Kupka–Smale diffeomorphisms at the boundary of uniform hyperbolicity: a model

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Received 24 June 2016, revised 26 June 2017
Accepted for publication 10 July 2017
Published 15 September 2017

Abstract
We construct an explicit example of a family of non-uniformly hyperbolic diffeomorphisms, at the boundary of a set of uniformly hyperbolic systems, with one orbit of cubic heteroclinic tangency. One of the leaves involved in this heteroclinic tangency is periodic, and there is a Cantor set of choices of the second one. For a non-countable subset of these choices, the second leaf is not periodic and the diffeomorphism is Kupka–Smale: every periodic point is hyperbolic and the intersections of stable and unstable leaves of periodic points are transverse. The bifurcating system is Hölder-conjugated to a subshift of finite type; thus every Hölder potential admits a unique equilibrium state associated with it.

Keywords: Kupka–Smale diffeomorphisms, non-uniform hyperbolicity, equilibrium states
Mathematics Subject Classification numbers: 37D05, 37D25, 37D35

1. Introduction

1.1. Background

In this work we present an explicit example of a family of diffeomorphisms of a plane having heteroclinic cubic tangency inside the limit set. More precisely, there is a point of cubic tangency between stable and unstable foliations, which is accumulated by periodic points of the
system. Each diffeomorphism of the family is at the boundary of the set of uniformly hyperbolic diffeomorphisms. All the periodic points are hyperbolic, and the tangency is associated with a periodic leaf and a second leaf that can be chosen to be periodic or not.

The construction is part of a project to study the dynamic and ergodic properties of bifurcating systems. One considers systems at the boundary of uniformly hyperbolic ones in such a way that the lack of uniform hyperbolicity appears as a localized phenomenon, and the system satisfies a weaker notion of hyperbolicity (partial hyperbolicity, dominated splitting or non-uniform hyperbolicity).

One can reach the boundary of hyperbolicity by starting with a (uniformly) hyperbolic system, and perturbing it in a controlled way, so that the hyperbolicity degenerates. For instance, one can change the system in the neighborhood of a fixed point, in order to reduce the strength of expansion (or contraction), creating a neutral eigendirection for the derivative of the map. Or, as is the case here, one can perturb the system to continuously decrease the angle between stable and unstable foliations. In dimensions higher than two, it is possible also to perturb the system to create a heterodimensional cycle. These bifurcations are more easily achieved if the region of perturbation is far from other recurrent points, but to realize them inside the limit set requires extra work. One has to control the returns of the region to itself in order to guarantee that the hyperbolicity is not lost before the intended bifurcation.

For bifurcating systems, one is interested in features such as the Hausdorff dimension, conformal measures and equilibrium states, which are very well understood for uniformly hyperbolic systems, but still not established for systems at the boundary of hyperbolicity. The literature is very rich in that domain and we refer the reader to some references and other references therein: [4, 5, 17, 18] for general and complete surveys of bifurcation problems, [2–3, 12] for specific problems relating to SRB measures, and [7–9, 16, 19, 22] for thermodynamic formalism and Hausdorff dimensions.

In previous works of the authors, the mentioned problems were addressed by studying explicit examples or classes of examples. The general case proved not to be treatable in a single approach, since the answers to some of the questions (for instance, the uniqueness of equilibrium states) are intimately related to the cause of the bifurcation: there are indications that the loss of transversality is not as bad as the weakening of expansion and contraction rates. Many of the results for uniformly hyperbolic systems can be reproduced in the first case (see [13, 14]), but not in the second case (see [15]).

In [13] and [14], the authors considered a bifurcating horseshoe displaying quadratic (internal) homoclinic tangency associated with a fixed hyperbolic saddle. Since the manifolds of the saddle were accumulated by leaves of the set on just one side, it was possible to obtain the tangency as the first bifurcation. Most of the leaves of the horseshoe, though, were accumulated by other leaves on both sides, and a quadratic tangency involving those leaves could not be reached as the first bifurcation of a one-parameter family.

The strategy here is to start from a hyperbolic horseshoe and perturb it in a certain region in order to create a cubic tangency between stable and unstable foliations, which will be a first bifurcation. The topological transversality of the stable and unstable foliations is preserved at the moment of bifurcation, and there is a single orbit of tangency. This orbit is contained in the intersection of the unstable manifold of a fixed saddle and some leaves of the stable foliation. When the second leaf is not periodic, the system is Kupka–Smale; every periodic point is hyperbolic and the intersections of stable and unstable leaves of periodic points are transverse.

Systems with cubic heteroclinic tangencies were studied in [6, 10] and [11]. In the first two papers, the authors find cubic tangencies as first bifurcations of Anosov systems. In the third work, the authors prove that the cubic tangencies associated with non-periodic points are somehow prevalent as first bifurcations of Anosov systems, and they also find them in the
context of horseshoes. With their results, they prove a conjecture proposed by Bonatti and Viana (see conjecture 3.33 in [5]), that these bifurcations are abundant. In view of their results, it is even more important to understand these systems.

Although the context here and that of the work of Horita et al are quite similar, there are big differences between the two works. For instance, Horita et al find the prevalence of Kupka–Smale systems using parameter exclusion techniques. Here we have explicit diffeomorphisms, and complete control of the itinerary of the tangency. This provides a model with some important elements for the pursuit of ergodic properties.

We point out that there is a Cantor set of choices for one of the leaves involved in the heteroclinic tangency, and for a non-countable number of them, it is non-periodic. The possible choices of leaves involved in the bifurcation include periodic leaves and leaves with chaotic behavior. This allows a very rich set of possible dynamic phenomena.

Since the systems studied here are not uniformly hyperbolic, one of the main difficulties is to prove the existence of well-defined expanding and contracting directions (for points outside the critical orbit), and to set the existence and regularity of invariant manifolds. This is done through the construction of hyperbolic conefields for points that are not in the tangency orbit, and a geometric study of the return maps. This amount of hyperbolicity is enough to prove the existence and uniqueness of equilibrium states for Hölder-continuous potentials. As in the case of quadratic tangencies (see [14]), one important potential, the unstable Jacobian, is not in this class. This potential is being considered in a work in progress.

There are two questions we would like to raise. First, are there Kupka–Smale diffeomorphisms of a plane with quadratic (non-periodic) tangencies? Second, through our attempt to find a cubic tangency allowing good control of hyperbolicity estimates, we ruled out many possibilities. It seems that low-degree terms involving $x$, $y^3$ and $yx$ must be the only ones present up to degree 3. Different choices led to increased difficulties in producing the necessary estimates. Is there an intrinsic reason for that, or is it just an inadequacy of the method? In other words, does the term in $x^2$ affect the delicate balance between the time needed to return to the critical region and the rates of expansion/contraction?

1.2. Statement of results

Our main result is as follows:

**Theorem.** Any diffeomorphism $F$ of the family $A$, to be defined later, satisfies the following properties:

1. $F$ is at the boundary of the set of uniformly hyperbolic diffeomorphisms, having an orbit of heteroclinic tangency associated with two points in the limit set. This special orbit is said to be critical.
2. It is topologically conjugated to a subshift of finite type with nine symbols and the transition matrix given in (2).
3. Every point $x$ in the maximal invariant set $\Lambda := \bigcap_{k \in \mathbb{Z}} F^{-k}([0, 1]^2)$ and outside of the orbit of tangency is hyperbolic: there is invariant splitting $T_x \mathbb{R}^2 = E^u(x) \oplus E^s(x)$, where non-zero vectors in $E^u(x)$ expand exponentially, and vectors in $E^s(x)$ contract exponentially.
4. These sub-bundles are integrable in the sense that every $x \in \Lambda$ admits stable and unstable manifolds tangential to $E^u(x)$ and $E^s(x)$, respectively.
5. The critical orbit is the tangent intersection $W^u(P) \cap W^s(Q)$. The point $P$ is a fixed point for $F$ and the point $Q \in \Lambda$ can be chosen in a Cantor set of stable leaves.
The methods here differ from the method in [11]. There, they use parameter exclusion techniques applied to a family of bifurcating systems, to select those with no tangencies between periodic leaves. Here, we give the explicit form of the perturbation, describing how it produces a cubic tangency. The main difficulty in our case is to prove hyperbolicity via the construction of conefields. Our strategy follows the approach in [20] for the construction of the map and the conefields, and that in [13] to get the local stable and unstable manifolds. This explicit construction allows us to choose some properties for the orbit of \( Q \). It is, for instance, possible to choose whether that point is to be periodic or non-periodic.

As a consequence of the exponential rates of expansion and contraction along the invariant manifolds, the conjugacy with the subshift of finite type is a Hölder-continuous homeomorphism; then we have the following result.

**Corollary.** For every Hölder-continuous potential \( \phi : \Lambda \to \mathbb{R} \), there exists a unique equilibrium state supported by \( \Lambda \).

This paper is organized as follows. In section 2 we define the maps \( F \) which depend on several parameters. In section 3 we construct stable and unstable conefields and show that the map is (non-uniformly) hyperbolic. In section 4 we prove the existence of stable and unstable manifolds. In section 5 we prove that the family is in the boundary of non-uniformly hyperbolic diffeomorphisms.

**2. The map and the parameters**

**2.1. The initial map \( F_0 \)**

In this section, we define the maps \( F \). First, choose \( 0 < \lambda < \frac{1}{2} \), \( \sigma > \rho > 3 \) such that

\[
-1.2 < \frac{\log \lambda}{\log \rho} < \frac{\log \lambda}{\log \sigma} < -1. \tag{1}
\]

Note that this implies \( \lambda \rho < 1 \) and \( \lambda \sigma < 1 \). For the purpose of the present work, one can consider the case \( \sigma = \rho \). For future uses, though, we keep the two parameters independent. We conjecture that the different rates of expansion might interfere with properties concerning the uniqueness of equilibrium states for suitable potentials.

Consider a piecewise linear horseshoe map \( F_0 \) in \( Q = [0, 1]^2 \) with three components, as in figure 1. The rectangles in the intersection \( F_0(Q) \cap F_0^{-1}(Q) \) (they will be called the rectangles of the first generation) are labeled \( R_1 \) to \( R_9 \), from left to right and from top to bottom.
The horizontal and vertical directions are invariant by $DF_0$. The horizontal factor of contraction is $\lambda$ for all rectangles, and the vertical expansion is $\rho$ for $R_1, R_2$ and $R_3$, and $\sigma$ for the other rectangles of the first generation.

All rectangles of the first generation have the same horizontal length, denoted by $\tilde{l}_0$. Due to different expansions in the vertical directions, their vertical sizes are different. We use $l_0$ to denote the upper bound for all these vertical sizes. Then, we use $d$ to denote the lower bound for the distance between all the rectangles of the first generation.

We emphasize that

$$3l_0 + 2d \approx 1, \quad 3\tilde{l}_0 + 2d \approx 1.$$ 

By increasing $\lambda^{-1}, \sigma$ and $\rho$ we can choose $d$ that is as close to $1/2$ as desired and $l_0$ and $\tilde{l}_0$ that are as close to 0 as desired.

Note that each rectangle $R_i$ is mapped on a vertical strip that crosses three rectangles. The transition matrix for this partition is given by

$$A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},$$

(2)

where $A_{ij} = 1$ iff $F_0(R_i) \cap R_j \neq \emptyset$.

This horseshoe-like map is uniformly hyperbolic and is conjugated to the subshift in the space of sequences of nine symbols allowed by the transition matrix $A$. For any $k, n \in \mathbb{N}$, the maximal invariant set in $Q$ is contained in the intersection of the horizontal strips $H_0^k = \{ \tau \in Q : F_0^i(\tau) \in Q, \forall i, 0 \leq i \leq k \}$ of generation $k$ with the vertical strips $V_0^n = \{ \tau \in Q : F_0^{-i}(\tau) \in Q, \forall i, 0 \leq i \leq n \}$ of generation $n$. Each connected component of this intersection will be called a rectangle of generation $(n, k)$, and the set of all connected components will be denoted by $G_{0,n}^k$.

Each rectangle in $G_{0,n}^k$ has a horizontal size $\lambda^n$ and vertical size between $\rho^{-k}$ and $\sigma^{-k}$. More precisely a horizontal strip $H_0^k$ has a vertical length $\sigma^{-k_1}\rho^{-k_2}$, where $k_1$ and $k_2$ are the respective numbers of visits of points of the strip $H_0^k$ to the most and least significantly expanding zones, $k_1 + k_2 = k$ (note that these numbers are the same for all points in $H_0^k$).

Roughly speaking, points in $H_0^k$ have the same itinerary for the next forward $k$ iterates of $F_0$ and points in $V_0^n$ have the same itinerary for the next $n$ backward iterates. Points in some rectangles of $G_{0,n}^k$ have the same itinerary from $-n$ to $+k$.

The image by $F_0$ of one rectangle in $G_{0,n}^k$ intersects exactly three rectangles of the same generation, and contains (determines) three rectangles of generation $(n + 1, k)$.

We recall that $d$ is the lower bound for the horizontal and vertical gaps between the elements of the partition $G_{0,1}^1$; $l_0$ is the upper bound for the horizontal and vertical lengths of the elements of $G_{0,1}^1$. 

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2.2. Definition of the map $F$

In this section, we define the map $F$, based on the parameters $\lambda$, $\sigma$, and $\rho$ and some others to be introduced. We are going to assume some conditions on the parameters (to be stated as we go along), always respecting the conditions stated until now, including equation (1).

