Toward Generalized Entropy Composition with Different $q$ Indices and H-Theorem

Kenichi SASAKI* and Masahiro HOTTA**

Department of Physics, Tohoku University, Sendai 980-8578

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An attempt is made to construct composable composite entropy with different $q$ indices of subsystems and address the H-theorem problem of the composite system. Though the H-theorem does not hold in general situations, it is shown that some composite entropies do not decrease in time in near-equilibrium states and factorized states with negligibly weak interaction between the subsystems.

KEYWORDS: Tsallis statistics, H-theorem, compositability, bi-compositability

§1. Introduction

For the recent decade, generalization of the Boltzmann-Gibbs statistical mechanics has gradually attracted much attention. Using the $q$-modified entropy, so called Tsallis entropy:

$$S_q = -\frac{1}{1-q} \left[ 1 - \sum_i (p_i)^q \right],$$

many anomalous phenomena, including multifractals, self-gravitating systems, pure electron plasma respectively, may be described in "equilibrium" physics language. The index $q$ is a parameter unknown a priori and widely believed to be fixed by dynamical details beyond thermodynamical feature of the systems. Here the Tsallis probability variables $p_i$ are related with the physical probability $P_i$ as follows.

$$P_i = \frac{(p_i)^q}{\sum_k (p_k)^q}.$$ (2)

The entropy is also expressed in terms of the physical variables such that

$$S_q = -\frac{1}{1-q} \left[ 1 - \left( \sum_i (P_i)^\frac{1}{q} \right)^{-q} \right].$$ (3)

* e-mail:sasaki@tuhep.phys.tohoku.ac.jp
** e-mail:hotta@tuhep.phys.tohoku.ac.jp
Expectation value of energy $E$ is just given in the standard form as

$$\langle E \rangle = \sum_i E_i P_i.$$  \hfill (4)

Though recently a subtlety is pointed out when $P_i$ is treated as the fundamental variable, we in this paper focus on formulation by use of the variable $P_i$. It is also possible to discuss a composite system of two subsystems $A$ and $B$ with the same $q$ indices. The entropy is just extended like

$$S_{A+B} = -\frac{1}{1-q} \left[ 1 - \left( \sum_i \sum_j (P_{ij})^{\frac{1}{q}} \right)^{-q} \right],$$  \hfill (5)

and shows the following non-extensivity for factorized probability distribution $P_{ij} = P_i^A P_j^B$, where $P_i^A (P_j^B)$ is probability for the subsystem $A (B)$.

$$S_{A+B} = S_A + S_B + (1-q)S_A S_B.$$  \hfill (6)

Based on the modified entropy one can construct the canonical statistical theory and compare its predictions with experimental data of some exotic systems. It is known, for example, that peculiar distribution in the quasi-equilibrium state of the electron plasma is well reproduced by the theory with $q \sim 0.5$.

Although the generalized statistical theory seems to succeed in phenomenological descriptions of many exotic systems in which the standard statistics does not work, we must say that their theoretical foundation has not been fully understood. For example, it is still unclear whether the second law of the thermodynamics of composite systems with different $q$ subindices holds or not. Moreover it is impossible so far to discuss composite entropy of a Tsallis system with $q \neq 1$ and a thermometer made of ordinary matter ($q = 1$). The essential reason of the poor status comes from a fact that guiding principles for determination of the composite entropy have not yet been established.

In this paper, as one of possibilities, we investigate composable composite entropy of two Tsallis subsystems $A$ and $B$ with different indices $q_A$ and $q_B$. It is shown that bi-composability, especially, may possess a powerful ability for restriction of the entropy form. Here the bi-composability means that two kinds of composability are simultaneously satisfied. One of the composabilities is just an ordinary one:

$$S_{A+B}(P_{ij} = P_i^A P_j^B) = \lambda(S_A, S_B).$$  \hfill (7)

Thus the total entropy for a factorized distribution $P_{ij} = P_i^A P_j^B$ must be computed using only the values of subentropies $S_A$ and $S_B$. By virtue of the composability we are able to take a variation of the composite entropy directly using the subsystem variations as

$$\delta S_{A+B} = \frac{\partial \lambda}{\partial S_A} \delta S_A + \frac{\partial \lambda}{\partial S_B} \delta S_B.$$  \hfill (8)
Hence it is automatically ensured that the variation of the composite entropy vanishes:

$$\delta S_{A+B} = 0$$  \hspace{1cm} (9)

under a deviation as

$$\delta P_{ij} = \delta(P_A^i P_B^j) = \bar{P}_j^B \delta P_i^A + \bar{P}_i^A \delta P_j^B,$$  \hspace{1cm} (10)

where the distribution $$\bar{P}_i^A(\bar{P}_j^B)$$ gives extremum of the subentropy $$S_A(S_B)$$. This property will be helpful in building concept of thermal equilibrium even in the extended theory. Here the important point is that even if subsystems are statistically independent, there is no guarantee that the composablity should hold. In fact the counter examples can be seen in [11, 12, 13]. Another composability is introduced when one constructs a grand composite system $$(A + B) + (A + B)'$$ of two composite systems $$(A + B)$$ and $$(A + B)'$$. Let us impose the following property on their entropies.

$$S_{(A+B)+(A+B)'} = \Lambda(S_{(A+B)}, S_{(A+B)'})$$,  \hspace{1cm} (11)

where $$\Lambda$$ is an arbitrary function and no need to coincide with the functional form of $$\lambda$$. This implies that the value of the grand entropy is fixed only by information of the composite entropies $$S_{(A+B)}$$ and $$S_{(A+B)'}$$. The bi-composability is defined by simultaneous realization of the above two composabilities (7) and (11). Recall here that the Tsallis entropy satisfies the bi-composability when the $$q$$ indices of the subsystems are the same. Actually the first composability holds, just seen in eqn(6). Moreover a simple computation shows that the second composability is also satisfied as follows.

$$S_{(A+B)+(A+B)'} = S_{(A+B)} + S_{(A+B)'} + (1 - q)S_{A+B}S_{(A+B)'}.$$  \hspace{1cm} (12)

Meanwhile it may be instructive to note that some generalized entropies satisfy one of the composabilities (7) and (11) but the other not, as seen later. Therefore the bi-composableness is not automatically induced, even assuming a single composability (7) or (11) and statistical independence of the subsystems.

