Inelastic Dissipation in a Freely Rotating Body. 
Application to Cosmic-Dust Alignment.

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ABSTRACT

Motivated by a recent study by Lazarian and Draine, which showed that a high degree of grain alignment of the paramagnetic dust is achievable if the rates of internal relaxation are controlled by the Barnett relaxation process, we undertake a study of an alternative mechanism of internal dissipation, namely, the inelastic dissipation of energy in oblate dust grains. We find that deformations at double frequency that were disregarded in earlier studies dominate the inelastic relaxation. Our results indicate that for sufficiently prolate and oblate grains, e.g. for grains with 4:1 axis ratio, or/and grains formed by agglomeration inelastic relaxation dominates the Barnett relaxation within large ($a > 0.1 \mu$m) grains. For grains with axis ratio less than 1:2 the inelastic relaxation is dominant for suprathermally rotating grains.

Subject headings: Magnetic fields; Galaxies: magnetic fields, ISM: molecular clouds, magnetic fields

1. Introduction

The problem of inelastic dissipation of a freely rotating body emerges naturally in the context of interstellar grain alignment.

It is well known that the polarization of starlight is caused by the alignment of interstellar dust grains. This was discovered by Hiltner and Hall (see Hall 1949 and references therein) who showed that the polarization was entailed by the linear dichroism,
i.e., by the differential extinction, by clouds of aligned grains, of electromagnetic waves of different polarizations.

Various mechanisms of grain alignment (see table 1 in Lazarian, Goodman & Myers 1997) depend on the internal dissipation of energy (Purcell 1979). This dissipation aligned the axis of the maximal moment of inertia and the angular momentum. This changes grain dynamics and should be accounted for in the theory of grain alignment. Therefore the grain alignment depends on the internal relaxation. In some cases, e.g. the alignment of thermally rotating grains, the alignment is not sensitive to the exact rate of internal relaxation. However for grains rotating suprathermally, more detailed knowledge of the internal relaxation is usually required (see more in the discussion section of this paper).

It is normally believed that the Barnett dissipation is the dominant process of internal relaxation within interstellar grains (Purcell 1979, Spitzer & McGlynn 1979, Lazarian 1994, Draine 1996). It happens due to the time-dependent (due to the precession of angular velocity in grain body axes) remagnetisation of the sample by means of the Barnett effect. Indeed, the Barnett effect develops magnetization parallel to the angular velocity of a rotating body because a share of the angular momentum gets transferred to the spins (see Landau and Lifshitz 1984). In the cause of the angular-velocity precession about the angular momentum (in accordance with the Euler equations), the magnetization direction is also precessing in the body-axes frame, and this entails dissipation. A recent study of paramagnetic alignment of suprathermally rotating grains (or “Purcell alignment”) has shown that grains can be nearly perfectly aligned by the interstellar magnetic field if the internal dissipation is controlled by the Barnett effect (Lazarian & Draine 1997).

Another process responsible for dissipation, known as inelastic relaxation happens as time-dependent stresses emerge in a body when it rotates about an arbitrary axis. In this paper we subject this process to scrutiny, as the existing treatment (Purcell 1979) has a number of deficiencies that we discuss in the paper.

A rigorous treatment of internal relaxation is important: if the inelastic relaxation is more efficient than the Barnett relaxation then it is the former, not the latter, that determines the dynamics of crossovers and, thereby, the degree of alignment achievable by suprathermally rotating grains (see Lazarian & Draine 1997).

In what follows we briefly discuss the rotation of an oblate grain (Section II); calculate the acceleration experienced by a point inside the grain (Section III). Then we compute the stresses caused by grain precession (Section IV), the rate of energy dissipation (Section V) and the rate of internal-dissipation-caused alignment of the major-inertia axis of the grain to its angular momentum. We also discuss whether the obtained formulae could be easily
altered in the case of prolate grains (Section VI). Our results are discussed in Section VII.

2. Notations and Assumptions

We consider a freely rotating cosmic-dust grain, employing two Cartesian coordinate systems, each with its origin at the center of mass of the body. The inertial frame \((X, Y, Z)\), with unit vectors \(e_X, e_Y, e_Z\), is chosen so that its \(Z\) axis is parallel to the (conserved) angular momentum \(\mathbf{J}\) (see Fig. 1). Coordinates with respect to this frame are denoted by the same capital letters: \(X, Y,\) and \(Z\). We also use the system which we call body frame and associate with the three principal axes of inertia: 1, 2, and 3, with coordinates \(x, y,\) and \(z\) along these. The appropriate unit vectors are \(e_1, e_2, e_3\).

Without loss of generality, one may take \(I_3 \geq I_1\) and \(I_3 \geq I_2\), where \(I_i\) are the principal moments of inertia.

Following Purcell (1979), we denote the angular velocity by \(\Omega\) and reserve \(\omega\) for the rate of precession. It is a well-known fact that, as a perfectly elastic body rotates, the extremity of the angular velocity vector \(\Omega\) describes a curve that is the intersection of two ellipsoids. One of those comes from the angular-momentum conservation:

\[
I_1^2 \Omega_1^2 + I_2^2 \Omega_2^2 + I_3^2 \Omega_3^2 = J^2 = \text{const}, \tag{1}
\]

where \(\Omega_{1,2,3}\) are the body-frame-related components of \(\Omega\). Another one, known as the Poinsot ellipsoid, is defined by the kinetic-energy conservation:

\[
I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 = J \Omega = 2 E_{\text{rot}} = \text{const} \tag{2}
\]

In the presence of inner dissipation, the Poinsot ellipsoid will be distorting in the cause of time, and will eventually degenerate into a line of length \(I_3^{-1/2}\) oriented along the axis of maximal moment of inertia. That will happen after the body aligned its angular velocity along its maximal-inertia axis: \(\Omega_1 = \Omega_2 = 0, \; |\Omega_3| = |J|/I_3\). This mode minimizes the kinetic energy, with the angular momentum \(\mathbf{J}\) fixed.

In what follows we model the grain by an oblate symmetric top, e.g. a disk

\[
I_3 > I_1 = I_2 \equiv I. \tag{3}
\]

In the grain-alignment studies such an approximation is customary (Purcell 1979, Spitzer & McGlynn 1979). For such a body the Euler equations read:

\[
\dot{\Omega}_1 + \frac{I_3 - I}{I} \Omega_2 \Omega_3 = 0, \tag{4}
\]
\[ \dot{\Omega}_2 + \frac{I - I^3}{I} \Omega_3 \Omega_1 = 0 , \]  
\[ \dot{\Omega}_3 = 0 , \]  
where \( \Omega_i \) is the body-frame components of the angular velocity. As well known, the angular velocity \( \Omega \) nutates about the principal axis 3 at a constant rate

\[ \omega = (h - 1)\Omega_3, \quad h \equiv I_3/I \]  

However from the point of view of the inertial observer it is rather the principal axis 3 that is precessing about \( J \) (which is conserved in the inertial frame). Therefore angle \( \theta \) is constant. Now, let \( \alpha \) be the angle described by the precession of vector \( \Omega \) about axis 3 (or of axis 3 about \( \Omega \)). It is easy to see that

\[ \tan \theta = (I \| \Omega \| \sin \alpha)/(I_3 \| \Omega \| \cos \alpha) = h \tan \alpha ; \]  

hence \( \alpha \) is also constant. As \( \Omega_3 = \Omega \cos \alpha \), the angular velocity of precession given by (7), except a special case of \( \Omega \) and \( J \) orthogonal or almost orthogonal to the maximal-inertia axis 3, is typically of the order of \( |\Omega| \). Hence one may call the rotation and precession “fast motions”, implying that the relaxation is a slow process (which means that the rate of alignment: \( \dot{\theta} \ll \omega \)).

