LIOUVILLE THEOREMS FOR AN INTEGRAL EQUATION OF CHOQUARD TYPE

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Abstract. We establish sharp Liouville theorems for the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} dz dy, \quad x \in \mathbb{R}^n,$$

where $0 < \alpha, \beta < n$ and $p > 1$. Our results hold true for positive solutions under appropriate assumptions on $p$ and integrability of the solutions. As a consequence, we derive a Liouville theorem for positive $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ solutions of the higher fractional order Choquard type equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{n-\beta}} * u^p\right) u^{p-1} \quad \text{in } \mathbb{R}^n.$$ 

1. Introduction. In this paper, we study Liouville theorems, i.e., nonexistence of positive solutions to the following integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} dz dy, \quad x \in \mathbb{R}^n,$$

where $0 < \alpha, \beta < n$ and $p > 1$. This equation is closely related to the following higher fractional order Choquard type equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{n-\beta}} * u^p\right) u^{p-1} \quad \text{in } \mathbb{R}^n.$$ 

One should observe that both the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}} u$ and the Choquard type nonlinearity $\left(\frac{1}{|x|^{n-\beta}} * u^p\right) u^{p-1}$ are nonlocal in our equation (2). This phenomenon is quite different from most of the known results in previous literature and makes (2) more difficult to investigate.

In recent years, problems involving the fractional Laplacian have attracted much attention. They appear in several physical phenomena, such as quasi-geostrophic flows, anomalous diffusions in plasmas, flames propagation, chemical reactions in liquids and geographical fluid dynamics (see [5,6,12,27]). The fractional Laplacian also has various applications in probability and finance. In particular, it can be

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understood as the infinitesimal generator of a stable Lévy diffusion process and is used to model American options in finance (see [1, 4] and the references therein).

Equation of type (2) is analogous to the well known Choquard-Pekar equation

$$-\Delta u + u = \frac{1}{2} \left( \frac{1}{|x|} * u^2 \right) u \quad \text{in } \mathbb{R}^3,$$

which can date back to [25] in 1954, where the equation was used to describe a polaron at rest in quantum theory. In 1977, Lieb [19] built the model of an electron trapped in its own hole. Equation (3) was also proposed to characterize the particle moving in its own gravitational field (see [22], where it is called Schrödinger-Newton equation). Since then, more and more researchers have focused their attention on the existence of solutions for equation (2) and related problems using variational methods. Without any intention to provide a survey about the subject, we would like to refer the reader to the papers [2, 3, 21, 23, 29] and the references therein.

Liouville theorems for equation (2) were also studied by several authors in the last decade. Moroz and J.V. Schaftingen [24] studied nonexistence of weak supersolutions of (2) in exterior domains when $\alpha = 2$ and various sufficient conditions were listed. However, it seems difficult to investigate directly the nonexistence of positive solutions of (1) and all positive $H^\frac{2}{n-2} (\mathbb{R}^n)$ solutions of (2). This classification result was proved in [11, 16, 28] for $\alpha = \beta$, in [7, 13, 20] for $\beta = n - 2\alpha$ and in [14, 15] for more general cases.

Liouville theorems for positive solutions were also established by some authors in the subcritical case $p < \frac{n+\beta}{n-\alpha}$ under additional assumptions on $\alpha$ and $\beta$. By using a Pohozaev identity in integral forms, Xu-Lei [28] proved that (1) has no positive $L^{\frac{n(n-1)}{\alpha-1}} (\mathbb{R}^n)$ solutions if $\beta = \alpha \in (1, n)$ and $p < \frac{n+\beta}{n-\alpha}$. Similar result was established by Lei [16] for $\alpha = \beta = 2$ and by Cao-Dai [7] for $n - \beta = 2\alpha = 8$ using the moving plane method. According to [17], if $\beta = \alpha \in (1, n)$, then the solutions $u$ of (1) have better regularity, i.e., $u \in C^1 (\mathbb{R}^n)$. This smoothness of $u$ is essential for applying the Pohozaev identity to deduce nonexistence result for (1).

Motivated by above works, in this paper, we prove some Liouville theorems for positive solutions of (1) and (2) in the general case $0 < \alpha, \beta < n$. Our first nonexistence result is the following one, which can be applied to positive supersolutions of more general equations.

**Theorem 1.** Assume that $0 < \alpha, \beta < n$, $p > 1$, $q > 0$ and $p + q \leq \frac{n+\beta}{n-\alpha}$. Then there is no positive function $u \in L^p_{\text{loc}} (\mathbb{R}^n)$ such that

$$u(x) \geq \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} dy \quad \text{for a.e. } x \in \mathbb{R}^n.$$  

