Analytical analysis for non-homogeneous two-layer functionally graded material

1 Introduction

Since traditional materials do not meet the current needs of industry, the production of a new generation of materials, such as functionally graded materials (FGMs), is expected to meet the recent needs of the industry. FGMs have a heterogeneous but isotropic microstructure and their mechanical properties change gradually and continuously. FGM is preferred in terms of mechanical behavior compared to materials with a fibrous structure. The gradual change in properties with respect to their dimensions has led to the growth of the application of these materials [1]. It is anticipated that global industry would benefit significantly from research into FGM materials. Therefore, the investigation of these materials is crucial for the design of structures. Recently, materials having functionally graded characteristics have been widely studied. Miyamoto et al. [2] published the first book on FGM. Various studies have been conducted on FGM [3–6]. Elkafrawy et al. [3] investigated the linear eigenvalue buckling of uniaxially loaded FGM plates. Their study determined how the critical buckling load is impacted by the size and geometry of the openings in the FGM plate. A free vibration of 2D FGM beams with continuous material property variation was analyzed by Aminbaghai et al. [5]. Nonlinear strain–displacement relations of a sheet with FGM was studied by Praveen and Reddy [6].

Some research studies have presented a mathematical solution [7–17]. A mathematical model was provided by Njim et al. [7] to assess the buckling stress of FGM rectangular plates. Cheng and Batra [8] studied the buckling and vibrations of a FGM sheet analytically. The critical load and the natural frequencies of the sheet were examined in their study. Wang [9] presents an analytical solution for the piezoelectric cylindrical construction and scaled function. This article examines the electro-elastic behavior with a changing Young’s modulus along the radius. An analytical solution using functionally graded plates was presented by Bouazza et al. [10]. A comprehensive review of FGM behavior was described by Swaminathan et al. [11] and both analytical and numerical
approaches were considered in their study. A comprehensive review of multi-layered plates in terms of wave transmission features was presented by Zarastvand et al. [12]. They also focused on the fundamental equations of composite shell structure. They proposed an analytical solution for the first time to determine the vibration response of composite shell structure with a double curvature [13]. In another work, Zarastvand et al. [14] provided a solution strategy for acoustic equations which were solved along the stiffened shell equations.

Some researchers studied FGM in cylindrical forms [9,15–19]. Wang [9] obtained an analytical solution for functionally graded piezoelectric cylindrical structures. Their results illustrated that non-homogeneity of material has main effect on the electro-elastic field. Sharma and Kaur [16] presented a numerical study by using finite element method. Tokovyy and Ma [18] conducted an analytical approach for radially inhomogeneous geometry. Their method was based on the direct integration approach. The stresses and strains produced by a cylinder of FGM were determined by Fukui and Yamanaka [20]. Wang and Shao [21] conducted a three-dimensional analysis of FGM with a limited length and subjected to non-uniform loads. The steady-state thermo-elasticity of a functionally graded panel has been theoretically investigated by Ootao and Tanigawa [22]. They presented a precise solution. Oral and Anlas [23] have investigated the stress distribution in a heterogeneous anisotropic cylinder.

Most research studies analytical analysis of FGM material only with one-layer in only one direction. In this work, the analytical analysis of a two-layer geometry made of FGM is studied. Derivation of governing equation in two-layer FGM is more challenging than one-layer FGM. The novelty of this work is to propose an analytical solution for two-layer FGM graded materials in two dimensions. Since there is a lack of study in two-layer FGM with the material properties graded along \((r, \theta)\) coordinates in both \(r\) and \(\theta\) directions, this work fulfils the gap in the existing literature. Navier’s equations primarily help us to better describe the mechanics of the engineering phenomena. Navier’s equations are derived by combining the stress–strain, strain–displacement and stress equilibrium equations and a direct method is presented to solve these equations analytically.

### 2 Equation derivations

In most research, the potential function is a commonly used approach for investigating the corresponding stresses in FGM in two dimensions and three dimensions. It should be mentioned that this method has some limitations in choosing boundary conditions. Therefore, it is important to apply a method that does not have these limitations. The direct method is a common method for solving equations which does not have these limitations. At present, the displacement and stress equations are obtained by solving the Navier’s equations using the direct method.

#### 2.1 Mathematical modeling

The governing equations for the mechanical behavior of FGM are different from the governing equations for isotropic materials, so it is necessary to determine their equations according to the type of function which is appropriate for their properties. In this study, the investigated problem is a two-layer cylinder made of FGM. An analytical solution is presented for axially symmetric thermal stresses. The analysis is performed based on the theory of small deformations and the assumption of a very large length in the plane strain mode. In this work, combined power-exponential models are used to express the changes in the properties of FGM. First, the heat transfer equations are obtained and then the equilibrium equations in terms of displacement, known as Navier’s equations, are derived by combining the stress–strain, strain–displacement and stress equilibrium equations. Second, displacement and stress equations are obtained by solving the Navier’s equations. The direct method is applied to solve the Navier’s equations.

