ON THE PRIMARY COVERINGS OF FINITE SOLVABLE AND 
SYMMETRIC GROUPS

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Abstract. A primary covering of a finite group $G$ is a family of proper sub-
groups of $G$ whose union contains the set of elements of $G$ having order a 
prime power. We denote with $\sigma_0(G)$ the smallest size of a primary covering 
of $G$, and call it the primary covering number of $G$. We study this number 
and compare it with its analogous $\sigma(G)$, the covering number, for the classes 
of groups $G$ that are solvable and symmetric.

1. Introduction

A covering of a finite group $G$ is a family of proper sub-
groups of $G$ whose union equals $G$. The covering number of $G$ is defined as the minimal size of a covering of 
$G$, and it is denoted by $\sigma(G)$ (by Cohn [1]). A group admits a covering unless it is 
cyclic, in which case it is generally set $\sigma(G)$ to be equal to $\infty$ (with the convention 
that $n < \infty$ for every integer $n$). The covering number has been studied by many 
authors and in particular $\sigma(G)$ was determined when $G$ is solvable by Tomkinson 
[12, Theorem 2.2] and when $G$ is symmetric by Maróti [9, Theorem] (see also [11], 
[7] and [10]).

Given a subset $\Pi$ of $G$ we may be interested in the minimal number of proper 
subgroups of $G$ whose union contains $\Pi$. In this paper we focus on the set of pri-
mary elements, complementing the work done in [3]. A primary element of $G$ is 
an element of $G$ whose order is some prime power. We define $G_0$ to be the set 
of primary elements of $G$ and a primary covering of $G$ to be a family of proper 
subgroups of $G$ whose union contains $G_0$. We set $\sigma_0(G)$ the smallest size of a pri-
mary covering of $G$, and call it the primary covering number of $G$. Observe that $G$ 
admits primary coverings if and only if $G$ is not a cyclic $p$-group for no prime $p$, so 
in this case we define $\sigma_0(G) = \infty$, with the convention that $n < \infty$ for every integer 
n. Clearly, we always have $\sigma_0(G) \leq \sigma(G)$. Moreover, a deep result ([2, Theorem 
1]) shows that a primary covering of any finite group is never a unique conjugacy 
class of a proper subgroup.

In this paper we study $\sigma_0(G)$ when $G$ is solvable and when $G$ is a symmetric 
group $S_n$.

Our main result about solvable groups is the following.

Theorem 1. Let $G$ be a finite solvable group which is not a cyclic $p$-group, for every 
prime $p$. If $G/G'$ is not a $p$-group, then $\sigma_0(G) = 2$. Otherwise, $\sigma_0(G) = \sigma(G)$.

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Our results on the primary covering number for $S_n$ can be summarized in the following statement.

**Theorem 2.** The following hold for $n \geq 3$.
1. $\sigma_0(S_3) = 4$, $\sigma_0(S_6) = 7$ and $33 \leq \sigma_0(S_{10}) \leq 46$ (see Lemmas 8, 11 and 16).
2. If $n = 2^a$ for some $a > 1$, then $\sigma_0(S_n) = 1 + \frac{n}{2(n/2)}$ (see Proposition 7).
3. If $n \neq 10$ and $n \neq 3 \cdot 2^a$, for $\epsilon \in \{0, 1\}$ and $a > 1$, then $\sigma_0(S_n) = 1 + \binom{n}{2}$, where $n_2$ denotes the maximum power of 2 that divides $n$ (see Proposition 7).
4. If $n = 3 \cdot 2^a$, with $a \geq 2$, then $c_1 \leq \sigma_0(S_n) \leq c_2$, where
   
   \[
   c_1 = \begin{cases} 
   117 & \text{if } a = 2, \\
   1 + \frac{n-1}{2^a-1} & \text{if } a \geq 3,
   \end{cases}
   \]

   \[
   c_2 = 2 + \frac{n-1}{2^a-1} + \sum_{i=2}^{2^a+1} \binom{n-i}{2^a-1}.
   \]

(see Proposition 7).

The group $S_{10}$ needs a separate treatment. Here the problematic elements are the ones of cycle structure $(4, 4, 2)$. Such elements are efficiently covered by the intransitive subgroups $S_8 \times S_2$, however with our techniques we could not exclude the appearance of the maximal imprimitive subgroups $S_8 \wr S_2$ in a minimal primary cover, since such subgroups cover individually more elements of type $(4, 4, 2)$ than the intransitive ones. It is reasonably easy to show that $33 \leq \sigma_0(S_{10}) \leq 46$. Our conjecture is that $\sigma_0(S_{10}) = 46$, a minimal primary cover being conjecturally given by the alternating group $A_{10}$ and the $\binom{10}{2} = 45$ intransitive subgroups $S_8 \times S_2$.

2. **Solvable groups**

In this section we determine the primary covering number for every finite solvable group $G$.

We start with the following trivial observations that hold in a general context.

**Lemma 1.** Assume that $G$ is a finite group.
1. If $G/G'$ is not a $p$-group, for some prime $p$, then $\sigma_0(G) = 2$.
2. If $N \leq G$, then $\sigma_0(G) \leq \sigma_0(G/N)$. Moreover, if $N$ is contained in the Frattini subgroup of $G$, then $\sigma_0(G) = \sigma_0(G/N)$.
3. If $M$ is a maximal subgroup of $G$ such that $\sigma_0(M) > \sigma_0(G)$, then $M$ belongs to every minimal primary covering of $G$.

**Proof.** (1) We trivially have that $G_0 \subseteq H \cup K$, where $H$ and $K$ are maximal subgroups containing $G'$ of coprime indices.

(2) Since any primary covering of $G/N$ lifts to a primary covering of $G$, we have $\sigma_0(G) \leq \sigma_0(G/N)$. Also, any subgroup in a primary covering of $G$ can be replaced by a maximal subgroup containing it. Thus when $N \leq \Phi(G)$ we have $\sigma_0(G) = \sigma_0(G/N)$.

(3) Let $M$ be a maximal subgroup of $G$ such that $\sigma_0(G) < \sigma_0(M)$ and let $H_1, \ldots, H_n$ be any primary covering of $G$ of size $n = \sigma_0(G)$. Of course, the family $\{H_i \cap M\}_{i=1}^n$ covers $M_0$ (the set of primary elements of $M$), so since $\sigma_0(M) > n$ we deduce that there exists $i \in \{1, \ldots, n\}$ such that $H_i \cap M = M$, in other words $M \leq H_i$. Being $M$ a maximal subgroup of $G$ we deduce $M = H_i$. \hfill $\Box$
Lemma 2. Let $N$ be a complemented minimal normal subgroup of a solvable group $G$ and let $b$ be the number of complements of $N$ in $G$. If $b > 1$ then $b \geq |N|$.  

Proof. Let $E = \text{End}_G(N)$. By a result of Gaschütz, [4, Satz 3], we know that $b = |N|^|E|^{\beta - 1}$ where $\varepsilon$ is 0 or 1 according to whether $N$ is central or not in $G$, and $\beta$ is the number of non-Frattini chief factors $G$-isomorphic to $N$, in a chief series of $G$ starting with $N$. It follows that $b \geq |N|$ if $N$ is not central, and, if $N$ is central then $|N| = p$ is a prime number and $E$ is the field with $p$ elements, so that $\beta \neq 1$ (being $b > 1$ by hypothesis) hence $b \geq |E| = p = |N|$.

