The Monty Hall Problem: Switching is Forced by the Strategic Thinking

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To switch or not to switch, that is the question . . .

1 The MHP

The Monty Hall Problem is often called paradox. The layman, trapped by the alleged symmetry between two unrevealed doors, readily overlooks the advantage of the switching action. This is not surprising as even the great mathematical minds were not immune to fallacies of the symmetry. Leibniz believed that “with two dice it is as feasible to throw a total 12 points as to throw a total 11, as either one can only be achieved in one way” [11]. Erdős was reluctant in admitting utility of switching the doors in the MHP [7].

In the past twenty years the Monty Hall paradox made its way from the pages of popular magazines to numerous introductory texts on probability theory [1 6]. Dozens of references are found in Wikipedia, thousands more on the Web [13]. YouTube broadcasts funny animations and academic explanations. A comprehensive source for the MHP is the book by Rosenhouse [8] which traces the history, analyzes some mathematical variations and points to the literature from distinct areas of science.

Recently, Richard Gill from Leiden University devoted a series of publications to the MHP [3 4 5]. Upon due analysis of the Wikipedia struggle [14] between simplists and conditionalists, who dispute about the question if a rigorous solution does require the conditional probabilities, Gill condemned
the academic explanations of the paradox as “classical example of solution
driven science”. A major focus of his criticism is that the unspecified proba-
bility distributions are commonly assumed uniform to make the problem well
posed and amenable for analysis by tools of the probability theory.

As a new line of thinking Gill suggested to view the MHP as a game in
which two actors, Host and Contestant, employ two-action decision strate-
gies. This is a very attractive approach, since no a priori assumptions on
the distribution of unknown factors are made, rather the randomness is in-
troduced as a feature of the game-theoretic solution. In particular, Contes-
tant’s strategy “choose a door uniformly at random, then switch” appears as
a mixed minimax strategy, which ensures the winning probability 2/3, equal
to the value of the zero-sum game.

In this paper we elaborate details of the game-theoretic approach. Our
main point is that the fundamental principle of eliminating the dominated
strategies provides a convincing explanation of the advantage of the switching
action to the man from the street, as compared with the more sophisticated
arguments based on decision trees, conditional probabilities and Bayes’ the-
orem. Every Contestant’s policy “choose door Y and stay with it” is out-
performed by a policy “choose door $Y' \neq Y$ then switch”, no matter what
Host does. Once the man from the street adopts strategic thinking and real-
izes that there is a two-step action, the comparison of alternatives becomes
obvious and, moreover, free of any probability considerations.

To conceive another twist in the switch-versus-not-switch dilemma let us
take in this introduction a simplistic approach, that is disregard which par-
ticular door is revealed by Host after Contestant’s choice. Let $X$ denote the
door hiding the car. Consider three Contestant’s strategies:

- $A$: choose door 1, do not switch,
- $B$: choose door 1 then switch,
- $C$: choose door 2 then switch.

Strategy $A$ wins if $X = 1$, while strategy $B$ wins if $X \in \{2, 3\}$, so they
cannot win simultaneously. The odds 1 : 2 are against $A$ if the values of $X$
are assumed equally likely. More generally, the statistical reasoning assigns
probabilities to the values of $X$ and leads to the familiar conclusion that $B$
should be preferred to $A$ under the condition that the probability of $X = 1$ is
less than the probability of $X \in \{2, 3\}$. Nothing new so far, but now including
C into the consideration we observe that strategy $C$ wins for $X \in \{1, 3\}$, so if $A$ wins then $C$ wins too, and there is a situation when $C$ wins while $A$ fails. Thus strategy $C$ is not worse than $A$, and it is strictly better if door 3 sometimes hides the car. This provides a universal ground to avoid $A$, and for similar reason the other strategies which do not switch.

2 The zero-sum game

Regarding the rules of the Monty Hall show, we shall follow the conventions which Rosenhouse [8] calls canonical or classic, with the only amendment that Host has the freedom to choose a door hiding the car. A natural model to start with is a pure competition of the actors. Whether it is realistic or not, this instance answers the question what Contestant can achieve under the least favorable circumstances.

