Gaussian wave packet states of relic gravitons

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1. Introduction

In the last few decades, the study of quantum physical systems in curved spacetime has attracted a lot of attention of physicists [1–16]. These systems deserve detailed investigation because they are of considerable theoretical and experimental interest and certainly constitute an essential element to construct a consistent theory that combines quantum physics with gravity. As examples of investigations concerning these systems, we can mention those related to the determination of the vacuum expectation value of the energy-momentum tensor [1], the problem of creation of particles in a external time-dependent background, like a FRW cosmological model [2,5,10], the construction of squeezed quantum states [8–10] and those connected with relativistic quantum mechanics in different background spacetimes [17,18].

In recent papers, some authors [11–13], have studied the particle creation in an external time-dependent background (for a review see [16]), and have related this problem to that of a time-dependent harmonic oscillator. In this context, Grishchuk and Sidorov, in a remarkable paper [4], have shown that relic gravitons can be created from the vacuum quantum fluctuations of the gravitational field during the cosmological expansion and may provide extremely valuable information on the physical conditions in the very early universe. To obtain their results, these authors have related the problem of relic gravitons in a time-dependent cosmological background to that of a time-dependent harmonic oscillator.

The motivation of the present Letter is to investigate quantum effects of relic gravitons in an external time-dependent background, that is, in an FRW cosmological background from a Schrödinger-picture point of view. In doing so, we reduce the problem to that of a generalized time-dependent harmonic oscillator and solve exactly the corresponding Schrödinger equation with the help of linear invariants and of the invariant operator method. Afterwards, we construct Gaussian wave packet states and calculate the quantum dispersions as well as the quantum correlations for each mode of the quantized field.

This Letter is structured as follows. In Section 2, we reduce the problem of relic gravitons in the FRW background to that of the generalized time-dependent harmonic oscillator. Next, in Section 3, we use exact linear invariant and the dynamical invariant operator method to solve the Schrödinger equation for the problem. Section 4 describes the construction of Gaussian wave packet states and shows how one can calculate the quantum dispersions and correlations for each mode of the quantized field. Finally, we conclude the Letter in Section 5.

2. Relic gravitons in a FRW background

In this section we are interested in reducing the problem of relic gravitons in a FRW cosmological background to that of a generalized time-dependent harmonic oscillator. For this purpose, let us first consider the FRW metric [14]

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]

where \( a(t) \) is the dimensionless scale factor and \( \kappa = -1, 0, 1 \) is the curvature of the spacetime. When \( \kappa = -1 \) we say that the universe is open; when \( \kappa = 0 \) the universe is flat and when \( \kappa = 1 \) we say that the universe is closed.
Next, following Ref. [10], we decompose the gravitational waves on a complete basis \( u_k(\vec{x}, t) \) as

\[
u_k(\vec{x}, t) = \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} \mu_k(t), \quad (2)
\]

where \( \mu_k(t) \) is a real field and we consider normalization to a finite volume \( V \). What is more, each polarization component of the classical gravity-wave field satisfies the curved spacetime D'Alembert equation. Hence, when we apply the time-dependent mode functions \( u_k(\vec{x}, t) \) in the curved D'Alembert equation, we obtain the second-order differential equation [4]

\[
\ddot{\mu}_k + \left( k^2 - \frac{\dot{a}}{a} \right) \mu_k = 0,
\]

where overdots denote differentiation with respect to time and \( k^2 = k_x^2 + k_y^2 + k_z^2 \). Thus, from the latter equation, we see that for each mode \( k \) we could have seen interpreted as the ground state of the generalized harmonic oscillator by making the change \( \mu_k(t) \rightarrow \mu_k e^{i\omega_k t} \) where \( \omega_k = \sqrt{k^2 - \frac{\dot{a}}{a}} \). Hence, from Eqs. (5) and (7) we obtain the Hamiltonian of the system whose associated Hamiltonian is

\[
\hat{H}_k = \frac{1}{2} \hat{p}_k^2 + \frac{1}{2} \hat{q}_k^2 + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \hat{\mu}_k^2 - \frac{1}{2} \hat{p}_k^2 \hat{\mu}_k^2,
\]

where \( \hat{p}_k = -\sqrt{\frac{\delta}{\hbar}} \partial_{\mu_k} \hat{\mu}_k \). The conjugate momentum. Thus, the Hamiltonian of the system is

\[
H = \sum_k \hat{H}_k.
\]

In what follows, let us consider the generalized time-dependent harmonic oscillator. This system is described by the Hamiltonian [19–22].