Recall Tomkinson’s result ([4, Theorem 2.2]) which states that if $G$ is a finite solvable group then $\sigma(G) = q + 1$ where $q$ is the order of the smallest chief factor of $G$ with more than one complement.

Proof of Theorem 2.

Let $G$ be a non-cyclic finite solvable group. If $|G/G'|$ is not a prime power, then by Lemma 1 (1) we have $\sigma_0(G) = 2$. Thus assume that $G/G'$ is a $p$-group for some prime $p$ and define $\alpha$ to be the smallest order of a chief factor of $G$ admitting more than one complement (this is well-defined because $G$ is not a cyclic $p$-group).

We need to show that $\sigma_0(G) = 1 + \alpha$. By the aforementioned result of Tomkinson, $\sigma_0(G) \leq \sigma(G) = 1 + \alpha$. Let $N$ be a normal subgroup of $G$ such that $\sigma_0(G) = \sigma_0(G/N)$ with $|G/N|$ minimal with this property. Let $K/N$ be a minimal normal subgroup of $G/N$, then $\sigma_0(G/N) < \sigma(G/K)$, hence by Lemma 1 (2) $K/N$ admits a complement $M/N$ in $G/N$ which is a maximal subgroup. Being $\sigma_0(G/N) < \sigma_0(G/K) = \sigma_0(M/N)$, we deduce that $M/N$ appears in every minimal primary covering of $G/N$, hence all $b$ complements of $K/N$ in $G/N$ appear in a fixed minimal primary covering of $G/N$. However, no element of $K/N$ belong to any complement of $K/N$, hence $\sigma_0(G/N) \geq 1 + b$. If $b \neq 1$, then $b \geq |K/N|$ by Lemma 2 and this implies the result. Assume now $b = 1$. Then $G/N$ is a direct product $K/N \times M/N$ hence $K/N$, being a central chief factor, is cyclic of prime order and, being an epimorphic image of the $p$-group $G/G'$, we deduce that $|K/N| = p$. Moreover $M/N$ is nontrivial, because $G/N \not\cong C_p$ (being $\sigma_0(G) = \sigma_0(G/N)$), so $M/N$ has nontrivial abelianization. Since $G/G'$ is a $p$-group it follows that $M/N$ projects onto $C_p$. This $C_p$ is therefore a chief factor above $K/N$ which is $G$-isomorphic to $K/N$. In particular, the number of non-Frattini chief factors $G$-isomorphic to $C_p$ is $\beta \geq 2$. This contradicts the formula in the proof of Lemma 2.

\[\square\]

3. Symmetric groups

We introduce some notation that will be frequently used. Let $n$ be a positive integer and set $\Omega := \{1, 2, \ldots, n\}$ on which the symmetric group $S_n$ acts naturally. Given natural numbers $k \geq 1$ and $x_1 \geq x_2 \geq \ldots \geq x_k \geq 1$ such that $\sum_{i=1}^k x_i = n$, we denote with $(x_1, x_2, \ldots, x_k)$ both the partition of $n$ whose parts are precisely the $x_i$ and the set of all permutations in $S_n$ having as cyclic type this partition. In particular, $(n)$ denotes the set of $n$-cycles of $S_n$.

The maximal subgroups of $S_n$ split in three different classes, according to their action on $\Omega$: intransitive, imprimitive and primitive subgroups.

Any intransitive maximal subgroup is the setwise stabilizer of a set of cardinality $m$, for some $1 \leq m < n/2$. In particular, any such subgroup is conjugate to the
stabilizer of the set \( \{1, 2, \ldots, m\} \subseteq \Omega \), which we denote with \( X_m \). It is therefore isomorphic to \( S_m \times S_{n-m} \) and its index is \( \binom{n}{m} \). We set
\[
X_m = \{ \text{conjugates of } X_m \simeq S_m \times S_{n-m} \}
\]and
\[
X' = \bigcup_{1 \leq m < n/2} X_m = \{ \text{intransitive maximal subgroups of } S_n \}.
\]
The imprimitive maximal subgroups of \( S_n \) are the stabilizers of partitions of \( \Omega \) into equal-sized subsets. If \( d \) is any proper nontrivial divisor of \( n \) we set \( W_d \) the stabilizer of the partition
\[
\{\{1, 2, \ldots, d\}, \{d+1, d+2, \ldots, 2d\}, \ldots, \{n-d+1, n-d+2, \ldots, n\}\}.
\]
Note that \( W_d \) is isomorphic to the wreath product \( S_d \wr S_{n/d} \), and it has index \( n! (d!)^{n/d} \cdot (n/d)! \). Also note that any imprimitive maximal subgroup of \( S_n \) is conjugate to \( W_d \), for some proper nontrivial divisor \( d \) of \( n \). We set
\[
W_d = \{ \text{conjugates of } W_d \simeq S_d \wr S_{n/d} \}
\]and
\[
W = \bigcup_{1 < d | n, d \neq n} W_d = \{ \text{imprimitive maximal subgroups of } S_n \}.
\]
Finally set
\[
\mathcal{P} = \{ \text{proper primitive maximal subgroups of } S_n \}
\]where proper means that both \( S_n \) and \( A_n \) are not members of \( \mathcal{P} \).

**Lemma 3.** If \( a > b \) are positive integers then
\[
a^b!b! \geq b^a!a!
\]
with equality if and only if \( b = 1 \).

**Proof.** If \( b = 1 \) we have equality, now assume \( b \geq 2 \), so that \( a \geq 3 \). Since the stated inequality is equivalent to
\[
\frac{\ln(a!)}{a-1} \geq \frac{\ln(b!)}{b-1}
\]
it is enough to prove that the function \( \ln(n!)/(n-1) \) is increasing, hence we may assume \( b = a-1 \). So we need to prove that \( a^{a-2} \geq (a-1)^{a-1} \) which is equivalent to \( a^{a-1} \geq a! \), which is actually a strict inequality being \( a \geq 3 \). \( \square \)

The following lemmas are part of [8, Corollary 1.2 and Lemma 2.1].

**Lemma 4** (Maróti Lemma 2.1, On the orders of primitive groups). Let \( m > 1 \) be an integer and suppose \( m = a_1 b_1 = a_2 b_2 \) with \( a_1, a_2, b_1, b_2 \) positive integers at least 2, \( b_1 \geq a_1, b_2 \geq a_2, a_1 \leq a_2 \) and (consequently) \( b_1 \geq b_2 \). Then
\[
b_1^{p_1} a_1! \geq b_2^{p_2} a_2!
\]
with equality if and only if \( a_1 = a_2 \) and \( b_1 = b_2 \). Moreover if \( p \) is the smallest prime divisor of \( m \) and \( d \) is any divisor of \( m \) with \( 1 < d < m \) then
\[
(m/p)!p! \geq (m/d)!d!
\]
with equality if and only if \( d = p \).
Proof. Observe that since $b_2 \geq a_2 \geq 2$ we have $b_2^{b_2} \geq b_2!b_2 \geq b_2!a_2$.

\[
\begin{align*}
b_1!^{a_1}a_1! & \geq b_2!^{a_1}(b_2 + 1)^{a_1} \cdots b_2^{a_2}a_1! \\
& \geq b_2!^{a_1}a_1!b_2^{a_1(b_1 - b_2)} \\
& \geq b_2!^{a_1}a_1!(b_2!a_2)^{(a_1/b_2)(b_1 - b_2)} \\
& = b_2!^{a_1}a_1!(b_2!a_2)^{(a_2 - a_1)} \\
& \geq b_2!^{a_1}a_1!b_2^{a_2 - a_1}(a_1 + 1) \cdots a_2 \\
& = b_2!^{a_2}a_2!.
\end{align*}
\]

