The beginning of string theory: a historical sketch

Paolo Di Vecchia\textsuperscript{1} and Adam Schwimmer\textsuperscript{2}

\textsuperscript{1} Nordita, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark, divecchi@nbi.dk
\textsuperscript{2} Weizmann Institute, Rehovot 76100, Israel, adam.schwimmer@weizmann.ac.il

Summary. In this note we follow the historical development of the ideas that led to the formulation of String Theory. We start from the inspired guess of Veneziano and its extension to the scattering of $N$ scalar particles, then we describe how the study of its factorization properties allowed to identify the physical spectrum making the string worldsheet manifest and finally we discuss how the critical values of the intercept of the Regge trajectory and of the critical dimension were fixed to 1 and 26.

1 Introduction

The purpose of this note is to follow the historical development of the ideas that led to the formulation of String Theory. As we will discuss, the story consists of a remarkable succession of inspired insights. First Veneziano guessed the form of the four point function \cite{Veneziano}. This was followed by its extension to amplitudes with an arbitrary number of external legs. At this point the dual resonance model was constructed. The subsequent analysis of its factorization properties allowed one to identify the full target Hilbert space of physical states and its critical dimension by the use of various consistency conditions. The natural interpretation of the structure uncovered was that of a string propagating in Minkowski space-time.

We want to stress that all this was achieved without the use of a Lagrangian formulation but by implementing the basic principles of S-matrix theory directly on the scattering amplitudes in a model containing an infinite number of zero width resonances. The new, additional requirement was the Dolen-Horn-Schmid (DHS) duality \cite{Dolen} i.e. that the sum of resonances in one channel represents correctly the resonances in the other channel.

As a result, the basic framework of Perturbative String Theory at the operational level was well understood by 1971. Further progress was achieved through the discovery of the Superstring and Space-time Supersymmetry which led to tachyon free theories. Later some basic concepts used before
at a heuristic level like the origin of the first class constraints necessary for making the spectrum unitary and Lorentz invariant were put on a firm ground starting from the action used in Ref. [3].

Further conceptual developments like the connection between world sheet conformal invariance and target space equations of motion were only partially understood and had to wait for the first String Revolution to get a more complete formulation. Finally the relation between different String Theories through dualities was the result of the second String Revolution.

In this note we will concentrate on the developments during the period 1969-1972.

As we mentioned above three components entering the basic structure of perturbative string theory i.e.:

- the string world sheet
- the physical spectrum and vertex operators
- the critical dimension

were all correctly identified by the end of 1972 and in this short note we will limit ourselves to the description of the evolution of their understanding. We will not cover other very important developments during the same period like e.g. fermionic degrees of freedom on the worldsheet (the Neveu-Schwarz-Ramond formalism [4, 5]), compact degrees of freedom on the world sheet leading to internal symmetries [6] and String Field Theory in its light-cone formulation [7].

We will follow the evolution of the ideas which led to the understanding of the three basic concepts above outlining the most important conceptual jumps. Just the essential formulae will be given referring for the detailed derivations to the accompanying paper [8]. We will try to put in perspective the evolution of the ideas by translating the guesses and insights in today’s language and understanding, as presented in the standard modern textbooks [9]. We start with a brief reminder of the developments on which the three breakthroughs mentioned above were based.

2 Prehistory: The discovery of the Dual Scattering Amplitudes

The first step which started the evolution of String Theory was the Veneziano Formula [1]. By a historical accident Veneziano’s formula refers to what is today Open String Theory. The analogous formula for Closed String Theory guessed by Virasoro [10] was generalized [11] and analysed later [12] when the basic structure of the open string was already understood. We will follow the historical path and discuss only Open String Theory.

The formula guessed by Veneziano corresponds to what we call today the 2 to 2 scattering amplitude of the bosonic open string tachyons:
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\[ A(s, t, u) = A(s, t) + A(s, u) + A(t, u) \]  (1)

where

\[ A(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} = \int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1} \]  (2)

and

\[ \alpha(s) = \alpha_0 + \alpha's \]  (3)

is a linearly rising Regge trajectory.

The appearance of the free parameter \( \alpha_0 \) instead of the usual value 1 will be discussed below. Moreover, in the Veneziano amplitude, as written above, there is no requirement that the external particles are the spin 0 particles on the leading trajectory \( \alpha(s) \). Nevertheless we will continue to call the external particles "tachyons" because they have negative mass squared if we require them to be on the leading trajectory for \( \alpha_0 = 1 \).

In Veneziano's original approach the amplitude was supposed to describe scattering of mesons due to strong interactions. The physical principles guiding Veneziano in his guess were the usual analyticity and crossing symmetry requirements of the scattering amplitudes and a new principle, the DHS duality [2].

