ON A GENERIC SYMMETRY DEFECT HYPERSURFACE

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Abstract. Let \( f : X \rightarrow Y \) be a dominant polynomial mapping of affine varieties. For generic \( y \in Y \) we have \( \text{Sing}(f^{-1}(y)) = f^{-1}(y) \cap \text{Sing}(X) \). As an application we show that symmetry defect hypersurfaces for two generic members of the irreducible algebraic family of \( n \)-dimensional smooth irreducible subvarieties in general position in \( \mathbb{C}^{2n} \) are homeomorphic and they have homeomorphic sets of singular points. In particular symmetry defect curves for two generic curves in \( \mathbb{C}^{2} \) of the same degree have the same number of singular points.

1. Introduction

Let \( X \subset \mathbb{C}^{2n} \) be a smooth algebraic variety. In [6] we have investigated the central symmetry of \( X \) (see also [1], [2], [3]). For \( p \in \mathbb{C}^{2n} \) we have introduced a number \( \mu(p) \) of pairs of points \( x, y \in X \), such that \( p \) is the center of the interval \( xy \). Recall that the subvariety \( X \subset \mathbb{C}^{2n} \) is in a general position if there exist points \( x, y \in X \) such that \( T_x X \oplus T_y Y = \mathbb{C}^{2n} \).

We have showed in [6] that if \( X \) is in general position, then there is a closed algebraic hypersurface \( B \subset \mathbb{C}^{2n} \), called symmetry defect hypersurface of \( X \), such that the function \( \mu \) is constant (non-zero) exactly outside \( B \). Here we prove that the symmetry defect hypersurfaces for two generic members of an irreducible algebraic family of \( n \)-dimensional smooth irreducible subvarieties in general position in \( \mathbb{C}^{2n} \) are homeomorphic.

Moreover, we prove a version of Sard theorem for singular varieties (section 2), which implies that additionally the symmetry defect hypersurfaces for two generic members of an irreducible algebraic family of \( n \)-dimensional smooth irreducible subvarieties in general position in \( \mathbb{C}^{2n} \) have homeomorphic sets of singular points. In particular symmetry defect curves for two generic curves in \( \mathbb{C}^{2} \) of the same degree have the same number of singular points.

2. Generalized Sard’s Theorem

Let \( X \) be an irreducible affine variety. Let \( \text{Sing}(X) \) denote the set of singular points of \( X \). Let \( Y \) be another affine variety and consider a dominant morphism \( f : X \rightarrow Y \). If \( X \) is smooth then by Sard’s Theorem a generic fiber of \( f \) is smooth. We show that in a general case the following theorem holds:

**Theorem 2.1.** Let \( f : X^k \rightarrow Y^l \) be a dominant polynomial mapping of affine varieties. For generic \( y \in Y \) we have \( \text{Sing}(f^{-1}(y)) = f^{-1}(y) \cap \text{Sing}(X) \).

**Proof.** We can assume that \( Y \) is smooth. Since there exists a mapping \( \pi : Y^l \rightarrow \mathbb{C}^l \) which is generically etale, we can assume that \( Y = \mathbb{C}^l \). Let us recall that if \( Z \) is an algebraic variety, then a point \( z \in Z \) is smooth, if and only if the local ring \( \mathcal{O}_z(Z) \) is regular. This
is equivalent to the fact that \( \dim \mathbb{C} m/m^2 = \dim Z \), where \( m \) denotes the maximal ideal of \( \mathcal{O}_Z(Z) \).

Let \( y = (y_1, ..., y_l) \in \mathbb{C}^l \) be a sufficiently generic point. Then by Sard’s Theorem the fiber \( Z = f^{-1}(y) \) is smooth outside \( \text{Sing}(X) \) and \( \dim Z = \dim X - l = k - l \). Since the generic fiber of \( f \) is reduced (we are in characteristic zero!), then also a generic particular fiber is reduced. Hence we can assume that \( Z \) is reduced. It is enough to show that every point \( z \in Z \cap \text{Sing}(X) \) is singular on \( Z \).

