ON RANDOM FOURIER-HERMITE TRANSFORM

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Abstract. Motivated by the work of Z. Liu and S. Liu, we have introduced random Fourier-Hermite transform \( \sum_{n=0}^{\infty} c_n A_n(\omega) \lambda_n \phi_n(t) \), where \( c_n \) are Fourier-Hermite coefficient of a function \( f \) in \( L^p(-\infty, \infty) \), \( A_n(\omega) \) are specific type of random variables associated with symmetric stable process \( X(s, \omega) \) of index \( \gamma \), \( \frac{1}{4} < \gamma \leq 2 \), \( \lambda_n \) are the eigen values of the conventional Fourier transform and \( \phi_n(t) \) are \( n^{th} \) order normalized Hermite-Gaussian function. It is shown that this series converges to a stochastic integral, whose existence is also proved. Random fractional Fourier transform \( \sum_{n=0}^{\infty} c_n A_n(\omega) \lambda_n^\alpha \phi_n(t) \) of rational order \( \alpha \) is also introduced.

1. Introduction and Preliminaries

Random Fourier transform (RFT) was introduced by G. A. Hunt in 1951 [4]. After a long gap, in the beginning of 21st century, RFT of the type \( \sum c_n \lambda_n R \phi_n(t) \) appears in the literature of optics, where \( c_n \) are the Fourier-Hermite coefficient of a function \( f \) in \( L^p(-\infty, \infty) \), \( \phi_n(t) \) are the \( n^{th} \) order normalized Hermite-Gaussian function. Here \( \lambda_n^R \) are the randomly choosen values in the unit circle which are fractional eigen values of the fractional Fourier transform of irrational order.

The eigen values and eigen functions of the Fourier transform

\[
F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt,
\]

play important role in image processing [2]. In the operator notation we can write \( F = F(f) \) and call it as conventional Fourier transform. The normalized Hermite-Gaussian function defined by

\[
\phi_n(t) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} H_n(t)e^{-t^2/2},
\]

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are the eigen functions of the Fourier transform ($F(f)$) of a continuous signal (function) $f$, for $n \in \mathbb{N}_0$, $\mathbb{N}_0 = 0, 1, 2,...$ and this satisfy the eigen value equation,

(1.3) \[ F(\phi_n(t)) = \lambda_n \phi_n(t), \]

where $\lambda_n = e^{-\frac{\pi n^2}{2}}$ are the eigen values of the conventional Fourier transform (1.1), which are of periodicity 4 and also lie on the unit circle. Here the Hermite polynomials $H_n(t)$ are defined by the equations $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$. The functions $\phi_n(t)$ form a complete orthonormal system in $L^2(-\infty, \infty)$. Since $\phi_n(t)$ belongs to $L^p(-\infty, \infty)$ spaces for all $p$, $1 < p < \infty$, the generalized Fourier coefficients (called as Fourier-Hermite coefficient) of $f(t)$ in $L^p$ for any $p$ is defined as,

(1.4) \[ c_n = \int_{-\infty}^{\infty} f(t)\phi_n(t)dt. \]

For each function $f \in L^p$ we have an associated expansion

(1.5) \[ f(t) \sim \sum_{n=0}^{\infty} c_n H_n(t). \]

A function $f e^{-t^2} \in L^2(-\infty, \infty)$ can be represented by the Hermite polynomial expansion as follows,

(1.6) \[ f(t) = \sum_{n=0}^{\infty} a_n h_n(t), \]

where

(1.7) \[ a_n = \int_{-\infty}^{\infty} f(s)h_n(s)e^{-s^2} ds \]

and $h_n = \frac{1}{\sqrt{2^{n!} \pi}} H_n(t)$ is the $n^{th}$ normalized polynomial that is $\int_{-\infty}^{\infty} h_n(t)h_m(t)e^{-t^2} = \delta_{nm}$ for $n, m \in \mathbb{N}_0$. Furthermore $\phi_n$ defined in equation (1.2) are the orthonormal Hermite function that is $\int_{-\infty}^{\infty} \phi_n(t)\phi_m(t)dt = \delta_{nm},$ $n, m \in \mathbb{N}_0$. The Hermite function representation of $f(t)$ in terms of $\{\phi_n(t)\}$ takes the form

(1.8) \[ f(t) = \sum_{n=0}^{\infty} c_n \phi_n(t), \]

where

(1.9) \[ c_n = \int_{-\infty}^{\infty} f(s)\phi_n(s)ds. \]