DHS duality was abstracted from a phenomenological study of hadronic reactions and stated that the scattering amplitude could be decomposed alternatively into a set of s-channel or t-channel poles, each decomposition being complete, and containing, by analytic continuation, the other. This was expressed by the pictorial identity [13] in Fig. 1.

\[ \text{Fig. 1. The duality diagram contains both s and t channel poles} \]

In today’s language it is qualitatively clear that the DHS requirement is fulfilled if the amplitude is related to the correlator of four vertex operators in a conformal field theory. The two different decompositions which make explicit the pole structure can be represented graphically by two “duality diagrams” related by a continuous deformation and correspond to the two possible decompositions in conformal blocks of the conformal correlator. This happens if
the conformal block is translated into poles in Lorentz invariants constructed from the space-time momenta. This basic feature of String Theory to which DHS duality led, is very far from its phenomenological origin. Ironically, it seems that present hadron scattering data [14] are not anymore in agreement with DHS duality which was a feature related to the energy range available at the time.

For the $N$-point function the DHS duality is generalized by requiring that, for a fixed ordering of the external particles, the amplitude can be represented by any one of the deformations of the respective $N$-point duality diagram. As described in Ref. [8], one way to understand the mechanism by which $A(s,t)$ satisfies the DHS duality is to study its integral representation and identify the two mutually exclusive integration domains which produce the poles in the $s$ and in $t$ channel respectively. This is generalized for the $N$-point function by writing it as a sum of terms, each one corresponding to a given ordering of the external legs. Each term has a $N-3$-dimensional integral representation. The different deformations of the duality diagram are obtained from the singular contributions to the integral representation of mutually exclusive $N-3$ dimensional integration regions.

Based on this idea the unique $N$-point function was constructed in Ref. [15]:

$$B_N = \prod_{i=2}^{N-2} \left[ \int_0^1 du_i u_i^{-\alpha(s_i)-1} (1-u_i)^{\alpha_0-1} \right] \prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} (1-x_{ij})^{2\alpha'} p_i \cdot p_j$$

where

$$s_i \equiv s_{1i} \quad ; \quad x_{ij} = u_i u_{i+1} \ldots u_{j-1}$$

$$s_{ij} = -(p_i + p_{i+1} + \ldots + p_j)^2$$

and $p_i, i = 1, 2, \ldots, N$, are the external momenta. One requires that the external scalar lies on the leading trajectory as explained in Ref. [8]. Starting from this expression Koba and Nielsen [16] put it in the more symmetric $SL(2,R)$ invariant form (see Ref. [8] for details)

$$B_N = \int_{-\infty}^{\infty} dV(z) \prod_{(i,j)} (z_i, z_{i+1}, z_j, z_{j+1})^{-\alpha(s_{ij})-1}$$

where

$$dV(z) = \prod_{i=1}^{N} \left[ \theta(z_i - z_{i+1}) dz_i \right] \prod_{i=1}^{N} (z_i - z_{i+2}) dV_{abc}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}$$

and the variables $z_i$ are integrated along the real axis in a cyclically ordered way: $z_1 \geq z_2 \ldots \geq z_N$ with $a,b,c$ arbitrarily chosen.
The \( SL(2, R) \) group mentioned above acts on the integration variables \( z_i \) as a Möbius transformation:

\[
z_i \rightarrow \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \quad ; \quad i = 1 \ldots N \quad ; \quad \alpha \delta - \beta \gamma = 1 \quad (9)
\]

Using the transformation in Eq. (9) for a fixed ordering, one can relate amplitudes corresponding to circularly permuted kinematical invariants and then adding terms for different orderings one can show that all the requirements of crossing symmetry are fulfilled. As we understand it today, the Möbius transformations are related to globally defined reparametrizations of the disk which leave invariant the metric up to a conformal factor. This was the first manifestation of the conformal symmetry underlying the world sheet action of String Theory which played an essential role in the understanding of the theory.

The expression in Eq. (7) which was guessed as following from the principles mentioned above, coincides (for \( \alpha_0 = 1 \)) with the tree level scattering amplitude of \( N \) open string tachyons obtained from calculating the open string path integral on a disk with the insertion of \( N \)-tachyon vertex operators after mapping the disk to the upper half plane.

