A CONJECTURE OF EIGENVALUES OF THRESHOLD GRAPHS

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Abstract. Let $A_n$ be the anti-regular graph of order $n$. It was conjectured that among all threshold graphs on $n$ vertices, $A_n$ has the smallest positive eigenvalue and the largest eigenvalue less than $-1$. Recently, in [1] was given partial results for this conjecture and identified the critical cases where a more refined method is needed. In this paper, we deal with these cases and confirm that conjecture holds.

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1. Introduction

A simple graph $G = (V,E)$ is a threshold graph if there exists a function $w : V(G) \rightarrow [0, \infty)$ and a real number $t \geq 0$ called the threshold such that $uv \in E(G)$ if and only if $w(u) + w(v) \geq t$. This class of graphs was introduced by Chvátal and Hammer [4] and Henderson and Zalcstein [6] in 1977. They are an important class of graphs because of their numerous applications in many areas such as computer science and psychology [11].

One way to characterize threshold graphs is through an iterative process which starts with an isolated vertex, and where, at each step, either a new isolated vertex is added, or a dominating vertex is added. We represent a threshold graph $G$ on $n$ vertices using a binary string $(b_1, \ldots, b_n)$. Here $b_i = 0$ if vertex $v_i$ was added as an isolated vertex, and $b_i = 1$ if $v_i$ was added as a dominating vertex. We call our representation a creation sequence, and always take $b_1$ to be zero. If $n \geq 2$, $G$ is connected if and only if $b_n = 1$.

There is a considerable body of knowledge on the spectral properties of threshold graphs. For example, all eigenvalues except $-1$ and $0$ are main, meaning that the entries in the associated eigenvector do not sum to zero (see [12], Theorem 7.5). With the exception of $-1$ and $0$, all eigenvalues of threshold graphs are simple [7]. In [8] was proved that no threshold graph has eigenvalues in the interval $(-1, 0)$. For more spectral properties we suggest consulting the articles [1 2 3 5 9 10].

A distinguished subclass of threshold graphs is the family of anti-regular graphs $A_n$ which are the graphs with only two vertices of equal degrees. The Figure 1 shows the graph $A_{16}$. In [2], a nearly complete characterization of the eigenvalues of anti-regular graphs is given, and was proposed some
conjectures about it. Among them, we consider in this work the following one:

**Conjecture 1.** For each $n$, the anti-regular graph $A_n$ has the smallest positive eigenvalue and the largest negative eigenvalue less than $-1$ among all threshold graphs on $n$ vertices.

Recently in [1], was given partial results for this conjecture and identified the critical cases where a more refined method is needed. More exactly, the conjecture was proved for all threshold graphs on $n$ vertices except for $n − 2$ critical cases where the interlacing method fails. In this paper, we deal with these cases and confirms that conjecture holds.

The paper is organized as follows. The main tool used to prove the conjecture, and some known results are reviewed in Section 2. In Section 3, we present some auxiliaries results and finally in Section 4 we confirm that conjecture holds for the remaining cases.

![Figure 1. The anti-regular graph $A_{16}$](image)

2. **Background Results**

Recall that two matrices $R$ and $S$ are **congruent** if there exists a nonsingular matrix $P$ such that $R = P^T SP$. An important tool used in [7] was an algorithm for constructing a diagonal matrix $D$ congruent to $A + xI$, where $A$ is the adjacency matrix of a threshold graph, and $x$ is an arbitrary scalar. The algorithm is shown in Figure 2. The diagonal elements are stored in the array $d$, and the graph’s initial representation is stored in $b$.