Before pursuing with the calculational part of our article, let us once again dwell upon the physical mechanism that changes \( \Omega_3 \) over the spans of time comparable with the internal relaxation time. In terms of energy, everything is clear: the inelastic body aligns so as to minimize its rotational energy and still to conserve its angular momentum. This obvious mechanism does not seem to be instilled in the Euler equations (4), (5), (6), for the torques remain to be zero, and nothing seems to be able to shift the solutions \( \Omega_{1,2,3}(t) \) from the form that yields \( \alpha \) constant. In reality, however, this would-be contradiction is easily resolved if one recalls that inelastic relaxation stems from the inner displacements of grain material. These displacements result in changes of grain moments of inertia. Therefore in the cause of the grain’s wobbling, its principle moments of inertia fluctuate, and terms \( \Omega_i \dot{I}_i \neq 0 \) should be accounted for in Euler equations. Inelasticity results in a phase shift between the angular velocity and \( \dot{I}_i \) and this causes internal alignment\[.\]

Since \( \dot{I}_i \) depend on the grain elasticity, while the phase offset of \( \dot{I}_i \) and \( \Omega \) is determined by inelastic effects, then the parameters that determine elasticity and dissipation within the material should enter our final formulae for the inelastic relaxation.

\[ ^1 \text{The latter statement is equivalent to the statement that there is a phase shift between the stresses that act upon the grain, and the deformation of the grain.} \]
3. Acceleration of a Point inside an Oblate Grain \((I_3 \geq I_1 = I_2)\)

Our goal is to calculate \(\dot{\theta}\), the rate of the maximal-inertia axis' approaching the direction of angular momentum \(J\). To achieve this goal, one has to know the rate of energy losses entailed by the inelastic deformation. To calculate the deformation, we shall have to know the acceleration experienced by a particle located inside the grain at a point \((x, y, z)\). Note that we address the proper acceleration, i.e. that with respect to the inertial frame \((X, Y, Z)\), but we shall express it in terms of coordinates \(x, y, z\) of the body frame \((1, 2, 3)\). The fast processes (revolution and precession of a symmetric oblate grain) are described by the Euler equations (4 - 6) whose solution, in neglect of the slow relaxation, will read

\[
\Omega_1 = \Omega_\perp \cos \omega t, \quad \Omega_2 = \Omega_\perp \sin \omega t, \quad \Omega_3 = \text{const}
\]

where

\[
\Omega_\perp = \Omega \sin \alpha, \quad \Omega_3 = \Omega \cos \alpha
\]

and

\[
\Omega_\perp/\Omega_3 = \tan \alpha = h \tan \theta.
\]

Besides, we shall need formulae connecting the components of \(\Omega\) with the absolute values of the angular momentum:

\[
\Omega_3 = \frac{J_3}{I_3} = \frac{J}{I_3} \cos \theta, \quad \Omega_\perp = \frac{J}{I_3} h \sin \theta
\]

We denote the position, velocity and acceleration relative to the body frame as: \(r, v, a\), while those related to the body frame \((1, 2, 3)\) will be called \(r'', v'', a''\), where we shall keep in mind that \(r = r''\). The acceleration \(a\) in the inertial frame looks:

\[
a = a'' + \dot{\Omega} \times r'' + 2 \\Omega \times v'' + \Omega \times (\Omega \times r'')
\]

where the first and the third terms vanish, as the values of relative deformations \(\delta l/l\) are minute in a solid body. The values of \(v'' \approx \delta l/\tau\) and \(a'' \approx \delta l/\tau^2\), where \(\tau\) is the period of rotation, are negligible compared to velocities and accelerations in the inertial system of reference that by the order of magnitude are, respectively \(l/\tau\) and \(l/\tau^2\). The second and the fourth term in Eq. (13), referred to the grain frame with unit vectors \((e_1, e_2, e_3)\), will read:

\[
\dot{\Omega} \times r'' = \left(\frac{\dot{\Omega}_2}{\Omega} z - \dot{\Omega}_3 y\right) + \left(\frac{\dot{\Omega}_3}{\Omega} x - \dot{\Omega}_1 z\right) + \left(\frac{\dot{\Omega}_1}{\Omega} y - \dot{\Omega}_2 x\right) = e_1 \omega z \Omega_1 + e_2 \omega z \Omega_2 + e_3 \left(-\omega y \Omega_2 - \omega x \Omega_1\right)
\]

and

\[
\Omega \times (\Omega \times r'') = \Omega \left(\Omega \cdot r''\right) - r'' \Omega^2 =
\]
\[ e_1 \{ \Omega_1 ( \Omega_1 x + \Omega_2 y + \Omega_3 z ) - x \Omega^2 \} + \]
\[ e_2 \{ \Omega_2 ( \Omega_1 x + \Omega_2 y + \Omega_3 z ) - y \Omega^2 \} + \]
\[ e_3 \{ \Omega_3 ( \Omega_1 x + \Omega_2 y + \Omega_3 z ) - z \Omega^2 \} . \]  \tag{15}

All in all,
\[ a = e_1 \{ - x \Omega^2 - x \Omega_3^2 + y \Omega_2 \Omega_1 + z \Omega_3 \Omega_1 + z \omega \Omega_\perp \cos \omega t \} \]
\[ + e_2 \{ x \Omega_2 \Omega_1 - y \Omega_1^2 - y \Omega_3^2 + z \Omega_3 \Omega_2 + z \omega \Omega_\perp \sin \omega t \} \]
\[ + e_3 \{ x \Omega_1 \Omega_3 + y \Omega_3 \Omega_2 - z \Omega_\perp^2 - \omega \Omega_\perp (x \cos \omega t + y \sin \omega t) \} . \]  \tag{16}

We had to present our calculation in great detail because the latter formula considerably differs from the one in (Purcell 1979), eq. 28. To be doubly sure of our formula being correct, we derived it also in the intermediate, “Eulerian”, system of reference (see Appendix A).

4. Stresses and Strains Caused by the Precession

In this section we derive the stresses and strains produced in the rotating body by the time-dependent terms in (16), as only these terms influence the rate of energy dissipation in the grain.

