expression:

$$B^{2\alpha} = C_F^2 \chi_2 w \int d\alpha_{1,2} d\beta_{1,2} d\kappa_{1,2} \frac{N^{(2)}}{k_{1,2}^2 k_{2,1}^2 \kappa_{1,2}^2} \delta ((q + k_2)^2) \delta ((k_1 - k_2)^2) \delta ((p - k_1)^2),$$ \hspace{1cm} \text{(16)}

where

$$\chi_2 = \frac{\alpha_s^2}{8\pi}.$$ \hspace{1cm} \text{(17)}
and
\[
N^{2a} = \frac{1}{2} Tr \left[ \hat{p} \gamma_\lambda, \hat{k}_1 \gamma_\lambda \hat{k}_2 \hat{p} (\hat{q} + \hat{\lambda}) \right] = 2k^2_1 Tr \left[ \hat{k}_2 (\hat{k}_1 - \hat{p}) \hat{q} \right].
\]
\[
= 2k^2_1 (w + 2pk_2) Tr \left[ \hat{k}_2 (\hat{k}_1 - \hat{p}) \hat{q} \right].
\]
(18)

We represent it as the sum of \( N_1^{(a)} \) and \( N_2^{(a)} \):
\[
N^{2a} = N_1^{(a)} + N_2^{(a)}
\]
with
\[
N_1^{2a} = -4k^2_1 ((2pk_2)^3 + w(2pk_2)^2)
\]
and
\[
N_2^{2a} = 4k^2_1 [(k^2_1 + k^2_2)((2pk_2)^2 + w(2pk_2)) - k^2_1 k^2_2 (2pk_2) - wk^2 k^2_2],
\]
(21)

In Eqs. \( 20,21 \) we have used the quark density matrix
\[
\hat{\rho}(p) = \frac{1}{2} \hat{p}
\]
(22)

and made use of the \( \delta \)-functions of Eq. (16). They yield that \( 2k_1 k_2 = k^2_1 + k^2_2 \) and \( 2pk_1 = k^2_2 \). It turns out that the leading contributions comes from \( N_1^{2a} \), so first of all we consider it. Throughout the paper we will use dimensionless variables \( z_{1,2} \) instead of \( k^2_{1,2\perp} \):
\[
z_1 = k^2_{1\perp}/w, \quad z_2 = k^2_{2\perp}/w, \quad z = z_1 + z_2.
\]
(23)

It is also convenient to use the variable \( l \) defined as follows:
\[
l = \beta_1 - \beta_2.
\]
(24)

Using the \( \delta \)-functions to integrate (16) over \( \alpha_1,2 \) and \( \beta_2 \) and replacing \( N^{2a} \) by \( N_1^{2a} \) we are left with three more integrations:
\[
B^{2a} \approx 4C^2_F x_2 \int_{\lambda}^1 \frac{dz_1}{z_1} \int_{\lambda}^1 \frac{dz_2}{z_2} \int_z^1 dl \left[ -\frac{z^3}{l^2(1+\eta)^2} + \frac{z^2}{l(l+\eta)^2} \right],
\]
(25)

with \( \eta \) defined as follows:
\[
\eta = \frac{z(x + z_2)}{z_2}.
\]
(26)

Details of calculation in Eq. (25) can be found in Appendix A. Let us remind that throughout this paper we focus on the small-\( x \) region. The most important contributions in Eq. (25) at small \( x \) are \( \sim 1/x \). Retaining them only and integrating (25) with logarithmic accuracy, we arrive at
\[
B^{2a} \approx C^2_F \gamma^{(2)} x^{-1},
\]
(27)

with
\[
\gamma^{(2)} = 4x_2 \rho \ln 2,
\]
(28)
where $\chi_2$ and $\rho$ defined in \([17]\) and

$$\rho = \ln(w/\mu^2),$$ \hspace{1cm} (29)

with $\mu$ being an infrared cut-off. Contribution to $B$ of graph (b) in Fig. 2 is given by the following expression:

$$B^{2b} = \frac{C_F}{2} \chi_2 w \int d\alpha_{1,2} d\beta_{1,2} d\alpha_{1,2} \delta \left( \frac{N}{k_1^2 k_2^2} \right) \delta \left( (q + k_2)^2 \right) \delta \left( (k_1 - k_2)^2 \right) \delta \left( (p - k_1)^2 \right),$$ \hspace{1cm} (30)

where

$$N^{2b} = p_\mu p_\nu Tr \left[ \gamma_{\nu} \left( \hat{q} + \hat{k}_2 \right) \gamma_{\mu} k_2 \gamma_{\lambda'} \left( \hat{k}_1 - \hat{k}_2 \right) \gamma_{\sigma'} \hat{k}_2 \right] (p_\lambda k_1 + k_1 p_{1\sigma'}).$$ \hspace{1cm} (31)

Apart from the color factor $C_F/2$, the integrand in Eq. (30) coincides with the integrand of Eq. (16), so we obtain the same leading contribution:

$$B^{2b} \approx \left( C_F/2 \right) x^{-1} \gamma^{(2)},$$ \hspace{1cm} (32)

where $\gamma^{(2)}$ is given by Eq. (28). Our analysis of non-ladder graphs shows that they do not bring the factor $1/x$ because they do not contain $(k_2^2)^2$ in denominators. Therefore, the total leading contribution $B^{(2)}_q$ to $B_q$ in the second loop is

$$B^{(2)}_q = \left( C_F^2 + \frac{C_F}{2} \right) \gamma^{(2)} x^{-1}.$$ \hspace{1cm} (33)

