Superconformal Field Theory with Boundary:
Spin Model

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Abstract

GSO projected Superconformal field theory (Spin Model) with boundary is considered. There were written the boundary states. For this model were derived one-point structure constants and "bootstrap" equations for boundary-bulk structure constants.

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1 Introduction

Superconformal Field Theory on manifold with boundary plays an important role in open superstring theories and are the basic ingredient for the construction of the open superstring theory. Perhaps it can be also essential for some two dimensional exactly solvable models and their critical phenomenon.

Here we recall basic facts from the superconformal field theory adapted to our case and establish our notation. The basic ideas of superconformal field theory can be found in refs [1], [2].

Supersymmetric extensions of Virasoro algebra are obtained by generalizing conformal transformations to superconformal transformations of supercoordinates $\hat{z} = (z, \theta)$. The generators of superconformal transformations

$$
\delta z = u + \theta \epsilon; \quad \delta \theta = \epsilon + \frac{1}{2} \theta u_z;
\delta \bar{z} = \bar{u} + \bar{\theta} \bar{\epsilon}; \quad \delta \bar{\theta} = \bar{\epsilon} + \frac{1}{2} \bar{\theta} \bar{u}_{\bar{z}}
$$

are super stress-energy tensor $G(z, \theta) = \frac{1}{2} S(z) + \theta T(\bar{z})$. The operators $L_n$ and $S_r$ (Laurent coefficients of $T$ and $S$) generate analytic coordinate and supersymmetry transformations respectively and obey the algebra,

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{\epsilon}{8} m(m^2 - 1) \delta_{m+n}
\{S_r, S_s\} = 2L_{r+s} + \frac{\epsilon}{8} (r^2 - \frac{1}{4}) \delta_{r+s}
[L_m, S_r] = (\frac{m}{2} - r)S_{m+r}
$$

The algebra has a $Z_2$ symmetry, so there are two possible moding for the fermionic generator $S_r$, either half-integer ($r \in 1/2 + Z$) or integer ($r \in Z$) giving the Neveu-Schwarz (NS) and Ramond (R) algebras respectively. Highest weight states $| h \rangle$ of the NS and R algebras satisfy

$$
L_n | h \rangle = S_r | h \rangle = 0, \quad n, r > 0
L_0 | h \rangle = h | h \rangle
$$

Representation are built up by applying the raising operators $L_n, S_r$ with $n, r > 0$ to the highest weight state $| h \rangle$. In the Ramond sector superconformal current has zero mode, which form two dimensional Clifford Algebra with the Fermion Number Operator $\Gamma = (-)^F$, commuting with the $L_0$. As a result, we have double degeneration of the ground state $| 0 \rangle$. In this space we can choose the following orhogonal basis $| h^+ \rangle = R_h^+(0) | 0 \rangle$, $| h^- \rangle = R_h^-(0) | 0 \rangle$ (where $R^\pm$-Ramond spin fields):

$$
|h^-\rangle = S_0 | h^+ \rangle
$$
where $| h^+ \rangle$ and $| h^- \rangle$ are eigenvectors of operator $(-)^F$ with eigenvalues $+1$ and $-1$ respectively having the same conformal weight $h$. Using commutation relations (2) we can obtain:

$$S_0 | h^- \rangle = S_0^2 | h^+ \rangle = (L_0 - \frac{c}{16}) | h^+ \rangle = (h - \frac{c}{16}) | h^+ \rangle$$

Thus, if one normalizes $| h^+ \rangle$ as, $\langle h^+ | h^+ \rangle = 1$, then from (5) it follows, that $\langle h^- | h^- \rangle = h - \frac{c}{16}$. In case, when $h \neq \frac{c}{16}$, it can be chosen basis $| h' \rangle$ such, that $S_0 | h' \rangle = \sqrt{h - \frac{c}{16}} | h' \rangle$ and which is orthonormal. In further we will use the basis (4). Let us note, that if $h = \frac{c}{16}$, then $| h^- \rangle$ becomes 0-vector and decouples from representation of algebra. Hence chiral symmetry of the ground state is destroyed and the global supersymmetry is restored.

