Following Weyl on Quantum Mechanics: the contribution of Ettore Majorana

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Abstract

After a quick historical account of the introduction of the group-theoretical
description of Quantum Mechanics in terms of symmetries, as proposed by
Weyl, we examine some unpublished papers by Ettore Majorana. Remarkable
results achieved by him in frontier research topics as well as in physics teaching
point out that the Italian physicist can be well considered as a follower of Weyl
in his reformulation of Quantum Mechanics.

Keywords: Majorana and Quantum Mechanics, Weyl reformulation of Quantum
Mechanics, Group Theory and Quantum Mechanics, Symmetries in Physics
1 INTRODUCTION

The important role of symmetries (and, then, of Group Theory) in Quantum Mechanics was established in the third decade of the XX century, when it was discovered the special relationships concerning systems of identical particles, reflection and rotational symmetry or translation invariance. In particular it was realized that a given symmetry of (the Hamiltonian of) a physical system leads to the partition of its states into terms of different symmetrical behavior of their eigenfunctions and then to selection rules. The systematic theory of symmetry resulted to be just a part of the mathematical theory of groups (see, for example, [1]).

In many cases the dependence of the eigenfunctions on the variables involved in the symmetry is explicit, and thus the partitions into systems of different symmetry is straightforward. However, this method fails if the number of variables is large. An alternative approach is to characterize the behavior of the eigenfunctions by means of transformation matrices; for example, for a rotation along a given axis of an angle \( \alpha \), the corresponding matrix is:

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\]

Matrices of this kind corresponding to a given group of transformations form the “representation of the group by means of linear transformations”, and the sets of terms in the partitions correspond to the “irreducible” representations of the group of the covering operations of the given physical system. This form of Group Theory was first applied to the quantum theory by E. Wigner in 1926-1927 [2]. The inclusion of the spin (and other internal symmetries) into the game was, then, possible when H. Weyl discovered the two-valued representations of the rotation group [1] and used them to describe the atomic states with spin.

The final form of such applications of Group Theory to Quantum Mechanics was established in the books by Weyl [1], Wigner [3] and van der Waerden [4]. Nevertheless, despite these successes, the group-theoretical description of Quantum Mechanics in terms of symmetries was ignored by almost all theoretical physicists and it received attention by physics textbooks only in recent times. It is also noticeable that, although in the 1950’s almost every physicist had a copy of the Weyl book [5], the extensive use of Group Theory in physics research started only in nuclear and particle physics in 1950’s.

Such an indifference, however, does not apply to Ettore Majorana. Although he was in Germany for a short period [6], he never met Weyl, but the brilliant works on Group Theory and its applications of this mathematician, as well as those of Wigner, left an unambiguous and fruitful mark on the work of the Italian physicist. This is not easily recognizable by looking at the very few published papers by Majorana (just 9 different articles, one of which posthumous) but clearly emerges from his unpublished manuscripts (see also the comment of N. Cabibbo in [7]), most of which are deposited at the Domus Galilaeana in Pisa (Italy) [6].

Majorana was introduced to Physics at the beginning of 1928 by E. Segré, and graduated with Fermi on July 6, 1929. He thus went on to collaborate with the famous group in Rome created by Enrico Fermi and Franco Rasetti. During 1933 Majorana spent about six months in Leipzig with W. Heisenberg, and then, for some
unknown reasons, stopped participating in the activities of Fermi’s group. He even ceased publishing the results of his research, except for his paper on the “symmetric theory of electrons and positrons” [8], which (ready since 1933) Majorana was persuaded by his colleagues to remove from a drawer and publish just prior to the 1937 Italian national competition for full-professorships. With respect to the last point, it is remarkable that, on the recommendation of the judging committee (Fermi being one of the members), the Italian Minister of National Education installed Majorana as professor of Theoretical Physics at the University of Naples because of his “great and well-deserved fame”, independently of the competition itself [6]. Fermi was, probably, the only great physicist of that time that adequately recognized the genius of Majorana, as can be deduced from his own words: “Because, you see, in the world there are various categories of scientists: people of a secondary or tertiary standing, who do their best but do not go very far. There are also those of high standing, who come to discoveries of great importance, fundamental for the development of science. But then there are geniuses like Galileo and Newton. Well, Ettore was one of them. Majorana had what no one else in the world had...” [9]. Unfortunately, Ettore Majorana disappeared during rather mysteriously on March 26, 1938, and was never seen again.

