ON THE ZERO FORCING NUMBER OF CORONA AND LEXICOGRAPHIC PRODUCT OF GRAPHS

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Abstract. The zero forcing number of a graph $G$, denoted by $Z(G)$, is the minimum cardinality of a set $S$ of black vertices (where vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of “the color change rule”: a white vertex is turned black if it is the only white neighbor of a black vertex. In this paper, we study the zero forcing number of corona product, $G \odot H$ and lexicographic product, $G \circ H$ of two graphs $G$ and $H$. It is shown that if $G$ and $H$ are connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$ respectively, then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2+1)^{k-1}Z(H)$, where $G \odot^k H = (G \odot^{k-1} H) \odot H$. Also, it is shown that for a connected graph $G$ of order $n \geq 2$ and an arbitrary graph $H$ containing $l \geq 1$ components $H_1, H_2, \ldots, H_l$ with $|V(H_i)| = m_i \geq 2$, $1 \leq i \leq l$, $(n-1)l + \sum_{i=1}^{l} m_i \leq Z(G \circ H) \leq n(\sum_{i=1}^{l} m_i) - l$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple, undirected, connected graph with $|V(G)| \geq 2$. The number of vertices and edges of $G$ are called the order and the size of $G$ respectively. The degree of a vertex $v \in V$, denoted by $deg_G(v)$, is the number of edges incident to the vertex $v$ in $G$. If there is no ambiguity, we will use the notation $deg(v)$ instead of $deg_G(v)$. An end vertex is a vertex of degree one. Given $u, v \in V$, $u \sim v$ means that $u$ and $v$ are adjacent vertices and $u \sim v$ means that $u$ and $v$ are not adjacent. We define the open neighborhood of a vertex $v$ in $G$, $N_G(v) = \{u \in V(G) : u \sim v\}$ and the closed neighborhood of $v$, $N_G[v] = N_G(v) \cup \{v\}$. If there is no ambiguity, we will simply write $N(v)$ or $N[v]$. If $u \in N_G(v)$ then $u$ is said to be a neighbor of $v$. We denote a path, cycle, complete graph and empty graph on $n$ vertices by $P_n$, $C_n$, $K_n$ and $\overline{K}_n$ respectively. All graphs considered in this paper are non

Key words and phrases. Zero forcing number, zero forcing sets, corona product of graphs, lexicographic product of graphs.

2010 Mathematics Subject Classification. 05C50

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trivial unless otherwise stated.

The notion of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced in [1] to bound the minimum rank of associated matrices for numerous families of graphs. Let each vertex of a graph $G$ be given one of two colors, “black” and “white” by convention. Let $S$ denote the initial set of black vertices of $G$. The color-change rule converts the color of a vertex $u_2$ from white to black if the white vertex $u_2$ is the only white neighbor of a black vertex $u_1$; we say that $u_1$ forces $u_2$, which we denote by $u_1 \to u_2$. And a sequence, $u_1 \to u_2 \to \cdots \to u_i \to u_{i+1} \to \cdots \to u_t$, obtained through iterative applications of the color-change rule is called a forcing chain. The set $S$ is said to be a zero forcing set of $G$ if all the vertices of $G$ will be turned black after finitely many applications of the color-change rule. The zero forcing number of $G$, denoted by $Z(G)$, is the minimum of $|S|$ over all zero forcing sets $S \subseteq V(G)$. A zero forcing set of cardinality $Z(G)$ is called a forcing basis for $G$. For surveys on the zero forcing parameter, see [9, 10]. For more on the zero forcing parameter in graphs, see [2, 3, 5, 6, 7, 12].

If $F$ is a field, $M_n(F)$ denotes the set of all $n \times n$ matrices over $F$. An $n$-square matrix $A$ is said to be a symmetric matrix if $A^T = A$. The set of all real symmetric $n$-square matrices is denoted by $S_n$. To a given graph $G$ with vertex set $\{1, 2, \cdots, n\}$, we associate a class of real, symmetric matrices as follows:

$$S(G) = \{A = [a_{ij}] | A \in S_n, \text{ for } i \neq j, a_{ij} \neq 0 \iff ij \in E(G)\}.$$  

Note that there is no restriction on the value of $a_{ii}$ with $i = 1, 2, \cdots, n$ and the adjacency matrix $A(G)$ belongs to $S(G)$, where the adjacency matrix of a graph $G$ is a square $(0, 1)$-matrix of size $n$, whose $(i, j)$-th entry is 1 if and only if $v_i$ is adjacent to $v_j$, since there are no loops in the graph, the diagonal entries of the adjacency matrix are zero. On the other hand, the graph of an $n$-square symmetric matrix $A$, denoted by $\tilde{G}(A)$, is the graph with vertices $\{1, 2, \cdots, n\}$ and the edge set

$$\{ij|a_{ij} \neq 0, 1 \leq i \neq j \leq n\}.$$  

The minimum rank of $G$ is defined to be

$$mr(G) = \min \{\text{rank}(A) | A \in S(G)\},$$

while the maximum nullity of $G$ is defined as

$$M(G) = \max \{\text{null}(A) | A \in S(G)\}.$$  

We have

$$mr(G) + M(G) = |V(G)|.$$
The underlying idea for the zero forcing set of a graph is that a black vertex is associated with a coordinate in a vector that is required to be zero, while a white vertex indicates a coordinate that can be either zero or nonzero. Changing a vertex from white to black is essentially noting that the corresponding coordinate is forced to be zero if the vector is in the kernel of a matrix in $S(G)$ and all black vertices indicate coordinates assumed to be or previously forced to be 0. Hence the use of the term “zero forcing set”, see [1].

