The Open Quantum Brownian Motion and continual measurements

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Abstract

This article is a mathematical analysis of the Open Quantum Brownian Motion. This object was introduced in [BBT14] as the limit of a family of Open Quantum Random Walks on the graph Z. We prove the convergence for the three possible descriptions of this object: the quantum trajectory satisfying a Belavkin Equation, the unitary evolution on the Fock space satisfying a quantum Langevin Equation, and the Lindbladian evolution. We introduce a very general framework for the continual measurement of
non-demolition observables, which is applied to the measurement of the position of the Open Quantum Brownian Motion, and we probe some questions related to the convergence of processes in this context.

1 Introduction

1.1 General introduction

Open Quantum Random Walks (OQW) are a quantum generalization of discrete Markov chains and were introduced by Attal, Petruccione, Sabot and Sinayskyi in [ASPS12]. They consist into a particle moving randomly on a discrete graph with transition probability depending on its internal quantum state. They model a quantum system subject to dissipation or repeated measurement with control, and are used for example as a toy model to study coherence in photosynthetic cells [AIK13]; they have been the subject of extensive mathematical study, see the end of Paragraph 1.2 for more references. While OQW are defined on discrete graph and on discrete space, the Open Quantum Brownian Motion (OQBM) was introduced in [BBT14] to model a particle moving on the line in continuous time. It was defined as the limit of a family of OQW on \( \mathbb{Z} \), with a diffusive normalization, i.e. with a time scale \( \tau \) going to zero and a space scale \( \delta = \sqrt{\tau} \).

The obtained process depends in two operators \( N \) and \( H \); in the trivial case where \( N = H = 0 \) the classical Brownian motion is recovered. The Open Quantum Brownian Motion has been derived from a microscopic physical model in [SP13] and [SP16]. A mathematically interesting phenomenon was observed on the OQBM, namely the transition from diffusive to ballistic behavior as the parameters \( N \) and \( H \) are changed [BBT13] with the appearance of so-called spikes in the ballistic regime [TBB15] [BBT15], which were then studied in the context of more general stochastic differential equations [BB17], [KL19] and [BCC18].

As for OQWs, the OQBM has three different descriptions. It can be seen a has a Lindblad evolution \( \rho_t = \Lambda_S (\rho_0) \) on the Hilbert space \( \mathcal{H}_G \otimes L^2(\mathbb{R}) \), where \( \mathcal{H}_G \) represents the internal state of the particle. The second description is a Stinespring dilation \( \rho_{\text{tot},t} = \mathcal{U}_t (\rho_0 \otimes |\Omega\rangle \langle \Omega|) \mathcal{U}_t^* \) on \( \mathcal{H}_G \otimes L^2(\mathbb{R}) \otimes \Phi \), where \( \Phi \) is the Fock space and \( \mathcal{U}_t \) satisfies a Quantum Stochastic Differential Equation (QSDE) called the Hudson-Parthasarathy Equation. This representation is more complete than the Lindbladian one, since it allows to compute the quantum correlation between the events at two different times. Finally, upon the continual measure of the position of the particle, it admits a quantum trajectories unraveling, that is a random process \( (\eta_t, X_t)_{t \in \mathbb{R}} \) where \( \eta_t \) is a random state on \( \mathcal{H}_G \) and \( X_t \in \mathbb{R} \) is a random position. When \( \mathcal{H}_G \) is of finite dimension this process obeys a classical stochastic differential equation, often called the diffusive Belavkin Equation [BGM04] [Bel92].

In the original article on the OQBM [BBT14], most results where derived formally but not rigorously proved. The main purpose of this article is to explicit the mathematical meaning of the statements of [BBT14], pointing out some of the mathematical issues and completing the proofs.

In the second part of the introduction, we introduce OQWs and the formal definition of the OQBM, and the mathematical problems raised by this definition, which are tackled in the rest of the article. Besides the problem of the convergence, a mathematical issue appears in the description of the Lindbladian: for an OQW, the evolution projects the states on the set of diagonal state, i.e. states of the form \( \rho = \sum_{x \in V} \rho(x) \otimes |x\rangle \langle x| \in \mathcal{S}(\mathcal{H}_G \otimes L^2(V)) \), where \( V \) is the set of vertices of the graph on which the particle is moving. In the continuous case, diagonal operators are replaced by multiplication operators of the form \( \int_{\mathbb{R}} \rho(x) d|x\rangle \langle x| \), which cannot be trace class and hence cannot be a state. Hence, the discrete object which converges to the continuous OQBM is actually not an OQW in the strict meaning of the term, though it coincides with an OQW on a microscopic physical model in [SP13] and [SP16]. A mathematically interesting phenomenon was observed on the OQBM, namely the transition from diffusive to ballistic behavior as the parameters \( N \) and \( H \) are changed [BBT13] with the appearance of so-called spikes in the ballistic regime [TBB15] [BBT15], which were then studied in the context of more general stochastic differential equations [BB17], [KL19] and [BCC18].

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In the second section, we introduce the repeated measurement model and the quantum stochastic calculus and we prove the convergence of the discrete models for the OQBM to the continuous one in each description: for the unraveled process, we prove a convergence in distribution in the Skorokhod space as a direct consequence of a theorem of Pellegrini [Pc03]. For the unitary dilations, the strong convergence of the unitary operators is proved from a theorem of Attal and Pautrat [AP06]. This strong convergence allows to prove the strong convergence for the Lindblad operators.

In the third section we look into another claim of the article [BBT14], in which the unraveled process \( (\eta_t, X_t)_{t \in [0,T]} \) is obtained from the continual measurement of an observable under the evolution by the unitary operators \( \mathcal{U}_t \). This makes use of the quantum filtering theory [Gon97] [BGM04] [Bel92] and the notion of non-demolition measurement. We introduce rigorously the continual measurement of non-demolition observables in a way which is equivalent to the quantum filtering approach but we believe is more adapted to the Schrödinger picture of the evolution, and we apply it to the case of the OQBM. Finally, we ponder the
relation between the convergence of the unitary operators $U_t$ and the convergence in distribution of the unraveling, obtaining only an incomplete result which generates a few open questions.

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1.2 Open Quantum Random Walks and the Open Quantum Brownian Motion

In this subsection we introduce the notion of Open and Unitary Quantum Walks (OQW and UQW) and we describe the formal definitions of the Open Quantum Brownian Motion (OQBM) and the related mathematical issues.

1.2.1 General notations

The basic object in quantum mechanics is a separable Hilbert space $H$ (all Hilbert spaces are implicitly supposed to be separable in this article). Let us gather some of the notations and definitions we will use:

- The identity operator on $H$ (respectively $H_A$ and $\mathbb{C}^n$) is written $I_H$ (respectively $I_A$ and $I_n$) or simply $I$ when it does not cause confusion. If $H_A$ and $H_B$ are two spaces and $A$ is an operator on $H_A$, we shall denote by $A$ the operator $A \otimes I_B$ on $H_A \otimes H_B$.
- A vector $v \in H$ may also be written $|v\rangle$, and the corresponding linear form is denoted $\langle v|$, so that $|v\rangle \langle v|$ is the orthogonal projection on $\mathbb{C}v$. In any tensor space $H_A \otimes H_B$, the partial trace with respect to $H_B$ is written $\text{Tr}_B$ or $\text{Tr}_{H_B}$.
- The algebra of bounded operators on $H$ is written $\mathcal{B}(H)$, endowed with the operator norm $\|A\|$ (sometimes written $\|A\|_\infty$ to avoid confusion with other norms). The space of compact operators on $H$ is written $\mathcal{B}_c(H)$. An operator on $\mathcal{B}(H)$ is called a super-operator.
- The adjoint of an operator $A$ is denoted $A^\ast$.
- The Schatten space of order $p$ is the space $S^p(H)$ of bounded operators $A$ such that $\text{Tr}(|A|^p) < +\infty$, endowed with the norm $\|A\|_p = \text{Tr}(|A|^p)^{1/p}$. In particular, $S^1(H)$ is the space of trace-class operators.
- The $\sigma$-weak (or ultraweak) topology on $\mathcal{B}(H)$ is the topology generated by the seminorms
  $$\|A\|(u_i)_{i \in \mathbb{N}}, (v_i)_{i \in \mathbb{N}} := \sum_{i \in \mathbb{N}} \langle u_i, Av_i \rangle$$
  where the $u_i$ and $v_i$’s are vectors in $H$ with $\sum_{i \in \mathbb{N}} \|u_i\|^2 + \|v_i\|^2 < +\infty$.
- For a measured space $(\mathcal{X}, \mathcal{F}, \mu)$ we write the corresponding $L^p$ space as $L^p(\mathcal{X}, \mathcal{F}, \mu)$ or when it does not cause confusion $L^p(\mathcal{X})$ or even $L^p$.
- For any Banach space $B$, we write $L^2(\mathcal{X}, B, \mu)$ the space of $L^2$ function from $\mathcal{X}$ to $B$, and the Sobolev space of functions $f : \mathbb{R}^n \to B$ with distributional derivatives $f^{(k)} \in L^p$ for $k < l$ is written $W^{l,p}(\mathbb{R}^n, B)$. For $p = 2$ and $B = H$ a Hilbert space, it is itself a Hilbert space and is written $H^l(\mathbb{R}^n)$. It is isomorphic to $H \otimes H^l(\mathbb{R}^n)$ and injected to a dense subset of $L^2(\mathbb{R}^n, H) = H \otimes L^2(\mathbb{R}^n)$. We write $X$ the position operator defined by $Xf(x) = xf(x)$, and $P = -i\partial_x$ the impulsion operator with domain $H^1(\mathbb{R}, \text{Leb})$.
- On the space $L^2(\mathcal{X}, \mu)$, for any measurable function $f : \mathcal{X} \to \mathbb{C}$ we write $M_f$ the operator of multiplication by $f$, defined by $M_fg(x) = f(x)g(x)$ for any $g$ such that $fg \in L^2(\mathcal{X}, \mu)$.
- We write $1_A$ the indicator function of the set $A$, and $1 = 1_\mathcal{X}$.
- We write $\otimes_{alg}$ the algebraic tensor product and $\otimes$ the completed tensor product of Hilbert spaces.
- We generally use the letter $\mathcal{G}$ for isometries or unitary operators whose role is to identify a space as the subspace of another, or to identify two representations of the same space. This type or map is often implicit in the literature of quantum mechanics, so there is no standard notation; we chose the letter $\mathcal{G}$ because it evokes the curved arrow $\hookrightarrow$ used for injections in category theory.
1.2.2 Unitary and Open Quantum Walks

Unitary quantum walks are generally called simply “quantum random walks”, we add “unitary” to distinguish them from open quantum walks. They were formally introduced in [ADZ93] as quantum version of classical random walks on graphs, and they were extensively studied, notably in relation to quantum computing: universal quantum computation can be obtained with UQW [LCE+09] and has been used to develop quantum algorithm, generally for the search of marked node in graphs (see [CNAO16] [Koc17] among many other articles). See the comprehensive review [VA12] on unitary quantum random walks.

For the sake of completeness, let us briefly describe unitary quantum walks. A UQW represents a quantum particle moving on a graph $G = (V, E)$, where the set of vertices $V$ is countable or finite. The internal state of the particle is described by a space $H_G$ (which is called the “gyroscope” in this article; it is also called the “quantum coin”, the internal space or the chirality space in the literature). The Hilbert space of the position of the particle is described by a space $H$ (which is not unique).

\[ H = (H_G)^{\otimes \infty} \]

\[ \rho \to U^\rho \]

\[ \sum_{x \in V} U(x \rightarrow y) \otimes \ket{y} \bra{x} \]

\[ U = \sum_{(y \rightarrow x) \in E} U(x \rightarrow y) \otimes \ket{y} \bra{x} \]

An operator of the form of Equation (1.1) is unitary if and only if for any $x, y \in V$ we have

\[ \sum_{x \in x_i \rightarrow y_i} U(x \rightarrow y)^* U(y \rightarrow x) = \delta_{x,y} \mathbb{I}_{H_G} \]

The most classical example is the one of translation-invariant UQW on the graph $\mathbb{Z}$ (with edges between nearest neighbours). It can always be written of the form

\[ U = B_- \otimes D^* + B_+ \otimes D \]

where $D$ is the translation to the right

\[ D = \sum_{i \in \mathbb{Z}} |i + 1 \rangle \langle i| \]

and $B_-$ and $B_+$ are operators such that $B^2 B_- + B^*_+ B_+ = I_G$ and $B^* B_+ = 0$ (or equivalently if $H_G$ is of finite dimension, there exists an orthogonal projector $P$ and a unitary $V$ on $H_G$ such that $B_- = VP$ and $B_+ = V(I_G - P)$).

Open quantum Walks where introduced in [ASPS12] as another dynamics on $H_G \otimes H_z$, which is not unitary as for UQWs but completely positive, meaning that it corresponds to the dynamics of an open quantum system. Let us briefly describe this concept. We look at a system described by a Hilbert space $H_S$ in interaction with an exterior system $H_p$, which are initially independent (that is in a state of the form $\rho_S \otimes \rho_B$) and evolve during some time $\tau$, with an evolution described by a unitary $V$. The state on $H_S$ after the evolution is then described by $\text{Tr}_{H_p}(U(\rho_S \otimes \rho_B)V^*)$. This leads to the following definition:

**Definition 2.** We call quantum channel on $H_S$ a linear map $\Lambda$ on $\mathcal{S}(H_S)$ which is of the form

\[ \Lambda(\rho) = \text{Tr}_{H_p}(V(\rho \otimes \rho_p)V^*) \]

for some space $H_p$ and some state $\rho_p$ on $H_p$ and some unitary $V$ on $H_S \otimes H_p$.

Quantum channels can be characterized as the completely positive, trace-preserving and $\sigma$-weakly continuous maps on bounded operators (see for example Chapter 6 of [A11]). Alternately, they are the maps which are of the form

\[ \Lambda(\rho) = \sum_{k=1}^r K_k \rho K_k^* \]

where $r \in \mathbb{N} \cup \{+\infty\}$, the $K_k$'s are bounded operators on $H_A$ with $\sum_{k=1}^r K_k^* K_k = I_A$ and are called the Krauss operators of $\Phi$. The tripe $(H_p, V, \rho_p)$ corresponding to $\Lambda$ is called a Stinespring dilation of the channel (it is not unique).

We can now define open quantum walks:
Definition 3. An Open Quantum Walk (OQW) on the graph $G = (\mathcal{V}, E)$ with gyroscope space $\mathcal{H}_G$ is a quantum channel $\Lambda$ on $\mathcal{H}_G \otimes \mathcal{H}_z$ which is of the form

$$\Lambda(\rho) = \sum_{(y \leftarrow x) \in E} \left( K_{(y \leftarrow x)} \otimes |y\rangle \langle x| \right) \rho \left( K^*_{(y \leftarrow x)} \otimes |y\rangle \langle x| \right)$$

for some operators $K_{(y \leftarrow x)}$ satisfying

$$\sum_{y \in x_+} K^*_{(y \leftarrow x)} K_{(y \leftarrow x)} = I_G$$

for all $x \in \mathcal{V}$.

A peculiar feature of OQWs is that $\Lambda(\rho)$ is always block-diagonal with respect to the basis $(|x\rangle)_{x \in \mathcal{V}}$ of $\mathcal{H}_z$, that is, we can write

$$\Lambda(\rho) = \sum_{x \in \mathcal{V}} \rho_x \otimes |x\rangle \langle x|$$

for some family of positive semi-definite operators $(\rho_x)_{x \in \mathcal{V}}$.

To any OQW corresponds a stochastic process called the quantum trajectories of the OQW:

**Definition 4.** The quantum trajectory of an OQW $\Lambda$ is the process $(X_n, q_n)_{n \in \mathbb{N}}$ with $X_n \in \mathcal{V}$ and $q_n \in \mathcal{S}(\mathcal{H}_G)$ such that $(X_{n+1} \leftarrow X_n) \in E$ for all $n$ and with the following transition probabilities:

$$P(X_{n+1} = y | X_n = x) = Tr \left( K_{(y \leftarrow x)} q_n K^*_{(y \leftarrow x)} \right)$$

(1.2)

$$q_{n+1} = \frac{K_{(X_{n+1} \leftarrow X_n)} q_n K^*_{(X_{n+1} \leftarrow X_n)}}{Tr \left( K_{(X_{n+1} \leftarrow X_n)} q_n K^*_{(X_{n+1} \leftarrow X_n)} \right)}.$$  

(1.3)

When we fix for initial state $X_0 = x$ and $q_0 = \rho$ the quantum trajectory is related to the OQW by the formula

$$E (g_n \otimes |X_n\rangle \langle X_n|) = \Lambda^n (\rho \otimes |x\rangle \langle x|).$$

This direct relation between the OQW and a random walk on the graph makes it closer to classical random walks than UQW. The concept of OQW has attracted significant interest; a central limit Theorem on the trajectories of translation invariant OQW on $\mathbb{Z}^n$ has been proved in [AGPS15] and extended to more general lattices in [CPT14, KKSY18] and completed by a large deviation principles in [CPT14], while criterions for the ergodic properties of the quantum channel $\Lambda$ where proved in [CP15]. Two notable generalisation of OQW have been defined, one which interpolates between OQW and UQW [XY12], and another which considers continuous-time OQW [Pel14], still on discrete graphs.

### 1.2.3 The formal definition of the Open Quantum Brownian Motion

The idea of the Open Quantum Brownian Motion is to define a dynamics which is similar to the OQW dynamic, but in continuous time and continuous space (with a particle moving on the line). It is defined as a limit of a family of OQW on the graph $\mathbb{Z}$, with a time scale $\tau$ and a space scale $\delta = \sqrt{\tau}$ going to zero.

**In the rest of the article, we will write $\delta = \sqrt{\tau}$ the space scale.** Let us define formally the Open Quantum Brownian Motion, following [BBT14]. We consider the graph $\delta \mathbb{Z}$ (with nearest neighbours edges) and $\mathcal{H}_{\tau, z} = l^2(\delta \mathbb{Z})$ and a gyroscope space $\mathcal{H}_G$. We define the OQW $\Lambda_\tau$ on $\mathcal{S}(\mathcal{H}_G \otimes \mathcal{H}_{\tau, z})$ by

$$\Lambda_\tau(\rho) = \sum_{x \in \delta \mathbb{Z}} (B_{\tau, -} \otimes |x - \delta\rangle \langle x|) \rho (B^*_{\tau, -} \otimes |x - \delta\rangle \langle x - \delta|) + (B_{\tau, +} \otimes |x + \delta\rangle \langle x|) \rho (B^*_{\tau, +} \otimes |x + \delta\rangle \langle x + \delta|)$$

(1.4)

where the Krauss operators $B_{\tau, +}$ and $B_{\tau, -}$ satisfy

$$B_{\tau, \pm 1} = \frac{1}{\sqrt{2}} \left( I \pm \delta N + \tau \left(-iH - \frac{1}{2}N^*N \pm M\right) \right) + O(\tau^{3/2}).$$

(1.5)

for some bounded operators $N, H, M$ on $\mathcal{H}_G$ with $H$ self-adjoint. It is argued in [BBT14] that it is the only choice of $B_{\tau, \pm}$ such that $\Lambda_\tau[t/\tau]$ converges for all $t$ as $\tau = \delta^2 \to 0$. Let us derive formally the limit: consider the state $\rho(n) = \Lambda_\tau^n(\rho)$. For any $n > 0$ it is of the form

$$\rho(n) = \sum_{x \in \delta \mathbb{Z}} \rho(n, x) \otimes |x\rangle \langle x|.$$
for some positive semi-definite operators $\rho(n, x)$ on $\mathcal{H}_G$. By the definition of $\Lambda_\tau$ we have

$$\rho(n + 1, x) = \frac{\rho(n, x + \delta) + \rho(n, x - \delta)}{2} + \frac{\rho(n, x + \delta) - \rho(n, x - \delta)}{2} N + \tau \left( \mathcal{L} \left( \frac{\rho(n, x + \delta) + \rho(n, x - \delta)}{2} \right) + M \frac{\rho(n, x + \delta) - \rho(n, x - \delta)}{2} M^* \right) + O(\tau^2)$$

(1.6)

where the super-operator $\mathcal{L}$ on $\mathcal{S}(\mathcal{H}_G)$ is defined by

$$\mathcal{L}(\rho) = -i[H, \rho] - \frac{1}{2} \{ N^* N, \rho \} + N \rho N^* .$$

(1.7)

Assume that $(t, x) \mapsto \rho([t/\tau], x)$ converges as $\tau \to +\infty$ to some function $(t, x) \mapsto \rho(t, x)$ from $[0, +\infty) \times \mathbb{R}$ to $\mathcal{S}(\mathcal{H}_G)$. We have the formal, non-rigorous estimates

$$\frac{\rho(t + \tau, x) - \rho(t, x)}{\tau} \approx \frac{\partial}{\partial t} \rho(t, x)$$

(1.8)

$$\frac{\rho(t, x + \delta) - \rho(t, x - \delta)}{2\delta} \approx \frac{\partial}{\partial x} \rho(t, x)$$

(1.9)

$$\frac{\rho(t, x + \delta) - \rho(t, x - \delta)}{\delta^2} \approx \frac{\partial^2}{\partial x^2} \rho(t, x)$$

(1.10)

Assuming that these estimates are justified and replacing them in Equation (1.6) we obtain

$$\frac{\partial}{\partial t} \rho(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho(t, x) + N \frac{\partial}{\partial x} \rho(t, x) + \frac{\partial}{\partial x} (\rho(t, x) N^* + \mathcal{L}(\rho(t, x))) .$$

(1.11)

This equation defines the dynamics of the Open Quantum Brownian Motion. We call it the Lindblad Equation for the OQBM as it represents the generator of a continuous-time dynamics $t \mapsto \Lambda^t(\rho)$ where $\Lambda^t$ is a quantum channel for all $t \in [0, +\infty)$. This raises several problems, but before listing them let us describe the two other descriptions of the OQBM. The first is the stochastic process of quantum trajectories. Let $(X_n, \varrho_n)_{n \in \mathbb{N}}$ be the quantum trajectories of the Open Quantum Walk $\Lambda_\tau$ (with $X_n \in \delta \mathbb{Z}$), then another formal estimate gives a stochastic differential equation for the limit of $(X_{[t/\tau]}, \varrho_{[t/\tau]})_{t \in [0, +\infty)}$ as $\tau \to 0$:

$$\left\{ \begin{array}{ll}
 d\varrho_t &= \mathcal{L}(\varrho_t) dt + (N \varrho_t + \varrho_t N^* - \varrho_t T(\rho_t)) dB_t \\
 dX_t &= T(\varrho_t) dt + dB_t
 \end{array} \right.$$  

(1.12)

where $B_t$ is a Brownian motion and $dB_t$ is its Itô differential, and $T(\rho) = \text{Tr} ((N + N^*) \rho)$.

