ALL GENERATING SETS OF ALL PROPERTY T VON NEUMANN ALGEBRAS HAVE FREE ENTROPY DIMENSION $\leq 1$

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ABSTRACT. Suppose that $N$ is a diffuse, property T von Neumann algebra and $X$ is an arbitrary finite generating set of selfadjoint elements for $N$. By using rigidity/deformation arguments applied to representations of $N$ in ultraproducts of full matrix algebras, we deduce that the microstate spaces of $X$ are asymptotically discrete up to unitary conjugacy. We use this description to show that the free entropy dimension of $X$, $\delta_0(X)$, is less than or equal to 1. It follows that when $N$ embeds into the ultraproduct of the hyperfinite $\text{II}_1$-factor, then $\delta_0(X) = 1$ and otherwise, $\delta_0(X) = -\infty$. This generalizes the earlier results of Voiculescu, and Ge, Shen pertaining to $SL_n(\mathbb{Z})$ as well as the results of Connes, Shlyakhtenko pertaining to group generators of arbitrary property $T$ algebras.

INTRODUCTION

In [22] and [23], Voiculescu introduced the notion of free entropy dimension. For $X$ a finite set of self-adjoint elements of a tracial von Neumann algebra, $\delta_0(X)$ is a kind of asymptotic Minkowski dimension of the set of matricial microstates for $X$. These notions led to the solution of several old operator algebra problems (see [24] for an overview). Closely tied to this is the invariance question for $\delta_0$ which asks the following. If $X$ and $Y$ are two finite sets of selfadjoint elements generating the same tracial von Neumann algebra, then is it true that $\delta_0(X) = \delta_0(Y)$?

For certain $X$ one can compute $\delta_0(X)$ and answer the invariance question in the affirmative. Suppose that $N = W^*(X)$ is diffuse and embeds into the ultraproduct of the hyperfinite $\text{II}_1$-factor. Then $\delta_0(X) = 1$ when $N$ has property $\Gamma$, or has a Cartan subalgebra, or is nonprime, or can be decomposed as an amalgamated free product of these algebras over a common diffuse subalgebra (see [10, 12, 14, 23]).

Another class of algebras to investigate in regard to possible values of $\delta_0(X)$ and the invariance question are those with Kazhdan’s property T ([6, 15, 18]). These first appeared in the von Neumann algebra context in Connes’ seminal work [5]. In recent years, Popa introduced the technique of playing the rigidity properties of such algebras against deformation results; this has led to a number of significant advances in the theory of von Neumann algebras. ([18], [19], [11]).

Voiculescu made the first computations of $\delta_0$ for property $T$ factors by showing that if $x_1, \ldots, x_n$ are diffuse, selfadjoint elements in a tracial von Neumann algebra such that for each $1 \leq i \leq n - 1$, $x_ix_{i+1} = x_{i+1}x_i$, then $\delta_0(x_1, \ldots, x_n) \leq 1$. For $n \geq 3$, there exists a finite set of generators $X_n$ for the group algebra $\mathbb{C}SL_n(\mathbb{Z})$ with this property (this was first used in the context of measurable equivalence relations by Gaboriau [8] to prove that their cost is at most 1). Hence $L(SL_n(\mathbb{Z}))$ has a set of generators $X$ for which $\delta_0(X) \leq 1$. This was generalized in [10] (see also [9] and references therein) where Ge and Shen weakened the conditions on the generators $x_i$ and in particular obtained the stronger statement that $\delta_0(Y) \leq 1$ for any other set $Y$ of self-adjoint generators of the von Neumann algebra. However, all of these results rely on the special algebraic properties of certain generators (e.g. in $SL_n(\mathbb{Z})$) and thus do not apply to the more general property $T$ groups or von Neumann algebras.

2000 Mathematics Subject Classification. Primary 46L54; Secondary 52C17.
Research supported in part by the NSF.
In [7] a notion of $L^2$ cohomology for von Neumann algebras was introduced, and the values of the resulting $L^2$ Betti numbers were connected with free probability and the value of $\delta_0$. Indeed, using cohomological ideas, it was proved in [7] that if $X \subset \mathbb{C}\Gamma$ is an arbitrary set of generators, then

$$\delta_0(X) \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1.$$  

Here $\beta_j^{(2)}(\Gamma)$ are the Atiyah-Cheeger-Gromov $\ell^2$-Betti numbers of $\Gamma$ (see e.g. [16]). This inequality is quite complicated to prove; indeed, one first proves the same inequality with $\delta_0$ replaced by its “non-microstates” analog $\delta^*$, and then uses a highly nontrivial result of Biane, Capitaine, Guionnet [11] that implies $\delta_0 \leq \delta^*$.

