Formation of superscar waves in plane polygonal billiards

Eugene Bogomolny
Université Paris-Saclay, CNRS, LPTMS, 91405 Orsay, France
E-mail: eugene.bogomolny@lptms.u-psud.fr
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Abstract
Polygonal billiards constitute a special class of models. Though they have zero Lyapunov exponent their classical and quantum properties are involved due to scattering on singular vertices with angles $\approx \pi / n$ with integer $n$. It is demonstrated that in the semiclassical limit the multiple singular scattering on such vertices, when optical boundaries of many scatters overlap, leads to the vanishing of quantum wave functions along straight lines built by these scatters. This phenomenon has an especially important consequence for polygonal billiards where periodic orbits (when they exist) form pencils of parallel rays restricted from the both sides by singular vertices. Due to the singular scattering on boundary vertices, waves propagated inside a periodic orbit pencil tend in the semiclassical limit to zero along pencil boundaries thus forming weakly interacting quasi-modes. Contrary to scars in chaotic systems, the discussed quasi-modes in polygonal billiards become almost exact for high-excited states and for brevity they are designated as superscars. Many pictures of eigenfunctions for a triangular billiard and a barrier billiard which have clear superscar structures are presented in the paper. Special attention is given to the development of quantitative methods of detecting and analysing such superscars. In particular, it is demonstrated that the overlap between superscar waves associated with a fixed periodic orbit and eigenfunctions of a barrier billiard is distributed according to the Breit-Wigner distribution typical for weakly interacting quasi-modes. For special subclass of rational polygonal billiards called Veech polygons where all periodic orbits can be calculated analytically it is argued and checked numerically that their eigenfunctions are fractal in the Fourier space.

1. Introduction

The largest part of this work has been prepared during 2003–2004 but an implacable illness of Charles Schmit had permitted to publish uniquely its short account [1]. It is only now that I collect different fragments of performed investigations and organise them in a readable form.

The paper examines the structure of eigenfunctions for a special class of quantum models, namely two-dimensional polygonal billiards whose boundaries are straight lines. Classical mechanics of these problems corresponding to the ray propagation with the specular reflection from the boundaries is intricate, surprisingly rich, and notorious difficult (see, e.g., [2] and references therein).

The most investigated is the case of rational or pseudo-integrable billiards where all (internal) billiard angles $\theta_j$ are rational fractions of $\pi$

$$\theta_j = \frac{m_j}{n_j}$$

with co-prime integers $m_j$ and $n_j$. A characteristic property of such billiards is that their classical trajectories belong to 2-dimensional surfaces of finite genus $g$ related with angles (1) as follows (see, e.g., [3])

$$g = 1 + \frac{N}{2} \sum_j \frac{m_j - 1}{n_j}$$

where $N$ is the least common multiple of all $n_j$.

* In memory of Charles Schmit.
Two particular examples of such models discussed in the paper are depicted in figure 1. The first model is a right triangular billiard with angles $\pi/8$, $3\pi/8$, $\pi/2$. The second one is a rectangular billiard of sides $2a$ and $b$ with a barrier of height $b/2$ in the middle. This model has 6 corners with angles $\pi/2$ and one corner with angle $2\pi$.

In a billiard where all angle numerators $m_j=1$ trajectories belong to tori with $g=1$ and the model is classically integrable. The list of 2-dimensional integrable polygonal billiards is limited. It includes rectangular (square) billiards and 3 triangular billiards with angles $[\pi/4, \pi/4, \pi/2], [\pi/3, \pi/3, \pi/3], [\pi/6, \pi/3, \pi/2]$.

If at least one numerator is bigger than 1, classical trajectories lie on a surface of genus $g \geq 2$ and such models are genuine pseudo-integrable models. The billiards at figure 1 have genus $g = 2$. The values of genus can be obtained by the explicit unfolding of the initial billiard table. For example, at figure 2 the unfolding of the right triangular billiard with angle $\pi/8$ is performed. By reflections one gets a surface with the shape of the regular octagon whose opposite parallel sides are identified. Topologically the resulting surface is a sphere with 2 handles which is the canonical image of genus 2 surfaces.

The quantisation of billiards consists in finding eigenvalues and eigenfunctions of the wave equation

$$(\Delta + E_n)\Psi_n(x) = 0$$

provided the eigenfunctions obey certain boundary conditions along the billiard boundaries. In the paper the Dirichlet boundary conditions are chosen

$$\Psi_n(x)|_{\text{boundaries}} = 0.$$  

For all integrable polygonal billiards cited above the solution of the quantum problem is well known (see e.g. [3]). Their eigenvalues and eigenfunctions depend on two integers. Eigenvalues are quadratic functions of these integers and eigenfunctions are finite combination of trigonometric functions. Even the inverse theorem is valid:
the list of billiards whose all eigenfunctions are finite combinations of trigonometric functions is exhausted by the above integrable billiards [4].

The structure of eigenvalues and eigenfunctions of polygonal billiards are much more complicated and only partial results are available.

In quantum chaos there are two big conjectures concerning the spectral statistics of generic integrable and fully chaotic systems. For integrable models the spectral statistics (after unfolding) coincides with the Poisson statistics of independent random variables [5] and for chaotic systems it corresponds to the eigenvalue statistics of the standard random matrix ensembles depended only on symmetries [6]. The difference between these two types of universal statistics is clearly seen in the behaviour of the nearest-neighbour distribution, \( p(s) \), which gives the probability that two nearest levels are separated by a distance \( s \) (see e.g. [7, 8]). For integrable systems \( p(s) = \exp(-s) \) which implies the absence of the level repulsion \( p(0) = 0 \) and an exponential decrease of correlations at large distances. For chaotic systems \( p(s) \) is well approximated by the Wigner ansatz \( p(s) = a^s \exp(-bs^2) \) where \( b = 1, 2, 4 \) with \( a, b \) being constants determined from normalisation conditions.

Contrary to integrable models, fully chaotic systems are characterised by the level repulsion \( p(0) = 0 \) and a quadratic falloff of \( p(s) \) at large distances \( p(s) \xrightarrow{s\rightarrow\infty} \exp(-s^2) \).

The numerical calculations of pseudo-integrable billiards [3], [9–18] demonstrate that their spectral properties are in-between these two universal distributions. Namely, their nearest-neighbour distribution reveals a linear level repulsion \( p(s) \xrightarrow{s\rightarrow0} s \) as for the random matrix ensemble with \( \beta = 1 \) but at large distances \( p(s) \) decreases exponentially as for the Poisson statistics.

The purpose of this paper is to investigate certain properties of eigenfunctions for plane polygonal billiards. The principal difficulty in treating such problems is the fact that in polygonal billiards the vertices with angles \( \pm\pi/n \) with integer \( n \) are singular points for the classical motion. If a parallel pencil of rays hits such vertex it splits discontinuously into two different pencils (cf figure 3).

Quantum mechanics has to smooth these singularities and leads to the notion of singular diffraction. The exact solution for the scattering on wedge has been obtained long time ago by Sommerfeld [19] (cf also [20]). The simplest case of such diffraction corresponds to the scattering on a half-plane with, e.g., the Dirichlet boundary conditions, see figure 4(a).

The exact solution for this problem has been found by Sommerfeld in 1896 [19] and in polar coordinates it reads

\[
\Psi(r, \theta) = e^{-ikr \cos(\theta_2 - \theta_1)} \left( -\sqrt{2kr} \cos \frac{\theta_2 - \theta_1}{2} \right) - e^{-ikr \cos(\theta_2 + \theta_1)} \left( -\sqrt{2kr} \cos \frac{\theta_2 + \theta_1}{2} \right) \tag{5}
\]

where \( F(u) \) is the Fresnel integral

\[
F(u) = \frac{e^{iu^2/4}}{\sqrt{\pi}} \int_u^\infty e^{-t^2} \, dt. \tag{6}
\]

From the expansion of \( \Psi(\vec{r}) \) at large distances one finds that the total wave splits into two contributions, the incident plane wave and the out-going cylindrical wave

\[
\Psi(\vec{r}) = e^{ikr} + \frac{D(\theta_2, \theta_1)}{\sqrt{8\pi kr}} e^{ikr - 3\pi/4} \tag{7}
\]

where \( D(\theta_2, \theta_1) \) is the diffraction coefficient

\[
D(\theta_2, \theta_1) = \frac{1}{\cos \frac{\theta_2 - \theta_1}{2}} - \frac{1}{\cos \frac{\theta_2 + \theta_1}{2}}. \tag{8}
\]
Sommerfeld [19] also found the exact solution for the scattering on an arbitrary wedge with the Dirichlet boundary conditions as at figure 4(c). In this case the diffraction coefficient has the following form

$$D(\theta_f, \theta_i) = \frac{2 \sin \frac{\pi}{\gamma}}{\gamma} \left[ \frac{1}{\cos \frac{\gamma}{2} \cos \frac{\theta_f + \theta_i}{\gamma}} - \frac{1}{\cos \frac{\gamma}{2} \cos \frac{\theta_f - \theta_i}{\gamma}} \right]$$

where $\gamma = \alpha/\pi$ and $\alpha$ is the wedge angle.

The main feature of such diffraction coefficients is the existence of certain lines where diffraction coefficients formally blow-up. These lines are called optical boundaries and they correspond to zeros of the denominators in the above formulas. For the scattering on a half-plan they appear when

$$\theta_f = \pi \pm \theta_i,$$

Physically these lines separate regions with different numbers of geometrical rays and are a manifestation of the discontinuous character of classical ray motion, cf figure 4(b). As in quantum mechanics wave fields are continuous, the separation of the exact field into a sum of the free motion (plane wave) plus a small reflected field is not possible in a vicinity of such optical boundaries which forces the diffraction coefficient to diverge. Consequently, the diffractive coefficient description cannot be applied in parabolic regions near optical boundaries where the dimensionless arguments of the $F$-functions are of the order of 1, $u = \sqrt{k \sigma} \sin \frac{\Delta \varphi}{2} \approx 1$, and $\Delta \varphi$ is the angle of deviation from optical boundaries, cf figure 4(b).

