Global Frictional Equilibrium via Stochastic, Local Coulomb Frictional Slips

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Abstract Based on the assumption that fault slip dominates in the stress relaxation in the brittle crust, Coulomb theory allows for the crustal stress estimation with an empirical frictional coefficient. However, natural variability of fault friction and slip uncertainty exist in the Earth's crust. To address the extent to which heterogeneous frictional slips influence crustal stress and its evolution, we establish a quasi-static, 2D model to represent the fractured crustal rock mass. The model consists of randomly oriented fractures with heterogeneous distribution of frictional coefficients. The global mechanical response is quantitatively related to the cascades of local frictional slips under specific boundary conditions. The temporal evolution of stress is explicitly modeled by an iterative process where a simple slip law is assumed for critical fractures. We particularly illustrate the stress evolution in a normal faulting stress regime, considering different distributions of frictional coefficients. All cases indicate that the decrease in differential stress manifests as a self-organized process, eventually leading to the frictional equilibrium of the fractured rock mass. The final stress state upon equilibrium jointly depends on the orientation and frictional coefficient of all fractures therein. The model informs that the global stress state of a stochastic system can depart substantially from a deterministic estimation via an empirical frictional coefficient. This model quantitatively corroborates and extends the notion of frictional equilibrium, and reveals far more profound influence of system heterogeneity on the local and global stress evolution.

Plain Language Summary Knowledge of the stress level in the Earth's upper crust is of fundamental importance to a wide range of scientific and engineering problems. Earthquakes take place when the stress level exceeds the crustal strength. It is generally assumed that the strength of the crust is uniform everywhere, limiting the stress magnitudes therein. Based on ample direct and indirect observations, we challenge this general assumption and use a model to simulate the stress evolution of a nonuniform crust. We show that the stresses differ substantially from that of the uniform assumption and fit better to the observations. Our finding underscores the potential error that may be introduced if the empirical, uniform assumption is adhered to without discretion.

1. Introduction

Fault slip (and often the consequent seismicity) is one of the dominant mechanisms for stress release in the Earth's upper crust. The state of stress in the fractured upper crust is generally interpreted to be under "frictional equilibrium" (M. D. Zoback & Townend, 2001). While exceptions certainly exist (e.g., Cornet et al., 2007; Ma & Zoback, 2017), ongoing tectonic/gravity loading and the resulting ubiquitous fault slips (and seismicity) suggest that such a dynamic equilibrium status maintained by frictional resistance is widely representative. Via the simple Coulomb frictional failure theory, the limiting state of stress can thus be conveniently expressed as (Jaeger & Cook, 1979):

\[
\sigma_1 / \sigma_3 = \left( \frac{\sqrt{\mu^2 + 1} + \mu}{\sqrt{\mu^2 + 1} - \mu} \right)^2
\]

(1)

where \(\sigma_1\) and \(\sigma_3\) are the effective major and minor principal stresses, respectively, and \(\mu\) is the frictional coefficient. Adopting laboratory-derived frictional coefficient values (\(\mu = 0.6-1.0\)) (Byerlee, 1978), Equation 1 has been frequently used to estimate in situ stress, or at best, its bounding values (Brace & Kohlstedt, 1980; Townend & Zoback, 2000). On the other hand, Equation 1 enables the quantification of fault criticality, provided with the knowledge of \(\sigma_1\) and \(\sigma_3\) in situ. However, the nuance between the two applications is that the former often refers to the far-field or global stress state that is characteristic of a representative
size of a crustal rock mass, while the latter concerns the local stress state near a particular fault. Although Equation 1 is able to attain a first-order agreement with in situ stress measurements at various locations and depths worldwide, to what spatial and temporal scale a deterministic use of Equation 1 applies to is often unclear, and may incur misuse and mis-interpretations. For example, measured stress values at local scales that are significantly perturbed by nearby discontinuities should not be employed to estimate the general criticality of the rock mass that is far away from such perturbations. Regarding temporal mismatch, Equation 1 may lead to overestimation of the differential stress that can be accommodated in certain formations. For example, in some compliant lithologies such as shales and carbonates, significant time-dependency in deformation and stress evolution drive the differential stress well below the nominally interpreted frictional limit (e.g., Gunzburger & Cornet, 2007; Ma & Zoback, 2020; Sone & Zoback, 2014).

For the other key parameter in Equation 1, the frictional coefficient $\mu$ is subject to high variability. The difference in $\mu$ between different rock types has been well evidenced by numerous laboratory friction experiments (Byerlee, 1978). Even for the same lithology, repeated laboratory characterization reveals appreciable dispersion (e.g., Blanpied et al., 1995). Such differences and dispersion add substantial uncertainty to the deterministic use of Equation 1. At larger scales, faults exhibit diverse frictional properties due to their complex geometry and the variations of the constituent materials therein (Ben-Zion & Sammis, 2003; Collettini et al., 2019; Luo & Ampuero, 2018). These factors contribute to the spatial heterogeneity of $\mu$. The frictional coefficient $\mu$ also varies greatly in a temporal sense, corroborated by both laboratory and field observations (e.g., Marone, 1998). Although Equation 1 has been generally applied in a static manner, the temporal heterogeneity in $\mu$ and its interplay with the stress state should not be overlooked.

In situ state of stress is typically ruled by tectonic loading and local perturbations (M. L. Zoback, 1992). Ample evidence suggests a relatively uniform regional stress field over a certain scale, for example, tens of kilometers or beyond (Heidbach et al., 2018), and remarkable heterogeneities over shorter wavelengths (Day-Lewis et al., 2010; Schoenball & Davatzes, 2017). With a simple application of Equation 1, variations of fault frictional properties can induce such complex and spatially heterogeneous crustal stress distributions at local scales. Variations in the style of frictional slips suggest that the heterogeneous distribution of frictional coefficients and heterogeneous stresses locally on faults likely accompany each other (e.g., Iio et al., 2017; Rivera & Kanamori, 2002). These field observations underscore that such variability in a natural system must be considered. An absence of such considerations in Equation 1 may incur erroneous estimation of in situ stress and fault stability. To reflect such variability, recent attempts in stress estimation and/or fault slip analysis incorporated uncertainties in geomechanical parameters with a probabilistic approach (e.g., Hosseini et al., 2018; Shen et al., 2019; Walsh & Zoback, 2016), which offers more insights than a purely deterministic application of Equation 1.

Another aspect worth investigating is how such system variability and heterogeneity influence the temporal evolution of the in situ stress. Equation 1 implicitly refers to the final, stable or equilibrated state of stress. Not only its deterministic application is subject to debate (e.g., Walsh & Zoback, 2016), but also how its global scale significance is built on the relatively micromechanical fault slip mechanism is vague (Ma et al., 2020). The temporal process that leads to such dynamic equilibrium state can plausibly reconcile existing discrepancies and better illuminate the notion of frictional equilibrium. In the context of a heterogeneous system, the dynamic frictional equilibrium not only concerns the influence of the far-field stress on the local fault slip, but also requires the feedback from the local slip to the global stress release. Given heterogeneities at the local scale, such interactions are expected to incur a diffuse and stochastic process for stress evolution.

