Distributed Zeroth-Order Stochastic Optimization in Time-varying Networks

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Abstract

We consider a distributed convex optimization problem in a network which is time-varying and not always strongly connected. The local cost function of each node is affected by some stochastic process. All nodes of the network collaborate to minimize the average of their local cost functions. The major challenge of our work is that the gradient of cost functions is supposed to be unavailable and has to be estimated only based on the numerical observation of cost functions. Such problem is known as zeroth-order stochastic convex optimization (ZOSCO). In this paper we take a first step towards the distributed optimization problem with a ZOSCO setting. The proposed algorithm contains two basic steps at each iteration: i) each unit updates a local variable according to a random perturbation based single point gradient estimator of its own local cost function; ii) each unit exchange its local variable with its direct neighbors and then perform a weighted average. In the situation where the cost function is smooth and strongly convex, our attainable optimization error is $O\left(\frac{1}{T}\right)$ after $T$ iterations. This result is interesting as $O\left(\frac{1}{T}\right)$ is the optimal convergence rate in the ZOSCO problem. We have also investigate the optimization error with the general Lipschitz convex function, the result is $O\left(\frac{1}{T^{1/4}}\right)$.

I. INTRODUCTION

Distributed optimization is an attractive problem in various domain. A set of distributed nodes need to minimize some global cost function, while each node can only have access to information of local cost function. For example, in the Internet of Things applications, each device collects local data to serve the whole network. Most of the existing work in this area requires the knowledge of gradient information.

In this work, we consider the distributed optimization problem with more realistic yet challenging settings: 1) the network is time-varying and not always strongly connected; 2) the local cost functions are disturbed by some stochastic process; 3) the gradient information is unavailable and each node is only able to access to a numerical value of its local function at each time. Such problem belongs to zeroth-order stochastic convex optimization (ZOSCO), it can be also named as derivative-free stochastic convex optimization or bandit convex optimization. Using the numerical value of objective function, one can only has some biased estimation of its gradient. It is thus not surprising that there is a fundamental performance gap between ZOSCO and Stochastic Approximation (SA) [17], as in SA an unbiased gradient estimation is available. Several lower bounds of the convergence rate have been derived in [9], [23], [4] to state that, without the knowledge of gradient, the optimization error cannot be better than $O\left(T^{-0.5}\right)$ after $T$ iterations. Note that when the gradient is available, the convergence rate can be as fast as $O\left(T^{-1}\right)$ given that the objective function is strongly concave [17]. Recently, ZOSCO is receiving an increasing interest in various applications. For example, [3] considered the management of fog computing in IoT, i.e., nodes in the fog layer need to collaboratively process some data requests to minimize the processing delay. Due to the unpredictable network congestion, the closed form expression of delay can not be available. As a result, one can only apply ZOSCO. An example related to sensor selection for parameter estimation was considered in [13]. The main goal is to find the optimal tradeoff between sensor activations and the estimation accuracy. In order to evaluate the gradient of the objective function, the calculation of inverse matrices is necessary, while a zeroth-order method proposed [13] helps to avoid matrix inversion.

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Zeroth-order methods are also interesting for solving adversarial machine learning problem, see [14] as an example.

Although both distributed optimization and ZOSCO are popular problems in recent years, the distributed optimization problem in a ZOSCO setting is rarely investigated to the best our knowledge. In this work, we make a first step towards such problem. Similar to the typical distributed optimization algorithm, our algorithm contains a gradient step and a communication step in each iteration. While the major difference is that, in our gradient step, each node has to estimate the gradient of its local function by using a single numerical value of the function.

We consider a time-varying and not always strongly connected network and derive the convergence rate of the proposed algorithm. We have obtained a $O(T^{-0.5})$ convergence rate of the optimization error, under the assumption that the global function is strongly convex and each local function is smooth. This result is remarkable as it is the same as the lower bound of the convergence rate in the centralized ZOSCO problem. We have also considered the case with general Lipschitz convex function (non smooth and non strongly convex), the resulted attainable convergence rate is $O(T^{-0.25})$. While similar assumptions related to network graph have been considered in [16], the major difference is that gradient was supposed to be available in [16]. The main challenge in our work comes from the fact that our zeroth-order approach leads to biased estimation of the gradient and therefore we have to analyze an additional term associated to the gradient estimation bias.

**Related work.** There exist significant amount of work related to distributed optimization aiming at improving convergence rate, while most of the work consider the direct availability of gradient information, e.g., [10], [22], [21]. ZOSCO or bandit convex optimization problem has also attracted much interests recently. Several works have based on two-point gradient estimator (TPGE) [1], [4], [15], [6], [18], which requires two successive observation of the cost function $F(\theta_t; \xi_t)$ and $F(\theta_t'; \xi_t)$ under the same stochastic parameter $\xi_t$. Several recent work [24], [20], [19], [6] have applied such TPGE to address distributed zeroth-order convex optimization. However, TPGE cannot be realized in the situation where the value of $\xi_t$ change fast (e.g., in an i.i.d. manner), as it is impossible to observe both $F(\theta_t; \xi_t)$ and $F(\theta_t'; \xi_t)$. Such a situation arises widely in various applications in practice, e.g. in wireless networks. In this work, we focus on single point gradient estimator, which has been barely addressed in distributed ZOSCO problem. In fact, it is possible to have a biased estimation of gradient by using a single realization of the cost function [5]. In [23], the gradient estimator is shown to be unbiased for quadratic functions, while it is a biased estimation for general nonlinear function. In the framework of ZOSCO, there exist several work that provide advanced methods trying to achieve the optimal convergence rate for general convex functions [7], [8], [2], however, within a centralized setting as the proposed algorithms contains operations of vectors and matrices which requires to be handled by a central controller.

In our previous work [11], [10], [12], we considered a different framework where each node controls its own action to perform ZOSCO. The network model was also different: each node has a constant probability to communicate with any other node of the network. However, in this work, each node aim to optimize a common vector and we consider a time-varying communication matrix between the nodes.

The rest of the paper is organized as follows. Section II describes the problem setting and the basic assumptions. Section III presents the proposed distributed optimization algorithm. In Section IV, we provide the convergence rate of the proposed algorithm for smooth and strongly-convex function. Whereas general Lipschitz convex function is considered in Section V. Section VI concludes this paper.

**II. Problem setting and assumptions**

In this paper, matrices are in boldface upper case letters and vectors are in boldface lower case letters. Calligraphic font denotes set. We denote by $\|x\|$ the Euclidean norm of any vector $x$. $I_M$ denotes a identity matrix of size $M \times M$ and $0_M$ denotes a zero matrix of size $M \times M$. 

A. Problem setting

Consider a set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and a time-varying directed graph $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$ at any timeslot $t$. The edge set $\mathcal{E}_t$ denotes a collection of pairs having direct communication. We can introduce a time-varying communication matrix $A(t) = [A_{i,j}(t)]_{i,j \in \mathcal{V}}$, with $A_{i,j}(t) \equiv (0,1)$ only if $(i,j) \in \mathcal{E}_t$, otherwise $A_{i,j}(t) = 0, \forall (i,j) \notin \mathcal{E}_t$.

For any $\theta \in \mathcal{K} \subseteq \mathbb{R}^M$ and some random vector $\xi_i \in S_i$, denote $F_i(\theta; \xi_i) : \mathcal{K} \times S_i \rightarrow \mathbb{R}$ as the local cost function of node $i$. Note that we use $\xi_i$ to describe some non-additive stochastic process, which is assumed to be i.i.d. and ergodic. Introduce the expected local cost function $f_i(\theta) = \mathbb{E}_\xi [F_i(\theta; \xi_i)]$. The problem is to find the optimal value of $\theta$ to minimize the average cost function, i.e.,

$$
\min_{\theta \in \mathcal{K}} f(\theta) = \frac{1}{N} \sum_{i=1}^{N} f_i(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_\xi [F_i(\theta; \xi_i)],
$$

under the challenging condition that:

1) each node $i$ only knows the numerical value of $F_i$ rather then the closed form expression of $F_i$;

2) each node $i$ can only communicate with its neighbors at each time, the global information of the network is not available.

We denote the minimizer of $f(\theta)$ as $\theta^*$, i.e.,

$$
\theta^* = \arg \min_{\theta \in \mathcal{K}} f(\theta) = \frac{1}{N} \sum_{i=1}^{N} f_i(\theta).
$$

A typical strategy to solve the distributed optimization problem (1) is to make each node $i$ update a local variable $\theta_i$ aiming to minimize its local cost function, then nodes need communication to have an agreement of their local variables. An alternative way to write problem (1) is the following

$$
\min_{\theta_1, \ldots, \theta_N \in \mathcal{K}, N} \frac{1}{N} \sum_{i=1}^{N} f_i(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_\xi [F_i(\theta; \xi_i)],
$$

s.t. $\theta_1 = \theta_2 = \ldots = \theta_N$.

(3)

B. Assumptions

In this section we present the assumptions that will be considered in this paper. In terms of the communication matrix, we assume that:

A1: i). $A(t)$ is doubly-stochastic, i.e., $\sum_{j \in \mathcal{N}} A_{ij}(t) = \sum_{j \in \mathcal{N}} A_{ji}(t) = 1, \forall i \in \mathcal{N}$; ii). every non-zero element of $A(t)$ is always equal or larger than a constant $\alpha > 0$; iii). two arbitrary nodes can directly communicate with each other at least once every $\tau > 0$ consecutive time slots, in other words, $(\mathcal{V}, \bigcup_{t=t_0}^{t_0+\tau-1} \mathcal{E}_t)$ is strongly connected for any $t_0 \geq 0$.

Throughout this paper, we have the following assumptions in terms of the basic properties of cost function.

A2: The feasible set $\mathcal{K}$ is bounded, convex, and contains the zero point 0. For any $\theta \in \mathcal{K}$, we have $\|\theta\| \leq \bar{R} < +\infty$.

A3: The value of $F_i$ is bounded, we have $|F_i(\theta; \xi_i)| \leq C < +\infty$ for any $i \in \mathcal{V}$, $\theta \in \mathcal{K}$, and $\xi_i \in S_i$.

Specifically, in Section IV, we perform the analysis assuming that the cost functions are smooth and strongly convex as described in what follows.

A4: The expected global cost function $f(\theta)$ is $\mu-$strongly convex. For any $i \in \mathcal{V}$, $f_i(\theta)$ is $L_i$-smooth, i.e.,

$$
\|\nabla f_i(\theta) - \nabla f_i(\theta')\| \leq L_i \|\theta - \theta'\|, \quad \forall \theta, \theta' \in \mathcal{K}, \forall i \in \mathcal{V},
$$

denote $L = \frac{1}{N} \sum_{i=1}^{N} L_i$. 


