Observables in quantum field theory on a collapsing black hole background

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Covariant quantities like the energy density and flux associated with a quantum field theory depend on the trajectory of the observer’s motion. In a gravitational collapse scenario, they also depend on the details of the collapsing classical matter which determines the background spacetime. We explore the separation between the contribution associated entirely by the Schwarzschild metric exterior to the classical matter and the contribution that depends on details of how the matter collapsed. We also examine acceleration effects such as the temperature a detector experiences in light of this separation, and discuss the contribution of quantum scalar fields to the backreaction problem which is dominated near the apparent singularity by a collapse-independent contribution.

I. INTRODUCTION

The physics of quantum field theory on a background of classical matter collapsing to form a black hole is interesting for several reasons. It is well-known that the concept of particle is more complicated when one extends quantum field theory in Minkowski space to the more general situation involving curvature of spacetime. Particle observers define their notion of particles, and the associated particle number operator, according to a field mode decomposition which may depend on an observer’s trajectory. These general results have also been appreciated in special cases such as the Rindler spacetime where an accelerated observer’s vacuum is seen to be different from the vacuum of an inertial observer even in Minkowski spacetime. The thermal spectrum seen by the accelerated observer in the vacuum defined by the inertial observer is known as the Unruh effect.

The most famous result involving spacetime curvature is the Hawking radiation, in which an observer in the Schwarzschild metric (in the spherically symmetric case) outside infalling matter, sees a thermal spectrum at asymptotic spatial infinity at late times. This observer sees a thermal flux of radiation as determined by considering quantum field theory defined on the spacetime of the collapsing matter. The physics is covered extensively in textbooks and reviews. More generally, one can ask what another observer might experience in the form of energy density, flux, and pressure. These scalar quantities can be defined in a properly covariant way, whereas the components of energy-momentum tensor is given with respect to a certain coordinate system. The scalar quantities derived from the energy-momentum tensor depend on the nature of the gravitational collapse as well as the nature of the observer’s trajectory. For example, what is the experience of an observer free-falling past the event horizon that has formed from collapsing matter? Or what is the experience of an observer who is maintaining a position at fixed radial coordinate outside the event horizon? Finally, what features of the energy density, flux, and pressure are universal and independent of the details of the collapse? It should also be remembered that these quantities depend on the position and velocity of the observer but not her acceleration. So there may not by themselves capture all important aspects. For example, the acceleration gives essential contributions to how a particle detector will respond.

It is known that in quantum field theory one can obtain negative energy densities. Near the event horizon a free-falling observer experiences negative energy density. One may ask how general is this phenomenon? Other observers will experience a net flux which contains contributions from energy density flowing out and from energy density flowing in. Only the flux flowing out will depend on the history of the collapsing matter since ray tracing indicates only the outgoing ray has experienced the spacetime region in which the collapsing matter is present. Two collapse scenarios, one involving the collapse of a null shell and the other in the Oppenheimer-Snyder model of a collapsing dust ball, have obtained remarkable similar results for the energy density and flux for points both outside and inside the horizon. In the latter example, points within the dust ball itself can be probed. The results are qualitatively very similar, and it invites an investigation to determine what aspects of process depend on the nature of the collapsing matter and what aspects are universal.

Another question of interest is the nature of the divergence of these physical quantities as an observer approaches the singularity. The energy density of the quantum field can be compared to that of the collapsing matter. Since the renormalized energy-momentum tensor (or renormalized stress-energy tensor, RSET) should contribute as a source in Einstein’s equations, it should be included in a full solution. If the energy density of the quantum field dominates, then the neglect of the back reaction of the quantum field on the geometry can result in a lack of confidence in the standard calculations and the nature of the singularity. Again, one is interested in isolating the contributions to the energy density that might dominate inside the event horizon and determining whether these dominant contributions are independent of the details of the gravitational collapse. Furthermore, interest in a quantum atmosphere for a black hole...
may encourage us to consider the region just outside a black hole and the properties of the RSET there \cite{20,21}.

In this work we concentrate attention on separating the universal contributions to the physical observables from the collapse-dependent ones for the simplified but important case of (1+1) dimensions. This model is thought to capture the important physical aspects of the quantum field theory on a background collapsing to form a black hole. Some recent results have obtained expressions for energy density and flux for regions both inside and outside collapsing matter. Clearly if one is inside the region where the matter is collapsing to form a black hole, then one cannot obtain unique answers because they depend on the details of the matter distribution and its history. On the other hand, if one restricts attention to the exterior region which uniquely has the Schwarzschild metric, it can be shown that one obtains results that are independent of the particular details of the collapse for all observers. Previous papers have concentrated attention of free-falling and stationary observers, but some features involving the energy density and flux are common for all observers. It is well-known that the RSET for the scalar field depends on more than the local metric and is influenced by the entire history of the gravitational collapse. In this paper we highlight that some observables can be shown to be independent of the details of the collapse and depend only on the Schwarzschild metric. The difference between the energy density and pressure has the same form for all observers, is related to the conformal anomaly, and depends only on the local geometry (it is a unique function of the radial coordinate \( r \) and varies smoothly across the horizon). The difference between the flux and the energy density, which represents incoming scalar field modes, depends on the Schwarzschild metric and the trajectory of the observer, but not the past history of the gravitational collapse. For a static observer sitting at fixed \( r \), this quantity diverges at the horizon as might be expected as it is increasingly blueshifted. For any observer crossing the horizon, and in particular for the observer who is free-falling through the horizon, one obtains a finite but collapse-independent result.