The Koba-Nielsen form of the \( N \)-point function was the starting point for the crucial developments which started in 1969. There was a general feeling among the workers in the field that the set of \( N \)-point functions represent the result of a unique and consistent underlying theory. While attempts to use the functions to fit hadronic data continued, the search for this theory became the major theoretical challenge. One aspect which became immediately obvious was the necessity to "unitarize" the theory: the presence of zero width poles in the \( N \)-point functions showed that the amplitudes should be considered, at best, as "tree diagrams" of an underlying, unknown theory and "loop" diagrams should be added to them. A first attempt [17] to write loop diagrams was by using again a generalized form of the DHS principle requiring a singularity structure of the amplitudes consistent with deformations of duality diagrams involving loops. The existence of rather involved integrals, found in Ref [17], which fulfill the constraints, reinforced the belief in the existence of an underlying theory. On the other hand, the ambiguities in the amplitudes constructed originating in what we call today "the measure factors" and the impossibility to verify the unitarity, reinforced the necessity of understanding the basic underlying theory.

The approaches used were conditioned by the development of the theoretical techniques at the time. Though the path integral formulation of Quantum Field Theory existed, it was not well developed as a calculational tool. This was the case especially for gauge theories where the correct treatment of gauge symmetries achieved a few years later by Faddeev-Popov did not exist. As a consequence Lagrangian methods based on an action were not very precise and involved some guess work at different stages. On the other hand, operatorial methods were well developed and through the Gupta-Bleuler treatment of
QED as a prototype, even the correct impositions of constraints corresponding to a gauge fixing (at least for the case when the ghosts are decoupled in today’s language) were understood. We can roughly divide the search for the underlying theory as the ”Lagrangian approach” and the ”operatorial approach”.

Since we will discuss later in more detail the operatorial approach we start with a description of the evolution of the ”Lagrangian” ideas. Researchers following this path tried to guess the underlying Lagrangian which would lead to the N-point functions. This line was open by Nambu, Nielsen and Susskind. Nambu [18] and Susskind [19] proposed that the underlying dynamics of the dual N-point functions corresponds to a generalization of the Schwinger proper time formalism where a relativistic string is propagating in proper time. The equation of motion satisfied by the string coordinates was the two-dimensional D’Alembert equation following from a linearized Lagrangian. Using plausible arguments they obtained expressions similar to the N-point (tree) amplitudes.

Then Nielsen [20] and immediately after Fairlie and Nielsen [21] used this linearized Lagrangian for constructing the ”analogue model”. The basic observation was that the momentum dependence of the integrands in the Koba-Nielsen amplitudes and their loop generalizations is related to the energy of two dimensional electrostatic problems where the momenta are ”charges” located on the boundary. Then the electrostatic problem is solved on a disk for the tree amplitude or on an higher genus two dimensional surface described by the duality diagram corresponding to the respective loop amplitude. We understand this result today as a simple consequence of the fact that the $i k X(\sigma)$ factor in the exponential of the vertex operator acts as a source for the string coordinates whose propagator is the two dimensional Coulomb kernel. Though the measure was not correctly reproduced, the ”analogue model” is important since it is the first appearance of the two-dimensional world sheet in a mathematical role rather than just as a picture in the duality diagram. This model is the precursor of the path integral formulation of string theory that was understood completely only later. Furthermore, the ”analogue model” motivated the generalization [11] of the Virasoro amplitude [10] and therefore the formulation of the Closed String Theory by simply putting electrostatic sources on a sphere instead on the boundary of a disk.

A non-linear action, proportional to the area spanned by the string, generalizing the non-linear one for the pointlike particle, was also proposed by Nambu and Goto in Ref.s [22, 23]. But the consequences of its non-linear structure, implying the invariance under an arbitrary reparametrization of the world sheet coordinates, were only clarified few years later with the treatment of Ref. [24] that provides a rigorous derivation of the properties of the generalized Veneziano model, though our present understanding of string theory is mostly based on the action used in Ref. [3].

The second approach that we will describe in detail in the next section, is based instead on the construction of an operator formalism that made transparent the most important properties of the model as the spectrum of
physical states and their scattering amplitudes and that historically has been essential for relating it to string theory in a completely satisfactory way.

3 The String World Sheet through Factorization of the N-point amplitudes

The basic observation used in order to uncover the underlying theory in the operatorial approach was that, having a set of N-point functions satisfying the DHS duality, crossing symmetry and tree level analyticity, does not define a consistent set of S-matrix elements unless the different poles in the various channels can be shown to come from the same set of physical states, the residues being factorized. This means that one should find a set of states and a set of three point couplings between these states such that any expansion of a given ordering contribution to any of the N point functions is reproduced by the same set of states and couplings.