We have also considered a more general situation in Section $\text{V}$ that each local cost function is Lipschitz:

\textbf{A5}: For any $i \in \mathcal{V}$, $f_i(\theta)$ is $\ell_i$-smooth, i.e.,
\[ \|f_i(\theta) - f_i(\theta')\| \leq \ell_i \|\theta - \theta'\|, \quad \forall \theta, \theta' \in \mathcal{K}, \forall i \in \mathcal{V}. \]

III. DISTRIBUTED ZERO-ORDER STOCHASTIC OPTIMIZATION ALGORITHM

In this section, we first introduce a general single point gradient estimator and then present a simple algorithm to solve the distributed optimization problem as well as its fundamental properties.

A. A general single point gradient estimator

We consider a gradient estimator which is based on a single point evaluation of local cost function. At each time $t$, node $i$ estimates the gradient of its local cost function with
\[ \hat{g}_i(t) = \frac{\nu_{i,t} f_i(\theta_i(t) + \beta_t \nu_{i,t}; \xi_{i,t})}{\beta_t}, \quad (5) \]
where $\beta_t > 0$ represents the step-size and $\nu_{i,t} \in [-V, V]^M \subseteq \mathbb{R}^M$ is a random perturbation vector which is independently generated by each node. Two different settings will be considered in Sections $\text{IV}$ and $\text{V}$ respectively.

\textbf{S1}: For any $i \in \mathcal{V}$ and $t \in \{1, \ldots, T\}$, $\|\nu_{i,t}\| \leq \sqrt{M} V < +\infty$, each element of $\nu_{i,t}$ is i.i.d. such that $\mathbb{E}[\nu_{i,t}] = 0$ and $\mathbb{E}[\nu_{i,t} \nu_{i,t}^*] = \mathbf{I}_M$.

\textbf{S2}: For any $i \in \mathcal{V}$ and $t \in \{1, \ldots, T\}$, $\nu_{i,t}$ is a random unit vector such that $\mathbb{E}[\nu_{i,t}] = 0$ and $\|\nu_{i,t}\| = 1$.

Clearly, S1 is a relaxed setting compared with S2: $\nu_{i,t}$ has to be a unit vector in S2, which is not the case in S1. Providing a single point gradient estimator under S1 is an interesting extension of ZOSCO, which can be useful also in some other frameworks where the perturbation vector has to be generated in a distributed manner, i.e., each distributed node only generates some coordinate of the vector and the resulted entire perturbation vector cannot be unit without a control center.

In what follows, we can show that $\hat{g}_i(t)$ is a reasonable estimator of $\nabla f_i$ with either S1 or S2, as stated in Lemma $[1]$ and Lemma $[2]$.

\textbf{Lemma 1.} Suppose that the random perturbation vector $\nu_{i,t}$ is generated according to S1, then under Assumption A4, we can derive that
\[ \nabla f_i(\theta_i(t)) = \mathbb{E}[\hat{g}_i(t) | \theta_i(t)] - b_i(t), \quad (6) \]
where $b_i(t)$ represents an estimation bias of the gradient estimator with the following property
\[ \|b_i(t)\| \leq \frac{1}{2} M^2 \gamma^2 L_i \beta_t = O(\beta_t) \quad \forall i \in \mathcal{V}, t \in \mathbb{N}^+ \quad (7) \]

Lemma $[1]$ can be proved by applying Taylor’s theorem and mean value theorem, the proof detail can be found in Appendix $[?]$. In general, the gradient estimator $\hat{g}_i(t)$ introduces a non-zero bias term $b_i(t)$. Nevertheless, we can see from Lemma $[1]$ that $\|b_i(t)\|$ decreases with the step-size $\beta_t$, if $\beta_t$ is vanishing. Lemma $[1]$ will be a basis of our analysis concerning smooth and strongly convex functions in Section $\text{IV}$.

\textbf{Lemma 2.} Suppose that $\nu_{i,t}$ is generated according to S2, introduce a smoothed version of function $f_i$ for any $i \in \mathcal{N}$, i.e.,
\[ \tilde{f}_i(\theta) = \mathbb{E}_{\nu \in \mathbb{R}^M: \|\nu\| \leq 1} [f_i(\theta + \beta \nu)], \]
then we have
\[ \frac{\beta}{M} \nabla \tilde{f}_i(\theta) = \mathbb{E}_{\nu_{i,t} \in \mathbb{R}^M: \|\nu_{i,t}\| = 1} [f_i(\theta + \beta \nu_{i,t}) \nu_{i,t}] = \mathbb{E} [\hat{g}_i(t) | \theta]. \]

Lemma $2$ is a fundamental result presented in $[5]$, it states that the gradient estimator $\hat{g}_i(t)$ is in fact the exact gradient of a smoothed version of $f_i$ when $\nu_{i,t}$ satisfies S2. Our analysis related to general Lipschitz convex functions will rely on Lemma $[2]$. 


B. Distributed Algorithm

By using the gradient estimator $\hat{g}_i(t)$ as defined in (5), at time $t$ each node $i$ needs to observe the value of $F_i$ at the point $\theta_i(t) + \beta_t \nu_{i,t}$. In order to have $\theta_i(t) + \beta_t \nu_{i,t} \in \mathcal{K}$, the value of $\theta_i(t)$ should belong to the set

$$\mathcal{K}_t = \{ x \in \mathbb{R}^M : x + \beta_t V 1 \in \mathcal{K} \},$$

where $1$ denotes a vector of ones with dimension $M \times 1$. Recall that $\nu_{i,t} \in [-V, V]^M$. A simple solution is to use the projection operator defined as

$$\Pi_{\mathcal{K}_t}(x) = \arg \min_{x' \in \mathcal{K}_t} \| x - x' \|, \quad \forall x \in \mathbb{R}^M.$$  \hspace{1cm} (8)

at each time $t$. The distributed algorithm considered in this paper is given by

$$\theta_i(t + 1) = \Pi_{\mathcal{K}_{t+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right),$$ \hspace{1cm} (9)

where $\alpha_t > 0$ is another step-size. In order to make the algorithm converge with a fast speed, both $\alpha_t$ and $\beta_t$ should be chosen wisely. We will provide a detailed discussion in the next sections. The basic idea of (9) is to make each node update $\theta_i(t)$ according to the gradient estimator of its local cost function, as well as the weighted sum of $\theta_j(t)$ from neighbors. The detailed algorithm is described in Algorithm 1.

**Algorithm 1** D-ZOSCO Algorithm for each node $i$

1) Initialize $t = 1$ and $\theta_i(t) = 0$.
2) Generate a random perturbation vector $\nu_{i,t}$ and observe the value of local cost function $F_i$ at point $\theta_i(t) + \beta_t \nu_{i,t}$.
3) Update the gradient estimator $\hat{g}_i(t) = \nu_{i,t} F_i(\theta_i(t) + \beta_t \nu_{i,t}; \xi_{i,t}) / \beta_t$.
4) Communicate with all neighbors, i.e., broadcast the value of $\theta_i(t)$ and receive $\theta_j(t)$ from each neighboring node $j$ (with $A_{i,j}(t) > 0$). Compute the weighted average $\sum_{j=1}^{N} A_{i,j}(t) \theta_j(t)$.
5) Update $\theta_i(t + 1)$ according to (9), i.e., $\theta_i(t + 1) = \Pi_{\mathcal{K}_{t+1}}(\sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t))$.
6) $t = t + 1$, go to 2.

We introduce an auxiliary variable

$$\overline{\theta}(t) = \frac{1}{N} \sum_{i=1}^{N} \theta_i(t),$$

which is the average of local variables $\theta_i(t)$ at time $t$. Since an essential objective of our distributed algorithm is to make $\theta_1(t) \approx \ldots \approx \theta_N(t)$ according to the problem setting (4), it is desirable to have $\theta_i(t) \rightarrow \overline{\theta}(t)$ for any $i \in N$ as $t$ is large. The following lemma presents an upper bound of $\| \theta_i(t) - \overline{\theta}(t) \|$, from which we can see the disagreement of local variables $\theta_i(t)$ from a quantitative point of view.

**Lemma 3.** Suppose that Assumptions A1, A2, A3 hold, besides $\| \nu_{i,t} \| \leq \sqrt{MV}$ and the step-sizes are vanishing such that $\alpha_t = \alpha_0 t^{-c_1}$, $\beta_t = \beta_0 t^{-c_2}$, and $\alpha_t / \beta_t = \alpha_0 \beta_0 t^{-c_3}$, then we have $\| \theta_i(t) - \overline{\theta}(t) \| \leq \delta_t$ with

$$\delta_t = \left(2^{c_1 + c_2} + N \rho \left(1 + \frac{c_3}{1 - \eta} \right) \right) \sqrt{MV C \alpha_t / \beta_t},$$ \hspace{1cm} (10)

where $\rho = 2^{1 + \frac{N-1}{N-1}}$ and $\eta = (1 - a(N-1)r)^{(N-1)r}$. If $\| \nu_{i,t} \| = 1$ and the step-sizes are constant such that $\alpha_t = \alpha$ and $\beta_t = \beta$, then we have

$$\| \theta_i(t) - \overline{\theta}(t) \| \leq \delta = \left(2 + N \rho \frac{\eta}{1 - \eta} \right) C \frac{\alpha}{\beta}.$$ \hspace{1cm} (11)

The proof detail is provided in Appendix ??.
IV. Convergence rate for smooth strongly convex function

A. Main result

In this section, we present the convergence rate of average optimization error for a special class of convex function.

Denote \( \bar{\theta}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \theta_i(t) / T \). The aim of this section is to find an upper bound for the average optimization error, defined as \( \mathbb{E}[f(\bar{\theta}^{(T)})] - f(\theta^*) \).

**Theorem 4.** Suppose that Assumptions A1-A4 hold, \( \theta^* \in \mathcal{K}_0 \) and each node \( i \) updates \( \theta_i(t) \) according to (9) with perturbation vector \( \nu_{i,t} \) satisfying S1 and step-sizes \( \alpha_t = \frac{3}{2} t^{-1} \) and \( \beta_t = \left( 6\lambda_2^{-1}\lambda_3 \right)^{\frac{1}{3}} t^{-\frac{1}{3}} \), then

\[
\mathbb{E} \left[ f \left( \bar{\theta}^{(T)} \right) - f(\theta^*) \right] \leq \frac{L}{N} \Psi^2 \left( \frac{1}{\sqrt{T}} + \frac{\lambda_2}{\mu \sqrt{6\lambda_3}} \log \left( \frac{T}{\beta} \right) \right) + \left( \frac{\lambda_2}{\mu \sqrt{6\lambda_3}} + \frac{\lambda_2^2}{4\mu^2 \lambda_3} \right) \frac{1}{T} 
\]

(12)

where the constant terms are defined as

\[
\begin{align*}
\lambda_1 &= \sqrt{NM^2V^2L} \\
\lambda_2 &= 2L\sqrt{NMVC} \left( 2^{\frac{2}{3}} + N \rho \left( \frac{10 - 7\eta}{4(1-\eta)^2} \right) \right) \\
\lambda_3 &= 2NMVC^2 \left( 2^{\frac{2}{3}} + N \rho \left( \frac{10 - 7\eta}{4(1-\eta)^2} \right)^2 + 1 \right) \\
\Psi &= \max \left\{ 4\sqrt{6NR^2} \left( \lambda_1 + \frac{\lambda_2 \lambda_3}{10 \sqrt{\lambda_3} \lambda_2^2} \right)^{-2} \right\}
\end{align*}
\]

Note that we introduce the constant terms \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) mainly to lighten the notations. We explain the basic steps of our analysis in the rest of this section.

