ASYMPTOTIC POLYBALANCED KERNELS
ON EXTREMAL KÄHLER MANIFOLDS

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In honor of Professor Ngaiming Mok’s 60th birthday

ABSTRACT. In this paper, improving a result in [12], we obtain asymptotic polybalanced kernels associated to extremal Kähler metrics on polarized algebraic manifolds. As a corollary, we strengthen a result in [15] on asymptotic relative Chow-polystability for extremal Kähler polarized algebraic manifolds. Finally, related to the Yau-Tian-Donaldson Conjecture for extremal Kähler metrics, we shall discuss the difference between strong relative K-stability (cf. [19]) and relative K-stability.

1. Introduction

In this paper, we fix once for all a polarized algebraic manifold $(M, L)$ which is by definition a pair of a nonsingular irreducible complex projective variety $M$ of dimension $n$ and a very ample holomorphic line bundle $L$ on $M$. By taking the identity component $\text{Aut}^{0}(M)$ of the group of all biholomorphisms of $M$, we consider the maximal connected linear algebraic subgroup $H$ of $\text{Aut}^{0}(M)$. Hence $\text{Aut}^{0}(M)/H$ is an abelian variety. For the identity component $Z$ of the center of a maximal compact connected subgroup $K$ of $H$, we take its complexification $Z_{C}$ in $H$. Let $\mathfrak{z} := \text{Lie}(Z)$ and $\mathfrak{z}_{C} := \text{Lie}(Z_{C})$ be the associated Lie algebras. Then the infinitesimal action of $\mathfrak{z}_{C}$ on $M$ lifts to an infinitesimal bundle action of $\mathfrak{z}_{C}$ on $L$. For

$$V_{m} := H^{0}(M, \mathcal{O}(L^{\otimes m})), \quad m = 1, 2, \ldots,$$

we view $\mathfrak{z}_{C}$ as a Lie subalgebra of $\mathfrak{sl}(V_{m})$ by taking the traceless part for each element of $\mathfrak{z}_{C}$. In view of the infinitesimal action of $\mathfrak{z}_{C}$, $V_{m}$ is expressible as a direct sum of $K$-invariant subspaces,

$$V_{m} = \bigoplus_{\alpha=1}^{\nu_{m}} V_{m,\alpha},$$

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where $V_{m,\alpha} := \{ \tau \in V_m : Y\tau = \chi_{m,\alpha}(Y)\tau \text{ for all } Y \in \mathfrak{g}_\mathbb{C} \}$ with mutually distinct characters $\chi_{m,\alpha} \in \mathfrak{g}_\mathbb{C}^\ast$, $\alpha = 1, 2, \ldots, \nu_m$. Let $h$ be a $K$-invariant Hermitian metric for $L$ such that the associated first Chen form $\omega = c_1(L; h)$ is Kähler. Define a Hermitian metric $\rho_m(h)$ for $V_m$ by

$$\rho_m(h)(\tau, \tau') := \int_X (\tau, \tau')_h \omega^n \quad \tau, \tau' \in V_m,$$

where $(\tau, \tau')_h$ denotes the pointwise Hermitian pairing of $\tau, \tau'$ in terms of the Hermitian metric $h$. If $\tau$ and $\tau'$ coincide, then we write $(\tau, \tau)_h$ simply as $|\tau|^2_h$. By setting $q := 1/m$, we now define

$$B_{m,\alpha}(h) := n! q^n \sum_{i=1}^{\nu_m} |\tau_{\alpha,i}|^2_h, \quad B_m(h) := \sum_{\alpha=1}^{\nu_m} B_{m,\alpha}(h),$$

where $\{\tau_{\alpha,i}; i = 1, 2, \ldots, n_{\alpha}\}$ is an orthonormal basis for the subspace $V_{m,\alpha}$ of the Hermitian vector space $(V_m, \rho_m(h))$. Note that, if $M$ admits an extremal Kähler metric $\omega_0$ in the class $c_1(L)$, then in view of [2], by choosing $K$ to be the identity component of the group of isometries for $(M, \omega_0)$ in $H$, we may assume that the extremal Kähler vector field $V$ (cf. [3]) belongs to $\mathfrak{g}$. Let $\sigma_{\omega_0}$ be the scalar curvature of $\omega_0$, and define a real constant $C_0$ by

$$C_0 := \{2c_1(L)^n[M]\}^{-1} \left\{ \int_M \sigma_{\omega_0} \omega_0^n + \sqrt{-1} \int_M h_0^{-1}(\mathcal{V} h_0) \omega_0^n \right\}.$$

Then by setting $\mathcal{Y}_0 := \sqrt{-1} \mathcal{V} / 2$ and $\varphi_0 := 0$, we obtain

**Main Theorem:** Suppose that $L$ admits a Hermitian metric $h_0$ such that $\omega_0 := c_1(L; h_0)$ is an extremal Kähler metric with extremal Kähler vector field $V \in \mathfrak{g}$. Then there exist vector fields $\mathcal{Y}_k \in \sqrt{-1} \mathfrak{g}$, smooth real-valued $K$-invariant functions $\varphi_k$, real constants $C_k$, $k = 1, 2, \ldots,$ on $M$ such that

$$\sum_{\alpha=1}^{\nu_m} \{1 - \chi_{m,\alpha}(\mathcal{Y}(\ell))\} B_{m,\alpha}(h(\ell)) = 1 + C(\ell) + O(q^{\ell+2}), \quad \ell = 0, 1, 2, \ldots, \tag{1.1}$$

with $\mathcal{Y}_k, \varphi_k, C_k$ independent of $q$ and $\ell$, where $\mathcal{Y}(\ell) := \sum_{k=0}^{\ell} q^{k+2} \mathcal{Y}_k$, $h(\ell) := h_0 \exp(-\sum_{k=0}^{\ell} q^k \varphi_k)$, and $C(\ell) := \sum_{k=0}^{\ell} C_k q^{k+1}$.

Here for every integer $r$, we mean by $O(q^r)$ a quantity whose $C^j$-norm for every nonnegative integer $j$ is bounded by $\kappa_j q^r$ for some positive constant $\kappa_j$ independent of $q$ and $\alpha$. In the above Main Theorem, let $\ell \to \infty$. Then the formal expression of the left-hand side of (1.1) is called the asymptotic polybalanced kernel for $(M, L)$.

Let $T$ be an arbitrary algebraic torus in $H$ satisfying $Z_\mathbb{C} \subset T$. As a corollary of Main Theorem, we obtain
Corollary: If $M$ admits an extremal Kähler metric in the class $c_1(L)$, then $(M, L)$ is asymptotically Chow-polystable relative to $T$, i.e., $(M, L^\otimes m)$, $m \gg 1$, are Chow-polystable relative to $T$.

In the last section, in view of [19] and a recent result of Yotsutani-Zhou [29], we shall discuss extremal Kähler versions of the Yau-Tian-Donaldson Conjecture from various points of view.

For preceding related works, see Apostolov-Huang [1], Donaldson [5], [6], Futaki [7], Hashimoto [9], Lu [10], Ono-Sano-Yotsutani [21], Phong-Sturm [22], Sano-Tipler [23], Székelyhidi [24], [25], Tian [26], [27], Zelditch [30] and Zhang [31]. I owe much to these works.

Parts of this paper were announced in Pacific Rim Conference on Complex and Simplectic Geometry XI at Heifei in July, 2016. Afterwards, during the preparation of this paper, I heard that R. Seyyedali showed asymptotic Chow-stability, relative to a maximal algebraic torus in $\text{Aut}^0(M)$, for an extremal Kähler polarized algebraic manifold $(M, L)$.

