Abstract

It is known that orbit reduction can be performed in one or two stages and it has been proven that the two processes are symplectically equivalent. In the context of orbit reduction by one stage we shall write an expression for the reduced two-form in the general case and obtain the equations of motion derived from this theory. Then we shall develop the same process in the case in which the symmetry group has a normal subgroup to get the reduced symplectic form by two stages and the consequent orbit reduced equations. In both cases we shall illustrate the method with the example of a rigid body with rotors and compare the obtained equations with the ones given by other authors in different frameworks.
1 Introduction

Reduction theory in the presence of symmetry is a very important tool in the study of mechanical systems from the geometric point of view in relation with the degrees of freedom decrease. Symmetries are often associated with the concept of integrals of motion.

There has been a long history of works about the dynamics of mechanical systems using their symmetries. These studies mainly have their origin in the works of Arnold [2] and Smale [37]. A good exposition of reduction techniques can be found in the book of Abraham and Marsden [1].

In the field of classical mechanics, we can deal with reduction theory from two different points of view: Lagrangian and Hamiltonian formalisms, which are connected by Legendre transform.

A frame for Lagrangian reduction by stages was developed by Cendra, Marsden and Ratiu in [9], presenting Lagrange-Poincaré equations and describing a context in which reduction by stages can be performed. The case in which the configuration space coincides with the symmetry group is known as Euler-Poincaré reduction (see [6], [7], [22], and [27]) and, if the group is abelian, it reduces to the procedure known as Routh reduction (see [22], [26], [27]). In the framework of Lagrange-Poincaré reduction, Cendra and Díaz in [5] generalize the process to the case in which the symmetry group has a chain of normal subgroups, obtaining an expression for Lie bracket by several stages and the corresponding Lagrange-Poincaré equations.

The Hamiltonian version of Lagrange-Poincaré reduction was obtained variationally in [8]. This Hamilton-Poincaré theory is connected with the one known as Poisson cotangent bundle reduction (see [20] and [29]).

For studies in the framework of cotangent bundle reduction we shall mention the references [12], [19], [21] and [28].

The Dirac procedure of reduction includes Lagrangian, Hamiltonian, and variational points of view simultaneously, and enlarges pre-symplectic and Poisson reduction. Dirac reduction also encloses the case of implicit systems and degenerate Lagrangians. In the frame of Dirac theory for symplectic and Poisson cases, we can mention the references [11], [14], [23]. The case of cotangent bundles of Lie groups was studied by Marsden and Yoshimura in [42], and generalized by the same authors in [41]. The context for Dirac reduction by stages was presented in [10] and [41].

A symmetry of a system is usually encoded by a Lie algebra action which is associated with a function from the phase space to the dual of the Lie algebra, named the momentum map, whose level sets are preserved by the dynamics (see [17], [18], [22], [37], [38], [39], [40]).

The structure of the quotient of the preimage of the momentum map of a point for compact Lie groups acting over manifolds of finite dimension was developed in [36]. If the coadjoint orbit is locally closed in the dual of the symmetry group, this structure of the quotient was extended for proper actions of Lie groups to the quotient of the preimage of the momentum map of the orbit by Bates and Lerman in [3].

The theory of symplectic reduction by stages was originated by the study of semidirect product that was developed in [15], [24], [25], [38], [39], [45]. In [20] the authors generalize the theory of semidirect product reduction, considering a symmetry group acting over a symplectic manifold as an extension of one of its normal subgroups, to study reduction by two stages. A Lagrangian analogous of the reduction for group extensions was done by Cendra, Marsden and Ratiu in [9].

In the framework of symplectic reduction, an approach considers the preimage of the momentum map of the coadjoint orbit of a fix point over the symmetry group as the reduced space, while the setting known as orbit reduction passes to quotient the preimage of the momentum map of the point by the isotropy group.
In this paper, we shall study orbit reduction theory to obtain with purely symplectic techniques an expression in coordinates for the two-form, with the aim of getting in this context associated equations of motion considering a unique Lie group of symmetry. Then, inspired in the way in which Cendra and Díaz in [5] have written equations by stages in coordinates in the Lagrangian framework, we study the case in which the symmetry group has a normal subgroup to get equations in coordinates by two stages in the symplectic orbit approach.

The paper is organized as follows. In section 2 we shall obtain an expression in coordinates for the reduced two-form and then, using this expression, we shall write equations in the context of orbit reduction. In section 3 we shall repeat the process in the case in which the symmetry group has a normal subgroup to get equations in coordinates by two stages in the symplectic orbit approach.

2 Orbit reduced equations

In this section we shall recall the process of orbit reduction in one stage and write an expression of the reduced two-form in coordinates. With this expression, we will obtain the reduced equations of motion considering a unique Lie group of symmetry. Then, inspired in the way in which Cendra and Díaz in [5] have written equations by stages in coordinates in the Lagrangian framework, we study the case in which the system has a symmetry and apply these results to study the classical example of a rigid body with rotors, getting orbit reduced equations in one and two stages for this example.

2.1 Orbit reduction theory

Now we will gather some results of the theory of orbit reduction related to our work in this paper. The reader can find the proofs and more details in the references [9, 19, 31] and [32].

Let us consider a connected symplectic manifold \((M, \omega)\) and a free proper canonical (left) action \(\Phi : G \times M \to M\) of a Lie group \(G\) over \((M, \omega)\). We shall note \(\Phi(g, m) = \Phi_g(m) = \Phi^m(g)\), indistinctly. Being \(\mathfrak{g}\) the Lie algebra of \(G\), each element \(\xi \in \mathfrak{g}\) defines a complete vector field \(\xi_M\) on \(M\) known as the infinitesimal generator associated to \(\xi\). If \(f\) is a differential function on \((M, \omega)\), its associated Hamiltonian vector field \(X_f\) is defined as the unique vector field such that \(i_{X_f} \omega = df\).

The map \(J : M \to \mathfrak{g}^*\) defined by the relation \((J(m), \xi) = J^\xi(m)\), for all \(\xi \in \mathfrak{g}\) and \(m \in M\), is called the momentum map of the \(G\)-action if \(J^\xi \in C^\infty(M)\) is the Hamiltonian function of the infinitesimal generator \(\xi_M\). The map \(J\) is said equivariant if \(J(\Phi(g, m)) = Ad^*_g J(m)\), for all \(g \in G\) and \(m \in M\), where \(Ad^*\) denotes the coadjoint action of \(G\) on \(\mathfrak{g}^*\).

