Semi-regular varieties and variational Hodge conjecture

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1 Introduction

The aim of this article is to study examples of semi-regular varieties. The semi-regularity for a curve on a surface was first introduced by [Sev44]. This was later generalized to arbitrary divisors on a complex manifold by Kodaira-Spencer in [KS59]. In [Blo72], Bloch extended the notion to cycles corresponding to local complete intersection subschemes. This was further generalized by Buchweitz and Flenner in [BF03].

Abstract

Following [Blo72, BF03] we know that semi-regular sub-varieties satisfy the variational Hodge conjecture i.e., given a family of smooth projective varieties π : X → B, a special fiber X_o and a semi-regular subvariety Z ⊂ X_o, the cohomology class corresponding to Z remains a Hodge class (as X_o deforms along B) if and only if Z remains an algebraic cycle. In this article, we investigate examples of such sub-varieties. In particular, we prove that any smooth projective variety Z of dimension n is a semi-regular sub-variety of a smooth projective hypersurface in \( \mathbb{P}^{2n+1} \) of large enough degree.

Résumé

D’après [Blo72, BF03] nous savons que sous-variétés semi-régulières satisfont la conjecture de Hodge variationnelle, c’est-à-dire, donné une famille de variétés projectives, lisses π : X → B, une fibre spéciale X_o et un semi-régulière sous-variété Z ⊂ X_o, la classe de cohomologie correspondant à Z reste une classe Hodge (comme X_o déforme le long B) si et seulement si Z reste un le cycle algébrique. Dans cet article, nous étudions des exemples de tels sous-variétés. En particulier, nous prouvons que toute lisse variété projective Z de dimension n est une sous-variété semi-régulière d’une hypersurface projective lisse dans \( \mathbb{P}^{2n+1} \) du grand degré suffisant.
One of the motivations for the study of semi-regular varieties comes from the variational Hodge conjecture, namely these varieties satisfy the variational Hodge conjecture. In particular, Bloch in [Blo72] and Buchweitz and Flenner in [BF03] noticed that for a smooth projective variety $X$ and a semi-regular local complete intersection subscheme $Z$ in $X$, any infinitesimal deformation of $X$ lifts the cohomology class of $Z$ (which is a Hodge class) to a Hodge class if and only if $Z$ lifts to a local complete intersection subscheme (in the deformed scheme).

In the case of a smooth hypersurface $X$ in $\mathbb{P}^3$, an effective divisor $C$ in $X$ is said to be semi-regular if $h^1(\mathcal{O}_X(C)) = 0$. If $C$ is smooth and $\deg(X) > \deg(C) + 4$ then Serre duality implies that $h^1(\mathcal{O}_X(C)) = h^1(\mathcal{O}_X(-C)(d-4))$ which is equal to zero because the Castelnuovo-Mumford regularity of $C$ is at most $\deg(C)$. But the description of the semi-regularity for subschemes which are not divisors is more complicated, as we see below in §2. The main result of this article generalizes the above case of divisors to higher codimension subvarieties (see §3). In particular, we prove

**Theorem 1.1.** Let $Z$ be a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n + 1$. Then for $d \gg 0$, there exists a smooth degree $d$ hypersurface in $\mathbb{P}^{2n+1}$ containing $Z$ such that $Z$ is semi-regular in $X$.

We finally observe in Remark 3.5 that for such a choice of $Z$ and $X$, the cohomology class of $Z$ in $H^{n,n}(X, Z)$ satisfies the variational Hodge conjecture for a family of degree $d$ hypersurfaces in $\mathbb{P}^{2n+1}$ with a special fiber $X$.

## 2 Bloch’s Semi-regularity map

2.1. In [Blo72], Bloch generalizes the above definition of semi-regularity for divisors to any local complete intersection subscheme in a smooth projective variety over an algebraically closed field. We briefly recall the definition. Let $X$ be a smooth projective variety of dimension $n$ and $Z$ be a local complete intersection subscheme in $X$ of codimension $q$. Consider the composition morphism

\[ \Omega_X^{n-q+1} \times \bigwedge^{q-1} \mathcal{N}_{Z|X}^\vee \xrightarrow{1+\bigwedge^{q-1} \tilde{d}} \Omega_X^{n-q+1} \times \Omega_X^{q-1} \otimes \mathcal{O}_Z \xrightarrow{\bigwedge} K_X \otimes \mathcal{O}_Z \]

where

\[ \tilde{d} : \mathcal{N}_{Z|X}^\vee \cong \mathcal{I}_{Z|X}^\vee / \mathcal{I}_{Z|X}^2 \rightarrow \Omega_X^1 \otimes \mathcal{O}_Z \]

