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1 Overview

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2 Linear and time-invariant systems

We consider a system with a single input and a single output

Input signal $x(t)$, output signal $y(t)$

Such systems are called SISO systems

SISO stands for Single Input Single Output

If a system has Multiple Inputs and Multiple Outputs it is called a MIMO system

We restrict ourselves to SISO systems
The action of the system on the input signal $x(t)$ is described by the system operator $S$. We write

$$y(t) = S\{x(t)\}$$

In this course we are particularly interested in systems that are **Linear** and **Time-Invariant**

Such systems are called LTI systems
2 Linear and time-invariant systems

**Linearity** Suppose we have two input signals \( x_1(t) \) and \( x_2(t) \). Denote the corresponding output signals by \( y_1(t) \) and \( y_2(t) \):

\[
y_1(t) = S\{x_1(t)\} \quad \text{and} \quad y_2(t) = S\{x_2(t)\}
\]

The system is called *linear* if

\[
y(t) = S\{\alpha x_1(t) + \beta x_2(t)\} \\
= \alpha S\{x_1(t)\} + \beta S\{x_2(t)\} \\
= \alpha y_1(t) + \beta y_2(t)
\]

for any two constants \( \alpha \) and \( \beta \)
Any linear combination of input signals produces the same linear combination of their corresponding output signals.

Taking $\beta = 0$, it follows from the above definition that

$$y(t) = S\{\alpha x_1(t)\} = \alpha S\{x_1(t)\} = \alpha y_1(t)$$

In other words, for a linear system, if you scale the input signal by a factor $\alpha$, the output signal will scale with the same factor.
Example Consider a SISO system with input signal $x(t)$ and an output signal given by

$$y(t) = \frac{1}{T} \int_{\tau=t-T}^{t} x(\tau) \, d\tau + B,$$

where $B$ is a constant. Such a system is called a biased averager (can you see why?)

Scaling the input signal by a factor $\alpha$, we obtain the output signal

$$\frac{\alpha}{T} \int_{\tau=t-T}^{t} x(\tau) \, d\tau + B,$$

which is not equal to $\alpha y(t)$ unless $B = 0$. The averager is nonlinear for $B \neq 0$ and linear for $B = 0$
2 Linear and time-invariant systems

**Time-Invariance** Let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

The system is called *time-invariant* if

$$y(t - \tau) = S\{x(t - \tau)\}$$

for any time shift $\tau \in \mathbb{R}$

In words: shifting your input signal produces an equally time-shifted output signal
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3 The impulse response and the convolution integral

Let the Dirac delta function be the input signal of an LTI system.

The corresponding output signal is written as \( h(t) \) and is called the impulse response:

\[
h(t) = S\{\delta(t)\}
\]

We claim that if you know the impulse response of an LTI system then you know the response to any other input signal!
To show this, let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

Because of the sifting property of the delta function, we have

$$x(t) = \int_{\tau=\infty}^{\infty} x(\tau)\delta(t - \tau) \, d\tau$$

The right-hand side of the above expression can be seen as a continuous weighted summation of shifted Dirac distributions.
Substitution gives

\[ y(t) = S\left\{ \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t - \tau) \, d\tau \right\} \]

Now note that \( S \) is a linear system operator and acts on functions that depend on time \( t \)

This allows us to write

\[ y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)S\{\delta(t - \tau)\} \, d\tau \]

Since the system is time-invariant as well, we have

\[ h(t - \tau) = S\{\delta(t - \tau)\} \]
We arrive at

\[ y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t - \tau) \, d\tau \]

Knowing the impulse response \( h(t) \), we can determine the response \( y(t) \) to any input signal \( x(t) \) by evaluating the above integral.

This integral is called the *convolution integral* or *convolution product* of the signals \( x \) and \( h \).

