Higher Algebraic $K$-theory for Twisted Laurent Series Rings Over Orders and Semisimple Algebras

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Abstract Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-order in a semisimple $F$-algebra $\Sigma$, $\alpha$ an $R$-automorphism of $\Lambda$. Denote the extension of $\alpha$ to $\Sigma$ also by $\alpha$. Let $\Lambda_\alpha[T]$ (resp. $\Sigma_\alpha[T]$) be the $\alpha$-twisted Laurent series ring over $\Lambda$ (resp. $\Sigma$). In this paper we prove that (i) There exist isomorphisms $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \cong \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$ for all $n \geq 1$. (ii) $G^\text{pr}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \cong G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is an $l$-complete profinite Abelian group for all $n \geq 2$. (iii) $\text{div} G^\text{pr}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$ for all $n \geq 2$. (iv) $G_n(\Lambda_\alpha[T]) \longrightarrow G^\text{pr}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is injective with uniquely $l$-divisible cokernel (for all $n \geq 2$). (v) $K_{-1}(\Lambda), K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups.

Keywords $K$-theory · Twisted Laurent series rings · Semisimple algebras · Orders · Virtually infinite cyclic group

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1. Introduction

Let $R$ be the ring of integers in a number field $F$. The initial motivation for this work was a desire to obtain results on higher $K$-theory of the groupring $RV$ of a virtually infinite cyclic group of the form $V = G \rtimes_\alpha T$, where $G$ is a finite group, $\alpha$...
an automorphism of $G$ and the action of the infinite cyclic group $T = (t)$ on $G$ is given by $\alpha(g) = t g t^{-1}$ for all $g \in G$.

Note that understanding the $K$-theory of $RV$ is fundamental to the Farrell-Jones conjecture which asserts that $K$-theory of an arbitrary discrete group $H$ should have as “building blocks” the $K$-theory of virtually cyclic subgroups of $H$ (see [8]). A group $V$ is virtually cyclic if it is either finite or virtually infinite cyclic (i.e., contains a finite index subgroup that is infinite cyclic). For results on higher $K$-theory of groupings of finite groups see [15, chapter 7] and associated references. There are two types of virtually infinite cyclic groups — one type of the form $V = G \cong T$ as described above and the other of the form $V = G_0 \ast H G_1$, where the groups $G_0, G_1, H$ are finite and $[G_0 : H] = [G_1 : H] = 2$. For some results on higher $K$-theory of both types of groups see [15, 7.5] or [16]. In this paper, we obtain results on higher $K$-theory of twisted Laurent series ring that translate into results on groupings $RV$, $V = G \cong T$, as we now explain.

If $\alpha$ is an automorphism of a finite group $G$, we also denote by $\alpha$ the automorphism induced on $RG$ by $\alpha$ and observe that for $V = G_\alpha \cong T$, $RV = (RG)_\alpha (T) = (RG)_\alpha [t, t^{-1}]$ is the $\alpha$-twisted Laurent series ring over the group $RG$. Now, $RG$ is an $R$-order in the semi-simple $F$-algebra $FG$ and so, we endeavour in this paper to obtain general results on higher $K$-theory of $\Lambda_\alpha (T)$ where $\Lambda$ is an arbitrary $R$-order in a semi-simple $F$-algebra $\Sigma$ so that results on $(RG)_\alpha (T)$ become examples and applications of our results.

Note also that an automorphism of $\Lambda$ extends to an $F$-automorphism of $\Sigma$ which we also denote by $\alpha$. We also study higher $K$-theory of $\Sigma_\alpha (T)$ and prove in Theorem 1(b) that there exist isomorphisms

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha (T)) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha [T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha [T])$$

for all $n \geq 2$. Hence $\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq \mathbb{Q} \otimes K_n(FV)$ for all $n \geq 2$. Since we have shown in Theorem 1(a) that $G_n(\Lambda_\alpha [T])$ is a finitely generated Abelian group for all $n \geq 1$, it follows that $K_n(\Lambda_\alpha [T]), K_n(\Sigma_\alpha [T])$ and hence $K_n(RV), K_n(FV)$ have finite torsion-free ranks for all $n \geq 2$.

We next investigate under what conditions $G_n(\Lambda_\alpha [T])$ could actually be a finite group and show in Theorem 6 that when $F$ is a totally real number field with ring of integers $R$ and $\Lambda$ any $R$-order in a semi-simple $F$-algebra, then $G_{2(m+1)}(\Lambda_\alpha [T])$ is finite for all odd $m \geq 1$. Hence $G_{2(m+1)}(RV)$ is finite.

In Section 3, we study profinite higher $K$-theory of $\Lambda_\alpha [T]$ and prove that $G^p_n(\Lambda_\alpha [T], \hat{\mathbb{Z}}_l) = G_n(\Lambda_\alpha [T], \hat{\mathbb{Z}}_l)$ are $l$-complete profinite Abelian groups; $\delta G^p_n(\Lambda_\alpha [T], \hat{\mathbb{Z}}_l) = 0$; and that the map $G_n(\Lambda_\alpha [T]) \rightarrow G^p_n(\Lambda_\alpha [T], \hat{\mathbb{Z}}_l)$ is injective with uniquely divisible cokernel. Corresponding results follow when we replace $\Lambda_\alpha [T]$ by $RV$.

In a final section, we prove that if $F$ is an algebraic number field with ring of integers $R$ and $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$, then $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha [T])$ are finitely generated Abelian groups; $NK_{-1}(\Lambda, \alpha) = 0$ and $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. That $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha [T])$ are finitely generated for arbitrary $R$-orders $\Lambda$ generalizes similar results by D. Carter for $K_{-1}(RG)$ $(G$ a finite group, see [4]) resp. by Farrell/Jones for $K_{-1}(\mathbb{Z}V)$ (see [9]).

