O-MINIMAL FLOWS ON ABELIAN VARIETIES.

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Abstract. Let $A$ be an abelian variety over $\mathbb{C}$ of dimension $n$ and $\pi: \mathbb{C}^n \rightarrow A$ be the complex uniformisation. Let $X$ be an unbounded subset of $\mathbb{C}^n$ definable in a suitable o-minimal structure. We give a description of the Zariski closure of $\pi(X)$.

1. Introduction.

Let $A$ be a complex abelian variety of dimension $n$. Write $A = \mathbb{C}^n/\Lambda$ where $\Lambda \subset \mathbb{C}^n$ is a lattice and let $\pi: \mathbb{C}^n \rightarrow A$ be the uniformisation map.

A subvariety $V$ of $A$ is called weakly special if $V = P + B$ where $P$ is a point of $A$ and $B$ is an abelian subvariety. The abelian Ax-Lindemann-Weierstrass theorem is the following.

Theorem 1.1. Let $Y$ be a complex algebraic subset of $\mathbb{C}^n$. The components of the Zariski closure of $\pi(Y)$ are weakly special subvarieties.

This theorem is due to Ax (see [1] and [2]) and plays an important role in the new proof by Pila and Zannier of the Manin-Mumford conjecture [7]. Note that the paper [7] provides a different proof of the abelian Ax-Lindemann-Weierstrass theorem. For a proof close in spirit to the contents of this paper, see Section 9 of [5]. In reality, in this statement, $Y$ can be taken to be only semialgebraic ($\mathbb{C}^n$ being identified with $\mathbb{R}^{2n}$).

The aim of this paper is to investigate the Zariski closure of the sets $\pi(X)$ where $X$ is definable in an o-minimal structure which is a much wider class of objects. We refer to the book [12] for the notion of a set definable in an o-minimal structure, in particular the structures $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$ (this last structure is actually defined and studied in [4]). Just recall that $\mathbb{R}_{an}$ is the o-minimal structure generated by the restricted analytic functions and $\mathbb{R}_{an,exp}$ is additionally generated by the graph of the real exponential. For a subset $\Sigma$ of $A$, we denote by $Zar(\Sigma)$ its Zariski closure.

To be able to prove anything, we will need to make certain additional assumptions. Firstly, the set $X$ will be assumed to be unbounded.
The necessity of this condition can be demonstrated by the following example. Let $\mathcal{F}$ be a connected bounded fundamental domain for the action of $\Lambda$ on $\mathbb{C}^n$. The restriction of $\pi$ to $\mathcal{F}$ is definable in $\mathbb{R}_{an}$. Let $V$ be any algebraic subvariety of $A$ and let $\tilde{V} = \pi^{-1}(V) \cap \mathcal{F}$. Then $\tilde{V}$ is definable in $\mathbb{R}_{an}$ and $\text{Zar}(\pi(\tilde{V})) = V$.

However, when $X$ is an unbounded real analytic manifold, we prove the following.

**Theorem 1.2.** Let $X$ be an unbounded real analytic manifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in an o-minimal structure which is an extension of $\mathbb{R}_{an}$.

Let $V = \text{Zar}(\pi(X))$. For any point $P$ of $\pi(X)$ there is a positive dimensional abelian subvariety $B_P$ of $A$ such that $P + B_P$ is contained in $V$.

In particular, $V$ contains a Zariski dense set of positive dimensional weakly special subvarieties.

To investigate general definable sets $X$, we will also impose some mild restrictions on the o-minimal structure. Let $\mathcal{S}$ be an o-minimal structure over $\mathbb{R}$, containing $\mathbb{R}_{an}$ and whose definable sets admit an analytic stratification (as defined in [12], Chapter 3). This condition holds for most ‘usual’ o-minimal structures, for example $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$ (see [4]). We fix such a structure $\mathcal{S}$ and in what follows and by definable, we will mean ‘definable in $\mathcal{S}$’.

