ANTIPODE FORMULAS FOR COMBINATORIAL HOPF ALGEBRAS

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ABSTRACT. Motivated by work of Buch on set-valued tableaux in relation to the K-theory of the Grassmannian, Lam and Pylyavskyy studied six combinatorial Hopf algebras that can be thought of as K-theoretic analogues of the Hopf algebras of symmetric functions, quasisymmetric functions, noncommutative symmetric functions, and of the Malvenuto-Reutenauer Hopf algebra of permutations. They described the bialgebra structure in all cases that were not yet known but left open the question of finding explicit formulas for the antipode maps. We give combinatorial formulas for the antipode map in these cases.

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1. INTRODUCTION

A Hopf algebra is a structure that is both an associative algebra with unit and a coassociative coalgebra with counit. The algebra and coalgebra structures are compatible, which makes it a bialgebra. To be a Hopf algebra, a bialgebra must have a special anti-endomorphism called the antipode, which must satisfy certain properties.

Hopf algebras arise naturally in combinatorics. Notably, the symmetric functions (Sym), quasisymmetric functions (QSym), noncommutative symmetric functions (NSym), and the Malvenuto-Reutenauer algebra of permutations (MR) are Hopf algebras, which can be arranged as in the following diagram.

```
Sym ---             --- Sym
    |                        |
    v                        v
NSym ---             --- QSym
    |                        |
    v                        v
MR ---             --- MR
```

Date: January 6, 2015.
1991 Mathematics Subject Classification. Primary 05E99,
Horizontal lines in this diagram denote Hopf duality.

Through the work of Lascoux and Schützengerger [6], Fomin and Kirillov [2], and Buch [1], symmetric functions known as stable Grothendieck polynomials were discovered and given a combinatorial interpretation in terms of set-valued tableaux. They originated from Grothendieck polynomials, which serve as representatives of $K$-theory classes of structure sheaves of Schubert varieties. The stable Grothendieck polynomials play the role of Schur functions in the $K$-theory of Grassmannians. They also determine a $K$-theoretic analogue of the symmetric functions, which we call the multi-symmetric functions and denote $m\text{Sym}$.

In [5], Lam and Pylyavskyy extend the definition of $P$-partitions to create $P$-set-valued partitions, which they use to define a new $K$-theoretic analogue of the Hopf algebra of quasisymmetric functions called the Hopf algebra of multi-quasisymmetric functions. The entire diagram may be extended to give:

$$
\begin{array}{c}
\mathfrak{MR} \\
m\text{MR} \\
\downarrow \\
\mathfrak{MNSym} \\
m\text{QSym} \\
\downarrow \\
\mathfrak{MSym} \\
m\text{Sym}
\end{array}
$$

Using Takeuchi’s formula [12], they give a formula for the antipode for $\mathfrak{MR}$ but leave open the question of an antipode for the remaining Hopf algebras. In this paper, we give new combinatorial formulas for the antipode maps of $\mathfrak{MNSym}$, $m\text{QSym}$, $m\text{Sym}$, and $\mathcal{M}\text{Sym}$.

After a brief introduction to Hopf algebras, we introduce the Hopf algebra $m\text{QSym}$ in Section 3. Next, we introduce $\mathfrak{MNSym}$ in Section 4. We present results concerning the antipode map in $\mathfrak{MNSym}$ and $m\text{QSym}$, namely Theorems 4.7 and 4.9. In Section 5, we present an additional basis for $m\text{QSym}$, give analogues of results in [5] for this new basis, and give an antipode formula in $m\text{QSym}$ involving the new basis in Theorem 5.6. Lastly, we introduce the Hopf algebras of multi-symmetric functions, $m\text{Sym}$, and of Multi-symmetric functions, $\mathcal{M}\text{Sym}$ in Sections 6 and 7. We end with Theorems 8.2, 8.3, and 8.7 which describe antipode maps in these spaces.

### 2. Hopf Algebra Basics

#### 2.1. Algebras and coalgebras

First we build a series of definitions leading to the definition of a Hopf algebra. For more information, see [4, 8, 9, 13].

In this section, $k$ will usually denote a field, although it may also be a commutative ring. In all later sections we take $k = \mathbb{Z}$. All tensor products are taken over $k$.

**Definition 2.1.** An associative $k$-algebra $A$ is a $k$-vector space with associative operation $m : A \otimes A \to A$ (the product) and unit map $\eta : k \to A$ with $\eta(1_k) = 1_A$ such that the following diagrams commute:

$$
\begin{array}{c}
A \otimes A \otimes A \\
\downarrow m \otimes 1 \\
\downarrow m \\
A \otimes A \\
\downarrow m
\end{array}
$$

$$
\begin{array}{c}
k \otimes A \\
\eta \otimes 1 \\
\downarrow m \\
A \otimes k
\end{array}
$$
where we take the isomorphisms sending $a \otimes k$ to $ak$ and $k \otimes a$ to $ka$.

The first diagram tells us that $m$ is an associative product and the second that $\eta(1_k) = 1_A$.

**Definition 2.2.** A co-associative coalgebra $C$ is a $k$-vector space with $k$-linear map $\Delta : C \rightarrow C \otimes C$ (the coproduct) and a counit $\epsilon : C \rightarrow k$ such that the following diagrams commute:

![Diagram](image)

The diagram on the left indicates that $\Delta$ is co-associative. Note that these are the same diagrams as in the Definition 2.1 with all of the arrows reversed.

It is often useful to think of the product as a way to combine two elements of an algebra and to think of the coproduct as a sum over ways to split a coalgebra element into two pieces. When discussing formulas involving $\Delta$, we will use Sweedler notation as shown below:

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2.$$  

This is a common convention that will greatly simplify our notation.

**Example 2.3.** To illustrate the concepts just defined, we give the example of the shuffle algebra, which is both an algebra and coalgebra.

Let $I$ be an alphabet and $\bar{I}$ be the set of words on $I$. We declare that words on $\bar{I}$ form a $k$-basis for the shuffle algebra.

Given two words $a = a_1 a_2 \cdots a_t$ and $b = b_1 b_2 \cdots b_n$ in $\bar{I}$, define their product, $m(a \otimes b)$, to be the shuffle product of $a$ and $b$. That is, $m(a \otimes b)$ is the sum of all $(t+n)$ ways to interlace the two words while maintaining the relative order of the letters in each word. For example,

$$m(a_1 a_2 \otimes b_1) = a_1 a_2 b_1 + a_1 b_1 a_2 + b_1 a_1 a_2.$$  

We may then extend by linearity. It is not hard to see that this multiplication is associative.

The unit map for the shuffle algebra is defined by $\eta(1_k) = \emptyset$, where $\emptyset$ is the empty word. Note that $m(a \otimes \emptyset) = m(\emptyset \otimes a) = a$ for any word $a$.

For a word $a = a_1 a_2 \cdots a_t$ in $\bar{I}$, we define

$$\Delta(a) = \sum_{i=0}^{t} a_1 a_2 \cdots a_i \otimes a_{i+1} a_{i+2} \cdots a_t$$

and call this the cut coproduct of $a$. For example, given a word $a = a_1 a_2$,

$$\Delta(a) = \emptyset \otimes a_1 a_2 + a_1 \otimes a_2 + a_1 a_2 \otimes \emptyset.$$  

The counit map is defined by letting $\epsilon$ take the coefficient of the empty word. Hence for any nonempty $a \in \bar{I}$, $\epsilon(a) = 0$. 

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2.2. Morphisms and bialgebras. The next step in defining a Hopf algebra is to define a bialgebra. For this, we need a notion of compatibility of maps of an algebra \((m, \eta)\) and maps of a coalgebra \((\Delta, \epsilon)\). With this as our motivation, we introduce the following definitions.

**Definition 2.4.** If \(A\) and \(B\) are \(k\)-algebras with multiplication \(m_A\) and \(m_B\) and unit maps \(\eta_A\) and \(\eta_B\), respectively, then a \(k\)-linear map \(f : A \to B\) is an algebra morphism if \(f \circ m_A = m_B \circ (f \otimes f)\) and \(f \circ \eta_A = \eta_B\).

**Definition 2.5.** Given \(k\)-coalgebras \(C\) and \(D\) with comultiplication and counit \(\Delta_C, \epsilon_C, \Delta_D, \text{and } \epsilon_D\), \(k\)-linear map \(g : C \to D\) is a coalgebra morphism if \(\Delta_D \circ g = (g \otimes g) \circ \Delta_C\) and \(\epsilon_D \circ g = \epsilon_C\).

Given two \(k\)-algebras \(A\) and \(B\), their tensor product \(A \otimes B\) is also a \(k\)-algebra with \(m_{A \otimes B}\) defined to be the composite of

\[
A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes T \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,
\]

where \(T(b \otimes a) = a \otimes b\). For example, we have

\[
m_{A \otimes B}((a \otimes b) \otimes (a' \otimes b')) = m_A(a \otimes a') \otimes m_B(b \otimes b').
\]

The unit map in \(A \otimes B\), \(\eta_{A \otimes B}\), is given by the composite

\[
k \xrightarrow{k \otimes k} A \otimes B.
\]

Similarly, given two coalgebras \(C\) and \(D\), their tensor product \(C \otimes D\) is a coalgebra with \(\Delta_{C \otimes D}\) the composite of

\[
C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes T \otimes 1} C \otimes D \otimes C \otimes D,
\]

and the counit \(\epsilon_{A \otimes B}\) is the composite

\[
C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} k \otimes k \xrightarrow{k} k.
\]

**Definition 2.6.** Given \(A\) that is both a \(k\)-algebra and a \(k\)-coalgebra, we call \(A\) a \(k\)-bialgebra if \((\Delta, \epsilon)\) are morphisms for the algebra structure \((m, \eta)\) or equivalently, if \((m, \eta)\) are morphisms for the coalgebra structure \((\Delta, \epsilon)\).

**Example 2.7.** The shuffle algebra is a bialgebra. We can see, for example, that

\[
\Delta \circ m_{A}(a_1 \otimes b_1) = \Delta(a_1 b_1 + b_1 a_1)
\]

\[
= \emptyset \otimes a_1 b_1 + a_1 \otimes b_1 + a_1 b_1 \otimes \emptyset + \emptyset \otimes b_1 a_1 + b_1 \otimes a_1 + b_1 a_1 \otimes \emptyset
\]

\[
= \emptyset \otimes (a_1 b_1 + b_1 a_1) + b_1 \otimes a_1 + a_1 \otimes b_1 + (a_1 b_1 + b_1 a_1) \otimes \emptyset
\]

\[
= m_{A}(\emptyset \otimes \emptyset) \otimes m_{A}(a_1 \otimes b_1) + m_{A}(\emptyset \otimes b_1) \otimes m_{A}(a_1 \otimes \emptyset) + m_{A}(a_1 \otimes \emptyset) \otimes m_{A}(\emptyset \otimes b_1)
\]

\[
+ m_{A}(a_1 \otimes b_1) \otimes m_{A}(\emptyset \otimes \emptyset)
\]

\[
= m_{A \otimes A}((\emptyset \otimes a_1 + a_1 \otimes \emptyset) \otimes (\emptyset \otimes b_1 + b_1 \otimes \emptyset))
\]

\[
= m_{A \otimes A} (\Delta(a_1) \otimes \Delta(b_1)).
\]

This is evidence that the coproduct, \(\Delta\), is an algebra morphism.
2.3. The antipode map. A Hopf algebra is a bialgebra equipped with an additional map called
the antipode map. On our way to defining the antipode map, we must first introduce an algebra
structure on \( k \)-linear algebra maps that take coalgebras to algebras.

