LINE-BUNDLES ON STACKS OF RELATIVE MAPS

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Abstract. We study line-bundles on the moduli stacks of relative stable and rubber maps that are used to define relative Gromov-Witten invariants in the sense of J. Li [11, 12]. Relations between these line-bundles yield degeneration formulae which are used in [7]. In addition we prove the Trivial Cylinders Theorem, a technical result also needed in [7].

1. Introduction

Relative Gromov-Witten invariants [11, 12] are intersections numbers on a stack parameterizing stable maps to a projective manifold \(Z\) relative to a smooth divisor \(D\). One looks at stable maps to \(Z\) where all points of intersections of the map with \(D\) are marked and multiplicities at these points are specified. To obtain a proper moduli stack of such maps, one must allow the target to degenerate to \(kZ = Z \sqcup D P_1 \sqcup \cdots \sqcup D P_k\), that is, \(Z\) union a number of copies of \(P = \mathbb{P}(N_{D/Z} \oplus 1_D)\) the projective completion of the normal bundle to \(D\) in \(Z\). Maps with a non-smooth target are said to be split maps. Li constructed a moduli stack of relative maps called \(\mathcal{M}(Z, \Gamma)\) for \(\Gamma\), a certain kind of graph, and constructed its virtual fundamental cycle. This stack has an evaluation map

\[
\text{Ev}_{\mathcal{M}Z} : \mathcal{M}(Z, \Gamma) \to Z^m \times D^r
\]

where \(m\) and \(r\) are the number of interior and boundary marked points, respectively. Relative Gromov-Witten invariants are given by evaluating pullbacks of cohomology classes by \(\text{Ev}\) against the virtual cycle.

It is natural to break the target \(kZ\) as the union of \(iZ = Z \sqcup D P_i \sqcup \cdots \sqcup D P_l\) and \(k-l-1P = P_{l+1} \sqcup \cdots \sqcup D P_k\). In fact, such splitting is necessary to parameterize fixed loci in \(\mathbb{C}^*\)-localization in the sense of [8] and [4] in the relative framework [5]. If we set \(X = D\), and \(L = N_{D/Z}\), the normal bundle to \(D\) in \(Z\), one is led to study stable maps into the projectivization of a line bundle \(P = \mathbb{P}_X(L \oplus 1_X)\) relative to the zero and infinity sections, \(D_0\) and \(D_\infty\) where two stable maps are declared equivalent if they can be related by a \(\mathbb{C}^*\)-factor dilating the fibers of \(P \to X\). One can construct a moduli stack of such maps, \(\mathcal{M}(A, \Gamma)\) and its virtual cycle. This moduli stack has certain natural line bundles, called the target cotangent line bundles, \(L^0\) and \(L^\infty\) and has an evaluation map

\[
\text{Ev}_{\mathcal{M}A} : \mathcal{M}(A, \Gamma) \to X^m \times X^{r_0} \times X^{r_\infty}
\]

The rubber invariants are obtained by evaluating pullbacks of cohomology by \(\text{Ev}\) map and powers of \(c_1(L^\infty)\) against the virtual cycle.

One can take fiber products of stacks \(\mathcal{M}(Z, \Gamma_Z)\) and \(\mathcal{M}(A, \Gamma_A)\) to parameterize split maps in a bigger moduli stack \(\mathcal{M}(Z, \Gamma_Z \ast \Gamma_A)\). Such split maps form divisors

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that whose fundamental classes are (up to multiplicity) the first Chern classes of particular line-bundles on $\mathcal{M}(Z, \Gamma_Z \ast \Gamma_A)$. By finding relations between such line-bundles, one is able to prove degeneration formulae that allow us to rewrite Gromov-Witten invariants of one moduli stack in terms of Gromov-Witten invariants are smaller moduli stacks. The approach of this paper is to do an in-depth study of natural line-bundles on stacks.

In section 2, we introduce some technical definitions.

In section 3, we review the construction of $\mathcal{M}(Z, \Gamma_Z)$ and $\mathcal{M}(A, \Gamma_A)$. This material is essentially a rephrasing of parts of [11, 12].

In section 4, we introduce line-bundle systems. These are line-bundles defined on a sequence of spaces that obey certain transition properties under inclusions and a group action. In more esoteric language, line-bundle systems are line-bundles on the stacks of degenerations $Z_{rel}^r, A_{rel}^l$ and their universal targets.

In section 5, we explain how line-bundle systems naturally induce line-bundles on $\mathcal{M}(Z, \Gamma_Z)$ and $\mathcal{M}(A, \Gamma_A)$. We describe line-bundles on $\mathcal{M}(Z, \Gamma)$: Dil and $L_{i,ext}$; and line-bundles on $\mathcal{M}(A, \Gamma)$: Split, $L^0$, $L^\infty$, $L_{i,not\ top}$, $L_{i,not\ bot}$. These line-bundles have geometric meaning: $L_{i,ext}$ is a line-bundle which has a section whose zero-stack consists of maps $f : C \to kZ$ so that the $i$th marked point is not mapped to $Z \subset kZ$ (counted with multiplicity); Split is a line-bundle whose zero-stack is all split maps; $L_{i,not\ top}$, where $i$ is the label of interior marked point, is a line-bundle whose zero-stack consists of all split maps $f : C \to kP$ where $i$th marked point is not mapped to $P_k$; $L_{i,not\ bot}$ is its upside-down analog. These line-bundles satisfy certain relations. On $\mathcal{M}(Z, \Gamma_Z)$:

$$ev_i^*\mathcal{O}(D) = L_{i,ext};$$

and on $\mathcal{M}(A, \Gamma_A)$:

$$L^0 \otimes L^\infty = \text{Split}$$
$$L^0 \otimes ev_i^* L^\vee = L_{i,not\ top}$$
$$L^\infty \otimes ev_i^* L = L_{i,not\ bot}.$$

In section 6, we prove degeneration formulae involving these line-bundles. The degeneration formulae are the precise mathematical statements of the geometric interpretations described above.

In section 7, we prove a technical result called the Trivial Cylinder Theorem which is necessary to the formalism of [7].

This paper draws most directly on the Relative Gromov-Witten Invariants constructed by J. Li [11, 12]. Other approaches to relative invariants include those of Gathmann [3], Ionel and Parker [6], and A.-M. Li and Ruan [10].

Because of the technical nature of this paper, it is not to be read independently of [7]. It is our sincere hope that making this paper available will be useful to other researchers.

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2. Definitions

2.1. Schemes with the action of a group. We need to work in the category of schemes with the action of a group, or swags.

Definition 2.1.1. A scheme with the action of a group is a triple $(X, G, m)$ where $X$ is a scheme, $G$ is a group scheme, and $m$ is a morphism $m : G \times X \to X$ that gives a group action of $G$ on $X$.

Definition 2.1.2. A morphism of swags $(X, G, m) \to (X', G', m')$ is a pair $(f, f_*)$ where $f : X \to X'$ is a morphism, $f_* : G \to G'$ is a group scheme homomorphism and the following diagram commutes.

\[
\begin{array}{ccc}
G \times X & \xrightarrow{m} & X \\
\downarrow{f \times f} & & \downarrow{f} \\
G' \times X' & \xrightarrow{m'} & X'
\end{array}
\]

Definition 2.1.3. Given a morphism of swags $(f, f_*) : (X, G, m) \to (X', G', m')$, the swag-structure on $X'$ induced by $(f, f_*)$ is the swag $(X', G, m'')$ whose group action is given by the following composition

\[
G \times X' \xrightarrow{f_* \times \text{id}} G' \times X' \xrightarrow{m'} X'
\]

Definition 2.1.4. A vector bundle on a swag $(X, G, m)$ is an equivariant bundle $E$ on $X$.

Definition 2.1.5. Given a morphism of swags $(f, f_*) : (X, G, m) \to (X', G', m')$ and a vector bundle $E$, $X'$, the pullback, $f^* E$ is defined to be the equivariant pullback of $E$ under the $G$-equivariant map $f : (X, G, m) \to (X', G, m'')$ where $(X', G, m'')$ is the swag-structure on $X'$ induced by $(f, f_*)$.

2.2. Convention for Projectivizations. If $E$ is a vector-bundle on a scheme $X$, let $\mathbb{P}_X(E)$ denote the projectization of $E$ where we use the old-fashioned geometric notation where points in the projectivization represent lines. In this case, if $1_X$ is the trivial bundle on $X$, $\mathbb{P}_X(E \oplus 1_X)$ is the projective completion of $E$. The scheme $\mathbb{P}_X(E \oplus 1_X) \setminus E$ is called the infinity section.

3. Construction of moduli stacks

In this section, we construct moduli stacks $\mathcal{M}(\mathbb{Z}, \Gamma_Z)$, $\mathcal{M}(A, \Gamma_A)$. The material in this chapter is a straightforward adaptation of [11] and [12]. While J. Li does not construct $\mathcal{M}(A, \Gamma_A)$, our construction directly parallels his. We do change some notation from [11] to suit our purposes.
Consider a projective manifold $Z$ with a smooth divisor $D$. $\mathcal{M}(Z, \Gamma)$ is the moduli stack of stable maps of curves to $Z$ relative to $D$. That is, we look at stable maps where we specify multiplicities at $D$ together with the usual genus and degree information. This data is summarized in a relative graph $\Gamma$. Because the target $Z$ may degenerate, we are forced to introduce $\mathbb{A}^n$-schemes $Z[n]$ which measure degenerations. The schemes $Z[n]$ are constructed by an inductive procedure where $Z[0] = Z$, and the inductive step is similar to deformation to the normal cone. Now, $Z[n]$ admits a natural $G[n] = (\mathbb{C}^*)^n$-action where we will consider two degenerations of $Z$ equal if they are related by the $G[n]$-action. We introduce a stack, $Z^{rel}$ that models families that are locally isomorphic to $Z[n] \to \mathbb{A}^n$. Following that, we introduce maps of families of curves to families in $Z^{rel}$. We impose the conditions of pre-deformability and stability on these maps to ensure they form a proper Deligne-Mumford stack $\mathcal{M}(Z, \Gamma)$. We recall the requisite results from [12] to construct a tangent obstruction complex and a virtual cycle.

For $\mathcal{M}(A, \Gamma)$, we consider a related geometric situation. Let $X$ be a projective manifold and $L$ a line bundle on $X$. Let $P = \mathbb{P}_X(L \oplus 1_X)$, the projective completion of $L$ which is a $\mathbb{P}^1$-bundle. $P$ has two important divisors, $D_0$ and $D_\infty$, the zero and infinity sections. We study stable maps to $P$ relative to $D_0$ and $D_\infty$ where we mod out by a $\mathbb{C}^*$-factor that dilates the fibers. Again, the target $P$ may degenerate, we need to introduce a sequence of $\mathbb{A}^n$-schemes $A[n]$ where $A[0] = P$. $A[n]$ admits a natural $G[n] = (\mathbb{C}^*)^{n+1}$-action where one of $\mathbb{C}^*$ factor comes from dilating the fibers of $P \to X$, and the others are analogous to those on $Z[n]$. We introduce a stack of degenerations, $A^{rel}$ and a proper Deligne-Mumford stack $\mathcal{M}(A, \Gamma)$ of maps to $P$ modulo dilation of the fibers. This stack is called the stack of maps to rubber and carries a virtual cycle.

We should explain our top/bottom convention. In $Z$, moving towards $D$ is considered moving towards the top. In $P$, $D_0$ is considered the top while $D_\infty$ is the bottom. This slightly odd convention makes sense in that the most natural choice for $(X, L)$ is $(D, N_{D/Z})$. In this case, the zero section of $P$ is identified with $D$ and the normal bundle to $D_0$ in $P$ is equal to the normal bundle to $D$ in $Z$. Therefore, $D_0 \subset P$ like $D \subset Z$ is on top.

3.1. The spaces $Z[n]$. We will study maps to a projective manifold $Z$ relative to a divisor $D$. Because of the nature of relative invariants, we must allow the target $Z$ to degenerate. Let $L = N_{D/Z}$ be the normal bundle to $D$ in $Z$. Let $P = \mathbb{P}(L \oplus 1_D)$ be the projective completion of $L$. $P$ has two distinguished divisors, $D_0$ and $D_\infty$, the zero and infinity sections of $L$.

**Definition 3.1.1.** Let $kZ$ be the union of $Z$ with $k$ copies of $P$, $Z \sqcup D P_1 \sqcup \cdots \sqcup D P_k$, the scheme given by identifying $D \subset Z$ with $D_\infty \subset P_1$ and $D_0 \subset P_i$ with $D_\infty \subset P_{i+1}$ for $i = 0, 1, \ldots, k-1$. $kZ$ has a distinguished divisor $D = D_0 \subset P_k$.

We will construct a sequence of pairs of projective manifolds and smooth divisors, $(Z[n], D[n])$ that map to $\mathbb{A}^n$ so that the fiber over different closed points in $\mathbb{A}^n$ is $(kZ, D)$ for $k = 0, \ldots, n$.

**Definition 3.1.2.** Let $Z[0] = Z$ and $D[0] = D$. 


(Z[n], D[n]) is defined inductively. Let

1. \( Z[n] = \text{Bl}_{D[n-1 \times \{0\}}(Z[n-1] \times \mathbb{A}^1) \).
2. \( D[n] \) is the proper transform of \( D[n-1] \times \mathbb{A}^1 \).

**Definition 3.1.3.** Let \([n]\) denote the set of integers \(\{1, 2, \ldots, n\}\).

**Definition 3.1.4.** Let \( G[n] \) denote the group \((\mathbb{C}^\ast)^n\). An element \( \sigma \in G[n] \) can be written as an n-tuple, \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \)

\( G[n] \) acts on \( \mathbb{A}^n \), \( G[n] \times \mathbb{A}^n \to \mathbb{A}^n \) by

\[
(\sigma_1, \sigma_2, \ldots, \sigma_n) \cdot (t_1, t_2, \ldots, t_n) \mapsto (\sigma_1^{-1}t_1, \sigma_2^{-1}t_2, \ldots, \sigma_n^{-1}t_n).
\]

\((Z[n], D[n])\) has the following properties:

1. There exists a morphism \( p : Z[n] \to \mathbb{A}^n \).
2. \( Z[n] \) admits a \( G[n] \)-action fixing \( D[n] \) point-wise, so that \( p \) is equivariant.
3. \( D[n] \) is given as the zero-scheme of a section \( s[n] \) of a line-bundle \( V[n] \).
4. For any order-preserving inclusion \( i : [k] \hookrightarrow [n] \), there is an injective homomorphism \( G[k] \hookrightarrow G[n] \), an inclusion of swags \( \mathbb{A}^k \hookrightarrow \mathbb{A}^n \), \( Z[k] \hookrightarrow Z[n] \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Z[k] & \rightarrow & Z[n] \\
\downarrow{p} & & \downarrow{p} \\
\mathbb{A}^k & \rightarrow & \mathbb{A}^n
\end{array}
\]

The maps induce an isomorphism \( Z[k] \cong Z[n] \times_{\mathbb{A}^n} \mathbb{A}^k \).
5. There is a morphism \( c : Z[n] \to Z \) such that for any inclusion as in (3), the following diagram commutes

\[
\begin{array}{ccc}
Z[k] & \rightarrow & Z[n] \\
\downarrow{c} & & \downarrow{c} \\
Z & & \\
\end{array}
\]

Let us address (3). Since \( Z \) is smooth \( D \) is represented by a Cartier divisor \((V, s)\) where \( V \) is a line-bundle and \( s \) is a section of \( V \) whose zero scheme is \( D \).

\( Z[n] = \text{Proj}_{Z[n-1] \times \mathbb{A}^1} \left( \bigoplus_{n \geq 0} T^n \right) \) for a sheaf of ideals \( I \). Let \( \pi : Z[n] \to Z[n-1] \times \mathbb{A}^1 \to Z[n-1] \) be the projection. \( V[n] = \pi^*V[n-1] \otimes \mathcal{O}(1) \) and \( s[n] \) is \( \pi^*s[n-1] \) considered as a degree one element. Therefore, we have,

**Lemma 3.1.5.** \( D[n] \) is represented by a Cartier divisor of a section \( s[n] \) of a line-bundle \( V[n] \).

The contraction map \( c : Z[n] \to Z \) is given by the composition

\[
c : Z[n] \to Z[n-1] \to \ldots \to Z[1] \to Z[0].
\]
The projection $p : Z[n] \to \mathbb{A}^n$ is defined inductively by the composition
$$p : Z[n] \to Z[n-1] \times \mathbb{A}^1 \to \mathbb{A}^{n-1} \times \mathbb{A}.$$

### 3.2. Effective Maps

Let us give a geometric description of $Z[n]$. The construction of $Z[n]$ is similar to that of the deformation of the normal cone in [2]. Let $N$ be the normal bundle to $D$ in $Z$. Let $P = \mathbb{P}_X(N \oplus 1)$ be the projective completion of the total space of $N$. $Z[1]$ is an $\mathbb{A}^1$-scheme. The fibers of this map over $t \in \mathbb{A}^1$, $t \neq 0$ is $Z$ while the fiber over $t = 0$ is $Z \cup D$, that is, $Z$ union $P$ along $D \subset Z$ which is isomorphic to $D_\infty$, the infinity divisor in $P$. The $G[1]$ group action permutes the fibers over $\mathbb{C} \setminus 0$ and dilates $P$, considered as a $\mathbb{P}^1$-bundle.

The fiber of $Z[n]$ over $t \in \mathbb{A}^n$ is $kZ$. $k$ is the number of 0’s among the coordinates of $t$. If the $i$th zero in $t$’s coordinate occurs in the $j$th place, then the $j$th factor of $\mathbb{C}^*$ in $G[n]$ dilates the fibers of $P_k$ for all $k \geq i$. We will show this in 3.4

**Definition 3.2.1.** Let $H_l \subset \mathbb{A}^n$ be the hyperplane in $\mathbb{A}^n$ given by the equation $t_l = 1$.

Let $i$ be the morphism of swags given by
$$i : \mathbb{A}^{n-1} = H_l \hookrightarrow \mathbb{A}^n$$
$$i_* : G[n-1] \to G[n]$$
$$i_* : (\sigma_1, \ldots, \sigma_{n-1}) \mapsto (\sigma_1, \ldots, \sigma_{l-1}, 1, \sigma_{l+1}, \ldots, \sigma_{n-1})$$

Note that this inclusion makes $Z[n]$ a $G[n-1]$-scheme. We call this the **standard inclusion**.

**Lemma 3.2.2.** $Z[n] \times_{\mathbb{A}^n} H_l = Z[n-1]$ as $G[n-1]$-schemes

Let $i : [k] \to [n]$ be an order preserving inclusion. This induces an inclusion,
$$\mathbb{A}^k \hookrightarrow \mathbb{A}^n$$
where is $(s_1, \ldots, s_k)$ are the coordinates on $\mathbb{A}^k$ and $(t_1, \ldots, t_n)$ are the coordinates on $\mathbb{A}^n$, $t_{i(m)} = s_m$ for $1 \leq m \leq k$ and the other coordinates on $\mathbb{A}^n$ are set to 1. This inclusion can be factored as the inclusion of hyper-planes. Given an order-preserving inclusion $i : [k] \to [n]$, there is an inclusion $Z[k] \hookrightarrow Z[n]$, given as the composition of inclusions as above.

