Sharing a Measure of Maximal Entropy in Polynomial Semigroups

Fedor Pakovich*

Department of Mathematics, Ben Gurion University of the Negev, 8410501, Israel

*Correspondence to be sent to: e-mail: pakovich@math.bgu.ac.il

Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two that are not simultaneously conjugate to monomials or to Chebyshev polynomials, and $S$ the semigroup under composition generated by $P_1, P_2, \ldots, P_k$. We show that all elements of $S$ share a measure of maximal entropy if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

1 Introduction

In the recent paper by Jiang and Zieve [10], the authors showed that a semigroup of polynomials under composition generated by two complex polynomials $P_1$ and $P_2$ of degrees $n_1 \geq 2$ and $n_2 \geq 2$ is not free if and only if it is isomorphic to the semigroup generated by $z^{n_1}$ and $\epsilon z^{n_2}$, where $\epsilon$ is a root of unity. This implies in particular that whenever $S = \langle P_1, P_2 \rangle$ is not free there exists $r > 0$ for which $P_1^{or}$ and $P_2^{or}$ commute. Since commuting polynomials can be described explicitly ([18],[19]), the last property is sufficient to classify all pairs of polynomials $P_1$ and $P_2$ for which $S = \langle P_1, P_2 \rangle$ is not free.

Combined with the description of pairs of rational functions sharing a measure of maximal entropy obtained in [11], [12], the result of [10] implies the following criterion: a semigroup $S = \langle P_1, P_2 \rangle$ generated by two polynomials $P_1$ and $P_2$ of degree at least two that are not simultaneously conjugate to monomials or to Chebyshev polynomials is...
not free if and only if all elements of $S$ share a measure of maximal entropy. The problem of characterization of semigroups of polynomials satisfying the last property has been studied in the recent papers [7], [17], where several equivalent characterizations of such semigroups in semigroup-theoretic terms were given. Among such characterizations we mention the right amenability and the absence of free subsemigroups. The result of [10] provides yet another characterization of such semigroups, in terms of freeness, if considered semigroups are generated by two polynomials.

It is not hard to see that the result of [10] and the corresponding characterization of semigroups whose elements share a measure of maximal entropy do not allow for a direct generalization to a greater number of generators. For example, for arbitrary polynomials $R$ and $X$ setting

$$P_1 = R \circ z^n, \quad P_2 = X, \quad P_3 = \varepsilon X,$$

where $\varepsilon$ satisfies $\varepsilon^n = 1$, we obtain a semigroup $S = \langle P_1, P_2, P_3 \rangle$, which is not free since

$$P_1 \circ P_2 = P_1 \circ P_3.$$

However, it is clear that in general $P_1, P_2,$ and $P_3$ do not share a measure of maximal entropy.

In this note, we provide a generalization of the result of [10] to arbitrary finitely generated semigroups of polynomials replacing the non-freeness condition by another condition, which is however equivalent to the condition that $S$ is not free if the number of generators equals two. We also provide a characterization of finitely generated semigroups of polynomials whose elements share a measure of maximal entropy.

To formulate our results explicitly, we introduce some notation. We say that two semigroups of polynomials $S_1$ and $S_2$ are conjugate if there exists $\alpha \in Aut (\mathbb{C})$ such that

$$\alpha \circ S_1 \circ \alpha^{-1} = S_2.$$

We denote by $\mathcal{Z}$ the semigroup of polynomials consisting of monomials $az^n$, where $a \in \mathbb{C}^*$ and $n \geq 1$, and by $\mathcal{T}$ the semigroup consisting of polynomials of the form $\pm T_n$, $n \geq 1$, where $T_n$ stands for the Chebyshev polynomial of degree $n$. Finally, we denote by $\mathcal{Z}^U$ the subsemigroup of $\mathcal{Z}$ consisting of polynomials of the form $\omega z^n$, where $\omega$ is a root of unity.

