M-theory branes and their interactions

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Abstract

In recent years there has been some progress in understanding how one might model the interactions of branes in M-theory despite not having a fundamental perturbative description. The goal of this review is to describe different approaches to M-theory branes and their interactions. This includes: a review of M-theory branes themselves and their properties; brane interactions; the self-dual string and its properties; the role of anomalies in learning about brane systems; the recent work of Basu and Harvey with subsequent developments; and how these complementary approaches might fit together.

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1 Introduction

In 1995 it was realised that the strong coupling limit of the IIA string is an eleven dimensional theory whose low energy limit is eleven dimensional supergravity [1,2]. This unknown mysterious eleven dimensional theory became known as M-theory. Considering that it was a string theory at strong coupling, the properties of M-theory were somewhat surprising. The critical dimension was eleven not ten. The extended objects were no longer strings but membranes and five-branes. All the different string theories were different compactification limits of this single theory, as such M-theory unified string theories. The five different versions of string theory were just M-theory expanded around different vacua. This M-theory web then explained the nonperturbative dualities that had been conjectured in string theory some years previously. Most elegantly perhaps, the IIB SL(2,Z) strong weak duality was a simple consequence of M-theory diffeomorphism invariance [3]. Work took place in understanding how the branes in M-theory fitted in with the branes of string theory and a dictionary was uncovered between string theory objects and their M-theory counterparts. This was very successful and yet there were fundamental holes in our understanding of M-theory. Of course, M-theory was not described by some fundamental description; only a definition of the low energy limit was really known. A true formulation of M-theory away from the low energy limit is still a far away dream that will not be discussed here.

Yet even in the low energy limit, strong evidence emerged that there was more to M-theory than eleven dimensional supergravity. This evidence came from examining the branes in M-theory.

String theory was revolutionised by the discovery of D-branes [4,5]. The understanding of these nonperturbative objects allowed a deep connection to be uncovered between non-Abelian gauge theories and string theory. This would result in the Maldacena correspondence [6] where a fascinating duality between gauge theories and string/gravitational theories has resulted in untold theoretical riches. The origin of the non-Abelian degrees of freedom came from the open strings extending
between different D-branes becoming massless when the D-branes coincided. This was a fundamentally stringy effect that was not apparent (at least not immediately) from analysing the D-branes as supergravity solutions. In fact these massless non-Abelian degrees of freedom were apparent in the solutions in the following indirect ways. The thermodynamic properties of the brane solutions were analysed a la Hawking and their entropies calculated [7]. N coincident D-branes were shown to have an entropy that scaled as $N^2$, consistent with the number of non-Abelian, U(N), gauge degrees of freedom. The low energy scattering cross section was calculated and again an $N^2$ scaling was found [8]. Finally anomaly cancellation arguments also revealed the $N^2$ scaling of degrees of freedom. All the above could be calculated without recourse to the underlying string theory and yet the degrees of freedom themselves were of stringy origin.

These techniques were available to analyse the branes in M-theory. The supergravity solutions were known so one followed the D-brane example and calculated the thermodynamics or cross section scattering or in the case of the five-brane, its R-symmetry anomaly. All these results pointed to the same answer. The theory describing N coincident membranes has an $N^{\frac{3}{2}}$ scaling of its world volume degrees of freedom and the theory describing N coincident five-branes an $N^3$ scaling. This teased the question, what are these degrees of freedom?

For the five-brane there was the guess that the origin must be in open membranes. The membrane becomes a string after compactification and the origin of the non-Abelian degrees of freedom on D-branes were open strings thus these open strings were actually open membranes when lifted to M-theory [9, 10].

With this motivation amongst others there was a study of how membranes may end on five-branes. This has lead to a description of open membranes as soliton like objects on the five-brane world volume. This soliton is known as the self-dual string [11]. Its properties have been analysed using similar techniques as for the supergravity brane solutions. These include low energy scattering and anomaly calculations and even the

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2 Whether these are really independent checks is not clear, there is a discussion of the interrelatedness of these results in [12]
development of a Maldacena style limit on the brane.

Despite this, the real origin of the $N^3$ degrees of freedom of the five-brane is still to be understood as is the $N^3/2$ scaling of the membrane. Yet the membrane naively seems to give greater hope. The number of degrees of freedom is less than a gauge theory so one might speculate that it is possible to view the membrane theory as some sort of matrix valued field theory with constraints that restrict the real degrees of freedom. (This is similar to many matrix models of condensed matter systems for example.) Some eleven years after the inception of M-theory a recent attempt was made to construct such a model of non-Abelian membranes. [13]. This was inspired by the seminal work of Basu and Harvey [14] and subsequent developments [16,17]. This theory of non-Abelian membranes is truely novel. In order to be supersymmetric the theory has a nonassociative product for the fields and in order for the supersymmetry algebra to close there is a new gauge symmetry for the membrane fields. The evidence for this work is still speculative but it does have the one key property of having $N^3/2$ degrees of freedom.

This review will follow the above narrative. We begin with the membrane and five-brane as supergravity solutions and describe how analysing those solutions gives us the mysterious $N^3/2$ and $N^3$ scaling of the degrees of freedom. We will then move to world volume descriptions and the self-dual string as a description of membranes ending on a five-brane. We shall discuss its properties. Then we move to describe the recent membrane models and in passing discuss new ideas inspired by this search for a description of interacting M-theory branes. Unfortunately, this story does not have a final ending. Despite eleven years of progress and the many new ideas discussed here, the true description of M-theory branes remains unknown.

