On the Non-Abelian Stokes Theorem

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Abstract

We present the non-Abelian Stokes theorem for the Wilson loop in various forms and discuss its meaning. Its validity has been recently questioned by Faber, Ivanov, Troitskaya and Zach. We demonstrate that all points of their criticism are based on mistakes in mathematics. Finally, we derive a variant of our formula for the Wilson loop in lattice regularization.

1 Introduction

One of the main objects in the Yang–Mills theory is the Wilson loop or holonomy; it is defined as a path-ordered exponent,

$$W_r = \frac{1}{d(r)} \text{Tr P exp } i \oint d\tau \frac{dx^\mu}{d\tau} A_\mu^a T^a,$$

where $x^\mu(\tau)$ with $0 \leq \tau \leq 1$ parametrizes the closed contour, $A_\mu^a$ is the Yang–Mills field (or connection) and $T^a$ are the generators of the gauge group in a given representation $r$ whose dimension is $d(r)$.

It is generally believed that in three and four dimensions the average of the Wilson loop in a pure Yang–Mills quantum theory exhibits an area behavior for large and simple contours (like flat rectangular). This should be true not for all representations but those with ‘$N$-ality’ nonequal zero; in the simplest case of the $SU(2)$ gauge group these are representations with half-integer spin $J$. 
One of the difficulties in proving the asymptotic area law for the Wilson loop in half-integer representations (and proving that in integer representations it is absent) is that the Wilson loop is a complicated object by itself: it is impossible to calculate it analytically in a general non-Abelian background field. Meanwhile, it is sometimes easier to average a quantity over an ensemble than to calculate it for a specific representative. However, in case of the Wilson loop the path-ordering is a serious obstacle on that way.

A decade ago we have suggested a formula for the Wilson loop in a given background belonging to any gauge group and any representation [1]. In this formula the path ordering along the loop is removed, but at the price of an additional functional integration over all gauge transformations of the given non-Abelian background field. This formula is discussed below, in section 2. Furthermore, the Wilson loop can be presented in a form of a surface integral [2], see section 3. We call this representation the non-Abelian Stokes theorem. It is quite different from previous interesting statements [3, 4, 5, 6] also called by their authors ‘non-Abelian Stokes theorems’ but which involve surface ordering. Our formula has no surface ordering. A classification of ‘non-Abelian Stokes theorems’ for arbitrary groups and their representations has been given recently by Kondo et al. [7] who used the naturally arising techniques of flag manifolds.

Though these formulae usually do not facilitate finding Wilson loops in particular backgrounds, they can be used to average over ensembles of Yang–Mills configurations, and in more general settings, see e.g. [8, 9, 7, 10].

Our formula for the Wilson loop have been recently questioned by Faber, Ivanov, Troitskaya and Zach (FITZ) [11]. In section 4 we show that all points of their criticism are due to mistakes in mathematics, which we thoroughly locate, one by one. An alternative formula for the Wilson loop proposed by FITZ is also mathematically inconsistent, and we pinpoint their concrete errors.

Finally, in section 5 we present a variant of our formula for the Wilson loop in lattice regularization. It appears to be very similar to that presented in section 2.

## 2 Formula for the Wilson loop

Let \( \tau \) parametrize the loop defined by the trajectory \( x^\mu(\tau) \) and \( A(\tau) \) be the tangent component of the Yang–Mills field along the loop in the fundamental representation of the gauge group, \( A(\tau) = A^\mu_\mu(t^a dx^\mu/d\tau), \) \( \text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab} \). The gauge transformation of \( A(\tau) \) is

\[
A(\tau) \to S^{-1}(\tau) A(\tau) S(\tau) + iS^{-1}(\tau) \frac{d}{d\tau} S(\tau).
\]  

(2)

Let \( H_i \) be the generators from the Cartan subalgebra \( (i = 1, \ldots, r; \ r \text{ is the rank of the gauge group}) \) and the \( r \)-dimensional vector \( m \) be the highest weight of the representation \( r \) in which the Wilson loop is considered. The formula for the Wilson loop derived in ref. [1] is a path integral over all gauge transformations \( S(\tau) \) which should be periodic along the contour:

\[
W_r = \int DS(\tau) \exp i \int d\tau \text{ Tr } \left[ m_i H_i (S^{-1} A S + i S^{-1} S_d) \right].
\]  

(3)

Let us stress that eq. (3) is manifestly gauge invariant, as is the Wilson loop itself. For example, in the simple case of the \( SU(2) \) group eq. (3) reads:
where \( \tau_3 \) is the third Pauli matrix and \( J = \frac{1}{2}, 1, \frac{3}{2}, \ldots \) is the ‘spin’ of the representation of the Wilson loop considered.

The path integrals over all gauge rotations (3,4) are not of the Feynman type: they do not contain terms quadratic in the derivatives in \( \tau \). Therefore, a certain regularization is understood in these equations, ensuring that \( S(\tau) \) is sufficiently smooth. For example, one can introduce quadratic terms in the angular velocities \( iS^\dagger \dot{S} \) with small coefficients eventually put to zero; see ref. [1] for details. In ref. [1] eq. (4) has been derived in two independent ways: i) by direct discretization and ii) by using the standard Feynman representation of path integrals as a sum over all intermediate states, in this case that of an axial top supplemented by a ‘Wess–Zumino’ type of the action. Another discretization but leading to the same result has been used recently by Kondo [7]. A similar formula has been used by Alekseev, Faddeev and Shatashvili [12] who derived a formula for group characters to which the Wilson loop is reduced in case of a constant \( A \) field actually considered in [12]. In ref. [13] eq. (3) has been rederived in an independent way specifically for the fundamental representation of the \( SU(N) \) gauge group. Finally, at the end of this paper we consider the Wilson loop in lattice regularization and derive a formula similar to eq. (4).

The second term in the exponent of eqs. (3,4) is in fact a ‘Wess–Zumino’-type action, and it can be rewritten not as a line but as a surface integral inside a closed contour. Let us consider for simplicity the \( SU(2) \) gauge group and parametrize the \( SU(2) \) matrix \( S \) from eq. (4) by Euler’s angles,

\[
S = \exp(-i\gamma \tau_3/2) \exp(-i\beta \tau_2/2) \exp(-i\alpha \tau_3/2)
\]

\[
= \begin{pmatrix}
\cos \frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}} & -\sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}} \\
\sin \frac{\beta}{2} e^{-i\frac{\alpha-\gamma}{2}} & \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}}
\end{pmatrix}.
\]

The derivation of eq. (4) implies that \( S(\tau) \) is a periodic matrix. It means that \( \alpha \pm \gamma \) and \( \beta \) are periodic functions of \( \tau \), modulo \( 4\pi \). The second term in the exponent of eq. (4) which we denote by \( \Phi \) is then

\[
\Phi = \int d\tau \ Tr(\tau_3 i S^\dagger \dot{S}) = \int d\tau \dot{\alpha}(\cos \beta + \dot{\gamma})
\]

\[
= \int d\tau [\dot{\alpha}(\cos \beta - 1) + (\dot{\alpha} + \dot{\gamma})] = \int d\tau \dot{\alpha}(\cos \beta - 1).
\]

The last term is a full derivative and can be actually dropped because \( \alpha + \gamma \) is 4\pi-periodic, therefore even for half-integer representations \( J \) it does not contribute to eq. (4). Notice that \( \alpha \) can be 2\pi-periodic if \( \gamma \) (which drops from eq. (3)) is \( 2\pi, 6\pi, \ldots \)-periodic. If \( \alpha(0) = \alpha(0) + 2\pi k \) we shall say that \( \alpha(\tau) \) makes \( k \) windings. Integration over all possible \( \alpha(\tau) \) implied in eq. (4) can be divided into distinct sectors with different winding numbers \( k \).