The abstraction of the mathematical Game Theory is based on a number of concepts such as strategy, payoff and common knowledge. For these and basic propositions used below we refer to the online tutorial by Ferguson [2].

2.1 Strategies and the payoff matrix

To introduce formally the possible actions of actors and the rules it will be convenient to label the doors 1, 2, 3 in the left-to-right order. The game in extensive form has four moves:

(i) Host chooses a door $X$ out of 1, 2, 3 to hide the car. The choice is kept in secret.

(ii) Contestant picks a door $Y$ out of 1, 2, 3 and announces her choice. Now both actors know $Y$, and they label the doors distinct from $Y$ Left and Right in the left-to-right order.

(iii) If $Y = X$, so the choice of Contestant fell on the door with the car, Host chooses door $Z$ to reveal from Left and Right doors. In the event of mismatch, $Y \neq X$, Host reveals the remaining door $Z$ (distinct from $X$ and $Y$), which is either Left or Right depending on $X, Y$.

(iv) Contestant observes the revealed door $Z$ and makes a final decision: she can choose between Switch and Notswitch from $Y$ to another un-
revealed door (so distinct from $Y$ and $Z$). Contestant wins if the final choice yields $X$ and loses otherwise.

Host’s action on step (iii), when he has some freedom, may depend on the initial Contestant’s choice $Y$. Contestant’s final action in (iv) depends on both $Y$ and $Z$. The rules of the game is a common knowledge.

To put the game in the matrix form we label the admissible pure strategies of the actors. The pure strategies of Host are

1L, 1R, 2L, 2R, 3L, 3R.

For instance, according to strategy 2L the car is hidden behind door $X = 2$, then if the outcome of (ii) is $Y = 2$, Host will reveal Left door (which is door 1); if $Y = 1$ Host will reveal Right door (which is door 3); and if $Y = 3$ Host will reveal Left door (which is door 1).

The pure strategies of Contestant are

1SS, 1SN, 1NS, 1NN, 2SS, 2SN, 2NS, 2NN, 3SS, 3SN, 3NS, 3NN.

The first symbol is a value of $Y$, while SS, SN, NS, SS encode how Contestant’s second action depends on $Y$ and Left/Right door opened. For instance, 1NS means that door $Y = 1$ is initially chosen, then Contestant plays Notswitch if Host reveals Left door; and she plays Switch if Host reveals Right door.

The game is played as if Host and Contestant have chosen their two-step pure strategies before the Monty Hall show starts. For this purpose they may ask friends for advice, or employ random devices like spinning a roulette or rolling dice. After the choices are made the actors simply follow their plans. The choices could be communicated to a referee who announces the then pre-determined outcome of the game. For example, if Contestant and Host chose profile (2SN, 1R) the show proceeds as follows:

(i) Host hides the car behind door 1.

(ii) Contestant picks door 2, thus the actors label door 1 as Left and door 3 as Right.

(iii) Host observes a mismatch hence he reveals the remaining door 3.

(iv) Contestant observes opened door 3, which is Right, hence she plays Notswitch - meaning that she stays with door 2 (and loses).

The zero-sum game is assumed to have two distinguishable outcomes – Contestant either wins the car (payoff 1) or not (payoff 0). In the zero-sum game the payoff of one actor is the negative of the payoff of another.
Contestant is willing to win the car while Host aims to avoid this. All possible outcomes of the game are summarized in matrix $\mathbf{C}$ with the payoffs of Contestant:

|     | 1L | 1R | 2L | 2R | 3L | 3R |
|-----|----|----|----|----|----|----|
| 1SS | 0  | 0  | 1  | 1  | 1  | 1  |
| 1SN | 0  | 1  | 0  | 0  | 1  | 1  |
| 1NS | 1  | 0  | 1  | 1  | 0  | 0  |
| 1NN | 1  | 1  | 0  | 0  | 0  | 0  |
| 2SS | 1  | 1  | 0  | 0  | 1  | 1  |
| 2SN | 0  | 0  | 0  | 1  | 1  | 1  |
| 2NS | 1  | 1  | 1  | 0  | 0  | 0  |
| 2NN | 0  | 0  | 1  | 1  | 0  | 0  |
| 3SS | 1  | 1  | 1  | 1  | 0  | 0  |
| 3SN | 0  | 0  | 1  | 1  | 0  | 1  |
| 3NS | 1  | 1  | 0  | 0  | 1  | 0  |
| 3NN | 0  | 0  | 0  | 0  | 1  | 1  |