Now let us consider some important technical details concerning Eqs. (27) (the same reasoning holds for Eq. (32)). This result stems from the terms in Eq. (18) where momenta $k_2$ are coupled with the external momenta $p$ and $q$. The other terms in Eq. (18) (i.e. the ones $\sim k_2^2$) either cancel $k_2^2$ in the denominator of Eq. (18), preventing appearance of the factor $1/x$, or cancel $1/k_1^2$, killing in $w$. Hence, the first step to calculate the trace in Eq. (18) can be reducing the trace down to $Tr[\hat{p}\hat{k}_2\hat{p}\hat{k}_2]$. Obviously, it corresponds to neglecting the factor $2pk_1$ in $\hat{k}_1\hat{p}\hat{k}_1$:

$$\hat{k}_1\hat{p}\hat{k}_1 = 2pk_1\hat{k}_1 - k_1^2\hat{p} \approx -k_1^2\hat{p}.$$ \hspace{1cm} (34)

This observation allows us to develop a strategy to select most important contributions to $B$ in arbitrary orders in $\alpha_s$. In other words, the non-singlet component of $F_L$ can be calculated in DLA in the straightforward way, without evolution equations.

### B. Contributions to $B$ for DIS off gluons

The second-loop contributions to the DIS off the initial gluon correspond to the ladder graphs (c,d) in Fig. 2. We calculate their contributions $B^g$ to $F_L$. Obviously,

$$B^{2c} = C^{(2)}_g \chi^{(2)} \int dz_{1,2} d\beta_{1,2} d\alpha_{1,2} \frac{N^{2c}}{k_1^2 k_2^2} \delta \left( (p - k_1)^2 \right) \delta \left( (k_1 - k_2)^2 \right) \delta \left( (q + k_2)^2 \right),$$ \hspace{1cm} (35)

where $\chi^{(2)}$ is defined in Eq. (17) and $C^{(2)}_g$ is the color factor:

$$C^{(2)}_g = \frac{1}{N} Tr[t^a t^b] N \delta_{ab} = NC_F.$$ \hspace{1cm} (36)

The term $N^{2c}$ is defined as follows:

$$N^{2c} = p_\mu p_\nu Tr \left[ \gamma_{\nu} \left( \hat{q} + \hat{k}_2 \right) \gamma_{\mu} \hat{k}_2 \gamma_{\lambda'} \left( \hat{k}_1 - \hat{k}_2 \right) \gamma_{\sigma'} \hat{k}_2 \right] H_{\lambda'\sigma'}.$$ \hspace{1cm} (37)
with

$$H_{\lambda'\sigma'} = H_{\lambda'\lambda\sigma}\rho_{\lambda\sigma}. \quad (38)$$

In Eq. (38) the notation $H_{\lambda'\lambda\sigma}$ stands for the ladder gluon rung while $\rho_{\lambda\sigma}$ denotes the gluon density matrix for the initial gluons which we treat as slightly virtual:

$$H_{\lambda'\lambda\sigma} = -\frac{(2k_1 - p_1)\lambda g_{\lambda'\tau} + (2p_1 - k_1)\lambda' g_{\lambda\tau} + (-k_1 - p_1)\tau g_{\lambda'\lambda}}{[2(k_1 - p_1)\sigma g_{\tau\tau} + (2p_1 - k_1)\sigma' g_{\sigma\tau} + (-k_1 - p_1)\sigma g_{\sigma\lambda}].} \quad (39)$$

The terms $\sim p_\lambda, p_\sigma$ in (39) can be dropped because of the gauge invariance. We use the Feynman gauge for the initial gluons:

$$\rho_{\lambda\sigma} = \frac{1}{2} g_{\lambda\sigma}. \quad (40)$$

As a result we obtain

$$H_{\lambda'\sigma'} = 8p_\lambda p_{\sigma'} - 4(p_\lambda k_{1\sigma'} + k_{1\lambda} p_{\sigma'}) + 2k_{1\lambda'} k_{1\sigma'} + 3 g_{\lambda'\sigma'} k_{1}^2. \quad (41)$$

We have used in the last term of (41) that $2pk_1 \approx k_1^2$. DL contributions to the gluon ladder come from the kinematics where $\lambda' \in R_L, \sigma' \in R_T$ or vice versa (The symbols $R_L$ and $R_T$ denote the longitudinal and transverse momentum spaces respectively). Therefore, the leading term in (39) in DLA is

$$H^{DL}_{\lambda'\sigma'} = -4(p_\lambda k_{1\sigma'} + k_{1\lambda} p_{\sigma'}) \quad (42)$$

while $2k_{1\lambda} k_{1\sigma'}$ brings corrections to it. The first term in (41) contain the longitudinal momenta only and the last term vanishes at $\lambda' \neq \sigma'$. Substituting (42) in (37) we obtain

$$N^{2c} = Tr \left[ \hat{p} \left( \hat{q} + \hat{k}_2 \right) \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_{1\perp} \hat{k}_2 \right] + Tr \left[ \hat{p} \left( \hat{q} + \hat{k}_2 \right) \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_2 \right] \quad (43)$$

$$\approx Tr \left[ \hat{p} \hat{q} \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_{1\perp} \hat{k}_2 \right] + Tr \left[ \hat{p} \hat{q} \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_2 \right]$$

$$= w Tr \left[ \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_{1\perp} \hat{k}_2 \right] + w Tr \left[ \hat{p} \hat{k}_2 \hat{p} \left( \hat{k}_1 - \hat{k}_2 \right) \hat{k}_2 \right]$$