Next we introduce the notion of essential Zariski closure. Let $X$ be an unbounded definable set as before. For $R > 0$, let $B(0,R)$ be the open unit ball of centre 0 and radius $R$. The variation of the sets $\pi(X \cap B(0,R))$ when $R$ varies is what we call an o-minimal flow. We show that for $R$ large enough, the Zariski closure of the set $\pi(X \setminus (X \cap B(0,R)))$ is constant. We call this the essential Zariski closure of $\pi(X)$ and denote it by $\text{Zaress}(\pi(X))$.

For an abelian subvariety $B$ of $A$, write $V_B \subset \mathbb{C}^n$ for the tangent space to $B$ at the origin and $p_B$ for the projection $\mathbb{C}^n \to V_B$.

We prove the following:

**Theorem 1.3.** Let $X$ be an unbounded definable subset of $\mathbb{C}^n$. Let $V$ be $\text{Zaress}(\pi(X))$.

For each point $P$, in $\pi(X)$, there exists a positive dimensional abelian subvariety $B_P$ of $A$ such that $P + B_P$ is contained in $V$.

In particular, $V$ contains a Zariski dense set of positive dimensional weakly special subvarieties.

We prove a characterisation of subvarieties of an abelian variety containing a Zariski dense set of weakly special subvarieties (see proposition 4.1). Let $V$ be such a subvariety. Our proposition 4.1 shows that
there exist abelian subvarieties $B$ and $B'$ of $A$ such that $A = B + B'$ and $B \cap B'$ is finite, $V = B + V'$ where $V'$ is a subvariety of $B'$.

We deduce the following.

**Theorem 1.4.** Assume that $X$ is a definable subset of $\mathbb{C}^n$ such that for all abelian subvarieties $B$ of $A$, $p_B(X)$ is unbounded. Then components of $\text{Zaress}(\pi(X))$ are weakly special.

The strategy of the proof of the theorem relies on the theory of o-minimality and Pila-Wilkie counting theorem. Let $X$ be as in the statement and $V$ be the Zariski closure of $\pi(X)$. Using a suitable definable set and applying Pila-Wilkie theorem, we show that there exists a positive dimensional semi-algebraic set $W \subset \mathbb{C}^n = \mathbb{R}^{2n}$ such that $X + W$ is contained in $\pi^{-1}(V)$. Applying the Ax-Lindemann-Weierstrass theorem, we then show that for any $P$ of $\pi(X)$, there exists a weakly special subvariety $P + B_p \subset V$.

Finally, we would like to point out one possible application of our theorem.

Recall the following theorem of Bloch-Ochiai (see Chapter 9 of [3]) which is proved using Nevanlinna theory.

**Theorem 1.5.** Let $A$ be an abelian variety and $f : \mathbb{C} \longrightarrow A$ be a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.

Theorem 1.4 implies some cases of theorem 1.5.

Consider for example $A = \mathbb{C}^n/\Lambda$ (where $\Lambda$ is a lattice such that $A$ is a simple abelian variety) and $f : \mathbb{C} \longrightarrow A$ given by $f(z) = (z, \ldots, z, e^z, \ldots, e^z) \mod \Lambda$ with $s$ factors of $z$ and $r$ times of $e^z$ with $r + s = n$. Then consider the set $X \subset \mathbb{C}^n$ given by

$$X = \{(x + iy, \ldots, x + iy, e^x e^{iy}, \ldots, e^x e^{iy}) : x \in \mathbb{R}, y \in [0, 2\pi]\}.$$ 

Clearly $X$ is unbounded and definable in $\mathbb{R}_{an,exp}$ and its image in $A$ is contained in $f(\mathbb{C})$. By theorem 1.4, the Zariski closure of $f(\mathbb{C})$ is $A$ (since $A$ is simple).

It is not however always possible to “extract” such a definable unbounded set $X$ from $f(\mathbb{C})$ as the example of $(e^z, e^{iz}) \subset \mathbb{C}^2$ shows. Indeed, in this example, for any subset $Y \subset \mathbb{C}$ such that $f(Y)$ is definable, both the real and imaginary parts of $z \in Y$ must be bounded.

Another (counter)-example is the following. Define the iterated exponential function $\exp_n(x)$ by $\exp_1 = \exp$ and $\exp_n = \exp \circ \exp_{n-1}$. By Proposition 9.10 of [4], a definable function in $\mathbb{R}_{an,exp}$ is bounded by $\exp_n(x^n)$ for some $n, m$. Therefore a graph of a function which ‘grows faster’ than any $\exp_n$ will not satisfy the assumptions of our theorems.
Note that it is a long-standing open problem whether there exists an o-minimal structure containing a “super-exponential” function.