During 1969 there was an intensive activity in this program of finding the universal set of states and couplings leading to factorization. We will describe in words the main steps in historical succession and then describe the complete solution as formulated in Ref. [25] at the end of 1969. Through an explicit analysis of the residues of a given pole in Ref.s [26, 27] it was shown that factorization can be achieved by having an infinite number of intermediate states. An essential step was made in Ref. [28] where it was proven that the spectrum is the Fock space of an infinite number of harmonic oscillators. The authors of Ref. [28] gave general formulae for the masses of the states in terms of occupation numbers and for the couplings of the external tachyons to arbitrary pairs of states in terms of matrix elements of vertex operators depending on the harmonic oscillator degrees of freedom. An important result of Ref. [28] was the discovery of the existence of the Hagedorn temperature in the theory, a basic feature characterizing String Theories.

We describe now the solution of the factorization problem following Ref. [25]. One starts defining the operator $Q_\mu(z)$ by:

$$Q_\mu(z) = Q_\mu^+(z) + Q_\mu^0(z) + Q_\mu^-(z)$$  \hspace{1cm} (10)

where

$$Q_\mu^+ = i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} z^{-n} ; \quad Q_\mu^- = -i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_n^\dagger}{\sqrt{n}} z^n$$

$$Q_\mu^0 = \hat{q} - 2i\alpha'\hat{p}\log z$$  \hspace{1cm} (11)

and the vertex operator by

$$V(z;p) =: e^{ip\cdot Q(z)} := e^{ip\cdot Q_\mu^0(z)} e^{ip\cdot \hat{q} + 2i\alpha'\hat{p}\log z} e^{ip\cdot Q_\mu^+(z)}$$  \hspace{1cm} (12)
Then it was shown [25] that the integrand of the Koba-Nielsen $N$-point function is related to the Fock space vacuum matrix element of the product of vertex operators:

$$\langle 0,0 \prod_{i=1}^{N} V(z_i,p_i)|0,0 \rangle = \prod_{i>j} (z_i - z_j)^{2\alpha'} p_i \cdot p_j (2\pi)^4 \delta^{(4)}(\sum_{i=1}^{N} p_i)$$  \hspace{1cm} (13)

In order to obtain exactly the Koba-Nielsen expression one has to deal carefully with the fixing of three of the $z$ variables. This is done by extracting the $z$ dependence of the vertex operators using the identity:

$$z^{L_0} V(1,p) z^{-L_0} = V(z,p) z^{\alpha_0}$$  \hspace{1cm} (14)

where $L_0$ is the operator:

$$L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} n a_n ^\dagger \cdot a_n$$  \hspace{1cm} (15)

Choosing three consecutive values of $z_i$ to be fixed:

$$z_a = z_1 = \infty ; \hspace{0.5cm} z_b = z_2 = 1 ; \hspace{0.5cm} z_c = z_N = 0$$  \hspace{1cm} (16)

the Koba-Nielsen amplitude can be rewritten in the operator language as:

$$A_N \equiv \langle 0,p_1 | V(1,p_2) D V(1,p_3) \ldots D V(1,p_{N-1}) | 0,p_N \rangle$$  \hspace{1cm} (17)

where the "propagator" $D$ is equal to:

$$D = \int_{0}^{1} dx x^{L_0 - 1 - \alpha_0} (1 - x)^{\alpha_0 - 1} = \frac{\Gamma(L_0 - \alpha_0) \Gamma(\alpha_0)}{\Gamma(L_0)}$$  \hspace{1cm} (18)

and the states (using what we understand today as "operator-state correspondence") are defined as:

$$\lim_{z \to 0} V(z;p) | 0,0 \rangle \equiv | 0;p \rangle ; \hspace{0.5cm} \langle 0;0 | \lim_{z \to \infty} z^{2\alpha_0} V(z;p) = \langle 0,p |$$  \hspace{1cm} (19)

The $z_i$ integrations of the Koba-Nielsen formula which were absorbed in the definition in Eq. (18) are translated into integrations over the "proper times" $x_i$ appearing in the propagators.

This provides an explicit solution to the factorization. In fact, one can insert between each $V$ and $D$ a complete set of states of the space spanned by the harmonic oscillators (Fock space) appearing in $Q(z)$. Since $D$ is diagonal in the basis of occupation numbers, poles will appear at $\alpha(s) = 0, 1, 2, \ldots$ with factorized residues related in a universal fashion to the matrix elements of the vertex operators.