With aid of (9) one can easily split (16) into a time-independent and time-dependent parts:
\[ a = a_0 + a_t , \]  \tag{17}
where
\[ a_0 = - ( e_1 x + e_2 y ) ( \Omega_3^2 + \frac{1}{2} \Omega_\perp^2 ) - e_3 z \Omega_\perp^2 , \]  \tag{18}
and
\[ a_t = e_1 \{ \frac{1}{2} \Omega_\perp^2 x \cos 2\omega t + \frac{1}{2} \Omega_\perp^2 y \sin 2\omega t + z \Omega_\perp \Omega_3 h \cos \omega t \} \]
\[ + e_2 \{ \frac{1}{2} \Omega_\perp^2 x \sin 2\omega t - \frac{1}{2} \Omega_\perp^2 y \cos 2\omega t + z \Omega_\perp \Omega_3 h \sin \omega t \} \]
\[ + e_3 \{ \Omega_\perp \Omega_3 (2 - h) (x \cos \omega t + y \sin \omega t) \} . \]  \tag{19}

Now we see that dissipation of energy will be taking place at two modes one of which will be of double frequency. Hereafter we shall be taking into consideration only time-dependent, i.e. produced by $a_t$, inputs in the stresses and strains.
Following Purcell, we shall model the oblate grain by a prism of sizes $2a \times 2a \times 2c$, $(c < a)$. The stresses vanishing on the boundaries of our rectangular prism read

\begin{align}
\sigma_{xx} &= \frac{\rho \Omega^2}{4} (x^2 - a^2) \cos 2\omega t , \quad \sigma_{yy} = -\frac{\rho \Omega^2}{4} (y^2 - a^2) \cos 2\omega t , \quad \sigma_{zz} = 0 , \\
\sigma_{xy} &= \frac{\rho}{2} \Omega_\perp \Omega_3 \left[ h (z^2 - c^2) + (2 - h) (x^2 - a^2) \right] \sin \omega t , \\
\sigma_{xz} &= \frac{\rho}{2} \Omega_\perp \Omega_3 \left[ h (z^2 - c^2) + (2 - h) (y^2 - a^2) \right] \cos \omega t , \\
\sigma_{yz} &= \frac{\rho}{2} \Omega_\perp \Omega_3 \left[ h (z^2 - c^2) + (2 - h) (y^2 - a^2) \right] \sin \omega t ,
\end{align}

which leads to forces per unit volume:

\begin{align}
F_x &= \partial_i \sigma_{xi} = \frac{\rho}{2} \Omega_\perp^2 x \cos 2\omega t + \frac{\rho}{2} \Omega_\perp^2 y \sin 2\omega t + \rho \Omega_\perp \Omega_3 h z \cos \omega t , \\
F_y &= \partial_i \sigma_{yi} = \frac{\rho}{2} \Omega_\perp^2 x \sin 2\omega t - \frac{\rho}{2} \Omega_\perp^2 y \cos 2\omega t + \rho \Omega_\perp \Omega_3 h z \sin \omega t , \\
F_z &= \partial_i \sigma_{zi} = \rho \Omega_\perp \Omega_3 (2 - h) x \cos \omega t + \rho \Omega_\perp \Omega_3 (2 - h) y \sin \omega t ,
\end{align}

in full accordance with (19). Here $\rho$ is the density of the grain material. For a $2a \times 2a \times 2c$ prism, the moment of inertia $I_3$ and the parameter $h$ are function of the half-sizes $a$ and $c$:

\begin{align}
I_3 &= \frac{16}{3} \rho a^4 c \\
h &\equiv \frac{I_3}{I} = \frac{2}{1 + (c/a)^2} , \quad 2 - h = \left( \frac{c}{a} \right)^2 \frac{2}{1 + (c/a)^2} .
\end{align}

According to (3.12), (3.13),

\begin{align}
\Omega_3 &= \Omega_0 \cos \theta , \quad \Omega_\perp = \Omega_0 h \sin \theta ,
\end{align}

$\Omega_0 \equiv J/I_3$ being the typical angular velocity of a grain. Our knowledge of the stresses should be sufficient for computing the rate of energy losses in the body.

Our expressions for the stress-tensor components differ very significantly from the appropriate expressions derived in (Purcell 1979). There are two reasons for it. The first reason is the afore-mentioned Purcell’s miscalculation of the acceleration experienced by a point inside the wobbling body. (Because of that Purcell completely missed the double-frequency terms, and they do not vanish on the boundaries of the grain, which they should do.)
double-frequency contribution to the stress tensor. Later we shall show that this input is of the leading order.) The second reason is that Purcell forgot to impose proper boundary conditions upon the stress-tensor components: his stresses fail to vanish on the boundaries of the body.

Our goal now is to calculate the average deformation-caused energy, stored in the tumbling body, and to estimate the energy dissipation, using an appropriate quality factor. We shall take into account that the deformation of the grain is neither purely elastic nor purely plastic, but is a superposition of the former and the latter. It is then to be described by the tensor $\epsilon_{ij}$ of viscoelastic strains and by the velocity tensor consisting of the time-derivatives $\dot{\epsilon}_{ij}$. The stress tensor will now be separated into two components: the elastic stress and the plastic (viscous) stress:

$$\sigma_{ij} = \sigma^{(e)}_{ij} + \sigma^{(p)}_{ij}, \quad (30)$$

where the components of the elastic stress tensor are interconnected with those of the strain tensor (Landau and Lifshitz 1976):

$$\epsilon_{ij} = \delta_{ij} \frac{Tr \sigma^{(e)}_{ij}}{9K} + \left( \sigma^{(e)}_{ij} - \frac{1}{3} \delta_{ij} \frac{Tr \sigma^{(e)}_{ij}}{2\mu} \right), \quad (31)$$

$$\sigma^{(e)}_{ij} = K \delta_{ij} Tr \epsilon + 2 \mu \left( \epsilon_{ij} - \frac{1}{3} \delta_{ij} Tr \epsilon \right), \quad (32)$$

$\mu$ and $K$ being the isothermal shear and bulk moduli, and $Tr$ standing for the trace of a tensor. Components of the plastic stress are connected with the strain derivatives as

$$\dot{\epsilon}_{ij} = \delta_{ij} \frac{Tr \sigma^{(p)}_{ij}}{9\zeta} + \left( \sigma^{(p)}_{ij} - \frac{1}{3} \delta_{ij} \frac{Tr \sigma^{(p)}_{ij}}{2\eta} \right), \quad (33)$$

$$\sigma^{(p)}_{ij} = \zeta \delta_{ij} Tr \dot{\epsilon} + 2 \eta \left( \dot{\epsilon}_{ij} - \frac{1}{3} \delta_{ij} Tr \dot{\epsilon} \right), \quad (34)$$

where $\eta$ and $\zeta$ are the shear and stretch viscosities.