We have analyzed the physics work by Majorana before and after the appearance of the Weyl’s book, and the results of this research are presented here. We find that Majorana can be well considered as a faithful follower (and, probably, the only one) of the Weyl thought who, however, went beyond the track drawn by Weyl himself. In the next section we outline the Weyl proposal for an adequate description of Quantum Mechanics in terms of Group Theory and discuss the results reached by the author himself. Instead in Sect. 3 we consider the work of Majorana in the direction tracked by Weyl, while in the subsequent section we compare the two approaches and the corresponding results obtained. Finally in Sec. 5 we give our conclusions.

2 WEYL PROPOSAL

According to Wigner [3], it was probably M. von Laue who first recognized the significance of Group Theory as the natural tool with which to obtain a first orientation in problems of Quantum Mechanics. However, the program for a description of Quantum Mechanics in terms of Group Theory is clearly stated for the first time by Weyl in the introduction of his well-known book [1]:

“...it has recently been recognized that Group Theory is of fundamental importance for quantum physics; it here reveals the essential features which are not contingent on a special form of the dynamical laws nor on special assumptions concerning the forces involved. We may well expect that it is just this part of quantum physics which is most certain of a lasting place. Two groups, the group of rotations in 3-dimensional space and the permutation group, play here the principal role... The investigation of groups first becomes a connected and complete theory in
the theory of the representations of groups by linear transformations, and it is exactly this mathematically most important part which is necessary for an adequate description of the quantum mechanical relations...

As an illustration, let us consider the problem of the splitting of the spectral lines of an atom, placed in a homogeneous magnetic field $B$ in the direction of the $z$ axis: for simplicity we neglect the spin interaction with the magnetic field (Zeeman effect). The non group-theoretical description of the phenomenon proceeds as follows. Let us consider, at first, one-electron atoms; the Hamiltonian describing the system is then:

$$H = H_0 + W = H_0 + \mu_B B L_z,$$

where $H_0$ is the undisturbed Hamiltonian of the electron and the second term $W$ accounts for the “perturbation” introduced by the magnetic field. Here $\mu_B = e\hbar/2mc$ is the Bohr magneton for the electron and $L_z$ is the $z$ component of the angular momentum operator divided by $\hbar$. The energy eigenfunctions of the system coincide with the eigenstates $\psi_m$ of the operator $L_z$, whose eigenvalues are the integers $m$. Thus the energy terms results to be

$$E = E_0 + \mu_B B m,$$

and the frequencies $\omega$ of the spectral lines corresponding to the transition $E \rightarrow E'$ are given by:

$$\hbar \omega = E - E' = (E_0 - E'_0) + \mu_B B (m - m').$$

Each spectral line is, then, broken up into the lines associated with all possible transitions $m \rightarrow m'$. Since $m$ may assume the $2\ell + 1$ values between $-\ell$ and $+\ell$, the integer $\ell$ being the orbital quantum number associated to the eigenvalues of the $L^2$ angular momentum operator, in general we can expect a splitting into $(2\ell + 1)(2\ell' + 1)$ lines. Nevertheless the transition probabilities are proportional to the squared modulus of the matrix elements of the components $q_x, q_y, q_z$ of the dipole moment of the atom along the coordinate axes, given by:

$$
\begin{align*}
(q_x + iq_y)_{m'm} & = \int \psi_{m'}^* (q_x + iq_y) \psi_m \, d^3\vec{r}, \\
(q_x - iq_y)_{m'm} & = \int \psi_{m'}^* (q_x - iq_y) \psi_m \, d^3\vec{r}, \\
(q_z)_{m'm} & = \int \psi_{m'}^* q_z \psi_m \, d^3\vec{r}.
\end{align*}
$$

By expressing the integrand functions in polar coordinates $(r, \theta, \phi)$, we see that the one in $(q_x + iq_y)_{m'm}$ contains the factor $\exp \{i(-m' + 1 + m)\phi\}$ which, integrated with respect to $\phi$, gives zero unless $m' = m + 1$. Similarly, for the second and third integrand we have $m' = m - 1$ and $m' = m$, respectively. We thus find a splitting into only three lines, corresponding to the selection rules $\Delta m = m' - m = 0, \pm 1$, as observed experimentally in the normal Zeeman effect.