is the map induced by the differential $d : \mathcal{I}_{Z|X} \rightarrow \Omega_X^1$, with $\mathcal{I}_{Z|X}$ denoting the ideal sheaf of $Z$ in $X$. By adjunction, this induces a map,

\[ \Omega_X^{n-q+1} \rightarrow \bigwedge^{q-1} \mathcal{N}_{Z|X} \otimes K_X \cong \mathcal{N}_{Z|X} \otimes K_Z^0, \]

where $K_Z^0 := \bigwedge^q \mathcal{N}_{Z|X} \otimes K_X$ is the dualizing sheaf. Dualizing the induced map in cohomology,

\[ H^{n-q-1}(X, \Omega^{n-q+1}) \rightarrow H^{n-q-1}(Z, \mathcal{N}_{Z|X}^\vee \otimes K_Z^0), \]

gives us $\pi : H^1(\mathcal{N}_{Z|X}) \rightarrow H^{q+1}(X, \Omega_X^{q-1})$. 

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Definition 2.2. The map $\pi$ is called the semi-regularity map and if it is injective we say that $Z$ is semi-regular.

3 Proof of Theorem 1.1 and an application

3.1. Before we come to the final result of this article we recall a result by Kleiman and Altman which tells us given a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n+1$ there exist a smooth hypersurface in $\mathbb{P}^{2n+1}$ containing it.

Notation 3.2. Let $Z$ be a projective subscheme in $\mathbb{P}^{2n+1}$. Denote by $Z_e := \{ z \in Z | \dim \Omega^1_{Z,z} = e \}$.

Theorem 3.3 ([KA79, Theorem 7]). If for any $e > 0$ such that $Z_e \neq \emptyset$ we have that $\dim Z_e + e$ is less than $2n+1$ then there exists a smooth hypersurface in $\mathbb{P}^{2n+1}$ containing $Z$. Moreover, if $Z$ is $d-1$-regular (in the sense of Castelnuovo-Mumford) then there exists a smooth degree $d$ such hypersurface containing $Z$.

We need the following proposition:

Proposition 3.4. Let $Z$ be a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n+1$ and $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^{2n+1}$ containing $Z$ for some $d \gg 0$. Then, for any integers $2 \leq i < n$, $h^n \left( \Lambda^{i-1} T_Z \otimes \Lambda^{n-i} N_{Z \mid X}(d-4) \right) = 0$.

Proof. Since $X$ is a hypersurface in $\mathbb{P}^{2n+1}$, $N_{X \mid \mathbb{P}^{2n+1}}$ is isomorphic to $\mathcal{O}_X(d)$. Under this identification, we get the following normal short exact sequence,

$$
0 \to N_{Z \mid X} \to N_{Z \mid \mathbb{P}^{2n+1}} \to \mathcal{O}_Z(d) \to 0.
$$

This gives rise to the following short exact sequence for $0 \leq i \leq n$:

$$
0 \to \bigwedge^{n-i} N_{Z \mid X} \to \bigwedge^{n-i} N_{Z \mid \mathbb{P}^{2n+1}} \to \left( \bigwedge^{n-i-1} N_{Z \mid X} \right) \otimes \mathcal{O}_Z(d) \to 0.
$$

Denote by $\mathcal{F}_{j,k} := \bigwedge^j T_Z \otimes \mathcal{O}_X(k)$ for some $j, k \in \mathbb{Z}_{\geq 0}$. Since $Z$ and $X$ are smooth, $\mathcal{F}_{j,k}$ is $\mathcal{O}_Z$-locally free and hence $\mathcal{O}_Z$-flat. Tensoring the previous short exact sequence by $\mathcal{F}_{j,k}$ then gives us the following short exact sequence,

$$
0 \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i} N_{Z \mid X} \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i} N_{Z \mid \mathbb{P}^{2n+1}} \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i-1} N_{Z \mid X}(d) \to 0.
$$

By Serre’s vanishing theorem, for $d \gg 0$, $l > 0$ and $m \geq 1$, $H^m \left( \mathcal{F}_{j,l-4} \otimes \bigwedge^{n-i} N_{Z \mid \mathbb{P}^{2n+1}} \right) = 0$, hence

$$
H^m \left( \mathcal{F}_{j,l-4} \otimes \bigwedge^{n-i} N_{Z \mid X}(d) \right) \cong H^{m+1} \left( \mathcal{F}_{j,l-4} \otimes \bigwedge^{n-i} N_{Z \mid X} \right).
$$