We conclude this introduction with an open question in the spirit of [10]. It concerns the topological closure of $\pi(X)$ rather than Zariski closure. Recall from [10] that a real weakly special subvariety is defined to be a translate of a real subtorus of $\mathbb{A}$ (hence not necessarily algebraic).

**Conjecture 1.6.** Let $X$ be, as before, an unbounded definable real analytic manifold. We denote by $\overline{\pi(X)}$ the topological closure of $\pi(X)$.

There exists a real analytic submanifold $V$ of $\mathbb{A}$ containing a dense subset of real weakly special subvarieties such that

$$\overline{\pi(X)} = \pi(X) \cup V.$$  

In section 4, we prove a characterisation of subvarieties of abelian varieties containing a Zariski dense subset of weakly special subvarieties, namely that such a subvariety is a union of weakly special ones. We believe this result and our argument to be of independent interest.

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2. Proof of theorem 1.2.

In this section we assume that $X$ is an unbounded real analytic submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in some o-minimal structure which contains $\mathbb{R}_{an}$. Let $V$ be the Zariski closure of $\pi(X)$ in $\mathbb{A}$.

2.1. A definable set and point counting. The contents of this section are essentially a reproduction of the arguments of Orr from Section 9 of [5] with slight adjustments.

In this section we define a certain definable set associated with $X$ and, using Pila-Wilkie theorem, show that this set contains a positive dimensional semi-algebraic subset.

Choose a fundamental set $\mathcal{F}$ for the action of $\Lambda$ on $\mathbb{C}^n$ such that $X \cap \mathcal{F}$ is non-empty. We choose $\mathcal{F}$ to be an open connected subset of $\mathbb{C}^n$ such that $\overline{\mathcal{F}}$ is compact and $\Lambda$-translates of $\overline{\mathcal{F}}$ cover $\mathbb{C}^n$. The set $\mathcal{F}$
is an ‘open parallelepided’. Since $F$ is an open subset of $\mathbb{C}^n$, we have that $\dim(X \cap F) = \dim(X)$. Let $\tilde{V}$ be $F \cap \pi^{-1}V$. This is a definable set since the o-minimal structure contains $\mathbb{R}_{an}$ and $\pi$ restricted to $F$ is definable in $\mathbb{R}_{an}$.

Consider the definable set
$$\Sigma = \{ x \in \mathbb{C}^n : \dim(X + x) \cap \tilde{V} = \dim(X) \}.$$  

We have the following lemma:

**Lemma 2.1.** If $\lambda \in \Lambda$ and $X \cap (F - \lambda) \neq \emptyset$, then $\lambda \in \Sigma$.

**Proof.** From $\Lambda$-invariance of $\pi^{-1}V + \lambda = \pi^{-1}V$, we see that for $\lambda$ as in the statement (in particular for $\lambda \in \Lambda$), $X + \lambda \subset \pi^{-1}V$.

It follows that
$$(X + \lambda) \cap \tilde{V} = (X + \lambda) \cap F.$$  

As $F - \lambda$ is an open subset of $\mathbb{C}^n$, we see that
$$\dim(X \cap (F - \lambda)) = \dim(X) = \dim((X + \lambda) \cap F)$$  

The conclusion follows. \qed

Fix a basis $\lambda_1, \ldots, \lambda_{2n}$ of $\Lambda$. Then $\Lambda \otimes \mathbb{Q}$ is identified with $\mathbb{Q}^{2n}$. We define the height of an element $\lambda = \sum a_i \lambda_i \in \Lambda$ ($a_i \in \mathbb{Z}$) as
$$H(\lambda) = \max(|a_1|, \ldots, |a_{2n}|).$$

This height thus coincides with the usual height on $\mathbb{Q}^n$.