In this notation, our first result is following.
Theorem 1.1. Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two. Then the semigroup $S = \langle P_1, P_2, \ldots, P_k \rangle$ is isomorphic to a subsemigroup of $\mathbb{Z}^U$ if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

It is easy to see that for $k = 2$ the condition

$$SP_1 \cap SP_2 \cap \cdots \cap SP_k \neq \emptyset \quad (1)$$

is equivalent to the condition that $S$ is not free. Indeed, any semigroup of rational functions is right cancellative. Therefore, if there exist two different words in the letters $\{P_1, P_2\}$ representing the same element in $S = \langle P_1, P_2 \rangle$, then cancelling their common suffix we obtain two different words representing the same element with different ending letters. Since one of these letters is $P_1$ and the other one is $P_2$, this implies that $SP_1 \cap SP_2 \neq \emptyset$. Notice, however, that for $k > 2$ condition (1) is clearly stronger than merely the requirement that $S$ is not free.

Our second result is following.

Theorem 1.2. Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two such that $S = \langle P_1, P_2, \ldots, P_k \rangle$ is not conjugate to a subsemigroup of $\mathbb{Z}$ or $\mathcal{T}$. Then all elements of $S$ share a measure of maximal entropy if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

Notice that since in the polynomial case having the same measure of maximal entropy is equivalent to having the same Julia set, Theorem 1.2 can be viewed as a characterization of polynomials $P_1, P_2, \ldots, P_n$ sharing a Julia set via existence of a relation of the form

$$A_1 P_1 = A_2 P_2 = \cdots = A_n P_n,$$

where $A_i$, $1 \leq i \leq n$, are words in $P_1, P_2, \ldots, P_n$.

The assumption that $S$ is not conjugate to a subsemigroup of $\mathbb{Z}$ or $\mathcal{T}$ is not essential for the “if” part of Theorem 1.2, but essential for the “only if” part. Indeed, for instance, polynomials $z^n$ and $b z^m$, $b \in \mathbb{C}^*$, share a measure of maximal entropy whenever $|b| = 1$, but generate a free group, unless $b$ is a root of unity.

Finally, notice that since semigroups of polynomials whose elements share a measure of maximal entropy admit many equivalent descriptions (see [17]), Theorem 1.2 also can be formulated in many equivalent forms. In particular, under the assumptions
of Theorem 1.2, condition (1) is equivalent to the condition that there exists a polynomial $T$ of the form $T = z^r R(z^l)$, where $R \in \mathbb{C}[z]$, $l \geq 1$, and $0 \leq r < l$, such that

$$P_i = \omega_i T^{|li|}, \quad 1 \leq i \leq k,$$

for some $l_i \geq 1$, $1 \leq i \leq k$, and $l$th roots of unity $\omega_i$, $1 \leq i \leq k$.

2 Proof of Theorem 1.1

Let us recall that for every complex polynomial $P_1$ of degree $n_1 \geq 2$ there exists a series

$$\beta = \sum_{i=-1}^{\infty} c_i z^{-i}, \quad c_{-1} \neq 0,$$

called the Böttcher function, which makes the diagram

$$\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{P_1} & \mathbb{CP}^1 \\
\downarrow{\beta} & \quad & \downarrow{\beta} \\
\mathbb{CP}^1 & \xrightarrow{z^{n_1}} & \mathbb{CP}^1
\end{array}$$

commutative. Having its roots in complex dynamics (see [5], [14]), the Böttcher function is widely used for studying functional relations between polynomials and related problems (see [1–4, 8, 10, 18, 20]).

