Large book–cycle Ramsey numbers

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Abstract

Let $B^{(k)}_n$ be the book graph which consists of $n$ copies of $K_{k+1}$ all sharing a common $K_k$, and let $C_m$ be a cycle of length $m$. We determine the exact value of $r(B^{(2)}_n, C_m)$ for $9n/10 \leq m \leq 10n/9$ and $n$ large enough. This gives an answer to a question by Faudree, Rousseau and Sheehan in a stronger form when $m$ and $n$ are large. Furthermore, we are able to determine the asymptotic value of $r(B^{(k)}_n, C_n)$ for each fixed integer $k \geq 3$. Namely, we prove for each $k \geq 3$, $r(B^{(k)}_n, C_n) = (k + 1 + o(1))n$.

The proofs are mainly built upon results on the (weakly) pancyclic properties of graphs and a refined version of regularity lemma by Conlon.

Keywords: Ramsey number; Regularity method; Pancyclic; Book; Cycle

1 Introduction

For graphs $H_1$ and $H_2$, the Ramsey number $r(H_1, H_2)$ is the minimum integer $N$ such that every red-blue edge coloring of $K_N$ contains either a red $H_1$ or a blue $H_2$. Following from Ramsey’s theorem [32], we know $r(H_1, H_2)$ exists for each $H_1$ and $H_2$. The Ramsey theory became popular after Erdős and Szekeres [19] obtained the classical bound $r(K_k, K_n) \leq \binom{n+k-2}{k-1}$. For $k = n$, this implies $r(K_n, K_n) \leq O(4^n/\sqrt{n})$, and later Erdős [13] proved $r(K_n, K_n) \geq 2^{n/2}$. There are not too many improvements on these bounds over the past 70 years. The best upper bound due to Conlon [10] improves the one by Thomason [42], while the best lower bound is due to Spencer [39] by using the Lovász Local Lemma. Usually, finding the exact or the asymptotic value of $r(H_1, H_2)$ is not easy. The above one is such a notable example.

In this paper, we consider the Ramsey numbers involving large books. The book $B^{(k)}_n$ is a graph which consists of $n$ copies of $K_{k+1}$ all sharing a common $K_k$. When $k = 2$, we write $B_n$ instead of $B^{(2)}_n$ for convenience. Since Rousseau and Sheehan [34] discovered a link between book Ramsey numbers and the theory of strongly regular graphs, the Ramsey numbers involving books have received a great deal of attention. For example, Rousseau and Sheehan [35], and subsequently Burr et al. [8] and Erdős et al. [18] studied the book-tree Ramsey numbers.

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Recently, answering a question of Erdős et al. [17], Conlon [11] established an asymptotic version of a conjecture of Thomason [41] by showing

$$r(B_n^{(k)}, B_n^{(k)}) = (2^k + o(1))n.$$  

For more results on Ramsey numbers of books, we refer the reader to [21, 29, 30, 31] and other related references.

Let $C_m$ be a cycle of order $m$. The study of Ramsey numbers for books versus cycles goes back to [34] by Rousseau and Sheehan. In particular, they proved $r(B_n, C_3) = 2n + 3$ for $n > 1$. In [20], Faudree, Rousseau and Sheehan proved some results of $r(B_n, C_4)$, and further in [22], they determined the value of $r(B_n, C_5)$ for all $n$ and generally,

$$r(B_n, C_m) = \begin{cases} 2m - 1 & \text{if } m \geq 2n + 2, \\ 2n + 3 & \text{if } m \geq 5 \text{ is odd and } n \geq 4m - 13. \end{cases}$$

Improving upon the result in [22], Shi [38] showed $r(B_n, C_m) = 2m - 1$ for $m > \frac{6n+7}{4}$. In the same paper, the author also showed $r(B_n^{(3)}, C_m) = 3m - 2$ for $m > \max\{\frac{6n+7}{4}, 70\}$. In [26], Liu and Li proved that for fixed $k \geq 1$ and fixed odd $m \geq 3$, $r(B_n^{(k)}, C_m) = 2(n+k-1)$ holds for large $n$. One can easily see that $r(B_n, C_m) > 3(n-1) \geq \max\{2m-1, 2n+3\}$ for $6 \leq n \leq m \leq \frac{3m}{2} - 1$, and $r(B_n, C_m) > 3(m-1) \geq \max\{2m-1, 2n+3\}$ for $\frac{2m}{3} + 2 \leq m \leq n$. This indicates that the formula for $r(B_n, C_m)$ varies when $m$ and $n$ change, especially when $m$ and $n$ are nearly equal. As mentioned in [22], “the problem of computing $r(B_n, C_m)$ when $m$ is odd and $m$ and $n$ are nearly equal provides an unanswered test of strength”.

The goal of this paper is to study the Ramsey numbers $r(B_n^{(k)}, C_m)$ when the size $n$ of the book is as large as $m$. Firstly, we determine the exact value of $r(B_n, C_m)$ for $\frac{9n}{10} \leq m \leq \frac{10n}{9}$ and $n$ large, which provides an answer to the question of Faudree, Rousseau and Sheehan [22] in a stronger form when $m$ and $n$ are large.