The last representation of the OQBM is the “dilated” one, it consists in a unitary evolution $(\mathcal{U}_t)_{t \in [0, +\infty)}$ on a space $\mathcal{H}_G \otimes \mathcal{H}_z \otimes \Phi$ where $\Phi$ is the bosonic Fock space on $L^2(\mathbb{R})$. It satisfies a Hudson-Parthasarathy Equation (whose formalism is introduced later in the article):

$$d\mathcal{U}_t = \left( (-iH - \frac{1}{2} N^* N + \frac{1}{2} \frac{\partial^2}{\partial x^2} - \partial_x N) dt + (N - \partial_x) da_0^1(t) + (-N^* - \partial_x) da_0^1(t) \right) \mathcal{U}_t$$

(1.13)

where $da_0^1(t)$ is the creation operator of $\mathbb{1}_{[0,t]}$ and $da_0^1(t) = (a_0^1(t))^*$ is the corresponding annihilation operator. It is related to the Lindblad dynamics of the OQBM through the partial trace,

$$\Lambda^t(\rho) = \text{Tr}_\Phi (\mathcal{U}_t (\rho \otimes |\Omega\rangle \langle \Omega|) \mathcal{U}_t^*)$$

and it is related with the quantum trajectories through the concept of continual measurement described in the third section of this article.

The purpose of this article is to adress the many mathematical problems aroused by these definitions, as listed below.

1. A first problem is the projection on diagonal states. Indeed, for any $t \geq \tau$ the state $\rho_{\tau, t} = \Lambda^{t/\tau}(\rho)$ is of the form

$$\sum_{x \in \delta \mathbb{Z}} \rho(t, x) \otimes |x\rangle \langle x|$$

6
In this section, we outline some of the mathematical objects of quantum mechanics, describing states on
2 The Open Quantum Brownian motion
show that the discrete OQBM can be seen as a repeated interactions model supplemented with a quantum
some von Neumann algebras, measurement, the repeated interactions setup and the Belavkin Equation. We
6. The OQBM can be generalized in various ways, as noted in [BBT14], and related problems are listed
5. One conceptual problem is to relate quantum trajectories and the dilation
4. We show the strong convergence of \( \tilde{\Lambda} \) (Theorem 26).
3. For the convergence to the unitary dilation \( \mathcal{U}_t \), we construct a unitary dilation \( (\mathcal{U}_{\tau,n})_{n \in \mathbb{N}} \) of the discrete-time semigroup \( (\Lambda^\tau_n)_{n \in \mathbb{N}} \), and show its strong convergence to a unitary \( \mathcal{U}_t \) satisfying the Hudson-Parthasarathy Equation \( \{1.13\} \) (Theorem 21). For this, we use a theorem of Attal and Pautrat [AP06]; this theorem is designed to work with bounded operators, while the operator \( \partial_x \) is unbounded. This problem is bypassed by considering the restriction to the space \( D_F = \{ f \in L^2(\mathbb{R}) \mid \text{supp}(\mathcal{F} f) \subset [-C, C] \} \)
2. The convergence for the quantum trajectories is the less problematic. We prove it in the case where \( \mathcal{H}_G \)
is of finite dimension, on the time interval \( [0, T] \) for some fixed \( T \), as a convergence in distribution in the Skorokhod space of continuous functions (Proposition 14). The convergence is a direct consequence of a theorem of Pellegrini [Pel08].
1. We show the strong convergence of \( \tilde{\Lambda} \) to the quantum channel

The problem is then to show that \( \Lambda_t \) satisfies indeed Equation \( \{1.11\} \), provided \( \rho \) is a state on
the space of Sobolev states, show an extended version of Equation \( \{1.11\} \) for Sobolev states on \( \mathcal{B}(\mathcal{H}_G \otimes \mathcal{H}_z) \) (Theorem 25) with the use of the quantum stochastic calculus on \( \mathcal{U}_t \), and restrict this equation to states on \( \mathcal{B}(\mathcal{H}_G \otimes \mathcal{H}_z) \) to obtain Equation \( \{1.11\} \)
(Theorem 26).

5. One conceptual problem is to relate quantum trajectories and the dilation \( \mathcal{U}_t \). This is the object of the third section of this article, where we expose the formalism of continual measurement of non-demolition evolution. We prove a partial result relating the convergence of quantum trajectories and of the dilation; this theorem is redundant in the case of the OQBM since the convergence of quantum trajectories can be proved by other ways, but it applies to more general evolution under continual measurement.

6. The OQBM can be generalized in various ways, as noted in [BBT14], and related problems are listed
at the end of this article.

2 The Open Quantum Brownian motion

In this section, we outline some of the mathematical objects of quantum mechanics, describing states on
some von Neumann algebras, measurement, the repeated interactions setup and the Belavkin Equation. We
show that the discrete OQBM can be seen as a repeated interactions model supplemented with a quantum
description of a pointer linked with some repeated measurement.
2.1 von Neumann algebras and quantum states

The notion of standard measured space is crucial in the mathematical definition of measure, since for every Hilbert space $\mathcal{H}$ there exists a standard measured space $(\mathcal{X}, \mathcal{F}, \nu)$ such that $\mathcal{H}$ is isomorphic to $L^2(\mathcal{X}, \nu)$. This also allows to study commutative von Neumann algebras, and to relate the notion of quantum state to classical probabilities and the measurement of observables.

2.1.1 Standard measured space

Standard measured spaces form a very large class of measured space; notably, two spaces of special interest in this article are $\mathcal{X} = \mathbb{R}$ with the Lebesgue measure, and $\mathcal{X} = W([0, +\infty))$ the Wiener space on $[0, +\infty)$ equipped with the Wiener measure (i.e. the space of continuous functions on $[0, +\infty)$ equiped with the measure corresponding to the Brownian motion). Standard measured spaces have many different characterizations, see the chapter on Lebesgue-Rohlin spaces in Bogachev II [Bog06]. Let us describe two of them:

**Definition 5.** Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a measured space (every measured are nonnegative in this article). It is called a standard measured space if it satisfies one of the following equivalent properties:

1. There exists a measure $\nu$ on $\mathbb{R}$ of the form $\nu = \nu_1 + \sum_{i \in \mathbb{N}} c_i \delta_i$, where $\nu_1$ is absolutely continuous with respect to the Lebesgue measure, the $\delta_i$ are the Dirac distributions at $i$ and the $c_i$ are nonnegative numbers, such that $(\mathcal{X}, \mathcal{F}, \mu)$ is almost isomorphic to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, that is, there exists sets of full measure $A \subset \mathcal{X}$ and $B \subset \mathbb{R}$ and a measure-preserving isomorphism between $(A, \mu)$ and $(B, \nu)$.

2. There exists a complete metric $d$ on a set of full measure $D \subset \mathcal{X}$ such that $\mathcal{F}|_D$ is the $\sigma$-algebra generated by open sets for $d$ and $\mu$ is a Radon measure for this topology.

Note that standard measured spaces are necessarily almost separated (i.e. for almost every $x \neq y \in \mathcal{X}$ there exists two disjoint measurable sets $A, B \in \mathcal{F}$ with $x \in A$ and $y \in B$). More importantly, if $\mathcal{F}_1 \subset \mathcal{F}$ is another $\sigma$-algebra, the measured space $(\mathcal{X}, \mathcal{F}_1, \mu)$ is standard if and only if $\mathcal{F}_1 = \mathcal{F}$. If $\mathcal{F}_1 \neq \mathcal{F}$, we make $(\mathcal{X}, \mathcal{F}_1, \mu)$ into a standard probability space by quotient:

**Definition 6.** For any standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ with a sub-$\sigma$-algebra $\mathcal{F}_1$, let $\mathcal{X}/\mathcal{F}_1$ the quotient of $\mathcal{X}$ by the relation: $x \sim y$ if every set $A \in \mathcal{F}_1$ containing $x$ also contains $y$. There is a surjective map $s_{\mathcal{F}_1} : \mathcal{X} \to \mathcal{X}/\mathcal{F}_1$, we endow $\mathcal{X}/\mathcal{F}_1$ with the image of $\mathcal{F}_1$ by $s_{\mathcal{F}_1}$ and the push-forward measure of $\mu$ by $s_{\mathcal{F}_1}$, which we still write $\mathcal{F}_1$ and $\mu$. The space $(\mathcal{X}/\mathcal{F}_1, \mathcal{F}_1, \mu)$ is a standard measured space, called the quotient of $(\mathcal{X}, \mathcal{F}, \mu)$ by $\mathcal{F}_1$.

There exists many different maps $r_{\mathcal{F}_1} : \mathcal{X}/\mathcal{F}_1 \to \mathcal{X}$ such that $s_{\mathcal{F}_1} \circ r_{\mathcal{F}_1} = 1_{\mathcal{X}_1}$. Each of them gives an identification of $\mathcal{X}_1$ with a subspace of $\mathcal{X}$, and we have a map $c = r_{\mathcal{F}_1} \circ s_{\mathcal{F}_1} : \mathcal{X} \to \mathcal{X}$ onto this subspace.

An extension of a standard measured space $(\mathcal{X}_1, \mathcal{F}_1, \mu_1)$ is another standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ with a surjective measurable map $s : \mathcal{X} \to \mathcal{X}_1$ such that the push forward measure $s_* \mu$ of $\mu$ by $s$ is $\mu_1$.

These notions are useful in the description of commutative von Neumann algebras.

2.1.2 Commutative von Neumann algebras

The set of quantum observables of a system is described by a von Neumann algebra on $\mathcal{H}$, i.e. a unital subalgebra of $\mathcal{B}(\mathcal{H})$ which is stable by adjoint and closed for the strong topology. This article does not involve most of the subtleties of von Neumann algebra theory, since we are essentially interested in the simplest cases: the full algebra $\mathcal{B}(\mathcal{H})$, the commutative von Neumann algebras and the tensor products of these. Let us recall a few facts about commutative von Neumann algebras:

1. For any standard probability space $(\mathcal{X}, \mathcal{F}, \mu)$ and any sub-$\sigma$-algebra $\mathcal{F}_1 \subset \mathcal{F}$ the space $L^\infty(\mathcal{X}, \mathcal{F}_1, \mu)$ is identified with a commutative von Neumann algebra on $L^2(\mathcal{X}, \mathcal{F}, \mu)$ by $f \mapsto M_f$ (the operator of multiplication by $f$).

2. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a commutative von Neumann algebra. Then there exists a standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$, a sub-$\sigma$-algebra $\mathcal{F}_1 \subset \mathcal{F}$ and a unitary operator $\pi$ from $L^2(\mathcal{X}, \mathcal{F}, \mu)$ to $\mathcal{H}$ such that $\mathcal{A} = \pi^* L^\infty(\mathcal{X}, \mathcal{F}_1, \mu) \pi$. Thus, if we consider the quotient $\mathcal{X}_1 = \mathcal{X}/\mathcal{F}_1$, then $\mathcal{A}$ is isomorphic (as a C*-algebra) to $L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu)$. The algebra $\mathcal{A}$ is a maximal commutative von Neumann algebra if and only if $\mathcal{F}_1 = \mathcal{F}$ (up to measure-zero sets). It is called “discrete” if $\mathcal{X}_1$ is countable or finite, the $\sigma$-algebra $\mathcal{F}_1$ is then called “coarse”.

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1the term “discrete” $\sigma$-algebra often refers to the $\sigma$-algebra of all subsets of $\mathcal{X}$, so we use coarse to avoid confusion.
3. Let $\mathcal{A}_1 \subset \mathcal{A}_2$ be two commutative von Neumann algebras on a von Neumann algebra with two isomorphisms of $C^*$-algebras $\psi_1 : \mathcal{A}_1 \to L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu_1)$ and $\psi_2 : \mathcal{A}_2 \to L^\infty(\mathcal{X}_2, \mathcal{F}_2, \mu_2)$. Then there exists a measurable map $\eta : \mathcal{X}_2 \to \mathcal{X}_1$ such that $\mu_1$ is absolutely continuous with respect to the push forward measure $\eta_*\mu_2$ and for any $f \in L^\infty(\mathcal{A}_1, \mathcal{F}_1, \mu_1)$ we have $\psi_2 \circ \psi_1^{-1}(f) = f \circ \eta$.

See Takesaki’s book [Tak01], notably Theorem 8.21 and Lemma 8.22. An application of the last fact is that if $U$ is an isometry of $\mathcal{H}$ with $U\mathcal{A}_1U^* \subset \mathcal{A}_2$ then its action on $\mathcal{A}_1$ can be implemented by some map $\eta$ between the underlying spaces $\mathcal{X}_1$ and $\mathcal{X}_2$.

A full study of a non-maximal commutative von Neumann algebra involves direct integrals of Hilbert spaces. We don’t need it here, so let us just give a taste of it: if $\mathcal{A} \simeq L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu)$ then we can decompose $\mathcal{H}$ as $\int_{\mathcal{X}_1} \mathcal{H}(x) d\mu(x)$ where $x \mapsto \mathcal{H}(x)$ is a measurable field of Hilbert spaces, and the elements of $\mathcal{A}$ are operators of the form $\int_{\mathcal{X}_1} f(x) I_{\mathcal{H}(x)} d\mu(x)$.

### 2.1.3 Quantum states

Let us describe states on several von Neumann algebras. The state of a quantum system with observables in a von Neumann algebra $\mathcal{A}$ is modeled the following way:

**Definition 7.** A (normal) state on a von Neumann algebra $\mathcal{M}$ is a linear form $\rho$ on $\mathcal{M}$ which is:

- **positive**, i.e. $\rho(A) \geq 0$ for any positive semi-definite operator $A \in \mathcal{M}$.
- **normed**, i.e. $\rho(I) = 1$
- **normal**, i.e. continuous for the $\sigma$-weak topology, or equivalently for any sequence of mutually orthogonal projections $(P_n)_{n \in \mathbb{N}} \in \mathcal{M}^\oplus$ we have $\sum_{n \in \mathbb{N}} \rho(P_n) = \rho(\sum_{n \in \mathbb{N}} P_n)$.

The set of states on $\mathcal{M}$ is written $\mathcal{G}(\mathcal{M})$ or simply $\mathcal{G}(\mathcal{H})$ if $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Let us consider the two cases of maximal commutative von Neumann algebra and of the full von Neumann algebra:

**States on $\mathcal{A} = L^\infty(\mathcal{X}, \mathcal{F}, \mu)$:** any state $\rho$ on $\mathcal{A}$ is of the form

$$\rho(f) = \int_{\mathcal{X}} f(x)p(x)d\mu(x)$$

where $p$ is a positive function on $\mathcal{X}$ with $\int_{\mathcal{X}} p(x)d\mu(x) = 1$. Hence the set $\mathcal{G}(L^\infty(\mathcal{X}, \mu))$ can be identified with the set of probability measures which are absolutely continuous with respect to $\mu$.

**States on $\mathcal{B}(\mathcal{H})$:** any state $\rho$ on the full algebra is the form

$$\rho(A) = \text{Tr}(AT_{\rho})$$

where $T_{\rho}$ is a positive semi-definite trace-class operator on $\mathcal{H}$ with $\text{Tr}(T_{\rho}) = 1$. By convention, we use the letter $\rho$ for both the state and the corresponding trace-class operator, and we identify the set $\mathcal{G}(\mathcal{H})$ with the set of positive semi-definite trace-class operators of trace 1.

**States on $\mathcal{B}(\mathcal{H}) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu)$:** This is the mix of the two previous situations: a state $\rho$ on $\mathcal{B}(\mathcal{H}) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu) \subset \mathcal{B}(\mathcal{H} \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu))$ is of the form

$$\rho(A \otimes f) = \int_{\mathcal{X}} \text{Tr}(AQ_{\rho}(x)) f(x)d\mu(x)$$

where $x \mapsto Q_{\rho}(x)$ is a measurable function from $\mathcal{X}$ to the set of positive semi-definite trace-class operators on $\mathcal{H}$ such that $\int_{\mathcal{X}} \text{Tr}(Q_{\rho}(x)) d\mu(x) = 1$. We call $Q_{\rho}$ the density matrix function.

**Remark 1.** 1. If $\mathcal{M}_1 \subset \mathcal{M}_2$ are two von Neumann algebras, we may extends states on $\mathcal{M}_1$ to states on $\mathcal{M}_2$, and restrict states on $\mathcal{M}_2$ to states on $\mathcal{M}_1$. In particular, if $\mathcal{M}_1 = L^\infty(\mathcal{X}, \mu)$ and $\mathcal{M}_2 = \mathcal{B}(L^2(\mathcal{X}, \mu))$, a state on $\mathcal{M}_1$ can be extended in many different ways to a state on $\mathcal{M}_2$, notably we can make it a pure state: take $f = \sqrt{p}$ where $p$ is the probability density of the state with respect to $\mu$, and consider the state $|f\rangle \langle f|$ on $\mathcal{M}_2$. We may also be tempted to take the multiplication operator $M_p$ as another extension, but this operator may not be trace class when $\mathcal{X}$ is not coarse.
2. Another important example is the case of a bipartite system. If $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and we are given a state $\rho$ on $\mathcal{M}_2 = \mathcal{B}(\mathcal{H})$, its restriction to $\mathcal{M}_1 = \mathcal{B}(\mathcal{H}_A) \otimes \{1_B\}$ has for density matrix the partial trace of $\rho$ with respect to $B$, that is $\rho_B = \operatorname{Tr}_B(\rho)$.

3. With $\mathcal{H}_B = L^2(\mathcal{X}, \mathcal{F}, \mu)$ and $\mathcal{M}_1 = \mathcal{B}(\mathcal{H}_A) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu)$ and $\mathcal{M}_2 = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ the situation is more subtle. A state $\rho$ on $\mathcal{M}_2$ can always be described by a kernel $(x,y) \mapsto K_\rho(x,y)$ from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{S}^1(\mathcal{H})$, such that for any function $f \in L^2(\mathcal{X}, \mathcal{H}_A) = \mathcal{H}_A \otimes L^2(\mathcal{X}, \mathcal{F}, \mu)$ we have

$$\langle \rho f \rangle(x) = \int_X K_\rho(x,y) f(y) \, d\mu(y)$$

(where we see $\rho$ as an operator on $\mathcal{H}$). To describe the state $\rho_{\mathcal{M}_1}$ on $\mathcal{M}_1$ it seems natural to take for density matrix function $Q_{\rho,\mathcal{M}_1}(x) = K_\rho(x,x)/\operatorname{Tr}(K_\rho(x,x))$. Unless $K$ is continuous with respect to some metric, this requires technicalities since the diagonal $\{(x,x) | x \in \mathcal{X}\}$ is possibly of measure zero in $(\mathcal{X} \times \mathcal{X}, \mu \otimes \mu)$. This issue can be solved with the help of the Lebesgue differentiation theorem, either by averaging on small rectangles (see Brislawn [Bri91]) or with the notion of virtual continuity (see Vershik et al. [VZ10]).