Note that each rectangle $R_i$ is a union of horizontal lines each consisting of points with the same future itinerary by $F_0$ (some of them are sent outside $Q$ by some future iterates).

2.2.1. The critical region.

Now we perturb $F_0$ in a small region contained in $R_4$ (to be referred to as the critical region) to obtain the map $F$. This small region is sent to $R_8$, and its image will be called the post-critical region.

We choose a point $\xi'$ belonging to the right-hand side of $R_8$. We choose it such that

1. its future itinerary never intersects $R_7 \cup R_4$,
2. its future itinerary intersects infinitely many times the set $R_5 \cup R_6$, and
3. its future itinerary intersects infinitely many times the set $R_1 \cup R_2 \cup R_3$.

Its pre-image $\xi = (0, \xi_2)$ belongs to the left-hand side of $R_4$, and all backward iterates of $\xi$ are in the left-hand side of $R_7$, converging to the fixed point $(0, 0)$.

Notation. We use $O(\xi)$ to denote the critical orbit, i.e.

$$O(\xi) := \{ F^k(\xi), \ k \in \mathbb{Z} \}.$$ 

We set $O^\pm(\xi)$ for the forward or backward orbit of $\xi$.

We change the map $F_0$ in the rectangle $\mathcal{R} = [0, \alpha_{\text{max}}] \times [\xi_2 - \beta_{\text{max}}, \xi_2 + \beta_{\text{max}}]$ (see figure 2). This region is referred to as the critical region, and $\xi$ is the critical point. The constants $\alpha_{\text{max}}$ and $\beta_{\text{max}}$ must satisfy several conditions, to be stated in later paragraphs.

In relation to these quantities we introduce two integers $n_c$ and $k_c$. We make sure $\mathcal{R}$ contains exactly one rectangle of $G_{0}^{n_c+1,k_c+1}$, namely $G_{0}^{n_c+1,k_c+1}(\xi)$, and has empty intersection with all the other rectangles of $G_{0}^{n_c+1,k_c+1}$.

It also has to be contained in a rectangle $G_{0}^{n_c,k}$ (see figure 2). For this, we choose a value of $k$ such that $F_0^k(\xi) \in R_3 \cup R_6$. By doing this, we can be sure that $G_{0}^{n_c+1,k_c+1}(\xi)$ is contained in the interior of the rectangle $G_{0}^{n_c,k}(\xi)$, except for the left boundary of $G_{0}^{n_c+1,k_c+1}(\xi)$, which is contained in the interior of the left boundary of $G_{0}^{n_c,k}(\xi)$. Other conditions on $n_c$ and $k_c$ will also be stated later.

Figure 2. Choice of critical region.
We consider \(k_1\) and \(k_2\) such that \(k_1 + 1 + k_2 = k_c + 1\) corresponding to the number of visits into the less and more significantly expanding zones of \(G_{n_c+1}^{k_c+1}\) based on the forward \(k_c + 1\) iterations of \(F_0\). By construction, the first expansion is \(\sigma\). The alternation of big (\(\rho\)) and small (\(\sigma\)) expansions is similar for every point in \(G_{n_c+1}^{k_c+1}\). The condition \(G_{n_c+1}^{k_c+1} = R \subset G_{n_c+1}^{k_c+1}(\xi)\) and \(G_{n_c+1}^{k_c+1}(M) \cap R = \emptyset\) if \(G_{n_c+1}^{k_c+1}(\xi) \neq G_{n_c+1}^{k_c+1}(M)\) can be realized if \(l_0\sigma^{-k_1} \rho^{-k_2} \leq \beta_{\text{max}} \leq d\sigma^{-k_1} \rho^{-k_2}\), \((3a)\)

\[\alpha_{\text{max}} = \tilde{l}_0 \lambda^{k_c}.\] \((3b)\)

These two conditions are referred to as conditions (3).

Figure 3. Perturbation of the dynamics.

We emphasize that \(k_c\) gives bounds to \(\beta_{\text{max}}\), and \(n_c\) gives bounds to \(\alpha_{\text{max}}\). The converse also holds.

Remark 1. We emphasize that \(k_c\) gives bounds to \(\beta_{\text{max}}\), and \(n_c\) gives bounds to \(\alpha_{\text{max}}\). The converse also holds.

2.2.2. Dynamics in the critical region. The map \(F\) is equal to \(F_0\) outside a neighborhood of \(R\). In \(R\) it is defined as follows (see figure 3). The vertical lines in \(R\) are sent over cubic curves and the horizontal lines are sent over verticals. This is done in such a way that \(F(\xi)\) is in the same horizontal line as \(F_0(\xi) = \xi'\). As \(\xi'\) never comes back to \(R_4\) by iteration of \(F_0\), for every
\( n \geq 0 \), \( F^n(F(\xi)) = F_0^n(F(\xi)) \) and then \( F^{n+1}(\xi) \) and \( F_0^{n+1}(\xi') \) always belong to the same \( R_j \). We check \( F(\xi) \) is centered with respect to the vertical strip \( \mathcal{V}_{\alpha_j}^{n+2}(\xi') \) in the horizontal direction (again, see figure 3). For \( \xi + (x, y) \in \mathcal{R} \), we set

\[
F(\xi + (x, y)) = F(\xi) + (\varepsilon_1, b\varepsilon_2 - cy(y^2 + x)).
\]

The constants \( c > 1, \varepsilon_1 < 1 \) and \( b \) are positive parameters of the map. For simplicity we set \( \mathcal{F}(x, y) = (-\varepsilon_1, b\varepsilon_2 - cy(y^2 + x)) \). In other words, we have

\[
F(\xi + (x, y)) = F(\xi) + \mathcal{F}(x, y).
\]

We assume that the parameters satisfy the following conditions:

\[
1 < 3c\beta_{\max} < \frac{1}{\varepsilon_1}, \quad (4a)
\]

\[
b = 2c\beta_{\max}, \quad (4b)
\]

\[
1 \ll \frac{d}{24\beta_{\max}}, \quad (4c)
\]

\[
1 \ll \frac{c}{8\varepsilon_1}. \quad (4d)
\]

These conditions will be referred to as system of conditions (4). Note that (4b) fixes the value of \( b \) if \( c \) and \( \beta_{\max} \) are fixed. Both conditions (4c) and (4d) will be used later for the construction of unstable conefields (see section 3.2). Condition (4a) will be used to control the Jacobian of \( \mathcal{F} \).

To control the future dynamics of \( F \) using the dynamics of \( F_0 \) we make sure that the image by \( F \) of \( \mathcal{R} \) stays in the interior of the vertical band \( \mathcal{V}_{\alpha_0}^{n+2}(F_0(\xi)) \) (see figure 3). We also check that \( F(\mathcal{R}) \) only intersects one element of \( \mathcal{G}_{\alpha_0}^{n+2} \) and that this intersection occurs in a special way, which we describe below.

All the cubic curves we consider are graphs over the horizontal interval of length \( 2\varepsilon_1\beta_{\max} \) centered at \( F(\xi) \). On the left-hand side these curves finish in a vertical segment, which is the image of the horizontal segment \( [0, \alpha_{\max}] \times \{\xi_2 + \beta_{\max}\} \). Its points are of the form \(( -\varepsilon_1\beta_{\max}, bx - c\beta_{\max}(\beta_{\max}^2 + x) \), where \( x \in [0, \alpha_{\max}] \). Similarly, on the right-hand side there is the vertical segment image of the horizontal segment \( [0, \alpha_{\max}] \times \{\xi_2 - \beta_{\max}\} \). Its points are of the form \(( \varepsilon_1\beta_{\max}, bx + c\beta_{\max}(\beta_{\max}^2 + x) \), where \( x \in [0, \alpha_{\max}] \).

These two segments are not at the same vertical position. If \( \alpha_{\max} \) is chosen to be very small, equality (4b) shows that the segment from the right has its bottom higher than the top of the segment from the left. We want the horizontal line \( H_0 (F_0(\xi)) \) to pass through these two vertical segments.

These conditions are realized if the following system holds:

\[
-c\beta_{\max}^3 > \\frac{-2d}{3} \sigma^{-k_1 + 1} \rho^{-k_2} + c\beta_{\max}^3 + 3c\beta_{\max}\alpha_{\max} < \\frac{-2d}{3} \sigma^{-k_1 + 1} \rho^{-k_2}, \quad (5a)
\]

\[
2\varepsilon_1\beta_{\max} < \tilde{\lambda}_0 \lambda_0^{n+1}, \quad (5b)
\]

\[
c\beta_{\max}\alpha_{\max} - c\beta_{\max}^3 < -l_0 \sigma^{-k_1 + 1} \rho^{-k_2} < l_0 \sigma^{-k_1 + 1} \rho^{-k_2} < c\beta_{\max}^3. \quad (5c)
\]

These conditions are referred to as conditions (5). Condition (5c) is the one to ensure exactly that the two vertical segments are torn apart by the horizontal strip \( H_0 (F_0(\xi)) \). Condition (5b)
means that the image $F(R)$ can be included in the vertical band $V_{n+2}^{m+2}(F_0(\xi))$. Condition (5a) ensures that the image $F(R)$ does not intersect other rectangles of $G_{n+2,k}^m$. Moreover, as the conditions hold with strict inequalities, the perturbation $F$ of $F_0$ (for the $C^0$-topology) can be made as regular as desired.

2.2.3. Dynamics outside the critical region. To finish describing the map in $R_4$ we have to explain how it is defined inside the gaps between the elements of $G_{n+1,k}^m$ of $G_{n}^m(\xi)$. Note that perturbation is conducted only on a small neighborhood of $G_{n+1,k}^m(\xi)$ which does not intersect other elements of $G_{n+1,k}^m$.

This neighborhood is however close to three other rectangles of $G_{n+1,k}^m$, the one at the right-hand side of $G_{n+1,k}^m(\xi)$, the one directly above and the one directly below. These last two rectangles are laminated by vertical lines, indexed by $x$. Their images by $F = F_0$ are also laminated by vertical lines, still indexed by $x$ (up to a linear scaling). In $G_{n+1,k}^m(\xi)$, the corresponding vertical lines are sent on cubics. They get ‘out’ of $F([0, \alpha_{\text{max}}] \times [-\beta_{\text{max}}, \beta_{\text{max}}])$ by the two vertical segments described above; they are still indexed by $x$. Then, we connect the line of index $x$ to the cubic with the same index, and do it such that the global envelope stays in the vertical band $V_{n+2}^m$ (see figure 3). This ensures that points which are not in the horizontal strip $H_{n+2}^m(\xi)$ have the same image by $F$ or $F_0$.

We still use this vertical lamination to explain the perturbation of the dynamics on both lateral sides of $G_{n+1,k}^m(\xi)$ and inside the perturbed zone in figure 2.

We require that points between the gaps of rectangles of $G_{n+2,k}^m$ do not belong to the image $F(Q)$.

Figure 4. Image of the perturbation of the dynamics outside $G_{n+1,k}^m(\xi)$ and inside the perturbed zone.

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2.3. Set of parameters

We can now prove that the set of parameters satisfying our conditions is large.

**Lemma 2.1.** The four series of conditions—(1), (3)–(5)—are available for an open set of parameters. This set of parameters allows $c$, $\sigma$ and $\rho$ to increase, and $\varepsilon_1$ and $\lambda$ to decrease.

**Proof.** We set

$$\beta_{\text{max}} = \frac{d}{2} \sigma^{-k_1} \rho^{-k_2}. \quad (6)$$

**Remark 2.** We emphasize that now, fixing $\beta_{\text{max}}$ determines $k_c$. Based on remark 1, this is equivalent to fixing $k_c$ or to fixing $\beta_{\text{max}}$. \qed

We assume that $\sigma$, $\rho$ and $\lambda^{-1}$ are very large. Remember that $\hat{h}_0, l_0 \approx \lambda$ and $d \approx 1/2$. This means that (3a) holds (and one can increase $\sigma$, $\rho$ and $\lambda^{-1}$). Moreover, our condition on the existence of infinitely many forward iterates for $F_0$ such that $F_0^k(\xi)$ belongs to $R_5$ or $R_6$ shows that $k_c$ can be taken to be as large as desired. Hence, $\beta_{\text{max}}$ can be chosen to be as small as desired.

Due to (3b), there exists a unique integer $n_c$ such that

$$\frac{\lambda \beta_{\text{max}}^2}{10} < \alpha_{\text{max}} \leq \frac{\beta_{\text{max}}^2}{10}. \quad (7)$$

This fixes $\alpha_{\text{max}}$ as soon as $\beta_{\text{max}}$ is fixed. Recalling remark 1, we assume that $\beta_{\text{max}}$ is chosen to be sufficiently small such that $n_c$ is larger than 5. We also need to choose $\beta_{\text{max}}$ to be sufficiently large such that (4c) holds.

Now, we set

$$\sigma = c \beta_{\text{max}}^2. \quad (8)$$

We recall that condition (4b) fixes the value of $b$ as soon as $c \beta_{\text{max}}$ is chosen. Therefore, we point out that at that stage of the construction, all the parameters, $c$, $b$, $k_c$, $\alpha_{\text{max}}$ and $n_c$, are fixed as soon as $\beta_{\text{max}}$ is determined.

Our choice for $\alpha_{\text{max}}$ yields

$$c \beta_{\text{max}}^3 < \frac{2d}{3} \sigma^{-k_1+1} \rho^{-k_2}. \quad (9)$$

We let the reader check that this, plus inequality (7), yields that inequalities (5a) and (5c) hold.