Algorithm **Diagonalize** works bottom up. For a graph of order $n$, it makes $n − 1$ passes. Each diagonal element, except the first and last, participates in two iterations. During each iteration, the assignment to $d_m$ produces a final diagonal element. On the last iteration, when $m = 2$, the assignment to $d_{m−1}$ also produces a final diagonal element at the top.
Algorithm Diagonalize\( (G, x) \)

initialize \( d_i \leftarrow x \), for all \( i \)

for \( m = n \) to 2

\( \alpha \leftarrow d_m \)

if \( b_{m-1} = 1 \) and \( b_m = 1 \)

if \( \alpha + x \neq 2 \) //subcase 1a

\( d_{m-1} \leftarrow \frac{\alpha x - 1}{\alpha x^2 - 2} \)
\( d_m \leftarrow \alpha + x - 2 \)

else if \( x = 1 \) //subcase 1b

\( d_{m-1} \leftarrow 1 \)
\( d_m \leftarrow 0 \)

else //subcase 1c

\( d_{m-1} \leftarrow 1 \)
\( d_m \leftarrow -(1 - x)^2 \)
\( b_{m-1} \leftarrow 0 \)

else if \( b_{m-1} = 0 \) and \( b_m = 1 \)

if \( x = 0 \) //subcase 2a

\( d_{m-1} \leftarrow 1 \)
\( d_m \leftarrow -1 \)

else //subcase 2b

\( d_{m-1} \leftarrow \alpha - \frac{1}{x} \)
\( d_m \leftarrow x \)
\( b_{m-1} \leftarrow 1 \)

end loop

Figure 2. Algorithm Diagonalize.

Note when \( b_m = 0 \), the algorithm does nothing and moves to the next step. Also note that the values in \( b \) can change. In each iteration, the algorithm executes one of the five subcases. It should be noted that subcase 1a and subcase 2b are the normal cases, and the other three subcases represent singularities. Executing subcase 1b requires \( x = 1 \), executing subcase 2a requires \( x = 0 \), and executing subcase 1c requires \( \alpha + x = 2 \).

The next result from [7] will be used throughout the paper.

Theorem 1. Let \( G \) be a threshold graph and let \((d_v)_{v \in G}\) be the sequence produced by Diagonalize \( (G, -x) \). Then the diagonal matrix \( D = \text{diag}(d_v)_{v \in G} \) is congruent to \( A(G) - xI \), so that the number of (positive - negative - zero) entries in \((d_v)_{v \in G}\) is equal to the number eigenvalues of \( A(G) \) that are (greater than \( x \) - small than \( x \) - equal to \( x \)).

Lemma 1. If algorithm Diagonalize executes subcase 1c, then it will leave both a permanent negative and positive number on the diagonal.

Proof. The assignment \( d_m \leftarrow -(1 - x)^2 \) produces a negative number. The positive number written occurs with \( d_{m-1} \leftarrow 1 \). Normally, assignments to
of positive and negative eigenvalues of $G$ multiplicities of 0 and $-1$. The triple $(n_+(G), n_0(G), n_-(G))$ is called the inertia of $G$.

The following result is due to Bapat [8].

**Theorem 2.** In a connected threshold graph $G$ represented with $b$, $n_-(G)$ is the number of 1’s in $b$, and $n_0(G)$ is the number of substrings 00 in $b$.

Note that in the creation sequence of a connected threshold graph, every zero must be followed by a zero or one. So $u$ the number of zeros, equals $u_{00}$ the number of substrings 00, plus $u_{01}$ the number of substrings 01. If $v$ is the number of ones in the sequence, $n = v + u = v + u_{00} + u_{01}$. Therefore, $n_+(G) = n - n_-(G) - n_0(G) = n - v - u_{00} = u_{01}$. That is,

**Theorem 3.** In a connected threshold graph $G$, the number of occurrences of the substring 01 in its creation sequence equals $n_+(G)$, and $n_1(G)$ is the number of substrings 11 in $b$.

3. Basic Results

Throughout this section we let $G$ be a connected threshold graph of order $n \geq 3$, whose $\lambda(G)$ is a simple eigenvalue $\lambda(G) \neq -1, 0$.

**Lemma 2.** If $G$ is a threshold graph on $n$ vertices with $x = -\lambda(G)$, then Diagonalize$(G, x)$ produces a zero at the top of the diagonal.

