GLUING SEMI-ORTHOGONAL DECOMPOSITIONS

SARAH SCHEROTZKE, NICOLÒ SIBILLA, AND MATTIA TALPO

Abstract. We introduce preordered semi-orthogonal decompositions (psod-s) of dg-categories. We show that homotopy limits of dg-categories equipped with compatible psod-s carry a natural psod. This gives a way to glue semi-orthogonal decompositions along faithfully flat covers, extending the main result of [4]. As applications we will construct semi-orthogonal decompositions for root stacks of log pairs \((X, D)\) where \(D\) is a \((not necessarily simple)\) normal crossing divisor, generalizing results from [17] and [3]. Further we will compute the Kummer flat K-theory of general log pairs \((X, D)\), generalizing earlier results of Hagihara and Nizioł in the simple normal crossing case [15], [23].

1. Introduction

In this paper we study conditions under which semi-orthogonal decompositions \((sod-s)\) of dg-categories can be glued together to yield global semi-orthogonal decompositions. We formulate our results in terms of general homotopy limits of dg-categories under appropriate compatibility assumptions on the structure functors. Our main technical result is, roughly, that a limit of dg categories equipped with sod-s and compatible functors between them carries a natural sod (Theorem A in this introduction).

Making use of the homotopy theory of dg categories, the proof of Theorem A is not difficult. However this result has several significant consequences. We will describe them briefly here, while referring the reader to the remainder of the introduction for a fuller summary of the contents of the paper.

(1) As a consequence of Theorem A we recover one of the main results of the interesting recent article [4], namely what the authors call conservative descent. The proof given in [4] is framed in the language of classical triangulated categories, and depends on rather sophisticated arguments. Leveraging the formalism of \(\infty\)-categories, however, we can give a very simple proof of this result. Indeed in section 3.1 we will show that conservative descent follows immediately from our Theorem A.

(2) Root stacks of normal crossing divisors \(D \subset X\) have been much studied in algebraic geometry. In particular in [17] and [3] it is proved that their derived category carries a natural semi-orthogonal decomposition. These prior results assume \(D\) to be simple normal crossing. As an application of Theorem A we drop this assumption. We construct semi-orthogonal decompositions on categories of perfect complexes of root stacks of general normal crossing divisors \(D \subset X\). A new feature emerges: whereas in the simple normal crossing case the semi-orthogonal summands are given by categories of perfect complexes on the strata of \(D\), without the simplicity assumption the summands correspond to perfect complexes on the normalization of the strata.

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Although the previous two applications are of general geometric import, we were motivated by log geometry and the theory of parabolic sheaves. Here are two applications of log geometric nature:

(3) The infinite root stack, introduced in [33], is an important construction in log geometry. Using (2) above, we construct sod-s for infinite root stacks in the general normal crossing case. This improves on a result from our previous paper [27, Section 4], where we worked under restrictive assumptions on the ground field (and we used a highly non-trivial invariance of the derived category of infinite root stacks under log blow-ups). By the results of [33], we also obtain sod-s on derived categories of parabolic sheaves of general normal crossing divisors (with rational weights).

(4) Hagihara and Nizioł [15, 23] established an important structure theorem for Kummer flat K-theory of log schemes with divisorial log structure \((X,D)\), where \(D\) is simple normal crossing. They proved that Kummer flat K-theory splits as an infinite direct sum labeled by the strata. As a consequence of (3) we extend their description of Kummer flat K-theory to log pairs \((X,D)\) where \(D\) is general normal crossing.

Preordered semi-orthogonal decompositions and gluing. Dg-categories can be viewed as objects inside homotopically enriched categories. In technical terms, we say that dg-categories form a model category or an \(\infty\)-category. This yields meaningful notions of (homotopy) limits and colimits of dg-categories. This is a key difference with the classical theory of triangulated categories, where limits and colimits are poorly behaved.

In algebraic geometry, descent properties of sheaves can be encoded via homotopy limits of dg-categories. If \(U \to X\) is faithfully flat cover, the category of perfect complexes \(\text{Perf}(X)\) can be computed as a limit of the cosimplicial diagram of dg-categories determined by the Čech nerve of \(U \to X\). Limits and colimits of dg-categories arise also in other geometric contexts. For instance, the Fukaya category of exact symplectic manifolds localizes, and therefore can be calculated as a limit of Fukaya categories of open patches.

For this reason, it is important to have structure theorems that allow us to deduce properties of the limit category from the behaviour of the dg-categories appearing as vertices of the limit. In this paper we prove a result of this type for semi-orthogonal decompositions (sod-s). These are categorified analogues of direct sum decompositions of abelian groups and have long played a key role in algebraic geometry, see [20] for a survey of results. In fact, it is more convenient to work with the slightly more sophisticated concept of preordered semi-orthogonal decompositions (psod-s), where the factors \(C_w\) of a dg-category \(C\) are labeled by elements of a preorder \((P,\leq)\). We use the notation \((C, P)\) to indicate a dg-category \(C\) with with a psod indexed by \(P\).

We introduce a notion of ordered structure on an exact functor \(F\): \((C_1, P_1) \to (C_2, P_2)\): this is the datum of an order-reflecting map \(\phi_F\): \(P_2 \to P_1\) keeping track of the way the psod-s on \(C_1\) and \(C_2\) interact via \(F\).

**Theorem A** (Theorem 3.13). Assume that for all \(i \in I\), \(C_i = (C_i, P_i)\) is equipped with a psod, and that for all morphisms \(f: i \to j\) in \(I\), \(\alpha(f): C_i \to C_j\) is ordered. Assume additionally that the colimit of indexing preorders \(P = \varinjlim_{i \in I} P_i\) is finite and directed. Then the limit category \(C = \varprojlim_{i \in I} C_i\) carries a psod with indexing preordered set \(P\), \(C = (C_w, w \in P)\), such that for all \(w \in P\) we have

\[
C_w \simeq \varprojlim_{i \in I} \bigoplus_{z \in \phi_i^{-1}(w)} C_{i,z},
\]

where \(\phi_i: P_i \to P\) is the natural map.
The proof of Theorem A is conceptually clear and not difficult. It reduces to relatively straightforward manipulations in the \( \infty \)-category of dg categories. This simplicity is one of the main assets of our approach. As it is often the case, leveraging the power of \( \infty \)-categories allows for simpler and more conceptual arguments. As an example, we will show how Theorem A immediately implies an interesting recent result of Bergh and Schnürer from [4]. One of their main theorems is, roughly, a gluing result akin to Theorem A, but limited to the geometric setting, which is called conservative descent. Their approach is interesting in itself, but requires rather sophisticated arguments based on the classical theory of triangulated categories. However from a dg point-of-view conservative descent admits a simple proof. Indeed in section 3.1 we will show that it can be recovered as a special case of Theorem A.

Since the precise setting of conservative descent is somewhat complicated, we prefer not to reproduce that result here but refer the reader directly to section 3.1 in the main text for more details.

**Perfect complexes on root stacks.** Root stacks were first studied systematically by Cadman in [9]. They carry universal roots of line bundles equipped with a section, and in [9] were used to compactify moduli of stable maps. Root stacks have since found applications in many different areas of geometry including enumerative geometry, quantum groups [26], the theory of Néron models [11], and more. From our perspective, root stacks are an essential tool to probe the geometry of log schemes.

Taking the root stack of a divisor is a fundamental geometric operation akin to blowing-up. In fact these two operations are often combined, as in the notion of stacky blow-up proposed by Rydh [25]. From the view-point of the derived category classical blow-ups have a very simple description: the surgery replacing a smooth subscheme with the projectivization of its normal bundle becomes, at the level of derived categories, the addition of semi-orthogonal summands to the derived category of the ambient space. It is a natural question, with many important applications, whether this picture extends to stacky blow-ups. Semi-orthogonal decompositions associated to root stacks of normal crossing divisors were studied in [17] and [3]. However these results require the normal crossing divisor to be simple.

The assumption of simplicity is artificial from the viewpoint of the underyling geometry. One of the chief goals of this paper is to lift the simplicity assumption and extend these semi-orthogonal decompositions to the general normal crossing case. The definition of the root stack of a general normal crossing divisor requires some care, but the geometry is clear: the isotropy along the strata of the divisor keeps track of their codimension.

We formulate a version of our result as Theorem B below, but we refer the reader to the main text for a sharper and more general statement (see Corollary 4.3). Let \( D \subset X \) be a normal crossing divisor. The divisor \( D \) determines a stratification of \( X \) where the strata are intersections of local branches of \( D \). Let \( S \) be the set of strata and let \( M \) be the top codimension of the strata. For every \( 0 \leq j \leq M \), let \( S_j \) be the set of \( j \)-codimensional strata. If \( S \) is a stratum in \( S \) we denote by \( \bar{S} \) its normalization.

**Theorem B** (Corollary 4.3). The dg-category of perfect complexes of the \( r \)-th root stack \( \sqrt{r}(X,D) \) admits a semi-orthogonal decomposition

\[
\text{Perf}(\sqrt{r}(X,D)) = \langle A_M, \ldots, A_0 \rangle
\]

having the following properties:

1. \( A_0 \simeq \text{Perf}(X) \),

2. for every \( 1 \leq j \leq M \), \( A_j \) decomposes as a direct sum \( A_j \simeq \bigoplus_{S \in S_j} B_S \), and
for every $S \in S_j$ the category $B_S$ carries a semi-orthogonal decomposition composed of $r \cdot j$ semi-orthogonal factors, which are all equivalent to $\text{Perf}(\tilde{S})$.

**Applications to log geometry: infinite root stacks and Kummer flat $K$-theory.** Log schemes are an enhancement of ordinary schemes which was introduced in the 80’s in the context of arithmetic geometry. In recent years log techniques have become a mainstay of algebraic geometry and mirror symmetry: for instance, log geometry provides the language in which the Gross–Siebert program in mirror symmetry is formulated [14]. In [33] Talpo and Vistoli explain how to associate to a log scheme its infinite root stack, which is a projective limit of usual (i.e. finite-index) root stacks. This assignment gives rise to a faithful functor from log schemes to stacks: log information is converted into stacky information without any data loss. Additionally, the authors prove in [33] that the Kummer flat topos of a log scheme $X$ is equivalent to the flat topos of its infinite root stack.

Using Theorem B and passing to the limit on $r$, we obtain an infinite sod on $\text{Perf}(\sqrt{\infty}(X,D))$. This result is stated as Theorem 5.3 in the main text. The passage to the limit actually requires a careful construction of nested sod-s on categories of perfect complexes of root stacks, which was explained in our previous work [27], to which we refer the reader. It follows from results of [33] that $\text{Perf}(\sqrt{\infty}(X,D))$ is equivalent to the category of parabolic sheaves with rational weights on $(X,D)$: thus our result can be read as the construction of a sod on $\text{Par}^Q(X,D)$.

