WOHLFAHRT’S THEOREM AND INDEX FORMULA FOR ELEMENTARY MATRIX GROUPS AND $SL(2, \mathcal{O})$

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Abstract. The present article determines the indices of the principal congruence subgroups of the Bianchi groups $B_d$, $SL(2, \mathcal{O})$ and elementary matrix group $E$ and extends Wohlfahrt’s Theorem to $B_d$, $SL(2, \mathcal{O})$ and $E$, where $\mathcal{O}$ is the ring of integers of some number fields.

1. Introduction

1.1. Elementary matrix groups and $\Gamma = SL(2, \mathcal{O})$. Let $\mathcal{O}$ be the ring of integers of a number field and let $\{w_1 = 1, w_2, \ldots, w_n\}$ be an integral basis of $\mathcal{O}$. The elementary matrix group studied by P. M. Cohn is the subgroup $E$ of $\Gamma$ generated by (see [C1], [C2], and Section 4.2 of [F])

$$T_i = T_{w_i} = \begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : 1 \leq i \leq n.$$ (1.1)

Let $\pi$ be a nontrivial ideal of $\mathcal{O}$. The principal congruence subgroups $E(\pi)$ and $\Gamma(\pi)$ of $E$ and $\Gamma$ respectively are defined by

$$E(\pi) = \{(a_{ij}) \in E : a_{11} - 1, a_{22} - 1, a_{12}, a_{21} \in \pi\},$$

$$\Gamma(\pi) = \{(a_{ij}) \in \Gamma : a_{11} - 1, a_{22} - 1, a_{12}, a_{21} \in \pi\}.$$ (1.2) (1.3)

A subgroup $K$ of $E$ is called congruence if it contains $E(\pi)$ for some $\pi \subseteq \mathcal{O}$. Congruence subgroups for $\Gamma$ and $PSL(2, \mathcal{O})$ can be defined analogously (see 3.3 for $PSL(2, \mathcal{O})$).

1.2. Index formula. The first part of the present article studies the index of the congruence subgroup $E(\pi)$ of $E$. As a byproduct, the index of the principal congruence subgroup $\Gamma(\pi)$ of $\Gamma$ can be recovered simultaneously (see [F], [P] when the class number of $\mathcal{O}$ is one).

Theorem 2.6. Let $(1) \neq \pi \subseteq \mathcal{O}$ be an ideal and let $E$ and $\Gamma$ be given as in subsection 1.1. Then

$$[E : E(\pi)] = [\Gamma : \Gamma(\pi)] = N(\pi)^3 \prod (1 - N(P)^{-2})$$ (1.4)

Theorem 2.6 is equivalent to $g : E \to SL(2, \mathcal{O}/\pi)$ defined by $g(X) = X \mod \pi$ is surjective. Unlike the modular group case (Lemma 1.38 of [Sh]), we give an indirect proof by studying the normal series of $E(\pi) \triangleleft E$ that consists of principal congruence subgroups (Section 2). Our study shows that $[\Gamma : \Gamma(\pi)] = [E : E(\pi)]$ is multiplicative (Lemma 2.1). In the case $\pi = \mathcal{P}^n$, where $\mathcal{P}$ is a prime ideal, $\Gamma/\Gamma(\mathcal{P}) \cong E/E(\mathcal{P}) \cong SL(2, \mathcal{O}/\mathcal{P})$ and $\Gamma(\mathcal{P}^r)/\Gamma(\mathcal{P}^{r+1}) \cong E(\mathcal{P}^r)/E(\mathcal{P}^{r+1})$ is elementary abelian of order $N(\mathcal{P})^3$ for every $r \geq 1$ (Lemmas 2.3 and 2.4). Consequently, our approach determines the order and group structure of $\Gamma/\Gamma(\pi)$ and $E/E(\pi)$.

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Proposition 5.3. Suppose that $O$ has level $m$ in $\Gamma = PSL(2,\mathbb{O})$. Then $\Gamma(m) \subseteq K$.

Theorem 3.2. Let $K$ be a subgroup of finite index of $E$. Suppose that $K$ has level $m$. Then $K$ is congruence if and only if $\Gamma(m) \subseteq K$.

Theorem 3.4. Let $K$ be a subgroup of finite index of $\Gamma$. Suppose that $K$ has level $m$. Then $K$ is congruence if and only if $\Gamma(m) \subseteq K$.

Theorem 3.4 has been proved by Fine and Petersen when $O$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$ where $-d = 1, 2, 3, 7, 11, 19, 43, 67, 163$ (Theorem 4.7.3 of [F], Theorem 3.1.1 of [P]). As an immediate application of Serre’s result [S] and Theorem 3.4, one has

Corollary 3.5. Suppose that $O$ has a unit of infinite order and that $K$ is normal of index $m$ in $\Gamma = PSL(2,\mathbb{O})$. Then $\Gamma(m) \subseteq K$.

Lemma 5.2. Let $S$ be a subgroup of $PSL(2,\mathbb{O})$ of index $g$, level $n$. Suppose that $O \neq \mathbb{Z}$. Suppose further that $S/[S,S]$ has a subgroup $N$ of prime index $q$ such that $q$ is inert or $q \geq 5$ is split, and $gcd(q, |SL(2,\mathbb{O}/n)|/g) = 1$. Then $N$ is a non-congruence subgroup of $PSL(2,\mathbb{O})$ of index $gq$. In particular, if the rank of $S/[S,S]$ is one or more, then $S$ contains a non-congruence subgroup of $PSL(2,\mathbb{O})$ of finite index.

1.4. Congruence subgroup problem. We say $G = PSL(2,\mathbb{O})$ has congruence subgroup property (CSP) if every subgroup of finite index is congruence. Applying the well known results of Serre [S], $PSL(2,\mathbb{O})$ does not have CSP if and only if $O$ is either $\mathbb{Z}$ or the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. In the latter case, the group is known as the Bianchi group $B_d$. Equivalently, $B_d = PSL(2,\mathbb{O}_d)$, where $\mathbb{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. The third part of the present article is to study the commutator and power subgroups of $B_d$. By Theorem 3.4, we have

Proposition 6.2 and Proposition 6.3. Suppose that $d \not\equiv 5 \pmod{8}$. Then $B_d^3 = \langle x^3 : x \in B_d \rangle$ is non-congruence if and only if $|B_d/B_d^2| \geq 8$. In particular, if the class number of $\mathbb{O}_d$ is three or more, then $B_d^2$ is non-congruence. In the case $d \equiv 5 \pmod{8}$, $B_d$ is non-congruence if and only if $d \neq -3$.