In Appendix A, we give the most general form of the bi-composable composite entropy. In later discussions, in order to argue as clearly as possible, we often explain by use of an attractive toy model, as an example, called Tsallis-type bi-composable entropy which satisfies a simple pseudo-additivity relation:

$$S_{(A+B)+(A+B)'} = S_{(A+B)} + S_{(A+B)'} + (1 - Q)S_{A+B}S_{(A+B)'}$$,  \hspace{1cm} (13)

even when the subindices take different values: $$q_A \neq q_B$$. The property (13) is apparently a straightforward extension of the result (12) in the same $$q$$ index case. The parameter $$Q$$ may be regarded as grand $$q$$ index of the composite systems $$(A + B)$$ and $$(A + B)'$$. It will be often noticed
in later sections that a lot of properties which a certain Tsallis-type model has remain still valid for more extended bi-composable entropies.

In this paper we also address the H-theorem problem. Though several works on the H-theorem for Tsallis statistics have been already performed, the analysis for composite systems with different q indices has not yet been discussed anywhere. It is proven in Section 3 that there exist some probability distributions in which time-derivative of any composable entropy which satisfies eqn (7) takes negative value. Meanwhile, it is also analytically shown that the H-theorem for any state near equilibrium exactly holds for some bi-composable entropies. Moreover, for factorized probability distributions with interaction between subsystems negligibly weak, the entropies do not decrease in time.

Noncomposable forms of the composite entropy may be also the other interesting alternative, but it looks fairly difficult so far to deal with the entropy systematically, just in a technical sense. It is beyond scope of this paper and we expect progress of the investigation in the near future.

§2. Composability and Composite Entropy

Let us start to write down the most general entropy form for a composite system $A + B$ of subsystems $A$ and $B$. The probability is denoted by $P_{ij}$ and the subscript $i$ and $j$ are associated with states of $A$ and $B$. The number of the states of $A(B)$ is denoted by $N(M)$ in the following.

Note here that for the subsystem $A(B)$ we must respect permutation symmetry in the subscripts $i(j)$ for the system $A(B)$. Even if the numbering of the $A(B)$ states are rearranged in an arbitrary way, the entropy must reproduce the same value as before. Hence the general form reads

$$S_{A+B} = S_0(R_1^A, R_1^B, R_2^A, R_2^B, R_3^A, R_3^B, \cdots)$$  \hspace{1cm} (14)

where

$$R_{\mu}^A = \sum_{j=1}^{M} \phi_{\mu}^{A} \left( \sum_{i=1}^{N} \psi_{\mu}^{A}(P_{ij}) \right),$$  \hspace{1cm} (15)

$$R_{\mu}^B = \sum_{i=1}^{N} \phi_{\mu}^{B} \left( \sum_{j=1}^{M} \psi_{\mu}^{B}(P_{ij}) \right),$$  \hspace{1cm} (16)

and $\phi_{\mu}^{A}, \psi_{\mu}^{A}, \phi_{\mu}^{B}$ and $\psi_{\mu}^{B}$ are arbitrary functions. Here $\mu$ is a label of function.

Next we construct the most general composable entropy which is reduced into a function of two subsystem Tsallis entropies $S_A$ and $S_B$ defined as

$$S_A = -\frac{1}{1-q_A} \left[ 1 - \left( \sum_{i=1}^{N} (P_i^A)^{\frac{1}{q_A}} \right)^{-q_A} \right],$$  \hspace{1cm} (17)

$$S_B = -\frac{1}{1-q_B} \left[ 1 - \left( \sum_{j=1}^{M} (P_j^B)^{\frac{1}{q_B}} \right)^{-q_B} \right].$$  \hspace{1cm} (18)
Here $q_A(q_B)$ denotes the $q$ index of the subsystem $A(B)$. In order to construct the entropy from eqn (14), it is worthwhile to point out that Tsallis entropies of the subsystems have probability dependence only through

$$C_A = \sum_{i=1}^{N} (P_i^A)^{r_A} = \left[ 1 + (1 - q_A)S_A \right]^{-\frac{1}{q_A}}, \quad (19)$$

$$C_B = \sum_{j=1}^{M} (P_j^B)^{r_B} = \left[ 1 + (1 - q_B)S_B \right]^{-\frac{1}{q_B}}, \quad (20)$$

where

$$r_A = \frac{1}{q_A}, \quad (21)$$

$$r_B = \frac{1}{q_B}. \quad (22)$$

Taking account of this fact it is easily noticed that the first composability (7) requires that

$$S_{A+B}(P_i^AP_j^B) = S_1 \left( C_A, C_B, \sum_{i=1}^{N} P_i^A, \sum_{j=1}^{M} P_j^B \right), \quad (23)$$

where $S_1$ is some function and $\sum P_i^A$ and $\sum P_j^B$ will be one after substitution of physical probabilities because of unitarity. The relation (23) gives a rather strong restriction to the functions $R^A_{\mu}$ and $R^B_{\mu}$. Let us explain the situation using the $R^A_{\mu}$ case. Substitution of $P_{ij} = P_i^AP_j^B$ into $R^A_{\mu}$ yields

$$\sum_{j=1}^{M} \phi_{i\mu}^A \left( \sum_{i=1}^{N} \psi_{i\mu}^A \left( P_i^AP_j^B \right) \right) = R^A_{\mu} \left( C_A, C_B, \sum_{i=1}^{N} P_i^A, \sum_{j=1}^{M} P_j^B \right). \quad (24)$$

Because there appears a sum over the subscript $j$ on the left-hand-side edge of the left-hand-side term in eqn (24), $R^A_{\mu}$ must be always a linear superposition of

$$\sum_{j=1}^{M} (P_j^B)^{r_B} f_{1\mu}^A \left( \sum_{i=1}^{N} (P_i^A)^{r_A}, \sum_{i=1}^{N} P_i^A, \right) \quad (25)$$

and

$$\sum_{j=1}^{M} P_j^B f_{2\mu}^A \left( \sum_{i=1}^{N} (P_i^A)^{r_A}, \sum_{i=1}^{N} P_i^A, \right). \quad (26)$$