Let us prove that eq. (4) can be written as an integral over any surface spanned on the contour (we shall call it a ‘disk’),

\[
\Phi = \frac{1}{2} \int d\tau d\sigma e^{abc} \epsilon_{ij} n^a \partial_i n^b \partial_j n^c, \quad i, j = \tau, \sigma,
\]

(7)
where \( \mathbf{n} \) is a unit 3-vector,

\[
n^a = \frac{1}{2} \text{Tr} \left( S^a \tau^a S_3 \right) = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta).
\]  

(8)

It is understood that \( \mathbf{n} \) or \( \alpha \) and \( \beta \) are continued inside the disk. We denote the second coordinate by \( \sigma \) such that \( \sigma = 1 \) corresponds to the edge of the disk coinciding with the contour and \( \sigma = 0 \) corresponds to the center of the disk. Let us establish what conditions should the continuation \( \mathbf{n}(\sigma, \tau) \) satisfy.

Eq. (7) can be identically rewritten as

\[
\Phi = \iint d\tau d\sigma \left[ \partial_\tau \alpha \partial_\sigma (\cos \beta - 1) - \partial_\sigma \alpha \partial_\tau (\cos \beta - 1) \right]
\]

\[
= \iint d\tau d\sigma \left\{ \partial_\sigma \left[ \partial_\tau \alpha (\cos \beta - 1) \right] - \partial_\tau \left[ \partial_\sigma \alpha (\cos \beta - 1) \right] \right\}
\]

\[
= \int_0^1 d\tau \left[ \partial_\tau \alpha (\cos \beta - 1) \right]_{\sigma=1} - \int_0^1 d\sigma \left[ \partial_\sigma \alpha (\cos \beta - 1) \right]_{\tau=1}.
\]

(9)

The second term here is zero, for the following reasons. Let \( \alpha(\tau) \) belong to the sector with winding number \( k \). The periodicity property of \( \alpha(\sigma, \tau) \) cannot change abruptly as we continue it inside the disk. We can write \( \alpha(\sigma, \tau) = 2\pi k \tau + \tilde{\alpha}(\sigma, \tau) \) where \( \tilde{\alpha} \) is strictly periodic, as well as \( \cos \beta \). Therefore, the integrand of the second term in eq. (9) is zero by periodicity.

Let us now consider the first term in eq. (9). We want the surface integral (7) to reproduce the contour integral (6), up to a possible contribution of \( 4\pi l \) with integer \( l \), coming from the center of the disk, \( \sigma = 0 \). Such a contribution is allowed since it does not affect eq. (4) even for half-integer \( J \). The contribution of the first term in eq. (9) at \( \sigma = 0 \) is

\[
- \int_0^1 d\tau \left[ 2\pi k + \partial_\tau \tilde{\alpha}(0, \tau) \right] [\cos \beta(0, \tau) - 1] = 2\pi k [\cos \beta(0) - 1] = 4\pi l.
\]

(10)

where we have used that \( \tilde{\alpha} \) is periodic while \( \beta(0, \tau) \) is in fact independent of \( \tau \), otherwise \( n^3 = \cos \beta \) would be not uniquely defined at the center of the disk. Eq. (10) means that either \( \alpha(\tau) \) belongs to the winding number \( k = 0 \) sector, then \( \beta(0) \) is arbitrary, or, if \( k \neq 0 \) then \( \beta = 0, \pi \) meaning that \( n^3 = \pm 1 \) at the center of the disk. Notice that for \( |n^3| \neq 1 \) the components \( n^{1,2} \) are nonzero and are varying very rapidly as function of \( \tau \) at the point \( \sigma = 0 \). These conditions can be summarized as the condition that \( \mathbf{n}(\sigma, \tau) \) should be smooth and uniquely defined inside the disk.

Let us note that if the surface is closed or infinite the r.h.s. of eq. (7) is the integer topological charge of the \( \mathbf{n} \) field on the surface:

\[
Q = \frac{1}{8\pi} \int d\sigma d\tau \epsilon^{abc} \epsilon_{ij} n^a \partial_i n^b \partial_j n^c.
\]

(11)

Eq. (7) can be also rewritten in a form which is invariant under the reparametrizations of the surface. Introducing the invariant element of a surface,

\[
d^2 S^{\mu\nu} = d\sigma d\tau \left( \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma} - \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \right) = e^{\mu\nu} d(\text{Area}),
\]

(12)

one can rewrite eq. (7) as
\[
\Phi = \frac{1}{2} \int d^2 \Sigma^{\mu\nu} \epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c.
\]  

(13)

We get for the Wilson loop

\[
W_J = \int Dn(\sigma, \tau) \exp \left[ i J \int d\tau (A^a \tau) + \frac{i J}{2} \int d^2 \Sigma^{\mu\nu} \epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c \right].
\]  

(14)

The interpretation of this formula is obvious: the unit vector \(n\) plays the role of the instant direction of the color ‘spin’ in color space; however, multiplying its length by \(J\) does not yet guarantee that we deal with a true quantum state from a representation labelled by \(J\) – that is achieved only by introducing the ‘Wess–Zumino’ term in eq. (14): it fixes the representation to which the probe quark of the Wilson loop belongs to be exactly \(J\).

3 Non-Abelian Stokes theorem

We can now rewrite the exponent in eq. (14) so that both terms appear to be surface integrals

\[
W_J = \int Dn(\sigma, \tau) \exp \left[ \frac{i J}{2} \int d^2 \Sigma^{\mu\nu} (-F^a_{\mu\nu} n^a + \epsilon^{abc} n^a (D_\mu n)^b (D_\nu n)^c) \right],
\]  

(15)

where \(D^a_\mu = \partial_\mu \delta^a_\mu + \epsilon^{acb} A^c_\mu\) is the covariant derivative and \(F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu\) is the field strength. Indeed, expanding the exponent of eq. (15) in powers of \(A^a\) one observes that the quadratic term cancels out while the linear one is a full derivative reproducing the \(A^a n^a\) term in eq. (14); the zero-order term is the ‘Wess–Zumino’ term (7) or (8). Note that both terms in eq. (15) are explicitly gauge invariant. We call eq. (13) the non-Abelian Stokes theorem. We stress that it is different from previously suggested Stokes-like representations of the Wilson loop, based on ordering of elementary surfaces inside the loop [3, 4, 5, 6]. It is understood that the functional integration measure in eq. (15) is such that \(W_J = 1\) for zero field.