Notes on notation If $\alpha$ is an automorphism of a ring $A$, we shall write $A_\alpha [T] = A_\alpha [t, t^{-1}]$ for the $\alpha$-twisted Laurent series ring over $A$. Note that additively $A_\alpha [T] = \bigoplus重心 Springer
\( A_\alpha[t, t^{-1}] \) with multiplication given by \((at^i) \cdot (bt^j) = a\alpha^{-1}(b)t^{i+j} \) for \( a, b \in A \). \( A_\alpha[t] \) (resp. \( A_\alpha[r^{-1}] \)) is the subring of \( A_\alpha[T] \) generated by \( A \) and \( t \) (resp. \( A \) and \( t^{-1} \)). Call \( A_\alpha[t] \) the \( \alpha \)-twisted polynomial ring over \( A \). We also have inclusion maps \( i^+: A \to A_\alpha[T], i^+: A \to A_\alpha[t] \) and \( i^-: A \to A_\alpha[T] \).

The augmentation map \( \varepsilon: A_\alpha[t] \to A \) induces a group homomorphism \( \varepsilon_\alpha: K_n(A_\alpha[t]) \to K_n(A) \) and we put \( NK_n(A, \alpha) := \ker \varepsilon_\alpha \). Since \( \varepsilon \) is split by \( i^+ \), we have \( K_n(A_\alpha[t]) \cong K_n(A) \oplus NK_n(A, \alpha) \).

If \( B \) is an additive Abelian group and \( m \) is a positive integer, we shall write \( B/m \) for \( B/mB \) and \( B[m] \) for the set of elements \( x \) of \( B \) such that \( mx = 0 \). We write \( \text{div} B \) for the subgroup of divisible elements of \( B \). If \( l \) is a rational prime, we write \( B_l \) for the \( l \)-primary subgroup of \( B \). Note that \( B_l = \bigcup B[F] = \varprojlim B[F] \).

2. Higher K-theory of \( \Lambda_\alpha[T], \Sigma_\alpha[T] \) (\( \Lambda \) Arbitrary Orders)

2.1. \( K_n(\Lambda_\alpha[T]), G_n(\Lambda_\alpha[T]), K_n(\Sigma_\alpha[T]) \)

2.1.1 Let \( R \) be the ring of integers in a number field \( F \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \), \( \alpha \) an \( R \)-automorphism of \( \Lambda \). Then \( \alpha \) can be extended to an \( F \)-automorphism of \( \Sigma \) (since \( \Sigma = \Lambda \otimes_R F \)). The aim of this section is to prove the following theorem.

**Theorem 1** Let \( F \) be an algebraic number field with ring of integers \( R \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \), \( \alpha \) an \( R \)-automorphism of \( \Lambda \). Denote the extension of \( \alpha \) to \( \Sigma \) also by \( \alpha \). Let \( \Lambda_\alpha[T] \) (resp. \( \Sigma_\alpha[T] \)) be the \( \alpha \)-twisted Laurent series ring over \( \Lambda \) (resp. \( \Sigma \)). Then we have

(a) \( G_n(\Lambda_\alpha[T]) \) is a finitely generated Abelian group for all \( n \geq 1 \).
(b) \( \text{There exist isomorphisms:} \)

\[
\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \cong \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \cong \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])
\]

for \( n \geq 2 \).

Before proving Theorem 1 we state the following consequence of the result.

**Corollary 1** Let \( V = G \rtimes_{\alpha} T \) be the virtually infinite cyclic subgroup where \( G \) is a finite group, \( \alpha \in \text{Aut}(G) \) and the action of \( T \) on \( G \) is given by \( \alpha(g) = t g t^{-1} \), for all \( g \in G \). Then,

(a) \( G_n(RV) \) is a finitely generated Abelian group for al \( n \geq 1 \).
(b) \( \mathbb{Q} \otimes K_n(RV) \cong \mathbb{Q} \otimes G_n(RV) \cong \mathbb{Q} \otimes K_n(FV) \) for all \( n \geq 2 \).

The proof of Theorem 1(b) will proceed in several steps (see Theorems 3, 4, 5 below). However, we first recall the following result: Theorem 2.

**Theorem 2** ([15, Theorem 7.3.2] or [16]) Let \( R \) be the ring of integers in a number field \( F \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \). If \( \alpha: \Lambda \to \Lambda \) is an \( R \)-automorphism, then there exists an \( R \)-order \( \Gamma \subset \Sigma \), such that

1. \( \Lambda \subset \Gamma \);
2. \( \Gamma \) is \( \alpha \)-invariant;
3. \( \Gamma \) is (right) regular ring. In fact \( \Gamma \) is (right) hereditary.
Theorem 3 Let \( R \) be the ring of integers in a number field \( F \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra, \( \alpha : \Lambda \to \Lambda \) and \( \Gamma \)-automorphism of \( \Lambda \), \( \Gamma \) an \( \alpha \)-invariant order containing \( \Lambda \) as in Theorem 2, \( \Lambda_\alpha[T] \) (resp. \( \Gamma_\alpha[T] \)) the \( \alpha \)-twisted Laurent series ring over \( \Lambda \) (resp. \( \Gamma \)). \( \varphi : \Lambda_\alpha[T] \to \Gamma_\alpha[T] \) the map induced by the inclusion \( \Lambda \to \Gamma \). Then the induced homomorphisms \( \varphi_n : K_n(\Lambda_\alpha[T]) \to K_n(\Gamma_\alpha[T]) \) has torsion kernel and cokernel. Hence for all \( n \geq 2 \) we have \( \mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T]) \).

Proof There exists a positive integer \( s \) such that \( s\Gamma \subset \Lambda \) (see [18] or [15]). Put \( q = s\Gamma \). Then \( q \) is an ideal of \( \Gamma \) and \( \Lambda \). Put \( B = \Lambda/q, B' = \Gamma/q \). Then we have cartesian squares

\[
\begin{array}{ccc}
\Lambda & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
\]  

and

\[
\begin{array}{ccc}
\Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\
\downarrow & & \downarrow \\
B_\alpha[T] & \longrightarrow & B'_\alpha[T].
\end{array}
\]

So, by [5] and [19], we have a long exact sequence

\[
\cdots \to K_{n+1}(B'_\alpha[T]) \left( \frac{1}{s} \right) \to K_n(\Lambda_\alpha[T]) \left( \frac{1}{s} \right) \to K_n(\Gamma_\alpha[T]) \left( \frac{1}{s} \right) \oplus K_n(B_\alpha[T]) \left( \frac{1}{s} \right) \to K_n(B'_\alpha[T]) \left( \frac{1}{s} \right) \to \cdots
\]

