Strongly Nil ∗-Clean Rings

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Abstract

A ∗-ring $R$ is called strongly nil ∗-clean if every element of $R$ is the sum of a projection and a nilpotent element that commute with each other. In this paper we prove that $R$ is a strongly nil ∗-clean ring if and only if every idempotent in $R$ is a projection, $R$ is periodic, and $R/J(R)$ is Boolean. For any commutative ∗-ring $R$ with $\mu^* = \mu$, $\eta^* = \eta \in R$, the algebraic extension $R[i] = \{a + bi \mid a, b \in R, i^2 = \mu i + \eta\}$ is strongly nil ∗-clean if and only if $R$ is strongly nil ∗-clean and $\mu \eta$ is nilpotent. We also prove that a ∗-ring $R$ is commutative, strongly nil ∗-clean and every primary ideal is maximal if and only if every element of $R$ is a projection.

2010 Mathematics Subject Classification : 16W10, 16U99
Key words: rings with involution; strongly nil ∗-clean ring; algebraic extension; ∗-Boolean-like ring.

1 Introduction

Let $R$ be an associative ring with unity. A ring $R$ is called strongly nil clean if every element of $R$ is the sum of an idempotent and a nilpotent that commute. These rings are first discovered by Hirano-Tominaga-Yakub [11] and were referred to as [E-N]-representable rings. In [8] and [9], Diesl refers to this class as strongly nil clean and studies their properties. Studying strongly nil cleanness is also relevant for Lie algebra. The decomposition of matrices as in the definition of strongly nil cleanness over a field must be the Jordan-Chevalley decomposition in Lie theory. An involution of a ring $R$ is an operation $*: R \rightarrow R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring $R$ with involution $*$ is called a ∗-ring. An element $p$ in a ∗-ring $R$ is called a projection if $p^2 = p = p^*$ (see [2]). Recently the concept of strongly clean rings are considered for any ∗-ring. Vaš [15] calls a ∗-ring $R$ strongly ∗-clean if each of its elements is the sum of a projection and a unit that commute with each other (see also [14]).

In this paper, we adapt strongly nil cleanness to ∗-rings. We call a ∗-ring $R$ strongly nil ∗-clean if every element of $R$ is the sum of a projection and a nilpotent element that commute with each other.

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commute. The paper consists of three parts. In Section 2, we characterize the class of strongly nil *-clean rings on several different ways. For example, we show that a ring $R$ is a strongly nil *-clean ring if and only if every idempotent in $R$ is a projection, $R$ is periodic, and $R/J(R)$ is Boolean. In Section 3, we prove a result related to the strongly nil *-cleanness of a commutative *-ring and its algebraic extension. For a commutative *-ring $R$ with $\mu^* = \mu$, $\eta^* = \eta \in R$, $R[i] = \{a + bi | a, b \in R, i^2 = \mu i + \eta\}$ is strongly nil *-clean if and only if $R$ is strongly nil *-clean and $\mu \eta$ is nilpotent. Foster [10] introduced the concept of Boolean-like rings as a generalization of Boolean rings in the commutative case. We adopt the concept of Boolean-like rings to rings with involution and prove that a *-ring $R$ is *-Boolean-like if and only if $R$ is strongly nil *-clean and $\alpha \beta = 0$ for all nilpotent elements $\alpha, \beta$ in $R$. In the last section, we investigate submaximal ideals ([12]) of strongly nil *-clean rings; and also define *-Boolean rings as *-rings over which every element is a projection and characterize them in terms of strongly nil *-cleanness.

Throughout this paper all rings are associative with unity (unless otherwise noted). We write $J(R)$, $N(R)$ and $U(R)$ for the Jacobson radical of a ring $R$, the set of all nilpotent elements in $R$ and the set of all units in $R$, respectively. The ring of all polynomials in one variable over $R$ is denoted by $R[x]$.

2 Characterization Theorems

The main purpose of this section is to explore the structure of strongly nil *-clean rings. A ring $R$ is called uniquely nil clean if, for any $x \in R$, there exists a unique idempotent $e \in R$ such that $x - e \in N(R)$ [8]. If, in addition, $x$ and $e$ commute, $R$ is called uniquely strongly nil clean [11]. Strongly nil cleanness and uniquely strongly nil cleanness are equivalent by [11, Theorem 3]. Analogously, for a *-ring, we define uniquely strongly nil *-clean rings by replacing “idempotent” with “projection” in the definition of uniquely strongly nil clean rings.

We will use the following lemma frequently.

Lemma 2.1 [14, Lemma 2.1] Let $R$ be a *-ring. If every idempotent in $R$ is a projection, then $R$ is abelian, i.e. every idempotent in $R$ is central.

Proposition 2.2 Let $R$ be a *-ring. Then the following are equivalent.

(i) $R$ is strongly nil *-clean;

(ii) $R$ is uniquely nil clean and every idempotent in $R$ is a projection;

(iii) $R$ is uniquely strongly nil *-clean.
Proof If $R$ is strongly nil $*$-clean, then $R$ is strongly $*$-clean. For, if $x \in R$, then there exist a projection $e \in R$ and $w \in N(R)$ such that $2 - x = e + w$ and $ew = we$. This gives that $x = (1 - e) + (1 - w)$ where $1 - e$ is a projection and $1 - w \in U(R)$. If $R$ is strongly $*$-clean, then every idempotent in $R$ is a projection by [14, Theorem 2.2]. By [11, Theorem 3], the proof is completed. □

We note that the condition “every idempotent in $R$ is a projection” in Proposition 2.2 is necessary as the following example shows.