Here functions $f_{1\mu}^A$ and $f_{2\mu}^A$ are not determined yet. It should be here stressed again that the partial probabilities $P_i^A$ and $P_j^B$ must enter into the expressions (25) and (26) as a simple product $P_i^AP_j^B$. Therefore the form of $R^A_{\mu}$ is severely constrained and we argue that there exist essentially only the following four candidates to take. From eqn (25) the following two components come out:

$$\sum_{j=1}^{M} (P_j^B)^{r_B} \left( \sum_{i=1}^{N} (P_i^A)^{r_A} \right) \quad (27)$$
\[
\sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{i}^{A} P_{j}^{B})^{r_{A}} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{ij})^{r_{A}} \right)^{r_{B}} = X_{1},
\]

(27)

\[
\sum_{j=1}^{M} (P_{j}^{B})^{r_{B}}^{r_{A}} \left( \sum_{i=1}^{N} P_{i}^{A} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} P_{i}^{A} P_{j}^{B} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} P_{ij} \right)^{r_{B}} = X_{2}.
\]

(28)

In the same way the following components can be found using eqn (26):

\[
\sum_{j=1}^{M} P_{j}^{B} \left( \sum_{i=1}^{N} (P_{i}^{A})^{r_{A}} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{i}^{A} P_{j}^{B})^{r_{A}} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{ij})^{r_{A}} \right)^{r_{B}} = X_{3},
\]

(29)

\[
\sum_{j=1}^{M} P_{j}^{B} \left( \sum_{i=1}^{N} P_{i}^{A} \right)^{r_{B}}^{r_{A}} = \sum_{j=1}^{M} \sum_{i=1}^{N} P_{ij} = 1.
\]

(30)

The last one is trivial and we have finally three-type building blocks \(X_{a}\) \((a = 1, 2, 3)\). The same argument is also possible for \(R_{\mu}^{B}\) and we conclude that the most general form of the composable composite entropy is given as a function \(\Omega\) in terms of six independent building blocks \(X_{a}\) and \(X_{b}\) as follows.

\[
S_{A+B} = \Omega(X_{a}, X_{b}; r_{A}, r_{B}),
\]

(31)

where

\[
X_{1} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{ij})^{r_{A}} \right)^{r_{B}}^{r_{A}},
\]

(32)

\[
X_{2} = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} P_{ij} \right)^{r_{B}},
\]

(33)
\[
X_3 = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} (P_{ij})^{r_A} \right)^{\frac{1}{r_A}},
\]
\[
\bar{X}_1 = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} (P_{ij})^{r_B} \right)^{\frac{r_A}{r_B}},
\]
\[
\bar{X}_2 = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} P_{ij} \right)^{r_A},
\]
\[
\bar{X}_3 = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} (P_{ij})^{r_B} \right)^{\frac{1}{r_B}}.
\]

Here permutation symmetry between the subsystems \(A\) and \(B\) should be requested in \(\Omega\), that is,
\[
\Omega(\bar{X}_b, X_a; r_B, r_A) = \Omega(X_a, \bar{X}_b; r_A, r_B).
\]

So far we have imposed only the first composability (7) on the entropy. It is an interesting problem to look for the most general bi-composable form which satisfies the two composabilities (7) and (11). The analysis has been achieved in Appendix A and the result is as follows.

\[
S_{A+B} = F(\Delta; r_A, r_B),
\]

where \(F(x; r_A, r_B)\) is an arbitrary function and
\[
\Delta = \prod_{a=1}^{3} (X_a)^{-\nu_a} \prod_{b=1}^{3} (\bar{X}_b)^{-\bar{\nu}_b} > 0,
\]
and the exponents \(\nu_a\) and \(\bar{\nu}_b\) are arbitrary constants. Apparently a lot of entropies which are composable but not bi-composable can be found. Actually it is possible to construct some entropies with the form (31) which dependence of \(X_a\) and \(\bar{X}_b\) occurs not only through \(\Delta\). This simple fact stresses again that the notion of bi-composability is completely independent of a single composability. Moreover if one assumes the pseudo-additivity (13) for the entropy (39), it is proven that the entropy must take the following form.

\[
S_{A+B} = -\frac{1 - \Delta}{1 - Q},
\]

The proof is given also in Appendix A. The form (31) is called Tsallis-type bi-composable in this paper because double composabilities (7) and (11) are simultaneously satisfied and \(\Lambda\) in eqn (11) shows the Tsallis-type non-extensivity with its index \(Q\). The Tsallis-type entropy is just a toy model, but as one of the simplest examples it is quite useful to draw what can happen in the composing procedure.

Dynamical details beyond the macroscopic features of the system are expected to determine, in principle, the functional form \(\Omega\), or \(F\), or the values of the indices \(q_A, q_B\) (and \(Q\) for the Tsallis-type...
model), and the exponents $\nu_a, \bar{\nu}_b$ of the bi-composable entropy, just like determination of the index $q$ in the case with $q_A = q_B = q$. So far we do not know the theoretical method to compute them explicitly. However, once one discovers certain composite systems which are described by canonical ensembles based on the entropies, it is evidently possible that they are measured by parameter fitting analysis of $P_{ij}$. The canonical ensemble will be given by extremization of the action $S$ defined as

$$S = S_{A+B} - \alpha \left( \sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} - 1 \right) - \beta \left( \sum_{i=1}^{N} \sum_{j=1}^{M} E_{ij} P_{ij} - \langle E \rangle \right),$$

where $\alpha$ and $\beta$ are Lagrange multipliers and generates the unitarity condition:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} = 1,$$

and the total energy constraint of canonical ensemble:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} E_{ij} P_{ij} = \langle E \rangle,$$

where $\langle E \rangle$ is expectation value of the total energy and $E_{ij}$ is energy of $(ij)$ state. For example, for the Tsallis-type bi-composable case, the equilibrium equation from eqn (42) just reads

$$\frac{\Delta}{Q - 1} \left[ \sum_{a=1}^{3} \frac{\nu_a}{X_a} \partial X_a \right] + \sum_{b=1}^{3} \frac{\bar{\nu}_b}{X_b} \partial X_b = \alpha + \beta E_{ij}.$$

The eqns (43),(44) and (45) generate the probability distribution $P_{ij}$ as a function of $q_A, q_B, Q, \nu_a$ and $\bar{\nu}_b$. Consequently one can find out the best fit values of the indices and the exponents from the experimental data which might discover in the future.