The universality of results only require that the observer be in the Schwarzschild or vacuum region of spacetime. This region remains outside the collapsing matter but extends inside the Schwarzschild event horizon. Thus physically relevant issues such as the size of the quantum fields contribution to the source of Einstein’s equation can be addressed independent of any assumptions about the nature of the gravitational collapse. While the size of the backreaction, for example, can call into question the validity of the semiclassical approximation near the singularity, the stability of the calculation can be addressed at and inside the horizon for the situation where one believes the results of the semiclassical calculation at all (it must be called into question if there is a firewall at the horizon).

II. DOUBLE NULL COORDINATE

A collapse scenario has a spacetime region where the matter is collapsing and an exterior region where, if the collapse is spherically symmetric, the metric is Schwarzschild. In the Schwarzschild region the (1+3) wave-equation for the radial component (after separation of variables) is

\[
\left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - V_l(r) \right) f_l(t, r) = 0 ,
\]

where

\[
V_l(r) = \left( 1 - \frac{1}{r} \right) \left[ \frac{l(l+1)}{r^2} + \frac{1}{r^3} \right] .
\]

We have chosen units for which the Schwarzschild radius \( r_s = 2M = 1 \). The approximations typically made are to consider only s-wave and to neglect the remaining term (“residual barrier”).

A common theoretical laboratory for studying gravitational collapse scenarios is to work in (1+1) dimensions with a minimally coupled massless scalar field. In this particular case the renormalized and conserved energy momentum tensor \( T_{\mu\nu} \) of the scalar field has been calculated exactly and involves a Schwarzian derivative of a conformal factor. The relevant factor is a redshift or scaling factor that represents the transformation from null coordinates used to define the vacuum to null coordinates used to express the (in this case) Schwarzschild metric outside the collapsing matter. In attempting to extrapolate the results to the (1+3) dimensional case, it should be remembered that even though it is known that the s-wave contributions dominates, the residual barrier should influence the behavior near the singularity at \( r = 0 \). Thus the (1+1) toy model becomes increasingly unreliable as one approaches the singularity at \( r = 0 \) for understanding the (1+3) dimensional case, so attempts to rescale physical quantities by multiplying by powers of the radial coordinate (as is a common practice) need to be examined carefully.

We can exploit double null coordinates as done in Refs. \cite{14,16,22} where the matching between the Schwarzschild metric and the metric inside the collapsing matter has been performed to obtain explicit expressions for physical quantities like the energy density and flux experienced by free-falling and stationary observers. For example, the Vaidya spacetime which involves a collapsing null shell has been considered and the collapsing dust ball (Robertson-Walker) has been treated in detail. The general form of these equations exhibits features which are independent of the collapse scenario assumed. This results from understanding the dependence of these covariant physical quantities (energy density and flux) on ray tracing. The double null coordinates involve one coordinate which is sensitive to ray tracing through the collapsing matter and one which is not. This results in a rather simple general conclusion which can be drawn in terms of the physical quantities for any observer in
any state of motion (not just stationary or following a geodesic) and at any position.

The double null coordinates are defined as in Fig. 1. For a massless scalar field the influence of the collapse can be determined by ray tracing. The RSET at point $x$ is sensitive to the collapse through coordinate $V^-$ while it is insensitive to the coordinate $V^+$. When described in terms of the coordinates $u$ and $v$ appropriate for $I^+$, one sees that one can take $V^+=v$, but there is in general a scaling factor connecting $V^-$ to $u$. So one has a special case of a conformal transformation where only one of the two directions is involved. This simplified situation leads to an RSET for the quantum scalar field for which the only component sensitive to the details of the collapse is $\langle T^- \rangle$ (in $(1+1)$ dimension $\langle T^+ \rangle$ is proportional to the trace, which is completely determined by the conformal anomaly). The physical measurables separate into parts which depend only on the local metric and parts which depend on the collapse through a Schwarzian derivative.

The combination $F - U$ depend only on the radial coordinate $r$ and the state of motion of the observer. The combination $P - U$ is universal, depending only on the Schwarzschild metric and is entirely independent of the motion of the observer. All of these physical quantities diverge at the $r = 0$ singularity for all observers.

It is clear from the null directions indicated in red in the diagram, and from the fact that the scalar field is massless (so that all its modes travel at the speed of light), that the physics is given simply by ray tracing. More generally, there will be a mixing of modes from either a nonzero scalar mass, from including modes beyond s-wave, and from extending to the full $(1+3)$ dimensions.

It is well-known in the subject of quantum field theory on a curved background that different observer can consider different physical states to be the vacuum devoid of (real) particle. For example, even in Minkowski space an accelerating observer sees the usual Minkowski vacuum as a state with particle in a thermal spectrum. So, in general, the experience of a physical observer depends on what he considers a vacuum. The question emerges as to what is the best or most relevant vacuum state to understand the properties of the spacetime region outside matter collapsing to form a black hole (but not necessarily outside the horizon). As has been argued previously, the "in"-vacuum defined on $I^-$ is an ideal choice which captures the details of how the collapse happened and has a useful physical interpretation.

