Early universe cosmological solutions in exponential gravity

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May 18, 2018

Abstract

In this work we present some cosmologically relevant solutions using the spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime in metric $f(R)$ gravity where the form of the gravitational Lagrangian is given by $\frac{1}{2} e^{\alpha R}$. In the low curvature limit this theory reduces to ordinary Einstein-Hilbert Lagrangian together with a cosmological constant term. Precisely because of this cosmological constant term this theory of gravity is able to support nonsingular bouncing solutions in both matter and vacuum background. Since for this theory of gravity $f'$ and $f''$ is always positive, this is free of both ghost instability and tachyonic instability. Moreover, because of the existence of the cosmological constant term, this gravity theory also admits a de-Sitter solution. Lastly we hint towards the possibility of a new type of cosmological solution that is possible only in higher derivative theories of gravity like this one.

1 Introduction

Investigation of non-singular bouncing cosmological solutions to Einstein field equations has a history that dates back to first half of the twentieth century, and can be attributed to various works of Lemaitre, Tolman, Friedmann and even Einstein himself(see, e.g. ref. [1] for early histories of bouncing cosmology). But it was then largely considered just as an alternative solution to Einstein equations and did not have any physical motivation as such. The most accepted paradigm about the universe evolution then was the big-bang paradigm, which is based upon the existence of a curvature singularity in the past. However, big-bang singularity was plagued with several other problems. Inflationary scenario came into picture in the later half of the nineteenth century as a very promising candidate to solve these problems. This paradigm was pioneered by the works of Guth([2]) and Linde([3]). There are many models in the literature so far that realize an inflationary scenario(see [4] for a comprehensive review of all the inflationary models).

However successful the inflationary paradigm be to solve various issues, this is still plagued with the issue of singularity[5] which, by definition, is a state of physical lawlessness. When we try to describe our universe with the available physical theories, we usually do not want a singularity to come into the picture. This was the physical motivation which refuelled the interest in nonsingular bouncing scenarios. There are various ways to realize a bouncing scenario(see [6] or[7] for a comprehensive review).

If one tries to realize a bouncing solution for spatially flat FLRW metric in general relativity (GR), one needs to invoke null energy condition (NEC) violating matter components like ghost fields, ghost condensates or Galileons. If one does not wish to invoke such exotic matter components and still wants to realize a bouncing solution in spatially flat FLRW metric, then he/she has to resort to modified gravity. Modifications to general relativity at high curvature regime near a curvature singularity can indeed be expected. When quantum corrections or string theory motivated effects are taken into account, then the effective low energy gravitational action indeed admits higher order curvature...
invariant terms (8,9). The simplest of such modifications is when the correction terms depend only on the Ricci scalar $R$. In such cases the Einstein-Hilbert Lagrangian $R$ is modified to $f(R)$, a function of $R$ (see 10 and 12 for beautiful reviews on $f(R)$ gravity).

There has been many attempts to realize bouncing scenario in $f(R)$ theories of gravity. It was first pointed out in Ref. 13 that $R + \alpha R^2$ gravity with a negative $\alpha$ can give rise to a bouncing scenario in spatially flat FLRW metric. Bouncing cosmology for quadratic and cubic polynomial $f(R)$ gravity theory was more recently worked out in 14 and 15 respectively, from both Jordan frame and Einstein frame point of view. It was shown in 14 that for spatially flat FLRW metric a bounce in the Jordan frame is never accompanied by a bounce in the Einstein frame when hydrodynamic matter in the Jordan frame satisfies the condition $\rho + P \geq 0$. Here $\rho$ and $P$ specifies the energy density and pressure of matter in the Jordan frame. Working solely in the Jordan frame, it was shown in 16 that quadratic gravity theories of the form $\lambda + R + \alpha R^2$ with a negative $\alpha$ and monomial gravity theories of the form $R^{1+\delta}$ can also produce bouncing cosmologies. Carloni et. al. in 17 presented the bouncing conditions in $f(R)$ gravity and also analyzed the conditions for $R^{1+\delta}$, $R + \alpha R^m$ and $\exp \alpha R$ type of gravity.

Two necessary conditions for the physical viability of any $f(R)$ theory are $f'(R) > 0$ and $f''(R) > 0$, violation of which leads to ghost instability and tachyonic instability respectively. As we will see in a later section, all the other $f(R)$ theories except $e^{\alpha R}(\alpha > 0)$ that has commonly been considered in the literature so far to realize a bounce in spatially flat FLRW metric can not have $f'(R) > 0$ and $f''(R) > 0$ simultaneously for all $R$. So exponential gravity may be the only physically viable candidate for achieving a bounce. Carloni et. al. in 17 concludes that this gravity theory can give rise to a bouncing solution only in a closed FLRW universe, but as we will show later on, this theory can also produce a bounce in the flat FLRW universe.

In this paper we present a new solution of $f(R)$ gravity theories which admits a cosmological bounce. The new solution is related to more degrees of freedom of $f(R)$ theories. Unlike GR, $f(R)$ theories depend on the second time derivative of the Hubble parameter and one can tune the cosmological development of a model by specifying various values of $\dot{H}$ at some specific time. These new solutions can produce interesting new model universes. We present a new solution in bouncing $f(R)$ theories where the universe transits from an decelerated expansion phase to a normal expansion phase vis a contraction phase. The new result presented in the present paper is general and we have given some specific examples using exponential gravity as an example.

