Magnetohydrodynamic Stability at a Separatrix: Part I

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(Dated: February 20, 2009)

Abstract

The rapid deposition of energy by Edge Localised Modes (ELMs) onto plasma facing components, is a potentially serious issue for large Tokamaks such as ITER and DEMO. The trigger for ELMs is believed to be the ideal Magnetohydrodynamic Peeling-Ballooning instability, but recent numerical calculations have suggested that a plasma equilibrium with an X-point - as is found in all ITER-like Tokamaks, is stable to the Peeling mode. This contrasts with analytical calculations (G. Laval, R. Pellat, J. S. Soule, Phys Fluids, 17, 835, (1974)), that found the Peeling mode to be unstable in cylindrical plasmas with arbitrary cross-sectional shape. However the analytical calculation only applies to a Tokamak plasma in a cylindrical approximation. Here, we re-examine the assumptions made in cylindrical geometry calculations, and generalise the calculation to an arbitrary Tokamak geometry at marginal stability. The resulting equations solely describe the Peeling mode, and are not complicated by coupling to the ballooning mode, for example. We find that stability is determined by the value of a single parameter $\Delta'$ that is the poloidal average of the normalised jump in the radial derivative of the perturbed magnetic field's normal component. We also find that near a separatrix it is possible for the energy principle's $\delta W$ to be negative (that is usually taken to indicate that the mode is unstable, as in the cylindrical theory), but the growth rate to be arbitrarily small.

PACS numbers: 52.55.Tn,52.30.Cv,52.55.Fa,52.35.Py

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I. INTRODUCTION

Thermonuclear fusion requires plasmas with a pressure of at least an atmosphere, and temperatures in excess of 100 million degrees Kelvin. These conditions can be achieved in Tokamaks such as JET[^1], but the plasmas are subject to a number of instabilities, the consequences of which range from benign to structurally damaging. By understanding the instabilities that can occur, they can be avoided or mitigated. A class of instabilities that are only partly understood are Edge Localised Modes (ELMs)[^2]. ELMs can lead to a rapid deposition of energy onto plasma facing components, and this is a potentially serious issue for proposed large tokamak devices such as ITER[^3].

Our present understanding of ELMs is based on the linear ideal Magnetohydrodynamic Peeling-Ballooning instability (Wilson et al[^4], Gimblett et al[^5]), which is thought to trigger ELMs, that subsequently evolve non-linearly. The studies upon which this understanding were based considered Tokamak equilibria with a smoothly shaped magnetic flux-surface at the plasma-vacuum boundary. In contrast, modern Tokamak plasmas have a cross-section in which the outermost flux surface is redirected onto divertor plates, forming a separatrix with a sharp “X-point” where the magnetic topology changes from closed (confined plasma), to open field lines along which plasma can flow to the divertor plates[^1].

The first numerical evidence for a stabilising effect from the separatrix was found by Medvedev et al[^6]. More recently, numerical studies of the Peeling-Ballooning instability in these X-point plasmas (Huysmans[^7]), have found that as the plasma’s outermost flux surface is made increasingly close to that of a separatrix with an X-point, the Peeling mode becomes stabilised. Crucially the stabilisation appeared to happen before the plasma formed a separatrix with an X-point, shaping alone appeared to be sufficient to stabilise the mode. This appears to be contrary to theoretical work by Laval et al[^8], which has indicated that the peeling mode is unstable in cylindrical plasmas with an arbitrarily shaped cross-section. In addition the ELITE code (Wilson et al[^9]) has recently been used to examine Peeling mode stability as the outermost flux surface approaches the separatrix. It was found that although the growth rate reduced in size as the boundary more closely approximated a separatrix, it did so increasingly slowly, and its asymptotic behaviour was uncertain (Saarelma[^10]). To help understand and reconcile these results, here in the first part of this two part paper we re-examine the assumptions made in the derivation of the peeling mode stability criterion.
for a cylinder, and generalise the calculation so that it applies to a toroidal Tokamak plasma.

As with the original studies of the Peeling mode in a cylinder, for simplicity we will firstly consider marginal stability, and generalise the condition for Peeling mode stability in a straight cylinder to a condition for Peeling mode stability in an arbitrary cross-section Tokamak plasma. The resulting equations only describe Peeling mode stability, and are not complicated by coupling to the Ballooning mode instability, for example.

This paper generalises previous analytic calculations in a number of ways. Firstly it applies to axisymmetric toroidal geometries, as opposed to the cylindrical geometry in which the Peeling mode has been extensively studied (for example see Laval et al[8], Lortz[15], Connor et al[14]). It allows for equilibrium poloidal currents at the plasma edge, in addition to the toroidal current that is solely included in previous analytic studies. The skin currents that are induced by a plasma perturbation are related to the difference between the magnetic field in the plasma and the vacuum, and it is found that at marginal stability a plasma perturbation induces a skin current that is parallel and proportional to the equilibrium edge current and proportional to the amplitude of the radial plasma displacement. The complicated-looking plasma-vacuum boundary condition that is usually found in association with the energy principle may be expressed as a simple relationship between the normal components of the plasma and vacuum magnetic fields. This is used to relate the generalised equations for Peeling mode stability at marginal stability to the energy principle, and this allows us to define the Peeling mode in terms of the energy principle’s $\delta W$. With this energy principle for the Peeling mode we can consider the trial function used by Laval et al[8], finding that a single parameter $\Delta'$ determines the sign and magnitude of $\delta W$. Finally the instability’s growth rate is considered.

II. BACKGROUND

Laval et al[8] considered a large aspect ratio ordering that neglects toroidal effects, and also neglects any equilibrium poloidal current at the plasma edge. The work suggests that the Peeling mode will be unstable for a non-zero edge current, regardless of the plasma cross section. Later we will reconsider Peeling mode stability for arbitrary cross section Tokamak plasmas, but firstly we consider some properties of the trial function considered by Laval et al[8].
FIG. 1: A plot of the trial function used by Laval et al.[8], that for the most unstable modes
with $m \sim nq$ has $\xi \sim e^{inq\theta}$, with $\theta$ the usual straight field line angle[11]. In the second part to
this paper we will show that the length $l$ along a flux surface in the poloidal cross section may be
parameterised by $\alpha$, with $\alpha = -\pi..\pi$, as can $\theta$. It is also shown that a physically reasonable model
for $\theta$ has $\theta(\alpha) \simeq \frac{1}{2} \int_{-\pi}^{\alpha} \left( \frac{d\alpha}{2} + \epsilon^2 \right) \frac{1}{4} d\alpha / \oint \left( \frac{d\alpha}{2} + \epsilon^2 \right)^{1/4} d\alpha$. Therefore
by parameterising both $\theta(\alpha)$ and $l(\alpha)$, then the figure plots $\theta$ (vertical axis) versus $l/l(\pi)$ (horizontal
axis), for $\epsilon = 0.001$ and $n = 20$.

Note that for an element of length $dl$ along a flux surface in the poloidal plane,
$$\frac{dl}{B_p} = \frac{J_x B_p}{B_p^2} d\chi = \frac{\nu R^2}{I} d\chi$$ with $J_x$ the Jacobian, $B_p$ the poloidal field, $\nu = \frac{I J_x}{R^2}$ is the
local field-line pitch, and $\chi, \phi, \psi$ an orthogonal toroidal co-ordinate system (for ex-
ample, see Freidberg[12] for details). Thus the poloidal angle used in Laval et al.,
$$2\pi \int_0^1 \frac{dl}{B_p} / \oint \frac{dl}{B_p} = 2\pi \int \nu d\chi' / \oint \nu d\chi' = \theta$$ is the same as the usual straight field line angle[11],
and $q = \frac{1}{2\pi} \oint \nu d\chi = \frac{1}{2\pi} \oint \nu d\chi / \oint \nu d\chi$ is the perturbation they consider has a plasma displacement
$\xi \sim e^{im\theta}$, so if we plot $\xi$ versus the length along a flux surface in the poloidal plane, then near
the separatrix in an X-point equilibrium $\xi$ will oscillate arbitrarily rapidly as we approach
the X-point (see figure 1). The most unstable modes have $m \simeq nq$, so in the figure we plot a
mode with $m \simeq nq$, for which $e^{im\theta} \approx e^{-im\theta}$. Alternately, when $m \ll nq$ or $m \gg nq$, then
the mode localises poloidally in the vicinity of the X-point, and is approximately constant
elsewhere. The rapid oscillation of $\xi$ near the X-point makes it questionable whether it is
physically acceptable, and other terms beyond ideal MHD need to be considered, but it cer-
tainly means that the closer we approach the separatrix the greater the number of Fourier modes required (since \( m \approx nq \)), and the smaller a computer code’s mesh spacing would need to be to represent the mode. Therefore as we approach the separatrix it will be increasingly difficult for a numerical calculation to represent the mode.