(1)
Using Serre’s vanishing theorem again for $d \gg 0$ and $i \geq 1$, 
$h^n \left( \bigwedge^{i-1} T_Z((n-i+1)d-4) \right) = 0$. Hence, using the isomorphism (1) recursively, we get for $j = i - 1$,
\begin{align*}
h^n \left( \bigwedge T_Z \otimes \bigwedge^{n-i} N_{Z|X}((d-4)) \right) &= h^{n-1} \left( \bigwedge T_Z \otimes \bigwedge^{n-i-1} N_{Z|X}(2d-4) \right) = \ldots \\
&= \ldots = h^i \left( \bigwedge T_Z((n-i+1)d-4) \right) = 0.
\end{align*}

This proves the proposition. \(\square\)

**Proof of Theorem 1.1.** The existence of a smooth hypersurface in $\mathbb{P}^{2n+1}$ containing $Z$ for $d \gg 0$ follows from Theorem 3.3. It suffices to prove that there exists a hypersurface $X$ in $\mathbb{P}^{2n+1}$ of degree $d \gg 0$ containing $Z$ such that the morphism from $H^{n-1}(\Omega_{X}^{n+1} \otimes O_Z)$ to $H^{n-1}(N_{Z|X}^{\vee} \otimes N_{Z|X} \otimes K_X)$, which is the dual to the semi-regularity map $\pi$ (see 2.1), is surjective.

Consider the short exact sequence,
\[ 0 \rightarrow T_Z \rightarrow T_X \otimes O_Z \rightarrow N_{Z|X} \rightarrow 0. \]

Consider the associated filtration,
\[ 0 = F^n \subset F^{n-1} \subset \ldots \subset F^0 = \bigwedge (T_X \otimes O_Z) \text{ satisfying } F^p/F^{p+1} \cong \bigwedge T_Z \otimes \bigwedge N_{Z|X} \]
for all $p$. Taking $p = 0$ we get the following short exact sequence
\[ 0 \rightarrow F^1 \rightarrow \bigwedge (T_X \otimes O_Z) \rightarrow \bigwedge N_{Z|X} \rightarrow 0. \]

Tensoring this by $K_X$ and looking at the associated long exact sequence, we get
\[ \ldots \rightarrow H^{n-1}(\Omega_{X}^{n+1} \otimes O_Z) \rightarrow H^{n-1}(N_{Z|X}^{\vee} \otimes \bigwedge N_{Z|X} \otimes K_X) \rightarrow H^n(F^1(d-4)) \rightarrow \ldots \]

It therefore suffices to prove that $h^n(F^1(d-4)) = 0$.

We claim that it is sufficient to prove $h^n(F^{n-1}(d-4)) = 0$. Indeed, suppose $h^n(F^{n-1}(d-4)) = 0$. By Proposition 3.4 for any integer $2 \leq i \leq n - 1$, we have
\[ h^n \left( \bigwedge^{i-1} T_Z \otimes \bigwedge^{n-i} N_{Z|X}(d-4) \right) = 0. \]

Consider the following short exact sequence, where $2 \leq p \leq n - 1$,
\[ 0 \rightarrow F^p \rightarrow F^{p-1} \rightarrow \bigwedge T_Z \otimes \bigwedge N_{Z|X} \rightarrow 0 \]

Tensoring (2) by $K_X \cong O_X(d-4)$ and considering the corresponding long exact sequence, we can conclude $h^n(F^{n-2}(d-4)) = 0$ (substitute $p = n - 1$). Recursively substituting
\[ p = n - 2, n - 3, \ldots, 2 \text{ in (2)}, \] we observe that \( h^n(F_i(d - 4)) = 0 \) for \( i = 1, \ldots, n - 2 \). In particular \( h^n(F^1(d - 4)) = 0 \). Hence, it suffices to prove \( h^n(F^{n-1}(d - 4)) = 0 \).

Note that, \( F^{n-1} \cong \bigwedge^{n-1} T_Z \) does not depend on the choice of \( X \), hence independent of \( d \). Therefore, by Serre’s vanishing theorem, \( h^n(F^{n-1}(d - 4)) = 0 \) for \( d \gg 0 \). This completes the proof of the theorem. \( \square \)

**Remark 3.5.** Notations as in Theorem [1.1]. We now note that the theorem implies a very special case of the variational Hodge conjecture. Indeed, consider a family \( \pi : \mathcal{X} \to S \) of smooth degree \( d \) hypersurfaces in \( \mathbb{P}^{2n+1} \) with \( X \) as a special fiber. Denote by \( \gamma \) the cohomology class of \( Z \) in \( X \). Then, using [Blo72, Theorem 7.1] notice that \( \gamma \) remains a Hodge class if and only if \( Z \) remains an algebraic variety as \( X \) deforms along \( S \).

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