Effective Equations on the 3-Brane World from Type IIB String

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Abstract

The effective field equations on a 3-brane are established considering the massless bosonic sector of the type IIB string compactified on $S^5$. The covariant embedding formalism in a space endowed with $Z_2$-symmetry is applied. Recently the derivation of effective equations on the 3-brane, where only gravity penetrates in the bulk has been performed by Shiromizu, Maeda, and Sasaki [23]. We extend this analysis to the situation when the bulk contains a set of fields given by the type IIB string. The notion of the Einstein-Cartan space is considered in order to avoid extra suppositions about the embedding of these fields. The interactions between the brane and the bulk fields are understood in a purely geometric way, which fixes the form of these interactions. Finally, we present the dynamically equivalent effective equations have expressed completely in Riemannian terms and make conclusions.

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INTRODUCTION AND MOTIVATIONS

Since the papers of Kaluza and Klein [22] it has been suggested a possibility that there exist extra dimensions beyond those of Minkowski space-time. In recent years, ideas of extra dimensions become much more compelling. According to [23, 24, 29] it has been understood that additional dimensions could have a quite distinct nature from those of Kaluza and Klein. In other words, ordinary matter would confine to our 4-dimensional world while gravity would penetrate in the extended space (bulk). An intriguing motivation for considering these models comes from string/M-theory. For instance, the 10-dimensional type $IIB$ theory can admit brane worlds solutions [13, 14]. The type $IIB$ theory have a rich massless spectrum. Hence this theory illustrates an enlarged complexity due to the supplementary fields in addition to gravity and even scalar fields (see [14, 15, 27] for previous work). This situation is more curious since it seems unlikely that only gravity can penetrate in extra dimensions. In this paper, considering the massless bosonic sector of the type $IIB$ theory have compactified on a sphere $S^5$ we derive effective field equations on the 3-brane in a 5-dimensional bulk endowed with $Z_2$-symmetry.

Our approach to this problem is based on the covariant embedding formalism, which gives a coordinate independent derivation of the dynamics on the brane. In a typical brane worlds setup, where only gravity penetrates in the bulk, the derivation of effective field equations was performed in [23, 24]. We extend this analysis to the situation, which assumes the bulk to contain a set of fields given by the type $IIB$ theory. The problem that arises is to find an appropriate description of the embedding procedure in presence of additional bulk fields in order to avoid extra suppositions about the embedding of these fields. In this paper, we consider one possible candidate for such description, which is based on the notion of the Einstein-Cartan space. Roughly speaking this space is a generalization of the Riemannian space by considering the torsion tensor [16, 33]. The status of torsion in the framework of 4-dimensional gravity and cosmology remains open today [16, 17, 19, 20, 21]. However such spaces have recently become relevant due to the relationship to many extended theories like superstrings, supergravity, and e.t.c. [2, 3, 4, 5, 17, 18]. Hence it has been expected that the embedding formalism developed for the Einstein-Cartan space can give an unified and transparent understanding of the dynamics on the brane. At the same time we stress that one can express the resulting effective equations in several dynamically equivalent forms.
Thus to consider the Riemannian form supplied with extra terms of non-Riemannian origin.

The organization of the article is as follows. In the first section we start with basic facts about the type IIB theory and the Einstein-Cartan space. We establish the connection between the torsion and a combination of bulk fields. In the second section taking account of the covariant embedding formalism constructed in the Einstein-Cartan space (see Appendix A) we derive the effective field equations on the brane. Finally, in the last section we make conclusions and discuss the results.

I. MODEL BUILDING

In fact, the strings can be formulated in curved spaces in presence of massless background fields. In this case, conformal invariance conditions for a closed bosonic string become equivalent to equations of motion for the following background fields: the antisymmetric tensor field $B_{AB}$, the dilaton field $\Phi$, and the metric $G_{AB}$. These equations can be derived from the following action

$$S_b = \frac{1}{2k^2} \int d^{26}X \sqrt{|G|} e^{-2\Phi} \left( \hat{R} - \frac{1}{12} H_{ABC}H^{ABC} + 4\partial A\Phi\partial A\Phi \right); \quad H_{ABC} \equiv 3\partial [A_{BC}], \quad (1.1)$$

where $\hat{R}$ is the curvature scalar computed from the metric $G_{AB}$ and $k^2$ is the gravitational constant. On the other hand, the massless bosonic part of the action of superstring theories can be expressed in the form

$$S_{superstring} = S_{universal} + S_{model} + S_{interactions}; \quad (1.2)$$

where $S_{universal}$ does not depend on which of superstring theories we consider and it has the form (1.1) taken at $D = 10$. The $S_{model}$ part depends on the superstring theory we consider and it contains the Ramond fields: $\{A_C, W_{ABC}\}$ (type IIA string) or $\{\Psi, A_{BC}, W_{ABCD}\}$ (type IIB string). The last term in (1.2) is the Chern-Simons like term.

Motivated by brane worlds ideology and 4-dimensional cosmology the bulk should be the almost-$AdS_5$ space, which represents a space that is $AdS_5$ like on a large scale but allows for generic inhomogeneities on a small scale. Hence a suitable compactification of a specified

1 Where the uppercase Latin indices $A, B = 0, \ldots, D - 1$.

2 $A_{[a_1, \ldots, a_p]} \equiv \frac{1}{p!} \sum (-1)^p A_{a_{\pi 1} \cdots a_{\pi p}}$. 
superstring theory should be considered. Such situation is possible in the framework of the type \( \text{IIB} \) theory compactified on \( S^5 \). Incidentally, the massless bosonic sector of the type \( \text{IIB} \) theory can not exactly be described by a covariant action \([11]\), but the covariant equations of motion exist \([12]\). This situation arises whenever the dimensions of space-time is \( D = 4n + 2 \). In this case, the \( 2n + 1 \) index field strength may be self-dual in locally Minkowski spaces. Hence the set of equations should contain the self-duality condition. Nevertheless these equations can be derived from the following action \([6]\)

\[
S_\sigma = \frac{1}{2k_{10}^2} \int d^{10} \sqrt{|G|} \left[ e^{-2\Phi} \left( (10) \hat{R} - \frac{1}{12} H_{ABC} H^{ABC} + 4 \partial_A \tilde{\Phi} \partial^A \tilde{\Phi} \right) - \frac{1}{2} \partial_A \Psi \partial^A \Psi - \frac{1}{12} \tilde{F}_{ABC} \tilde{F}^{ABC} - \frac{1}{480} \tilde{Y}_{ABCDE} \tilde{Y}^{ABCDE} \right]; \quad (1.3)
\]

\[
H_{ABC} = 3 \partial_{[A} B_{BC]}, \quad F_{ABC} = 3 \partial_{[A} A_{BC]}, \quad \tilde{F}_{ABC} = F_{ABC} - \Psi H_{ABC};
\]

\[
Y_{ABCDE} = 5 \partial_{[A} W_{BCDE]}, \quad \tilde{Y}_{ABCDE} = Y_{ABCDE} - 5 A_{[AB} H_{CDE]} + 5 B_{[AB} F_{CDE]};
\]

except for a term \( \tilde{Y}^2 \) in one of equations and for the self-duality condition \( \tilde{Y}_{ABCDE} = *\tilde{Y}_{ABCDE} \). Taking account of these equations the compactification on \( S^5 \) proposed in \([7]\) has widely used \([30]\). On the other hand, the action \((1.3)\) has derived in the \( \sigma \)-model frame. Such frame should be not confused with the coordinate frame in general relativity. The string theory is insensitive to local redefinitions of background fields \([9]\). Therefore in order to obtain a standard normalized Einstein action (in the 5-dimensional case) one should redefine the metric as follows

\[
G_{AB} = e^{\frac{4\Phi}{D-2}} \tilde{G}_{AB}, \quad \text{where} \quad \tilde{\Phi} \equiv \Phi + \frac{5}{8} \tilde{\Phi}. \quad (1.4)
\]

It can straightforwardly be checked that the action \((1.3)\) becomes

\[
S_E = \frac{1}{2k_{10}^2} \int d^{10} \sqrt{|\tilde{G}|} \left[ e^{-\frac{\Phi}{4} - \frac{\Phi}{2}} \left( \hat{R} - e^{-\Phi} \frac{1}{12} H^2 - \frac{1}{2} \partial_A \Phi \partial^A \Phi - \frac{5}{8} \partial_A \Phi \partial^A \tilde{\Phi} + \frac{25}{16} \partial_A \tilde{\Phi} \partial^A \tilde{\Phi} - e^{\frac{\Phi}{2} + \frac{\Phi}{4}} \partial_A \Psi \partial^A \Psi - e^{\Phi + \frac{\Phi}{4}} \frac{\tilde{F}^2}{12} - \frac{e^{\frac{\Phi}{2} + \frac{\Phi}{4}}}{480} \tilde{Y}^2 \right) \right]. \quad (1.5)
\]

By the frame \((1.4)\) we have increased the number of independent fields by one. This introduces an additional invariance and implies that one of equations derived from this action has kinematically related to the remaining set of equations. Hence this equation can be eliminated. Notice that by the compactification ansatz given below this new field plays
the role of moduli. The equations of motion obtained from those considered in \[7\] by the redefinition \((1.4)\) can readily be derived from the action \((1.5)\) taking account of the term \(\tilde{Y}^2\) and the self-duality condition. These equations are listed in the Appendix C [see (C1)].

Further, taking into account the action \((1.5)\) instead of \((1.3)\) we apply the ansatz \([7]\)

\[
\begin{align*}
    ds^2_{10} &= \tilde{G}_{AB}dX^AdX^B = g_{ab}(x)dx^adx^b + e^{\rho(x)/2}g_{ab}(\theta)d\theta^ad\theta^b, \\
    \Phi(X) &= \phi(x), \quad \tilde{\Phi}(X) = \rho(x), \quad B_{AB}(X) = B_{ab}(x)\delta^{ab}_{AB}, \quad \Psi(X) = \chi(x), \quad A_{AB}(X) = A_{ab}(x)\delta^{ab}_{AB}, \\
    W_{ABCD}(X) &= W_{abcd}(x)\delta^{abcd}_{ABCD} + \rho(\theta)W_{abcd}(\theta)\delta^{abcd}_{ABCD} \quad \text{such that} \quad 5\partial_{[a}\tilde{W}_{bcde]} = \eta_{abcde},
\end{align*}
\]

where \(x^a\) are 5-dimensional coordinates, \(\theta^a\) parametrizes the sphere \(S^5\), \(\delta^{C1\ldots D}_{A1\ldots B}\equiv\delta^C_A\ldots\delta^D_B\), \(\eta_{abcde}\) is the volume form defined on \(S^5\), and \(\rho\) is the Freund-Rubin parameter. We stress that in order to retain the direct relationship between the different dimensional theories one should have the truncation to the low-dimensional subset of fields be a consistent one. In other words, the solutions of 5-dimensional equations of motion should give the solutions of original 10-dimensional theory. It has been widely believed \([13, 14]\) that every sphere compactification is a consistent one. However according to \([8, 15]\) the situation seems to be more complicate. Fortunately, the authors of \([8]\) have shown the strong consistency of \(AdS_5 \times S^5\) compactification. Hence we suppose that the compactification based on \(almost-AdS_5 \times S^5\) can admit a consistent truncation too. The structure of the \(almost-AdS_5\) space is understood as follows. Suppose that all the background fields except \(\tilde{G}_{AB}\) and \(W_{ABCD}\) can be neglected; then equations of motion can be solved. The solution describes an infinite \(D3\)-brane \([10]\), which metric close to the horizon is given by

\[
ds^2_{10} = \frac{r^2}{L^2} \left( -dt^2 + \sum_{i=1}^{3} dx^i dx^i \right) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2, \quad L = const.
\]

This metric describes exactly the \(AdS_5 \times S^5\) space. In general, the existence of the \(almost-AdS_5\) background in the type \(IIB\) theory should be proven. However taking account of above arguments we conjecture it.