**Proof.** Since $-x$ is an eigenvalue, by Theorem 1 we must obtain a zero on the diagonal. An inspection of the algorithm shows that since $x \neq 0, 1$, a zero can be written only during the algorithm’s last iteration, to the top of the diagonal. □

**Lemma 3.** Let $G$ and $H$ be two threshold graphs on $n$ vertices with their respective eigenvalues $\lambda(G), \lambda(H) \neq -1, 0$.

i: If the number of negative entries in Diagonalize$(H, x)$ exceeds the number of negative entries in Diagonalize$(G, x)$ by one, where $x = -\lambda(G)$, and $\lambda(G), \lambda(H) < -1$ then $\lambda(H) < \lambda(G)$

ii: If the number of positive entries in Diagonalize$(H, x)$ exceeds the number of positive entries in Diagonalize$(G, x)$ by one, where $x = -\lambda(G)$, and $0 < \lambda(G), \lambda(H)$ then $\lambda(G) < \lambda(H)$.

**Proof.** We check item (i). Since that the number of negative entries in Diagonalize$(G, x)$, where $x = -\lambda(G)$ corresponds to the number of eigenvalues of $G$ that are smaller than $\lambda(G)$, by Theorem 1 and Diagonalize$(H, x)$ exceeds this number by one, then the largest eigenvalue of $H$ less than $-1$ is smaller than $\lambda(G)$, that is, $\lambda(H) < \lambda(G)$. The item (ii) is similar. □
During diagonalization \((G, x)\), where \(x = -\lambda(G)\), it is impossible for the algorithm to enter subcase 1b or subcase 2a because \(x \neq 0, 1\). As we will show in the next section Diagonalize\((G, x)\) does not execute the subcase 1c. Then it must enter subcase 1a or subcase 2b initially, and remain in one of these two subcases. The key to solve our problem is to understand the behavior of the following functions:

\[
g(\alpha) = \frac{\alpha x - 1}{\alpha + x - 2} \tag{1}
\]

\[
f(\alpha) = \frac{1}{x} \tag{2}
\]

These functions, of course, are used in subcase 1a and subcase 2b, respectively. We regard \(x\) as fixed and \(\alpha\) is an indetermined.

During the execution of Diagonalize \((G, x)\), there is a sequence of \(n\) values calculated right to left

\[
\alpha_{G,x} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n = x) \tag{3}
\]

that are temporarily assigned to the diagonal, we call the \(\alpha\)-sequence. Except for the final value \(\alpha_1\), each gets overwritten. They are computed:

\[
\alpha_{i-1} = h_i(\alpha_i), \quad 2 \leq i \leq n
\]

where

\[
h_i(\alpha) = \begin{cases} 
  g(\alpha), & \text{se } b_{i-1} = 1 \\
  f(\alpha), & \text{se } b_{i-1} = 0
\end{cases}
\]

and \(g\) and \(f\) are defined in (1) and (2). As compositions we have

\[
\alpha_1 = h_2 \circ h_3 \circ \ldots \circ h_i(\alpha_i) \tag{4}
\]

for \(2 \leq i \leq n\). The sequence of \(h_i\) depends only on the original \(b_i\).

The Figure 3 illustrates the functions \(f\) and \(g\) for \(x = 4\).

\[\text{Figure 3. The functions } f \text{ and } g\]
Lemma 4. Both $f$ and $g$ are continuous and increasing on $(2 - x, +\infty)$.

*Proof.* Their derivatives $\frac{df}{d\alpha} = 1$ and $\frac{dg}{d\alpha} = \frac{(x-1)^2}{(\alpha + x - 2)^2}$ are positive. \hfill \Box

Lemma 5. The following properties hold for $f$ and $g$:

i: $f(\alpha) = 0$ if and only if $\alpha = \frac{1}{x}$.

ii: If $\alpha < \frac{1}{x}, x > 0$ and $\alpha + x - 2 < 0$ then $f(\alpha) < 0$ and $g(\alpha) > 0$.

iii: If $0 < \frac{1}{x} < \alpha < 2$ and $\alpha + x - 2 > 0$ then $f(\alpha) < g(\alpha)$.

iv: If $\alpha > 2$ then $g(\alpha) < f(\alpha)$.