Difficult problems appear when inside these intermediate regions there are other points of singular diffractions which is inevitable for plane polygonal billiards. For a finite number of singular diffraction vertices it is possible to develop the uniform approximations which give a good description of the multiple singular diffraction in the semiclassical limit $k \to \infty$ (see, e.g., [21] and references therein). For an infinite number of singular diffractions the situation is less clear. To understand the behaviour of waves scattered on many singular scatters where optical boundaries strongly overlap three interrelated approaches are discussed in section 2. All these methods demonstrate that the multiple singular diffraction in the semiclassical limit of high energy scattering leads to a non-perturbative effect of (almost) vanishing of eigenfunctions along straight lines passed through singular scatters (vertices with angles $\approx \pi/n$). Consequently, a wave scattered with a small incident angle from many singular scatters arranged along a line will be reflected from them as from a mirror with the Dirichlet boundary condition although the mirror itself does not exist. The importance of this phenomenon for polygonal billiards is related with the fact that periodic orbits in such billiards (when they exist) form families of parallel trajectories, cf figure 6(c). When unfolded each family constitutes an infinite pencil (or a channel) restricted from the both sides by singular scatters. Such configuration is exactly the one which permits the propagation of a plane wave with (approximately) the Dirichlet boundary conditions along two fictitious mirrors built by singular scatters. The validity of such approximation becomes better in the semiclassical limit of high energy. Therefore we propose to call these waves 'superscars' to distinguish them from the scar phenomena in chaotic systems [22–24] where individual scar amplitudes decrease in the semiclassical limit.

Numerous examples of numerically computed high-excited eigenfunctions with clear superscar structures for the triangular and the barrier billiards depicted at figure 1 are presented in section 3. An additional confirmation of the applicability of the superscar picture is the very good agreement of the true eigen-energies of such states with the superscar energies computed analytically from the knowledge of periodic orbit parameters.

To get a more quantitative information about the formation of superscar waves the overlaps between consecutive barrier billiard eigenfunctions and specific folded superscar waves are investigated in section 4. It is observed that in a small vicinity of all superscar energies there exist true eigenstates having large overlaps with the corresponding superscar wave. In a finite energy window the values of the overlap fluctuate according to the

![Figure 4](https://example.com/figure4.png)

**Figure 4.** (a) Scattering on a half-plane. (b) Optical boundaries (shade regions) for the scattering on a half-plane. (c) Scattering on a wedge with angle $\alpha$. 

where $\alpha$ is the wedge angle.
Breit-Wigner distribution whose parameters agree with the ones calculated analytically in section 2. Another useful approach discussed in the same section is the Fourier-type expansion method. It consists in the expansion of true eigenfunctions in a series of convenient basis functions. The existence of superscars manifests as anomalously large values of certain expansion coefficients.

If a periodic orbit family exists in a given polygonal billiard it may and will support superscar waves. But for generic polygonal billiards little is known about periodic orbits. Only for a special sub-class of pseudo-integrable billiards called Veech polygons one can find all periodic orbits analytically. Billiards considered in the paper belong to this class. For Veech polygons it is possible to calculate analytically the level compressibility which is practically the only one spectral characteristic accessible to analytical calculations. It is believed (and confirmed numerically for many different models, see e.g. [26]) that systems with non-trivial compressibility should have eigenfunctions with non-trivial fractal dimensions. For pseudo-integrable billiards the above mentioned strong fluctuations of Fourier coefficients mean that eigenfunctions in the momentum space may have fractal dimensions. In section 5 it is numerically demonstrated that indeed eigenfunctions of the barrier billiard do have non-trivial fractal dimensions. Section 6 contains a brief summary of obtained results. Appendix is devoted to the investigation of periodic orbit pencils in the barrier billiards and the folding of corresponding superscar waves.

2. Singular multiple diffraction

The purpose of this section is to present different approaches to the multiple singular scattering on a periodic array of singular vertices (wedges with angles $\frac{\pi}{n}$ with integer $n$) arranged along a straight line as indicated at figure 5. The simplest method consists in the construction of the Kirchhoff-type approximation to this problem. It has been done in [21] and briefly reviewed in section 2.1. It is known that the condition of applicability of the Kirchhoff approximation is not easy to be rigorously established. To get more precise information of this process, the exact solution for the scattering on staggered periodic set of half-planes as indicated at figure 6(a) derived by Carlson and Heins in 1947 [27] and analysed in the semiclassical limit in [28] is discussed in section 2.2. Section 2.3 is devoted to numerical investigation of wave propagation inside periodic array of slits depicted at figure 6(b).

The main result established in that sections is the fact that the small-angle high-energy multiple scattering on singular wedges is equivalent to the much simpler specular (i.e. mirror) reflection from a straight line passing through the wedge apexes though the line itself does not constitute a physical boundary. In section 2.4 it is demonstrated that this result applied to polygonal billiards proofs the existence of special weekly interacting quasi-modes corresponding to plane waves propagating inside periodic orbit channels (when they exist). These quasi-modes called in the paper superscars are a specific feature of polygonal billiards. They do not exist neither in integrable nor in chaotic systems and are a non-perturbative consequence of the multiple singular diffraction inherent for polygonal billiards.

2.1. The Kirchhoff approximation

A direct approach to multiple singular scattering consists in the construction of the uniform approximation based on the Kirchhoff approach (see e.g. [19]) which corresponds to the calculation of semiclassical
contributions of piecewise linear trajectories indicated at figure 5. In this approximation the role of wedges is reduced to the restriction of integration domains to half-lines \((x, 0)\), \((x, +\infty)\) (cf figure 5). This problem has been investigated in [21] where it has been proved that the contribution to the trace formula from such trajectories for a finite number (\(n\)) of wedges (i.e. \((n + 1)\)-fold integral over all \(x\)) at figure 5 can be calculated analytically and the result is

\[
I_{\text{diff}}(E) = -\frac{d}{16\pi k} A_n e^{i \frac{d}{n}} + \text{c.c.}, \quad A_n = \frac{1}{\pi} \sum_{q=1}^{n-1} \frac{1}{q(n-q)}.
\]  

For large numbers of scatters the sum over \(q\) can be substituted by the integral and [28]

\[
A_n \underset{n \to \infty}{\longrightarrow} 1 + \frac{2\zeta(1/2)}{\pi \sqrt{n}}
\]  

where \(\zeta(s)\) is the Riemann zeta function (\(\zeta(1/2) = -1.460354\)).

It has been demonstrated in [28] that this result for the multiple scattering on the periodic set of wedges is equivalent to the specular reflection (with the Dirichlet boundary condition) of the incident wave from a straight (fictitious) mirror which passes through the apexes of all wedges. For small incident angle \(\phi\) with respect to that (fictitious) mirror the effective reflection coefficient for high-energy scattering determined from equation (12) is the following

\[
R_0 = -1 - \sqrt{\frac{kd}{\pi}} (1 - i) \zeta(1/2) \phi, \quad k = \sqrt{E}.
\]  

Heuristically this result can be understood as follows. By construction, the exact wave function for the scattering on a system of wedges (with the Dirichlet boundary conditions) equals zero at wedge apexes. When many wedges are aligned and the incident angle is small the visible distances between apexes are also small and in the semiclassical limit this discrete set of points can be approximated by a straight line which explains the dominance of specular scattering with the Dirichlet condition.

### 2.2. Scattering on staggered periodic set of half-planes

Though the Kirchoff approximation discusses in the preceding section does indicate that the multiple singular diffraction leads to an effective scattering from a (fictitious) mirror formed by singular scatters it is difficult, in general, to prove rigorously the applicability of this approximation. In this section an exact solution of a similar problem of scattering of a plane wave with incident angle \(\phi\)

\[
\Psi^{(\text{inc})}(z, x) = e^{i \frac{d}{z} (\cos \phi - x \sin \phi)}
\]  

on a periodic set of half-planes separated by perpendicular distance \(a\) is discussed. The apexes of all half-planes belong to a straight line and planes are inclined with respect to this line by angle \(\alpha\) (cf figure 6(a)).

The field at large distances is the sum of the reflected (into the upper half-plane) and transmitted (into the lower half-plane) fields. The total reflected field is the sum of finite number of reflected plane waves

\[
\Psi^{(\text{ref})}(z, x) = \sum_n R_n e^{i \frac{k}{z} (\cos \phi_n + x \sin \phi_n)}
\]  

where \(R_n\) are the reflection coefficients and \(\phi_n\) are the reflected angles determined due to the periodicity from the grating equation

\[
k d (\cos \phi - \cos \phi_n) = 2\pi n, \quad n = \text{integer}, \quad -Q \sin^2 \frac{\phi}{2} \leq n \leq Q \cos^2 \frac{\phi}{2}.
\]
Here \( d \) is the distance between the apexes of half-planes, \( d = a / \cos \alpha \), and \( Q = kd / \pi \) is the dimensionless momentum.

The total transmitted field is the same as inside straight tubes built by half-planes with the Dirichlet boundary conditions

\[
\Psi^{\text{trans}} (z, x) = \sum_{n=1}^{\text{max}} T_n e^{-i \omega_n (x \sin \alpha + z \cos \alpha)} \sin \left( \frac{\pi m}{a} (x \cos \alpha - z \sin \alpha) \right), \quad m_{\text{max}} = \left\lfloor \frac{ka}{\pi} \right\rfloor
\]

where frequencies of transmitted waves \( \omega_n = \sqrt{k^2 - \left(\pi m / a\right)^2} \) and \( T_n \) are transmission coefficients.

