Twin-width and Limits of Tractability of FO Model Checking on Geometric Graphs

Petr Hliněný
Masaryk University, Brno, Czechia
Filip Pokrývka
Masaryk University, Brno, Czechia

Abstract
The complexity of the problem of deciding properties expressible in FO logic on graphs – the FO model checking problem (parameterized by the respective FO formula), is well-understood on so-called sparse graph classes, but much less understood on hereditary dense graph classes. Regarding the latter, a recent concept of twin-width [Bonnet et al., FOCS 2020] appears to be very useful. For instance, the question of these authors [CGTA 2019] about where is the exact limit of fixed-parameter tractability of FO model checking on permutation graphs has been answered by Bonnet et al. in 2020 quite easily, using the newly introduced twin-width. We prove that such exact characterization of hereditary subclasses with tractable FO model checking naturally extends from permutation to circle graphs (the intersection graphs of chords in a circle). Namely, we prove that under usual complexity assumptions, FO model checking of a hereditary class of circle graphs is in \( \text{FPT} \) if and only if the class excludes some permutation graph. We also prove a similar excluded-subgraphs characterization for hereditary classes of interval graphs with FO model checking in \( \text{FPT} \), which concludes the line a research of interval classes with tractable FO model checking started in [Ganian et al., ICALP 2013].

The mathematical side of the presented characterizations – about when subclasses of the classes of circle and permutation graphs have bounded twin-width, moreover extends to so-called bounded perturbations of these classes.

2012 ACM Subject Classification Mathematics of computing \( \rightarrow \) Graph theory; Theory of computation \( \rightarrow \) Fixed parameter tractability

Keywords and phrases twin-width, FO model checking, circle graph, interval graph, FPT

Funding Supported by the Czech Science Foundation, project no. 20-04567S. The Dagstuhl seminar 21391 Sparsity in Algorithms, Combinatorics and Logic (September 2021) significantly contributed to part of these results.

1 Introduction

So-called algorithmic meta-theorems receive considerable attention in theoretical computer science. One particular challenge in this direction is about tractability of the model checking problem for first-order logic (FO) on graphs – given a graph \( G \) and an FO formula \( \phi \), the task to decide whether \( G \) satisfies \( \phi \) (written as \( G \models \phi \)). This task is trivially solvable in time \( |V(G)|^{O(|\phi|)} \). “Efficient solvability”, or tractability, hence in this context often means fixed-parameter tractability (FPT); that is, solvability in time \( f(|\phi|) \cdot |V(G)|^{O(1)} \) for some computable function \( f \).

While for the FO model checking problem on sparse graph classes we know a full answer—Grohe, Kreutzer and Siebertz [17] proved that FO model checking is FPT on nowhere dense graph classes, while it is intractable (under usual complexity assumptions) on all monotone somewhere dense classes—much less is known about the problem on hereditary dense graph classes. Research in this direction has recently received a new strong stimulus in the form of the notion of twin-width, introduced in 2020 by Bonnet, Kim, Thomassé and Watrigant [8].

The basic definition of twin-width, in a condensed form, is as follows.
A trigraph is a simple graph $G$ in which some edges are marked as red, and we then naturally speak about red neighbors and red degree in $G$. We denote the set of red neighbors of a vertex $v$ by $N_r(v)$. For a pair of (possibly not adjacent) vertices $x_1, x_2 \in V(G)$, we define a contraction of the pair $x_1, x_2$ as the operation creating a trigraph $G'$ which is the same as $G$ except that $x_1, x_2$ are replaced with a new vertex $x_0$ whose full neighborhood is the union of neighborhoods of $x_1$ and $x_2$ in $G$, that is, $N(x_0) = (N(x_1) \cup N(x_2)) \setminus \{x_1, x_2\}$, and the red neighbors $N_r(x_0)$ of $x_0$ inherit all red neighbors of $x_1$ and $x_2$ and those in $N(x_1) \triangle N(x_2)$, that is, $N_r(x_0) = ((N_r(x_1) \cup N_r(x_2)) \setminus \{x_1, x_2\}) \cup (N(x_1) \triangle N(x_2))$. A contraction sequence of a trigraph $G$ is a sequence of successive contractions turning $G$ into a single vertex, and its width is the maximum red degree of any vertex in any trigraph of the sequence. The twin-width is the minimum width over all possible contraction sequences (where for an ordinary graph, we start with the same trigraph having no red edges).

This new notion has already found many very interesting applications, which span from efficient parameterized algorithms and algorithmic metatheorems, through finite model theory, to classical combinatorial questions. See the (still growing) series of follow-up papers [2, 7, 9].

In particular, graph classes of bounded twin-width have FO model checking in FPT, assuming the input graph comes with a suitable contraction sequence. The input assumption is crucial, since finding the exact value of twin-width is para-NP-hard [2], and we so far have no efficient approximation or asymptotic algorithm for it. Generally, bounded twin-width (of hereditary graph classes) does not characterize tractability of FO model checking; a prominent example are the graphs of bounded degree with tractable FO model checking [20] and unbounded twin-width [4]. However, for some natural types of graphs, such an exact characterization is true. For instance, [8] have proved this for the permutation graphs; a hereditary class of permutation graphs has FO model checking in FPT if and only if it has bounded twin-width, which is if and only if it excludes some permutation graph. This provided a full answer to a question in an earlier paper by these authors [18].

Directly inspired by the permutation-graph case, we extend the result to circle graphs (the intersection graphs of chords in a circle) – they again have bounded twin-width and FO model checking in FPT, if and only if they exclude some fixed permutation graph. The characterization of bounded twin-width (but not the complexity part) extends even to a certain asymptotic generalization of circle graphs. We also investigate interval graphs, which represent one of the early examples of dense graph classes on which FO model checking was systematically studied [16]. For them we prove an exact characterization of tractable FO model checking analogous to the circle-graph case, and again extend the bounded twin-width characterization to an asymptotic generalization of interval graphs. This answers some of the questions left open in [16], and in [14, 15] specifically for interval graphs.

It is worth to note that in both directions which we study here, boundedness of twin-width is more or less explicitly related to expressibility of some permutations in the classes. In related direction, Bonnet et al. [2] proved that a graph class is of bounded twin-width, if and only if the class is an FO transduction of a proper class of permutations. This general asymptotic result is not directly comparable with the characterizations we prove here. On the other hand, another current paper of Bonnet et al. [3] independently deals with related aspect of twin-width of interval graphs, and we discuss this below in details.

1.1 Outline of the paper

In Section 2 we give an overview of the necessary concepts from graph theory and logic; namely about intersection graphs (permutation, interval and circle graphs), the twin-width measure and its basic properties, and about FO interpretations and transductions.
In Section 3, we present a unified handling of interval and circle intersection representations, and prove technical claims for use within the coming main results.

In Section 4, we focus on the circle graphs, and prove the following.

(Theorem 4.1) In a hereditary class $C$ of circle graphs, these are equivalent: that (i) $C$ is of bounded twin-width, (ii) $C$ excludes some (fixed) permutation graph, and (iii) FO model checking on $C$ is in FPT (under the assumption of ETH).

(Theorem 4.3) In a hereditary class $C$ of graphs which are “bounded perturbations” of circle graphs, these are equivalent: that (i) $C$ is of bounded twin-width, and (ii) $C$ excludes some (fixed) permutation graph.

In Section 5, we deal with interval graphs. For every permutation $\pi$, we define a finite set of graphs which are said to expose $\pi$, and prove the following.

(Theorem 5.2) In a hereditary class $C$ of interval graphs, these are equivalent: that (i) $C$ is of bounded twin-width, (ii) $C$ excludes all graphs which expose some (fixed) permutation, and (iii) FO model checking on $C$ is in FPT (under the assumption of ETH).

(Theorem 5.3) In a hereditary class $C$ of graphs which are “bounded perturbations” of interval graphs, these are equivalent: that (i) $C$ is of bounded twin-width, and (ii) $C$ excludes all graphs which expose some (fixed) permutation.