B. Proof sketch of Theorem 4

Thanks to the convexity and smoothness of the cost functions, we can show that (with proof detail in Appendix ??)

\[
\mathbb{E} \left[ f \left( \bar{\theta}^{(T)} \right) - f(\theta^*) \right] \leq \frac{L}{2NT} \sum_{t=1}^{T} \mathbb{E} \left[ \sum_{i=1}^{N} \left\| \theta_i(t) - \theta^* \right\|^2 \right] 
\]

(13)

from which we find that the convergence speed of \( \mathbb{E}[f(\bar{\theta}^{(T)})] - f(\theta^*) \) is the same as that of the average divergence defined as

\[
d_t = \mathbb{E} \left[ \sum_{i=1}^{N} \left\| \theta_i(t) - \theta^* \right\|^2 \right].
\]

(14)

Thus we can focus on the upper bound of \( d_t \). Our first step is to find the relation between \( d_{t+1} \) and \( d_t \), as described in the following lemma.

**Lemma 5.** Suppose that Assumptions A1-A4 hold, \( \theta^* \in \mathcal{K}_0 \), and each node \( i \) updates \( \theta_i(t) \) according to (9), and \( \alpha_t = \alpha_0 t^{-1}, \beta_t = \beta_0 t^{-\frac{1}{3}} \). Consider the constants \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) as defined in Theorem 4, we have

\[
d_{t+1} \leq (1 - 2\mu \alpha_t) d_t + \left( \lambda_1 + \lambda_2 \frac{\alpha_t}{\beta_t^2} \right) \alpha_t \beta_t \sqrt{d_t} + \lambda_3 \frac{\alpha_t^2}{\beta_t^2}
\]

(15)

The proof of Lemma 5 uses the results presented in Lemma 1 and Lemma 3, we need to investigate both \( \| \bar{\theta}(t) - \theta_i(t) \| \) and \( \| \bar{\theta}(t) - \theta^* \| \). Please refer to Appendix ?? for the proof details.
Based on Lemma 5, the next step is to show the upper bound of $d_t$ by using induction.

**Lemma 6.** Consider $\alpha_t = \alpha_0 t^{-1}$ and $\beta_t = \beta_0 t^{-\frac{1}{2}}$. Suppose that Assumptions A1-A6 hold, $\theta^* \in K_0$, and $\alpha_0 > \frac{3}{4\mu}$. Introduce $t_0 = \lceil 2\mu \alpha_0 \rceil$, then

$$d_t \leq \Psi \left( \lambda_1 + \lambda_{11} \frac{\alpha_0 \beta_0}{\beta_0^2} t^{-\frac{1}{2}} \right)^2 t^{-\frac{1}{2}}, \quad \forall t \geq 1$$

with the constant term

$$\Psi = \max \left\{ \frac{4NR^2 t_0}{\left( \lambda_1 + \lambda_{11} \frac{\alpha_0 \beta_0}{\beta_0^2} t_0^{-\frac{1}{2}} \right)^2}, \left( \frac{\alpha_0 \beta_0 + \sqrt{\frac{\alpha_0^2 \beta_0^2}{4}} + 2(4\mu \alpha_0 - 3) \left( \frac{2 \lambda_{11}^2 \beta_0^2}{\alpha_0 \beta_0} \right)}{4\mu \alpha_0 - 3} \right)^2 \right\}$$

The proof of Lemma 6 can be found in Appendix ???. We find that it is possible to have $d_t = O \left( t^{-\frac{1}{2}} \right)$. The constant term $\Psi$ is a function of $\alpha_0$ and $\beta_0$ with complicated form, it is necessary to properly choose $\alpha_0$ and $\beta_0$ to make $\Psi$ as tight as possible. The following lemma provides a reasonable solution. The proof detail is provided in Appendix ???.

**Lemma 7.** If $\alpha_0 > \frac{3}{4\mu}$ and $\beta_0 > 0$, we have

$$\frac{\alpha_0 \beta_0 + \sqrt{\frac{\alpha_0^2 \beta_0^2}{4}} + 2(4\mu \alpha_0 - 3) \left( \frac{2 \lambda_{11}^2 \beta_0^2}{\alpha_0 \beta_0} \right)}{4\mu \alpha_0 - 3} \geq \left( \frac{6}{\lambda_1^2 \lambda_{11}} \right)^{\frac{1}{2}} \mu^{-1}$$

with equality as $\alpha_0^* = \frac{3}{\mu}$ and $\beta_0^* = \left( \frac{6}{\lambda_1^2 \lambda_{11}} \right)^{\frac{1}{2}}$.

We can introduce the values of $\alpha_0^*$ and $\beta_0^*$ into Lemma 6 to get

$$d_t \leq \Psi^* \left( \lambda_1 + \frac{3\lambda_1 \lambda_{11}}{\mu \sqrt{6\lambda_{11}}} t^{-\frac{1}{2}} \right)^2 t^{-\frac{1}{2}}$$

with $\Psi^*$ is defined in Theorem 4.

Finally, we can turn to evaluate the average optimization error $\mathbb{E}[f(\bar{\theta}^{(T)}) - f(\theta^*)]$ by using (13) and the upper bound of $d_t$, i.e.,

$$\mathbb{E} \left[ f \left( \bar{\theta}^{(T)} \right) - f \left( \theta^* \right) \right] \leq \frac{L}{2NT} \sum_{t=1}^{T} d_t \leq \frac{L}{2NT} \Psi^* \lambda_1^2 \sum_{t=1}^{T} \left( t^{-\frac{1}{2}} + \frac{\sqrt{6} \lambda_{11}}{2 \mu \sqrt{\lambda_{11}}} \frac{1}{T} \right) \leq \frac{L}{N} \Psi^* \lambda_1^2 \left( \frac{1}{\sqrt{T}} + \frac{\sqrt{6} \lambda_{11}}{2 \mu \sqrt{\lambda_{11}}} \frac{\log(T)}{T} + \frac{9 \lambda_{11}^2}{4 \mu^2 \lambda_{11}} \frac{1}{T} \right),$$

which is obtained by using the bounds $\sum_{t=1}^{T} t^{-\frac{1}{2}} < \int_{x=0}^{T} x^{-\frac{1}{2}} dx = 2\sqrt{T}$, $\sum_{t=1}^{T} t^{-1} < 1 + \int_{x=1}^{T} x^{-1} dx = \log(T) + 1$, and $\sum_{t=1}^{T} t^{-\frac{1}{2}} < 1 + \int_{x=1}^{\infty} x^{-\frac{1}{2}} dx = 3$.

**C. Discussion**

In this section, we have shown that the average optimization error after $T$ iterations can be $O(T^{-1/2})$ for smooth and strongly convex function. Such result is nice as the optimal convergence rate for a centralized ZOSCO problem is also $O(T^{-1/2})$. Recall that the challenge of our problem not only comes from the ZOSCO setting, but also caused by the imperfect network topology: in a distributed setting, a node only has the local information of the global function. As derived in Lemma 6 and Lemma 7 the estimation bias of gradient decreases with $\beta_t = O(t^{-1/4})$, while $\| \bar{\theta}(t) - \theta^* \|$ caused by communication decreases with a faster speed $\frac{\alpha_t}{\beta_t} = O(t^{-3/4})$. This explains that the convergence rate of our problem is more sensitive to the performance of the biased gradient estimator.
V. CONVERGENCE RATE FOR LIPSCHITZ CONVEX FUNCTION

In this section, we consider a more general case where \( f \) is neither smooth nor strongly convex. In the situation where the cost functions are non-smooth, we have to use the random unit perturbation vector to ensure that the gradient estimator works well.

Our main result is stated as follows.

**Theorem 8.** Suppose that Assumptions A1, A2, A3, A5 hold, and the perturbation vector satisfy S2. If each node \( i \) updates \( \theta_i(t) \) according to (9) with constant step-sizes

\[
\begin{aligned}
\alpha_t &= \alpha^* = \sqrt{\frac{2\sqrt{M}R^3}{C T (7 + 5N \rho \frac{\eta}{1 - \eta} + (N \rho \frac{\eta}{1 - \eta})^2) T^{-\frac{3}{4}}}} \\
\beta_t &= \beta^* = \sqrt{\frac{M \rho C}{\ell} \sqrt{14 + 10N \rho \frac{\eta}{1 - \eta} + 2 \left( N \rho \frac{\eta}{1 - \eta} \right)^2 T^{-\frac{3}{4}}}}
\end{aligned}
\]

and \( \theta^* \in \tilde{\mathcal{K}} = \{ \theta \in \mathcal{K} : \theta + \beta \nu \in \mathcal{K}, \forall \nu \text{ s.t. } ||\nu|| = 1 \} \), then the average optimization error is bounded by

\[
\mathbb{E} \left[ f \left( \overline{\theta}^T \right) - f \left( \theta^* \right) \right] 
\leq 4 \sqrt{M \rho C} \sqrt{14 + 10N \rho \frac{\eta}{1 - \eta} + 2 \left( N \rho \frac{\eta}{1 - \eta} \right)^2 T^{-\frac{3}{4}}} + \sqrt{2} R \ell T^{-\frac{1}{2}}
\]

\[
= O \left( \sqrt{MT}^{-\frac{3}{4}} \right) + O \left( T^{-\frac{1}{2}} \right)
\]

The proof sketch of Theorem 8 is presented in the rest of this section.

