Metrizability of Mahavier products indexed by partial orders

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Abstract

Let $X$ be separable metrizable, and let $f \subseteq X^2$ be a non-trivial relation on $X$. For a given partial order $\langle P, \leq \rangle$, the Mahavier product $M \langle X, f, P \rangle \subseteq X^P$ (also known as a generalized inverse limit) collects functions such that $x(p) \in f(x(q))$ for all $p \leq q$. Clontz and Varaona previously showed for well orders $P$ that $M \langle X, f, P \rangle$ is separable metrizable exactly when $P$ is countable and $f$ satisfies condition $\Gamma$; we extend this result to hold for all partial orders.

1 Introduction

Let $X$ be a separable metrizable topological space, let $f \subseteq X^2$ be a relation on $X$, and let $Q$ be a set preordered by (reflexive and transitive) $\preceq$.

Extending work done in e.g. [1, 2] we consider the subspace $M \langle X, f, Q \rangle$ of $X^Q$ where $x(p) \in f(x(q))$ for all $p \preceq q$. Such subspaces are often known as generalized inverse limits, or as we will refer to them, Mahavier products. For an introduction to such structures, often considered in the context of continuum theory, we direct the reader to [3]. Much of the literature on this subject considers only simple indices, particularly $Q = \mathbb{N}$ or $\mathbb{Z}$ with its usual order.

It is immediate that any subspace of $X^Q$ is separable metrizable whenever $Q$ is countable. In [4, 5] the authors consider whether $M \langle X, f, Q \rangle$ might be metrizable when $Q$ is an uncountable well-order. It turns out that, except in trivial situations, the answer is no.

We aim to extend this result to the more general case where the index set of the Mahavier product is any partial order, that is, any antisymmetric preorder. To do this, we utilize the notion of a partially-ordered topological space (POTS) originally defined in [6], a generalization of the class of linearly ordered topological spaces (LOTS) studied broadly throughout general topology.

2 Partially-Ordered Topological Spaces

For convenience, we formally define the notion of a preorder and partial order here.

Definition 1. A preorder $\langle Q, \preceq \rangle$ is a set paired with a reflexive ($x \preceq x$) and transitive ($x \preceq y \preceq z \Rightarrow x \preceq z$) relation.

We adopt the convention of using $Q$ for preorders as they are sometimes known as quasi-orders.
**Definition 2.** A partial order \( \langle P, \leq \rangle \) is a preorder that is antisymmetric \((x < y \Rightarrow y \not< x)\).

We note that every preorder admits a natural partial order.

**Proposition 3.** Let \( Q \) be preordered by \( \preceq \), and set \( p \sim q \) if and only if \( p \preceq q \) and \( q \preceq p \). Then \( \sim \) is an equivalence relation, and its set of equivalence classes \( P = \{ [p] : p \in Q \} \) ordered by \( A \leq B \) if and only if \( p \preceq q \) for all \( p \in A, q \in B \) is a partial order.

**Notation 4.** For each preorder \( \langle Q, \preceq \rangle \) we denote the partial order given in the previous proposition by \( \langle Q^*, \leq \rangle \).

As we will see, for our purposes we can use the partial order \( Q^* \) in place of any preorder \( Q \), so we now only consider partial orders.

**Notation 5.** Let \( \langle P, \leq \rangle \) be a partial order. Then we adopt the following notation similar to that commonly used for linear orders; for example:

\[
\begin{align*}
(\leftarrow, p) &= \prec p = \{ r \in P : r < p \} \\
[p, \rightarrow) &= p^\leq = \{ r \in P : r \geq p \} \\
(p, q] &= p^\leq \cap q^\leq \\
p^\lessdot &= P \setminus p^\leq
\end{align*}
\]

Of course, if a partial order is total, that is, \( p \leq q \) or \( q \leq p \) for all \( p, q \), then we have the usual idea of a total order or linear order, and e.g. \( \lessdot p = p^\prec \). A wide class of topological spaces known as LOTs are defined in terms of linear orders; we characterize them for all partial orders as follows.

**Definition 6.** A partially ordered topological space, or \( POTS \), is a topological space \( X \) partially ordered by \( \leq \) with a subbasis \( \{ \lessdot x : x \in X \} \cup \{ x^\lessdot : x \in X \} \).

The reader may verify that if \( \leq \) is a linear order, then this subbasis yields the usual basis of open intervals \{ \((x, y) : x, y \in X \cup \{\leftarrow, \rightarrow\}\)\}. However, the next example illustrates that a POTS may not even be Hausdorff (though it will always be \( T_1 \)).

