Γ-CONVERGENCE FOR FUNCTIONALS DEPENDING ON VECTOR FIELDS. I. INTEGRAL REPRESENTATION AND COMPACTNESS.

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Abstract. Given a family of locally Lipschitz vector fields $X(x) = (X_1(x), \ldots, X_m(x))$ on $\mathbb{R}^n$, $m \leq n$, we study functionals depending on $X$. We prove an integral representation for local functionals with respect to $X$ and a result of Γ-compactness for a class of integral functionals depending on $X$.

1. Introduction

In this paper we will deal with the Γ-convergence, with respect to $L^p(\Omega)$-topology, for integral functionals $F, F_1 : L^p(\Omega) \to [0, \infty]$, $1 < p < \infty$, defined by

\begin{equation}
F(u) := \begin{cases}
\int_{\Omega} f(x, Xu(x)) \, dx & \text{if } u \in C^1(\Omega) \\
\infty & \text{otherwise}
\end{cases}
\end{equation}

and

\begin{equation}
F_1(u) := \begin{cases}
\int_{\Omega} f(x, Xu(x)) \, dx & \text{if } u \in W^{1,1}_{\text{loc}}(\Omega) \\
\infty & \text{otherwise}
\end{cases},
\end{equation}

where $X(x) := (X_1(x), \ldots, X_m(x))$ is a given family of first linear differential operators, with Lipschitz coefficients on a bounded open set $\Omega \subset \mathbb{R}^n$, that is,

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x) \partial_i \quad j = 1, \ldots, m$$

with $c_{ji}(x) \in Lip(\Omega)$ for $j = 1, \ldots, m$, $i = 1, \ldots, n$ and where $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a Borel function. In the following, we will refer to $X$ and $f$ as $X$-gradient and integrand function, respectively. As usual, we will identify...
each \( X_j \) with the vector field \((c_{j1}(x), \ldots, c_{jn}(x)) \in \text{Lip}(\Omega, \mathbb{R}^n)\). Moreover, we set

\[
C(x) = \left[ c_{ji}(x) \right]_{j=1, \ldots, m}^{i=1, \ldots, n},
\]

and we will call \( C(x) \) the coefficient matrix of the \( X \)-gradient.

Throughout the paper the class of integrand functions will typically satisfy the following structural conditions:

1. \((I_1)\) for every \( \eta \in \mathbb{R}^m \), the function \( f(\cdot, \eta) : \Omega \to [0, \infty] \) is Borel measurable on \( \Omega \);
2. \((I_2)\) for a.e. \( x \in \Omega \), the function \( f(x, \cdot) : \mathbb{R}^m \to [0, \infty) \) is convex;
3. \((I_3)\) there exists constants \( c_1 > c_0 \geq 0 \) such that

\[
c_0 |\eta|^p \leq f(x, \eta) \leq c_1 (|\eta|^p + 1),
\]

for a.e. \( x \in \Omega \) and for each \( \eta \in \mathbb{R}^m \).

We will denote by \( I_{m,p}(\Omega, c_0, c_1) \) the class of such integrand functions. Notice that both functionals \((\text{1})\) and \((\text{2})\) always admit an integral representation with respect to the Euclidean gradient. Indeed, for instance, functional \((\text{1})\) can be represented as follows

\[
F(u) = \int_{\Omega} f_e(x, Du) \, dx \quad \text{for each } u \in C^1(\Omega)
\]

where \( f_e : \Omega \times \mathbb{R}^n \to [0, \infty] \) now denotes the Euclidean integrand defined as

\[
f_e(x, \xi) := f(x, C(x) \xi) \quad \text{for a.e. } x \in \Omega, \text{ for each } \xi \in \mathbb{R}^n.
\]

Notice also that, in general, we cannot reverse this representation (see Counterexample 3.15). Moreover the representation with respect to the Euclidean gradient could yield a loss of coercivity. Indeed, for instance, let us consider as \( X \)-gradient the Grushin and Heisenberg vector fields in Example 2.2 (ii) and (iii), respectively, and let \( f(x, \eta) = |\eta|^2 \). Then, it is easy to see that there are no positive constants \( c > 0 \) such that the associated Euclidean integrand \( f_e(x, \xi) = f(x, C(x) \xi) = |C(x) \xi|^2 \) satisfies

\[
f_e(x, \xi) \geq c |\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n,
\]

if the open set \( \Omega \subset \mathbb{R}^2 \) contains some segment of the line \( \{x_1 = 0\} \), for the Grushin vector field, and for each open set \( \Omega \subset \mathbb{R}^3 \), for the Heisenberg vector fields. Nonetheless, we will show that, by replacing the Euclidean gradient with the \( X \)-gradient, we can get rid of this drawback.

Functional \((\text{1})\) was studied in \[\text{FSSC1}\] as far as its relaxation and in connection with the so-called Meyers-Serrin theorem for Sobolev spaces associated with the \( X \)-gradient, denoted \( W_X^{1,p}(\Omega) \) (see Definition 2.3 and \[\text{FS}\]). As a consequence, the following characterization of relaxed functionals \( \bar{F} \) and \( \bar{F}_1 \) can be given (see \[\text{22}\] and Theorem 3.1): if \( f \in I_{m,p}(\Omega, c_0, c_1) \) with \( c_1 \geq c_0 > 0 \) and \( F^* : L^p(\Omega) \to [0, \infty] \) denotes the functional

\[
F^*(u) := \begin{cases} 
\int_{\Omega} f(x, Xu(x)) \, dx & \text{if } u \in W_X^{1,p}(\Omega), \\
\infty & \text{otherwise}
\end{cases}
\]

2
then
\begin{equation}
\tilde{F}(u) = \widetilde{F}_1(u) = F^*(u) \quad \forall u \in L^p(\Omega).
\end{equation}

By (6) and a well-known property of \(\Gamma\)-convergence (see [DM, Proposition 6.11]), the characterization of \(\Gamma\)-limits for functionals of type (1) or (2), associated to integrand functions in \(I_{m,p}(\Omega, c_0, c_1)\), can be reduced to the one for functionals of type (5) still associated to integrand functions in \(I_{m,p}(\Omega, c_0, c_1)\). For getting such a characterization, the following structure assumption on the \(X\)-gradient turns out to be a key point.

1.1. **Definition.** We say that the family of vector fields \(X(x) = (X_1(x), \ldots, X_m(x))\) on an open set \(\Omega \subset \mathbb{R}^n\) satisfies the **linear independence condition** (LIC) if there exists a closed set \(N_X \subset \Omega\) such that \(|N_X| = 0\) and, for each \(x \in \Omega_X := \Omega \setminus N_X\), \(X_1(x), \ldots, X_m(x)\) are linearly independent as vectors of \(\mathbb{R}^n\).

Let us point out that (LIC) condition embraces many relevant families of vector fields studied in literature (see Example 2.2). In particular neither the \(\ddot{\text{H}}\)ormander condition for \(X\), that is, vector fields \(X_j\)’s are smooth and the rank of the Lie algebra generated by \(X_1, \ldots, X_m\) equals \(n\) at any point of \(\Omega\), nor the (weaker) assumption that the \(X\)-gradient induces a Carnot-Carathéodory metric in \(\Omega\) is requested. An exhaustive account of these topics can be found in [BLU].

The main results of this paper are the following (see Theorems 3.12 and 4.11).

- Assume that the \(X\)-gradient satisfies (LIC) on \(\Omega\) and let us denote by \(A\) the class of open sets contained in \(\Omega\). Then an integral representation result, with respect to the \(X\)-gradient, is provided for a local functional \(F : L^p(\Omega) \times A \to [0, \infty]\) satisfying suitable assumptions.
- Assume that the \(X\)-gradient satisfies (LIC) on \(\Omega\), and let \(F^*_h : L^p(\Omega) \to [0, \infty]\) (\(h = 1, 2, \ldots\)) be a sequence of integral functionals of the form (3) with \(f_i \equiv f_h\), where \((f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)\) for given constants \(0 < c_0 \leq c_1\). Then, up to a subsequence, \((F^*_h)_h\) \(\Gamma\)-converges, in \(L^p(\Omega)\)-topology, to a functional \(F^* : L^p(\Omega) \to [0, \infty]\), and \(F^*\) can be still represented as in (5), for a suitable integrand function \(f \in I_{m,p}(\Omega, c_0, c_1)\).

We will also single out two significant integrand function subclasses \(J_i \subset I_{m,p}(\Omega, c_0, c_1)\) \((i = 1, 2)\) for which the associated functionals in (5) are still compact with respect to \(\Gamma\)-convergence with respect to \(L^p(\Omega)\)-topology (see Theorem 4.20).

The techniques for showing the integral representation Theorem 3.12 rely on the analogous classical integral representation result for the Euclidean gradient (see [DM, Theorem 20.1] ), together with a characterization of integral functionals depending on the Euclidean gradient which can be also represented with respect to a given \(X\)-gradient (see Theorem 3.5). Let us stress that we cannot here exploit, as in the case of the Euclidean gradient,
the approximation by piecewise-affine functions in classical Sobolev space $W^{1,p}(\Omega)$, since it could not work in Sobolev space $W^{1,p}_{X}(\Omega)$ (see section 2.3).

The strategy for showing the $\Gamma$-compactness Theorem 4.11 will consists of two steps.

1st step. By applying classical results contained in [DM], we will prove the following result (see Theorem 4.18): let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$, let $(F_h)_h$ be a sequence of integral functionals on $L^p(\Omega) \times A$, $1 < p < \infty$, of the form

$$F_h(u, A) := \begin{cases} \int_A f_{h,e}(x, Du(x)) \, dx & \text{if } A \in A, \ u \in W^{1,1}_{loc}(A) \\ \infty & \text{otherwise} \end{cases},$$

where

$$f_{h,e}(x, \xi) := f_h(x, C(x)\xi) \quad x \in \Omega, \ \xi \in \mathbb{R}^n.$$

Then, up to a subsequence, there exists $F : L^p(\Omega) \times A \to [0, \infty]$ such that

$$F(\cdot, A) = \Gamma(L^p(\Omega)) - \lim_{h \to \infty} F_h(\cdot, A) \quad \text{for each } A \in A,$$

and $F$ can be represented by an integral form on $W^{1,p}(A)$ by means of an Euclidean integrand function, that is,

$$F(u, A) := \int_A f_e(x, Du(x)) \, dx$$

for every $A \in A$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}(A)$ for a suitable Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$.

2nd step. We will show that the class $I_{m,p}(\Omega, c_0, c_1)$ satisfies the following closure property with respect to $\Gamma(L^p(\Omega))$-convergence (see Theorem 4.19): assume that $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ and (9) and (10) hold, then $F$ satisfies the assumptions of the integral representation Theorem 3.12. Thus $F$ can be also represented in the integral form (5), by means of an integrand function $f \in I_{m,p}(\Omega, c_0, c_1)$.

Eventually let us point out that the $\Gamma$-convergence for functionals such as in (1) have been studied in the framework of Dirichlet forms [MR, Fu], but for special integrand functions $f$ and $X$-gradient satisfying the Hörmander condition, (see, for instance, [Mo, BT] and references there in). Other variational convergences, such as homogenization and $H$-convergence for subelliptic PDEs have been also widely studied, always assuming the $X$-gradient satisfying the Hölder condition (see, for instance, [BMT, BPT, BPT2, FT, FG, FT, BPT, BPTT, BPTTT] and the references there in). In the subsequent paper [FPS] we will be concerned with relationships between $\Gamma$-convergence of functionals depending on vector fields and convergence of their minimizers. Thus, we will refer to [FPS] for a comparison among our results with those already present in literature.

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2. Vector fields and Sobolev spaces depending on vector fields

2.1. Notation and definitions. Through this paper $\Omega \subset \mathbb{R}^n$ is a fixed open set and $\mathbb{R} = [-\infty, \infty]$. If $v, w \in \mathbb{R}^n$, we denote by $|v|$ and $\langle v, w \rangle$ the Euclidean norm and the scalar product, respectively. If $\Omega$ and $\Omega'$ are subsets of $\mathbb{R}^n$ then $\Omega' \Subset \Omega$ means that $\Omega'$ is compactly contained in $\Omega$. Moreover, $B(x, r)$ is the open Euclidean ball of radius $r$ centered at $x$. Sometimes we will denote by $B^k(x, r)$ the open Euclidean ball of radius $r$ centered at $x \in \mathbb{R}^k$ in $\mathbb{R}^k$. If $A \subset \mathbb{R}^n$ then $\chi_A$ is the characteristic function of $A$, $|A|$ is its $n$-dimensional Lebesgue measure $\mathcal{L}^n$ and by notation $a.e. ~x \in A$, we will simply mean $\mathcal{L}^n$-a.e. $x \in A$.

In the sequel we denote by $C^k(\Omega)$ the space of $\mathbb{R}$-valued functions $k$ times continuously differentiable and by $C^k_{C}(\Omega)$ the subspace of $C^k(\Omega)$ whose functions have support compactly contained in $\Omega$.

We will use spherically symmetric mollifiers $\rho_\epsilon$ defined by $\rho_\epsilon(x) := e^{-\|x\|^2} \rho(\epsilon^{-1}\|x\|)$, where $\rho \in C^\infty_{c}([-1, 1])$, $\rho \geq 0$ and $\int_{0}^{1} \rho(t) dt = |B(0, 1)|^{-1}$.

For any $u \in L^1(\Omega)$ define $Xu$ as an element of $\mathcal{D}'(\Omega; \mathbb{R}^m)$ as follows

$$ Xu(\psi) := (X_1 u(\psi_1), \ldots, X_m u(\psi_m)) $$

$$ = - \int_{\Omega} u \left( \sum_{i=1}^{n} \partial_{x_i} (c_{1,i} \psi_1), \ldots, \sum_{i=1}^{n} \partial_{x_i} (c_{m,i} \psi_m) \right) dx $$

$\forall \psi = (\psi_1, \ldots, \psi_m) \in C^\infty_{c}(\Omega; \mathbb{R}^m)$. If we set $X^T \psi := (X_1^T \psi_1, \ldots, X_m^T \psi_m)$ with

$$ X_j^T \varphi := \int_{\Omega} \sum_{i=1}^{n} \partial_{x_i} (c_{j,i} \varphi) dx \quad \forall \varphi \in C^\infty_{c}(\Omega), \forall j = 1, \ldots, m, $$

the aspect of the definition is even more familiar

$$ Xu(\psi) := - \int_{\Omega} u X^T \psi dx \quad \forall \psi \in C^\infty_{c}(\Omega; \mathbb{R}^m). $$

2.1. Remark. If $X = (X_1, \ldots, X_m)$ satisfies (LIC) on an open set $\Omega \subset \mathbb{R}^n$, then $m \leq n$. Moreover, by the well-known extension result for Lipschitz functions, without loss of generality, we can assume that vector fields' coefficients $c_{ji} \in Lip_{loc}(\mathbb{R}^n)$ for each $j = 1, \ldots, m$, $i = 1, \ldots, n$.

