Stability and boundedness in AdS/CFT with double trace deformations II: Vector Fields

William Cottrell¹, Akikazu Hashimoto², Andrew Loveridge², and Duncan Pettengill²

¹ Institute for Theoretical Physics Amsterdam, University of Amsterdam
1098 XH Amsterdam, The Netherlands

² Department of Physics, University of Wisconsin, Madison, WI 53706, USA

Abstract

We extend the analysis of boundedness and stability, initiated for scalar fields in anti de Sitter space in a previous work, to the case of vector fields. We show that the double trace deformation of Marolf and Ross is distinct from the double trace deformation of Witten. The former gives rise to an \( SL(2, \mathbb{R}) \) family of theories whereas the latter gives rise to an independent \( SL(2, \mathbb{Z}) \). We analyze the finite temperature two-point correlation function of current operators and infer the susceptibility and spectrum of low lying states. We discuss various physical features exhibited by these theories.
1 Introduction

In a recent article \cite{1}, we surveyed the double trace deformation of scalar fields in an anti de Sitter Schwarzschild background in the probe approximation and mapped out the regions of dynamical and thermodynamic stability in the theory space. The main conclusions reported in \cite{1} are that

1. The space of theories is naturally parameterized by the group manifold of $SL(2, \mathbb{R})$ which can be visualized as $AdS_3$.

2. The full $SL(2, \mathbb{R})$ structure is required once one includes $S$ and $T$ deformations (corresponding to Legendre transform and the double trace deformation, respectively) as generators of deformations in the space of theories.

3. The three generators of $SL(2, \mathbb{R})$ can be interpreted as parameterizing the double trace deformation, overall rescaling, and contact term deformation.

4. Observables such as the free energy and susceptibility depend on the full set of $SL(2, \mathbb{R})$ parameters including the contact term, but the spectrum of small fluctuations is independent of the contact term. We referred to the positivity of the susceptibility observable as “boundedness” whereas the term “stability” referred to the absence of tachyonic fluctuations.

5. The issue of boundedness is directly related to the observables being distributed as a normalizable distribution in the path integral.

These conclusions can be summarized succinctly in a picture which we illustrated in figure 2 of \cite{1}.

In this article, we examine an extension of the analysis in \cite{1}, which was strictly in the context of scalar fields, to a setup involving vector fields. It was already noted in \cite{1} that parallel issues were described in \cite{2,3} and one might think that there is nothing left to discuss. Upon closer examination, one finds that \cite{2} and \cite{3} are discussing double trace deformations of different kinds, giving rise to different physics. The goal of this paper, therefore, is to spell out the differences in the physics of \cite{2} and \cite{3} and to further compare the physics to that of the scalar case considered in \cite{1}. It should be stressed that the physics of Witten’s double trace deformation \cite{2} is intimately connected to the $SL(2, \mathbb{Z})$ modular transform of Chern-Simons theory and has been studied extensively in the context of studying the quantum Hall effect. A sample of such work includes \cite{4-8}. We will show that the Marolf-Ross deformation
will naturally give rise to an $SL(2,\mathbb{R})$ space of theories. These appear to have been studied more in the context of generalizing boundary dynamics [9–12].

The organization of this paper is as follows. In section 2, we review the boundedness and stability issues for the scalars [1] and establish some notation. In section 3, we formulate the double trace and contact term deformation for the vector fields generalizing [1]. We emphasize that the prescription of Marolf and Ross [3] and the prescription of Witten [2] give rise to distinct deformations, and work out the finite temperature two-point correlation function in each of these cases. In section 4, we comment on the physical features exhibited by these deformations and present various concluding remarks. In appendix A, we review the basic formulation of AdS/CFT involving vector fields.

2 Boundedness and stability of double trace deformations for scalar fields

In this section, we will briefly review the analysis and the conventions for the scalars that was done in [1]. Readers are referred to [1] for a more detailed account. We have also included a review of standard AdS/CFT conventions in appendix A. Let us start by recalling the action

$$S = \frac{1}{2e^2} \int dz \; d^d x \; \sqrt{g} \left( \frac{1}{2} g^{zz} (\partial_z \phi(x,z))^2 + \frac{1}{2} g^{ii} (\partial_i \phi(x,z))^2 + \frac{1}{2} m^2 R^2 \phi(x,z)^2 \right)$$

(2.1)

where $m^2$ is dimensionless and we assign dimension $(3 - d)/2$ to $\phi$ so that $\phi$ has dimension one. We will regulate the volume of $\mathbb{R}^{d-1} \times S_1$ by treating it as $T^{d-1} \times S_1$ where $T^{d-1}$ has the volume of $V_{d-1}$ with all of the cycles in $T^{d-1}$ being much larger than that of the $S_1$. We will also concentrate on the zero momentum component of $\phi$ along $T^{d-1}$ by defining

$$\tilde{\phi}(z) = \frac{T}{V_{d-1}} \int d^d x \; \phi(x,z), \quad [\tilde{\phi}] = 1$$

(2.2)

in terms of which the action becomes

$$S = \frac{R^{d-1} V_{d-1}}{2e^2 T} \int_0^{z_0} dz \; z^{-d-1} \left( \frac{1}{2} f z^2 (\partial_z \tilde{\phi}(z))^2 + \frac{1}{2} m^2 \tilde{\phi}(z)^2 \right).$$

(2.3)

It is convenient to scale out $z_0$ and work in units where $z_0$ is set to one,

$$S = \mathcal{N} \frac{2^{d-1}}{z_0^{2+d}} \int_0^1 dz \; z^{-d-1} \left( \frac{1}{2} f z^2 (\partial_z \tilde{\phi}(z))^2 + \frac{1}{2} m^2 \tilde{\phi}(z)^2 \right),$$

(2.4)

---

1 We have however reparameterized $u = z_0/z$.

2 In [1], we assigned dimension $(d - 2)/2$ to $\phi$ and treated $e^2$ as dimensionless.
where \[ \mathcal{N} = \frac{R^{d-1}V_{d-1}}{2e^2T\zeta_0^{2+d}} \] (2.5)
is dimensionless. We can now reproduce the generating function given in (3.11) of [1] as follows
\[
Z[J] = \int [D\rho][D\tilde{\phi}(z_c)][D\tilde{\phi}]_{\tilde{\phi}(z_c)} \exp \left[ -\mathcal{N} \int_{z_c}^1 dz \ z^{-d-1} \left( \frac{1}{2} f z^2 (\partial_z \tilde{\phi}(z))^2 + \frac{1}{2} m^2 \tilde{\phi}(z)^2 \right) - \frac{\mathcal{N}}{2} \Delta_- z_c^{-d} \tilde{\phi}(z_c)^2 + \mathcal{N}(\Delta_+ - \Delta_-) \left( \alpha z_c^{-2\Delta_+} \tilde{\phi}(z_c)^2 + z_c^{-\Delta_-} \beta \tilde{\phi}(z_c) J + \frac{\beta^2 z_c^{-2\Delta_-}}{4\gamma} \rho^2 \right) \right]_{z_c \to 0} \] (2.6)
where \( z_c \) is the UV cut-off for implementing holographic renormalization [13], and the term in the second line is the standard holographic renormalization counter-term. Also, we have
\[ \Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2} \] (2.7)
The measure \([D\tilde{\phi}]_{\tilde{\phi}(z_c)}\) indicates integrating over \( \tilde{\phi}(z) \) with the boundary value at \( z = z_c \) fixed in the path integral, although we are also explicitly integrating over \( \tilde{\phi}(z_c) \) as well. Variation of \( Z[J] \) with respect to \( J \) will encode the correlation function of operators sourced by \( J \).