Retaining in (43) the terms $\sim (pk_2)^2$ and $\sim (pk_2)^3$, we obtain the leading contribution to $N^{2c}_g$:

$$N^{2c} \approx 4(w + 2pk_2)(2pk_2)^2k_{1\perp}^2 = 4w^3(\alpha_2)^2k_{1\perp}^2 \quad (44)$$

which coincides with $N^{2c}_1$. Substituting $N^{2c}_g$ in (35), representing $B_g$ as

$$B^{2c} = C_g^{(2)} \chi^{(2)} I_g \quad (45)$$

and then integrating over $\alpha_2$, we arrive at

$$I_{g}^{(c)} = \int_\lambda^1 \frac{dz_1}{z_1} \int_\lambda^1 \frac{dz_2}{z_2^2} \int_\lambda^1 d\eta \left[ -\frac{\eta^3}{(l + \eta)^2} + \frac{\eta^2}{l(l + \eta)^2} \right] \quad (46)$$

with $z, z_1, z_2, l$ and $\eta$ defined in Eqs. (103) and (26) respectively. The integral in Eq. (46) coincides with the integral bringing the leading contribution to $B_q^{(c)}$ in (25), obtained for the quark ladder graph and calculated in Appendix A. So, we arrive at the leading contribution to $B$:
where we have used the gluon density matrix of Eq. (40). Retaining the terms with \( p_k \) containing \( \alpha \) and in the \( B \) with \( \eta \) growth.

Contribution \( B^{2d}_g \) to \( B_g \) of graph (d) in Fig. 2 is different color factors. Combining Eqs. (33, 53) with (10) demonstrate that \( F \) defined in Eq. (26). Comparison of (51) with Eq. (25) shows that the leading contribution, \( B^{2d}_L \) to \( B \) coincides with \( B^{(2a)}_q \):

\[
B^{2d}_q = B^{(2a)}_q = 4C_F^2 x^{-1} \gamma^{(2)}.
\]

Therefore, the total leading contribution \( B^{(2)}_g \) to \( B_g \) in the second loop is

\[
B^{(2)}_g = (C_F^2 + NC_F) \gamma^{(2)} x^{-1}.
\]

Eqs. (27, 32, 47) and (52) demonstrate explicitly that the only difference between leading contributions of all ladder graphs in Fig. 2 is different color factors. Combining Eqs. (33, 53) with (10) demonstrate that \( F_L \) in the \( \alpha^2 \)-order decreases at \( x \to 0 \) slower than the first-order result (11). Nevertheless, there are no growth of \( F_L \) in the \( \alpha^2 \)-order and in the \( \alpha^3 \)-order as shown in Ref. (12). It suggests that only all-order resummations can provide \( F_L \) with some growth.

**C. Remark on leading contributions of the ladder graphs in higher loops**

Contribution \( B^{(n)}_q \) of the quark ladder graph to \( B \) in the \( n^{\text{th}} \) order of the perturbative expansion can be written as follows:

\[
B^{(n)}_q = \chi_n C_n^q w^{n-1} \int dk_1^2 \ldots dk_n^2 \int d\alpha_1 d\beta_1 \ldots d\alpha_n d\beta_n \frac{N^{(n)}_q}{k_1^2 k_2^2 \ldots k_n^2} \delta ((q + k_n)^2) \delta ((k_n - k_{n-1})^2) \ldots \delta ((p - k_1)^2),
\]

with \( \gamma^{(2)} \) defined in Eq. (28). Now calculate contribution \( B^{2d}_g \) to \( B_g \) of graph (d) in Fig. 2. It is given by the following expression:

\[
B^{2d}_g = -C_F^2 \chi_2 \int dz_1 dz_1 \int d\alpha_1 d\beta_1 \delta ((p - k_1)^2) \delta ((k_1 - k_2)^2) \delta ((q + k_2)^2),
\]

where \( \chi_2 \) is defined in Eq. (17) and

\[
N^{2d} = \frac{1}{2} Tr \left[ \hat{p} \left( \hat{q} + \hat{k}_2 \right) \gamma_\rho k_1 \gamma_\lambda \left( \hat{k}_1 - \hat{p} \right) \gamma_\lambda k_1 \gamma_\rho k_2 \right] = 2 (w + 2pk_2) Tr \left[ \hat{p} \hat{k}_2 \hat{k}_1 \left( k_1 - \hat{p} \right) \hat{k}_1 \hat{k}_2 \right],
\]

where we have used the gluon density matrix of Eq. (40). Retaining the terms with \( pk_2 \) and neglecting other terms containing \( k_2 \), we obtain

\[
N^{2d} \approx 2 (w + 2pk_2) k_1^2 Tr \left[ \hat{p} \hat{k}_2 \hat{k}_1 \right] = 4 (w + 2pk_2) (2pk_2)^2 k_1^2.
\]

Substituting Eq. (50) in (48), introducing variables \( l, z_{1,2} \), then accounting for the \( \delta \)-functions, we arrive at

\[
B^{2d}_g \approx C_F^2 \chi_2 \int \frac{1}{\lambda} \frac{dz_1}{z_1} \int \frac{1}{\lambda} \frac{dz_2}{z_2} \int \frac{1}{\lambda} \frac{dl}{(l + \eta)^2} \left[ -\frac{z_1^2}{l^2} + \frac{z_2^2}{l} \right],
\]

with \( \eta \) defined in Eq. (26). Comparison of (51) with Eq. (25) shows that the leading contribution, \( B^{2d}_L \) to \( B \) coincides with \( B^{(2a)}_q \):

