AN ADJUNCTION CRITERION IN ALMOST-COMPLEX 4-MANIFOLDS

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Abstract. The adjunction inequality is a key tool for bounding the genus of smoothly embedded surfaces in 4-manifolds. Using gauge-theoretic invariants, many versions of this inequality have been established for both closed surfaces and surfaces with boundary. However, these invariants generally require some global geometry, such as a symplectic structure or nonzero Seiberg-Witten invariants. In this paper, we extend previous work on trisections and the Thom conjecture to obtain adjunction information in a much larger class of smooth 4-manifolds. We introduce polyhedral decompositions of almost-complex 4-manifolds and give a criterion in terms of this decomposition for surfaces to satisfy the adjunction inequality.

1. Introduction

The goal of this paper is to extract the essential core of the techniques from \cite{LC20, Lam20}, where trisections of symplectic 4-manifolds were used to reprove the Thom and Symplectic Thom conjectures. The key bound on the genus of a smoothly embedded surface \( K \) is the adjunction inequality

\[
\chi(K) \leq \langle c_1(J), K \rangle - K^2
\]

where \( J \) is an almost-complex structure tamed by the symplectic form. This inequality is sharp for complex curves in Kähler surfaces and for symplectic surfaces in symplectic 4-manifolds. Therefore these minimize genus in their homology classes. Other forms of the adjunction inequality hold where \( J \) is replaced by a Spin\(^C\)-structure that is a basic class for Donaldson, Seiberg-Witten or Heegaard-Floer invariants.

In \cite{LC20, Lam20}, trisections are used to bootstrap slice genus bounds in \( B^4 \) – expressed in terms of the slice-Bennequin inequality – to bound the genus of closed, smooth surfaces. However, the formalism of trisections of 4-manifolds and bridge trisections of knotted surfaces is too restrictive. It is possible to extend these ideas to almost-complex 4-manifolds admitting more complicated decompositions, which we call polyhedral decompositions, and to surfaces that satisfy a weak geometric positivity condition with respect to this decomposition. Moreover, we do not require a global symplectic form. We only require that each component of the polyhedral decomposition admit a symplectic form that tames the (global) almost-complex structure over that component. Consequently, we are able to give minimal genus information in a much larger class of smooth 4-manifolds.

The first component of the adjunction criterion is a class of geometric decompositions of almost-complex 4-manifolds.

Definition 1.1. Let \( (X, J) \) be a compact, almost-complex 4-manifold. A polyhedral decomposition \( \mathcal{P} \) of \( (X, J) \) is a Whitney stratification of \( X \) satisfying the following criteria. For notational purposes, let \( H_P \) denote the union of the closures of the 3-dimensional strata and let \( \Sigma_P \) denote the union of the closures of the 2-dimensional strata.

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(1) The closure of each open stratum is a compact manifold with corners.
(2) $J$ is integrable in an open neighborhood of $\Sigma_P$.
(3) The field of $J$-complex tangencies along each 3-dimensional stratum $H_\lambda$ is an integrable 2-plane field.
(4) Each 4-dimensional stratum has convex corners. Specifically, let $Z_\lambda$ be a 4-dimensional stratum and $z \in \partial Z_\lambda$ a point that is contained in $\Sigma_P$. Then there exists an open neighborhood $U$ of $x$ on which $J$ is integrable and holomorphic functions $f_1, \ldots, f_i$ on $U$ such that
$$Z \cap U = \{x \in U : \max(|f_1(x)|, \ldots, |f_i(x)|) < 1\}$$
(5) Each 4-dimensional sector $Z_\lambda$ admits a symplectic form $\omega_\lambda$ that tames $J$ (meaning $\omega_\lambda(v, Jv) > 0$ for all tangent vectors $v$).

The standard trisection of $\mathbb{C}P^2$ and Weinstein trisections of symplectic 4-manifolds are polyhedral decompositions. Toric 4-manifolds also admit natural polyhedral decompositions. Many examples of polyhedral decompositions are given in Section 2.2.

Since each 4-dimensional sector of a polyhedral decomposition has convex corners (Condition (4)), its boundary can be smoothed (Proposition 2.17) so that the field of $J$-complex tangencies along the boundary is a positive confoliation and $(X_\lambda, \omega_\lambda)$ is a weak symplectic fillings (Condition (5)). This is the necessary condition to apply the slice-Bennequin inequality (Theorem 3.5). Unless this confoliation is the foliation on $S^2 \times S^1$ by 2-spheres, the result of Eliashberg-Thurston [ET98] implies that the confoliation can be $C^0$-perturbed to a positive contact structure. We say that a polyhedral decomposition is aspherical if none of the smoothed boundary components admits this $J$-holomorphic foliation by 2-spheres.

The second component of the criterion is a class of surfaces that have a certain amount of geometric positivity along their spine, which is the codimension-1 portion of the polygonal decomposition.

**Definition 1.2.** Let $(X, J)$ be a compact, almost-complex 4-manifold with polygonal decomposition $P$. An immersed, oriented surface $K \subset X$ is homotopically transverse to $P$ if

1. $K$ intersects each stratum transversely
2. $K$ has complex bridge points. Specifically, there exists an open neighborhood $U$ of $\Sigma_P$ such that $J$ is integrable on $U$ and the intersection $K \cap U$ is $J$-holomorphic.
3. for each dimension 3 stratum in $H_\lambda$, the tangle $\tau = K \cap H_\lambda$ is homotopically transverse. That is, $\tau$ is homotopic rel endpoints to a tangle that is everywhere positively transverse to the foliation on $H$ induced by the field of $J$-complex tangencies.

