Fast algorithms for morphological operations using run-length encoded binary images

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Abstract
This paper presents innovative algorithms to efficiently compute erosions and dilations of run-length encoded (RLE) binary images with arbitrary shaped structuring elements. An RLE image is given by a set of runs, where a run is a horizontal concatenation of foreground pixels. The proposed algorithms extract the skeleton of the structuring element and build distance tables of the input image, which are storing the distance to the next background pixel on the left and right hand sides. This information is then used to speed up the calculations of the erosion and dilation operator by enabling the use of techniques which allow to skip the analysis of certain pixels whenever a hit or miss occurs. Additionally the input image gets trimmed during the preprocessing steps on the base of two primitive criteria. Experimental results show the advantages over other algorithms. The source code of our algorithms is available in C++.

Keywords: binary image, dilation, erosion, filtering algorithms, image analysis, image denoising, computer vision, morphological operators, RLE, run-length encoding

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1. Introduction

Mathematical morphology is a general method for the analysis of spatial structures which aims at analysing the shape and form of objects [1]. In a variety of industrial computer vision applications, ranging from barcode scanning to the placement of chips in semiconductor industry, mathematical morphology is being used to process images and filter noise. We are interested in analysing binary images since these can be represented as sets. Therefore it enables us to use set-theoretical tools to process these images. Morphological operators which are used for noise filtering are constructed using two basic operators, namely erosion and dilation. In this paper we propose fast erosion and dilation algorithms on two-dimensional run-length encoded (RLE) binary images. A run $R = (lx, rx, y)$ is a concatenation of pixels in horizontal direction, where $y$ indicates the $y$–coordinate of these pixels, $lx$ the $x$–coordinate of the leftmost pixel and $rx$ the $x$-coordinate of the rightmost pixel, formally $R = (lx, rx, y) = \{(lx, y), (lx + 1, y), \ldots , (rx, y)\}$. Then a two-dimensional binary image $Z$ can also be described by its run-length representation $Z = \bigcup_{n=1}^{N} R_n$.

Previously, several very different ideas for the fast computation of erosion and dilation have been presented,

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such as using the decomposition of a rectangular-shaped structuring element [2], basing the algorithm on set-theoretical investigations [3, 4], solely operating on the contours of the image [5], or applying methods that are similar to string-matching techniques [6]. Inspired by some of the ideas introduced in [3, 4, 6] we propose two algorithms that extend these ideas by a couple of new theorems.

An outline of our paper is as follows: In Section 3 we provide the mathematical investigation of the erosion operator on RLE images. We come up with various ideas which can be used to formulate a fast algorithm. In Section 4 we take use of the duality between erosion and dilation to port our ideas to not only work with erosion but also with dilation. In Section 5 we propose fast algorithms based on the theorems given in the previous sections and in Section 6 we analyse the complexity of these algorithms. Finally we present runtime experiments and comparisons with other algorithms in Section 7 and our conclusion can be found in Section 8.

2. Preliminaries

As described in the Introduction, morphological operators are constructed using the two basic operators: erosion and dilation. A binary image is given by a set of pixels. Assuming \( X \subseteq \mathbb{Z}^d \) and the structuring element (SE) \( B \subseteq \mathbb{Z}^d \) to be binary images, these operators are defined as:

\[
\varepsilon_B(X) = \left\{ p \in \mathbb{Z}^d \mid B_p \subseteq X \right\},
\]

\[
\delta_B(X) = \left\{ p \in \mathbb{Z}^d \mid (B^t)_p \cap X \neq \emptyset \right\},
\]

where \( \varepsilon_B(X) \) denotes the erosion of \( X \) by \( B \), \( \delta_B(X) \) the dilation of \( X \) by \( B \), \( B^t \) the reflection about the origin and \( B_p \) the translation of \( B \) by a vector \( p \in \mathbb{Z}^d \). Instead of \( \varepsilon_B(X) \) and \( \delta_B(X) \) we also write \( X \ominus B \) and \( X \oplus B \) whenever it is convenient. A two-dimensional binary image \( Z \) can also be described by its run-length representation \( Z = \bigcup_{n=1}^{N} R_n \), where \( N \) denotes the number of runs and \( R_n \) the \( n \)th run of \( Z \). A compact representation of an RLE image is given when concatenated pixels are defined as a single run (not divided into several runs) and when runs are not overlapping. The proposed algorithms require compact RLE binary images as input and also return the eroded or dilated image in compact RLE representation.

3. Erosion Using RLE

In this chapter, we develop a theory to speed up the calculation of the erosion operator. This is done in five steps. In Section 3.1 we formulate a theorem which allows us to translate the structuring element before calculating the erosion such that the translated structuring element contains the origin and describe, how this improves the efficiency of our algorithm. In Section 3.2 we formulate a primitive criterion which
allows us to remove all runs of the input image that are shorter than the longest run within the structuring element. This is achieved in linear runtime complexity before calculating the erosion. Section 3.3 on one hand summarizes the results of [6] which are needed to prove the lemmata and theorems stated in Sections 3.4 and 3.5, on the other hand it also describes an efficient way to build the erosion transform tables and skeletons of a given image. In Section 3.4 we extend the Jump-Miss Theorem by formulating the Jump-Hit Theorem and in Section 3.5 we give another – easy to apply – criterion, which makes the investigation of a big number of pixels redundant (depending on the length of the shortest run of the structuring element).

