Path integrals for spinning particles, stationary phase and the Duistermaat-Heckman theorem.

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Abstract

We examine the problem of the evaluation of both the propagator and of the partition function of a spinning particle in an external field at the classical as well as the quantum level, in connection with the asserted exactness of the saddle point approximation (SPA) for this problem. At the classical level we argue that exactness of the SPA stems from the fact that the dynamics (on the two–sphere $S^2$) of a classical spinning particle in a magnetic field is the reduction from $\mathbb{R}^4$ to $S^2$ of a linear dynamical system on $\mathbb{R}^4$. At the quantum level, however, and within the path integral approach, the restriction, inherent to the use of the SPA, to regular paths clashes with the fact that no regulators are present in the action that enters the path integral. This is shown to lead to a prefactor for the path integral that is strictly divergent except in the classical limit. A critical comparison is made with the various approaches to the same problem that have been presented in the literature. The validity of a formula given in literature for the spin propagator is extended to the case of motion in an arbitrary magnetic field.
I Introduction.

Since the early days of path integration, how to do a path integral for spinning particles was recognized as one of the major difficulties of the formalism. Schulman (but see also [3]) made a first attempt towards a formulation of a path integral for spinning particles, one which was however rather a related path integral, namely that for a spinning top.

Much progress has been made since with the systematic use, initiated by Klauder [4, 5], of the resolution of the identity associated with spin–coherent states [6, 7] in the discretized-time (or time-sliced) approach to the path integral. Path integral quantization using coherent states will be discussed extensively in the sequel. Other versions of the path integral not making an explicit use of coherent states have been discussed instead in [8, 9] (see also [10]).

Already at the classical level, a spin (classically a vector \( \vec{S} \) of fixed magnitude \( s \)) presents some peculiarities as a dynamical system. While the Hamiltonian description is essentially straightforward if one assumes the Poisson brackets:

\[
\{ S_i, S_j \} = \epsilon_{ijk} S_k
\]  

(1)

among the components of the spin vector, it has been pointed out by Balachandran et al. [11] that no global Lagrangian description can be given as long as one sticks to the natural configuration space of the spin, which is the compact two-sphere \( S^2 \). The same authors have shown that a global Lagrangian can be associated with a classical spin by lifting the description of the system from \( S^2 \) to the group manifold of SU(2), which is the three-sphere \( S^3 \), as follows. Considering the usual spin-1/2 representation of SU(2), one can define a vector \( \vec{S} \) in \( S^2 \) via the Hopf map, i.e.:

\[
\vec{S} = \vec{S} \cdot \vec{\sigma} = sg\sigma_3g^{-1},
\]  

(2)

with \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) the Pauli matrices. Then it can be shown [11] that the (global) Lagrangian on TSU(2):

\[
L = is\text{Tr}(\sigma_3g^{-1}\dot{g}) - \frac{\mu}{2}\text{Tr}(\vec{S} \vec{B})
\]  

(3)

where \( \mu \) is the Bohr magneton and \( \vec{B} = \vec{B} \cdot \vec{\sigma} \), yields the correct equations of motion for a classical spin in an external magnetic field \( \vec{B} \). The same
equations can be derived of course at the Hamiltonian level using the Poisson brackets (1) and the Hamiltonian:

\[ H = \mu \vec{S} \cdot \vec{B}. \]  

(4)

As it is clear from the fact that the spin is recovered via the Hopf projection (2), this approach introduces an extra, nondynamical U(1) gauge degree of freedom. Indeed, under: \( g \rightarrow g \exp[i\gamma \sigma_3/2], \) (2) is invariant while \( L \) changes by a total time derivative (i.e. it is “weakly invariant” (11)):

\[ L \rightarrow L - s \dot{\gamma}. \]  

(5)

This is also evident if we parametrize SU(2) with the Euler angles as

\[ g = e^{-i\phi \sigma_3/2}e^{-i\theta \sigma_2/2}e^{-i\gamma \sigma_3/2} \]  

(6)

\((0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \gamma \leq 4\pi),\) which yields the (local) Lagrangian:

\[ L = s(\dot{\phi} \cos \theta + \dot{\gamma}) + \mu \vec{S} \cdot \vec{B}. \]  

(7)

Although it can be shown [12] that this extra gauge degree of freedom has interesting consequences for the path integral quantization of (3), and namely that it leads in a straightforward way to spin quantization (i.e.: \( 2s/\hbar = \text{integer} \)), we would like to stress here the fact that (3) (or (7) for that matter) being linear in the time derivatives, a spinning particle is described as a constrained dynamical system. Canonical quantization requires then the use of Dirac’s theory [13] of constraints, and it has been shown by Balachandran et al. [11] that this does indeed yield the correct quantum mechanical description of the spin including spin quantization. As pointed out, e.g., in Ref. [9], it turns out that the (semiclassical) Bohr-Sommerfeld quantization is exact for a spin Hamiltonian. This is of course a strong indication that the saddle point, or stationary phase, approximation (henceforth referred to as SPA) to the path integral for, say, the propagator should be exact as well.

Concerning the path integral approach and the legitimacy of evaluating the path integral within the SPA, which is the main problem which the present paper addresses to, a Lagrangian of the form (1) poses another problem, and namely that, the classical equations of motion being first-order in time, the saddle point problem becomes over determined, as the solutions of
the saddle point equations have to obey two boundary conditions and not a single one. Different ways to overcome this problem have been proposed in the literature, and notably by Klauder [4, 5], Keski-Vakkuri et al. [14] and Suzuki [15].

This paper is devoted mainly to the discussion of two questions, namely to that of the legitimacy of some approximations that are currently made in the coherent state formulation of the path integral for spins and of the apparently surprising fact, that has also been widely discussed in the literature, that the SPA to the path integral yields the exact result for both the propagator and/or the partition function of a quantum spin. While the SPA is almost trivially exact for quadratic Hamiltonians, here the apparently surprising fact is that it turns out to be exact also for the Hamiltonian (4) and/or the Lagrangian (7) that are far from being quadratic. Exactness of the SPA implies of course that we discuss the applicability to the present problem of the Duistermaat-Heckman (DH) theorem which provides a rigorous framework for the discussion of the exactness of the SPA.

The paper is organized as follows. In the Appendix we briefly review the DH theorem and the SPA. In Section II we apply the latter to the calculation of the partition function of a classical spin and show that the deep reason of the validity of such an approximation is that a classical spin, when viewed as a dynamical system, can be shown to result from the reduction of a linear dynamical system from $\mathbb{R}^4$ to $S^2$. In Section III we begin by discussing briefly the use of spin–coherent state path integral approach to the calculation of the propagator and/or of the partition function of a quantum spin. We show there that the approximation currently used in literature restricting the paths in the functional integral to be continuous leads to incorrect and diverging prefactors. Then we review briefly the approaches of Klauder and Suzuki to the same problem. We clarify the origin of some apparently mysterious terms that are added to the action in Klauder’s approach. Then we analyze why the SPA yields the exact result in Suzuki’s approach, by arguing that his main result stems simply from the exactness of the Ehrenfest theorem in the present context. In Section IV we discuss in more detail the coherent state approach by using the holomorphic representation for the latter, applying the complex SPA to the discrete version of the spin path integral. The final Section V is devoted to a general discussion and to some conclusions.
II The classical spin.

Let us start our considerations from a classical (nonrelativistic) spin in a time-independent magnetic field \( \vec{B} = (B_1, B_2, B_3) \).

