We study the problem of finding a point of maximal electrostatic potential inside an arbitrary triangle with homogeneous surface charge distribution. In this article we show that such point is indeed unique and we derive several synthetic and analytic relations for its location in the plane. Moreover, this point satisfies the definition of a triangle center, different from any of 5622 currently known centers from Clark Kimberling’s encyclopedia.

1. Introduction.

The topic we are about to discuss was initiated by a concrete and practical question in physics that has eventually revealed its unexpectedly interesting geometrical flavor. Let us begin with a statement of this theoretical problem and postpone applied motivation to the end of this section.

Problem. Suppose that a planar triangle $T$ is a continuous source of charge, which is homogeneously distributed over its surface, i.e. the charge density is constant over the triangle. At which point in the same plane the electrostatic potential of $T$ attains its maximum value?

The above terminology should not be a source of discomfort for a reader with preference for pure mathematical material. All physical notions will be accompanied with their precise definitions and the discussion will soon turn into elementary geometrical considerations. Let us only recall that the potential of a point source with charge $q$ evaluated at a point that is $r$ units apart is given by $V(r) = kq/r$. This is merely a restatement of Coulomb’s law and the constant $k$ is not important for us. By “superposition principle” for multiple charges it is therefore reasonable to define the potential generated by the whole triangle $T$ as

$$V(P) = \int\int_T \frac{d\lambda(Q)}{|PQ|}$$

for any point $P$ in the plane. Here $\lambda$ denotes the two-dimensional Lebesgue measure (i.e. the area measure), $Q$ is an integration variable, and $|PQ|$ denotes the distance between points $P$ and $Q$. We are careless about the multiplicative constant or the charge density and we even omit them from writing. In Cartesian coordinates the above formula becomes simply

$$V(x, y) = \int\int_T \frac{dx'dy'}{\sqrt{(x' - x)^2 + (y' - y)^2}}.$$

It will be shown in proposition 1 below that $V$ is indeed a well-defined function on the whole plane. One can draw contour graphs of (1) for various choices of triangles using the Mathematica command ContourPlot and the level sets will look as those in figure 1. Such drawings can make us suspect that $V$ has the shape of a single “mountain peak,” but this certainly could not pass as a rigorous argument. It is not immediately clear from the formula that there even exist a point $P_{\text{max}}$ inside $T$ where $V$ attains its maximum and it is certainly not obvious that such point should be unique for every triangle.

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What would be the physical meaning of the maximum potential point? It is the point where the electrostatic field $\vec{E}$ generated by $T$ stabilizes. Let us perform a simple thought experiment. Assume that $T$ is charged positively and place a negative point charge somewhere in the plane. It will necessarily be driven by electrostatic forces unless it is placed at a point where it “feels perfectly stable.” Figure 2 illustrates several integral curves of the vector field $\vec{E}$, which are also known in physics as lines of force or field lines. Observe that they all meet at the same point inside $T$. This experiment is once again very far from a rigorous proof. Existence of the maximum potential point will be shown in the next section, while its uniqueness will come as a byproduct of several attempts to specify its location throughout the rest of the paper.

We have just mentioned the notion of electrostatic field, so what would that field be in the case of our charged triangle $T$? It can be defined simply as $\vec{E} = -\nabla V$ at any point where the potential $V$ is differentiable. In physics, the electric field is sometimes (but not always) given before the potential. We have intentionally ordered things this way, simply because the potential of $T$ was easier to define mathematically. Going back to a point source, an easily derived and well-known formula is $\vec{E} = kq\hat{r}/r^3$. Here $\hat{r}$ denotes a directed line segment from the source to a point where the field is computed. Using the superposition principle once again we suspect that the correct corresponding expression is

$$\vec{E}(P) = \int_T \frac{Q\vec{P}}{|PQ|^3} d\lambda(Q) = -\int_T \frac{\vec{PQ}}{|PQ|^3} d\lambda(Q),$$

(2)
or coordinate-wise with $\vec{i}$ and $\vec{j}$ being the standard unit vectors,

$$\vec{E}(x, y) = -\iint_T \frac{(x' - x)\vec{i} + (y' - y)\vec{j}}{((x' - x)^2 + (y' - y)^2)^{3/2}} dx' dy'.$$

However, the double integral in the above formula will not be absolutely convergent unless $P(x, y)$ lies outside $T$. To explain the difficulty, assume that $P$ is contained in the triangle interior, together with a “small” disk $D_\varepsilon(P)$ of radius $\varepsilon$ around it. We insert absolute values inside the double integral and only integrate over this disk. Changing to a polar coordinate system centered at $P$ we obtain

$$\iint_{D_\varepsilon(P)} \frac{|\vec{PQ}|}{|PQ|^3} d\lambda(Q) = \int_0^\varepsilon \int_0^{2\pi} \frac{r}{r^3} r dr d\varphi = +\infty,$$

because $\int_0^\varepsilon \frac{dr}{r}$ diverges.

So is there a valid formula for $\vec{E}$ that would hold for points $P$ in the interior of $T$? One simply has to observe that the contributions $\frac{|\vec{PQ}|}{|PQ|^3}$ of points $Q \in D_\varepsilon(P)$ cancel out each other completely, as the opposite vectors add up to $\vec{0}$, see figure 3. Therefore,

$$\vec{E}(P) = -\iint_{T \setminus D_\varepsilon(P)} \frac{\vec{PQ}}{|PQ|^3} d\lambda(Q)$$

(3)

should hold for $P$ inside the triangle. Indeed, one can even let $\varepsilon \to 0$, obtaining the expression called the principal value of the integral:

$$\vec{E}(P) = -\text{p.v.} \iint_T \frac{\vec{PQ}}{|PQ|^3} d\lambda(Q).$$

(4)

Things remain problematic for points $P$ at the boundary, because the same argument shows that the expression for $\vec{E}(P)$ does not converge in any usual sense. Indeed, the potential is continuous but not differentiable at those points.

The main source of motivation for the problem comes from implementation of a certain type of boundary element method (BEM) for electrostatic problems [1], [5], [8], [12]. Boundary element methods are usually formulated by surface elements of a three-dimensional object and these elements are in turn most often represented by triangles. In the case of an electrostatic problem, a single triangle potential could be evaluated either at vertices, or at a certain interior point, depending on the formulation of the method. In the later case, it is common to take the center of mass (i.e. the centroid), but there is no reason or evidence why this would be the best choice. Indeed, one can argue that using the maximum potential point provides better results, but such discourse is out of the scope of this paper. Calculating its coordinates and discovering its properties proved to be challenges on their own.
Hoping the reader got interested in the topic, we turn to purely mathematical discussion. Our potential is a particular instance of the so-called fractional integral,

\[
(I_p f)(x, y) = \iint \left( (x' - x)^2 + (y' - y)^2 \right)^{p/2} f(x', y') \, dx' \, dy',
\]

which is also known as the Riesz potential \( I_1 \) when \(-2 < p < 0\) and when it is properly normalized. In order to obtain \( I_1 \), one only has to take \( p = -1 \) and choose \( f \) to be the indicator function of \( T \).