**A. Proof sketch of Theorem 8**

The proof of Theorem 8 is based on the application of Lemma 2, which states that our gradient estimator is the exact gradient of a smoothed version of the objective function

\[
\tilde{f}(\theta) = \frac{1}{N} \sum_{i \in N} \tilde{f}_i(\theta) = \frac{1}{N} \sum_{i \in N} \mathbb{E}_{\nu \in \mathbb{R}^M : ||\nu|| \leq 1} [f_i(\theta + \beta \nu)]
\]

with \( \theta \in \tilde{\mathcal{K}} \subseteq \mathcal{K} \). It is worth mentioning that \( \tilde{\mathcal{K}} \) is a subset of \( \mathcal{K} \) to ensure \( \theta + \beta \nu \in \mathcal{K} \), i.e., \( \tilde{\mathcal{K}} = \{ \theta \in \mathcal{K} : \theta + \beta \nu \in \mathcal{K}, \forall \nu \text{ s.t. } ||\nu|| = 1 \} \). Introduce \( \tilde{\theta}^* \) as the minimizer of \( \tilde{f}(\theta) \), i.e.,

\[
\tilde{\theta}^* = \arg \min_{\theta \in \tilde{\mathcal{K}}} \frac{1}{N} \sum_{i \in N} \tilde{f}_i(\theta).
\]

First, we can build a relation between the optimization error \( \mathbb{E}[f(\overline{\theta}^T) - f(\theta^*)] \) of the original cost function and that of the smoothed function, as stated in the following lemma.

**Lemma 9.** Suppose that Assumptions A1, A2, A3, and A5 hold. Each node \( i \) updates \( \theta_i(t) \) according to (9) with perturbation vector satisfying S2 and constant step-sizes \( \alpha_t = \alpha \) and \( \beta_t = \beta \). Then we have

\[
\mathbb{E} \left[ f \left( \overline{\theta}^T \right) - f \left( \theta^* \right) \right] \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbb{E} \left[ \tilde{f}_i(\theta_i(t)) \right] - \tilde{f}_i(\theta^*) \right) + 2\ell \beta + \ell \delta.
\]

recall that \( \delta \propto \alpha / \beta \) is defined in (11).
The gap of the two optimization errors mainly comes from the differences between \( \tilde{f}(\theta) \) and \( f(\theta) \), as well as the difference between \( \tilde{f}(\theta^*) \) and \( f(\theta^*) \). Please refer to the proof details of Lemma 9 in Appendix ??.

Clearly, our next step is to find the upper bound of the optimization error of the smoothed function \( \tilde{f} \).

**Lemma 10.** Under the same condition of Lemma 9, we have

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbb{E} \left[ \tilde{f}_i (\theta_i (t)) \right] - \tilde{f}_i (\theta^*) \right) \leq \frac{2MR^2}{\alpha T} + M \frac{\delta^2}{\alpha} + MC^2 \frac{\alpha}{\beta^2} + MC \frac{\delta}{\beta}.
\] (23)

The proof of Lemma 10 is presented in Appendix ??.

From (22), (23), and consider \( \delta = C \left( 2 + N \rho \frac{\eta}{1 - \eta} \right)^\frac{\alpha}{2} \), we can obtain an upper bound of \( \mathbb{E} \left[ f(\theta^{(T)}) - f(\theta^*) \right] \) as the function of the step-sizes \( \alpha \) and \( \beta \), i.e.,

\[
\mathbb{E} \left[ f(\theta^{(T)}) - f(\theta^*) \right] \\
\leq \frac{2MR^2}{\alpha T} + MC^2 \left( 7 + 5N \rho \frac{\eta}{1 - \eta} + \left( N \rho \frac{\eta}{1 - \eta} \right)^2 \right) \frac{\alpha}{\beta^2} + 2\ell \beta + C\ell \left( 2 + N \rho \frac{\eta}{1 - \eta} \right) \frac{\alpha}{\beta}.
\] (24)

\[
\propto O \left( \frac{1}{\alpha T} \right) + O \left( \frac{\alpha}{\beta^2} \right) + O \left( \beta \right) + O \left( \frac{\alpha}{\beta} \right)
\] (25)

The final task is to find the reasonable values of \( \alpha \) and \( \beta \), to make the above upper bound as tight as possible. Intuitively, the desirable values of \( \alpha \) and \( \beta \) should be small and can be seen as some function of \( T \). In this situation \( \frac{\alpha}{\beta^2} \) should be much larger than \( \frac{\alpha}{\beta} \), which means that the upper bound (25) is not dominated by \( O \left( \frac{\alpha}{\beta} \right) \). For this reason, we focus on the minimization of the remaining term \( O \left( 1/(\alpha T) \right) + O \left( \frac{\alpha}{\beta^2} \right) + O \left( \beta \right) \). We have the following achievable lower bound

\[
2MR^2 \frac{1}{\alpha T} + MC^2 \left( 7 + 5N \rho \frac{\eta}{1 - \eta} + \left( N \rho \frac{\eta}{1 - \eta} \right)^2 \right) \frac{1}{\beta^2} + 2\ell \beta
\] (26)

\[
\geq 2MR \sqrt{14 + 10N \rho \frac{\eta}{1 - \eta} + 2 \left( N \rho \frac{\eta}{1 - \eta} \right)^2} \frac{1}{\sqrt{T} \beta} + 2\ell \beta
\] (27)

where both (26) and (27) come from the fact that \( ax + bx^{-1} \geq 2\sqrt{ab} \), \( \forall a, b, x \in \mathbb{R}^+ \) and the equality holds if and only if \( ax = bx^{-1} \). Thus the equality of (26) and (27) hold only if

\[
\begin{cases}
2MR^2 \frac{1}{\alpha T} = MC^2 \left( 7 + 5N \rho \frac{\eta}{1 - \eta} + \left( N \rho \frac{\eta}{1 - \eta} \right)^2 \right) \frac{1}{\beta^2} \\
2MR \sqrt{14 + 10N \rho \frac{\eta}{1 - \eta} + 2 \left( N \rho \frac{\eta}{1 - \eta} \right)^2} \frac{1}{\sqrt{T} \beta} = 2\ell \beta
\end{cases}
\] (28)
which can be solved to get \( \alpha^* \propto T^{-\frac{3}{4}} \) and \( \beta^* \propto T^{-\frac{1}{4}} \) with exact expression given in (19). With \( \alpha^* \) and \( \beta^* \), we can see that the last term \( O(\alpha/\beta) = O(T^{-\frac{1}{2}}) \), which is indeed much smaller than \( O(T^{-\frac{3}{4}}) \) when \( T \) is large. In fact

\[
C\ell \left( 2 + N\rho \frac{\eta}{1-\eta} \right) \frac{\alpha^*}{\beta^*} = R\ell \sqrt{\frac{2 \left( 2 + N\rho \frac{\eta}{1-\eta} \right)^2}{\left( 1 + \left( 2 + N\rho \frac{\eta}{1-\eta} \right)^2 + \left( 2 + N\rho \frac{\eta}{1-\eta} \right)^2 \right) T^{-\frac{1}{4}}}} \leq \sqrt{2}R\ell T^{-\frac{1}{4}}.
\]

Introduce (27) and (29) into (24), we get

\[
E_{\|} \text{ is large. In fact } T_{\|} \text{ which can be solved to get } \alpha = \frac{\sqrt{4\rho^2 + 1}}{1 - \eta} \text{ for any } \eta < 1. \text{ Note that such convergence rate is optimal in the } ZOSCO \text{ problem.}
\]

\[
\text{B. Discussion}
\]

In this section, we have investigated the convergence rate of the proposed algorithm for general Lipschitz convex function. Our result can be seen as an extension of the classical work [5], which has shown that the optimization error is \( O(\sqrt{MT^{-\frac{1}{4}}}) \) in a centralized setting.

VI. CONCLUSION

In this work, we have addressed a distributed optimization problem in a time-varying network with non always strong connectivity. Unlike the classical problems where gradient information is directly available, we consider a zeroth-order stochastic convex optimization setting. Each node can only use a numerical observation of its local cost function to get a biased estimation of gradient. We take a first step towards this challenging problem. A simple distributed algorithm is considered, with the best attainable convergence rate of \( O(T^{-\frac{1}{2}}) \) after \( T \) iterations, under the assumption that the global cost function is strongly convex and local cost functions are smooth. Note that such convergence rate is optimal in the ZOSCO problem.

APPENDIX

We first evaluate the expected value of \( \hat{g}_i(t) \) for a given \( \theta_i(t) \), in order to find out its difference to the gradient \( \nabla f_i(\theta_i(t)) \). We have

\[
\mathbb{E} [g_i(t) | \theta_i(t)] = \mathbb{E}_{\nu_{i,t}, \xi_{i,t}} [\beta_t^{-1} \nu_{i,t} f_i(\theta_i(t)) + \beta_t \nu_{i,t} | \theta_i(t)] = \beta_t^{-1} \mathbb{E}_{\nu_{i,t}} [\nu_{i,t} f_i(\theta_i(t)) + \beta_t \nu_{i,t}] | \theta_i(t) \]
\[
\text{(30)}
\]

\[
\beta_t^{-1} \mathbb{E}_{\nu_{i,t}} [\nu_{i,t} f_i(\theta_i(t)) + \beta_t \nu_{i,t} \nabla f_i(\theta_i(t)) + \frac{1}{2} \beta_t^2 \nu_{i,t} \nabla^2 f_i(\theta_i(t)) \nu_{i,t}] | \theta_i(t) = \beta_t^{-1} \mathbb{E} [\nu_{i,t} \nabla f_i(\theta_i(t))] + \mathbb{E} [\nu_{i,t} \nu_{i,t}^T \nabla^2 f_i(\theta_i(t)) + b_i(t)] \text{ (31)}
\]

\[
= \nu_i(\theta_i(t)) + b_i(t) \text{ (32)}
\]

\[
\text{Note that (30) is by } f_i(\theta) = \mathbb{E}_{\xi_{i,t}} [F_i(\theta; \xi_{i,t})] \text{ for any } \theta \in \mathcal{K}, \text{ (31) can be obtained by applying Taylor’s theorem and mean-valued theorem, where } \tilde{\theta}_i(t) \text{ locates between } \theta_i(t) \text{ and } \theta_i(t) + \beta_t \nu_{i,t}. \text{ In (32), we introduce}
\]

\[
b_i(t) = \frac{1}{2} \beta_t \mathbb{E}_{\nu_{i,t}} [\nu_{i,t} (\nu_{i,t}^T \nabla^2 f_i(\theta_i(t)) \nu_{i,t})] | \theta_i(t) \text{ (34)}
\]

\[
\text{(33) comes from the statistical property of } \nu_{i,t}, \text{ i.e., } \mathbb{E} [\nu_{i,t}] = 0 \text{ and } \mathbb{E} [\nu_{i,t} \nu_{i,t}^T] = I_M. \text{ One can clearly see that } b_i(t) \text{ defined in (34) is exactly the difference between } \mathbb{E} [\hat{g}_i(t) | \theta_i(t)] \text{ and } \nabla f_i(\theta_i(t)). \text{ Thus } b_i(t) \text{ can be named as the estimation bias of the gradient estimator.}
Since

\[ f_i(\theta_i(t) + \beta_t \nu_{i,t}) \geq f_i(\theta_i(t)) + \beta_t \nabla f_i(\theta_i(t)); \quad (35) \]

\[ f_i(\theta_i(t) + \beta_t \nu_{i,t}) \leq f_i(\theta_i(t)) + \beta_t \nabla f_i(\theta_i(t)) + \frac{1}{2} L_i \| \beta_t \nu_{i,t} \|^2. \quad (36) \]