Various vacuum states have been used to define the energy-momentum tensor:

(i) the Boulware vacuum is defined with respect to the exterior double null coordinates $u$ and $v$ for which the metric is

$$ds^2_{\text{ext}} = - \left(1 - \frac{1}{r}\right) du dv.$$  

In this vacuum the RSET is independent of the collapse by definition. It gives Schwarzschild-metric-dependent contributions to the energy-momentum tensor.

(ii) the Hartle-Hawking vacuum is defined as the vacuum state associated with the Kruskal coordinates. Since the Kruskal modes don’t reduce to the standard Minkowski modes asymptotically, there is a thermal flux at the Hawking temperature. This thermal bath can be viewed as being in equilibrium with the black hole. Hence, the Kruskal extension is often referred to as an eternal black hole.

(iii) the "in"-vacuum is defined in terms of the global double null coordinates $V^+$ and $V^-$. The ray tracing in the figure makes it evident that the components of $\langle T_{\mu\nu} \rangle$ are greatly simplified since only the coordinate $V^-$ has a collapse-dependent and in general complicated relationship to the exterior coordinate $u$. In general, one has $V^+ = v$. This makes the RSET in this vacuum have some generic features in comparison to the Boulware vacuum. Since the component $\langle T^{+-} \rangle$ arises from the conformal anomaly, it gives a universal result, not only as a function of the radial coordinate, but for any observer executing any timelike trajectory.

(iv) the Unruh state is defined with respect to $V^+$ and the outgoing Kruskal coordinate. At late times the coordinate $V^-$ behaves like the outgoing Kruskal coordinate.
So the Unruh vacuum describes the late-time limit of the “in”-vacuum, and this vacuum has often been used to study the late-time thermal radiation.

In this paper we utilize the RSET in the uniquely defined “in”-vacuum. The observer sees no particles in the $V^\pm$ modes defined on $I^-$ which can be interpreted as the spacetime region which is causally preceding the gravitational collapse.

The effects of the collapse depend entirely on a scaling or redshift factor defined by integrating through the collapsing matter region. Looking from outside the collapse at a spacetime point $(x$ in Fig. 1) back through the matter by ray tracing to $r = 0$ and then back to $I^-$, one obtains a net scaling factor. Naturally, one expects in a collapse scenario that the gravitational well the null ray climbed out of on its way out is larger than the well the same ray previously had followed in to $r = 0$. This integrated effect connects modes at $I^-$ to $I^+$. The important feature here is that only one of the null rays connecting back to the null coordinates $V^-$ and $V^+$ passes through the collapsing matter, namely $V^-$. This means the scaling factor is only in one dimension. In general, $V^+ = v$, where $v$ is the exterior Schwarzschild null coordinate. One can always redefine the ingoing null coordinate $v$; for example, one can rescale it by some factor which results in different components of the energy momentum tensor in the new redefined coordinate. This dependence on coordinates is unphysical, and it is more appropriate to consider only the covariant quantities which can be obtained from the components of the RSET. These are the energy density $U$, flux $F$, and pressure $P$ that an observer experiencing some motion experiences. In (1+1) dimensions the pressure and the energy density are related by the trace anomaly. A trivial change of coordinates, cancels out in properly defined covariant quantities like the energy density and flux for any observer. The ray tracing makes clear that the nontrivial physical effect can be isolated in only the $(T_{--})$ component.

A collapse scenario is characterized in the external (Schwarzschild) spacetime by a connection between two sets of coordinates which arises from an integrated redshift from a past-directed null ray looking through the collapse back to $V^-$ defined on $I^-$. The exterior double null coordinates are given in Eq. (3). The same metric described in the global double null coordinates is

$$ ds^2_{ext} = -(1 - \frac{1}{r}) S(V^-) dV^- dV^+ , $$

As described above the $S(V^-)$ is a scaling or redshift factor describing the connection with the Boulware $u$ coordinate only. We have

$$ du = S(V^-) dV^- , $$

$$ dv = dV^+ . $$

If one is calculating in the Boulware vacuum, this conformal factor is simply the Schwarzschild factor $1 - 1/r$. For the “in”-vacuum, whose modes are defined by the coordinates $V^\pm$ defined on $I^+$, the correct conformal factor is the product of the Schwarzschild factor and the scaling factor, $C = (1 - 1/r)S$. In general, the components $\langle T_{--} \rangle$ and $\langle T_{++} \rangle$ involve the Schwarzian derivative and $\langle T_{+-} \rangle$ is related to the two-dimensional trace and therefore to the anomaly. The RSET’s components involve overall factors of $S$. These result because of the homogeneous (“log-like”) derivatives guarantee that an overall factor can be separated out as additional terms in the RSET. In fact, as we will see in the next section there is only one collapse-dependent function (involving derivatives of the scaling factor $S$) which enters into covariant physical quantities like the energy density and flux of the quantum scalar field.

### III. COLLAPSE SCENARIOS

The past-directed ray tracing produces a scaling factor $S$ which depends in detail on the history of the collapse as the null ray traversed that region of spacetime. Nevertheless, one can separate the physical effects which depend on $S$ from those that only depend on the spacetime outside the collapse being Schwarzschild. Discussions of the separation into universal characteristics of
the Schwarzschild metric and the collapse-dependent redshift factor \( S \) should involve the properly covariant quantities such as the energy density \( \mathcal{U} \), the flux \( \mathcal{F} \), and the pressure \( \mathcal{P} \). Some dependence on \( S \) by the RSET is simply a result of a particular coordinate choice (for \( V^- \)), and does not enter into these three physical quantities (only two of these quantities are independent since the trace of the RSET is given by the conformal anomaly).