**Theorem 2.1.** Let \(\Phi : G \times M \to M\) be a free proper canonical action of the Lie group \(G\) on a connected symplectic manifold \((M, \omega)\). Suppose that this action has an equivariant associated momentum map \(J : M \to \mathfrak{g}^*\). Let \(O_\mu := G.\mu \subset \mathfrak{g}^*\) be the \(G\)-orbit of the point \(\mu \in \mathfrak{g}^*\) with respect to the coadjoint action of \(G\) on \(\mathfrak{g}^*\). Then:

1. The set \(J^{-1}(O_\mu)/G\) is a regular quotient symplectic manifold with the symplectic form \(\omega_{O_\mu}\) uniquely characterized by the relation \(i_{\xi} \omega = \pi^{\mathfrak{g}^*}_* \omega_{O_\mu} + J^\xi_\mu \omega^+_\mu\), where \(J^\xi_\mu\) is the restriction of \(J\) to \(J^{-1}(O_\mu)\) and \(\omega^+_\mu\) is the +symplectic structure on the coadjoint orbit \(O_\mu\) given by \(\omega^+_\mu(\nu, \eta_\mu(\nu)) = (\nu, [\xi, \eta])\), for arbitrary \(\nu \in O_\mu\) and \(\xi, \eta \in \mathfrak{g}\).

The maps \(i_{O_\mu} : J^{-1}(O_\mu) \hookrightarrow M\) and \(\pi_{O_\mu} : J^{-1}(O_\mu) \to J^{-1}(O_\mu)/G\) are the natural injection and projection, respectively. The pair \((J^{-1}(O_\mu)/G, \omega_{O_\mu})\) is called the symplectic orbit reduced space.
2. Let $H$ be a $G$-invariant Hamiltonian. The Hamiltonian flow $F_t$ of $H$ leaves the connected components of $J^{-1}(O_\mu)$ invariant and commutes with the $G$-action, so it induces a flow $F_t^O_{\mu}$ on $J^{-1}(O_\mu)/G$, uniquely determined by $\pi_{O_\mu} \circ F_t \circ i_{O_\mu} = F_t^O_{\mu} \circ \pi_{O_\mu}$.

3. The vector field $X_{h_{O_\mu}}$ generated by the flow $F_t^O_{\mu}$ on $(J^{-1}(O_\mu)/G, \omega_{O_\mu})$ is Hamiltonian with associated reduced Hamiltonian function $h_{O_\mu} \in C^\infty(J^{-1}(O_\mu)/G)$ defined by $h_{O_\mu} \circ \pi_{O_\mu} = H \circ i_{O_\mu}$.

Now, let us consider a left smooth action $\Phi : G \times Q \to Q$ of a Lie group $G$ on a manifold $Q$, and its associated lifted left action $G \times T^*Q \to T^*Q$ given by $g.\alpha_q = T^*_g\Phi^{-1}(\alpha_q)$ for $g \in G$ and $\alpha_q \in T^*_Q$, where $T^*_g\Phi$ denotes the dual application of the tangent map of the diffeomorphism $\Phi$. If $\omega$ is the canonical symplectic structure on $M = T^*Q$, this $G$-action admits an equivariant momentum map $J : T^*Q \to g^*$ given by $\langle J(\alpha_q), \xi \rangle = \alpha_q(\xi_{Q}(q))$, for all $\xi \in g$.

It is assumed that the Lie group $G$ acts freely and properly on the manifold $Q$, and the $G$-action defines a principal bundle $\pi : Q \to Q/G$. Let us consider the adjoint bundle of $Q$, namely $\tilde{g} := (Q \times g)/G$, which is the associated vector bundle defined by the adjoint representation of $G$ over its Lie algebra $g$ whose base is $Q/G$ and its fiber is isomorphic to $g$.

From a principal connection $A$ on the principle bundle $\pi : Q \to Q/G$, one can define the vector bundle isomorphism $\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{g}$ given by $\alpha_A([v_q]) := T_q\pi(v_q) \oplus [q, A(q)](v_q)$, where corner brackets denote equivalence classes in the respective quotient manifolds. Thus, $(\alpha_A^{-1})^* : (T^*Q)/G \to T^*(Q/G) \oplus \tilde{g}^*$ is a diffeomorphism where $\tilde{g}^*$ denotes the coadjoint bundle of $Q$.

Given $\mu \in g^*$, let $O_\mu \subset g^*$ be the $G$-coadjoint orbit of $\mu$, it can be verified that $$(\alpha_A^{-1})^*(J^{-1}(O_\mu)/G)) = T^*(Q/G) \times_{Q/G} \tilde{O}_\mu \subset T^*(Q/G) \oplus \tilde{g}^*,$$ where $\tilde{O}_\mu := (Q \times O_\mu)/G \to Q/G$ is the associated fiber bundle to $Q$ considering the coadjoint action of $G$ on $O_\mu$ and $$T^*(Q/G) \times_{Q/G} \tilde{O}_\mu := \{([q], [q, v]_G) : \alpha_{[q]} \in T^*_{[q]}(Q/G), q \in Q, v \in O_\mu\}.$$ Therefore, the reduced symplectic form $\omega_{O_\mu}$ of the orbit reduced space $J^{-1}(O_\mu)/G$ (see Theorem 2.1) is pushed forward by $(\alpha_A^{-1})^*$ to a symplectic form $\omega_{red}$ on $T^*(Q/G) \times_{Q/G} \tilde{O}_\mu$. This form can be written as (see [19, 32]) $$\omega_{red} = \omega_{Q/G} - \beta,$$ where $\omega_{Q/G}$ is the canonical symplectic form on the cotangent bundle of the quotient $T^*(Q/G)$ and $\beta$ is the unique two-form on $\tilde{O}_\mu$ determined by $\pi_2^*\beta = d\alpha + \pi_2^*\omega_{\mu}$, where $\pi_2 : Q \times O_\mu \to \tilde{O}_\mu$ is the projection on the second factor, and $\alpha$ is the one-form on $Q \times O_\mu$ given by $\alpha(q, v)(v_q, -ad_{\pi_q^*}(v)) = \langle \nu, A(q)(v_q) \rangle$ for $q \in Q, \nu \in O_\mu, v_q \in T_qQ, -ad_{\pi_q^*}(v) \in T_qO$, and $\xi' \in g$. As it has been shown in [32], $d\alpha$ has the expression $$d\alpha(q, \nu)((u_q, -ad_{\pi_q^*}(v)), (v_q, -ad_{\pi_q^*}(\nu))) = \langle \nu, [\eta', \xi] + [\eta, \xi'] + [\xi, \eta] + B(q)(u_q, v_q) \rangle,$$ where $B$ is the curvature of $A$, $q \in Q, \nu \in O_\mu, \xi, \xi', \eta, \eta' \in g, u_q, v_q \in T_qQ, \xi$ and $\eta$ are the elements that induce the vertical components of $u_q$ and $v_q$ with respect to the principal connection $A$. 

4
2.2 Reduced two-form in coordinates

In this section, we shall obtain a coordinate expression of the 2-form $\omega_{\text{red}}$ needed to write equations in the orbit reduction framework.