2.2. Example (Relevant vector fields).

(i) (Euclidean gradient) Let $X = (X_1, \ldots, X_n) = D := (\partial_1, \ldots, \partial_n)$. In this case the coefficients matrix $C(x)$ of $X$ is a $n \times n$ matrix and

$$ C(x) = I_n \quad \forall x \in \mathbb{R}^n, $$

denoting $I_n$ the identity matrix of order $n$.

(ii) (Grushin vector fields) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as

$$ X_1(x) := \partial_1, \quad X_2(x) := x_1 \partial_2 \quad \text{if} \quad x = (x_1, x_2) \in \mathbb{R}^2. $$
In this case the coefficients matrix $C(x)$ of $X$ is a $2 \times 2$ matrix and

\[(12) \quad C(x) := \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \]

(iii) (Heisenberg vector fields) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^3$ defined as

\[X_1(x) := \partial_1 - \frac{x_2}{2} \partial_3, \quad X_2(x) := \partial_2 + \frac{x_1}{2} \partial_3 \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3.\]

In this case the coefficients matrix $C(x)$ of $X$ is a $2 \times 3$ matrix and

\[(13) \quad C(x) := \begin{bmatrix} 1 & 0 & -x_2/2 \\ 0 & 1 & x_1/2 \end{bmatrix} \]

Notice that all three families of vector fields satisfy (LIC) respectively in $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{R}^3$. Indeed, it suffices to take $\Omega_X = \Omega$ in (i) and (ii) and $\Omega_X = \Omega \setminus N_X$ with $N_X := \{(0, x_2) : x_2 \in \mathbb{R}\}$ in (ii). Moreover they are locally Lipschitz continuous in $\Omega$.

2.3. Definition. For $1 \leq p \leq \infty$ we set

\[W^{1,p}_X(\Omega) := \{ u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \ldots, m \} \]

\[W^{1,p}_{X;\text{loc}}(\Omega) := \{ u : u|_{\Omega'} \in W^{1,p}_X(\Omega') \text{ for every open set } \Omega' \subset \Omega \} \]

2.4. Remark. Since vector fields $X_j$ have locally Lipschitz continuous coefficients, $\partial_i c_{j,i} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ for each $j = 1, \ldots, m$, $i = 1, \ldots, n$, thus, by definition, it is immediate that, for each open bounded set $\Omega \subset \mathbb{R}^n$,

\[(14) \quad W^{1,p}(\Omega) \subset W^{1,p}_X(\Omega) \quad \forall p \in [1, \infty], \]

and for any $u \in W^{1,p}(\Omega)$

\[(15) \quad X u(x) = C(x) Du(x) \quad \text{for a.e. } x \in \Omega, \]

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W^{1,p}_X(\Omega)$ associated to $X = D := (\partial_{x_1}, \ldots, \partial_{x_n})$ (see Example 2.2 (i)). Moreover it is easy to see that inclusion (14) can be strict and turns out to be continuous. As well, there is the inclusion

\[(16) \quad W^{1,p}_{\text{loc}}(\Omega) \subset W^{1,p}_{X;\text{loc}}(\Omega) \quad \forall p \in [1, \infty], \]

The following Proposition is proved in [FS]

2.5. Proposition. $W^{1,p}_X(\Omega)$ endowed with the norm

\[\|u\|_{W^{1,p}_X(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^m \|X_j u\|_{L^p(\Omega)} \]

is a Banach space, reflexive if $1 < p < \infty$.

2.6. Remark. The following properties hold for functions in $W^{1,p}_{X;\text{loc}}(\Omega)$:
(i) let $u \in L^p(\Omega)$ and assume there exists an open set $A \subset \Omega$ such that $u|_A \in W^{1,p}_{X;\text{loc}}(A)$. Then, for every open set $A' \subset A$, there exists
\begin{equation}
 w \in W^{1,p}_X(\Omega) \text{ such that } u|_{A'} = w|_{A'}.
\end{equation}
Indeed, there exists a cut-off function $\varphi \in C^1_c(A)$ such that $\varphi \equiv 1$ in $A'$. If
\[ w(x) := u(x) \varphi(x) \text{ if } x \in \Omega, \]
then it is easy to see that $w$ satisfies \( (17) \).

(ii) Let $\{A_1, \ldots, A_N\}$ be a finite family of open subsets of $\Omega$ and let $u \in L^p(\Omega)$. If $u|_{A_i} \in W^{1,p}_X(A_i)$ for all $i = 1, \ldots, N$ then $u \in W^{1,p}_X(\bigcup_{i=1}^N A_i)$. Consider a partition of unity subordinate to the covering $\{A_1, \ldots, A_N\}$, i.e., nonnegative functions $\{\eta_1, \ldots, \eta_N\} \subset C^\infty_c \left( \bigcup_{i=1}^N A_i \right)$ such that each $\eta_j$ has support in some $A_i$ and $\sum_{j=1}^N \eta_j(x) = 1$ for all $x \in \bigcup_{i=1}^N A_i$. Set $u_j = u \eta_j$. Since the support of $\eta_j$ is contained in some $A_i$, it is clear that $u_j \in W^{1,p}_X \left( \bigcup_{i=1}^N A_i \right)$. The conclusion follows observing that $u = \sum_{j=1}^N u_j$.

(iii) Let $A \subset \Omega$ be an open subset and let $u \in L^p(A)$ be such that there exists $M > 0$, $\|u\|_{W^{1,p}_X(A')} \leq M$ for any $A' \subset A$, then $u \in W^{1,p}_X(A)$. It is easy to see that $u$ admits the weak gradient $Xu$. Consider a sequence of open subsets of $A$, $\{A_i\}_{i \in \mathbb{N}}$ with $A_i \subset A_{i+1}$ and $A \subset \bigcup_{i=1}^\infty A_i$
\[ \int_A |Xu|^p \, dx \leq \int_{\bigcup_{i=1}^\infty A_i} |Xu|^p \, dx = \lim_{i \to \infty} \int_{A_i} |Xu|^p \, dx \leq M \]
and the conclusion follows.

(iv) Let $A \subset \Omega$ be an open subset and $u \in W^{1,p}_X(A)$, then $u|_B \in W^{1,p}_X(B)$ for any open set $B \subset A$. The thesis follows easily observing that $C^\infty_c(B) \subseteq C^\infty_c(A)$.

2.2. Approximation by regular functions. Let us recall in this section some results of approximation by regular functions in these anisotropic Sobolev spaces. In particular the analogous of the celebrated Meyers-Serrin theorem, proved, independently, in [FSSC1] and [GN]. Analogous results (under some additional assumptions) in the weighted cases are proved in [FSSC2], see also [APS] for a generalization to metric measure spaces.

Here and in the sequel, if $u : \Omega \to \mathbb{R}$, we will denote by $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ its extension to the whole $\mathbb{R}^n$ being 0 outside of $\Omega$.

2.7. Proposition. Assume $u \in W^{1,p}_X(\Omega)$ for $1 \leq p < \infty$. Then if $\Omega' \subset \Omega$
\[ \lim_{\epsilon \to 0} \|\bar{u} \ast \rho_\epsilon - u\|_{W^{1,p}_X(\Omega')} = 0, \]
where $\rho_\epsilon(x) = \epsilon^{-n} \rho(\epsilon^{-1}|x|)$ is a mollifier supported in $B(0,\epsilon)$. 7
2.8. **Definition.** For $1 \leq p \leq \infty$ we set

$$H_{X}^{p}(\Omega) := \text{closure of } C^{1}(\Omega) \cap W_{X}^{1,p}(\Omega) \text{ in } W_{X}^{1,p}(\Omega)$$

As for the usual Sobolev spaces $H_{X}^{p}(\Omega) \subset W_{X}^{1,p}(\Omega)$. The classical result $H = W'$ of Meyers and Serrin ([MS]) still holds for these anisotropic Sobolev spaces.

2.9. **Theorem.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $1 \leq p < \infty$. Then

$$H_{X}^{p}(\Omega) = W_{X}^{1,p}(\Omega).$$

The proofs of Proposition 2.7 and Theorem 2.9 can be found in [FSSC1] and [GN].

Let us collect below some well-known properties about approximation by convolution and convex functions.

2.10. **Proposition.** (i) Let $(u_h)_h$ and $u$ be in $L^p_{\text{loc}}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that

$$u_h \to u \text{ in } L^1_{\text{loc}}(\Omega) \text{ as } h \to \infty.$$  

Then, for each open set $\Omega' \Subset \Omega$, for given $0 < \epsilon < \text{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\rho_\epsilon * u_h \to \rho_\epsilon * u \text{ uniformly on } \Omega', \text{ as } h \to \infty.$$  

(ii) Let $f : \mathbb{R}^m \to [0, \infty)$ be a convex function and let $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. Then, for each bounded open sets $\Omega'$ and $\Omega$ with $\Omega' \Subset \Omega$, for each $0 < \epsilon < \text{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\int_{\Omega'} f(\rho_\epsilon * w) \, dx \leq \int_{\Omega} f(w) \, dx.$$

**Proof.** (i) See, for instance, [DM] Proof of Theorem 23.1  
(ii) See, for instance, [DM] (23.5)].

2.3. **Approximation by piecewise affine functions.** It is well known (see, for instance, [ET] Chap. X, Proposition 2.9) that the class of piecewise affine functions is dense in the classical Sobolev space $W^{1,p}(\Omega)$, provided that $\Omega$ is a bounded open set with Lipschitz boundary. This result is crucial in the proof of the classical integral representation theorem with respect to the Euclidean gradient (see, for instance, [DM] Theorem 20.1). The aim of this section is to prove that no results of this kind are available for a general family $X = (X_1, \ldots, X_m)$ in $\mathbb{R}^n$, by extending, in a natural way, the notion to be affine with respect to the $X$-gradient. We say that $u \in C^\infty(\mathbb{R}^n)$ is $X$-affine if there exists $c \in \mathbb{R}^n$ such that $Xu(x) = c$ for all $x \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be open. We say that $u : \Omega \to \mathbb{R}$ is $X$-affine if it is the restriction to $\Omega$ of a $X$-affine function over $\mathbb{R}^n$. Moreover, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is $X$-piecewise affine if it is continuous and there is a partition of $\mathbb{R}^n$ into a negligible set and a finite number of open sets on which $u$ is $X$-affine. We prove that for Grushin and Heisenberg vector fields the approximation of functions in $W_{X}^{1,p}(\Omega)$ using $X$-piecewise affine functions does not hold.
It is easy to see that, if $X = (X_1, X_2)$ is the Heisenberg vector field on $\mathbb{R}^3$ (see Example 2.2 (iii)), then a function $u \in C^\infty(\mathbb{R}^3)$ is $X$–affine if and only if

\begin{equation}
(19) \quad u(x) = c_1 x_1 + c_2 x_2 + c_3 \quad \text{for each } x = (x_1, x_2, x_3) \in \mathbb{R}^3,
\end{equation}

for suitable constants $c_i \in \mathbb{R}$ $i = 1, 2, 3$. Indeed, it is trivial that a function $u$ in (19) is $X$–affine. Conversely, if $X_1 u = c_1$ and $X_2 u = c_2$ on $\mathbb{R}^3$, for some $u \in C^\infty(\mathbb{R}^3)$, then the commutator $[X_1, X_2] u := (X_1 X_2 - X_2 X_1) u = \partial_3 u = 0$ on $\mathbb{R}^3$, which gives $u(x) = c_1 x_1 + c_2 x_2 + c_3$ for each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, for some $c_3 \in \mathbb{R}$.

Let $u(x) = x_3$, then $u \in W^{1,p}_X(\Omega)$ whenever $|\Omega| < \infty$. Since any $X$–piecewise affine function does not depend on $x_3$, there cannot be any sequence of $X$–piecewise affine functions $(u_h)_h$ such that $u_h(x_1, x_2, x_3) \rightarrow u(x_1, x_2, x_3)$ for a.e. $(x_1, x_2, x_3) \in \Omega$.

Let $X = (X_1, X_2)$ be the Grushin vector fields on $\mathbb{R}^2$ (see Example 2.2 (ii)). Let $u \in C^\infty(\mathbb{R}^2)$ be such that $X_1 u = c_1$ and $X_2 u = c_2$ on $\mathbb{R}^2$. Then it is easy to prove, arguing as before, that $u(x) = c_1 x_1 + c_3$ for each $x = (x_1, x_2) \in \mathbb{R}^2$, for some $c_3 \in \mathbb{R}$. The conclusion follows as in the previous case taking $u(x_1, x_2) = x_2$, which belongs to $W^{1,p}_X(\Omega)$ for any $p \geq 1$ and any bounded open set $\Omega \subset \mathbb{R}^2$.

3. RELAXATION AND CHARACTERIZATION OF INTEGRAL FUNCTIONALS DEPENDING ON VECTOR FIELDS

In the study of the $\Gamma$–convergence it will be helpful to consider $F$ and $F_1$ as local functionals. Namely, according to [DM] Chap. 15], we will consider the functionals $F, F_1 : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$

\begin{equation}
(20) \quad F(u, A) := \begin{cases} 
\int_A f(x, Xu(x)) dx & \text{if } A \in \mathcal{A} \text{ and } u \in C^1(A) \cap L^p(A) \\
\infty & \text{otherwise}
\end{cases}
\end{equation}

\begin{equation}
(21) \quad F_1(u, A) := \begin{cases} 
\int_A f(x, Xu(x)) dx & \text{if } A \in \mathcal{A} \text{ and } u \in W^{1,1}_{\text{loc}}(A) \cap L^p(A) \\
\infty & \text{otherwise}
\end{cases}
\end{equation}

For future use, we denote by $\mathcal{A}_0$ the class of all open sets compactly contained in $\Omega$.

3.1. Characterization of the relaxed functional and its finiteness domain. We are going to characterize the relaxed functionals of $F$ in (11) and $F_1$ in (2) with respect to the topology of $L^p(\Omega)$. Let us recall that the relaxed functional of a given functional $G : L^p(\Omega) \rightarrow [0, \infty]$ is defined as follows (see, for instance, [13]):

\begin{equation}
(22) \quad \bar{G}(u) := \inf \left\{ \liminf_{h \rightarrow \infty} G(u_h) : (u_h)_h \subset L^p(\Omega), u_h \rightarrow u \text{ in } L^p(\Omega) \right\}.
\end{equation}
Then it is well known (see, for instance, [B]) that $\bar{G}$ is the greatest $L^p(\Omega)$-lower semicontinuous functional smaller or equal to $G$.