Using Eqs. (3), (4), and (5), the RSET is found to be

\[
(T_-) = \frac{k^2}{48\pi} \left[ \left( \frac{3}{r^4} - \frac{4}{r^3} \right) S^2 - 16S^{1/2}\partial_S S^{-1/2} \right],
\]

\[
(T_{++}) = \frac{k^2}{48\pi} \left( \frac{3}{r^4} - \frac{4}{r^3} \right),
\]

\[
(T_{+-}) = \frac{k^2}{12\pi} \left( \frac{1}{r^4} - \frac{1}{r^3} \right) S.
\]

We have defined the overall scale \( \kappa = (2r_s)^{-1} = 1/2 \). For example, the component \( (T_{++}) \), related to the conformal anomaly, depends on \( S \) associated with its one minus index. In covariant expressions this factor will disappear. It just represents our freedom to rescale the null coordinate \( V^- \). The component \( (T_-) \), similarly contains a factor \( S^2 \), but also a collapse-dependent contribution involving the Schwarzschild derivative. The collapse-independent parts are the terms that appear in the components of \( (T_{\mu\nu}) \) in the Boulware vacuum. See, for example, Eq. (22) of Ref. 24 or Eq. (5.88) of Ref. 3. The metric in the global null coordinates is the product of the scaling factor and the Schwarzschild factor. Since the derivatives are homogeneous, each factor contributes as one term in a sum in the physical quantities.

IV. STATIC AND FREE-FALLING OBSERVERS

The energy density, flux, and pressure depend on the trajectory of the observer. These quantities depend on the the global properties of spacetime: (1) the Schwarzschild metric which is the unique spherically symmetric spacetime outside the collapsing matter according to the Birkhoffs theorem, and (2) the details of the collapsing matter which forms the black hole. In Ref. 10 the authors discussed the case of the static observer maintaining a position at fixed radial coordinate \( r \) outside the horizon, and the case of the free-falling (geodesic) observer who can cross the event horizon. We reproduce some results for each of these cases of observer, and then we can make some conclusions for other observers executing an arbitrary timelike trajectory. So-called null observers, either directed in or directed out, isolate a null component of \( (T_{\mu\nu}) \). An outgoing null observer is sensitive only to \( (T_{++}) \) whereas an ingoing null observer can only see \( (T_-) \). This follows simply from the causal nature of the conformal diagram. Since all modes move at the speed of light for a massless scalar field, all light rays can be easily ordered. The anomaly, on the other hand, has physical effects for timelike observers.

The observer’s motion is described by a four-velocity \( u^\mu \). One defines a normal \( n^\mu \) in the outward radial direction such that \( n_\mu u^\mu = 0 \). Then the energy density, flux, and pressure are given by the covariant expressions

\[
\mathcal{U} = (T_{\mu\nu}) u^\mu n^\nu,
\]

\[
\mathcal{F} = -(T_{\mu\nu}) u^\mu n^\nu,
\]

\[
\mathcal{P} = (T_{\mu\nu}) n^\mu n^\nu.
\]

A. Static Observer

For a static observer at fixed \( r > 1 \), one has

\[
V^+ = \sqrt{\frac{r}{r-1}},
\]

\[
V^- = \sqrt{\frac{r}{r-1} S^{-1}},
\]

so that, for the energy density and flux, one obtains

\[
\mathcal{U} = \langle T_{++} \rangle \langle V^+ \rangle^2 + 2 \langle T_{+-} \rangle V^+ V^- + \langle T_- \rangle \langle V^- \rangle^2 = \frac{k^2}{24\pi} \left( \frac{r}{r-1} \right)^2 \left( 8 \frac{r}{r^3} - 8S^{-3/2}\partial^2 S^{-1/2} \right),
\]

\[
\mathcal{F} = -\langle T_{++} \rangle \langle V^+ \rangle^2 + \langle T_- \rangle \langle V^- \rangle^2 = \frac{-k^2}{24\pi} \left[ 8 \left( \frac{r}{r-1} \right)^2 S^{-3/2}\partial^2 S^{-1/2} \right],
\]

This observer must be accelerating away from the event horizon, and this acceleration must be increasingly large to maintain the observer’s position as the fixed radius approaches the horizon. At the horizon, the acceleration diverges. Furthermore, the covariant quantities depend on a universal Schwarzschild contribution together with a collapse-dependent flux. The Schwarzschild derivative is multiplied by an overall factor \( S^{-2} \) and the Schwarzschild factor \( r/(r-1) \) in the metric, Eq. (11). Consequently, the combination \( \mathcal{F} - \mathcal{U} \) is independent of the details of the collapse, represents the incoming modes, and assumes a universal dependence on \( r \) associated with the Schwarzschild metric:

\[
\mathcal{F} - \mathcal{U} = \frac{k^2}{24\pi} \left( \frac{1}{r-1} \right)^2 \left( \frac{8}{r^2} - 7 \right).
\]