Proposition 5.3. Let $q \in \mathbb{N}$ be a prime. Suppose that $q$ is inert or $q \geq 5$ is split in $\mathbb{O}_d$ and that $d \neq -1, -3$. Then $B_d$ has a normal non-congruence subgroup of level $q$, index $q$.

Normal non-congruence subgroups for $B_{-1}$ and $B_{-3}$ can be found by applying Lemma 5.2 (see Proposition 5.4). Consequently, the fact that $B_d$ does not have CSP can be viewed as a consequence of Wohlfahrt’s Theorem. Suppose that $G = PSL(2,\mathbb{O})$ has CSP. It is well known that $S/[S,S]$ is finite if $[G : S]$ is finite (Section 16 of [BMS]). With the help of Wohlfahrt’s Theorem, one has the following slightly better result.

Proposition 7.1. Suppose that $G = PSL(2,\mathbb{O})$ has CSP. Let $S$ be a subgroup of $G$ of index $g$, level $n$. Then $S/[S,S]$ is finite. Let $p$ be a prime divisor of $|S/[S,S]|$. Suppose that $p$ is
inert or \( p \geq 5 \) is split. Then \( p \) divides \(|\text{SL}(2, \mathcal{O}/n)|/g\). In particular, if \( p \) is a prime divisor of \(|G/[G,G]| < \infty\), then \( p \) is ramified or split. In the case \( p \) is split, \( p = 2, 3\).

Applying Proposition 7.1, we show that if \( G = \text{PSL}(2, \mathcal{O}_d)\) has CSP, then \( G/[G,G] \) is of order \( 2^a 3^b\) for some \( a, b \) (Proposition 7.2). This makes groups with CSP very different from the Bianchi groups \( B_d\) as \( B_d\) do not have CSP and \( B_d/[B_d, B_d] \) is finite only of \( d = -1, -3 \) (Theorem 4.1 and Table 1).

1.5. Discussion. Our main results (Theorem 2.6 and 3.2. 3.4) are in line with the modular group case and their proofs are elementary (compare to the works we listed in the references). However, it seems worth recording as both the index formula and Wohlfahrt’s Theorem are of importance in the study of the Bianchi groups, \( SL(2, \mathcal{O})\) and the elementary matrix groups.

2. INDEX FORMULA OF \([\Gamma : \Gamma(\pi)]\) AND \([E : E(\pi)]\)

The purpose of this section is to determine the indices \([\Gamma : \Gamma(\pi)]\) and \([E : E(\pi)]\) simultaneously. Recall that \( g : E \to SL(2, \mathcal{O}/A)\) is defined by \( g(X) = X \pmod{A} \) and that a key fact of the study of the index \([E : E(A)]\) is to prove that \( g \) is surjective (see subsection 1.2). We find this difficult as a matrix \( \sigma \in SL(2, \mathcal{O})\) satisfies \( \sigma \equiv x\), where \( x \in SL(2, \mathcal{O}/A)\), is not necessarily a member of \( E \) as \( E \neq SL(2, \mathcal{O})\) if the class number of \( \mathcal{O} \) is two or more. As a consequence, we have to prove the surjectivity of \( g \) in a different way.

2.1. Upper and lower bounds for \([\Gamma : \Gamma(\pi)]\). It is clear that

\[ [E : E(\pi)] = [\Gamma \cap E : \Gamma(\pi) \cap E] \leq [\Gamma : \Gamma(\pi)]. \tag{2.1} \]

Hence \([E : E(\pi)]\) is an lower bound of \([\Gamma : \Gamma(\pi)]\). To start off our study of an upper bound of \([\Gamma : \Gamma(\pi)]\), we consider the homomorphism \( f : \Gamma \to SL(2, \mathcal{O}/\pi)\) defined by \( f(X) = X \pmod{\pi} \). It is clear that the kernel of \( f \) is \( \Gamma(\pi) \). As a consequence,

\[ [\Gamma : \Gamma(\pi)] \leq [SL(2, \mathcal{O}/\pi)]. \tag{2.2} \]

Consider the prime factorisation \( \pi = \prod \pi_i^{n_i} \) and the homomorphism \( g : SL(2, \mathcal{O}/\pi) \to \prod SL(2, \mathcal{O}/\pi_i^{n_i}) \) defined by \( g(X) = (X \pmod{\pi_1^{n_1}}, X \pmod{\pi_2^{n_2}}, \ldots, X \pmod{\pi_r^{n_r}}) \). It is clear that \( g \) is injective. As a consequence,

\[ [\Gamma : \Gamma(\pi)] \leq |SL(2, \mathcal{O}/\pi)| \leq \prod |SL(2, \mathcal{O}/\pi_i^{n_i})|. \tag{2.3} \]

Similar to the modular group case (see Section 1.6 of [Sh]), the order of \( SL(2, \mathcal{O}/\pi_i^{n_i}) \) can be determined. It is

\[ |SL(2, \mathcal{O}/\pi_i^{n_i})| = N(\pi_i)^{3n_i}(1 - N(\pi_i)^{-2}), \tag{2.4} \]

where \( N(\pi_i) \) is the absolute norm of \( \pi_i \). As a consequence, \([\Gamma : \Gamma(\pi)] \leq N(\pi)^3 \prod(1 - N(\pi_i)^{-2}) \). In summary, one has

\[ [E : E(\pi)] \leq [\Gamma : \Gamma(\pi)] \leq N(\pi)^3 \prod(1 - N(\pi_i)^{-2}). \tag{2.5} \]

The rest of this section is to study \([E : E(\pi)]\). See Theorem 2.6 for the main results.

2.2. The index \([E : E(\pi)]\) is multiplicative. We prove the following useful lemma which will be used later in our study of the concept level of Wohlfahrt’s (Lemma 3.1).

**Lemma 2.1.** Let \( K \) be a normal subgroup of \( E \). Suppose that \( T_i \in K \) for all \( 1 \leq i \leq n \). Then \( K = E \). In particular, let \( A \) and \( B \) be two ideals of \( \mathcal{O} \) such that \( A + B = \mathcal{O} \). Then \( E(A)E(B) = E, [E : E(AB)] = [E(A) : E(AB)][E(B) : E(AB)] = [E : E(A)][E : E(B)] \).