The partial sum of these series (1.6) and (1.15) are respectively given by

\[ f_n(t) = \sum_{k=0}^{n} a_k h_k(t) = \int_{-\infty}^{\infty} H_n(t, s)f(s)e^{-s^2} ds \]
with $\mathcal{H}_n(t, s) = \sum_{k=0}^{n} h_k(t)h_k(s)$ and

$$f_n(t) = \sum_{k=0}^{n} c_k \phi_k(t) = \int_{-\infty}^{\infty} \Phi_n(t, s) f(s) ds$$

where $\Phi_n(t, s) = \sum_{k=0}^{n} \phi_k(t) \phi_k(s)$.

Pollard [14] proved that the series (1.3) converges to $f$ in $L^2$ norm if $f$ is in $L^2(-\infty, \infty)$, that is

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \left| f(t) - \sum_{k=0}^{n} c_k H_k(t) \right|^2 e^{-t^2} dt = 0.$$

Askey and Wainger [1] extended this result to the class of functions in $L^p(-\infty, \infty)$ for $\frac{4}{3} < p < 4$ that is,

$$\| f - \sum_{k=0}^{n} c_k \phi_k(t) \| \to 0,$$

if $f(t)e^{-\frac{t^2}{2}} \in L^p(\mathbb{R})$. However in general, for all $p$ the series fails to converge in the $L^p$ norm unless a suitable summation method is applied.

Pawlak and Stadtmüller [13] obtained that when $\Phi_n(t, s) = e^{-\frac{(t^2+s^2)}{2}} \mathcal{H}_n(t, s)$,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \Phi_n(t, s) f(s) ds = f(t),$$

for $t \in \mathbb{R}$ a.e., $f \in L_r(\mathbb{R})$, $1 < r < \infty$ and

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \mathcal{H}_n(t, s) f(s) e^{-s^2} ds = f(t),$$

for $t \in \mathbb{R}$ a.e. $f \in L_r(e^{-s^2})$ and $1 < r < \infty$.

Now the Fourier transform of $f(t)$ can be given by

$$\mathcal{F}[f(t)] = \sum_{n=0}^{\infty} c_n \lambda_n \phi_n(t).$$

Bultheel and Martunez [2] have worked on Fourier transform and fractional Fourier transform (FrFT). The eigen values of the FrFT of rational order $\alpha$ are the $\alpha$th power of the eigen values of the conventional Fourier transform, which are finite in number on the unit circle that is

$$\mathcal{F}^{\alpha}[\phi_n(t)] = e^{-\frac{\alpha \pi n}{2}} \phi_n(t).$$
Then the FrFT of rational order $\alpha$ of an arbitrary function, $f$ in $L^2(-\infty, \infty)$ can be expressed as,

\begin{equation}
\mathcal{F}_\alpha[f(t)] = \sum_{n=0}^{\infty} c_n \lambda_n^\alpha \phi_n(t).
\end{equation}

This can also be written in integral form as

\begin{equation}
\mathcal{F}_\alpha[f(t)] = \int_{-\infty}^{\infty} f(s) K_\alpha(t, s) ds,
\end{equation}

with kernel

\begin{equation}
K_\alpha(t, s) = \sum_{n=0}^{\infty} \phi_n(s) \phi_n(t) \lambda_n^\alpha = \sqrt{1 - i \cot \frac{\alpha \pi}{2}} \left[ i \pi \left( \frac{s^2 + t^2}{\tan \frac{\alpha \pi}{2}} - \frac{2st}{\sin \frac{\alpha \pi}{2}} \right) \right].
\end{equation}

The FrFT (1.13) and (1.14) are randomized and are applied in image encryption and decryption.

The FrFT (1.13) is randomized by randomizing the eigen values $\lambda_n^\alpha$ [7]. Hence the FrFT (1.13) of rational order $\alpha$ is extended to irrational order, giving rise to RFT $\sum_{n=0}^{\infty} c_n \lambda_n^R \phi_n(t)$ where the eigen values $\lambda_n^R$ of this RFT are now randomly choosen values on the unit circle.