Extreme points of “regularized” versions of \( I_p f \) when \( p \) is a real number and \( f \) is the characteristic function of a general convex set \( S \) (in several dimensions) have already been studied in the literature. They were named “radial centers” by M. Moszyńska [9], who seems to be the first to establish their existence and uniqueness for a certain range of exponents \( p \), while the remaining cases were studied by I. Herburt [3]. The later paper contains a mistake that was fixed by J. O’Hara [10] and S. Sakata [13], who called these points \( r^p \) centers. We remark that our corollaries 1 and 2 below follow from general results of Moszyńska [9] and Herburt [2]. However, we are interested in a very special case when \( p = -1 \) and when the set \( S \) is a triangle, so fortunately we are able to keep the exposition simple and self-contained. Finally, Herburt, Moszyńska, and Peradzyński [4] give physical interpretations of radial centers, mentioning gravitational and electrostatic potentials, but do not specialize the discussion to triangles. On the other hand, we need to mention an unpublished text by K. Shibata [14] on a similarly defined but different point in a triangle, called the illuminating center, which we discuss briefly in the last section.

2. Properties of the potential.

We write \( \text{Int}(T) \), \( \text{Ext}(T) \), and \( \text{Bd}(T) \) for interior, exterior, and boundary of \( T \) respectively. Whenever \( T \) is mentioned, we understand that it contains its boundary, but we will always be precise, just for any case. Let us also write “dist” for the Euclidean distance function, i.e. \( \text{dist}(P, Q) = |PQ| \). Likewise, “area” will sometimes be used in place of \( \lambda \).

**Proposition 1.**

(a) Potential \( V \) is finite and continuous on the whole plane.
(b) \( V(P) \to 0 \) uniformly as \( \text{dist}(P, T) \to \infty \).
(c) Potential \( V \) is differentiable both in the interior and in the exterior of \( T \).
(d) Field \( \vec{E} = -\nabla V \) is given by (2) for \( P \in \text{Ext}(T) \) and by (3) or (4) for \( P \in \text{Int}(T) \).
(e) Potential \( V \) cannot attain local maxima in the exterior or on the boundary of \( T \).

A reader who finds these facts intuitive enough to take them for granted may skip freely to more elementary material in the next section. However, as mathematicians we are obliged to provide formal proofs. Those who prefer to clarify all details will probably stay with us through a few pages of technicalities.

**Proof of proposition [7] Part (a).** Take any point \( P_0 \) and choose a radius \( R > 0 \) large enough such that the disk \( D_R(P_0) \) contains \( T \) in its interior. By changing to polar coordinates,

\[
\iint_{D_R(P_0)} \frac{d\lambda(Q)}{|P_0Q|^2} = \int_0^R \int_0^{2\pi} \frac{r \, dr \, d\varphi}{r} = 2\pi R < \infty,
\]

which implies that \( V(P_0) \) is also finite.

If we take another point \( P \) in the plane, we can write

\[
V(P) = \iint_{T} \frac{d\lambda(Q)}{|P_0Q + PP_0|^2} = \iint_{T + \overline{PP_0}} \frac{d\lambda(Q)}{|P_0Q|^2} = \iint \frac{1_{T + \overline{PP_0}}(Q)}{|P_0Q|^2} \, d\lambda(Q),
\]
where \( T + \overrightarrow{PP_0} \) denotes the translate of triangle \( T \) by vector \( \overrightarrow{PP_0} \) and \( 1_S \) stands for indicator function of a set \( S \). Since

\[
\lim_{P \to P_0} 1_{T + \overrightarrow{PP_0}}(Q) = \begin{cases} 
1 & \text{for } Q \in \text{Int}(T), \\
0 & \text{for } Q \in \text{Ext}(T)
\end{cases}
\]

and we just saw that \( Q \mapsto 1/|PP_0| \) is a locally integrable function, the dominated convergence theorem implies \( \lim_{P \to P_0} V(P) = V(P_0) \), which is precisely the desired continuity of \( V \).

Part (b). Choose a disk \( D_R(P_0) \) as in part (a). If \(|PP_0| > R\), then the angle \( \theta \) under which \( D_R(P_0) \) can be seen from \( P \) satisfies

\[
sin\left(\frac{\theta}{2}\right) = \frac{R}{|PP_0|}.
\]

This gives

\[
\theta = 2\arcsin\left(\frac{R}{|PP_0|}\right) \leq 2\arcsin\left(\frac{R}{\text{dist}(P, T)} - R\right).
\]

The statement follows by letting \( \text{dist}(P, T) \to \infty \) and using the continuity of \( \text{arcsine function} \) at 0.

Parts (c) and (d). Fix a point \( P_0 \in \text{Int}(T) \) and choose \( 0 < \varepsilon < \frac{1}{2}\text{dist}(P_0, \text{Bd}(T)) \). We need to show that \( V \) is differentiable at \( P_0 \) and that \( \nabla V(P_0) = -\vec{E}(P_0) \), where \( \vec{E}(P_0) \) is given by formula (3). Take any point \( P \) such that \(|PP_0| < \varepsilon\). Parts of the integrals in the expression \( V(P) - V(P_0) \) corresponding to \( D_{\varepsilon}(P_0) \cap D_{\varepsilon}(P) \) cancel out by symmetry, so this difference is equal to

\[
\int_{T \setminus (D_{\varepsilon}(P_0) \cup D_{\varepsilon}(P))} \left( \frac{1}{|PQ|} - \frac{1}{|P_0Q|} \right) d\lambda(Q).
\]

Using

\[
|P_0Q|^2 - |PQ|^2 = 2\overrightarrow{P_0Q} \cdot \overrightarrow{P_0P} - |P_0P|^2,
\]

it can be rewritten as

\[
V(P) - V(P_0) = \int_{T \setminus (D_{\varepsilon}(P_0) \cup D_{\varepsilon}(P))} \frac{2\overrightarrow{P_0Q} \cdot \overrightarrow{P_0P} - |P_0P|^2}{|P_0Q||PQ||(|P_0Q| + |PQ|)} d\lambda(Q).
\]