2. Proof of Main Theorem

In this section, we prove Main Theorem by induction on $\ell$. For each $K$-invariant Hermitian metric $h$ for $L$, by setting $|\tau_{\alpha,i}|^2 := \tau_{\alpha,i} \bar{\tau}_{\alpha,i}$, we consider

$$
(2.1) \quad \Psi_m(h) := n! q^m \sum_{\alpha=1}^{n} \sum_{i=1}^{n_\alpha} |\tau_{\alpha,i}|^2,
$$

where $\{\tau_{\alpha,i} : i = 1, 2, \ldots, n_\alpha\}$ is an orthonormal basis for $(V_m, \rho_{m}(h))$ as in the introduction. Then the left-hand side of (1.1) is the real-valued function on $X$ obtained as the contraction

$$
(1 - \mathcal{Y}(\ell)) \Psi_m(h(\ell))
$$

of $(1 - \mathcal{Y}(\ell)) \Psi_m(h(\ell))$ with $h(\ell)^m$ (see [11], (1.4.1), for the definition of $\mathcal{Y}(\ell)|_{\tau_{\alpha,i}}$). Hence the proof of (1.1) (and hence Main Theorem) is reduced to showing the following for all nonnegative integers $\ell$:

$$
(2.2) \quad h(\ell)^m \cdot \{(1 - \mathcal{Y}(\ell)) \Psi_m(h(\ell))\} = 1 + C(\ell) + O(q^{\ell+2}).
$$

Note that $\mathcal{Y}(\ell)$ acts on the antiholomorphic section $\bar{\tau}_{\alpha,i}$ trivially. Let $\mathcal{D}_0$ be the Lichnérowicz operator as defined in [2], (2.1), for the extremal Kähler manifold $(M, \omega_0)$, where we write $\omega_0$ as

$$
\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j
$$
in terms of a system \((z^1, \ldots, z^n)\) of holomorphic local coordinates on \(M\). Let \(\mathcal{S}\) be the space of all real-valued smooth \(K\)-invariant functions \(\varphi\) on \(M\) such that \(\int_M \varphi \omega_0^n = 0\). Since \(\mathcal{D}_0\) maps \(\mathcal{S}\) into itself, the restricted operator
\[
\mathcal{D}_0 : \mathcal{S} \to \mathcal{S}
\]
is denoted also by \(\mathcal{D}_0\) whose kernel in \(\mathcal{S}\) is written simply as \(\text{Ker} \mathcal{D}_0\). Then we have an isomorphism
\[
eq_0 : \text{Ker} \mathcal{D}_0 \cong \mathfrak{g}, \quad \varphi \leftrightarrow \varphi(\varphi) := \text{grad}_\omega^C \varphi,
\]
where \(\text{grad}_\omega^C \varphi := (1/\sqrt{-1})\Sigma_{i,j} g^{ij}(\partial \varphi/\partial z^j)\partial/\partial z^i\). By the inner product
\[
(\varphi, \psi)_\omega := \int_M \varphi \psi \omega_0^n, \quad \varphi, \psi \in \mathcal{S},
\]
we write \(\mathcal{S}\) as an orthogonal direct sum \(\text{Ker} \mathcal{D}_0 \oplus (\text{Ker} \mathcal{D}_0)^\perp\). We then consider the orthogonal projection \(\text{pr}_1 : \mathcal{S} \to \text{Ker} \mathcal{D}_0\) to the first factor. The proof of (2.2) is divided into two steps:

**Step 1:** In this step, we shall show that (2.2) is true for \(\ell = 1\). Note that \(h(0) = h_0\). In view of Lu [10], the Tian-Yau-Zelditch asymptotic expansion \([20, 30] ; \text{see also } 3\) is written in the form
\[
h_0^m \cdot \Psi_m(h(0)) \quad ( = B_m(h(0))) = 1 + \frac{\sigma_{\omega_0}}{2} q + O(q^2).
\]
By \(Y(0) = q^2 \sqrt{-1} V/2\), we have \(m Y(0) = q \sqrt{-1} V/2\). Take the infinitesimal action of \(Y(0)\) on (2.4). Dividing it further by (2.4), we obtain
\[
h_0^{-1} \sqrt{-1} (V/2) h_0 + \Psi_m(h(0))^{-1}\{Y(0) \Psi_m(h(0))\} = O(q^3).
\]
On the other hand, by [12], p.579, we have \(h_0^{-1} \sqrt{-1} (V/2) h_0 = C_0 - (\sigma_{\omega_0}/2)\). Hence (2.5) is rewritten as
\[
\Psi_m(h(0))^{-1}\{Y(0) \Psi_m(h(0))\} = - q \{C_0 - (\sigma_{\omega_0}/2)\} + O(q^3).
\]
This together with (2.4) implies that
\[
h_0^m \cdot \{Y(0) \Psi_m(h(0))\} = - q \{C_0 - (\sigma_{\omega_0}/2)\} + O(q^2).
\]
Subtracting this from (2.4), we obtain the required equality:
\[
h_0^m \cdot \{(1 - Y(0)) \Psi_m(h(0))\} = (1 + q C_0) + O(q^2) = 1 + C(0) + O(q^2).
\]

**Step 2:** Note that the left-hand side of (2.2) is
\[
\{h(\ell)^m \cdot \Psi_m(h(\ell))\} \left(\frac{1 - Y(\ell)}{\Psi_m(h(\ell))}\right) = B_m(h(\ell)) \left\{ 1 - \frac{Y(\ell) \Psi_m(h(\ell))}{\Psi_m(h(\ell))} \right\}.
\]
For a positive integer \(\ell\), by assuming the induction hypothesis
\[
B_m(h(\ell - 1)) \left\{ 1 - \frac{Y(\ell - 1) \Psi_m(h(\ell - 1))}{\Psi_m(h(\ell - 1))} \right\} = 1 + C(\ell - 1) + O(q^{\ell + 1}),
\]
we have only to find \( Y_\ell, \varphi_\ell \) and \( C_\ell \) such that \( Y(\ell) := Y(\ell - 1) + q^{\ell + 2}Y_\ell, \)

\[ h(\ell) := h(\ell - 1)e^{-q^\ell \varphi_\ell} \quad \text{and} \quad C(\ell) := C(\ell - 1) + C_\ell q^{\ell + 1} \]
satisfy

\[
B_m(h(\ell)) \left\{ 1 - \frac{Y(\ell)\Psi_m(h(\ell))}{\Psi_m(h(\ell))} \right\} = 1 + C(\ell) + O(q^{\ell + 2}).
\]

By setting \( \omega(\ell) := c_1(L; h(\ell)) \) and \( \omega(\ell - 1) := c_1(L; h(\ell - 1)) \), we have \( \omega(\ell) = \omega(\ell - 1) + (\sqrt{-1}/2\pi)q^\ell \partial \bar{\partial} \varphi_\ell \). Let \( C^\infty(M)_K^\mathbb{R} \) be the space of all real-valued \( K \)-invariant functions on \( M \). To each \( (Y_\ell, \varphi_\ell, C_\ell) \) in \( \sqrt{-1} \mathbb{I} \times C^\infty(M)_K^\mathbb{R} \times \mathbb{R} \), we assign a real-valued \( K \)-invariant function \( \Phi(q; Y_\ell, \varphi_\ell, C_\ell) \) by

\[
\Phi(q; Y_\ell, \varphi_\ell, C_\ell) := B_m(h(\ell)) \left\{ 1 - \frac{Y(\ell)\Psi_m(h(\ell))}{\Psi_m(h(\ell))} \right\} = B_m(h(\ell - 1)e^{-q^\ell \varphi_\ell}) \left\{ 1 - \frac{(Y(\ell - 1) + q^{\ell + 2}Y_\ell)\Psi_m(h(\ell - 1)e^{-q^\ell \varphi_\ell})}{\Psi_m(h(\ell - 1)e^{-q^\ell \varphi_\ell})} \right\}.
\]

By the induction hypothesis (2.6), there exists a real-valued \( K \)-invariant function \( u_\ell \) on \( M \) such that

\[
\Phi(q; 0, 0, 0) \equiv 1 + C(\ell - 1) + u_\ell q^{\ell + 1}, \quad \text{modulo } q^{\ell + 2}.
\]

In view of the variation formula for the scalar curvature (see for instance [12], (2.5)), we see that, modulo \( q^{\ell + 2} \),

\[
\left\{ \begin{array}{l}
B_m(h(\ell)) - B_m(h(\ell - 1)) \equiv (q/2)\{\sigma_\omega(\ell) - \sigma_\omega(\ell - 1)\} \\
\equiv q^{\ell + 1}(-D_0 + \sqrt{-1}V)(\varphi_\ell/2).
\end{array} \right.
\]

Put \( I_1 := \{Y(\ell)\Psi_m(h(\ell))/\Psi_m(h(\ell))\} / \Psi_m(h(\ell)) \), \( J := \{Y(\ell - 1)\Psi_m(h(\ell))/\Psi_m(h(\ell))\} / \Psi_m(h(\ell - 1)) \) and \( I_2 := \{Y(\ell - 1)\Psi_m(h(\ell - 1))/\Psi_m(h(\ell - 1))\} / \Psi_m(h(\ell - 1)) \). In view of (2.4), we have \( h_0^m \cdot \Psi_m(h(\ell)) \equiv 1 \) modulo \( q \). Hence, modulo \( q^{\ell + 2} \),

\[
I_1 - J \equiv q^{\ell + 2}\{Y_\ell\Psi_m(h(\ell))/\Psi_m(h(\ell))\} \equiv -q^{\ell + 1}h_0^{-1}(Y_\ell h_0).
\]