Liu and Liu [6] has randomized the FrFT by multiplying the random input phase $P(s, t)$ and conjugate of it as output phase $P^*(s, t)$ in the kernel giving rise to the random fractional Fourier transform (RFrFT).

\begin{equation}
\mathcal{F}_\alpha[f(t)] = \int_{-\infty}^{\infty} P(s, t) K_\alpha(s, t) P^*(s, t) f(t) dt.
\end{equation}

These motivated us to extended these ideas to random series of functions $F$ defined on $\mathbb{R} \times \Omega$, where $\Omega$ is a sample space. We have Considered random series of the type

\begin{equation}
\sum_{n=0}^{\infty} c_n r_n(\omega) \phi_n(t),
\end{equation}

where $c_n$ are the Fourier-Hermite coefficient of a function $f \in L^p(\mathbb{R})$, $\frac{4}{3} < p \leq 2$, $r_n(\omega)$ are random variables and $\phi_n$ are normalized Hermite-Gaussian process.

We know that the stochastic integral $\int_{a}^{b} f(s) dX(s, \omega)$ is defined in the sense of probability and is a random variable [9, p. 148], where $X(s, \omega)$ for $s \in \mathbb{R}$ is a continuous stochastic process with independent increments and $f$ is a continuous function in $[a, b]$. If $X(s, \omega)$ is a symmetric stable process of index $\gamma \in [1, 2]$, then the stochastic integral $\frac{1}{2\pi} \int_{0}^{2\pi} f(s) dX(s, \omega)$ converges in probability for $f \in L^p[0, 2\pi]$, $p \geq \gamma$ [11]. Further if $X(s, \omega)$, is a symmetric stable process with independent increment of index $\gamma \in (1, 2]$, then
the integral \( \int_{a}^{b} f(s) dX(s, \omega) \) is defined in the sense of convergence in mean [5]. If \( X(s, \omega) \) is a symmetric stable process of index \( \gamma \in (1, 2] \), then the random Fourier (RF) series

\[
\sum_{n=-\infty}^{\infty} a_n A_n(\omega)e^{int}
\]

converges in the mean to the stochastic integral

\[
\frac{1}{2\pi} \int_{0}^{2\pi} f(s) dX(s, \omega),
\]

where \( A_n(\omega) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ins} dX(s, \omega) \) and \( a_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{int} dt \), for \( f \in L^p(0, 2\pi] \), \( p \geq \gamma \), \( n \in \mathbb{Z} \) \[12\].

These works lead us to establish the existence of the stochastic integral

\[
\int_{-\infty}^{\infty} f(s) dX(s, \omega)
\]

in the sense of convergence in mean where \( f(t)e^{-\frac{t^2}{2}} \in L^p(\mathbb{R}) \), \( \frac{4}{3} < p \leq 2 \) (see theorem 2).

Hence in particular the stochastic integrals

\[
(1.18) \quad A_n(\omega) = \int_{-\infty}^{\infty} \phi_n(s) dX(s, \omega)
\]

exists, where \( \phi_n(s) \) is the nth order normalized Hermite-Gaussian function. These random variables \( A_n(\omega) \) are the Fourier-Hermite Stieltjes coefficient of \( X(s, \omega) \).

In this work we have chosen the random variables \( r_n(\omega) \) in (1.17) as \( A_n(\omega) \) which are associated with symmetric stable process \( X(t, \omega) \) of index \( \frac{4}{3} < \gamma \leq 2 \) and have shown that the series

\[
(1.19) \quad \sum_{n=0}^{\infty} c_n A_n(\omega) \phi_n(t),
\]

converges to the stochastic integral

\[
(1.20) \quad \int_{-\infty}^{\infty} f(s, t) dX(s, \omega),
\]

in the sense of mean, if \( c_n \) are the Fourier-Hermite coefficient (1.9) of \( f(s)e^{-\frac{s^2}{2}} \in L^r(\mathbb{R}) \)(see theorem 4). We call the series (1.19) as random Fourier-Hermite Stieltjes (RFHS) series and the sum (1.20) is denoted as \( F(t, \omega) \).

Similar to the expression (1.11) we consider the series,

\[
(1.21) \quad \sum_{n=0}^{\infty} c_n A_n(\omega) \lambda_n \phi_n(t).
\]

It is established that the sum of this series exists in the sense of mean (cf: theorem 5). We call this random series (1.21) as random Fourier-Hermite transform (RFHT) and denote it as \( \mathcal{F}[f(t, \omega)] \).
Further it is shown that the series
\[ \sum_{0}^{\infty} c_n A_n(\omega) \lambda_n^\alpha \varphi_n(t) \]
exists in the sense of mean for \( \alpha \) rational. We introduce this series as random fractional Fourier-Hermite transform (RFrFHT) and denote it as \( \mathcal{F}^\alpha[f(t, \omega)] \).