The group-theoretical description of the same phenomenon does not need the previous simplification of one-electron atoms, but considers the system as a whole without resolving it into individual electrons. This is justified by the Wigner observation in 1927 that it is always possible to choose the eigenfunctions $\psi_m$ of the unperturbed Hamiltonian $H_0$ in such a way that a rotation over an angle $\phi$ about the $z$ axis transforms $\psi_m$ into $\exp \{-im\phi\} \psi_m$, with integer $m$. Given the rotational invariance and that $L_z$ is the infinitesimal generator of rotations about the $z$ axis,
we then have, once more, $L_z \psi_m = m \psi_m$, so that Eq. (3) again follows from Eq. (2).

As far as the transition probabilities are concerned, we note that the component $q_z$ has to remain unchanged when a rotation over an angle $\phi$ about the $z$ axis is performed, so that $(q_z)_{m'}_m$ must be diagonal. On the other hand, the quantities $q_x \pm i q_y$ acquire a factor $\exp \{ \mp i \phi \}$ when a rotation is performed, while $\psi^*_m$ and $\psi_m$ account for a global factor of $\exp \{ i (m' - m) \phi \}$. From rotational invariance, then, the selection rules immediately follow:

$$
(q_x + i q_y)_{m'}_m : \quad \Delta m = +1, \quad (7)
$$
$$
(q_x - i q_y)_{m'}_m : \quad \Delta m = -1, \quad (8)
$$
$$
(q_z)_{m'}_m : \quad \Delta m = 0. \quad (9)
$$

When the spin interaction with the magnetic field is taken into account, the anomalous Zeeman effect arises, whose explanation requires to change the perturbation term $W$ in the Hamiltonian (2) and the formalism of relativistic Quantum Mechanics should be used. Nevertheless the selection rules for the quantum number $m$ established above have been obtained from fundamental concepts of Group Theory, so that they are valid in all cases of the Zeeman effect. It is then possible to deal with the splitting of the spectral lines occurring when the symmetry is decreased; in the case considered above, we have the transition from the spherical symmetry of the undisturbed $H_0$ to the axial symmetry of the perturbed $H$. In particular, Weyl recognized (see, for example, Sect. A of Chapter IV in [1]) that the breaking up of the energy levels, due to the axially symmetric perturbation, parallels the reduction of irreducible representations of the rotation group, when this is restricted to the group of rotations about the $z$ axis. The problem considered above is, then, a typical example which well illustrates the above-quoted Weyl proposal.

The Weyl book [1] is practically a sandwich of mathematical formalism and physics applications (starting from mathematics); nowadays it can be considered quite a successful attempt for an “adequate” group-based description of the quantum mechanical phenomena known at that time. Nevertheless it was readily recognized a great reluctance among physicists toward accepting the group-theoretical point of view, they called it the “group pest” (see, for example, the prefaces of Refs. [3] and [1] in their English translations). Remarkably, in the second edition of his book, Weyl did not go further in his program but “followed the trend of the times, as far as justifiable, in presenting the group-theoretic portions in as elementary a form as possible”. However the word “elementary” used by Weyl is of dubious meaning; indeed, the (assumed) didactic sense of this sentence is contradicted by a careful look inside the revised book (for another interpretation of the Weyl thought, in terms of his elementary mathematics, see [10]). After the publication of the second edition of his book, Weyl abandoned his proposal in order to improve the formulation of Group Theory and render it a more suitable tool for Theoretical Physics; the “group pest” was, however, cut out from fundamental physics for many years, until the discovery of peculiar symmetries in nuclear and particle physics. Furthermore we must wait up to recent times for group-based descriptions of Quantum Mechanics in physics textbooks.