\[
H(t) = \frac{1}{2} \left[ X(t) q^2 + Y(t) (pq + qp) + Z(t) p^2 \right],
\]

where \( p \) and \( q \) are canonical variables and \( X(t), Y(t) \) and \( Z(t) \) are time-dependent real functions. Hence, from Eqs. (5) and (7) we can see that for each mode \( k \), the Hamiltonian (5) corresponds to a generalized time-dependent harmonic oscillator by making the changes

\[
q = \mu_k, \quad p = \hat{p}_k
\]

and

\[
X(t) = k^2, \quad Y(t) = \frac{\dot{a}}{a}, \quad Z(t) = 1.
\]

Therefore, the problem of relic gravitons in a FRW cosmological model characterized by a scalar factor \( a(t) \) can be mapped into that of the generalized time-dependent harmonic oscillator. In this case, the frequency \( \Omega_k^2(t) \) of the oscillator is given by

\[
\Omega_k^2(t) = \left( k^2 - \frac{\dot{a}}{a} \right).
\]

In the next section, the Schrödinger equation associated with the Hamiltonian (5) will be exactly solved.

3. Relic gravitons and Schrödinger equation

We begin with the Schrödinger equation for the system described by the Hamiltonian (5) given by

\[
\frac{\partial \Psi(\mu_k, t)}{\partial t} = \hat{H}(\mu_k, t),
\]

with

\[
\hat{H} = \left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial \mu_k^2} + \frac{k^2}{2} \mu_k^2 - \frac{\hbar}{2} \frac{\dot{a}}{a} \mu_k - i \hbar \frac{\partial}{\partial \mu_k} \right].
\]

where \( p_k = -\hbar \partial / \partial \mu_k \) has been used and \( [\mu_k, \hat{p}_k] = i \hbar \delta_{k,k} \). We are interested in the analytical exact solution to this equation which could have been interpreted as the ground state of the generalized oscillator at some time \( t = t_0 \) [16]. To find a class of solutions of Eq. (11) we use the invariant operator method devised by Lewis and Riesenfeld [23,24]. According to this method, the solutions of the Schrödinger equation (11) are related to the eigenfunctions of a Hermitian invariant operator \( I_k(t) \) which satisfies the equation

\[
\frac{dI_k(t)}{dt} = i \hbar \{ I_k, \hat{H} \} + \frac{\partial I_k}{\partial t} = 0.
\]

If such a invariant exist and if it does not contain time-derivative operators, the condition (13) allow us to construct solutions the Schrödinger equation (11) in the form

\[
| \phi_\lambda(\mu_k, t) \rangle = e^{i\Omega(t)\Omega(t)} \phi_\lambda(\mu_k, t),
\]

where \( \phi_\lambda(\mu_k, t) \) are orthonormalized eigenfunctions of \( I_k(t) \) with time-independent eigenvalues \( \lambda \)

\[
| \hat{I}_k(\phi_\lambda, \mu_k, t) \rangle = \delta(\Omega - \lambda),
\]

and \( \hat{\phi}_\lambda \) is a phase function to be determined by the condition

\[
\hbar \frac{d\phi_\lambda}{dt} = (\hat{\phi}_\lambda | i \hbar \frac{\partial}{\partial t} - \hat{H} | \phi_\lambda \rangle).
\]

In Eqs. (14)–(17) we have suppressed the index \( k \) for the eigenfunctions and the phase function.

Quadratic invariants satisfying Eq. (13) are innumerable [24], however, as already commented, we are interested in dealing with a linear Hermitian invariant operator of the form

\[
I_k(t) = \alpha_k(t) \mu_k + \beta_k(t) p_k + \gamma_k(t),
\]

where \( \alpha_k(t), \beta_k(t) \) and \( \gamma_k(t) \) are time-dependent real functions to be determined. Now, \( I_k(t) \) must satisfy the condition (13). So, inserting expression (18) into Eq. (13) we obtain that

\[
\frac{\dot{\alpha}_k(t)}{\alpha_k(t)} = -\frac{\dot{\beta}_k(t)}{\beta_k(t)} + 2k^2,
\]

\[
\frac{\dot{\beta}_k(t)}{\beta_k(t)} = -\frac{\dot{\alpha}_k(t)}{\alpha_k(t)} - \alpha_k(t),
\]

and

\[
\frac{\dot{\gamma}_k(t)}{\gamma_k(t)} = 0.
\]