Since the only degree of freedom for a spin is its direction, we describe it by means of a three dimensional vector \((S_1, S_2, S_3) \in \mathbb{R}^3\) of fixed norm: \( S_1^2 + S_2^2 + S_3^2 = s^2 \), where the three classical variables \( S_j \) \((j = 1, 2, 3)\) satisfy the Poisson brackets \([\text{II}]\). Thus the phase space for a classical spin is the two-dimensional manifold \( S^2(s) \), equipped with the symplectic two-form

\[
\Omega = \frac{1}{2s^2} \epsilon_{ijk} S_i dS_j \wedge dS_k .
\] (8)

The Hamiltonian that describes a classical spin in a magnetic field is given by \([\text{II}]\). For simplicity we will set \( \mu \equiv 1 \) in the following so that

\[
H = \vec{B} \cdot \vec{S} .
\] (9)

The Hamiltonian vector field \( \Delta \) associated to (9), and determined by the symplectic form \( \Omega \), is given by

\[
\Delta = \epsilon_{ijk} B_j S_k \frac{\partial}{\partial S_i} ,
\] (10)

so that the classical equations of motion, \( i_\Delta \Omega = dH \), read

\[
\dot{S}_i = \epsilon_{ijk} B_j S_k .
\] (11)

Without any loss of generality, we will set \( \vec{B} = (0, 0, B) \) \((B > 0)\) from now on. In this case, the Hamiltonian \([\text{II}]\) and the equations of motion \([\text{II}]\) assume a very simple form if we use spherical coordinates \( S_1 = s \sin \theta \cos \phi \), \( S_2 = s \sin \theta \sin \phi \), \( S_3 = s \cos \theta \). They become respectively

\[
H = sB \cos \theta
\] (12)

and

\[
\cos \theta \dot{\phi} = 0
\] (13)

\[
\sin \theta (\dot{\phi} - B) = 0 .
\]
The latter can be easily integrated and one sees that the classical orbits are
circles parallel to the equator, the spin precessing about the magnetic field
with a period \( \tau = \frac{2\pi}{B} \).

In spherical coordinates, it is also very easy to compute the \textit{exact} partition
function for the Hamiltonian (9):

\[
Z_{cl}(\beta) = \int_{S^2(s)} \Omega e^{-\beta H}.
\]  

(14)

Indeed one gets:

\[
Z_{cl}(\beta) = s \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta e^{-\beta s B \cos \theta} \quad (15)
\]

\[
= \frac{2\pi}{\beta B} \left( e^{\beta s B} - e^{-\beta s B} \right).
\]

(15)

From (15), one recognizes that the classical partition function can be written
as the weighted sum of two terms, each given by the evaluation of \( e^{-\beta H} \)
at the two critical points \( \theta = \pi \) and \( \theta = 0 \) of the Hamiltonian for which
\( \Delta \vec{s}_{(0,0,\pm s)} = 0 \). In addition, it is not hard to check that the weights \( \pm \frac{2\pi}{\beta B} \)
are exactly the ones coming from the calculation of the contributions to the
integral (14) of the gaussian fluctuations around the stationary points of \( H \).

Everybody is familiar with such a result whenever dealing with (multidi-
mensional) harmonic oscillators or in general with a quadratic Hamiltonian
on the linear manifold \( \mathbb{R}^{2n} \). Even if the Hamiltonian (9) in not of this kind,
the SPA is exact as well. A spin in a magnetic field is in fact the simplest
(nontrivial) application of the Duistermaat-Heckman theorem [16, 17], which
establishes under which conditions a phase-space integral, such as (14), can
be evaluated exactly in the SPA. We refer to the Appendix for a review of
the Duistermaat-Heckman theorem and for the proof of its applicability to
the system of a spin in a magnetic field. Here we recall only that this result
holds essentially for the two following geometrical reasons:

1) the Hamiltonian (12) is invariant under an \( U(1) \)-action, given by rotations
about the third-axis (i.e. the axis of the constant magnetic field);
2) the associated Hamiltonian vector field, given by

\[
\Delta = B \left( -S_2 \frac{\partial}{\partial S_1} + S_1 \frac{\partial}{\partial S_2} \right)
\]

(16)
is proportional to the generator of this $U(1)$-action. This is clear in spherical coordinates, where $\Delta = B \frac{\partial}{\partial \phi}$. However while (16) defines it globally on $S^2(s)$, the spherical coordinate representation becomes singular at $\theta = 0, \pi$.

The Duistermaat-Heckman theorem gives some abstract mathematical conditions for the SPA to be exact. In the following we will show that for a spin in a magnetic field there is however a deeper reason why this holds: the dynamical system that describes a spin in magnetic field is the reduction of a bigger system which is described by a quadratic Hamiltonian on the linear manifold $\mathbb{R}^4$. Such a situation has already been considered in the literature (see [19] and references therein), even if not much in the context of the SPA, but mainly in the context of integrable systems, for which there exists the conjecture [19] that every integrable system is the reduction of a bigger linear dynamical system. To show what happens in the case of a spin, let us consider the linear manifold $\mathbb{R}^4 = \mathbb{C}^2$, equipped with the symplectic two-form $\tilde{\Omega} = \frac{1}{2s}(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = \frac{1}{2s}(dz_1^* \wedge dz_1 + dz_2^* \wedge dz_2)$.

There is a natural action of SU(2) on $\mathbb{R}^4 = \mathbb{C}^2$, which is simply given by left multiplication:

$$\begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \alpha z_1 - \beta^* z_2 \\ \beta z_1 + \alpha^* z_2 \end{bmatrix},$$

where $\begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}$, with $|\alpha|^2 + |\beta|^2 = 1$, is an element of SU(2) in the fundamental representation. This action is symplectic and Hamiltonian [20], and its Lie algebra is spanned by the vector fields:

$$\begin{align*}
\tilde{\chi}_1 &= x_4 \partial_1 - x_3 \partial_2 + x_2 \partial_3 - x_1 \partial_4 \\
\tilde{\chi}_2 &= -x_3 \partial_1 - x_4 \partial_2 + x_1 \partial_3 + x_2 \partial_4 \\
\tilde{\chi}_3 &= x_2 \partial_1 - x_1 \partial_2 - x_4 \partial_3 + x_3 \partial_4.
\end{align*}$$

Hence it is easy to prove that the linear vector field $\Delta = \sum_{j=1}^{3} B_j \tilde{\chi}_j$ is a Hamiltonian vector field with a quadratic Hamiltonian, namely:

$$\tilde{H} = \sum_{j=1}^{3} B_j f_j,$$
where

\[ f_1 = \frac{1}{2s}(x_1 x_3 + x_2 x_4) = \frac{1}{2s} \text{Re} z_1^* z_2 \]

\[ f_2 = \frac{1}{2s}(x_1 x_4 - x_2 x_3) = \frac{1}{2s} \text{Im} z_1^* z_2 \]

\[ f_3 = \frac{1}{4s}(x_1^2 + x_2^2 - x_3^2 - x_4^2) = \frac{1}{4s} \left( |z_1|^2 - |z_2|^2 \right). \]  

(19)

Here we can identify \( \mathbb{R}^4 \) with \( T^* \mathbb{R}^2 \) with canonical coordinates \( x_1, x_3 \) and momenta \( p_1 = \frac{x_2}{2s} \) and \( p_2 = \frac{x_4}{2s} \).