We hypothesize that the stress state and its evolution of a heterogeneous, stochastic system can differ significantly from its homogeneous, deterministic counterpart. To test this hypothesis, we implement a straightforward model to simulate how local fault slips leads to global stress relaxation in the heterogeneous crustal rock masses. A quasi-static, 2D stochastic model is employed in this instance. The model aims to capture processes over a representative volume of the crust, consistent with the scale to which the concept of frictional equilibrium typically pertains. We explicitly consider frictional coefficient heterogeneity, as a proxy of the intrinsic system uncertainties and variabilities. The temporal stress evolution due to Coulomb frictional slips and the final stress state allowed by the system, or frictional equilibrium, are realized by an iterative process. In this model, we emphasize the interactions between the global and local stress and strain, and
explore its stochastic and temporal characteristics through iterations. Specifically, this paper is organized as follows. We first introduce the model configuration and iterative methods for stress relaxation in Section 2. In Sections 3.1 and 3.2, we present and explicitly compare the results of the deterministic (featuring homogeneous frictional coefficient) and stochastic (heterogeneous frictional coefficient) cases. We further investigate the influence of frictional coefficient heterogeneity on stress uncertainties via Monte Carlo simulations in Section 3.3. Finally, we offer several lines of discussion on different aspects of the modeling results.

2. Methods

2.1. Model Configuration and Basic Assumptions

The model we present is a fractured, elastic matrix configured under plane strain condition (Figure 1a). The embedded fractures are linear, planar, and cohesion-less. They are perpendicular to the plane section and through-going (Figure 1b). The fractures are spatially distributed and only characterized by their orientations, so that their actual locations in the plane are irrelevant. This treatment follows the seminal work by Wiebols and Cook (1968) and others on effective medium (e.g., Davy et al., 2018; Kachanov, 1992). Deemed essential to our model, the embedded fractures differ in their frictional coefficient $\mu$, which can follow any arbitrary distribution (Figure 1c), as a proxy for the inherent heterogeneity in the system. The elastic matrix is simply characterized by its elastic parameters, for example, shear modulus $G$ and Poisson's ratio $\nu$. As a quasi-static model for stress relaxation of much longer time and geometry scales, complex fracture behavior such as fracture initiation, propagation, and termination are considered as secondary (Scholz, 1998) and therefore not addressed.

Given a remote (effective) stress tensor $\sigma$ applied at the model boundary, local shear and normal stresses ($\sigma_n$ and $\tau$) acting on individual fractures can be resolved by tensor transformation:

\[ \sigma_n = n \cdot \sigma \cdot n \]

\[ \tau = n^T \cdot \sigma \cdot s \]
where $\mathbf{n}$ is the unit normal vector and the unit shear vector $\mathbf{s}$ is taken as positive when the fracture has a right-lateral shear motion (Figure 1b). The fracture orientation is differentiated by the angle $\theta$ between the fracture normal vector and the positive horizontal axis of the model’s global coordinates, as shown in Figure 2. In this study, we regulate the fracture orientations to be randomly distributed.

2.2. Local Displacement and Global Strain Field

We adopt the classic Coulomb frictional failure criterion to determine whether slip occurs on a fracture. If $\tau > \mu \cdot \sigma_n$, the fracture is identified as critical and frictional slip takes place, otherwise the fracture stays perfectly bonded, behaving as part of the elastic matrix with no relative displacement occurring between opposite fracture sides. We assume that the shear stress on the fracture will drop to its frictional resistance after the slip, so in that sense the shear stress difference $\Delta \tau = \tau - \mu \cdot \sigma_n$ drives the relative displacement across the fracture.

Based on elastic crack theory (Pollard & Segall, 1987), the normal and shear displacements ($u_n$ and $u_s$) on opposite sides of a fracture that are associated with the slip can be analyzed conveniently in the local fracture coordinates ($x_n$, $x_s$) (Figure 1b). Specifically, they are:

$$ u_n = \Delta \tau \frac{1 - 2v}{2G} x_s $$

(4a)

$$ u_s^\pm = \pm \Delta \tau \frac{1 - v}{G} \sqrt{a^2 - x_s^2} $$

(4b)

where $x_s \in [-a, a]$, $a$ is the fracture half-length, and the superscript “±” of $u_s$ refers to displacement along the upper and lower fracture side ($x_s = \pm 0$), respectively. The average relative shear displacement between opposite sides $d_s$ is then obtained from integrating the relative shear displacement ($u_s^+ - u_s^-$) across the whole fracture length (2a):

$$ d_s = \frac{1}{a} \int_0^a \Delta \tau \frac{2(1-v)}{G} \sqrt{a^2 - x_s^2} dx_s = \left( \frac{\Delta \tau}{2G} \right) s $$

(5a)

To reflect shear-induced dilatancy that is commonly observed in the brittle rock masses (Fielding et al., 2009; Scholz, 1974), we utilize dilatancy factor $\beta$ to relate $d_n$ to the average relative normal displacement $d_n^*$:

$$ d_n = \beta |d| n = \beta \left( \frac{\Delta \tau}{2G} \right) n $$

(5b)

We further invoke Gaussian theorem (Hill, 1963; Kachanov, 1992) to relate the contribution of local displacement incurred by individual frictional slip to the global strain increment at the model boundary $\Delta \varepsilon$:

$$ \Delta \varepsilon = \frac{a}{A} (\bar{\mathbf{d}} \otimes \mathbf{n} + \mathbf{n} \otimes \bar{\mathbf{d}}) $$

(6)

where $A$ is the cross-section area of the model and the total displacement vector $\mathbf{d} = d_n^* + d_n$. This procedure is graphically illustrated in Figure 3.
The total global strain at the model boundary comprises the strain of the intact elastic matrix $\varepsilon^m$ and, if critical fractures exist, the summed strain increment $\Sigma \Delta \varepsilon$ induced by a cascade of slips on critical fractures:

$$\varepsilon_i = \varepsilon^m + \sum_{j} \frac{d_i}{A} \left( \mathbf{d}_i \otimes \mathbf{n}_j + \mathbf{n}_j \otimes \mathbf{d}_i \right)$$

(7)

where subscript $i$ denotes the $i$th critical fracture in a cascade of slips. It is noteworthy that, the shear strain component at either boundary can be canceled out provided that fractures are randomly distributed. The intact elastic matrix strain $\varepsilon^m$ is simply regulated by Hooke’s law under plane strain condition:

$$\varepsilon^m = \begin{bmatrix} \varepsilon_{11}^m \\ \varepsilon_{22}^m \\ \varepsilon_{12}^m \end{bmatrix} = \begin{bmatrix} 1 - \nu / (1 - \nu) & -\nu / (1 - \nu) & 0 \\ -\nu / (1 - \nu) & 1 & 0 \\ 0 & 0 & 1 / (1 - \nu) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{13} \end{bmatrix}$$

(8)

The summation of individual contribution of fracture slips, that is, the second right-hand term in Equation 7, is essentially equivalent to the incremental insertion of fractures into the model. In general, the effective medium theory treats this issue by two methods, depending on whether or not fracture interactions are considered. The Non-Interaction Approximation (NIA) assumes that each fracture is inserted into an undamaged matrix (Bristow, 1960). Therefore, its displacement vector can be calculated by Equation 5 with original elastic moduli ($G$ and $\nu$) of the matrix. Alternatively, the fracture interactions can be approximated by adding one isolated fracture in a damaged effective matrix, for example, self-consistent theory (O’Connell, 1974) and differential scheme (DS) (Hashin, 1960). Specifically, the DS assumes a gradually damaged medium with incrementally inserted fractures. When implementing the DS, the global stress and strain of the model should be updated after inserting each critical fracture, and the effective elastic properties are updated accordingly following Hooke’s law, based on the assumption of a homogeneous effective medium. By contrast with the NIA, the DS employs consecutively updated elastic moduli in Equation 5 after each critical fracture is inserted.