In Section 6, we conclude our findings, state open questions and outline future research directions on the studied topic.

Independently of our research, Bonnet, Chakraborty, Kim, Köhler, Lopes and Thomassé [3] have just now come with a result closely related to our Theorem 5.2. In their terminology, they prove that interval graphs are efficiently delineated (see [3, Theorem 23]), which means that for every hereditary class $C$ of interval graphs has bounded twin-width (with an efficient algorithm to get a contraction sequence), if and only if the class $C$ does not transduce all finite graphs (see Section 2.3 for transductions). This is equivalent to our Theorem 5.2 without the explicit obstructions in part (c) (although, their proof reveals similar asymptotic obstructions, which is quite natural given the result). The details of the proofs in [3] and here are not much similar, and [3] do not include bounded perturbations of interval graphs.

## 2 Preliminaries and formal definitions

A (simple) graph is a pair $G = (V, E)$ where $V = V(G)$ is the finite vertex set and $E = E(G)$ is the edge set – a set of unordered pairs of vertices $\{u, v\}$, shortly $uv$. For a set $Z \subseteq V(G)$, we denote by $G[Z]$ the subgraph of $G$ induced on the vertices of $Z$. The neighborhood of a vertex $v \in V(G)$ (implicitly in the graph $G$) is denoted by $N(v)$, and two vertices $u, v \in V(G)$ are twins if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ (contracting twin vertices, hence, does not create new red edges in the above definition of twin-width). A graph is twin-free if it contains no twin pair of vertices.

We will also deal with matrices (as combinatorial objects); a $P \times Q$ matrix is a matrix whose rows and columns are indexed by the linearly ordered sets $P$ and $Q$, respectively. A submatrix of a matrix $A = (a_{i,j} : i \in P, j \in Q)$ is, for any $P_1 \subseteq P$ and $Q_1 \subseteq Q$, the matrix $A' = (a_{i,j} : i \in P_1, j \in Q_1)$ which we shortly denote by $A' = A[P_1, Q_1]$. If $\pi$ is a permutation of a set $M$, then the permutation matrix of $\pi$ is (in an implicit order on $M$) the $M \times M$ matrix $P_{\pi} = (p_{i,j} : i, j \in M)$ such that $p_{i,j} = 1$ if $\pi(i) = j$ and $p_{i,j} = 0$ otherwise.
2.1 Intersection graphs

The intersection graph $G$ of a finite collection of sets $\{S_1, \ldots, S_n\}$ is a simple graph in which each set $S_i$ is associated with a vertex $v_i \in V(G)$ (then $S_i$ is the representative of $v_i$), and each pair $v_i, v_j$ of vertices is joined by an edge if and only if the corresponding sets have a non-empty intersection, i.e., $v_i v_j \in E(G) \iff S_i \cap S_j \neq \emptyset$.

A traditional example of intersection graphs are interval graphs, which are the intersection graphs of intervals on the real line. Another example we are interested in are circle graphs \cite{12}, which are the intersection graphs of chords of a circle. A special subcase of circle graphs are permutation graphs, which are the intersection graphs of chords which join the lower semicircle with the upper semicircle. The more traditional definition relates a permutation graph $G$ to a permutation $\pi$ on the set $V(G)$, such that $uv \in E(G) \iff \pi(u) \leq \pi(v)$.

One can easily see equivalence of these definitions in the permutation of the $x$-axis orders of the chord ends in the lower and upper semicircle.

For our paper, it will be useful to note the following alternative definition of circle graphs as interval-overlap graphs, see Figure 1. An interval-overlap representation of a graph is a set of intervals on the real line such that two vertices are adjacent, if and only if their intervals intersect, but one is not strictly contained in the other (strictly with respect to both ends). The latter strictness condition means that two intervals sharing an endpoint (at any end) do make an edge. This condition is usually formulated the other way round, but the presented formulation is convenient with respect to coming Definition 3.1. Of course, it does not change the represented class since we can always perturb the representation to make the ends pairwise distinct in the right direction. As Figure 1 shows, there is a trivial correspondence between circle and interval-overlap representations of the same graph.

2.2 Twin-width

In addition to the previous brief introduction of twin-width through red edges of graphs, we formally present the definition of twin-width based on matrices, as taken from \cite[Section 5]{8}. The advantage of it is that it applies (through a matrix representation) to arbitrary binary relational structures of finite signature, and gives a useful asymptotic characterization in Theorem 2.1. See also an illustration in Figure 2.

Let $A$ be a matrix with entries from a finite set (e.g., $\{0, 1, r\}$ for graphs) and let $R$ and $C$ be the set indexing rows and columns of $A$. The special entry $r$ is called a red entry, and the red number of a matrix $A$ is the maximum number of red entries over all columns and rows in $A$. Contraction of two rows (resp. columns) $k$ and $l$ results in the matrix obtained.
by deleting the row (resp. column) $\ell$, and replacing entries of the row (resp. column) $k$ by $r$ whenever they differ from the corresponding entries in the row (resp. column) $\ell$. A sequence of matrices $A = A_n, \ldots, A_1$ is a contraction sequence of $A$, if $A_1$ is a matrix and for all $1 \leq i < n$, the matrix $A_i$ is a contraction of matrix $A_{i+1}$. The twin-width of the matrix $A$ is the minimum integer $d$, such that there exists a contraction sequence $A = A_n, \ldots, A_1$, such that for all $1 \leq i \leq n$, the red number of the matrix $A_i$ is at most $d$.

For example, in case of a graph $G$, we apply the previous to the adjacency matrix of $G$, and we require symmetric contractions, meaning that the contraction of rows $k, \ell$ is always immediately followed by the contraction of the columns $k, \ell$. Then the red entries of the matrices along this contraction sequence correspond to the red edges in the graph, except that graphs do not have red loops corresponding to red entries on the main diagonal. Hence the graph- and the matrix-based values of twin-width need not be exactly the same, but we are in this paper anyway interested in whether the twin-width is asymptotically bounded or not, for which the minor differences are not relevant.

Consider now a fixed linear order $\preceq$ on the row and column indices $R$ and $C$ of $A$. A division of $A$ (wrt. implicit $\preceq$) is a set partition $(R_1, \ldots, R_a)$ of $R$ and $(C_1, \ldots, C_b)$ of $C$ into nonempty consecutive parts in $\preceq$ (for any integers $a, b \geq 1$), and the corresponding collection of submatrices $A[R_i, C_j], 1 \leq i \leq a$ and $1 \leq j \leq b$, which are called the zones of this division. A $k$-mixed minor in $A$ is a division of $A$ in to $k \times k$ parts (i.e., $a = k = b$ above) such that every zone of it contains two distinct rows and two distinct columns. The following is a very useful characterization:

*Theorem 2.1* (Bonnet et al. [8]). a) If a matrix $A$ has twin-width at most $t$, then there is an ordering of the rows and columns of $A$ such that it contains no $(2t + 2)$-mixed minor. b) If there is an ordering of the rows and columns of a matrix $A$ such that it contains no $k$-mixed minor, then the twin-width of $A$ is $2^{2^{O(k)}}$, and the corresponding contraction sequence of $A$ can be computed from the ordering in quadratic time.

### 2.3 FO logic and transductions

A relational signature $\Sigma$ is a finite collection of relational symbols $R_i$, each with associated arity $r_i$. A relational structure $A$ with signature $\Sigma$ (or shortly a $\Sigma$-structure) is defined by a domain $A$ and relations $R_i[A] \subseteq A^{r_i}$ for each relation symbol $R_i \in \Sigma$ (the relations interpret...
the relational symbols). For example, a graph is a structure with the domain \( V \) and a single binary relational symbol \( E \).

