ALGEBRAIC INDEPENDENCE OF THE CARLITZ PERIOD AND ITS HYPERDERIVATIVES.

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Abstract. This paper deals with the fundamental period $\tilde{\pi}$ of the Carlitz module. The main theorem states that the Carlitz period and all its hyperderivatives are algebraically independent over the base field $\mathbb{F}_q(\theta)$. Our approach also reveals a connection of these hyperderivatives with the coordinates of a period lattice generator of the tensor powers of the Carlitz module which was already observed by M. Papanikolas in a yet unpublished paper. Namely, these coordinates can be obtained by explicit polynomial expressions in $\tilde{\pi}$ and its hyperderivatives. Papanikolas also gave various presentations of these expressions which we also prove here.

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Introduction

Periods of Drinfeld modules and Anderson $t$-modules play a central role in number theory in positive characteristic, and questions about their algebraic independence are of major interest. The most prominent period is the Carlitz period

$$\tilde{\pi} = \lambda_\theta \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in K_{\infty}(\lambda_\theta),$$

where $\lambda_\theta \in K^{\text{sep}}$ is a $(q-1)$-th root of $-\theta$. Here, $K = \mathbb{F}_q(\theta)$ is the rational function field over the finite field $\mathbb{F}_q$, $K^{\text{sep}}$ its separable algebraic closure, and $K_{\infty} = \mathbb{F}_q((\frac{1}{\theta}))$ is the completion of $K$ with respect to the absolute value $| \cdot |_{\infty}$ given by $|\theta|_{\infty} = q$.

The Carlitz period is the function field analog of the complex number $2\pi i$, and it was already proven by Wade in 1941 that $\tilde{\pi}$ is transcendental over $K$ (see [Wad41]).

On the field $K_{\infty}(\lambda_\theta)$, one can consider the hyperdifferential operators with respect to $\theta$, denoted by $\partial_{\theta}^{(n)}$, $n \geq 0$, which are defined for $\sum_{i=i_0}^{\infty} c_i \theta^{-i} \in K_{\infty} = \mathbb{F}_q((\frac{1}{\theta}))$ by

$$\partial_{\theta}^{(n)} \left( \sum_{i=i_0}^{\infty} c_i \theta^{-i} \right) = \sum_{i=i_0}^{\infty} c_i \left( -i \right)_{n} \theta^{-i-n},$$

and are uniquely extended to $K_{\infty}(\lambda_\theta)$.

Brownawell and Denis considered hyperderivatives of periods and quasi-periods of Drinfeld modules, and showed linear independence and algebraic independence
results for them (see [Den95], [Den07], [Bro99], [Den00], [BD00]). With respect to the hyperderivatives of the Carlitz period $\hat{\pi}$, they proved the following.

**Theorem.** (1) (see [BD00, Thm. 1.1], case $d = 1$)

The elements $1, \hat{\pi}, \partial^{(1)}_{\theta}(\hat{\pi}), \partial^{(2)}_{\theta}(\hat{\pi}), \ldots$ are $K$-linearly independent, where $K$ is an algebraic closure of $K$.

(2) (see [Den00, Thm. 1])

The elements $\hat{\pi}, \partial^{(1)}_{\theta}(\hat{\pi}), \ldots, \partial^{(p-1)}_{\theta}(\hat{\pi})$ are algebraically independent over $K$, where $p = \text{char}(\mathbb{F}_q)$ is the characteristic of $\mathbb{F}_q$.

In this paper, we prove a much stronger result.

**Theorem 0.1.** (see Theorem 2.1)

The Carlitz period $\hat{\pi}$ is hypertranscendental over $K = \mathbb{F}_q[\theta]$, i.e. the set \{ $\partial^{(n)}_{\theta}(\hat{\pi})$ \mid $n \geq 0$\} is algebraically independent over $K$.

Recently, Namoijam proved an even more general result on hypertranscendence of periods and quasi-periods of Drinfeld modules using differential algebraic geometry in positive characteristic (see [Nam21, Thm. 1.1.5]). In the case of the Carlitz module, her Theorem 1.1.5 specializes to Theorem 2.1. In the second main theorem of [Nam21], Namoijam extended Theorem 1.1.5 further to also include logarithms of algebraic points.

Our proof of Theorem 2.1 is similar to the proof of Theorem 8.1 in [Mau18] where we proved algebraic independence for the coordinates of a fundamental period $\bar{\pi}_n = (z_1, \ldots, z_n)^{\theta}$ of the $n$-th Carlitz tensor power $C^\otimes n$ ($n \geq 1$) if $n$ is prime to the characteristic. Furthermore, these two proofs reveal a link between the hyperderivatives of $\hat{\pi}$ and the coordinates of $\bar{\pi}_n$.

**Theorem 0.2.** (see Thm. 3.1)

The coordinates $z_1, \ldots, z_n$ belong to the $K$-vector space generated by the set of "monomials" \{ $\prod_{j=1}^{n} \partial^{(m_j)}_{\theta}(\hat{\pi})$ \mid $0 \leq m_j \leq n - 1$\}.

The proof even provides an explicit description for these coordinates.

Such a link between the coordinates of $\bar{\pi}_n$ and the hyperderivatives of $\hat{\pi}$ has already been discovered by Papanikolas using [Mau18, Lemma 8.3] and results from a yet unpublished manuscript [Pap15]. Papanikolas obtained the following explicit description. For stating it, we consider the $t$-linear extensions of the hyperdifferential operators $\partial^{(n)}_{\theta}$ to $K_\infty(\lambda_\theta)(t)$ which we still denote by $\partial^{(n)}_{\theta}$.