Following the paradigm of the zero-sum games, we shall look for minimax solutions. Quick inspection of $\mathbf{C}$ shows that there are no saddle points in pure strategies. We turn therefore to randomized, or mixed strategies. A mixed strategy of Contestant is a row vector $\mathbf{P}$ of twelve probabilities that are assigned to her pure strategies. Similarly, a mixed strategy of Host is a row vector $\mathbf{Q}$ with six components. When profile $(\mathbf{P}, \mathbf{Q})$ is played by the actors, the expected payoff of Contestant, equal to the winning probability, is computed by the matrix multiplication as $\mathbf{PCQ}^T$, where $^T$ denotes transposition. We stress that this way to compute the winning probability presumes that actors’ choices of pure strategies are independent random variables, which may be simulated by their private randomization devices.

### 2.2 The dominance

The search of solution is largely facilitated by a simple reduction process based on the idea of dominance. Actor’s strategy $B$ is said to be dominated by strategy $A$ if anything the actor can achieve using strategy $B$ can be achieved at least as well using $A$ (that dominates $B$), no matter what the
opponent does. Contestant is willing to maximize her payoff, hence she will have no disadvantage by excluding the dominated strategies.

The principle of eliminating the dominated strategies is a theorem which asserts that removal of dominated rows (or columns) does not affect the value of the game. This enables us to reduce the game matrix by noticing that 1SS dominates 2SN and 2NN

\[
\begin{array}{c|cccccc}
& 1L & 1R & 2L & 2R & 3L & 3R \\
\hline
1SS & 0 & 0 & 1 & 1 & 1 & 1 \\
2SN & 0 & 0 & 0 & 1 & 1 & 1 \\
2NN & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

and that 3SS dominates 2NS

\[
\begin{array}{c|cccccc}
& 1L & 1R & 2L & 2R & 3L & 3R \\
\hline
3SS & 1 & 1 & 1 & 1 & 0 & 0 \\
2NS & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

Similarly, all YNS, YNN and YSN strategies are dominated for \( Y = 1, 2, 3 \).

After the row elimination the original game matrix \( C \) is reduced to a smaller matrix

\[
\begin{array}{c|cccc}
& 1L & 1R & 2L & 2R \\
\hline
1SS & 0 & 0 & 1 & 1 \\
2SS & 1 & 1 & 0 & 0 \\
3SS & 1 & 1 & 1 & 0 \\
\end{array}
\]

Note that the strategies involving Nonswitch action are all gone!

Continuing the reduction process, we observe that columns XR and XL of the reduced matrix are identical for \( X = 1, 2, 3 \), hence using the dominance, now from the perspective of Host, the matrix can be further reduced to the square matrix \( c \)

\[
\begin{array}{c|ccc}
& 1L & 2L & 3L \\
\hline
1SS & 0 & 1 & 1 \\
2SS & 1 & 0 & 1 \\
3SS & 1 & 1 & 0 \\
\end{array}
\]
2.3 A minimax solution

Matrix $c$ is the structure of payoffs in a game in which each actor has only three pure strategies. The matrix has no saddle point, thus we turn to actors’ mixed strategies $p, q$ which we write as vectors of size three.