The SU(2)–action we are considering leaves the three–dimensional spheres \( S^3(R) = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\} = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = R^2\} \) invariant so that we can restrict the dynamics from the full \( \mathbb{R}^4 \) to the submanifolds \( S^3(R) \) on which the classical orbits lie. In the following we choose \( R = 2s \) and work on \( S^3(2s) \).

The functions (19) and hence the Hamiltonian (18) have an additional symmetry, being invariant under the action of U(1) given by:

\( (z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2) \quad \theta \in [0, 2\pi[. \)

This allows us to project the Hamiltonian (18) from \( S^3(2s) \) down to the two-dimensional manifold \( S^3(2s)/\text{U}(1) \), which is homeomorphic to the two-dimensional sphere. Indeed, the three functions \( (x_1, x_2, x_3, x_4) \mapsto S_j \equiv f_j(x_1, x_2, x_3, x_4) \quad (j = 1, 2, 3) \) given in (19) are the components of the projection map from the three-sphere \( S^3(2s) \) to the two-sphere \( S^2(s) \) of the Hopf bundle \( \text{U}(1) \to S^3 \to S^2 \) [21].

On \( S^2(s) \) the Hamiltonian (18) becomes simply \( H = \sum_{j=1}^3 B_j S_j \) and therefore coincides with the Hamiltonian (8) for a spin in a magnetic field. This shows that the latter is the reduction of a quadratic Hamiltonian defined on a linear manifold [1].

\[ ^1 \text{To prove this rigorously, we should show also that the restriction to } S^3(2s) \text{ of the symplectic two-form } \Omega \text{ is the pull-back of the symplectic two-form } \Omega \text{ we defined on } S^2(s). \]\n
The proof of this statement is straightforward, so we will omit details here.
III Coherent state path integrals for spin.

Let us consider now the quantum mechanics of a spinning particle, described by the Hamiltonian

\[ \hat{H} = \vec{B} \cdot \hat{\vec{S}} , \]  

(20)

where the spin operators \( \hat{S}_j \) \((j = 1, 2, 3)\) satisfy the usual commutation relations \((\hbar = 1)\):

\[ [\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk} \hat{S}_k . \]  

(21)

We have already mentioned in the Introduction that the semiclassical Bohr-Sommerfeld quantization turns out to be exact for this problem and this seems to suggest that the SPA to the path integral for the propagator and/or the partition function could be exact as well. In addition, for such a simple problem, one can evaluate the partition function exactly. Its expression

\[ Z(\beta) = \text{Tr}\{ e^{-\beta \hat{H}} \} = \frac{e^{\beta Bs}}{1 - e^{-\beta B}} + \frac{e^{-\beta Bs}}{1 - e^{\beta B}} \]  

(22)

can be thought of as the sum of two terms, each corresponding to one of the two poles of the sphere \((S_3 = \pm s)\), similarly to what happens in the classical case.

In this Section we will briefly review the existing literature on the evaluation of the partition function for a quantum spin in a magnetic field and in particular on the validity of the SPA applied to this problem.

Let’s begin by considering a set \(|l\rangle\) of generalized coherent states [4, 7] labeled by one or more continuous variables that we denote collectively as \(l\). The \(|l\rangle\)’s will be assumed to be normalized. They are however overcomplete, because although there is a resolution of the identity associated with them:

\[ 1 = \int dl |l\rangle\langle l| \]  

(23)

with “\(dl\)” a suitable measure, in general they fail to be an orthonormal set: \(\langle l|l'\rangle \neq 0\).

Here we are interested in the group SU(2), which is generated by the spin-operator algebra (21). Coherent states [7] for a spin \(s\) (2\(s\) being an integer) can be constructed as

\[ |\theta, \phi\rangle = e^{-i\phi \hat{S}_3} e^{-i\theta \hat{S}_2} |0\rangle , \]  

(24)
where $|0\rangle$ denotes the highest-weight state of the spin-$s$ representation of $SU(2)$ ($\hat{S}_3|0\rangle = s|0\rangle$) and $\theta$, $\phi$ are the two angular coordinates parametrizing $S^2$: $0 \leq \theta \leq \pi$; $0 \leq \phi \leq 2\pi$. It is well known [6, 7] that:

$$\langle \theta', \phi' | \theta, \phi \rangle = \left[ \cos \frac{\theta'}{2} \cos \frac{\theta}{2} e^{i(\phi' - \phi)/2} + \sin \frac{\theta'}{2} \sin \frac{\theta}{2} e^{-i(\phi' - \phi)/2} \right]^{2s} \tag{25}$$

and that the resolution of the identity associated with spin–coherent states is:

$$1 = \frac{2s + 1}{4\pi} \int_0^{2\pi} \int_0^\pi |\theta, \phi\rangle \langle \theta, \phi| \sin \theta d\theta d\phi \,. \tag{26}$$

Going back to the general case, let $\hat{H}$ be the Hamiltonian of a quantum system. The matrix element of the propagator

$$K(l_F, l_I, T) = \langle l_F|e^{-i\hat{H}T}|l_I \rangle \tag{27}$$

can be represented as the following path integral:

$$K(l_F, l_I, T) = \lim_{\epsilon \to 0} \int \prod_{k=0}^N (dl_k) \prod_{k=0}^N \langle l_{k+1}|e^{-i\epsilon \hat{H}}|l_k \rangle \,, \tag{28}$$

where $\epsilon = T/N$ and $|l_0\rangle \equiv |l_I\rangle, |l_{N+1}\rangle \equiv |l_F\rangle$. Eq. (28) can be rewritten as:

$$K(l_F, l_I; T) = \lim_{\epsilon \to 0} \int \prod_{k=1}^N (dl_k) \prod_{k=0}^N e^{iA(l_{k+1}, l_k)} \,, \tag{29}$$

where

$$A(l_{k+1}, l_k) = -i \ln \langle l_{k+1}|l_k \rangle - \epsilon H(l_{k+1}, l_k) \tag{30}$$

and

$$H(l_{k+1}, l_k) = \frac{\langle l_{k+1}|\hat{H}|l_k \rangle}{\langle l_{k+1}|l_k \rangle} \,. \tag{31}$$

It is usually assumed that, for $\epsilon$ small, $|l_{k+1}\rangle$ is so close to $|l_k\rangle$ that one is allowed to expand the former around the latter to leading order in $\epsilon$. This leads to

$$H(l_{k+1}, l_k) \sim H(l_k) = \langle l_k|\hat{H}|l_k \rangle \tag{32}$$

and to

$$\ln \langle l_{k+1}|l_k \rangle \sim \langle \delta l_k |l_k \rangle \,, \tag{33}$$
where $|\delta l_k\rangle = |l_{k+1}\rangle - |l_k\rangle$. Hence the continuum version of the path integral reads

$$
\int \mathcal{D}l \exp \left[ i \int_0^T dt \left[ i \langle l|\dot{l} - H(l) \right] \right]
$$

(34)

where $|\dot{l}\rangle = d|l\rangle / dt$. This procedure is justified when the Hamiltonian contains an explicit kinetic term that can act as a regulator concentrating the functional measure on continuous paths. This is the case of Wiener integral, where \cite{22, 23} the measure is concentrated on paths $q(t)$ satisfying the Lipschitz condition: $|q(t + \epsilon) - q(t)| = O(\epsilon^{1/2})$. The same holds true for the Feynman path integrals for massive particles in not too singular potentials \cite{1}.