The gauge-invariant field strength appearing in eq. (15),

\[
G_{\mu\nu} = F^a_{\mu\nu} n^a - \epsilon^{abc} n^a (D_\mu n)^b (D_\nu n)^c,
\]  

(16)

coincides in form with the field strength introduced by Polyakov [14] and ’t Hooft [15] in connection with monopoles. In that case the unit-vector field \(n^a\) has the meaning of the direction of the elementary Higgs field, \(\phi^a/|\phi|\).

One can introduce a ‘monopole current’,

\[
j^a_\mu = \frac{1}{8\pi} \epsilon_{\mu\kappa\lambda} \partial_\nu G_{\kappa\lambda}.
\]  

(17)

Using equations

\[
\epsilon_{\mu\kappa\lambda} D^a_\mu F^b_\kappa = 0, \quad \epsilon_{\mu\kappa\lambda} \epsilon^{abc} (D_\nu n)^a (D_\kappa n)^b (D_\lambda n)^c = 0,
\]

\[
\epsilon_{\mu\kappa\lambda} (D_\nu D_\kappa n) = 0, \quad n^a (D_\kappa n)^a = 0,
\]
one can easily check that this current is zero. In mathematical language it is the statement that any exact two-form is complete. There is a subtlety here, however. Let us consider a configuration which we shall call the ‘Wu–Yang monopole’, \( n^a = x^a/r \) (not to be confused with the monopoles of the \( A_\mu \) field). Let us take a contour of unit radius about the origin lying in the \( z = 0 \) plane, the equator. In terms of angles the monopole corresponds to \( \alpha(\tau) = \phi = 2\pi \tau, \beta = \theta \), see eq. (8). At the equator \( \theta = \pi/2 \), and we have from the line form\( \Phi = \int_0^1 d\tau \partial_\tau (\cos \beta - 1) = \int_0^1 d\tau 2\pi (-1) = -2\pi \).

Let us now compute \( \Phi \) in the surface form: first, over the upper hemisphere whose edge is the equator, second, over the lower hemisphere. For the upper hemisphere we introduce the variable \( \sigma = 1 - \cos \theta \), so that \( \sigma = 0 \) at the north pole and \( \sigma = 1 \) on the equator. Then \( \cos \beta - 1 = -\sigma \), and eq. (7) reads
\[
\Phi = \int_0^1 \int_0^1 d\tau d\sigma \partial_\tau (2\pi \tau) \partial_\sigma(-\sigma) = -2\pi,
\]
in agreement with eq. (18). For the lower hemisphere we define the variable \( \sigma = 1 + \cos \theta \) so that \( \sigma = 0 \) at the south pole and \( \sigma = 1 \) on the equator. Then \( \cos \beta - 1 = \sigma - 2 \), and eq. (7) reads
\[
\Phi = \int_0^1 \int_0^1 d\tau d\sigma \partial_\tau (2\pi \tau) \partial_\sigma(\sigma - 2) = +2\pi.
\]
There is no arithmetical contradiction here since in deriving the surface form (9) we have allowed for the contribution from \( \sigma = 0 \) to be a multiple of \( 4\pi \), and it is exactly what has happened here. Simultaneously, it is in correspondence with the fact that the integral over the full sphere around a monopole (see eq. (11)) is \( 4\pi \). In both cases the contribution of the ‘Wu–Yang monopole’ configuration to the functional integral (15) is
\[
\exp \pm 2\pi i J = (-1)^2 J,
\]
irrespective of whether the surface is drawn above or below the singularity of the \( n \) field. Notice that eq. (21) makes a clear distinction between integer and half-integer representations.

If, however, we take the surface which passes exactly through the center of the monopole, we shall be in trouble. For example, choosing the equatorial plane we have \( n^3 = 0 \) everywhere on the plane, hence the surface integral (9) is zero, in contradiction with eq. (18). It is because we have drawn the surface directly through the singularity and have thus violated the condition needed for the applicability of the surface form (7). Does it mean that singular configurations of the type \( n^a = x^a/r \) are altogether forbidden in the functional integral (15)? Probably not, as long as such singularities have a zero-measure probability to hit the surface.

Let us integrate the ‘monopole charge density’ \( j_0^* \) over a volume surrounded by a closed surface. According to the Gauss theorem,
\[
8\pi \int d^3V \varepsilon_{\mu\nu\rho} \partial_\mu n^\mu \partial_\nu n^\nu \partial_\rho n^\rho = -8\pi Q,
\]
where \( Q \) is the integer winding number \([\mathbb{I}]\) of the \( n \) field over the closed 2-surface surrounding the ‘monopole’; in the concrete case of the field \( n^a = x^a/r \) one has \( Q = 1 \). Since the surface can be drawn as close to the center as one wishes, it is tempting to say that the ‘Wu–Yang monopole’ has \( j_0^* = -\delta(3)(r) \). Since computing \( j_0^* \) involves singular operations, one can regularize the singularity, for example by replacing \( n^a \rightarrow x^a/(r^2 + \delta^2)^{1/2}, \ \delta \rightarrow 0 \). This regularization, indeed, leads to \( j_0^* = -3\delta^2/(r^2 + \delta^2)^{5/2}/(4\pi) \rightarrow -\delta(3)(r) \). It should be stressed, however, that this regularization violates the condition \( n^2 = 1 \) which has been crucial to prove that \( j_\mu^* = 0 \).

To conclude this discussion: The Wilson loop is defined as a contour integral, so when one writes it in a surface form one has to take care that it does reproduce the contour form. Generally, singular \( n(\sigma, \tau) \) leading, after regularization, to nonzero \( j_\mu^* \) imply that the result depends on how one draws the surface, which is unacceptable. However, e.g. a ‘gas’ of singularities of the type \( n^a = x^a/r \) with quantized \( 4\pi \) flux (in \( d = 4 \) it is a ‘gas’ of worldlines) is still admissible since the probability that the singularity occurs exactly on the surface has zero measure.