2. Definitions

2.1. Definition. A sequence of random variables \( X_n(t, \omega) \) is said to converge in mean to a random variable \( X \), if \( E(|X_n - X| > \epsilon) = 0 \), for any \( \epsilon > 0 \).

3. Results

To prove our theorems we need the following lemmas -:

Lemma 1. \([12]\) If \( X(t, \omega) \) is a symmetric stable process with independent increment of index \( \gamma \), for \( 1 < \gamma \leq 2 \) and \( f \in L^p[a, b] \), \( p \geq \gamma \), then the following inequality holds:

\[
E\left( \left| \int_a^b f(t)dX(t, \omega) \right| \right) \leq \frac{4}{\pi(\gamma - 1)} \int_a^b |f(t)|^\gamma dt + \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp(-|u|^\gamma \int_a^b |f(t)|^\gamma dt)}{u^2} du.
\]

Theorem 2. If \( X(t, \omega) \) is a symmetric stable process of index \( \gamma \), where \( 1 < \gamma \leq 2 \) and \( f \in L^\gamma(\mathbb{R}) \), then the stochastic integral \( \int_{-\infty}^\infty f(t)dX(t, \omega) \) exist in the sense of convergence in mean.

Proof. Let us consider the two random variables \( Y_n = \int_{-n}^n f(t)dX(t, \omega) \) and \( Y_m = \int_{-m}^m f(t)dX(t, \omega) \), which exist in the sense of convergence in mean.

By using lemma \([12]\), we get

\[
E[Y_n - Y_m] = E\left[ \int_{-n}^n f(t)dX(t, \omega) - \int_{-m}^m f(t)dX(t, \omega) \right]
\]

\[
= E\left[ \int_{-n}^{-m} f(t)dX(t, \omega) + \int_{-m}^n f(t)dX(t, \omega) \right]
\]

\[
\leq E\left[ \int_{-n}^{-m} f(t)dX(t, \omega) \right] + E\left[ \int_{-m}^n f(t)dX(t, \omega) \right]
\]

\[
\leq \frac{4}{\pi(\gamma - 1)} \int_{-n}^{-m} |f(t)|^\gamma dt + \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp(-|u|^\gamma \int_{-n}^{-m} |f(t)|^\gamma dt)}{u^2} du
\]

\[
+ \frac{4}{\pi(\gamma - 1)} \int_{-m}^{m} |f(t)|^\gamma dt + \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp(-|u|^\gamma \int_{-m}^{m} |f(t)|^\gamma dt)}{u^2} du.
\]
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The integrand in the 2\textsuperscript{nd} and 4\textsuperscript{th} integral are dominated by the integrable function \( \frac{1}{u^2} \) over \((-\infty, 1]\) and \([1, \infty)\). Hence by dominated convergence theorem

\[
\lim_{m,n \to \infty} E[Y_n - Y_m] = 0.
\]

Now \( Y_n \) is a Cauchy sequence which will converge to \( Y \) (say) and denote \( Y = \int_{-\infty}^{\infty} f(t) dX(t, \omega) \).

This theorem remarks the existence of the stochastic integrals \( \int_{-\infty}^{\infty} f(t) dX(t, \omega) \) in the sense of mean. \( \square \)

**Lemma 3.** Let, \( f \) be any positive function in \( L^\gamma(\mathbb{R}) \) and \( X(t, \omega) \) be a symmetric stable process of index \( \gamma \) and \( 1 < \gamma \leq 2 \), then for all \( \epsilon > 0 \),

\[
E\left( \left| \int_{-\infty}^{\infty} f(t) dX(t, \omega) \right| > \epsilon \right) \leq \frac{4}{\pi(\gamma - 1)} \int_{-\infty}^{\infty} |f(t)|^\gamma dt + \frac{2}{\pi} \int_{|u| > 1} \frac{1 - \exp \left( -|u|^\gamma \int_{-\infty}^{\infty} |f(t)|^\gamma dt \right)}{u^2} du.
\]