Probably the failure of the Weyl program should be looked for in the involved mathematical presentation of the subject, which seemed too much hard to swallow.
to the physicists of that time [11]. In this depressing framework it appears even more relevant the enthusiasm by Majorana for Weyl’s proposal about group-theoretical foundation of Quantum Mechanics.

3 MAJORANA ON GROUP THEORY AND ITS APPLICATIONS

Among the very few books owned by E. Majorana (about 15 volumes kept by his nephew Ettore in Rome) it appears the Weyl one “Gruppentheorie und Quantenmechanik” in its first German edition (1928). As testified by the large number of unpublished manuscript pages (and also some published articles) of the Italian physicist, the Weyl approach greatly influenced the scientific thought and work of Majorana. Here we focus primarily on his 5 orderly notebooks, known as the “Volumetti” [12], written between 1927 and 1932. Besides a surprising clarity and linearity of exposition, two peculiar features of the “Volumetti” are relevant for us: they are dated by the author himself (each notebook was written during a period of time of about one year, starting from 1927) and explicit references to the Weyl book are included \(^1\). These features allow us to reconstruct the line of thinking of our author. In fact, just by looking at the table of contents of the notebooks (this table was written down by Majorana himself) we are able to realize the impact of Weyl’s work.

The first two notebooks, whose “closing dates” correspond to March 1927 and April 1928, respectively, deal mainly with electromagnetism, thermodynamics and atomic physics arguments, accounting for a total of 81 sections, and no reference to Group Theory or some related application occurs. The situation changes starting from the third notebook, whose closing date is June 1929: a number of sections are devoted to the crucial topic under consideration.

In the following we report the list of such sections pertaining to the last three notebooks (out of a total of 64 sections):

VOLUMETTO III

13. The group of proper unitary transformations in two variables

14. Exchange relations for infinitesimal transformations in the representations of continuous groups

16. The Group of Rotations \(O(3)\)

17. The Lorentz Group

18. Dirac matrices and the Lorentz Group

20. Characters of \(D_j\) and reduction of \(D_j \times D'_j\)

\(^1\)In about 500 pages, only 5 bibliographical references appears, 3 of which correspond to the Weyl book of 1928.
21. Intensity and selection rules for a central field

22. The anomalous Zeeman effect (according to the Dirac theory)

VOLUMETTO IV

7. Fundamental characters of the group of permutations of \( f \) objects

14. Cubic symmetry

25. Spherical functions with spin \((s=1)\)

29. Spherical functions with spin \((II)\)

VOLUMETTO V

1. Representations of the Lorentz group

6. Spinor transformations

7. Spherical functions with spin \((s=1/2)\)

8. Infinite-dimensional unitary representations of the Lorentz group

It is remarkable that the above list approximately mimics the sequence of arguments covered by the Weyl book [1]. Explicit reference to this book appear in Sec. 21 of Volumetto III; actually, the development of the subject of this section follows closely that of Sec. 3, chapter 4 of [1] (Intensity and selection rules). Likely as previous sections, it reiterates Weyl’s arguments, but the last section of the notebooks dealing with group-theoretical applications (last entry in the above list) is a preliminary study of what will be one of the most important (published) papers by Majorana on a generalization of the Dirac equation to particles with arbitrary spin [13]. We will get an insight into this and other subject in the following section.

4 CONFRONTING MAJORANA WITH THE WEYL APPROACH

Here we now compare the results achieved by the two authors on some relevant specific topics of Group Theory and its applications to Quantum mechanics.
4.1 Group Theory

As a starting point let us consider the discussion of the group of unitary transformations in two dimensions. Weyl approach is typical of a mathematical physicist: he gives an abstract description of the above-mentioned group as a subgroup of the group of linear transformations:

\[
\begin{align*}
    x \to x' &= \alpha x + \beta y, \\
y \to y' &= -\beta^* x + \alpha^* y,
\end{align*}
\]

with \( \alpha \alpha^* + \beta \beta^* = 1 \). Since \( \alpha \) and \( \beta \) are, in general, complex numbers, Weyl characterizes the transformations of the group by means of 4 real parameters (real and imaginary parts of \( \alpha \) and \( \beta \)) the sum of whose squares is 1. By using this representation, he then points out (without an explicit proof) that the composition of two such transformations corresponds to the multiplication rule for Hamilton quaternions.