The pressure is given by

\[
\mathcal{P} = \langle T_{++} \rangle \langle V^+ \rangle^2 - 2 \langle T_{+-} \rangle V^+ V^- + \langle T_- \rangle \langle V^- \rangle^2 = \frac{k^2}{24\pi} \left( \frac{1}{r-1} \right) \left[ 8 + 8rS^{-3/2}\partial^2 S^{-1/2} \right].
\]

In (1+1) dimensions the traceless classical energy-momentum tensor requires that \( \mathcal{P} = \mathcal{U} \). Since there is a conformal anomaly, one expects this equality to be broken in a universal way. In fact, the combination \( \mathcal{P} - \mathcal{U} \) is also independent of the details of the collapse:

\[
\mathcal{P} - \mathcal{U} = -4\langle T_{+-} \rangle V^+ V^- = \frac{k^2}{3\pi r^3}.
\]
In (1+1) dimensions the anomaly is proportional to the Ricci scalar, so it is a function of $r$ and there is no special behavior at the horizon. In the region outside the collapsing matter this physical observable is a geometric quantity determined entirely by the local metric. We have presented this here in the context of black hole collapse, but the result is generally true for any RSET. Note that classically, since the trace of the energy-momentum tensor vanishes for any conformal field, this means that the pressure and energy density are equal. In the RSET, the difference between these two (covariant) observables is entirely determined by the anomaly.

We note in passing that changing the sign of the anomaly would interchange the energy density and pressure. The correct sign can be checked in (1+1) dimensions by requiring that the RSET is conserved. This can be used to compensate for any inconsistent definition of the metric conformal factor $\mathcal{C}$ and the RSET calculated as in Eq. (6) has occurred.

**B. Free-falling Observer**

Another example for an observer that has been worked out in detail is the free-falling observer who begins at radial coordinate $r_i$. Then, solving the geodesic equation, one obtains

$$
\dot{V}^+ = \frac{r}{r-1} \left( E - \sqrt{E^2 - \frac{r-1}{r}} \right),
$$

$$
\dot{V}^- = \frac{r}{r-1} \left( E + \sqrt{E^2 - \frac{r-1}{r}} \right) S^{-1},
$$

where

$$
E = \left(1 - \frac{1}{r_i}\right)^{1/2}.
$$

So $E$ takes values from one for an observer beginning free-fall at asymptotic infinity to an arbitrarily small value as the starting point approaches the horizon. Considering the form of the RSET in Eq. (7), the only contribution to the covariant quantities (energy density, flux) involving the scaling factor $S$ arises from the $\langle T_{-\cdot} \rangle$ component. The contribution is again proportional to

$$
S^{-3/2} \partial^2 S^{-1/2} = -S^{-1/2} \partial^2 S^{1/2}.
$$

In the first expression we have $S^{-2}$ times the Schwarzian derivative in the $V^-$ coordinate. In the Schwarzschild coordinate $u$ we have the Schwarzian derivative of $V^-$. The covariant scalar quantities are

$$
U = \frac{\kappa^2}{24\pi} \left(\frac{r}{r-1}\right)^2 \left[ E^2 \left(\frac{6}{r^4} - \frac{8}{r^2}\right) + \frac{r-1}{r^5} - 8 \left( E + \sqrt{E^2 - \frac{r-1}{r}} \right) S^{-3/2} \partial^2 S^{-1/2} \right],
$$

$$
F = \frac{\kappa^2}{24\pi} \left(\frac{r}{r-1}\right)^2 \left[ E \sqrt{E^2 - \frac{r-1}{r}} \left(\frac{6}{r^4} - \frac{8}{r^2}\right) - 8 \left( E + \sqrt{E^2 - \frac{r-1}{r}} \right) S^{-3/2} \partial^2 S^{-1/2} \right],
$$

where the difference

$$
F - U = \frac{\kappa^2}{24\pi} \left(\frac{r}{r-1}\right)^2 \left[ \left( E \sqrt{E^2 - \frac{r-1}{r}} - E^2 \right) \left(\frac{6}{r^4} - \frac{8}{r^2}\right) - \frac{r-1}{r^5} \right],
$$

is $S$-independent. The pressure is

$$
P = \frac{\kappa^2}{24\pi} \left(\frac{r}{r-1}\right)^2 \left[ E^2 \left(\frac{6}{r^4} - \frac{8}{r^2}\right) + \frac{r-1}{r} \left(\frac{7}{r^4} + \frac{8}{r^2}\right) - 8 \left( E + \sqrt{E^2 - \frac{r-1}{r}} \right) S^{-3/2} \partial^2 S^{-1/2} \right],
$$

and again Eq. (13) is satisfied.

While the pressure is connected to the energy density in this universal way associated with the anomaly, a short argument can establish this result for any timelike observer is instructive. We find that $P - U$ is the same for static and infalling observers, since $V^+ V^-$ is the same for both cases. In fact, $P - U = \frac{\kappa^2}{3\pi r^4}$ for any timelike observer outside the collapsing matter. From Eq. (3), $\dot{v} \ddot{u} = (1 - \frac{1}{r})^{-1} = \frac{r}{r-1}$. Since $\dot{V}^+ = \dot{v}$ and

$$
\dot{V}^- = \frac{dV^-}{dt} = \frac{dV^-}{du} \frac{du}{dt} = \dot{u} S^{-1},
$$

we have

$$
P - U = -4 \langle T_{-\cdot} \rangle \dot{V}^+ \dot{V}^- = -4 \cdot \frac{\kappa^2}{12\pi} \cdot \frac{1}{r^4} \cdot S \dot{u} S^{-1} = \frac{\kappa^2}{3\pi r^4},
$$

