An Analytic Formula and an Upper Bound
for $\varepsilon'/\varepsilon$ in the Standard Model

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Abstract

Using the idea of the penguin box expansion we find an analytic expression for $\varepsilon'/\varepsilon$ in the Standard Model as a function of $m_t$, $m_s(m_c)$ and two non-perturbative parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$. This formula includes next-to-leading QCD/QED short distance effects calculated recently by means of the operator product expansion and renormalization group techniques. We also derive an analytic expression for the upper bound on $\varepsilon'/\varepsilon$ as a function of $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $B_K$ and other relevant parameters. Numerical examples of the bound are given.
This year [1] we have analyzed the CP violating ratio $\varepsilon'/\varepsilon$ in the Standard Model including leading and next-to-leading logarithmic contributions to the Wilson coefficient functions of the relevant local operators. Another next-to-leading order analysis of $\varepsilon'/\varepsilon$ can be found in [2].

Imposing the constraints from the CP conserving $K \to \pi\pi$ data on the hadronic matrix elements of these operators we have given numerical results for $\varepsilon'/\varepsilon$ as a function of $\Lambda_{\overline{MS}}$, $m_t$ and two non-perturbative parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ which cannot be fixed by the CP conserving data at present. These two parameters are defined by

$$\langle Q_6(m_c) \rangle_0 \equiv B_6^{(1/2)} \langle Q_6(m_c) \rangle_0^{(\text{vac})} \quad \langle Q_8(m_c) \rangle_0 \equiv B_8^{(3/2)} \langle Q_8(m_c) \rangle_2^{(\text{vac})}, \quad (1)$$

where

$$Q_6 = (\bar{s}_\alpha d_\beta)_{V-A} \sum_q (\bar{q}_\beta q_\alpha)_{V+A} \quad Q_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_q e_q (\bar{q}_\beta q_\alpha)_{V+A} \quad (2)$$

are the dominant QCD and electroweak penguin operators, respectively. The subscripts on the hadronic matrix elements denote the isospin of the final $\pi\pi$-state. The label “vac” stands for the vacuum insertion estimate of the hadronic matrix elements in question for which $B_6^{(1/2)} = B_8^{(3/2)} = 1$. The same result is found in the large $N$ limit [3, 4]. Also lattice calculations give similar results $B_6^{(1/2)} = 1.0 \pm 0.2$ [5, 6] and $B_8^{(3/2)} = 1.0 \pm 0.2$ [5, 6, 7, 8]. We have demonstrated in [1] that in QCD the parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ depend only very weakly on the renormalization scale $\mu$ when $\mu > 1$ GeV is considered. The $\mu$ dependence for the matrix elements in [1] is then given to an excellent accuracy by $1/m_s^2(\mu)$ with $m_s(\mu)$ denoting the running strange quark mass.

In the present letter we would like to cast the numerical results of [1] into an analytic formula which exhibits the $m_t$-dependence of $\varepsilon'/\varepsilon$ together with the dependence on $m_s$, $B_6^{(1/2)}$ and $B_8^{(3/2)}$. Such an analytic formula should be useful to those phenomenologists and experimentalists who are not interested in getting involved with the technicalities of ref. [1]. Combining the analytic
formula obtained here with the analytic lower bound on \( m_t \) derived recently by one of us \([1]\) we will be able to find an analytic upper bound on \( \varepsilon'/\varepsilon \).

In order to find an analytic expression for \( \varepsilon'/\varepsilon \) which exactly reproduces the results of \([1]\) we use the idea of the penguin-box expansion (PBE) suggested and developed in \([10]\). This method allows to express flavour-changing neutral current (FCNC) processes, in particular \( \varepsilon'/\varepsilon \), as linear combinations of universal \( m_t \)-dependent functions resulting from various penguin and box diagrams common to many FCNC processes. The \( m_t \)-independent coefficients of these functions are \( \mu \) and renormalization scheme independent. They depend in the case at hand only on \( \Lambda_{\overline{MS}}, m_s, B_6^{(1/2)} \) and \( B_8^{(3/2)} \).

The basic ideas of PBE have been discussed at length in ref. \([10]\) and will not be repeated here. The simplest method for transforming the results of ref. \([1]\) into the analytic formula given below is presented at the end of section 3 of ref. \([10]\). In this letter we give only the final formula for \( \varepsilon'/\varepsilon \).