*Proof.* Properties i and ii are easily verified. To see iii, we note that $f(\alpha) < g(\alpha)$ we must show that

$$\alpha - \frac{1}{x} = \frac{\alpha x - 1}{x} < \frac{\alpha x - 1}{\alpha + x - 2},$$

which is equivalent to $\frac{1}{x} < \frac{1}{\alpha + x - 2}$, which holds since $\alpha + x - 2 < x$. Using similar argument we prove the item iv. \hfill \Box

Lemma 6. Let $H$ be a connected threshold graph obtained from $G$ by changing a single $b_l, 1 < l < n$, from 1 to 0, and consider the execution of Diagonalize($H,x$).

i: If $\frac{1}{x} < \alpha_{l+1} < 2$, for $x > 0$, and $h_{l+1} = g$ is replaced by $f$, then $\alpha_1'$ will decrease.

ii: If $\alpha_{l+1} < \frac{1}{x}$, for $x > 0$, and $h_{l+1} = g$ is replaced by $f$, then $\alpha_1'$ will decrease.

iii: If $\frac{1}{x} < \alpha_{l+1}$, for $x < 0$, and $h_{l+1} = g$ is replaced by $f$, then $\alpha_1'$ will increase.

*Proof.* We prove item (i). Assuming subcase 1c is avoided, the new alpha sequence

$$\alpha_{H,x} = (\alpha_1', \ldots, \alpha_{l}', \alpha_{l+1}, \ldots, \alpha_k = x)$$

(5)

is computed exactly the same, except $h_{l+1}$ will change from $g$ to $f$. Since $\frac{1}{x} < \alpha_{l+1} < 2$, by Lemma 5 (part iii) $\alpha_l = f(\alpha_{l+1}) < g(\alpha_{l+1})$. Let $h = h_2 \circ h_3 \circ \ldots \circ h_{l}$ be the remaining composition in (4). By Lemma 4 each $h_i$ is continuous and increasing on $(2 - x, +\infty)$, so the composition must be continuous and increasing, and we have: $\alpha_1' = h(\alpha_k') < h(\alpha_k) = \alpha_1$. The proof is similar for items (ii) and (iii). \hfill \Box

Analogously, the following result can be verified.

Lemma 7. Let $H$ denote the connected threshold graph obtained from $G$ by changing a single $b_l, 1 < l < k$, from 0 to 1. Consider the execution of Diagonalize($H,x$),

i: If $\alpha_{l+1} > 2$, for $x > 0$ and $h_{l+1} = f$ is replaced by $g$, then $\alpha_1'$ will decrease.

ii: If $\alpha_{l+1} < \frac{1}{x}$, for $x < 0$ and $h_{l+1} = f$ is replaced by $g$, then $\alpha_1'$ will increase.
4. The proof of Conjecture

The connected anti-regular graph on \( n \) vertices, denoted by \( A_n \), is a threshold graph with binary sequence \( b = (0101 \ldots 01) \) when \( n \) is even and \( b = (00101 \ldots 01) \) when \( n \) is odd. It was proved in [2] (see, also [7]) that \( A_n \) has simple eigenvalues and moreover has inertia \( i(A_{2k}) = (k,0,0) \) if \( n = 2k \) is even and \( i(A_{2k+1}) = (k,1,k) \) if \( n = 2k+1 \) is odd, and therefore \( \lambda_{k+1}(A_{2k+1}) = 0 \) and \( \lambda_k(A_{2k}) = -1 \).

The following result is due [1].

**Proposition 1.** The interval \( \Omega = [-\frac{1-\sqrt{2}}{2},-\frac{1+\sqrt{2}}{2}] \) does not contain any eigenvalue \( \lambda \neq -1,0 \) of any threshold graph.

Now, let introduce the \( n-2 \) critical threshold graphs. According cited in [1] they are identified having binary sequence \( G = (0^{s_1}1^{t_1} \ldots 0^{s_k}1^{t_k}) \) such that

- If \( n = 2k+2 \) then either \( s_1 = 2 \) and exactly one of \( s_2, \ldots, s_k, t_1, \ldots, t_k \) is also equal to two and all others are one, or \( s_1 = 3 \) and all other \( s_i = t_i = 1 \).
- If \( n = 2k+1 \) then either \( s_1 = 1 \) and only one of \( s_2, \ldots, s_k, t_1, \ldots, t_k \) equals two and all others equal one.