III. CYLINDRICAL PLASMAS

Here we outline the derivation of the marginal stability condition for an arbitrarily large aspect ratio (cylindrical) equilibrium with the Tokamak ordering (Freidberg [12], Wesson [1]). We start from the usual force balance equation \( \vec{J} \wedge \vec{B} = \nabla p \), and take the curl of both sides, expanding to give,

\[
0 = \vec{B} \cdot \nabla \vec{J} - \vec{J} \cdot \nabla \vec{B}
\]

(1)

Linearising the equation then gives for the equilibrium quantities

\[
0 = \vec{B}_0 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_0
\]

(2)

and for the perturbed quantities

\[
0 = \vec{B}_0 \cdot \nabla \vec{J}_1 + \vec{B}_1 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_1 - \vec{J}_1 \cdot \nabla \vec{B}_0 + \xi \cdot \nabla \left( \vec{B}_0 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_0 \right)
\]

(3)

Because Eq. 2 holds everywhere, with \( \vec{B}_0 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_0 \) having the constant value of zero, the last term in Eq. 3 that arises from the displacement of the plasma surface by \( \xi \), is zero. In the large aspect ratio approximation (Freidberg [12], Wesson [1]), Eq. 3 further simplifies to

\[
0 = \vec{B}_0 \cdot \nabla \vec{J}_1 + \vec{B}_1 \cdot \nabla \vec{J}_0
\]

(4)

Again, in the large aspect ratio (cylindrical) approximation we may write \( \vec{B}_1 = \vec{e}_z \wedge \nabla \tilde{\psi} \), for which

\[
\vec{J}_1 = \nabla \wedge \left( \vec{e}_z \wedge \nabla \tilde{\psi} \right)
= \vec{e}_z \nabla^2 \tilde{\psi} - \vec{e}_z \cdot \nabla \nabla \tilde{\psi}
\]

(5)

With this same ordering (Freidberg [12], Wesson [1]), \( \vec{J}_0 \) is taken to be parallel to \( \vec{e}_z \). Therefore the \( \vec{e}_z \) component of Eq. 4 gives

\[
0 = \vec{B}_0 \cdot \nabla \left( \vec{J}_1 \cdot \vec{e}_z \right) + \vec{B}_1 \cdot \nabla J_{0z}
\]

(6)
Substituting for $\vec{J}_1$ then gives

$$0 = \vec{B}_0 \cdot \nabla \left( \nabla^2 \tilde{\psi} \right) + \vec{B}_1 \cdot \vec{e}_r \frac{dJ_{0z}}{dr}$$

(7)

Symmetry with respect to both the axial and the poloidal coordinates, means that we need only consider a single mode,

$$\tilde{\psi} = e^{ikz+im\theta}\tilde{\psi}_m(r)$$

(8)

where $r$, $\theta$, and $z$ are the usual cylindrical coordinates, $k$ is a dimensional mode number, and $m$ is a non-dimensional poloidal mode number. Hence we have

$$\nabla \tilde{\psi} = e^{ikz+im\theta} \left\{ \left[ ik\vec{e}_z + \frac{im}{r}\vec{e}_\theta \right] \tilde{\psi}_m(r) + \vec{e}_r \frac{d\tilde{\psi}_m}{dr} \right\}$$

(9)

and because $\vec{B}_1 = \vec{e}_z \wedge \nabla \tilde{\psi}$, then $\vec{B}_1 \cdot \nabla = -\vec{e}_\theta \cdot \nabla \tilde{\psi}$, giving

$$\vec{B}_1 \cdot \vec{e}_r = -e^{ikz+im\theta} \frac{im}{r} \tilde{\psi}_m$$

(10)

We also have that

$$\nabla^2 \tilde{\psi} = \left\{ \left( -k^2 - \frac{m^2}{r^2} \right) \tilde{\psi}_m + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \tilde{\psi}_m}{\partial r} \right\} e^{ikz+im\theta}$$

(11)

Hence Eq. (11) gives

$$0 = \left( ikB_z + im\frac{B_p}{r} \right) \left\{ \left( -k^2 - \frac{m^2}{r^2} \right) \tilde{\psi}_m + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \tilde{\psi}_m}{\partial r} \right\} + B_1 \frac{dJ_{0\phi}}{dr}$$

(12)

$\nabla \cdot \vec{B} = 0$ requires the normal component of $\vec{B}$ to be continuous across the plasma-vacuum surface, that with $\vec{B}_1 = \vec{e}_z \wedge \nabla \tilde{\psi}$ requires $\tilde{\psi}_m$ to be continuous across the surface. The perturbed field from $d\tilde{\psi}_m/dr$ may be discontinuous however. Also $J_{0z} = J_a$ just inside the plasma, but $J_{0z} = 0$ in the vacuum outside the plasma, so if we integrate from an arbitrarily small distance inside the plasma surface to an arbitrarily small distance outside the surface, we get

$$0 = \left( ikB_z + im\frac{B_p}{r} \right) \left[ \left\{ d\tilde{\psi}_m \right\} \right] - \frac{im}{r} \tilde{\psi}_m \left[ \left| J_{0\phi} \right| \right]$$

(13)

where $\left| f \right|$ indicates the difference between $f$ evaluated in the vacuum just outside of the plasma, and $f$ evaluated just inside the plasma ($\left| f \right|$ does not equal the integral of $f$ from just inside to just outside the plasma). Writing $k = -\frac{n}{R}$ and $q = rB_z/ RB_p$, then gives

$$0 = \left( \frac{m - nq}{m} \right) \left[ \left| \frac{r d\tilde{\psi}_m}{dr} \right| \right] + \frac{rJ_a}{B_p}$$

(14)

as the condition for marginal stability.
IV. TOKAMAK PLASMAS

A. Assumptions

Here we re-examine the assumptions made in going from Eq. 12 to Eq. 13. To obtain Eq. 13 we allowed the perturbed fields to be discontinuous, but required the equilibrium fields to be continuous across the plasma-vacuum boundary. Because a discontinuous magnetic field requires a skin current, we will allow perturbed skin currents, but the continuous equilibrium fields imply zero equilibrium skin currents. Therefore we assume that: (a) there are no equilibrium skin currents, but \( \vec{J}_0 \) can be discontinuous at the plasma-vacuum interface, and (b) perturbations to the magnetic field induce surface skin currents. The first of these assumptions means that with the exception of \( \vec{J}_0 \), the equilibrium quantities will be continuous across the plasma-vacuum boundary. So if we integrate along the unit normal to the plasma surface, from a distance \( \epsilon \) just inside the surface to a distance \( \epsilon \) just outside the plasma, then we will get

\[
\int_{l-\epsilon}^{l+\epsilon} f(\eta(l'))dl' = \epsilon f(\eta(l)) \to 0 \quad \text{as} \quad \epsilon \to 0
\]

(15)

where \( \eta(l') \) is a path parameterised by \( l' \), that is parallel to the unit normal to the plasma, and where \( f \) is a continuous equilibrium quantity. We also get

\[
\int_{l-\epsilon}^{l+\epsilon} \frac{df}{dl'}dl' = f(l + \epsilon) - f(l - \epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0
\]

(16)

However, because the equilibrium current is discontinuous at the plasma surface being non-zero within the surface but zero in the vacuum, its derivatives act like delta functions, and hence

\[
\int_{l-\epsilon}^{l+\epsilon} \frac{d\vec{J}_0}{dl'} dl' = \vec{J}_0(l + \epsilon) - \vec{J}_0(l - \epsilon) \equiv \left[ \vec{J}_0 \right]
\]

(17)

and similarly

\[
\int_{l-\epsilon}^{l+\epsilon} f \frac{d\vec{J}_0}{dl'} dl' = f(l) \left[ \vec{J}_0 \right]
\]

(18)

The second of these assumptions, that the perturbation will produce skin currents at the plasma surface (i.e. that the perturbed magnetic field can be discontinuous at the perturbed plasma surface), means that for a perturbed current \( \vec{J}_1 \)

\[
\int_{l-\epsilon}^{l+\epsilon} \vec{J}_1(l')dl' \equiv \frac{\vec{\sigma}}{RB_p} \neq 0
\]

(19)
i.e. the skin current acts like a delta function at the plasma surface. The reason for including the factor of $1/RB_p$ will be clear later. Similarly if $f$ is a continuous function at the plasma surface,

$$
\int_{l-\epsilon}^{l+\epsilon} f \tilde{J}_1(l') dl' = f(l) \frac{\tilde{\sigma}}{RB_p} \tag{20}
$$

A few other remarks are worthwhile. Firstly the terms in Eq. 3 are evaluated at the equilibrium surface positions, but give the value of the perturbed force balance equation at the perturbed surface. Secondly, the perturbed unit normal $\vec{n} = \vec{n}_0 + \vec{n}_1$, with $\vec{n}_1 \sim |\xi| \ll 1$. Hence we have that

$$
\int_{l-\epsilon}^{l+\epsilon} f \frac{d\tilde{J}_0}{d\psi} dl' = \frac{1}{RB_p} \int_{\psi_-}^{\psi_+} f \frac{d\tilde{J}_0}{d\psi} d\psi + O(\xi) \tag{21}
$$

or equivalently,

$$
\int_{l-\epsilon}^{l+\epsilon} f \frac{d\tilde{J}_0}{d\psi} dl' = \frac{1}{RB_p} \int_{\psi_-}^{\psi_+} f \frac{d\tilde{J}_0}{d\psi} d\psi + O(\xi) \tag{22}
$$

where $\int_{\psi_-}^{\psi_+}$ indicates an integral from just inside the last closed flux surface, to just outside it. Similarly because $\nabla \psi \cdot \vec{n} = \nabla \psi \cdot \vec{n}_0 + O(\xi) = RB_p + O(\xi)$, then

$$
\int_{l-\epsilon}^{l+\epsilon} f dl = \frac{1}{RB_p} \int_{\psi_-}^{\psi_+} f d\psi + O(\xi) \tag{23}
$$

and

$$
\tilde{\sigma} = \int_{\psi_-}^{\psi_+} J_1 d\psi + O(\xi^2) \tag{24}
$$

Bearing the above remarks in mind, we now integrate Eq. 3 across the plasma surface, distinguishing perturbed quantities by a subscript of 1 and the equilibrium currents and magnetic field by a subscript of 0, to get

$$
0 = \vec{B}_0 \cdot \nabla \int_{\psi_-}^{\psi_+} \tilde{J}_1 d\psi + \int_{\psi_-}^{\psi_+} \left( \vec{B}_1 \cdot \nabla \psi \right) \frac{\partial \tilde{J}_0}{\partial \psi} d\psi - \left( \int_{\psi_-}^{\psi_+} \tilde{J}_1 d\psi \right) \cdot \nabla \vec{B}_0 + O(\xi^2) \tag{25}
$$