\[
B^{2d}_q = B^{(2a)}_q = 4C_F^2 x^{-1} \gamma^{(2)}.
\]
with

\[ \chi_n = 2\varepsilon^2 \left( -\frac{\alpha_s}{2\pi^2} \right)^n = 2\varepsilon^2 \left( -\frac{\alpha_s}{4\pi} \right)^n. \]

and

\[ N_q^{(n)} = \frac{1}{2} Tr \left[ \gamma_{\lambda_1} \hat{k}_1 ... \gamma_{\lambda_{n-1}} \hat{k}_{n-1} \gamma_{\lambda_n} \hat{k}_n \left( \hat{q} + \hat{k}_n \right) \hat{p} \hat{k}_n \gamma_{\lambda_n} \hat{k}_{n-1} \gamma_{\lambda_{n-1}} ... \hat{k}_1 \gamma_{\lambda_1} \hat{p} \right] \]

\[ = -(w + 2pk_n) Tr \left[ \hat{k}_1 ... \gamma_{\lambda_{n-1}} \hat{k}_{n-1} \gamma_{\lambda_n} \hat{k}_n \hat{p} \hat{k}_n \gamma_{\lambda_n} \hat{k}_{n-1} \gamma_{\lambda_{n-1}} ... \hat{k}_1 \hat{p} \right]. \tag{56} \]

We have used in (56) the quark density matrix given by Eq. (22). We are going to calculate \( B_q^{(n)} \) in DLA. In order to select appropriate contributions in the trace in (56), we generalize the approximation of Eq. (34) to \( k_i \), with \( i = 1, 2, ..., n - 1 \):

\[ \hat{k}_i \hat{p} \hat{k}_i = 2pk_i \hat{k}_i - k_i^2 \hat{p} \approx -k_i^2 \hat{p}. \tag{57} \]

Doing so we arrive at the DL contribution \( N_q^{DL} \):

\[ N_q^{DL} = (-2)^{n-1}k_1^2 ... k_{n-1}^2 (w + 2pk_n) Tr[\hat{p} \hat{k}_n \hat{p} \hat{k}_n] \approx 2^{n-1}k_1^2 ... k_{n-1}^2 Tr[\hat{p} \hat{k}_n \hat{p} \hat{k}_n]. \tag{58} \]

Substituting (56), we arrive at \( B_q^{(n)} \) in DLA. The integration region in DLA was found in [27]:
\[ B_{q,g}(x, Q^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (w/\mu^2)^\omega f_{q,g}(\omega, Q^2), \]

with \( \rho \) defined in Eq. (29). The transform inverse to (60) is

\[ f_{q,g}(\omega, Q^2) = \frac{1}{\omega} \int_{\mu^2}^{\infty} \frac{dw}{w} (w/\mu^2)^{-\omega} B_{q,g}(w, Q^2), \]

The same transforms we will use for amplitude \( A \) in Sect. VI. It is convenient to use beyond the Born approximation the logarithmic variables

\[ \rho = \ln (w/\mu^2), \quad y = \ln (Q^2/\mu^2). \]

We have used the mass scale \( \mu \) of Eq. (11) as an IR cut-off for simplicity reason, in order to avoid introducing extra parameters. One can choose another IR cut-off. Throughout the paper we will address the logarithmic variables

\[ \ln \omega \rightarrow \ln \left( \frac{w}{\mu^2} \right), \quad y \rightarrow \ln \left( \frac{Q^2}{\mu^2} \right). \]

In order to specify \( C_{ik} \) with \( i, k = q, g \) can be found in Appendix B. Eqs. (63) exhibit a certain similarity to the DGLAP equations. Indeed, the l.h.s. of Eqs. (63) are derivatives with respect to \( \ln Q^2 \) while the r.h.s. involve anomalous dimensions \( h_{ik} \). The difference between Eqs. (63) and DGLAP equations is that all anomalous dimensions in Eqs. (63) are calculated in DLA whereas the DGLAP equations operate with the anomalous dimensions calculated in several fixed orders in \( \alpha_s \). General solution to Eq. (63) is:

\[ f_q(\omega, y) = e^{-\omega y} \left[ C_{(+)} e^{\Omega_y} + C_{(-)} e^{-\Omega_y} \right], \]

\[ f_g(\omega, y) = e^{-\omega y} \left[ C_{(+)} \frac{h_{gg} - h_{qq} + \sqrt{R}}{2 h_{gg}} e^{\Omega_y} + C_{(-)} \frac{h_{gg} - h_{qq} - \sqrt{R}}{2 h_{gg}} e^{-\Omega_y} \right], \]

where \( C_{(\pm)}(\omega) \) are arbitrary factors whereas \( \Omega_{(\pm)} \) are expressed through \( h_{ik} \):

\[ \Omega_{(\pm)} = \frac{1}{2} \left[ h_{gg} + h_{qq} \pm \sqrt{R} \right], \]

with

\[ R = (h_{gg} + h_{qq})^2 - 4(h_{qq}h_{gg} - h_{gg}h_{qq}) = (h_{gg} - h_{qq})^2 + 4h_{gg}h_{qq}. \]

