Behavior near walls in the mean-field approach to crowd dynamics*

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Abstract

This paper introduces a system of stochastic differential equations (SDE) of mean-field type that by means of sticky boundaries and boundary diffusion accounts for the possibility of pedestrians to spend time at, and to move along, walls. As an alternative to Neumann-type boundary conditions, sticky boundaries and boundary diffusion have a ‘smoothing’ effect on pedestrian motion. When these effects are active, pedestrian paths are semimartingales with first-variation part absolutely continuous with respect to Lebesgue measure \( dt \), rather than an increasing processes (which in general induces a measure singular with respect to \( dt \)) as is the case under Neumann boundary conditions. We show that the proposed mean-field model for pedestrian motion admits a unique weak solution and that it is possible to control the system in the weak sense, using a Pontryagin-type maximum principle. We also relate the mean-field type control problem to the social cost minimization in an interacting particle system. We study the novel model features numerically and we confirm empirical findings on pedestrian crowd motion in congested corridors.

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1 Introduction

Models for pedestrian motion in confined domains must consider interaction with solid obstacles such as pillars and walls. The pedestrian response to a restriction of movement has been included into crowd models either as boundary

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conditions or repulsive forces. Up to now, the Neumann condition and its variants (no-flux) have been especially popular among the boundary conditions. The Neumann condition suffers from a drawback related to its microscopic (pathwise) interpretation. A Neumann condition on the crowd density corresponds to pedestrian paths reflecting in the boundary. In reality, pedestrians do not bounce off walls in the manner of classical Newtonian particles, but their movement is slowed down by the impact and a positive amount of time is needed to choose a new direction of motion. It is natural to think that whenever a pedestrian is forced (or decides) to make contact with a wall, she stays there for some time. During this time, she can move and interact with other pedestrians, before re-entering the interior of the domain.

### 1.1 Mathematical modeling of pedestrian-wall interaction

Today there is more than one conventional approach to the mathematical modeling of pedestrian motion. This section aims to summarize how they incorporate the interaction between pedestrians and walls.

Microscopic force-based models, among which the social force model has gained the most attention, describes pedestrians as Newton-like particles. From the initial work [22] and onward, the influence a wall has on the pedestrian is modelled as a repulsive force. The shape of the corresponding potential has been studied experimentally, for example in [28]. The cellular automata is another widely used microscopic approach to pedestrian crowd modeling. Walls are modeled as cells to which pedestrians cannot transition, already the original work [25] considers this viewpoint. In the continuum limits of cellular automata, as for example in [11, 10], boundary conditions are often set to no-flux conditions of the same type as (1) below.

The focus of macroscopic models is the global pedestrian density, either in a stationary or a dynamic regime. Inspired by fluid dynamics [24] treats the crowd as a ‘thinking fluid’ that moves at maximum speed towards a common target while taking environmental factors into account, such as the congestion of the crowd. In this category of models, boundary conditions at impenetrable walls are most often implemented as Neumann conditions for the pedestrian density. The pathwise interpretation of a Neumann boundary condition is instantaneous reflection. A nonlocal projection of pedestrian velocity in normal and tangential direction of the boundary respectively is suggested in [5] and implemented in [6], allowing for nonlocal interaction with boundaries.

Mean-field games and mean-field type control/games are macroscopic models of rational pedestrians with the ability to anticipate crowd movement, and adapt accordingly. These models can capture competition between individuals as well as crowd/sub-crowd cooperation. In the mean-field approach to pedestrian crowd modeling pedestrian-to-pedestrian interaction is assumed to be symmetric and weak, thus plausibly replaced by an interaction with a mean field (typically a functional of the pedestrian density). One of the most attractive features of the mean-field approach is that it has both a macroscopic (pedestrian density) and a microscopic (pedestrian path) interpretation. The
microscopic interpretation is valuable from the applied point of view, it allows us to study the crowd density and draw conclusions about individual behavior and vice versa. In what follows, the crowd density is denoted by $m$. In [26], the density is subjected to $n(x) \cdot \nabla m(t, x) = 0$ at walls, where $n(x)$ is the outward normal at $x$. Under this constraint, pedestrian velocity is zero at any wall. Taking conservation of probability mass into account, [9] derives the following boundary condition

$$- n(x) \cdot (\nabla m(t, x) - G(m)v(t, x)) = 0,$$

(1)

where $G(m)v$ is a general form of the pedestrian velocity. The constraint (1) represents reflection at the boundary since in the corresponding microscopic interpretation pedestrians make a classical Newtonian bounce whenever they hit the boundary. The same type of constraint is used in [2]. The case of several interacting populations in a bounded domain with reflecting boundaries has been studied in the stationary and dynamic case [13, 1, 4]. In these papers, the crowd density at walls is constrained by

$$n(x) \cdot (\nabla m(t, x) + m(t, x) \partial_p H(x, \nabla u)) = 0.$$

The term $-\partial_p H(x, \nabla u)$ is the velocity when the mean-field equilibrium strategy is implemented by the pedestrians, the constraint is a reflection.

1.2 Sticky reflected stochastic differential equations

It is shown in [19] a particular system of SDEs, called sticky reflected Brownian motion, admits a unique weak solution (but no strong solution) and that the corresponding coordinate process is sojourned at the boundary; the boundary is ’sticky’. The fact that the system has no strong solution has consequences for how optimal control of the system can be approached, as we will see in this paper.

Interacting systems of sticky reflected Brownian motions are considered in [20]. Interaction is introduced via a Girsanov transformation. See [20, Sect. 3.2] for the construction. Under assumptions on the ’shape’ of the interaction and integrability of the Girsanov kernel, the interacting system is well-defined. The boundary behavior is shown to be sticky in the sense that the process spends a positive time on the boundary.

Sticky reflected SDEs with boundary diffusion are considered in [21]. The paths defined by such a system are allowed to move on the (sufficiently smooth) boundary $\Gamma$ of some bounded domain $\mathcal{D} \subset \mathbb{R}^d$. Under smoothness conditions on $\Gamma$, the authors show that this type of equations has a unique weak solution. Furthermore, an interacting system is studied, where interaction is introduced via a Girsanov transformation.

1.3 Synopsis

In this paper, the sticky reflected SDE with boundary diffusion of [21] is proposed as a model for pedestrian crowd motion in confined domains. We begin by...
considering a (non-transformed) sticky reflected SDE with boundary diffusion on \( D \), a non-empty bounded subset of \( \mathbb{R}^n \) with \( C^2 \)-smooth boundary \( \Gamma := \partial D \) (see Section 2.2 below) and outward normal \( n \),

\[
\begin{align*}
dX_t = (1_D(X_t) + 1_\Gamma(X_t) \pi(X_t)) dB_t - 1_\Gamma(X_t) \left( \frac{1}{2} \frac{n(X_t)}{\gamma} + \frac{\kappa(X_t)}{2} \right) n(X_t) dt, 
\end{align*}
\]

where \( \pi \) is the projection operator on the tangential direction of \( \Gamma \), \( \kappa \) is the mean curvature of \( \Gamma \), and \( \gamma \) is a positive constant representing the stickiness of \( \Gamma \), cf. Remark 1 in Section 3 below. All relevant technical details can be found in Section 2. Equation (2) admits a unique weak solution, but no strong solution. To control an equation admitting only a weak solution is to control the underlying probability measure, under which the state process \( X \cdot := \{X_t\}_{t \in [0,T]} \) is interpreted as the coordinate process. The Girsanov theorem can be applied if all admissible measures are absolutely continuous with respect the reference measure \( \mathbb{P} \). This accounts for the case of controlled drift. In the controlled diffusion case, admissible measures are all singular with \( \mathbb{P} \) and with one another (for different controls), and the control problem is in fact a robustness problem over all admissible measures which leads to the so-called second order backward SDE framework [30]. In this paper we treat the case with controlled drift, the controlled diffusion case will be treated elsewhere. A mean-field dependent drift \( \beta \) is introduced into the coordinate process through the Girsanov transformation

\[
\begin{align*}
d\mathbb{P}^u_t \big| \mathcal{F}_t &= L_t^u := \mathcal{E}_t \left( \int_0^t \beta(t, X, \mathbb{P}^u(t), u_t)^* dB_t \right),
\end{align*}
\]

where \( \mathbb{P}^u(t) := \mathbb{P}^u \circ X_t^{-1} \) is the marginal distribution of \( X_t \) under \( \mathbb{P}^u \), \( \beta^* \) denotes the transpose of \( \beta \), and \( \mathcal{E} \) is the Doléans-Dade exponential defined for a continuous local martingale \( M \) as

\[
\mathcal{E}_t(M) := \exp \left( M_t - \frac{1}{2} \langle M \rangle_t \right). \tag{3}
\]