(4)
Remark 1. The upper bound of $p+q$ in Theorem 1 is optimal. Indeed, taking any $p > 1$, $q > 0$ such that $p + q > \frac{n + \beta}{n - \alpha}$, $\frac{\beta}{p} < \frac{\alpha + \beta}{p+q-1} < \frac{\beta}{p}$ and let

$$u(x) = \frac{\lambda}{|x|^k},$$

where $k = \frac{\alpha + \beta}{p + q - 1}$ and $\lambda > 0$ will be determined later. Then

$$\beta < kp < n \quad \text{and} \quad \alpha < k(p + q) - \beta < n.$$

For any $y \in \mathbb{R}^n \setminus \{0\}$, we have

$$\int_{\mathbb{R}^n} \frac{u^p(z)}{|y - z|^{n-\beta}} dz = \int_{\mathbb{R}^n} \frac{\lambda^p}{|y - z|^{n-\beta}|z|^k} dz$$

$$= \left\{ \begin{array}{l}
\int_{|z| < |y|}\int_{|y - z| < \frac{|y|}{2}} + \int_{|y - z| \geq \frac{|y|}{2}} + \int_{|y - z| < |z|} \frac{\lambda^p}{|y - z|^{n-\beta}|z|^k} dz \\
\int_{|y - z| < |y|} 2^{n-\beta}\lambda^p|z|^k dz + \int_{|y - z| < \frac{|y|}{2}} 2^{kp}\lambda^p|y|^k dz \\
+ \int_{|z| \geq \frac{|y|}{2}} \frac{\lambda^p}{|z|^{n+kp-\beta}} dz + \int_{|y - z| \geq \frac{|y|}{2}} \frac{\lambda^p}{|y - z|^{n+kp-\beta}} dz
\end{array} \right\}$$

$$\leq \frac{C \lambda^p}{|y|^{kp-\beta}}.\quad \text{(5)}$$

From (5) and by a similar argument, we deduce for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$\int_{\mathbb{R}^n} \frac{u^p(y)}{|x - y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y - z|^{n-\beta}} dy \leq C \int_{\mathbb{R}^n} \frac{\lambda^{p+q}}{|x - y|^{n-\alpha}|y|^{k(p+q) - \beta}} dy$$

$$\leq \frac{C \lambda^{p+q}}{|x|^{k(p+q) - \alpha - \beta}} = C \lambda^{p+q-1} u(x).\quad \text{(6)}$$

Inequality (6) implies that $u$ satisfies (4) for $\lambda > 0$ sufficiently small.

As a consequence of Theorem 1, we have the following corollary.

Corollary 1. Assume that $0 < \alpha, \beta < n$ and $1 < p \leq \frac{2n-\alpha+\beta}{2n-2\alpha}$. Then equation (1) has no positive solution in $L^P_{\text{loc}}(\mathbb{R}^n)$.

When $p > \frac{2n-\alpha+\beta}{2n-2\alpha}$, equation (1) may still have no solution satisfying some global integrability conditions as pointed out in our next result.

Theorem 2. Assume that $0 < \alpha, \beta < n$, $\frac{2n}{2n-\alpha-\beta} < p < \frac{n+\beta}{n-\alpha}$ and $p \geq 2$. Then equation (1) has no positive solution in $L^{\frac{2n(p-1)}{\alpha + \beta}}(\mathbb{R}^n)$.

Remark 2. The assumption $p < \frac{n+\beta}{n-\alpha}$ in Theorem 2 is sharp in the sense that (1) may have positive solutions in $L^{\frac{n+\beta}{n-\alpha}}(\mathbb{R}^n)$ if $p = \frac{n+\beta}{n-\alpha}$. Indeed, if $p = \frac{n+\beta}{n-\alpha}$, then $\frac{2n(p-1)}{\alpha + \beta} = \frac{2n}{n-\alpha}$ and all positive solutions $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ of (1) have been classified in [14] as

$$u(x) = c \left( \frac{t}{t^2 + |x - x_0|^2} \right)^{\frac{n-\alpha}{2}},$$
where \( c, t > 0 \) and \( x^0 \in \mathbb{R}^n \).

Now we deal with Liouville theorems for equation (2). The positive solutions \( u \) of (2) are defined in the distributional sense, i.e., \( u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n) \) and satisfies

\[
\int_{\mathbb{R}^n} (\Delta)^{\frac{\alpha}{2}} u (\Delta)^{\frac{\alpha}{2}} \phi \, dx = \int_{\mathbb{R}^n} \left( \frac{1}{|x|^{n-\beta}} + u^p \right) u^{p-1} \phi \, dx
\]  

for any \( \phi \in C^\infty_0(\mathbb{R}^n) \), where the fractional Laplacian is defined by Fourier transform as in [26]

\[
(\Delta)^{\frac{\alpha}{2}} u = \mathcal{F}^{-1} \left( |\xi|^\frac{\alpha}{2} \mathcal{F} u \right),
\]

and \( H^{\frac{\alpha}{2}}(\mathbb{R}^n) \) is the inhomogeneous Sobolev space with the norm

\[
||u||_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \left( ||u||_{L^2(\mathbb{R}^n)}^2 + ||u||_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}},
\]

while the homogeneous Sobolev norm

\[
||u||_{H^{\frac{\alpha}{2}}_0(\mathbb{R}^n)} = ||(\Delta)^{\frac{\alpha}{2}} u||_{L^2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\xi|^n |\mathcal{F} u|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

Here, as usual,

\[
\mathcal{F} u(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2\pi i x \cdot \xi} \, dx
\]

denotes the Fourier transform of \( u \).

By using Corollary 1, Theorem 2 and exploiting the relation between equations (1) and (2), we establish the following Liouville theorem for (2).