We consider an element of $T^*(Q/G) \times_{Q/G} \widetilde{O}_\mu$ as a triple $(x, \alpha, \varpi)$. That is, $x = [q]_G \in Q/G$, $\alpha \in T^*_x(Q/G)$, $\nu \in \mathcal{O}_\mu$ and $\varpi = [q, \nu]_G \in \widetilde{O}_\mu$ is the adjoint element of $\nu$.

From now on, we shall note $\mathcal{O}$ instead of $\mathcal{O}_\mu$ the coadjoint orbit of a fixed element $\mu$.

Let $(\dot{x}_1, \dot{\alpha}_1, \dot{\varpi}_1)$ and $(\dot{x}_2, \dot{\alpha}_2, \dot{\varpi}_2)$ be two vectors belonging to $T_{(x, \alpha, \varpi)} \left( T^*(Q/G) \times_{Q/G} \widetilde{O} \right)$. That is, $(\dot{x}_i, \dot{\alpha}_i) \in T_{(x, \alpha)} T^*(Q/G)$ and $\dot{\varpi}_i = \frac{D}{Dt} \varpi_i \in T_{\varpi} \widetilde{O}$ is the covariant derivative of a curve $\varpi_i$ on $\widetilde{O}$, for $i = 1, 2$.

To obtain an expression in coordinates for the two-form $\omega_{\text{red}}$, since [2] and the fact that $\omega_{Q/G}$ is the canonical form over $T^*(Q/G)$ given by the equation

$$\omega_{Q/G}(x, \alpha)((\dot{x}_1, \dot{\alpha}_1), (\dot{x}_2, \dot{\alpha}_2)) = (dx \wedge d\alpha)((\dot{x}_1, \dot{\alpha}_1), (\dot{x}_2, \dot{\alpha}_2)) = \langle \dot{\alpha}_2, \dot{x}_1 \rangle - \langle \dot{\alpha}_1, \dot{x}_2 \rangle,$$

we only need to know an explicit expression of $\beta$ which is defined over $\widetilde{O}$ by

$$\pi^*_G \beta = d\alpha + \pi^*_2 \omega^+ \varpi.$$

(4)

We can see from [2] that if we have a connection $\mathcal{A}$ on a principal bundle $Q \rightarrow Q/G$, we are allowed to split a tangent vector $\varpi \in \widetilde{O}$ in its horizontal and vertical part, and write

$$\dot{\varpi} = [\dot{q}^h, \nu]_G + [q, \dot{\nu} + \mathcal{A}(\dot{q})\nu]_G,$$

where $\dot{q}^h$ is the horizontal lift of the tangent vector of the curve $[q(t)]_G$ at the point $q \in Q$ and $\dot{\nu}$ is a tangent vector of $\nu \in \mathcal{O}$.

Thus, we can obtain $\beta$ defined over $\widetilde{O}$ as

$$\beta(\dot{\varpi}_1, \dot{\varpi}_2) = \beta([\dot{q}^h_1, \nu]_G + [q, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu]_G, [\dot{q}^h_2, \nu]_G + [q, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu]_G) =$$

$$\beta(T \pi_G(\dot{q}^h_1, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu), T \pi_G(\dot{q}^h_2, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu)) = \pi^*_G \beta((\dot{q}^h_1, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu), (\dot{q}^h_2, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu)) =$$

$$(d\alpha + \pi^*_2 \omega^+ \varpi)((\dot{q}^h_1, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu), (\dot{q}^h_2, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu)) = d\alpha((\dot{q}^h_1, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu), (\dot{q}^h_2, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu)) +$$

$$\omega^+ \varpi((\dot{q}^h_1, \dot{\nu}_1 + \mathcal{A}(\dot{q}_1)\nu), (\dot{q}^h_2, \dot{\nu}_2 + \mathcal{A}(\dot{q}_2)\nu))$$

where $\dot{q}^h_i$ is the horizontal lift of the tangent vector of the curve $[q_i(t)]_G$ at the point $q \in Q$ for $i = 1, 2$.

Since the vertical part of $\dot{q}^h_1$ and $\dot{q}^h_2$ vanish, from [3] we have that

$$\beta(\dot{\varpi}_1, \dot{\varpi}_2) = \langle \nu, B(q)(\dot{q}^h_1, \dot{q}^h_2) \rangle + \langle \nu, [\varepsilon_1 + \mathcal{A}(\dot{q}_1), \varepsilon_2 + \mathcal{A}(\dot{q}_2)] \rangle,$$

where $-\text{ad}^*_i \nu = \dot{\nu}_i$ with $i = 1, 2$.

Finally, we can write the reduced two-form as follows:

$$\omega_{\text{red}}((\dot{x}_1, \dot{\alpha}_1, \dot{\varpi}_1), (\dot{x}_2, \dot{\alpha}_2, \dot{\varpi}_2)) = \langle \dot{\alpha}_2, \dot{x}_1 \rangle - \langle \dot{\alpha}_1, \dot{x}_2 \rangle - \langle \nu, B(q)(\dot{q}^h_1, \dot{q}^h_2) \rangle - \langle \nu, [\varepsilon_1 + \mathcal{A}(\dot{q}_1), \varepsilon_2 + \mathcal{A}(\dot{q}_2)] \rangle,$$

(6)

where $-\text{ad}^*_i \nu = \dot{\nu}_i$ with $i = 1, 2$.

Note that this expression of the reduced form is the one obtained by Cendra, Marsden, Pekarsky and Ratiu in [3] but it has been obtained here by symplectic techniques.
Particular cases

- If $Q = G$, then the fiber bundle $T^*(Q/G) \times_{Q/G} \tilde{O}$ can be identified with $O$. So it is clear that

$$\omega_{\text{red}}(-\text{ad}^*_{\varepsilon_1} \nu, -\text{ad}^*_{\varepsilon_2} \nu) = -\langle \nu, [\varepsilon_1, \varepsilon_2] \rangle.$$

- If $G$ is abelian, the space $T^*(Q/G) \times_{Q/G} \tilde{O}$ can be thought as $T^*(Q/G)$. Thus, the reduced two-form is given by

$$\omega_{\text{red}}((\dot{x}_1, \dot{\alpha}_1), (\dot{x}_2, \dot{\alpha}_2)) = \langle \dot{\alpha}_2, \dot{x}_1 \rangle - \langle \dot{\alpha}_1, \dot{x}_2 \rangle - \langle \nu, B(q)(\dot{q}_1^h, \dot{q}_2^h) \rangle.$$

Both results coincide with the expressions shown in [32].

2.3 Reduced equations

Now we shall write the equations of motion in the context of orbit reduction using the expression (6) obtained in coordinates of the reduced symplectic form.