If equality holds then all of the above inequalities are equalities and it is easy to deduce that $a_1 = a_2$ and consequently $b_1 = b_2$. To deduce the last statement, observe that it is trivial if $m = p$ so assume this is not the case, so that $p^2 \leq m$. Choose $a_1 = p$, $b_1 = m/p$, then $a_1 \leq b_1$ and the inequality $a_1 \leq a_2$ will be true for every choice of a divisor $a_2 > 1$ of $m$, by minimality of $p$. If $d^2 \leq m$ then choose $a_2 = d$, proving the strict inequality with equality if and only if $d = p$. If $d^2 > m$ then $d > m/d$ hence Lemma 5 implies that $d!^{m/d}(m/d)! \geq (m/d)!d!$, so it is enough to show that $(m/p)!p! \geq d!^{m/d}(m/d)!$, which follows from the above choosing $a_2 = m/d$, and again by Lemma 5 equality does not hold in this case being $m/d \neq 1$.

The orders of the different primitive and imprimitive maximal subgroups of $S_n$ can be compared as in the following lemma.

Lemma 5. Let $n \geq 2$, then the following hold.

1. For every proper nontrivial divisor $d$ of $n$, we have

   \[
   \left| W_{n/p} \right| = \left(\frac{n}{p}\right)!p! \geq \left(\frac{n}{d}\right)!d! = \left| W_d \right|,
   \]

   where $p$ is the smallest prime divisor of $n$, and equality holds if and only if $d = p$;

2. If $P \in \mathcal{P}$, then $|P| < 3^n$, and when $n > 24$, then $|P| < 2^n$.

In particular, for $n \geq 12$ every $M \in W \cup \mathcal{P}$ has order

\[
|M| \leq 2(\lfloor n/2 \rfloor)!(n - \lfloor n/2 \rfloor)!
\]

with equality if and only if $M \in W_{n/2}$.

Proof. (1) follows from Lemma 4 and (2) is [5 Corollary 1.2].

We now prove the last part of the Lemma.

Let first $M \in \mathcal{P}$. By (2) the order of $M$ is bounded above by $3^n$. By using $k! > e(k/e)^k$ which holds for every $k \geq 2$, we have that

\[
2(\lfloor n/2 \rfloor)!(n - \lfloor n/2 \rfloor)! \geq g(n) = \begin{cases} 
2e^2 \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is even}, \\
2e^2(n + 1)(\frac{n - 1}{2e})^{n-1} & \text{if } n \text{ is odd}.
\end{cases}
\]

Computation shows that for every $n \geq 14$ the function $g(n) \geq 3^n$. The cases $n = 12$ and $n = 13$ can be done by inspection and the proof for the primitive case is completed.

Assume now that $M \in W$, in particular $m$ is not a prime number. Then by (1) we have $|M| \leq |W_{n/p}|$, where $p$ is the smallest prime number that divides $n$. The
result is therefore trivial if \( n \) is even. Let \( n \) be odd. We need to prove that

\[
R = \frac{(n+1)((n-1)/2)!}{(n/p)!p!} > 1.
\]

Observe that, being \( p^2 \leq n \), the map \( x \mapsto x^{n-1}/n^x \) is an increasing function in the interval \( 1 \leq x \leq p \), so that \( p^{n-1}/n^p \geq 3^{n-1}/n^3 \). Using the inequalities

\[
e(n/e)^n \leq n! \leq e(n/e)^n
\]

we see that

\[
R \geq e^2 \cdot \frac{(n+1)(n-1)^{n-1}}{2n} \cdot \frac{(n-1)}{2n} \cdot \frac{p}{n^p} \cdot \left( \frac{n-1}{2n} \right)^n
\]

> \( 2e^2 \cdot \frac{3^{n-1}}{n^3} \cdot \left( \frac{n-1}{2n} \right)^n \) whenever \( n \geq 22 \). The case \( 12 \leq n \leq 21 \) can be done by inspection. \( \square \)

In the sequel we will need the following result.

**Lemma 6.** Let \( n \) be even and \( W \in \mathcal{W}_{n/2} \). Assume that \( n = \sum_{i=1}^k 2^{a_i} \) is a partition of \( n \) with \( a_1 \geq a_2 \geq \ldots \geq a_k \geq 1 \) that does not contain subpartitions of \( n/2 \), and let \( \Pi \) be the conjugacy class of elements of \( S_n \) of type \( (2^{a_1}, 2^{a_2}, \ldots, 2^{a_k}) \). Then

\[
|W \cap \Pi| = \frac{||\Pi|}{|S_n : W|} \cdot 2^{k-1}.
\]

In particular, when \( n \) is a power of 2 and \( \Pi = (n) \), the set of \( n \)-cycles, then

\[
|W \cap \Pi| = \frac{|W|}{n} = (n/2)!/(n/2-1)!
\]

**Proof.** Double counting the size of the set

\[
\{ (x, W) \mid x \in \Pi \cap W, \ W \in \mathcal{W}_{n/2} \},
\]

we find that

\[
|W \cap \Pi| = \frac{||\Pi|}{[G : W]} \cdot r,
\]

where \( r \) is the number of elements of \( \mathcal{W}_{n/2} \) containing a fixed element of \( \Pi \). By the assumption on the partition defining \( \Pi \), the elements of \( \Pi \cap W \) move the two imprimitivity blocks of \( W \), hence \( r = 2^{m-1} \).

When \( n \) is a power of 2 and \( \Pi = (n) \), by Lemma 5(1), we have that

\[
|W \cap (n)| = \frac{|W|}{n} \leq \frac{|\mathcal{W}_{n/2}|}{n} = (n/2)!/(n/2-1)!
\]

which completes the proof. \( \square \)

Moreover, we will make use of the following notation and terminology introduced in \([9]\).

**Definition 1.** Let \( \Pi \) be a set of permutations of \( S_n \). We will say that a collection \( \mathcal{H} = \{ H_1, \ldots, H_m \} \) of \( m \) proper subgroups of \( S_n \) is definitely unbeatable on \( \Pi \) if the following three conditions hold:

1. \( \Pi \subseteq \bigcup_{i=1}^m H_i \),
2. \( \Pi \cap H_i \cap H_j = \emptyset \) for every \( i \neq j \),
(3) $|M \cap \Pi| \leq |H_i \cap \Pi|$ for every $1 \leq i \leq m$ and every proper subgroup $M$ of $S_n$ not belonging to $\mathcal{H}$.

If $\mathcal{H}$ is definitely unbeatable on $\Pi$, then $|\mathcal{H}| = \sigma(\Pi)$, where $\sigma(\Pi)$ denotes the least integer $m$ such that $\Pi$ is a subset of the union of $m$ proper subgroups of $S_n$. Moreover, we also say that $\mathcal{H}$ is strongly definitely unbeatable on $\Pi$ if the three conditions above hold and the third one always holds with strict inequalities. Note that in the case when $\mathcal{H}$ consists of maximal subgroups and it is strongly definitely unbeatable on $\Pi$, then $\mathcal{H}$ is the unique minimal cover of the elements of $\Pi$ that uses only maximal subgroups (see also [11, Lemma 3.1]).

We start our considerations on the primary covering number of $S_n$ by producing a general upper bound. Here and in the following if $p$ is any prime, we define $n_p$ to be the $p$-part of $n$, that is the maximum power of $p$ that divides $n$.