**Theorem 1** For all sufficiently large $n$,

$$r(B_n, C_m) = \begin{cases} 3m - 2 & \text{if } \frac{9n}{10} \leq m \leq n, \\ 3m - 1 & \text{if } m = n + 1, \\ 3n & \text{if } n + 2 \leq m \leq \frac{10n}{9}. \end{cases}$$

Furthermore, we obtain the asymptotic value of $r(B_n^{(k)}, C_m)$ for each fixed integer $k \geq 3$.

**Theorem 2** Let $k \geq 3$ be a fixed integer. We have

$$r(B_n^{(k)}, C_n) = (k + 1 + o(1))n.$$  

We write $m \sim n$ if $\lim_{n \to \infty} m/n = 1$. The following is an immediate corollary of Theorem 2.

**Corollary 1** Let $k \geq 3$ be a fixed integer. If $m \sim n$, then

$$r(B_n^{(k)}, C_m) = (k + 1 + o(1))n.$$  

The rest of the paper is organized as follows. In Section 2, we will collect several results which will be used in proofs of our main results. In Section 3, we will present proofs of Theorem 1 and Theorem 2. We will conclude the paper with some open questions.
2 Preliminaries

In this section, we collect a number of previous results which are needed for our proofs. The crucial tool we use is a refined version of the original regularity lemma, and we will first state it and its related results as follows.

2.1 Regularity method

Let $G$ be a graph defined on vertex set $V = V(G)$. For $X, Y \subseteq V$, denote $e(X, Y)$ by the number of edges between $X$ and $Y$ of $G$. The ratio

$$d_G(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

is called the edge density of $(X, Y)$, which can be referred as the probability that a pair $(x, y)$ selected randomly from $X \times Y$ is an edge.

For $\epsilon > 0$, a pair $(U, W)$ of nonempty disjoint sets $U, W \subseteq V$ is called $\epsilon$-regular if

$$|d_G(X, Y) - d_G(U, W)| \leq \epsilon$$

for every $X \subseteq U, Y \subseteq W$ such that $|X| \geq \epsilon |U|$ and $|Y| \geq \epsilon |W|$.

A pair $(U, W)$ is called $(\epsilon, d)$-regular if it is $\epsilon$-regular and $d_G(U, W) \geq d$. We say a subset $U$ is $\epsilon$-regular if the pair $(U, U)$ is $\epsilon$-regular.

A partition $V(G) = \bigsqcup_{i=1}^m V_i$ of the vertex set of a graph $G$ is equitable if $||V_i| - |V_j|| \leq 1$ for all $i$ and $j$.

We will use the following lemma from Conlon [11, Lemma 3], which is a refined version of the original regularity lemma [40].

Lemma 1 For every $0 < \epsilon < 1$ and natural number $m_0$, there exists a natural number $M$ such that every graph $G$ with at least $m_0$ vertices has an equitable partition $V(G) = \bigsqcup_{i=1}^m V_i$ with $m_0 \leq m \leq M$ parts and subsets $W_i \subset V_i$ such that $W_i$ is $\epsilon$-regular for all $i$ and, for all but $em^2$ pairs $(i, j)$ with $1 \leq i \neq j \leq m$, $(V_i, V_j)$, $(W_i, V_j)$ and $(W_i, W_j)$ are $\epsilon$-regular with $|d(W_i, V_j) - d(V_i, V_j)| \leq \epsilon$ and $|d(W_i, W_j) - d(V_i, V_j)| \leq \epsilon$.

The following is a standard counting lemma, see, e.g., [33, Theorem 18] by Rödl and Schacht.

Lemma 2 For any $\delta > 0$, there is $\eta > 0$ such that if $U_1, \ldots, U_k$ are (not necessarily distinct) vertex sets with $(U_i, U_j)$ $\eta$-regular of density $d_{ij}$ for all $1 \leq i \leq j \leq k$, then there are

$$\prod_{i < j} d_{ij} \prod_{i=1}^k |U_i| \pm \delta \prod_{i=1}^k |U_i|$$

copies of $K_k$ with vertex $u_i \in U_i$ for each $1 \leq i \leq k$.

We will use the following form of the counting lemma from Conlon [11, Lemma 5].

Lemma 3 For any $\delta > 0$ and any natural integer $k$, there is $\eta > 0$ such that if $U_1, \ldots, U_k$, $U_{k+1}, \ldots, U_{k+\ell}$ are (not necessarily distinct) vertex sets with $(U_i, U_i')$ $\eta$-regular of density $d_{i,i'}$
for all $1 \leq i < i' \leq k$ and $1 \leq i \leq k < i' \leq k + \ell$ and $d_{i,i'} \geq \delta$ for all $1 \leq i < i' \leq k$, then there is a copy of $K_k$ with vertex $i \in U_i$ for each $1 \leq i \leq k$ which is contained in at least

$$\sum_{j=1}^{\ell} \left( \prod_{i=1}^{k} d_{i,k+j} - \delta \right) |U_{k+j}|$$

copies of $K_{k+1}$ with vertex $k + 1$ in $\bigcup_{j=1}^{\ell} U_{k+j}$.

The next lemma due to Benevides and Skokan [3] is a slightly stronger version compared to the original one established by Luczak [27, Claim 3].