2.1.4 Measure of an observable

Let $A$ be a self-adjoint operator on $\mathcal{H}$ (which is not necessarily bounded). Assume that the system is in the state $\rho$. The measurement of $A$ is mathematically described the following way: the von Neumann algebra $\mathcal{A}$ generated by $A$ is commutative, so there exists a unitary operator $\pi : \mathcal{H} \rightarrow L^2(\mathcal{X}, \mu)$ for some standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ and a measurable function $g : \mathcal{X} \rightarrow \mathbb{R}$ such that $\pi^* A \pi = M_g$. Let $\rho$ be the state on the system, then $\pi^* \rho \pi$ restricts to a state on $L^\infty(\mathcal{X}, \mu)$, that is, a probability measure $\mathbb{P}_\rho$ on $\mathcal{X}$ which is absolutely continuous with respect to $\mu$. This makes $(\mathcal{X}, \mathbb{P}_\rho)$ a probability space. The result of the measurement is then the random variable $\tilde{A}_\rho$ on $(\mathcal{X}, \mathbb{P}_\rho)$ defined by the function $g$.

Note that for a commuting family of self-adjoint operators $(A_\alpha)_{\alpha \in I}$ we can consider their joint spectral theory: there exists a unitary operator $U : \mathcal{H} \rightarrow L^2(\mathcal{X}, \mu)$ with $U^* A_\alpha U = M_{g_\alpha}$ for a family of functions $(g_\alpha)_{\alpha \in I}$. Thus, we can consider the family of random variables $A_{\alpha, \rho}$ on the same probability space $(\mathcal{X}, \mathbb{P}_\rho)$. However, if $A$ and $B$ are not commuting, there is no consistent way to consider jointly $\tilde{A}_\rho$ and $\tilde{B}_\rho$ as random variables on the same probability space.

Now, it is not always possible to describe the quantum mechanical state of $\rho$ after the exact measurement. In the case where $A$ has only pure point spectrum, it is possible and we do it as follows.

**Definition 8** (State after the measurement). Let $A$ be an observable of the form

$$A = \sum_{a \in \operatorname{sp}(A)} a P_a$$

where the $P_a$ are mutually orthogonal projections. Write $\mathcal{A}$ the commutative von Neumann algebra generated by $A$, it is isomorphic to $L^\infty(\operatorname{sp}(A), \sum_\delta \delta_a)$. We endow $\operatorname{sp}(A)$ with the probability $\mathbb{P}_\rho(a) = \operatorname{Tr}(\rho P_a)$. The state after the measurement of $A$ is the random variable $\rho_{\mathcal{A}}$ on $(\operatorname{sp}(A), \mathbb{P})$ defined by

$$\rho_{\mathcal{A}}(a) = \frac{P_a \rho P_a}{\operatorname{Tr}(P_a \rho)}.$$  

We may also write $\rho_{\mathcal{A}} := \operatorname{Tr}_B(\rho_{\mathcal{A}})$, and to shorten notation we will often use the variant calligraphy $\rho$ for a random density matrix corresponding to a deterministic density matrix $\rho$.

The action of not reading the result of the measurement consists in discarding the random variable $\tilde{A}_\rho$ and replacing $\rho_{\mathcal{A}}(a)$ by its expectancy $\rho' = \mathbb{E}(\rho_{\mathcal{A}})$. The operator $\rho' = \sum_{a \in \operatorname{sp}(A)} P_a \rho P_a$ is in $\mathcal{S}(\mathcal{H})$. It carries all the information which can be obtained without the knowledge of $\tilde{A}_\rho$, since $\mathbb{E}(\operatorname{Tr}(\rho_{\mathcal{A}} B)) = \operatorname{Tr}(\rho' B)$ for any observable $B \in \mathcal{B}(\mathcal{H})$.

If $A$ has singular spectrum it is no more possible to describe the state after the exact measurement as a random variable on $\mathcal{S}(\mathcal{H})$. For example, if we measure the position observable $X$ on $L^2(\mathbb{R}, \operatorname{Leb})$ the state of the system after the measurement should correspond to the Dirac measure $\delta_{\xi}$ on the algebra $L^\infty(\mathbb{R}, \operatorname{Leb})$, but it is not possible since states on this algebra are absolutely continuous with respect to the Lebesgue measure. This is linked to the fact that every repeatable instrument is discrete, see Ozawa [Oza85].
This is not really a physical problem since no real-life measurement is exact, hence we only measure discrete observables in real life. Though, it is always better to have an idealization of the measure of continuous observables, and we show a way to circumvent these issues below.

2.1.5 The quantum state after the measurement of a continuous observable and indirect measurement

This part is not used before Section 3 but it introduces the notion of “pointer unitary operator” which helps the understanding of the OQBM. The idea to describe the state after the exact measurement is to restrict the state to some subalgebra of $\mathcal{B}(\mathcal{H})$. The case we consider is the following:

- The space $\mathcal{H}$ is the tensor product of two Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_B$.
- We want to measure a family of mutually commuting operators $(B_\alpha)_{\alpha \in I}$ acting on $\mathcal{H}_B$. Write $\mathcal{A}$ the von Neumann algebra generated by the $B_\alpha$’s.
- We are interested on the state after the measurement on $\mathcal{B}(\mathcal{H}_G)$ only. It will be written $\rho_{G|A}$.

We will see that concentrating on the state on $\mathcal{H}_G$ and ignoring the full picture on $\mathcal{H}_G \otimes \mathcal{H}_B$ allows us to get a rigorous definition of $\rho_{G|A}$. This setup is geared to describe indirect measurement.

Since the $B_\alpha$ are commuting, we can identify $\mathcal{H}_B$ with $L^2(\mathcal{X}, \mu)$ for some standard measured space space $(\mathcal{X}, \mathcal{F}, \mu)$ such that there exists measurable functions $g_\alpha$ with $B_\alpha = M_{g_\alpha}$. We want to define $\rho_{G|A}$ as a random variable with values in $\mathfrak{S}(\mathcal{H}_G)$ on the probability space generated by the random variables $g_\alpha = \tilde{B}_\alpha$.\footnote{\textit{Note that $u_\mathcal{X} : \rho \mapsto \varsigma$ is an isometry, contrarily to the map $\rho \mapsto \varrho$.}}

**Theorem 9.** Let $\rho$ be a state on $\mathcal{B}(\mathcal{H}_G \otimes L^2(\mathcal{X}, \mu))$. Then there exists a measurable map $\varsigma$ from $\mathcal{X}$ to $S^1(\mathcal{H}_B)$ such that for any $f \in L^\infty(\mathcal{X}, \mu)$ and for any observable $A \in \mathcal{B}(\mathcal{H}_G)$ we have

$$\text{Tr}(\rho \otimes M_f) = \int_{\mathcal{X}} \text{Tr}(\varsigma(x) A) f(x) d\mu(x).$$

It is unique (up to a $\mu$-negligible set), and $\varsigma(x)$ is a positive semi-definite and satisfies

$$\text{Tr}(\varsigma(x)) = \frac{d\mathbb{P}_\rho}{d\mu}(x)$$

for $\mu$-ae $x$. It is called the unnormalized state on $\mathcal{H}_G$ associated to $(\mathcal{X}, \mu)$. Note that its trace depends on the measure $\mu$ which is chosen.

Now, consider a sub-$\sigma$-algebra $\mathcal{F}_1 \subset \mathcal{F}$ and let $A = L^\infty(\mathcal{X}, \mathcal{F}_1, \mu)$. Let $\mathbb{P}_\rho$ the probability measure induced by $\rho$ on $\mathcal{X}$. Then there exists a random variable $\rho_{G|A}$ on $(\mathcal{X}, \mathcal{F}_1, \mathbb{P}_\rho)$ with values in $\mathfrak{S}(\mathcal{H}_G)$ such that for any operator $A \in \mathcal{B}(\mathcal{H}_G)$ and any random variable $f \in L^\infty(\mathcal{X}, \mathcal{F}_1, \mathbb{P}_\rho)$ we have

$$\text{Tr}(\rho A \otimes M_f) = \mathbb{E}_\rho \left( \text{Tr}(\rho_{G|A} A) f \right)$$

where on the right $f$ is seen as a random variable. The random variable $\rho_{G|A}$ is unique up to a set of probability zero, and for $\mathbb{P}$-almost $x \in \mathcal{X}$ we have $\rho_{G|A}(x) = \varsigma(x)/\mathbb{P}_\rho(x)$.

We will often write $\varrho$ for $\rho_{G|A}$ when it does not cause confusion, and we write $\varsigma = u_\mathcal{X}(\rho)$ (or $u_{\mathcal{X}, \mu}(\rho)$ when the measure needs to be precised).

Note that $u_\mathcal{X} : \rho \mapsto \varsigma$ is an isometry, contrarily to the map $\rho \mapsto \varrho$.

**Proof.** The function $\varsigma$ is just the matrix density function of the restriction of $\rho$ to $\mathcal{M} = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}$, so its existence is just a consequence of the Riesz theorem.

We have $\int_{\mathcal{X}} f(x) \text{Tr}(\varsigma(x)) d\mu(x) = \rho(M_f) = \mathbb{E}_\rho(f)$ so $\text{Tr}(\varsigma(x)) = d\mathbb{P}_\rho/d\mu$, and so $\text{Tr}(\varsigma(x))$ is nonzero $\mathbb{P}_\rho$-almost surely. We now define

$$R(x) = \frac{\varsigma(x)}{\text{Tr}(\varsigma(x))}$$

on $x$ such that $\varsigma(x) \neq 0$. It is a random variable on $(\mathcal{X}, \mathbb{P}_\rho)$. Now we take the conditional expectation with respect to the $\sigma$-algebra $\mathcal{F}$ generated by the $g_\alpha$ on $\mathcal{X}$:

$$\rho_{G|A} = \mathbb{E}(R \mid \mathcal{F}).$$

It is easy to show that it fits the requirement of the theorem.

The uniqueness is straightforward.
Remark 2. 1. With this approach, we clearly separate the quantum superposition, described by a density matrix, and the classical randomness on the probability space \((X, \mathcal{F}, P, \rho)\). It is frequent in quantum filtering theory to define \(\rho\) as a state on the commutant of \(A\), which is in general bigger than \(B(H_B) \otimes L^\infty(X)\), but this does not define \(\rho\) explicitly as a random variable on some probability space.

2. Note that the state \(\rho_{G,A}\) contains more information that \(\rho_G = \text{Tr}_{H_B}(\rho)\) since \(\rho_G = \mathbb{E}_{\rho_G, \mu}(\rho_{G,A})\). Thus, we have three descriptions of the state of the system, containing less and less information: the full state \(\rho\) on \(H_G \otimes H_B\), the random state \(\rho_{G,A}\) and the state \(\rho_G\). We could define a fourth description between \(\rho_{G,A}\) and \(\rho_G\) by using the theory of direct integral: if \(A\) is the set of decomposable operators on \(H = \int_X^H H(x) dp(x)\) we may consider a random state \(\rho(X)\) on the random Hilbert space \(H(X)\). This level of precision is not needed for our purpose.

As an application of this theorem, we can model the indirect measurement of an observable; it is a framework often called von Neumann measurement of an observable in the literature \([\text{Be94}], [\text{BM91}], [\text{Gou}]\). Let us describe the measurement of the observable \(X\) on \(H_B = L^2(\mathbb{R}, \text{Leb})\). We couple the system with the pointer of some measurement device, described by \(H_B = L^2(\mathbb{R}, \text{Leb})\). We call \(H_B\) the pointer space (think of it as the needle of a weighting scale or a seismometer). We move the pointer depending on the value of \(X\), which has the effect of applying a unitary operator \(Z\) on \(H_B \otimes H_B = L^2(\mathbb{R}^2, \text{Leb}_2)\) which is defined by

\[
(Zf)(x,a) = f(x, a - x) .
\]

Then, we perform the measurement of the pointer: we measure \(A = M_{a\rightarrow a}\) on \(H_B\). The result is a random variable \(\bar{A}\) and the state after the measurement is \(\rho_{G,A}\) (where \(A\) is the algebra generated by \(A\)). Note that the noise is described by the initial state of the pointer. For example, if the system is in the pure state \(f \in L^2(\mathbb{R}, \text{Leb})\) and the pointer in the pure state \(g \in L^2(\mathbb{R}, \text{Leb})\), the probability density of \(\bar{A}\) is

\[
p(a) = \int_\mathbb{R} |f(x)|^2 |g(a-x)|^2 dx = |f|^2 \ast |g|^2(a)
\]

and for any \(a \in \mathbb{R}\) the state \(\rho_{G,A}(a)\) is the pure state \(|fa\rangle \langle fa|\) where

\[
fa(x) = \frac{f(x)g(a-x)}{p(a)} .
\]

This really corresponds to a classical noisy measurement: if \(X\) is a random variable with density \(|f|^2\) and \(B\) a random variable with density \(|g|^2\) then \(p\) is the density of \(X + B\) and \(|fa|^2\) is the density of \(X\) conditioned to \(X + B = a\). Note however that this situation is truly quantum: if we do not perform the measurement, the density matrix of the system after the evolution is

\[
\rho_G' = \mathbb{E}(\rho_G) = \text{Tr}_B(Z(\rho_G \otimes \rho_B)Z^*)
\]

which is of kernel

\[
K_{\rho_G'}(x,y) = f(x)f(y)\int_\mathbb{R} g(a-x)g(a-y)da = f(x)f(y)C_g(x-y) .
\]

where

\[
C_g(z) = \int_\mathbb{R} g(a-z)\overline{g(a)}da .
\]

It is no more a pure state.

A more general version of this process is the following:

**Definition 10.** Let \(H_G\) be a Hilbert space and \(A\) a commutative von Neumann algebra on \(H_G\), with an isometry \(G : L^2(X, \mu) \to H_G\) implementing an isomorphism \(A \simeq L^\infty(X, \mu)\). Consider an auxiliary space \(H_B = L^2(Y, \nu)\). A pointer map is some measurable function \(\psi : X \times Y \to X\) such that for all \(x \in X\) the map \(\psi(x,\bullet)\) is a measure-preserving bijection on \(Y\). The pointer unitary operator \(Z_\psi\) on \(H_G \otimes H_B\) corresponding to \(\psi\) is the operator defined as \(Z_\psi = G\tilde{Z}_\psi G^*\) where \(\tilde{Z}_\psi\) is the unitary on \(L^2(X \times Y, \mu \times \nu)\) defined by

\[
(\tilde{Z}_\psi f)(x, y) = f(x, \psi(x,y)) .
\]

The indirect measurement corresponding to \(\psi\) is the measurement of the algebra \(L^\infty(Y, \nu)\) on \(H_B\), resulting in the random value \(Y \in Y\) of the pointer and the random state \(\rho_G|Y\in \mathcal{S}(H_B)\).
This is a little more restrictive than the processes considered by Belavkin [Bel94], in which the unitary operator $Z$ (written $S$ by Belavkin) is only assumed to commute with elements of $L^\infty(\mathcal{X}, \mu) \otimes \{I_B\}$. This restrictive definition has the advantage of making it more explicit.

This definition include the perfect measurement of a discrete observable $A$: take $\mathcal{X} = \mathcal{Y} = \text{sp}(A)$ with $\mu$ the counting measure and fix an initial state $a_0 \in \mathcal{Y}$, choose $\rho_B = |\delta_{a_0}\rangle \langle \delta_{a_0}|$ and any pointer function $\psi$ such that $\psi(a, a_0) = a$.

### 2.2 Repeated measurement process and the trajectories of the OQBM

In this section we introduce repeated interactions and repeated measurement processes, and we show how the discrete OQBM can be seen as an extension of these measurement. We use this picture to show the convergence of the quantum trajectories of the discrete OQBM, thanks to a theorem of Pellegrini [Pel08].

#### 2.2.1 The repeated measurement process

The repeated measurement model relates to many experimental protocols, notably with the experiments of Haroche’s team. It describes a process on discrete time, and we are interested in its continuous-time limit.

We consider a Hilbert space $\mathcal{H}_G$ describing a system of interest in the state $\rho_0 \in \mathcal{S}(\mathcal{H}_G)$, and a space modeling a probe $\mathcal{H}_p$ in the fixed pure state $\rho_p = |0\rangle \langle 0|$. In this article the probe space is always $\mathcal{H}_p = \mathbb{C}^2$.

Make it evolve according to some unitary operator $V$ on $\mathcal{H}_G \otimes \mathcal{H}_p$ and measure some observable $A \in \mathcal{B}_a(\mathcal{H}_p)$. Then take a copy of $\mathcal{H}_G$, also in the state $\rho_p = |0\rangle \langle 0|$, and repeat this procedure again and again. What we obtain is a stochastic process $(\varrho_n)_{n \in \mathbb{N}}$ where $\varrho_n \in \mathcal{S}(\mathcal{H}_G)$ is the state of the system after the $n$-th measurement, together with another process $(\Delta_n)_{n \in \mathbb{N}}$ where $\Delta_n \in \mathbb{R}$ is the result of the $n+1$-th measurement of $A$. Since the probe space $\mathcal{H}_p$ is constantly renewed, $(\varrho_n, \Delta_n)_{n \in \mathbb{N}}$ is a Markov process. We can also note that for any $n$ the state $\varrho_n$ deterministically depends in the sequence $(D_k)_{k < n}$, since if $P_d$ is the spectral projection for the eigenvalue $d$ of $A$, we have

$$
\varrho_{n+1} = \frac{\text{Tr}_B(P_{\Delta_n}V(\varrho_n \otimes \rho_p)V^*P_{\Delta_n})}{\text{Tr}(P_{\Delta_n}V(\varrho_n \otimes \rho_p)V^*P_{\Delta_n})}.
$$

It is also interesting to study the evolution when the result of the measurement is discarded, that is, the evolution of $\rho_n = \mathbb{E}(\varrho_n)$. We have

$$
\rho_{n+1} = \text{Tr}_B (V(\varrho_n \otimes \rho_p)V^*)
$$

The evolution of $\rho_n$ is called a quantum dynamical system, and its description as the interaction of the system with a bath is called a repeated interactions model [AP06].

#### 2.2.2 The Belavkin diffusive Equation and the Lindblad Equation

We want to study the continuous time limit of this type of process. Thus, we will consider that each step of the process lasts a time $\tau > 0$ and we make $\tau$ go to zero with suitable normalization. The case we consider is the following:

1. We take $\mathcal{H}_p = \mathbb{C}^2$ with $\rho_p = |0\rangle \langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

2. The unitary evolution $V_\tau$ on $\mathcal{H}_G \otimes \mathcal{H}_p$ is described as follows: fix a self-adjoint bounded operator $H \in \mathcal{B}(\mathcal{H}_G)$ and a bounded operator $N \in \mathcal{B}(\mathcal{H}_G)$ and take

$$
V_\tau = \exp\left(-i\tau H + \sqrt{\tau} \begin{pmatrix} 0 & N^* \\ -N & 0 \end{pmatrix}\right) = I + \sqrt{\tau} \begin{pmatrix} 0 & N^* \\ -N & 0 \end{pmatrix} + \tau \left(-iH - \frac{1}{2}\begin{pmatrix} N^*N & 0 \\ 0 & NN^* \end{pmatrix}\right) + O(\tau^{3/2}) .
$$

3. We measure the observable $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 

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4. The process of states obtained is written \((\rho_{\tau,n})_{n \in \mathbb{N}}\), and the result of the \((n + 1)\)-th measurement is written \(\Delta_{\tau,n} \in \{-1, +1\}\). We also define

\[
W_{\tau,n} = \sqrt{\tau} \sum_{k=0}^{n-1} \Delta_{\tau,k} .
\]

The normalization in \(\sqrt{\tau}\) to define \(W_{\tau,n}\) corresponds to a diffusive limit in physics, where the time scale \(\tau\) is proportional to the square of the space scale. In the rest of the article, we will write \(\delta = \sqrt{\tau}\) the space scale.

In this setup, the eigenvectors for the eigenvalues \(\pm 1\) of \(A\) are

\[
|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle) ,
\]

and we have

\[
\rho_{\tau,n+1} = \frac{K_{\tau,\Delta_n} \rho_{\tau,n} K_{\tau,\Delta_n}^*}{\text{Tr} \left( K_{\tau,\Delta_n} \rho_{\tau,n} K_{\tau,\Delta_n}^* \right)}
\]

where

\[
K_{\tau,\pm 1} = \frac{1}{\sqrt{2}} \left( I \pm \delta N + \tau \left( -iH - \frac{1}{2} N^* N \right) \right) + O(\tau^{3/2}) .
\]

The following theorem describes the limit in distribution of this process as \(\tau \to 0\). It was proved by Pellegrini.

