I. INTRODUCTION

Topological defects play an important role in condensed matter physics. The first and the most famous example is vortex lines, defining critical properties of type II superconductors in the external magnetic field. In two dimensions, the role of defects is even more noticeable. So an interaction of vortices in the $O(2)$ model leads to emergence of a quasi-long-range order and a Berezinskii-Kosterlitz-Thouless (BKT) transition. In $O(2)$ symmetric systems with the additional twofold degeneracy of the ground state, such as Josephson junction arrays in the magnetic field or XY helimagnets, vortex excitations with fractional charges lead to a phase transition on domain walls, and as a consequence to separation of a BKT and Ising (chiral) transition. The appearance of so-called $\mathbb{Z}_2$-vortices corresponds to exceptional thermal properties of two-dimensional frustrated magnets with isotropic spins (see Refs. and Refs. therein). The superlattice structure observed in magnets and multiferroics with the Dzyaloshinskii-Moria interaction in the magnetic field is believed to be related to vortex-like excitations, called skyrmions. The similar skyrmion structures appear in the quantum Hall system.

In this paper we discuss topological defects in quantum ferromagnets. It is known that the usual $O(N)$ model, describing ferromagnets, has different types of topological defects. The case $N = 1$ corresponds to the Ising model, where line-like defects are domain walls. The case $N = 2$ has been mentioned above in a context of point-like vortices and a BKT transition. At $N = 3$, point-like defects of another type are present, distinct from vortices of the $O(2)$ model. They can be obtained as static classical solution of the $O(3)$ sigma model describing low-temperature properties of ferromagnets,

$$H = A \int d^2r \nabla_\mu \varphi^{a} \nabla_\mu \varphi^{a}, \quad \varphi \in S^2. \quad (1)$$

Taking into account the isotropic condition at spatial infinity $\varphi(\infty) = \varphi_0$, the field $\varphi$ becomes a map $\varphi : \mathbb{R}^2 \cup \{ \infty \} \simeq S^2 \to S^2$, which is characterized by an integer number $Q$, the topological degree of the map $\varphi$. The topological charge $Q$ has an explicit representation as the integral

$$Q = \frac{1}{8\pi} \int d^2r \epsilon_{\mu\nu} \epsilon_{ab\gamma} \nabla_\mu \varphi^{b} \nabla_\nu \varphi^{c}. \quad (2)$$

For each value of the topological charge, the conditions of energy minimum is satisfied on a family of field configurations, being a solution of the first order Bogomol'ny equations

$$\nabla_\mu \varphi^{a} \pm \epsilon_{\mu\nu} \epsilon_{ab\gamma} \nabla_\nu \varphi^{c} = 0, \quad (3)$$

where $E_Q = 2\pi A|Q|$ is a minimum energy of a configuration with a charge $Q$.

The families of solutions consist of configurations related to each other by global field-rotations and coordinate transformations. The later symmetry includes rescalings (dilatations), that is specific to the two-dimensional sigma model, which is conformal invariant. As consequence, a size of defects is not defined by the energy minimum conditions, in accordance with the Derrick theorem. However, fixing of the size is necessary to stabilize defect in lattice formulations of the model. To fix the size of defects, the sigma model is routinely supplemented by high-order derivatives of the field (so-called Skyrme’s terms), the anisotropy term like $(\varphi^3)^2$, and/or the interaction with the external field $h^a \varphi^a$. Various approaches to such stabilization have been discussed.

The finite value of energy $E_0$ for topologically nontrivial configurations $Q \neq 0$ is an important difference to vortices in the $O(2)$ model, where energy of the vortex diverges logarithmically. The topological defects thus can be regarded as massive excitations of the $O(3)$ sigma model, in addition to usual massless (Goldstone) perturbative excitations. At non-zero temperature, topological defects are produced and destroyed by the thermal fluctuations, and the topological charge $Q$ is not conserved.
From the beautiful multi-skyrmion solution\textsuperscript{[24,36,39]} it follows that the system of skyrmions at non-zero temperature is equivalent to the two-dimensional Coulomb gas in the gapped phase. At low temperatures, the processes of production and destroying of skyrmions are suppressed by the factor \(\exp(-E_Q/T)\). At zero temperature, the skyrmions become stable.

The point \(T = 0\) is the quantum critical point of two-dimensional ferromagnets. One may expect that a topologically non-trivial classic background changes a character of quantum excitations, due to symmetry arguments.