5. Dynamics of a tumbling body

The kinetic energy of a rotating body reads, according to (2), (3), (9), and (12):

$$E_{rot} = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2) = \frac{1}{2} \left[ I_1 \Omega_1^2 + I_3 \Omega_3^2 \right] =\frac{1}{2} \left[ \frac{1}{I} \sin^2 \theta + \frac{1}{I_3} \cos^2 \theta \right] J^2 \quad (35)$$
wherefrom
\[
\frac{dE_{\text{rot}}}{d\theta} = \frac{J^2}{I_3} (h - 1) \sin \theta \cos \theta = \omega J \sin \theta .
\] (36)

Formula (36) provides an insight in how the change of rotational energy yields a change of \(\theta\). If we calculate the rate of energy losses, \(\dot{E}_{\text{rot}}\), it will be easy to find the rate of alignment \(\dot{\theta}\) as \((dE_{\text{rot}}/d\theta)^{-1}\dot{E}_{\text{rot}}\). How to compute \(\dot{E}_{\text{rot}}\)? The rotational energy changes via the inelastic dissipation, so that
\[
\dot{E}_{\text{rot}} = \dot{\dot{W}}
\] (37)

Hence what we have to find is the rate of the elastic-energy losses \(\dot{W}\). Then, with aid of (36) we shall calculate the rate of alignment:
\[
\frac{d\theta}{dt} = \left(\frac{dE_{\text{rot}}}{d\theta}\right)^{-1} \frac{dE_{\text{rot}}}{dt} = (\omega J \sin \theta)^{-1} \dot{W}
\] (38)

In our case, dissipation is taking place at two modes:
\[
\dot{W} = W^{(\omega)} + W^{(2\omega)} = \omega \frac{W^{(\omega)}}{Q^{(\omega)}} + 2 \omega \frac{W^{(2\omega)}}{Q^{(2\omega)}} \approx \frac{\omega}{Q} \left\{ W^{(\omega)} + 2W^{(2\omega)} \right\}
\] (39)

where we used the fact that the quality factor is almost frequency-independent: \(Q^{(2\omega)} \approx Q^{(\omega)} \approx Q\).

6. Rate of Energy Dissipation at low temperatures

Before pursuing to our calculations of the energy-dissipation rate \(\dot{W}\), several prefatory notes will be in order. As well known, at low temperatures materials are fragile: when the deformations exceed some critical threshold, the body will rather break than flow. At the same time, at these temperatures the materials are elastic, provided the deformations are beneath the said threshold: the sound absorption, for example, is almost exclusively due to the thermal conductivity rather than to the viscosity. These facts may be summarized like this: at low temperatures, the viscosity coefficient \(\eta\) has, effectively, two values: one value - for small deformations (and this value is almost exactly zero); another value - for larger-than-threshold deformations (and that value is high\(^3\)).

At high temperatures materials become plastic, which means that the shear viscosity \(\eta\) gets its single value, deformation-independent in the first approximation. On the one

\(^3\)Effectively it may be put infinity because, as explained above, the body will rather crack than demonstrate fluidity.
hand, this value will be far from zero (so that the scattering of vibrations will now be predominantly due to the viscosity terms, not due to the thermal conductivity). On the other hand, this value will not be that high: a plastic body will rather yield than break. All this is certainly valid for the stretch viscosity $\zeta$ as well.

Since at low temperatures the bodies manifest, for small displacements, no viscosity ($\omega\eta \ll \mu \sim K$), the stress tensor will be approximated, to a high accuracy, by its elastic part: instead of the system (30) - (34) we shall simply write:

$$\epsilon_{ij} = \delta_{ij} \frac{Tr \sigma}{9K} + \left( \sigma_{ij} - \frac{1}{3} \delta_{ij} Tr \sigma \right) \frac{1}{2 \mu} , \quad (40)$$

This will enable us to derive an expression for the elastic energy stored in a unit volume of the precessing body:

$$dW/dV = \frac{1}{2} \epsilon_{ij} \sigma_{ij} = \frac{1}{4} \left\{ \left( \frac{2 \mu}{9K} - \frac{1}{3} \right) (Tr \sigma)^2 + \sigma_{ij} \sigma_{ij} \right\} \approx$$

$$\approx \frac{1}{4\mu} \left\{ -\frac{1}{5} (Tr \sigma)^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + 2 \left( \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \right) \right\} \quad (41)$$

where we made use of the expressions connecting the shear and bulk moduli with the Young modulus $E$ and Poisson’s ratio $\sigma$: since $K = E/[3(1 - 2\sigma)]$ and $\mu = E/[2(1 + \sigma)]$ then $2\mu/(9K) - 1/3 = -\sigma/(1 + \sigma)$. As for frozen solids Poisson’s ratio $\sigma$ is typically about 0.25, we put $2\mu/(9K) - 1/3 \approx -1/5$. Mind that the body is trembling at two frequencies: $\omega$ and $2\omega$. Anticipating the different rates of dissipation at these two modes, we shall split the total elastic energy into two parts:

$$dW/dV = dW^{(\omega)}/dV + dW^{(2\omega)}/dV \quad (42)$$

where, according to (20) - (23),

$$dW^{(\omega)}/dV = \frac{1}{2\mu} \left\{ \sigma_{yz}^2 + \sigma_{zx}^2 \right\} \quad (43)$$

and

$$dW^{(2\omega)}/dV = \frac{1}{4\mu} \left\{ -\frac{1}{5} (Tr \sigma)^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + 2\sigma_{xy}^2 \right\} \quad (44)$$

From now on we shall be interested in the energies averaged over several periods of the precession. Therefore we shall substitute $\sin^2 \omega \ldots$ and $\cos^2 \omega \ldots$ by $1/2$, and shall omit expressions $\sin \ldots \cos \ldots$, $\sin \omega t \sin 2\omega t$ and $\cos \omega t \cos 2\omega t$. With the above reservation being beared in mind, expressions (20) - (23) for the stresses will yield:

$$(Tr \sigma)^2 = \left( \frac{\rho \Omega_1^2}{4} \right)^2 a^4 \left( x_1^2 - y_1^2 \right)^2 \frac{1}{2} \quad (45)$$
\[
\sigma_{xx}^2 = \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 a^4 \left( x_1^2 - 1 \right)^2 \frac{1}{2}, \\
\sigma_{yy}^2 = \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 a^4 \left( y_1^2 - 1 \right)^2 \frac{1}{2}, \\
\sigma_{xy}^2 = 2 \gamma \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 a^4 \left( x_1^2 + y_1^2 - 2 \right)^2, \\
\sigma_{xz}^2 = 2 \gamma \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 a^4 \left( x_1^2 + z_1^2 - 2 \right)^2, \\
\sigma_{yz}^2 = 2 \gamma \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 a^4 \left( y_1^2 + z_1^2 - 2 \right)^2,
\]

where

\[x_1 \equiv x/a, \quad y_1 \equiv y/a, \quad z_1 \equiv z/a,\]

and

\[\gamma \equiv \left( \frac{\Omega_3}{\Omega_\perp} \right)^2 (2 - h)^2 = \left( \frac{\Omega_3}{\Omega_\perp} \right)^2 h^2 \left( \frac{c}{a} \right)^4 = \frac{(c/a)^4}{\tan^2 \theta} .\]

According to (39), what we shall need is the sum \( W(\omega) + 2W(2\omega) \). To get it, we shall integrate over the volume:

\[W(\omega) + 2W(2\omega) = \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-c}^{c} dz \left\{ dW(\omega)/dV + 2 dW(2\omega)/dV \right\} = \]

\[= a^2 c \int_{-1}^{1} dx_1 \int_{-1}^{1} dy_1 \int_{-1}^{1} dz_1 \frac{1}{2 \mu} \left\{ - \frac{1}{5} (Tr \sigma)^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + 2\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zz}^2 \right\} .\]