In the general superconformal theory the full operator algebra of $NS$ superfields and $R^\pm$ spin fields is nonlocal [1]. There are two possibility for projecting onto a local set of fields. First one, keeping only the $NS$-sector giving the usual algebra of superfields, a fermionic model. The second one, we can get a local field theory the ”spin model” restricting in superconformal field theory by $\Gamma = 1$ sector. In this paper we are going to consider ”Spin Model” with boundary defined on the upper half plane (the ”Fermionic Model” there was studied in ref.[3]).

It is easy to see, that the requirement of preservation of the geometry gives strong limitations on parameters of superconformal transformation. One can see that the expansion coefficients of parameters must be real. Therefore holomorphic and anti–holomorphic transformations are not independent. So, let’s make analytical continuation of $T$ and $S$ on to lower half plane.

$$T(z) = \bar{T}(z); \quad S(z) = \bar{S}(z); \quad \text{for} \quad Imz < 0$$

It means that now we have only one algebra (2) in opposite to ”bulk” theory, there were two, holomorphic and anti–holomorphic algebras, which is consistent with the fact, that in theory with boundary we have only one set of coefficient in expansion of parameters.. Then for $\langle X \rangle = \langle R^\pm(z_1, \bar{z}_1) ... R^\pm(z_n, \bar{z}_n) \rangle$ correlation function from (6) (using bulk OPE) follows, that in contrast to bulk Ward Identity where $T(z)$ and $S(z)$ acts only on $(z_1, ..., z_n)$, in theory with boundary the action of $T(z)$ and $S(z)$ is extended to $(\tilde{z}_1, ... , \tilde{z}_n)$ and hence, in the relations of the boundary Ward Identity the doubling of terms on the right hand sides takes place due to terms with $z'_i = \tilde{z}_i$. So,
correlation function for Ramond fields \(\langle X(z_1, \bar{z}_1, ...z_n, \bar{z}_n)\rangle_B\) in our geometry satisfies the same differential equation as does bulk correlation function of Ramond fields \(\langle X(z_1, \bar{z}_1, ...z_2n, \bar{z}_2n)\rangle\).

2 Boundary States

Further we will construct boundary states of theories defined on the upper half plane or strip, which one can also interpretate as a world sheet of an open superstring. Mapping of the upper half plane on to strip is given by the conformal transformation \(z = e^{t+i\sigma}\), where \((t, \sigma)\) are coordinates on strip \((0, \pi)\).

In general superconformal field theory with boundary, the unique requirement on boundary condition is the superconformal invariance:

\[
T(z = e^t) = \bar{T}(\bar{z} = e^{\bar{t}}) \quad T(z = e^{t+i\pi}) = \bar{T}(\bar{z} = e^{-i\pi}) \\
S(z = e^t) = \bar{S}(\bar{z} = e^{\bar{t}}) \quad S(z = e^{t+i\pi}) = \bar{S}(\bar{z} = e^{-i\pi}) \quad NS - sector \\
S(z = e^t) = \bar{S}(\bar{z} = e^{\bar{t}}) \quad S(z = e^{t+i\pi}) = -\bar{S}(\bar{z} = e^{-i\pi}) \quad R - sector
\]

If one compactifies \(t\) by mod \(2\pi Im\tau\) (\(\tau\) is purely imaginary) he obtains the theory defined on a cylinder with radius \(Im\tau\). Then partition functions with boundary conditions \(\alpha, \beta\) at the ends of cylinder can be written (for antiperiodic and periodic boundary condition in time direction) as follows,

\[
Z^{NS}_{\alpha\beta} = Tr e^{2\pi i\tau H^{open}_{\alpha\beta}}; \quad Z^{(-)NS}_{\alpha\beta} = Tr(-1)^F e^{2\pi i\tau H^{open}_{\alpha\beta}} \\
Z^R_{\alpha'\beta'} = Tr e^{2\pi i\tau H^{open}_{\alpha'\beta'}}; \quad Z^{(-)R}_{\alpha'\beta'} = Tr(-1)^F e^{2\pi i\tau H^{open}_{\alpha'\beta'}}
\]