In this paper we consider the spectrum of spin-waves in the presence of a single skyrmion. Considerable theoretical efforts were devoted in recent years to the analysis of dynamics of magnetic fluctuations in ferromagnets with skyrmions. In all these studies however the consideration was done for the case of skyrmion forming a lattice, which is apparently stabilized by additional interactions.\textsuperscript{[35,42,44]} The price for this stabilization was the analytical intractability of the appearing equations. In the present study we choose a different route and consider pure Heisenberg exchange Hamiltonian with a single skyrmion as the (metastable) ground state. The advantage we obtain is the entirely analytical form of intermediate expressions for the skyrmion’s shape. The analysis of one skyrmion is helpful, because we better understand the novelties in the spin dynamics, produced already at the level of single topological object. Keeping in mind the observation\textsuperscript{[24,31,35]} that the skyrmion lattice can be well approximated by a superposition of individual skyrmions, we hope that our analysis may be a helpful building block in the study of dynamics of skyrmion lattice configurations. We note that a similar analysis of one skyrmion in antiferromagnets was performed in\textsuperscript{[10]}

The plan of the paper is as follows. We briefly remind the derivation of the skyrmion configuration in the continuum model of isotropic ferromagnet in Sec. I. The derivation of the same results in the Heisenberg model on a lattice is presented in Sec. II. The equation for the spectrum of magnons is discussed in Sec. IV. The supersymmetric partner for the magnon spectrum equation is introduced in Sec. V which allows us to obtain a particular explicit solution for the magnon wave function. The unusual zero modes for this equation are analyzed in Sec. VI, their connection to the conformal symmetries is demonstrated. We present our concluding remarks in Sec. VII. The explicit form of large appearing matrices is shown in the Appendix A.

II. SKYRMION IN CONTINUUM MODEL

Consider a system of spins lying in a 2D plane and coupled by the ferromagnetic exchange interaction. We are interested in the fluctuations of local magnetization, \(\mathbf{M}\), on the scale much larger than the lattice parameter. In this case one usually defines the normalized spin density, \(\varphi = \mathbf{M}/|\mathbf{M}|\), with the unit vector \(\varphi = (\sin \alpha \sin \beta, \cos \alpha \sin \beta, \cos \beta)\) characterizing the direction of local magnetization. The exchange interaction is then written in the form of Eq. (1) with \(A > 0\) - spin stiffness constant tending to align spins along the same (arbitrary) direction.

We will refer to the expression (1) as the classical exchange energy, because the quantum-mechanical expression for the square of spin reads \(s^2 = s(s + 1)\), so that for the normalized quantity \(s^2/s^2 = 1 + 1/s \neq 1 = \varphi^2\). In this sense, Eq. (1) corresponds to the limit \(s \to \infty\).

Clearly, the uniform magnetization, \(\nabla \varphi = 0\), delivers the absolute minimum to the energy, \(H = 0\). It was demonstrated in\textsuperscript{[23]} that the extremum to the energy may also be provided by non-trivial skyrmion configurations. To see it, one rewrites the integrand in (1) in the form

\[
\nabla_\mu \varphi^\alpha \nabla_\mu \varphi^\alpha = (\nabla \beta)^2 + \sin^2 \beta (\nabla \alpha)^2. \tag{4}
\]

Seeking for extremum of \(H_{cl}\), we perform a variation in \(\alpha\) and \(\beta\), which gives a set of coupled equations

\[
\nabla (\sin \beta \cos \beta \nabla \alpha) = 0, \quad -\nabla^2 \beta + \sin \beta \cos \beta (\nabla \alpha)^2 = 0. \tag{5}
\]

These equations are general ones, and we focus on a centrosymmetric solution. We use the parametrization \(\mathbf{r} = r (\cos \phi, \sin \phi)\) and impose the additional constraints

\[
\frac{d \beta}{d \phi} = 0, \quad \frac{d \alpha}{d \phi} = \ell, \quad \frac{d \alpha}{d r} = 0 \tag{6}
\]

with integer \(\ell\). First non-trivial solutions are given by \(\ell = 1\) (skyrmion) and \(\ell = -1\) (antiskyrmion). Taking either of these values we arrive at the solution (see Appendix)

\[
\beta_0 = 2 \arctan \left( \frac{r_0}{r} \right), \tag{7}
\]

which leads to the explicit expressions

\[
\varphi^1 = i \varphi^2 = 2 r r_0 e^{i \ell (\phi - \alpha_0)}, \quad \varphi^3 = \frac{r^2 - r_0^2}{r^2 + r_0^2} \tag{8}
\]

with \(\ell = \pm 1\) in the exponent. The arbitrariness in the choice of \(r_0, \alpha_0\) is discussed below.