One may guess and then check that if Contestant plays the mixed strategy with probability vector $p^* = (1/3, 1/3, 1/3)$ then her probability of win is $2/3$ no matter what Host does. It is sufficient to check this for three products $p^*cq^T$, where $q$ is one of the pure strategies of Contestant $(1,0,0), (0,1,0), (0,0,1)$. Similarly, if Host plays mixed strategy $q^* = (1/3, 1/3, 1/3)$ then Contestant’s winning probability is $2/3$ no matter what she does. Contestant can guarantee winning chance $2/3$, and Host can guarantee that the chance is not higher, therefore $V = 2/3$ is the minimax value of the reduced game, i.e.

$$\max_p \min_q pcq = \min_q \max_p pcq^T = p^*cq^T = 2/3.$$ 

Instead of guessing the minimax probability vectors $p^*$ and $q^*$ we could use various techniques applicable in our situation:

(a) The principle of indifference (see [2], Theorem 3.1) says that $p^*$ equalizes the outcome of the game in which Host uses any of pure strategies that enter his minimax strategy with positive probability. This leads to a system of linear equations for $p^*$ . Similarly for $q^*$ .

(b) Matrix $c$ is a square nonsingular matrix, thus the approach based on a general formula involving the inverse matrix can be tried (see [2], Theorem 3.2).

(c) Subtracting constant matrix with all entries equal 1 reduces to the game with diagonal matrix,

\[
\begin{array}{c|ccc}
& 1L & 2L & 3L \\
1SS & -1 & 0 & 0 \\
2SS & 0 & -1 & 0 \\
3SS & 0 & 0 & -1
\end{array}
\]

for which a formula (see [2], Section 3.3 and Exercise 3.1) applies to give

\[
V - 1 = \left( \frac{1}{-1} + \frac{1}{-1} + \frac{1}{-1} \right)^{-1} = -\frac{1}{3}.
\]
(d) Observe that the square matrix is invariant under simultaneous permutations of rows and columns (relabelling doors 1, 2, 3). Using the principle of invariance (see [2], Theorem 3.4) it is easy to deduce that \( p^* \) is the uniform distribution on the set of three pure strategies in game \( c \). Similarly for \( q^* \).

Going back to the original matrix \( C \), we conclude that \( V = 2/3 \) is the value of the game, and that the profile

\[
P^* = \left( \frac{1}{3}, 0, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0 \right), \quad Q^*_{1,1,1} = \left( \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0 \right)
\]

is a solution to the game. The subscript of \( Q^*_{1,1,1} \) will be explained soon. According to this solution Host hides the car uniformly at random, and always reveals Left door when there is a freedom of second action. Contestant selects door \( Y \) uniformly at random and always plays Switch.

A curious feature of this solution is that the preference of Host to Left door sometimes gives strong confidence for Contestant’s decision. When Host reveals Right door he signals that Left could not be opened, so Contestant learns the location of the car and her Switch action bears no risk.

2.4 All minimax solutions

The reader might have noticed that strategy \( Q^*_{1,1,1} \) disagrees with the commonly assumed Host’s behaviour, which corresponds to the uniform distribution over all possible choices,

\[
Q^*_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right).
\]

According to \( Q^*_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} \), Host hides the car uniformly at random, and for the second choice between Left and Right (if there is a freedom) a fair coin is flipped. This strategy is also minimax.

What are all minimax strategies? To answer this question we need to trace back what was lost in the elimination process. By the column elimination we may remove any of two pure strategies \( XL, XR \) for each \( X = 1, 2, 3 \). This yields eight minimax solutions \( Q^*_{0,0,0}, Q^*_{0,0,1}, \ldots, Q^*_{1,1,1} \), where in position \( X = 1, 2, 3 \) of the index we write 0 if \( XL \) is never used, and we write 1 if
XR is never used. Mixtures of these minimax strategies are again minimax, and each such mixture can be uniquely represented in the form

\[ Q_{\lambda_1, \lambda_2, \lambda_3}^* = \left( \frac{\lambda_1}{3}, \frac{1 - \lambda_1}{3}, \frac{\lambda_2}{3}, \frac{1 - \lambda_2}{3}, \frac{\lambda_3}{3}, \frac{1 - \lambda_3}{3} \right) \]

where \( 0 \leq \lambda_X \leq 1 \). Parameter \( \lambda_X \) has a transparent interpretation: this is the conditional probability that Host will reveal Left door when the car is hidden behind \( X \) and a match \( Y = X \) occurs.