Note here that, by setting

\[
C := \{c_1(L)^n[M]\}^{-1} \int_M h_0^{-1}(Y_\ell h_0) \omega_0^8,
\]

we obtain \( h_0^{-1}(Y_\ell h_0) = C + c_0^{-1}(\sqrt{-1}Y_\ell) \) (see for instance [11]). On the other hand, we have the following:

\[
J - I_2 = Y(\ell - 1)\log \left\{ \frac{\Psi_m(h(\ell))}{\Psi_m(h(\ell - 1))} \right\}
= Y(\ell - 1) \left( q^{\ell - 1} \varphi_\ell + \log \left\{ \frac{h(\ell)^m \cdot \Psi_m(h(\ell))}{h(\ell - 1)^m \cdot \Psi_m(h(\ell - 1))} \right\} \right)
= Y(\ell - 1) \left( q^{\ell - 1} \varphi_\ell + \log \{B_m(h(\ell))/B_m(h(\ell - 1))\} \right).
\]
Since \( B_m(h(\ell)) \equiv B_m(h(\ell - 1)) \equiv 1 \) modulo \( q \), we see from (2.9) that
\[
\log \{ B_m(h(\ell))/B_m(h(\ell - 1)) \} \equiv 0 \text{ modulo } q^{\ell + 1}.
\]
Moreover, by \( \ell \geq 1 \), we obtain \( \mathcal{Y}(\ell - 1) \equiv q^2(\sqrt{-1}/2)\mathcal{V} \) modulo \( q^3 \). It then follows that
\[
(2.11) \quad J - I_2 \equiv q^{\ell + 1}\sqrt{-1}\mathcal{V}(\varphi_{\ell}/2), \quad \text{modulo } q^{\ell + 2}.
\]
On the other hand, since \( h_0^m \cdot \Psi_m(h(\ell)) \equiv 1 \) modulo \( q \), we obtain
\[
(2.12) \quad \begin{cases}
I_1 & \equiv q^2(\sqrt{-1}\mathcal{V}/2)\Psi_m(h(\ell))/\Psi_m(h(\ell)) \\
& \equiv - (\sqrt{-1}/2) q h_0^{-1}(\mathcal{V} h_0) \equiv 0, \quad \text{modulo } q.
\end{cases}
\]
Now by (2.10) and (2.11), \( I_1 - I_2 \equiv q^{\ell + 1}\{ \sqrt{-1}\mathcal{V}(\varphi_{\ell}/2) - h_0^{-1}(\mathcal{V} h_0) \} \equiv q^{\ell + 1}\{ \sqrt{-1}\mathcal{V}(\varphi_{\ell}/2) - C - e_0^{-1}(\sqrt{-1}\mathcal{V} \ell) \} \) modulo \( q^{\ell + 2} \). Thus by setting \( B_1 := B_m(h(\ell)) \) and \( B_2 := B_m(h(\ell - 1)) \), we see from (2.9) and (2.12) the following:
\[
(2.13) \quad \begin{cases}
\Phi(q; \mathcal{Y}, \psi, C) - \Phi(q; 0, 0, 0) & = B_1(1 - I_1) - B_2(1 - I_2) \\
& = (B_1 - B_2)(1 - I_1) - B_2(I_1 - I_2) \\
& \equiv q^{\ell + 1}\{ (-D_0 + \sqrt{-1}\mathcal{V})(\varphi_{\ell}/2) \\
& - \sqrt{-1}\mathcal{V}(\varphi_{\ell}/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{V} \ell) \} \\
& \equiv q^{\ell + 1}\{ -D_0(\varphi_{\ell}/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{V} \ell) \}, \quad \text{modulo } q^{\ell + 2}.
\end{cases}
\]
Since the function \( u_\ell - \mu \ell \) belongs to \( S \) for \( \mu \ell := \{ c_1(L)^n[M] \}^{-1} \int_M u_\ell \omega^n_0 \), we can write \( u_\ell \) as a sum
\[
(2.14) \quad u_\ell = \mu_\ell + u_\ell' + u_\ell'',
\]
where \( u'_\ell := (1 - \text{pr}_1)(u_\ell - \mu_\ell) \in (\text{Ker } D_0)^{\perp} \) and \( u''_\ell := \text{pr}_1(u_\ell - \mu_\ell) \in \text{Ker } D_0 \).
Let \( \varphi_{\ell} \) be the unique element of \( (\text{Ker } D_0)^{\perp} \) such that \( D_0(\varphi_{\ell}/2) = u'_\ell \). Put
\[
(2.15) \quad \mathcal{Y}_\ell := \sqrt{-1} e_0(u''_\ell) \quad \text{and} \quad C_\ell := \mu_\ell + C.
\]
Then by (2.3), \( \mathcal{Y}_\ell \in \sqrt{-1} \mathbb{I} \), while \( C_\ell \) is a real constant. Since \( D_0(\varphi_{\ell}/2) = u'_\ell \), it follows from (2.8), (2.13), (2.14) and (2.15) that, modulo \( q^{\ell + 2} \),
\[
\Phi(q; \mathcal{Y}, \psi, C) \equiv \Phi(q; 0, 0, 0) + q^{\ell + 1}\{ -D_0(\varphi_{\ell}/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{V} \ell) \}
\equiv 1 + C(\ell - 1) + q^{\ell + 1}\{ u_\ell - D_0(\varphi_{\ell}/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{V} \ell) \}
= 1 + C(\ell - 1) + q^{\ell + 1}\{ (u'_\ell - D_0(\varphi_{\ell}/2)) + (\mu_\ell + C) + (u''_\ell + e_0^{-1}(\sqrt{-1}\mathcal{V} \ell)) \}
= 1 + C(\ell - 1) + q^{\ell + 1}C_\ell \equiv 1 + C(\ell),
\]
which shows (2.7), as required. \( \square \)

Remark 2.16: The preceding work in [12], Theorem B, is obtained from Main Theorem above by replacing the left-hand side of (1.1) by
\[
(2.17) \quad \sum_{\alpha=1}^{\nu_m} \exp\{ -\chi_{m,\alpha}(\mathcal{Y}(\ell)) \} B_{m,\alpha}(h(\ell)).
\]
The point is the following: The coefficients $1 - \chi_{m,\alpha}(Y(\ell)), \alpha = 1, 2, \ldots, \nu_m,$ in (1.1) are linear in $Y(\ell)$, while the coefficients $\exp\{-\chi_{m,\alpha}(Y(\ell))\}, \alpha = 1, 2, \ldots, \nu_m,$ in (2.17) aren’t. This linearity is essential in the proof of the asymptotic relative Chow-polystability.

3. Relative Chow-polystability

In this section, we fix an algebraic torus $T$ in $\text{Aut}^0(M)$. Then for $K$ in the introduction, replacing $T$ by its conjugate group, we may assume that the maximal compact subgroup $T_c$ of $T$ sits in $K$. Put $t_c := \text{Lie}(T_c)$. Note that the infinitesimal action of the Lie algebra $t$ of $T$ lifts to an infinitesimal bundle action of $t$ on $L$. For each positive integer $m$, let $V_m := H^0(M, \mathcal{O}(L^\otimes m))$, and we view $t$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$ by considering the traceless part. Define a symmetric bilinear form $\langle , \rangle_m$ on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y \rangle_m := \text{Tr}(XY)/m^{n+2}, \quad X, Y \in \mathfrak{sl}(V_m).$$

Let $\mathfrak{z}_m$ be the centralizer of $t$ in $\mathfrak{sl}(V_m)$, and $\mathfrak{g}_m$ be the orthogonal complement of $t$ in $\mathfrak{z}_m$, i.e.,

$$\begin{align*}
\mathfrak{z}_m &:= \{X \in \mathfrak{sl}(V_m) ; [X, Y] = 0 \text{ for all } Y \in t\}, \\
\mathfrak{g}_m &:= \{X \in \mathfrak{z}_m ; \langle X, Y \rangle_m = 0 \text{ for all } Y \in t\}.
\end{align*}$$