First-order logic (abbreviated as FO) applies to \( \Sigma \)-structures as follows. The standard language of first-order logic—including the equality predicate \( x = y \), usual logical connectives \( \land, \lor, \rightarrow \), and quantifiers \( \forall \), \( \exists \) over the domain \( A \)—is used together with the relational symbols \( R_i \in \Sigma \) with the meaning \( A \models R_i(\vec{x}) \iff \vec{x} \in R_i[A] \).

An FO interpretation \( \iota \) of \( \Gamma \)-structures in \( \Sigma \)-structures is a mapping from \( \Sigma \)-structures to \( \Gamma \)-structures defined by an FO formula \( \varphi_0(x) \) and an FO formula \( \varphi_R(x_1, \ldots, x_k) \) for each relation symbol \( R \in \Gamma \) with arity \( k \) (these formulas use the relational symbols of \( \Sigma \)). Given a \( \Sigma \)-structure \( A \), \( \iota(A) \) is the \( \Gamma \)-structure whose domain \( B \) contains all elements \( a \in A \) such that \( A \models \varphi_0(a) \), and in which every relation symbol \( R \in \Gamma \) of arity \( k \) is interpreted as the set of tuples \((a_1, \ldots, a_k) \in B^k \) satisfying \( A \models \varphi_R(a_1, \ldots, a_k) \).

An FO transduction \( \tau \) (here in a simplified non-copying version), on the other hand, maps from \( \Sigma \)-structures to subsets of \( \Gamma \)-structures. In the first step of \( \tau \) — called parameter expansion, one maps a \( \Sigma \)-structure \( A \) into a set \( \mathcal{A}^+ \) of \( \Sigma^+ \)-structures, where \( \Sigma^+ = \Sigma \) extended with a finite number of unary relation symbols and \( \mathcal{A}^+ \) consists of all possible interpretations of the new symbols. In the second step, \( \tau \) is composed of an FO formula \( \alpha \) (informally, \( \alpha \) ‘marks’ valid structures) and an FO interpretation \( \iota \) in \( \mathcal{A}^+ \). Precisely, the outcome of the transduction is the set \( \tau(A) = \{ \mu(B) : B \in \mathcal{A}^+ \land \mu \models \alpha \} \). For a class of structures \( C \), we define \( \tau(C) = \bigcup_{A \in C} \tau(A) \).

### 3 Common matrix representation of interval-like graphs

In this section, we study a special matrix representation that can capture in the same way (the “combinatorial side” of) both circle and interval intersection representations of graphs. For this purpose, we consider the interval-overlap view of a circle representation.

**Definition 3.1.** An interval-like graph representation — of a simple graph \( G \) — is a quadruple \((S, \leq, \eta, \phi)\), where

- \((S, \leq)\) is a linearly ordered set (\( S \) is seen as the set of distinct interval ends),
- \( \eta \subseteq S \times S \) is a binary relation such that \( (s_1, s_2) \in \eta \) implies \( s_1 \leq s_2 \) (the pairs \( (s_1, s_2) \in \eta \) are seen as the represented intervals, and it may happen that \( s_1 = s_2 \)), where the pairs of \( \eta \) then form the vertex set of \( G \) as \( V(G) := \eta \), and
- \( \phi \) is a predicate of four variables over \( S \) determining whether the vertices represented by \((s_1, s_2)\) and \((t_1, t_2)\) form an edge of \( G \): for \( s_1, s_2, t_1, t_2 \in S \) such that \( (s_1, s_2), (t_1, t_2) \in \eta \) and \( (s_1, s_2) \neq (t_1, t_2) \), we have \({\{s_1, s_2\}, \{t_1, t_2\}} \in E(G) \iff \phi(s_1, s_2, t_1, t_2) \) holds true.

We say that this is an FO interval-like representation if \( \phi \) can be expressed as an FO formula.

**Definition 3.2.** Consider an interval-like graph representation \((S, \leq, \eta, \phi)\), and the set \( R := \eta \cup \{(t, t) : t \in S\} \). An il-representation matrix (‘il’ stands for interval-like) is an \( R \times S \) matrix \( A = (a_{i,j} : i \in R, j \in S) \) with entries from \( \{0, 1, 2\} \); such that the columns of \( A \) are ordered by \( \leq \) (of \( S \)) and the rows are ordered by the lexicographic power of \( \leq \). For \( i = (s_1, s_2) \in R \) and \( j \in S \), we have \( a_{i,j} = 2 \) iff \( j \leq s_1 \) in \((S, \leq)\), and \( a_{i,j} = 1 \) iff \( i \in \eta \) and \( j = s_2 \) (which does not conflict with the previous since \( i = (s_1, s_2) \in \eta \) implies \( s_2 \geq s_1 \)).

We also say that \( A \) is an il-representation matrix of the graph \( G \) which is represented by \((S, \leq, \eta, \phi)\) on the vertex set \( V(G) = \eta \) as in Definition 3.1.

Both definitions are illustrated in Figure 3.
(which is, of course, different for each of the two classes). For interval graphs, it is (right) the il-representation matrix (common to both graphs). The “dummy” matrix rows (with no every interval graph (resp., interval-overlap graph)

Therefore, there are traditional polynomial time algorithms for the recognition and construction of an intersection representation of interval graphs and the 6-representation matrix – such that representations. Given an interval or interval-overlap graph

Lemma 3.3. The classes of interval and of interval-overlap graphs have FO interval-like representations. Given an interval or interval-overlap graph G, its interval-like representation and the il-representation matrix – such that ≤ is the same as the order of interval ends in a (respectively) intersection representation of G, can be computed in polynomial time.

Proof. There are traditional polynomial time algorithms for the recognition and construction of an intersection representation of interval graphs and interval-overlap graphs. Furthermore, in both cases it is straightforward to possibly modify the obtained representation so that it does not contain multiple copies of the same interval. The representation immediately gives the linearly ordered set (S, ≤) of interval ends, the relation η of the intervals, and then the il-representation matrix A.

It remains to give the formula φ determining the edges of G from the representation (which is, of course, different for each of the two classes). For interval graphs, it is

\[ φ(s_1, s_2, t_1, t_2) ≡ α(s_1, s_2, t_1, t_2) \lor α(t_1, t_2, s_1, s_2) \text{ where } α(s_1, s_2, t_1, t_2) ≡ (s_1 ≤ t_1 ≤ s_2). \]

For interval-overlap graphs, we simply replace α with α’ where

\[ α'(s_1, s_2, t_1, t_2) ≡ (s_1 ≤ t_1 ≤ s_2 ≤ t_2). \]

Regarding FO logic, the matrix A of Definition 3.2 is viewed as a relational structure with the domain R ∪ S and two binary predicates A_1 and A_2, where A_k(i, j) holds for i, j ∈ R ∪ S and k = 1, 2, if and only if i ∈ R, j ∈ S and a_{i,j} = k in the matrix. We have:

Lemma 3.4. There exist FO interpretations i_1 and i_0 such that the following holds. For every interval graph (resp., interval-overlap graph) G and its il-representation matrix A, the interpretation i_1(A) (resp., i_0(A)) results in a graph isomorphic to G.