Note that because the terms in Eq. 3 are of order $\xi$ (it describes a linearised perturbation to the plasma), the $O(\xi)$ corrections that arise when integrating along the normal to the surface produce terms of order $\xi^2$ and are neglected. Therefore at leading order we have

$$
0 = \vec{B}_0 \cdot \nabla \tilde{\sigma}_1 + B_1^\psi \left[ \tilde{J}_0 \right] - \tilde{\sigma}_1 \nabla \vec{B}_0 \tag{26}
$$
where $B_1^\psi = \vec{B}_1 \cdot \nabla \psi$. This is the generalised force balance equation for Peeling mode marginal stability, valid for an arbitrary cross-section Tokamak plasma. The last term is a new term that is not present in a circular cross-section cylindrical geometry.

The procedure of integrating across the plasma-vacuum boundary is clearer if we project out the components before integrating across the surface. We have done this as a check, but it is algebraically cumbersome, and obscures the physical arguments that are clearer in the presentation above.

### B. Ampere’s law at the surface

Before proceeding it is worth examining some relationships between the magnetic fields and the skin currents. At the plasma-vacuum interface, $\nabla \cdot \vec{B} = 0$ and Ampere’s law imply that

$$\left[ \vec{n} \cdot \vec{B} \right] = 0$$

(27)

$$\vec{n} \wedge \left( \vec{B}_V - \vec{B} \right) = \frac{\vec{\sigma}}{R B_p}$$

(28)

with $\vec{\sigma} = \int_{\psi_-}^{\psi_+} \vec{J}_1 d\psi$ as before, $\vec{n}$ denotes the unit normal to the surface, and $\vec{B}_V$ the magnetic field in the vacuum. Because we assume zero equilibrium skin currents

$$\frac{\vec{\sigma}_0}{R B_p} = \vec{n}_0 \wedge \left( \vec{B}^V_0 - \vec{B}_0 \right) = 0$$

(29)

and therefore because Eq. 27 gives $\vec{n}_0 \cdot \left( \vec{B}^V_0 - \vec{B}_0 \right) = 0$, it then follows that $\vec{B}^V_0 - \vec{B}_0$ is not parallel to $\vec{n}_0$, and hence the only way to satisfy Eq. 29 is if $\vec{B}^V_0 = \vec{B}_0$.

Considering the lowest order perturbation, we have

$$\frac{\vec{\sigma}_0}{R B_p} = \vec{n}_1 \wedge \left( \vec{B}^V_0 - \vec{B}_0 \right) + \vec{n}_0 \wedge \left( \vec{B}^V_1 - \vec{B}_1 \right) + \xi \cdot \nabla \left( \vec{n}_0 \wedge \left( \vec{B}^V_0 - \vec{B}_0 \right) \right)$$

(30)

which using $\vec{B}^V_0 = \vec{B}_0$ and $\nabla \psi = R B_p \vec{n}_0$, gives

$$\vec{\sigma} = \nabla \psi \wedge \left( \vec{B}^V_1 - \vec{B}_1 \right)$$

(31)

and hence

$$\nabla \psi \cdot \vec{\sigma} = 0$$

(32)
In the $\psi$, $\chi$, $\phi$ coordinate system it is possible to directly evaluate
$\bar{\sigma} = \int_{l-\epsilon}^{l+\epsilon} \nabla \wedge \vec{B}_1 dl'$, giving

$$\bar{\sigma} = R^2 \nabla \phi \left[ \vec{B}_p, \vec{B}_1 \right] - R^2 \vec{B}_p \left[ \nabla \phi, \vec{B}_1 \right]$$

(33)

C. Skin currents at marginal stability

Using $\nabla \psi \cdot \bar{\sigma} = 0$, while projecting out components of Eq. 26 gives

$$0 = \vec{B}_0, \nabla \left( \bar{\sigma}, \vec{B}_p \right) + \vec{B}_1, \nabla \psi \left[ \vec{B}_p, \vec{J}_0 \right] - \bar{\sigma}, \nabla B_p$$

$$0 = \vec{B}_0, \nabla \left( \bar{\sigma}, \nabla \phi \right) + \vec{B}_1, \nabla \psi \left[ \nabla \phi, \vec{J}_0 \right] + 2I \frac{\sigma R}{R^2}$$

(34)

Because $2I \frac{\sigma R}{R^2} = -\sigma, \nabla \left( \frac{l}{R^2} \right) = -\left( \frac{\sigma B_p}{B_p^2} \right) \vec{B}_p, \nabla \left( \frac{l}{R^2} \right)$, then we can rewrite Eq. 34 as

$$0 = \vec{B}_0, \nabla \left( \frac{\sigma B_p}{B_p^2} \right) + \vec{B}_1, \nabla \psi \left[ \vec{B}_p, \vec{J}_0 \right] - \left( \frac{\sigma B_p}{B_p^2} \right) \vec{B}_p, \nabla \left( \frac{l}{R^2} \right)$$

$$0 = \vec{B}_0, \nabla \left( \bar{\sigma}, \nabla \psi \right) + \vec{B}_1, \nabla \psi \left[ \nabla \phi, \vec{J}_0 \right] - \left( \frac{\sigma B_p}{B_p^2} \right) \vec{B}_p, \nabla \left( \frac{l}{R^2} \right)$$

(35)

Next we note that for $\vec{B}_0 = I(\psi) \nabla \phi + \nabla \phi \wedge \nabla \psi$, then

$$\frac{\vec{B}_p, \vec{J}_0}{B_p^2} = -I'$$

$$\nabla \phi, \vec{J}_0 = -p - \frac{I' l}{R^2}$$

(36)

giving

$$\left[ \frac{\vec{B}_p, \vec{J}_0}{B_p^2} \right] = I'_a$$

$$\left[ \nabla \phi, \vec{J}_0 \right] = p_a + \frac{l I'_a}{R^2}$$

(37)

Using $\vec{B}_1, \nabla \psi = \vec{B}_0, \nabla \xi$ and also noting that $I = I(\psi)$, Eqs. 35 now become

$$0 = \vec{B}_0, \nabla \left( \frac{\sigma B_p}{B_p^2} + I'_a \xi \right)$$

$$0 = \vec{B}_0, \nabla \left( \bar{\sigma}, \nabla \phi + \xi \left( p'_a + \frac{l I'_a}{R^2} \right) \right) - \vec{B}_0, \nabla \left( \frac{l}{R^2} \right) \left( \frac{\sigma B_p}{B_p^2} + I'_a \xi \right)$$

(38)

Hence we have that

$$\frac{\sigma B_p}{B_p^2} = -I'_a \xi + f \left( \phi - \int^\chi \nu d\chi' \right)$$

$$\bar{\sigma}, \nabla \phi = - \left( p'_a + \frac{l I'_a}{R^2} \right) \xi + \frac{l}{R^2} f \left( \phi - \int^\chi \nu d\chi' \right)$$

(39)

where $f$ is some function of $\phi - \int^\chi \nu d\chi'$, so that $\vec{B}_0, \nabla f = 0$. Therefore using Eq. 36 to write Eqs. 39 in terms of the equilibrium current $\vec{J}_0$, we have the skin currents at marginal stability given by

$$\bar{\sigma} = \vec{J}_0, \xi \psi + \vec{B}_0 f \left( \phi - \int^\chi \nu d\chi' \right)$$

(40)
If we require that $\sigma = 0$ for $\xi \psi = 0$, then because $\vec{B}_0 \nabla \xi \psi \neq 0$, we simply have that

$$\sigma = \vec{J}_0 \xi \psi$$

(41)
i.e. that the skin current due to a perturbation at marginal stability is equal to the product of the equilibrium current at the edge and the radial displacement of the plasma.

Notice that at this point we have not used any expansion, e.g. in terms of straight field line co-ordinates, and we have not neglected any poloidal dependencies of the equilibrium quantities. We have implicitly assumed the existence of the $\chi, \phi, \psi$ coordinate system, but it is possible to re-express the coordinates in terms of arc length along a flux surface, and the consequent relations expressed in Eqs. 33 for example, remain valid (except at the point of zero size that is the X-point). Hence the results appear valid at the separatrix.