The path of a typical pedestrian in the interacting crowd is then (under \( \mathbb{P}^u \)) described by

\[
\begin{align*}
dX_t = 1_D(X_t) \left( \beta(t, X, \mathbb{P}^u(t), u_t) dt + dB^u_t \right) \\
+ 1_\Gamma(X_t) \left( \pi(X_t) \beta(t, X, \mathbb{P}^u(t), u_t) - \frac{n(X_t)}{2\gamma} \right) dt \\
+ 1_\Gamma(X_t) dB^\Gamma_{t,u} \\
\text{d}B^u_{t,u} = \pi(X_t) dB^u_t - \frac{1}{2} \kappa(X_t) n(X_t) dt,
\end{align*}
\]

where \( B^u \) is a \( \mathbb{P}^u \)-Brownian motion. We provide a proof of the existence of the controlled probability measure \( \mathbb{P}^u \) based on a fixed-point argument involving the total variation distance (cf. [16]).
Pedestrians are assumed to be cooperating and controlled by a rational central planner. This setup is used in the modeling of the most cooperative crowds. For a discussion on the degrees of cooperation in a pedestrian crowd, see [14]. The central planner’s goal is to minimize the cost functional

\[ J(u) := E^u \left[ \int_0^T f(t,X_u(t),u_t) \, dt + g(X_T,P^u(T)) \right]. \]  

(5)

The minimization of (5) subject to (4) is equivalent to the following mean-field type control problem, stated in the strong sense in the original probability space with measure \( P_u \),

\[
\begin{align*}
\inf_{u \in \mathcal{U}} E \left[ \int_0^T L^u_t f(t,X_u(t),u_t) \, dt + L^u_T g(X_T,P^u(T)) \right], \\
\text{s.t. } dL^u_t = L^u_t \beta(t,X_u(t),u_t) \, dB_t, \quad L^u_0 = 1.
\end{align*}
\]  

(6)

Problem (6) is nowadays a standard mean-field type control problem and a stochastic maximum principle yielding necessary conditions for an optimal control can be found in [8]. Solving the general problem (6) with a Pontryagin-type maximum principle poses some practical difficulties, the main one being the necessity of a second order adjoint process. However, most difficulties can be tackled by imposing assumptions plausible for the application in pedestrian crowd motion. With the aim to replicate the pedestrian behavior observed in the empirical studies [33] and [34], we consider here a special case of (6) where \( P^u(t) \) is replaced by \( E^u[r_\phi(X_t)] \) in \( \phi \in \{b,f,g\} \) and where \( u_t \) takes values in a convex set.

1.4 Paper contribution and outline

The main contribution of this paper is a new approach to boundary conditions in pedestrian crowd modeling. Sticky reflected SDEs of mean-field type with boundary diffusion is proposed as an alternative to reflected SDEs of mean-field type to model pedestrian paths in optimal-control based models. Sticky boundaries and boundary diffusion allow the pedestrian to spend time and move along the boundary (walls, pillars, etc.), in contrast to reflected SDE-based models where pedestrians are immediately reflected. Existence and uniqueness of the mean-field type version of the sticky reflected SDE with boundary diffusion is treated. The model can be optimally controlled (in the weak sense) and a Pontryagin-type stochastic maximum principle is applied to derive necessary optimality conditions. Furthermore, the mean-field type control problem has a microscopic interpretation in the form of a system of interacting sticky reflected SDEs with boundary diffusion. The new features of sticky boundaries and boundary diffusion yield more flexibility when modeling pedestrian behavior at boundaries. A scenario of unidirectional pedestrian flow in a long narrow corridor is studied numerically to highlight these novel characteristics and to replicate experimental findings as a first step in model validation.
The rest of the paper is organized as follows. Section 2 defines notation and summarizes relevant background theory. Section 3 introduces sticky reflected SDEs of mean-field type with boundary diffusion. Conditions under which the equation has a unique weak solution are presented. In Section 4 the finite horizon optimal control of the state equation introduced in Section 3 is considered. The relationship to the minimization of the social cost of an interacting (non-mean-field) particle system is discussed. Finally, Section 5 presents analytic examples and numerical results concerning unidirectional flow in a long narrow corridor.

2 Preliminaries

The domain \( D \) is a non-empty bounded subset of \( \mathbb{R}^d \) with \( C^2 \)-smooth boundary \( \Gamma := \partial D \). The closure of \( D \) is denoted \( \bar{D} \). The Euclidean norm is denoted \( |\cdot| \).

2.1 The coordinate process and probability metrics

Let \( \Omega := C([0, T]; \mathbb{R}^d) \) be endowed with the uniform metric, \( |\omega|_T := \sup_{t \in [0, T]} |\omega(t)| \) for \( \omega \in \Omega \). Denote by \( \mathcal{F} \) the Borel \( \sigma \)-field over \( \Omega \). Given \( t \in [0, T] \) and \( \omega \in \Omega \), put \( X_t(\omega) = \omega(t) \) and denote by \( \mathcal{F}_t^0 := \sigma(X_s; s \leq t) \) the filtration generated by \( X \). \( X \) is the so-called coordinate process. For any \( m \in \mathcal{P}(\Omega) \) (the set of probability measures on \( \Omega \)) we denote by \( \mathcal{F} := (\mathcal{F}_t; t \in [0, T]) \) the completion of \( \mathcal{F}_0 := (\mathcal{F}_t^0; t \in [0, T]) \) with the \( m \)-null sets of \( \Omega \). It will be clear from the context which \( m \) is meant.

Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) and let \( \mathcal{B}(\mathbb{R}^d) \) be the Borel algebra on \( \mathbb{R}^d \). The total variation metric on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) is

\[
d_{TV}(\mu, \nu) := 2 \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |\mu(A) - \nu(A)|. \tag{7}
\]

On the filtration \( \mathcal{F} \), the total variation metric between \( m, m' \in \mathcal{P}(\Omega) \) is

\[
D_t(m, m') := 2 \sup_{A \in \mathcal{F}_t} |m(A) - m'(A)|, \quad 0 \leq t \leq T, \tag{8}
\]

and satisfies \( D_s(\mu, \nu) \leq D_T(\mu, \nu) \) for \( 0 \leq s \leq t \). Consider the coordinate process \( X \), then for \( m, m' \in \mathcal{P}(\Omega) \),

\[
d_{TV}(m \circ X_t^{-1}, m' \circ X_t^{-1}) \leq D_t(m, m'), \quad 0 \leq t \leq T.
\]

Endowed with the metric \( D_T \), \( \mathcal{P}(\Omega) \) is a complete metric (Polish) space. The total variation metric is connected to the Kullback-Leibler divergence through the Csiszár-Kullback-Pinsker inequality,

\[
D^2_t(m, m') \leq 2E[m \log (dm/dm')], \quad 0 \leq t \leq T. \tag{9}
\]
where $E^m$ denotes expectation with respect to $m$.