Short-hand notation:

\[ y = x \ast h \quad \text{or} \quad y(t) = x(t) \ast h(t) \]
For two real numbers $a$ and $b$, we have $ab = ba$

The product of two real numbers commutes

Is this also true for the convolution product? In other words, do we have $x * h = h * x$?

The answer is yes. Let’s check it.

$$y(t) = x * h = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau) \, d\tau \quad p = t - \tau \quad \int_{p=-\infty}^{\infty} x(t-p)h(p) \, dp$$

$$= \int_{p=-\infty}^{\infty} h(p)x(t-p) \, dp = h * x$$

Conclusion: the convolution product of two signals commutes
For the product of real numbers, there exists an identity element called “one” and written as 1 for which \( a = a \cdot 1 = 1 \cdot a \).

What is the identity element for the convolution product?

We already know the answer to this question. It is the Dirac delta function!

\[ x = x \ast \delta = \delta \ast x \]

The convolution product is also **associative**, that is, for three signals \( u \), \( v \), and \( w \), we have (check this yourself)

\[ (u \ast v) \ast w = u \ast (v \ast w) \]
This property can be exploited to determine the total impulse function of two LTI systems interconnected in cascade

System 1: input signal $x(t)$, impulse function $h_1(t)$, output signal $y_1(t)$

System 2: input signal $y_1(t)$, impulse function $h_2(t)$, output signal $y(t)$

We assume that System 2 does not “load” System 1
Response of the total system:

\[ y = y_1 \ast h_2 = (x \ast h_1) \ast h_2 = x \ast (h_1 \ast h_2) = x \ast h \]

where we have introduced the impulse function of the total system as

\[ h = h_1 \ast h_2 = h_2 \ast h_1 \]

Note that since the convolution product of two signals commute, we can interchange the order of the subsystems without affecting the output signal \( y(t) \) (provided both systems do not “load” each other)
Finally, if the support of a signal $x$ is $(\ell_x, u_x)$ and the support of a signal $h$ is $(\ell_h, u_h)$ then

the support of $y(t) = x(t) \ast h(t)$ is $(\ell_x + \ell_h, u_x + u_h)$

Verify this statement!
Examples

Exercise 1. Let $p(t)$ denote the rectangular pulse signal. Determine $x(t) = p(t) * p(t)$.

Exercise 2. (13-12-2023) Determine the signal $y(t) = h(t) * x(t)$, where $h(t) = u(t - 1)$, $x(t) = u(t - 2)$, and $u(t)$ is the Heaviside unit step function.

Exercise 3. (19-07-2021) Given the signal $x(t) = te^{-\alpha t}u(t)$ with $\alpha > 0$. Determine the convolution $y(t) = x(t) * x(t)$ directly using the convolution integral.
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6. BIBO stability
Up till now we have been looking at fairly general systems whose action on the input signal is described by some operator $S$.

Let us now be more specific and consider systems described by the linear ordinary differential equation

$$
\left( a_N \frac{d^N}{dt^N} + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} + \ldots + a_1 \frac{d}{dt} + a_0 \right) y(t) =
$$

$$
\left( b_M \frac{d^M}{dt^M} + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} + \ldots + b_1 \frac{d}{dt} + b_0 \right) x(t)
$$

which holds for $t > 0$ and $N$ and $M$ are nonnegative integers.
In the above differential equation, \( x(t) \) is the prescribed (known) input signal and \( y(t) \) is the desired output signal.

To obtain the output signal \( y(t) \), we also need the \( N \) initial conditions

\[
y(0) \quad \text{and} \quad \left. \frac{d^k y(t)}{dt^k} \right|_{t=0} \quad \text{for} \quad k = 1, 2, \ldots, N - 1
\]

RLC circuits, mechanical systems, etc. can all be described by a differential equation of the above form.
Further on we will show how to solve the differential equation using the Laplace transform

Here, we state that the solution (output signal) can be written as

\[ y(t) = y_{zs}(t) + y_{zi}(t) \]

- \( y_{zs}(t) \) is called the zero-state response. This is the solution exclusively due to the input signal with initial conditions set to zero

- **Special case:** The impulse response \( h(t) \) of the system is a zero-state response of the system in case the input signal is the Dirac impulse function: \( x(t) = \delta(t) \)

- \( y_{zi}(t) \) is called the zero-input response. This is the solution exclusively due to the initial conditions with the input set to zero
For vanishing initial conditions the system is linear and time-invariant.