Considering that fracture interactions are ubiquitous and profound in rock masses, the NIA appears to be oversimplified compared with the DS. However, Kachanov (1992) argued that the NIA is in fact more applicable in this instance since mutually opposite effects of stress shielding and amplification near fractures tend to balance out globally, especially when fractures are randomly oriented. The NIA has been validated by either numerical simulations (Grechka & Kachanov, 2006) or laboratory observations (Katz & Reches, 2004), and has been successfully adopted to quantify the effective properties of fractured rock masses (e.g., Healy, 2008; Davy et al., 2018). In the Supporting Information (Text S1), we show the comparison between the NIA and DS in terms of the stress-strain curves and elastic moduli of the model under compression. It is found that the difference between the NIA and DS treatment is trivial. For computational convenience, we adopt the NIA in this study to incorporate the contribution of frictional fractures. In the case of randomly oriented fractures, the NIA is applicable even at a relatively high fracture density level ($\sim$0.5), which is defined as the quadratic sum of the half-length in terms of all fractures per unit area. As shown in Figure 12, the initial fracture density ($\sim$0.5) supports the application of NIA. In addition, the number of critical fractures in each iteration further decreases as iterations continue, justifying the applicability of the NIA in this stress relaxation context.

2.3. Slip Cascades and Iterative Time Steps

If stress and/or strain is mandated constant at the model boundary, the contribution of local fracture slips requires adjustments of stress/strain in the intact elastic matrix. The updated global stress in turn modifies the criticality of individual fractures locally, necessitating a frequent re-evaluation of the fracture criticality. To this end, we impose an iterative process to regulate cascades of fracture slips into successive time steps, as shown by the flowchart in Figure 4. This is to accommodate the interplay between local fracture criticality, global stress and strain, and fractured matrix. Within each time step, a cascade of slips is set to occur given the fracture criticality evaluation under the current stress condition, and it is regulated to finish so that the next time step embraces the updated stress condition for re-evaluation of fracture criticality. For those
critical fractures identified in individual time steps, static elastic fields defined in Section 2.2 are utilized so that the temporal process can be modeled in a quasi-static manner. For computational efficiency, complex dynamic effects (e.g., stress drop, seismic energy release, etc.) are not considered. This simplification is contradictory to the behavior of natural discontinuities, but one would expect that such dynamic and comparatively transient complexity is secondary in terms of the long-term frictional equilibrium. In addition, the seismic energy release (associated with even the major earthquakes) is insignificant to the energy required to sustain the brittle crust under frictional equilibrium. In Section 4.1, we provide further discussion on such assumptions. Specifically, the iterative process is implemented as follows (illustrated in Figure 4).

Separating cascades of slips into discrete time steps is a heuristic approach to enable iterative processes. We are fully cognizant that the slip rate is variable, which depends on the past slip history and stress state (Dieterich, 1979; Ruina, 1980; Sleep, 2006). To reflect this in the iterations, a simple linear relationship between fracture slip rate and shear stress difference $\Delta \tau$ is proposed:

$$\frac{d\bar{d}_s}{d\tau} = \eta \Delta \tau$$  \hspace{1cm} (9a)

$$\frac{d\bar{d}_n}{d\tau} = \beta \cdot \eta \Delta \tau$$  \hspace{1cm} (9b)

with $\eta$ defined as the slip rate parameter with the dimension of length/(stress · time).

In discrete form, the relative shear and normal displacement increments can be further expressed as:

$$\Delta [\bar{d}_s]_j = (\eta \cdot \Delta \tau_j) \cdot \Delta t$$  \hspace{1cm} (10a)

$$\Delta [\bar{d}_n]_j = (\beta \cdot \eta \cdot \Delta \tau_j) \cdot \Delta t$$  \hspace{1cm} (10b)
where the time step is given as:

\[ \Delta t = t_j - t_{j-1}, j = 1, 2, 3, \ldots \]

It is hypothesized that the slip rate and shear stress difference of each critical fracture are constant over the time step interval \( \Delta t \). If frictional slip is assumed to occur completely and reach the final steady state at the end of each time step, the slip rate parameter \( \eta_j \) can be simply specified according to Equation 5 as:

\[ \eta_j = \frac{a \pi (1 - v_{j-1})}{2G_{j-1}} \]  

This implies that the updated results at the end of the previous time step \((j-1)\) serve as the new input. Note that, the time step interval \( \Delta t \) is set to be equal to one another in the iterative process. The slip rate is varying one time step after another, dependent on the slip rate parameter and shear stress difference, as indicated by Equations 10 and 11. This treatment incorporates the effect of slip history to some extent, although the frictional coefficient is kept constant. In Section 4.1, we offer more arguments about this simple treatment of slip behavior. With the displacements of fractures obtained at the end of the \( j \)th time step, the fracture-induced strain is calculated according to Equation 7 with the adoption of the NIA. It requires adjustments of stress and strain in the intact elastic matrix due to the constant strain and stress boundary conditions, respectively.

We then take the normal faulting stress regime \((\sigma_v = \sigma_1 > \sigma_h = \sigma_3)\) as an example to illustrate the updates of the global stress and strain. In this context, the vertical stress \( \sigma_v = \sigma_1 \) is assumed to be constant while the horizontal strain \( \varepsilon_3 = \varepsilon_3 \) at boundary is maintained constant (Figure 5a). Taking compression as positive, fracture slips induce positive and negative strain increments in the \( \sigma_1 \) and \( \sigma_3 \) directions, respectively, if there is no horizontal strain boundary (Figure 5b). In order to maintain the prescribed constant global horizontal strain \( \varepsilon_{3,0} \), the horizontal strain of the elastic matrix \( \varepsilon_{3}^{\text{in}} \) needs to increase to accommodate the slip-induced increment \( \Delta \varepsilon_3 \), which is self-regulated by an increase in \( \sigma_3 \) as illustrated in Figure 5c.