The bulk superconformal algebra is the tensor product of two algebras, therefore natural chirality operator is \(\Gamma = (-1)^{F_{tot}}\), where \(F_{tot} = F + \bar{F}\) is the fermion number of the full algebra. The projection of boundary SCFT is analogous to the GSO projection of the bulk SCFT with the difference that in the boundary theory only one chirality operator \(\Gamma = (-1)^F\) is defined, since for boundary case there is just one algebra. The projection to local theory in \(NS\) and \(R\) sectors is given by \(\Gamma = 1\).

Let’s note that summarizing partition functions in each sector \(Z^{NS}_{\alpha\beta} + Z^{(-)NS}_{\alpha\beta}\) and \(Z^R_{\alpha'\beta'} + Z^{(-)R}_{\alpha'\beta'}\) are just projecting into subspace having even fermion number.
From the other side, the same partition function can be considered as a propagation of closed superstring on \( \sigma \) direction between boundary states \( \langle \alpha |, | \beta \rangle \),

\[
Z_{\alpha\beta} = \langle \alpha | e^{-\pi H_{cyl}} | \beta \rangle = \langle \alpha | e^{-\pi \operatorname{Im} \tau(L_{cyl}^0 + \bar{L}_{cyl}^0)} | \beta \rangle
\]

where \( H_{cyl} \) is the Hamiltonian for closed superstring, \( L_{cyl}^0, \bar{L}_{cyl}^0 \) are generators of Virasoro and \( | \alpha \rangle, | \beta \rangle \) satisfy to conditions (7), which can be rewritten as

\[
T_{cyl}(\zeta) = \bar{T}_{cyl}(\bar{\zeta}) | \zeta = e^{-it} \rangle = \bar{T}_{cyl}(\bar{\zeta}) \]

\[
S_{cyl}(\zeta) = -i\bar{S}_{cyl}(\bar{\zeta}) | \zeta = e^{-it} \rangle = -i\bar{S}_{cyl}(\bar{\zeta}) \]

\[
S_{cyl}(\zeta) = i\bar{S}_{cyl}(\bar{\zeta}) | \zeta = e^{-it} \rangle = i\bar{S}_{cyl}(\bar{\zeta}) \]

where \( \zeta = e^{-i(t + i\sigma)} \). One can rewrite conditions (10), in the form:

\[
(L_n - \bar{L}_{-n}) | B_\pm \rangle = 0
\]

\[
(S_r \pm i\bar{S}_{-r}) | B_\pm \rangle = 0
\]

where \( r \in \mathbb{Z} \) or \( r \in \mathbb{Z} + \frac{1}{2} \).

It is easy to see from (10), (11) that one should choose “+” boundary states (or “−”) for both ends of the cylinder for propagation of Neveu-Schwarz and “+−” (or “−+”) for propagation of Ramond states in open string channel. Of course “+” and “−” states are not essentially different. For our purposes we will fix “+” boundary states for \( \sigma = 0 \) end of cylinder and vary “+” and “−” for the other end.

One of the basic aims of this paper is to find solutions (11) in each irreducible representation of superconformal algebra. The solution of conditions (11) in NS sector is given by the following anzats [4],

\[
| h_{+}^{NS} \rangle = \sum_{s \in \mathbb{Z} + \mathbf{i}/2} | h, s \rangle \otimes U_{+}^{NS} | h, s \rangle
\]

where \( U_{NS}^{\pm} \) is an anti-unitary operators, satisfying the following conditions:

\[
L_n U_{NS}^{\pm} = U_{NS}^{\pm} L_n
\]

\[
U_{NS}^{\pm} S_r = \mp i S_r U_{NS}^{\pm} (-)^F
\]

One can see that equations (13) yield
\begin{equation}
U_{\pm}^{NS} | h, s \rangle = \frac{1-i}{2} (1 \pm i(-)^F) | h, s \rangle \quad (14)
\end{equation}