3.1. Translating the SE such that it contains the origin

When the origin \( o = (0, \ldots, 0) \in \mathbb{Z}^d \) is contained in \( B (o \in B) \) expression (1) obviously reduces to

\[
\varepsilon_B(X) = \left\{ p \in X \mid B_p \subseteq X \right\}. \tag{3}
\]

The following theorem allows us to translate \( B \) by a vector \( q \) such that \( o \in B_q \) before calculating the erosion.

**Theorem 1.** Let \( X, B \subseteq \mathbb{Z}^d \) and \( q \in \mathbb{Z}^d \). Then we have that:

\[
\varepsilon_B(X) = [\varepsilon_{B_q}(X)]_q.
\]

**Proof.** Obviously

\[
[\varepsilon_{B_q}(X)]_q = \left\{ p \mid B_{p+q} \subseteq X \right\}_q = \{ p + q \mid B_{p+q} \subseteq X \} = \{ \tilde{p} \mid B_{\tilde{p}} \subseteq X \}
\]

by substitution \( \tilde{p} := p + q \).

Since our image \( X \) is given in RLE representation, we are able to look up all \( p \in X \) efficiently.

3.2. Removing short runs before eroding

Let \( X = \bigcup_{n=1}^N R^X_n \) and \( B = \bigcup_{m=1}^M R^B_m \) be two run-length encoded binary images in compact representation. The article [3] provided an elegant method to calculate the erosion directly using the RLE representations of \( B \) and \( X \):

\[
\varepsilon_B(X) = \bigcap_{m=1}^M \bigcup_{n=1}^N \varepsilon_{R^m_n}(R^X_n). \tag{4}
\]

While the erosion of runs can be computed very efficiently, the intersections and unions of runs take more effort. A runtime comparison with an algorithm which is based on this method can be found in Section 7.
We are able to use this result to prove the following theorem. It allows us to remove all runs of $X$ that are shorter than the shortest run within $B$ before calculating the erosion.

**Theorem 2.** Let $X = \bigcup_{n=1}^{N} R^X_n$ and $B = \bigcup_{m=1}^{M} R^B_m$ be two binary images in compact RLE representation and

\[
L_{\text{min}} := \min_{m \in \{1, \ldots, M\}} \{|R^B_m|\} \quad \text{as well as}

X_{L_{\text{min}}} := \bigcup_{1 \leq n \leq N} R^X_n \quad \text{for} \quad |R^X_n| \geq L_{\text{min}}.
\]

Then we have that:

\[\varepsilon_B(X) = \varepsilon_B(X_{L_{\text{min}}}).\]

**Proof.** For all runs $R^X_n$ with $|R^X_n| < L_{\text{min}}$ and all $m \in \{1, \ldots, M\}$ the following equality obviously holds:

\[\varepsilon_{R^B_m}(R^X_n) = \emptyset. \quad (5)\]

In the following, let $E^m_n := \varepsilon_{R^B_m}(R^X_n)$. By applying (4) we get:

\[
\varepsilon_B(X) = \bigcap_{m=1}^{M} \bigcup_{n=1}^{N} E^m_n
\]

\[
= \bigcap_{m=1}^{M} \left( \left[ \bigcup_{1 \leq n \leq N} E^m_n \bigcup \left[ \bigcup_{1 \leq n \leq N} E^m_n \right] \right] \right)
\]

\[
= \bigcap_{m=1}^{M} \bigcup_{1 \leq n \leq N} E^m_n \quad \text{using (5)}
\]

\[
= \varepsilon_B(X_{L_{\text{min}}}).
\]

### 3.3 Jump-Miss Theorem

Let $X, A \subseteq \mathbb{Z}^d$ be two binary images with $|X| < \infty$, and $|A| > 1$. Then the $n$-fold erosion is given by

\[
X \ominus_n A := \begin{cases} 
X & \text{for } n = 0, \\
(X \ominus_{n-1} A) \ominus A & \text{for } n > 0.
\end{cases}
\]
This allows us to introduce the erosion transform by

\[
 f^A_X(p) := \begin{cases} 
 \max_{n > 0} \{ n \mid p \in X \ominus n - 1 A \} & \text{if } p \in X, \\
 0 & \text{otherwise}.
\end{cases}
\]

The skeleton \( S_B^A \) of \( B \) with respect to \( A \) (where \(|A| > 1 \) and \( o \in A \)) is the set of all points \( p \in B \) which satisfy the following inequality (see [6, Proposition 2.7]):

\[
 \max_{e \in A} \{ f^A_B(p + e) \} \leq f^A_X(p). 
\] (6)

According to the following theorem when checking whether a structuring element \( B \) is contained in \( X \) at a position \( h \in \mathbb{Z}^d \) it is enough to compare the values \( f^A_B(h + s) \) and \( f^A_X(h + s) \) at all points of the skeleton \( s \in S_B^A \).

**Theorem 3.** [6, Theorem 3.3] Let \( A, B, X \subseteq \mathbb{Z}^d, |A|, |B|, |X| < \infty \) with \( o \in A, |A| > 1, \) and \( h \in \mathbb{Z}^d \). Then the following are equivalent:

1. \( h \in \varepsilon_B(X) \)
2. \( B_h \subseteq X \)
3. \( \forall s \in S_B^A: f^A_B(h + s) \leq f^A_X(h + s) \).

In practice, for most structuring elements \( B \) we have that \(|S_B^A| \ll |B| \). So once the erosion transform and skeleton are computed, the runtime of the actual erosion reduces to \( \mathcal{O}(|X||S_B^A|) \). Furthermore it was shown that it is also possible to skip the analysis of certain pixels whenever \( f^A_B(h + s) - f^A_X(h + s) > 0 \). This fact is described by the Jump-Miss Theorem which will be discussed next.