The subclass of Host’s strategies with the second action independent of the first given \( X = Y \) consists of strategies with equal probabilities \( \lambda_1 = \lambda_2 = \lambda_3 \). This mode of Host’s behavior is classified in \( [8] \) as Version Five of the MHP. More general strategies \( Q_{\lambda_1, \lambda_2, \lambda_3}^* \) were considered in \( [9] \).

We need to further check if some minimax strategies of Contestant were lost in the course of row elimination. The verification is necessary because the deleted dominated strategies \( YNN, YNS \) and \( YSN \) are only weakly dominated, meaning that in some situations they perform equally well as the strategies \( Y’SS \) which dominate them. Examples of games can be given showing that weakly dominated strategies may be minimax (see \( [2] \), Section 2.6, Exercise 9).

Recall that best response is a strategy optimal for an actor knowing which particular strategy the opponent will use. Every minimax strategy \( P \) is necessarily a best response to every minimax strategy of Host, yielding the expected payoff equal to the value \( PCQ_{\lambda_1, \lambda_2, \lambda_3}^* = \frac{2}{3} \). Suppose for a time being that minimax \( P \) assigns nonzero probability \( p > 0 \) to the pure strategy \( 2SN \), and let \( P’ \) be a strategy obtained from \( P \) by removing the \( 2SN \)-component but adding weight \( p \) to the \( 1SS \)-component. Recalling the pattern

\[
\begin{array}{c|ccccccc}
 & 1L & 1R & 2L & 2R & 3L & 3R \\
\hline
1SS & 0 & 0 & 1 & 1 & 1 & 1 \\
2SN & 0 & 0 & 0 & 1 & 1 & 1 \\
2NN & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

we obtain

\[ P’CQ_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^* = PCQ_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^* + \frac{p}{6} > PCQ_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^* \]

which means that \( P’ \) strictly improves \( P \) in the combat against the minimax strategy \( Q_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^* \). But this is a contradiction with the assumed minimax property of \( P \), thus \( 2SN \) cannot have positive probability in \( P \). In the same way
it is shown that 2NN does not enter \( P \), and by symmetry among the doors we conclude that none of the dominated strategies enters \( P \). Thus nothing was lost by the row elimination.

A crucial property of \( Q_1^{\lambda_1,\lambda_2,\lambda_3} \) we just used is that this strategy gives nonzero probability to each of six pure strategies of Host. We shall call mixed strategy \( Q \) fully supported if every pure strategy has a positive probability in \( Q \). In particular, \( Q_1^{\lambda_1,\lambda_2,\lambda_3} \) is fully supported if and only if \( 0 < \lambda_X < 1 \) for \( X = 1, 2, 3 \), and minimaxity of any of these precludes minimaxity of every (weakly) dominated strategy of Contestant.

To compare, let us examine strategy \( Q_{1,1,1}^* \) which always reveals Left door by a match. Pure strategy 1NS is a best response to \( Q_{1,1,1}^* \), with the winning chance \( 2/3 \), like for any other minimax strategy of Contestant. If Contestant were ensured that Host will play \( Q_{1,1,1}^* \) then she may, in principle, choose 1NS. However 1NS versus \( Q_{1,1,1}^* \) would be an unstable profile, since Host could drop Contestant’s winning chance by swapping to \( Q_2^{\lambda_1,\lambda_2,\lambda_3} \).

We summarize our analysis of the zero-sum game in the following theorem:

**Theorem** Strategy \( P^* \), which is the uniform mixture of 1SS, 2SS, 3SS is the unique minimax strategy of Contestant. Every strategy \( Q_1^{\lambda_1,\lambda_2,\lambda_3} \) with \( 0 \leq \lambda_X \leq 1 \), \( X = 1, 2, 3 \) is a minimax strategy of Host. The value of the game is \( V = 2/3 \).

We see that in the setting of zero-sum games any rational behaviour of Host keeps Contestant away from employing strategies with Notswitch action.