in agreement with Eq. (13). The Eddington-Finkelstein coordinate $u$ does not extend across the event horizon as
it diverges there. However, we can use it for our purposes in two patches for inside and outside the horizon, and the Schwarzsian derivative varies smoothly across the horizon. So, as advertised, the covariant quantity $P - U$ has a universal expression for all timelike observers outside the collapsing matter and doesn’t depend on the details of the collapse or on the quantum state. Often, for this reason, the pressure is not considered as an additional covariant quantity to discuss. However, some recent ideas to define an effective Tolman temperature by implementing a Stefan-Boltzmann law by properly partitioning the contribution from the anomaly equally to the energy density and pressure (see Ref. [31]), highlight the role the trace anomaly might play. In this line of argument, a temperature $T$ can be consistently defined according to
\begin{align}
    U &= \gamma T^2 - \frac{1}{2} T^\mu \mu , \\
    P &= \gamma T^2 + \frac{1}{2} T^\mu \mu ,
\end{align}
where $\gamma = \pi/6$ is the (1+1) dimensional Stefan-Boltzmann constant. In the presence of the trace anomaly, this modified definition of temperature gives a finite value for the Tolman temperature even at the horizon.

C. Null Observer

Also of interest is the experience of the null observer. This observer is sensitive to only the $\langle T_{--} \rangle$ or the $\langle T_{++} \rangle$ component, depending on whether the observer is traveling inward or outward, respectively. The null geodesics are described by affine parameters, and the conformal anomaly is present only for observers following timelike trajectories.

D. FRW Collapse Example

In Ref. [16] the authors examined a collapsing dust ball, where they were able to study the covariant energy density and flux both inside and outside the collapsing matter. In this example, the scaling factor takes the form
\begin{equation}
S = \frac{dB/dU|_{U+\chi_0}}{dA/dU|_{U-\chi_0}} ,
\end{equation}
for the example of a collapsing dust ball in the Oppenheimer-Snyder model [32] which consists of a collapsing Friedmann-Robertson-Walker (FRW) dust ball surrounded by the Schwarzschild geometry,
\begin{equation}
ds_{\text{int}}^2 = a^2(\eta) (-d\eta^2 + d\chi^2) .
\end{equation}
Here $B(U)$ is a function obtained from matching the FRW metric to the exterior Schwarzschild metric for when the null ray exits the dust ball, whereas $A(U)$ is obtained by matching at the (earlier) point when the null ray enters the dust ball. The detailed formulas for these functions can be found in Ref. [16]. These two factors represent the connection of the double null coordinate $V^-$ to the Schwarzschild coordinates $u$ through an intermediate coordinate $U$ describing the region within the dust ball (FRW) [33]. The parameter $\chi_0$ is the value of the Robertson-Walker coordinate at the surface of the dust ball.

Another example of a collapse for which the scaling factor $S$ has been worked out in detail is the Vaidya collapse, a null shell collapsing inward. The results for this second example can be found in Ref. [14, 15]. See also Ref. [34].

In Fig. 2 the scaling factor is plotted versus the dust ball parameter $U$ for the case of $\chi_0 = \pi/6$. The scaling factor is positive in the spacetime region outside the event horizon, rising to infinite value at the horizon. Inside the event horizon it is negative, falling to zero at the singularity. The transition through infinity is the behavior of an Eddington-Finkelstein coordinate at the event horizon which occurs at $U = \pi - 3\chi_0 = \pi/2$. The scaling factor is greater than unity (i.e. a net redshift) for null rays that escape through the collapse just outside the event horizon. However, earlier null rays (with smaller $U$) explore the part of the Robertson-Walker spacetime where the dust ball is expanding. So it is possible to have some values of $S < 1$, which is a net blueshift. The equations for the covariant physical quantities apply not only to pure collapse scenarios, but to any (1+1) dimensional problem where only one of the two null rays explores spacetime that is not Schwarzschild.

![FIG. 2. The scaling factor $S$ for the FRW dust ball model for the model parameter $\chi_0 = \pi/6$. It diverges at the event horizon $U = \pi - 3\chi_0 = \pi/2$ and changes sign as expected for the matching of the dust ball coordinate $U$ with the Eddington-Finkelstein coordinate $u$ there (the coordinate diverges at the horizon).](image-url)

In Fig. 3 the Schwarzian factor is plotted versus the dust ball null coordinate. It is smooth through the horizon and diverges at the singularity.

In fact, one can dispense with the gravitation collapse...
entirely. As long as the classical background has only one null coordinate sensitive (by ray tracing) to the classical matter history, one can consider cases where a black hole diverges as expected.

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For static observers this, for late times, gives the Tolman temperature, other quantities such as the effective temperature measured in terms of null coordinates the Minkowski vacuum. This phenomenon is known as the Unruh effect. Since in general the particle perception of an observer depends on the acceleration, other quantities such as the effective temperature have been introduced that more properly characterize this effect. The effective temperature measured by a static observer is given in terms of the scaling factor, for static observers this, for late times, gives the Tolman redshifted temperature.