**Theorem 4.** (Jump-Miss Theorem) [6, Theorem 3.5] Let \( A, B, X \subseteq \mathbb{Z}^d \) with \( o \in A \) and \(|A|, |B|, |X| < \infty, |A| > 1 \) as well as \( s \in S_B^A \) and \( h \in \mathbb{Z}^d \). Then

\[
 f^A_X(h + s) < f^A_B(h + s)
\]

implies that there are at least

\[
 k := f^A_B(h + s) - f^A_X(h + s)
\]

points \( g \in \mathbb{Z}^d \) such that \( B_g \not\subseteq X \). These \( k \) points are given by \( g_i := h + i \cdot e \) for all \( i \in \{0, \ldots, k - 1\} \) and \( e \in A' \).

Now the question arises whether the erosion transform \( f^A_B \) needs to be recomputed for every given \( h \). The following lemma states that it is sufficient to only determine \( f^A_B \).
Lemma 5. [6, Proposition 3.2] Let $A, B \subseteq \mathbb{Z}^d$ with $|A| > 1$, $o \in A$ and $|A|, |B| < \infty$. Then for all $h \in \mathbb{Z}^d$ and $s \in S_B^A$:

$$f_B^A(s) = f_B^A(h + s).$$

When thinking of a convenient set $A$ to work with on run-length encoded images, one immediately comes up with $A = \{(-1, 0), (0, 0)\}$ or a similarly shaped form. This set ensures fast computation of $f_A^X$, $f_B^X$ and allows one to determine $S_B^A$ very efficiently. With this set the erosion transform $f_A^X$ evaluates to 0 outside of $X$, to 1 for the left-most pixel of a given run $R_n^X \subseteq X$ and increments its value for every following pixel (in horizontal direction) within the run, such that $f_A^X$ evaluates to $|R_n^X|$ for the rightmost pixel. The following theorem proves this statement.

Theorem 6. Let $A, X \subseteq \mathbb{Z}^2$, $|X| < \infty$, $h \in X$, and $A = \{(-1, 0), (0, 0)\}$. Then

$$f_A^X(h) = f_A^X(h + (-1, 0)) + 1.$$ 

Proof. We distinguish between two cases:

1. Case 1 ($h + (-1, 0) \notin X$):

   Then by definition we have that $f_A^X(h + (-1, 0)) = 0$, $h \in X = X \ominus_0 A$ and obviously

   $$X \ominus_1 A = \{p \mid \{(-1, 0), (0, 0)\}_p \subseteq X\} = \{p \mid \{(-1, 0) + p, (0, 0) + p\} \subseteq X\} \ni h$$

   because by assumption $((-1, 0) + h) \notin X$. Thus $f_A^X(h) = 1 = f_A^X(h + (-1, 0)) + 1$.

2. Case 2 ($h + (-1, 0) \in X$):

   Without loss of generality let $f_A^X(h + (-1, 0)) = \tilde{n}$. First we prove the following implication by induction:

   $$(h \in X \text{ and } h + (-1, 0) \in X \ominus_{\tilde{n} - 1} A) \implies h \in X \ominus_{\tilde{n}} A. \quad (7)$$

   (a) Base case ($\tilde{n} = 1$): When $h, h + (-1, 0) \in X$, obviously also $h \in X \ominus A = \{p \mid \{(-1, 0), (0, 0)\}_p \subseteq X\} = \{p \mid \{p + (-1, 0), p\} \subseteq X\}$.

   (b) Induction step ($\tilde{n} + 1$): Let $h \in X$ and $h + (-1, 0) \in X \ominus_{\tilde{n}} A$. Since $o \in A$, erosion is an anti-extensive operator [1], thus also $h + (-1, 0) \in X \ominus_{\tilde{n} - 1} A$ and by induction hypothesis $h \in X \ominus_{\tilde{n}} A$.

   We conclude $X \ominus_{\tilde{n} + 1} A = \{p \mid \{(p + (-1, 0), p)\} \subseteq X \ominus_{\tilde{n}} A\} \ni h$. This proves (7).

   From $f_A^X(h + (-1, 0)) = \tilde{n}$ we conclude that $h + (-1, 0) \in X \ominus_{\tilde{n} - 1} A$ and $h + (-1, 0) \notin X \ominus_{\tilde{n}} A$. Thus

   $$X \ominus_{\tilde{n} + 1} A = \{p \mid \{(p + (-1, 0), p)\} \subseteq X \ominus_{\tilde{n}} A\} \ni h$$
and using (7), we get \( h \in X \ominus \tilde{n}A \). Putting all together, this implies \( f_A^X(h) = \tilde{n} + 1 \).

Note that (6) and Theorem 6 imply that the skeleton can be retrieved by taking the rightmost pixel of each run.

### 3.4. Jump-Hit Theorem

In the previous section we described a criterion which allows us to skip the analysis of certain pixels whenever the algorithm finds a pixel \( h \) such that \( h \notin \varepsilon_B(X) \). We also call this a miss. Now we formulate a criterion which applies for hits. A hit denotes the occurrence of a \( h \) such that \( h \in \varepsilon_B(X) \).

**Lemma 7.** Let \( X \subseteq \mathbb{Z}^2 \) be a binary image, \( |X| < \infty \), \( p \in X \) and \( A = \{(-1,0),(0,0)\} \). Then the condition \( f_A^X(p) = n \) implies

\[
\forall i \in \{0, \ldots, n - 1\} : f_A^X(p + (i,0)) = f_A^X(p) + i.
\]

**Proof.** Obviously this statement holds for \( i = 0 \). Let \( i \in \{1, \ldots, n - 1\} \). For the given set \( A \) the equality \( f_A^X(p) = n \) implies that for all \( i \in \{1, \ldots, n - 1\} \) also \( (p + (i,0)) \in X \). By applying Theorem 6 multiple times we get \( f_A^X(p + (i,0)) = f_A^X(p) + i \).