With \( \alpha = \gamma = 0 \) and \( \beta = 1 \), this generating function is interpretable as the Legendre transform, where we integrate over the boundary value \( \tilde{\phi}(z_c) \), of the usual AdS/CFT prescription in the Dirichlet setup, so it is the Neumann generating function. Parameters \( \alpha \) and \( \gamma \) deform this generating function. The parameter \( \alpha \) corresponds to the double trace deformation, while the parameter \( \gamma \) controls the coupling between \( J \) and an auxiliary field \( \rho \) with vanishing kinetic term. This gives rise to a contact-term, which can be made more manifest by integrating out \( \rho \) to write an effective generating function of the form
\[
Z[J] = \int [D\tilde{\phi}(z_c)] [D\tilde{\phi}]_{\tilde{\phi}(z_c)} \exp \left[ -\mathcal{N} \int_{z_c}^1 dz \ z^{-d-1} \left( \frac{1}{2} f z^2 (\partial_z \tilde{\phi}(z))^2 + \frac{1}{2} m^2 \tilde{\phi}(z)^2 \right) - \frac{\mathcal{N}}{2} \Delta_- z_c^{-d} \tilde{\phi}(z_c)^2 + \mathcal{N}(\Delta_+ - \Delta_-) \left( \alpha z_c^{-2\Delta_+} \tilde{\phi}(z_c)^2 + z_c^{-\Delta_-} \beta \tilde{\phi}(z_c) J + z_c^{2\Delta_-} J^2 \right) \right]_{z_c \to 0} \] (2.8)
The reason we introduce the auxiliary field \( \rho \) in (2.6) is to satisfy the requirement of the fluctuation dissipation theorem that the coupling to the source \( J \) be linear, but nothing prevents us from integrating out \( \rho \) for the purpose of computing the correlation functions, giving rise to a term quadratic in \( J \) in the generating functional which is manifestly a contact term.
The small $z$ asymptotics of $\tilde{\phi}(z)$ can be parameterized as

$$\tilde{\phi}(z) = p_1 z^{\Delta_-} (1 + O(z)) - p_2 z^{\Delta_+} (1 + O(z)).$$

(2.9)

In terms of these expressions, one can read off the boundary condition by varying with respect to $\tilde{\phi}(z_c)$, and the degree of freedom sourced by $J$ by varying with respect to $J$. This can be expressed in the form

$$\frac{1}{(\Delta_+ - \Delta_-) N^\delta} \delta \delta J = \left( \beta - \frac{4 \alpha \gamma}{\beta} \right) p_1 + \frac{2 \gamma}{\beta} p_2$$

(2.10)

$$J = -\frac{2 \alpha}{\beta} p_1 + \frac{1}{\beta} p_2,$$

(2.11)

which simplifies to the expected Neumann expression for $\alpha = \gamma = 0$ and $\beta = 1$.

One important feature implicit in (2.10) and (2.11) is the fact that the expressions in terms of $\alpha$, $\beta$, and $\gamma$ appearing on the right hand side have a natural $SL(2, \mathbb{R})$ parameterization under the map

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta - \frac{4 \alpha \gamma}{\beta} & \frac{2 \gamma}{\beta} \\ -\frac{2 \alpha}{\beta} & \frac{1}{\beta} \end{pmatrix}.$$ 

(2.12)

The factor of $(\Delta_+ - \Delta_-)$ is a bit annoying, but note that this is a positive number of order one. Taking slight liberty in the normalization conventions, we can define the “operator”

$$O \equiv \frac{1}{(\Delta_+ - \Delta_-) N^\delta} \delta$$

and infer that the normalized correlation function is

$$\chi = (\Delta_+ - \Delta_-) N \langle O^2 \rangle = \frac{\delta O}{\delta J} = \frac{a p_1 + b p_2}{c p_1 + d p_2}.$$ 

(2.14)

We refer to this quantity as susceptibility. For the Dirichlet and Neumann theories, this correlation function simplifies to

$$\chi_N = \frac{p_1}{p_2}, \quad \chi_D = -\frac{p_2}{p_1}$$

(2.15)

and so we can write the susceptibility for the general case in the form

$$\chi_\Lambda = \frac{a \chi_N + b}{c \chi_N + d} = \frac{a - b \chi_D}{c - d \chi_D}.$$ 

(2.16)

In other words, the susceptibility transforms modularly.

Note that despite formally being an expectation value of the square of an operator, the susceptibility $\chi_\Lambda$ can be positive or negative. In fact, the fact that $\chi_N$ and $\chi_D$ are related
by an $S$ transformation of $SL(2, \mathbb{R})$ requires that one be positive and the other be negative. The point of [1] was to shed light on this issue.

It is straightforward to extend this analysis to the case when momentum along $T^{d-1}$ is non-vanishing. This is explained in detail in section 3 of [1]. One can also infer the spectrum of small fluctuations by examining the poles of the two-point function. For finite $T$, the spectrum is gapped as is expected. For some set of boundary conditions in the $SL(2, \mathbb{R})$ parameter space, the spectrum will include a normalizable tachyon. These features are summarized in figure 2 of [1].

3 Boundedness and stability of double trace deformation for vector fields

We are now ready to consider the generalization of [1] to vector fields. A useful and concrete starting point is to write the analogue of (2.8). Already, at this point, the differences between the construction of Witten [2] and Marolf and Ross [3] emerge.

3.1 Marolf-Ross v.s. Witten

Following the template of [1], consider the generating function

$$Z[K_i] = \int [D\alpha_i][D\alpha_i]\exp\left[ -S[A_i] + S_{CT}[\alpha_i] + \int d^d x \ (d-2)(\alpha a_i a_i + \beta a_i K_i + \gamma K_i K_i) \right]$$

(3.1)

where $S[A_i]$ is the positive definite Maxwell action (A.8) and $S_{CT}$ is the holographic renormalization counter-term. We could, if desired, integrate back in an auxiliary vector field $\rho_i$ to make the coupling to the source $K_i$ linear. The coupling constant $e^2$ in (A.8) has dimensions

$$[e^2] = 3 - d$$

(3.2)

so that the gauge field $A_i$ and the source $K_i$ have dimensions

$$[A_i] = 1, \quad [K_i] = d - 1$$

(3.3)

and

$$[\alpha] = d - 2, \quad [\beta] = 0, \quad [\gamma] = 2 - d.$$ 

(3.4)

Aside from the contact term parameterized by $\gamma$, this is essentially the type of generating function considered in [3].
The formulation of Witten’s double trace deformation \cite{2} is different. It is a formulation which only exists when \( d = 3 \), and can be written in the form

\[
Z[K_i] = \int [Da_i][DA_i]_a \exp \left[ -S[A_i] + i \int d^3x \left( \alpha \epsilon^{ijk} a_i \partial_j a_k + \beta \epsilon^{ijk} a_i \partial_j K_k + \gamma \epsilon^{ijk} K_i \partial_j K_k \right) \right].
\] (3.5)

This time, the dimensions are such that

\[
[\alpha] = [\beta] = [\gamma] = 0, \quad [A] = [a] = [K] = 1.
\] (3.6)

Although (3.1) and (3.5) are similar in structure, and seem completely natural as the generalization of (2.8), they are manifestly different in the details. The goal of this section is to explain the features such as stability, boundedness, and the role of contact terms, for both of these scenarios.

### 3.2 Double trace deformation of Marolf and Ross

Let us now take a closer look at the structure of the generating function (3.1). When \( \alpha = \gamma = 0 \) and \( \beta = 1 \), this is interpretable as the Legendre transform from the standard Dirichlet formulation to the Neumann one. One can think of this as gauging the \( U(1) \) global symmetry of the original CFT and coupling the \( U(1) \) gauge field \( a_i \) to the source \( K_i \). To the extent that the Legendre transform involves a path integral over a vector field \( a_i(x) \) on the boundary, one must formally prescribe the quotient of the gauge orbit. On the first pass, this may seem problematic in that the double trace term \( \alpha a_i a_i \) is manifestly not gauge invariant. This can be remedied by thinking of the double trace term as arising from a Stueckelberg action \cite{14}

\[
\alpha a_i a_i \rightarrow \alpha (\partial_i \varphi - a_i)(\partial_i \varphi - a_i)
\] (3.7)

and including the path integral over the Stueckelberg field \( \varphi \). Then, one can formally define the quotient by gauge orbit using the standard Fadeev-Popov procedure. For the purpose of working at the classical level, we can also just as well chose the gauge where \( \varphi = 0 \).