Let us briefly discuss gauge groups higher than \( SU(2) \): for that purpose we have to return to our eq. \( (3) \). Eq. \( (3) \) is valid for any group and any representation. It is easy to present it also in a surface form. We denote the combination of Cartan generators \( m_iH_i \) where \( m_i \) is the highest weight of a given representation \( r \) by \( \mathcal{H}_r \). Using the identity

\[
\epsilon_{ij} \partial_i \text{Tr} \mathcal{H}_r S^{-1} \nabla_j S = \epsilon_{ij} \text{Tr} \mathcal{H}_r \left[ S^{-1} (\nabla_i S \nabla_j S) + \left( S^{-1} \nabla_i \right) (\nabla_j S) \right] = \epsilon_{ij} \text{Tr} \mathcal{H}_r \left[ -\frac{i}{2} (S^{-1} F_{ij}) + \left( S^{-1} \nabla_i \right) (\nabla_j S) \right],
\]

we can present eq. \( (3) \) in a surface form:

\[
W_r = \int DS(\sigma, \tau) \exp i \int dS^{\mu \nu} \text{Tr} \mathcal{H}_r \left[ -\frac{i}{2} (S^{-1} F_{\mu \nu} S) + \left( S^{-1} \nabla_\mu \right) (\nabla_\nu S) \right].
\]

Actually, eqs. \( (23,24) \) depend not on all parameters of the gauge transformation but only on those which do not commute with the Cartan combination \( \mathcal{H}_r = m_i H_i \). In the \( SU(2) \) case one has \( m_i H_i = J_{3i} \), \( J = 1/2, 1, 3/2, \ldots \), since \( SU(2) \) is of rank 1, and there is only one Cartan generator. Therefore, in the \( SU(2) \) case one integrates over the coset \( SU(2)/U(1) \) for any representation; this coset can be parametrized by the \( n \) field as described above.

In case of higher groups the particular coset depends on the representation of the Wilson loop. For example, in case the Wilson loop is considered in the fundamental representation of the \( SU(N) \) group the combination \( m_i H_i \) is proportional to one particular generator of the Cartan subalgebra, which commutes with the \( SU(N-1) \times U(1) \) subgroup. [In case of \( SU(3) \) this generator is the Gell-Mann \( \lambda_8 \) matrix or a permutation of its elements.] Therefore, the appropriate coset for the fundamental representation of the \( SU(N) \) group is \( SU(N) \) / \( SU(N-1) \) / \( U(1) = CP^{N-1} \). A possible parametrization of this coset is given by a complex \( N \)-vector \( u^a \) of unit length, \( u_\alpha^a u^a = 1 \). To be concrete, the Cartan combination in the fundamental representation can be always set to be \( m_i H_i = \text{diag}(1,0,\ldots,0) \) by rotating the axes and subtracting the unit matrix. In such a basis \( u^a \) is just the first column of the unitary matrix \( S \) while \( u^\dagger_\alpha \) is the first row of \( S^\dagger \). Unitarity of \( S \) implies that \( u_\alpha^a u^\dagger_\alpha = 1 \).
In this parametrization eq. (3) can be written as

\[ W_{\text{fund}}^{SU(N)} = \int Du Du^\dagger \delta(u_\alpha^\dagger u^\alpha - 1) \exp i \int d\tau \frac{dx^\mu}{d\tau} u_\alpha^\dagger (i\nabla_\mu)^\alpha_\beta u^\beta. \]  

Using the identity,

\[
\begin{align*}
\epsilon_{ij} \partial_i \left( u^\dagger \nabla_j u \right) &= \epsilon_{ij} \left[ (\nabla_i u)^\dagger (\nabla_j u) + u^\dagger \nabla_i \nabla_j u \right] \\
&= \epsilon_{ij} \left[ -\frac{i}{2} (u^\dagger F_{ij} u) + (\nabla_i u)^\dagger (\nabla_j u) \right],
\end{align*}
\]  

we can present eq. (25) in a surface form:

\[ W_{\text{fund}}^{SU(N)} = \int Du Du^\dagger \delta(|u|^2 - 1) \exp i \int dS^{\mu\nu} \left[ \frac{1}{2} (u^\dagger F_{\mu\nu} u) + i (\nabla_\mu u)^\dagger (\nabla_\nu u) \right], \]  

where \( F_{\mu\nu} \) is the field strength in the fundamental representation. Eq. (24) has been first published in ref. [13] however with an unexpected overall coefficient 2 in the exponent. Eq. (27) presents the non-Abelian Stokes theorem for the Wilson loop in the fundamental representation of \( SU(N) \). In the particular case of the \( SU(2) \) group transition to eq. (15) is achieved by identifying the unit 3-vector: 

\[ n^a = u_\alpha^\dagger (\tau^a)^\alpha_\beta u^\beta \]

where \( u^{\alpha} = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i \frac{\phi}{2}} \\ \sin \frac{\theta}{2} e^{i \frac{\phi}{2}} \end{pmatrix}, \quad 2i u^\dagger \partial_\tau u = \dot{\alpha}(\cos \beta - 1) + (\dot{\alpha} + \dot{\gamma}). \]  

It should be mentioned that the quantity

\[ \int d\sigma d\tau \epsilon_{ij} i \partial_i u_\alpha^\dagger \partial_j u^\alpha = 2\pi Q \]

appearing in eq. (27) is the topological charge of the 2-dimensional \( CP^{N-1} \) model. For closed or infinite surfaces \( Q \) is an integer. For example, in case of the ‘Wu–Yang monopole’ \( u^\alpha \) can be chosen in two forms compatible with periodicity:

\[ u^{\alpha} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} e^{i\phi} \quad \text{or} \quad \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} \end{pmatrix}. \]  

They are regular everywhere except the negative (positive) \( z \) axis. One can regularize the string singularity of \( u^\alpha \) relaxing the condition \(|u|^2 = 1\). The coefficient \( 2\pi \) guarantees that eq. (27) does not actually depend on the choice of the surface even for singular \( u^\alpha \)’s, cf. the discussion around eq. (24) above. At the same time it means that \( CP^{N-1} \) ‘instantons’ of an effective 2-dimensional model, having integer topological charges, can hardly be relevant to the area behavior of the Wilson loop, as conjectured recently in ref. [7]. The coefficient with which the topological charge enters the formula for the Wilson loop (27) corresponds to the ‘instanton angle’ \( \theta = 2\pi \), hence it is unobservable from the point of view of the 2-dimensional instantons. Only configurations with half-integer topological charges (like that given by eq. (30) or its gauge equivalents) stand a chance of being of relevance to confinement.

In case the Wilson loop is taken in the adjoint representation of the \( SU(N) \) gauge group the combination \( m_i H_i \) in eq. (8) is the highest root. Only group elements of the
form \( \exp(i \alpha_i H_i) \) commute with this combination, belonging to the maximal torus subgroup \( U(1)^{N-1} \). Hence, in case of the adjoint representation one in fact integrates over the maximal coset \( SU(N)/U(1)^{N-1} = F^{N-1} \), i.e. over flag variables \([16, 7]\).

4 Reply to criticism by Faber, Ivanov, Troitskaya and Zach

In a recent paper [11] Faber, Ivanov, Troitskaya and Zach (FITZ) have questioned the validity of our formula for the Wilson loop, eq. (4). Their points can be summarized as follows:

1. A direct calculation using eq. (4) in two simple cases where the Wilson loop is known, i.e. for a pure gauge potential and for a vortex field, produces zero results instead of correct ones.

2. The regularized evolution operator for an axial top used by us to derive the formula for the Wilson loop, when computed directly, is zero.