Therefore, from Eqs. (19) and (20) we find that

\[
\frac{\dot{\beta}_k(t) + \Omega_k^2(t)}{\beta_k(t)} = 0,
\]

with \( \Omega_k(t) \) given by Eq. (10). Thus, using Eq. (20) we can write the invariant operator (18) as

\[
I_k(t) = \beta_k(t) p_k - \left( \beta_k(t) - \frac{\dot{\beta}_k(t)}{\beta_k(t)} \right) \mu_k,
\]

where without loss of generality, we have set \( \gamma_k(t) = \text{const} = 0 \). Then, once that Eq. (22) is solved the invariant (23) is completely determined.
The next step is to find the eigenstates \( \phi_{\lambda}(\mu_k, t) \) of \( I_k(t) \). For this purpose, we must solve the eigenvalue equation (15) with \( I_k(t) \) given by (23). After a straightforward calculation, we find that the eigenstate of \( I_k(t) \) are of the form

\[
\phi_{\lambda}(\mu_k, t) = \frac{1}{\sqrt{2\pi\hbar\beta_k(t)}} \times \exp\left\{ \frac{i\lambda}{\hbar\beta_k(t)} \mu_k^2 + \frac{i\lambda}{\hbar\beta_k(t)} \mu_k \right\}. \tag{24}
\]

On the other hand, the phase function \( \eta_{\lambda}(t) \) may be obtained from Eq. (17). After performing some algebra we arrive at the result

\[
\eta_{\lambda} = -\frac{\lambda^2}{2\hbar} \int_0^t \frac{1}{\beta_k^2(\tau)} d\tau. \tag{25}
\]

Therefore, we are finally able to write the solutions of the Schrödinger equation (11) as

\[
\psi_{\lambda}(\mu_k, t) = \frac{1}{\sqrt{2\pi\hbar\beta_k(t)}} \exp\left\{ i\eta_{\lambda}(t) + \frac{i\beta_k(t)-\frac{\lambda}{\hbar}\beta_k(t)}{2\hbar\beta_k(t)} \mu_k^2 + \frac{i\lambda}{\hbar\beta_k(t)} \mu_k \right\}. \tag{26}
\]

It is worth remarking that when \( \beta_k(t) \) vanishes the phase function \( \eta_{\lambda}(t) \) diverges. In spite of this divergence, one can prove that the wave functions (26) are always finite [25,26]. Here, we observe that this result does not depend on the scale factor \( a(t) \). Furthermore, the evolution of a general Schrödinger state can be written as

\[
\Psi(\mu_k, t) = \int_{-\infty}^{\infty} g(\lambda) \psi_{\lambda}(\mu_k, t) d\lambda, \tag{27}
\]

where \( g(\lambda) \) is a weight function which determines the states of the system.

### 4. Relic gravitons, Gaussian wave packet states, dispersions and correlations

In what follows, we show how one can construct Gaussian wave packet states of Eq. (11). For that matter, we first write the weight function \( g(\lambda) \) as

\[
g(\lambda) = \frac{\sqrt{b}}{(2\pi)^{1/4}} \exp\left( -\frac{b^2/4}{\lambda^2} \right), \tag{28}
\]

where \( b \) is a positive real constant. Thus, inserting Eqs. (26) and (28) into Eq. (27) and performing the integration we obtain the Gaussian packet states as

\[
\Psi(\mu_k, t) = \left(\frac{2B}{\pi}\right)^{1/4} e^{i\lambda t/2} \frac{1}{\sqrt{\hbar b^2}} e^{-B\mu_k^2}, \tag{29}
\]

Moreover, the time-dependent probability density associated with this Gaussian wave packet state is Gaussian for all times

\[
|\psi(\mu_k, t)|^2 = \frac{1}{\sqrt{\pi} \sigma_k(t)} e^{-\mu_k^2/(2\sigma_k^2(t))}, \tag{34}
\]

with the time-dependent width \( \sigma_k(t) \) given by

\[
\sigma_k(t) = \frac{\sqrt{\hbar^2 b^2 (\Delta \mu_k)^2}}{2} \left(1 + \frac{2f_k^2}{\hbar b^4}\right). \tag{35}
\]

Therefore, the center of the wave packet remains at \( \langle \mu_k \rangle = 0 \), while its width changes in time, as expected for oscillatory behaviour [27,28]. Furthermore, it is also readily verified that the Gaussian state (29) is normalized and the time-dependent probability density is conserved

\[
\int_{-\infty}^{\infty} |\psi(\mu_k, t)|^2 d\mu_k = 1. \tag{36}
\]