**Lemma 3.2.3.** Given an order-preserving inclusion $i : [k] \to [n]$, the induced maps fit into the following commutative diagram

$$\begin{array}{ccc}
Z[k] & \longrightarrow & Z[n] \\
p & & p \\
\mathbb{A}^k & \longleftarrow & \mathbb{A}^n
\end{array}$$

We have an isomorphism
$$Z[n] \times_{\mathbb{A}^n} \mathbb{A}^k \cong Z[k].$$
The spaces $Z[n]$ are models of families of degenerations of $Z$.

**Definition 3.2.4.** If $S$ is any scheme then an **effective family of** $(Z, D)$ **over** $S$ **is**

$$\tau : S \to \mathbb{A}^n$$

**where**

$$\tilde{Z} = \tau^* Z[n] = Z[n] \times_{\mathbb{A}^n S} S$$

$$\tilde{D} = \tau^* D[n] = D[n] \times_{\mathbb{A}^n S} S$$

**Definition 3.2.5.** Given $\rho : S \to G[n]$ and an effective family $(\tau, \tilde{Z}, \tilde{D})$ **over** $S$, the **action of** $\rho$ **on** $(\tau, \tilde{Z}, \tilde{D})$ **is** the family $(\tau^\rho, \tilde{Z}^\rho, \tilde{D}^\rho)$ **induced by the map**

$$\tau^\rho : S \xrightarrow{\rho \times \tau} G[n] \times \mathbb{A}^n \xrightarrow{m} \mathbb{A}^n$$

**Definition 3.2.6.** Given two effective families,

$$\xi_1 = (\tau_1, \tilde{Z}_1, \tilde{D}_1)$$

$$\xi_2 = (\tau_2, \tilde{Z}_2, \tilde{D}_2)$$

**associated to morphisms**

$$\tau_1 : S \to \mathbb{A}^{n_1}$$

$$\tau_2 : S \to \mathbb{A}^{n_2}$$

**where** $n_1 \leq n_2$, **then an effective map** $a$ **from** $\xi_1$ **to** $\xi_2$,

$$a : \xi_1 \to \xi_2$$

**consists of**

1. An inclusion $i : \mathbb{A}^{n_1} \to \mathbb{A}^{n_2}$ **associated to an order preserving inclusion**

$$[n_1] \to [n_2]$$

2. A morphism $\rho : S \to G[n_2]$

**so that**

$$(i \circ \tau_1)^\rho = \tau_2.$$ 

It is clear that the composition of two effective maps is an effective map.

**Definition 3.2.7.** Two effective families,

$$\xi_1 = (\tau_1, \tilde{Z}_1, \tilde{D}_1)$$

$$\xi_2 = (\tau_2, \tilde{Z}_2, \tilde{D}_2)$$

are **said to be related by an effective arrow**

$$F : \xi_1 \to \xi_2$$

if there is a **dominating** family,

$$\xi = (\tau, \tilde{Z}, \tilde{D})$$

**together with effective maps**

$$a_1 : \xi_1 \to \xi$$

$$a_2 : \xi_2 \to \xi$$
One can easily show that the composition of effective arrows is an effective arrow by constructing a family which dominates all the families in the composition.

**Lemma 3.2.8.** An effective family \((\tau, \tilde{Z}, \tilde{D})\) has a collapsing map to \((Z, D)\),

\[ c_{\tilde{Z}} : \tilde{Z} \to Z \]

so that \(c(\tilde{D}) \subset D\)

**Proof.** The collapsing is induced from the composition \(c \circ \tau : S \to Z[n] \to Z\)

\[ \square \]

**Lemma 3.2.9.** [11], Lemma 4.3 Let \(\xi_i = (\tau_i, \tilde{Z}_i, \tilde{D}_i)\) be effective families over \(S\). Suppose there is an isomorphism \(f : (\tilde{Z}_1, \tilde{D}_1) \to (\tilde{Z}_2, \tilde{D}_2)\) that fits into a commutative diagram

\[ \tilde{Z}_1 \quad \cong \quad \tilde{Z}_2 \]

\[ \downarrow c \times \pi \quad \downarrow c \times \pi \]

\[ Z \times S \]

Let \(p \in S\). Then there is an open neighborhood \(U\) of \(p\) in \(S\) so that over \(U\),

\[ f|_U : \tilde{Z}_1 \times_S U \to \tilde{Z}_2 \times_S U \]

is induced by an effective arrow between \(\xi_1 \times_S U\) and \(\xi_2 \times_S U\).

**Definition 3.2.10.** If \(S\) is a scheme, a family of relative pairs over \(S\) is a triple \((\tilde{Z}, \tilde{D}, \tilde{\pi})\) where \(\tilde{\pi} : Z \to Z \times S\) is a morphism and \(\tilde{D}\) is a Cartier divisor of \(\tilde{Z}\) such that the following property holds: There exists an open covering \(\{U_\alpha\}\) of \(S\) such that for all \(\alpha\), \((\tilde{Z} \times_S U_\alpha, \tilde{D} \times_S U_\alpha)\) is isomorphic to an effective family \((\tau_\alpha, \tilde{Z}_\alpha, \tilde{D}_\alpha)\) over \(U_\alpha\) by a morphism \(f\) such that the following diagram commutes

\[ (\tilde{Z} \times_S U_\alpha, \tilde{D} \times_S U_\alpha) \quad \cong \quad (\tilde{Z}_\alpha, \tilde{D}_\alpha) \]

\[ \downarrow \pi \quad \downarrow c \times \pi \]

\[ Z \times S \]

**Definition 3.2.11.** Let \(Z^{\text{rel}}\) be the category whose objects are relative families of \((Z, D)\) over schemes. For \(\xi_1, \xi_2 \in \text{Obj}(Z^{\text{rel}})\) such that \(\xi_i\) is a family over \(S_i\), \(\text{Mor}(\xi_1, \xi_2)\) consists of pairs \((h, f)\) where

\[ h : S_1 \to S_2 \]

and \(f\) is an isomorphism fitting into the following commutative diagram

\[ \xi_1 \quad \overset{f}{\rightarrow} \quad h^* \xi_2 \]

\[ \downarrow \quad \downarrow \]

\[ Z \times S_1 \]

Define \(\mathcal{P} : Z^{\text{rel}} \to (\text{Sch})\) to be the functor that sends families over \(S\) to \(S\). \((Z^{\text{rel}}, \mathcal{P})\) is a groupoid.
Proposition 3.2.12. \((\mathcal{Z}^{rel}, \mathcal{P})\) is a stack.

3.3. **The spaces** \(A[n]\). We construct a sequence of spaces \(A[n]\), called the rubber target spaces that are used to study curves in the projectivization of a line bundle \(L\) over a projective manifold \(X\). These spaces will be used to construct the rubber invariants. These spaces have properties analogous to the \(Z[n]\)'s.

Let \(X\) be a projective manifold and \(L\) a line bundle on \(X\). Let \(P = \mathbb{P}_X (L \oplus 1_X)\), and let \(X_0\) and \(X_\infty\) denote the zero and infinity sections. We study stable maps to \(P\) relative to \(X_0\) and \(X_\infty\) where we mod out by a \(\mathbb{C}^*\)-factor that dilates the fibers. Again, the target \(P\) may degenerate.

**Definition 3.3.1.** Let \(kP\) be the union of \(k + 1\) copies of \(P\),

\[ kP = P_0 \sqcup_X P_1 \sqcup \cdots \sqcup_X P_k \]

gluing \(X_0 \subset P_i \) to \(X_\infty \subset P_{i+1}\) for \(i = 0, \ldots, k - 1\).

\(kP\) has distinguished divisors \(D_\infty = X_\infty \subset P_0\) and \(D_0 = X_0 \subset P_k\). We will construct a sequence of projective manifolds and divisors \((A[n], D_0[n], D_\infty[n])\). \(A[n]\) will map to \(\mathbb{A}^n\) with fiber over closed points equal to \(kP\) for varying \(k\).

**Definition 3.3.2.** Let \(\widetilde{G}[n]\) denote the group \((\mathbb{C}^*)^{n+1}\). An element \(\sigma \in \widetilde{G}[n]\) can be written as an \(n + 1\)-tuple, \(\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)\)

In what follows, we will have \(\widetilde{G}[n]\) act on \(\mathbb{A}^n\), \(\widetilde{G}[n] \times \mathbb{A}^n \to \mathbb{A}^n\) as

\[ (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n) \cdot (t_1, t_2, \ldots, t_n) \mapsto (\sigma_1^{-1}t_1, \sigma_2^{-1}t_2, \ldots, \sigma_n^{-1}t_n). \]

**Definition 3.3.3.** The spaces \((A[n], D_0[n], D_\infty[n])\) are defined as \(\widetilde{G}[n]\)-schemes as follows

1. \(A[0] = P = \mathbb{P}_X (L \oplus 1_X)\) where
   a. \(D_0[0]\) is the zero-section in \(A[0]\), which is seen as the projective completion of \(L\), alternatively as a divisor of the line bundle \(\mathcal{O}_P(1) \otimes \pi^*L\).
   b. \(D_\infty[0]\) is the infinity section of \(A[0]\), that is the divisor of the line bundle \(\mathcal{O}_P(1)\).
   c. The \(\widetilde{G}[0] = \mathbb{C}^*\)-action is given by
      \[ \sigma_0 \cdot [l : t] = [\sigma_0 l : t] \]

2. \(A[n]\) is defined inductively from \(A[n-1]\) as follows. Consider \(A[n-1] \times \mathbb{A}^1\) under the group action
   \[ (\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot (z, t_n) = ((\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) \cdot z, \sigma_n^{-1}t_n) \]

Let \(A[n] = \text{Bl}_{D_0[n-1] \times \{0\}}(A[n-1] \times \mathbb{A}^1)\).
   a. \(D_0[n]\) is the proper transform of \(D_0[n-1] \times \mathbb{A}^1\).
   b. \(D_\infty[n]\) is the proper transform of \(D_\infty[n-1] \times \mathbb{A}^1\).

\((A[n], D_0[n], D_\infty[n])\) has the following properties
(1) There exists a morphism \( p : A[n] \to \mathbb{A}^n \).

(2) \( A[n] \) admits a \( \mathcal{G}[n] \)-action fixing \( D_0[n] \) and \( D_\infty[n] \) making \( p \) equivariant.

(3) For any order-preserving inclusion \([k] \hookrightarrow [n]\), there is an injective homomorphism \( \mathcal{G}[k] \hookrightarrow \mathcal{G}[n] \), an inclusion of swags \( \mathbb{A}^k \hookrightarrow \mathbb{A}^n \), \( A[k] \to A[n] \), such that the following diagram commutes:

\[
\begin{array}{ccc}
A[k] & \longrightarrow & A[n] \\
\downarrow p & & \downarrow p \\
\mathbb{A}^k & \longrightarrow & \mathbb{A}^n
\end{array}
\]

Moreover, the maps induce an isomorphism \( A[k] \cong A[n] \times_{\mathbb{A}^n} \mathbb{A}^k \).

(4) There is a natural projection \( \pi : A[n] \to X \)

so that for any inclusion as in (3), the following diagram commutes

\[
\begin{array}{ccc}
A[k] & \longrightarrow & A[n] \\
\downarrow \pi & & \downarrow \pi \\
X & & X
\end{array}
\]

(5) There is a morphism of swags \( t : A[n] \to A[0] \) where \( t_* : \mathcal{G}[n] \to \mathcal{G}[0] \) is given by

\[
t_* : (\sigma_0, \sigma_1, \ldots, \sigma_n) \to (\sigma_0 \sigma_1 \ldots \sigma_n).
\]

For any inclusion as in (3), the following diagram commutes

\[
\begin{array}{ccc}
A[k] & \longrightarrow & A[n] \\
\downarrow t & & \downarrow t \\
A[0] & & A[0]
\end{array}
\]

\( t \) is called the top morphism.

(6) There is a morphism of swags \( b : A[n] \to A[0] \) where \( b_* : \mathcal{G}[n] \to \mathcal{G}[0] \) is given by

\[
b_* : (\sigma_0, \sigma_1, \ldots, \sigma_n) \to (\sigma_0).
\]

For any inclusion as in (3), the following diagram commutes

\[
\begin{array}{ccc}
A[k] & \longrightarrow & A[n] \\
\downarrow b & & \downarrow b \\
A[0] & & A[0]
\end{array}
\]

\( b \) is called the bottom morphism.

Properties (1-4) follow from proofs analogous to those for \( Z[n] \).

**Definition 3.3.4.** The map \( b : A[n] \to A[0] \) is given as the composition of blow-downs and projections

\[
b : A[n] \to A[n-1] \times \mathbb{A}^1 \to A[n-1] \to \ldots \to A[0] \times \mathbb{A}^1 \to A[0].
\]
Definition 3.3.5. We can construct $A[n]$ by repeatedly blowing up $D_{\infty}$ instead of $D_0$. The map $t : A[n] \rightarrow A[0]$ is given as a composition of blow-downs and projections analogous to $b$.

The terminology for the top and bottom maps is as follows. Given a point $x \in \mathbb{A}^n$, the fiber in $A[n]$ over $x$ is

$$iP = P_0 \sqcup X P_1 \sqcup X \cdots \sqcup X P_l$$

where $l$ is the number of zeroes among $x$’s coordinates. The map $b$ restricted to the fiber over $x$ is the map $b : lP \rightarrow P$ so that

1. On $P_0$, it is the identity $P_0 \rightarrow P$.
2. On $P_i$ for $i \geq 1$, it is the projection $P_i \rightarrow X = D_0 \subset P$.

Likewise, the map $t : A[n] \rightarrow A[0]$ restricts to $lP$ as

1. On $P_l$, it is the identity $P_l \rightarrow P$.
2. On $P_i$ for $i \leq l - 1$, it is the projection $P_i \rightarrow X = D_{\infty} \subset P$.

The definition for effective arrows for $A[n]$ is analogous to that of $Z[n]$ with $Z$’s replaced by $A$’s and $G[n]$’s replaced by $\tilde{G}[n]$’s. We have families of rubber modelled on maps to $S \rightarrow \mathbb{A}^n$ analogous families of relative pairs 3.2.10

In particular, we have

Lemma 3.3.6. Let $\xi_i = (\tau_i, \tilde{A}_i, \tilde{D}_{0i}, \tilde{D}_{\infty i})$ be rubber families over $S$. Suppose there is an isomorphism $f : (\tilde{A}_1, \tilde{D}_{01}, \tilde{D}_{\infty 1}) \rightarrow (\tilde{A}_2, \tilde{D}_{02}, \tilde{D}_{\infty 2})$ that fits into a commutative diagram

$$\begin{array}{c}
\tilde{A}_1 \\
\downarrow \pi \times q \\
X \times S \\
\downarrow \pi \times q \\
\tilde{A}_2
\end{array}$$

Let $p \in S$. Then there is an open neighborhood $U$ of $p$ in $S$ so that over $U$,

$$f|_U : \tilde{A}_1 \times_S U \rightarrow \tilde{A}_2 \times_S U$$

is induced by an effective arrow between $\xi_1 \times_S U$ and $\xi_2 \times_S U$.

Definition 3.3.7. Let $A^{\text{rel}}$ be the category whose objects are relative families of $(X, L)$, modelled on $A[n] \rightarrow \mathbb{A}^n$ over schemes. For $\xi_1, \xi_2 \in \text{Obj}(A^{\text{rel}})$ such that $\xi_i$ is a family over $S_i$, $\text{Mor}(\xi_1, \xi_2)$ consists of pairs $(h, f)$ where

$$h : S_1 \rightarrow S_2$$

and $f$ is an isomorphism fitting into the following commutative diagram
Define $\mathcal{P} : A^{\text{rel}} \to (\text{Sch})$ to be the functor that sends families over $S$ to $S$. $(A^{\text{rel}}, \mathcal{P})$ is a groupoid.

**Proposition 3.3.8.** $(A^{\text{rel}}, \mathcal{P})$ is a stack.

### 3.4. Degenerate Fibers

We identify the fibers of $Z[n] \to \mathbb{A}^n$. Let $N = N_{D/Z}$ be the normal bundle to $D$ in $Z$. Let $A[n]$ be the spaces constructed from $(D, N)$.

Let $K_k \subset \mathbb{A}^n$ denote the subset of $\mathbb{A}^n$ given by the equation $t_k = 0$. We need to understand the fiber of $K_k$ under $p : Z[n] \to \mathbb{A}^n$.

Consider $Z[k-1] \times \mathbb{A}^{n-k}$ with the $G[n]$-action given by

$$(\sigma_1, \ldots, \sigma_n) \cdot (z, (t_{k+1}, \ldots, t_n)) \mapsto ((\sigma_1^{-1}t_1, \ldots, \sigma_{k-1}^{-1}t_{k-1}), (\sigma_1 \ldots \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) \cdot z)$$

Note that this scheme has a divisor $D[k-1] \times \mathbb{A}^{n-k}$.

Let us also consider $\mathbb{A}^{k-1} \times A[n-k]$ with the $G[n]$-action given by

$$(\sigma_1, \ldots, \sigma_n) \cdot (t_1, \ldots, t_{k-1}, z) \mapsto ((\sigma_1^{-1}t_1, \ldots, \sigma_{k-1}^{-1}t_{k-1}), (\sigma_1 \ldots \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) \cdot z)$$

Note that this scheme has a divisor given by $\mathbb{A}^{k-1} \times D_{\infty}[n-k]$. Note that the two divisors are isomorphic to $D \times \mathbb{A}^{n-1}$ and that they have the same $G[n]$-action.

**Lemma 3.4.1.** The fiber over $K_k$ is given by the following isomorphism of $G[n]$-schemes.

$$Z[n] \times_{\mathbb{A}^n} K_k = (Z[k-1] \times \mathbb{A}^{n-k}) \sqcup_{D \times \mathbb{A}^{n-1}} (\mathbb{A}^{k-1} \times A[n-k])$$

where $\sqcup_D$ denotes union identifying the divisor on each scheme.

**Proof.** This follows from standard facts about blow-ups and by induction. \qed

By induction, we can give a description of fibers over closed points. Let $t \in \mathbb{A}^n$ have $l$ zeroes among its coordinates. Then the fiber of $Z[n]$ over $t$ is $Z$.