By comparing (35)-(36) with (31), we get

\[ \theta_{i,j}(t) = \frac{1}{\beta_t} \sum_{j=1}^{\nu_{i,t}} f_i(\tilde{\theta}_i(t)) \nu_{i,t} \]

Thus

\[ \sum_{j=1}^{\nu_{i,t}} \| \nu_{i,t} \|^2 \leq C K N \sum_{j=1}^{\nu_{i,t}} \| \nu_{i,t} \|^2 = M V^2 L_i. \quad (37) \]

To get an upper bound of \( \| b_i(t) \| \), we have

\[ \| b_i(t) \| = \frac{1}{2} \beta_t \sqrt{\mathbb{E}_{\nu_{i,t}} \left[ \| \nu_{i,t} \|^2 \| \nabla^2 f_i(\tilde{\theta}_i(t)) \nu_{i,t} \|^2 | \theta_i(t) \right]} \]

\[ \leq \frac{1}{2} \beta_t \sqrt{\mathbb{E}_{\nu_{i,t}} \left[ \| \nu_{i,t} \|^2 \| \nabla^2 f_i(\tilde{\theta}_i(t)) \nu_{i,t} \|^2 | \theta_i(t) \right]} \]

\[ \leq \frac{1}{2} \beta_t \sqrt{\mathbb{E}_{\nu_{i,t}} \left[ \| \nu_{i,t} \|^2 \| \nabla^2 f_i(\tilde{\theta}_i(t)) \nu_{i,t} \|^2 | \theta_i(t) \right]} \]

Thus

\[ \Delta_i(t) = \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \Pi_{K_{i+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right). \]

Since \( \theta_j(t) \in K_t \subseteq K_{t+1}, \forall j \), we have \( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) \in K_{i+1} \) as \( K_{i+1} \) is convex. On the other hand, we also have \( \Pi_{K_{i+1}}(x) \in K_{t+1} \). Thus

\[ \| \Delta_i(t) \| = \left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \Pi_{K_{i+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right) \right\| \]

\[ \leq \left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right) \right\| \]

\[ = \alpha_t \| \hat{g}_i(t) \| = \frac{\alpha_t}{\beta_t} \left| F_i(\theta_i(t) + \beta_t \Phi_i(t) ; \xi_i,t) \right| \| \nu_{i,t} \| \]

\[ \leq \frac{\alpha_t}{\beta_t} C \| \nu_{i,t} \|. \quad (40) \]

Recall that \( |F_i| \leq C \) by Assumption A3. From the definition of \( \Delta_i(t) \), we evaluate

\[ \theta_i(t) = \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t - 1) - \Delta_i(t - 1) \]

\[ = \sum_{j=1}^{N} A_{i,j}(t - 1) \left( \sum_{j=2}^{N} A_{j_1,j_2} (t - 2) \theta_{j_2} (t - 2) - \Delta_{j_1} (t - 2) \right) - \Delta_i (t - 1) = \cdots \]

\[ = \sum_{j=1}^{N} \left[ \Phi(t - 1, 0) \right]_{i,j} \theta_i(0) - \sum_{s=1}^{t-1} \sum_{s=1}^{N} \left[ \Phi(t - 1, t - s) \right]_{i,j} \Delta_j(t - s) - \Delta_i(t - 1) \]

\[ = \sum_{j=1}^{N} \left[ \Phi(t - 1, 0) \right]_{i,j} \theta_i(0) - \sum_{s=1}^{t-1} \sum_{s=1}^{N} \left[ \Phi(t - 1, s) \right]_{i,j} \Delta_j(s - 1) - \Delta_i(t - 1) \]
Meanwhile
\[
\tilde{\theta}(t) = \frac{1}{N} \sum_{i=1}^{N} \theta_i(t) = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} A_{i,j} (t-1) \theta_j(t-1) - \Delta_i(t-1) \right)
\]
\[
= \frac{1}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} A_{i,j}(t-1) \right) \theta_j(t-1) - \frac{1}{N} \sum_{i=1}^{N} \Delta_i(t-1) = \tilde{\theta}(t-1) - \frac{1}{N} \sum_{i=1}^{N} \Delta_i(t-1)
\]
\[
= \ldots = \frac{1}{N} \sum_{j=1}^{N} \theta_j(0) - \sum_{s=1}^{t-1} \sum_{j=1}^{N} \frac{1}{N} \Delta_j(s-1) - \frac{1}{N} \sum_{j=1}^{N} \Delta_j(t-1)
\]

Now we can have the difference between \( \theta_i(t) \) and \( \tilde{\theta}(t) \), since \( \theta_j(0) = 0 \), \( \forall j \):
\[
\left\| \theta_i(t) - \tilde{\theta}(t) \right\| \leq \left\| \Delta_i(t-1) + \frac{1}{N} \sum_{j=1}^{N} \Delta_j(t-1) \right\|
\]
\[
+ \sum_{s=1}^{t-1} \sum_{j=1}^{N} \left[ \Phi(t-1,s) \right]_{i,j} - \frac{1}{N} \left\| \Delta_j(s-1) \right\|
\]
(41)

Under Assumption A1, according Proposition 1 in the reference [16], we have
\[
\left[ \Phi(t,s) \right]_{i,j} - \frac{1}{N} \leq \rho \eta^{t-s}
\]

where \( \rho = 2 \frac{1 + c(N-1)\tau}{1 - \alpha(N-1)\tau} \) and \( \eta = \left(1 - \alpha(N-1)\tau\right)^{\frac{1}{(N-1)\tau}} \) as introduces in Lemma 3.

Consider the first case where \( \|\nu_{i,t}\| \leq \sqrt{MV} \) and the step-sizes are vanishing. From (41) and (40), we have
\[
\left\| \theta_i(t) - \tilde{\theta}(t) \right\|
\]
\[
\leq \left\| \Delta_i(t-1) \right\| + \frac{1}{N} \sum_{j=1}^{N} \left\| \Delta_j(t-1) \right\| + \sum_{s=1}^{t-1} \sum_{j=1}^{N} \rho \eta^{t-s-1} \left\| \Delta_j(s-1) \right\|
\]
(42)
\[
\leq 2C \|\nu_{i,t}\| \frac{\alpha_{t-1}}{\beta_{t-1}} + N \rho C \|\nu_{i,t}\| \sum_{s=0}^{t-2} \frac{\alpha_s}{\beta_s} \eta^{t-s-2}
\]
\[
\leq \sqrt{MV} C \left(2^{1+c_3} + N \rho \left(\frac{1 + c_3}{1 - \eta} + \frac{c_3}{\eta (1 - \eta)^2}\right)\right) \frac{\alpha_t}{\beta_t}
\]
(43)

Denote \( \gamma_s = \alpha_s/\beta_s = \gamma_0 s^{-c_3} \), then (43) comes from \( \frac{\gamma_{t-1}}{\gamma_t} \leq 2^{c_3} \) and \( \sum_{s=0}^{t-2} \frac{\alpha_s}{\beta_s} \eta^{t-s-2} \leq \gamma_t \left(\frac{1 + c_3}{1 - \eta} + \frac{c_3}{(1 - \eta)^2}\right) \)
which can be proved by the following:

\[
\sum_{s=0}^{t-2} \eta^{t-s-2} \alpha_s \beta_s = \gamma_t \sum_{s=0}^{t-2} \eta^{t-s-2} \gamma_s = \gamma_t \sum_{s=0}^{t-2} \eta^{t-s-2} \left( \frac{s}{t} \right)^{c_3} \\
\leq \gamma_t \left( \eta^{t-0-2} + \sum_{s=1}^{t-2} \eta^{t-s-2} \left( 1 + c_3 \frac{t-s}{s} \right) \right) \\
\leq \gamma_t \left( \sum_{s=0}^{t-2} \eta^{t-s-2} + c_3 \sum_{s=1}^{t-2} \eta^{t-s-2} (t-s) \right) \\
= \gamma_t \left( \sum_{s=0}^{t-2} \eta^{s} + c_3 \sum_{s=2}^{t-1} \eta^{s-1} \right) = \gamma_t \left( \sum_{s=0}^{t-2} \eta^{s} + \frac{c_3}{\eta} \frac{d}{d\eta} \left( \sum_{s=2}^{t-1} \eta^{s} \right) \right) \\
= \gamma_t \left( \frac{1}{1-\eta} - \frac{c_3}{\eta} \frac{2 \eta + \eta^2}{(1-\eta)^2} \right) = \gamma_t \left( \frac{1+c_3}{1-\eta} + \frac{c_3}{(1-\eta)^2} \right),
\]

note that (44) is by \((1+x)^{c_3} \leq 1 + c_3 x\) for any \(x > 0\) and \(c_3 < 1\).

In the end, we consider the more simple case where \(\|\nu_{i,t}\| = 1\) and the step-sizes are constant. With similar steps as (43), we have

\[
\|\theta_i(t) - \overline{\theta}(t)\| \leq \frac{\alpha}{\beta} C \left( 2 + N \rho \sum_{s=0}^{t-2} \eta^{t-s-2} \right) = \frac{\alpha}{\beta} C \left( 2 + N \rho \eta^{t-1} \right) \\
\leq \frac{\alpha}{\beta} C \left( 2 + N \rho \frac{\eta}{1-\eta} \right),
\]

which concludes the proof.