**Theorem 0.3.** (Papanikolas)

Let $\bar{\pi}_n$ be chosen such that its last coordinate $z_n$ equals $\bar{\pi}_n$. For $l \geq 1$, let $\alpha_l(t) \in \mathbb{F}_q[\theta, t]$ be the $l$-th Anderson-Thakur polynomial, and $\Gamma_l$ the $l$-th Carlitz factorial. Then for all $j = 0, \ldots, n - 1$, one has

$$z_{n-j} = \partial^{(j)}_{\theta} \left( \frac{\alpha_l(t)}{\Gamma_l}, \bar{\pi}_n \right)_{t=\theta}.$$

(See Definition 4.6 for precise definitions).

The proof of this identity, however, is quite long, and we give a shorter proof for it in Section 4. The main improvement with respect to Papanikolas’ proof is the following new identity in the ring $K\{\{t - \theta\}\}$ of convergent power series in $(t - \theta)$.

**Theorem 0.4.** (see Thm. 4.8 (1))

For $m \geq 0$, let $\tau_m(t) = \prod_{k=1}^{m} (\theta^m - t^k) \in \mathbb{F}_q[\theta, t]$, and $D_m = \prod_{k=0}^{m-1} (\theta^m - \theta^k) \in \mathbb{F}_q[\theta]$. Further, let

$$\eta = \sum_{m=1}^{\infty} \left( 1 + \frac{(t - \theta)^{\gamma^m}}{\theta^m - \theta} \right) \in K\{\{t - \theta\}\}.$$
Then
\[\sum_{j=0}^{\infty} \frac{\gamma_j(t)}{D_j} \cdot t^j = 1 \in K\{t - \theta\}\].

Other improvements were obtained by a consequent use of Taylor series expansions and identities for such expansions.

In Section 1, we set up the notation and explain the basics on hyperdifferential operators and Taylor series expansions which are used later. Section 2 is devoted to our main theorem and its proof. We continue by providing the connection to the coordinates of the period of Carlitz tensor powers in Section 3. In Section 4, we present our proofs of the explicit expressions for these period coordinates.

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1. Notation and calculation rules

1.1. Basic rings and fields. Let $\mathbb{F}_q$ be the finite field with $q$ elements and characteristic $p$, and $K = \mathbb{F}_q(\theta)$ the rational function field in the variable $\theta$. We equip $K$ with the absolute value $| \cdot |_\infty$ which is given by $|\theta|_\infty = q$. We further denote by $t$ a second indeterminate. All rings and fields occurring will be extensions of $K$ or of $K[t]$. These are

- $K_{\infty} = \mathbb{F}_q((\frac{1}{t}))$ the completion of $K$ at this infinite place,
- $K_{\text{sep}}^\infty$ a separable algebraic closure of $K_{\infty}$ with the extension of the absolute value $| \cdot |_\infty$,
- $K_{\text{sep}}$ the separable algebraic closure of $K$ inside $K_{\text{sep}}^\infty$,
- $K(t)$ the rational function field over $K$,
- $K[t, t-\theta]$ the localization of $K[t]$ at the prime ideal $(t - \theta)$, i.e. the rational functions in $K(t)$ which are regular at $t = \theta$,
- $K_{\text{sep}}^\infty[t]$ the power series ring in $t$ with coefficients in $K_{\text{sep}}^\infty$,
- $T_\theta$ the subring of $K_{\text{sep}}^\infty[t]$ consisting of series which converge on the closed disc of radius $|\theta|_\infty = q$.
- $E$ the subring of $K_{\text{sep}}^\infty[t]$ consisting of entire functions, i.e. of those series $f(t)$ for which $f(x)$ converges for any $x \in K_{\text{sep}}^\infty$, and whose coefficients lie in a finite extension of $K_{\infty}$,
- $K\{t - \theta\}$ the ring of power series $\sum_{n=0}^{\infty} a_n(t - \theta)^n$ with coefficients in $K$, and whose coefficients $a_n$ tend to zero as $n$ goes to infinity.

For elements $f$ in all these rings, we sometimes write $f(t)$ to emphasize the dependence on $t$, or even $f(\theta, t)$, to emphasize both dependencies. If we replace the variable $t$ of such a function $f$ by $z$ for some element $z$, we will denote this by $f|_{t=z}$ or by $f(\theta, z)$. Similarly, if we replace the variable $\theta$ by $z$, we write $f|_{\theta=z}$ or $f(z, t)$.

1.2. Hyperdifferential operators. In this subsection, we give a short presentation of hyperdifferential operators, also called higher derivations (see e.g. [Mat89, §27], or [Con00]).

**Definition 1.1.** A higher derivation on an $\mathbb{F}_q$-algebra $R$ is a family of $\mathbb{F}_q$-linear operators $(\partial^{(n)} : R \rightarrow R)_{n \geq 0}$ such that

1. $\partial^{(0)} = \text{id}_R$,  
2. $\forall n \geq 0, \forall f, g \in R: \partial^{(n)}(f \cdot g) = \sum_{i+j=n} \partial^{(i)}(f) \cdot \partial^{(j)}(g)$ (generalized Leibniz rule)

**Remark 1.2.** By definition, the hyperdifferential operator $\partial := \partial^{(1)}$ is a derivation, and $\partial^{(n)}(f)$ corresponds to $\frac{1}{n!} \partial^n(f)$ in characteristic zero.
For many calculations, it is much more convenient to consider the corresponding map

$$\mathcal{D} : R \rightarrow R[[X]], f \mapsto \sum_{n \geq 0} \partial^{(n)}(f) X^n,$$

also called the (generic) Taylor series expansion. Namely, the $\mathbb{F}_q$-linearity and the general Leibniz rule imply that the map $\mathcal{D}$ is a homomorphism of $\mathbb{F}_q$-algebras (actually is equivalent to that condition). In particular, $\mathcal{D}$ is determined by the images of generators of the $\mathbb{F}_q$-algebra $R$.