Let us now calculate the quantum dispersions for each mode of the quantized \( \mu_k \) field in the Gaussian state \( \Psi(\mu_k, t) \). After some algebra, we find that

\[
(\Delta \mu_k) = \sqrt{\langle \mu_k^2 \rangle - \langle \mu_k \rangle^2} = \frac{1}{2\sqrt{B}}. \tag{37}
\]

and

\[
(\Delta p_k) = \sqrt{\langle p_k^2 \rangle - \langle p_k \rangle^2} = \hbar \sqrt{\frac{b^2 + C^4}{B}}. \tag{38}
\]

Here, we note that in terms of the dispersions in \( \mu_k \) the width \( \sigma_k(t) \) of the Gaussian packet (34) is given by \( \sqrt{2\Delta \mu_k} \). Further, the uncertainty product is given by

\[
(\Delta \mu_k)(\Delta p_k) = \frac{\hbar}{2} \left(1 + \frac{C^4}{B} \right). \tag{39}
\]

This product attains its minimum value \( \frac{\hbar}{2} \) for a given time \( \tau \) such that \( C(\tau) = 0 \). This condition may be explicitly written as

\[
\beta_k(\tau) = -\frac{4f_k(\tau)}{\beta_k(\tau)[\hbar^2 b^4 + 4f_k^2(\tau)]} + \frac{\dot{A}}{A} \beta_k(\tau). \tag{40}
\]

Thus, if we start with a minimum uncertainty packet, \( (\Delta \mu_k) \times (\Delta p_k) = \hbar/2 \) for \( \tau = 0 \), then the above condition is obviously reduced to \( \beta_k(0) \) (note that by definition \( f_k(0) = 0 \)). Moreover, \( \beta_k(0) \) is related to the initial width of the Gaussian packet (see (35)). Hence, the initial conditions needed to solve Eq. (22) are completely established.

Next, we calculate the quantum correlations for the quantized field modes. They are defined as [29]

\[
C_{1,1} = \frac{1}{2} \langle \left[ (\mu_k - \langle \mu_k \rangle), (p_k - \langle p_k \rangle) \right] \rangle, \tag{41}
\]

where \( \left\lbrack , \right\rbrack \) represents the anti-commutator. Hence, after minor algebra using Eqs. (29) and (41), we find that the quantum correlations are given by

\[
C_{1,1} = \frac{\hbar^2}{2B \tau}. \tag{42}
\]

From this expression, we can see that even when the initial state is uncorrelated, quantum correlations develop as time go on. Furthermore, the appearance of the correlations comes with an increase in the uncertainty. Actually, the uncertainty and the quantum correlations are directly related as
\[(\Delta \mu_k)(\Delta p_k) = \frac{\hbar}{2\sqrt{1 + \left(\frac{2C_{11}}{\hbar}\right)^2}}, \quad (43)\]

which clearly implies that the uncertainty is minimum when the correlation vanishes. Finally, we remark that the absence of correlation in the minimum uncertainty product is not unexpected at all, since correlations impose a constraint on the minimization of the uncertainty product. Consequently, the uncertainty product is not able to be minimum [30]. In fact, the generalized uncertainty product is given by [31]

\[(\Delta \mu_k)^2(\Delta p_k)^2 \geq \frac{1}{4} \left|\left[\mu_k, p_k\right]\right|^2 + C_{11}^2. \quad (44)\]

Thus, for the Gaussian wave packet state (29), the inequality (44) becomes the equality (43).

5. Conclusion

In this work, we have studied the problem of relic gravitons in an external time-dependent background, that is, a FRW cosmological background from a Schrödinger-picture point of view. We have mapped the problem into that of the generalized time-dependent harmonic oscillator and have solved the corresponding Schrödinger equation by using exact linear invariants and the invariant operator method. We have obtained the time-dependent Schrödinger states of the system as a function of the solution of the second order ordinary differential equation (22). Furthermore, this latter equation is completely determined by the scale factor \(a(t)\) of the metric. In addition, we have constructed Gaussian wave packet states whose probability density, quantum dispersions, quantum correlations and uncertainty product were determined also as function of solutions of Eq. (22). We have also seen that the linear invariants is readily diagonalized and that our Schrödinger state allow one easily to derive Gaussian wave packet states. Here, let us recall that the Gaussian packets are very common and the expectation value of physical quantities in these states are, in principle, easily evaluated. Yet, it seems that we would not have any problem in using the results of the present Letter to analyze specific phases of the evolution of the universe. We hope to report on this possibility in a future paper. Finally, we would like to point out that

the approach developed in this work can be useful to investigate quantum effects in others problems involving creation of particles in a non-trivial specific external time-dependent background.

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