We have

\[
f\left( \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \theta_i(t) \right) - f(\theta^*) \leq \frac{1}{T} \sum_{t=1}^{T} f\left( \frac{1}{N} \sum_{i=1}^{N} \theta_i(t) \right) - f(\theta^*) \\
= \frac{1}{T} \sum_{t=1}^{T} \left( f(\overline{\theta}(t)) - f(\theta^*) \right) = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} \left( f_i(\overline{\theta}(t)) - f_i(\theta^*) \right) \\
\leq \frac{1}{T} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} \frac{L_i}{2} \|\overline{\theta}(t) - \theta^*\|^2 = \frac{1}{T} \sum_{t=1}^{T} \frac{L}{2T} \|\overline{\theta}(t) - \theta^*\|^2 \\
\leq \frac{L}{2T} \sum_{t=1}^{T} \sum_{i=1}^{N} \|\theta_i(t) - \theta^*\|^2,
\]

where (46) is by the convexity of \(f\); (47) is by the assumption that \(f_i\) is \(L_i\)-smooth and \(L = \frac{1}{N} \sum_{i=1}^{N} L_i\); (48) comes from

\[
\|\overline{\theta}(t) - \theta^*\|^2 = \left\| \frac{1}{N} \sum_{i=1}^{N} (\theta_i(t) - \theta^*) \right\|^2 \leq \frac{1}{N} \sum_{i=1}^{N} \|\theta_i(t) - \theta^*\|^2.
\]

By taking expectation on both sides of the inequality, we get (13).
We need to evaluate
\[
\sum_{i=1}^{N} \|\theta_i(t+1) - \theta^*\|^2 = \sum_{i=1}^{N} \left\| \Pi_{K_{t+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right) - \theta^* \right\|^2 \\
\leq \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) - \theta^* \right\|^2 \\
= \sum_{i=1}^{N} \left\| (\theta(t) - \theta^*) + \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) - \theta(t) \right) \right\|^2 \\
= N \|\theta(t) - \theta^*\|^2 + \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \theta(t) - \alpha_t \hat{g}_i(t) \right\|^2 \\
- 2\alpha_t \sum_{i=1}^{N} \left\langle \theta(t) - \theta^*, \hat{g}_i(t) \right\rangle + 2 \sum_{i=1}^{N} \left\langle \theta(t) - \theta^*, \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \theta(t) \right\rangle \\
(50)
\]
where (50) holds as \(\theta^* \in K_0 \subseteq K_{t+1}\) and \(\Pi_{K_{t+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right) \subseteq K_{t+1}\), thus
\[
\left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) - \theta^* \right\| \geq \left\| \Pi_{K_{t+1}} \left( \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \alpha_t \hat{g}_i(t) \right) - \theta^* \right\| .
\]

We then need to investigate the term \(\sum_{i=1}^{N} \langle \theta_i(t) - \theta^*, \hat{g}_i(t) \rangle\). For any \(i \in \mathcal{V}\), we evaluate
\[
\hat{g}_i(t) = \mathbb{E} [\hat{g}_i(t) | \theta_i(t)] + \hat{g}_i(t) - \mathbb{E} [\hat{g}_i(t) | \theta_i(t)] = \nabla f_i(\theta_i(t)) + b_i(t) + e_i(t)
\]
where we denote \(e_i(t) = \hat{g}_i(t) - \mathbb{E} [\hat{g}_i(t) | \theta_i(t)]\) and recall that \(b_i(t) = \mathbb{E} [\hat{g}_i(t) | \theta_i(t)] - \nabla f_i(\theta_i(t))\).

Thus we have
\[
\left\langle \theta(t) - \theta^*, \hat{g}_i(t) \right\rangle = \left\langle \theta(t) - \theta^*, \nabla f_i(\theta_i(t)) \right\rangle + \left\langle \theta(t) - \theta^*, b_i(t) + e_i(t) \right\rangle \\
= \left\langle \theta(t) - \theta^*, \nabla f_i(\theta(t)) \right\rangle + \left\langle \theta(t) - \theta^*, \nabla f_i(\theta_i(t)) - \nabla f_i(\theta(t)) \right\rangle \\
+ \left\langle \theta(t) - \theta^*, b_i(t) + e_i(t) \right\rangle \\
\geq \left\langle \theta(t) - \theta^*, \nabla f_i(\theta(t)) \right\rangle + \left\langle \theta(t) - \theta^*, e_i(t) \right\rangle \\
- \left\| \theta(t) - \theta^* \right\| \left( \left\| \nabla f_i(\theta_i(t)) - \nabla f_i(\theta(t)) \right\| + \|b_i(t)\| \right) \\
\geq \left\langle \theta(t) - \theta^*, \nabla f_i(\theta(t)) \right\rangle + \left\langle \theta(t) - \theta^*, e_i(t) \right\rangle \\
- \left\| \theta(t) - \theta^* \right\| \left( L_i \|\theta_i(t) - \theta(t)\| + \|b_i(t)\| \right) \\
(52)
\]
where (52) is by \(|\langle a_1, a_2 \rangle| \leq \|a_1\| \|a_2\|\) for any vectors \(a_1\) and \(a_2\) of the same dimension; (53) is by the assumption that each average local cost function \(f_i\) is \(L_i\)-smooth. Based on (53), we have
\[
\sum_{i=1}^{N} \left\langle \theta(t) - \theta^*, \hat{g}_i(t) \right\rangle \geq \left( \theta(t) - \theta^*, \sum_{i=1}^{N} \nabla f_i(\theta(t)) \right) + \sum_{i=1}^{N} \left\langle \theta(t) - \theta^*, e_i(t) \right\rangle \\
- \left\| \theta(t) - \theta^* \right\| \sum_{i=1}^{N} \left( L_i \|\theta_i(t) - \theta(t)\| + \|b_i(t)\| \right) \\
\geq N\mu \left\| \theta(t) - \theta^* \right\|^2 - N\left( \frac{M^2}{2}V^2\beta_t + L\beta_t \right) \left\| \theta(t) - \theta^* \right\| + \sum_{i=1}^{N} \left\langle \theta(t) - \theta^*, e_i(t) \right\rangle \\
(54)
\]
where \( \sum_{i=1}^{N} \| \mathbf{b}_i (t) \| \leq \frac{1}{2} N M^2 V^2 L \beta_t \) by using Lemma 1; \( \sum_{i=1}^{N} L_i \| \theta_i (t) - \overline{\theta} (t) \| \leq \sum_{i=1}^{N} L_i \delta_t = N L \delta_t \) by Lemma 3; and

\[
\left\langle \overline{\theta} (t) - \theta^*, \sum_{i=1}^{N} \nabla f_i (\overline{\theta} (t)) \right\rangle = N \left\langle \overline{\theta} (t) - \theta^*, \nabla f (\overline{\theta} (t)) \right\rangle \geq N \mu \| \overline{\theta} (t) - \theta^* \| ^2 ,
\]

which can be got by the assumption that \( f \) is \( \mu \)-strongly convex (Assumption A4), i.e.,

\[
\left\langle \overline{\theta} (t) - \theta^*, \nabla f (\overline{\theta} (t)) - \nabla f (\theta^*) \right\rangle \geq \mu \| \overline{\theta} (t) - \theta^* \| ^2 \tag{55}
\]

and

\[
\left\langle \overline{\theta} (t) - \theta^*, \nabla f (\theta^*) \right\rangle \geq 0 \tag{56}
\]

since \( \theta^* \) is minimizer of the convex function \( f \).

Back to (51), we still need to derive

\[
\sum_{i=1}^{N} \left\langle \overline{\theta} (t) - \theta^*, \sum_{j=1}^{N} A_{i,j} (t) \theta_j (t) - \overline{\theta} (t) \right\rangle = \left\langle \overline{\theta} (t) - \theta^*, \sum_{j=1}^{N} \left( \sum_{i=1}^{N} A_{i,j} (t) \right) \theta_j (t) - N \overline{\theta} (t) \right\rangle = \left\langle \overline{\theta} (t) - \theta^*, \sum_{j=1}^{N} \theta_j (t) - N \frac{1}{N} \sum_{i=1}^{N} \theta_i (t) \right\rangle = 0 , \tag{57}
\]

thanks to the assumption that \( \mathbf{A} (t) \) is doubly stochastic, \( \sum_{i=1}^{N} A_{i,j} (t) = 1 \). Meanwhile

\[
\sum_{i=1}^{N} \left\| \sum_{j=1}^{N} A_{i,j} (t) \theta_j (t) - \overline{\theta} (t) - \alpha_t \hat{\mathbf{g}}_i (t) \right\| ^2 \\
\leq 2 \sum_{i=1}^{N} \left( \left\| \sum_{j=1}^{N} A_{i,j} (t) \theta_j (t) - \overline{\theta} (t) \right\| ^2 + \| \alpha_t \hat{\mathbf{g}}_i (t) \| ^2 \right) \\
\leq 2 \sum_{i=1}^{N} \left( \left( \sum_{j=1}^{N} A_{i,j} (t) \| \theta_j (t) - \overline{\theta} (t) \| \right) ^2 + \| \alpha_t \hat{\mathbf{g}}_i (t) \| ^2 \right) \\
\leq 2 \sum_{i=1}^{N} \left( \sum_{j=1}^{N} A_{i,j} (t) \delta_t \right) ^2 + 2 \alpha_t ^2 \sum_{i=1}^{N} \frac{M V^2 C^2}{\beta_t ^2} \leq 2 N \delta_t ^2 + 2 M N V^2 C^2 \alpha_t ^2 \beta_t ^2 \tag{58}
\]
Introduce (54), (57), and (58) into (51), we get

\[
\sum_{i=1}^{N} \| \theta_i(t+1) - \theta^* \|^2 \\
\leq N (1 - 2 \mu \alpha_t) \left\| \bar{\theta}(t) - \theta^* \right\|^2 + N L \alpha_t \left( M \bar{\lambda} V^2 \beta_t + 2 \delta_t \right) \left\| \bar{\theta}(t) - \theta^* \right\|
\]

\[
+ 2 N \delta_t^2 + 2 M N V^2 C^2 \frac{\alpha_t^2}{\beta_t^2} - 2 \alpha_t \sum_{i=1}^{N} \langle \bar{\theta}(t) - \theta^*, e_i(t) \rangle
\]

\[
\leq (1 - 2 \mu \alpha_t) \sum_{i=1}^{N} \| \theta_i(t) - \theta^* \|^2 + \sqrt{N} L \alpha_t (M \bar{\lambda} V^2 \beta_t + 2 \delta_t) \sum_{i=1}^{N} \| \theta_i(t) - \theta^* \|^2
\]

\[
+ 2 N \delta_t^2 + 2 M N V^2 C^2 \frac{\alpha_t^2}{\beta_t^2} - 2 \alpha_t \sum_{i=1}^{N} \langle \bar{\theta}(t) - \theta^*, e_i(t) \rangle,
\]

recall that we have shown \( \| \bar{\theta}(t) - \theta^* \| \leq \frac{1}{N} \sum_{i=1}^{N} \| \theta_i(t) - \theta^* \| \) in (49).

We can take expectation on both sides of (59), considering the fact that \( e_i(t) \) has zero-mean by definition and

\[
\mathbb{E} \left[ \sqrt{\sum_{i=1}^{N} \| \theta_i(t) - \theta^* \|^2} \right] \leq \sqrt{\mathbb{E} \left[ \sum_{i=1}^{N} \| \theta_i(t) - \theta^* \|^2 \right]} = \sqrt{d_t},
\]

we get

\[
d_{t+1} \leq (1 - 2 \mu \alpha_t) d_t + \sqrt{N} L \alpha_t (M \bar{\lambda} V^2 \beta_t + 2 \delta_t) \sqrt{d_t} + 2 N \delta_t^2 + 2 M N V^2 C^2 \frac{\alpha_t^2}{\beta_t^2}.
\]

By considering the definition of \( \lambda_i, \lambda_{II}, \) and \( \lambda_{III} \) in Theorem 4, the inequality (60) can be written as (15), which concludes the proof.