It has been shown in [27] that the above problem is solvable by the Wiener-Hopf method but the calculations in that article were performed only for low values of momenta. In [28] this problem has been reconsidered in the semiclassical limit \( Q \to \infty \) and it was demonstrated that in that limit infinite products inherent in the Wiener-Hopf method and, consequently, reflection and transmission coefficients can be obtained analytically. The most difficult (and the most interesting for us) is the case of the small-angle scattering when the incident angle \( \varphi \to 0 \) as within the optical boundary of one scatter there exist many other scatters.

The main conclusions of [28] for this problem in the limit \( \sqrt{Q} \varphi \ll 1 \) and \( Q \to \infty \) are as follows:

- The reflection coefficient with \( n = 0 \) in equation (16) corresponding to the specular (mirror-like) reflection, \( \varphi_0 = \varphi \), is special

\[
R_0 = -1 - \sqrt{Q} (1 - i) \zeta (1/2) \varphi.
\]  

Notice that this expression coincides with equation (13) obtained in the Kirchhoff approximation.

- Reflection coefficients when \( n > 0 \) in equation (16) is kept fixed and \( Q \to \infty \) corresponding to small reflection angle, \( \varphi_0 \approx 2 \sqrt{n / Q} \) (independent on incident angle \( \varphi \)) provided \( \sqrt{Q} \varphi \ll 1 \) are small and proportional to \( \sqrt{Q} \varphi \)

\[
R_n^{\text{small}} = \sqrt{Q} \varphi \, r_n, \quad |r_n|^2 = \frac{\epsilon^{2-\pi \zeta (1/2) - 2}}{n} \prod_{m=1}^{\infty} \frac{1 + \sqrt{n/m}}{1 - \sqrt{n/m}} \epsilon^{-2 \sqrt{n/m}}.
\]  

- When \( \pi / 2 < \alpha < \pi \) the transmission is negligible and the large-angle reflection coefficients dominate when \( n \) is close to \( n^* = Q \sin^2 \alpha \) and \( \varphi_0 \) is nearly to \( 2 \pi - 2 \alpha \) (as for the specular reflection from the full inclined plane). For small \( \varphi \) these coefficients are proportional to \( \varphi \)

\[
R_n^{\text{large}} = \frac{\varphi}{\sin 2 \alpha} r(u_n(\alpha)), \quad |r(u)|^2 = \epsilon^{2(1/2) n} \prod_{m=1}^{\infty} \left( 1 + \frac{u}{\sqrt{m}} \right)^2 \left( 1 + \frac{u^2}{m} \right) \epsilon^{-2u/\sqrt{m}}
\]  

where

\[
u_n(\alpha) = \frac{n - n^*}{\sqrt{Q} \sin 2 \alpha}.
\]  

- When \( 0 < \alpha < \pi / 2 \) the large-angle reflection coefficients are negligible and the transmission coefficients are

\[
T_n = \varphi \, t(u_n(\pi - \alpha)), \quad |t(u_n(\pi - \alpha))|^2 = 2 |r(u_n(\pi - \alpha))|^2
\]  

with the same functions \( r(u) \) and \( u_n(\alpha) \) as in (20) and (21).

The main conclusion from the above expressions is that for the sliding-type multiple scattering when the incident angle is small, \( \sqrt{Q} \varphi \ll 1 \), and \( Q \to \infty \) the dominant contribution to the reflected field comes only from one term with \( n = 0 \) (\( R_0 \approx -1 \) and \( \varphi_0 = \varphi \)). This term corresponds to the specular reflection from a fictitious mirror built from a straight line passing through singular scatters (indicated by dashed lines at figure 6)

\[
\Psi(z, x) \approx \epsilon^{ikz \cos \varphi} \left[ e^{-i kx \sin \varphi} - e^{i kx \sin \varphi} \right] + \delta \Psi(z, x)
\]  

where \( \delta \Psi(z, x) \) is small when \( \varphi \sqrt{Q} \ll 1 \). For \( \pi / 2 < \alpha < \pi \)

\[
\delta \Psi(z, x) = \varphi \sqrt{Q} \left\{ \sum_{n=0}^{\infty} r_n e^{ikz \cos \varphi_0 + i kx \sin \varphi_0} + \frac{\varphi}{\sin 2 \alpha} \sum_{u_n} r(u_n) e^{ikz \cos \varphi_0 + i kx \sin \varphi_0} \right\}
\]  

The formation of a quasi-mirror boundary where in the semiclassical limit the total field tends to zero is a non-perturbative effect of the small-angle multiple scattering on singular scatters (i.e. vertices with angle \( \approx \pi / n \)).
The existence of the exact solution permits also to find a small leakage of the specular reflected wave after one scattering into other channels. The modulus of the amplitude of that wave deviates from unity by a small amount (when $\sqrt{Q} \varphi \ll 1$) as follows

$$|R_0|^2 = 1 - C\sqrt{d} \varphi, \quad C = \frac{2 \zeta(1/2)}{\sqrt{\pi}} \approx 1.65. \quad (25)$$

2.3. Scattering on a periodic array of slits

To investigate this phenomenon further it is instructive to investigate the propagation of waves inside periodic array of slits with Dirichlet boundary conditions as indicated at figure 3(b). The problem corresponds to finding the solution of the Helmholtz equation $(\Delta + k^2)\Phi(z, x) = 0$ which vanishes at indicated slits and is generated by the plane wave along $z$-axis.

In the Kirchhoff approximation (see e.g. [19]) the wave $\Phi(d, x)$ at distance $d$ from the origin is related with the wave $\Phi(0, x')$ by the following relation valid provided the width $w$ is much smaller than the distance between slits $d$

$$\Phi(d, x) = e^{i(kd - zw/4)} \int_{-\pi/2}^{\pi/2} e^{ik(x - x')^2/2d} \Phi(0, x')dx'. \quad (26)$$

Periodicity of the slits requires that $\Phi(d, x) = \lambda \Phi(0, x)$ where $\lambda$ determines the propagation and attenuation due to scattering on slits. Therefore the considered problem is reduced to the following equation

$$e^{i(kd - zw/4)} \int_{-\pi/2}^{\pi/2} e^{ik(x - x')^2/2d} \Psi(y)dy = \lambda \Psi(x). \quad (27)$$

Here $\Psi(x) \equiv \Phi(0, x)$ is the value of the wave inside a slit.

No analytical solution of this (simplified) equation is known. Nevertheless, as it has been discussed above, in the semiclassical limit $k \to \infty$ its solution should be close to waves propagating inside a rectangular slab restricted by straight lines passing through corners of the slits (denoted by dashed lines at figure 6(b))

$$\Psi_n^{(approx)}(z, x) \sim \sin(p_n(x + w/2))e^{i(kz - p_n^2)(w/2)}, \quad p_n = \frac{\pi n}{w}, \quad n = 1, 2, \ldots \quad (28)$$

To check this statement the numerical calculation of equation (27) was performed. To simplify numerics all space variables were rescaled in units of $w/2$ and the Wick rotation has been performed. It leads to a simpler equation

$$\int_{-1}^{1} e^{-ik(x - y)^2/2d} \Psi(y)dy = \Lambda_n \Psi_n(x) \quad (29)$$

where the dimensionless variable $\kappa = -ikw^2/8d$ and

$$\Lambda_n = e^{i(\pi^2 - p_n^2 - k^2)d}. \quad (30)$$

Eq. (29) is the Fredholm integral equation of the first kind with symmetric kernel and it has a discrete set of eigenvalues $\Lambda_n$ ($\Lambda_1 \geq \Lambda_2 \geq \ldots$) and eigenfunctions $\Psi_n(x)$ which were determined numerically. Due to the symmetry solutions are either even or odd with respect to coordinate inversion: $\Psi_n(-x) = (-1)^n + i \Psi_n(x)$. At figure 7 ten largest eigenvalues of this equation are represented for different values of parameter $\kappa$. At figure 8 a few corresponding eigenfunctions are plotted for $\kappa = 200$.

The main conclusion of these (and others) calculations is that in the semiclassical limit $\kappa \to \infty$ eigenfunctions of (29) are indeed well described by the simple waves as in equation (28). An important characteristics of such waves is the requirement that they vanish as $\kappa \to \infty$ at boundaries of effective slab which implies the quantisation of transverse momentum. At figure 8(b) the values of true eigenfunctions at the boundary, $\Psi_n(1)$, are plotted. It is clearly seen that with fixed $p_n$ and with increasing of $\kappa$ this value indeed tends to zero. It was observed that these values are well described by the following asymptotic formula

$$\Psi_n(1) \to \sim \left| \tilde{p}_n \right| \left( \frac{1}{2 \sqrt{\kappa}} - \frac{1}{8 \kappa} \right) \quad (31)$$

where the dimensionless transverse momentum $\tilde{p}_n = \pi n/2$.