This solution to the factorization problem was the crucial step in the development of String Theory since, from now on, the $N$-point functions were
clearly related to a theory in which the set of space-time fields is labeled by the states in the Fock space on which the $Q_\mu$ fields are realized. The $Q_\mu$ fields are, of course, the open string coordinate fields $X^\mu(\sigma, \tau)$ in $d$ space-time dimensions for $\mu = 0, 1, 2, \ldots, d-1$, computed at the endpoint of the string coordinate $\sigma = 0$, where $z$ is related to the other string coordinate $\tau$ by $z = e^{i\tau}$. They are Heisenberg operators, their dependence on the world-sheet coordinates $\sigma$ and $\tau$ follows from the fact that they are solutions of an equation of motion following from a free linearized Lagrangian. However, as it is described above the Lagrangian was not used in the derivation, the various expressions being obtained by a rewriting of the $N$-point amplitudes. While the linear spacing between the poles of the Veneziano formula was suggestive of some underlying harmonic oscillator type structure only the solution of the factorization problem unveiled the true structure of the theory, i.e. an infinite number of oscillators assembled into a set of fields $Q_\mu$ living on a two-dimensional world sheet.

The vertex operators for the emission of tachyons represent insertions on the boundary (for open string theories) of the two-dimensional world sheet. Of course the relation in Eq. (17) is the way in which scattering amplitudes are obtained in String Theory starting from the matrix element of products of vertex operators. The historical way was exactly the opposite i.e. given the Koba-Nielsen formula, the operators whose matrix elements reproduce the formula were correctly guessed identifying the Hilbert space. Now the fulfillment of the DHS requirements became natural: the $Q_\mu$ are massless two dimensional fields defining a two-dimensional conformal theory and the $N$-point functions are related to integrals of correlators of the vertex operators in the $SL(2, R)$ invariant vacuum. The integration over the $z$ variables required by the Koba-Nielsen formula, in order to produce the poles in $\alpha(s_{ij})$, is related to the integration over the "proper times" after the mapping of the disk into the upper half plane. The fact that this particular expression is special to a particular gauge (at that time called the orthonormal gauge) was already understood during the first period of String Theory, but it became more transparent and rigorous after Polyakov’s seminal paper [3].

Having the decomposition of the amplitudes in "vertices" and "propagators" allows the calculation of loop diagrams by gluing them and taking traces for the loops. The loop diagrams are necessary for producing an $S$-matrix consistent with unitarity. In this way, one obtained already in 1970 the correct expression in the Schottky parametrization of quantities defined on a Riemann surface as the period matrix, the abelian differentials and the Green’s functions [29, 30, 31]. However, the correct measure of integration in the multiloops was not known at the time since it requires the understanding of ghost contributions. It is clear now that these operatorial expressions in the covariant gauge are the same as those obtained by performing the path integral of the string Lagrangian over the appropriate world sheet.

We know today that the restrictions on the operators $V$ and $D$ which can be used follow from a correct gauge fixing of the string Lagrangian. In the
absence of a Lagrangian again the correct restrictions on $V$ and $D$ were found by a rather tortuous path (from today point of view) which we are going now to describe.

The expressions used above differ from the ones used in the modern formulation in two respects:

i) The vertex operators used were defined for a conformal weight $\alpha'k^2$. This value, related to the mass squared of the open string tachyon, is given in terms of the arbitrary parameter $\alpha_0$: $\alpha_0 = \alpha'k^2$.

ii) The dimension $d$ of space-time i.e. the number of string coordinates was left free.

4 The Virasoro Conditions

We start this section by reminding the reader how the two points mentioned at the end of the previous section are understood today. The starting point today for the bosonic string theory is the $\sigma$-model action (the action used in Ref. [3]) that, at the classical level, couples the string coordinates to the two dimensional world sheet metric in a diffeomorphism and Weyl invariant manner. Then the requirement that these two “gauge symmetries” (diffeomorphism and Weyl) are not anomalous in the quantum theory fixes the space-time dimension to the value $d = 26$ for the bosonic string.

Once the two “gauge symmetries” are respected at the quantum level, the standard Faddeev-Popov procedure can be applied, in principle in an arbitrary gauge, and a consistent quantization can be performed giving the physical states/operators in the gauge chosen. The states/operators in different gauges are isomorphic leading to the same results when gauge invariant correlators are calculated. In particular, by choosing a covariant gauge the Lorentz invariance of the theory follows automatically, while the unitarity of the theory is not obvious. On the other hand, by choosing an explicitly unitary gauge (the light-cone gauge) the unitarity of the theory is completely manifest, while the Lorentz invariance has to be checked. In the covariant gauge the physical states correspond to operators with dimension 1 for the open string and (1, 1) for the closed string. This fixes the leading Regge trajectories to have intercept $\alpha_0 = 1$ or $\alpha_0 = 2$ for the open and closed strings, respectively. In a “physical” gauge, as the light-cone gauge, the states which are now ”transverse” correspond to cohomologically equivalent families in the covariant gauge.

Actually the BRST approach outlined above is essential for the bosonic string only in order to compute, with manifestly Lorentz covariant methods, the correct integration measure for multiloop amplitudes. The understanding reached at the end of 1972 was almost complete including tree and loop diagrams.