The functional relationship of the material properties must be known in order to develop the equations. The mathematical models that can be used to express the changes in the mechanical properties of FGM materials include the exponential model, the power model, and the power–exponential model. In this work, a combined power-exponential model is used to express the changes in the mechanical properties of FGM. The material properties are asymmetric and are assumed in Eq. (1) [24,25] to be graded along \((r, \theta)\) coordinates. The modulus of elasticity and the heat conductivity coefficient are assumed to be described with a power-exponential model.

\[
p(r, \theta) = p_0 r^a e^{b\theta}, \tag{1}
\]

where \(a\) and \(b\) are power-exponential model indices of the material that can be positive or negative, \(p(x)\) and \(p_0\) are material properties and material constant, respectively.
2.2 Heat conduction problem in two-layer FGM

Two-dimensional strain–displacement relationships in the radial and circumferential directions are defined. The radial, circumferential strain, and angular strain are obtained from Eq. (2) [21]:

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \text{,} \\
\varepsilon_{\theta \theta} &= \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \text{,} \\
\varepsilon_{r \theta} &= \frac{1}{r} \frac{\partial v}{\partial r} - \frac{\partial u}{\partial \theta} + \frac{u}{r} \text{.}
\end{align*}
\]

The strain components are obtained in the radial and circumferential directions. The equation of stress and strain are calculated in the \((r, \theta, Z)\) coordinate system. The equilibrium equations with consideration of unit thickness are obtained as follows:

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta} + \frac{\sigma_{rr}}{r} - \sigma_{\theta \theta} &= 0, \\
\frac{\partial \sigma_{r \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{2}{r} (\sigma_{r \theta} - \sigma_{\theta r}) &= 0.
\end{align*}
\]

According to the law of conservation of energy, we have

\[
\frac{\partial q_r}{\partial r} + q_r + \frac{\partial q_\theta}{\partial \theta} = 0.
\]

Derivation of heat transfer equations in two dimensions:

By taking the derivative of the law of the conservation of energy [21], and assuming \(k = k(r)\) we have

\[
\frac{\partial^2 T}{\partial r^2} + \left( \frac{k'}{k} + \frac{1}{r} \right) \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0.
\]

Using the theory of linear elasticity [21]

\[
\begin{align*}
e_{ij} &= \frac{1}{2} (u_{ij} + u_{ji}) \text{,} \\
e_{ij} &= \frac{1}{2} \left( \varepsilon_{ij} - \frac{\lambda}{3\mu} \sigma_{kk} \delta_{ij} \right) + a(T) \delta_{ij},
\end{align*}
\]

Using the elasticity constants \(E = \frac{3(3\lambda + 2\mu)}{\lambda + \mu}\) and \(v = \frac{\mu}{3\lambda + 2\mu}\), \(\lambda\) and \(\mu\) are the Lamé coefficients related to \(E\), and by substituting \(\sigma_{kk}\) which is \(\sigma_{kk} = (3\lambda + 2\mu)(\varepsilon_{kk} - 3\alpha(T))\), in Eq. (7) we have

\[
\sigma_{ij} = 2\mu e_{ij} + \lambda (\varepsilon_{kk} - 3\lambda + 2\mu \alpha(T)) \delta_{ij}.
\]

By using the constants, \(\delta_{ij} = \delta_{jj} = \delta_{\theta \theta} = 1, \delta_{ij} = \delta_{ij} = 0\), \(\varepsilon_{kk} = (\varepsilon_{rr} + \varepsilon_{\theta \theta})\) and also \(e_{ij}(T) = a(T) \delta_{ij}\), the stress-strain relations for the FGM in the plane strain are calculated as follows:

\[
\begin{align*}
\sigma_{rr} &= (\lambda + 2\mu)\varepsilon_{rr} + \lambda \varepsilon_{\theta \theta} - (3\lambda + 2\mu)\alpha(T) (r, \theta) \text{,} \\
\sigma_{\theta \theta} &= (\lambda + 2\mu)\varepsilon_{\theta \theta} + \lambda \varepsilon_{rr} - (3\lambda + 2\mu)\alpha(T) (r, \theta) \text{,} \\
\sigma_{r \theta} &= 2\mu \varepsilon_{r \theta}.
\end{align*}
\]

The equations of heat transfer are derived in first and second layer FGM. The geometry of the problem is a cylinder with an inner radius \(r_1\), an outer radius of the first layer \(r_2\) and the outer radius of the second layer \(r_3\).

In general, in this problem we have

\[
\begin{align*}
\frac{1}{r} (k r T)_{rr} + \frac{1}{r^2} (k T)_{\theta \theta} &= 0, \\
\eta &\leq r \leq r_3, \quad 0 \leq \theta \leq 2\pi,
\end{align*}
\]

where \(k\) is the heat conductivity coefficient, \(C_{ij}\) is a constant parameter, \(f_1(\theta)\) and \(f_2(\theta)\) are the known functions of the problem. With the boundary conditions as Eq. (11)

\[
\begin{align*}
C_{ii} T(n_1, \theta) + C_{jij} T(n_2, \theta) &= f_1(\theta), \\
C_{ii} T(r_2, \theta) + C_{jij} T(r_3, \theta) &= f_2(\theta).
\end{align*}
\]

2.2.1 Equation for the first layer

For the first layer \(k(r, \theta) = k_0 e^{m_0 r} e^{m_1 \theta}\), where \(k_0, m_3\) and \(m_6\) are material parameters. Based on the Eqs. (10) and (11), we obtain the following:

\[
T_{,rr} + (m_3 + 1) T_{,r} + \frac{1}{r^2} (m_6 T_{,\theta \theta} + T_{,\theta}) = 0.
\]

By considering \(T(r, \theta)\) in the form of a complex Fourier series, \(T(r, \theta) = \sum_{n=-\infty}^{\infty} T_n(r) e^{i n \theta} \text{ and } T_n(r) = \frac{1}{2\pi} \int_{0}^{2\pi} T(r, \theta) e^{-i n \theta} d\theta\), and substituting in the heat transfer equation, Euler’s equation is determined

\[
T_n''(r) + (m_3 + 1) T_n'(r) + \frac{1}{r^2} (m_6 T_n - n^2 T_n) = 0.
\]

To solve Eq. (13), \(T_n(r) = A_n e^{\beta n} \text{ and } m_3 + m_6 \beta + (m_6 n^2 - n^2) = 0\), the roots of the equation are determined as \(\beta_{n,2} = \frac{-m_3 + \sqrt{m_3^2 - 4 m_6 n^2}}{2 m_6}\).