An analogous result holds for $A[n]$. Let us put the following $\tilde{G}[n]$-structure on $A[k-1] \times \mathbb{A}^{n-k}$

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot (z, (t_{k+1}, \ldots, t_n)) \mapsto ((\sigma_0, \sigma_1, \ldots, \sigma_{k-1}^{-1}t_{k-1}), (\sigma_0 \sigma_1 \ldots \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) \cdot z)$$

Note that this scheme has a divisor $D_0[k-1] \times \mathbb{A}^{n-k}$. Let us also consider $\mathbb{A}^{k-1} \times A[n-k]$ with the $G[n]$-action given by

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot (t_1, \ldots, t_{k-1}, z) \mapsto ((\sigma_1^{-1}t_1, \ldots, \sigma_{k-1}^{-1}t_{k-1}), (\sigma_0 \sigma_1 \ldots \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) \cdot z)$$

Note that this scheme has a divisor given by $\mathbb{A}^{k-1} \times D_{\infty}[n-k]$. 

\[\xi_1 \xrightarrow{f} h^*\xi_2 \xrightarrow{h} X \times S_1\]
Lemma 3.4.2. The fiber over $K_k$ is given by the following isomorphism of $\widehat{\mathbb{G}[n]}$-schemes.

$$A[n] \times_{\mathbb{A}^n} K_k = (A[k-1] \times \mathbb{A}^{n-k}) \sqcup \mathbb{A}^{n-k}$$

Let $t \in \mathbb{A}^n$ have $l$ zeroes among its coordinates. Then the fiber of $A[n]$ over $t$ is $lP$.

3.5. Pre-deformable Families. Note that $\text{Sing}(kZ)$, the singular locus of $kZ$ is the disjoint union of $k$ copies of $D$, which we label $D_1, \ldots, D_{k-1}$ where $D_i = D_{\infty} \subset P_i$.

Definition 3.5.1. A morphism $f : C \to kZ$ is said to be pre-deformable if $f^{-1}(D_i)$ is the union of nodes so that for $p \in f^{-1}(D_i)$ ($i = 1, 2, \ldots, k$), the two branches of the node map to different irreducible component of $kZ$ and that the order of contact to $D_i$ are equal.

An obvious analogous definition exists for morphisms to $kP$.

There is a precise notion of pre-deformable families of morphisms given in terms of local models. See [11] for details.

Definition 3.5.2. A morphism of a pre-stable family with marked points over a scheme $S$ consists

1. A family of targets,
   $$Y \in \text{Obj}(\mathcal{Z}^{\text{rel}}(S))$$
2. A pre-stable family $\mathcal{X} \to S$.
3. A morphism $f$ fitting into
   $$\begin{tikzcd}
   \mathcal{X} \ar{r}{f} \ar{dr} & Y \\
   & S
   \end{tikzcd}$$
4. Morphisms $\gamma_1, \ldots, \gamma_k : S \to \mathcal{X}$

so that in a neighborhood $S_{\alpha} \to S$ where $Z \times_S S_{\alpha}$ is given by a morphism to $\mathbb{A}^n$, the diagram looks as follows

$$\begin{tikzcd}
\mathcal{X} \ar{r}{f} \ar{dr}{\gamma_1 \ldots \gamma_k} & Z[n] \\
& S_{\alpha} \ar{r} & \mathbb{A}^n
\end{tikzcd}$$

where

1. $f$ is a pre-deformable morphism.
(2) $\gamma_1, \ldots, \gamma_k$ are disjoint section of $\mathcal{X} \to S$ whose image does not intersect the nodes in the fibers of $\mathcal{X} \to S$.

The image of $\gamma$ are marked points. We will often be distinguishing a particular marked point and write the following diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & Z[n] \\
\downarrow \gamma_i & & \downarrow \\
S & \rightarrow & \mathbb{A}^n
\end{array}
$$

where $\gamma_2, \ldots, \gamma_k$ are present, but are suppressed in our notation. We will often refer to a distinguished marked point in the above sense.

**Definition 3.5.3.** Two morphisms of pre-stable families over $S$, indexed by $i = 1, 2$

$$
\begin{array}{ccc}
\mathcal{X}_i & \xrightarrow{f_i} & Z_i \\
\downarrow \gamma_{i,1}, \ldots, \gamma_{i,k} & & \downarrow \\
S & \rightarrow & 
\end{array}
$$

are isomorphic if there exists an isomorphism over $S$

$$
\rho : \mathcal{X}_1 \to \mathcal{X}_2
$$

and an arrow in $Z_{rel}(S)$

$$
\tau : Z_1 \to Z_2
$$

so that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{f_1} & Z_1 \\
\downarrow \rho & & \downarrow \tau \\
\mathcal{X}_2 & \xrightarrow{f_2} & Z_2
\end{array}
$$

and

$$
\rho \circ \gamma_{1,j} = \gamma_{2,j}
$$

for $j = 1, \ldots, k$

**Definition 3.5.4.** A morphisms of families over $S$ is said to be stable if for every point $s \in S$, the automorphisms of the family restricted to $s$ is finite.

As in absolute Gromov-Witten theory, a contracted curve (one mapped to a point) occurring as a component in a family is unstable unless it has genus greater than or equal to two, at least one marked point if genus is equal to 1, at least three marked points if genus is equal to 0. In the relative case, we have to consider the phenomenon of curves mapping to a degenerated target $iZ$ where we have a $\mathbb{C}^*$ dilating the fiber of each copy of $P$. It is shown in ([11], lemma 3.2) that a morphism $f : C \to iZ$ that does not involve any unstable contracted components is stable if and only if for every $i$, there is an irreducible component of $C$ mapped into $P_i$ by $f$ that is not a trivial component, that is, a rational curve lying in a fiber of $P \to X$, totally ramified at points mapping to $D_0$ and $D_\infty$.
Analogous definitions hold for morphisms to $A[n]$. For a map $f : C \to \mathcal{P}$, if $f$ does not involve any unstable contracted components, then $f$ is stable if and only if for every $i$, some irreducible component of $C$ that is not a trivial component is mapped into $P_i$.

3.6. The Moduli Stacks. We need to specify the appropriate data for the moduli stacks $\mathcal{M}(Z, \Gamma_Z)$, $\mathcal{M}(A, \Gamma_A)$. Let us begin with $\mathcal{M}(Z, \Gamma)$. Here we consider stable maps to $Z$ with specified tangency to $D$ together with marked points, called interior marked points whose image is not mapped to $D$ by $f$. We will assume that the points that are mapped to $D$ by $X$ are also marked. Those marked points will be called boundary marked points.

Definition 3.6.1. A relative graph $\Gamma$ is the following data:

1. A finite set of vertices $V(\Gamma)$
2. A genus assignment for each vertex $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$
3. A degree assignment for each vertex $d : V(\Gamma) \to B_1(Z) = A_1(Z)/\sim_{\text{alg}}$
   that assigns the class of a curve modulo algebraic equivalence to each vertex.
4. A finite set $R = \{1, \ldots, r\}$ labelling boundary marked points together with a function assigning boundary marked points to vertices $a_R : R \to V(\Gamma)$
5. A multiplicity assignment for each boundary marked point $\mu : R \to \mathbb{Z}_{\geq 1}$
6. A finite set $M = \{1, \ldots, m\}$ labelling interior marked points together with an assignment to vertices $a_M : M \to V(\Gamma)$

Definition 3.6.2. Given two relative graphs $\Gamma$, $\Gamma'$ are said to be isomorphic if there is a bijection $q : V(\Gamma) \to V(\Gamma')$

such that

1. $g(v) = g'(q(v))$
2. $d(v) = d'(q(v))$
3. $\mu_R(i) = q(a_R(i))$
4. $\mu'R = \mu R$
5. $a_M(i) = q(a_M(i))$.

Definition 3.6.3. Let $\Gamma$ be a relative graph. A morphism of families of type $\Gamma$ over $S$ consists of a stable morphism of a pre-stable family with marked points where $Z \in \text{Obj}(\mathcal{Z}_{\text{rel}}(S))$
such that for any closed point \( s \in S \), the fiber \( \mathcal{X}_s = \mathcal{X} \times_S s \) obeys

1. \( \mathcal{X}_s \) can be written as a disjoint union of pre-stable curves \((\mathcal{X}_s)_v \to S\)
2. \((\mathcal{X}_s)_v \) is a connected curve of arithmetic genus \( g(v) \).
3. The map \((c \circ f)_v : (\mathcal{X}_s)_v \to Z \to Z\)
   has \((c \circ f)_v((\mathcal{X}_s)_v)_s = d(v)\). Note that the map \( Z \to Z \) is induced from the map \( Z[n] \to Z \).
4. \( \gamma_i(s) \subset (\mathcal{X}_s)_v \) for \( v = a_M(i) \). These are the interior marked points.
5. \( \delta_i(s) \subset (\mathcal{X}_s)_v \) for \( v = a_R(i) \). These are the boundary marked points.
6. There is the following multiplicity condition

\[ f^* D[n] = \sum_{i \in R} \mu(i) \delta_i(s) \]

**Definition 3.6.4.** A morphism of families of type \( \Gamma \) is said to be a *nice family* if the target \( Z \) is an effective family, that is the the morphism be expressed as

\[ \mathcal{X} \to \mathcal{Z}[n] \]

**Definition 3.6.5.** The category \( \mathcal{M}(Z, \Gamma) \) is the groupoid over (Sch) so that for a scheme \( S \), the objects of \( \mathcal{M}(Z, \Gamma)(S) \) are morphisms of families of type \( \Gamma \). Given a morphism \( \sigma : S \to T \), families \( \xi_1 \in \mathcal{M}(Z, \Gamma)(S) \), \( \xi_2 \in \mathcal{M}(Z, \Gamma)(T) \), an arrow \( \xi_1 \to \xi_2 \) is an isomorphism over \( S \) of \( \xi_1 \) with \( \sigma^* \xi_2 \).

**Theorem 3.6.6.** [11] \( \mathcal{M}(Z, \Gamma) \) is a proper Deligne-Mumford stack.

Given a family

\[ \mathcal{X} \to \mathcal{Z}[n] \]

we have morphisms \( S \to Z \) given by

\[(c \circ f \circ \gamma_i) : S \to \mathcal{X} \to \mathcal{Z}[n] \to Z \]
\[(c \circ f \circ \delta_i) : S \to \mathcal{X} \to D[n] \to D \]

These maps extend to \( \mathcal{M}(Z, \Gamma) \) giving evaluation maps \( \text{ev}_i : \mathcal{M}(Z, \Gamma) \to Z \) at the interior and boundary marked points.
Definition 3.6.7. The evaluation map on $\mathcal{M}(Z, \Gamma)$ is

$$\text{Ev} : \mathcal{M}(Z, \Gamma) \to Z^n \times D^r$$

Definition 3.6.8. An étale nice family $S \to \mathcal{M}(Z, \Gamma Z)$ is said to be a nice chart.

Theorem 3.6.9. ([11], Theorem 3.10) $\mathcal{M}(Z, \Gamma Z)$ has an atlas that is a union of nice charts.

The case for $cm(A, \Gamma)$ is analogous with only a few modifications.

Definition 3.6.10. A rubber graph $\Gamma$ is the following data:

1. A finite collection of vertices $V(\Gamma)$
2. A genus assignment for each vertex $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
3. A degree assignment for each vertex $d : V(\Gamma) \to B_1(X) = A_1(X)/_{\text{alg}}$
   that assigns the class of a curve modulo algebraic equivalence to each vertex.
4. A finite sets $R_0 = \{1, \ldots, r_0\}, R_\infty = \{1, \ldots, r_\infty\}$ labelling boundary marked points together with a function assigning boundary marked points to vertices $a_0 : R_0 \to V(\Gamma)$

5. A multiplicity assignment for boundary marked points $\mu^0 : R_0 \to \mathbb{Z}_{\geq 1}$
6. $\mu^\infty : R_\infty \to \mathbb{Z}_{\geq 1}$

7. A finite set $M = \{1, \ldots, m\}$ labelling interior marked points together with an assignment to vertices $a_M : M \to V(\Gamma)$

Definitions of morphisms of rubber families are analogous with $Z$’s replaced with $A$’s and the following modifications. The degree assignment is

$$d(v) \in B_1(X)$$

Instead of sections $\delta_1, \ldots, \delta_r$, we have sections $\delta^0_1, \ldots, \delta^0_{r_0} : S \to X$

$\delta^\infty_1, \ldots, \delta^\infty_{r_\infty} : S \to X$

so that

$$f^* D^0_0[n] = \sum_{i \in R_0} \mu^0(i) \delta^0_i(s)$$

$$f^* D^\infty_\infty[n] = \sum_{i \in R_\infty} \mu^\infty(i) \delta^\infty_i(s)$$
Theorem 3.6.11. For a rubber graph $\Gamma$, $\mathcal{M}(A, \Gamma)$ is a proper Deligne-Mumford stack.

We have analogous evaluation maps $\text{ev}_i$ at the interior and boundary marked points (mapping to $D_0$ and $D_{\infty}$).

Definition 3.6.12. The evaluation map on $\mathcal{M}(A, \Gamma)$ is

$$\text{Ev} : \mathcal{M}(A, \Gamma) \to X^n \times X^{r_0} \times X^{r_{\infty}}$$

Now, we introduce some notation which will be essential in the sequel.

Definition 3.6.13. A map $f : C \to kZ$ in $\mathcal{M}(Z, \Gamma)$ is said to be split if $k \geq 1$. The irreducible components of $C$ that are mapped to $P_i \subset kZ$ are said to be extended components.

An analogous situation occurs for $\mathcal{M}(A, \Gamma)$.

Definition 3.6.14. A split map in $\mathcal{M}(A, \Gamma)$ is a map $f : C \to kP$ where $k \geq 1$, that is, the target is not smooth.

Definition 3.6.15. For a map $f : C \to kP$ in $\mathcal{M}(A, \Gamma)$, the irreducible components of $C$ that are mapped to $P_k$ are said to be the top components while the components of $C$ that are mapped to $P_0$ are said to be the bottom components.

3.7. Gluing Moduli Stacks. Consider a projective manifold $Z$, together with a smooth divisor $D$. We will consider the relative moduli stack $\mathcal{M}(Z, \Gamma_Z)$ corresponding to $(Z, D)$. Let us look at the rubber moduli space corresponding to $(X = D, L = N_{D/Z})$ where $N_{D/Z}$ is the normal bundle to $D$ in $Z$. $\mathcal{M}(A, \Gamma_A)$ corresponds to certain maps that are added to maps in $\mathcal{M}(Z, \Gamma_Z)$ to compactify.

There is a specific way to join $\mathcal{M}(Z, \Gamma_Z)$ to $\mathcal{M}(A, \Gamma_A)$ if some conditions are met. Likewise we can join some $\mathcal{M}(A, \Gamma_b)$ to $\mathcal{M}(A, \Gamma_t)$ if similar conditions are met. We make these conditions precise below.

Definition 3.7.1. Let $\Gamma_Z$ be a relative graph and $\Gamma_A$ be a rubber graph. Suppose that $L : RZ \to RA_{\infty}$ is a bijection from the labelling sets for boundary marked points in $\Gamma_Z$ to the labelling sets for boundary marked points mapping to $D_{\infty}$ in $\Gamma_A$ so that

$$\mu_Z(q) = \mu_A^\infty(L(q)).$$

Let

$$J : M_Z \sqcup M_A \to \{1, \ldots, |M_Z| + |M_A|\}$$

be a bijection between the labelling sets of the interior marked points and a set of $|M_Z| + |M_A|$ elements. We call the data $(\Gamma_A, \Gamma_Z, L, J)$ a graph-join quadruple.

Colloquially, we’ve matched boundary marked points on $\Gamma_Z$ and $\Gamma_A$ with the same multiplicity.

Definition 3.7.2. Define the graph join $\Gamma_A *_{L, J} \Gamma_Z$ to be the following relative graph. Let the graph $\Delta$ be obtained by taking as vertices the vertices of $\Gamma_Z$ and $\Gamma_A$ and for every $q \in RZ$, place an edge between the vertices corresponding to $q$
and $L(q)$. Let $\Gamma_A \ast_{L,J} \Gamma_Z$ be given as follows. The vertices of $\Gamma = \Gamma_A \ast_{L,J} \Gamma_Z$ are the connected components of $\Delta$. Let $b_Z : V(\Gamma_Z) \to V(\Gamma)$, and $b_A : V(\Gamma_A) \to V(\Gamma)$ be the functions taking vertices of $\Gamma_A$ and $\Gamma_A$ to the components in $\Delta$ containing them. For $v \in V(\Gamma)$, let $\Delta_v$ be the connected component of $\Delta$ corresponding to $v$. Define the data for $\Gamma_A \ast_{L,J} \Gamma_Z$ as follows:

1. $g(v) = (\sum w \in b_Z^{-1}(v) g(w)) + (\sum w \in b_A^{-1}(v) g(w)) + \dim(h^1(\Delta_v))$
2. $d(v) = (\sum w \in b_Z^{-1}(v) d(w)) + (\sum w \in b_A^{-1}(v) \sigma, \tau) = (\sum w \in b_Z^{-1}(v) d(w))$ where $i : X \to Z$ is the inclusion and $i_* : B_1(X) \to B_1(Z)$ is the induced map.
3. $R = RA_0$ with $a_R : R \to V(\Gamma)$ given by $a_R = b_A \circ a_0$
4. $\mu : R \to Z_{\geq 1}$ given by $\mu_R = \mu^0$
5. $M = \{1, \ldots, |M_Z| + |M_A|\}$ with assignment function $a : M \to V(\Gamma)$ given for $k \in J(M_Z)$ by
   
   \[ a(k) = b_Z \circ a_{M_Z} \circ J^{-1} \]
   
   while for $k \in J(M_A)$ by
   
   \[ a(k) = b_A \circ a_{M_A} \circ J^{-1} \]

Given $(\Gamma_Z, \Gamma_A, L, J)$ as above, consider the evaluation map at the boundary marked points on $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$ followed by a map $L_* : D^r \to D^r$ which reorders the products of $D^r$ according to $L$:

\[ L_* \circ Ev_R : \mathcal{M}(\mathcal{Z}, \Gamma_Z) \to D^r \to D^r \]

and the evaluation map at the boundary marked points mapping to $D_\infty \cong X$ on $\mathcal{M}(A, \Gamma_A)$,

\[ Ev_{R_\infty} : \mathcal{M}(A, \Gamma_A) \to D^r \]

**Theorem 3.7.3.** [11] There is a morphism

\[ \Phi_{\Gamma_Z, \Gamma_A, L, J} : \mathcal{M}(A, \Gamma_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_Z) \to \mathcal{M}(\mathcal{Z}, \Gamma_A \ast_{L,J} \Gamma_Z). \]

**Proof.** We look at the morphisms in charts and glue curves together to form nodes. \(\Box\)

**Definition 3.7.4.** Define the stack $\mathcal{M}(A \cup \mathcal{Z}, \Gamma_A \cup_{L,J} \Gamma_Z)$ as the image stack of $\Phi$ in $\mathcal{M}(\mathcal{Z}, \Gamma_A \ast_{L,J} \Gamma_Z)$.