In the case that $\Gamma$ has property T, the first $\ell^2$ Betti number vanishes ([3]). So for $\Gamma$ an infinite group, one has $\delta_0(X) \leq 1$ for any finite generating set $X \subset \mathbb{C}\Gamma$. However, even in this case, an “elementary” proof of this bound was not available and, moreover, it was not known whether $\delta_0(X) \leq 1$ for any finite generating set $X \subset L(\Gamma)$.

Our result settles the question of the value of $\delta_0(X)$ for an arbitrary set of self-adjoint generators of a property T factor in full generality:

**Theorem.** Suppose that $N$ is a property T diffuse von Neumann algebra with a finite set of self-adjoint generators $X$, and let $R^\omega$ be an ultrapower of the hyperfinite $\text{II}_1$ factor. Then $\delta_0(X) \leq 1$. Moreover, if $N$ has an embedding into $R^\omega$, then $\delta_0(X) = 1$, and if $N$ has no embedding into $R^\omega$, then $\delta_0(X) = -\infty$.

Note that this result shows that the value of the free entropy dimension $\delta_0$ is independent of the choice of generators of $N$. In particular, one gets as a corollary that if $\Gamma$ is any infinite discrete group with property T, and $X$ is any set of self-adjoint generators of the group von Neumann algebra $L(\Gamma)$ (we do not make the assumption that $X \subset \mathbb{C}\Gamma$ here), then $\delta_0(X) = 1$ or $-\infty$, depending on whether $\Gamma$ embeds into the unitary group of $R^\omega$.

The proof of the main theorem relies on a deformation/rigidity argument in the style of Popa, which is used to prove that the set of unitary conjugacy classes of embeddings of a property T von Neumann algebra $N$ into the ultrapower of the hyperfinite $\text{II}_1$ factor is discrete. This fact can then be employed to show that if $X \subset N$ is a set of self-adjoint generators, then any $k \times k$ matrical microstate for $X$ essentially lies in the unitary orbit of a certain discrete set $S$, all of whose elements are at least a certain fixed distance apart. One then turns this into an estimate for the packing dimension of the microstate space for $X$. We prove, effectively, that the packing dimension of the microstate set is essentially the same as that of a small number of disjoint copies of the $k$-dimensional unitary group.

## 1. Property T, Embeddings, and Unitary Orbits

Throughout this section and the next we fix a property T finite von Neumann algebra $N$ and a finite $p$-tuple of self-adjoint generators $X \subset N$. $\| \cdot \|_2$ denotes the $L^2$-norm induced by a specified trace on a von Neumann algebra. $M_k^{sa}(\mathbb{C})$ denotes the set of self-adjoint $k \times k$ matrices and $M_k(\mathbb{C})$ denotes the set of $k \times k$ matrices. $tr_k$ is the trace on $M_k(\mathbb{C})$. If $\xi = \{y_1, \ldots, y_p\}$ and $\eta = \{z_1, \ldots, z_p\}$ are $p$-tuples in a von Neumann algebra and $u, w$ are element in a tracial von Neumann algebra, then $\xi - \eta = \{y_1 - z_1, \ldots, y_p - z_p\}$, $u_\eta w = \{uy_1w, \ldots, uy_pw\}$, and $\|\xi\|_2 = (\sum_{i=1}^p \|y_i\|^2)^{1/2}$. $R > 0$ will be a fixed constant greater than any of the operator norms of the elements in $X$. $\Gamma_R(X; m, k, \gamma)$ will denote the standard microstate spaces introduced in [22].

The following theorem, stated for the reader’s convenience, is by now among the standard results in the theory of rigid factors. Such deformation-conjugacy arguments have played a fundamental role in the recent startling results of Popa and others ([11], [17], [19], [20]).
**Theorem 1.1.** Let $X$ and $N$ be as above. Then for any $t > 0$ there exists a corresponding $r_t > 0$ so that if $(M, π)$ is a tracial von Neumann algebra and $π, σ : N → M$ are normal faithful trace-preserving *-homomorphisms such that for all $x ∈ X$, $\|π(x) − σ(x)\| < r_t$, then there exist projections $e ∈ π(N)' ∩ M$, $f ∈ σ(N)' ∩ M$, a partial isometry $v ∈ M$ such that $v^*v = e$, $vv^* = f$, $π(e) > 1 − t$, and for all $x ∈ N$, $vπ(x)ev^* = fσ(x)f$.