Recall that \( \lambda^-(G) \) denotes the largest eigenvalue of a critical threshold graph \( G \) less than \(-1\) and \( \lambda^+(G) \) denotes the smallest positive eigenvalue of \( G \). For completing the proof of Conjecture [1] we need to show that \( \lambda^-(G) \leq \lambda^-(A_n) \) if \( n \) is even, and \( \lambda^+(A_n) \leq \lambda^+(G) \) if \( n \) is odd.

4.1. The case \( n \) is even.

**Lemma 8.** Let \( G \) be a graph having the largest eigenvalue \( \lambda^-(G) \) among all \( n-2 \) critical threshold graphs with \( s_1 = 2 \).

i: If \( G \) has binary sequence \( G = (b_1, b_2, \ldots, b_{i-1}, 0, 0, 1, 0, 1, \ldots, 0, 1) \) then after processing \( b_{i+1} = 0 \) and \( b_{i+2} = 1 \) by \texttt{Diagonalize}(\( G, x \)) the assignment is \( \alpha < \frac{1}{x} \), where \( x = -\lambda^-(G) \).

ii: If \( G \) has binary sequence \( G = (b_1, b_2, \ldots, b_{i-1}, 1, 1, 0, \ldots, 0, 1) \) then after processing \( b_{i+1} = 1 \) and \( b_{i+2} = 0 \) by \texttt{Diagonalize}(\( G, x \)) has assignment \( \frac{1}{x} < \alpha \), where \( x = -\lambda^-(G) \).

**Proof.** First we note that during execution of \texttt{Diagonalize}(\( G, x \)) where \( x = -\lambda^-(G) \), we must have \( \alpha = \frac{1}{x} \) only in the step \( m = 2 \), according to Lemma 2 and Lemma 5 (part (i)). Now, we check the item (i).

Let \( G \) having binary sequence \( G = (b_1, b_2, \ldots, b_{i-1}, 0, 0, 1, 0, 1, \ldots, 0, 1) \). We assume that in the \((i+1)\)-th iteration of \texttt{Diagonalize}(\( G, x \)) has assigned \( \alpha > \frac{1}{x} \). Let \( H \) be the threshold graph obtained from \( G \) by changing a single \( b_{i+1} \) from 0 to 1, and consider \texttt{Diagonalize}(\( H, x \)), where \( x = -\lambda^-(G) \). It is easy to see that in \( i \)-th iteration of \texttt{Diagonalize}(\( G, x \)) we will have \( f(\alpha) > 0 \) while that in \texttt{Diagonalize}(\( H, x \)) we will have \( g(\alpha) > 0 \), such that \( 0 < f(\alpha) < g(\alpha) \). Taking into account remaining elements of binary
sequence are equal follows Diagonalize\((H, x)\) will assigned \(\alpha' > \alpha_1 = 0\). Since \(G\) and \(H\) have the same number of substrings 01 (and therefore have the same number of positive eigenvalues) follows \(\lambda^-(G) < \lambda^-(H)\), what is a contradiction. The proof for item (ii) is similar.

Let \(G\) be a threshold graph having binary sequence \(G = (0^{s_1}1^{t_1} \ldots 0^{s_k}1^{t_k})\) and let \(\lambda^-(G)\) be the largest eigenvalue less than \(-1\). Let \(\delta_d(G)\) denotes the signal of the final diagonal of Diagonalize\((G, x)\), where \(x = -\lambda^-(G)\).

It follows from the Lemma 8 the following result.

**Theorem 4.** Let \(A_n\) be the anti-regular graph on \(n\) vertices and let \((d_i)\) be the final diagonal of Diagonalize\((A_n, y)\). If \(n \geq 3\) then

\[
\delta_d(A_n) = \begin{cases} 
(+, +, -, \ldots, -, +) & \text{if } n \text{ is even and } -y \in (\lambda^-(A_n), -1) \\
(-, +, +, \ldots, -, +) & \text{if } n \text{ is odd and } -y \in (\lambda^-(A_n), 0)
\end{cases}
\]

**Remark:** Let \(G\) be one of \(n - 2\) critical threshold graphs. Note that during execution of Diagonalize\((G, x)\), with \(x = -\lambda^-(G)\) the subcase 1c cannot occur for \(m = 2\) nor for \(m = 3\). If it occurs for an intermediate step then implies each substring of type \((1010)\) has final sign equal to \((+, +, +, -)\) contrary to Theorem 4.