In order to obtain these equations, we shall consider a $G$-invariant Hamiltonian $H : T^* Q \to \mathbb{R}$ and its corresponding reduced Hamiltonian $h_O$ (see theorem 2.1) that we will note $h$. If $X_h \in \mathcal{X}(T^*(Q/G) \times_{Q/G} \tilde{O})$ is the Hamiltonian vector field associated to the reduced Hamiltonian $h : T^*(Q/G) \times_{Q/G} \tilde{O} \to \mathbb{R}$, then the reduced equations are given by (see [1])

$$i_{X_h} \omega_{\text{red}} = dh.$$

We can write the Hamiltonian vector field in coordinates as $X_h = (\dot{x}, \dot{\alpha}, \dot{\nu})$. So applying the expression (6) for $\omega_{\text{red}}$ we shall obtain the following system of orbit reduced equations

$$\begin{align*}
\frac{\partial h}{\partial x} &= -\dot{\alpha} - \langle \nu, B(q)(\dot{q}^h, .) \rangle - \langle \nu, [\varepsilon + A(\dot{q}), .] \rangle \\
\frac{\partial h}{\partial \alpha} &= \dot{x} \\
\frac{\partial h}{\partial \nu} &= -\langle \nu, [\varepsilon + A(\dot{q}), .] \rangle = -\text{ad}^*_{\varepsilon + A(\dot{q})} \nu
\end{align*}$$

(7)

where $-\text{ad}^*_{\varepsilon} \nu = \dot{\nu}$.

Orbit reduced equations vs. Hamilton-Poincaré equations. With variational techniques, Cendra, Marsden, Pekarsky and Ratiu in [8] have developed a process of reduction for a Hamiltonian system and obtained the well known Hamilton-Poincaré equations which are

$$\begin{align*}
\frac{Dy}{Dt} &= -\frac{\partial h}{\partial x} - \langle \pi, \tilde{B}(\dot{x}, .) \rangle \\
\dot{x} &= \frac{\partial h}{\partial y} \\
\dot{\pi} &= \frac{\partial h}{\partial \pi} \\
\frac{D\pi}{Dt} &= \text{ad}^*_{\pi} \pi
\end{align*}$$

(8)
where $\tilde{D}$ denotes the reduced curvature of the connection.

It can be seen that systems (7) and (8) are equivalent. To do that, we may consider the canonical identification between elements in the Lie algebra and tangent vectors of the coadjoint orbit, and the fact that $\pi$ in (8) can be identified with the element $\varepsilon + A(q)$ in system (7).

In spite of system (7) and (8) are equivalent, it is clear that they are different because they hold in distinct spaces.

Since Lagrange-Poincarè equations are equivalent to Hamilton-Poincarè system (see [8]), orbit reduced equations (7) result also equivalent to Lagrange-Poincarè equations.

**Remark.** According to [4] and [16], the equations (7) can be easily adapted to the case of nonholonomic systems with horizontal symmetries taking restrictions to the constraint spaces.

**Local version.** If we consider $Q \rightarrow Q/G$ as a trivial bundle, we have that $\tilde{O}$ which is identified with $O$ is also a trivial bundle, and we can obtain the following local expression for the symplectic reduced form

$$\omega_{\text{red}}((\dot{x}_1, \dot{\alpha}_1, \dot{\nu}_1, \dot{x}_2, \dot{\alpha}_2, \dot{\nu}_2)) = \langle \dot{\alpha}_2, \dot{x}_1 \rangle - \langle \dot{\alpha}_1, \dot{x}_2 \rangle - \langle \nu, B(q)(\dot{q}_1^h, \dot{q}_2^h) \rangle - \langle \nu, [\varepsilon_1, \varepsilon_2] \rangle$$

where $\varepsilon_i \in \mathfrak{g}$ is such that $-\text{ad}_{\varepsilon_i}^* \nu = \dot{\nu}_i$, with $i = 1, 2$. Having this expression for $\omega_{\text{red}}$, we can write local orbit reduced equations for Hamiltonian systems with symmetry as follows

$$\begin{align*}
\frac{\partial h}{\partial x} &= -\dot{\alpha} - \langle \nu, B(q)(\dot{q}_1^h, \cdot) \rangle \\
\frac{\partial h}{\partial \alpha} &= \dot{x} \\
\frac{\partial h}{\partial \nu} &= -\langle \nu, [\varepsilon, \cdot] \rangle = -\text{ad}_{\varepsilon}^* \nu
\end{align*}$$

being $\varepsilon \in \mathfrak{g}$ the unique element such that $\dot{\nu} = -\text{ad}_{\varepsilon}^* \nu$.

### 2.4 Rigid body with rotors as an example

In this section, we are going to study as an illustrative example the case of a rigid body with three rotors in our scheme, with the aim of getting the orbit reduced equations for this system.

Let $Q = SO(3) \times S^1 \times S^1 \times S^1$ be the configuration space of a rigid body with three rotors and let $G = SO(3)$ be the symmetry group of the system. Let us consider the left action of an element $A \in G$ given by $A(B, \theta_1, \theta_2, \theta_3) = (AB, \theta_1, \theta_2, \theta_3)$ being $(B, \theta_1, \theta_2, \theta_3)$ an arbitrary element of $Q$. It is clear that this action is proper and free. Thus, we have that $Q \rightarrow Q/G = S^1 \times S^1 \times S^1$ is a trivial bundle which admits a trivial connection.

If $\mu \in \mathfrak{so}(3)^* \equiv \mathbb{R}^3$, then its $G$-coadjoint orbit $O$ is a sphere with ratio $\mu$. Then, the tangent space in a point $\nu \in O$ is given by its orthogonal complement. The bundle $T^*(Q/G) \times O$ can be identified with $T^*(S^1 \times S^1 \times S^1 \times S^2)$. We shall denote $(\theta_1, \dot{y}_1, \dot{\nu}_1)$ and $(\theta_2, \dot{y}_2, \dot{\nu}_2)$ two tangent vectors on a given element $(\theta, y, \nu) \in T^*(S^1 \times S^1 \times S^1 \times S^2)$. So, the reduced two-form $\omega_{\text{red}}$ can be written as follows

$$\omega_{\text{red}}((\dot{\theta}_1, \dot{y}_1, \dot{\nu}_1), (\dot{\theta}_2, \dot{y}_2, \dot{\nu}_2)) = \langle \dot{y}_2, \dot{\theta}_1 \rangle - \langle \dot{y}_1, \dot{\theta}_2 \rangle - \langle \nu, \varepsilon_1 \times \varepsilon_2 \rangle,$$
where $\epsilon_i = \dot{\nu}_i \times \nu$ for $i = 1, 2$.

The reduced hamiltonian of this system $h : T^*(S^1 \times S^1 \times S^1) \times S^2 \to \mathbb{R}$ is given by (see [8])

$$h(y, \nu) = \frac{1}{2} \sum_{r=1}^{3} \left( \frac{(\nu_r - y_r)^2}{I_r} \right) + \frac{1}{2} \sum_{r=1}^{3} \left( \frac{y_r^2}{K_r} \right)$$

where $I_r$ and $K_r$ are the tensors of inertia.