Definition 2.8. Given coalgebra \( C \) and algebra \( A \), we form an associative algebra structure on
the set of \( k \)-linear maps from \( C \) to \( A \), \( \text{Hom}_k(C, A) \), called the convolution algebra as follows: for \( f \) and \( g \) in \( \text{Hom}_k(C, A) \), define the product, \( f \ast g \), by
\[
(f \ast g)(c) = m \circ (f \otimes g) \circ \Delta(c) = \sum f(c_1)g(c_2),
\]
where \( \Delta(c) = \sum c_1 \otimes c_2 \).

Note that \( \eta \circ \epsilon \) is the two-sided identity element for \( \ast \). We can easily see this in the shuffle
algebra from Example 2.7 if we remember that \((\eta \circ \epsilon)(a) = \eta(0) = 0\) for all words \( a \neq \emptyset \). Let \( c \)
be a word in the shuffle algebra, then
\[
(f \ast (\eta \circ \epsilon))(c) = \sum f(c_1)(\eta \circ \epsilon)(c_2) = f(c) = \sum (\eta \circ \epsilon)(c_1)f(c_2) = ((\eta \circ \epsilon) \ast f)(c)
\]
because \( c_1 = c \) when \( c_2 = \emptyset \) and \( c_2 = c \) when \( c_1 = \emptyset \).

If we have a bialgebra \( A \), then we can consider this convolution structure to be on \( \text{End}_k(A) := \text{Hom}_k(A, A) \).

Definition 2.9. Let \((A, m, \eta, \Delta, \epsilon)\) be a bialgebra. Then \( S \in \text{End}_k(A) \) is called an antipode for
bialgebra \( A \) if
\[
id_A \ast S = S \ast id_A = \eta \circ \epsilon,
\]
where \( id_A : A \rightarrow A \) is the identity map.

In other words, the endomorphism \( S \) is the two-sided inverse for the identity map \( id_A \) under
the convolution product. Equivalently, if \( \Delta(a) = \sum a_1 \otimes a_2 \),
\[
(S \ast id_A)(a) = \sum S(a_1)a_2 = \eta(\epsilon(a)) = \sum a_1S(a_2) = (id_A \ast S).
\]
Because we have an associative algebra, this means that if an antipode exists, then it is unique.

Example 2.10. In the shuffle algebra, we define the antipode of a word by \( S(a_1a_2 \cdots a_t) = (-1)^t a_1a_{t-1} \cdots a_2a_1 \) and extend by linearity. We can see an example of the defining property by computing
\[
(id \ast S)(a_1a_2) = m(id(\emptyset) \otimes S(a_1a_2)) + m(id(a_1) \otimes S(a_2)) + m(id(a_1a_2) \otimes S(\emptyset))
\]
\[
= -a_2a_1 - m(a_1 \otimes a_2) + a_1a_2
\]
\[
= -a_2a_1 - (a_1a_2 + a_2a_1) + a_1a_2
\]
\[
= 0
\]
\[
= \eta(\epsilon(a_1a_2)).
\]

We end this section with two useful properties that we use in later sections. The first is a
well-known property of the antipode map for any Hopf algebra.

Proposition 2.11. Let \( S \) be the antipode map for Hopf algebra \( A \). Then \( S \) is an algebra anti-
endomorphism: \( S(1) = 1 \), and \( S(ab) = S(b)S(a) \) for all \( a, b \) in \( A \).

The second property allows us to translate antipode formulas between certain Hopf algebras.
Lemma 2.12. Suppose we have two bialgebra bases, \(\{A_\lambda\}\) and \(\{B_\mu\}\), that are dual under a pairing and such that the structure constants for the product of the first basis are the structure constants for the coproduct of the second basis and vice versa. In other words, \(\langle A_\lambda, B_\mu \rangle = \delta_{\lambda,\mu}\), \(A_\lambda A_\mu = \sum \nu f^\nu_{\lambda,\mu} A_\nu\) and \(\Delta(B_\lambda) = \sum \mu, \nu f^\lambda_{\mu,\nu} B_\mu \otimes B_\nu\), and \(\Delta(A_\lambda) = \sum \mu, \nu h^\lambda_{\mu,\nu} A_\mu \otimes A_\nu\) and \(B_\lambda B_\mu = \sum \nu h^\nu_{\lambda,\mu} B_\nu\). If

\[
S(A_\lambda) = \sum \mu e_{\lambda,\mu} A_\mu
\]

for \(S\) satisfying \(0 = \sum \mu, \nu h^\lambda_{\mu,\nu} S(A_\mu) A_\nu\), then

\[
S(B_\mu) = \sum \lambda e_{\lambda,\mu} B_\lambda
\]

satisfies \(\sum \mu, \nu f^\lambda_{\mu,\nu} S(B_\mu) B_\nu = 0\).

Proof. Indeed,

\[
\left\langle \sum \mu, \nu f^\lambda_{\mu,\nu} S(B_\mu) B_\nu, A_\tau \right\rangle = \left\langle \sum \mu, \nu, \gamma f^\lambda_{\mu,\nu,\gamma} k_{\gamma,\mu} B_\gamma B_\nu, A_\tau \right\rangle
\]

\[
= \left\langle \sum \mu, \nu, \gamma f^\lambda_{\mu,\nu,\gamma} k_{\gamma,\mu} h^\mu_{\gamma,\nu} B_\mu, A_\tau \right\rangle
\]

\[
= \sum \mu, \nu, \gamma f^\lambda_{\mu,\nu,\gamma} k_{\gamma,\mu} h^\tau_{\gamma,\nu}
\]

\[
= \left\langle B_\lambda, \sum \rho, \mu, \nu, \gamma h^\rho_{\gamma,\nu} k_{\gamma,\mu} f^\lambda_{\mu,\nu} A_\rho \right\rangle
\]

\[
= \left\langle B_\lambda, \sum \mu, \nu, \gamma h^\rho_{\gamma,\nu} k_{\gamma,\mu} A_\mu A_\nu \right\rangle
\]

\[
= \left\langle B_\lambda, \sum \nu, \gamma h^\rho_{\gamma,\nu} S(A_\gamma) A_\nu \right\rangle
\]

\[
= 0 \text{ by assumption.}
\]

3. The Hopf algebra of multi-quasisymmetric functions

The multi-quasisymmetric functions (\(m\)\(Q\)Sym) are a \(K\)-theoretic analogue of the Hopf algebra of quasisymmetric functions (\(Q\)Sym), which was introduced by Gessel [3] and stemmed from work of Stanely [11]. An understanding of \(Q\)Sym is useful for understanding \(m\)\(Q\)Sym. We recommend [5], [9], and [10] for exposition on \(Q\)Sym and its Hopf algebra structure.

In what follows, we say that a set \(\{A_\lambda\}\) continuously spans space \(A\) if everything in \(A\) can be written as a (possibly infinite) linear combination of \(A_\lambda\)'s. Here, we assume that \(\{A_\lambda\}\) comes with a natural filtration and that each filtered component is finite. Then we may talk about continuous span with respect to the topology induced by the filtration. A continuous basis for
A allows elements to be written as arbitrary linear combinations of the basis elements. We say that a linear function \( f : A \rightarrow A \) is continuous if it respects arbitrary linear combinations of elements in \( A \).

3.1. \((P, \theta)\)-set-valued partitions. Following [5], we define \( \text{mQSym} \), the Hopf algebra of multi-quasisymmetric functions, by defining the continuous basis of multi-fundamental quasisymmetric functions, \( \hat{L}_\alpha \). We start with a finite poset \( P \) with \( n \) elements and a bijective labeling \( \theta : P \rightarrow [n] \). Let \( \hat{P} \) denote the set of nonempty, finite subsets of the positive integers. If \( a \in \hat{P} \) and \( b \in \hat{P} \) are two such subsets, we say that \( a < b \) if \( \max(a) < \min(b) \). Similarly, \( a \leq b \) if \( \max(a) \leq \min(b) \).

We next define the \((P, \theta)\)-set-valued partition. The definition is almost identical to that of the more well-known \((P, \theta)\)-set partitions except that \( \sigma \) now assigns a nonempty, finite subset of positive integers to each element of the poset instead of assigning a single positive integer.

**Definition 3.1.** Let \((P, \theta)\) be a poset with a bijective labeling. A \((P, \theta)\)-set-valued partition is a map \( \sigma : P \rightarrow \hat{P} \) such that for each covering relation \( s < t \) in \( P \),

1. \( \sigma(s) \leq \sigma(t) \) if \( \theta(s) \leq \theta(t) \),
2. \( \sigma(s) < \sigma(t) \) if \( \theta(s) > \theta(t) \).

**Example 3.2.** The diagram on the left shows an example of a poset \( P \) with a bijective labeling \( \theta \). We identify elements of \( P \) with their labeling. The diagram on the right shows a valid \((P, \theta)\)-set-valued partition \( \sigma \). Note that since \( 3 < 2 \) in the poset, we must have the strict inequality \( \max(\sigma(3)) = 6 < \min(\sigma(2)) = 30 \).

\[
\begin{array}{c}
\sigma(1) = \{1, 3, 6\} \\
\sigma(3) = \{6\} \\
\sigma(4) = \{7, 100\} \\
\sigma(2) = \{30, 31, 32\}
\end{array}
\]

We denote the set of all \((P, \theta)\)-set-valued partitions for given poset \( P \) by \( \hat{A}(P, \theta) \). For each element \( i \) in \( P \), let \( \sigma^{-1}(i) = \{ x \in P | i \in \sigma(x) \} \). Now define \( \hat{K}_{P, \theta} \in \mathbb{Z}[x_1, x_2, \ldots] \) by

\[
\hat{K}_{P, \theta} = \sum_{\sigma \in \hat{A}(P, \theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \ldots .
\]

For example, the \((P, \theta)\)-set-valued partition in the previous example contributes

\[
x_1x_3x_6^2x_7x_{30}x_{31}x_{32}x_{100}
\]

to \( \hat{K}_{P, \theta} \). Note that \( \hat{K}_{P, \theta} \) will be of unbounded degree for any nonempty poset \( P \).