Consider $RZ$ as part of the data of $\Gamma_Z$. An automorphism of $RZ$ is a permutation

\[ \sigma : RZ \to RZ \]

so that $\mu_Z(\sigma(i)) = \mu_Z(i)$ and $a_{RZ}(\sigma(i)) = a_{RZ}(i)$. The group of all such automorphisms is denoted by $\text{Aut}_{\Gamma_A}(RZ)$. Likewise, we may define $\text{Aut}_{\Gamma_A}(RA_0)$ and $\text{Aut}_{\Gamma_A}(RA_\infty)$. Given $L : RZ \to RA_\infty$, we may define $\text{Aut}_{\Gamma_A, RZ, L}(RZ, RA_\infty)$ as the subgroup of $\text{Aut}_{\Gamma_A}(RZ) \times \text{Aut}_{\Gamma_A}(RA_\infty)$ such that for $(\sigma, \tau) \in \text{Aut}_{\Gamma_A}(RZ) \times \text{Aut}_{\Gamma_A}(RA_\infty)$ we have $L(\sigma(i)) = \tau(L(i))$ for $1 \leq i \leq |RZ|$. 


Lemma 3.7.5. ([11], Proposition 4.13) $\Phi$ is finite and étale onto its image. It has degree equal to

$$|\text{Aut}_{\Gamma_A, \Gamma_Z, L}(RZ, RA_\infty)|$$

at every integral substack of $\mathcal{M}(A \sqcup Z, \Gamma_A \sqcup L, J \Gamma_Z)$.

Definition 3.7.6. Two quadruples are said to be join-equivalent if they give the same image under $\Phi$. By straightforward combinatorics, we get that there are

$$|M_Z||M_A|!\frac{(|RZ|)^2}{|\text{Aut}_{\Gamma_A, \Gamma_Z, L}(RZ, RA_\infty)|}$$

graph-join quadruples in $(\Gamma_Z, \Gamma_A, L, J)$’s equivalence class.

Corollary 3.7.7. Consider a join equivalence class of quadruples $[\Upsilon] = [((\Gamma_Z, \Gamma_A, L, J))]$. Let

$$N = \mathcal{M}(A \sqcup MZ, \Upsilon) \times \prod_{(\Gamma'_Z, \Gamma'_A, L', J')} \mathcal{M}(A, \Gamma'_A \times D_r \mathcal{M}(Z, \Gamma'_Z))$$

where the disjoint union is over quadruples join-equivalent to $\Upsilon$. Then $\Phi_{[\Upsilon]} : M \to N$ is an étale map of degree

$$|M_Z||M_A|!|RZ|!^2$$

Likewise, we may define graph-join for rubber graphs, $\Gamma_t, \Gamma_b$ (where $t$ and $b$ stand for top and bottom). Let $L : R_{b0} \to R_{t\infty}$ be a bijective function satisfying

$$\mu_0^b(q) = \mu_t^\infty(L(q)).$$

Let

$$J : M_b \sqcup M_t \to \{1, \ldots, |M_b| + |M_t|\}$$

be a bijective map. Then we define the graph join, a rubber graph $\Gamma = \Gamma_t *_{L, J} \Gamma_b$ as above, except that instead of condition (3) above, we have

$$R_0 = R_{t0}, \quad a_0 = b_{A_t} \circ a_{0t}$$

$$\mu_0 = \mu_t^0$$

$$R_\infty = R_{b\infty}, \quad a_\infty = b_{A_b} \circ a_{\infty b}$$

$$\mu_\infty = \mu_b^\infty.$$ 

Now, let $r = |R_{b0}| = |R_{t\infty}|$. Exactly as above, we have

Theorem 3.7.8. There is a morphism

$$\Phi : \mathcal{M}(A, \Gamma_{A_t}) \times D_r \mathcal{M}(A, \Gamma_{A_b}) \to \mathcal{M}(Z, \Gamma_{A_b} *_{L, J} \Gamma_{A_t}).$$

that is of finite, étale of degree $|\text{Aut}_{\Gamma_{A_t}, \Gamma_{A_b}, L}(RA_{b0}, RA_{t\infty})|$ onto its image.
Corollary 3.7.9. Consider a moduli stack \(N = \mathcal{M}(A, \Gamma_{A_0} \cup_{L,J} \Gamma_{A_1})\). Let
\[
M = \coprod_{\left(\Gamma'_A, \Gamma'_t \mid L, J, L', J'\right)} \mathcal{M}(A, \Gamma'_{A_0}) \times_{D^r} \mathcal{A}(A, \Gamma'_{A_1})
\]
where the disjoint union is of quadruples which give \(\Phi\) with image \(N\). The \(M \to N\) is an étale map of degree
\[
|M_{A_0}|! |M_{A_1}|!/ (|R_{A_0,0}|)!
\]

3.8. Virtual Cycles. In [12], Li constructs a virtual cycle on \(\mathcal{M}(Z, \Gamma)\). This is accomplished by first constructing a perfect obstruction theory. This obstruction theory is defined over charts as a particular two-term complex. The complex over charts is constructed by considering charts of \(\mathcal{M}(Z[n], \Gamma)^{st}\), a moduli stack of stable maps of type \(\Gamma\) satisfying the pre-deformability and the stability conditions but not quotiented by the \(G[n] = (\mathbb{C}^*)^{n}\)-action. Li then uses a theorem ([12], Theorem 2.2) to quotient the complex by the action induced by the Lie algebra of \(G[n]\). This complex gives a perfect obstruction theory on a chart on \(\mathcal{M}(Z, \Gamma)\). This yields a perfect obstruction theory on \(\mathcal{M}(Z, \Gamma)\).

It is completely straightforward to apply this construction to \(\mathcal{M}(A, \Gamma)\). Instead of using \(\mathcal{M}(Z[n], \Gamma)^{st}\), one uses \(\mathcal{M}(A[n], \Gamma)^{st}\). One modifies the complex to enforce particular multiplicities to \(D_0\) and \(D_\infty\) instead of \(D\). Then one applies Theorem 2.2 of [12] to quotient the complex. One uses \(\tilde{G}[n]\) instead of \(G[n]\), but other than that, the proof is unchanged. One then obtains a perfect obstruction theory on \(\mathcal{M}(A, \Gamma)\). The virtual cycle follows from Li’s very general construction.

4. Line-Bundle Systems

We introduce the notion of a line-bundle system. This is a sequence of line-bundles \(L[n]\) on \(\mathbb{A}^n, Z[n]\), or \(A[n]\) with certain transition properties. They will induce line-bundles on the stack of relative stable and rubber morphisms. In more esoteric language, line-bundle systems are line-bundles on the stacks of degenerations \(Z^{rel}, A^{rel}\) and their universal targets. We’ve chosen to use more primitive notions for the sake of readability.

4.1. Definition of line-bundle Systems.

Definition 4.1.1. A base of a line-bundle system is a triple \((B[n], H[n], \{i\})\) where

(1) \(B[n]\) is a sequence of schemes.
(2) \(H[n]\) is a group acting on \(B[n]\).
(3) \(\{i\}\), a set of morphisms of swags \(i : B[n-1] \to B[n]\) for varying \(n\), called standard inclusions.

The bases that we shall are the following:

(1) \(\mathbb{A} = (\mathbb{A}^n, G[n], \{i\})\) where \(\{i\}\) are the standard inclusions.
(2) $\mathcal{Z} = (Z[n], G[n], \{i\})$.
(3) $\mathcal{B} = (\mathbb{A}^n, G[n], \{i\})$.
(4) $\mathcal{A} = (A[n], G[n], \{i\})$.

**Definition 4.1.2.** A line-bundle system on $(B[n], G[n], \{i\})$ is a sequence of $G[n]$-equivariant line-bundles $L[n]$ on $B[n]$ for each non-negative integer $n$ together with a line-bundle isomorphism $i^* : L[n-1] \cong i^* L[n]$ for every standard inclusion $i$ such that

1. $L[n]$ is an equivariant line-bundle on $B[n]$ under group action $H[n]$.
2. The group action on $L[n]$ commutes with standard inclusions. That is, given a standard inclusion
   
   $$ i : B[n-1] \rightarrow B[n] $$
   
   $$ i_* : H[n-1] \rightarrow H[n] $$
   
   then $i^* L[n]$ is isomorphic to $L[n-1]$ as $H[n-1]$-equivariant bundles.

**4.2. Induced Line-Bundle Systems.**

**Proposition 4.2.1.**

1. A line-bundle system $L[n]$ on $\mathbb{A}^n$ naturally induces a line-bundle system on $\mathcal{Z}$.
2. A line-bundle system $L[n]$ on $\mathcal{B}$ naturally induces a line-bundle system on $\mathcal{A}$.

**Proof.** This follows from the fact that the morphisms

$$ p : Z[n] \rightarrow \mathbb{A}^n $$

$$ p : A[n] \rightarrow \mathbb{A}^n. $$

are equivariant and commute with standard inclusions. \qed

There is a natural notion of a map between line-bundle systems. Let $B[n]$ be the base of a line-bundle system with group $H[n]$.

**Definition 4.2.2.** A map between two line-bundle systems $L[n]$ and $M[n]$ over a base $B[n]$ is a sequence of line-bundle isomorphisms $f : L[n] \rightarrow M[n]$ such that

1. $f$ is an equivariant map under $H[n]$.
2. The map $f$ commutes with standard inclusions.

There is also a natural, obvious notion of tensor product of line-bundle systems.

We can also define sections of line-bundle systems.

**Definition 4.2.3.** A sequence of sections $s[n] : B[n] \rightarrow L[n]$ is a section of the line-bundle system if

1. $s[n]$ is equivariant under $H[n]$.
2. $s[n]$ commutes with standard inclusions.
4.3. **Reference Line-Bundle Systems.** We have several a line-bundle system that will be very important in the sequel, the reference line-bundle systems.

**Proposition 4.3.1.** There is a line-bundle system, $V[n]$ on $Z$ together which a section $s[n]$ that when restricted to $Z[n]$ has $D[n]$ as its zero divisor.

**Proof.** $(V[n], s[n])$ is defined by Lemma 3.1.5. It is straightforward to verify that $V[n]$ is a line-bundle system and $s[n]$ is a section. □

There is an analogous result for $A[n]$.

**Proposition 4.3.2.** There are line-bundle systems, $V_{\infty}[n], V_0[n]$ together with sections $s_{\infty}[n], s_0[n]$ on $A$ whose corresponding divisor on $A[n]$ are $D_{\infty}[n], D_0[n]$.

**Proof.** Let us first construct $V_{\infty}[0], s_{\infty}[0]$.

Let $V_{\infty}[0] = \mathcal{O}(1)$ with linearization dual to $\mathcal{O}(-1)$ with

$$\sigma_0 \cdot (l, t) \mapsto (\sigma_0 l, t).$$

$L_{\infty}[0]$ has a canonical section $s_{\infty}[0]$ which is dual to a section $s_{\infty}[0]^\vee$ given by

$$s_{\infty}[0]^\vee : [l : t] \mapsto \left(\frac{l}{t}, 1\right).$$

$s_{\infty}[0]$ is an equivariant section with zero scheme $D_{\infty}$. Define $V_{\infty}[n], s_{\infty}[n]$ by

$$V_{\infty}[n] = b^* V_{\infty}[0], \quad s_{\infty}[n] = b^* s_{\infty}[0].$$

Likewise $L_0[0]$ to be the line-bundle on $A[0]$ dual to $\mathcal{O}(-1) \otimes L^\vee$ with the linearization

$$\sigma_0 \cdot ((l, t) \otimes a) = (\sigma_0 l, t) \otimes \sigma_0^{-1} a$$

$s_0[0]$ is defined as the section dual to

$$s_0[0]^\vee : [l : t] \mapsto (l, t) \otimes a_l$$

where $a_l \in L^\vee$ is defined by $a_0(l) = 1$. $s_0[0]$ is an equivariant section with zero scheme $D_0$. Define $V_0[n], s_0[n]$ by

$$V_0[n] = t^* V_0[0], \quad s_0[n] = t^* s_0[0].$$

□
5. Line-Bundles on the Moduli Stacks

In this section, we will define bundles on $\mathcal{M}(\mathcal{Z}, \Gamma_{\mathcal{Z}})$ and $\mathcal{M}(\mathcal{A}, \Gamma_{\mathcal{A}})$. We begin with the bundles on $\mathcal{M}(\mathcal{Z}, \Gamma_{\mathcal{Z}})$. They are

(1) Dil, a line-bundle that has a section whose zero scheme is supported on all split curves.

(2) $L_{i, \text{ext}}$ where $i$ is an interior marked point (3.6.3), a line-bundle that has a section whose zero scheme is supported on split curves where $i$ is mapped to an extended component (3.6.13) of the target.

5.1. Line-Bundles on Stacks. Let us recall some definitions from [13].

**Definition 5.1.1.** Let $F$ be an algebraic stack. A line-bundle $\mathcal{L}$ on $F$ can be given by the following data

1. A particular atlas $U$ together with a line-bundle $\mathcal{L}_U$ on $U$.
2. For the fiber product

$$
\begin{array}{c}
\text{U} \\
\downarrow p_1 \\
\text{U} \\
\downarrow p_2 \\
\text{U} \\
\end{array}
\quad \quad
\begin{array}{c}
\text{U} \\
\downarrow p_1 \times_F U \\
\downarrow p_2 \times_F U \\
\text{U} \\
\end{array}
$$

we have an isomorphism $\alpha : p_1^* \mathcal{L}_U \rightarrow p_2^* \mathcal{L}_U$ that satisfies the cocycle condition. That is, on $U \times_F U \times_F U$ with $p_{ij} : U \times_F U \times_F U \rightarrow U \times_F U$ ($i, j \in \{1, 2, 3\}$, $i \neq j$) being projection onto pairs of factors, we have

$$
p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha : p_1^* \mathcal{L}_U \rightarrow p_3^* \mathcal{L}_U
$$

We will use atlases which have nice properties over which it will be simple to define line-bundles.

**Definition 5.1.2.** A property (P) of a morphism from a scheme to an algebraic stack $F$ is said to be preserved under pullback if given any morphism $S \rightarrow F$, from a scheme to $F$ with property $P$ and any morphism of schemes $T \rightarrow S$ then the composition

$$
T \rightarrow S \rightarrow F
$$

has property (P)

An example of a property that is preserved under pullback is for a morphism $S \rightarrow \mathcal{M}(\mathcal{Z}, \Gamma)$ to be written as the disjoint union $S = \bigsqcup S_\alpha$ so that each $S_\alpha \rightarrow \mathcal{M}(\mathcal{Z}, \Gamma)$ is a nice family. Another example which we will meet later (Definition 6.1.1) is that of a family being the disjoint union of $i$-admissible families.

Let us suppose that we have an atlas with a property (P) that is preserved under pullback. Now, instead of specifying the line-bundle on a particular atlas, we can specify it, a fortiori, over all morphisms $S \rightarrow F$ with property (P) provided that it satisfies certain pull-back and transition properties.
5.2. **line-bundles on** \( \mathcal{M}(\mathcal{Z}, \Gamma) \) **from Line-Bundle Systems.** Now, we restrict to the case where the stack in question is \( \mathcal{M}(\mathcal{Z}, \Gamma) \). The results are equally true for \( \mathcal{M}(\mathcal{A}, \Gamma_A) \) if we replace \( \mathcal{Z} \)'s with \( \mathcal{A} \)'s and \( G[n] \)'s with \( \tilde{G}[n] \)'s. Property (P) will be that a morphism \( S \to \mathcal{M}(\mathcal{Z}, \Gamma) \) can be written as a disjoint union of nice families

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \gamma \\
Z[n_{\alpha}] \\
\downarrow \\
S_{\alpha} \\
\downarrow h_{\alpha} \\
\mathbb{A}^n
\end{array}
\]

**Theorem 5.2.1.** A line-bundle system on \( \mathcal{h} \) induces a line-bundle on \( \mathcal{M}(\mathcal{Z}, \Gamma) \)

**Proof.** As above, we set property (P) to be that for \( S \to \mathcal{M}(\mathcal{Z}, \Gamma) \), \( S \) can be written as the disjoint union of \( S_{\alpha} \) where each \( S_{\alpha} \) is a nice family.

Given a nice family on \( \mathcal{M}(\mathcal{Z}, \Gamma) \),

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \gamma \\
Z[n] \\
\downarrow \\
S \\
\downarrow h \\
\mathbb{A}^n
\end{array}
\]

define a line-bundle \( L \) on \( S \) as \( h^*L[n] \). This is well-behaved under pull-backs.

Now, we have to consider the transitions. Since the definition only depends on \( h \), not on \( f \), we need only consider isomorphisms \( S \times_{\mathbb{A}^n_1} Z[n_1] \to S \times_{\mathbb{A}^n_2} Z[n_2] \) which can be written locally as effective arrows. We can factor these as transitions under standard inclusions and the action of \( \rho : S \to G[n] \). But this transition data is exactly specified by the definition of a line-bundle system. It is standard to verify that the cocycle condition is satisfied. \( \square \)

Given an interior marked point, we can study the line-bundle on \( \mathcal{M}(\mathcal{Z}, \Gamma) \) induced by a line-bundle system on \( \mathcal{Z} \).

**Theorem 5.2.2.** A line-bundle system on \( \mathcal{Z} \) together with the choice of an interior marked point induces a line-bundle system on \( \mathcal{M}(\mathcal{Z}, \Gamma) \).

**Proof.** Consider the nice chart

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \gamma \\
Z[n] \\
\downarrow \\
S \\
\downarrow h \\
\mathbb{A}^n
\end{array}
\]

where \( \gamma \) is the section corresponding to the interior marked point. We define a bundle \( L \) on \( S \) by

\[
L = (f \circ \gamma)^*L[n].
\]

\( \square \)
We have the following simple facts

**Lemma 5.2.3.** Given line-bundle systems on $\mathbb{A}$ or $\mathbb{Z}$, $L[n], M[n], L[n] \otimes M[n]$, then if $L, M, N$ are the line-bundles induced by $\mathcal{M}(Z, \Gamma)$, respectively then

$$N = L \otimes M$$

**Lemma 5.2.4.** A section of a line-bundle system on $\mathbb{A}$ or on $\mathbb{Z}$ gives a section of the induced line-bundle on $\mathcal{M}(Z, \Gamma)$.

5.3. The Dilation Bundle. In this section, we define the dilation line-bundle, Dil on $\mathcal{M}(Z, \Gamma)$ induced from a line-bundle system. The name "dilation bundle" comes from the dilation of the fibers of $P = P_D(N \oplus 1)$ where $N$ is the normal bundle to $D$ in $Z$ and is not related to the dilation equation in Gromov-Witten theory with descendants.