(3)

Now, \( \Gamma, B, B' \) are quasi-regular rings, so are \( \Gamma_\alpha[T], B_\alpha[T] \) and \( B'_\alpha[T] \) (see [9]). If we write \( A \) for \( B_\alpha[T] \) or \( B'_\alpha[T] \), \( JA \) for the Jacobson’s radical of \( A \), then by [19] \( K_n(A, JA) \) is \( s \)-torsion since \( s \) annihilates \( A \) and so from the relative sequence

\[
\cdots \to K_n(A, JA) \to K_n(A) \to K_n(A/JA) \to \cdots
\]

we have \( K_n(A) \left( \frac{1}{s} \right) \simeq K_n(A/JA) \left( \frac{1}{s} \right) \). We now claim that \( K_n(A) \left( \frac{1}{s} \right) \simeq K_n(A/JA) \left( \frac{1}{s} \right) \) is torsion.

Proof of the claim Note that \( A/JA \simeq (A'/JA')_\alpha[T] \) is a regular ring (see [9]) where \( A'/JA' \) is a finite semi-simple ring which is a finite direct product of matrix algebras over finite fields. Hence \( K_n((A'/JA')_\alpha[T]) \) is a finite direct sum of \( K \)-groups of the form \( K_n((F_i)_\alpha[T]) \) where \( F_i \) is a finite field. Also, \( (F_i)_\alpha[T] \) is a regular ring and so \( K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T]) \). 

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Now, for each $F_i$, we have by [15, Theorem 7.5.3(iii)] or [16], that there exists a long exact sequence

$$
\cdots \to G_n(F_i) \to G_n(F_i) \to G_n((F_i)_\alpha[T]) \to G_{n-1}(F_i) \to G_{n-1}(F_i) \to \cdots
$$

(4)

where each $G_n(F_i)$ is a finite Abelian group for $n \geq 2$ — by [15, Theorem 7.1.12] or by Quillen’s result. So, from Eq. 4 above, $G_n((F_i)_\alpha[T])$ is finite for all $n \geq 2$, i.e. $K_n((F_i)_\alpha[T])$ is a finite Abelian group. Hence $(K_n((A'/JA')_\alpha[T]))$ is a finite direct sum of Abelian groups of the form $K_n(F_i)_\alpha[T]$ is a finite group. Hence $K_n((A'/JA')_\alpha[T])(\frac{1}{2})$ is torsion. So, for $A = B_\alpha(T)$ or $B'_\alpha[T]$, $K_n(A)(\frac{1}{2}) \simeq K_n((A/JA)(\frac{1}{2}))$ is torsion and $\mathbb{Q} \otimes K_n(A)(\frac{1}{2}) = 0$.

So, by tensoring the Mayer-Vietoris exact sequence Eq. 3 with $\mathbb{Q}$ we get an isomorphism

$$
\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])
$$

for all $n \geq 2$. \hfill \Box

**Theorem 4** Let $R$, $F$, $\Lambda$, $\alpha$; $\Gamma$, $\Lambda_\alpha[T]$, $\Gamma_\alpha[T]$ be as in Theorem 3. Let $\varphi_n : G_n(\Gamma_\alpha[T]) \to G_n(\Lambda_\alpha[T])$ be the homomorphism induced by the exact functor $\mathcal{M}(\Gamma_\alpha[T]) \to \mathcal{M}(\Lambda_\alpha[T])$ given by ‘restriction of scalars’. Then for all $n \geq 2$, $\varphi_n$ has finite kernel and torsion cokernel and hence induces an isomorphism

$$
\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])
$$

**Proof** First note that the exact functor $\mathcal{M}(\Gamma) \to \mathcal{M}(\Lambda)$ given by ‘restriction of scalars’ yields group homomorphisms $\delta_n : G_n(\Gamma) \to G_n(\Lambda)$. Now, by replacing the maximal order $\Gamma$ in the proof of [15, Theorem 7.2.3, p. 146] or [16] with the $\alpha$-invariant order $\Gamma$ containing $\Lambda$, as in Theorem 2, we have that for all $n \geq 1$, $\delta_n : G_n(\Gamma) \to G_n(\Lambda)$ has finite kernel and cokernel. The proof in [15, Theorem 7.2.3] works for this $\Gamma$ also. Now from [15, Theorem 7.5.3(b)] or [16], we have the following horizontal exact sequence and hence a commutative diagram

$$
\begin{array}{cccccc}
G_n(\Gamma) & \xrightarrow{1-\alpha_\epsilon} & G_n(\Gamma) & \xrightarrow{\delta_n} & G_n(\Gamma_\alpha[T]) & \xrightarrow{\varphi_n} & G_{n-1}(\Gamma) & \xrightarrow{1-\alpha_\epsilon} & G_{n-1}(\Gamma) \\
\downarrow \delta_n & & \downarrow \delta_n & & \downarrow \varphi_n & & \downarrow \delta_{n-1} & & \downarrow \delta_{n-1} \\
G_n(\Lambda) & \xrightarrow{1-\alpha_\epsilon} & G_n(\Lambda) & \xrightarrow{\delta_n} & G_n(\Lambda_\alpha[T]) & \xrightarrow{\varphi_n} & G_{n-1}(\Lambda) & \xrightarrow{1-\alpha_\epsilon} & G_{n-1}(\Lambda)
\end{array}
$$

(5)

By taking kernels and cokernels of vertical arrows in Eq. 5, we have a top (resp. bottom) horizontal exact sequence consisting of kernels (resp. cokernels) of the vertical maps. Since we saw above that $\delta_n$ has finite kernels and cokernels, we then have that $\phi_n : G_n(\Gamma_\alpha[T]) \to G_n(\Lambda_\alpha[T])$ has finite kernel and cokernel for each $n \geq 2$. Hence $\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$. But $\Gamma_\alpha[T]$ is regular. Hence

$$
\mathbb{Q} \otimes K_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]).
$$
Theorem 5 Let \( R, F, \Sigma, \Lambda, \alpha, T \) be as in Theorem 1. Then for all \( n \geq 2 \), the map \( \theta_n : G_n(\Lambda_\alpha[T]) \rightarrow G_n(\Sigma_\alpha[T]) \) induced by the canonical map \( \Lambda_\alpha[T] \rightarrow \Sigma_\alpha[T] \) has finite kernel and torsion cokernel. Hence
\[
\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]).
\]

Proof Note that the canonical (inclusion) map \( \Lambda \rightarrow \Sigma \) induces a group homomorphism \( \rho_n : G_n(\Lambda) \rightarrow G_n(\Sigma) \simeq K_n(\Sigma) \) (note that \( G_n(\Sigma) \simeq K_n(\Sigma) \) since \( \Sigma \) is regular).