On the other hand, from (3),

\[
\vec{E}(P_0) \cdot \overrightarrow{P_0P} = -\int_{T \setminus D_{\varepsilon}(P_0)} \frac{\overrightarrow{P_0Q} \cdot \overrightarrow{P_0P}}{|P_0Q|^3} d\lambda(Q).
\]

After simple algebraic manipulations and by splitting

\[
D_{\varepsilon}(P_0) = (D_{\varepsilon}(P_0) \cup D_{\varepsilon}(P)) \setminus (D_{\varepsilon}(P) \setminus D_{\varepsilon}(P_0)),
\]

\[\text{Figure 4. Angle that determines the range of integration.}\]
we arrive at
\[
\frac{1}{|P_0P|} \left( V(P) - V(P_0) + \vec{E}(P_0) \cdot \overrightarrow{P_0P} \right) = J_1 - J_2 - J_3,
\]
where
\[
J_1 = \iint_{T \setminus (D_\varepsilon(P_0) \cup D_\varepsilon(P))} \frac{\overrightarrow{PQ} \cdot \overrightarrow{P_0Q}}{|\overrightarrow{PQ}| |\overrightarrow{P_0P}|} \frac{2|P_0Q|}{\lambda(T)} d\lambda(Q),
\]
\[
J_2 = \iint_{T \setminus (D_\varepsilon(P_0) \cup D_\varepsilon(P))} \frac{|\overrightarrow{P_0P}|}{|\overrightarrow{P_0Q}| |\overrightarrow{P_0P}|} d\lambda(Q),
\]
\[
J_3 = \iint_{D_\varepsilon(P) \setminus D_\varepsilon(P_0)} \frac{\overrightarrow{P_0Q} \cdot \overrightarrow{P_0P}}{\lambda(T)|\overrightarrow{P_0Q}|^2} d\lambda(Q).
\]
Using $|P_0Q| \geq \varepsilon$, $|PQ| \geq \varepsilon$, and $|P_0Q| - |PQ| \leq |P_0P|$ the first integral is easily bounded as
\[
|J_1| \leq \frac{2}{\varepsilon^3} \lambda(T)|P_0P|
\]
and similarly we get
\[
|J_2| \leq \frac{1}{2\varepsilon^3} \lambda(T)|P_0P|, \quad |J_3| \leq \frac{1}{\varepsilon^2} \lambda(D_\varepsilon(P) \setminus D_\varepsilon(P_0)).
\]
Letting $P \to P_0$ we conclude
\[
\lim_{P \to P_0} \frac{V(P) - V(P_0) + \vec{E}(P_0) \cdot \overrightarrow{P_0P}}{|\overrightarrow{P_0P}|} = 0,
\]
which is precisely what we needed.

For points $P_0$ in the exterior of $T$ the proof can follow the same lines. Moreover, an even shorter proof can be given for such $P_0$ by entirely standard arguments of interchanging limits and integrals, as the integral in (2) is an absolutely convergent one.

Part (e). Begin by taking $P_0 \in \text{Ext}(T)$. Informally saying, the field does not vanish at $P_0$ since it has to “point” away from $T$. More rigorously, let $l$ be any line passing though $P_0$ and containing $T$ entirely in one of the two corresponding half-planes, see figure 5. If $\vec{n}$ is a vector normal to $l$ and oriented in the opposite direction, then formula (2) yields
\[
\vec{E}(P_0) \cdot \vec{n} = \iint_{T} \frac{QF_0 \cdot \vec{n}}{|P_0Q|^3} d\lambda(Q) > 0.
\]
Consequently, $(\nabla V)(P_0) \neq \vec{0}$, so $P_0$ cannot be a stationary point for $V$. 

![Figure 5. Treatment of exterior points.](image-url)
The same argument "almost works" for points at the triangle boundary. Even though \( \vec{E}(P_0) \) does not exist, we can imagine that it is a vector of infinite length pointing outwards. The reader can modify the proof of parts (c) and (d) to show that
\[
\lim_{h \to 0} \frac{V(P_0 - h\vec{n}) - V(P_0)}{h} = +\infty
\]
holds for the same choice of \( \vec{n} \). Once again, we conclude that \( P_0 \) is not a local maximum point for \( V \). \( \square \)

Corollary 1. Potential \( V \) attains its maximum at some point inside triangle \( T \). At each such point \( P \) one has \( \vec{E}(P) = \vec{0} \).

Proof of corollary 1. By positivity and parts (a) and (b) of proposition 1 it follows that \( V \) is bounded and has a maximum that is attained at some (finite) point in the plane. By part (e) we know that any such point must lie in the interior of \( T \). Finally, the second assertion is a consequence of parts (c) and (d). \( \square \)

We need to remark that an explicit formula for \( V \) can be computed, although it is rather complicated and not practically useful. It could only make the above proof more elementary, but also significantly less elegant. Instead, it will be more useful to transform formula (3) for \( \vec{E}(P) \) in the next section. Moreover, an advantage of our proof is that it also works for general convex polygons and, with minor modifications, even for arbitrary compact convex sets.

3. Geometric relations.

Throughout this section suppose that \( P \) is a stationary point inside a positively oriented triangle \( T = \triangle ABC \), i.e. the corresponding vector field \( \vec{E} \) vanishes at \( P \). Denote its distances from vertices \( A, B, C \) respectively by
\[
 r_A = |PA|, \quad r_B = |PB|, \quad r_C = |PC|.
\]
Let us also introduce convenient notation for the several angles it determines,
\[
\alpha_1 = \angle BAP, \quad \beta_1 = \angle CBP, \quad \gamma_1 = \angle ACP, \\
\alpha_2 = \angle PAC, \quad \beta_2 = \angle PBA, \quad \gamma_2 = \angle PCB,
\]
as in figure 6. Finally, we use standard notation for triangle sidelengths and angles:
\[
a = |BC|, \quad b = |CA|, \quad c = |AB|, \quad \alpha = \angle BAC, \quad \beta = \angle CBA, \quad \gamma = \angle ACB.
\]

![Figure 6. Convenient notation.](image)

The following theorem gives two remarkably simple relations that enable us to locate such point \( P \) in the plane.
Theorem 1. If $P$ is a point inside triangle $ABC$ such that $\vec{E}(P) = \vec{0}$, then

$$
\left(\frac{r_B + r_C - a}{r_B + r_C + a}\right)^{1/a} = \left(\frac{r_C + r_A - b}{r_C + r_A + b}\right)^{1/b} = \left(\frac{r_A + r_B - c}{r_A + r_B + c}\right)^{1/c} \tag{6}
$$

and

$$
\left(\frac{\tan \beta_1}{2} \frac{\tan \gamma_2}{2}\right)^{\frac{1}{\sin \alpha}} = \left(\frac{\tan \gamma_1}{2} \frac{\tan \alpha_2}{2}\right)^{\frac{1}{\sin \beta}} = \left(\frac{\tan \alpha_1}{2} \frac{\tan \beta_2}{2}\right)^{\frac{1}{\sin \gamma}}. \tag{7}
$$

In particular, equations (6) and (7) also hold for any maximum point of potential $V$. We have formulated the theorem in a seemingly more general way, as we do not yet know that this point is unique.