Dil is induced from a line-bundle system on $\mathbb{A}$. We define the line-bundle system $\text{DIL}[n]$ as follows. Let $\text{DIL}[n]$ be the trivial line-bundle, $1_{\mathbb{A}^n}$ on $\mathbb{A}^n$ with $G[n]$ action given by

$$(\sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_1 \ldots \sigma_n)^{-1}s$$

**Proposition 5.3.1.** The line-bundle system $\text{DIL}[n]$ has a section.

**Proof.** Consider the section of $\text{DIL}[n]$ over $\mathbb{A}^n$ defined by

$$s(x_1, \ldots, x_n) = x_1 \ldots x_n.$$ 

Dil has a simple geometric interpretation which will be proved later: $c_1(\text{Dil})$ on $\mathcal{M}(Z, \Gamma)$ is (counted with multiplicity) the locus of split maps. It is easy to see that is a reasonable fact. Dil has a section whose zero scheme are split maps: this section is induced from a section of the line-bundle system $\text{DIL}[n]$ on $\mathbb{A}^n$; the zero-scheme of this section is the the union of hyper-planes in $\mathbb{A}^n$ consisting of points with at least one coordinate equal to 0; the fiber of $\mathbb{Z}[n] \to \mathbb{A}^n$ over any closed point in this scheme consists of a non-smooth target.

5.4. Definition of $L_{i,\text{ext}}$. Consider $\mathcal{M}(Z, \Gamma)$ with at one distinguished interior marked point, which we will call $i$. We define a line-bundle $L_{i,\text{ext}}$ which will have a section whose zero stack is supported on split maps where the marked point $i$ is mapped to an extended component of the target. This line-bundle is induced from a line-bundle system on $\mathcal{Z}$.

We define $LE[n]$ on $\mathcal{Z}[n]$. Recall that $\mathcal{Z}[0] = Z$. $D \subset Z$ is a Cartier divisor and is the zero-section of a section $s$ of a line-bundle $V$ on $Z$. Define $LE[0] = V$, $LE[n] = c^*LE[0]$ where $c$ is the collapsing map

$$c : \mathcal{Z}[n] \to Z.$$ 

**Lemma 5.4.1.** The line-bundle system $LE[n]$ has a section.
Proof. Define the section by
\[ se[n] = c^* s. \]
\[ \square \]

Note that if we take a closed point \( p \in A^n \) then \( s^* c \) is zero on the extended components of \( Z[n] \times \mathbb{A}^n p \).

**Theorem 5.4.2.** \( L_{i,ext} \) is canonically isomorphic to \( ev_i^* L \).

**Proof.** On \( S \to \mathcal{M}(Z, \Gamma) \), a family of \( \mathcal{M}(Z, \Gamma) \), we have the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & Z[n] \\
\downarrow \gamma & & \downarrow c \\
S & \xrightarrow{\pi} & \mathbb{A}^n
\end{array}
\]

But, \( ev_i^* L = (c \circ f \circ \gamma)^* L \) which is canonically the line-bundle induced on \( S \) by the line-bundle system \( LE[n] \).
\[ \square \]

5.5. **Line-Bundles on \( \mathcal{M}(A, \Gamma) \) from Line-Bundle Systems.** In this subsection, we define the following bundles on \( \mathcal{M}(A, \Gamma) \) which are induced from line-bundle systems:

1. \( L^0 \), the target cotangent line-bundle at \( X_0 \). \( c_1(L^0) = \Psi_0 \) is the target \( \Psi \) class of \([1]\).
2. \( L^\infty \), the target cotangent line-bundle at \( X_\infty \) which is \( L^0 \)'s upside-down analog. \( c_1(L^\infty) = \Psi_\infty \).
3. Split, the Split bundle which has a section whose zero stack is supported on all split maps (Definition 3.6.14).
4. \( L_{i,not\ top} \), the not-top bundle with respect to a distinguished interior marked point \( i \). This bundle has a section whose zero stack is supported on split maps where the \( i \)th marked point is not on a top component.
5. \( L_{i,not\ bot} \), the not-bottom bundle with respect to a interior marked point \( i \). This bundle has a section whose zero stack is supported on split maps where the \( i \)th marked point is not on a bottom component.

We prove the following relations among bundles which will be used in the sequel.

\[
L^0 \otimes L^\infty = \text{Split} \\
L^0 \otimes ev_i^* L^\vee = L_{i,not\ top} \\
L^\infty \otimes ev_i^* L = L_{i,not\ bot}
\]

5.6. **Line-bundles on \( \mathcal{M}(A, \Gamma) \) from line-bundle systems.** Line-bundles can be defined on \( \mathcal{M}(A, \Gamma) \) by defining them over nice families, or more generally on charts with a property (P) preserved under pullback (5.1.2) so that there is an atlas with property (P).
5.7. **Definition of $L^0$ and $L^\infty$.** $L^0$ is induced from a line-bundle system on $B$.

We define the line-bundle system $T[n]$ as follows. Let $T[n]$ be the trivial line-bundle, $1_{\mathbb{A}^n}$ on $\mathbb{A}^n$ with $\tilde{G}[n]$ action given by

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_0 \sigma_1 \ldots \sigma_n)^{-1} s$$

Therefore the maps in the following diagram

$$
\begin{array}{ccc}
T[n] & \xrightarrow{\sigma} & T[n] \\
\downarrow & & \downarrow \\
\mathbb{A}^n & \xrightarrow{\sigma} & \mathbb{A}^n
\end{array}
$$

are given as

$$s \mapsto (\sigma_0 \sigma_1 \ldots \sigma_n)^{-1} s$$

$$(x_1, \ldots, x_n) \mapsto (\sigma_1^{-1} x_1, \ldots, \sigma_n^{-1} x_n)$$

The transition map under an effective inclusion $i$ is trivial. We call the induced line-bundle $L^0$.

The definition of $L^\infty$ is similar to that of $L^0$. It is induced by a line-bundle system $B[n]$. $B[n]$ is the trivial line-bundle $1_{\mathbb{A}^n}$ on $\mathbb{A}^n$ under the $\tilde{G}[n]$-action

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = \sigma_0 s$$

and trivial transition map under effective inclusion.

$L^0$ and $L^\infty$ can be given an interpretation in the stack of rational sausages, the substack of $\mathfrak{M}_{0,2}$ consisting of pre-stable curves so that the two marked points lie on different sides of every node. $L^0$ and $L^\infty$ are equal to the pullbacks of the cotangent line classes at the two marked points. See [5] for an elaboration.

5.8. **The split bundle.** We define Split to be the bundle on $M(A, \Gamma)$ induced from the line-bundle system on $B$ given by the trivial bundle $S[n] = 1_{\mathbb{A}^n}$ on $\mathbb{A}^n$ with the trivial transition map under effective inclusion and the following $\tilde{G}[n]$-action:

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_1 \ldots \sigma_n)^{-1} s$$

**Proposition 5.8.1.** The line-bundle system $S[n]$ has a section.

*Proof.* The proof is identical to the one for $DIL[n]$, (Proposition 5.3.1). \qed

**Proposition 5.8.2.** $L^0 \otimes L^\infty = \text{Split}$

*Proof.* Because $T[n] \otimes B[n] = S[n]$ as line-bundle systems, the induced bundles are equal. \qed
Split has a simple geometric interpretation analogous to that of Dil. On $\mathbb{A}^n$ under $G[n]$, Split has a section given by $x_1 x_2 \ldots x_n$. The zero-scheme of this section is the union of hyper-planes in $\mathbb{A}^n$ consisting of points with at least one coordinate equal to 0. We know that the fiber of $A[n] \to \mathbb{A}^n$ over any closed point in this scheme consists of a chain of more than one P's. These sections glue together to form a section of Split on $\mathcal{M}(\mathbb{A}, \Gamma)$. It follows that $c_1(\text{Split})$ on $\mathcal{M}(\mathbb{A}, \Gamma)$ is (counted with multiplicity) the locus of split maps.

If we consider the stack of rational sausages where $L^0$ and $L^\infty$ are the restriction of $\psi$ classes on $\mathcal{M}_0^{g,2}$, this relation is the pullback of the genus 0 recursion relation of Lee and Pandharipande [9].

5.9. Definition of $L_i, \text{not top}$. Consider $\mathcal{M}(\mathbb{A}, \Gamma)$ a moduli stack of rubber maps with a distinguished interior marked point, $i$. We will define two bundles $L_i, \text{not top}$ and $L_i, \text{not bot}$ on $\mathcal{M}(\mathbb{A}, \Gamma)$ with very clear geometric meanings. $c_1(L_i, \text{not top})$ will be supported on the substack of relative stable maps consisting of split maps where the $i$th interior marked point does not lie on the top component of the target while $c_1(L_i, \text{not bot})$ will be the substack where the $i$th interior marked point does not lie on the bottom component. Many geometric results follow from equations relating $L_i, \text{not top}$, $L_i, \text{not bot}$ to other line-bundles.

$L_i, \text{not top}$ and $L_i, \text{not bot}$ are induced from line-bundle systems on $\mathbb{A}, NT[n]$ and $NB[n]$, respectively.

Let $NT[0]$ be the equivariant bundle on $A[0] = \mathbb{P}(L \oplus 1)$

which is dual to $\mathcal{O}(-1)$ with the $G[0]$-action

$$\sigma_0 \cdot (l, t) = (\sigma_0 l, t).$$

Let $NT[n] = t^* NT[0]$ where $t : A[n] \to A[0]$ is the map given in Definition 3.3.5

Note also that $NT[0]$ has a section $s$ whose zero-scheme is $D_\infty$ so $NT[n]$ has a section $t^* s$. If we look at $P$, the fiber over a point $x \in \mathbb{A}^n$, we see that this section is zero on $P_0, \ldots, P_{l-1}$ and is non-zero on $P_l$ away from $D_\infty$.

For $i$, an interior marked point for $\mathcal{M}(\mathbb{A}, \Gamma)$, $L_i, \text{not top}$ is the line-bundle on $\mathcal{M}(\mathbb{A}, \Gamma)$ induced by $NT[n]$ and $i$.

$L_i, \text{not bot}$ is defined similarly. Let $NB[0]$ be the equivariant bundle on $A[0] = \mathbb{P}(L \oplus 1)$ dual to $\mathcal{O}(-1) \otimes L^\vee$ with the $G[0]$-action

$$\sigma_0 \cdot ((l, t) \otimes a) = (\sigma_0 l, t) \otimes \sigma_0^{-1} a.$$

$NB[n]$ is given by $b^* NB[0]$.

For $i$, an interior marked point for $\mathcal{M}(\mathbb{A}, \Gamma)$, $L_i, \text{not bot}$ is the line-bundle on $\mathcal{M}(\mathbb{A}, \Gamma)$ induced by $NB[n]$ and $i$. 
5.10. **The Reference Line-Bundles.** Earlier, we defined line-bundle systems \( V_0[n], V_\infty[n] \) on \( \mathcal{A} \).

**Proposition 5.10.1.** The line-bundles induced by \( V_0[n], V_\infty[n] \) on \( \mathcal{M}(\mathcal{A}, \Gamma) \) are trivial.

*Proof.* We produce a canonical nowhere-zero section of the induced line-bundle by \( V_0[n] \). The case of \( V_\infty[n] \) is analogous. Recall that the line-bundles are defined on each nice chart

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A[n] \\
\downarrow & & \downarrow \\
S & \xrightarrow{\gamma} & \mathbb{A}^n
\end{array}
\]

\( V_0 \) is defined on \( S \) to be

\[ V_0 = (f \circ \gamma)^* V_0[n]. \]

It has a section \((f \circ \gamma)^* s_0[n]\). Since \( s_0[n] \) is non-zero away from \( D_0[n] \) and interior marked points are not mapped to \( D_0[n] \), the pulled-back section is non-zero. It is easy to verify that this section transformers properly. \( \square \)

5.11. **The Evaluation Map and Line-bundle System.**

**Lemma 5.11.1.** Let \( i \) be an interior marked point on \( \mathcal{M}(\mathcal{A}, \Gamma) \), and let \( M \) be a line-bundle on \( X \). Then there is a line-bundle system \( M[n] \) on \( \mathcal{A} \) so that the line-bundle it induces on \( \mathcal{M}(\mathcal{A}, \Gamma) \) at \( i \) is \( \text{ev}_i^* M \)

*Proof.* Given a nice chart, we have the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A[n] \\
\downarrow & & \downarrow \\
S & \xrightarrow{\gamma} & \mathbb{A}^n
\end{array}
\]

Let \( M[n] = \pi^* M \). \( M[n] \) is easily seen to be a line-bundle system, and since \( \pi \circ f \circ \gamma = \text{ev}_i \), induced line-bundle is \( \text{ev}_i^* L \). \( \square \)

5.12. **The Marked Point Interpretation of the Top Bundle.** By relating the various bundles described in this section, we can find significant geometric facts. There is one geometric interpretation of the top bundle that will be supremely useful in the sequel. This interpretation was first advanced by Andreas Gathmann in conversation with the author.

Let \( \mathcal{M}(\mathcal{A}, \Gamma) \) be the moduli stack of rubber maps with a distinguished interior marked point, \( i \). Consider the evaluation map

\[ \text{ev}_i : \mathcal{M}(\mathcal{A}, \Gamma) \to X \]
at $i$.

**Theorem 5.12.1.** $L^0 \otimes e_i^* L^\vee = L_{i, not\ top}$

**Proof.** It suffices to prove that $L^0 \otimes e_i^* L \otimes L_{i, not\ top}$ is the trivial bundle. To do this, we will show that the line-bundle system on $A$ that induces $L^0 \otimes e_i^* L \otimes L_{i, not\ top}$ is line-bundle system-isomorphic to $V_0[n]$ which induces the trivial bundle on $M(A, \Gamma)$.

Consider the following diagram of swags where $X$ is equipped with the trivial group action $A[n] \xrightarrow{\pi} A[0] \xrightarrow{\pi} X$.

Consider also the map of swags $p : A[n] \to \mathbb{A}^n$.

Let us review the bundles involved. They are pullbacks of bundles from $A[0]$ by $t^*$.

1. $L^0 \otimes e_i^* L^\vee$ is induced by the bundle $\theta[n] = p^* T[n]^{\vee}$ on $A[n]$, where $T[n]^{\vee}$ is the trivial bundle on $\mathbb{A}^n$ with the $G[n]$-linearization $(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_0 \sigma_1 \ldots \sigma_n)s$.

   Note that $\theta[n] = t^* \theta[0]$ where $\theta[0]$ is the trivial bundle on $A[0]$ under the action $\sigma_0 \cdot s = \sigma_0 s$.

2. $e_i^* L$ is induced by the line-bundle system $\pi^* L$ on $A[n]$. But $\pi^* L = t^* \pi^* L$.
3. $L_{i, not\ top}$ is induced by $NT[n] = t^* NT[0]$.
4. A trivial bundle is induced by $L_0[n] = t^* L_0[0]$.

Therefore, to prove $\theta[n] \otimes e^* L \otimes NT[n] = L_0[n]$, as a line-bundle system on $A[n]$, it suffices to verify $\theta[0] \otimes e_0^* L \otimes NT[0] = L_0[0]$ on $A[0] = P$ which is straightforward. 

Likewise,

**Theorem 5.12.2.** $L^\infty \otimes e_i^* L = L_{i, not\ bot}$

**Proof.** The proof is exactly analogous to the above.
5.13. **Boundary $\Psi$ class interpretation of $L^0$.** There is an interpretation of $L^0$ in terms of $\psi$ classes at boundary marked points. Consider $\mathcal{M}(\mathcal{A}, \Gamma)$. Let $j$ be a boundary marked point evaluating to $X_0$ with multiplicity $m$. Let $L_j$ be the tangent space at the $j$th marked point. Then, we will show $L^0 = (L_j^\vee)^m \otimes \text{ev}_j^* L$.

This relation tells us that nothing new can be found in rubber theory with $\psi$ classes at the boundary marked point. Also, since this formula is independent of $i$, we can relate $\psi$ classes at different boundary marked points.

**Theorem 5.13.1.** $L^0 = (L_j^\vee)^m \otimes \text{ev}_j^* L$

**Proof.** We prove this by exhibiting a regular, nowhere vanishing section of this bundle.

We need to consider the line-bundle on $\mathcal{M}(\mathcal{A}, \Gamma)$ induced from a line-bundle system on $\mathcal{A}$ by a boundary marked point. This is identical to a line-bundle induced by an interior marked point except for the following modification. Given a line-bundle system $L[n]$, consider the nice chart on $\mathcal{M}(\mathcal{A}, \Gamma)$

$$
\begin{array}{ccc}
\chi & \longrightarrow & A[n] \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathbb{A}^n \\
\delta & \longleftarrow & \gamma
\end{array}
$$

where $\delta$ is the section corresponding to the boundary marked point evaluating to $D_0$. Then, set $L = (f \circ \delta)^* L[n]$. This is seen to give a line-bundle on $\mathcal{M}(\mathcal{A}, \Gamma)$. Let $V$ be the line-bundle induced by $V_0[n]$ at $j$.

**Lemma 5.13.2.** $V = L^0 \otimes \text{ev}_j^* L$

**Proof.** We show that $V_0[n] \otimes p^* T[n]^\vee \otimes \pi^* L$ has an equivariant section that does not vanish near the image of $(f \circ \gamma)$.

Let us recall the morphism of swags

$$
t : A[n] \rightarrow A[0] \\
p : A[n] \rightarrow \mathbb{A}^n \\
\pi : A[n] \rightarrow X
$$

Note that $L^0 \otimes p^* T[n]^\vee$ is induced at the boundary marked point $i$ from the line-bundle system $p^* T[n]^\vee$. Note also that $V_0[n] = t^* V_0[0]^\vee$, $\pi^* L = t^* \pi^* L$, and $p^* T[n]^\vee = t^* p^* T[0]^\vee$. Therefore, it suffices to show $L_0[0]^\vee \otimes p^* T[0]^\vee \otimes \pi^* L$ has an equivariant section that does not vanish near $D_0$.

Now, $L_0[0]^\vee$ is the bundle on $A[0]$ equal to $\mathcal{O}(-1) \otimes \pi_0^* L^\vee$ with the linearization $\sigma_0 \cdot ((l, t) \otimes a) = (\sigma_0 l, t) \otimes \sigma_0^{-1} a$.