The resulting analytic expression for \( \varepsilon'/\varepsilon \) is then given as follows

\[
\varepsilon'/\varepsilon = 10^{-4} \left[ \frac{\text{Im} \lambda_t}{1.7 \times 10^{-4}} \right] F(x_t),
\]

(3)

where

\[
F(x_t) = P_0 + P_X X(x_t) + P_Y Y(x_t) + P_Z Z(x_t) + P_E E(x_t).
\]

(4)

Next

\[
\text{Im} \lambda_t = |V_{ub}| |V_{cb}| \sin \delta = \eta \lambda^5 A^2
\]

(5)
in the standard parameterization of the CKM matrix \([11]\) and in the Wolfenstein parameterization \([12]\), respectively. Here

\[
\lambda = |V_{us}| = 0.22 \quad |V_{cb}| = A \lambda^2
\]

(6)

and

\[
\eta = R_b \sin \delta \quad R_b = \frac{1}{\lambda} \frac{|V_{ub}|}{|V_{cb}|}.
\]

(7)
The basic $m_t$-dependent functions are given by

$$X(x_t) = \frac{x_t}{8} \left[ \frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t \right]$$

$$Y(x_t) = \frac{x_t}{8} \left[ \frac{x_t - 4}{x_t - 1} + \frac{3x_t}{(x_t - 1)^2} \ln x_t \right]$$

$$Z(x_t) = -\frac{1}{9} \ln x_t + \frac{18x_t^4 - 163x_t^3 + 259x_t^2 - 108x_t}{144(x_t - 1)^3} + \frac{32x_t^4 - 38x_t^3 - 15x_t^2 + 18x_t}{72(x_t - 1)^4} \ln x_t$$

$$E(x_t) = -\frac{2}{3} \ln x_t + \frac{x_t^2(15 - 16x_t + 4x_t^2)}{6(1 - x_t)^4} \ln x_t + \frac{x_t(18 - 11x_t - x_t^2)}{12(1 - x_t)^3}$$

with $x_t = m_t^2/M_W^2$. In the range $100 \text{ GeV} \leq m_t \leq 300 \text{ GeV}$ these functions can be approximated to better than 3% accuracy by the following expressions

$$X(x_t) = 0.650 x_t^{0.59} \quad Y(x_t) = 0.315 x_t^{0.78} \quad Z(x_t) = 0.175 x_t^{0.93} \quad E(x_t) = 0.570 x_t^{-0.51}.$$  

The coefficients $P_i$ contain the physics below the $M_W$-scale. They are given in terms of $B_6^{(1/2)} = B_6^{(1/2)}(m_c)$, $B_8^{(3/2)} = B_8^{(3/2)}(m_c)$ and $m_s(m_c)$ as follows

$$P_i = r_i^{(0)} + \left[ \frac{150 \text{ MeV}}{m_s(m_c)} \right]^2 \left( r_i^{(6)} B_6^{(1/2)} + r_i^{(8)} B_8^{(3/2)} \right).$$

The $P_i$ are $\mu$-independent and renormalization scheme independent. They depend however on $\Lambda_{\overline{MS}} = \Lambda_{\overline{MS}}^{(4)}$. In the table below we give the numerical values of $r_i^{(0)}$, $r_i^{(6)}$ and $r_i^{(8)}$ for different values of $\Lambda_{\overline{MS}}$ at $\mu = m_c = 1.4 \text{ GeV}$ and the NDR renormalization scheme. The coefficients $r_i^{(0)}$, $r_i^{(6)}$ and $r_i^{(8)}$ do not depend on $m_s(m_c)$ as this dependence has been factored out. $r_i^{(0)}$ does, however, depend on the particular choice for the parameter $B_2^{(1/2)}$ in the parameterization of the matrix element $\langle Q_2 \rangle_0$. The values given in the table correspond to the central value for $B_2^{(1/2)} = 6.7 \pm 0.9$ as determined from
the $CP$ conserving data in ref. [1]. Variation of $B^{(1/2)}_2$ in the full allowed range introduces an uncertainty of at most 18% in the $r_i^{(0)}$ column of the table. Since the parameters $r_i^{(0)}$ give only subdominant contributions to $\varepsilon'/\varepsilon$ keeping $B^{(1/2)}_2$ and $r_i^{(0)}$ at their central values is a very good approximation.