**Lemma 7.**

1. If $n$ is a power of $2$, then $\sigma_0(S_n) \leq 1 + \frac{1}{2}(\frac{n}{2})$.
2. If $n$ is not a power of $2$, then $\sigma_0(S_n) \leq 1 + \frac{1}{2}\binom{n}{2}$.

**Proof.** Note that in any case the alternating groups $A_n$ contains every permutation of odd order, therefore in order to exhibit a primary covering for $S_n$, we may add to $\{A_n\}$ those subgroups that contain 2-elements (that are odd permutations).

1. Assume $n = 2^a$ with $a \geq 2$. Then every 2-element of $S_n$ stabilizes a 2-block partition of $\Omega$ and therefore every 2-element belongs to a intransitive maximal subgroup of type $W_{n/2}$. Since the number of such partitions (and subgroups) is $\frac{1}{2}\binom{n}{2}$ the lemma is proved in this case.

2. As fairly known, the number of subsets of order $n_2$ of the set $\Omega = \{1, 2, \ldots, n\}$ is $\binom{n}{n_2}$ which is an odd number, and it is easy to prove by induction that every 2-element of $S_n$ belongs to some stabilizer $S_\Delta$ with $\Delta$ a subset of $\Omega$ of cardinality $n_2$, in other words if we write $n$ as a sum of distinct powers of 2 then there exists a subsum that equals $n_2$. Since the number of these stabilizers is exactly $\binom{n}{n_2}$ also this point is proved. \[\square\]

We already have enough ingredients to complete the proof in the case $n = 2^a$, with $a \geq 2$.

**Proposition 1.** If $n = 2^a \geq 4$ then $\sigma_0(S_n) = 1 + \frac{1}{2}(\frac{n}{2})$ and a minimal primary covering is given by $\{A_n\} \cup W_{n/2}$.

**Proof.** By Lemma 7 we know that $\sigma_0(S_n) \leq 1 + \frac{1}{2}(\frac{n}{2})$. Assume that $H$ is a maximal subgroup of $S_n$. Then either $H \cap (n) = \emptyset$, or $H$ is an imprimitive subgroup, or $H = \text{PGL}(2, q)$ with $q = 2^a - 1$ a Mersenne prime, by [6, Theorem 3]. To see this observe that from the equality $2^a = (q^d - 1)/(q - 1)$ one easily deduces that $d = 2$ so that $2^a = q + 1$, now $q$ cannot be a square since $2^a - 1 \equiv 3 \mod 4$, and if $q = p^m$ is an odd power of the prime $p$, the usual factorization of $x^m + 1 = (x + 1)(x^{m-1} - x^{m-2} + \ldots + 1)$ implies that $q$ must be a prime. By Lemma 4 and the fact the elements of order $n$ in $\text{PGL}(2, q)$ are in number of $2^{a-2}(2^a - 1)(2^a - 2)$ (use [5, Satz 7.3]), we deduce in any case that $|H \cap (n)| \leq (n/2)!(n/2 - 1)!$. 


with equality if and only if $H \in \mathcal{W}_{n/2}$. This shows that the set $\mathcal{W}_{n/2}$ is strongly definitely unbeatable on $\Pi = (n)$, and therefore we obtain that

$$\sigma_0(S_n) \geq \frac{1}{2} \binom{n}{n/2}.$$

To complete this case, assume that $\sigma_0(S_n) = \frac{1}{2} \binom{n}{n/2}$. Then, by the above, the collection $\mathcal{W}_{n/2}$ must be the unique minimal primary covering for $S_n$. By Bertrand’s postulate there is a prime number $p$ between $n/2$ and $n$; we reach a contradiction by noting that $p$-cycles do not belong to imprimitive subgroups of type $\mathcal{W}_{n/2}$. \hfill $\Box$

We assume now that $n \not\in \{3^r \cdot 2^a \mid \epsilon \in \{0, 1\}, a \geq 0\}$. We deal separately with the case $n = 5$.

**Lemma 8.** For $n = 5$ we have that $\sigma_0(S_5) = 1 + \binom{5}{1} = 6$ and $\{A_5\} \cup \mathcal{X}_1$ is the unique minimal primary covering of $S_5$.

**Proof.** We already know that $\{A_5\} \cup \mathcal{X}_1$ is a primary covering for $S_5$ and therefore $\sigma_0(S_5) \leq 6$. Assume by contradiction that $\mathcal{C}$ is a primary covering of smaller cardinality. Inside $S_5$ there are six subgroups of order 5, therefore, as $|\mathcal{C}| \leq 5$, there exists one element of $\mathcal{C}$ containing at least two different Sylow 5-subgroups. But the only proper subgroup of $S_5$ containing more than one subgroup of order 5 is $A_5$. Thus $A_5 \in \mathcal{C}$ and the remaining members of $\mathcal{C}$ cover all of the odd 2-elements of $S_5$, which are thirty 4-cycles and ten 2-cycles. Any maximal subgroup isomorphic to $S_4$ contains precisely six 4-cycles and six 2-cycles, any $S_3 \times S_2$ contains no 4-cycles and four 2-cycles and any Frobenius group $5 : 4$ contains ten 4-cycles and no 2-cycles. Therefore, if we assume that $\mathcal{C}$ contains respectively $a_1$ subgroups in $\mathcal{X}_1$ (that is isomorphic to $S_4$), $a_2$ subgroups in $\mathcal{X}_2$ (that is isomorphic to $S_3 \times S_2$), and $a_3$ primitive subgroups isomorphic to $5 : 4$, we obtain the following system of Diophantine inequalities

\[
\begin{align*}
& a_1 + a_2 + a_3 \leq 4 \\
& 3a_1 + 5a_3 \geq 15 \\
& 6a_1 + 4a_2 \geq 10.
\end{align*}
\]

The only integer solution of this system is $(a_1, a_2, a_3) = (2, 0, 2)$, but then if $\text{Stab}_{S_5}(i)$ and $\text{Stab}_{S_5}(j)$ are the two elements of $\mathcal{X}_1$ in $\mathcal{C}$ we have that the permutation $(ij)$ is not covered by elements of $\mathcal{C}$, which is a contradiction. \hfill $\Box$

Let $n \geq 7$ and $n \not\in \{3^r \cdot 2^a \mid \epsilon \in \{0, 1\}, a \geq 0\}$ and write the 2-adic expansion of $n$ as

$$n = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_t},$$

where $a_1 > a_2 > \ldots > a_t \geq 0$ and $t \geq 2$. Note that $n_2 = 2^{a_t}$ and also that when $t = 2$ then $a_1 \geq a_2 + 2$.

We define $\Pi$ to be the following conjugacy class of permutations

\[
\Pi = \begin{cases} (2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}) & \text{if } n \not\equiv t \pmod{2}, \\
(2^{a_1-1}, 2^{a_1-1}, 2^{a_2}, \ldots, 2^{a_t}) & \text{if } n \equiv t \pmod{2}.
\end{cases}
\]

The set $\Pi$ consists of odd permutations, that is $A_n \cap \Pi = \emptyset$.

The computation of $\sigma_0(S_n)$ in this case depends on the following proposition.
Proposition 2. Assume that \( n \notin \{3^e \cdot 2^e | e \in \{0, 1\}, a \geq 1\} \). If \( n \) is odd and \( n \geq 15 \) or if \( n \) is even, \( n \geq 22 \) and \( n \neq 40 \), the collection of subgroups \( \mathcal{X}_{n/2} \) is strongly definitely unbeatable on \( \Pi \).