Now, to satisfy (5b), (4a) and (4d), it is sufficient that the following holds (remember (3b)):

$$2 \varepsilon_1 \beta_{\text{max}} < \lambda \alpha_{\text{max}},$$

$$3 \sigma \varepsilon_1 < \beta_{\text{max}},$$

$$\varepsilon_1 \beta_{\text{max}}^2 < \frac{1}{8}.$$  

All these conditions hold if, for a given $\beta_{\text{max}}$, $\varepsilon_1$ is chosen to be sufficiently small.

To conclude the proof, note that $c$ can increase as desired (if $\beta_{\text{max}}$ decreases), $\sigma$ can increase (then $\beta_{\text{max}}$ and $\lambda$ decrease and $\rho$ increases), and $\varepsilon_1$ can decrease. Integers $k_c$ and $n_c$ can be chosen to be as large as desired, up to the restriction on $k_c$ indicating that $F_0^k(\xi) \in R_5 \cup R_6$. \qed
Remark 3. Later we will impose more conditions on the parameters. In particular we shall require the ratio $\frac{c}{\varepsilon_1}$ to be high enough, with respect to other parameters. As we pointed out, this can always be done by increasing the value of $c$, keeping $c\beta_{\max}^2$ constant or decreasing $\varepsilon_1$. ■

3. Conefields, expansion and contraction

In this section we construct unstable and stable conefields and prove expansion and contraction in the hyperbolic splitting. The construction is stated for points in $Q \cap F(Q) \cap F^{-1}(Q)$ and outside the critical orbit. The unstable conefield is invariant by $DF$, in the sense that the image of the cone at $M$ by $DF(M)$ is included in the cone at $F(M)$. Analogously, the stable conefield is invariant by $DF^{-1}$. The properties of the expansion and contraction of vectors inside the unstable and stable cones will be proved only along pieces of orbits that stay inside $Q$ long enough. As a consequence, there is invariant splitting into stable and unstable subspaces of the tangent space at points of $\Lambda$ outside of the critical orbit.

3.1. Dynamical partition for $F$

Now consider the sets
\[
H^k = \{ \tau \in Q : F^i(\tau) \in Q, \forall i, 0 \leq i \leq k \},
\]
\[
V^n = \{ \tau \in Q : F^{-i}(\tau) \in Q, \forall i, 0 \leq i \leq n \},
\]
and $G^{n,k}$, the set of connected components of $H^k \cap V^n$. Note that they are defined using the perturbed map $F$.

Lemma 3.1. Let $k_c$ and $n_c$ be as in the choice of region $\mathcal{R}$. Then $Q \setminus H^{k_c}$ contains $Q \setminus H^0$. Similarly, $Q \setminus V^{n_c+2}$ contains $Q \setminus V^0$.

Proof. Let us consider a point $M$ in $Q \setminus H^0$. By definition, there exists $j < k_c$ such that $F_{-j}(M)$ belongs to $Q$ but $F_{-j}(M) \notin Q$.

This first shows that $M$ does not belong to $H^0_0(\xi)$; otherwise we could iterate $F_0$ for at least $k_c$ times. In particular $F(M) = F_0(M)$, and $F_0(M)$ belongs to $Q \setminus H^0_0(\xi)$. Again, this shows that $F(M) = F_0(M)$ does not belong to $H^0_0(\xi)$, and $F^2(M) = F^2_0(M)$. By induction, we finally get $F^i(M) = F^i_0(M) \notin Q$.

The proof for $Q \setminus V^{n_c+2}$ is the same, up to the fact that we iterate $F^{-1}$ and we have to consider the vertical band $V^{n_c+2}_0(F_0(\xi))$. □

Now consider the set $\Lambda := \bigcap_{n \in \mathbb{Z}} F^n([0, 1]^2)$.

Corollary 3.2. A point $M \in \mathcal{R} \cap \Lambda$ has a first return time in $\mathcal{R}$ greater than or equal to $k_c + 1$.

Proof. Pick a point $M$ in $\mathcal{R}$. If its image by $F$ is outside $G^{n_c+2,k_c}_0(F(\xi))$, then it belongs to $Q \setminus H^0_c$ and by lemma 3.1 it does not belong to $\Lambda$.

If it belongs to $G^{n_c+2,k_c}_0(F(\xi))$, then $F_0'(F(M))$ belongs to the same $R_j$ as $F_0'(\xi)$ for $j \leq k_c - 1$. Therefore $F_0'(F(M))$ does not come back to $R_j$ for $j \leq k_c - 1$ and $R_j$ is the unique place where $F$ and $F_0$ are different. Thus $F^{j+1}(M)$ coincides with $F_j'(F(M))$ for $j \leq k_c - 1$, and the first return time in $R_j$ is at least $k_c + 1$. □
Remark 4. An immediate corollary of lemma 3.1 is that rectangles of $G^{n,2,k}$ are inside rectangles of $G^{n,2,k}$.

3.2. Unstable conefield at the post-critical region

We introduce a new parameter which is a positive real number. We set

$$ A = \frac{c}{8\varepsilon_1}, $$

where $A$ is the parameter introduced above. This cone is centered on the vertical direction, and contains the tangent direction at $F_k$ to $M$. For $0 \leq k \leq k_0 - 1$, $F_k(M)$ belongs to the region where $F$ is linear. Then $DF_k(M)\cdot v := \left( \frac{\lambda^{k_0}}{A\sigma^{(i)}\beta^{(2)} u(3y^2 + x)} \right)$, where $k_0^{(1)}$ and $k_0^{(2)}$ are respectively the number of visits to the less and more significantly expanding rectangles. The derivative at $\xi + (\alpha, \beta) = F_k(M)$ is

$$ \begin{pmatrix} 0 \\ b - c\beta \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 \\ -c(3\beta^2 + \alpha) \end{pmatrix}, $$

We thus get...
$$DF^{k_0+1}(M) \cdot v = \begin{pmatrix} 0 & -\varepsilon_1 \\ b - c\beta & -c(3\beta^2 + \alpha) \end{pmatrix} \cdot \begin{pmatrix} \lambda^{k_0} \\ A\sigma^{(1)} \rho^{(2)}u(3y^2 + x) \end{pmatrix}.$$ 

$$= \begin{pmatrix} \lambda^{k_0}(b - c\beta) - c(3\beta^2 + \alpha)A\sigma^{(1)} \rho^{(2)}u(3y^2 + x) \\ -\varepsilon_1 \cdot A\sigma^{(1)} \rho^{(2)}u(3y^2 + x) \end{pmatrix} .$$ \hspace{1cm} (13)

Our goal is to prove that the right-hand side member of (13) belongs to \( C^0(F(\xi + (\alpha, \beta))) \). It is thus sufficient to prove that with our parameters, inequality

$$\frac{\lambda^{k_0}(b - c\beta)}{\sigma^{(1)} \rho^{(2)}A(3y^2 + x) \varepsilon_1} - \frac{c}{\varepsilon_1} (3\beta^2 + \alpha) \geq A(3\beta^2 + \alpha)$$

holds. The main idea is to prove that if the parameters satisfy conditions (4), then the first term in the left-hand side of (14) is smaller than half of the second term in the left-hand side. That is, we want to prove that our assumptions yield

$$0 \leq \frac{\lambda^{k_0}(b - c\beta)}{\sigma^{(1)} \rho^{(2)}A(3y^2 + x) \varepsilon_1} \leq \frac{c}{2\varepsilon_1} (3\beta^2 + \alpha).$$ \hspace{1cm} (15)

Indeed, assuming that (15) holds, (10b) (equivalent to (4d)) shows that (14) also holds because the left-hand side term is at least \( \frac{c}{2\varepsilon_1} (3\beta^2 + \alpha) \) and the right-hand side is \( \frac{c}{8\varepsilon_1} (3\beta^2 + \alpha) \).

Let us now prove that (15) holds. The fact that \( k_0 \) is the first visit for \( M \) close to the critical point (i.e. in \( \mathcal{R} \)) yields some additional inequalities. First, we claim that the vertical distance between \( F^{k_0}(M) \) and \( F^{k_0+1}(\xi) \) is strictly larger than \( d \); otherwise these two points would belong to the same \( R_i \). This is impossible because the forward orbit of the critical point never returns to the element \( R_i \). This yields

$$\sigma^{(1)} \rho^{(2)} |bx - cy(y^2 + x)| > d.$$ \hspace{1cm} (16)

Similarly, the backward orbit of the critical point never returns to the element \( R_i \); hence the horizontal distance between \( F^{-k_0}(\xi) \) and \( F^{-k_0}(\xi + (\alpha, \beta)) \) is larger than \( d \). This yields

$$\frac{\alpha}{\lambda^{k_0}} > d.$$ \hspace{1cm} (17)

Note that (17) is equivalent to \( \lambda^{k_0} < \frac{1}{d}\alpha \). Reporting this in \( \frac{\lambda^{k_0}(b - c\beta)}{\sigma^{(1)} \rho^{(2)}A(3y^2 + x) \varepsilon_1} \), we reach our goal if \( \sigma^{(1)} \rho^{(2)}A(3y^2 + x) \) is uniformly large (we obviously have \( \alpha \leq 3\beta^2 + \alpha \)).

3.2.1. The case of \( y \leq 0 \). As \( x \) is non-negative, inequality (16) is equivalent to

$$\sigma^{(1)} \rho^{(2)} (bx + c|y(y^2 + x)|) > d.$$ 

This and (4b) imply that we get \( \sigma^{(1)} \rho^{(2)} (2c\beta_{\max}x + c\beta_{\max}(y^2 + x)) > d \). This finally yields

$$\sigma^{(1)} \rho^{(2)} (3y^2 + x) > \frac{d}{3c\beta_{\max}}.$$ \hspace{1cm} (18)

This implies the inequality \( \sigma^{(1)} \rho^{(2)} A|u|(3y^2 + x) > \frac{Ad}{3c\beta_{\max}} \), and we finally get
Hence, we want to check
\[
\left( \frac{3c\beta_{\max}}{d} \right)^2 \frac{1}{\sigma_k(1)\rho_k(2)} < c \frac{1}{\varepsilon_1 A}.
\]  
This inequality is the same as the one in (18). Hence, at that point, the sufficient conditions are the same.

3.3. Extending the unstable conefield

3.3.1. Extending to \( \Lambda \setminus \mathcal{O}(\xi) \). For a point of the form \( M = F(\xi + (x, y)) \) (image by \( F \) of point \( \xi + (x, y) \) of the critical region), \( C^u(M) \) is already defined.

**Notation.** Let \( M \) be a point in the critical zone. We write \( M = \xi + (x, y) \) and set \( n_- = n_-(M) := \sup\{k, x \leq d\lambda^k\} \leq +\infty \) and \( n_+ = n_+(M) := \sup\{k, \sigma^k\rho|x - cy|\leq d\} \leq +\infty \).

**Definition 3.4.** The piece of orbit \( F^{-n_-}(M), \ldots, M, \ldots, F^{n_+}(M) \) is called the critical tube for \( M \).

Due to the fact that the forward orbit of \( \xi \) never returns to \( \mathcal{R} \), the critical tubes for an orbit are disjoint. Therefore, we can decompose an orbit into critical tubes and 'free' iterates (see figure 5).
We recall that the definition of \( \alpha_{\text{max}} \) implies that for \( M = \xi + (x, y) \) in the critical zone, \( F^{-k}(M) \) belongs to \( R_1 \) for \( k \in [1; 5] \). As we choose \( d \) that is close to \( 1/2, l_0 \) is very small and then \( n_- (M) \geq 5 \).

For a point of the form \( M = F^n (\xi + (x, y)) \) (image by \( F^n \) of point \( \xi + (x, y) \) of the critical zone) with \( 1 \leq n \leq n_+ (M) \), we set
\[
C^u (M) := DF^{n+1} (F]\xi + (x, y)) \cdot C^u (F\xi + F(x, y))
\]

For a point of the form \( \xi + (x, y) \) we set
\[
C^u (\xi + (x, y)) = \{ v = (v_x, v_y), |v_x| \geq \frac{\sigma^n - Ad}{3c \beta_{\text{max}}} |v_y| \}.
\]

Note that \( n_- = +\infty \) is possible if (and only if) \( x = 0 \). In that case the unstable cone is just the vertical line.

For a point of the form \( M = F^{-i} (\xi + (x, y)) \) with \( 1 \leq j \leq n_- (\xi + (x, y)) \), we set
\[
C^u (M) := DF^{-j} (C^u (F(M))).
\]

Lastly, for any other point \( M \), we set
\[
C^u (\xi + (x, y)) = \{ v = (v_x, v_y), |v_y| \geq \frac{Ad}{3c \beta_{\text{max}}} |v_x| \}.
\]

**Proposition 3.5.** The unstable conefield satisfies, for every point \( M \in \Lambda \setminus O(\xi), \)
\[
DF (M) \cdot C^u (M) \subset C^u (F(M)).
\]

**Proof.** Some inclusions are direct consequences of the definition: this is the case if \( M \) is of the form \( M = F^{-k} (M') \), with \( M' \) located in the critical zone and \( k \leq n_- (M') \). This also holds if \( M \) is of the form \( F^{k+1} (M') \), with \( M' \) located in the critical zone and \( k \leq n_- (F(M')) \).