**Proof.** Since \( S^4 = 1, T_1 \in K \triangleleft E, (ST_1)^4 \in K \). Note that the order of \( T_1S \) is three. Hence

\[ T_1S = (T_1S)^3 \in K. \tag{2.6} \]
Hence $S, T_i \in K$. It follows that $K = E$. This completes the proof of the first part of the lemma. Let $A$ and $B$ be two ideals of $\mathcal{O}$ and let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be integral bases of $A$ and $B$ respectively. Since $A + B = \mathcal{O}$, there exists $x_i, y_j \in \mathbb{Z}$ such that
\[\sum x_i a_k + \sum y_j b_k = w_i\] (2.7)
for all $1 \leq i \leq n$. Pass to $E(A)$ and $E(B)$, (2.7) implies that (see Remark 2.2 for quadratic case)
\[T_{w_i} = T_i \in E(A)E(B)\] for all $i$. (2.8)
Apply the first part of our lemma, $E = E(A)E(B)$. Since $E(A) \cap E(B) = E(AB)$, by second isomorphism theorem, we have $[E : E(AB)] = [E(A) : E(AB)][E(B) : E(AB)]$. □

Remark 2.2. Let $A$ and $B$ be two ideals of $\mathcal{O}$, where $\mathcal{O} = (1, w)$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Z}$ (square free). Let $\{a_0, a_1 + a_2w\}$ and $\{b_0, b_1 + b_2w\}$ be integral bases of $A$ and $B$ respectively. Since $A + B = \mathcal{O}$, there exists $x_i, y_j \in \mathbb{Z}$ such that $x_0a_0 + x_1(a_1 + a_2w) + x_2b_0 + x_3(b_1 + b_2w) = 1$ and $y_0a_0 + y_1(a_1 + a_2w) + y_2b_0 + y_3(b_1 + b_2w) = w$. As a consequence,
\[T_1^{x_0a_0}(T_1^{b_1}T_2^{b_2})^{-x}T_1^{y_0a_0}(T_1^{b_1}T_2^{b_2})^{-y} = T_1, T_1^{x_0a_0}(T_1^{b_1}T_2^{b_2})^{x}T_1^{y_0a_0}(T_1^{b_1}T_2^{b_2})^{y} = T_w.\] (2.9)
Note that $T_1^{x_0a_0}, T_1^{x_0a_0} \in E(A)$ and that $T_1^{b_0}, T_1^{b_0} \in E(B)$. As a consequence, (2.10) implies that $T_1, T_w \in E(A)E(B)$.

2.3. The index $[E : E(\pi^m)]$, where $\pi$ is a prime ideal. We shall first study the index $[E : E(\pi)]$. It is clear that $E(\pi)/E(\pi)$ is a subgroup of $SL(2, \mathcal{O}/\pi)$. Note that $\mathcal{O}/\pi$ is a finite field of characteristic $p$ where $p$ is the smallest positive rational prime in $\pi$.

Lemma 2.3. Let $\pi \subseteq \mathcal{O}$ be a prime ideal and let $p$ be the smallest positive rational prime in $\pi$. Then $E(\pi)/E(\pi)$ isomorphic to $SL(2, \mathcal{O}/\pi)$, where $\mathcal{O}/\pi$ is a finite field of characteristic $p$ and $|SL(2, \mathcal{O}/\pi)| = N(\pi)^3(1 - N(\pi)^{-2})$.

Proof. Since $\{w_1, w_2, \ldots, w_n\}$ is an integral basis of $\mathcal{O}$ (see subsection 1.1) and $S, T_i \in E$ for all $i$. $E(\pi)$ contains all the elementary matrices.
\[S, T_x = \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix}, L_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in E(\pi)\] (2.10)
for all $x, y \in \mathcal{O}/\pi$. Note that $\mathcal{O}/\pi$ is a finite field. One may perform elementary row and column operations to show that every matrix in $SL(2, \mathcal{O}/\pi)$ can be written as a word in $S, T_i$ and $L_y$. As a consequence, $E(\pi)/E(\pi)$ isomorphic to $SL(2, \mathcal{O}/\pi)$. This completes the proof of the lemma. □

Lemma 2.4. Let $\pi \subseteq \mathcal{O}$ be a prime ideal, $m \geq 1$. Then $E(\pi^m)/E(\pi^{m+1})$ is an elementary abelian $p$-group of order $N(\pi)^3$, where $p$ is the smallest positive rational prime in $\pi$.

Proof. Suppose that $N(\pi) = \mathcal{O}/\pi = p^r$ for some $r$. It follows that $\pi^m/\pi^{m+1}$ is an elementary abelian $p$-group of order $p^r$. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of generators of $\pi^m/\pi^{m+1}$. Then $x_i \in \pi^m \setminus \pi^{m+1}$ and $\{x_1, x_2, \ldots, x_r\}$ is a $\mathbb{Z}_p$-independent set. Equivalently, if $\sum_{i=1}^r a_ix_i \in \pi^{m+1}$, where $a_i \in \mathbb{Z}$, then $p|a_i$ for all $i$. Since $T_i \in E$ for all $i$, it is clear that
\[X_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \in E(\pi^m) \setminus E(\pi^{m+1})\] (2.11)
for all $x_i$. Set $Y_i = SX_iS^{-1}, Z_i = T_1SX_iS^{-1}T_i^{-1}$ (see (1.1) for the definition of $T_i$). The matrix forms of $X_i, Y_i, Z_i$ are given as follows.
\[\begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -x_i & 1 \end{pmatrix}, \begin{pmatrix} 1 - x_i & x_i \\ -x_i & 1 + x_i \end{pmatrix} \in E(\pi^m) \setminus E(\pi^{m+1})\] (2.12)
Write the above matrices into the form $I + A_i$ for some $A_i$. Since $x_ix_j \in \pi^{m+1}$ for all $i$ and $j$, one sees easily that...
(i) \((I + A_i)p \equiv I \pmod{\pi^{m+1}}\),

(ii) \((I + A_i)(I + A_j) \equiv I + (A_i + A_j) \pmod{\pi^{m+1}}\).

As a consequence, the above matrices modulo \(\pi^{m+1}\) generate an elementary abelian \(p\)-group.

Set

\[
A = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & x_i \\ -x_i & 1 + x_i \end{pmatrix}
\]  

(2.13)

Since \(\{x_1, x_2, \cdots, x_r\}\) is a \(\mathbb{Z}_p\)-independent set, one sees easily that \(A\) modulo \(\pi^{m+1}\) is elementary abelian of order \(p^{2r}\), \(B\) modulo \(\pi^{m+1}\) is elementary abelian of order \(p^{r}\) and \(A \cap B = \{1\}\) modulo \(\pi^{m+1}\) (see Remark 2.5). Hence \(|E(\pi^{m+1})/E(\pi^{m+1})| \geq p^{3r} = N(\pi)^3\). Applying (2.5), \(|E(\pi^m/E(\pi^{m+1}))| \leq N(\pi)^3\). This completes the proof of the lemma.