Let $(X, d)$ be a Polish metric space and let $p \in [1, \infty)$. Let $\mathcal{M}(\mu, \nu)$ denote the collection of all measures on $X \times X$ with marginals $\mu$ and $\nu$. The Wasserstein distance of order $p$ between $\mu, \nu \in \mathcal{P}_p(X)$ is defined by the formula

$$W_p(\mu, \nu) := \left( \inf_{\Pi \in \mathcal{M}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} d(x, y)^p d\Pi(x, y) \right)^{1/p}$$

$$= \inf \left\{ E[d(X, Y)^p]^{1/p} : \text{law}(X) = \mu, \text{law}(Y) = \nu \right\}.$$  \hspace{1cm} (10)

Let $\mathcal{P}_p(X)$ denote the set of all probability measures $\mu \in \mathcal{P}(X)$ such that $\|\mu\|_p^p = \int d(y_0, y)^p \mu(dy) < \infty$ for an arbitrary $y_0 \in X$. Then $W_p$ defines a finite distance on $\mathcal{P}_p(X)$ and $(\mathcal{P}_p(X), W_p)$ is a Polish space. Being defined by an infimum, the Wasserstein distance between $\mu$ and $\nu$ is bounded from above by any coupling $\Pi$ between $\mu$ and $\nu$:

$$W_p(\mu, \nu) \leq E[d(X, Y)^p],$$

for $X$ and $Y$ that are $\mu$- and $\nu$-distributed, respectively. The Wasserstein distance is controlled by the total variation on bounded sets, if $\mu_B, \nu_B$ are probability measures on a bounded set $B \subset \mathbb{R}^d$ then

$$W_p(\mu_B, \nu_B) \leq C d_{TV}(\mu_B, \nu_B).$$

where $C$ depends on $p$ and $\sup_{b \in B} |b|$ [31, Thm. 6.18].

### 2.2 Boundary diffusion

In this subsection we introduce the boundary diffusion $B^\Gamma$ and review the necessary parts of the background theory presented in [21, Sect. 2].

**Definition 1.** $\Gamma$ is Lipschitz continuous (resp. $C^k$-smooth) if for every $x \in \Gamma$ there exists a neighborhood $V \subset \mathbb{R}^d$ of $x$ such that $\Gamma \cap V$ is the graph of a Lipschitz continuous (resp. $C^k$-smooth) function and $\mathcal{D} \cap V$ is located on one side of the graph, i.e., there exists new orthogonal coordinates $(y_1, \ldots, y_d)$ given by an orthogonal map $T$, a reference point $z \in \mathbb{R}^{d-1}$, real numbers $r, h > 0$, and a Lipschitz continuous (resp. $C^k$-smooth) function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

(i) $V = \{ y \in \mathbb{R}^d : |y_d - z| < r, |y_d - \varphi(y_d)| < h \}$

(ii) $\mathcal{D} \cap V = \{ y \in V : -h < y_d - \varphi(y_d) < 0 \}$

(iii) $\Gamma \cap V = \{ y \in V : y_d = \varphi(y_d) \}$

**Definition 2.** For $y \in V$, let

$$\vec{n}(y) := \frac{(-\nabla \varphi(y_d), 1)}{\sqrt{|\nabla \varphi(y_d)|^2 + 1}}.$$  \hspace{1cm} (i)

Let $x \in \Gamma$ and $T \in \mathbb{R}^{d \times d}$ be the orthogonal transformation from Definition [4]. Then the outward normal vector at $x$ is defined by $n(x) := T^{-1}\vec{n}(Tx)$.  \hspace{1cm} (ii)
Definition 3. Let $x \in \Gamma$ and $\pi(x) := E - n(x)n(x)^* \in \mathbb{R}^{d \times d}$, where $E$ is the identity matrix. $\pi(x)$ is the orthogonal projection on the tangent space at $x$.

Note that for $z \in \mathbb{R}^d$, $\pi(x)z = z - (n(x), z)n(x)$.

Definition 4. Let $f \in C^1(\overline{D})$ and $x \in \Gamma$. Whenever $\Gamma$ is sufficiently smooth at $x$, $\nabla \Gamma f(x) := \pi(x)\nabla f(x)$ and if $f \in C^2(\overline{D})$, $\Delta \Gamma f(x) := \text{Tr}(\nabla^2 f(x))$. If $n$ is differentiable at $x$ the mean curvature of $\Gamma$ at $x$ is $\kappa(x) := \text{div}n(x) = (\pi(x)\nabla) \cdot n(x)$.

In [21] it is noted that whenever $\Gamma$ is $C^2$-smooth, $(\pi \nabla)^* \pi = -\kappa n$.

A Brownian motion $B^\Gamma$ on a smooth boundary $\Gamma$ is a $\Gamma$-valued stochastic process generated by $\frac{1}{2} \Delta \Gamma$. This is in analogy with the standard Brownian motion on $\mathbb{R}^d$, in the sense that $B^\Gamma$ solves the martingale problem for $(\frac{1}{2} \Delta \Gamma, C^\infty(\Gamma))$. A solution to the Stratonovich SDE
$$
\frac{d B^\Gamma_t}{dB_t} = \frac{1}{2} \kappa(B^\Gamma_t)n(B^\Gamma_t)dt + \pi(B^\Gamma_t)dB_t.
$$

where $B.$ is a standard Brownian motion on $\mathbb{R}^d$, is a Brownian motion on $\Gamma$ [23 Chap. 3, Sect. 2]. By the Itô-Stratonovich transformation rule, the Brownian motion on $\Gamma$ solves
$$
\frac{d B_t^\Gamma}{dB_t} = -\frac{1}{2} \kappa(B_t^\Gamma)n(B_t^\Gamma)dt + \pi(B_t^\Gamma)dB_t.
$$

3 Sticky reflected SDEs of mean-field type with boundary diffusion

In this section we provide conditions for the existence and uniqueness of a weak solution to the sticky reflected SDE of mean-field type with boundary diffusion.

Consider the reflected sticky SDE with boundary diffusion,

\[
\begin{cases}
    dX_t = -1_{\Gamma}(X_t)\frac{1}{2} \left( \frac{1}{\gamma} + \kappa(X_t) \right) n(X_t)dt \\
    \quad \quad \quad \quad + (1_{D}(X_t) + 1_{\Gamma}(X_t)\pi(X_t)) dB_t, \\
    X_0 = x_0 \in \bar{D},
\end{cases}
\]

which from now on will be written in short-hand notation as

\[
    dX_t = a(X_t)dt + \sigma(X_t)dB_t,
\]

where $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are bounded functions over $[0, T] \times \bar{D}$, defined as

\[
a(x) := -1_{\Gamma}(x)\frac{1}{2} \left( \frac{1}{\gamma} + \kappa(x) \right) n(x), \quad \sigma(x) := 1_{D}(x) + 1_{\Gamma}(x)\pi(x).
\]

By [21] there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ under which the coordinate process satisfies (12) and is $C([0, T]; \bar{D})$-valued $\mathbb{P}$-a.s.
Remark 1. Note that the coordinate process is composed of three essential parts:

- A Brownian motion in the interior $\pi_t(X_t)dB_t$.
- Boundary diffusion
  $$1_t(X_t)(\pi(X_t)dB_t - \frac{1}{2}(\kappa u)(X_t)dt) = 1_t(X_t)d\Gamma_t,$$
- Normal sticky reflection
  $$-\frac{1}{2}\Gamma_t(X_t)\frac{1}{2}(\gamma n)(X_t)dt.$$

The constant $\gamma$ is connected to the level of stickiness of the boundary $\Gamma$. It is related to the invariant distribution of the coordinate processes’ $\mathbb{R}^d$-valued time marginal. Let $\lambda$ and $s$ denote the Lebesgue measure on $\mathbb{R}^d$ and the surface measure on $\Gamma$, respectively. Consider the measure
$$\rho := 1_D^\alpha \lambda + 1_\Gamma^\alpha' s,$$ where $\alpha, \alpha' \in \mathbb{R}$.