**Lemma 4** For every $0 < \beta < 1$, there exists an $n_0$ such that for every $n > n_0$ the following holds: Let $G$ be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = n$. Furthermore, let the pair $(V_1, V_2)$ be $\epsilon$-regular with density at least $\beta$ for some $\epsilon$ satisfying $0 < \epsilon < \beta/100$. Then for every $\ell, 1 \leq \ell \leq n - 5\beta n/\beta$, and for every pair of vertices $v' \in V_1, v'' \in V_2$ satisfying $d(v'), d(v'') \geq 4\beta n/5$, $G$ contains a path of length $2\ell + 1$ connecting $v'$ and $v''$.

We will also need the following lemma which will be used to find long odd cycles in graphs.

**Lemma 5** Suppose that $(V_i, W_j)$ and $(W_j, V_k)$ are $\epsilon$-regular pairs with density at least $\beta$, here $\epsilon$ is sufficiently small. If $|V_i| = |V_k| = n$ and $|E(W_j)| \geq |W_j|^2/5$, then there is an edge $uv \in E(W_j)$ such that $u$ has at least $4\beta n/5$ neighbors in $V_i$ and $v$ has at least $4\beta n/5$ neighbors in $V_k$.

**Proof:** Let $H$ be the $2|W_j|$-core of the subgraph induced by $W_j$, i.e., $H$ is the maximum induced subgraph of $W_j$ with minimum degree at least $2|W_j|$. As $|E(W_j)| \geq |W_j|^2/5$ and $\epsilon$ is small enough, we can see $H$ is not empty and $|V(H)| \geq 2\epsilon|W_j|$. Since $(V_i, W_j)$ is an $\epsilon$-regular pair, all but at most $\epsilon|W_j|$ vertices in $H$ have at least $4\beta n/5$ neighbors in $V_i$. We assume $u$ is such a vertex. The definition of $H$ gives $N_H(u) \geq 2\epsilon|W_j|$. Now, $(W_j, V_k)$ being an $\epsilon$-regular pair yields that there is a vertex $v \in N_H(u)$ having at least $4\beta n/5$ neighbors in $V_k$. The edge $uv$ is a desired one and the proof is complete. \qed

The following lemma by Luczak [27, Claim 7] is a key ingredient in the proof of Theorem 2.

**Lemma 6** For every $0 < \delta < 10^{-15}, \alpha > 2\delta$ and $n \geq \exp(\delta^{-16}/\alpha)$ the following holds. Each graph $G$ on $n$ vertices which contains no odd cycles longer than $\alpha n$ contains subgraphs $G'$ and $G''$ such that:

1. $V(G') \cup V(G'') = V(G), V(G') \cap V(G'') = \emptyset$ and each of the sets $V(G')$ and $V(G'')$ is either empty or contains at least $\alpha \delta n/2$ vertices;
2. $G'$ is bipartite;
3. $G''$ contains not more than $\alpha n |V(G'')|/2$ edges;
4. all except no more than $\delta n^2$ edges of $G$ belong to either $G'$ or $G''$.

### 2.2 Pancyclic properties of graphs

We will also need results on the (weakly) pancyclic properties of graphs for the proofs of our results. Let us recall some useful definitions. For a graph $G$, we use $g(G)$ and $c(G)$ to denote its *girth* and *circumference*, i.e., the length of a shortest cycle and a longest cycle of $G$. 


A graph is called weakly pancyclic if it contains cycles of every length between its girth and its circumference. A graph is pancyclic if it is weakly pancyclic with girth 3 and circumference \( n = |G| \).

We say a graph is 2-connected if it remains connected after the deletion of any vertex. The following result is due to Dirac [12].

**Lemma 7** Let \( G \) be a 2-connected graph of order \( n \) with minimum degree \( \delta \). Then \( c(G) \geq \min\{2\delta, n\} \).

Dirac’s result tells us that the circumference of a 2-connected graph can not be too small. In particular, if \( \delta = \delta(G) \geq n/2 \), then \( c(G) = n \). This is the famous condition for a graph to be hamiltonian. The following result by Bondy [5] is an extension of Dirac’s result.

**Lemma 8** If a graph \( G \) of order \( n \) has minimum degree \( \delta(G) \geq n/2 \), then \( G \) is pancyclic, or \( n = 2r \) and \( G = K_{r,r} \).

There are many extensions of the above results, and we will use the following one on graphs to be weakly pancyclic by Brandt, Faudree, and Goddard [7].

**Lemma 9** Let \( G \) be a 2-connected nonbipartite graph of order \( n \) with minimum degree \( \delta(G) \geq n/4 + 250 \). Then \( G \) is weakly pancyclic unless \( G \) has odd girth 7, in which case it has every cycle from 4 up to its circumference except the 5-cycle.

### 3 Proofs of main results

In this section, we will give proofs for our main results. Let graph \( G \) be defined on vertex set \( V \). For any subset \( U \subseteq V \), we use \( G[U] \) to denote the subgraph induced by \( U \). For any vertex \( v \in V \), let \( N_G(v) \) and \( d_G(v) \) be its neighborhood and degree respectively. Write \( N_G(v, U) = N_G(v) \cap U \) and \( d_G(v, U) = |N_G(v, U)| \).

In the following, when considering a red-blue edge coloring of \( K_N \), we always use \( R \) and \( B \) to denote the subgraphs induced by red and blue edges, respectively. We also suppose that \( n \) is large enough.