**Theorem C** (Corollary 5.5). Let $(X,D)$ be a log stack given by an algebraic stack $X$ equipped with a normal crossing divisor $D$. Then the Kummer flat $K$-theory spectrum of $X$ decomposes as a direct sum

$$K_{\text{Kfl}}(X,D) \simeq K(X) \bigoplus \bigoplus_{S \in S^*_D} \left( \bigoplus_{\chi \in (Q/Z)_S} K(\tilde{S}) \right).$$

We refer the reader to the main text for the definition of all the terms appearing in the formula. Our result extends to the general normal crossing case the structure theorem in Kummer flat $K$-theory due to Hagihara and Nizioł [15], [23]. A notable difference from those results is the appearance of the $K$-theory of the normalization of the strata, rather than of the strata themselves. In fact the statement of Corollary 5.5 in the main text is considerably more general than Theorem C: it is not limited to $K$-theory but holds across all Kummer flat additive invariants.

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**Conventions.** We will work over an arbitrary ground commutative ring $\kappa$. We use the definition of algebraic stacks given in [29, Tag 026O]. In the following all algebraic stacks will be of finite type. All functors between derived categories of sheaves or categories of perfect complexes are implicitly derived.

## 2. Preliminaries

### 2.1. Categories

We will work with *dg-categories*, that is, $\kappa$-linear differential graded categories in the sense of [18] and [12]. If $C$ is a dg-category, and $A$ and $B$ are in $C$, we denote by $\text{Hom}_C(A,B)$ the Hom-complex between $A$ and $B$. We will be mostly interested in *triangulated dg-categories* which are defined for instance in Section 3 of [5]. The homotopy category of a triangulated dg-category is a triangulated, Karoubi-complete category.
The category of dg-categories and exact functors carries a model structure, which was studied in [31] and [35], where weak equivalences are Morita equivalences. A Morita equivalence is a dg-functor \( F: A \to B \) such that the associated derived functor is an equivalence. Localizing this model category at weak equivalences yields an \((\infty, 1)\)-category, which we denote \( \text{dgCat} \). Our reference for the theory of \((\infty, 1)\)-categories is given by Lurie’s work [21, 22]. In the rest of the paper we will refer to \((\infty, 1)\)-categories simply as \( \infty \)-categories. The category \( \text{dgCat} \) has a symmetric monoidal structure given by the tensor product of dg-categories, see [35].

We will be interested in taking limits and colimits of diagrams \( \alpha: I \to \text{dgCat} \), where \( I \) is an ordinary category and \( i \) is a pseudo-functor, in the sense for instance of Definition 4.1 of [6]. All limits and colimits are to be understood as homotopy limits and colimits for the Morita model structure: equivalently, they are (co)limits in the \((\infty, 1)\)-categorical sense. Since every pseudo-functor \( \alpha: I \to \text{dgCat} \) can be strictified up to Morita equivalence, and this does not affect homotopy (co)limits, the reader can assume that all diagrams \( \alpha: I \to \text{dgCat} \) in the paper are strict.

Throughout the paper, we will say that a square of dg-categories is commutative if there is an invertible natural transformation \( \alpha: KG \Rightarrow HF \)

\[
\begin{array}{c}
\begin{array}{c}
C_1 \\
G
\end{array}
\begin{array}{c}
\alpha \\
\swarrow
\end{array}
\begin{array}{c}
\begin{array}{c}
C_2 \\
H
\end{array}
\end{array}
\end{array}
\]

Whenever only the existence of a natural transformation making the diagram commute will be needed, and not its explicit definition, we will omit \( \alpha \) and the 2-cell notation from the diagram. We will say that \( \alpha \) commutes strictly if \( \alpha \) is the identity natural transformation, and we will sometimes denote this as

\[
\begin{array}{c}
\begin{array}{c}
C_1 \\
G
\end{array}
\begin{array}{c}
\swarrow
\end{array}
\begin{array}{c}
\begin{array}{c}
C_2 \\
H
\end{array}
\end{array}
\end{array}
\]

Let \( I \) be a small category. Let \( \gamma_1, \gamma_2: I \to \text{dgCat} \) be diagrams in \( \text{dgCat} \). For all \( i \in I \) set \( D_i := \gamma_1(i) \) and \( C_i := \gamma_2(i) \). Let

\[
T: \gamma_1 \Rightarrow \gamma_2: I \to \text{dgCat}
\]

be a pseudo-natural transformation, given by the following data:

- for all \( i \in I \), a functor \( T(i): D_i \to C_i \),
- for all \( i, j \in I \), and for all maps \( i \xrightarrow{a_{ij}} j \), an invertible natural transformation \( \alpha_{i,j} \)

\[
\begin{array}{c}
\begin{array}{c}
D_i \\
\gamma_1(a_{ij})
\end{array}
\begin{array}{c}
\swarrow
\end{array}
\begin{array}{c}
T(j)
\end{array}
\begin{array}{c}
\gamma_2(a_{ij})
\end{array}
\end{array}
\]

Denote by \( D \) and \( C \) the limits \( \lim_{i \in I} D_i \) and \( \lim_{i \in I} C_i \) respectively, and let \( T \) be the limit of the functors \( T(i) \)

\[
T = \lim_{i \in I} T(i): D \to C.
\]

**Lemma 2.1.** Assume the following.

(1) For all \( i \in I \), \( T(i) \) admits a left adjoint \( T(i)^L \), \( T(i)^L \dashv T(i) \).
(2) The following Beck–Chevalley condition is satisfied: for all $i, j \in I$, and for all maps $i \overset{\alpha_{i,j}}{\rightarrow} j$, the canonical natural transformation

\[ \alpha_{i,j}^L : T(j)^L \circ \gamma_2(\alpha_{i,j}) \Rightarrow \gamma_1(\alpha_{i,j}) \circ T(i)^L \]

induced by $\alpha_{i,j}$ is invertible.

Then $T^L = \lim_{i \in I} T(i)^L$ is the left adjoint of $T$.

**Remark 2.2.** Recall that, by definition, $\alpha_{i,j}^L$ is given by the composite

\[ T(j)^L \circ \gamma_2(\alpha_{i,j}) \xrightarrow{(a)} T(j)^L \circ \gamma_2(\alpha_{i,j}) \circ T(i) \circ T(i)^L \xrightarrow{(b)} T(j)^L \circ \gamma_1(\alpha_{i,j}) \circ T(i)^L \xrightarrow{(c)} \gamma_1(\alpha_{i,j}) \circ T(i)^L \]

where (a) and (c) are given by the counit of $T(i)^L \vdash T(i)$ and the unit of $T(j)^L \vdash T(j)$, while (b) is given by $\alpha_{i,j}$.

**Proof of Lemma 2.1.** This is a well-known fact in the general setting of $\infty$-categories. We give a proof based on [1, Appendix D]. With small abuse of notation we keep denoting by $I$ also the nerve $\infty$-category of $I$. Via the Grothendieck construction we can write $\gamma_1$ and $\gamma_2$ as Cartesian fibrations over $I^\text{op}$. Then $T$ yields a morphism of cartesian fibrations over $I^\text{op}$

\[
\begin{array}{ccc}
\int_I \gamma_1 & \xrightarrow{T} & \int_I \gamma_2 \\
\downarrow & & \downarrow \\
I^\text{op} & & I^\text{op}
\end{array}
\]

By assumption (1), $T$ has a relative left adjoint: this follows from Lemma D.3 of [1]. Now by Lemma D.6 of [1] relative adjunctions induce adjunctions between $\infty$-categories of sections. Further, if the relative left adjoint preserves cartesian edges (which is the case by assumption (2)), this restricts to an adjunction between the full subcategories of cartesian sections: this gives the desired adjunction $T^L \dashv T$ and concludes the proof. $\square$

We also state the analogue of Lemma 2.1 for right adjoints.

**Lemma 2.3.** Assume the following.

(1) For all $i \in I$, $T(i)$ admits a right adjoint $T(i)^R$, $T(i) \vdash T(i)^R$.

(2) The following Beck–Chevalley condition is satisfied: for all $i, j \in I$, and for all maps $i \overset{\alpha_{i,j}}{\rightarrow} j$, the canonical natural transformation

\[ \alpha_{i,j}^R : \gamma_1(\alpha_{i,j}) \circ T(i)^R \Rightarrow T(j)^R \circ \gamma_2(\alpha_{i,j}) \]

induced by $\alpha_{i,j}$ is invertible.

Then $T^R = \lim_{i \in I} T(i)^R$ is the right adjoint of $T$.

2.1.1. Categories of sheaves. If $X$ is an algebraic stack, we denote by $\text{Qcoh}(X)$ the triangulated dg-category of quasi-coherent sheaves on $X$. The tensor product of quasi-coherent sheaves equips $\text{Qcoh}(X)$ with a symmetric monoidal structure, and $\text{Perf}(X)$, the dg-category of perfect complexes, is defined as the full subcategory of dualizable objects (see [2]). By [13, Theorem 1.3.4], the dg-category of quasi-coherent sheaves $\text{Qcoh}(-)$ satisfies faithfully flat descent: given a faithfully flat cover $Y \rightarrow X$, if $Y^\bullet$ is the semi-simplicial object given by the Čech nerve of $Y \rightarrow X$, then $\lim \text{Qcoh}(Y^\bullet) \simeq \text{Qcoh}(X)$. Passing to dualizable objects on both sides, we obtain that $\text{Perf}(-)$ also satisfies faithfully flat descent.
2.1.2. Exact sequences. Let $\mathcal{C}$ be a triangulated dg-category. We say that two objects $A$ and $A'$ are equivalent if there is a degree 0 map $A \to A'$ that becomes an isomorphism in the homotopy category of $\mathcal{C}$. If $\iota : \mathcal{C}' \to \mathcal{C}$ is a fully faithful functor, we often view $\mathcal{C}'$ as a subcategory of $\mathcal{C}$ and identify $\iota$ with an inclusion. Accordingly, we will usually denote the image under $\iota$ of an object $A$ of $\mathcal{C}'$ simply by $A$ rather than $\iota(A)$. We will always assume that subcategories are closed under equivalence. That is, if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$, $A$ is an object of $\mathcal{C}'$, and $A'$ is an object of $\mathcal{C}$ which is equivalent to $A$, we will always assume that $A'$ lies in $\mathcal{C}'$ as well.

Recall that if $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, $(\mathcal{D})^\perp$ denotes the right orthogonal of $\mathcal{D}$, i.e. the full subcategory of $\mathcal{C}$ consisting of the objects $B$ such that the Hom-complex $\text{Hom}_\mathcal{C}(A,B)$ is acyclic for every object $A \in \mathcal{D}$. Let $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be a finite collection of triangulated subcategories of $\mathcal{C}$ such that, for all $1 \leq i < j \leq n$, $\mathcal{C}_i \subset (\mathcal{C}_j)^\perp$. Then we denote by $\langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle$ the smallest triangulated subcategory of $\mathcal{C}$ containing all the subcategories $\mathcal{C}_i$. An exact sequence of triangulated dg-categories is a sequence

$$(2) \quad A \xrightarrow{F} B \xrightarrow{G} C$$

which is both a fiber and a cofiber sequence in $\text{dgCat}$. This is an analogue of classical Verdier localization of triangulated categories in the dg setting. Exact sequences of dg-categories are detected at the homotopy level: it can be shown that $(2)$ is an exact sequence if and only if the localizations of triangulated categories in the dg setting. Exact sequences of dg-categories are

$$\text{Ho}(A) \xrightarrow{\text{Ho}(F)} \text{Ho}(B) \xrightarrow{\text{Ho}(G)} \text{Ho}(C)$$

is a classical Verdier localization of triangulated categories.