**Remark 2.5.** Since \(A\) and \(B\) are abelian groups, elements in \(A\) and \(B\) take the following simple forms respectively.

\[
I + \sum_{i=1}^{r} \begin{pmatrix} 0 & a_ix_i \\ -b_ix_i & 0 \end{pmatrix}, \quad I + \sum_{i=1}^{r} \begin{pmatrix} -c_ix_i & c_ix_i \\ -c_ix_i & c_ix_i \end{pmatrix}
\]  

(2.14)

2.4. The main result. Apply (2.5) and Lemmas 2.3 and 2.4, we have the following results.

**Theorem 2.6.** Let \(1 \neq \pi \subseteq \mathcal{O}\) be an ideal. Then

\[
[\Gamma : \Gamma(\pi)] = [E : E(\pi)] = N(\pi)^3 \prod (1 - N(P)^{-2}),
\]  

(2.15)

where the product is over the set of all prime ideals \(P\) that divide \(\pi\) and \(N(P)\) is the absolute norm of \(P\).

**Proof.** Since the index \([E : E(\pi)]\) is multiplicative (Lemma 2.1), one may apply Lemmas 2.3 and 2.4 to conclude that \([\Gamma : \Gamma(\pi)] = [E : E(\pi)] = N(\pi)^3 \prod (1 - N(P)^{-2})\).

**Corollary 2.7.** The homomorphisms \(f : \Gamma \to SL(2, \mathcal{O}/\pi), g : E \to SL(2, \mathcal{O}/\pi)\) defined by \(f(X) = X, g(X) = X \bmod \pi\) are surjective.

3. Wohlfahrt’s Theorem

3.1. Wohlfahrt’s Theorem for \(E\). Denoted by \(N(E(mn), m)\) the normal closure of \(E(mn)\) and \(T_i^{m_1} (1 \leq i \leq n)\) in \(E\). The following lemma (Lemma 3.1) shows that \(N(E(mn), m) = E(m)\). It follows immediately from this lemma that Wohlfahrt’s Theorem can be extended to the elementary matrix groups (Theorem 3.2).

**Lemma 3.1.** \(N(E(mn), m) = E(m)\).

**Proof.** Let \(x\) be the smallest positive rational integer such that \(E(x) \subseteq N(E(mn), m) \subseteq E(m)\). Suppose that \(x > m\). Since \(m\) is a divisor of \(x\), one has \(x = mn_0p\), where \(n_0, p \in \mathbb{N}\) and \(p\) is a prime. Set \(q = mn_0\). Then \(x = pq\). Since \(E(pq) = E(x) \subseteq N(E(mn), m)\), it follows that

\[
N(E(pq), q) \subseteq N(E(mn), m).
\]  

(3.1)

**Case 1.** \(\gcd(p, q) = 1\). Let \(X = N(E(pq), q)\). It is clear that \(X \subseteq E(q)\). We consider the group \(XE(p)\). Since \(\gcd(p, q) = 1\), \(T_i^{m} \in E(p)\) and \(T_i^q \in X\) for all \(i\), we have \(T_i \in XE(p)\) for all \(i\). This implies that \(XE(p) = E\) (Lemma 2.1). Hence

\[
X/E(pq) \times E(p)/E(pq) \cong E/E(pq) \cong E(p)/E(pq) \times E(q)/E(pq).
\]  

(3.2)

Note that \(X \subseteq E(q)\). It is now clear that (3.2) implies that \(X = E(q)\). By (3.1), \(E(q) = N(E(pq), q) \subseteq N(E(mn), m)\). This contradicts the minimality of \(x\). Hence \(x = m\).

Equivalently, \(N(E(mn), m) = E(m)\).

**Case 2.** \(\gcd(p, q) \neq 1\). It follows that \(p|q\). By Theorem 2.6, \([E(q) : E(pq)] = N(\pi)^3\). Since \(N(E(pq), q) \subseteq E(q)\),

\[
[N(E(pq), q) : E(pq)] \leq N(\pi)^3.
\]  

(3.3)
It is clear that \( T_i^3, ST_i^2 S^{-1}, T_i ST_i^2 S^{-1} T_i^{-1} \in E(q) \). The matrix form of the above matrices are given as follows.

\[
\begin{pmatrix}
1 & qw_i \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
-qw_i & 1
\end{pmatrix}, \begin{pmatrix}
1 - qw_i & qw_i \\
-qw_i & 1 + qw_i
\end{pmatrix}.
\] (3.4)

Similar to Lemma 2.4, the above matrices modulo \( E(pq) \) generate an elementary abelian \( p \)-group of order \( N(p)^3 \). Since these matrices are conjugates of \( T_i^3 \), it follows that \( [N(E(pq), m) : E(pq)] \) is at least \( N(p)^3 \). By (3.3), \( [N(E(pq), m) : E(pq)] = N(p)^3 \). Hence \( N(E(pq), m) = E(q) \). By (3.1), \( E(q) \subseteq N(E(mn), m) \). This contradicts the minimality of \( x \). Hence \( x = m \). Equivalently, \( N(E(mn), m) = E(m) \). This completes the proof of the lemma. \( \square \)

**Theorem 3.2.** (Wohlfahrt’s Theorem) Let \( K \) be a subgroup of \( E \) of finite index. Suppose that the level of \( K \) is \( m \). Then \( K \) is congruence if and only if \( E(m) \subseteq K \).

**Proof.** Suppose that \( K \) is congruence. Then \( E(n) \subseteq K \) for some \( n \). Hence \( E(mn) \subseteq K \). Since the level of \( K \) is \( m \), it follows that \( N(E(mn), m) \subseteq K \). By Lemma 3.1, \( E(m) \subseteq K \). The converse is clear. \( \square \)

3.2. Wohlfahrt’s Theorem for \( \Gamma = SL(2, \mathcal{O}) \). Denoted by \( N(\Gamma(mn), m) \) the normal closure of \( \Gamma(mn) \) and \( T_i^{an} \) (\( 1 \leq i \leq n \)) in \( \Gamma \). The following lemma is essential in our study of Wohlfahrt’s Theorem for \( \Gamma \).