Now, by [15, Theorem 7.5.3(b)] or [16], we have the following horizontal exact sequences and hence a commutative diagram
\[
\begin{array}{ccccccccc}
G_n(\Lambda) & \xrightarrow{1-\alpha_n} & G_n(\Lambda) & \xrightarrow{\rho_n} & G_n(\Lambda_\alpha[T]) & \xrightarrow{\delta_n} & G_n-1(\Lambda) & \xrightarrow{\rho_{n-1}} & G_n-1(\Lambda) \\
\downarrow{\rho_n} & & \downarrow{\rho_n} & & \downarrow{\delta_n} & & \downarrow{\rho_{n-1}} & & \downarrow{\rho_{n-1}} \\
G_n(\Sigma) & \xrightarrow{1-\alpha_n} & G_n(\Sigma) & \xrightarrow{\theta_n} & G_n(\Sigma_\alpha[T]) & \xrightarrow{\beta_n} & G_n-1(\Sigma) & \xrightarrow{\beta_n} & G_n-1(\Sigma) \\
\end{array}
\]

(6)

Now, from the commutative diagram
\[
\begin{array}{ccc}
G_n(\Lambda) & \xrightarrow{\rho_n} & G_n(\Sigma) \\
\downarrow{\delta_n} & & \downarrow{\beta_n} \\
K_n(\Gamma) & & \\
\end{array}
\]

(7)

we have
\[
0 \rightarrow \ker \delta_n \rightarrow \ker \beta_n \rightarrow \ker \rho_n \rightarrow \coker \delta_n \rightarrow \coker \beta_n \rightarrow \coker \rho_n \rightarrow 0
\]

Now, by the proof of Theorem 4, \( \ker \delta_n \) and \( \coker \delta_n \) are finite. Also by [15, Theorem 7.2.2] or [12], \( \ker \beta_n \) is finite and \( \coker \beta_n \) is torsion for all \( n \geq 2 \). Hence from diagram Eq. 7 above, \( \ker \rho_n \) is finite and \( \coker \rho_n \) is torsion for all \( n \geq 2 \). It then follows from the diagram Eq. 6 above that \( \ker \theta_n \) is finite and \( \coker \theta_n \) is torsion.

Proof of Theorem 1 (a) From [15, Theorem 7.5.3(b)] or [16], we have an exact sequence
\[
G_n(\Lambda) \xrightarrow{1-\alpha_n} G_n(\Lambda) \rightarrow G_n(\Lambda_\alpha[T]) \rightarrow G_n-1(\Lambda) \xrightarrow{1-\alpha_n} G_n-1(\Lambda)
\]

Also by [15, Theorem 7.1.13] or [10] \( G_n(\Lambda) \) is a finitely generated Abelian group for all \( n \geq 1 \). Hence \( G_n(\Lambda_\alpha[T]) \) is finitely generated for all \( n \geq 2 \). (b) That \( \mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \) follow from Theorem 2 i.e. \( \mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]) \) and Theorem 3 i.e. \( \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]) \).

Remark 1 Since by Theorem 1(a), \( G_n(\Lambda_\alpha[T]) \) is finitely generated Abelian group for all \( n \geq 2 \), it follows that \( K_n(\Lambda_\alpha[T]) \) and \( K_n(\Sigma_\alpha[T]) \) have finite torsion free rank just like \( G_n(\Lambda_\alpha[T]) \).

Hence if \( V = G \rtimes_\alpha T \) is a virtually infinite cyclic group, then \( K_n(RV) \), \( K_n(FV) \) have finite torsion-free rank for \( n \geq 2 \).

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2.2. Finiteness of $G_{2(m+1)}(\Lambda_a[T])$

In this subsection, we investigate under what circumstances $G_n(\Lambda_a[T])$ could actually be a finite group. We prove below (see Theorem 6) that if $F$ is a totally real field, then the group $G_{2(m+1)}(\Lambda_a[T])$ is finite for all odd positive integers $m$. We state this formally:

**Theorem 6** Let $R$ be the ring of integers in a totally real number field $F$, $\Lambda$ an $R$-order in a semi-simple $F$-algebra, $\alpha : \Lambda \to \Lambda$ and $R$-automorphism. Then for all odd positive integers $m$, $G_{2(m+1)}(\Lambda_a[T])$ is a finite group. Hence in the notation of Theorem 1, $G_{2(m+1)}(RV)$ is finite.

The proof of Theorem 6 will make use of the following:

**Theorem 7** Let $F$ be a number field with ring of integers $R$, $\Lambda$ and $R$-order in a semi-simple $F$-algebra $\Sigma$. Then (a) For all $n \geq 1$, $G_{2n}(\Lambda)$ is a finite group. (b) If $F$ is totally real, then $G_{2m+1}(\Lambda)$ is also finite for all odd $m \geq 1$.

**Proof** Part (a) is proved in [15] and [14]. See [15, Theorem 7.2.7].

If $F$ is a totally real number field with ring of integers $O_F$, a similar proof works. We only have to show that $K_{2m+1}(\Gamma)$ is finite if $\Gamma$ is a maximal order in a central division algebra $D$ over a totally real number field $F$ with ring of integer $O_F$. Let the dimension of $D$ over $F$ be $s^2$. We know from [15, Theorem 7.1.11] or [11] that $K_{2m+1}(\Gamma)$ is finitely generated. We only need to show that $K_{2m+1}(\Gamma)$ is torsion. Let $\text{tr} : K_{2m+1}(\Gamma) \to K_{2m+1}(O_F)$ be the transfer map and $i : K_{2m+1}(O_F) \to K_{2m+1}(\Gamma)$ the map induced by the inclusion map $O_F \to \Gamma$. Let $x \in K_{2m+1}(\Gamma)$. Then $i \circ \text{tr}(x) = x^{s^2}$. But $K_{2m+1}(\Gamma)$ is finite since it is also finitely generated. (See [2] for the proof that $K_{2m+1}(O_F)$ is torsion).