The relaxed functionals $\bar{F}$ and $\bar{F}_1$ can be characterized as follows:

3.1. **Theorem.** Let $p > 1$ and let $\Omega$ be an open subset of $\mathbb{R}^n$; let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be an integrand function in $I_{m,p}(\Omega, c_1, c_0)$ with $c_1 \geq c_0 > 0$. Then

(i) $\text{dom } \bar{F} := \{ u \in L^p(\Omega) : \bar{F}(u) < \infty \} = W_{X}^{1,p}(\Omega)$;
(ii) $\bar{F}(u) = \int_{\Omega} f(x, Xu(x)) \, dx$ for every $u \in W_{X}^{1,p}(\Omega)$;
(iii) $\bar{F}(u) = \bar{F}_1(u)$ for each $u \in L^p(\Omega)$.

**Proof.** Claims (i) and (ii) are proved in [FSSC1, Theorem 3.3.1]. Let us prove (iii). Let $u \in L^p(\Omega)$ and $(u_h)_h \subset C^1(\Omega) \cap L^p(\Omega)$ with $u_h \to u$ in $L^p(\Omega)$. Since, in particular, $(u_h)_h \subset W_{loc}^{1,1}(\Omega) \cap L^p(\Omega)$ we get

\begin{equation}
\bar{F}_1(u) \leq \liminf_{h \to \infty} \bar{F}_1(u_h) = \liminf_{h \to \infty} \int_{\Omega} f(x, Xu_h(x)) \, dx = \liminf_{h \to \infty} F(u_h)
\end{equation}

which implies

\begin{equation}
\bar{F}_1(u) \leq \bar{F}(u).
\end{equation}

Let $F^* : L^p(\Omega) \to [0, \infty]$ denote the functional in (5). By [B, Theorem 2.3.1], $F^*$ is $L^p(\Omega)$-lower semicontinuous. Let $u \in \text{dom}(F_1) := \{ v \in W_{loc}^{1,1}(\Omega) \cap L^p(\Omega) \mid F_1(u) < \infty \}$, then, by $(I_3)$, we have

\[ c_0 \int_{\Omega} |Xu|^p \, dx \leq F_1(u) < \infty, \]

thus $u \in W_{X}^{1,p}(\Omega)$ and $\text{dom } F_1 \subset W_{X}^{1,p}(\Omega)$. This implies $F^* \leq F_1$ on $L^p(\Omega)$ and consequently $F^* \leq \bar{F}_1$ on $L^p(\Omega)$. Using (24) and (ii) we conclude

\[ F^* \leq \bar{F}_1 \leq \bar{F} = F^* \quad \text{on } \quad L^p(\Omega) \]

which completes the proof.

When $p = 1$ the domain of relaxed functional $\bar{F}$ gives rise to the space of functions of bounded variation function associated to $X, BV_X(\Omega)$ (see [FSSC1, Theorem 3.2.3]).

3.2. **A characterization of functionals depending on vector fields.**
We are going to study when a local functional $F : C^1(\Omega) \times A \to [0, \infty]$ can be equivalently represented both with respect to a family of vector fields $X$ and the Euclidean gradient $D$.

We already stressed that the functional $F$ in (11) can be always represented with respect to the Euclidean gradient on $C^1(\Omega)$ by means of the Euclidean integrand $\mathcal{F}$.
Then, it is clear that, for each $A \in \mathcal{A}$ and $u \in C^1(A)$,

\begin{equation}
F(u, A) = \int_A f(x, Xu) \, dx = \int_A f(x, C(x)Du) \, dx
= \int_A f_e(x, Du) \, dx.
\end{equation}

Viceversa, we are going to study when, given a $X$-gradient and a functional $F : C^1(\Omega) \times \mathcal{A} \to [0, \infty)$

\begin{equation}
F(u, A) = \int_A f_e(x, Du) \, dx \quad u \in C^1(A),
\end{equation}

there exist a function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ such that

\begin{equation}
F(u, A) = \int_A f(x, Xu) \, dx.
\end{equation}

Let us begin with some preliminaries of linear algebra.

In the following, we identify the space of real matrices of order $m \times n$ with $\mathbb{R}^{mn}$ or $L(\mathbb{R}^m, \mathbb{R}^n)$, where $L(\mathbb{R}^m, \mathbb{R}^n)$ denotes the class of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ endowed with its operator norm. Given a matrix $A = [a_{ij}]$ of order $m \times n$ its operator norm is defined as

\begin{equation}
\|A\| := \sup_{|z|=1} |Az|
\end{equation}

and its Hilbert-Schmidt norm as

\begin{equation}
\|A\|_{\mathbb{R}^{mn}} := \sqrt{\sum_{i,j} a_{ij}^2}
\end{equation}

(see [La] Chap. 7). Since the norms are equivalent, we can also identify the spaces

\begin{equation}
C^0(\Omega_X, \mathbb{R}^{mn}) \equiv C^0(\Omega_X, L(\mathbb{R}^m, \mathbb{R}^n)),
\end{equation}

where we recall that $\Omega_X = \Omega \setminus N_X$. For each $x \in \Omega$, let $L_x : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map

\begin{equation}
L_x(v) := C(x)v \text{ if } v \in \mathbb{R}^n
\end{equation}

where $C(x)$ denotes the matrix in (3). Let $N_x$ and $V_x$ respectively denote the subspaces of $\mathbb{R}^n$ defined as

\begin{equation}
N_x := \ker(L_x), \quad V_x := \{C(x)^T z : z \in \mathbb{R}^m\}.
\end{equation}

It is well-known that $N_x$ and $V_x$ are orthogonal complements in $\mathbb{R}^n$, that is

\begin{equation}
\mathbb{R}^n = N_x \oplus V_x.
\end{equation}

Moreover, for each $x \in \Omega$ and $\xi \in \mathbb{R}^n$, let us define $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$ as the unique vectors of $\mathbb{R}^n$ such that

\begin{equation}
\xi = \xi_{N_x} + \xi_{V_x}
\end{equation}
and let $\Pi_x : \mathbb{R}^n \to V_x \subset \mathbb{R}^n$ be the projection
\[ (32) \quad \Pi_x(\xi) := \xi_{V_x}. \]

3.2. **Proposition.** Assume that the family $X$ of vector fields satisfies (LIC) on $\Omega$. Let $C(x)$ be the matrix in $(3)$ and $L_x$ be the map in $(28)$. Then $L_x : V_x \to \mathbb{R}^m$ is invertible and the map $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ defined as
\[ (33) \quad L^{-1}(x) := L^{-1}_x \text{ if } x \in \Omega_X \]
belongs to $C^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

Before giving the proof of Proposition 3.2, let us prove a preliminary technical lemma.

3.3. **Lemma.** Under the same assumptions of Proposition 3.2,

(i) $\dim V_x = m$ for each $x \in \Omega_X$ and $L_x(V_x) = \text{range}(L_x) = \mathbb{R}^m$ where \text{range}(L_x) denotes the range of $L_x$, that is, $\text{range}(L_x) := \{ L_x(v) : v \in \mathbb{R}^n \}$. In particular $L_x : V_x \to \mathbb{R}^m$ is an isomorphism.

(ii) Let
\[ (34) \quad B(x) := C(x)C^T(x) \quad x \in \Omega. \]
Then, for each $x \in \Omega_X$, $B(x)$ is a symmetric invertible matrix of order $m$. Moreover the map $B^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, defined as
\[ (35) \quad B^{-1}(x)(z) := B(x)^{-1}z \quad \text{if } z \in \mathbb{R}^m, \]
is continuous.

(iii) For each $x \in \Omega_X$, the projection $\Pi_x$ in $(32)$ can be represented as
\[ \Pi_x(\xi) = \xi_{V_x} = C(x)^TB(x)^{-1}C(x)\xi, \quad \forall \xi \in \mathbb{R}^n. \]
If $m = n$, then, $\Pi_x = \text{Id}_n : \mathbb{R}^n \to \mathbb{R}^n$, the identity map in $\mathbb{R}^n$.

3.4. **Remark.** Using the definition of $V_x$ it is easy to see that
\[ V_x = \text{span}_{\mathbb{R}} \{ X_1(x), \ldots, X_m(x) \}, \]
i.e., the so-called horizontal bundle, denoted also by $H_x$.

**Proof.** (i) The claim is a well-known result of basic linear algebra.

(ii) It is straightforward that $B(x)$ a symmetric matrix of order $m$ for each $x \in \Omega$. We have only to show that it is invertible for each $x \in \Omega_X$ or, equivalently, that
\[ (36) \quad \text{if } B(x)z = 0 \text{ for some } z \in \mathbb{R}^m, \text{ then } z = 0. \]
Let $z^T$ denotes the transpose of a column vector $z \in \mathbb{R}^m$. If $B(x)z = 0$, then
\[ (37) \quad 0 = z^TB(x)z = z^TC(x)C^T(x)z \]
\[ = |C^T(x)z|_{\mathbb{R}^m}^2 \iff C^T(x)z = 0. \]
By (LIC), since
\[ \text{rank } C(x) = \text{rank } C^T(x) = m \quad \forall x \in \Omega_X, \]
from (37) we get that \( z = 0 \) and (36) follows. Let us now prove that the map (35) is continuous. Let us recall that, given a matrix \( A \in C^0(\Omega_X, \mathbb{R}^{m^2}) \), by the definition of determinant (see, for instance, [La, Chap. 3, Theorem 6]), the determinant map

\[
\det A : \Omega_X \to \mathbb{R}, \quad (\det A)(x) := \det(A(x))
\]

is continuous. Moreover

\( A(x) \) is invertible \( \iff \) \( \det A(x) \neq 0 \).

By Cramer’s rule (see, for instance, [La, Chap. 3, Theorem 7]), if \( B(x)^{-1} = [b_{ij}^*(x)] \), then

\[
b_{ij}^*(x) = (-1)^{i+j} \frac{\det B_{ij}(x)}{\det B(x)} \quad x \in \Omega_X, \ i, j = 1, \ldots, m,
\]

where \( B_{ij} \) is the \((m - 1) \times (m - 1)\) matrix obtained by striking out the \(i\)th row and \(j\)th column of \( B \), i.e., the \((ij)\)th minor of \( B \). This implies that \( B^{-1} \in C^0(\Omega_X, \mathbb{R}^{m^2}) \equiv C^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)) \).

(iii) We have

\[
(38) \quad \Pi_x(\xi) = \xi_{V_x} = C(x)^T w
\]

for a suitable (unique) \( w = w(x, \xi) \in \mathbb{R}^m \) depending on \( x \) and \( \xi \). On the other hand, by (35),

\[
(39) \quad C(x)\xi = L_x(\xi) = L_x(\xi_{N_x}) + L_x(\xi_{V_x})
\]

\[
= C(x)\xi_{V_x} = C(x)C(x)^T w = B(x)w.
\]

Since \( B(x) \) is invertible, by (39), we get the desired conclusion. \( \square \)

**Proof of Proposition 3.2.** The fact that the map \( L_x : V_x \to \mathbb{R}^m \) is invertible follows from Lemma 3.3 (i). Let us now prove that

\[
(40) \quad L_x^{-1}(z) = C^T(x)B(x)^{-1}z \quad \forall z \in \mathbb{R}^m,
\]

where \( B(x) \) is the matrix in (34). Let us fix \( z \in \mathbb{R}^m \) and let \( v = L_x^{-1}(z) \in V_x \). By Lemma 3.3 (iii), there exists \( w \in \mathbb{R}^m \) such that \( v = C^T(x)w \). Thus

\[
z = L_x(v) = C(x)C^T(x)w = B(x)w.
\]

By Lemma 3.3 (ii), it holds \( w = B(x)^{-1}z \). Therefore we get

\[
(41) \quad L_x^{-1}(z) = v = C^T(x)B(x)^{-1}z
\]

and (40) follows. Let us define

\[
A(x) := C^T(x)B(x)^{-1} \quad \text{if} \ x \in \Omega_X.
\]

Then, from Lemma 3.3 (ii), \( A \in C^0(\Omega_X, \mathbb{R}^{mn}) \equiv C^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \). Thus, by (41), we get the desired conclusion. \( \square \)
3.5. **Theorem.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and assume that \( X \) satisfies (LIC) on \( \Omega \). Let \( F : C^1(\Omega) \times A \to [0, \infty] \) be the functional in (26) with \( f_e : \Omega \times \mathbb{R}^n \to [0, \infty] \) a Borel measurable function satisfying
\[
(42) \quad \text{for each } \xi \in \mathbb{R}^n, \ f_e(\cdot, \xi) \in L^1_{loc}(\Omega)
\]
and
\[
(43) \quad f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty) \text{ convex for a.e. } x \in \Omega.
\]
Define \( f : \Omega \times \mathbb{R}^n \to [0, \infty) \) as
\[
(44) \quad f(x, \eta) := \begin{cases} f_e(x, L^{-1}(x)(\eta)) & \text{if } (x, \eta) \in \Omega_X \times \mathbb{R}^m, \\ 0 & \text{otherwise} \end{cases},
\]
where \( L^{-1} : \Omega_X \to L(\mathbb{R}^m, \mathbb{R}^n) \) is the map in (33). Then, \( f \) is a Borel measurable function satisfying
\[
(45) \quad f(x, \cdot) : \mathbb{R}^m \to [0, \infty) \text{ convex for a.e. } x \in \Omega.
\]
Moreover,
\[
(46) \quad F(u, A) = \int_A f_e(x, Du) \, dx
\]
\[
= \int_A f(x, Xu) \, dx \quad \forall A \in \mathcal{A}, \ u \in C^1(A)
\]
if and only if
\[
(47) \quad f_e(x, \xi) = f_e(x, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n,
\]
where \( \{V_x : x \in \Omega_X\} \) is the distribution of \( m \)-planes in \( \mathbb{R}^n \) defined in Proposition (42) and \( \Pi_x : \mathbb{R}^n \to V_x \) denotes the projection of \( \mathbb{R}^n \) on \( V_x \) in (42).

In addition, the function \( f \) for which (46) holds is unique, that is, if there exists another Borel measurable function \( f^* : \Omega \times \mathbb{R}^m \to [0, \infty) \) satisfying \( f^*(x, \cdot) : \mathbb{R}^m \to [0, \infty) \) convex a.e. \( x \in \Omega \) and (46) holds, then \( f(x, \eta) = f^*(x, \eta) \) for a.e. \( x \in \Omega \) and \( \eta \in \mathbb{R}^m \).