Note that Lemma 7 does not hold for arbitrary sets \( A \).

**Theorem 8. (Jump-Hit Theorem)** Let \( B,X \subseteq \mathbb{Z}^2 \), \( A = \{(-1,0),(0,0)\} \) and \( (h_x,h_y) = h \in \mathbb{Z}^2 \). If for all \( s \in S_A^B \) we have that \( f_B^A(h + s) \leq f_A^X(h + s) \), then the complete eroded run is given by

\[
\langle h_x, h_x + n - 1, h_y \rangle \subseteq \varepsilon_B(X),
\]

where \( n := \min_{s \in S_A^B} \left\{ f_A^X(h + s) \right\} \).

**Proof.** Using Theorem 3 we get \( h \in \varepsilon_B(X) \) and Lemma 7 implies for all \( s \in S_A^B \) and all \( i \in \{0, \ldots, n - 1\} \) that

\[
f_A^X(h + s + (i,0)) = f_A^X(h + s) + i \quad \text{with Lemma 7}
\]

\[
\geq f_A^X(h + s) \quad \text{for } i \geq 0
\]

\[
\geq f_B^A(h + s), \quad \text{by assumption}
\]

from which we conclude – again with Theorem 3 – that \( (h + (i,0)) \in \varepsilon_B(X) \) for all \( i \in \{0, \ldots, n - 1\} \) and therefore \( \langle h_x, h_x + n - 1, h_y \rangle \subseteq \varepsilon_B(X) \).

We want to emphasize that, in case of a miss, the length of a jump is limited by the width of the structuring element. When having a hit, however, the length of the jump is always maximal and determined by just checking the values of \( f_A^X(h + s) \) at its starting pixel \( h \) along the pixels \( s \) of the skeleton \( S_A^B \).
3.5. Skipping the analysis of additional pixels

The goal of this section is to find a binary image $X_{\text{cut}} \subseteq X_{\text{Lmin}}$ such that $|X_{\text{cut}}| \ll |X_{\text{Lmin}}|$ and

$$\varepsilon_B(X) = \left\{ p \in X_{\text{cut}} \mid B_p \subseteq X_{\text{Lmin}} \right\}.$$ 

In the upcoming lemma we investigate a very special class of structuring elements. These elements include a run whose rightmost element is the origin. In a next step we are going to extend the lemma such that this condition is no longer needed.

**Lemma 9.** Let $X = \bigcup_{n=1}^{N} R_n^X$ and $B = \bigcup_{m=1}^{M} R_m^B$ be two run-length encoded binary images in compact representation, $\hat{m} \in \{1, \ldots, M\}$ such that $R_{\hat{m}}^B = \langle lx_{\hat{m}}, 0, 0 \rangle$, and $\tilde{X} := \bigcup_{n=1}^{N} \tilde{R}_n^X$, where

$$\tilde{R}_n^X := \begin{cases} \langle lx_n + (|R_{\hat{m}}^B| - 1), rx_n, y_n \rangle & \text{if } |R_{\hat{m}}^B| \leq |R_n^X|, \\ \emptyset & \text{else} \end{cases}$$

for all $n = 1, \ldots, N$. Then we have that:

$$\varepsilon_B(X) = \left\{ p \in \tilde{X} \mid B_p \subseteq X_{\text{Lmin}} \right\},$$

where $X_{\text{Lmin}}$ is defined as in Theorem 2.

**Proof.** Let $R_n^X$ be an arbitrary run of $X$, $A := \{(-1, 0), (0, 0)\}$, and $h \in R_n^X$. Then, since $R_{\hat{m}}^B = \langle lx_{\hat{m}}, 0, 0 \rangle$, using Theorems 1 and 6, the following equality holds:

$$f_{B_{\hat{m}}}^A(h + (0, 0)) = f_{B_{\hat{m}}}^A(h) = f_B^A((0, 0)) = |R_{\hat{m}}^B|.$$

We distinguish between two cases:

1. Case 1: Let $|R_{\hat{m}}^B| > |R_n^X|$. Then, by using the previous equality and Theorem 6, we get:

$$f_{B_{\hat{m}}}^A(h + (0, 0)) = f_{B_{\hat{m}}}^A(h) = f_B^A((0, 0)) = |R_{\hat{m}}^B| > |R_n^X| \geq f_X^A(h).$$

Theorem 3 implies $h \notin \varepsilon_B(X)$ for all $h \in R_n^X$.

2. Case 2: Let $|R_{\hat{m}}^B| \leq |R_n^X|$. Using Theorem 6 we conclude that for the first $|R_{\hat{m}}^B| - 1$ pixels $h$ of run $R_n^X$ we have that $f_X^A(h) < |R_{\hat{m}}^B|$. This leads to:

$$f_{B_{\hat{m}}}^A(h + (0, 0)) = f_B^A((0, 0)) = |R_{\hat{m}}^B| > f_X^A(h)$$

for $h \in \langle lx_n, lx_n + |R_{\hat{m}}^B| - 2, y_n \rangle$. Again Theorem 3 implies $h \notin \varepsilon_B(X)$ for all $h \in \langle lx_n, lx_n + |R_{\hat{m}}^B| - 2, y_n \rangle$. 