By choosing $\alpha = \bar{\alpha}/\lambda(D)$ and $\alpha' = (1 - \bar{\alpha})/s(\Gamma)$, $\bar{\alpha} \in [0, 1]$, $\rho$ becomes a probability measure on $\mathbb{R}^d$ with support in $\bar{D}$ and $\rho$ is in fact the invariant distribution of (12) whenever
$$\gamma = \bar{\alpha} s(\Gamma)/(1 - \bar{\alpha}) \lambda(D).$$

Hence $\bar{\alpha} \to 1$ as $\gamma \to 0$ and the invariant distribution of (12) concentrates on the interior $D$. But as $\gamma \to \infty$, it concentrates on the boundary $\Gamma$. We say that the more probability mass that $\rho$ locates on $\Gamma$, the stickier $\Gamma$ is.

Next, we introduce mean-field interactions and a control process in (12) through a Girsanov transformation.

Definition 5. Let the set of control values $U$ be a compact subset of $\mathbb{R}^d$. The set of admissible controls is
$$U := \{u : [0, T] \times \Omega \to U : u \text{ $\mathbb{F}$-prog. measurable}\}.$$ (14)

Let $Q(t) := Q \circ X_t^{-1}$ denote the $t$-marginal distribution of the coordinate process under $Q \in \mathcal{P}(\Omega)$. Let $\beta$ be a measurable function from $[0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U$ into $\mathbb{R}^d$ such that

Assumption 1. The process $(\beta(t, X_t, Q_t, u_t))_{t \in [0, T]}$ is progressively measurable for every $Q \in \mathcal{P}(\Omega)$ and $u \in U$.

Assumption 2. For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$,
$$|\beta(t, \omega, \mu, u)| \leq C(1 + |\omega|_T + \int_{\mathbb{R}^d} |y|\mu(dy)).$$

Assumption 3. For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$,
$$|\beta(t, \omega, \mu, u) - \beta(t, X_t, \mu', u)| \leq C_{TV}(\mu, \mu').$$

Given $Q \in \mathcal{P}(\Omega)$ and $u \in U$, let
$$L_t^{u, Q} := E_t \left( \int_0^t \beta(s, X_s, Q(s), u_s)dB_s \right),$$ (15)
where $E$ is the Doléans-Dade exponential (cf. (3)).
Lemma 1. The positive measure $\mathbb{P}^{u,\varnothing}$ defined by $d\mathbb{P}^{u,\varnothing} = L_t^{u,\varnothing} d\mathbb{P}$ on $\mathcal{F}_t$ for all $t \in [0, T]$, is a probability measure on $\Omega$. Moreover, under $\mathbb{P}^{u,\varnothing}$ the coordinate process satisfies

$$X_t = x_0 + \int_0^t (\sigma(X_s) \beta(s, X_s, Q(s), u_s) + a(X_s)) ds + \int_0^t \sigma(X_s) dB_s^Q, \quad (16)$$

where $B^Q$ is a standard $\mathbb{P}^{u,\varnothing}$-Brownian motion.

Proof. Assume that $\varphi$ is a process such that $\mathbb{P}^\varphi$, defined by $d\mathbb{P}^\varphi = L_t^\varphi d\mathbb{P}$ on $\mathcal{F}_t$ where $L_t^\varphi := E_t(\int_0^\infty \varphi_s dB_s)$, is a probability measure on $\Omega$. By Girsanov’s theorem, the coordinate process under $\mathbb{P}^\varphi$ satisfies

$$dX_t = (\sigma(X_t) \varphi_t + a(X_t)) dt + \sigma(X_t) dB_t^\varphi,$$

where $B^\varphi$ is a $\mathbb{P}^\varphi$-Brownian motion. $C^2$-smoothness of the boundary $\Gamma$ grants a bounded orthogonal projection on $\Gamma$’s tangent space and a bounded mean curvature of $\Gamma$, and by the Burkholder-Davis-Gundy inequality

$$E^\varphi[|X|^p] \leq E^\varphi \left[ C \left( |X_0|^p + \int_0^T |\sigma(X_s) \varphi_s|^p ds + \int_0^T |a(X_s)|^p ds 
+ \left| \int_0^T \sigma(X_s) dB_s^\varphi \right|^p_T \right) \right]
\leq C \left( 1 + \int_0^T E^\varphi[|\varphi_s|^p] ds \right), \quad (17)$$

where $E^\varphi$ denotes expectation taken under $\mathbb{P}^\varphi$. Consider the truncated drift

$$\beta_n(t, \omega, \mu, u) := \beta(t, \omega, \mu, u) 1_{|\omega|_t \leq n}.$$

By Assumption (3) it holds for every $t \in [0, T]$, $\omega \in \Omega$, $\mu \in \mathcal{P}(\mathbb{R}^d)$, and $u \in U$ that

$$|\beta_n(t, \omega, \mu, u)| \leq C \left( d_{TV}(\mu, \mathbb{P}(t)) + |\beta_n(t, \omega, \mathbb{P}(t), u)| \right), \quad (18)$$

Recall that under $\mathbb{P}$, the coordinate process is almost surely $C([0, T]; \bar{D})$-valued. This, (18), the fact that the total variation between two probability measures is uniformly bounded, and Assumption (3) yields that $\beta_n$ is bounded,

$$|\beta_n(t, \omega, \mu, u)| \leq C \left( 1 + C \left[ 1 + |\omega|_t + \int_{\mathbb{R}^d} |y| \mathbb{P}(t)(dy) \right] 1_{|\omega|_t \leq n} \right)
\leq C \left( 1 + |\omega|_t 1_{|\omega|_t \leq n} + E \|X_t\| \right) \leq C \left( n, \sup_{y \in \bar{D}} |y| \right).$$

Consider the probability measure $\mathbb{P}_{n,\varnothing}^{u,\varnothing}$ given (on $\mathcal{F}_t$) by the Girsanov transformation

$$d\mathbb{P}_{n,\varnothing}^{u,\varnothing} = \mathcal{E}_t \left( \int_0^t \beta_n(s, X_s, Q(s), u_s) dB_s \right) d\mathbb{P}, \quad (19)$$
and let $E^{u,Q}_n$ denote expectation under $\mathbb{P}^{u,Q}_n$. Since $\beta_n$ is bounded, the Doléans-Dade exponential in (19) an $(\mathcal{F}_t, \mathbb{P})$-martingale, so $\mathbb{P}^{u,Q}_n$ is indeed a probability measure, i.e. $\mathbb{P}^{u,Q}_n \in \mathcal{P}(\Omega)$.

In view of (17), Assumption 2 and 3, and the fact that the total variation between two probability measures is uniformly bounded,

$$E^{u,Q}_n [ |X|_T^p ] \leq C \left( 1 + \int_0^T E^{u,Q}_n [ |\beta(s, X(s), u_s)|^p ] ds \right) \leq C \left( 1 + \int_0^T E^{u,Q}_n [ |\beta(s, X(s), \mathbb{P}^{u,Q}_n(s), u_s)|^p ] ds \right) \leq C \left( 1 + \int_0^T E^{u,Q}_n [ |X|^p_s + E^{u,Q}_n [ |X|^p_s ] ] ds \right) \leq C \left( 1 + \int_0^T E^{u,Q}_n [ |X|^p_s ] ds \right).$$

By Grönwall’s lemma,

$$E^{u,Q}_n [|X|^p_T] \leq C(p),$$

where $C(p)$ depends only on $p, T$, the Lipschitz and the linear growth constant of $\beta$, and $\sup_{y \in \bar{D}} |y|$. Thus, by the same lines as the proof of Proposition (A.1) in [18], the likelihood $L^{u,Q}$ defined by (15) is a martingale for every $u \in \mathcal{U}$ and $Q \in \mathcal{P}(\Omega)$, hence $\mathbb{P}^{u,Q} \in \mathcal{P}(\Omega)$. Finally, by Girsanov’s theorem the coordinate process under $\mathbb{P}^{u,Q}$ satisfies (16).