We want to stress here once more that none of the ideas based on the BRST invariant approach (including the $\sigma$-model action) were known in the early days of string theory. The Nambu-Goto action was known, but it was
not really understood how to use it rigorously for deriving all the properties obtained using the operator formalism. One had to use alternative methods which amazingly enough led to the correct results. This is what we are going to explain below.

Before we proceed, let us notice that, from the present point of view, the description done in the previous section involved just a conformal theory of $d$ massless fields. Of course in such a theory any vertex operator is legal and the correlators of vertex operators on the $SL(2,R)$ invariant vacuum have the block decomposition properties even after integrating over their "proper time" coordinates. Interestingly, even without the understanding that a consistent String Theory should be the gauge fixed version of a Weyl anomaly free theory the way to make the theory consistent by restricting i) and ii) was correctly guessed. This was done by looking for some "gauge" conditions that could help in decoupling the negative norm states, required by manifest Lorentz covariance, from the spectrum of the physical states pretty much in analogy with what was known to happen in QED. We start discussing the way in which the correct gauge conditions were discovered.

In Ref. [26] it was pointed out that the residues of the poles on which the amplitude is factorized are not positive definite simply due to the presence of the time components of the oscillators which in the operator formulation lead to a negative contribution to the scalar product. As a possible way out from this inconsistency of the theory, linear relations between the residues were uncovered leading to the decoupling of some Fock space states from the amplitude. The basic driving idea was that the situation here was analogous to the Gupta-Bleuler quantization of QED. As in QED where the Lorentz condition was imposed to characterize the subspace of the physical states, here also some "gauge" conditions, that later on were understood to be due to some first class constraints, were imposed on the spectrum which would eliminate the negative norm states.

In this way one managed to get the correct result without having to fix the gauge of the diffeomorphisms and Weyl invariance and to introduce the $b,c$ ghost system. This has been possible because the ghosts are decoupled from the string coordinates. As a consequence, the nontrivial BRST cohomology can be realized in terms of the string coordinates only, the ghost ground state not being excited and, for tree diagrams at least, one can calculate consistently using the string coordinates restricted by the first class constraints.

The correct final answer was reached following a rather tortuous, but physical and at that time intuitive path.

We start describing the linear relations [32] mentioned above. In the operator formalism there is a realization [33, 34] of the Möbius transformations in Eq. (9) in terms of the infinite set of harmonic oscillators. This $SL(2,R)$ algebra has a simple action on the vertex operators and annihilates the vacuum. Its generators $L_1, L_0, L_{-1}$ are:
\[ L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n ; \quad L_1 = \sqrt{2\alpha'\hat{p}} \cdot a_1 + \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1}^\dagger \cdot a_n \] (20)

and

\[ L_{-1} = L_1^\dagger = \sqrt{2\alpha'\hat{p}} \cdot a_1^\dagger + \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_n^\dagger \cdot a_{n+1} \] (21)

We recognize, of course, the central extension free \( SL(2, R) \) subalgebra of the Virasoro algebra which acts as a symmetry on an arbitrary CFT correlator, provided it is evaluated on the \( SL(2, R) \) invariant vacuum. We remind the reader, however, that the algebra of the Virasoro operators and more generally two dimensional conformal field theories were not known at the time. Their understanding was a result of the developments we are describing. The \( SL(2, R) \) subalgebra generates the Möbius group of the finite transformations of \( z \):

\[ z' = \frac{\alpha z + \beta}{\gamma z + \delta} \] (22)

where \( \alpha \delta - \beta \gamma = 1 \). The vertex operators have the standard transformation properties under the Möbius group corresponding to the weight \( L_0 = \alpha' \hat{p}^2 \).

In the expectation value in Eq. (17) the information that \( z_a \) is fixed appears only through the "bra" vector on the l.h.s. of the matrix element. Therefore the r.h.s. has a residual symmetry, the subgroup of the Möbius group which leaves the fixed \( z_b = 1, z_c = 0 \) unchanged:

\[ z' = \frac{z}{1 - \alpha(z - 1)} = z + \alpha(z^2 - z) + o(\alpha^2) \] (23)

This subgroup is generated by

\[ W_1 = L_1 - L_0 \] (24)