In Condition (3), the sign of the intersection of $\tau$ with the foliation is determined as follows. The surface $K$ is oriented and the tangent planes to the foliation are oriented by $J$. We require that the intersection of $\tau$ with a leaf, viewed as an intersection of $K$ with this leaf, is a positive transverse intersection.

Using this setup, we obtain the following adjunction criterion in almost-complex 4-manifolds.

**Theorem 1.3 (Adjunction Inequality).** Let $(X, J)$ be a compact, almost-complex 4-manifold with aspherical polygonal decomposition $P$. Let $K$ be a closed, oriented, embedded surface $K$ that is homotopically transverse to $P$. Then
$$\chi(K) \leq \langle c_1(J), K \rangle - K^2.$$
criterion gives new information about the minimal genus of smoothly embedded surfaces. In a sequel paper, it is applied to get minimal genus bounds on some homology classes in connected sums of rational surfaces.

The same arguments also imply a relative adjunction criterion in the form of a generalized slice-Bennequin inequality for properly embedded surfaces with transverse boundary. The original slice-Bennequin inequality was established by Rudolph for properly embedded surfaces in $B^4$ [Rud95]. It was then extended to Stein surfaces by Lisca-Matic [LM98] and to weak symplectic fillings by Mrowka-Rollin [MR06].

We say that a properly embedded, oriented surface $D$ in $(X, J)$ has transverse boundary if $\partial D$ is positively transverse to the field of $J$-tangencies along $\partial X$. In this case, the self-linking number $sl(\partial D, D)$ is well-defined (Definition 3.1).

**Theorem 1.4** (Slice-Bennequin inequality). Let $(X, J)$ be a compact, almost-complex 4-manifold with aspherical polygonal decomposition $\mathcal{P}$. Let $D$ be a properly embedded surface that is homotopically transverse to $\mathcal{P}$ and has transverse boundary. Then

$$sl(\partial D, D) \leq -\chi(D).$$

Importantly, there is no convexity requirement for the boundary of $\partial X$. For example, it can be applied to symplectic cobordisms with at least one concave end.

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2. **Polyhedral decomposition**

2.1. **Examples of polyhedra.** The basic example motivating the definition of symplectic polygons are analytic polyhedra in complex analysis. Let $X$ be a complex manifold and let $\Omega \subset X$ be a domain. An analytic polyhedron in $\Omega$ is a domain of the form

$$P = \{z \in \Omega : |f_1(z)|, \ldots, |f_k(z)| < 1\}$$

where $f_1, \ldots, f_k$ are holomorphic functions on $\Omega$.

**Example 2.1.** Polydisks. The unit polydisk in $\mathbb{C}^n$ is the subset

$$\Delta_n = \{z \in \mathbb{C}^n : |z_i| \leq 1 \text{ for } i = 1, \ldots, n\}$$

In particular, it is the analytic polyhedron defined by the coordinate functions.

**Example 2.2.** Holomorphic branched covers of the polydisk. Now suppose there is a finite, proper, holomorphic map

$$\pi : X \to \mathbb{C}^n$$

The preimage of the unit polydisk is an analytic polyhedron

$$\pi^{-1}(\Delta_n) = \max \{x \in X : |z_i \circ \pi(x)| \leq 1 \text{ for } i = 1, \ldots, n\}$$

**Example 2.3.** Lefschetz fibrations over the disk. A Lefschetz fibration of a symplectic 4-manifold over the disk is a smooth map $\pi : X \to D^2$ satisfying

1. the set $A$ of critical values in $D^2$ is finite and over the complement $D^2 \setminus A$, the map $\pi$ is a locally trivial fibration by compact surfaces with boundary. In particular, all regular fibers are homeomorphic to a fixed surface $F$. 

(2) each singular fiber (i.e. \( f^{-1}(a_i) \) for \( a_i \in A \)) has a unique critical point that is a Lefschetz singularity. Specifically, there are local \( \mathbb{C} \)-coordinates \((z, w)\) of such that \( \pi(z, w) = z^2 + w^2 \).

The boundary of a Lefschetz fibration over \( D^2 \) splits into two pieces:

1. a local trivial fibration over the boundary of the disk:
\[
F \to Y_1 \to \partial D^2
\]
2. a trivial fibration by circles over the disk:
\[
\partial F \to Y_2 \to D^2
\]
or equivalently a trivial fibration by disks over the circle:
\[
D^2 \to Y_2 \to S^2
\]

A Lefschetz fibration admits an almost-complex structure \( J \) such that fibers are \( J \)-holomorphic. In particular, we can choose \( J \) such that near \( \partial X \), the surface fibers of \( Y_1 \) are \( J \)-holomorphic and the meridional disks of \( Y_2 \) are \( J \)-holomorphic, with the two components meeting at a convex corner.

**Example 2.4.** Polygonal bidisk. Let \( \mu = \{a_i, b_i\} \) be a collection of pairs of nonnegative real numbers. Define
\[
\Delta_\mu = \{(z_1, z_2) \in \mathbb{C}^2 : a_i|z_i| + b_i|z_2| \leq 1\}
\]
For example, the standard bidisk is defined by \( \mu = \{(1, 0), (0, 1)\} \). The polygonal bidisk defined by \( \mu = \{(1, 0), (3/2, 3/2), (0, 1)\} \) will have three boundary components (see Figure 1).