It’s easy to show, that (12) satisfies to conditions (11). For this purpose we just have to check, that for any basic vector $\langle i \rangle \otimes | j \rangle$, following relations are valid,
\begin{equation}
\begin{aligned}
\langle i \rangle \otimes | j \rangle | (L_n - \bar{L}_{-n}) \rangle & = 0, \\
\langle i \rangle \otimes | j \rangle | (S_r \pm \bar{S}_{-r}) \rangle & = 0.
\end{aligned} \quad (15)
\end{equation}

It is more interesting Ramond sector. For the beginning let us consider the case $h \neq c/16$. We can use the same anzats (12) to solve (11),
\begin{equation}
| h_{\pm} \rangle = \sum_{q \in \mathbb{Z}^+} | h, q \rangle \otimes U_{\pm}^R | h, q \rangle = \\
\sum_{p \in \mathbb{N}} | h^+, p \rangle \otimes U_{\pm}^R | h^+, p \rangle + \sum_{p \in \mathbb{N}} | h^-, p \rangle \otimes U_{\pm}^R | h^-, p \rangle \quad (16)
\end{equation}

where $U_{\pm}^R$ is anti-unitary operators, satisfying to conditions:
\begin{equation}
L_n U_{\pm}^R = U_{\pm}^R L_n \\
U_{\pm}^R S_r = \mp i S_r U_{\pm}^R (-)^F \quad (17)
\end{equation}

Since the ground state is now non–trivial, we have freedom in a definition of the action $U_{\pm}^R$ on this space. And we have the only restriction on $U_{\pm}^R$:
\begin{equation}
(U_{\pm}^R S_0 \pm i S_0 U_{\pm}^R (-)^F) | h^\pm \rangle = 0 \quad (18)
\end{equation}

In representation, where
\begin{equation}
| h^+ \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad | h^- \rangle = \begin{pmatrix} 0 \\ \sqrt{h - c/16} \end{pmatrix}
\end{equation}

$S_0$ and $(-)^F$ can be represented as
\begin{equation}
S_0 = \sqrt{h - \frac{c}{16}} \sigma_x; \quad (-)^F = \sigma_z \quad (19)
\end{equation}

where $\sigma_x$ and $\sigma_z$ are Pauli matrixes. Using (18) and representation (19), we get:
\begin{equation}
U_{\pm}^R = \begin{pmatrix} a & \mp i c \\ c & \mp i a \end{pmatrix} \quad (20)
\end{equation}
where \( a \) and \( c \) satisfy anti-unitary condition: \( aa^* + cc^* = 1 \) and \( ac^* + a^*c = 0 \). According to latter equations there are two independent choices for \( U^R_\pm \):

\[
U^R_\pm = \begin{pmatrix} 1 & 0 \\ 0 & \mp i \end{pmatrix} \quad \text{or} \quad U^R_\pm = \begin{pmatrix} 0 & \mp i \\ 1 & 0 \end{pmatrix}
\]  

(21)

It is interesting to note, that for \( h = \frac{c}{16} \), the uniqueness of \( U^R_\pm \) is recovered. The nature of this degeneration is very interesting but we will not analyze it. We only note that it is sufficient to restrict to first choice of \( U^R_\pm \).

The partition functions (8) of the theory defined on compactified cylinder can be expressed as a linear combination of characters since instead of holomorphic and atiholomorphic algebras (in the bulk) now there is just one algebra:

\[
Z^{NS}_{\alpha\beta} = \sum n^i_{\alpha\beta} \chi^i_{NS}(q), \quad Z^{(-)NS}_{\alpha\beta} = \sum n^i_{\alpha\beta} \chi^i_{(-)NS}(q);
\]

\[
Z^R_{\alpha\beta} = \sum m^i_{\alpha\beta} \chi^i_{R}(q), \quad Z^{(-)R}_{\alpha\beta} = \sum m^i_{\alpha\beta} \chi^i_{(-)R}
\]