**Theorem 11** (Theorem 8 of [Pel08].) Assume that \(\mathcal{H}_G\) is finite-dimensional. Fix some \(T > 0\). Then the process \((\rho_{\tau,[t/\tau]}, W_{\tau,[t/\tau]})_{0 \leq t \leq T}\) described above converges in distribution as \(\tau \to 0\) (in the space of bounded functions with the uniform norm) to a process \((\rho_t, W_t)_{0 \leq t \leq T}\) satisfying the following stochastic equation (in the Itô sense):

\[
\begin{align*}
\frac{d\rho_t}{dt} &= \mathcal{L}(\rho_t) dt + (N\rho_t + \rho_t N^* - \rho_t \mathcal{T}(\rho_t)) dB_t \\
\frac{dW_t}{dt} &= \mathcal{T}(\rho_t) dt + dB_t
\end{align*}
\]

(2.18)

where \((B_t)\) is a standard Wiener process, \(\mathcal{L}\) is the super-operator defined by

\[
\mathcal{L}(\rho) = -i[H, \rho] + N\rho N^* - \frac{1}{2} (N^* N\rho + \rho N^* N)
\]

(2.19)

and

\[
\mathcal{T}(\rho) = \text{Tr}((N + N^*)\rho) .
\]

This theorem was proved with methods of classical stochastic process, notably the Kurtz-Protter’s theorem. Importantly, the proof is still valid with \(K_{\tau,\pm}\) replaced by \(K_{\tau,\pm} + o(\tau)\) with some rest \(o(\tau)\) uniformly small.

If we discard the probes before measuring it, the state of the system is the deterministic density matrix

\[
\rho_{G,\tau,n} = \mathbb{E}(\rho_{\tau,n}) .
\]

It follows a quantum dynamical semigroup, with \(\rho_{G,\tau,n+1} = \Lambda_G, \tau(\rho_{G,\tau,n})\) where

\[
\Lambda_G, \tau(\rho) = K_{\tau,+1} \rho K_{\tau,+1}^* + K_{\tau,-1} \rho K_{\tau,-1}^*
\]

(2.20)

\[
= \rho + \tau \mathcal{L}(\rho) + O(\tau^{3/2})
\]

(2.21)

where \(\mathcal{L}\) is defined in Equation (2.19). Thus \(\rho_{G,\tau,[t/\tau]}\) converges to some limit \(\rho_{G,t}\) satisfying the so-called Lindblad Equation

\[
\frac{d}{dt} \rho_{G,t} = \mathcal{L}(\rho_{G,t}) .
\]

The family of super-operators \(\Lambda^t = e^{t\mathcal{L}}\) is called a Lindblad semigroup. Note that \(\rho_t = \mathbb{E}(\rho_t)\), which can be seen both by the above convergence or by using the fact that the term in \(dB_t\) in Equation (2.18) is of expectancy zero.
2.2.3 A dilation of the discrete OQBM

Let us consider a gyroscopic space $\mathcal{H}_G$ and the position space $\mathcal{H}_{\tau,z} = l^2(\delta \mathbb{Z})$ (for $\delta = \sqrt{\tau} > 0$), and fix some bounded operators $N$ and $H$ on $\mathcal{H}_G$ with $H$ self-adjoint, and consider the operators $B_{\tau,+}$ and $B_{\tau,-}$ defined as in the introduction (with $M = 0$ since the effects of $M$ are negligible). We consider the two channels defining the OQBM: the one corresponding to the OQW definition:

$$\Lambda_\tau(\rho) = \sum_{x \in \mathbb{Z}} (B_{\tau,-} \otimes |x-\delta\rangle \langle x|) \rho (B_{\tau,-}^* \otimes |x-\delta\rangle \langle x|) + (B_{\tau,+} \otimes |x+\delta\rangle \langle x|) \rho (B_{\tau,+}^* \otimes |x+\delta\rangle \langle x|)$$

and the one with only two Krauss operators:

$$\tilde{\Lambda}_\tau(\rho) = (B_{\tau,-} \otimes D_{\tau}^*) \rho (B_{\tau,-}^* \otimes D_{\tau}) + (B_{\tau,+} \otimes D_{\tau}) \rho (B_{\tau,+}^* \otimes D_{\tau})$$

where $D_{\tau}$ is the right translation of distance $\delta$ on $\mathcal{H}_{\tau,z}$. It is easily checked that $\Lambda_\tau^*$ and $\tilde{\Lambda}_\tau^*$ coincide on the algebra $\mathcal{B}(\mathcal{H}_G) \otimes L^\infty(\delta \mathbb{Z})$, and we concentrate on the study of $\tilde{\Lambda}_\tau$ from now on. To make the link with the repeated measurement process and the Belavkin Equation, we define one Stinespring dilation of $\tilde{\Lambda}_\tau$.

**Lemma 12.** We have

$$\tilde{\Lambda}_\tau(\rho) = \text{Tr}_{\mathcal{H}_p} (R_\tau V_\tau (\rho \otimes |0\rangle \langle 0|) V_\tau^* R_\tau^*) + O(\tau \sqrt{\tau})$$

(2.22)

where $\mathcal{H}_p = \mathbb{C}^2$ and $V_\tau$ is the unitary operator on $\mathcal{H}_G \otimes \mathcal{H}_p$ defined by Equation (2.14) and $R_\tau$ is the operator on $\mathcal{H}_{\tau,z} \otimes \mathcal{H}_p$ defined by

$$R_\tau = D_{\tau} \otimes |+\rangle \langle +| + D_{\tau}^* \otimes |\rangle \langle |\rangle$$

(2.23)

where $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ and $|\rangle \langle |\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$.

This lemma proved by a straightforward computation, and a consequence of the equality $L_{\tau,\pm 1} = B_{\tau,\pm}$ where $L_{\tau,\pm}$ is defined in the repeated measurement procedure (2.17).

The notation $O(\tau \sqrt{\tau})$ is meant uniform in $\rho$, in the sense that there exists a constant $C$ such that for all $\tau > 0$ small enough and all $\rho \in \mathcal{S}(\mathcal{H}_G \otimes \mathcal{H}_{\tau,z})$ we have $\|O(\tau \sqrt{\tau})\| \leq C \sqrt{\tau}$.

As a consequence, we see that $\tilde{\Lambda}_\tau$ is an extension of the quantum dynamics $\Lambda_{G,\tau}$ on $\mathcal{H}_G$ generated by repeated interactions:

**Corollary 13.** For any initial state $\rho \in \mathcal{S}(\mathcal{H}_G \otimes \mathcal{H}_{\tau,z})$ we have

$$\text{Tr}_{\mathcal{H}_{\tau,z}} (\tilde{\Lambda}_\tau(\rho)) = \Lambda_{G,\tau} (\text{Tr}_{\mathcal{H}_{\tau,z}}(\rho))$$

(2.24)

(where $\Lambda_{G,\tau}$ is defined by the repeated measurement process, Equation (2.20)).

**Proof.** This can be proved by direct computation, but it is also a consequence of Lemma 12. Indeed, $R_\tau$ does not act on $\mathcal{H}_G$ so it commutes with any operator $E$ on $\mathcal{H}_G$ and we have

$$\text{Tr} (\tilde{\Lambda}_\tau(\rho) E) = \text{Tr} (R_\tau V_\tau (\rho \otimes |0\rangle \langle 0|) V_\tau^* R_\tau^* E)$$

$$= \text{Tr} (V_\tau (\rho \otimes |0\rangle \langle 0|) V_\tau^* E)$$

$$= \text{Tr} (\Lambda_{G,\tau}(\rho) E)$$

which proves the corollary.

2.2.4 Convergence of the quantum trajectories

The following theorem is a direct consequence of Pellegrini’s theorem [11] and of the picture of the discrete OQBM as an extension of the repeated interactions process:

**Proposition 14.** Let $(\varrho_0, X_0) \in \mathcal{S}(\mathcal{H}_G) \times \mathbb{R}$ be a random variable. For any $\tau > 0$ let us consider the process $(\varrho_{\tau,n}, X_{\tau,n})_{n \in \mathbb{N}}$ describing the quantum trajectories of the OQBM (defined in (1.21)) with initial state $(\varrho_0, \delta [X_0/\delta])$. Then for any $T > 0$ the family of processes $(\varrho_{\tau,[t/T]}, X_{\tau,[t/T]})_{t \in [0,T]}$ converges in distribution as $\tau \to 0$ to a process $(\varrho_t, X_t)_{t \in [0,T]}$ satisfying the following differential equation:

$$\begin{cases}
    d\varrho_t = \mathcal{L}(\varrho_t)dt + (N \varrho_t + \varrho_t N^* - \varrho_t T(\rho_t)) dB_t \\
    dX_t = T(\varrho_t)dt + dB_t
\end{cases}$$

(2.24)

where $B_t$ is a Wiener process.
Proof. We have $B_{\tau,\pm} = K_{\tau,\pm} + O(\tau \sqrt{\tau})$; we can ignore the rest $O(\tau \sqrt{\tau})$, since in the proof of Theorem 11 (as exposed in [Pel08]) does not depends on the terms which are of order $o(\tau)$. Thus, the process $(\varrho_{\tau,\pm}(t/\tau), X_{\tau,\pm}(t/\tau) - X_0)_{t\in[0,T]}$ has the same limit in distribution as the process $(\varrho_{\tau,\pm}(t/\tau), W_{\tau,\pm}(t/\tau))_{t\in[0,T]}$, which satisfies Equation (2.22).

The fact that $B_{\tau,\pm} = K_{\tau,\pm} + O(\tau \sqrt{\tau})$ can be directly computed, but it is also a consequence of Lemma 12 the unitary $R_{\tau}$ converts the measurement of the observable $\Lambda$ into the measurement of the increasing of the position. Thus, the quantum trajectories of the OQBM are nothing more than the trajectories of the Belavkin Equation; the OQBM is truly different from the quantum dynamics arising from $V_{\tau}$ when we consider the position $X_{\tau}$ of the particle as a quantum observable, that is in the Lindbladian and the QSDE versions of the OQBM.

2.3 Quantum Stochastic Calculus for the Open Quantum Brownian Motion

A fully quantum view on the OQBM which encompass the quantum correlations between the events at different times is obtained with the Quantum Stochastic Calculus on the Fock space. We will briefly introduce the Fock space and quantum stochastic calculus, by approaching it by the repeated interactions process.

2.3.1 Repeated interaction process and the Toy Fock space

In the definition of the repeated interactions process, a new probe space $H_p$ is introduced at every iteration. The so called Toy Fock space is the Hilbert space $T\Phi$ obtained when considering all these probe spaces at once. Formally, $T\Phi = \bigotimes_{n\in\mathbb{N}_*} H_p$. More concretely, it is the Hilbert space which generated by the vectors $\otimes_{n\in\mathbb{N}_*} e_n$ where the vectors $e_n$ are unit vectors of $H_p$, which are all equal to $|0\rangle$ except for a finite number of indexes. It has a distinguished unit vector $|\Omega\rangle = \bigotimes_{n\in\mathbb{N}_*} |0\rangle$, and for each $n \in \mathbb{N}^*$ it can be naturally decomposed as

$$T\Phi \simeq H_p^\otimes n \otimes T\Phi.$$ 

This identification is implicit in the following.

The evolution correspondent to the $n$-th interaction is described by the operator $V_n$ acting on the $n$-th copy of $H_p$, i.e. the operator $V_{\tau,n} = 1_{H_p^\otimes(n-1)} \otimes V_{\tau} \otimes 1_{T\Phi}$, and the evolution from time zero to time $n$ is represented by the unitary $U_{\tau,n} = V_{\tau,n} V_{\tau,n-1} \cdots V_{\tau,1}$.

For each $n \in \mathbb{N}^*$, the space $T\Phi_d$ contains a copy of $H_p$ given by the isometry $G_n : H_p \rightarrow T\Phi_d$

$$v \mapsto \left( \bigotimes_{k=1}^{n-1} |0\rangle \right) \otimes v \otimes \left( \bigotimes_{k=n+1}^{+\infty} |0\rangle \right).$$

We can obtain the random state $\varrho_n$ by performing the simultaneous measurement of all the observables $A_k = G_n A G^*_n$ when in the total state

$$\rho_{\text{tot.} \tau,n} = U_{\tau,n} (\varrho_{\tau,0} \otimes |\Omega\rangle \langle \Omega|) U^*_{\tau,n}.$$ 

The position of the particle is then

$$X_{\tau,n} = X_{\tau,0} + \delta \sum_{k=1}^{n} \hat{A}_k$$

where $\hat{A}_k = \pm 1$ is the result of the measurement of $A_k$.

2.3.2 The Fock space

Before studying the convergence of $T\Phi$ as $\tau \rightarrow 0$, let us describe its limit, the Fock space $\Phi = \bigotimes_{t\in\mathbb{R}} H_p$. This space and its interpretation as an infinite tensor product is well known, see Parthasarathy’s book [Par92] for example, or Attal’s lecture in the second book of [AJP00], and we refer to these lectures for a more complete introduction to the Fock space. Let us briefly recall two of its descriptions. Here, we only treat the case where $H_p = \mathbb{C}^2$, but the case where $H_p = \mathbb{C}^n$ or even $H_p$ is infinite-dimensional are similar.
The Guichardet interpretation: Let us consider the set $\mathcal{P}$ of increasing sequences of $\mathbb{R}_+$ of finite length (including the empty sequence ($\emptyset$)). We have $\mathcal{P} = \cup_{n \in \mathbb{N}} \mathcal{P}_n$ where $\mathcal{P}_n \subset (\mathbb{R}_+)^n$ is the set of increasing sequence of length $n$. This set inherits the Lebesgue measure on $(\mathbb{R}_+)^n$ (and $\mathcal{P}_0 = \{ (\emptyset) \}$ has the Dirac measure), so we can endow $\mathcal{P}$ with the sum of these measure, which we write $\lambda$. The Fock space in the Guichardet interpretation is $\Phi_G = L^2(\mathcal{P}, \lambda)$.

It can be interpreted as an infinite tensor product. Indeed, if we write $\mathcal{P}_{[s,t]}$ the space of finite sequences in $[s,t]$ and $\Phi_{G,[s,t]} = L^2(\mathcal{P}_{[s,t]}, \lambda)$, we have $\Phi_{G,[s,t]} \otimes \Phi_{G,[t,u]} = \Phi_{G,[s,u]}$. There is a distinguished vector $|\Omega\rangle = 1_{\mathcal{P}_0}$. We identify $\Phi_{G,[s,t]}$ to the subspace $\{ |\Omega_{[0,n]}\rangle \} \otimes \Phi_{G,[s,t]} \otimes \{ |\Omega_{(t,\infty)}\rangle \}$ of $\Phi_G$.

The probabilistic interpretation from the Brownian motion: This interpretation has been introduced by Attal and Meyer [SM93]. See [Att05] for more details. We consider the Wiener space $(\mathcal{W}, \mathcal{F})$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ with the Wiener measure $\mu$ corresponding to the Brownian motion. We then take $\Phi_W = L^2(\mathcal{W}, \mu)$ the space of $L^2$ random variables on $(\mathcal{W}, \mu)$. There is a distinguished vector $|\Omega\rangle = 1$ (the constant random variable equal to 1). If $W([s,t])$ is the space of functions from $[s,t]$ to $\mathbb{R}$, we can define $\Phi_{W,[s,t]} = L^2(W([s,t]), \mu)$, and we have $\Phi_{W,[s,t]} \otimes \Phi_{W,[t,u]} = \Phi_{W,[s,u]}$.

These two interpretation are equivalent: we can construct an unitary operator $G_{G,W} : \Phi_G \rightarrow \Phi_W$ such that $G_{G,W} \Phi_{G,[s,t]} = \Phi_{W,[s,t]}$ and $G_{G,|\Omega\rangle} = |\Omega\rangle$. To describe it, let us write $(W_t)_{t \in \mathbb{R}_+}$ the Brownian motion and $dW_t$ the Itô differential. For any function $f \in L^2(\mathcal{P}_n, \lambda)$, the random variable $X = G_{G,W} f$ is defined as the successive Itô integrals

$$G_{G,W} f = X = \int_{0 < t_1 < t_2 < \cdots < t_n} f(t_1, \cdots, t_n) dW_{t_1} dW_{t_2} \cdots dW_{t_n}$$

(and if $n = 0$ then $G_{G,W} f$ is the deterministic variable equal to $f(\emptyset)$).

By the Itô isometry formula, we have

$$\|X\|^2 = \mathbb{E}(\|X\|^2) = \int_{0 < t_1 < t_2 < \cdots < t_n} |f(t_1, \cdots, t_n)|^2 dt_1 \cdots dt_n = \|f\|^2,$$

so $G_{G,W}$ is an isometry, and the chaotic representation property ensure that it is surjective (see [Att05]).

From now on, we will write $\Phi$ the Fock space, and either the Guichardet or the probabilistic interpretation depending on the context. There exists many more probabilistic interpretations, one for each normal martingale. We concentrate on the Brownian interpretation in this article.

To complete this picture, we need to approximate the Toy Fock space by the Fock space. This was done by Attal [Attu03] and developed by Attal and Pautrat [AP06]. Let us first design an isometry of $T \Phi$ into $\Phi$. The idea is the following: for each $\tau$, we have

$$\Phi = \bigotimes_{n \in \mathbb{N}} \Phi_{[\tau n, \tau(n+1)]}$$

(where the infinite tensor product is taken with respect to $|\Omega_{[\tau n, \tau(n+1)]}\rangle$ as in the construction of the toy Fock space). Thus, it is sufficient to define an isometry from $H_p = \mathbb{C}^2$ to $\Phi_{[\tau n, \tau(n+1)]} = \Phi_{[0,\tau]}$ and to extend it by tensor product to $T \Phi = \otimes_{n \in \mathbb{N}}$. We choose the isometry

$$G_{n,\tau} : H_p \rightarrow \Phi_{[\tau n, \tau(n+1)]}$$

$$|0\rangle \mapsto |\Omega_{[\tau n, \tau(n+1)]}\rangle$$

$$|1\rangle \mapsto \frac{1}{\sqrt{n}} (W_{\tau(n+1)} - W_{\tau n})$$

which tensorise to $G_\tau = \otimes_{n \in \mathbb{N}} G_{n,\tau} : T \Phi \rightarrow \Phi$.

Let us write $P_\tau = G_\tau G^*_\tau$ the projection on the image of $G_\tau$, and $T_\tau \Phi$ this image. Then $P_\tau$ strongly converge to the identity on $\Phi$ as $\tau \rightarrow 0$. In this sense, the Toy Fock space approximate the Fock space, but this is not sufficient; we also need some more precise convergence on operators in $B(\Phi)$. But first, we need to study the operators in the Fock space.

2.3.3 Quantum Stochastic Calculus on the Fock space

The quantum stochastic calculus is thoroughly described in Parthasarathy [Par92] and in [Att05], [AP06]. We give it a very short introduction geared for this article.
The operators on $\mathcal{H}_p$ are all linear combinations of the four operators $|j\rangle \langle i|$ for $i, j \in \{0, 1\}$. In the toy Fock space, they translate as the operators

$$a_j^*(n) = \mathcal{G}_n(|j\rangle \langle i|)\mathcal{G}_n^*.$$  

Thus, the algebra $\mathcal{B}(T\Phi)$ is generated by the operators $a_j^*(n)$ for $n \in \mathbb{N}^*$ and $i, j \in \{0, 1\}$. Under suitable renormalization, they converge as $\tau \to 0$. Using the isometry $\mathcal{G}_\tau$ in the Fock space, we define the operator

$$a_j^*(\tau, k, l) = \mathcal{G}_\tau \prod_{n=k}^{l} a_j^*(n)\mathcal{G}_\tau$$

then there exists closed operators $a_j^*(t)$ on $\Phi$ such that there is strong convergence

$$\tau^{\varepsilon_{j,i}} a_j^*(\tau, 0, [t/\tau]) \rightarrow a_j^*(t)$$

where

$$\tau^{\varepsilon_{j,i}} = \begin{cases} \tau & \text{if } i = j = 0 \\ \sqrt{\tau} & \text{if } (i, j) = (0, 1) \text{ or } (i, j) = (1, 0) \\ 1 & \text{if } i = j = 1 \end{cases}.$$  

The operator $a_j^0(t)$ is just the multiplication by $t$, while $a_j^0(t)^* = a_j^0(t)$ and $a_j^1(t)$ is self-adjoint (they are respectively the creation, annihilation and number operator on $\Phi_H$). We write $a_j^*(n, t) = a_j^*(t) - a_j^*(s)$: we have

$$a_j^*(\tau, n) = \tau^{-\varepsilon_{j,i}} \mathcal{P}_\tau a_j^*(|\tau(n + 1), \tau n\rangle)\mathcal{P}_\tau.$$  

See [Att03] or [AP06] for more details on these operators. We will now explain how to integrate with respect to these operators, in a way parallel to the Itô Stochastic integration. First, we need to define the set of coherent vectors. For any function $u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we define the coherent vector $\varepsilon(u)$ in the Guichardet interpretation by

$$\varepsilon(u)(t_1, \cdots, t_n) = u(t_1)u(t_2)\cdots u(t_n)$$

(the empty product being considered to be 1). In the probabilistic interpretation, it corresponds to exponential martingale: writing $Y_t = \varepsilon(u1_{[0,t]})$ it verifies the (classical) SDE

$$dY_t = u(t)Y_tdW_t$$

Thus, writing $H_\infty = \int_0^{+\infty} u(s)dW_s$ and $[H]_\infty = \int_0^{+\infty} |u(s)|^2 ds$ we have

$$\varepsilon(u) = \exp \left( H_\infty - \frac{1}{2}[H]_\infty \right).$$

We have $\|u\|^2 = e^{\|u\|_2^2}$. Hence, $\varepsilon$ is continuous; it is clearly not linear.