We point out that for \( F^{n+} (M) \), the slope of the unstable cone is given by the expression
\[
\frac{\sigma^{n_1} \rho^{k_2}}{\lambda^n - A(3y^2 + x)}, \text{ where } k_2 \text{ and } k_1 \text{ satisfy } n_+ = k_1 + k_2 \text{ and are related to the time spent in } R_1 \cup R_2 \cup R_3 \text{ or other rectangles. It satisfies}
\]
\[
\frac{\sigma^{n_1} \rho^{k_2}}{\lambda^n - A(3y^2 + x)} \geq \frac{Ad}{\lambda^n + 3c \beta_{\text{max}}} > \frac{Ad}{3c \beta_{\text{max}}},
\]
\[
(20)
\]
since we have
\[
d < \sigma^{n_1} \rho^{k_2} |bx - cy (y^2 + x)| < \sigma^{n_1} \rho^{k_2} 3c \beta_{\text{max}} (3y^2 + x).
\]

If \( M \) is not of that form and neither belongs to the critical zone, then the result of the proposition is just a consequence of the linearity of the map \( F \) at \( M \).

If \( M = \xi + (x, y) \) is in \( \Omega \), we recall that the image of vertical vector \((0, 1)\) by \( DF(M) \) is \(- (\varepsilon_1, c(3y^2 + x))\), which belongs to the interior of the unstable cone at \( F(M) \). Now, we leave it to the reader to check that the same kind of computation as in the proof of lemma 3.3 shows \( DF (M) \cdot C^u (M) \subset C^u (F(M)). \) Just check that we get
\[
|b - cy| \lambda^n - 3c \beta_{\text{max}} \frac{c}{\varepsilon_1} \sigma^n - Ad < \frac{c}{2\varepsilon_1} (3y^2 + x),
\]
since (19) holds. \( \square \)
3.3.2. Extending the unstable conefield to $\mathcal{Q} \setminus \mathcal{O}(\xi)$. We want to point out that the construction of the unstable conefield does not require that the point lies in $\Lambda$. Actually, the conefield is firstly defined in the post-critical zone, and then extended to the rest of $\Lambda \setminus \mathcal{O}(\xi)$. The construction only requires that the piece of orbit we consider stays in $\mathcal{Q}$. We can thus extend the unstable conefield to every point in $\mathcal{Q} \setminus \mathcal{O}(\xi)$ and it will still satisfy $DF(M) \cdot C^u(M) \subset C^u(F(M))$ provided that $F(M)$ belongs to $\mathcal{Q}$.

3.4. Expansion and unstable directions

Here, we prove that vectors in the unstable conefield are expanded by forward iterations of $DF$. We get from this the existence of the unstable direction. We recall that for $v := (v_x, v_y)$ we set $\|v\|_\infty = \max(|v_x|, |v_y|)$. Let us start with a technical lemma.

**Lemma 3.6.** Let $M$ be located in the critical zone. Let $n < +\infty$ and $m < +\infty$ be respectively $n_-(M)$ and $n_+(M)$. Then for every $v$ in $C^u(F^{-n}(M))$ we get

$$\|DF^{n+1+m}(F^{-n}(M)) \cdot v\|_\infty \geq \rho^{\frac{1}{2}(n+1+m)}.$$

**Proof.** Let us set $M = \xi + (\alpha, \beta)$. Note that our assumptions $n, m < +\infty$ yield that neither $\alpha$ nor $\beta$ vanishes. Let us pick $(v_x, v_y) \neq 0$ in $C^u(F^{-n}(M))$. We recall $\|v\|_\infty = \max(|v_x|, |v_y|)$. We also consider the vector $w = (1, A(3\beta^2 + \alpha))$ in $C^u(F(M))$.

By definition of the conefield $C^u$, $|v_y| \geq \frac{Ad}{3c\beta_{\max}}|v_x| \geq |v_x|$, where the last inequality follows from our choice of parameters (see inequality (10a)). Hence, there is exponential expansion for $v$ in the piece of orbit between $F^{-n}(M)$ and $M$ because $DF \equiv DF_0$, which expands verticals and contracts horizontals.

Our strategy is to consider two cases. Either $m$ is ‘small’ with respect to $n$ and the expansion during the entrance phase (from $-n$ to 0) is sufficient to ensure global exponential expansion between $-n$ and $m$, or $m$ is large, and then there also is exponential growth in the unstable direction between times 0 and $m$.

By the construction of the unstable cone, the vector $w$ is more horizontal than $DF^{n+1}(x) \cdot v$, and hence is less significantly expanded (remember that $F \equiv F_0$ for these points). Moreover, the vertical component of $DF^{n}(F(M)) \cdot w$ is larger (in terms of the absolute value) than the horizontal one (its slope is at least $\frac{1}{3}\frac{Ad}{3c\beta_{\max}}$, which is larger than $\frac{1}{\epsilon_1}$ based on inequality (4c)).

Therefore, the expansion (between times $F(M)$ and $F^n(M)$) of any vector in the unstable cone $C^u(F(M))$ is greater than $\sigma^n A(3\beta^2 + \alpha) \geq \sigma^n A(\beta^2 + \alpha) \geq \sigma^n A\alpha$.

On the other hand, the matrix of $DF(M)$ on a canonical basis is

$$\begin{pmatrix} 0 & -\epsilon_1 \\ b - c\beta & -c(3\beta^2 + \alpha) \end{pmatrix}.$$

By the definition of the map we get

$$\tilde{v} := DF^n(F^{-n}(M)) \cdot v = (\lambda^n v_x, \sigma^n v_y),$$

and the vertical component of this vector is larger than the horizontal one. Therefore we get

$$\|DF(M)\tilde{v}\|_\infty \geq \epsilon_1 \|\tilde{v}\|_\infty,$$

which yields

$$\text{on the one hand, } \|DF^{n+1}(M)\tilde{v}\|_\infty \geq \rho^{\frac{1}{2}} \sigma^n A\epsilon_1 \|\tilde{v}\|_\infty,$$

(22a)
and on the other hand, due to (20), \[\|DF^{m+1}(M)\mathbf{v}\|_\infty \geq \frac{A\varepsilon_1}{3\varepsilon \beta_{\max}} \cdot \|\mathbf{v}\|_\infty.\]

(22b)

3.4.1. First case: \(m + 1 \leq 2n\). We use inequality (22b). We recall that inequality (10a) means

\[\frac{A\varepsilon_1}{3\varepsilon \beta_{\max}} > 1,\]

which shows that there is expansion (though not necessarily exponential) in the unstable direction between \(M\) and \(F^m(M)\). Therefore, the expansion for \(\mathbf{v}\) by the derivative \(DF^{m+1+n}(F^{-n}(M))\) is greater than \(\sigma^n\) (the expansion between \(F^{-n}(M)\) and \(M\)).

Hence, the logarithmic expansion between \(F^{-n}(M)\) and \(F^m(M)\) is greater than

\[\frac{n}{n + 1 + m} \log \sigma \geq \frac{1}{3} \log \sigma \geq \frac{1}{5} \log \sigma \geq \frac{1}{5} \log \rho.\]

3.4.2. Second case: \(m + 1 > 2n\). We use inequality (22a). Using inequality (1), the expansion between \(M\) and \(F^m(M)\) in the unstable direction is greater than

\[\rho^n A_{\varepsilon_1} \alpha \geq \rho^{1.2n + 0.82} A_{\varepsilon_1} \alpha \geq \rho^{1.2n + 0.82} - 1 A_{\varepsilon_1} \alpha \geq \alpha\frac{\varepsilon}{\lambda^5} A_{\varepsilon_1} \rho^{0.4n - 1 - \varepsilon} \geq \rho^{0.4n - 1 - \varepsilon},\]

(note that \(m \geq 9\) since \(n \geq 5\)). Now, we use inequality (17) and again (4c) and (4a) to conclude that the logarithmic expansion between \(F^{-n}(M)\) and \(F^m(M)\) in the unstable direction is greater than

\[\frac{n + \frac{1}{5}(m + 1)}{n + 1 + m} \log \rho \geq \frac{1}{5} \log \rho.\]

The proof is finished. \(\Box\)

We can now prove expansion for vectors in the unstable cone field.

**Proposition 3.7.** Let \(M\) be located in \(\Lambda \setminus O(\xi)\). Then for every \(0 \neq \mathbf{v} \in C^0(M)\), we get

\[\liminf_{n \to +\infty} \frac{1}{n} \log \|DF^m(M) \cdot \mathbf{v}\|_\infty \geq \frac{1}{5} \log \rho.\]

**Proof.** The result is obvious if \(M\) returns only a finite number of times to the critical zone. We thus focus on the difficult case, where \(M\) visits infinitely many times the critical zone by iterations of \(F\). For simplicity we assume that \(M\) belongs to the critical zone. Then we use \(n_i\) to denote the time of the \(i\)th visit to the critical zone. We also use \(n_-(i)\) and \(n_+(i)\) to denote the integers \(n_-(F^n(M))\) and \(n_+(F^n(M))\).

We point out that, due to our choice of \(d \approx 1/2\) and \(l_0 \approx \lambda^{-1}\), \(F^{n_j}(M)\) belongs to \(R_j\) for \(0 < j < n_-(i)\). On the other hand \(F^{n_j+1}(M)\) belongs to the same \(R_i\) as \(F^j(\xi)\). Since \(\xi\) never comes back to \(R_i \cup R_j\) by forward iterates, it is sure that

\[n_i + n_+(i) < n_{i+1} - n_-(i + 1).\]

Lemma 3.6 shows that for every \(i \geq 1\), the logarithmic expansion between \(F^{n_-(i)}(M)\) and \(F^{n_i+1+n_+(i)}(M)\) is greater than \((n_-(i) + 1 + n_+(i))\frac{1}{5} \log \rho\).
Moreover, we get for every \( i \geq 1, n_i + 1 + n_+(i) \leq n_{i+1} - n_-(i + 1) \). By the construction of the unstable conefield, the logarithmic expansion between \( F_{n_+ + n_-(i+1)}(M) \) and \( F_{n_+ + n_-(i+1)}(M) \) is greater than \( \log \rho \) (considering the norm given by the greatest coordinate in its absolute value).

Lemma 3.6 and the splitting of orbits as performed in the proof of proposition 3.7 show the next two results. To see this we need to introduce a new notion:

**Definition 3.8.** Let \( M \) be located in \( \bigcap_{k=0}^N F^{-k}(Q) \). We say that the \( n+1 \)-sized piece of orbit \( \ldots, F^n(M) \) is complete if the time interval \([0, n]\) contains a full number of critical tubes.

Then lemma 3.6 yields:

**Proposition 3.9.** Let \( \ldots, F^n(M) \) be a complete piece of orbit. Then, for every \( v \) in \( C^u(M) \),

\[
\|DF^{n+1}(M) \cdot v\|_\infty \geq \rho^n \|v\|_\infty.
\]

We propose that lemma 3.6 also yields the next proposition, which is very common in the so-called Pesin theory: expansion and contraction are on a uniform exponential scale up to a multiplicative term which depends on the point (see figure 6). Note that we do not require point \( M \) to be in the critical zone.

**Proposition 3.10.** Let \( M \) be located in a critical tube, say \([-N_-, N_+]\). Then, there exist two positive numbers \( l^+(M) \) and \( l^-(M) \) such that the following conditions hold:

1. For any \( n > N_+ \), if \( F^{n+1}(M), \ldots, F^n \) is complete, then for every vector \( v \) in \( E^u(M) \),

\[
\|DF^n(M) \cdot v\|_\infty \geq l^+(M) \rho^n \|v\|_8.
\]

2. For any \( n > N_- \), if \( F^{-n}, \ldots, F^{-N_- - 1}(M) \) is complete, then for every vector \( v \) in \( E^u(M) \),

\[
\|DF^{-n}(M) \cdot v\|_\infty \leq l^-(M) \rho^{-n} \|v\|_8.
\]

Moreover, \( l^+(M) = l^+(M') \) if \( M' \) belongs to \( B_{N_+ + 1}(M, l_0) \) and \( l^-(M) = l^-(M') \) if \( M' \) belongs to \( \bigcap_{k=0}^{N_- + 1} F^k B(F^{-k}(M), l_0) \).

Now, we can prove the existence of the unstable direction.
Proposition 3.11. For $M \in \Lambda \setminus \mathcal{O}(\xi)$, $\bigcap_{n \geq 0} DF^n(\mathcal{C}^u(F^{-n}(M)))$ is a single direction in $\mathbb{R}^2$. It is called the unstable direction at $M$ and denoted by $E^u(M)$.

Proof. The set $\bigcap_{n \geq 0} DF^n(\mathcal{C}^u(F^{-n}(M)))$ is a decreasing intersection of cones; thus it has a non-empty intersection. If $v$ and $w$ are two normalized vectors in this intersection, we consider two renormalized pre-images $v_n$ and $w_n$ by $DF^{-n}(M)$. We get

$$|\det DF^n| = \frac{|\sin(\angle(v, w))|}{|\sin(\angle(v_n, w_n))|} \cdot \|DF^n(v_n)\| \cdot \|DF^n(w_n)\|.$$ 

We point out that inequalities (4a) and (1) show that $|\det DF| < 1$ everywhere. Then

$$|\sin(\angle(v, w))| \leq \frac{|\det DF^n|}{\|DF^n(v_n)\| \cdot \|DF^n(w_n)\|}$$

and proposition 3.10 shows that $\angle(v, w)$ decreases exponentially with $n$ as $n \to +\infty$. □

Remark 5. Note that $E^u(M)$ is actually well defined for every point whose full backward orbit is well defined.

3.5. Stable direction

In this section we prove the existence of the stable direction and the exponential contraction of vectors in that direction.