3. The evaluation of this evolution operator by introducing small regulator moments of inertia, as suggested by us, is prohibited because it changes the symmetry from \( SU(2) \) to that of \( U(2) \).

4. Another formula for the Wilson loop is proposed which is similar to but different from our.

We consider these points below, one by one, and show that all four are groundless as due to errors in mathematics.

1. In sections 4,5 of their paper FITZ attempt to compute the Wilson loop in two simple cases: i) in a pure gauge background and ii) in a background of a vortex. In both cases the Wilson loop is known. In this calculation FITZ use our discretization of the path integral though without the regulator terms which, as we have stressed in the original paper [1], are important to get the correct result. Nevertheless, it could be a useful exercise, were it not for mistakes in mathematics.

   In this calculation FITZ use several relations for group characters citing a paper by Bars [17] “modified for our case”. The modification is not performed properly. Ref. [17] deals with the \( GL(N) \) group, and averaging is performed over the \( U(N) \) group, not \( SU(N) \) which is the case in question. The characters of a group are unambiguously defined for the elements of that group. Meanwhile, FITZ intensively use such ambiguous quantities as \( SU(2) \) characters of the non-\( SU(2) \) elements like \( t_3, t_3^2 = 1/4 \), etc. \( t_3 \) is one half of the Pauli matrix \( \tau_3 \). Though no explicit definition of the characters of non-\( SU(2) \) elements is given in the paper, from eqs. (64) and (A3) of ref.[11] one infers that FITZ implicitly use the definition:

\[
\chi_j[(t_3)^n U] = \sum_{m=-j}^{j} m^n D_{mm}^{i}(U) \tag{31}
\]
where \( D_{mn}^j(U) \) are Wigner’s finite rotation matrices, \( \sum_m D_{mn}^j(U) = \chi_j[U] \). [The properties of \( D \)-functions are given, e.g., in ref. [18].]

The key formula of FITZ calculation is their decomposition formula (51),

\[
\exp z \text{Tr}[t_3 U] = \sum_j a_j(z) \chi_j[t_3 U] = \sum_j a_j(z) \sum_{m=-j}^j m D_{mm}^j(U),
\]

and its inverse (eq. (53) of [11]),

\[
a_j(z) = \frac{3}{j(j+1)} \int dU \chi_j[t_3 U^\dagger] \exp (z \text{Tr}[t_3 U]).
\]

The quantities \( \chi_j[t_3 U] \) do not form a complete set of functions for the space of \( SU(2) \) matrices, therefore there are no reasons why the decomposition (32) is at all possible, for whatever coefficients \( a_j(z) \). To see that it is in fact wrong let us present the decomposition of the l.h.s. of eq. (32) using well-defined characters of group elements only. We make use of the fact that \( t_3 = (-i/2)(i\tau_3) = (-i/2)\exp(i\pi t_3) \) where the last factor is definitely an element of \( SU(2) \).

We have

\[
\exp z \text{Tr}[t_3 U] = \exp \left\{ (-iz/2)\text{Tr}[e^{i\pi t_3 U}] \right\} = \sum_j \tilde{a}_j(z) \chi_j[e^{i\pi t_3 U}]
\]

\[
= \sum_j \tilde{a}_j(z) \sum_{m=-j}^j e^{i\pi m} D_{mm}^j(U),
\]

\[
\tilde{a}_j(z) = e^{-i\pi j (2j+1)} \frac{2J_{2j+1}(z)}{z},
\]

the coefficients \( \tilde{a}_j(z) \) being well-known from the lattice strong-coupling expansion [20].

Eq. (34) differs significantly from eq. (32) suggested by FITZ; there exists no choice of coefficients \( a_j(z) \) for which eq. (32) coincides with the correct one, eq. (34). Eq. (34) and its inverse, eq. (33), (eqs. (51) and (53) of [11]) result from uncritical ‘modification’ of ref. [17] and are wrong.

Furthermore, assuming the decomposition (32) to hold true FITZ arrive to a “completeness condition” for their coefficients \( a_j(z) \) (see their eq.(59)):

\[
a_0^2(z) + \sum_{j>0} \frac{1}{3} j(j+1)a_j^2(z) = 1,
\]

where (see eq.(A6) of [11])

\[
a_j(z) = \begin{cases} 
\frac{2}{j(j+1)} J_1(z), & j = 0, \\
\frac{3(2j+1)}{J_0(z)} J_{2j+1}(z), & j = 1, 3/2, 5/2, \ldots, \\
0, & j = 1, 2, \ldots.
\end{cases}
\]

We plot the l.h.s. of (35) as function of \( z \) in Fig.1: it does not look as being identically unity, as claimed by FITZ. It is another manifestation of that eqs. (34,33) are wrong. \(^1\)

\(^1\)Ironically, FITZ have checked their “completeness condition” themselves but only at one value of \( z = 1 \) where the l.h.s. of eq. (35) is still rather close to unity being equal to 0.986. Having computed this value summing up the series on the l.h.s. up to \( j = 5/2 \) FITZ note: “Thus, the series converges slowly to unity” not paying attention to that their \( j = 5/2 \) contribution is \( \sim 10^{-4} \) and the next contribution at \( j = 7/2 \) is \( \sim 10^{-9} \). The series converges rather rapidly but not to unity.
Unfortunately, this “completeness condition” together with the erroneous coefficients $a_j(z)$ are at the heart of the calculation of the Wilson loop by FITZ both for the pure gauge and vortex cases, which cannot be, thus, considered as correct.

2. In section 6 of their paper FITZ attempt to compute the (regularized) evolution operator for the ‘Wess–Zumino’ action, following directly our approach. This calculation has been presented in some detail in the original paper [1], however, FITZ seem to be dissatisfied by it and present their own. Their final answer (eqs. (98) and (112) of ref. [11]), which differs from our, is a result of several mistakes.

First, going from eq.(83) to eq.(87) FITZ use a strange relation,

$$
\exp \left( \sum_{n=0}^{N} (-i) \frac{I_\perp}{2\delta} (-4) \right) = \exp \left( iN(N+1) \frac{I_\perp}{\delta} \right),
$$

instead of the correct (and trivial) $\exp(i2(N+1)I_\perp/\delta)$, where $I_\perp$ and $\delta$ are constants and $N$ is the number of pieces in which one divides the contour.