The constants of the problem calculated by Cramer’s method:

\[
A_n = \frac{1}{2\pi} \int_{0}^{2\pi} \left[(C_{21} b_{n1} + C_{22} b_{n2} b_{n1}) f_1(\theta) \right. \\
- \left. (C_{11} b_{n1} + C_{12} b_{n2} b_{n1}) f_2(\theta)\right] e^{-i n \theta} d\theta / (\tilde{C}_1 - \tilde{C}_2)
\]
The roots of the equation are \( \beta_{1,2} \), where \( \beta_{1,2} \)

\[ A_n = \frac{1}{2\pi} \int_0^{2\pi} [(C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})f_2(\theta)
- (C_1b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1})f_1(\theta)]e^{-in\theta}d\theta/(C_1 - C_2), \]

where \( C_1 = (C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})(C_2b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1}) \) and \( C_2 = (C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})(C_2b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1}) \).

2.2.2 Equation for the second layer

For the second layer \( k(r, \theta) = k_0 \hat{m}_3 e^{\hat{m}_3} \), where \( k_0, \hat{m}_3 \) and \( \hat{m}_e \) are material parameters. Finally, the heat transfer equation is obtained

\[ T_{rr} + \left( \tilde{m}_3 + 1 \right) \frac{1}{r} T_r + \left( \frac{1}{r^2} \right) (\hat{m}_e n - n^2) T_\theta \theta = 0. \tag{16} \]

Just as in the first layer, by considering \( T(r, \theta) \) in the form of a complex Fourier series, Euler’s equation obtained:

\[ T_n''(r) + \left( \tilde{m}_3 + 1 \right) \frac{1}{r} T_n'(r) + \left( \frac{1}{r^2} \right) (\hat{m}_e n - n^2) T_n(r) = 0. \tag{17} \]

To solve the above equation, \( T_n(r) = A_n r^{\hat{m}_3} \) and \( \hat{m}_3 \).

\[ T(r, \theta) = \sum_{n=-\infty}^{\infty} (A_n r^{\hat{m}_3} + A_{n'} r^{\hat{m}_{n'}})e^{in\theta}. \tag{18} \]

The constants of the problem are calculated by Cramer’s method:

\[ A_{n1} = \frac{1}{2\pi} \int_0^{2\pi} [(C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})f_1(\theta)
- (C_1b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1})f_1(\theta)]e^{-in\theta}d\theta/(C_1 - C_2), \]

\[ A_{n2} = \frac{1}{2\pi} \int_0^{2\pi} [(C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})f_2(\theta)
- (C_1b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1})f_2(\theta)]e^{-in\theta}d\theta/(C_1 - C_2), \]

where \( C_1 = (C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})(C_2b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1}) \) and \( C_2 = (C_1a^{\beta_{1}}} + C_2\beta_{1,2}a^{\beta_{1,2}-1})(C_2b^{\beta_{1}}} + C_2\beta_{1,2}b^{\beta_{1,2}-1}) \).

In the equations for \( T(r, \theta) \), Eqs. (14) and (18), we have four unknowns \( (A_{n1}, A_{n2}, A_{n3}, A_{n4}) \) which are obtained by applying the boundary conditions and continuity between layers.

\[ T_j(r_1, \theta) = \sum_{n=-\infty}^{\infty} (A_{n1} r_1^{\beta_{1}}} + A_{n2} r_1^{\beta_{1,2}})e^{in\theta} = f_1(\theta) \]

\[ T_j'(r_1, \theta) \text{ first layer} = T_j'(r_2, \theta) \text{ second layer} \]

2.3 Navier’s equations

By combining the equations of stress, strain, displacement, and equilibrium in terms of stress, the equations of equilibrium in terms of displacement, known as Navier’s equations, are obtained.

For the first layer, the modulus of elasticity and heat expansion coefficient with the power-exponential model are assumed as \( E(r, \theta) = E_0 r^{m_1}e^{m_2} \) and \( \alpha(r, \theta) = \alpha_0 r^{m_1}e^{m_2} \) where \( m_1, m_2, m_3, m_4 \) are the power-exponential model indices of the material and \( E_0 \) and \( \alpha_0 \) are material constants. Variables \( A \) and \( B \) are defined as \( A = \frac{v}{(1+v)(1-2v)} \) and \( B = \frac{1}{2(1+v)} \). By simplifying the stress equations and substituting into the first equilibrium equation we have
Farhad Belalpour Dastjerdi and Mohsen Jabbari

The Navier’s equation is obtained as follows:

\[ u_{rr} + (m_1 + 1) \frac{1}{r} u_r + \left[ \frac{A}{A + 2B} \right] m_1 \frac{1}{r^2} u = \frac{B}{A + 2B} m_1 \frac{1}{r^2} u_{\theta \theta} \]