In order to specify \( C_{(\pm)}(\omega) \), we use the matching:

\[ f_q(\omega, y)|_{y=0} = \tilde{f}_q, \quad f_g(\omega, y)|_{y=0} = \tilde{f}_g, \]

where \( \tilde{f}_{q,g} \) correspond again to \( B_{q,g} \) but when the photon is (nearly) on-shell. They have to be found independently. Combining Eqs. (67) and (64) lead us to the algebraic system:

\[ \tilde{f}_q = C_{(+)} + C_{(-)}, \]

\[ \tilde{f}_g = C_{(+)} \left( \frac{h_{gg} - h_{qq} + \sqrt{R}}{2 h_{gg}} \right) + C_{(-)} \left( \frac{h_{gg} - h_{qq} - \sqrt{R}}{2 h_{gg}} \right), \]
which makes possible to express $C_{(\pm)}$ through $\tilde{f}_{1,2}$:

$$C_{(+)} = \frac{-\tilde{f}_q \left( h_{gg} - h_{qq} - \sqrt{R} \right) + \tilde{f}_g 2 h_{qq}}{2 \sqrt{R}},$$

$$C_{(-)} = \frac{\tilde{f}_q \left( h_{gg} - h_{qq} + \sqrt{R} \right) - \tilde{f}_g 2 h_{qq}}{2 \sqrt{R}}.$$  \hfill (69)

The next step is to calculate $\tilde{f}_{1,2}$. They can be found through constructing and solving IREEs for them. The IREEs for $\tilde{f}_{1,2}$ are

$$g_q + h_{qq}(\omega)\tilde{f}_q(\omega) + (\omega + h_{qq}(\omega))\tilde{f}_q(\omega) = 0,$$

$$g_g + (\omega + h_{qq}(\omega))\tilde{f}_q(\omega) + h_{gg}(\omega)\tilde{f}_g(\omega) = 0,$$  \hfill (70)

where $g_{q,g}$ stand for the inputs. Solution to Eq. (70) is

$$\tilde{f}_q = \frac{-g_q(h_{gg} - \omega) + g_g h_{qq}}{\Delta},$$

$$\tilde{f}_g = \frac{g_a g_{qq} - g_q(h_{qq} - \omega)}{\Delta},$$  \hfill (71)

where

$$\Delta = (\omega - h_{qq})(\omega - h_{qq}) - h_{qq} h_{gg}.$$  \hfill (72)

Substituting (71) in (69) allows us to represent $C_{(\pm)}$ through $h_{ik}$ and inputs $g_{q,g}$. We write $C_{(\pm)}$ in the following form:

$$C_{(+)} = g_q G_q^{(+)} + g_g G_g^{(+)},$$

$$C_{(-)} = g_q G_q^{(-)} + g_g G_g^{(-)},$$  \hfill (73)

where

$$G_q^{(+)} = \frac{(h_{gg} - \omega) \left( h_{gg} - h_{qq} - \sqrt{R} \right) + 2 h_{qq} h_{gg}}{2 \Delta \sqrt{R}},$$

$$G_g^{(+)} = \frac{-h_{gg} \left( h_{gg} - h_{qq} - \sqrt{R} \right) - 2 h_{gg} (h_{qq} - \omega)}{2 \Delta \sqrt{R}},$$

$$G_q^{(-)} = \frac{-(h_{qq} - \omega) \left( h_{gg} - h_{qq} + \sqrt{R} \right) - 2 h_{qq} h_{gg}}{2 \Delta \sqrt{R}},$$

$$G_g^{(-)} = \frac{h_{gg} \left( h_{gg} - h_{qq} + \sqrt{R} \right) + 2 h_{qq} (h_{qq} - \omega)}{2 \Delta \sqrt{R}}.$$  \hfill (74)

Combining Eqs. (71), (73) and (74) leads to expressions for $f_{q,g}$ in terms of $h_{ik}$ and $g_{q,g}$. We remind that explicit expressions for $h_{ik}$ can be found in Appendix B. They are known in DLA for both spin-dependent DIS structure function $g_1$ (see Ref. [23]) and for $F_1$ as well (see Ref. [22]). The other ingredients in Eq. (73) are inhomogeneous terms $g_{q,g}$. They first appeared in Eq. (70). Let us compare Eq. (70) for $\tilde{f}_{q,g}(\omega)$ and Eq. (63) for $f_{q,g}(\omega, y)$. The first difference between them is that Eq. (70) does not contain the derivative $\partial / \partial y$. The reason is that $\tilde{f}_{q,g}$ do not depend on $y$. The second difference is the presence of terms $g_q$ and $g_g$ in (70). These terms stand for the inputs, i.e. for the starting point of the evolution. Specifying them is necessary for obtaining explicit expressions for $f_{q,g}$. Below we consider this issue in detail.
V. SPECIFYING INPUTS $g_q$ AND $g_g$ FOR AMPLITUDES $B_{q,g}$

Similarly to other evolution equations, IREEs evolve inputs. These inputs are usually defined as the Born contributions. For instance, it is true for amplitude $A_q$ defined in Eq. (3). Specifying an input $g_q^A$ for amplitude $A_q$ was done in Ref. [22]. However, such choice is not optimal for amplitude $B$ defined in Eq. (4). In what follows we firstly remind how the input $g_q^A$ was specified and then proceed to specifying inputs $g_q^B$ for amplitude $B$.

A. Input for amplitude $A_q$

Amplitude $A_q$ was calculated with DL accuracy in Ref. [22]. The evolution in this case starts with the Born contribution:

$$A_q^{(Born)} = \delta(x - 1),$$  \hspace{1cm} (75)

where we have neglected the quark mass. $A_q^{(Born)}$ in the region (1) does not depend on $\mu$, so it vanishes when differentiated over $\mu$. It is the reason why inhomogeneous terms are absent in IREEs for $f_{q,g}(x,Q^2)$. In the case of smaller $Q^2$, where

$$Q^2 = \mu^2$$  \hspace{1cm} (76)

$$\tilde{A}_q^{(Born)} = \delta(1 - \mu^2/w).$$  \hspace{1cm} (77)