Let $Z_m$ and $G_m$ be the connected reductive algebraic subgroups in $\text{SL}(V_m)$ associated to $\mathfrak{z}_m$ and $\mathfrak{g}_m$, respectively. By the infinitesimal $t$-action on $V_m$, we can write $V_m$ as a direct sum of $T_c$-invariant subspaces,

$$V_m = \bigoplus_{\gamma=1}^{\eta_m} \hat{V}_{m,\gamma},$$

where $\hat{V}_{m,\gamma} = \{\sigma \in V_m ; Y\sigma = \hat{\chi}_{m,\gamma}(Y)\sigma \text{ for all } Y \in t\}$ with mutually distinct characters $\hat{\chi}_{m,\gamma} \in t^*$, $\gamma = 1, 2, \ldots, \eta_m$. We now consider the algebraic subgroup $R_m$ of $\text{SL}(V_m)$ defined by

$$R_m := \prod_{\gamma=1}^{\eta_m} \text{SL}(\hat{V}_{m,\gamma}),$$

where each $\text{SL}(\hat{V}_{m,\gamma})$ fixes $\hat{V}_{m,\gamma'}$ if $\gamma' \neq \gamma$. Then the centralizer $H_m$ of $R_m$ in $\text{SL}(V_m)$ consists of all diagonal matrices in $\text{SL}(V_m)$ acting on each $\hat{V}_{m,\gamma}, \gamma = 1, 2, \ldots, \eta_m,$ by constant scalar multiplication. Note that $t$ viewed as a Lie subalgebra of $\mathfrak{sl}(V_m)$ sits in the Lie algebra $\mathfrak{h}_m$ of $H_m$. Let

$$t^\perp_m := \{X \in \mathfrak{h}_m ; \langle X, Y \rangle_m = 0 \text{ for all } Y \in t\}.$$

be the orthogonal complement of $t$ in $\mathfrak{h}_m$. Let $T^\perp_m$ denote the corresponding algebraic torus sitting in $H_m$. Since $Z_m = H_m \cdot R_m$, it follows that

$$G_m = T^\perp_m \cdot R_m.$$
Let $M_m$ be the image of $M$ under the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^\otimes m|$ on $M$. For the degree $d_m$ of $M_m$ in $\mathbb{P}^*(V_m)$, we consider the space

$$W_m := \{\text{Sym}^{d_m}(V_m)\}^\otimes n+1,$$

where $\text{Sym}^{d_m}(V_m)$ denotes the $d_m$-th symmetric tensor product of $V_m$. For the dual space $W^*_m$ of $W_m$, let $0 \neq \tilde{M}_m \in W^*_m$ denote the Chow form for the irreducible reduced algebraic cycle $M_m$ on $\mathbb{P}^*(V_m)$, so that the associated point $[\tilde{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for $M_m$.

For relative stability, we showed in [13] (see also [15]) that, if $c_1(L)$ admits an extremal Kähler metric, then the orbit $R_m \cdot \tilde{M}_m$ is closed in $W^*_m$ for all sufficiently large $m$. In [24] (see also [25]), Székelyhidi introduced the following stronger stability concept:

**Definition 3.2.** (1) A polarized algebraic manifold $(M, L^\otimes m)$ is called Chow-polystable relative to $T$, if $G_m \cdot \tilde{M}_m$ is closed in $W^*_m$.

(2) $(M, L)$ is called asymptotically Chow-polystable relative to $T$, if $(M, L^\otimes m)$ is Chow-polystable relative to $T$ for $m \gg 1$.

**Definition 3.3.** A polarized algebraic manifold $(M, L^\otimes m)$ is called Chow-stable relative to $T$, if the following conditions are satisfied:

(1) $G_m \cdot \tilde{M}_m$ is closed in $W^*_m$.

(2) The isotropy subgroup of $G_m$ at $[\tilde{M}_m]$ is finite.

### 4. Proof of Corollary

In this section, using the same notation as in the preceding sections, we assume that $M$ admits an extremal Kähler metric $\omega_0 = c_1(L; h_0)$ in the class $c_1(L)$, where $h_0$ is a Hermitian metric for $L$. Following the arguments in [14], we shall show that $(M, L^\otimes m)$ are Chow-polystable relative to $T$ for $m \gg 1$.

As in the introduction, we may assume that $K$ is the identity component of the group of isometries for $(M, \omega_0)$. Put $\mathfrak{k} := \text{Lie}(K)$. Let $[n/2]$ be the largest integer which does not exceed $n/2$. By applying Main Theorem to

$$\ell := [n/2] + 3,$$

we obtain a $K$-invariant Kähler metric $\omega(\ell) = c_1(L; h(\ell))$ in the class $c_1(L)$ such that (1.1) holds. For the compact group

$$K_m := \text{SU}(V_m; \rho_m(h(\ell))) \cap G_m,$$

we can view $G_m$ as its complexification. Then for the $G_m$-action on $W^*_m$, the isotropy subgroup of $K_m$ at $\tilde{M}_m$ has the Lie algebra $\mathfrak{k}_0$ sitting in $\mathfrak{k}$. Since
\( Z \subset T \), the isotropy subgroup of \( K \) at \([\tilde{M}_m]\) has the same Lie algebra \( \mathfrak{k}_0 \).

For \( \mathfrak{g}_m := \text{Lie}(G_m) \), we define the Lie subalgebras \( \mathfrak{p}_m \) and \( \mathfrak{p} \) by

\[
\mathfrak{p}_m := \sqrt{-1} \mathfrak{t}_m \quad \text{and} \quad \mathfrak{p} := \sqrt{-1} \mathfrak{t}_0,
\]

where \( \mathfrak{t}_m := \text{Lie}(K_m) \). Put \( n_\gamma := \dim \hat{V}_{m,\gamma} \). By choosing an orthonormal basis \( \{ \sigma, i : i = 1, 2, \ldots, n_\gamma \} \) for \((\hat{V}_{m,\gamma}, \rho_m(h(\ell)))\), we set

\[
(4.1) \quad j(\gamma, i) := i + \sum_{\beta=1}^{\gamma-1} n_\beta, \quad i = 1, 2, \ldots, n_\gamma; \; \gamma = 1, 2, \ldots, \eta_m,
\]

where the right-hand side denotes \( i \) in the special case \( \gamma = 1 \). Then for each \( \gamma \) and \( i \) as above, we put

\[
(4.2) \quad \hat{\sigma}_{\gamma, i} := \sqrt{1 - \hat{\chi}_{m,\gamma}(\mathcal{V}(\ell))} \sigma_{\gamma, i}.
\]

By writing \( \hat{\sigma}_{\alpha, i}, \sigma_{\alpha, i} \) as \( \hat{\sigma}_{j(\alpha, i)}, \sigma_{j(\alpha, i)} \), by abuse of terminology, we have bases

\[
(4.3) \quad \{ \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m} \} \quad \text{and} \quad \{ \sigma_1, \sigma_2, \ldots, \sigma_{N_m} \},
\]

respectively, for \((V_m, \rho_m(h(\ell)))\). Let \( \Phi_m : M \hookrightarrow \mathbb{P}^{N_m - 1}(\mathbb{C}) = \mathbb{P}^*(V_m) \) be the associated Kodaira embedding defined by

\[
(4.4) \quad \Phi_m(p) := (\hat{\sigma}_1(p) : \hat{\sigma}_2(p) : \cdots : \hat{\sigma}_{N_m}(p)), \quad p \in M.
\]

Here \( V_m \) and \( \mathbb{C}^{N_m} \) are identified by the basis \( \{ \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m} \} \). Let \( g_{\text{euc}} \) be the Euclidean metric for the space \( \mathbb{C}^{N_m} = \{(z_1, z_2, \ldots, z_{N_m})\} \). Define the Fubini-Study form \( \omega_{\text{FS}} \) on \( \mathbb{P}^*(V_m) \) (= \( \{(z_1 : z_2 : \cdots : z_{N_m})\} \) by

\[
\omega_{\text{FS}} := (\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\Sigma_{j=1}^{N_m} |z_j|^2).
\]

For each \( X \in \mathfrak{p}_m \), let \( \mathcal{V}_X \) be the associated holomorphic vector field on \( \mathbb{P}^*(V_m) \). We then have a unique real-valued function \( \varphi_X \) on \( \mathbb{P}^*(V_m) \) satisfying

\[
\int_{\mathbb{P}^{N_m - 1}(\mathbb{C})} \varphi_X \omega_{\text{FS}}^{N_m - 1} = 0 \quad \text{and} \quad i_{\mathcal{V}_X}(\omega_{\text{FS}}/m) = (\sqrt{-1}/2\pi) \partial \bar{\partial} \varphi_X.
\]

Let us consider the real-valued function \( \zeta = \zeta(x) \) on \( \mathbb{R} \) defined by \( \zeta(x) := x(e^x + e^{-x})/(e^x - e^{-x}), \; x \in \mathbb{R} \). In view of \( M_m = \Phi_m(M) \), we define a positive semidefinite \( K \)-invariant inner product \((\; , \;)_m\) on \( \mathfrak{p}_m \) by

\[
(X, Y)_m = \sqrt{-1} \int_{M_m} \partial \varphi_Y \wedge \bar{\partial} \varphi_X \wedge n \omega_{\text{FS}}^{n-1}, \quad X, Y \in \mathfrak{p}_m.
\]