The material in this paper is presented in the following way. In the next section we present the formalism of $f(R)$ theory in the two conformal frames, the Jordan frame and the Einstein frame. The relation between the frames is presented in the this sections. The bouncing scenario is described in section 3. Section 4 specifies bounce in exponential gravity. In this section we present the numerical results depicting various kinds of bounces. Two exact solutions of exponential gravity is presented in section 5. We discuss the new solutions in $f(R)$ gravity theories in section 6. The next section concludes the article by summarizing the results obtained.

2 The cosmological set up in the two frame

In this section we present the formal structure of the theory we will pursue in this article. The field equations in the two conformal frames and the formulæ connecting the frames are briefly specified in this section.

2.1 The Jordan frame

In the Jordan frame the field equations for $f(R)$ gravity is given in the tensorial form as follows,

$$G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{\kappa T_{\mu \nu}}{f'(R)} + g_{\mu \nu} \left[ \frac{f'(R) - R f'(R)}{2 f'(R)} \right]$$

$$+ \frac{\nabla_\mu \nabla_\nu f'(R) - g_{\mu \nu} \Box f'(R)}{f'(R)} .$$

(1)
For the flat FLRW metric given by
\[ ds^2 = -dt^2 + a^2(t)d\bar{x}^2, \] (2)
and for a perfect fluid given by the energy-momentum tensor
\[ T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \] (3)
the modified Friedmann equations for \( f(R) \) gravity are:
\[ 3H^2 = \frac{\kappa}{f'(R)}(\rho + \rho_{\text{eff}}), \] (4)
\[ 3H^2 + 2\dot{H} = \frac{-\kappa}{f(R)}(P + P_{\text{eff}}), \] (5)
where \( \rho_{\text{eff}} \) and \( P_{\text{eff}} \) are related to energy density and pressure arising from curvature. They are defined by the following expressions,
\[ \rho_{\text{eff}} = \frac{Rf' - f}{2\kappa} - \frac{3H\dot{R} f''(R)}{\kappa}, \] (6)
\[ P_{\text{eff}} = \frac{\dot{R}^2 f''' + 2H\dot{R} f'' + \ddot{R} f''}{\kappa} - \frac{Rf' - f}{2\kappa}. \] (7)
For a barotropic fluid,
\[ P = \omega \rho, \] (8)
where \( \omega = 0 \) corresponds to dust and \( \omega = \frac{1}{3} \) corresponds to radiation. It must be noted that the 4-velocity \( u_\mu \) in Eq. (3) is the normalized 4-velocity of a fluid element. The other relevant dynamical equation is the continuity equation for the hydrodynamic matter component
\[ \dot{\rho} + 3H\rho(1 + \omega) = 0. \] (9)

### 2.2 The Einstein frame
The equivalent Einstein frame description of \( f(R) \) gravity is defined in terms of a conformally related metric
\[ \tilde{g}_{\mu\nu} = F(R)g_{\mu\nu}, \] (10)
where
\[ F(R) \equiv \frac{df(R)}{dR}. \]
Let us define in Einstein frame a scalar field \( \phi \) and the potential \( V(\phi) \) as
\[ \phi \equiv \sqrt{\frac{3}{2\kappa}} \ln F, \quad V(\phi) = \frac{RF - f}{2\kappa F^2}. \] (11)
Gravitational dynamics in Einstein frame is affected by the dynamics of the scalar field which comes into existence in the Einstein frame due to the conformal transformation. Under the conformal transformation in Eq. (10), the energy-momentum tensor transforms as
\[ \tilde{T}_\mu^\nu = \frac{T_\mu^\nu}{F^2}. \] (12)
Specifically, we have the relations
\[ \tilde{\rho} = \frac{\rho}{F^2}, \quad \tilde{P} = \frac{P}{F^2}, \quad \tilde{u}_\mu \equiv \sqrt{F} u_\mu. \] (13)
In the Jordan frame there is only one hydrodynamic matter component, where as in the Einstein frame there is also a scalar field defined above, which has a potential given by Eq. (11), which couples non-minimally with the hydrodynamic
matter component. Since in the Einstein frame the gravitational theory is GR, we can write the field equations in the tensorial form as

\[ \tilde{R}^\mu_\nu - \frac{1}{2} \delta^\mu_\nu \tilde{R} = \kappa \left( \tilde{T}^\mu_\nu + \tilde{T}^\nu_\mu \right) . \tag{14} \]

where \( \tilde{T}^\mu_\nu \) is the energy-momentum tensor of the hydrodynamic fluid in the Einstein frame

\[ \tilde{T}^\mu_\nu = (\tilde{\rho} + \tilde{P}) \tilde{u}_\mu \tilde{u}_\nu + \tilde{P} \tilde{g}^\mu_\nu , \tag{15} \]

and \( \tilde{T}^\nu_\mu \) is the energy-momentum tensor of the scalar field \( \phi \). Here \( \tilde{R}^\mu_\nu \) is the corresponding Ricci tensor in the Einstein frame. We can recast the conformally related Einstein frame metric \( \tilde{g}^\mu_\nu \) as an FLRW metric

\[ d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 d\bar{x}^2 , \tag{16} \]

by using the redefinitions

\[ d\tilde{t} = \sqrt{F(\tilde{R})} dt , \quad \tilde{a}(t) = \sqrt{F(\tilde{R})} a(t) . \]