#### 3.1 Proof of Theorem 1

**I** \( \frac{9n^3}{16} \leq m \leq n \)

The lower bound \( r(B_n, C_m) > 3m - 3 \) holds since the graph with three disjoint copies of \( K_{m-1} \) contains no \( C_m \) and its complement contains no \( B_n \). Thus it suffices to prove the upper bound. Let \( N = 3m - 2 \), and consider a red-blue edge coloring of \( K_N \).

If there is a vertex \( v \) with \( d_R(v) \geq 2n \), then we take a subset \( X \subset N_R(v) \) with \( |X| = 2n \). If further there exists a vertex \( u \in X \) with \( d_R(u, X) \geq n \), then there is a red \( B_n \). Thus we assume \( d_B(u, X) \geq n \) for each \( u \in X \). By Lemma 8, \( B[X] \) is pancyclic, or \( B[X] = K_{n,n} \). There will be a blue \( C_m \) if \( B[X] \) is pancyclic, so we assume \( B[X] = K_{n,n} \) with bipartition \( (X_1, X_2) \). If there exists a vertex \( y \in V \setminus (X \cup \{v\}) \) such that \( d_B(y, X_i) \geq 1 \) for each \( i \in \{1, 2\} \), then \( G[X \cup \{y\}] \) contains blue cycles of length between 3 and \( 2n + 1 \) and it definitely contains a blue
$\text{Case 1. } B - u \text{ contains three components.}$

Denote $V_1, V_2$ and $V_3$ by the vertex sets of these three components of $B - u$. We claim $|V_i| = m - 1$ for each $i \in \{1, 2, 3\}$. Otherwise, assume that $V_1$ is the largest component in $B - u$. Since each component has size at least $3m - 2n - 2$, we have

$$m \leq |V_1| \leq 3m - 3 - 2(3m - 2n - 2) = 4n - 3m + 1.$$ 

Thus $\delta(B[V_1]) \geq 3m - 2n - 3 > |V_1|/2$ and so $B[V_1]$ is pancyclic by Lemma 8, which forces a blue $C_m$. Moreover, each $V_i$ induces a blue clique $K_{m-1}$. Indeed, if there is a red edge in some $V_i$, then we can find a red $B_{2n-2}$ and hence a red $B_n$ since all edges between these components are red. Note that $d_B(u) \geq 3m - 2n - 2 \geq 4$, it follows that $u$ has at least two blue neighbors in some $V_i$, and then a blue $C_m$ will appear.

$\text{Case 2. } B - u \text{ contains exactly two components.}$

Denote $V_1$ and $V_2$ by the vertex sets of these two components with $|V_1| \leq |V_2|$. We claim $3m - 2n - 2 \leq |V_1| \leq m - 1$. Otherwise, $m \leq |V_1| \leq \frac{3m-3}{2}$ and so $\delta(B[V_1]) \geq 3m - 2n - 3 > |V_1|/2$. This again gives a blue $C_m$ by Lemma 8. Thus $2m - 2 \leq |V_2| \leq 2n - 1$.

We claim $B[V_2]$ is nonbipartite. Otherwise, suppose $B[V_2]$ is a bipartite graph with bipartition $(X,Y)$, where $|X| \geq |Y|$. Since $X$ induces a red clique and all edges between $V_1$ and $X$ are red, it follows

$$|V_1| + |X| \geq |V_1| + \frac{N - 1 - |V_1|}{2} \geq 3m - n - 3 \geq n + 2.$$ 

Clearly, there exists a red $B_n$ in $V_1 \cup X$. Moreover, if $B[V_2]$ is 2-connected, then the existence of a blue $C_m$ in $B[V_2]$ follows from Lemma 7 and Lemma 9 since $\delta(B[V_2]) \geq 3m - 2n - 3 \geq \frac{|V_2|}{4} + 250$.

Therefore, we assume $B[V_2]$ contains a cut vertex $w$. Denote $V'_2$ and $V''_2$ by the vertex sets of these two components of $B[V_2] - w$. We may assume $|V'_2|, |V''_2| \leq m - 1$. Otherwise, similar to Case 1, Lemma 8 gives a blue $C_m$. Note that $|V_1 \cup V'_2 \cup V''_2| = 3m - 4$, and $|V_1|, |V'_2|, |V''_2| \leq m - 1$. Thus we get $m - 2 \leq |V_1|, |V'_2|, |V''_2| \leq m - 1$. We claim each of $V_1, V'_2$ and $V''_2$ induces a blue
clique. Otherwise, we can easily get a red $B_{2m-3}$ by noting that all edges between any two sets of $V_1, V_2', V_2''$ are red, which contains a red $B_n$.

If $|V_1| = m - 2$, then $|V_2'| = |V_2''| = m - 1$. Since $\delta(B[V_2]) \geq 3m - 2n - 3 \geq 4$, we have either $d_B(w, V_2') \geq 2$ or $d_B(w, V_2'') \geq 2$ and hence either $V_2' \cup \{w\}$ or $V_2'' \cup \{w\}$ gives a blue $C_m$. If $|V_1| = m - 1$, then without loss of generality we can assume $|V_2'| = m - 1$ and $|V_2''| = m - 2$. If $d_B(u, V_1) \geq 2$, then there is a blue $C_m$. Thus $d_B(u, V_1) \geq m - 2$. Similarly, $d_B(u, V_2') \geq m - 2$. Further, if $u$ has a red neighbor $x$ in $V_2''$, then the subgraph induced by $\{u, x\} \cup V_1 \cup V_2''$ contains a red $B_{2m-4}$ and so a red $B_n$. Thus $u$ is completely blue-adjacent to $V_2''$. Similarly, we can show $w$ is completely blue-adjacent to $V_2''$. Now, the subgraph induced by $\{u, w\} \cup V_2''$ contains a blue $C_m$.