Example 2.3 Let $R = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \}$ where $0, 1 \in \mathbb{Z}_2$. Define $*: R \to R, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a + b & b \\ a + b + c + d & b + d \end{pmatrix}$. Then $R$ is a commutative $*$-ring with the usual matrix addition and multiplication. In fact, $R$ is Boolean. Thus, for any $x \in R$, there exists a unique idempotent $e \in R$ such that $x - e \in R$ is nilpotent. But it is not strongly nil $*$-clean because the only projections are the trivial projections and there does not exist a projection $e$ in $R$ such that $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - e$ is nilpotent.

On the other hand, in [11, Theorem 3], it is proved that $R$ is strongly nil clean if and only if $N(R)$ is an ideal and $R/N(R)$ is Boolean. Also, $R$ is uniquely nil clean if and only if $R$ is abelian, $N(R)$ is an ideal and $R/N(R)$ is Boolean [11, Theorem 4]. So if we adopt these results to rings with involution, immediately we have the following theorem by using Proposition 2.2. But we give a new proof to the necessity.

Theorem 2.4 Let $R$ be a $*$-ring. Then $R$ is strongly nil $*$-clean if and only if

1. Every idempotent in $R$ is a projection;
2. $N(R)$ forms an ideal;
3. $R/N(R)$ is Boolean.

Proof Assume that $R$ is strongly nil $*$-clean. In view of Lemma 2.1 for any $x \in R$, there exist an idempotent $g \in R$ and a nilpotent element $v \in R$ such that $x = g + v$ and $gv = vg$. Thus, $x^2 = g + (2g + v)v$, and so $x - x^2 = (1 - 2g - v)v \in N(R)$. Write $(x - x^2)^m = 0$, and so $x^m \in x^{m+1}R$. This shows that $R$ is strongly $\pi$-regular. According to [11, Theorem 3], $N(R)$ forms an ideal of $R$. Further, $x - x^2 \in N(R)$, and so $R/N(R)$ is Boolean. The converse is obvious by [11, Theorem 3]. □

A ring $R$ is called strongly $J*$-clean if for any $x \in R$ there exists a unique projection $e \in R$ such that $x - e \in J(R)$ [7].
Lemma 2.5 Let $R$ be a $*$-ring. Then $R$ is strongly nil $*$-clean if and only if

(1) $R$ is strongly $J$-$*$-clean;
(2) $J(R)$ is nil.

Proof Suppose that $R$ is strongly nil $*$-clean. In view of Theorem 2.4, $N(R)$ forms an ideal of $R$, and this gives that $N(R) \subseteq J(R)$. On the other hand, for any $x \in J(R)$, there exists a projection $e \in R$ such that $x - e \in N(R)$. Then $e = x - (x - e) \in J(R)$. This shows that $e = 0$, and so $x$ is nilpotent. That is $J(R)$ is nil, and so $N(R) = J(R)$. In view of Proposition 2.2, we can see that there exists a unique projection $e \in R$ such that $x = e + w$ and $xe = ex$. As $J(R)$ is nil, $w \in R$ is nilpotent. Therefore $R$ is strongly nil $*$-clean.

Conversely, assume that (1) and (2) hold. In view of [7, Proposition 2.1], $R$ is strongly $*$-clean. Thus, $R$ is abelian. Let $x \in R$. By virtue of [7, Theorem 3.3], there exist a projection $e \in R$ and a $w \in J(R)$ such that $x = e + w$ and $xe = ex$. As $J(R)$ is nil, $w \in R$ is nilpotent. Therefore $R$ is strongly nil $*$-clean. □

From Lemma 2.5 and [7, Proposition 2.1], it follows that

$$\{\text{strongly nil } *\text{-clean}\} \subset \{\text{strongly } J\text{-*\-clean}\} \subset \{\text{strongly } *\text{-clean}\}. $$

The first inclusion is strict, because, for example, the power series ring $\mathbb{Z}_2[[x]]$ is strongly $J$-$*$-clean but not strongly nil $*$-clean where $*$ is the identity involution by [4, Example 2.5(5)]. The second inclusion is also strict by [7, Example 2.2(2)].

We should note that a strongly nil clean ring may not be strongly $J$-clean (see [4, Example on p. 3799]). Hence strongly nil clean and strongly nil $*$-clean classes have different behavior when compared to classes of strongly $J$-clean and strongly $J$-$*$-clean classes respectively.

Lemma 2.6 Let $R$ be a $*$-ring. Then $R$ is strongly nil $*$-clean if and only if

(1) Every idempotent in $R$ is a projection;
(2) $J(R)$ is nil;
(3) $R/J(R)$ is Boolean.