After cumbersome calculations (similar to \([7]\)) one yields the 5-dimensional equations of
motion (C3) listed in the Appendix C, which can be derived from the following action

$$S_0 = \frac{1}{2k^2} \int d^5x \sqrt{|g|} \left( \hat{R} - \frac{1}{2} \left( \partial_a \phi \partial^a \phi + \frac{5}{8} \partial_a \rho \partial^a \rho + \frac{5}{4} \partial_a \phi \partial^a \rho + e^{2\phi + \frac{7}{8} \rho} \partial_a \chi \partial^a \chi \right) ight)$$

$$- \frac{1}{12} \left( e^{\phi + \frac{7}{8} \rho} F^2 + e^{-\phi} H^2 \right) - \frac{e^{\frac{7}{8} \rho}}{480} \bar{Y}^2 + e^{-\frac{1}{2} \rho} \alpha \right); \quad (1.7)$$

$$F_{abc} = 3 \partial_{[a} A_{bc]}, \quad H_{abc} = 3 \partial_{[a} B_{bc]}, \quad \bar{F}_{abc} = F_{abc} - \chi H_{abc};$$

$$Y_{abcdef} = 5 \partial_{[a} W_{bcdef]}, \quad \bar{Y}_{abcdef} = Y_{abcdef} - 5 A_{[ab} H_{cdef]} + 5 B_{[ab} F_{cdef]},$$

except for the terms $\bar{Y}^2$ and for consequences of the self-duality condition in the form

$$\varrho = \sqrt{|g|} e^{\frac{7}{8} \rho} \bar{Y}_{01234} = - \frac{1}{\sqrt{|g|}} e^{-\frac{1}{2} \rho} \bar{Y}_{01234} = const. \text{ hence }$$

$$\bar{Y}^{abcde} = - \sqrt{|g|} \eta^{abcde} \bar{Y}_{01234} = - \varrho \eta^{abcde} e^{-\frac{1}{2} \rho} ,$$

$$\bar{Y}_{abcde} = \frac{1}{\sqrt{|g|}} \eta_{abcde} \bar{Y}_{01234} = - \varrho \eta_{abcde} e^{\frac{7}{8} \rho} ,$$

and $$\bar{Y}_{abcde} \bar{Y}^{abcde} = - \bar{Y}^{abcde} \bar{Y}_{abcde} = -5! \varrho^2 . \quad (1.8)$$

Notice that $\alpha = const.$ is the scalar curvature of $S^5$, the 5-dimensional scalar curvature $\hat{R}$ has computed from $g_{ab}$, and $\eta_{abcde}$ is the volume form defined on the almost-AdS$_5$ space.

Finally, $k$ is the 5-dimensional coupling constant related to the string scale as follows

$$\frac{1}{k^2} = \frac{\text{volume of } S^5}{k_{10}},$$

such that $k^2 = 8\pi M_p^{-3}$, where $M_p$ is the fundamental 5-dimensional Planck mass.

Actually one particular combination of Neveu-Schwarz and Ramond fields can safely be interpreted as the bulk torsion in the framework of Einstein-Cartan (EC) space (see [2, 3, 4, 5, 20, 25] for related work). Indeed, recall the following definitions [16, 33]

$$\Gamma^a_{bc} = \Gamma^a_{(bc)} + \Gamma^a_{[bc]} = \Gamma^a_{bc} + (\Upsilon^a_{bc} + \Upsilon^a_{bc} + \Upsilon^a_{cb}) \equiv \Gamma^a_{bc} - Q^a_{cb} \quad \text{and } \quad (1.9)$$

$$\nabla_c g_{ab} \equiv \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0; \quad \nabla_c g_{ab} \equiv \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0,$$

where $\Gamma^a_{abc}$ is the Christoffel connection, $Q_{abc} = -Q_{acb}$ is the contortion, and $\Upsilon_{abc} \equiv \Gamma_{a[bc]}$ is the Cartan’s torsion tensor, a purely affine quantity. If torsion vanishes, we naturally

3 Where the lowercase Latin indices $a, b = 0, 1, 2, 3, 4$; the Greek indices $\mu, \nu = 0, 1, 2, 3$; and $x_4 \equiv y$. 
recover the Riemannian space. The contortion can covariantly be split into irreducible parts

\[ Q_{abc} = \frac{1}{(D-1)} (Q_b g_{ac} - Q_c g_{ab}) + L_{abc} + Q_{[abc]}, \tag{1.10} \]

where \( Q_a \equiv Q^b_{\ ab} \) is the trace, \( L_{abc} \) is a tensor constrained by \( L^a_{\ ba} = 0, L_{abc} \eta^{abc} i^{i D-3} = 0, \) and \( \eta_{i_1 \ldots i_D} \) is the volume form.

Now instead of the action (1.7) defined in the Riemannian bulk consider the following action defined in the EC bulk

\[ S = \frac{1}{2 k^2} \int d^5 x \sqrt{|g|} \left[ R - \frac{1}{2} \left( \partial_a \phi \partial^a \phi + \frac{5}{8} \partial_a \rho \partial^a \rho + \frac{5}{4} \partial_a \phi \partial^a \rho + e^{2 \phi + \frac{5}{2} \rho} \partial_a \chi \partial^a \chi \right) - \frac{1}{12} e^{-\phi} H^2 - \frac{e^{-\frac{5}{2} \rho}}{480} Y^2 + e^{-\frac{1}{2} \rho} \alpha \right], \tag{1.11} \]

where the scalar curvature is constructed from \( \{ g_{ab}, Q_{abc} \} \). Using (1.9) and (1.10) one can readily compare equations of motion derived from (1.7) and (1.11). The conclusion is that these sets of equations coincide whenever

\[ Q_{[abc]} = \frac{1}{2 \sqrt{3}} e^{\frac{1}{2} \rho} (\phi + \frac{5}{2} \rho)^{\frac{1}{2}} \tilde{F}_{abc} \equiv \Theta_{abc}, \quad Q_a = 0, \quad L_{abc} = 0. \tag{1.12} \]

In other words, the theory given by (1.7) in the Riemannian space is dynamically equivalent to the theory given by (1.11) in the EC space. A combination of Neveu-Schwarz and Ramond fields is identified with the \( Q_{[abc]} \) part of the bulk contortion. Throughout the paper we shall deal with the action (1.11), which underlie the idea of EC space. Notice that in the EC space one should distinguish two classes of extremal curves. The autoparallel curves and the geodesic curves. However these two classes coincide whenever the contortion is totally antisymmetric [16]. Incidentally, this is exactly the case considered in this paper.

In our brane world scenario the 4-dimensional world \((q_{\mu\nu}, (4)\Theta_{\alpha\beta\gamma}, \psi_A)^5\) is understood as a brane in the 5-dimensional EC bulk \((g_{ab}, \Theta_{abc}, W_{abcd})\) characterized by the contortion (1.12). The brane matter sector \( L(\psi_A, (4)\nabla_\mu \psi_A, q_{\mu\nu}) \) consists of a continuous spinning medium \( \psi_A \) minimally coupled to the induced metric \( q_{\mu\nu} \) and the induced contortion \((4)\Theta_{\lambda\mu\nu}\). The spinning medium (i.e., Maxwell and Yang-Mills fields, the Proca field, the Dirac field, and the Weyssenhoff-Raabe spin fluid) is a matter, which possess the intrinsic spin, that is the

\[ R^a_{\ bcd} = \partial_a \Gamma^a_{\ bd} - \partial_b \Gamma^a_{\ ad} + \Gamma^a_{\ fd} \Gamma^f_{\ bd} - \Gamma^a_{\ fd} \Gamma^f_{\ ad}, \quad R_{ac} = R^b_{\ acb}, \quad R = g^{ab} R_{ab}. \]

\[ 5 \quad \text{Further, the uppercase Latin indices denote the matter sector of the brane.} \]
irreducible spin of elementary particles (see [17, 19, 21], where the Lagrangian description in the EC spaces is given). Taking account of minimal coupling the brane Lagrangian can be decomposed as follows

\[ L(\psi_A, (4)\nabla_\mu \psi_A, q_{\mu\nu}) = \hat{L}(\psi_A, (4)\nabla_\mu \psi_A, q_{\mu\nu}) + \tilde{L}(q_{\mu\nu}, (4)\Theta_{\alpha\beta\gamma}, \psi_A), \]  

(1.13)

where \( \hat{L} \equiv L|_{(\alpha)\Theta_{\alpha\beta\gamma} \to 0} \) and \( \tilde{L} \) describes the interaction with the bulk fields \( (4)\Theta_{\alpha\beta\gamma} \).

The equations of motion that describe the compactified type IIB theory in presence of brane are listed in Appendix B [see (B5)]. Nevertheless we observe that these equations can be derived from the following action

\[ S = \frac{1}{2k^2} \int d^5x \sqrt{|g|} \left[ R - \frac{1}{2} \left( \partial_\alpha \phi \partial^\alpha \phi + \frac{5}{8} \partial_\alpha \rho \partial^\alpha \rho + \frac{5}{4} \partial_\alpha \phi \partial^\alpha \rho + e^{2\phi + \frac{5}{2}\rho} \partial_a \chi \theta^a \chi \right) \right. \]

\[ - \frac{1}{12} e^{-\phi} H_{abc} H^{abc} - \frac{e^\phi \rho}{480} \tilde{Y}_{abcde} \tilde{Y}^{abcde} + e^{-\frac{1}{2}\rho} \alpha \left. \right], \]

\[ S_c = \int d^5x \sqrt{|g|} (-\Lambda), \quad S_{cbr} = \int d^4x \sqrt{|q|} (-\lambda), \quad S_{br} = \int d^4x \sqrt{|q|} L(q_{\mu\nu}, \psi_A, (4)\nabla_\mu \psi_A), \]

except for the terms \( \tilde{Y}^2 \) and for consequences of the self-duality condition [see (B6)]. Also, \( \Lambda (\lt 0) \) is the bulk cosmological constant and \( \lambda (\gt 0) \) is the intrinsic brane tension.

The appearance of the negative bulk cosmological constant in the action (1.14) does not follow directly from the type IIB theory. Recall that the structure of almost-AdS_5 space supposes the existence of the negative cosmological constant, which shows up on a large scale. Hence the introduction of one additional negative constant does not change the picture. The \( \mathbb{Z}_2 \)-symmetry of the fields \( (g_{ab}, \rho, \phi, \chi, A_{ab}, B_{ab}) \) is also an interesting topic. In our case, the compactification procedure given in the Appendix does not imposes some special symmetries on the fields \( (g_{ab}, \rho, \phi, \chi, A_{ab}, B_{ab}) \) but supplies every point of the bulk with a 5-sphere scaled by \( \rho(x) \) [see (1.6a)]. To make the \( \mathbb{Z}_2 \)-symmetry be compatible with the sphere compactification one must combine the flip in \( y \) with an orientation-reversing of spheres. In other words, we must operate with an extended notion of the \( \mathbb{Z}_2 \)-transformation that means: \( y \rightarrow -y \) as well \( \rho \rightarrow -\rho \). At the same time the bulk metric \( g_{ab} \) can always be taken \( \mathbb{Z}_2 \)-symmetric (in the extended sense) by considering the patches of almost-AdS_5 space glued together along the brane world volume in a \( \mathbb{Z}_2 \)-symmetric manner. The situation with \( (\rho, \phi, \chi, A_{ab}, B_{ab}) \) is less optimistic. Nevertheless it is believed [13] that in the
framework of the type IIB theory exists a mode-locking mechanism that projects these fields into a \( Z_2 \)-invariant subspace. Therefore throughout the paper we shall consider the fields \((g_{ab}, \rho, \phi, \chi, A_{ab}, B_{ab})\) being symmetric in the extended sense of the \( Z_2 \)-transformation.

II. FIELD EQUATIONS ON THE BRANE

Further, we impose a gauge: \( A_{4\gamma} = 0, B_{4\gamma} = 0 \) and assume that the hypersurface \( y = 0 \) coincides with the brane world. Suppose that the vector normal to the brane is given by \( n_c dx^c = dy \) and \( g_{ab} n^a n^b = 1 \); then

\[
 ds_5^2 = q_{\mu \nu} dx^\mu dx^\nu + dy^2. \tag{2.1}
\]

Consider the Gauss like equation \([A16]\) in a space with nontrivial torsion

\[
 (4) R_{\alpha \beta} = R_{cd} e^c_{\alpha} e^d_{\beta} + K_{4\alpha \beta} K - K_{4\alpha \gamma} K_{4\beta} - R^f_{\ cde} e^c_{\alpha} e^d_{\beta} n^e f, \tag{2.2}
\]

where \( K_{4\alpha \beta} = -e^c_{\alpha} e^d_{\beta} \nabla_d n_c \equiv K_{4\alpha \beta} + \Theta_{4\alpha \beta} \) and \( K \equiv K_{4\lambda} \).