Namely, we have i_1 = (σ_0, σ_E) and i_0 = (σ_0, σ_E') where

- σ_0(x) ≡ \exists r A_1(x, r) in both interpretations,
- σ_E(x, y) is the irreflexive and symmetric closure of the formula \[ ∀r [A_1(x, r) \lor A_2(y, r)] \land \forall c [A_1(x, c) \lor \neg A_2(y, c)], \]
- and similarly σ_E'(x, y) is the irreflexive and symmetric closure of \[ ∀r [\neg A_2(x, r)] \land A_2(y, r) \land \forall c [A_1(x, c) \lor \neg A_1(y, c)] \land \forall c, c' [(A_1(x, c) \land A_1(y, c')) \implies \forall r (A_2(r, c) \implies A_2(r, c'))]. \]
Twin-width and Limits of Tractability of FO Model Checking

Proof. The formula $σ₀$ correctly identifies the ground set $V(G) = η$ by Definition 3.2. Considering $x = (s₁, s₂) ∈ η$ and $y = (t₁, t₂) ∈ η$, we have that $∀c[A_2(x, c) ∨ A_2(y, c)]$ if and only if $s₁ ≤ t₁$, and $∀c[A_1(x, c) → ¬A_2(y, c)]$ if and only if $t₁ ≤ s₂$, which both directly follow from Definition 3.2. A bit more complicated is the meaning of the last subformula $∀c, c′[A_1(x, c) ∧ A_1(y, c′)] → ∀r(A_2(r, c) → A_2(r, c′))]$; this claims that for the columns $c = s₂$ and $c′ = t₂$ which are, by Definition 3.2, uniquely identified by entries 1 of $A$ in the rows $x$ and $y$, we have $s₂ ≤ t₂$ again from Definition 3.2 (the “$(t, t)$-rows” are important).

The rest follows by the formulas $α$ and $α'$ from the proof of Lemma 3.3.

One particularly useful consequence of the previous claims is that FO model checking is in FPT for graphs which have il-representation matrices of bounded twin-width. However, one must be careful with right formulation of the opposite direction; since the aforementioned algorithms for interval and interval-overlap representations do not guarantee to give a representation leading to an ordered matrix without large mixed minors, and one can usually come with il-representation matrices of large mixed minors even if those without them exist as well. We overcome this formulation problem with the following technical definition.

In an interval-like representation $(S, ≤, η, φ)$ of a graph $G$ is condensed if there is no legal unification of consecutive ends in $S$ such that the resulting representation $(S′, ≤, η′, φ)$ would represent a graph isomorphic to $G$ (the condensed form may not be unique, especially when the represented graph has twin vertices). An il-representation matrix is condensed if it comes from a condensed interval-like representation. Using Lemmas 3.4 and 3.5, we obtain:

Lemma 3.5. Assume that there is an integer constant $b$, such that every condensed il-representation matrix of an interval or interval-overlap graph $G$ has no $b$-mixed minor. Then the FO model checking problem of $G$ can be solved in FPT time with respect to the formula size.

Proof. We compute an arbitrary interval-like representation of $G$ using Lemma 3.3. Then we greedily make it condensed, which is trivial in polynomial time, and construct the il-representation matrix $A$ of $G$ by Definition 3.2. Then $A$ is of bounded twin-width by Theorem 2.1 and, moreover, by [8, Section 5] we can in polynomial time construct a contraction sequence for $A$ of bounded red degree. Then, by [8, Section 7], we solve FO model checking on $A$ in FPT time with respect to $b$ and the checked formula size.

Now we turn the attention to the FO model checking problem of our graph $G$. Given an interval or interval-overlap graph $G$ and a formula $φ$ on $G$, we take the respective interpretation $ι ∈ \{ι₁, ι₂\}$ of Lemma 3.4 and syntactically translate $φ$ into an FO formula $ι(φ)$ such that $(G ⊨ ι(A)) \models φ ↔ A \models ι(φ)$. Then we call the FPT model-checking algorithm on $A$. For the sake of completeness, we outline the standard translation of $φ$ to $ι(φ)$ where $ι = (σ₀, σ_δ)$; every occurrence of a quantifier $∃v(α)$ in $φ$ is replaced with $∃v(σ₀(v) ∧ α)$, and every occurrence of the edge predicate $edge(v, w)$ is replaced with $σ_δ(v, w)$.

In the opposite situation to Lemma 3.5 when some il-representation matrix $A$ of our graph $G$ contains a large (unbounded in the asymptotic setting) mixed minor, we will “extract” from $A$ a substructure which will, in the coming sections, serve as an explicit certificate of
unbounded twin-width of $G$. This may seem contradictory at the first sight, as mentioned above, since we may come up with an il-representation matrix $A$ of $G$ of large mixed minor, and still may possibly have another il-representation matrix of $G$ without such, which would certify bounded twin-width of $G$ via Lemma 3.4 and [8]. However, an additional assumption of a condensed il-representation matrix will make this approach sound. We will hence (later) implicitly show that the twin-width of various condensed il-representation matrices of the same graphs cannot “range from bounded to unbounded”.

Before we get to the specific obstructions, we derive from Theorem 2.1 the following tool:

**Lemma 3.6.** Let $G$ be an interval or interval-overlap graph. If we have an il-representation matrix $A$ of $G$ with a $(2p + 1)$-mixed minor, then, for every permutation $\pi$ of $p$ elements, there exists a $p \times p$ submatrix $A'$ of $A$ equal to $P_\pi$, such that for every two distinct rows of $A'$ indexed by $(s_1, s_2)$ and $(t_1, t_2)$ (cf. Definition 3.2) we have $s_1 \neq t_1$.

**Proof.** In the assumed $R \times S$ il-representation matrix $A$ of $G$ (cf. Definition 3.2), we pick a $(2p + 1)$-division $D = (P, Q)$ of $A$ such that each zone is mixed – in particular, each zone of the division $D$ contains a nonzero entry. In this division, $P = (P_1, \ldots, P_{2p}, P_{2p+1})$ is a consecutive partition of the row index set $R$, and $Q = (Q_0, Q_1, \ldots, Q_{2p})$ is a consecutive partition of the column index set $S$. Observe that if any zone which is not among the lowest ones or among the leftmost ones, contained an entry 2, then the lowest leftmost zone $A[2p+1, Q_0]$ in $A$ would be homogeneous of all entries 2 by Definition 3.2, a contradiction to it being mixed. Hence all zones $A[P_i, P_j]$ such that $1 \leq i, j \leq 2p$ must contain only entries 0 or 1.

Assume $\pi$ to be a permutation of $\{1, \ldots, p\}$. Then every zone $A[P_{2k}, Q_{2\pi(k)}]$, $k = 1, \ldots, p$, contains a nonzero entry $a_{r_k, c_k} = 1$, and we choose the submatrix $A' := (a_{i, j} : i \in \{r_1, \ldots, r_p\}, j \in \{c_1, \ldots, c_p\})$. We have that every row $k$ of $A'$ contains only one entry 1 by Definition 3.2 and so it is our $a_{r_k, c_k} = 1$ and, indeed, $A' = P_\pi$. It remains to verify the claimed “index condition” on the rows of $A'$. Assume the contrary, that for rows $r_k = (s_1, s_2)$ and $r_{k'} = (t_1, t_2)$ of $A'$ we have $k < k'$ and $s_1 = s_2$. However, then by the lexicographic ordering of row indices of $A$ from Definition 3.2 we get that the zone $A[P_{2k'-1}, Q_{2\pi-1}]$ where $\ell = \min(\pi(k), \pi(k'))$ cannot contain any nonzero entry again by Definition 3.2 (the row indices of this zone are actually of the form $(s_1, t)$ where $\min(c_{k'}, c_{k'}) < t < \max(c_k, c_{k'})$).

This contradicts the fact that $A[P_{2k'-1}, Q_{2\pi-1}]$ is mixed. The statement is proved.

4 Circle graphs

We now formulate and prove our first main result.

**Theorem 4.1.** Let $\mathcal{C}$ be a hereditary class of circle graphs, and let $\mathcal{A}$ be the class of all possible il-representation matrices of graphs from $\mathcal{C}$. Then the following are equivalent:

- a) $\mathcal{C}$ is of bounded twin-width,
- b) there is an integer $b$ such that no ordered matrix of $A$ contains a $b$-mixed minor,
- c) $\mathcal{C}$ does not contain all permutation graphs,
- d) $\text{FO model checking on } \mathcal{C} \text{ is in FPT (under the assumption FPT} \neq \text{AW}[^*], \text{ or usual ETH)}$.

**Proof.** Recall that every permutation graph is also a circle graph, and that the class of circle graphs is the same as the class of interval-overlap graphs (with a direct translation between the representations, cf. Figure 1). We first show the equivalence of (a), (b) and (c).