Proof Assume that (1), (2) and (3) hold. For any $x \in R$, $x + J(R) = x^2 + J(R)$. As $J(R)$ is nil, every idempotent in $R$ lifts modulo $J(R)$. Thus, we can find an idempotent $e \in R$ such that $x - e \in J(R) \subseteq N(R)$. By Lemma 2.1, $xe = ex$, and so the result follows. The converse is by Theorem 2.4 and Lemma 2.5. □

Recall that a ring $R$ is periodic if for any $x \in R$, there exist distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. With this information we can now prove the following.
Theorem 2.7 Let \( R \) be a \( *- \)ring. Then \( R \) is strongly nil \( * \) -clean if and only if

1. Every idempotent in \( R \) is a projection;
2. \( R \) is periodic;
3. \( R/J(R) \) is Boolean.

Proof Suppose that \( R \) is strongly nil \( * \) -clean. By virtue of Lemma 2.6, every idempotent in \( R \) is a projection and \( R/J(R) \) is Boolean. For any \( x \in R \), \( x - x^2 \in N(R) \). Write \((x - x^2)^m = 0\), and so \( x^m = x^{m+1}f(x) \), where \( f(x) \in \mathbb{Z}[x] \). According to Herstein’s Theorem (cf. [3, Proposition 2]), \( R \) is periodic. Conversely, \( J(R) \) is nil as \( R \) is periodic. Therefore the proof is completed by Lemma 2.6. □

Proposition 2.8 A \( * \) -ring \( R \) is strongly nil \( * \) -clean if and only if

1. \( R \) is strongly \( * \) -clean;
2. \( N(R) = \{ x \in R \mid 1 - x \in U(R) \} \).

Proof Suppose that \( R \) is strongly nil \( * \) -clean. By the proof of Lemma 2.5, \( N(R) = J(R) \). Since \( R \) is strongly \( J \) -clean, \( N(R) = \{ x \in R \mid 1 - x \in U(R) \} \) by [7, Theorem 3.5].

Conversely, assume that (1) and (2) hold. Let \( a \in R \). Then we can find a projection \( e \in R \) such that \( (a - 1) - e \in U(R) \) and \( e(a - 1) = (a - 1)e \). That is, \( 1 - (a - e) \in U(R) \). As \( 1 - (a - e) \in U(R) \), by hypothesis, \( a - e \in N(R) \). In addition, \( ea = ae \). Accordingly, \( R \) is strongly nil \( * \) -clean. □

Let \( R \) be a \( * \) -ring. Define \( * : R[x]/(x^n) \rightarrow R[x]/(x^n) \) by \( a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mapsto a_0^* + a_1^*x + \cdots + a_{n-1}^*x^{n-1} + (x^n) \). Then \( R[x]/(x^n) \) is a \( * \) -ring (cf. [7]).

Corollary 2.9 Let \( R \) be a \( * \) -ring. Then \( R \) is strongly nil \( * \) -clean if and only if so is \( R[x]/(x^n) \) \( (n \geq 1) \).

Proof One direction is obvious. Conversely, assume that \( R \) is strongly nil \( * \) -clean. Clearly, \( N(R[x]/(x^n)) = \{ a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mid a_0 \in N(R), a_1, \cdots, a_{n-1} \in R \} \). In view of Proposition 2.8, \( N(R[x]/(x^n)) = \{ a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mid 1 - a_0 \in U(R), a_1, \cdots, a_{n-1} \in R \} \). Also note that \( R \) is abelian. Thus, it can be easily seen that every element in \( R[x]/(x^n) \) can be written as the sum of a projection and a nilpotent element that commute. □
3 Algebraic Extensions

Let $R$ be a commutative $*$-ring, and let $\mu, \eta \in R$ with $\mu^* = \mu$ and $\eta^* = \eta$. Let $R[i] = \{ a + bi \mid a, b \in R, i^2 = \mu i + \eta \}$. Then $R[i]$ is a $*$-ring, where the involution is $*: R[i] \to R[i], a + bi \mapsto a^* + b^*i$. The aim of this section is to explore the algebraic extensions of a strongly nil $*$-clean ring.

Proposition 3.1. Let $R$ be a commutative $*$-ring with $\mu^* = \mu, \eta^* = \eta \in R$. Then $R[i]$ is strongly nil $*$-clean if and only if

1. $R$ is strongly nil $*$-clean;
2. $\mu \eta$ is nilpotent.

Proof Suppose that $R[i]$ is strongly nil $*$-clean. Then every idempotent in $R$ is a projection. Since $R$ is commutative, $N(R)$ forms an ideal. For any $a \in R$, we see that $a - a^2 \in N(R[i])$, and so $a - a^2 \in N(R)$. Thus, $R/N(R)$ is Boolean. Therefore $R$ is strongly nil $*$-clean by Theorem 2.3. As $R[i]/N(R[i])$ is Boolean, $i - i^2 \in N(R[i])$. This shows that $\eta + (\mu - 1)i \in N(R[i])$, and so $\mu \eta + (\mu^2 - \mu)i \in N(R[i])$. As $\mu^2 - \mu \in N(R)$, we see that $\mu \eta \in N(R[i])$. Thus, $\mu \eta$ is nilpotent.

Conversely, assume that (1), (2) hold. As $R$ is commutative, $N(R[i])$ forms an ideal of $R[i]$. Let $a + bi \in R[i]$ be an idempotent. Then we can find projections $e, f \in R$ and nilpotent elements $u, v \in R$ such that $a = e + u, b = f + w$. Then $a - a^2, b - b^2 \in N(R)$. This shows that $(a + bi) - (a + bi)^* = (a - a^*) + (b - b^*)i \in N(R[i])$. As $a + bi, (a + bi)^* \in R[i]$ are idempotents, we see that $((a + bi) - (a + bi)^*)^3 = ((a + bi) - 2(a + bi)(a + bi)^* + (a + bi)^*)(a + bi) - (a + bi)^*) = (a + bi) - (a + bi)^*$. Hence, $((a + bi) - (a + bi)^*)(1 - ((a + bi) - (a + bi)^*)^2) = 0$, therefore $(a + bi) - (a + bi)^* = 0$. That is, $a + bi \in R[i]$ is a projection.