3.2. **The multi-fundamental quasisymmetric functions.** A composition of \( n \) is an ordered arrangement of positive integers that sum to \( n \). For example, \((3), (1, 2), (2, 1), \) and \((1, 1, 1)\) are all of the compositions of \( 3 \).

If \( S = \{s_1, s_2, \ldots, s_k\} \) is a subset of \([n-1]\), we associate a composition, \( \mathcal{C}(S) \), to \( S \) by \( \mathcal{C}(S) = \{s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_k\} \). To composition \( \alpha \) of \( n \), we associate \( S_\alpha \subset [n-1] \) by letting \( S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_{k-1}\} \). We may extend this correspondence to permutations by letting \( \mathcal{C}(w) = \mathcal{C}(\text{Des}(w)) \), where \( w \in \mathfrak{S}_n \) and \( \text{Des}(w) \) is the descent set of \( w \). For example, if \( S = (1, 4, 5) \subset [6-1] \), \( \mathcal{C}(S) = (1, 4, 1, 5, 4, 6, 5) = (1, 3, 1, 1) \). Conversely,
given composition \( \alpha = (1, 3, 1, 1) \), \( S_\alpha = (1, 1 + 3, 1 + 3 + 1) = (1, 4, 5) \). For \( w = 132 \in \mathfrak{S}_3 \), \( \text{Des}(w) = (2) \) and \( \mathcal{C}(w) = (2, 1) \).

Given a composition \( \alpha \) of \( n \), we write \( w_\alpha \) to denote any permutation in \( \mathfrak{S}_n \) with \( \mathcal{C}(w_\alpha) = \alpha \).

We may now define the multi-fundamental quasisymmetric function \( \tilde{L}_\alpha \) indexed by composition \( \alpha \).

**Definition 3.3.** Let \( P \) be a finite chain \( p_1 < p_2 < \ldots < p_k \), \( w \in \mathfrak{S}_k \) a permutation, and \( \mathcal{C}(w) = \alpha \) the composition of \( n \) associated to the descent set of \( w \). We label \( P \) using \( w \) with \( \theta(p_i) = w_i \). Then

\[
\tilde{L}_\alpha = \tilde{K}_{(P,w)} = \sum_{\sigma \in \tilde{A}(P,w)} x_1^{\# \sigma^{-1}(1)} x_2^{\# \sigma^{-1}(2)} \ldots.
\]

It is easy to see that \( \tilde{K}_{(P,w)} \) depends only on \( \alpha \). Note that this is an infinite sum of unbounded degree. The sum of the lowest degree terms in \( \tilde{L}_\alpha \) gives \( L_\alpha \), the fundamental quasisymmetric function in \( \text{QSym} \).

**Example 3.4.** Let \( \alpha = (2, 1) \) and \( w_\alpha = 231 \). We consider all \( (P,w_\alpha) \)-set-valued partitions on the chain below.

```
1

3

2
```

We can have set-valued partitions \( \sigma_1(2) = 1 \), \( \sigma_1(3) = 1 \), and \( \sigma_1(1) = 2 \), \( \sigma_2(2) = 1 \), \( \sigma_2(3) = 1 \), and \( \sigma_2(1) = 3 \), \( \sigma_3(2) = 1 \), \( \sigma_3(3) = 2 \), and \( \sigma_3(1) = 3 \), \( \sigma_4(2) = \{1, 2\} \), \( \sigma_4(3) = 2 \), and \( \sigma_4(1) = \{3, 4\} \), \( \sigma_5(2) = \{1, 2\} \), \( \sigma_5(3) = \{2, 3\} \), and \( \sigma_5(1) = 4 \), \( \sigma_6(2) = \{5, 6, 7\} \), \( \sigma_6(3) = \{7, 100\} \), and \( \sigma_6(1) = 101 \), \( \sigma_8(2) = \{5, 6, 7\} \), \( \sigma_8(3) = 7 \), and \( \sigma_8(1) = \{100, 101\} \), etc. So

\[
\tilde{L}_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + 2x_1 x_2^2 x_3 x_4 + 2x_5 x_6 x_7^2 x_{100} x_{101} + \ldots,
\]

an infinite sum of unbounded degree.

**Definition 3.5.** Given a poset \( P \) with \( n \) elements, a linear multi-extension of \( P \) by \( [N] \) is a map \( e: P \rightarrow 2^{[N]} \) for \( N \geq n \) such that

1. \( e(x) < e(y) \) if \( x < y \) in \( P \),
2. each \( i \in [N] \) is in \( e(x) \) for exactly one \( x \in P \), and
3. no set \( e(x) \) contains both \( i \) and \( i + 1 \) for any \( i \).

We then define the multi-Jordan-Holder set \( \tilde{J}(P,\theta) \) to be the union of the sets

\[
\tilde{J}_N(P,\theta) = \{\theta(e^{-1}(1))\theta(e^{-1}(2)) \ldots \theta(e^{-1}(N))\}.
\]

Note that elements in \( \tilde{J}_N(P,\theta) \) are \( m \)-permutations of \( [n] \) with \( N \) letters, where we define an \( m \)-permutation of \( [n] \) to be a word in the alphabet \( 1, 2, \ldots, n \) such that no two consecutive letters are equal.
Example 3.6. Consider again the labeled poset below.

\[ \begin{array}{ccc}
3 & & 4 \\
\downarrow & & \downarrow \\
2 & & 3 \\
\downarrow & & \downarrow \\
1 & & \\
\end{array} \]

We can define a linear multi-extension of \( P \) by \( e(1) = 1, e(3) = \{3, 5\} e(4) = \{2, 4, 6\}, \) and \( e(2) = 7 \). This linear multi-extension contributes the \( m \)-permutation \( 1434342 \) to \( \tilde{\mathcal{J}}(P, \theta) \).

The following result is proven in \([5]\) by giving an explicit weight-preserving bijection between \( \mathcal{A}(P, \theta) \) and the set of pairs \( (w, \sigma') \) where \( w \in \tilde{\mathcal{J}}_N(P, \theta) \) and \( \sigma' \in \mathcal{A}(C, w) \), where \( C = \{c_1 < c_2 < \ldots < c_r\} \) is a chain with \( r \) elements. One can easily recover this bijection from the bijection given in the proof of Theorem 5.4 by restricting to \( \mathcal{A}(P, \theta) \).

Theorem 3.7 (\([5]\), Theorem 5.6). We can write

\[ \tilde{K}_{(P, \theta)} = \sum_{n \geq N} \sum_{w \in \mathcal{J}_N(P, \theta)} \tilde{L}_{C(w)}. \]

We now describe how to express \( \tilde{L}_\alpha \) as an infinite linear combination of \( L_\alpha \)'s, where \( L_\alpha \) is the fundamental quasisymmetric function in \( \text{QSym} \). Let \( L^{(i)}_\alpha \) denote the homogeneous component of \( \tilde{L}_\alpha \) of degree \( |\alpha| + i \).

Given \( D \subset [n-1] \) and \( E \subset [n+i-1] \), an injective, order-preserving map \( t : [n-1] \to [n+i-1] \) is an \( i \)-extension of \( D \) to \( E \) if \( t(D) \subset E \) and \((E \setminus t(D)) = ([n+i-1] \setminus t([n-1])) \). In other words, \( E \) is the union of the image of \( D \) and the elements not in the image of \( t \). Thus \( |E| = |D| + i \).

Let \( T(D, E) \) denote the set of \( i \)-extensions from \( D \) to \( E \). For example, if \( D = \{1, 2\} \subset [2] \) and \( E = \{1, 2, 3\} \subset [3] \), then \( |T(D, E)| = 3 \). On the other hand, if we have \( D' = \{1, 2\} \subset [3] \) and \( E' = \{1, 3, 4\} \subset [4] \), then \( |T(D', E')| = 0 \). The proof of the following theorem is similar to that of Theorem 5.11.

Theorem 3.8 (\([5]\), Theorem 5.12). Let \( \alpha \) be a composition of \( n \) and \( D = D(\alpha) \) be the corresponding descent set. Then for each \( i \geq 0 \), we have

\[ L^{(i)}_\alpha = \sum_{E \subset [n+i-1]} |T(D, E)| L_{C(E)}. \]

3.3. Hopf structure. Next we describe the bialgebra structure of \( \text{mqSym} \) using the continuous basis of multi-fundamental quasisymmetric functions. The first step is to define the multishuffle of two words in a fixed alphabet. To that end, we give the following definition.

Definition 3.9. Let \( a = a_1 a_2 \cdots a_k \) be a word. We call \( w = w_1 w_2 \cdots w_r \) a multiword of \( a \) if there exists non-decreasing, surjective map \( t : [k] \to [r] \) such that \( w_j = a_{t(j)}. \)

As an example, consider the permutation 1342 as a word in \( \mathbb{N} \). Then 11333422 and 1342 are both multiwords of 1342, while 34442 and 113344 are not multiwords of 1342.

Definition 3.10. Let \( a = a_1 a_2 \cdots a_k \) and \( b = b_1 b_2 \cdots b_n \) be words with distinct letters. We say that \( w = w_1 w_2 \cdots w_m \) is a multishuffle of \( a \) and \( b \) if the following conditions are satisfied:
(1) $w_i \neq w_{i+1}$ for all $i$

(2) when restricted to $\{a_i\}$, $w$ is a multiword of $a$

(3) when restricted to $\{b_j\}$, $w$ is a multiword of $b$.

Eventually, we would like to multishuffle two permutations, which will not have distinct letters. To remedy this, given a permutation $w = w_1w_2 \cdots w_k$, define $w[n] = (w_1 + n)(w_2 + n) \cdots (w_k + n)$ to be the word obtained by adding $n$ to each digit entry of $w$. For example, for $w = 21$, $w[4] = 65$.

Starting with permutations $u = 1342$ and $w = 21$, we see that $v = 161613346252$ is a multishuffle of $u = 1342$ and $w[4] = 65$, where we shift $w$ by 4 since 4 is the largest letter in $u$. If we restrict to the letters in $u$, $v|_u = 11133422$ is a multiword of $u$, and similarly $v|_w[4] = 6665$ is a multiword of $w[4]$.

**Proposition 3.11** ([5], Proposition 5.9). Let $\alpha$ be a composition of $n$ and $\beta$ be a composition of $m$. Then

\[
\Delta(\tilde{L}_\alpha \tilde{L}_\beta) = \sum_{u \in \text{Sh}^m(w_\alpha, w_\beta[n])} \tilde{L}_{C(u)},
\]

where the sum is over all multishuffles of $w_\alpha$ and $w_\beta[n]$.