**Proposition 2.2.** There exists $T_0 \geq 0$ such that for all $T \geq T_0$,
$$|\{ x \in \Sigma \cap \Lambda : H(x) \leq T \}| \geq T/2.$$  

**Proof.** This is essentially Lemma 9.1 of [5].

The first observation is that if $x_1$ and $x_2$ are two points of $\Lambda$ such that $X \cap (F - x_1)$ and $X \cap (F - x_2)$ are both non-empty, then $\Sigma \cap \Lambda$ contains at least one point of height $h$ for every $h$ between $H(x_1)$ and $H(x_2)$.

Note that $X$ is path-wise connected in the Euclidean topology. Let $C$ be a path from a point in $X \cap (F - x_1)$ to a point in $X \cap (F - x_2)$.

When $C$ crosses over from $F - u_1$ to an adjacent domain $F - u_2$, the heights of $u_1$ and $u_2$ change by at most one.

It follows that for any $h$ between $H(x_1)$ and $H(x_2)$, there is a $u \in \Lambda$ of height $\leq h$ such that $X \cap (F - u)$ is not empty. This $u$ belongs to $\Sigma \cap X$.

By assumption $X$ is unbounded. Thus as $x$ varies in $\Lambda$ such that $X \cap F - x$ is non-empty, $h(x)$ goes to infinity.
It follows that there is an \( h_0 \) such that for any \( h > h_0 \), \( \Sigma \cap \Lambda \) contains at least one point of height \( h \).

Take \( T_0 = 2h_0 \).

\[ \square \]

**Remark 2.3.** The referee has pointed out to us that Tsimerman, in [11], has made a similar observation. Namely, that in a similar setting an unbounded analytic set should intersect ‘a lot of fundamental domains’.

We now use the following theorem of Pila and Wilkie ([6], Theorem 1.8).

For a definable subset \( \Theta \subset \mathbb{R}^n \), we define \( \Theta^{alg} \) to be the union of all positive dimensional semi-algebraic subsets contained in \( \Theta \). We define \( \Theta^{tr} \) to be \( \Theta \setminus \Theta^{alg} \).

**Theorem 2.4** (Pila-Wilkie). Let \( \Theta \) be a subset of \( \mathbb{R}^n \) definable in an \( \alpha \)-minimal structure. Let \( \epsilon > 0 \). There exists a constant \( c = c(\Theta, \epsilon) \) such that for any \( T \geq 0 \),

\[ |\{ x \in \Theta^{tr} \cap \mathbb{Q}^n : H(x) \leq T \}| \geq cT^\epsilon. \]

From Proposition 2.2 it now follows that \( \Sigma^{alg} \cap \Lambda \) is not empty.

Let \( W \) be a connected positive dimensional semi-algebraic subset contained in \( \Sigma \). For each \( w \) in \( W \), \( \dim((X + w) \cap \tilde{V}) = \dim(X) \) and hence an analytic component of \((X + w) \cap \mathcal{F} \) is contained in \( \pi^{-1}V \). By analytic continuation, we see that \( X + w \subset \pi^{-1}V \). We have proved:

**Proposition 2.5.** With the notations and assumptions of this section, there exists a positive dimensional semialgebraic subset \( W \) such that

\[ X + W \subset \pi^{-1}V. \]

2.2. **Final argument.** We use the following lemma whose proof can for example be found in [5], Lemma 8.1.

**Lemma 2.6.** Let \( \mathcal{Z} \) be a connected complex analytic subset of \( \mathbb{C}^g \). Let \( \mathcal{X} \) be a connected irreducible semialgebraic set contained in \( \mathcal{Z} \). Then there is a complex algebraic variety \( \mathcal{Y} \) such that \( \mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \).

By proposition 2.5 and the above lemma, we see that for any \( x \in X \), there exists a positive dimensional complex algebraic subset \( Y_x \) containing \( X \) and contained in \( \pi^{-1}(V) \). By the abelian Ax-Lindemann-Weierstrass theorem 1.1, the Zariski closure of \( \pi(Y_x) \) is a union of weakly special subvarieties of \( V \). Therefore, \( V \) contains a subvariety of the form \( P + B_P \) where \( P = \pi(x) \) and \( B_P \) is a positive dimensional abelian subvariety of \( A \). This finishes the proof of theorem 1.2.
3. **Cell decomposition and essential closure.**

In this section we consider an unbounded definable set \( X \subset \mathbb{C}^n \). We refer to section 8 of [4] for the definition of a real analytic cell. What is relevant to us is that a real analytic cell in \( \mathbb{R}^n \) is a definable real analytic submanifold, definable-analytically isomorphic to \( \mathbb{R}^m \) for some \( m \leq n \). By Theorem 8.9 of [4], there is a finite number of analytic cells \( X_1, \ldots, X_k \) such that \( X \) is a disjoint union of the \( X_k \).