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Since \( o \in B \) by assumption, (3) and Theorem 2 state that \( \varepsilon_B(X) = \{ p \in X_{L_{\min}} \mid B_p \subseteq X_{L_{\min}} \} \). Additionally we just proved that for all \( n \in \{1, \ldots, N\} \) and \( h \in \langle lx_n, lx_n + |R_{m_n}^B| - 2, y_n \rangle \) where \( \langle a, b, c \rangle := \emptyset \) for \( a > b \) also \( h \notin \varepsilon_B(X) \). Thus the statement holds.

Obviously it makes sense to translate \( B \) by a vector \( q \) such that the rightmost pixel of the longest run within \( B \) is placed at the origin. This translation is described in the next theorem.

**Theorem 10.** Let \( X = \bigcup_{n=1}^{N} \mathcal{R}_n^X \) and \( B = \bigcup_{m=1}^{M} \mathcal{R}_m^B \) be two run-length encoded binary images in compact representation and \( X_{\text{cut}} := \bigcup_{n=1}^{N} \tilde{R}_n^X \), where \( L_{\text{max}}^B := \max_{m \in \{1, \ldots, M\}} \{|R_{m}^B|\} \) and

\[
\tilde{R}_n^X = \begin{cases} 
(lx_n + (L_{\text{max}}^B - 1), rx_n, y_n) & \text{if } L_{\text{max}}^B \leq |R_{n}^X|, \\
\emptyset & \text{else}
\end{cases}
\]

for all \( n = 1, \ldots, N \). Let \( R_{m}^B \subseteq B \) be any run such that \( |R_{m}^B| = L_{\text{max}}^B \). Then \( q := (rx_{m}^B, y_{m}^B) \) denotes the rightmost pixel of the longest run \( R_{m}^B \) within \( B \) and we have that:

\[
\varepsilon_B(X) = [\varepsilon_{B-q}(X)]_{-q} = \{ p \in X_{\text{cut}} | B_{p-q} \subseteq X_{L_{\min}} \}_{-q},
\]

where \( X_{L_{\min}} \) is defined as in Theorem 2.

**Proof.** Theorem 1 allows us to translate the structuring element \( B \) by \( q = (rx_{m}^B, y_{m}^B) \). So \( o \in B-q \) and \( (R_{m}^B)_{-q} = \langle lx_{m}^B - rx_{m}^B, 0, 0 \rangle \). Hence the claim immediately follows by using Lemma 9.

Although we can reduce our investigation to the points of \( X_{\text{cut}} \), we still need to evaluate \( f_A^{X_{L_{\min}}} \) and \( f_A^{X_{L_{\min}}} \). Figure 1 demonstrates the efficiency of this theorem.

4. Dilation Using RLE

By taking use of the duality between erosion and dilation [1]

\[
\delta_B(X) = \left[ \varepsilon_B(X^c) \right]^c,
\]

we are able to propose a fast dilation algorithm based on the previously developed erosion algorithm in a straightforward way. But we have to make some additional thoughts, because obviously \( |X^c| = \infty \) when \( |X| < \infty \).

Consider the rectangle \( \text{REC}^{l,r,t,b} \) with \( l, r, t, b \in \mathbb{Z} \) given by:

\[
\text{REC}^{l,r,t,b} := \{(x, y) \in \mathbb{Z}^2 \mid l \leq x \leq r, t \leq y \leq b\}.
\]
Then we denote the smallest rectangle which includes $X$ by $\text{REC}_{X}^{\text{min}}$.

Obviously (by considering the definition of dilation (2)) there exists a sufficiently big rectangle $\text{REC}^{\delta}$ such that $\delta_B(X)$ does not contain any points outside this rectangle:

$$\delta_B(X) \setminus \text{REC}^{\delta} = \left(\left[\varepsilon_{B^t}(X^c)^c\right]^c\right) \setminus \text{REC}^{\delta} = \emptyset.$$ 

Therefore we get:

$$\left[\varepsilon_{B^t}(X^c)^c\right]^c \cap \text{REC}^{\delta} = \left[\varepsilon_{B^t}(X^c)^c\right]^c.$$ 

Because of $|\text{REC}^{\delta}| < \infty$ we are also allowed to restrict $X^c$ by a rectangle $\text{REC}^c$, such that

$$\left[\varepsilon_{B^t}(X^c \cap \text{REC}^c)^c\right]^c \cap \text{REC}^{\delta} = \left[\varepsilon_{B^t}(X^c)^c\right]^c.$$ \hfill (8)

Still we have to find suitable sizes for the rectangles $\text{REC}^c$ and $\text{REC}^{\delta}$. Our goal is to make them as small as possible while being able to compute their sizes efficiently.

Because of Theorem 1 we can assume that $o \in B$. If this is not the case, we are allowed to translate $B$ by a suitable vector $q$ such that $o \in B_q$. Since

$$\delta_B(X) = \{p \in \mathbb{Z}^2 \mid (B^t)_p \cap X \neq \emptyset\}$$

one can easily see, that a suitable rectangle $\text{REC}^{\delta}$ is received by adding a border to $\text{REC}_{X}^{\text{min}}$ which has the same width as the structuring element on the left and right hand sides and the same height as the structuring element on the top and bottom. Now we only need to find a rectangle $\text{REC}^c$ such that (8) holds. Therefore we just double the border of $\text{REC}^{\delta}$ to obtain $\text{REC}^c$.