**Proof.** Recall (see [15]) that there exist $K, ε_0 > 0$, and a finite set $F ⊂ N$ such that if $0 < δ ≤ ε_0$ and $H$ is a correspondence of $N$ with a vector $ξ ∈ H$ satisfying $\|zξ − ξz\| < δ$, $z ∈ F$, then there exists a vector $η ∈ H$ which is central for $M$ and $\|η − ξ\| < Kδ$. Choose $r_t$ so small so that if $ρ_1, ρ_2 : N → M$ are any two faithful, normal trace preserving *-homomorphisms such that for all $x ∈ X$, $\|ρ_1(x) − ρ_2(x)\| < r_t$, then for all $z ∈ F$, $\|ρ_1(z) − ρ_2(z)\| < \min\{t, ε_0\} · (4K)^{-1}$. This can be done because $X$ generates $N$.

Suppose $π, σ : N → M$ are two normal, faithful trace-preserving *-homomorphisms such that for all $x ∈ X$, $\|π(x) − σ(x)\| < r_t$. Consider $L^2(M)$ as an $N − N$ bimodule where for any $ξ ∈ L^2(M)$, $x, y ∈ N$, $xξy = π(x)Jσ(y)^*Jξ$. Denote by $1_M$ the vector associated to the unit of $M$. The hypothesis on $π$ and $σ$ guarantee that for all $x ∈ F$, $\|x 1_M − 1_Mx\| = \|π(x) − σ(x)\| < \min\{t, ε_0\} · (4K)^{-1}$ which in turn implies the existence of a central vector $η_0 ∈ L^2(M)$ for $N$ such that $\|η_0 − 1_M\| < t/4$. Regard $η_0$ as an unbounded operator on $L^2(M)$ by its left action. If $η_0 = u|η_0|$ is the polar decomposition of $η_0$, then $u ∈ M$ and $\|η_0 − 1_M\| < t/4$ implies $\|u − 1_M\| < t/2 \Rightarrow \|u^*u − 1_M\| < t$. On the other hand, since for any $x ∈ N$, $xη_0 = η_0x$, one concludes in the usual way that $xu = ux$. Consequently, $uu^* ∈ π(N)'$, $u^*u ∈ σ(N)'$. Set $e = uu^* ∈ π(N)' ∩ M$ and $f = u^*u ∈ σ(N)' ∩ M$. It follows that for all $x ∈ N$, $u^*eπ(x)ev^* = fσ(x)f$. Finally, $τ(e) = τ(f) > 1 − t$.

For each $t > 0$, we now choose a critical $r = r_t > 0$ dependent on $t$ as in Theorem 1.1.

We now need some notation.

**Notation 1.2.** (a) If $η ∈ (M_k^{sa}(C))^p$ and $r > 0$, then

$$Ω_r(η) = \{ξ ∈ (M_k^{sa}(C))^p : \text{for some } u ∈ U_k, \|ξ − u^*ηu\| < r\}.$$ 

(b) If $η ∈ (M_k^{sa}(C))^p$ and $k, s > 0$, then $G_{k,s}(η)$ consist of all $p$-tuples $ξ$ such that there exists projections $e, f ∈ M_k^{sa}(C)$ and a $w ∈ M_k(C)$ such that $w^*w = e$, $ww^* = f$, $tr_k(e) = tr_k(f) > s$ and $\|weξw^* − fηf\| < k$.

**Lemma 1.3.** For any $k, t > 0$ there exist an $m ∈ N$ such that if $ξ, η ∈ Γ_R(X; m, k, m^{-1})$ and $ξ ∈ Ω_{r_t}(η)$, then $ξ ∈ G_{k,1−t}(η)$.

**Proof.** We proceed by contradiction. Assume that there exists some $κ_0, t_0 > 0$ such that for each $m ∈ N$ there are $m_k, m_w ∈ N$ and $ξ_m, η_m ∈ Γ_R(X; m, k, m^{-1})$ such that $ξ ∈ Ω_{r_t}(η)$ and $ξ ∉ G_{κ_0,1−t}(η)$.