Proof of theorem 4. Take $P$ to be the origin of the coordinate system and change to polar coordinates. Let us denote by $M_\varphi$ the point at the intersection of the polar ray determined by an angle $\varphi \in [0, 2\pi)$ with the boundary of $\triangle ABC$. Furthermore, let us write $R(\varphi) = |PM_\varphi|$. For $\varepsilon > 0$ small enough formula (3) becomes

$$
\vec{E}(P) = -\int_0^{\varphi} \int_0^{2\pi} \frac{r(\cos \varphi)\vec{i} + r(\sin \varphi)\vec{j}}{r^3} r dr d\varphi
$$

$$
= -\int_0^{2\pi} \left(\log R(\varphi) - \log \varepsilon\right) \left((\cos \varphi)\vec{i} + (\sin \varphi)\vec{j}\right) d\varphi.
$$

and then using $\int_0^{2\pi} \cos \varphi \, d\varphi = 0 = \int_0^{2\pi} \sin \varphi \, d\varphi$ we get

$$
\vec{E}(P) = -\int_0^{2\pi} \log R(\varphi) \left((\cos \varphi)\vec{i} + (\sin \varphi)\vec{j}\right) d\varphi.
$$

For the rest of the proof it will be convenient to represent vectors by complex numbers, i.e. to work in the complex plane. Using $e^{i\varphi} = \cos \varphi + i \sin \varphi$ the condition $\vec{E}(P) = \vec{0}$ becomes simply

$$
\int_0^{2\pi} \log R(\varphi) e^{i\varphi} d\varphi = 0. \tag{8}
$$

The next step is to find an expression for $\log R(\varphi)$. Let vertices $A, B, C$ have complex coordinates

$$
r_A e^{i\varphi_A}, r_B e^{i\varphi_B}, r_C e^{i\varphi_C}
$$

and let vectors $\overrightarrow{CB}, \overrightarrow{AC}, \overrightarrow{BA}$ be represented by complex numbers

$$
ae^{i\theta_a}, be^{i\theta_b}, ce^{i\theta_c}.
$$

Without loss of generality suppose that $M_\varphi$ lies on side $AB$ of $\triangle ABC$, which is the same as saying $\varphi_A < \varphi < \varphi_B$, where we possibly need to adjust the angles by adding appropriate multiples of $2\pi$. Let $d_c$ denote the distance from $P$ to the line $AB$ and let $\psi$ denote the angle $\angle BM_\varphi P$. From figure 7 we see that $\psi = \varphi - \varphi_A + \alpha_1$ and $R(\varphi) = d_c / \sin \psi$, i.e.

$$
\log R(\varphi) = \log d_c - \log \sin \psi.
$$

Observing that $\psi$ ranges from $\alpha_1$ to $\pi - \beta_2$ we get

$$
\int_{\varphi_A}^{\varphi_B} \log R(\varphi) e^{i\varphi} d\varphi = \log d_c \int_{\varphi_A}^{\varphi_B} e^{i\varphi} d\varphi - \int_{\alpha_1}^{\pi - \beta_2} (\log \sin \psi) e^{i(\psi + \varphi_A - \alpha_1)} d\psi.
$$

First, we use an immediate formula

$$
\int_\eta^\varphi e^{i\varphi} d\varphi = \left(-ie^{i\varphi}\right)\bigg|_{\varphi=\eta}^{\varphi=\varphi}.
$$

(9)
Next, it is an easy exercise in integration by parts to obtain

$$\int_{\eta}^{\vartheta} (\log \sin \psi) \cos \psi \, d\psi = \left( \left( \log \sin \psi - 1 \right) \sin \psi \right) \bigg|_{\psi=\eta}^{\psi=\vartheta}$$

and

$$\int_{\eta}^{\vartheta} (\log \sin \psi) \sin \psi \, d\psi = \left( - \left( \log \sin \psi - 1 \right) \cos \psi + \log \tan \frac{\psi}{2} \right) \bigg|_{\psi=\eta}^{\psi=\vartheta}$$

for angles $0 < \eta < \vartheta < \pi$. Combining we get

$$\int_{\eta}^{\vartheta} (\log \sin \psi) e^{i\psi} \, d\psi = \left( -i \left( \log \sin \psi - 1 \right) e^{i\psi} + i \log \tan \frac{\psi}{2} \right) \bigg|_{\psi=\eta}^{\psi=\vartheta}. \quad (10)$$

From formulas (9), (10) we obtain

$$\int_{\varphi_A}^{\varphi_B} \log R(\varphi) e^{i\varphi} d\varphi = -i \log d_c e^{i\varphi_B} + i \log d_c e^{i\varphi_A}$$

$$+ ie^{i(\varphi_A-\alpha_1+\pi-\beta_2)} (\log \sin \beta_2 - 1) - ie^{i\varphi_A} (\log \sin \alpha_1 - 1)$$

$$- ie^{i(\varphi_B-\alpha_1)} \log \tan \frac{\pi-\beta_2}{2} + ie^{i(\varphi_B-\alpha_1)} \log \tan \frac{\alpha_1}{2},$$

and then using

$$d_c/\sin \alpha_1 = r_A, \quad d_c/\sin \beta_2 = r_B, \quad \varphi_A-\alpha_1+\pi-\beta_2 = \varphi_B, \quad \varphi_A-\alpha_1 = \theta_c$$

we get

$$\int_{\varphi_A}^{\varphi_B} \log R(\varphi) e^{i\varphi} d\varphi = -ie^{i\varphi_B} (\log r_B - 1) + ie^{i\varphi_A} (\log r_A - 1)$$

$$+ ie^{i\theta_c} \left( \log \tan \frac{\alpha_1}{2} - \log \cot \frac{\beta_2}{2} \right).$$

Adding this one and the two analogous relations, applying (8), and observing cancellations of

$$ie^{i\varphi_A} (\log r_A - 1)$$

and the two alike terms gives

$$e^{i\theta_c} \log(\tan \frac{\alpha_1}{2} \tan \frac{\beta_2}{2}) + e^{i\theta_b} \log(\tan \frac{\beta_1}{2} \tan \frac{\gamma_2}{2}) + e^{i\theta_a} \log(\tan \frac{\gamma_1}{2} \tan \frac{\alpha_2}{2}) = 0.$$
To see (11) one only has to observe $\overrightarrow{AC} = -\overrightarrow{BA} - \overrightarrow{CB}$ and make use of linear independence of $\overrightarrow{BA}$ and $\overrightarrow{CB}$. If we apply the law of sines and exponentiate (11), we will complete the proof of (7).