Given the classical solution (9), or “vacuum”, we may proceed further with consideration of small fluctuations of \(\varphi\) around this vacuum.\textsuperscript{[10]} Expanding in powers of these fluctuations, \(\delta \varphi\), we obtain first non-vanishing term as the quadratic form in \(\delta \varphi\). This quadratic form should be positively defined and can be expressed via its eigenmodes. Given the eigenset of this quadratic form, the subsequent quantization of the fluctuations becomes straightforward, see e.g.\textsuperscript{[23]}

However, we find the above recipe for the quantization of the spin fluctuations rather artificial and inconvenient. One reason against using the model (1) as the basic one, is that it is already a long-range approximation for the underlying lattice models. On the other hand, the lattice spin models allow the standard and convenient description of bosonic spin-wave excitations, Eq. (25) below.
The existence of this standard representation of lattice spin operators largely facilitates an eventual analysis of non-linear effects, related to the non-quadratic terms in the expansion of fluctuations.

Therefore, two reasons to abandon the continuum model (1) and to return to the lattice spin model are (i) the greater flexibility of lattice Hamiltonians with respect to inclusion of additional interactions and (ii) the existence of the standard Maleyev-Dyson representation for the spin operators with the relative ease of the anticipated analysis of magnon-magnon interaction.

The shortcoming of the Maleyev-Dyson representation for our task is that it explicitly assumes the direction of all local spins along $\hat{z}$-axis in spin space. This is natural for the uniform ferromagnet, but is obviously violated in the skyrmion configuration. We discuss a way to overcome this difficulty in the next section.

III. DERIVATION OF THE SKYRMION IN THE LATTICE MODEL

Consider the exchange lattice Hamiltonian

$$H = \sum_{i,j} J(r_i - r_j) S_r S_r$$

(9)

We assume that the ground state is characterized by the non-collinear Skyrmion ground state. Our aim is to rewrite the Hamiltonian (9) in such local basis, where the average local spin is directed along the $\hat{z}$-axis. The transition to this local basis, $S_r = \hat{U}(r) S_r$, is given by the position-dependent 3×3 matrix $\hat{U}(r)$:

$$\hat{U} = e^{i\sigma_3} e^{i\sigma_1} e^{i\gamma_3}$$

(10)

with $\sigma_3, \sigma_1$ generators of SO(3) group, and $\alpha, \beta, \gamma$ Euler angles. In the new basis the Hamiltonian (9) takes the form:

$$H = \sum_{r,n} J(n) \hat{S}_r \hat{R} (r, n) \hat{S}_{r+n}$$

(11)

$$n = r_i - r_j, \hat{R} (r, n) = \hat{U}^{-1}(r) \hat{U}(r + n).$$

We assume that the exchange $J(n)$ rapidly decreases with distance $n$, it is then convenient to expand the matrix $R_{ab}(r, n)$ in a series:

$$R_{ab}(r, n) = \delta_{ab} + \chi_{ab}^{1,\mu}(r)n^{\mu} + \frac{1}{2} \chi_{ab}^{2,\mu\nu}(r)n^{\mu}n^{\nu} + \ldots$$

(12)

with $\delta_{ab}$ Kronecker symbol and $\chi_{ab}^{1,\mu}, \chi_{ab}^{2,\mu\nu}$.

$$\chi_{1,\mu}(r) = U^{ca}(r)\nabla^{\mu}U^{cb}(r), \chi_{2,\mu\nu}(r) = U^{ca}(r)\nabla^{\mu}\nabla^{\nu}U^{cb}(r).$$

(13)

Here and below we assume the summation over the repeated tensorial indices.

The equilibrium state of the spin configuration implies that the total field induced by the neighboring spins is parallel to the direction of the spin at a given site, i.e. along the direction $\hat{e}_3$. This results in a double condition

$$\sum_n J(n) R^{a3}_n (r, n) = 0, \quad \sum_n J(n) R^{3a}_n (r, n) = 0,$$

(14)

with $a = 1, 2$.