It follows that $L_0[0]^\vee \otimes p_0^* T[0]^\vee \otimes \pi_0^* L$ is $\mathcal{O}(-1)$ with the linearization $\sigma_0 \cdot ((l, t)) = (\sigma_0 l, t)$.
But $O(-1)$ has an invariant rational section that is well-defined and nonzero near $D_0 = \{[0 : t]\}$. □

**Lemma 5.13.3.** There is a nowhere vanishing regular section of $(L_j)^\otimes m \otimes V$.

**Proof.** We observe that on a nice chart, $\delta^*(\Omega_{X/S}^m \otimes f^*V_0[n])$ has a $\widetilde{G[n]}^S$-equivariant, nowhere vanishing, regular section. Since $X/S$ is smooth near $\gamma(S)$, we have

$$\delta^*(\Omega_{X/S}^m \otimes f^*V_0[n]) = \mathcal{H}om(N_{\delta(S)/X}^m, N_{D_0[n]}/A[n]).$$

But this bundle has a section given by the projection of the $n$th formal derivative of $f$ to $N_{D_0[n]}/A[n]$. □

## 6. Degeneration Formulae

In the previous section, we have proved equations relating the line-bundle

$$\text{Dil, Split, } L_{i, \text{not top}}, L_{i, \text{not bot}}$$

to other line bundles on $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$ and $\mathcal{M}(\mathcal{A}, \Gamma_A)$. The first Chern classes of these line-bundles represent specific geometric situations involving split curves. For example, $c_1(\text{Split})$ is a substack of $\mathcal{M}(\mathcal{A}, \Gamma)$, that is, in a virtual sense, all split curves. $c_1(L_{i, \text{not top}})$ virtually consists of all split curves in which the $i$th marked point is not on the topmost component. Of course, we are counting certain curves with multiplicity and there are virtual issues that make geometric interpretations inaccurate. Fortunately, by closely adapting the arguments in [12], we can make arguments compatible with virtual cycle constructions. This allows us to write the cap product of a first Chern class of one of our bundles with the virtual cycle in terms of the virtual cycles of smaller moduli spaces.

We will express the first Chern class of various line-bundles geometrically by adapting Li’s argument [12]. The argument is in several stages and we state it only in the case $\mathcal{M}(\mathcal{Z}, \Gamma)$ noting that the case for $\mathcal{M}(\mathcal{A}, \Gamma)$ is exactly analogous:

1. For $\Gamma$, consider quadruples $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$ so that the graph join, $\Gamma_Z \ast_{L,J} \Gamma_A$ is isomorphic to $\Gamma$. We can define a line bundle $L_\Upsilon$ on $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$.
2. We show that $c_1(L_\Upsilon) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{\vir} = m(\Upsilon)[\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)]^{\vir}$, where $m(\Upsilon)$ is defined in Definition 6.5.1 and $[\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)]^{\vir}$ is an appropriately defined virtual cycle.
(3) Given the joining morphism
\[ \Phi : \mathcal{M}(\mathcal{A}, \Gamma_A) \times D^r \mathcal{M}(\mathcal{Z}, \Gamma_Z) \rightarrow \mathcal{M}(\mathcal{Z}, \Gamma_A \sqcup \mathcal{L}) \]
and the diagram
\[ \begin{array}{ccc}
\mathcal{M}(\mathcal{A}, \Gamma_A) \times D^r \mathcal{M}(\mathcal{Z}, \Gamma_Z) & \longrightarrow & \mathcal{M}(\mathcal{A}, \Gamma_A) \times \mathcal{M}(\mathcal{Z}, \Gamma_Z) \\
\downarrow & & \downarrow \\
D^r & \longrightarrow & D^r \times D^r
\end{array} \]
where \( \Delta \) is the diagonal map. We have
\[ \Phi_* \Delta^! ([M(\mathcal{A}, \Gamma_A)]^{\text{vir}} \times [M(\mathcal{Z}, \Gamma_Z)]^{\text{vir}}) = [M(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)]^{\text{vir}}. \]

(4) Given a line-bundle \( L = \text{Dil} \) or \( L = L^{i, \text{ext}} \), we exhibit a set of join-equivalence classes \( \Omega \) so that
\[ L = \otimes_{\Upsilon \in \Omega} L^{i, \Upsilon}. \]

(5) Consequently
\[ c_1(L) \cap [M(\mathcal{Z}, \Gamma)]^{\text{vir}} = \sum_{\Upsilon \in \Omega} m(\Upsilon) \Phi_* \Delta^! ([M(\mathcal{A}, \Gamma_A)]^{\text{vir}} \times [M(\mathcal{Z}, \Gamma_Z)]^{\text{vir}}) \]

To modify this argument to work for \( \mathcal{M}(\mathcal{Z}, \Gamma) \), replace all pairs \((\Gamma_A, \Gamma_Z)\) with \((\Gamma_t, \Gamma_b)\) and replace \( Z \) with \( A \). (4) is the only item significantly different from [12] to warrant much explanation.

6.1. **Local Interpretation of \( L^{i, \text{bot}} \).** We study a section of \( L^{i, \text{bot}} \) in the interest of proving (4) above. Recall that \( L^{i, \text{bot}} \) is induced from a line bundle system on \( \mathcal{A} \) called \( NB[n] \). For the bottom map \( b : A[n] \rightarrow A[0] \), \( NB[n] = b^* NB[0] \) where \( NB[0] = V_0[0] \). Therefore, \( NB[0] \) is given on \( A[0] = \mathbb{P}_X(L \oplus 1) \) by a bundle dual to \( O(-1) \otimes L^\vee \) under the linearization
\[ \sigma_0 \cdot ((l, t) \otimes a) = ((\sigma_0 l, t) \otimes \sigma_0^{-1} a). \]
Therefore, \( NB[0] \) has an equivariant section \( s[0] \) given by
\[ s[0]^{\vee} : [l : t] \rightarrow (l, t) \otimes a_l \]
where \( a_l \in L^\vee \) satisfies \( a_l(l) = 1 \).

Now, \( s[n] = b^* s[0] \) forms a line bundle system section of \( NB[n] \) and therefore induces a section of \( L^{i, \text{bot}} \) on \( \mathcal{M}(\mathcal{A}, \Gamma) \).

We need to describe how \( s \) looks in charts.

Recall that \( K_k \) is the hyperplane on \( \mathbb{A}^n \) cut out by \( t_k = 0 \) and that
\[ A[n] \times \mathbb{A}^n \; K_k = (A[k - 1] \times \mathbb{A}^{n-k}) \sqcup D (\mathbb{A}^{k-1} \times A[n-k]) \]
Definition 6.1.1. A chart with a marked point, \( i \),

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{f} A[n] \\
\downarrow_{\gamma} \\
S \xrightarrow{h} \mathbb{A}^{n}
\end{array}
\]

is said to be \( i \)-admissible for \( l \in \{0, 1, \ldots, n\} \) if

1. if \( l \geq 1 \) then \( S \times K_l \) is nonempty
2. For \( k \in \{1, 2, \ldots, n\} \) so that \( S \times \mathbb{A}^{n} K_k \) is non-empty, the image of every closed point under

\[
f \circ \gamma : S \times \mathbb{A}^{n} K_k \to A[n] \times \mathbb{A}^{n} K_k = (A[k - 1] \times \mathbb{A}^{n-k}) \sqcup D(A[k-1] \times A[n-k])
\]

lies in
   (a) \( \mathbb{A}^{k-1} \times A[n-k] \) for \( k \leq l \)
   (b) \( A[k-1] \times \mathbb{A}^{n-k} \) for \( k \geq l + 1 \).

The property of \( i \)-admissibility has a simple geometric explanation. Suppose a chart is \( i \)-admissible for \( l \in \{0, 1, \ldots, n\} \). Let \( s \in S \) be a closed point, \( t = h(s) \). Let \( \{a(1), a(2), \ldots, a(m)\} \) be a subset of \( \{1, \ldots, n\} \) corresponding to the coordinates of \( t \) which are zero. We have a morphism

\[
f : C = \mathcal{X} \times_S s \to A[n] \times \mathbb{A}^{n} t
\]

Then \( A[n] \times \mathbb{A}^{n} t = mP = P_0 \sqcup D_1 \cdots \sqcup D_m P_m \). The \( i \)th marked point is mapped into the component of \( P_i \setminus D_{a-1(l)} \) that contains \( X_0 \).

It is easy to see that the property of a morphism \( S \to M(A, \Gamma) \) being \( i \)-admissible is preserved under pullbacks in the sense of 5.1.2.

Lemma 6.1.2. Give a chart

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{f} A[n] \\
\downarrow_{\gamma} \\
S \xrightarrow{h} \mathbb{A}^{n}
\end{array}
\]

and a closed point \( p \in S \), then there is a Zariski neighborhood \( U_p \subseteq S \) of \( p \) so that

\[
\begin{array}{c}
\mathcal{X} \times_S U_p \xrightarrow{f} A[n] \\
\downarrow_{\gamma} \\
U_p \xrightarrow{h} \mathbb{A}^{n}
\end{array}
\]

is \( i \)-admissible.
Proof. The family over $p$,

\[
\begin{array}{ccc}
X \times_S \mathbb{A}^n & \xrightarrow{f} & \mathbb{A}^n \\
\downarrow \gamma & & \downarrow \\
\downarrow p & & \downarrow \rho \\
\end{array}
\]

is $i$-admissible for some $l \in \{0, 1, \ldots, n\}$.

Consequently, we can find a neighborhood $U_k$ of $p$ so that image of closed points under

\[
f \circ \gamma : U_k \times \mathbb{A}^n K_k \to A[n] \times \mathbb{A}^n = (A[k-1] \times \mathbb{A}^{n-k}) \sqcup \mathbb{A}^{k-1} \times A[n-k]
\]

lies in

\[
(1) \quad \mathbb{A}^{k-1} \times A[n-k] \text{ for } k \leq l
\]

\[
(2) \quad A[k-1] \times \mathbb{A}^{n-k} \text{ for } k \geq l + 1.
\]

Let $U_p = U_1 \cap U_2 \cap \cdots \cap U_n$. \qed

Corollary 6.1.3. $\mathcal{M}(A, \Gamma)$ has étale atlas $U \to \mathcal{M}(A, \Gamma)$ that has the property that $U$ can be written as a disjoint union, $U = \bigsqcup U_\alpha$ where each $U_\alpha$ is $i$-admissible.

Proof. We can refine an atlas of $\mathcal{M}(A, \Gamma)$ that is the disjoint union of nice charts. \qed

Note that if

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A[n] \\
\downarrow \gamma & & \downarrow \\
\downarrow S & & \downarrow \rho \\
\end{array}
\]

is $i$-admissible for an $l \in \{1, \ldots, n\}$ and if $j : [n] \to [N]$ is an order-preserving inclusion, then if $j_* : A[n] \to A[N]$, $J : \mathbb{A}^n \to \mathbb{A}^N$ are the induced inclusions, then

\[
\begin{array}{ccc}
X & \xrightarrow{j \circ f} & A[n] \\
\downarrow \gamma & & \downarrow \\
\downarrow S & & \downarrow J \circ \rho \\
\end{array}
\]

is $i$-admissible for $j(l)$ if $l \geq 1$ and $i$-admissible for 0 otherwise.

Note also that the property of $i$-admissibility for $l \in \{0, 1, \ldots, n\}$ is invariant under the action of $\rho : S \to G[n]$.

Lemma 6.1.4. Let $S$ be an $i$-admissible chart for $l \in \{0, 1, \ldots, n\}$. Then $L_{i, \text{not bot}}|S \cong 1_S$, the topologically trivial bundle under the $G[n]$-action

\[
(s_0, s_1, \ldots, s_n)[s] = [(s_1 \ldots s_l)^{-1} s]
\]
Furthermore, there is a constant $c$ so that the section is given on $S$ as $h^*(cx_1 \ldots x_l)$.

**Proof.** We need to define $\tilde{G}[n]$-equivariant line bundles $C_i[n]$ on $\mathbb{A}^n$. Let $C_i[n]$ as $1_{\mathbb{A}^n}$, the topologically trivial bundle on $\mathbb{A}^n$ with linearization

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = \sigma_i^{-1}s$$

$C_i$ has a $\tilde{G}[n]$-equivariant section given by $x_i$.

We will show that given an $i$-admissible atlas on $\mathcal{M}(A, \Gamma)$, $\{X_\alpha, S_\alpha, f_\alpha, h_\alpha, \gamma_\alpha\}$, so that

$$X_\alpha \xrightarrow{f} A[n]$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$S_\alpha \xrightarrow{h} \mathbb{A}^n$$

is $i$-admissible for $l_\alpha \in \{0, 1, \ldots, n\}$ then

$$(f_\alpha \circ \gamma_\alpha)^*NB[n] \otimes h_\alpha^*(C_1[n]^\vee \otimes \cdots \otimes C_{l_\alpha}[n]^\vee)$$

is canonically isomorphic to the trivial bundle on $S_\alpha$. Therefore, we conclude that

$$h_\alpha^*(C_1[n] \otimes \cdots \otimes C_{l_\alpha}[n])$$

induces a line-bundle on $\mathcal{M}(A, \Gamma)$ that is isomorphic to $L_{i, \text{not bot}}$. The following lemma gives a description of this bundle.

**Lemma 6.1.5.** There is line bundle on $\mathcal{M}(A, \Gamma)$ defined over an $i$-admissible chart $S_\alpha \to \mathcal{M}(A, \Gamma)$ by

$$h_\alpha^*(C_1[n] \otimes \cdots \otimes C_{l_\alpha}[n])$$

which has a section

$$h_\alpha^*(x_1x_2 \ldots x_{l_\alpha})$$

**Proof.** We just have to check that the line-bundles are well-behaved under the group action and standard inclusions. This is standard in light of the observation preceding Lemma 6.1.4. □

Recall that $\tilde{G}[n]^S$ is the group of morphisms $\rho : S \to \tilde{G}[n]$ under point-wise multiplication. To obtain the triviality of $(f_\alpha \circ \gamma_\alpha)^*NB[n] \otimes h_\alpha^*(C_1[n]^\vee \otimes \cdots \otimes C_{l_\alpha}[n]^\vee)$, we have the following.

**Claim 6.1.6.** For any family

$$X \xrightarrow{f} A[n]$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$S \xrightarrow{\gamma} \mathbb{A}^n$$

that is $i$-admissible for $l \in \{0, 1, \ldots, n\}$, the bundle

$$(f \circ \gamma)^*(NB[n] \otimes C_1[n]^\vee \otimes \cdots \otimes C_l[n]^\vee)$$

has a nowhere zero section $u[n]$ that obeys the following properties:
Then, it is easy to see by induction that we have the following
isomorphism of line-bundles,
\[
\pi^*\bigl((N_B[N])^\vee\bigr) \cong (J \circ \gamma)^*\bigl((N_B[N])^\vee\bigr)
\]
that is \(G[n]^S = (j_* G[n])^S\)-equivariant and that takes \(u[N]\) to \(u[n]\).

**Claim 6.1.7.** Given an effective inclusion induced by \(j: [n] \hookrightarrow [N]\), \(\rho: S \to \widetilde{G[N]}\) then there is an isomorphism
\[
(j_* \circ f \circ \gamma)^*\bigl((N_B[N])^\vee\bigr) \cong (f \circ \gamma)^*\bigl((N_B[n])^\vee\bigr)
\]

so if \(\sigma\) bundle
\[
(j_* \circ f \circ \gamma)^*\bigl((N_B[N])^\vee\bigr) \cong (f \circ \gamma)^*\bigl((N_B[n])^\vee\bigr)
\]

that is \(G[n]^S = (j_* G[n])^S\)-equivariant and that takes \(u[N]\) to \(u[n]\).

We first relate \(N_B[n]\) to \(V_0[n] = t^* V_0[0]\) which has a canonical equivariant section \(s[n] = t^* s[0]\). We note that the bottom map can be factored as a morphism of swags into blow-downs and projections as follows:

\[
b: A[n] \to A[n - 1] \times A^1 \to A[n - 1] \to \cdots \to A[1] \to A[0] \times A^1 \to A[0]
\]

so if

\[
\pi_n: A[n] \quad \stackrel{\beta}{\longrightarrow} \quad A[n - 1] \times A^1 \quad \stackrel{q}{\longrightarrow} \quad A[n - 1]
\]

we have

\[
N_B[n] = \pi_n^* N_B[n - 1] = \beta^* q^* N_B[n - 1]
\]

On the other hand, \(V_0[n]\) is the proper transform of \(q^* V_0[n - 1]\). Consequently, if \(E_n = O(1)\) is the line bundle whose zero scheme is exceptional divisor of \(\beta\) then

\[
V_0[n] = \pi_n^* V_0[n - 1] \otimes E_n^\vee.
\]

Let \(\pi_{n,i}: A[n] \to A[i]\) be given by the composition

\[
\pi_{n,i} = \pi_{i+1} \circ \cdots \circ \pi_n: A[n] \to A[n - 1] \cdots A[i+1] \to A[i]
\]

with the group homomorphism given by

\[
\pi_{n,i*}: (\sigma_0, \sigma_1, \ldots, \sigma_n) \to (\sigma_0, \sigma_1, \ldots, \sigma_i).
\]

Then, it is easy to see by induction that we have the following \(\widetilde{G[n]}\)-equivariant isomorphism of line-bundles,

\[
N_B[n] = V_0[n] \otimes \pi_{n,1}^* E_1 \otimes \pi_{n,2}^* E_2 \otimes \pi_{n,n-1}^* E_{n-1} \otimes E_n
\]

Since \(V_0[n]\) induces the trivial bundle on \(\mathcal{M}(A, \Gamma)\), \(N_B[n]\) induces the same bundle as

\[
N_B[n] \otimes L_0[n]^\vee = \pi_{n,1}^* E_1 \otimes \pi_{n,2}^* E_2 \otimes \pi_{n,n-1}^* E_{n-1} \otimes E_n.
\]

The theorem then follows as a consequence of the following lemma.

**Claim 6.1.7.** Consider an \(i\)-admissible atlas as above. On each \(S_\alpha\), consider the bundle

\[
(f \circ \gamma_\alpha)^* \bigl(\pi_{n_{\alpha,1}}^* E_1 \otimes \pi_{n_{\alpha,2}}^* E_2 \otimes \cdots \otimes \pi_{n_{\alpha,n-1}}^* E_{n-1} \otimes E_n\bigr)
\]

\[
\otimes h_\alpha^* (C_1[n]^\vee \otimes C_2[n]^\vee \otimes \cdots \otimes C_{l_\alpha}[n]^\vee)
\]

(1) \(u[n]\) is \(G[n]^S\)-equivariant
(2) Given an effective inclusion induced by \(j: [n] \hookrightarrow [N]\), \(\rho: S \to \widetilde{G[N]}\) then there is an isomorphism

\[
(j_* \circ f \circ \gamma)^*\bigl((N_B[N])^\vee\bigr) \cong (f \circ \gamma)^*\bigl((N_B[n])^\vee\bigr)
\]
on $S$. These bundles descend to a section to $\mathcal{M}(A, \Gamma)$. Moreover, they possess a nowhere zero section that descends to a section on $\mathcal{M}(A, \Gamma)$.

This claim follows from examining blow-ups in local coordinates.

There is the analog for $L_{\text{i, not top}}$ which has a similarly defined canonical section.

**Lemma 6.1.8.** Let $S$ be an $i$-admissible chart for $l \in \{0, 1, \ldots, n\}$. Then

$$L_{\text{i, not top}} \mid S \cong 1_S,$$

the topologically trivial bundle under the $\widehat{G}[n]$-action

$$(\sigma_0, \sigma_1, \ldots, \sigma_n)[s] = [(\sigma_{l+1} \ldots \sigma_n)^{-1}s]$$

Furthermore, $L_{\text{i, not top}}$ has a canonical section that is given on $S$ as $h^*(x_{l+1} \ldots x_n)$.