D. Marginal stability

To relate this to the previous cylindrical condition for marginal stability, Eq. 14, we consider $\nabla \cdot \vec{B}$, which may be written

$$\nabla \cdot \vec{B} = \frac{1}{J_\chi} \left( \frac{\partial}{\partial \psi} \left( J_\chi \nabla \psi \cdot \vec{B} \right) + \frac{\partial}{\partial \chi} \left( J_\chi \nabla \chi \cdot \vec{B} \right) + \frac{\partial}{\partial \phi} \left( J_\chi \nabla \phi \cdot \vec{B} \right) \right)$$

(42)

with $J_\chi$ the Jacobian. Then we evaluate $||\nabla \cdot \vec{B}||$, the difference between $\nabla \cdot \vec{B}$ evaluated in the vacuum just outside the plasma and $\nabla \cdot \vec{B}$ evaluated just inside the plasma.

Noting that $J_\chi \nabla \chi \cdot \vec{B}_1 = J_\chi \frac{1}{J_\chi B_p} \nabla \chi \cdot \vec{B}_1 = \frac{\vec{B}_p \cdot \vec{B}_1}{B_p^2}$, then $||\nabla \cdot \vec{B} = 0||$ requires that

$$0 = \left[ \frac{1}{J_\chi} \frac{\partial}{\partial \psi} J_\chi \nabla \psi \cdot \vec{B}_1 \right] + \frac{1}{J_\chi} \frac{\partial}{\partial \chi} \left[ \frac{\vec{B}_p \cdot \vec{B}_1}{B_p^2} \right] + \frac{\partial}{\partial \phi} \left[ \nabla \phi \cdot \vec{B}_1 \right]$$

(43)

Eq. 33 gives

$$-\frac{\bar{\sigma} \vec{B}_p}{R^2 B_p^2} = \left[ \nabla \phi \cdot \vec{B}_1 \right]$$

$$\bar{\sigma} \cdot \nabla \phi = \left[ \vec{B}_p \cdot \vec{B}_1 \right]$$

(44)

So we have

$$0 = \left[ \frac{1}{J_\chi} \frac{\partial}{\partial \psi} J_\chi \nabla \psi \cdot \vec{B}_1 \right] + \frac{1}{J_\chi} \frac{\partial}{\partial \chi} \left( \bar{\sigma} \cdot \nabla \phi \right) + \frac{\partial}{\partial \phi} \left( -\frac{\bar{\sigma} \vec{B}_p}{R^2 B_p^2} \right)$$

(45)

To simplify this we consider a limit of high toroidal mode number $n$ and note that

$$\frac{1}{J_\chi} \frac{\partial}{\partial \chi} = \vec{B}_0 \cdot \nabla - \frac{1}{R^2} \frac{\partial}{\partial \phi}$$. Then we note that $\vec{B}_0 \cdot \nabla$ is of order 1 to prevent a large stabilising
contribution from field-line bending, but \( \frac{\partial}{\partial \phi} \) is of order \( n \). We also have \( \frac{\partial}{\partial \psi} \nabla \psi \vec{B}_1 \sim n \).

Because the system is axisymmetric, we need only consider a single Fourier mode in the toroidal angle, and take \( \vec{\sigma} \sim e^{-in\phi} \). Then we have

\[
0 = \left[ \frac{1}{J_\chi} \frac{\partial}{\partial \psi} J_\chi \nabla \psi \vec{B}_1 \right] + \frac{in}{R^2 B_p^2} \left( I \vec{\sigma} \cdot \nabla \phi + \vec{\sigma} \cdot \vec{B}_p \right) + O(1) \tag{46}
\]

Later in Section V we will find that \( ||\nabla \psi \cdot \vec{B}_1|| = 0 \), so there will be zero contribution from the terms involving \( \partial J_\chi / \partial \psi \). Rearranging the resulting equation leaves

\[
R^2 B_p^2 \left[ \frac{i}{n} \frac{\partial}{\partial \psi} \nabla \psi \vec{B}_1 \right] = \vec{B}_0 \vec{\sigma} + O \left( \frac{1}{n} \right) \tag{47}
\]

Eq. 47 is our generalised criterion for marginal stability to the Peeling mode at high-\( n \), with \( \vec{\sigma} \) given by Eq. 41.

**E. Cylindrical limit**

Using Eq. 41 (\( \vec{\sigma} = \vec{\xi}_0 \vec{J}_0 = \vec{J}_0 \nabla \psi \vec{\xi} \)), then for large aspect ratio (cylindrical) geometry \( \vec{\sigma} = \vec{J}_0 (RB_p) \vec{\xi}_r \) and Eq. 47 becomes

\[
\frac{RB_p}{r B_0} \left[ \frac{i}{n} \frac{db_r}{dr} \right] = \frac{\vec{B}_0 \cdot \vec{J}_0}{B_0} \frac{\vec{\xi}_r}{r} \tag{48}
\]

Noting that \( b_r = \vec{B}_0 \cdot \nabla \vec{\xi}_r \), taking \( \vec{\xi} \sim e^{im\theta - in\phi} \), and using \( \frac{1}{nq} = \frac{1}{m} + \frac{m-nq}{nq} \) with \( m \simeq nq \), this gives

\[
0 = \frac{r J_\parallel}{B_p} + \frac{m-nq}{m} \left[ \frac{\left| \frac{db_r}{dr} \right|}{b_r} \right] \tag{49}
\]

where we have also used that for a cylinder \( q = \frac{r B_0}{R_0 B_p} \). Finally the plasma vacuum boundary conditions (e.g. see Freidberg[12]), of

\[
\vec{n}_0 \vec{B}'_1 = \vec{B}_0 \cdot \nabla \left( \vec{n}_0 \cdot \vec{\xi} \right) - \left( \vec{n}_0 \cdot \vec{\xi} \right) \vec{n}_0 \cdot \nabla \vec{B}_0 \tag{50}
\]

for a cylinder simplify to give \( b'_r = \vec{B}_0 \cdot \nabla \vec{\xi}_r = b_r \). So we regain the usual condition for marginal stability to the Peeling mode in cylindrical geometry, of

\[
\Delta' a + J_\parallel = 0 \tag{51}
\]

with \( \Delta' = \left[ \frac{\left| \frac{db_r}{dr} \right|}{b_r} \right] \) and \( \Delta_a = (1 - \frac{nq}{m}) \).
V. THE PLASMA-VACUUM BOUNDARY CONDITIONS

The question now arises: what are the equivalent terms in the energy principle that correspond to our ordering for the Peeling mode? Before addressing this question, we first show that the plasma-vacuum boundary condition that is usually given in conjunction with the energy principle simply requires continuity of the normal component of the perturbed magnetic field, evaluated with the equilibrium normal to the surface at the equilibrium surface position. This is shown to agree with a simpler and more intuitive derivation that is given later.

We start from the boundary condition usually given in conjunction with the energy principle (Freidberg[12]), of

\[ \vec{n}_0 \cdot \vec{B}^V = \vec{B}_0 \cdot \nabla \left( \vec{n}_0 \cdot \vec{\xi} \right) - \left( \vec{n}_0 \cdot \vec{\xi} \right) \left( \vec{n}_0 . (\vec{n}_0 . \nabla ) \vec{B}_0 \right) \]  

(52)

with \( \vec{n}_0 \) the equilibrium unit normal. We will rewrite the expression in terms of \( \nabla \psi = \vec{n}_0 R B_p \), and then simplify the result. Firstly we multiply by \( R B_p \), and rearrange the equation to get

\[ \nabla \psi . \vec{B}^V = R B_p \vec{B}_0 . \nabla \left( \vec{n}_0 \cdot \vec{\xi} \right) - \left( \vec{n}_0 \cdot \vec{\xi} \right) \vec{B}_0 \cdot \nabla \left( R B_p \right) \]

\[ = \vec{B}_0 . \nabla \left( R B_p \vec{n}_0 . \vec{\xi} \right) - \left( \vec{n}_0 \cdot \vec{\xi} \right) \vec{B}_0 \cdot \nabla \left( R B_p \right) \]

\[ - \frac{\psi \cdot \vec{\xi}}{R^2 B_p^2} \left[ \nabla \psi . \nabla \left( \vec{B}_0 . \nabla \psi \right) - \vec{B}_0 . \nabla \psi . \nabla \psi \nabla \psi \right] \]

\[ = \vec{B}_0 . \nabla \left( \nabla \psi . \vec{\xi} \right) - \frac{\psi \cdot \vec{\xi}}{R^2 B_p^2} \vec{B}_0 . \nabla \left( R B_p \right) - \frac{\psi \cdot \vec{\xi}}{R^2 B_p^2} \left[ -\vec{B}_0 . \nabla \left( \frac{R^2 B_p^2}{2} \right) \right] \]

(53)