Similar asymptotic formulas are also established for eigenvalues $\Lambda_n$ of equation (29)

$$\Lambda_n \to \sim 1 - \frac{\tilde{p}_n^2}{4\kappa} + .206 \frac{\tilde{p}_n^2}{\kappa^{3/2}}. \quad (32)$$

At figure 7 the comparison of this formula with numerical calculations is performed and good agreement have been found for large $\kappa$ and a fixed momenta $\tilde{p}_n$. 
The first term of the series (32) corresponds to the large \( k \) expansion of equation (30). The second term can be interpreted as a complex shift of wave energy when propagating inside the slits. Expanding longitudinal momenta \( \sqrt{k^2 + \delta E - p_n^2} \) in the exponent of equation (30) and comparing coefficients with (32) one gets (in the original units)

\[
\delta E = C(1 + i)p_n^2 \sqrt{\frac{d}{kw^2}}, \quad p_n = \frac{\pi n}{w}
\]  

(33)

where the constant \( C \approx 1.65 \) is numerically the same as in equation (25), \( C = -2\zeta(1/2)/\sqrt{\pi} \). The equality of these two constants can be explained as follows. The appearance of the imaginary part of the wave energy physically means that propagating wave escapes into other channels. The modulus squared of this wave after passing the distance \( L \) decreases by \( \Im \delta E L/k \). According to equation (25) after each collision with slits this quantity decreases by \( C \sqrt{k \delta E} \varphi \). When a wave propagates with the incident angle \( \varphi \) along the distance \( L \) it has approximately \( L/(w/\varphi) \) collisions. Therefore the total leakage is

\[
\frac{L}{k} \Im \delta E = C \sqrt{k \delta E} \varphi^2 \frac{L}{w}
\]  

(34)

As \( \varphi \approx p_n/k \) one reproduces the imaginary part of equation (33).
2.4. Application to polygonal billiards

The multiple scattering on singular wedges with $\alpha \neq \pi/n$ is, in general, a complicated problem, especially when optical boundaries of different scatters overlap. The above discussion proves that in the semiclassical limit when singular scatters are arranged along a straight line and the incident wave is inclined with a small angle with respect to this line the reflected wave dominates by a specular reflection from that line though the line itself does not constitute a physical boundary. The Kirchhoff approximation discussed in section 2.1 clearly demonstrates that this result is independent of wedge shapes.

Such non-perturbative effect is especially important for polygonal billiards where classical periodic orbits appear in families which after unfolding form infinite periodic pencils (or channels) limited from the both sides by singular vertices, cf figure 2. Consider one pencil corresponding to a primitive periodic orbit with period $L_p$ and let $w$ be its width, see figure 6(c). Of course, the horizontal pencil boundaries do not exist but they are constituted by singular scatters. Due to multiple singular diffraction a wave propagating inside such pencil approximately vanishes at effective horizontal boundaries and therefore will take the form of a plane wave as in equation (28)

$$\Psi(z, x) \sim \sin \left( p(x + w/2) \right) e^{iqz}, \quad p = \frac{n}{w}, \quad n = 1, 2, \ldots$$

(35)

where due to periodicity, $\Psi(z + L_p, x) = \pm \Psi(z, x)$, longitudinal momenta $q$ is also quantised

$$q = \frac{n}{L_p}, \quad m = \text{integer}.$$  

(36)

The energy of such wave is

$$E_{m,n} = \frac{\pi^2 m^2}{L_p} + \frac{\pi^2 n^2}{w^2}.$$  

(37)

It is plain that such wave is only an approximation to (a much more complicated) exact solution. The validity of this approximation is governed by the dimensionless perturbation parameter $\lambda = \varphi \sqrt{kL_p/\pi}$ where $\varphi$ is the angle between the wave direction and the horizontal boundaries. For the plane wave (35) with a small transverse momentum $\varphi \approx p/k$, $p = \pi n/w$, and $k \approx q = \pi m/L_p$. Therefore the wave (35) will be a good approximation provided the following dimensionless parameter is small (the smaller the better)

$$\lambda \approx \frac{nL_p}{\sqrt{\pi w}},$$  

(38)

The values of integer $n$ are also restricted

$$1 \leq n \leq \sqrt{\frac{k}{\pi L_p}}.$$  

(39)

The requirement that $n \geq 1$ leads to the conclusion that at fixed energy not all periodic orbit pencils can support propagating waves. As $w L_p = \gamma A$ where $A$ is the billiard area and $\gamma = O(1)$ is the fraction of the billiard area covered by a given family of periodic trajectories, the length of a propagating channel is restricted as follows

$$L_p \leq L_{\max} = \delta k^{1/3}, \quad \delta = (A\gamma/\sqrt{\pi})^{1/3}.$$  

(40)

Long-period channels with $L_p > L_{\max}$ are closed and cannot support propagating waves.

An important property of discussed propagating waves is that they become more visible (i.e., more isolated from other states) when the parameter (38) is decreasing. But for a given periodic orbit (i.e., fixed $L_p$ and $w$) when transverse momentum $p$ (i.e., $n$) is kept fixed but energy (i.e., $m$) increases this parameter goes to zero. Consequently in the semiclassical limit any periodic orbit pencil may and will support such propagating quasi-modes. That phenomenon resembles the formation of scars around of unstable periodic orbits in chaotic systems [22, 23, 29] but contrary to the usual scars the discussed quasi-modes become practically exact in the semiclassical limit. It explains the name superscars proposed for these quasi-modes. In the next section many examples of such superscars are presented for the billiards depicted at figure 1.

The problem considered in this section looks similar to the bouncing ball scattering in the stadium billiard (see, e.g., [30]) but the above arguments cannot be applied to the bouncing ball case. The discussed non-perturbative superscar formation is based on the existence of strong diffraction on singular vertices, i.e., points where classical ray dynamics is discontinuous. But the tangent to the stadium boundary is continuous along the boundary. Therefore classical ray dynamics in the stadium is also continuous and singular vertices are absent. The bouncing ball pencil in the stadium billiard is restricted by vertices with the boundary discontinuity only in the second derivative which generate merely a conventional diffraction whose analysis is beyond the scope of the paper.
3. Examples of superscars in triangular and barrier billiards

Consider the billiard in the shape of the right triangle with one angle equals $\pi/8$ as at figure 1(a). One of the simplest periodic orbit family of this billiard corresponds to trajectories perpendicular to the both catheti as indicated at figure 2(b). When unfolded this family fills the rectangular pencil shown at figure 2(c). The length of this rectangle (i.e., the periodic orbit length) equals twice the length of the largest cathetus and its width is the length of the smallest cathetus. According to the above discussed multiple scattering on singular points the superscar wave should propagate inside this rectangle with the Dirichlet boundary conditions on horizontal boundaries. As vertical boundaries are a part of the triangle boundaries the wave have to vanish on these boundaries as well. Taking into account symmetry of the problem one concludes that the unfolded superscar wave in the semiclassical limit obeys the Dirichlet boundary conditions on all sides of the rectangle indicated at figure 9 and has the form

$$
\Psi_{m,n}(z, x) = \frac{2}{\sqrt{ab}} \sin \left(\frac{\pi}{a} mz\right) \sin \left(\frac{\pi}{b} nz\right) \Theta(z) \Theta(b - z)
$$

(41)

where $a, b$ are lengths of respectively the largest and the smallest cathetus ($b = \tan(\pi/8)a$). Two Heaviside $\Theta$-functions ($\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$) are introduced to stress that this expression exists only inside the rectangle. The energy of such state is

$$
\mathcal{E}_{m,n} = \frac{\pi^2}{a^2} m^2 + \frac{\pi^2}{b^2} n^2.
$$

(42)

According to the previous section such superscar approximation is valid provided that the perturbation parameter $\lambda_p$ (with $L_p = a$ and $w = b$) is restricted. To emphasise the importance of this parameter, its value for the corresponding superscar wave is indicated in the captions of eigenfunction figures below.

The superscar wave is simple (cf equation (41)) only after unfolding. When folding inside the original triangle it takes the following form

$$
\Psi^{(\text{superscar})}(z, x) = \Psi_{m,n}(z, x) - \Psi_{m,n} \left( \frac{z + x}{\sqrt{2}}, \frac{z - x}{\sqrt{2}} \right) + \Psi_{m,n} \left( \frac{z - x}{\sqrt{2}}, \frac{z + x}{\sqrt{2}} \right) - \Psi_{m,n}(x, z)
$$

(43)

where $\Psi_{m,n}(z, x)$ is given by equation (41) (with $\Theta$-functions included).

To examine the correspondence between (approximate) superscar waves and the true quantum eigenfunctions numerical calculations of high-excited states were performed. The area of the billiard is normalised to $4\pi$ in order that the mean distance between consecutive high-energy levels equals 1. To find what numerically calculated (true) eigenfunctions resemble to superscar waves the following procedure has been used. First, values of integers $m$ and $n$ with $n \ll m$ were chosen and the superscar energy was calculated from equation (42). Then from numerically calculated eigenvalues the one closest to the superscar energy has been plotted. Notice the characteristic picture of propagating wave fronts. At figure 10(a) the folded superscar wave (43) with $m = 50$ and $n = 1$ is plotted. Notice the characteristic picture of propagating wave fronts. At figure 10(b) the exact eigenfunction with energy $E_{\text{exact}} = 407.4$ which differs from the superscar energy by 0.2 is shown. Though the energy is not too big this eigenfunction resembles well the superscar wave. At figure 11 and 12 the exact eigenfunctions corresponding to superscar waves with $(m,n)$ equal respectively $(79, 1), (110, 1), (148, 1),$ and $(201, 1)$ are presented. These eigenfunctions clearly have the same structure that the superscar waves and the indicated exact energies agree well with superstar energies calculated from equation (42). Figure 10–12 clearly validate the formation of superscar states around the simplest periodic orbit pencil in the triangular billiard. But even that orbit requires 5 scatterings from the boundary (cf figure 2(b)) and the folded superscar function is
complicated (cf figure 10(a)). Longer periodic orbit pencils will necessarily be more elaborate and, consequently, the structure of corresponding superscar functions would be less clear.