**Proposition 1.3.** (see [Con00, Thm. 5 & 6])

Higher derivations can be uniquely extended to localizations and to separable algebraic field extensions.

If $R$ is equipped with an absolute value, and all $\partial^{(n)}$ are continuous, the $\partial^{(n)}$ and also $\mathcal{D}$ can be continuously extended to the completion of $R$ with respect to the absolute value in a unique way.

The higher derivations/hyperdifferential operators that are relevant in this paper are the following.

**Example 1.4.** On the polynomial ring $\mathbb{F}_q[\theta, t]$, we consider two families of hyperdifferential operators, the hyperdifferential operators $(\partial^{(n)}_\theta)_{n \geq 0}$ with respect to $\theta$ – with corresponding generic Taylor series expansion $\mathcal{D}_\theta$ –, and the hyperdifferential operators $(\partial^{(n)}_t)_{n \geq 0}$ with respect to $t$ – with corresponding generic Taylor series expansion $\mathcal{D}_t$. They are determined by

$$\mathcal{D}_\theta(\theta) = \theta + X, \mathcal{D}_\theta(t) = t \quad \text{and} \quad \mathcal{D}_t(\theta) = \theta, \mathcal{D}_t(t) = t + X.$$

Explicitly, $(\partial^{(n)}_\theta)_{n \geq 0}$ and $(\partial^{(n)}_t)_{n \geq 0}$ are given by

$$\partial^{(n)}_\theta \left( \sum_{i,j} c_{ij} \theta^it^j \right) = \sum_{i,j} \binom{i}{n} c_{ij} \theta^{-n} t^j,$$

$$\partial^{(n)}_t \left( \sum_{i,j} c_{ij} \theta^it^j \right) = \sum_{i,j} \binom{j}{n} c_{ij} \theta^k t^{-n}.$$
(2) If we can evaluate $t$ at $\theta$, i.e. $R$ is a subring of $K[[t^{-\theta}]]$, $\mathbb{T}_\theta$ or $K[[t-\theta]]$, then for all $n \geq 0$:

$$\partial^{(n)}_{\theta}(f(\theta, t)) = \sum_{i+j=n} \partial^{(i)}_{\theta}\bigg(\partial^{(j)}_{\theta}(f(\theta, t))\bigg)|_{t=\theta}.$$  

Equivalently, in terms of $\mathcal{D}_\theta$ and $\mathcal{D}_t$:

$$\mathcal{D}_\theta(f(\theta, t)|_{t=\theta}) = \mathcal{D}_t(\mathcal{D}_\theta(f(\theta, t))|_{t=\theta}).$$

Proof. We check the equations using $\mathcal{D}_t$ and $\mathcal{D}_\theta$. For the first part one has:

$$\mathcal{D}_\theta(\mathcal{D}_t(f(\theta, t))) = \mathcal{D}_\theta(f(\theta, t + X)) = f(\theta + X, t + X) = \mathcal{D}_t(\mathcal{D}_\theta(f(\theta, t))).$$

For the second part:

$$\mathcal{D}_\theta(f(\theta, t)|_{t=\theta}) = f(\theta + X, \theta + X) = f(\theta + X, t + X)|_{t=\theta} = \mathcal{D}_t(\mathcal{D}_\theta(f(\theta, t))|_{t=\theta}).$$

We will often need the hyperdifferential operators just up to some bound $n$, i.e. $\partial^{(0)}, \partial^{(1)}, \ldots, \partial^{(n)}$. For this it is convenient to transform the corresponding Taylor series homomorphism $\mathcal{D}$ to a homomorphism into a matrix ring.

**Definition 1.7.** (cf. [Mau18] or [MP19])

Let $\mathcal{D} : R \rightarrow R[X]$ be the Taylor series homomorphism corresponding to a higher derivation $(\partial^{(k)})_{k \geq 0}$ on $R$, let $n \geq 0$, and let $N \in \text{Mat}_{(n+1) \times (n+1)}(R)$ be the nilpotent matrix

$$N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}.$$  

We define the ring homomorphism $\rho_{[n]} : R \rightarrow \text{Mat}_{(n+1) \times (n+1)}(R)$ to be the composition of $\mathcal{D}$ with the evaluation homomorphism replacing $X$ by $N$, i.e.

$$\rho_{[n]}(f) := \mathcal{D}(f)|_{X=N} = \begin{pmatrix}
f & \partial^{(1)}(f) & \cdots & \partial^{(n)}(f) \\
0 & f & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \partial^{(1)}(f) \\
0 & \cdots & \cdots & 0 & f
\end{pmatrix}. $$

In case of the hyperdifferential operators with respect to $\theta$ and $t$, we add the variable as subscript, i.e. we denote

$$\rho_{\theta,n}(f) := \mathcal{D}_\theta(f)|_{X=N} \quad \text{as well as} \quad \rho_{t,n}(f) := \mathcal{D}_t(f)|_{X=N}.$$

2. Hypertranscendence of the Carlitz period

The main result of this article is on the Carlitz period

$$\tilde{\pi} = \lambda_0^q \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in K_\infty(\lambda_0),$$

where $\lambda_0 \in K^{sep}$ is a $(q - 1)$-th root of $-\theta$, as well as on its hyperderivatives with respect to $\theta$.

**Theorem 2.1.** The Carlitz period $\tilde{\pi}$ is hypertranscendental over $K = \mathbb{F}_q(\theta)$, i.e. the set $\{\partial^{(n)}_{\theta}(\tilde{\pi}) \mid n \geq 0\}$ is algebraically independent over $K$. 