2.2. Examples of polyhedral decompositions.

**Example 2.5.** The standard trisection of \( \mathbb{CP}^2 \). Consider \( \mathbb{CP}^2 \) with homogeneous coordinates \( z = [z_0 : z_1 : z_2] \). For \( i = 1, 2, 3 \), define subsets
\[
Z_i = \{z : |z_j| \leq |z_i| \text{ for } j \neq i\}
\]
This subset is precisely the unit bidisk \( \mathbb{D} \times \mathbb{D} \) in the affine chart given by setting \( z_i = 1 \).

**Example 2.6.** Complex projective space. Projective space of dimension \( n \) admits a polygonal decomposition into \( n + 1 \) polydisks. Define subsets
\[
Z_i = \{z : |z_j| \leq |z_i| \text{ for all } j \neq i\}
\]

![Figure 1](image-url). The polygonal bidisk determined by \( \mu = \{(1, 0), (3/2, 3/2), (0, 1)\} \), viewed in the \((|z_1|, |z_2|)\)-plane.
This subset is the unit polydisk in the affine chart given by setting $z_i = 1$.

**Example 2.7.** Projective surfaces. Given a projective surface $X$, we can find a holomorphic branched covering

$$\pi : X \to \mathbb{CP}^2$$

and pull back the standard trisection $\mathbb{CP}^2 = Z_1 \cup Z_2 \cup Z_3$ by $\pi$ to get a decomposition of $X$ into three analytic polyhedra.

**Example 2.8.** Let $X$ be a compact, $n$-dimensional complex manifold, let $\{L_i\}$ be a collection of $m$ Hermitian line bundles with meromorphic sections $\{s_i\}$. Define subsets

$$X_i := \{x \in X : |s_j(x)| \leq |s_i(x)| \text{ for all } j \neq i\}$$

Then $X_i$ is equivalently the subset where $\max_{i=1,\ldots,m}(|s_j/s_i|) = 1$.

**Example 2.9.** Symplectic trisections. If $(X, \omega)$ is a closed symplectic 4-manifold, it admits a Weinstein trisection with compatible almost-complex structure $J$ [LCMS21] (see also [Lam20]). In particular, this is a decomposition

$$(X, \omega) = \bigcup \lambda (X_\lambda, \omega_\lambda, \rho_\lambda, \phi_\lambda)$$

into three subcritical Weinstein domains. It is possible to choose an $\omega$-tame almost-complex structure so that each $H_\lambda = X_\lambda \cap X_{\lambda-1}$ is foliated by $J$-holomorphic curves.

As in Example 2.7, these are constructed as branched covers of the standard trisection of $\mathbb{CP}^2$. In particular, this symplectic trisection decomposition is a polygonal decomposition of $(X, J)$.

**Example 2.10.** Toric symplectic 4-manifolds. By definition, a toric symplectic 4-manifold $(X, \omega)$ admit a moment map

$$\mu : X \to \mathbb{R}^2$$

that commutes with an effective $T^2$-action on $(X, \omega)$. The image $\mu(X)$ is a convex polytope in the plane. If $x \in \mathbb{R}^2$ lies in a dimension $i$ face of the polytope, the preimage $\mu^{-1}(x)$ is the $i$-torus $(\mathbb{S}^1)^i$.

Suppose that the moment polytope $\mu(X)$ has $k$ exterior faces. Let $\Gamma$ be a tree with $k$ exterior vertices and choose a proper embedding of $\Gamma$ in $\mu(X)$ such that each exterior vertex goes to a unique exterior face of the polytope. This decomposes $\mu(M)$ into $k$ components. For topological reasons, the preimage of each component is a $B^4$ with piecewise-smooth boundary. Moreover, we can choose $J$ compatible with the moment map and identify each with a polygonal bidisk. See Figure 2.

![Figure 2](image-url)

**Figure 2.** The moment polytope of $S^2 \times S^2 \# \mathbb{CP}^2$ is a pentagon. A proper embedding of the trivalent graph $\Gamma$ into the moment polytope determined a decomposition into five polygonal bidisks.
Example 2.11. Stein trisections. A Stein trisection of a complex surface (possibly compact with boundary) is a (relative) trisection into analytic polyhedra. Consequently, it is a symplectic polyhedral decomposition. The standard trisection of \( \mathbb{C}P^2 \) is a Stein trisection [LCM0]. Forthcoming work of Zupan shows that the projective surfaces \( V_d = \{ w^d + x^{d-1}y + y^{d-1}z + z^{d-1}x = 0 \} \) admit Stein trisections for all \( d > 0 \). An infinite family of distinct Stein trisections of the 4-ball was constructed in [Lam21].

Example 2.12. Pinwheels. Applying Symmington and McDuff’s results on \( k \)-fold sums of symplectic 4-manifolds [Sym98, MS96], Fintushel and Stern used pinwheel decompositions of 4-manifolds to construct exotic smooth structures on \( \mathbb{C}P^2 \# k \mathbb{C}P^2 \) for small values of \( k \) [FS11]. These decompositions can be interpreted as polyhedral decompositions, where the codimension-2 strata are all tori and the 3-dimensional strata are of the form \( F \times S^1 \), where \( F \) is a compact surface with boundary. In particular, the product foliations of the latter are taut.