6.2. **Local Interpretation of $L_{\text{i, ext}}$.** We can define $i$-admissible charts on $\mathcal{M}(Z, \Gamma_Z)$ exactly as we did on $\mathcal{M}(A, \Gamma_A)$. Recall that $L_{\text{i, ext}}$ is defined as $\text{ev}_i^*L$ where $L$ is a line bundle on $Z$ with section $s$ whose zero-scheme is $D$. $L_{\text{i, ext}}$ has a canonical section $\text{ev}_i^*s$.

**Lemma 6.2.1.** Let $S$ be an $i$-admissible chart for $l \in \{0, 1, \ldots, n\}$. Then

$$L_{\text{i, not bot}} \mid S \cong 1_S,$$

the topologically trivial bundle under the $\widehat{G}[n]$-action

$$(\sigma_1, \ldots, \sigma_n)[s] = [(\sigma_1 \ldots \sigma_l)^{-1}s]$$

Furthermore, the canonical section is given on $S$ as $h^*(x_1 \ldots x_l)$.

The proof is exactly analogous to the one given for $L_{\text{i, not bot}}$. It carries through word-for-word with $G[n]'s$ substituted for $\widehat{G}[n]$ and $Z$'s substituted for $A$'s.

6.3. **$\Gamma_1 \sqcup \Gamma_2$-admissible charts.** Let $\Gamma$ be a relative graph. Consider the moduli space $\mathcal{M}(Z, \Gamma)$.

**Definition 6.3.1.** Let $(\Gamma_Z, \Gamma_A, L, J)$ be a quadruple whose graph-join, $\Gamma_Z \ast_{L, J} \Gamma_A$ is isomorphic to $\Gamma$. Consider a nice chart in $\mathcal{M}(Z, \Gamma)$,

![Diagram]

Let

$$T = S \times_{\mathcal{M}(Z, \Gamma)} \mathcal{M}(Z \sqcup A, \Gamma_Z \sqcup_{L, J} \Gamma_A).$$

The chart $S$ is said to be $\Gamma_Z \sqcup_{L, J} \Gamma_A$-admissible if one of the following happens.
(1) $T$ is empty. In this case, $S$ is said to be trivially admissible.

(2) There exists an integer $l \in \{1, \ldots, n\}$ so that $T \to S$ factors as $T \to S \times_{A^n} K_l$ so that

$$f: X \times_{A^n} K_l \to Z[n] \times_{A^n} K_l = (Z[l - 1] \times A^{n-l}) \sqcup_D (A^{l-1} \times A[n - l])$$

can be written as

$$g_Z \times h_A \sqcup h_Z \times g_A: Y_Z \sqcup D_Z Y_A \to (Z[l - 1] \times A^{n-l}) \sqcup_D (A^{l-1} \times A[n - l])$$

where $\sqcup_D$ refers to gluing the following families along $f^{-1}(D_l)$ to form nodes:

$$
\begin{array}{c}
X \downarrow \downarrow f \rightarrow Z[n] \\
\gamma, \delta \downarrow \\
S \times_{A^n} K_l \downarrow h_Z \rightarrow A^{l-1}
\end{array}
$$

and

$$
\begin{array}{c}
Y_A \downarrow \downarrow f_A \rightarrow Z[n - l] \\
\gamma_A, \delta^n, \delta^* \downarrow \\
S \times_{A^n} K_l \downarrow h_A \rightarrow A^{n-l}
\end{array}
$$

where this splitting is described by a quadruple that is join-equivalent to $(\Gamma_Z, \Gamma_A, L, J)$.

In this case, the chart is said to be admissible for $l$.

Again, everything holds for $\mathcal{M}(A, \Gamma)$ if we replace $(\Gamma_Z, \Gamma_A)$ by $(\Gamma_b, \Gamma_l)$. By [12], there exists an atlas consisting of $(\Gamma_Z, \Gamma_A, L, J)$-admissible charts.

**Definition 6.3.2.** Given a quadruple $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$, and a nice chart in $\mathcal{M}(Z, \Gamma)$,

$$
\begin{array}{c}
X \downarrow \downarrow f \rightarrow Z[n] \\
\gamma, \delta \downarrow \\
S \downarrow h \rightarrow A^n
\end{array}
$$

that is $\Gamma_Z \sqcup_{L,J} \Gamma_A$-admissible, define $L_{\Upsilon}$ on $S$ as follows:

(1) If $S$ is trivially admissible, $L_{\Upsilon} = 1_S$, a topologically trivial bundle with trivial $G[n]$-action.

(2) If $S$ is admissible for $l \in \{1, \ldots, n\}$, define $L_{\Upsilon}$ on $S$ as the $G[n]$-equivariant line bundle $h^*1^n_A$ where $1^n_A$ is the topologically trivial bundle with group action

$$(\sigma_1, \ldots, \sigma_n) \cdot s = \sigma_l^{-1}s$$

Now, $L_{\Upsilon}$ has a canonical section given by $s_{\Upsilon} = 1$ in trivial charts and $s_{\Upsilon} = h^*(x_l)$ for non-trivial charts. It is standard to verify by arguments similar to 6.1.5 that $(L_{\Upsilon}, s_{\Upsilon})$ globalizes to a line bundle on $\mathcal{M}(Z, \Gamma)$ (see [12], Lemma 3.4).
Once we fix $\Gamma$, there are finitely many join-equivalence classes of quadruples $(\Gamma_Z, \Gamma_A, L, J)$ so that $\Gamma_Z \ast_L J \Gamma_A = \Gamma$. Therefore, we can construct an atlas that is admissible with respect to every possible join-equivalence class and is admissible with respect to every interior marked point.

6.4. Interpretation of Bundles. Let us rewrite the bundles $Dil, L_{i, \text{ext}}, L_{i, \text{not top}}, L_{i, \text{not bot}}$ as tensor products of $L_\Upsilon$'s on $\mathcal{M}(Z, \Gamma)$ and $\mathcal{M}(A, \Gamma)$.

On $\mathcal{M}(Z, \Gamma)$ where $i$ is the label for an interior marked point,

1. $\Omega_{Dil} = \{ \Upsilon = (\Gamma_A, \Gamma_Z, L, J) \}$ the set of all join-equivalence classes of quadruples $\Upsilon = (\Gamma_A, \Gamma_Z, L, J)$.
2. $\Omega_{L_{i, \text{ext}}} = \{ (\Gamma_A, \Gamma_Z, L, J) \mid i \in J(M_A) \}$.

while on $\mathcal{M}(A, \Gamma)$ where $i, j$ are labels for interior marked points,

1. $\Omega_{\text{Split}} = \{ (\Gamma_t, \Gamma_b, L, J) \}$.
2. $\Omega_{L_{i, \text{not bot}}} = \{ (\Gamma_t, \Gamma_b, L, J) \mid i \in J(M_t) \}$.
3. $\Omega_{L_{i, \text{not top}}} = \{ (\Gamma_t, \Gamma_b, L, J) \mid i \in J(M_b) \}$.

Theorem 6.4.1. For $L = Dil, L_{i, \text{ext}}, \text{Split}, L_{i, \text{not bot}}, L_{i, \text{not top}}$,

$$L = \bigotimes_{[\Upsilon] \in \Omega_L} L_\Upsilon.$$  

where $[\Upsilon]$ denotes a join-equivalence class and $\Upsilon$ a representative element.

Proof. Let us prove (2) on $\mathcal{M}(A, \Gamma)$. The other cases are similar. Consider a chart $S$ that is admissible with respect to every $\Gamma_b, \Gamma_t$ decomposition and with respect to every interior marked point:

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & Z[n] \\
\downarrow & & \downarrow \\
S & \xrightarrow{\gamma} & \mathbb{A}^n
\end{array}$$

where $\gamma$ is the section corresponding to the interior marked point $i$. Suppose that $S$ is $i$-admissible for $l \in \{0, 1, \ldots, n\}$. The for any $k \in \{1, \ldots, n\}$ so that $S \times_{\mathbb{A}^n} K_k$ is not empty,

$$f : \mathcal{X} \times_{\mathbb{A}^n} K_k \to A[n] \times_{\mathbb{A}^n} K_k = (A[k-1] \times \mathbb{A}^{n-k}) \sqcup_D (\mathbb{A}^{k-1} \times A[n-k])$$

is given by a join-equivalence class $[\Upsilon]$ where $\Upsilon = (\Gamma_b, \Gamma_t, L, J)$. Following from $i$-admissibility, we have the image of every closed point under

$$(f \circ \gamma) : S \to (A[k-1] \times \mathbb{A}^{n-k}) \sqcup_D (\mathbb{A}^{k-1} \times A[n-k])$$

lying in $\mathbb{A}^{k-1} \times A[n-k]$ for $k \leq l$ and lying in $A[k-1] \times \mathbb{A}^{n-k}$ for $k \geq l+1$. This implies that $J^{-1}(i) \in M_t$ for $k \leq l$ and $J^{-1}(i) \in M_b$ for $k \geq l+1$. Now, let
$B = \{ k \in \{1, \ldots, l \} | S \times_{A^n} K_k = \emptyset \}$ and define $L_\emptyset$ as a topologically trivial bundle on $A^n$ with linearization given by

$$(\sigma_0, \sigma_1, \ldots, \sigma_n)_s = (\prod_{k \in B} \sigma_{k}^{-1})_s.$$ 

$L_\emptyset$ is canonically isomorphic to the trivial bundle with the trivial linearization. Therefore,

$$\bigotimes_{[\Upsilon] \in \Omega_{i, \text{not bot}}} L_i \otimes L_\emptyset = 1$$

with the linearization

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_1 \ldots \sigma_l)^{-1}_s.$$ 

But this is just local interpretation of $L_i$. 

6.5. Splitting of Moduli Stacks. We need to cite a number of results from [12]. These results were proved for a different moduli stack, $M(W)$, but because of the parallels between that space and $M(A, \Gamma)$ and $M(Z, \Gamma)$, the proofs can be modified in straightforward fashion.

**Definition 6.5.1.** Let $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$ be a quadruples describing a decomposition in $M(Z, \Gamma_Z \ast L, J \Gamma_A)$. Define $m(\Upsilon)$ by

$$m(\Upsilon) = \prod_{i \in RZ} \mu_Z(i).$$

**Theorem 6.5.2.** We have the following equality among cycle classes

$$c_1(L_\Upsilon, s_\Upsilon) \cap [M(Z, \Gamma_Z \ast L, J \Gamma_A)]^{vir} = m(\Upsilon)[M(Z \sqcup A, \Upsilon)]^{vir}$$

Let $[\Upsilon]$ be $\Upsilon$'s join-equivalence class. Then we have as a consequence of Corollary 3.7.7,

**Theorem 6.5.3.** If

$$M_{[\Upsilon]} = \prod_{(\Gamma_Z', \Gamma_A', L, J) \in [\Upsilon]} M(A, \Gamma_A') \times_{D^r} M(Z, \Gamma_Z')$$

is given the virtual cycle of a disjoint union, then

$$\Psi_{[\Upsilon]} : M_{[\Upsilon]} \to M(Z, \Gamma_Z \ast L, J \Gamma_A)$$

induces

$$\Psi_{[\Upsilon]*}[M]^{vir} = |MZ|! |MA|! |(RZ)|^2 [M(Z \sqcup A, \Upsilon)]^{vir}$$

Now, we have the following fiber square for a quadruple $(\Gamma_Z, \Gamma_A, L, J)$,

$$
\begin{array}{ccc}
M(A, \Gamma_A) \times_{D^r} M(Z, \Gamma_Z) & \rightarrow & M(A, \Gamma_A) \times M(Z, \Gamma_Z) \\
\downarrow_{\Delta} & & \downarrow_{\Delta} \\
D^r & \rightarrow & D^r \times D^r
\end{array}
$$
where downward pointing maps are induced from evaluation at the boundary marked points of \( \mathcal{M}(Z, \Gamma_Z) \) and the boundary marked points at \( D_\infty \) on \( \mathcal{M}(A, \Gamma_A) \) and \( \Delta \) is the diagonal.

**Theorem 6.5.4.** We have the equality of cycle classes

\[
\Delta^!(\mathcal{M}(A, \Gamma_A))^{\text{vir}} \times \mathcal{M}(Z, \Gamma_Z))^{\text{vir}} = \mathcal{M}(A, \Gamma_A) \times_{D_\infty} \mathcal{M}(Z, \Gamma_Z))^{\text{vir}}
\]

**Corollary 6.5.5.** If we define

\[
P_\Upsilon = \prod_{(\Gamma'_Z, \Gamma'_A, L, J) \in [\Upsilon]} \mathcal{M}(A, \Gamma'_A) \times \mathcal{M}(Z, \Gamma'_Z)
\]

then

\[
c_1(L_\Upsilon, s_\Upsilon) \cap \mathcal{M}(Z, \Gamma_Z *_{L, J} \Gamma_A))^{\text{vir}} = \frac{m(\Upsilon)}{|MZ|!|MA|!(|RZ|)!^2} \Delta^!(P_\Upsilon)
\]

\( L \) together with \( i : X \to Z \) induces a morphism

\[
\Lambda : (X^{[MA]} \times X^{[RA_0]} \times Z^{[MZ]} \to Z^M \times X^R
\]

where \( M = |MZ| + |MA| \) and \( R = |RA_0| \) are the number of interior and boundary marked points in \( \Gamma_A *_{L, J} \Gamma_Z \)

We have morphisms

\[
X^{[MA]+|RA_0|} \times X^{[RZ]} \times Z^{[MZ]} \xrightarrow{\tilde{\Delta}} X^{[MA]+|RA_0|} \times X^{[RA_\infty]} \times Z^{[MZ]} \times X^{[RZ]}
\]

\[
\downarrow p
\]

\[
(X^{[MA]} \times X^{[RA_0]} \times Z^{[MZ]}
\]

where \( \tilde{\Delta} \) is induced by \( \Delta : X^{[RZ]} \to X^{[RA_\infty]} \times X^{[RZ]} \) and \( p \) is the projection.

Therefore, for \( c \in H^*(Z^{[MZ]} \times X^{[RZ]}), \)

\[
\text{deg}((\text{Ev}^*(c) \cup c_1(L_\Upsilon)) \cap \mathcal{M}(Z, \Gamma_A *_{L, J} \Gamma_Z))^{\text{vir}}
\]

\[
= \frac{m(\Upsilon)}{\text{Aut}_{\Gamma_Z, \Gamma_A, L}(RZ, RA_\infty)} \text{deg}(\text{Ev}^*(\tilde{\Delta}^!(p^*\Lambda^*c) \cap ([\mathcal{M}(A, \Gamma_A)]^{\text{vir}} \times [\mathcal{M}(Z, \Gamma_Z)]^{\text{vir}})
\]

Again, there are obvious degeneration formulae on \( \mathcal{M}(A, \Gamma_b *_{L, J} \Gamma_t) \) obtained by replacing \( \Gamma_Z \) with \( \Gamma_b \) and \( \Gamma_A \) with \( \Gamma_t \).

By writing

\[
c_1(L_\Omega) = \sum_{[\Upsilon] \in \Omega} c_1(L_\Upsilon)
\]

we obtain formulas for expressing

\[
\text{deg}((\text{Ev}^*(C) \cup c_1(L_\Omega)) \cap \mathcal{M}(Z, \Gamma_Z *_{L, J} \Gamma_A))^{\text{vir}}
\]

and

\[
\text{deg}((\text{Ev}^*(C) \cup c_1(L_\Omega)) \cap \mathcal{M}(A, \Gamma_b *_{L, J} \Gamma_t))^{\text{vir}}.
\]
6.6. Normal Bundle to Split Maps. The split maps in $\mathcal{M}(Z, \Gamma)$ for a divisor in $\mathcal{M}(Z, \Gamma)$. It is natural to ask what the normal bundle to such a divisor is. Since a localization computation will have split maps as fixed loci, knowledge of the normal bundle will be indispensable for localization as in [5]. This section is logically independent from the rest of this paper, but is included as reference.

Consider $(Z, D)$, a projective manifold $Z$ and a smooth divisor $D \subset Z$. Let $(X, L)$ be given by $X = D$, $L = N_{D/Z}$. Let $\Gamma_Z$ and $\Gamma_A$ be relative and rubber graphs, respectively. Let $L, J$ be joining data as in 3.7.2. Consider $L_\Upsilon$ for the quadruple $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$. Then $c_1(L_\Upsilon)$ is a substack of $\mathcal{M}(Z, \Gamma_Z \ast_{L, J} \Gamma_A)$.

Consider the moduli stacks $\mathcal{M}(Z, \Gamma_Z)$, $\mathcal{M}(A, \Gamma_A)$, $\mathcal{M}(Z, \Gamma_Z \ast_{L, J} \Gamma_A)$, and the inclusion

$$\Phi : \mathcal{M}(A, \Gamma_A) \times_{D^r} \mathcal{M}(Z, \Gamma_Z) \to \mathcal{M}(Z, \Gamma_Z \ast_{L, J} \Gamma_A).$$

Proposition 6.6.1. \(\Phi^* L_\Upsilon = p_2^* \text{DIL}^\vee \otimes p_A^* L^\infty\).

**Proof.** Consider an $\Upsilon$ admissible chart for $l \in \{1, \ldots, n\}$,

$$\begin{array}{ccc}
X & \xrightarrow{f} & Z[n] \\
\gamma, \delta \downarrow & & \downarrow \\
S & \xrightarrow{h} & \mathbb{A}^n
\end{array}$$

Then $L_\Upsilon$ is given as $h^*1_{\mathbb{A}^n}$ where $1_{\mathbb{A}^n}$ is the topologically trivial bundle with group action

$$(\sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_l)^{-1} s.$$ 

Let $K_l \subseteq \mathbb{A}^n$ be the subset given by $t_l = 0$. Let $\phi : K_l \to \mathbb{A}^n$ be the inclusion.