The Hubble parameters of the Jordan frame metric and the Einstein frame metric are now related by the equation

\[ H = \sqrt{F} \left( \tilde{H} - \frac{\kappa}{6} \phi' \right) , \tag{17} \]

where \( \tilde{H} = \frac{\tilde{a}'(\tilde{t})}{\tilde{a}(\tilde{t})} \), prime now stands for \( d/d\tilde{t} \). We can now write the Friedmann equations in the Einstein frame

\[ \tilde{H}^2 = \frac{\kappa}{3} (\rho_\phi + \tilde{\rho}) , \tag{18} \]

\[ \tilde{H}' = -\frac{\kappa}{2} [\phi'^2 + (1 + \omega)\tilde{\rho}] , \tag{19} \]

where

\[ \tilde{\rho} = \frac{1}{2} \phi'^2 + V(\phi) . \tag{20} \]

The other relevant dynamical equations in Einstein frame are the Klein-Gordon equation for the scalar field

\[ \phi'' + 3\tilde{H}\phi' + \frac{dV}{d\phi} = \sqrt{\frac{\kappa}{6}} (1 - 3\omega)\tilde{\rho} , \tag{21} \]

and the continuity equation for the hydrodynamic matter component

\[ \tilde{\rho}' + \sqrt{\frac{\kappa}{6}} (1 - 3\omega)\tilde{\rho}\phi' + 3\tilde{H}\tilde{\rho}(1 + \omega) = 0 . \tag{22} \]

It is seen from the above equations that the scalar field and hydrodynamic matter components in the Einstein frame satisfy coupled differential equations.

3 Description of a bouncing scenario

In this section we describe the scenario of a cosmological bounce in Jordan frame from the point of view of both the frames. A cosmological bounce for the homogeneous and isotropic FLRW universe is defined mathematically by the conditions

\[ H_b = 0 , \quad \text{and} \quad \dot{H}_b > 0 . \tag{23} \]

where the subscript \( b \) on a time dependent quantity denotes the value of the quantity at the time of the bounce. Using Eq. \([4]\), the first bouncing condition in the Jordan frame becomes

\[ \rho_b + \frac{R_b f_b^\prime - f_b}{2\kappa} = 0 . \tag{24} \]
Throughout the article we will assume that the matter component satisfies the conditions
\[ \rho \geq 0, \quad \rho + P \geq 0, \]
in the Jordan frame. If matter satisfies the above condition then the second bouncing condition in the Jordan frame becomes
\[ \dot{R}_b^2 f'' + \ddot{R}_b f'_b < 0. \] (26)

Solving the modified dynamical equations, Eq. (5) and Eq. (9), in the Jordan frame, while keeping in mind the constraint, Eq. (4), requires three conditions \( H(t = 0), \dot{H}(t = 0) \) and \( \ddot{H}(t = 0) \). If we choose \( t = 0 \) to be the bouncing moment, then the bouncing conditions require
\[ H(0) = 0, \quad \dot{H}(0) > 0. \] (27)

However, \( \ddot{H}(0) \) can be positive, negative or zero. If \( \ddot{H}(0) = 0 \), then we have a completely symmetric bounce, i.e., the contracting phase of the bounce is a mirror image of the expanding phase of the bounce. If \( \ddot{H}(0) > 0 \) (\( \ddot{H}(0) < 0 \)) then the evolution of the scale factor is steeper in the expansion (contraction) phase.

Let us now try to visualize how a Jordan frame bounce looks like from the Einstein frame. Any cosmological evolution in the Einstein frame is governed by the nature of the potential \( V(\phi) \) and how the scalar field moves on the potential. From the definition of the scalar field \( \phi \), we see that
\[ \phi_b = \sqrt{\frac{3}{2\kappa}} \ln f'(R_b) = \sqrt{\frac{3}{2\kappa}} \ln f'(6\dot{H}_b), \]
and
\[ \frac{d\phi}{dt}_b = \sqrt{\frac{3}{2\kappa}} \frac{\dot{R}_b f''(R_b)}{f^{3/2}(R_b)} = \sqrt{\frac{3}{2\kappa}} \frac{\dot{H}_b f''(6\dot{H}_b)}{f^{3/2}(6\dot{H}_b)}. \] (29)

Therefore the Jordan frame bouncing conditions applied on \( \dot{H}_b \) and \( \ddot{H}_b \) determines the Einstein frame values of \( \phi_b \) and \( \phi'_b \) at \( \dot{t} = 0 \). Note that if the viability conditions \( f' > 0, f'' > 0 \) of an \( f(R) \) gravity is respected, then the sign of \( \dot{H}_b \) determines the sign of \( \phi'_b \), which is in turn related to the time symmetrical or asymmetrical nature of the bounce. The \( H_b = 0 \) condition, from Eq. (17) implies
\[ \dot{H}_b = \frac{\kappa}{6} \phi'_b. \] (30)