From the expression (3.1) for the generating function, we can infer the operator and expectation value relation

\[
\frac{1}{\mathcal{N}(d-2)} \delta \quad = \quad \frac{\beta^2 - 4\alpha \gamma}{\beta} \tilde{a}_i - \frac{2\gamma}{\beta} \left( \frac{1}{e^2} \sqrt{g} \right) z^i \partial_z \tilde{A}_i \quad \frac{1}{(d-2)} \left( \frac{2\gamma}{\beta} \right) \left( \frac{1}{e^2} \sqrt{g} \right) z^i \partial_z \tilde{A}_i \quad (3.8)
\]

\[
\tilde{K}_i = - \frac{2\alpha}{\beta} \tilde{a}_i - \frac{1}{\beta} \left( \frac{1}{e^2} \sqrt{g} \right) z^i \partial_z \tilde{A}_i \quad (3.9)
\]

We recognize the same \( \text{SL}(2, \mathbb{R}) \) structure we saw for the scalars by identifying

\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\beta^2 - 4\alpha \gamma}{\beta} & \frac{2\gamma}{\beta} \\ \frac{2\alpha}{\beta} & \frac{1}{\beta} \end{pmatrix} \quad (3.10)
\]
Upon identifying
\[ p_{1i} = \tilde{a}_i, \quad p_{2i} = -\frac{1}{(d-2)} \frac{1}{e^z} \sqrt{g^{zz}} g^{ii} \partial_z \tilde{A}_i \] (3.11)
we see that
\[ \tilde{A}_i = p_{1i} - p_{2i} z^{d-2} + \ldots \] (3.12)
and that
\[ \Delta_{ij} = \frac{a p_{1i} + b p_{2i}}{c p_{1j} + d p_{2j}} = \left( a \delta_{ik} - b \Delta^D_{ik} \right) \left( c \delta_{jk} - d \Delta^D_{jk} \right)^{-1} \] (3.13)
where
\[ \Delta^D_{ik} = -\frac{\partial p_{2i}}{\partial p_{1j}}. \] (3.14)
The factor of \((d - 2)\) is the analogue of \((\Delta_+ - \Delta_-)\) which we encountered in the scalar case. We see that the case \(d = 2\) is special because the expansion (3.12) degenerates. This is an indication of strongly coupled physics in the IR and can be addressed by including logarithmic terms and a holographic renormalization counterterm [15,16].

The conclusion of this analysis is that \(\Delta_{ij}\) transforms modularly as a matrix. Focusing on the components enumerated in (A.31) and using the fact that \(J = H = 0\) for the Dirichlet pure Maxwell theory, we find that the \(SL(2, \mathbb{R})\) orbit has \(F_\Lambda\) and \(G_\Lambda\) of the form
\[ F_\Lambda = \frac{a - b F_D}{c - d F_D}, \quad G_\Lambda = \frac{a - b G_D}{c - d G_D}. \] (3.15)
In other words, \(F_\Lambda\) and \(G_\Lambda\) transform separately and modularly. We can now use the result of the computation of \(F_D\) and \(G_D\) in appendix A and infer the susceptibility and the spectrum of normalizable states for each of the theories in the space of theories parameterized by \(SL(2, \mathbb{R})\).

We can summarize the essential features of the \(SL(2, \mathbb{R})\) family of models arising from the Marolf-Ross deformations by producing a plot similar to figure 2 of [1]. This is provided in figure 1. To avoid cluttering the image, we have only illustrated the cross-section, parameterized by \(\tau\) and \(z\), of the \(SL(2, \mathbb{R})\) theory space along the plane where \(\phi = 0\) and \(\phi = \pi\) where \(\tau, \phi, \) and \(\rho = \tanh^{-1} z\) are the standard cylindrical coordinates on \(AdS_3\). Figure 1.a illustrates the features of the longitudinal components encoded in \(F_\Lambda(p^2, \omega)\), for \(\omega = 0\). Just as in [1], the red shaded region is where the susceptibility \(\chi = F_\Lambda(0)\) is negative and consequently, the correlation function is unbounded. The shaded blue region is where the spectrum contains a tachyon. Along the red line, where the shaded blue region and the shaded red region share a boundary, one expects a normalizable massless fluctuation. Figure 1 is in fact identical to what was drawn in figure 2 of [1]. Figure 1.b, on the other hand, illustrates the same features but for the transverse components encoded in \(G_\Lambda(p^2, \omega)\),
again for $\omega = 0$. This time, the red line intersects the Neumann point. This is where one might expect to find a massless vector in the spectrum. We will comment on the potential subtleties involved with approaching the massless point as well as working in the tachyonic region when we discuss the physics of these models in the following section.

### 3.3 Double trace deformation of Witten

Let us now examine the behavior of generating function (3.5) for the double trace deformation of Witten type [2]. This is a formulation which only exists in $d = 3$ because that is the only dimension in which an anti-symmetric 3 form exists as an invariant tensor. The action (3.5) should be viewed as an effective action, obtained by considering a general theory of the type

$$i \int d^3x \left( \frac{n_1}{4\pi} \epsilon^{ijk} A_i \partial_j A_k + \frac{1}{2\pi} \epsilon^{ijk} A_i \partial_j B_k + \frac{n_2}{4\pi} \epsilon^{ijk} B_i \partial_j B_k + \frac{1}{2\pi} \epsilon^{ijk} B_i \partial_j K_k \right)$$  \hspace{1cm} (3.16)

where $n_1$ and $n_2$ are integers, as is required by the quantization of Chern-Simons level. One can also consider repeating this manipulation for any finite set sequence of integers $(n_1, n_2, \ldots n_N)$. Integrating out all fields except $A$ (and recalling that $K$ is not a field), one finds that

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{2\pi(\beta^2 - 4\alpha\gamma)}{\beta} & \frac{2\gamma}{\beta} \\ \frac{-2\alpha}{\beta} & \frac{1}{2\pi\beta} \end{pmatrix}$$  \hspace{1cm} (3.17)

must be an element of $SL(2, \mathbb{Z})$. This implies that $\alpha$, $\beta$, and $\gamma$ can take on fractional values. This is fine as long as one understands that this is an effective theory. See [17–20] for related discussions. One should also regard (3.16) as the microscopically well defined prescription for introducing the auxiliary field (which we refer generally as $\rho$) in order for the source $K$ to couple linearly to the fields in order to satisfy the assumptions of the fluctuation dissipation theorem.

In order to understand the $SL(2, \mathbb{Z})$ transformation of the susceptibility and the two-point function, we need the analogue of (3.8) and (3.9). It is not very difficult to show that (3.8) and (3.9) for (3.5) becomes

$$\frac{1}{N(d-2)} \frac{\delta}{\delta K_i} = \frac{\alpha^2 - 4\alpha\gamma}{\beta} \epsilon_{ijk} p_j \tilde{a}_k - \frac{2\gamma}{\beta (d-2)} \left( \frac{1}{e^2} \sqrt{g} g^{zz} g^{ii} \partial_z \tilde{A}_i \right) \hspace{1cm} (3.18)$$

$$\epsilon_{ijk} p_j \tilde{K}_k = -\frac{2\alpha}{\beta} \epsilon_{ijk} p_j \tilde{a}_k - \frac{1}{\beta (d-2)} \left( \frac{1}{e^2} \sqrt{g} g^{zz} g^{ii} \partial_z \tilde{A}_i \right). \hspace{1cm} (3.19)$$

With a little bit of algebra, one can show that the quantity

$$\sigma_{ij} = P_{mn} p_k \epsilon_{kjm} \Delta_{in}$$  \hspace{1cm} (3.20)
Figure 1: Cross section plot of $SL(2, \mathbb{R})$ theory space for the Marolf-Ross family of theories. Plot (a) encodes the longitudinal component and plot (b) encodes the transverse component for $\omega = 0$. The black dot corresponds to the Neumann point and the green dot corresponds to the Dirichlet point. Inside the red shaded region, the susceptibility is negative and we say that the model is unbounded. Inside the blue region, the spectrum of small fluctuation includes a tachyon. The tachyons for the longitudinal component of the vector fluctuations in $\mathbb{R}^{d-1}$ are ghostlike when continued to Minkowski signature $\mathbb{R}^{1,d-2} \times S_1$. A similar plot for the scalar $SL(2, \mathbb{R})$ family of scalar theories can be found in figure 2 of [1].
transforms modularly, in terms of which
\[ \Delta_{ij} = -\sigma_{il}\epsilon_{mj}p_m. \] (3.21)

The quantity \(2\pi\sigma_{ij}\) indeed can be interpreted as the conductivity tensor which is known to transform modularly \([4, 5, 7, 8]\). One can therefore start with \(\Delta_{ij}\) for the Dirichlet theory, map to \(2\pi\sigma_{ij}\), perform the modular transformation, and map back to \(\Delta_{ij}\). This time, the \(SL(2, \mathbb{Z})\) transform can give rise to non-vanishing \(H\) in (A.31) even if \(H = 0\) for the Dirichlet theory. Curiously, the symmetric component \(J\) appears to remain zero for these classes of models.