With these thoughts we are able to reformulate the dilation algorithm using the proposed erosion algorithm:

1. We receive run-length encoded images $X$ and $B$ as input.
2. We calculate $B^t$ and the compact run-length representation of $X^c \cap \text{REC}^c$.
3. These images can be used to retrieve $\varepsilon_{B^t}(X^c \cap \text{REC}^c)$ by executing $\text{GenErosionTransX2cut}(X^c \cap \text{REC}^c, B^t)$.
4. In the last step we take the complement of the erosion and restrict the image to the size of the rectangle $\text{REC}^{\delta}$:

$$\delta_B(X) = \left[\varepsilon_{B^t}(X^c \cap \text{REC}^c)^c\right]^c \cap \text{REC}^{\delta}.$$ 

Please note that 2) and 4) can also be done on the fly as described in Section 5.2.
Algorithm 1: GenerateSkeletonB

1: function GenerateSkeletonB(B) ➷ (01)
2: \( S_B^A \leftarrow \emptyset \)
3: \( L_{\text{min}}^B \leftarrow \infty \)
4: for all \( R^m_B \subseteq B \) do ➷ (02)
5: \( S_B^A \leftarrow S_B^A \cup \{(r_{x_m}^m, r_{y_m}^m)\} \) ➷ (03)
6: \( f_B^A((r_{x_m}^m, r_{y_m}^m)) \leftarrow |R^m_B| \) ➷ (04)
7: \( L_{\text{min}}^B \leftarrow \min \{L_{\text{min}}^B, |R^m_B|\} \) ➷ (05)
8: end for
9: return (List\((S_B^A), f_B^A, L_{\text{min}}^B) \)) ➷ (06)
10: end function

comments:
(01): returns List\((S_B^A), f_B^A, L_{\text{min}}^B) \)
(02): \( R^m_B \) is defined as \( R^m_B = \langle l_{x_m}^m, r_{x_m}^m, y_{m}\rangle \)
(03): adds the rightmost element to \( S_B^A \)
(04): erosion transform of this element equals the length of the run
(05): keeps track of the shortest run within \( B \)
(06): List\((S_B^A) \) means that \( S_B^A \) is stored as a list

5. Proposed Algorithms

In this section we are going to describe fast erosion and dilation algorithms. Reference [7] provides several figures that demonstrate the principles of these algorithms.

5.1. Proposed Erosion Algorithm

Observation (3) and Theorem 1 allow us to focus our investigations at those pixels of \( X \), which can be looked up very efficiently for RLE images. Due to Theorem 2, we are allowed to drop all runs of \( X \) during preprocessing which are shorter than the shortest run of \( B \). Theorem 10 states that it is enough to investigate the pixels of \( X_{\text{cut}} \), which is obtained by removing the first \( L_{\text{max}}^B - 1 \) pixels of every run of \( X \), where \( L_{\text{max}}^B \) denotes the length of the longest run of \( B \). Of course, runs that are shorter than \( L_{\text{max}}^B \) vanish completely. In case of a miss, the Jump-Miss Theorem enables us to skip the analysis of certain pixels and whenever a hit occurs, the full eroded run can immediately be added by applying the Jump-Hit Theorem.

A pseudocode that implements all of these methods is given by the Algorithms 1, 2, and 3.

5.2. Proposed Dilation Algorithm

By adapting the described erosion algorithm, all required operations (calculating the complement, intersecting with rectangles) can be done on the fly. Recall that we obtained the following formula in Section 4:

\[ \delta_B(X) = \left[ \varepsilon_B((X^c \cap \text{REC}^c))^c \right] \cap \text{REC}^d. \]
Algorithm 2 GenErosionTransX2cut

1: function GenErosionTransX2cut(X, L_Bmin, L_Bmax) \> (01)
2: for all z \in M_X do \> (02)
3: \( f_X^{A}_{L_{min}}(z) \leftarrow 0 \)
4: \( f_X^{A}_{t_{min}}(z) \leftarrow 0 \)
5: end for
6: \( X_{cut} \leftarrow \emptyset \)
7: for all \( R_X^N \subseteq X \) do \> (03)
8: if \( |R_X^N| \geq L_{Bmin} \) then \> (04)
9: \( X_{cut} \leftarrow X_{cut} \cup \langle lx_X^N + L_B^{max} - 1, rx_X^N, y_X^N \rangle \)
10: \( j \leftarrow 1 \)
11: for i \leftarrow lx_X^N to rx_X^N do \> (05)
12: \( f_X^{A}_{L_{min}}((i, y_X^N)) \leftarrow j \)
13: \( f_X^{A}_{t_{min}}((i, y_X^N)) \leftarrow |R_X^N| - j + 1 \)
14: \( j \leftarrow j + 1 \)
15: end for
16: end if
17: end for
18: return \( (f_X^{A}_{L_{min}}, f_X^{A}_{t_{min}}, X_{cut}) \)
19: end function

comments:
(01): returns \( f_X^{A}_{L_{min}}, f_X^{A}_{t_{min}} \) and \( X_{cut} \)
(02): initializes \( f_X^{A}_{L_{min}} \) and \( f_X^{A}_{t_{min}} \); \( M_X \) denotes a \( k \times l \) array where \( k := \text{width}(X) \) and \( l := \text{height}(X) \)
(03): visits every run in \( X \); \( R_X^N \) is defined as \( R_X^N = \langle lx_X^N, rx_X^N, y_X^N \rangle \)
(04): only considers runs with length \( \geq L_{Bmin} \)
(05): where \( (a, b, c) := \emptyset \) when \( a > b \)
(06): visits every point within the given run and calculates the according erosion transform values

By replacing line 5 in GenerateSkeletonB with \( S_B^A \leftarrow S_B^A \cup \{(-lx_{B_{min}}^B, -y_{B_{min}}^B)\} \) we are able to generate the skeleton of \( B^t \) on the fly. In GenErosionTransX2cut we immediately generate the erosion transform of \( X^c \cap \text{REC}^c \) by adding the sequences between two runs \( R_X^N \subseteq X \) instead of the runs themselves. Additionally we need to add the border with width and height of \( B \). Last but not least, we can modify GetErosion2cut by adding the misses instead of the hits to the dilated image.