The support of a vector $x = (x_i)$, denoted by $Supp(x)$, is the set of indices $i$ such that $x_i \neq 0$. Let $Z$ be a zero forcing set of $G$ and $A \in S(G)$. If $x \in null(A)$ and $Supp(x) \cap Z = \phi$, then $x = 0$, stated in [1, 14]. Also from [1, 14], we have $M(G) \leq Z(G)$ for a graph $G$.

In this paper, we consider corona product and lexicographic product of graphs in the context of zero forcing number. This paper consists of three sections. Section 1 includes introduction. Sections 2 and 3 include several results related to the zero forcing number of corona and lexicographic product of graphs, respectively.

2. CORONA PRODUCT OF GRAPHS

Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$ respectively. The corona product of $G$ and $H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. We will denote by $V = \{v_1, v_2, \ldots, v_{n_1}\}$, the set of vertices of $G$ and by $H_i = (V_i, E_i)$, the $i$-th copy of $H$, where $V_i = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$, such that $v_i \sim u_k^i$ for every $u_k^i \in V_i$. Note that the subgraph of $G \circ H$ induced by $V_i$ is $H_i$ and the corona graph $K_1 \circ H$ is isomorphic to the join graph $K_1 + H$. For any integer $k \geq 2$, we define the graph $G \circ^k H$ recursively from $G \circ H$ as $G \circ^k H = (G \circ^{k-1} H) \circ H$. It is also noted that $|G \circ^{k-1} H| = n_1(n_2 + 1)^{k-1}$ and $|G \circ^k H| = |G \circ^{k-1} H| + n_1n_2(n_2 + 1)^{k-1}$.

We call the copies of $H$ in $G \circ H$ as the copies of $H$ in 1st-corona, the newly added copies of $H$ in $G \circ H$ to obtain $G \circ^2 H$ as the copies of $H$ in 2nd-corona and generally the newly added copies of $H$ in $G \circ^{k-1} H$ to obtain $G \circ^k H$ as the copies of $H$ in $k^{th}$-corona.

In $G \circ^k H$ for any positive integer $k$, we name the vertices in $G \circ^{k-1} H$ as the root vertices of the copies of $H$ in $k^{th}$-corona, that are joined to these vertices in $G \circ^k H$.

As one can color the vertices of $G \circ^k H$ in more than one ways, but in this paper for a disconnected graph $H$ (containing isolated vertices) of order at least two, we will consider the zero forcing set of $G \circ^k H$, that contains only the vertices of $H$ but not the vertices of $G$. 

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In Figure 1, the graph with grey vertices is $G = P_2$, the copies of $H$ with black vertices are the copies of $H$ in first corona and with white vertices are the copies of $H$ in 2nd corona. The black and grey vertices are the root vertices of the corresponding copies of $H$ with white vertices.

![Figure 1. $P_2 \odot^2 P_2$](image)

We first recall the useful result obtained in [8].

**Proposition 2.1.** [8] Let $G$ be a connected graph of order $n \geq 2$. Then

(a) $Z(G) = 1$ if and only if $G = P_n$,

(b) $Z(G) = n - 1$ if and only if $G = K_n$.

Note that for a connected graph $G$ of order $n$, we have

$$1 \leq Z(G) \leq n - 1. \quad (1)$$

**Lemma 2.2.** Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a graph of order at least two. Let $H_i$ be the subgraph of $G \odot H$ corresponding to the $i^{th}$-copy of $H$.

(i) If $S$ is a zero forcing set of $G \odot H$, then $V_i \cap S \neq \phi$ for every $i \in \{1, ..., n\}$.

(ii) If $H$ is a connected graph and $S$ is a zero forcing set of $G \odot H$, then for every $i \in \{1, ..., n\}$, $S \cap V_i$ is a zero forcing set of $H_i$.

**Proof.** (i) Suppose $V_i \cap S = \phi$, for some $i$. Then the vertex $v_i \in V_i$, initially black or forced to black in the zero forcing process, will not force any vertex in $V_i$ to turn black because it has more than one white neighbors, a contradiction.

(ii) Suppose contrary that $S \cap V_i$ is not a zero forcing set for $H_i$. Then there exists a black vertex say $u_i^j \in V_i$, that has more than one white neighbors in graph $H_i$, so no forcing situation can occur. Note that the vertex $u_i^j \in V_i$ also has more than one white neighbors in $G \odot H$, a contradiction. \qed
Here we introduce some terminology related to the following theorem. We assume that $B' = \{v_1, v_2, \ldots, v_t\}$ is a forcing basis for $G$ and by using $B'$ one can color all the other vertices of $G$ by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, \ldots, v_{n_1}$, with appropriate indexing of vertices. We denote the vertices of $i$-th copies of $H$ in $l$-th corona $(1 \leq l \leq k)$ by $u_{i,j_1,j_2,\ldots,j_l}$, where $1 \leq j_1 \leq n_2$ and $1 \leq j_p \leq n_2 + 1$ for each $2 \leq p \leq l$. Assume that $Z(H) = m$ and a forcing basis for $i^{th}$-copy $H_i$ of $H$ in 1st-corona is denoted by $B_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,m}\}$ and forcing basis for $i$-th copies of $H$ in $l^{th}$-corona $(2 \leq l \leq k)$ by $B_{i,j_1,j_2,\ldots,j_l} = \{u_{i,j_1,j_2,\ldots,j_l}, u_{i,j_2,j_3,\ldots,j_l}, \ldots, u_{i,m,j_2,\ldots,j_l}\}$, where $1 \leq j_2, \ldots, j_l \leq n_2 + 1$. We denote the collection of forcing basis of all copies of $H$ in first corona by $B_1$, i.e. $\bigcup_{i=1}^{n_1} B_i = B_1$, similarly the collection of forcing basis of all copies of $H$ in $l$-th corona by $B_l$ i.e $\bigcup_{i=1}^{n_1} \left( \bigcup_{j_1=1}^{n_2+1} \cdots \left( \bigcup_{j_{l-1}=1}^{n_2+1} B_{i,j_1,\ldots,j_l} \right) \right) = B_l$ and $|B_l| = n_1(n_2 + 1)^{l-1}Z(H)$. FIGURE 2 helps in understanding the indices as mentioned.