One can in fact write \(\sigma\) for the components of (A.31) in the form
\[ \sigma = \frac{1}{p} \begin{pmatrix} pH & -F \\ G & pH \end{pmatrix}. \] (3.22)

Note that in the zero temperature limit, \(F\) and \(G\) become the same, and \(\sigma\) can be viewed as
\[ H1 + \frac{1}{p}F(i\sigma_2) \leftrightarrow \frac{1}{2\pi}(w + it) \] (3.23)
in the notation of \([2]\). It is \(w + it\) which transformed modularly in \([2]\). We can understand our computation as simply the finite temperature generalization of the computation of \([2]\).

It is straightforward to read off the form of \(F\), \(G\), and \(H\) for any of the models labeled by elements of \(SL(2, \mathbb{Z})\). We find that
\[
\begin{align*}
F_A(p^2) &= \frac{p^2 F_D(p^2)}{c^2 p^2 + 4\pi^2 d^2 F_D(p^2) G_D(p^2)} \quad (3.24) \\
G_A(p^2) &= \frac{p^2 G_D(p^2)}{c^2 p^2 + 4\pi^2 d^2 F_D(p^2) G_D(p^2)} \quad (3.25) \\
H_A(p^2) &= \frac{acp^2 + 4\pi^2 bd F_D(p^2) G_D(p^2)}{2\pi^2 p^2 + 8\pi^3 d^2 F_D(p^2) G_D(p^2)}. \quad (3.26)
\end{align*}
\]

Note that although strictly speaking \(\{a, b, c, d\}\) should form a discrete set consistent with the \(SL(2, \mathbb{Z})\) structure, the formulas for \(F_A(p^2)\), \(G_A(p^2)\), and \(H_A(p^2)\) depend smoothly on them, as if they were defined on \(SL(2, \mathbb{R})\). This is not uncommon for a physical observable in fractionally structured systems \([21]\).

One might consider drawing a diagram similar to figure [1] for the Witten theory, but it appears that in this class of theories we never encounter models containing tachyons. Curiously, the susceptibility
\[ \chi_A = -\frac{1}{c^2 + 4\pi^2 d^2} \] (3.27)
is strictly negative. We will comment on what these things might possibly mean physically in the next section.
4 Physical features of Marolf-Ross and Witten deformations

Now that we have spelled out the technical aspects of Marolf-Ross and Witten deformations of vector fields in anti de Sitter space, let us comment on their physical features.

First and foremost, we wish to reiterate that the two deformations are distinct from one another. That of Marolf and Ross gives rise to an $SL(2, \mathbb{R})$ set of allowed deformed theories, whereas Witten’s gives rise to a discrete set corresponding to $SL(2, \mathbb{Z})$. The way in which the modular transform acts is also different. The two deformations give rise to different boundary conditions, and they ultimately give rise to different correlation functions. Clearly, they are physically distinct.

Following our own treatment of the scalars in [1], we formulated our analysis in Euclidean signature where the boundary is topologically $\mathbb{R}^{d-1} \times S_1$ (or $\mathbb{T}^{d-1} \times S_1$ for the purpose of providing an IR cut-off). It is worth noting at this point that in contemplating real time physics, one can either consider Wick rotating along one of the $\mathbb{R}^{d-1}$ directions or the $S_1$ direction. We will make some comments on both.

Since the $S_1$ is compact and considered to be small, it is natural to restrict to the zero mode sector in this direction and set $\omega = 0$ for many issues. If so, $A_0$ essentially becomes a scalar in $\mathbb{R}^{d-1}$. The susceptibility, computed for the Dirichlet theory, turns out to be negative. This is in agreement with what we found for scalars in our previous work [1].

For the case of scalars, the $SL(2, \mathbb{R})$ transform was found to potentially push the susceptibility into positive values. It is natural to contemplate what the status of this issue is for the $A_0$ field. Here, there is already a difference between Marolf-Ross and Witten. In the Marolf-Ross case, the susceptibility transforms modularly as shown in (3.15). As such, one expects to find a class of theories in the $SL(2, \mathbb{R})$ theory space with positive susceptibility. In fact, the Neumann theory has positive susceptibility.

The situation is quite different in the Witten case. Looking at (3.24) combined with the fact that $F_D(p^2)$ and $G_D(p^2)/p^2$ at $p^2 = 0$ are finite and negative implies that $F_A(0)$ is negative definite for all of the models parameterized by $SL(2, \mathbb{Z})$.

It is interesting to compare this result to the charge susceptibility of Reissner-Nordstrom black holes in anti de Sitter space. In canonical ensemble at fixed charge, one defines the free energy

$$F(Q)$$

which in the presence of background chemical potential is

$$F(Q) - \mu Q.$$
The system seeks to adjust $Q$ to minimize this quantity, and so the stability is encoded in the condition that

$$\frac{d^2 F(Q)}{dQ^2} = \frac{d\mu}{dQ} > 0$$

(4.3)

which is equivalent to

$$\frac{dQ}{d\mu} > 0$$

(4.4)

in grand canonical ensemble. The fact that large charged black holes are stable (see e.g. [22]) appears to be at odds with our observation that the susceptibility of $A_0$ is negative. This apparent conflict can be resolved upon realizing that in order to recover the black hole physics, we analytically continue $A_0$ and therefore $Q$. If $F(Q) \sim -Q^2_E$ but $Q_E = iQ$, then $F(Q) \sim Q^2$ with a positive coefficient.

The susceptibility for the transverse mode encoded in $G_D(p^2)$ appears to be automatically zero for the Dirichlet theory. In the Marolf-Ross setup, since the susceptibility $G_A$ also transforms modularly according to (3.15), we will find a region in $SL(2, \mathbb{R})$ theory space where the $G_A(0)$ is positive and a complementary region where $G_A(0)$ is negative. The regions of positive and negative susceptibilities were illustrated in figure 1. In the Witten setup, one sees according to (3.25) that $G_A(0)$ is zero for all the theories in the $SL(2, \mathbb{Z})$ theory space. We are beginning to see the trend that while the behavior of the theories is sensitive to the parameters in the Marolf-Ross setup, the Witten setup in contrast is much more rigid.

Another class of physical features explored for the scalars in [1] is the spectrum of small oscillations. The most straightforward case is to restrict to the $\omega = 0$ sector and identify the poles in the correlation function as a function of $p^2$ as corresponding to normalizable fluctuation modes. In the case of scalars, there was indeed a set of gapped normalizable states as long as the radius of $S_1$ was kept finite. It was also observed that tachyons can appear in the spectrum as the boundary condition is varied. The precise region in the $SL(2, \mathbb{R})$ parameter space where the spectrum includes a tachyon was illustrated in figure 2 of [1]. An especially interesting feature we can infer from the analysis of small fluctuation of this type is the fact that one can find massless normalizable fluctuations precisely at the boundary of the tachyonic region. When embedded into a non-linear setup, the appearance of a normalizable massless mode is precisely identifiable as the field responsible for the long range correlation at criticality. Near the onset of such a criticality, the tachyon is a signal of unstable vacua giving rise to second order phase transitions.

It is natural to contemplate what the status of analogous issues is for the vectors. If one contemplates the $A_0$ component of the gauge field from the $\mathbb{R}^{d-1}$ point of view, the Marolf-Ross version of the story is very similar to what we found for the scalars. Perhaps
this should not come as a surprise by now. \( A_0 \) is a Kaluza-Klein scalar from the \( \mathbb{R}^{d-1} \times S_1 \) point of view.