If $\mathcal{C}$ contains all permutation graphs, then $\mathcal{C}$ is of unbounded twin-width since the class of all permutation graphs has unbounded twin-width [8], establishing (a)$\Rightarrow$(c). For (b)$\Rightarrow$(a),
we start with Theorem 2.1 deriving that the unordered matrix family \( A \) is of bounded twin-width. Then we use that the property of bounded twin-width is preserved under FO interpretations and more generally transductions of relational structures by [8] Section 8. Hence, if \( A \) is of bounded twin-width, then so is \( \mathcal{C} \) by Lemma 3.3.

We are left with the implication (c)⇒(b), for which we assume that no such bound \( b \) exists and choose any permutation \( \pi \) of a \( p \)-element set for some \( p \). Thanks to our assumption, we can pick an \( R \times S \) matrix \( A \in \mathcal{A} \) (an il-representation of a graph \( G \in \mathcal{C} \)) such that \( A \) contains a \( p \times p \) submatrix \( A' = P_{\pi^{-1}} \) as claimed by Lemma 3.6 (note the inverse permutation \( \pi^{-1} \)). Let \( P' \subseteq R \) be the row subset of \( A' \) and \( Q' \subseteq S \) be the column subset of it. Since every row of \( A' \) has an entry 1, we actually get \( P' \subseteq V(G) \).

For any two distinct vertices \( x, y \in P' \) giving entries \( a_{i,j} = 1 \) and \( a_{i',j'} = 1 \) in \( A' \), we have \( i = (s, j) \) and \( i' = (s', j') \) where \( s, s' \in S \) by Definition 3.2, and \( s \leq s' \) up to symmetry. Furthermore, \( j \geq s' \) since \( A' \) has no entry 2. By the definition of an interval-overlap graph, we get that \( xy \in E(G) \) if and only if \( j \leq j' \), which means that the pair \( i, i' \) is not an inversion in \( \pi^{-1} \). In other words, \( xy \in E(G) \) if \( i \) and \( i' \) have \( x \) in \( E(G) \) and \( y \) in \( E(G) \). The induced subgraph \( G[P'] \) is the permutation graph of \( \pi \). Since \( \pi \) is hereditary, it thus contains all permutation graphs.

Lastly, we look at (d). The implication (b)⇒(d) has been proved in Lemma 3.5. Conversely, (d)⇒(c) follows since there exists an FO transduction from the class of all permutation graphs to the class of all graphs [9][18], and FO model checking on the class of all graphs is \( AW[*]\)-complete [13]. The fine assumption \( \text{FPT} \neq AW[*] \) is, moreover, implied by the more usual complexity assumption of the Exponential Time Hypothesis (ETH).

Theorem 4.1 has a notable extension in Theorem 4.3 (unfortunately not yet satisfactory in algorithmic sense). In order to formulate it, we need to define perturbations of graphs. An elementary perturbation of a (simple) graph \( G \) is the operation which chooses an arbitrary subset of vertices \( X \subseteq V(G) \), and then complements the edge relation on the pairs from \( X \). An \( r \)-bounded perturbation of \( G \) performs a sequence of at most \( r \) elementary perturbations on \( G \).

The concept of bounded perturbations is not widespread in graph theory, however, it is closely related to low-rank perturbations of matroids in the matroid-minor structure theory. In fact, bounded perturbations of circle graphs seem to play a very important role in the ongoing investigation of the structural theory of vertex-minors of graphs, see [19] (in this regard, Theorem 4.3 thus may find applications in studying the twin-width of proper vertex-minor closed classes of graphs).

We make use of the following easy technical claim:

**Lemma 4.2.** Let \( r \in \mathbb{N} \) and \( s = 2^r \). Consider a finite set \( Y \) and take the Cartesian power \( Z := Y^s \). For any collection of sets \( X_1, \ldots, X_r \subseteq Z \), there exist \( 1 \leq \ell \leq s \), and \( a_i \in Y \) where \( i = 1, \ldots, \ell - 1 \), and \( b_j \in Y \) where \( j = \ell + 1, \ldots, s \) and \( k \in Y \), such that the following holds: For the set \( Z_0 = \{ (a_1, \ldots, a_{\ell-1}, y, b_{\ell+1}, \ldots, b_k) : y \in Y \} \subseteq Z \) and every \( i = 1, \ldots, r \), we have \( Z_0 \subseteq X_i \) or \( Z_0 \cap X_i = \emptyset \).

In the setting of Lemma 4.2, we call \( Z_0 \) a homogeneous set with prefix \( (a_1, \ldots, a_{\ell-1}) \).

**Proof.** We color each element \( z \in Z = Y^s \) by one of at most \( s \) colors based on the sets among \( X_1, \ldots, X_r \) that \( z \) belongs to (we thus aim for monochromatic \( Z_0 \)). Then we forget \( X_1, \ldots, X_r \) and prove the statement by induction on \( s \), where the base case of \( s = 1 \) is trivial.

For \( s > 1 \), we consider the sets \( A_x = \{ (x, c_2, \ldots, c_s) : c_2, \ldots, c_s \in Y \} \) where \( x \in Y \). If each of the sets \( A_x \) contains an element of color (say) 1, then we are done by setting \( \ell = 1 \). Otherwise, some set \( A_x \) for \( x \in Y \) does not contain any element of color 1, and we apply
induction with \(s - 1\) colors and the set \(A_x\) in place of \(Z\). This way we get a homogeneous set \(Z_1\) with prefix \((a_2, \ldots, a_{r-1})\), and we add \(a_1 := x\) to this prefix in order to finish the claim.

\[\textbf{Theorem 4.3.}\ Let r \in \mathbb{N} and \mathcal{C} be a hereditary class of graphs, such that every graph in \mathcal{C} is an \(r\)-bounded perturbation of a circle graph. Then \mathcal{C} is of bounded twin-width, if and only if \mathcal{C} does not contain all permutation graphs.\]

\[\textbf{Proof.}\ If \mathcal{C} contains all permutation graphs (which themselves have unbounded twin-width, as shown already in Theorem 4.1), then \mathcal{C} is of unbounded twin-width.

Conversely, let \(\mathcal{C}_0\) be a hereditary class of circle graphs, such that every graph in \(\mathcal{C}\) is an \(r\)-bounded perturbation of a graph in \(\mathcal{C}_0\), and that some \(r\)-bounded perturbation of every graph in \(\mathcal{C}_0\) falls into \(\mathcal{C}\). Since an \(r\)-bounded perturbation, for constant \(r\), can be straightforwardly (although a bit technically) expressed as an FO transduction, bounded twin-width of \(\mathcal{C}_0\) would imply bounded twin-width of \(\mathcal{C}\) by \([8\text{ Section 8}]\). Therefore, if \(\mathcal{C}\) is of unbounded twin-width, then \(\mathcal{C}_0\) contains all permutation graphs by Theorem 4.1. To finish that \(\mathcal{C}\) contains every permutation graph \(H\), too, we construct from \(H\) a suitable (much larger) permutation graph \(H^+ \in \mathcal{C}_0\) such that every \(r\)-bounded perturbation of \(H^+\) contains \(H\).

Let \(H\) be a \(p\)-vertex permutation graph determined by a permutation \(\pi_1\) of \(\{1, \ldots, p\}\), and let \(\pi_2\) be the permutation obtained by concatenation (as an order) of \(\pi_1\) followed by the inverse of \(\pi_1\) on \(\{p + 1, \ldots, 2p\}\). Hence, the permutation graph \(H_2\) of \(\pi_2\), on \(V(H_2) = \{1, \ldots, 2p\}\), is the disjoint union of \(H\) and of the complement of \(H\). The permutation \(\pi_2\) is determined by two linear orders; \(1 \leq 1 \leq 2 \leq \ldots \leq 1 \leq 2p\) and \(\pi_2(1) \leq 2 \leq \ldots \leq 2 \leq \pi_2(2p)\). Let \(Z := \{1, \ldots, 2p\}^*\) where \(s = 2^r\), and let the linear orders \(\leq 1\) and \(\leq 2\) on \(Z\) be the standard lexicographic powers of \(\leq 1\) and \(\leq 2\) on \(\{1, \ldots, 2p\}\). Then \(\leq 1\) and \(\leq 2\) on \(Z\) determine a permutation \(\rho\).