The proof of this theorem will take up the whole section. For the proof, we need the rigid analytic trivialization of the dual Carlitz motive. This is the $\Omega$-function,

$$\Omega(t) = \lambda_q^{-q} \prod_{j \geq 1} (1 - \frac{t}{\theta^q}) \in K_\infty(\lambda_q)[[t]],$$

which is an entire function, and satisfies

$$\Omega(\theta) = -\frac{1}{\pi}.$$

**Lemma 2.2.** For all $j \geq 0$, there is $b_j \in K[[t]]_{l_\theta}$ such that $\partial_{\theta}^{(j)}(\Omega^{-1}) = b_j \cdot \Omega^{-1}$. These $b_j$ are given as $b_0 = 1$, and for $j > 0$,

$$b_j = \prod_{i=1}^{l-1} (\theta^q - t) \cdot \partial_{\theta}^{(j)} \left( \prod_{i=1}^{l-1} (\theta^q - t)^{-1} \right),$$

where $l = \lfloor \log_q(j) \rfloor + 1$.\textsuperscript{1}

**Proof.** The case $j = 0$ is trivial. So let $j > 0$ and let $l \in \mathbb{N}$ such that $q^l > j$ (e.g. $l = \lfloor \log_q(j) \rfloor + 1$). As $\partial_{\theta}^{(j)}(f^q) = 0$ for all $f \in K_\text{sep}^\ast$, $0 < i < q^l$\textsuperscript{2}, one obtains from the generalized Leibniz rule

$$\partial_{\theta}^{(j)}(\Omega^{-1}) = \partial_{\theta}^{(j)} \left( \lambda_q^{-q} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q}) \cdot \lambda_q^q \prod_{i=1}^{\infty} (1 - \frac{t}{\theta^q})^{-1} \right)$$

$$= \partial_{\theta}^{(j)} \left( \lambda_q^{-q} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q})^{-1} \right) \cdot \lambda_q^q \prod_{i=1}^{\infty} (1 - \frac{t}{\theta^q})^{-1}$$

$$= \partial_{\theta}^{(j)} \left( \lambda_q^{-q} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q}) \right) \cdot \lambda_q^{q^l-q} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q}) \cdot \Omega^{-1}.$$

As $q^l - q$ is divisible by $q - 1$, we have $\lambda_q^{q^l-q} = (-\theta)^{\frac{q^l-q}{q-1}} \in K$, and hence, $b_j = (-\theta)^{\frac{q^l-q}{q-1}} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q}) \cdot \partial_{\theta}^{(j)} \left( (-\theta)^{\frac{q^l-q}{q-1}} \prod_{i=1}^{l-1} (1 - \frac{t}{\theta^q})^{-1} \right) \in K[[t]]_{l_\theta}$. Indeed, we can write $b_j$ more simply, since

$$\prod_{i=1}^{l-1} \left( 1 - \frac{t}{\theta^q} \right) = \frac{\prod_{i=1}^{l-1} \theta^{q^l-t}}{\theta^{q^{l-1}+q^l-1}} = \frac{\prod_{i=1}^{l-1} (\theta^q - t)}{\theta^{q^{l-1}-t}}.$$

Hence, after canceling out the $(-1)^{\frac{q^l-q}{q-1}}$, we obtain

$$b_j = \prod_{i=1}^{l-1} (\theta^q - t) \cdot \partial_{\theta}^{(j)} \left( \prod_{i=1}^{l-1} (\theta^q - t)^{-1} \right).$$

The connection of the hyperderivatives $\partial_{\theta}^{(j)}(\Omega^{-1})$ with the hyperderivatives $\partial_{\theta}^{(j)}(\bar{\pi})$ is given by the following proposition.

**Proposition 2.3.** For all $n \geq 0$, the sum $\partial_{\theta}^{(n)}(\bar{\pi}) + \partial_{\theta}^{(n)}(\Omega^{-1}) \vert_{t=\theta}$ lies in the $K$-span of the elements $\partial_{\theta}^{(j)}(\Omega^{-1}) \vert_{t=\theta}$ for $0 \leq j < n$.\textsuperscript{1}

\textsuperscript{1}As usual, empty products are always considered to be equal to 1.

\textsuperscript{2}This follows immediately from $\mathcal{D}_{\theta}(f^q) = \mathcal{D}_{\theta}(f)^q \in K_\text{sep}^\ast[X^q]$. 
Proof. Combining Lemma 1.6 and Lemma 2.2, and the identity $\Omega^{-1}|_{t=\theta} = -\tilde{\pi}$, one computes:

$$-\partial^\iota(t)(\tilde{\pi}) = \sum_{i+j=n} \partial^\iota(t)(\partial^\iota(t)(\Omega)) |_{t=\theta}$$

$$= \sum_{i+j=n} \partial^\iota(t)(b_j \cdot \Omega^{-1}) |_{t=\theta}$$

$$= \left( \sum_{i+j=n} \sum_{i_1+i_2=i} \partial^\iota(t)(b_{i_1}) \partial^\iota(t)(\Omega^{-1}) \right) |_{t=\theta}$$

$$= \partial^\iota(t)(\Omega^{-1}) |_{t=\theta} + \left( \sum_{i_1+i_2=n-j} \partial^\iota(t)(b_{i_1}) \partial^\iota(t)(\Omega^{-1}) \right) |_{t=\theta}.$$

Remark 2.4. We can rewrite the last line using $\mathcal{D}_t$ and $\mathcal{D}_\theta$. For that we write $B(X) := \mathcal{D}_t(\Omega^{-1}) \cdot \Omega = \sum_{j=0}^{\infty} b_j X^j \in K[t][t-\theta][X]$, where $b_j \in K[t][t-\theta]$ as in Lemma 2.2. Then:

$$-\mathcal{D}_\theta(\pi) = \mathcal{D}_t(\mathcal{D}_\theta(\Omega^{-1})) |_{t=\theta} = \mathcal{D}_t(B(X) \cdot \Omega^{-1}) |_{t=\theta} = \mathcal{D}_t(B(X)) \cdot \mathcal{D}_t(\Omega^{-1}) |_{t=\theta}.$$

Proof of Thm. 2.1. For the proof, we show that for any $n \geq 0$, the transcendence degree of $K(\tilde{\pi}, \partial^\iota(\tilde{\pi}), \ldots, \partial^\iota(\tilde{\pi}))$ over $K$ is $n + 1$.