**Proposition 3.1.** The essential closure \( \text{Zaress}(\pi(X)) \) is the union of \( \text{Zar}(\pi(X_i)) \) where \( X_i \)'s are the unbounded cells.

**Proof.** We start with a lemma.

**Lemma 3.2.** Let \( Z \) be a real analytic manifold in \( \mathbb{C}^n \) and \( U \subset Z \) an open subset.

Then

\[
\text{Zar}(\pi(U)) = \text{Zar}(\pi(Z))
\]

In particular, if \( Z \) is an analytic unbounded submanifold of \( \mathbb{C}^n \), then

\[
\text{Zaress}(\pi(Z)) = \text{Zar}(\pi(Z))
\]

**Proof.** One inclusion is obvious.

Write \( \text{Zar}(\pi(U)) \subset \mathbb{P}^m \) for some \( m \) and let \( s \in H^0(\mathbb{P}^m, \mathcal{O}(l)) \) for \( l \geq 1 \) such that \( s \) is zero on \( \pi(U) \). Then \( s \circ \pi \) is zero on \( U \) and by analytic continuation \( s \circ \pi \) is zero on \( Z \). It follows that \( s \) is zero on \( \pi(Z) \), hence \( \text{Zar}(\pi(Z)) \subset \text{Zar}(\pi(U)) \).

Let \( X = X_1 \amalg \cdots \amalg X_k \) be a cell decomposition of \( X \). For \( R \) large enough, \( X \cap B(0, R) \) contains the union of all the bounded cells in the above decomposition.

We have

\[
\text{Zaress}(\pi(X)) = \bigcup_{\{i: X_i \text{ unbounded}\}} \text{Zaress}(\pi(X_i)).
\]

By Lemma 3.2, for an unbounded cell \( X_i \),

\[
\text{Zaress}(\pi(X_i)) = \text{Zar}(\pi(X_i)).
\]

The result follows.

4. **Characterisation of subvarieties containing a dense set of weakly special subvarieties.**

In this section we prove a proposition which we believe to be of independent interest.
Let $A$ be an abelian variety and $V$ a subvariety of $A$. Define the stabiliser of $V$ as
\[
\text{Stab}(V) = \{ P \in A : P + V = V \}.
\]

Recall that for an abelian subvariety $B$ of $A$, there exists an abelian subvariety $B'$ such that $A = B + B'$ and $B \cap B'$ is finite. We always refer to $B$ and $B'$ as above.

**Proposition 4.1.** Let $V$ be an irreducible subvariety of $A$.

1. Assume $\dim \text{Stab}(V) > 0$.
   Then there exists abelian subvarieties $B$ and $B'$ of $A$ such that $A = B + B'$ and $V = B + V'$ where $V'$ is a subvariety of $B'$.

2. Assume that $\text{Stab}(V)$ is finite. Then the set of positive dimensional weakly special subvarieties contained in $V$ is not Zariski dense.

3. Assume again that $\text{Stab}(V)$ is finite. Let $\Sigma$ be the set of all positive dimensional weakly special subvarieties contained in $V$.
   For an abelian subvariety $B \subset A$, denote by $B'$ an abelian subvariety such that $A = B + B'$.
   There exists a finite set $B_1, \ldots, B_r$ of abelian subvarieties of $A$ and $W_1, \ldots, W_r$ of subvarieties of $B'_i$ such that
   \[
   \text{Zar}(\Sigma) = \bigcup_{i=1}^r B_i + W_i.
   \]

**Proof.** Assume $\dim \text{Stab}(V) > 0$ and let $B$ be the neutral component of $\text{Stab}(V)$.

Let $B'$ be an abelian subvariety such that $A = B + B'$ and let $\psi: A \rightarrow A/B$ be the quotient. Let $V'$ be $\psi|_{B'}^{-1}(\psi(V))$. Then
\[
V = \{ B + x : x \in V \} = \{ B + x : x \in V' \} = B + V'.
\]

This proves (1).