In order to derive (6), we use trigonometric half-angle formulas, the law of cosines, and some factoring:

$$\tan^2 \frac{\alpha_1}{2} = \frac{1 - \cos \alpha_1}{1 + \cos \alpha_1} = \frac{1 - (r_A^2 + c^2 - r_B^2)/2r_Ac}{1 + (r_A^2 + c^2 - r_B^2)/2r_Ac} = \frac{(r_A + r_B - c)(r_B - r_A + c)}{(r_A + r_B + c)(r_A - r_B + c)}.$$  

Multiplying this one with an analogous expression for $\tan \frac{\beta_2}{2}$ and taking square roots gives

$$\tan \frac{\alpha_1}{2} \tan \frac{\beta_2}{2} = \frac{r_A + r_B - c}{r_A + r_B + c},$$

so that (11) becomes

$$\frac{1}{c} \log \left( \frac{r_A + r_B - c}{r_A + r_B + c} \right) = \frac{1}{a} \log \left( \frac{r_B + r_C - a}{r_B + r_C + a} \right) = \frac{1}{b} \log \left( \frac{r_C + r_A - b}{r_C + r_A + b} \right).$$  

(12)

Exponentiation proves relation (6). \qed

4. Cartesian coordinates and uniqueness.

Theorem 1 is a nice theoretical result, but how does one determine the exact location of $P$? The starting point are equalities (6), i.e. their logarithmic version (12). It is easy to see that these expressions are less than 0, so it is natural to consider their negatives. Multiply them further by the semiperimeter $s = \frac{1}{2}(a + b + c)$ of triangle $ABC$ in order to make them “dimensionless” and denote the obtained common value by $\lambda$:

$$-\frac{s}{a} \log \left( \frac{r_B + r_C - a}{r_B + r_C + a} \right) = -\frac{s}{b} \log \left( \frac{r_C + r_A - b}{r_C + r_A + b} \right) = \frac{s}{c} \log \left( \frac{r_A + r_B - c}{r_A + r_B + c} \right) = \lambda.$$  

Concentrating on only one expression at a time, we can now write

$$\frac{r_B + r_C - a}{r_B + r_C + a} = e^{-\lambda/s},$$

so that

$$r_B + r_C = a \frac{1 + e^{-\lambda/s}}{1 - e^{-\lambda/s}} = a \frac{e^{\lambda/2s} + e^{-\lambda/2s}}{e^{\lambda/2s} - e^{-\lambda/2s}} = a \coth \frac{\lambda}{2s}$$

and similarly

$$r_C + r_A = b \coth \frac{\lambda}{2s}, \quad r_A + r_B = c \coth \frac{\lambda}{2s}.$$  

Let us agree to write

$$u = a \coth \frac{\lambda}{2s}, \quad v = b \coth \frac{\lambda}{2s}, \quad w = c \coth \frac{\lambda}{2s}$$  

(13)

in all that follows. Hence,

$$r_A = \frac{1}{2}(u + w - u), \quad r_B = \frac{1}{2}(w + u - v), \quad r_C = \frac{1}{2}(u + v - w).$$  

(14)

Now is the time to observe that the distances $r_A, r_B, r_C$ are not independent. The simplest equation relating them can be derived from

$$\text{area} (\triangle PBC) + \text{area} (\triangle PCA) + \text{area} (\triangle PAB) = \text{area} (\triangle ABC)$$

using Heron’s formula:

$$\sqrt{s_a(s_a - a)(s_a - r_B)(s_a - r_C)} + \sqrt{s_b(s_b - b)(s_b - r_C)(s_b - r_A)} + \sqrt{s_c(s_c - c)(s_c - r_A)(s_c - r_B)} = \sqrt{s(s - a)(s - b)(s - c)},$$

where $s_a = s - b - c$. Thus

$$u(v + w - u) + v(w + u - v) + w(u + v - w) = \frac{1}{2}[(v + w - u) + (w + u - v) + (u + v - w)](u + v + w)$$

and

$$\lambda = \log \left( \frac{u + v + w}{3} \right) = \log \left( \frac{u + v + w}{u + v + w} \right) = 0.$$  

(15)
with \(s_a, s_b, s_c, s\) being semiperimeters of the the four triangles respectively. Substituting (14), multiplying by 4, and simplifying we obtain
\[
\sqrt{(u^2 - a^2)(a^2 - (v-w)^2)} + \sqrt{(v^2 - b^2)(b^2 - (w-u)^2)} + \sqrt{(w^2 - c^2)(c^2 - (u-v)^2)} = 2(2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)).
\]  
(15)
This is a nonlinear equation for \(\lambda\) and then \(r_A, r_B, r_C\) are determined by (13) and (14).

It remains to explain how to express coordinates of \(P(x_P, y_P)\) from its distances to triangle vertices \(A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)\). Using the formula for Euclidean distance in Cartesian coordinates we get an overdetermined quadratic system for \(x_P\) and \(y_P\),
\[
(x_P - x_A)^2 + (y_P - y_A)^2 = r_A^2, \\
(x_P - x_B)^2 + (y_P - y_B)^2 = r_B^2, \\
(x_P - x_C)^2 + (y_P - y_C)^2 = r_C^2.
\]
Subtracting the third equation from the first two leads to a linear system
\[
2(x_C - x_A)x_P + 2(y_C - y_A)y_P = x_C^2 - x_A^2 + y_C^2 - y_A^2 + v(w - u), \\
2(x_C - x_B)x_P + 2(y_C - y_B)y_P = x_C^2 - x_B^2 + y_C^2 - y_B^2 + u(w - v),
\]
which can be quickly solved as
\[
x_P = \frac{(x_A^2 + y_A^2 - uv)(y_B - y_C) + (x_B^2 + y_B^2 - uv)(y_C - y_A) + (x_C^2 + y_C^2 - uv)(y_A - y_B)}{2x_A(y_B - y_C) + 2x_B(y_C - y_A) + 2x_C(y_A - y_B)},
\]
\[
y_P = \frac{(x_A^2 + y_A^2 - uv)(x_B - x_C) + (x_B^2 + y_B^2 - uv)(x_C - x_A) + (x_C^2 + y_C^2 - uv)(x_A - x_B)}{2y_A(x_B - x_C) + 2y_B(x_C - x_A) + 2y_C(x_A - x_B)}. 
\]
(16) (17)
That way we have established the following theorem.