By [Mau18, Prop. 6.2], the matrix

$$\Psi = \rho_{t, [n]}(\Omega) = \begin{pmatrix} \Omega & \partial^1(t)(\Omega) & \partial^2(t)(\Omega) & \cdots & \partial^n(t)(\Omega) \\ 0 & \Omega & \partial^1(t)(\Omega) & \cdots & \partial^2(t)(\Omega) \\ \vdots & \ddots & \ddots & \ddots & \partial^2(t)(\Omega) \\ \vdots & \ddots & \ddots & \ddots & \partial^2(t)(\Omega) \\ 0 & \cdots & \cdots & \cdots & \Omega \end{pmatrix}$$

is the rigid analytic trivialization of the dual $t$-motive corresponding to the $n$-th prolongation of the Carlitz module.

The transcendence degree of $K(t)(\Psi)$ over $K(t)$ is the same as the one when adjoining to $K(t)$ the entries of $\rho_{t, [n]}(\omega) = \rho_{t, [n]}(\frac{1}{(t-\theta)\omega} ) = \rho_{t, [n]}(t-\theta)^{-1} \cdot \Psi^{-1}$.

This transcendence degree equals $n + 1$ by [Mau18, Thm. 7.2].

By a theorem of Papanikolas (cf. [Pap08, Thm. 5.2.2] and its proof; see also a refinement of Chang in [Cha09, Thm. 1.2(2)]), the transcendence degree of $K(t)(\Psi)$ over $K(t)$ is the same as the transcendence degree of $\Psi(\theta)$ over $K$. The latter is the same when adjoining the entries of

$$\Psi(\theta)^{-1} = \rho_{t, [n]}(\Omega^{-1}) |_{t=\theta},$$

i.e. adjoining the elements $\Omega^{-1}|_{t=\theta}, \partial^1(t)(\Omega^{-1}) |_{t=\theta}, \ldots, \partial^n(t)(\Omega^{-1}) |_{t=\theta}$. Finally by Prop. 2.3, the elements $\tilde{\pi}, \partial^1(\tilde{\pi}), \ldots, \partial^n(\tilde{\pi})$ span the same $K$-vector space, hence the fields generated by these sets of elements are the same, and we conclude that the transcendence degree of $K(\tilde{\pi}, \partial^1(\tilde{\pi}), \ldots, \partial^n(\tilde{\pi}))$ over $K$ is indeed $n + 1$. □
3. Period coordinates of Carlitz tensor powers

Let $n$ be a positive integer, and let $C^\otimes_n$ denote the $n$-th Carlitz tensor power. In [Mau18], we showed an algebraic independence result for the coordinates of a fundamental period $\tilde{\pi}_n = (z_1, \ldots, z_n)^\text{tr}$ of $C^\otimes_n$. It was obtained in a similar manner as for the hyperderivatives of $\tilde{\pi}$ in the previous section. Besides using Papanikolas’ theorem [Pap08, Thm. 5.2.2], the main point was the following identity, where we chose the fundamental period such that its last coordinate $z_n$ equals $\tilde{\pi}^n$:

$$\rho_{t,[n-1]}(\Omega^{-n})|_{t=\theta} = (-1)^n \cdot \begin{pmatrix} z_n & z_{n-1} & \cdots & z_1 \\ 0 & z_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & z_n \end{pmatrix}.$$  \hspace{1cm} (3)

As one can already imagine, this provides a link between these coordinates and the hyperderivatives of $\tilde{\pi}$.

**Theorem 3.1.** The coordinates $z_1, \ldots, z_n$ belong to the $K$-vector space generated by the set of “monomials” $\left\{ \prod_{j=1}^n \partial^{(m_j)}(\tilde{\pi}) \right\} \forall j : 0 \leq m_j \leq n - 1$.

**Proof.** Equation (3) can be rewritten as:

$$z_n + z_{n-1}X + \ldots + z_1X^{n-1} \equiv (-1)^n \mathcal{D}_t(\Omega^{-n})|_{t=\theta} \mod X^n.$$  

By Remark 2.4, we further have:

$$(-1)^n \mathcal{D}_t(\Omega^{-n})|_{t=\theta} = \mathcal{D}_t(-\Omega^{-n})|_{t=\theta} = \mathcal{D}_t(B(X))^{-n} \mathcal{D}_\theta(\tilde{\pi})^n|_{t=\theta} = \mathcal{D}_t(B(X))^{-n}|_{t=\theta} \cdot \mathcal{D}_\theta(\tilde{\pi})^n.$$  

As the coefficients of $B(X)$ are in $K[t]|_{(t-\theta)}$, hence $\mathcal{D}_t(B(X))^{-n}|_{t=\theta} \in K[X]$, and as the non-zero entries of $\mathcal{D}_\theta(\tilde{\pi})$ up to the coefficients of $X^{-n}$ are the elements $\tilde{\pi}, \partial^{(1)}(\tilde{\pi}), \ldots, \partial^{(n-1)}(\tilde{\pi})$, this completes the proof. \blacksquare

4. Expressions for the period coordinates

We are going to derive nicer expressions for the coefficients $\mathcal{D}_t(B(X)^{-n})|_{t=\theta}$ occurring in the proof of Theorem 3.1. We start by deriving a nicer expression for $\mathcal{D}_t(B(X))|_{t=\theta} \mod X^n$.