3.6. **Remark.** If the \( X \)-gradient does not satisfy (LIC) condition, the uniqueness of representation (46) may trivially fail. For instance, let \( X = (X_1, X_2) := (\partial_1, 0) \) be the family of vector fields on \( \Omega = \mathbb{R}^2 \) and let \( f(\eta) := \eta_1^2 + g(\eta_2) \) and \( f^*(\eta) := \eta_1^2 + g^*(\eta_2) \) for each \( \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \), where \( g, g^* : \mathbb{R} \to [0, \infty) \) are convex functions satisfying \( g(0) = g^*(0) = 0 \), but \( g \neq g^* \). Then it clear that \( f \) and \( f^* \) are integrand functions of the same functional \( F \) defined in (46), even though \( f \neq f^* \).

3.7. **Remark.** Notice that, in the case \( m = n \) and \( X \) satisfies (LIC) on \( \Omega \), condition (47) always holds, since, by Lemma 3.3(iii), \( \Pi_x \equiv \text{Id}_n \).

**Proof. 1st step.** Let us prove that \( f \) is Borel measurable. Let \( \Psi : \Omega_X \times \mathbb{R}^m \to \Omega_X \times \mathbb{R}^n \) denote the map
\[
(\Psi(x, \eta) := (x, L^{-1}(x)(\eta))) \quad \text{if } (x, \eta) \in \Omega_X \times \mathbb{R}^m.
\]
By Proposition 3.2, $\Psi$ is continuous, then it is also Borel measurable. Since $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$ is Borel measurable, the composition $f = f_e \circ \Psi : \Omega_X \times \mathbb{R}^m \to [0, \infty]$ is still Borel measurable.

To prove (45) it is sufficient to notice that $f(x, \cdot) = f_e(x, \cdot) \circ L^{-1}(x) \quad \forall x \in \Omega_X$

indeed $f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is convex for a.e. $x \in \Omega$ and $L^{-1}(x) : \mathbb{R}^m \to \mathbb{R}^n$ is linear for $x \in \Omega_X$.

2nd step. Let us prove the uniqueness of representation in (46). Assume that

$$
\int_A f(x, Xu) \, dx = \int_A f^*(x, Xu) \, dx \quad (48)
$$

for given Borel measurable functions $f, f^* : \Omega \times \mathbb{R}^m \to [0, \infty)$, with $f(x, \cdot), f^*(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ convex a.e. $x \in \Omega$. Let us choose as

$$
u(x) = u_\xi(x) := \langle \xi, x \rangle \quad x \in \mathbb{R}^n,
$$

for fixed $\xi \in \mathbb{Q}^n$, in the previous equality. By (48) and (42), it follows that the functions

$$
\Omega \ni y \mapsto f(y, C(y)\xi) \quad \text{and} \quad \Omega \ni y \mapsto f^*(y, C(y)\xi) \quad \text{are in} \quad L^1_{\text{loc}}(\Omega).
$$

Choosing $A = B(x, r)$ in (48), by Lebesgue’s differentiation theorem, we get that there exists a negligible set $N_\xi \subset \Omega$ such that $\forall x \in \Omega \setminus N_\xi$

$$
f(x, L_x(\xi)) = f(x, C(x)\xi) = f^*(x, C(x)\xi)
$$

(50)

If $N := \bigcup_{\xi \in \mathbb{Q}^n} N_\xi$, then (50) holds for each $x \in \Omega \setminus N$ and $\xi \in \mathbb{Q}^n$. Since, for each $x \in \Omega \setminus N$, $f(x, \cdot), f^*(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ are continuous, it follows that (50) holds for each $x \in \Omega \setminus N$ and $\xi \in \mathbb{R}^n$. Being the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$ onto, we get the desired conclusion.

3rd step. Let us assume (47). To prove (46) it is sufficient to prove that, for each $A \in \mathcal{A}, u \in C^1(A)$

$$
f(x, Xu(x)) = f_e(x, Du(x)) \quad \text{a.e.} \quad x \in \Omega.
$$

(51)

Given $A \in \mathcal{A}$ and $u \in C^1(A)$, let us recall that

$$
Xu(x) = C(x)Du(x) \quad \forall x \in A.
$$
Thus, by (47), Lemma 3.3 (iii) and the definition of $V_x$, a.e. $x \in \Omega$, if $v_x := Du(x)$

$$f(x, Xu(x)) = f(x, C(x)v_x) = f(x, L_x(\Pi_x(v_x)))$$

(52)

$$= f_e(x, L_x^{-1}(L_x(\Pi_x(v_x)))) = f_e(x, \Pi_x(v_x))$$

$$= f_e(x, v_x) = f_e(x, Du(x))$$

and (51) follows. On the other hand, let us assume that for every $A \in \mathcal{A}$ and $u \in C^1(A)$

$$\int_A f_e(x, Du) \, dx = \int_A f(x, Xu) \, dx$$

where $f$ is the function in (44). By (52), for every $A \in \mathcal{A}$ and $u \in C^1(A)$,

$$f(x, Xu(x)) = f_e(x, \Pi_x(Du(x))) \quad \forall x \in A,$$

which implies

$$\int_A f(x, Xu(x)) \, dx = \int_A f_e(x, \Pi_x(Du(x))) \, dx.$$ 

Thus, for every $A \in \mathcal{A}$ and $u \in C^1(A)$,

$$\int_A f_e(x, \Pi_x(Du(x))) \, dx = \int_A f_e(x, Du) \, dx$$

and the conclusion now follows by proceeding as in the second step of the proof. □

3.8. Remark. Observe that (51) actually holds for each $u \in W^{1,p}(A)$. As a consequence, (46) holds for each $A \in \mathcal{A}$ and $u \in W^{1,p}(A)$.

3.3. Integral representation for local functionals with respect to vector fields. Let us recall, for reader’s convenience, some notation about set functions on $\mathcal{A}$ and local functionals on $L^p(\Omega) \times \mathcal{A}$, according to [DM]. Let $\Omega \subset \mathbb{R}^n$ be an open set.

3.9. Definition. Let $\alpha : \mathcal{A} \to [0, \infty]$ be a set function. We say that:

(i) $\alpha$ is increasing if $\alpha(A) \leq \alpha(B)$, for each $A, B \in \mathcal{A}$ with $A \subseteq B$;

(ii) $\alpha$ is inner regular if $\alpha(A) = \sup \{ \alpha(B) : B \in \mathcal{A}, B \subseteq A \}$ for each $A \in \mathcal{A}$;

(iii) $\alpha$ is subadditive if $\alpha(A) \leq \alpha(A_1) + \alpha(A_2)$ for every $A, A_1, A_2 \in \mathcal{A}$ with $A \subseteq A_1 \cup A_2$;

(iv) $\alpha$ is superadditive if $\alpha(A) \geq \alpha(A_1) + \alpha(A_2)$ for every $A, A_1, A_2 \in \mathcal{A}$ with $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$;

(v) $\alpha$ is a measure if there exists a Borel measure $\mu : B(\Omega) \to [0, \infty]$ such that $\alpha(A) = \mu(A)$ for every $A \in \mathcal{A}$.

3.10. Remark. Let us recall that, if $\alpha : \mathcal{A} \to [0, \infty]$ is an increasing set function, then it is a measure if and only if it is subadditive, superadditive and inner regular (see [DM Theorem 14.23]).
3.11. Definition. Let

\[ F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]. \]

We say that:

(i) \( F \) is increasing if, for every \( u \in L^p(\Omega) \), \( F(u, \cdot) : \mathcal{A} \to [0, \infty] \) is increasing as set function;

(ii) \( F \) is inner regular (on \( \mathcal{A} \)) if it is increasing and, for each \( u \in L^p(\Omega) \), \( F(u, \cdot) : \mathcal{A} \to [0, \infty] \) is inner regular as set function;

(iii) \( F \) is a measure, if for every \( u \in L^p(\Omega) \), \( F(u, \cdot) : \mathcal{A} \to [0, \infty] \) is a measure as set function;

(iv) \( F \) is local if \( F(u, A) = F(v, A) \) for each \( A \in \mathcal{A}, u, v \in L^p(\Omega) \) such that \( u = v \) a.e. on \( A \);

(v) \( F \) is lower semicontinuous (lsc), if for every \( A \in \mathcal{A}, F(\cdot, A) : L^p(\Omega) \to [0, \infty] \) is lower semicontinuous.

3.12. Theorem. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and assume that \( X \) satisfies (LIC) on \( \Omega \). Let \( p > 1 \) and

\[ F : L^p(\Omega) \times \mathcal{A} \to [0, \infty] \]

be an increasing functional satisfying the following properties:

(a) \( F \) is local;

(b) \( F \) is a measure;

(c) \( F \) is lsc;

(d) \( F(u + c, A) = F(u, A) \) for each \( u \in L^p(\Omega), A \in \mathcal{A} \) and \( c \in \mathbb{R} \);

(e) there exist a nonnegative function \( a \in L^1_{\text{loc}}(\Omega) \) and a positive constant \( b \) such that

\[ 0 \leq F(u, A) \leq \int_A (a(x) + b |Xu|(x)|^{p}) \, dx \]

for each \( u \in C^1(A), A \in \mathcal{A} \).

Then, there exists a Borel function \( f : \Omega \times \mathbb{R}^m \to [0, \infty] \) such that:

(i) for each \( u \in L^p(\Omega) \), for each \( A \in \mathcal{A} \) with \( u|_A \in W^{1,p}_{X,\text{loc}}(A) \), we have

\[ F(u, A) = \int_A f(x, Xu(x)) \, dx; \]

(ii) for a.e. \( x \in \Omega \), \( f(x, \cdot) : \mathbb{R}^m \to [0, \infty) \) is convex;

(iii) for a.e. \( x \in \Omega \),

\[ 0 \leq f(x, \eta) \leq a(x) + b |\eta|^p \quad \forall \eta \in \mathbb{R}^m. \]

In order to prove Theorem 3.12, we need two auxiliary key lemmas. The former is well-known (see, for instance, [Ro, Theorem 12.1]). Let us recall that an affine function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is a function

\[ \varphi(\xi) = \langle z, \xi \rangle + k \quad \forall \xi \in \mathbb{R}^n, \]

for a suitable \( z \in \mathbb{R}^n \) and \( k \in \mathbb{R} \).
3.13. **Lemma.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then

$$g(\xi) = \sup \{ \varphi(\xi) : \varphi \text{ affine, } \varphi(\xi) \leq g(\xi) \quad \forall \xi \in \mathbb{R}^n \}.$$ 

The latter will turn out to be a key result through the paper and provides when a Euclidean integrand can be represented as an integrand respect to $X$-gradient.

3.14. **Lemma.** Let $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$ be a Borel measurable function. Suppose that

(i) for a.e. $x \in \Omega$, $f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is convex;

(ii) there exist a non-negative function $a \in L^{1,\text{loc}}(\Omega)$ and a positive constant $b$ such that for a.e. $x \in \Omega$

$$f_e(x, \xi) \leq a(x) + b|C(x)\xi|^p \quad \forall \xi \in \mathbb{R}^n,$$

where $C(x)$ denotes the coefficient matrix of $X$-gradient in \([3]\).

Then, $f_e$ satisfies \((47)\).

**Proof.** Let us prove that, for a.e. $x \in \Omega$,

\[(53) \quad f_e(x, \xi_N + \zeta) = f_e(x, \zeta) \quad \forall \xi, \zeta \in \mathbb{R}^n,\]

according to notation in section 3.2. Notice that \((53)\) is equivalent to \((47)\), that is, for a.e. $x \in \Omega$

$$f_e(x, \xi) = f_e(x, \xi_V) \quad \forall \xi \in \mathbb{R}^n.$$ 

By our assumptions, we can assume that, for a.e. $x \in \Omega$, $g := f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is a convex function and (ii) holds with $a = a(x) \in [0, \infty)$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be affine with $\varphi(\xi) = \langle z, \xi \rangle + k$ and $\varphi(\xi) \leq g(\xi)$ for each $\xi \in \mathbb{R}^n$. Let us prove that

\[(54) \quad \langle z, v \rangle = 0 \quad \forall v \in N_x.\]

Let $v \in N_x \setminus \{0\}$ be given, then also $tv \in N_x$ for each $t \in \mathbb{R}$. In particular, $C(x)tv = 0$ for each $t \in \mathbb{R}$. Then, by (ii)

$$\varphi(tv) = t\langle z, v \rangle + k \leq g(tv) \leq a \quad \forall t \in \mathbb{R}.$$ 

The previous inequality implies \((54)\). From \((54)\), we get that

\[(55) \quad \varphi(\xi_N + \zeta) = \langle a, \xi_N + \zeta \rangle + b = \langle a, \zeta \rangle + b = \varphi(\zeta) \quad \forall \xi, \zeta \in \mathbb{R}^n.$$ 

From Lemma 3.13, \((53)\) follows. \(\square\)

**Proof Theorem 3.12.** Let us first observe that inequality in assumption (e) can be extended to each $u \in W^{1,p}_X(A)$, $A \in \mathcal{A}$. Let us recall that, if $A \in \mathcal{A}$, by Proposition 2.7, given $(\rho_e)_e$ a family of mollifiers, then, for each $u \in W^{1,p}_X(A)$, denoting by $\bar{u}$ its extension to $\mathbb{R}^n$ being 0 outside $\Omega$, if

$$u_e(x) := \bar{u} * \rho_e(x) \quad x \in \mathbb{R}^n,$$
for each \( A' \in \mathcal{A} \) with \( A' \subseteq A \), we have
\[
(56) \quad u_\varepsilon \to u \text{ in } L^p(\Omega);
\]
\[
(57) \quad u_\varepsilon|_{A'} \in W^{1,p}_X(A') \text{ and } u_\varepsilon \to u \text{ in } W^{1,p}_X(A').
\]
Let \( u \in L^p(\Omega) \) be such that \( u|_A \in W^{1,p}(A) \) for some \( A \in \mathcal{A} \). For each \( A' \subseteq A \), by assumption (c), (56) and (57), it follows that
\[
F(u, A') \leq \liminf_{\varepsilon \to 0} F(u_\varepsilon, A') \leq \lim_{\varepsilon \to 0} \left( \int_{A'} (a(x) + Xu_\varepsilon(x)|^p) \, dx \right)
= \int_{A'} (a(x) + Xu(x)|^p) \, dx.
\]
Since \( F(u, \cdot) \) is a measure, it is also inner regular (see Remark 3.10). Thus, taking the supremum on all \( A' \in \mathcal{A} \) with \( A' \subseteq A \), we get the desired conclusion. We will now divide the proof in three steps.