This proof is by induction. Consider first the situation where \( t \leq t_0 = [2 \mu \alpha_0] \). Since \( \| \bar{\theta} \| \leq R \) for any \( \bar{\theta} \in \mathcal{K} \), we have, \( \forall t \leq t_0, \)

\[
d_t = \mathbb{E} \left[ \sum_{i=1}^{N} \| \theta_i(t) - \theta^* \|^2 \right] \leq \mathbb{E} \left[ \sum_{i=1}^{N} (\| \theta_i(1) \| + \| \theta^* \|) \right] \leq 4 N R^2.
\]

Thus \( d_t \leq \Psi \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t^{-\frac{1}{2}} \right)^2 t^{-\frac{1}{2}} \) holds for all \( t \leq t_0 \) if

\[
\Psi \geq \max_{t \leq t_0} \left( 4 N R^2 \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t^{-\frac{1}{2}} \right)^{-2} t^{\frac{1}{2}} \right) = 4 N R^2 \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t_0^{-\frac{1}{2}} \right)^{-2} t_0^{-\frac{1}{2}}.
\]

The next step is to show that, for all \( t \geq t_0, \sqrt{d_t} \leq \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0} t^{-\frac{1}{2}} \right) t^{-\frac{1}{4}} \sqrt{\Psi} \) leads to \( \sqrt{d_{t+1}} \leq \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0} (t+1)^{-\frac{1}{2}} \right) (t+1)^{-\frac{1}{4}} \sqrt{\Psi} \) if \( \Psi \) satisfies (17).

With \( \alpha_t = \alpha_0 t^{-1}, \beta_t = \beta_0 t^{-\frac{1}{4}}, \) and \( \sqrt{d_t} \leq \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t^{-\frac{1}{2}} \right) t^{-\frac{1}{4}} \sqrt{\Psi} \), we have the following bound according to (15):

\[
d_{t+1} \leq (1 - 2 \mu \alpha_0 t^{-1}) d_t + \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t^{-\frac{1}{2}} \right) \alpha_0 \beta_0 t^{-\frac{1}{4}} \sqrt{d_t} + \lambda_{III} \frac{\alpha_0^2}{\beta_0^2} t^{-\frac{3}{2}}
\]

\[
\leq (1 - 2 \mu \alpha_0 t^{-1}) \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0} t^{-\frac{1}{2}} \right)^2 t^{-\frac{1}{2}} \Psi + \left( \lambda + \lambda_{II} \frac{\alpha_0}{\beta_0^2} t^{-\frac{1}{2}} \right)^2 \alpha_0 \beta_0 t^{-\frac{3}{4}} \sqrt{\Psi} + \lambda_{III} \frac{\alpha_0^2}{\beta_0^2} t^{-\frac{3}{2}}.
\]
Note that as \( t \geq t_0 = [2\mu \alpha_0] \geq 2\mu \alpha_0 \), we have
\[
1 - 2\mu \alpha_0 t^{-1} \geq 1 - 2\mu \alpha_0 (2\mu \alpha_0)^{-1} = 0.
\]
A sufficient condition to have \( \sqrt{d_{t+1}} \leq \left( \lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} (t + 1)^{-\frac{1}{2}} \right) (t + 1)^{-\frac{t}{2}} \) is to have the following satisfied
\[
(1 - 2\mu \alpha_0 t^{-1}) \left( \lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}} \right)^2 \Psi + \left( \lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}} \right)^2 \Psi_0 t^{-\frac{1}{2}} \Psi + \lambda_{III} \frac{\alpha_0^2}{\rho_0^2} t^{-\frac{1}{2}} \Psi.
\]
In other words, we need to show that there exists \( \Psi < \infty \) such that (65) holds for any \( t \geq t_0 \).

We can rewrite (65) as,
\[
\left( t - \left( \frac{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} (t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}} \right) (t + 1)^{-\frac{1}{2}} \right) \Psi
\]
\[
+ \Psi_0 t^{-\frac{1}{2}} \Psi + \frac{\lambda_{III} \frac{\alpha_0^2}{\rho_0^2}}{\left( \lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}} \right)^2} \leq 0.
\]
We need to use the bounds
\[
\frac{\lambda_{III} \frac{\alpha_0^2}{\rho_0^2}}{\left( \lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}} \right)^2} \leq \frac{1}{\lambda_1 \frac{\rho_0^2}{\rho_0^2}},
\]
and
\[
t - \left( \frac{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} (t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}} \right)^2 (t + 1)^{-\frac{1}{2}} t^{\frac{3}{2}}
\]
\[
= t - \left( 1 + \frac{\lambda_{II} \frac{\alpha_0}{\rho_0} (t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}} \right)^2 (t + 1)^{-\frac{1}{2}} t^{\frac{3}{2}}
\]
\[
= t - \left( 1 + 2\lambda_{II} \frac{\alpha_0}{\rho_0} (t + 1)^{-\frac{1}{2}} - t^{-\frac{1}{2}} \right)^2 (t + 1)^{-\frac{1}{2}} t^{\frac{3}{2}}
\]
\[
\leq \left( t^{\frac{1}{2}} - (t + 1)^{-\frac{1}{2}} \right) t^{\frac{3}{2}} + 2\lambda_{II} \frac{\alpha_0}{\rho_0} \frac{t - (t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}} (t + 1)^{-\frac{1}{2}}
\]
\[
\leq \frac{1}{2} + \frac{\alpha_0}{\rho_0} \frac{(t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}}
\]
\[
\leq \frac{1}{2} + \frac{\alpha_0}{\rho_0} \frac{(t + 1)^{-\frac{1}{2}}}{\lambda_1 + \lambda_{II} \frac{\alpha_0}{\rho_0} t^{-\frac{1}{2}}} = \frac{1}{2} + \sqrt{\frac{t}{t + 1}} \leq \frac{3}{2},
\]
where (68) is by
\[
\left( t^{-\frac{1}{2}} - (t + 1)^{-\frac{1}{2}} \right) t^{\frac{3}{2}}
\]
\[
= \left( t^{-\frac{1}{2}} (t + 1)^{\frac{1}{2}} - 1 \right) t^{\frac{3}{2}} (t + 1)^{-\frac{1}{2}} = \left( \left( 1 + \frac{1}{t} \right)^{\frac{1}{2}} - 1 \right) t^{\frac{3}{2}} (t + 1)^{-\frac{1}{2}}
\]
\[
\leq \frac{1}{2} \left( 1 - 1 \right) t^{\frac{3}{2}} (t + 1)^{-\frac{1}{2}} = \frac{1}{2} \left( \frac{t}{t + 1} \right)^{\frac{1}{2}} \leq \frac{1}{2} \quad (70)
\]
Using (67) and (69), we get
\[
\left( t - \left( \frac{\lambda_1 + \lambda_{II} \alpha_0}{\lambda_0} (t + 1)^{-\frac{1}{2}} \right)^2 (t + 1)^{-\frac{1}{2}} t^{\frac{3}{2}} - 2 \mu \alpha_0 \right) \Psi + \alpha_0 \beta_0 \sqrt{\Psi} + \frac{\lambda_{III} \alpha_0^2}{\lambda_1^2 \beta_0^2}
\]
\[
\leq \left( \frac{3}{2} - 2 \mu \alpha_0 \right) \Psi + \alpha_0 \beta_0 \sqrt{\Psi} + \frac{\lambda_{III} \alpha_0^2}{\lambda_1^2 \beta_0^2} \quad (71)
\]
Then (66) can be ensured if the following holds
\[
\left( -\frac{3}{2} + 2 \mu \alpha_0 \right) \Psi - \alpha_0 \beta_0 \sqrt{\Psi} - \frac{\lambda_{III} \alpha_0^2}{\lambda_1^2 \beta_0^2} \geq 0. \quad (72)
\]
As \( \sqrt{\Psi} \geq 0 \), We can solve (72) to get:
\[
\sqrt{\Psi} \geq \frac{\beta_0 + \sqrt{\beta_0^2 + 2 \left( 4 \mu \alpha_0 - 3 \right) \frac{\lambda_{III}}{\lambda_1^2 \beta_0}}}{4 \mu - \frac{3}{\alpha_0}} \quad (73)
\]
Therefore, for all \( t \geq t_0 \), \( \sqrt{d_{t+1}} \leq \left( \lambda_1 + \lambda_{II} \alpha_0 \right) (t + 1)^{-\frac{1}{2}} (t + 1)^{-\frac{1}{2}} \sqrt{\Psi} \) can be deduced under condition \( \sqrt{d_t} \leq \left( \lambda_1 + \lambda_{II} \alpha_0 \right) t^{-\frac{1}{2}} \sqrt{\Psi} \), as long as (73) holds.
As a conclusion, we have shown (16) when \( \Psi \) satisfies both (62) and (73).