As in the paper [10], our proof of the “if” part of Theorem 1.1 uses the Böttcher function and the following lemma proved in the paper [8]. Following [8], for a non-zero element $T = b_0 + b_1 z + b_2 z^2 + \ldots$ of $\mathbb{C}[z]$, we define $\text{Ord}_0(T)$ as the minimum number $i \geq 0$ such that $b_i \neq 0$, and $l_0(T)$ as the difference $m - \text{Ord}_0(T)$, where $m$ is the minimum number greater than $\text{Ord}_0(T)$ such that $b_m \neq 0$. If $T$ is a monomial, we set $l_0(T) = \infty$. The parameter $l_0(T)$ possesses certain properties making it useful for studying functional relations between powers series (see [8], Lemma 2.6). Below we need only the properties listed in the following statement, which can be checked by a direct calculation.

**Lemma 2.1.** Let $X$ be an element of $\mathbb{C}[z]$ such that $l_0(X) < \infty$. Then for any element $T$ of $\mathbb{C}[z]$ with $l_0(T) < \infty$ the inequality

$$l_0(T \circ X) \geq \min \left( l_0(X), \text{Ord}_0(X)l_0(T) \right)$$

(2)
holds, and the equality is attained whenever $l_0(X) \neq \text{Ord}_0(X)l_0(T)$. On the other hand, if $l_0(T) = \infty$, then the equalities

$$l_0(X \circ T) = \text{Ord}_0(T)l_0(X), \tag{3}$$

$$l_0(T \circ X) = l_0(X) \tag{4}$$

hold.

Lemma 2.1 implies the following corollary.

**Corollary 2.2.** Let $T_i, 1 \leq i \leq k$, be elements of $z^2C[[z]]$ such that $l_0(T_1) = l < \infty$, and

$$l_0(T_1) \leq l_0(T_i), \quad 2 \leq i \leq k. \tag{5}$$

Assume that $A$ is a word in $T_1, T_2, \ldots T_k$, and $X$ is an element of $z^2C[[z]]$. Then $l_0(AX) = l$ if $l_0(X) = l$, and $l_0(AX) > l$ if $l_0(X) > l$.

**Proof.** In the both cases, the proof is by induction on the length of $A$. If $A$ is empty, then the corollary is trivially true. Further, in the first case, the induction step reduces to the following statement: if $X \in z^2C[[z]]$ satisfies $l_0(X) = l$, then

$$l_0(T_iX) = l, \quad 1 \leq i \leq k.$$  

In turn, the last statement follows from formulas (2), (4) taking into account that inequality (5) implies the inequality

$$\text{Ord}_0(X)l_0(T_i) \geq \text{Ord}_0(X)l_0(T_1) \geq 2l > l, \quad 1 \leq i \leq k. \tag{6}$$

Similarly, in the second case, we must prove that if $l_0(X) > l$, then

$$l_0(T_iX) > l, \quad 1 \leq i \leq k. \tag{7}$$

If $l_0(X) < \infty$, then (7) follows from (2) and (4) taking into account the inequalities $l_0(X) > l$ and (6). On the other hand, if $l_0(X) = \infty$, then either $l_0(T_i) = \infty$ and $l_0(T_iX) = \infty > l$, or
\[ l_0(T_i) < \infty \text{ and} \]
\[ l_0(T_i X) = \text{Ord}_0(X) l_0(T_i) > l \]
by (3) and (6).

We deduce Theorem 1.1 from the following result.

**Theorem 2.3.** Let \( Q_i, 1 \leq i \leq k \), be elements of \( \mathbb{Z}^2 \mathbb{C}[[z]] \), and \( R \) the semigroup generated by \( Q_1, Q_2, \ldots, Q_k \). Assume that \( Q_k \) is contained in \( \mathbb{Z}^U \). Then

\[ RQ_1 \cap RQ_2 \cap \cdots \cap RQ_k \neq \emptyset \]  \hspace{1cm} (8)

if and only if every \( Q_i, 1 \leq i \leq k - 1 \), is contained in \( \mathbb{Z}^U \).