An important property is that if $\mathcal{M} \subset L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$, then the vector space $\text{Vect}(\varepsilon(\mathcal{M}))$ is dense in $\Phi$. Thus, it is often sufficient to define an object on coherent vectors to fix it.

Now, the objects that we can integrate are the adapted process of operators. We give here a restrictive definition taken from Parthasarathy [Par92]. A more general definition was produced by Attal and Lindsay [AL04], but it is not needed here.

**Definition 15** (Adapted process of operators). A dense subspace $\mathcal{M} \subset L^2(\mathbb{R})$ is called adapted if for any $0 \leq s \leq t \leq \infty$, the space $\mathcal{M}([s, t]) := \{ f \in \mathcal{M} | f = 1_{[s,t]}f \}$ is dense in $L^2([s, t])$.

Consider some Hilbert space $\mathcal{H}_S$. A family of (possibly unbounded) operators $(H_t)_{t \in \mathbb{R}^+}$ on $\mathcal{H}_S \otimes \Phi$ is called adapted if there exists a dense subspace $\mathcal{D}$ and an adapted subspace $\mathcal{M} \subset L^2(\mathbb{R})$ such that for all $t$ the domain of $H_t$ contains $\mathcal{D} \otimes_{\text{alg}} \varepsilon(\mathcal{M})$, and there is an operator $\tilde{H}_t$ on $\mathcal{H}_S \otimes \Phi$ with domain $\mathcal{D} \otimes_{\text{alg}} \varepsilon(\mathcal{M})$ such that $H_t = \tilde{H}_t \otimes I_{\Phi_{[t, +\infty]}}$ on $\mathcal{D} \otimes_{\text{alg}} \varepsilon(\mathcal{M})$.

Now, for an adapted process of operators $(H_t)_{t \in \mathbb{R}^+}$, we want to define the operator

$$\int_0^t H_sda_j^*(s).$$
which would correspond to the limit of
\[
\frac{1}{\tau} \sum_{k=0}^{[t/\tau]} H_{k\tau} (a_j^1(\tau(k+1)) - a_j^1(\tau k)) \ .
\] (2.25)

Note that \(a_j^1(\tau(k+1)) - a_j^1(\tau k)\) only acts on \(\Phi_{[\tau k, \tau(k+1)]}\) so it commutes with \(H_{k\tau}\), and the order of the operators in the above formula is not important. The concrete way we define the integral is the following:

**Definition 16.** Let \((H_t)_{t \in T}\) be an adapted process of operators on \(\mathcal{H}_S \otimes \Phi\), with domain containing \(\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})\) where \(\mathcal{M}\) is adapted and \(\mathcal{D}\) is dense. Let \(T\) be an operator on \(\mathcal{H}_S \otimes \Phi\). We say that the formula
\[
T = \int_0^t H_t da_j^1(t)
\]
is true on \(\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})\) if for any \(a, b \in \mathcal{D}\) and \(u, v \in \mathcal{M}\) the following formula is meaningful and true:
\[
\langle a \otimes \varepsilon(u), T b \otimes \varepsilon(v) \rangle = \int_0^t u_j(s)u_i(s) \langle a \otimes \varepsilon(u), H_s b \otimes \varepsilon(v) \rangle ds
\]
(2.26)
where \(u_i(s) = 1\) if \(i = 0\) and \(u_i(s) = u(s)\) if \(i = 1\), and by “meaningful” we mean that the integral is absolutely convergent.

If \(T_t = \int_0^t H_s da_j^1(s)\) for all \(t\) we will write \(dT_t = H_t da_j^1(t)\). A more general formula exists to compute \(Tf\) for some vector \(f\), see [Att05]. Note that the existence of an operator \(\int_0^t H_t da_j^1(t)\) is not guaranteed. If \(H_t\) is bounded locally uniformly in \(t\), it is at least possible to define \(\int_0^t H_t dt\) on the space generated by \(\mathcal{H}_S \otimes \mathcal{D}_B\), where \(\mathcal{D}_B\) is the vector space generated by \(\varepsilon(L^2(\mathbb{R}) \otimes L^\infty(\mathbb{R}))\). The obtained operator may still be unbounded.

It is easy to check that in the case where \(H_t\) is constant on the intervals \(t \in [\tau k, \tau(k+1)]\) this formula corresponds to the Riemann sum (2.25). In particular,
\[
a_j^1(t) = \int_0^t da_j^1(s) \ .
\]
The case of \(a_j^0(t) = t\) is simple, the integral being just the integral with respect to \(dt\) in the Banach space \(\mathcal{B}(\mathcal{H}_S)\).

The case of \(a_j^0(t)\) and \(a_j^1(t)\) is more subtle, and it actually generalize the Itô integral, as shown by the following proposition.

**Proposition 17.** Let \((f_t)_{t \in \mathbb{R}_+}\) be a process of random variables in \(L^\infty(\mathcal{W}, \mu)\), adapted in the sense of Itô, and such that \(\int_0^t \mathbb{E}(|f_s|^2) ds < \infty\). Let
\[
g = \int_0^t f_s dW_s \ .
\]
Consider the operators \(H_s = M_{f_s}\) and \(T = M_g\) on multiplication by \(f_s\) on \(\Phi\). Then we have
\[
T = \int_0^t H_s(da_j^0(t) + da_j^1(t))
\]
on the domain \(\varepsilon(L^2(\mathbb{R}))\). Thus, in terms of operators, we can write \(dW_t = da_j^0(t) + da_j^1(t)\).

By the predictable representation property (see [Att05]), this implies that the commutative von Neumann algebra \(\mathcal{A}([0, t]) = L^\infty(\mathcal{W}([0, t]), \mu)\) is generated by the operators \(a_j^0(s) + a_j^1(s)\) for \(s \leq t\). Note that the observable measure we use in the definition of the OQBM is \(A = \{0\} |1\rangle + |1\rangle \langle 0\} \), so the observable \(A(\tau, n) = \mathcal{G}_\tau A(n) \mathcal{G}_\tau^*\) is
\[
A(\tau, n) = \frac{1}{\sqrt{\tau}} P_{\tau} (a_j^1([\tau n, \tau(n+1)]) + a_j^0([\tau n, \tau(n+1)])) P_{\tau} \ .
\]
Thus, the algebra generated by the \(A(\tau, k)\) for \(k \leq n\) is \(P_{\tau} L^\infty(\mathcal{W}([0, t]), \mu)P_{\tau}\), which is the reason why the Brownian representation of \(\Phi\) is adapted to the study of the OQBM.

The product of two quantum stochastic integrals is itself a quantum stochastic integral under some regularity conditions.
**Proposition 18** (Quantum Itô product formula). Let \((A_t)_{t \in \mathbb{R}_+}\) and \((B_t)_{t \in \mathbb{R}_+}\) be two adapted processes of operators, with domains containing the dense adapted domains \(D \otimes_{\text{alg}} \mathcal{E}(\mathcal{M})\) in \(\mathcal{H}_S \otimes \Phi\). Assume that \((A_t^*)_{t \in \mathbb{R}_+}\) is also an adapted process with domain containing \(D \otimes_{\text{alg}} \mathcal{E}(\mathcal{M})\) and that the following integrals are well defined, on \(D \otimes_{\text{alg}} \mathcal{E}(\mathcal{M})\):

\[
T_t = \int_0^t A_s da^j_s(s) \quad S_t = \int_0^t A_s^* da^j_s(s) \quad U_t = \int_0^t B_s da^j_s(s) .
\]

Moreover, assume that for all \(s\) the operators \(A_s U_s, T_s B_s\) and \(A_s B_s\) are defined on a domain containing \(D \otimes_{\text{alg}} \mathcal{E}(\mathcal{M})\) and that the following integrals are well defined on this domain:

\[
\int_0^t A_s U_s da^j_s(s) \quad \int_0^t T_s B_s da^j_s(s) \quad \int_0^t \delta_{i,j} \delta_{t \neq 0} A_s B_s da^j_s(s) .
\]

Then the following formula is satisfied on \(D_B \otimes \mathcal{E}(\mathcal{M}_B)\):

\[
T_t U_t = \int_0^t A_s U_s da^j_s(s) + T_s B_s da^j_s(s) + \delta_{i,j} \delta_{t \neq 0} A_s B_s da^j_s(s) .
\]

This proposition was proved by Hudson and Parthasarathy, see Proposition 25.26 of Parthasarathy’s book [Par92].

Writing \(da^j_s(s) da^k_s(s) = \delta_{i,j} \delta_{k \neq 0} da^j_s(s)\), this formula can be used as

\[
d(T_t U_t) = T_t dU_t + (dT_t) U_t + (dT_t)(dU_t) .
\]

Note that in particular, if \(A_t = B_t = a_{11}^0(t) + a_{12}^0(t)\) we have

\[
d(A^2(t)) = 2A(t) dA(t) + dt
\]

which is actually the formula \(d(W_t^2) = 2W_t dW_t + dt\) for the Brownian motion.

We are now ready to present the theorem of convergence of the repeated interactions of Attal and Pautrat.

### 2.3.4 Hudson-Parthasarathy Equations and Attal-Pautrat convergence

The Attal-Pautrat limit [AP06] was devised in the context of repeated interactions processes. The idea is to show that \(G_t U_{[t/t]} G_t^*\) converge to some limit \(U_t\) as \(t\) goes to 0, which satisfies a quantum stochastic differential equation. We only present the case which is needed here.

First, we need to describe what will be the limit. It is a family of unitary following the so called quantum Langevin Equations (or Hudson-Parthasarathy Equations).

**Theorem 19.** Let \(H\) and \(N\) be two bounded operators on \(\mathcal{H}_G\) with \(H\) self-adjoint. Write

\[
G = -iH - \frac{1}{2} N^* N .
\]

Then there exists an adapted process of unitary operators \(U_t\) on \(\mathcal{H}_G \otimes \Phi\) which satisfies the following quantum stochastic equation on \(\mathcal{H}_G \otimes_{\text{alg}} \mathcal{E}(L^2(\mathbb{R}))\):

\[
dU_t = (G dt + N da_1^0(t) - N^* da_1^0(t)) U_t . \tag{2.27}
\]

The adjoint operator \(U_t^*\) satisfies the adjoint equation. With the condition \(U_0 = I\), it is unique.

This theorem is proved in [Par92]; the idea is to make Picard iterations on Equation (2.27) starting from \(U_0^0 = I\), applying Formula (2.27) to show that at each iteration the obtained operators are still unitary.

Attal and Pautrat proved the following theorem (in a more general setup).

**Theorem 20.** Let \((U_{\tau,n})_{n \in \mathbb{N}}\) be a family of operators on \(\mathcal{H}_G \otimes T \Phi\) defined as in Paragraph 2.3.4, and write \(w_{\tau,n} = G_\tau U_{\tau,n} G_\tau^*\) the isometry on \(\mathcal{H}_G \otimes \Phi\) corresponding to \(U_{\tau,n}\). Then for any \(t \geq 0\) the operator \(w_{\tau,[t/\tau]}\) converges strongly to the unitary operator \(U_t\) solution of the Hudson-Parthasarathy Equation of Theorem 19.

This theorem is proved in [AP06] in a more general context where there may be some term in \(da_1^0(t)\) in the equation and the space \(\mathcal{H}_p\) is of arbitrary dimension.
2.3.5 Convergence to the continuous OQBM

We are now ready to prove the convergence of the discrete OQBM. We consider the unitary operator \( R_\tau V_\tau \) of the discrete OQBM built in Paragraph 2.2.3. The isometry \( G_\tau \) converts it in a unitary on \( \mathcal{H}_G \otimes \mathcal{H}_{\tau,z} \otimes \Phi \). The goal is to show the convergence to a unitary on \( \mathcal{H}_G \otimes L^2(\mathbb{R}) \otimes \Phi \), so we need to see \( \mathcal{H}_{\tau,z} \) as a subspace of \( \mathcal{H}_z = L^2(\mathbb{R}) \). For each \( \tau \) we define an isometry of \( \mathcal{H}_{\tau,z} \) into a subspace of \( L^2(\mathbb{R}) \).

\[
G_{\delta,\mathbb{R}} : \mathcal{H}_{\tau,z} \longrightarrow L^2(\mathbb{R}) \\
x \longmapsto \frac{1}{\delta} \mathbb{1}_{[x, x+\delta]}
\]

The image of this isometry is the space of functions which are constant on each interval \([x, x+\delta]\), which we identify with \( \mathcal{H}_{\tau,z} \) in the following of the article, and we write \( \mathcal{H}_z = L^2(\mathbb{R}) \). We define \( P_{\delta,\mathbb{R}} = G_{\delta,\mathbb{R}}^* G_{\delta,\mathbb{R}} \) the orthogonal projection on this space. By the Lebesgue differentiation Theorem, it strongly converges to the identity as \( \delta \to 0 \). In this sense, the space \( \mathcal{H}_{\tau,z} \) converges to \( L^2(\mathbb{R}) \) as \( \tau \to 0 \).

Moreover, the translation operator \( D_\tau \in \mathcal{B}(\mathcal{H}_z) \) is transformed into

\[
G_{\delta,\mathbb{R}} D_\tau G_{\delta,\mathbb{R}}^* = P_{\delta,\mathbb{R}} e^{-\delta \partial_x}
\]

since \( e^{-\delta \partial_x} = e^{-iP} \) is the translation operator on \( L^2(\mathbb{R}) \).

Let us write

\[ l_{\tau,n} = G_\tau G_n R_\tau V_\tau G_n^* G_\tau^* \]

and define the OQBM isometry

\[ \mathfrak{U}_{\tau,n} = l_{\tau,n} l_{\tau,n-1} \cdots l_{\tau,1} . \]

We have the following convergence theorem.

**Theorem 21.** For each \( t \geq 0 \) the operator \( g u_{\tau,[t/\tau]} \) converge strongly to some unitary operator \( \mathfrak{U}_t \) solution of the equation

\[
d\mathfrak{U}_t = \left( (-iH - \frac{1}{2} N^* N + \frac{1}{2} \partial_x^2 - \partial_x N) dt + (N - \partial_x) da_0^0(t) + (-N^* - \partial_x) da_1^0(t) \right) \mathfrak{U}_t \quad (2.28)
\]

on the set \( \mathcal{H}_G \otimes_{\text{alg}} H^2(\mathbb{R}) \otimes_{\text{alg}} \varepsilon(L^2(\mathbb{R})) \).

**Remark 3.**

1. This theorem can probably be generalized to cases where \( N \) and \( H \) depends on the position \( x \), but this would require to extend non-trivially the theorem of Attal and Pautrat, the issue of the non-boundedness of \( \partial_x \) being harder to bypass when \( N \) and \( \partial_x \) are not commuting.

2. Equation (2.13) is a Hudson-Parthasarathy Equation of the form of Theorem 19 with \( N \) replaced by \( \tilde{N} = N - \partial_x \) and \( H \) replaced by \( \tilde{H} = H - \frac{1}{2}(N^* \partial_x + \partial_x N) \).

3. The operator \( \partial_x \) is unbounded, so we cannot directly apply Theorem 19 to show the existence of a solution \( U_t \), neither Theorem 20 to show the convergence. Instead, we will break \( \mathfrak{U}_{\tau,n} \) in two parts: one which is solution of a Hudson-Parthasarathy Equation with bounded coefficients, and one which is solution of a Hudson-Parthasarathy Equation with unbounded coefficients but which is very simple.

4. Hudson-Parthasarathy Equations with unbounded coefficients have been studied by Fagnola in [FR00] and Fagnola and Will in [FW03] in more complex cases and with more general method.

**Proof.** We treat the convergence of \( R_\tau \) and of \( V_\tau \) separately. First, let us consider the isomorphism \( R_\tau = D_\tau \otimes P_\tau + P_\tau \otimes D_\tau \) defined in Paragraph 2.2.3. We write \( R_{\tau,n} = G_n R_\tau G_n^* \) the corresponding operator acting on the toy Fock space, and \( r_{\tau,n} = (G_{\delta,\mathbb{R}} \otimes G_\tau) R_{\tau,n} (G_{\delta,\mathbb{R}} \otimes G_\tau)^* \). Let us consider their product

\[ z_{\tau,n} = r_{\tau,n} r_{\tau,n-1} \cdots r_{\tau,1} . \]

Note that \( V_\tau \) is not acting on \( \mathcal{H}_z \) and \( Z_\tau \) is not acting on \( \mathcal{H}_G \), so \( G_n^* Z_\tau G_n \) commutes with \( G_k^* V_\tau G_k \) for any \( n > k \). Thus we have

\[ \mathfrak{U}_{\tau,n} = z_{\tau,n} u_{\tau,n} . \]

We already know that \( u_{\tau,[t/\tau]} \) converges to some operator \( U_t \) by Theorem 20. Let us consider the limit of the operator \( z_{\tau,n} \).

**The pointer process** \( Z_t \):
Proof. Note that 
\[ \frac{\partial}{\partial Z} \text{formula yields the stochastic Equation (1.13).} \]

Remark 4. Proposition 22.

For any \( t \in \mathbb{R}_+ \) the operator \( z_{\tau, [t/\tau]} \) strongly converges to a unitary operator \( Z_t \). The process \( (Z_t)_{t \in \mathbb{R}_+} \) satisfies the following quantum SDE on the space \( H^2(\mathbb{R}) \otimes_{alg} \mathcal{E}(L^2(\mathbb{R})) \).

\[
dZ_t = \left( -\frac{1}{2} \frac{\partial^2}{\partial Z^2} dt - \frac{1}{2} \frac{\partial}{\partial Z} \left( dZ_0(t) + dZ_0^0(t) \right) \right) Z_t .
\]

(2.29)

In the probabilistic representation, \( Z_t \) is explicit: for any function \( f \in L^2(\mathbb{R}) \) and any random variable \( A \in L^2(\mathcal{W}, \mu) \) we have

\[
(Z_t f A)(x) = f(x-W_t) A .
\]

Proof. Note that \( G_{\delta Z} D_r G_{\delta Z}^* = e^{-\delta x} P_{\delta Z} \), since \( e^{-\delta x} \) is the operator of translation by \( \delta \) on \( L^2(\mathbb{R}) \). Moreover,

\[
G_r P_{\delta Z} G_r^* = \left( \frac{1}{2} (a_0^0(\tau, \tau) + a_0^1(\tau, \tau) \pm a^0_1(\tau, \tau) \pm a_0^0(\tau, n)) \right)
\]

so we have

\[
r_{\tau, n} = \frac{e^{-\delta x} + e^{\delta x}}{2} (a_0^0(\tau, \tau) + a_0^1(\tau, \tau)) P_{\delta Z} + \frac{e^{-\delta x} - e^{\delta x}}{2} (a_0^1(\tau, \tau) + a_0^0(\tau, n)) P_{\delta Z}.
\]

We want to write \( e^{-\delta x} \approx I - \delta x + \frac{1}{2} \delta^2 x^2 \). Since \( \delta x \) is unbounded, it cannot be done directly. Let us consider the space \( D_C \subset L^2(\mathbb{R}) \) of C-bandlimited functions for \( C > 0 \) : Writing \( \mathcal{F} \) the Fourier transform, the space \( D_C \) is defined as

\[
D_C = \{ f \in L^2(\mathbb{R}) \mid \mathcal{F} f \text{ is supported in } [-C, C] \} .
\]

This space is stable by \( \partial_x \) and \( \bigcup_{C > 0} D_C \) is dense in \( L^2(\mathbb{R}) \). Restricted to \( D_C \), the operator \( \partial_x \) is bounded, so we can expand the exponential. However, the space \( D_C \) is not stable by \( P_{\delta Z} \), so we introduce

\[
r_\tau, n = \frac{e^{-\delta x} + e^{\delta x}}{2} (a_0^0(\tau, \tau) + a_0^1(\tau, \tau)) P_{\delta Z} + \frac{e^{-\delta x} - e^{\delta x}}{2} (a_0^1(\tau, \tau) + a_0^0(\tau, n)) P_{\delta Z}.
\]

so that \( r_{\tau, n} = r_\tau, n P_{\delta Z} \). We also write \( \tilde{r}_{\tau, n} = \tilde{r}_{\tau, n} r_{\tau, n} - \cdots r_{\tau, 1} \). Since \( P_{\delta Z} \) commutes with \( \tilde{r}_{\tau, k} \) for all \( k \), we have that \( z_{\tau, n} = \tilde{r}_{\tau, n} P_{\delta Z} \). The space \( D_C \) is stable by \( \tilde{r}_\tau \), and on this space, since \( \partial_x \) is bounded we have

\[
r_\tau, n = \tilde{r}_\tau, n P_{\delta Z}.
\]

With \( \delta = \sqrt{\mathbb{R}} \), this sets us under the hypothesis of Theorem 20 with \( K = 0 \) and \( L = -\partial_x \). Thus, \( \tilde{z}_{\tau, [t/\tau]} \) converges strongly (on \( D_C \)) to a unitary operator \( Z_t^C \) which is solution of (2.29). All the \( Z_t^C \)’s coincide on their common domain of definition, and they are unitary, so we can extend them to \( H^2(\mathbb{R}) \) and \( L^2(\mathbb{R}) \). They commute with \( \partial_x \), so they are also unitary for the space \( H^2(\mathbb{R}) \). Since \( \tilde{r}_{\tau, [t/\tau]} \) are unitary and converge to \( Z_t \) strongly on a dense subspace, they converge strongly on the full space. Moreover, \( P_{\delta Z} \) converges strongly to \( I \), so \( z_{\tau, [t/\tau]} \mid P_{\delta Z} \) also converge strongly to \( Z_t \).