Second, and more important, both equations in (91) are erroneous, they do not follow from eq.(89) from where they are derived. Passing from eq.(89) to eq.(91) one gets:

$$
\begin{align*}
\text{Tr}(R_n R_{n+1}^\dagger) &= 2 - \frac{1}{4} \left[ \delta \alpha_n^2 + \delta \beta_n^2 + \delta \gamma_n^2 + 2\delta \alpha_n \delta \gamma_n \cos \beta_n \right], \\
\text{Tr}(R_n R_{n+1}^\dagger \tau_3) &= i(\delta \alpha_n + \delta \gamma_n \cos \beta_n), \\
\delta \alpha_n &= \alpha_{n+1} - \alpha_n, \quad \delta \beta_n = \beta_{n+1} - \beta_n, \quad \delta \gamma_n = \gamma_{n+1} - \gamma_n.
\end{align*}
$$

FITZ have written these formulae without the crucial factors $\cos \beta_n$. Because of this mistake the subsequent integration over Euler angles $\alpha, \beta, \gamma$ becomes gaussian, and the evolution operator for the axial top, as computed by FITZ, in fact becomes that of a free particle, which is definitely wrong. The factors $\cos \beta_n$ being reinstalled, the derivation returns to that of our paper [1].
3. The last objection by FITZ is to our alternative (and in fact equivalent) derivation of the evolution operator, this time through the standard Feynman representation for the path integral as a sum over intermediate states. FITZ quote our result for the evolution operator of an axial top with the ‘Wess–Zumino’ term, evolving from its orientation given by a unitary matrix $R_1$ at time $t_1$ to orientation $R_2$ at time $t_2$:

$$Z_{\text{Reg}}(R_2, R_1) = (2J + 1)D_{J,J}^I(R_2 R_1^\dagger) \exp \left[-i(t_2 - t_1)\frac{J}{2I_\perp}\right] \quad (39)$$

where $I_\perp$ is a regulator moment of inertia, $I_\perp \to 0$. Apart from a nontrivial dependence on the orientation matrices $R_{1,2}$ coming through the Wigner $D$-function, this expression contains a phase factor $\exp(-i(t_2 - t_1)\ldots)$. It is an overall factor independent of the external field: it can and should be absorbed into the integration measure to make the evolution operator unity for the trivial case $R_2 = R_1 = 1$. Indeed, dividing the time interval into $N$ pieces of small length $\delta$, $N\delta = t_2 - t_1$, one can write this factor as a product,

$$\exp \left[-i(t_2 - t_1)\frac{J}{2I_\perp}\right] = \prod_{k=1}^{N} \exp \left[-i\delta\frac{J}{2I_\perp}\right], \quad (40)$$

where, according to the regularization prescription of ref. [1], $\delta/I_\perp \ll 1$ so that each factor is close to unity. Each factor can be now absorbed into the integration measure $dR(t_k)$ in the functional-integral representation for the evolution operator (39). The fact that the factor is complex is irrelevant; moreover, it is typical for the path-integral representation of the evolution operators to have a complex measure, see the classical Feynman’s book [19]. However, FITZ write: “...a removal of the fluctuating factor is prohibited since this leads to the change of the starting symmetry of the system from SU(2) to U(2)”. It may seem that FITZ believe that an absorption of a constant factor into the integration measure changes the number of variables over which one integrates.

4. FITZ make an attempt to derive another formula for the Wilson loop (see section 2 of ref. [11]); this derivation is, however, inconsistent. Their resulting path-integral representation for the Wilson loop is (see eq.(30) of [11]; we simplify their notations to make the formulae better readable):

$$W_J = \frac{1}{(2J + 1)^2} \int D\Omega(\tau) \sum_{\{m(\tau)\}} (2J + 1) \exp \left(i \oint_C d\tau \frac{m(\tau)}{2}\right) \sqrt{2\text{Tr} [A^\Omega(\tau)A^\Omega(\tau)\dagger]}, \quad (41)$$

where $A^\Omega = \Omega A^\dagger \Omega^\dagger - i\dot{\Omega} \Omega^\dagger$ is the gauge-transformed Yang–Mills field tangent to the contour parametrized by $\tau$, $0 \leq \tau \leq 1$, such that $A = A_\mu dx^\mu/d\tau$, $\dot{\Omega} = \partial_\mu \Omega dx^\mu/d\tau$. One integrates over all gauge transformations $\Omega(\tau)$ in a given representation of the SU(2) group, labelled by spin $J$, and sums over the projections $m$ from $-J$ to $+J$ at all points of the loop.

Integration over a variable $m(\tau)$ assuming only integer or half-integer values is an unusual construction and one can question whether such an integral has a limiting value as one takes the number of discretization points $N$ to infinity. We have checked that eq. (41) does not possess a finite limiting value at least for the zero field, $A = 0$. However, this is not our main point. The main problem is the derivation of eq. (41) by FITZ.
If one looks into how eq. (41) has been derived by FITZ one finds that it is a result of two mistakes in mathematics.

FITZ start from using the following discretized form of the Wilson loop (see their eqs. 14,15):

\[
W_r = \frac{1}{d_r^2} \lim_{n \to \infty} \int \ldots \int d\Omega(\tau_1) \ldots d\Omega(\tau_N)
\]

\[
d_r \chi_r \left[ \Omega(\tau_N)(1 + iA \Delta\tau)\Omega^\dagger(\tau_{N-1}) \right] \ldots d_r \chi_r \left[ \Omega(\tau_1)(1 + iA \Delta\tau)\Omega^\dagger(\tau_N) \right], \quad \Delta\tau = \tau_k - \tau_{k-1},
\]

(42)

where gauge transformations \( \Omega \) and the gauge field \( A \) are taken in a given representation \( r \) with dimension \( d_r \); \( \chi_r \) is the group character. Indeed, integrating over gauge transformations at all points along the loop \( \Omega(\tau_k) \) and using the relation

\[
\int d\Omega(\Omega^\dagger)_{a_1b_1}(\Omega)_{a_2b_2} = \frac{1}{d_r} \delta_{a_1b_2} \delta_{b_1a_2},
\]

one recovers the Wilson loop as a path-ordered product of the factors \( (1 + iA\Delta\tau) \) along the loop. However at this point FITZ depart from a consistent derivation.

It is tempting to say that, since the discretization points on the contour can be taken as close to one another as one wishes, the gauge transformations \( \Omega \) at neighboring points are also close. It should be emphasized that this is, generally, wrong: one needs a fully independent integration over \( \Omega \)'s at neighbor points – otherwise the unity matrix on the r.h.s. of eq. (43) is not achieved. In other words the unitary matrix \( \Omega(\tau_k)\Omega^\dagger(\tau_{k-1}) \) is, generally, not close to the unity matrix, even though the points \( \tau_k \) and \( \tau_{k-1} \) are close. The derivative \( \dot{\Omega} \) does not exist in a strict sense since taking the neighbor points closer does not change the fact that \( \Omega(\tau_k) \) and \( \Omega(\tau_{k-1}) \) are independent integration variables.

