Logarithmic vector fields along smooth divisors in projective spaces

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Abstract

We show that a smooth divisor in a projective space can be reconstructed from the isomorphism class of the sheaf of logarithmic vector fields along it if and only if its defining equation is of Sebastiani–Thom type.

1 Introduction

Let $D$ be a smooth divisor in $\mathbb{P}^n$ defined by a homogeneous polynomial $f$ of degree $k$. We say that $f$ is of Sebastiani–Thom type if $f$ can be represented as the sum

$$f(x_0, \ldots, x_n) = f_1(x_0, \ldots, x_l) + f_2(x_{l+1}, \ldots, x_n)$$

for a choice of a homogeneous coordinate $(x_i)_{i=0}^n$ of $\mathbb{P}^n$ and some $0 \leq l \leq n-1$.

We study the Torelli problem for logarithmic vector fields in the sense of Dolgachev and Kapranov [1]. For a divisor $D$ in the projective space $\mathbb{P}^n$, the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ of logarithmic vector fields along $D$ is defined as the subsheaf of the tangent sheaf $\mathcal{T}_{\mathbb{P}^n}$ whose section consists of vector fields tangent to $D$. It is the sheafification of

$$D_0(- \log f) = \{ \delta \in \text{Der}_A \mid \delta f = 0 \},$$

where $A$ is a homogeneous coordinate ring of $\mathbb{P}^n$ and $f \in A$ is the defining polynomial of $D$. A divisor $D$ is said to be Torelli if the isomorphism class of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ as an $\mathcal{O}_{\mathbb{P}^n}$-module determines $D$ uniquely. The main theorem of Dolgachev and Kapranov [1] is a condition for an arrangement of sufficiently many hyperplanes to be Torelli.

The main result in this paper is the following:

Theorem 1. A smooth divisor in a projective space is Torelli if and only if its defining equation is not of Sebastiani–Thom type.
The strategy for the proof is the following:

1. The Jacobi ideal of a smooth divisor $D$ of degree $k$ is determined by the set of divisors $E$ of degree $k - 1$ such that the dimension of $H^0(T_{\mathbb{P}^n}(-\log D)(-1)|_E)$ jumps up.

2. A smooth divisor is determined by its Jacobi ideal if and only if it is not of Sebastiani–Thom type.

3. A divisor $D$ is not Torelli if its defining equation is of Sebastiani–Thom type.

As a corollary of Theorem 1, we give another proof of the main theorem of [2] that a smooth plane cubic curve is Torelli if and only if its $j$-invariant does not vanish.

2 Jacobi ideals from logarithmic vector fields

Let $D$ be a smooth divisor of degree $k$ in $\mathbb{P}^n$ defined by a homogeneous polynomial $f$, and $T_{\mathbb{P}^n}(-\log D) \subset T_{\mathbb{P}^n}$ be the sheaf of logarithmic vector fields along $D$. We have an exact sequence

$$0 \to T_{\mathbb{P}^n}(-\log D) \to T_{\mathbb{P}^n} \to N_{D/\mathbb{P}^n} \to 0,$$

where $N_{D/\mathbb{P}^n}$ is the normal bundle of $D$ in $\mathbb{P}^n$, and an isomorphism

$$df : N_{D/\mathbb{P}^n} \sim \bigcup \cup X \mapsto Xf$$

of $\mathcal{O}_{\mathbb{P}^n}$-modules. By the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to T_{\mathbb{P}^n}(-1) \to 0,$$

the space $H^0(T_{\mathbb{P}^n}(-1))$ of global sections of $T_{\mathbb{P}^n}(-1)$ is spanned by $\{\partial / \partial x_i\}_{i=0}^n$. The image of the map

$$H^0(T_{\mathbb{P}^n}(-1)) \to H^0(\mathcal{O}_D(k - 1))$$

induced by the composition

$$T_{\mathbb{P}^n}(-1) \to N_{D/\mathbb{P}^n}(-1) \to \mathcal{O}_D(k - 1)$$

is the restriction to $D$ of the degree $k - 1$ part

$$J(f)_{k-1} = \text{span}\{\partial f / \partial x_i\}_{i=0}^n$$

of the Jacobi ideal $J(f)$ of $f$. 