Instead, in the Volumetti, Majorana gives a very detailed description of the group considered, reporting a simple proof of the above-mentioned relationship with the quaternions. He writes the transformation in Eq. (10) in matrix form:

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
k + i \lambda & -\mu + i \nu \\
\mu + i \nu & k - i \lambda
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix},
\]

(11)

where \( k, \lambda, \mu, \nu \) are 4 real parameters (in practice, according to Weyl, the real and imaginary parts of the complex numbers \( \alpha \) and \( \beta \) above). The composition of two unitary transformations \( T_A \cdot T_B \) is then described by matrix multiplication \( AB \):

\[
A = \begin{pmatrix}
k + i \lambda & -\mu + i \nu \\
\mu + i \nu & k - i \lambda
\end{pmatrix}, \quad B = \begin{pmatrix}
k' + i \lambda' & -\mu' + i \nu' \\
\mu' + i \nu' & k' - i \lambda'
\end{pmatrix},
\]

(12)

\[
AB = \begin{pmatrix}
k'' + i \lambda'' & -\mu'' + i \nu'' \\
\mu'' + i \nu'' & k'' - i \lambda''
\end{pmatrix},
\]

so that the correspondence with the multiplication rule for quaternions immediately follows:

\[
\begin{align*}
k'' &= kk' - \lambda \lambda' - \mu \mu' - \nu \nu', \\
\lambda'' &= k\lambda' + \lambda k' - \mu \nu' + \nu \mu', \\
\mu'' &= k\mu' + \lambda \nu' + \mu k' - \nu \lambda', \\
\nu'' &= k \nu' - \lambda \mu' + \mu \lambda' + \nu k'.
\end{align*}
\]

(13)

A general derivation of the generators of the group is reported in the Volumetti as well. Moreover, keeping in mind physical applications, Majorana also gives explicit expressions for matrix representations corresponding to the index (angular momentum) \( j = 0, 1/2, 1, 3/2, 2 \) (even in modern texts, only \( j = 0 \) and \( j = 1/2 \) representations are reported).

Similar considerations hold for the study of both the group of rotations and the Lorentz group.

Instead, for what concerns the group of permutations, relevant for applications to a system of identical particles, Majorana completes and generalizes the analysis by Weyl, by giving the explicit expressions for the fundamental characters of the group. He also discusses cubic symmetry, not considered in [1], by considering the group
of permutations of 4 objects, and its relationship with the 24 (proper) rotations transforming the $x, y, z$ axes into themselves.

4.2 Applications to Quantum Mechanics

Several applications of Group Theory to quantum mechanical phenomena are considered by Weyl as well as Majorana. We have already pointed out in Sec. 3 that the question of selection rules for atomic transitions in a central field is tackled in the same way by the two authors, with quite similar results. Here we focus on another typical problem of atomic physics, namely that of the anomalous Zeeman effect, that Weyl considers as a “simple” application of the theory of the group of rotations. He evaluates the Landé $g$-factor (for one-electron as well as many-electron atoms) entering in the anomalous splitting of the spectral lines just as the result of the mean expectation values of some angular momentum operators (adopting a “physical language”, according to Weyl himself), showing the power of the group-theoretical methods (but even its uncomfortable use for physicists). On the contrary Majorana deals with the problem as with an exercise of physics, starting from the Dirac equation for the physical system considered. The appropriate Hamiltonian matrix describing the system is, then, deduced and the energy eigenvalues are obtained with the help of Group Theory (following the Weyl method). Majorana reports detailed predictions of the physical results for the energy eigenvalues in the weak field limit as well as for strong magnetic fields and, moreover, he considers the transition from the anomalous Zeeman effect to the Paschen-Back effect by changing the size of the magnetic field acting on the atom. His results for the shift of the spectral lines are sketched in Figure 1. It appears, by looking at the notebooks, that such transition is studied only on qualitative grounds; nevertheless some details in Sec. 22 of Volumetto III seem to suggest some specific calculations which, to our knowledge, cannot be performed by using perturbation theory but only numerically.