The explicit dependence of $U, \chi_{1,\mu}, \chi_{2,\mu\nu}$ on the Euler angles is known. The conditions (14) then determine the dependence of these angles on $r$. We implicitly assume here that the average spin has the same absolute value, which assumption should be checked afterwards, see below.

We immediately notice that (14) does not pose any restriction on $\chi_{1,\mu}$, when we deal with centrosymmetric situation, in which case $J(n)$ is an even function of $n$ and $\sum_n J(n) = 0$. In the long wavelength limit, which is our primary interest, we can expand the Fourier transform

$$J(q) = \sum_n e^{iqn} J(n) \approx J(0) + \frac{1}{2} C q^2$$

(15)

with $J(0) < 0$ for ferromagnetic exchange and $C > 0$.

It is useful to write the Hamiltonian in Fourier components

$$H = N^{-2} \sum_{q, q'} \tilde{S}_q^a \tilde{S}_{q'}^b \sum_{r, n} e^{i(q+q')r} J(n) e^{iqn} R^{ab}_n (r, n)$$

(16)

with $N$ the number of lattice sites. Taking into account the expansion (12) and the properties

$$\sum_n n^{\mu} e^{iqn} J(n) = -i \frac{d}{dq^{\mu}} J(q) = -i C q^{\mu}$$

(17)

$$\sum_n n^{\mu} n^{\nu} e^{iqn} J(n) = -\frac{d^2}{dq^{\mu} dq^{\nu}} J(q) = -C \delta_{\mu\nu}$$

we write the inner sum in (16) in general as

$$\left[ \delta(q + q') \delta_{ab} - i \chi_{1,\mu}(q + q') \frac{d}{dq^{\mu}} 
- \frac{1}{2} \chi_{2,\mu\nu}^{ab}(q + q') \frac{d^2}{dq^{\mu} dq^{\nu}} \right] J(q)$$

(18)

$$= \delta(q + q') \delta_{ab} J(q) - i \chi_{1,\mu}(q + q') C q^{\mu}$$

$$- \frac{1}{2} \chi_{2,\mu\nu}^{ab}(q + q') C$$

This expression leads to a simplification for the first equilibrium condition in (14). Putting $\hat{S}_q^b = s_3 \delta_{3b} \hat{q}(q)$ we obtain

$$\sum_{\mu} \chi_{2,\mu\nu}(r) = 0, \quad a = 1, 2$$

(19)

which might be represented as a condition for complex-valued quantity

$$\chi_2^+ = \chi_{2,13}^+ + i \chi_{2,23}^+ = 0$$
we obtain then (see Appendix A)
\[ \chi_2^+ = e^{i\gamma} \left[ 2 \cos \beta \nabla \alpha \nabla \beta + \sin \beta \nabla^2 \alpha \right. \\
\left. + i(\nabla^2 \beta - \sin \beta \cos(\nabla\alpha)^2) \right] = 0. \] (20)

The real and imaginary parts of the expression in square brackets here are proportional to the variational derivatives of \((\nabla \varphi)^2\), Eq. (5), over \(\alpha\) and \(\beta\), respectively.

Demanding \(\chi_2^+ = 0\) and using the conditions (6) with \(\ell = 1\) for single skyrmion with a center at the origin, \(r = 0\), we come to the set of equations
\[ \beta = 2 \arctan \left( \frac{r_0}{r} \right), \quad \alpha + \alpha_0 = \phi \] (21)
with \(r_0\) Skyrmion radius. It translates to the well-known explicit dependence of the local spin direction on the coordinates
\[ S^1 + iS^2 = s \frac{2rr_0}{r^2 + r_0^2} e^{i(\phi - \alpha_0)}, \quad S^3 = s \frac{r^2 - r_0^2}{r^2 + r_0^2}. \] (22)

Notice that \(r_0\) and the phase \(\alpha_0\) cannot be determined in this calculation.