Since $R$ is strongly nil $*$-clean, it follows from Theorem 2.4 that $2 - 2^2 \in N(R)$, and so $2 \in N(R)$. For any $a + bi \in R[i]$, it is easy to verify that

\[
(a + bi) - (a + bi)^2 = (a - a^2) - 2abi + bi - b^2i^2
\]

\[
\equiv -b^2\eta + (b - b^2\mu)i
\]

\[
\equiv -b\eta + b(1 - \mu)i \pmod{N(R[i])}.
\]

This shows that

\[
((a + bi) - (a + bi)^2)^2 \equiv b^2\eta^2 - 2b^2\eta(1 - \mu)i + b^2(1 - \mu)^2i^2
\]

\[
\equiv b\eta^2 + b(1 - \mu)^2(\mu i + \eta)
\]

\[
\equiv b\eta + b(1 - \mu)(\mu i + \eta)
\]

\[
\equiv 2b\eta - b\mu \eta + b(\mu - \mu^2)i
\]

\[
\equiv -b\mu \eta
\]

\[
\equiv 0 \pmod{N(R[i])}.
\]
Hence, \((a + bi) - (a + bi)^2 \in N(R[i])\). That is, \(R[i]/N(R[i])\) is Boolean. According to Theorem 2.4 we complete the proof. \(\square\)

As an immediate consequence, we deduce that a commutative \(*\)-ring \(R\) is strongly nil \(*\)-clean if and only if so is \(R[i]\) where \(i^2 = -1\).

We now consider a subclass of strongly nil \(*\)-clean rings consisting of rings which we call \(*\)-Boolean-like. First recall that a ring \(R\) is called Boolean-like [10] if it is commutative with unit and is of characteristic 2 with \(ab(1 + a)(1 + b) = 0\) for every \(a, b \in R\). Any Boolean ring is clearly a Boolean-like ring but not conversely (see [10]). Any Boolean-like ring is uniquely nil clean by [10, Theorem 17]. Also, \(R\) is Boolean-like if and only if (1) \(R\) is commutative ring with unit; (2) It is of characteristic 2; (3) It is nil clean; (4) \(ab = 0\) for every nilpotent element \(a, b\) in \(R\) [10, Theorem 19].

**Definition 3.2** A \(*\)-ring \(R\) is said to be \(*\)-Boolean-like provided that every idempotent in \(R\) is a projection and \((a - a^2)(b - b^2) = 0\) for all \(a, b \in R\).

By using the following theorem, we can see that \(*\)-Boolean-like rings are commutative rings. For, let \(x, y \in R\). In view of Theorem 3.4, \(x - e\) and \(y - f\) are nilpotent for some projections \(e, f \in R\). Again by Theorem 3.4, \((x - e)(y - f) = 0 = (y - f)(x - e)\). Since \(R\) is abelian, it follows that \(xy = yx\). Hence \(R\) is commutative.

The following is an example of a \(*\)-Boolean-like ring.

**Example 3.3** Let \(R = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}\). Define \(\begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & a + a' \end{pmatrix}, \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} aa' + ab' + ba' \\ ca' + ac' & aa' \end{pmatrix}\) and \(* : R \rightarrow R, \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & a \end{pmatrix}\). Then \(R\) is a \(*\)-ring. Let \(\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in R\) be an idempotent. Then \(a = a^2\) and \((2a - 1)b = (2a - 1)c = 0\). As \((2a - 1)^2 = 1\), we see that \(b = c = 0\), and so the set of all idempotents in \(R\) is \(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}\). Thus, every idempotent in \(R\) is a projection. For any \(A, B \in R\), we see that \((A - A^2)(B - B^2) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0\). Therefore \(R\) is \(*\)-Boolean-like.

**Theorem 3.4** Let \(R\) be a \(*\)-ring. Then \(R\) is \(*\)-Boolean-like if and only if

1. \(R\) is strongly nil \(*\)-clean;
2. \(\alpha \beta = 0\) for all nilpotent elements \(\alpha, \beta \in R\).
Proof Suppose that $R$ is $*$-Boolean-like. Then every idempotent in $R$ is a projection; hence, $R$ is abelian. For any $a \in R$, $(a - a^2)^2 = 0$, and so $a^2 = a^3f(a)$ for some $f(t) \in \mathbb{Z}[t]$. This implies that $R$ is strongly $\pi$-regular, and so it is $\pi$-regular. It follows from [1, Theorem 3] that $N(R)$ forms an ideal. Further, $a - a^2 \in N(R)$. Therefore $R/N(R)$ is Boolean. According to Theorem 2.4, $R$ is strongly nil $*$-clean. For any nilpotent elements $\alpha, \beta \in R$, we can find some $m, n \in \mathbb{N}$ such that $\alpha^m = \beta^n = 0$. Since $\alpha^2 = \alpha^3g(\alpha)$ for some $g(t) \in \mathbb{Z}[t]$, $\alpha^2 = 0$. Likewise, $\beta^2 = 0$. This shows that $\alpha\beta = (\alpha - \alpha^2)(\beta - \beta^2) = 0$.

Conversely, assume that (1) and (2) hold. By Theorem 2.4, every idempotent is a projection, and for any $a \in R$, $a - a^2$ is nilpotent. Hence for any $a, b \in R$, $(a - a^2)(b - b^2) = 0$. Therefore $R$ is $*$-Boolean-like. □

Corollary 3.5 Let $R$ be a commutative $*$-ring with $\mu^* = \mu, \eta^* = \eta \in R$. If $\mu \in U(R)$, then $R[\mu^*]$ is $*$-Boolean-like if and only if

(1) $R$ is $*$-Boolean-like;
(2) $\eta$ is nilpotent.