More specifically, treating the model as an effective medium and invoking Hooke’s law, we have the horizontal strain at the \( j \)th time step as:

\[ \varepsilon_{3,j} = \frac{1 - v_{j-1}}{2G_{j-1}} \left( \frac{-v_{j-1}}{1 - v_{j-1}} \sigma_{1,j-1} + \left( \sigma_{3,j-1} - \Delta \sigma_{3,j} \right) \right) + \Delta \varepsilon_{3,j} = \varepsilon_{3,0} \]  

which further gives:
\[
\Delta \sigma_{3,j} = \frac{2G_{j-1}}{1 - v_{j-1}} \Delta \varepsilon_{3,j}
\]

and

\[
\sigma_{3,j} = \sigma_{3,j-1} - \Delta \sigma_{3,j} = \sigma_{3,j-1} - \frac{2G_{j-1}}{1 - v_{j-1}} \Delta \varepsilon_{3,j}
\]

It leads to a monotonic increase of \(\sigma_3\) with each time step given a dilational \(\Delta \varepsilon_{3,j}\). Accordingly, the updated strain \(\varepsilon_{1,j}\) can be calculated by:

\[
\varepsilon_{1,j} = \varepsilon_{1,j-1} + \frac{v_{j-1}}{2G_{j-1}} \Delta \sigma_{3,j} + \Delta \varepsilon_{1,j}
\]

where the second right-hand term reflects the Poisson effect induced by the increase of \(\sigma_3\) and the third term is the slip-contributed strain increase. It should be noted that, the vertical stress is set to be constant in the normal faulting regime, that is, \(\sigma_i = \sigma_{i,0} > \sigma_3 = \sigma_3\). Assuming an overall homogeneous effective medium, effective elastic moduli \((G_j, v_j)\) can be updated following Hooke’s law with the updated global stress and strain at the boundary at the end of the \(j\)th time step. Then the updated values act as the input in the \((j+1)\)th time step while the iterative process continues.

3. Results

In this section, the stress evolution processes in the deterministic case (featuring homogeneous frictional coefficient, Section 3.1) and the stochastic cases (with heterogeneous distributions of frictional coefficient, Section 3.2) are presented consecutively and compared. The stress uncertainties associated with the stochastic systems are further explored via Monte Carlo simulations, as presented in Section 3.3.

3.1. Model Experimentation and Example for a Deterministic Case

We start experimenting our model by simulating the simple scenario of the stable intra-plate region with normal faulting stress environment \((\sigma_i = \sigma_{i,0} > \sigma_3 = \sigma_3)\). We assign the model with size \(A = 100 \times 100\) (in unit length), and 10,000 fractures with randomly distributed orientation (Figure 2) and equal length \((a = 1\) unit length). All fractures have the same values of frictional coefficient \((0.6)\). The dilatancy factor of fractures \(\beta\) is set as 0.05. The (intact) elastic matrix is assigned shear modulus \(G = 20\) GPa and Poisson’s ratio \(\nu = 0.3\). We apply an effective stress tensor \(\sigma\) \((\sigma_{3,0} = 20\) MPa, and \(\sigma_{i,0} = 100\) MPa) at the model boundary instantaneously at initial time \(t_0\). The large differential stress is selected arbitrarily, well exceeding the nominal frictional limit with empirical frictional coefficient values, so that the stress relaxation process can be initiated and distinctly illustrated. As indicated in Section 3.3, the final stress state of the model depends only on the system characteristics, regardless of the choice of the initial boundary stresses. Since no fracture slip occurs at \(t_0\), \(\varepsilon_{1,0}\) and \(\varepsilon_{3,0}\) at the model boundary are only related to the elastic matrix response, that is, \(\varepsilon_{1,0} = \varepsilon_1^m\) and \(\varepsilon_{3,0} = \varepsilon_3^m\).

An initial evaluation of fracture criticality allows the iterative slip process to begin (Figure 6a). The slip of critical fractures reduces the shear stresses on themselves to their frictional resistance, that is, linear failure line with \(\mu = 0.6\), as shown in Figure 6b. Upon the end of a time step, that is, a cascade of slips, boundary stress \(\sigma_{3,0}\) increases to \(\sigma_{3,1}\) to maintain constant lateral strain. Based on the new stress state, fracture criticality is re-evaluated and slips on new critical fractures further update the stress state (Figures 6c and 6d). With a starting differential stress substantially above the possible equilibrium state, the stress evolution undergoes multiple time steps \((1, 2, \ldots, j, \ldots)\) before it terminates when the shear stress difference \(\Delta \tau\) of the most critical fracture is below 0.01 MPa.

During this iteration, as the vertical stress \(\sigma_1\) is held constant, the horizontal stress \(\sigma_{i,j}\) increases, or the differential stress \((\sigma_1 - \sigma_{3,j})\) relaxes monotonically. Such response is characteristic of the absence of lateral...
tectonic loading. The stress evolution at the model boundary manifests itself as a series of contracting Mohr diagrams with one end fixed (Figure 6e). The model’s final global stress state, or frictional equilibrium, is attained when iterations terminate. The most critical fracture in the system can be identified as the one that keeps slipping until the termination. Retrospectively, the stress state and frictional resistance of the most critical fracture through the iterations can be traced, as illustrated in Figure 6e. Evidently, the fracture keeps slipping as long as its shear stress (red circle) is larger than but converges toward its frictional resistance (green circle on the dashed failure line).

On closer inspection, it is found that the final stress state of the system is manifested as a Mohr circle tangent to the failure line at the convergence point of the most critical fracture. That is to say, the maximum differential stress that the system can sustain is limited only by the frictional strength of fractures, as described by Equation 1. Fracture criticality in this deterministic case thus depends only on fracture orientations. The most critical fracture is thus the one that is most optimally oriented for failure, as indicated by Mohr diagram. This simulation gives a quantitative interpretation on the classic notion of frictional equilibrium (M. D. Zoback & Townend, 2001) and the process that leads to it. It also confirms the very control of the most critical fracture(s) on the global stress state.