**Theorem 2.3.** Let $G$ and $H$ be connected graphs of order $n_1$ and $n_2$ respectively, then

$$Z(G \circ^k H) = Z(G \circ^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H).$$
Proof. We prove the result by mathematical induction. For \( k = 1 \), we have to show that
\[
Z(G \odot H) = Z(G) + n_1Z(H). \tag{2}
\]
First, we show that
\[
Z(G \odot H) \leq Z(G) + n_1Z(H). \tag{3}
\]
We define \( B = B' \cup B_1 \) and \( |B| = Z(G) + n_1Z(H) \). We claim that \( B \) is a zero forcing set of \( G \odot H \). To prove the claim, first assume that \( B \) is initially colored black and we color all the vertices of \( H_i \) with \( 1 \leq i \leq t \) which are associated with the vertices of \( B' \) using the corresponding sets \( B^i \). Now all the vertices in \( H_i \) associated with \( v_i \), \( 1 \leq i \leq t \) are colored black. Note that there is a vertex \( v_i \) belonging to \( B' \) that has only one white neighbor \( v_{t+1} \). Thus \( v_i \to v_{t+1} \). Then we color all the vertices in \( H_{t+1} \) by using the black vertices in \( B^{t+1} \). Continuing this process, we can color all the vertices of \( G \odot H \).

Note that the degree of each vertex of \( G \) is increased by \( n_2 \) and the degree of each vertex of \( H \) is increased by 1 in \( G \odot H \). Let \( v_i \in B' \) and consider the corresponding copy \( H_i \) of \( H \). Note that at least \( Z(H) + 1 \) vertices are required as initially colored black to start the zero forcing process in each of these \( H_i \)'s, \( 1 \leq i \leq t \). Then \( v_i \to v_{t+1} \) and to continue the process at least \( Z(H) \) more vertices are required in \( H_{t+1} \) as initially colored black. Continue the process until all the vertices are turned black. Hence,
\[
Z(G \odot H) \geq Z(G) + n_1Z(H). \tag{4}
\]
By (3) and (4), (2) holds.

Suppose that the result is true for \( k - 1 \), i.e
\[
Z(G \odot^{k-1} H) = Z(G \odot^{k-2} H) + n_1(n_2 + 1)^{k-2}Z(H). \tag{5}
\]
Now we have to show that the result is true for \( k \), i.e
\[
Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H). \tag{6}
\]
We define \( B^o = B' \cup B_1 \cup B_2 \cup \cdots \cup B_{(k-1)} \) and \( |B^o| = Z(G) + n_1Z(H) + n_1(n_2 + 1)Z(H) + \cdots + n_1(n_2 + 1)^{k-2}Z(H) = \alpha \). Then \( B^o \) is a zero forcing set of \( G \odot^{k-1} H \) by (3). Therefore, we color all the vertices of \( G \odot^{k-1} H \) using \( B^o \). Now all the vertices from 1st-corona to \((k-1)^{th}\)-corona in \( G \odot^k H \) are colored black. It suffices to show that \( n_1n_2(n_2 + 1)^{k-1} \) vertices in the copies of \( H \) in \( k^{th}\)-corona in \( G \odot^k H \) will be colored black by taking \( n_1(n_2 + 1)^{k-1}Z(H) \) more vertices as initially colored black.
First, we show that
\[
Z(G \odot^k H) \leq Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H). \tag{7}
\]
We define \( B = B^\circ \cup B_k \) where \(|B| = \alpha + n_1(n_2 + 1)^{k-1}Z(H)\). We claim that \( B \) is a zero forcing set of \( G \odot^k H \). Let \( B \) is initially colored black. Note that the degree of each vertex in \( H_i \) in \( k^{th} \)-corona in \( G \odot^k H \) is increased by one. We color all the vertices of copies of \( H \) in \( k^{th} \)-corona by using \( B_{j_2 \cdots j_k} \) and the corresponding root vertex in \((k - 1)^{th}\)-corona. We obtain the derived set of all black vertices in \( G \odot^k H \) resulting from repeatedly applying the color-change rule. Hence, (7) holds.

Note that in each copy of \( H \) in \( k^{th} \)-corona at least \( Z(H) \) vertices are required as initially colored black to continue the zero forcing process. Hence,
\[
Z(G \odot^k H) \geq Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H). \tag{8}
\]
By (7) and (8), (6) holds. Hence, the result is true for any positive integer \( k \). \( \square \)

By using Theorem 2.3 and Proposition 2.1, we have the following immediate corollaries:

**Corollary 2.4.** Let \( G \) and \( H \) be connected graphs of order \( n_1, n_2 \geq 2 \) respectively. Then \( Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} \) if and only if \( H \cong P_{n_2} \).

**Corollary 2.5.** Let \( G \) and \( H \) be connected graphs of order \( n_1, n_2 \geq 2 \) respectively. Then \( Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}(n_2 - 1) \) if and only if \( H \cong K_{n_2} \).