Assume that \( z \in Z \cap \text{Sing}(X) \) is smooth on \( Z \). Let \( f : X \to \mathbb{C}^l \) be given as \( f = (f_1, ..., f_l) \), where \( f_i \in \mathbb{C}[X] \). Then \( \mathcal{O}_Z(Z) = \mathcal{O}_Z(X)/(f_1 - y_1, ..., f_l - y_l) \). In particular if \( m' \) denotes the maximal ideal of \( \mathcal{O}_Z(Z) \) and \( m \) denotes the maximal ideal of \( \mathcal{O}_Z(X) \) then \( m' = m/(f_1 - y_1, ..., f_l - y_l) \). Let \( \alpha_i \) denote a class of the polynomial \( f_i - y_i \) in \( m/m^2 \). Let us note that

\[
(1) \quad m'/m^2 = m/(m^2 + (\alpha_1, ..., \alpha_l)).
\]

Since the point \( z \) is smooth on \( Z \) we have \( \dim m'/m^2 = \dim Z = \dim X - l \). Take a basis \( \beta_1, ..., \beta_{k-l} \) of the space \( m'/m^2 \) and let \( \beta_i \in m/m^2 \) correspond to \( \beta_i \) under the correspondence \( (1) \). Note that vectors \( \beta_1, ..., \beta_{k-l}, \alpha_1, ..., \alpha_l \) generates the space \( m/m^2 \). This means that \( \dim \mathbb{C} m/m^2 \leq k - l + l = k = \dim X \). Hence the point \( z \) is smooth on \( X \), a contradiction. \( \square \)

**Example 2.2.** Note that the general singularities of generic fibers of \( f : X \to Y \) are not necessarily simpler than those of \( X \). Indeed, if \( X = Z \times \mathbb{C}^k \) and \( f : X \to \mathbb{C}^k \) is a projection, then the singularities of \( X \) and the fiber \( f^{-1}(y) \) are of the same type.

3. **Bifurcation set**

Let \( k = \mathbb{C} \) or \( k = \mathbb{R} \) and let \( X, Y \) be affine varieties over \( k \). Recall the following (see [4], [5]):

**Definition 3.1.** Let \( f : X \to Y \) be a generically-finite (i.e. a generic fiber is finite) and dominant (i.e. \( \overline{f(X)} = Y \)) polynomial mapping of affine varieties. We say that \( f \) is finite at a point \( y \in Y \), if there exists an open neighborhood \( U \) of \( y \) such that the mapping \( f|_{f^{-1}(U)} : f^{-1}(U) \to U \) is proper.

If \( k = \mathbb{C} \) it is well-known that the set \( S_f \) of points at which the mapping \( f \) is not finite, is either empty or it is a hypersurface (see [4], [5]). We say that the set \( S_f \) is the set of non-properness of the mapping \( f \).

**Definition 3.2.** Let \( k = \mathbb{C} \). Let \( X, Y \) be smooth affine \( n \)-dimensional varieties and let \( f : X \to Y \) be a generically finite dominant mapping of geometric degree \( \mu(f) \). The bifurcation set of the mapping \( f \) is the set

\[
B(f) = \{ y \in Y : \#f^{-1}(y) \neq \mu(f) \}.
\]

We have the following fundamental theorem:

**Theorem 3.3.** Let \( k = \mathbb{C} \). Let \( X, Y \) be smooth affine complex varieties of dimension \( n \). Let \( f : X \to Y \) be a polynomial dominant mapping. Then the set \( B(f) \) is either empty (so \( f \) is an unramified topological covering) or it is a closed hypersurface.

**Proof.** Let us note that outside the set \( S_f \) the mapping \( f \) is a (ramified) analytic cover of degree \( \mu(f) \). By the Lemma [3,4] below if \( y \not\in S_f \) we have \( \#f^{-1}(y) \leq \mu(f) \). Moreover, since \( f \) is an analytic covering outside \( S_f \) it is well known that the fiber \( f^{-1}(y) \) counted with
multiplicity has exactly $\mu(f)$ points. In particular, if $y \in K_0(f)$, the set of critical values of $f$, then $\#f^{-1}(y) < \mu(f)$.