The proof for this case is complete. \(\square\)

(II) $n = m - 1$

Denote $3K_{n-1}$ by the three disjoint copies of $K_{n-1}$. Let $K_1 + 3K_{n-1}$ be the graph obtained from $3K_{n-1}$ by adding a new vertex $u$ and all edges between $u$ and the vertices of $3K_{n-1}$. The lower bound $r(B_n, C_{n+1}) > 3n - 2$ follows from the fact that $K_1 + 3K_{n-1}$ contains no $C_{n+1}$ and its complement contains no $B_n$. The proof of the upper bound $r(B_n, C_{n+1}) \leq 3n - 1$ is very similar to the case of $m = n$ and we omit it.

(III) $n + 2 \leq m \leq \frac{10n}{3}$

The lower bound $r(B_n, C_m) > 3n - 1$ can be seen as follows. Let $K_{n-1}, n-1, n-1$ be the complete tripartite graph with color classes $V_1, V_2$ and $V_3$, and let $u$ and $w$ be two new vertices. Define a graph $G$ from $K_{n-1,n-1,n-1}$ by adding all edges between $u$ and $V_1$, and all edges between $w$ and $V_3$. It is clear that $G$ contains no $B_n$ and its complement contains no $C_m$.

Now, let $N = 3n$, and consider a red-blue edge coloring of $K_N$ on vertex set $V$. Since the proof for this case is similar to that in (I), we only sketch the proof as follows.

If there is a vertex $v$ with $d_R(v) \geq 2m$, then repeating arguments in (I) gives either a red $B_n$ or a blue $C_m$. Thus we suppose $\delta(B) \geq 3n - 2m$. By the same argument as the one in (I), $B$ contains a cut vertex $u$ such that $B - u$ has at most three components. If $B - u$ contains three components, then we can find a red $B_n$ since the largest component has size at least $n$ and all edges between different components are red. So we assume that $B - u$ contains exactly two components. Denote $V_1$ and $V_2$ by the vertex sets of these two components with $|V_1| \leq |V_2|$. Similarly, we can show $B[V_2]$ contains a cut vertex $w$, which implies the blue subgraph induced by $V \setminus \{u, w\}$ contains three components. We are done as there is a red $B_n$ in $V \setminus \{u, w\}$. This completes the proof for this case and so the proof of Theorem 1. \(\square\)

3.2 Proof of Theorem 2

We note for $k \geq 3$, the graph with $k + 1$ disjoint copies of $K_{n-1}$ contains no $C_n$ and its complement contains no $B_{n}^{(k)}$, so we have $r(B_n^{(k)}, C_n) \geq (k + 1)(n - 1) + 1$. Therefore, it suffices to establish the upper bound. The proof is by induction on $k \geq 3$. The base case where $k = 3$ is built upon an induction idea and the case of $k = 2$.

**Step 1:** $B_n^{(3)}$ versus $C_n$

Let $0 < \xi < 1/10$, and $N = \lceil (4 + \xi)n \rceil$. Consider a red-blue edge coloring of $K_N$ on vertex set $V$. If there is a vertex $v \in V$ with $d_R(v) \geq 3n - 2$, then by Theorem 1, we can find either a
red $B_n$ or a blue $C_n$. In the former case, the red $B_n$ together with $v$ form a red $B_n^{(3)}$. We are done in either case. Thus we assume $\delta(B) > (1 + \xi)n$.

We may assume $B$ is nonbipartite. Otherwise, one of its color classes induces a red clique of size at least $N/2 \geq n + 3$ which will give a red $B_n^{(3)}$.

Moreover, we may assume $B$ is 2-connected. Otherwise, suppose there exists a vertex $u$ such that $B - u$ is disconnected. Then $B - u$ has two or three components. Suppose that $V_1 \subseteq V$ induces the smallest component of $B - u$. It is clear that

$$(1 + \xi)n \leq |V_1| \leq (N - 1)/2 \leq (2 + \xi/2)n.$$  

Since $\delta(B[V_1]) \geq (1 + \xi)n - 1 > |V_1|/2$, it follows from Lemma 8 that $B[V_1]$ is pancyclic, which implies that there is a blue $C_n$.

Now, for any $0 < \xi < 1/10$, if we take $n_0 = \lceil 1000/(3\xi) \rceil$, then $\delta(B) \geq (1 + \xi)n \geq N/2 + 250$ for all $n \geq n_0$. Therefore, we can find a blue $C_n$ by Lemma 7 and Lemma 9. The proof for $k = 3$ is complete.

**Step 2: $B_n^{(k)}$ versus $C_n$ for $k \geq 4$**

For $k = 3$, it has been verified particularly for all sufficiently large $n$ in Step 1. Suppose the assertion holds for small $k \geq 3$, and we will prove it also holds for $k + 1$.