In the case of spin, the Hamiltonian (20) contains no regulators and the straightforward application of (32) and (33) is questionable. We will see indeed that it may lead to serious problems. Let’s be more specific. Taking again the magnetic field along the positive z-axis we have, in terms of spin–

$$
\langle l|\hat{H}|l\rangle \rightarrow \langle \theta, \phi|\hat{H}|\theta, \phi\rangle = sB \cos \theta
$$

and

$$
i\langle l|\dot{l}\rangle \rightarrow i\langle \theta, \phi|\dot{d}/dt|\theta, \phi\rangle = s \cos \theta \dot{\phi} .
$$

(36)

So we obtain the path integral:

$$
\langle \theta_F, \phi_F|e^{-i\mu B S_3 T}|\theta_I, \phi_I\rangle = \int \mathcal{D}\Omega \exp \left[ i \int_0^T \left[ s \cos \theta \dot{\phi} - s \mu B \cos \theta \right] dt \right] ,
$$

(37)

where $\mathcal{D}\Omega = \sin \theta \mathcal{D}\theta \mathcal{D}\phi$. From (37) we can obtain the canonical partition function $Z(\beta)$ by setting $T = -i\beta$, $|\theta_F, \phi_F\rangle = |\theta_I, \phi_I\rangle$ and by tracing over the angles. Following the standard procedure and Ref. \cite{8}, we have evaluated $Z(\beta)$ by time-slicing the path integral and using the resolution of the identity (26) at each intermediate (Euclidean) time. Denoting with $Z_N(\beta)$ the result for $N$ time slices, we find for it without any further approximation the expression \cite{12}:

$$
Z_N(\beta) = \left( 1 + \frac{1}{2s} \right)^N Z(\beta) ,
$$

(38)
where \( Z(\beta) \) in the right hand side is the exact partition function \( (22) \).

As one can easily notice by simple inspection, in the limit \( N \to \infty \), \( Z_N \) coincides with the exact partition function up to a diverging prefactor. Such divergency disappears only in the classical limit, namely \( \hbar \to 0, s \to \infty \) with \( \hbar s = \text{const.} \). On the other hand the only approximations we used are those given by eqs. \( (32) \) and mainly \( (33) \) which again are exact in the classical limit. In the following of the paper we shall come back to this point.

We discuss now briefly two different approaches \[5, 15\] that lead to the conclusion that in the case of a spinning particle in a magnetic field the SPA to the path integral for the propagator does indeed yield the exact result. This holds again up to a normalization factor not taken into account in \[5, 15\], since the calculation is done in the continuum.

Klauder \[5\] has proposed a modified form for the action that appears in the path integral \( (34) \), redefining the latter as:

\[
K(l_F, l_I; T) = \lim_{\epsilon \to 0} \int \mathcal{D}l \exp \left[ i \int_0^T dt \left( i \langle l| \dot{l} \rangle + \frac{1}{2} i \epsilon \langle l| \dot{l} \rangle (1 - \langle l | \langle l \rangle \rangle \langle \dot{l} | \dot{l} \rangle) - H(l) \right) \right].
\]

The prescription here is that the limit for \( \epsilon \to 0 \) should be taken after having evaluated the path integral in the SPA. For the case of a spin in a magnetic field, \( (39) \) becomes:

\[
K(\theta_F, \phi_F, \theta_I, \phi_I; T) = \lim_{\epsilon \to 0} \int \mathcal{D}\Omega \exp \left[ i \int_0^T dt \left( s \cos \theta \dot{\phi} + \frac{1}{4} i \epsilon \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 - s \mu B \cos \theta \right) \right]
\]

and the action in \( (40) \) becomes essentially that of a particle of charge \( s \) and mass \( m = 1/2s\epsilon \) moving on the two-sphere \( S^2 \), coupled both with a magnetic monopole of unit strength located at the centre of the sphere and with a constant electric-type field directed along the z-axis. In this context, Dirac’s quantization condition \( (24) \) becomes identical with the spin quantization condition, i.e.: \( 2s = \text{integer} \).

\[\text{In order to include paths turning around north pole any number of times we need to extend, at every time slice, the domain of integration of the variable } \phi \text{ from } [0, 2\pi] \text{ to } [-\infty, +\infty]. \text{ In fact without following such procedure one would get, as in Ref. } [5], \text{ a completely wrong partition function.} \]
The classical equations of motion that can be derived from the action in (40) are now second-order in time for $\epsilon \neq 0$, and hence are not plagued by the already mentioned problem of overdetermination. Klauder has proved that they can be solved explicitly and that the resulting SPA to the path integral (or, better, what he calls the “dominant SPA”, namely approximating the path integral with $e^{iS_{cl}}$, without any prefactor originating from the integration of gaussian fluctuations) is indeed exact. Actually, only the free-spin case $B = 0$ has been considered in Ref. [5], but the extension to $B \neq 0$ is straightforward [12]. It should be noted, incidentally, that the term that Klauder has added to the action is in the form of a kinetic energy term, thus providing the required regulator justifying the assumption of continuously-varying paths.

The origin of the additional kinetic-type term, that looks somewhat mysterious in Klauder’s original paper [4], can be clarified as follows. If we push the expansion of the logarithm in (33) one step further beyond first order we obtain, with $|\delta l| \sim \epsilon |\dot{l}| + \epsilon^2/2|\ddot{l}|$:

$$\ln\langle l_{k+1}|l_k \rangle \sim \epsilon \langle l_k|\dot{l}_k \rangle + \frac{\epsilon^2}{2} \langle \ddot{l}_k|l_k \rangle - \frac{\epsilon^2}{2} (\langle \dot{l}_k|l_k \rangle)^2,$$

leading, in the continuum limit to

$$\lim_{\epsilon \to 0} \exp \left[ i \int_{t_i}^{t_f} dt \left[ i \langle l|\dot{l} \rangle - \frac{1}{2} i \epsilon \left( -\langle \dot{l}|l \rangle + \langle \dot{l}|\dot{l} \rangle \rangle \right) \right] \right]$$

where we have used $\langle l|\dot{l} \rangle = -\langle \dot{l}|l \rangle$. A final integration by parts of the term containing the second derivative yields precisely Klauder’s additional term.

Suzuki [15] has adopted quite a different approach. Introducing two additional resolutions of the identity, the propagator

$$K(l_F, l_I, T) = \int_{l(t_i)=l_F}^{l(T)=l_I} Dl e^{iA_{FI}(l)} ,$$

where

$$A_{FI}(l) = \int_0^T dt \left[ i \langle l|\dot{l} \rangle - H(l) \right]$$

is rewritten in Ref. [15] in the form:

$$K(l_F, l_I, T) = \int \int d\omega d\eta \langle l_F|l_I \rangle \langle l_I|\omega \rangle \langle \omega|\eta \rangle e^{-iH_T|\eta|l_I}$$

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In general the parameter(s) $l$ labeling a coherent state can be defined in terms of the expectation values of a suitable set of operators: position and momentum for the free particle and the harmonic oscillator, spin components in the case of spins. Taking then the latter as Cauchy data for the canonical equations one can determine how they evolve classically in time, thus determining a “classical” coherent state $|l(t)\rangle$ as the coherent state labeled by $l(t)$, the evolved at time $t$ of any given initial parameter set $l$. Then, Suzuki has argued that the SPA to the last integral leads to

$$
\int_{l(0)=l_i}^{l(T)=l_f} \mathcal{D}l e^{iA_{fi}(l)} \sim \delta(l_f - l_i(T)) e^{iA_{fi}},
$$

where $l_i(T)$ is the evolved at time $T$ of $l_i$, and that the final result of the application of the SPA is the semiclassical propagator

$$
K_{sc}(l_F, l_I, T) = \int dl_i \langle l_F|l_i(T)\rangle \langle l_i|l_I\rangle e^{iA_{fi}},
$$

where $A_{fi}$ is the classical action evaluated along the classical path leading from $l(0) = l_i$ to $l(T)$. Once again, the semiclassical propagator (47) yields the exact result for the propagator of a spinning particle.