**Example 7.** Let \( [\omega]^{<\aleph_0} = \{ F \subseteq \omega : F \text{ is finite} \} \) be partially ordered by \( \subseteq \).

We then let \( F \subseteq \omega \) be finite, and thus for any \( x \not\in F \), the singleton set \( \{ x \} \in F^\lessdot \). Consider now the subbasic open set \( F^\lessdot \). If \( F = \emptyset \) then \( F^\lessdot = \emptyset \) as well, and if \( F = \{ y \} \) then \( \{ x \} \in F^\lessdot \) for all \( x \in \omega \setminus \{ y \} \). Finally, if \(|F| > 1\), then \( \{ x \} \in F^\lessdot \) for all \( x \in \omega \). Thus, any two non-empty subbasic open sets have infinite intersection, showing \( \langle [\omega]^{<\aleph_0}, \subseteq \rangle \) cannot be Hausdorff.

To address this, Ward [6] gives the following definition for “continuous” partial orders; some authors [7] require this as a condition of being a “pospace.”

**Definition 8.** A POTS is said to be continuous if for each \( p \not\leq q \) there exist open sets \( U, V \) such that \( p \in U, q \in V \) and \( r \not\leq s \) for all \( r \in U, s \in V \).

Note that the definition immediately implies the Hausdorff property, and may be recharacterized as follows.

**Definition 9.** A subset \( A \) of a partial order \( \leq \) is said to be downward if for all \( p \in A, p^\lessdot \subseteq A \). A subset \( A \) of a partial order is said to be upward if for all \( p \in A, p^\leq \subseteq A \).
Lemma 10 ([6]). The following are equivalent for a given POTS $\langle P, \leq \rangle$.

- $P$ is continuous.
- For each $p \not\leq q$, there exists an upward open neighborhood $U$ of $p$ and a downward open neighborhood $V$ of $q$ such that $U, V$ are disjoint.

3 Mahavier Products with Partial Orders

We may define the Mahavier product as a subspace of the usual Tychonoff product. By convention, we will treat relations $f \subseteq X^2$ as set-valued functions, that is, $f(x) = \{y : \langle x, y \rangle \in f\}$.

Definition 11. Let $X$ be a topological space. A relation $f \subseteq X^2$ is said to be a $V$-relation if it is closed, idempotent ($f(f(x)) = f^2(x) = f(x)$), surjective ($\forall y \in X \exists x \in X(y \in f(x))$), and serial ($\forall x \in X \exists y \in X(y \in f(x))$).

Definition 12. Let $X$ be a topological space, $f \subseteq X^2$ be a $V$-relation, and $Q$ be a preorder. Then

$$M(X, f, Q) = \{x \in X^Q : x(p) \in f(x(q)) \text{ for all } p \leq q\}$$

is the Mahavier product, considered as a subspace of the usual Tychonoff product $X^Q$.

When $Q = \mathbb{N}$ and $f$ is singleton-valued, then $M(X, f, Q)$ is the standard topological inverse limit with bonding map $f$. This definition may be also contrasted with other interesting subspaces of $X^Q$ studied in general topology, such as the $\Sigma$-product. (We pose a question relating to compact subspaces of $\Sigma$-product, known as Corson compacta, at the end of the paper.)

In [5], the first author and Varagona showed the following.

Definition 13. A relation $f \subseteq X^2$ satisfies condition $\Gamma$ if there exist distinct $x, y \in X$ such that $\langle x, x \rangle, \langle x, y \rangle, \langle y, y \rangle \in f$.

Lemma 14. Let $X$ be weakly countably compact and $f \neq i = \{\langle x, x \rangle : x \in X\}$ (that is, $f$ is nontrivial) be a $V$-relation. Then $f$ satisfies condition $\Gamma$.

To this end, we will concentrate on the following particular Mahavier product.

Definition 15. Let $X$ contain distinct elements $0, 1$. Then $\gamma_X \subseteq X^2$ satisfies $\gamma_X(0) = X$ and $\gamma_X(x) = \{1\}$ otherwise.

Of course, if $X$ is $T_1$ then $\gamma_X$ is a $V$-relation that satisfies condition $\Gamma$. When $X$ is assumed from context, we simply write $\gamma$.

Proposition 16. Let $Y \subseteq X$ and $g \subseteq f$. Then $M(Y, g, Q) \subseteq M(X, f, Q)$.