**Theorem 3.** Assume that \( 0 < \alpha, \beta < n \). Then equation (2) has no positive solution in \( H^{\frac{\alpha}{2}}(\mathbb{R}^n) \) provided that one of the following conditions holds

(i) \( 1 < p \leq \frac{2n-\alpha+\beta}{2n-2\alpha} \),
(ii) \( \frac{2n-\alpha-\beta}{2n-\alpha-\beta} < p < \frac{n+\beta}{n-\alpha} \) and \( p \geq \max\{ \frac{n+\alpha+\beta}{n}, 2 \} \).

The rest of our paper is organized as follows. In Section 2, we prove Theorem 1 via the method of integral estimates. In Section 3, we employ the method of moving planes in integral forms to prove Theorem 2. The proof of Theorem 3 is given in Section 4. Throughout the paper, we denote by \( C \) the generic positive constant whose value may change from line to line or even in the same line.

### 2. Liouville theorem for positive super-solutions

In this section, we prove Theorem 1 by following the ideas in [8].

**Proof of Theorem 1.** Suppose that such function \( u \) exists. Let us define

\[
v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\beta}} \, dy.
\]

Choosing any \( R > 1 \). Then we have for a.e. \( x \in \mathbb{R}^n \),

\[
u(x) \geq \int_{B_R(0)} \frac{u^q(y)v(y)}{|x-y|^{n-\alpha}} \, dy \geq \frac{C}{(R + |x|)^{n-\alpha}} \int_{B_R(0)} u^q(y)v(y) \, dy.
\]  

(8)
From (8), we deduce that
\[ \int_{B_R(0)} u^p(x)dx \geq \int_{B_R(0)} \frac{C}{(R + |x|)^{p(n-\alpha)}} \left( \int_{B_R(0)} u^q(y)v(y)dy \right)^p \geq CR^{n-p(n-\alpha)} \left( \int_{B_R(0)} u^q(y)v(y)dy \right)^p. \] (9)

This indicates \( u^q \in L^1_{\text{loc}}(\mathbb{R}^n) \). Since \( R > 1 \), one can observe that (8) also implies
\[ u(x) \geq \frac{C}{(R + |x|)^{n-\alpha}}. \] (10)

Using
\[ v(x) \geq \int_{B_R(0)} \frac{u^p(y)}{|x-y|^{n-\beta}}dy \geq \frac{C}{(R + |x|)^{n-\beta}} \int_{B_R(0)} u^p(y)dy \] and (10), we also deduce
\[ \int_{B_R(0)} u^q(x)v(x)dx \geq \int_{B_R(0)} \frac{C}{(R + |x|)^{(q(n-\alpha)+n-\beta)}} \int_{B_R(0)} u^p(y)dy \geq CR^{\beta-q(n-\alpha)} \int_{B_R(0)} u^p(y)dy. \] (12)

From (9) and (12), we have
\[ \int_{B_R(0)} u^q(x)v(x)dx \geq CR^{n+\beta-(p+q)(n-\alpha)} \left( \int_{B_R(0)} u^q(y)v(y)dy \right)^p, \]
which implies
\[ 0 < \int_{B_R(0)} u^q(x)v(x)dx \leq CR^{\frac{(p+q)(n-\alpha)-(n+\beta)}{p-1}}. \] (13)

If \( p + q < \frac{n+\beta}{n-\alpha} \), we reach a contradiction by letting \( R \to \infty \) in (13).

It remains to consider the case \( p + q = \frac{n+\beta}{n-\alpha} \). In this case, (13) leads to \( u^q v \in L^1(\mathbb{R}^n) \) if we let \( R \to \infty \).

Multiplying (11) by \( u^q(x) \) and integrating on \( A_R = B_R(0) \setminus B_{R/2}(0) \), we still have
\[ \int_{A_R} u^q(x)v(x)dx \geq \int_{A_R} \frac{Cu^q(x)}{(R + |x|)^{n-\beta}} \int_{B_R(0)} u^p(y)dy \geq \int_{A_R} \frac{C}{(R + |x|)^{(q(n-\alpha)+n-\beta)}} \int_{B_R(0)} u^p(y)dy \geq CR^{\beta-q(n-\alpha)} \int_{B_R(0)} u^p(y)dy \geq C \left( \int_{B_R(0)} u^q(y)v(y)dy \right)^p. \]

Letting \( R \to \infty \) and noting that \( u^q v \in L^1(\mathbb{R}^n) \), we obtain \( \|u^p v\|_{L^1(\mathbb{R}^n)} = 0 \), which is impossible. \( \square \)
3. Liouville theorem for positive integrable solutions. Throughout this section, we always denote
\[ \gamma = n + \beta - p(n - \alpha). \]

If \( u \) is a positive solution of (1), we denote by \( \overline{u} \) the Kelvin transformation of \( u \) centered at the origin. That is, \( \overline{u}(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^2} \right) \) for \( x \in \mathbb{R}^n \setminus \{0\} \).