Example 2.13. Toric multisections. Islambouli and Naylor introduced multisections of smooth 4-manifolds [IN20], which are generalizations of trisections that have an arbitrary number of sectors. In the case when the central surface has genus 1, the multisection is determined by a sequence of slopes \( \alpha_1, \ldots, \alpha_n \in H_1(T^2) \) satisfying

\[
\langle \alpha_i, \alpha_{i+1} \rangle = \pm 1
\]

These 4-manifolds were essentially classified by Orlik and Raymond [OR70, OR74] and are diffeomorphic to

\[
a \mathbb{C}P^2 \# b \mathbb{C}P^2 \# c S^2 \times S^2
\]

for some integers \( a, b, c \) satisfying \( a + b + 2c + 2 = n \). When the sequence \( \{\alpha_1, \ldots, \alpha_n\} \) satisfies the stronger condition

\[
\langle \alpha_i, \alpha_{i+1} \rangle = 1
\]

it is possible to define an almost-complex structure compatible with the toric structure. This is described in the sequel paper.

It is also possible to find polygonal decompositions of compact 4-manifolds with contact-type boundary, including when one or more boundary components is concave.

Example 2.14. Stein cobordisms via Lefschetz fibrations. A Stein cobordism \( X \) from \( (Y_1, \xi_1) \) to \( (Y_2, \xi_2) \) can be built as the union of a Lefschetz fibration over the annulus with a Stein \( \sqcup S^1 \times B^3 \) corresponding to the binding of an open book decomposition.

Suppose that \( (Y_1, \xi_1) \) admits an open book decompositon with page \( F \) and that \( (Y_2, \xi_2) \) is obtained by a collection of Legendrian surgeries along knots that lie in the page \( F \). Then the corresponding Stein cobordism \( X \) has a decomposition

\[
X = \hat{X} \cup \bigcup \nu(B) \times [0, 1]
\]

where \( \hat{X} \) is a Lefschetz fibration over the annulus \( S^1 \times [0, 1] \) with smooth fiber \( F \) and the restriction of the fibration to \( S^1 \times \{0\} \) and \( S^1 \times \{1\} \) are the open book decompositions of \( (Y_1, \xi_1) \) and \( (Y_2, \xi_2) \), respectively, and \( \nu(B) \) is a neighborhood of the binding.

To get a polygonal decomposition, we cut the annulus into two pieces:

\[
S^1 \times [0, 1] \cong [0, \pi] \times [0, 1] \cup [\pi, 2\pi] \times [0, 1]
\]

and pull back this decomposition to get a decomposition \( \hat{X} = \hat{X}_a \cup \hat{X}_b \). In particular, both \( \hat{X}_a \) and \( \hat{X}_b \) are Lefschetz fibrations over the disk.
Finally, we take \( \tilde{B} = \mathbb{D} \times \nu(\partial F) \) with the split (integrable) almost-complex structure and attach it to \( \tilde{X} \) by attaching the component \( S^1 \times \nu(\partial F) \) of \( \partial \tilde{B} \) to \( \tilde{X} \) along the fibration over \( S^1 \times \{0\} \) using the identify map on \( \nu(\partial F) \).

\section*{2.3. Geometrically transverse surfaces.}

A stronger version of homotopic transversality (Definition 1.2) is also useful.

\begin{definition}
Let \((X, J)\) be a compact, almost-complex 4-manifold with polygonal decomposition \(\mathcal{P}\). An immersed, oriented surface \(K \subset X\) is geometrically transverse (to \(\mathcal{P}\)) if

1. \(K\) intersects each stratum transversely,
2. \(K\) has complex bridge points, and
3. for each codimension 1 stratum \(H\), the tangle \(\tau - K \cap H\) is positively transverse to the foliation on \(H\) induced by the field of \(J\)-complex tangencies.

Up to a perturbation, being geometrically transverse is equivalent to the spine of the surface being \(J\)-holomorphic.

\end{definition}

\begin{lemma}
Let \(K\) be geometrically transverse to \(\mathcal{P}\). There exists a homotopy of \(J\), fixing the \(J\)-holomorphic foliation on \(H_{\mathcal{P}}\), such that a neighborhood of the spine \(S_K\) is \(J\)-holomorphic.
\end{lemma}

\begin{proof}
By assumption, the surface \(K\) has complex bridge points and so is already \(J\)-holomorphic in a neighborhood of the codimension-2 strata. Along the codimension-1 strata, the surface \(K\) is positively transverse to the foliation \(F\) on \(H_{\mathcal{P}}\) by \(J\)-holomorphic leaves. In particular, for each point \(x \in \tau = K \cap H_{\mathcal{P}}\), we have a splitting \(T_x X = T_x F \oplus T_x K\). The tangent plane \(T_x F\) is \(J\)-holomorphic and we can then homotope \(J\) so that \(T_x K\) is also \(J\)-holomorphic, then extend this to some tubular neighborhood of the spine of \(K\).
\end{proof}

\section*{2.4. Smoothing convex corners.}

The following key proposition is a generalization of \([LC20, \text{Lemma 3.5, Proposition 3.8}]\) and \([Lam20, \text{Proposition 4.5}]\).

\begin{proposition}
Let \(X_\lambda\) be a sector of an aspherical polygonal decomposition \(\mathcal{P}\). There exists an exhaustion

\[ \cdots \subset \tilde{X}_{\lambda,N-1} \subset \tilde{X}_{\lambda,N} \subset \tilde{X}_{\lambda,N} \subset \cdots \subset X_\lambda \]

by compact 4-manifolds with smooth boundary and a \(C^0\)-small perturbation of \(J\) such that

1. for \(N\) sufficiently large, the boundary \(\tilde{Y}_{\lambda,N} = \partial \tilde{X}_{\lambda,N}\) is \(C^0\)-close to \(\partial X_\lambda\),
2. for every fixed open neighborhood \(U\) of \(\Sigma\) and \(N\) sufficiently large, the hypersurface \(\tilde{Y}_{N,\lambda}\) is \(C^\infty\)-close to \(\partial X_\lambda\) outside of \(U\).
3. the field \(\tilde{\xi}_{\lambda,N}\) of \(J\)-complex tangencies is a positive contact structure with weak symplectic filling \((\tilde{X}_{\lambda,N}, \omega_\lambda)\),
4. if \(K\) is a geometrically transverse surface, then for \(N\) sufficiently large, the intersection \(K_\lambda = K \cap \tilde{Y}_{\lambda,N}\) is a transverse link.