**Proof.** Since \( \int_{-n}^{n} f(t) dX(t) \) are bounded for each integer \( n \), bounded convergence theorem implies

\[
E\left( \left| \int_{-\infty}^{\infty} f(t) dX(t, \omega) \right| > \epsilon \right) = E\left( \left| \lim_{n \to \infty} \int_{-n}^{n} f(t) dX(t, \omega) \right| > \epsilon \right)
\]

\[
= \lim_{n \to \infty} E\left( \left| \int_{-n}^{n} f(t) dX(t, \omega) \right| > \epsilon \right)
\]

\[
\leq \frac{4}{\pi(\gamma - 1)} \int_{-\infty}^{\infty} |f(t)|^\gamma dt + \frac{2}{\pi} \int_{|u| > 1} \frac{1 - \exp \left( -|u|^\gamma \int_{-\infty}^{\infty} |f(t)|^\gamma dt \right)}{u^2} du.
\]

\( \square \)

**Theorem 4.** If \( X(s, \omega), s \in \mathbb{R} \) is a symmetric stable process of index \( \frac{3}{2} < \gamma \leq 2 \), then for all measurable function \( f \), such that \( f(t) e^{-\frac{t^2}{2}} \in L^\gamma(\mathbb{R}) \), the series \( (1.19) \), converges in the mean to the stochastic integral \( (1.20) \) where \( c_n \) and \( A_n(\omega) \) are defined in \( (1.9) \) and \( (1.18) \) respectively.

**Proof.** Let,

\[
S_n(s, \omega) = \sum_{k=0}^{n} c_k A_k(\omega) \phi_k(t)
\]

be the \( n \textsuperscript{th} \) partial sum of the series \( (1.19) \).

Consider the partial sum of the Fourier-Hermite series \( \sum_{n=0}^{\infty} c_n \phi_n(t) \) as

\[
f_n(t) = \sum_{k=0}^{n} c_k \phi_k(t)
\]
Since \( \phi_n(t) \) are bounded, the series \( \sum_{n=0}^{\infty} c_n \phi_n(s) \phi_n(t) \) converges. Denote it as \( f(s, t) \)

Denote its partial sum as

\[
f_n(s, t) = \sum_{k=0}^{n} c_n \phi_n(s) \phi_n(t).
\]

In integral form,

\[
S_n(s, \omega) = \sum_{k=0}^{n} c_k A_k(\omega) \phi_k(t)
\]

\[
= \sum_{k=0}^{n} c_k \left( \int_{-\infty}^{\infty} \phi_k(s) dX(s, \omega) \right) \phi_k(t)
\]

\[
= \int_{-\infty}^{\infty} \left( \sum_{k=0}^{n} c_k \phi_n(s) \phi_n(t) \right) dX(s, \omega)
\]

\[
= \int_{-\infty}^{\infty} f_n(s, t) dX(s, \omega).
\]

By theorem (2) we know that \( \int_{-\infty}^{\infty} f(s, t) dX(s, \omega) \) exist in the sense of convergence in mean. Now,

\[
E \left( \left| \int_{-\infty}^{\infty} f(s, t) dX(s, \omega) - S_n(s, \omega) \right| > \epsilon \right)
\]

\[
= E \left( \left| \int_{-\infty}^{\infty} f(s, t) dX(s, \omega) - \int_{-\infty}^{\infty} f_n(s, t) dX(s, \omega) \right| > \epsilon \right)
\]

\[
= E \left( \left| \int_{-\infty}^{\infty} (f(s, t) - f_n(s, t)) \right| \right)
\]

\[
\leq \frac{4}{\pi(\gamma - 1)} \int_{-\infty}^{\infty} |(f(s, t) - f_n(s, t)|^\gamma ds
\]

\[
+ \frac{2}{\pi} \int_{|u|>1} 1 - \exp \left( - |u|^\gamma \int_{-\infty}^{\infty} |(f(s, t) - f_n(s, t)|^\gamma ds \right) du.
\]

Since the integrand in the second integral is dominated by the integrable function \( \frac{1}{u^2} \), the two integrals tends to 0 as \( n \to \infty \) by (1.10). Hence the theorem is proved.

We call the series (1.19) as RFHS series and denote the sum function (1.21) as \( F(t, \omega) \).