Since the considered connection is trivial, we have that the covariant are usual derivatives and then we can write the orbit reduced equations as follows

$$\begin{cases}
\dot{y}_0 = 0 \\
\dot{\theta}_r = \frac{y_r - \nu_r}{I_r} + \frac{y_r}{K_r} \\
- \sum_{r=1}^{3} \nu_r (\epsilon_0 \times \epsilon_j)_r = \sum_{r=1}^{3} \frac{\nu_r - y_r}{I_r}
\end{cases}$$

with $1 \leq r \leq 3$, $1 \leq j \leq 3$, and $\nu \in \mathcal{O} = S^2$.

These equations coincides with the ones obtained in [8] with variational techniques but we are focusing in the symplectic setting to get equations by two stages for this example.

### 3 Orbit reduced equations by two stages

In the case in which the symmetry group of a Hamiltonian system has a normal subgroup we can apply the results of section 2.2 thinking of the normal subgroup as the symmetry group of the system. So we can obtain the symplectic form in which we shall call the first reduced space and the consequent system of orbit reduced equations. After doing that we still have a remainder symmetry, given by the quotient of the stabilizer (by the big group) of a distinguished coadjoint orbit over the normal subgroup, with which a second stage of reduction process can be performed to obtain orbit reduced equations by two stages.

#### 3.1 Setting of orbit reduction by stages

In this subsection, we will recall the principal result of orbit reduction by stages and some isomorphisms between reduced spaces, which will allow us to take coordinates in order to write the reduced symplectic form by two stages.

As in section 2.1, we assume that the Lie group $G$ acts on $Q$ and the action has a canonical lifting on $(T^*Q, \omega)$ in a free, proper, and symplectic way.

We also assume that the group $G$ has a normal subgroup $N$, so $N$ also acts freely, proper and symplectically on $T^*Q$.

Defining $\nu = i^*\mu \in \mathfrak{n}^*$, where $i : \mathfrak{n} \to \mathfrak{g}$ is the inclusion of the Lie algebra $\mathfrak{n}$ of the subgroup $N$ in $\mathfrak{g}$, and considering the action of $N$ on $T^*Q$ instead of the action of $G$ in the theorem 2.1 we can think in the momentum map $J_N : T^*Q \to \mathfrak{n}^*$ and the reduced space $J_N^{-1}(\mathcal{O}_\nu)/N \subset T^*Q/N$ where $\mathcal{O}_\nu$ denotes the $N$-coadjoint orbit of $\nu$.

Following the techniques shown in [19], we shall consider $G_{\mathcal{O}_\nu} = \{ g \in G : g.\mathcal{O}_\nu \subset \mathcal{O}_\nu \}$ the stabilizer of the orbit $\mathcal{O}_\nu$ with respect to the $G$-action on $\mathfrak{n}^*$, which is a closed Lie subgroup of $G$ containing $N$ as
a normal subgroup. Then, we have that the Lie subgroup $G_{O_{\nu}}$ leaves $J^{-1}_{N}(O_{\nu})$ invariant and hence, the Lie group $H = G_{O_{\nu}}/N$ acts on $J^{-1}_{N}(O_{\nu})/N$ in a free, proper, and symplectic way. So we can consider the momentum map of the $H$-action on $J^{-1}_{N}(O_{\nu})/N$, namely $J_{O_{\nu}} : J^{-1}_{N}(O_{\nu})/N \to \mathfrak{h}^*$ where $\mathfrak{h}^*$ is the dual of the Lie algebra of the group $H$. For any element $\rho \in \mathfrak{h}^*$ we shall note $O_{\rho}$ the $H$-coadjoint orbit of $\rho$.

In this context the following theorem holds (see [9]).

**Theorem 3.1.** (Orbit Reduction by Stages Theorem) If the orbit $O_{\mu} \subset \mathfrak{g}^*$ satisfies the stages hypothesis and let $\rho \in \mathfrak{h}^*$ defined by the relation $(\rho, T_{e}r_{\mathcal{O}_{\nu}}(\xi)) = (\mu, \xi) - (\tilde{\nu}, \xi)$, for every $\xi \in \mathfrak{g}_{O_{\nu}}$ and being $r_{\mathcal{O}_{\nu}} : \mathfrak{g}_{O_{\nu}} \to \mathfrak{g}$ the inclusion, where $\mathfrak{g}_{O_{\nu}}$ denotes the Lie algebra of $G_{O_{\nu}}$ and $\tilde{\nu}$ is an arbitrary linear extension of $\nu$ from $\mathfrak{n}^*$ to $\mathfrak{g}^*$, then there is a symplectic diffeomorphism between $J^{-1}(O_{\mu})/G$ and $J^{-1}_{O_{\nu}}(O_{\rho})/H$.

If we consider the principal bundle $Q \to Q/G_{O_{\nu}}$, we have a $G_{O_{\nu}}$-invariant metric on $Q$ and then, we can define a connection $A_{G_{O_{\nu}}}$ given by this metric, see [9]. Let $A_{N}$ be the $G_{O_{\nu}}$-invariant connection on the principal bundle $\pi_{N} : Q \to Q/N$ associated to the $G_{O_{\nu}}$-invariant metric on the principal bundle $Q \to Q/G_{O_{\nu}}$, as it is also explained in [9]. As before (see (1)), we have an isomorphism $(a^{-1}_{A_{N}})^* : J^{-1}_{N}(O_{\nu})/N \to T^*(Q/N) \times_{Q/N} \bar{O}_{\nu}$ where $\bar{O}_{\nu} = (Q \times O_{\nu})/N$ is the adjoint bundle of the $N$-coadjoint orbit of $\nu$. We shall call $T^*(Q/N) \times_{Q/N} \bar{O}_{\nu}$ the first reduced space.

Considering the action of $H$ on $Q/N$, we can choose a connection $A_{H}$ on the principal bundle $Q/N \to (Q/N)/H$ such that, for all $q \in Q$ and all $v \in TQ$, we have $A_{G_{O_{\nu}}}(v) = 0$ if and only if $A_{N}(v) = 0$ and $A_{H}(T\pi_{N}(v)) = 0$ (see [9]).

Now we shall see that $J^{-1}_{O_{\nu}}(O_{\rho})/H$ can also be identified with a bundle in which we can take local coordinates to write the symplectic form by two stages.