**Lemma 3.3.** \( N(\Gamma(mn), m) = \Gamma(m) \).

**Proof.** Let \( X = N(\Gamma(mn), m) \). An easy study of the group diagram of the following six groups \( \Gamma(mn) \subseteq X \subseteq \Gamma(m) \) and \( \Gamma(mn) \cap E(m) \subseteq X \cap E(m) \subseteq \Gamma(m) \cap E(m) \) implies that

(i) \( \Gamma(m) : X \geq \Gamma(mn) \cap E(m) : X \cap E(m) = [E(m) : X \cap E(m)] \),

(ii) \( [X : \Gamma(mn)] \geq [X \cap E(m) : \Gamma(mn) \cap E(m)] = [X \cap E(m) : E(mn)] \),

(iii) \( \Gamma(m) : \Gamma(mn) = [E(m) : E(mn)] \) (Theorem 2.6).

Applying (iii) of the above to (i) and (ii), it follows that the inequalities are actually equalities. Hence (i) of the above becomes

\[
\Gamma(m) : X = \Gamma(mn) \cap E(m) : X \cap E(m) = [E(m) : X \cap E(m)].
\] (3.5)

It is clear that \( N(E(mn), m) \subseteq X \cap E(m) \). By Lemma 3.1, \( N(E(mn), m) = E(m) \). Hence the third term of (3.5) is 1. This implies that \( \Gamma(m) : X = 1 \). Equivalently, \( \Gamma(m) = N(\Gamma(mn), m) \).

Similar to Theorem 3.2, we may extend Wohlfahrt’s Theorem to \( \Gamma \) as follows.

**Theorem 3.4.** (Wohlfahrt’s Theorem) Let \( K \) be a subgroup of \( \Gamma \) of finite index. Suppose that the level of \( K \) is \( m \). Then \( K \) is congruence if and only if \( \Gamma(m) \subseteq K \).

**Corollary 3.5.** Suppose that \( \mathcal{O} \) has a unit of infinite order and that \( K \) is normal of index \( m \) in \( \Gamma = SL(2, \mathcal{O}) \). Then \( \Gamma(m) \subseteq K \).

**Proof.** Since \( \mathcal{O} \) has a unit of infinite order, every subgroup of \( \Gamma \) of finite index is congruence (Serre [S]). Since \( K \) is normal of index \( m \), the level of \( K \) is a divisor of \( m \). It follows from Theorem 3.4 that \( \Gamma(m) \subseteq K \). \( \square \)

3.3. Discussion. Let \( G = PSL(2, \mathcal{O}) \) or \( PE = E/\{ \pm I \} \). Define the principal congruence subgroup \( G(\pi) \) to be \( \{ x \in G : x \equiv \pm I \ (mod \ \pi) \} \). A subgroup \( K \) of \( G \) is a congruence subgroup if \( K \) contains \( G(\pi) \) for some \( \pi \subseteq \mathcal{O} \). Let \( K \) be a subgroup of \( G \) of finite index. The level of \( K \) is the smallest positive integer \( m \) such that the normal closure of the subgroup generated by

\[
\left\{ \pm \begin{pmatrix} 1 & mw_i \\ 0 & 1 \end{pmatrix} : 1 \leq i \leq n \right\}
\] (3.6)

is contained in \( K \). Theorems 3.2 and 3.4 can be extended easily to \( G \) as follows.
Theorem 4.1. (Fine [F]) imply that Bianchi groups $K$ of finite index. Suppose that the level of $Wohlfahrt$'s Theorem) Theorem 3.6. $d < 0$. Let $\{1, w\}$ be an integral basis of $O_d$. Then $PE_d$ is generated by

$$T_1 = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \quad T_w = \left( \begin{array}{cc} 1 & w \\ 0 & 1 \end{array} \right), \quad S = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).$$

(4.1)

In the case $-d = 1, 2, 3, 7, 11$, Cohn's results [Cl] implies that $PE_d = B_d$. The group structure for such $B_d$ is completely known (Theorem 4.3.1 of Fine [F]). In the case $-d \neq 1, 2, 3, 7, 11$, $PE_d = \{a, t, u : a^2 = (at)^3 = [t, u] = 1\}$ (Theorem 4.8.1 of [F]). Set $\Omega = \{-1, -2, -3, -7, -11\}$. As an easy application of their results, one has the following.

Table 1. $PE_d/[PE_d, PE_d]$ and $PE_d/PE_d^2$

| $d$   | $PE_d/[PE_d, PE_d]$ | $PE_d/PE_d^2$ | Remark |
|-------|---------------------|---------------|--------|
| $-1$  | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $B_d = PE_d$ |
| $-2$  | $\mathbb{Z} \times \mathbb{Z}_2$ | $\mathbb{Z} \times \mathbb{Z}_2$ | $B_d = PE_d$ |
| $-3$  | $\mathbb{Z}_3$ | $1$ | $B_d = PE_d$ |
| $-7$  | $\mathbb{Z} \times \mathbb{Z}_2$ | $\mathbb{Z} \times \mathbb{Z}_2$ | $B_d = PE_d$ |
| $-11$ | $\mathbb{Z} \times \mathbb{Z}_3$ | $\mathbb{Z}_2$ | $B_d = PE_d$ |
| $d \notin \Omega$ | $\mathbb{Z} \times \mathbb{Z}_6$ | $\mathbb{Z} \times \mathbb{Z}_2$ | $B_d \neq PE_d$ |

where $B_d^2 = \langle x^2 : x \in B_d \rangle$ and $PE_d^2 = \langle x^2 : x \in PE_d \rangle$. We now turn our attention to the case $d \notin \Omega = \{-1, -2, -3, -7, -11\}$. Works of Swan and others (see Theorem 6.2.2 and pp. 195 of Fine [F]) imply that

Theorem 4.1. Suppose that $-d \neq 1, 2, 3, 7, 11$. Then the rank $r$ of $B_d/[B_d, B_d]$ is finite. Further, $r \geq h_d$, where $h_d$ is the class number of $O_d$. In particular, $\infty > |B_d/B_d^2| \geq 2^r$.

5. First application: Non-congruence subgroups of small indices

The main purpose of this section is to construct normal non-congruence subgroups of small indices and levels of the Bianchi groups. Throughout the section, $d < 0$ is square free and $B_d(x)$ is the principal congruence subgroup associate to $x$.

Definition 5.1. Consider the prime decomposition $(q) = \mathcal{P}_1^{e_1} \mathcal{P}_2^{e_2} \cdots \mathcal{P}_s^{e_s}$ in $\mathcal{O}$. $q$ is inert if $s = 1, e_1 = 1$, $q$ is split if $s \geq 2, e_1 = e_2 = \cdots e_s = 1$, $q$ is ramified if $e_i \geq 2$ for some $i$.