By substituting the relevant terms in the second equilibrium equation, the second Navier’s equation is obtained as follows:

\[ v_{rr} + (m_1 + 1) \frac{1}{r} v_r - (m_1 + 1) \frac{1}{r^2} v = \frac{2 - 2\nu}{1 - 2\nu} \frac{m_1}{r} v_{\theta \theta} \]

\[ + \frac{2 - 2\nu}{1 - 2\nu} \frac{m_1}{r} v_{r \theta} + \left( \frac{m_1}{r} + 3 \right) \frac{2\nu}{1 - 2\nu} \frac{1}{r^2} u \]

\[ - \frac{\bar{A}}{\bar{A} + 2\bar{B}} \frac{m_1}{r^2} \bar{u} \theta \theta = \frac{\bar{B}}{\bar{A} + 2\bar{B}} \frac{m_1}{r^2} \bar{v}, \]

\[ + \frac{\bar{A} + \bar{B}}{\bar{A} + 2\bar{B}} \frac{1}{r^2} \bar{v}_{r \theta} + \left[ \frac{\bar{A}}{\bar{A} + 2\bar{B}} (m_1 - 1) \right] \]

\[ - \frac{3\bar{B}}{\bar{A} + 2\bar{B}} \frac{1}{r^2} \bar{v}_{\theta \theta} = \frac{3\bar{A} + \bar{B}}{\bar{A} + 2\bar{B}} \bar{a}_{\theta \theta} e^{m_0 \theta}(m_1 + m_2) r^{m_1 - 1} T + r^{m_2} T_{r \theta}. \]

The Navier’s equation is obtained as Eq. (27)

\[ u_{rr} + (m_1 + 1) \frac{1}{r} u_r + \frac{1}{v - m_1 - 1} \frac{1}{r^2} u \]

\[ + \frac{1 - 2\nu}{2 - 2\nu} \frac{m_1}{r^2} u_{\theta \theta} + \frac{1 - 2\nu}{2 - 2\nu} \frac{m_1}{r} u_{r \theta} \]

\[ + \frac{1 - 2\nu}{2 - 2\nu} \frac{m_1}{r} v_{r \theta} + \left( \frac{m_1}{r} + 3 \right) \frac{2\nu}{1 - 2\nu} \frac{1}{r^2} u_{\theta \theta} = \frac{1 + \nu}{1 - \nu} \bar{a}_{\theta \theta} e^{m_0 \theta}(m_1 + m_2) r^{m_1 - 1} T + r^{m_2} T_{r \theta}. \]

By substituting the relevant terms in the second equilibrium equation, the second Navier’s equation is obtained as follows:

\[ v_{rr} + (m_1 + 1) \frac{1}{r} v_r - (m_1 + 1) \frac{1}{r^2} v + \frac{2 - 2\nu}{1 - 2\nu} \frac{m_1}{r} v_{\theta \theta} \]

\[ + \frac{2 - 2\nu}{1 - 2\nu} \frac{m_1}{r} v_{r \theta} \]

\[ + \frac{1}{1 - 2\nu} \frac{1}{r} u_{r \theta} + \left( \frac{m_1}{r} + 3 \right) \frac{2\nu}{1 - 2\nu} \frac{1}{r^2} u_{\theta \theta} = \frac{2 + 2\nu}{1 - 2\nu} \bar{a}_{\theta \theta} e^{m_0 \theta}(m_1 + m_2) T + T_{r \theta}. \]

### 3 Results

The displacement equations and stress equations are presented by solving the Navier’s equations.

#### 3.1 Solutions of Navier’s equations in two-layer FGM

To solve the Navier’s equations, we expand the displacement components in the form of a Fourier series, \( u(r, \theta) = \sum_{n=-\infty}^{\infty} u_n(r) e^{i n \theta}, \ v(r, \theta) = \sum_{n=-\infty}^{\infty} v_n(r) e^{i n \theta}, \ u_{\theta}(r) = \)
where

\[ u_0 = \frac{1}{2r} \int_0^{2\pi} u(r, \theta) e^{i(m + m_0) \theta} d\theta, \quad \text{and} \quad v_0 = \frac{1}{2r} \int_0^{2\pi} v(r, \theta) e^{i(m + m_0) \theta} d\theta, \]

and substitute in Eqs. (24) and (27):

\[ u''_n + \left( m_1 + 1 \right) \frac{1}{r} u'_n + \left( \frac{vm_n}{1 - v} - 1 + \frac{2 - \nu}{2 - \nu} (in + m_2) m_4 + \frac{1 - 2\nu}{2 - \nu} (in + m_2)^2 \right) \frac{1}{r^2} u_n + \left( \frac{1 - 2\nu}{2 - \nu} m_4 + \frac{1}{2 - \nu} \frac{(in + m_2)}{1 - \nu} \right) v'_n + \left[ \frac{\nu(m_1 - 1)}{1 - \nu} - \frac{3 - 6\nu}{2 - 2\nu} \right] (m_1 + m_0) \]

\[ \left( \frac{vm_n}{1 - v} - 1 + \frac{2 - \nu}{2 - \nu} \right) \frac{1}{r^2} v_n + \frac{1}{2 - 2\nu} \frac{\nu m_n}{1 - v} A_{n1} \beta^{m_{n1} + m_{n2} - 1} + (m_1 + m_2 + \beta^m_2) A_{n2} \beta^{m_{n1} + m_{n2} - 1}, \]