Finally, by the classical Itô formula, for any \( C^2 \) function

\[
df(x-W_t) = f(x) - \int_0^t \partial_x f(x-W_s) dW_s + \frac{1}{2} \int_0^t \partial_x^2 f(x-W_s) ds .
\]

Thus, if we write \( (Z_t f A)(x) = f(x-W_t) A \) for any \( f \in L^2(\mathbb{R}) \), the processes \( \tilde{Z}_t \) and \( Z_t \) follow the same quantum SDE on \( C^2 \) functions. Since they have the same initial state \( Z_0 = I \), this implies that they are equal.

As a consequence of this proposition, the operators \( \mathcal{U}_{\tau, [t/\tau]} \) converges to \( \mathcal{U}_t := Z_t U_t \) and the Itô product formula yields the stochastic Equation (1.13).

Remark 4. 1. It is also possible to prove Theorem 21 by using the Attal-Pautrat Theorem directly on \( U_t \) restricted to \( \mathcal{H}_G \otimes_{alg} D_C \otimes_{alg} \Phi \) since \( \mathcal{H}_G \otimes_{alg} D_C \) is stable by \( H \) and \( N \). However, we believe that the pointer unitary operator \( Z_t \) has its own interest.

2. Note that \( Z_t \) does not commute with \( U_t \), we only have the commutation of \( U_s \) and \( Z_{t,s} := Z_t Z_s^* \) for \( s \leq t \). The formula \( \mathcal{U}_t = Z_t U_t \) is consistent with the construction of the discrete OQBM: we make the system evolve according to the unitary operator \( U_t \), and we apply the operator \( Z_t \) which implements the translation by \( W_t \) to the position of the quantum particle.
2.4 From the Hudson-Parthasarathy Equation to the Lindblad Equation

The family of operators \((\mathcal{U}_t)_{t \geq 0}\) and of states \(\rho_{\text{tot}, t} = \mathcal{U}_t(\rho_{S} \otimes |\Omega\rangle\langle\Omega|)\) consists in the most complete description of the OQBM. In this subsection, we show how the Lindbladian picture of the OQBM result from this unitary description.

Let us consider \(\hat{A}_\tau\) as a quantum channel on \(\mathcal{H}_G \otimes \mathcal{H}_z\) with the help of the isometry \(\mathcal{G}_{\delta S, z}: l^2(\delta \mathbb{Z}) \to L^2(\mathbb{R})\); for \(\rho \in \mathcal{H}_G \otimes L^2(\mathbb{R})\) we write

\[
\Lambda_{S, \tau}(\rho) = \mathcal{G}_{\delta S, z}(\mathcal{G}_{\delta S, z}^* \rho \mathcal{G}_{\delta S, z}) \mathcal{G}_{\delta S, z}^*
\]

Now we are ready to study the convergence as \(\tau \to 0\).

**Proposition 23.** For any initial state \(\rho \in \mathcal{S}(\mathcal{H}_{G} \otimes \mathcal{H}_z)\) and for any \(t \geq 0\) the state \(\Lambda^{(t/\tau)}_{S, \tau}(\rho)\) converges in trace norm to

\[
\Lambda^t_S(\rho) = \text{Tr}_\Phi (\mathcal{U}_t (\rho \otimes |\Omega\rangle\langle\Omega|) \mathcal{U}_t^*)
\]

**Proof.** Because of the dilation of \(\hat{A}_\tau\) (Lemma 22) we have

\[
\Lambda^t_{S, \tau}(\rho) = \text{Tr}_\Phi (\mathcal{U}_{\tau, n}(\rho \otimes |\Omega\rangle\langle\Omega|) \mathcal{U}_{\tau, n}^*)
\]

But by Theorem 21 the state \(\mathcal{U}_{\tau, n}(\rho \otimes |\Omega\rangle\langle\Omega|)\) converges in trace norm to \(\mathcal{U}_t (\rho \otimes |\Omega\rangle\langle\Omega|)\), and the convergence is preserved by applying the partial trace. \(\square\)

The semigroup \(\Lambda^t_S\) is strongly continuous in \(t\), but not continuous for the trace norm, so its generator is not defined on the whole space \(\mathcal{S}(\mathcal{H}_G \otimes \mathcal{H}_z)\) but only on the space of Sobolev states, as defined below.

**Definition 24.** For any Hilbert space \(\mathcal{H}\) and any \(k \in \mathbb{N}\) the set \(W^k \mathcal{S}(\mathcal{H}, \mathcal{H}_z)\) is the set of states \(\rho\) on \(\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_z)\) which admits a kernel \((x, y) \in \mathbb{R}^2 \mapsto K_\rho(x, y) \in \mathcal{S}^1(\mathcal{H})\) which is in the Sobolev space \(W^{k, 1}(\mathbb{R}^2, \mathcal{S}^1(\mathcal{H}))\). Equivalently, it is the space of states \(\rho\) in \(\mathcal{S}(\mathcal{H} \otimes \mathcal{H}_z)\) such that for any \(n \leq k\) the operator \([\rho, [\partial_x|^n)]\) is a bounded operator on \(\mathcal{H} \otimes W^{2, k}(\mathbb{R})\).

The set \(W^k \mathcal{S}(\mathcal{H}, L^\infty(\mathbb{R}))\) is the set of states \(\rho\) on \(\mathcal{B}(\mathcal{H} \otimes L^\infty(\mathbb{R}))\) which admits a kernel \(x \in \mathbb{R} \mapsto K_\rho(x) \in \mathcal{S}^1(\mathcal{H})\) which is in the Sobolev space \(W^{k, 1}(\mathbb{R}, \mathcal{S}^1(\mathcal{H}))\).

We can now express the Lindblad Equation for \(\Lambda^t_S\).

**Theorem 25.** For any initial state \(\rho \in W^2 \mathcal{S}(\mathcal{H}_G, \mathcal{H}_z)\) the state \(\rho_S(t) = \Lambda^t_S(\rho)\) is in \(W^2 \mathcal{S}(\mathcal{H}_G, \mathcal{H}_z)\) for all \(t > 0\). Moreover, it satisfies the following equation:

\[
\frac{d}{dt}\rho_S(t) = \mathcal{L}(\rho_S(t))
\]

where

\[
\mathcal{L}(\rho) = -i[\hat{H}, \rho] + L \rho L^* - \frac{1}{2} \{L^*L, \rho\}
\]

with \(L = N - \partial_x\) and \(\hat{H} = H - \frac{i}{2}\partial_x (N + N^*)\).

Writing \(K_i(x, y)\) the kernel of \(\rho(t)\) this equation becomes

\[
\frac{d}{dt} K_i(x, y) = \mathcal{L}(K_i(x, y)) + \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 K_i(x, y) - N \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) K_i(x, y) - \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) K_i(x, y) N^*.
\]

Equation (2.31) can be formally obtained by writing \(e^{-\delta \hat{A}_\tau} \approx I - \delta \hat{A}_\tau + \frac{\delta^2}{2} \hat{A}_\tau^2\), but let us prove it from the dilation of \(\Lambda^t_S\).

**Proof.** The operator \(\mathcal{U}_t\) preserves the space \(\mathcal{S}_2(\mathcal{H}_G \otimes \Phi, \mathcal{H}_z)\):

The operator \(\mathcal{U}_t\) commutes with \(\partial_x\) (since \(Z_t\) and \(S_t\) both commute with \(\partial_x\)) so for any operator \(\rho_{\text{tot}} \in W^2 \mathcal{S}(\mathcal{H}_G \otimes \Phi)\) we have \([\mathcal{U}_t \rho \mathcal{U}_t^*, [\partial_x[^n]] = \mathcal{U}_t[\rho, [\partial_x[^n]]]\mathcal{U}_t^*\).

Thus, if \(\rho \in W^2 \mathcal{S}(\mathcal{H}_G, \mathcal{H}_z)\) then \(\mathcal{U}_t (\rho \otimes |\Omega\rangle\langle\Omega|) \mathcal{U}_t^* \in W^2 \mathcal{S}(\mathcal{H}_G \otimes \Phi, \mathcal{H}_z)\) and so \(\rho_S(t) \in W^2 \mathcal{S}(\mathcal{H}_G, \mathcal{H}_z)\).

To obtain the Lindblad Equation we use the Heisenberg representation: for any observable \(A \in W^{2, 1}(\mathbb{R}^2, \mathcal{B}(\mathcal{H}_G))\) we have

\[
\text{Tr} (\rho_S(t) A) = \text{Tr} (\rho \langle\Omega| \mathcal{U}_t^* (A \otimes I_\Phi) \mathcal{U}_t |\Omega\rangle).
\]
Using the Itô formula applied to $\Omega^*\mathcal{U}_t$ on the domain $\mathcal{H}_G \otimes_{alg} H^2(\mathbb{R}) \otimes_{alg} E(L^2(\mathbb{R}))$, we obtain that

$$\Omega^*\mathcal{U}_t = A + \int_0^t \Omega^*\hat{\mathcal{L}}^*(A)\,ds + R_t$$

where $R_t$ is a quantum stochastic integral with respect of terms of the form $da^*_j(t)$ with $(i,j) \neq (0,0)$, so that $\langle \Omega | R_t | \Omega \rangle = 0$. Thus

$$\text{Tr} (\rho_{S,t} A) - \text{Tr} (\rho A) = \int_0^t \text{Tr} \left( \Omega^*(\rho \otimes |\Omega\rangle \langle \Omega|)\Omega^*\hat{\mathcal{L}}^*(A) \right) \, ds$$

$$= \int_0^t \text{Tr} \left( \hat{\mathcal{L}}(\rho_{S,t}(s))A \right) \, ds$$

which implies Equation (2.33) by density of $W^{1,1}(\mathbb{R}^2, B(\mathcal{H}_G))$ in $B(hh_G \otimes \mathcal{H}_z)$. The equation on the kernel $K_\rho$ is obtained by using the following formulas: if $\rho \in W^1(\mathcal{S}(H_G, \mathcal{H}_z))$ then $\partial_x \rho$ and $\rho \partial_x$ are kernel operators, with

$$K_{\rho \partial_x}(x,y) = -\frac{\partial}{\partial y} K_\rho(x,y)$$

$$K_{\partial_x \rho}(x,y) = \frac{\partial}{\partial x} K_\rho(x,y) \quad (2.32)$$

These formulas are obtained through integrations by parts. \[\square\]

Let us consider the restriction on the algebra $\mathcal{M} = B(\mathcal{H}_G) \otimes L^2(\mathbb{R})$, in order to obtain the Lindblad Equation on “diagonal states” (Equation (1.11)). As noted in the introduction, we consider states restricted to $\mathcal{M}$ rather than states whose density matrix is in $\mathcal{M}$, since $\mathcal{M}$ contains no non-trivial trace-class operators.

**Theorem 26.** There exists a semigroup of super-operators $(\Lambda^t_M)_{0 \leq t} \subset \mathcal{S}(\mathcal{M})$ such that for any state $\rho \in \mathcal{S}(\mathcal{H}_S)$ with restriction $\rho_M$ to $\mathcal{M}$, the restriction to $\mathcal{M}$ of the state $\rho_t = \Lambda^t_S(\rho)$ is $\Lambda^t_M(\rho_M)$.

If a state $\rho_M$ admits a kernel $x \mapsto Q_\rho(x)$ which is in $W^{2,1}(\mathbb{R}, S^1(\mathcal{H}_G))$ then $\rho_{t,M} = \Lambda^t_M(\rho_M)$ also admits a kernel $Q_t \in W^{2,1}(\mathbb{R}, S^1(\mathcal{H}_G))$ and we have

$$\frac{d}{dt} Q_t(x) = \mathcal{L}(Q_t(x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} Q_t(x) - \left( N \frac{\partial}{\partial x} Q_t(x) + \left( \frac{\partial}{\partial x} Q_t(x) \right) N^* \right) \quad (2.34)$$

**Proof.** Note that $\Omega^*\mathcal{U}_t \subset \mathcal{M} \otimes B(\Phi)$ (because of the form of $Z_t$) so that for any $A \in \mathcal{M}$ the expectancy $\text{Tr} \left( \Lambda^t_S(\rho)A \right) = \text{Tr} \left( (\rho \otimes |\Omega\rangle \langle \Omega|)\Omega^*\mathcal{U}_t A \right)$ only depends on the restriction of $\rho$ to $\mathcal{M}$. This proves the existence of $\Lambda^t_M$. Equation (2.34) is proved exactly the same way as Proposition 25. \[\square\]

### 2.5 Hierarchy of the descriptions of the OQBM

With the OQBM, we have many views on the same object, carrying more or less informations:

a) The state $\rho_{tot,t} = \mathcal{U}_t(\rho_{S,t} \otimes |\Omega\rangle \langle \Omega|)\mathcal{U}_t^*$ on $\mathcal{H}_G \otimes \mathcal{H}_z \otimes \Phi$ offers the most complete description.

b) The state $\rho_{tot,G,t} = U_t(\rho_{G,t} \otimes |\Omega\rangle \langle \Omega|)U_t^* = \text{Tr}_{\mathcal{H}_z}(\rho_{tot,t})$ ignores the position of the particle, though its translation $W_t$ is still registered in $\Phi$.

c) The random state $\rho_t$ with the random position $X_t$ ignores the quantum aspect of the position, but keeps tracks of the classical correlations between two different times.

d) The state $\rho_{S,t} = \text{Tr}_{\mathcal{H}_z}(\rho_{tot,t}) = \Lambda^t_S(\rho_S)$ on $\mathcal{B}(\mathcal{H}_S)$ forgets about correlations between different times and the precise distribution of $\rho_t$, but conserves a quantum view on the position.

e) The restriction of $\rho_{S,t}$ to $\mathcal{M} = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}_G$ with matrix density function $Q_t(x) = E(\rho_t | X_t = x)$: it forgets the correlations between different times and has only the classical information about the position. This is the smallest description where we have a closed equation for the evolution (Equation (2.34)) and which allows to compute the distribution of $X_t$.

f) The state $\rho_G = \text{Tr}_{\mathcal{H}_z \otimes \Phi}(\rho_{tot,G,t}) = \int_{x \in \mathbb{R}} Q_t(x) \, dx$ evolves according to the Lindbhadian $\mathcal{L}$ and it completely ignores the position $X_t$. 

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The descriptions a), c), d), e) are really dealing with the OQ BM, while b) and f) are only considering the evolution on $H_G$. They can be obtained one from another by partial traces, restriction and conditional expectancy according to the following hierarchy:

Figure 1: Hierarchy between the descriptions of the Open Quantum Brownian Motion

The way we can obtain the process $(\rho_t, X_t)$ directly from the unitary description is the subject of the second section of this article.

3 Non-demolition measured evolution applied to the Open Quantum Brownian Motion

In the first section, we described the process $(\rho_{\tau,n}, X_{\tau,n})_{n \in \mathbb{N}}$ as the result of a succession of unitary evolution by $L_{\tau}$ and measure of the position $X_{\tau,n} \in \delta \mathbb{Z}$. In continuous time this picture is harder to obtain, since the the measure and the evolution are happening at the same time. In this section we construct a general framework to deal with simultaneous measurement and evolution, using the crucial idea of non-demolition measurement introduced by Belavkin [Bel94].

3.1 Evolution and measurement

The evolution of a system after a measurement may be impossible to describe. Let us assume that the system evolves according to a unitary operator $U$ on $H_G \otimes L^2(\mathcal{X}, \mu)$. We may measure the algebra $\mathcal{A} = L^\infty(\mathcal{X}, \nu)$ before or after applying $U$, obtaining a random variable $X \in \mathcal{X}$ and random states $\rho_0 = \rho_{\mathcal{G}|\mathcal{A}}$ and $\rho_1 = (U\rho U^*)_{\mathcal{S}|\mathcal{A}}$. However, it is not clear how to describe the measurement before and after applying $U$. There may be two issues there:

1. The state $\rho_{\mathcal{G}|\mathcal{A}}$ is well defined only if $\mathcal{A}$ is discrete, else we only have the partial state $\rho_{\mathcal{G}|\mathcal{A}}$. Thus, we cannot define $U\rho_{\mathcal{G}|\mathcal{A}}U^*$.

2. Even if $\mathcal{A}$ is discrete, the measurement before applying $U$ modifies the state of the system, so $(U\rho_{\mathcal{G}|\mathcal{A}}U^*)_{\mathcal{S}|\mathcal{A}}$ may not have the same distribution as $(U\rho U^*)_{\mathcal{S}|\mathcal{A}}$.

The restriction to so called non-demolition evolutions allows to bypass these two issues in the general context of measurement under evolution.

**Definition 27.** Let $H_G$ and $H_B$ be two Hilbert spaces, let $I \subset \mathbb{R}$ be a set of times and $(U_t)_{t \in I}$ be a family of unitary operators on $H_G \otimes H_B$ with $U_0 = I$ if $0 \in I$ and let $(\mathcal{A}_t)_{t \in I}$ be a family of commutative von Neumann algebras on $H_B$. Write $U_{t,s} = U_t U_s^*$ for any $s, t \in I$. We say that the process $(U_t, \mathcal{A}_t)_{t \in I}$ is a $H_G$-non demolition evolution if for any $s \leq t \in I$ we have

$$U_{t,s} \mathcal{A}_s U_{t,s}^* \subset \{ I_G \} \otimes \mathcal{A}_t'$$

where $\mathcal{A}_t'$ is the commutant of $\mathcal{A}_t$.

In most cases the family of algebras will be increasing ($\mathcal{A}_s \subset \mathcal{A}_t$ for $s \leq t$) but we do not require it.
The non-demolition condition can be divided in two parts: the condition $U_{t,s}^*A_tU_{t,s} \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_t'$ is here to ensure that the measure of $\mathcal{A}_t$ does not disturb the measure of $\mathcal{A}_t'$ after evolution, while the condition $U_{t,s}^*A_tU_{t,s} \subseteq (I_G \otimes \mathcal{B}(\mathcal{H}))$ ensure that the random state at time $t$ is well defined. Let us describe more precisely how the random evolution can be defined.

Let us consider an $\mathcal{H}_G$-non demolition evolution $(U_t, \mathcal{A}_t)_{t \in I}$ and a state $\rho_0 \in \mathcal{S}(\mathcal{H}_G \otimes \mathcal{H}_B)$. We make the assumption that $I$ is upper bounded\footnote{this assumption is actually not necessary but it allows to use more concrete notations} by some $T \in I$. We fix some identifications $\mathcal{A}_t \simeq L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$, implemented by some isometries $G_t : L^2(\mathcal{X}_t, \mathcal{F}_t, \mu_t) \to \mathcal{H}_B$. We want to define a probability space $(\Omega, \mathcal{P})$ with a stochastic process $(X_t, \eta_t)_{t \in I}$ with $X_t \in \mathcal{X}_t$ and $\eta_t \in \mathcal{S}(\mathcal{H}_G)$ obtained by simultaneously measuring $\mathcal{A}_t$ at time $t$ and making evolve the system according to $U_t$. We construct it as follows.

- Let $\mathcal{A}_t^U$ be the smallest von Neumann algebra containing all the algebras $U_{t,s}^*A_sU_{t,s} \subseteq \mathcal{A}_t'$ for $s \leq t$. It is commutative and contained in $I_G \otimes \mathcal{B}(\mathcal{H}_B)$ by the $\mathcal{H}_G$-non demolition hypothesis. We fix an identification $\mathcal{A}_t^U \simeq L^\infty(\mathcal{X}_t^U, \mathcal{F}_t^U, \mu_t^U)$ implemented by an isometry $G_t^U : L^2(\mathcal{X}_t^U, \mathcal{F}_t^U, \mu_t^U) \to \mathcal{H}_B$.