Now let $y \in S_f$. There are two possible cases:

a) $\#f^{-1}(y) = \infty$.

b) $\#f^{-1}(y) < \infty$.

In case b) let $U$ be an affine neighborhood of $y$ over which the mapping $f$ is quasi-finite. Let $V = f^{-1}(U)$. By Zariski Main Theorem in the version given by Grothendieck, there exists a normal variety $\overline{V}$ and a finite mapping $\overline{f}: \overline{V} \to U$, such that

1) $V \subset \overline{V}$,

2) $\overline{f}|_V = f$.

Since $y \in \overline{f}(\overline{V} \setminus V)$, it follows by the Lemma 3.4 below, that $\#f^{-1}(y) < \mu(f)$. Consequently, if $y \in S_f$, we have $\#f^{-1}(y) < \mu(f)$. Finally we have $B(f) = K_0(f) \cup S_f$.

Now we show that the set $B(f) = K_0(f) \cup S_f$ is a hypersurface. Let $J(f)$ be the set of singular points of $f$. The set $J(f)$ is a hypersurface (because locally it is the zero set of the Jacobian of $f$). Denote by $J_i$ the irreducible components of $J(f)$. Let $W_i = f(J_i)$. If all $W_i$ are hypersurfaces then the theorem is true. If, for example, $\dim W_1 < n_1$, then the mapping $f : J_1 \to W_1$ has non-compact generic fiber, this means in particular that $W_1 \subset S_f$. Thus the set $\bigcup W_i \cup S_f$ is a hypersurface. But $B(f) = \bigcup W_i \cup S_f$ (note that $B(f)$ is closed).

Moreover, if $B(f) = \emptyset$, then $f$ is a surjective topological covering. \hfill \qedsymbol

**Lemma 3.4.** Let $X, Y$ be affine normal varieties of dimension $n$. Let $f : X \to Y$ be a finite mapping. Then for every $y \in Y$ we have $\#f^{-1}(y) \leq \mu(f)$.

**Proof.** Let $\#f^{-1}(y) = \{x_1, \ldots, x_r\}$. We can choose a function $h \in \mathbb{C}[X]$ which separates all $x_i$ (in particular we can take as $h$ the equation of a general hyperplane section). Since $f$ is finite we have a monic polynomial $T^s + a_1(f)T^{s-1} + \ldots + a_s(f) \in f^*\mathbb{C}[Y][T]$, $s \leq \mu(f)$. If we substitute $f = y$ to this equation we get the desired result. \hfill \qedsymbol

### 4. A Super General Position

In this section we describe some properties of a variety $X^n \subset \mathbb{C}^{2n}$ which implies that $X$ is in a general position. Recall that the subvariety $X^n \subset \mathbb{C}^{2n}$ is in a general position if there exist points $x, y \in X^n$ such that $T_xX \oplus T_yY = \mathbb{C}^{2n}$.

**Definition 4.1.** Let $X^n \subset \mathbb{C}^{2n}$ be a smooth algebraic variety. We say that $X$ is in very general position if there exists a point $x \in X$ such that the set $T_xX \cap X$ has an isolated point (here we consider $T_xX$ as a linear subspace of $\mathbb{C}^{2n}$).

We consider also a slightly stronger property:

**Definition 4.2.** Let $X^n \subset \mathbb{C}^{2n}$ be a smooth algebraic variety and let $S = \overline{X} \setminus X \subset \pi_\infty$ be the set of points at infinity of $X^n$. We say that $X$ is in super general position if there exists a point $x \in X$ such that $T_xX \cap S = \emptyset$ (here we consider $T_xX$ as a linear subspace of $\mathbb{P}^{2n} = \mathbb{C}^{2n} \cup \pi_\infty$).

We have the following:

**Proposition 4.3.** If $X$ is in a super general position, then it is in a very general position.
Proposition 4.4. Let \( x \in X \) be a point such that \( T_x X \cap S = \emptyset \). Take \( R = T_x X \cap X \). Then the set \( R \) is finite, since otherwise the point at infinity of \( R \) belongs to \( T_x X \cap S = \emptyset \).