Since the "ket" on the r.h.s. is left invariant by the subgroup in Eq. (23) we obtain:

\[ W_1 |p_{(1,M)}\rangle = 0 \] (25)

where:

\[ |p_{(1,M)}\rangle = V(1, p_M)D \ldots V(1, p_2)|p_1, 0\rangle \] (26)

independently on the number of \( VD \) insertions. Clearly, one gauge condition \( W_1 \) is not enough to project out all the negative norm states and additional conditions were searched for. We remark that Eq. (25) is not a consequence of any gauge symmetry, being valid in any CFT for vertex operators of arbitrary dimensions provided the vertex operators are inserted at the value \( z = 1 \).
Nevertheless following the pattern that led to Eq. (24), Virasoro [35] realized that, if $\alpha_0 = 1$, the state in Eq. (26) is annihilated by an infinite set of "gauge" operators:

$$W_n | p_{1,M}\rangle = 0 \quad ; \quad n = 1, 2, 3, \ldots$$

(27)

where

$$W_n = L_n - L_0 - (n - 1)$$

(28)

with:

$$L_n = \sqrt{2\alpha' n \hat{p} \cdot a_n} + \sum_{m=1}^{\infty} \sqrt{m(n + m)} a_{n + m} \cdot a_m +$$

$$+ \frac{1}{2} \sum_{m=1}^{n} \sqrt{m(n - m)} a_{m - n} \cdot a_m \quad ; \quad n \geq 0 \quad L_{-n} = L_n$$

(29)

The "gauge" conditions in Eq. (27) imply the following equations for the on shell physical states of the generalized Veneziano model [36]

$$(L_0 - 1)|\text{Phys}\rangle = L_n |\text{Phys}\rangle = 0 \quad ; \quad n = 1, 2, \ldots$$

(30)

These are exactly the constraints following from the diffeomorphism and Weyl symmetry of the action in presence of a two dimensional metric after the gauge fixing which eliminates completely the metric. These constraints annihilate the intermediate states in Eq. (17), that are not physical, as we know from the now standard gauge fixing-BRST procedure [9]. We postpone the discussion of the exact conditions under which the constraints eliminate the negative norm states to the next section since it is closely tied to the recognition of the critical dimension. In conclusion, the correct results were obtained at the tree level without needing to know the underlying Lagrangian and to introduce the ghost degrees of freedom. What is more amazing is that also the one-loop measure was correctly obtained by using the Brink-Olive operator, that projected in the subspace of physical states [37]. The correct measure for the multiloop amplitudes was determined much later, although it would have been possible, in principle, to determine it by extending the procedure of Brink and Olive to multiloops.

Once the intercept $\alpha_0$ got fixed to 1 it became clear that the first state on the leading trajectory is a tachyon; its consistent removal was achieved only with the discovery of the superstring and the GSO projection [38]. Imposing the infinite set of Virasoro constraints on the vertex operators corresponds in today’s language, that was already used in Ref. [39], to the requirement that the vertex operators should be primary fields with dimension 1 [39]. Projecting from the Fock space the states which are annihilated by all the Virasoro constraints and eliminating the zero norm states following the procedure explained in Ref. [36], defines the physical Hilbert space which should have positive norm.
Shortly after Virasoro found the constraints (28) it was realized that the \( L_n \) operators are the generators of the conformal group in \( d = 2 \) \cite{32}. The full algebra of the group including the central extension present in the commutator of \( L_n \) with \( L_{-n} \) was correctly worked out only somehow later \cite{40} \(^3\). In this way the algebra of the Virasoro operators was established and became the basic algebraic structure underlying two-dimensional CFT and String Theory. The central extension discovered by Weis \cite{40} which is understood today as a manifestation of the conformal anomaly \cite{3}, has far reaching consequences which we are going to discuss now.

5 The Critical Dimension

The discovery of the critical dimension with its various manifestations shows the serendipity characteristic of this first period of String Theory. Since, as we know it today, the existence of the critical dimension is a consequence of the conformal anomaly cancellation between the string coordinates fields and the \( h, c \) ghost system, it is clear that in the absence of the understanding of the coupling to two dimensional metrics and its gauge fixing which leads to the ghosts, the critical dimension could manifest itself only through its "side effects" i.e. various consistency conditions of the theory. The first calculation pointing to the existence of the critical dimension was done by Lovelace \cite{42}. He calculated the non-planar loop with a number of tachyons as external particles, represented in Fig. 2.

![Fig. 2. The doubly twisted open string diagram.](image)

This diagram was proposed earlier \cite{43} as a model for the "Pomeron" which dominates the high energy elastic scattering amplitude of hadrons and therefore according to the lore of the time was described as the Regge pole in the \( t \)-channel with the highest intercept. The diagram was first calculated in \( d=4 \) \cite{41} and the analytic structure showed the presence of a branch cut in the \( t \)-channel. In Lovelace’s calculation the dimension of space-time \( d \) and the effective number of dimensions going around in the loop \( d' \), were left as free parameters. It was understood at the time that only the physical degrees of freedom which obey the Virasoro gauge conditions circulate in the loops but

\(^3\) See note added in proof of Ref. \cite{32}.
the exact way to implement this fact was not understood. The result of the calculation showed that the singularity in the $t$-channel became a pole only when $d = 26$ and $d' = 24$ and in this case the intercept of the "Pomeron" Regge trajectory is 2. We understand this result today as a consequence of the conformal invariance of the theory: by a continuous deformation of the world sheet the diagram in Fig. 2 can be brought to the form in Fig. 3.