Then this inner product is positive definite when restricted to \( \mathfrak{p} \). Hence as a vector space, \( \mathfrak{p}_m \) is written as an orthogonal direct sum \( \mathfrak{p} \oplus \mathfrak{p}^\perp \), where \( \mathfrak{p}^\perp \) is the orthogonal complement of \( \mathfrak{p} \) in \( \mathfrak{p}_m \). Define an open neighborhood

\[
U_m := \{ X \in \mathfrak{p}^\perp ; \zeta(\text{ad } X)p \cap \mathfrak{p}^\perp = \{0\} \}.
\]
of the origin in $p^\perp$. Let $0 \neq X \in p^\perp$. Since $X$ belongs to $g_m$, by choosing a suitable orthonormal basis $\{\sigma_{\gamma,i} : i = 1, 2, \ldots, n\}$ for $(\hat{V}_{m,\gamma}, \rho_m(h(\ell)))$, we obtain real constants $b_j$ such that

\[
X \tilde{\sigma}_j = b_j \tilde{\sigma}_j, \quad j = 1, 2, \ldots, N_m,
\]

where $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{N_m}\}$ is the basis for $(V_m, \rho_m(h(\ell)))$ as in (4.3). Define a real one-parameter subgroup $\lambda_X : \mathbb{R}_+ \to G_m$ of $G_m$ by $\lambda_X(e^t) := \exp(tX)$, $t \in \mathbb{R}$. We then consider a real-valued function $f_{X,m}(t)$ on $\mathbb{R}$ defined by

\[
f_{X,m}(t) := \log \|\lambda_X(e^t) \cdot \tilde{M}_m\|_{CH(g_{eucl})}, \quad t \in \mathbb{R},
\]

where $W_m^* \ni w \mapsto \|w\|_{CH(g_{eucl})} \in \mathbb{R}_{\geq 0}$ is the Chow norm by Zhang [31] (see also [12]). Put $\dot{f}_{X,m}(t) := (d/dt)f_{X,m}$ and $\ddot{f}_{X,m}(t) := (d^2/dt^2)f_{X,m}$. Let

\[
\delta_0 := \frac{q}{\sqrt{\sum_{j=1}^{N_m} b_j^2}}.
\]

Then by Lemma 3.4 in [14], the proof of Corollary is reduced to showing that, if $m \gg 1$, then for every $0 \neq X \in p^\perp$,

\[
\dot{f}_{X,m}(t_m) = 0 < \ddot{f}_{X,m}(t_m) \quad \text{and} \quad t_m \cdot X \in U_m,
\]

where $t_m$ is a suitable real number satisfying $|t_m| < \delta_0$. Now the proof is divided into the following three steps:

**Step 1.** Put $\bar{b} := \max_j |b_j|$. Let $t$ be an arbitrary real number satisfying

\[
|t| \leq \delta_0.
\]

It then follows that $|t| \leq q/\bar{b}$. Put $\lambda_t := \lambda_X(e^t)$ and $M_{m,t} := \lambda_t(M_m)$. Let $T\PP^*(V_m)$ and $TM_{m,t}$ denote the holomorphic tangent bundles of $\PP^*(V_m)$ and $M_{m,t}$, respectively. From now on, by $C_i$, $i = 0, 1, 2, \ldots$, we mean positive real constants independent of the choice of the triple $(m, t, X)$. Let $m \gg 1$. Then for each integer $k \geq 0$, the argument in [14], Step 1, shows that

\[
\|\omega_0 - (1/m)\Phi^*_m \lambda_t^* \omega_{FS}\|_{C^k(M, \omega_0)} \leq C_0.
\]

Here $C_1$ possibly depends on $k$. From now on, $X$ viewed as a holomorphic vector field on $\PP^*(V_m)$ will be denoted by $\mathcal{X}$. Metrically, we identify the normal bundle of $M_{m,t}$ in $\PP^*(V_m)$ with the subbundle $TM_{m,t}^\perp$ of $T\PP^*(V_m)|_{M_{m,t}}$ obtained as the the orthogonal complement of $TM_{m,t}$ in $T\PP^*(V_m)|_{M_{m,t}}$. Hence $T\PP^*(V_m)|_{M_{m,t}}$ is differentiably written as the direct sum $TM_{m,t} \oplus TM_{m,t}^\perp$. Associated to this, the restriction $\mathcal{X}|_{M_{m,t}}$ of $\mathcal{X}$ to $M_{m,t}$ is written as

\[
\mathcal{X}|_{M_{m,t}} = \mathcal{X}_{TM_{m,t}} \oplus \mathcal{X}_{TM_{m,t}^\perp}.
\]
where $\mathcal{X}_{TM_{m,t}}$ and $\mathcal{X}_{TM_{m,t}^\perp}$ are $C^\infty$ sections of $TM_{m,t}$ and $TM_{m,t}^\perp$, respectively. Then the second derivative $\ddot{f}_{X,m}(t)$ is given by
\[ \ddot{f}_{X,m}(t) = \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|_{\omega_{FS}}^2 \omega_{FS}^n \geq 0. \]

In view of (4.8), it follows from the argument of Phong and Sturm [22] that (cf. [12], p.235; see also [5])
\begin{align}
\begin{cases}
\int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|_{\omega_{FS}}^2 \omega_{FS}^n & \geq C_1 q \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|_{\omega_{FS}}^2 \omega_{FS}^n, \\
\ddot{f}_{X,m}(t) & \geq C_2 q \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|_{\omega_{FS}}^2 \omega_{FS}^n \geq C_2 q \int_{M} \Theta \Phi^*_m \omega_{FS}^n,
\end{cases}
\end{align}
where $\Theta := (\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2)^{-2} \{ (\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2)(\Sigma_{j=1}^{N_m} b_j^2 |\hat{\sigma}_j|^2) - (\Sigma_{j=1}^{N_m} a_j |\hat{\sigma}_j|^2)^2 \} \geq 0$.

Moreover, by (3.4.2) in [31] (see also [12]),
\[ \dot{f}_{X,m}(0) = (n + 1) \int_{M} \Sigma_{j=1}^{N_m} b_j |\hat{\sigma}_j|^2 |h(\ell)| \Phi^*_m \omega_{FS}^n. \]

**Step 2.** For the basis $\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m}\}$ as above satisfying (4.2), we can write $\Psi(m,h(\ell))$ in (2.1) applied to $h = h(\ell)$ in the following form:
\[ \Psi_m(h(\ell)) = n! q^n \sum_{j=1}^{N_m} |\sigma_j|^2 = n! q^n \sum_{\gamma=1}^{m} \sum_{i=1}^{n_\gamma} |\sigma_{\gamma,i}|^2. \]

Then the left-hand side of (1.1) is expressible as
\[ h(\ell)^m \cdot \{(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))\} = n! q^n \sum_{\gamma=1}^{m} \sum_{i=1}^{n_\gamma} \{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))\} |\sigma_{\gamma,i}|^2 |h(\ell)| \]
\[ = n! q^n \sum_{j=1}^{N_m} |\hat{\sigma}_j|^2 |h(\ell)|, \]
and hence (1.1) is written as
\[ n! q^n \sum_{j=1}^{N_m} |\hat{\sigma}_j|^2 |h(\ell)| = 1 + C(\ell) + O(\ell^{q+2}). \]

By taking $(\sqrt{-1}/2\pi) \partial \bar{\partial} \log$ of both sides of (4.11), we obtain
\[ \omega_{FS} - m \omega(\ell) = O(\ell^{q+2}). \]

For each $\gamma$ and $i$, we put $a_{\gamma,i} := \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))$, where $a_{\gamma,i}$ is obviously independent of the choice of $i$. Then by $\hat{\chi}_{m,\gamma} = O(q^{-1})$ and $\mathcal{Y}(\ell) = O(q^2)$, we see that $a_{\gamma,i} = O(q)$, i.e., $|a_{\gamma,i}| \leq \kappa q$ for some positive constant $\kappa$ independent of the choice of $(m, \gamma, i)$. We also obtain
\[ |\hat{\sigma}_{\gamma,i}|^2 |h(\ell)| = \{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))\} |\sigma_{\gamma,i}|^2 |h(\ell)| = (1 - a_{\gamma,i}) |\sigma_{\gamma,i}|^2 |h(\ell)|. \]
In terms of (4.1), we write $a_{\gamma,i}$ as $a_{j(\gamma,i)}$. Note that $X$, as an element of $p^\perp$, belongs to $g_m$, so that $X$ sits in $\mathfrak{s}\mathfrak{l}(V_m)$. This together with (3.1) implies

$$
\sum_{j=1}^{N_m} a_j = \sum_{j=1}^{N_m} b_j a_j = 0.
$$

Hence, by setting

$$
I_m := \sum_{j=1}^{N_m} b_j (1 - a_j)|\sigma_j|^2_{h(0)}
$$

and $\xi_1 := (n + 1)!\{1 + C(\ell)\}^{-1}$, we see from (4.10), (4.11), (4.12) and (4.13) that $\dot{f}_{X,m}(0)$ is expressible as

$$
\begin{align*}
\left\{
(n + 1) & \int_M \{ n!q^n \sum_{j=1}^{N_m} |\tilde{\sigma}_j|^2_{h(0)} \}^{-1} (n!q^n \Phi_{m,\omega}n_F) \\
= & \xi_1 \int_M \{ 1 + O(q^{\ell + 2}) \} \omega(\ell)_{n_F}
\right. \\
& = \left. \xi_1 \{ \sum_{j=1}^{N_m} b_j - \sum_{j=1}^{N_m} b_j a_j \} + \int_M O(q^{\ell + 2}) I_m \omega(\ell)_{n_F} \}
\right].
\end{align*}
$$