Alternately, \( \nabla . \vec{B} = 0 \) at the surface requires \( \vec{n} . \vec{B} = \vec{n} . \vec{B}_V \), with \( \vec{n} = \vec{n}_0 + \vec{n}_1 \), where \( \vec{n}_0 \) is the unit normal to the equilibrium magnetic flux surfaces (and hence also the equilibrium plasma surface) and \( \vec{n}_1 \) is the unit normal perturbation from the equilibrium unit normal \( \vec{n}_0 \) (and hence also the perturbation from the equilibrium unit normal to the plasma surface). Then \( \vec{n} . \vec{B} = \vec{n} . \vec{B}_V \) requires

\[ (\vec{n}_0 + \vec{n}_1) . \left( \vec{B}_0 + \vec{B}_1 \right) + \vec{\xi} . \nabla \left( \vec{n}_0 \cdot \vec{B}_0 \right) = (\vec{n}_0 + \vec{n}_1) . \left( \vec{B}_0^V + \vec{B}_1^V \right) + \vec{\xi} \cdot \nabla \left( \vec{n}_0 \cdot \vec{B}_0^V \right) \]  

(54)

where \( \vec{\xi} \) is the displacement of the plasma from its equilibrium position, and where all quantities are evaluated at their equilibrium positions. Similarly at equilibrium it is required
that
\[ \vec{n}_0 \cdot \vec{B}_0 = \vec{n}_0 \cdot \vec{B}_0^V \] (55)
again with the quantities evaluated at their equilibrium positions. Therefore because
\[ \vec{n}_0 \cdot \vec{B}_0 = \vec{n}_0 \cdot \vec{B}_0^V = 0 \] everywhere, then \( \vec{\xi} \cdot \nabla \left( \vec{n}_0 \cdot \vec{B}_0 \right) = \vec{\xi} \cdot \nabla \left( \vec{n}_0 \cdot \vec{B}_0^V \right) = 0 \), and Eq. 56 may be simplified and rearranged as
\[ \vec{n}_1. \left( \vec{B}_0 - \vec{B}_0^V \right) + \vec{n}_0. \left( \vec{B}_1 - \vec{B}_1^V \right) = 0 \] (56)
Assuming there are no skin currents at equilibrium, then as shown previously in the main text, we have \( \vec{B}_0 = \vec{B}_0^V \), so Eq. 56 becomes
\[ \vec{n}_0. \left( \vec{B}_1 - \vec{B}_1^V \right) = 0 \] (57)
or upon multiplying both sides by \( RB_p \)
\[ \nabla \psi. \left( \vec{B}_1 - \vec{B}_1^V \right) = 0 \] (58)
as above.

VI. THE ENERGY PRINCIPLE

Now we return to the relationship between our equations for marginal stability of the Peeling mode, and the energy principle. We start from the high mode number formulation for \( \delta W = \delta W_F + \delta W_S + \delta W_V \), given in Connor et al[14]. Looking firstly at \( \delta W_S \), then later \( \delta W_V \), we have
\[ \delta W_S = \pi \int d\chi \frac{\xi^*}{n} J_\chi B k || \left[ \frac{R^2 B_p^2}{B^2} \frac{1}{n} \frac{\partial}{\partial \psi} \left( J_\chi B k || \xi \right) - \frac{\vec{B} \cdot \vec{J}}{B^2} \xi \right] \] (59)
with \( J_\chi B k || = i J_\chi \vec{B} \cdot \nabla \). Note that Connor et al[14] took \( \xi \sim e^{+i\phi} \), however here we have \( \xi \sim e^{-i\phi} \), which is reflected by \( n \rightarrow -n \) in the expression for \( \delta W \). We integrate by parts, noting that \( \xi \sim e^{-i\phi} \) but \( \xi^* \sim e^{i\phi} \), and also replace \( J_\chi B k || \) with \( i J_\chi \vec{B} \cdot \nabla \) to give
\[ \delta W_S = -\pi \int J_\chi d\chi \left( \frac{i}{n} \vec{B} \cdot \nabla \xi^* \right) \left[ \frac{R^2 B_p^2}{B^2} \frac{i}{n} \frac{\partial}{\partial \psi} \left( J_\chi \vec{B} \cdot \nabla \xi \right) - \frac{\vec{B} \cdot \vec{J}}{B^2} \xi \right] \] (60)
Using \( \vec{B} \cdot \nabla \xi^* = \nabla \psi. \vec{B}_1^* \), then we get
\[ \delta W_S = \pi \int J_\chi d\chi \left( \frac{i}{n} \nabla \psi. \vec{B}_1^* \right) \left[ -\frac{R^2 B_p^2}{B^2} \frac{i}{n} \frac{1}{J_\chi} \frac{\partial}{\partial \psi} \left( J_\chi \nabla \psi. \vec{B}_1 \right) + \frac{\vec{B} \cdot \vec{J}}{B^2} \xi \right] \] (61)
To obtain the vacuum solution for $\vec{B}_1^V = \nabla V$, we need to solve $\nabla^2 V = 0$, which requires

$$0 = \frac{1}{J_x} \left\{ \frac{\partial}{\partial \psi} (J_x \nabla \psi \cdot \nabla V) + \frac{\partial}{\partial \chi} (J_x \nabla \chi \cdot \nabla V) + \frac{\partial}{\partial \phi} (J_x \nabla \phi \cdot \nabla V) \right\}$$  \hspace{1cm} (62)

Using $\frac{1}{J_x} \frac{\partial}{\partial \chi} = (\vec{B} \cdot \nabla - \frac{I}{R^2} \frac{\partial}{\partial \phi})$, $|\nabla \chi|^2 = \frac{1}{J_x^2 B_p^2}$, and $V \sim e^{-in \phi}$, we can rewrite this as

$$0 = \frac{1}{J_x} \frac{\partial}{\partial \psi} (J_x \nabla \psi \cdot \nabla V) + \left( \vec{B} \cdot \nabla + in \frac{I}{R^2} \right) \left( \frac{1}{B_p^2} \left( \vec{B} \cdot \nabla - in \frac{I}{R^2} \right) V - \frac{n^2}{R^2} \right)$$  \hspace{1cm} (63)

Then using $\vec{B} \cdot \nabla \sim 1$, and taking the high-$n$ limit, gives

$$0 = \frac{1}{J_x} \frac{\partial}{\partial \psi} (J_x \nabla \psi \cdot \nabla V) - \frac{n^2}{2} \frac{I^2}{R^4 B_p^2} V - \frac{n^2}{2} \frac{1}{R^2} V$$  \hspace{1cm} (64)

which using $\vec{B}_1^V = \nabla V$, rearranges to give the result that for $n \gg 1$,

$$V = \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_x} \frac{\partial}{\partial \psi} \left( J_x \nabla \psi \cdot \vec{B}_1^V \right)$$  \hspace{1cm} (65)

The vacuum contribution to $\delta W$ is

$$\delta W_V = \frac{1}{2} \int \vec{d}r \ |\vec{B}_1^V|^2$$  \hspace{1cm} (66)

Using $\nabla^2 V = 0$, $\vec{B}_1 = \nabla V$, and Gauss’ theorem, this may be written as an integral over the plasma surface, with

$$\delta W_V = -\frac{1}{2} \int V \vec{d}S \cdot \nabla V^*$$  \hspace{1cm} (67)

Using $\vec{d}S = \left( \frac{\nabla \psi}{R B_p} \right) (J_x B_p d\chi) (R d\phi)$, and integrating with respect to $\phi$ gives

$$\delta W_V = -\pi \oint J_x d\chi V \nabla \psi \cdot \nabla V^*$$  \hspace{1cm} (68)

Using the expression Eq. 65 for $V$ at high-$n$, and the boundary condition $\nabla \psi \cdot \vec{B}_1^V = \nabla \psi \cdot \vec{B}_1$ (that Section IV shows to be equivalent to the boundary condition that is usually given in formulations of the energy principle), we get

$$\delta W_V = -\pi \oint J_x d\chi \left( \nabla \psi \cdot \vec{B}_1 \right) \left[ \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_x} \frac{\partial}{\partial \psi} \left( J_x \nabla \psi \cdot \vec{B}_1^V \right) \right]$$  \hspace{1cm} (69)

Inserting $-1 = i^2$ into $\delta W_V$, we get

$$\delta W_S + \delta W_V = \pi \oint J_x d\chi \left( \frac{i}{n} \right) \left( \nabla \psi \cdot \vec{B}_1^* \right) \left[ \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_x} \frac{\partial}{\partial \psi} \left( J_x \nabla \psi \cdot \vec{B}_1^V \right) \right] - \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_x} \frac{\partial}{\partial \psi} \left( J_x \nabla \psi \cdot \vec{B}_1 \right)$$  \hspace{1cm} (70)
Because $\nabla \psi. \vec{B}_1^* = \nabla \psi. \vec{B}_1$, the terms involving $\partial J / \partial \psi$ will cancel. Then using the notation $|[f]|$ to denote the difference between $f$ evaluated just outside and just inside the plasma, gives

$$\delta W_S + \delta W_V = \pi \oint J \, d\chi \left( \frac{i}{n} \right) \left( \nabla \psi. \vec{B}_1^* \right) B^2 \left\{ R^2 B_p^2 \left[ \frac{\partial}{\partial \psi} \left( \nabla \psi. \vec{B}_1 \right) \right] \left( \frac{i}{n} \right) \vec{B}. \nabla \xi \psi + \vec{B}. \vec{J} \xi \psi \right\}$$

(71)

The term in {} is exactly Eq. 47 with $\tilde{\sigma} = \tilde{J}_0 \xi \psi$ as given by Eq. 41. Therefore marginal stability of the Peeling mode corresponds (at high-$n$) to taking the plasma’s contribution to $\delta W$, $\delta W_F \equiv 0$, and then solving $\delta W_S + \delta W_V = 0$. This suggests we should define the high-$n$ Peeling mode as a mode (represented by a trial function in this analysis), that allows us to neglect $\delta W_F$ compared to $\delta W_S + \delta W_V$ (by for example being sufficiently localised), and whose subsequent stability is determined by $\delta W_S$ and $\delta W_V$.