Note that this is an infinite sum whose lowest degree terms are exactly those of $L_\alpha L_\beta$, the product of the two corresponding fundamental quasisymmetric functions.

To define the coproduct, we need the following definition.

**Definition 3.12.** Let $w = w_1w_2 \cdots w_k$ be a permutation. Then $\text{Cuut}(w)$ is the set of terms of the form $w_1w_2 \cdots w_i \otimes w_{i+1}w_{i+1} \cdots w_k$ or of the form $w_1w_2 \cdots w_i \otimes w_iw_{i+1} \cdots w_k$ for $i \in [0, k]$.

For example, $\text{Cuut}(132) = \{\emptyset \otimes 132, 1 \otimes 132, 1 \otimes 32, 13 \otimes 32, 13 \otimes 2, 132 \otimes 2, 132 \otimes \emptyset\}$. Note how this compares to the cut coproduct of the shuffle algebra described in Section 2 to understand the strange spelling.

**Proposition 3.13** ([5], Proposition 5.10). We have that

\[
\Delta(\tilde{L}_\alpha) = \tilde{L}_\alpha(x, y) = \sum_{u \otimes u' \in \text{Cuut}(w_\alpha)} \tilde{L}_{C(u)}(x) \otimes \tilde{L}_{C(u')}(y).
\]

**Example 3.14.** Let $\alpha = (1)$ and $\beta = (2, 1)$ with $w_\alpha = 1$ and $w_\beta = 231$. Then

\[
\tilde{L}_\alpha \tilde{L}_\beta = \tilde{L}_{(3,1)} + \tilde{L}_{(1,2,1)} + \tilde{L}_{(2,2)} + \tilde{L}_{(2,1,1)} + \tilde{L}_{(2,2,1,1)} + \tilde{L}_{(2,2,1,2)} + \cdots,
\]

where the terms listed correspond to the multishuffles $1342, 3142, 3412, 3421, 13421, 131421,$ and $3414212$ of $w_\alpha$ and $w_\beta[1]$. We also compute

\[
\Delta(\tilde{L}_\beta) = \emptyset \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(1,1)} + \tilde{L}_{(2)} \otimes \tilde{L}_{(1,1)} + \tilde{L}_{(2)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \emptyset.
\]

We give a combinatorial formula for the antipode map in $\mathfrak{m}QSym$ in Theorem 4.9. In Section 5 we give an antipode map in terms of a new basis introduced within the section.

4. **The Hopf algebra of Multi-noncommutative symmetric functions**

The Hopf algebra of noncommutative symmetric functions ($\text{NSym}$) is dual to that of quasisymmetric functions. We next describe a $K$-theoretic analogue called the Multi-noncommutative symmetric functions or $\mathfrak{mNSym}$. We recall its bialgebra structure as given in [5] and develop a combinatorial formula for its antipode map.
4.1. Multi-noncommutative ribbon functions and bialgebra structure. \(\mathcal{MNSym}\) has a basis \(\{\tilde{R}_\alpha\}\) of Multi-noncommutative ribbon functions indexed by compositions, which is an analogue to the basis of noncommutative ribbon functions \(\{R_\alpha\}\) for \(\mathcal{NSym}\). There is a bijection between compositions and ribbon diagrams sending \(\alpha = (\alpha_1, \ldots, \alpha_k)\) to the skew diagram \(\lambda/\mu\) with \(k\) rows where row \(k-i\) has \(\alpha_{i+1}\) squares and there is exactly one column of overlap between adjacent rows. Thinking of \(\{\tilde{R}_\alpha\}\) as being indexed by ribbon diagrams will be useful.

Example 4.1. The ribbon diagram on the left corresponds to \(\alpha = (2, 1, 3)\), and the diagram on the right corresponds to \(\beta = (1, 1, 2)\).

\[
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\end{array}
\]

We first introduce a product structure on \(\{\tilde{R}_\alpha\}\) as given in [5].

**Proposition 4.2** ([5], Proposition 8.1). Let \(\alpha = (\alpha_1, \ldots, \alpha_k)\) and \(\beta = (\beta_1, \ldots, \beta_m)\) be compositions. Then

\[
\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \lessdot \beta} + \tilde{R}_{\alpha \cdot \beta} + \tilde{R}_{\alpha \ominus \beta},
\]

where \(\alpha \lessdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)\), \(\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1 - 1, \beta_2, \ldots, \beta_m)\), and \(\alpha \ominus \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m)\).

Example 4.3. It is helpful to think of the product using ribbon diagrams. From the statement above, we have

\[
\tilde{R}_{(2, 2)} \bullet \tilde{R}_{(1, 2)} = \tilde{R}_{(2, 2, 1, 2)} + \tilde{R}_{(2, 2, 2)} + \tilde{R}_{(2, 3, 2)}.
\]

In pictures, this is

\[
\begin{array}{ccc|c|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \quad + \quad \begin{array}{c|c|c|c}
\cdot & \cdot & \cdot & \cdot \\
\end{array} \quad + \quad \begin{array}{c|c|c|c}
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

In contrast to the product in \(\mathcal{mQSym}\), the product in \(\mathcal{MNSym}\) is a finite sum whose highest degree terms are those of the corresponding product \(R_\alpha R_\beta\) in \(\mathcal{NSym}\).

**Proposition 4.4.** The coproduct of a basis element is

\[
\Delta(\tilde{R}_\alpha) = \sum_{w_\beta \in \text{Sh}^n(w_\beta, w_\delta[i])} \tilde{R}_\beta \otimes \tilde{R}_\delta,
\]

where \(i \in \mathbb{N}\) and \(w_\beta \in \mathfrak{S}_i\).

Note that since multishuffles of \(w_\beta\) and \(w_\delta[i]\) may not have adjacent letters that are equal, we may define the descent set of a multishuffle of \(w_\beta\) and \(w_\delta[i]\) in the usual way.

**Example 4.5.** In general, computing the coproduct in \(\mathcal{MNSym}\) is not an easy task. However, for compositions with only one part, we have

\[
\Delta(\tilde{R}_{(n)}) = \tilde{R}_{(n)} \otimes 1 + \tilde{R}_{(n-1)} \otimes \tilde{R}_{(1)} + \tilde{R}_{(n-2)} \otimes \tilde{R}_{(2)} + \ldots + \tilde{R}_{(1)} \otimes \tilde{R}_{(n-1)} + 1 \otimes \tilde{R}_{(n)}
\]

because the only way a multishuffle of two permutations results in an increasing sequence is for it to be the concatenation of two increasing permutations. We use this fact in the proof of the antipode in \(\mathcal{MNSym}\).
4.2. Antipode map for \( \mathfrak{MNSym} \). Suppose we have a ribbon shape corresponding to \( \alpha \), a composition of \( n \). We say that ribbon shape \( \beta \) is a merging of ribbon shape \( \alpha \) if we can obtain shape \( \beta \) from shape \( \alpha \) by merging pairs of boxes that share an edge. The order in which the pairs are merged does not matter, only set of boxes that were merged. Let \( M_{\alpha,\beta} \) be the number of ways to obtain shape \( \beta \) from shape \( \alpha \) by merging. We will label each box in the ribbon shape to keep track of our actions.

**Example 4.6.** Let \( \alpha = (2,2,1) \) and \( \beta = (2,1) \). Then \( M_{\alpha,\beta} = 3 \). The labeled ribbon shape \( \alpha \) and the three mergings resulting in shape \( \beta \) are shown below.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 1 \\
245 & 3 & 123 \\
4 & 5 & 4 & 235 \\
\end{array}
\]

**Theorem 4.7.** Let \( \alpha \) be a composition of \( n \). Then

\[
S(\tilde{\mathcal{R}}_\alpha) = (-1)^n \sum_{\beta} M_{\omega(\alpha),\beta} \tilde{R}_\beta,
\]

where we sum over all compositions \( \beta \).

Note that only finitely many terms will be nonzero because \( M_{\omega(\alpha),\beta} = 0 \) if \( |\beta| > |\alpha| \).

**Proof.** We prove this by induction on the number of parts of the composition \( \alpha \).

We compute directly that

\[
0 = S(\tilde{\mathcal{R}}_{(1)}) = S(\tilde{\mathcal{R}}_{(1)}) \cdot 1 + 1 \cdot S(\tilde{\mathcal{R}}_{(1)}) = S(\tilde{\mathcal{R}}_{(1)}) + \tilde{R}_{(1)}
\]

by Definition 2.11 so \( S(\tilde{\mathcal{R}}_{(1)}) = -\tilde{R}_{(1)} \).

Now assume that \( S(\tilde{\mathcal{R}}_{(k)}) = (-1)^k \sum_{i=0}^{k-1} \binom{k-1}{i} \tilde{R}_{1+i} \) for all \( k < n \). Then, using Example 4.5 and Definition 2.9 we see that

\[
0 = \tilde{R}_{(n)} + S(\tilde{\mathcal{R}}_{(n)}) + \sum_{i=1}^{n-1} S(\tilde{\mathcal{R}}_{(i)}) \cdot \tilde{R}_{(n-i)}
\]

\[
= \tilde{R}_{(n)} + S(\tilde{\mathcal{R}}_{(n)}) + \sum_{i=1}^{n-1} \left( (-1)^i \sum_{j=0}^{i-1} \binom{i-1}{j} \tilde{R}_{(1+i)} \right) \cdot \tilde{R}_{(n-i)}
\]

\[
= \tilde{R}_{(n)} + S(\tilde{\mathcal{R}}_{(n)}) + \sum_{i=1}^{n-1} (-1)^i \sum_{j=0}^{i-1} \binom{i-1}{j} \tilde{R}_{(1+i,n-i)} + \tilde{R}_{(1,n-i+1)} + \tilde{R}_{(1,n-i)}.
\]

There are five types of terms that show up in this sum.

(1) \( \tilde{R}_{(1^s,m)} \), where \( s = n - m \). The coefficient of this term is

\[
(-1)^{n-m} \binom{n-m-1}{k-1} + (-1)^{n-m-1} \binom{n-m}{k} = 0.
\]

(2) \( \tilde{R}_{(m)} \), where \( 1 < m < n \). The coefficient of this term is

\[
(-1)^{n-m} \binom{n-m-1}{0} + (-1)^{n-m+1} \binom{n+m}{0} = 0.
\]
(3) \( \tilde{R}_{(1^s,m)} \), where \( s < n - m \), and \( m > 0 \). The coefficient of this term

\[
(-1)^{n-m} \binom{n-m-1}{0} + (-1)^{n-m-1} + (-1)^{n-m+1} \binom{n-m}{1} = 0.
\]

(4) \( \tilde{R}_{(1^k)} \), where \( k \leq n \). The coefficient of this term is

\[
(-1)^{n-1} \binom{n-2}{k-2} + (-1)^{n-1} \binom{n-2}{k-1} = (-1)^{n-1} \binom{n-1}{k-1}.
\]

(5) \( \tilde{R}_{(n)} \). The coefficient of this term is

\[
(-1)^1 \binom{0}{0} = -1.
\]

Thus \( 0 = S(\tilde{R}_{(n)}) + (-1)^{n-1} \sum_{s=1}^{n} \binom{n-1}{s-1} \tilde{R}_{1^s} \), and so \( S(\tilde{R}_{(n)}) = (-1)^n \sum_{s=0}^{n-1} \binom{n-1}{s} \tilde{R}_{(1^{s+1})} \).