Let $0 < \xi < 1/10$ and $N = [(k + 2)(1 + \xi)n]$, and consider a red-blue edge coloring of $K_N$ on vertex set $V$. In the following, we will omit the ceiling and floor as it does not affect the result. We choose constants $0 < \delta < 10^{-15}$ and $\beta > 0$ which are sufficiently smaller than $\xi$ and $1/k^2$. Lemma 3 with $\delta$ and $k + 1$ gives us a constant $\eta$. Let $\epsilon < \eta$ be small enough compared with $\beta$. Take

$$\alpha = \frac{1}{k + 2} - \beta - \sqrt{\epsilon}.$$  

To be clear, let us note that relationships between parameters $\xi, \epsilon, \delta$ and $\beta$ are as follows:

$$\sqrt{\epsilon} \ll \beta \ll \min\{\xi, 1/k^2\} \text{ and } \sqrt{\delta} \ll \min\{\xi, 1/k^2\}. \quad (1)$$

Let $M$ be given by Lemma 1 with $\epsilon$ and large $m_0$. We apply Lemma 1 to the red subgraph $R$ and obtain an equitable partition $V(G) = \bigsqcup_{i=1}^{m_0} V_i$ and subsets $W_i \subseteq V_i$ such that $W_i$ is $\epsilon$-regular for all $i$ and, for all but $cm^2$ pairs $(i, j)$ with $1 \leq i \neq j \leq m$, $(V_i, V_j), (W_i, V_j)$ and $(W_i, W_j)$ are $\epsilon$-regular with $|d_R(W_i, V_j) - d_R(V_i, V_j)| \leq \epsilon$ and $|d_R(W_i, W_j) - d_R(V_i, V_j)| \leq \epsilon$. For convenience, we will assume $|V_i| = \frac{n}{m}$ for all $1 \leq i \leq m$. If $n$ is large enough, then $\frac{n}{m} \geq \frac{n}{m} \geq \max\{n_1, n_2\}$, where $n_1$ is given by Lemma 4 with $\beta - \epsilon$, and $n_2$ is given by Lemma 6 with $\delta$ and $\alpha$. Note that a partition obtained by applying Theorem 1 for $R$ is also such a partition for $B$.

Let $F$ be the reduced graph defined on $\{v_1, v_2, \ldots, v_m\}$, in which $v_i$ and $v_j$ are non-adjacent in $F$ if the pairs $(V_i, V_j), (W_i, V_j)$ and $(W_i, W_j)$ are not all $\epsilon$-regular with $|d_R(W_i, V_j) - d_R(V_i, V_j)| \leq \epsilon$ and $|d_R(W_i, W_j) - d_R(V_i, V_j)| \leq \epsilon$. Then the number of edges of $F$ is at least $(1 - \epsilon)m^2$. Therefore, by deleting at most $\sqrt{\epsilon}m$ vertices, we may assume that each vertex is adjacent to at least $(1 - \sqrt{\epsilon})m$ vertices. In what follows, when referring to the reduced graph, we will assume that these vertices have been removed. Color an edge $v_iv_j$ red if $d_R(V_i, V_j) \geq 1 - \beta$, or blue if $d_R(V_i, V_j) \geq \beta$. For each remaining vertex $v_i$, we color $v_i$, either red or blue, depending on which color has higher density inside $W_i$, breaking ties arbitrarily. Let $F_R$ and $F_B$ be the subgraphs induced by red edges and blue edges of $F$, respectively.
For each red vertex $v_a$, we have the red density $d_R(W_a) \geq 1/2 > \delta$. If $d_{F_B}(v_a) \geq \frac{(1-0.5\xi)m}{k+2}$, then we apply Lemma 3 with $U_i = W_a$ for $1 \leq i \leq k+1$ and $U_{k+1+s}$ equal to each of the $V_s$ for which $(W_a, V_s)$ is $\epsilon$-regular and the edge $v_a v_s$ is red. We conclude that there is a red $K_{k+1}$ which is contained in at least

$$\sum_s (d_R(W_a, V_s)^{k+1} - \delta) |V_s| \geq \left((1 - \beta - \epsilon)^{k+1} - \delta\right) \frac{N}{m} \cdot \frac{(1-0.5\xi)m}{k+2}$$

$$\geq \left((1 - \beta - \epsilon)^{k+1} - \delta\right) \left(1 + \frac{\xi}{3}\right)n$$

red $K_{k+2}$. We notice this quantity is at least $n$ since $\beta, \epsilon$ and $\delta$ are sufficiently small in terms of $1/k^2$ and $\xi$ by the assumption (1), so we are through as there is a red $B_n^{(k+1)}$. Therefore, for the rest of the proof, we assume $d_{F_B}(v_a) < \frac{(1-0.5\xi)m}{k+2}$ for each red vertex $v_a$. Equivalently,

$$d_{F_B}(v_a) \geq \left(\frac{k + 1 + 0.5\xi}{k + 2} - \sqrt{\epsilon}\right)m$$

(2)

for each red vertex $v_a$.