\[ v''_n + \left( m_1 + 1 \right) \frac{1}{r} v'_n + \left( \frac{2 - \nu}{1 - \nu} \right) \frac{1}{r^2} v_n + \frac{2 - \nu}{1 - \nu} (in + m_2) m_4 - (m_1 + 1) \]

\[ + \frac{2 - \nu}{1 - \nu} (in + m_2)^2 \frac{1}{r^2} v_n + \left( \frac{2vm_n}{1 - v} + \frac{(in + m_2)}{1 - \nu} \right) \frac{1}{r^2} u'_n + \left( \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right) (m_1 + m_0) \]

\[ + \left( \frac{2 - 2\nu}{1 - \nu} m_4 \right) \frac{1}{r^2} u_n = \left( \frac{2 - 2\nu}{1 - \nu} \right) \frac{\nu m_n}{1 - v} A_{n1} \beta^{m_{n1} + m_{n2} - 1} + (m_4 + m_2) \frac{1}{1 - \nu} \frac{\nu m_n}{1 - v} A_{n2} \beta^{m_{n1} + m_{n2} - 1} \]

Eqs. (29) and (30) are a system of ordinary differential equations with general and particular solutions. To solve

\[ \left[ \eta(n - 1) + (m_1 + 1) \eta + \frac{vm_n}{1 - v} + \frac{1 - 2\nu}{2 - 2\nu} (m_1 + m_0) m_4 \right] + \left[ \frac{1 - 2\nu}{2 - 2\nu} (m_1 + m_0)^2 \right] B + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]

\[ \left[ \frac{vm_n}{1 - v} + \frac{(m_1 + m_0)}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] + \left[ \frac{(m_1 + 3) + \frac{2 - \nu}{1 - \nu}}{2 - 2\nu} \right] \left[ \eta(n - 1) + (m_1 + 1) \eta \right] \]
\[ d_{1n}r^\beta_{n1} + d_{2n}r^\beta_{n2} + d_{3n}r^\beta_{n3} + d_{4n}r^\beta_{n4} \]

\[ + d_{1m}r^\beta_{m1} + d_{2m}r^\beta_{m2} + d_{3m}r^\beta_{m3} + d_{4m}r^\beta_{m4} \]

\[ + d_{1m}r^\beta_{n1} + d_{2m}r^\beta_{n2} + d_{3m}r^\beta_{n3} + d_{4m}r^\beta_{n4} \]

\[ + d_{1m}r^\beta_{m1} + d_{2m}r^\beta_{m2} + d_{3m}r^\beta_{m3} + d_{4m}r^\beta_{m4} \]

where \( d_1 \) to \( d_{12} \) are the defined constants. Equating the power coefficients and using Cramer’s method:

\[ D_n = \frac{d_{1n}}{d_{1n} - d_{1d}} \]

\[ D_m = \frac{d_{2m}}{d_{2m} - d_{1d}} \]

\[ D_n = \frac{d_{1n}}{d_{1n} - d_{1d}} \]

\[ D_m = \frac{d_{2m}}{d_{2m} - d_{1d}} \]

\[ D_n = \frac{d_{1n}}{d_{1n} - d_{1d}} \]

\[ D_m = \frac{d_{2m}}{d_{2m} - d_{1d}} \]

\[ D_n = \frac{d_{1n}}{d_{1n} - d_{1d}} \]

\[ D_m = \frac{d_{2m}}{d_{2m} - d_{1d}} \]

The complete solutions are obtained as:

\[ u(r, \theta) = \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} B_{nj}r^{\beta_{n1} + m_2} + D_{nj}r^{\beta_{n2} + m_2} \right] e^{i(n + m_2)\theta} \]

\[ v(r, \theta) = \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} N_{nj}r^{\beta_{n1} + m_2} + D_{nj}r^{\beta_{n2} + m_2} \right] e^{i(n + m_2)\theta} \]

Therefore, by using Eq. (35) we calculated \( \varepsilon_{rr} \), \( \varepsilon_{\theta \theta} \), \( \varepsilon_{r \theta} \), \( \sigma_{rr} \), \( \sigma_{\theta \theta} \), and \( \sigma_{r \theta} \).

\[ \varepsilon_{rr} = \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} \eta_{nj}B_{nj}r^{\beta_{n1} + m_2} + D_{nj}r^{\beta_{n2} + m_2} \right] e^{i(n + m_2)\theta} \]

\[ \varepsilon_{\theta \theta} = \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} ((in + m_2)N_{nj} + 1)B_{nj}r^{\beta_{n1} + m_2} \right] e^{i(n + m_2)\theta} \]

\[ \varepsilon_{r \theta} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} ((in + m_2) + (\eta_{nj} - 1)N_{nj})B_{nj}r^{\beta_{n1} + m_2} \right] e^{i(n + m_2)\theta} \]

and,

\[ \sigma_{rr} = \frac{E_0}{(1 + \nu)(1 - 2\nu)} \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} ((1 + \nu)\nu_{nj} + \nu)D_{nj}e^{i(n + m_2)\theta} \right] \]

Now for the second layer, we repeat the same above calculation.