For different $\mu$ and renormalization schemes the numerical values in the table change without modifying the values of the $P_i$'s as it should be. To this end also $B^{(1/2)}_6$ and $B^{(3/2)}_8$ have to be modified as they depend on the renormalization scheme and albeit weakly on $\mu$.

\[ \Delta S = 1 \text{ PBE coefficients for various } \Lambda_{\overline{\text{MS}}}. \]

| $i$ | $r_i^{(0)}$ | $r_i^{(6)}$ | $r_i^{(8)}$ | $r_i^{(0)}$ | $r_i^{(6)}$ | $r_i^{(8)}$ |
|-----|------------|------------|------------|------------|------------|------------|
| 0   | -4.323     | 9.388      | 2.109      | -4.406     | 10.660     | 1.957      |
| X   | 0.988      | 0.016      | 0          | 0.958      | 0.019      | 0          |
| Y   | 0.763      | 0.078      | 0          | 0.729      | 0.086      | 0          |
| Z   | 0.235      | -0.014     | -10.072    | 0.297      | -0.015     | -10.899    |
| E   | 0.359      | -1.207     | 0.411      | 0.339      | -1.327     | 0.462      |

| $i$ | $r_i^{(0)}$ | $r_i^{(6)}$ | $r_i^{(8)}$ | $r_i^{(0)}$ | $r_i^{(6)}$ | $r_i^{(8)}$ |
|-----|------------|------------|------------|------------|------------|------------|
| 0   | -4.484     | 12.038     | 1.792      | -4.557     | 13.573     | 1.609      |
| X   | 0.933      | 0.022      | 0          | 0.910      | 0.026      | 0          |
| Y   | 0.701      | 0.094      | 0          | 0.676      | 0.103      | 0          |
| Z   | 0.361      | -0.017     | -11.779    | 0.427      | -0.019     | -12.737    |
| E   | 0.319      | -1.449     | 0.515      | 0.299      | -1.579     | 0.573      |

The inspection of the table shows that the terms involving $r_0^{(6)}$ and $r_Z^{(8)}$ dominate the ratio $\varepsilon'/\varepsilon$. The function $Z(x_t)$ represents a gauge invariant combination of $Z^0$- and $\gamma$-penguins [10] which increases rapidly with $m_t$ and due to $r_Z^{(8)} < 0$ suppresses $\varepsilon'/\varepsilon$ strongly for large $m_t$ [13, 14]. The term $r_0^{(6)}$ which also plays some role represents the contributions of $(V - A) \otimes (V - A)$
QCD penguins. The positive contributions of $X(x_t)$ and $Y(x_t)$ represent to large extent the $(V - A) \otimes (V - A)$ electroweak penguins. Finally $E(x_t)$ describes the residual $m_t$-dependence of the QCD-penguins which is only a very small correction to the full expression for $\varepsilon'/\varepsilon$.

Combining the formula (3) with the analytic lower bound on $m_t$ from $\varepsilon_K$ derived recently in [9] we can find an analytic upper bound on $\varepsilon'/\varepsilon$. Indeed the lower bound of [9] corresponds to $\sin \delta = 1$ at which $\text{Im} \lambda_t$ is maximal.

Since $F(x_t)$ decreases with increasing $x_t$ we find

$$[\varepsilon'/\varepsilon]_{\text{max}} = 10^{-4} \left[ \frac{R_b \lambda^5 A^2}{1.7 \times 10^{-4}} \right] F((x_t)_{\text{min}}) \quad (11)$$

where according to [3]

$$(x_t)_{\text{min}} = \left[ \frac{1}{2} A^2 \left( \frac{1}{A^2 B_K R_b} - 1.2 \right) \right]^{1.316} \quad (12)$$

with the renormalization group invariant $B_K$ giving the size of the hadronic matrix element $\langle \bar{K}^0 |(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}|K^0 \rangle$. Formula (11) gives the maximal value for $\varepsilon'/\varepsilon$ in the Standard Model consistent with the measured value of $\varepsilon_K$ as a function of $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $B_K$, $B_6^{(1/2)}$, $B_8^{(3/2)}$, $m_s$ and $\Lambda_{\overline{\text{MS}}}$. It should be noted that this formula does not involve $m_t$ directly as $m_t$ has been eliminated by means of the lower bound of [9]. It is evident that the upper bound increases with increasing $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $B_K$, $B_6^{(1/2)}$, $\Lambda_{\overline{\text{MS}}}$ and with decreasing $m_s$ and $B_8^{(3/2)}$.

In fig. 1 we show $(\varepsilon'/\varepsilon)_{\text{max}}$ as a function of $|V_{cb}|$ for different values of $B_K$, three choices of $(B_6^{(1/2)}, B_8^{(3/2)})$, $\Lambda_{\overline{\text{MS}}} = 300 \text{ MeV}$, $m_s(m_c) = 150 \text{ MeV}$ or equivalently $m_s(1 \text{ GeV}) = 175 \text{ MeV}$ and $|V_{ub}/V_{cb}| = 0.10$. The range for $|V_{cb}|$ has been chosen as in [4] in accordance with the analyses of [13, 16, 17, 18] and the increased $B$-meson life-time $\tau_B = 1.49 \pm 0.03 \text{ ps}$ [19].