The stable cone $\mathcal{C}^s(M)$ is defined as the closure of the complement (in $\mathbb{R}^2$) of the unstable cone. It is defined in the same domain as the unstable conefield. It satisfies

$$DF^{-1}(\mathcal{C}^s(M)) \subset \mathcal{C}^s(F^{-1}(M))$$

provided $F^{-1}(M)$ belongs to $\mathcal{Q}$.

Proposition 3.12. For every $M \in \bigcap_{n \geq 0} F^{-n}(\mathcal{Q}) \setminus \mathcal{O}(\xi)$, $\bigcap_{n \geq 0} DF^{-n}(\mathcal{C}^s(F^n(M)))$ is a single direction, called the stable direction at $M$ and denoted by $E^s(M)$.

Moreover, let $\Delta := \max |\det DF|$. Then there exists some universal constant $C$ such that for every complete piece of orbit $M, \ldots, F^n(M)$,

$$|DF^n_{|E^s(M)}| < C.\Delta^n\rho^{-\frac{n}{5}}.$$ 

Proof. If there is $n_0$ such that $F^n(M)$ does not belong to the critical zone $\forall n > n_0$, then the result follows the hyperbolic case, for $F^n(M)$, and the unique stable direction for this point is iterated back to define the stable direction for $M$.

For the rest of the proof we assume that the forward orbit of $M$ visits the critical zone infinitely many times. We can thus split this forward orbit into critical tubes and free moments. Note that we have seen in the proof of proposition 3.7 that there necessarily exists at least one free moment between two consecutive critical tubes. Then, we can always assume that $M$ belongs to the end of a critical tube and that $F(M)$ belongs to a free moment.
We claim that $DF(M)C^s(M)$ is uniformly far from the border of $C^u(F(M))$. At the end of a critical tube, inequality (20) shows that the unstable cone is strictly inside the cones defined by $\{v = (v_1, v_2), \ |v_2| \geq \frac{Ad}{3c_{\beta_{\text{max}}}} |v_1| \}$. Then its image by the linear map $DF(M)$ is strictly inside $C^u(F(M)) = \{v = (v_1, v_2), \ |v_2| \geq \frac{Ad}{3c_{\beta_{\text{max}}}} |v_1| \}$ and uniformly far from its border.

This shows that, if $F^k(M)$ is in a free moment, then $\forall v \in \bigcap_{n \geq 0} DF^{-n}(C^s(F^{n+1}(M)))$ and $\forall w \in DF \cdot C^u(F^{k-1}(M))$, the angle between $v$ and $w$ is uniformly bounded from below away from zero. So are their sines, and we choose $C$ that is a positive lower bound.

Consider $v$ in $\bigcap_{n \geq 0} DF^{-n}(C^s(F^{n+1}(M)))$ and $w$ in $DF \cdot C^u(F(M))$. Let $v_{n+1}$ and $w_{n+1}$ be normalized and collinear respectively to $DF^{n+1}(M) \cdot v$ and $DF^{n+1}(M) \cdot w$. Again we use the equality

$$\det DF = \frac{\sin(\angle(v_{n+1}, w_{n+1}))}{\sin(\angle(v, w))} \cdot \|DF^{n+1}(M)\| \cdot \|DF^{n+1}(M) \cdot w\|.$$ 

The left-hand-side term is exponentially small, $\|DF^{n+1}(M) \cdot w\|$ is exponentially large (due to proposition 3.10), and the two sines are bounded above by one and below away from zero, if $n$ is chosen such that $F^{n+1}(M)$ belongs to a free moment (which happens for infinitely many integers $n$).

Therefore, for every $n$ and $v$ as above

$$\|DF^{n+1}(M)\| \leq C \Delta^n \rho^{-\frac{\Delta}{\rho}} \|v\|. \quad (23)$$

We propose that inequality (23) shows that $\bigcap_{n \geq 0} DF^{-n}(C^s(F^{n+1}(M)))$ is a single vector. Otherwise, let us pick $v$ and $v'$ in $\bigcap_{n \geq 0} DF^{-n}(C^s(F^{n+1}(M)))$ and adjust them such that $v' = v + w$, with $w$ in $DF \cdot C^u(F(M))$. Pick $n$ as above. We get

$$DF^{n+1}(F) \cdot v' = DF^{n+1}(M) \cdot v + DF^{n+1}(M) \cdot w.$$

Two of these terms are exponentially small and the last one is exponentially large. This is impossible.

**Corollary 3.13.** Every ergodic $F$-invariant measure is hyperbolic: it has one (unstable) positive Lyapunov exponent $\lambda^s \geq \frac{1}{2} \log \rho$ and one (stable) negative Lyapunov exponent $\lambda^s \leq \log \Delta - \lambda^s$.

**Proof.** If $\mu$ is any ergodic $F$-invariant measure, it is clearly hyperbolic. Moreover

$$\lambda^s + \lambda^u = \lim_{n \to +\infty} \frac{1}{n} \log |DF^n(M)|$$

keeps $\mu$ constant a.e., which yields $\lambda^s \leq \log \Delta - \lambda^s$. \qed
4. Local stable and unstable manifolds

The goal of this section is to construct the local unstable and stable manifolds. The construction has three steps. In the first step we construct a candidate as the local unstable manifold (see proposition 4.2). In the second step we construct a candidate as the local stable manifold (see proposition 4.5). In the third subsection, we prove that they are indeed the local unstable and stable sub-manifolds. Actually, the main obstacle to the construction is that we have to consider complete critical tubes. This makes it more complicated to show that points in the stable (or unstable) manifold are characterized by the fact that the respective distance goes to zero along forward (or backward) orbits.

4.1. Invariant family of local unstable graphs

Here we construct an invariant family of local unstable graphs for points which return infinitely many times to $\mathfrak{R}$ by iterations of $F^{-1}$. For other points, their backward orbit eventually stays in the uniformly hyperbolic zone; hence the construction of unstable manifolds is standard.

We consider in the critical zone a family of vertical curves. Using the local coordinates, each curve is going to be written as

$$x = x(y).$$

We define the family $\mathcal{V}$ of curves as those satisfying

$$\begin{cases} 1 & \text{any curve in } \mathcal{V} \text{ that joins the top and bottom sides of } \mathfrak{R}, \\ 2 & |x'(y)| \leq \frac{1}{6\beta_{\max}}(3y^2 + x(y)), \\ 3 & |x''(y)| \leq D, \end{cases}$$

where $D$ is a fixed positive real number.

Figure 7. Image of relatively vertical curves.


**Lemma 4.1.** For an open set of parameters and for any sufficiently large D, the set of graphs \( \mathcal{V} \) is stable by the first return map in the critical zone.

**Proof.** Let us consider \( M \in \mathcal{R} \) such that its first return to \( \mathcal{R} \) occurs for the \((n+1)\)th iterate. Let \( C \) be a curve in \( \mathcal{V} \) containing \( M \); consider its local parametrization \((x(y), y)\), where \( y \) runs \([-\beta_{\text{max}}, \beta_{\text{max}}]\), and let \( M = (x(y_0), y_0) \). We want to prove that the connected component of \( F^{n+1}(C) \cap \mathcal{R} \) which contains \( F^n(M) \) is again in \( \mathcal{V} \).

The image of the curve by \( F \) is the set
\[
\{(x, y) : y \in [-\beta_{\text{max}}, \beta_{\text{max}}]\}.
\]

This is a graph over the horizontal interval \([-\varepsilon_1 \beta_{\text{max}}, \varepsilon_1 \beta_{\text{max}}]\) joining the vertical interval image by \( F \) of the bottom side of \( \mathcal{R} \) to the vertical interval image by \( F \) of the top side of \( \mathcal{R} \). By construction, this graph crosses the rectangle \( G_{0,n+2k}(F(\xi)) \) (see figure 7).

After \( n \) iterates of the linear map \( F \), this graph is sent to
\[
\{(x, y) : y \in [-\beta_{\text{max}}, \beta_{\text{max}}]\}.
\]

In particular, it is a curve joining the bottom side to the top side of \( \mathcal{R} \).

By definition, we are considering a curve in the critical zone defined by this parametrization. We set \( t = -\frac{\lambda^n \varepsilon_1 y}{x} \); then \( y = -\frac{t}{\lambda^n \varepsilon_1 y} \). We use this new parametrization to compute the derivative and the curvature of this curve. We want the first coordinate in the function of the second but here we have the contrary. We recall the formulas for the derivative of an inverse map:
\[
(\varphi^{-1})'(\varphi(t)) = \frac{1}{\varphi'(t)}, \quad (\varphi^{-1})''(\varphi(t)) = -\frac{\varphi''(t)}{(\varphi'(t))^3}.
\]

Setting \( \varphi(t) := \alpha^k \rho(\beta) : (bx(y) - cy(y^2 + x(y))) \), we get
\[
(\varphi^{-1})'(\varphi(t)) = \frac{1}{\lambda^n \varepsilon_1}[(b - cy)x'(y) - c(3y^2 + x(y))] \quad (25)
\]
\[
\varphi''(t) = \frac{2c}{\lambda^n \varepsilon_1}x'(y) - (b - cy)x'(y) + (\frac{6c}{\lambda^n \varepsilon_1})y',
\]
\[
(\varphi^{-1})'(\varphi(t)) = \frac{\alpha^k \rho(\beta)}{\lambda^n \varepsilon_1}[-(b - cy)x''(y) + 6cy + 2cx'(y)]
\]
\[
\times \frac{(\lambda^n \varepsilon_1)^3}{\alpha^k \rho(\beta)}((b - cy)x'(y) - c(3y^2 + x(y)))^3. \quad (26)
\]

To prove that the curve belongs to the family \( \mathcal{V} \), it is necessary to check that the derivative and the curvature respectively satisfy properties 2 and 3 in (24). As before, we use the local coordinates \((\alpha, \beta)\) for a return to the critical zone.
• Concerning property 2 (which deals with the first derivative), note that 

\( (b - cy) x'(y) \leq 3c\beta_{\max} \frac{1}{6c\beta_{\max}} (3y^2 + x(y)) \).

Then, due to (25), it is sufficient to check that

\[
\frac{2\lambda^\varepsilon_1}{\sigma^k \rho^k (c(3y^2 + x(y)))} \leq \frac{1}{6\beta_{\max}} (3\beta^2 + \alpha)
\]

holds. Now (15) yields

\[
\lambda^\varepsilon_1 c\beta_{\max} \leq \frac{c}{2} (3\beta^2 + \alpha).
\]

We recall equality (9):

\[
A = \frac{c}{8\varepsilon_1}.
\]

Thus,

\[
\frac{2\lambda^\varepsilon_1}{\sigma^k \rho^k (c(3y^2 + x(y)))} = \frac{2\lambda^\varepsilon_1}{\sigma^k \rho^k c(3y^2 + x(y))} \leq \frac{c}{2} (3\beta^2 + \alpha) \frac{2\varepsilon_1}{c^2\beta_{\max}} = \frac{1}{8\beta_{\max}} (3\beta^2 + \alpha),
\]

and (27) holds.

• Now, let us focus on control of the curvature. To satisfy condition 3 on the curvature, (26) shows it is sufficient to get

\[
\frac{8\lambda^\varepsilon_1}{\sigma^k \rho^k c(3y^2 + x(y)))^3} \left[ 3c\beta_{\max} D + 6c\beta_{\max} + \frac{c^2}{3c\beta_{\max}} (y^2 + x(y)) \right] \leq D,
\]

Inequality (1) yields \( \lambda < \frac{1}{\sigma} \) and \( \lambda < \frac{1}{\rho} \), and then it is sufficient to get

\[
\frac{8\varepsilon_1}{\sigma^k \rho^k c^3 (3y^2 + x(y)))^3} \left[ 3c\beta_{\max} D + 6c\beta_{\max} + \frac{c^2}{3c\beta_{\max}} (y^2 + x(y)) \right] \leq D.
\]

Inequality (18) shows that it is sufficient to get

\[
\frac{8(3\varepsilon_1^3 \beta_{\max})^3}{c^3 d^3} \left[ 3c\beta_{\max} D + 6c\beta_{\max} + \frac{c^2}{3c\beta_{\max}} (\beta_{\max}^2 + \varepsilon_{\alpha}) \right] \leq D.
\]

Now, the conditions stated before on the parameters mean that the term on the left-hand side is an affine term in \( D \) with slope \( \frac{2^3 3^4}{d^3} \beta_{\max}^2 \varepsilon_1 \). As \( d \) is very close to \( \frac{1}{2} \), this slope is strictly smaller than 1, if \( \beta_{\max} \) or \( \varepsilon_1 \) is sufficiently small, which is allowed. We assume that this holds, then, depending on the choices of parameters, there exists \( D_0 \) such that for \( D > D_0 \), the term on the left-hand side is smaller than the one on the right-hand side. This shows that (28) holds if \( D \) is chosen to be sufficiently large.

\[ \square \]

**Remark 6.** If \( \beta_{\max} \) and \( \varepsilon_1 \) are sufficiently small,

\[
D_0 := \frac{2^3 3.7 (19\beta_{\max}^2 + \varepsilon_{\alpha})}{d^3} \varepsilon_1 \frac{\varepsilon_1}{d^2 \beta_{\max}^2 \varepsilon_1}
\]

is as close to 0 as desired.  

\[ \blacksquare \]
Proposition 4.2. Let us consider a point \( M \in \mathbb{R} \) returning infinitely many times to the critical zone by iterations of \( F^{-1} \). Let \( n_k, k = 1, 2, \ldots \) be these return times in \( \mathbb{R} \). Then, for each \( k \), there exists a curve \( C^u_k \) in \( V \) containing \( F^{-n_k}(M) \), such that for every \( k < j \)

\[
F^{-(n_j-n_k)}(C^u_k) \subset C^u_j.
\]

Proof. Let us consider a point \( M \) close to the critical value and returning infinitely many times to the critical zone by iterations of \( F^{-1} \). Let \( (n_k)_{k \geq 1} \) be the sequence of positive integers such that \( M_k := F^{-n_k}(M) \) belongs to the critical zone (and \( n_0 := 0 \)).