\[ d_{1n}r^\beta_{n1} + d_{2n}r^\beta_{n2} + d_{3n}r^\beta_{n3} + d_{4n}r^\beta_{n4} \]

\[ + d_{1m}r^\beta_{n1} + d_{2m}r^\beta_{n2} + d_{3m}r^\beta_{n3} + d_{4m}r^\beta_{n4} \]

Based on Cramer’s method, the complete solutions are calculated:

\[ u(r, \theta) = \sum_{n=-\infty}^{\infty} \left[ \sum_{j=1}^{4} B_{nj}r^{\beta_{n1} + m_2} + D_{nj}r^{\beta_{n2} + m_2} \right] e^{i(n + m_2)\theta} \]
$$v(r, \theta) = \sum_{n=-\infty}^{+\infty} \left[ \sum_{j=1}^{4} \sum_{m=-n}^{n} \bar{B}_{nj} \bar{B}_{mj}^\ast r^{m} + \bar{D}_{nj} \bar{B}_{mj}^\ast r^{m+1} \right] e^{i(m+n)\theta}.$$ 

Accordingly, $\varepsilon_{r\theta}$, $\varepsilon_{\theta\theta}$, $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are calculated by using Eq. (39). It should be noted that parameters $B_{nj}$ and $B_{mj}$ are determined by considering the boundary conditions for displacements and stresses $u(r_1, \theta) = g_1(\theta)$, $u(r_3, \theta) = g_2(\theta)$, $v(r_1, \theta) = g_3(\theta)$, $v(r_3, \theta) = g_4(\theta)$, $\sigma_{r1}(r_1, \theta) = g_5(\theta)$, $\sigma_{r1}(r_3, \theta) = g_6(\theta)$, $\sigma_{\theta1}(r_1, \theta) = g_7(\theta)$, and $\sigma_{\theta1}(r_3, \theta) = g_8(\theta)$, as well as four continuity conditions $\sigma_{r1}(r_2, \theta)_1 = \sigma_{r1}(r_2, \theta)_2$, $\sigma_{\theta1}(r_2, \theta)_1 = \sigma_{\theta1}(r_2, \theta)_2$, $u(r_2, \theta)_1 = u(r_2, \theta)_2$, and $v(r_2, \theta)_1 = v(r_2, \theta)_2$.

Therefore, we have eight unknowns ($B_{n1}$, $B_{n2}$, $B_{n3}$, $B_{n4}$, $\bar{B}_{n1}$, $\bar{B}_{n2}$, $\bar{B}_{n3}$, $\bar{B}_{n4}$) and eight equations. A complex Fourier series $(g_j(\theta) = \sum_{m=-\infty}^{+\infty} G_j(n)e^{i(m\theta)}$, $j = 1, \ldots, 4$) is implemented for solving the equations, where $m_5 = m_6$ and $m_4 = \bar{m}_4$.

### 3.2 Example

We assumed the following inputs: inner radius $r_1 = 1m$ and outer radius $r_2 = 1.4m$ and the radius of the middle layer $r_3 = 1.2 m$. Assume that Poisson’s ratio, modulus of elasticity, and coefficient of heat expansion of the first layer are $\nu = 0.3$, $E_1 = 200$ GPa, and $\alpha_1 = 1.2 \times 10^{-6}$ 1/°C, and for the second layer, $\nu = 0.2$, $E_2 = 70$ GPa and $\alpha_2 = 1.5 \times 10^{-6}$ 1/°C, respectively.

The stress distributions in the functionally graded two-layer cylinder under the heat boundary conditions are presented to evaluate the solution of the applied method under the defined boundary conditions. There are six independent power-exponential parameters in the equations, which indicates that the parametric solution of the problem has not been obtained for the specific case. Since the obtained equations are general, different parameters can be applied in any desired range for the material. Due to the existence of six independent parameters such as $m_1$, $m_2$, $m_3$, etc., choosing their value without observing a specific basis for the ratio between the parameters enables many possible situations.

For example, consider a stress-free inner boundary with temperature distribution of $T(r_1, \theta) = 60 \sin \theta$. The temperature of the outer boundary is assumed to be zero. Therefore, by using four boundary conditions and four continuity conditions for stress and displacement, it can be concluded $\sigma_{r1}(r_3, \theta) = 0$, $\sigma_{\theta1}(r_3, \theta) = 0$, $u(r_3, \theta) = 0$, $v(r_3, \theta) = 0$, $\sigma_{r1}(r_2, \theta)_1 = \sigma_{r1}(r_2, \theta)_2$, $\sigma_{\theta1}(r_2, \theta)_1 = \sigma_{\theta1}(r_2, \theta)_2$, $u(r_2, \theta)_1 = u(r_2, \theta)_2$, $v(r_2, \theta)_1 = v(r_2, \theta)_2$, $T_f(r_2, \theta) = f_1(\theta)$, $T_f(r_3, \theta) = f_2(\theta)$, $T_f(r_2, \theta)_{\text{firstlayer}} = T_f(r_2, \theta)_{\text{secondlayer}}$, and $T_f(r_2, \theta)_{\text{secondlayer}} = T_f(r_2, \theta)_{\text{secondlayer}}$. Heat distribution coefficients are obtained by placing the boundary conditions and continuity of stress and displacement and using the heat boundary conditions.