For the proof of Proposition 2 we need the following number theoretic result.

Lemma 9. Using the above assumptions and notation, define
\[
s = \begin{cases} 
\sum_{i=1}^t a_i + a_1 - 1 & \text{if } n \equiv t \pmod{2} \\
\sum_{i=1}^t a_i & \text{if } n \not\equiv t \pmod{2}
\end{cases}
\]
and
\[
f(n) = 2^{s+1} \left( \frac{n}{n/2} \right).
\]
Then
\[
f(n) < 1
\]
holds if and only if either \( n \) is odd and \( n \geq 15 \) or \( n \) is even, \( n \geq 22 \) and \( n \neq 40 \).

Proof. Since \( a_i \leq a_1 - (i-1) \) for every \( i = 1, 2, \ldots, t \), we have
\[
s \leq \sum_{i=1}^t a_i + a_1 - 1 \leq ta_1 + a_t - 1 - \frac{(t-2)(t-1)}{2}.
\]
The coefficient \( a_t \) satisfies \( a_t \leq a_1 + 1 - t \) and therefore
\[
s \leq \begin{cases} 
ta_1 & \text{if } n \text{ is odd}, \\
(t+1)a_1 - 2 & \text{if } n \text{ is even},
\end{cases}
\]
where we used the fact that \( (t^2 - 3t + 2)/2 \geq 0 \) and \( (t^2 - t + 2)/2 \geq 2 \) for every \( t \geq 2 \). Set \( l = \log_2(n) \geq a_1 \). Being \( t \leq a_1 - a_t + 1 \), we have that \( t \leq l \) if \( n \) is odd and \( t \leq l - 1 \) if \( n \) is even. Thus we obtain
\[(2) \quad s \leq \begin{cases} 
l^2 & \text{if } n \text{ is odd}, \\
l^2 - 2 & \text{if } n \text{ is even}.
\end{cases}
\]
By considering the binomial expansion of \( 2^n \) we have that \( \left( \frac{n}{[n/2]} \right) \geq \frac{2^n}{n+1} \), that is, being \( l \leq \log_2(n+1) \leq l + 1 \),
\[(3) \quad \left( \frac{n}{[n/2]} \right) \geq 2^{n-(l+1)}.
\]
We distinguish now the different cases.

Case \( n \) odd.
Then \( \binom{n}{n/2} = n = 2^t \) and \( 2 \) and \( 3 \) imply that
\[
f(n) \leq 2^{t^2/2t + 2-n}.
\]
We have that \( t^2 + 2t + 2 - n < 0 \) for every \( n \geq 43 \). The cases \( 15 \leq n \leq 41 \) can be checked by direct computation. (Note that \( f(n) > 1 \) for \( 5 \leq n \leq 13 \).)
Case $n$ even.
Assume first that $t \geq 4$.
Then $n \geq 2^{a_t}(1 + 2 + \ldots + 2^{t-1}) = n_2(2^{t} - 1) \geq 15n_2$, hence $n_2 \leq \lfloor n/15 \rfloor$ and
\[
\binom{n}{n_2} \leq \binom{n}{\lfloor n/15 \rfloor}.
\]
Using the upper bound $\binom{n}{k} < (ne/k)^k$ and the fact that for every real number $x$ we have $x - 1 \leq \lfloor x \rfloor \leq x$, we deduce that
\[
\binom{n}{n_2} \leq \left( \frac{15en}{n - 15} \right)^{n/15} = 2^{\frac{n}{15} \log_2(n/15)}.
\]
Combining \ref{eq:2}, \ref{eq:3} and \ref{eq:4} we have that
\[
f(n) \leq 2^{l^2 + l - n(1 - \frac{1}{15} \log_2 \left( \frac{15en}{n - 15} \right))}.
\]
Note that $f(n) < 1$ if $l^2 + l - n \left( 1 - \frac{1}{15} \log_2 \left( \frac{15en}{n - 15} \right) \right) < 0$, which is true for every $n \geq 72$. The cases $22 \leq n \leq 70$, with $t \geq 4$, can be checked by a direct computation.

Let now $t = 2$ or $t = 3$.
In both cases we have that the value of $s$ is bounded from above by $3l - 3$, because when $t = 2$ then $a_1 \geq a_2 + 2$ and so
\[
s = 2a_1 + a_2 - 1 \leq 3a_1 - 3 \leq 3l - 3
\]
and when $t = 3$ then
\[
s = a_1 + a_2 + a_3 \leq a_1 + a_1 - 1 + a_1 - 2 \leq 3l - 3.
\]
Moreover, when $t = 2$, since $a_1 \geq a_2 + 2$, we have that $n_2 \leq \lfloor n/5 \rfloor$. Therefore, by considerations analogous to the ones above, we have that $f(n) < 1$ if
\[
4l - 1 - n \left( 1 - \frac{1}{l} \log_2 \left( \frac{7en}{n - l} \right) \right) < 0 \quad \text{when } t = 3,
\]
\[
4l - 1 - n \left( 1 - \frac{1}{5} \log_2 \left( \frac{5en}{n - 5} \right) \right) < 0 \quad \text{when } t = 2.
\]
Now \ref{eq:5} holds for every $n \geq 62$, while \ref{eq:6} holds for every $n \geq 114$.
As above, the intermediate cases can be checked by computation, the only exceptions being all $n$ even with $n \leq 20$ and $n = 40$. \hfill $\Box$

**Proof of Proposition** \ref{prop:finite}. To prove that $X_{n_2}$ is strongly definitely unbeatable on $\Pi$ we need to show that the three conditions of the Definition \ref{def:proof} are satisfied. Conditions (1) and (2) are straightforward (note that when $t = 2$ we have that $a_1 - 1 > a_2$).

We show condition (3), that is, for every $X_{n_2} \in X_{n_2}$ and every maximal subgroup $M$ of $S_n$, which is not the stabilizer of a $n_2$-subset, the proportion
\[
c(M) = \frac{|M \cap \Pi|}{|X_{n_2} \cap \Pi|} = \frac{|M|}{|X_{n_2}|} m_M < 1,
\]
where $m_M$ denotes the number of conjugates of $M$ containing a fixed element of $\Pi$.

Assume first that $M$ is an intransitive maximal subgroup not in $X_{n_2}$.
If $M \cap \Pi \neq \emptyset$ we necessarily have that $M$ is the stabilizer of a union of disjoint 2-power sized subsets, say $A_i$ for some $i \geq 1$. Since $|M \cap \Pi| \leq |\text{Stab}(A_1) \cap \Pi|$, we can assume that $M$ coincides with the stabilizer of a single subset of cardinality
some power of 2, say $2^c$, with $n_2 < 2^c < n/2$, being $2^{a_1} \geq n/2$, and therefore $c \in \{a_2, \ldots, a_{t-1}\}$ when $n \not\equiv t \pmod{2}$ and $c \in \{a_1 - 1, a_2, \ldots, a_{t-1}\}$ when $n \equiv t \pmod{2}$. Now, except in the case $n \equiv t \pmod{2}$ and $c = a_1 - 1$, we have that $m_M = 1$ and

$$c(M) = \frac{n}{n_2} \times \frac{n_2}{2^c} < 1,$$

since $n_2 = 2^{a_1} < 2^c$. Otherwise $m_M = 2$ and, since $a_t \leq a_1 - 2$, we obtain

$$c(M) = 2 \cdot \frac{n}{2^{a_1-1}} \leq 2 \cdot \frac{n}{2^{a_1-2}} < 1. \tag{1}$$