We have also:

**Proposition 4.4.** Let \( X \subset \mathbb{C}^{2n} \) be in a super general position. Then for a generic point \( x \in X \) we have \( T_x X \cap S = \emptyset \).

**Proof.** It is easy to see that the set \( \Gamma = \{(s, x) \in S \times X : s \in T_x X\} \) is an algebraic subset of \( S \times X \). Let \( \pi : \Gamma \ni (s, x) \rightarrow x \in X \) be a projection. It is a proper mapping. Since the variety \( X \) is in a very general position, we see that at least one point \( x_0 \in X \) is not in the image of \( \pi \). Thus almost every point of \( X \) is not in the image of \( \pi \), because the image of \( \pi \) is a closed subset of \( X \).

Finally we have:

**Theorem 4.5.** If \( X \subset \mathbb{C}^{2n} \) is in a very general position, then it is in a general position, i.e., there exist points \( x, y \in X \) such that \( T_x X \oplus T_y X = \mathbb{C}^{2n} \). In fact for every generic pair \( (x, y) \in X \times X \) we have \( T_x X \oplus T_y X = \mathbb{C}^{2n} \).

**Proof.** Let \( x_0 \in X \) be the point such that the set \( T_{x_0} X \cap X \) has an isolated point. The space \( T_{x_0} X \) is given by \( n \) linear equations \( l_i = 0 \). Let \( F : X \ni x \rightarrow (l_1(x), \ldots, l_n(x)) \in \mathbb{C}^n \). By the assumption the fiber over 0 of \( F \) has an isolated point, in particular the mapping \( F \) is dominant. Now by the Sard Theorem almost every point \( x \in X \) is a regular point of \( F \). This means that \( T_x X \) is complementary to \( T_{x_0} X \), i.e., \( T_{x_0} X \oplus T_x X = \mathbb{C}^{2n} \). If we consider the mapping \( \Phi : X \times X \ni (x, y) \rightarrow x + y \in \mathbb{C}^{2n} \), we see that it has the smooth point \((x_0, x)\). In particular almost every pair \((x, y)\) is a smooth point of \( F \), which implies that for every generic pair \((x, y)\) we have \( T_x X \oplus T_y X = \mathbb{C}^{2n} \).

We shall use in the sequel the following:

**Proposition 4.6.** Let \( X^n \subset \mathbb{C}^{2n} \) be a generic smooth complete intersection of multi-degree \( d_1, \ldots, d_n \). If every \( d_i > 1 \), then \( X \) is in a super general position.

**Proof.** We can assume that \( X \) is given by \( n \) smooth hypersurfaces \( f_i = a_i + f_{1i} + \ldots + f_{id_i} \) (where \( f_{ik} \) is a homogenous polynomial of degree \( k \)), which have independent coefficients (see section below). The tangent space is described by polynomials \( f_{1i}, i = 1, \ldots, n \) and the set \( S \) of points at infinity of \( X \) is described by polynomials \( f_{id_i}, i = 1, \ldots, n \). Since these two families of polynomials have independent coefficients, we see that generically the zero sets at infinity of these two families are disjoint. In particular such a generic \( X \) is in a super general position.

5. **Algebraic families**

Now we introduce the notion of an algebraic family.

**Definition 5.1.** Let \( M \) be a smooth affine algebraic variety and let \( Z \) be a smooth irreducible subvariety of \( M \times \mathbb{C}^n \). If the restriction to \( Z \) of the projection \( \pi : M \times \mathbb{C}^n \rightarrow M \) is a dominant map with generically irreducible fibers of the same dimension, then we call the collection \( \Sigma = \{Z_m = \pi^{-1}(m)\}_{m \in M} \) an algebraic family of subvarieties in \( \mathbb{C}^n \). We say that this family is in a general position if a generic member of \( \Sigma \) is in a general position in \( \mathbb{C}^n \).