Then by (4.12) and (4.15) together with the second line of (4.9), we see that

$$
\begin{align*}
\dot{f}_{X,m}(\delta_0) & \geq \dot{f}_{X,m}(0) + C_2 \delta_0 q \int_M \Theta \Phi_{m,\omega}n_F \\
& \geq \int_M \{ O(q^{\ell + 2}) I_m + C_3 \delta_0 q^{1-n} \Theta \} \omega(\ell)_{n_F}, \\
\dot{f}_{X,m}(-\delta_0) & \leq \dot{f}_{X,m}(0) - C_2 \delta_0 q \int_M \Theta \Phi_{m,\omega}n_F \\
& \leq \int_M \{ O(q^{\ell + 2}) I_m - C_3 \delta_0 q^{1-n} \Theta \} \omega(\ell)_{n_F}.
\end{align*}
$$

Now as in [14], Remark 4.25, the inclusion $t_m X \in U_m$ follows from $|t_m| < \delta_0$, where in the proof, we use the basis $\{ \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{N_m} \}$ in place of $\{ \sigma_1, \ldots, \sigma_{N_m} \}$. Hence, in order to prove (4.7), it suffices to show the following for $m \gg 1$:

$$
\int_M \Theta \omega(\ell)_n > 0 \quad \text{and} \quad \frac{\int_M q^{\ell + 2} |I_m| \omega(\ell)_{n_F}}{\int_M \delta_0 q^{1-n} \Theta \omega(\ell)_{n_F}} \ll 1.
$$

Step 3. Put $B_0 := \sum_{j=1}^{N_m} |\tilde{\sigma}_j|^2_{h(0)}$, $B_1 := \sum_{j=1}^{N_m} b_j^2 |\tilde{\sigma}_j|^2_{h(0)}$, $B_2 := \sum_{j=1}^{N_m} b_j |\tilde{\sigma}_j|^2_{h(0)}$. By setting $\theta_1 := \int_M (B_1/B_0) \omega(\ell)_{n_F}$ and $\theta_2 := \int_M (B_2/B_0^2) \omega(\ell)_{n_F}$, we obtain

$$
\int_M \Theta \omega(\ell)_n = \theta_1 - \theta_2,
$$

where both $\theta_1$ and $\theta_2$ are obviously nonnegative. Moreover, for $m \gg 1$,

$$
\theta_1 = \int_M (n!q^n B_0)^{-1} n!q^n B_1 \omega(\ell)_{n_F}
$$

and

$$
\begin{align*}
& \geq n!q^n (1 - \varepsilon) \sum_{j=1}^{N_m} b_j^2,
\end{align*}
$$

(4.17)
where $\varepsilon > 0$ is a small real constant independent of $m$. On the other hand, we see from (4.14) that

\begin{equation}
\int_M q^{\ell+2} |I_m| \omega(\ell)^n \leq q^{\ell+2} \sum b_j (1 - a_j). \tag{4.18}
\end{equation}

Now the following cases are possible:

Case 1: $\theta_1 > 2\theta_2$,  \hspace{1cm} Case 2: $\theta_1 \leq 2\theta_2$.

Suppose Case 1 occurs. Then $\theta_1 - \theta_2 = (1/2)\theta_1 + (1/2)(\theta_1 - 2\theta_2) > 0$, i.e., the first inequality in (4.16) holds. Let $m \gg 1$. Then by $a_j = O(q)$,

$$0 < 1 - a_j < 1 + \varepsilon,$$

where $\varepsilon$ is as above. Let L.H.S. denotes the left-hand side of the second inequality in (4.16). Then in view of (4.17) and (4.18), we obtain

$$\text{L.H.S.} = \frac{\int_M q^{\ell+2} |I_m| \omega(\ell)^n}{\delta_0 q^{1-n} \Theta \omega(\ell)^n} \leq \frac{q^{\ell+2} \sum b_j (1 - a_j)}{\delta_0 q^{1-n} (\theta_1 - \theta_2)} \leq \frac{q^{\ell+2} (1 + \varepsilon) \sum b_j}{(1/2) \delta_0 q(1 - \varepsilon) \sum b_j} \leq \frac{q^{\ell}(1 + \varepsilon) N_m^{1/2}}{n! (1 - \varepsilon) \sum b_j^{1/2}},$$

where the Schwarz inequality $\sum b_j \leq N_m^{1/2} \sqrt{\sum b_j^{1/2}}$ is used in the last inequality. Hence by $N_m = O(m^n)$, it follows that

$$\text{L.H.S.} \leq O(q^{\ell-\frac{n}{2}}).$$

In view of the definition $\ell := [n/2] + 3$, we have $\ell - \frac{n}{2} > 0$, and therefore L.H.S. $\ll 1$, as required. Thus in this case (4.7) holds.

Next we consider the situation where Case 2 occurs. Note that, for $0 \neq X \in p^\perp$, we can write

\begin{equation}
\Phi_m^* \varphi_X = \frac{\sum b_j |\sigma_j|^2}{m \sum |\sigma_j|^2} = \frac{B_2}{m B_0}, \tag{4.19}
\end{equation}

Let $c_X$ be the real constant such that the function $\phi_X := c_X + \Phi_m^* \varphi_X$ on $M$ satisfies $\int_M \phi_X \tilde{\omega}^n = 0$, where $\tilde{\omega} := \Phi_m^*(q \omega_{FS})$. Then

$$\|\phi_X\|_{L^2(M, \tilde{\omega})} \leq C_4 \|	ilde{\phi}_X\|_{L^2(M, \tilde{\omega})} = C_4 \|\Phi_m^* \chi_{TM_m}\|_{L^2(M, \tilde{\omega})}$$

for some $C_4$, while by the first inequality in (4.9) applied to $t = 0$,

$$\|\Phi_m^* \chi_{TM_m}\|_{L^2(M, \tilde{\omega})} \leq C_1^{-1} q^{-1} \|\Phi_m^* \chi_{TM_m}\|_{L^2(M, \tilde{\omega})}.$$
In view of these inequalities, we obtain

\[ \|\phi_X\|_{L^2(M, \omega)}^2 \leq C_4 C_1^{-1} q^{-1} \|\Phi^*_m X_{TM_m}\|_{L^2(M, \omega)}^2. \]

Let \( m \gg 1 \). By (4.10) and (4.19),

\[ |c_X| = \left( \int_M \omega^n \right)^{-1} \left| \int_M (\Phi^*_m \varphi X) \omega^n \right| = \frac{q^{n+1} |\tilde{f}_{X,m}(0)|}{(n+1) c_1 (L)^n [M]}. \]

In view of (4.12) and (4.15), there exist \( C_5 \) and \( C_6 \) satisfying

\[ |\tilde{f}_{X,m}(0)| \leq C_5 q^{\ell+2} \|I_m\|_{L^2(M, \omega(\ell))} \leq C_6 q^{\ell+2} \|I_m\|_{L^2(M, \omega)}, \]

while by (4.11), (4.14) and (4.19), we obtain \( C_7 \) such that

\[ \|I_m\|_{L^2(M, \omega)} \leq C_7 m^{n+1} \|\Phi^*_m \varphi X\|_{L^2(M, \omega)}. \]

Hence for some \( C_8 \), it follows that

\[ |\tilde{f}_{X,m}(0)| \leq C_8 q^{\ell-n+1} \|\Phi^*_m \varphi X\|_{L^2(M, \omega)}. \]