### VII. X-POINT PLASMAS

Now we consider the stability of Peeling modes to the trial function considered by Laval et al.[8], that consists of a single Fourier mode with $\xi = \xi_m(\psi)e^{i m \theta - i n \phi}$, where $\theta = \frac{1}{q} \int \nu \, d\chi'$ is the usual straight field-line poloidal coordinate. After taking Eq. 71 and using $\nabla \psi. \vec{B}_1^* = \nabla \psi. \vec{B}_1 = \vec{B}. \nabla \xi \psi$, we have

$$\delta W_S + \delta W_V = \pi \oint J \, d\chi \left( \frac{i}{n} \right) \vec{B}. \nabla \xi \psi \left\{ R^2 B_p^2 \left[ \frac{\partial}{\partial \psi} \left( \nabla \psi. \vec{B}_1 \right) \right] \left( \frac{i}{n} \right) \vec{B}. \nabla \xi \psi + \vec{B}. \vec{J} \xi \psi \right\}$$

(72)

Substituting the trial function into Eq. 72 gives

$$\delta W = -2\pi^2 \left| \xi_m \right|^2 \Delta \left( \Delta \hat{\Delta}' + \hat{J} \right)$$

(73)

where

$$\Delta \equiv \frac{m - n q}{n q}$$

(74)

$$\hat{J} \equiv \frac{1}{2\pi} \oint dl \frac{I}{R^2 B_p} \frac{\vec{J} \vec{B}}{B^2}$$

(75)

$$\hat{\Delta}' \equiv \left[ \frac{1}{2\pi} \oint dl R_0 B_p \frac{I^2}{R^2 B_p^2} \frac{\partial}{\partial \psi} \left( \frac{\nabla \psi. \vec{B}_1}{\nabla \psi. \vec{B}_1} \right) \right]$$

(76)

and with $dl = J \chi B_p d\chi$ an element of arc length in the poloidal cross-section, and $R_0$ a typical measure of the major radius such as it’s average for example. Note that because $\xi_m$ is a
Fourier component of $\tilde{\xi}\nabla \psi \sim \xi(RB_p)$, the dimensions of $|\xi_m|^2/R_0$ are energy. Equation 78 may easily be minimised for $\Delta$ (or equivalently, minimised with respect to choice of toroidal mode number), with

$$\Delta = -\frac{\hat{j}}{2\Delta'}$$

(77)

giving

$$\delta W = \left(\frac{\pi}{2}\right)^2 \frac{|\xi_m|^2}{R_0} \left(\frac{2\hat{j}^2}{\Delta'}\right)$$

(78)

If $\Delta$ is chosen to maximise the growth rate, then a similar but different value will be found. Similarly, there is no reason why there should not be a more unstable mode than the trial function we have considered. However, our primary interest is stability to the trial function that was found to be unstable by Laval et al.[8].

A calculation of $\delta W$ requires the evaluation of $\Delta'$ for a plasma equilibrium with a separatrix. The calculation of $\Delta'$ is the main subject of the second part to this paper. As an introduction to this, we note that for a circular cross section plasma, an estimate for $\Delta'$ may be found by approximating the perturbation to the magnetic field near the edge of the plasma as being the same as for a vacuum, then solving Laplace’s equation both inside and outside the plasma, and matching the solutions at the plasma-vacuum boundary. Then for a circular cross-section $\Delta' = -2m \approx -2nq$.

Observe that for $\vec{J}.\nabla \phi = 0$, $\vec{B}_p.\vec{J} = -I'B^2_p$, for which

$$\hat{j} = \frac{1}{2\pi} \oint \frac{dl}{R^2B_p} \left(-I'\right) \frac{B^2_p}{B^2} \sim 1$$

(79)

and as discussed above we will have $\delta W \to 0$ as $q \to \infty$. However, for $\vec{J}.\nabla \phi \neq 0$, then $\vec{B}.\vec{J} = -Ip' - B^2I'$ and therefore

$$\hat{j} = \frac{1}{2\pi} \oint \frac{dl}{R^2B_p} \frac{(-Ip' - B^2I')}{B^2}$$

$$\approx \left(R_0 \frac{\vec{r}\vec{B}}{R^2B_p}\right) \oint \frac{dl}{R^2B_p}$$

$$= R_0 \left(\frac{\vec{r}\vec{B}}{R^2B_p}\right) q$$

(80)

For which if $\Delta' \sim -nq$ as is the case for a circular cross-section, then $\hat{j}^2/\Delta'$ would be of order $-q$, and $\delta W < 0$, suggesting that the mode would be unstable.

Although the sign of $\delta W$ is usually taken to indicate whether a mode is unstable or not, the growth rate determines how unstable the mode is (i.e. how rapidly it develops). For example if our trial function $\xi_m(\psi)e^{im\theta} = \xi_\psi = \nabla \psi.\vec{\xi}$ had been $\xi_m(\psi)e^{im\theta} = \nabla \psi.\vec{\xi}/RB_p$ so
that it had dimensions of length as opposed to dimensions of length times \( RB_p \), then we would no longer have \( \hat{J} \sim q \), despite our model only depending on the poloidal structure of the mode. The dependence of \( \delta W \) on the normalisation of the plasma perturbation does not affect the calculation of the growth rate however, for which the consequences of the normalisation of the plasma perturbation will cancel. The growth rate is discussed later.

Are there any reasons why a computer code might fail to find an unstable mode? One possibility is that the need for \( m \sim nq \) will require very high poloidal mode numbers as \( q \to \infty \), and this could potentially prevent a numerical code from seeing the instability. Also important, is the need to consider the most unstable mode. Minimising \( \delta W \) with respect to the toroidal mode number gave \( \Delta = -\hat{J}/2\Delta' \). If \( \Delta' \simeq -2nq \) and \( \hat{J} \simeq qR_0(\vec{J}.\vec{B})/B^2 \) (for \( \vec{J}.\nabla \phi \neq 0 \)), this would require

\[
\Delta \simeq -\frac{\hat{J}}{2\Delta'} = R_0\frac{\vec{J}.\vec{B}}{B^2} \left( \frac{1}{4n} \right)
\]

that is independent of \( q \) and the poloidal mode number. A final possibility is that \( \Delta' \) might diverge more rapidly than \( \hat{J} \) as we approach the separatrix. The second part to this paper calculates \( \Delta' \) analytically, thereby avoiding the numerical problems associated with an X-point.

Another, less obvious reason why an unstable mode might not be found in computer calculations is that despite \( \delta W < 0 \) indicating that it is energetically favourable for the mode to be unstable, the growth rate can still be vanishingly small. This possibility is explored next.

VIII. THE GROWTH RATE

So far we have only considered \( \delta W \), because its sign is usually presumed to be sufficient to indicate whether a mode is stable or not. The growth rate \( \gamma \) for a mode with \( \xi \sim e^{\gamma t} \) is obtained from \( \gamma^2 = -\delta W/\frac{1}{2} \int \rho_0|\xi|^2d\vec{r} \), where \( \frac{1}{2} \int \rho_0|\xi|^2d\vec{r} \) is the kinetic energy term. Next we will estimate the kinetic energy term, so as to estimate the growth rate. The surprising result that we will find is that even if \( \delta W < 0 \), indicating it is energetically favourable for an instability, the kinetic energy term can diverge so strongly that although it may be energetically favourable for a mode to be unstable, its growth rate is vanishingly small. An alternative complementary calculation to the one given below, with the same conclusions, is
given in Appendix A.

To estimate $\int \rho_0 |\vec{\xi}|^2 d\vec{r}$, we write

$$\vec{\xi} = \xi_\psi \nabla_\psi + \xi_B \frac{\vec{B}}{B} + \xi_\perp \frac{\vec{B} \wedge \nabla_\psi}{R^2 B_p^2 B^2}$$

(82)

for which

$$|\vec{\xi}|^2 = \frac{|\xi_\psi|^2}{R^2 B_p^2} + \frac{|\xi_B|^2}{B^2} + \frac{|\xi_\perp|^2}{R^2 B_p^2 B^2}$$

(83)

It is convenient to write $\vec{\xi}$ in the form of Eq. 82 so that we can use the results from a high-$n$ ordering (e.g. see Webster and Wilson [19]), that gives

$$\xi_\perp = \frac{i}{n} \nabla_\psi \cdot \nabla_\psi = \frac{i}{n} R^2 B_p^2 \frac{\partial \xi_\psi}{\partial \psi}$$

(84)

and

$$\vec{B} \cdot \nabla_\psi + \frac{\vec{B} \cdot \nabla B^2}{B^2} = \xi_\psi \frac{\partial}{\partial \psi} (2p + B^2) + \frac{I \xi_\perp}{R^2 B_p^2 B^2} \vec{B} \cdot \nabla B^2$$

(85)

Before continuing further we make some observations on the high-$n$ ordering that for $\nabla . \vec{\xi} = 0$ usually leads to $\xi_\perp = \frac{i}{n} R^2 B_p^2 \frac{\partial \xi_\psi}{\partial \psi}$. This analysis is used in the derivation of $\delta W$ used in Section VI onwards, and to derive the equations solved by ELITE [9]. The ordering implicitly assumes that $\frac{\partial \xi_\psi}{\partial \psi} \gg \frac{\xi_\psi}{J_x}$, which is the case in the plasma core where $\frac{1}{n} \frac{\partial J_x}{\partial \psi} \sim 1$, because $\frac{\partial \xi_\psi}{\partial \psi} \sim n \gg 1$. Whereas a sufficiently large $n$ can always be found to ensure $\frac{1}{n} \frac{\partial J_x}{\partial \psi} \ll 1$, and terms of this type will often be negligible order one contributions anyhow, future calculations would be improved by including them. For example $\xi_\perp$ would then become $\xi_\perp = \frac{i}{n} R^2 B_p^2 \frac{\partial (J_x \xi_\psi)}{\partial \psi}$. For the present we will continue to use the ordering employed by Connor [14], that is also used to derive the equations solved by ELITE.