§3. H-theorem

In this section let us address the H-theorem problem for the composable entropies. Here we assume the positivity of the subindices: $r_A > 0, r_B > 0$. Time evolution of the entropy is now calculated by using the master equation:

$$\frac{dP_{ij}}{dt} = \sum_{k=1}^{N} \sum_{l=1}^{M} (\Gamma_{ij;kl} P_{kl} - \Gamma_{kl;ij} P_{ij}),$$

where $\Gamma_{ij;kl}$ is transition rate from $(kl)$ state to $(ij)$ state, thus takes positive value by definition.

Unfortunately the H-theorem does not hold in general situations. It is noticed after rather tedious calculation that there exist some probability configurations, governed by a special master equational dynamics, in which the whole composable entropy takes negative values of its time derivative.
as long as $q_A \neq q_B$. Thus the H-theorem is actually broken around the configurations. In order to explain how it fails, let us write down time derivative of the composable entropy (41), assuming the detailed balance situation:

$$\Gamma_{ij;kl} = \Gamma_{kl;ij}, \quad \text{(47)}$$

and using the master equation (40) as follows.

$$\frac{dS_{A+B}}{dt} = \sum_{ijkl} \Gamma_{ij;kl} S_{ij;kl}, \quad \text{(48)}$$

where

$$S_{ij;kl} = -\frac{1}{2}(P_{ij} - P_{kl}) \times \left[ \sum_a \frac{\partial \Omega}{\partial X_a} \left( \frac{\partial X_a}{\partial P_{ij}} - \frac{\partial X_a}{\partial P_{kl}} \right) + \sum_b \frac{\partial \Omega}{\partial \bar{X}_b} \left( \frac{\partial \bar{X}_b}{\partial P_{ij}} - \frac{\partial \bar{X}_b}{\partial P_{kl}} \right) \right]. \quad \text{(49)}$$

Next let us calculate the $S_{11;22}$ component in eqn (49) for the following configuration in the case with $(N, M) = (4, 4)$.

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} = \begin{bmatrix} x + \epsilon & 0 & y_{13} & y_{14} \\ 0 & x - \epsilon & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & 1 - 2x - \sum y_{cd} \end{bmatrix}. \quad \text{(50)}$$

What is firstly noticed here is that $S_{11;22}$ vanishes if we take exactly $\epsilon = 0$. As the result, expansion of $S_{11;22}$ in terms of $\epsilon$ behaves as

$$S_{11;22} = -\epsilon K + o(\epsilon^2). \quad \text{(51)}$$

Consequently if $K$ does not vanish in eqn (51), one can always take negative value of $S_{11;22}$ by taking the same sign of the infinitesimal parameter $\epsilon$ as that of $K$:

$$S_{11;22} \sim -\epsilon K < 0. \quad \text{(52)}$$

Therefore next let us prove the non-vanishingness of $K$. When we take $\epsilon = 0$, there exist twelve independent parameters $x$ and $y_{cd}$ in the expression of $P_{ij}$ (50). Hence the following twelve variables also are functions of the twelve parameters:

$$X_a = X_a(x, y_{cd}) \quad (a = 1, 2, 3), \quad \text{(53)}$$

$$\bar{X}_b = \bar{X}_b(x, y_{cd}) \quad (b = 1, 2, 3), \quad \text{(54)}$$

$$Y_a = \frac{\partial X_a}{\partial P_{11}}(x, y_{cd}) - \frac{\partial X_a}{\partial P_{22}}(x, y_{cd}) \quad (a = 1, 2, 3), \quad \text{(55)}$$

$$\bar{Y}_b = \frac{\partial \bar{X}_b}{\partial P_{11}}(x, y_{cd}) - \frac{\partial \bar{X}_b}{\partial P_{22}}(x, y_{cd}) \quad (b = 1, 2, 3). \quad \text{(56)}$$
After a tedious calculation, it can be shown that the Jacobian:

$$J = \left| \frac{\partial (X_a, \bar{X}_b, Y_a, \bar{Y}_b)}{\partial (x, y)} \right|$$

(57)
does not vanish as long as \( q_A \neq q_B \). Thus we can regard \( X_a, \bar{X}_b, Y_a \) and \( \bar{Y}_b \) as free parameters in the expression of \( K \). Because the explicit form of \( K \) is given as

$$K = \sum_{a'} \frac{\partial \Omega}{\partial X_{a'}} (X_a, \bar{X}_b) Y_{a'} + \sum_{y'} \frac{\partial \Omega}{\partial \bar{X}_{y'}} (X_a, \bar{X}_b) \bar{Y}_{y'},$$

(58)
it is trivial that one can always choose the independent parameters to achieve \( K \neq 0 \). Therefore we conclude that there exist the configurations for which \( S_{11;22} \) is negative. Consequently if we set transition rates \( \Gamma_{ij;kl} \) to zero except

$$\Gamma_{11;22} = \Gamma_{22;11} = 1,$$

(59)
decrease of the entropy for the configuration:

$$\frac{dS_{A+B}}{dt} < 0$$

(60)
is easily shown due to eqn (58). Hence we have found breakdown of the H-theorem for any composable entropy (31) in the configuration (50).

This failure might give a negative impression for the composable entropy itself, but we do not think that the situation is so serious. This is because the q-modified statistical picture itself might be inadequate to describe the physics for configurations like eqn (50). There is a possibility that the entropy permits the H-theorem to hold for physically relevant configurations like near-equilibrium states, where the q-deformed statistical description is able to work enough. In fact we can find some Tsallis-type bi-composable models as the examples in which the H-theorem rigidly holds around the equilibrium. When the detailed balance transition rate (47) is adopted, the equilibrium distribution of the model is just equal-weight profile:

$$\bar{P}_{ij} = \frac{1}{NM}.$$  

(61)

Near the equilibrium the probability is expressed as follows.

$$P_{ij} = \bar{P}_{ij} + \epsilon_{ij},$$

(62)

where \( \epsilon_{ij} \) are infinitesimal variables and satisfy the unitary condition:

$$\sum_{ij} \epsilon_{ij} = 0.$$  

(63)