The following theorem establishes the existence of the RFH transform.
Theorem 5. If $X(s, \omega), s \in \mathbb{R}$ is a symmetric stable process of index $1 < \gamma \leq 2$, $F$ is the Fourier transform operator, then for all $f(t)e^{-\frac{s^2}{2}} \in L^\gamma(\mathbb{R})$, the series (1.21) converges in the mean to the stochastic integral

$$\int_{-\infty}^{\infty} F(f(s, t))dX(s, \omega),$$

where $\lambda_n = e^{-\frac{in\pi}{2}}$ is the eigen value of the Fourier transform $F(f)$, where $c_n$ and $A_n(\omega)$ are defined in (1.3) and (1.18) respectively.

Proof. Let,

$$S_n(s, \omega) = \sum_{k=0}^{n} c_k \lambda_k A_k(\omega) \phi_k(t)$$

be the $n^{th}$ partial sum of the series (1.21).

Denote the partial sum of the Fourier transform $F(f(t))$ as

$$F[f_n(t)] = \sum_{k=0}^{n} c_k \lambda_k \phi_k(t)$$

and let

$$F[f_n(s, t)] = \sum_{k=0}^{n} c_k \lambda_k \phi_k(s) \phi_k(t).$$

Since $\lambda_n$ lies in the unit circle, we have that the series $F[f_n(t)]$ converges to $F[f(t)]$ in $L^\gamma$, for function $f$ such that $fe^{-\frac{s^2}{2}} \in L^\gamma$.

In integral form,

$$S_n(s, \omega) = \sum_{k=0}^{n} c_k \lambda_k A_k(\omega) \phi_k(t)$$

$$= \sum_{k=0}^{n} c_k \lambda_k \left( \int_{-\infty}^{\infty} \phi_k(s)dX(s, \omega) \right) \phi_k(t)$$

$$= \int_{-\infty}^{\infty} F[f_n(s, t)]dX(s, \omega).$$
Now,
\[
E \left( \left| \int_{-\infty}^{\infty} \mathcal{F}[f(s, t)]dX(s, \omega) - S_n(s, \omega) \right| > \epsilon \right)
\]
\[
= E \left( \left| \int_{-\infty}^{\infty} \mathcal{F}(f[s, t])dX(s, \omega) - \int_{-\infty}^{\infty} \mathcal{F}[f_n(s, t)]dX(s, \omega) \right| > \epsilon \right)
\]
\[
= E \left( \left| \int_{-\infty}^{\infty} (\mathcal{F}[f(s, t)] - \mathcal{F}[f_n(s, t)])dX(s, \omega) \right| > \epsilon \right)
\]
\[
\leq \frac{4}{\pi} \int_{-\infty}^{\infty} \left| \mathcal{F}[f(s, t)] - \mathcal{F}[f_n(s, t)] \right| \gamma ds
\]
\[
+ \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp \left( -\frac{|u| \gamma}{u^2} \right)}{|u|^2} \mathcal{F}[f(s, t) - f_n(s, t)]dS \]

The integrand of the second integral is dominated by $\frac{1}{u^2}$ if $|u| > 1$ and hence the two integrals tends to 0 as $n \to \infty$ by (1.10). Hence, it is proved that the random series (1.21) convergence in mean to $\int_{-\infty}^{\infty} \mathcal{F}[f(s, t)]dX(s, \omega)$.

We call the series (1.21) as RFHT and denote the sum function (3.1) as $\mathcal{F}[f(s, t)]$. □

The following theorem on RFrFHT can be proved in the same way as the proof of the previous theorem.

**Theorem 6.** If $X(s, \omega), s \in \mathbb{R}$ is a symmetric stable process of index $\frac{4}{3} < \gamma \leq 2$, $\mathcal{F}$ is the Fourier transform operator, then for all $f \in L^\gamma(\mathbb{R})$, the series

\[
\sum_{n=0}^{\infty} c_n A_n(\omega) \lambda_n^\alpha \phi_n(t)
\]

converges in the mean to the stochastic integral

\[
\int_{-\infty}^{\infty} \mathcal{F}^\alpha[f(s, t)]dX(s, \omega),
\]

where $\lambda_n^\alpha = e^{-\max x_n}$ is the eigen value of order rational order $\alpha$ of the Fourier transform $\mathcal{F}^\alpha[f(s, t)]$. $A_n(\omega)$ and $c_n$ are defined as in (1.18) and (1.9) respectively.

We call series (3.2) as RFrFHT and denote its sum function (3.3) as $\mathcal{F}^\alpha[f(t, \omega)]$.

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