The situation with regards to the vectors in \( \mathbb{R}^{d-1} \) is more interesting. For generic \( SL(2, \mathbb{R}) \) elements, the spectrum is gapped, as one would expect from the presence of thermal \( S_1 \) \cite{23}. The lightest state can have positive or negative mass squared. It is easy to interpret the positive mass squared perturbations as corresponding to physical, massive, vector fluctuations when \( \mathbb{R}^{d-1} \) is continued to Minkowski signature \( \mathbb{R}^{1,d-2} \).

As we explore the \( SL(2, \mathbb{R}) \) theory space and modify the boundary condition for the bulk gauge fields accordingly, the mass of the lowest mass state can be pushed to zero, and beyond. In fact, in the Marolf-Ross setup, the Neumann boundary condition precisely corresponds to the case where the vectors on \( \mathbb{R}^{d-1} \) is massless. This is also consistent with the observation of \cite{3} that massless normalizable modes were found for vectors in anti de Sitter space in global coordinates with Neumann boundary conditions. In both the global anti de Sitter space and in thermally compactified Poincaré AdS geometry, there is an explicit IR cut-off and the bulk modes are expected to normalizable \cite{23}. The Neumann boundary condition appears to precisely support such a normalizable mode.

There is, however, a subtlety with continuing the squared mass of vector fields to zero and and to negative values in the Lorentzian setup where the boundary is \( \mathbb{R}^{1,d-2} \times S_1 \). The issue concerns the longitudinal mode, which is a perfectly normalizable, gapped state when the mass squared is positive. In the massless limit, however, this state becomes null, and when mass squared is negative, the state has negative norm. This issue is easier to understand in the context of the equivalent Goldstone description \cite{24,25}, where one uses the Stueckelberg field instead of the longitudinal photon to parameterize the degree of freedom. The \( m^2 \to 0 \) limit then makes the Stueckelberg action \cite{3,7} strongly coupled, and negative \( m^2 \) makes the action negative. For Abelian theories, the strict \( m^2 \to 0 \) limit may be safe, but continuation to negative \( m^2 \) looks to be problematic. With non-linearities included, one expects even more serious singularities in taking the massless limit, at least in flat space \cite{26,27}. This pathology in taking massless limit of a vector field\(^3\) appears to be consistent with the difficulty in generalizing the standard Landau-Ginzburg description of second order phase transition as condensation of scalars to the case of vectors. In fact, attempts to stabilize the tachyons via quartic interactions \cite{31} appear to lead to an irrelevant operator, the sign of which signals tension with unitarity and analyticity \cite{32}. This also sits well with the empirical observation that symmetry breaking via vector fields acquiring a vacuum expectation value has not yet been seen in particle physics or condensed matter contexts, when other features such as superconductivity and the quantum Hall effect are prevalent. We suspect similar

\(^3\)A slightly different story \cite{28,30} emerges when the cosmological constant of the boundary is non-zero.
issues are present when attempting to induce normalizable massless spin 2 modes by tuning the double trace deformations \[6,10,33–35\]. Another interesting issue to explore is whether the dichotomy between \(SL(2, \mathbb{R})\) and \(SL(2, \mathbb{Z})\) exists also in the gravity sector. We believe that the formulation (3.1) we adopt in classifying the boundary condition, which explicitly accounts for the double trace and contact term deformations, is useful for studying these issues systematically.

Another useful application of the parameterization (3.1) is to explicitly implement a Thirring \(J^2\) type deformation in the Dirichlet boundary condition. A path integral expression for a generating function for such a setup where one Legendre transforms from Dirichlet, to Neumann, and back to Dirichlet again, can be written explicitly as follows

\[
Z[\mu_i] = \int [Da_i][DA_i]_{a_i}[DJ_i] \exp \left[ -S[A_i] + (d - 2)(\beta a_i J_i + \gamma J_i J_i - \beta J_i \mu_i) \right]
= \int [Da_i][DA_i]_{a_i} \exp \left[ -S[A_i] + \frac{(d - 2)\beta^2}{4\gamma} (a_i - \mu_i)^2 \right].
\]

Here, \(\mu_i\) is playing the role of the Dirichlet source. One can think of this model as an element \(\Lambda\) in the \(SL(2, \mathbb{R})\) family of theories where

\[
\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & \frac{2\gamma}{\beta^2} \end{pmatrix}.
\]

As such, this is precisely in the class of double trace deformed Dirichlet theories discussed in [1]. Note that the act of adding a \(J^2\) term can be seen as effectively inducing a Gaussian width for the path integral of \(a_i\). This deformation does induce a normalizable, propagating, spin 1 degree of freedom although for a generic value of \(\gamma/\beta^2\), the state is massive. Perhaps this way of thinking is closely related to how a \(T\overline{T}\) deformation induces a normalizable gravitational fluctuation in the analysis of [35]. It would be interesting if this type of Gaussian deformation could be related to an explicit Dirichlet wall along the lines considered in [34,35]. It is also worth noting that the limit \(\gamma \to 0\) may also be subtle since it involves integrating out a vector field \(J_i\) in the process.

Not surprisingly, the Witten version (3.5) of the story for the spectrum of small fluctuations (aside from the well established application to quantum Hall effect) is much less exciting. Massless and tachyonic modes do not appear to arise. Perhaps one can understand this simply as a reflection of the fact that Chern-Simons terms are a reliable mechanism for generating a mass gap for vector fields in 2+1 dimensions.

Let us close this paper by considering few possible interesting extensions of this work. A main ingredient in the construction of the models with double trace and contact term deformations of the type (3.1) and (3.5) is the gauging of a \(U(1)\) global symmetry of a CFT
which admits a AdS/CFT type dual. It might be interesting to consider a setup where several CFTs with $U(1)$ global symmetry are gauged together so that they interact via the gauge sector. This will give rise to a holographic dual which has several distinct IR branches which communicate via the degrees of freedom localized in the UV. The various IR sectors are approximately decoupled in the IR limit, and that corresponds to different branches of the bulk solution. Presumably, upon increasing the strength of the gauge interaction, the different branches of the gravity dual will gradually merge, starting from the UV end. It would be interesting to make this idea more concrete. Constructions similar to this have also been considered in the context of Randall-Sundrum model building [36].

Another interesting issue to consider is whether it is possible to construct a situation where a theory in three dimensions with a two dimensional interface is arranged so that one side of the interface is a Witten-type theory and the other side is some other theory in the $SL(2, \mathbb{Z})$ family. One might try the same exercise in the Marolf-Ross setup but the fact that different theories in $SL(2, \mathbb{R})$ are smoothly deformable seems inconsistent with the existence of an interface with a good UV description, unless the interface itself has a continuous deformation parameter. These objects are not strictly speaking ‘domain walls’ since the existence or absence of such objects requires more information than is intrinsic to the theory itself. One way in which it would be possible to provide a concrete context for these objects is to require that a bulk description exists with consistent microscopic physics. This, presumably, can be related to the existence of a concrete UV complete description of the interface on the boundary side. A setup like this is very strongly reminiscent of what was considered in [37,38], and it would be of interest to explore the range of possible dynamics one

\[4\] Of course, there are other ways to couple the AdS systems at the boundary that one could consider.
can engineer in this approach. If the physics of the interface can be understood sufficiently thoroughly, perhaps one might even consider letting several boundary theories meet at a common interface and give rise to even richer physics.