Since $J^{-1}_{O_{\nu}}(O_{\rho}) \subseteq J^{-1}_{N}(O_{\nu})/N \simeq T^*(Q/N) \oplus \bar{O}_{\nu}$, we have that $J^{-1}_{O_{\nu}}(O_{\rho})/H \subseteq (T^*(Q/N) \oplus \bar{O}_{\nu})/H$. So we can consider the isomorphism induced by the one defined in [9] for a Lagrange-Poincaré bundle $W$ and an action of a symmetry group $S$ given by

$$\alpha_{A_{S}} := \alpha_{A_{S}} \oplus \text{id}_{V/H} : T((Q/N) \oplus V)/S \to T((Q/N)/S) \oplus \bar{s} \oplus V/S$$

where $\alpha_{A_{S}}$ is the vector bundle isomorphism $\alpha_{A_{S}} : T((Q/N)/S) \to T(Q/N)/S \oplus \bar{s}$ and $\bar{s}$ is the adjoint bundle of the Lie algebra $\mathfrak{s}$ of $S$. Thus, taking $V = \bar{O}_{\nu}$, $W = T(Q/N) \oplus \bar{O}_{\nu}$, and $S = H$, we obtain an isomorphism

$$(\alpha_{A_{H}})^{-1} : (T^*(Q/N) \oplus \bar{O}_{\nu})/H \to T^*((Q/N)/H) \oplus \bar{O}_{\rho} \oplus (\bar{O}_{\nu}/H)$$

being $\bar{O}_{\rho} = (Q/N \times O_{\rho})/H$ the adjoint bundle of the $H$-coadjoint orbit of $\rho$. We will call $T^*((Q/N)/H) \oplus \bar{O}_{\rho} \oplus (\bar{O}_{\nu}/H)$ the second reduced space and then we think the quotient $J^{-1}_{O_{\nu}}(O_{\rho})/H$ as included in the second reduced space, that is,

$$J^{-1}_{O_{\nu}}(O_{\rho})/H \subseteq T^*((Q/N)/H) \oplus \bar{O}_{\rho} \oplus (\bar{O}_{\nu}/H).$$

### 3.2 Symplectic form by two stages

Now we will obtain a coordinate version of the symplectic form by two stages, that is, the corresponding form on the second reduced space.
According to the results of the section 2.2, if we think in the symmetry of $N$ and the principal bundle $\pi_N : Q \rightarrow Q/N$, we can write the reduced two-form in the first reduced space, named $\omega_1$, as follows.

We shall consider $\nu \in \mathfrak{n}^*$ and $T^*(Q/N) \times_{Q/N} \mathcal{O}_\nu$ with $\mathcal{O}_\nu$ the $N$-coadjoint orbit of $\nu$, an element $(x, \alpha, \nu) \in T^*(Q/N) \times_{Q/N} \mathcal{O}_\nu$, i.e., $x = [q]_{N} \in Q/N$, $\alpha \in T^*_x(Q/N)$ and $\nu = [q, \nu]_{N} \in \mathcal{O}_\nu$, and two tangent vectors $(\dot{x}_1, \dot{\alpha}_1, \vec{\nu}_1)$ and $(\dot{x}_2, \dot{\alpha}_2, \vec{\nu}_2)$ on the point $(x, \alpha, \nu)$. Then, we can write a coordinate expression for the two form on the first reduced space as (see (2))

$$
\omega_1((\dot{x}_1, \dot{\alpha}_1, \vec{\nu}_1), (\dot{x}_2, \dot{\alpha}_2, \vec{\nu}_2)) = \langle \dot{\alpha}_2, \dot{x}_1 \rangle - \langle \dot{\alpha}_1, \dot{x}_2 \rangle - \langle \nu, B_N(q)(\dot{q}_1^h, \dot{q}_2^h) \rangle - \langle \nu, [\varepsilon_1 + \mathcal{A}_N(\dot{q}_1), \varepsilon_2 + \mathcal{A}_N(\dot{q}_2)] \rangle,
$$

where $B_N$ is the curvature associated with the connection $\mathcal{A}_N$ on $\pi_N : Q \rightarrow Q/N$, $\dot{q}_i^h$ is the horizontal lift of the tangent vector of the curve $[q_i(t)]_N$ in this principal bundle and $\varepsilon_i \in \mathfrak{n}$ is such that $\dot{\nu}_i = -\text{ad}^*_x \nu$, for $1 \leq i \leq 2$.

In order to write an explicit expression for the reduced form by two stages, we shall consider a point

$$
(y, \gamma, [x, \tau], [\eta]_{\mathcal{H}}) \in T^*((Q/N)/H) \times_{(Q/N)/H} \mathcal{O}_{\rho} \times \mathcal{O}_{\nu}/H,
$$

with $x \in Q/N$, $\gamma \in T^*((Q/N)/H)$, $\tau \in \mathcal{O}_{\rho} \subset \mathfrak{h}$ and $\eta \in \mathcal{O}_{\nu} \subset \mathfrak{n}^*$, and two tangent vectors $(\dot{y}, \dot{\gamma}, \dot{\tau}, [\dot{\eta}]_{\mathcal{H}})$ and $(\dot{y}_1, \dot{\gamma}_1, \dot{\tau}_1, [\dot{\eta}_1]_{\mathcal{H}})$ belonging to $T_{(y, \gamma, [x, \tau], [\eta]_{\mathcal{H}})}((Q/N)/H) \times_{(Q/N)/H} \mathcal{O}_{\rho} \times \mathcal{O}_{\nu}/H$.

We will denote $\omega_2$ the form defined over the second reduced space $T^*((Q/N)/H) \times \mathcal{O}_{\rho} \times \mathcal{O}_{\nu}/H$ as the image of the induced form $\omega_1$ on the quotient space via the isomorphism $((\alpha_{\mathcal{H}}^W)^{-1})^*$ (see (3)).

Keeping in mind that $\vec{\nu}_i \in T_{\mathcal{R}_{\rho}}$ and $\vec{\eta}_i \in T_{\mathcal{R}_{\nu}}$ for $i = 1, 2$, we can write

$$
\omega_2((y_1, \dot{\gamma}_1, \dot{\tau}_1, [\dot{\eta}_1]_{\mathcal{H}}), (y_2, \dot{\gamma}_2, \dot{\tau}_2, [\dot{\eta}_2]_{\mathcal{H}})) = 
\omega_1 
\begin{pmatrix}
(\dot{x}_1, T^*_{y_1} \pi_h(\dot{\gamma}_1) + \mathcal{A}_H(\dot{x}_1)(\dot{\tau}_1 + \mathcal{A}_H(\dot{x}_1)\tau)), [\dot{q}_1^h, \dot{\eta}_1 + \mathcal{A}_N(\dot{q}_1)]_{\mathcal{H}} \\
(\dot{x}_2, T^*_{y_2} \pi_h(\dot{\gamma}_2) + \mathcal{A}_H(\dot{x}_2)(\dot{\tau}_2 + \mathcal{A}_H(\dot{x}_2)\tau)), [\dot{q}_2^h, \dot{\eta}_2 + \mathcal{A}_N(\dot{q}_2)]_{\mathcal{H}}
\end{pmatrix}
$$

and identify this image with

$$
\omega_1 
\begin{pmatrix}
\langle \dot{x}_1, T^*_{y_1} \pi_h(\dot{\gamma}_1) + \mathcal{A}_H(\dot{x}_1)(\dot{\tau}_1 + \mathcal{A}_H(\dot{x}_1)\tau) \rangle, [\dot{q}_1^h, \dot{\eta}_1 + \mathcal{A}_N(\dot{q}_1)]_{\mathcal{H}} \\
\langle \dot{x}_2, T^*_{y_2} \pi_h(\dot{\gamma}_2) + \mathcal{A}_H(\dot{x}_2)(\dot{\tau}_2 + \mathcal{A}_H(\dot{x}_2)\tau) \rangle, [\dot{q}_2^h, \dot{\eta}_2 + \mathcal{A}_N(\dot{q}_2)]_{\mathcal{H}}
\end{pmatrix}
$$