For showing the conjecture, we first need to prove the following result.

**Theorem 5.** Let \(G\) be one of \(n - 2\) critical threshold graphs on \(n = 2k\) vertices. Then holds:

\[
i: \lambda^-(0^2101 \ldots 0^2101 \ldots 01) < \lambda^-(0^2101 \ldots 01^201 \ldots 01) \text{ for each } (s_k, t_k) \text{ and } \lambda^-(0^2101 \ldots 01^201 \ldots 01) < \lambda^-(0^2101 \ldots 010^21 \ldots 01) \text{ for each } (t_k, s_{k+1}) \text{ and } k > \frac{n}{2}.
\]

\[
ii: \lambda^-(0^2101 \ldots 10^2101 \ldots 01) < \lambda^-(0^2101 \ldots 0^210 \ldots 01) \text{ for each } (s_k, t_k) \text{ and } \lambda^-(0^2101 \ldots 0^2101 \ldots 01) < \lambda^-(0^2101 \ldots 01^201 \ldots 01) \text{ for each } (t_{k-1}, s_k), \text{ and } k \leq \frac{n}{2}.
\]

**Proof.** Let denotes by \(G_1 = (0^2101 \ldots 0^2101 \ldots 01)\) with \(s_k = 2\) and \(G_2 = (0^2101 \ldots 0^210 \ldots 01)\) with \(t_k = 2\) for \(k > \frac{n}{2}\). We show the inequality \(\lambda^-(G_1) < \lambda^-(G_2)\). According to Lemma 3 it is suffices to show that number of negative entries in Diagonalize\((G_1, x)\), exceed by one the number of negative entries in Diagonalize\((G_2, x)\), where \(x = -\lambda(G_2)\).

We consider the Diagonalize\((G_2, x)\), where \(x = -\lambda^-(G_2)\). Since that entries positive corresponds to the number of substrings 01, 11 and 00 in the creation sequence, and using the Lemma 2

\[
\delta_d(G_2) = (0, +, +, -, +, \ldots, -, +, +, +, -, +, \ldots, -)
\]

(6)

Now, we consider the Diagonalize\((G_1, x)\), where \(x = -\lambda(G_2)\). Since \(G_1\) and \(G_2\) have the same sequence \(b_i\) for \(i > t_k\) then they have the same values and therefore the same signs. Let \(\alpha\) be the most recent assignment common to both graphs. If \(G_2\) is the graph with largest \(\lambda^-(G)\) then by Lemma 8 we have \(\frac{1}{2} < \alpha\) and \(\alpha < 2\), since that subcase 1a was executed. Furthermore
$G_1$ can be obtained from $G_2$ by changing a single $b_i$ from 1 to 0. By Lemma 6 (item $i$) we will have the final value $\alpha'_1 < \alpha_1 = 0$, that is
\[
\delta_d(G_1) = (-, +, +, -, +, \ldots, -, +, -) \quad (7)
\]
Therefore thus, comparing the signs of final diagonal of both graphs in (6) and (7) we have $\lambda^-(G_1) < \lambda^-(G_2)$. The proof is similar for the others items.

**Corollary 1.** Among all threshold graphs of order $n = 2k$, the anti-regular graph $A_n$ has the largest eigenvalue less than $-1$.

**Proof.** Let $G_1, G_2, G_3$ and $A_n$ be threshold graphs having binary sequence $G_1 = (0^21^201 \ldots 0101 \ldots 01), G_2 = (0^2101 \ldots 01001 \ldots 01), G_3 = (0^2101 \ldots 0101 \ldots 01^2)$ and $A_n = (010101 \ldots 0101)$. We claim that
\[
\lambda^-(G_1) < \lambda^-(G_2) < \lambda^-(A_n) \quad (8)
\]
and
\[
\lambda^-(G_3) < \lambda^-(A_n) \quad (9)
\]
The inequality on the left in (8) is proved by similar way to Theorem 5 above. Now, we check the inequality on the right in (9).