Before we close this section we will like to present a short discussion on the difficulties of attaining a bouncing solution in \( f(R) \) theory of gravity. One may work with \( f(R) = AR^{1+\delta} \) where \( A \) and \( \delta \) are constants. For stability \( f' = A(1+\delta)R^{\delta} > 0 \) and \( f'' = A\delta(1+\delta)R^{\delta-1} > 0 \). From the bouncing condition in the Jordan frame one can easily check that for a successful bounce one requires \( A\delta \leq 0 \) if we assume that \( \rho \geq 0 \) for the hydrodynamic fluid. These conditions make \( f'' < 0 \) near the bounce where we know that \( R_b > 0 \) and consequently \( f(R) = AR^{1+\delta} \) cannot describe a gravitationally stable bounce. In the quadratic model we have \( f(R) = \lambda + R + \alpha R^2 \) where \( \lambda, \alpha \) are constants and for a bounce \( \alpha < 0 \). In this case it is obvious that \( f'' < 0 \) throughout the bounce making the theory unstable. Even cubic gravity models, where \( f(R) = R + \alpha R^2 + \beta R^3 \) where \( \alpha, \beta \) are constants and for bounce \( \alpha < 0 \) and \( \beta > 0 \), can accommodate cosmological bounces. One can tune the parameters in such a way that \( f' > 0 \) for all values of \( R \) [18] by fixing the values of the parameters, but then it is seen that there are two branches corresponding to positive and negative values of \( f'' \) [15]. The unstable branch remains a reality in such theories and the theory becomes more complex as there appears a point where \( f'' = 0 \) which is a singular point in \( f(R) \) gravity. In this way one can go on to show that even \( f(R) = R + \alpha R^m \) where \( \alpha \) and \( m \) are real constants and \( m \) is an integer greater than one gives rise to an unstable bounce. It was shown in Ref. [15] that no polynomial \( f(R) \) gravity can simultaneously void both ghost and tachyonic instability for all \( R \). All these examples show that it is very difficult to find a \( f(R) \) which gives rise to a stable cosmological bounce. In this regard we show that the exponential \( f(R) \) gravity theory, as chosen in the present article, can produce perfectly stable cosmological bounces. Our model is explained in the next section.

\[ ^2 \text{We are not calling this condition as the weak energy condition as the energy conditions are generally stated in the Einstein frame.} \]

\[ ^3 \text{We want to remind the reader at this point that a prime on } f \text{ implies a derivative with respect to } R \text{ where as a prime on } \phi \text{ implies a derivative with respect to } \dot{t}. \]
4 Bounce in exponential gravity

In this section we will study cosmological bounce in $f(R)$ gravity where the form of $f(R)$ is given by

$$f(R) = \frac{1}{\alpha} \exp(\alpha R)$$  \hspace{1cm} (31)

where $\alpha > 0$. Note that for this gravity theory

$$f'(R) = e^{\alpha R} > 0, \text{ and } f''(R) = \alpha e^{\alpha R} > 0,$$

implying that this theory is free from both ghost instability and tachyonic instability for all values of $R$. Moreover,

$$Rf' - f = \left( R - \frac{1}{\alpha} \right) e^{\alpha R},$$  \hspace{1cm} (33)

so we can see from the bouncing condition in Eq. (24) that in this case both matter and matter less bounce is possible. For bounce in presence of matter,

$$R_b < \frac{1}{\alpha},$$  \hspace{1cm} (34)

whereas for a matter less bounce

$$R_b = \frac{1}{\alpha}.$$  \hspace{1cm} (35)

The cosmological constant term plays a crucial role in producing the bounce If we want to remove the cosmological constant term by taking a theory like $f(R) = (1/\alpha)\exp(\alpha R) - 1$ with positive $\alpha$, then we have

$$Rf' - f = \frac{1}{\alpha}[(\alpha R - 1)e^{\alpha R} + 1],$$  \hspace{1cm} (36)

which is always positive for any positive value of $R$. This implies that this theory of gravity cannot support a cosmological bounce in either matter or vacuum background. It is precisely the cosmological constant term that helps to achieve a bounce.