For the configuration (62) we are able to compute \( S_{ij;kl} \) of the Tsallis-type bi-composable model (41) as follows.

$$S_{ij;kl} = -\frac{NM}{2 (Q - 1)} (\epsilon_{ij} - \epsilon_{kl})$$

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\begin{align*}
&\times \left\{ r_B(r_A - 1)\nu_1 + r_A(r_B - 1)\bar{\nu}_1 + (r_A - 1)\nu_3 + (r_B - 1)\bar{\nu}_3 \right\}(\epsilon_{ij} - \epsilon_{kl}) \\
&+ \left\{ r_B(r_B - r_A)\nu_1 + r_B(r_B - 1)\nu_2 - (r_A - 1)\nu_3 \right\} \frac{1}{N} \sum_{s=1}^{N} (\epsilon_{sj} - \epsilon_{sl}) \\
&+ \left\{ r_A(r_A - r_B)\nu_1 + r_A(r_A - 1)\bar{\nu}_2 - (r_B - 1)\bar{\nu}_3 \right\} \frac{1}{M} \sum_{s=1}^{M} (\epsilon_{js} - \epsilon_{ks}) \\
&+ o(\epsilon^3). \quad (64)
\end{align*}

Therefore if one chooses the grand index \( Q \) and the exponents \( \nu_a, \bar{\nu}_b \) of the model to allow the relations:
\begin{align*}
g &\equiv \frac{r_B(r_A - 1)\nu_1 + r_A(r_B - 1)\bar{\nu}_1 + (r_A - 1)\nu_3 + (r_B - 1)\bar{\nu}_3}{1 - Q} > 0, \quad (65) \\
r_B(r_B - r_A)\nu_1 + r_B(r_B - 1)\nu_2 - (r_A - 1)\nu_3 = 0, \quad (66) \\
r_A(r_A - r_B)\bar{\nu}_1 + r_A(r_A - 1)\bar{\nu}_2 - (r_B - 1)\bar{\nu}_3 = 0, \quad (67)
\end{align*}
all the coefficients \( S_{ij,kl} \) take non-negative values as
\begin{equation}
S_{ij,kl} \sim \frac{1}{2} g N M \Delta (\epsilon_{ij} - \epsilon_{kl})^2 \geq 0. \quad (68)
\end{equation}

Consequently it is concluded using both eqn \((48)\) and positivity of \( \Gamma_{ij;kl} \) that the value of the entropy does not decrease in time around the equilibrium:
\begin{equation}
\frac{dS_{A+B}}{dt} \geq 0.
\end{equation}

It is also possible to construct a simple and regular toy model of the Tsallis-type entropy satisfying eqns \((65), (66)\) and \((67)\), and it is given in Appendix II.

Furthermore, beyond the Tsallis-type model, let us introduce a more general bi-composable entropy as
\begin{equation}
S_{A+B} = G \left( \frac{1 - \Delta}{1 - Q} \right) \quad (69)
\end{equation}
which satisfies
\begin{equation}
\frac{d}{dx} G(x) > 0, \quad (70)
\end{equation}
and eqns \((65), (66)\) and \((67)\). Then it is trivial that the \( H \)-theorem around the equilibrium also holds for the entropy \((69)\) due to monotonically increasingness of \( G \).

It is quite remarkable that the bi-composable entropy \((69)\) does not decrease in time when interaction between the subsystems is negligibly weak and initial probability takes factorized form as
\begin{equation}
P_{ij}(0) = P^A_i(0) P^B_j(0). \quad (71)
\end{equation}
Due to the omission of the interaction, the transition rates can be written as
\[ \Gamma_{ij;k} = \Gamma^A_{i;k} \delta_{jl} + \delta_{ik} \Gamma^B_{j;l}, \]  
(72)
where \( \Gamma^A_{i;k}(\Gamma^B_{j;l}) \) is transition rate for the subsystem \( A(B) \). The factorized initial condition (71) and eqn (72) enable the master equation to be reduced into two subsystem equations as
\[ \frac{dP^A_i}{dt} = \sum_{k=1}^{N} (\Gamma^A_{i;k} P^A_k - \Gamma^A_{k;i} P^A_i), \]  
(73)
\[ \frac{dP^B_j}{dt} = \sum_{l=1}^{M} (\Gamma^B_{j;l} P^B_l - \Gamma^B_{l;j} P^B_j), \]  
(74)
and guarantees the factorized time-evolution form:
\[ P_{ij}(t) = P^A_i(t)P^B_j(t). \]  
(75)

Here it is convenient to introduce a Rényi-type entropy as
\[ S_R = \frac{1}{1 - Q} \ln \Delta. \]  
(76)
Then the time-derivative of the entropy (69) is related with that of the Rényi-type entropy (76) as follows.
\[ \frac{dS_{A+B}}{dt} = \Delta G' \left( -\frac{1 - \Delta}{1 - Q} \right) \frac{dS_R}{dt}. \]  
(77)

Also it is a useful property that the Rényi-type entropy (76) has additivity up to factors like that
\[ S_R(P^A_i P^B_j) = c_A S_{R,A} + c_B S_{R,B}, \]  
(78)
where \( S_{R,r_A} \) and \( S_{R,r_B} \) are the standard Rényi entropies defined by
\[ S_{R,A} = \frac{1}{1 - \bar{r}_A} \ln \left( \sum_{i=1}^{N} (P^A_i)^{r_A} \right), \]  
(79)
\[ S_{R,B} = \frac{1}{1 - \bar{r}_B} \ln \left( \sum_{j=1}^{M} (P^B_j)^{r_B} \right), \]  
(80)
and \( c_A \) and \( c_B \) are constants defined as
\[ c_A = \frac{1 - \bar{q}_A}{1 - Q} \left( \bar{r}_B \nu_1 + r_A \bar{\nu}_1 + r_A \bar{\nu}_2 + \nu_3 \right), \]  
(81)
\[ c_B = \frac{1 - \bar{q}_B}{1 - Q} \left( \bar{r}_B \nu_1 + r_A \bar{\nu}_1 + r_B \nu_2 + \bar{\nu}_3 \right). \]  
(82)