The wheel graph of order \( n + 1 \) is defined as \( W_{1,n} = K_1 \odot C_n \), where \( K_1 \) is the singleton graph. Any three pairwise adjacent vertices of a wheel form a zero forcing set of \( W_{1,n} \).

**Remark 2.6.** Let \( W_{1,n}, n \geq 3 \), be a wheel graph. Then \( Z(W_{1,n}) = 3 \).

The fan graph \( F_{n_1,n_2} \) is defined as the join graph \( K_{n_1} + P_{n_2} \). The case \( n_1 = 1 \) corresponds to the usual fan graph \( F_{1,n_2} \). Note that \( F_{1,n_2} = K_1 \odot P_{n_2} \), where \( K_1 \) is the singleton graph. Two adjacent vertices of \( P_{n_2} \) where one must be an end vertex form a zero forcing set of \( F_{1,n_2} \).

**Remark 2.7.** Let \( F_{1,n}, n \geq 2 \), be a fan graph. Then \( Z(F_{1,n}) = 2 \).

**Theorem 2.8.** Let \( G \) be a connected graph of order \( n_1 \geq 2 \) and let \( H \) be a graph of order \( n_2 \geq 2 \). Then
\[
Z(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}Z(K_1 \odot H). \tag{7}
\]
Proof. We denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all the vertices of $H_i$. Let $A_i$ be a forcing basis for $K_1 \odot H_i$ and $A = \bigcup_{i=1}^{n_1} A_i$ with $|A| = n_1 Z(K_1 \odot H)$. We show that $A$ is a zero forcing set of $G \odot H$. Now there are two cases:

Case 1: $H$ contains no isolated vertex. We have that $v_i \in A_i$. So $A$ contains all the vertices of $G$. Note that any black vertex in $H_i$ has only one white neighbor, so after finite many applications of the color-change rule all the vertices in $H_i$ for each $i = 1, 2, \ldots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Case 2: $H$ contains isolated vertices, so there exists at least one vertex $x \in H$ such that $x \sim u$, for all $u \in H$. We have that $v_i$ does not belong to the zero forcing set of minimum cardinality of $K_1 \odot H_i$. So $A$ does not contain any vertex from $G$.

Subcase 2.1: $H$ has only one isolated vertex. Let $x_i$ is the isolated vertex of $H_i$ also $x_i \in A_i$ and $x_i \sim u'_i$ for any $u'_i \in V_i$. Thus $x_i \rightarrow v_i$, $1 \leq i \leq n_1$. Now all the vertices of $G$ are forced to black. Note that any black vertex in $H_i$ has only one white neighbor, so after finite many applications of the color-change rule all the vertices in $H_i$ for each $i = 1, 2, \ldots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Subcase 2.2: $H$ has more than one isolated vertices, then all isolated vertices in $H_i$ belong to $A_i$ except one, say $y_i$ does not belong to $A_i$, then $v_i$ will be forced by any isolated black vertex in $H_i$, $1 \leq i \leq n_1$. Now all the vertices of $G$ are forced to black and after finite iterative applications of the color-change rule for connected subgraph of $H_i$ all the vertices in these graphs are turned black for each $i = 1, 2, \ldots, n_1$, and then $v_i \rightarrow y_i$. Hence $Z(G \odot H) \leq n_1 Z(K_1 \odot H)$. Therefore, the result follows. \qed

Corollary 2.9. Let $G$ be a connected graph of order $n_1 \geq 2$ and $H$ be a disconnected graph of order $n_2 \geq 2$. Then

$$Z(G \odot^k H) = n_1(n_2 + 1)^{k-1} Z(K_1 \odot H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong \overline{K_{n_2}}$.

Proof. Suppose $H \cong \overline{K_{n_2}}$. For $k = 1$, we have to show that $Z(G \odot H) = n_1(n_2 - 1)$. We define $B_i = V_i - \{u'_i\}$, for any $1 \leq l \leq n_2$, and for each $i = 1, 2, \ldots, n_1$ and $B = \bigcup_{i=1}^{n_1} B_i$. We claim that $B$ is a zero forcing set of $G \odot H$ with $|B| = n_1(n_2 - 1)$. To prove the claim, we first assume that $B$ is initially colored black. Note that every initial black vertex of $H_i$ has single white neighbor $v_i$, so $u'_i \rightarrow v_i$. Now there is only one white vertex $u'_i$ in each
Theorem 2.12. Let $v$ of emvs of $G$ of degree at least three be a vertex of degree two such that the terminal degree of a major vertex $v$ in $T$, denoted by $\text{ter}_T(v)$, is the number of terminal vertices of $v$. A major vertex $v$ of $G$ is an exterior major vertex (emv) if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of $G$ and let $ex(G)$ denote the number of emvs of $G$. We further define an exterior degree two vertex to be a vertex of degree two that lies on a path from a terminal vertex to its major vertex and an interior degree two vertex to be a vertex of degree two such that the shortest path to any terminal vertex includes a major vertex.

Theorem 2.10. If $T$ is a tree that is not a path, then $\dim(T) = \sigma(T) - ex(T)$.

Theorem 2.11. For any tree $T$, we have $Z(T) = \dim(T)$ iff $T$ has no interior degree two vertices and each major vertex $v$ of $T$ satisfies $\text{ter}_T(v) \geq 2$.