Proof If $R[\mu^*]$ is $*$-Boolean-like, then $R$ is $*$-Boolean-like. Also $\mu\eta \in R$ is nilpotent by Proposition 3.1 and Theorem 3.4. Since $\mu$ is unit and $N(R)$ is an ideal, $\eta$ is nilpotent.

Conversely, assume that (1) and (2) hold. Then $R[\mu^*]$ is strongly nil-$*$-clean by Proposition 3.1. In addition, $2 \in N(R)$. Let $a + bi \in R[\mu^*]$ be nilpotent. We claim that $a$ and $b$ are nilpotent. As $2 \in N(R)$, we can find some $n \in \mathbb{N}$ such that $(a + bi)^{2n} = a^{2n} + b^{2n}i^{2n} = 0$. We claim that $i^{2n} = ci + d$ for some $c \in U(R), d \in N(R)$. This is true for $n = 1$ by hypothesis. Assume that this holds for $n = k(k \geq 1)$. Write $i^{2k} = \alpha \beta$ with $\alpha \in U(R), \beta \in N(R)$. Now assume that $n = k + 1$. Then $i^{2(k+1)} = i^{2k}(\mu i + \eta) = (\alpha \mu^2 + \alpha \eta + \beta \mu)i + (\alpha \mu \eta + \beta \eta)$ with $\alpha \mu^2 + \alpha \eta + \beta \mu \in U(R), \alpha \mu \eta + \beta \eta \in N(R)$. Therefore $(a + bi)^{2n} = a^{2n} + b^{2n}i^{2n} = a^{2n} + b^{2n}(ci + d) = (a^{2n} + b^{2n}d) + b^{2n}ci$. This implies that $a^{2n} + b^{2n}d = 0 = b^{2n}c$. As $c \in U(R)$, we get $b^{2n} = a^{2n} = 0$. That is, $a, b \in R$ are nilpotent. Now let $\alpha + \beta i \in R[\mu^*]$ be another nilpotent element. Similarly, $\alpha, \beta \in R$ are nilpotent. Thus, $(a + bi)(\alpha + \beta i) = (a\alpha - b\beta) + (a\beta + b\beta)i = 0$. By Theorem 3.4, $R[\mu^*]$ is $*$-Boolean-like.

Example 3.6 Let $R$ be the ring
\[
\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \},
\]
where $0, 1 \in \mathbb{Z}_2$. Define $*: R \to R, A \mapsto A^T$, the transpose of $A$. Then $R$ is a $*$-ring in which $(a - a^2)(b - b^2) = 0$ for all $a, b \in R$. Further, $\alpha\beta = 0$ for all nilpotent elements $\alpha, \beta \in R$. But $R$ is not $*$-Boolean-like.
We end this section with an example showing that strongly nil clean ring need not be strongly nil $\ast$-clean.

Example 3.7 Consider the ring

$$R = \left\{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_4 \right\}.$$ 

Then for any $x, y \in R$, $(x - x^2)(y - y^2) = 0$. Obviously, $R$ is not commutative. This implies that $R$ is not a $\ast$-Boolean-like ring for any involution $\ast$. Accordingly, $R$ is not strongly nil $\ast$-clean for any involution $\ast$; otherwise, every idempotent in $R$ is a projection, a contradiction (see Lemma 2.1). We can also consider the involution $\ast : R \to R$, \( \left( \begin{array}{cc} a & 2b \\ 0 & c \end{array} \right) \mapsto \left( \begin{array}{cc} c & -2b \\ 0 & a \end{array} \right) \) and the idempotent \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) which is not a projection. On the other hand, since $(x - x^2)^2 = 0$ and so $x - x^2 \in N(R)$ for all $x \in R$, we get that $R$ is strongly nil clean by [11, Theorem 3].

4 Submaximal Ideals and $\ast$-Boolean Rings

An ideal $I$ of a ring $R$ is called a submaximal ideal if $I$ is covered by a maximal ideal of $R$. That is, there exists a maximal ideal $J$ of $R$ such that $I \subseteq J \subseteq R$ and for any ideal $K$ of $R$ such that $I \subseteq K \subseteq J$ then we have $I = K$ or $K = J$. This concept was firstly introduced to study Boolean-like rings (cf. [12]).

A $\ast$-ring $R$ is called a $\ast$-Boolean ring if every element of $R$ is a projection.

The purpose of this section is to characterize submaximal ideals of strongly nil $\ast$-clean rings, and $\ast$-Boolean rings by means of strongly nil $\ast$-cleanness. We begin with the following lemma.

Lemma 4.1 Let $R$ be strongly nil $\ast$-clean. Then an ideal $M$ of $R$ is maximal if and only if

1. $M$ is prime;
2. For any $a \in R, n \geq 1$, $a^n \in M$ implies that $a \in M$.