For nonvanishing initial conditions, the system is no longer an LTI system.

**Example.** Consider a circuit consisting of a resistor $R$ in series with an inductor $L$ and a voltage source $v(t) = Bu(t)$. The initial current in the inductor is $I_0$. The input signal of the system is $v(t)$, the current $i(t)$ is the output signal.
The output signal is given by

\[ i(t) = i_{zs}(t) + i_{zi}(t) \quad \text{for } t > 0 \]

with

\[ i_{zs}(t) = \frac{B}{R}(1 - e^{-t/\tau}), \quad i_{zi}(t) = I_0 e^{-t/\tau}, \quad \text{and} \quad \tau = L/R \]

- If we double the amplitude of the input signal the output signal becomes \( i(t) = 2i_{zs}(t) + i_{zi}(t) \) with \( i_{zs}(t) \) and \( i_{zi}(t) \) as above.
- Clearly, the output is not doubled, since \( i_{zi}(t) \) does not vanish: the system is not linear.
- However, for \( I_0 = 0 \) (vanishing initial condition) we have \( i_{zi}(t) = 0 \) and the system is linear.
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Consider a continuous-time system with input signal $x(t)$ and output signal $y(t)$. The system is *causal*

- if the output signal $y(t) = 0$ for a vanishing input signal and vanishing initial conditions
- if the output $y(t)$ does not depend on future inputs

An LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0$$

Indeed, for an LTI system we have the convolution integral

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t - \tau) \, d\tau$$
Writing the integral as

\[ y(t) = \int_{\tau=\infty}^{t} x(\tau) h(t - \tau) \, d\tau + \int_{\tau=t}^{\infty} x(\tau) h(t - \tau) \, d\tau \]

we observe that in the second term on the right-hand side integration takes place over future inputs.

For a causal system, these inputs cannot contribute to the output signal at time instant \( t \).

Consequently, for a causal system we must have \( h(t - \tau) = 0 \) for \( t < \tau < \infty \) or \( h(t) = 0 \) for \( t < 0 \).
In case the LTI system is causal, we are left with

\[ y(t) = \int_{\tau=-\infty}^{t} x(\tau) h(t - \tau) \, d\tau \]

In addition, if the input signal \( x(t) \) also vanishes for \( t < 0 \) then the convolution integral simplifies even further to

\[ y(t) = \int_{\tau=0}^{t} x(\tau) h(t - \tau) \, d\tau \]
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Finally, we discuss the concept of BIBO stability.

BIBO stands for Bounded Input Bounded Output.

We are given a bounded input signal $x(t)$, that is, a signal that satisfies

$$|x(t)| \leq M$$

for some positive $M$.

We ask: Under what conditions is the output $y(t)$ of an LTI system also bounded?
6 BIBO Stability

Consider

\[ |y(t)| = \left| \int_{\tau=-\infty}^{\infty} x(t-\tau)h(\tau) \, d\tau \right| \]
\[ \leq \int_{\tau=-\infty}^{\infty} |x(t-\tau)| |h(\tau)| \, d\tau \leq M \int_{\tau=-\infty}^{\infty} |h(\tau)| \, d\tau \]

From this last inequality it follows that if

\[ \int_{\tau=-\infty}^{\infty} |h(\tau)| \, d\tau < \infty \]

then the output signal \( y(t) \) is bounded

**Conclusion:** If the impulse response \( h(t) \) is absolutely integrable (its action is finite) then the LTI system is BIBO stable