Figure 3. The diagram of Fig. 2 in the closed string channel.

Now it is clear that one has a tree diagram, in the $t$-channel a closed string (the cylinder) being exchanged with the open string tachyons being coupled to the upper and lower disks. As a consequence in the $t$-channel one should have only poles. However, the conformal deformation of the world sheet on which the above expectation is based is valid only when conformal transformations act as expected classically i.e. no anomaly is present implying $d = 26$. In addition, we know today that the $b, c$ ghosts circulating in the loop cancel the contribution of two of the space-time string coordinates leading to $d' = 24$. Finally the intercept 2 is the one required by the correct gauge fixing for the closed string. We identify nowadays the trajectory in the $t$-channel with the graviton and not the Pomeron though the connection may come back to haunt us [44]. In the critical case the couplings of the open strings can be factorized and a consistent open-closed theory can be constructed [45, 46].

Further evidence for the existence of the critical dimension came from a close examination of the physical spectrum i.e. the Hilbert space left after the infinite set of Virasoro conditions are imposed on the Fock space. In Ref.s [47, 48] it was shown that the physical spectrum i.e. the ensemble of Fock space states which satisfy the conditions in Eq. (30) has a positive definite scalar product (it is "ghost free") only when $d \leq 26$. Of course, if the spectrum is ghost free for $d = 26$, it is a fortiori so also for $d < 26$. In order to prove the

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4 This was clarified few years later by Brink and Olive [37] inserting in the loop the operator that projected into the space of physical states.
"no ghost theorem" for $d = 26$ the manipulations used in Ref. [47] are very similar to the modern ones based on the BRST formalism and which are valid provided that the BRST operator $Q$ obeys at the quantum level $Q^2 = 0$ which requires $d = 26$. As a corollary of their proof Goddard and Thorn showed that the DDF [49] states form a basis for the physical Hilbert space.

This leads to a third manifestation [24] of the critical dimension which is already very close to our modern understanding. Though the starting point in Ref. [24] is the Nambu-Goto action the final results correspond to a correct quantization in light-cone [9] and in covariant gauge [9] of the $\sigma$-model action. The DDF states are isomorphic to the states in the light-cone gauge which live in a Hilbert space which has an explicitly positive definite scalar product. The light-cone gauge is, therefore, unitary, however Lorentz invariance is not explicit. On the other hand, in the covariant gauge Lorentz invariance is explicit but unitarity is valid only on the physical Hilbert space after the imposition of the conditions in Eq. (30). In our modern understanding the two gauges being equivalent at the critical dimension insures without further proof that the spectrum is both unitary and Lorentz invariant. However, at the time one had to prove explicitly that on the spectrum in the light-cone gauge the Lorentz algebra is fully realized. By constructing all the Lorentz generators in Ref. [24] it was shown that the algebra closes correctly only if $d = 26$. The treatment in Ref. [24] is already very close to the modern one and completely satisfactory for computing string diagrams in the light-cone gauge. The modern BRST treatment from a calculational point of view just fixes the measure for the multiloop amplitudes.

We mention finally an interesting interpretation of the central extension (and implicitly of the critical dimension) given by Brink and Nielsen in Ref. [50]. They related the central extension to the Casimir energy of the string. In our understanding of today this is simply the fact that, transforming $L_0$ to the strip (or cylinder for the closed string) coordinates, an additional term proportional to the central extension appears. This argument was later generalized to an arbitrary CFT in Ref. [51] giving a relation between the central extension and energies on finite geometries.

6 Conclusions

In this history-oriented note we briefly reviewed some of the developments that led to what we call today "String Theory". At the end of 1972 a complete theory existed (as summarized in Ref. [24]) which, except for the existence of the tachyon, was consistent. Its perturbative spectrum and the precise rules for calculating perturbatively scattering amplitudes were completely understood in the operator formalism. The theory is unitary and Lorentz invariant for $\alpha_0 = 1$ and $d = 26$. All this was obtained starting from a rather strange physical motivation and involved a long chain of beautiful conceptual insights and guesses. The impressive theoretical structure created in the years 1969-1972
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and further intensively developed during the last twenty five years continues to be at the forefront of Theoretical Physics. We dedicate this contribution to Gabriele Veneziano who played a leading role in the developments we described.

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