(22)

where \( \chi^i_{NS}(q) = q^{-\epsilon/16} Tr_i q^{L_0} \), \( \chi^i_{(-)NS}(q) = q^{-\epsilon/16} Tr_i (-1)^F q^{L_0} \) and \( \chi^R(q) = q^{-\epsilon/16} Tr_i q^{L_0} \), \( \chi^i_{(-)R} = Tr_i (-1)^F \) are the characters of the superconformal algebras in NS and R sectors respectively. For the last character note that R fermion has zero energy on the cylinder at the supersymmetric ground state (\( h = \frac{c}{16} \)). By non-negative integer \( n^i_{\alpha\beta}, m^i_{\alpha\beta} \) denoted the number of times that representation \( i \) occurs in the spectrum of \( H^{open}_{\alpha\beta} \).

The character formulas for the NS and R algebra have been derived by Goddard, Kent and Olive [7] and by Kac and Wakimoto [8] and under the modular transformation \( \tau \to -1/\tau \) the characters for the ”spin model” transform linearly [3],

\[
\chi^i_{NS}(q) = \sum (S^i_{NS})^{j}_{i} \chi^{j}_{NS}(q),
\]

\[
\chi^i_{(-)NS}(q) = \sum (S^i_{NS})^{j}_{i} \chi^{j}_{(-)NS}(q),
\]

\[
\chi^i_{R}(q) = \sum (S^i_{R})^{j}_{i} \chi^{j}_{R}(q),
\]

(23)

which leads to

\[
Z^{NS}_{\alpha\beta} = \sum n^{i}_{\alpha\beta} (S^{NS})^{j}_{i} \chi^{j}_{NS}(\tilde{q})
\]

\[
Z^{(-)NS}_{\alpha\beta} = \sum n^{i}_{\alpha\beta} (S^{(-)NS})^{j}_{i} \chi^{j}_{(-)NS}(\tilde{q})
\]

\[
Z^{R}_{\alpha\beta} = \sum m^{i}_{\alpha\beta} (S^{R})^{j}_{i} \chi^{j}_{R}(\tilde{q})
\]

\[
Z^{(-)R}_{\alpha\beta} = \sum m^{i}_{\alpha\beta} \chi^{i}_{(-)R}
\]

(24)
where $\tilde{q} = e^{-2\pi i/\tau}$. In order to have complete set of boundary states defined by equation (12), we have to consider diagonal bulk theory. Following to Cappelli and Kastor [9] there are different superconformal theories corresponding to different modular invariant combination of characters

$$Z_{NS,R} = \sum_{i,j} F_{i,j} N_{i,j} \chi_i(q) \bar{\chi}_j(q) \tag{25}$$

here the factor $F$ is equal to 2 for the nonsupersymmetric $R$ highest weight states, which one twofold degenerated, and is equal to 1 otherwise. $N_{i,j}$ is the number of highest weight states $(h_i, \tilde{h}_j)$ in the bulk theory which one obeys to the sum rules, requiring modular invariance of $Z_{NS}(q) = Z_{NS}(q)$, $Z_R(q) = Z_{NS}^{(-1)}(q)$, $Z_{NS}^{(-2)}(q) = Z_{R}(q)$ we can get

$$\sum N_{nm,kl} \sin \frac{\pi m}{p} \sin \frac{\pi m'}{p+2} \sin \frac{\pi k}{p} \sin \frac{\pi k'}{p+2} = \frac{p(p+2)}{16} N_{n'm',k'l'}$$

$\sum N_{nm,kl} Y_{nm,kl} \sin \frac{\pi m'}{p} \sin \frac{\pi m'l'}{p+2} \sin \frac{\pi k}{p} \sin \frac{\pi k'l'}{p+2} = \frac{(p-1)^2}{16} N_{n'm',k'l'}$

$\sum N_{nm,kl} (-1)^{\alpha} \sin \frac{\pi m}{p} \sin \frac{\pi m'}{p+2} \sin \frac{\pi k}{p} \sin \frac{\pi k'}{p+2} = \frac{p(p+2)}{16} y^{-1} N_{n'm',k'l'} \tag{26}$