The functor $F$ admits a right adjoint $F^R$ exactly if $G$ admits a right adjoint $G^R$, and similarly for left adjoints, see e.g. [19, Proposition 4.9.1]. If $F$ (or equivalently $G$) admits a right adjoint we say that $(2)$ is a split exact sequence. In this case the functor $G^R$ is fully faithful and we have that $B = \langle G^R(\mathcal{C}) , A \rangle$. As we indicated earlier, since $G^R$ is fully faithful we will drop it from our notations whenever this is not likely to create confusion: thus we will denote $G^R(\mathcal{C})$ simply by $\mathcal{C}$, and write $B = \langle \mathcal{C} , A \rangle$.

2.2. Root stacks. For the convenience of the reader, we include a brief reminder about root stacks of Cartier divisors in an algebraic stack. More details can be found in [27, Section 2.1] and references therein.

Let $X$ be a scheme. Then a Cartier divisor $D \subset X$ is said to be simple (or strict) normal crossings if for every $x \in D$ the local ring $\mathcal{O}_{X,x}$ is regular, and there exist a regular sequence $f_1, \ldots, f_k \in \mathcal{O}_{X,x}$ such that $I_x = (f_1, \ldots, f_k) \subset \mathcal{O}_{X,x}$ for some $k \leq n$. Moreover, $D$ is said to be normal crossings if étale locally on $X$ it is simple normal crossings. These notions are naturally extended to algebraic stacks by working on an atlas.

Given an algebraic stack $X$ with a normal crossings divisor $D \subset X$ one can form a root stack $\sqrt{(X,D)}$ for every $r \in \mathbb{N}$. If $D$ is simple normal crossings, this has a simple functorial description as the stack parametrizing tuples $(L_1, s_1), \ldots, (L_k, s_k)$ of line bundles with global sections, with isomorphisms $(L_i, s_i)^{\times r} \cong (\mathcal{O}(D_i), s_{D_i})$, where $D_i$ are the irreducible components of $D$ and $s_{D_i}$ is the canonical section of the line bundle $\mathcal{O}(D_i)$, whose zero locus is $D_i$. Passing to $\sqrt{(X,D)}$ has the effect of attaching a stabilizer $\mu_k^r$ to points in the intersection of exactly $k$ irreducible components of $D$.

When $D$ is only normal crossings, this description is not correct, because it doesn’t distinguish the branches of $D$ at points where an irreducible components self-intersects (as for example in the node of an irreducible nodal plane cubic). In this case we have to use the notion of a root stack of a
logarithmic scheme (see [8, 32]). In this particular case, we can think about \( \sqrt{r}(X,D) \) as being the gluing of the \( r \)-th root stacks \( \sqrt{r}(U,D|_{U}) \), where \( \{ U \to X \} \) is an étale cover of \( X \) where \( D \) becomes simple normal crossings.

The root stacks \( \sqrt{r}(X,D) \) form an inverse system. Indeed, if \( r | r' \) there is a natural projection \( \sqrt{r}(X,D) \to \sqrt{r'}(X,D) \). The inverse limit \( \lim_{\to} \sqrt{r}(X,D) \) is the infinite root stack \([33]\) of \( (X,D) \). It is a pro-algebraic stack that embodies the “logarithmic geometry” of the pair \( (X,D) \) in its stacky structure, and it is an algebraic analogue of the so-called “Kato-Nakayama space” \([10, 34]\). In particular, quasi-coherent sheaves on it correspond exactly to parabolic sheaves \([8]\) on the pair \( (X,D) \), and finitely presented sheaves can be identified with finitely presented sheaves on the Kummer-flat site of \( (X,D) \) (and also on the Kummer-étale site, if the base ring has characteristic 0).

3. Preordered semi-orthogonal decompositions

In this section we introduce preordered semi-orthogonal decompositions (psod-s). We will study limits of dg-categories equipped with compatible psod-s. This concept was also considered in \([3]\). Once the set-up is in place the proof of the main result (Theorem 3.13) is not difficult, relaying as it does on general properties of limits in the \( \infty \)-category of dg-categories. As an application of our theory, we will obtain gluing results for semi-orthogonal decompositions along appropriate faithfully flat covers. In the setting of classical triangulated categories a related result, called conservative descent, has been obtained in the recent paper \([4]\). The advantage of our set-up is that conservative descent follows in a straightforward way from the general formalism, and thus the proof that we will give is considerably easier than the one contained in \([4]\).

We will follow closely the account of psod-s contained in the previous paper of the authors \([27]\). We refer the reader to \([20]\) for a comprehensive survey of the classical theory of semi-orthogonal decompositions.

Let \( \mathcal{C} \) be a triangulated dg-category, and let \( P \) be a preordered set. Consider a collection of full triangulated subcategories \( \iota_{x}: \mathcal{C}_{x} \to \mathcal{C} \) indexed by \( x \in P \).

**Definition 3.1.**

- The subcategories \( \mathcal{C}_{x} \) form a preordered semi-orthogonal decomposition (psod) of type \( P \) if they satisfy the following three properties.
  1. For all \( x \in P \), \( \mathcal{C}_{x} \) is a non-zero admissible subcategory: that is, the embedding \( \iota_{x} \) admits a right adjoint and a left adjoint, which we denote by \( r_{x}: \mathcal{C} \to \mathcal{C}_{x} \) and \( l_{x}: \mathcal{C} \to \mathcal{C}_{x} \).
  2. If \( y \leq_{P} x \), i.e. \( y \leq_{P} x \), and \( x \neq y \), then \( \mathcal{C}_{y} \subseteq \mathcal{C}_{x}^{\perp} \).
  3. \( \mathcal{C} \) is the smallest stable subcategory of \( \mathcal{C} \) containing \( \mathcal{C}_{x} \) for all \( x \in P \).
- We say that the subcategories \( \mathcal{C}_{x} \) form a pre-psod of type \( P \) if they satisfy only properties (1) and (2).

Note that from condition (2) it follows that if we have both \( y \leq_{P} x \) and \( x \leq_{P} y \), then

\[
\langle \mathcal{C}_{x}, \mathcal{C}_{y} \rangle = \langle \mathcal{C}_{y}, \mathcal{C}_{x} \rangle \cong \mathcal{C}_{x} \oplus \mathcal{C}_{y}
\]

**Definition 3.2.** If \( \mathcal{C} \) is equipped with a psod of type \( P \), we write \( \mathcal{C} = (\mathcal{C}_{x}, x \in P) \). We sometimes denote the category \( \mathcal{C} \) by \( (\mathcal{C}, P) \) to make explicit the role of the indexing preordered set \( P \).

**Remark 3.3.** If \( \mathcal{C} \) is the zero category, then it carries a psod indexed by the empty preordered set.
We will be interested in gluing psod-s along limits of dg-categories. This requires introducing an appropriate notion of exact functor compatible with psod-s. We do so after introducing some preliminary concepts.

**Definition 3.4.** Let \( P \) and \( Q \) be preordered sets. We say that a map of sets \( \phi: Q \to P \) is order-reflecting if for all \( x, y \) in \( Q \) we have \( \phi(x) \leq_P \phi(y) \implies x \leq_Q y \).

We denote by \( \text{PSet}^{\text{refl}} \) the category of preordered sets and order-reflecting maps between them. Let us summarize some of its basic properties. Note first that the forgetful functor to sets

\[
U: \text{PSet}^{\text{refl}} \to \text{Sets}
\]

admits a left and a right adjoint, \( U^L \dashv U \dashv U^R \). The functor \( U^L \) sends a set \( S \) to the preordered set \((S, \leq)\) such that \( x \leq y \) for all \( x, y \in S \), with the obvious definition on morphisms. We call \( U^L(S) \) the complete preorder on the set \( S \). The functor \( U^R \) sends a set \( S \) to the preordered set \((S, \leq)\) such that \( x \leq y \), if and only if \( x = y \), with the obvious definition on morphisms. We call \( U^R(S) \) the discrete preorder on the set \( S \).

The category \( \text{PSet}^{\text{refl}} \) admits all small limits and colimits. Since \( U \) has a right and a left adjoint, they are computed by the underlying set-theoretic limits and colimits. In particular, the coproduct of a collection of partially ordered sets \( \{(P_i, \leq_{P_i})\}_{i \in I} \) is given by the disjoint union \( P = \coprod_{i \in I} P_i \) equipped with a preorder \( \leq_P \) defined as follows: let \( i, j \in I \) and let \( x \) be in \( P_i \) and \( y \) in \( P_j \), then

- if \( i = j \), we have \( x \leq_P y \) if and only if \( x \leq_{P_i} y \);
- if \( i \neq j \), we have \( x \leq_P y \).

Let us describe next the push-out of preordered sets in \( \text{PSet}^{\text{refl}} \),

\[
\begin{array}{ccc}
(P_1, \leq_{P_1}) & \xleftarrow{\pi_1} & (P_3, \leq_{P_3}) \\
\pi_2 \uparrow & & \pi_3 \\
(P_2, \leq_{P_2}) & \xrightarrow{\pi_4} & (P_3, \leq_{P_3}).
\end{array}
\]

The set \( P \) is the push-out of the underlying sets, and is equipped with a preorder \( \leq_P \) defined as follows: let \( z \) and \( z' \) be in \( P \), then \( z \leq_P z' \) if and only if

- for all pairs \( x, x' \in P_1 \) such that \( \pi_1(x) = z \) and \( \pi_1(x') = z' \), we have \( x \leq_{P_1} x' \), and
- for all pairs \( x, x' \in P_2 \) such that \( \pi_2(x) = z \) and \( \pi_2(x') = z' \), we have \( x \leq_{P_2} x' \).

**Remark 3.5.** Let \( \phi: (P, \leq_P) \to (Q, \leq_Q) \) be an order-reflecting map. Then, for all \( x \in Q \), the preordered set \( \phi^{-1}(x) \subseteq (P, \leq_P) \) is equipped with the complete preorder. Indeed, since \( \phi \) is order-reflecting, for all \( y \) and \( y' \) in \( \phi^{-1}(x) \) we must have that \( y \leq_P y' \) and \( y' \leq_P y \).

Next, we introduce a notion of compatible functor between categories equipped with a psod.