**Definition 4.1.** For $l \geq 0$, we let

$$L_l = \prod_{m=1}^l (\theta^q^m - \theta) \in K \quad \text{and} \quad \mathcal{L}_l(t) = \prod_{m=1}^l (\theta^q^m - t) \in K[t].$$  

Furthermore, we define the elements

$$\eta_l(t) = \prod_{m=1}^l \frac{t^q^m - \theta}{\theta^q^m - \theta} = \prod_{m=1}^l \left( 1 + \frac{(t - \theta)q^m}{\theta^q^m - \theta} \right) \in K[t]|_{(t-\theta)},$$

as well as their limit in $K\{ (t-\theta) \}$,

$$\eta(t) := \lim_{l \to \infty} \eta_l(t) = \prod_{m=1}^\infty \left( 1 + \frac{(t - \theta)q^m}{\theta^q^m - \theta} \right) \in K\{ (t-\theta) \}.$$
Remark 4.2. The element $\eta(t)$ is the inverse of $\xi_C(t)$ in [Pap15] where the formula above was a consequence of a different definition (cf. [Pap15, Prop. 7.2.2]). Papanikolas’ definition in terms of $\eta(t)$ merely is

$$\eta(t) = \frac{\Omega(t, \theta)}{\Omega(\theta, \theta)} = -\tilde{\pi} \cdot \Omega(t, \theta)$$

viewed in some appropriate ring. This definition shows its close connection to the Carlitz module. Note that the roles of $t$ and $\theta$ in the numerator are interchanged in that definition.

Using Equation (3) and a swap of $t$ and $\theta$, Papanikolas deduced from this definition the equation

$$z_{n-j} = \partial_\theta^{(j)} (\eta^{-n} \cdot \tilde{\pi}^n) |_{t=\theta}.$$ 

This is the limit version of our identity in Corollary 4.5 below.

Lemma 4.3. For all $l \in \mathbb{N}$:

$$\mathcal{D}_\theta(\eta_l)|_{t=\theta} = \frac{\mathcal{D}_\theta(L_l(t))|_{t=\theta}}{\mathcal{D}_\theta(L_l)}. $$

Proof. On one hand, we have

$$\mathcal{D}_\theta(\eta_l)|_{t=\theta} = \mathcal{D}_\theta \left( \prod_{m=1}^l \frac{\theta^m - \theta}{\theta^m - \theta} \right) |_{t=\theta} = \prod_{m=1}^l \frac{\theta^m - \theta - X}{\theta^m + X \theta^m - \theta - X}. $$

On the other hand,

$$\frac{\mathcal{D}_\theta(L_l(t))|_{t=\theta}}{\mathcal{D}_\theta(L_l)} = \frac{\prod_{i=1}^l \mathcal{D}_\theta(\theta^{i'} - t)}{\prod_{i=1}^l \mathcal{D}_\theta(\theta^{i'} - \theta)} |_{t=\theta} = \frac{\prod_{i=1}^l \left( \theta^{i'} - (t + X) \right)}{\prod_{i=1}^l \left( \theta^{i'} - \theta - X \right)} |_{t=\theta} = \prod_{m=1}^l \frac{\theta^m - \theta - X}{\theta^m + X \theta^m - \theta - X}. $$



Proposition 4.4. If $q^l \geq n$, then

$$\mathcal{D}_t(B(X))|_{t=\theta} \equiv \mathcal{D}_\theta(\eta_{l-1})|_{t=\theta} \mod X^n. $$

Proof. Let $l \in \mathbb{N}$ such that $q^l \geq n$. Then from Equation (2), we see that

$$B(X) \equiv L_{l-1}(t) \cdot \mathcal{D}_\theta(L_{l-1}(t)^{-1}) = L_{l-1}(t) \cdot \mathcal{D}_\theta(L_{l-1}(t))^{-1} \mod X^n, $$

and hence modulo $X^n$ we obtain

$$\mathcal{D}_t(B(X))|_{t=\theta} \equiv \mathcal{D}_t(L_{l-1}(t))|_{t=\theta} \cdot \mathcal{D}_t(L_{l-1}(t))^{-1}|_{t=\theta} = \frac{\mathcal{D}_t(L_{l-1}(t))|_{t=\theta}}{\mathcal{D}_\theta(L_{l-1})} = \mathcal{D}_\theta(\eta_{l-1})|_{t=\theta}, $$

where we used Lemma 1.6 in the second last step.

Applying this congruence to the formula obtained in the proof of Thm. 3.1, we get the following corollary.
Corollary 4.5. Let \(z_1, \ldots, z_n\) be the coordinates of the period of \(C^\otimes n\) as in Equation (3), and \(l \in \mathbb{N}\) such that \(q^l \geq n\). Then
\[
\begin{pmatrix}
z_n & z_{n-1} & \cdots & z_1 \\
0 & z_n & \ddots & \vdots \\
\vdots & \ddots & \ddots & z_{n-1} \\
0 & \cdots & 0 & z_n
\end{pmatrix} = \rho_\emptyset[n^{-1}] (\eta_{n-1}^{-1} \tilde{\pi})^n \big|_{t=\theta} = \rho_\emptyset[n^{-1}] (\eta_{n-1}^{-1})^n \big|_{t=\theta} \rho_\emptyset[n^{-1}] (\tilde{\pi})^n,
\]
or equivalently for \(0 \leq j \leq n - 1\):
\[
z_{n-j} = \delta^{(j)} \eta_{n-j}^{-\alpha} \big|_{t=\theta}.
\]
Papanikolas discovered another nice expression using Anderson-Thakur polynomials and Carlitz factorials (see Thm. 0.3 or the second equation in Corollary 4.10). He derived it from the equation that we state as first equation in Corollary 4.10. His proof for that equation, however, is quite long and we managed to shorten it immensely. This is due to the first identity in Thm. 4.8 (already stated in the introduction as Thm. 0.4) from which the rest is deduced almost instantly. Nevertheless, the proof of Thm. 4.8 builds on ideas from Papanikolas’ proof.