Assume now that $M$ is a primitive or an imprimitive maximal subgroup of $S_n$. Then by Lemma 5,

$$|M \cap \Pi| \leq |M| \leq \frac{2n!}{n^{\lfloor n/2 \rfloor}},$$

while

$$|X_{n_2} \cap \Pi| = \frac{n_2!(n-n_2)!}{2^{s^2}},$$

where $s = \sum_{i=1}^t a_i$ if $n \not\equiv t \pmod{2}$, and $s = \sum_{i=1}^t a_i + a_1 - 1$ if $n \equiv t \pmod{2}$. Therefore

$$c(M) = \frac{|M \cap \Pi|}{|X_{n_2} \cap \Pi|} \leq \frac{|M|}{|X_{n_2} \cap \Pi|} \leq 2^{s+1} \cdot \frac{n}{n_2} \cdot \frac{n_2}{\lfloor n/2 \rfloor}. \tag{2}$$

Lemma 8 proves that $c(M) < 1$, whenever $n$ is odd and $n \geq 15$, or $n$ is even and $n \geq 22$ and $n \not\equiv 40$. \hfill \Box

We treat now the case when $n \leq 20$ or $n = 40$.

**Proposition 3.** For $n \in \{5, 7, 9, 10, 11, 13, 14, 18, 20, 40\}$, the collection of subgroups $X_{n_2}$ is strongly definitely unbeatable on $\Pi$ if and only if $n \notin \{5, 10\}$.

**Proof.** As in the proof of Proposition 2, the conditions (1) and (2) of Definition 1 are straightforward to prove, so we limit ourselves to show that condition (3) holds.

In the sequel, denote by $M$ a maximal subgroup of $S_n$ not in $X_{n_2} \cup \{A_n\}$.

If $M$ is intransitive the inequality $|M \cap \Pi| \leq |X_{n_2} \cap \Pi|$ holds in all cases, with equality if and only if $M \in X_{n_2}$ (see the proof of Proposition 2). So we may concentrate on the imprimitive and primitive maximal subgroups. We treat the various cases separately.

Case $n = 5$. Then $\Pi = (4, 1)$ and if we take $M$ to be a primitive subgroup isomorphic to the Frobenius group $5 : 4$, we have that

$$|M \cap \Pi| = 10 > 6 = |X_{n_2} \cap \Pi|.$$ 

Therefore $X_{n_2}$ is not unbeatable on $\Pi$.

Case $n = 7$. Then $\Pi = (2, 2, 2, 1)$ and $|X_{n_2} \cap \Pi| = 15$. There is only one class of transitive (primitive) maximal subgroups of $G = S_7$ (not containing $A_7$). They have order 42 and their intersection with $\Pi$ has size 7.

Case $n = 9$. Then $\Pi = (8, 1)$ and $|X_{n_2} \cap \Pi| = 7!$, which is larger than 432, the maximal size of a primitive subgroup of $S_9$ not containing $A_9$. Since subgroups in $W_3$ have trivial intersection with $\Pi$, we have that $X_{n_2}$ is unbeatable on $\Pi$.

Case $n = 10$. Then $\Pi = (4, 4, 2)$ and if $M \in W_5$ we have, by Lemma 6

$$|M \cap \Pi| = 1.800 > 1.260 = |X_{n_2} \cap \Pi|,$$
which shows that \( X_{n_2} \) is not unbeatable on \( \Pi \).

Case \( n = 11 \). We have that \( \Pi = (4,4,2,1) \) and \( |X_{n_2} \cap \Pi| = 56.700 \), which is larger than 110, the maximal size of a primitive subgroup of \( S_{11} \) not containing \( A_{11} \).

Case \( n = 13 \). Then \( \Pi = (4,4,4,1) \) and \( |X_{n_2} \cap \Pi| \geq 1.4 \cdot 10^6 \). This number is larger than the maximal size of a primitive subgroup of \( S_{13} \) not containing \( A_{13} \), which is 156.

Case \( n = 14 \). Then \( \Pi = (8,4,2) \) and \( |X_{n_2} \cap \Pi| \geq 1.4 \cdot 10^7 \). If \( M \) is a primitive maximal subgroup of \( S_{14} \) not containing \( A_{14} \) then \( |M \cap \Pi| < |M| < 3^{14} < |X_{n_2} \cap \Pi| \).

Assume that \( M \) is imprimitive. If \( M \in \mathcal{W}_2 \), then
\[
|M \cap \Pi| \leq |M| = 2^7 \cdot 7! = 645.120 < |X_{n_2} \cap \Pi|.
\]
If \( M \in \mathcal{W}_7 \) then by Lemma [5]
\[
|M \cap \Pi| = \frac{|\Pi|}{|G : M|} \cdot 2^2 = 3.175.200 < |X_{n_2} \cap \Pi|.
\]

Case \( n = 18 \). Then \( \Pi = (8,8,2) \) and \( |X_{n_2} \cap \Pi| = \frac{|\Pi|}{2^7} \cdot 3! \geq 9.8 \cdot 10^{11} \). If \( M \) is a primitive maximal subgroup of \( S_{18} \) not containing \( A_{18} \), then
\[
|M \cap \Pi| < |M| < 3^{18} = 387.420.489 < |X_{n_2} \cap \Pi|.
\]
If \( M \) is an imprimitive maximal subgroup of \( S_{18} \), not in \( \mathcal{W}_9 \), then
\[
|M \cap \Pi| < |M| \leq (6!)^3 \cdot 3! \leq 2,24 \cdot 10^9 < |X_{n_2} \cap \Pi|.
\]
If \( M \in \mathcal{W}_9 \) then by Lemma [5]
\[
|M \cap \Pi| = \frac{|\Pi|}{|G : M|} \cdot 2^2 \leq 4,12 \cdot 10^9 < |X_{n_2} \cap \Pi|.
\]

Case \( n = 20 \). Then \( \Pi = (8,8,4) \) and \( |X_{n_2} \cap \Pi| = \frac{|\Pi|}{2^7} \cdot 3! \geq 9.8 \cdot 10^{11} \). If \( M \) is a primitive maximal subgroup of \( S_{20} \) not containing \( A_{20} \), then
\[
|M \cap \Pi| \leq 3^{20} \leq 3,49 \cdot 10^9 < |X_{n_2} \cap \Pi|.
\]
If \( M \) is an imprimitive maximal subgroup of \( G \), not in \( \mathcal{W}_{10} \), then
\[
|M \cap \Pi| \leq |M| \leq |S_{10} \wr S_4| = (5! \cdot 4! \leq 4,98 \cdot 10^9 < |X_{n_2} \cap \Pi|.
\]
If \( M \in \mathcal{W}_{10} \) then by Lemma [5]
\[
|M \cap \Pi| = \frac{|\Pi|}{|G : M|} \cdot 2^2 \leq 2,06 \cdot 10^{11} < |X_{n_2} \cap \Pi|.
\]

Case \( n = 40 \). Then \( \Pi = (16,16,8) \) and \( |X_{n_2} \cap \Pi| = \frac{|\Pi|}{2^{32}} \cdot 7! \geq 2,59 \cdot 10^{36} \). If \( M \) is a primitive maximal subgroup of \( S_{40} \) not containing \( A_{40} \) then \( |M \cap \Pi| < |M| \leq 2^{40} \approx 1,10 \cdot 10^{12} < |X_{n_2} \cap \Pi| \). If \( M \) is an imprimitive maximal subgroup of \( G \), not in \( \mathcal{W}_{20} \), then
\[
|M \cap \Pi| < |M| \approx |S_{10} \wr S_4| = 10! \cdot 4! \delta 4,17 \cdot 10^{27} < |X_{n_2} \cap \Pi|.
\]
If \( M \in \mathcal{W}_{20} \) then by Lemma [6]
\[
|M \cap \Pi| = \frac{|\Pi|}{|G : M|} \cdot 2^2 \leq 1,16 \cdot 10^{34} < |X_{n_2} \cap \Pi|.
\]

The proof is now complete. \( \Box \)

We can now complete this case.
Proposition 4. Assume that \( n \neq 10 \) and \( n \neq 3 \cdot 2^a \), for every \( \epsilon \in \{0, 1\} \) and every \( a \geq 0 \). Then \( \sigma_0(S_n) = 1 + \binom{n}{n_2} \).