We would like to show here that the possible exactness of Eq. (47) has very little to do with the path integral formalism itself, and that it follows rather from a single assumption, one that amounts basically to assuming that the Ehrenfest theorem be applicable in the present case. Let $\hat{l}$ be the set of operators whose expectation values label a given coherent state. To each coherent state $|l\rangle$ we can associate two different time dependent states:

- the “classically” time evolved state $|l(t)\rangle$ defined above and therefore such that

$$
\langle l(t)|\hat{l}|l(t)\rangle = \delta(t),
$$

$\delta(t)$ being the classical trajectory,

- the quantum time evolved state $|l, t\rangle$ obtained through the application of the full quantum evolution operator:

$$
|l, t\rangle = e^{-i\hat{H}t}|l\rangle.
$$

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We require now the expectation values on the quantum evolved state to evolve in time according to the classical equations of motion:

\[ \langle l, t | \hat{l}, t \rangle = l_{cl}(t) . \]  \hspace{1cm} (50)

In other words we are assuming, as anticipated, the validity of the Ehrenfest theorem. A sufficient condition for this to happen is that the Heisenberg equations of motion be linear equations. This is certainly true for quadratic Hamiltonians with conventional canonical coordinates and momenta and the standard Poisson brackets among them. For spins, the Hamiltonian (4) is not of the quadratic type, but the Poisson brackets (1) lead nonetheless to linear equations of the motion and hence to the validity of the Ehrenfest theorem. If we now assume that the set of operators \( \hat{l} \) act irreducibly on the Hilbert space of states (which is true, e.g., for harmonic oscillators and for spin systems) the two states \( |l, t\rangle \) and \( |l(t)\rangle \) will differ at most by a phase, i.e.:

\[ |l, t\rangle = e^{i\chi(t)} |l(t)\rangle . \]  \hspace{1cm} (51)

Differentiating with respect to time we obtain

\[ \langle l(t) | \frac{d}{dt} |l(t)\rangle = -i \dot{\chi} + \langle l, t | \frac{d}{dt} |l, t\rangle \]  \hspace{1cm} (52)

and hence

\[ \langle l|\dot{l})_{cl} = -i \dot{\chi} - i \langle l, t | \hat{H} |l, t\rangle . \]  \hspace{1cm} (53)

If, as it is presently the case, the Hamiltonian does not contain time derivatives, we obtain furthermore:

\[ \dot{\chi} = i \langle l|\dot{l})_{cl} - \langle l(t) |\hat{H} |l(t)\rangle = i \langle l|\dot{l})_{cl} - H(l(t)) \]  \hspace{1cm} (54)

and eventually, integrating the last equation and up to an irrelevant constant phase:

\[ \chi(t) = \int_0^t \left[ i \langle l|\dot{l})_{cl} - H(l) \right] dt' . \]  \hspace{1cm} (55)

The r.h.s. of (55) is exactly the classical action, hence \( \chi(t) = A_{cl}(t) \). Then Suzuki’s result follows at once from:
\[ K(l_F, l_I, T) = \langle l_F | e^{-i\hat{H}T} | l_I \rangle \]
\[ = \int dl_i \langle l_F | e^{-i\hat{H}T} | l_i \rangle \langle l_i | l_I \rangle \]
\[ = \int dl_i \langle l_F | l_i(T) \rangle \langle l_i | l_I \rangle e^{i\phi(T)} \]
\[ = \int dl_i \langle l_F | l_i(T) \rangle \langle l_i | l_I \rangle e^{iA_{cl}}, \quad (56) \]

which is the desired result.

Let’s analyze (56) in the specific case of a spinning particle. As already pointed out the Hamiltonian (4) leads to linear Heisenberg equations of motion and hence to the validity of formula (56). It seems to us that this is the ultimate reason for the apparently surprising fact that SPA to path integrals and/or semiclassical quantization leads to the exact result for the quantum propagator of a spinning particle. We would like to stress here that this is also true for the motion in a magnetic field with an arbitrary dependence on time. Formula (56) reduces therefore the quantum problem of the calculation of the spin propagator in an arbitrary magnetic field to the solution of the classical equation of motion.

The expression for the propagator can of course be worked out explicitly only when the classical equation of motion can be solved analytically. With a constant magnetic field along the z-axis, formula (57) reads:
\[ K(\theta_F, \phi_F, \theta_I, \phi_I; T) = \frac{2s+1}{4\pi} \int d\Omega_i \langle \theta_F, \phi_F | \theta_i(T), \phi_i(T) \rangle \langle \theta_i, \phi_i | \theta_I, \phi_I \rangle e^{iA_{cl}}. \]

(57)

Since the solutions of the classical equations of motion are \( \theta_i(T) = \theta_i; \phi_i(T) = \phi_i + \mu BT \), while \( A_{cl} = 0 \), the above expression becomes:
\[ K(\theta_F, \phi_F, \theta_I, \phi_I; T) = \frac{2s+1}{4\pi} \int d\Omega_i \langle \theta_F, \phi_F | \theta_i, \phi_i + \mu BT \rangle \langle \theta_i, \phi_i | \theta_I, \phi_I \rangle. \]

(58)

The integral in (58) can be done easily by noting that \( \langle \theta_F, \phi_F | \theta_i, \phi_i + \mu BT \rangle = \langle \theta_F, \phi_F - \mu BT | \theta_i, \phi_i \rangle \) and using the resolution of the identity. We obtain eventually
\[ K(\theta_F, \phi_F, \theta_I, \phi_I; T) = \langle \theta_F, \phi_F - \mu BT | \theta_I, \phi_I \rangle, \]

(59)

which indeed gives the exact propagator, upon using formula (25) for the overlap of two coherent states.

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IV The complex saddle point.

We devote this Section to another approach to path integral for spin. It has been used for example in [14] and in [25], where however only continuous and hence formal expressions for the action and the path integral have been studied.

Here we will study the path integral discrete version for the propagator (34). The method is based on a different choice of the spin–coherent states. Instead of (24), we consider the following states, labeled by a complex parameter \( \mu \):

\[
|\mu\rangle = e^{\mu S^-} |0\rangle, \tag{60}
\]

where we have set \( \hat{S}^\pm = \hat{S}_1 \pm i \hat{S}_2 \). To be more precise, \( \mu \) parametrizes the homogeneous space \( SU(2)/U(1) = S^2 \) by means of the stereographic projection of \( S^2 \) onto \( \mathbb{R}^2 \). With respect to the spherical coordinates \( \theta \) and \( \phi \) used previously, one has \( \mu = \tan\frac{\theta}{2}e^{i\phi} \) and it is also not difficult to check that the state (60) coincides up to a phase with the state \( |\theta, \phi\rangle \) defined in (24).