**1st step.** Let us first prove that there exists an integral representation of \( F \) with respect to a Euclidean integrand, that is, there exists a Borel function \( f_e : \Omega \times \mathbb{R}^n \to [0, \infty] \) and a positive constant \( b_2 \) such that
\[
(58) \quad F(u, A) = \int_A f_e(x, Du) \, dx,
\]
for each \( u \in L^p(\Omega) \), \( A \in \mathcal{A} \) with \( u|_A \in W^{1,p}_{\text{loc}}(A) \);
\[
(59) \quad \text{for a.e. } x \in \Omega, \ f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty) \text{ is convex};
\]
\[
(60) \quad \text{for a.e. } x \in \Omega, \ 0 \leq f_e(x, \xi) \leq a(x) + b_2 |\xi|^p \quad \forall \xi \in \mathbb{R}^n;
\]
\[
(61) \quad (57) \text{ holds, that is, for a.e. } x \in \Omega \ f_e(x, \xi) = f_e(x, \Pi_x(\xi)) \quad \forall \xi \in \mathbb{R}^n.
\]
By (15), if \( u \in W^{1,p}(\Omega) \), then, for a.e. \( x \in \Omega \), we have that
\[
(62) \quad |X u(x)|^p \leq \sup_{x \in \Omega} ||C(x)||^p |Du(x)|^p = b_2 |Du(x)|^p,
\]
with \( b_2 < \infty \), since the coefficients of \( X \)-gradient are Lipschitz on \( \Omega \). By (62) and assumption (e), it follows that
\[
(63) \quad 0 \leq F(u, A) \leq \int_A (a(x) + b_2 |Du(x)|^p) \, dx,
\]
for each \( u \in W^{1,p}(\Omega) \), for every \( A \in \mathcal{A} \). Therefore by (a), (b), (c), (d) and (63), by applying [DM, Theorem 20.1], there exists a Borel function \( f_e : \Omega \times \mathbb{R}^n \to [0, \infty] \) satisfying (58), (59) and (60). Observe now that, by (58) and assumption (e), if \( u = u_\xi \), if follows that, for each \( x \in \mathbb{R}^n \),
\[
\int_A f_e(x, \xi) \, dx \leq \int_A (a(x) + b|C(x)|^p) \, dx \quad \forall A \in \mathcal{A}.
\]
From this integral inequality, arguing as in section 3.2, we can infer the pointwise inequality, that is, there exists a negligible set $N \subset \Omega$, such that,

$$f_e(x, \xi) \leq a(x) + b|C(x)\xi|^p \quad \forall \xi \in \mathbb{R}^n, \quad \text{(64)}$$

From (59), (64) and Lemma 3.14, (61) holds.

2nd step. Let us prove that there exists a Borel function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ such that

$$(65) \quad F(u, A) = \int_A f(x, Xu) \, dx,$$

for each $A \in \mathcal{A}$, $u \in C^1(A)$ satisfying claims (ii) and (iii). By (59), (60) and (61), we can apply Theorem 3.5 and (65) follows at once with $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ defined as in (44), which also satisfies claim (ii).

From assumption (e) and (65) with $u = u_\xi$, it follows that

$$0 \leq \int_A f(y, C(y)\xi) \, dy \leq \int_A (a(y) + b|C(y)\xi|^p) \, dy \quad \text{for each } A \in \mathcal{A}, \xi \in \mathbb{R}^n.$$  

Taking $A = B(x, r)$, applying Lebesgue’s differentiation theorem and arguing as before, from the previous inequality, we can get the following pointwise estimate: for a.e. $x \in \Omega$ it holds that

$$0 \leq f(x, C(x)\xi) \leq a(x) + b|C(x)\xi|^p \quad \forall \xi \in \mathbb{R}^n.$$  

Observe now that, by (LIC), for a.e. $x \in \Omega$, the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$, $L_x(\xi) := C(x)\xi$, is surjective. Then claim (iii) also follows.

3rd step. Let us prove that the integral representation in (65) can be extended to functions $u \in W^{1,p}_{X, loc}(\Omega)$. Therefore claim (i) will follow.

Let us begin to observe that, given $A \in \mathcal{A}_0$, the functional

$$(66) \quad W^{1,p}_X(A) \ni u \mapsto \int_A f(x, Xu) \, dx$$

is (strongly) continuous.

Indeed, since for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is continuous and claim (iii) holds, we can apply the Carathéodory continuity theorem (see, for instance, [DM, Example 1.22]).

Let $u \in W^{1,p}_X(\Omega)$ and let $A, A' \in \mathcal{A}$ with $A' \subseteq A$. Since $F(\cdot, A') : L^p(\Omega) \to [0, \infty)$, by (59), it follows that

$$F(u, A') \leq \inf_{\varepsilon \to 0^+} \int_{A'} f(x, Xu_\varepsilon) \, dx = \int_{A'} f(x, Xu) \, dx.$$

As $F$ is a measure, taking the limit as $A' \uparrow A$, we get

$$(67) \quad F(u, A) \leq \int_A f(x, Xu) \, dx,$$

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$.

Let us fix $w \in W^{1,p}_X(\Omega)$ and let us consider the functional $G : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$

$$G(u, A) := F(u + w, A).$$
It is easy to show that $G$ still satisfies assumptions (a)-(e). Thus, by the second step, there exists a Borel function $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfying claims (ii) and (iii) with $f \equiv g$, for suitable $a \in L^1_{\text{loc}}(\Omega)$ and $b > 0$ such that

\[(68)\quad G(u, A) = \int_A g(x, Xu) \, dx,\]

for each $A \in \mathcal{A}$, $u \in C^1(A)$ and

\[(69)\quad G(u, A) \leq \int_A g(x, Xu) \, dx,\]

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$. Moreover, arguing as in (66), one can prove that, for each $A \in \mathcal{A}_0$, the functional

\[(70)\quad W^{1,p}_X(A) \ni u \mapsto \int_A g(x, Xu) \, dx\]

is (strongly) continuous.

Let

\[w_\varepsilon := \bar{w} * \rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}\]

and fix $A \in \mathcal{A}$. Then, for every $A' \in \mathcal{A}$ with $A' \Subset A$, as $\varepsilon \to 0^+$,

\[w_\varepsilon \to w \text{ in } L^p(\Omega) \text{ and } w_\varepsilon \to w \text{ in } W^{1,p}_X(A').\]

Thus, by (66), (67), (68), (69), (70) we obtain

\[
\int_{A'} g(x, 0) \, dx = G(0, A') = F(w, A') \leq \int_{A'} f(x, Xw) \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{A'} f(x, Xw_\varepsilon) \, dx = \lim_{\varepsilon \to 0^+} F(w_\varepsilon, A')
\]

\[
= \lim_{\varepsilon \to 0^+} G(w_\varepsilon - w, A') = \lim_{\varepsilon \to 0^+} \int_{A'} g(x, Xw_\varepsilon - Xw) \, dx
\]

\[
= \int_{A'} g(x, 0) .
\]

This implies that

\[F(w, A') = \int_{A'} f(x, Xw) \, dx\]

for each $A' \in \mathcal{A}$ with $A' \Subset A$.

Taking the limit as $A' \uparrow A$ in the previous identity, we get that

\[(71)\quad F(w, A) = \int_A f(x, Xw) \, dx\]

for each $w \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$.

If $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $u|_A \in W^{1,p}_{X,\text{loc}}(A)$ then, for every $A' \in \mathcal{A}$ with $A' \Subset A$, by Remark 2.6, there exists $v \in W^{1,p}_X(\Omega)$ such that

\[u|_{A'} = v|_{A'}.\]
Since $F$ is local, by (71), we obtain that
\[
F(u, A') = F(w, A') = \int_{A'} f(x, Xu) \, dx.
\]
Taking the limit as $A' \uparrow A$ we get
\[
F(u, A) = \int_A f(x, Xu) \, dx,
\]
which concludes the proof. □

3.15. **Counterexample.** If $X$ agrees with the Euclidean gradient (Example 2.2 (i)), there are well-known examples that, dropping one of the assumptions among (a)-(e) in Theorem 3.12, then the conclusion may fail (see, for instance, [B]). Let $X$ be the Heisenberg vector fields in $\mathbb{R}^3$ (Example 2.2 (iii)), let $\Omega \subset \mathbb{R}^3$ be a bounded open set containing the origin and $p = 2$. Then we give an instance that, dropping assumption (e), the conclusion of Theorem 3.12 may fail. Let $F : L^2(\Omega) \times A \rightarrow [0, \infty]$ be the local functional defined as
\[
F(u, A) := \begin{cases} 
\int_A |Du|^2 \, dx & \text{if } u \in W^{1,2}(A) \\
\infty & \text{otherwise}
\end{cases}
\]
Then, it is clear that $F$ satisfies (a)-(d). Let us prove that functional $F$ cannot satisfy claim (i). Indeed, by contradiction, if there is some integrand $f : \Omega \times \mathbb{R}^2 \rightarrow [0, \infty]$ for which (i) holds, then, by Theorem 3.5, the compatibility condition (47) must be satisfied, that is,
\[
|\xi|^2 = f_e(x, \xi) = f(x, C(x)\xi) = f_e(x, \Pi_x(\xi)) = |\Pi_x(\xi)|^2
\]
for a.e. $x \in \Omega, \forall \xi \in \mathbb{R}^3$. Since, by Lemma 3.3 (iii), function $\Omega \ni x \mapsto \Pi_x(\xi)$ is continuous, the previous identity must hold for each $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Let $x = 0$, then a simple calculation yields that $\Pi_0(\xi) = (\xi_1, \xi_2, 0)$ for each $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Thus, if we choose $\xi = (0, 0, 1)$, the previous identity is not satisfied and then we have a contradiction. This example also shows that the correspondence which maps integrand $f(x, \eta)$ to Euclidean integrand $f_e(x, \xi)$ cannot be reversed.

4. **$\Gamma$-convergence for integral functionals depending on vector fields**

In this section we are going to show some results concerning $\Gamma$-convergence of integral functionals depending on vector fields, in the strong and weak topology of $W^{1,p}_X(\Omega)$ and in the strong one of $L^p(\Omega)$. In particular, we will prove a $\Gamma$-compactness result for a class of integral functionals depending on vector fields with respect to $L^p(\Omega)$-topology (see Theorem 4.11).

Let us first recall some notions and results concerning $\Gamma$-convergence theory, which are contained in the fundamental monograph [DM] and to which we will refer through this section. We also recommend monograph [Bra] as exhaustive account on this topic, containing also interesting applications of $\Gamma$-convergence.
Let \((X, \tau)\) be a topological space and let \((F_h)_h\) be a sequence of functionals from the space \((X, \tau)\) to \(\mathbb{R}\). Let \(U(x)\) be the family of open neighborhoods of \(x \in X\). Then we pose for every \(x \in X\)

\[
\Gamma(\tau) - \liminf_{h \to \infty} F_h(x) = \sup_{U \in U(x)} \liminf_{h \to \infty} F_h.
\]

\[
\Gamma(\tau) - \limsup_{h \to \infty} F_h(x) = \sup_{U \in U(x)} \limsup_{h \to \infty} F_h.
\]

They are called, respectively, the \(\Gamma\)-lower limit and \(\Gamma\)-upper limit of the sequence \((F_h)_h\) in the topology \(\tau\).

Then, we give the following definition.

4.1. Definition. Let \((F_h)_h\) and \(F\) be functionals from space \((X, \tau)\) to \(\mathbb{R}\). We say that \((F_h)_h\) \(\Gamma(\tau)\)-converges to \(F\), or also that \((F_h)_h\) \(\Gamma\)-converges to \(F\) in the topology \(\tau\), at \(x \in X\), if

\[
\Gamma(\tau) - \liminf_{h \to \infty} F_h(x) = \Gamma(\tau) - \limsup_{h \to \infty} F_h(x) = F(x)
\]

and we write

\[
F(x) = \Gamma(\tau) - \lim_{h \to \infty} F_h(x).
\]

Let us recall below some relevant properties concerning \(\Gamma\)-convergence that we will need later.

4.2. Theorem. Let \(F_h\) and \(F\) be functionals from space \((X, \tau)\) to \(\mathbb{R}\).

(i) ([DM Proposition 6.1]) If \((F_h)_h\) \(\Gamma(\tau)\)-converges to \(F\), then each of its subsequence \((F_{h_k})_k\) still \(\Gamma(\tau)\)-converges to \(F\).

(ii) ([DM Proposition 6.3]) Let \(\tau_i\), \(i = 1, 2\), be two topologies on \(X\) and suppose that \(\tau_1\) is weaker than \(\tau_2\). If \((F_h)_h\) \(\Gamma(\tau_1)\)-converges to \(F_1\) and \(\Gamma(\tau_2)\)-converges to \(F_2\), then \(F_1 \leq F_2\).

(iii) ([DM Theorem 7.8]) (Fundamental Theorem of \(\Gamma\)-convergence) Assume that the sequence \((F_h)_h\) is equicoercive (on \(X\)), that is, for each \(t \in \mathbb{R}\) there exists a closed countably compact \(K_t \subset X\) such that

\[
\{x \in X : F_h(x) \leq t\} \subset K_t \quad \text{for each } h.
\]

Let us also assume that \((F_h)_h\) \(\Gamma(\tau)\)-converges to \(F\). Then \(F\) is coercive and

\[
\min_{x \in X} F(x) = \liminf_{h \to \infty} F_h(x).
\]

(iv) ([DM Proposition 8.1]) Assume that \((X, \tau)\) satisfies the first countability axiom. Then \((F_h)_h\) \(\Gamma(\tau)\)-converges to \(F\) if and only if the following two conditions hold:

1. (\(\Gamma - \liminf\) inequality) for any \(x \in X\) and for any sequence \((x_h)_h\) converging to \(x\) in \(X\) one has

\[
F(x) \leq \liminf_{h \to \infty} F_h(x_h);
\]
(2) ($\Gamma - \lim$ equality) for any $x \in X$, there exists a sequence $(x_h)_h$ converging to $x$ in $X$ such that

$$F(x) = \lim_{h \to \infty} F_h(x_h).$$

(v) ([DM Theorem 8.5]) Assume that $(X, \tau)$ satisfies the second countability axiom, that is, there is a countable base for the topology $\tau$. Then every sequence $(F_h)_h$ of functionals from $X$ to $\mathbb{R}$ has a $\Gamma(\tau)$-convergent subsequence.

4.3. Remark. It is well-known that inequality in Theorem 4.2 (ii) can be strict, even in the case of a (infinite dimensional) Banach space $X$, $\tau_1 \equiv$ weak topology of $X$ and $\tau_2 \equiv$ strong topology of $X$ (see, for instance, [DM, Example 6.6]). An instance of such a phenomenon can occur in the case of non-coercive quadratic integral functionals [ACM].