$$
= \langle \dot{x}_1, T^*_{y_1} \pi_h(\dot{\gamma}_1) + \mathcal{A}_H(\dot{x}_1)(\dot{\tau}_1 + \mathcal{A}_H(\dot{x}_1)\tau) \rangle - \langle \dot{x}_2, T^*_{y_2} \pi_h(\dot{\gamma}_2) + \mathcal{A}_H(\dot{x}_2)(\dot{\tau}_2 + \mathcal{A}_H(\dot{x}_2)\tau) \rangle 
- \langle \dot{\gamma}_1, B_N(q)(\dot{q}_1^h, \dot{q}_2^h) \rangle - \langle \dot{\gamma}_2, B_N(q)(\dot{q}_1^h, \dot{q}_2^h) \rangle - \langle \dot{\eta}_1, [\varepsilon_1 + \mathcal{A}_N(\dot{q}_1), \varepsilon_2 + \mathcal{A}_N(\dot{q}_2)] \rangle,
$$

where $\varepsilon_i \in \mathfrak{n}$ are such that $\dot{\nu}_i = -\text{ad}^*_x \nu$, for $1 \leq i \leq 2$.

Finally, we obtain a coordinate expression for the symplectic form by two stages as follows

$$
\omega_2((y_1, \dot{\gamma}_1, \dot{\tau}_1, [\dot{\eta}_1]_{\mathcal{H}}), (y_2, \dot{\gamma}_2, \dot{\tau}_2, [\dot{\eta}_2]_{\mathcal{H}})) = \langle \dot{x}_1, T^*_{y_1} \pi_h(\dot{\gamma}_1) + \mathcal{A}_H(\dot{x}_1)(\dot{\tau}_1 + \mathcal{A}_H(\dot{x}_1)\tau) \rangle 
- \langle \dot{x}_2, T^*_{y_2} \pi_h(\dot{\gamma}_2) + \mathcal{A}_H(\dot{x}_2)(\dot{\tau}_2 + \mathcal{A}_H(\dot{x}_2)\tau) \rangle
$$

where $T^*_x \pi_h$ is a slight abuse of notation to indicate the tangent map of $T^*_x \pi_h$. Also $\mathcal{A}_H(x)^*$ denotes the tangent map of $\mathcal{A}_H(x)^*$ which is the dual of the map $\mathcal{A}_H(x) : T_x(Q/N) \rightarrow \mathfrak{h}$. 

10
3.3 Reduced equations by two stages

In this subsection, we will write orbit reduced equations in coordinates by stages with the expression of the two form obtained above.

We shall consider a given reduced Hamiltonian \( h_2 : T^*((Q/N)/H) \times \tilde{O}_\rho \times \tilde{O}_\nu / H \to \mathbb{R} \) and its associated Hamiltonian vector field \( X_{h_2} = (\dot{y}, \dot{\gamma}, \dot{\tau}, [\dot{\eta}]) \in \mathfrak{x}(T^*((Q/N)/H) \times \tilde{O}_\rho \times \tilde{O}_\nu / H) \). Then the equations of motion in the second reduced space are given by the equation \( i_{X_{h_2}} \omega_2 = dh_2 \) that becomes the system of orbit reduced equations by two stages as follows

\[
\begin{align*}
\frac{\partial h_2}{\partial y} &= -T^*_y \pi_H(\dot{\gamma}) - A_H(x)^*(\dot{\tau} + A_H(\dot{x})\tau) - \langle \eta, B_N(q)(q^b,.) \rangle \\
\frac{\partial h_2}{\partial \gamma} &= \dot{y} \\
\frac{\partial h_2}{\partial \tau} &= A_H(x)(\dot{x}) \\
\frac{\partial h_2}{\partial [\dot{\eta}]_H} &= -\langle \eta, [\varepsilon + A_N(\dot{q}),.] \rangle = -ad^*_{\varepsilon + A_N(\dot{q})}\eta
\end{align*}
\]

where \( \varepsilon \in \mathfrak{n} \) such that \( \dot{\eta} = -ad^*_{\varepsilon}\eta \).

Equivalently, taking into account that the notation \( A_H(x)^* \) denotes the derivative of the map \( A_H(x)^* \), we can write the term \( A_H(x)^*(\dot{\tau}) \) as \( \langle \dot{\tau}, B_H(x)(\dot{x},.) + [A_H(x)(\dot{x}), A_H(x)(.]_H) \rangle \) and finally obtain the system

\[
\begin{align*}
\frac{\partial h_2}{\partial y} &= -T^*_y \pi_H(\dot{\gamma}) - \langle \dot{\tau} + A_H(\dot{x})\tau, B_H(x)(\dot{x},.) \rangle - \langle \eta, B_N(q)(q^b,.) \rangle \\
\frac{\partial h_2}{\partial \gamma} &= \dot{y} \\
\frac{\partial h_2}{\partial \tau} &= A_H(x)(\dot{x}) \\
\frac{\partial h_2}{\partial [\dot{\eta}]_H} &= -\langle \eta, [\varepsilon + A_N(\dot{q}),.] \rangle = -ad^*_{\varepsilon + A_N(\dot{q})}\eta
\end{align*}
\]

where \( \varepsilon \in \mathfrak{n} \) such that \( \dot{\eta} = -ad^*_{\varepsilon}\eta \).

Let us note that in the case in which \( G_{O_{\nu}} = G \) this system could be seen as equivalent to the decomposition of Hamilton-Poincaré equations, considering the decomposition of the Lie algebra of the symmetry group in the Lie algebra of its normal subgroup and the Lie algebra of the quotient (see the decompositions of the Lie bracket and the principal connection in [5]).