### 3 Nonzero-sum games

Could the strategies with Notswitch action be rational if the goals of actors are not antagonistic? The answer is trivially “yes”. For suppose Host is sympathetic to the extent that he is most happy when Contestant wins the car. The profile (1NN, 1L) is then optimal for both actors: Host will “hide” the car behind door 1, and Contestant will “guess” the prize there. Every actor knows that unilateral stepping away from (1NN, 1L) cannot increase private payoffs, thus the profile is an acceptable solution for everybody.

To treat the MHP within the framework of the general-sum game theory we need to assume some Host’s payoff matrix \( H \) of the same dimensions as \( C \). Both matrices can be conveniently written as one bimatrix with the generic
entry \((c, h)\) specifying two payoffs for a given profile of pure strategies.

Thinking of the real-life Monty Hall show there is no obvious candidate for \(H\). With some degree of plausibility, if Host is antagonistic we may take \(h \equiv -c\) (zero-sum game), if sympathetic \(h \equiv c\), and if indifferent to the outcome \(h \equiv 0\). In fact, if the only Host’s concern is whether car won or not, these three instances cover all essentially different possibilities.

A central solution concept for bimatrix games is Nash equilibrium. A bimatrix entry \((c', h')\) corresponds to a pure Nash equilibrium if \(c'\) is a maximum in the \(C\)-component of the column of \((c', h')\), and \(h'\) is a maximum in the \(H\)-component of the row of \((c', h')\). Similarly, a profile of mixed strategies \((P', Q')\) is a mixed Nash equilibrium if

\[
P' C Q'^T = \max_P P C Q'^T \quad \text{and} \quad P' H Q'^T = \max_Q P' H Q'^T.
\]

The first equation says that \(P'\) is a best response to \(Q'\), that is \(P'\) maximizes Contestant’s expected payoff when the opponent plays \(Q'\). The second equation says that \(Q'\) is a best response to \(P'\). A general theorem due to John Nash ensures that at least one such Nash equilibrium exists.

In every Nash equilibrium Contestant will have the winning probability not less than her minimax value \(V = 2/3\). Higher chance might be possible, unless the game is strictly antagonistic.

3.1 Some examples

The examples of game matrices to follow are designed for the sake of instruction, and do not pretend to any degree of realism.

(a) Sympathetic Host is modelled by the bimatrix \(H = C\) (that is \(h \equiv c\)). Every entry \((1, 1)\) corresponds then to a pure Nash equilibrium.

(b) Indifferent Host has payoff matrix with identical entries, for instance \(h \equiv 0\). Every Host’s strategy \(Q\) and best Contestant’s response \(P' = P'(Q)\) to \(Q\) make up a Nash equilibrium.

(c) Maverick Host with such payoff might want to disprove the advantage of Switch action:
The remaining rows are completed by requiring exchangeability among the doors.

Observe that both profiles (1NS,1L) and (1NN,1L) are pure Nash equilibria with distinct payoff profiles (1,5) and (1,4), respectively. This contrasts with zero-sum games, where all minimax solutions result in the same payoff (the value of the game).

Naturally, Host would prefer outcome (1,5) to (1,4), but he has no means to force Contestant playing 1NN in place of 1NS even though she will have no disadvantage. Paradoxes of this kind, congenial with the famous Prisoner’s Dilemma, are inherent to the noncooperative games.

Among Contestant’s strategies, rows 1NS and 1NN are dominated, but discarding them we lose two Nash equilibria, one of which entails for Host the highest possible payoff.

(δ) Antagonistic and superstitious Host. Suppose Host loses a point when the car is won, and loses another point for opening Right door in case of match:

The (the matrix is completed by requiring the exchangeability of doors). Columns XR are dominated in the $H$-component and can be removed
without loss of Nash equilibria. Removing the columns we reduce to a zero-sum game whose unique solution is \((P^*, Q^*_{1,1,1})\) already encountered in Section 2.3.

### 3.2 Best responses

In a Nash equilibrium \((P', Q')\), strategy \(P'\) is a best response to \(Q'\). This feature and the dominance are the keys to the question in the epigram to this paper.

