**The $D\pi$ form factors from analyticity and unitarity**

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**Abstract.** We study the shape parameters of the $D\pi$ scalar and vector form factors using as input dispersion relations and unitarity for the moments of suitable heavy-light correlators evaluated with Operator Product Expansions, including $O(\alpha_s^2)$ terms in perturbative QCD. For the scalar form factor, a low energy theorem and phase information on the unitarity cut are implemented to further constrain the shape parameters. We finally determine points on the real axis and isolate regions in the complex energy plane where zeros of the form factors are excluded.

1. Introduction

The knowledge of shape parameters of the $D\pi$ form factors is of interest for the determination of the element $|V_{cd}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix entering precision tests of the Standard Model. The branching fractions of the semileptonic decays $D \to \pi l\nu$ and $D \to Kl\nu$ have recently been analysed by the CLEO collaboration [1, 2], which renewed the interest in the theoretical study of these processes.

The knowledge on the form factors can be further improved using analyticity and unitarity techniques, in particular, the method of unitarity bounds [3, 4]. Using this framework, one can obtain bounds on the form factors using as input an integral of the modulus squared of the form factor along the unitarity cut, known from unitarity and a dispersion relation for a suitable correlator of the same current, which can be evaluated by Operator Product Expansion (OPE) in the spacelike region. Using standard mathematical techniques, one can then correlate the values of the form factor and its derivatives at different points inside the analyticity domain. A review of the method was presented recently in [5]. In this work, we apply the above technique to the $D\pi$ form factors. The work is motivated in part by the progress in perturbative QCD calculations, which now yield the heavy-light correlators of interest to order $\alpha_s^2$ [6] for a massless light quark and partly form the new experimental analyses of the semileptonic $D$ decays.

We investigate the shape parameters entering the Taylor expansion around $t = 0$,

$$f_k(t) = f_k(0) \left(1 + \lambda'_k \frac{t}{M^2_\pi} + \frac{1}{2} \lambda''_k \frac{t^2}{M^4_\pi} + \cdots \right), \ k = +, 0, \ (1)$$
and derive allowed ranges for the slopes $\lambda_k'$ and the curvatures $\lambda_k''$. Here, $f_+(t)$ and $f_0(t)$ are respectively the vector and scalar form factors and are related to each other by the relation,

$$f_0(t) = f_+(t) + \frac{t}{M_D^2 - M_\pi^2} f_-(t), \quad t = q^2 = (p - p')^2. \quad (2)$$

We further find points on the real $t$-axis and isolate regions in the complex $t$-plane where zeros of the form factors are excluded. The knowledge of the zeros are of interest, for instance, for the dispersive methods based on phase (Omnès-type representations) and for testing specific models of the form factors.

2. Review of the method

To start with, we have the heavy-light invariant amplitudes $\Pi_+(q^2)$ and $\Pi_0(q^2)$ defined by the vector-vector correlation function,

$$-(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_+(q^2) + q^\mu q^\nu \Pi_0(q^2) = i \int d^4x e^{iqx} (0|TV^\mu(x)V^\nu|0), \quad (3)$$

where $V_{\mu} = \bar{d}\gamma_\mu c$.

We then consider the moments of the invariant amplitudes at $q^2 = 0$,

$$\chi_k^{(n)} \equiv \frac{1}{n!} \frac{d^n}{d q^2^n} [\Pi_k(q^2)]_{q^2=0}, \quad k = +, 0, \quad (4)$$

which satisfy dispersion relations of the form,

$$\chi_k^{(n)} = \frac{1}{\pi} \int_{t_+}^\infty dt \frac{\text{Im}\Pi_k(t + i\epsilon)}{t^{n+1}}, \quad k = +, 0, \quad (5)$$

where $t_\pm = (M_D \pm M_\pi)^2$.

It is well-known from QCD that the amplitude $\Pi_+(q^2)$ satisfies a once subtracted dispersion relation, while for $\Pi_0(q^2)$ an unsubtracted relation converges. The quantities $\chi_+^{(n)}$ and $\chi_0^{(n)}$ are hence defined for $n \geq 1$ and $n \geq 0$, respectively. These relations are connected to the form factors $f_+(t)$ and $f_0(t)$ by means of unitarity,

$$\text{Im}\Pi_+(t + i\epsilon) \geq \frac{3}{2} \frac{1}{48\pi} \frac{[(t - t_+)(t - t_-)]^{3/2}}{t^3} |f_+(t)|^2, \quad (6)$$

$$\text{Im}\Pi_0(t + i\epsilon) \geq \frac{3}{2} \frac{t_+ + t_-}{16\pi} \frac{[(t - t_+)(t - t_-)]^{1/2}}{t^3} |f_0(t)|^2, \quad (7)$$

respectively for the vector and scalar form factor, which hold for $t > t_+$. The above expressions are the unitarity sums for the spectral functions, restricted to the contribution of the $D\pi$ states in the isospin limit.