**Proof of Theorem 6** Assume that $m$ is an odd positive integer. The we have an exact sequence

$$\cdots \to G_{2m+2}(\Lambda) \xrightarrow{1-a_n} G_{2m+2}(\Lambda) \xrightarrow{\beta} G_{2m+2}(\Lambda_a[T]) \xrightarrow{\gamma} G_{2m+1}(\Lambda) \to \cdots$$

where $G_{2m+2}(\Lambda)$ is finite by Theorem 7(a) and $G_{2m+1}(\Lambda)$ is finite by Theorem 7(b). So $G_{2m+2}(\Lambda_a[T])/\text{Im} \beta \cong \text{Im} \gamma$.

But $\text{Im} \beta$ is finite and $\text{Im} \gamma$ is also finite as a subgroup of the finite group $G_{2m+1}(\Lambda)$. Note that $\text{Im} \beta$ is finite as a homomorphic image of the finite group $G_{2m+2}(\Lambda)$. Hence $G_{2m+2}(\Lambda_a[T])$ is finite for all odd positive integers $m$.

**3. Mod-$l^s$ and Profinite Higher K-theory of $\Lambda_a(T)$**

**3.1. Mod-$l^s$ Theory**

**3.1.1** Let $C$ be an exact category, $l$ a rational prime, $s$ a positive integer, $M_{p+1}^n$ the $(n+1)$-dimensional mod-$l^s$-space, i.e. the space obtained from $S^n$ by attaching an $(n+1)$-cell via a map of degree $l^s$ (see [3, 15, 17]).

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If $X$ is an $H$-space, let $[M^n_{p+1}, X]$ be the set of homotopy classes of maps from $M^n_{p+1}$ to $X$. We shall write $\pi_{n+1}(X, \mathbb{Z}/l^n)$ for $[M^n_{p+1}, X]$. If $C$ is an exact category and we put $X = BQC$, we write $K_n(C, \mathbb{Z}/l^n)$ for $\pi_{n+1}(BQC)$, we write $K_n(C, \mathbb{Z}/l^n)$ for $\pi_{n+1}(C, \mathbb{Z}/l^n)$ and $K_0(C, \mathbb{Z}/l^n)$ for $K_0(C) \otimes \mathbb{Z}/l^n$. We shall refer to $K_n(C, \mathbb{Z}/l^n)$ as mod-$l^n$ $K$-theory of $C$.

3.2.1 From [15, 8.1.2] or [13], we have an exact sequence

$$K_n(C) \xrightarrow{l^n} K_n(C) \xrightarrow{\rho} K_n(C, \mathbb{Z}/l^n) \xrightarrow{\beta} K_{n-1}(C) \rightarrow K_{n-1}(C)$$

and hence a short exact sequence for all $n \geq 2$

$$0 \rightarrow K_n(C)/l^n \rightarrow K_n(C, \mathbb{Z}/l^n) \rightarrow K_n(C)[l^n] \rightarrow 0$$

where $K_n(C)[l^n] = \{ x \in K_n(C) \mid l^n x = 0 \}$.

**Example 1**

(i) Let $A$ be a ring with identity and $\mathcal{P}(A)$ the category of finitely generated projective $A$-modules. We write $K_n(A, \mathbb{Z}/l^n)$ for $K_n(\mathcal{P}(A), \mathbb{Z}/l^n)$. We are interested in $A = \Lambda_\omega(T)$. Note that $K_n(A, \mathbb{Z}/l^n)$ is also $\pi_n(BGL(A)^+, \mathbb{Z}/l^n)$.

(ii) Let $A$ be a Noetherian ring and $\mathcal{M}(A)$ the category of finitely generated $A$-modules. We write $G_n(A, \mathbb{Z}/l^n)$ for $K_n(\mathcal{M}(A), \mathbb{Z}/l^n)$.

(iii) Let $Y$ be a scheme, $C = \mathcal{P}(Y)$ the category of locally free sheaves of $O_Y$-modules of finite rank. We write $K_n(X, \mathbb{Z}/l^n)$ for $K_n(\mathcal{P}(Y), \mathbb{Z}/l^n)$ and observe that for $Y = \text{Spec}(A)$, $A$ a commutative ring, we recover $K_n(A, \mathbb{Z}/l^n)$ as in (i).

(iv) Let $Y$ be a Noetherian scheme and $\mathcal{M}(Y)$ the category of coherent sheaves of $O_Y$-modules. We write $G_n(Y, \mathbb{Z}/l^n)$ for $K_n(\mathcal{M}(Y'), \mathbb{Z}/l^n)$ and when $Y = \text{Spec}(A)$, where $A$ is commutative, then we recover $G_n(A, \mathbb{Z}/l^n)$ as in (ii) above.

(v) It follows from Section 3.1.2 that we have exact sequences

$$0 \rightarrow K_n(\Lambda_\omega[T])/l^n \rightarrow K_n(\Lambda_\omega[T], \mathbb{Z}/l^n) \rightarrow K_n(\Lambda_\omega[T])[l^n] \rightarrow 0$$

and

$$0 \rightarrow G_n(\Lambda_\omega[T])/l^n \rightarrow G_n(\Lambda_\omega[T], \mathbb{Z}/l^n) \rightarrow G_n(\Lambda_\omega[T])[l^n] \rightarrow 0$$

3.2.2 Profinite Higher $K$-theory

3.2.1 Let $C$ be an exact category, $l$ a rational prime, $s$ a positive integer $M^n_{p+1} = \lim_{s \to \infty} M^n_{p+1}$. We define the profinite $K$-theory of $C$ by $K^n_{pr}(C, \mathbb{Z}_l) := [M^n_{p+1}, BQC]$. We write $K_n(C, \mathbb{Z}_l)$ for $\lim_{s \to \infty} K_n(C, \mathbb{Z}/l^n)$.