We show that the ideals \( I(Z_m) \subset \mathbb{C}[x_1, \ldots, x_n] \) of a generic member of \( \Sigma \) depend in a parametric way on \( m \in M \).
Lemma 5.2. Let $\Sigma$ be an algebraic family given by a smooth variety $Z \subset M \times \mathbb{C}^n$. The ideal $I(Z) \subset \mathbb{C}[M][x_1, ..., x_n]$ is finitely generated, let the polynomials $\{f_1(m, x), ..., f_s(m, x)\}$ form its set of generators. The ideal $I(Z_m) \subset \mathbb{C}[x_1, ..., x_n]$ of a generic member $Z_m := \pi^{-1}(m) \subset \mathbb{C}^n$ of $\Sigma$ is equal to $I(Z_m) = (f_1(m, x), ..., f_s(m, x))$.

Proof. Let $\dim Z = p$ and $\dim M = q$. Thus the variety $M \times \mathbb{C}^n$ has dimension $n + q$. Choose local holomorphic coordinates on $M$. Since the variety $Z$ is smooth we have

$$\text{rank } \begin{bmatrix} \frac{\partial f_1}{\partial m_1}(m, x) & \ldots & \frac{\partial f_1}{\partial m_q}(m, x) & \frac{\partial f_1}{\partial x_1}(m, x) & \ldots & \frac{\partial f_1}{\partial x_n}(m, x) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_s}{\partial m_1}(m, x) & \ldots & \frac{\partial f_s}{\partial m_q}(m, x) & \frac{\partial f_s}{\partial x_1}(m, x) & \ldots & \frac{\partial f_s}{\partial x_n}(m, x) \end{bmatrix} = n + q - p$$

on $Z$. Let us consider the projection $\pi : Z \ni (m, x) \mapsto m \in M$. By Sard’s theorem a generic $m \in M$ is a regular value of the mapping $\pi$. For such a regular value $m$ we have that $\ker d_{(m, x)}\pi$ is disjoint from $T_{(m, x)}Z$ for every $x$ such that $(m, x) \in Z$. In local coordinates on $M$ this is equivalent to

$$\text{rank } \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ * & \ldots & * & \frac{\partial f_1}{\partial x_1}(m, x) & \ldots & \frac{\partial f_1}{\partial x_n}(m, x) \\ \vdots & & \vdots & & \vdots \\ * & \ldots & * & \frac{\partial f_s}{\partial x_1}(m, x) & \ldots & \frac{\partial f_s}{\partial x_n}(m, x) \end{bmatrix} = n + 2q - p.$$

Consequently for $(m, x) \in Z$ and $m$ a regular value of $\pi$ we have

$$\text{rank } \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(m, x) & \ldots & \frac{\partial f_1}{\partial x_n}(m, x) \\ \vdots & & \vdots \\ \frac{\partial f_s}{\partial x_1}(m, x) & \ldots & \frac{\partial f_s}{\partial x_n}(m, x) \end{bmatrix} = n + q - p.$$

Note that $n + q - p = \text{codim } Z_m$ (in $\mathbb{C}^n$). This means that the ideal $(f_1(m, x), ..., f_s(m, x))$ locally coincide with $I(Z_m)$, because it contains local equations of $Z_m$. Hence it also coincides globally, i.e., $(f_1(m, x), ..., f_s(m, x)) = I(Z_m)$. □

Remark 5.3. This can be also obtained by a computation of a scheme theoretic fibers of $\pi$ and using the fact that such generic fibers are reduced.

Example 5.4. a) Let $N := (n + d_1) \subset \mathbb{C}^N \times \mathbb{C}^n$ be given by equations $Z = \{(a, x) \in \mathbb{C}^N \times \mathbb{C}^n : \sum_{|\alpha| \leq d_1 a_\alpha x^\alpha = 0}\}$. The projection $\pi : Z \ni (a, x) \mapsto a \in \mathbb{C}^N$ determines an algebraic family of hypersurfaces of degree $d$ in $\mathbb{C}^n$. If $n = 2$ and $d > 1$ this family is in general position in $\mathbb{C}^2$.