In view of (4.21) and (4.22), \( |c_X| \leq C_9 q^{\ell+2} \|\Phi^*_m \varphi X\|_{L^2(M, \omega)} \) for some \( C_9 \). Then by the definition of \( \phi_X \) together with (4.12), we obtain

\[ \left\{ \begin{array}{l}
\|\phi_X\|_{L^2(M, \omega)} \geq \|\Phi^*_m \varphi X\|_{L^2(M, \omega)} - |c_X| \|L^2(M, \omega) \\
\geq (1 - C_9 q^{\ell+2} (c_1 (L)^n [M])^{1/2}) \|\Phi^*_m \varphi X\|_{L^2(M, \omega)} \\
\geq C_{10} \|\Phi^*_m \varphi X\|_{L^2(M, \omega)} \geq C_{11} \|\Phi^*_m \varphi X\|_{L^2(M, \omega(\ell))}
\end{array} \right. \]

for some \( C_{10} \) and \( C_{11} \). Since \( \theta_2 = \int_M (B_2 / B_0)^2 \omega(\ell) = m^2 \|\Phi^*_m \varphi X\|_{L^2(M, \omega(\ell))}^2 \), we see from (4.20) and (4.23) that

\[ \|\Phi^*_m X_{TM_m}\|_{L^2(M, \omega)}^2 \geq C_{12} q^3 \theta_2 \]

for some \( C_{12} \). Note that \( q^{1/2} |X_m|_{\omega_{FS}} = |X_m|_{\omega} \geq |X_{TM_m}|_{\omega}. \) In view of the second line in (4.9), it follows from (4.24) that, for \( |t| \leq \delta_0 \),

\[ \left\{ \begin{array}{l}
\tilde{f}_{X,m}(t) \geq C_2 \int_{M_m} (q^{1/2} |X_m|_{\omega_{FS}}) \omega^n_{FS} \\
\geq C_2 q^{-n} \|\Phi^*_m X_{TM_m}\|_{L^2(M, \omega)} \geq C_{13} q^{3-n} \theta_2
\end{array} \right. \]

for some \( C_{13} \). Therefore, by (4.22) and (4.25), we obtain

\[ \tilde{f}_{X,m}(\delta_0) \geq R \quad \text{and} \quad \tilde{f}_{X,m}(-\delta_0) \geq -R, \]

where \( R := C_{13} \delta_0 q^{3-n} \theta_2 - C_8 q^{\ell-n+1} \|\Phi^*_m \varphi X\|_{L^2(M, \omega(\ell))}. \) Moreover, if \( R > 0 \), then \( \theta_2 > 0 \), and hence by (4.25), \( \tilde{f}_{X,m}(t) > 0 \) for \( |t| \leq \delta_0 \). It now suffices to show \( R > 0 \) for \( m \gg 1 \). By \( \theta_2 = m^2 \|\Phi^*_m \varphi X\|_{L^2(M, \omega(\ell))}^2 \), we can write \( R \) as

\[ R = q^{3-n} \sqrt{\theta_2} \{- C_8 q^{\ell-1} + C_{13} \delta_0 \sqrt{\theta_2} \}. \]

Recall that, by our assumption of Case 2, we have \( \theta_1 \leq 2 \theta_2 \). Hence by (4.17) together with the definition of \( \delta_0 \), we obtain

\[ \delta_0 \sqrt{\theta_2} \geq \delta_0 \sqrt{\theta_1} / \sqrt{2} \geq \sqrt{n!(1 - \varepsilon) q^{n/2} + 1} = C_{14} q^{(n/2) + 1}, \]
where \( C_{14} := \sqrt{n!(1 - \varepsilon)} \). In view of the definition \( \ell := \lfloor n/2 \rfloor + 3 \),
\[
(n/2) + 1 < \lfloor n/2 \rfloor + 2 = \ell - 1.
\]
Therefore \( R > 0 \) for \( m \gg 1 \), as required. \( \square \)

Remark 4.26: Assume that \((M, L^\otimes m)\) is Chow-polystable relative to a maximal algebraic torus \( T_{\text{max}} \) in \( \text{Aut}^0(M) \). Then the arguments in [19], Step 2, which uses [20] allow us to obtain finiteness of the isotropy subgroup of \( G_m \) at \([\tilde{M}_m] \). Hence in this case, \((M, L^\otimes m)\) is Chow-stable relative to \( T_{\text{max}} \).

5. Polybalanced metrics

As in the introduction, we consider a \( K \)-invariant Hermitian \( h \) metric for \( L \) such that \( \omega = c_1(L; h) \) is Kähler. Let \( m \) be a positive integer. In (3.2), we choose an orthonormal basis \( \{\sigma_{\gamma,i} ; i = 1, 2, \ldots, n_\gamma\} \) of the space \((V_{m,\gamma}, \rho_{m}(h))\) for each \( \gamma \). In this section, we discuss the result in [16] from a slightly different point of view. For recent related works, see [23] and [9].

Definition 5.1. For a polarized algebraic manifold \((M, L)\), \( \omega \) is called an \( m \)-th polybalanced metric relative to \( T \), if for some \( Y \in \sqrt{-1} t \) satisfying \( 1 - \hat{\chi}_{m,\gamma}(Y) > 0 \) for all \( \gamma \), there exists a positive real constant \( C \) such that

\[
\sum_{\gamma=1}^{\eta_m} \{1 - \hat{\chi}_{m,\gamma}(Y)\} |\sigma_{\gamma,i}|_h^2 = C.
\]

This concept is closely related to relative Chow-polystability. For brevity, we use the notation in Section 4 freely until the end of this section.

Theorem 5.3: \((M, L^\otimes m)\) is Chow-polystable relative to \( T \) if and only if \((M, L)\) admits an \( m \)-th polybalanced metric relative to \( T \).

Proof: The proof of “only if” part follows from Theorem C and (3.7) in [16]. For “if” part, we give a proof as follows: Let

\[
\{\hat{\sigma}_{\gamma,i} ; i = 1, 2, \ldots, n_\gamma, \gamma = 1, 2, \ldots, \eta_m\}
\]

be the basis for \( V_m \) obtained from \( \{\sigma_{\gamma,i} ; i = 1, 2, \ldots, n_\gamma, \gamma = 1, 2, \ldots, \eta_m\} \) by replacing \( Y(\ell) \) by \( Y \) in (4.2). Then by (4.4), we have the Kodaira embedding \( \Phi_m : M \hookrightarrow \mathbb{P}^{N_m-1}(\mathbb{C}) \). Also by (4.6), we have the function \( f_{X,m}(t) \) for the orbit through \( \tilde{M}_m \) of the one-parameter group \( \exp(tX), t \in \mathbb{R} \), generated by \( 0 \neq X \in p^\perp \). Since (5.2) is written as \( \sum_{j=1}^{\eta_m} |\hat{\sigma}_{j}|_h^2 = C \) in terms of the notation (4.1), by taking \( (\sqrt{-1}/2\pi) \partial \bar{\partial} \log \) of both sides of (5.2), we obtain

\[
\Phi_m^* \omega_{FS} = m \omega.
\]
Put $a_{\gamma,i} := \hat{\chi}_{m,\gamma}(Y)$. Then by $Y \in \sqrt{-1}t_e \subset \mathfrak{t}$ and $X \in \mathfrak{p}^\perp \subset \mathfrak{g}_m \subset \mathfrak{sl}(V_m)$, in view of (3.1), it follows that $\sum_{j=1}^{N_m} b_j = \sum_{j=1}^{N_m} a_j = 0$, where $b_j$ is as in (4.5). Now by replacing $h(\ell)$ by $h$ in (4.10), we obtain

$$\dot{f}_{X,m}(0) = (n+1) \int_M \frac{\sum_{j=1}^{N_m} b_j |\sigma_j|_h^2}{\sum_{j=1}^{N_m} |\sigma_j|_h^2} (m\omega)^n$$

$$= \frac{m^n(n+1)}{C} \int_M \sum_{j=1}^{N_m} b_j (1 - a_j |\sigma_j|_h^2 \omega^n$$

$$= \frac{m^n(n+1)}{C} \sum_{j=1}^{N_m} b_j (1 - a_j) = 0.$$

By this together with the convexity $\dot{f}_{X,m}(t) \geq 0$, the function $f_{X,m}$ attains a minimum at the origin. Therefore every special one-parameter subgroup of $G_m$ has a closed orbit through $\tilde{M}_m$ in $W^*_m$, and hence the orbit $G_m \cdot \tilde{M}_m$ is closed in $W^*_m$ (cf. [12], p.568), as required.

□

**Remark 5.4:** If $(M, L)$ is asymptotically Chow-polystable relative to $T$, then for $m \gg 1$, there exists an $m$-th polybalanced metric $\omega$ such that

$$|\hat{\chi}_{m,\gamma}(Y)| = O(q),$$

i.e., the inequality $|\hat{\chi}_{m,\gamma}(Y)| \leq C'q$ holds for some positive constant $C'$ independent of $m$, $\gamma$ and $\gamma'$ (see [16], Theorem A).