4.4. Definition ($\bar{\Gamma}$-convergence for local functional on $L^p(\Omega) \times A$). Let $F_h : L^p(\Omega) \times A \to [0, \infty]$ $(h = 1, 2, \ldots)$ be a sequence of increasing functionals. We say that the sequence $(F_h)_h \bar{\Gamma}$-converges to a functional $F : L^p(\Omega) \times A \to [0, \infty]$, and we will write $F = \bar{\Gamma} - \lim_{h \to \infty} F_h$, if $F$ is increasing, inner regular and lsc and the following conditions are satisfied:

[\text{\bar{\Gamma} - \liminf inequality}] for each $u \in L^p(\Omega)$, for every $A \in \mathcal{A}$ and $(u_h)_h \subset L^p(\Omega)$ converging to $u$ in $L^p(\Omega)$, it holds

$$F(u, A) \leq \liminf_{h \to \infty} F_h(u_h, A);$$

[\text{\bar{\Gamma} - \limsup inequality}] for each $u \in L^p(\Omega)$, for each $A, B \in \mathcal{A}$ with $A \subseteq B$, there exists $(u_h)_h \subset L^p(\Omega)$ converging to $u$ in $L^p(\Omega)$ with

$$F(u, B) \geq \limsup_{h \to \infty} F_h(u_h, A).$$

4.5. Remark. Let us consider a sequence of increasing functionals $F_h : L^p(\Omega) \times A \to [0, \infty]$ $(h = 1, 2, \ldots)$. Assume that there exists a measure functional $F : L^p(\Omega) \times A \to [0, \infty]$ such that $(F_h(\cdot, A))_h \bar{\Gamma}$-converges to $F(\cdot, A)$ for each $A \in \mathcal{A}$. Then $(F_h)_h \bar{\Gamma}$-converges to $F$. Indeed, being $F$ a $\Gamma$-limit, it is lsc (see [DM Proposition 6.8]) and it is increasing and inner regular, because it is a measure. Moreover the $\Gamma - \liminf$ and $\Gamma - \limsup$ inequalities immediately follows by the characterization of $\Gamma$-limit in Theorem 4.2 (iv).

4.6. Definition. Let $F : L^p(\Omega) \times A \to [0, \infty]$ be a non-negative functional. We say that $F$ satisfies the fundamental estimate if, for every $\varepsilon > 0$ and for every $A', A''$, $B \in \mathcal{A}$, with $A' \subseteq A''$, there exists a constant $M > 0$ with the following property: for every $u, v \in L^p(\Omega)$, there exists a function $\varphi \in C_c^\infty(A'')$, with $0 \leq \varphi \leq 1$ on $A''$, $\varphi = 1$ in a neighborhood of $A'$, such that

$$F(\varphi u + (1 - \varphi)v, A' \cup B) \leq (1 + \varepsilon)\left(F(u, A'') + F(v, B)\right) +$$

$$+ \varepsilon\left(||u||_{L^p(S)}^p + ||v||_{L^p(S)}^p + 1\right) + M||u - v||_{L^p(S)},$$

24
where \( S = (A'' \setminus A') \cap B \). Moreover, if \( \mathcal{F} \) is a class of non-negative functional on \( L^p(\Omega) \times A \), we say that the fundamental estimate holds uniformly in \( \mathcal{F} \) if each element \( F \) of \( \mathcal{F} \) satisfies the fundamental estimate with \( M \) depending only on \( \varepsilon, A', A'', B \) while \( \varphi \) may depend also on \( F, u, v \).

4.7. **Remark.** Let us recall that, if \( F = \Gamma - \lim_{h \to \infty} F_h \) and \( F_h : L^p(\Omega) \times A \to [0, \infty] \) are measures, then \( F \) need not be a measure (see [DM], Examples 16.13 and 16.14). If the sequence \( (F_h)_h \) satisfies the fundamental estimates uniformly with respect to \( h \), then \( F \) is a measure (see [DM] Theorem 18.5).

Let us now state a result which assures the coincidence between the \( \Gamma - \lim F_h \) and \( \Gamma - \lim F_h \) for a sequence of local functional \( F_h : L^p(\Omega) \times A \to [0, \infty] \), provided that the fundamental estimate holds uniformly for the sequence \( (F_h)_h \) [DM] Theorem 18.7.

4.8. **Theorem.** Let \( (F_h)_h \) be a sequence of non-negative increasing functionals on \( L^p(\Omega) \times A \) which \( \Gamma \)-converges to a functional \( F \). Assume that there exist two constants \( c_1 \geq 1 \) and \( c_2 \geq 0 \), a non-negative increasing functional \( G : L^p(\Omega) \times A \to [0, \infty] \), and a non-negative Radon measure \( \mu : \mathcal{B}(\Omega) \to [0, \infty] \) such that

\[
G(u, A) \leq F_h(u, A) \leq c_1 G(u, A) + c_2 \| u \|_{L^p(A)}^p + \mu(A)
\]

for every \( u \in L^p(\Omega) \), \( A \in A \) and \( h \in \mathbb{N} \). Assume, in addition, that \( G \) is a lower semicontinuous measure and that the fundamental estimate holds uniformly for the sequence \( (F_h)_h \). Then, \( (F_h(\cdot, A))_h \) \( \Gamma \)-converges in \( L^p(\Omega) \) to \( F(\cdot, A) \) for every \( A \in A \) such that \( \mu(A) < \infty \).

4.1. **Convergence of integrands and \( \Gamma \)-convergence for integral functionals depending on vector fields.** In this section we will deal with integral functionals \( F : W^{1,p}_X(\Omega) \to \mathbb{R} \), with \( \Omega \) bounded open subset of \( \mathbb{R}^n \) and \( p > 1 \), of the form

\[
(72) \quad F(u) := \int_{\Omega} f(x, Xu) \, dx
\]

where the integrand \( f : \Omega \times \mathbb{R}^m \to \mathbb{R} \) belongs to class \( I_{m,p}(\Omega, c_0, c_1) \) (i.e., \( f \) satisfies \((I_1), (I_2) \) and \((I_3) \) in the Introduction).

It is easy to show, taking [DM] Proposition 5.12] into account, that the following \( \Gamma \)-convergence results still hold.

4.9. **Proposition.** Let \( (f_h)_h \) and \( f \) be functions in \( I_{m,p}(\Omega, 0, c_1) \). Let \( F_h, F : W^{1,p}_X(\Omega) \to \mathbb{R} \) be the corresponding integral functionals in (72). Assume that

\[
(73) \quad F_h \to F \text{ (pointwise) in } W^{1,p}_X(\Omega).
\]

Then \( (F_h)_h \) \( \Gamma \)-converges to \( F \) in \( W^{1,p}_X(\Omega) \), i.e.,

\[
(74) \quad F(u) = \lim_{h \to \infty} F_h(u) \quad \forall u \in W^{1,p}_X(\Omega).
\]
The following theorem, in particular, shows that the pointwise convergence of the integrands also implies the $\Gamma$-convergence of the corresponding integral functionals in the weak topology of $W^{1,p}_X(\Omega)$.

4.10. **Theorem.** Let $(f_h)_h$ and $f$ be functions in $I_{m,p}(\Omega, 0, c_1)$. Let $F_h$, $F : W^{1,p}_X(\Omega) \to \mathbb{R}$ be the corresponding integral functionals in (72). Assume that
\begin{equation}
 f_h(\cdot, \eta) \to f(\cdot, \eta) \text{ a.e. in } \Omega, \text{ for each } \eta \in \mathbb{R}^m.
\end{equation}
Then
\begin{equation}
 F(u) = (\Gamma(W^{1,p}_X(\Omega)-\text{weak}) - \lim_{h\to\infty} F_h)(u) \quad \forall u \in W^{1,p}_X(\Omega),
\end{equation}
i.e., $(F_h)_h$ $\Gamma$-converges to $F$ in the weak topology of $W^{1,p}_X(\Omega)$.

The scheme of the proof trivially follows the one of [DM, Theorem 5.14] and we omit it.

4.2. **$\Gamma$-compactness results for integral functional depending on vector fields.** The main result of this section is the following.

4.11. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X = (X_1, \ldots, X_m)$ satisfy (LIC) on $\Omega$. Let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ and, for each $h$, let $F^*_h : L^p(\Omega) \times A \to [0, \infty]$ be the local functional defined as
\begin{equation}
 F^*_h(u, A) := \begin{cases} 
 \int_A f_h(x, Xu(x))dx & \text{if } A \in A, u \in W^{1,p}_X(A) \\
 \infty & \text{otherwise}
 \end{cases}
\end{equation}
Then, up to a subsequence, there exist a local functional $F : L^p(\Omega) \times A \to [0, \infty]$ and $f \in I_{m,p}(\Omega, c_0, c_1)$ such that
(i) (9) holds;
(ii) $F$ admits the following representation
\begin{equation}
 F(u, A) := \begin{cases} 
 \int_A f(x, Xu(x))dx & \text{if } A \in A, u \in W^{1,p}_X(A) \\
 \infty & \text{otherwise}
 \end{cases}
\end{equation}
Let us begin to recall a fundamental result about the representation of the $\Gamma$-limit with respect to a Euclidean integrand [DM, Theorem 20.3], which applies to a large class of integral functionals. Let $c_1, c_2, c_3$ be real numbers with $c_i \geq 0$ $i = 1, 2, 3$. Let us denote by $\mathcal{H} = \mathcal{H}(p, c_1, c_2, c_3)$ the class of all local functionals $F : L^p(\Omega) \times A \to [0, \infty]$ for which there exist two Borel functions $f_c, g : \Omega \times \mathbb{R}^n \to [0, \infty)$ (depending on $F$) such that
(a) $F(u, A) := \begin{cases} 
 \int_A f_c(x, Du)dx & \text{if } A \in A, u \in W^{1,1}_{\text{loc}}(A) \\
 \infty & \text{otherwise}
 \end{cases}$
(b) $g(x, \xi) \leq f_c(x, \xi) \leq c_1 (g(x, \xi) + 1)$;
(c) $0 \leq g(x, \xi) \leq c_2 (|\xi|^p + 1)$;
(d) $g(x, \cdot)$ is convex on $\mathbb{R}^n$;
(e) $g(x, 2\xi) \leq c_3 (g(x, \xi) + 1)$,
for every \( u \in L^p(\Omega) \), \( x \in \Omega \), \( \xi \in \mathbb{R}^n \).

4.12. **Theorem.** For every sequence \((F_h)_h\) of functionals of the class \( \mathcal{H} \) there exist a subsequence \((F_{h_k})_k\) and an increasing functional \( F : L^p(\Omega) \times \mathcal{A} \to [0, \infty] \) such that \((F_{h_k})_k\) \( \bar{\Gamma}\)-converges to \( F \). The functional \( F \) can be represented in integral form by a Euclidean integrand, that is, there exists a Borel function \( f_c : \Omega \times \mathbb{R}^n \to [0, \infty] \) verifying

\[
  (i) \ f_c(x, \cdot) \text{ is convex on } \mathbb{R}^n; \\
  (ii) \ 0 \leq f_c(x, \xi) \leq c_1(c_2 + 1) + c_1 c_2 |\xi|^p \text{ for a.e. } x \in \Omega, \text{ for each } \xi \in \mathbb{R}^n,
\]

such that \( (10) \) holds.

Let us also recall an useful criterion for proving that a class of local functionals on \( L^p(\Omega) \times \mathcal{A} \) satisfies the fundamental estimate uniformly [DM, Theorem 19.4] and a \( \bar{\Gamma} \)-compactness result in this class [DM, Theorem 19.5].

4.13. **Theorem.** Let \( c_i \ (i = 1, 2, 3, 4) \) be non negative real numbers and let \( \sigma : \mathcal{A} \to [0, \infty] \) be a superadditive increasing set function such that \( \sigma(A) < \infty \) for each \( A \in \Omega \). Let \( \mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4) \) be the class of all non-negative increasing local functionals \( F : L^p(\Omega) \times \mathcal{A} \to [0, \infty] \) with the following properties: \( F \) is a measure and there exists a non-negative increasing local functional \( G : L^p(\Omega) \times \mathcal{A} \to [0, \infty] \) (depending on \( F \)) such that \( G \) is a measure and

\[
  G(u, A) \leq F(u, A) \leq c_1 G(u, A) + c_2 \|u\|_{L^p(A)}^p + \sigma(A); \tag{79}
\]

\[
  G(\varphi u + (1 - \varphi)v, A) \leq c_4 (G(u, A) + G(v, A)) + c_3 c_4 \max_{\Omega} |D\varphi|^p \|u - v\|_{L^p(A)}^p + 2c_3 \sigma(A), \tag{80}
\]

for every \( u, v \in L^p(\Omega) \), \( A \in \mathcal{A} \), \( \varphi \in C_0^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \). Then, the fundamental estimate holds uniformly on \( \mathcal{F}' \).

4.14. **Theorem.** Let \( \mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4) \) be the class of local functionals defined in Theorem 4.13. For every sequence \((F_h)_h \subset \mathcal{F}'\), there exists a subsequence \((F_{h_k})_k\) which \( \bar{\Gamma} \)-converges to a lower semicontinuous functional \( F \in \mathcal{F}' \).

Let us now introduce some results concerning functionals depending on vector fields. Let us first prove a \( \bar{\Gamma} \)-compactness result (see Theorem 4.16) for a class of local functional on \( L^p(\Omega) \times \mathcal{A} \) satisfying suitable growth conditions with respect to the local functional \( \Psi_p : L^p(\Omega) \times \mathcal{A} \to [0, \infty] \) defined as

\[
  \Psi_p(u, A) := \begin{cases} 
  \int_A |Xu|^p \, dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\
  +\infty & \text{otherwise}
  \end{cases} \tag{81}
\]

As a consequence, we will get a \( \bar{\Gamma} \)-compactness result for a class of integral functionals represented with respect to Euclidean integrands, but still with growth condition with respect to to \( \Psi_p \) (see Theorem 4.17). The former is an extension of [DM, Theorem 19.6], the latter of [DM, Theorem 20.4].
4.15. Lemma. Let $p > 1$. Then $\Psi_p : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ is a measure and lsc.

Proof. Let us start by proving that for any $A \in \mathcal{A}$ the function $u \to \Psi_p(u, A)$ is $L^p-$lsc, i.e., for any $A \in \mathcal{A}$ and $(u_h)_h \subset L^p(\Omega)$, $u_h \to u$ in $L^p(\Omega)$, it satisfies

\begin{equation}
\Psi_p(u, A) \leq \liminf_{h \to \infty} \Psi_p(u_h, A).
\end{equation}

We can assume $\liminf_{h \to \infty} \Psi_p(u_h, A) < \infty$. Therefore, up to a subsequence, we can also assume that $\lim_{h \to \infty} \Psi_p(u_h, A)$ exists. Hence $(u_h)_h$ is bounded in $W^{1,p}_X(A)$ and, since $W^{1,p}_X(A)$ is reflexive (recall Proposition 2.25 and that $p > 1$), we get a subsequence $u_h \to u$ in $W^{1,p}_X(A)$ and, in particular, $Xu_h \to Xu$ in $L^p(A)$, which implies the conclusion, recalling the lower semicontinuity of the $L^p-$norm with respect to the weak convergence.