**Definition 3.6.** Let \( (C, P) \) and \( (D, Q) \) be triangulated dg-categories equipped with a pre-psod

\[
C = \langle C_x, x \in P \rangle \quad \text{and} \quad D = \langle D_y, y \in Q \rangle
\]

and let \( F: C \to D \) be an exact functor. A structure of ordered functor on \( F \) is the datum of a function \( \phi_F: Q \to P \) satisfying the following properties:
The function $\phi_F$ is an order-reflecting map, and for all $x$ in $P$ there is a strictly commutative square
\[
\begin{array}{c}
\mathcal{C}_x \xrightarrow{\iota_x} \mathcal{C} \\
F|_{\mathcal{C}_x} \uparrow \downarrow \quad \uparrow \downarrow \\
\bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y \xrightarrow{\alpha^K_x} \mathcal{D} \\
\end{array}
\]
\[
\begin{array}{c}
\langle D_y, y \in \phi_F^{-1}(x) \rangle \xrightarrow{=} D. \\
\end{array}
\]

For all $x \in P$, set $r_{\phi_F^{-1}(x)} := \bigoplus_{y \in \phi_F^{-1}(x)} r_y$, and $l_{\phi_F^{-1}(x)} := \bigoplus_{y \in \phi_F^{-1}(x)} l_y$,
\[
r_{\phi_F^{-1}(x)} : \mathcal{D} \rightarrow \bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y, \quad l_{\phi_F^{-1}(x)} : \mathcal{D} \rightarrow \bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y.
\]

Then the following Beck–Chevalley condition holds: there are commutative diagrams
\[
\begin{array}{c}
\mathcal{C}_x \xrightarrow{r_x} \mathcal{C} \\
F|_{\mathcal{C}_x} \uparrow \downarrow \quad \uparrow \downarrow \\
\bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y \xrightarrow{\alpha^K_x} \mathcal{D} \\
\end{array}
\quad
\begin{array}{c}
\mathcal{C}_x \xleftarrow{l_x} \mathcal{C} \\
F|_{\mathcal{C}_x} \uparrow \downarrow \quad \uparrow \downarrow \\
\bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y \xleftarrow{\alpha^K_x} \mathcal{D} \\
\end{array}
\]

where $\alpha^K_x$ and $\alpha^K_x$ are defined as in Remark 2.2.

**Remark 3.7.** Being ordered is a structure on a functor $F : (\mathcal{C}, P) \rightarrow (\mathcal{D}, Q)$ and not a property: there might be more than one function $\phi$ satisfying the properties of Definition 3.6.

**Remark 3.8.** As we noted in Remark 3.5, $\phi_F^{-1}(x)$ is equipped with the complete preorder. This yields a canonical equivalence $\bigoplus_{y \in \phi_F^{-1}(x)} \mathcal{D}_y \simeq \langle D_y, y \in \phi_F^{-1}(x) \rangle$ which we assumed implicitly in the formulation of property (2) of Definition 3.6. In particular, $r_{\phi_F^{-1}(x)}$ and $l_{\phi_F^{-1}(x)}$ are the right and left adjoints of the fully faithful functor $l_{\phi_F^{-1}(x)} : \langle D_y, y \in \phi_F^{-1}(x) \rangle \rightarrow \mathcal{D}$.

Let $I$ be a small category, and consider a diagram $\alpha : I \rightarrow \text{dgCat}$. For all $i \in I$ set $\mathcal{C}_i := \alpha(i)$. Assume that:

1. For all $i \in I$, the category $\mathcal{C}_i$ is equipped with a pre-psod $\mathcal{C}_i = (\mathcal{C}_i, P_i)$, and
2. For all arrows $f : i \rightarrow j$ in $I$, the functor $\alpha(f) : (\mathcal{C}_i, P_i) \rightarrow (\mathcal{C}_j, P_j)$ is equipped with an ordered structure $\phi_{\alpha(f)}$.

Assume additionally that the assignments
\[
i \in I \mapsto P_i \quad \text{and} \quad (f : i \rightarrow j) \in I \mapsto \phi_{\alpha(f)} : P_j \rightarrow P_i
\]
yield a well-defined functor $\mathcal{I}^P \rightarrow \text{PSets}^{\text{ord}}$. Let $P$ be the colimit of this diagram, and for all $i \in I$ let $\phi_i : P_i \rightarrow P$ be the structure maps.

**Proposition 3.9.** The limit category $\mathcal{C} = \lim_{i \in I} \mathcal{C}_i$ carries a pre-psod with indexing preordered set $P = \lim_{j \in J} P_i$ such that, for all $w \in P$ the subcategory $\mathcal{C}_w \subset \mathcal{C}$ is given by
\[
\mathcal{C}_w \simeq \lim_{i \in I} \bigoplus_{z \in \phi_i^{-1}(w)} \mathcal{C}_{i,z}.
\]

**Proof.** Every limit can be expressed in terms of products and fiber products. Thus it is sufficient to show the statement for these two classes of limits. Let us consider the case of products first. The product of dg-categories $\{\mathcal{C}_i\}_{i \in I}$ is the limit of the zero functors $(\mathcal{C}_i, P_i) \rightarrow (0, \emptyset)$. Thus, we
need to show that the product category $C = \prod_{i \in I} C_i$ carries a pre-psod indexed by $P = \prod_{i \in I} P_i$. If $x$ is in $P$, there is a $j \in I$ such that $x$ is in $P_j$, and we denote $\iota_x : C_x \to C$ the subcategory of $C$ given by

$$C_x := (C_j)_x \times (0) \to C_j \times \prod_{i \in I, i \neq j} C_i \xrightarrow{\cong} C.$$  

It is immediate to verify that the collection of subcategories $C_x$ for $x \in P$ satisfies properties (1) and (2) from Definition 3.1 and thus that it is a pre-psod.

Let us check next that the statement holds for fiber products. Let $P$ be a fiber product of triangulated dg-categories, such that $C_1$, $C_2$ and $C_3$ are equipped with a pre-psod

$$(C_1, P_1) = \{C_{1,x}, x \in P_1\}, \quad (C_2, P_2) = \{C_{2,y}, y \in P_2\}, \quad (C_3, P_3) = \{C_{3,z}, z \in P_3\},$$

and $G$ and $F$ are ordered functors. Let $P$ be the pushout of the $P_i$, and denote

$$\phi_K : P_1 \to P, \quad \phi_H : P_2 \to P$$

the corresponding order-reflecting maps. We also set $\phi_{FK} := \phi_K \circ \phi_F$ and $\phi_{GH} := \phi_H \circ \phi_G$. Note that, since $P$ is a push-out, $\phi_{FK} = \phi_{GH}$.

For all $w$ in $P$ we set

$$C_w := \{C_{1,x}, x \in \phi_K^{-1}(w)\} \times_{C_1} \{C_{2,y}, y \in \phi_H^{-1}(w)\}.$$  

Since $\{C_{1,x}, x \in \phi_K^{-1}(w)\}$ and $\{C_{2,y}, y \in \phi_H^{-1}(w)\}$ are full subcategories of $C_1$ and $C_2$, we have that $C_w$ is a full subcategory of $C = C_1 \times_{C_2} C_3$. Note that we can write $C_w$ equivalently as the fiber product

$$\{C_{1,x}, x \in \phi_K^{-1}(w)\} \times_{C_3} \{C_{2,y}, y \in \phi_H^{-1}(w)\},$$

since $\{C_{3,z}, z \in \phi_{GH}^{-1}(w)\}$ is a full triangulated subcategory of $C_3$, and the functors

$$\{C_{1,x}, x \in \phi_K^{-1}(w)\} \to C_3 \leftarrow \{C_{2,y}, y \in \phi_H^{-1}(w)\}$$

factor through it.

We will show that the collection of subcategories of $C$ given by $\{C_{w}, w \in P\}$ satisfies the properties of a pre-psod of type $P$. Property (1) follows from Lemma 2.1. Thus we are reduced to checking property (2). In order to do this, it is useful to use an explicit model for the fiber product of dg-categories, which can be found for instance in [12] Appendix IV. The category $C_1 \times_{C_2} C_2$ has

- as objects, triples $(A_1, A_2, u : F(A_1) \to G(A_2))$, where $A_1$ is in $C_1$, $A_2$ is in $C_2$, and $u$ is an equivalence,

- while the Hom-complex $\text{Hom}_C((A_1, A_2, u), (A_1', A_2', u'))$ is given by the cone of the map

$$\text{Hom}_{C_1}(A_1, A_1') \oplus \text{Hom}_{C_2}(A_2, A_2') \xrightarrow{u'F-Gu} \text{Hom}_{C_3}(F(A_1), G(A_2)).$$

If $w < w'$ are distinct elements of $P$, we have to show that $C_w \subseteq C_{w'}$. That is, we need to prove that if $(A_1, A_2, u)$ is in $C_w$ and $(A_1', A_2', u')$ is in $C_{w'}$, then $\text{Hom}_C((A_1', A_2', u'), (A_1, A_2, u)) \simeq 0$. This
however follows immediately by the calculation of the Hom-complexes in \( C \) given by (3). Indeed, since we have inclusions
\[
\bigoplus_{x \in \phi^{-1}_K(w)} C_{1,x} \subseteq \left( \bigoplus_{x \in \phi^{-1}_K(w')} C_{1,x} \right)^\perp, \quad \bigoplus_{y \in \phi^{-1}_H(w)} C_{2,y} \subseteq \left( \bigoplus_{y \in \phi^{-1}_H(w')} C_{2,y} \right)^\perp,
\]
the source of the morphism of complexes (3) vanishes. Further, we have that
\[
F(A_1') \in \langle C_{3,z}, z \in \phi^{-1}_{FK}(w') \rangle \quad \text{and} \quad G(A_2) \in \langle C_{3,z}, z \in \phi^{-1}_{GH}(w) \rangle,
\]
and as \( \phi_{GH} = \phi_{FK} \) is order-reflecting we have that
\[
\langle C_{3,z}, z \in \phi^{-1}_{GH}(w) \rangle \subseteq \left( \langle C_{3,z}, z \in \phi^{-1}_{FK}(w') \rangle \right)^\perp.
\]
Hence also the target of the morphism of complexes (3) vanishes and thus its cone is zero, and this concludes the proof. \( \Box \)

We will be interested in calculating limits of categories equipped with actual psod-s, rather than just pre-psod-s. However, in general we cannot conclude that the limit category \( C \) will carry a psod, as the admissible subcategories constructed in the proof of Proposition 3.9 might fail to generate \( C \). We clarify this point via an example in Example 3.10 below. Then, in Theorem 3.13, we give sufficient conditions ensuring that the limit category will carry an actual psod.

**Example 3.10.** Let \( \{C_n\}_{n \in \mathbb{N}} \) be a collection of triangulated dg-categories. We can equip them with a psod indexed by the trivial preorder \( P_n = \{\ast\} \) for all \( n \in \mathbb{N} \). Then Proposition 3.9 yields a pre-psod on \( D = \prod_{n \in \mathbb{N}} C_n \) indexed by the set \( \mathbb{N} \) equipped with the discrete preorder: the subcategories of \( C \) forming this pre-psod are given by
\[
D_n := C_n \times \langle 0 \rangle \longrightarrow C_n \times \prod_{m \in \mathbb{N}, m \neq n} C_m \xrightarrow{\simeq} C.
\]
It is easy to see that the collection \( \{D_n\}_{n \in \mathbb{N}} \) fails to be a psod. Indeed the category spanned by the \( D_n \) is \( \langle D_n, n \in \mathbb{N} \rangle \simeq \bigoplus_{n \in \mathbb{N}} D_n \), which is strictly contained in \( C \).

Let \( (\mathbb{N}, \leq) \) be the set of natural numbers equipped with their usual ordering.

**Definition 3.11.** A preordered set \((P, \leq_P)\) is **directed** if there exist an order-reflecting map
\[
(P, \leq_P) \rightarrow (\mathbb{N}, \leq).
\]

**Remark 3.12.** Note that if \((P, \leq_P)\) is a directed finite preorder, then we can number its elements \( \{p_0, \ldots, p_m\} \) by natural numbers in such a way that, if \( 0 \leq n < n' \leq m \), then \( p_n <_P p_{n'} \).