6. Runtime Analysis

6.1. Erosion

In Section 3.3 we stated that the runtime complexity of the actual erosion algorithm is \( O(|X||S_B^A|) \). Next we are going to investigate the preprocessing steps. To build the skeleton \( S_B^A \) we need to visit every run and store the length of the run as well as the rightmost pixel. Then the complexity is given by \( O(M) \), where \( M \) denotes the number of runs of \( B \) and obviously this leads to \( |S_B^A| = M \). To obtain \( f_X^{A}_{L_{min}} \) and
Algorithm 3 GetErosion2cut

1: function GetErosion2cut(X,B)  \hspace{1cm} (01): returns $\varepsilon_B(X)$
2: $L^B_{\text{max}} \leftarrow 0$
3: for all $R^B_m \subseteq B$ do  \hspace{1cm} (02): determines the length and the position
4: \hspace{1cm} if $f^B_{\text{max}} < |R^B_m|$ then \hspace{1cm} of the rightmost pixel of the longest
5: \hspace{1cm} $L^B_{\text{max}} \leftarrow |R^B_m|$ \hspace{1cm} run within $B$
6: \hspace{1cm} $q \leftarrow (x^B_m, y^B_m)$ \hspace{1cm} (03): sets $q$ to the rightmost pixel of the
7: \hspace{1cm} end if \hspace{1cm} longest run
8: \hspace{1cm} end for \hspace{1cm} (04): $B_{-q}$ contains the origin, which is the
9: $B_{-q} \leftarrow \text{TranslateImage}(B,-q)$ \hspace{1cm} rightmost pixel of the longest run of $B$
10: List($S^A_{B_{-q}}$), $f^A_{B_{-q}}$, $L^B_{\text{min}} \leftarrow \text{GenerateSkeletonB}(B_{-q})$ \hspace{1cm} (05): visits every run of $X_{\text{cut}}$; $R^X_n$ is defined
11: \hspace{1cm} $f^A_{X_{\text{min}}}$, $f^A_{X_{\text{max}}}$, $X_{\text{cut}} \leftarrow \text{GenErosionTransX2cut}(X,L^B_{\text{min}},L^B_{\text{max}})$ \hspace{1cm} as $R^X_n = \langle l^X_n, r^X_n, y^X_n \rangle$
12: $\varepsilon^X_{B_{-q}}(X) \leftarrow \emptyset$ \hspace{1cm} (06): stores the $x$-coordinate of point $h$: we
13: for all $R^X_n \subseteq X_{\text{cut}}$ do \hspace{1cm} are checking if $B_{h_{-q}} \subseteq X$
14: \hspace{1cm} $x \leftarrow l^X_n$ \hspace{1cm} (07): breaks the loop once our $x$ lies outside
15: \hspace{1cm} while $x \leq r^X_n$ do \hspace{1cm} the given sequence
16: \hspace{1cm} miss \leftarrow \text{false} \hspace{1cm} (08): variable needed for breaking the loop
17: \hspace{1cm} $s \leftarrow \text{head}(\text{List}(S^A_{B_{-q}}))$ \hspace{1cm} in case of a miss according to the
18: \hspace{1cm} while \hspace{1cm} (not(miss)) AND (exists(\text{next}(\text{List}(S^A_{B_{-q}})))) do \hspace{1cm} \textbf{Jump-Miss Theorem}
19: \hspace{1cm} $s \leftarrow \text{next}(\text{List}(S^A_{B_{-q}}))$ \hspace{1cm} (09): $s$ points at the head of List($S^A_{B_{-q}}$)
20: \hspace{1cm} Diff \leftarrow f^A_{B_{-q}}(s) - f^A_{X_{\text{min}}}(s + (x, y^X_n)) \hspace{1cm} (10): loop will be continued as long as there
21: \hspace{1cm} while (\text{Diff} > 0) AND (x \leq r^X_n) do \hspace{1cm} is no miss and there are more $s \in S^A_{B_{-q}}$
22: \hspace{1cm} $x \leftarrow x \text{+ Diff}$ \hspace{1cm} which have not been visited yet
23: \hspace{1cm} Diff \leftarrow f^A_{B_{-q}}(s) - f^A_{X_{\text{min}}}(s + (x, y^X_n)) \hspace{1cm} (11): gets the next element within
24: \hspace{1cm} miss \leftarrow \text{true} \hspace{1cm} \text{List}(S^A_{B_{-q}})$
25: \hspace{1cm} end while \hspace{1cm} (12): $B_{h_{-q}} \not\subseteq X$
26: \hspace{1cm} end while \hspace{1cm} (13): jump by Diff number of pixels
27: \hspace{1cm} if (not(miss)) then \hspace{1cm} (14): breaks the outer loop since $B_{h_{-q}} \not\subseteq X$
28: \hspace{1cm} minDist \leftarrow \infty \hspace{1cm} (15): $B_{h_{-q}} \subseteq X$, so the Jump-Hit Theorem
29: \hspace{1cm} for all $s \in S^A_{B_{-q}}$ do \hspace{1cm} can be applied
30: \hspace{1cm} minDist \leftarrow \text{min}(\text{minDist}, f^A_{X_{\text{min}}}(s + (x, y^X_n))) \hspace{1cm} (16): determines the minimum distance to
31: \hspace{1cm} end for \hspace{1cm} the right to be able to apply the
32: \hspace{1cm} $\varepsilon_{B_{-q}}(X) \leftarrow \varepsilon_{B_{-q}}(X) \cup (x, x + \text{minDist} - 1, y^X_n)$ \hspace{1cm} \textbf{Jump-Hit Theorem}
33: \hspace{1cm} $x \leftarrow x + \text{minDist + 1}$ \hspace{1cm} (17): Jump-Hit Theorem
34: \hspace{1cm} end if \hspace{1cm} (18): next point $(h - q)$ to look at will be at
35: \hspace{1cm} end while \hspace{1cm} the $x$-coordinate $(x + \text{minDist + 1})$
36: \hspace{1cm} end for \hspace{1cm} (19): we retrieve $\varepsilon_B(X)$ by applying
37: $\varepsilon_B(X) \leftarrow \text{TranslateImage}(\varepsilon_{B_{-q}}(X),-q)$ \hspace{1cm} theorems
38: \hspace{1cm} return ($\varepsilon_B(X)$)
39: \hspace{1cm} end function