There is actually one concrete realization of an interface of this type that one might consider. This consists of considering the $\mathcal{N} = 6$ ABJM system $AdS_4 \times CP_3$ with $k$ units of RR2 flux through $CP_1$ \[39\]. Such a theory gives rise to a gauge field in $AdS_4$ with gauge group $SU(4) \times U(1)$ which contains $U(1)^4$ as an Abelian subgroup, corresponding to the global symmetry of the boundary CFT. One can in fact think of this system as a concrete, UV complete, realization of the CFT with global $U(1)$ symmetries for which one might consider various double trace and contact term deformations. A particularly convenient $U(1)$ to consider is the baryonic $U(1)_J$ current, which \[39\] refers to as $A_J$, which for $k^5 \gg N \gg k$ essentially corresponds to the M-theory 3-form $C_3$ integrated over $CP_1 \subset CP_3$ giving rise to a 1-form gauge field on $AdS_4$. The IIA Chern-Simons term then essentially induces a $\theta F \wedge F$ term in the bulk $AdS_4$ where $\theta$ is the period of $B_{NSNS}$ on $CP_1$. The bulk $F \wedge F$ term is the same thing as the boundary Chern-Simons term. One can then think of a defect which shifts $\theta$ as connecting two theories related by a Witten-type $T$ transformation. Since the value of $B_{NSNS}$ corresponds to the D4 Page charge \[40\], a D4 brane wrapped on $CP_1$ and extended along a surface of codimension one at the boundary and the radial coordinate will correspond precisely to a magnetic source shifting the D4 Page charge. Some aspect of a construction like this is reminiscent of \[41\]. It would also be interesting to identify the defect corresponding to the $S$-transformation which presumably is related to something resembling a Janus configuration \[42\]. All such interfaces should exhibit specific localized dynamics which would be interesting to map out.

It would also be interesting to include the effects of gravitational back reaction as we tune the $SL(2, \mathbb{R})$ and $SL(2, \mathbb{Z})$ boundary parameters. It is interesting to note that the $SL(2, \mathbb{Z})$ structure of Witten is somewhat sensitive to quantum issues via incorporation of the constraint of quantization of instanton number. It is possible that the $SL(2, \mathbb{R})$ of Marolf and Ross also experiences some kind of discretization from effects which we have not yet accounted for.

Finally, let us note that in this article, we demonstrated that there are at least four distinct deformations, which one might call $S_{MR}$, $T_{MR}$, $S_W$, and $T_W$, for the $S$ and $T$ transformations of Marolf-Ross and Witten formulations of double trace and contact deformations. The fact that $S_{MR}$ and $T_{MR}$ do not commute gave rise to the $SL(2, \mathbb{R})$ family of theories as the maximally allowed possible deformations in the Marolf-Ross setup, and similar consideration for the Witten setup gave rise to the $SL(2, \mathbb{Z})$ structure for the set of theories. However, it is also clear that $S_{MR}$ and $T_W$ shouldn’t commute. It could be interesting to
explore the maximal set generated by the requirement that all four generators $S_{MR}$, $T_{MR}$, $S_W$, and $T_W$ are included. Presumably, this will be some kind of an affine group anticipated in [1].

We hope to address some of these issues in a very near future.

Acknowledgements

We thank Yang Bai and Josh Berger for discussions. This work is supported in part by the DOE grant de-sc0017647.

A Review of AdS/CFT Correspondence for bulk vector fields

The AdS/CFT correspondence for vector fields was analyzed systematically as far back as [43–45]. Let us review some of the basic setup here so that the conventions and notations are explicit. We will attempt to follow the conventions of [1] as closely as possible.

A.1 Background geometry

Let us consider a Euclidean Schwarzschild $AdS_{d+1}$ background with the metric of the form

$$ds^2 = \frac{r^2}{R^2} (f(r)d\tau^2 + d\vec{x}^2) + \frac{R^2}{f(r)r^2} dr^2$$

(A.1)

where

$$f(r) = 1 - \frac{d}{r^d}$$

(A.2)

and $\vec{x}$ has $d - 1$ components. We find it convenient to also introduce the radial coordinate

$$z = \frac{R^2}{r}$$

(A.3)

so that the metric becomes

$$ds^2 = \frac{R^2}{z^2} \left( f(z)d\tau^2 + d\vec{x}^2 + \frac{1}{f(z)} dz^2 \right)$$

(A.4)

with

$$f(z) = 1 - \frac{d}{z_0}, \quad z_0 = \frac{R^2}{r_0}.$$  

(A.5)

As usual,

$$T = \frac{d}{4\pi R^2 r_0} = \frac{d}{4\pi z_0}.$$  

(A.6)
and in these coordinates, the boundary is $\mathbb{R}^{d-1} \times S_1$ with the $\tau$ coordinate being periodic

$$\tau \sim \tau + \frac{1}{T}.$$  \hfill (A.7)

To this background, we add a free Maxwell field whose action reads

$$S = -\frac{1}{4e^2} \int d^{d+1}x \sqrt{g} g^{\alpha \mu} F_{\mu \nu} g_{\nu \lambda} F_{\lambda \sigma}$$  \hfill (A.8)

where the sign is chosen so that $S$ is positive definite in Euclidean signature. Here, $e^2$ has mass dimension $(3 - d)$ and $A_\mu$ has dimension one. In order to state the usual AdS/CFT interpretation, we need to deal with the issue of gauge redundancy in the bulk. A standard practice is to impose the radial gauge

$$A_z = 0$$  \hfill (A.9)

and further impose the residual condition (adopting a convention that Greek indices e.g. $\mu$ take values $0 \ldots d$ whereas Roman indices e.g. $i$ take values $0 \ldots (d - 1)$ excluding the $z$ coordinate)

$$\partial_i A_i|_{z=0} = 0$$  \hfill (A.10)

at the boundary $z = 0$ in order to fix the gauge completely. Then, we can interpret the remaining $A_i$ as corresponding to an operator $O_i(x)$. We say that the $U(1)$ gauge invariance of $A_\mu$ in the bulk is manifested as the $U(1)$ global symmetry of the field theory dual, and that the $O_i(x)$ is the corresponding conserved current operator.

In a realistic AdS/CFT construction which follows from string theory, $A_\mu$ couples to other fields, especially the gravitons. In this article, we will only discuss the toy model where the $A_\mu$ field in the bulk is free. For the full dynamics, for concrete gauge gravity duals, the interactions definitely matter, but they can be analyzed separately. Our aim here is to classify the double trace and related boundary conditions. We will also restrict attention to the two-point functions which are not as sensitive to these non-linear issues.

### A.2 Equations of motion

In order to carry on with the computation of the correlation function of $O_i(x)$’s, it is useful to solve the bulk wave equation as a function of the radial variable $z$ in the momentum space basis for the boundary coordinates $(t, \vec{x})$. Without any loss of generality, we can orient the momentum to be of the form

$$\vec{p} = (\omega, p, \vec{0})$$  \hfill (A.11)

using the residual $SO(d - 1)$ symmetry of $\mathbb{R}^{d-1} \times S_1$. 18
It is convenient to express the action and the equation of motion in terms of momentum modes with normalization and dimensions explicitly specified. Let us define

$$\tilde{A}_i(z) = \frac{T}{V_{d-1}} \int d^d x \, e^{ipx} A_i(x, z)$$

(A.12)

so that

$$[\tilde{A}_i] = [A_i] = 1.$$  

(A.13)

The $p$ dependence of $\tilde{A}_i(z)$ will be suppressed to prevent clutter.

In these conventions, the equations of motion inferred from the variation of (A.8) read

$$\omega \tilde{A}_0''(z) + p f(z) \tilde{A}_1'(z)$$

(A.14)

$$\tilde{A}_0''(z) - \frac{(d - 3)\tilde{A}_0'(z)}{z} - \frac{p^2 \tilde{A}_0(z)}{f(z)} + \frac{\omega p \tilde{A}_1(z)}{f(z)}$$

(A.15)

$$\tilde{A}_1''(z) + \tilde{A}_0'(z) \left( \frac{3 - d}{z} + \frac{f'(z)}{f(z)} \right) - \frac{\omega^2 \tilde{A}_1(z)}{f(z)^2} + \frac{\omega p \tilde{A}_0(z)}{f(z)^2}$$

(A.16)

$$\tilde{A}_1''(z) + \tilde{A}_0'(z) \left( \frac{3 - d}{z} + \frac{f'(z)}{f(z)} \right) - \frac{\tilde{A}_1(z)(p^2 f(z) + \omega^2)}{f(z)^2}.$$  

(A.17)