Let’s recall the necessary definitions. We will use the notation as in Papanikolas’ manuscript which differ from the notation in [AT90].

Definition 4.6. For \(m \geq 0\), let
\[
\gamma_m(t) = \prod_{k=1}^{m} (\theta q^m - \theta^k) \in \mathbb{F}_q[\theta, t]
\]
where the empty product (case \(m = 0\)) is defined to be 1, as usual. Further, we let
\[
D_m = \prod_{k=0}^{m-1} (\theta q^m - \theta^k) \in \mathbb{F}_q[\theta],
\]
and \(\Gamma_m \in \mathbb{F}_q[\theta]\) be the Carlitz factorial, defined by
\[
\Gamma_m = \prod_{j=0}^{r} D_j^{m_j}
\]
where \(m = m_0 + m_1 q + \ldots + m_r q^r\) in base-\(q\) expansion (i.e. with \(m_j \in \{0, \ldots, q-1\}\)).

The Anderson-Thakur polynomials \(\alpha_n(t) \in K[t]\) are then defined by the generating series
\[
\sum_{n=1}^{\infty} \frac{\alpha_n(t)}{\Gamma_n} x^{n-1} = \left(1 - \sum_{j=0}^{\infty} \frac{\gamma_j(t)}{D_j} x^{q^j}\right)^{-1} \in K[t][x].
\]

Remark 4.7. From the definition of the Anderson-Thakur polynomials, one obtains a recursive formula for computing these polynomials: \(\alpha_1(t) = 1\), and for \(n \geq 2\),
\[
\frac{\alpha_n(t)}{\Gamma_n} = \sum_{j=0}^{\ell_{n-1}} \frac{\gamma_j(t)}{D_j} \frac{\alpha_{n-q^j}(t)}{\Gamma_{n-q^j}},
\]
where \(\ell_{n-1} := \lfloor \log_q(n-1) \rfloor\).

Furthermore, one has for \(n \geq 1\),
\[
\frac{\alpha_n(t)}{\Gamma_n} \equiv \left(\frac{\alpha_n(t)}{\Gamma_n}\right)^q
\]
(see [Pap15, Cor. 7.1.10]).

Theorem 4.8. Let \(\eta = \prod_{m=1}^{\infty} \left(1 + \frac{(t-\theta)^m}{p q^{m-\theta}}\right) \in K\{\theta \}\) as in Definition 4.1. We have the following identities in \(K\{\theta \}\):
\[
\frac{\alpha_n(t)}{\Gamma_n} = \left(\frac{\alpha_n(t)}{\Gamma_n}\right)^q
\]
Proof. The series in (1) is well defined in $K\{(t - \theta)\}$, since $\gamma_m(t)$ is divisible by $\theta^q - t^q = (\theta - t)^q$. First, we observe that for $m \geq 1$,

$$
\frac{\gamma_m(t)}{D_m} = \frac{\prod_{k=1}^{m}(\theta^q - t^q)}{\prod_{k=1}^{m}(\theta^q - \theta^q)} = \frac{(\theta - t)^q}{\theta^q - \theta} \cdot \prod_{k=1}^{m-1}(1 + \frac{\theta^q - t^q}{\theta^q - \theta^q})
$$

So the left hand side of the equation in (1) is

$$
\eta - \sum_{j=1}^{\infty} (t - \theta)^{\eta^j} \cdot \prod_{k=1}^{j-1}(1 + \frac{(t - \theta)^{\eta^j}}{\theta^q - \theta^q}) \cdot \prod_{m=1}^{\infty}(1 + \frac{(t - \theta)^{\eta^{m+j}}}{\theta^q - \theta^q})
$$

$$
= \eta - \sum_{j=1}^{\infty} (t - \theta)^{\eta^j} \cdot \prod_{k=1}^{j-1}(1 + \frac{(t - \theta)^{\eta^j}}{\theta^q - \theta^q}) \cdot \prod_{k=j+1}^{\infty}(1 + \frac{(t - \theta)^{\eta^j}}{\theta^q - \theta^q})
$$

$$
= \prod_{m=1}^{\infty}(1 + \frac{(t - \theta)^{\eta^m}}{\theta^q - \theta^q}) \cdot \sum_{j=1}^{\infty} (t - \theta)^{\eta^j} \cdot \prod_{k=1}^{s}(1 + \frac{(t - \theta)^{\eta^m}}{\theta^q - \theta^q})
$$

Hence, the only powers of $(t - \theta)$ that occur in this expression are $(t - \theta)^0$ (with coefficient 1) and $(t - \theta)^n$ where $n$ is of the form $n = q^{m_1} + q^{m_2} + \ldots + q^{m_s}$ with $s \geq 1$ and $0 < m_1 < m_2 < \ldots < m_s$. The coefficient of such an $n$ is:

$$
\prod_{i=1}^{s} \frac{1}{(\theta^q^{m_i} - \theta)} \cdot \sum_{i=1}^{s} \frac{(t - \theta)^{\eta^i}}{\theta^q - \theta^q} \prod_{k=1}^{s}(1 + \frac{(t - \theta)^{\eta^i}}{\theta^q - \theta^q})
$$

By Lemma 4.9 below, this coefficient is zero.