For more details on these constructions and their properties, see [15, Chapter 8] or [13].

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Example 2

(i) For $\mathcal{C} = \mathcal{P}(A)$ as in Example 1(i), we shall write $K_n^{pr}(A, \hat{\mathbb{Z}}_d)$ for $K_n^{pr}(\mathcal{P}(A), \hat{\mathbb{Z}}_d)$ and $K_n(A, \hat{\mathbb{Z}}_d)$ for $K_n(\mathcal{P}(A), \hat{\mathbb{Z}}_d)$.

(ii) For $\mathcal{C} = \mathcal{M}(A)$ as in Example 1(ii), we shall write $G_n^{pr}(A, \hat{\mathbb{Z}}_d)$ for $K_n^{pr}(\mathcal{M}(A), \hat{\mathbb{Z}}_d)$ and $G_n(A, \hat{\mathbb{Z}}_d)$ for $K_n(\mathcal{M}(A), \hat{\mathbb{Z}}_d)$.

(iii) For $\mathcal{C} = \mathcal{P}(Y)$ as in Example 1(iii) we shall write $K_n^{pr}(Y, \hat{\mathbb{Z}}_d)$ for $K_n^{pr}(\mathcal{P}(Y), \hat{\mathbb{Z}}_d)$ and $K_n(Y, \hat{\mathbb{Z}}_d)$ for $K_n(\mathcal{P}(Y), \hat{\mathbb{Z}}_d)$.

(iv) For $\mathcal{C} = \mathcal{M}(Y)$ as in Example 1(iv), we shall write $G_n^{pr}(Y, \hat{\mathbb{Z}}_d)$ for $K_n^{pr}(Y, \hat{\mathbb{Z}}_d)$ and $G_n(Y, \hat{\mathbb{Z}}_d) = K_n(\mathcal{M}(Y), \hat{\mathbb{Z}}_d)$.

Remark 2 From the results obtained earlier by this author for general exact categories, (see [15, Chapter 8] or [13]) we can already deduce the following for $\mathcal{P}(\Lambda_a[T])$ and $\mathcal{M}(\Lambda_a[T])$.

(i) From [15, Lemma 8.2.1], we have the following exact sequences for $n \geq 1$.

(a) $0 \rightarrow \lim_{\leftarrow s} K_{n+1}(\Lambda_a[T], \mathbb{Z}/p) \rightarrow K_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d) \rightarrow K_n(\Lambda_a[T], \hat{\mathbb{Z}}_d) \rightarrow 0$.

(b) $0 \rightarrow \lim_{\leftarrow s} G_{n+1}(\Lambda_a[T], \mathbb{Z}/p) \rightarrow G_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d) \rightarrow G_n(\Lambda_a[T], \hat{\mathbb{Z}}_d) \rightarrow 0$.

(ii) From [15, Theorem 8.2.2] we have for all $n \geq 2$.

(a) $\lim_{\leftarrow s} K_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)[p] = 0$; $\lim_{\leftarrow s} K_{n+1}(\Lambda_a[T], \mathbb{Z}/p) = \text{div } K_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)$;

(b) $\lim_{\leftarrow s} G_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)[p] = 0$; $\lim_{\leftarrow s} G_{n+1}(\Lambda_a[T], \mathbb{Z}/p) = \text{div } G_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)$.

(iii) From [15, Lemma 8.2.2] or [13], we have

(a) $\lim_{\leftarrow s} K_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)/p \simeq K_n(\Lambda_a[T], \hat{\mathbb{Z}}_d)$;

(b) $\lim_{\leftarrow s} G_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d)/p \simeq G_n(\Lambda_a[T], \hat{\mathbb{Z}}_d)$.

3.3. Some Computations

3.3.1 The aim of this subsection is to prove Theorem 8 below. Before stating the result, we first explain the construction of map $\varphi$ in Theorem 8(c) below.

Note that for any exact category $\mathcal{C}$, the natural map $M_{l_{\infty}}^{n+1} \rightarrow S^{n+1}$ induces a map

$$[S^{n+1}, BQC] \xrightarrow{\varphi} [M_{l_{\infty}}^{n+1}, BQC], \quad \text{i.e.,}$$

$$K_n(\mathcal{C}) \xrightarrow{\varphi} K_n^{pr}(\mathcal{C}, \hat{\mathbb{Z}}_d).$$

So when $\mathcal{C} = \mathcal{M}(\Lambda_a[T])$ we have a map

$$\varphi : G_n(\Lambda_a[T]) \rightarrow G_n^{pr}(\Lambda_a[T], \hat{\mathbb{Z}}_d).$$
Theorem 8 Let \( R \) be the ring of integers in a number field \( F \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \), \( \alpha : \Lambda \to \Lambda \) an \( R \)-automorphism of \( \Lambda \), \( \Lambda_a[T] \) the \( \alpha \)-twisted Laurent series ring over \( \Lambda \). Then, for all \( n \geq 2 \):

(a) \( \text{div } G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) = 0. \)

(b) \( G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \cong G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \) is an \( l \)-complete profinite Abelian group.

(c) The map \( G_n(\Lambda_a[T]) \to G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \) is injective with uniquely \( l \)-divisible cokernel.

Proof (a) From Remark 2(ii)(b), we have

\[
\lim_{s}^{1} G_{n+1}(\Lambda_a[T], \mathbb{Z}/l^s) = \text{div } G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l), \tag{8}
\]

for all \( n \geq 2 \). Now, by Theorem 1(a) \( G_n(\Lambda_a[T]) \) is finitely generated for all \( n \geq 1 \). Hence \( G_n(\Lambda_a[T], \mathbb{Z}/l^s) \) is finite for all \( n \geq 1 \). In particular, \( G_{n+1}(\Lambda_a[T], \mathbb{Z}/l^s) \) is finite for all \( n \geq 2 \) and so \( \lim_{s} G_{n+1}(\Lambda_a[T], \mathbb{Z}/l^s) = 0 \) for all \( n \geq 2 \). Hence from Eq. 8, \( \text{div } G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) = 0 \) for all \( n \geq 2 \).