Not all of these equations are independent. We see that the $d - 2$ transverse components $\tilde{A}_\perp(z)$ are decoupled. The independence of the remaining three equations (A.14), (A.15), and (A.16) can be made more explicit by changing variables to gauge and longitudinal components

$$\tilde{A}_g(z) = \frac{\omega \tilde{A}_0 + p \tilde{A}_1}{\sqrt{\omega^2 + p^2}}, \quad \tilde{A}_l(z) = \frac{p \tilde{A}_0 - \omega \tilde{A}_1}{\sqrt{\omega^2 + p^2}}.$$  

(A.18)

Then, we see that (A.14) - (A.16) is equivalent to

$$\tilde{A}_0''(z) - \frac{d - 3}{z} \tilde{A}_1'(z) + \frac{\omega^2 f'(z)}{f(z)(p^2 f(z) + \omega^2)} \tilde{A}_1'(z) - \frac{(p^2 f(z) + \omega^2)}{f(z)^2} \tilde{A}_1(z)$$

(A.19)

$$\tilde{A}_g'(z) = \frac{\omega p (f(z) - 1) \tilde{A}_1'(z)}{p^2 f(z) + \omega^2}.$$  

(A.20)

which decouples the longitudinal mode into a second order differential equation, and $\tilde{A}_g'(z)$ becomes a first order equation. The initial condition for $\tilde{A}_g'(z)$ is fixed by the residual gauge condition (A.10) at $z = 0$, so once $\tilde{A}_l(z)$ is determined, $\tilde{A}_g(z)$ is determined uniquely in this gauge. It is from the analysis of $\tilde{A}_g(z)$ where one infers that

$$\langle \partial_i O_i(x) \rangle = 0$$  

(A.21)

in the usual AdS/CFT dictionary, confirming that the operator $O_i(x)$ is conserved.
In order to proceed with the conventional analysis of the correlation function, we first observe that near the AdS boundary at \( z = 0 \), we expect\footnote{In a subtle way, I claim this is consistent with the convention of (1.5) of \cite{1}.}

\[
\tilde{A}_i(\tau, \vec{x}) = \tilde{a}_i + \tilde{O}_i z^{d-2} \tag{A.22}
\]

where

\[
\tilde{O}_i = \frac{T}{V_{d+1}} \int d^d x \, e^{ipx} O_i(\tau, x) . \tag{A.23}
\]

In the standard AdS/CFT context, we can associate the operator with the variation with respect to the source

\[
\tilde{O}_i \leftrightarrow \frac{1}{\mathcal{N}(d-2)} \frac{\delta}{\delta \tilde{a}_i} \tag{A.24}
\]

where the factor of \( \mathcal{N} \) also shown in (2.5) can be traced in a manner analogous to what we considered for the scalar. At this point, dependence on \( V_{d+1} \) and \( R \) has been scaled entirely into \( \mathcal{N} \). We can therefore treat all dimensionful objects as being dimensionless in units where \( z_0 \) is set to one, or restore dimensions in appropriate powers of \( z_0 \), as needed.

We also need to impose the regularity of the fluctuating fields near the horizon at \( z = z_0 \). Solving (A.17) and (A.19) near \( z = z_0 \) reveals two linearly independent behaviors

\[
\tilde{A}_i \sim (z_0 - z)^{\pm \omega} h(z_0 - z) \tag{A.25}
\]

where \( h(z_0 - z) \) is analytic in \((z_0 - z)\). For regularity, then, we chose the solution which decays at \( z = z_0 \). On Euclidean \( \mathbb{R}^{d-1} \times S_1 \), \( \omega \) is quantized as

\[
\omega = \frac{d}{2z_0} n, \quad n \in \mathbb{Z} . \tag{A.26}
\]

This makes

\[
\tilde{A}_i \sim (z_0 - z)^{n/2} h(z_0 - z) \tag{A.27}
\]

which corresponds precisely to the discretization which makes \( \tilde{A}_i \) smooth locally near the horizon. When analytically continued to Lorentzian signature in the \( S_1 \) direction, the condition for picking \( \pm \) corresponds to picking in-falling versus out-going boundary condition at the horizon.

With the regularity condition imposed at the horizon, the solutions to the second order differential equations (A.17) and (A.19) are uniquely specified upon determining the \( \tilde{a}_i \)'s at \( z = 0 \). This means the \( \tilde{O}_i \) are determined in terms of the \( \tilde{a}_i \)'s. Furthermore, to the extent that we are working in the probe approximation where the equations of motion are linear,
the dependence of $\tilde{O}_i$ on $\tilde{a}_i$ is linear. The two-point correlation function is determined by computing

$$
\langle \tilde{O}_i(p)\tilde{O}_j(-p) \rangle = \frac{1}{N(d-2)} \frac{\delta}{\delta \tilde{a}_i(-p)} \langle \tilde{O}_j(-p) \rangle .
$$

(A.28)

One can further use the space-time symmetries to constrain the form of this correlation function. Had we been working on $\mathbb{R}^d$, we would expect the normalized two-point function

$$
\Delta_{ij} = N(d-2)\langle \tilde{O}_i(p)\tilde{O}_j(-p) \rangle = F(p^2)P_{ij}
$$

(A.29)

to depend on a single momentum dependent function $F(p^2)$ where the projection operator is

$$
P_{ij} = \left( \delta_{ij} - \frac{p_ip_j}{p^2} \right).
$$

(A.30)

When working instead on $\mathbb{R}^{d-1} \times S_1$, we expect $\Delta_{ij}$ to break up into gauge, longitudinal, and transverse components. One can parameterize the non-trivial components of $\Delta_{ij}$ in a $2 \times 2$ matrix of the form

$$
\begin{bmatrix}
\Delta_{ll} & \Delta_{l\perp} \\
\Delta_{\perp l} & \Delta_{\perp\perp}
\end{bmatrix}
= \begin{bmatrix}
F & pJ + pH \\
pJ - pH & G
\end{bmatrix}
$$

(A.31)

and encode all the non-trivial components of the two-point function without loss of generality. The factor of $p$ in front of $J$ and $H$, corresponding to the symmetric and anti-symmetric off-diagonal components, respectively, is for future convenience. So, we find that the symmetry of $\mathbb{R}^{d-1} \times S_1$ allows up to four independent components of $\Delta_{ij}$, corresponding to four components of a $2 \times 2$ matrix.

For the pure Maxwell system whose equations of motion are (A.17) and (A.19), the longitudinal and the transverse components are decoupled, so $J = H = 0$. The only non-trivial components are $F$ and $G$. (In the $T \rightarrow 0$ limit, $F$ and $G$ become identical and reduces to (A.29).)

All that remains is to determine $F$ and $G$ by solving (A.17) and (A.19). Unfortunately, these equations are not analytically solvable, but can easily be solved numerically and analyzed in the small $p^2$ limit. The result of this analysis, for the case where we set $d = 3$ and $\omega = 0$, is illustrated in figure 3.

Note that both $F(p^2)$ and $G(p^2)$ are negative. These plots should be viewed as the analogue of figure 3 of [1] except that here, we are plotting the correlation function for the Dirichlet, rather than the Neumann theory. In fact, $F(p^2)$ behaves qualitatively the same way as the scalar two-point function. This is not unexpected, in that $A_0$ behaves as a Kaluza-Klein scalar on $\mathbb{R}^{d-1} \times S_1$. The transverse correlation function encoded by $G(p^2)$ is different from $F(p^2)$, in that $G(p^2) = 0$ for $p = 0$.  

21
Figure 3: Correlation function $F = \Delta_{ll}(p^2, \omega)$ and $G = \Delta_{\perp\perp}(p^2, \omega)$ plotted as a function of $p$ with $\omega = 0$ for the Maxwell field in AdS-Schwarzschild background with standard Dirichlet boundary condition in units where $z_0 = 1$.

We can also compute the small $p^2$ asymptotic behavior

$$F_D(p^2 = 0) = -1 + \mathcal{O}(p^2), \quad G_D(p^2 = 0) = -p^2 + \mathcal{O}(p^4).$$  \hspace{1cm} (A.32)

Explicit computations reveal that the coefficient of the constant term in $F_D$ and the $p^2$ term in $G_D$ are precisely $-1$. As was noted shortly below (A.24), $F$ and $G$ can be treated as being dimensionless by working in units where $z_0 = 1$.