b) More generally let $N_1 := (n + d_1), N_2 := (n + d_2), N_n := (n + d_n)$ and let $Z \subset \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_n} \times \mathbb{C}^{2n}$ be given by equations $Z = \{(a_1, a_2, ..., a_n, x) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_n} \times \mathbb{C}^{2n} : \sum_{|\alpha| \leq d_1} a_1 a_\alpha x^\alpha = 0, \sum_{|\alpha| \leq d_2} a_2 a_\alpha x^\alpha = 0, ..., \sum_{|\alpha| \leq d_n} a_n a_\alpha x^\alpha = 0\}$. The projection $\pi : Z \ni (a_1, a_2, ..., a_n, x) \mapsto (a_1, a_2, ..., a_n) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_n}$ determines an algebraic family $\Sigma(d_1, d_2, ..., d_n, 2n)$ of complete intersections of multi-degree $d_1, d_2, ..., d_n$ in $\mathbb{C}^{2n}$. 
If \( d_1, d_2, \ldots, d_n > 1 \), then this family is in general position in \( \mathbb{C}^{2n} \). This follows from Proposition 4.6.

6. Main result

Let us recall that a following result is true (see e.g. [6]):

**Lemma 6.1.** Let \( X, Y \) be complex algebraic varieties and \( f : X \to Y \) a polynomial dominant mapping. Then two generic fibers of \( f \) are homeomorphic.

**Proof.** Let \( X_1 \) be an algebraic completion of \( X \). Take \( X_2 = \text{graph}(f) \subseteq X_1 \times Y \), where \( Y \) is a smooth algebraic completion of \( Y \). We can assume that \( X \subset X_2 \). Let \( Z = X_2 \setminus X \). We have an induced mapping \( \overline{f} : X_2 \to Y \), such that \( \overline{f}|_X = f \).

There is a Whitney stratification \( \mathcal{S} \) of the pair \((X, Z)\). For every smooth strata \( S_i \in \mathcal{S} \) let \( B_i \) be the set of critical values of the mapping \( f|_{S_i} \). Take \( B = \bigcup B_i \). Take \( X_3 = X_2 \setminus f^{-1}(B) \) and \( Z_1 = Z \setminus f^{-1}(B) \). The restriction of the stratification \( \mathcal{S} \) to \( X_3 \) gives a Whitney stratification of the pair \((X_3, Z_1)\). We have a proper mapping \( f_1 : X_3 \to Y \setminus B \) which is submersion on each strata. By the Thom first isotopy theorem there is a trivialization of \( f_1 \), which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping \( f : X \setminus f^{-1}(B) \to Y \setminus B \). \( \square \)

**Definition 6.2.** Let \( X \) be an affine variety. Let us define \( \text{Sing}^k(X) := \text{Sing}(X) \) for \( k := 1 \) and inductively \( \text{Sing}^{k+1}(X) := \text{Sing}(\text{Sing}^k(X)) \).

As a direct application of the Lemma 6.1 and Theorem 2.1 we have:

**Theorem 6.3.** Let \( f : X^n \to Y^l \) be a dominant polynomial mapping of affine varieties. If \( y_1, y_2 \) are sufficiently general then \( f^{-1}(y_1) \) is homeomorphic to \( f^{-1}(y_2) \) and \( \text{Sing}(f^{-1}(y_1)) \) is homeomorphic to \( \text{Sing}(f^{-1}(y_2)) \). More generally, for every \( k \) we have \( \text{Sing}^k(f^{-1}(y_1)) \) is homeomorphic to \( \text{Sing}^k(f^{-1}(y_2)) \).

Now we are ready to prove:

**Theorem 6.4.** Let \( \Sigma \) be an algebraic family of \( n \)-dimensional algebraic subvarieties in \( \mathbb{C}^{2n} \) in general position. Symmetry defect hypersurfaces \( B_1, B_2 \) for generic members \( C_1, C_2 \in \Sigma \) are homeomorphic and they have homeomorphic singular parts i.e., \( \text{Sing}(B_1) \cong \text{Sing}(B_2) \). More generally, for every \( k \) we have \( \text{Sing}^k(B_1) \) is homeomorphic to \( \text{Sing}^k(B_2) \).