Plugging of (45) - (50) in the above expression entails:

\[W(\omega) + 2W(2\omega) = 2^{-5} \frac{a^6 \rho}{\mu} \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 \int_{-1}^{1} dx_1 \int_{-1}^{1} dy_1 \int_{-1}^{1} dz_1 \left\{ - \frac{1}{10} \left( x_1^2 - y_1^2 \right)^2 + \right. \]

\[+ \frac{1}{2} \left( x_1^2 - 1 \right)^2 + \frac{1}{2} \left( y_1^2 - 1 \right)^2 + \left( x_1^2 + y_1^2 - 2 \right)^2 + \]

\[+ 2 \gamma \left( x_1^2 + z_1^2 - 2 \right)^2 + 2 \gamma \left( y_1^2 + z_1^2 - 2 \right)^2 \right\} \approx \]

\[\approx \frac{a^6 \rho}{\mu} \left( \frac{\rho \Omega_\perp^2}{4} \right)^2 2^{-5} (63 \gamma + 20)\]

where the numerics was performed by means of the Maple software. The product \( \rho^2 \Omega_\perp^4 \) emerging in the above expression may be cast in a form wherein its temperature-dependence becomes manifest. According to (27), (28), and (29),

\[\rho^2 \Omega_\perp^4 = \frac{a^{-8} c^{-2}}{[1 + (c/a)^2]^4} \frac{9}{4} (\beta kT_{gas})^2 \sin^4 \theta \]

(55)
where \( k \) signifies the Boltzmann constant and \( \beta \) stands for the parameter of suprathermality:

\[
\frac{I_3 \Omega_2^2}{2} = \frac{J^2}{2I_3} = \beta k T_{\text{gas}}
\]  

(56)

The cosmic dust is thermal when \( \beta = 1 \). This way of writing the energies enables one to express these in terms of the temperature of the surrounding gas, \( T_{\text{gas}} \):

\[
W^{(\omega)} + 2 W^{(2\omega)} = \frac{3^2 2^{-7} (63 \gamma + 20)}{\mu} \frac{a^{-2} c^{-1}}{[1 + (c/a)^2]^4} (\beta k T_{\text{gas}})^2 \sin^4 \theta
\]  

(57)

Together, (38), (39) and (57) give:

\[
d\theta/dt = -\frac{3^2 2^{-7} (63 \gamma + 20)}{\mu Q J} \frac{a^{-2} c^{-1}}{[1 + (c/a)^2]^4} (\beta k T_{\text{gas}})^2 \sin^3 \theta
\]  

(58)

According to (58) and (27),

\[
J = (2 \beta k T_{\text{gas}} I_3)^{1/2} = \left(\frac{32}{3} \beta k T_{\text{gas}} \rho a^4 c\right)^{1/2} = 2^{5/2} 3^{-1/2} a^2 c^{1/2} \rho^{1/2} (\beta k T_{\text{gas}})^{1/2}
\]  

(59)

Substitution of the latter in the former will give us the final expression for the alignment rate:

\[
d\theta/dt = -\frac{a^{-5.5}}{[1 + (c/a)^2]^4} \left(\frac{a}{c}\right)^{3/2} (\beta k T_{\text{gas}})^{3/2} \frac{\sin^3 \theta}{\mu Q \rho^{1/2}} \left(63 (c/a)^4 \cot^2 \theta + 20\right) 3^{2.5} 2^{-9.5}
\]  

(60)

Simply from looking at this formula one can conclude that the major-inertia axis slows down its alignment for \( \theta \) approaching zero. This feature looks physically reasonable.

On physical grounds, one may also expect that the alignment rate vanishes for \( \hbar \) approaching unity (i.e., when the body lacks an axis of maximal inertia). Besides, one may expect the major axis of the grain to linger in the position \( \theta \approx \pi/2 \), as if “hesitating” to whether to start aligning along or opposite its angular momentum. However, none of the latter two features seems to be instilled into (60): it may seem from this formula that \( d\theta/dt \) is \( \hbar \)-independent, and that the major axis leaves the initial position \( \theta = \pi/2 \) with a finite angular velocity. To dispel these illusions, simply recall that our treatment is valid only for as long as the rotation and precession are fast motions compared to the alignment: \( \dot{\theta} \ll \omega \).

All these subtleties are anyway irrelevant when one merely wants to estimate the relaxation time, i.e., the time required for the maximal-inertia axis of an oblate cosmic-dust grain to be considerably shifted toward alignment with the angular momentum:

\[
t_i \simeq \left( \left\langle \frac{d\theta}{dt} \right\rangle \right)^{-1} \simeq -\int_{\pi/2}^{\delta} \frac{d\theta}{d\theta/dt}.
\]  

(61)
where \( \delta \) is introduced to avoid the divergence associated with the “slow finish”. (One can take, for example, \( \delta = \pi/8 \).) A particular choice of \( \delta \) will bring into the expression for \( t_i \) some numerical factor of order unity. Since we want nothing more but a rough estimate for \( t_i \), we shall approximate \( t_i \) simply by the inverse \( d\theta/dt \) evaluated in the middle of the interval, at \( \theta = \pi/4 \). Then

\[
t_i \approx a^{5.5} \left[ 1 + (c/a)^2 \right]^4 \left( \frac{c}{a} \right)^{3/2} \left( \beta k T_{\text{gas}} \right)^{-3/2} \mu Q \rho^{1/2} \frac{2^{11} 3^{-2.5}}{63(c/a)^4 + 20}
\]

The quantity \( (c/a)^{3/2} \left[ 1 + (c/a)^2 \right]^4 \left[ 63(c/a)^4 + 20 \right]^{-1} \) is a steep function of parameter \( (c/a) \). For \( c/a = 1/2 \) it equals to \( 4.3 \times 10^{-2} \), while for \( c/a = 1/10 \) it will be \( 1.6 \times 10^{-3} \).

The above formula for the typical time of alignment is the main result of our article. Now we must think of the possible values for the material parameters involved. The values may depend both on the temperature \( T_{\text{grain}} \) of the cosmic-dust grains, and on the frequency of the precession. To start with, temperature- and frequency-caused variations of the density \( \rho \) may be neglected, as they are to be small anyway. This way, we can use the (static) densities appropriate to the room temperature and pressure.

Now, consider the isothermal shear modulus \( \mu \). The tables of physical quantities would provide its values for the room temperature and atmospheric pressure, and for quasistatic regimes solely. As for the possible frequency-related effects in materials (the so-called ultrasonic attenuation), these become noticeable only at frequencies higher than \( 10^8 \) Hz (see section 17.7 in Nowick and Berry 1972). Another fortunate circumstance is that the pressure-dependence of the elastic moduli is known to be weak (Ahrens 1995). Besides, the elastic moduli of solids are known to be insensitive to temperature variations, as long as these variations are far enough from the melting point. The value of \( \mu \) may increase by several percent when the temperature is drops from the room temperature to 10 K. Dislocations don’t affect the elastic moduli either. Solute elements have very little effect on moduli in quantities up to a few percent\(^4\). As for the role of the possible porosity, the elastic moduli scale as the square of the relative density. For porosities up to about 20 %, this is not of much relevance for our estimates\(^5\).