To visualise better superscar structures, it is convenient to investigate the barrier billiard as at figure 1(b) where short-period orbits are simpler (see below). In numerical calculations only symmetric modes of this billiard were considered. Now the problem is reduced to the solving the Helmholtz equation

\[ (\Delta + k^2)\Psi(x, y) = 0 \]

where \( \Psi(x, y) = 0 \) at all boundaries of the desymmetrised rectangle indicated at figure 1(c) except the segment \( x = 0, b/2 < y < b \) where \( \partial\Psi(x, y) / \partial x = 0 \).

In calculations the aspect ratio of the barrier billiard, \( b/a \), is chosen equal to \( \sqrt{5} + 1 \approx 1.8 \) and the area of the billiard is normalised to \( 4\pi \). A bunch of high excited eigenfunctions around the 10000th level for this billiard was obtained numerically and eigenfunctions corresponding to a few superscar waves were selected as it has been discussed above. For clarity at certain figures below nodal domains of these eigenfunctions were plotted. Black (white) regions correspond to points where \( \Psi(y, x) > 0 \) and \( \Psi(y, x) < 0 \) respectively. At other figures it was more convenient to show grey images of the eigenfunction modulus.

Figure 10. (a) The folded superscar function given by equation (43) for \( m = 50 \) and \( n = 1 \) with energy \( E_{50,1} = 407.6 \). (b) The numerically calculated eigenfunction with energy \( E_{\text{exact}} = 407.4 \). The perturbation parameter (38) \( \lambda = 0.34 \). Reprinted with permission from [1]. Copyright (2004) by the American Physical Society.

Figure 11. (a) The same as at figure 10(b) but with energy \( E_{\text{exact}} = 1015.9 \). The corresponding superscar energy \( E_{50,1} = 1016.12 \). The perturbation parameter (38) \( \lambda = 0.27 \). Reprinted with permission from [1]. (b) The same but with energy \( E_{\text{exact}} = 1968.97 \). The superscar energy \( E_{50,1} = 1969.15 \). The perturbation parameter (38) \( \lambda = 0.23 \). Reprinted with permission from [1], Copyright (2004) by the American Physical Society.

Figure 12. (a) The same as at figure 10(b) but with energy \( E_{\text{exact}} = 3563.91 \). The superscar energy \( E_{50,1} = 3563.88 \). The perturbation parameter (38) \( \lambda = 0.20 \). (b) The same but with energy \( E_{\text{exact}} = 6572.47 \). The superscar energy \( E_{50,1} = 6572.63 \). The perturbation parameter (38) \( \lambda = 0.17 \).
The structure of periodic orbit pencils in the barrier billiard is discussed in detail in Appendix. Any primitive periodic orbit in such billiard is characterised by 2 co-prime integers \( n_a \) and \( n_b \) which count the shifts by \( 2a \) and \( 2b \) in horizontal and vertical directions respectively on the unfolded rectangular lattice. Below such orbit is denoted by \((n_a - n_b)\). The length of such orbit is \( L_p = \sqrt{(2an_a)^2 + (2bn_b)^2} \). In the barrier billiard periodic orbit pencil with even \( n_a \) has the width \( w = 4ab/L_p \) and such pencil fills the whole rectangle. For odd \( n_a \) there exit two different pencils of width \( w = 2ab/L_p \). Both pencils may support superscar waves but the one with odd longitudinal quantum number \( m \) and the other one with even \( m \). The difference is due to different phases of reflection on boundaries with the Dirichlet boundary conditions.

One of the simplest periodic orbit of the barrier billiard corresponds to the horizontal motion inside the rectangle (i.e., the \((0 - 1)\) periodic orbit). The superscar wave associated with this motion should have the form (the axes are indicated at figure 1(c))

\[
\psi^{(\text{superscar})}(y, x) = \frac{2}{\sqrt{ab}} \sin\left(\frac{\pi}{b} y\right) \sin\left(\frac{\pi}{a} x\right).
\]

(44)

When \( n = 1 \) it has vertical wavefronts as at figure 13(a). For larger \( n \) the wavefronts have same form but with additional \( n - 1 \) equidistant horizontal lines where the function vanishes.

The energy of such horizontal bouncing ball is \((1 \ll n \ll m)\)

\[
E_{m,n}^{(0-1)} = \frac{\pi^2}{a^2} m^2 + \frac{\pi^2}{b^2} n^2.
\]

(45)

At figure 13(b), 14(a) and 14(b) eigenfunctions related with such superscar waves with \( m = 152 \) and \( n=1,2 \) with \( m = 153 \) and \( n = 3 \) are shown. The superscar structures are clear visible on these figures. The superscar energies (obtained from equation (45)) indicated in figure captions are also very close to numerically calculated energies for that states.

Another simple periodic orbit of the barrier billiard is the \((1 - 0)\) orbit i.e. the vertical bouncing ball. There are two types of such orbits. The first is related with the motion between two Dirichlet boundaries \((0 < y < b/2)\) and the second is associated with the motion between the Dirichlet and Neumann boundaries \((b/2 < y < b)\), cf figure 15.

The superscar waves propagating inside these two pencils are

\[
\psi^{(\text{superscar})}_{\text{DD}}(y, x) = \frac{2}{\sqrt{ab}} \sin\left(\frac{\pi}{a} m x\right) \sin\left(\frac{2\pi}{b} n y\right) \Theta(b/2 - y),
\]

(46)

\[
\psi^{(\text{superscar})}_{\text{DN}}(y, x) = \frac{2}{\sqrt{ab}} \cos\left(\frac{\pi}{a} (m - 1/2) x\right) \sin\left(\frac{2\pi}{b} n y\right) \Theta(y - b/2)
\]

(47)

and their energies are as follows \((1 \ll n \ll m)\)

\[
E_{m,n}^{\text{DD}} = \frac{\pi^2}{a^2} m^2 + \frac{4\pi^2}{b^2} n^2,
\]

(48)

\[
E_{m,n}^{\text{DN}} = \frac{\pi^2}{a^2} (m - 1/2)^2 + \frac{4\pi^2}{b^2} n^2.
\]

(49)

At figure 16(a) the nodal domain of a numerically calculated eigenfunction of the barrier billiard with energy \( E_{\text{exact}} = 10209.55 \) is presented. Its structure consists of two different parts. The one corresponds to regular waves.
propagating between the left part of the billiard as it should be for the Dirichlet-Dirichlet vertical bouncing ball (cf figure 15(a) and (46)). The second part is built from irregular waves with much smaller amplitudes. If superscar picture would be exact, this part should be exactly zero but as the superscar wave is only an approximation such regions have to be constituted of small-amplitude waves with irregular nodal domains. Such co-existence of two different parts of eigenfunctions is typical for superscar waves propagating in pencils with odd $n_a$ (see below). The calculated energy of the corresponding superscar with $m = 85$ and $n = 1$ $E_{DD}^{85,1} = 10209.65$ is very close to the exact energy.

With the increasing of the perpendicular momentum $n$ superscar waves become less pronounced as the parameter (38) which controls the validity of superscar approximation grows. Nevertheless the vertical bouncing ball structure remains visible even for $n = 6$ and $m = 84$ as shown at figure 16(b). Notice that the
second part of the this picture is not irregular as at figure 16(a) but contains a (deformed) wave corresponding to the Dirichlet-Neumann vertical bouncing ball structure. It can be explained by the fact that \( E_{\text{exact}} = 10089.70 \). The superscar energy \( E_{\text{DN}}^{(1)} = 10089.91 \). The perturbation parameter \( \lambda = 0.12 \).

At figure 17 and 18 a few images of eigenfunctions with clear structure of the Dirichlet-Neumann vertical bouncing balls are presented. The corresponding superscar waves correspond to \( m = 85 \) and \( n = 1, 2, 3, 8 \) with energies (noted in the figure captions) very close to the exact energies.

The horizontal and vertical bouncing balls are the only periodic motions inside the barrier billiard which do not require the folding of periodic orbits pencils. Other orbits are more complicated and should be folded inside the billiard. The simplest of such orbits is the \((1 - 1)\) orbit indicated at figure 19(a) with the length \( L_p = \sqrt{(2a)^2 + (2b)^2} \). When folding back to the original barrier billiard this orbit gives rise to two periodic orbit pencils as shown at figure 19(b). In the usual rectangular billiard these two pencils may be continuously transformed one into another but in the barrier billiard they are restricted by singular vertices and constitute two different pencils which should be treated separately. The superscar waves propagating in the pencils have energies given by the expression

\[
E_{m,n}^{(1-1)} = \frac{\pi^2 m^2}{L_p^2} + \frac{\pi^2 n^2}{w^2}, \quad L_p = \sqrt{(2a)^2 + (2b)^2}, \quad w = \frac{2ab}{L_p}. \tag{50}
\]

The existence of different pencils manifests in different phases accumulated by a wave when propagating inside the pencils. It is plain that even and odd \( m \) correspond to the pencils indicated at figure 19(b).

At figure 20 and 21 certain examples of superscar waves associated with the \((1 - 1)\) orbit are presented. The first figure corresponds to \( m = 348 \) and \( m = 347 \) with \( n = 1 \) and the second one shows eigenfunctions with the same values of \( m \) but \( n = 2 \). The effect of switching from one pencil to another for even and odd \( m \) is clear visible.

Figure 17. Nodal domains of numerically calculated eigenfunctions of the barrier billiard. (a) with \( E_{\text{exact}} = 10089.70 \). The superscar energy \( E_{\text{DN}}^{(1)} = 10089.91 \). The perturbation parameter \( \lambda = 0.12 \). (b) with \( E_{\text{exact}} = 10094.44 \). The superscar energy \( E_{\text{DN}}^{(1)} = 10095.15 \). The perturbation parameter \( \lambda = 0.24 \).