**Theorem 2.** Suppose that \(P\) is a point inside \(\triangle ABC\) satisfying \(\vec{E}(P) = 0\). Its Cartesian coordinates satisfy (16) and (17), where \(u, v, w\) are defined by (13) and \(\lambda > 0\) is a solution of equation (15).

Turning back to equation (15), we might want to know the number of its positive solutions. We claim that the left-hand side is a strictly decreasing function of \(\lambda \in (0, \infty)\). Since
\[
\lambda \mapsto u^2 - a^2 = a^2(\coth^2 \frac{a\lambda}{2s} - 1)
\]
is obviously strictly decreasing, it remains to show that
\[
\lambda \mapsto |v - w| = |b \coth \frac{b\lambda}{2s} - c \coth \frac{c\lambda}{2s}|
\]
increases and that its values stay below \(a\). Without loss of generality suppose \(b \geq c\). It is an easy calculus exercise to see that \(t \mapsto t \coth t\) is increasing, so the expression inside the last modulus is always positive. Define
\[
g(t) = b \coth bt - c \coth ct,
\]
so that
\[
g'(t) = -\frac{b^2}{\sinh^2 bt} + \frac{c^2}{\sinh^2 ct}.
\]
Inequality \(g'(t) \geq 0\) is equivalent with
\[
\frac{\sinh bt}{b} \geq \frac{\sinh ct}{c},
\]
which can also be verified easily, using the fact that \(t \mapsto (\sinh t)/t\) increases. Finally, we observe that
\[
\lim_{t \to \infty} g(t) = b - c < a,
\]
by the triangle inequality.

Therefore, (15) can have at most one positive solution \( \lambda \), which combines nicely with theorem 2 to prove the fact that there can be only one point \( P \in \text{Int}(T) \) such that \( \vec{E}(P) = 0 \). This leads us to the long promised uniqueness result.

**Corollary 2.** There is a unique stationary point of field \( \vec{E} \) inside \( T \). Consequently, there is a unique maximum point of potential \( V \).

From now on we denote this unique maximum potential point by \( P_{\text{max}}(x_{\text{max}}, y_{\text{max}}) \). One could name it the electrostatic center of \( T \), although the term gravitational center has already been used in the literature [4], in the study of general convex bodies in \( \mathbb{R}^n \). When we actually want to solve equation (15) for \( \lambda \), we do not know how to do it analytically, so we need to use numerical techniques. The following mini-code in Mathematica [16] successfully evaluates the sought values of \( \lambda, u, v, w \) to an arbitrary precision.

```mathematica
u := a*Coth[a*lambda/(a+b+c)];
v := b*Coth[b*lambda/(a+b+c)];
w := c*Coth[c*lambda/(a+b+c)];
lambda = lambda /. FindRoot[ Sqrt[(u^2-a^2)(a^2-(v-w)^2)]
+ Sqrt[(v^2-b^2)(b^2-(w-u)^2)]
+ Sqrt[(w^2-c^2)(c^2-(u-v)^2)]
== Sqrt[2(a^2*b^2+b^2*c^2+c^2*a^2)-(a^4+b^4+c^4)],
{lambda,1}, WorkingPrecision->30];
```

For instance, by taking \( A(-1,0), B(2,0), \) and \( C(0,2) \) we almost instantly get

\[ \lambda_{\text{max}} = 4.010297207243007522718690055346\ldots, \]

and then from [16] and [17],

\[ x_{\text{max}} = 0.272557906914867702024319226991\ldots, \]
\[ y_{\text{max}} = 0.70414818972307702017531030875\ldots. \]

Even though equation (15) does not seem to be solvable in terms of elementary functions, we do not really have a rigorous proof of this fact.

**Open problem 1.** Is it possible to express Cartesian coordinates of \( P_{\text{max}} \) (or equivalently its parameter \( \lambda_{\text{max}} \)) as elementary functions of triangle sides \( a, b, c \)?

If one desires to write the coordinates of \( P_{\text{max}} \) as explicitly as possible, it will perhaps be easier to do so using a series expansion. We still require that each term of the series is given by an elementary formula.

**Open problem 2.** Is it possible to express Cartesian coordinates of \( P_{\text{max}} \) as two convergent series, \( x_{\text{max}} = \sum_{n=1}^{\infty} x_n \) and \( y_{\text{max}} = \sum_{n=1}^{\infty} y_n \), where both \( x_n \) and \( y_n \) are elementary functions of \( a, b, c, \) and \( n \)?

Our desire to obtain a series expansion is motivated by a common practice in theoretical physics. We have to remark once again that numerical schemes for solving (15) actually do lead to approximations of \( x_{\text{max}} \) and \( y_{\text{max}} \) by sequences or series. However, in that case \( (x_n)_{n=1}^{\infty} \) and \( (y_n)_{n=1}^{\infty} \) are defined recursively, still without giving us a single explicit formula that would hold for each \( n \).

5. Trilinear coordinates.

The point \( P_{\text{max}} \) deserves to be called a triangle center, as purely physical reasons suggest that it always occupies the same relative position in any member of a family of mutually similar triangles. However, the notion of triangle center was rigorously defined in [7]. Let us
begin by introducing a convenient choice of relative homogeneous coordinates with respect to a given triangle \(ABC\). **Trilinear coordinates** of a point \(P\) inside \(\Delta ABC\) are any real numbers \(\tau_a : \tau_b : \tau_c\) such that

\[
\frac{\tau_a}{d_a} = \frac{\tau_b}{d_b} = \frac{\tau_c}{d_c},
\]

where \(d_a, d_b, d_c\) are (directed) distances from \(P\) to triangle sides \(BC, CA, AB\) respectively. Some of these distances are regarded as negative if \(P\) lies in triangle exterior, but we will not consider such points anyway. From figure 8 we see that

\[
\frac{\tau_a}{\text{area}(\Delta PBC)/a} = \frac{\tau_b}{\text{area}(\Delta PCA)/b} = \frac{\tau_c}{\text{area}(\Delta PAB)/c},
\]

so \(a\tau_a : b\tau_b : c\tau_c\) are indeed the barycentric coordinates of \(P\).

![Figure 8. Trilinear coordinates.](image)

A real valued function \(f\) defined on the set of all possible triples of triangle side lengths \((a, b, c)\) is called a **triangle center function** if it has the following properties.