The proof is by applying the lower bounds of two fundamental functions. We have
\[
\frac{\beta_0 + \sqrt{\beta_0^2 + 2 \left( 4 \mu \alpha_0 - 3 \right) \frac{\lambda_{III}}{\lambda_1^2 \beta_0}}}{4 \mu - \frac{3}{\alpha_0}} \geq \frac{3^\frac{3}{4} \left( 2 \left( 4 \mu \alpha_0 - 3 \right) \frac{\lambda_{III}}{\lambda_1^2 \beta_0} \right)^{\frac{1}{4}}}{4 \mu - \frac{3}{\alpha_0}}
\]
\[
= 3^\frac{3}{4} \left( 2 \lambda_1^{-2} \lambda_{III} \right)^{\frac{1}{4}} \left( 4 \mu \alpha_0 - 3 \right)^{-\frac{3}{4}} \alpha_0
\]
\[
\geq 3^\frac{3}{4} \left( 2 \lambda_1^{-2} \lambda_{III} \right)^{\frac{1}{4}} 3^{-\frac{1}{2}} \mu^{-1} \alpha_0
\]
where (74) is obtained by fact that for any \( a > 0 \) and \( \beta_0 > 0 \), the function \( h_1 (\beta_0) = \beta_0 + \sqrt{\beta_0^2 + \vartheta \beta_0^{-2}} \) takes its minimum value when \( \beta_0^* = 3^{-\frac{1}{2}} \vartheta^{\frac{1}{2}} \) and
\[
h_1 (\beta_0) \geq h_1 (\beta_0^*) = \left( 3^{-\frac{1}{2}} + \sqrt{3^\frac{1}{2} + 3^{-\frac{1}{2}}} \right) \vartheta^{\frac{1}{2}} = 3^\frac{3}{4} \vartheta^{\frac{1}{2}},
\]
here \( \vartheta = 2 \left( 4 \mu \alpha_0 - 3 \right) \frac{\lambda_{III}}{\lambda_1^2} \) and the equality of (74) holds when
\[
\beta_0^* = \left( 2 \left( 4 \mu \alpha_0 - 3 \right) \frac{\lambda_{III}}{3 \lambda_1^2} \right)^{\frac{1}{4}} \quad (76)
\]
\[ f(\bar{\theta}(t)) = f \left( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \theta_i(t) \right) \leq \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} \theta_i(t) \right) = \frac{1}{T} \sum_{t=1}^{T} f(\bar{\theta}(t)) \]

\[ = \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} f_i(\bar{\theta}(t)) \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( f_i(\theta_i(t)) + |f_i(\bar{\theta}(t)) - f_i(\theta_i(t))| \right) \]

\[ \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( f_i(\theta_i(t)) + \ell_i \|\bar{\theta}(t) - \theta_i(t)\| \right) \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} f_i(\theta_i(t)) + \ell \delta \]  

which is by the assumption that \( f_i \) is \( \ell_i \)-Lipschitz (A5). Recall that \( \ell = \frac{1}{N} \sum_{i=1}^{N} \ell_i \). We have also used the bound \( \|\bar{\theta}(t) - \theta_i(t)\| \leq \delta \) that is presented in Lemma 3.

From (78), we can deduce that

\[ \mathbb{E} \left[ f(\bar{\theta}(T)) - f(\theta^*) \right] \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbb{E} [f_i(\theta_i(t))] - f_i(\theta^*) \right) + \frac{1}{N} \sum_{i=1}^{N} \ell_i \delta \]

\[ = \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbb{E} [\tilde{f}_i(\theta_i(t))] - \tilde{f}_i(\tilde{\theta}) \right) + \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \tilde{f}_i(\tilde{\theta}) - f_i(\theta^*) \right) \]

\[ + \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbb{E} \left[ f_i(\theta_i(t)) - \tilde{f}_i(\theta_i(t)) \right] + \ell \delta. \]  

Clearly, our next step is to find upper bounds of \( f_i(\theta_i(t)) - \tilde{f}_i(\theta_i(t)) \) and of \( \tilde{f}_i(\tilde{\theta}) - f_i(\theta^*) \). By definition of \( \tilde{f}_i \), we know that

\[ \tilde{f}_i(\theta_i(t)) = \mathbb{E}_{\omega \in \mathbb{R}^M : \|\omega\| \leq 1} \left[ f_i(\theta_i(t)) + \beta \omega \right] \]

\[ \geq \mathbb{E}_{\omega \in \mathbb{R}^M : \|\omega\| \leq 1} \left[ f_i(\theta_i(t)) - |f_i(\theta_i(t)) + \beta \omega| - f_i(\theta_i(t)) \right] \]

\[ \geq f_i(\theta_i(t)) - \mathbb{E}_{\omega \in \mathbb{R}^M : \|\omega\| \leq 1} [\ell_i \|\beta \omega\|] \geq f_i(\theta_i(t)) - \beta \ell_i, \]  

leading to

\[ \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbb{E} \left[ f_i(\theta_i(t)) - \tilde{f}_i(\theta_i(t)) \right] \leq \ell \beta. \]  

Similarly to (80), we can also deduce that \( \tilde{f}_i(\theta) \leq f_i(\theta) + \ell_i \beta \), which implies

\[ \frac{1}{N} \sum_{i \in N} \tilde{f}_i(\tilde{\theta}) = \min_{\theta \in \mathbb{K}} \frac{1}{N} \sum_{i \in N} \tilde{f}_i(\theta) \leq \min_{\theta \in \mathbb{K}} \frac{1}{N} \sum_{i \in N} f_i(\theta) + \beta. \]
By assuming that $\theta^* \in \tilde{K}$, we also have $f(\theta^*) = \min_{\theta \in \tilde{K}} \frac{1}{N} \sum_{i \in N} f_i(\theta)$. Thus

$$\frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( f_i(\tilde{\theta}) - f_i(\theta^*) \right) \leq \ell \beta. \quad (82)$$

We can introduce (81) and (82) into (79) to obtain (22), which concludes the proof.

Denote

$$\tilde{D}_t = \sum_{i=1}^{N} \left\| \theta_i(t) - \tilde{\theta}^* \right\|^2$$

Using the similar steps as in (51), (57), and (58), we can get

$$\tilde{D}_{t+1} = \sum_{i=1}^{N} \left\| \theta_i(t+1) - \tilde{\theta}^* \right\|^2$$

$$= \sum_{i=1}^{N} \left\| \tilde{\theta}(t) - \tilde{\theta}^* \right\|^2 + \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \tilde{\theta}(t) - \alpha \tilde{g}_i(t) \right\|^2$$

$$- 2\alpha \sum_{i=1}^{N} \langle \tilde{\theta}(t) - \tilde{\theta}^*, \tilde{g}_i(t) \rangle + 2 \sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \tilde{\theta}^*, \sum_{j=1}^{N} A_{i,j}(t) \theta_j(t) - \tilde{\theta}(t) \right\rangle$$

$$\leq \tilde{D}_t + 2N\delta^2 + 2NC^2 \frac{\alpha^2}{\beta^2} - 2\alpha \sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \tilde{\theta}^*, \tilde{g}_i(t) \right\rangle. \quad (83)$$

Note that in this part of analysis, the step-sizes are constant $\alpha$ and $\beta$, besides the perturbation vector is a unit vector so that $||\tilde{g}_i(t)|| \leq C/\beta$.

Now we need to find a lower bound of $\sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \tilde{\theta}^*, \tilde{g}_i(t) \right\rangle$. Again, we introduce the stochastic noise $e_i(t)$ such that $e_i(t) = \tilde{g}_i(t) - \mathbb{E}[\tilde{g}_i(t) | \theta_i(t)]$. We have

$$\sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \tilde{\theta}^*, \tilde{g}_i(t) - e_i(t) \right\rangle = \sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \tilde{\theta}^*, \mathbb{E}[\tilde{g}_i(t) | \theta_i(t)] \right\rangle$$

$$= \sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \theta_i(t) + \theta_i - \tilde{\theta}^*, \mathbb{E}[\tilde{g}_i(t) | \theta_i(t)] \right\rangle$$

$$= \sum_{i=1}^{N} \left\langle \theta_i(t) - \tilde{\theta}^*, \nabla f_i(\theta_i(t)) \frac{1}{M} \right\rangle + \sum_{i=1}^{N} \left\langle \tilde{\theta}(t) - \theta_i(t), \mathbb{E}[\tilde{g}_i(t) | \theta_i(t)] \right\rangle$$

$$\geq \frac{1}{M} \sum_{i=1}^{N} \left( f_i(\theta_i(t)) - \tilde{f}_i(\tilde{\theta}^*) \right) - \sum_{i=1}^{N} \left\| \tilde{\theta}(t) - \theta_i(t) \right\| \left\| \mathbb{E}[\tilde{g}_i(t) | \theta_i(t)] \right\|$$

$$\geq \frac{1}{M} \sum_{i=1}^{N} \left( f_i(\theta_i(t)) - \tilde{f}_i(\tilde{\theta}^*) \right) - NC \frac{\delta}{\beta}. \quad (84)$$

in which (84) can be proved by applying Lemma [2], i.e.,

$$\mathbb{E}[\tilde{g}_i(t) | \theta_i(t)] = \mathbb{E}_{\nu_{i,t} \xi_{i,t}} \left[ \frac{\nu_{i,t} F_i(\theta_i(t) + \beta \nu_{i,t} \xi_{i,t})}{\beta} \right] = \frac{1}{\beta} \mathbb{E}_{\nu_{i,t}} \left[ \nu_{i,t} f_i(\theta_i(t) + \beta \nu_{i,t}) \right]$$

$$= \frac{1}{\beta} \beta \frac{1}{M} \nabla \tilde{f}_i(\tilde{\theta}^*) = \frac{1}{M} \nabla \tilde{f}_i(\tilde{\theta}^*);$$
(85) comes from the fact that $\tilde{f}_i(\theta_i(t))$ is a convex function of $\theta_i(t)$; we use Lemma 3 to get $\|\tilde{\theta}(t) - \theta_i(t)\| \leq \delta$ and also the fact that $\|\tilde{g}_i(t)\| \leq C/\beta$, thus (86) can be obtained.

By substituting (3) into (83), we have

$$\tilde{D}_{t+1} \leq \tilde{D}_t + 2N\delta^2 + 2NC^2\frac{\alpha^2}{\beta^2} - 2\alpha \frac{1}{M} \sum_{i=1}^{N} \left( \tilde{f}_i(\theta_i(t)) - \tilde{f}_i(\theta^*) \right)$$

$$+ 2\alpha NC\frac{\delta}{\beta} - 2\alpha \sum_{i=1}^{N} \left( \tilde{\theta}(t) - \tilde{\theta}^* , e_i(t) \right).$$

(87)

Denote $\tilde{d}_t = \mathbb{E}[\tilde{D}_t]$, we can take expectation on both sides of (87) to get

$$\sum_{i=1}^{N} \left( \mathbb{E} \left[ \tilde{f}_i(\theta_i(t)) \right] - \tilde{f}_i(\theta^*) \right) \leq \frac{M}{2\alpha} \left( \tilde{d}_t - \tilde{d}_{t+1} \right) + NM\frac{\delta^2}{\alpha} + NMC^2\frac{\alpha}{\beta^2} + NMC\frac{\delta}{\beta}.$$  (88)

We perform the summation of (88) from $t = 1$ to $t = T$:

$$\frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbb{E} \left[ \tilde{f}_i(\theta_i(t)) \right] - \tilde{f}_i(\theta^*) \right)$$

$$\leq \frac{M}{2\alpha T} \sum_{t=1}^{T} \left( \tilde{d}_t - \tilde{d}_{t+1} \right) + NM\frac{\delta^2}{\alpha} + NMC^2\frac{\alpha}{\beta^2} + NMC\frac{\delta}{\beta}$$

$$\leq \frac{M}{2\alpha T} \tilde{d}_1 + NM\frac{\delta^2}{\alpha} + NMC^2\frac{\alpha}{\beta^2} + NMC\frac{\delta}{\beta} \leq \frac{2MR^2}{\alphaT} + M\frac{\delta^2}{\alpha} + MC^2\frac{\alpha}{\beta^2} + MC\frac{\delta}{\beta},$$

which concludes the proof.

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