We now prove that for any $u \in L^p(\Omega)$ the function $\Psi_p(u, \cdot) : \mathcal{A} \to [0, \infty]$ is a measure, i.e., there exists a Borel measure $\mu_u : \mathcal{B}(\Omega) \to [0, \infty]$ such that $\Psi_p(u, A) = \mu_u(A)$ for every $A \in \mathcal{A}$. Since, by Remark 3.10, $\Psi_p(u, \cdot)$ is nonnegative, increasing and such that $\Psi_p(u, \emptyset) = 0$, it suffices to prove that $\Psi_p(u, \cdot)$ is subadditive, superadditive and inner regular.

$\Psi_p(u, \cdot)$ is subadditive, namely for every $A, A_1, A_2 \in \mathcal{A}$ with $A \subseteq A_1 \cup A_2$

\begin{equation}
\Psi_p(u, A) \leq \Psi_p(u, A_1) + \Psi_p(u, A_2).
\end{equation}

We can assume $u \in W^{1,p}_X(A_1) \cap W^{1,p}_X(A_2)$ and $A_1, A_2 \in \mathcal{A}$, otherwise the conclusion is trivial. Remark 2.6 (ii) gives $u \in W^{1,p}_X(A_1 \cup A_2)$, therefore

\[ \Psi_p(u, A_1 \cup A_2) = \int_{A_1 \cup A_2} |Xu|^p \, dx \]

and (83) follows.

\[ \Psi_p(u, A) \geq \Psi_p(u, A_1) + \Psi_p(u, A_2). \]

We can assume $u \in W^{1,p}_X(A)$ and $A \in \mathcal{A}$, otherwise the conclusion is trivial. Remark 2.6 (iv) gives $u \in W^{1,p}_X(B)$ for any open set $B \subseteq A$. Let $A, A_1, A_2 \in \mathcal{A}$, $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$. Then

\[ \Psi_p(u, A_1) + \Psi_p(u, A_2) = \int_{A_1 \cup A_2} |Xu|^p \, dx \leq \int_A |Xu|^p \, dx \]

and (84) follows.

$\Psi_p(u, \cdot)$ is inner regular, namely for every $A \in \mathcal{A}$

\begin{equation}
\Psi_p(u, A) = \sup \{ \Psi_p(u, B) : B \in \mathcal{A}, B \subseteq A \}.
\end{equation}

Let $M := \sup \{ \Psi_p(u, B) : B \in \mathcal{A}, B \subseteq A \} \in [0, +\infty]$. If $M = +\infty$, there exists $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$, $B_i \subseteq A$ such that $\Psi_p(u, B_i) \to \infty$ as $i \to \infty$ and the conclusion follows by observing that for all $i \in \mathbb{N}$, $\Psi_p(u, B_i) \leq \Psi_p(u, A)$. If $M \in [0, \infty)$, then $\|u\|_{W^{1,p}_X(B)} \leq M$ for any $B \in \mathcal{A}, B \subseteq A$. Then, Remark 2.6 (iii) gives $u \in W^{1,p}_X(A)$ and, by definition, $\Psi_p(u, A) = \int_A |Xu|^p \, dx$. For
any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |Xu|^p \, dx \leq \varepsilon$ for any $E \in \mathcal{A}$ with $|E| \leq \delta$. Let $B \subset A$ such that $|A \setminus B| \leq \delta$, then
\[
\int_A |Xu|^p \, dx = \int_B |Xu|^p \, dx + \int_{A \setminus B} |Xu|^p \, dx \leq \int_B |Xu|^p \, dx + \varepsilon
\]
and the thesis follows. $\square$

4.16. Theorem. Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $c_1 \geq c_0 > 0$. Denote by $\mathcal{M} = \mathcal{M}(p, c_0, c_1)$ the class of local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $F$ is a measure and
\[
\begin{align*}
&c_0 \psi_p(u, A) \leq F(u, A) \leq c_1 \left( \psi_p(u, A) + \|u\|_{L^p(A)}^p + |A| \right) \quad \text{for every } u \in L^p(\Omega) \text{ and for every } A \in \mathcal{A}. \\
&\text{Then, the fundamental estimate holds uniformly in } \mathcal{M} \text{ and every sequence } (F_h)_h \subset \mathcal{M} \text{ has a subsequence } (F_{h_k})_k \text{ which } \Gamma\text{-converges to a functional } F \text{ of the class } \mathcal{M}. \text{ Moreover, } (F_{h_k}(\cdot, A))_k \text{ } \Gamma\text{-converges to } F(\cdot, A) \text{ in } L^p(\Omega) \text{ and}
\end{align*}
\]
\[
\text{dom } F(\cdot, A) := \{u \in L^p(\Omega) : F(u, A) < \infty\} = W^{1,p}_X(A)
\]
for every $A \in \mathcal{A}$.

Proof. Let us begin to prove that the fundamental estimate holds uniformly in $\mathcal{M}$. Let
\[
g(x, \xi) := c_0 |C(x)\xi|^p \quad \text{if } x \in \Omega, \xi \in \mathbb{R}^n.
\]
Notice that, since the entries of matrix $C(x)$ are Lipschitz continuous functions,
\[
g(x, \xi) \leq c_0 \sup_{\xi \in \mathbb{R}^n} \|C(x)\|_p \|\xi\|^p = c_2 |\xi|^p \quad \text{if } x \in \Omega, \xi \in \mathbb{R}^n,
\]
\[
g(x, 2\xi) = 2^{p-1} 2g(x, \xi) = c_3 2g(x, \xi) \quad \text{if } x \in \Omega, \xi \in \mathbb{R}^n
\]
and
\[
g(x, \cdot) \text{ is convex on } \mathbb{R}^n.
\]
Thus, from (89), (90) and (91), arguing as in [DM (19.6)], it follows that
\[
g(x, t\xi + (1-t)\eta + \zeta) \leq c_3 (g(x, \xi) + g(x, \eta)) + c_2 |\zeta|^p
\]
for every $x \in \Omega$, $t \in [0, 1]$, $\xi, \eta \in \mathbb{R}^n$. We are going to apply Theorem 4.13.

Observe that, choosing $G = c_0 \psi_p$, from (86) and (79) immediately holds with
\[
c_1 \equiv \frac{c_1}{c_0}, \ c_2 \equiv c_1, \ \sigma(A) = c_1 |A|.
\]
Let us show (80). By (92), it follows that
\[
G(\varphi u + (1 - \varphi)v, A) = \int_A g(x, \varphi Du + (1 - \varphi)Dv + (u - v)D\varphi) \, dx
\]
\[
\leq \int_A [c_3(g(x, Du) + g(x, Dv)) + c_2|D\varphi|^p|u - v|^p] \, dx
\]
\[
\leq c_3(G(u, A) + G(v, A)) + c_2 \left( \max_\Omega |D\varphi|^p \right) \|u - v\|^p_{L^p(A)}
\]
for each \( u, v \in L^p(\Omega), A \in \mathcal{A}, \varphi \in C_0^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \). Thus (80) holds with
\[
c_4 \equiv c_3 \text{ and } c_3c_4 \equiv c_2.
\]
Thus we get the desired conclusion. From Theorem 4.14 every sequence \((F_h)_h \subset \mathcal{M}\) has a subsequence \((F_{h_k})_k\) \(\Gamma\)-converging to a functional \(F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]\) which is a measure. As each functional \(F_h\) satisfies (80), the functional \(F\) satisfies (80), since \(\Psi_p\) is lsc and inner regular by Lemma 4.13 and Remark 3.10. By applying Theorem 4.8, we get that \((F_{h_k}(\cdot, A))_k\) \(\Gamma\)-converges to \(F(\cdot, A)\) in \(L^p(\Omega)\) for each \(A \in \mathcal{A}\), since \(\Omega\) is bounded. Finally, by (80), (87) follows. \(\square\)

Let \(p > 1\) and let \(c_1 \geq c_0\), let \(\Omega \subset \mathbb{R}^n\) be a bounded open set. Let us denote by \(\mathcal{I} = \mathcal{I}(p,c_0,c_1)\) the class of local functionals \(F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]\) for which there exists a Borel function \(f_e : \Omega \times \mathbb{R}^n \to [0, \infty)\) such that

(i) claim (a) of properties defining \(\mathcal{H}\) holds;
(ii) \(c_0 |C(x)\xi|^p \leq f_e(x, \xi) \leq c_1 (|C(x)\xi|^p + 1)\) a.e. \(x \in \Omega\), for each \(\xi \in \mathbb{R}^n\).

4.17. Theorem. For every sequence \((F_h)_h \subset \mathcal{I}\) there exist a subsequence \((F_{h_k})_k\) and a measure functional \(F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]\) such that \((F_{h_k}(\cdot, A))_k\) \(\Gamma\)-converges to \(F(\cdot, A)\) in \(L^p(\Omega)\) and (87) holds for every \(A \in \mathcal{A}\). Moreover there exists a Borel function \(f_e : \Omega \times \mathbb{R}^n \to [0, \infty)\), convex in the second variable and satisfying (ii), for which (10) holds.

Proof. By Theorem 4.10 for each \((F_h)_h \subset \mathcal{I}\) there exist a subsequence \((F_{h_k})_k\) and an inner regular functional \(F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]\) such that \((F_{h_k}(\cdot, A))_k\) \(\Gamma\)-converges to \(F(\cdot, A)\) in \(L^p(\Omega)\) for every \(A \in \mathcal{A}\). Moreover, since \(\Psi_p\) is lsc and inner regular, for each \(u \in L^p(\Omega), A \in \mathcal{A}\),

\[
c_0 \Psi_p(u, A) \leq F(u, A) \leq c_1 (\Psi_p(u, A) + |A|)
\]

where \(\Psi_p\) is the local functional in (81). If \(g(x, \xi)\) is as in (88), \(\mathcal{I}(p,c_0,c_1) \subset \mathcal{H}(p,c_{1}'_1,c_{1}'_2,c_{1}'_3)\), for suitable \(c_{1}'_i\) \((i = 1, 2, 3)\). From Theorem 4.12, there exists a Borel function \(f_e : \Omega \times \mathbb{R}^n \to [0, \infty)\), also convex in the second variable, for which (10) holds.

Let us now prove that (ii) of properties defining \(\mathcal{I}\) holds. Let \(u_\varepsilon\) be the function in (49). From (93), it follows that
\[
c_0 \int_A |C(x)\xi|^p \, dx \leq \int_A f_{u_\varepsilon}(x, \xi) \, dx \leq c_1 \left( |A| + \int_A |C(x)\xi|^p \, dx \right)
\]
for each $\xi \in \mathbb{R}^n$ and $A \in \mathcal{A}$. By means of the usual procedure, we can infer that there exists a negligible set $\mathcal{N} \subset \Omega$ such that, for each $x \in \Omega \setminus \mathcal{N}$,

$$c_0 |C(x)\xi|^p \leq f_e(x, \xi) \leq c_1 (|C(x)\xi|^p + 1) \quad \forall \xi \in \mathbb{Q}^n.$$ 

Then, since $f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is continuous a.e. $x \in \Omega$, we can extend the previous inequality to all $\xi \in \mathbb{R}^n$. 

4.18. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ and, for each $h$, let $F^*_h : L^p(\Omega) \times A \to [0, \infty]$ be the local functional defined in (77). Then, there exist a subsequence $(F^*_h)_k$ and a measure functional $F : L^p(\Omega) \times A \to [0, \infty]$ such that $(F^*_h(\cdot, A))_k$ $\Gamma$-converges to $F(\cdot, A)$ in $L^p(\Omega)$ and (87) holds for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which (10) holds.

**Proof.** Let $(f_{h,e})_h$ denote the sequence of Euclidean integrands in (8) and let $(F_h)_h$ be the sequence of local functionals in (7). Since $(f_{h,e})_h \subset \mathcal{I}$, by applying Theorem 4.17 there exist a subsequence $(F^*_h)_k$ and a measure functional $F : L^p(\Omega) \times A \to [0, \infty]$ such that $(F^*_h(\cdot, A))_k$ $\Gamma$-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii), for which (10) holds.

By Theorem 3.11 (iii), it follows that, for each $h \in \mathbb{N}$, $A \in \mathcal{A}$,

$$F^*_h(\cdot, A) = \tilde{F}_h(\cdot, A)$$

where $\tilde{F}_h(\cdot, A) : L^p(\Omega) \to [0, \infty]$ denotes the relaxed functional of $F_h(\cdot, A) : L^p(\Omega) \to [0, \infty]$ with respect to the $L^p(\Omega)$ topology (see (22)). By (79) and a well-known property of $\Gamma$-convergence (see [DM, Proposition 6.11]), we also get that $(F^*_h(\cdot, A))_k$ $\Gamma$-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. 

4.19. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X = (X_1, \ldots, X_m)$ satisfy (LIC) on $\Omega$. Let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ and, for each $h$, let $F^*_h : L^p(\Omega) \times A \to [0, \infty]$ be the local functional defined in (77). Assume that:

(i) there exists a measure functional $F : L^p(\Omega) \times A \to [0, \infty]$ such that $(F^*_h)_h$ $\Gamma$-converges to $F(\cdot, A)$ in $L^p(\Omega)$ and (87) holds for each $A \in \mathcal{A}$;

(ii) there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which $F$ admits the integral representation in (10).

(iii) (87) holds for every $A \in \mathcal{A}$.

Then, there exists $f \in I_{m,p}(\Omega, c_0, c_1)$ for which $F$ admits the integral representation (78).

**Proof.** Let us first notice that $f_e$ satisfies the assumptions of Lemma 3.14. Thus we can assume that it satisfies (77).
Let \( f : \Omega \times \mathbb{R}^m \to [0, \infty] \) be the function in (44). Let us prove that \( f \in I_{m,p}(\Omega, c_0, c_1) \). Properties \((I_1)\) and \((I_2)\) follow from Theorem 3.5. Since \( f_e \) satisfies (ii) of properties defining class \( I \), from (100), we can infer \((I_3)\).