\end{proposition}

\begin{proof}
The compact exhaustion \(\{\tilde{X}_{\lambda,N}\}\) is obtained by taking a collar neighborhood of \(\partial X_\lambda\) and smoothing the corners. In particular, since by assumption \(J\) is integrable in a neighborhood of the corners and the boundary is locally defined by the continuous plurisubharmonic function \(\max(|f_1|, \ldots, |f_i|)\), the smoothing can be accomplished by convolving with a mollifer \([Ric68]\) (see also \([CE12, \text{Chapter 3.2}]\)). This ensures the boundary of \(\tilde{X}_{\lambda,N}\) is \(C^0\)-close to \(\partial X_\lambda\) near the corners and \(C^\infty\)-close everywhere else.
\end{proof}
After smoothing, the field of $J$-complex tangencies along $\hat{Y}_{\lambda,N}$ is a positive confoliation, as it is positive contact structure at the smoothed corners and integrable everywhere else. By the theorem of Eliashberg-Thurston [ET98], it is possible to $C^0$-approximate this positive confoliation by a positive contact structure. Moreover, the perturbed $J$ remains $\omega_\lambda$-tame, hence this contact structure is weakly fillable.

To prove part (4), we can choose a neighborhoods $U \subset V$ of $\Sigma$ such that the intersection $K \cap V$ consists of $J$-holomorphic curves. Now apply part (2) of the proposition with respect to $U$. Consequently, positivity of intersection for $J$-complex lines implies that the intersection $\hat{Y}_{\lambda,N} \cap U \cap K$ consists of transverse arcs. Moreover, $\hat{Y}_{\lambda,N}$ is $C^\infty$-close to $\partial X_\lambda$ outside $U$. Therefore, since $K$ intersects $\partial X_\lambda$ along transverse arcs, it also intersects $\hat{Y}_{\lambda,N}$ along transverse arcs. \[\square\]

3. Self-linking and Slice-Bennequin

3.1. Self-linking number. Let $(X, J)$ be a compact, almost-complex 4-manifold with boundary, let $\xi$ denote the field of $J$-complex tangencies along $\partial X$, let $L \subset \partial X$ be a link transverse to the 2-plane field $(\partial X, \xi)$ and let $D \subset X$ be a properly immersed, oriented surface bounded by $L$.

Since $L$ is transverse to the plane field $\xi$, we can choose an identification $N D|_L \cong \xi|_L$ between the normal bundle and the contact structure along $L$. In addition, since $\xi$ is the field of $J$-complex tangencies along $\partial X$, the outward-pointing normal vector $\partial_t$ determines an isomorphism $\det_C(TX)|_L \cong \xi|_L$.

**Definition 3.1.** The self-linking number of $L$ with respect to the surface $D$ is

$$sl(L, D) = e(N D, s) - c_1(\det_C(TX)|_D, s)$$

where $s$ is any nonvanishing section of $\xi|_L$.

Note that since the self-linking number is defined as the difference of two relative obstructions, it does not depend on the choice of section of $\xi|_L$.

**Lemma 3.2.** Let $L$ be a transverse link in the boundary of $(X, J)$. If $D$ is a properly embedded, $J$-holomorphic curve with boundary $L$, then

$$sl(L, D) = -\chi(D).$$

**Proof.** Note that the definition of the self-linking number is additive over the connected components of $D$.

If $D$ has a closed component $K$, then the adjunction formula implies that

$$\chi(K) = \langle c_1(J), K \rangle - K^2$$

and the statement holds for this component.

Now suppose that every component of $D$ has boundary. Then we can choose a nonvanishing normal vector field $s$ along $D$ and a tangent vector field $v$ along $D$ that points outward along $\partial D$. Therefore $s$ determines a section of $N D$ and $\det(s, v)$ determines a section of the determinant bundle $\det(TX)$ over $D$. Since $D$ is $J$-holomorphic, the sections $s$ and $\det(s, v)$ determine that same sections of $\xi|_{\partial D}$ under the identifications. Therefore,

$$e(N D, s) = 0$$

$$c_1(\det(TX)|_D, s) = \chi(D)$$
and the lemma follows by definition.

\[ \square \]

3.2. Geometrically transverse surfaces. Suppose that \( L \) is geometrically transverse to a polygonal decomposition \( \mathcal{P} \) of \((X, J)\). By Proposition 2.17, we can approximate the boundary of each \( X_\lambda \) by a smooth \( \tilde{X}_{\lambda,N} \) such that for \( N \) sufficiently large, the link \( L_{\lambda,N} = L \cap \tilde{X}_{\lambda,N} \) is a transverse to the contact structure \( \tilde{\xi}_{\lambda,N} \). From here on, we will drop the subscript \( N \) and implicitly assume it has been chosen sufficiently large.