There are at least two series of solutions to the above sum rules. One of these the diagonal (or scalar) solution of the superconformal sum rules are given by $N_{nm,kl} = \delta_{nk} \delta_{md}$ in $NS, R$ sectors. For the diagonal theory the constructed states $| j_{\pm}^{R,NS} \rangle$ give complete set in the space of all boundary states and we can therefore write

$$| \alpha_\pm \rangle = \sum_j | j_{\pm} \rangle \langle j_{\pm} | \alpha_\pm \rangle = \sum_j | j_{\pm}^{NS} \rangle \langle j_{\pm}^{NS} | \alpha_{\pm} \rangle + \sum_j | j_{\pm}^{R} \rangle \langle j_{\pm}^{R} | \alpha_{\pm} \rangle \tag{27}$$

Using these representations we can rewrite (4)

$$Z_{\alpha_+,\beta_+}(q) = \sum \langle \alpha_+ | i_+^{NS} \rangle \langle i_+^{NS} | \beta_+ \rangle \chi_i^{NS}(\tilde{q}) + \sum \langle \alpha_+ | i_+^{R} \rangle \langle i_+^{R} | \beta_+ \rangle \chi_i^{R}(\tilde{q})$$

$$Z_{\alpha_-,\beta_+}(q) = i \sum \langle \alpha_- | i_-^{NS} \rangle \langle i_-^{NS} | \beta_+ \rangle \chi_i^{(-)NS}(\tilde{q}) + i \sum \langle \alpha_- | i_-^{R} \rangle \langle i_-^{R} | \beta_+ \rangle \chi_i^{(-)R} \tag{28}$$

For such theories, when each representation occurs just once in the spectrum of bulk $H$, we have linearly independent different characters, therefore comparing last relations and (24), namely $Z_{\alpha_+,\beta_+}(q) = Z_{\alpha_+}^{NS}(q) + Z_{\alpha_+}^{(-)NS}(q)$ and $Z_{\alpha_-,\beta_+}(q) = Z_{\alpha_-}^{R}(q) + Z_{\alpha_-}^{(-)R}(q)$ we can get immedietly relations
\[
\sum_j (S_{NS}^{i+})_j n_{\alpha_+ \beta_+}^j = \langle \alpha_+ | i_+^{NS} \rangle \langle i_+^{NS} | \beta_+ \rangle \\
\sum_j (S_{R}^{i+})_j n_{\alpha_+ \beta_+}^j = \langle \alpha_+ | i_+^{R} \rangle \langle i_+^{R} | \beta_+ \rangle \\
\sum_j (S_{NS}^{i+})_j m_{\alpha_+ \beta_-}^j = i \langle \alpha_+ | i_+^{NS} \rangle \langle i_-^{NS} | \beta_- \rangle \\
m_{\alpha_+ \beta_-}^j = i \langle \alpha_+ | h = c/16 \rangle \langle h = c/16 | \beta_- \rangle 
\]

Thus, solving equations for coefficients of boundary states \(|\alpha_+\rangle\) (in the same way for \(|\alpha_-\rangle\)), we can write finally particulary for \(|\alpha_+\rangle\) following expression

\[
| \tilde{k} \rangle = \sum_j \frac{(S_{NS}^{i+})_j}{| (S_{NS}^{i+})_0 |^{1/2} } | j_+^{NS} \rangle + \sum_j \frac{(S_{R}^{i+})_j}{| (S_{R}^{i+})_0 |^{1/2} } | j_+^{R} \rangle
\]

These states have property that \(n_{0 \tilde{k}}^i = \delta_\tilde{k}^i\) which means that the representation \(k\) appears in the spectrum of \(H_{0 \tilde{k}}\).