We consider the Diagonalize($G_2, x$), where $x = -\lambda^- (A_n)$. Since $G_2$ and $A_n$ have the same $b_i$ for $i \geq n - 3$ then
\[
\delta_d(G_2) = (\delta(d_1), \delta(d_2), \delta(d_3), +, -, +, \ldots, -) \quad (10)
\]
We will show that $d_1 < 0$ and $d_2, d_3 > 0$. Since $b_1 = b_2 = b_3 = 0$ and $b_4 = 1$ the subcase $2b$ occurs for the last three steps of algorithm, then $d_2, d_3 > 0$. To see $d_1 < 0$, we consider the assignment $\alpha$ of Diagonalize($A_n, x$), where $m = 3$. Note we must have in $m = 2$ the assignment $\frac{1}{x}$ in Diagonalize($A_n, x$), according Lemma 5 (item $i$), and subcase $1a$ was executed in the previous step, follows that $\alpha = \frac{2}{x+1}$ is the assignment in $m = 3$. Since $\frac{1}{x} < \alpha < 2$ and, $G_2$ can be obtained from $A_n$ by changing $b_2$ from 1 to 0. Therefore thus, by Lemma 5 (item $i$) follows that $\alpha'_1 < \alpha_1 = 0$. Finally, comparing the signs of final diagonal of both graphs we have $\lambda^-(G_2) < \lambda^- (A_n)$.

Now, let $(d_i)$ be the final diagonal of Diagonalize($G_3, x$), with $x = -\lambda^- (A_n)$. We claim that
\[
\delta_d(G_3) = (-, +, +, -, +, \ldots, +, -) \quad (11)
\]
The subcase $1a$ occurs in the first step of Diagonalize($G_3, x$). Since that $x > 1$ we have that $d_n = 2(x-1) > 0$ and $\alpha = \frac{x+1}{x+2}$. Now, we have a subgraph isomorphic to anti-regular graph $A_{n-1}$ and assignment $\frac{x+1}{x+2} < x = -\lambda^-(A_n)$. Follows each substring 01, left a positive value and each substring 10, left a negative value. It remains to check only the sign of the last iteration. Since subcase $2b$ occurs in the last iteration, we have that $d_2 > 0$. We suppose that $d_1 > 0$. It implies that $A_{n-1}$ has $\lfloor \frac{n-1}{2} \rfloor + 2$ eigenvalues greater than $-\frac{n+1}{2}$. Since $A_{n-1}$ has exactly $\lfloor \frac{n-1}{2} \rfloor$ positive eigenvalues, and 0 is a simple
eigenvalue, follows that \( A_{n-1} \) has an eigenvalue in the interval \((-\frac{1-\sqrt{2}}{2}, 0)\), what is a contradiction. Thus, we must have \( d_1 < 0 \), and comparing the signs of final diagonal of both graphs we have that \( \lambda^{-}(G_3) < \lambda^{-}(A_n) \) as desired.

\[ \square \]

4.2. The case \( n \) is odd. We now treat the case \( n \) odd. Using a procedure similar to Theorem \( 5 \) with Lemma \( 6 \) (part iii) and Lemma \( 7 \) (part ii), it can be verified that:

- \( \lambda^+(01^201\ldots01) < \lambda^+(010^201\ldots01) < \ldots < \lambda^+(01\ldots10^2\ldots01) \) for \( s_k = 2, t_k = 2 \) and \( k \leq \lfloor \frac{n}{2} \rfloor - 1 \)
- \( \lambda^+(0101\ldots01^2) < \lambda^+(0101\ldots01^21) < \ldots < \lambda^+(01\ldots10^2\ldots01) \) for \( s_k = 2, t_k = 2 \) and \( k \geq \lfloor \frac{n}{2} \rfloor - 1 \)

**Corollary 2.** Among all threshold graphs of order \( n = 2k + 1 \), the anti-regular graph \( A_n \) has the smallest positive eigenvalue.