The link \( L_\lambda \) bounds a properly embedded surface \( D_\lambda = L \cap \tilde{X}_\lambda \). This induces a decomposition

\[ L = S_L \cup \cup_\lambda D_\lambda \]

where \( S_L \) denotes a tubular neighborhood of the spine \( L \cap H_\mathcal{P} \) of the surface \( L \). It follows immediately that

\[ \chi(K) = \chi(S_L) + \sum_\lambda \chi(D_\lambda). \]

Lemma 3.3. Let \( L \) be a geometrically transverse surface. There exists a homotopy of \( J \), fixing the field of \( J \)-complex tangencies along \( H_\mathcal{P} \) and each \( \tilde{X}_\lambda \), such that the spine \( S_L \) is \( J \)-holomorphic.

**Proof.** By assumption, the almost-complex structure \( J \) is integrable in an open neighborhood \( U \) of \( \Sigma_\mathcal{P} \) and the surface \( L \) has complex bridge points. Also, the hypersurfaces \( \tilde{X}_\lambda \) are \( C^\infty \)-close to \( H_\mathcal{P} \) in the complement of \( U \). On the complement of \( U \), the surface \( L \) is geometrically transverse, so the tangent planes to \( L \) are positively transverse to the field of \( J \)-tangencies along \( H_\mathcal{P} \). Consequently, we can homotope \( J \) so that \( L \) is \( J \)-holomorphic in a neighborhood of \( H_\mathcal{P} \).

\[ \square \]

Furthermore, the total self-linking number of the links \( \{L_\lambda\} \) is determined by the homology class of \([L]\), up to a correct term given by the Euler characteristic of its spine.

**Proposition 3.4.** Suppose that \( L \) is an immersed surface that is geometrically transverse to a polygonal decomposition \( \mathcal{P} \) of \((X, J)\). Then

\[ \sum_\lambda \text{sl}(L_\lambda, D_\lambda) = e(NL) - (c_1(J), L) + \chi(S_L) \]

**Proof.** Since \( L \) is geometrically transverse, we can choose a section \( s \) of \( \xi \) over \( \partial S_L \) and the total obstruction class decomposes into the sum over relative obstructions:

\[ e(NL) = e(NS_L, s) + \sum_\lambda e(ND_\lambda, s) \]

\[ c_1(\det(TX)|_L) = c_1(\det(TX)|_{S_L}, s) + \sum_\lambda c_1(\det(TX)|_{D_\lambda}, s) \]

We can compute the self-linking numbers of the transverse links \( \{L_\lambda\} \) using this section to obtain the equation

\[ \sum_\lambda \text{sl}(L_\lambda, D_\lambda) = \sum_\lambda (e(ND_\lambda, s) - c_1(\det(TX)|_{D_\lambda}, s)) \]

By Lemma 3.3, the spine \( S_L \) is \( J \)-holomorphic and therefore by Lemma 3.2

\[ e(NS_L, s) - c_1(\det(TX)|_{S_L}, s) = \text{sl}(\partial S_L, S_L) = -\chi(S_L) \]

Combining these computations gives the calculation of the total self-linking number.

\[ \square \]
3.3. **Slice-Bennequin inequality.** Recall that a symplectic 4-manifold \( (X, \omega) \) is a weak filling of a contact structure \( (\partial X, \xi) \) if \( \omega|_\xi > 0 \).

**Theorem 3.5.** Let \( L \) be a transverse link in a contact 3-manifold \( (Y, \xi) \), given as the boundary of a weak filling \( (X, \omega) \). Then the self-linking number \( sl(L, D) \) of \( L \) with respect to a properly embedded surface \( D \) satisfies the inequality

\[
sl(L, D) \leq -\chi(D)
\]

**Proof.** Approximate \( L \) by a Legendrian link \( K \). The Legendrian link \( K \) has a preferred contact framing \( s \) induced by the contact structure \( \xi \). In addition, the outward-pointing normal vector to \( \partial X \) determines an isomorphism between \( \xi \) and \( \text{det}_C(TX) \).

The surface \( D \) induces two numerical invariants of \( K \):

1. the *Thurston-Bennequin number* \( tb(K, D) \), which is the relative obstruction to extending the contact framing to a nonvanishing section of \( N\overline{D} \), and
2. the *rotation number* \( rot(K, D) \), which is the obstruction to extending the tangent vector field to \( K \), viewed as a section of \( \xi|_L \), to a nonvanishing section of \( \text{det}_C(TX) \) on \( D \).

The Legendrian approximation \( K \) of the transverse link \( L \) is not unique, however all Legendrian approximations satisfy

\[
sl(L, D) = tb(K, D) - rot(K, D)
\]

Suppose that \( L \) has \( k \) components. We can attaching a Stein 2-handle along each component, with framing 1 less than the contact framing. The result is a weak filling \( (X', \omega') \) containing a surface \( \overline{D} \) obtained by capping the surface \( D \) off by the cores of the Stein 2-handles. In particular,

\[
\chi(\overline{D}) = k + \chi(D)
\]

Moreover, we have that

\[
\overline{D} \cdot \overline{D} = tb(L, D) - k
\]

and

\[
\langle c_1(J), [\overline{D}] \rangle = rot(L, D)
\]

Applying [Eli04, Etn04], we can cap off \( (X', \omega') \) to get a closed symplectic 4-manifold \( (X, \omega) \). The adjunction inequality for closed symplectic 4-manifolds implies that

\[
\chi(\overline{D}) \leq \langle c_1(J), \overline{D} \rangle - \overline{D}^2
\]

Therefore

\[
-sl(L, D) = rot(K, D) - tb(K, D)
\]

\[
= \langle c_1(J), [\overline{D}] \rangle + k - \overline{D}^2
\]

\[
\geq \chi(\overline{D}) - k
\]

\[= \chi(D) \]

\(\square\)
The adjunction inequality (Theorem 1.3) and the generalized slice-Bennequin inequality (Theorem 1.4) are immediate consequences of the slice-Bennequin inequality (Theorem 3.5) combined with the following proposition, which is a summary of the results in this section.