For a given $u \in \mathcal{U}$, consider the map

$$\Phi : \mathcal{P}(\Omega) \ni Q \mapsto \mathbb{P}^{u,Q} \in \mathcal{P}(\Omega).$$

**Proposition 1.** The map $\Phi$ is well-defined and admits a unique fixed point. Moreover, for every $p \geq 2$, the fixed point, denoted $\mathbb{P}^u$, belongs to $\mathcal{P}_p(\Omega)$, i.e.

$$E^u [|X|^p_T] \leq C_p < \infty,$$

where the constant $C_p$ depends only on $p, T$, the Lipschitz and the linear-growth constant of $\beta$, and $\sup_{y \in \bar{D}} |y|$.

**Proof.** By Lemma 1, the mapping is well defined. We show the contraction property of the map $\Phi$ in the complete metric space $\mathcal{P}(\Omega)$, endowed with the total variation distance $D_T$. The proof is an adaptation of the proof of [12, Thm. 8]. For each $t \in [0, T]$, let $\beta^Q_t := \beta(t, X(t), Q(t), u_t)$. Given $Q, \tilde{Q} \in \mathcal{P}(\Omega)$, the Csiszár-Kullback-Pinsker inequality [9] and the fact that $\int_0^T (dB_s - \beta^Q_s ds)$ is
a martingale under $\Phi(Q)$ yields

$$D_T^2 \left( \Phi(Q), \Phi(\tilde{Q}) \right) \leq 2E^{\Phi(Q)} \left[ \log \left( \frac{L_Q T}{L_{\tilde{Q} T}} \right) \right]$$

$$= 2E^{\Phi(Q)} \left[ \int_0^T \left( \beta_Q^s - \beta_{\tilde{Q}}^s \right) dB_s - \frac{1}{2} \int_0^T \left( \beta_Q^s \right)^2 - \left( \beta_{\tilde{Q}}^s \right)^2 \right]$$

$$= 2E^{\Phi(Q)} \left[ \int_0^T \left( \beta_Q^s - \beta_{\tilde{Q}}^s \right) \beta_Q^s - \frac{1}{2} \left( \beta_Q^s \right)^2 + \frac{1}{2} \left( \beta_{\tilde{Q}}^s \right)^2 \right]$$

$$= \int_0^T E^{\Phi(Q)} \left[ \left( \beta_Q^s - \beta_{\tilde{Q}}^s \right)^2 \right] ds$$

$$\leq C \int_0^T d_{TV}(Q(s), \tilde{Q}(s)) ds \leq C \int_0^T D_s^2(Q, \tilde{Q}) ds.$$  \hfill (22)

Iterating the inequality, we obtain for every $N \in \mathbb{N}$,

$$D_T^2 \left( \Phi^N(Q), \Phi^N(\tilde{Q}) \right) \leq \frac{C^N T^N}{N!} D_T^2(Q, \tilde{Q}),$$

where $\Phi^N$ denotes the $N$-fold composition of $\Phi$ and $E^{\Phi(Q)}$ denotes expectation under $\Phi(Q)$. Hence $\Phi^N$ is a contraction for $N$ large enough, thus admitting a unique fixed point.

Under $P^u$, the fixed point of $\Phi$ given $u \cdot \in \mathcal{U}$, the coordinate process satisfies

$$dX_t = (\sigma(X_t) \beta(t, X_t, P^u(t), u_t) + a(X_t)) dt + \sigma(X_t) dB^u_t,$$

where $B^u$ is a $P^u$-Brownian motion. Following the calculations from Lemma 1 that lead to (21), we get the estimate

$$\|P^u\|_p = E^u [\|X\|_p^p] \leq C_p \left( 1 + E^u \left[ \int_0^T |X|^p_s ds \right] \right),$$

where $C_p$ depends only on $p$, $T$, the Lipschitz and the linear-growth constant of $\beta$, and $\sup_{y \in \bar{D}} |y|$. Grönwall’s inequality then yields $E^u [\|X\|_p^p] \leq C_p < \infty$. $\square$

From now on, we will denote the Brownian motion corresponding to $P^u$ by $B^u$. To summarize this section, we have proved the following result under Assumption 1.

**Theorem 1.** Given $u \in \mathcal{U}$, there exists a unique weak solution to the sticky reflected SDE of mean-field type with boundary diffusion

$$dX_t = (\sigma(X_t) \beta(t, X_t, P^u(t), u_t) + a(X_t)) dt + \sigma(X_t) dB^u_t.$$  \hfill (23)

Under $P^u$ the t-marginal distribution of $X$ is $P^u(t)$ for $t \in [0,T]$ and $X$ is almost surely $C([0,T]; \bar{D})$-valued. Furthermore, $P^u \in \mathcal{P}_p(\Omega)$. 12
Remark 2. The drift component $\beta$ is projected in the tangential direction of the boundary by $\sigma$ whenever the process is at the boundary (cf. (12)). The drift component $a$ is not effected by the transformation. From a modeling perspective, the interpretation is that the pedestrian’s tangential movement is partially controllable but also influenced by other pedestrians through the mean field. The normal direction is an uncontrolled delayed reflection.

4 A mean-field type optimal control model

Let $E^u$ denote expectation taken under $\mathbb{P}^u$. To apply the stochastic maximum principle of [7], we make the assumption that the mean-field type Girsanov kernel $\beta$ depends linearly on $\mathbb{P}^u$.

Assumption 4. Let $\tilde{\beta} : [0, T] \times \Omega \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ and let $r_\beta : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and assume that
\[
\beta(t, X, \mathbb{P}^u(t), u_t) = \tilde{\beta}(t, X, E^u[r_\phi(X_t)], u_t). 
\]
(24)

With some abuse of notation, we will continue to denote the Girsanov kernel by $\beta$, although from now this refers to $\tilde{\beta}$. Let $f : [0, T] \times \Omega \times \mathbb{R}^d \times U \rightarrow \mathbb{R}$, $g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $r_f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and $r_g : \mathbb{R}^d \rightarrow \mathbb{R}^d$.

Assumption 5. For every $u \in \mathcal{U}$, the process $(f(t, X, E^u[r_f(X_t)], u_t))_t$ is progressively measurable and $g(X_T, E^u[r_g(X_T)])$ is $\mathcal{F}_T$-measurable.