$B_6^{(1/2)} = B_8^{(3/2)} = 1$ corresponds to the leading $1/N$ result or central values of lattice calculations. $B_6^{(1/2)} = 2$, $B_8^{(3/2)} = 1$ is representative for the case advocated in ref. [20]. Finally, $B_6^{(1/2)} = B_8^{(3/2)} = 2$ corresponds effectively to
the first case with a smaller value of $m_s$. Comparing fig. 1 with the most recent messages from NA31 and E731 collaborations \[21, 22\]

\[
\text{Re}(\varepsilon'/\varepsilon) = \begin{cases} 
(23 \pm 6.5) \cdot 10^{-4} & \text{NA31} \\
(7.4 \pm 6.0) \cdot 10^{-4} & \text{E731}
\end{cases}
\]  

(13)

we observe that whereas the results of E731 are fully compatible with the cases considered here, the NA31 data lie above the bounds of fig. 1 if $|V_{cb}| \leq 0.040$. This is in particular the case for $B_6^{(1/2)} = B_8^{(3/2)} = 1$ which we advocate.

In fig. 1 we only show $(\varepsilon'/\varepsilon)_{\text{max}} \geq 0$. We observe that for $B_K \leq 0.5$ the upper bound is very low and for $B_6^{(1/2)} = B_8^{(3/2)} = 1$ and $|V_{cb}| \leq 0.039$ the ratio $\varepsilon'/\varepsilon$ becomes negative. This is simply the result of the high value of $m_t$ required by $\varepsilon_K$ (see eq. (12)) when $B_K$ and $|V_{cb}|$ are low. For $B_K = 0.7 \pm 0.2$ obtained in the $1/N$ and lattice calculations, $(\varepsilon'/\varepsilon)_{\text{max}}$ is in the ball park of the E731 result. We note however that for $|V_{cb}| \leq 0.040$, $B_K \leq 0.8$ and $B_6^{(1/2)} = B_8^{(3/2)} = 1$ one has $(\varepsilon'/\varepsilon)_{\text{max}} \leq 5 \times 10^{-4}$. Since $|V_{ub}/V_{cb}|$ is expected to be smaller than 0.10 \[23, 24\] we should indeed be prepared for $(\varepsilon'/\varepsilon)_{\text{max}} \approx \text{few} \times 10^{-4}$ in the Standard Model if $1/N$ calculations and lattice results are the full story. This has been already emphasized in \[1, 2\] but the bounds in fig. 1(c) make this point even stronger. On the other hand we should keep in mind the strong dependence of the upper bound (11) on $B_K$, $B_6^{(1/2)}$, $B_8^{(3/2)}$, $m_s$, $|V_{cb}|$ and $|V_{ub}/V_{cb}|$. Future improvements on these parameters may bring surprises.

Finally it should be mentioned that below the upper bound ($\sin \delta < 1$) the dependence of $\varepsilon'/\varepsilon$ on $B_K$, $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ for fixed $m_t$ is more complicated and different than in (11). $\varepsilon'/\varepsilon$ increases with decreasing $B_K$ and $|V_{cb}|$ at fixed $|V_{ub}/V_{cb}|$ until the bound is reached. It increases (decreases) with decreasing $|V_{ub}/V_{cb}|$ for $\pi/2 \leq \delta \leq \pi$ ($0 \leq \delta \leq \pi/2$) at fixed $m_t$, $B_K$ and $|V_{cb}|$.

We hope that the analytic formula for $\varepsilon'/\varepsilon$ and the analytic upper bound on this important ratio presented in eqs. (13) and (11), respectively, should facilitate the future phenomenological analyses once the values for $|V_{cb}|$,
\[ |V_{ub}/V_{cb}|, B_K, B_\delta^{(1/2)}, B_\delta^{(3/2)}, m_s, A_{\overline{\text{MS}}} \text{ and } m_t \text{ have been improved and most importantly } \varepsilon'/\varepsilon \text{ accurately measured.} \]

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Figure 1: $(\varepsilon'/\varepsilon)_{\text{max}}$ as a function of $|V_{cb}|$ for $|V_{ub}/V_{cb}| = 0.10$ and various choices of $B_K$. The three $(\varepsilon'/\varepsilon)_{\text{max}}$ plots correspond to hadronic parameters (a) $B_6^{(1/2)}(m_c) = B_8^{(3/2)}(m_c) = 2$, (b) $B_6^{(1/2)}(m_c) = 2$, $B_8^{(3/2)}(m_c) = 1$ and (c) $B_6^{(1/2)}(m_c) = B_8^{(3/2)}(m_c) = 1$, respectively.