**Local reduced equations by two stages.** If we again consider \( Q \to Q/G \) as a trivial bundle, we get that the coadjoint bundles of the orbits \( \tilde{O}_\nu \) and \( \tilde{O}_\rho \) are trivials, so we can take a point \((y, \gamma, \tau, [\nu]_H) \in T^*((Q/N)/H) \times G_{O_{\nu}} \times O_{\nu}/H \) and two tangent vectors, for \( i = 1, 2, \)
\((\dot{y}_1, \dot{\gamma}_1, \bar{\tau}_1, [\bar{v}_1]_H) \in T_{(\bar{y}, \dot{\gamma}, \bar{\tau}, [\bar{v}]_H)}(T^*((Q/N)/H) \times O_{\nu} \times O_{\nu}/H). \) Then we can obtain the two stages-reduced form as follows

\[\omega_2((y_1, \gamma_1, \tau_1, [v_1]_H), (y_2, \gamma_2, \tau_2, [v_2]_H)) = \langle \dot{x}_1, T^*_{\pi_H}(\gamma_2) + \mathcal{A}_H(x)^{\ast}(\tau_2) \rangle - \langle \dot{x}_2, T^*_{\pi_H}(\gamma_1) + \mathcal{A}_H(x)^{\ast}(\tau_1) \rangle - \langle \eta, B_N(q)(\dot{q}^h_2, q^h_2) \rangle - \langle \eta, [\epsilon_1, \epsilon_2] \rangle.\]

So, in this case we get the following system of local orbit reduced equations by two stages

\[
\begin{align*}
\frac{\partial h_2}{\partial y} &= -T^*_{x\pi_H}(\gamma) - \mathcal{A}_H(x)^{\ast}(\hat{\tau}) - \langle \eta, B_N(q)(\dot{q}^h, .) \rangle \\
\frac{\partial h_2}{\partial \gamma} &= \dot{y} \\
\frac{\partial h_2}{\partial \tau} &= \mathcal{A}_H(x)(\dot{x}) \\
\frac{\partial h_2}{\partial [\eta]_H} &= -\langle \eta, [\epsilon, .] \rangle = -\text{ad}_\epsilon^* \eta
\end{align*}
\]

Remembering again the notation \(\mathcal{A}_H(x)^{\ast}\) for the derivative of the map \(\mathcal{A}_H(x)^{\ast}\), we can write the system as

\[
\begin{align*}
\frac{\partial h_2}{\partial y} &= -T^*_{x\pi_H}(\gamma) - \langle \hat{\tau}, B_H(x)(\dot{x}, .) \rangle - \langle \eta, B_N(q)(\dot{q}^h, .) \rangle \\
\frac{\partial h_2}{\partial \gamma} &= \dot{y} \\
\frac{\partial h_2}{\partial \tau} &= \mathcal{A}_H(x)(\dot{x}) \\
\frac{\partial h_2}{\partial [\eta]_H} &= -\langle \eta, [\epsilon, .] \rangle = -\text{ad}_\epsilon^* \eta
\end{align*}
\]

This system of equations is in correspondence via Legendre transform to local Lagrange-Poincaré equations by two stages obtained in \([5]\) in the comparable case, that is, \(G_{O_{\nu}} = G\).

### 3.4 Rigid body with rotors revisited

In this subsection, we shall again consider the rigid body with three rotors and the same configuration space as before but a different group of symmetry, to write equations of motion in the context of orbit reduction by two stages.

Let \(Q = SO(3) \times S^1 \times S^1 \times S^1\) and \(\dot{G} = Q\) acting by the left action given by \((B, \alpha_1, \alpha_2, \alpha_3)(A, \theta_1, \theta_2, \theta_3) = (B.A, \alpha_1 + \theta_1, \alpha_2 + \theta_2, \alpha_3 + \theta_3)\). We shall consider \(N = SO(3)\) as a normal subgroup of \(G\) and the \(N\)-coadjoint orbit of an element \(\nu \in n^*\) denoted \(O_{\nu} \subset n^*\). Thus, we have that \(O_{\nu} \simeq S^2\). Then, we can write \(\bar{O}_{\nu} = (Q \times O_{\nu})/N = (SO(3) \times (S^1)^3 \times S^2)/SO(3) = (S^1)^3 \times S^2\). In this case, we can obtain that the stabilizer \(G_{O_{\nu}}\) is the group \(G\) and so \(H = G_{O_{\nu}}/N = G/N = (S^1)^3\).
If we consider an element $\rho \in O_\nu \simeq S^2$ satisfying the condition in the theorem 3.1, since the action of $H$ on $Q/N = (S^1)^3$ leaves it invariant, we have that the $H$-coadjoint orbit of $\rho$ can be identified with the element $\rho$ itself. Thus, $\tilde{O}_\rho = (G/N \times O_\rho)/H = ((S^1)^3 \times \{\rho\})/(S^1)^3 \equiv O_\rho = \{\rho\}$.

The $H$-invariant reduced Hamiltonian of the system $h_1 : T^*(S^1)^3 \times \tilde{S}^2 \to \mathbb{R}$ is given by (see 3)

$$h_1(y, \nu) = \frac{1}{2} \sum_{r=1}^{3} \frac{(\nu_r - y_r)^2}{I_r} + \frac{1}{2} \sum_{r=1}^{3} \frac{\nu_r^2}{K_r}.$$ 

We can consider the second stage of reduction of the Hamiltonian, that is, the reduction of $h_1$ by $H$ that we shall note $h_2 : \left( T^*(S^1)^3 \times \tilde{S}^2 \right) / (S^1)^3 \simeq \tilde{S}^2 \to \mathbb{R}$ given by

$$h_2(\nu) = h_1(0, \nu) = \frac{1}{2} \sum_{r=1}^{3} \frac{\nu_r^2}{I_r}.$$ 

On the other hand, we know that $T \left( T^* ((Q/N)/H) \times \tilde{O}_\rho \times \tilde{O}_\nu / H \right) = T \left( T^* ((S^1)^3 / (S^1)^3) \times \{\rho\} \times S^2 \right) \simeq TS^2$. So, $\omega_2$ is a 2-form over $S^2$. Then if $\nu \in S^2$, from formula 10 for this particular case, we have $\omega_2(\nu)(\dot{\nu}_1, \dot{\nu}_2) = -\nu_1 [\varepsilon_1, \varepsilon_2]$ where $\dot{\nu}_i = \text{ad}^*_{\varepsilon_i} \nu$, for $1 \leq i \leq 2$. In other words, $\omega_2(\nu)(\dot{\nu}_1, \dot{\nu}_2) = -\langle \nu, \varepsilon_1 \times \varepsilon_2 \rangle$ where $\varepsilon_i = \dot{\nu}_i \times \nu$, for $1 \leq i \leq 2$. Then, the orbit reduced equation by two stages for this example is given by

$$-\text{ad}^*_{\nu} \nu = \dot{\nu} = \nu \times \varepsilon,$$

or equivalently

$$\begin{align*}
\frac{\nu_1}{I_1} &= pr_1(\nu \times \varepsilon) \\
\frac{\nu_2}{I_2} &= pr_2(\nu \times \varepsilon) \\
\frac{\nu_3}{I_3} &= pr_3(\nu \times \varepsilon)
\end{align*}$$

where $pr_i$ is the projection on the $i$-th factor.

**Possible future research.** This work could naturally suggest further investigations in different directions. In the first place we can mention the study of other examples of interest in mechanics to write orbit reduced equations, process that can give additional information about the systems under consideration. Another interesting topics are the generalization of these techniques to the case of non-equivariant momentum map by considering its associated cocycle, and to the situation in which the symmetry group has a chain of normal subgroups leading to a theory of orbit reduction by several stages.

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