Applying Eq. (61) to $\tilde{A}_q^{(Born)}$, we obtain

$$(\tilde{f}_q^A)^{Born} = 1/\omega,$$  \hspace{1cm} (78)

which leads immediately to

$$g_q^A \equiv \omega(f_q^A)^{Born} = 1.$$  \hspace{1cm} (79)

B. Inputs for amplitudes $B_q, B_g$

It is impossible to evolve amplitudes $B_{q,g}$ from their Born values onwards because $(B_q)_{Born} = (B_g)_{Born} = 0$. Then, one could try to use the first-loop contributions as inputs. However, the IR-evolution, by definition, deals with logarithms, which cannot generate the leading second-loop contributions $B_{q,g}^{(2)}$ which are $\sim 1/x$. On the other hand, in Sect. II we demonstrated that these factors, having appeared in the second loop, hold in all higher orders while logarithmic contributions are controlled by the evolution. It prompts us to suggest that $B_{q,g}^{(2)}$ should be used as the inputs in IREEs for $B_{q,g}$ while the Born and the first-loop contributions should be added by hands. Both of $B_{q,g}$ are $\sim 1/x$. In order to write it explicitly we write $B_{q,g}$ as follows:

$$B_q^{(2)} = b_q^{(2)}/x,$$  \hspace{1cm} (80)

$$B_g^{(2)} = b_g^{(2)}/x.$$  \hspace{1cm} (81)

Therefore, we suggest that

$$g_q = b_q^{(2)}/x,$$  \hspace{1cm} (82)

$$g_g = b_g^{(2)}/x.$$  \hspace{1cm} (83)

Strictly speaking, accepting this suggestion takes us out of the conventional form of DLA, where Born amplitudes have been invariably considered as the starting point of evolution.
VI. BEHAVIOUR OF $B_{q,g}$ AT SMALL $x$

Eq. (73) is linear in $g_{q,g}$. In order to extract the overall factor $1/x$ we re-define $C_\pm$ participating in (64). Namely, we introduce $C_\pm'$ so that

$$C_\pm = x^{-1}C_\pm'. \quad (82)$$

$C_\pm'$ obey the system

$$C_\pm' = b_q^{(2)}G_q^{(\pm)} + b_g^{(2)}G_g^{(\pm)}, \quad (83)$$

with $G_{q,g}^{(\pm)}$ defined in Eq. (74). Using this form for $C_\pm'$, substituting Eq. (81) in (73), then proceeding to (64) and (60), we obtain explicit expressions for $B_{q,g}$. Using Eq. (6), we arrive at the following expressions for $F_L^{(q,g)}$:

$$F_L^{(q)} = 4x \int^{\infty}_{-\infty} \frac{d\omega}{2\pi i} x^{-\omega} \left[ C_\pm' e^{\Omega_\pm y} \right], \quad (84)$$

$$F_L^{(g)} = 4x \int^{\infty}_{-\infty} \frac{d\omega}{2\pi i} x^{-\omega} \left[ C_\pm' e^{\Omega_\pm y} \right].$$

The overall factor $4x$ at Eq. (84) is the product of the factor $4x^2$ of Eq. (6) and the factor $1/x$ appearing when $C_\pm$ are replaced by $C_\pm'$. Eq. (84) represents the contributions to $F_L^{(q,g)}$ most essential at small $x$ only. For instance, it does not include the first-loop contribution decreasing $\sim x^2$ at small $x$ (see (11)). Despite the small factors $x$ at the integrals in Eq. (84), actually both $F_L^{(q)}$ and $F_L^{(g)}$ rise when $x$ is decreasing, albeit this does not look obvious. In order to make it seen clearly we consider below the small-$x$ asymptotics of $F_L^{(q,g)}$, which look much simpler than the parent expressions in Eq. (84).

A. Small- $x$ asymptotics of $B_{q,g}$

At $x \to 0$, $F_L^{(q,g)}$ can be approximated by their small-$x$ asymptotics which we denote $\left( F_L^{(q,g)} \right)_{AS}$. Technology of calculating the asymptotics is based on the saddle-point method and the whole procedure is identical to the one for $F_1$. So, we can use the appropriate results of Ref. [22]. After the asymptotics of $F_L^{(q,g)}$ have been calculated and convoluted with the parton distributions $\Phi_{q,g}$ (see Eq. (7)), the small-$x$ asymptotics of $F_L$ is obtained:

$$(F_L)_{AS} \sim \frac{\Pi}{\ln^{1/2}(1/x)} x^{1-\omega_0} \left( Q^2/\mu^2 \right)^{\omega_0/2}, \quad (85)$$

where the factor $\Pi$ includes both numerical factors of perturbative origin and values of the quark and gluon distributions in the $\omega$-space at $\omega = \omega_0$. In any form of QCD factorization $\Pi$ does not contain any dependency on $Q^2$ or $x$ (see [22] for detail). Then, $\omega_0$ is the Pomeron intercept calculated with DL accuracy. This intercept was first calculated in Ref. [22]. We remind that it has nothing in common with the BFKL intercept. It is convenient to represent $\omega_0$ as follows:

$$\omega_0 = 1 + \Delta^{(DL)}. \quad (86)$$

Numerical estimates for $\Delta^{(DL)}$ depend on accuracy of calculations. When $\alpha_s$ is assumed to be fixed$^2$.

$^2$ we use here the value $\alpha_s = 0.24$ according to prescription of Ref. [23]
\[ \Delta_{f_{1x}}^{(DL)} = 0.29 \]  
and

\[ \Delta^{(DL)} = 0.07, \]  
when the \( \alpha_s \) running effects are accounted for. Substituting either (87) or (88) in Eq. (85), one easily finds that \( F_L \sim x^{-\Delta^{(DL)}} \) at \( x \to 0 \). The asymptotics of \( F_1 \) was calculated in Ref. [22] showed that asymptotically \( F_1 \sim x^{-\omega_0} \) and therefore \( F_L \sim 2xF_1 \).