The decomposition of curvatures into Riemannian and non-Riemannian parts is as follows

\[
 R_{ab} \equiv \hat{R}_{ab} + \tilde{R}_{ab} = \hat{R}_{ab} + \left( \nabla_c \Theta^c_{ab} - \Theta_{acd} \Theta^d_{b} \right), \quad R \equiv \hat{R} + \tilde{R} = \hat{R} - \Theta^2. \tag{2.3}
\]

The terms \( R_{cde} n_a e^c_{\alpha} e^d_{\beta} n_b, R^c_{\ def} n_c n^d n^e n^f, \) and \( R^f_{\ abc} n_a n^b e^c_{\mu} \) are given by equations

\[
 e^a_{\gamma} \nabla_{\mu} \nabla_{\nu} n_a = e^a_{\gamma} e^c_{\mu} n^d \nabla_c \nabla_d n_a + \frac{1}{2} e^a_{\gamma} \left( \nabla_c n^c - \nabla_4 e^c_{\mu} \right) \nabla_c n_a = \frac{1}{2} R^f_{\ cde} n_a n^b e^c_{\mu},
\]

\[
 = \frac{1}{2} e^a_{\gamma} n^c \nabla_c \left( e^\lambda_{\mu} K_{4\lambda \mu} \right) = \frac{1}{2} \left( \partial_4 K_{4\gamma \mu} + K_{4\lambda \mu} K_{4\lambda \gamma} \right); \tag{2.4a}
\]

\[
 n^a \nabla_{\mu} \nabla_{\nu} n_a = n^a e^c_{\mu} n^d \nabla_c \nabla_d n_a + \frac{1}{2} n^a \left( \nabla_c n^c - \nabla_4 e^c_{\mu} \right) \nabla_c n_a = \frac{1}{2} R^f_{\ cde} n_a n^b n^c_{\mu},
\]

\[
 = 0; \tag{2.4b}
\]

\[
 n_{a} \nabla_{[4} \nabla_{d]} n^a = n_{a} n^c \nabla_{c} \left( n^d \nabla_{d} n^a \right) = \frac{1}{2} R^a_{\ cde} n_{a} n^f n^c n^d,
\]

\[
 = 0. \tag{2.4c}
\]

Following methods of \([23, 24]\) we evaluate effective field equations not exactly on the brane but at the limiting values of \( y \). In other words, we consider two sets of effective equations denoted by the \( \pm \) sign and have taken at \( y \to \pm 0 \). In this case, the delta function vanishes;
then using (B5a), (2.2), (2.3), and (2.4c) it can straightforwardly be checked that

\[
^{(4)}G_{(\mu \nu)}^\pm = \left|^{(4)}R_{(\mu \nu)} - \frac{1}{2} q_{\mu \nu}^{(4)} R - 2 \Theta_{\mu \alpha \beta} \Theta_{\nu}^{\alpha \beta} \right|^\pm
\]

\[
= \frac{2}{3} k^2 \left[ T_{\mu \nu} + \left( T_{44} - \frac{1}{4} T_e^c \right) q_{\mu \nu} \right] - C_{4 \mu \nu}
\] 

\[
+ K_{4(\mu \nu)} K - K_{4(\mu \gamma)} K_{4\nu}^{\gamma} - \frac{1}{2} q_{\mu \nu} \left( K^2 - K_{4 \alpha \beta} K^{4 \alpha \beta} \right)
\] 

\[- \frac{1}{4} \Theta_{\alpha \beta \gamma} \Theta_{\nu}^{\alpha \beta \gamma} q_{\mu \nu} - \Theta_{\mu \alpha \beta} \Theta_{\nu}^{\alpha \beta} + \frac{1}{4} \Theta_{4 \alpha \beta} \Theta^{4 \alpha \beta} q_{\mu \nu} + 3 \Theta_{4 \alpha \beta} \Theta^{4 \alpha \beta} \right|^\pm,
\]

where \( C_{abcd} \equiv \hat{R}_{abcd} + \frac{2}{3} \left( g_{a[d} R_{c]b} + g_{b[c} R_{d]a} \right) + \frac{1}{6} \hat{R} g_{a[c} g_{d]b} \)
is by definition the 5-dimensional Weyl tensor \([31, 33]\). The \( Z_2 \)-symmetry implies that

\[
K_{4(\mu \nu)} = -K_{4(\mu \nu)}^\perp, \quad H_{4 \mu \nu} = -H_{4 \mu \nu}^\perp, \quad F_{4 \mu \nu} = -F_{4 \mu \nu}^\perp,
\]

\[
\partial_4 \rho^\perp = -\partial_4 \rho^\perp, \quad \partial_4 \phi^\perp = -\partial_4 \phi^\perp, \quad \partial_4 \chi^\perp = -\partial_4 \chi^\perp.
\]

Hence we can drop the \( \pm \) sign and evaluate the equation (2.5) on the one side only.

Taking into account (1.8), (1.6) the effective energy-momentum tensor can be expressed as

\[
T_{\mu \nu} + \left( T_{44} - \frac{1}{4} T_e^c \right) q_{\mu \nu} = \frac{3}{2} \left( -\frac{1}{2} \Lambda q_{\mu \nu} + k^{-2} T_{\mu \nu}^{\Sigma} + k^2 T_{\mu \nu}^{\perp} \right), \text{ where} \quad (2.7)
\]

\[
T_{\mu \nu}^{\Sigma} = \frac{1}{3} \partial_\mu \phi \partial_\nu \phi + \frac{5}{12} \partial_\mu \phi \partial_\nu \rho + \frac{5}{24} \partial_\mu \rho \partial_\nu \rho + \frac{e^{2 \phi + \frac{3}{2} \rho}}{3} \partial_\mu \chi \partial_\nu \chi + \left( \frac{e^{-\phi}}{6} H_{\mu \alpha \beta} H_{\nu}^{\alpha \beta} ight)
\]

\[-q_{\mu \nu}^{(4)} \left( 5 \partial_\alpha \phi \partial^\alpha \phi + \frac{25}{4} \partial_\alpha \phi \partial^\alpha \rho + \frac{25}{8} \partial_\alpha \rho \partial^\alpha \rho + 5 e^{2 \phi + \frac{3}{2} \rho} \partial_\alpha \chi \partial^\alpha \chi + \frac{3 e^{-\phi}}{2} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} \right);
\]

\[
T_{\mu \nu}^{\perp} = k^{-4} \left[ \frac{e^{-\phi}}{3} H_{\mu \alpha \beta} H_{\nu}^{\alpha \beta} + \frac{q_{\mu \nu}^{(4)}}{24} \left( 3 \partial_\alpha \phi \partial_\alpha \phi + \frac{15}{4} \partial_\alpha \phi \partial_\alpha \rho + \frac{15}{8} \partial_\alpha \rho \partial_\alpha \rho + 3 e^{2 \phi + \frac{3}{2} \rho} \partial_\alpha \chi \partial_\alpha \chi \right.ight.
\]

\[-e^{\phi} \left. H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} \right) - \frac{1}{8} q_{\mu \alpha} e^{\phi} \partial_\alpha \chi + \frac{1}{4} q_{\mu \alpha} e^{-\phi} \chi \right].
\]

Clearly, that the effective equation (2.5) is not closed in four dimensions due to the presence of extrinsic quantities \( \Theta_{\alpha \beta \gamma}, C_{4 \mu \nu}, \) and \( T_{\mu \nu}^{\perp} \). However both the equations (B5a) - (B5f) and the assumed continuity of fields \( (g_{ab}, \rho, \phi, \chi, A_{ab}, B_{ab}) \) lead to Israel’s like junction conditions \([32]\). These conditions give a possibility to decrease the number of extrinsic quantities.

The mentioned junction conditions can be expressed as follows

\[
\{g_{ab} \} = 0; \quad \{\rho \} = 0; \quad \{\phi \} = 0; \quad \{\chi \} = 0; \quad \{B_{ab} \} = 0; \quad \{A_{ab} \} = 0,
\]
\{K_{4(\mu\nu)}\} = k^2 \left( S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S^A_{\lambda} \right), \quad (2.8a)

\{\partial_4 \phi\} + \{\partial_4 \rho\} = \frac{k^2}{2\sqrt{3}} e^{\frac{1}{2} (\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} F_{\alpha\mu\nu}, \quad (2.8b)

\{\partial_4 \phi\} + \frac{5}{8} \{\partial_4 \rho\} = \frac{k^2}{4\sqrt{3}} e^{\frac{1}{2} (\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} \tilde{F}_{\alpha\mu\nu}, \quad (2.8c)

\{\partial_4 \chi\} = \frac{k^2}{2\sqrt{3}} e^{\frac{1}{4} (3\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} H_{\alpha\mu\nu}, \quad (2.8d)

\{F_{4\mu\nu}\} - \chi \{H_{4\mu\nu}\} = 2\sqrt{3} k^2 e^{-\frac{1}{4} (\phi + \frac{5}{4} \rho)} \left[ \theta^{\mu\nu} \gamma \left( \phi + \frac{5}{4} \rho \right) + 2 \sigma_{\mu\nu} \right], \quad (2.8e)

\chi \{F_{4\mu\nu}\} - \left( e^{-2\phi - \frac{5}{4} \rho} + \chi^2 \right) \{H_{4\mu\nu}\} = \sqrt{3} k^2 e^{-\frac{1}{4} (\phi + \frac{5}{4} \rho)} \left[ \chi \sigma_{\mu\nu} + \theta^{\mu\nu} \gamma \left( \partial_\gamma \chi + \frac{1}{2} \chi \partial_\gamma \phi + \frac{5}{8} \chi \partial_\gamma \rho \right) \right]. \quad (2.8f)

where \{\mathcal{V}\} \equiv \lim_{y \to +0} \mathcal{V} - \lim_{y \to -0} \mathcal{V} \equiv \mathcal{V}^+ - \mathcal{V}^- \text{ denotes the discontinuity between two sides of the brane. The } Z_2 \text{-symmetry } (2.6) \text{ implies that we can once again drop the } ^+ \text{ sign and evaluate quantities on the brane by taking only the limit } y \to +0. \text{ The junction conditions } (2.8) \text{ become}

K_{4(\mu\nu)} = \frac{k^2}{2} \left( S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S^A_{\lambda} \right), \quad (2.9a)

\partial_4 \rho = \frac{k^2}{3\sqrt{3}} e^{\frac{1}{4} (\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} F_{\alpha\mu\nu}, \quad (2.9b)

\partial_4 \phi = \frac{k^2}{12\sqrt{3}} e^{\frac{1}{2} (\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} F_{\alpha\mu\nu}, \quad (2.9c)

\partial_4 \chi = \frac{k^2}{4\sqrt{3}} e^{-\frac{1}{4} (3\phi + \frac{5}{4} \rho)} \theta^{\mu\nu} H_{\alpha\mu\nu}, \quad (2.9d)

H_{4\mu\nu} = \frac{\sqrt{3}}{2} k^2 e^{\frac{1}{4} (3\phi + \frac{5}{4} \rho)} \left( 3 \chi \sigma_{\mu\nu} - \theta_{\mu\nu} \gamma \Pi_\gamma \right), \quad (2.9e)

F_{4\mu\nu} = \frac{\sqrt{3}}{2} k^2 e^{\frac{1}{4} (3\phi + \frac{5}{4} \rho)} \left[ \theta^{\mu\nu} \gamma \left( 2 e^{-(2\phi + \frac{5}{4} \rho)} \tilde{\Pi}_\gamma - \chi \Pi_\gamma \right) + \sigma_{\mu\nu} \left( 4 e^{-(2\phi + \frac{5}{4} \rho)} + 3 \chi^2 \right) \right], \quad (2.9f)
where \( \Pi_{\gamma} \equiv \left( \partial_{\gamma} \chi - \frac{3}{2} \chi \partial_{\gamma} \phi - \frac{15}{8} \chi \partial_{\gamma} \rho \right) \); \( \hat{\Pi}_{\gamma} \equiv \partial_{\gamma} \left( \phi + \frac{5}{4} \rho \right) \).

In other words, these conditions mean that quantities \((K_{4(\mu\nu)}, H_{4\mu\nu}, F_{4\mu\nu}, \partial_{4\rho}, \partial_{4\phi}, \partial_{4\chi})\) depend on the brane matter content. In addition, these conditions make possible to express \(\Theta_{4\mu\nu}\) and \(T_{\perp\mu\nu}\) as follows

\[
K_{4[\mu\nu]} = \Theta_{4\mu\nu} = \frac{1}{2} k^2 \left( \theta^{\mu\nu}_{\gamma} \hat{\Pi}_{\gamma} + 2 \sigma_{\mu\nu} \right),
\]

(2.10)

\[
T_{\perp\mu\nu} = e^{\phi + \frac{5}{4} \rho} \left[ \frac{e^{\phi}}{4} \left( \theta_{\alpha\lambda}^{\mu} \theta_{\nu}^{\lambda} \Pi_{\alpha} \Pi_{\beta} - 6 \chi \theta_{(\mu}^{\lambda} \sigma_{\nu)} \lambda \Pi_{\alpha} + 9 \chi^2 \sigma_{\mu\lambda} \sigma_{\nu}^{\lambda} \right) - \frac{e^{-2\phi - \frac{5}{4} \rho}}{64} q_{\mu\nu} \left( \theta^{\alpha\beta} H_{\gamma\alpha\beta} \right)^2 - \frac{1}{8} q_{\mu\nu} e^{\frac{5}{4} \rho} \gamma^{\alpha\beta} + \frac{1}{4} q_{\mu\nu} e^{-\frac{5}{4} \rho} \tilde{\alpha} \right],
\]

(2.11)

where \(\tilde{\alpha} \equiv \alpha k^{-1}\) and \(\tilde{\rho} \equiv \rho k^{-2}\).

Further, we conclude that the equation (2.5) contains extrinsic quantities only in \(C_{4\mu4\nu}\) term. It is a projection of the bulk Weyl tensor, which vanishes whenever the bulk is exactly \(AdS_5\).