The power-exponential indices of material properties at $\theta = \frac{\pi}{2}$ and $r = [1, 1.4]$; for the first layer $m_1 = m_2 = m_3 = m$, $m = [-1, 0.0001, 1]$, $m_5 = m_5 = m_6 = 0$, and also for the second layer $m_1 = m_2 = m_3 = m$, $m_4 = m_4 = m_5 = 0$. The above conditions are assumed for the material properties change along the radial direction without considering the distribution of the material properties in the circumferential direction.

$T$ values are shown in Table 1 in different radii for three different $m$ values.

Heat distribution is also shown in Figure 1. Figure 1 shows that the amount of heat distribution decreases with the increase in the parameter $m$.

| $r/r_1$ | $T$ °C | $m = 1$ | $m = 0.0001$ | $m = -1$ |
|--------|--------|--------|-------------|--------|
| 1.00   | 60.00  | 60.00  | 60.00       |        |
| 1.04   | 51.47  | 52.79  | 54.01       |        |
| 1.08   | 43.65  | 45.93  | 48.10       |        |
| 1.12   | 36.44  | 39.37  | 42.26       |        |
| 1.16   | 29.77  | 33.10  | 36.47       |        |
| 1.20   | 23.58  | 27.08  | 30.73       |        |
| 1.24   | 17.91  | 21.29  | 24.92       |        |
| 1.28   | 12.79  | 15.70  | 18.96       |        |
| 1.32   | 8.13   | 10.30  | 12.82       |        |
| 1.36   | 3.88   | 5.07   | 6.50        |        |
| 1.40   | 0.00   | 0.00   | 0.00        |        |

![Figure 1: Heat distribution in radial direction.](image-url)
Thermal stress component distributions are shown in Table 2 and Figure 3.

Our findings indicate that according to Figure 3, by increasing \( m \) values, the value of the stress components \( \sigma_{\theta \theta}, \sigma_{rr}, \) and \( \sigma_{r \theta} \) increases.

### 4 Conclusion

Due to the benefits of gradually varying properties, FGM can be employed in a variety of engineering applications, such as building materials in civil engineering. This study presents an analytical solution in a two-layered geometry made of FGM. In this research, the direct method and power-exponential model are implemented to determine the solution of the governing equations instead of using other approaches such as the potential function method. The advantage of the direct method in comparison with the potential function method is its generality and mathematical ability to apply different boundary conditions. The approach implemented helps us to better describe the mechanics of the engineering phenomena. In addition, direct method gives us precise answers. However, for more complex equations it might be better to implement other methods such as iteration method.

Since increasing the values of the parameter \( m \), the values of modulus of elasticity increases, it enables us to grade the geometry from a low modulus of elasticity to a high modulus of elasticity. According to our results by increasing \( m \) values, the value of circumferential displacement \( V \) decreases and the value of radial displacement \( U \) increases. Moreover, by increasing \( m \) values, the value of the stress components \( \sigma_{\theta \theta}, \sigma_{rr}, \) and \( \sigma_{r \theta} \)

Displacement components are shown in Figure 2. According to this figure by increasing \( m \) values, the value of \( V \) decreases and the value of \( U \) increases.

![Figure 2](image)

(a)

(b)

**Figure 2:** Displacement components in radial direction, (a) circumferential displacement and (b) radial displacement.

| \( r/r_1 \) | \( \sigma_{\theta \theta} \times 10^8 \) (Pa) | \( \sigma_{rr} \times 10^8 \) (Pa) | \( \sigma_{r \theta} \times 10^8 \) (Pa) |
|-----------|---------------------------------|---------------------------------|---------------------------------|
| \( m = 1 \) | \( m = 0.0001 \) | \( m = -1 \) | \( m = 1 \) | \( m = 0.0001 \) | \( m = -1 \) | \( m = 1 \) | \( m = 0.0001 \) | \( m = -1 \) |
| 1.00 | -6.40 | -6.35 | -6.33 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1.04 | -5.92 | -5.78 | -5.64 | -1.23 | -1.20 | -1.17 | -1.97 | -1.92 | -1.86 |
| 1.08 | -5.38 | -5.18 | -4.97 | -1.82 | -1.75 | -1.70 | -2.91 | -2.80 | -2.72 |
| 1.12 | -4.85 | -4.58 | -4.32 | -2.19 | -2.10 | -2.01 | -3.50 | -3.36 | -3.22 |
| 1.16 | -4.37 | -4.06 | -3.76 | -2.42 | -2.30 | -2.20 | -3.86 | -3.68 | -3.52 |
| 1.18* | -3.90 | -3.56 | -3.25 | -2.46 | -2.35 | -2.25 | -3.94 | -3.76 | -3.60 |
| 1.20 | -2.24 | -1.92 | -1.63 | -2.49 | -2.39 | -2.29 | -3.98 | -3.82 | -3.66 |
| 1.24 | -1.81 | -1.49 | -1.21 | -2.47 | -2.37 | -2.27 | -3.95 | -3.79 | -3.63 |
| 1.28 | -1.35 | -1.07 | -0.83 | -2.43 | -2.33 | -2.23 | -3.89 | -3.73 | -3.57 |
| 1.32 | -0.91 | -0.69 | -0.50 | -2.34 | -2.25 | -2.16 | -3.75 | -3.59 | -3.45 |
| 1.36 | -0.41 | -0.29 | -0.19 | -2.23 | -2.13 | -2.05 | -3.57 | -3.41 | -3.27 |
| 1.40 | 0.00 | 0.00 | 0.00 | -2.10 | -1.99 | -1.90 | -3.35 | -3.18 | -3.03 |

*This value is 1.19 for \( \sigma_{\theta \theta} \).
increases. For further work, it would also be beneficial to apply other methods to solve the governing equations. This enables us to compare results of different approaches to achieve a better understanding of the problem.