Consider the finite horizon mean-field type cost functional $J : \mathcal{U} \rightarrow \mathbb{R},$

\[
J(u) := E^u \left[ \int_0^T f(t, X, E^u[r_f(X_t)], u_t) \, dt + g(X_T, E^u[r_g(X_T)]) \right].
\]

The control problem considered in this section is the minimization of $J$ with respect to $u \in \mathcal{U}$ under the constraint that the coordinate process for any given $u$ satisfies (23). Integrating by parts,

\[
J(u) = E \left[ \int_0^T L^u_t f(t, X, E[L^u_t r_f(X_t)], u_t) \, dt + L^u_T g(X_T, E[L^u_T r_g(X_T)]) \right] 
\]
(25)

with $E$ being expectation taken under the original probability measure $\mathbb{P}$ and $L^u$ is the controlled likelihood process, given by the SDE of mean-field type

\[
dL^u_t = L^u_t \beta(t, X, E[L^u_t r_\beta(X_t)], u_t) \, dB_t, \quad L^u_0 = 1.
\]
(26)

4.1 Necessary optimality conditions

After making one final assumption about the regularity of $\beta$, $f$, and $g$ (Assumption 6 below), the stochastic maximum principle of [7, Thm 2.1] yields necessary conditions on an optimal control for the minimization of (25) subject to (26). Assumption [4] and 6 are stated in their current form for the sake of technical, not conceptual, simplicity and may be relaxed.
Assumption 6. The functions \((t, x, y, u) \mapsto (f, \beta)(t, x, y, u)\) and \((x, y) \mapsto g(x, y)\) are twice continuously differentiable with respect to \(y\). Moreover, \(\beta, f\) and \(g\) and all their derivatives up to second order with respect to \(y\) are continuous in \((y, u)\), and bounded.

Theorem 2. Assume that \((\hat{u}, L^\hat{u})\) solves the optimal control problem \([25]-[26]\. Then there are two pairs of \(\mathbb{F}\)-adapted processes, \((p, q)\) and \((P, Q)\), that satisfy the first and second order adjoint equations

\[
\begin{aligned}
dp_t &= - \left( q_t \beta_t^u + E \left[ q_t L_t^u \nabla_y \beta_t^u \right] r_\beta(X_t) - f_t^u - E \left[ L_t^u \nabla_y f_t^u \right] r_f(X_t) \right) dt + q_t dB_t, \\
p_T &= - g_T^u - E \left[ L_T^u \nabla_y g_T^u \right] r_g(X_T), \\
dP_t &= - \left( \beta_t^P + E \left[ L_t^P \nabla_y \beta_t^P \right] r_\beta(X_t) \right)^2 P_t \\
&\quad + 2Q_t \left( \beta_t^P + E \left[ L_t^P \nabla_y \beta_t^P \right] r_\beta(X_t) \right) dt + Q_t dB_t, \\
P_T &= 0,
\end{aligned}
\]

where \(\nabla_y\) denotes differentiation with respect to the \(\mathbb{R}^d\)-valued argument. Furthermore, \((p, q)\) and \((P, Q)\) satisfy

\[
E \left[ \sup_{t \in [0,T]} |p_t|^2 + \int_0^T |q_t|^2 dt \right] < \infty, \quad E \left[ \sup_{t \in [0,T]} |P_t|^2 + \int_0^T |Q_t|^2 dt \right] < \infty,
\]

and for every \(u \in U\) and a.e. \(t \in [0,T]\), it holds \(\mathbb{P}\)-a.s. that

\[
\mathcal{H} \left( L_t^u, u, p_t, q_t \right) - \mathcal{H} \left( L_t^\hat{u}, \hat{u}_t, p_t, q_t \right) + \frac{1}{2} \left[ \delta (L\beta) (t) \right]^T P_t \left[ \delta (L\beta) (t) \right] \leq 0, \quad (27)
\]

where \(\mathcal{H}(L_t^u, u, p_t, q_t):= L_t^u \beta_t^u q_t - L_t^u f_t^u \) and

\[
\delta(L\beta)(t) := L_t^u \left( \beta \ (t, X_t, E[L_t^u r_\beta(X_t)], u) - \beta_t^u \right).
\]

When \(U\) is a convex set the following useful sufficient condition for \([27]\) is at hand. If \(U\) is convex and if

\[
\mathcal{H} \left( L_t^u, u, p_t, q_t \right) - \mathcal{H} \left( L_t^\hat{u}, \hat{u}_t, p_t, q_t \right) \leq 0, \quad \mathbb{P}\text{-a.s.}
\]

for all \(u \in U\) and a.e. \(t \in [0,T]\), then \(\hat{u}\) is an optimal control for \([25]-[26] [32]\. [32].

Remark 3. Sufficient conditions for weak optimal controls will seldom be satisfied since they typically require the Hamiltonian to be convex (or concave) in at least state \((L_t^u)\) and control \((u)\). This is false even for the simplest version of our problem. Assume that \(\beta(t, \omega, y, u) = u\) and \(f = 0\), then \((\ell, u) \mapsto \mathcal{H}(\ell, u, p, q) = f u q\), which is neither convex nor concave. However, necessary optimality conditions can be useful as we will see in Section 7.
4.2 Microscopic interpretation of the mean-field type control problem

A system of interacting sticky reflected SDEs with boundary diffusion is considered [21]. Such a system can be used to give a microscopic interpretation of the mean-field type control problem (6) in the form of an interacting particle system (collaboratively) minimizing the social cost.

Consider the following system of \( N \in \mathbb{N} \) sticky reflected Brownian motions with boundary diffusion,

\[
dX^i_t = a(X^i_t)dt + \sigma(X^i_t)dB^i_t, \quad X^i_0 = x_i, \quad i = 1, \ldots, N, \tag{28}
\]

with the functions \( a \) and \( \sigma \) defined as in (13). There is a unique probability measure \( \mathbb{P}^N \) on \( (\Omega, \mathcal{F}) \), where \( \Omega := C([0,T]; \mathbb{R}^d) \) and \( \mathcal{F} \) is the corresponding filtration (cf. Section 2), under which the coordinate process \( X^i = (X^i_1, \ldots, X^i_T) \) satisfies (28) and is \( \mathbb{P}^N \)-a.s. Furthermore, \( \mathbb{P}^N \in \mathcal{P}_p(\Omega) \).

Given a strategy vector \( u := (u_1, \ldots, u_N) \in U^N \), let

\[
dL^u_{i,t} = L^u_{i,t} \beta(t, X^i_t, \mu^N_t, u^i_t) dB^i_t, \quad \mu^N_{t,0} = 1, \quad i = 1, \ldots, N,
\]

where \( \mu^N_t \) is the empirical measure of the coordinate process \( X^i \) at time \( t \),

\[
\mu^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}, \quad \mathcal{P}(\Omega).
\]

Let \( L^N_{i,u} := \prod_{i=1}^N L^u_{i,t} \). Then \( L^N_{i,u} \) yields a Girsanov-type change of measure from \( \mathbb{P}^N \) to \( \mathbb{P}^{N,u} \) by the arguments of Lemma 1. Under \( \mathbb{P}^{N,u} \) the coordinate processes are given by

\[
\begin{cases}
  dX^i_t = (\sigma(X^i_t) \beta(t, X^i_t, \mu^N_t, u^i_t) + a(X^i_t)) dt + \sigma(X^i_t) dB^i_t, \\
  X^i_0 = x^i,
\end{cases} \quad i = 1, \ldots, N, \tag{29}
\]

where \( (B^1,u, \ldots, B^N,u) \) is a \( \mathbb{P}^{N,u} \)-Brownian motion.

Let \( E^{N,u} \) denote expectation taken under \( \mathbb{P}^{N,u} \) and consider the social cost

\[
J_N(u) := \frac{1}{N} \sum_{i=1}^N E^{N,u} \left[ \int_0^T f(t, X^i_t, \mu^N_t, u^i_t) dt + g(X^i_T, \mu^N_T) \right] \tag{30}
\]

Results concerning the convergence of particle system (29) in the large population limit \( N \to \infty \) are based on the convergence properties of relaxed controls. In [17] the case of standard SDEs is treated and in [29] mean-field type SDEs without control. The case of controlled SDEs of mean-field type was recently treated in [27]. A minor modification of the proof of [27, Prop. 5.2], making use of the boundedness of \( D \) and \( U \), provides us with the microscopic interpretation: a mean-field type optimal control is \( \epsilon(N) \)-optimal for the social cost minimization problem (29)-(30), with \( \epsilon(N) \to 0 \) as \( N \to \infty \). We do not push further this analysis.
5 Examples

As a first step in model validation, experimental results on pedestrian speed profiles in a long narrow corridor are replicated in this section. The application of the proposed approach also displays the new features it offers regarding behavior near walls. From the necessary optimality conditions we derive an expression for the optimal control valid in following two toy examples and the corridor scenario.