Lemma 5.2. Let $S$ be a subgroup of $PSL(2, \mathcal{O})$ of index $g$, level $n$. Suppose that $\mathcal{O} \neq \mathbb{Z}$. Suppose further that $S/[S, S]$ has a subgroup of prime index $q$ such that $q$ is inert or $q \geq 5$ is split, and $gcd(q, \{|S|, \mathcal{O}/n|/g\} = 1$. Then $N$ is a non-congruence subgroup of $PSL(2, \mathcal{O})$ of index $qg$. In particular, if the rank of $S/[S, S]$ is one or more, then $S$ contains a non-congruence subgroup of $PSL(2, \mathcal{O})$ of finite index.
Proof. Set $G = PSL(2, \mathcal{O})$. Consider the prime decomposition $(q) = \prod \pi_i$, $\pi_i \neq \pi_j$ if and only if $i \neq j$. Since $\mathcal{O} \neq \mathbb{Z}$, $\mathcal{O}/\pi_i$ is a field of at least 4 elements. It follows that $PSL(2, \mathcal{O}/\pi_i)$ is non-abelian simple,

$$gcd(q, |SL(2, \mathcal{O}/n)|/q) = 1$$

(5.1)

and that $S$ has a normal subgroup $N$ of index $q$. It is clear that the level of $N$ is a divisor of $nq$. Suppose that $N$ is congruence. By Theorem 3.6, $G(nq) \subseteq N$ and $G(n) \subseteq S$, where $G(m)$ is the principal congruence subgroup of level $m$. We have two cases to consider.

Case 1. $G(n) \subseteq N$. It follows that $N/G(n) = PSL(2, \mathcal{O}/n)$ of index $q$. An easy study of the indices of the groups $G(n) \subseteq N \subseteq S \subseteq G$ implies that $q$ is a divisor of $|SL(2, \mathcal{O}/n)|/q$. A contradiction. Hence $N$ is non-congruence.

Case 2. $G(n)$ is not a subgroup of $N$. This implies that $(N \cap G(n))/G(nq)$ is a normal subgroup of $G(n)/G(nq) \cong SL(2, \mathcal{O}/q)$ of index $q$. We consider the composition factors of the following normal series.

$$1 \triangleleft (N \cap G(n))/G(nq) \triangleleft G(n)/G(nq) \cong SL(2, \mathcal{O}/q).$$

(5.2)

$PSL(2, \mathcal{O}/\pi_i)$ is a composition factor of (5.2). Since $[G(n) : N \cap G(n)] = q$ is a prime, $PSL(2, \mathcal{O}/\pi_i)$ is not a composition factor of $(N \cap G(n))/G(nq) \triangleleft G(n)/G(nq)$. Hence $PSL(2, \mathcal{O}/\pi_i)$ is a composition factor of $1 \triangleleft (N \cap G(n))/G(nq)$ for all $i$. By (5.2), the order of $G(n)/G(nq)$ is a multiple of $q|PSL(2, \mathcal{O}/q)|$. A contradiction. Hence $N$ in non-congruence.

Note that $\cap_n xN x^{-1}$ is normal in $G$. \hfill \Box

Proposition 5.3. Let $q \in \mathbb{N}$ be a prime. Suppose that $q$ is invert or $q \geq 5$ is split in $O_d$ and that $d \neq -1, -3$. Then $B_d$ has a normal non-congruence subgroup of level $q$, index $q$. Similarly, $PE_d$ has a normal non-congruence subgroup of level $q$, index $q$.

Proof. Let $S = B_d$. Since $d \neq -1, -3$, $S/[S, S] = B_d/B_d'$ has positive rank (Theorem 4.1 and Table 1). Let $q$ be given as in the proposition let $N$ be a normal subgroup of index $q$ in $S = B_d$. By Lemma 5.2, $N$ is normal non-congruence in $S = B_d$. Note that the argument works for $PE_d$. \hfill \Box

Proposition 5.4. Let $d = -1, -3$. Then $B_d = PE_d$ has normal non-congruence subgroups.

Proof. (i) $d = -1$. Let $S = B_d''$ be the third commutator subgroup of $B_d$. Then the level of $S$ is a divisor of $12$, $[B_d : S] = 76$, $S$ has 384 generators and $S/[S, S]$ is infinite (Theorem 5.3.1 of Fine [F]). By Lemma 5.2, $B_d$ has a normal non-congruence subgroup.

(ii) $d = -3$. Let $H_2 = B_d(\sqrt{-3})$. Applying results of Fine (Theorem 4.4.4 of Fine [F]), $H_2 = B_d''$ and $S = \langle x^2 : x \in H_2 \rangle$ is characteristic of index $3$ in $H_2$. Further,

$$S = \langle w_1, w_2, w_3, w_4 : (w_1w_3)^3 = (w_1w_3)^3 = (w_1, w_2) = (w_3, w_4) = 1 \rangle.$$

Note that $S/[S, S] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. By Lemma 5.2, $B_d$ has a normal non-congruence subgroup. \hfill \Box

6. Second Application : The Power Subgroups

6.1. Now we turn our attention to the power group $B_d^2 = \langle x^2 : x \in B_d \rangle$. We shall first prove a lemma that will be useful for our study. Throughout the section, $d < 0$ is square free, $B_d(x)$ is the principal congruence subgroup associated to $x$. The main purpose of this section is to show that the number of $d$'s such that $B_d^2$ is congruence is finite (Propositions 6.2 and 6.3).

Lemma 6.1. Let $G = PSL(2, O_d/2)$ and $G^2 = \langle x^2 : x \in G \rangle$. Suppose that (2) is a prime ideal of $O_d$. Then $G = G^2$. Suppose that (2) is not a prime ideal of $O_d$. Then $G/G^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Suppose that (2) is a prime ideal. Then $PSL(2, O/2) \cong A_5$ is non-abelian simple. Hence $G = G^2$. In the case (2) is split, $G \cong S_3 \times S_3$ and the lemma holds. We may therefore assume that (2) is ramified. Let $\pi$ be the prime ideal that divides $2 = \pi^2$. It follows that
\(|G| = 48\) and \(G\) is an extension of \(B_d(\pi) / B_d(2) = A \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) by \(B_d / B_d(\pi) = X \cong S_3\).