Theorem 2.12. Let $T$ be a tree of order $n \geq 3$, that has no interior degree two vertices and each major vertex $v$ of $T$ satisfies $\text{ter}_T(v) \geq 2$, then

$$Z(T \odot^k K_1) = \begin{cases} \sigma(T), & k = 1, \\ 2^{k-2}n, & k \geq 2. \end{cases}$$
Proof. Since $T \odot^k K_1$ is a tree, with no interior degree two vertex and each major vertex $v$ satisfies $ter_T(v) \geq 2$. Now for $k = 1$, $\sigma(T \odot K_1) = n$ and $ex(T \odot K_1) = n - \sigma(T)$, $dim(T \odot K_1) = \sigma(T) = Z(T \odot K_1)$ by Theorem 2.10 and Theorem 2.11 we obtain the result. Since we have $\sigma(T \odot^2 K_1) = 2n$, $ex(T \odot^2 K_1) = n$, so we obtain the result for $k = 2$ by Theorem 2.10 and Theorem 2.11 $dim(T \odot^2 K_1) = n = Z(T \odot^2 K_1)$.

Let $\alpha$ be the number of connected components of a graph $H$. Let us denote the connected components of $H$ by $C_l$, where $1 \leq l \leq \alpha$.

Theorem 2.13. Let $G$ be a connected graph of order $n_1$ and $H$ be a graph of order $n_2$. Let $\alpha$ be the number of connected components of $H$ of order greater than one and let $\beta$ be the number of isolated vertices of $H$. Then

$$Z(G \odot^k H) \leq \begin{cases} n_1(n_2 + 1)^{k-1} \sum_{i=1}^{\alpha} Z(C_i) + n_1(n_2 + 1)^{k-1}(\beta - 1), & \alpha \geq 1, \beta \geq 2, \\ Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} \sum_{i=1}^{\alpha} Z(C_i), & \alpha \geq 1, \beta = 0, \\ n_1(n_2 + 1)^{k-1} \sum_{i=1}^{\alpha} Z(C_i) + n_1(n_2 + 1)^{k-1} - 1, & \alpha \geq 1, \beta = 1, \\ n_1(n_2 + 1)^{k-1}(n_2 - 1), & \alpha = 0, \beta \geq 2. \end{cases}$$

Proof. We define $K_i^l$, $1 \leq l \leq \alpha$, be a forcing basis for connected component $C_i^l$ of $H_i$, $1 \leq i \leq n_1$.

We suppose $\alpha \geq 1, \beta \geq 2$. We define $P_i$ to be the set of vertices of $G \odot H$ formed by all but one of the isolated vertices of $H_i$, $1 \leq i \leq n_1$. Let us show that $B = \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_i^l \cup P_i)$ is a zero forcing set of $G \odot H$ with $|B| = n_1 \sum_{i=1}^{\alpha} Z(C_i) + n_1(\beta - 1)$. Let $B$ is initially colored black. Note that $p_i^j \in P_i$ has only one white neighbor $v_i \in V$, so $p_i^j \rightarrow v_i$ for each $i$, $1 \leq i \leq n_1$. Now all the vertices of $G$ are colored black. We color all the vertices of connected components $C_i^l$, $1 \leq l \leq \alpha$, of $H_i$ using $K_i^l$ and the corresponding vertex $v_i \in V$, $1 \leq i \leq n_1$. Note that the vertex $v_i$ has only one white neighbor $p_i^j$, the isolated vertex of $H_i$ not belonging to $P_i$, so $v_i \rightarrow p_i^j$, $1 \leq i \leq n_1$, and we have the derived set of all black vertices in $G \odot H$. As a consequence, $Z(G \odot H) \leq n_1 \sum_{i=1}^{\alpha} Z(C_i^l) + n_1(\beta - 1)$. Therefore, the result follows.

Now suppose $\alpha \geq 1$, and $\beta = 0$. Let $B = B' \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_i^l)$, where $B' = \{v_1, v_2, \ldots, v_t\}$ is a forcing basis for $G$ and by using $B'$ one can color all the other vertices of $G$ by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, \ldots, v_n$, with appropriate indexing of vertices. We show that $B$ is a zero forcing set of $G \odot H$. Consider iterative applications of the color-change
rule with initial black set $B$. We color all the vertices of $H_i$ with $1 \leq i \leq t$ which are associated with the vertices of $B'$ using the corresponding sets $K'_i$, $1 \leq l \leq \alpha$. Now all the vertices in $H_i$ associated with $v_i, 1 \leq i \leq t$ are colored black. Note that there is a vertex $v_i$ belonging to $B'$ that has only one white neighbor $v_{t+1} \in V$. Thus $v_i \rightarrow v_{t+1}$. Then we color all the vertices in $H_{t+1}$ using the black vertices in $K'_{t+1}$, $1 \leq l \leq \alpha$. Continuing this process, we can color all the vertices of $G \odot H$. So $Z(G \odot H) \leq Z(G) + n_1 \sum_{i=1}^{\alpha} Z(C_i)$. Therefore, the result follows.

Now suppose $\alpha \geq 1, \beta = 1$. Let $R$ be the set of all isolated vertices in each $H_i, 1 \leq i \leq n_1$, except one say $r_{n_1}$. We define $B = \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K'_i) \cup R$. We show that $B$ is a zero forcing set of $G \odot H$. Let $B$ is initially colored black. Note that the vertex $r_i \in R, 1 \leq i \leq n_1 - 1$, has only one white neighbor $v_i \in V$, so $r_i \rightarrow v_i, 1 \leq i \leq n_1 - 1$. We color all the vertices of $H_i, 1 \leq i \leq n_1 - 1$, which are associated with $v_i, 1 \leq i \leq n_1 - 1$ using the corresponding sets $K'_i$, $1 \leq l \leq \alpha$. Now all the vertices in $H_i$ associated with $v_i, 1 \leq i \leq n_1 - 1$ are colored black. Now $v_{n_1-1}$ has only one white neighbor $v_{n_1}$, so $v_{n_1-1} \rightarrow v_{n_1}$. We color $C'_i$ in $H_{n_1}$ using $K'_{n_1}, 1 \leq l \leq \alpha$. Now $v_{n_1}$ has only one white neighbor $r_{n_1} \in V_{n_1}$, so $v_{n_1} \rightarrow r_{n_1}$ and we have the derived set of all black vertices. So $Z(G \odot H) \leq n_1 \sum_{i=1}^{\alpha} Z(C_i) + n_1 - 1$. Therefore, the result follows.