In particular, let $2 \subseteq X$ and $f$ satisfy condition $\Gamma$ (that is, $\gamma = \gamma_2 \subseteq f$). Then $M(2, \gamma, Q) \subseteq M(X, f, Q)$.

Recalling our motivation, we want to characterize the separable metrizability of the Mahavier product $M(X, f, P)$ (for separable metrizable $X$ and nontrivial $f$ satisfying condition $\Gamma$). Since $M(X, f, P)$ is trivially separable and metrizable when $P$ is countable, we will consider the case for uncountable $P$. In particular, we will show that the subspace $M(2, \gamma, P)$ fails to be second-countable in this case.

First consider the following simplification for preorders that allows us to concentrate on partial orders. We use the symbol $\cong$ to denote homeomorphic spaces.
Proposition 17. Let $Q$ be preordered by $\le$. Then $M(2,\gamma,Q) \cong M(2,\gamma,Q^*)$.  

Proof. Note that for any $x \in M(2,\gamma,Q)$, if $x(p) = 1$ then $x(q) = 1$ for all $q \sim p$, due to $q \le p$. Likewise, if $x(p) = 0$, then $x(q) = 0$ for all $q \sim p$, due to $p \le q$. Thus $h : M(2,\gamma,Q) \to M(2,\gamma,Q^*)$ defined by $h(x)([p]) = x(p)$ (where $[p] \in Q^*$ is the equivalence class of $p \in Q$) is a well-defined homeomorphism. 

Many properties of $M(X,f,P)$ are inherited from its superspace $X^P$; for example, it is $T_3$ provided $X$ is. The following result shows that compactness is also preserved.

Proposition 18. Let $f \subseteq X^2$ be closed. Then $M(X,f,P)$ is a closed subspace of $X^P$. 

Proof. Let $l \not\in M(X,f,P)$. Then there exist $p < q$ such that $l(p) \not\in f(l(q))$. So choose open subsets $U,V$ of $X$ such that $(l(q),l(p)) \in U \times V \subseteq X^2 \setminus f$. Then $\prod_{r \in P} U_r$ defined by $U_p = V$, $U_q = U$, and $U_r = X$ otherwise, is an open neighborhood of $l$ and disjoint from $M(X,f,P)$.

Corollary 19. If $X$ is compact and $f$ is a $V$-relation, then $M(X,f,P)$ is compact.

4 Two illustrative examples

The following is an example where $X$ is a compact metrizable space, $P$ is a linear order, and $M(X,\gamma,P)$ is a familiar continuum (albeit non-metrizable, as will follow later from the fact that $P$ is uncountable).

Example 20. Let $X = [0,1]$ and $P = [0,1]$ with the usual (linear) order. Then $M(X,\gamma,P)$ is a copy of the lexicographic square.

Proof. The lexicographic square is given by $L = [0,1]^2$ where $(x,y) < (w,z)$ if and only if $x < w$, or both $x = w$ and $y < z$. This space is well-known to be non-metrizable (this follows from the fact that it is compact but non-separable).

For $f \in M(X,\gamma,P)$ let $x_f = \inf\{x : f(x) = 0\}$. It follows that $f(x) = 1$ for $x < x_f$ and $f(x) = 0$ for $x > x_f$, and $f = q$ if and only if $x_f = x_q$ and $f(x_f) = g(x_q)$.

So let $\theta : M(X,\gamma,P) \to L$ be defined by $\theta(f) = (x_f,f(x_f))$. It follows that $\theta$ is a bijection. We will show that $\theta$ is a homeomorphism.

To see this, consider a nonempty subbasic open set $U = \prod_{x \in [0,1]} U_x$ containing $f$, where each $U_x = [0,1]$ except for a single $x' \in X$. First assume $U_{x'} = (y',1]$. It follows that $f(x) = 1$ for all $x < x'$. We claim $\theta(U) = ((x',y'),\rightarrow)$.

To see this, first note that $\theta(f) > (x',y')$ follows from $x_f \ge x'$, and if $x_f = x'$ then $f(x_f) = f(x') > y'$. Therefore $\theta(U) \subseteq ((x',y'),\rightarrow)$.