**Lemma 6.6.2.** There is an isomorphism of swags, $K_l \cong \mathbb{A}^{l-1} \times \mathbb{A}^{n-l}$ where $\mathbb{A}^{l-1}$ is considered a swag under $G[l-1]$, $\mathbb{A}^{n-l}$ is considered a swag under $\tilde{G}[n-l]$. Under this isomorphism,

$$\phi^* L_\Upsilon \cong \pi_1^* \text{DIL}[l-1]^{\vee} \otimes \pi_2^* B[n-l]^{\vee}$$

where $\pi_1$ and $\pi_2$ are defined as

$$\begin{array}{ccc}
& K_l & \\
\pi_1 \downarrow & \downarrow \pi_2 & \\
\mathbb{A}^{l-1} & \mathbb{A}^{n-l}
\end{array}$$

**Proof.** The isomorphism is given by

$$H_l \to \mathbb{A}^{l-1} \times \mathbb{A}^{n-l}$$

$$(t_1, \ldots, t_{l-1}, 0, t_{l+1}, \ldots, t_n) \mapsto (t_1, \ldots, t_{l-1}, (t_{l+1}, \ldots, t_n)$$

$$(t_1, \ldots, t_{l-1}, (t_{l+1}, \ldots, t_n)) \mapsto (t_1, \ldots, t_{l-1}) \cdot (t_{l+1}, \ldots, t_n)$$

$$(\sigma_1, \ldots, \sigma_n) \mapsto (\sigma_1, \ldots, \sigma_{l-1}), (\sigma_1 \sigma_2 \ldots \sigma_1, \sigma_{l+1}, \ldots, \sigma_n)$$
Note that in the above, $\sigma_1 \ldots \sigma_l$ gets mapped into the zeroth place in $G[l-n]$. Therefore, $\pi^1_1 Djour[l-1]^\vee \otimes \pi^* B[l-n]^\vee$, is the topologically trivial bundle on $H_l$ with the group action 

$$(\sigma_1, \ldots, \sigma_n) \cdot s = (\sigma_1 \ldots \sigma_{l-1})(\sigma_1 \ldots \sigma_l)^{-1}s = \sigma_l^{-1}s$$

The above isomorphism is canonical and globalizes giving the conclusion.

7. The Trivial Cylinder Theorem

7.1. Trivial Cylinders. We will single out certain connected components of curves parameterized by $M(\mathcal{A}, \Gamma)$. These are the trivial cylinders which will be significant in [7].

Definition 7.1.1. Let $\Gamma$ be a rubber graph. A vertex $v$ is said to correspond to a trivial cylinder if

1. $g(v) = 0$.
2. $d(v) = 0$
3. $a_0^{-1}(v)$ is a single point.
4. $a_{\infty}^{-1}(v)$ is a single point.
5. $\mu^0(a_0^{-1}(v)) = \mu^\infty(a_{\infty}^{-1}(v))$.
6. $A^{-1}_{\mathbb{A}}(v)$ is empty.

A trivial cylinder corresponds to a component $X_v$ in a family. This component is given over a point $p \in S$ by a map $f$ from $C$ to $\mathbb{P}^1$ where $C$ is a chain of $P^1$'s. $f$ takes $C$ into a fiber over some point $x \in X$, and there is an integer $d$ so that $f$ is given on each $\mathbb{P}^1$ by

$$z \mapsto z^d.$$ 

Note that if $\Gamma$ consists of a single vertex corresponding to a trivial cylinder, then a morphism of a family of type $\Gamma$,

$$
\begin{array}{c}
\mathcal{X} \\
\delta_{\mathbb{A}}^n \\
S \\
\mathbb{A}^n
\end{array}
\xymatrix{
\mathcal{X} \ar[r]^f & A[n] \\
\mathbb{A}^n}
$$

would be invariant under composing $f$ with the action of $(\sigma_0, 1, \ldots, 1) \in G[n]$ and pre-composing with the appropriate $S$-isomorphism on $\mathcal{X}$. Consequently a trivial cylinder is not stable. This does not rule out morphisms of type $\Gamma$ which has a component which is a trivial cylinder.
There is a straightforward way of comparing relative Gromov-Witten invariants involving graphs with trivial cylinders to those without. It follows from the following theorem.

**Theorem 7.1.2.** Let $\Gamma$ be some rubber graph. Let $\Gamma_1$ be $\Gamma$ together with a vertex corresponding to a trivial cylinder. There is a natural map

$$v : \mathcal{M}(\mathcal{A}, \Gamma_1) \to \mathcal{M}(\mathcal{A}, \Gamma) \times \mathcal{X}$$

so that

$$v_*[\mathcal{M}(\mathcal{A}, \Gamma_1)]^{vir} = \frac{1}{r}[\mathcal{M}(\mathcal{A}, \Gamma)]^{vir} \times [\mathcal{X}]$$

and

$$v^*(L^\infty) = L^\infty.$$

Consequently if $\Gamma$ has $m$ interior marked points and $r_0 + r_\infty$ boundary marked points then we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(\mathcal{A}, \Gamma_1) & \xrightarrow{Ev} & X^m \times X^{r_0+1} \times X^{r_\infty+1} \\
\downarrow & & \downarrow h \\
\mathcal{M}(\mathcal{A}, \Gamma) \times \mathcal{X} & \xrightarrow{Ev \times \Delta} & X^m \times X^{r_0} \times X^{r_\infty} \times (\mathcal{X} \times \mathcal{X})
\end{array}$$

where $\Delta : \mathcal{X} \to \mathcal{X}^2$ is the diagonal and the morphism $h$ reorders the products of $\mathcal{X}$ so that the copies of $\mathcal{X}$ corresponding to the $r_0 + 1$st and $r_\infty + 1$st boundary marked points are taken to the image of the diagonal morphism. Then for classes $A \in H^*(X^m \times X^{r_0+1} \times X^{r_\infty+1})$, we have

$$\deg(Ev^*(A) \cap \mathcal{M}(\mathcal{A}, \Gamma_1))^{vir} = \frac{1}{r} \deg((Ev \times \Delta) \circ h^{-1})^*(A) \cap (\mathcal{M}(\mathcal{A}, \Gamma))^{vir} \times [\mathcal{X}]$$

There is a heuristic argument for why this is true. We can think of a trivial cylinder in $A[0]$ as a map $\mathbb{P}^1 \to \mathbb{P}^1$ given by

$$z \mapsto z^r$$

Note that this map $r$ automorphisms given by

$$z \mapsto \omega^a z$$

where $\omega_r$ is an $r$th root of unity and $0 \leq a \leq r - 1$. Therefore, we would like to say

$$\mathcal{M}(\mathcal{A}, \Gamma_1) = \mathcal{M}(\mathcal{A}, \Gamma) \times (X/\mathbb{Z}/r)$$

where $X/\mathbb{Z}/r$ is the stack consisting of $X$ quotiented by a cyclic group of order $r$, acting trivially. We should even hope for the virtual cycles of each space to agree. This, unfortunately, is not true as stated.

The main obstacle for this fact being the case is that the target of a map in $\mathcal{M}(\mathcal{A}, \Gamma)$ may not be $\mathbb{P}^1$, but $lP$ for $l \geq 1$. Therefore, we may have split maps. This gives an automorphism group of $(\mathbb{Z}/r)^{l+1}$ where we can multiply by a different power of $\omega_r$ on each $P$. Moreover, this splitting phenomenon gives a non-reduced scheme where points corresponding to maps into a chain of length $l$ have multiplicity $r^l$. Therefore, we’d like to define a map

$$v : \mathcal{M}(\mathcal{A}, \Gamma_1) \to \mathcal{M}(\mathcal{A}, \Gamma) \times \mathcal{X}$$
If we have
\[ v_*([\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}}) = \frac{1}{d+1} [\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}} \times X \]
then we can conclude the theorem.

The argument below was suggested by Jun Li.

7.2. Stacks of Trivial Cylinders. We need to define a stack \( \mathcal{M}_T \) that parameterizes map rubber maps of cylinders. Since trivial cylinders are not stable, this moduli stack is not Deligne-Mumford.

Let \( \Delta_r \) be a rubber graph corresponding to a degree \( r \) trivial cylinder. Consider \( \mathcal{M}_T = \mathcal{M}(\mathcal{A}, \Delta_r) \) constructed as above but where we do not impose the stability condition on families. Instead, we just impose that for a family over any point,

\[
\begin{array}{ccc}
C & \xrightarrow{f} & A[n] \\
\downarrow & & \downarrow \\
P & \xrightarrow{p} & \mathbb{A}^n
\end{array}
\]

if \( A[n] \times_{\mathbb{A}^n} p \) is a chain of \( P \)’s,

\[ P_0 \sqcup_D P_1 \sqcup_D \cdots \sqcup_D P_l \]

then \( C \) consists of a chain

\[ C_0 \cup C_1 \cup \cdots \cup C_l \]

where \( f \) maps \( C_i \) to a fiber of \( \pi : P_l \rightarrow X \) by a map of the form

\[ z \rightarrow z^r \]

There is a natural (although not representable) map \( v : \mathcal{M}_T \rightarrow \mathbb{A}^n \times X \) taking each family of trivial cylinders to its family of target schemes together with the point in \( X \) in whose fiber it lies.

We can get an explicit handle on the morphism \( v \) by pulling it back by the morphism \( \mathbb{A}^n \rightarrow \mathcal{A}^{\text{rel}} \) given by the family of targets \( A[n] \rightarrow \mathbb{A}^n \). Now \( T = \mathcal{M}_T \times_{\mathcal{A}^{\text{rel}}} \mathbb{A}^n \) has an explicit description. Define the scheme \( TC^n \) by

\[ TC^n = \text{Spec } \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x - y_1^r, \ldots, x - y_n^r). \]

\( TC^n \times X \) has a natural morphism to \( T \). The map to \( \mathbb{A}^n \) is given by

\[ TC^n \rightarrow \mathbb{A}^n = \text{Spec } \mathbb{K}[x_1, \ldots, x_n]. \]

The map to \( \mathcal{M}_T \) comes from the following family: the domain is given by

\[ A[n] \times_{\mathbb{A}^n \times X} TC^n \]

where the map \( TC^n \rightarrow \mathbb{A}^n \) is

\[ TC^n \rightarrow \mathbb{A}^n = \text{Spec } \mathbb{K}[y_1, \ldots, y_n] \]

and \( A[n] \rightarrow \mathbb{A}^n \times X \) is the usual projection \( p \times \pi \). The target is given by

\[ A[n] \times_{\mathbb{A}^n \times X} TC^n \]
where the map $TC^n \to \mathbb{A}^n$ is

$$TC^n \to \mathbb{A}^n = \text{Spec } \mathbb{K}[x_1, \ldots, x_n].$$

The stable morphism of families, $f$ is given by constructing nodal normal neighborhoods $w_1w_2 = x_1$ and $z_1z_2 = y_1$ and setting

$$f^*w_i = z_i^r.$$

This induces a degree $r$ covering of the fibers.

This morphism has the automorphisms

$$TC^n \times_T TC^n = (\mathbb{Z}/r)^{n+1} \times TC^n.$$

The first projection is

$$(a_0, a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

where $a_i \in \mathbb{Z}/r$. The second projection is

$$(a_0, a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n, \omega_{r}^{a_1} y_1, \ldots, \omega_{r}^{a_n} y_n)$$

The fact that $y$ becomes nilpotent at $x = 0$ come from the fact that when we consider a split curve, we must smooth the node to $r$th order before we can move the curve out of its singular target fiber. The $\mathbb{Z}/r$ factors come from rotating different components of split maps.

Consider the map

$$v : \mathcal{MT} \times_{\mathcal{A}^{\text{rel}}} \mathbb{A}^n \to (\mathbb{A}^n \times_{\mathcal{A}^{\text{rel}}} \mathcal{A}^{\text{rel}}) \times X = \mathbb{A}^n \times X.$$

It is clear that for any point $q \in \mathbb{A}^n \times X$, we have $\deg(v^{-1}(q)) = \frac{1}{r}$. This follows from the fact that degree and local multiplicity are properties that can be checked at generic geometric points.

Note that given any map $\text{Spec } \mathbb{K} \to \mathcal{A}^{\text{rel}}$, we may find an $n$ so that the map factors as $\text{Spec } \mathbb{K} \to \mathbb{A}^n \to \mathcal{A}^{\text{rel}}$. Consequently, $\deg(\text{Spec } \mathbb{K} \times_{\mathcal{A}^{\text{rel}}} \mathcal{MT}) = \frac{1}{r}$.

7.3. **Comparing $\mathcal{M}(\mathcal{A}, \Gamma_1)$ to $\mathcal{M}(\mathcal{A}, \Gamma)$**.

**Theorem 7.3.1.** $\mathcal{M}(\mathcal{A}, \Gamma_1) = \mathcal{M}(\mathcal{A}, \Gamma) \times_{\mathcal{A}^{\text{rel}}} \mathcal{MT}$

**Proof.** This is a matter of unwinding the definition of a fiber product in the category of stacks. Let $p_{\mathcal{M}_{\mathcal{A}}} : \mathcal{M}(\mathcal{A}, \Gamma), \mathcal{MT} \to \mathcal{A}^{\text{rel}}$ be the projections. A family $S \to \mathcal{M}(\mathcal{A}, \Gamma) \times_{\mathcal{A}^{\text{rel}}} \mathcal{MT}$ consists of a triple $(h_{\mathcal{M}_{\mathcal{A}}}, h_T, a)$ where

$$h_{\mathcal{M}_{\mathcal{A}}} : S \to \mathcal{M}(\mathcal{A}, \Gamma)$$

$$h_T : S \to \mathcal{MT}$$
and $a$ is an arrow in $\mathcal{A}^{rel}$, $a : p_A(h_{MA}) \to p_T(h_T)$. If we shrink $S$ to make $h_A$ and $h_T$ a nice family, we get diagrams

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f_*} & A[n_1] \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathbb{A}^{n_1}
\end{array}
$$

and

$$
\begin{array}{ccc}
T & \xrightarrow{f_T} & A[n_2] \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathbb{A}^{n_2}
\end{array}
$$

By shrinking $S$ further, we can find an effective arrow between $S \rightarrow \mathbb{A}^{n_1}$, $S \rightarrow \mathbb{A}^{n_2}$, that is an integer $N$, standard inclusions $[n_1] \rightarrow [N]$, $[n_2] \rightarrow [N]$ and maps $\rho_1, \rho_2 : S \rightarrow \overline{G[N]}$ so that the following compositions are equal

$$
S \rightarrow S \times \mathbb{A}^{n_1} \rightarrow S \times \mathbb{A}^N \rightarrow \overline{G[N]} \times \mathbb{A}^N \rightarrow \mathbb{A}^N
$$

$$
S \rightarrow S \times \mathbb{A}^{n_2} \rightarrow S \times \mathbb{A}^N \rightarrow \overline{G[N]} \times \mathbb{A}^N \rightarrow \mathbb{A}^N.
$$

This allows us to combine the two families

$$
\begin{array}{ccc}
\mathcal{X} \sqcup T & \xrightarrow{f_* \sqcup f_T} & A[N] \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathbb{A}^N
\end{array}
$$

to obtain a family in $\mathcal{M}(\mathcal{A}, \Gamma)$. Showing that this map from $\mathcal{M}(\mathcal{A}, \Gamma) \times_{\mathcal{A}^{rel}} \mathcal{M} \rightarrow \mathcal{M}(\mathcal{A}, \Gamma)$ is an isomorphism is a straightforward verification. \hfill \Box

**Corollary 7.3.2.** There is a natural morphism $\mathcal{M}(\mathcal{A}, \Gamma) \rightarrow \mathcal{M}(\mathcal{A}, \Gamma) \times X$

**Proof.** Consider $v : \mathcal{M} \rightarrow \mathcal{A}^{rel} \times X$. This induces

$$
v : \mathcal{M}(\mathcal{A}, \Gamma) = \mathcal{M}(\mathcal{A}, \Gamma) \times_{\mathcal{A}^{rel}} \mathcal{M} \rightarrow \mathcal{M}(\mathcal{A}, \Gamma) \times_{\mathcal{A}^{rel}} (\mathcal{A}^{rel} \times X) = \mathcal{M}(\mathcal{A}, \Gamma) \times X.
$$

\hfill \Box

**7.4. Virtual Cycles.** In this section we will show how to compare the virtual cycle of $\mathcal{M}(\mathcal{A}, \Gamma)$ to that of $\mathcal{M}(\mathcal{A}, \Gamma) \times X$.

Let us first note that $\mathcal{M}$ has no obstructions. Let $V \rightarrow \mathcal{A}^{rel}$ be some family. Then consider the Deligne-Mumford stack $\mathcal{M} \times_{\mathcal{A}^{rel}} V$. We can write down the tangent/obstruction complex for $\mathcal{M} \times_{\mathcal{A}^{rel}} V$ following the recipe in [12]. We see that $h^2(E^\bullet) = 0$, so

$$
[\mathcal{M} \times_{\mathcal{A}^{rel}} V]^{vir} = \mathcal{M} \times_{\mathcal{A}^{rel}} V
$$
Now let us look at the stack $\mathcal{M}(\mathcal{A}, \Gamma) \times_{\text{A}^{\text{rel}}} \mathcal{M}T$. We consider a particular kind of étale chart. Let $S \to \mathcal{M}(\mathcal{A}, \Gamma)$ be étale. Then let $W \to S \times_{\text{A}^{\text{rel}}} \mathcal{M}T$ be an étale chart. Then we have the following composition of étale morphisms

$$W = S \times_S W \to S \times_S (S \times_{\text{A}^{\text{rel}}} \mathcal{M}T) = S \times_{\text{A}^{\text{rel}}} \mathcal{M}T \to \mathcal{M}(\mathcal{A}, \Gamma) \times_{\text{A}^{\text{rel}}} \mathcal{M}T$$

If we shrink $W$ as above, we can obtain nice families

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f_*} & \mathbb{A}[N] \\
\downarrow & & \downarrow \\
W & \xrightarrow{h} & \mathbb{A}^N
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{f_T} & \mathbb{A}[N] \\
\downarrow & & \downarrow \\
S & \xrightarrow{h} & \mathbb{A}^N
\end{array}$$

Now, the tangent/obstruction complex on $W = S \times_S W$ splits into contributions from $S \to \mathcal{M}(\mathcal{A}, \Gamma)$ and $W \to \mathcal{M}T$. Moreover, the induced tangent/obstruction complex is identical to the one given by considering $W \to \mathcal{M}(\mathcal{A}, \Gamma)$. Therefore, we obtain a cone cycle $C \in Z_*(E = E_\mathcal{A} \oplus E_T)$ over $W$ where

$$C = C_\mathcal{A} \times_S C_T$$

where $C_\mathcal{A} \in Z_*(E_\mathcal{A})$ is the cone cycle on $S$ and $C_T \in Z_*(E_T)$ on $W$. By the naturality of the Gysin map (see [2] 6.5),

$$s_E^i(C_\mathcal{A} \times_S C_T) = (s_E^i C_\mathcal{A}) \times_S W$$

Globally, this construction tells us that

$$[\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}} = [\mathcal{M}(\mathcal{A}, \Gamma) \times_{\text{A}^{\text{rel}}} \mathcal{M}T]^{\text{vir}} = [\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}} \times_{\text{A}^{\text{rel}}} [\mathcal{M}T]$$

Therefore, if we consider the morphism,

$$v : \mathcal{M}(\mathcal{A}, \Gamma) \to \mathcal{M}(\mathcal{A}, \Gamma) \times \mathcal{X}$$

induced from the degree $\frac{1}{r}$ morphism

$$\mathcal{M}T \to \mathcal{A}^{\text{rel}} \times \mathcal{X},$$

we have

$$v_*([\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}}) = v_*([\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}} \times_{\text{A}^{\text{rel}}} [\mathcal{M}T]) = \frac{1}{r} [\mathcal{M}(\mathcal{A}, \Gamma)]^{\text{vir}} \times [\mathcal{X}].$$

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