In the statement of Theorem 3.13 below we use the same notations as in Proposition 3.9 and make the same assumptions that were made there.

**Theorem 3.13.** Assume that for all \( i \in I \), \( C_i = (C_i, P_i) \) is equipped with a psod. Assume also that the colimit of indexing preorders \( P = \lim_{i \in I} P_i \) is finite and directed. Then the limit category \( C = \lim_{i \in I} C_i \) carries a psod with indexing preordered set \( P \), \( C = (C_w, w \in P) \), such that for all \( w \in P \)
\[
C_w \simeq \lim_{i \in I} \bigoplus_{z \in \phi^{-1}_i(w)} C_{i,z}.
\]
Proof. We use the same notations we introduced in the proof of Proposition 3.9. In particular, Proposition 3.9 yields a collection of subcategories $\mathcal{C}_w$ for $w \in P$ satisfying properties (1) and (2) of a psod in Definition 3.1. We only need to show that these subcategories generate $\mathcal{C}$.

We denote by $\alpha_i: \mathcal{C} \to \mathcal{C}_i$ the universal functors from the limit category. Note that by construction, these are ordered functors. Using the directedness of $P$ we can choose a numbering of its elements $\{w_0, \ldots, w_m\}$ having the property discussed in Remark 3.12. Recall also that for $w \in P$ we denote by $r_w: \mathcal{C} \to \mathcal{C}_w$ the right adjoint of the inclusion $\iota_w: \mathcal{C}_w \to \mathcal{C}$. Let us pick a non-zero object $A \in \mathcal{C}$, and show that it belongs to the subcategory $(\mathcal{C}_w, w \in P)$. Since $\mathcal{C}_{wm}$ is right-admissible, there is a triangle
\[ t_{wm} r_{wm} A \to A \to A_{\perp_{wm}} \]
where $A_{\perp_{wm}}$ has the property that $r_{wm}(A_{\perp_{wm}}) \simeq 0$. Now set $A_1 := A_{\perp_{wm}}$, and consider the analogous triangle for $A_1$, using $\mathcal{C}_{wm-1}$ instead of $\mathcal{C}_{wm}$.
\[ t_{wm-1} r_{wm-1} A_1 \to A_1 \to A_{\perp_{wm-1}}. \]

Note that $r_{wm-1}(A_{\perp_{wm-1}}) \simeq 0$, and also $r_{wm}(A_{\perp_{wm-1}}) \simeq 0$ as
- $r_{wm}(t_{wm-1} r_{wm-1} A_1) \simeq 0$, because $t_{wm-1} r_{wm-1} A_1 \subseteq \mathcal{C}_{wm-1} \subseteq A_{\perp_{wm}}$,
- and $r_{wm}(A_1) \simeq 0$, because by construction $A_1 \subseteq A_{\perp_{wm}}$.

Next we set $A_2 := A_{\perp_{wm-1}}$, and we can iterate the construction above, this time with respect to $\mathcal{C}_{wm-2}$.

Since $P$ is finite, in this way we construct inductively an object $A_{m+1} \in \mathcal{C}$ having the property that $r_{w}(A_{m+1}) \simeq 0$ for all $0 \leq i \leq m$. Since the functors $\alpha_i$ are ordered, this implies that $r_{x} \alpha_i(A_{m+1}) \simeq 0$ in $\mathcal{C}_{i,x}$ for every $i$ and every $x \in P_i$. As each of the categories $(\mathcal{C}_i, P_i)$ is generated by the subcategories making up their psod-s, this implies that $\alpha_i(A_{m+1}) \simeq 0$ in $\mathcal{C}_i$ for all $i$ in $I$. Thus $A_{m+1} \simeq 0$. As a consequence $A$ can be realized as an iterated cone of objects lying in the subcategories $\mathcal{C}_w$ and therefore it lies in $(\mathcal{C}_w, w \in P)$, as we needed to show. \qed

Remark 3.14. For all $i \in I$, denote by $\pi_i: P_i \to P$ and $\alpha_i: \mathcal{C} \to \mathcal{C}_i$ the universal morphisms. Then it follows from the proof of Proposition 3.9 that if $w$ is in $P$, then $A \in \mathcal{C}_w$ lies in $\mathcal{C}_w \subseteq \mathcal{C}$ if and only if, for all $i \in I$ the image $\alpha_i(A)$ lies in the subcategory $(\mathcal{C}_i, x, x \in \pi_i^{-1}(w)) \subseteq \mathcal{C}_i$.

3.1. Gluing psod-s and conservative descent. A formalism for gluing semi-orthogonal decompositions along faithfully flat covers was proposed in [4, Theorem B]. The authors call their theory conservative descent. The proof given in [4] depends on rather subtle arguments. The key difference with our approach is that in that paper, the authors work with the classical theory of triangulated categories, for which there is no well-behaved notion of limits and colimits. Using the full power of the $\infty$-category of dg categories we can sidestep these difficulties, and give a simple and conceptual proof of conservative descent. From our perspective, conservative descent becomes a special case of the general structure result for limits of categories equipped with a psod given by Theorem 3.13.

More precisely, we will show that our Proposition 3.9 and Theorem 3.13 imply Theorem B from [4]. We start by briefly recalling the setting of [4], referring the reader to the original reference for full details. Let $S$ be an algebraic stack. Let $X$ and $Z_1, \ldots, Z_m$ be algebraic stacks over $S$, and let $S' \to S$ a faithfully flat map. If $T$ is an algebraic stack, we denote by $D_{qc}(T)$ the classical derived category of quasi-coherent sheaves over $T$. Note that the category $D_{qc}(T)$ is the homotopy category of $\text{Qcoh}(T)$.

The set-up of Theorem B requires to consider
- functors $\Phi_i: D_{qc}(Z_i) \to D_{qc}(X)$ which are of Fourier-Mukai type (in the sense of Definition 3.3 of [4]),
- forgetful functors $\phi_i: D_{qc}(Z_i) \to D_{qc}(S)$, and
- the six-functor formalism of [4].
and their base change along \( S' \to S \). If we set \( X' = X \times_S S' \), and \( Z'_i = Z_i \times_S S' \), then the base change of \( \Phi_i \) is a functor \( \Phi'_i : D_{qc}(Z'_i) \to D_{qc}(X') \).

Then Theorem B breaks down as the following two statements:

1. If the functors \( \Phi'_i \) are fully-faithful, then the functors \( \Phi_i \) are also fully-faithful. Under this assumption, if the subcategories \( \operatorname{Im}(\Phi'_i) \) are semi-orthogonal in \( D_{qc}(X') \),

\[
\text{i.e. } \operatorname{Im}(\Phi'_j) \subset \operatorname{Im}(\Phi'_j)^\perp \text{ if } j < j',
\]

then \( \operatorname{Im}(\Phi_i) \) are semi-orthogonal in \( D_{qc}(X) \).

2. Moreover if the subcategories \( \operatorname{Im}(\Phi'_i) \) generate \( D_{qc}(X') \), then the subcategories \( \operatorname{Im}(\Phi_i) \) generate \( D_{qc}(X) \).

Let us sketch how to recover these results from Proposition 3.9 and Theorem 3.13. We will break this explanation down into several steps. For the sake of clarity we will gloss over some technical details, which will be left to the reader.

- We set \( C := \text{Qcoh}(X) \). For all \( k \in \mathbb{N} \), we denote by \( C_k \) the dg-category of quasi-coherent sheaves over the \( k \)-th iterated fiber product of \( X \) and \( S' \) over \( S \)

\[
C_k := \text{Qcoh}(X \times_S S' \times_S \ldots \times_S S').
\]

- For all \( k \in \mathbb{N} \) we denote by \( C_{k,i} \) the dg-category of quasi-coherent sheaves over the \( k \)-th iterated fiber product of \( Z_i \) and \( S' \) over \( S \)

\[
C_{k,i} := \text{Qcoh}(Z_i \times_S S' \times_S \ldots \times_S S').
\]

- Since \( \Phi_i \) is of Fourier–Mukai type it lifts to a functor between the dg-enhancements of the derived categories of quasi-coherent sheaves. We keep denoting these functors \( \Phi_i \)

\[
\Phi_i : \text{Qcoh}(Z_i) \to \text{Qcoh}(X).
\]

By base change for all \( k \in \mathbb{N} \) we get functors \( \Phi_{k,i} : C_{k,i} \to C_k \).

- As in Theorem B from [4] we assume that for all \( i \in \{ 1, \ldots, m \} \) \( \Phi_{1,i} = \Phi'_i \) is fully-faithful, and that \( C_{1,i} \simeq \operatorname{Im}(\Phi'_i) \) are semi-orthogonal in \( C_1 = \text{Qcoh}(X') \). Equivalently, this can be formulated by saying that the subcategories \( C_{1,i} \) form a pre-psod of \( C_1 \) of type \( P \), where \( P \) is the set \( \{ 1, \ldots, m \} \) with the usual ordering. One can show that this implies that, for all \( k \in \mathbb{N} \), the subcategories \( C_{k,i} \) form a pre-psod on \( C_k \) of type \( P \).

- By faithfully flat descent, \( \text{Qcoh}(X) \) can be obtained as the limit

\[
\text{Qcoh}(X) \to \left[ \text{Qcoh}(X') \Rightarrow \text{Qcoh}(X' \times_S S') \Rightarrow \text{Qcoh}(X' \times_S S' \times_S S') \ldots \right].
\]

In formulas we will write \( C = \lim_{\longleftarrow k \in \mathbb{N}} C_k \). For every \( k \in \mathbb{N} \) we denote by

\[
a_{k,1}, \ldots, a_{k,k+1} : C_k \to C_{k+1}
\]

the \( k+1 \) structure maps from diagram [4].

Let \( k \in \mathbb{N} \), and let \( j \in \{ 1, \ldots, k + 1 \} \). It is easy to see that the identity \( \text{id} : P \to P \) is an order-reflecting map and equips \( a_{k,j} : (C_k, P) \to (C_{k+1}, P) \) with a structure of ordered functor.

- Proposition 3.9 then implies that \( C = \lim_{\longleftarrow k \in \mathbb{N}} C_k \) carries a pre-psod of type

\[
\lim_{\longleftarrow k \in \mathbb{N}} P = P
\]
with semi-orthogonal factors given by \( \lim_{\leftarrow} C_{k,i} \simeq \text{Qcoh}(Z_i) \). This recovers statement (1) of Theorem B. Now assume that \( C_{1,i} \) is actually a psod of \( C_1 = \text{Qcoh}(X') \). This implies that for all \( k \in \mathbb{N} \), \( C_{k,i} \) is a psod of type \( P \) of \( C_k \). Then Theorem 3.13 implies statement (2) of Theorem B.