Figure 6. Global stress evolution for the deterministic case (uniform frictional coefficient $\mu = 0.6$) in normal faulting stress regime. (a–d) Identification of critical fractures and stress evolution in the first two time steps. Beginning each time step, critical fractures are identified and marked as red dots on the Mohr diagram; ending each time step, slip of all critical fractures takes place and the shear stresses on the fractures fall on their frictional strength (black dashed line). The following time step begins with the updated global stress state (black Mohr diagram). (e) Complete process of stress evolution illustrated by a series of contracting Mohr diagrams. The colormap of the Mohr diagrams is shown with color scaled to time steps. The red circle traces the resolved stress state on the most critical fracture through time steps, while the green circle is its frictional resistance. The subscript of $\sigma_i$ (i.e., 0, 1, 2, ..., final) indicates the corresponding time step.
To demonstrate the critical influence of system heterogeneity, we further simulate three more stochastic cases with inhomogeneous friction property. Each features different frictional coefficient distribution as shown in Figure 1c: (a) normal distribution $N(0.6, 0.05^2)$, (b) normal distribution $N(0.8, 0.06^2)$, and (c) Weibull distribution with scale parameter $\lambda = 0.8$ and shape parameter $k = 10$. As an example, Figure 7 illustrates the first two time steps of stress evolution for the stochastic case of $N(0.6, 0.05^2)$ (see Figures A1 and A2 for the iterations of the other two cases). Similar to the deterministic case, slips of critical fractures also reduce the shear stresses on themselves to their frictional resistance. However, due to heterogeneous distribution of frictional coefficients, the shear stress on the fracture after each time step, or each cascade of slips, varies significantly from one to another. This is represented by the color scale in Figures 7b and 7d. This can be understood in the context of local stress heterogeneity in the system, which is discussed at length in Section 4.3. Unlike the deterministic case wherein there is only one definitive shear failure envelope of certain constant value of $\mu$ to examine the criticality of all fractures, in the stochastic case the criticality of each fracture is evaluated against its own frictional strength, that is, its characteristic $\mu$. Because of $\mu$ heterogeneity, we could not conveniently estimate the equivalent frictional strength/coefficient of the stochastic system without executing all iterations of slip cascades. Since fracture criticality is related to the orientation of the fracture in question, which is spatially random, and also the stress state, which updates over each time step, these uncertainties will result in a highly uncertain evolution of the global stress state. This leads to significant uncertainty in the stochastic system's equivalent frictional strength/coefficient, which is synonymous with the final global stress state in equilibrium.

Figure 8a shows the evolved stress state through all iterations for the stochastic case of $N(0.6, 0.05^2)$. Compared with the deterministic case, the stochastic treatment demands far more iterative time steps (specifically, 18,594 vs. 98) for the model to reach the final global stress state in equilibrium. For comparison, the time when the deterministic case terminates is marked in Figure 8b. It appears that the self-organized slip cascades in the stochastic system prolong the otherwise brief temporal evolution in the deterministic system. The final global stress state is bounded by the equivalent frictional coefficient $\mu = 0.43$, which is neither the mean in the system's $\mu$ distribution nor its lowest value, but rather a value within its lower range. This
suggests that the equivalent frictional strength/coefficient of the system is jointly controlled by the heterogeneity in both \( \mu \) distribution and the randomness of fracture orientations. We expand this argument in Section 3.3.

It is worth noting that the number of fracture slips and the amount of relaxed stress diminish significantly, one time step after another (Figure 8a). This is shown more clearly by the evolving lateral stress \( (\sigma_3) \) in Figure 8b. Normalized by time step, \( \sigma_3 \) increases quickly, corresponding to the relaxing differential stress, although the rate of increase continuously decreases so that it eventually plateaus after several thousands of time steps. Accompanying the stress relaxation, the vertical strain \( (\varepsilon_1) \) gradually accumulates in the

Figure 8. Overview of the temporal evolution of the model with frictional coefficient \( \mu \) distribution following normal distribution \( N(0.6, 0.05^2) \). (a) The stress relaxation due to cascades of fracture slips is illustrated by a series of contracting Mohr diagrams. The final stress state rests on the frictional envelope \( (\mu = 0.43) \), controlled by the frictional strength of the most critical fracture in the system. Red and green circles represent the resolved stress state and frictional resistance on the most critical fracture at each time step, respectively. The subscript of \( \sigma_3 \) (0 and final) indicates the initial and final time steps. (b) Evolution of global stress \( (\sigma_3) \) and strain \( (\varepsilon_1) \), and effective Young's modulus \( (E) \) and effective Poisson's ratio \( (\nu) \) of the fractured system through iterations. Note that the rate of change in these parameters gradually diminishes toward the final stress state. The state upon which the model of the deterministic case terminates is marked. (c) Evolution of slip rate parameter \( (\eta) \), slip rate, and shear stress difference \( (\Delta \tau) \) of the most critical fracture. Background color in (b and c) corresponds to time steps (logarithmic) and the color of the Mohr diagrams in (a).
same manner. In addition, elastic moduli are also plotted against the logarithm of time step. During the relaxation, the model displays decreasing effective Young’s modulus (\(E\)) and increasing effective Poisson’s ratio (\(\nu\)). As illustrated in Figure 8b, the mechanical data first show approximately linear changes with the logarithm of time step, and then followed by horizontal asymptote values. This feature indicates that there is a characteristic time scale for the time-dependent process, which refers to the duration for the relaxation process to reach a relatively steady state. It can be determined by the intersection point of the linearly changing part and the horizontal asymptote, as depicted by the yellow dashed lines in Figure 8b. The linear trends corresponding to the characteristic time could be quantified by the functions with self-similar features, such as power-law functions. In Section 4.2, we discuss more on the self-similar behavior manifested in the time-dependent stress relaxation process.

Similar to the deterministic case, we also show the stress evolution of the most critical fracture in this stochastic case in Figure 8a. Based on this, more information of the most critical fracture can be further found. In Figure 8c, we first show its shear stress difference \(\Delta\tau\) and slip rate parameter \(\eta\) at each time step. Both show drastic rates of change within the characteristic time scale (several thousands of time steps), which become trivial in the following tens of thousands time steps. Further, the shear slip rate of the most critical fracture is plotted by Equation 9a. Since \(\Delta\tau\) is of the order of 0.01–10 MPa, much larger than \(\eta\left(10^{-11}\right)\), it is expected that the shear slip rate has the same changing trend with time step as \(\Delta\tau\).

3.3. Stress Uncertainties Associated With Stochastic Systems

Figure 9 summarizes the final global stress states for the three stochastic cases. For each case, the reference dash line shows the state of stress if the frictional coefficient is invariantly uniform (i.e., the deterministic case) and equals the median of the corresponding stochastic case. Apparently, the state of stress for the stochastic case (solid failure line and Mohr circle in each case) departs significantly from that of the corresponding deterministic case with the median of frictional coefficient distribution (gray dashed line). In other words, the issue of stress uncertainty arises once the system becomes heterogeneous. We found in Figure 9 that the case with normal distribution \(N(0.6, 0.05^2)\) has the smallest discrepancy in the equivalent frictional strength (\(\mu = 0.43\) vs. \(\mu = 0.6\) had this been a deterministic system), while the case with Weibull distribution (\(\lambda = 0.8\) and \(k = 10\)) has the largest discrepancy (\(\mu = 0.34\) vs. the deterministic \(\mu = 0.77\)). It seems that the extent that the final stress state of these stochastic systems departs from the deterministic counterpart is closely associated with its degree of heterogeneity. Similarly, within a stochastic system, the uncertainty of the final stress state also exists, since the fracture orientations and frictional coefficients are randomly generated, albeit under a given distribution.

In order to understand such uncertainty, we conduct Monte Carlo simulations (10,000 times) to characterize each stochastic system. Primarily, we quantify the probability of the frictional coefficient and orientation of the most critical fracture, which, as we demonstrated earlier, controls the final global state of stress of the system. More specifically, for each simulation, the frictional coefficient and orientation of the fractures are randomly generated according to the respective distribution. For all Monte Carlo simulations, the same initial differential stress (\(\sigma_c = \sigma_{s0} = 100\) MPa > \(\sigma_h = \sigma_{s0} = 20\) MPa) in the normal faulting scenario is applied. The most critical fracture in the system is determined by identifying the largest starting shear stress difference \(\Delta\tau\).