It is clear that there are \( \binom{n-1}{s} \) mergings of \( \omega(\alpha) = (1^n) \) that result in shape \( (1^{s+1}) \) since we are choosing \( s + 1 \) of the \( n - 1 \) border edges to merge to remain intact.

Now suppose \( S(\tilde{R}_{\alpha}) = (-1)^n \sum_{\beta} M_{\omega(\alpha),\beta} \tilde{R}_{\beta} \) holds for all compositions \( \alpha \) with up to \( k - 1 \) parts, and let \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \) be a composition with \( k \) parts. We know that

\[
\tilde{R}_{\beta} = \tilde{R}_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1})} \cdot \tilde{R}_{(\beta_k)} - \tilde{R}_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1}+\beta_k)} - R_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1}+\beta_k-1)},
\]

and so

\[
S(\tilde{R}_{\beta}) = S(\tilde{R}_{(\beta_k)}) \cdot S(\tilde{R}_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1})}) - S(\tilde{R}_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1}+\beta_k)}) - S(\tilde{R}_{(\beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1}+\beta_k-1)}).
\]

In the image below, let the thin rectangle represent all mergings of \( \omega(\beta_k) \) and the square represent all mergings of \( \omega(\beta_1, \ldots, \beta_{k-1}) \). Then the image labeled (1) represents all mergings obtained by adding the last part of a merging of \( \omega(\beta_k) \) to the first part of a merging of \( \omega(\beta_1, \ldots, \beta_{k-1}) \). The image labeled (2) represents all mergings obtained by merging the topmost box in a merging of \( \omega(\beta_k) \) with the bottom leftmost box of a merging of \( \omega(\beta_1, \ldots, \beta_{k}) \). These two mergings with multiplicities are exactly the shapes we want in \( S(\tilde{R}_{\beta}) \).

The imaged labeled (3) represents all mergings obtained by concatenating a merging of \( \omega(\beta_k) \) with a merging of \( \omega(\beta_1, \ldots, \beta_{k-1}) \). We do not want these mergings to appear in \( S(\tilde{R}_{\beta}) \) because it is impossible for boxes that are side by side in \( \omega(\beta) \) to be stacked one on top of the other in a merging of \( \omega(\beta) \).

We use the fact that \( S(\tilde{R}_{\beta_k}) \cdot S(\tilde{R}_{(\beta_1, \ldots, \beta_{(k-1)})}) \) results in all mergings of type (1), (2), and (3), \( S(\tilde{R}_{(\beta_1, \ldots, \beta_{k-1}+\beta_k)}) \) gives all mergings of type (2) and (3), and \( S(\tilde{R}_{(\beta_1, \ldots, \beta_{k-1}+\beta_k-1)}) \) contains exactly those mergings of type (2). The parity of the sizes of the compositions provides the necessary cancellation and leaves us with all mergings of type (1) and (2), as desired.
Example 4.8. Consider $S(\tilde{R}_{(1,2)}) = S(\tilde{R}_{(1,1)}) \cdot S(\tilde{R}_{(1)}) - S(\tilde{R}_{(1,1,1)}) - S(\tilde{R}_{(1,1)}).$ The image below shows all of the mergings in $S(\tilde{R}_{(1,1)}) \cdot S(\tilde{R}_{(1)})$ in the first line with the proper sign, subtracts mergings of $S(\tilde{R}_{(1,1,1)})$ in the second line, and subtracts mergings of $S(\tilde{R}_{(1,1)})$ in the third line. The box labeled 1 represents the top box in mergings of $\omega(\beta_k) = \omega(2) = (1,1)$, and the box labeled 2 represents the bottom leftmost box in mergings of $\omega(\beta_1, \ldots, \beta_{k-1}) = \omega(1) = (1)$.

\begin{align*}
&= \begin{array}{cccc}
\begin{array}{c}
2
\end{array} & \begin{array}{c}
1
\end{array} & \begin{array}{c}
2
\end{array} & \begin{array}{c}
12
\end{array} \\
\begin{array}{c}
2
\end{array} & \begin{array}{c}
1
\end{array} & \begin{array}{c}
2
\end{array} & \begin{array}{c}
12
\end{array}
\end{array} \\
&+ \begin{array}{cc}
\begin{array}{c}
2
\end{array} & \begin{array}{c}
1
\end{array}
\end{array} + \begin{array}{c}
12
\end{array} + \begin{array}{c}
12
\end{array} \\
&- \begin{array}{c}
12
\end{array} - \begin{array}{c}
12
\end{array}
\end{align*}

4.3. **Antipode map for mQSym.** We know from [5, Theorem 8.4] that the bases $\{\tilde{L}_\alpha\}$ and $\{\tilde{R}_\alpha\}$ satisfy the criteria in Lemma 2.12. Extending the definition below by continuity gives the following antipode formula in mQSym.

**Theorem 4.9.** Let $\alpha$ be a composition of $n$. Then

$$S(\tilde{L}_\alpha) = \sum_\beta (-1)^{|\beta|} M_{\beta, \omega(\alpha)} \tilde{L}_\beta,$$

where the sum is over all compositions $\beta$.

Note that while $S(\tilde{R}_\alpha)$ is a finite sum of Multi-noncommutative ribbon functions for any $\alpha$, $S(\tilde{L}_\alpha)$ is an infinite sum of multi-fundamental quasisymmetric functions for any $\alpha$. Since any arbitrary linear combination of multi-fundamental quasisymmetric functions is in mQSym, this is an admissible antipode formula.
5. A new basis for $\mathfrak{mQSym}$

5.1. $(P, \theta)$-multiset-valued partitions. To create a new basis for $\mathfrak{mQSym}$, which will be useful in finding antipode formulas, we extend the definition of a $(P, \theta)$-set-valued partition to what we call a $(P, \theta)$-multiset-valued partition in the natural way. In a $(P, \theta)$-multiset-valued partition $\sigma$, we allow $\sigma(p)$ to be a finite multiset of $\mathbb{P}$, keeping all other definitions the same. An example of a $(P, \theta)$-multiset-valued partition is shown below.

\[
\begin{array}{c}
1 & 2 \\
4 & 3
\end{array}
\]

Now define $\hat{\mathcal{A}}(P, \theta)$ to be the set of all $(P, \theta)$-multiset-valued partitions. For each element $i \in \mathbb{P}$, let $\sigma^{-1}(i)$ be the multiset $\{x \in P | i = \sigma(x)\}$. In the example shown above, $\sigma^{-1}(1) = \{1, 1, 3\}$. Now define $\hat{K}_{P,\theta} \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

\[\hat{K}_{P,\theta} = \sum_{\sigma \in \hat{\mathcal{A}}(P,\theta)} x_1^{#\sigma^{-1}(1)} x_2^{#\sigma^{-1}(2)} \cdots.\]

Using this multiset analogue of our definitions, we define

\[\hat{L}_\alpha = \hat{K}_{(P,w)} = \sum_{\sigma \in \hat{\mathcal{A}}(P,w)} x_1^{#\sigma^{-1}(1)} x_2^{#\sigma^{-1}(2)} \cdots,\]

where $P = p_1 < \ldots < p_k$ is a finite linear order and $w \in \mathfrak{S}_k$.

5.2. Properties. Recall the definition of $T(D(\alpha), E)$ from Section 3.

**Theorem 5.1.** We have that

\[\hat{L}_\alpha^{(i)} = \sum_{E \subseteq [n+i-1]} |T(D(\omega(\alpha)), E)| L_{\omega(E)}|L_{\omega(C(E))}.\]

**Proof.** Let $\omega = \omega_\alpha$ and consider the subset $\hat{\mathcal{A}}_i(C, \omega) \subset \hat{\mathcal{A}}(C, \omega)$ consisting of multiset-valued $(C, \omega)$-partitions $\sigma$ of size $|\sigma| = n+i$. We must show that the generating function of $\hat{\mathcal{A}}_i(C, \omega)$ is equal to

\[\sum_{E \subseteq [n+i-1]} |T(D(\omega(\alpha)), E)| L_{\omega(E)}|L_{\omega(C(E))}.\]

Here, we define $L_\alpha$ by

\[L_\alpha = \sum_{\sigma \in \hat{\mathcal{A}}(P,w)} x_1^{#\sigma^{-1}(1)} x_2^{#\sigma^{-1}(2)} \cdots,\]

where $\mathcal{A}(P, w)$ is the set of all $(P, w)$-partitions as in \[3\]. Indeed, for each pair $t \in T(D(\omega(\alpha)), E)$ for some $E$, the function $L_{\omega(E)}|L_{\omega(C(E))}$ is the generating function of all $\sigma \in \hat{\mathcal{A}}_i(C, \omega)$ satisfying $|\sigma(c_i)| = t(n+i-j) - t(n+i-(j+1))$, where $t(0) = 0$ and $t(n) = n+i$. Letting $C' = c'_1 < c'_2, \ldots, c'_{n+i}$ be a chain with $n+i$ elements, we obtain a $(C', \omega(C(E)))$-partition $\sigma' \in \mathcal{A}(C', \omega(C(E)))$ by assigning the elements of $\sigma(c_i)$ in increasing order to $c'_{t(i)+1}, \ldots, c'_{t(i)}$. \[\square\]
Example 5.2. Letting $\alpha = (1, 2, 1, 2)$, we see that
\[ \hat{L}^{(i)}_\alpha = L_{(2,2,1,2)} + 2L_{(1,3,1,2)} + L_{(1,2,2,2)} + 2L_{(1,2,1,3)}. \]
In this example, the coefficient of $L_{(1,3,1,2)}$ is 2 because
\[ |T(D(\omega(1, 2, 1, 2)), \{1, 4, 5\}) \setminus \{n+1-1 = 6\}| = |T(D(1, 3, 2), \{1, 4, 5\})| = |T(\{2, 4\}, \{1, 4, 5\})| = 2. \]

Given the basis of multi-quasisymmetric functions, $\{\tilde{L}_\alpha\}$, the set $\{\hat{L}_\alpha\}$ is natural to consider because of the following proposition. In the following sections, $\omega$ will denote the fundamental involution of the symmetric functions defined by $\omega(e_n) = h_n$ for all for elementary symmetric function $e_n$ and complete homogeneous symmetric function $h_n$ and for all $n$.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of $n$, define $rev(\alpha) = (\alpha_k, \ldots, \alpha_1)$. Now let $\omega(\alpha)$ be the unique composition of $n$ whose partial sums $S_{\omega(\alpha)}$ form the complementary set within $[n-1]$ to the partial sums $S_{rev(\alpha)}$. Alternately, we may think of the ribbon shape $\alpha$ this as $\lambda/\mu$ for tableaux $\lambda$ and $\mu$. Then $\omega(\alpha)$ to be the ribbon shape $\lambda'/\mu'$, where $\lambda'$ and $\mu'$ are the transposes of $\lambda$ and $\mu$, respectively. The number of blocks in each row of $\omega(\alpha)$ reading from bottom to top corresponds to the number of blocks in each column of $\alpha$ reading from right to left. For example, if $\alpha = (2, 1, 1, 3)$, $\omega(\alpha) = (1, 1, 4, 1)$. It is known that $\omega(L_\alpha) = L_{\omega(\alpha)}$ in $\text{QSym}$.