- For any $s \leq t$ we have $\mathcal{A}_t \simeq \mathcal{A}_t^U$ so there exists a map $\phi_t : \mathcal{X}_t^U \to \mathcal{X}_s^U$ such that for any $f \in L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$ we have $G_t^U M_{f \circ \phi_t}(G_t^U)^* = G_t M_f G_t^*$. \footnote{\textit{Definition 28}. Any process $(X_t, \eta_t)_{t \in I}$ obtained as above is called a measured evolution obtained from the $\mathcal{H}_G$-non demolition evolution $(U_t, \mathcal{A}_t)_{t \in I}$ and the state $\rho_0$. This way of define the stochastic process should seem natural; a first motivation is that for all $t$ the variable $X_t$ has the same distribution as the result of the measure of $\mathcal{A}_t$ in the state $U_t^*\rho U_t$, indeed for any function $f \in L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$ we have $E(f) = E(f \circ \phi_t \circ \eta_{t,T})$}

\begin{align*}
E(f) &= E(f \circ \phi_t \circ \eta_{t,T}) \\
&= \text{Tr} \left( U_T U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* G_t M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right).
\end{align*}

However, this is only the distribution of $X_t$ at one time, and it does not justifies the joint distribution of the $X_t$’s for $t \in I$. We will use the indirect measurement (\textit{Definition 10}) to make a more complete and useful argument.

\begin{align*}
E(f) &= E(f \circ \phi_t \circ \eta_{t,T}) \\
&= \text{Tr} \left( U_T U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* G_t M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right).
\end{align*}

\begin{align*}
E(f) &= E(f \circ \phi_t \circ \eta_{t,T}) \\
&= \text{Tr} \left( U_T U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* U_{t,T}^* G_t^U M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right) \\
&= \text{Tr} \left( U_t^* \rho U_t^* G_t M_{f \circ \phi_t \circ \eta_{t,T}} G_t^U \right).
\end{align*}
the result of the measurement of \(L^\infty(\mathcal{Y}_0)\) for the state \(Z_0 = (U_{t_0} \rho U_{t_0}) \otimes \sigma_0) Z_0^*\), and \(\varrho \in B(t_0)\) the state on \(\mathcal{H}_S \otimes \mathcal{H}_B\) after the measurement; then, define \(Y_1\) the result of the measurement of \(L^\infty(Y_0)\) for the state \(Z_1 = (U_{t_1} \rho \otimes \sigma_1) Z_1^*\), and define successively \(Y_2, \ldots, Y_n\) the same way. We obtain a random process \((Y_k)_{0 \leq k \leq n}\) on the space \(\prod_{k=0}^n \mathcal{Y}_k\) and a family of random states \(\varrho^Y_k (\cdot) = \text{Tr}_B (\rho \otimes \varrho (U_{t_k} Z_k^*))\).

Note that we can perform this type of indirect measurement even if the property of \(\mathcal{H}_G\)-non demolition is missing. The non-demolition property makes these indirect measurements to be consistent with the process described above, as follows.

**Proposition 30** (Consistency of the unraveling). Consider any indirect measurement of \((\mathcal{A}_t)_{t \in I}\) under the evolution \((U_t)_{t \in I}\) described as above. Assume that the \(\mathcal{H}_G\)-non demolition property is satisfied. Consider the random state \(\varrho_t\) and the random variables \(X_t \in \mathcal{X}_t\) defined above on the universe \(\mathcal{X}_\text{tot}\). Add to this universe a family of random variables \((Y^0_k)_{k \in I}\) with probability distribution \(p_k d\nu_k\), where \(p_k\) is the probability density corresponding to the state \(\sigma_k\) on \(L^2(\mathcal{Y}_k, \nu_k)\). Assume that they are mutually independent and independent of \((X_t)_{t \in I}\) and define

\[
\tilde{Y}_k = \psi(X_{t_k}, Y^0_k)
\]

\[
\varrho^Y_k = \mathbb{E}(\varrho_t((X_s)_{s \in I}) \mid (\tilde{Y}_k)_{0 \leq k \leq n})
\]

Then \((\tilde{Y}_k, \varrho^Y_k)_{0 \leq k \leq n}\) has the same distribution as the process \((Y_k, \varrho^Y_k)_{0 \leq k \leq n}\) defined by the indirect measurement.

**Proof.** Let us write

\[
W_k = Z_k U_{t_k,t_{k-1}} \cdots U_{t_1,t_0} Z_0 U_{t_0} \otimes \nu_0
\]

and let \(\mathcal{A}^Y_k = L^\infty(\prod_{l \leq k} \mathcal{Y}_l, \otimes_{l \leq k} \nu_l)\). Then

\[
\varrho^Y_k = (W_k (\rho \otimes \sigma) W_k^*)_{\mathcal{A}^Y_k}
\]

Moreover, for any function \(f \in \mathcal{A}^Y_k\) and operator \(A \in \mathcal{B}(\mathcal{H}_S)\) we have

\[
\mathbb{E}(\text{Tr} (\varrho^Y_k) f(Y_0, \ldots, Y_k)) = \text{Tr} (W_k (\rho \otimes \sigma) W_k^* A \otimes M_f)
\]

Similarly, for all \(l \leq k\) the random variable \(\tilde{Y}_l\) corresponds to the measurable map \(\psi_l(\varrho_t \circ \eta_{t_l,t_0} (x_{t_l}), y_l)\) on \(\mathcal{A}^Y_{t_l}\) and so the random variable \(f(Y_0, \ldots, \tilde{Y}_k)\) can be seen as the measurable map \(g \in L^\infty(\mathcal{A}^U_{t_k} \times \prod_{l \leq k} \mathcal{Y}_l)\) defined by

\[
g(x_{t_k}, y_0, \ldots, y_k) = f(\psi_0(\varrho_{t_0} \circ \eta_{t_0,t_k} (x_{t_k}), y_0), \ldots, \psi_k(\varrho_{t_k} (x_{t_k}, y_k)))
\]

and by the construction of \(\varrho_t\) and \(\tilde{Y}_k\) we have

\[
\mathbb{E}(\text{Tr} (\varrho^Y_k) f(Y_0, \ldots, \tilde{Y}_k) = \text{Tr} \left((U_{t_k} \rho U_{t_k}^* \otimes \sigma) A \otimes G^U_{t_k} M_g (G^U_{t_k})^*\right)
\]

Now, by the definition of \(\eta_{t,t_k}\) and of \(W_k\) we have

\[
G^U_{t_k} M_g (G^U_{t_k})^* = U_{t_k} W_k^* f W_k U_{t_k}^*
\]

by the definition of the \(Z_k\) and \(\varrho_{t_k,\eta_{t_k}}\). Thus,

\[
\mathbb{E}(\text{Tr} (\varrho^Y_k) f(Y_0, \ldots, Y_k)) = \text{Tr} \left((U_{t_k} \rho U_{t_k}^* \otimes \sigma) A U_{t_k} W_k^* f W_k U_{t_k}^*\right)
\]

\[
= \text{Tr} (W_k (\rho \otimes \sigma) U_{t_k} A U_{t_k} W_k^* f) .
\]

Now, by \(\mathcal{H}_G\)-non demolition, since \(A\) is in the commutator of \(I_G \otimes \mathcal{B}(\mathcal{H}_B)\) for any \(l \leq k\) we have \(U_{t_{k+1}}^* A U_{t_{k+1}}^* \otimes \mathcal{A}'_{t_{k+1}}\) and in particular \(U_{t_{k+1}}^* A U_{t_{k+1}}^* \otimes \mathcal{A}'_{t_{k+1}}\) commutes with \(Z_k^*\). Thus, we have

\[
U_{t_k}^* A U_{t_k} W_k^* = U_{t_k}^* (U_{t_{k+1}}^* A U_{t_{k+1}}^* \otimes \mathcal{A}'_{t_{k+1}}) Z_0 \cdots U_{t_{k+1}}^* Z_{t_{k+1}}^* U_{t_{k+1}}^* \otimes \mathcal{A}'_{t_{k+1}} Z_k^* Z_k^* U_{t_k}^* A U_{t_k} W_k^*
\]

and with successive commutations we get

\[
U_{t_k}^* A U_{t_k} W_k^* = W_k^* A .
\]

Thus we have

\[
\mathbb{E}(\text{Tr} (\varrho^Y_k) f(Y_0, \ldots, \tilde{Y}_k)) = \mathbb{E}(\text{Tr} (\varrho^Y_k) f(Y_0, \ldots, Y_k)) .
\]

This proves the equality in distribution.
3.1.1 The example of OQWs

Open Quantum Random Walks are our first example of measured evolution. Let us consider any OQW \((B_z)_{z \in E}\) on a countable graph \((V,E)\). It consists in the succession of evolution by the quantum channel \(\varphi(\rho) = \sum_{(x,y) \in E} B_{(x,y)}^* \rho B_{(x,y)} \otimes |x\rangle \langle x| \otimes |y\rangle \langle y|\) and of measure of the algebra \(A_V = l^\infty(V)\). As such, it does not need the formalism of measured evolution to be defined since \(A_V\) is discrete, but it is a good demonstrator of measured evolution.

Let us construct the auxiliary space \(H_p = l^2(V)\). In the article [ASPI12] in which OQW where first defined, a unitary operator \(U\) on \(H_G \otimes l^2(V) \otimes H_p\) is constructed the following way: we fix a point \(x_0 \in V\). For any \(x \in V\) we consider a unitary operator \(V(x)\) such that for all \(y \in V\) we have

\[
\langle y | H_p \rho (x) | x_0 \rangle_{H_p} = \mathbb{1}_{(x,y) \in E} B_{(x,y)} \cdot
\]

It exists because of the condition \(\sum_y \text{ with } (x,y) \in E B_{(x,y)}^* B_e = I\). Write \(V(x)y_{yz} = \langle y | H_p V(x) | z \rangle_{H_p}\). We put

\[
U = \sum_{x,y,z \in V} V(x)_{yz} \otimes |y\rangle \langle x| \otimes |z\rangle.
\]

Consider the Toy Fock space \(T \Phi_V = \bigotimes_{n \in \mathbb{N}*} H_p\) with respect to \(|x_0\rangle\), and write \(|\Omega\rangle = \bigotimes_{n \in \mathbb{N}*} |x_0\rangle\). We consider the unitary operator \(U(n,n-1) = G_n^* U G_n\) on \(H_G \otimes l^2(V) \otimes T \Phi_V\) and define \(U(n) = U(n,n-1)U(n-1,n-2) \cdots U(2,1)\). The system \((U(n),\mathcal{A}_n)_{n \in \mathbb{N}*}\) is \(H_G\)-non demolition, indeed \(U(I_G \otimes \mathcal{A}_V)U^* \subset I_G \otimes \mathcal{A}_V \otimes B(H_p)\). More precisely, for any \(f = \sum_{x \in V} f(x) |x\rangle \langle x| \in \mathcal{A}_z\) we have

\[
UFU^* = \sum_{x,y,z \in V} f(x) U(x)_{yz} U(x)_{y'z} \otimes |y'\rangle \langle x| \otimes |z\rangle
\]

Moreover, we have

\[
\text{Tr}_{T \Phi_V} (U(n)(\rho \otimes |\Omega\rangle \langle \Omega|))U(n)^* = \varphi^n(\rho),
\]

where \(\varphi\) is the quantum channel defined by the OQW. By Proposition [9] this means that the OQW has the same distribution that the process \((\rho_t,X_t)_{n \in \mathbb{N}^*}\) given by the measured evolution of \((U(n),\mathcal{A}_V)_{n \in \mathbb{N}*}\) with initial state \(\rho \otimes |\Omega\rangle \langle \Omega|\). Let us just make explicit the algebras \(A^V_t\) and the maps \(\phi_t\) and \(\eta_{n,t}\) used in the definition of the measured evolution.

Writing \(\mathcal{A}_n = l^\infty(V^n)\) the algebra generated by the operators \(|x_1\rangle \langle x_1| \otimes \cdots \otimes |x_n\rangle \langle x_n| \otimes I\) on \(T \Phi_V\) we have

\[
A^V_t = \mathcal{A}_z \otimes \mathcal{A}_n = l^\infty(V \times V^n).
\]

The operator \(\phi_n : V \times V^n \rightarrow V\) is simply the projection on the first coordinate, and for \(m < n\) the operator \(\eta_{m,n} : V \times V^n \rightarrow V \times V^m\) is defined by

\[
\eta_{m,n}(x,x_1,\cdots,x_n) = (x_n,x_1,\cdots,x_m).
\]

3.2 Application to the Open Quantum Brownian Motion

With the measured evolution setup, we are able to obtain the process \((\rho_t,X_t)_{0 \leq t \leq T}\) satisfying the diffusive Belavkin Equation directly from the unitary operator \(U_t\) and no more as the limit of a discrete-time repeated measurement setup. First, we just consider the system \((U_t,\mathcal{A}_t)_{0 \leq t \leq T}\) where \(\mathcal{A}_t = L^\infty(W([0,t]) \subset B(\Phi)\). Second, we apply this to the measured evolution of \((\Omega_t,\mathcal{A}_z)_{0 \leq t \leq T}\) where \(\mathcal{A}_z = L^\infty(\mathbb{R}) \subset B(H_z)\).

3.2.1 Measured evolution for the Hudson-Parthasarathy process

In this part we studied the measured evolution \((U_t,\mathcal{A}_t)_{0 \leq t \leq T}\) on \(H_G \otimes \Phi\). The setup is quite simple in this case, because \(A^G_t = A_t\) and \(\eta_{n,t}\) is just the map \((w_n)_{0 \leq u \leq t} \mapsto (w_n)_{0 \leq u \leq s}\). This allows to study it in a less contrived way that the measured evolution described above, and the following result is well known in quantum filtering theory (see Theorem 7.1 and Corollary 7.2 in [BVHIJ07]).

**Proposition 31.** The system \((U_t,\mathcal{A}_t)_{0 \leq t \leq T}\) is \(H_G\)-non demolition. If \(H_G\) is finite-dimensional it admits a measured evolution process \((\rho_t,\mathcal{A}_z)_{0 \leq t \leq T}\) corresponding to the initial state \(\rho \otimes |\Omega\rangle \langle \Omega|\) which satisfies the diffusive Belavkin Equation (2.10).
Proof. Note that for any $s$ the process of operators $(U_{s,t})_{s \leq t \leq T}$ satisfies the Hudson-Parthasarathy Equation [2.27] and $U_{s,s} = I$. Thus, $U_{s,t}$ does not act on $\mathbb{P}_{[0,s]}$, in particular for any $f \in \mathcal{A}_s$ we have $U_{s,t}f U_{s,t}^* = f$. This proves the non-demolition. Since $U_{s,t}A U_{s,t}^* = A_s$ we have $\mathcal{A}_U = \mathcal{A}_s$, we can take $\phi_t$ the identity map on $\mathcal{W}([0,t])$, and $\eta_{s,t} : \mathcal{W}([0,t]) \to \mathcal{W}([0,s])$ is just the restriction to $[0,s]$. Thus, the state $\eta_t$ satisfies

$$\mathbb{E}_\eta(\text{Tr} (\eta_t A) f ((W_u)_{u \leq t})) = \text{Tr} (U_t (\rho \otimes |\Omega\rangle \langle \Omega|) U_t^* A \otimes f)$$

for any observable $A \in \mathcal{B}(\mathcal{H}_G)$ and function $f \in \mathcal{A}_t$. We study first the unnormalized state $\zeta_t = u_{\mathcal{W}([0,t])} (U_t (\rho \otimes |\Omega\rangle \langle \Omega|) U_t^*)$. It is defined as in Theorem 9, by

$$\text{E}_\mu (\text{Tr} (\zeta_t A) f ((W_u)_{u \leq t})) = \text{Tr} (U_t (\rho \otimes |\Omega\rangle \langle \Omega|) U_t^* A \otimes f)$$

(where $\mu$ is the measure on $\mathcal{W}([0,T])$ under which $(W_t)_{0 \leq t \leq T}$ is the Wiener process). We compute the Equation for $\zeta_t$ using the Itô formula. First, we use the Heisenberg representation:

$$\text{Tr} (U_t (\rho \otimes |\Omega\rangle \langle \Omega|) U_t^* A \otimes f) = \text{Tr} (\rho (U_t^* (A \otimes f) U_t) \Omega) \ .$$

Let us write $f_s = \text{E}_\mu (f |\mathcal{F}_s)$ (where $\mathcal{F}_s$ is the $\sigma$-algebra generated by $(W_u)_{u \leq s}$). It is a martingale: $f_s$ is bounded for all $s$ since $f$ is bounded, and by the predictable representation theory there exists an adapted process $(g_s)_{s \leq t}$ such that

$$df_s = g_s dW_s$$

or in terms of quantum SDE, $f_s = f_0 + \int_0^s g_s (da_0(s) + da_1^*(s))$ on $\mathcal{E}(\mathbb{L}^2(\mathbb{R}))$. We apply the quantum Itô formula two times to the product $U_t^* (A \otimes f_s) U_s$; since we are interested in $\langle \Omega | U_t^* (A \otimes f_s) U_s | \Omega \rangle$ we can ignore the terms which are not in $dt$. We obtain

$$U_t^* (A \otimes f_t) U_s = A \otimes f_0 + \int_0^t U_s^* L(A) f_s U_s ds + \int_0^t U_s^* (N^* A + AN) U_s g_s ds + R_t ,$$

where $R_t$ is a quantum Itô integral with only terms in $da_0(s)$ and $da_1^*(s)$. This implies that

$$\text{E}_\mu (\text{Tr} (\zeta_t A) f ((W_u)_{u \leq t})) = f_0 \text{Tr} (\rho A) + \int_0^t \text{Tr} (U_s (\rho \otimes |\Omega\rangle \langle \Omega|) U_s (L(A) f_s + (N^* A + AN) g_s)) ds$$

$$= f_0 \text{Tr} (\rho A) + \int_0^t \mathbb{E}_\mu (\text{Tr} (L(\zeta_s) A) f_s + \text{Tr} ((N \zeta_s + \zeta_s N^*) g_s)) ds$$

$$= f_0 \text{Tr} (\rho A) + \mathbb{E}_\mu \left( \left( \int_0^t L(\zeta_s) ds + \int_0^t (N \zeta_s + \zeta_s N^*) dW_s \right) A \right) f$$

the last equality being a consequence of the classical Itô formula. This implies that, for $\mathcal{H}_G$ of finite-dimension,

$$d\zeta_t = L(\zeta_t) dt + (N \zeta_t + \zeta_t N^*) dW_t \ .$$

(3.35)

It is now time to go back to $\zeta_t = \zeta_t / \text{Tr} (\zeta_t)$, and to compute the measure $\mathbb{P}$ with $d\mathbb{P} = \text{Tr} (\zeta_t) d\mu$. First, note that Equation (3.35) has linear coefficients, so $\zeta_t$ is bounded in $\mathcal{E}(\mathbb{L}^2(\mathbb{R}))$. Write $p_t = \text{Tr} (\zeta_t)$. Since $\text{Tr} (L(A)) = 0$ for any operator $A$, conditioned in $p_t \neq 0$ we have

$$dp_t = \text{Tr} (N \zeta_t + \zeta_t N^*) dW_t = p_t T(\phi) dW_t .$$

Thus, $p_t$ is the exponential martingale

$$p_t = \exp \int_0^t T(\phi_t) ds - \frac{1}{2} \int_0^t T(\phi_t)^2 ds .$$

Note that $\mathbb{E}_\mu (p_T) = 1$ by definition of $\zeta_t$, so it is indeed a martingale. By the Girsanov Theorem, under the distribution $p_T d\mu$ there exists a Wiener process $B_t$ defined by

$$B_0 = 0$$

$$dB_t = -T(\phi_t) dt + dW_t .$$

(3.36) (3.37)

This is the second line of Equation (2.18). To compute the equation for $\phi_t$, note that

$$\frac{d}{p_t} = d \exp - \int_0^t T(\phi_t) ds + \frac{1}{2} \int_0^t T(\phi_t)^2 ds = \frac{1}{p_t} (T(\phi_t)^2 dt - T(\phi_t) dW_t)$$

so with $\phi_t = \phi \phi_t^{-1}$ the Itô formula yields the first line of Equation (2.18).
This derivation can be extended to more general Hudson-Parthasarathy Equations, and has also been studied in the case where the state on $\Phi$ is not $|\Omega\rangle$ but a more complex, single-photon state, with a resulting non-markovian Belavkin Equation \cite{GJN12}.