Figure 18. (a) The same as at figure 17 but with \( E_{\text{exact}} = 10102.03 \). The superscar energy \( E_{\text{DN}}^{(1)} = 10103.88 \). The perturbation parameter \( \lambda = 0.36 \). (b) Grey image of the modulus of the eigenfunction with \( E_{\text{exact}} = 10192.82 \). The superscar energy \( E_{\text{DN}}^{(1)} = 10199.93 \). The perturbation parameter \( \lambda = 0.96 \).
At figure 22 the unfolded and folded \((2-1)\) orbit is plotted. The pencil of this orbit has the length 
\[ L_p = \sqrt{(4a)^2 + (2b)^2} \] and the width 
\[ w = 4ab/L_p. \] When folded it covers the whole barrier billiard surface. The superscar energies for such orbit are
At figure 23 and 24(a) eigenfunctions corresponding to superscar waves with \( n = 1 \) and \( m = 227, 228, 229 \) are represented. Notice exceptionally regular shape of nodal domains of these high-excited eigenfunctions and excellent agreement of exact eigenvalues with superscar energies (51).
The next example corresponds to the \((3 - 1)\)-orbit, see figure 25. In this case there are two symmetric channels with different parity of the longitudinal quantum number \(m\). The length of each of such channels is 
\[ L_p = \frac{\pi^2 m^2}{L_p^2} + \frac{\pi^2 n^2}{w^2}, \]
and their width is 
\[ w = \frac{2ab}{L_p}. \]

The superscar waves propagating in these channels have the energy equal to 
\[ \mathcal{E}^{(3-1)}_{m,n} = \frac{\pi^2 m^2}{L_p^2} + \frac{\pi^2 n^2}{w^2}. \]

At figure 26 two examples of eigenfunctions corresponding to odd and even \(m\) are presented. The structure of propagating superscar waves is clearly visible and the exact energies are very close to the superscar ones.

The last example is the \((3 - 2)\) orbit corresponding to a superscar wave with odd \(m\) and its wavefronts. The second pencil with even \(m\) occupies the complimentary part of the billiard.

4. Quantitative characteristics of superscars

Numerous pictures of the superscar waves formation were presented in the previous section. But such pictures are useful only to illustrate a few superscar waves associated with short-period orbit pencils. To get a quantitative information about the whole structure of eigenfunctions in plane polygonal billiards it is convenient to calculate numerically the overlap between an exact eigenfunction with energy \(E\) and a superscar wave propagated in a fixed periodic orbit pencil

\[ C_{m,n}(E) = \int \Psi_{m,n}(x, y) \Psi_{E}(x, y) \, dx \, dy. \]

Here \(m\) and \(n\) are integers corresponding to longitudinal and transverse quantum numbers of the superscar wave and the integration is performed over the whole billiard surface. Both functions in this equation are assumed to be normalised so 
\[ 0 \leq |C_{m,n}(E)| \leq 1. \]
When the exact energy differs considerably from the superscar energy this overlap should be small. It means that for fixed \(m\) only one peak appears when \(E \approx \mathcal{E}_{m,n}^{(3-1)}\).
In calculations the transverse quantum number \( n \) (which exists only due to the singular diffraction) is kept fixed but the longitudinal quantum number \( m \) (denoted below by \( m(E) \)) has been adjusted for different energies \( E \) in such a way that the energy difference \( |E - E_{m,n}| \) is minimal

\[
m(E) = \left\lceil \frac{L_p}{\pi} \sqrt{E - \frac{\pi^2 n^2}{w^2}} \right\rceil
\]

where \( \lceil x \rceil \) denotes the integer closest to \( x \).

A technical difficulty in this approach is the calculation of the folded superscar wave, \( \psi_{\text{superscar}}(\chi, y) \). The superscar wave is simple when it is unfolded. Due to the folding back of periodic orbit pencils, superscar waves inside the original billiard become complicated. For simple orbits the folded wave can be directly calculated as it has been done in equation (43). In Appendix it is shown how to calculate folded superscar function associated with an arbitrary periodic orbit pencil.

The overlaps between all eigenstates in the interval \( 2000 < E \leq 4000 \) and all superscar waves propagating inside the \((0 - 1)\) pencil (horizontal bouncing ball), the \((1 - 0)\) pencil (left vertical bouncing ball), and the \((1 - 1)\) pencil are presented at figure 28(a)–30(a). Each of these figures shows the overlap for the four lowest transverse quantum numbers, \( n = 1, \ldots, 4 \).

Every time when the energy of a true eigenstate is close to the superscar energy the corresponding eigenfunction has a considerable overlap with the superscar wave. As expected, small \( \delta E \) leads to higher values of the overlap. To analyse quantitatively the structure of overlap peaks it is instructive to calculate their local density for each fixed \( n \) defined as follows (\( \delta E \) is the difference between the true energy \( E_0 \) and the best superscar energy \( E_{m(E_0), n} \))

\[
\rho_n(\delta E) = \left\langle \sum_{\lambda} |C_{m,n}(E_0)|^2 \delta(\delta E - E_\lambda + E_{m,n}) \right\rangle
\]

where the averaging is taken over all peaks in a given energy interval \([E - e, E + e]\) where \( e \ll E \). For \( n = 1, 2, 3, 4 \) this local density is plotted at figure 28(b)–30(b).

As has been discussed above superscar waves can be considered as long-lived states which interact weekly due to residual interactions governed by parameter (38). From general considerations [31–35], it is expected that in such situation the local density should be well approximated by the Breit-Wigner distribution

\[
\rho_n(\delta E) \approx \frac{\Gamma_n(E)}{2\pi[(\delta E - \epsilon_n(E))^2 + \Gamma_n^2(E)]/4}
\]

where \( \epsilon_n(E) \) and \( \Gamma_n(E) \) are certain parameters (depending on the energy interval) which, in principle, could be calculated from perturbation series. In figure 28(b)–30(b) it is demonstrated that such fits indeed approximate well the local densities for all considered cases. In the previous section it was argued that the width \( \Gamma_n(E) \) asymptotically should have the form indicated in equation (33)

\[
\Gamma(E) = C \frac{\pi n^2}{w^2} \sqrt{\frac{d}{kw^2}}, \quad C = -\frac{2\zeta(1/2)}{\sqrt{\pi}} \approx 1.65.
\]

Here \( w \) is the width of a periodic orbit pencil and \( d \) is the distance between singular vertices along the pencil boundary.
Numerical fits confirm this estimation. For example, for the \((1 - 1)\) orbit the data on figure 30(b) are fitted well by expression \(\Gamma_n(E) \approx 3.5n^2/\sqrt{k}\). When calculating analytically from equation (57) one gets \(\Gamma_n(E) = 3.52n^2/\sqrt{k}\).

Another method to obtain quantitative measure of the superscar phenomenon in plane polygonal billiards consists in the Fourier-type expansion of eigenfunctions. For the full barrier billiard as at figure 1(b) with the Dirichlet boundary conditions along the rectangle \((a, b)\) it is natural to represent eigenfunctions in the following basis

\[
\Psi(x, y) = \sum_{q, k=1}^{\infty} F_{q,k} \sin \left( \frac{\pi q(x + a)}{2a} \right) \sin \left( \frac{\pi k y}{b} \right) \tag{58}
\]

where \(q\) and \(k\) are integers. As the basis trigonometric functions in the right-hand part of this expansion are orthogonal inside the rectangle \((2a, b)\) the expansion coefficients \(F_{q,k}\) can be calculated by the inverse Fourier transform.
\[
F_{p,k} = \frac{2}{ab} \int_0^{2a} dx \int_0^b dy \Psi(x, y) \sin \left( \frac{\pi q(x + a)}{2a} \right) \sin \left( \frac{\pi ky}{b} \right).
\]

For the desymmetrised barrier billiard as at figure 1(c) no preferential system of expansion exists. Expansion (58) inside the desymmetrised barrier billiard gives rise to two different series depended on the parity of \(k\). For odd \(p = 2q - 1\) the function \(\Psi(x, y)\) is even, \(\Psi(-x, y) = \Psi(x, y)\)

\[
\Psi_{\text{even}}(x, y) = \sum_{q,k=1}^{\infty} f_{p,k} \cos \left( \frac{\pi q - 1/2}{a} \right) \sin \left( \frac{\pi ky}{b} \right)
\]

and for even \(p = 2q\) the function \(\Psi(x, y)\) is odd, \(\Psi(-x, y) = -\Psi(x, y)\)

\[
\Psi_{\text{odd}}(x, y) = \sum_{q,k=1}^{\infty} g_{q,k} \sin \left( \frac{\pi qx}{a} \right) \sin \left( \frac{\pi ky}{b} \right).
\]

Formally the both series can be used on equal footing as inside the desymmetrised barrier billiard these two expansions are equivalent because

\[
\sin \left( \frac{\pi mx}{a} \right) = \sum_{n=1}^{\infty} A_{mn} \cos \left( \frac{\pi (n - 1/2)x}{a} \right)
\]

where matrix \(A_{mn}\) is an orthogonal matrix

\[
A_{mn} = \frac{1}{\pi} \left( \frac{1}{m + n - 1/2} + \frac{1}{m - n + 1/2} \right).
\]

This possibility of re-expansion constitutes is a kind of the Gibbs phenomenon as the series (62) is only conditionally converges.