Fix a free ultrafilter $ω$, and consider the ultraproduct

$$R^ω = \prod_{m ∈ N} M_{m_k}(C) = \{\langle x_m⟩_{m=1}^∞ : \lim_{m} tr_{m_k}(x_mx_m^*) = 0\}.$$ 

Denote by $Q : \prod_{m} M_{m_k} → R^ω$ the quotient map. Set $ξ = ⟨ξ_m⟩_{m=1}^∞$ and $η = ⟨η_m⟩_{m=1}^∞$.

For each $m$ we can find a $k_m × k_m$ unitary $u_m$ such that $\|u_m^*ξ_mu_m − η\| < r$. Set $u = ⟨u_m⟩_{m=1}^∞$. It follows that there exist two normal faithful trace-preserving *-homomorphisms $π, σ : N → R^ω$ such that $π(X) = Q(U)^*Q(ξ)Q(U)$ and $σ(X) = Q(η)$. Clearly $\|π(X) − σ(X)\| < r$. By Theorem 1.1 there exist projections $e ∈ π(N)' ∩ R^ω$, $f ∈ σ(N)' ∩ R^ω$ and a partial isometry $v ∈ R^ω$ with initial domain $e$ and final range $f$ such that for all $x ∈ N$, $vπ(x)ev^* = fσ(x)f$ and $τ(e) = τ(f) > 1 − t_0$. There exist sequences of projections $⟨e_m⟩_{m=1}^∞$ and $⟨f_m⟩_{m=1}^∞$ such that for each $m$, $e_m, f_m ∈ M_{m_k}(C)$ and $Q(⟨e_m⟩_{m=1}^∞) = e$, $Q(⟨f_m⟩_{m=1}^∞) = f$. 

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Similarly there exists a sequence of partial isometries \( \langle v_m \rangle_{m=1}^{\infty} \) such that for each \( m, v_m \in M_{k_m}(\mathbb{C}) \) and \( Q(\langle v_m \rangle_{m=1}^{\infty}) = v \). We can also arrange it so that for each \( m, \) \( v_m v_m^* = f_m \) and \( v_m^* v_m = e_m \). Now, the equation \( v e x v^* = f \), \( x \in M \) implies in particular, that for some \( \lambda_0 \in \omega \)
\[ \|v_{m\lambda_0} e_{m\lambda_0} \xi_{m\lambda_0} e_{m\lambda_0} v_{m\lambda_0}^* - f_{m\lambda_0} \eta_{m\lambda_0} f_{m\lambda_0}\|_2 < \kappa_0 \]
and that the normalized trace of both \( f_{m\lambda_0} \) and \( e_{m\lambda_0}\) is strictly greater than \( 1 - t_0 \). But this means that \( \xi_{m\lambda_0} \in G_{\kappa_0,1-t_0}(\eta) \) which contradicts our initial assumption.

\[\Box\]

**Remark 1.4.** Observe that in Lemma 2.1 the quantity \( r_i \) is independent of \( \kappa \).

2. The Main Estimate

In this section we maintain the notation for \( K_\epsilon \) introduced in [13] taken now with respect to the microstate spaces with the operator norm cutoffs. Set \( K = \|X\|_2 \). We first state a technical lemma on the covering numbers for the spaces \( G_{\kappa,s}(\eta) \).

**Lemma 2.1.** If \( \eta \in (M^a_k(\mathbb{C}))^p \) and \( \epsilon, \kappa, s > 0 \) with \( \epsilon > \kappa \), then there exists an \( 5K\epsilon \)-net for \( G_{\kappa,s}(\eta) \) with cardinality no greater than

\[
\left( \frac{2\pi}{\epsilon} \right)^{2k^2+2s^2k^2} \cdot \left( \frac{K + 1}{\epsilon} \right)^{4(1-s)^2k^2}.
\]

**Proof.** Find the smallest \( m \in \mathbb{N} \) such that \( sk \leq m \leq k \). Denote by \( V \) the set of partial isometries in \( M_k(\mathbb{C}) \) whose range has dimension \( m \). Denote by \( P_{m} \) the set of projections of trace \( mk^{-1} \). It follows from [21] that there exists an \( \epsilon \)-net for \( P_{m} \) (with respect to the operator norm) with cardinality no greater that \( (\frac{2\pi}{\epsilon})k^2 - m^2 - (k-m)^2 \). There exists again by [21] an \( \epsilon \)-net for the unitary group of \( M_{m}(\mathbb{C}) \) (with respect to the operator norm) with cardinality no greater than \( (\frac{2\pi}{\epsilon})m^2 \). These two facts imply that there exists an \( \epsilon \)-net \( \langle v_{jk} \rangle_{j \in J_k} \) for \( V \) with respect to the operator norm such that

\[
\# J_k < \left( \frac{2\pi}{\epsilon} \right)^{2km-m^2}.
\]