**Proposition 5.3.** We have $\omega(\tilde{L}_\alpha) = \hat{L}_{\omega(\alpha)}$, and the set of $\hat{L}_\alpha$’s form a basis for $\mathfrak{mQSym}$.

**Proof.** Using Proposition 5.1
\[ \omega(\tilde{L}_\alpha) = \omega \left( \sum_{E \subset [n+1-i-1]} |T(D(\alpha), E)|L_{C(E)} \right) = \sum_{E \subset [n+1-i-1]} |T(D(\alpha), E)|L_{\omega(C(E))} = \hat{L}_{\omega(\alpha)}. \]

We have an analogue of Stanley’s Fundamental Theorem of P-partitions for our new basis of $\tilde{L}_\alpha$’s. The proof of this result follows closely that of Theorem 3.7 given in [5].

**Theorem 5.4.** We have
\[ \hat{K}_{P,\theta} = \sum_{N \geq n} \sum_{w \in \mathcal{J}_N(P, \theta)} \hat{L}_{C(w)}. \]

**Proof.** We prove this result by giving an explicit weight-preserving bijection between $\hat{A}(P, \theta)$ and the set of pairs $(w, \sigma')$ where $w \in \mathcal{J}_N(P, \theta)$ and $\sigma' \in \hat{A}(C, \sigma)$ where $C = (c_1 < c_2 < \cdots < c_l)$ is a chain with $l = \ell(w)$ elements. Let $\sigma \in \hat{A}(P, \theta)$. For each $i$, let $\sigma^{-1}(i)$ denote the submultiset of $[n]$ via $\theta$, and let $w^{(i)}$ denote the word of length $|\sigma^{-1}(i)|$ obtained by writing the elements of $\sigma^{-1}(i)$ in increasing order. Note that it is possible for $w^{(i)} = w^{(i+1)}$. This will occur when the letter $i$ appears more than once in some $\sigma(s)$ for $s \in P$.

Let $w$ denote the unique $\mathfrak{m}$-permutation such that $w_\sigma := w^{(1)}w^{(2)}\cdots$ is a multiword of $w$ and $t: \ell(w_\sigma) \rightarrow \ell(w)$ be the associated function as in Definition 3.9. We know that $w_\sigma$ is a finite word because $\sigma^{-1}(r) = \emptyset$ for sufficiently large $r$. Note that $w_\sigma$ uses all letters $[n]$. Now define $\sigma' \in \hat{A}(C, \sigma)$ by
\[ \sigma'(c_i) = \{r^k \mid r \in \mathbb{P} \text{ and } w^{(r)}_\sigma \text{ contributes } k \text{ letters to } w_\sigma|_{t^{-1}(i)} \} \]
where $w_\sigma|_{t^{-1}(i)}$ is the set of letters in $w_\sigma$ at the positions in the interval $t^{-1}(i)$. We will show that this defines a map $\alpha : \sigma \mapsto (w, \sigma')$ with the required properties.

First, $w$ is the multi-permutation associated to the linear multi-extension $e_w$ of $P$ by $\ell(w)$ defined by the condition that $e_w(x)$ contains $j$ if and only if $w_j = \theta(x)$. It follows from the
for example and the corresponding composition $C_\sigma w$ is a linear multi-extension. To check that $\sigma'$ is a multiset-valued $(C, w)$ partition, we note that $\sigma'(c_i) \leq \sigma'(c_{i+1})$ because the function $t$ is non-decreasing. Moreover, if $w_i > w_{i+1}$, then $\sigma'(c_i) < \sigma'(c_{i+1})$ because each $w^{(r)}_\sigma$ is increasing.

We define the inverse map $\beta : (w, \sigma') \mapsto \sigma$ by the formula
\[
\sigma(x) = \bigcup_{j \in e_w(x)} \sigma'(c_j).
\]
The $(P, \theta)$-multiset-valued partition $\sigma$ respects $\theta$ because $e_w$ is a linear multi-extension. Thus if $x < y$ in $P$ and $\theta(x) > \theta(y)$, then $\sigma(x) < \sigma(y)$ since $e_w(x) < e_w(y)$ and there is a descent in $w$ between the corresponding entries of $\theta(x)$ and $\theta(y)$.

Then $\beta \circ \alpha = \text{id}$ follows immediately. For $\alpha \circ \beta = \text{id}$, consider a subset $\sigma'(c_j) \subseteq \sigma(x)$. One checks that this subset gives rise to $|\sigma'(c_j)|$ consecutive letters all equal to $\theta(x)$ in $w_\sigma$ and that this is a maximal set of consecutive repeated letters. This shows that one can recover $\sigma'$. To see that $w$ is recovered correctly, one notes that if $\sigma'(c_j)$ and $\sigma'(c_{j+1})$ contain the same letter $r$ then $w_j < w_{j+1}$ so by definition $w_j$ is placed correctly before $w_{j+1}$ in $w^{(r)}_\sigma$.

\[\Box\]

**Example 5.5.** Let $\theta$ be the labeling
\[
\begin{array}{ccc}
3 & 4 & 5 \\
1 & 2 \\
\end{array}
\]
of the shape $\lambda = (3, 2)$. Take the $(\lambda, \theta)$-partition
\[
\begin{array}{cccccc}
112 & 23 & 345 & \\
45 & 667 & \\
\end{array}
\]
in $\hat{\mathcal{A}}(\lambda, \theta)$. Then we have
\[
w_\sigma = (3, 3; 3, 4; 4, 5; 1, 5; 1, 5; 2, 2; 2),
\]
where for example, $w^{(1)}_\sigma = (3, 3)$ since the cells labeled 3 contains two copies of the number 1 in $\sigma$. Therefore
\[
w = (3, 4, 5, 1, 5, 1, 5, 2)
\]
and the corresponding composition $C(w)$ is $(3, 2, 2, 1)$. Then $\sigma'$ written as sequence is
\[
\{1, 1, 2\}, \{2, 3\}, \{3\}, \{4\}, \{4\}, \{5\}, \{5\}, \{6, 6, 7\}.
\]
For example $\sigma'(c_1) = \{1, 1, 2\}$ since $w^{(1)}_\sigma$ contributes two 3’s and $w^{(2)}_\sigma$ contributes one 3 to the beginning of $w_\sigma$.

To obtain the inverse map, $\beta$, read $w$ and $\sigma'$ in parallel and place $\sigma'(c_i)$ into cell $\theta_\sigma^{-1}(w_i)$. For example, we put $\{1, 1, 2\}$ into the cell labeled 3, and we put $\{2, 3\}$ into the cell labeled 4.

The linear multi-extension, $e_w$ in this example can be represented by the filling below.
\[
\begin{array}{cccc}
1 & 2 & 357 \\
46 & 8 \\
\end{array}
\]

5.3. **Antipode.** Recall that in $\text{QSym}$, $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)} = (-1)^{|\alpha|} \omega(L_\alpha) = \omega(L_\alpha(-x_1, -x_2, \ldots))$. Using the set $\{\hat{L}_\alpha\}$, we have a similar result in $\text{mQSym}$.

**Theorem 5.6.** In $\text{mQSym}$,
\[
S(\hat{L}_\alpha) = \hat{L}_{\omega(\alpha)}(-x_1, -x_2, \ldots).
\]

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Proof. Using Theorem 3.7 and the antipode in QSym, we see that

\[ S(\tilde{L}_\alpha) = S \left( \sum_{E \subset [n+i-1]} |T(D,E)| L_{C(E)} \right) \]

\[ = \sum_{E \subset [n+i-1]} |T(D,E)| S(L_{C(E)}) \]

\[ = \sum_{E \subset [n+i-1]} |T(D,E)|(−1)^{|C(E)|} L_{\omega(C(E))} \]

\[ = \hat{L}_{\omega(\alpha)}(-x_1, -x_2, \ldots). \]

\[ \square \]

6. The Hopf Algebra of Multi-Symmetric Functions

We next describe the space of multi-symmetric functions, mSym. We refer the reader to [5] for details. As in previous sections, familiarity with the Hopf structure of the ring of symmetric functions, Sym, is helpful. We refer the reader to [10] for background on Sym.

6.1. Set-valued tableaux. Let \( P \) be the poset of squares in the Young diagram of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \) and \( \theta_s \) be the bijective labeling of \( P \) obtained from labeling \( P \) in row reading order, i.e. from left to right the bottom row of \( \lambda \) is labeled 1, 2, \ldots, \( \lambda_t \), the next row up is labeled \( \lambda_t + 1, \ldots, \lambda_t + \lambda_{t-1} \) and so on. Note that \( \tilde{K}_{\lambda, \theta_s} \) is the Schur function \( s_\lambda \). We define

\[ \text{mSym} = \prod_{\lambda} \mathbb{Z}\tilde{K}_{\lambda, \theta_s} \]

to be the subspace of mQSym continuously spanned by the \( \tilde{K}_{\lambda, \theta_s} \), where \( \lambda \) varies over all partitions. From this point forward, we will write \( \tilde{K}_\lambda \) in place of \( \tilde{K}_{\lambda, \theta_s} \) call a \((\lambda/\mu, \theta_s)\)-set-valued partition a set-valued tableau of shape \( \lambda/\mu \).

Example 6.1. For \( \lambda = (2, 1) \), we have \( \tilde{K}_\lambda = x_1^2x_2 + 2x_1x_2x_3 + x_1^2x_2^2 + 3x_1^2x_2x_3 + 8x_1x_2x_3x_4 + \ldots \), corresponding to the following labeled poset:

```
3
/ \
2-->1
```

6.2. Basis of stable Grothendieck polynomials. We next introduce another (continuous) basis for mSym, the stable Grothendieck polynomials. Stable Grothendieck polynomials originated from the Grothendieck polynomials of Lascoux and Schützenberger [6], which served as representatives of \( K \)-theory classes of structure sheaves of Schubert varieties. Through the work of Fomin and Kirillov [2] and Buch [1], the stable Grothendieck polynomials, a limit of the Grothendieck polynomials, were discovered and given the combinatorial interpretation in the theorem below. These symmetric functions play the role of Schur functions in the \( K \)-theory of Grassmannians.