Now suppose $\alpha = 0, \beta \geq 2$. Here $H$ is an empty graph so the result followed by Corollary 2.9.

3. Lexicographic Product of Graphs

Let $G$ and $H$ be two graphs. The lexicographic product of $G$ and $H$, denoted by $G \odot H$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) \mid a \in V(G)$ and $v \in V(H)\}$, where $(a, v)$ is adjacent to $(b, w)$ whenever $ab \in E(G)$ or $a = b$ and $vw \in E(H)$. For any vertex $a \in V(G)$ and $b \in V(H)$, we define the vertex set $H(a) = \{(a, v) \in V(G \odot H) \mid v \in V(H)\}$ and $G(b) = \{(v, b) \in V(G \odot H) \mid v \in V(G)\}$. It is clear that the graph induced by $H(a)$, called a layer $H(a)$, is isomorphic to $H$ and the graph induced by $G(b)$, called a layer $G(b)$, is isomorphic to $G$, denoted by $H(a) \cong H$ and $G(b) \cong G$ respectively. We write $H(a) \sim H(b)$ when each vertex of $H(a)$ is adjacent to all vertices of $H(b)$ and vice versa, and $H(a) \not\sim H(b)$ means that no vertex of $H(a)$ is adjacent to any vertex of $H(b)$ and vice versa.

Let $G$ be a connected graph and $H$ be a non-trivial graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ with $|V(H_j)| \geq 2$ for each $j = 1, 2, \ldots, k$. For any vertex $a \in V(G)$ and $1 \leq i \leq k$, we define the vertex set $H_i(a) = \{(a, v) \in
$V(G \circ H) \{v \in V(H_i)\}$. Let $|V(H_i)| = m_i$, $1 \leq i \leq k$. From the definition of $G \circ H$, it is clear that for every $(a, v) \in V(G \circ H)$, $\text{deg}_{G \circ H}(a, v) = \text{deg}_G(a) \cdot |V(H)| + \text{deg}_H(v)$. If $G$ is a disconnected graph having $k \geq 2$ components $G_1$, $G_2$, $\ldots$, $G_k$, then $G \circ H$ is also a disconnected graph having $k$ components such that $G \circ H = G_1 \circ H \cup G_2 \circ H \cup \ldots \cup G_k \circ H$ and each component $G_i \circ H$ is the lexicographic product of connected component $G_i$ of $G$ with $H$, therefore throughout this section, we will assume $G$ to be connected.

**Observation 3.1.** For any $a, b \in V(G)$, either $H(a) \sim H(b)$ or $H(a) \not\sim H(b)$ in $G \circ H$.

First, we give a general lower bound on the zero forcing number of lexicographic product of graphs. Note that given any connected graph $G$, then $Z(G) = 1$ if and only if $G \cong P_n$, $n \geq 2$. So, if $Z(G \circ H) = 1$ for some graph $H$, then clearly $G \circ H$ is a path graph, i.e $G$ is the trivial graph $K_1$ and $H$ is a path or viceversa. So, we have the following result:

**Remark 3.2.** If $G$ and $H$ are non trivial graphs, then $Z(G \circ H) \geq 2$.

**Lemma 3.3.** Let $G$ be a connected graph on $n$ vertices. There exists a forcing basis $S$ for $G + K_1$ such that $S \subseteq V(G)$.

**Proof.** Let $V(G + K_1) = V(G) \cup \{v\}$. If $v \notin S$ we have nothing to prove. Suppose that $v \in S$. Since $G$ is connected and $\text{deg}_{G+K_1}(v) = n$ and also $v$ is initially colored black so by equation (1), there exists at least one white vertex $x \in N_{G+K_1}(v)$ such that $(S \setminus \{v\}) \cup \{x\}$ is a forcing basis for $G + K_1$. \hfill $\Box$

**Theorem 3.4.** Let $G$ be a connected graph and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \ldots, H_k$ and $m_i \geq 2$. Let $Z$ be a zero forcing set of $G \circ H$. For any vertex $a \in V(G)$, if $Z_i(a) = Z \cap H_i(a)$ for every $i \in \{1,2,\ldots,k\}$, then $Z_i(a) \neq \emptyset$. Moreover, if $B_i$ is a forcing basis for $H_i$, then $|Z_i(a)| \geq |B_i|$.

**Proof.** Suppose that for some $i \in \{1,2,\ldots,k\}$ there exists a vertex $a \in V(G)$ such that $Z_i(a) = \emptyset$. Then, by Observation 3.1 any vertex in $H_i(a)$ cannot be forced by any vertex in $H_j(b)$, $i \neq j$ for any $a \neq b \in V(G)$, a contradiction.