It has been already known that the Rényi entropy for positive index monotonically increases by the dynamics of the master equation:
\[ \frac{dS_{R,A}}{dt} \geq 0, \]  
(83)
\[ \frac{dS_{R,B}}{dt} \geq 0. \]  
(84)
By using eqns (66) and (67) in the expressions of \(c_A\) and \(c_B\), it can be shown that

\[
c_A > 0, \tag{85}
\]
\[
c_B > 0 \tag{86}
\]

if eqns (65) holds. Therefore it turns out that time-derivative of the entropy (69) entropy takes non-negative value:

\[
\frac{dS_{A+B}}{dt} \geq 0, \tag{87}
\]

because the inequality:

\[
\frac{dS_R}{dt} = c_A \frac{dS_{R,A}}{dt} + c_B \frac{dS_{R,B}}{dt} \geq 0 \tag{88}
\]

holds and

\[
\Delta G' \left( \frac{1 - \Delta}{1 - Q} \right) > 0 \tag{89}
\]

is guaranteed due to eqn (70). Consequently it has been proven that there exists the composable entropies which satisfy the H-theorem for both the near-equilibrium case and the uncorrelated subsystems case, and an explicite example is the form (69).

It may be interesting to comment, along Abe’s argument (also see[13]), on the thermal balance relation generated by the entropy (69), though the discussion is still in a rather heuristic level. It turns out that the nonextensivity of the entropy (69) is given as follows.

\[
S_{A+B} = G \left( -\frac{1 - \Delta}{1 - Q} \right),
\]

\[
\Delta = [1 + (1 - q_A)S_A]^{r_B
\nu_1 + \nu_3 + r_A\tilde{\nu}_1 + r_A\tilde{\nu}_2}
\times [1 + (1 - q_B)S_B]^{r_A\tilde{\nu}_1 + \nu_3 + r_B\nu_1 + r_B\nu_2}. \tag{90}
\]

Using the H-theorem conditions (66) and (67), the variation of the entropy can be written as

\[
\delta S_{A+B} = \frac{r_B(r_B - 1)(\nu_1 + \nu_2) + r_A(r_A - 1)(\tilde{\nu}_1 + \tilde{\nu}_2)}{1 - Q} \Delta G'
\]

\[
\times \left[ \frac{q_A}{1 + (1 - q_A)S_A} \delta S_A + \frac{q_B}{1 + (1 - q_B)S_B} \delta S_B \right]. \tag{91}
\]

Then we take \(\delta S_{A+B} = 0\) under the total energy conservation relation:

\[
\delta E_A + \delta E_B = 0 \tag{92}
\]

to get the thermal balance relation. The result is given as follows, independent of the functional form \(G(x)\) and the value of \(Q\).

\[
\frac{q_A}{1 + (1 - q_A)S_A} \frac{\delta S_A}{\delta E_A} = \frac{q_B}{1 + (1 - q_B)S_B} \frac{\delta S_B}{\delta E_B}. \tag{93}
\]
Note that this relation includes the Abe’s balance relation\(^{(12)}\) as a special case. Actually when \(q_A = q_B = q\) is taken

\[
\frac{1}{1 + (1 - q)S_A} \frac{\delta S_A}{\delta E_A} = \frac{1}{1 + (1 - q)S_B} \frac{\delta S_B}{\delta E_B} \tag{94}
\]

is exactly reproduced. Also eqn (93) is consistent with a guessed relation\(^{(6)}\) by Tsallis for the different \(q\) case. Though we have a subtlety that the canonical thermal state of the composite noninteracting system is not the product of the states of subsystems for the generalized entropy, there is a possibility that the above result \((93)\) remains unchanged for thermodynamic-limit situations just as in the same \(q\) case\(^{(12)}\) (also see\(^{(6)}\)).

If the system \(A\) is taken as an ordinary Boltzmann-Gibbs system \((q_A = 1)\), the subentropy is reduced into the BG form \(S_{BG:A}\). Then the relation \((93)\) is re-expressed as

\[
\frac{\delta S_{BG:A}}{\delta E_A} = \frac{q_B}{1 + (1 - q_B)S_B} \frac{\delta S_B}{\delta E_B}. \tag{95}
\]

Even in the exotic situation, the fact can be considered robust that physical temperature \(T_{phys}\) can be introduced for the system \(A\) as follows.

\[
\frac{1}{T_{phys}} = \frac{\delta S_{BG:A}}{\delta E_A}. \tag{96}
\]

Therefore observable temperature \(T_B\) of the Tsallis system \(B\) should be defined as

\[
\frac{1}{T_B} = \frac{q_B}{1 + (1 - q_B)S_B} \frac{\delta S_B}{\delta E_B}, \tag{97}
\]

so as to preserve the zero-th law of thermodynamics:

\[
\frac{1}{T_{phys}} = \frac{1}{T_B}. \tag{98}
\]

Here we want to mention that before our analysis no one argues explicitly presence of the numerator \(q_B\) of the prefactor in the right-hand-side term of eqn (97). Due to the definition (97), the original relation \((93)\), in which \(q_A\) is not needed to take unit, can be interpreted as a generalized thermal balance as follows.

\[
\frac{1}{T_A} = \frac{1}{T_B}. \tag{99}
\]

The transitivity relation \((99)\) looks quite plausible and attractive, though the derivation seems still nonunique.

In summary, though the H-theorem of all the composable entropies \((31)\) does not hold for some probability distributions like in eqn \((51)\), it has been rigorously proven that there exists a rather general entropy \((69)\) which does not decrease in time for both any states around the equipartion
equilibrium and the factorized distributions without interaction between the subsystems. We also
derive the thermal balance relation for the different-indices case from a plausible but still heuristic
argument.

Though analysis for noncomposable composite entropy has not been included in this paper,
the possibility remains, of course, still alive and may be interesting. However the systematic
investigation seems complicated and seems to need some innovation in the future.