(b) We saw in (a) above that \( G_n(\Lambda_a[T], \mathbb{Z}/l^s) \) is a finite group for all \( n \geq 1 \). Hence in the exact sequence

\[
0 \to \lim_{s}^{1} G_{n+1}(\Lambda_a[T], \mathbb{Z}/l^s) \to G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \to G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \to 0
\]

we have \( \lim_{s}^{1} G_{n+1}(\Lambda_a[T], \mathbb{Z}/l^s) = 0 \). Hence,

\[
G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \cong G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l). \tag{9}
\]

Now, by Remark 2(ii)(b),

\[
G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l)/l^s \cong G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l). \tag{10}
\]

So, from Eqs. 9 and 10 \( G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l)/l^s \cong G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \) i.e. \( G^{pr}_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \cong G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) \) is \( l \)-complete. It is profinite since \( G_n(\Lambda_a[T], \hat{\mathbb{Z}}_l) = \lim_{s} G_n(\Lambda_a[T], \mathbb{Z}/l^s) \) where each \( G_n(\Lambda_a[T], \mathbb{Z}/l^s) \) is a finite group.

(c) Since for all \( n \geq 1 \), \( G_n(\Lambda_a[T]) \) is a finitely generated Abelian group (see 2.1.1(a)), it follows that \( G_n(\Lambda_a[T]) \) is a finite group for each \( n \). Hence \( G_n(\Lambda_a[T]) \) has no non-trivial divisible subgroups. Hence by [15, Corollary 8.2.1] or [13], kernel and cokernel of \( \phi \) are uniquely \( l \)-divisible. But \( G_n(\Lambda_a[T]) \) is finitely generated and so, \( \ker \phi = \text{div } \ker \phi = 0 \), as subgroups of \( G_n(\Lambda_a[T]) \). \( \square \)

4. \( K_{-1}(\Lambda), K_{-1}(\Lambda_a[T]), \Lambda \) Arbitrary Orders

4.1. Finite Generation of \( K_{-1}(\Lambda), K_{-1}(\Lambda_a[T]) \).

Let \( R \) be the ring of integers in a number field \( F \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \), \( \alpha : \Lambda \to \Lambda \) and \( R \)-automorphism of \( \Lambda \), \( \Lambda_a[T] \), the \( \alpha \)-twisted Laurent polynomial ring over \( \Lambda \). We prove in this section that \( K_{-1}(\Lambda) \) and \( K_{-1}(\Lambda_a[T]) \) are finitely generated Abelian groups for arbitrary \( R \)-orders \( \Lambda \) in semi-simple \( F \)-algebras. Note that the proof in [9] by Farrell/Jones is for \( \Lambda = \mathbb{Z}G \), \( G \) a finite group.
Also D. Carter shows in [4] that \( K_{-1}(RG) \) is finitely generated and here we show that this result also holds more generally for arbitrary orders.

Finally we prove also that \( NK_{-1}(\Lambda, \alpha) = 0 \) and so, \( K_{-1}(\Lambda, [t]) \simeq K_{-1}(\Lambda) \).

**Theorem 9** Let \( F \) be an algebraic number field with ring of integers \( R \), \( \Lambda \) any \( R \)-order in a semi-simple \( F \)-algebra \( \Sigma \), \( \alpha : \Lambda \to \Lambda \) an \( R \)-automorphism of \( \Lambda \), \( \Lambda_{\alpha}[T] \) the \( \alpha \)-twisted Laurent series ring over \( \Lambda \). Then

(a) \( K_{-1}(\Lambda) \) is a finitely generated Abelian group.
(b) \( K_{-1}(\Lambda_{\alpha}[T]) \) is a finitely generated Abelian group.
(c) \( K_{-1}(\Lambda) \simeq K_{-1}(\Lambda_{\alpha}[T]) \).

**Proof** (a) Let \( \Gamma \) be a maximal \( R \)-order containing \( \Lambda \). Then, there exists a positive integer \( s \) such that \( s\Gamma \subseteq \Lambda \). Then \( q = s\Gamma \) is an ideal of \( \Lambda \) and \( \Gamma \). Put \( B = \Lambda/q \), \( B' = \Gamma/q \). Then we have a cartesian square

\[
\begin{array}{ccc}
\Lambda & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
\]

and hence a Mayer-Vietoris sequence

\[
\cdots \to K_1(B') \to K_0(\Lambda) \to K_0(\Gamma) \oplus K_0(B) \to K_0(B') \\
\to K_{-1}(\Lambda) \to K_{-1}(\Gamma) \oplus K_{-1}(B) \to \cdots \tag{11}
\]

Now by [1, Prop. 10.1, p. 685], \( K_{-i}(A) = 0 \) for \( i \geq 1 \) and any quasi-regular ring \( A \). Note that \( B, B' \) are finite rings and hence quasi-regular. Also \( \Gamma \) is quasi-regular. Hence for \( A = B, B' \) or \( \Gamma, K_{-i}(A) = 0 \) for \( i \geq 1 \). So the sequence Eq. 11 becomes

\[
\cdots \to K_0(\Lambda) \to K_0(\Gamma) \oplus K_0(B) \to K_0(B') \to K_{-1}(\Lambda) \to 0. \tag{12}
\]

To show that \( K_{-1}(\Lambda) \) is finitely generated it suffices from Eq. 12 to show that \( K_0(B') \) is finitely generated. Now \( B' \) is a finite Artinian ring and so, by [1, p. 465], \( K_0(B') \simeq K_0(B'/JB') \) where \( JB' \) = radical of \( B' \). But \( B'/JB' \) is a finite semi-simple ring and so, \( K_0(B') \simeq K_0(B'/JB') \) is a finite direct sum of \( K_0 \) of (finite) fields each of which is isomorphic to \( \mathbb{Z} \). Hence \( K_0(B') \) is a (free) Abelian group of finite rank and hence is finitely generated. Hence \( K_{-1}(\Lambda) \) is finitely generated.