The two vertical sides of \( \mathbb{R} \) are curves in \( V \), and point \( M_k \) is between them. Following the proof of lemma 4.1, \( M_{k+1} \) is between two curves in \( V \) which are images by \( F^{n_k} \) of these two vertical segments. Moreover, these two curves are located between the two sides of \( \mathbb{R} \).

Doing this by induction, we get two families of curves, \( (C^l_k)_k \) and \( (C^r_k)_k \), images by \( F^{n_k} \) of the vertical sides of \( \mathbb{R} \), where \( C^l_k \) is always on the left of \( M \) and \( C^r_k \) is always on the right of \( M \).

With orientation from the left to the right, the family of curves \( (C^l_k)_k \) is an increasing family (each one has a parametrization \( x_{k,l}(y) \), with \( x_{k+1,l} > x_{k,l} \)) and the family \( (C^r_k)_k \) is a decreasing family.

Lemma 4.1 shows that the limit curves \( C^\infty,l \) and \( C^\infty,r \) are located in \( V \). We prove by contradiction that the two limit curves are actually equal. If not, lemma 4.1 shows that there must be some ‘bubbles’ between these two curves (see figure 8). Let \( B \) denote such a bubble. This bubble has a positive Lebesgue measure. By construction, for every point \( M' \) in the bubble and for every integer \( n, F^{-n}(M') \) belongs to the ball \( B(F^{-n}(M), 1) \).

On the other hand, if \( |\det DF| < 1 \), then the volume of \( F^{-n}(B) \) goes to \( +\infty \). This produces a contradiction and proves that there cannot be bubbles between the two curves. In other words, the two curves coincide. This defines the curve \( C^u_0 \).

Doing the same for each \( M_k \), we get a family of curves \( C^u_k \) in \( V \). By construction, if \( k < j \),

\[
F^{-(n_j-n_k)}(C^u_k) \subset C^u_j.
\]

□
4.2. Invariant family of local stable graphs

Now, we construct a family of local stable manifolds in the post-critical zone. We prove they are uniformly long and relatively horizontal curves. The construction holds only for points which return (forward) infinitely many times to the critical zone. For other points, the construction of the stable manifold is standard and follows from the fact that these points are (forward) uniformly hyperbolic.

4.2.1. Notation. Let \( M \) be a point in \( \mathcal{R} \) whose first return to \( \mathcal{R} \) is \( n + 1 \). We denote by \( F_{F(M)}^{-n}(\mathcal{R}) \) the connected component of \( F^{-n}(\mathcal{R}) \cap \mathcal{Q} \) which contains \( F(M) \). It is a horizontal band in \( \mathcal{Q} \) (note that shadowing \( M, F^{-n} \) is linear).

Definition 4.3. Let \( M \) be a point in \( \mathcal{R} \) whose first return to \( \mathcal{R} \) is \( n + 1 \). A curve containing \( F(M) \) is said to be an approximate stable curve if it is a \( C^1 \)-graph over the whole interval \([0, 1]\) in \( F_{F(M)}^{-n}(\mathcal{R}) \) with the slope at every point in the stable conefield (when it makes sense).

Lemma 4.4. Let \( M \) be a point in \( \mathcal{R} \). Assume that \( n + 1 \) is its first return time in \( \mathcal{R} \) for \( F \). If \( C \) is an approximate stable curve containing \( F^{n+2}(M) \), then the connected component in \( \mathcal{Q} \) of \( F^{-n-1}(C) \) which contains \( F(M) \) is again an approximate stable curve.

Proof. Assume that the curve has a parametrization which has the form \( (\eta, \zeta(\eta)) \) in \( G_{0, n+2}(F(\xi)) \). This curve must intersect the two cubics, images by \( F \) of the left-hand and right-hand sides of \( \mathcal{R} \). This proves that the connected component of \( F^{-1}(C) \) which contains \( F^{n+1}(M) \) is a \( C^1 \) curve joining the left-hand side to the right-hand side of \( \mathcal{R} \). By construction, the direction of the tangent to this curve at each point is contained in the stable conefield at this point. Applying \( F^{-n} \), we get in \( F_{F(M)}^{-n}(\mathcal{R}) \) a \( C^1 \) curve joining the left-hand side of \( \mathcal{Q} \) to its right-hand side (note that \( n > k \), because \( F(M) \) is in \( F(\mathcal{R}) \)), still with the tangent direction in the stable conefields. By definition, the slopes of the tangent lines are bounded from above, and the curve must be a graph. By construction, it is a graph over the whole interval \([0, 1]\) contained in the band \( F_{F(M)}^{-n}(\mathcal{R}) \).

Proposition 4.5. Let \( M \) be located in \( \mathcal{R} \) and returning infinitely many times to \( \mathcal{R} \) by iterations of \( F \). Let \( n_k, k = 1, 2, \ldots \) be the positive return times, \( n_0 := 0 \). Then, for each \( k \) there exists a curve \( C_j^k \) containing \( F^{1+n_k}(M) \) and having the slope in the stable conefield such that for every \( k < j \),

\[
F^{n_{j-k}}(C_{k}^j) \subset C_{j}^j.
\]

Proof. We consider a family of positive integers \( n_k \), such that \( n_1 + 1 + n_2 + 1 + \ldots + n_k + 1 \) is the \( k \)th return time in \( \mathcal{R} \). For simplicity we denote by \( M_k \) the \( k \)th return of \( M \) to \( \mathcal{R} \). Thus we have

\[
M_{k+1} = F^{n_{k+1}}(F(M_k)),
\]

and \( F(M_k) \) belongs to \( F(\mathcal{R}) \).

We construct a family of curves which accumulate on the local stable manifold. Consider a large \( k \). \( F(M_k) \) belongs to \( F(\mathcal{R}) \), and \( F(\mathcal{R}) \) crosses the horizontal band \( F_{F(M_k)}^{-n_{k+1}}(\mathcal{R}) \) (see figure 9).
Let us consider two vertical segments. The first one starts at the intersection of the top horizontal line of $F^{-nk_1}(\mathcal{R})$ with the right-hand-side cubic, and goes upward until it reaches the left-hand-side cubic. The second one starts at the intersection of the bottom horizontal line of $F^{-nk_1}(\mathcal{R})$ with the left-hand-side cubic, and goes downward until it reaches the right-hand-side cubic (see figure 10). The region bounded by these two segments and the cubic curves will be referred to as the zone with double fins.

By construction, this zone is located entirely in $F(\mathcal{R})$ and contains $F(M_k)$. Its pre-image by $F$ gives a rectangle $R_k$: the two vertical segments are sent over the horizontal ones, and the two cubics are sent over the vertical segments (see figure 11).

This rectangle has the same horizontal length as $\mathcal{R}$ (namely $\alpha_{\text{max}}$) but has lower height and is inside $\mathcal{R}$. It also contains $M_k$. The images by $F^{-1}$ of the top and bottom sides of $F^{-n}\mathcal{R}$ intersecting with the zone with double fins are curves joining one edge of the rectangle to the border of the other vertical side (see figure 11). Using Cardan’s method to solve degree-3 equations we see that the curves are actually graphs of a $C^1$ map over the horizontal side of rectangle $R_k$, with the slope in the stable cone. Lemma 4.4 shows that their components by the push-backward $F^{-nk_{1}}$ in $F^{F(M_k)}(\mathcal{R})$ are two approximate stable curves in $F^{F(M_k)}(\mathcal{R})$. We can reproduce the construction of the double fins but with a thinner rectangle (see figure 12).

The image by $F^{-1}$ of these curves yields four curves with the slope in the stable cone in $\mathcal{R}$ as described in figure 13.

Then we get, in $F^{-nk_{1}-2}(\mathcal{R})$, four approximate stable curves, and $F(M_k)$ is below two and above two of these curves. The two extremal curves are the ones obtained by performing the
construction we described (double-fin structure) but starting from point $F(M_{k-1})$, and the two intermediate curves are the ones obtained by our first construction (and pushed backward).

Performing this construction by induction, we get $F(F(M))$, a sequence of decreasing graphs and a sequence of increasing graphs, say $C_{k,+}$ and $C_{k,-}$. Point $F(M)$ is above every $C_{k,-}$ curve and below every $C_{k,+}$ one. Each curve, $C_{k,+}$ or $C_{k,-}$, is a pre-image of the horizontal part of one of the fins close to $F(M_k)$.

Again, lemma 4.4 shows that all these curves are approximate stable curves, and thus have bounded slopes. In other words, we have an increasing and a decreasing sequence of equi-continuous functions. Ascoli’s theorem shows that both converge to two curves, say $C_{\infty,+}$ and $C_{\infty,-}$. Their slopes are in the stable cones.

To finish the proof, we have to prove that these two curves are the same. We copy and adapt the proof for the unstable manifold. The two curves are continuous; thus, if they do not coincide, the set where they are different is open and this generates ‘bubbles’.

Figure 11. Pre-image of double-fin zone.

Figure 12. Second and thinner double-fin zone for thinner rectangle.

Figure 13. Four curves in $\mathfrak{N}$.
Assume that there is a bubble between these two curves.

Pushing forward, the image of the bubble is approaching the forward orbit of $M$, since there is exponential contraction in the stable direction (see proposition 3.12). Moreover, by construction, the forward orbit of the bubble stays in $\mathcal{Q}$, and points in it have a well-defined unstable cone over them. We can thus take in the initial bubble a curve with the slope in the unstable cones. By iterating forward, the length of the images of these curves grows exponentially fast (see proposition 3.10). This contradicts the fact that, for all points in the bubble, their forward image is always very close to the forward images of $M$.

Let $C_{\infty,+} = C_0^+$ and consider, for each $F^n(M)$, a curve $C_0^{+j}$ constructed in the same way: an approximate stable curve containing $F^{1+n}(M)$. By construction, if $k < j$,

\[ F^{n-k}(C_0^{+j}) \subset C_0^{+k}. \]

We emphasize that at that step, the curves defined in proposition 4.5 are Lipschitz graphs with bounded slopes: $|DF^n(M) \cdot E^s(M') - DF^{n+k}(M') \cdot E^u(M')|$.

### 4.3. Local (un)stable manifolds and product structure

Consider a point $M$ in $\Lambda \cap \mathcal{R}$. Assume it returns to $\mathcal{R}$ infinitely many times in the past and in the future. Let $n_k^-$ and $n_k^+$ be the $k$th return times for forward and backward iterates into $\mathcal{R}$. By proposition 4.2 there exists a local manifold $C_0^+$ such that for every $M'$ in this manifold, the distance between $F^{-n_k^+}(M')$ and $F^{-n_k^-}(M)$ converges to 0.

Similarly, proposition 4.5 yields a local stable manifold $C_0^-$ such that for every $F(M')$ in it, the distance between $F_0^{n_k^+ - 1}(M')$ and $F_0^{-n_k^- - 1}(M)$ converges to 0.

By construction we clearly have, for $M'$ in $C_0^+$, $\lim_{n \to +\infty} |F^{-n}(M) - F^{-n}(M')| = 0$ and, for $M'$ in $C_0^-$, $\lim_{n \to +\infty} |F^n(M) - F^n(M')| = 0$. Nevertheless, we are interested in a converse and more general result: if $M'$ satisfies $|F^{-n}(M') - F^{-n}(M)| \to 0$ as $n \to +\infty$, does this imply that, for some $n$, $F^n(M') \in C_0^+(F^n(M))$? This is the purpose of proposition 4.9 and the definition of the local product structure which separates points.

#### 4.3.1. Definition of $W^{0, 0}_{\text{loc}}(M)$

We start by recalling that the construction of local manifolds $W^{0, 0}_{\text{loc}}(M)$ is standard for points which return only finitely many times to $\mathcal{R}$. We recall that the stable manifold is defined as long as the forward orbits stay in $\mathcal{Q}$, and the unstable manifold is defined as long as the backward orbits stay in $\mathcal{Q}$.

For $\xi$, we recall that its unstable manifold is naturally defined. It is also the unstable manifold for $\{0, 0\} \in \mathcal{Q}$ (bottom left point). Its unstable manifold is also well defined: actually the stable local manifold for $F(\xi)$ is the horizontal line which contains $F(\xi)$ in $\mathcal{Q}$. Then $W^{0, 0}_{\text{loc}}(M)$ is well defined for any $M$ in $O(\xi)$.

We thus now focus on points which return infinitely many times to $\mathcal{R}$.

**Definition 4.6.** Let $M$ be a point that returns infinitely many times to $\mathcal{R}$ by iteration of $F$. Let $n$ be the smallest non-negative integer such that $F^n(M)$ belongs to $\mathcal{R}$, and let $C^n(F^{n+1}(M))$ be the curve constructed in proposition 4.5 and containing $F^{n+1}(M)$.

The connected component of $F^{-n-1}(C^n(F^{n+1}(M)))$ in $\mathcal{Q}$ which contains $M$ is called the local stable manifold of $M$ and is denoted by $W^{0, 0}_{\text{loc}}(M)$.