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Appendix A: Bi-Composable Entropy

In this appendix, let us prove first that the most general bi-composable entropy takes the following
form.

\[ S_{A+B} = F(\Delta; r_A, r_B), \]

where the function \( F \) is arbitrary and \( \Delta \) is defined in eqn (40). Recall here that the most general
composable entropy is given in eqn (31). Thus our remaining task is just imposition of another
composability (11) on \( \Omega \) in eqn (31). To achieve this, it is very helpful to note a fact that

\[
\sum_{i=1}^{N} \sum_{i' = 1}^{N'} \left[ \sum_{j=1}^{M} \sum_{j' = 1}^{M'} \left( P_{ij}^{(A+B)} P_{i'j'}^{(A+B)'} \right)^{\alpha} \right]^{\beta} = \left( \sum_{i=1}^{N} \left( P_{ij}^{(A+B)} \right)^{\alpha} \right)^{\beta} \left( \sum_{i'=1}^{N'} \left( P_{i'j'}^{(A+B)'} \right)^{\alpha} \right)^{\beta}.
\]

(A.1)

Thus all the six components \( X_a \) and \( \bar{X}_b \) are completely factorized for the probability forms as

\[ P_{ijkl} = P_{ij}^{(A+B)} P_{kl}^{(A+B)'} \]

(A.2)

Consequently we obtain the following parametric relation among \( S_{(A+B)}, S_{(A+B)'} \) and
\( S_{(A+B)+(A+B)'} \cdot \)

\[ S_{(A+B)+(A+B)'} = \Omega(s_a t_a, \bar{s}_b \bar{t}_b; r_A, r_B), \]

(A.3)

\[ S_{(A+B)} = \Omega(s_a, \bar{s}_b; r_A, r_B), \]

(A.4)

\[ S_{(A+B)'} = \Omega(t_a, \bar{t}_b; r_A, r_B), \]

(A.5)

where

\[ s_a = X_a^{(A+B)}, \]

(A.6)
\[ t_a = X_a^{(A+B)'}, \]
\[ \bar{s}_b = \bar{X}_b^{(A+B)}, \]
\[ \bar{t}_b = \bar{X}_b^{(A+B)''}. \]

can be regarded as free parameters of the expression, independent from each other by taking \( N, N', M \) and \( M' \) largely enough. Using the relations (A.3)~(A.5), it is possible to write down a lot of parametric expressions of \( S_{(A+B)+(A+B)'} \) in terms of \( S_{(A+B)} \) and \( S_{(A+B)'} \). For example, we obtain an expression by taking

\[ s_1 = s_2 = s_3 = \bar{s}_1 = \bar{s}_2 = \bar{s}_3 = \phi, \]
\[ t_1 = t_2 = t_3 = \bar{t}_1 = \bar{t}_2 = \bar{t}_3 = \theta, \] \hspace{1cm} (A.10)

as follows.

\[ S_{(A+B)+(A+B)'} = \Omega(\phi \theta, \phi \theta, \phi \theta, \phi \theta; r_A, r_B), \] \hspace{1cm} (A.12)
\[ S_{(A+B)} = \Omega(\phi, \phi, \phi, \phi; r_A, r_B), \] \hspace{1cm} (A.13)
\[ S_{(A+B)'} = \Omega(\theta, \theta, \theta, \theta; r_A, r_B). \] \hspace{1cm} (A.14)

The above relations (A.12)~(A.14) indicate that \( S_{(A+B)+(A+B)'} \) is given as a function of \( S_{(A+B)} \) and \( S_{(A+B)'} \) implicitly via the two parameters \( \phi \) and \( \theta \).

Next we must find explicitly the general solution \( \Omega \) of eqns (A.3)~(A.5). In order to do that, let us consider first the following relation obtained from eqns (A.4) and (A.13).

\[ \Omega(s_a, \bar{s}_1, \bar{s}_2, \bar{s}_3; r_A, r_B) = \Omega(\phi, \phi, \phi, \phi; r_A, r_B). \] \hspace{1cm} (A.15)

By solving eqn (A.15) for \( \bar{s}_3 \), we can define a function \( \sigma \) as

\[ \bar{s}_3 = \sigma(\phi, s_a, \bar{s}_1, \bar{s}_2). \] \hspace{1cm} (A.16)

Substituting eqn (A.16) into eqn (A.3) and using eqn (A.12) yield

\[ \Omega(s_a, \bar{s}_1, \bar{s}_2, \bar{s}_3; r_A, r_B) = \Omega(\phi \theta, \phi \theta, \phi \theta, \phi \theta; r_A, r_B). \] \hspace{1cm} (A.17)

Then it is soon noticed that eqn (A.17) means

\[ \sigma(\phi, s_a, \bar{s}_1, \bar{s}_2) = Z(s_a, \bar{t}_1, \bar{t}_2, \phi \theta), \] \hspace{1cm} (A.18)

where \( Z \) is a certain function. Eqn (A.18) looks very complicated, but fortunately can be solved by treating the variables as six pairs: \( (s_a, t_a), (\bar{s}_1, \bar{t}_1), (\bar{s}_2, \bar{t}_2) \) and \( (\phi, \theta) \). For each pair we solve the
equation independently. Actually for an arbitrary pair \((\rho_s, \rho_t)\) in the six, eqn (A.18) is regarded as
\[
\sigma(\rho_s)\tilde{\sigma}(\rho_t) = Z(\rho_s\rho_t). \tag{A.19}
\]
From eqn (A.19) the following relations are straightforwardly obtained by taking \(\rho_s = 1\) or \(\rho_t = 1\).
\[
\sigma(\rho) = \frac{1}{\tilde{\sigma}(1)} Z(\rho), \tag{A.20}
\]
\[
\tilde{\sigma}(\rho) = \frac{1}{\sigma(1)} Z(\rho). \tag{A.21}
\]
Substituting eqns (A.20) and (A.21), eqn (A.19) is replaced into
\[
Z(\rho_s)Z(\rho_t) = Z(1)Z(\rho_s\rho_t). \tag{A.22}
\]
By taking differentiation with respect to \(\rho_t\) in eqn (A.22) and setting \(\rho_t = 1\) and \(\rho_s = \rho\), the following equation appears.
\[
\rho \frac{dZ}{d\rho} = \frac{Z'(1)}{Z(1)} Z(\rho), \tag{A.23}
\]
where \(Z'(1)\) is derivative of \(Z\) at \(\rho = 1\). This equation is easily integrated and leads us to the following relation:
\[
\sigma(\rho) = \frac{1}{\tilde{\sigma}(1)} Z(\rho) \propto \rho^\xi, \tag{A.24}
\]
where \(\xi\) is a constant. After the same procedure has been performed for all the six pairs, it is finally found that
\[
\bar{s}_3 = \sigma(\phi, s_a, \bar{s}_1, \bar{s}_2) \propto \phi^{\xi_\phi} \prod_{a=1}^3 (s_a)^{\xi_a} \prod_{b=1}^2 (\bar{s}_b)^{\xi_b}; \tag{A.25}
\]
where the exponents \(\xi_\phi, \xi_a\) and \(\xi_b\) are constants. Solving eqn (A.25) for \(\phi\) leads to the following result.
\[
\phi = C_o \prod_{a=1}^3 (s_a)^{-\nu_a} \prod_{b=1}^2 (\bar{s}_b)^{-\bar{\nu}_b} = C_o \Delta, \tag{A.26}
\]
where \(C_o, \nu_a\) and \(\bar{\nu}_b\) are also constants. By introducing a new function \(F\) as
\[
F(x; r_A, r_B) = \Omega(C_o x, C_o x, C_o x, C_o x, C_o x, C_o x; r_A, r_B), \tag{A.27}
\]
the final result (39) is achieved using eqns (A.15) and (A.26).