This important circumstance is neglected by FITZ who say: “Due to the infinitesimality of the segments we can omit the path ordering operator” and write (see their eq. (18))

\[
\Omega(\tau_k)(1 + iA\Delta\tau)\Omega^\dagger(\tau_{k-1}) = \exp i \int_{\tau_{k-1}}^{\tau_k} d\tau A^\Omega(\tau).
\]

(44)

This equation is erroneous for reasons explained above. Only if \( \Omega(\tau) \) has a finite derivative one can expand \( \Omega(\tau_k)\Omega^\dagger(\tau_{k-1}) = 1 + \dot{\Omega}(\tau_{k-1})\Omega^\dagger(\tau_{k-1}) \Delta\tau \) but then the exponent on the r.h.s. of eq. (44) should be expanded too. If the derivative is large so that one cannot expand the exponent (and FITZ keep it), then eq. (44) is simply wrong. To see it more clearly let us take an example of a zero field, \( A = 0, \quad A^\Omega = -i\dot{\Omega} \Omega^\dagger \). Apparently,

\[
\Omega(\tau_k)\Omega^\dagger(\tau_{k-1}) \neq \exp \int_{\tau_{k-1}}^{\tau_k} d\tau \dot{\Omega} \Omega^\dagger
\]

(45)

if the l.h.s is an arbitrary unitary matrix. The path ordering on the r.h.s. is absolutely necessary to restore the l.h.s.

Even if one accepts the wrong eq. (44) the way FITZ proceed further is questionable. According to eq. (42) FITZ need to compute the character of eq. (44). Denoting the exponent in eq. (44) by an \( su(2) \) matrix \( \Phi \) in representation \( J \),

\[
\Phi = \int d\tau A^\Omega = \Phi^a T^a = \int d\tau \left[ A^\Omega(\tau) \right]^a T^a,
\]

(46)
the character can be written as
\[
\chi_J[\exp i\hat{\Phi}] = \sum_{m=-J}^{J} \exp im\sqrt{\Phi^a\Phi^a} = \sum_{m=-J}^{J} \exp im\sqrt{\int d\tau_1 [A^\Omega(\tau_1)]^a \int d\tau_2 [A^\Omega(\tau_2)]^a}
\]
\[
= \sum_{m=-J}^{J} \exp im\sqrt{\int \int d\tau_1 d\tau_2 c_J \text{Tr} [A^\Omega(\tau_1)A^\Omega(\tau_2)]},
\]
\[c_J = \frac{3}{J(J+1)(2J+1)}.\]

Meanwhile, FITZ use the following formula (their eqs.(19), (21) and (22))
\[
\chi_J[\exp i\hat{\Phi}] = \sum_{m=-J}^{J} \exp im\int \sqrt{2 \text{Tr} [A^\Omega(\tau)A^\Omega(\tau)]}.\]

Leaving aside the incorrect numerical coefficient, the square root of a product of integrals is not equal to the integral of the square root of the product of integrands. In this way FITZ arrive to their formula for the Wilson loop, eq. (41). Their \(SU(3)\) generalization (section 3 of [11]) is based on the same manipulation and is wrong from the start.

To conclude this section: all points of the FITZ’ criticism of our formula for the Wilson loop are based on their errors in mathematics, and the alternative formula suggested by these authors is mathematically inconsistent, too.

Finally, we would like to comment on two sentences from FITZ’ paper. In the Conclusion they write: “The use of the erroneous path integral representation for Wilson loops [meaning our formula] has led to the conclusion that at large distances the average value of the Wilson loops shows area-law falloff for any irreducible representation of \(SU(N)\)...” and “...has led to result supporting the hypothesis of maximal Abelian projection”. We are not aware of any work where either of the statements has been mathematically derived using our formula.

Obviously on the contrary, the non-Abelian Stokes theorem of section 3 stresses the difference between various representations for Wilson loops: in \(SU(2)\) there is a clear distinction between integer and half-integer representations (see, e.g., eq. (21)); in higher groups the number of integration variables itself depends on the representation (see the end of section 3).

5 Lattice-regularized formula for the Wilson loop

The Wilson loop on a lattice is just the trace of the product of link variables in a given representation along the discretized contour. Else, it is the character of the product of link variables,
\[
W_J = \frac{1}{2J+1} \chi_J (U_{N,N-1}U_{N-1,N-2}...U_{1,N})
\] (we consider the \(SU(2)\) gauge group for simplicity). The aim of this section is to derive a representation for eq. (49) analogous to the continuum eq. (4), using lattice regularization. It has the form:
\[ W_J = \mathcal{N}^{-1} \int \prod_{k=1}^N dS_k \exp \left( \frac{z}{2} \text{Tr} (S_k^\dagger U_{k,k-1} S_{k-1} \tau_3) \right), \]  

where \( z \) is a function of \( J \) and \( \mathcal{N} \) is a normalization coefficient, both of them to be determined below. \( U_{k,k-1} \) are link variables which in the continuum limit become \( U_{k,k-1} \approx 1 + iaA_k \) where \( a \) is the lattice spacing and \( A_k \) is the component of the Yang–Mills field along the link at point \( k \). In eq. (50) one integrates over all gauge transformations \( S_k \) at lattice sites \( k \) along the loop, with the condition that \( S_0 = S_N \) where \( N \) is the total number of links in the loop. Integration is over the Haar measure normalized to unity. We shall see that eq. (50) is valid for large Wilson loop with number of links \( N \gg 1 \). Notice that on cubic lattices \( N \) is always even for closed loops.

In the limit \( aA_k \ll 1 \) and of smoothly varying \( A_k \) and \( S_k \) eq. (50) coincides with eq. (4) provided one takes \( z = 2J \). However, eq. (4) has been derived from another regularization of the functional integral over gauge transformations, and there is no \textit{a priori} reason to expect that in the lattice regularization \( z \) should be the same.

Let us expand the exponent in eq. (50) according to eq. (34):

\[ W_J = \mathcal{N}^{-1} \int \prod_{k=1}^N dS_k \sum_j \sum_{m_k=-j_k}^{j_k} e^{i\pi m_k} D_{m_k m_k}^{j_k}(S_k^\dagger U_{k,k-1} S_{k-1}), \]  

\[ \tilde{a}_j(z) = e^{-i\pi j} (2j + 1) \frac{1}{2} J_{2j+1}(z). \]

Every matrix \( S_k \) enters twice this expression. The integration can be performed as follows:

\[ \int dS_k D_{m_k m_k}^{j_k+1}(AS_k) D_{m_k m_k}^{j_k}(S_k^\dagger B) = \frac{1}{2j_k + 1} \delta_{j_k,j_k+1} \delta_{m_k m_k+1} D_{m_k m_k}^{j_k+1}(AB). \]

Using eq. (53) we get:

\[ W_J = \mathcal{N}^{-1} \sum_j [b_j(z)]^N e^{i\pi N(j-m)} \sum_m D_{mm}^{j}(U_{N,N-1}U_{N-1,N-2} \cdots U_{1,N}), \]

\[ b_j(z) = \frac{2}{z} J_{2j+1}(z). \]

The factor \( e^{i\pi N(j-m)} = 1 \) for any \( j, m \) since \( j-m \) is an integer and \( N \) is even. Consequently,

\[ W_J = \mathcal{N}^{-1} \sum_j [b_j(z)]^N \chi_j (U_{N,N-1}U_{N-1,N-2} \cdots U_{1,N}), \]

\[ \mathcal{N} = \sum_j (2j + 1) [b_j(z)]^N, \]

where we have chosen the normalization factor \( \mathcal{N} \) such that \( W_J = 1 \) for unity link variables.