- There exists a real constant \(\nu\) such that \(f(ta, tb, tc) = t^\nu f(a, b, c)\) for \(t > 0\), i.e. \(f\) is homogeneous of order \(\nu\).
- Equality \(f(a, c, b) = f(a, b, c)\) holds for any triple in the domain of \(f\).
- \(f\) is not identically 0.

A **triangle center** associated to \(f\) is then the point given by trilinear coordinates

\[
(f(a, b, c) : f(b, c, a) : f(c, a, b)).
\]

(18)

We need to remark that the same center can be associated to many different center functions \(f\).

What can we say about our point \(P_{\text{max}}\)? Calculations from the previous section immediately give

\[
\frac{\tau_a}{\text{area}(\Delta PBC)/a} = \sqrt{(u/a)^2 - 1)(a^2 - (v - w)^2)}
\]

\[
\frac{\tau_b}{\text{area}(\Delta PCA)/b} = \sqrt{(v/b)^2 - 1)(b^2 - (w - u)^2)},
\]

so we see that a good choice of triangle center function for \(P_{\text{max}}\) is

\[
f(a, b, c) = \sqrt{\left(\coth^2 a^2 - 1\right)} \left(a^2 - (b\coth b - c\coth c)^2\right),
\]

where \(\lambda_{\text{max}}\) is the unique positive solution to (15). Also, \(f\) obviously fulfills all three requirements above (with \(\nu = 1\)). One only has to observe that \(\lambda_{\text{max}}\) remains the same if the triangle is scaled by a factor \(t\). This proves the announced assertion that \(P_{\text{max}}\) is a non-trivial triangle center.
All interesting triangle centers are being collected systematically in C. Kimberling’s encyclopedia [7], which contains 5622 entries \( X_1 - X_{5622} \) at the moment of writing. Trilinear coordinates are given for these characteristic points, justifying their worth to be mentioned. In order to detect new centers, the encyclopedia also offers the search among the existing ones using the numerical value of

\[
d_a = \text{dist}(P, BC) = \frac{2\tau_a \text{area}(ABC)}{a\tau_a + b\tau_b + c\tau_c}
\]

in the particular case of triangle with sides \( a = 6, b = 9, c = 13 \). For point \( P_{\text{max}} \) it is now easy to compute this value to 30 decimal digits:

\[
d_a = 2.1107317966902891774598368888182 \ldots
\]

and realize that it does not appear in the list. Browsing the centers manually for points satisfying similar relations, one can stumble across the isoperimetric point \( X_{175} \), introduced by G. R. Veldkamp [15]. It is defined as the point for which triangles \( PBC, PCA, PAB \) have equal perimeters, which is somewhat reminiscent of our formula (6), but turns out to be a “very different” property.

Trilinear coordinates for \( P_{\text{max}} \) are implicit due to the fact that \( \lambda_{\text{max}} \) is not explicitly given. Just in the case that the first open problem we stated turns out to have a positive answer, it will be interesting to see if the trilinear coordinates can be algebraic functions of triangle sides. Once again we are quite sceptical about that possibility.

**Open problem 3.** Prove that \( P_{\text{max}} \) is a transcendental triangle center, i.e. it does not have a trilinear representation \([18]\), with \( f \) being an algebraic function of \( a, b, c \).

### 6. Approximation for the Parameter.

It remains to say a few words on estimation of \( \lambda_{\text{max}} \). Equation \([15]\) degenerates for an equilateral triangle simply to

\[
3a^2 \sqrt{\coth^2 \frac{\lambda}{3} - 1} = a^2 \sqrt{3},
\]

which is easily solved as \( \lambda_0 = 3 \log(2 + \sqrt{3}) \). An interesting fact we obtained “experimentally” is that the exact value of \( \lambda_{\text{max}} \) for a general triangle \( ABC \) is “quite correlated” with the quantity

\[
t = \log \frac{s^2}{27 \rho^2} = \log \frac{s^3}{27(s-a)(s-b)(s-c)} \geq 0,
\]

where \( \rho \) is radius of the inscribed circle. The first sign of such dependence can be noticed in figure [9], where many random choices of triangles are investigated using Mathematica [16]. Figure [10] then sketches graph of the ratio \((\lambda_{\text{max}} - \lambda_0)/t\) as a function of two angles \( \alpha \) and \( \beta \). It is obtained using Plot3D command in Mathematica [16]. Note that it is enough to restrict the domain to \( 0 < \alpha, \beta < \pi/2 \), because every triangle has at least two acute angles. Both figures illustrate that the ratio is always between (say) \( 1/2 \) and \( 1 \), although it is not so easy to prove the corresponding inequalities rigorously. The moral of this remark could be that there are some wise choices of the initial approximation to \( \lambda \) when solving \([15]\) numerically.

Another interesting observation is related to formulas \([13]\), \([16]\), and \([17]\) for Cartesian coordinates of point \( P \). If we now “free” the variable \( \lambda \) and treat it simply as a parameter that runs over interval \((0, \infty)\), then the point \( P \) traces some planar curve. Each specific choice of \( \lambda \) theoretically corresponds to a triangle center. As a simple exercise, the reader can try to find limiting positions of \( P \) as \( \lambda \to 0 \) and \( \lambda \to \infty \). We only hint that both answers are among the four classical triangle centers.
It is interesting to investigate extreme points of more general convolution potentials, such as \( p \) for parameter \( p \) taking values other than \(-1\). Observe that the integral in (5) is not singular for \( p \geq 0 \), while it even diverges for \( p \leq -2 \). In the later case one can still define potential difference (or “voltage”) between two interior points, simply by cutting out small congruent disks around those two points. More precisely, the expression

\[ V_p(P) - V_p(P') = \int_{T \setminus D_\varepsilon(P)} |PQ|^p d\lambda(Q) - \int_{T \setminus D_\varepsilon(P')} |P'Q|^p d\lambda(Q) \]

is well-defined for \( P, P' \in \text{Int}(T) \) and \( \varepsilon > 0 \) small enough and determines function \( V_p \) up to an additive constant. Our definition is a simpler equivalent to more common approaches of, either subtracting the singular part from the limit as \( \varepsilon \to 0 \), or expanding the integral over \( T \setminus D_\varepsilon(P) \) into a Laurent series in \( \varepsilon \) and taking the constant term. Case \( p = 0 \) is completely degenerate as \( V_p \) is just a constant function, so let us assume that \( p \neq -1, 0 \). We are looking for the maximum point of \( V_p \) when \( p < 0 \) and for the minimum point when \( p > 0 \).