To be more accurate, pick \(w \in \pi \setminus \{2\}\), then \(A = \langle x, y, z \rangle\), where

\[
x = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 + w & w \\ w & 1 + w \end{pmatrix}.
\]

(6.1)

One can show easily that the group generated by \(x, y\) and \(z\) is elementary abelian of order 8 modulo \(2\) (see (i) and (ii) of Lemma 2.4). Similarly, \(S\) generate a symmetric group of order 6 modulo \(2\). Let \(w\) be the subgroup of \(A\) generated by \(xz\) and \(yz\).

\[
B = \langle xz = \begin{pmatrix} 1 + w & 0 \\ w & 1 + w \end{pmatrix}, yz = \begin{pmatrix} 1 + w & w \\ 0 & 1 + w \end{pmatrix} \rangle.
\]

(6.3)

\(B\) is invariant under the conjugation of \(R\). Hence \(B\) and \(R\) generate a subgroup of \(G\) of order 12. It is clear that \(\langle z, S \rangle \cap \langle B, R \rangle = 1\) and that \(z\) and \(S\) normalise \(D = \langle B, R \rangle\). Hence \(D\) is a normal subgroup of \(G\) and that

\[
G / D \cong \langle z, S \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

(6.4)

This implies that \(G^2 \subseteq D\). Suppose that \(G^2 \neq D\). Since \(G / G^2\) is an elementary abelian 2-group and \(|G| = 48\), the order of \(G^2\) is either 3 or 6. In both cases, one concludes that the Sylow 3-subgroup of \(G^2\) is a normal subgroup of \(G\). This is a contradiction as \(\langle R \rangle\) is not normal. Hence \(D = G^2\). This completes the proof of the lemma. \(\square\)

**Proposition 6.2.** Suppose that \(d \equiv 5 \mod 8\). Then \(B_d^e\) and \(B_d^o\) are non-congruence if and only if \(d \neq -3\).

**Proof.** Since \(d \equiv 5 \mod 8\), \(2 \subseteq O_d\) is a prime ideal. It follows that \(\text{PSL}(2, O_d/2) \cong A_5\).

(i) \(d \neq -3\). \(B_d^e\) is of level 2. Suppose that \(B_d^e\) is congruence. By Theorem 3.6, \(B_d(2) \subseteq B_d^o\).

Hence \(B_d^o / B_d(2)\) is a normal subgroup of \(B_d / B_d(2) \cong \text{PSL}(2, O_d/2)\). By Lemma 6.1, \(B_d^o = B_d\). This is a contradiction (Theorem 4.1 and Table 1). Hence \(B_d^e\) and \(B_d^o\) are not congruence.

(ii) \(d = -3\). Let \(w = \sqrt{-3}\). \(B_{-3} / B_{-3}(w) \cong A_4\) has a unique normal subgroup \(U / B_{-3}(w)\) of index 3. In particular, \(B_{-3} / U\) is abelian. Hence \(B_{-3}' / U\) is abelian. Then \(B_{-3}' / U \cong \mathbb{Z}_3\). This implies that \(U\) and \(B_{-3}^e\) have the same index. Hence \(U = B_{-3}'\). As a consequence, \(B_{-3}'(w) \subseteq U = B_{-3}\). Hence \(B_{-3}'\) is congruence of level 3. Note that \(B_{-3}' / B_{-3} = B_{-3}^o\).

\(\square\)

**Proposition 6.3.** Suppose that \(d \not\equiv 5 \mod 8\). Then \(B_d^e\) is non-congruence if and only if \(|B_d / B_d^o| \geq 8\). In particular, if the class number of \(O_d\) is three or more, then \(B_d^e\) is non-congruence.

**Proof.** The level of \(B_d^o\) is 2. Suppose that \(B_d^o\) is congruence. By Theorem 3.6, \(B_d^o(2) \subseteq B_d^o\).

Note that \(B_d / B_d(2) \cong \text{PSL}(2, O_d/2)\). Since \(d \not\equiv 5 \mod 5\), \(2\) is not a prime. By Lemma 6.1, \(B_d / B_d(2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). In particular, \(|B_d / B_d^o| < 8\). Conversely, suppose that \(|B_d / B_d^o| < 8\).

Then \(|B_d / B_d^o| = 1, 2, 4\). Let \(U \subseteq \text{PSL}(2, O_d) = B_d\) be the pre-image of \(D\) (see (6.4) of Lemma 6.1). Then \(B_d / U \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). This implies that \(B_d^o = U\). Since \(B_d(2) \subseteq U\), \(B_d^o\) is congruence. This completes the proof of the first part of the proposition.

Suppose that the class number of \(O_d\) is three or more. By Theorem 4.1, \(|B_d / B_d^o| \geq 8\). Suppose that \(B_d^o\) is congruence. Then \(B_d(2) \subseteq B_d^o\) (Theorem 3.6) and \(|B_d / B_d^o| < B_d / B_d(2) \cong \text{PSL}(2, O_d/2)\). In the case \(d \equiv 5 \mod 8\), \(\text{PSL}(2, O_d/2) \cong A_5\) is simple. Hence \(B_d^o = B_d\) or \(B_d(2)\). This contradicts the fact that \(B_d / B_d^o\) is abelian of order 8 or more. Hence \(B_d^o\) is non-congruence. In the case \(d \not\equiv 5 \mod 8\), \(B_d^o\) is not congruence by the first part of the present proposition. \(\square\)
Discussion 6.4. The set of $d$’s ($d < 0$ is square free) such that the class number of $O_d$ is 2 or less is $\{-1, -2, -3, -7, -11, -19, -43, -67, -163\} \cup \{-5, -6, -10, -13, -15, -22, -35, -37, -51, -58, -91, -115, -123, -187, -235, -267, -403, -427\}$.  

6.2. One may apply the proof of Proposition 6.2 and conclude immediately that

**Proposition 6.5.** Suppose that $d \equiv 5 \pmod{8}$. Then $PE_d'$ and $PE_d^2$ are non-congruence if and only if $d \neq -3$.

In the case $d \equiv 5 \pmod{8}$, one has the following.