Next let us prove that the Tsallis-type bi-composable entropy takes the form (41). Firstly let us impose the grand pseudo additivity (13) on the bi-composable entropy (38). Then it turns out that the function \(F\) possesses the following property:
\[
F(xy; r_A, r_B) = F(x; r_A, r_B) + F(y; r_A, r_B) + (1 - Q)F(x; r_A, r_B)F(y; r_A, r_B). \tag{A.28}
\]
for arbitrary parameters $x$ and $y$. Differentiating eqn (A.28) with respect to $y$ and taking $y = 1$ yield

$$x \frac{\partial F}{\partial x} = F'(1; r_A, r_B) + (1 - Q)F'(1; r_A, r_B)F(x; r_A, r_B),$$

where $F'(1; r_A, r_B)$ is derivative of $F$ at $x = 1$. Eqn (A.29) can be integrated and we get the following solution:

$$F(x; r_A, r_B) = \frac{1}{1 - Q} + C'x^\eta,$$

where $C'$ and $\eta$ are constants. Substituting eqn (A.30) into eqn (A.28) fix the constant $C'$ as

$$C' = \frac{1}{1 - Q}$$

and finally the function $F$ is determined as

$$F(x; r_A, r_B) = -\frac{1}{1 - Q}[1 - x^\eta].$$

Therefore, redefining as

$$\eta \nu_a \to \nu_a,$$

$$\eta \tilde{\nu}_b \to \tilde{\nu}_b$$

in the definition of $\Delta$, it is concluded that the most general form of the Tsallis-type bi-composable entropy is given by eqn (41).

**Appendix B: A Regular Example of Tsallis-Type Bi-Composable Toy Model**

Let us discuss a plain case that the grand index $Q$ and the exponents $\nu_a$, $\tilde{\nu}_b$ behave as functions only in terms of $r_A$ and $r_B$. Then in the functions the permutation symmetry between the subsystems $A$ and $B$ should be preserved:

$$Q(r_A, r_B) = Q(r_B, r_A),$$

$$\tilde{\nu}_b(r_A, r_B) = \nu_b(r_B, r_A), \quad (b = 1, 2, 3).$$

Furthermore when we take $r_A = r_B$, that is, $q_A = q_B$, the composite entropy must be reduced into the original Tsallis entropy (5). Hence we require the following boundary conditions:

$$Q(r, r) = \frac{1}{r} = q,$$

$$\nu_1(r, r) = \frac{1}{2r} = \frac{q}{2},$$

$$\nu_2(r, r) = \nu_3(r, r) = 0.$$

Then let us introduce a entropy form as

$$\hat{S}_{A+B} = -\frac{r_A(r_A - 1)^2 + r_B(r_B - 1)^2}{(r_A - 1)(r_B - 1)(r_A + r_B - 2)}.$$
\[
\times \left[ 1 - \left( X_1^{r_B-1} X_1^{r_A-1} \left[ \frac{X_2}{X_2} \right]^{r_A-r_B} \right)^{\frac{2-r_A-r_B}{2(r_A-1)^2 + r_B(r_B-1)^2}} \right].
\]

(B.6)

Clearly if \( r_A = r_B \) is taken, the entropy in eqn (B.6) is reduced into the original Tsallis entropy. Also it can be pointed out that eqns (65), (64) and (67) certainly hold in eqn (B.6), thus the entropy satisfies the H-theorem for near equilibrium and factorized states.

Here one might worry about a "singularity" at \( r_A + r_B = 2 \) (B.7) in eqn (B.6). But fortunately it is explicitly checked that if one takes \( r_B \to 2 - r_A \) limit, any singularity does not take place and the entropy really has well-defined finite form as

\[
\lim_{r_B \to 2 - r_A} \hat{S}_{A+B} = -\frac{1}{2(1-r_A)} \ln \left( \frac{X_1}{X_1} \left( \frac{X_2}{X_2} \right)^2 \right).
\]

(B.8)

The limit of \( r_A \to 1 \) is also obtained as a regular form.

\[
\lim_{r_A \to 1} \hat{S}_{A+B} = -\frac{X_1}{2X_2} - \frac{1}{2} \bar{X}_{2,I} + \frac{1}{2(1-r_B)} \ln X_2 + \frac{1}{2(1-r_B)} \ln \bar{X}_3,
\]

(B.9)

where

\[
X_{1:2} = \lim_{r_A \to 1} \frac{X_1 - X_2}{r_A - 1}
\]

\[
= -r_B \sum_{j=1}^{M} \left( \sum_{i=1}^{N} P_{ij} \right)^{r_B} \ln \left( \sum_{i'=1}^{N} P_{i'j} \right) + r_B \sum_{j=1}^{M} \left( \sum_{i=1}^{N} P_{ij} \right)^{r_B-1} \ln \left( \sum_{i'=1}^{N} P_{i'j} \ln P_{i'j} \right),
\]

\[
\bar{X}_{2,I} = \lim_{r_A \to 1} \frac{\bar{X}_2 - 1}{r_A - 1}
\]

\[
= \sum_{i=1}^{N} \left( \sum_{j=1}^{M} P_{ij} \right) \ln \left( \sum_{j'=1}^{M} P_{ij'} \right).
\]

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