Thus, eq. (50) is actually a weighted sum of Wilson loops in all representations \( j \). However at \( N \gg 1 \) the sum in eq. (56) is dominated by the term with \( j \) maximizing the coefficient \( b_j(z) \). Then only one term survives both in the numerator and the denominator of eq. (53) and we obtain:
\[
W_j \xrightarrow{N \gg 1} \frac{1}{2J+1} \chi_J(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) \tag{58}
\]
as it should be for the Wilson loop in representation \(J\).

Let us choose \(z(J)\) from the requirement that \(b_j(z)\) is maximal at \(j = J\). We find that for \(J \leq 2\) one can choose

\[z(J) = 2(J + 1). \tag{59}\]

With this value of \(z(J)\) the Wilson loop in the needed representation \(J\) is reproduced by eq. (50) at large number of links. [We would like to mention on this occasion that one can obtain eq. (59) also in a continuum formulation where the regularization is performed by introducing small moments of inertia \(I_\perp, I_\parallel\), as in ref. \[1\]. In that paper we used the following limiting procedure: first \(I_\parallel \to 0\), then \(I_\perp \to 0\), and obtained \(z(J) = 2J\). Had we chosen the opposite order of limits we would get eq. (59).]

In fact, one is not bound to take simple values like given by eq. (59) but choose the values of \(z(J)\) for \(J = \frac{1}{2}, 1, \frac{3}{2} \ldots\) such that \(b_j(z(J))\) is \textit{maximum maximorum} at \(j = J\). The result of such ‘fine tuning’ is presented in Table 1, where we give the best value of \(z\) for given \(J\), the maximum value of \(b_{\text{best}}\) and the ratio of \(b_j(z)\) to \(b_{\text{best}}\) for the most ‘dangerous’ competitor at \(j \neq J\).

| \(J\) | \(\text{best } z\) | \(b_{\text{best}}\) | \(b_j(z)/b_{\text{best}}\) | \(j\) |
|------|-----------------|-----------------|-----------------|------|
| \(\frac{1}{2}\) | 3.25103 | 0.296 | 0.50 | (0) |
| 1 | 4.36765 | 0.198 | 0.60 | (1/2) |
| \(\frac{3}{2}\) | 5.46564 | 0.146 | 0.86 | (0) |
| 2 | 6.55104 | 0.114 | 0.83 | (1/2) |
| \(\frac{5}{2}\) | 7.62728 | 0.093 | 0.83 | (3) |
| 3 | 8.69644 | 0.078 | 0.85 | (7/2) |

If, for example, we wish to get the Wilson loop in representation \(J = \frac{5}{2}\), we choose the coefficient \(z(5/2) = 7.62728\). The largest contribution to eq. (50) will be from \(j = J = \frac{5}{2}\). The next largest but parasite contribution will be from \(j = 3\); however, the Wilson loop in this representation will be suppressed as \(0.83^N\) as compared to the needed \(j = \frac{5}{2}\) contribution. If the number of links in the loop is \(N = 16\) the suppression factor is 0.051, for \(N = 24\) it is 0.011.

The lattice version of the ‘non-Abelian Stokes theorem’ is trivial: one takes any surface (realized as a collection of plaquettes) inside a given discretized contour (realized as a chain of links) and writes in the exponent a sum of the terms \(\text{Tr} \left[S_{k}^1 U_{k,k-l} S_{k-l} \tau_3\right]\) for each link belonging to all plaquettes inside the contour. Since all internal links are encountered twice in this sum, once going in a positive and once in a negative direction, all links cancel except those lying at the border of the surface, that is on the contour itself. The cancellation of links is due to the fact that \(\text{Tr}(V \tau_3) = -\text{Tr}(V^\dagger \tau_3)\) for a unitary matrix \(V\). The lattice version of the non-Abelian Stokes theorem reads:
\[ W_J = N^{-1} \int \left( \prod_{k \in \text{surf}} dS_k \right) \exp \left\{ \frac{z}{2} \sum_{\text{plaq} \in \text{surf}} \sum_{\text{links} \in \text{plaq}} \text{Tr} (S^k U_{k,k-l} S_{k-l} \tau_3) \right\}. \] (60)

We would like to stress that any Stokes theorem is trivial when written in discretized form. At the same time, if we first assemble links belonging to one plaquette and ascribe to the quantities \( S \) and \( U \) arguments corresponding to the plaquette centers, we recover, in the appropriate continuum limit, the non-Abelian Stokes theorem for the continuum, eq. (15).

If one uses eq. (50) or eq. (60) for the average Wilson loop in the lattice formulation of gauge theory there is no need to perform explicitly integration over gauge transformations as these are included in the integration over link variables. Therefore, one can use the following formula for the average Wilson loop on the lattice:

\[ \langle W_J \rangle = N^{-1} \prod_l \int dU_l \exp \left[ \text{lattice action} \right] \exp \left[ \frac{z(J)}{2} \sum_{l \in C} \text{Tr}(U_l \tau_3) \right]. \] (61)

Actually, this formula has been suggested in ref. [8], however with the coefficient \( z(J) = 2J \) arising from another regularization. This is not too consistent: in lattice calculations one has to use lattice regularization for the Wilson loop. Therefore, one has either to put \( z(J) = 2(J + 1) \) (for small \( J \)) or, better, pick it up from Table 1.

Finally, we note that the choice of the matrix \( \tau_3 \) in eq. (61) is arbitrary: one can take any rotated matrix as well. This fact can be used to increase many times the statistics in numerical computation of the average Wilson loop from eq. (61). It may be interesting to study the dependence of the r.h.s of eq. (61) on the coefficient \( z \) for a fixed but large loop as it can provide new information on the difference of integer and half-integer representations in respect to confinement.

6 Conclusions

We have formulated the non-Abelian Stokes theorem for the Wilson loop. The path-ordering is replaced by an ordinary exponent of a surface integral, but at the price of an additional functional integration over all gauge transformations of the Yang–Mills potential, actually, over a coset depending on the representation in which the Wilson loop is considered. We have given several forms of this representation, and discussed requirements on the continuation of the fields inside the contour.

Since the validity of our formula for the Wilson loop, which is key to the non-Abelian Stokes theorem, has been questioned recently in ref. [11] we had to reply to that criticism. We have demonstrated that it is groundless as due to mistakes in mathematics, which we have thoroughly pinpointed, one by one.

As the lattice regularization of functional integrals is one of the most popular we have included the derivation of the formula for the Wilson loop in lattice regularization. The resulting eq. (50) is very similar to the continuum formula (4). The corresponding lattice version of the Stokes theorem is almost trivial.

A formula analogous to the non-Abelian Stokes theorem can be also derived in gravity theory for holonomies in curved spaces [21].
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