Let \(H^+\) be the permutation graph of \(\rho\) on the vertex set \(V(H^+) = Z = V(H_2)^*\), and consider a graph \(G \in \mathcal{C}_0\) such that \(H^+ \subseteq G\) (up to isomorphism). Let \(G' \in \mathcal{C}\) and \(X_1, \ldots, X_r \subseteq V(G)\) be the sets to which an \(r\)-bounded perturbation of \(G\) has been applied, restricted to \(V(H^+)\), such that \(G'\) is the perturbed graph obtained from \(G\). We apply Lemma 4.2 to \(Y := V(H_2)\) (and \(Z = V(H^+)\) as chosen above); this gives us a homogeneous set \(Z_0 \subseteq Z\) with some prefix \((a_1, \ldots, a_{r-1})\). Since the linear orders \(\leq 1\) and \(\leq 2\) on \(Z\) have been obtained as lexicographic powers of \(\leq 1\) and \(\leq 2\) on \(Y\), their restriction to \(Z_0\) is isomorphic to \(\leq 1\) and \(\leq 2\) on \(Y\). In particular, the induced subgraph \(G[Z_0]\) is isomorphic to \(H_2\). Moreover, by Lemma 4.2 the complementations of the \(r\)-bounded perturbation leading to \(G'\) have been applied either to none or to all vertices of \(Z_0\), and so \(G'[Z_0]\) is either \(G[Z_0]\) or its complement. In any case, \(G'[Z_0]\) contains an induced subgraph isomorphic to \(H\), and so does \(G' \in \mathcal{C}\).

Since \(H\) has been chosen as an arbitrary permutation graph, and \(\mathcal{C}\) is hereditary, we get that \(\mathcal{C}\) contains all permutation graphs.

\[\textbf{5 Interval graphs}\]

Our second main result concerns tractability of FO model checking on interval graphs, and it is a bit more complicated to formulate since its “excluded-something” condition deals with finite collections of induced subgraphs to be excluded instead of individual excluded permutation graphs.

Consider a permutation \(\pi\) of \(\{1, \ldots, p\}\), and disjoint sets \(W = \{w_1, \ldots, w_p\}\) and \(W_1, W_2\) where \(|W_1| = |W_2| = p = |W|\). Let \(H\) be a graph on \(W \cup W_1 \cup W_2\) such that every vertex \(w_i \in W\) has nonempty neighborhoods \(N_1(w_i)\) in \(W_1\) and \(N_2(w_i)\) in \(W_2\). Moreover, \(H\) is
such that $N_1(w_1) \subseteq N_1(w_2) \subseteq \ldots \subseteq N_1(w_p)$ and $N_2(w_{\pi(1)}) \subseteq N_2(w_{\pi(2)}) \subseteq \ldots \subseteq N_2(w_{\pi(p)})$. Edges within each of $W$, $W_1$ and $W_2$, and between $W_1$ and $W_2$ can be arbitrary in general. We then say that the graph $H$ exposes the permutation $\pi$. It is easy to observe that for every permutation $\pi$, there exists an interval graph exposing $\pi$, in particular one in which each of the sets $W, W_1, W_2$ induces a clique and there is no edge from $W_1$ to $W_2$ – its interval representation can be simply constructed by arranging the left and right ends of the intervals of $W_1$ according to the permutation $\pi$.

Observe, moreover, that for a graph $H$ which exposes $\pi$, each of the set $N_1(w_i) \setminus N_1(w_{i-1})$, $2 \leq i \leq p$, has to consist of a single element of $W_1$ which we for reference call the mate of $w_i$ in $W_1$. The mate of $w_1$ is the single vertex of $N_1(w_1)$. We analogously define the mate of $w_i$ in $W_2$ as the singleton in $N_2(w_{\pi(i)}) \setminus N_2(w_{\pi(i-1)})$ or $N_2(w_{\pi(1)})$. Altogether, we call the set $W_1 \cup W_2$ the mates of $W$.

With respect to the previous definition, we remark that each of the induced subgraphs $H[W \cup W_1]$ and $H[W \cup W_2]$ is tightly related to FO definable linear orders in graphs, and such subgraphs are sometimes named “ladders” in papers on FO logic of graphs (however, “ladders” often additionally require that the subgraphs induced on $W$ and $W_1$ ($W_2$) are cliques or independent, which is not convenient for us).

Let $R^+_\pi$ denote the (finite) collection of all non-isomorphic graphs $H$ which expose the particular permutation $\pi$, and let $R_\pi \subseteq R^+_\pi$ be the subset of those which are interval graphs. Before the main result, we need a technical claim:

**Lemma 5.1.** There exists an FO transduction $\tau$ from graphs to permutations such that, for every permutation $\pi$, the following holds. If $G$ is a graph containing some member $H \in R^+_\pi$ as an induced subgraph, then the transduction image $\tau(G)$ contains the permutation $\pi$.

**Proof.** In the parameter-expansion part of the desired transduction $\tau$, we “guess” three marks $m, m_1, m_2$ such that $m$ is intended to represent in $V(G)$ the set $W$ and $m_i$, $i = 1, 2$, the set $W_i$ of the graph $H$ (cf. the definition of $H$ exposing $\pi$). In the subsequent FO interpretation, we restrict the domain (of $\pi$) with $\varphi_0(x) \equiv m(x)$, and FO interpret two intended linear orders $\leq_1$ and $\leq_2$ on this domain, for $j = 1, 2$, as $\leq_j(x, y) \equiv \forall z(m_j(z) \rightarrow (\text{edge}(x, z) \rightarrow \text{edge}(y, z)))$ (expressing that $N_j(x) \subseteq N_j(y)$). We can also routinely express in FO logic that $\leq_j, j = 1, 2$, is a linear order; $\alpha_j \equiv \forall x, y(x \leq_j y \lor y \leq_j x) \land (x \leq_j y \land y \leq_j x \rightarrow x = y) \land \forall x, y, z((x \leq_j y \land y \leq_j z) \rightarrow x \leq_j z)$, and so $\tau(G)$ consists of permutations (on subsets of $V(G)$).

Now, if the marks $m, m_1, m_2, m_3$ exactly coincide with the partition $V(H) = W \cup W_1 \cup W_2$ in whole $G$, then the interpreted relations $\leq_1$ and $\leq_2$ are indeed linear orders defining $\pi$ as exposed by $H$, and so $\pi \in \tau(G)$.

**Theorem 5.2.** Let $\mathcal{C}$ be a hereditary class of interval graphs, and let $\mathcal{A}$ be the class of all condensed $\pi$-representation matrices of the twin-free graphs from $\mathcal{C}$. Then the following are equivalent:

\begin{itemize}
  \item[a)] $\mathcal{C}$ is of bounded twin-width,
  \item[b)] there is an integer $b$ such that no ordered matrix of $\mathcal{A}$ contains a $b$-mixed minor,
  \item[c)] for some permutation $\pi$, the class $\mathcal{C}$ excludes all graphs in the collection $R^+_{\pi}$ exposing $\pi$,
  \item[d)] FO model checking on $\mathcal{C}$ is in FPT (under the assumption FPT $\neq$ AW[*], or usual ETH).
\end{itemize}

**Proof.** We proceed along the same steps as in the proof of Theorem 4.1 first showing the equivalence of (a), (b) and (c). Let $\mathcal{C}'$ be the subclass of twin-free members of $\mathcal{C}$.