According to (Ryan and Blevins 1987), at \( T \sim 20 K \), the share modulus value

\[4\]Beyond that, one might assume that the moduli vary linearly with substitutional impurities (in which the atoms of the impurity replace those of the hosts). However hydrogen is not like that: it enters the interstices between the atoms of the host, and has marginal effect on modulus.

\[5\]We are deeply thankful to Michael Ashby and Michael Aziz, who consulted us on all these subtle topics.
for silicate $\mu \approx 10^{10} \ Pa$ and density is $\rho_{\text{silicate}} \approx 2500 \ kg \ m^{-3}$. The temperature of subthermal rotating that according to Lazarian and Draine (1997) is important during crossovers is $T_{\text{rot}} \approx 20 \ K$.

Now, several words on the choice of values of the $Q$–factor will be in order. Purcell refers to (Krause 1973) wherein a review of acoustic dissipation in silicates is presented. According to (Krause 1973), for frequencies varying from $50 \ kHz$ to $27 \ GHz$, and temperatures varying from $10 \ K$ to $50 \ K$, the measured values of $Q$ range between 400 and 2000. We would tend to believe that interstellar grains are almost certainly have plenty of cracks and defects. As known from seismology, for real silicate rocks the $Q$–factor is typically between 150 and 300. Therefore we believe that $Q_{\text{sil}} < 400$. We are not familiar with the measurements of $Q$ for carbonatious materials, but it is likely that $Q$ for them will be lower than for silicates.

Therefore, for the silicate grains with axis ratio 1:2 we get:

$$t_i \approx 8 \times 10^9 \frac{1}{\beta^{3/2}} \left(\frac{a}{10^{-7} \ m}\right)^{5.5},$$

while for graphite grain we assume that the axis ratio is 1:10 $Q = 100$ and obtain

$$t_i \approx 7 \times 10^6 \frac{1}{\beta^{3/2}} \left(\frac{a}{10^{-7} \ m}\right)^{5.5},$$

where the values of $a$ are supposed to be in $metros$ while $t$ is in $seconds$ (so that for example for $a = 10^{-7} \ m$, $c/a = 1/2$ and $\beta = 1$ the typical time will be $8 \times 10^9 \ s$).

We have not attempted to calculate the dissipation rates for grains formed by loose aggregates of smaller particles. The inelastic relaxation within such grains may be orders of magnitude more efficient due to friction between parts of it ("effective viscosity"). It can well dominate the Barnett relaxation even for much larger grains.

7. Dynamics of Prolate Cosmic-Dust Grains: Libration.

At first glance, the dynamics of a freely-spinning prolate body obeys the same principles as the dynamics of an oblate one: the axis of maximal inertia will tend to align itself parallel to the angular momentum. If we model a prolate body with a symmetric top, it will be once again convenient to choose it be a prism of dimensions $2a \times 2a \times 2c$, though this time half-size $c$ are larger than $a$, and therefore $I_3 = I_2 > I_1$. Then all our calculations formally remain in force, up to formula (63): since now the factor $h - 1 = [1 - (c/a)^2]/[1 + (c/a)^2]$ becomes negative, the right-hand side in (63) will change
its sign:

\[
\frac{dE_{\text{rot}}}{d\theta} = \frac{J^2}{I_3} (h - 1) \sin \theta \cos \theta = -\omega J \sin \theta .
\]  

(65)

Thereby formula (38) will also acquire a “minus” sign in its right-hand side:

\[
\frac{d\theta}{dt} = \left( \frac{dE_{\text{rot}}}{d\theta} \right)^{-1} \frac{dE_{\text{rot}}}{dt} = - (\omega J \sin \theta)^{-1} \dot{W}
\]  

(66)

Formulae (37) and (39) will remain unaltered. Eventually, by using (66) and (39), we shall arrive to a formula that differs from (60) only by a sign, provided we keep the notation \( \theta \) for the angle between \( J \) and the body-frame axis 3 (parallel to dimension 2c):

\[
\frac{d\theta}{dt} = -a^{-5.5} \frac{(a/c)^{3/2} (\beta kT_{\text{gas}})^{3/2}}{[1 + (c/a)^2]^4} \frac{\sin^3 \theta}{\mu Q \rho^{1/2}} 3^{2.5} 2^{-9.5} \left( 63 (c/a)^4 \cot^2 \theta + 20 \right)
\]  

(67)

This looks like axis 3 tends to stand orthogonal to \( J \), which seems to be so natural since axis 3 is now not the maximal-inertia but the minimal-inertia axis.

Alas, all this nice extrapolation of the oblate-body-applicable approach to a prolate-body case is of no practical interest, because in reality an infinitesimally small deviation between the values of \( I_2 \) and \( I_3 \) leads to a considerably different type of wobble: the so-called libration (Synge & Griffith 1959). This phenomenon will be comprehensively discussed in our next article.

8. Discussion

8.1. Comparison with the Barnett relaxation

Another important mechanism of internal relaxation, i.e. the Barnett relaxation, dissipates the energy via oscillating magnetization that arising from angular velocity precession in grain body coordinates. Lazarian & Draine (1997) provide an estimate \( t_B \approx A/\omega^2 \), where \( A \approx 7.1 \times 10^{17} (a/10^{-5} \text{ cm})^2 \text{ s}^{-1} \) for an oblate grain with 2:1 axis ratio. For such grains the Barnett dissipation dominates for grains with \( a < 2 \times 10^{-6} \) m. However, this is not true for a larger axis ratio. Indeed, according to Lazarian & Draine (1997) for \( a/c \gg 1 \) \( t_B \) scales as \( (a/c)^6 \), while we found above that \( t_i \) scales as \( (c/a)^{3/2} \). Therefore fore grains with 4:1 axis ratio the inelastic relaxation dominates if \( a > 10^{-7} \) m. For grains less than \( 3 \times 10^{-8} \) m the inelastic relaxation becomes more important. It also dominates for grains constituted by loose aggregates of smaller particles. The exact value of the \( Q \)-factor and therefore of relaxation rates would depend on the structure and properties of the aggregate.
For grains with axis ratio 2:1 and radii $10^{-7} \text{ m} < a < 2 \times 10^{-6} \text{ m}$ the Barnett relaxation is dominant when grains rotate thermally. Suprathermally rotating grains stochastically undergo spin-ups and spin-downs and during crossovers rotate with subthermal velocities. Although short, in terms of grain alignment, crossovers, are the most important moments of grain dynamics. Grains are marginally susceptible to the randomization via gaseous bombardment when they rotate suprathermally (Purcell 1979). Our calculations of the inelastic relaxation efficiency show that the Barnett relaxation is the dominant process that determines internal dissipation during crossovers within grains with $a > 10^{-7} \text{ m}$. Therefore results on paramagnetic and mechanical alignment obtained for such grains in Lazarian & Draine (1997) stay unaltered provided that $Q$ factors of interstellar grains are as high as they were chosen in this paper.

In this paper we assumed that grains are axially symmetric (and oblate). For grains of arbitrary shape accelerations will be higher as it tumbles. Therefore the inelastic relaxation is bound to increase and a relaxation time to decrease by a numerical factor that will depend on the grain shape. This subject is beyond the scope of the present paper.

Our treatment of the inelastic relaxation ignored thermal fluctuations in grain material. In reality, for finite grain temperatures, $\theta$ fluctuates and the thermal distribution proportional to the Boltzmann factor $\exp(-E_{\text{kin}}(\theta)/kT)$ is established as $t \to \infty$. To describe the transient processes of alignment one can solve Fokker-Planck equation as it is done in Lazarian & Roberge (1997) in the case of the Barnett relaxation, but to use coefficients derived in Appendix $E$.