Proof Suppose that $M$ is maximal. Obviously, $M$ is prime. Let $a \in R$ and $a^n \in M$. If $a \not\in M$, $RaR + M = R$. Thus, $\overline{RaR} = \overline{R}$ where $\overline{R} = R/M$ and $\overline{a} = a + M$. Clearly, $R$ is an abelian clean ring, and so it is an exchange ring by [5 Theorem 17.2.2]. This implies that $R/M$ is an abelian exchange ring. As in the proof of [5 Proposition 17.1.9], there exists an idempotent $e + M \in R/M$ such that $\overline{R}(e + M)\overline{R} = \overline{R}$ and $e + M \in \overline{aR}$. Thus, $1 - e \in M$. Hence, $1 - ar \in M$ for some $r \in R$. This implies that $a^{n-1} - ar^n \in M$, and so $a^{n-1} \in M$. By iteration of this process, we see that $a \in M$, as required.
Conversely, assume that (1) and (2) hold. Assume that $M$ is not maximal. Then we can find a maximal ideal $I$ of $R$ such that $M \nsubseteq I \nsubseteq R$. Choose $a \in I$ while $a \notin M$. By hypothesis, there exist an idempotent $e \in R$ and a nilpotent $u \in R$ such that $a = e + u$. Write $u^m = 0$. Then $u^m \in M$. By hypothesis, $u \in M$. This shows that $e \notin M$. Clearly, $R$ is abelian. Thus $eR(1 - e) \subseteq M$. As $M$ is prime, we deduce that $1 - e \in M$. As a result, $1 - a = (1 - e) - u \in M$, and so $1 = (1 - a) + a \in I$. This gives a contradiction. Therefore $M$ is maximal.

Lemma 4.2 Let $I$ be an ideal of a strongly nil $*$-clean ring, and let $x \in R$ be such that $x \notin I$. If $xP \notin I$, then there exists a maximal ideal $J$ of $R$ such that $I \subseteq J$ and $x \notin J$.

Proof Let $\Omega = \{K \mid I \subseteq K, xP \notin K\}$. Then $\Omega \neq \emptyset$. Given $K_1 \subseteq K_2 \subseteq \cdots$ in $\Omega$, we set $Q = \bigcup_{i=1}^{\infty} K_i$. Then $Q$ is an ideal of $R$. If $Q \notin \Omega$, then $xP \in Q$, and so $xP \in K_i$ for some $i$. This gives a contradiction. Thus, $\Omega$ is inductive. By using Zorn’s Lemma, there exists an ideal $J$ of $R$ which is maximal in $\Omega$. Let $a, b \in R$ such that $a, b \notin J$. By the maximality of $J$, we see that $RaR + J, RbR + J \notin \Omega$. This shows that $xP \in (RaR + J) \cap (RbR + J)$. Hence, $xP = xP_2 \in RaRbR + J$. This yields that $aRb \notin J$; otherwise, $xP \in J$, a contradiction. Hence, $J$ is prime. Assume that $J$ is not maximal. Then we can find a maximal ideal $M$ of $R$ such that $J \nsubseteq M \nsubseteq R$. Clearly, $R$ is abelian. By the maximality, we see that $xP \in M$, and so $1 - xP \notin M$. This implies that $1 - xP \notin J$. As $xP(1 - xP) = 0 \subseteq J$, we have that $xP \in J$, a contradiction. Therefore $J$ is a maximal ideal, as asserted. \qed

Proposition 4.3 Let $R$ be strongly nil $*$-clean. Then the intersection of two maximal ideals is submaximal and it is covered by each of these two maximal ideals. Further, there is no other maximal ideals containing it.

Proof Let $I_1$ and $I_2$ be two distinct maximal ideals of $R$. Then $I_1 \cap I_2 \subseteq I_1$. Suppose $I_1 \cap I_2 \subseteq J \nsubseteq I_1$. Then we can find some $x \in I_1$ while $x \notin J$. Write $x^n = 0$. Then $x^n \in I_1$. In light of Lemma 4.1, $x^n \in I_1$. Likewise, $x^n \in I_2$. Thus, $x^n \in I_1 \cap I_2 \subseteq J$. This shows that $xP \notin J$. By virtue of Lemma 4.2 there exists a maximal ideal $M$ of $R$ such that $J \subseteq M$ and $x \notin M$. Hence, $I_1 \cap I_2 \subseteq M$ and $I_1 \neq M$. If $I_2 \neq M$, then $I_2 + M = R$. Write $t + y = 1$ with $t \in I_2, y \in M$. Then for any $z \in I_1$, $z = zt + zy \in I_1 \cap I_2 + M = M$, and so $I_1 = M$. This gives a contradiction. Thus $I_2 = M$, and then $J \subseteq M \subseteq I_2$. As a result, $J \subseteq I_1 \cap I_2$, and so $I_1 \cap I_2 = J$. Therefore $I_1 \cap I_2$ is a submaximal ideal of $R$. We claim that $I_1 \cap I_2$ is semiprime. If
\(K^2 \subseteq I_1 \cap I_2\), then for any \(a \in K\), we see that \(a^2 \in I_1 \cap I_2\). In view of Lemma 4.1, \(a \in I_1 \cap I_2\). This implies that \(K \subseteq I_1 \cap I_2\). Hence, \(I_1 \cap I_2\) is semiprime. Therefore \(I_1 \cap I_2\) is the intersection of maximal ideals containing \(I_1 \cap I_2\). Assume that \(K\) is a maximal ideal of \(R\) such that \(I_1 \cap I_2 \subseteq K\). If \(K \neq I_1, I_2\), then \(I_1 + K = I_2 + K = R\). This implies that \(I_1 \cap I_2 + K = R\), and so \(K = R\), a contradiction. Thus, \(K = I_1\) or \(K = I_2\), and so the proof is completed.

We call a local ring \(R\) **absolutely local** provided that for any \(0 \neq x \in J(R)\), \(J(R) = RxR\).