**Proof.** Let \( Q_i = a_{i,1} z^{n_i} + a_{i,2} z^{n_i+1} + \cdots \), \( 1 \leq i \leq k \), where \( a_{i,1} \neq 0 \). Assume that (8) holds, but not all \( Q_i, 1 \leq i \leq k \), are monomials. Without loss of generality we may assume that for some \( s, 1 \leq s < k \), the series \( Q_{s+1}, \ldots, Q_k \) are monomials, while the series \( Q_1, Q_2, \ldots, Q_s \) are not, and that

\[ l_0(Q_1) \leq \cdots \leq l_0(Q_s). \]

By the condition, there exist words \( A_1, A_2, \ldots, A_k \) in \( Q_1, Q_2, \ldots, Q_k \) such that

\[ A_1 Q_1 = A_2 Q_2 = \cdots = A_k Q_k. \]  \hspace{1cm} (9)

Applying the first part of Corollary 2.2 to the word \( A_1 Q_1 \), we obtain that

\[ l_0(A_1 Q_1) = l_0(Q_1). \]

On the other hand, applying the second part of Corollary 2.2 to the word \( A_{s+1} Q_{s+1} \), we obtain that

\[ l_0(A_{s+1} Q_{s+1}) > l_0(Q_1). \]
Since $A_1Q_1 = A_{s+1}Q_{s+1}$, we obtain a contradiction, which shows that all $Q_1, Q_2, \ldots, Q_s$ are monomials. In particular, equality (9) reduces to the equality

$$A_1a_{1,1}z^{n_1} = A_2a_{2,1}z^{n_2} = \ldots = A_k a_{k,1}z^{n_k},$$

(10)

where $A_i, 1 \leq i \leq k$, are words in $a_{1,1}z^{n_1}, a_{2,1}z^{n_2}, \ldots, a_{k,1}z^{n_k}$.

Clearly, (10) implies an equality of the form

$$U_1z^N = U_2z^N = \ldots = U_kz^N,$$

where $U_i, 1 \leq i \leq k$, are monomials in $a_{1,1}, a_{2,1}, \ldots, a_{k,1}$, and $N$ is a natural number. To finish the proof of the “only if” part of the theorem it is enough to show that whenever $i \neq j, 1 \leq i, j \leq k$, the inequality

$$\deg_{a_{i,1}} U_i > \deg_{a_{i,1}} U_j$$

(11)

holds. Indeed, in this case making in the equality

$$U_1 = U_2 = \ldots = U_k$$

all possible cancellations, we obtain an equality of the form

$$a_{1,1}^{s_1} = a_{2,1}^{s_2} = \ldots = a_{k,1}^{s_k},$$

where $s_i \geq 1, 1 \leq i \leq k$, implying that all $a_{i,1}, 1 \leq i \leq k - 1$, are roots of unity.

It is clear that the minimum value of $\deg_{a_{i,1}} U_i$ is attained if the word $A_i$ contains no letter $a_{i,1}z^{n_i}$ at all, implying that

$$\deg_{a_{i,1}} U_i \geq \frac{N}{n_i}.$$ 

Thus, to prove (11) it is enough to show that

$$\deg_{a_{i,1}} U_j < \frac{N}{n_i}.$$

Let $r$ be the number of appearances of $a_{i,1}z^{n_i}$ in $A_j$. It is easy to see that the maximum value of $\deg_{a_{i,1}} U_j$ is attained if these appearances occur in the last $r$ letters of $A_j$,
implying that
\[ \deg_{a_{i,1}} U_j \leq \frac{N}{n_j n_i} + \frac{N}{n_j n_i^2} + \frac{N}{n_j n_i^3} + \cdots + \frac{N}{n_j n_i^r}. \]

Taking into account that \( n_i \geq 2, n_j \geq 2 \), this implies that
\[ \deg_{a_{i,1}} U_j < \frac{N}{n_j n_i} \sum_{l=0}^{\infty} \frac{1}{n_i^l} \leq \frac{N}{2n_i} 1 - \frac{1}{n_i} \leq \frac{N}{n_i}. \]