OPE on the other hand allows for the calculation of $\chi_k^{(n)}$ as the sum of perturbative and nonperturbative contributions. While the perturbative parts of the moments of heavy-light correlators for $n \leq 7$ were calculated up to order $\alpha_s^2$ in [6], the leading non-perturbative contribution of the quark and gluon condensates were obtained from [7], the corresponding expressions of which are presented in [8]. For the vector and scalar form factors, we specifically use the dispersion relations for the moments $\chi_+^{(1)}, \chi_+^{(2)}, \chi_+^{(3)}$ and $\chi_0^{(0)}, \chi_0^{(1)}, \chi_0^{(2)}$ respectively, as a result of which we obtain, for each form factor, a family of three different constraints. The final
allowed domain for the parameters of interest will be the intersection of the three individual domains.

From the dispersion relations (5) and the unitarity conditions given in eqns.(6) and (7), it follows that each form factor \( f_k(t) \) satisfies a set of integral inequalities written as,

\[
\frac{1}{\pi} \int_{t_+}^{\infty} dt \, \rho_k^{(n)}(t)|f_k(t)|^2 \leq \chi_k^{(n)}, \quad k = +, 0,
\]

where the weights \( \rho_k^{(n)}(t) \) are the product of \( 1/t^{n+1} \) with the phase space factors entering the unitarity relations. Using the fact that the form factors \( f_+(t) \) and \( f_0(t) \) are analytic functions in the complex \( t \)-plane cut along the real axis from \( t_+ \) to \( \infty \), we apply standard techniques to derive from eqn.(8) constraints on their values, in particular on the shape parameters and on the regions in the real and complex energy plane where zeros are excluded. Specifically, we cast the problem into a canonical form via a conformal map, and derive a determinant which is central to our investigations for obtaining bounds on the shape-parameters \([5, 8, 9]\) and for finding the exclusion regions of the zeros. The quantity \( f_k(0) \) defined in eqn.(1) goes as an important input to our work. Besides, additional information for the scalar form factor provided by a low-energy soft-pion theorem namely the Callan-Treiman relation \([10]\), is exploited to obtain more stringent bounds on the shape parameters of the scalar form factor. For the Dπ case, the corresponding expression is given below,

\[
f_0(\Delta_{D\pi}) = f_D/f_π,
\]

where \( \Delta_{D\pi} = M_D^2 - M_π^2 \) is the relevant Callan-Treiman point and \( f_D \) and \( f_π \) are the meson decay constants.

Even more stringent constraints on the shape parameters can be obtained if we also have some information on \( f_k(t) \) on the unitarity cut, in particular, if the phase \( \delta_k(t) \) defined as,

\[
f_k(t + i\epsilon) = |f_k(t)|e^{i\delta_k(t)}, \quad k = +, 0,
\]

is known along a low-energy interval, \( t_+ \leq t \leq t_{in} \). This information on the phase can be implemented using the technique of generalized Lagrange multipliers and involves the solution of an integral equation. A review of the method and more references are given in \([5]\).

3. Inputs and Results

The perturbative and nonperturbative contributions to the moments \( \chi_k^n \) have been tabulated in \([8]\). The value of the inputs used for \( f_k(0) \) and \( f_0(\Delta_{D\pi}) \) which reads \( 0.67^{+0.10}_{-0.07} \) and \( 1.58 \pm 0.07 \) respectively are obtained from a recent work based on analyticity \([11]\). We roughly estimate the phase of the Dπ form factors from the masses and widths of the resonances dominant at low energies, namely, from the relativistic Breit-Wigner parametrization given by,

\[
\delta(t) = \arctan \left( \frac{M_R \Gamma(t)}{M_R^2 - t} \right),
\]

where \( M_R \) is the mass and \( \Gamma(t) \) the energy-dependent width -

\[
\Gamma(t) = \left( \frac{q(t)}{q(M_R^2)} \right)^{2J+1} \frac{M_R}{\sqrt{t}} \Gamma_R,
\]