Likewise if $(a,b) > (x',y')$, note $x' \le a$ and $x' = a \Rightarrow b > y'$, and define $g$ by

$$g(x) = \begin{cases} 1 & \text{if } x < a \\ b & \text{if } x = a \\ 0 & \text{if } x > a \end{cases}$$

Then $x_g = a$, and $\theta(g) = (x_g,g(x_g)) = (a,b)$. Finally $g \in U$ since $x' < a$ implies $g(x') = 1 \ge y'$ and $x' = a$ implies $g(x') = g(a) = b > y'$, thus $\theta(U) \supseteq ((x',y'),\rightarrow)$, completing the proof of our claim.

This argument may be adapted to show $U_{x'} = [0,y')$ implies $\theta[U] = ((-,y'),\rightarrow)$. Therefore, $\theta$ is an open continuous map, so $\theta$ is a homeomorphism. 

4
Figure 1: An element of $\mathbf{M} \langle [0, 1], \gamma, 2^{<\omega} \rangle$ with a typical neighborhood

The following example illustrates an interesting space produced as the Mahavier product of a non-linear partial order; since $P$ is countable, this is a metrizable subspace of $[0, 1]^P$.

**Example 21.** Let $2^{<\omega}$ be the Cantor tree of finite sequences of 0 and 1 ordered by extension. An illustration of a typical element $f \in \mathbf{M} \langle [0, 1], \gamma, 2^{<\omega} \rangle$ and its neighborhood is given in Figure 1. Each filled dot represents $s \in 2^{<\omega}$, where each 0 in the sequence moves northwest and each 1 in the sequence moves northeast. If the tree meets the dot for $s$ then $f(s) = 1$; if the tree meets $s$ but not $s \downarrow \langle n \rangle$ then $0 \leq f(s \downarrow \langle n \rangle) < 1$. A neighborhood allows for $\varepsilon$ error on each coordinate not determined by $\gamma$ (represented by the open dots and thinner lines).

5 Second-Countability and the POTS of Downward Subsets

In order to demonstrate that $\mathbf{M} \langle 2, \gamma, P \rangle$ is second-countable exactly when $P$ is countable, we will show that it is homeomorphic to a POTS derived from $P$.

Recall the well-known compactification $\hat{L} = \{ A \subseteq L : A$ is closed and downward $\}$ of a given LOTS $L$, that is, its Dedekind completion. The requirement that $A$ be closed is necessary for $\{ \leq l : l \in L \} \cong L$ to be dense in $\hat{L}$. But by dropping that requirement, we will obtain our desired copy of $\mathbf{M} \langle 2, \gamma, P \rangle$.

**Definition 22.** Let $P$ be a partial order. Then we denote the POTS of downward subsets by $\check{\mathbf{P}} = \{ A \subseteq P : A$ is downward $\}$ partially ordered by the subset relation $\subseteq$.

**Lemma 23.** Let $x \in P$, then $x^\preceq$ is downward and $\preceq x$ is upward.

**Proof.** To show $x^\preceq$ is downward, let $x \in P$, $b \in x^\preceq$ and $a < b$. We will show that $a \in x^\preceq$, that is, $a \npreceq x$. If $a \geq x$, then $x \preceq a < b$ implies $x \preceq b$ by transitivity, contradicting the fact that $b \in x^\preceq$. The proof that $\preceq x$ is upward is similar.

**Proposition 24.** Let $P$ be a partial order. Then $\langle \check{P}, \subseteq \rangle$ is a continuous POTS.

**Proof.** It is immediate that $\subseteq$ is a partial order. To see that it is continuous on $\check{P}$, we will verify the last bullet of Lemma 10. Let $X, Y \in \check{P}$ with $X \nsubseteq Y$ and $x \in X \setminus Y$. Note $x \preceq x^\preceq \in \check{P}$, and furthermore $X \in \check{\mathbf{M}} \langle x^\preceq \rangle$ since $x \in X$, and $Y \in \langle \subseteq x \rangle^\preceq$ since $x \notin Y$.
By the previous lemma these sets are upward and downward respectively. So we now show these two open sets are disjoint. If $A \in (\leq x)_{\mathbb{E}} \cap (\leq x)_{\mathbb{F}}$, then $A$ misses some some $a \leq x$ and contains some $b \geq x$. But since $b \geq x \geq a$, this means $A$ is not downward, a contradiction.