The Einstein frame scalar field for exponential gravity is

$$\phi = \sqrt{\frac{3}{2\kappa}} \ln f' = \sqrt{\frac{3}{2\kappa}} \alpha R,$$  \hspace{1cm} (37)
Throughout the article, the potential crosses the $0 < R \phi$ values of the second time derivative of the Hubble parameter in the Jordan frame. In all of the bounces we have that if we want to impose the Einstein frame intermediate conditions at the time of bounce in the Jordan frame for a bounce in vacuum background in exponential gravity found with the initial conditions $\phi(0) = 0, \dot{\phi}(0) = 0, \ddot{H}(0) = \sqrt{\kappa/6\phi'}(0)$. The shape of the above potential is shown in Fig. 1. The value of $\alpha = 10^{12}$ in our system of units where all the dimensional parameters are ultimately expressed in units of the Planck mass $M_P$. For convenience we assume $M_P = 1$ throughout the article. The potential crosses the $V(\phi)$ axis at $\phi = 0$ and the $\phi$ axis at $\phi = \sqrt{3/2\kappa}$. As the gravitational part of the theory, describing the bounce, becomes essentially GR in the Einstein frame, it is easier and interesting to start from the Einstein frame description and track the bounce as the movement of the scalar field on the scalar potential. Firstly, note that the values $R = 0$ and $R = 1/\alpha$ in the Jordan frame corresponds to the values $\phi = 0$ and $\phi = \sqrt{3/(2\kappa)}$ in the Einstein frame. Since $R_b$ has a limited range for bounce in presence of matter where the range is $0 < R_b < 1/\alpha$ in the Jordan frame, $\phi_b$ is limited in the range $0 < \phi_b < \sqrt{3/(2\kappa)}$ in the Einstein frame. This means that if we want to impose the Einstein frame intermediate conditions at the time of bounce in the Jordan frame ($t = \tilde{t} = 0$), then the scalar field in the Einstein frame can start from only a very small part of the curve in Fig. 1 which is in the fourth quadrant.

We may recall from our discussion of the previous section that the sign of $\phi''$, i.e., whether the field is rolling up or down the potential when the bounce has happened is related to whether the time evolution of the scale factor is completely symmetric or steeper on one side and flatter on the other. The dependence of the nature of the time evolution on the initial conditions at bounce is elaborated by Figs. 2, 3. In both the figures we show a symmetric bounce, where the intermediate conditions are specified on the figure captions. As exponential gravity can accommodate bounces in presence of hydrodynamic matter and vacuum the two figures show two different kind of universes, one filled with radiation and the other devoid of matter. For symmetric bounces we see that $\phi'(0) = 0$ which translates to $\ddot{H}(0) = 0$ in the Jordan frame. For purely symmetrical bounces one must have $\ddot{H}(0) = H(0) = 0$ and $\ddot{H}(0) > 0$ in the Jordan frame. The last statement is true for all kinds of bounces in metric $f(R)$ theory.

One can also model asymmetric bounces in $f(R)$ theories. In our case we show two asymmetric bounces in Fig. 4 and Fig. 5. The bounce depicted in Fig. 4 is assisted by radiation where as the other bounce takes place in vacuum. In the case of bounce in presence of radiation one can easily calculate from the intermediate conditions that $\ddot{H}(0) = (2/(3\epsilon^2)) \times 10^{-19}$ and for vacuum bounce one gets $\ddot{H}(0) = (2\sqrt{\epsilon}/3) \times 10^{-19}$. In these cases one has non-zero values of the second time derivative of the Hubble parameter in the Jordan frame. In all of the bounces we have discussed in this section $\phi$ in the Einstein frame remains positive in sign which implies that $R > 0$ for all the bounces.

\[ V(\phi) = \frac{1}{2\kappa\alpha} \left( \sqrt{\frac{2\kappa}{3}} \phi - 1 \right) e^{-\sqrt{2\kappa/3}\phi}. \] (38)

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in the Jordan frame. In the effective theory approach exponential gravity becomes similar to GR near small $R$ values when one can neglect the higher powers (starting from the quadratic one) of $R$ in the exponential $f(R)$. The theory presented in this paper is consistent when we use the theory for positive non-zero values of $R$ in the Jordan frame. During the end stages of the bouncing scenario, in all the cases, $R$ tends to zero by which one is very near the GR limit. In the bouncing scenarios presented in this article we do not specify how the effective $f(R)$ theory can transform to GR near $R \sim 0$, we hope some new physics is involved during this phase.

On the other hand if we do not take the $f(R)$ theory presented in our article as some form of effective theory, which can change cosmology only for high values of the Ricci scalar, then something interesting happens. If the bounce is not due to some effective change in the gravitational part of the Lagrangian, and the $f(R)$ modifications are valid for all values of $R$ including $R = 0$ then there are new possible dynamical configurations of the universe. These new configurations depend upon the intermediate conditions applied to produce the dynamics of the bounce.

In the next section we will discuss some interesting exact solutions in exponential gravity. The solutions plotted in this section are obtained numerically, rarely in $f(R)$ gravity theory we have the privilege of having exact solutions \[19,20\]. The solutions given in the next section do not define the form of $f(R)$, the solutions are exact solutions of exponential gravity.