**Lemma 4.** The Kelvin transformation \( \overline{u} \) satisfies the integral equation
\[ \overline{u}(x) = \int_{\mathbb{R}^n} \frac{\overline{u}^{-1}(y)}{|x-y|^{n-\alpha}|y|^\gamma} \int_{\mathbb{R}^n} \frac{\overline{u}^p(z)}{|y-z|^{n-\beta}|z|^\gamma} \, dy \, dz, \quad x \in \mathbb{R}^n \setminus \{0\}. \]  
(14)

**Proof.** Using the fact
\[ d \left( \frac{x}{|x|^2} \right) = \frac{dx}{|x|^{2n}} \quad \text{and} \quad \frac{x - y}{|x|^2} = \frac{|x - y|}{|x|} \cdot \frac{1}{|y|}, \]
we have
\[
\overline{u}(x) = \frac{1}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^{-1}(y)}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} \, dy \, dz
= \frac{1}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^{-1} \left( \frac{y}{|y|^2} \right) |x|^{n-\alpha}}{|x|^{n-\alpha} |y|^{2n}} \int_{\mathbb{R}^n} \frac{u^p \left( \frac{z}{|z|^2} \right)}{|y-z|^{n-\beta} |z|^{2n}} \, dy \, dz
= \frac{1}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^{-1} \left( \frac{y}{|y|^2} \right) |x|^{n-\alpha}}{|x-y|^{n-\alpha} |y|^{n+\alpha}} \int_{\mathbb{R}^n} \frac{u^p \left( \frac{z}{|z|^2} \right)}{|y-z|^{n-\beta} |z|^{n+\beta}} \, dy \, dz
= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha} |y|^{\gamma}} \int_{\mathbb{R}^n} \frac{\overline{u}^p(z)}{|y-z|^{n-\beta} |z|^{\gamma}} \, dz \, dy.
\]
That is, (14). \( \square \)

To prove Theorem 2, we exploit the method of moving planes in integral forms (see [9,10]) to integral equation (1) in the \( x_1 \) direction. For this purpose, we need some definitions. For any \( \lambda \in \mathbb{R} \), let
\[ T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \} \]
be the moving plane,
\[ \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \} \]
be the region to the left of the plane and
\[ x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n) \]
be the reflection of the point \( x = (x_1, x_2, \ldots, x_n) \) about the plane \( T_\lambda \).

For each positive solution \( u \) of (1), we also define
\[
\overline{u}_\lambda(x) := \overline{u}(x^\lambda), \quad w_\lambda(x) := \overline{u}_\lambda(x) - \overline{u}(x),
\]
\[
v(x) := \int_{\mathbb{R}^n} \frac{\overline{u}^p(y)}{|x-y|^{n-\beta} |y|^{\gamma}} \, dy, \quad v_\lambda(x) := v(x^\lambda)
\]
and

$$\Sigma^-_\lambda = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \}.$$

**Lemma 5.** For any $\lambda < 0$ and $x \in \Sigma^-_\lambda$, we have

$$w_\lambda(x) \geq (p - 1) \int_{\Sigma^-_\lambda} \frac{w_{\lambda^2}(y)v(y)w_\lambda(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} dy$$

$$+ p \int_{\Sigma^-_\lambda} \frac{w_{\lambda^2}(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} \int_{\Sigma^-_\lambda} \frac{w_{\lambda}(z)}{|y - z|^{n - \beta} |z|^{\gamma}} dz dy.$$

**Proof.** One can observe from Lemma 4 that, for any $x \in \Sigma_\lambda$,

$$w_\lambda(x) = \pi_\lambda(x) - \pi(x)$$

$$= \int_{\mathbb{R}^n} \frac{\pi_{\lambda^2}(y)v(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} dy - \int_{\mathbb{R}^n} \frac{\pi_{\lambda^2}(y)v(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} dy + \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \left( \frac{\pi_{\lambda^2}(y)v_\lambda(y)}{|y|^{\gamma}} - \frac{\pi_{\lambda^2}(y)v(y)}{|y|^{\gamma}} \right) dy$$

$$= \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \left( \frac{\pi_{\lambda^2}(y)v_\lambda(y)}{|y|^{\gamma}} - \frac{\pi_{\lambda^2}(y)v(y)}{|y|^{\gamma}} \right) v(y) dy$$

$$+ \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \frac{\pi_{\lambda^2}(y)}{|y|^{\gamma}} \left( v_\lambda(y) - v(y) \right) dy + \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \frac{\pi_{\lambda^2}(y)}{|y|^{\gamma}} \left( v_\lambda(y) - v(y) \right) dy$$

(15)

$$= \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \left( \frac{\pi_{\lambda^2}(z)}{|z|^{\gamma}} - \frac{\pi^\lambda(z)}{|z|^{\gamma}} \right) \left( \frac{\pi_{\lambda}(z)}{|z|^{\gamma}} - \frac{\pi^\lambda(z)}{|z|^{\gamma}} \right) \left( \frac{\pi_{\lambda^2}(y)}{|y|^{\gamma}} - \frac{\pi_{\lambda^2}(y)}{|y|^{\gamma}} \right) dy$$

Using the mean value theorem, we get from (15) that, for any $\lambda < 0$ and $x \in \Sigma^-_\lambda$,

$$w_\lambda(x) \geq \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \pi_{\lambda^2}(y) - \pi_{\lambda^2}(y) v(y) dy$$

$$+ \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \pi_{\lambda^2}(y) \left( v_\lambda(y) - v(y) \right) dy$$

$$\geq (p - 1) \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \pi_{\lambda^2}(y) w_\lambda(y) dy$$

$$+ p \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) \pi_{\lambda^2}(y) \left( v_\lambda(y) - v(y) \right) dy$$

$$\geq (p - 1) \int_{\Sigma_\lambda} \frac{w_{\lambda^2}(y)v(y)w_\lambda(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} dy + p \int_{\Sigma_\lambda} \frac{w_{\lambda^2}(y)}{|x - y|^{n - \alpha} |y|^{\gamma}} \int_{\Sigma_\lambda} \frac{w_{\lambda}(z)}{|y - z|^{n - \beta} |z|^{\gamma}} dz dy.$$
This completes the proof of Lemma 5.