**Proposition 6.6.** Suppose that $d \not\equiv 5 \pmod{8}$. Then $PE_d^2$ is congruence.

*Proof.* By Table 1, $PE_d/PE_d^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 6.1, $PE_d/PE_d(2) \cong PSL(2, O_d/2)$ has a normal subgroup $D$ such that $PSL(2, O_d/2)/D \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Denoted by $V$ the pre-image of $D$ in $PE_d$. Then $PE_d/V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $PE_d^2 \cong V$. Note that $PE_d(2) \subseteq V$. Hence $PE_d^2$ is congruence. \hfill \qed

7. Third Application : Congruence Subgroup Property (CSP)

The main purpose of this section is to show that if $G = PSL(2, O_d)$ has CSP, then $G/[G, G]$ is a finite group of order $2^a 3^b$ for some $a, b$ (Proposition 7.2). Note that $B_d$ do not have CSP and $B_t/[B_d, B_d]$ is finite only if $d = -1, -3$.

**Proposition 7.1.** Suppose that $G = PSL(2, \mathcal{O})$ has CSP. Let $S$ be a subgroup of $G$ of index $g$, level $n$. Then $[S/[S, S]]$ is finite. Let $q$ be a prime divisor of $[S/[S, S]]$. Suppose that $q$ is inert or $q \geq 5$ is split. Then $q$ divides $|S/[S, S]|$. In particular, if $q$ is a prime divisor of $G/[G, G] < \infty$, then $q$ is either ramified or split. In the case $q$ is split, $q = 2$ or 3.

*Proof.* Since $G$ has CSP, $S/[S, S]$ is finite (Section 16 of [BMS]). The second part of the proposition follows immediately by applying Lemma 5.2. \hfill \qed

**Proposition 7.2.** Let $O_d$ be the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$ and let $G = PSL(2, O_d)$. Let $q \in \mathbb{N}$ be a prime. Suppose that $q$ divides $|G/[G, G]|$. Then $q$ is ramified or split and $q = 2$ or 3. In particular, $|G/[G, G]| = 2^a 3^b$.

*Proof.* Since $\mathbb{Q}(\sqrt{d})$ has a unit of infinite order, $G$ has CSP (Serre [S]). By our assumption, $G$ has a normal subgroup of index $q$. By Lemma A1, $q$ is ramified or split and $q \leq 5$. \hfill \qed

Proposition 7.2 suggests the following.

**Conjecture 7.3.** Suppose that $G = PSL(2, \mathcal{O})$ has CSP. Let $q$ be a prime divisor of $|G/[G, G]|$. Then $q$ is ramified or split and $q = 2$ or 3. In particular, $|G/[G, G]| = 2^a 3^b$.

8. Appendix A

Let $q \geq 5$ be a rational prime that is ramified in $O_d$, the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$. Let $\pi$ be the prime ideal that divides $(q, x)$ and let $(q, x)$ be an integral basis of $q$. Let $G = PSL(2, O_d)$. Then $G/G(q)$ is an extension of $G(\pi)/G(q)$ by $G/G(\pi) \cong PSL(2, O_d/\pi) = PSL(2, q)$, where $G(\pi)/G(q) \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$.

$$G(\pi)/G(q) \cong \left\langle r = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, s = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 1-x & x \\ -x & 1+x \end{pmatrix} \right\rangle. \tag{A1}$$

Let $t = rs^{-1}u$. $t \pmod{q}$ can be found in (A2). It is clear that $\langle r, s, u \rangle = \langle r, s, t \rangle \pmod{q}$.

$$G/G(\pi) \cong \left\langle S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, t \equiv \begin{pmatrix} 1-x & 0 \\ 0 & 1+x \end{pmatrix}. \tag{A2}$$

Set $r^g = grg^{-1}$. Direct calculation shows that (see (i) and (ii) of Lemma 2.4 for matrix multiplication of $G(\pi)/G(q)$)

$$r^S = s^{-1}, s^S = r^{-1}, t^S = t^{-1}, r^T = rs^{-1}t^{-1}, s^T = s, t^T = s^2 t. \tag{A3}$$
Lemma A1. Suppose that $G = PSL(2, O_d)$ has CSP. Let $q$ be a rational prime. Suppose that $q \in O_d$ is inert or $q \geq 5$. Then $G$ has no normal subgroups of index $q$.

Proof. Suppose that $G$ has a normal subgroup $K$ of index $q$. Since $G$ has CSP, $G(q) \subseteq K$ (Theorem 3.6). In the case $q$ is inert, $G/K$ is a normal subgroup of the simple group $G/G(q) \cong PSL(2, q^2)$. A contradiction. In the case $q \geq 5$ is split, $G/K$ is a normal subgroup of index $q$ of $G/G(q) \cong (SL(2, q) \times SL(2, q))/\mathbb{Z}_2$. Again, a contradiction. Hence we may assume that $q \geq 5$ is ramified. Let $\pi$ be the prime ideal that divides $(q) = \pi^2$. Since $q \geq 5$, $PSL(2, O_d/\pi)$ is non-abelian simple. Hence $G(\pi)$ is not a subgroup of $K$. Hence $N = K/G(q) \cap G(\pi)/G(q)$ is a normal subgroup of $G/G(q)$ of order $q^2$. Take $1 \neq \sigma = r^a s^t c^d \in N \subseteq G(\pi)/G(p)$. There are four cases to consider.

Case 1. $a \neq b$. Recall that $N$ is normal. $\sigma^S \sigma = r^{a-b}s^{b-a}c \in N$. Hence $rs^{-1} \in N$. It follows that $(rs^{-1})^{-1}(rs^{-1})^T = (st)^{-1} \in N$. Hence $(st)(st)^{-1}T = s^{-2}t \in N$. It follows that $s \in N$. By (A3) and the fact that $N$ is normal, $r, t \in N$. Hence $N$ has order $q^3$. A contradiction.

Case 2. $a = b = 0$. Hence $t^c \in N$. This implies that $t \in N$. By (A3) and the fact that $N$ is normal, $r, s \in N$. Hence $N$ has order $q^3$. A contradiction.

Case 3. $a = b \neq 0$ and $2c/a - 1 = 0 \pmod{q}$. Hence $r^a s^t c^d \in N$. It follows that $rst^d \in N$, where $d = c/a$. Hence $(rst^d)^{-1}(rst^d)^T = t^{-1} \in N$. Similar to Case 2, this is a contradiction.

Case 4. $a = b \neq 0$ and $2c/a - 1 \neq 0 \pmod{q}$. Similar to Case 3, $rst^d \in N$, where $d = c/a$. Hence $(rst^d)^{-1}(rst^d)^T = s^{2d-4}t^{-1} \in N$. It follows that $(s^{2d-4}t^{-1})S(s^{2d-4}t^{-1}) = r^{1-2b^2}a^2d-1 \in N$. Hence $rs^{-1} \in N$. Similar to Case 1, this is a contradiction.

Hence $G$ has no normal subgroups of index $q$.

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