From Theorem 3.5 and Remark 3.8, \( F \) admits the integral representation (78), but only for functions \( u \in W^{1,p}(A) \). We are going to extend this representation to all functions \( u \in W^{1,p}_X(A) \) by means of Theorem 3.12 about the integral representation of local functionals with respect to \( X \)-gradient. Being \( F \) a \( \Gamma \)-limit, it is lsc (see [DM, Proposition 6.8]) and, by [DM, Proposition 16.15], it is also local and, by assumptions, a measure. Thus assumptions (a), (b) and (c) of Theorem 3.12 are satisfied. Let us prove assumption (d). For every \( h \in \mathbb{N} \), we have \( F^*_h(u + c, A) = F^*_h(u, A) \) whenever \( u \in L^p(\Omega) \), \( c \in \mathbb{R} \). Then it is easy to see that this property also holds for the \( \Gamma \)-limit \( F \). Let us now prove assumption (e). By the integral representation (10) and Remark 3.8, it follows that, for each \( A \in \mathcal{A} \), \( u \in W^{1,p}_X(A) \)

\[
F(u, A) = \int_A f_e(x, Du) \, dx = \int_A f(x, Xu) \, dx 
\]

\[
\leq c_1 \left( \int_A |Xu|^p + |A| \right) 
\]

which implies property (e). Thus there exists a Borel function \( f^* : \Omega \times \mathbb{R}^m \to [0, \infty] \) satisfying property (i) and (ii) of Theorem 3.12. In particular, for each \( A \in \mathcal{A} \), \( u \in W^{1,p}_X(A) \)

\[
F(u, A) = \int_A f^*(x, Xu) \, dx .
\]

By (95) and Theorem 3.5 we get that \( f(x, \eta) = f^*(x, \eta) \) for a.e. \( x \in \Omega \) and for each \( \eta \in \mathbb{R}^m \). This concludes the proof. \( \square \)

**Proof of Theorem 4.11** The proof immediately follows from Theorems 4.18 and 4.19. \( \square \)

We now introduce two integrand function subclasses \( J_i \subset I_{m,p}(\Omega, c_0, c_1) \) \((i = 1, 2)\) for which the associated functionals in (5) are still compact with respect to \( \Gamma \)-convergence in \( L^p(\Omega) \)-topology. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let us fix \( 0 < c_0 \leq c_1 \).

- \( J_1 \equiv J_1(\Omega, c_0, c_1) \) is the subclass of \( I_{m,2}(\Omega, c_0, c_1) \) composed of integrand functions \( f \in I_{m,2}(\Omega, c_0, c_1) \) which are quadratic forms with respect to \( \eta \), that is,

\[
f(x, \eta) = \langle a(x) \eta, \eta \rangle = \sum_{i,j=1}^m a_{ij}(x) \eta_i \eta_j \quad \text{a.e. } x \in \Omega, \forall \eta \in \mathbb{R}^m ,
\]

with \( a(x) = [a_{ij}(x)] \) \( m \times m \) symmetric matrix.
• The subclass $J_2 \equiv J_2(\Omega, c_0, c_1)$ is composed by integrand functions $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, $f$ is independent of $x$.

4.20. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X = (X_1, \ldots, X_m)$ satisfy \((LIC)\) on $\Omega$. Let $(f_h)_h \subset J_1(\Omega, c_0, c_1)$ \((i = 1, 2)\) and, for each $h$, let $F^*_h : L^p(\Omega) \times A \rightarrow [0, \infty]$ be the local functional defined in \((77)\). Then, up to a subsequence, there exist a local functional $F : L^p(\Omega) \times A \rightarrow [0, \infty]$ and $f \in J_i(\Omega, c_0, c_1)$ such that

(i) \((9)\) holds;
(ii) $F$ admits representation \((78)\).

**Proof.** _1st case._ Let us first show the conclusion for the subclass $J_1$.

Let $(f_h)_h \subset J_1$. By definition, we can assume that

$$f_h(x, \eta) := \langle a_h(x) \eta, \eta \rangle \quad x \in \Omega, \eta \in \mathbb{R}^m,$$

where $a_h(x) = [a_{h,ij}(x)]$ is a $m \times m$ symmetric matrix satisfying

$$c_0 |\eta|^2 \leq \langle a_h(x) \eta, \eta \rangle \leq c_1 \left(|\eta|^2 + 1\right) \quad \text{a.e.} \ x \in \Omega, \ \forall \eta \in \mathbb{R}^m$$

and

$$a_{h,ij} \in L^{\infty}(\Omega) \quad \text{for each} \ i,j = 1, \ldots, m, \ h \in \mathbb{N}.$$  

Applying Theorem 4.11 up to a subsequence, there exist a local functional $F : L^p(\Omega) \times A \rightarrow [0, \infty]$ and $f \in I_{m,2}(\Omega, c_0, c_1)$ such that \((9)\) holds and $F$ admits representation \((78)\). We have only to prove that

$$f \in J_1.$$  

Notice that we can also assume that $F$ admits representation \((10)\) with

$$f_e(x, \xi) := f(x, C(x)\xi) \quad \text{for a.e.} \ x \in \Omega, \ \forall \xi \in \mathbb{R}^n.$$  

Moreover, by Theorem 3.3 (see \((44)\) and \((40)\)), it also holds the opposite representation, that is, for each $x \in \Omega_X$,

$$f(x, \eta) = f_e(x, L^{-1}_x(\eta)) \quad \forall \eta \in \mathbb{R}^m,$$

with

$$L^{-1}_x(\eta) := C(x)^T B(x)^{-1} \eta.$$  

Let us now consider the sequence of Euclidean integrands

$$f_{h,e}(x, \xi) := f_h(x, C(x)\xi) = \langle a_h(x) C(x)\xi, C(x)\xi \rangle$$

and the related local functionals $F_h : L^p(\Omega) \times A \rightarrow [0, \infty]$ defined in \((7)\). Since $F_h(u, A) = F^*_h(u, A)$ for each $u \in W^{1,1}_{\text{loc}}(A)$, by using well-known results of $\Gamma$-convergence for quadratic functionals (see \([DM\] Theorem 22.1\) and Remark 4.5 one can easily prove that there exists a $n \times n$ symmetric matrix $a_e(x) = [a_{e,ij}(x)]$, with $a_{e,ij} \in L^\infty(\Omega)$ for each $i,j = 1, \ldots, n$ such that

$$f_e(x, \xi) = \langle a_e(x) \xi, \xi \rangle \quad \text{a.e.} \ x \in \Omega, \ \forall \xi \in \mathbb{R}^n.$$  

33
By (99), for each \( x \in \Omega \),

\[
\tilde{f}(x, \eta) := f_e(x, L_x^{-1}(\eta)) = \langle a_e(x)C(x)^T B(x)^{-1} \eta, C(x)^T B(x)^{-1} \eta \rangle = \langle (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1} \eta, \eta \rangle = \langle a(x) \eta, \eta \rangle
\]

(100)

with

\[
a(x) := (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1},
\]

\( m \times m \) symmetric matrix. Then \( f(x, \cdot) \) turns out to be a quadratic form on \( \mathbb{R}^m \), induced by the matrix \( a(x) \) for a.e. \( x \in \Omega \). Thus (98) follows.

2nd case. Let us now deal with the subclass \( J_2 \). Let \( (f_h)_h \subset J_2 \). Notice that \( f_h : \mathbb{R}^m \to [0, \infty) \), \( h \in \mathbb{N} \), is a sequence of locally bounded, convex functions. Thus, by a well-known result (see, for instance, [DM, Proposition 5.11]), we can infer that \( (f_h)_h \) is also locally equi-Lipschitz continuous. From Ascoli-Arzelà’s theorem, we can assume that, up to a subsequence, there exists \( f \in J_2 \) such that

\[
f_h \rightharpoonup f \text{ uniformly on bounded sets of } \mathbb{R}^n \text{ as } h \to \infty.
\]

Let us now prove that, for each \( A \in \mathcal{A} \),

\[
\lim_{h \to \infty} F^*_h(u, A) = \tilde{F}(u, A) \quad \forall u \in W^{1,p}_X(A).
\]

(102)

Let us fix \( A \in \mathcal{A} \) and \( u \in W^{1,p}_X(A) \). Since \( |Xu(x)| < \infty \) for a.e. \( x \in A \), by (101), it follows that

\[
\lim_{h \to \infty} f_h(Xu(x)) = f(Xu(x)) \text{ for a.e. } x \in A.
\]

(103)

On the other hand, as

\[
0 \leq f_h(Xu(x)) \leq c_1(1 + |Xu(x)|^p)
\]

for a.e. \( x \in A \), for each \( h \), by (103) and the dominated convergence theorem, (102) follows. We have only to prove that

\[
F(u, A) = \tilde{F}(u, A) \quad \forall A \in \mathcal{A}, \forall u \in L^p(\Omega)
\]

(104)

in order to get our desired conclusion. By (87), it is sufficient to prove (104) for each \( A \in \mathcal{A} \) and for each \( u \in W^{1,p}_X(A) \). The inequality

\[
F(u, A) \leq \tilde{F}(u, A) \quad \forall A \in \mathcal{A}, \forall u \in W^{1,p}_X(A),
\]

(105)

follows by noticing that, for each \( u \in W^{1,p}_X(A) \), by \( \Gamma - \lim \inf \) inequality and (102),

\[
F(u, A) \leq \lim \inf_{h \to \infty} F^*_h(u, A) = \tilde{F}(u, A).
\]

Let us now prove the opposite inequality

\[
F(u, A) \geq \tilde{F}(u, A) \quad \forall A \in \mathcal{A}, \forall u \in W^{1,p}_X(A).
\]

(106)
Let us first recall that, for each $A \in \mathcal{A}$, by (102) and Proposition 4.9,
\begin{equation}
\tilde{F}(u, A) = (\Gamma(W^{1,p}_X(A)) - \lim_{h \to \infty} F^*_h(u)) \quad \forall u \in W^{1,p}_X(A).
\end{equation}

Fix $A \in \mathcal{A}$ and let $u \in L^p(\Omega)$ with $u|_A \in W^{1,p}_X(A)$. By the $\Gamma$-lim equality, there exists a sequence $(u_h)_h \subset L^p(\Omega)$ such that
\begin{equation}
u_h \to u \text{ in } L^p(\Omega), \text{ as } h \to \infty
\end{equation}
and
\begin{equation}
\lim_{h \to \infty} F^*_h(u_h, A) = F(u, A) < \infty.
\end{equation}
By (109), we can assume that
\begin{equation}
(u_h|_A)_h \subset W^{1,p}_X(A).
\end{equation}

Let $A' \in \mathcal{A}$ with $A' \subset A$. From Proposition 2.10 (ii), if $w := \overline{Xu_h} : \mathbb{R}^n \to \mathbb{R}^m$, that is, $\overline{Xu_h} = Xu_h$ on $A$ and $\overline{Xu_h} = 0$ outside, for each $0 < \varepsilon < \text{dist}(A', \mathbb{R}^n \setminus A)$
\begin{equation}
\int_{A'} f_h(\rho_\varepsilon * \overline{Xu_h}) \, dx \leq \int_A f_h(\overline{Xu_h}) \, dx \text{ for each } h.
\end{equation}
By (108), (110) and Proposition 2.10 (i), for given $0 < \varepsilon < \text{dist}(A', \mathbb{R}^n \setminus A)$,
\begin{equation}
X(\rho_\varepsilon * \overline{u_h}) \to X(\rho_\varepsilon * \bar{u}) \text{ uniformly on } A' \text{ as } h \to \infty
\end{equation}
and
\begin{equation}
\rho_\varepsilon * \overline{Xu_h} \to \rho_\varepsilon * \overline{Xu} \text{ uniformly on } A' \text{ as } h \to \infty.
\end{equation}
In particular,
\begin{equation}
\rho_\varepsilon * \overline{u_h} \to \rho_\varepsilon * \bar{u} \text{ in } W^{1,p}_X(A') \text{ as } h \to \infty.
\end{equation}
Observe now that, by (111), for each $0 < \varepsilon < \text{dist}(A', \mathbb{R}^n \setminus A)$, for each $h$,
\begin{equation}
F^*_h(\rho_\varepsilon * \overline{u_h}, A') = \int_{A'} f_h(X(\rho_\varepsilon * \overline{u_h})) \, dx
\end{equation}
\begin{equation}
= \int_{A'} f_h(\rho_\varepsilon * \overline{Xu_h}) \, dx + \int_{A'} (f_h(X(\rho_\varepsilon * \overline{u_h})) - f_h(\rho_\varepsilon * \overline{Xu_h})) \, dx
\leq \int_{A} f_h(\overline{Xu_h}) \, dx + \int_{A'} (f_h(X(\rho_\varepsilon * \bar{u})) - f_h(\rho_\varepsilon * \overline{Xu_h})) \, dx
= F^*_h(u_h, A) + R_{\varepsilon, h}.
\end{equation}
From (101), (112) and (113), it follows that, for given $0 < \varepsilon < \text{dist}(A', \mathbb{R}^n \setminus A)$
\begin{equation}
\lim_{h \to \infty} R_{\varepsilon, h} = R_{\varepsilon} := \int_{A'} (f(X(\rho_\varepsilon * \bar{u})) - f(\rho_\varepsilon * \overline{Xu})) \, dx.
\end{equation}
For given $0 < \varepsilon < \text{dist}(A', \mathbb{R}^n \setminus A)$, by (107), (109), (114), and (116), passing to the limit in (115) as $h \to \infty$, it follows that

$$
\tilde{F}(\rho_\varepsilon \ast \bar{u}, A') \leq \liminf_{h \to \infty} F^*_h(\rho_\varepsilon \ast \bar{u}_h, A') \\
\leq \lim_{h \to \infty} F^*_h(u_h, A) + \lim_{h \to \infty} R_{\varepsilon,h} = F(u, A) + R_\varepsilon.
$$

(117)

Let us now show that

$$
\lim_{\varepsilon \to 0^+} R_\varepsilon = 0.
$$

(118)

Indeed

$$X(\rho_\varepsilon \ast \bar{u}) \to Xu \quad \text{and} \quad \rho_\varepsilon \ast X\bar{u} \to Xu \quad \text{in} \quad L^p(A'), \quad \text{as} \quad \varepsilon \to 0^+
$$

and

$$f(X(\rho_\varepsilon \ast \bar{u})) \leq c_1(1 + |X(\rho_\varepsilon \ast \bar{u})|^p) \quad \text{and} \quad f(\rho_\varepsilon \ast X\bar{u}) \leq c_1(1 + |\rho_\varepsilon \ast X\bar{u}|^p) \quad \text{a.e. in} \quad A'.
$$

Since $f$ is continuous, from Vitali's convergence theorem, (118) follows. By the semicontinuity of $\tilde{F}$, with respect to the $L^p$-topology, and by (118), we can pass to the limit as $\varepsilon \to 0^+$ in (117) and we get

$$\tilde{F}(u, A') \leq \lim_{\varepsilon \to 0^+} \tilde{F}(\rho_\varepsilon \ast \bar{u}, A') \leq F(u, A) \quad \text{for each} \quad A' \in \mathcal{A}.
$$

Finally, taking the supremum in (119) on all $A' \in \mathcal{A}$ with $A' \in \mathcal{A}$, we get

$$
\text{(106)}.
$$

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