5 Two exact solutions in exponential gravity

In this section we show that $(1/\alpha)e^{\alpha R}$ gravity has both an exact bouncing solution and a solution with constant Ricci curvature. We will show that the constant curvature (de Sitter point) solution admits an inflationary scale-factor. We have not worked out the cosmology around the de Sitter point and the analogy with the inflationary scale-factor may turn out to be purely formal. Since out of the three equations Eq. (4), Eq. (5) and Eq. (9), only two are independent, we can choose to work with equations Eq. (4) and Eq. (9) for the sake of convenience. Using the standard solution $\rho = \rho_0 a^{-3(1+\omega)}$ obtained from equation Eq. (9) where $\rho_0$ and $\omega$ are constants, the constraint equation Eq. (4), can be written as

$$6H^2 f' = 2\kappa \rho_0 e^{-3(1+\omega) \ln a} + R f' - f - 6H \dot{R} f''.$$  \hspace{1cm} (39)

For $(1/\alpha)e^{\alpha R}$ gravity, this becomes

$$6H^2 - R + 6\alpha \dot{H} + \frac{1}{\alpha} = 2\kappa \rho_0 e^{-3(1+\omega) \ln a + \alpha R}.$$  \hspace{1cm} (40)

Using the above equation let us now prove the existence of exact exponential bouncing solution and exact de-Sitter solution.
5.1 Exact exponential bouncing solution

In this subsection we choose a scale-factor \( a(t) = e^{Ct^2} \), where \( C \) is a positive constant. In such a case we get

\[
H(t) = 2Ct, \quad R = 12C(1 + 4Ct^2), \quad \dot{R} = 96C^2t.
\]

Using all these information and the modified constraint equation, Eq. (40), we get

\[
\left( \frac{1}{\alpha} - 12C \right) + 24C^2t^2(48\alpha C - 1) = 2\kappa\rho_0 \exp \left[ -12\alpha C - 3Ct^2(1 + \omega + 16\alpha C) \right].
\]

(41)

If \( a(t) = e^{Ct^2} \) is an exact solution of \((1/\alpha)e^{\alpha R}\) gravity theory, then the above equation has to be satisfied for all values of \( t \). This condition can be satisfied when

\[
\alpha C = \frac{1}{48}, \quad \omega = -\frac{4}{3}.
\]

We see that for these values of the parameters \( \alpha, C \) and \( \omega \), \( \rho_0 \) is given by

\[
\rho_0 = \frac{3e^{1/\alpha}}{8\kappa\alpha},
\]

which is always positive. Exponential gravity can yield a bouncing universe with exponential scale-factor in presence of hydrodynamic matter which can have negative pressure but whose energy density is positive definite. The causal nature of the universe where the scale-factor is as given in this section is worked out in [21].

5.2 Exact de-Sitter solution

A de-Sitter solution is a vacuum solution of constant positive curvature in GR. We use the same terminology and name our solution as de Sitter solution, as our solution in \( f(R) \) gravity is a constant positive curvature solution in presence of positive vacuum energy. We assume that the scale-factor of the universe as \( a(t) = e^{Ht} \), where \( H \) is a positive constant and consequently

\[
R = 12H^2.
\]

Using the above information in the modified constraint equation, Eq. (40), and setting \( \rho_0 = 0 \), we get

\[
H^2 = \frac{1}{6\alpha} \text{ or } R = \frac{2}{\alpha}.
\]

(42)

Therefore \((1/\alpha)e^{\alpha R}\) gravity has an exact de Sitter solution in which the constant value of the Ricci scalar is given by \( R = 2/\alpha \). In the Einstein frame this value of \( R \) correspond to \( \phi = 2\sqrt{3/(2\kappa)} \), the point at which the potential \( V(\phi) \) assumes it’s maximum.

As the de-Sitter point lies at the top of the Einstein frame potential one can intuitively conclude that the de Sitter solution is an unstable solution. In the rest of the section we will show that the de Sitter solution is indeed an unstable solution. To analyze the stability of this solution, we resort to a dynamical system analysis in the Jordan frame in terms of normalized, dimensionless dynamical variables as formulated in references [22–25]. Let us define the dimensionless dynamical variables

\[
u_1 = \frac{\alpha\dot{R}}{H}, \quad \nu_2 = \frac{R}{6H^2}, \quad \nu_3 = \frac{1}{6\alpha H^2}.
\]

(43)

Eq. (40) then implies the following constraint between them

\[
u_1 - \nu_2 + \nu_3 = -1,
\]

(44)

which implies only two of them are independent. Note that for the de-Sitter solution

\[
u_1 = 0, \quad \nu_2 = 2, \quad \nu_3 = \frac{1}{6\alpha H^2}.
\]

(45)
Figure 6: Phase space plot in the $u_1 u_2$ plane of isotropic vacuum solutions in exponential gravity, where $u_1$ is along the $x-$axis and $u_2$ is along the $y-$axis. The de-Sitter solution is given by the point $(0, 2)$, which is at the center of the figure. The arrows show the direction of the solution flow. The phase flows clearly show that the de-Sitter solution is a saddle fixed point.

Therefore for the constraint equation to be satisfied one must have $H^2 = 1/(6\alpha)$, which is consistent with what we previously obtained. Without loss of generality, we can take $u_1$ and $u_2$ as the two independent dynamical variables. With the help of the Eq. (5), we can find the dynamical equations for $u_1, u_2$ in terms of the dimensionless logarithmic time variable $N \equiv \ln a$ as:

\[
\frac{du_1}{dN} = 4 + 3u_1 - 2u_2 - u_1^2 - u_1 u_2, \tag{46}
\]

\[
\frac{du_2}{dN} = -u_1 - u_1^2 + 4u_2 + u_1 u_2 - 2u_2^2. \tag{47}
\]

It is straightforward to check that the de-Sitter point given by the coordinates $(0, 2)$ in the $u_1 u_2$ plane is a fixed point, i.e., at this point

\[
\frac{du_1}{dN} = \frac{du_2}{dN} = 0.
\]

The eigenvalues of the Jacobian matrix at this point are

\[
\frac{1}{2}(-3 \pm \sqrt{17}).
\]

The first eigenvalue is positive and the second is negative, meaning the de-Sitter point is a saddle point in the space of isotropic vacuum solutions in exponential gravity. Fig. (6) shows the flows of the solutions in the phase space.