**Proof.** It is sufficient to show the following inequalities

\[ \lambda^+(00101\ldots01) < \lambda^+(01^201\ldots01) \] (12)

and

\[ \lambda^+(00101\ldots01) < \lambda^+(0101\ldots01^2) \] (13)

Let denotes \( G = (01^201\ldots01) \) and \( A_n = (00101\ldots01) \). We consider the Diagonalize\((G,x)\), where \( x = -\lambda^+(A_n) \). Since \( G \) and \( A_n \) have the same \( b_i \), for \( i \geq 3 \), in their creation sequence, then

\[ \delta_d(G) = (\delta(d_1), \delta(d_2), \delta(d_3), +, -, +, -, \ldots, +, -) \] (14)

We will show that \( d_1 > 0 \) and \( d_2, d_3 < 0 \). We consider the assignment \( \alpha \) of Diagonalize\((A_n, x)\), where \( m = 3 \). Note we must have in \( m = 2 \) the assignment \( \frac{1}{2} \) in Diagonalize\((A_n, x)\), according Lemma \( 5 \) (item i.), follows that \( \alpha = \frac{2}{x} \) is also the assignment in \( m = 3 \) in Diagonalize\((G,x)\). Then subcase 1a gives the following assignments: \( d_3 = \frac{2}{x} + x - 2 < 0 \) and \( d_2 = \frac{1}{2/x + x - 2} \in (-1, 0) \). Then the subcase 2b occurs for the last iteration and gives: \( d_2 = x < 0 \), and \( d_1 = \alpha - \frac{1}{x} > 0 \), since that \( \alpha \in (-1, 0) \) and \( x < 0 \).

Finally, comparing the signs of final diagonal of both graphs we have that \( \lambda^+(A_n) < \lambda^+(G) \). For the inequality \( 13 \) use the same procedure of second part of Corollary \( 1 \) by changing the signs of graphs.

\[ \square \]

**References**

[1] C.O. Aguilar, M. Ficarra, N. Schurman, B. Sullivan, The role of the anti-regular graph in the spectral analysis of threshold graphs, Linear Algebra and its Applications, 588 (2020) 210–223.

[2] C.O. Aguilar, J. y. Lee, E. Piato, B.J. Schweitzer, Spectral characterizations of anti-regular graphs, Linear Algebra and its Applications, 557 (2018) 84–104.

[3] R. B. Bapat, On the adjacency matrix of a threshold graph, Linear Algebra and its Applications, 439 (2013) 3008–3015.
[4] V. Chvátal, P. L. Hammer, Aggregation of inequalities in integer programming, in Studies in Integer Programming, P. L. Hammer, et al., Eds., Annals of Discrete Mathematics, 1, 145–162, North-Holland, Amsterdam, 1977.

[5] E. Ghorbani, Eigenvalue-free interval for threshold graphs, Linear Algebra and its Applications. 583 (2019) 300–305.

[6] P. B. Henderson, Y. Zalcstein, A graph-theoretic characterization of the PV class of synchronizing primitives, SIAM Journal on Computing 6 (1977) 88–108.

[7] D. P. Jacobs, V. Trevisan, F. Tura, Eigenvalue location in threshold graphs, Linear Algebra and its Applications 439 (2013) 2762–2773.

[8] D. P. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, Linear Algebra and its Applications 465 (2015) 412–425.

[9] J. Lazzarin, O.F. Márquez, F. C. Tura, No threshold graphs are cospectral, Linear Algebra and its Applications 560 (2019) 133–145.

[10] Z. Lou, J. Wang, Q. Huang, On the eigenvalues distribution in threshold graphs, Graphs and Combinatorics 35 (2019) 867–880.

[11] N. V. R. Mahadev, U. N. Peled, Threshold graphs and related topics, Elsevier, 1995.

[12] I. Sciriha, S. Farrugia, On the spectrum of threshold graphs, ISRN Discrete Mathematics (2011). doi:10.5402/2011/108509.

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