Let us assume now that \( Q_i, 1 \leq i \leq k \), are contained in \( \mathbb{Z}^U \), and show that then (8) holds. Let \( l \geq 1 \) be a number such that all \( a_{i,1}, 1 \leq i \leq k \), are \( l \)th roots of unity. Setting
\[ F_1 = Q_1 \circ Q_2 \circ \cdots \circ Q_k, \quad F_2 = Q_2 \circ Q_3 \circ \cdots \circ Q_1, \quad \ldots \quad F_k = Q_k \circ Q_1 \circ \cdots \circ Q_{k-1} \]
and observing that
\[ \deg F_1 = \deg F_2 = \cdots = \deg F_k, \]
we see that for every \( j \geq 1 \) there exists an \( l \)th root of unity \( \omega_j \) such that
\[ F_1^{\circ j} = \omega_j F_2^{\circ j}. \]

The pigeonhole principle yields that there exists an infinite subset \( K_1 \) of \( \mathbb{N} \) and an \( l \)th root of unity \( \delta_1 \) such that for every \( j \in K_1 \) the equality
\[ F_1^{\circ j} = \delta_1 F_2^{\circ j} \]
holds, implying that for every \( j_1, j_2 \in K_1 \) with \( j_2 > j_1 \) the equality
\[ F_1^{\circ j_2} = F_1^{\circ j_1} \circ F_2^{\circ (j_2-j_1)} \]
holds. Similarly, there exists an infinite subset \( K_2 \) of \( K_1 \) and an \( l \)th root of unity \( \delta_2 \) such that for every \( j \in K_2 \) the equality
\[ F_1^{\circ j} = \delta_2 F_3^{\circ j} \]
holds, and for every \( j_1, j_2 \in K_2 \) with \( j_2 > j_1 \) the equality

\[
F_{j_2}^{j_1} = F_{j_1}^{j_1} \circ F_{3}^{(j_2-j_1)}
\]

holds. Continuing in the same way, we will find natural numbers \( j_2 \) and \( j_1 \) such that \( j_2 > j_1 \) and

\[
F_{j_2}^{j_1} = F_{j_1}^{j_1} \circ F_{2}^{(j_2-j_1)} = F_{j_1}^{j_1} \circ F_{3}^{(j_2-j_1)} = \ldots = F_{j_1}^{j_1} \circ F_{k}^{(j_2-j_1)}.
\]

Thus,

\[
F_{j_2}^{j_1} \in RQ_1 \cap RQ_2 \cap \cdots \cap RQ_k,
\]

implying (8).

**Proof of Theorem 1.1.** Let \( P_1, P_2, \ldots, P_k \) be polynomials of degree at least two. Then the Böttcher function \( \beta \) for \( P_k \) provides an isomorphism \( \psi \) between the semigroup \( S = \langle P_1, P_2, \ldots, P_k \rangle \) and the semigroup of power series \( R \) generated by the power series \( z^{\deg P_k} \) and

\[
Q_i = \beta \circ P_i \circ \beta^{-1}, \quad 1 \leq i \leq k - 1.
\]

Therefore, if (1) holds, then (8) also holds implying by Theorem 2.3 that \( S \) is isomorphic to a subsemigroup of \( \mathbb{Z}^U \). In the other direction, if \( S \) is isomorphic to a subsemigroup of \( \mathbb{Z}^U \), then for the images \( Q_1, Q_2, \ldots, Q_k \) of \( P_1, P_2, \ldots, P_k \) under this isomorphism condition (8) holds by Theorem 2.3. Therefore, for \( P_1, P_2, \ldots, P_k \) condition (1) holds.