**Proof.** Let \( \Sigma \) be given by a variety \( Z \subset M \times \mathbb{C}^{2n} \). The ideal \( I(Z) \subset \mathbb{C}[M][x_1, \ldots, x_{2n}] \) is finitely generated. Choose a finite set of generators \( \{f_1(m, x), \ldots, f_s(m, x)\} \) (see Lemma 5.2). Let us define \( R = \{(m, x, y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} : f_i(m)(x) = 0, i = 1, \ldots, s \} \) \& \( f_i(m)(y) = 0, i = 1, \ldots, s \} \).

The variety \( R \) is a smooth irreducible subvariety of \( M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \) of codimension 2n. Indeed, for given \( (m, x, y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \) choose polynomials \( f_1, \ldots, f_s, f_{j_1}, \ldots, f_{j_n} \) such that \( \text{rank } \frac{\partial f_i}{\partial x_j}(m, x) = 1, \ldots, n, i = 1, \ldots, n \) and \( \text{rank } \frac{\partial f_i}{\partial x_j}(m, x) = 1, \ldots, n, i = 1, \ldots, n \). Since \( Z \) is a smooth variety of dimension \( M + n \), we have that \( Z \) locally near \( (m, x) \) is given by equations \( f_{i_1}, \ldots, f_{i_n} \) and near \( (m, y) \) is given by equations \( f_{j_1}, \ldots, f_{j_n} \). Hence the variety \( R \) near the point \( (m, x, y) \) is given as

\[
\{(m, x, y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} : f_i(m)(x) = 0, l = 1, \ldots, n \} \& \ f_j(m)(y) = 0, l = 1, \ldots, s \}.
\]
In particular \( R \) is locally a smooth complete intersection, i.e., \( R \) is smooth.

Moreover we have a projection \( R \to M \) with irreducible fibers which are products \( Z_m \times Z_m, \ m \in M \). This means that \( R \) is irreducible. Note that \( R \) is an affine variety. Consider the following morphism

\[
\Psi : R \ni (m, x, y) \mapsto (m, \frac{x + y}{2}) \in M \times \mathbb{C}^{2n}.
\]

By the assumptions the mapping \( \Psi \) is dominant. Indeed for every \( m \in M \) the fiber \( Z_m \) is in a general position in \( \mathbb{C}^{2n} \) and consequently the set \( \Psi(R) \cap m \times \mathbb{C}^{2n} \) is dense in \( m \times \mathbb{C}^{2n} \).

We know by Theorem 3.3 that the mapping \( \Psi \) has constant number of points in the fiber outside the bifurcation set \( B(\Psi) \subset M \times \mathbb{C}^{2n} \). This implies that \( B(Z_m) = m \times \mathbb{C}^{2n} \cap B(\Psi) \).

In particular the symmetry defect hypersurface of the variety \( Z_m \) coincide with the fiber over \( m \) of the projection \( \pi : B(\Psi) \ni (m, x) \mapsto m \in M \). Now we conclude the proof by Theorem 6.3. \( \square \)

**Corollary 6.5.** Symmetry defect sets \( B_1, B_2 \) for generic curves \( C_1, C_2 \subset \mathbb{C}^2 \) of the same degree \( d > 1 \) are homeomorphic and they have the same number of singular points.

**Corollary 6.6.** Let \( C_1, C_2 \) be two smooth varieties, which are generic complete intersection of multi-degree \( d_1, d_2, \ldots, d_n \) in \( \mathbb{C}^{2n} \) (where all \( d_i > 1 \)). Then symmetry defect hypersurfaces \( B_1, B_2 \) of \( C_1, C_2 \), are homeomorphic and they have homeomorphic singular parts (i.e., \( \text{Sing}(B_1) \cong \text{Sing}(B_2) \)). More generally, for every \( k \) we have \( \text{Sing}^k(B_1) \) is homeomorphic to \( \text{Sing}^k(B_2) \).

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