With the identified most critical fractures of all Monte Carlo simulations, Figure 10 shows the probability density distribution of their frictional coefficient and orientation. It is found in Figures 10a–10c that the frictional coefficient of the most critical fracture has a distribution differing from case to case, but invariably falls at the lower end of the corresponding distribution (see Figure 1c for comparison). In the Supporting Information (Text S2), we further provide similar results of Monte Carlo simulations on Weibull distribution with different parameters. Repeated Monte Carlo simulations confirm our hypothesis outlined in the single experimentation of the stochastic model. In particular, the case with Weibull distribution has the widest distribution range (0.15–0.45), while the case with normal distribution \(N(0.6, 0.05^2)\) has the narrowest one (0.34–0.48). Obviously, the uncertainty of the frictional coefficient of the most critical fracture is closely associated with its degree of heterogeneity, as we inferred from the single experimentation. Figures 10d–10f show that the most critical fractures are conjugate and favorably oriented, with their normal at around 60° or 120° angle to horizontal. This is generally consistent with what the classic Coulomb criterion prescribes:
for frictional coefficient $\mu \approx 0.6$, the normal of the most critical fracture should be oriented approximately $60^\circ$ or $120^\circ$ with respect to the most compressive stress component. In the stochastic system, due to the variations of individual fracture frictional coefficient $\mu$, a favorably oriented fracture might be associated with high $\mu$ so that a slightly less favorably oriented fracture can become even more critical if it features substantially lower $\mu$. It is this joint requirement that introduces the uncertainty of the most critical fracture and the stochastic system. In short, these results reveal that the most critical fracture is determined jointly by its frictional coefficient and orientation with respect to the global stress tensor.

In addition, an alternative way to represent the marked uncertainty of the final global stress state of the stochastic system is to show all possible frictional failure lines that bounds the final Mohr circles upon equilibrium, as shown in Figure 9. Each pair of failure line and the corresponding Mohr circle represents the result...
out of 10,000 Monte Carlo simulations. Note that the frictional coefficient characterized by each failure line is equivalent to that of the most critical fracture shown in Figure 10. Although the distribution of the failure lines is somewhat wide, the stochastic case still departs clearly from the deterministic case with the median of frictional coefficient distribution. Among them, the Weibull distribution features substantially widest uncertainty of those possible failure lines compared with the other two normal distribution cases. This is attributed to wide spread towards the lower bound in the Weibull distribution (see Figure 1c). As the frictional coefficient distribution reaches lower values, variations in the orientation of those “weak” (or low μ) fractures can subject the most critical fracture to great uncertainty. This results in more possibilities for the equivalent frictional strength or the final global stress state upon equilibrium. Since the vertical stress

Figure 10. Probability density function of (a–c) the frictional coefficient and (d–f) the orientation of the most critical fracture based on Monte Carlo simulations for each frictional coefficient distribution. The color intensity of the histogram in (a–c) scales to its probability density (which also applies to Figure 9).
(σ₁) stays constant, the resulting σ₃ becomes highly uncertain. Figure 11 further summarizes the σ₃ probabilities for all three stochastic cases. Evidenced by such Monte Carlo simulations, the attempts to infer the state of stress in situ assuming an invariant frictional coefficient (i.e., the deterministic use of Equation 1) are subject to uncertainty, therefore are very concerning (Ma et al., 2020).

For all cases including the deterministic and stochastic cases, we also experimented the model in the context of reverse faulting regime (see Text S3 in the Supporting Information for more details). We apply the same initial differential stress conditions but \( \sigma_v = \sigma_{3,0} = 20 \) MPa < \( \sigma_h = \sigma_{1,0} = 100 \) MPa. The observations are consistent with those of the normal faulting scenario. The comparison between the normal and reverse faulting scenarios reveals that the equivalent frictional strength or the final global stress state of the model is independent of the applied boundary conditions, but characteristic of the stochastic nature of the fractures therein.

4. Discussion

4.1. Justifications of the Model Assumptions Associated With the Stress Evolution

Several major assumptions are made in this model to facilitate the implementation of the stress evolution process. While they are deemed reasonable and necessary in the context of this study, they certainly result in simplifications that depart from the real complexities. These affect our interpretation on the modeling results to different extents, which are discussed here.

The first is that frictional slips in the system are regulated in a series of time steps with equal interval. This discretization of time steps allows for explicit modeling of the time-dependent stress evolution by successive quasi-static processes. Despite the computational convenience associated with the time step, the “pseudo” time step is by no means equivalent to real time. Therefore, the interpretation of the temporal evolution should be executed with caution. Nonetheless, the modeled time-dependent stress relaxation and matrix response are practically reasonable and consistent with the first-order observations (as discussed in Section 4.2). Due to the lack of real time in the model, we were unable to experiment boundary conditions with prescribed strain or stress rate, which are necessary to realistically representing the stress evolution in tectonically active regions.

Second, presenting a temporal evolution requires adequate consideration of time-dependent parameters. It is well acknowledged that friction is dependent on slip rate and slip history (Dieterich, 1979; Ruina, 1980). Although we prescribe that the frictional coefficient in the model is independent of time, the effect of the latter is implicitly incorporated to some extent. We show that the global stress evolution and the changes in the effective elastic properties of the system accompany each other. Changes in both lead to the varying displacements of individual fractures, as indicated by Equation 5, and changing slip rates, as indicated by Equations 10 and 11, one time step after another. The treatment of varying slip rates in our model is in fact faithful to the classic formalism for rate- and state-dependent friction, which implies that slip rate depends on the stresses and state variable (Sleep, 2006). It is demonstrated that our implicit treatment is able to reproduce complex time-dependent behavior without explicitly varying the frictional coefficient. By contrast with the assumption of the time-independent friction, one would expect similar simulation results if the rate- and state-dependent friction is explicitly incorporated in this model, since the final steady value that the friction attains after slip is usually not significantly different from the its first-order reference value (e.g., Marone, 1998).

Finally, the use of static elastic fields around individual fractures within each time step circumvents complex dynamic effects (e.g., fracture initiation and propagation, fracture interactions, stress drop, seismic energy release, etc.) at local scales. As stated earlier, this treatment is frequently adopted in the context of effective medium, where relevant validations are abundant (Kachanov, 1992, 1993). It is worth noting that

![Figure 11. Probability density distribution of the final minimum principal stress \( \sigma_3 \) of the models with different frictional coefficient distributions. The possible \( \sigma_3 \) distribution corresponds to the fuzzy Mohr diagrams in Figure 9.](image-url)
processes related to the dynamic complexities can further dissipate the stored energy in the system, leading to more pronounced stress relaxation than the base case we present here. In the absence of these effects, nonetheless, the time-dependent stress evolution could be considered as the most conservative case. Further work is certainly warranted to consider more realistic fracture behavior (e.g., rate-and-state friction, fracture propagation/interaction, etc.) and matrix properties (e.g., viscoelasticity, viscoplasticity, etc.), if specific spatial and temporal scales of stress relaxation are of concern.