Now fix \( j \in J_k \). Denote by \( G(\eta, j) \), the set of all \( \xi \in (M^a_k(\mathbb{C}))^p \) such that \( \|\xi\|_2 \leq K \) and \( \|v_{jk}(e_{jk} \xi e_{jk}) v_{jk}^* - f_{jk} \eta f_{jk}\|_2 < 5K\epsilon \) where \( e_{jk} = v_{jk} v_{jk}^* \) and \( f_{jk} = v_{jk}^* v_{jk} \). There exists a 2\( \epsilon \)-cover \( \langle \xi_{ijk} \rangle_{i \in \theta(j)} \) such that \( \# \theta(j) < \left( \frac{K + 1}{\epsilon} \right)^{4(1-s)^2k^2} \).

Consider the set \( \langle \xi_{ijk} \rangle_{i \in \theta(j), j \in J_k} \). It is clear that this set has cardinality no greater than

\[
\left( \frac{2\pi}{\epsilon} \right)^{2km-m^2} \cdot \left( \frac{K + 1}{\epsilon} \right)^{4(1-s)^2k^2}.
\]

It remains to show that this set is a \( 5K\epsilon \)-cover for \( G_{\kappa,s}(\eta) \). Towards this end suppose \( \xi \in G_{\kappa,s}(\eta) \). Then there exists a partial isometry \( v \in M_k(\mathbb{C}) \) such that \( v^* v = e, vv^* = f, \|ve\xi ev^* - f\eta f\|_2 < \kappa \), and \( tr_k(e) = tr_k(f) > s \). By cutting the domain and range of the projection, we can assume that \( e \) and \( f \) are projections onto subspaces of dimension exactly \( m \) and we can assume that the inequality with tolerance \( \kappa \) is preserved. Obviously then \( v \in V \), whence there exists a \( j_0 \in J_k \) such that \( \|v_{j_0} - v\| < \epsilon \). This condition immediately implies that \( \|v_{j_0} e_{j_0} \xi_{j_0} - ve\|, \|f_{j_0} - f\| < 2\epsilon \) and thus

\[
\|v_{j_0} e_{j_0} \xi_{j_0} - f_{j_0} \eta f_{j_0}\|_2 < 4\epsilon K + \|ve\xi ev^* - f\eta f\|_2 < 5K\epsilon.
\]

By definition, \( \xi \in G(\eta, j_0) \). Thus, there exists some \( i_0 \) such that \( i_0 \in \theta(j_0) \) and \( \|\xi_{i_0j_0} - \xi\|_2 < 5K\epsilon \).

We can now prove the main result of the paper:
Theorem 2.2. Let $N$ be a property $T$ diffuse von Neumann algebra with a finite set of selfadjoint generators $X$, and let $R^\omega$ be an ultrapower of the hyperfinite II$_1$ factor.

(a) If $N$ has an embedding into $R^\omega$, then $\delta_0(X) = 1$. (b) If $N$ has no embedding into $R^\omega$, then $\delta_0(X) = -\infty$.

Proof. Fix $1 > a > 0$. For any $\epsilon > 0$, setting $\kappa = \epsilon$ and $t = 1 - a$ in Lemma [13] shows that there exists an $m \in \mathbb{N}$, $m > p^2$, such that if $\xi, \eta \in \Gamma_R(X; m, k, m^{-1})$ and $\xi \in \Theta_{r_a}(\eta)$, then $\xi \in G_{\epsilon,1-a}(\eta)$. Consider the ball $B_k$ of $(M_k^e(\mathbb{C}))^p$ of $\|\cdot\|_2$-radius $K + 1$. For each $k$ find an $r_a$-net $(\eta_{jk})_{j,k}^p$ of $\Gamma_R(X; m, k, m^{-1})$ with minimal cardinality such that each element of the net lies in $\Gamma(X; m, k, m^{-1})$. The standard volume comparison test of this set with $B_k$ (remember that $\Gamma(X; m, k, m^{-1}) \subset (M_k^e(\mathbb{C}))^p$) implies that