We need to visit every point of $X$, thus its complexity is given by $O(|X|)$. Further $X_{\text{cut}}$ is built while
determining $f^A_{X_{\text{min}}}$ and $f^A_{X_{\text{min}}}^{t}$ and does not affect the given complexity. Putting all together, we get a
complexity of

$$O(|X| M + M + |X|) = O(|X|M).$$

(9)
Table 1. Abbreviations of the tested algorithms

| algorithm | source |
|-----------|--------|
| RLE       | erosion as in [4] |
| Machado   | erosion as in [6] |
| OpenCV    | erosion as in [8] |
| eEJMH     | erosion algorithm as in Sec. 5.1 |
| dRLE      | dilation as in [4] |
| dOpenCV   | dilation as in [8] |
| dEJMH     | dilation algorithm as in Sec. 5.2 |

6.2. Dilation

Since this algorithm is taking use of the duality between erosion and dilation, and building the complements is done on the fly, we get the following complexity (compare with (9)):

\[ \mathcal{O}(|X^c \cap \text{REC}^c| \cdot M). \]

7. Experimental Results

In this section we are going to compare the runtimes of the proposed algorithms. Implementations of these algorithms are available in C++ and can be found at https://numerical-analysis.uibk.ac.at/g.ehrensperger. We are comparing them with the implementations of the free library OpenCV (developed by Intel) v2.4.2 and with the algorithms proposed in [4] and [6]. The latter two got implemented by Machado and the source code is available at http://score.ime.usp.br/~dandy/mestrado.php. This implementations are also used in our tests. Additionally he implemented various variants of his own algorithm. In the following plots we took the pointwise minimum of the runtimes of his algorithms. The conversion of \( X \) and \( B \) into the used input format of the various algorithms is not part of the given runtimes.\(^1\) All tests were carried out on a workstation with Intel Core i7-3770K (for the following tests only one core was used), 32 GB DDR3-1333 RAM and OS Ubuntu 13.04 64 Bit. CPU-stepping, overlocking settings, and various energy saving options had been disabled. The test environment was compiled with GCC (the GNU Compiler Collection) v4.7.3 (official project URL: http://gcc.gnu.org/) and compiler flag O2. The following results are the arithmetic mean values of three iterations. We used the image in Figure 1 to compare the runtimes of the algorithms listed in Table 1. The results are given in Figure 2. Note that the plots’ y–axes are scaled logarithmically. As can be seen, the algorithms proposed in this paper tremendously improve the runtime of both, the erosion and

\(^1\)In [6] the conversion of the raster graphics \( X \) and \( B \) into the used input format of the algorithms was part of the given runtimes.
dilation operator, over the compared algorithms on the test image given in Figure 1. We observe that the execution times of RLE, dRLE, OpenCV, and dOpenCV grow with the size of the structuring element. In contrary eEJMH and Machado benefit from the size of the structuring element. This can be explained, since the bigger the structuring element the larger the jumps allowed by the Jump-Miss Theorem. The clear advantage of eEJMH over Machado results from the various new methods described in Section 3. Although dEJMH is based on the same methods as eEJMH, we observe that the runtime still grows with the size of the structuring element. This is because we work with the inverted input image $X^c \cap \text{REC}^c$ and the rectangle $\text{REC}^c$ depends on the size of the structuring element. Still eEJMH and dEJMH seem to provide a better alternative in most cases.

8. Conclusions

In this paper we developed new ideas to speed up the calculation of erosion and dilation. We also proposed fast algorithms to perform these operations with arbitrary structuring elements. Further we determined the runtime complexity of our algorithms which is given by $O(|X| M)$ for erosion and by $O(|X^c \cap \text{REC}^c | \cdot M)$ for dilation, where $X$ is the input image, $B$ the structuring element, $M$ denotes the number of runs of $B$, and $\text{REC}^c$ is a rectangle of the size of $X$ plus a border that is twice the size of $B$. Finally experiments confirmed that our algorithms provide huge speedup compared to some other well known implementations.
Figure 2. Upper left: Erosion with a square-shaped structuring element. Upper right: Dilation with a square-shaped structuring element. Lower left: Erosion with a diamond-shaped structuring element. Lower right: Dilation with a diamond-shaped structuring element.

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