Suppose that there is a vertex $v_a$ in $F$ satisfying $m' := d_{F_B}(v_a) \geq \frac{(k+1}{k+2} + \beta)m$. Note that an edge $v_ar_j$ in $F$ is red if and only if $d_R(V_i, V_j) \geq 1 - \beta$, by average there exists a vertex $u \in V_a$ such that

$$d_R(u) \geq (1 - \beta) \frac{N}{m} \cdot m' = (1 - \beta)(k+2)(1 + \xi) \left(\frac{k + 1}{k + 2} + \beta\right)n > (k + 1)(1 + \xi)n,$$

since $\beta$ is sufficiently small in terms of $1/k^2$ and $\xi$. Thus by induction on $k$, there is either a blue $C_n$ or a red $B_n^{(k)}$ which together with $u$ gives a red $B_n^{(k+1)}$. We are through for this case. Therefore, we suppose each vertex $v$ in the reduced graph $F$ satisfying

$$m'' := d_{F_B}(v) \geq \left(\frac{1}{k + 2} - \beta - \sqrt{\epsilon}\right)m,$$

If there is an odd blue cycle of length $2\ell + 1 \geq (\frac{1}{k + 2} - \beta - \sqrt{\epsilon})m$ in $F_B$, then we can apply techniques in Luczak [27, Claim 4] and Lemma 4 with $\beta' = \beta - \epsilon$ to show that the blue graph $B$ contains all cycles of length from $4\ell$ to $2\ell(1 - 15\epsilon/\beta) \geq (1 + \zeta)n$ for some positive real number $\zeta$ by the assumption (1). Hence we can find a blue $C_n$ as desired. Therefore, we assume $F_B$ contains no odd cycle with length at least $(\frac{1}{k + 2} - \beta - \sqrt{\epsilon})m$.

Let $C$ be a longest even cycle in $F_B$. Suppose $|C| = d$ and $d \geq m''$ by the famous Erdős–Gallai theorem. Without loss of generality, we assume the vertices of $C$ are $v_1, v_2, \ldots, v_d$.

If there is a blue vertex $v_b \in C$, then we note $d_B(V_{b-1}, W_b) \geq \beta - \epsilon$ and $d_B(W_b, V_{b+1}) \geq \beta - \epsilon$ by the choice of $W_b$. Here the addition is under modulo $d$. Now, we apply Lemma 5 with $V_i = V_{b-1}, W_j = W_b, V_k = V_{b+1}$ and $\beta' = \beta - \epsilon$ to get a blue edge $xy \in E(W_b)$ such that $x$ has at least $\frac{1}{k + 2} \cdot \frac{N}{m}$ vertices in $V_{b-1}$ and $y$ has at least $\frac{1}{k + 2} \cdot \frac{N}{m}$ vertices in $V_{b+1}$. Now, we again apply techniques in Luczak [27, Claim 4] and Lemma 4 together with the blue edge $xy$ to obtain that the blue graph $B$ contains all cycles of length from $2d$ to $(1 + \zeta')n$ for some $\zeta' > 0$. Hence we can find a blue $C_n$ as desired.
Therefore, we get each vertex \( v_i \in C \) is red. Consequently, by noting (2), the number of edges in \( F_B \) can be bounded from below as follows:

\[
|E(F_B)| \geq \sum_{i=1}^{d} d_{F_B}(v_i) - \left( \frac{d}{2} \right) \geq \left( \frac{k + 1 + 0.5\xi}{k + 2} - \sqrt{\epsilon} \right) dm - \left( \frac{d}{2} \right).
\]

Since \( d \geq m'' \) and the right hand side above is increasing on \( d \) when \( d \geq \left( \frac{1}{k+2} - \beta - \sqrt{\epsilon} \right)m \), it follows that

\[
|E(F_B)| \geq \left( \frac{k + 1 + 0.5\xi}{k + 2} - \sqrt{\epsilon} \right) \left( \frac{1}{k + 2} - \beta - \sqrt{\epsilon} \right) m^2 - \frac{1}{2} \left( \frac{1}{k + 2} - \beta - \sqrt{\epsilon} \right)^2 m^2
\]

\[
\geq \left( \frac{2k + 1}{2(k + 2)^2} + \frac{\xi}{3(k + 2)^2} \right) m^2,
\]
by noting that \( \beta \) and \( \sqrt{\epsilon} \) are sufficiently small in terms of \( 1/k^2 \) and \( \xi \).

We next apply Lemma 6 to obtain an upper bound on \( |E(F_B)| \). Actually, we will apply Lemma 6 with \( G = F_B \), \( \delta \), and \( \alpha = \frac{1}{k+2} - \beta - \sqrt{\epsilon} \) defined as above. Then there are two subgraphs \( G' = F'_{B} \) and \( G'' = F''_{B} \) satisfying all properties listed in the lemma. If \( F'_{B} \) is empty, then we get

\[
|E(F_B)| \leq \left( \frac{1}{k+2} - \beta - \sqrt{\epsilon} \right) m^2 + \delta m^2,
\]
which clearly is a contradiction to the lower bound on \( |E(F_B)| \) from (3). If \( F'_{B} \) is not empty, then we assume \( A \) and \( B \) are the two color classes of \( F'_{B} \). We claim

\[
\max\{|A|, |B|\} \leq k + 1 + \frac{(1 - 0.5\xi)m}{k + 2}.
\]