The connection between the coordinates gives \( \dot{u} = S(V^-)V^- \), we have

\[
\frac{\dot{V}^-}{V^-} = -\frac{\dot{S}}{S} + \frac{\dot{u}}{u},
\]

(24)

For any observer one can separate the temperature into this same contribution and one associated with the motion which we might call an Unruh contribution:

\[
T_- = \frac{1}{2\pi} \left| \frac{d}{d\tau} \ln S - \frac{d}{d\tau} \ln \dot{u} \right|.
\]

(25)

An analogous expression (and temperature) exists for the coordinates associated with the incoming null coordinate. It is less interesting for us since it is universal and does not have a dependence on the scaling factor \( S \). Here, following the interpretation in Ref. \[36\], the Unruh effect is associated with the acceleration of the observer with respect to the asymptotic region (as opposed to the local reference frame associated with the free-fall trajectory). The expression for \( T_- \) demonstrates the clean separation of the contributions to the temperature into a collapse-dependent Hawking contribution and a contribution that depends only on the observers trajectory. It exhibits a clear separation of the effects of Hawking radiation and the Unruh effect. It can be usefully deployed in situations where one is interested in the particle perception of an observer and depends explicitly on the the observer’s acceleration.

One can define a perceived vacuum \( \langle \tilde{0} \rangle \) for any observer in terms of null coordinates \( \tilde{U} \) and \( \tilde{V} \) in which the metric has the locally Minkowski form \( ds^2 = -d\tilde{U}d\tilde{V} \) for every point along the observer’s trajectory. Since the trajectory is in general not a geodesic, such a vacuum allows for the definition of physical observables which may be sensitive to the acceleration of the observer. This has led some authors \[36, 37\] to introduce a “perception renormalized stress-energy tensor (PeRSET)”. The PeRSET is defined by subtracting from the usual renormalized energy-momentum tensor evaluated in the vacuum state of the field its value in a vacuum naturally associated with the observer,

\[
\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle - \langle \tilde{0} \rangle [T_{\mu\nu}]_{\tilde{0}}.
\]

(26)

This observer-dependent quantity depends on acceleration, allows one to construct covariant physical quantities as one does with the RSET, and offers the possibility of a separation of the radiation perceived by the observer into two distinct forms: Hawking and Unruh. The result is

\[
T_- = \frac{1}{24\pi} \left( \frac{1}{2} \kappa_-^2 + \kappa_-^2 \right) (V^-)^{-2},
\]

\[
T_+ = \frac{1}{24\pi} \left( \frac{1}{2} \kappa_+^2 + \kappa_+^2 \right) (V^+)^{-2},
\]

\[
T_{++} = 0,
\]

(27)

where \( \kappa_- \) and \( \kappa_+ \) are effective temperature functions for \( V^- \) and \( V^+ \) sectors, respectively,

\[
\kappa_- = \frac{\dot{V}^-}{V^-},
\]

\[
\kappa_+ = -\frac{\dot{V}^+}{V^+}.
\]

(28)
A comparison of our results to the PeRSET formalism, one finds a similar separation of the physical effects into a Hawking-like effect and an Unruh-type effect. For example, this separation is seen in the results for the temperature where one contribution comes from the collapsing matter scaling factor $S$ and the other factor comes from the motion of the observer and depends on her trajectory. Since the conformal anomaly is universal, the PeRSET has vanishing trace (and the energy density and pressure are equal to each other). The PeRSET interpretation removes the anomaly since it is a purely geometric quantity. So Eq. (27) holds for all observers following any trajectory.

One can use $T_{\mu\nu}$ to define the associated energy density $\tilde{\mathcal{U}}$, flux $\tilde{\mathcal{F}}$, and pressure $\tilde{\mathcal{P}}$ by equations analogous to those in Eq. (30). This yields

$$\tilde{\mathcal{U}} = \tilde{\mathcal{P}} = T_{++}(\tilde{V}^+)^2 + T_{--}(\tilde{V}^-)^2,$$

$$\tilde{\mathcal{F}} = -T_{++}(\tilde{V}^+)^2 + T_{--}(\tilde{V}^-)^2. \quad (29)$$

Inserting our results one obtains for a general collapse

$$\tilde{\mathcal{U}} = \tilde{\mathcal{P}} = \frac{1}{48\pi} \left[ \left( \frac{d}{d\tau} \ln S \right)^2 + \left( \frac{d}{d\tilde{v}} \ln \tilde{v} \right)^2 + 2 \frac{d^2}{d\tau^2} \ln \frac{S}{\tilde{u}} \right],$$

$$\tilde{\mathcal{F}} = \frac{1}{48\pi} \left[ \left( \frac{d}{d\tau} \ln S \right)^2 - \left( \frac{d}{d\tilde{v}} \ln \tilde{v} \right)^2 + 2 \frac{d^2}{d\tau^2} \ln \frac{\tilde{S}}{\tilde{u}} \right]. \quad (30)$$

For the special case of the static observer, these expressions reduce to one involving only the scaling factor

$$\tilde{\mathcal{U}} = \tilde{\mathcal{F}} = \tilde{\mathcal{P}} = \frac{1}{48\pi} \left[ \left( \frac{d}{d\tau} \ln S \right)^2 + 2 \frac{d^2}{d\tau^2} \ln S \right], \quad (31)$$

The interpretation of this result is that the version of these scalar quantities obtained using the PeRSET contains only on the collapse-dependent Hawking contribution and the Unruh effect is absent for this kind of observer.