At this stage we apply the relation (1.12) in order to consider one dynamically equivalent to (2.5) equation on the Riemannian space instead of EC space.

\[
(4) \ G^{\perp}_{\mu\nu} \equiv (4) G_{\mu\nu} - \frac{1}{2} q_{\mu\nu} \Theta_{\alpha\beta\gamma} \Theta^{\alpha\beta\gamma} + 3 \Theta_{\mu\alpha\beta} \Theta_{\nu}^{\alpha\beta} = \frac{2}{3} k^2 \left[ T_{\mu\nu} + \left( T_{44} - \frac{1}{4} T_{c}^{c} \right) q_{\mu\nu} \right] - C_{4\mu4\nu}
\]

(2.12)

\[
+ K_{4\mu\nu} \tilde{K} - K_{4\mu\nu} \tilde{K}_{4\gamma}^{\nu} - \frac{1}{2} q_{\mu\nu} \left( K^{2} - K_{4\alpha\beta} \tilde{K}_{4\alpha\beta} \right) - \frac{3}{4} \Theta_{\alpha\beta\gamma} q_{\mu\nu} + 2 \Theta_{\mu\alpha\beta} \Theta_{\nu}^{\alpha\beta} - \frac{1}{4} \Theta_{\alpha\beta\gamma} \Theta_{\nu}^{\alpha\beta} q_{\mu\nu} + 4 \Theta_{\mu\gamma} \Theta_{\nu}^{\gamma}.
\]

This equation can be compared to the restricted result \(23, 24\), where only gravity exists in the bulk. Recall that the brane energy-momentum tensor \(\tau_{\mu\nu}\) is decomposed into a sum of \(\tau^{*}_{\mu\nu}\) and \(\tau_{\mu\nu}\) parts given by (1.13), where \(\tau_{\mu\nu}\) describes interactions with bulk fields localized on the brane. Taking account of (2.9), (1.12), (2.10), and (2.7) the equation (2.12) becomes

\[
(4) \ G^{*}_{\mu\nu} = -\tilde{\Lambda} q_{\mu\nu} + 8 \pi G_N \tau^{*}_{\mu\nu} + \tilde{T}_{\mu\nu}^{*} + k^4 \left( \tilde{\tau}_{\mu\nu} + \tilde{T}_{\mu\nu}^{*} \right) - E_{\mu\nu}^{g},
\]

(2.13)
The main difference of (2.13) from Einstein gravity resides in the presence of terms \( \pi_{\mu\nu} \), \( \hat{T}^\Sigma_{\mu\nu} \), \( \tilde{T}^\perp_{\mu\nu} \), and \( E_{\mu\nu}^g \) which can not be obtained by a Lagrangian description. It is seen that in the limit whenever \( \lambda \) is of high energy scale the terms \( \pi_{\mu\nu} \) and \( \tilde{T}^\perp_{\mu\nu} \) can safely be neglected.

Now let us examine the behaviour of the brane energy-momentum tensor \( S_{\mu\nu} \) [see (B5a)]. Consider the Codacci like equation (A17) in a space with nontrivial torsion

\[
R_{ab} n^a e^b_\mu = (4\sqrt{\lambda} K - (4\sqrt{\lambda} K^4_\mu - 2\Theta_{\lambda\mu\gamma} K^4_{\lambda\mu} + R_{bcd} n_a n^b n^c) \cd (2.14)
\]

Using (2.4b), (2.3) decompose (2.14) and \( G_{(ab)} \) as follows

\[
R_{ab} n^a e^b_\mu = (4\sqrt{\lambda} K - (4\sqrt{\lambda} K^4_\mu - (4\sqrt{\lambda} \Theta_{4\mu} - e^b_\mu \Theta_{4\mu} - \Theta_{\alpha\beta} \Theta_{4\alpha\beta} = (4\sqrt{\lambda} K - (4\sqrt{\lambda} K^4_\mu, \cd G_{(ab)} n^a e^b_\mu = (G_{ab} - 3\Theta_{a\alpha\beta} \Theta_{b\alpha\beta} \cd n^a e^b_\mu = R_{ab} n^a e^b_\mu - 3\Theta_{a\alpha\beta} \Theta_{b\alpha\beta} \cd n^a e^b_\mu.
\]

These equations together with (1.12), (B5a), and (2.9) lead to the relation

\[
(4\sqrt{\lambda} S^0_\mu = -2k^{-2} \left( P_{ab} + \frac{1}{4} e^{\phi \sqrt{\lambda}} \rho \tilde{F}_{a\beta\gamma} \Theta_{\mu} \left( \frac{5}{4} \rho - \phi \right) \right) \cd n^a e^b_\mu = e^{\frac{1}{4} \rho \sqrt{\lambda}} \left[ \frac{1}{12 \sqrt{3}} \rho \tilde{F}_{\alpha\beta\gamma} \Theta_{\mu} \left( \frac{5}{4} \rho - \phi \right) \right] \cd - \frac{\sqrt{3}}{4} (3\sigma_{\alpha\beta} - \Theta_{\alpha\beta} \Pi_{\gamma}) H^\alpha_{\beta\gamma} = \frac{\sqrt{3}}{2} \left( \Theta_{\alpha\beta} \Pi_{\gamma} + 2\sigma_{\alpha\beta} \right) \cd \Theta_{\mu} \cd \cd J_{\mu} \cd (2.15)
\]
This relation indicates that the brane energy-momentum tensor $S_{\mu\nu}$ is non-conserved. The non-conservation reflects an exchange of energy-momentum between the brane and bulk fields localized on the brane (see [27] for related work). If one imposes $J_\mu = 0$; then there is no such exchange and the brane vacuum state remains stable. However the effective brane energy-momentum tensor given by the right part of (2.13) is conserved and the effective vacuum remains stable. Let us remark that throughout the literature, the true brane energy-momentum tensor is understood ambiguously.

By the same arguments, the equations (B5b) - (B5f) must also be taken at the limiting values of $y$. In this case, the situation is simpler. Taking into account (A14) one see that

\[
(5)^* \nabla^c (5) \nabla^c \phi = (4) \nabla^c (4) \nabla^\lambda \phi - K \partial_4 \phi + \partial_4 \partial_4 \phi,
\]

\[
(5)^* \nabla^c F^{c\mu\nu} = (4)^* \nabla^\lambda F^{\lambda\mu\nu} - K F^{4\mu\nu} + \partial_4 F^{4\mu\nu};
\]

then using (2.9), (B5g) and (B6) equations (B5b) - (B5f) become

\[
(4)^* \nabla^\lambda (\rho + \phi) - e^\frac{\phi}{4} \rho \left( e^{2\phi} \partial_\gamma \partial^\gamma \chi + \frac{e^\phi}{6} F_{\alpha\beta\gamma} \tilde{F}^{\alpha\beta\gamma} \right) + e^\frac{\phi}{4} g^2 - \frac{4}{5} e^{-\frac{\phi}{2}} \rho \alpha + (E_\phi + E_\rho) = \frac{k^4}{16\sqrt{3}} \left( \theta^{\alpha\mu\nu} H_{\alpha\mu\nu} \right)^2 + \frac{3}{2} \left( \theta_{\alpha\beta} \theta^{\alpha\beta\lambda} \tilde{\Pi}_\lambda \tilde{\Pi}_\lambda + 4 \theta^{\alpha\beta\gamma} \tilde{\Pi}_\gamma \sigma_{\alpha\beta} + 4 \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) + \frac{e^\frac{\phi}{4} (\phi + \frac{\rho}{2})}{24\sqrt{3}} S^\lambda_{\alpha\beta\gamma} \theta^{\alpha\mu\nu} \tilde{F}_{\alpha\mu\nu},
\]

(2.16a)

\[
(4)^* \nabla^\gamma (4) \nabla^\gamma \left( \phi + \frac{5}{8} \rho \right) - e^\frac{\phi}{4} \rho \left( e^{2\phi} \partial_\gamma \partial^\gamma \chi + \frac{e^\phi}{12} F_{\alpha\beta\gamma} \tilde{F}^{\alpha\beta\gamma} \right) + e^{-\phi} \frac{12}{12} H_{\alpha\beta\gamma} \theta^{\alpha\beta\gamma} + \left( E_\phi + \frac{5}{8} E_\rho \right) = k^4 \left[ \frac{3}{4} \left( 6 \theta^{\alpha\beta\gamma} \tilde{\Pi}_\gamma \sigma_{\alpha\beta} - \theta_{\alpha\beta} \theta^{\alpha\beta\lambda} \tilde{\Pi}_\gamma \tilde{\Pi}_\lambda - 9 \chi^2 \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) + \frac{e^\frac{\phi}{4} (\phi + \frac{\rho}{2})}{48\sqrt{3}} S^\lambda_{\alpha\beta\gamma} \theta^{\alpha\mu\nu} \tilde{F}_{\alpha\mu\nu} + \frac{3}{4} \left( \theta_{\alpha\beta} \theta^{\alpha\beta\lambda} \tilde{\Pi}_\lambda \tilde{\Pi}_\lambda + 4 \theta^{\alpha\beta\gamma} \tilde{\Pi}_\gamma \sigma_{\alpha\beta} + 4 \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) + \frac{e^{-\phi}}{16\sqrt{3}} \left( \theta^{\alpha\mu\nu} H_{\alpha\mu\nu} \right)^2 \right],
\]

(2.16b)

\[
(4)^* \nabla^\gamma (4) \nabla^\gamma \left( \phi + \frac{5}{8} \rho \right) - e^\frac{\phi}{4} \rho \left( e^{2\phi} \partial_\gamma \partial^\gamma \chi + \frac{e^\phi}{6} F_{\alpha\beta\gamma} \tilde{F}^{\alpha\beta\gamma} \right) + e^{-\phi} \frac{1}{12} H_{\alpha\beta\gamma} \theta^{\alpha\beta\gamma} + \left( E_\phi + \frac{5}{8} E_\rho \right) = k^4 \left[ \frac{3}{4} \left( 6 \theta^{\alpha\beta\gamma} \tilde{\Pi}_\gamma \sigma_{\alpha\beta} - \theta_{\alpha\beta} \theta^{\alpha\beta\lambda} \tilde{\Pi}_\gamma \tilde{\Pi}_\lambda - 9 \chi^2 \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) + \frac{e^\frac{\phi}{4} (\phi + \frac{\rho}{2})}{48\sqrt{3}} S^\lambda_{\alpha\beta\gamma} \theta^{\alpha\mu\nu} \tilde{F}_{\alpha\mu\nu} + \frac{3}{4} \left( \theta_{\alpha\beta} \theta^{\alpha\beta\lambda} \tilde{\Pi}_\lambda \tilde{\Pi}_\lambda + 4 \theta^{\alpha\beta\gamma} \tilde{\Pi}_\gamma \sigma_{\alpha\beta} + 4 \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) + \frac{e^{-\phi}}{16\sqrt{3}} \left( \theta^{\alpha\mu\nu} H_{\alpha\mu\nu} \right)^2 \right],
\]

(2.16c)
\begin{align}
(4) \nabla^* \tilde{F}^{\gamma \mu \nu} + \partial_\gamma \left( \phi + \frac{5}{4} \rho \right) \tilde{F}^{\gamma \mu \nu} + E^{\mu \nu}_F = k^4 \left[ \frac{\sqrt{3}}{4} e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} \eta^{\mu \nu \alpha \beta} \left( 3 \chi \sigma_{\alpha \beta} - \theta_{\alpha \beta} \Pi_\gamma \right) \tilde{\varrho} \right. \\
- \frac{1}{3} \left( \theta^{\mu \nu \gamma} \tilde{\Pi}_\gamma + 2 \sigma^{\mu \nu} \right) \left( \frac{\sqrt{3}}{2} e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} S^\lambda_{\alpha \beta \gamma} + \theta^{\beta \alpha \gamma} \tilde{F}_{\alpha \beta \gamma} \right) \right], \tag{2.16d}
\end{align}

\begin{align}
(4) \nabla^* \left( e^{-\phi} H^{\gamma \mu \nu} - \chi e^{\phi + \frac{5}{4} \rho} \tilde{F}^{\gamma \mu \nu} \right) + e^{-\phi} E_H^{\mu \nu} = \chi e^{\phi + \frac{5}{4} \rho} E^{\mu \nu}_F \\
= k^4 e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} \left[ \frac{1}{12} \left( \sqrt{3} S^\lambda_{\alpha \beta \gamma} - \frac{1}{2} e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} \theta^{\alpha \beta \gamma} \tilde{F}_{\alpha \beta \gamma} \right) \left( \theta^{\mu \nu \gamma} \Pi_\gamma - 3 \chi \sigma^{\mu \nu} \right) \\
+ \left( \frac{1}{4} e^{-\phi} (3 \phi + \frac{5}{4} \rho) \theta^{\alpha \beta \gamma} H_{\alpha \beta \gamma} - \frac{\chi}{3} e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} \theta^{\alpha \beta \gamma} \tilde{F}_{\alpha \beta \gamma} + \frac{\chi}{2 \sqrt{3}} S^\lambda_{\alpha \beta \gamma} \right) \left( \theta^{\mu \nu \gamma} \Pi_\gamma + 2 \sigma^{\mu \nu} \right) \\
+ \frac{\sqrt{3}}{4} \tilde{\varrho} \eta^{\mu \nu \alpha \beta} e^{\phi} \left[ \theta_{\alpha \beta} \gamma \left( 2 e^{-(2 \phi + \frac{5}{4} \rho) \tilde{\Pi}_\gamma - \chi \Pi_\gamma} \right) + \sigma_{\alpha \beta} \left( 4 e^{-(2 \phi + \frac{5}{4} \rho) + 3 \chi^2} \right) \right] \right], \tag{2.16e}
\end{align}

where \( E_\phi \equiv \partial_4 \partial_4 \phi, \ E_\rho \equiv \partial_4 \partial_4 \rho, \ E_\chi \equiv \partial_4 \partial_4 \chi, \ E^{H}_H^{\mu \nu} \equiv \partial_4 H_{4 \mu \nu}, \ E^{F}_F^{\mu \nu} \equiv \partial_4 \tilde{F}_{4 \mu \nu}, \)

and \( \eta_{\alpha \beta ; \gamma} \) is the brane volume form [see (B6)]. The equations (2.13), (2.16a) - (2.16e), and (B5h) are the effective equations on the brane.