$$\#J_k \leq \left(\frac{K + 2}{r_a}\right)^{pk^2}.$$ \hfill ($\Box$

For each such $j \in J_k$ find a $5K\epsilon$-net $\langle \xi_{ij} \rangle_{i \in \theta(j)}$ for $G_{\epsilon,1-a}(\eta_{jk})$ where $\theta(j)$ is an indexing set satisfying

$$\#\theta(j) \leq \left(\frac{2\pi}{\epsilon}\right)^{2k^2-(1-a)^2k^2} \cdot \left(\frac{K + 2}{\epsilon}\right)^{4a^2k^2}.$$ \hfill ($\Box$

Consider now the set $\langle \xi_{ij} \rangle_{i \in \theta(j), j \in J_k}$. It is clear that this set has cardinality no greater than

$$\left(\frac{K + 2}{r_a}\right)^{pk^2} \cdot \left(\frac{2\pi}{\epsilon}\right)^{(1+2a-a^2)k^2} \cdot \left(\frac{K + 2}{\epsilon}\right)^{4a^2k^2}.$$ \hfill ($\Box$

Moreover, if $\xi \in \Gamma_R(X; m, k, m^{-1})$, then there exists some $j_0 \in J_k$ such that $\|\xi - \eta_{j_0}\|_2 < r_a$. Clearly then, $\xi \in \Theta_{r_a}(\eta)$ which implies that $\xi \in G_{\epsilon,1-a}(\eta_{j_0})$. Consequently there exists some $i_0 \in \theta(j_0)$ such that $\|\xi - \xi_{i_0}\|_2 < 5K\epsilon$. Therefore, $\langle \xi_{ij} \rangle_{i \in \theta(j), j \in J_k}$ is a $5K\epsilon$-net for $\Gamma_R(X; m, k, m^{-1})$.

The preceding paragraph implies that for $\epsilon > 0$,

$$\mathbb{K}_{5K\epsilon}(X) \leq \limsup_{k \to \infty} k^{-2} \log \left[\left(\frac{K + 2}{r_a}\right)^{pk^2} \cdot \left(\frac{2\pi}{\epsilon}\right)^{(1+2a-a^2)k^2} \cdot \left(\frac{K + 2}{\epsilon}\right)^{4a^2k^2}\right]$$

$$= p \log r_a + (1 + 2a - a^2) \log \epsilon + \log \left[(2\pi)^2(K + 2)^{p+4}\right].$$

Keeping in mind that $a$ and $\epsilon$ are independent it now follows from [13]

$$\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(X)}{\log \epsilon}$$

$$= \limsup_{\epsilon \to 0} \frac{\mathbb{K}_{5K\epsilon}(X)}{\log \epsilon}$$

$$\leq \limsup_{\epsilon \to 0} p \cdot \frac{\log r_a}{\log \epsilon} + 1 + 2a - a^2 + \frac{\log ((2\pi)^2(K + 2)^{p+4})}{\log \epsilon}$$

$$= 1 + 2a - a^2.$$ \hfill ($\Box$

As $1 > a > 0$ was arbitrary, $\delta_0(X) \leq 1$. The rest of the assertions follow from [12].

Remark 2.3. For $\epsilon > 0$ consider the set $X + \epsilon S = \{x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n\}$ where $\{s_1, \ldots, s_n\}$ is a semicircular family free with respect to $X$. [2] shows that for sufficiently small $\epsilon > 0$ the von Neumann algebras $M^e$ generated by $X + \epsilon S$ are not isomorphic to the free group factors and yet, if $X''$ embeds into the ultraproduct of the hyperfinite II$_1$-factor, then $\chi(X + \epsilon S) > -\infty$. Theorem 2.2 implies that if $X''$ embeds into the ultraproduct of the hyperfinite II$_1$-factor, then $M^e$ cannot have property $T$. Also observe that the usual rigidity/deformation argument shows that for sufficiently small $\epsilon > 0$, there exists a II$_1$ property $T$ subfactor $N^e$ of $M^e$.\hfill ($\Box$
Remark 2.4. Unfortunately, we were not able to settle the question of whether $N$ must be strongly \(1\)-bounded in the sense of [13].

Acknowledgments. The authors would like to thank Adrian Ioana, Jesse Peterson, and Sorin Popa for useful conversations.

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