Theorem 6.2 ([1], Theorem 3.1). The stable Grothendieck polynomial \( G_{\lambda/\mu}(x) \) is given by the formula

\[ G_{\lambda/\mu}(x) = \sum_T (-1)^{|T|-|\lambda/\mu|} x^T, \]
where the sum is taken over all set-valued tableaux of shape \( \lambda/\mu \).

The stable Grothendieck polynomials are related to the \( \tilde{K}_\lambda \) by

\[
\tilde{K}_\lambda(x_1, x_1, \ldots) = (-1)^{|\lambda|} G_{\lambda}(-x_1, -x_2, \ldots).
\]

**Remark 6.3.** In [1], Buch studied a bialgebra \( \Gamma = \bigoplus \mathbb{Z} G_\lambda \) spanned by the set of stable Grothendieck polynomials. Note that the bialgebra \( \Gamma \) is not the same as \( \mathfrak{m} \text{Sym} \). In particular, the antipode formula given in Theorem 8.2 is valid in \( \mathfrak{m} \text{Sym} \) but not in \( \Gamma \) as only finite linear combinations of stable Grothendieck polynomials are allowed in \( \Gamma \).

### 6.3. Weak set-valued tableaux

The following definition is needed to introduce one final basis for \( \mathfrak{m} \text{Sym}, \{J_\lambda\} \).

**Definition 6.4.** A weak set-valued tableau \( T \) of shape \( \lambda/\nu \) is a filling of the boxes of the skew shape \( \lambda/\nu \) with finite, non-empty multisets of positive integers so that

1. the largest number in each box is strictly smaller than the smallest number in the box directly to the right of it, and
2. the largest number in each box is less than or equal to the smallest number in the box directly below it.

In other words, we fill the boxes with multisets so that rows are strictly increasing and columns are weakly increasing. For example, the filling of shape \( (3, 2, 1) \) shown below gives a weak set-valued tableau, \( T \), of weight \( x_T = x_1^2 x_3^2 x_4^2 x_5 x_6 x_7 \).

\[
\begin{array}{ccc}
12 & 33 & 46 \\
223 & 4 \\
57 &
\end{array}
\]

Let \( J_{\lambda/\nu} = \sum T x^T \) be the weight generating function of weak set-valued tableaux \( T \) of shape \( \lambda/\nu \).

**Theorem 6.5** ([5], Proposition 9.22). For any skew shape \( \lambda/\nu \), we have

\[
\omega(K_{\lambda/\nu}) = J_{\lambda/\nu}.
\]

### 7. The Hopf algebra of Multi-symmetric functions

#### 7.1. Reverse plane partitions

We next introduce the big Hopf algebra of Multi-symmetric function, \( \mathfrak{M} \text{Sym} \), with basis \( \{g_\lambda\} \). \( \mathfrak{M} \text{Sym} \) is isomorphic to \( \text{Sym} \) as a Hopf algebra, but the basis \( \{g_\lambda\} \) is distinct from the basis of Schur functions, \( \{s_\lambda\} \) for \( \text{Sym} \).

**Definition 7.1.** A reverse plane partition \( T \) of shape \( \lambda \) is a filling of the Young diagram of shape \( \lambda \) with positive integers such that the numbers are weakly increasing in rows and columns.

Given a reverse plane partition \( T \), let \( T(i) \) denote the numbers of columns of \( T \) which contain the number \( i \). Then \( x^T := \prod_{i \in \mathbb{P}} x_i^{T(i)} \). Now we may define the dual stable Grothendieck polynomial

\[
g_\lambda(x_1, x_2, \ldots) = \sum_{\text{sh}(T) = \lambda} x^T,
\]

where we sum over all reverse plane partitions of shape \( \lambda \). For a skew shape \( \lambda/\mu \), we may define \( g_{\lambda/\mu} \) analogously, summing over reverse plane partitions of shape \( \lambda/\mu \).
Example 7.2. We use the definition of \( g_\lambda \) to compute
\[
g_{(2,1)} = 2x_1x_2x_3 + 2x_1x_3x_4 + \ldots + x_1^2x_2 + x_1^2x_3 + \ldots + x_3^3 + \ldots + x_1x_2 + x_1x_3 + \ldots\]
corresponding to fillings of type
\[
\begin{array}{ccc}
1 & 2 & \\
3 & 2 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & \\
2 & 1 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & \\
2 & 1 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & \\
1 & 1 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & \\
& & \\
\end{array}
\]

7.2. Valued-set tableaux. We introduce one more basis for \( \mathbb{M}_{\text{Sym}} \), \( \{ j_\lambda \} \), which is the continuous Hopf dual of \( \{ \tilde{J}_\lambda \} = \{ (-1)^{|\lambda|} J_\lambda(-x_1, -x_2, \ldots) \} \).

Definition 7.3. A valued-set tableau \( T \) of shape \( \lambda/\mu \) is a filling of the boxes of \( \lambda/\mu \) with positive integers so that
1. the transpose of the filling of \( T \) is a semistandard tableau, and
2. we have a decomposition of the shape into a disjoint union of groups of boxes, \( \lambda/\mu = \bigsqcup A_j \), so that each \( A_i \) is connected, contained in a single column, and each box in \( A_i \) contains the same number.

Given such a valued-set tableau, \( T \), let \( a_i \) be the number of groups \( A_j \) which contain the number \( i \). Then \( x^T := \prod_{i \geq 1} x_i^{a_i} \). Finally, let \( j_{\lambda/\mu} := \sum_T x^T \), where the sum is over all valued-set tableaux of shape \( \lambda/\mu \).

Example 7.4. The image below shows an example of a valued-set tableau. This tableau contributes the monomial \( x_1x_2x_3x_5x_6^2 \) to \( j_{(4,3,1,1)} \).

Proposition 7.5 \( (\text{[5]}, \text{Proposition 9.25}) \). We have
\[
\omega(g_{\lambda/\mu}) = j_{\lambda/\mu}.
\]

8. Antipode results for \( m_{\text{Sym}} \) and \( \mathbb{M}_{\text{Sym}} \)

As with \( m_{\text{QSym}} \) and \( \mathbb{M}_{\text{NSym}} \), there is a pairing \( \langle g_\lambda, G_\mu \rangle = \delta_{\lambda,\mu} \) with the usual Hall inner product for \( \text{Sym} \) defined by \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \) and the structure constants satisfy the conditions of Lemma 2.12. See Theorem 9.15 in \( \text{[5]} \) for details. It follows that \( \langle \omega(g_\lambda), \omega(G_\mu) \rangle = \langle j_\lambda, \tilde{J}_\mu \rangle = \delta_{\lambda,\mu} \) and \( \langle \tilde{J}_\lambda, K_\mu \rangle = \langle (-1)^{|\lambda|} g_\lambda(-x_1, -x_2, \ldots), (-1)^{|\mu|} G_\mu(-x_1, -x_2, \ldots) \rangle = \delta_{\lambda,\mu} \). We will use these facts to translate antipode results between \( m_{\text{Sym}} \) and \( \mathbb{M}_{\text{Sym}} \).

Using results from Section 5, the following lemma will allow us to easily prove results regarding the antipode map in \( m_{\text{Sym}} \).

Lemma 8.1. We can expand \( J_\lambda = \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \hat{L}_{\omega(C(w))} \).

Proof. We know from Theorem 3.7 that
\[
\tilde{K}_{(P, \theta)} = \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \hat{L}_{\tilde{C}(w)},
\]
$$J_\lambda = \omega(\tilde{K}_\lambda) = \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \omega(\tilde{L}_C(w)) = \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \hat{L}_{\omega(C(w))}. \square$$

Recall that in $\text{Sym}$, $S(s_\lambda) = (-1)^{|\lambda|} \omega(s_\lambda)$, so one may expect similar behavior from $\tilde{K}_\lambda$ and $G_\lambda$. Indeed, we obtain the theorem below.

**Theorem 8.2.** In $m\text{Sym}$, the antipode map acts as follows.

(a) $S(\tilde{K}_\lambda) = J_\lambda(-x_1, -x_2, \ldots) = (-1)^{|\lambda|} \omega(G_\lambda)$, and
(b) $S(G_\lambda) = (-1)^{|\lambda|} J_\lambda = (-1)^{|\lambda|} \omega(\tilde{K}_\lambda)$.

**Proof.** For the first assertion, we have that

$$S(\tilde{K}_\lambda) = S(\sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \tilde{L}_C(w))$$

$$= \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} S(\tilde{L}_C(w))$$

$$= \sum_{n \geq N} \sum_{w \in \tilde{J}_N(P, \theta)} \hat{L}_{\omega(C(w))}(-x_1, -x_2, \ldots)$$

$$= J_\lambda(-x_1, -x_2, \ldots).$$

And for the second assertion,

$$S(G_\lambda) = S((-1)^{|\lambda|} \tilde{K}_\lambda(-x_1, -x_2, \ldots))$$

$$= (-1)^{|\lambda|} S(\tilde{K}_\lambda(-x_1, -x_2, \ldots)$$

$$= (-1)^{|\lambda|} J_\lambda.$$

By Lemma 2.12, we immediately have the following results in $\mathfrak{M}\text{Sym}$.

**Theorem 8.3.** We have

(a) $S(\tilde{j}_\lambda) = (-1)^{|\lambda|} g_\lambda$, where $\tilde{j}_\lambda = (-1)^{|\lambda|} j_\lambda(-x_1, -x_2, \ldots)$, and
(b) $S(j_\lambda) = g_\lambda(-x_1, -x_2, \ldots)$.

Next, we work toward expanding $S(G_\lambda)$ and $S(\tilde{j}_\lambda)$ in terms of $\{G_\mu\}$ and $\{\tilde{j}_\mu\}$, respectively. We introduce two theorems of Lenart as well as the notion of a hook-restricted plane partitions.

Given partitions $\lambda$ and $\mu$ with $\mu \subset \lambda$, define an elegant filling of the skew shape $\lambda/\mu$ to be a semistandard filling such that the numbers in row $i$ lie in $[1, i - 1]$. Now let $f^\mu_\lambda$ denote the number of elegant fillings of $\lambda/\mu$ for $\mu \subset \lambda$ and set $f^\mu_\lambda = 0$ otherwise.