written in terms of the angular momentum \( J \), the width \( \Gamma_R \) and the centre of mass momentum \( q(t) = \sqrt{(t - t_-(t - t_+)/4t} \).
The central values of the masses and widths of the lowest charged $D\pi$ vector and scalar resonances, namely the $D^*$ and $D^*_0$ respectively are listed in [12]. We note that the vector resonance $D^*$ is very close to the threshold so that a reasonable expression of the phase cannot be obtained from a Breit-Wigner parametrization, but in the scalar case we can assume that the phase $\delta_0(t)$ is reliably described by the expression (11) with $J = 0$. For our analysis, we consider the phase up to the point $t_{\text{in}} = (2.6 \text{ GeV})^2$, close to the first inelastic $D\eta$ channel opening at 2.42 GeV.

\[\text{Figure 1.} \text{ Constraints on the slope and curvature of the vector form factor obtained using as input different moments of the correlator. The point indicates the slope and curvature of the pole ansatz given in [13].}\]

\[\text{Figure 2.} \text{ Constraints on the slope and curvature of the scalar form factor obtained with the moment } \chi_0^{(0)} \text{, from the standard bound and by including the phase and the low-energy theorem. The point indicates the slope and curvature of the pole ansatz given in [13].}\]

The results of our analysis are presented in Fig.1, where the the interior of the ellipses represent the allowed domains in the slope-curvature plane for the vector form factor, obtained with three moments $\chi_{+}^{(n)}$ of the vector correlator with just $f_+(0)$ and no other input. While the smallest or in other words the best domain is obtained from the lowest moment, we find that the higher moments tend to slightly reduce the domain, since one must take the intersection of all the domains in order to fulfill simultaneously the constraints. We also indicate the slope and curvature of the simple pole ansatz [13], which is seen to satisfy the unitarity constraints coming from our analysis. In Fig.2, we show the bounds derived from the lowest moment $\chi_0^{(0)}$ of the scalar form factor where we also use information on the phase and the soft pion theorem. The effect of these additional constraints can clearly be seen in the figure. We also show the slope and curvature from the pole ansatz [13] which satisfies our constraints. It may be noted that
the above bounds were obtained with the central values of the input parameters. By varying simultaneously all the input values we can obtain more conservative regions, which are slightly larger than the domains shown in the figures.

Figure 3. Domain without zeros for the vector form factor, obtained from the lowest moment $\chi_{+}^{(1)}$ and the input $f_{+}(0)$.

We now present results on the zeros of the scalar and vector $D\pi$ form factors. We have considered in the following only the constraints imposed by the lowest moments, corresponding to $n = 1$ in the vector case and $n = 0$ in the scalar one, using $f_{+}(0)$ as input quantity. For the vector form factor, real zeros are excluded in the range $(-1.0, 0.80)$ GeV$^2$. While for the scalar case it is $(-2.51, 1.55)$ GeV$^2$ for the standard bounds. With the inclusion of the low-energy constraint in the case of the scalar form factor, the range is increased to $(-3.55, 3.89)$ GeV$^2$. We present in Figs. 3 and 4 the regions of excluded zeros in the complex $t$-plane for the vector and scalar form factors respectively. In the scalar case the zeros are excluded in a larger domain if the low-energy constraint is also imposed.

Figure 4. Domain without zeros for the scalar form factor, obtained from the lowest moment $\chi_{0}^{(0)}$ and the input $f_{0}(0)$ (smaller region) and using in addition the CT constraint (larger region).

4. Conclusions

In this work, we have used analyticity and unitarity techniques to derive information on the $D\pi$ form factors, which might be of interest for the determination of the element $|V_{cd}|$ of the CKM matrix. We have applied the formalism of unitarity bounds and have derived a family of constraints to be satisfied by the shape parameters of both the scalar and vector form factors at
$t = 0$ and explored regions on the real axis as well as in the complex plane where the form factors cannot have zeros. The exclusion regions for the zeros that we have isolated basically cover a significant part of the entire low energy region. They are of practical interest, for instance, for the dispersive methods based on phase (Omnès-type representations) and for testing specific models of the form factors. The technique can be extended also for deriving new parameterizations for the $D\pi$ form factors in the semileptonic region, similar to those proposed in [14] for the semileptonic $B \to \pi l \nu$ decay. The new parametrizations properly implement the singularities related to the lowest charmed resonances and are useful for the description of the semileptonic data (see Ref.[8] for more details).

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