Let us recall the following Hardy-Littlewood-Sobolev inequality that will be used in the method of moving planes.

**Lemma 6** (Hardy-Littlewood-Sobolev inequality [18, 26]). Let $0 < \alpha < n$ and $p, q > 1$ be such that $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$. Then we have

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right\|_{L^q(\mathbb{R}^n)} \leq C_{n,\alpha,p}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

**Proof of Theorem 2.** Assume that equation (1) has a positive solution $u \in L^s(\mathbb{R}^n)$, where $s = \frac{2n(p-1)}{\alpha + p}$. From Corollary 1, it suffices to consider the case $p > \frac{2n-\alpha+\beta}{2n-2\alpha}$.

We denote

$$h = \frac{2n}{s} - n + \alpha \quad \text{and} \quad t = \frac{ns}{pn - \beta s}.$$  

One can easily check that

$$\gamma = h(p - 1) > 0, \quad \frac{nt}{n + \beta t} = \frac{s}{p} > 1, \quad \frac{ns}{n + \alpha s} > 1$$

and

$$\frac{1}{t} = \frac{p - \beta}{n} = \frac{\alpha}{n} \frac{p - 2}{s}.$$  

Now we observe that

$$\int_{\mathbb{R}^n} \frac{\pi^s(x)}{|x|^h} dx = \int_{\mathbb{R}^n} \frac{1}{|x|^{2n}} u^s \left( \frac{x}{|x|^2} \right) dx = \int_{\mathbb{R}^n} u^s(x) dx < \infty$$

and

$$\int_{\mathbb{R}^n} \frac{v^t(x)}{|x|^ht} dx = \int_{\mathbb{R}^n} \frac{1}{|x|^{2n-ht}} \left( \int_{\mathbb{R}^n} \frac{\pi^p(y)}{|x-y|^{n-\beta}|y|^\gamma} dy \right)^t dx$$

$$= \int_{\mathbb{R}^n} \frac{1}{|x|^{2n-ht}} \left( \int_{\mathbb{R}^n} \frac{u^p \left( \frac{y}{|y|^\gamma} \right)}{|x-y|^{n-\beta}|y|^{\gamma + p(n-\alpha)}} dy \right)^t dx$$

$$= \int_{\mathbb{R}^n} \frac{1}{|x|^{2n-ht}} \left( \int_{\mathbb{R}^n} \frac{u^p(y)}{\left| \frac{x}{|x|^2} - \frac{y}{|y|^\gamma} \right|^{n-\beta}|y|^{2n-\gamma - p(n-\alpha)}} dy \right)^t dx$$

$$= \int_{\mathbb{R}^n} \frac{1}{|x|^{2n-ht-t(n-\beta)}} \left( \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\beta}|y|^{2n-\gamma - p(n-\alpha)-(n-\beta)}} dy \right)^t dx$$

$$\leq \left\| \frac{u^p(y)}{|x-y|^{n-\beta}} dy \right\|_{L^t(\mathbb{R}^n)^t} \leq C\|u^p\|_{L^t(\mathbb{R}^n)}^t = C\|u\|_{L^t(\mathbb{R}^n)}^t < \infty.$$  

That is,

$$\frac{\pi}{|x|^h} \in L^s(\mathbb{R}^n) \quad \text{and} \quad \frac{v}{|x|^t} \in L^t(\mathbb{R}^n).$$

(16)
From Lemma 5 and the Hardy-Littlewood-Sobolev inequality, we have

$$
\|w_\lambda\|_{L^r(\Sigma^-)} \leq C \left( \frac{\|p^{p-2}vw_\lambda\|}{|x|^{\gamma}} \right)_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} + C \left( \frac{\|p^{p-1}\lambda w_\lambda\|}{|x|^{\gamma}} \right)_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)}.
$$

(17)

By Hölder inequality, we obtain for $p > 2$

$$
\left\| \frac{p^{p-2}vw_\lambda}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} \leq \left\| \frac{p^{p-2}}{|x|^{(p-2)\alpha}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} \left\| v \right\|_{L^r(\Sigma^-)} \|w_\lambda\|_{L^r(\Sigma^-)}.
$$

(18)

While for $p = 2$, we have

$$
\left\| \frac{p^{p-2}vw_\lambda}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} = \left\| \frac{vw_\lambda}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} \leq \left\| \frac{v}{|x|^{\gamma}} \right\|_{L^r(\Sigma^-)} \|w_\lambda\|_{L^r(\Sigma^-)}.
$$

(19)

On the other hand,

$$
\left\| \frac{\|p^{p-1}\lambda w_\lambda\|}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} \leq \left\| \frac{\|p^{p-1}\lambda w_\lambda\|}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)} \left\| \frac{p^{p-1}w_\lambda}{|x|^{\gamma}} \right\|_{L^{\frac{a_n}{a_n-\nu}}(\Sigma^-)}
$$

(20)

Substituting (18), (19) and (20) into (17), we arrive at the following estimate

$$
\|w_\lambda\|_{L^r(\Sigma^-)} \leq C \left( \left\| \frac{p^{p-2}}{|x|^{\gamma}} \right\|_{L^r(\Sigma^-)} + \left\| \frac{p^{p-1}}{|x|^{\gamma}} \right\|_{L^r(\Sigma^-)} \right) \|w_\lambda\|_{L^r(\Sigma^-)}.
$$

(21)

With the aid of (21), we are able to start moving the plane $T_\lambda$ from near $\lambda = -\infty$ to the right until it reaches the limiting position in order to derive symmetry. This procedure contains two steps.