\begin{proof}

Consider Theorem 27. Let $\tilde{B} \in f$, then it follows that (since $A$ was earlier shown to be a bijection. We now consider $\gamma, R$ for some $r \in R$, therefore $A \in B (\{r\}, 0) \subseteq R_{\mathbb{E}}$. Similarly, if $A \in R_{\mathbb{F}}$, then there exists some $r \in R$ with $r \notin A$, showing that $A \in B (0, \{r\}) \subseteq R_{\mathbb{F}}$. Therefore $R_{\mathbb{E}}$ and $R_{\mathbb{F}}$ are open in the topology generated by the sets $B (T, F)$. \hfill \Box

\begin{corollary}

$\tilde{P} \cong M (2, \gamma, P)$.

\begin{proof}

The map $h : \tilde{P} \to M (2, \gamma, P)$ defined by

\[
h(A)(p) = \begin{cases} 
0 & p \notin A \\
1 & p \in A 
\end{cases}
\]

was earlier shown to be a bijection. We now consider $h [B (T, F)]$. As $T \subseteq A$ and $F \cap A = \emptyset$ for all $A \in B (T, F)$, $h$ will map the set $B (T, F)$ to the set of sequences which are restricted to $\{1\}$ on $T$, and $\{0\}$ on $F$.

Thus, let $U_p = \{1\}$ if $p \in T$, $U_p = \{0\}$ if $p \in F$, and $U_p = 2$ otherwise. Then $h[B (T, F)] = \prod_{p \in B} U_p$, showing $h$ is an open map. This also shows that $h$ is continuous; given nonempty $\prod_{p \in B} U_p$, let $T = \{p \in P : U_p = \{1\}\}$ and $F = \{p \in P : U_p = \{0\}\}$; then $h[B (T, F)] = \prod_{p \in B} U_p$. \hfill \Box

\end{proof}

\end{corollary}

\end{proof}
Due to this homeomorphism, the following theorem completes our characterization of separable/compact metrizable $M\langle X, f, P \rangle$.

**Theorem 29.** $\hat{P}$ is second-countable if and only if $P$ is countable.

**Proof.** If $P$ is countable, $2P$ is second-countable, and therefore its subspace $M\langle 2, \gamma, P \rangle \cong \hat{P}$ is second-countable.

Suppose $P$ is uncountable and $\mathcal{B}$ is a basis for $\hat{P}$. Then for all $p \in P$, fix $B_p \in \mathcal{B}$ such that $\hat{z}p \in B_p \subseteq B\langle \{p\}, \emptyset \rangle$. Now let $p \neq q$, and without loss of generality assume $p \preceq q$. Therefore, $\hat{z}q \notin B\langle \{p\}, \emptyset \rangle$, and so $q \preceq \notin B_p$. By construction we have $q \preceq \in B_q$, and so it must be that $B_p \neq B_q$. Thus, each $B_p$ is unique, and so $\mathcal{B}$ must be uncountable, showing that $\hat{P}$ is not second-countable.

**Corollary 30.** Let $X$ be separable metrizable, $f$ satisfy condition $\Gamma$, and $P$ be a partial order. Then $M\langle X, f, P \rangle$ is separable metrizable if and only if $P$ is countable.

**Proof.** If $P$ is countable, $X^P$ is separable metrizable, and therefore its subspace $M\langle X, f, P \rangle$ is separable metrizable.

If $P$ is uncountable, $M\langle X, f, P \rangle$ contains a non-second-countable subspace $\hat{P}$, and therefore cannot be second-countable itself.

Note that Corollary 30 is a generalization of [5, Theorem 5.1], as $\hat{\alpha} = \alpha + 1$ for all ordinals $\alpha$. We also have the following.

**Corollary 31.** Let $X$ be compact metrizable, $f$ satisfy condition $\Gamma$, and $P$ be a partial order. Then $M\langle X, f, P \rangle$ is compact metrizable if and only if $P$ is countable.

Of course these corollaries cannot extend to preorders $(Q, \preceq)$: if $p \preceq q$ and $q \preceq p$ for all $p, q \in Q$, then $M\langle 2, \gamma, Q \rangle \cong 2$ regardless of the cardinality of $Q$. In [5] it was also observed that the Corson compactness of $M\langle X, f, \alpha \rangle$ was linked to the countability of the ordinal $\alpha$ (as the same holds for $\hat{\alpha} = \alpha + 1$). But the following generalization remains open.

**Question 32.** Let $P$ be a partial (or linear) order. Is $\hat{P}$ Corson compact if and only if $P$ is countable?

If so, then for Corson compact $X$ and $f$ satisfying condition $\Gamma$, $M\langle X, f, P \rangle$ would be Corson compact if and only if $P$ is countable.

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