For the same reason, we can apply the relation (1.12) in order to consider these equations on the Riemannian space instead of EC space. This gives a set of dynamically equivalent equations of motion. At the same time notice that (2.13), (2.16a) - (2.16e) have completely described in Riemannian terms. The equation (B5h) can readily be expressed as follows

\begin{align}
\frac{\partial L}{\partial \psi_A} - (4) \nabla^*_\lambda \frac{\partial L}{\partial (4 \nabla^*_\lambda \psi_A)} = \frac{\sqrt{3}}{2} e^{\frac{1}{2} \left( \phi + \frac{5}{4} \rho \right)} \left( \partial_{[\alpha} A_{\beta \lambda]} - \chi \partial_{[\alpha} \varPi_{\beta \lambda]} \right) \Omega_{B]}^{A \alpha \beta} \frac{\partial L}{\partial (4 \nabla^*_\lambda \psi_B)}. \tag{2.17}
\end{align}

III. CONCLUSIONS

In this paper we have derived effective field equations on the 3-brane motivated by the massless bosonic sector of the type IIB string. This setup supposes that fields of this sector can penetrate in the extra dimension. The brane worlds motivated by string/M-theory have become an widely investigated topic in the literature. However in our case, we have introduced the Einstein-Cartan space in order to induce interactions between the brane and bulk fields. By the developed embedding procedure these interactions are understood in a purely geometric way as the induced brane contortion. Hence this approach allows us to
avoid additional suppositions about the form of interactions. Finally, we have presented the dynamically equivalent effective equations expressed completely in Riemannian terms.

It has been expected that the effective theory is not closed in 4-dimensions. The considered approach does not give a possibility to establish effective equations in terms of quantities defined on the brane only. In order to obtain one complete set we should derive the equations of motion for extrinsic quantities \( \{ E_\phi, E_\rho, E_\chi, E_\mu^H, E_\mu^\tilde{F}, E_\mu^g \} \) in addition to (2.13), (2.16a) - (2.16e), and (2.17). However the established effective equations (as well as the bulk equations) appears very complex to be solved. Hence it seems reasonable to impose some special symmetries on the bulk fields. The imposed symmetries can eliminate extrinsic quantities or make them negligible. On the other hand, in the papers [23], where only gravity exists in the bulk the evolution of extrinsic quantities it was found and it was shown that these contributions are subtle. This result makes us to believe that one similar conclusion can hold in our case. The situation with the non-conservation of the brane energy-momentum tensor \( S_{\mu\nu} \) is also curious. We noticed that the effective brane energy-momentum tensor given by the right part of (2.13) conserves and the effective vacuum remains stable. Nevertheless if \( S_{\mu\nu} \) is understood as the true brane energy-momentum tensor; then additional suppositions about the asymptotical structure of brane are necessary (see [23, 27, 28] for related work). The examination of all these problems we leave for a forthcoming paper.

We stress that in the framework of brane worlds exist articles, where the authors have also focused on the massless bosonic sector of the type IIB theory. We only mention the paper [30], where the brane matter sector consists of a Born-Infeld action constructed from the induced metric and projected antisymmetric bulk fields. One motivation to consider such action is the magnetogenesis in early universe. Let us also remark articles, where the scenario assumes the torsion. First notice the papers [25], where the torsion has given by the Kalb-Ramond field that coexists with gravity in a 6-dimensional bulk. The 4-dimensional effective theory is constructed. However this approach is based on the Kaluza-Klein like reduction instead of the covariant embedding formalism. Apparently, the Gauss and Codacci like formulas obtained in our paper can easily be adapted for the 6-dimensional case, and some new ideas can be suggested. Finally notice the papers [26], where the presence of bulk fermions induces the torsion. The authors have shown that the induced contact interaction on the brane is suppressed only by the square of the fundamental scale, which could be of the observed order. The authors as well do not apply the embedding formalism.
Anyway the arguments in this paper extend the results in the literature on the brane worlds motivated by the string/M-theory.

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**APPENDIX A: EMBEDDING OF A BRANE**

The geometry of embedded branes has been widely investigated in the literature. In this appendix we construct the embedding formalism for spaces with nontrivial torsion (see [18] for related work). Let \( x^a(z, y) \) be the coordinates of the bulk, \( z^\lambda \) the coordinates of the brane, and \( y^i \) the coordinates of a space normal to the brane\(^6\). First we define the metric on the brane (\( \Sigma \))

\[
q_{\mu\nu} = g_{ab} e^a_\mu e^b_\nu, \tag{A1}
\]

where \( e^a_\mu = \frac{\partial x^a}{\partial z^\mu} \equiv \partial_\mu x^a \) is a frame associated with the intrinsic coordinates. Properties of a non-intrinsic character encode the unit vectors \( n^i_a = \frac{\partial x^i}{\partial y^j} \equiv \partial_i x^a \) normal to the brane such that \( g_{ab} n^a_i e^b_\mu = 0 \). Now we define the metric of space normal to the brane (\( \perp \))

\[
q_{ij} = g_{ab} n^a_i n^b_j. \tag{A2}
\]

The decomposition of a bulk vector is as follows

\[
V^a = n^a_k n^b_k V^b + e^a_\mu e^b_\nu V^b \equiv V^a_\perp + V^a_\Sigma \equiv n^a_k v^k + e^a_\nu v^\nu;
\]

\[
\text{then } \quad e^a_\mu e^\mu_b = \delta^a_b - n^a_k n^b_k \equiv h^a_b; \quad \text{where } \quad e^\mu_a \equiv g_{ab} e^b_\nu q^{\mu\nu}; \quad n^i_a \equiv g_{ab} n^b_j q^{ij},
\]

\[
\text{and } \quad h_{ab} e^a_\mu e^\nu_b = q_{\mu\nu}; \quad h^a_b h^b_c = h^a_c; \quad h_{ab} = g_{ab} - n_{ak} n^k_b.
\]

These considerations mean that \( h_{ab} \) is the induced metric on the brane [31]. Let us perform the parallel transport of tangent vector to the brane world volume \( V^\Sigma_a \) by means \( z^\gamma \to z^\gamma + dz^\gamma \) taking into account the bulk connection \( \Gamma^a_{bc} \)

\[
V^\Sigma_a(z^\gamma + dz^\gamma) = V^\Sigma_a(z^\gamma) + \Gamma^c_{ab} V^\Sigma_c(z^\gamma) dx^b = V^\Sigma_a(z^\gamma) + e^b_\gamma(z^\gamma) \Gamma^c_{ab} V^\Sigma_c(z^\gamma) dz^\lambda. \tag{A4}
\]

---

\(^6\) Where the Latin indices \( a, b = 0, \ldots, D-1; i, j = D_\Sigma, \ldots, D-1; \) and the Greek indices \( \mu, \nu = 0, \ldots, D_\Sigma - 1. \)
In the basis \((n_i^a; e^a_i)\) the brane projection of \((A4)\) at \(\mathcal{O}(dz^2)\) becomes

\[
\text{left part : } v_{||\mu}(z^\gamma + dz^\gamma)e^a_\mu(z^\gamma + dz^\gamma) \simeq v_{||\mu}(z^\gamma + dz^\gamma)e^a_\mu(z^\gamma) + v_{\mu}(z^\gamma)\partial_\gamma e^a_\mu(z^\gamma)dz^\gamma,
\]

\[
\text{right part : } v_{\mu}(z^\gamma)e^a_\mu(z^\gamma) + e^a_\gamma(z^\gamma)e^b_\lambda(z^\gamma)\Gamma^c_{ab}v_{\mu}(z^\gamma)dz^\lambda;
\]

\[
\text{finally : } v_{||\mu} = v_{\mu} + e^a_\mu e^b_\gamma e^c_\lambda\Gamma^e_{ab}v_{\nu}dz^\gamma - e^a_\mu v_\nu\partial_\gamma e^a_\mu dz^\gamma.
\]  

(A5)

At the same time \((A5)\) defines the parallel transport rule of brane world volume vector \(v_{\mu}\) with respect to the brane connection \(\Lambda_{\mu\nu}^\lambda\) in the form

\[
v_{||\mu} = v_{\mu} + \left(\epsilon^a_\mu e^b_\nu e^c_\lambda\Gamma^e_{ab} - e^a_\nu\partial_\mu e^c_\lambda\right) v_\lambda dz^\nu \equiv v_{\mu} + \left(\Sigma^\lambda_{\mu\nu}v_\lambda dz^\nu\right),
\]

with

\[
\left(\Sigma^\lambda_{\mu\nu}\right) = e^a_\nu e^b_\gamma e^c_\lambda\Gamma^e_{ab} - e^a_\nu\partial_\mu e^c_\lambda = e^a_\nu \left(\epsilon^b_\mu e^c_\lambda\Gamma^e_{ab} + \partial_\nu e^c_\lambda\right) = g_{cd}e^d_\gamma q^\gamma_\nu\nabla_\nu e^c_\mu,
\]

where \(\nabla_\nu \equiv e^a_\nu \nabla_a\) is the covariant derivative along the brane world volume direction. The orthogonal projection of \((A4)\) defines the brane second fundamental form as follows

\[
-K_{i\mu\nu}v^\mu dz^\nu \equiv n^a_i(z^\gamma + dz^\gamma) V^i_{||a}(z^\gamma + dz^\gamma)
\]

\[
\simeq e_{c\nu}(z^\gamma) v^\mu(z^\gamma)\partial_\nu n^a_i(z^\gamma)dz^\nu + n^a_i(z^\gamma)\Gamma^c_{ab}e_{c\nu}(z^\gamma) v^\mu(z^\gamma) e^b_\nu(z^\gamma)dz^\nu
\]

\[
= g_{cd}e^d_\mu \left(\partial_\nu n^a_i + \Gamma^c_{ab}n^a_i e^c_\nu\right) v^\mu dz^\nu = g_{cd}e^d_\mu \nabla_\nu n^a_i v^\mu dz^\nu \quad \text{(A7a)}
\]

\[
= -n^c_i \left(\partial_\nu e_{c\mu} - \Gamma^a_{cb}e_{a\mu} e^b_\nu\right) v^\mu dz^\nu = -g_{cd}n^a_i \nabla_\nu e^a_i v^\mu dz^\nu. \quad \text{(A7b)}
\]

As above, let us perform the parallel transport of normal vector to the brane world volume \(V^a_{||i}\) taking into account the bulk connection \(\Gamma^a_{bc}\)

\[
V^i_{||a}(z^\gamma + dz^\gamma) = V^i_{a\nu}(z^\gamma) + \Gamma^c_{ab}V^i_{c\nu}(z^\gamma)dx^b = V^i_{a\nu}(z^\gamma) + e^b_\gamma(z^\gamma)\Gamma^c_{ab}V^i_{c\nu}(z^\gamma)dz^\gamma.
\]  

(A8)

The reader will have no difficulty in showing that the projections of \((A8)\) in the basis \((n^a_i; e^a_i)\) defines the brane connection \((\Sigma^j_{i\lambda}) \equiv g_{cd}n^q_{i\lambda}q^{kj}\nabla_j n^c_i\) and the second fundamental form \(K_{i\mu\nu}\) exactly as \((A7)\). Notice that \((\Sigma^\lambda_{\mu\nu}), (\Sigma^j_{i\lambda}), \text{and } K_{i\mu\nu}\) being the projections of \(\nabla_\nu e^a_\mu, \nabla_\nu n^a_i\) can be related via Gauss-Weingarten like equations as follows

\[
\nabla_\nu e^a_\mu = g_{cd}d^a_\nu \nabla_\nu e^c_\mu = g_{cd}c^a_{\nu\lambda} e^c_\mu + n^a_i d^i_{\nu\lambda} \nabla_\nu e^c_\mu = \left(\Sigma^\lambda_{\mu\nu}e^a_\mu + K^i_{\mu\nu}n^a_i; \quad (A9a)\right)
\]

\[
\nabla_\nu n^a_i = g_{cd}d^a_\nu \nabla_\nu n^c_i = g_{cd}c^a_{\nu\lambda} n^a_i + n^a_j d^j_{\nu\lambda} \nabla_\nu n^c_i = \left(\Sigma^j_{i\lambda} n^a_i - K^i_{\lambda\nu} e^a_\lambda; \quad (A9b)\right)
\]