If, for every permutation $\pi$, the class $\mathcal{C}$ contains a graph isomorphic to a member of $R_{\pi}$, then the transduced class $\tau(\mathcal{C})$ by Lemma 5.1 contains all permutations. Then $\tau(\mathcal{C})$ is of unbounded twin-width by [0], and since the property of bounded twin-width is preserved
under FO transductions of relational structures by \[8\] Section 8, we get (a)⇒(c); that \(C\) is of unbounded twin-width. We similarly (as in \[4,1\]) deduce (b)⇒(a); from Theorem \[2,1\] we get that \(A\) as a family of unordered matrices is of bounded twin-width, and then so is \(C\) which is FO interpreted in \(A\) by Lemma \[3,4\] and each graph in \(C\) is obtained from a graph in \(C\) by adding twin vertices which does not raise the twin-width.

We are left with the implication (c)⇒(b), for which we assume that no such bound \(b\) exists and choose any permutation \(\pi\) of a \(p\)-element set for some \(p\). Thanks to our assumption, we can pick an \(R \times S\) matrix \(A \in A\) (an il-representation of a graph \(G \in C\)) such that \(A\) contains a \(p \times p\) submatrix \(A' = P_\pi\) as claimed by Lemma \[3,6\]. Let \(P' \subseteq R\) be the row subset of \(A'\) and \(Q' \subseteq S\) be the column subset of it. Since every row of \(A'\) has an entry 1, we actually get \(P' \subseteq V(G)\).

Now comes the main difference from the proof of Theorem \[4,1\] we have and will use that \(A \in A\) is a condensed il-representation matrix of twin-free \(G\) (the proof would not go through without this assumption). Pick a vertex \(x \in V(G)\) with the entry \(a_{i,j} = 1\) in \(A\), and so \(i = (s, j)\) for some \(s \in S\); then \(s\) will be called the left end and \(j \in S\) the right end of \(x\) (which naturally corresponds to the geometric image of an interval representation of \(G\) described by \(A\)). We first observe that since \(A'\) has no entry 2, by Definition \[3,2\] all left ends of vertices in \(P'\) are to the left (or equal) of all right ends of these vertices.

Since \(A\) is condensed (and it represents a twin-free graph), the left end of \(x\) must also be the right end of some other vertex \(y\) of \(G\) (but \(y \notin P'\)), as otherwise we could unify this left end with the next higher one in \(S\) without changing the represented interval graph. For reference, we denote by \(l(x) := y\). Symmetrically, the right end of \(x\) must also be the left end of some other vertex \(r(x) = z\) of \(G\) (where \(z \notin P'\)). Let \(H\) be the subgraph of \(G\) induced by the set \(X \cup \{l(x), r(x) : x \in X\}\). Then, as one may routinely verify from the definition of an interval representation by \(A\), we have \(H \subseteq R_\pi\). Since \(C\) is hereditary, it thus contains a graph from \(R_\pi\) (exposing \(\pi\)) for every permutation \(\pi\).

Lastly, we look at (d). The implication (b)⇒(d) has again been proved in Lemma \[3,5\]. Conversely, (d)⇒(c) follows since we have got the transduced class \(\tau(C)\) of all permutations and there exists an FO transduction from the class of all permutations to the class of all graphs \[9\]. As before, FO model checking on the class of all graphs is AW\[*]-complete \[13\], and the fine assumption FPT ≠ AW\[*] is implied by the ETH.

We can again extend the non-algorithmic part of Theorem \[5,2\] to bounded perturbations of interval graphs, using an approach similar to the proof of Theorem \[4,3\].

\[\textbf{Theorem 5.3.}\ Let \(r \in \mathbb{N}\) and \(C\) be a hereditary class of graphs, such that every graph in \(C\) is an \(r\)-bounded perturbation of an interval graph. Then \(C\) is of bounded twin-width, if and only if, for some permutation \(\pi\), the class \(C\) excludes all graphs in the collection \(R_\pi^+\) exposing \(\pi\).

\textbf{Proof.}\ If \(C\) contains some graph from \(R_\pi^+\) for every permutation \(\pi\), then, by Lemma \[5,1\] the transduced class \(\tau(C)\) contains all permutations and hence is of unbounded twin-width, exactly as in the proof of Theorem \[5,2\].

Conversely, let \(C_0\) be a hereditary class of interval graphs, such that every graph in \(C\) is an \(r\)-bounded perturbation of a graph in \(C_0\), and that some \(r\)-bounded perturbation of every graph in \(C_0\) falls into \(C\). Again, since an \(r\)-bounded perturbation can be expressed as an FO transduction, bounded twin-width of \(C_0\) would imply bounded twin-width of \(C\) by \[8\]. Therefore, if \(C\) is of unbounded twin-width, then \(C_0\) contains a graph from \(R_\pi^+\) for every permutation \(\pi\) by Theorem \[5,2\]. To finish, we are going to show that the same holds
also for \( F \), by constructing a suitable (much larger) permutation \( \varrho \) from \( \pi \), such that if we take a graph from \( R_\varrho \), then every \( r \)-bounded perturbation of it contains a graph from \( R_\varrho^+ \).

So, let \( \pi = \pi_1 \) be a permutation of \( \{1, \ldots, p\} \) for some \( p \) and, same as in the proof of Theorem 4.3, let \( \pi_2 \) be the permutation obtained by concatenation (as an order) of \( \pi_1 \) followed by the inverse of \( \pi_1 \) on \( \{p + 1, \ldots, 2p\} \). We further “double” (for technical reasons) \( \pi_2 \) to form a permutation \( \pi_2' \) of the set \( T = \{1, \ldots, 4p\} \), such that \( \pi_2' \) is determined by the following two linear orders; \( 1 \leq 2 \leq \ldots \leq 1, 4p \) and \( 2 \pi_2(1) - 1 \leq 2 \pi_2(1) \leq 2 \pi_2(2) - 1 \leq 2 \pi_2(2) \leq \ldots \leq 2 \pi_2(2p) - 1 \leq 2 \pi_2(2p) \). Let \( U := T^4 \), and let the linear orders \( \leq_1 \) and \( \leq_2 \) on \( U \) be the standard lexicographic powers of \( \leq_1 \) and \( \leq_2 \) on \( T \). Furthermore, let \( Z = U^s \) where \( s = 2r \), and \( \leq_1 \) and \( \leq_2 \) on \( Z \) again be the standard lexicographic powers. Finally, let \( \leq_1 \) and \( \leq_2 \) on \( Z \) determine the permutation \( \varrho \) of \( Z \).

By the previous, there exists a graph \( H^+ \in \mathcal{C}_0 \cap \mathcal{R}_\varrho \), where \( V(H^+) = Z \cup Z_1 \cup Z_2 \) where \( Z_1 \cup Z_2 \) are the mates of \( Z \) in it. Let \( H' \in \mathcal{C} \) and \( X_1, \ldots, X_r \) be the sets to which an \( r \)-bounded perturbation of \( H^+ \) has been applied, such that \( H' \) is the resulting perturbed graph. We apply Lemma 4.2 to \( Y := U \) (and to \( Z \) and the sets \( X_1, \ldots, X_r \)); this gives us a homogeneous set \( Z_0 \subseteq Z \) with some prefix \( (a_1, \ldots, a_{\ell-1}) \). Since the linear orders \( \leq_1 \) and \( \leq_2 \) on \( Z \) have been obtained as lexicographic powers of \( \leq_1 \) and \( \leq_2 \) on \( Y = U \), their restriction to \( Z_0 \) is isomorphic to \( \leq_1 \) and \( \leq_2 \) on \( U \). Abusing the notation for simplicity, we hence set \( Z_0 = U \), and then denote by \( U_1, U_2 \) the corresponding mates of vertices of \( U \) within \( H^+ \).