Acknowledgments: The authors are grateful for the reviewer’s valuable comments that improved the manuscript.

Funding information: Authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: Authors state no conflict of interest.

References

[1] Zhang N, Khan T, Guo H, Shi S, Zhong W. Functionally graded materials: An overview of stability, buckling, and free vibration analysis. Adv Mater Sci Eng. 2019;2019:1354150.

[2] Miyamoto Y, Kaysser WA, Rabin BH, Kawasaki A, Ford RG. The characterization of properties: functionally graded materials. Boston (MA): Springer; 1999.

[3] Elkafrawy M, Alashkar A, Hawileh R, AlHamaydeh M. FEA investigation of elastic buckling for functionally graded material (FGM) thin plates with different hole shapes under uniaxial loading. Buildings. 2022;12(6):802.

[4] Sharma D, Kaur R, Sharm H. Investigation of thermo-elastic characteristics in functionally graded rotating disk using finite element method. Nonlinear Eng. 2021;10(1):312–22.

[5] Aminbaghai M, Murin J, Kutilš V. Modal analysis of the FGM beams with continuous transversal symmetric and longitudinal variation of material properties with effect of large axial force. Eng Struct. 2012;34:314–29.

[6] Praveen GN, Reddy JN. Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates. Int J Solids Struct. 1998;35:4457–76.

[7] Njm EK, Bakhy SH, Al-Waily M. Analytical and numerical investigation of buckling load of functionally graded materials with porous metal of sandwich plate. Mater Today Proc. 2021.

[8] Cheng ZQ, Batra RC. Exact correspondence between eigenvalues of membrane and functionally graded simply supported polygonal plates. J Sound Vib. 2000;229(4):879–95.

[9] Wang HM. Elastic analysis of exponentially graded piezoelectric cylindrical structures as sensors and actuators. J Mech Sci Technol. 2012;26(12):4047–53.

[10] Bouazza M, Tounsi A, Adda-Bedia EA, Megeuni A. Thermoelastic stability analysis of functionally graded plates: An analytical approach. Comput Mater Sci. 2010;49(4):865–70.

[11] Swaminathan K, Naveenkumar DT, Zenkour AM, Carrera E. Stress, vibration and buckling analyses of FGM plates—A state-of-the-art review. Compos Struct. 2015;120:10–31.

Figure 3: Stress component distributions in radial direction, (a) circumferential stress ($\sigma_{\theta\theta}$), (b) radial stress ($\sigma_r$), and (c) shear stress ($\sigma_{r\theta}$).
Zarastvand MR, Ghassabi M, Talebitooti R. Prediction of acoustic wave transmission features of the multilayered plate constructions: A review. J Sandw Struct Mater. 2022;24(1):218–93.

Rahmatnezhad K, Zarastvand MR, Talebitooti R. Mechanism study and power transmission feature of acoustically stimulated and thermally loaded composite shell structures with double curvature. Compos Struct. 2021;276:114557.

Zarastvand MR, Asadijafari MH, Talebitooti R. Acoustic wave transmission characteristics of stiffened composite shell systems with double curvature. Compos Struct. 2022;292:115688.

Dai HL, Hong L, Fu YM, Xiao X. Analytical solution for electromagnetothermoelastic behaviors of a functionally graded piezoelectric hollow cylinder. Appl Math Model. 2010;34(2):343–57.

Sharma D, Kaur R. Thermoelastic analysis of FGM hollow cylinder for variable parameters and temperature distributions using FEM. Nonlinear Eng. 2020;9(1):256–64.

Nie GJ, Batra RC. Material tailoring and analysis of functionally graded isotropic and incompressible linear elastic hollow cylinders. Compos Struct. 2010;92:265–74.

Tokovyy YV, Ma CC. Analysis of 2D non-axisymmetric elasticity and thermoelasticity problems for radially inhomogeneous hollow cylinders. J Eng Math. 2008;61:171–84.

Shojaeefard MH, Najibi A. Nonlinear transient heat conduction analysis of hollow thick temperature-dependent 2D-FGM cylinders with finite length using numerical method. J Mech Sci Technol. 2014;28(8):3825–35.

Fukui Y, Yamanaka N. Elastic analysis for thick-walled tubes of functionally graded material subjected to internal pressure. JSME Int J. 1991;35:379–85.

Wang TJ, Shao ZS. Three-dimensional solutions for the stress fields in functionally graded cylindrical panel with finite length and subjected to thermal/mechanical loads. Int J Solids Struct. 2006;43(13):3856–74.

Ootao Y, Tanigawa Y. Two-dimensional thermoelastic analysis of a functionally graded cylindrical panel due to nonuniform heat supply. Mech Res Commun. 2005;32:429–43.

Oral A, Anlas G. Effects of radially varying moduli on stress distribution of nonhomogeneous anisotropic cylindrical bodies. Int J Solids Struct. 2005;42:5568–88.

Chi SH, Chung YL. Mechanical behavior of functionally graded material plates under transverse load–Part I: Analysis. Int J Solids Struct. 2006;43(13):3657–74.

Tutuncu N. Stresses in thick-walled FGM cylinders with exponentially-varying properties. Eng Struct. 2007;29(9):2032–5.