The growth of \( F_L \) and \( xF_1 \) at small \( x \) is caused by the Pomeron behaviour of the parton-parton amplitudes \( f_{ik} = 8\pi^2 h_{ik} \sim x^{-\omega_0} \). Amplitudes \( f_{gg} \) and \( f_{gq} \), being convoluted with \( \Phi_g \) and \( \Phi_q \), form the gluon distribution in the initial hadron, which we denote \( G_h \):

\[
G_h = h_{gg} \otimes \Phi_g + h_{gq} \otimes \Phi_q.
\]  
(89)

So, at small \( x \)

\[
F_L \sim xG_h.
\]  
(90)

Another interesting observation following from Eq. (85) is that

\[
2 \frac{\partial \ln F_L}{\partial \ln Q^2} + \frac{\partial \ln F_L}{\partial \ln x} \to 1
\]

(91)
at \( x \to 0 \). We think that it would be interesting to check this relation with analysis of available experimental data.

To conclude discussion of the asymptotics, we notice that the asymptotics (85) should be used within its applicability region, otherwise one should use the expressions of Eq. (84). The estimate obtained in Ref. [22] states that (85) can be used at \( x \leq 10^{-6} \).

### B. Remark on \( F_L \) at arbitrary \( Q^2 \)

The expressions in Eq. (84) are valid in the kinematic region (1) where \( Q^2 \) is large. However, it is easy to generalize Eq. (84) to small \( Q^2 \). It was proved in Refs. [22, 24] that such a generalization is achieved with replacement of \( Q^2 \) by \( Q^2 + \mu^2 \). When this shift has been done, \( F_L^{(q)} \) and \( F_L^{(g)} \) of Eq. (84) depend on new variables \( \bar{x}, \bar{Q}^2 \):

\[
\bar{Q}^2 = Q^2 + \mu^2, \quad \bar{x} = \bar{Q}^2/w.
\]  
(92)

Thus, one can universally apply \( F_L(\bar{x}, \bar{Q}^2) \) at both large and small \( Q^2 \).

### VII. CONCLUSIONS

Our results predict that \( F_L \) grows at small \( x \) despite the very small factor \( x^2 \) at \( B \) in Eq. (6). First, we re-calculated with logarithmic accuracy the available in the literature second-loop contributions \( B^{(2)}_q \) and \( B^{(2)}_g \), each contains the large power factor \( 1/x \) in contrast to the Born and first-loop contributions. This calculation allowed us to conclude that \( 1/x \) will be present in higher-loop expressions and cannot disappear or be replaced by another power factor. We demonstrated that most important contributions coming from higher orders are double logarithms. Accounting for DL contributions to all orders in \( \alpha_s \), we calculated the \( x \) and \( Q^2 \)-evolution of \( B^{(2)}_{q,g} \) in DLA. This evolution proved to be similar to the evolution of the structure function \( F_1 \). Eventually we obtained Eq. (84) for the partonic components \( F^{(q)}_L \) and \( F^{(g)}_L \) of \( F_L \). The both these components rise at small \( x \) though complexity of expressions in Eq. (84) prevents to see the rise. To make the rise be clearly seen, we calculated the small-\( x \) asymptotics of \( F_L \), which proved to be of the Regge type. The asymptotics make obvious that the synergic effect of the factor \( 1/x \) and the total resummation...
of double logarithms overcomes smallness of the factor $x^2$ at $B$ in Eq. (6) and ensures the rise of $F_L$ at small $x$, see Eq. (85). Then in Eq. (90) we noticed that the rise of $F_L$ and the gluon distributions in the hadrons at small $x$ are identical. We also suggested in Eq. (91) the simple relation between derivatives of logarithm of $F_L$. This relation could be checked with analysis of experimental data, so such check could test correctness of our reasoning. It was presumed in IREEs considered in Sect. V that $Q^2 \geq \mu^2$. However, it is easy to extend the expressions in Eq. (84) to the region $Q^2 < \mu^2$, applying to them the results of Ref. [22] obtained for the structure function $F_1$ at small $Q^2$.

VIII. ACKNOWLEDGEMENT

We are grateful to V. Bertone, N.Ya. Ivanov and Yuri V. Kovchegov for useful communications.

IX. APPENDIX

A. Integration in Eq. (25)

We write Eq. (25) in the following form:

$$B^{2a} \approx 4C^2_\chi^2 \chi^2 \left[ I^{2a}_{1} + I^{2a}_{2} \right],$$

(93)

with $I^{2a}_{1,2}$ defined as integrals over the transverse momenta $z_1$:

$$I^{2a}_{1} = \int_\lambda^{1} \frac{dz_1}{z_1} J^{2a}_{1},$$

(94)

$$I^{2a}_{2} = \int_\lambda^{1} \frac{dz_1}{z_1} J^{2a}_{2},$$

where $J^{2a}_{1,2}$ involve integration over $z_2$:

$$J^{2a}_{1} = \int_\lambda^{1} \frac{dz_2}{z_2} \frac{z_3^3}{z_2^2} \tilde{J}^{2a}_{1},$$

(95)

$$J^{2a}_{2} = \int_\lambda^{1} \frac{dz_2}{z_2} \frac{z_2^2}{z_2^2} \tilde{J}^{2a}_{2}.$$ 