Throughout the rest of this section it is assumed that the compact set $U$ is convex and sufficiently large so that all optimal control in the following analytical expressions are admissible. Furthermore, it is assumed that $(\hat{u}, L)^\alpha$ is optimal for the mean-field type control problem \cite{25-26}. We recall the first order adjoint equation,

\begin{equation}
\begin{cases}
dp_t = -\left(q_t \beta_t^{\hat{u}} + E \left[q_t L_t^{\hat{u}} \nabla g^{\hat{u}}_t \right] r_{\hat{u}}(X_t) \right)
- f_t^{\hat{u}} - E \left[L_t^{\hat{u}} \nabla f_t^{\hat{u}} \right] r_{\hat{u}}(X_t)) dt + q_t dB_t, \\
p_T = -g_T^{\hat{u}} - E \left[L_t^{\hat{u}} \nabla g_T^{\hat{u}} \right] r_{\hat{u}}(X_T).
\end{cases}
\end{equation}

Rewriting $E[L_t^{\hat{u}} Y_t] = E[\hat{u}]$ and changing measure to $\mathbb{P}^{\hat{u}}$, \eqref{eq:adjoint} becomes

\begin{equation}
\begin{cases}
dp_t = -A_t dt + q_t dB_t^\alpha, \\
p_T = -g_T^{\hat{u}} - E^{\hat{u}} \left[\nabla g_T^{\hat{u}} \right] r_{\hat{u}}(X_T),
\end{cases}
\end{equation}

where $A_t := E^{\hat{u}} \left[q_t \nabla \beta_t^{\hat{u}} \right] r_{\hat{u}}(X_t) - f_t^{\hat{u}} - E^{\hat{u}} \left[\nabla f_t^{\hat{u}} \right] r_{\hat{u}}(X_t)$. Since $(p, q)$ solves the backward SDE \eqref{eq:adjoint}, $p$ can be written as the conditional expectation

\begin{equation}
p_t = -E^{\hat{u}} \left[\nabla g_t^{\hat{u}} + E^{\hat{u}} \left[\nabla g_T^{\hat{u}} \right] r_{\hat{u}}(X_T) \mid \mathcal{F}_t \right] + E^{\hat{u}} \left[\int_t^T A_s ds \mid \mathcal{F}_t \right].
\end{equation}

Let

$$
\phi(X_t, \bar{g}_1(t), \bar{g}_2(t)) := g(X_t, \bar{g}_1(t)) + \bar{g}_2(t) r_{\hat{u}}(X_t),
$$

with $\bar{g}_1(t) := E^{\hat{u}}[r_{\hat{u}}(X_t)]$ and $\bar{g}_2(t) := E^{\hat{u}}[\nabla g_T^{\hat{u}}]$. By Dynkin’s formula,

\begin{align*}
E^{\hat{u}}[\phi(X_T, \bar{g}_1(T), \bar{g}_2(T)) \mid \mathcal{F}_t] &= \phi(X_t, \bar{g}_1(t), \bar{g}_2(t)) \\
&+ \int_t^T E^{\hat{u}} \left[(\mathcal{G} + \partial_s) \phi \mid \mathcal{F}_s \right] ds,
\end{align*}

where $\mathcal{G}$ is the generator of the coordinate process and $\partial_s$ denotes differentiation with respect to time, working on the two remaining arguments of $\phi$. Hence, by applying Itô’s formula on $p$ in \eqref{eq:conditional}, where only $X$ contributes to the diffusion part, and matching the diffusion parts of that and $p$ from \eqref{eq:adjoint}, we get

\begin{equation}
q_s = -\nabla_x \phi(X_s, \bar{g}_1(s), \bar{g}_2(s)) \sigma(X_s).
\end{equation}
The necessary optimality conditions state that for almost every \( t \in [0, T] \),
\[
q_t \nabla u_\tilde{u}_t = \nabla u_f, \quad \mathbb{P}\text{-a.s.} \tag{35}
\]
Since \( \mathbb{P}_\tilde{u} \) is absolutely continuous with respect to \( \mathbb{P} \), the equality above also holds for almost every \( t \in [0, T] \) \( \mathbb{P}_\tilde{u} \)-a.s. We have now at hand an expression for the optimal control whenever we can solve (34)-(35) for \( \tilde{u} \).

5.1 Linear-quadratic problems with convex \( U \)

5.1.1 A non-mean-field example
Let \( D \subset \mathbb{R}^d \) be an admissible domain and \( \mathbb{P} \) the probability measure on the space of continuous paths under which the coordinate process solves (12). Consider the following LQ problem on \( D \),
\[
\begin{align*}
\min_{u \in U} & \quad \frac{1}{2} \int_0^T L^u_t |u_t|^2 dt + L^u_T |X_T - x_T|^2 ,
\text{s.t.} & \quad dL^u_t = L^u_t u^*_t dB_t, \quad L^u_0 = 1,
\end{align*}
\]
where \( B \) is a \( \mathbb{P} \)-Brownian motion. The necessary optimality condition (35) yields
\[
\hat{u}_t = q^*_t, \quad \mathbb{P}\text{-a.s., a.e. } t \in [0, T]. \tag{36}
\]
Matching the diffusion coefficients gives us the optimal control,
\[
\hat{u}_t = -\sigma(X_t) (X_t - x_T), \quad \mathbb{P}\text{-a.s., a.e. } t \in [0, T]. \tag{37}
\]
The corresponding likelihood process solves
\[
dL_t^\hat{u} = -L_t^\hat{u} (X_t - x_T)^* \sigma(t, X_t) dB_t, \quad L_0^\hat{u} = 1,
\]
and under \( \mathbb{P}_\hat{u} \), the optimally controlled path distribution, the coordinate process solves
\[
\begin{align*}
dX_t &= a(X_t) dt + \sigma(X_t) dB_t \\
&= a(X_t) dt + \sigma(X_t) \left( -\sigma(X_t) (X_t - x_T) dt + dB_t^\hat{u} \right) \\
&= (a(X_t) - \sigma(X_t) (X_t - x_T)) dt + \sigma(X_t) dB_t^\hat{u}.
\end{align*}
\]
We have used the fact that \( \pi^2 = \pi = \pi^* \), which holds since \( \pi \) is an orthogonal projection.

5.1.2 A mean-field example
Consider now on some admissible domain \( D \subset \mathbb{R}^d \) the mean-field type optimal control problem
\[
\begin{align*}
\min_{u \in U} & \quad \frac{1}{2} \int_0^T L^u_t |u_t|^2 dt + L^u_T |X_T - E[L_T^u X_T]|^2 ,
\text{s.t.} & \quad dL^u_t = L^u_t u^*_t dB_t, \quad L_0^u = 1.
\end{align*}
\]
As before, $B$ is a $\mathbb{P}$-Brownian motion, where $\mathbb{P}$ is a probability measure on the path space under which the coordinate process solves (12). Then $\tilde{g}_2(t) = 0$, so
\[
\nabla_x \phi (X_t, \tilde{g}_1(t), \tilde{g}_2(t)) = (X_t - \tilde{g}_1(t))^\dagger.
\]
and (35) yields $\dot{u}_t = -\sigma(X_t)(X_t - E^u[X_t]) \mathbb{P}$-a.s. for almost every $t \in [0, T]$. Under $\mathbb{P}^u$ the coordinate process solves
\[
dX_t = \left( a(X_t) - \sigma(X_t)(X_t - E^u[X_t]) \right) dt + \sigma(X_t)dB_t^u.
\]

5.2 Unidirectional pedestrian motion in a corridor

Experimental studies have been conducted on the impact of proximity to walls on pedestrian speed. Pedestrian speed profiles heavily depend on circumstances like location, weather, and congestion. In this section, we will replicate two scenarios of unidirectional motion in a confined domain with the proposed mean-field type optimal control model. Especially, we are interested in how the proposed model behaves on the boundary and if boundary movement characteristics can be influenced through the running cost $f$. Sticky boundaries and boundary diffusion grants our pedestrians controlled movement at the boundary. By altering the internal parameters of these effect, we are able to shape the mean speed profile at the boundary.