**Theorem 8.4** ([7], Theorem 2.7). For a partition $\lambda$, we have

$$s_\lambda = \sum_{\mu \geq \lambda} f^\lambda_\mu G_\mu,$$

where $f^\lambda_\mu$ is the number of elegant fillings of $\lambda/\mu$.

For the second theorem, let $r_{\lambda/\mu}$ be the number of elegant fillings of $\lambda/\mu$ such that both rows and columns are strictly increasing. We will refer to such fillings as strictly elegant.
Theorem 8.5 ([7], Theorem 2.2). We can expand the stable Grothendieck polynomial $G_{\lambda}$ in terms of Schur functions as follows

$$G_{\lambda} = \sum_{\mu \supset \lambda} (-1)^{|\mu/\lambda|} r_{\lambda\mu} s_{\mu}.$$ 

Given two partitions, $\lambda$ and $\mu$, we now define the number $P_{\lambda}^{\mu}$. First, $P_{\lambda}^{\mu} = 0$ if $\mu \not\subset \lambda$, and $P_{\lambda}^{\mu} = 1$ if $\lambda = \mu$. If $\mu \subset \lambda$, then $P_{\lambda}^{\mu}$ is equal to the number of hook restricted plane partitions of the skew shape $\lambda/\mu$. A hook restricted plane partition is a filling of the boxes of $\lambda/\mu$ with positive integers such that the numbers are weakly decreasing along rows and columns with the following restrictions.

1. If box $b$ in shape $\lambda/\mu$ shares an edge with a box in shape $\mu$, then the number in box $b$ must lie in $[1, h(b)]$, where $h(b)$ is the number of boxes in $\mu$ lying above $b$ in the same column as $b$ or lying to the left of box $b$ in the same row as $b$. We may think of $h(b)$ as being a reflected hook length of $b$.
2. If box $b$ in shape $\lambda/\mu$ does not share an edge with a box in shape $\mu$, let $a_1$ and $a_2$ denote the boxes in $\lambda/\mu$ directly above and directly to the left of $b$. It is possible that one of these two boxes does not exist. Define $h(b)$ to be the minimum of $h(a_1)$ and $h(a_2)$, where $h(a_i) = \infty$ if box $a_i$ does not exist. Then the number in box $b$ must lie in $[1, h(b)]$.

Example 8.6. The diagram on the left shows $h(b)$ for each box $b$ in the shape $(5,5,5)/(4,2)$ and is also an example of a hook restricted plane partition on $(5,5,5)/(4,2)$. The diagram on the right shows another hook restricted plane partition on $(5,5,5)/(4,2)$.

|   |   |   |   |   |
|---|---|---|---|---|
| 4 |   |   |   |   |
| 3 | 3 | 3 |   |   |
| 2 | 2 | 2 | 2 | 2 |

|   |   |   |   |   |
|---|---|---|---|---|
|   |   |   |   |   |
| 3 | 3 | 3 |   |   |
| 2 | 2 | 2 | 1 | 1 |

Theorem 8.7. Let $\lambda$ and $\mu$ be partitions. Then

(a) $S(G_{\mu}) = (-1)^{|\mu|} \sum_{\lambda} P_{\lambda}^{\mu} G_{\lambda}$, and

(b) $S(\tilde{J}_{\lambda}) = (-1)^{|\lambda|} \sum_{\mu} P_{\lambda}^{\mu} \tilde{J}_{\mu}$.

Proof. We will focus on part (a), and part (b) will follow from Lemma 2.12.

From Theorem 8.2 we know that

$$S(G_{\lambda}) = (-1)^{|\lambda|} J_{\lambda},$$

so it remains to expand $J_{\lambda}$ in terms of stable Grothendieck polynomials.

From Theorem 8.5 it easily follows that we can write

$$\tilde{K}_{\lambda} = \sum_{\mu \supset \lambda} r_{\lambda\mu} s_{\mu}.$$ 

Applying $\omega$ to both sides, we have

$$\tilde{J}_{\lambda} = \sum_{\mu \supset \lambda} r_{\lambda\mu} s_{\mu}.$$ 

Now we can use Theorem 8.4 to write

$$\tilde{J}_{\lambda} = \sum_{\substack{\mu \supset \lambda \\mu \supset \mu^t}} r_{\lambda\mu} f_{\mu^t} G_{\mu^t}.$$
Thus the coefficient of $G_\nu$ in $\tilde{J}_\lambda$ is \[
\sum_{\mu \text{ such that } \mu \supset \lambda \text{ and } \mu \vdash \nu} r_{\lambda \mu} f_{\nu}(t) .
\]

We describe a bijection between partitions of shape $\nu^t$ which contain some $\mu \supset \lambda$ such that the filling of $\mu/\lambda$ is strictly elegant and boxes in $\nu^t/\mu$ are filled such that the transpose is an elegant filling of $\nu^t/\mu^t$ and hook-restricted plane partitions of $\nu/\lambda^t$. Note that if we have a hook-restricted plane partition of $\nu^t/\lambda$, then its transpose is a hook-restricted plane partition of $\nu/\lambda^t$.

We first define a map $\phi$ from pairs consisting of a strictly elegant filling and the transpose of an elegant filling to a hook restricted plane partitions. Suppose we have such a filling of shape $\nu^t$ and some $\mu$ with $\lambda \subseteq \mu \subseteq \nu^t$. For any box $b$ in $\nu^t$, let $d(b)$ denote the southwest to northeast diagonal that contains box $b$. If box $b$ is in row $i$ and column $j$, then $d(b) = i + j - 1$. Let $c(b)$ denote the column that contains box $b$ and $e_b$ denote the integer in box $b$. To obtain a hook-restricted plane partition follow these steps:

1. if box $b$ is in $\mu$, fill the corresponding box in the hook-restricted plane partition with $\phi(b) = d(b) - e_b$, and
2. if box $b$ is in $\nu^t/\mu$, fill the corresponding box in the hook-restricted plane partition with $\phi(b) = c(b) - e_b$.

It is easy to see that the parts of the hook-restricted plane partition corresponding to shape $\mu$ and to $\nu^t/\mu$ are weakly decreasing in rows and columns. We now check that entries are weakly decreasing along the seams. If box $b$ is in $\mu$, then $e_b \leq i_b - 1$, where $b$ is in row $i_b$ and column $j_b$. Therefore $\phi(b) = d(b) - e_b \geq i_b + j_b - 1 - (i_b - 1) = j_b$. If box $a$ is in $\nu^t/\mu$, then $1 \leq \phi(a) = e_a \leq j_a - 1$, so $1 \leq c(a) - e_a = j_a - e_a \leq j_a - 1$. If $b$ and $a$ are adjacent, then $j_b \leq j_a$, so $\phi(b) \geq \phi(a)$.

Next, we check that for all boxes in $\nu^t/\lambda$ that are adjacent to $\lambda$, $\phi(b) \in [1, h(b)]$, so the resulting filling is indeed a hook-restricted plane partition. Let box $b$ be in $\mu$ in row $i$ and column $j$ with $l$ boxes above $b$ contributing to $h(b)$ and $k$ boxes to the left of $b$ contributing to $h(b)$. (See Example 8.8 below.) Then since rows and columns are strictly increasing, $e_b \geq j - k + (i - l - 1)$. If follows that $\phi(b) = d(b) - e_b \leq d(e) - (j - k) - (i - l - 1) = l + k = h(b)$, as desired.

**Example 8.8.** In the figure below, boxes in $\lambda$ are marked with a dot. For box $b$, we have $i = 2$, $j = 4$, $l = 1$, and $k = 2$.

```
\[
\begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & b \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
```

Next suppose box $b$ described above is in $\nu^t/\mu$. Because the transpose of the filling of $\nu^t/\mu$ is an elegant filling, $e_b \geq j - k$. Then we have that $\phi(b) = c(b) - e_b \leq k \leq k + l = h(b)$. Note that since rows and columns of the image of $\phi$ are weakly decreasing, we have shown that $\phi(b) \in [1, h(b)]$ for all boxes $b$.

Beginning with a hook-restricted plane partition of $\nu^t/\mu$, we define a map, $\psi$, to recover $\mu$ and the fillings of $\mu/\lambda$ and $\nu^t/\mu$ as follows. If the integer in the box in row $i$ and column $j$ is greater than or equal to $j$, then that box is in $\mu$ and $\psi(b) = d(b) - e_b$. Note that since $e_b \geq j$, $\psi(b) = (i + j - 1) - e_b \leq i - 1$, as is required to be strictly elegant. If the entry is less than $j$, that box is in $\nu^t/\mu$, and $\psi(b) = c(b) - e_b$. Note here that $e_b \leq j$ implies that $\psi(b) = c(b) - e_b \leq j - 1$, which is necessary to have an elegant filling. It is easy to see that rows and columns in $\mu$ will be strictly increasing in the image of $\psi$ and that in $\nu^t/\mu$, rows will be strictly increasing and columns
will be weakly increasing. Thus the image of $\psi$ is a strictly elegant filling of $\mu \supset \lambda$ and an elegant filling of $\nu/\mu^t$. Clearly the composition of $\phi$ and $\psi$ is the identity, so they are indeed inverses.

Note that the antipode applied to $G_{\lambda}$ gives an infinite sum of stable Grothendieck polynomials (see Remark 6.3) while applying $S$ to $\tilde{j}_{\lambda}$ can be written as a finite sum of $\tilde{j}$’s. This implies that while the space spanned by stable Grothendieck polynomials, $\Gamma$, is not a Hopf algebra, the space spanned by $\tilde{j}$’s is a Hopf algebra.

**Example 8.9.** To illustrate the bijection described above, consider $\lambda = (3, 2, 1)$, $\mu = (3, 3, 2, 2)$, and $\nu^t = (5, 4, 4, 3)$. The figure on the left is a filling such that $\mu/\lambda$ is strictly elegant and the transpose of $\nu^t/\mu$ is elegant. The entries in $\mu/\lambda$ are in bold. The figure on the right is the corresponding hook-restricted plane partition of $\nu^t/\lambda$.

|   |   |   |   |
|---|---|---|---|
| 2 | 4 |   |   |
| 1 | 3 |   |   |
| 1 | 1 | 3 |   |
| 2 | 3 | 2 |   |

If $b$ is the box in the bottom left corner of the partition on the left, we see that $\phi(b) = d(b) - e_b = 4 - 2 = 2$. If $a$ is the box in the upper right corner of the partition on the left, we have $\phi(a) = c(a) - e_a = 5 - 4 = 1$. In the hook-restricted plane partition on the left, we can see that the boxes in positions (4, 1), (3, 2), (4, 2), and (2, 3) are in $\mu/\lambda$ in the image of $\psi$ since in these boxes $e_b \geq j_b$.

**Acknowledgements**

The author thanks Pasha Pylyavskyy and Vic Reiner for many helpful conversations.

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