Further, the angle \(\gamma\) is arbitrary and not defined from (20). The rotation by \(\gamma\) in (10) corresponds to transformation \(\tilde{S}^3 \rightarrow \tilde{S}^3 e^{i\gamma}\) in Eq. (25) below, which reduces to \(a_j \rightarrow a_j e^{i\gamma}\) in terms of bosons. We observe that our solution (21) corresponds to the following form of \(\tilde{U}\):
\[ \tilde{U} = \begin{pmatrix} \cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \\ -\sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = 0, \]
\[ = \begin{pmatrix} \cos(\alpha - \gamma) & -\sin(\alpha - \gamma) & 0 \\ -\sin(\alpha - \gamma) & -\cos(\alpha - \gamma) & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \beta = \pi, \] (23)
at \(r \to \infty\) and \(r \to 0\), respectively. It means that in order to provide the natural choice \(\tilde{U} = 1\) at \(r \to \infty\), we should demand \(\gamma = -\alpha = \alpha_0 - \phi\), which in turn results in discontinuity of \(\tilde{U}\) at \(r \to 0\). Alternatively, the continuous character of \(\tilde{U}\) at the origin, \(r = 0\), means that \(\alpha - \gamma = \text{cst}\) and it translates to the double full rotation, \(\tilde{U} = \exp[2i(\phi - \alpha_0)\sigma_3]\) at \(r \to \infty\). We adopt below the latter choice,
\[ \gamma = \alpha = \phi - \alpha_0, \] (24)
which provides us with the continuity at \(r = 0\) and somewhat simplifies the equations for magnon spectrum.

**IV. MAGNON SPECTRUM EQUATION**

Knowing the form of \(\tilde{U}\) via the \(r\)-dependent \(\alpha, \beta, \gamma\) given by (20) and (21), we obtain the explicit expressions \(\chi_1^{ab}(r)\) and \(\chi_2^{a\mu}(r)\). We then use these expressions and Maleyev-Dyson representation for spin operators, preserving the spin commutation relations, \([\tilde{S}^a, \tilde{S}^b] = i\epsilon_{abc}\tilde{S}^c\):
\[ \tilde{S}^3_j = s - a_j^+ a_j, \quad \tilde{S}^+ = \sqrt{2} s a_j \]
\[ \tilde{S}^- = \sqrt{2} s \left( a_j^+ - \frac{1}{2s} a_j^+ a_j a_j^+ \right) \] (25)

here \(s\) the value of spin, \(\tilde{S}^\pm = \tilde{S}^1 \pm i \tilde{S}^2\) and \([a_j, a_j^+] = 1\). We thus express our Hamiltonian (11) in bosonic representation.

We already made one simplification, by considering the long-wavelength limit of \(J(q)\) in Eq. (15), which corresponds to continuum limit of initially lattice Hamiltonian. Another simplification which we make now is the semiclassical limit of large spin \(s\). Assuming \(s \gg 1\), we keep those largest-in-\(s\) terms, which will provide us with the spectrum of magnon excitations. We write to this end
\[ \tilde{S}^3_q = \frac{1}{2s} (\tilde{S}^3_q + \tilde{S}^3_q) \]
\[ \simeq \frac{\sqrt{2s}}{2} \sum_j e^{iqr_j} (a_j^+ + a_j) = \frac{\sqrt{2s}}{2} (a_{-q}^+ + a_q) \]
\[ \tilde{S}^3_q = \frac{1}{2s} (\tilde{S}^3_q - \tilde{S}^3_q) \]
\[ \simeq \frac{\sqrt{2s}}{2i} \sum_j e^{iqr_j} (a_j - a_j^+ ) = \frac{\sqrt{2s}}{2i} (a_{q} - a_{-q}^+) \] (26)
\[ \tilde{S}^3_q = s\delta(q) - \sum_k a_{q+k} a_k \]

Putting these expressions into the Hamiltonian we obtain the classical energy of the magnet in the order \(s^2\) of the form
\[ s^2 \int d\mathbf{r} \left( -J(0) + C \frac{4r_0^2}{(r^2 + r_0^2)^2} \right), \] (27)
Given the absence of translational and mirror symmetry, the terms in the Hamiltonian of order of tors via eigenmodes: Eq. (28) assumes the above choice of our paper. Let us analyze it in more detail. First of all, the first term, \( \sum_i \nu_i \) describes the magnons in the presence of the skyrmion. The terms linear-in-skyrmion with \( s \) value, \( \nu \), motion effects, and the average spin has its saturated ferromagnets. As a result, we have the absence of zero-modes, \( \sum_i \nu_i \), \( \psi \) creates zero modes \( (31) \), we can transform the equation \( (30) \) into the form of \( (33) \) with \( A = \frac{d}{dx} + W(x) \), \( A^\dagger = -\frac{d}{dx} + W(x) \), and \( W(x) = -\frac{d \ln \Phi_0}{dx} = \frac{m+1/2}{x} + \frac{2x}{x + 1/2} \).
Importantly, if we find a solution to Eq. (34), then the solution to the equation with the same energy is given by \( \Phi_r = A^\dagger \Phi_e \).