Proof. The case \( n = 5 \) has been done in Lemma 8.

By Propositions 2 and 3 we have that

\[ \sigma_0(S_n) \geq \sigma(\Pi) = |\mathcal{X}_{n_2}| = \binom{n}{n_2}. \]

Moreover, if it were \( \sigma_0(S_n) = \binom{n}{n_2} \) then, by the strongly definitely unbeatable property, \( \mathcal{X}_{n_2} \) would be a primary covering for \( S_n \), which is impossible by [2, Theorem 1], or simply because this collection does not cover the primary elements acting fixed-point-freely. Therefore, \( \sigma_0(S_n) > \binom{n}{n_2} \) and then Lemma 7 completes the proof.

Note that in this situation a minimal primary covering is given by \( \{A_n\} \cup \mathcal{X}_{n_2} \). \( \square \)

Lemma 10. For \( n = 10 \) we have that \( 33 \leq \sigma_0(S_{10}) \leq 46 \).

Proof. We already know that \( \{A_{10}\} \cup \mathcal{X}_2 \) is a primary covering and therefore \( \sigma_0(S_{10}) \leq 46 \). Assume by contradiction that \( \mathcal{C} \) is a primary covering of smaller cardinality. \( A_{10} \) belongs to \( \mathcal{C} \) because the maximal intersection of a maximal subgroup of \( S_{10} \) distinct from \( A_{10} \) with the set of the 72,576 elements of cycle structure \( (5,5) \) is 576, realized by the class of subgroups \( \mathcal{W}_5 \), and 72,576/576 = 126 > 46 \( \geq \sigma_0(S_{10}) \). The remaining members of \( \mathcal{C} \) cover all of the odd 2-elements. Now the conjugacy class \( \Pi_1 = (4,4,2) \) consists of 56,700 elements and we have that the maximal subgroups that contain most elements of \( \Pi_1 \) are the ones in \( \mathcal{W}_5 \), each of which contains exactly 1,800 such elements. Since 56,700/1,800 = 31,5 we conclude that \( \mathcal{C} \) has at least 1 + 32 elements.

Finally, assume now that \( n = 3 \cdot 2^a \) for some \( a \geq 0 \). The case \( n = 3 \) is trivial, thus assume that \( a \geq 1 \). We first deal with the case \( n = 6 \).

Lemma 11. The primary covering number for \( S_6 \) is 7. A minimal primary covering is (a conjugate to) the following

\[ \mathcal{C} = \left\{ A_6, X_1, X_1^{(12)}, X_1^{(13)}, P_1, P_1^{(34)}, P_1^{(35)} \right\}, \]

where \( X_1 = \text{Stab}_{S_6}(\{1\}) \in \mathcal{X}_1 \) and \( P_1 = \langle (3465), (123)(456) \rangle \) belongs to the family \( \mathcal{P} \) of primitive maximal subgroups isomorphic to \( S_5 \).

Proof. A direct check with GAP shows that \( \mathcal{C} \) is a covering for the set of primary elements of \( S_6 \).

Assume by contradiction that \( \mathcal{D} \) is a primary covering (consisting of maximal subgroups of \( G \) and) containing less than seven elements.

We first show that \( A_6 \in \mathcal{D} \). If this is not the case, then the class \( \Pi_0 \) of 5-cycles should be covered by at most six maximal subgroups, which are either 1-point stabilizers, that is elements of \( \mathcal{X}_1 \), or primitive maximal subgroups, that is elements of \( \mathcal{P} \), and in both cases they are all isomorphic to \( S_5 \). Note that \( |\Pi_0| = 6 \cdot 24 \), and that for every \( S_1 \neq S_2 \in \mathcal{X}_1 \) and every \( P_1 \neq P_2 \in \mathcal{P} \) we have:

- \( |\Pi_0 \cap S_1| = |\Pi_0 \cap P_1| = 24 \),
- \( |\Pi_0 \cap S_1 \cap S_2| = |\Pi_0 \cap P_1 \cap P_2| = 0 \),
- \( |\Pi_0 \cap S_1 \cap P_1| = 4 \).
(the second equation is trivial for the 1-point stabilizers, and it holds for the members of \( \mathcal{P} \) too, since there is an outer involutory automorphism of \( S_6 \) that interchanges 1-point stabilizers with the members of \( \mathcal{P} \).

By applying an inclusion/exclusion argument there are only two ways to cover \( \Pi_0 \) with no more than six of these proper subgroups, either using all of the six elements of \( X_1 \), or all of the elements of \( \mathcal{P} \). It follows that \( \mathcal{D} \) is either \( X_1 \) or \( \mathcal{P} \). In both cases we have a contradiction, since \( X_1 \) does not cover the 2-elements of type \( \Pi_1 = (2, 2, 2) \), while \( \mathcal{P} \) does not cover the 2-cycles. We proved therefore that \( A_6 \in \mathcal{D} \).

We set \( \mathcal{D}_1 = \mathcal{D} \setminus \{ A_6 \} \). The collection \( \mathcal{D}_1 \) consists of at most five subgroups, which should cover the set of odd 2-elements, that is the set \( \Pi_1 \cup \Pi_2 \cup \Pi_3 \), where \( \Pi_1 = (2, 2, 2) \), \( \Pi_2 = (4, 1, 1) \) and \( \Pi_3 = (2, 1, 1, 1, 1) \). The following table shows the sizes of the intersections of these classes with the maximal subgroups (different form \( A_6 \)).

| \( \Pi_i \) | \( |\Pi_i| \) | \( |\Pi_i \cap X_1| \) | \( |\Pi_i \cap X_2| \) | \( |\Pi_i \cap \mathcal{W}_3| \) | \( |\Pi_i \cap \mathcal{W}_2| \) | \( |\Pi_i \cap \mathcal{P}| \) |
|----------|---------|----------------|----------------|----------------|----------------|----------------|
| (2, 2, 2) | 15      | 0              | 3              | 6              | 7              | 10             |
| (4, 1, 1) | 90      | 30             | 6              | 0              | 6              | 30             |
| (2, 1, 1, 1, 1) | 15 | 10             | 7              | 6              | 3              | 0              |

We claim that in order to cover the class \( \Pi_2 \) of 4-cycles we need to take either at least three different elements of \( X_1 \) or at least three different elements of \( \mathcal{P} \). This comes from the fact that, for every \( S_1 \neq S_2 \in X_1 \) and every \( P_1 \neq P_2 \in \mathcal{P} \), the following holds