### 3.2.2 The measured evolution applied to the Open Quantum Brownian Motion

The measured evolution of $(g_{t,t}, A_z)_{t \leq T}$ is a little more subtle than the one of $(U_t, A_t)_{t \leq T}$, but it can be reduced to this last one by using the formula

$$U_{s,t} = Z_{s,t} U_{s,t} U_{s,t}^* Z_{s,t}^*,$$

which is the operator of multiplication by the function $\tilde{f}_{s,t}(x, (w_u)_{0 \leq u \leq t}) = f(x - w_t + w_t)$. Hence the system $(U_t, A_t)_{t \leq T}$ is $\mathcal{H}_G$-non-demolition, and we have $A_t^u = A_t \otimes A_t = L^\infty(\mathbb{R} \times \mathcal{W}([0, t]), \text{Leb} \otimes \mu)$. We choose the map $\phi_t : \mathbb{R} \times \mathcal{W}([0, t]) \to \mathbb{R}$ as the projection on the first coordinate, and for $s \leq t$ we take the map $\eta_{s,t} : \mathbb{R} \times \mathcal{W}([0, t]) \to \mathbb{R} \times \mathcal{W}([0, s])$ defined by

$$\eta_{s,t}(x, (w_u)_{0 \leq u \leq t}) = (x - w_t + w_s, (w_u)_{0 \leq u \leq s}) .$$

Let $(\gamma_t, X_t)_{0 \leq t \leq T}$ be the random measured process corresponding to these maps. Write $h = (|\Omega\rangle \otimes |\Omega\rangle) |\Omega\rangle^* \otimes |\Omega\rangle^*$ (it is a random variable on $\mathbb{R} \otimes \mathcal{W}([0, t])$), then $\gamma_t$ is the random variable on $\mathbb{R} \otimes \mathcal{W}([0, T])$ defined by

$$\phi_t(x, (w_u)_{0 \leq u \leq T}) = h(x - w_t + w_t, (w_u)_{0 \leq u \leq T}) .$$

For any $x \in \mathbb{R}$ consider the random variable on $\mathcal{W}([0, T])$ obtained by conditioning $\gamma_t$ to $X_0 = x$. This random variable is $\nu_t(x) = \phi_t(x + w_t, (w_u)_{0 \leq u \leq T})$. By definition of $Z$ it is actually equal to

$$(U_t |\nu_0(x)\rangle \otimes |\Omega\rangle \langle\Omega| U_t^*)_{\mathcal{G} \mid A_t} .$$

Thus, $(\nu_t(x), W_t)_{0 \leq t \leq T}$ is the random variable corresponding to the measured evolution of $(U_t, A_t)_{0 \leq t \leq T}$ with initial state $\nu_0(x)$, and by the definition of $\eta_t$ we have $X_t = X_T - W_T + W_t = X_0 + W_T$ so Proposition \ref{prop:21} yields Equation \ref{eq:38}.

### 3.3 Towards general convergence theorems for measured evolution

The convergence of $\rho_{t,T} = \Lambda_T^{\frac{t}{T}}(\rho)$ to $\rho_t = \Lambda_T^{\frac{t}{T}}(\rho)$ was obtained directly from the strong convergence of $U_{t,T}$ to $\mathcal{U}_t$. On the contrary, the convergence in distribution of $(\gamma_{t,t}, X_{t,t})_{0 \leq t \leq T}$ to $(\gamma_t, X_t)_{0 \leq t \leq T}$ was shown as a consequence of Pellegrini’s Theorem \ref{prop:21} which was proved by classical probabilistic methods without any reference to the operators $U_t$ on the Fock space and on the measured evolution.

A natural question is: can we prove the convergence in distribution of a family of processes $(\gamma_{t,t}, X_{t,t})_{0 \leq t \leq T}$ coming from a measured evolution $(U_t, A_t)_{0 \leq t \leq T}$ just from the strong convergence of $U_{t,t}$ and of the algebras $A_{t,t}$ to some operator $U_t$ and some algebra $A_t$?

This question turns out to be rather difficult, since the algebra $A_{t,t}^u$ also depends in $(U_{t,t}, A_{t,t})_{0 \leq t \leq T}$. In what follows we present some results in this direction.

A first result can be obtained when there is no evolution and we are only considering one measurement.

**Proposition 33.** Let $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}_G \otimes L^2(\mathcal{X}, \mu))$ be two states and let $\mathcal{A} = L^\infty(\mathcal{X}, \mu)$. For $i = 1, 2$ define the random variables $\phi_i = (\rho_i)_{\mathcal{G} \mid \mathcal{A}}$ on $(\mathcal{X}, \mathcal{F}_i)$ where $d\phi_i = p_i d\mu_i$ are defined as in Theorem \ref{prop:13}. Then

$$\|\phi_1 - \phi_2\|_{L^1(\mathcal{X}, \mu)} \leq \|\rho_1 - \rho_2\|_{S^1(\mathcal{H}_G \otimes L^2(\mathcal{X}))} \tag{3.38}$$

$$E_{\phi_i}(\|\phi_1 - \phi_2\|_{S^1(\mathcal{H}_G)}) \leq 2\|\rho_1 - \rho_2\|_{S^1(\mathcal{H}_G \otimes L^2(\mathcal{X}))} \tag{3.39}$$
Proof. Write \( h_i = u_X(\rho_i) \) the unnormalized states corresponding to \( \rho_i \). Then \( p_i(x) = \mathrm{Tr}(h_i(x)) \) for \( \mu \)-almost every \( x \in X \) so
\[
\|p_1 - p_2\|_{L^1(X,\mu)} \leq \int_X \mathrm{Tr}(|h_1(x) - h_2(x)|) \, d\mu(x) \leq \|\rho_1 - \rho_2\|_{S^1}
\]
the last inequality being a consequence of the fact that \( h_i \) is the restriction to \( E(H_G) \otimes A \) of the state \( \rho_i \). Thus,
\[
E_{P_n}(\|\varphi_1 - \varphi_2\|_{S^1(H_G)}) = \int_X \mathrm{Tr}(|\varphi_1(x) - \varphi_2(x)|) \, p_1 d\mu(x)
\leq \int X \mathrm{Tr}((p_1(x)\varphi_1(x) - p_2(x)\varphi_2(x))) \, d\mu(x) + \int X \mathrm{Tr}((p_1(x) - p_2(x))\varphi_2(x)) \, d\mu(x)
\leq 2\|\rho_1 - \rho_2\|_{S^1}.
\]

As a consequence we have the following:

Corollary 34. Let \((\rho_n)_{n \in \mathbb{N}}\) be a sequence of states on \( H_G \otimes L^2(X,\mu) \) converging in \( S^1(H_G \otimes H_B) \) to some state \( \rho \). Consider the sequence of random variables \( \varrho_n = \rho_{G|A} \) defined as in Theorem 7. Then \( \varrho_n \) converges to \( \varrho \) in distribution and in \( L^1(X,S^1(H_S),p_\mu d\mu) \).

Note that it would make no sense to ask that \( \varrho_n \) converge to \( \varrho \) in probability or almost surely since they are attached to different probability measures on \( X \). The convergence in \( L^1(X,S^1(H_S),p_\mu d\mu) \) is already a little strange from a probabilistic point of view though it is mathematically meaningful: the random state \( \varrho_n \) is in \( L^1(X,S^1(H_S),p_\mu d\mu) \) since it is bounded in \( S^1(H_S) \) and \( p_\mu d\mu \) is a probability measure.

In the case of measured evolutions, we were only able to obtain the following partial result, in which the convergence of the process \((X_{t_{a_i,1}},t_{a_i})_{t_{a_i} \in I_n}\) is obtained, but not the convergence of the random state.

Proposition 35. Let \( X = \mathbb{R}^d \) with Borelian algebra \( \mathcal{F} \) and a Radon measure \( \mu \). For each \( n \in \mathbb{N} \) let \( \mathcal{F}_n \) be a coarse sub-\( \sigma \)-algebra of \( \mathcal{F} \). Assume \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) for each \( n \) and write \( X_n = \mathbb{R}^d/\mathcal{F}_n \). Identify with subsets of \( \mathbb{R}^d \) such that \( X_n \subset X_{n+1} \subset X \). We fix some time set \( I = [0,T] \) upper-bounded by some \( T \in \mathbb{R} \) and some finite set \( I_n \subset I \) with \( I_n \subset I_{n+1} \).

Consider some Hilbert spaces \( H_G \) and \( H_C \) and write \( H_B = L^2(X,\mathcal{F},\mu) \otimes H_C \). Consider \( A = L^\infty(X,\mathcal{F},\mu) \) and let \((U_t,A)_{t \in I} \) be an \( H_G \)-non demolition measured evolution and \( \rho \) a state on \( H_G \otimes H_B \). We write \((X_t)_{t \in I} \subset X^I \) and \((\varrho_t)_{t \in I} \) the random variables obtained by measuring \( A \) under the evolution.

For each \( n \in \mathbb{N} \) fix a closed subspace \( H_{n,C} \subset H_C \) with \( H_{n,C} \subset H_{n+1,C} \). Write \( H_n = L^2(X,\mathcal{F}_n,\text{Leb}) \otimes H_{n,C} \) and let \( P_n \) the orthogonal projection on \( H_n \). Note that \( P_n \) commutes with every elements of \( A \), we define \( A_n = P_n A \) and \( X_n = \mathbb{R}^d/\mathcal{F}_n \). Consider a process of unitary operators \((U_{n,t})_{t \in I_n} \) on \( H_G \otimes H_n \) (that we may see as partial isometries on \( H_G \otimes H_B \)), and a state \( \rho^n \) on \( H_G \otimes H_n \) (that we may see as a state on \( H_G \otimes H_B \)). Assume that \((U_{n,t},A_n)_{t \in I_n} \) is \( H_G \)-non demolition for all \( t \). Define the process \((X_{n,t})_{t \in I_n} \) with values in \( X_n \) and \((\varrho_{n,t})_{t \in I_n} \) obtained by the measured evolution of \( A_n \) under the evolution \( U_{n,t} \) with initial state \( \rho^n \). We still write \( t \in I \rightarrow X_{n,t} \) the extension of \( t \in I_n \rightarrow X_{n,t} \) to \( I \) by linear interpolation, and the same for \( \varrho_{n,t} \).

We make the following assumptions:

Assumption 1. Writing \( I_n = \{t_{1,n},\cdots,t_{k_n,n}\} \) (in increasing order) we assume that
\[
l_n = \max\{t_{i+1,n} - t_{i,n} | 1 \leq i \leq k_n\}
\]
converges to 0 as \( n \rightarrow \infty \).

Assumption 2. For any \( x \in \mathbb{R}^d \) write
\[
C_{\mathcal{F}_n}(x) = \bigcap_{A \in \mathcal{F}_n, x \in A} A.
\]
Then we assume that
\[
\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \text{diam}(C_{\mathcal{F}_n}(x)) = 0.
\]

Assumption 3. The sequence of processes \((X_{n,t})_{t \in I} \) is tight for the topology of the uniform convergence on the set of continuous functions on \( I \), and \((X_t)_{t \in I} \) is almost surely continuous.
Assumption 4. The sequence of projections \((P_n)_{n \in \mathbb{N}}\) strongly converges to the identity and the state \(\rho^n\) converges to \(\rho\) in \(\mathcal{B}^1\) as \(n \to 0\) and for all sequence \((t_n)_{n \in \mathbb{N}}\) with \(t_n \in I_n\) converging to some \(t \in I\) the operator \(U_{n,t_n}\) strongly converge to \(U_t\) on \(\mathcal{H}_C \otimes \mathcal{H}_B\).

Then \((X_{n,t})_{t \in I}\) converges in distribution (in the topology of uniform convergence) to \((X_t)_{t \in I}\).

In the case of the OQBM, we choose a sequence \(\tau_n\) such that \(\delta_n/\delta_{n+1} \in \mathbb{N}\). We have \(\mathcal{X} = \mathbb{R}\) and \(\mathcal{X}_n = \delta_n\mathbb{Z}\), the algebra \(\mathcal{F}_n\) being generated by the sets \([\delta k, \delta (k+1)]\) and we take \(\mathcal{H}_C = \Phi\) and \(\mathcal{H}_{C,n} = T_{\tau_n}\Phi\). Upon proving the tightness assumption \(^3\) this theorem together with Theorem \(^3\) provides an alternative proof of the convergence of \((X_{n,t}^\epsilon)_{t \in [0,T]}\) to a process solution of \(\text{(1.12)}\). However, it is very incomplete since we do not prove the convergence of \(\eta_{n,t}^\epsilon\).

Note that Assumption \(^3\) depends on the maps \(\eta_{n,t}\) and \(\phi_t\) chosen in the construction of the process, which are only defined up to a set of measure zero.

Proof. We separate the dependency on \(I_n\) and \(A_n\) on the one hand and on \(U_{n,t}\) on the other hand. For \(k \leq n\) and any \(t \in nI_k\) we write

\[ X_{k,n,t} = C_{F_k}(X_{n,t}) \]

and we consider the \(\sigma\)-algebra \(\mathcal{F}_{k,n}\) generated by \((X_{k,n,t})_{t \in I_k}\) and define

\[ \theta_{k,n,t} = \mathbb{E}(\theta_{n,t}|\mathcal{F}_{k,n}) . \]

Then \((\theta_{k,n,t}, X_{k,n,t})_{t \in I_k}\) is a measured evolution corresponding to the system \((U_{n,t}, A_k)_{t \in I_k}\). We also write

\[ X_{k,\infty,t} = C_{F_k}(X_t) \]

and \(\mathcal{F}_{k,\infty}\) the \(\sigma\)-algebra generated by \((X_{k,\infty,t})_{t \in I_k}\), and \(\theta_{k,\infty,t} = \mathbb{E}(\theta_t|\mathcal{F}_{k,\infty})\), so that \((\theta_{k,\infty,t}, X_{k,\infty,t})_{t \in I_k}\) is a measured evolution corresponding to the system \((U_t, A_k)_{t \in I_k}\). We extend all these functions to \(I\) by linear interpolation.

We prove the convergence in distribution of \((X_{n,t})_{t \in I}\). Let \(f\) be a bounded Lipschitz function on the space \(\mathcal{D}\) of continuous functions from \([0,T]\) to \(\mathcal{X}\). We want to show that \(\mathbb{E}(f((X_{n,t})_{t \in [0,T]})) \) converges to \(\mathbb{E}(f((X_t)_{t \in [0,T]}))\) as \(n \to \infty\).

We fix \(\epsilon > 0\). For any \(k\) sufficiently large, we have \(\text{diam}(C_{F_k}(x)) \leq \epsilon\) for all \(x \in \mathcal{X}\). By the tightness assumption, there is \(C > 0\) such that for any \(n\) sufficiently large, with probability higher than \(1 - \epsilon\) we have \(\|X_{n,t} - X_{n,s}\| \leq \epsilon\) if \(|t - s| \leq C\). Since \(d_k \to 0\) as \(n \to \infty\) this implies that for all \(n\) and \(k\) large enough we have \(\|X_{k,n,t} - X_{n,t}\| \leq 2\epsilon\) for all \(t\) with probability higher than \(1 - \epsilon\). Writing \(M = \max{|f|}\) and \(L\) the Lipschitz constant for \(f\), this means that there is \(K \in \mathbb{N}\) such that for any \(n, k \geq K\),

\[ |\mathbb{E}(f((X_{n,t})_{t \in I})) - \mathbb{E}(f((X_{n,t})_{t \in I}))| \leq \epsilon M + 2\epsilon L . \]

The crucial point is that this bound is uniform in \(n\). The same reasoning shows that for any \(k\) large enough we have

\[ |\mathbb{E}(f((X_{k,\infty,t})_{t \in I})) - \mathbb{E}(f((X_{k,t})_{t \in I}))| \leq \epsilon M + 2\epsilon L . \]
strongly to $W$ as $n_2 \to n$ and $\rho_k$ converges to $\rho$ in $\mathcal{B}^1$ so $\rho_{W, k}$ converges to $\rho_{W}$ in $\mathcal{B}^1$, so by Proposition 33, the process $\{Y_{n, t}\}_{t \in I_n}$ converges in distribution to the process $\{Y_t\}_{t \in I_n}$.

This implies that for $n$ large enough,

$$|E(f((X_{n, \infty})_{t \in I})) - E(f((X_{n, n, t}))_{t \in I}))| \leq \varepsilon.$$  

But $k$ was fixed large enough so that $|E(f((X_{n, n, t}))_{t \in I})) - E(f((X_{n, t}))_{t \in I}))| \leq \varepsilon$ this implies that

$$|E(f((X_{n, t}))_{t \in I})) - E(f((X_{t}))_{t \in I}))| \leq 3\varepsilon$$

thus proving the convergence in distribution of $(X_{n, t})_{t \in I}$. □

The key point is the inequality 33, which is uniform in $n$. Such a uniform estimate could not be obtained for $\varrho_{k, n, t}$. Indeed, even if the $\sigma$-algebra $\mathcal{F}_{k, n}$ is very close to the full $\sigma$-algebra for $k$ large enough, this does not implies that $\varrho_{k, n, t} = E(\varrho_{n, t} | \mathcal{F}_{k, n})$ is close to $\varrho_{n, t}$ for $k$ large enough uniformly in $n$.

The hypothesis that $\mathcal{A}_n$ is coarse and $I_n$ is finite is actually not necessary. To go without it, we may use coarse subalgebras $\mathcal{A}_{k, n}$ of $\mathcal{A}_n$ and finite subsets $I_{n, k} \subset I_n$ and look at the measured evolutions of $(U_{n, t, \mathcal{A}_{k, n}})_{t \in I_{n, k}}$.

### 3.4 Open questions and prospects

Three questions are left open in Theorem 35:

1. Are the assumptions sufficient to ensure the convergence in distribution of $(\varrho_{n, t})_{t \in I}$?

2. On what condition does an $\mathcal{H}_G$-non-demolition system $(U_t, \mathcal{A}_t)_{0 \leq t \leq T}$ admit a measured evolution process $(\varrho_t, X_t)_{0 \leq t \leq T}$ which is almost surely continuous in time? It is the case for $U_t$ defined by the Hudson-Parthasarathy Equation and $\mathcal{A}_t = L^\infty(\mathcal{W}([0, t]), \mu)$, but it is not the case when $\mathcal{A}_t$ is the algebra generated by the $a_1^s(t)$ for $s \leq t$ (the measured evolution has jumps in this case, see for example [Pel10]).

3. Considering a family of $\mathcal{H}_G$-non demolition systems $(U_{\tau, t}, \mathcal{A}_{\tau})_{t \in I}$, with measured evolutions $(\varrho_{\tau, t}, X_{\tau, t})_{0 \leq t \leq T}$. Is there any condition on the unitaries and algebras to ensure the tightness of the family of processes in the space of continuous functions?

Some questions concern the OQBM more specifically.

4. In the trajectories of the Open Quantum Brownian Motion, there is no back-action of the position $X_t$ on the state $\varrho_t$, which satisfies a closed equation. This framework is insufficient in the context of quantum control, where we would want $X_t$ to represent some control function which depends on the history of the trajectory. What if $N$ and $H$ depends on the position $X_t$? We may expect that under some regularity condition on the functions $x \mapsto N(x)$ and $x \mapsto H(x)$ (for example, Schwartz functions), there exists an inhomogeneous OQBM, whose unitary operator $U_t$ is solution of the equation

$$dU_t = \left( (-iH - \frac{1}{2} M_N - \frac{1}{2} \partial_x^2 - \partial_x M_N)dt + (M_N - \partial_x)da_1^0(t) + (-M_N^* - \partial_x)da_1^0(t) \right) U_t,$$

where $M_N$ is the operator on $\mathcal{H}_G \otimes \mathcal{H}_z = L^2(\mathbb{R}, \mathcal{H}_G)$ defined by $M_N f(x) = N(x) f(x)$. This idea was raised in the original article on the OQBM, [BBT14]. Formally, everything works the same way as the homogeneous OQBM, the equation for the measured evolution being expected to be of the form

$$\left\{ \begin{array}{l}
  d\varrho_t = \mathcal{L}_{X_t}(\varrho_t) dt + \langle N(X_t) \varrho_t + \varrho_t N(X_t)^* - \varrho_t \mathcal{L}_{X_t}(\varrho_t) \rangle dB_t \\
  dX_t = \mathcal{T}_{X_t}(\varrho_t) dt + dB_t.
\end{array} \right.$$  

However, proving the existence of $U_t$ is far more complex than for the homogeneous OQBM, since the operators $\partial_x$ and $M_N$ are no more commuting, and the space of bandlimited functions $D_C$ is no more preserved. The existence of solutions of Hudson-Parthasarathy Equations with unbounded coefficients have been studied in [FR06] and [FW03], but the convergence of discretisations in the toy Fock space in the spirit of Attal and Pautrat has never been studied for unbounded operators.

5. The generalization of the homogeneous OQBM to higher dimensions is straightforward. Going further, we may study an inhomogeneous OQBM on a manifold. With an Einstein manifold for example, this may provide a semiclassical model for a relativistic quantum particle, in the spirit of the relativistic Brownian motion [Ang16], [FLJ07].
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