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Lemma 2. For a divisor $E$ of $\mathbb{P}^n$ of degree $k - 1$, the dimension of $H^0(T_{\mathbb{P}^n}(- \log D)(-1)_{|E})$ jumps up if and only if the defining equation of $E$ is contained in the Jacobi ideal of $D$.

Proof. Since $D$ is smooth,

$$\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_D, \mathcal{O}_E) = 0$$

and we have an exact sequence

$$0 \to T_{\mathbb{P}^n}(- \log D)(-1)|_E \to T_{\mathbb{P}^n}(-1)|_E \to \mathcal{O}_{D \cap E}(k - 1) \to 0,$$

from which follows the long exact sequence

$$0 \to H^0(T_{\mathbb{P}^n}(- \log D)(-1)|_E) \to H^0(T_{\mathbb{P}^n}(-1)|_E) \to H^0(\mathcal{O}_{D \cap E}(k - 1)) \to H^1(T_{\mathbb{P}^n}(- \log D)(-1)|_E) \to \cdots.$$

Note that the image of the map

$$H^0(T_{\mathbb{P}^n}(-1)|_E) \to H^0(\mathcal{O}_{D \cap E}(k - 1))$$

is the restriction to $D \cap E$ of the degree $k - 1$ part of the Jacobi ideal of $D$. Since the dimension of $H^0(T_{\mathbb{P}^n}(-1)|_E)$ does not depend on $E$, the dimension of $H^0(T_{\mathbb{P}^n}(- \log D)(-1)|_E)$ jumps up if and only if the defining equation of $E$ is contained in the Jacobi ideal of $D$. 

3 Divisors from their Jacobi ideals

We prove the following in this section:

Lemma 3. If two smooth distinct divisors in $\mathbb{P}^n$ have identical Jacobi ideals, their defining equations are of Sebastiani–Thom type.

Proof. We divide the proof into steps. Let $f$ and $g$ be homogeneous polynomials of degree $k$ defining distinct smooth hypersurfaces such that their Jacobi ideals $J(f)$ and $J(g)$ are identical.

Step 1. The pencil over $f$ and $g$ contains a polynomial $F$ such that $\partial_0 F = \cdots = \partial_l F = 0$ and $\{\partial_i F\}_{i=l+1}^n$ is linearly independent for some integer $l$ and a suitable choice of a homogeneous coordinate $(x_i)_{i=0}^n$ of $\mathbb{P}^n$.

Indeed, any pencil of projective hypersurfaces contains a singular element $F$, and the assumption $J(f) = J(g)$ implies that $\partial_0 F, \ldots, \partial_l F$ are linearly dependent. Let $l$ be $n$ minus the dimension of the linear span of $\{\partial_i F\}_{i=0}^n$. Then we can choose a homogeneous coordinate so that $\partial_i F = 0$ for $i = 0, \ldots, l$ and $\{\partial_i F\}_{i=l+1}^n$ is linearly independent. Note that one has $l < n$ since the divisors defined by $f$ and $g$ are distinct.
Step 2. There exists a matrix \((a_{ij})_{i,j=0}^n\) such that \(\det(a_{ij})_{i,j=l+1}^n \neq 0\) and
\[
\frac{\partial F}{\partial x_i} = \sum_{j=0}^n a_{ij} \frac{\partial f}{\partial x_j}
\]
for \(i = 0, \ldots, n\).