Now suppose that $|Z_i(a)| < |B_i|$ and $Z_i(a) = \{(a, z_1), (a, z_2), \ldots, (a, z_t)\}$ for some forcing basis $B_i$ of $H_i$, where $\{z_1, z_2, \ldots, z_t\} \subset V(H_i)$. Then, each black vertex in $H_i(a)$ has more than one white neighbors and no vertex of $H_i(a)$ can be forced by any vertex in $H_j(v)$ for any $v \in V(G)$, $i \neq j$. Hence, $|Z_i(a)| \geq |B_i|$.

From above theorem, we have an immediate corollary:
Corollary 3.5. Let $G$ be a connected graph and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \ldots, H_k$ and $m_i \geq 2$. Let $Z(a) = \bigcup_{1 \leq i \leq k} Z_i(a)$ for $a \in V(G)$. Then $Z(a)$ is a zero forcing set of $H(a)$.

Proposition 3.6. Let $G$ be a connected graph and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ and $m_i \geq 2$. Let $a \in V(G)$ and $Z$ be a forcing basis for $G \circ H$. If $Z(a) = Z \cap H(a)$ and $\alpha(a) = |Z(a)|$. Then

$$\alpha(a) \leq \sum_{i=1}^{k} m_i.$$ 

Proof. For any $(b, x) \in V(G \circ H)$, $deg_{G \circ H}(b, x) = \sum_{b \sim v} |H(u)| + deg_H(x)$. Since $G$ is connected so for any $b \in V(G)$, there exist at least one vertex $v \in V(G)$ such that $b \sim v$ and $deg_{G \circ H}(b, x) \geq |H(v)| + deg_H(x)$. To start the zero forcing process at least all the vertices of $H(v)$ along with $deg_H(x)$ vertices are initially colored black. Since $Z$ is a forcing basis for $G \circ H$. Hence, $Z \cap H(v) = H(v)$ and $\alpha(a) \leq \sum_{i=1}^{k} m_i$ for any $a \in V(G)$. \hfill \square

Corollary 3.7. Let $G$ be a connected graph and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ and $m_i \geq 2$. Then there exists at least one vertex $x \in V(G)$ such that $\alpha(x) = \sum_{i=1}^{k} m_i$.

The projection of $S \subseteq V(G \circ H)$ onto $G$, denoted by $P_G(S)$, is the set of vertices $a \in V(G)$ for which there exists a vertex $(a, v) \in S$. Similarly, the projection of $S \subseteq V(G \circ H)$ onto $H$, $P_H(S)$, is the set of vertices $v \in V(H)$ for which there exists a vertex $(a, v) \in S$.

Lemma 3.8. Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ and $m_i \geq 2$. Let $Z$ be a forcing basis for $G \circ H$ and $Z_i = Z \cap V(G \circ H_i)$, where $G \circ H_i$ is the induced subgraph of $G \circ H$. Then $P_G(Z_i) = V(G)$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H_i) = \{v_1^i, v_2^i, \ldots, v_{m_i}^i\}$ for $1 \leq i \leq k$. Suppose $P_G(Z_i) \neq V(G)$, i.e there exists a vertex $u_j \in V(G)$ such that $u_j \notin P_G(Z_i)$. This implies $(u_j, v_p^i) \notin Z$ for any $v_p^i \in V(H_i)$ for $1 \leq p \leq m_i$. Hence, $H_i(u_j) \cap Z = \phi$, a contradiction by Theorem 3.4. \hfill \square

Lemma 3.9. Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ and $m_i \geq 2$. Then

$$Z(G \circ H) \leq n\left(\sum_{i=1}^{k} m_i\right) - k.$$ 

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Proof. Let \( V(G) = \{u_1, u_2, \cdots, u_n\} \) and \( V(H_i) = \{v^i_1, v^i_2, \cdots, v^i_{m_i}\} \) for \( 1 \leq i \leq k \). We define \( Z = V(G \circ H) \setminus \{(u_1, v^i_2) | 1 \leq i \leq k\} \) and \( |Z| = n(\sum_{i=1}^{k} m_i) - k \).

We claim that \( Z \) is a zero forcing set of \( G \circ H \). To prove the claim, assume that \( Z \) is initially colored black. Since for any \( i \neq j \), \( v^i_r \not\sim v^j_r \) in \( H \) hence \( (u_r, v^i_r) \not\sim (u_r, v^j_r) \), \( 1 \leq r \leq n \), in \( G \circ H \). Since for any \( i \), \( H_i \) is connected, so there exists at least one vertex \( v^i_r \) such that \( v^i_2 \sim v^i_r \) in \( H_i \) and hence \( (u_r, v^i_2) \sim (u_r, v^i_r) \), \( 1 \leq r \leq n \), in \( G \circ H \). Therefore, \( (u_1, v^i_1) \rightarrow (u_1, v^i_2) \) for \( 1 \leq r \leq n \). Hence, 

\[
Z(G \circ H) \leq n(\sum_{i=1}^{k} m_i) - k.
\]

This bound is sharp for \( G = K_n \) and \( H = K_{m_1}, K_{m_2}, \cdots, K_{m_k} \).

**Lemma 3.10.** Let \( G \) be a connected graph of order \( n \) and \( H \) be an arbitrary graph containing \( k \geq 1 \) components \( H_1, H_2, \cdots, H_k \) and \( m_i \geq 2 \). Then

\[
Z(G \circ H) \geq (n - 1)k + \sum_{i=1}^{k} m_i.
\]

**Proof.** The result follows from Corollary 3.7 and Lemma 3.8. \( \square \)

This bound is sharp for \( G = K_{1,n-1} \) and \( H = P_{m_1}, P_{m_2}, \cdots, P_{m_k} \).