**Proposition 4.1.** Let $\mathcal{P}$ be a symplectic polyhedral decomposition of $(X, J)$ and let $\mathcal{K}$ be an embedded surface that is homotopically transverse to $\mathcal{P}$ and has transverse boundary. There exists a codimension-0 submanifold $\tilde{X} \subset X$, a homotopy of $J$, an isotopy of $\mathcal{K}$, and a symplectic form $\tilde{\omega}$ on $\tilde{X}$ such that:

1. the field $\tilde{\xi}$ of $J$-complex tangencies along $\tilde{Y} = \partial \tilde{X}$ is a positive contact structure.
2. the pair $(\tilde{X}, \tilde{\omega})$ is a weak symplectic filling of $(\tilde{Y}, \tilde{\xi})$.
3. the intersection $\tilde{L} = \mathcal{K} \cap \tilde{Y}$ is a transverse link with respect to the contact structure $\tilde{\xi}$.
4. the subsurface $S_\mathcal{K} = \mathcal{K} \cap (X \setminus \tilde{X})$ is $J$-holomorphic.
5. if $\mathcal{K}$ is closed, the self-linking number of $\tilde{L}$ is given by the formula
   \[
   sl(\tilde{L}, \tilde{D}) = \mathcal{K}^2 - \langle c_1(J), \mathcal{K} \rangle + \chi(S_\mathcal{K})
   \]
   where $\tilde{D} = \mathcal{K} \cap \tilde{X}$.
6. if $\mathcal{K}$ has boundary, the self-linking number of $\tilde{L}$ is given by the formula
   \[
   sl(\tilde{L}, \tilde{D}) = sl(\partial \mathcal{K}, \mathcal{K}) + \chi(S_\mathcal{K})
   \]

**Proof of Theorems 1.3 and 1.4.** By Condition (2) of Proposition 4.1, we can apply the slice-Bennequin inequality and obtain

\[
sl(\tilde{L}, \tilde{D}) \leq -\chi(\tilde{D})
\]

When $\mathcal{K}$ is closed, applying the self-linking formula of Condition (5) gives

\[
\mathcal{K}^2 - \langle c_1(J), \mathcal{K} \rangle + \chi(S_\mathcal{K}) \leq -\chi(\tilde{D})
\]

which is equivalent to the adjunction inequality since $\chi(\mathcal{K}) = \chi(\tilde{D}) + \chi(S_\mathcal{K})$. Similarly, when $\mathcal{K}$ has boundary, applying the formula of Part (6) gives

\[
sl(\partial \mathcal{K}, \mathcal{K}) + \chi(S_\mathcal{K}) \leq -\chi(\tilde{D})
\]

which is equivalent to the slice-Bennequin inequality for the same reason as in the closed case above. □

### 4.1. Homotope $\mathcal{K}$ to $\mathcal{L}$

In order to simultaneously establish both the topological and geometric properties listed in Proposition 4.1, the isotopy of $\mathcal{K}$ is actually a regular homotopy consisting of $n$ finger moves which are then undone by the corresponding $n$ Whitney moves.

The first step is to homotope $\mathcal{K}$ to an immersed, geometrically transverse surface $\mathcal{L}$.

**Lemma 4.2.** There exists a regular homotopy of $\mathcal{K}$ to an immersed, geometrically transverse surface $\mathcal{L}$. Moreover, the regular homotopy is encoded by $n$ Whitney disks, each of which intersects $H_P$ in a single, connected arc.

**Proof.** By assumption, the surface $\mathcal{K}$ is homotopically transverse. In particular, for each 3-dimensional stratum $H$ of $H_P$, the tangle $\tau = \mathcal{K} \cap H$ can be homotoped rel endpoints to be geometrically transverse. In other words, there is a sequence of ambient isotopies and crossing changes of the tangle that make it geometrically transverse. Moreover, in the case where $\mathcal{K}$ has transverse boundary, each component of $\partial \mathcal{K} \cap H_P$ is already geometrically transverse.

The homotopy of $\tau$ extends to a regular homotopy of $\mathcal{K}$. Moreover, crossing changes of $\tau$ correspond to finger moves of the surface $\mathcal{K}$. This introduces a pair of self-intersection points of the surface. The
finger move can be undone by a Whitney move across a Whitney disk \( W \). This disk intersects \( J \) along an arc, which we call a Whitney arc, whose endpoints lie the tangle \( \tau \) and encode the crossing change.

\[ \square \]

4.2. Construct \( \tilde{X} \). The submanifold \( \tilde{X} \) is constructed as the union of the (smoothed) sectors of the polyhedral decomposition with several Weinstein 1-handles.

Recall that by Proposition 2.17, each sector \( X_\lambda \) of the polygonal decomposition can be arbitrarily approximated by some \( \tilde{X}_\lambda \). Define

\[ \tilde{X} = \bigsqcup_\lambda \tilde{X}_\lambda \]

to be the union of all these approximations. The boundary of \( \tilde{X} \) is the contact 3-manifold

\[ (\tilde{Y}, \tilde{\xi}) = \bigsqcup_\lambda (\tilde{Y}_\lambda, \tilde{\xi}_\lambda) \]

Finally, since \( L \) is geometrically transverse, the intersection \( \tilde{L} = L \cap \tilde{Y} \) is a transverse link.