To write down an explicit expression for the path integral in terms of the coherent states (60), we need the matrix elements

\[
\langle \lambda | \mu \rangle = \frac{(1 + \lambda^* \mu)^{2s}}{(1 + |\mu|^2)^s (1 + |\lambda|^2)^s}, \tag{61}
\]

\[
\langle \lambda | \hat{S}_3 | \mu \rangle = \langle \lambda | \mu \rangle s \frac{1 - \lambda^* \mu}{1 + \lambda^* \mu},
\]

\[
\langle \mu | \hat{\mu} \rangle = \frac{s}{1 + |\mu|^2} (\hat{\mu} \mu^* - \mu^* \hat{\mu})
\]

and the explicit formula for the resolution of the identity:

\[
\frac{2s + 1}{\pi} \int \frac{d^2 \mu}{(1 + |\mu|^2)^2} |\mu\rangle \langle \mu| = 1 \tag{62}
\]

where \( d^2 \mu = d(\text{Re} \mu)d(\text{Im} \mu) \). After some algebra, one finds that the path integral representation of the propagator is given by

\[
K(\mu_F^*, \mu_I; T) = \int_{\mu(0) = \mu_I}^{\mu(T) = \mu_F^*} \frac{d^2 \mu}{(1 + |\mu|^2)^2} e^{\mathcal{A}}, \tag{63}
\]
where the action for a spin in a magnetic field is
\[
A = is \int_0^T dt \left( i\dot{\mu} \mu^* - \mu \dot{\mu}^* - B \frac{1 - |\mu|^2}{1 + |\mu|^2} \right).
\]
(64)

Again the SPA equations that are derived by extremizing (64):
\[
\left\{ \begin{array}{l}
\dot{\mu} = iB\mu \\
\dot{\mu}^* = -iB\mu^*
\end{array} \right.
\]
(65)
are first-order, so that they do not admit of a solution for boundary conditions of the form:
\[
\left\{ \begin{array}{l}
\mu(0) = \mu_I \\
(\mu(T))^* = \mu_F^*
\end{array} \right.
\]
(66)
with arbitrary \( \mu_I \) and \( \mu_F \).

This problem does however admit of solutions [14, 26] if we enlarge our variable space from \( \mathbb{C} \) to \( \mathbb{C}^2 \) and look for saddle point solutions for which \( \mu(t) \) and \( \mu^*(t) \) are not complex-conjugate one of the other. In other words, we have to consider \( \mu \) and \( \mu^* \) as independent complex variables and look for solutions of (65) satisfying the boundary conditions
\[
\left\{ \begin{array}{l}
\mu(0) = \mu_I \\
(\mu(T))^* = \mu_F^*
\end{array} \right.
\]
(67)
These so called complex saddle point solutions are easily found to be
\[
\left\{ \begin{array}{l}
\mu(t) = e^{iBt} \mu_I \\
\mu^*(t) = e^{iB(T-t)} \mu_F^*
\end{array} \right.
\]
(68)
and, to complete the SPA, one has further to evaluate the contributions of the gaussian fluctuations around the solutions (68).

This has been done by the authors of [14, 25], who have concluded that one can obtain the exact propagator in this way. They have however worked with the continuous expression (64) and therefore they have performed only formal calculations. Funahashi et al. [27] have considered the discrete version of the integral but they have applied SPA to the calculation of the partition function only. This problem, as pointed out by the same authors, is simpler than the calculation of the propagator since the overdetermination problem can be overcome without enlarging the variable space from \( \mathbb{C} \) to \( \mathbb{C}^2 \).
Our aim in this Section is to examine whether the SPA to the path integral (63) is exact by performing the calculation in the discrete version of the path integral. The latter is given by (29), where now the generic index $l$ stands for the complex parameter $\mu$. By using (61) and rearranging the terms, one eventually gets:

$$K(\mu_F^*, \mu_I, T) = \mathcal{N} \int \prod_{j=1}^{N-1} \frac{d^2 \mu_j}{(1 + \mu_j^* \mu_j)^2} e^{A_{cl}}$$

with

$$\mathcal{N} = \left(\frac{2s + 1}{\pi}\right)^{N-1} \frac{1}{(1 + |\mu_F|^2)^s(1 + |\mu_I|^2)^s},$$

whereas the discretized action $A$ is given by

$$A = \sum_{j=1}^{N} [A_{j,j-1} + H_{j,j-1}] + 2s \ln(1 + \mu_F^* \mu_N)$$

$$A_{j,j-1} \equiv 2s \ln \frac{1 + \mu_j^* \mu_{j-1}}{1 + \mu_j^* \mu_j}$$

$$H_{j,j-1} = -iBse \frac{1 - \mu_j^* \mu_{j-1}}{1 + \mu_j^* \mu_{j-1}}.$$

The expression for $A_{j,j-1}$ given in (72) comes from the exponentiation of the overlaps $\langle \mu_j | \mu_{j-1} \rangle$ ($j = 1, \cdots, N$). Let us remark that we have not expanded the overlaps to first order in the difference $|\mu_j - \mu_{j-1}|$, as usually done to recover the continuum version (see (33)). As already pointed such an expansion would be correct only in the presence of a regulating term in the action. Thus we will work only with (72) and we will proceed now to evaluate the SPA to the multidimensional integral (69), according to the formula

$$K(\mu_F^*, \mu_I, T)_{sc} = \mathcal{N} \prod_{j=1}^{N-1} \left( \frac{1}{(1 + \mu_j^* \mu_j)^2} \right) e^{A_{cl}} \frac{\pi^{N-1}}{(\det M_N)^{1/2}},$$

where $M_N$ is the $2(N-1) \times 2(N-1)$ matrix of quadratic fluctuations around...
the classical solutions of the discretized action (74):

\[ A = A_{cl} + (\delta \mu_1, \delta \mu_1, \ldots, \delta \mu_{N-1}, \delta \mu_{N-1})M_N \begin{pmatrix} \delta \mu_1 \\ \delta \mu_1^* \\ \vdots \\ \delta \mu_{N-1} \\ \delta \mu_{N-1}^* \end{pmatrix} . \]  

The subscript “cl” means that the function (the action in this case) has to be evaluated on the classical solution, which satisfies the saddle point equations:

\[ \begin{cases} 
\frac{\mu_{j+1} - \mu_j^*}{1 + \mu_j^* \mu_j} &= -i \epsilon B \frac{\mu_{j+1}^*}{1 + \mu_{j+1}^* \mu_j} \\
\frac{\mu_j - \mu_{j-1}^*}{1 + \mu_{j-1}^* \mu_j} &= +i \epsilon B \frac{\mu_j^*}{1 + \mu_j^* \mu_{j-1}} \end{cases} . \]  

In (76), as well as in (74), we have explicitly written \(|\mu_j|^2\) as \(\mu_j \mu_j^*\) to stress the fact that in the search of the saddle point solutions we have to treat \(\mu_j\) and \(\mu_j^*\) as independent complex variables. Indeed, exactly as in the continuum, the equations (76) are incompatible with the boundary conditions

\[ \begin{cases} 
\mu_0 = \mu_I \\
\mu_N = \mu_F \end{cases} \]  

unless \(\mu_j\) and \(\mu_j^*\) are treated as independent variables.