References

[1] S. Casper, W. Cottrell, A. Hashimoto, A. Loveridge, and D. Pettengill, “Stability and boundedness in AdS/CFT with double trace deformations,” [1709.00445].

[2] E. Witten, “SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry,” [hep-th/0307041].

[3] D. Marolf and S. F. Ross, “Boundary conditions and new dualities: Vector fields in AdS/CFT,” JHEP 11 (2006) 085, [hep-th/0606113].

[4] A. Zee, “Quantum Hall fluids,” [cond-mat/9501022] [Lect. Notes Phys.456,99(1995)].

[5] C. P. Burgess and B. P. Dolan, “Particle vortex duality and the modular group: Applications to the quantum Hall effect and other 2d systems,” Phys. Rev. B63 (2001) 155309, [hep-th/0010246].

[6] R. G. Leigh and A. C. Petkou, “SL(2,Z) action on three-dimensional CFTs and holography,” JHEP 12 (2003) 020, [hep-th/0309177].

22
[7] S. A. Hartnoll and P. Kovtun, “Hall conductivity from dyonic black holes,” *Phys. Rev. D* **76** (2007) 066001, [0704.1160](https://arxiv.org/abs/0704.1160).

[8] M. Fujita, M. Kaminski, and A. Karch, “$SL(2,Z)$ Action on AdS/BCFT and Hall Conductivities,” *JHEP* **07** (2012) 150, [1204.0012](https://arxiv.org/abs/1204.0012).

[9] A. J. Amsel and D. Marolf, “Supersymmetric multi-trace boundary conditions in AdS,” *Class. Quant. Grav.* **26** (2009) 025010, [0808.2184](https://arxiv.org/abs/0808.2184).

[10] G. Compere and D. Marolf, “Setting the boundary free in AdS/CFT,” *Class. Quant. Grav.* **25** (2008) 195014, [0805.1902](https://arxiv.org/abs/0805.1902).

[11] T. Andrade and D. Marolf, “AdS/CFT beyond the unitarity bound,” *JHEP* **01** (2012) 049, [1105.6337](https://arxiv.org/abs/1105.6337).

[12] Ying Zhao, “Phase transition of a vector operator in AdS/CFT,” MIT Master’s Thesis, 2012.

[13] M. Bianchi, D. Z. Freedman, and K. Skenderis, “Holographic renormalization,” *Nucl. Phys. B* **631** (2002) 159–194, [hep-th/0112119](https://arxiv.org/abs/hep-th/0112119).

[14] H. Ruegg and M. Ruiz-Altaba, “The Stueckelberg field,” *Int. J. Mod. Phys. A* **19** (2004) 3265–3348, [hep-th/0304245](https://arxiv.org/abs/hep-th/0304245).

[15] L.-Y. Hung and A. Sinha, “Holographic quantum liquids in 1+1 dimensions,” *JHEP* **01** (2010) 114, [0909.3526](https://arxiv.org/abs/0909.3526).

[16] W. Cottrell, J. Hanson, A. Hashimoto, A. Loveridge, and D. Pettengill, “Intersecting D3-D3’-brane system at finite temperature,” *Phys. Rev. D* **95** (2017), no. 4, 044022, [1509.04750](https://arxiv.org/abs/1509.04750).

[17] O. Bergman, A. Hanany, A. Karch, and B. Kol, “Branes and supersymmetry breaking in three-dimensional gauge theories,” *JHEP* **10** (1999) 036, [hep-th/9908075](https://arxiv.org/abs/hep-th/9908075).

[18] D. Gaiotto and E. Witten, “S-duality of boundary conditions in $\mathcal{N} = 4$ Super Yang-Mills theory,” *Adv. Theor. Math. Phys.* **13** (2009), no. 3, 721–896, [0807.3720](https://arxiv.org/abs/0807.3720).

[19] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, “Contact Terms, Unitarity, and F-Maximization in Three-dimensional superconformal theories,” *JHEP* **10** (2012) 053, [1205.4142](https://arxiv.org/abs/1205.4142).

[20] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, “Comments on Chern-Simons contact terms in three dimensions,” *JHEP* **09** (2012) 091, [1206.5218](https://arxiv.org/abs/1206.5218).
[21] D. R. Hofstadter, “Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields,” *Phys. Rev.* B14 (1976) 2239–2249.

[22] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, “Holography, thermodynamics and fluctuations of charged AdS black holes,” *Phys. Rev.* D60 (1999) 104026, hep-th/9904197.

[23] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* 2 (1998) 505–532, hep-th/9803131.

[24] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, “Effective field theory for massive gravitons and gravity in theory space,” *Annals Phys.* 305 (2003) 96–118, hep-th/0210184.

[25] M. E. Peskin, “Lectures on the theory of the weak interaction,” 1708.09043.

[26] H. van Dam and M. J. G. Veltman, “Massive and massless Yang-Mills and gravitational fields,” *Nucl. Phys.* B22 (1970) 397–411.

[27] V. I. Zakharov, “Linearized gravitation theory and the graviton mass,” *JETP Lett.* 12 (1970) 312. [Pisma Zh. Eksp. Teor. Fiz.12,447(1970)].

[28] M. Porrati, “No van Dam-Veltman-Zakharov discontinuity in AdS space,” *Phys. Lett.* B498 (2001) 92–96, hep-th/0011152.

[29] I. I. Kogan, S. Mouslopoulos, and A. Papazoglou, “The $m \to 0$ limit for massive graviton in $dS(4)$ and $AdS(4)$: How to circumvent the van Dam-Veltman-Zakharov discontinuity,” *Phys. Lett.* B503 (2001) 173–180, hep-th/0011138.

[30] A. Karch, E. Katz, and L. Randall, “Absence of a VVDZ discontinuity in AdS(AdS),” *JHEP* 12 (2001) 016, hep-th/0106261.

[31] A. Hashimoto, “A note on spontaneously broken Lorentz invariance,” *JHEP* 08 (2008) 040, 0801.3266.

[32] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, “Causality, analyticity and an IR obstruction to UV completion,” *JHEP* 10 (2006) 014, hep-th/0602178.

[33] R. G. Leigh and A. C. Petkou, “Gravitational duality transformations on (A)dS(4),” *JHEP* 11 (2007) 079, 0704.0531.

[34] T. Andrade, W. R. Kelly, D. Marolf, and J. E. Santos, “On the stability of gravity with Dirichlet walls,” *Class. Quant. Grav.* 32 (2015), no. 23, 235006, 1504.07580.
[35] L. McGough, M. Mezei, and H. Verlinde, “Moving the CFT into the bulk with $T\bar{T}$,” 1611.03470

[36] A. Karch and L. Randall, “Locally localized gravity,” JHEP 05 (2001) 008, hep-th/0011156, [140(2000)].

[37] T. Takayanagi, “Holographic dual of BCFT,” Phys. Rev. Lett. 107 (2011) 101602, 1105.5165.

[38] M. Fujita, T. Takayanagi, and E. Tonni, “Aspects of AdS/BCFT,” JHEP 11 (2011) 043, 1108.5152.

[39] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 10 (2008) 091, 0806.1218.

[40] O. Aharony, A. Hashimoto, S. Hirano, and P. Ouyang, “D-brane charges in gravitational duals of 2+1 dimensional gauge theories and duality cascades,” JHEP 01 (2010) 072, 0906.2390.

[41] M. Fujita, C. M. Melby-Thompson, R. Meyer, and S. Sugimoto, “Holographic Chern-Simons Defects,” JHEP 06 (2016) 163, 1601.00525.

[42] D. Gaiotto and E. Witten, “Janus configurations, Chern-Simons couplings, and the $\theta$-angle in $\mathcal{N} = 4$ super Yang-Mills theory,” JHEP 06 (2010) 097, 0804.2907.

[43] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253–291, hep-th/9802150.

[44] E. D’Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, “Graviton and gauge boson propagators in AdS(d+1),” Nucl. Phys. B562 (1999) 330–352, hep-th/9902042.

[45] E. D’Hoker and D. Z. Freedman, “Gauge boson exchange in AdS(d+1),” Nucl. Phys. B544 (1999) 612–632, hep-th/9809179.