The existence of superscars manifests itself in the appearance of large coefficients in such Fourier-type expansions. A few examples of these expansions for the barrier billiard are presented at figure 31 and 32. It is plain that the energies of each large term in these expansions should be close to true energy

\[
E_{\text{exact}} \approx \frac{\pi^2}{b^2} k^2 + \frac{\pi^2}{a^2} \left( \frac{q - 1/2}{q} \right)^2, \quad \text{for } \Psi_{\text{even}}
\]

\[
E_{\text{exact}} \approx \frac{\pi^2}{b^2} k^2 + \frac{\pi^2}{a^2} q^2, \quad \text{for } \Psi_{\text{odd}}.
\]

It means that the energy conservation forces coefficients in these figures to be close to a quarter to the ellipse. Noticeable exception is seen at figure 31(b) where certain coefficients deviate considerably from the constant energy curve. It is plain that it corresponds to the above mentioned Gibbs phenomenon. If this eigenfunction is expanded into odd series (61) such large deviations would disappear. But for orbits with even \(M\) like the \((2 - 1)\) orbit indicated at figure 22(a) the expansion coefficients have Gibbs tails which could not be removed by a simple change of the basis (cf figure 32(a)).

It is clear that well isolated superscar states associated with short-period orbits are rare. Typical eigenfunctions may contain a certain number of large coefficients corresponded to a kind of superposition of many different superscar waves (see figure 32(b)).
5. Fractal dimensions

Everything discussed in the previous sections about a superscar wave propagating inside a periodic orbit pencil could be applied to an arbitrary polygonal billiards even with irrational angles where there exists at least one periodic orbit family. Unfortunately periodic orbits in generic polygonal billiards is an elusive subject and even the existence of one periodic orbit, in general, is not guaranteed. Only for a special sub-class of rational polygonal billiards called Veech polygons \[25\] where there exits a hidden group structure one can control all classical periodic orbits.

The knowledge of periodic orbits in such models permits to calculate analytically an important characteristic of their spectral statistics, namely the spectral compressibility, \(\chi\), \[15, 17\] which determines the linear growth of the number variance with the length of the interval

\[
\Sigma^2(L) \equiv \langle (n(L) - \bar{n}(L))^2 \rangle \xrightarrow{L \to \infty} \chi L. \tag{65}
\]

Here \(n(L)\) is the number of energy levels in an interval \(L\), \(\bar{n}(L)\) is the mean number of levels in this interval normalised to unit density, \(\bar{n}(L) = L\), and the average is taken over different intervals of length \(L\) in a small energy window. For the Poisson distribution typical for spectral statistics of integrable systems \(\chi = 1\) and for the standard random matrix ensembles which describe spectral statistics of chaotic systems \(\chi = 0\). The right triangular billiard with \(\pi/8\) angle has \(\chi = 5/9\) \[15\] and the barrier billiard considered in the paper has \(\chi = 1/2\) \[17\].

Spectral statistics of models with non-trivial compressibility, \(0 < \chi < 1\), are called intermediate statistics. Many pseudo-integrable billiards belongs to this class \[3, 9–17\]. The characteristic features of the intermediate statistics are (i) a level repulsion on small distances as for the usual random matrix ensembles, (ii) an exponential decrease of the nearest-neighbour distribution on large distances similar to the Poisson distribution.
properties have been observed in numerical calculations but have not been fully proved mathematically. Numerics (and certain analytical arguments [26]) also suggest that for models with intermediate spectral statistics eigenfunctions are fractal (in general, even multifractal).

The notion of multifractality (see, e.g., [36, 37] and references therein) is related with a natural question about the number of important components in eigenfunctions. Let an eigenfunction with eigenvalue \( E \) be written as an expansion in a certain basis

\[
\Psi(x, y) = \sum_{i=1}^{N} A_i(E) \phi_i(x, y), \quad A_0 = \int \Psi(x, y) \phi_0(x, y) dx dy, \quad \sum_{i=1}^{N} A_i^2 = 1.
\]

Here \( N \) is the total number of components.

The central question in the multifractal formalism is the scaling of the moments of expansion coefficients with \( N \). Define the moments with arbitrary \( q \) as follows

\[
M_q(E) = \sum_{i=1}^{N} \left| A_i(E) \right|^{2q}.
\]

The inverse of these moments, \( R_q = M_q^{-1} \), is called the participation ratio.

The multifractality means that moments of eigenfunctions (or their inverse) scale as a certain power of total number of wave function components

\[
M_q \xrightarrow{N \to \infty} N^{-\tau(q)}, \quad \tau(q) = (q - 1) D_q
\]

where \( D_q \) are called generalised fractal dimensions.

If only a finite number of components gives contributions to an eigenfunction (66) (which means that the state is localised) then \( D_q = 0 \). In the opposite case of completely extended states when all components are of the same order then from the normalisation it follows that \( A_i \sim N^{-1/2} \) and consequently \( D_q = 1 \).

In [38] the multifractality was observed in the 3-dimensional Anderson model at the metal-insulator transition and later it has been investigated in different matrix models [37].

For billiards the sum in (66) includes formally an infinite number of summands. For 2-dimensional billiards a natural basis consists of elementary trigonometric functions with fixed momentum. Physically it is clear that in the semiclassical limit \( k \to \infty \) the number of important (large) components should be of the order of the number of possible quantum cells on the constant momentum surface. For 2-dimensional billiards this surface is a circle of radius \( k \) and therefore \( N \sim k \) (as we are interested only in powers of \( N \) the precise pre-factor is irrelevant). Consequently, for billiards fractal dimensions determine the behaviour of the moments as function of the momentum

\[
M_q(E) \xrightarrow{k \to \infty} k^{-\tau(q)}, \quad k = \sqrt{E}.
\]

For systems with non-trival spectral compressibility, \( 0 < \chi < 1 \), numerical and partly analytical calculations [26] suggest that fractal dimensions in the Fourier space should be also non-trival, \( 0 < D_q < 1 \). To check these predictions numerical calculations of fractal dimensions were performed for high-excited states in the barrier billiard. Each eigenfunction has been expanded into the Fourier series

\[
\Psi(x, y) = \sum_{q,k=1}^{\infty} f_{q,k}(E) \cos \left( \frac{\pi(q - 1/2)x}{a} \right) \sin \left( \frac{\pi ky}{b} \right)
\]

and expansion coefficients \( A_i(E) \equiv f_{q,k}(E) \) were computed.

At figure 33(left) the participation ratio \( R_z \) for 3 energy intervals close to the 1000th, the 4000th, and the 10000th levels for the barrier billiard are presented. For comparison at this figure the same quantity but for the quarter of the (chaotic) stadium billiard with the same aspect ratio are shown for comparison. The area of the both billiards is \( 4\pi \) and the energies approximately equal the level numbers.

At figure 34(a) these data were used to calculate average values of the participation ratios for the barrier billiard and the stadium billiard. As expected, for the chaotic stadium billiard participation ratio scales linear with the momentum. The best fit gives \( R_z(E) = 0.75k \). But for the barrier billiard the best fit suggests that \( R_z(E) = 2.55\sqrt{k} \) which means that fractal dimension \( D_z \) is non-trivial, \( D_z = 0.5 \). At figure 34 the participation ratios \( R_z(E) \) and \( R_x(E) \) for all states till \( E = 5000 \) are plotted for the barrier billiard. The best fits (indicated by white lines at this figure) give \( R_z(E) \approx 2.52\sqrt{k} \) and \( R_x(E) \approx 4.7k \). Therefore, these results suggest that \( D_z \approx D_x \approx 0.5 \).

Of course, much more calculations should be done to establish correct values of fractal dimensions for pseudo-integrable billiards.
6. Summary

Wave functions are on the very basis of quantum mechanical calculations and the investigation of their properties are important for many applications. For generic classically chaotic systems Berry’s conjecture...
The purpose of this appendix is to present the main properties of periodic orbit pencils (POP) and the corresponding superscar waves (SW) for the rectangular billiard and to propose a method of the visualisation of folded supescar waves.

A rectangular billiard with sides \( a \) and \( b \) cover the whole plane under the group of reflections at its boundary. This group consists of 4 inversions: \( x \rightarrow \pm x \), \( y \rightarrow \pm y \) and integer translations: \( x \rightarrow x + 2ma, \ y \rightarrow y + 2nb \), \( m, n \in \mathbb{Z} \).
A non-oriented primitive periodic orbit in the folded rectangular lattice can be represented by a line which connects the origin with a point with coordinates $2Ma$ and $2Nb$ (here and below the coordinate system is chosen as at figure 1(c)) where $M$ and $N$ are positive co-prime integers (peculiarities for zero $M$ or $N$ are easy to take into account). The length of such orbit is $L_p = \sqrt{(2Ma)^2 + (2Nb)^2}$.

Introduction of a new coordinate system $(\xi, \eta)$ with the coordinate $\xi$ along this orbit and $\eta$ perpendicular to it gives

$$\xi = x \cos \theta + y \sin \theta, \quad \eta = -x \sin \theta + y \cos \theta, \quad \cos \theta = \frac{2Ma}{L_p}, \quad \sin \theta = \frac{2Nb}{L_p}. \tag{A1}$$

Any point of the plane $(x,y)$ by integer translations can be put inside the following rectangle (a fundamental domain of the group)

$$-\frac{1}{2}L_p \leq \xi < \frac{1}{2}L_p, \quad -w \leq \eta < w, \quad w = \frac{2ab}{L_p}. \tag{A2}$$

Explicitly such transformation is $x' = x + 2a(\eta \mu + M t)$, $y' = y + 2b(\eta \nu + N t)$ with

$$r = \left[\frac{1}{2} - \frac{-x \sin \theta + y \cos \theta}{2w}\right], \quad t = \left[\frac{1}{2} - \frac{x \cos \theta + y \sin \theta}{L_p}\right] - \frac{M \mu a^2 + N \nu b^2}{M^2 a^2 + N^2 b^2 + r^2}.$$  

Here $[z]$ is the largest integer lower than $z$, $z - 1 < [z] \leq z$ and $(\mu, \nu)$ is a pair of co-prime integers related with the pair of co-prime integers $(M,N)$ by the relation $M \nu - N \mu = 1$.