(b) Let \( \Gamma \) be an \( \alpha \)-invariant order containing \( \Lambda \) as in Corollary 1. Let \( s \) be a positive integer such that \( s\Gamma \subseteq \Lambda \) and put \( q = s\Gamma \), \( B = \Lambda/q \), \( B' = \Gamma/q \). Then we have cartesian squares

\[
\begin{array}{ccc}
\Lambda & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
\]

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and
\[\Lambda_\alpha[T] \longrightarrow \Gamma_\alpha[T]\]
\[\downarrow \quad \downarrow\]
\[B_\alpha[T] \longrightarrow B'_\alpha[T]\]
and hence a Mayer-Vietoris sequence
\[
\cdots \longrightarrow K_0(\Lambda_\alpha[T]) \longrightarrow K_0(\Gamma_\alpha[T]) \oplus K_0(B_\alpha[T]) \\
\longrightarrow K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T]) \longrightarrow 0.
\]
where \(\Gamma_\alpha[T], B_\alpha[T]\) and \(B'_\alpha[T]\) are quasi-regular (see [9]). If \(A = \Gamma_\alpha[T], B_\alpha[T]\) or \(B'_\alpha[T]\) and \(T^n\) is the free Abelian group of rank \(n\). Then by [1, Prop. 10.1], \(K_{-n}(A) = 0\) for \(n \geq 1\).

Also, by Serre’s theorem \(K_0(A) \rightarrow K_0(A[T^n])\) is an epimorphism (see [7]). Since \(K_{-n}(A)\) is a direct summand of the cokernel of \(K_0(A) \rightarrow K_0(A[T^n])\) we have \(K_{-n}(A) = 0\) for \(n \geq 1\). So from the exact sequence Eq. 11, we have \(K_{-n}(\Lambda_\alpha[T]) = 0\) for \(n \geq 2\) and \(K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T])\) is an epimorphism.

By mapping the Mayer-Vietoris sequence associated with the cartesian square Eq. 11 to the Mayer-Vietoris sequence associated with square Eq. 12, we have a commutative square
\[
\begin{array}{ccc}
K_0(B') & \longrightarrow & K_{-1}(\Lambda) \\
\downarrow & & \downarrow \\
K_0(B'_\alpha[T]) & \longrightarrow & K_{-1}(\Lambda_\alpha[T]).
\end{array}
\]
To prove that \(K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])\) is an epimorphism, it suffices to prove that \(K_0(B') \longrightarrow K_0(B'_\alpha[T])\) is an epimorphism in the commutative diagram
\[
\begin{array}{ccc}
K_0(B') & \longrightarrow & K_0(B'_\alpha[T]) \\
\downarrow & & \downarrow \\
K_0(B'/JB') & \longrightarrow & K_0((B'/JB')_\alpha[T])
\end{array}
\]
where the vertical maps are isomorphisms. Also by [7, Theorem 27], the map \(K_0(B'/JB') \longrightarrow K_0((B'/JB')_\alpha[T])\) is an epimorphism. Hence \(K_0(B') \longrightarrow K_0(B'_\alpha[T])\) is an epimorphism. So \(K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])\) is an epimorphism. Since by (a), \(K_{-1}(\Lambda)\) is finitely generated, then \(K_{-1}(\Lambda_\alpha[T])\) is also finitely generated.

(c) By definition, \(K_{-1}(\Lambda_\alpha[T]) \simeq K_{-1}(\Lambda) \oplus NK_{-1}(\Lambda, \alpha).\) So it suffices to show that \(NK_{-1}(\Lambda, \alpha) = 0.\)

Let \(\Lambda, \Gamma, B = \Lambda/q, B' = \Gamma/q\) be as in the proof of (a) (b). Then we have two cartesian squares
\[
\begin{array}{ccc}
\Lambda & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
\]
and
\[ \begin{array}{ccc}
\Lambda_\alpha[t] & \longrightarrow & \Gamma_\alpha[t] \\
\downarrow & & \downarrow \\
B_\alpha[t] & \longrightarrow & B'_\alpha[t]
\end{array} \]  \tag{18}

where $\Gamma_\alpha[t]$, $B_\alpha[t]$ and $B'_\alpha[t]$ are quasi-regular as well as $\Gamma$, $B$, $B'$. Hence we have Mayer-Vietoris sequences
\[ \cdots \rightarrow K_0(\Lambda_\alpha[t]) \rightarrow K_0(\Gamma_\alpha[t]) \oplus K_0(B_\alpha[t]) \rightarrow K_0(B'_\alpha[t]) \rightarrow K_{-1}(\Lambda_\alpha[t]) \rightarrow \cdots \]  \tag{19}

and
\[ \cdots \rightarrow K_0(\Lambda) \rightarrow K_0(\Gamma) \rightarrow K_0(B) \rightarrow K_0(B') \rightarrow K_{-1}(\Lambda) \rightarrow \cdots \]  \tag{20}

where for $A = \Gamma$, $B$, $B'$, $\Gamma_\alpha[t]$, $B_\alpha[t]$, $B'_\alpha[t]$, $K_{-i}(A) = 0$ for $i \geq 1$ (see [1, Prop. 10.1]). By mapping Eqs. 19 to 20 and taking kernels, we have that
\[ NK_{-1}(\Lambda, \alpha) = \text{coker}(NK_0(\Gamma, \alpha) \oplus NK_0(B, \alpha) \rightarrow NK_0(B', \alpha)). \]

So it suffices to show that $NK_0(B', \alpha) = 0$. Since $B', B'_\alpha[t]$ are quasi-regular, the result follows from [6, Lemma 2.4]. So $NK_{-1}(\Lambda, \alpha) = 0$ and hence $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. □

**Corollary 2** Let $R$ be the ring of integers in a number field $F$, $V = G \rtimes_\alpha T$ a virtually infinite cyclic group where $G$ is a finite group and the action of the infinite cyclic group $T$ on $G$ is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$. Then $K_{-1}(RV)$ is a finitely generated Abelian group.

**Corollary 3** Let $\alpha$ be an automorphism of a finite group $G$, $R$ the ring of integers in a number field $F$. Denote the induced automorphism on $RG$ also by $\alpha$. Then $K_{-1}(RG) \simeq K_{-1}((RG)_\alpha[t])$ is a finitely generated Abelian group.

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