4. Semi-orthogonal decompositions of root stacks

In this section we explain our main application. Let \( X \) be an algebraic stack and let \( D \) be a normal crossings divisor in \( X \). The root stack \( \sqrt[\mu_r]{(X,D)} \) of a pair \( (X,D) \), where \( D \subset X \) is a normal crossings divisor, has long been an important object in algebraic geometry. We refer the reader to the Introduction for more information on previous work in this area. We will construct semi-orthogonal decompositions on \( \text{Perf}(\sqrt[\mu_r]{(X,D)}) \). This generalizes earlier results by other authors. In \([17]\) and \([3]\) the authors construct sods on \( \text{Perf}(\sqrt[\mu_r]{(X,D)}) \), under the assumption that \( D \) is simple normal crossings. We drop the assumption of simplicity and work with general normal crossings divisors.

Remark 4.1. We want to stress a subtle point about this assumption: from \([3\), Definition 3.5\], it might seem that in that paper they do consider divisors that are merely normal crossings.

The point is that the root construction that they use in the non-simple case is not the “correct” one from the point of view of logarithmic geometry. For instance, if \( D \subset \mathbb{P}^2 \) is an irreducible nodal cubic, their \( r \)-th root construction would add a stabilizer \( \mu_r \) along all the points of \( D \), including the node. This does not take into account that locally around the node there are two distinct branches of \( D \). In this case, the “correct” construction of \( \sqrt[\mu_r]{(\mathbb{P}^2,D)} \) (and more generally when \( D \) is normal crossings but not simple) from our point of view is the one introduced in \([8]\), that adds a stabilizer \( \mu_r \) along all the points of \( D \) different from the node, and a stabilizer \( \mu_r^\circ \) at the node. A way to see that this is indeed the right notion is that, following the definition of \([8]\), the stack \( \sqrt[\mu_r]{(\mathbb{P}^2,D)} \) is smooth, just as the root stacks of smooth schemes along simple normal crossings divisors are; whereas with the definition of root stacks given in \([3\), Definition 3.5\], in the non-simple normal crossings case one obtains singular stacks.

The shape of the sods that we construct in the general case have interesting differences from the ones given in \([17]\) and \([3]\). Indeed whereas in the simple normal crossings case the factors of the sods are equivalent to perfect complexes on the strata, in the general case we need to work with the normalizations of the strata.

Our construction of psod-s for \( \text{Perf}(\sqrt{(X,D)}) \) relies in essential way on Theorem 3.13. The other main ingredient are the psod-s obtained in \([3]\) in the simple normal crossing case, and which will be reviewed in a slightly different formulation in Section 4.1 below.

4.1. Root stacks of normal crossing divisors. We start by introducing some notations. Let \( X \) be an algebraic stack and let \( D \subset X \) be a (non-necessarily simple) normal crossing divisor. Note that \( X \) carries a natural stratification given by locally closed substacks which can be obtained, locally, as intersections of the branches of \( D \). Equivalently this stratification can be defined as follows: let \( \bar{D} \) be the normalization of \( D \) and consider the locally closed substacks \( S \) which are maximal with respect to the following two properties: \( S \) is connected; the map \( \bar{D} \times_X S \to S \) is étale.

---

1Proposition 3.9 and Theorem 3.13 were formulated for small dg-categories, while here we are applying them to the large category \( \text{Qcoh}(X) \): however our results also hold, without variations, in the setting of large categories.
Let $S(D)$ denote the set of strata closures, that we will henceforth simply call “strata”\footnote{In the usual definition, strata are locally closed, and their boundary is a disjoint union of smaller dimensional strata. We look instead at closed strata, which are given by the closures of the locally closed ones.} For every $k \in \mathbb{N}$ we set

$$S(D)_k := \{ S \in S(D) \mid \text{codim}(S) = k \}, \quad \text{and} \quad S(D) := \bigcup_{S \in S(D)} S.$$  

We say that $S(D)_k$ is the $k$-codimensional skeleton of the stratification. For every stratum $S \in S(D)$ we denote by $\tilde{S}$ its normalization. We denote by $\tilde{S(D)}_k$ the normalization of $S(D)_k$: $\tilde{S(D)}_k$ is the disjoint union of the normalizations of the irreducible components of $D$; more generally, $\tilde{S(D)}_k$ is the disjoint union of the normalizations of the strata in $S(D)_k$. In formulas we can write

$$\tilde{S(D)}_k = \coprod_{S \in S(D)_k} \tilde{S}.$$  

Let $N_D$ be the maximal codimension of strata of $(X, D)$. We denote by $(N_D, \leq)$ the ordered set

$$\overline{N_D} := \{ N_D, N_D - 1, \ldots, 0 \} \quad \text{ordered by} \quad N_D < N_D - 1 < \ldots < 0.$$  

We set $N_D^* = N_D - \{ 0 \}$. For $r \in \mathbb{N}$, we denote by $\mathbb{Z}_r$ the group of residue classes modulo $r$ and set $\mathbb{Z}_r^* := \mathbb{Z}_r - \{ 0 \}$. As in \cite{27}, it is useful to identify $\mathbb{Z}_r$ and $\mathbb{Z}_r^*$ with the order $r$,\footnote{Note that $\mathbb{Z}_r^*$ has a psod of type $\mu_r \in \mathbb{Z}/r\mathbb{Z}$, where $\mu_r$ is the $r$th root stack of $(\mathbb{Z}/r\mathbb{Z})_r$.} $\mathbb{Z}_r \cong \left\{ \frac{r-1}{r}, \ldots, \frac{-1}{r}, 0 \right\} \subset \mathbb{Q} \cap (-1, 0]$, $\mathbb{Z}_r^* \cong \left\{ \frac{-r-1}{r}, \ldots, \frac{1}{r} \right\} \subset \mathbb{Q} \cap (-1, 0]$.

We equip $\mathbb{Z}_r$ and $\mathbb{Z}_r^*$ with the order $\leq$ given by the restriction of the order on $\mathbb{Q}$

$$-\frac{r-1}{r} < \frac{-r-2}{r} < \ldots < \frac{-1}{r} < 0.$$  

For all $k \in N_D^*$ we set $\mathbb{Z}_{k,r} := \bigoplus_{i=1}^k \mathbb{Z}_r$ and $\mathbb{Z}_{k,r}^* := \bigoplus_{i=1}^k \left( \mathbb{Z}_r - \{ 0 \} \right)$. Note that $\mathbb{Z}_{0,r}^* = \{ 0 \}$. We equip $\mathbb{Z}_{k,r}$ and $\mathbb{Z}_{k,r}^*$ with the product preorder. The group $\mathbb{Z}_{k,r}$ is canonically isomorphic to the group of characters of the group of roots of unity $\mu_{k,r} := \bigoplus_{i=1}^k \mu_r$. We denote by $(\mathbb{Z}_{D,r}, \leq)$ the set $\prod_{k=0}^{N_D} \mathbb{Z}_{k,r}^*$ equipped with the following preorder: let $\xi, \xi'$ be in $\mathbb{Z}_{D,r}$, with $\xi \in \mathbb{Z}_{k,r}$ and $\xi' \in \mathbb{Z}_{k',r}$, then

$$\xi \leq \xi' \text{ if } k < \overline{k} \text{ or } k = \overline{k}' \text{ and } \xi \leq_{\mathbb{Z}_{k,r}} \xi'.$$  

Let $\sqrt[\overline{N_D}]{(X, D)}$ be the $r$-th root stack of $(X, D)$. Note that $\sqrt[\overline{N_D}]{(X, D)}$ also carries a natural normal crossing divisor, which we denote $D_r \subset \sqrt[\overline{N_D}]{(X, D)}$, obtained as reduction of the preimage of $D$. All previous notations and definitions therefore also apply to the log pair $\left( \sqrt[\overline{N_D}]{(X, D)}, D_r \right)$.

**Proposition 4.2.** (1) The category $\text{Perf}(\sqrt[\overline{N_D}]{(X, D)})$ has a psod of type $(N_D, \leq)$, $\text{Perf}(\sqrt[\overline{N_D}]{(X, D)}) = \langle A_0, A_k \in \mathbb{N} \rangle$ where:

- $A_0 \simeq \text{Perf}(X)$.

- For all $k \in N_D^*$, the subcategory $A_k$ has a psod of type $(\mathbb{Z}_{k,r}^*, \leq)$

$$A_k = \langle A^k, \chi \in (\mathbb{Z}_{k,r}^*, \leq) \rangle$$  

and, for all $\chi \in \mathbb{Z}_{k,r}^*$, there is an equivalence $A^k_{\chi} \simeq \text{Perf}(\tilde{S(D)}_k)$.
(2) The category $\text{Perf}(\sqrt{(X,D)})$ has a psod of type $(\mathbb{Z}_{D,r}, \leq)$

$$\text{Perf}(\sqrt{(X,D)}) = \langle A^k_\chi, \chi \in (\mathbb{Z}_{D,r}, \leq) \rangle,$$

where $A_0 \simeq \text{Perf}(X)$ and for all $\chi \in \mathbb{Z}_{k,r}$, there is an equivalence $A^k_\chi \simeq \text{Perf}(\widetilde{S(D)}_k)$.

Before giving a proof of Proposition 4.2 we will make a few preliminary considerations.

If $D \subset X$ is a simple normal crossing divisor, Proposition 4.2 is a reformulation of Theorem 4.9 from [3]. In order to translate back to the statement of [3, Theorem 4.9], it is sufficient to note that $S_k$ decomposes as the disjoint union of strata of codimension $k$. Indeed, in the simple normal crossing setting, all strata are already normal. This yields an equivalence

$$A^k_\chi \simeq \bigoplus_{S \in S_k} \text{Perf}(S)$$

which recovers the semi-orthogonal factors given by [3, Theorem 4.9].

It will be useful to explain how the semi-orthogonal factors $A^k_\chi$ are constructed in the simple normal crossing case, and some of their basic properties. We refer to the treatment contained in Sections 3.2.1 and 3.2.2 of [27], and limit ourselves to statements without proof. For all $k \in \mathbb{N}$ there is a canonical $\mu_{k,r}$-gerbe $G_{k,r}(D) \to \widetilde{S(D)}_k$, that fits in a diagram

$$G_{k,r}(D) \xleftarrow{q} \widetilde{S(D)}_k \xrightarrow{i} \text{Perf}(\sqrt{(X,D)})$$

where $\widetilde{S(D)}_k$ is the normalization of the codimension $k$ skeleton of the stratification of $(\sqrt{(X,D)}, D_r)$. There is a natural splitting (as in Lemma 3.9 of [27])

$$\text{Perf}(G_{k,r}(D)) \simeq \bigoplus_{\chi \in \mathbb{Z}_{k,r}} \text{Perf}(G_{k,r}(D))_\chi \simeq \bigoplus_{\chi \in \mathbb{Z}_{k,r}} \text{Perf}(\widetilde{S(D)}_k).$$

For every $\chi \in \mathbb{Z}_{k,r}$, the subcategory $A^k_\chi$ is defined as the image of the composite

$$A^k_\chi = \text{Perf}(\widetilde{S(D)}_k) \simeq \text{Perf}(G_{k,r}(D))_\chi \xrightarrow{(\ast)} \text{Perf}(G_{k,r}(D)) \xrightarrow{i \ast q^e} \text{Perf}(\sqrt{(X,D)})$$

where arrow $(\ast)$ is the inclusion of the $\chi$-th factor.