Otherwise, suppose \( |A| \geq k + 1 + \frac{(1 - 0.5\xi)m}{k + 2} \) without loss of generality. Since all except no more than \( \delta m^2 \) edges of \( F_B \) belong to either \( F'_{B} \) or \( F''_{B} \), by (4) of Lemma 6, it follows that \( A \) contains at least \( \left( \frac{|A|}{2} \right) - (\epsilon + \delta)m^2 \geq (1 - \gamma)\left( \frac{|A|}{2} \right) \) red edges, here \( 0 \leq \gamma < 3(k+2)(\epsilon + \delta) \). By Turán’s Theorem, \( A \) contains a red \( K_{k+1} \). Relabelling vertices if necessary, we assume the vertices of this red \( K_{k+1} \) are \( v_1, \ldots, v_{k+1} \). By deleting at most \( \sqrt{|\gamma|} |A| \) vertices, we may assume that each vertex of \( A \) is red-adjacent to at least \( (1 - \sqrt{|\gamma|}|A|) \) vertices in \( A \). Let

\[
m''' = (1 - (k + 1)\sqrt{|\gamma|})|A| \geq \frac{(1 - 0.6\xi)m}{k + 2}.
\]

here we note \( \beta \) and \( \delta \) are sufficiently small in terms of \( \xi \) and \( 1/k^2 \). The greedy idea shows \( v_1, \ldots, v_{k+1} \) have at least \( m''' \) common red neighbors, say \( v_{k+2}, \ldots, v_{k+1+m'''\kern-2pt} \). Thus \( A \) contains a red book \( B_{m''}^{k+1} \). We apply Lemma 3 with \( U_i = V_i \) for \( 1 \leq i \leq k + 1 \) and \( U_{k+1+j} = V_{k+1+j} \) for \( 1 \leq j \leq m''' \), which will again give us a red \( B_{m''}^{k+1} \). This proves the claim.

Consequently, \( |E(F_B)| \leq \left( k + 1 + \frac{(1 - 0.5\xi)m}{k + 2} \right)^2 \), and the number of edges in \( F_B \) can be bounded from above as follows:

\[
|E(F_B)| \leq \left( k + 1 + \frac{(1 - 0.5\xi)m}{k + 2} \right)^2 + \frac{1}{2} \left( \frac{1}{k + 2} - \beta - \sqrt{\epsilon} \right) m^2 + \delta m^2
\]

\[
\leq \left( \frac{k + 4}{2(k + 2)^2} - \frac{\xi}{2(k + 2)^2} \right) m^2,
\]
(4)
by noting that $\beta, \sqrt{c}$ and $\delta$ are sufficiently small in terms of $1/k^2$ and $\xi$. This is a contradiction to the lower bound on $|E(F_B)|$ from (3) for $k \geq 3$ and large $m$.

This completes the proof of the induction step and hence Theorem 2. \hfill $\Box$

4 Concluding remarks

Theorem 1 gives the exact value of $r(B_n, C_m)$ when $\frac{9n}{10} \leq m \leq \frac{10n}{9}$ and $n$ large. A careful calculation can give us a slightly larger range for $m$. i.e., for any sufficiently small $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon)$ such that for all $n > n_0$, $r(B_n, C_m) = 3m - 2$ for $(8/9 + \epsilon)n \leq m \leq n$ and $r(B_n, C_m) = 3n$ for $n + 2 \leq m \leq (9/8 - \epsilon)n$. For the range $n/4 < m \leq 8n/9$ and $9n/8 \leq m \leq 3n/2$, the Ramsey number of $r(B_n, C_m)$ is much less known. Theorem 1 particularly implies that $r(B_n, C_n) = 3n - 2$ for $n$ large enough. It would be interesting to know for $k \geq 3$, does $r(B_n^{(k)}, C_n) = (k + 1)(n - 1) + 1$ hold for all large $n$?

We next discuss some related problems when the size of the book is small. When $n = 1$, $B_1^{(k-1)}$ is the complete graph $K_k$. The study of $r(C_\ell, K_n)$ is fruitful. In particular, the case of $r(K_3, K_n)$ has received a lot of attention, see [1, 4, 23, 25, 37] and other related references. For large $\ell$, Erdős, Faudree, Rousseau and Schelp [16] conjectured that for every $\ell \geq n \geq 3$, except for $\ell = n = 3$, $r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1$. (5)

Bondy and Erdős [6] verified it for $n > 3$ and $\ell \geq n^2 - 2$, which was slightly improved by Schiermeyer [36] and further by Nikiforov [28] for $\ell \geq 4n + 2$. Recently, Keevash, Long and Skokan [24] confirmed this conjecture in a stronger form by proving (5) holds for $\ell \geq c \log n / \log \log n$, where $c > 0$ is constant. This is best possible up to the constant factor $c$ since the authors also proved that for any $\epsilon > 0$, there exists $n_0(\epsilon)$ such that $r(C_\ell, K_n) > n \log n \gg (n - 1)(\ell - 1) + 1$ for all $n \geq n_0(\epsilon)$ and $3 \leq \ell \leq (1 - \epsilon) \log n / \log \log n$. It is challenging to determine the value of $r(C_\ell, K_n)$ for each fixed $\ell > 3$ which is widely open. Erdős asked if there exists a constant $\epsilon > 0$ such that $r(C_4, K_n) = o(n^{2-\epsilon})$, see [9, 15]. Let us point out that by using the “random shifting” technique, Alon and Rödl [2] have shown that $r(C_4, C_4, K_n) \geq \Omega(n^2/P(\log n))$. Here $P(\log n)$ is a polynomial of $\log n$, and $r(C_4, C_4, K_n)$ is the minimum integer $N$ such that every three edge coloring of $K_N$ contains a monochromatic $C_4$ in one of the first two colors or a monochromatic $K_n$ in the third color.

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