**Lemma 3.11.** Let \( G \) be a connected graph of order \( n \) and \( H_1, H_2, \cdots, H_k \), \( k \geq 2 \) are singleton components. Then \( Z(G \circ H) \leq nk - 2 \).

**Proof.** Let \( V(G) = \{u_1, u_2, \cdots, u_n\} \) and \( V(H_i) = \{x_i\} \) for \( 1 \leq i \leq k \). We define, for \( u_1 \sim u_2 \), \( Z = V(G \circ H) \setminus \{(u_1, x_k), (u_2, x_k)\} \) with \( |Z| = nk - 2 \).

Since \( x_1 \not\sim x_k \) so \( (u_1, x_1) \rightarrow (u_2, x_k) \). Similarly, \( (u_2, x_1) \rightarrow (u_1, x_k) \). Now all the vertices in \( G \circ H \) are colored black. Therefore, the result follows. \( \square \)

Now we study the zero forcing number of lexicographic product of graphs for some specific families of graphs and \( H \) contains one component only. Note that \( K_m \circ K_n \cong K_{mn} \), so \( Z(K_M \circ K_n) = mn - 1 \). Therefore, from now on we consider the graphs when at most one of the factors of the product is a complete graph.

**Lemma 3.12.** For any connected graph \( H \) of order \( m \), \( Z(K_n \circ H) = Z(H) + (n - 1)m \).

**Proof.** Note that for any \( a \in V(K_n) \), \( H(a) \sim H(b) \) for all \( b \in V(K_n) \setminus \{a\} \). Therefore all the vertices in \( H(b) \) for all \( b \in V(K_n) \setminus \{a\} \) are initially colored black. Now \( H(a) \cong H \) so \( Z(H) \) vertices are required as initially colored black to complete the zero forcing process. \( \square \)
Since the lexicographic product of graphs is not commutative, i.e \( K_n \circ H \neq H \circ K_n \). Therefore we study the case when the second factor is a complete graph.

**Lemma 3.13.** For any connected non complete graph \( G \) of order \( m \), \( m(n - 1) + 1 \leq Z(G \circ K_n) \leq nm - 2 \).

**Proof.** Suppose \( V(G) = \{u_1, u_2, \ldots, u_m\} \) and \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \). It is easy to check that \( Z = V(G \circ K_n) \setminus \{(u_1, v_2), (u_n, v_2)\} \) is a zero forcing set of \( G \circ K_n \). Hence \( Z(G \circ K_n) \leq nm - 2 \).

By Corollary 3.7 there exists at least one vertex \( a \in V(G) \) such that \( \alpha(a) = n \). Since \( H(v) \cong K_n \), therefore for \( m - 1 \) layers \( H(v) \) at least \( n - 1 \) vertices from each layer are required as initially colored black to color all the vertices of \( G \circ K_n \). Hence \( Z(G \circ K_n) \geq n + (m - 1)(n - 1) = m(n - 1) + 1 \). \( \square \)

Now we study the zero forcing number of \( P_n \circ H \), for \( n \geq 3 \) and a connected graph \( H \). Suppose \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \). Since \( H(u_i) \sim H(u_{i+1}) \), \( 1 \leq i \leq n - 1 \). Note that to color the vertices of \( H(u_1) \), \( Z(H) \) vertices in \( H(u_1) \) and all the vertices in \( H(u_2) \) are required as initially colored black and to color the vertices of \( H(v_3) \), \( Z(H) \) vertices in \( H(v_3) \) and all the vertices in \( H(v_4) \) are required as initially colored black. Continue the process until all the vertices in \( P_n \circ H \) are turned black.

**Proposition 3.14.** For a connected graph \( H \) and \( n \geq 3 \),

\[
Z(P_n \circ H) = \begin{cases} 
\frac{n(Z(H)+m)}{2}, & \text{n is even} \\
\frac{n(Z(H)+m)+Z(H)-m}{2}, & \text{n is odd}.
\end{cases}
\]

**Corollary 3.15.** For \( n, m \geq 3 \),

\[
Z(P_n \circ K_m) = \begin{cases} 
\frac{n(2m-1)}{2}, & \text{n is even} \\
\frac{nm - \frac{n+1}{2}}{2}, & \text{n is odd}.
\end{cases}
\]

Now we study the zero forcing number of \( C_n \circ H \), for \( n \geq 4 \) and a connected graph \( H \). Suppose \( V(C_n) = \{u_1, u_2, \ldots, u_n\} \). Since \( H(u_i) \sim H(u_{i+1}) \), \( 1 \leq i \leq n \) and \( u_{n+1} = u_1 \). Note that to color the vertices of \( H(u_1) \), \( Z(H) \) vertices in \( H(u_1) \) and all the vertices in \( H(u_2) \) and \( H(u_n) \) are required as initially colored black and to color the vertices of \( H(v_3) \), \( Z(H) \) vertices in \( H(v_3) \) and all the vertices in \( H(v_4) \) are required as initially colored black. Continue the process until all the vertices in \( C_n \circ H \) are turned black.

**Proposition 3.16.** For a connected graph \( H \) and \( n \geq 4 \),

\[
Z(C_n \circ H) = \begin{cases} 
\frac{n(Z(H)+m)}{2}, & \text{n is even} \\
\frac{n(m+Z(H)+m-Z(H))}{2}, & \text{n is odd}.
\end{cases}
\]
Corollary 3.17. For \( n \geq 4 \) and \( m \geq 3 \),

\[
Z(C_n \circ K_m) = \begin{cases} 
\frac{n(2m-1)}{2}, & n \text{ is even} \\
\frac{n(2m-1)+1}{2}, & n \text{ is odd}.
\end{cases}
\]

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