Because the trial function that we consider consists of a single Fourier mode, Eq. 85 may be solved for $\xi_B$, with

$$\xi_B = \frac{\xi_\psi \frac{\partial}{\partial \psi} (2p + B^2) + \frac{I \vec{B} \cdot \nabla B^2}{B^2} \left( \frac{1}{n} \frac{\partial \xi_\psi}{\partial \psi} \right)}{\frac{1}{R^2} (im - inq) - \frac{\vec{B} \cdot \nabla B^2}{B^2}}$$

(86)

giving

$$\frac{|\xi_B|^2}{B^2} = \frac{1}{B^2} \frac{|\xi_\psi|^2 \left( \frac{\partial}{\partial \psi} (2p + B^2) \right)^2 + \frac{1}{n} \frac{\partial \xi_\psi}{\partial \psi}^2 I^2 \left( \frac{\vec{B} \cdot \nabla B^2}{B^2} \right)^2}{\frac{R^2}{n^2} n^2 \Delta^2 + \left( \vec{B} \cdot \nabla B^2 B^2 \right)^2}$$

(87)

with $\Delta = (m - nq)/nq$ as before. Using Eq. 84 for $\xi_\perp$, and substituting this and 87 into Eq.
\[ |\vec{\xi}|^2 = |\xi_\psi|^2 \left\{ \frac{R^2}{B_p^2} \left[ \frac{\rho}{B_0} (2p + B^2) \right] \right\}^2 \left/ \left[ \frac{R^2}{B_p^2} n^2 \Delta^2 + \left( \frac{\vec{B} \cdot \nabla B^2}{B_p^2} \right)^2 \right] \right\} + \frac{1}{n^2} \left| \frac{\partial \xi_\psi}{\partial \psi} \right|^2 \left\{ \frac{R^2}{B_p^2} \left( \frac{\vec{B} \cdot \nabla B^2}{B_p^2} \right)^2 + \frac{R^2}{B_p^2} \right\} \left/ \left[ \frac{R^2}{B_p^2} n^2 \Delta^2 + \left( \frac{\vec{B} \cdot \nabla B^2}{B_p^2} \right)^2 \right] \right\} \] (88)

Noting that \( R^2 \frac{\vec{B} \cdot \nabla B^2}{B_p^2} \sim B_p^2 \) and \( \frac{R^2}{B_p^2} B_p^2 \sim B_p^2 \), whereas \( \left( \frac{R^2}{B_p^2} n^2 \Delta^2 + \left( \frac{\vec{B} \cdot \nabla B^2}{B_p^2} \right)^2 \right) \sim B_p^2 \) as \( B_p \to 0 \), with \( \Delta \sim 1/n \) as found previously, then we find

\[ |\vec{\xi}|^2 \sim |\xi_\psi|^2 \frac{R^2 B_p^2}{B_p^2} \frac{1}{n^2} \left| \frac{\partial \xi_\psi}{\partial \psi} \right|^2 \] (89)

In other words, we may neglect the term in \( |\xi_B|^2 \) compared with the \( |\xi_\psi|^2 \) and \( |\xi_\perp|^2 \) terms. We will consider each of these terms in turn. Firstly

\[ \int \rho_0 \frac{|\xi_\psi|^2}{R^2 B_p^2} \, dR = 2\pi \int \frac{dt}{B_p} d\psi \rho_0 \frac{|\xi_\psi|^2}{R^2 B_p^2} \]
\[ \sim \left. \rho_0 \right|_{\psi_s} \int |\xi_\psi|^2 d\psi \int \frac{dt}{B_p} \]
\[ \lesssim \left. \rho_0 \right|_{\psi_s} \int q' |\xi_\psi|^2 d\psi \] (90)

with \( \psi = \psi_s \) at the plasma surface, and where we used [18]

\[ q'(\psi) \simeq \frac{1}{2\pi} \oint \frac{d\chi}{B_p} \left[ \nabla \phi \vec{J} - \frac{\partial B_p^2}{\partial \psi} \right] d\chi \]
\[ \simeq \frac{\nabla \phi \vec{J}}{2\pi} \oint \frac{dt}{B_p} \] (91)

with \( dl = (J_B d\chi) \), and at large aspect ratio \( \nabla \phi \vec{J} \) is approximately a function of the poloidal magnetic flux. Although we have not considered the radial structure of the mode in this paper, to estimate these terms we will adopt the ansatz that near the plasma’s edge \( \xi_m \) can be approximated by a power law, with

\[ \xi_m = \xi_0 \left( \frac{\psi_a - \psi_s}{\psi_a - \psi} \right)^p \] (92)

where \( \psi = \psi_a \) at the separatrix, and \( \psi = \psi_s \) at the plasma surface. This is consistent with studies that do consider a mode’s radial structure [14, 15]. We also use the result found here [16] and elsewhere [20], that for a conventional X-point (as opposed to the X-point produced by a “snowflake” divertor [20]), near a separatrix we have

\[ q \simeq -q_0 \ln \left( \frac{\psi_a - \psi}{\psi_a} \right) \] (93)
for some constant $q_0 \sim 1$. Under these assumptions

$$\int q' |\xi_\psi|^2 d\psi \sim |\xi|^2$$

(94)
giving

$$\int \frac{q'}{B_\psi} \frac{|\xi|^2}{R^2 B_p^2 B^2} \sim \frac{q_0 r}{B(B_p)} |\xi|^2$$

(95)

where we took $I \nabla \phi \tilde{J} \sim B(B_p)/r$, with $r$ a measure of the plasma radius.

Next we consider,

$$\int \frac{q'}{B_\psi} \frac{|\xi|^2}{R^2 B_p^2 B^2} \sim \frac{\rho_0 \xi}{B(B_p)} |\xi|^2$$

(96)

For the trial function $\xi_\psi = \xi_m(\psi) e^{im\theta}$ with $\theta = \frac{1}{q} \int^\chi \nu d\chi'$, we have

$$\int \frac{q'}{B_\psi} \frac{|\xi_\psi|^2}{\partial \xi_\psi} d\psi \sim \int \left( \frac{d\xi_m}{d\psi} \right)^2 \left| \xi_m \right|^2 \left| \frac{\partial \theta}{\partial \psi} \right|^2 m^2 \right) d\psi$$

(97)

the last expression ignores terms in $\frac{d\xi_m}{d\psi}$ or $\frac{\partial \theta}{\partial \psi}$ because either $\left| \frac{d\xi_m}{d\psi} \right|^2$ or $\left| \xi_m \right|^2 \left| \frac{\partial \theta}{\partial \psi} \right|^2 m^2$. Taking $\xi_m$ as in Eq. (92) gives

$$\int \left| \frac{d\xi_m}{d\psi} \right|^2 d\psi \sim \left| \xi_0^2 q' \right|_{\psi=\psi_s}$$

(98)

Similarly, because $\int \nu d\chi = 2\pi q$ and $\theta = \frac{1}{q} \int^\chi \nu d\chi'$, we have

$$\left| \frac{\partial \theta}{\partial \psi} \right|^2 = \left[ \frac{-q'}{q} + \frac{1}{q} \int^\chi \frac{\partial \nu}{\partial \psi} d\chi' \right]^2 \sim \left( \frac{q'}{q} \right)^2$$

(99)

that with $m \simeq nq$ gives

$$\int m^2 \left| \xi_m \right|^2 \left| \frac{\partial \theta}{\partial \psi} \right|^2 \sim n^2 \int q'^2 \left| \xi_m \right|^2 \sim n^2\left| \xi_0^2 q' \right|_{\psi=\psi_s}$$

(100)

Therefore, because $\psi_s \sim \int R^2 B_p dl / \int dl$ we have

$$\int \frac{q'}{B_\psi} \frac{|\xi|^2}{R^2 B_p^2 B^2} \sim \frac{\rho_0 \frac{q}{B} \int dl}{B^2} (\psi_s q') \left| \xi_\psi \right|^2$$

(101)

Finally, taking $\nabla \phi \tilde{J} \neq 0$ and $\Delta' \simeq -2nq$ ($\Delta'$ is calculated in the second part to this paper), then gives

$$\gamma^2 = \frac{-\delta W}{\int d\psi r_0 |\xi|^2} \sim \gamma \left( \frac{R_0 \psi s q'}{B^2} \right)^2$$

(102)
with $\gamma_A \equiv B^2 / (\rho_0 R \oint dl)$. Because $q' \to \infty$ more rapidly than $q$ as we approach the separatrix, then for an outermost flux surface that is made increasingly close to that of a separatrix with $\psi_s \to \psi_a$, we have that $\gamma^2 \to 0$ and

$$\ln \left( \frac{\gamma}{\gamma_A} \right) = -\frac{1}{2} \ln \left( \frac{\psi_s q'}{q} \right) \quad (103)$$

with $\gamma_A$ the Alfven frequency, indicating that the growth rate $\gamma \to 0$ as $q' \to \infty$.