The second part is shown by induction on $n$.

If $n = 1$, we have $\frac{\alpha_n(t)}{\Gamma_n} = 1$ and

$$
\eta^{-1} \equiv 1 \mod (t - \theta)^q.
$$

Since $q \geq 2$, this shows the base case.

If $n > 1$, and $n$ is divisible by $q$, i.e. $n = q \cdot m$ with $m \geq 1$, then using the identity (5) and the induction hypothesis for $m$,

$$
\eta^{-n} - \frac{\alpha_n(t)}{\Gamma_n} = (\eta^{-m})^q = \left(\frac{\alpha_m(t)}{\Gamma_m}\right)^q = \left(\eta^{-m} - \frac{\alpha_m(t)}{\Gamma_m}\right)^q \equiv 0 \mod (t - \theta)^{(m+1)q}
$$

As $(m+1) \cdot q = qn + q > n + 1$, we are done in this case.
If \( n > 1 \) is not divisible by \( q \), we note that \( \ell_{n-1} = \lfloor \log_q(n-1) \rfloor = \lfloor \log_q(n) \rfloor = \ell_n \geq 0 \). By using the induction hypothesis for all \( m < n \), as well as the identities (1) and (4), we obtain:

\[
\eta^{-n} = \sum_{j=0}^{\ell_n-1} \frac{\gamma_j(t)}{D_j} q^{-(n-q^j)} = \sum_{j=0}^{\ell_n-1} \frac{\gamma_j(t)}{D_j} q^{-(n-q^j)} + \sum_{j=\ell_{n-1}+1}^{\infty} \frac{\gamma_j(t)}{D_j} q^{-(n-q^j)}
\]

\[
\equiv \sum_{j=0}^{\ell_n-1} \frac{\gamma_j(t)\alpha_{n-q^j}(t)}{\Gamma_{n-q^j}} \equiv \frac{\alpha_n(t)}{\Gamma_n} \mod (t-\theta)^m
\]

where \( m = \min\{q^{\ell_n+1}, q^{j}+(n-q^j)+1 \mid j = 0, \ldots, \ell_n \} = \min\{q^{\ell_n+1}, n+1 \} = n+1 \).

The third part is an immediate consequence of the second part, as \( \eta \equiv \eta \) modulo \( (t-\theta)^q \), and \( q > n+1 \).

\begin{lemma}
Let \( F \) be a field, and let \( a_1, \ldots, a_s, b \in F \) be pairwise distinct elements. Then

\[
\prod_{i=1}^{s} \frac{1}{(a_i-b)} - \sum_{i=1}^{s} \frac{1}{(a_i-b)} \prod_{k=1}^{s} \frac{1}{(a_k-a_i)} = 0.
\]

\end{lemma}

\begin{proof}
By Lagrange interpolation, we have the identity

\[
\sum_{i=1}^{s} \prod_{k=1}^{s} \frac{(a_k-x)}{(a_k-a_i)} = 1
\]

in the polynomial ring \( F[x] \), since \( a_1, \ldots, a_s \in F \) are pairwise distinct.

Dividing by \( \prod_{i=1}^{s} \frac{1}{(a_i-x)} \) and evaluating at \( b \notin \{a_1, \ldots, a_s\} \) leads to the desired result. \( \square \)

From the previous theorem and Corollary 4.5, we immediately get the following corollary

\begin{corollary}
For all \( n \geq 1 \) and \( 0 \leq j \leq n \), we have

\[
\partial_{\theta}^{(j)}(\eta^{-n}) \big|_{t=\theta} = \partial_{\theta}^{(j)}\left(\frac{\alpha_n(t)}{\Gamma_n}\right) \big|_{t=\theta},
\]

as well as

\[
z_{n-j} = \partial_{\theta}^{(j)}\left(\frac{\alpha_n(t)}{\Gamma_n} \tilde{\pi}^n\right) \big|_{t=\theta},
\]

where \( z_{n-j} \) is the coordinate of the period of Carlitz tensor power as given in Equation (3).

\end{corollary}

\begin{proof}
For all \( m > j \geq 0 \), and \( f \in K[[t-\theta]] \), the hyperderivative \( \partial_{\theta}^{(j)}((t-\theta)^m \cdot f) \) is divisible by \( (t-\theta) \), and so \( \partial_{\theta}^{(j)}((t-\theta)^m \cdot f) \big|_{t=\theta} = 0 \). Therefore, if two elements \( h, g \in K[[t-\theta]] \) are congruent modulo \( (t-\theta)^{n+1} \), then for all \( 0 \leq j \leq n \):

\[
\partial_{\theta}^{(j)}(h) \big|_{t=\theta} = \partial_{\theta}^{(j)}(g) \big|_{t=\theta}.
\]

Therefore, by Theorem 4.8, we have for all \( 0 \leq j \leq n \):

\[
\partial_{\theta}^{(j)}(\eta_{n-j}^{-1}) \big|_{t=\theta} = \partial_{\theta}^{(j)}(\eta^{-n}) \big|_{t=\theta} = \partial_{\theta}^{(j)}\left(\frac{\alpha_n(t)}{\Gamma_n}\right) \big|_{t=\theta},
\]

as well as (using also Corollary 4.5)

\[
z_{n-j} = \partial_{\theta}^{(j)}(\eta_{n-j}^{-1} \tilde{\pi}^n) \big|_{t=\theta} = \partial_{\theta}^{(j)}(\eta^{-n} \tilde{\pi}^n) \big|_{t=\theta} = \partial_{\theta}^{(j)}\left(\frac{\alpha_n(t)}{\Gamma_n} \tilde{\pi}^n\right) \big|_{t=\theta}
\]

where as before \( q' > n \).

\( \square \)
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