We will now prove (2). Assume that $\text{Stab}(V)$ is finite. We start by reducing to the case where $\text{Stab}(V) = \{0\}$. Let $A' = A/\text{Stab}(V)$ and let $\phi: A \rightarrow A'$ be the quotient map and let $V' = \phi(V)$. Note that $\phi^{-1}(V') = V + \text{Stab}(V) = V$. We claim that $\text{Stab}(V') = \{0\}$. Let $P \in \text{Stab}(V')$ and $Q \in \phi^{-1}(P)$. We have
\[
\phi(Q + V) = P + V' = V'
\]
It follows that $Q + V \subset \phi^{-1}(V') = V$ and for dimension reasons $Q + V = V$. Hence $Q \in \text{Stab}(V)$ and $P = \phi(Q) = 0$.

As the conclusion of (2) holds for $V$ if and only if it holds for $V'$, we may therefore assume that $\text{Stab}(V) = \{0\}$.
For \( m > 1 \), consider the map
\[
\phi_m : V^m \to A^{m-1}
\]
defined by
\[
\phi_m(x_1, \ldots, x_m) = (x_1 - x_2, \ldots, x_m - x_{m-1}).
\]
By [13], Lemma 3.1, there exists \( m > 1 \) such that the map \( \phi_m \) is a generic embedding.

Let \( P + B \) be a positive dimensional weakly special subvariety contained in \( V \). Then \( \phi_m((P + B)^m) = B^{m-1} \). The map \( \phi_m \) is therefore not injective on \( (P + B)^m \). Therefore \( V \) can not contain a Zariski dense set of positive dimensional subvarieties of the form \( P + B \). This proves (2).

Let us now prove (3). Let \( \Sigma \) as in the statement, the set of all positive dimensional weakly special subvarieties contained in \( V \) and let \( W \) be a component of \( \text{Zar}(\Sigma) \). Then \( W \) contains a Zariski dense set of weakly special subvarieties and by (2), \( \text{Stab}(W) \) is positive dimensional. It follows from (1) that \( W = B + W' \) where \( B \) is an abelian subvariety of \( A \) and \( W' \) a subvariety of \( B' \). Since \( \text{Zar}(\Sigma) \) has finitely many components, the conclusion of (3) follows.

\[\square\]

**Remark 4.2.** The geometric aspect of Lang’s conjecture predicts that given a variety of general type \( V \), the union of subvarieties, not of general type, is not Zariski dense. It is a known fact that a subvariety \( V \) of an abelian variety is of general type if and only if \( \text{Stab}(V) \) is finite. Therefore, our proposition 4.1 implies the geometric Lang’s conjecture for subvarieties of abelian varieties.

**Remark 4.3.** This proposition is an abelian analogue of the result of the first author (see [9]) in the hyperbolic case which is proved by completely different methods.

5. **Proof theorems 1.3 and 1.4**

In this section we deduce theorems 1.3 and 1.4 from the preceding results.

Let \( A \) and \( X \) be as in the assumptions of Theorem 1.3. Let \( V \) be a component of the essential Zariski closure of \( \pi(X) \).

In section 3 we have seen that \( \text{Zar}(\pi(X)) \) is a finite union of Zariski closures of sets of the form \( \pi(Y) \) where \( Y \) is an unbounded definable real analytic submanifold of \( \mathbb{C}^n \). Therefore, the conclusion of theorem 1.3 follows from theorem 1.2.

Let now \( X \) be as in 1.4. By theorem 1.3, \( V = \text{Zar}(X) \) contains a Zariski dense set of positive dimensional weakly special subvarieties.
From proposition 4.1, we deduce that $V$ is of the form $V = B + V'$ where $B$ is a positive dimensional abelian subvariety of $A$ and $V'$ is a subvariety of $B'$. Reiterating the argument with $B'$ and $V'$, we conclude that components of $V$ are weakly special.

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