### 3 One and three point boundary correlation functions

In the superstring theories we generally are interested in calculation of scattering amplitudes with both open and closed strings in the initial and final states. A string diagram with external open and closed string can be conformally mapped to the upper half plane. After this mapping the external open string are represented by vertex operators at finite points on the boundary, while the closed strings are represented by vertex operators at finite points on the upper half plane. All of this means that for construction open and closed superstring theories we are really interested in superconformal field theory with boundary (SCFT on half plane). One of the interesting question is how in the intermediate channel of string diagram (with external open and closed strings) closed string vertex can be expressed by open string vertex operators with given type of boundary condition. In a superconformal field theory (in which the boundary conditions do not break the superconformal symmetry) this can be represented as short distance expansion of bulk vertex operators.
operators near a boundary \[^{[3]}\]. There are two types of bulk fields: Ramond spin fields \( R(z, \bar{z}) \) and Neveu-Schwarz superfields

\[
\Phi(\hat{z}, \bar{z}) = \phi(z, \bar{z}) + \theta \Psi(z, \bar{z}) + \bar{\theta} \bar{\Psi}(z, \bar{z}) + \theta \bar{\theta} F(z, \bar{z}) \tag{31}
\]

where

\[
\begin{align*}
\Psi(z, \bar{z}) &= S_{-1/2} \phi(z, \bar{z}); \\
\bar{\Psi}(z, \bar{z}) &= \bar{S}_{-1/2} \phi(z, \bar{z}); \\
F(z, \bar{z}) &= S_{-1/2} \bar{S}_{-1/2} \phi(z, \bar{z})
\end{align*} \tag{32}
\]

one can write short distance expansion for \( \phi(z, \bar{z}) \) and \( R(z, \bar{z}) \) near boundary as follows

\[
\begin{align*}
\phi(z, \bar{z}) &= \sum_i (z - \bar{z})^{\Delta_{\phi}^B - \Delta_\phi} C_{\phi \phi \Phi}^B \phi_i^B(x) \\
R(z, \bar{z}) &= \sum_i (z - \bar{z})^{\Delta_{\phi}^R - \Delta_R} C_{\phi \phi \Phi}^B \phi_i^B(x)
\end{align*} \tag{33, 34}
\]

here \([\phi^B(x)]\), \(\Phi\)-are conformal class of boundary vertex operators \(\phi^B\) and \(C_{\phi \phi \Phi}^B\), \(C_{\phi \phi \Phi}^B\)-are OPE’s boundary structure constants of Neveu-Schwarz and Ramond fields respectively. From \(^{[32]}\) it is possible to obtain corresponding relations for \(\Psi\) and \(F\) fields.

Now let’s obtain these boundary structure constants. First of all note that for identity boundary operator corresponding structure constant is equal to constant factor of one point boundary correlation function. One point boundary correlation (with boundary conditions labelled by B) of NS and R fields with corresponding to superconformal invariance and boundary Ward identity can be written

\[
\begin{align*}
\langle \Phi(\hat{z}, \bar{z}) \rangle_B &= \frac{A_{\Phi}^B}{(z - \bar{z} - \theta \bar{\theta})^{\Delta_\phi}} \\
\langle R^+(z, \bar{z}) \rangle_B &= \frac{A_{\Psi}^B}{(z - \bar{z})^{\Delta_R}}
\end{align*} \tag{35}
\]

where \(A_{\phi}^B = C_{\phi \phi \Phi}^B, A_{\Psi}^B = C_{\phi \phi \Phi}^B\). It is easy to see that \(A_{\phi}^B = A_{\phi}^B\). Thus, according to the definition \(^{[3]}\) \(^{[4]}\),

\[
A_{\phi}^B = \frac{\langle \phi | B \rangle}{\langle 0 | B \rangle} \quad A_{\Psi}^B = \frac{\langle R^+ | B \rangle}{\langle 0 | B \rangle} \quad (36)
\]
and using the superconformal physical boundary states (30) we find

\[
A^\tilde{k}_\phi = \frac{\langle \phi | \tilde{k} \rangle}{\langle 0 | k \rangle} = \frac{[(S^{NS}_{00})^{01/2} (S^{NS}_{0k})^0]}{[(S^{NS}_{00})^{01/2} (S^{NS}_{0k})^0]} \frac{1}{2} \left[ (S^{NS}_{00})^{01} (S^{NS}_{0k})^0 \right]
\]

\[
A^\tilde{k}_R = \frac{\langle R^+ | \tilde{k} \rangle}{\langle 0 | \tilde{k} \rangle} = \frac{[(S^{NS}_{00})^{01/2} (S^{NS}_{0k})^1]}{[(S^{NS}_{00})^{01/2} (S^{NS}_{0k})^1]} \frac{1}{2} \left[ (S^{NS}_{00})^{01} (S^{NS}_{0k})^1 \right]
\]