Zanlungo et al. [33] observe that in a tunnel connecting a shopping center with a railway station in Osaka, Japan, pedestrians tend to lower their walking speed when walking close to the walls. The authors obtain a concave cross-section average speed profile from their experiment, with its maximum approximately at the center of the corridor. The average speed at the center of the corridor is about 10% higher than that of near-wall walkers.

Daamen and Hoogendoorn [15] on the other hand observe (in a controlled environment) pedestrian speeds that are higher at the boundary than in the interior of the domain. In their experiment, a unidirectional stream of pedestrians walk in a wide corridor that at a certain point, at a bottleneck, shrinks into a tight corridor. Upstream from the bottleneck, pedestrians close to the corridor walls move more freely due to less congestion, compared to those at the center of the corridor. The experiment results in a cross-section speed profile with more than twice as high average pedestrian speed in the low-density regions along corridor walls compared to the center of the corridor.

By modeling congestion with simple mean-dependent effects, we can replicate the overall shape of the average speed profiles of both [33] and [15] (not the density profile, to achieve this one needs a more sophisticated mean-field model). Our reason for implementing only mean-dependent effects, not of non-local distribution-dependent effects like those considered in for example [3], is solely to simplify the analysis.

Consider a long narrow corridor with walls parallel to the $x$-axis at $y = -0.1$ and $y = 0.1$. Our analysis requires $\mathcal{D}$ to be $C^2$-smooth, so the effective corridor (the corridor perceived by the pedestrians) has rounded corners. However, the corners will not have any substantial effect on the simulation results since the
crowd is initiated so far away from the target that under the chosen coefficient values, the pedestrians will not reach it ahead of the time horizon \( T = 1 \). On this domain, crowd behavior is modeled with the following optimal control problem

\[
\begin{align*}
\min_{u \in U} & \quad \frac{1}{2} E \left[ \int_0^1 \left( L^u_t f(t, x_t, E[L^u_t r_f(X_t)], u_t) \right) dt + L^u_T |X_T - x_T|^2 \right], \\
\text{s.t.} & \quad dL^u_t = L^u_t u_t dB_t, \quad L^u_0 = 1.
\end{align*}
\]

where \( B \) is a Brownian motion under \( \mathbb{P} \), the probability measure under which \( X \) solves \((12)\), and \( x_T \) is the location of an exit at the end of the corridor. The running cost \( f \) is of congestion-type,

\[ f(t, x, E[L^u_t r_f(X_t)], u_t) = C(X_t) \left( c_f + h(t, x, E^u [r_f(X_t)]) \right) u_t^2, \]

where \( c_f u^2, c_f > 0 \), is the cost of moving in free space, and \( hu^2 \) the additional cost to move in congested areas. The coefficient \( C(X_t) := c_T 1_D(X_t) + 1_{\Gamma}(X_t) \), \( c_T > 0 \), is used to monitor \( f \) (though it is not our control process) on the boundary \( \Gamma \). The cost of moving on the boundary is increasing with \( c_T \); we expect lower speed on the boundary. We know from \((35)-(34)\) that

\[ q^*_t = C(X_t) \left( c_f + h(t, x, E^u [r_f(X_t)]) \right) \hat{u}_t, \quad q_t = -(X_t - x_T)^* \sigma(X_t). \quad (39) \]

Matching the expressions in \((39)\) yields the optimal control

\[ \hat{u}_t = \frac{\sigma(X_t) (X_t - x_T)}{C(X_t) \left( c_f + h(t, x, E^u [r_f(X_t)]) \right)}. \]

It implements the following strategy: move towards the target location \( x_T \), but scale the speed according to the local congestion. Consider the two congestion penalties

\[ h_1 := c_f \left| X_2(t) - E^u [X_2(t)] \right|, \quad h_2 := \frac{1}{|X_2(t) - E^u [X_2(t)]|}, \quad (40) \]

where \( X_2(t) \) is the second (the \( y \)-)component of the coordinate process, i.e. the component in the direction perpendicular to the corridor walls. Stickiness is set to \( \gamma = 0.5 \). The choice of \( h \) in \((40)\) means that we have set \( r_f(X_t) = X_2(t) \).

The corridor is split into 9 segments parallel with the corridor walls. The mean speed is estimated in each segment for four different values of \( c_T \) and the results corresponding to congestion penalty \( h_1 \) and \( h_2 \) are presented in Figure \( 1 \) and \( 2 \) respectively. The profiles plotted in Figure \( 1 \) attains the concave shape observed by \(13\), mimicking the fast track in the middle of the lane. In Figure \( 2 \) the profiles follow the convex shape observed by \(15\), taking into account that movement in the crowded center (mean of the group) is costly. When \( c_T \) is small, the pedestrians can travel further on the boundary for the same cost. Heuristically, the higher \( \gamma \) is the longer it takes for the pedestrian to re-enter \( D \).
and therefore a high $\gamma$ combined with a small $c_\Gamma$ yields the highest boundary speed. This effect is evident in the figures, where smaller values of $c_\Gamma$ results in higher mean speed at the boundary. We note that we are able to shape the mean speed at the boundary by our choice of model parameters.

6 Conclusion and discussion

In this paper, we propose a to the best of our knowledge new variation of the mean-field approach to crowd modeling based on sticky reflected SDEs. The proposed model accounts for pedestrians that spend some time at the boundary and that have the possibility to choose a new direction of motion.

We provide conditions for the proposed dynamics to admit a unique weak solution, which is the best we can hope for (cf. [19]). Then, we consider mean-field type optimal control of the proposed dynamic model and give necessary conditions for optimality with a Pontryagin-type stochastic maximum principle. There is a microscopic interpretation of the model even on the boundary of the domain and thus it can be used to approximate optimal/equilibrium behavior of a pedestrian crowd on a microscopic (individual) level.
Figure 2: Mean speed in 9 segments of the corridor when $h = h_2$, estimated from 4000 realizations of the controlled coordinate process.

Pedestrians do often see and react to walls at a distance. This has been studied empirically, experiments are mentioned in the introduction. Force-based models can implement repulsing potential forces spiking to infinity at boundaries to keep the pedestrians away from the walls and inside the domain, effectively making it impossible for any pedestrian to reach a wall. A ranged, nonlocal, interaction with walls will have a smoothing effect on pedestrian density, just like nonlocal pedestrian-to-pedestrian interaction has, as is noted in [3]. Nonlocal interaction is an important aspect of pedestrian crowd modeling, but cannot give an answer to what will happen whenever a pedestrian actually reaches a wall. Interaction with walls at a distance can be included in our proposed model either in the drift, as is the case in force-based models, or through the cost functional, as in agent-based models.

An extension of the proposed framework would be to let the pedestrian control its stickiness, i.e. its motion in the normal direction of the boundary at the boundary. Stickiness is not necessarily a physical feature of the domain, but the time spent on the boundary may be subject to the pedestrian’s preference. This aspect cannot be described by the proposed model, since the Girsanov change of measure does not effect stickiness (cf. Remark 2). Another extension would be to consider the controlled diffusion case mentioned in the introduction.
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