After discussing the exact solutions in exponential gravity we discuss the extra solutions we get in exponential gravity if we allow exponential gravity to cross $R = 0$ value. In general we expect the theory to be similar to GR near $R = 0$ and the $f(R)$ effect becomes effective for high values of $R$. But if we assume that $f(R)$ gravity can be also used for small $R$ values then we come across new kind of solutions presented in the next section.
6 New solutions in exponential gravity

If we allow the cosmological model presented in this article to be valid for very small $R$ then we get new class of solutions which, to our understanding, was never reported before in any discussions about $f(R)$ gravity. The important feature of the solutions presented in this section is the generality of the discussion. Most of the results we present in this section is generally true for any $f(R)$ theory which accommodates a cosmological bounce. We present the results for exponential gravity in the present paper.

For a cosmological bounce one requires $H(0) = 0$ and $\dot{H}(0) > 0$ in the Jordan frame. In $f(R)$ gravity we can give another intermediate condition, and this condition is on $\ddot{H}(0)$. Suppose we specify $\ddot{H}(0) = \epsilon$ where $\epsilon$ is a real constant. Near the point where $t = 0$ we can then write approximately $\dot{H}(t) \approx \epsilon t + b$ where $b$ is a positive, real constant. The approximation remains valid as long as $\dot{H}$ varies linearly near the bouncing point. In such a case we can integrate once more and write the expression of the Hubble parameter near the bounce point as

$$H(t) \approx \frac{1}{2}\epsilon t^2 + bt,$$

the integration constant has been set to zero because the Hubble parameter vanishes at $t = 0$. The above equation shows that $H$ can be zero at two separate time instants, given by

$$t = 0, \quad t \approx -\frac{2b}{\epsilon},$$

The above statements are in general true for any $f(R)$ theory which accommodates a cosmological bounce. The system behavior, in the particular case of exponential gravity, is shown in the Fig. 7 which depicts an universe filled up radiation, and in Fig. 8 which depicts the properties of matter less universe. In both the cases we see that the Hubble parameter reaches zero value at two time instants, as predicted from Eq. (49). One of the points when the Hubble parameter is zero is placed at $t = 0$ where as the other time instant is approximately $-2 \times 10^5$, of the same order as predicted from Eq. (49). If one wants to see how the scale-factor of the universe behaves during this period then one can look at Fig. 9 and Fig. 10. From both the figures one can see that initially the universe was expanding and this expansion slowly stops and a brief period of contraction sets in, the period when the Hubble parameter turns negative, and then this contraction stops slowly and the universe again enters a period of expansion. It is to be noted that if we increase the value of $\epsilon$, or $\ddot{H}(0)$ in Jordan frame, then the temporal separation of the two roots in Eq. (49) decreases. This implies that the initial expansion phase ends near to the point where later expansion phase starts.
in the Jordan frame, still we are getting perfect bounces and not oscillatory behavior as expected from our present analysis. The reason for this is related to the time period used to study the asymmetric bounces in section 4 and the specific value of $\dot{H}(0)$ used there. From the conditions given for the asymmetric bounces one can easily show that the oscillatory behavior should have been evident if we presented the plot for a larger time period. In the time period of interest, only the bouncing region becomes prominent in Fig. 4 and Fig. 5.

Before we conclude this section we want to show another important property of the new solutions shown. For flat FLRW solutions one can write the Ricci scalar as

$$ R = 6(\dot{H} + 2H^2) = 6(\epsilon t + b) + 12 \left( \frac{\epsilon t^2}{2} + b t \right)^2, $$

close to the bounce point. From the above expression one can easily verify that

$$ R(0) = 6b, \quad R(t = -2b/\epsilon) = -6b, $$

which shows that the Ricci scalar has to change sign in between the two temporal values when the Hubble parameter vanishes. As a result of this fact, the universe has to cross the point when $R = 0$, between the two expanding regimes. This result is a general result and is true for any $f(R)$ theory which accommodates a cosmological bounce. If we want to work with large, positive values of $R$ and treat $f(R)$ theory as an effective gravitational theory then the new solutions become redundant as only a bounce exists, as analyzed in section 4. On the other hand if one wants to keep the effect of $f(R)$ bouncing cosmology for very small values of the Ricci scalar, then the new solutions will show up.