We emphasize that, by construction, $W^{0, 0}_{\text{loc}}(M)$ is a graph over $[0, 1]$ and is inside the horizontal band $H_0^1(M)$. Moreover its slope at any point is in the stable cone at this point. We have also seen it is a $C^{1,1}$ curve.

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Now, we construct long ‘vertical’ local unstable manifolds. Let $M$ be in the critical zone $\mathcal{R}$, and returning infinitely many times to $\mathcal{R}$ by iteration of $F^{-1}$. By construction, the curve $C^u(M)$ given by proposition 4.2 is a ‘vertical’ graph going from the bottom of $\mathcal{R}$ to the top of $\mathcal{R}$. Its image by $F$ is a graph going from the vertical segment on the left-hand side of $F(\mathcal{R})$ to the vertical segment of the right-hand side of $F(\mathcal{R})$ (see figure 7). This curve crosses the horizontal band $H^u_0(F_0(\xi))$, from the bottom to the top.

All the points in this band have the same itinerary as $F_0(\xi)$ by iteration of $F_0$ for the next $k_e$ iterations. By assumption, $F_0(\xi)$ never comes back to rectangles $R_7$ and $R_4$. This means that for all these points, their images by $F^j$ and by $F'_j$ coincide for $j \leq k_e$. Hence, $F_{k_c+1}(C^u(M))$ is a curve joining the bottom of the square $\mathcal{Q}$ to its top.

Moreover, this curve is a graph over the vertical segment $[0, 1]$ (the left side of $\mathcal{Q}$) with the slope in the unstable cone field. By construction it is contained in a vertical band $V^u_0$.

**Definition 4.7.** Let $M$ be a point which returns infinitely many times to $\mathcal{R}$ by iteration of $F^{-1}$. Let $n$ be the smallest positive integer such that $F^{-n}(M) \in \mathcal{R}$. The connected component of $F^n(C^u(F^{-n}(M)))$ in $\mathcal{Q}$ which contains $M$ is called the **local unstable manifold** of $M$. It is denoted by $W^u_{loc}(M)$.

We have seen above that $W^u_{loc}(M)$ is a graph over the vertical segment left side of $\mathcal{Q}$ with the slope in the unstable cone field, going from the bottom to the top of $\mathcal{Q}$ and included in the vertical band $V^u_0(M)$.

**Proposition 4.8.** For $M \in \Lambda \setminus \mathcal{O}(\xi)$, $TMW^u_{loc}(M) = E^u(M)$.

**Proof.** We recall that unstable and stable cone fields may be extended to points whose forward or backward orbit stays in $\mathcal{Q}$ (a weaker condition than that in $\Lambda$). By construction, this holds for points in any $W^u_{loc}$ and not only for points in $\Lambda$.

More precisely, for any point $M' \in W^u_{loc}(M)$, the backward orbit of $M'$ is well defined and so is the unstable cone field. We can consider $\cap_n D F^n(C^u(F^{-n}(M')))$: this is a single direction $E^u(M')$ (see proposition 3.11).

The proof of proposition 3.11 also shows that the tangent space to $W^u_{loc}(M)$ at $M'$ is $E^u(M')$. A similar result holds for $M'' \in W^u_{loc}(M)$ and $E^s(M'')$. □

An immediate consequence of propositions 3.7, 4.8 and corollary 3.13 is that for $M'$ in $W^u_{loc}(M)$,

$$d(F^{-n}(M), F^{-n}(M')) \to_{n \to +\infty} 0,$$

and for any $M'' \in W^u_{loc}(M)$,

$$d(F^n(M), F^n(M')) \to_{n \to +\infty} 0.$$

**4.3.2. The local product structure.** Roughly speaking, proposition 4.8 means that $F$ expands unstable manifolds and $F^{-1}$ expands stable manifolds. This, combined with the fact that the graphs are transversal, yields a local product structure which separates points:

**Proposition 4.9.** Let $M$ and $M'$ be located in $\Lambda$. Then, $W^s_{loc}(M) \cap W^u_{loc}(M')$ is a single point denoted by $[M, M']$, which is in $\Lambda$. Moreover,

- The point $M'$ satisfies $|F^{-n}(M) - F^{-n}(M')| \to_{n \to +\infty} 0$ if and only if there exists $m$ such that $F^m(M') \in W^s_{loc}(F^m(M))$.
- The point $M'$ satisfies $|F^n(M) - F^n(M')| \to_{n \to +\infty} 0$ if and only if there exists $m$ such that $F^m(M') \in W^s_{loc}(F^m(M))$. 

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Proof. The fact that $W_{\text{loc}}^u(M) \cap W_{\text{loc}}^s(M')$ is non-empty is a consequence of the fact that these two graphs cross $Q$ in all its extension, the first from the bottom to the top, and the second from the left to the right side of $Q$.

The intersection is a single point because at any intersection point, the slope of $W_{\text{loc}}^u$ is always strictly more vertical than the slope of $W_{\text{loc}}^s$, except for the countable set of points \( \{F^k(\xi), k \in \mathbb{Z} \} \), where both directions coincide.

By construction, point $M'' := [M, M']$ has its backward orbit converging to the backward orbit of $M$, and its forward orbit converging to the forward orbit of $M'$, both always in $Q$, and then it belongs to $\Lambda$.

Let us now prove the two last properties. For this, we assume that the orbit (backward or forward) of $M$ visits $\mathcal{R}$ infinitely many times; otherwise the proof is the same as in the uniformly hyperbolic setting.\footnote{Even if $M = \xi$.}

We assume just for now that $M$ belongs to $\mathcal{R}$. Let $n_k, k = 1, 2, \ldots$, be the sequence of positive return times to $\mathcal{R}$, and pick $P \in C^u(M)$, $P \neq M$.

Claim. There exists a positive integer $k$ such that $F^{n_k}(P) \in C^u(F^n(M))$ but $F^{n_{k+1}}(P) \notin C^u(F^{n_{k+1}}(M))$.

Proof of the claim. By proposition 3.9, for every $j$, $DF^{-n_j}$ acts as a contraction of a ratio of at least $(1 + M)^{-1} - \frac{\epsilon}{2}$ in $TW_{\text{loc}}^u(F^n(M))$. $\square$

Let us consider this largest integer $n_k$.

Claim. Let $n_k$ be as above. There exists $n > n_k$ such that $F^n(M)$ and $F^n(P)$ belong to two different horizontal bands $H_0^i$.\footnote{Even if $M = \xi$.}

Proof of the claim. The two points $F^{n_k+1}(P)$ and $F^{n_k+1}(M)$ belong to $F(\mathcal{R})$. If they have two different first return times to $R_4$, then either $F^{n_k+1}(P)$ does not belong to $H_0^i(\xi)$ (and the claim is proved) or it belongs to $H_0^i(\xi)$ but then it does not belong to $V_0^1(\xi)$ (namely it does not belong to $R_1 \cup R_2 \cup R_7$). Then, $F^{n_k+1}(M)$ and $F^{n_k+1}(P)$ belong to two different horizontal bands $H_0^i$.

If $F^{n_k+1}(P)$ belongs to $R_4$, it cannot belong to $\mathcal{R}$. If it does not belong to $H_0^i$, then $F^{n_k+1}(M)$ and $F^{n_k+1}(P)$ belong to two different horizontal bands $H_0^i$ (note that $F$ and $F_0$ coincide for times between $n_k + 1 + 2$ and $n_k + 2$). Finally, if $F^{n_k+1}(P)$ belongs to $H_0^i(\xi)$, then it does not belong to the same vertical band $V_0^1$. By construction, the first return time to $\mathcal{R}$ of a point in $\mathcal{R}$ is greater than $n_k + 2$, and this shows that there exists $n_k < n < n_{k+1}$ such that $F^n(M)$ and $F^n(P)$ belong to two different vertical bands $V_0$. Then, $F^{n_k+1}(M)$ and $F^{n_k+1}(P)$ belong to two different horizontal bands $H_0^i$. $\square$

Let us now finish the proof of proposition 4.9. Assume that $M$ and $M'$ satisfy

$$|F^n(M') - F^n(M)| \rightarrow_{n \rightarrow +\infty} 0.$$ 

Let consider $n_d$ such that for every $n > n_d$,

$$|F^n(M') - F^n(M)| < d,$$

and consider $P := [F^n(M), F^n(M')]$. By construction, for every $j \geq 0$, $F^j(P)$ and $F^{n+j}(M')$ belong to the same horizontal band $H_0^i$. The two claims show that if $P \neq M$, then for some $j$, $F^j(P)$ and $F^{n+j}(M)$ belong to two different horizontal bands. Then, for such $j$,
\[|F^{n+j}(M) - F^{n+j}(M')| > d,\]

which contradicts the assumption.

Therefore, \( P = M \) and \( F^n(M') \) belongs to \( W^s_{\text{loc}}(F^n(M)) \).

The proof of the other property is similar. \(\Box\)

### 4.3.3. Hölder conjugacy to a subshift of finite type

The Markovian property of the dynamical partition for \( F \) (see section 3.1) implies that every sequence \( m \in \Lambda \), the set of sequences of nine symbols and the transition matrix \( A \), corresponds to at least one point \( M \in \Lambda \) whose itinerary is \( m \).

In this subsection we show that the expansions in the unstable direction by iteration of \( F \) and in the stable direction by iteration of \( F^{-1} \) imply that \( F|_\Lambda \) is conjugated by a homeomorphism to the shift in \( \Sigma_9^A \). Moreover, the homeomorphism that assigns to each itinerary the corresponding point in \( \Lambda \) is Hölder-continuous, which clearly implies the last corollary in section 1.2.

Consider the defined metric by assigning to \( m \) and \( m' \) in \( \Sigma_9^A \), coinciding from positions \(-p\) to \( q \), \( d(m, m') = (1/2)^{\min(p, q)} \).

**Proposition 4.10.** The map \( F \) is expansive with expansivity constant \( d \).

**Proof.** Actually, the proof of proposition 4.9 shows a better result:

\[ M' \in W^u_{\text{loc}}(M) \iff \forall n \geq 1 |F^{-n}(M) - F^{-n}(M')| < d. \]

The same result holds for \( W^s_{\text{loc}} \) and forward iterates. This shows that \( d \) is an expansivity constant. \(\Box\)

This implies that each itinerary in \( m \in \Sigma_9^A \) corresponds to a unique point \( \Theta_F(m) = M \in \Lambda \).

**Proposition 4.11.** The map \( \Theta_F : \Sigma_9^A \to \Lambda \) is a Hölder-continuous homeomorphism.

**Proof.** Let \( M \) and \( M' \) be two points in \( \Lambda \). We consider the two maximal integers \( n_- \) and \( n_+ \) such that

\[ d(F^k(M), F^k(M')) < d \quad (29) \]

holds for every \( k \in [-n_-, n_+] \).

Hölder continuity for \( \Theta_F \) means that one can find \( C \) and \( 0 < \kappa < 1 \) (independent of \( M \) and \( M' \)) such that

\[ d(M, M') \leq C\kappa^{\min(n_-, n_+)} \quad (30) \]

holds. We emphasize that if \( M' \) belongs to \( W^u_{\text{loc}}(M) \), then \( n_- = +\infty \). Similarly if \( M' \) belongs to \( W^s_{\text{loc}}(M) \), then \( n_+ = +\infty \). Triangular inequality also yields

\[ d(M, M') \leq d(M, M'') + d(M'', M'), \]

where \( M'' = [M, M'] \). Consequently, it is sufficient to prove (30) for \( M' \in W^u_{\text{loc}}(M) \) and for \( M' \in W^s_{\text{loc}}(M) \).

- The case of \( M' \in W^u_{\text{loc}}(M) \). We claim that the result directly follows from lemma 3.6
  if \( [0, n_+] \) is a complete piece of orbit for both \( M \) and \( M' \). Therefore, we claim that it is sufficient to prove that (30) holds if \( [0, n_+] \) is included in a tube for \( M \) and \( M' \).

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Furthermore, the dynamics expands the unstable direction in the first part of the critical tube, until the points reach the critical zone $\mathcal{R}$. Then, we can only study the case where both $M$ and $M'$ belong to $W_{\text{loc}}^u(M)$, which is a ‘very’ vertical curve. Let us set $M = \xi + (x, y)$ and $M' = \xi + (x', y')$. We set $\delta := y' - y$. Therefore, $F(M)$ belongs to the vertical line $-\varepsilon_1y$ and $F(M')$ belongs to the vertical line $-\varepsilon y - \varepsilon\delta$. The set $F(W_{\text{loc}}^u(M))$ is a curve more vertical than the one obtained by taking the more horizontal possible vector of the unstable curve. In other words, the piece of local unstable manifold between $F(M)$ and $F(M')$ is above the integral curve of the vector field $A(3\beta^2 + \alpha)$ containing $F(M)$, with the horizontal coordinate going from $0$ to $-\varepsilon\delta$. This curve is above the one with vector field $3\alpha\beta^2$.

For the $n_\delta - 1$ iterates, the vertical distance between $F^{n_\delta - 1}(M)$ and $F^{n_\delta - 1}(M')$ has order $d$. We deduce from the above discussion the inequality
\[ A\rho^{k + n_\delta - 1}(\varepsilon_1\delta)^3 < d. \]
This yields (29).