We see that the partner equation at \( n = 0 \) corresponds to the free motion with shifted angular momentum. For other values of \( n \) the solutions to (34) are not obvious. It follows immediately, that for \( m = 0 \) we have

\[
\Phi_e(x) = x^{1/2} J_0(x\sqrt{\varepsilon})/\sqrt{\varepsilon},
\]

\[
\Phi_r(x) = x^{1/2} J_0(x\sqrt{\varepsilon}) - x^{3/2} J_1(x\sqrt{\varepsilon})/1 + x^2 \varepsilon,
\]

with \( J_n(x) \) the Bessel functions.

Returning to original variables and using the relations between the Bessel functions, we can represent our solutions in the form

\[
\psi_{m=0,E}(r) = \frac{r^2 J_0(\kappa r) - r^2 J_2(\kappa r)}{r^2 + r^2}.
\]

with the analog of wave-vector, \( \kappa = \sqrt{E/C_s} \). It is seen here, that the exact wave function smoothly interpolates between \( J_0(\kappa r) \) (i.e. free motion with \( m = 0 \)) at \( r \ll r_0 \) and \( J_2(\kappa r) \) (free motion with \( m = 2 \)) at \( r \gg r_0 \). Interestingly, the differential equation equivalent to our Hamiltonian (28) (with \( E \sim C_s k \)) and the exact wave function of the form (36) were obtained in (22) in the analysis of magnon dispersion for \( s = 1/2 \) antiferromagnet.

VI. ZERO MODES AND SYMMETRIES

It is known, that the zero modes correspond to transformations of the Hamiltonian, which do not change the energy of the classical field configuration. We have two such transformations readily available, one is the translation of the skyrmion in space. Another is the dilation of skyrmion, i.e. the change of its size, \( r_0 \rightarrow r_0' \), which also does not change the classical energy. The dilation may be combined with the rotation around the center of the skyrmion, which changes the angle \( \alpha_0 \) in (22). Less trivial is the special conformal symmetry transformation, consisting of inversion followed by translation followed by inversion

\[
\vec{r} \rightarrow \vec{r}', \quad \frac{\vec{r}'}{\vec{r}'} = \frac{\vec{r}}{\vec{r}^2} - \vec{b}
\]

Infinitesimal form of these three transformations is

\[
\vec{r}' = \vec{r} + \vec{b},
\]

\[
\vec{r}' = \vec{r} + \vec{r}(b_1 \hat{e}_r + b_2 \hat{e}_\phi),
\]

\[
\vec{r}' = \vec{r} + (2br - r^2 \vec{b}),
\]

with \( b_1, |b| \rightarrow 0 \). Here \( b_1 \neq 0 \) corresponds to dilations and \( b_1 = 0 \neq b_2 \) — to rotations.

According to our procedure, we incorporate the static rotations of spins \( \hat{U}_0(\vec{r}) \) into the definition of local basis.

The fluctuations around the static configuration, when considered at quantum level, are magnons. A seeming ambiguity in this approach arises when we consider static configurations of equal classical energy, which are transformed one into another by infinitesimal change in local basis. Evidently such transformation should manifest itself in the appearance of zero-energy, i.e. static fluctuations in the subsequent analysis. These solutions of zero energy are not well described in terms of second-quantized bosons, and it is more appropriate to characterize them in first-quantization description. We provide more details on this below.