## 7 Conclusion

In this article we presented some early universe solutions coming out from exponential $f(R)$ gravity. The primary reason for choosing exponential gravity is related to the stability of the theory. For positive cosmological constant one always gets $f'(R) > 0$ and $f''(R) > 0$ for all values of $R$. We presume the theory to be unstable when one moves away from bounce point as in the relevant time limit the Ricci scalar becomes negative and remains unbounded from below. This feature of the theory is well represented in the Einstein frame potential, as shown in Fig. 1. We will like to interpret the early universe results, coming out from exponential $f(R)$ theory, as an outcome from some effective theory of gravity which affects in the ultraviolet end.
In this paper we initially present the basics of cosmological dynamics in the two conformal frames, the Jordan frame where the original problem is posed and the Einstein frame which is used to calculate the dynamical development of the system. Our results related to bouncing cosmologies uses two conformal transformations. We show that it is difficult in $f(R)$ theory to have stable bounces. In this regard exponential gravity is an interesting exception as it satisfies the stability conditions for all values as $R$. In section 3 we present the bouncing condition in the Jordan frame and its corresponding condition in the Einstein frame. We want to specify here that in spatially flat FLRW spacetimes one in general does not get a simultaneous bounce in both the conformal frames when matter in Jordan frame satisfies $\rho + P \geq 0$ [14]. The conditions of bounce in Jordan frame, when written down in the Einstein frame, shows that the conditions depend upon the second time derivative of the Jordan frame Hubble parameter. Consequently, the bouncing intermediate conditions actually involves the values of all the relevant time derivatives of $H$ in the Jordan frame. It is to be noted that unlike $f(R)$ theories, GR based cosmology does not require the specification of $\ddot{H}$ at any instant of time. In this article we point out specifically how $\ddot{H}$ can affect cosmological dynamics in higher derivative gravity theory.

In the article we present the bouncing solutions in the Jordan frame. In the bouncing solution calculations we use the Einstein frame as an auxiliary frame where the main calculation is done and then we transport the solutions in the Jordan frame via a conformal transformation. The bouncing solutions presented in this paper involve both bounce in vacuum and bounce in the presence of hydrodynamic matter. The examples involve both symmetric and asymmetric bounces. The asymmetry of the bounces is related to finite values of $\ddot{H}(0)$ in the Jordan frame. For symmetric bounces $\ddot{H}(0) = 0$. The solutions presented in section 4 are computed numerically and they are well behaved in our time period of interest, $-10^5 \leq t \leq 10^5$. Outside the time window the solutions can show other features. All the bounces concerned in section 4 involves variation of positive values of the Ricci scalar in the Jordan frame. Treated $f(R)$ theory as an effective theory of gravity, the exponential gravity bounces become more plausible at higher positive values of $R$. As exponential gravity becomes similar to GR with positive cosmological constant for low $R$ values one may like to infer that GR effects becomes stronger as $R$ nears zero. The exact turn over from $f(R)$ to GR may involve new physics and is beyond the scope of this paper.

Another interesting property of exponential gravity, as discussed in the paper, is the existence of exact solutions. We present two such solutions in the article. One exact solution is a bouncing solution where the scale-factor of the universe is given by an exponential function of the square of Jordan frame time. This solution can be realized in an universe with matter having positive energy density and equation of state as $\frac{-P}{\rho} = 0 \frac{1}{3}$. With an equation of state lesser than $-1$, the matter does not satisfy $\rho + P \geq 0$ in the Jordan frame and consequently for such an universe one may expect simultaneous bounces in both the Jordan frame and the Einstein frame. The other exact solution is the exponential expansion solution with constant $H$ at a de Sitter point. In vacuum, exponential gravity allows such a solution to exist. One can easily show that such a de-Sitter point exists because exponential gravity involves a positive cosmological constant. If the form of $f(R)$ is suitably changed such that it does not contain any cosmological constant the de-Sitter point vanishes. Using the techniques of dynamical systems we have shown that a constant Hubble parameter solution at the de-Sitter point in exponential gravity is an unstable solution. In effect it is a saddle point solution in the phase space of suitable defined dimensionless phase space variables. We do not present the dynamical system approach in the other solutions in this paper because all the other solutions involve the values $H = 0$ and the phase space variables in our analysis always have the Hubble parameter in the denominator. We hope to construct a suitable dynamical systems approach to tackle bouncing problems in the near future.

We present a new solution in $f(R)$ gravity theory in the penultimate section. The new solution in $f(R)$ theories are allowed only if the theory is allowed to be unmodified in the low $R$ regime. As our theory is stable it can safely be extended to the low $R$ regime. The only cost one has to pay to attain these new solutions is that one has to reject the point of view that $f(R)$ gravity is an ultraviolet modification of GR effects. The new results are related to non-zero values of $\ddot{H}(0)$ in the Jordan frame when the other bouncing conditions hold. In such a case one can have a solution which represent decelerated expansion of the universe in the past. At some point in the past the decelerated expansion comes to a halt momentarily and contraction of the universe starts. This contraction does not lead to a spacetime
singularity. In time this contraction slows down and the universe comes to a static configuration momentarily after which again the universe starts to expand. This solution is practically not a bouncing solution, although one can get this solution with an extra intermediate condition on top of the bouncing conditions at $t = 0$. The new result which we obtain in this paper is a general result in $f(R)$ gravity which accommodates a bounce. We explicitly show the nature of the solutions in exponential gravity.

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