2.1 Proof of Theorem 1.2

Let us recall that if \( f \) is a rational function of degree \( n \geq 2 \), then by the results of Freire, Lopes, Mañé ([9]) and Lyubich ([13]) there exists a unique probability measure \( \mu_f \) on \( \mathbb{C}P^1 \), which is invariant under \( f \), has support equal to the Julia set \( J(f) \), and achieves maximal entropy \( \log n \) among all \( f \)-invariant probability measures. It is clear that the equality \( \mu_f = \mu_g \) implies the equality of the Julia sets \( J(f) = J(g) \). Moreover, for polynomials these conditions are equivalent. The problem of describing rational functions sharing a measure of maximal entropy and the problem of describing rational functions sharing a Julia set have been studied in [1–4, 11, 12, 15, 16, 20, 21].
For any rational functions $f$ and $g$ the equality

$$f^{o j_1} = f^{o j_2} \circ g^{os}$$  \hspace{1cm} (13)

for some $j_1, s \geq 1$ and $j_2 \geq 0$ implies that $f$ and $g$ share a measure of maximal entropy. Furthermore, the results of the papers [11] and [12] imply that if the functions $f$ and $g$ are neither Lattès maps nor conjugate to $z^{\pm n}$ or $\pm T_n$, then the equality $\mu_f = \mu_g$ holds if and only if equality (13) holds (see [16], [21] for more detail).

Proof of Theorem 1.2. In view of the isomorphism between semigroups $S$ and $R$, the proof of the “if” part of Theorem 2.3 shows that if (1) holds, then the polynomials

$$G_1 = P_1 \circ P_2 \circ \cdots \circ P_k, \quad G_2 = P_2 \circ P_3 \circ \cdots \circ P_1, \quad \ldots \quad G_k = P_k \circ P_1 \circ \cdots \circ P_{k-1}$$

along with the series $F_i$, $1 \leq i \leq k$, satisfy relations (12), implying that

$$J(G_1) = J(G_2) = \cdots = J(G_k).$$  \hspace{1cm} (14)

On the other hand, since the semiconjugacy relation

$$\mathbb{CP}^1 \xrightarrow{B} \mathbb{CP}^1 \xrightarrow{X} \mathbb{CP}^1 \xrightarrow{A} \mathbb{CP}^1$$

between rational functions of degree at least two implies that

$$X^{-1}(J(A)) = J(B)$$

(see e.g. [6], Lemma 5), the semiconjugacies

$$\mathbb{CP}^1 \xrightarrow{G_1} \mathbb{CP}^1, \quad \mathbb{CP}^1 \xrightarrow{G_{i+1}} \mathbb{CP}^1, \quad \mathbb{CP}^1 \xrightarrow{G_i} \mathbb{CP}^1.$$
1 \leq i \leq k - 1, imply that

\[ P_i^{-1}(J(G_i)) = J(G_{i+1}), \quad 1 \leq i \leq k - 1, \quad P_k^{-1}(J(G_k)) = J(G_1). \]

Thus, equality (14) implies that \( J(G_1) \) is a completely invariant set for \( P_i, 1 \leq i \leq k \). In turn, this implies that

\[ J(P_1) = J(P_2) = \cdots = J(P_k) = J(G_1) \]

(see [3], Lemma 8, or [15], Theorem 4). This proves the “if” part.

Finally, to prove the “only if” part, we observe that if all elements of \( S \) share a measure of maximal entropy, then by (13) for every \( i, 2 \leq i \leq k \), there exist \( t_i, s_i \geq 1 \) and \( r_i \geq 0 \) such that

\[ P_{t_i}^{r_i} = P_{t_i}^{s_i} \circ P_i, \quad 2 \leq i \leq k. \]

Therefore, for \( K = t_2 \ldots t_k \), we have:

\[ P_1^K = (P_{t_2}^{r_2} \circ P_{t_3}^{s_2})^{K/t_2} = (P_{t_3}^{r_3} \circ P_{t_4}^{s_3})^{K/t_3} = \cdots = (P_{t_k}^{r_k} \circ P_{t_1}^{s_k})^{K/t_k}, \]

implying (1). ■

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