**Step 1.** We show that, for $\lambda$ sufficiently negative,

$$
w_\lambda \geq 0 \quad \text{in } \Sigma^-.
$$

(22)
Indeed, from (16) and $p > 1$, we can choose $R > 0$ sufficiently large, such that for $\lambda \leq -R$, we have
\[
\left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-2} \left\| \frac{v}{|x|^p} \right\|_{L^p(\Sigma_\lambda^-)} + \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-1} \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\Sigma_\lambda^-)}^{p-1} \leq \frac{1}{2C},
\]
where the constant $C$ is the same as in (21).

Therefore, (21) and (23) imply that $\|w_\lambda\|_{L^p(\Sigma_\lambda^-)} = 0$ and hence $|\Sigma_\lambda^-| = 0$ for $\lambda \leq -R$. Furthermore, we can deduce from (15) that $w_\lambda(x) \geq 0$ for any $x \in \Sigma_\lambda$. Thus $\Sigma_\lambda^- = \emptyset$ and (22) holds for $\lambda \leq -R$. This completes Step 1.

Step 2. Let
\[
\lambda_0 = \sup\{\lambda \leq 0 \mid w_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}.
\]

In this step, we show that
\[
\lambda_0 = 0.
\]

Suppose on contrary that $\lambda_0 < 0$. One can observe that $w_{\lambda_0} \geq 0$. If $w_{\lambda_0} = 0$ in $\Sigma_{\lambda_0}$, then (15) becomes
\[
w_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_{\lambda_0} - y|^{n-\alpha}} \right) \left( \frac{1}{|y_{\lambda_0}|^\gamma} - \frac{1}{|y|^{\gamma}} \right) \frac{\pi^{p-1}(y)v(y)dy}{|y_{\lambda_0}|^{\gamma}} + \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_{\lambda_0} - y|^{n-\alpha}} \frac{\pi^{p-1}_{\lambda_0} (y)}{|y_{\lambda_0}|^{\gamma}} \right) \times \left\{ \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|y-z|^{n-\beta}} - \frac{1}{|y_{\lambda_0} - z|^{n-\beta}} \right) \left( \frac{1}{|z_{\lambda_0}|^{\gamma}} - \frac{1}{|z|^{\gamma}} \right) \frac{\pi^{p}(z)dz}{|z_{\lambda_0}|^{\gamma}} \right\} dy.
\]

Since $\gamma > 0$, the above identity implies $w_{\lambda_0}(x) > 0$ in $\Sigma_{\lambda_0}$, a contradiction.

Therefore, $w_{\lambda_0}$ is not identically zero in $\Sigma_{\lambda_0}$. Using again (15), we deduce that $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$. We will obtain a contradiction with (24) by showing the existence of an $\varepsilon > 0$ small enough such that $w_\lambda \geq 0$ in $\Sigma_\lambda$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$.

It can be clearly seen from (21) that, our primary task is to prove that, one can choose $\varepsilon > 0$ sufficiently small such that
\[
\left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-2} \left\| \frac{v}{|x|^p} \right\|_{L^p(\Sigma_\lambda^-)} + \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-1} \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\Sigma_\lambda^-)}^{p-1} \leq \frac{1}{2C}
\]
for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, where the constant $C$ is the same as in (21).

From (16) and $p > 1$, there exists $R > 0$ large enough such that
\[
\left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-2} \left\| \frac{v}{|x|^p} \right\|_{L^p(\mathbb{R}^n\setminus B_R(0))} + \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\mathbb{R}^n)}^{p-1} \left\| \frac{\pi}{|x|^p} \right\|_{L^p(\Sigma_\lambda^-)}^{p-1} \leq \frac{1}{4C}.
\]

Now fix this $R$, in order to derive (26), we only need to show that
\[
\lim_{\lambda \to \lambda_0^+} |\Sigma_\lambda^- \cap B_R(0)| = 0.
\]

To prove this, we define $E_\delta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) \mid w_{\lambda_0}(x) > \delta\}$ and $F_\delta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) \mid w_{\lambda_0}(x) \leq \delta\}$ for any $\delta > 0$, and let $D_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0)$ for any $\lambda > \lambda_0$. Then
\[
\lim_{\delta \to 0^+} |F_\delta| = 0, \quad \lim_{\lambda \to \lambda_0^+} |D_\lambda| = 0
\]
and
\[
\Sigma_\lambda^- \cap B_R(0) \subset \Sigma_\lambda^- \cap (E_\delta \cup F_\delta \cup D_\lambda) \subset (\Sigma_\lambda^- \cap E_\delta) \cup F_\delta \cup D_\lambda.
\]
Therefore, for an arbitrarily fixed η > 0, one can choose δ > 0 small enough such that |F_δ| ≤ η. For this fixed δ, we will point out that

$$\lim_{\lambda \to \lambda_0^+} |\Sigma^-_\lambda \cap E_\delta| = 0. $$ (31)