In the initial field configuration we have the local basis characterized by \( \hat{U}_0(\vec{r}) \) with the skyrmion described by Eqs. (A1), (21), (24) and centered at \( \vec{r} = 0 \). After infinitesimal change of coordinates \( \vec{r} \rightarrow \vec{r} + \vec{r}_1 \) we write

\[
\hat{U}_0(\vec{r} + \vec{r}_1) \simeq \hat{U}_0(\vec{r})(1 + r_1^\mu \hat{\chi}_{1,\mu}(\vec{r})).
\]

see (13). In the classical limit, \( s \rightarrow \infty \), the formulas (25) become

\[
\hat{S}^3 = s - \xi^2 - \eta^2, \quad \hat{S}^\pm = S^1 \pm is^2 = \sqrt{2}s(\xi \pm i\eta)
\]

We assumed above that \( \xi = \eta = 0 \), now we consider the second term in the combination \( 1 + r_1^\mu \hat{\chi}_{1,\mu}(\vec{r}) \) as a small static fluctuation. This results in

\[
\hat{S}^\pm = s + \hat{S}_1^\mu (\chi_{1,\mu}(\vec{r}) \pm i\chi_{1,\mu}(\vec{r}))
\]

Letting \( \vec{r}_1 = r_1(\cos \phi_1, \sin \phi_1) \), we obtain after some algebra from (42)

\[
\hat{S}^\pm = -2is \hat{r}_1 e^{i(\phi_1 - \phi_0)}.
\]

Three transformations (38) correspond to

\[
\begin{align*}
\vec{r}_1 &= b, & \phi_1 &= \text{cst}, \\
\vec{r}_1 &= br, & \phi_1 &= \phi, \\
\vec{r}_1 &= br^2, & \phi_1 &= 2\phi + \text{cst},
\end{align*}
\]

for translations, dilations and special conformal transformations, respectively. It becomes now evident that Eqs. (42), (43) correspond to (31) with \( m = 0, 1, 2 \).

VII. CONCLUSIONS

We discussed the spectrum of magnetic excitations in the ferromagnet, which is characterized by the isotropic Heisenberg exchange Hamiltonian and a single skyrmion in its ground state. This configuration is higher in classical energy and should be considered as metastable. In order to analyze the excitation spectrum, we employ the dully modified method of semiclassical quantization. We start from the lattice model and consider the equilibrium configuration of localized spins as a classical vacuum. The Maleyev-Dyson boson representation of spin
operators in the local bases is used afterwards. The part of the arising Hamiltonian which is quadratic in bosons corresponds to linear spin-wave theory, modified by the presence of the skyrmion. We obtain the Schrödinger equation for magnons in analytic form and find its explicit solution for the s-wave function. For the standard ferromagnetic uniform ground state the magnons correspond to usual plane waves, and the zero (Goldstone) mode wave function tends unity $e^{i\mathbf{q}\mathbf{r}} \to 1$. By contrast, the skyrmion ground state configuration is characterized by internal degrees of freedom, which are translations, rotation and dilatation, in addition to the spontaneous direction of magnetization at the infinity. As a result, one sees three zero modes in the equation for magnons, instead of one. This is quite unusual and we trace these modes to the conformal symmetries of our Hamiltonian.

We regards the results of this study as a first step in the direction of quantum analysis of the ferromagnets characterized by topologically non-trivial ground states. It is known that one possible way to stabilize the skyrmion background, is to include the Dzyaloshinskii-Moriya interaction and magnetic field. The problem becomes less tractable in this case, since one has to resort to numerical solution of the differential equations already for the classical configuration, and later use this solution for the spin-wave analysis. Another direction is the analysis of the interaction between magnons, in order to check whether the well-defined character of excitations is preserved. The most challenging seems to be the eventual calculation of the dynamic susceptibility tensor in case of skyrmion ground state. This tensor requires the knowledge of both the spectrum and wave functions, and is necessary for discussion of neutron scattering experiments.

Acknowledgments

We thank U.K. Rößler, S.V. Grigoriev, S.V. Maleyev and J. Schmalian for useful discussions. We acknowledge Saint-Petersburg State University for a research grants 11.38.636.2013 and 11.50.2514.2013.

Appendix A: explicit expressions for $U$, $\chi$

The explicit form of $\hat{U}$ characterizing the rotation to local basis where the average spin is directed along the $\hat{z} = \hat{e}_3$-axis is given by

\[
\hat{U} = \begin{pmatrix}
\cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & \sin \alpha \sin \beta \\
-\sin \alpha \cos \gamma - \cos \alpha \cos \beta \sin \gamma & \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & \cos \alpha \sin \beta \\
\sin \beta \sin \gamma & -\sin \beta \cos \gamma & \cos \beta 
\end{pmatrix}. \tag{A1}
\]

with $\alpha$, $\beta$, $\gamma$ position-dependent Euler angles.