Figure 12. Temporal variations of the number of critical fractures for the models with different frictional coefficient distributions. (a–c) The evolving number of critical fractures identified in each time step in the log-log space. (d–f) The cumulative number of critical fractures. The gray data beyond 1,000 time steps are discarded for power-law fitting due to its insignificant statistical importance.
Regarding the assumptions we made in this model, we fully acknowledge that they substantially simplify the complexities during the entire frictional equilibrium process. Admittedly, processes such as fracture interaction, local stress perturbations by the fractures, temporal variations of fracture frictional properties, all are very important as revealed by fault mechanics and earthquake science. However, we consider that these processes are secondary to the first-order issue in this study, that is, the comparison between the deterministic (featuring homogeneous frictional coefficient) and the stochastic case (heterogeneous frictional coefficient). Whether or not processes like fracture interaction and local perturbations are accounted for, the fundamental difference between the deterministic and stochastic cases, as revealed and discussed at length in this paper, will not be affected. Further to the second reason, we consider that it is appropriate to leave out the comparatively local, dynamic, short-term processes in the simulation of a quasi-static evolution of much broader spatial and temporal scale. Three facts are relevant here: (a) the earthquake cycles are much shorter than the time span required to attain the frictional equilibrium that is characteristic of the brittle crust as of today; (b) the strain energy release by individual earthquakes (even the major ones) is insignificant to the energy required to sustain the brittle crust under frictional equilibrium; (c) the local stress perturbation drastically diminishes at distance. These facts are fully honored in the hypothetical model considered in this paper. However, we are cognizant that the additional damage to the system (due to fracture propagation), if introduced to the model, will further relax the differential stress in the system. But again, both the deterministic and stochastic cases will be impacted, therefore not affecting the systematic differences between them. Also note that, the 2D model can be extended to 3D so that the influences of intermediate principal stress on the stress evolution can be explored.

4.2. Temporal Decay in the Number of Local Fracture Slips During Global Stress Relaxation

The interaction between local frictional slips and specific global boundary conditions drives this self-organized stress relaxation process (Figure 4). As local fracture slips continue to dissipate the stored energy, the damage accumulates within the model (e.g., Figure 8b) and the extent of global stress relaxation gradually diminishes. The former is indicated by the varying effective properties as shown in Figure 8b while the latter is manifested by the declining number of fracture slips within each cascade episode. In Figures 12a–12c, we plot the number of frictional slips (or synonymously critical fractures) within each time step for the three stochastic cases, respectively. As expected, the number of critical or slipped fractures drastically decreases one time step after another. Each data series exhibits approximately linear trend with time step in the log-log space. This statistical behavior can be well quantified by the universal scaling law. Specifically, its power-law decay with time (step) is analogous to what is prescribed by the Omori’s law. We adopt the commonly used version of Omori’s law modified by Utsu (1961):

\[
\frac{dN}{dt} = \frac{K}{(c + t)^p}
\]

where \(N\) is the seismic event number, \(t\) is the time that has elapsed, and \(K, c,\) and \(p\) are constants. We then apply Equation 16 to empirically fit each data set in Figures 12a–12c. For each case, the data (black) of the first 1,000 time steps can be well fitted by a straight line with a slope of 1, that is, \(p = 1\). This is quite similar to the statistics of aftershocks following natural main shocks (Reasenberg & Jones, 1989). For the following tens of thousands of time steps, the power-law fit starts to break down. This is because the number of critical fractures in these time steps has dropped below 5, which does not hold any statistical significance. For the reasons we described earlier, those few fractures will remain critical for quite some time steps going forward, due to the fact that extremely small stress relaxation by then is not sufficient to fully release the excessive shear stress difference on them. Hence, such data is not assigned with significance and is discarded for fitting (Figure 12). We also sum up the cumulative number of frictional slips and include it in Figures 12d–12f. The best-fit parameters \(K\) and \(c\) can therefore be calculated.

Although our simulations are aimed at stress relaxation at crustal scale, the statistical characteristics of the fracture slips bear interesting similarity to those of aftershocks following a major earthquake, where the empirical Omori’s law is typically applicable, and those of acoustic emissions in a laboratory relaxation experiment. Although the scale of the crustal rock mass surrounding individual fault zone and/or
the laboratory rock specimen is substantially smaller than the global scale of faulted crust considered in this model, the time-dependent relaxation observed in both scales is noteworthy. The boundary conditions applied to the model are in direct analogy to the rapid increase in stress and strain immediately after a major earthquake, or the loading to deform an intact rock specimen in a relaxation experiment. So, our simulations can also be understood as the relaxation of an excessive differential stress (accompanying the mainshock) with cascades of distributed local slips, which gradually leads to final equilibrium of the fractured system. Such similarities actually benefit from the scale-invariance of the model, which is applicable to any representative rock volume containing discontinuities of much smaller sizes therein. In the context of postseismic relaxation, if each critical fracture slip in the model is interpreted as a measurable seismic event, statistics of the modeled fracture slips and their temporal evolution offer implications for the stress relaxation that takes place in situ. On the other hand, the modeled temporal stress relaxation can be linked to the realistic estimates of frictional slips that actually occur. We notice that, the self-similar behavior validates the existence of a characteristic time scale for the stochastic temporal process, as implied by the yellow dash lines in Figure 8b. It could provide a means to calibrate the pseudo time in our model against real time. We would like to point out that such statistical characteristics are practically absent in the deterministic model. The fact that the statistical characteristics produced in our stochastic model resemble that of the field observations emphasizes the heterogeneity of fractures/faults properties in situ (e.g., Rivera & Kanamori, 2002; cf., Smith & Heaton, 2011), which should be treated so accordingly.

The modeled time-dependent stress relaxation process raises questions about the stage at which the current state of stress is and how it will evolve in the future. Such questions are particularly critical in consideration of seismicity and tectonic loading. This is informative to scientific and industrial applications such as stress estimation, fault slip tendency analysis, and hydraulic fracturing design, and is closely related to whether the currently measured data or designing scheme remain applicable in the future.

4.3. Local Friction Heterogeneity, Global Stress Uncertainty

In our model, we postulate that the shear stress on individual critical fractures differs from simple projection (i.e., Equation 3) of the global stresses due to the heterogeneous distribution of frictional coefficient. Complexities due to stress concentrations at fracture tips or stress perturbations induced by nearby fractures are neglected, for the reasons detailed in Section 2.2. Given homogeneous frictional strength in the deterministic case, the stresses on critical fractures can be well predicted since they are well defined by a straight frictional failure line (Figure 6). It is thus reasonable to expect a deterministic final state of global stress, as we presented before. In marked contrast, local stress uncertainty/heterogeneity emerges in the stochastic cases, as a result of heterogeneous distribution of frictional strength and consequently, cascades of frictional slips. In Figure 7 (and Figures A1 and A2), the stresses on the critical fractures clearly show the increasing uncertainty of local stress field as the frictional coefficient distribution departs further from homogeneity. The local stress uncertainty in our model is therefore completely attributed to the uncertainty of fracture frictional coefficient (i.e., frictional strength). Provided that the interactions of adjacent fractures are permitted, the local stress uncertainty can be even more pronounced.