Motivated by the ideology of this paper we consider only the totally antisymmetric part of the contortion understood as result of a covariant splitting into irreducible parts \((L10);\)
then $\Gamma_{abc} = \Gamma^*_{abc} + \Theta_{abc}$ such that $\Theta_{abc} = \Theta_{[abc]}$. Our next step is to decompose the brane connection and the second fundamental form into Riemannian and non-Riemannian parts:

$$\nabla_\nu e^a_\mu = \nabla_\nu e^a_\mu + \Theta^{a\nu}_{dc} e^d_c \equiv \nabla_\nu e^a_\mu + \Theta^a_{\mu \nu} , \quad (A10)$$

then the brane and the orthogonal projection of (A9a) becomes

$$e^a_\gamma \nabla_\nu e^a_\mu = (\Sigma) \Gamma_{\mu \nu} = e^a_\gamma \nabla_\nu e^a_\mu + e^a_\gamma \Theta^a_{\mu \nu} \equiv (\Sigma) \Gamma_{\mu \nu}^* + \Theta_{\mu \nu}^* \quad (A11a)$$

$$n^a_i \nabla_\nu e^a_\mu = K^j_{\mu \nu} = n^j_i \nabla_\nu e^a_\mu + n^j_i \Theta^a_{\mu \nu} \equiv K^j_{\mu \nu}^* + \Theta^j_{\mu \nu} \quad (A11b)$$

Further, taking account of (A11) and (A10) one can express (A9a) in the form

$$\nabla_\nu e^a_\mu + \Theta^a_{\mu \nu} = \left((\Sigma) \Gamma_{\mu \nu}^* + \Theta^\gamma_{\mu \nu}\right) e^a_\gamma + \left(K^j_{\mu \nu}^* + \Theta^j_{\mu \nu} n^a_j\right) \quad (A12)$$

The reader will easily prove that $e^a_\gamma (\Sigma) \Gamma_{\mu \nu}^* = e^a_\gamma (\Sigma) \Gamma_{\mu \nu}^*$ and similarly $n^a_i K^j_{\mu \nu}^* = n^a_j K^j_{\mu \nu}^*$; then acting via symmetrization/antisymmetrization on (A12) it can easily be checked that

$$\nabla_\nu e^a_\mu = (\Sigma) \Gamma_{\mu \nu}^* e^{a \gamma} + K^j_{\mu \nu}^* n^a_j, \quad \Theta^a_{\mu \nu} = \Theta^\gamma_{\mu \nu} e^{a \gamma} + \Theta^j_{\mu \nu} n^a_j. \quad (A13)$$

Considering the definition of bulk covariant derivative along the brane world volume direction one is able to define the brane covariant derivative as follows

$$(\Sigma) \nabla_\nu v_\mu \equiv \partial_\nu v_\mu - (\Sigma) \Gamma_{\mu \nu}^\lambda v_\lambda = \partial_\nu (e^a_\lambda V_a) - e^a_\lambda e^c_\lambda V_a \nabla_\nu e^c_\mu = e^a_\mu \nabla_\nu V_a + v_\lambda K^\lambda_{\mu \nu} \quad (A14a)$$

$$(\Sigma) \nabla_\nu v_\iota \equiv \partial_\nu v_\iota - (\Sigma) \Gamma_{\iota \nu}^j v^j = \partial_\nu (n^a_i V_a) - n^a_i n^a_j V_a \nabla_\nu n^c_i = n^a_i \nabla_\nu V_a - v_\lambda K^\lambda_{\iota \nu} \quad (A14b)$$

Further, considering the brane covariant derivative we can relate the bulk curvature tensor with the brane one. First notice a definition

$$\nabla_{[a} \nabla_{b]} V_c = \partial_{[a} \nabla_{b]} V_c - \Gamma^d_{[ba]} \nabla_d V_c - \Gamma^d_{c[a} \nabla_{b]} V_d = (\partial_{[a} \Gamma^d_{\mu \nu} + \Gamma^d_{c[b} \Gamma^a_{\mu c]} b) V_d - \Theta^d_{ba} \nabla_d V_c \quad (A15a)$$

$$= \frac{1}{2} R^d_{c b a} V_d - \Theta^d_{ba} \nabla_d V_c \quad \text{and} \quad (A15b)$$

$$\nabla_{[a} \nabla_{b]} V^c = \frac{1}{2} R^c_{d b a} V^d - \Theta^c_{ba} \nabla_d V^c. \quad (A15b)$$

This definition gives a possibility to derive the following properties of the curvature\footnote{These properties can be checked by the decomposition: $R_{abcd} = R_{abcd}^* + R_{abcd}^\#$.}

$$R_{abcd} = - R_{bacd} = - R_{abdc},$$

$$R_{f abc} + R_{f bca} + R_{f cab} = - 6 \left( \nabla_{[a} \Theta_{bc]} f + 2 \Theta^d_{[ab} \Theta_{c]d} f \right).$$
Now evaluate the commutator \( e_a^\gamma \nabla_{\mu} \nabla_{\nu} e_\alpha \)

\[
e_a^\gamma \nabla_{\mu} \nabla_{\nu} e_\alpha = e_a^\gamma e_\mu e_c \nabla_c \left( \Sigma \Gamma^\lambda_{[\alpha|\nu]} e_\lambda^a + K^i_{[\alpha|\nu]} n_i^a \right) = \partial_{\mu} \Sigma \Gamma^\gamma_{[\alpha|\nu]} + \Sigma \Gamma^\gamma_{\lambda|\mu} \Sigma \Gamma^\lambda_{[\alpha|\nu]} - K^{i\gamma}_{\mu} K_{i|\alpha|\nu} \]

\[
= \frac{1}{2} \left( \Sigma \Gamma^\gamma_{\alpha\mu
u} - K_{i\alpha\nu} K^i_{\gamma\mu} + K_{i\alpha\mu} K^i_{\gamma\nu} \right); \\
e_a^\gamma \nabla_{\mu} \nabla_{\nu} e_\alpha = e_a^\gamma \left( e_\mu^a e_\nu^d e_c \nabla_d e_\alpha^a + \Theta^d_{\nu\mu} \nabla_d e_\alpha^a \right) = \frac{1}{2} e_a^\gamma R^a_{\beta\mu\nu} e_\alpha^\beta e_\mu^d e_\nu^d.
\]

Therefore the Gauss like equation can be expressed as follows

\[
R^a_{\beta\mu\nu} e_\alpha^\beta e_\mu^d e_\nu^d = \left( \Sigma \Gamma^\gamma_{\alpha\mu\nu} - K_{i\alpha\nu} K^i_{\gamma\mu} + K_{i\alpha\mu} K^i_{\gamma\nu} \right). \tag{A16}
\]

Finally, evaluate the commutator \( n_a^i \nabla_{\mu} \nabla_{\nu} e_\alpha \)

\[
n_a^i \nabla_{\mu} \nabla_{\nu} e_\alpha = n_a^i e_\mu e_c \nabla_c \left( \Sigma \Gamma^\lambda_{[\alpha|\nu]} e_\lambda^a + K^i_{[\alpha|\nu]} n_i^a \right) = \frac{1}{2} \left( \Sigma \nabla_{\mu} K^i_{\alpha\nu} - \Sigma \nabla_{\nu} K^i_{\alpha\mu} - 2 \Theta^\lambda_{\mu\nu} K^i_{\alpha\lambda} \right); \\
n_a^i \nabla_{\mu} \nabla_{\nu} e_\alpha = n_a^i \left( e_\mu^a e_\nu^d e_c \nabla_d e_\alpha^a + \Theta^d_{\nu\mu} \nabla_d e_\alpha^a \right) = \frac{1}{2} n_a^i R^a_{\beta\mu\nu} e_\alpha^\beta e_\mu^d e_\nu^d.
\]

Therefore the Codacci like equation can be expressed as follows

\[
R^a_{\beta\mu\nu} n_a^i e_\alpha^\beta e_\mu^d e_\nu^d = \left( \Sigma \nabla_{\mu} K^i_{\alpha\nu} - \Sigma \nabla_{\nu} K^i_{\alpha\mu} - 2 \Theta^\lambda_{\mu\nu} K^i_{\alpha\lambda} \right). \tag{A17}
\]

**APPENDIX B: FIELD EQUATIONS IN THE BULK**

The variational formalism of relativistic conservative medium with internal degrees of freedom in the framework of EC space has been investigated in the literature \[17, 19, 21\]. The dynamics of compactified type IIB theory gives equations of motion listed in the Appendix \[C\] [see \[C3\]]. We modify these equations in order to include the brane world contribution. Consider these modifications at the level of action as follows

\[
S = \frac{1}{2k^2} \int d^5 x \sqrt{|g|} \left[ R - \frac{1}{2} \left( \partial_{\alpha} \phi \partial^{\alpha} \phi + \frac{5}{8} \partial_{\alpha} \rho \partial^{\alpha} \rho + \frac{5}{4} \partial_{\alpha} \phi \partial^{\alpha} \rho + e^{2\phi + \frac{5}{2} \rho} \partial_{\alpha} \chi \partial^{\alpha} \chi \right) \\
- \frac{1}{12} e^{-\phi} H_{abc} H^{abc} - \frac{e^{\frac{5}{2} \rho}}{480} \tilde{Y}^{abcde}_{abcd} + e^{-\frac{1}{2} \rho} \chi \right],
\]

\[
S_c = \int d^5 x \sqrt{|g|} \left( -\Lambda \right), \quad S_{cbr} = \int d^4 x \sqrt{|g|} \left( -\Lambda \right), \quad S_{br} = \int d^4 x \sqrt{|g|} L(q_{\mu\nu}, \psi_A, (4\nabla_{\mu} \psi_A), \psi_A),
\]

where

\[
(4\nabla_{\mu} \psi_A = \partial_{\mu} \psi_A - (4\Gamma_{\beta\mu}^{\alpha} \psi_A)_{\alpha}^2, \quad \psi_A)_{\alpha}^2 \equiv \phi_B \Omega_B A_{\alpha}^2. \tag{B1b}
\]

The term \( \Omega_B A_{\alpha}^2 \) is the generator of coordinate transformations and its explicit form depends on the order of the field \( \psi_A \). By construction, we suppose \( g_{ab} n^a n^b = 1, \ x^4 \equiv y, \) and \( e^a_\mu = \delta^a_\mu; \)
then the formalism constructed in the Appendix A can straightforwardly be applied. Further, we introduce a notation by taking the following variation

\[
\delta \int d^5x \sqrt{|g|} (\delta S) = \delta \int d^5x \sqrt{|g|} \left( \frac{5}{2} R - \Theta^2 \right)
\]

\[
= \int d^5x \sqrt{|g|} \left( \Theta_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} \Theta^2 - 3 \Theta_{\alpha\beta} \Theta_{\gamma\delta} \right) \delta g^{\alpha\beta}
\]

\[
= \int d^5x \sqrt{|g|} G_{\alpha\beta} \delta g^{\alpha\beta}, \quad \text{where}
\]

\[
G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (\delta S) = 2 \Theta_{\alpha\beta} \delta g^{\alpha\beta}; \quad \Theta_{abc} = \frac{\sqrt{3}}{2} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \left( \partial_{[a} A_{bc]} - \chi \partial_{[a} B_{bc]} - \chi \partial_{[a} B_{bc]} \right).
\]

The non-zero variations of \((S + S_{ab})\) gives the following expressions

\[
\delta S_{\alpha\beta} = -\delta \int d^5x \sqrt{|g|} \Lambda = \frac{1}{2} \int d^5x \sqrt{|g|} g_{\alpha\beta} \delta g^{\alpha\beta}, \quad (B3a)
\]

\[
\delta S_{\alpha\beta} = -\delta \int d^5x \sqrt{|g|} \Lambda \delta (\phi, \chi, A_{ab}, B_{ab}, \psi_A) \text{ gives respectively}
\]

\[
\delta S_{\alpha\beta} = \int d^5x \sqrt{|g|} \left[ \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \psi_A} \psi_A \right] \delta g^{\alpha\beta}.
\]

\[
\equiv -\frac{1}{2} \int d^5x \sqrt{|g|} \left( \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \psi_A} \psi_A \right) \delta g^{\alpha\beta}.
\]

\[
\delta S_{\alpha\beta} = -\frac{5}{32 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{8 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi.
\]

\[
\delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi.
\]

\[
\delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \tilde{F}_{\alpha\beta \gamma} \delta \phi.
\]

\[
\delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \chi \delta H_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \chi \delta H_{\alpha\beta \gamma} \delta \phi.
\]

\[
\delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \chi \delta H_{\alpha\beta \gamma} \delta \phi, \quad \delta S_{\alpha\beta} = -\frac{1}{4 \sqrt{3}} \int d^5x \sqrt{|g|} e^{\frac{i}{2} (\phi + \frac{5}{4} \rho)} \chi \delta H_{\alpha\beta \gamma} \delta \phi.
\]
Due to modifications given by (B1a) the equations of motion (C3) become

$$\frac{\delta S_{br}}{\delta \psi_A} = \int d^4x \sqrt{|g|} \left( \frac{\partial L}{\partial \psi_A} - \left( \frac{\partial L}{\partial \Gamma^\alpha_{\beta\lambda}} \frac{\partial \Gamma^\alpha_{\beta\lambda}}{\partial \psi_A} \right) \right) = 0,$$

where $\sigma_{\mu \nu} \equiv \left( \frac{\partial}{\partial \psi_A} \frac{\partial L}{\partial \psi_A} \right)_{\beta\lambda} \equiv \xi_{[\alpha\beta\lambda]}$, $\xi_{\alpha \beta\lambda} \equiv 2 \frac{\partial L}{\partial (\frac{\partial L}{\partial \Gamma^\alpha_{\beta\lambda}})} = -2 \frac{\partial L}{\partial (\frac{\partial L}{\partial \Gamma^\alpha_{\beta\lambda}})} \phi_B \Omega^B_{A\alpha}$, and $\{T_{\mu \nu}, \tau_{\mu \nu}\}$ tensors have generated by $\{L, \bar{L}\}$ parts of the brane Lagrangian [see (143)].