IX. SUMMARY

This paper re-explores the stability of the Peeling mode for toroidal Tokamak geometry. It starts from a simple approach to Peeling mode stability at marginal stability in cylindrical geometry, then generalises this to toroidal Tokamak equilibrium. In the process of doing so we find a number of interesting results, namely

1. At marginal stability, a plasma perturbation induces a skin current that is parallel and proportional to the equilibrium current at the edge, and proportional to the radial plasma displacement.

2. For zero equilibrium skin current the usual plasma-vacuum boundary conditions (Freidberg[12]), are identical to the requirement that $\vec{n}_0.\vec{B}_1 = \vec{n}_0.\vec{B}_V$, with the quantities evaluated at the equilibrium position.

3. The equilibrium conditions (force balance) for the Peeling mode at marginal stability and high toroidal mode number $n$, are identical to requiring $\delta W_S + \delta W_V = 0$, where $\delta W_S$ and $\delta W_V$ are the surface and vacuum contributions to the energy principle’s $\delta W = \delta W_F + \delta W_S + \delta W_V$, with $\delta W_F$ the plasma’s contribution to the energy principle (Freidberg[12]).

This suggests the Peeling mode be defined as a mode for which $\delta W_F \ll \delta W_S + \delta W_V$. For the trial function used by Laval et al[8], that consisted of a single Fourier mode in straight field line co-ordinates, we find that the most unstable choice of $\Delta$ gives $\delta W = \left( \frac{\pi}{2} \right)^2 \frac{|s_m|^2}{R_0} \left( \frac{\Delta'}{\Delta} \right)$. To evaluate $\delta W$ for this model, it is necessary to know $\Delta'$ for a plasma cross-section with a separatrix and X-point at the plasma-vacuum boundary. Doing this without making the usual approximations (i.e. with a discretisation of space), as are usually made in numerical calculations, is the subject of the second part to this paper[16].
Finally we considered the growth rate, and found that even with $\delta W < 0$, the growth rate can be vanishingly small. This is because the kinetic energy term was found to diverge like $q'$ as the outermost flux surface becomes increasingly close to a separatrix. When this divergence is sufficiently rapid (as would be the case for $\Delta' \simeq -2nq$), then $\ln(\gamma/\gamma_A)$ asymptotes to $\ln(\gamma/\gamma_A) = -\frac{1}{2} \ln(\psi_s q'/q)$, a result that may be compared with those from codes such as ELITE.

Acknowledgments

Thanks to Tim Hender for reading and commenting on an earlier draft of this paper. This work was jointly funded by the United Kingdom Engineering and Physical Sciences Research Council, and by the European Community under the contract of Association between EURATOM and UKAEA. The views and opinions expressed herein do not necessarily reflect those of the European commission.

APPENDIX A: ALTERNATIVE DERIVATION FOR THE GROWTH RATE

Here we provide an alternative derivation for the growth rate, to that given in Section VIII. In the following we will,

1. Continue to use the high-$n$ ordering of Connor et al[14], for which $\nabla \vec{\xi} = 0$ requires that

$$\xi_\perp = \frac{i}{n} R^2 B^2 \frac{\partial}{\partial \psi} \xi_\psi$$

(A1)

2. We will use the arguments from Section VIII that lead us to expect that $\partial \xi_\psi/\partial \psi \sim \frac{mq'}{q} |\xi_m| \sim nq' |\xi_m|$, for the trial function of Laval et al[8] with $\xi_\psi = \xi_m(\psi)e^{im\theta}$.

3. As in Section VIII we will continue to assume that near the separatrix,

$$q \simeq -q_0 \ln \left( \frac{\psi_a - \psi}{\psi_a} \right)$$

(A2)

for some constant $q_0 \sim 1$, with $\psi_a$ the value of $\psi$ at the separatrix. We will also continue to assume that near the separatrix we can approximate $|\xi_m(\psi)|$ as a power law, with

$$\xi_m = \xi_0 \left( \frac{\psi_a - \psi_s}{\psi_a - \psi} \right)^p$$

(A3)

where $\psi_s < \psi_a$ is the value of $\psi$ at the plasma-vacuum surface.
With the assumptions of \(1, 2, \) and \(3\) we require that
\[
\xi_\perp = \frac{i}{n} R^2 B^2_p \frac{\partial \xi_m}{\partial \psi} \\
\sim i R^2 B^2_p q' |\xi_m| \\
\sim i R^2 B^2_p \frac{q_0}{\psi_a - \psi} \xi_0 (\psi_a - \psi_s)^p
\] (A4)

However, we must have \(\xi_\perp \ll 1\) as \(\psi \to \psi_s\), and therefore we require \(\xi_0 = \hat{\xi}_0 (\psi_a - \psi_s)\) with \(\hat{\xi}_0 \ll 1\) a constant, so that as \(\psi \to \psi_a\) we have \(\xi_\perp \sim \hat{\xi}_0 \ll 1\). Therefore we also have as a consequence that
\[
\xi_\psi \sim \hat{\xi}_0 (\psi_a - \psi_s) (\psi_a - \psi_s)^p \\
\to \hat{\xi}_0 (\psi_a - \psi_s) \quad \text{as} \quad \psi \to \psi_a
\] (A5)

In the limit where the plasma surface tends to a separatrix, with \(\psi_s \to \psi_a\), we then must have \(\xi_\psi \sim \hat{\xi}_0 (\psi_a - \psi_s) \to 0\). Therefore \(\delta W\), for which \(\xi_m\) is evaluated at \(\psi = \psi_s\), has
\[
\delta W \sim -\frac{|\hat{\xi}_0|^2}{R_0} \left( \frac{R_0 J \cdot B}{B^2} \right) \left( \frac{q}{n} \right) \left( \frac{\psi_a - \psi_s}{\psi_a} \right)^2 \\
\to 0 \quad \text{as} \quad \psi_s \to \psi_a
\] (A6)

Next we consider the growth rate.

Using arguments from Section VIII, we expect
\[
\int \hat{\xi} \hat{\xi}^* \sim \int \frac{|\xi_\perp|^2}{R^2 B^2_p B^2}
\] (A7)

Now with \(\xi_\perp\) given by Eq. A4 we have
\[
\int \hat{\xi} \hat{\xi}^* \sim \int \frac{d\psi d\omega d\phi}{B_p} \frac{R^2 B^2_p B^2}{(\psi_a - \psi_s)(\psi_a - \psi)} \frac{(\psi_a - \psi_s)^p}{\psi_a} \\
\sim \left[ \hat{\xi}_0 \right]^2 \left( \frac{R_0 J \cdot B}{B^2} \right) \left( \frac{q}{n} \right) \int d\psi \frac{(\psi_a - \psi_s)}{\psi_a} \\
\sim \left[ \hat{\xi}_0 \right]^2 \left( \frac{\delta B}{B^2} \right) \left( \frac{\psi_a - \psi_s}{\psi_a} \right)
\] (A8)

where \(\langle \rangle\) denotes a poloidal average, and in the last line we used \(\psi_a \sim \langle B_p \rangle R^2\). Hence using Eqs. A6 and A8 we find
\[
\gamma^2 = \frac{\delta W}{\int d\psi p_0 |\xi|^2} \\
\sim -\frac{|\hat{\xi}_0|^2}{R_0} \left( \frac{R_0 J \cdot B}{B^2} \right) \left( \frac{q}{n} \right) \left( \frac{\psi_a - \psi_s}{\psi_a} \right)^2 / \left[ \hat{\xi}_0 \right]^2 \left( \frac{\delta B}{B^2} \right) \left( \frac{\psi_a - \psi_s}{\psi_a} \right) \\
\sim \gamma^2 \left( \frac{R_0 J \cdot B}{B^2} \right) \left( \frac{\psi_a - \psi_s}{\psi_a} \right) \left( \frac{q}{n} \right)
\] (A9)
with \( \gamma_A^2 = \langle B^2 \rangle / \rho_0 R_0 \oint dl \). Therefore as the outermost plasma surface more closely approximates a separatrix with \( \psi_s \rightarrow \psi_a \), we have that \( \gamma / \gamma_A \rightarrow 0 \). Note that because we have taken \( q \sim q_0 \ln \left( \frac{\psi_a - \psi}{\psi_a} \right) \), then \( q' \sim q_0 / (\psi_a - \psi) \), and hence \( (\gamma / \gamma_A)^2 \sim 1 / q' \) as found Section VIII.

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