Let \( H_1 = H^+[U \cup U_1 \cup U_2] \) be the induced subgraph of \( H^+ \) on \( U = Z_0 \), and likewise \( H_1' = H'[U \cup U_1 \cup U_2] \). By homogeneity of \( U \) (from Lemma 4.2), we have that each of the complemented sets \( X_i \) either contains \( U \) or is disjoint from it. Consequently, every vertex of \( U_1 \) either has the same neighborhood to \( U \) in \( H_1 \) as in \( H_1' \), or the exactly complementary neighborhood. Let \( X_1' \subseteq U \) be the set of those vertices whose mate in \( U_1 \) has the same neighborhood to \( U \) in \( H_1 \) as in \( H_1' \), and define analogously \( X_2' \subseteq U \) with respect to the mates in \( U_2 \). We apply Lemma 4.2 again, now to \( Y := T \), the set \( U \) in place of \( Z \), and to the sets \( X_1', X_2' \) \( (s = 2^r = 4) \). From that we get a homogeneous set \( U_0 \subseteq U \) (with a prefix which is formally appended to the prefix of \( Z_0 \) above), and we denote by \( T_1 \subseteq U_1 \) and \( T_2 \subseteq U_2 \) the sets of mates of the vertices of \( U_0 \). Abusing again the notation for simplicity, we set \( U_0 = T \).

By homogeneity of \( T = U_0 \), the edge set between \( T \) and \( T_1 \) is either the same in \( H_1 \) as in \( H_1' \), or the exact complement of it. The same holds for \( T \) and \( T_2 \). Now, the \( H_1' \)-neighborhoods in \( T_1 \) and \( T_2 \) define on \( T \) linear preorders \( \leq_1 \) and \( \leq_2 \), respectively (consecutive pairs of \( T \) may actually have equal neighborhoods in \( T_1 \) or \( T_2 \)). However, with appropriate choice of a representative of every pair of consecutive vertices of \( T \) (cf. the above definition of \( \pi_2 \) on \( T \)), giving \( T_0 \subseteq T \), and similar choice from \( T_1 \) and \( T_2 \), we get an induced subgraph \( H_1'' \subseteq H_1' \), such that the restricted orders \( \leq_1 \) and \( \leq_2 \) on \( T_0 \) (defined analogously by \( H_1'' \)-neighborhoods) give a permutation on \( T_0 \) isomorphic to the above permutation \( \pi_2' \) on \( \{1, \ldots, 2p\} \) or to its inversion. In any case, a suborder on \( T_0 \) defines a permutation isomorphic to \( \pi_1 \) we started with, and so \( H_1' \) contains as an induced subgraph a member of \( R_{\pi_1}^+ \).

Since \( \pi = \pi_1 \) has been chosen arbitrarily, and \( \mathcal{C} \ni H' \) is hereditary, we conclude that \( \mathcal{C} \) contains a graph from \( R_{\pi_1}^+ \) for every \( \pi \).

## 6 Conclusions

We have got precise characterizations of bounded twin-width by explicit obstructions (as induced subgraphs) in the classes of interval and circle graphs, and in the classes obtained by bounded perturbations from these graphs. In the case of interval and circle graphs alone, our obstructions also explicitly characterize fixed-parameter tractability of FO model checking.
under usual complexity assumptions. While it is relatively easy to relate the bounded twin-width property between a graph class and the class of its bounded perturbations, since bounded perturbations can be expressed by FO transductions in both directions, the fact that bounded perturbations do not change the class of the explicit obstructions is remarkable. It is perhaps worth to investigate to which level this finding can be generalized.

In order to extend our characterizations of tractability of FO model checking to bounded perturbations of the classes, we would need the following: Having a class $C$ of graphs of bounded twin-width with efficiently computable contraction sequences of bounded width, can we input a graph $G$ which is a bounded perturbation of a graph $G_1 \in C$ (without knowing $G_1$) and efficiently compute a contraction sequence of $G$ of bounded width? We propose this as the main open question of the paper.

In another direction, we propose to investigate possible extensions of the results of Section 5 to the classes of $k$-thin graphs [10] and possibly to $k$-mixed-thin graphs [1]. While we do not state the formal definition here, we remark that 1-thin graphs are exactly the interval graphs, and the mentioned classes naturally generalize interval graphs (and, in particular, proper $k$-mixed-thin graphs are of bounded twin-width [1]). An extension of our techniques to these classes seems possible, but not at all trivial.

References

1. Jakub Balabán, Petr Hliněný, and Jan Jedelský. Twin-width and transductions of proper $k$-mixed-thin graphs. CoRR, abs/2202.12536, 2022. Accepted to WG 2022.
2. Pierre Bergé, Édouard Bonnet, and Hugues Déprés. Deciding twin-width at most 4 is NP-complete. CoRR, abs/2112.08953, 2021.
3. Édouard Bonnet, Dibyayan Chakraborty, Eun Jung Kim, Noleen Köhler, Raul Lopes, and Stéphan Thomassé. Twin-width VIII: delineation and win-wins. CoRR, abs/2204.00722, 2022.
4. Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width II: small classes. In SODA, pages 1977–1996. SIAM, 2021.
5. Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width III: max independent set, min dominating set, and coloring. In ICALP, volume 198 of LIPIcs, pages 35:1–35:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
6. Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé. Twin-width IV: low complexity matrices. CoRR, abs/2102.03117, 2021.
7. Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, and Stéphan Thomassé. Twin-width VI: the lens of contraction sequences. In SODA, pages 1036–1056. SIAM, 2022.
8. Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. In FOCS, pages 601–612. IEEE, 2020.
9. Édouard Bonnet, Jaroslav Nešetřil, Patrice Ossona de Mendez, Sebastian Siebertz, and Stéphan Thomassé. Twin-width and permutations. CoRR, abs/2102.06880, 2021.
10. Flavia Bonomo and Diego de Estrada. On the thinness and proper thinness of a graph. Discret. Appl. Math., 261:78–92, 2019.
11. Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. J. Comput. Syst. Sci., 13(3):355–379, 1976.
12. André Bouchet. Reducing prime graphs and recognizing circle graphs. Comb., 7(3):243–254, 1987.
13. R. G. Downey, M. R. Fellows, and U. Taylor. The parameterized complexity of relational database queries and an improved characterization of W[1]. In First Conference of the Centre for Discrete Mathematics and Theoretical Computer Science, DMTCS 1996, pages 194–213. Springer-Verlag, Singapore, 1996.
14 Jakub Gajarský, Petr Hliněný, Daniel Lokshtanov, Jan Obdržálek, Sebastian Ordyniak, M. S. Ramanujan, and Saket Saurabh. FO model checking on posets of bounded width. In *FOCS*, pages 963–974. IEEE Computer Society, 2015.

15 Jakub Gajarský, Petr Hliněný, Daniel Lokshtanov, Jan Obdržálek, Sebastian Ordyniak, M. S. Ramanujan, and Saket Saurabh. FO model checking on posets of bounded width. *CoRR*, abs/1504.04115, 2015.

16 R. Ganian, P. Hliněný, D. Král’, J. Obdržálek, J. Schwartz, and J. Teska. FO model checking of interval graphs. In *ICALP 2013, Part II*, volume 7966 of *LNCS*, pages 250–262. Springer, 2013.

17 M. Grohe, S. Kreutzer, and S. Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017.

18 Petr Hliněný, Filip Pokrývka, and Bodhayan Roy. FO model checking on geometric graphs. *Comput. Geom.*, 78:1–19, 2019.

19 Rose McCarty. *Local Structure for Vertex-Minors*. PhD thesis, University of Waterloo, 2021. URL: [http://hdl.handle.net/10012/17633](http://hdl.handle.net/10012/17633)

20 D. Seese. Linear time computable problems and first-order descriptions. *Math. Structures Comput. Sci.*, 6(6):505–526, 1996.

21 Jeremy P. Spinrad. Recognition of circle graphs. *J. Algorithms*, 16(2):264–282, 1994.