We know that the classical solution has to fulfill the property \(\mu_{j+1}|_{cl} = \mu_j|_{cl} + O(\epsilon)\), so that we can approximate the denominators in the right hand side of (76), to get

\[ \begin{cases} 
\frac{\mu_{j+1} - \mu_j^*}{1 + \mu_j^* \mu_j} &= -i \epsilon B \frac{\mu_{j+1}^*}{1 + \mu_{j+1}^* \mu_j} + O(\epsilon^2) \\
\frac{\mu_j - \mu_{j-1}^*}{1 + \mu_{j-1}^* \mu_j} &= +i \epsilon B \frac{\mu_j^*}{1 + \mu_j^* \mu_{j-1}} + O(\epsilon^2) \end{cases} . \]  

To order \(\epsilon\), we find for the solutions satisfying the boundary conditions (77) and for the classical action the expressions:

\[ \begin{cases} 
\mu_j = (1 + i \epsilon B)^j \mu_I \\
\mu_j^* = (1 + i \epsilon B)^{N-j} \mu_F^* \end{cases} \]  

\[ A_{cl} = -i s BT . \]
To complete the calculation of the right hand side of (74) we have to compute the determinant of the gaussian fluctuation matrix $M_N$. As for the solutions of the classical equations (76), we evaluate the matrix elements to the order $O(\epsilon)$. Neglecting therefore all the terms of at least order $O(\epsilon^2)$ we find that the only non-zero matrix elements are

\[
\begin{align*}
\frac{\partial^2 A}{\partial \mu_j^* \partial \mu_j} &\equiv \left. \frac{\partial^2 A}{\partial \mu_j^* \partial \mu_j} \right|_{cl} = \frac{-2s}{(1 + \mu_j^* \mu_j)^2} \left|_{cl} = \frac{-2s}{(1 + (1 + i\epsilon B)^N \mu_F^\dagger \mu_J)^2}, \right. \quad (81) \\
\frac{\partial^2 A}{\partial \mu_j^* \partial \mu_j} &\equiv \left. \frac{\partial^2 A}{\partial \mu_j^* \partial \mu_j} \right|_{cl} = \frac{-2s}{(1 + (1 + i\epsilon B)^N \mu_F^\dagger \mu_J)^2}, \quad (82)
\end{align*}
\]

Defining

\[
\begin{align*}
A_i &= \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix}, \\
B_i &= \begin{pmatrix} 0 & 0 \\ 0 & b_i \end{pmatrix}, \\
C_i &= \begin{pmatrix} b_i & 0 \\ 0 & 0 \end{pmatrix}, \quad (83)
\end{align*}
\]

the matrix $M_N$ is given by

\[
M_N = \begin{pmatrix}
A_1 & B_1 \\
C_1 & A_2 & B_2 \\
& \\
& C_{N-3} & A_{N-2} & B_{N-2} \\
& C_{N-2} & A_{N-1} & \end{pmatrix}. \quad (84)
\]

Thus we have

\[
\text{det} M_N = a_1^2 \cdots a_{N-1}^2 = \left[ \frac{-2s}{(1 + (1 + i\epsilon B)^N \mu_F^\dagger \mu_J)^2} \right]^{2(N-1)}, \quad (85)
\]

which can be finally inserted, together with (80), in (74), yielding:

\[
K(\mu_F^\dagger, \mu, T)_{sc} = \left( 1 + \frac{1}{2s} \right)^{N-1} e^{-isBT(1 + \mu_F^\dagger \mu_F)(1 + iBT/N)^2s} \frac{(1 + |\mu_F|^2)^s(1 + |\mu|^2)^s}{(1 + |\mu_F|^2)^s(1 + |\mu|^2) \dagger s}. \quad (86)
\]
This is again the expected result up to the divergent normalization factor 
\((1 + 1/2s)^{N-1}\). Indeed since \(\lim_{N \to \infty} (1 + iBT/N)^N = e^{iBT}\), but for the prefactor, we obtain

\[
K(\mu_F^*, \mu_I, T)_{sc} = \frac{e^{-isBT}(1 + \mu_I \mu_F^* e^{iBT})^{2s}}{(1 + |\mu_F|^2)^s(1 + |\mu_I|^2)^s},
\]

which coincides with the exact propagator.

V Conclusions.

In this paper we have examined the problem of a spin in a magnetic field. We have seen that the SPA applied to the calculation of the classical partition function yields the correct result. In quantum mechanics the situation is more complicated. We have reexamined the different approaches that have been used in literature to prove the exactness of SPA in the calculation of the quantum propagator and partition function. All these methods, in particular those proposed by Klauder \[5\] and by Keski-Vakkuri et al. \[14\], make use of the continuum expression of the path integral and hence reproduce the correct result only formally.

To test the validity of the SPA for a spin system we decided to work with the very definition of the path integral, namely with its discrete version. We have written the discrete path integral for the partition function and the propagator in terms of spin coherent states and then considered two different kinds of approximations. In Section III we have expanded the overlaps \(\langle l_k|l_{k-1}\rangle\) to first order in \(\epsilon = T/N\) and then performed an exact integration. On the contrary, in Section IV we have made no expansion for the overlaps and applied instead the complex saddle point method to evaluate the SPA of the path integral. In both cases, we have found that the exact result is reproduced correctly only up to an infinite normalization factor, \(\lim_{N \to \infty} (1 + 1/2s)^N\), which goes to 1 in the classical limit \(s \to \infty\), provided the latter is performed first. Notice that this fact is not a drawback just of the spin Hamiltonian. Indeed one can repeat easily the calculation for \(H \equiv 0\) and get the same infinite constant.

All this seems to suggest that, for those Hamiltonians that do not contain a regulating term (such as a kinetic part), any approximation that restricts the class of quantum paths in phase space, by imposing some regularity
conditions on them, yields a wrong result. In our case this shows up in the appearance of an infinite prefactor.

We would like to conclude by commenting on the paper by Funahashi et al. [27], who have been able to reproduce the exact partition function (with the correct prefactor) by performing an SPA in the discrete path integral, written again in terms of spin coherent states. They have obtained this result by taking into account also the gaussian fluctuations coming from the factor \( \frac{1}{(1 + |\mu|^2)} = \sin \theta / 4 \) appearing in the integration measure. We do not want to explain this technique which is described in detail in [27]. Here we notice only that the inclusion of fluctuations coming from the measure induces a shift in the multiplicative factor appearing in front of the action from \( 2s \) to \( 2s + 1 \). Effectively, we can say that such a method amounts to choosing \( 2s + 1 \) as parameter for a semiclassical expansion. If so, we should not evaluate the path integral following (74), but according to:

\[
K(\mu_F^*, \mu_I, T)_{sc} = N \prod_{j=1}^{N-1} \left( \frac{e^{-\frac{\tilde{A}}{N-1}}}{(1 + \mu_j^2 \mu_j^*)} \right) \left( \pi^{N-1} e^{(2s+1)\tilde{A}_{cl}} \frac{\det \tilde{M}_N}{1/2} \right)^{1/2}, \tag{88}
\]

where \( \tilde{A} = A/2s \) and \( \tilde{M}_N \) is the matrix of gaussian fluctuations of \( (2s + 1)\tilde{A} \) around the classical solution, so that

\[
\det \tilde{M}_N = a_1^2 \cdots a_{N-1}^2 = \left[ \frac{-2(s + 1)}{(1 + (1 + i\epsilon B)^N \mu_{1F}^2)^2} \right]^{2(N-1)}. \tag{89}
\]

It is exactly the factor \( 2s + 1 \) in (89) which cancels the same factor in \( N \), yielding the correct propagator.

This is quite a remarkable result. In our opinion, however, the derivation presented in [27] needs some clarification. To us it seems to be inconsistent to include gaussian fluctuations of the measure factor in the calculation of the SPA, without considering also its contributions to the saddle point equations. In [27] this method works only because the first derivatives of the measure factor do vanish at the classical solution, which in this case correspond to \( \mu = 0 \) or \( \mu = \infty \) (\( \theta = 0, \pi \) in angular coordinates). But this would not be the case in slightly more complicated situations, for example for the calculation of the propagator or when considering complex saddle point solutions.
The Duistermaat-Heckman theorem.