Let us now deal with the case where $M$ and $M'$ belong to some $F^k(\mathcal{R})$ with $k > 0$. If the distance between both points is given by the difference of their vertical coordinates, then the result is obvious. By definition of the critical tube, both points are very close to the post-critical orbit; thus the map is linear at least until the separation time $n_\delta$. It expands the vertical distance, and thus we get some equality,
\[ d\sigma^{n_\delta} \approx d, \]
and this yields (30). We thus have to deal with the case where the distance is given by the difference of the horizontal coordinates. Let $\varepsilon_1\delta$ denote this difference (in absolute value). Then the way we construct the map $F$ shows that the distance between $F^{-k}(M)$ and $F^{-k}(M')$ is $\delta\lambda^{-k}$. Moreover these two points belong to $\mathcal{R}$ and to the same local unstable manifold. We are thus back to the above case. We finally get
\[ A\rho^{k + n_\delta - 1}(\varepsilon_1\delta\lambda^{-k})^3 < d. \]
This yields (30) since $\lambda < 1 < \rho$.

- The case of $M' \in W_{\text{loc}}^u(M)$. As for the previous case, the main difficulty is in dealing with $M$ and $M'$ in a critical tube, and with the (backward) separation time before both critical tubes end. Nevertheless, the stable cone is the complementary of the unstable cone, which shows that if $[-n_\beta, n_\beta]$ is a critical tube, the main difficulty arises as $M$ and $M'$ are located in part of the tube $[-n_\beta, 1]$. Indeed, there is uniform (backward) expansion for the stable direction in this part of the tube: $[1, n_\beta]$.

Let us first deal with the case of $M$ and $M'$ in $F(\mathcal{R})$. Note that the distance is mainly given by the horizontal components that one can write in the form $\varepsilon\Delta y$.

We also recall the expression for the differential (12):
\[
\begin{pmatrix}
0 & -\varepsilon_1 \\
\beta - c\beta & -c(3\beta^2 + \alpha)
\end{pmatrix}.
\]
It yields
\[ DF^{-1} = \begin{pmatrix} \frac{-c(3\beta^2 + \alpha)}{\varepsilon_1(b - c\beta)} & \frac{1}{b - c\beta} \\ -\frac{1}{\varepsilon_1} & 0 \end{pmatrix}. \]
(32)

If \( \mathbf{w} = \mathbf{w}_i + \mathbf{w}_j \) is tangential to a stable manifold in \( \mathcal{F}(\mathfrak{R}) \), then \( |\mathbf{w}_j| = t(\alpha, \beta)A(3\beta^2 + \alpha)|\mathbf{w}_i| \) with \( 0 \leq t(\alpha, \beta) \leq 1 \). Therefore, \( DF^{-1}(\mathbf{w}) \) is proportional to
\[ \frac{c}{b - c\beta}(1 - \frac{t(\alpha, \beta)}{8})(3\beta^2 + \alpha)\mathbf{i} + \mathbf{j}. \]
(33)

**Remark 7.** We emphasize an important consequence of (33): except for the critical point \( \xi \), no point can have a vertical stable direction.

To estimate the time that is needed to separate the two points \( M \) and \( M' \) (backward), one needs to estimate the slope of the stable leaves in \( \mathfrak{R} \).

We recall that the distance between \( M \) and \( M' \) is given by \( \varepsilon_1 \Delta y \). The more vertical the stable manifold is in \( \mathfrak{R} \), the more time it needs to separate points. On the other hand \( F^{-1}(M) \) and \( F^{-1}(M') \) may be quite far from each other (with large \( \Delta y \)). In other words, the more vertical the unstable manifolds are in \( \mathfrak{R} \), the less accurate any estimate for (30) is.

Note that unstable curves are less vertical than the integral curves of the ODE:
\[ \alpha' = \frac{7}{16} \frac{(3\beta^2 + \alpha)}{\beta_{\text{max}}}. \]
(34)

Let us set \( a := \frac{16\beta_{\text{max}}}{7} \). Solutions for (34) are of the form
\[ \alpha = (\alpha_0 - 6a^2)e^{\beta y} + 3\beta^3 + 6ay + 6a^2. \]

We point out that \( a \) has an order of \( \beta_{\text{max}} \) and that \( \alpha_{\text{max}} \) has an order of \( 1/10\beta_{\text{max}}^2 \). Thus, \( \alpha_0 \) (the position of the curve for \( \beta = 0 \)) is much smaller than \( 6a^2 \). By performing a Taylor expansion in the exponential, one shows that there are universal constants \( \kappa \) and \( \kappa' \) such that
\[ \alpha(\beta) - \alpha(\beta') = (\kappa + \kappa\alpha_0)\beta_{\text{max}}(\beta - \beta'). \]

As these curves are more vertical than the true unstable manifolds, we can conclude that there exists a universal constant \( D \) such that if \( M = F(\xi + (x, y)) \) and \( M' = F(\xi + (x', y')) \), then
\[ \Delta x := |x - x'| \geq D|y - y'| = \frac{D}{\varepsilon_1}d(M, M'). \]
(35)

Now \( n_+ \) satisfies \( \Delta x \approx d\lambda^{\alpha_-} \), which yields (30) in that case.

If \( M \) and \( M' \) belong to \( F^{-k}(\mathfrak{R}) \), for \( k \leq n_- \), then two sub-cases have to be considered. The distance is given either by the vertical difference, or by the horizontal one. In the first case, the previous computation can be used directly. In the second case, this is a standard computation since \( W^s \) is essentially horizontal and there is backward expansion in this direction.

The continuity of the inverse comes from the fact that \( \Theta_F \) is surjective and \( \Sigma^9_{\Delta} \) is compact.
5. Final remarks

Item 1 of the main theorem and the corollary stated in section 1.2 remain to be proven.

5.1. Heteroclinic tangency

In our construction, the backward itinerary of ξ is fixed, since ξ belongs to the unstable leaf of \( P_b := 0 \).

For the forward itinerary we recall that our unique assumptions are that the point never comes back to \( R_1 \cup R_4 \) and visits infinitely many times each of the two regions \( R_2 \cup R_3 \) and \( R_5 \cup R_6 \). This can be realized by a periodic or non-periodic point. Choose a point \( P_f \) with such a forward orbit, and take \( F(\xi) \) in \( W^s_{loc}(P_f) \).

Note that the second point \( P_f \) can be chosen in the set of points whose trajectories satisfy the assumptions above, and there is a (non-countable) Cantor set of choices.

5.2. Boundary of uniformly hyperbolic diffeomorphisms

Now we show that the map \( F \) is on the boundary of uniformly hyperbolic diffeomorphisms.

We consider a very small parameter \( \theta > 0 \) and then set

\[
F_\theta(\xi + (x,y)) = F(\xi) + (-\varepsilon_1 y, bx - cy(y^2 + x + \theta)).
\]

To get an idea why \( F_\theta \) is hyperbolic, we just point out that the parameter \( \theta \) straightens out the cubics in the post-critical zone, and thus improves the estimates. Now, we effectively prove it.

Close to the image of \( \xi \), the new unstable conefield is defined by

\[
C^u(F_\theta(x,y)) = \{v = (v_x, v_y), \ |v_y| \geq A(3y^2 + x + \theta)|v_x| \}.
\]

We recall that the main point for getting hyperbolicity for \( F \) is to prove that inequality (14) holds, i.e. for \( u \geq 1 \)

\[
\frac{\lambda^u(b - c\beta)}{\rho^{(1)} u \rho^{(2)} A(3y^2 + x + \theta) \varepsilon_1} - \frac{c}{\varepsilon_1} (3\beta^2 + \alpha) \geq A(3\beta^2 + \alpha).
\]

The way to prove this is to show that the first term on the left-hand side is smaller than half of the second term, and that half of the second term (on the left-hand side) is larger than the term on the right-hand side.

Introducing the parameter \( \theta > 0 \) we have to prove

\[
\frac{\lambda^u(b - c\beta)}{\rho^{(1)} u \rho^{(2)} A(3y^2 + x + \theta) \varepsilon_1} - \frac{c}{\varepsilon_1} (3\beta^2 + \alpha + \theta) \geq A(3\beta^2 + \alpha + \theta). \tag{36}
\]

Note that for \( \theta > 0 \), if

\[
0 \leq \frac{\lambda^u(b - c\beta)}{\rho^{(1)} u \rho^{(2)} A(3y^2 + x + \theta) \varepsilon_1} \leq \frac{c}{2\varepsilon_1} (3\beta^2 + \alpha)
\]

holds, then

\[
0 \leq \frac{\lambda^u(b - c\beta)}{\rho^{(1)} u \rho^{(2)} A(3y^2 + x + \theta) \varepsilon_1} \leq \frac{c}{2\varepsilon_1} (3\beta^2 + \alpha + \theta).
\]
also holds. Consequently (36) holds.

It is left to the reader to check that any other required condition holds for \( \theta \) that is sufficiently small, as \(|y| \leq \beta_{\max}\) is very small and we always consider strict inequalities to allow some small perturbation.

We also point out that exchanging \(-cyx\) with \(-c(x + \theta)y\) does not change the result of the uniqueness of the real root when we use Cardan’s method. In fact, it improves the result.

It now remains to be shown that \( F_\theta \) is uniformly hyperbolic. This follows from the following lemma:

**Lemma 5.1.** There exists \( N = N(\theta) \) such that for every \( M \) in \( \Lambda \) there exists \( 0 \leq n \leq N \) such that for every \( v \in C^u(M) \), \(|DF_\theta^n(M) \cdot v| \geq 2|v|\).

**Proof.** Note that any vector in some unstable cone is more vertical than \((1, A\theta)\). In the following, we consider \( N' \) such that \( A \theta \rho^{N'} \geq 2 \).

Now consider any point \( M \) and any vector \( v \in C^u(M) \). Assume \( M \) is very close to \( F(\xi) \). We call \( n_{\text{esc}}(M) \) the maximal integer such that \(|F^{j+1}(M) - F^{j+1}(\xi)| < d\) holds for every \( j \leq n_{\text{esc}}(M) \).

If \( n_{\text{esc}}(M) \geq N' \), then \(|DF_\theta^{n_{\text{esc}}}(M) \cdot v| \geq 2|v|\).

If \( n_{\text{esc}}(M) < N' \), after \( n_{\text{esc}}(M) \) iterates, inequality (22b) shows that \( v \) has been expanded by \( DF_\theta^{n_{\text{esc}}}(M) \) by a factor greater than \( \frac{Ad\varepsilon_1}{3c\beta_{\max}} \). This last term is supposed to be much larger than 1 (see condition (4c)) and can thus be assumed to be larger than 2.

If \( M \) is not in \( F(\mathcal{R}) \) and needs at least \( N' \) iterates to reach \( \mathcal{R} \), then we are finished. If \( M \) is not in \( F(\mathcal{R}) \) and needs fewer than \( N' \) iterates to reach \( \mathcal{R} \), we have to consider several cases.

If \( M \) is in the post-critical part of some critical tube, that is \( M = F^k(M') \), with \( k \leq n_{\text{esc}}(M') \) (see definition 3.4), then one of two situations occurs: either the vectors at the border of the unstable cones are contracted by \( DF(M) \), but when the point exits the critical time (which happens before \( N' \) iterates because \( F^{N'}(M) \) belongs to \( \mathcal{R} \) and \( \xi \) never returns to \( \mathcal{R} \)), the vectors have been expanded (case of \( n_{\text{esc}}(M) < N' \)); or the vectors at the border of the unstable cones are expanded and then the worst case is that we need \( N' \) more iterates after reaching \( \mathcal{R} \) to be sure we get an expansion greater than 2. Therefore, \( N(\theta) = 2N' \) satisfies the condition.
We claim that the factor $\theta$ implies that $DF_\theta(C^u)$ is uniformly included in the interior of $C^u$ as illustrated in figure 14. The security angle is uniformly proportional to $\theta$. Therefore, the angle between $E^u$ and $E^s$ is uniformly bounded away from zero. The assumption $|\det(DF_\theta)| < 1$ yields uniform contraction in the stable direction, and $F_\theta$ is uniformly hyperbolic. Moreover, the stable and unstable vector fields are Lipschitz-continuous when restricted to stable or unstable local manifolds.

This finishes the proof of the theorem.

5.3. Equilibrium states

We say that an $F$-invariant probability measure $\mu$ is an equilibrium state for $F$ w.r.t. a potential $\phi : \Lambda \rightarrow \mathbb{R}$ if it satisfies

$$h_\mu(F) + \int \phi \, d\mu = \sup_\eta \left( h_\eta(F) + \int \phi \, d\eta \right),$$

where the supremum is taken over all $F$-invariant probability measures.

The Hölder-continuous conjugacy $\Theta_F : \Sigma^F_A \rightarrow \Lambda$ allows us to define a Hölder-continuous potential $\tilde{\phi} : \Sigma^F_A \rightarrow \mathbb{R}$ by $\tilde{\phi}(\chi) = \phi(\Theta_F(\chi))$.

The existence and uniqueness of equilibrium states for subshifts of finite type with respect to Hölder-continuous potentials are a classical result of ergodic theory (see [21]). The conjugacy gives a correspondence between the invariant measures of the two systems. If $\mu_{\tilde{\phi}}$ is the unique equilibrium measure for the subshift, then $\Theta_F^*(\mu_{\tilde{\phi}})$ is the unique equilibrium state for $F$.

Acknowledgments

This work was partially supported by a CNRS–CNPq collaboration (FP7-IRSES 230844 DynEurBraz, FP7-IRSES 318999 BREUDS). The authors also want to thank J Rivera-Letelier for his help in finding the right explicit form of the cubic tangency, and Lorenzo Díaz for great suggestions on the text.

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