Lemma 4.3. The submanifold \( \tilde{X} = \tilde{X} \cup \bigcup \nu(W_i) \) is diffeomorphic to adding \( n \) topological 1-handles to \( \tilde{X} \).

Proof. By Lemma 4.2, the Whitney disk \( W_i \) intersects \( H_P \) along a single arc. Therefore \( \nu(W_i) \) intersects \( H_P \) along some \( B^3 \) neighborhood of this arc and intersects the complement of \( \tilde{X} \) along a 4-dimensional thickening of this 3-ball. In other words, taking the union with \( \nu(W_i) \) adds a \( B^3 \times [0, 1] \), which by definition is a 1-handle. Since the Whitney disks are disjoint, we can choose the tubular neighborhoods disjoint and therefore each Whitney disk contributes exactly one 1-handle to \( \tilde{X} \). \( \square \)

We can then modify \( J \) so that each \( \nu(W_i) \) is a Weinstein 1-handle and obtain the following proposition.

Proposition 4.4. There exists a homotopy of \( J \), fixed on \( \tilde{X} \), such that

1. the field of \( J \)-tangencies along \( \tilde{Y} = \partial \tilde{X} \) is a positive contact structure \( \tilde{\xi} \).
2. the contact structure \( (\tilde{Y}, \tilde{\xi}) \) is obtained from \( (\tilde{Y}, \tilde{\xi}) \) by \( n \) contact 0-surgeries.

In addition, this homotopy can be chosen so that \( S_L = L \cap (X \setminus \tilde{X}) \) remains \( J \)-holomorphic.

Moreover, \( \tilde{X} \) admits a symplectic form \( \tilde{\omega} \) that dominates \( \tilde{\xi} \). In particular, \( (\tilde{X}, \tilde{\omega}) \) is a weak symplectic filling of \( (\tilde{Y}, \tilde{\xi}) \).

Proof. The boundary of \( \nu(W_i) \) intersects \( \tilde{Y} \) along a pair of 2-spheres \( S_a \) and \( S_b \), which bound 3-balls \( B_a \) and \( B_b \). By a \( C^\infty \)-small perturbation, we can assume both spheres are convex surfaces with respect to the contact structure \( (\tilde{Y}, \tilde{\xi}) \). By Giroux’s criterion, the dividing set on \( S_a \) and \( S_b \) both consist of a single closed curve and by Giroux flexibility, we can further assume that there is a contactomorphism \( \phi : \nu(S_a) \to \nu(S_b) \) sending \( S_a \) to \( S_b \).

Topological 0-surgery consists of removing the pair \( B_a \cup B_b \) and gluing in \( S^2 \times [0, 1] \). Furthermore, this constitutes contact 0-surgery if we endow the cylinder with the vertically-invariant contact structure. Since we can choose \( J \) to be invariant in the normal direction along \( H_P \), this contact structure is homotopic to the field of \( J \)-complex tangencies along \( \partial \nu(W_i) \).
The final statements follow by viewing the interior of $\nu(W_i)$ as a Weinstein 1-handle, hence the symplectic structure extends across the handle and the effect on the boundary is precisely contact 0-surgery.

4.3. The link $\tilde{L}$. The final piece of the construction is the transverse link $\tilde{L}$ obtained as the intersection of $L$ with the boundary of $\tilde{X}$.

Lemma 4.5. Let $\tilde{L} = L \cap \tilde{Y}$. Then

(1) the link $\tilde{L}$ is transverse to the contact structure $(\tilde{Y}, \tilde{\xi})$, and

(2) there is a homotopy from $L$ to $K$ fixing $S_L$.

In particular, the intersection $K \cap \tilde{Y}$ is a transverse link.

Proof. Since the intersection of $L$ with $X \setminus \tilde{X}$ is $J$-holomorphic, its boundary must be transverse to the field of $J$-complex tangencies along $\tilde{Y}$. Secondly, since the Whitney disks encoding the homotopy from $L$ to $K$ are contained in $\tilde{X}$, we can assume that homotopy fixes $S_L$.

Lemma 4.6. Let $\tilde{L} = L \cap \tilde{Y} = K \cap \tilde{Y}$, let $\tilde{D} = K \cap \tilde{X}$ and let $S_K = S_L$ denote the complement of $\tilde{D}$ in $K$.

(1) if $K$ is closed, then

$$sl(\tilde{L}, \tilde{D}) = K^2 - \langle c_1(J), K \rangle + \chi(S_K)$$

(2) if $K$ has nonempty transverse boundary, then

$$sl(\tilde{L}, \tilde{D}) = sl(\partial K, K) + \chi(S_K)$$

Proof. In both cases, the surface $K$ splits into $\tilde{D}$ and $S_K$ along the link $\tilde{L}$.

Consequently, in the closed case, we have that

$$K^2 - \langle c_1(J), K \rangle = sl(\tilde{L}, \tilde{D}) + sl(\tilde{L}, S_K)$$

and in the case with transverse boundary we have that

$$sl(\partial K, K) = sl(\tilde{L}, \tilde{D}) + sl(\tilde{L}, S_K)$$

Since $S_K$ is $J$-holomorphic, Lemma 3.2 implies that

$$sl(\tilde{L}, S_K) = -\chi(S_K)$$

and both formulas follow immediately. □

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