4.3 Representations of the Lorentz Group

In [1] Weyl supposed that the general properties of finite-dimensional groups are preserved in when passing to infinite-dimensional ones. Majorana, instead, explored such a connection in a different way, and this is particularly evident in his treatment of the Lorentz group. This group underlies the Theory of Relativity and its representations are especially relevant for the Dirac equation in Relativistic Quantum Mechanics. In the Weyl book only a particular kind of such representations are considered (those related to the two-dimensional representations of the group of rotations, according to Pauli), but we must observe that an exhaustive study of this subject was still lacking at that time. The correspondence between the Dirac equation and the Lorentz transformations is obviously pointed out but, surprisingly, the group properties of this connection are not highlighted.
Figure 1: Transition from the anomalous Zeeman effect to the Paschen-Back effect, according to Majorana [12]. The parameter $\epsilon$ is the dimensionless Larmor frequency, parameterizing the perturbation on the system induced by the applied magnetic field. On the left axis the energy (in arbitrary units) of the spectral lines is reported while, on the right, the spin up or spin down component of the given line in the Paschen-Back effect is indicated.
Majorana treats the Lorentz group in his third and fifth notebook. confining ourselves to the same topics covered also by Weyl, again he gives a detailed deduction of the relationship between the representations of the Lorentz group and the matrices of the (special) unitary group in two dimensions, and a strict connection with the Dirac equation is always taken into account. Moreover, the explicit form of the transformations of every bilinear in the spinor field $\Psi$ is reported. For example, he obtains that some of such bilinears behave as the 4-position vector $(ct, x, y, z)$ or as the components of the rank-2 electromagnetic tensor $(E, H)$ under Lorentz transformations, according to the following rules:

\begin{align}
\Psi^{\dagger}\Psi & \sim -i\Psi^{\dagger}\alpha_x\alpha_y\alpha_z\Psi \sim ct, \\
-\Psi^{\dagger}\alpha_x\Psi & \sim i\Psi^{\dagger}\alpha_y\alpha_z\Psi \sim x, \\
-\Psi^{\dagger}\alpha_y\Psi & \sim i\Psi^{\dagger}\alpha_z\alpha_x\Psi \sim y, \\
-\Psi^{\dagger}\alpha_z\Psi & \sim i\Psi^{\dagger}\alpha_x\alpha_y\Psi \sim z,
\end{align}

\begin{align}
i\Psi^{\dagger}\beta\alpha_x\Psi & \sim E_x, \\
i\Psi^{\dagger}\beta\alpha_y\Psi & \sim E_y, \\
i\Psi^{\dagger}\beta\alpha_z\Psi & \sim E_z, \\
i\Psi^{\dagger}\beta\alpha_y\alpha_z\Psi & \sim H_x, \\
i\Psi^{\dagger}\beta\alpha_z\alpha_x\Psi & \sim H_y, \\
i\Psi^{\dagger}\beta\alpha_x\alpha_y\Psi & \sim H_z, \\
i\Psi^{\dagger}\beta\Psi & \sim \Psi^{\dagger}\beta\alpha_x\alpha_y\Psi \sim 1,
\end{align}

where $\alpha_x, \alpha_y, \alpha_z, \beta$ are Dirac matrices.

But, probably, the most important result achieved by Majorana on this subject is his discussion of infinite-dimensional unitary representations of the Lorentz group, giving also an explicit form for them. Note that such representations were independently discovered by Wigner in 1939 and 1948 [14] and were thoroughly studied only in the years 1948-1958 [15]. Lucky enough, we are able to reconstruct the reasoning which led Majorana to discuss the infinite-dimensional representations. In Sec. 8 of Volumetto V we read:

“The representations of the Lorentz group are, except for the identity representation, essentially not unitary, i.e., they cannot be converted into unitary representations by some transformation. The reason for this is that the Lorentz group is an open group. However, in contrast to what happens for closed groups, open groups may have irreducible representations (even unitary) in infinite dimensions. In what follows, we shall give two classes of such representations for the Lorentz group, each of them composed of a continuous infinity of unitary representations.”