Integrals $\tilde{J}^{2a}_{1,2}$ deal with integration over the longitudinal variable $l$:

$$\tilde{J}^{2a}_{1} = -\int_z^{1} \frac{dl}{l^2(l + \eta)^2},$$

(96)

$$\tilde{J}^{2a}_{2} = \int_z^{1} \frac{dl}{l((l + \eta)^2},$$

with $\eta$ defined in Eq. (26). Integration over $l$ in (96) yields

$$\tilde{J}^{2a}_{1} = \frac{1}{\eta^2} \left( 1 - \frac{1}{z} \right) - \frac{2}{\eta^2} \ln \left( \frac{1 + \eta}{z + \eta} \right) + \frac{1}{\eta^3} \left[ \frac{1}{1 + \eta} - \frac{1}{z + \eta} \right],$$

(97)

$$\tilde{J}^{2a}_{2} = \frac{1}{\eta^2} \left[ -\ln(1 + \eta) - \ln \left( \frac{(z + \eta)/z}{1 + \eta} \right) + \frac{\eta}{1 + \eta} - \frac{\eta}{z + \eta} \right],$$

and therefore
\[ J_1^{2a} = \int_{\lambda}^{1} dz_2 \left[ \frac{z - 1}{(z_2 + x)^2} - \frac{z_2}{(z_2 + x)^3} \ln U(z, z_2) \right. \]
\[ + \frac{z_2}{(z_2 + x)^3} \ln(2z_2 + x) + \frac{z_2}{(z_2 + x)^3} U(z, z_2) - \frac{z_2}{(z_2 + x)^3} (2z_2 + x) \left. \right] \]
\[ J_2^{2a} = \int_{\lambda}^{1} dz_2 \frac{1}{(z_2 + x)^2} \left[ \ln(2z_2 + x) - \ln U(z, z_2) - \frac{z_2}{U(z, z_2)} + \frac{z_2}{2z_2 + x} \right], \]

where

\[ U(z, z_2) = z_2 + z(z_2 + x). \]  

(99)

It is convenient to perform integration in (98), using the variable \( y = 1/(z_2 + x) \) instead of \( z_2 \). The most essential contributions in (98) at small \( x \) are the ones \( \sim 1/x \). Accounting for them only, we obtain

\[
J_1^{2a} = x^{-1} [\ln 2 - 1/2],
J_2^{2a} = x^{-1} (1/2).
\]

(100)

Substituting this result in (94), we obtain

\[ I_2^{2a} + I_2^{2a} = (\rho \ln 2) x^{-1}. \]

(101)

**B. Expressions for \( h_{ik} \)**

\[
\begin{align*}
  h_{qq} &= \frac{1}{2} \left[ \omega - Z - \frac{b_{qq} - b_{gg}}{Z} \right], \quad h_{gg} = \frac{b_{gg}}{Z}, \\
  h_{qq} &= \frac{1}{2} \left[ \omega - Z + \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{gg} = \frac{b_{gg}}{Z},
\end{align*}
\]

(102)

where

\[ Z = \frac{1}{\sqrt{2}} \sqrt{Y + W}, \]

(103)

with

\[ Y = \omega^2 - 2(b_{qq} + b_{gg}) \]

(104)

and

\[ W = \sqrt{\omega^2 - 2(b_{qq} + b_{gg})^2 - 4(b_{qq} - b_{gg})^2 - 16b_{gg}b_{gg}}, \]

(105)

where the terms \( b_{rr} \) include the Born factors \( a_{rr} \) and contributions of non-ladder graphs \( V_{rr} \):

\[ b_{rr} = a_{rr} + V_{rr}. \]

(106)

The Born factors are (see Ref. [24] for detail):

\[ a_{qq} = \frac{A(\omega)C_F}{2\pi}, \quad a_{gg} = \frac{A'(\omega)C_F}{\pi}, \quad a_{qq} = -\frac{A'(\omega)n_f}{2\pi}, \quad a_{gg} = \frac{2N\alpha(\omega)}{\pi}, \]

(107)

where \( A \) and \( A' \) stand for the running QCD couplings:
\[ A = \frac{1}{b} \left[ \frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty \frac{dze^{-\omega z}}{(z + \eta)^2 + \pi^2} \right], \quad A' = \frac{1}{b} \left[ 1 - \int_0^\infty dze^{-\omega z} \right], \]  

with \( \eta = \ln \left( \frac{\mu^2}{\Lambda_{QCD}^2} \right) \) and \( b \) being the first coefficient of the Gell-Mann-Low function. When the running effects for the QCD coupling are neglected, \( A(\omega) \) and \( A'(\omega) \) are replaced by \( \alpha_s \). The terms \( V_{rr} \) approximately represent the impact of non-ladder graphs on \( h_{rr} \) (see Ref. [24] for detail):

\[ V_{rr} = \frac{m_{rr}}{\pi^2} D(\omega), \quad V_{rr'} = \frac{m_{rr'}}{\pi^2} D'(\omega), \]

with

\[ m_{qq} = \frac{C_F}{2N}, \quad m_{gg} = -2N^2, \quad m_{qg} = \frac{n_f}{2}, \quad m_{gq} = -NC_F, \]

and

\[ D(\omega) = \frac{1}{2b^2} \int_0^\infty dze^{-\omega z} \ln \left( \frac{\omega}{\eta} \right) \left[ \frac{z + \eta}{(z + \eta)^2 + \pi^2} - \frac{1}{z + \eta} \right]. \]

Let us note that \( D = 0 \) when the running coupling effects are neglected. It corresponds the total compensation of DL contributions of non-ladder Feynman graphs to scattering amplitudes with the positive signature as was first noticed in Ref. [29]. When \( \alpha_s \) is running, such compensation is only partial.

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