Let us consider an oscillatory integral of the kind

$$I(t) \equiv \left( \frac{t}{2\pi} \right)^n \int_M \sigma e^{itf}$$  \hspace{1cm} (A1)

over a \((2n)\)-dimensional manifold \(M\) with volume form \(\sigma\). If \(M\) is a Riemannian manifold, under rather mild hypotheses, namely that the function \(f\) be a Morse function, i.e. that the Hessian matrix of \(f\) be non-singular at all critical points of \(f\) \((\det \text{Hess}_P(f) \neq 0 \text{ if } \nabla f(P) = 0)\), it is possible to show \([28]\) that for large values of the parameter \(t\), one has

$$I(t) = \sum_P c_P e^{itf(P)} + O(t^{-1})$$  \hspace{1cm} (A2)

where the sum ranges over all critical points of \(f\) and the coefficients are given in terms of the determinant of the gaussian fluctuations of \(f\) around the critical points:

$$c_P = \exp \left[ i\pi \text{sgnHess}_P(f) \right] \left[ \det \text{Hess}_P(f) \right]^{-\frac{1}{2}}.$$  \hspace{1cm} (A3)

Here the signature \(\text{sgn} A\) of a symmetric real-valued nonsingular matrix \(A\) is defined as the number of its positive eigenvalues minus the number of its negative eigenvalues.

Everybody is familiar with the elementary result that the remainder term \(O(t^{-1})\) vanishes identically if \(M\) is the linear manifold \(\mathbb{R}^{2n}\) with volume form \(\sigma = dx_1 \cdots dx_{2n}\) and the function \(f\) is a quadratic form: \(f = \frac{1}{2}Q\vec{x} \cdot \vec{x} - \vec{\xi} \cdot \vec{x}\), \(Q\) being any symmetric real-valued \((2n)\)-dimensional non-singular matrix. In this case the only critical point of \(f\) is \(\vec{x}_0 = Q^{-1}\vec{\xi}\) and \(\text{Hess}_{\vec{x}_0}(f) = Q\), so that \((A2)\) with \(O(t^{-1}) \equiv 0\) gives simply the formula for a gaussian integral.

The theorem of Duistermaat-Heckman \([16]\) and its generalization due to Berline and Vergne \([17]\) establish under which conditions an integral of the kind \((A1)\) can be exactly evaluated in the SPA, i.e. when \(O(t^{-1}) \equiv 0\).

Let \(M\) be a compact \((2n)\)-dimensional manifold with symplectic two-form \(\Omega\) and suppose \(M\) is acted upon by a compact Lie group \(G\), whose action is symplectic and Hamiltonian \([28]\). Let us denote with \(\chi_\eta\) the fundamental vector field on \(M\) generated by the action of the element \(\eta\) in the Lie algebra \(\hat{G}\) of \(G\) and with \(f_\eta\) the associated Hamiltonian function \((i_{\chi_\eta} \Omega = df_\eta)\), where
\(i\) denotes the contraction). Then, if \(\chi_\eta\) is non-degenerate, i.e. it is zero only at the fixed points of \(G\), the following results hold [28]:

1) \(f_\eta\) is a Morse function;

2) 
\[
\left(\frac{t}{2\pi}\right)^n \int_M \frac{\Omega^n}{n!} e^{itf_\eta} = \sum_P c_P e^{itf_\eta(P)}
\]

(A4)

where the sum ranges over the critical points of \(f\), i.e. over the points \(P\) such that \(\chi_\eta(P) = 0\) and the coefficients \(c_P\) are given by

\[
c_P = \frac{i^n}{\lambda_1 \lambda_2 \cdots \lambda_n},
\]

(A5)

the \(\lambda_j\)'s being the coefficients appearing in the matrix \(L_P\) of the derivatives at \(P\) of the components of the vector field \(\chi_\eta\), \([L_P]^{ij} = \left(\frac{\partial \chi^i}{\partial x^j}\right)_P\), which in a suitable positively oriented basis can always be written as

\[
\begin{bmatrix}
0 & \lambda_1 & 0 \\
\lambda_1 & 0 & \lambda_2 \\
0 & \lambda_2 & 0 \\
& & & \ddots \\
0 & \lambda_n & -\lambda_2 \\
& & & & \lambda_2 \\
& & & & & 0 \\
& & & & & & \lambda_1
\end{bmatrix}
\]

From a physical point of view, (A4) can be applied to classical statistical mechanics for the computation of the partition function:

\[
Z = \int_M \frac{\Omega^n}{n!} e^{-\beta H}.
\]

(A6)

In this case the vector field \(\chi_\eta\) is given by the Hamiltonian vector field \(\Delta_H\), where now \(H\) plays the role of the function \(f_\eta\). The assumptions of the Duistermaat-Heckman theorem require \(\Delta_H\) to be non-degenerate and to be the fundamental vector field associated with an element \(\eta\) of the Lie algebra \(\bar{G}\) of a compact Lie group \(G\) acting symplectically on the manifold \(M\). If \(M\) is equipped with a Riemannian metric, formula (A6) is seen to coincide with (A3).
these two conditions holds, we can apply formula (A4) to conclude that (we have set $\beta = -it$):

$$Z(\beta) = \left(\frac{2\pi}{i\beta}\right)^n \sum_P c_P e^{-\beta H(P)},$$  \hspace{1cm} (A7)

where

i) the sum ranges over the stationary points of the Hamiltonian, $\Delta H(P) = 0$;

ii) the coefficients $c_P$ are given by (A5) in terms of the $\lambda_j$’s, the latter being the coefficients of the matrix $[L_P]_{ij} = \left(\frac{\partial \Delta j}{\partial x_i}\right)_P$, written in a suitable positive oriented basis.

Let us go back now to the problem of a spin in a magnetic field. We want to show that the Duistermaat-Heckman theorem can be applied to this problem, so that the exact partition function can be calculated by means of formula (A7). The Hamiltonian vector field associated to the Hamiltonian (9) is

$$\Delta = B \left(-S_2 \frac{\partial}{\partial S_1} + S_1 \frac{\partial}{\partial S_2}\right)$$  \hspace{1cm} (A8)

and it is easy to recognize that it is proportional to the generator $\chi_q = \frac{\partial}{\partial \phi}$ of the rotations about the third axis (i.e. the axis along the constant magnetic field). Thus the Lie group that acts symplectically on the phase space manifold $S^2(s)$ is simply given by $U(1)$ in this case.

To apply (A7) we have first of all to find the critical points of the Hamiltonian, which are given by the North and the South poles of the sphere:

$$P_\pm \equiv (0,0,\pm s)$$  \hspace{1cm} (A9)

and then to compute the coefficients $c_{P_\pm}$ according to (A5). In the tangent space of $P_+$ and $P_-$ we choose to work with the positively oriented basis $(\frac{\partial}{\partial S_1}, \frac{\partial}{\partial S_2})$ and $(\frac{\partial}{\partial S_2}, \frac{\partial}{\partial S_1})$ respectively. With respect to these bases

$$L_{P_\pm}(\Delta) = \mp B \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$  \hspace{1cm} (A10)

so that

$$c_{P_\pm} = \mp \frac{i}{B}.$$  \hspace{1cm} (A11)
We can finally compute the partition function in the SPA as

\[ Z = \frac{2\pi}{\beta B} (e^{\beta Bs} - e^{-\beta Bs}) , \]

which, in agreement with the Duistermaat-Heckman theorem, coincides with the exact partition function (15) for a spin in a magnetic field.

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