For a general \( p \) it is more difficult to prove rigorously that potential \( V_p \) attains a unique global extreme inside \( T \). An interested reader can consult papers [9], [10], and [13], which together establish uniqueness for all possible cases of \( p \), although they discuss more general convex sets. One can still rather easily derive a formula analogous to (8). Similarly as in
sections 2 and 3 we conclude that any stationary point $P$ for $V_p$ in the interior of $T$ has to satisfy

$$
\int_0^{2\pi} R(\varphi)^{p+1} e^{i\varphi} d\varphi = 0. \tag{19}
$$

Here $R(\varphi)$ denotes length of the polar ray from $P$ to $\text{Bd}(T)$, i.e. we use the same notation as in the proof of theorem [1]. This equation even allows us to plug in $p = 0$, which corresponds to the logarithmic potential, see [10].

One can investigate points $P$ satisfying equation (19), but almost all choices of $p$ lead to unnamed triangle centers. Let us list several interesting exceptions.

Case $p = 2$. The point where $V_2$ attains its minimum is exactly the centroid $X_2$. In order to see that, we write

$$
0 = \int_0^{2\pi} R(\varphi)^3 e^{i\varphi} d\varphi = 3 \int_0^{2\pi} r e^{i\varphi} rdrd\varphi,
$$

i.e.

$$
\iint_T \overrightarrow{PQ} d\lambda(Q) = \vec{0},
$$

which confirms that $P$ is the center of mass of $T$.

Case $p = -4$. Related to the previous case, there is an interesting interpretation of the point $P$ satisfying (19) for $p = -4$. This time we write

$$
0 = \int_0^{2\pi} R(\varphi)^{-3} e^{i\varphi} d\varphi = 3 \int_0^{2\pi} r^{1/R(\varphi)} e^{i\varphi} rdrd\varphi.
$$

Let $\iota$ be the planar inversion with respect to the circle centered at $P$ and having radius 1. Then $\iota(\text{Bd}(T))$ constitutes a closed curve around $P$ and we denote the region it encloses by $S$. This set is depicted in figure 11. The last equation can be restated as

$$
\iint_S \overrightarrow{PQ} d\lambda(Q) = \vec{0},
$$

i.e. $P$ is the center of mass of the planar region $S$.

Case $p = -2$. This case turns out to be unexpectedly interesting and has already appeared in the literature. K. Shibata [14] considered the problem of choosing the position of a street

![Figure 11. Region S enclosed by inverted image of Bd(T).](image)
lamp in a triangular park, in a way that it maximizes the total brightness of the park. He further reformulates the problem as finding the maximum point of the potential \( V \) and names it the illuminating center of \( T \). Geometrical characterization of such point \( P \) inside \( \triangle ABC \) that was given in [14] can be restated as

\[
\frac{\angle BPC \ \text{area}(\triangle BPC)}{\angle CPA \ \text{area}(\triangle CPA)} = \frac{\angle APB \ \text{area}(\triangle APB)}{\angle APB}. 
\]

Shibata’s text does not contain a complete proof of this relation, but one can now derive it rather easily from (19). Using the notation from the proof of theorem 1 one first calculates

\[
\int_{\varphi_B}^{\varphi_A} R(\varphi)^{-1} e^{i\varphi} d\varphi = \int_{\alpha_1}^{\pi - \beta_2} \sin \psi \frac{d\psi}{d_c} e^{i(\psi + \varphi_A - \alpha_1)} d\psi 
\]

\[
= -ie^{i\varphi_B} \frac{e^{i\varphi_A} \cot \beta_2}{4r_B} + \frac{e^{i\varphi_B} \cot \alpha_1}{4r_A} - \frac{\angle APB}{2id_c} e^{i\theta_c},
\]

so that (19) gives

\[
\frac{e^{i\varphi_A} (\cot \alpha_1 + \cot \alpha_2)}{4r_A} + \frac{e^{i\varphi_B} (\cot \beta_1 + \cot \beta_2)}{4r_B} + \frac{e^{i\varphi_C} (\cot \gamma_1 + \cot \gamma_2)}{4r_C}
\]

\[
= \frac{\angle BPC}{2id_a} e^{i\theta_a} - \frac{\angle CPA}{2id_b} e^{i\theta_b} - \frac{\angle APB}{2id_c} e^{i\theta_c} = 0.
\]

Straightforward computation shows that the sum of the first three terms is 0 for just any point \( P \), so the above equality becomes

\[
\frac{\angle BPC}{d_a} e^{i\theta_a} + \frac{\angle CPA}{d_b} e^{i\theta_b} + \frac{\angle APB}{d_c} e^{i\theta_c} = 0,
\]

i.e.

\[
\frac{\angle BPC}{\text{area}(\triangle BPC)} \overrightarrow{CB} + \frac{\angle CPA}{\text{area}(\triangle CPA)} \overrightarrow{AC} + \frac{\angle APB}{\text{area}(\triangle APB)} \overrightarrow{BA} = \overrightarrow{0}.
\]

The reader will easily fill in the details.

One can fix a triangle and draw points \( P \) obtained by solving (19) as parameter \( p \) varies. We can also add the electrostatic potential point (corresponding to \( p = -1 \)) and even consider the limit cases \( p \to \pm\infty \). For an acute or right triangle we obtain a closed curve joining the incenter \( X_1 \) with the circumcenter \( X_3 \) (see figure 12), while for an obtuse triangle this curve joins the incenter with the midpoint of the longest side (see figure 13). Shibata [14] named this curve the *potential arc* and identified the limits as \( p \to \pm\infty \). They can be derived rigorously from a result in [10], which is formulated for more general convex sets. The curve

![Figure 12. Extreme points for various potentials \( V_p \) on an acute triangle.](image)
in question also passes through the centroid $X_2$, as we have already observed. However, there is a well-known curve that passes through all four classical centers, all triangle vertices, and all side midpoints. It is called the *Thomson cubic* and the reader can see its three branches in figure 14. Its equation in trilinear coordinates $\tau_a, \tau_b, \tau_c$ is given by

$$bc \tau_a (\tau_b^2 - \tau_c^2) + ca \tau_b (\tau_c^2 - \tau_a^2) + ab \tau_c (\tau_a^2 - \tau_b^2) = 0.$$  \hspace{1cm} (20)

Comparing illustrations 12 and 14 one might naively suspect that the two curves coincide, i.e. that all points satisfying (19) must also lie on the Thomson cubic. High-precision numerics reveals that this is generally not the case, but also suggests that it could still be true for some surprising choices, such as $p = -3$. We would be interested in rigorous results of this type.

**Open problem 4.** Determine (with a proof) all values of $p$ for which the extreme point of $V_p$ lies on the Thomson cubic (20) of the corresponding triangle.

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