(37)

To determine the boundary structure constants \( C^B_{\phi \phi} \), \( C^B_{R \phi} \), we use associativity of the boundary operator algebra which imposes global constraints on correlation functions. For this purpose, consider 2-point functions,

\[
\langle \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \rangle_B; \quad \langle R_i(z_1, \bar{z}_1) R_j(z_2, \bar{z}_2) \rangle_B
\]

(38)

in two channels. Of course corresponding correlation functions for \( \Psi \) and \( F \) can be restored from (38) by supersymmetry. We can evaluate these correlation functions using OPE in different crossing channels. Associativity of the operator algebra implies that correlation function of these two channels should give the same result (crossing symmetry),

\[
\sum_k C^\tilde{n}_{\phi \phi}^B C^\tilde{n}_{\phi \phi}^B F^{k}_{ii:ij} (1-\eta) = \sum_m C^\tilde{i}_{\phi \phi}^m A^\tilde{n}_{\phi \phi}^m F^{m}_{ij:ij} (\eta)
\]

(39)

\[
\sum_k C^\tilde{n}_{R \phi}^B C^\tilde{n}_{R \phi}^B F^{k}_{\rho \sigma:\rho \sigma} (1-\eta) = \sum_m C^\tilde{m}_{\rho \sigma}^m A^\tilde{n}_{\rho \sigma}^m F^{m}_{\rho \sigma:\rho \sigma} (\eta)
\]

(40)

here \( \eta = \frac{|z_1 - z_2|^2}{|z_1 - \bar{z}_2|^2} \) is cross-ratios, \( F^{m}_{ij:ij} (\eta) \), \( F^{m}_{\rho \sigma:\rho \sigma} (\eta) \), \( C^\tilde{i}_{\phi \phi}^m \), and \( C^\tilde{m}_{\rho \sigma}^m \) are conformal blocks and bulk structure constants respectivelly.

According to different basis of differential equations (to which obey conformal blocks) the solutions are expressed by each other linearly [10],

\[
F^{k}_{ij:ij} (\eta) = \sum \alpha \left[ \begin{array}{c} i \\ j \\ i \\ j \end{array} \right]_{kl}^{NS} F^{l}_{ii:jj} (1-\eta)
\]

\[
F^{k}_{\rho \sigma:\rho \sigma} (\eta) = \sum \alpha \left[ \begin{array}{c} \rho \\ \sigma \\ \rho \\ \sigma \end{array} \right]_{kl}^{R} F^{l}_{\rho \rho:\sigma \sigma} (1-\eta)
\]

(41)

using above equations from (39-40) we obtain immediately

\[
C^B_{\phi \phi} \tilde{C}^B_{\phi \phi} = \sum_m C^\tilde{i}_{\phi \phi} A^\tilde{n}_{\phi \phi} \alpha \left[ \begin{array}{c} i \\ j \\ i \\ j \end{array} \right]_{ml}^{NS}
\]

(42)

11
\[ C_{R^\rho \phi \sigma}^k C_{R^\sigma \phi}^l = \sum_m C_{\rho m A^k R^\rho \sigma}^\alpha \left[ \begin{array}{cc} \rho & \rho \\ \sigma & \sigma \end{array} \right]_m = \sum_m C_{\rho m A^k R^\rho \sigma}^\alpha \left[ \begin{array}{cc} \rho & \rho \\ \sigma & \sigma \end{array} \right]_m (43) \]

So, all boundary structure constants are expressed via well known bulk quantities.

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