Proof of Proposition 4.2. We will use the fact that, by [3, Theorem 4.9], the statement holds when $D$ is simple normal crossing.

Let $D \subset X$ be a general normal crossing divisor. Consider an étale covering $U \to X$ such that the pull-back to $U$ of the log structure of $(X,D)$ is induced by a simple normal crossings divisor $D|_U$. The construction of root stacks is compatible with base change, so that there are natural isomorphisms $\sqrt{(U,D|_U)} \simeq \sqrt{(X,D)} \times_X U$. Further, $\sqrt{(U,D|_U)} \to \sqrt{(X,D)}$ is an étale covering. For all $l \in \mathbb{N}$ we denote the $l$-fold iterated fiber product of $U$ over $X$ $U_1 := U \times_X \ldots \times_X U$ . For all $l \in \mathbb{N}$ the pull-back of the log structure on $(X,D)$ to $U_l$ is also induced by a simple normal crossing divisor, which we denote $D_l$.

Consider the semi-simplicial stack given by the nerve of the étale cover $\sqrt{(U,D|_U)} \to \sqrt{(X,D)}$

$$\ldots \implies \sqrt{(U_2,D_2)} \implies \sqrt{(U,D|_U)} \to \sqrt{(X,D)}.$$  

Note that for all $l$, the structure morphisms $p_1, \ldots, p_{l+1}$

$$p_1, \ldots, p_{l+1} : \sqrt{(U_{l+1},D_{l+1})} \to \sqrt{(U_l,D_l)}$$
map strata of codimension $k$ to strata of codimension $k$. By faithfully flat descent, we can realize $\text{Perf}(\sqrt{X,D})$ as the totalization of the induced semi-cosimplicial diagram of dg-categories, where the structure maps are given by pull-back functors:

\[
\text{Perf}(\sqrt{X,D}) \cong \lim_{\substack{\longrightarrow \cr \ell \in \mathbb{N}}} \text{Perf}(\sqrt{(U_\ell,D|_{U_\ell})}).
\]

We are going to prove the proposition by applying Theorem 3.13 to the limit of dg-categories (7), more precisely:

- Since $D_l$ is simple normal crossing, for every $l$ we can equip $\text{Perf}(\sqrt{(U_l,D|_{U_l})})$ with the psod of type $Z_{D_l,r}$ given by Proposition 4.2, where the semi-orthogonal factors are defined as in (6). Also, since $Z_{D_l,r} = Z_{D,r}$, we can write

\[
\text{Perf}(\sqrt{(U_l,D|_{U_l})}) \cong \langle A_{l,k}^{l+1}, \chi \in (Z_{D,r}, \leq) \rangle.
\]

- We equip the functors appearing in limit (7) with the ordered structure given by the identity map: that is, for every $j \in \{1, \ldots, l + 1\}$ we have

\[
p_j^* : \text{Perf}(\sqrt{(U_l,D_l)}) \longrightarrow \text{Perf}(\sqrt{(U_{l+1},D_{l+1})})
\]

and we set $\phi_{p_j} = \text{id}: Z_{D,r} \rightarrow Z_{D,r}$.

Note that $\lim Z_{D_l,r} = Z_{D,r}$ is a finite preorder. Thus, to apply Theorem 3.13, we only need to check that the identity map $\text{id}: Z_{D,r} \rightarrow Z_{D,r}$ does indeed induce a well-defined ordered structure on the functors appearing in (7). Namely, we have to prove the following two properties:

(a) We need to show that for all $j \in \{1, \ldots, l + 1\}$, for all $\chi \in Z_{D,r}$ there is a strictly commutative diagram

\[
\begin{array}{ccc}
A_{l,k}^{l+1} & \longrightarrow & \text{Perf}(\sqrt{(U_{l+1},D_{l+1})}) \\
p_j^* & \downarrow & \downarrow p_j \\
A_{l,k}^{l} & \longrightarrow & \text{Perf}(\sqrt{(U_l,D_l)}). \\
\end{array}
\]

(b) Additionally we need to show that the canonical 2-cells obtained via adjunction from (strictly) commutative diagram (8) are also invertible, giving rise to commutative diagrams

\[
\begin{array}{ccc}
A_{l,k}^{l+1} & \longrightarrow & \text{Perf}(\sqrt{(U_{l+1},D_{l+1})}) \\
l_\chi & \downarrow & r_\chi \\
A_{l,k}^{l} & \longrightarrow & \text{Perf}(\sqrt{(U_l,D_l)}). \\
\end{array}
\]

Let us start with property (a). The horizontal arrows in (8) are inclusions, thus we only need to check that $p_j^*$ maps $A_{l,k}^{l}$ to the subcategory $A_{l,k}^{l+1}$. Note that the map

\[
p_j : \sqrt{(U_{l+1},D_{l+1})} \rightarrow \sqrt{(U_l,D_l)}
\]
maps strata to strata. Thus we get a commutative diagram

\[
\begin{array}{ccc}
G_{k,r}(D_{l+1}) & \xleftarrow{q} & S(D_{l+1}, r)_k \\
\downarrow p_j & & \downarrow p_j \\
G_{k,r}(D_l) & \xleftarrow{q} & S(D_l, r)_k
\end{array}
\xrightarrow{t} \text{Perf}(\sqrt{(U_{l+1}, D_{l+1})})
\]

where all vertical arrows are étale (and in particular flat and proper), and both the left and the right square are fiber products. Property (a) follows because there is a natural equivalence

\[\tau_* q^* p_j^* \simeq p_j^* \tau_* q^* \]

given by the composite

\[\tau_* q^* p_j^* \simeq \tau_* p_j^* q^* \simeq p_j^* \tau_* q^*
\]

where equivalence (\(\ast\)) is given by flat base change.

Let us consider property (b) next. We will show that the Beck–Chevalley property holds with respect to right adjoints: that is, that the right square in (9) commutes. The case of left adjoints is similar. Note that \(r_\chi: \text{Perf}(\sqrt{(U_{l+1}, D_{l+1})}) \to \mathcal{A}_{\chi}^{l+1,k}\) is given by the composite

\[\text{Perf}(\sqrt{(U_{l+1}, D_{l+1})}) \xrightarrow{q_* \tau^!} G_{k,r}(D_{l+1}) \xrightarrow{pr_\chi} A_{\chi}^{l+1,k},\]

where \(pr_\chi\) is the projection onto the \(\chi\)-th factor, and similarly for \(r_\chi: \text{Perf}(\sqrt{(U_l, D_l)}) \to \mathcal{A}_{\chi}^{l,k}\).

Thus, in order to show that the right square in (9) commutes, it is enough to prove that the canonical natural transformation

\[q_* \tau^! p_j^* \Rightarrow p_j^* q_* \tau^! : \text{Perf}(\sqrt{(U_l, D_l)}) \to G_{k,r}(D_{l+1})\]

is invertible. This can be broken down as a composite of the base change natural transformations

\[q_* \tau^! p_j^* \Rightarrow q_* p_j^* \tau^! \Rightarrow p_j^* q_* \tau^!
\]

which are both equivalences: (i) is an equivalence by proper base change \((\tau^! p_j^* \simeq p_j^* \tau^!)(\), and (ii) by flat base change \((q_* p_j^* \simeq p_j^* q_*)\).

Since property (a) and (b) are satisfied we can apply Theorem 3.13 to our setting. Thus \(\text{Perf}(\sqrt{(X, D)})\) carries a psod of type \((\mathbb{Z}_{D,r}, \leq)\)

\[\text{Perf}(\sqrt{(X, D)}) = \langle \mathcal{A}_\chi^k, \chi \in (\mathbb{Z}_{D,r}, \leq) \rangle
\]

where for all \(\chi \in \mathbb{Z}_{D,r}\) we have that

\[\mathcal{A}_\chi^k = \lim_{l \in \mathbb{N}} A_{\chi}^{l,k} \simeq \lim_{l \in \mathbb{N}} \text{Perf}(S(D_l)_k).
\]

Note that the stacks \(\overline{S(D_l)_k}\), together with the structure maps between them, are the nerve of the étale cover \(\overline{S(D_U)_k} \to \overline{S(D)_k}\). Thus, by faithfully flat descent, we have an equivalence

\[\mathcal{A}_\chi^k \simeq \lim_{l \in \mathbb{N}} \text{Perf}(S(D_l)_k) \simeq \text{Perf}(\overline{S(D)_k}).
\]

This concludes the proof of part (2) of Proposition 4.2 which immediately implies part (1). \(\square\)
It will be useful to reformulate Proposition 4.2 in a way that is closer to the analogous statements in the simple normal crossing setting given in [3, Theorem 4.9] and [27, Proposition 3.12]. We do this in Corollary 4.3 below. This amounts to breaking down the factors $\mathcal{A}^S_\chi \simeq \text{Perf}(\tilde{S}(D)_k)$ into direct sums: since $S(D)_k$ is the disjoint union of the normalizations of the $k$-codimensional strata, we have a decomposition

$$\mathcal{A}^S_\chi \simeq \text{Perf}(\tilde{S}(D)_k) \simeq \bigoplus_{S \in S(D)_k} \text{Perf}(\tilde{S}).$$

Before stating this result we need to introduce some notations. We equip the set of strata $S(D)$ with the coarsest preorder satisfying the following two properties: let $S$ and $S'$ be in $S(D)$

- (1) if $S \subseteq S'$ then $S \leq S'$,
- (2) if $\dim(S) = \dim(S')$ then $S \leq S'$ and $S' \leq S$.

We denote $S(D)^* := S(D) - \{X\}$. For every $S \in S(D)$, if $\text{cod}(S)$ is the codimension of $S$ we set $|S| := \text{cod}(S)$. We denote by $(Z_{S(D)},r)\leq$ the set $\bigsqcup_{S \in S(D)} Z_{S,r}^*$ equipped with the following preorder: let $\xi,\xi'$ be in $Z_{S,D,r}$, with $\xi \in Z_{S,r}$ and $\xi' \in Z_{S',r}$, then $\xi < \xi'$

- if $|S| > |S'|$,
- or if $|S| = |S'|$ and $S \neq S'$,
- or if $S = S'$ and $\xi \prec_{Z_{S,r}} \xi'$.

Corollary 4.3 generalizes [3, Theorem 4.9] and [27, Proposition 3.12] to pairs $(X,D)$ where $D$ is normal crossing but necessarily simple.

**Corollary 4.3.**

1. The category $\text{Perf}(\sqrt{(X,D)})$ has a psod of type $(S(D),\leq)$, $\text{Perf}(\sqrt{(X,D)}) = (\mathcal{A}_S, S \in S(D))$ where:
   - $\mathcal{A}_X \simeq \text{Perf}(X)$.
   - For all $S \in S(D)^*$, the subcategory $\mathcal{A}_S$ has a psod of type $(Z_{S,r}^*, \leq)$
     $$\mathcal{A}_S = (\mathcal{A}_X^S, \chi \in (Z_{S,r}^*, \leq))$$
     and, for all $\chi \in Z_{S,r}^*$, there is an equivalence $\mathcal{A}_S^\chi \simeq \text{Perf}(S)$.

