Non-rationality of the symmetric sextic
Fano threefold

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Pour Gerard Van der Geer, en l’honneur de son 60ème anniversaire

Abstract. We prove that the symmetric sextic Fano threefold, defined by the equations \( \sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \) in \( \mathbb{P}^6 \), is not rational. In view of the work of Prokhorov \([P]\), our result implies that the alternating group \( \mathfrak{A}_7 \) admits only one embedding into the Cremona group \( \text{Cr}_3 \) up to conjugacy.

Résumé. Nous prouvons que le solide de Fano d’équations \( \sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \) dans \( \mathbb{P}^6 \) n’est pas rationnel. Grâce aux résultats de Prokhorov \([P]\), cela entraîne que le groupe alterné \( \mathfrak{A}_7 \) admet un seul plongement (à conjugaison près) dans le groupe de Cremona à 3 variables.

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Introduction

The symmetric sextic Fano threefold is the subvariety \( X \) of \( \mathbb{P}^6 \) defined by the equations

\[
\sum X_i = \sum X_i^2 = \sum X_i^3 = 0 .
\]

It is a smooth complete intersection of a quadric and a cubic in \( \mathbb{P}^5 \), with an action of \( \mathfrak{S}_7 \). We will prove that it is not rational.

Any smooth complete intersection of a quadric and a cubic in \( \mathbb{P}^5 \) is unirational \([E]\). It is known that a general such intersection is not rational: this is proved in \([B]\) (thm. 5.6) using the intermediate Jacobian, and in \([Pu]\) using the group of birational automorphisms. But neither of these methods allows to prove the non-rationality of any particular such threefold. Our method gives the above explicit (and very simple) counter-example to the Lüroth problem.

Our motivation comes from the recent paper of Prokhorov \([P]\), which classifies the simple finite subgroups of the Cremona group \( \text{Cr}_3 = \text{Bir}(\mathbb{P}^3) \). In view of this work our result implies that the alternating group \( \mathfrak{A}_7 \) admits only one embedding into \( \text{Cr}_3 \) up to conjugacy.

Our proof uses the Clemens-Griffiths criterion \((C-G, \text{Cor. 3.26})\): if \( X \) is rational, its intermediate Jacobian \( JX \) is the Jacobian of a curve, or a product of Jacobians. The presence of the automorphism group \( \mathfrak{S}_7 \), together with the celebrated bound \( \# \text{Aut}(C) \leq 84(g-1) \) for a curve \( C \) of genus \( g \), immediately implies
that $JX$ is not isomorphic to the Jacobian of a curve. To rule out products of Jacobians we need some more information, which is provided by a simple analysis of the representation of $S_7$ on the tangent space $T_0(JX)$.

**Proof of the result**

**Theorem.** The intermediate Jacobian $JX$ is not isomorphic to a Jacobian or a product of Jacobians. As a consequence, $X$ is not rational.

The second assertion follows from the first by the Clemens-Gri
gths criterion mentioned in the introduction. Since the Jacobians and their products form a closed subvariety of the moduli space of principally polarized abelian varieties, this gives an easy proof of the fact that a general intersection of a quadric and a cubic in $\mathbb{P}^5$ is not rational.

As mentioned in the introduction, the classification in [P] together with the theorem implies:

**Corollary.** Up to conjugacy, there is only one embedding of $\mathfrak{A}_7$ into the Cremona group $Cr_3$, given by an embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$.

(The embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$ is the composition of the standard representation $\mathfrak{A}_7 \to \text{SO}_6(\mathbb{C})$ and the double covering $\text{SO}_6(\mathbb{C}) \to \text{PGL}_4(\mathbb{C})$.

The intermediate Jacobian $JX$ has dimension 20. The group $S_7$ acts on $JX$ and therefore on the tangent space $T_0(JX)$; we will first determine this action.

**Lemma.** As a $S_7$-module $T_0(JX)$ is the sum of two irreducible representations, of dimensions 6 and 14.

**Proof.** Let $V$ be the standard (6-dimensional) representation of $S_7$, and put $\mathbb{P} := \text{P}(V)$; we will view $X$ as a subvariety of $\mathbb{P}$, stable under $S_7$.