Quantum frictionless trajectories have been defined as those lacking the Unruh effect in both the ingoing and outgoing radiation. For these trajectories it is interesting to consider the covariant quantities obtained by such observers from the energy-momentum tensor where presumably the Hawking contribution has been isolated. They would depend on the scaling factor $S$ in general.

VI. BACKREACTION

The consideration of backreaction effects have been of considerable interest since they inform one about the plausibility of the semiclassical solution. When one uses the renormalized energy-momentum tensor as a source for gravity by including it as a contribution to Einstein’s equation

$$G_{\mu\nu}(g_{\alpha\beta}) = 8\pi \langle T_{\mu\nu}(g_{\alpha\beta}) \rangle, \quad (32)$$

one requires knowledge of the expectation values as a function of the metric and knowledge of the appropriate physical state. In the absence of exact solutions, one is forced to adopt approximation schemes. With these limitations one can compare the size of the new contribution from the quantum field theory to the classical contribution from the infalling matter. In the event that the contribution from the field theory dominates over the classical matter, one expects the calculation to break down, and one cannot trust even the approximate validity of the solution. There has been much speculation that perhaps some other qualitative behavior emerges like the creation of a baby universe rather than a singularity.

Of considerable interest is the nature of the energy-momentum tensor near the singularity. For cases that have been studied in detail the divergence in the scalar field becomes dominant over the divergence in the collapsing classical matter for the energy density and flux. It is expected that these features survive in the more realistic (1+3) dimensional case. The calculations involved are much more complicated, but have been performed for a Robertson-Walker universe and studied numerically. We learn here that the singularity at $r = 0$ is nothing but the gravitational quantities $\mathcal{U}$, $\mathcal{F}$, and $\mathcal{P}$ all diverge like $1/r^3$ in (1+1) dimensions (at least outside the collapsing matter, i.e. in the Schwarzschild region). The question of the nature of the divergence within the collapsing matter has been investigated in Ref. with the conclusion it makes qualitatively little difference for quantities like $\mathcal{U}$ whether one is in the collapsing classical matter or outside it.

It should be stressed that the state assumed in our expressions for the energy density are with respect to the “in”-vacuum, which might not be the proper vacuum to use when including the RSET as a source in Einstein’s equations. In this state we are finding a negative energy density and a positive pressure near the singularity for the universal contribution which dominates over the model-dependent features of the collapsing classical matter. Finding the correct state to solve the backreaction problem even in (1+1) dimensions has proven intractable.

How does one expect these results to generalize and extend to the case of (1+3) dimensions, away then from the spherically symmetric case? The easiest approach is to include an overall $1/(4\pi r^2)$ factor in the energy-momentum tensor. This would indicate that the energy-momentum tensor diverges like $1/r^5$ near the singularity. Another point is that the extra term in the potential, Eq. (29), has been neglected and becomes increasingly important near the singularity (even in the s-wave approximation in the (1+3) dimensional case). So extrapolations of the (1+1) dimensional case to the realistic (1+3) should be approached with care.

VII. CONCLUSION AND DISCUSSION

The toy model of black hole collapse in (1+1) dimensions reduces the problem to one of ray tracing (for suf-
ficiently short wavelength), and only the outgoing rays have experienced the collapsing matter and give a contribution. While the qualitative features are expected to hold in the realistic case, certain generalizations can be considered: The higher $t$-waves can be included which represent a potential barrier. The residual barrier presumably cannot be ignored near the singularity even in $s$-wave approximation. A nonzero scalar mass will cause a mixing of modes as plane waves of different momenta will propagate with different velocities. However, none of these features of a more realistic black hole collapse in (1+3) dimensions obviates the fact that the outgoing modes are causally influenced by the collapse whereas the incoming modes are not. Some general conclusions can be made about the covariant quantities like energy density, flux, and pressure.

In the (1+1) dimensional black hole there are some covariant quantities that involve only the incoming contributions, and therefore depend only on the Schwarzschild metric and are therefore universal. One, $\mathcal{P} - \mathcal{U}$ is determined by the trace anomaly and all timelike observers agree on its values which is just a property of the Schwarzschild metric. Another example is $\mathcal{F} - \mathcal{U}$ which is sensitive to only the ingoing modes which do not pass through the collapsing matter. All observers will report values for this physical quantity which is a function of $r$, but will depend on the nature of his trajectory. The combination $\mathcal{F} + \mathcal{U}$ is determined by the outgoing modes which have passed through the collapsing matter and are sensitive to the details of how the spacetime scaled the Schwarzschild coordinate ($V^-$) before the null ray entered the collapse to the Schwarzschild coordinate ($u$) after the null ray exited the collapse. For a collapse which is monotonic, one expect a net redshift so that $S > 1$ outside the horizon, but one can also obtain $S < 1$ if there is a period where the collapsing matter is also expanding while the null ray is tracing a path through it. The FRW dust ball exhibits both situations.

In the (1+1) dimensional case, the backreaction of the quantum field does not ameliorate the singularity. This is independent of the collapse and scaling factor $S$, since the Schwarzschild contribution is dominant. Since the nature of the energy density and flux inside the event horizon has been examined, it is clear that this dominant feature is independent of any details about how the collapse occurred. This calls into question, in general, the reliability of the semiclassical approximation near the singularity even in the simplified setting of the (1+1) dimensional black hole.

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