### 8.2. Alignment and Internal dissipation

Several mechanisms of cosmic-dust alignment are known. They constitute three major types: mechanical, paramagnetic and via radiative torques. All of them appeal to internal relaxation that enables the alignment of the angular momentum with the axis of the maximal moment of inertia, henceforth the axis of major inertia. The degree of achievable alignment depends on whether grains rotate thermally $^6$ (the average rotational energy of a grain is of the order of the kinetic energy of the surrounding gas) or suprathermally

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$^6$In Landau and Lifshitz (1969, section 26) it was shown that the equilibrium position of an inelastic body corresponds to its maximal-inertia axis being parallel to the angular momentum. However in reality this statement remains true only up to thermal fluctuations: the rms of the angular deviation is $\Delta \theta = \sqrt{kT/|J|^2}$, $J$ and $I$ being the angular momentum and the moment of inertia (Lazarian 1994, Lazarian and Roberge 1997).
(spinning with energies much exceeding $kT$).

We note that all the types of mechanisms below provide the alignment of the grain angular momentum in respect to the magnetic field. This happens because grains swiftly precess about magnetic field lines. This precession is called into being by the interaction of the grain’s magnetic moment with the field. The said magnetic moment is generated by the Barnett effect and is thereby parallel to the angular velocity. For a typical cosmic-dust grain ($\sim 10^{-5}$ cm) in a typical interstellar magnetic field (5 $\mu$G) the period of Larmor precession is less than a week (Purcell 1979), which is much less than the typical time of alignment.

To relate the alignment of angular momentum to the polarization produced by grains, one has to know the alignment of grain axis, which is determined by the internal dissipation. However, the alignment of angular momentum, in its turn also depends on the alignment of grain axis. In the cases of thermally rotating grains, e.g. Davis & Greenstein (1951) and Gold (1952) alignment mechanisms it is sufficient to know that the rate of internal dissipation exceeds the rate of alignment. In cases of suprathermally rotating grains more precise estimates of the internal relaxation time are necessary. The latter include paramagnetic alignment of suprathermally rotating grains (Purcell 1979, Spitzer & McGlynn 1979, Lazarian & Draine 1979), crossover and cross sectional mechanical alignment (Lazarian 1995, Lazarian & Efroimsky 1996, Lazarian, Efroimsky & Ozik 1996) and the alignment via radiative torques (Draine & Weingartner 1996, 1997). Our finding that for some grains inelastic relaxation dominates is important and will be accounted in the quantitative studies of alignment elsewhere.

### 8.3. Comparison with the earlier work

Our treatment differs from that in (Purcell 1979) in a number of points.

I. We have obtained different expressions for stresses. In particular, our stresses do obey the boundary conditions: they vanish on the grain boundary (while the stresses in (Purcell 1979) fail to do so).

II. We have found energy dissipation at the double frequency. It is possible to show that the double-frequency dissipation provides a leading contribution. The dissipation at

---

7For example, in formula (67) the term $63(c/a)^4 \cot^2 \theta$ in the sum in brackets is due to damping at the precession frequency $\omega$, while the term $20$ is due to damping at the double frequency. Evidently, even for $c/a \approx 1/2$ the double frequency provides the leading input, while for $c/a \approx 1/10$ its input becomes absolutely overwhelming.
the double frequency was not considered in (Purcell 1979).

As a result, our final expression for the same \( Q \)-factor predicts higher inelastic relaxation efficiency than the corresponding one in (Purcell 1979). To make a comparison we rewrite our formulae in a form similar to that in Purcell (1979). Then our expression (60) would read:

\[
\left( \frac{d\theta}{dt} \right)_{\text{our result}} \approx - \frac{8}{5} \frac{\Omega_0^3 \rho a^2}{\mu Q [1 + (c/a)^2]^4} \left( 0.4 \frac{(c/a)^4}{4} \cos^2 \theta \sin \theta + 0.1 \right)
\] (68)

In the appropriate expression presented in Purcell’s article, the second term in brackets was missing (because Purcell missed the contribution from the dissipation at the double-frequency mode), while the first term is about 2.5 times larger (because Purcell’s calculation of the stress tensor ignored the boundary conditions upon stresses). One may see that the our calculations reveal that the inelastic relaxation being much more efficient than predicted in Purcell (1979). For example, if we define a typical alignment time \( t_i \) as the value of \( (d\theta/dt)^{-1} \) at \( \theta = \pi/4 \) then, for silicate grains with \( c/a \approx 1/2 \), “our” typical time derived from (68) will be 23 times less than the appropriate estimate derived from Purcell’s formula. For graphite grains with \( c/a \approx 1/10 \), “our” typical time will be almost 1200 times shorter than the estimate in Purcell (1979).

III. In reality, the difference between our calculation and the one by Purcell will be even bigger: due to imperfectness of the grain material, we would choose for the \( Q \)-factor lower values than the one suggested by Purcell. (See the end of Section VI for discussion.)

9. Conclusions

The principal results of this paper are as follows:

I. The stresses arising from tumbling of a grain deform it and change its moment of inertia, but this change lags behind grain angular velocity due to non-elastic effects. This process causes alignment of grain angular momentum with the grain axis of maximal moment of inertia.

II. Deformations in tumbling grain happen both at the frequency of precession and at a double frequency.

---

8 Comparing our results with that of Purcell, mind that the value of \( t_i \) obtained in (Purcell 1979) was for the degree of suprathermality \( \beta = 100 \).
III. Inelastic relaxation dominates the Barnett relaxation for (1) grains with large axis ratio, (2) grains with low $Q$ factor, i.e. fractal grains produced by coagulation, (3) grains with $a > 2 \mu m$, (4) suprathermally rotating grains.

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A. Intermediate system of reference

As pointed in Section III, our expression (16) for the acceleration of a point inside the spinning body considerably differs from an appropriate expression presented in the article (Purcell 1979). To be confident in our result, we shall now reproduce it by means of a two-step calculation.

To do so, we shall introduce an “intermediate” coordinate system \((X', Y', Z')\), with basis vectors denoted as \(e_{X'}\), \(e_{Y'}\), and \(e_{Z'}\). This system is obtained from \((X, Y, Z)\) by means of consequent rotations of axis \(X\) by the Euler angles \(\varphi\) and \(\theta\).

In terms of the said angles,

\[
\Omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi , \tag{A1}
\]
\[
\Omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi , \tag{A2}
\]
\[
\Omega_3 = \dot{\phi} \cos \theta + \dot{\psi} , \tag{A3}
\]

and

\[
\Omega_{\perp}^2 = \Omega_1^2 + \Omega_2^2 = \dot{\phi}^2 \sin^2 \theta + \dot{\theta} . \tag{A4}
\]

The position, velocity and acceleration of a point inside the grain, relative to the “intermediate” frame, will be denoted by \(r', v'\) and \(a'\). Evidently,

\[
r'' = r' = r , \quad e_3 = e_{Z'} . \tag{A5}
\]

Now, one will be able to calculate the acceleration \(a\) as

\[
a = a' + (e_z \ddot{\phi}) \times r' + 2 (e_z \dot{\phi}) \times v' + \dot{\phi}^2 e_z \times (e_z \times r') \quad (A6)
\]

and plug, instead of \(a'\), the expression

\[
a' = a'' + (e_3 \dddot{\psi}) \times r'' + 2 (e_3 \ddot{\psi}) \times v'' + \ddot{\psi}^2 e_3 \times (e_3 \times r'') \quad (A7)
\]

This will once again lead to \([16]\).