2. The category $\text{Perf}(\sqrt{(X,D)})$ has a psod of type $(Z_{S(D),r}^*, \leq)$

   $$\text{Perf}(\sqrt{(X,D)}) = (\mathcal{A}_X^S, \chi \in (Z_{S(D),r}^*, \leq)),$$

   where $\mathcal{A}_X^S \simeq \text{Perf}(X)$ and for all $\chi \in Z_{S,r}^*$ there is an equivalence $\mathcal{A}_X^\chi \simeq \text{Perf}(S)$.

**Example 4.4.** It might be useful to describe the sod given by Corollary 4.3 in a concrete example. Let $X = \mathbb{A}^2$ and let $D$ be an irreducible nodal cubic curve with node at the origin $o \in \mathbb{A}^2$. The stratification of $\mathbb{A}^2$ induced by $D$ has three strata: $o$, $D$ and $\mathbb{A}^2$. The normalization of $\tilde{D}$ of $D$ is isomorphic to $\mathbb{A}^1$. The preorder $(Z_{S(D),2}, \leq)$ is given by $o < D < \mathbb{A}^2$. Then Corollary 4.3 yields a sod of the form

$$\text{Perf}(\sqrt{(\mathbb{A}^2,D)}) = (\text{Perf}(o), \text{Perf}(\tilde{D}), \text{Perf}(\mathbb{A}^2)).$$

5. **Applications to logarithmic geometry**

In this last section we explain two consequences of our results in the context of logarithmic geometry and the theory of parabolic sheaves. In section 3 we construct an infinite sod on the category of perfect complexes over the infinite root stack $\sqrt{(X,D)}$ of a pair $(X,D)$ where $D$ is a general normal crossing divisor. Equivalently, this can be expressed by saying that we construct...
an infinite sod on the derived category of parabolic sheaves with rational weights on \((X, D)\). This improves earlier results that we obtained in [27]. Then in section 5.2 we show that this implies a generalization to the general normal crossings case of an important structure result in Kummer flat K-theory originally due to Hagihara and Nizioł.

5.1. **Infinite root stacks of normal crossing divisors.** Before proceeding it is useful to recall our results from [27], and explain in which way our current results improve on them.

1. In [27] we construct psod-s on Perf\(\sqrt[\infty]{(X, D)}\), under the assumption that \(D\) is simple normal crossings.
2. In [27] we also constructed psod-s on Perf\(\sqrt[\infty]{(X, D)}\) in the general normal crossing case. This however required, first of all, to work over a field of characteristic zero, and secondly involved using a highly non-trivial result on the invariance of Perf\(\sqrt[\infty]{(X, D)}\) under log blow-ups which was established in [28]. In particular the sod obtained in this way depended on a choice of a simple normal crossing model \((\tilde{X}, \tilde{D})\) obtained from \((X, D)\) by successive blow-ups along the strata of \(D\).

**Remark 5.1.** It is also important to note that the psod-s in the general normal crossing case constructed in [27] only exist for the infinite root stack, and not for finite root stacks. The argument in [27] requires to switch to a simple normal crossing birational model \((\tilde{X}, \tilde{D})\) of \((X, D)\). The key point is that the categories of perfect complexes of \(\sqrt[\infty]{(\tilde{X}, \tilde{D})}\) and \(\sqrt[\infty]{(X, D)}\) are equivalent by the main theorem of [28] but this is far from true for finite root stacks. Thus the psod-s on finite root stacks constructed in Proposition 4.2 are entirely new, even over a field of characteristic 0.

Let \((X, D)\) be a pair given by an algebraic stack equipped with a normal crossing divisor. Our main goal in this section is to use the results of Section 4 to construct psod-s (patterned after Corollary 4.3) on Perf\(\sqrt[\infty]{(X, D)}\) in the general normal crossing case, which are independent of the ground ring and of the choice of a desingularization \((\tilde{X}, \tilde{D})\) of \((X, D)\). This is given by Theorem 5.3 below. The proof strategy is the same as the one that was used to prove Theorem 3.16 of [27], whose statement exactly parallels Theorem 5.3: the only difference is that Theorem 3.16 of [27] assumes that \(D\) is simple normal crossings. For this reason we will limit ourselves to state our results, referring the reader to [27] for the proof.

Formulating Theorem 5.3 requires introducing some notations. Let \(S\) be a stratum in \(S(D)\). We denote

\[
(Q/Z)_{|S|} := \bigoplus_{i=1}^{|S|} Q/Z, \quad (Q/Z)^*_{|S|} := \bigoplus_{i=1}^{|S|} \left(Q/Z - \{0\}\right).
\]

There is a natural identification \((Q/Z)_{|S|} = Q^{|S|} \cap (-1, 0)^{|S|}\). We will equip \((Q/Z)_{|S|}\) with a total order, which we denote \(\leq^1\), that differs from the restriction of the usual ordering on the rational numbers.

First of all we define the order \(\leq^1\) on \(Z_{n!}\) recursively, as follows.

- On \(Z_{2!} = \{-\frac{1}{2}, 0\}\) we set \(-\frac{1}{2} <^1 0\).
- Having defined \(\leq^1\) on \(Z_{(n-1)!}\), let us consider the natural short exact sequence

\[
0 \to Z_{(n-1)!} \to Z_n \overset{\pi_n}{\to} Z_n \to 0,
\]

where \(Z_n = \{-\frac{n-1}{n}, \ldots, -\frac{1}{n}, 0\}\) is equipped with the standard order \(\leq\) described above. Given two elements \(a, b \in Z_n\), we set \(a <^1 b\) if either \(\pi_n(a) < \pi_n(b)\), or \(\pi_n(a) = \pi_n(b)\) and
5.2. Kummer flat K-theory of normal crossing divisors. The psod-s on infinite root stacks that we constructed in [27] are a categorification of structure theorems in Kummer flat K-theory of simple normal crossing divisors due to Hagihara and Nizioł [15], [23]. Our techniques allowed us to extend those structure theorems to a wider class of log stacks, including the case of general normal crossing divisors.

In this section we explain how the results obtained in Section 5.1 yield unconditional structure theorems.

Definition 5.2. Let
\[ \chi = \left( -\frac{p_1}{n!}, \ldots, -\frac{p_N}{n!} \right), \quad \chi' = \left( -\frac{q_1}{m!}, \ldots, -\frac{q_N}{m!} \right) \in \mathbb{Q}^{\leq 1} \]
for some \( p_1, \ldots, p_N \in \mathbb{N} \) where \( n \in \mathbb{N} \). This expression is unique if we require \( n \) to be as small as possible, and we call this the normal factorial form.

Definition 5.4. An additive invariant of dg-categories is a functor of \( \infty \)-categories
\[ H: dgCat \rightarrow \mathcal{P} \]
where \( \mathcal{P} \) is a stable and presentable \( \infty \)-category, satisfying the following properties:
1. \( H \) preserves zero-objects.
2. \( H \) sends split exact sequences of dg-categories to cofiber sequences in \( \mathcal{P} \).
3. \( H \) preserves filtered colimits.
Hochschild homology, algebraic K-theory and non-connective K-theory are all examples of additive invariants. As proved in [30, 7] (see also [24] and [16]) there exists a universal additive invariant

\[ U: \text{dgCat} \longrightarrow \text{Mot}. \]

The target category of the universal additive invariant, Mot, is called the category of additive noncommutative motives. If \( X \) is a stack, we denote \( U(\text{Perf}(X)) \) simply by \( U(X) \).

The following Corollary extends to the general normal crossing case Corollary 5.6 and 5.8 of [27]; its second half extends to the general normal crossing case Theorem 1.1 of [23].

Corollary 5.5. Let \((X, D)\) be a log stack given by an algebraic stack \( X \) equipped with a normal crossing divisor \( D \).

- There is an equivalence

\[ U(\text{Perf}(\sqrt[\infty](X, D))) \cong U(X) \bigoplus \bigoplus_{S \in \mathcal{S}(D)^*} \left( \bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S} U(\tilde{S}) \right). \]

Since \( U \) is universal, every additive invariant \( H(\text{Perf}(\sqrt[\infty](X, D))) \) decomposes as a direct sum patterned after (10).

- Denote by \( K_{\text{Kfl}}(X, D) \) the Kummer flat K-theory of \((X, D)\). Then there is a direct sum decomposition of spectra

\[ K_{\text{Kfl}}(X, D) \cong K(X) \bigoplus \bigoplus_{S \in \mathcal{S}(D)^*} \left( \bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S} K(\tilde{S}) \right). \]

Proof. The first part of the Corollary is proved exactly as Corollary 5.6 from [27]. As explained in Section 2.1.4 of [27], it follows from [33] that the Kummer flat K-theory of \((X, D)\) coincides with the algebraic K-theory of \( \text{Perf}(\sqrt[\infty](X, D)) \). Since K-theory is an additive invariant, the second half of the Corollary follows from the first. This concludes the proof. \( \square \)

Remark 5.6. In addition to Kummer flat K-theory one can define Kummer flat versions of all additive invariants, such as Hochschild homology, in the following way. If \( X \) is a log scheme, let \( \text{Perf}(X_{\text{Kfl}}) \) be the dg-category of perfect complexes over the Kummer flat topos of \( X \). Then for every additive invariant \( H \), we set \( H_{\text{Kfl}}(X) := H(\text{Perf}(X_{\text{Kfl}})) \). Corollary 5.5 implies that, if the log structure on \( X \) is given by a normal crossing divisor, \( H_{\text{Kfl}}(X) \) also decomposes as a direct sum patterned after (10).

Remark 5.7. Corollary 5.5 has an analogue for the Kummer étale topos of \((X, D)\): this, in particular, extends the second part of the statement of Theorem 1.1 of [23] and the Main Theorem of [15] to the general normal crossing setting. In characteristic zero there is no difference so this comment is relevant only if \( \kappa \) has positive or mixed characteristic, and, assuming that \( D \) is equicharacteristic as in [23], \( \mathbb{Q}/\mathbb{Z} \) has to be replaced by \( (\mathbb{Q}/\mathbb{Z})' = \mathbb{Z}_p/\mathbb{Z} \) (where \( p \) is the characteristic over which \( D \) lives) in the formulas above. The key observation is that, if \( X_{\text{K\acute{e}t}} \) is the Kummer étale topos, then \( \text{Perf}(X_{\text{K\acute{e}t}}) \) is equivalent to perfect complexes over a restricted version of the infinite root stack \( \sqrt[\infty'](X, D) \), where we take the inverse limit only of root stacks \( \sqrt[\infty](X, D) \) such that \( p \) does not divide \( r \). Then \( \text{Perf}(\sqrt[\infty'](X, D)) \) carries a psod which analogous to the one given by Theorem 5.3 except we work everywhere with indices which are coprime to \( p \). We leave the details to the interested reader.
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Sarah Scherotzke, Université du Luxembourg, Maison du Nombre, 6, Avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg

Email address: sarah.scherotzke@uni.lu

Nicolò Sibilla, SMSAS, University of Kent, Canterbury, Kent CT2 7NF, UK and SISSA, Via Bonomea 265, 34136 Trieste (TS), Italy

Email address: N.Sibilla@kent.ac.uk

Mattia Talpo, Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa (PI), Italy

Email address: mattia.talpo@unipi.it