Indeed, for all \( x \in \Sigma^-_\lambda \cap E_\delta \), we have \( \pi(x) - \pi(x^\lambda) = w_{\lambda_0}(x) - w_\lambda(x) > \delta \). It follows that \( \Sigma^-_\lambda \cap E_\delta \subset G^\lambda_\delta := \{ x \in B_R(0) | \pi(x) - \pi(x^\lambda) > \delta \} \). By Chebyshev inequality, we get

$$|G^\lambda_\delta| \leq \frac{1}{\delta} \int_{G^\lambda_\delta} \| \pi(x) - \pi(x^\lambda) \|^s dx$$

$$= \frac{1}{\delta} \int_{B_R(2\lambda_0 e_1)} \| \pi(x) - \pi(x + 2(\lambda_0 - \lambda)e_1) \|^s dx,$$

where \( e_1 = (1,0, \ldots, 0) \). Hence \( \lim_{\lambda \to \lambda_0^+} |G^\lambda_\delta| = 0 \), from which (31) follows.

Therefore, by (29), (30) and (31), we have

$$\lim_{\lambda \to \lambda_0^+} |\Sigma^-_\lambda \cap B_R(0)| \leq |F_\delta| \leq \eta.$$ 

This implies (28) since η > 0 is arbitrarily chosen. From (27) and (28), we arrive at (26).

Now we deduce from (21) and (26) that, there exists an \( \varepsilon > 0 \) sufficiently small such that \( |\Sigma^-_\lambda| = 0 \) for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \). Furthermore, by (15), we must have \( \Sigma^-_\lambda = \emptyset \). Hence \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \) for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \). This contradicts with the definition of \( \lambda_0 \) in (24). Therefore, (25) must hold and hence \( w_0(x) \geq 0 \) in \( \Sigma_0 \).

This completes Step 2.

Similar to previous steps, one can move the plane \( T_\lambda \) from \( +\infty \) to the left to get that \( w_0(x) \leq 0 \) in \( \Sigma_0 \). Hence, \( w_0 \equiv 0 \) and \( \pi \) is symmetric about \( T_0 \). Since we can repeat the previous arguments to any direction, we deduce that \( \pi \) is radially symmetric about 0. So is \( u \).

We may repeat above arguments for any Kelvin transformation of \( u \) to deduce that \( u \) is radially symmetric about any point in \( \mathbb{R}^n \). This only occurs if \( u \) is constant. However, this is absurd since a positive constant function does not satisfy equation (1).

Therefore, equation (1) has no positive solution in \( L^s(\mathbb{R}^n) \). \( \square \)

4. Liouville theorem for higher fractional order Choquard equations.

**Lemma 7.** Assume \( 0 < \alpha, \beta < n \) and \( p > 1 \). If \( u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n) \) is a positive solution of equation (2), then \( v = R^{n-2}_{n,\alpha} u \) is a solution of equation (1), and vice versa, where \( R_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{n+\alpha}{2}\right)} \) is the Riesz potential’s constant (see [26]).

**Proof.** Assume that \( u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n) \) is a positive solution of (2). For any \( \phi \in C_0^\infty(\mathbb{R}^n) \), let us define

$$\psi(x) = \int_{\mathbb{R}^n} \frac{R_{n,\alpha} \phi(y)}{|x-y|^{n-\alpha}} dy.$$ 

Then \( (-\Delta)^{\frac{\alpha}{2}} \psi = \phi \) and \( \psi \in H^\alpha(\mathbb{R}^n) \subset H^{\frac{\alpha}{2}}(\mathbb{R}^n) \). Testing (7) with this \( \psi \), we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} \psi(x) dx = \int_{\mathbb{R}^n} \left( \frac{1}{|x|^{n+\beta}} * u^p \right)(x) u^{p-1}(x) \psi(x) dx.$$
Integrating by parts of the left hand side and exchanging the order of integration of the right hand side, we obtain

\[ \int_{\mathbb{R}^n} u(x) \phi(x) \, dx = R_{n, \alpha} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} \, dz \right) \phi(x) \, dx. \]

Since this formula holds for any \( \phi \in C_0^\infty(\mathbb{R}^n) \), we deduce that

\[ u(x) = R_{n, \alpha} \int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\beta}} \, dz, \quad x \in \mathbb{R}^n, \]

which implies that \( v \) satisfies the integral equation (1).