**Proposition** Suppose Host uses strategy \(Q\) according to which the car is hidden behind door \(X\) with probability \(\pi_X\), for \(X = 1, 2, 3\). Then every Contestant’s best response to \(Q\) yields the winning probability \(1 - \min(\pi_1, \pi_2, \pi_3)\).

The proof of proposition is straightforward by dominance. Discarding dominated pure strategies does not diminish the winning chance in the combat with \(Q\). We are left with YSS, and the comparison of probabilities \(1 - \pi_Y\) for \(Y = 1, 2, 3\) is in favor of the minimizer of \(\pi_1, \pi_2, \pi_3\).

We stress that probability \(\pi_X\) is the cumulative probability of strategies \(XL\) and \(XR\). The behavior of Host when he has a freedom of the second action can be arbitrary.

To appreciate the method based on dominance, the reader may consult other approaches. Tijms (see [10], p. 217, problem 6.4) suggests to set up a chance tree, and Rosenhouse (see [8], Section 3.10) shows calculations with conditional probabilities; in both references the assumption that the second choice of Host occurs by unbiased coin-flipping is taken for granted.

Recall strategy \(Q^*_{1,1,1}\) which has preference to Left door. One best response to this is the (unique) minimax strategy \(P^*\). Another best response is the uniform mixture of 1NS, 2NS and 3NS, according to which Contestant chooses \(X\) uniformly and then plays Switch only if Right door is revealed.

Let us inspect conditions under which a best response strategy avoids Notswitch action. To ease notation suppose \(\pi_1 \geq \pi_2 \geq \pi_3\). Excluding the trivial case \(\pi_3 = 0\), we assume \(\pi_3 > 0\). Strategies YNN are not included in any best response to \(Q\), because they only achieve \(\pi_Y < 1 - \pi_3\). For other dominated pure strategies there is a simple exclusion rule: for \(X = 1, 2, 3\)

1. if \(XL\) enters \(Q\) with nonzero probability then every best response excludes XSN,
(II) if XR enters $Q$ with nonzero probability then every best response excludes XNS.

The rules are derived in the same way we used to show the uniqueness of minimax strategy $P^*$ in Section 2.4, from the patterns of $C$ like

|       | 1L | 1R | 2L | 2R | 3L | 3R |
|-------|----|----|----|----|----|----|
| 1SS   | 0  | 0  | 1  | 1  | 1  |    |
| 2SN   | 0  | 0  | 0  | 1  | 1  | 1  |

Looking at best response to a fully supported strategy we conclude:

**Theorem** Suppose in some Nash equilibrium $(P', Q')$ Host uses a strategy $Q'$ which gives nonzero probabilities to each of the admissible actions 1L, 1R, 2L, 2R, 3L, 3R. Then $P'$ is a mixture of strategies 1SS, 2SS, 3SS, which do not employ action Notswitch.

More precisely, assuming $\pi_1 \geq \pi_2 \geq \pi_3$ for the probabilities of $X = 1, 2, 3$ we have

1. If $\pi_1 \geq \pi_2 > \pi_3 \geq 0$ then $P'$ is the pure strategy 3SS.
2. If $\pi_1 > \pi_2 = \pi_3 \geq 0$ then $P'$ is a mixture of 2SS, 3SS.
3. If $\pi_1 = \pi_2 = \pi_3 = 1/3$ then $P'$ is the uniform mixture of 1SS, 2SS, 3SS.

Finally, we shall draw some conclusions about Host’s preferences when payoff matrix $H$ admits Nash equilibrium $(P', Q')$ with fully supported $Q'$. From the fact that $Q'$ is a best response to $P'$ follows that $P'$ equalizes all strategies of Host. That is to say, when Contestant plays $P'$ Host is indifferent which strategy to play, since the payoff $P' HQ$ is the same for all $Q$. In case (1) this is only possible if row 3SS of $H$ has equal entries. In case (2) a mixture of rows 2SS and 3SS of $H$ is a row with equal entries. In case (3) the equilibrium strategies are as in the zero-sum game, whence the average of rows 1SS, 2SS, 3SS of $H$ must be a row with equal entries.

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