By definition $T_0(JX)$ is $H^2(X,\Omega^1_X)$. Every $S_7$-module is isomorphic to its dual, so we can identify $T_0(JX)$ with $H^1(X,\mathcal{T}_X)$ by Serre duality. The exact sequence

$$0 \to T_X \to T_{\mathbb{P}|X} \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(3) \to 0$$

twisted by $\mathcal{O}_X(-1)$, gives a cohomology exact sequence

$$0 \to H^0(X,T_{\mathbb{P}}(-1)|_X) \to H^0(X,\mathcal{O}_X(1)) \oplus H^0(X,\mathcal{O}_X(2)) \to H^1(X,\mathcal{T}_X(-1)) \to H^1(X,T_{\mathbb{P}}(-1)|_X).$$

From the Euler exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \otimes \mathbb{C} V \to T_{\mathbb{P}|X} \to 0$ we deduce $H^1(X,T_{\mathbb{P}}(-1)|_X) = 0$ and an isomorphism $V \cong H^0(X,T_{\mathbb{P}}(-1)|_X)$. Thus we find an exact sequence

$$0 \to V \to H^0(X,\mathcal{O}_X(1)) \oplus H^0(X,\mathcal{O}_X(2)) \to T_0(JX) \to 0,$$
which is equivariant with respect to the action of $\mathfrak{S}_7$. As representations of $\mathfrak{S}_7$, $H^0(X, \mathcal{O}_X(1))$ is isomorphic to $V$ and $H^0(X, \mathcal{O}_X(2))$ to $S^2V/\mathbb{C}q$, where $q$ corresponds to the quadric containing $X$. On the other hand $S^2V = \mathbb{C} \oplus V \oplus V_{(5,2)}$, where $V_{(5,2)}$ is the irreducible representation of $\mathfrak{S}_7$ corresponding to the partition $(5,2)$ of $7$ ([F-H], exercise 4.19). Thus we get $T_0(JX) \cong V \oplus V_{(5,2)}$. Since $\dim T_0(JX) = 20$ and $\dim(V) = 6$ we find $\dim V_{(5,2)} = 14$.

Proof of the theorem. We first observe that $\mathfrak{A}_7$ cannot act non-trivially on the Jacobian $JC$ of a curve of genus $g \leq 20$. Indeed by the Torelli theorem we have $\text{Aut}(JC) \cong \text{Aut}(C)$ if $C$ is hyperelliptic and $\text{Aut}(JC) \cong \text{Aut}(C) \times \mathbb{Z}/2$ otherwise. Since $\mathfrak{A}_7$ is simple we find $\# \text{Aut}(C) \leq 2520$. On the other hand we have $\# \text{Aut}(C) \leq 84(g-1) \leq 1596$, a contradiction.

Now assume that $JX$ is a product $J^1 \times \ldots \times J^m$ of Jacobians. Such a decomposition is unique up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components, see [C-G], Cor. 3.23. Thus the group $\mathfrak{A}_7$ acts on $[1, m]$ by permuting the factors. Let $O_1, \ldots, O_\ell$ be the orbits of this action. For $1 \leq k \leq \ell$ we put $J^{(k)}_i := J^{m_k}_i$ with $m_k = \min O_k$; then for each $i$ in $O_k$ $J_i$ is isomorphic to $J^{(k)}_i$, so our decomposition can be written $JX \cong J^{(1)}_{O_1} \times \ldots \times J^{(\ell)}_{O_\ell}$.

Since $\#O_k \leq m \leq 20$, the orbit $O_k$ has $1, 7$ or $15$ elements ([D-M], thm. 5.2.A). If $\#O_k = 1$, $\mathfrak{A}_7$ acts on the Jacobian $J^{(k)}_i$; by the lemma this action is faithful, contradicting the beginning of the proof. Thus $\#O_k = 7$ or $15$ for each $k$, which contradicts the equality $\sum \#O_k \dim(J^{(k)}_i) = 20$.

Remarks. The same kind of argument gives the non-rationality of the threefold $\sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^5$, using the action of $\mathfrak{S}_6$. It also gives a simple proof of the non-rationality of the Klein cubic threefold, defined by $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ in $\mathbb{P}^4$ (and, by the same token, of the general cubic threefold). The automorphism group of the Klein cubic is $\text{PSL}_2(\mathbb{F}_{11})$, of order 660, while its intermediate Jacobian has dimension 5. It is easily seen as above that a 5-dimensional principally polarized abelian variety with an action of $\text{PSL}_2(\mathbb{F}_{11})$ cannot be a Jacobian or a product of Jacobians (see also [Z] for a somewhat analogous, though more sophisticated, proof).

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