### 6. Strong Relative K-stability

In this section, for a polarized algebraic manifold $(M, L)$, we consider an algebraic torus $T$ in Aut$^0(M)$. Let the group $\mathbb{C}^*$ act on the affine line $\mathbb{A}^1 := \{z \in \mathbb{C}\}$ by multiplication of complex numbers,

$$\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1, \quad (t, z) \mapsto tz.$$

By fixing a Hermitian metric $h$ for $L$ such that $\omega := c_1(L; h)$ is Kähler, we endow $V_m := H^0(X, L^\otimes m)$ with the Hermitian metric $\rho_m(h)$ as defined in the introduction. We then consider the Kodaira embedding

$$\Phi_m : X \hookrightarrow \mathbb{P}^*(V_m), \quad x \mapsto (\tau_1(x) : \tau_2(x) : \cdots : \tau_{N_m}(x)),$$

where $(\tau_1, \tau_2, \ldots, \tau_{N_m})$ is an orthonormal basis for $(V_m, \rho_m(h))$. For $G_m$ as in Section 3, we consider an algebraic group homomorphism

$$\psi : \mathbb{C}^* \to G_m$$

such that the maximal compact subgroup $S^1 \subset \mathbb{C}^*$ acts isometrically on the space $(V_m, \rho_m(h))$. Put $M_m := \Phi_m(M)$. Then by setting

$$M^\psi_m := \{z\} \times \psi(z)M_m, \quad z \in \mathbb{C}^*,$$
we consider the irreducible algebraic subvariety $\mathcal{M}^\psi$ of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ obtained as the closure of the subset
\[ \bigcup_{z \in C^*} \mathcal{M}^\psi_z \]
in $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$, where $\psi(z)$ in $G_m$ acts naturally on the space $\mathbb{P}^*(V_m)$ of all hyperplanes in $V_m$ passing through the origin. Let
\[ \pi : \mathcal{M}^\psi \to \mathbb{A}^1 \]
be the map induced by the projection of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ to the first factor $\mathbb{A}^1$. For the hyperplane bundle $\mathcal{O}_{\mathbb{P}^*(V_m)}(1)$ on $\mathbb{P}^*(V_m)$, we consider the pullback $\mathcal{L}^\psi := \text{pr}_2^* \mathcal{O}_{\mathbb{P}^*(V_m)}(1)|_{\mathcal{M}^\psi}$, where $\text{pr}_2 : \mathbb{A}^1 \times \mathbb{P}^*(V_m) \to \mathbb{P}^*(V_m)$ denotes the projection to the second factor. For the dual space $V_m^*$ of $V_m$, the $C^*$-action on $\mathbb{A}^1 \times V_m^*$ defined by
\[ C^* \times (\mathbb{A}^1 \times V_m^*) \to \mathbb{A}^1 \times V_m^*, \quad (t, (z, p)) \mapsto (tz, \psi(t)p), \]
induces $C^*$-actions on $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ and $\mathcal{O}_{\mathbb{P}^*(V_m)}(-1)$, where $G_m$ acts on $V_m^*$ by the contragradient representation. This then induces $C^*$-actions on $\mathcal{M}^\psi$ and $\mathcal{L}^\psi$, and hence $\pi : \mathcal{M}^\psi \to \mathbb{A}^1$ is a $C^*$-equivariant projective morphism with a relatively very ample line bundle $\mathcal{L}^\psi$ satisfying
\[ (\mathcal{M}^\psi_z, \mathcal{L}^\psi_z) \cong (M, L^{\otimes m}), \quad z \neq 0, \]
where $\mathcal{L}^\psi_z$ is the restriction of $\mathcal{L}^\psi$ to $\mathcal{M}^\psi_z := \pi^{-1}(z)$. Then a triple $(\mathcal{M}, \mathcal{L}, \psi)$ is called a test configuration for $(M, L)$, if we have both
\[ \mathcal{M} = \mathcal{M}^\psi \quad \text{and} \quad \mathcal{L} = \mathcal{L}^\psi. \]
Here $m$ is called the exponent of $(\mathcal{M}, \mathcal{L}, \psi)$. A test configuration $(\mathcal{M}, \mathcal{L}, \psi)$ is called trivial, if $\psi$ is a trivial homomorphism. Let $\mathbf{M}$ be the set of all sequences $\{\mu_j\}$ of test configurations
\[ \mu_j = (\mathcal{M}_j, \mathcal{L}_j, \psi_j), \quad j = 1, 2, \ldots, \]
for $(M, L)$ such that the exponent $m_j$ of the test configuration $\mu_j$ satisfies the following growth condition:
\[ m_j \to +\infty, \quad \text{as} \ j \to \infty. \]
In [17], to each $\{\mu_j\} \in \mathbf{M}$, we associated the Donaldson-Futaki invariant
\[ F_1(\{\mu_j\}) \in \mathbb{R} \cup \{-\infty\}, \]
which is viewed as a generalization of the Donaldson-Futaki invariant $DF(\mu)$ of a test configuration $\mu$. We can also define the following strong version of $K$-stability and $K$-semistability:

**Definition 5.1.** (1) A polarized algebraic manifold $(M, L)$ is called **strongly $K$-semistable relative to $T$**, if $F_1(\{\mu_j\}) \leq 0$ for all $\{\mu_j\} \in \mathbf{M}$.
A strongly K-semistable polarized algebraic manifold \((M, L)\) is called strongly K-stable relative to \(T\), if for every \(\{ \mu_j \} \in M\) with \(F_1(\{ \mu_j \}) = 0\), there exists \(j_0\) such that \(\mu_j\) are trivial for all \(j\) with \(j \geq j_0\).

7. Concluding Remarks

The Yau-Tian-Donaldson Conjecture for Kähler-Einstein cases was solved affirmatively by Chen-Donaldson-Sun [4] and Tian [28]. However, for general polarization cases or extremal Kähler cases, the conjecture is still open. In this paper, we discuss extremal Kähler versions of this conjecture by focussing on the difference between strong K-stability and K-stability. For an arbitrary polarized algebraic manifold \((M, L)\) as in the introduction, recall the following definition of K-stability [6] (cf. [27]):

**Definition 6.1.**
1. \((M, L)\) is called K-semistable, if \(DF(\mu) \leq 0\) for all test configurations \(\mu\) for \((M, L)\).
2. A K-semistable \((M, L)\) is called K-stable, if every test configuration \(\mu\) for \((M, L)\) with \(DF(\mu) = 0\) is trivial.

We now consider a maximal algebraic torus \(T_{\text{max}}\) in \(\text{Aut}^0(M)\). As to the existence of extremal Kähler metrics, it is natural to ask the following:

**Conjecture I:** A polarized algebraic manifold \((M, L)\) is K-stable relative to \(T_{\text{max}}\) if and only if \(c_1(L)\) admits an extremal Kähler metric.

**Conjecture II:** A polarized algebraic manifold \((M, L)\) is strongly K-stable relative to \(T_{\text{max}}\) if and only if \(c_1(L)\) admits an extremal Kähler metric.

By a result of Donaldson [5], every constant scalar curvature Kähler metric is approximated by a sequence of balanced metrics. In other words, a balanced metric can be viewed as a quantized version of a constant scalar curvature Kähler metric. Similarly, a polybalanced metric can be viewed as a quantized version of an extremal Kähler metric. Since existence of poly-balanced metrics corresponds to relative Chow-polystability, the following fact is viewed as a quantized version of the existence part of Conjecture II:

**Fact:** (cf. [19]) If a polarized algebraic manifold \((M, L)\) is strongly K-stable relative to an algebraic torus \(T\) in \(\text{Aut}^0(M)\), then \((M, L^\otimes m), m \gg 1\), are Chow-stable relative to \(T\).

Moreover, we expect that “if” part of Conjecture II is true. This will be discussed in a forthcoming paper [18]. In view of the above Fact, by assuming that “if” part of Conjecture II is true, we immediately obtain the case \(T = T_{\text{max}}\) of Corollary in the introduction.
For a polarized algebraic manifold \((M, L)\), let \(T_{\text{ex}}\) be the algebraic torus in \(\text{Aut}^0(M)\) generated by the extremal Kähler vector field. By a recent result of Yotsutani-Zhou [29], a smooth polarized toric Fano threefold
\[
\Pi := (E_4, K_{E_4}^{-1})
\]
is K-stable relative to \(T_{\text{max}}\), and is not asymptotically Chow-stable relative to \(T_{\text{ex}}\). Let \(Z_C\) be as in the introduction. We finally pose the following:

**Problem:** Check whether or not \(\Pi\) is asymptotically Chow stable relative to \(Z_C\). Or more generally, clarify whether there is an example of a polarized algebraic manifold \((M, L)\) which is K-stable relative to \(T_{\text{max}}\) and is not asymptotically Chow-stable relative to \(Z_C\).

If there is such an example of a polarized algebraic manifold \((M, L)\), then by Corollary in the introduction, \(c_1(L)\) admits no extremal Kähler metrics. In other words, this gives a counter-example to Conjecture I above.

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