Conversely, assume that \( u \in H_{N}^{\frac{n}{p}}(\mathbb{R}^n) \) is a positive solution of integral equation (1). Applying Fourier transform to both sides of (1) (see [26]), we get

\[ \mathcal{F}u(\xi) = \frac{1}{R_{n, \alpha}(2\pi|\xi|)^\alpha} \mathcal{F} \left[ \frac{1}{|x|^{n-\beta}} * u^p \right] u^{p-1} \left( \xi \right). \]

It follows that, for any \( \phi \in C_0^\infty(\mathbb{R}^n) \), we have

\[ \int_{\mathbb{R}^n} (-\Delta)^{\frac{p}{2}} u(-\Delta)^{\frac{p}{2}} \phi \, dx = \frac{1}{R_{n, \alpha}} \int_{\mathbb{R}^n} \mathcal{F} \left[ \frac{1}{|x|^{n-\beta}} * u^p \right] u^{p-1} \mathcal{F} \phi \, d\xi \]

\[ = \frac{1}{R_{n, \alpha}} \int_{\mathbb{R}^n} \left( \frac{1}{|x|^{n-\beta}} * u^p \right) u^{p-1} \phi \, dx, \]

which implies that \( u \) is also a weak solution to equation (2). \( \square \)

**Proof of Theorem 3.** By contradiction, assume that (2) has a positive solution \( u \in H_{N}^{\frac{n}{p}}(\mathbb{R}^n) \). From Lemma 7, we have that \( v = R_{n, \alpha}^{\frac{1}{p}} u \) is a positive solution of the integral equation (1).

In case (i), we have \( 1 < p \leq \frac{2n-\alpha+\beta}{2n-2\alpha} < \frac{2n}{n-\alpha} \). Hence, \( v \in H_{N}^{\frac{n}{p}}(\mathbb{R}^n) \subset L_{\text{loc}}^p(\mathbb{R}^n) \) by Sobolev embeddings. This contradicts Corollary 1.

Similarly, in case (ii), we have \( 2 \leq \frac{2n(p-1)}{n+\beta} < \frac{2n}{n-\alpha} \), which implies \( v \in H_{N}^{\frac{n}{p}}(\mathbb{R}^n) \subset L^{\frac{2n(p-1)}{n+\beta}}(\mathbb{R}^n) \). However, this contradicts Theorem 2. \( \square \)

**REFERENCES**

[1] D. Applebaum, Lévy processes - from probability to finance and quantum groups, Notices Amer. Math. Soc., 51 (2004), 1336–1347.

[2] P. d’Avenia, G. Siciliano and M. Squassina, On fractional Choquard equations, Math. Models Methods Appl. Sci., 25 (2015), 1447–1476.

[3] P. Belchior, H. Bueno, O. H. Miyagaki and G. A. Pereira, Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay, Nonlinear Anal., 164 (2017), 38–53.

[4] J. Bertoin, Lévy Processes, Cambridge University Press, 1996.

[5] J. P. Bouchard and A. Georges, Anomalous diffusion in disordered media: statistical mechanics, models and physical applications, Phys. Rep., 195 (1990), 127–293.

[6] L. Caffarelli and L. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Annals of Math., 171 (2010), 1903–1930.

[7] D. Cao and W. Dai, Classification of nonnegative solutions to a bi-harmonic equation with Hartree type nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A, 149 (2019), 979–994.

[8] G. Caristi, L’Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, Milan J. Math., 76 (2008), 27–67.

[9] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330–343.

[10] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations, 30 (2005), 59-65.
[11] W. Chen and C. Li, Classification of positive solutions for nonlinear differential and integral systems with critical exponents, Acta Math. Sci., 29B (2009), 949–960.

[12] P. Constantin, Euler Equations, Navier-Stokes Equations and Turbulence, Mathematical Foundation of Turbulent Viscous Flows, Vol. 1871 of lecture Notes in Math. 1–43, Springer, Berlin, 2006.

[13] W. Dai, Y. Fang, J. Huang, Y. Qin and B. Wang, Regularity and classification of solutions to static Hartree equations involving fractional Laplacians, Discrete Contin. Dyn. Syst., 39 (2019), 1389–1403.

[14] P. Le, Symmetry and classification of solutions to an integral equation of Choquard type, submitted for publication.

[15] P. Le, Liouville theorem and classification of positive solutions for a fractional Choquard type equation, Nonlinear Anal., 185 (2019), 123-141.

[16] Y. Lei, Liouville theorems and classification results for a nonlocal Schrödinger equation, Discrete Contin. Dyn. Syst., 38 (2018), 5351–5377.

[17] Y. Lei, On the regularity of positive solutions of a class of Choquard type equations, Math. Z., 273 (2013), 883–905.

[18] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math., 118 (1983), 349–374.

[19] E. Lieb, The Hartree-Fock theory for Coulomb systems, Comm. Math. Phys., 53 (1977) 185–194.

[20] S. Liu, Regularity, symmetry, and uniqueness of some integral type quasilinear equations, Nonlinear Anal., 71 (2009), 1796–1806.

[21] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Rational Mech. Anal., 195 (2010), 455–467.

[22] I. Moroz, R. Penrose and P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Classical Quantum Gravity, 15 (1998), 2733–2742.

[23] V. Moroz and J. V. Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl., 19 (2017), 773–813.

[24] V. Moroz and J. V. Schaftingen, Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains, J. Differential Equations, 254 (2013), 3089–3145.

[25] S. Pekar, Untersuchungen über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.

[26] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, New Jersey, 1970.

[27] V. Tarasov and G. Zaslavsky, Fractional dynamics of systems with long-range interaction, Comm. Nonl. Sci. Numer. Simul., 11 (2006), 885–889.

[28] D. Xu and Y. Lei, Classification of positive solutions for a static Schrödinger-Maxwell equation with fractional Laplacian, Applied Math. Letters, 43 (2015), 85–89.

[29] W. Zhang and X. Wu, Nodal solutions for a fractional Choquard equation, J. Math. Anal. Appl., 464 (2018), 1167–1183.

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