Analysis of entropy of \( XY \) Spin Chain

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Abstract. Entanglement in the ground state of the \( XY \) model on the infinite chain can be measured by the von Neumann entropy of a block of neighboring spins. We study a double scaling limit: the size of the block is much larger then 1 but much smaller then the length of the whole chain. In this limit, the entropy of the block approaches a constant. The limiting entropy is a function of the anisotropy and of the magnetic field. The entropy reaches minima at product states and increases boundlessly at phase transitions.
1. Introduction

Entanglement is a primary resource for quantum computation and information processing. It shows how much quantum effects we can use to control one system by another. Stable and large scale entanglement is necessary for scalability of quantum computation. The entropy of a subsystem as a measure of entanglement was discovered in [3]. Essential progress has been achieved in the understanding of entanglement in various quantum systems.

The importance of the XY model for quantum information was emphasized in [7, 20, 21, 22]. In this paper we consider the entropy of a block of \( L \) neighboring spins in the ground state of the XY model [on the infinite chain] in the limit \( L \to \infty \). We use the results of [23, 24, 25]. The Hamiltonian of the XY model is

\[
H = - \sum_{n=-\infty}^{\infty} (1 + \gamma) \sigma_n^x \sigma_{n+1}^x + (1 - \gamma) \sigma_n^y \sigma_{n+1}^y + h \sigma_n^z
\]

Here \( 0 < \gamma < 1 \) is the anisotropy parameter; \( \sigma_n^x, \sigma_n^y \) and \( \sigma_n^z \) are the Pauli matrices and \( 0 < h \) is the magnetic field. The model was solved in [26, 27, 28, 29]. The methods of Toeplitz determinants and integrable Fredholm operators were used for the evaluation of correlation functions, see [28, 30, 31, 32, 33, 34]. The idea to use the determinants for calculation of entropy was put forward in [8].

Solution of XY looks differently in three cases:

Case 1a is defined by the inequality \( 2\sqrt{1 - \gamma^2} < h < 2 \).

It describes moderate magnetic field.

Case 2 is defined by \( h > 2 \). This is strong magnetic field.

Case 1b is defined by \( 0 < h < 2\sqrt{1 - \gamma^2} \).

It is weak magnetic field, including zero magnetic field.
Analysis of entropy of XY Spin Chain

At the boundary between cases 1a and 1b \((h = 2\sqrt{1-\gamma^2})\) the ground state is doubly degenerated:

\[
|G_1\rangle = \prod_{n \in \text{lattice}} \left[ \cos(\theta)|\uparrow_n\rangle + (-1)^n \sin(\theta)|\downarrow_n\rangle \right],
\]

\[
|G_2\rangle = \prod_{n \in \text{lattice}} \left[ \cos(\theta)|\uparrow_n\rangle - (-1)^n \sin(\theta)|\downarrow_n\rangle \right]
\]

Here \(\cos^2(2\theta) = (1-\gamma)/(1+\gamma)\). The role of factorized states was emphasized in [30, 35, 36]. Let us mention that the rest of energy levels are separated by a gap and correlations decay exponentially. The boundary boundary between cases 1a and 1b is not a phase transition.

In general, we denote the ground state of the model by \(|GS\rangle\). We consider the entropy of a block of \(L\) neighboring spins: it measures the entanglement between the block and the rest of the chain [3, 20]. We treat the whole ground state as a binary system \(|GS\rangle = |A&B\rangle\). The block of \(L\) neighboring spins is subsystem A and the rest of the ground state is subsystem B. The density matrix of the ground state is \(\rho_{AB} = |GS\rangle \langle GS|\). The density matrix of the block is \(\rho_A = Tr_B(\rho_{AB})\). The entropy \(S(\rho_A)\) of the block is:

\[
S(\rho_A) = -Tr(\rho_A \ln \rho_A)
\]

Note that each of the ground states (2) is factorized and has no entropy.

To express the entropy we need the complete elliptic integral of the first kind,

\[
I(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}
\]

and the modulus

\[
\tau_0 = I(k')/I(k), \quad k' = \sqrt{1-k^2}
\]

In the paper [23] we used determinant representation for evaluation of entropy. Zeros of the determinant form an infinite sequence of numbers:

\[
\lambda_m = \tanh \left( m + \frac{1-\sigma}{2} \right) \pi \tau_0, \quad m \geq 0, \quad \lambda_{-m} = -\lambda_m
\]

here \(\sigma = 1\) in Case 1 and \(\sigma = 0\) in Case 2. Note \(0 < \lambda_m < 1\) and \(\lambda_m \to 1\) as \(m \to \infty\).
The magnetic field and anisotropy define $k$:

$$k = \begin{cases} 
\sqrt{(h/2)^2 + \gamma^2 - 1} / \gamma, & \text{Case 1a} \\
\sqrt{1 - \gamma^2 - (h/2)^2} / \sqrt{1 - (h/2)^2}, & \text{Case 1b} \\
\gamma / \sqrt{(h/2)^2 + \gamma^2 - 1}, & \text{Case 2}
\end{cases}$$

We represented the entropy as a convergent series in [23]:

$$S(\rho_A) = \sum_{m=-\infty}^{\infty} (1 + \lambda_m) \ln \frac{2}{1 + \lambda_m}$$  \hspace{1cm} (8)

I. Peshel using the approach of [11] also obtained the series (8) in cases of non-zero magnetic field, see [24]. He summed it up into:

$$S(\rho_A) = \frac{1}{6} \left[ \ln \left( \frac{k^2}{16k'} \right) + \left( 1 - \frac{k^2}{2} \right) \frac{4I(k)I(k')}{\pi} \right] + \ln 2, \hspace{1cm} \text{Case 1a}$$

$$S(\rho_A) = \frac{1}{12} \left[ \ln \frac{16}{(k^2k'^2)} + (k^2 - k'^2) \frac{4I(k)I(k')}{\pi} \right], \hspace{1cm} \text{Case 2}$$

We summed up the series (8) in case of weak magnetic field (including zero magnetic field) in the paper [23]:

$$S(\rho_A) = \frac{1}{6} \left[ \ln \left( \frac{k^2}{16k'} \right) + \left( 1 - \frac{k^2}{2} \right) \frac{4I(k)I(k')}{\pi} \right] + \ln 2, \hspace{1cm} \text{Case 1b},$$

The rigorous proof and the precise history is given in the paper [25].

Now we can study the range of variation of the limiting entropy. We find a local minimum $S(\rho_A) = \ln 2$ at the boundary between cases 1a and 1b ($h = 2\sqrt{1 - \gamma^2}$). This is the case of double degenerated ground state [2]. The absolute minimum is achieved at infinite magnetic field, where the ground state becomes ferromagnetic (i.e. all spins are parallel). The entropy diverges to $+\infty$, i.e has singularities, at the phase transitions: $h = 2$ [11] or $\gamma = 0$, see [8]. To show this behavior of the limiting entropy, we plot it as a function of the magnetic field $h$ at constant anisotropy $\gamma = 1/2$ in Fig. [4]. It is interesting to note that the critical behavior of the $XY$ model is similar to the Lipkin-Meshkov-Glick model [37].
Analysis of entropy of XY Spin Chain

Figure 1. The limiting entropy as a function of the magnetic field at constant anisotropy $\gamma = 1/2$. The entropy has a local minimum $S = \ln 2$ at $h = 2\sqrt{1 - \gamma^2}$ and the absolute minimum for $h \to \infty$ where it vanishes. $S$ is singular at the phase transition $h = 2$ where it diverges to $+\infty$. The three cases are marked.

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Analysis of entropy of XY Spin Chain

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