Local stress (or synonymously frictional strength) heterogeneity also substantially influences global stress uncertainty, indicated by our Monte Carlo simulations (see Figures 9 and 11). As the local frictional strength becomes more heterogeneous, the final state of stress allowable by a stochastic system becomes less predictable. Specifically, in the case of constant $\sigma_s$, the distribution of the final $\sigma_s$ suggests a clear departure from the deterministic estimation (Figure 11). Note that, the normal distribution $\mathcal{N}(0.8, 0.006^2)$ gives the frictional coefficient ranging from 0.6 to 1.0, which is similar to what was revealed by the laboratory friction experiments for most rock types (Byerlee, 1978). If such manner of distribution is characteristic of the Earth's upper crust, we can expect the corresponding global stress to be relatively confined, as shown in Figure 11. However, if the frictional coefficient distribution becomes even more heterogeneous, the global stress will adapt accordingly.

Compilation of local in situ stress data worldwide (e.g., Heidbach et al., 2018; M. L. Zoback, 1992) reveals relatively consistent global stress trend at the regional scale, although with some outliers. The uniformity
of the global stress field above certain scales has also been corroborated by earthquake focal mechanism inversions (e.g., Hardebeck, 2010; Iio et al., 2017; Luttrell & Hardebeck, 2021). Generally, we contend that while the heterogeneity of local stress and strength is ubiquitous, the global stress field in most cases can be seen as relatively uniform, albeit with some uncertainty. The synthetic tests conducted by Rivera and Kanamori (2002) indicated that a (perfectly) uniform global stress field and homogeneous frictional coefficients are mutually exclusive. It appears that our modeled results reasonably reconcile between a largely uniform global stress and a moderately heterogeneous distribution of frictional coefficients. According to our observations, we support the incorporation of some global stress uncertainty to justify the local stress heterogeneities, without compromising the assumption that a generally uniform global stress field prevails.

The local heterogeneous friction can significantly affect the final frictional equilibrium and its uncertainty. This causal relationship is also critical to the fault stability estimation, which requires local stress state and friction input. In the deterministic case, the projection of global stresses onto a fault gives a deterministic local stress field. In other words, the orientation of fault solely governs its criticality. In contrast, poorly oriented faults with low enough frictional coefficient can slip while those more optimally oriented but with higher frictional coefficient can be stable in the stochastic system. If the global stress uncertainty is considered, the fault stability estimation can deviate further from what is predicted by the deterministic treatment. Therefore, the deterministic application of Equation 1 with empirical values of \( \mu \) may lead to erroneous fault criticality estimations. Our results also imply that, ideally, global stress field and its uncertainty, frictional coefficient distribution and fault orientations must be considered simultaneously to identify the criticality of in situ faults. In fact, the uncertainties in stress, along with other geomechanical parameters have been considered in recent studies to analyze the fault slip potential (e.g., Schoenball et al., 2018; Shen et al., 2019; Walsh & Zoback, 2016). Such a probabilistic approach is more informative for decision making than a pure deterministic method, and is consistent with the efforts we present in this study.

5. Conclusions

In this paper, we present a simple quasi-static, stochastic 2D model to simulate the global stress evolution of fractured rock mass due to local frictional slips. The purpose of this model is mainly two-fold: (a) to explicitly incorporate system heterogeneity, and (b) to quantitatively relate the local fracture slips and global stress and strain. Through regulated iterations, we illustrate the stress evolution process as the interplay between cascades of local frictional slips and specific global boundary conditions. The Coulomb frictional failure criterion is adopted to evaluate the fracture criticality. The global stress evolution manifests itself as the decrease in differential stress through a self-organized process, which ultimately leads to frictional equilibrium of the rock mass.

The model is first experimented for a homogeneous system with uniform frictional coefficient, referred to as the deterministic case. The final global stress state of the deterministic case can be well described by the Coulomb frictional failure criterion, consistent with the classic notion of frictional equilibrium. In contrast, the stochastic cases, which feature a heterogeneous distribution of fracture frictional coefficient, are not precisely predicted by the nominal interpretation of the Coulomb frictional failure criterion. More significant time-dependence during the stress relaxation characterizes the stochastic cases. It is found that the more heterogeneous the friction distribution is, the longer time the stress relaxation will take and the greater uncertainty the final state of frictional equilibrium features. In addition, results of Monte Carlo simulations inform that the final frictional equilibrium of a stochastic system is jointly controlled by the orientation and frictional coefficient of all fractures therein. Basically, the lower range of frictional coefficient distribution and optimal orientation determine the most critical fracture(s). Therefore, the stress state allowable by a stochastic system can depart substantially from a deterministic estimation via a predicated, empirical frictional coefficient.

The stochastic treatments illuminate the interactive local-global process that ultimately leads to the stable stress state, extending the classic notion of frictional equilibrium. In a heterogeneous system such as the Earth’s upper crust, the frictional equilibrium can be understood as a dynamic, stochastic process that
spans from the very first frictional slip to the final one possibly allowed by the prevailing in situ differential stress. The deterministic use of the Coulomb criterion to estimate in situ stress and fault stability for an apparently heterogeneous system, although convenient, can incur significant uncertainty, therefore must be treated with caution. Again, we emphasize that the dominant mechanism of stress release in this context is frictional slip. Other mechanisms that may further reduce the differential stress are not considered here. Therefore, the presented stress relaxation can be considered as the most conservative case. Nonetheless, the fundamental differences between the deterministic and stochastic cases, as revealed, will still hold.

Appendix A: Stress Evolution for Two Other Stochastic Cases

Further to Section 3.2, the global stress evolution processes of the other two stochastic cases are illustrated: normal distribution \( N(0.8, 0.06^2) \) in Figure A1 and Weibull distribution \((\lambda = 0.8, k = 10)\) in Figure A2.

**Figure A1.** Global stress evolution for the stochastic case with frictional coefficient \( \mu \) distribution following normal distribution \( N(0.8, 0.06^2) \) in normal faulting stress regime. (a–d) Identification of critical fractures and stress evolution in the first two time steps. Beginning each time step, critical fractures are identified and marked as red dots on the Mohr diagram; ending each time step, slip of all critical fractures takes place and the shear stresses on the fractures fall on their frictional strength (scattered dots colored according to the frictional coefficient). The following time step begins with the updated global stress state. (e) Complete process of stress evolution illustrated by contracting Mohr diagrams. Red circle traces the resolved stress state on the most critical fracture through time steps, while green circle is its frictional resistance. The colormap of the Mohr diagrams is shown with color scaled to time step. The subscript of \( \sigma_3 \) indicates the corresponding time step.
Data Availability Statement

This is a theoretical study and contains no collected data. The MATLAB codes used to produce the results are publicly available at [https://doi.org/10.5905/ethz-1007-346](https://doi.org/10.5905/ethz-1007-346).

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