**Corollary 4.4** Let \(R\) be strongly nil \(\ast\)-clean, and let \(I\) be an ideal of \(R\). Then \(I\) is a submaximal ideal if and only if \(R/I\) is Boolean with four elements or \(R/I\) is absolutely local.

**Proof** Let \(I\) be a submaximal ideal of \(R\).

Case I. \(I\) is contained in more than a maximal ideal. Then \(I\) is contained in two distinct maximal ideals of \(R\). Since \(I\) is submaximal, there exists a maximal ideal \(J\) of \(R\) such that \(I\) is covered by \(J\). Thus, we have a maximal ideal \(J'\) such that \(J' \neq J\) and \(I \nsubseteq J'\). Hence, \(I \subseteq J \cap J' \subseteq J\). Clearly, \(J \cap J' \neq J + J' = R\), and so \(I = J \cap J'\).

In view of Proposition 4.3, there is no maximal ideal containing \(I\) except for \(J\) and \(J'\). This shows that \(R/I\) has only two maximal ideals covering \(\{0 + I\}\). For any \(a \in R\), it follows from Theorem 2.4 that \(a - a^2 \in R\) is nilpotent. Write \((a - a^2)^n = 0\). Then \((a - a^2)^n \in J\). According to Lemma 4.1, \(a - a^2 \in J\). Likewise, \(a - a^2 \in J'\). Thus, \(a - a^2 \in J \cap J'\), and so \(a - a^2 \in I\). This shows that \(R/I\) is Boolean. Therefore \(R/I\) is Boolean with four elements.

Case II. Suppose that \(I\) is contained in only one maximal ideal \(J\) of \(R\). Then \(R/I\) has only one maximal ideal \(J/I\). Clearly, \(R\) is an abelian exchange ring, and then so is \(R/I\). Let \(\overline{a} \in R/I\) be a nontrivial idempotent. Then \(I \subseteq I + ReR \subseteq J\) or \(I + ReR = R\). Likewise, \(I \subseteq I + R(1 - e)R \subseteq J\) or \(I + R(1 - e)R = R\). This shows that \(I + ReR = R\) or \(I + R(1 - e)R = R\). Thus, \((R/I)(e + I)(R/I) = R/I\) or \((R/I)(1 - e + I)(R/I) = R/I\), a contradiction. Therefore all idempotents in \(R/I\) are trivial. It follows from [5 Lemma 17.2.1] that \(R/I\) is local. For any \(\overline{0} \neq \overline{a} \in J/I\), we see that \(0 \neq I \subseteq RxR \subseteq J\). As \(I\) is submaximal, we deduce that \(J = RxR\). Therefore \(R\) is absolutely local.

Conversely, assume that \(R/I\) is Boolean with four elements. Then \(R/I\) has precisely two maximal ideals covering \(\{0 + I\}\), and so \(R\) has precisely two maximal ideals covering \(I\). Thus, we have a maximal ideal \(J\) such that \(I \nsubseteq J\). If \(I \subseteq K \subseteq J\). Then \(K = I\) or \(K = J\), and so \(K = J\). Consequently, \(I\) is submaximal. Assume that \(R/I\) is absolutely local. Then \(R/I\) has a uniquely maximal ideal \(J/I\). Hence, \(J\) is a maximal ideal of \(R\) such that \(I \nsubseteq J\). Assume that \(I \nsubseteq K \subseteq J\). Choose \(a \in K\) while \(a \notin I\). Then \(J = RaR \subseteq K\), and so \(K = J\). Therefore \(I\) is submaximal, as required.
Corollary 4.5 Let $R$ be strongly nil $*$-clean. If $I_1$ and $I_2$ are distinct maximal ideals of $R$, then $R/(I_1 \cap I_2)$ is Boolean.

Proof Since $I_1/(I_1 \cap I_2)$ and $I_2/(I_1 \cap I_2)$ are distinct maximal ideals, $R/(I_1 \cap I_2)$ is not local. In view of Proposition 4.3, $I_1 \cap I_2$ is a submaximal ideal of $R$. Therefore we complete the proof from Corollary 4.4. □

Recall that an ideal $I$ of a commutative ring $R$ is primary provided that for any $x, y \in R$, $xy \in I$ implies that $x \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$. Clearly, every maximal ideal of a commutative ring is primary. We end this article by giving the relation between strongly nil $*$-clean rings and $*$-Boolean rings.

Lemma 4.6 Let $R$ be a commutative strongly nil $*$-clean ring. Then the intersection of all primary ideals of $R$ is zero.

Proof Let $a$ be in the intersection of all primary ideal of $R$. Assume that $a \neq 0$. Let $\Omega = \{I \mid I$ is a primary ideal of $R$ such that $a \notin I\}$. Then $\Omega \neq \emptyset$ as $0 \in \Omega$. Given any ideals $I_1 \subseteq I_2 \subseteq \cdots$ in $\Omega$, we set $M = \bigcup_{i=1}^{\infty} I_i$. Then $M \in \Omega$. Thus, $\Omega$ is inductive. By using Zorn’s Lemma, we can find an ideal $Q$ which is maximal in $\Omega$. It will suffice to show that $Q$ is primary. If not, we can find some $x, y \in R$ such that $xy \in Q$, but $x \notin Q$ and $y^n \notin Q$ for any $n \in \mathbb{N}$. This shows that $a \in Q + (x)$, and so $a = b + cx$ for some $b \in Q, c \in R$.