Due to modifications given by (B1a), the equations of motion (C3) become

\[ G_{ab} = k^2 T_{ab}, \quad T_{ab} = -g_{ab} \Lambda + S_{ab} \delta^\alpha_{\beta} \delta^\nu_{\mu} \delta(y) + k^{-2} P_{ab}, \quad \text{where} \]

\[ P_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{5}{8} \partial_a \phi \partial_b \rho + \frac{5}{16} \partial_a \rho \partial_b \rho + \frac{e^{2\phi + \frac{3}{2} \rho}}{2} \partial_a \chi \partial_b \chi + \frac{e^{-\phi}}{4} H_{acd} H_{b}^{cd} + \frac{e^{2\phi + \frac{3}{2} \rho}}{96} \bar{Y}_{acdef} \bar{Y}_{b}^{e} - \frac{1}{2} g_{ab} \left( \frac{1}{2} \partial_c \phi \partial_c \phi + \frac{5}{8} \partial_c \phi \partial_c \rho + \frac{5}{16} \partial_c \rho \partial_c \rho \right) \]

\[ + \frac{e^{2\phi + \frac{3}{2} \rho}}{2} \partial_c \chi \partial_c \chi + \frac{e^{-\phi}}{12} H^2 - \frac{e^{-\frac{3}{2} \rho} \chi}{2} \right); \quad S_{\mu \nu} \equiv (-g_{\mu \nu} \rho + \tau_{\mu \nu}), \]

\[ \left( 5 \right) \bar{\nabla}_{c} \bar{\nabla}^{c} \rho + \left( 5 \right) \bar{\nabla}_{c} \bar{\nabla}^{c} \phi = e^{\frac{3}{2} \rho} \left( \frac{e^{2\phi \partial_a \chi \partial^a \chi} + \frac{e^{\phi}}{6} \bar{F}^2 + \frac{1}{120} \bar{Y}^2}{12} \right) - \frac{4}{5} e^{-\frac{3}{2} \rho} \chi, \]

\[ = \frac{k^2}{2 \sqrt{3}} e^{\frac{3}{2} \left( \phi + \frac{3}{2} \rho \right)} \theta^{\mu \nu \alpha} \bar{F}_{\alpha \mu \nu} \delta(y), \quad \text{(B5b)} \]

\[ \frac{e^{-\phi}}{12} H^2 - e^{2\phi + \frac{3}{2} \rho} \partial_a \chi \partial^a \chi - \frac{e^{2\phi + \frac{3}{2} \rho}}{12} \bar{F}^2 + \left( 5 \right) \bar{\nabla}_{a} \bar{\nabla}^{a} \phi + 5 \left( 5 \right) \bar{\nabla}_{a} \bar{\nabla}^{a} \rho \]

\[ = \frac{k^2}{4 \sqrt{3}} e^{\frac{3}{2} \left( \phi + \frac{3}{2} \rho \right)} \theta^{\mu \nu \alpha} \bar{F}_{\alpha \mu \nu} \delta(y), \quad \text{(B5c)} \]

\[ \left( 5 \right) \bar{\nabla}^{a} \left( e^{2\phi + \frac{3}{2} \rho} \partial_a \chi \right) + \frac{\phi^{\frac{3}{2} \rho}}{6} \bar{F}_{abc} \bar{H}^{abc} = \frac{k^2}{2 \sqrt{3}} e^{\frac{3}{2} \left( \phi + \frac{3}{2} \rho \right)} \theta^{\mu \nu \alpha} \bar{H}_{ab} \delta(y), \quad \text{(B5d)} \]

\[ 24 \left( 5 \right) \bar{\nabla}_{c} \left( e^{\phi + \frac{3}{2} \rho} \bar{F}_{cab} \right) + \frac{5}{8} \rho \bar{Y}_{abcdef} \bar{H}_{cde} + 3 \left( 5 \right) \bar{\nabla}_{c} \left( e^{\phi + \frac{3}{2} \rho} \bar{Y}_{cdeab} \bar{B}_{d} \right) \]

\[ = 12 \sqrt{3} k^2 e^{\frac{1}{2} \left( \phi + \frac{3}{2} \rho \right)} \left[ \theta^{\mu \nu \alpha} \partial_{\alpha} \left( \phi + \frac{5}{4} \rho \right) 2 \sigma^{\mu \nu} \right] \delta^{b}_{\rho} \delta^{b}_{\nu} \delta(y), \quad \text{(B5e)} \]

\[ 24 \left( 5 \right) \bar{\nabla}_{c} \left( e^{-\phi} \bar{H}_{cab} \right) - \left( 5 \right) \bar{\nabla}_{c} \left( e^{\phi + \frac{3}{2} \rho} \bar{Y}_{cdeab} \bar{A}_{d} \right) - e^{\frac{3}{2} \rho} \bar{Y}_{abcd} \bar{F}_{cde} \]

\[ = 24 \sqrt{3} k^2 e^{\frac{3}{2} \left( \phi + \frac{3}{2} \rho \right)} \left[ \theta^{\alpha \mu \nu} \left( \partial_{\alpha} \chi + \frac{1}{\chi} \partial_{\alpha} \phi \chi \partial_{\alpha} \rho \right) \right] \delta^{b}_{\rho} \delta^{b}_{\nu} \delta(y), \quad \text{(B5f)} \]
The relations (B5g) are the consequences of self-duality conditions, the ansatz (1.6), and the equation of motion (C3g). Hence throughout the paper we consider (B5g) instead of (C3g). On the other hand, on the brane survive only \( \tilde{Y}_{\alpha\beta\gamma\lambda4} \) components. It can be checked that

\[
\tilde{Y}_{\alpha\beta\gamma\lambda4} = -\sqrt{|q|} \eta_{\alpha\beta\gamma\lambda} \tilde{Y}_{01234} = -\varrho \eta_{\alpha\beta\gamma\lambda} e^{-\frac{\varrho}{4}} \rho,
\]

and \( \tilde{Y}_{\alpha\beta\gamma\lambda4} \tilde{Y}_{\alpha\beta\gamma\lambda4} = -4! \varrho^2 \), where \( \eta_{\alpha\beta\gamma\lambda} = \eta_{abcde} e^a e^b e^c e^d e^n \) (B6) is the brane volume form and \( \varrho \) is the Freund-Rubin parameter.

**APPENDIX C: EQUATIONS OF MOTION FOR THE TYPE IIB STRING**

We have noticed that the massless bosonic sector of the type IIB string can not be described by a covariant action [11] however the covariant equations of motion exists. The 10-dimensional equations considered in this paper are as follows

\[
0 = \tilde{R}_{AB} + \frac{5}{4} \nabla(A \nabla(B) \tilde{\Phi} - \frac{e^{-\Phi}}{4} H_{ACD} H_B^{CD} - \frac{1}{2} \partial_A \Phi \partial_B \Phi - \frac{5}{8} \partial(A \Phi \partial_B) \tilde{\Phi} - \frac{e^{2\Phi + \frac{5}{4} \Phi}}{2} \partial_A \Psi \partial_B \Psi
\]

\[
- \frac{e^{\varphi + \frac{5}{4} \Phi}}{4} \tilde{F}_{ACD} \tilde{F}_B^{CD} - \frac{d\varphi}{96} \tilde{Y}_{ACDEFG} \tilde{Y}_B^{CDEFG} - \frac{1}{2} G_{AB} \left( \frac{5}{2} \nabla_C \nabla^C \tilde{\Phi} - \frac{e^{-\Phi}}{12} H^2 \right)
\]

\[
- \frac{1}{2} \partial_C \Phi \partial^C \Phi - \frac{5}{8} \partial_C \Phi \partial^C \tilde{\Phi} - \frac{25}{16} \partial_C \Phi \partial^C \tilde{\Phi} - \frac{e^{2\Phi + \frac{5}{4} \Phi}}{2} \partial_C \Psi \partial^C \Psi - \frac{e^{\Phi + \frac{5}{4} \Phi}}{12} \tilde{F}^2 \right),
\]

\[
0 = \tilde{R} - \frac{e^{-\Phi}}{12} H^2 - \frac{1}{2} \partial_A \Phi \partial^A \Phi - \frac{25}{16} \partial_A \Phi \partial^A \tilde{\Phi} - \frac{1}{2} \nabla_A \nabla^A \Phi \Phi + \frac{5}{2} \nabla_A \nabla^A \tilde{\Phi},
\]

\[
0 = \frac{e^{-\Phi}}{12} H^2 - \frac{5}{4} \partial_A \tilde{\Phi} \partial^A \Phi - \frac{25}{32} \partial_A \tilde{\Phi} \partial^A \tilde{\Phi} - e^{2\Phi + \frac{5}{4} \Phi} \partial_A \Psi \partial^A \Psi
\]

\[
- \frac{e^{\varphi + \frac{5}{4} \Phi}}{12} \tilde{F}^2 + \frac{5}{8} \nabla_A \nabla^A \Phi + \frac{5}{8} \nabla_A \nabla^A \tilde{\Phi},
\]

\[
0 = \nabla_A \nabla^A \Psi + 2 \partial_A \Psi \partial^A \Phi + \frac{e^{-\Phi}}{6} \tilde{F}_{ABC} H^{ABC},
\]

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\[0 = 24 \, e^\Phi \left( \partial_C \Phi \, \tilde{F}^{CAB} + \nabla_C \tilde{F}^{CAB} \right) + \tilde{Y}^{ABCDE} H_{CDE} + 3 \nabla_C \left( \tilde{Y}^{CDEAB} B_{DE} \right), \quad \text{(C1e)}\]

\[0 = 24 \, e^{-\frac{2}{5} \Phi - \Phi} \left[ \partial_C \left( \Phi + \frac{5}{4} \tilde{\Phi} \right) H^{CAB} - \nabla_C H^{CAB} \right] + 3 \nabla_C \left( \tilde{Y}^{DECAB} A_{DE} \right) + 24 \, e^\Phi \left[ \nabla_C \left( \Psi \tilde{F}^{CAB} \right) + \Psi \tilde{F}^{CAB} \partial_C \Phi \right] + \tilde{Y}^{ABCDE} F_{CDE}, \quad \text{(C1f)}\]

\[0 = \nabla_E \tilde{Y}^{EABCD}, \quad \text{(C1g)}\]

and the imposed self-duality condition

\[\tilde{Y}_{ABCDEF} = *\tilde{Y}_{ABCDEF} = \frac{1}{5!} \eta_{ijklmnop} \tilde{Y}_{ijklmnop}, \quad \text{(C1h)}\]

We observe that these equations can be derived from the following action

\[S_E = \frac{1}{2k_{10}^2} \int d^{10}x \sqrt{|G|} e^{-\frac{2}{5} \tilde{\Phi} - \Phi} \left[ \frac{\tilde{R}}{12} - e^{-\Phi} H^2 - \frac{1}{2} \partial_A \Phi \partial^A \Phi - \frac{5}{8} \partial_A \Phi \partial^A \tilde{\Phi} + \frac{25}{16} \partial_A \tilde{\Phi} \partial^A \tilde{\Phi} - \frac{e^{2\Phi + \frac{2}{5} \tilde{\Phi}}}{2} \partial_A \Psi \partial^A \Psi - \frac{e^{\Phi + \frac{2}{5} \tilde{\Phi}}}{12} \tilde{F}^2 - \frac{e^\Phi}{480} \tilde{Y}^2 \right]; \quad \text{(C2)}\]

\[H_{ABC} = 3 \partial_{[A} B_{BC]}, \quad F_{ABC} = 3 \partial_{[A} A_{BC]}, \quad \tilde{F}_{ABC} = F_{ABC} - \Psi H_{ABC}; \]

\[Y_{ABCDEF} = 5 \partial_{[A} W_{BCDE]}, \quad \tilde{Y}_{ABCDEF} = Y_{ABCDEF} - 5 A_{[AB} H_{CDE]} + 5 B_{[AB} F_{CDE]};\]

except for vanishing of the term \(\tilde{Y}^2\) in \(\text{(C1a)}\) and the self-duality condition \(\text{(C1h)}\).