B. Diffusion coefficients

In the absence of gaseous bombardment grain angular momentum stays constant. The alignment of \(J\) in grain axes is being determined by internal relaxation and that tend to decrease \(\theta\) and thermal fluctuations that randomize \(\theta\). To quantify these processes Fokker-Planck equation may be used.
In the spherical coordinate system \( J, \theta, \phi \), where the polar axis is parallel to the principle axis of maximal inertia, the Fokker-Planck equation will read:

\[
\frac{\partial f}{\partial t} = -\nabla \cdot S,
\]  

(B1)

Here \( f = f(J) \) is the joint distribution for \( J, \theta \) and \( \phi \), while \( S \) is the probability current:

\[
S = A f - \frac{1}{2} \nabla \cdot (B f),
\]  

(B2)

where

\[
A \equiv \langle \frac{\Delta J}{\Delta t} \rangle,
\]  

(B3)

is the mean torque, and

\[
B \equiv \langle \frac{\Delta J \Delta J}{\Delta t} \rangle,
\]  

(B4)

where \( \langle .. \rangle \) denote ensemble averaging, is the diffusion tensor. Generally speaking, \( A \) and \( B \) include the cumulative effects of all the processes that change \( J \) in the body frame. Still, in what follows we shall consider the inelastic relaxation as if it were the sole factor contesting the gas damping. All the other mechanisms of orientation will be ignored.

We have written down the transport equation, and introduced the entities \( A \) and \( B \), in agreement with (Risken 1984). The difference between our definition of \( A \) and the definition presented in (Landau and Lifshitz 1981) stems from the fact that in (Risken 1984) the change of momentum of the grain is denoted as \( p + q \), whereas in (Landau and Lifshitz 1981, formula (21.1) and thereabout) it is denoted by \( p - q \). For this reason \( A \) in formula (21.5) for \( S \), in (Landau and Lifshitz 1981), appears with a negative sign.

The inelastic relaxation leaves \( J \) unaltered, in the inertial frame. In the body frame, which we are using here, the direction of \( J \) will vary, but its absolute value will be conserved. For this reason the components of \( A \) and \( B \), having \( J \) in their subscripts, will vanish. As for \( \phi \)-dependence, it would be nonexistent should the grain be an oblate ellipsoid rather than a prism. Still, in our approximation we may neglect the \( \phi \)-dependence: it is, in fact, not that relevant for our purposes whether the grain a circular or a square cross section. That is, for the purpose of our estimates (which are anyway not exhaustingly exact) we feel free to consider the grain either square, when we need to estimate \( t_a \) and \( d\theta/dt \), or circular, when we need to simplify the Fokker-Plank equation. In other words, let us assume that in our study of the relaxation over \( \theta \) the \( \phi \)-dependence is to be averaged out. Hence all the components of \( A \) and \( B \) with \( \phi \) in their subscripts will be thrown out either. Now the only remaining components will be \( A_{\theta}^{(ir)} \) and \( B_{\theta\theta}^{(ir)} \) where the extra superscript
is introduced to emphasize that only the effect of inelastic relaxation (versus the gas damping) is taken account of.

The expression for \( A_{\theta}^{(ir)} \) is straightforward from our expression for \( d\theta/dt \):

\[
A_{\theta}^{(ir)} = J d\theta/dt
\]  \hspace{1cm} (B5)

while \( B_{\theta\theta}^{(ir)} \) is to be found from the principle of detailed balance: in case the distribution function \( f(J) \) is the thermodynamical-equilibrium one \( (f = f_{TE}) \), the rate of every microscopic process equals that of its time-reversed counterpart, and hence the probability current must vanish at every point in the phase space. In particular,

\[
0 = S_{\theta} = \sin \theta f_{TE} A_{\theta}^{(ir)} - \frac{1}{2J} \frac{\partial}{\partial \theta} \{ \sin \theta f_{TE} B_{\theta\theta}^{(ir)} \} \]  \hspace{1cm} (B6)

where the thermodynamical-equilibrium distribution function is a Boltzmann one

\[
f_{TE}(J) = C \exp\{ -E_{rot}(\theta,J)/kT_{\text{grain}} \} ,
\]  \hspace{1cm} (B7)

\( C \) being a normalization constant, \( T_{\text{grain}} \) being the grain temperature, and \( E_{rot}(\theta,J) \) being the grain rotational energy (expressed by equation (37)). The solution to this first-order differential equation is:

\[
B_{\theta\theta}^{(ir)} = \frac{2J^2 \exp(\xi \sin^2 \theta)}{t_i \sin \theta} \int_{\pi/2}^{\theta} \sin^2 y \left( \tilde{A} \cos^2 y + \tilde{B} \sin^2 y \right) \exp(-\xi x^2) dx - \frac{J^2 \tilde{A}}{t_i} \exp\left( \frac{h}{h-1} \xi^2 \right) ,
\]  \hspace{1cm} (B8)

where

\[
\tilde{A} = \frac{2^{3/2} 63 (c/a)^4}{63 (c/a)^4 + 20} ,
\]  \hspace{1cm} (B9)

\[
\tilde{B} = \frac{2^{3/2} 20}{63 (c/a)^4 + 20}
\]  \hspace{1cm} (B10)

and

\[
\xi = \frac{(h - 1) J^2}{2 I_z k T_{\text{grain}}} .
\]  \hspace{1cm} (B11)

\[\text{In the case of a grain being in an equilibrium, we formally treat it as a small system placed in a bath of temperature } T_{\text{grain}}. \text{ This is certainly fair, since the number of rotational degrees of freedom of the grain (} = 3) \text{ is much less than the amount of its vibrational degrees of freedom (which is of the same order as the number of atoms in the grain).}\]
The solution \((B8)\) is obtained assuming that \(B_{\theta \theta}^{ir}(\pi/2, \xi) = -J^2 \tilde{A}/t_i\), to insure that \(B_{\theta \theta}^{ir}\) is smooth at \(\pi/2\) (compare with Lazarian & Roberge 1997):

\[
\lim_{\xi \to \infty} \frac{d^2 B_{\theta \theta}^{ir}(\pi/2, \xi)}{d\theta^2} < \infty .
\] (B12)

The coefficients \(A_\theta\) and \(B_{\theta \theta}^{ir}\) can be used to describe internal alignment in the presence of gaseous bombardment and inelastic relaxation as it has been done in Lazarian & Roberge (1995) for the case of Barnett relaxation.

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Fig. 1.— The coordinate system \((\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\) is associated with the inertial frame, so that the (conserved) angular momentum \(\mathbf{J}\) is aimed along \(\mathbf{e}_z\). The coordinate system \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\) is associated with the three principal axes of inertia of the body, so that \(\mathbf{e}_3\) points along the axis of major inertia.