Since $R$ is strongly nil $*$-clean, it follows from Theorem 2.7 that there are some distinct $k, l \in \mathbb{N}$ such that $y^k = y^l$. Say $k > l$. Then $y^i = y^{i+1}y^{k-l-1} = y^iy^{k-l-1} = y^{i+2}y^{2(k-l-1)} = \cdots = y^{2^n}y^{(k-l-1)}$. Hence, $y^{l(k-l)} = y^l(y^{l(k-l-1)}) = y^{2^n}y^{2(k-l-1)} = (y^{l(k-l)})^2$. Choose $s = l(k-l)$. Then $y^s$ is an idempotent. Write $y = yp + y_N$. Then $y^s - yp = (yp + y_N)^s - yp = y_N(yp + y_N^{s-1}) \in N(R)$. As $R$ is a commutative ring, we see that $(y^s - yp)^3 = y^s - yp$. This implies that $y^s = yp$. Since $xy \in Q$, we show that $xy^s \in Q$, and so $xy^s \in Q$. It follows from $a = b + cx$ that $ayp = byp + cxy_p \in Q$. Clearly, $y^s \notin Q$, and so $a \in Q + (yp)$. Write $a = d + ry_p$ for some $d \in Q, r \in R$. We see that $ay_p = dy_p + ry_p$, and so $ry_p \in Q$. This implies that $a \in Q$, a contradiction. Therefore $Q$ is primary, a contradiction. Consequently, the intersection of all primary ideal of $R$ is zero. □

Theorem 4.7 Let $R$ be a $*$-ring. Then $R$ is a $*$-Boolean ring if and only if

1. $R$ is commutative;
2. Every primary ideal of $R$ is maximal;
3. $R$ is strongly nil $*$-clean.
Proof Suppose that \( R \) is a \( * \)-Boolean ring. Clearly, \( R \) is a commutative strongly nil \( * \)-clean ring. Let \( I \) be a primary ideal of \( R \). If \( I \) is not maximal, then there exists a maximal ideal \( M \) such that \( I \subseteq M \subseteq R \). Choose \( x \in M \) while \( x \notin I \). As \( x \) is an idempotent, we see that \( xR(1-x) \subseteq I \), and so \((1-x)^m \in I \subseteq M \) for some \( m \in \mathbb{N} \). Thus, \( 1-x \in M \). This implies that \( 1 = x + (1-x) \in M \), a contradiction. Therefore \( I \) is maximal, as required.

Conversely, assume that (1), (2) and (3) hold. Clearly, every maximal ideal of \( R \) is primary, and so \( J(R) = \bigcap \{ P \mid P \text{ is primary} \} \). In view of Lemma 4.6, \( J(R) = 0 \). Hence every element is a projection i.e. \( R \) is \( * \)-Boolean. \( \square \)

**Corollary 4.8** A ring \( R \) is a Boolean ring if and only if

1. \( R \) is commutative;
2. Every primary ideal of \( R \) is maximal;
3. \( R \) is strongly nil clean.

**Proof** Choose the involution as the identity. Then the result follows from Theorem 4.7. \( \square \)

**Acknowledgement** The authors are indebted to the referee for his/her valuable comments. H. Chen is thankful for the support by the Natural Science Foundation of Zhejiang Province (LY13A010019).

**References**

[1] A. Badawi, *On abelian \( \pi \)-regular rings*, Comm. Algebra 25, (1997), 1009-1021.

[2] S.K. Berberian, *Baer \( * \)-Rings*, Springer-Verlag, Heidelberg, London, New York, 2011.

[3] M. Chacron, *On a theorem of Herstein*, Canadian J. Math. 21, (1969), 1348-1353.

[4] H. Chen, *On strongly \( J \)-clean rings*, Comm. Algebra, 38, (2010), 3790-3804.

[5] H. Chen, *Rings Related Stable Range Conditions*, Series in Algebra 11, World Scientific, Hackensack, NJ., 2011.

[6] H. Chen, *On strongly nil clean matrices*, Comm. Algebra, 41, (2013), 1074-1086.

[7] H. Chen, A. Harman and A.Ç. Özcan, *Strongly \( J \)-clean rings with involutions*, accepted in Proceedings Volume for the Ohio State -Denison Conference honoring T.Y. Lam (2012), arXiv:1207.0466v2 [math.RA] 7 Feb. 2013.
[8] A.J. Diesl, *Classes of Strongly Clean Rings*, University of California, Berkeley, USA, Ph.D. Thesis, 2006.

[9] A.J. Diesl, Nil clean rings, *J. Algebra*, 383, (2013), 197-211.

[10] A.L. Foster, *The theory of Boolean-like rings*, Trans. Amer. Math. Soc. 59, (1946), 166-187.

[11] Y. Hirano, H. Tominaga and A. Yaqub, *On rings in which every element is uniquely expressable as a sum of a nilpotent element and a certain potent element*, Math. J. Okayama Univ. 30, (1988), 33-40.

[12] V. Swaminathan, *Submaximal ideals in a Boolean-like rings*, Math. Seminar Notes 10, (1982), 529-542.

[13] H. Okamoto, J. Grosen and H. Komatsu, *Some generalizations of Boolean rings*, Math. J. Okayama Univ. 31, (1989), 125-133.

[14] C. Li and Y. Zhou, *On strongly ∗-clean rings*, J. Algebra Appl. 10, (2011), 1363-1370.

[15] L. Vaš, ∗-Clean rings; some clean and almost clean Baer ∗-rings and von Neumann algebras, J. Algebra 324, (2010), 3388-3400.

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