The existence of 4 different (in general) reflected points, $(\pm x, \pm y)$, shows that rectangle (A2) is the 4-fold covering of the original rectangle with sides $a$ and $b$, $(2wL = 4ab)$.

POP for the barrier billiard is defined as a set of parallel periodic orbits restricted from the both sides by singular vertices (i.e. images of vertex $(0,b/2)$).

There are 4 different types of POP in the barrier billiard depending on $M$ mod 4. For odd $M$ there exist 2 POP of width $w_p = w$ centred at the vertices with coordinates $(0,0)$ and $(0,b)$ respectively. For each even $M$ there is only one POP with width $w_p = 2w$. For $M = 4t + 2$ it is centred at the origin. For $M = 4t$ the centre of the corresponding POP can be chosen at the vertex with coordinates $(a,b)$.

Each POP can support SW propagating inside the pencil

$$\psi_{(\text{superscar})}(m,n)(\xi, \eta) = e^{i\epsilon m \xi/L_p} \varphi_{m,n}(\eta) \tag{A3}$$

where $m$ and $n$ are longitudinal and perpendicular quantum numbers (integers) which determine SW. The energy of such wave is $E_{m,n} = \pi^2 m^2 / L_p^2 + \pi^2 n^2 / w_p^2$.

Functions $\varphi_{m,n}(\eta)$ are non-zero only inside the interval $-\frac{w_p}{2} < \eta < \frac{w_p}{2}$ where they are given by the following expressions

$$\varphi_{2n}(\eta) = \sin \left(\frac{2n\pi}{w_p}\eta\right), \quad \varphi_{2n-1}(\eta) = \cos \left(\frac{(2n - 1)\pi}{w_p}\eta\right). \tag{A4}$$

Here $\eta$ is calculated from the central point of POP and $w_p$ is the width of POP.

During the motion inside POP classical rays cross different boundaries with the Dirichlet and Newman boundary conditions. The total phase determines the parity of the longitudinal quantum number, $m$, indicated in table 1. The width of corresponding POP and coordinates of its central point $(x_c, y_c)$ are also stated in this table for completeness.

| $M$  | $m_s$ | $w_p$ | $(x_c, y_c)$ |
|------|-------|-------|--------------|
| $2t+1$ | $2m$ | $w$ | $(0,0)$ |
| $2t+1$ | $2m-1$ | $w$ | $(0,b)$ |
| $4t+2$ | $2m-1$ | $2w$ | $(0,0)$ |
| $4t$ | $2m$ | $2w$ | $(a,b)$ |
• Equation (A3) represents an unfolded SW, \( \Psi(\xi, \eta) \). By definition, the wave folded back inside the original rectangular billiard, \( \Psi(x, y) \), is equal to the sum over all images of a point \((x, y)\) under the reflections on billiard boundaries weighted with the total phase accumulated along such reflections

\[
\Psi(x, y) = \sum_{j=1}^{4} e_j \Psi(\xi_j(x, y), \eta_j(x, y)), \quad e_j = \pm 1. \tag{A5}
\]

Here \((\xi_j(x, y), \eta_j(x, y))\) are images of 4 points \((\pm x, \pm y)\) inside the rectangle \((A2)\).

• The phase may conveniently be divided into two parts. The first corresponds to the phase when the Dirichlet boundary conditions are imposed along all rectangular boundaries. The second is the phase shift (additional \( -1 \)) related with the crossing of the Newman parts of the boundary. After unfolding the Newman segments occupy the segments \([b/2 + 2kb, 3b/2 + 2kb]\), at \(x_j = 2aj\), with \(k = 0, \ldots, N - 1, j = 0, \ldots, M - 1\). Let us calculate this phase for points on the periodic orbit passing through the origin whose equation is \(y = Nb/x/Ma\). For fixed \(j\) this orbit crosses the Newman part of the boundary iff there is an integer \(k\) which fulfills the inequalities

\[
\frac{r}{M} - \frac{3}{4} \leq k \leq \frac{r}{M} - \frac{1}{4}, \quad r = Nj \mod M. \tag{A6}
\]

The solution of the above equation exists if the fractional part of the rhs of this inequality is less or equal \(1/2\)

\[
\frac{M}{4} \leq r \leq \frac{3M}{4} \tag{A7}
\]

which permits easily to find numerically the necessary phase.

• When all 4 images are taken into account one can use complex expression \((A3)\). The inversion of \((\xi, \eta)\) corresponds to the inversion of \((x, y)\) coordinates. Due to the chosen boundary conditions the exact wave function should have a deﬁnite parity with respect to the coordinate refection around the central point of the rectangle

\[
\Psi(-\xi - \xi_c, -\eta - \eta_c) = \epsilon \Psi(\xi - \xi_c, \eta - \eta_c) \tag{A8}
\]

where \(\epsilon = +1\) for a vertex with 4 Dirichlet boundaries and \(\epsilon = -1\) for vertices where 2 Dirichlet and 2 Newman boundaries cross. Coordinates with subscript \(c\) are coordinates of the central point.

• Taking this symmetry into account and combining the above arguments one concludes that (non-normalised) superscar waves can be written as the following real expressions (with \(0 < \xi < L_p/2\))

1. For odd \(M\) and even \(m\), the central point is \(x_c, y_c = (0, 0)\),

\[
\Psi^{(odd M)}_{2m,0}(\xi, \eta) = \sin \left( \frac{2\pi m}{L_p} \frac{\xi}{\xi_c} \right) \sin \left( \frac{2\pi n}{w} \frac{\eta}{\eta_c} \right) \Theta \left( \frac{w^2}{4} - \eta^2 \right), \tag{A9}
\]

\[
\Psi^{(odd M)}_{2m,0-1}(\xi, \eta) = \cos \left( \frac{2\pi m}{L_p} \frac{\xi}{\xi_c} \right) \cos \left( \frac{2\pi (n-1)}{w} \frac{\eta}{\eta_c} \right) \Theta \left( \frac{w^2}{4} - \eta^2 \right). \tag{A10}
\]

2. For odd \(M\) and odd \(m\), the central point is \(x_c, y_c = (0, b)\)

\[
\Psi^{(odd M)}_{2m-1,0}(\xi, \eta) = \cos \left( \frac{\pi (2m - 1)}{L_p} \frac{\xi}{\xi_c} \right) \sin \left( \frac{2\pi n}{w} \frac{\eta}{\eta_c} \right) \Theta \left( \frac{w^2}{4} - \eta^2 \right), \tag{A11}
\]

\[
\Psi^{(odd M)}_{2m-1,0-1}(\xi, \eta) = \sin \left( \frac{\pi (2m - 1)}{L_p} \frac{\xi}{\xi_c} \right) \cos \left( \frac{\pi (2n - 1)}{w} \frac{\eta}{\eta_c} \right) \Theta \left( \frac{w^2}{4} - \eta^2 \right). \tag{A12}
\]

3. For even \(M\) but \(M \neq 0 \mod 4\), \(m\), has to be odd and the central point is \(x_c, y_c = (0, 0)\),

\[
\Psi^{(even M)}_{2m-1,0}(\xi, \eta) = \sin \left( \frac{\pi (2m - 1)}{L_p} \frac{\xi}{\xi_c} \right) \sin \left( \frac{\pi n}{w} \frac{\eta}{\eta_c} \right) \Theta \left( w^2 - \eta^2 \right), \tag{A13}
\]

\[
\Psi^{(even M)}_{2m-1,0-1}(\xi, \eta) = \cos \left( \frac{\pi (2m - 1)}{L_p} \frac{\xi}{\xi_c} \right) \cos \left( \frac{\pi (n - 1/2)}{w} \frac{\eta}{\eta_c} \right) \Theta \left( w^2 - \eta^2 \right). \tag{A14}
\]
4. For $M \equiv 0 \mod 4$, $m_s$ has to be even and the central point is $(x_c, y_c) = (a, b)$

$$
\Psi^{(even \, M)}_{2m,2n-1}(\xi, \eta) = \cos\left(\frac{2\pi m}{L_p}\xi\right)\cos\left(\frac{\pi(2n-1)}{2w}\eta\right)\Theta(w^2 - \eta^2),
$$

(A15)

$$
\Psi^{(even \, M)}_{2m,2n}(\xi, \eta) = \sin\left(\frac{2\pi m}{L_p}\xi\right)\sin\left(\frac{\pi n}{w}\eta\right)\Theta(w^2 - \eta^2).
$$

(A16)

Here $\Theta(x)$ is the Heaviside function ($\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$) introduced to stress that superscar waves are zero outside the indicated intervals. In the above formulas it is implicitly assumed that $x$ and $y$ in the definitions (A1) have to be calculated from the central vertices of the initial rectangle. For the coordinate system as at figure 1(c) they correspond to the substitution $x \rightarrow x - x_c$, $y \rightarrow y - y_c$.

- If the symmetric expressions for the superscar waves (A10)-(A16) are used only 2 images should be taken into account and

$$
\Psi(x, y) = c_1\Psi(\xi(x, y), \eta(x, w)) - c_2\Psi(-x, y), \eta(-x, y)).
$$

(A17)

For illustration at figure 35 and 36 a few examples of correct folding of superscar waves with total energy $E_s \approx 1000$ are presented. In the calculations $(x, y)$ coordinates were discretised into boxes with sides $\lambda/10$ where $\lambda = 2\pi/\sqrt{E}$ is the wavelength. All 4 different types of POP are represented at these figures. At all figures the rectangle has the area $4\pi$ and the aspect ratio $b/a = \sqrt{5} + 1$ as in the main text.

**ORCID iDs**

Eugene Bogomolny 📞 https://orcid.org/0000-0002-7627-8362
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