Applying the method based on the ansatz \(\text{(1.6)}\) one yields 5-dimensional equations of motion in the form

\[0 = (5)^* \rho_{ab} - \frac{e^{-\phi}}{4} H_{ac} H_{b}^{\cd} - \frac{1}{2} \partial_a \phi \partial_b \phi - \frac{5}{8} \partial_a \phi \partial_b \rho - \frac{5}{16} \partial_a \rho \partial_b \rho - \frac{e^{2\phi + \frac{3}{2} \rho}}{2} \partial_a \chi \partial_b \chi \]

\[- \frac{e^{\phi + \frac{3}{2} \rho}}{4} \tilde{F}_{ac} \tilde{F}_{b}^{\cd} - \frac{e^{\Phi + \frac{3}{2} \rho}}{96} Y_{acde} \tilde{Y}_{b}^{cdef} - \frac{1}{2} g_{ab} \left( (5)^* \tilde{R} - \frac{e^{-\phi}}{12} H^2 - \frac{1}{2} \partial_c \phi \partial^c \phi - \frac{5}{8} \partial_c \phi \partial^c \rho \right) \]

\[\quad - \frac{5}{16} \partial_c \rho \partial^c \rho - \frac{e^{2\phi + \frac{3}{2} \rho}}{2} \partial_c \chi \partial^c \chi - \frac{e^{\phi + \frac{3}{2} \rho}}{12} \tilde{F}^2 + e^{-\frac{\phi}{2}} \tilde{R}^{(1)} \right), \quad \text{(C3a)}\]

\[0 = (5)^* \nabla_c \left( e^{\frac{5}{2} \phi} \rho \right) + (5)^* \nabla_c \left( e^{\phi} \phi \right) - e^{\frac{3}{2} \rho} \left( e^{2\phi \partial_a \chi \partial^a \chi} + \frac{e^{\phi}}{6} \tilde{F}^2 + \frac{1}{120} \tilde{Y}^2 \right) - \frac{4}{5} e^{-\frac{1}{2} \rho} \tilde{R}^{(1)} . \quad \text{(C3b)}\]
\[ 0 = \frac{e^{-\phi}}{12} H^2 - e^{2\phi + \frac{5}{4} \phi} \partial_a \chi \partial^a \chi - \frac{e^{\phi + \frac{5}{4} \rho}}{12} \tilde{F}^2 + (5) \nabla^a (5) \nabla^a \phi + \frac{5}{8} \nabla^a (5) \nabla^a \rho, \quad (C3c) \]

\[ 0 = (5) \nabla^a (5) \nabla^a \chi + 2 \partial_a \chi \partial^a \phi + \frac{5}{4} \partial_a \chi \partial^a \rho + \frac{e^{-\phi}}{6} \tilde{F}_{abc} H^{abc}, \quad (C3d) \]

\[ 0 = 24 e^\phi \left[ \partial_c \left( \phi + \frac{5}{4} \rho \right) \tilde{F}^{cab} + (5) \nabla^c \tilde{F}^{cab} \right] + 3(5) \nabla^c \left( \tilde{Y}^{cdef} B_{de} \right) \]

\[ + \tilde{Y}^{abde} H_{cde} + \frac{15}{4} \partial_c \rho \tilde{Y}^{abde} B_{de}, \quad (C3e) \]

\[ 0 = 24 e^\phi \left[ \nabla^c \left( \chi \tilde{F}^{cab} \right) + \chi \tilde{F}^{cab} \partial_c \left( \phi + \frac{5}{4} \rho \right) \right] + 24 e^{\frac{5}{2} \rho - \phi} \left[ \partial_c \phi H^{cab} - (5) \nabla^c H^{cab} \right] \]

\[ + 3(5) \nabla^e \left( \tilde{Y}^{decab} A_{de} \right) + \frac{15}{4} \partial_e \rho \tilde{Y}^{decab} A_{de} + \tilde{Y}^{abde} F_{cde}, \quad (C3f) \]

\[ 0 = (5) \nabla^e \tilde{Y}^{eabcd} + \frac{5}{4} \partial_e \rho \tilde{Y}^{eabcd}, \quad (C3g) \]

where \((^1R = \text{const.})\) is the scalar curvature of \(S^5\).

We observe that these equations can be derived from the following action

\[ S_0 = \frac{1}{2k^2} \int d^5 x \sqrt{|g|} \left[ R - \frac{1}{2} \left( \partial_a \phi \partial^a \phi + \frac{5}{8} \partial_a \rho \partial^a \rho + \frac{5}{4} \partial_a \phi \partial^a \rho + e^{2\phi + \frac{5}{4} \rho} \partial_a \chi \partial^a \chi \right) \right. \]

\[ \left. - \frac{1}{12} \left( e^{\phi + \frac{5}{4} \rho} \tilde{F}^2 + e^{-\phi} H^2 \right) - \frac{e^{\frac{5}{2} \rho}}{480} \tilde{Y}^2 + e^{-\frac{1}{2} \rho} (^1R) \right]; \quad (C4) \]

\[ F_{abc} = 3\partial_{[a} A_{bc]}, \quad H_{abc} = 3\partial_{[a} B_{bc]}, \quad \tilde{F}_{abc} = F_{abc} - \chi H_{abc}, \]

\[ Y_{abcd} = 5\partial_{[a} W_{bcde]}, \quad \tilde{Y}_{abcd} = Y_{abcd} - 5A_{[ab} H_{cde]} + 5B_{[ab} F_{cde]}, \]

except for vanishing of the term \(\tilde{Y}^2\) in \((C3a)\), for the wrong coefficient of \(\tilde{Y}^2\) in \((C3b)\), and for consequences of the self-duality condition given by \((1.8)\).

[1] C. Lovelace, Strings in curved space, Phys. Lett. B 135, 75 (1984); E.S. Fradkin, A.A. Tseytlin, Quantum string theory effective action, Nucl. Phys. B 261, 1 (1985); C.G. Callan, D. Friedan, E.J. Martinec, M.J. Perry, Strings in background fields, Nucl. Phys. B 262, 593 (1985).
[2] T.L. Curtright, C.K. Zachos, Geometry, topology and supersymmetry in non-linear models, Phys. Rev. Lett. B 53, 1799 (1984).
[3] C.M. Hull, The Geometry of N=2 Strings with Torsion, Phys. Lett. B 387, 497 (1996), hep-th/9606190.
[4] R.T. Hammond, Strings in gravity with torsion, Gen. Rel. Grav. 32, 2007 (2000), gr-qc/9904033.
[5] I.L. Shapiro, Physical Aspects of the Space-Time Torsion, Phys. Rept. 357, 113 (2001), hep-th/0103093.
[6] J. Polchinski, String Theory, (Cambridge University Press, Cambridge 1998).
[7] M. Sato, A. Tsuchiya, Hamilton-Jacobi Method and Effective Actions of D-brane and M-brane in Supergravity, Nucl. Phys. B 671, 293 (2003), hep-th/0305090. M. Sato, A. Tsuchiya, Born-Infeld Action from Supergravity, Prog. Theor. Phys. 109, 687 (2003), hep-th/0211074.
[8] H. Nastase, D. Vaman, On the nonlinear KK reductions on spheres of supergravity theories, Nucl. Phys. B 583, 211 (2000), hep-th/0002028. M. Cvetic, H. Lu, C. N. Pope, Consistent Kaluza-Klein Sphere Reductions, Phys. Rev. D 62, 064028 (2000), hep-th/0003286.
[9] A. A. Tseytlin, Ambiguity in the effective action in string theories, Phys. Lett. B 176, 92 (1986); M. C. Bento, N. E. Mavromatos, Ambiguities in the low-energy effective actions of string theories with the inclusion of antisymmetric tensor and dilaton fields, Phys. Lett. B 190, 105 (1987).
[10] G. Horowitz and A. Strominger, Black Strings and p-branes, Nucl. Phys. B 360, 197 (1991); T. Mohaupt, Black Holes in Supergravity and String Theory, Class. Quant. Grav. 17, 3429 (2000), hep-th/0004098.
[11] N. Marcus and J.H. Schwarz, Field theories that have no manifestly Lorentz-invariant formulation, Phys. Lett. B 115, 111 (1982).
[12] J.H. Schwarz, Covariant field equations of chiral N = 2 D = 10 supergravity, Nucl. Phys. B 226, 269 (1983); P. Howe and P. West, The complete N = 2 D = 10 supergravity, Nucl. Phys. B 238, 181 (1984).
[13] M. Cvetic, M.J. Duff, James T. Liu, H. Lu, C.N. Pope, K.S. Stelle, Randall-Sundrum Brane Tensions, Nucl. Phys. B 605, 141 (2001), hep-th/0011167. M.J. Duff, James T. Liu, K.S. Stelle, A supersymmetric Type IIB Randall-Sundrum realization, J. Math. Phys. 42, 3027 (2001), hep-th/0007120.
[14] M.S. Bremer, M.J. Duff, H. Lü, C.N. Pope and K.S. Stelle, Instanton cosmology and domain walls from M-theory and string theory, Nucl. Phys. B 543, 321 (1999), hep-th/9807051; M. Cvetic, James T. Liu, H. Lu, C.N. Pope, Domain-wall Supergravities from Sphere Reduction Nucl. Phys. B 560, 230 (1999), hep-th/9905096.

[15] T. Shiromizu, D. Ida, H. Ochiai and T. Torii, Stability of $\text{AdS}_p \times S^n \times S^{q-n}$ Compactifications, Phys. Rev. D 64, 084025 (2001), hep-th/0106265; T. Torii, and T. Shiromizu, Cosmological constant, dilaton field and Freund-Rubin compactification, Phys. Lett. B 551, 161 (2003), hep-th/0210002.

[16] F.W. Hehl, P. Heide, G.D. Kerlick and J.M. Nester, General relativity with spin and torsion: Foundations and prospects, Rev. Mod. Phys. 48, 393 (1976).

[17] V. N. Ponomariov, A. Barvinsky and Yu. N. Obukhov, Geometrodynamical Methods and the Gauge Approach to the Gravitational Interactions (Energoatomizdat, Moscow, 1985).

[18] B. Sazdović, Torsion and nonmetricity in the stringly geometry, preprint hep-th/0304086

[19] A.V. Minkevich and F. Karakura, On the relativistic dynamics of spinning mater in space-time with curvature and torsion, J. Phys. A 16, 1409 (1983).

[20] G. Aprea, G. Montani, R. Ruffini, Test particles behavior in the framework of a lagrangian geometric theory with propagating torsion, Int. J. Mod. Phys. D 12, 1875 (2003), gr-qc/0401054; S. M. Carroll and G. B. Field, Consequences of Propagating Torsion in Connection-Dynamic Theories of Gravity Phys. Rev. D 50, 3867 (1994), gr-qc/9403058.

[21] C. G. Boehmer, The Einstein static universe with torsion and the sign problem of the cosmological constant, Class. Quant. Grav. 21, 1119 (2004), gr-qc/0310058.

[22] T. Kaluza, Zum Unitätsproblem der Physik, Sitz. Preuss. Akad. Wiss. Phys. Math. K 1, 966 (1921); O. Klein, The atomicity of electricity as a quantum theory law, Nature 118, 516 (1926).

[23] T. Shiromizu, K. Maeda, and M. Sasaki, The Einstein equations on the 3-Brane World, Phys. Rev. D 62, 024012 (2000), gr-qc/9910076; M. Sasaki, T. Shiromizu, and K. Maeda, Gravity, Stability and Energy Conservation on the Randall-Sundrum Brane-World, Phys. Rev. D 62, 024008 (2000), hep-th/9912233.

[24] E. Anderson and R. Tavakol, Reformulation and Interpretation of SMS Braneworld, Class. Quant. Grav. 20, L267 (2003), gr-qc/0305013.

[25] B. Mukhopadhyaya, S. Sen, S. Sen, S. SenGupta, Bulk Kalb-Ramond field in Randall Sundrum
scenario, Phys. Rev. D 70, 066009 (2004), [hep-th/0403098]. B. Mukhopadhyaya, S. Sen, S. SenGupta, Does a Randall-Sundrum scenario create the illusion of a torsion-free universe?, Phys. Rev. Lett. 89, 121101 (2002), [hep-th/0204242].

[26] O. Lebedev, Torsion Constraints in the Randall–Sundrum Scenario, Phys. Rev. D 65, 124008 (2002), [hep-ph/0201125]. L. N. Chang, O. Lebedev, W. Loinaz, T. Takeuchi, Universal Torsion-Induced Interaction from Large Extra Dimensions, Phys. Rev. Lett. 85, 3765 (2000), [hep-ph/0005236].

[27] A. Mennim and R. A. Battye, Cosmological expansion on a dilatonic brane-world, Class. Quant. Grav. 18, 2171 (2001), [hep-th/0008192]. C. Barcelo and M. Visser, Braneworld gravity: Influence of the moduli fields, J. High Energy Phys. 10, 019 (2000), [hep-th/0009032].

[28] D. Gonta, AdS$_5$ Brane World Cosmology, J. High Energy Phys. 12, 007 (2004), [hep-ph/0401117].

[29] L. Randall and R. Sundrum, A Large Mass Hierarchy from a Small Extra Dimension, Phys. Rev. Lett. 83, 3370 (1999), [hep-ph/9905221]. L. Randall and R. Sundrum, An Alternative to Compactification, Phys. Rev. Lett. 83, 4690 (1999), [hep-th/9906064].

[30] T. Shiromizu, K. Koyama, S. Onda, and T. Torii, Can We Live On A D-Brane? Effective Theory on a Selfgravitating D-Brane, Phys. Rev. D 68, 063506 (2003), [hep-th/0305253].

[31] S. W. Hawking and C. F. R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, 1973).

[32] W. Israel, Singular Hypersurfaces and Thin Shells in General Relativity, Nuovo Cimento B 44, 1 (1966).

[33] R. M. Wald, *General Relativity*, (Chicago University Press, Chicago, 1984).