- \( |\Pi_2 \cap S_1 \cap S_2| = |\Pi_2 \cap P_1 \cap P_2| = 6 \),
- \( |\Pi_2 \cap S_1 \cap P_1| = 10 \),
- \( |\Pi_2 \cap S_1 \cap S_2 \cap P_1| = |\Pi_2 \cap S_1 \cap P_1 \cap P_2| = 2 \),
- \( |\Pi_2 \cap S_1 \cap S_2 \cap P_1 \cap P_2| \leq 2 \),

hence

\[ |\Pi_2 \cap (S_1 \cup S_2 \cup P_1 \cup P_2)| \leq 76. \]

Assume that \( \mathcal{D}_1 \) contains three different elements of \( \mathcal{P} \), then, by looking a the last line of the Table, the class \( \Pi_3 \) should be covered using just two different subgroups, say \( A, B \in X_1 \cup X_2 \cup \mathcal{W}_3 \). this is impossible, since:

- if \( A, B \in X_1 \), then \( |\Pi_3 \cap A \cap B| = 6 \),
- if \( A \in X_1 \) and \( B \in X_2 \), then \( |\Pi_3 \cap A \cap B| \geq 4 \),
- if \( A \in X_1 \) and \( B \in \mathcal{W}_3 \), then \( |\Pi_3 \cap A \cap B| = 4 \).

The opposite case when \( \mathcal{D}_1 \) contains three different elements of \( X_1 \), follows immediately by using the duality of the outer automorphism of order two of \( S_6 \) (or with similar arguments applied to the first line of the table). \( \square \)

**Proposition 5.** Let \( n = 2^{a+1} + 2^a = 3 \cdot 2^a \), with \( a \geq 2 \). Then

\[ c_1 \leq \sigma_0(S_n) \leq c_2, \]
where
\[
\begin{align*}
c_1 &= \begin{cases} 
117 & \text{if } a = 2, \\
1 + \frac{n-1}{2^{a-1}} & \text{if } a \geq 3,
\end{cases} \\
c_2 &= 2 + \left(\frac{n-1}{2^a-1}\right) + \sum_{i=2}^{2^{a+1}} \left(\frac{n-i}{2^{a-1}-1}\right)
\end{align*}
\]

Proof. To prove the upper bound we consider the collection
\[
\mathcal{C}_2 = \bigcup_{i=0}^{m} \mathcal{M}_i,
\]
where \(m = 2^{a+1} + 1\) and
\[
\begin{align*}
\mathcal{M}_0 &= \{A_n\}, \\
\mathcal{M}_1 &= \{\text{Stab}_{S_n}(U)|1 \in U, |U| = 2^a\}, \\
\mathcal{M}_i &= \{\text{Stab}_{S_n}(V)|i \in V \subset \{i, \ldots, n\}, |V| = 2^{a-1}\}, \text{ for } i = 2, \ldots, 2^{a+1}, \\
\mathcal{M}_m &= \{\text{Stab}_{S_n}(\{m, \ldots, n\})\}.
\end{align*}
\]
The primary elements of odd order as well as the 2-elements that are even permutations are covered by \(A_n\). Let \(g\) be an odd 2-element. If \(g \in (2^a, 2^a, 2^a)\) then \(g\) is covered by a unique subgroup in \(\mathcal{M}_1\). Otherwise, since \(g\) is an odd permutation, there are at least two disjoint subsets of cardinality \(2^{a-1}\) on which \(g\) acts. If one of these two contains the point 1, then again \(g\) is covered by a unique element of \(\mathcal{M}_1\), otherwise \(g\) lies in the stabilizer of a subset of size \(2^{a-1}\) and not containing 1. In this case \(g\) is covered by a subgroup that lies in \(\bigcup_{i=2}^{m} \mathcal{M}_i\). Since \(|\mathcal{C}_2| = c_2\) the upper bound is proved.

We prove now that \(c_1\) is a lower bound for \(\sigma_0(S_n)\). Assume first that \(a = 2\), that is \(n = 12\). Arguing in a similar way as the cases \(n = 5, 6\) or 10, it is straightforward to prove that the alternating group \(A_{12}\) belongs to every minimal primary covering. Now, the maximal subgroups that contain most elements of type \(\Pi_1 = (4, 4, 4)\) are the ones in \(W_6\), each of which, by Lemma 5 contains exactly \(8|\Pi_1|(6!)^2/12!\) such elements. Since \(12!/8(6!)^2 = 115.5\) we conclude that a minimal primary covering has at least 117 elements.

Let \(a \geq 3\). We prove that \(c_1\) is a lower bound for \(\sigma_0(S_n)\) by showing that the collection \(\mathcal{M}_1\) is definitely unbeatable on \(\Pi_1 = (2^a, 2^a, 2^a)\).

Conditions (1) and (2) of Definition 1 follow immediately. Let us prove condition (3). Note that the only maximal subgroups \(M\) having nontrivial intersection with \(\Pi_1\) are either stabilizers of a set of cardinality \(2^a\), or imprimitive, or proper primitive maximal subgroups, that is elements of \(W \cup P\). For such \(M\) we define \(c(M) := |M \cap \Pi_1|/|M_1 \cap \Pi_1|\) for \(M_1 \in \mathcal{M}_1\), and we will prove that \(c(M) \leq 1\).

In the first case we have that
\[
|M \cap \Pi_1| = |M_1 \cap \Pi_1| = \frac{(2^a)!(2^{a+1})}{2^{5a+1}},
\]
for every \(M_1 \in \mathcal{M}_1\) and every \(M = \text{Stab}_{S_n}(U), 1 \notin U, |U| = 2^a\). Therefore \(c(M) = 1\).
Assume \( M \in W_{n/2} \). Then by Lemma 6,

\[
|M \cap \Pi_1| = \frac{4|\Pi_1||W_{n/2}|}{n!} = \frac{4}{3 \cdot 2^{3a}} ((n/2)!)^2,
\]

and therefore

\[
c(M) = \frac{8}{3} \cdot \frac{((n/2)!)^2}{(2a+1)!(2a)!} = \frac{8}{3} \cdot \frac{((2^a + 2^{a-1})!)^2}{(2a+1)!(2a)!}
\]

\[
= \frac{8}{3} \cdot \frac{(3t!)^2}{(4t)!(2t)!} = \frac{8}{3} \cdot \frac{6t}{(2t)}
\]

where \( t = 2^{a-1} \). Therefore \( c(M) < 1 \) for every \( t \geq 4 \), that is for every \( a \geq 3 \).

Assume \( M \in W_d \), with \( d \neq n/2 \). Then \( |M \cap \Pi_1| \leq |M| \leq |W_{n/3}| \) by Lemma 5, therefore

\[
c(M) < \frac{|W_{n/3}|}{|M \cap \Pi_1|} = \frac{(n/3!)^3 \cdot 6 \cdot 2^{3a+1}}{(2a+1)!(2^{a+1}+1)} < \frac{2^{3a+4}(2^{a+1}+1)}{2^{2a+6}} < \frac{2^{4a+6}}{2^{2a+1}} \leq 1,
\]

for every \( a \geq 4 \). The case \( a = 3 \) can be checked directly.

Finally assume that \( M \in P \). If \( a \geq 3 \), then \( |M| \leq 2^n \) by [8], hence

\[
c(M) < \frac{|M|}{|M \cap \Pi_1|} < \frac{2^{n+3a+1}}{(2a+1)!(2a)!} < 1.
\]

\[ \square \]

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