The two classes of representations correspond to integer and half-integer values for the representation index $j$ (angular momentum). Majorana begins by noting that the group of the real Lorentz transformations acting on the variables $ct, x, y, z$ can be constructed from the infinitesimal transformations associated to the matrices (not
taken into account by Weyl):

\[
S_x = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
S_y = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
S_z = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
T_x = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_y = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_z = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

from which he deduces the general commutation relations satisfied by the \(S\) and \(T\) operators acting on generic (even infinite) tensors or spinors:

\[
S_x S_y - S_y S_x = S_z,
T_x T_y - T_y T_x = -S_z,
S_x T_x - T_x S_x = 0,
S_x T_y - T_y S_x = T_z,
S_x T_z - T_z S_x = -T_y,
\]

Next he introduces the matrices

\[
a_x = i S_x, \quad b_x = -i T_x, \quad \text{etc.}
\]

which are Hermitian for unitary representations (and vice versa), and obey the following commutation relations:

\[
[a_x, a_y] = i a_z, \\
[b_x, b_y] = -i a_z, \\
[a_x, b_x] = 0, \\
[a_x, b_y] = i b_z, \\
[b_x, a_y] = i b_z,
\]

etc.

By using only these relations he then obtains (algebraically \(^2\)) the explicit expressions of the matrix elements for given \(j\) and \(m\) \([12][13]\). The non-zero elements of the infinite matrices \(a\) and \(b\), whose diagonal elements are labelled by \(j\) and \(m\), are as follows:

\[
<j, m \mid a_x - i a_y \mid j, m + 1> = \sqrt{(j + m + 1)(j - m)},
\]

\(^2\)The algebraic method to obtain the matrix elements in Eq. (25) follows closely the analogous one for evaluating eigenvalues and normalization factors for angular momentum operators, discovered by Born, Heisenberg and Jordan in 1926 and reported in every textbook on Quantum Mechanics (see, for example, \([16]\)).
The quantities on which a and b act are infinite tensors or spinors (for integer or half-integer \(j\), respectively) in the given representation, so that Majorana effectively constructs, for the first time, infinite-dimensional representations of the Lorentz group. In [13] the author also picks out a physical realization for the matrices \(a\) and \(b\) for Dirac particles with energy operator \(H\), momentum operator \(p\) and spin operator \(\sigma\):

\[
\begin{align*}
a_x &= \frac{1}{\hbar} (yp_z - zp_y) + \frac{1}{2} \sigma_x, \\
b_x &= \frac{1}{\hbar} x \frac{H}{c} + \frac{i}{2} \alpha_x, \quad \text{etc.}
\end{align*}
\]

where \(\alpha_x, \alpha_y, \alpha_z\) are the Dirac \(\alpha\)-matrices.

Further development of this material then brought Majorana to obtain a relativistic equation for a wave-function \(\psi\) with infinite components, able to describe particles with arbitrary spin (the result was published in 1932 [13]). By starting from the following variational principle:

\[
\delta \int \bar{\psi} \left( \frac{H}{c} + \alpha \cdot p - \beta mc \right) \psi \, d^4x = 0, 
\]

By requiring the relativistic invariance of the variational principle in Eq. (27), Majorana deduces both the transformation law for \(\psi\) under an (infinitesimal) Lorentz transformation and the explicit expressions for the matrices \(\alpha, \beta\). In particular, the transformation law for \(\psi\) is obtained directly from the corresponding ones for the variables \(ct, x, y, z\) by means of the matrices \(a\) and \(b\) in the representation (26). By using the same procedure leading to the matrix elements in (25), Majorana gets the following expressions for the elements of the (infinite) Dirac \(\alpha\) and \(\beta\) matrices:

\[
\begin{align*}
<j, m | a_x + ia_y | j, m - 1 > &= \sqrt{(j + m)(j - m + 1)}, \\
<j, m | a_x | j, m > &= m, \\
<j, m | b_x - ib_y | j + 1, m + 1 > &= -\frac{1}{2} \sqrt{(j + m + 1)(j + m + 2)}, \\
<j, m | b_x - ib_y | j - 1, m + 1 > &= \frac{1}{2} \sqrt{(j - m)(j - m + 1)}, \\
<j, m | b_x + ib_y | j + 1, m > &= \frac{1}{2} \sqrt{(j + m + 1)(j - m + 1)}, \\
<j, m | b_x + ib_y | j - 1, m > &= \frac{1}{2} \sqrt{(j + m)(j - m)}.
\end{align*}
\]
\[
\begin{align*}
\langle j, m | \alpha_x + i \alpha_y | j - 1, m - 1 \rangle &= \frac{1}{2} \sqrt{\frac{(j + m)(j + m - 1)}{(j - 1/2)(j + 1/2)}}, \\
\langle j, m | \alpha_z | j + 1, m \rangle &= -\frac{1}{2} \sqrt{\frac{(j + m)(j - m + 1)}{(j + 1/2)(j + 3/2)}}, \\
\langle j, m | \alpha_z | j - 1, m \rangle &= -\frac{1}{2} \sqrt{\frac{(j + m)(j - m)}{(j - 1/2)(j + 1/2)}},
\end{align*}
\]

The Majorana equation for particles with arbitrary spin has, then, the same form of the Dirac equation:
\[
\left( \frac{H}{c} + \alpha \cdot \mathbf{p} - \beta \gamma^0 \right) \psi = 0,
\]

but with different (and infinite) matrices \(\alpha\) and \(\beta\), whose elements are given in Eqs. (28). The rest energy of the particles thus described has the form:
\[
E_0 = \frac{mc^2}{s + 1/2},
\]

and depends on the spin \(s\) of the particle. We here stress that the scientific community of that time was convinced that only equations of motion for spin 0 (Klein-Gordon equation) and spin 1/2 (Dirac equation) particles could be written down. The importance of the Majorana work was first realized by van der Waerden [17] but, unfortunately, the paper remained unnoticed until recent times.

5 FINAL RESULTS

As emerged from the above, when Majorana became aware of the great relevance of the Weyl’s application of the Group Theory to Quantum Mechanics, he immediately grabbed the Weyl method and developed it in many applications. What outlined in the previous sections gives account of only a partial look of the question involving the adoption and application of the new method, that arose just after the appearance of the Weyl book. There is no trace of further work in this direction by Weyl after 1928-1931, as discussed in Sec. 2, while Majorana continued to use Group Theory in his scientific work (see Secs. 3 and 4) till up 1938 when he lectured on this subject. Indeed the Weyl proposal for a “novel” description of Quantum Mechanics was completely adopted by Majorana, who applied it in his lectures on Theoretical Physics at the University of Naples (January - March 1938). In these lectures [7] he followed Weyl in “sandwiching” mathematical formalism with physics applications but, differently from the author of *Gruppentheorie und Quantenmechanik*, he started from physics rather than mathematics, and the physical viewpoint was underlying throughout the lectures.
We finish by reporting some passages from the starting lecture held in Naples by Majorana, revealing an attitude which, some decades after the appearance of the Weyl proposal, became the usual one for authors dealing with Quantum Mechanics:

“In order to illustrate Quantum Mechanics in its present form, two nearly opposite methods could apply. The first one is the so-called historical method..., the other is the mathematical one... Both methods, if applied in an exclusive way, present very serious problems...

Then, if we simply illustrate the theory according to its historical appearance, we at first would make the listener feel ill-at-ease or create suspicion... that nowadays is no longer justified. The second method... does not at all allow to understand the genesis of the formalism... and, above all, it completely disappoints the desire of guessing its physical meaning in some manner.

The only way which can be followed..., without leaving off anything of the historical genesis of the ideas or the language itself which at the moment dominate, is to put before an as wide and clear as possible exposition of the essential mathematical tools...

Our only ambition will be to illustrate as clearly as possible the effective use of these tools made by physicist in over a decade, in which use - that never led to difficulties or ambiguities - is the source of their certainty.”

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