The existence of the matrix \((a_{ij})_{i,j=0}^n\) follows from the inclusion \(J(F) \subset J(f)\). We will show that if \(\det(a_{ij})_{i,j=l+1}^n\) vanishes, then the hypersurface defined by \(f\) is singular. Indeed, the vanishing of \(\det(a_{ij})_{i,j=l+1}^n\) and linear independence of \(\{\partial_i F\}_{i=l+1}^n\) imply that some linear combination of \(\{\partial_i f\}_{i=0}^n\) is a linear combination of \(\{\partial_i F\}_{i=l+1}^n\). Then one can choose a homogeneous coordinate so that \(\partial_0 f\) is a linear combination of \(\{\partial_j F\}_{j=l+1}^n\) and \(\{\partial_i f\}_{i=0}^n\) implies that some linear combination of \(\{\partial_j F\}_{j=l+1}^n\) is a linear combination of \(\{\partial_i F\}_{i=l+1}^n\). Assume that \(\deg f \geq 2\), since any linear form is of Sebastiani–Thom type. Note that \(F\) does not depend on \(\{x_i\}_{i=0}^l\) since \(\partial_i F = 0\) for \(i = 0, \ldots, l\). It follows that \([1 : 0 : \cdots : 0] \in \mathbb{P}^n\) is a singular point of the hypersurface defined by \(f\), since \(f(x_0, \ldots, x_n)\) is the sum of \(x_0\) times some linear combination of \(\{\partial_j F\}_{j=l+1}^n\) and terms which are zero at \(x_1 = \cdots = x_n = 0\).

Step 3. There is a homogeneous coordinate \((X_i)_{i=0}^n\) such that \(\partial_i F = 0\) for \(i = 0, \ldots, l\) and \(\partial_i f \in J(F)\) for \(i = l+1, \ldots, n\).

Since the \((n-l) \times (n-l)\) matrix \((a_{ij})_{i,j=l+1}^n\) is invertible, one can find an \((n-l) \times (n+1)\) matrix \((b_{ij})\) such that
\[
\sum_{j=0}^l b_{ij} \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_i} \in J(F)
\]
for \(i = l+1, \ldots, n\). Now make the projective coordinate transformation from \((x_i)_{i=0}^n\) to \((X_i)_{i=0}^n\) defined by
\[
x_j = \begin{cases} 
X_j + \sum_{i=l+1}^n b_{ij} X_i & 0 \leq j \leq l, \\
X_j & l+1 \leq j \leq n.
\end{cases}
\]

Then one has
\[
\frac{\partial f}{\partial X_i} = \sum_{j=0}^n \frac{\partial x_j}{\partial X_i} \frac{\partial f}{\partial x_j} = \begin{cases} 
\frac{\partial f}{\partial x_i} & i = 0, \ldots, l, \\
\sum_{j=0}^l b_{ij} \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_i} & i = l+1, \ldots, n.
\end{cases}
\]
This implies
\[ \frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial x_i} = 0 \]
for \( 0 \leq i \leq l \) and
\[ \frac{\partial f}{\partial X_i} \in J(F) \]
for \( l + 1 \leq i \leq n \).

**Step 4.** \( f \) is of Sebastiani–Thom type.

The fact
\[ \frac{\partial F}{\partial X_0} = \cdots = \frac{\partial F}{\partial X_l} = 0 \]
and
\[ \frac{\partial f}{\partial X_i} \in J(F) \]
for \( l + 1 \leq i \leq n \) shows
\[ \frac{\partial^2 f}{\partial X_i \partial X_j} = 0 \]
for \( 0 \leq i \leq l \) and \( l + 1 \leq j \leq n \). This implies that \( f \) is of Sebastiani–Thom type.

Since the isomorphism class of the sheaf of logarithmic vector fields along the divisor defined by \( \mu F(x_0, \ldots, x_l) + \nu G(x_{l+1}, \ldots, x_n) \) does not depend on the choice of \( (\mu, \nu) \in (\mathbb{C}^\times)^2 \), a divisor is not Torelli if its defining equation is of Sebastiani–Thom type.

### 4 Smooth plane cubic curves

Theorem 1 immediately yields the following:

**Theorem 4** ([2, Theorem 7]). A smooth plane cubic curve is Torelli if and only if its \( j \)-invariant does not vanish.

**Proof.** Since a smooth plane cubic curve has a vanishing \( j \)-invariant if and only if it is defined by the Fermat polynomial
\[ x^3 + y^3 + z^3 \]
for a suitable choice of a homogeneous coordinate, it suffices to show that any cubic polynomial of the form
\[ f(x) + g(y, z) \]
can be brought to the Fermat polynomial by a projective linear coordinate change, which is obvious.

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