Matrix with the first derivatives:

\[
\chi_{1,\mu} = \begin{pmatrix}
0 & \cos \beta \nabla^\mu \alpha + \nabla^\mu \gamma, & \sin \beta \cos \gamma \nabla^\mu \alpha - \sin \gamma \nabla^\mu \beta \\
-\cos \beta \nabla^\mu \alpha - \nabla^\mu \gamma, & 0, & \cos \gamma \nabla^\mu \beta + \sin \beta \sin \gamma \nabla^\mu \alpha \\
\sin \gamma \nabla^\mu \beta - \sin \beta \cos \gamma \nabla^\mu \alpha, & -\cos \gamma \nabla^\mu \beta - \sin \beta \sin \gamma \nabla^\mu \alpha, & 0 
\end{pmatrix}. \tag{A2}
\]

The matrix $\chi_{2,ab}^{\beta} \equiv \chi_{2,ab}^{\beta,\mu}$ appearing in long-wavelength expansion of the Hamiltonian has in general rather complicated form, not to be listed here in full. We present only a few components.

\[
\begin{align*}
\chi_{2}^{13} &= \sin \beta \cos \beta \sin \gamma (\nabla \alpha)^2 + 2 \cos \beta \cos \gamma \nabla \alpha \nabla \beta + \sin \beta \cos \gamma \nabla^2 \alpha - \sin \gamma \nabla^2 \beta, \\
\chi_{2}^{23} &= -\sin \beta \cos \beta \cos \gamma (\nabla \alpha)^2 + 2 \cos \beta \sin \gamma \nabla \alpha \nabla \beta + \sin \beta \sin \gamma \nabla^2 \alpha + \cos \gamma \nabla^2 \beta, \\
\chi_{2}^{33} &= -(\nabla \beta)^2 - (\nabla \alpha)^2 \sin^2 \beta.
\end{align*} \tag{A3}
\]

By construction, $\chi_{2}^{33}$ defines the contribution to the classical energy, cf. Eq. (5). Two other elements are conveniently combined as

\[
\chi_{2}^{13} \pm i \chi_{2}^{23} = e^{\pm i \gamma} \left( \pm i (\nabla^2 \beta - \sin \beta \cos \beta (\nabla \alpha)^2) + (\sin \beta \nabla^2 \alpha + 2 \cos \beta \nabla \alpha \nabla \beta) \right). \tag{A4}
\]

In cylindrical coordinates

\[
\begin{align*}
\nabla^2 \alpha &= \frac{d^2 \alpha}{dr^2} + \frac{1}{r} \frac{d \alpha}{dr} + \frac{1}{r^2} \frac{d^2 \alpha}{d\phi^2}, \\
\nabla \alpha \nabla \beta &= \frac{dx}{dr} \frac{d \beta}{dr} + \frac{1}{r^2} \frac{d \alpha}{d\phi} \frac{d \beta}{d\phi}.
\end{align*} \tag{A5}
\]
We have an equation $\chi_{2}^{13} \pm i \chi_{2}^{23} = 0$, where we seek a solution of the form \( \frac{\partial^2 \beta}{\partial \rho^2} = \ell^2 \sin \beta \cos \beta \) (A6). We see that the term in second parenthesis in \( \chi_{2} \) is identically zero and the first term leads to the static sine-Gordon equation

$$\beta = 2 \arctan(e^{\pm \rho}) = 2 \arctan((r/r_0)^{\pm 1})$$

(A7)

Out of these two solutions we choose one with $\beta = 0$ at $r \to \infty$. Substituting it to $\hat{\chi}_2$, we obtain

$$\hat{\chi}_2 = \begin{pmatrix} -\nabla^2 \gamma - 2a \nabla \alpha \nabla \gamma + \frac{1}{r}, & 0 \\ -\nabla^2 \gamma, & 0 \\ 2\nabla \beta \nabla \gamma \cos \gamma + \frac{4\ell^2}{r(r^2+1)^2} \nabla \alpha \nabla \gamma + \frac{2\nabla \beta \nabla \gamma \sin \gamma}{r(r^2+1)^2}, & -\frac{8}{r(r^2+1)^2} \end{pmatrix}.$$  

(A8)

Further simplification of this expression is possible on the assumptions $\nabla \beta \nabla \gamma = 0$, $\nabla^2 \gamma = 0$, $\nabla \alpha = \pm \nabla \gamma$, as suggested in the main text.

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