The informed reader may question the choice of topics chosen by the author in this review since many aspects and applications of M-theory are omitted. These include amongst others: M(atrix) theory [18]; Sieberg Witten theory from the five-brane [19,20]; $G_2$ compactifications [21] and; of course the basics of M-theory in unifying

\[ A \text{ similar observation concerning nonassociative models and the membrane was made in [17] } \]
the string theories.
This choice came about by trying to concentrate on aspects associated with the questions of interacting branes though inevitably the areas in which the author is more expert receive more attention. The final chapter on other ideas reviews areas not covered in detail in the rest of text and hopefully there are sufficient references throughout the review on questions not covered by the text.

2 M-theory Branes

2.1 Supergravity and brane solutions

We begin with branes as $\frac{1}{2}$ BPS solutions to eleven dimensional supergravity. The Bosonic sector of eleven dimensional supergravity (with the coupling set to one) is described by the following action [22, 23]:

$$S = \int d^{11}x \sqrt{-g} R - \frac{1}{2} F \wedge * F - \frac{1}{6} C \wedge F \wedge F,$$

where $F$ is the four form field strength of a three form potential $C$. Note the presence of the Chern-Simons like term. It is this term in the action that allows the possibility of membranes having boundaries that end on five-branes. The full consequences of this term and membrane boundaries will be discussed in the following section.

The supersymmetry transformation of the gravitini (in a purely Bosonic background) is given by:

$$\delta \psi_\mu = (\partial_\mu + \frac{1}{4} \omega_{\mu ab} \Gamma^{ab} + \frac{1}{288} (\Gamma^{\alpha\beta\gamma\delta}_\mu - 8 \Gamma^{\beta\gamma\delta}_\mu \delta^{\alpha}_\mu) F_{\alpha\beta\gamma\delta}) \epsilon.$$

(The full action and supersymmetry variation including Fermions is described amongst other places in [22, 23]).

The $\frac{1}{2}$ BPS brane solutions are found by imposing a projection on to a subspace of the spinor that generates the gravitini variation and then imposing that the supersymmetry variation vanish. This gives a relation between the fluxes and the spin connections. The projectors for the membrane and five-brane respectively are:

$$P_{M2} = \frac{1}{2} (1 + \Gamma^{012}) , \quad P_{M5} = \frac{1}{2} (1 + \Gamma^{012345}).$$

(3)
Solving these first order equations produces the brane solutions\textsuperscript{4}. The membrane metric is given by [24]:

\[ ds^2 = H^{-\frac{2}{3}}(r)(-dx^0 + dx^1 + dx^2) + H^\frac{2}{3}(r)(dr^2 + r^2d\Omega_2^2) \] (4)

with the flux given by

\[ F_{01r} = \partial_r H(r)^{-1} . \] (5)

The five-brane metric is given by [25]:

\[ ds^2 = H^{-\frac{4}{3}}(r)(-dx^0 + dx^1 + dx^2) + H^\frac{4}{3}(r)(dr^2 + r^2d\Omega_4^2) \] (6)

and its flux given by

\[ F_{\mu\nu\rho\sigma} = H(r)^{2/3}\epsilon_{\mu\nu\rho\sigma\tau}\partial^\tau H(r) . \] (7)

The $\epsilon$ tensor in the above refers to the antisymmetric tensor in the 5 dimensional space transverse to the five-brane. In each case the function $H(r)$ is a harmonic function on the tranverse space to the brane. Thus,

\[ H_{M2} = 1 + \frac{2\pi^2Q_{M2}^6}{r^6} \equiv 1 + \frac{R_{M2}^6}{r^6} \quad \text{and} \quad H_{M5} = 1 + \frac{\pi Q_{M5}^3}{r^3} \equiv 1 + \frac{R_{M2}^3}{r^3} . \] (8)

The charge, $Q$, in the harmonic function is the number of branes. This confirmed by simply integrating the flux (5) or (7) over the sphere at infinity transverse to the brane world volume. (The reader may wonder as to the origin of the constants of 2 and $\pi$ in the above harmonic functions. These are fixed by demanding the charge as the calculated by the flux through the sphere at infinity be integer as would be required by Dirac quantisation.) These harmonic functions carry much of the information of the brane physics and we will return to them frequently. In particular, we will see that the important information is contained in the relationship between, $R$, the length scale in the harmonic function and $Q$, the brane charge. Once we have the brane solutions then the next step is to determine their effective world volume description.

\textsuperscript{4}Actually one must use either the equations of motion for the flux or a component of the Einstein equation to fix the function $H$ to be Harmonic.
2.2 World volume descriptions

The low energy dynamics of branes may be captured by an effective world volume action. This is an action for the Goldstone modes that are present when, due to the presence of the brane solution, the symmetries of the supergravity action are broken. As the symmetries are only broken on the brane these Goldstone modes are restricted to lie in the brane world volume. Corrections to these actions will occur when the brane radius of curvature is of the order of the fundamental length scale of theory. For eleven dimension supergravity that is the Plank length. Thus, the following actions are good approximations provided the curvature is small in Plank units. This point of view is the most conservative, where the actions are simply low energy approximations. One might imagine that they are fundamental actions and will receive no corrections just as for the action of the fundamental string. The lack of renormalisability for an arbitrary background would seem to contradict this view.

The membrane is simple to describe from a world volume perspective. It is simply given by a Nambu-Goto style action (ie. induced volume) with minimal coupling to the background $C_3$ field. This gives [26]:

$$S_{M2} = \frac{1}{l_p^3} \int d^3\sigma (\sqrt{\det G_{\mu\nu}} + \partial_\mu X^I \partial_\nu X^J \partial_\rho X^K \epsilon^{\mu\nu\rho} C_{IJK})$$

(9)

where the induced metric is given by:

$$G_{\mu\nu} = \partial_\mu X^I \partial_\nu X^J g_{IJ}.$$  

(10)

Of course one can also use the Howe Tucker formulation to give:

$$S = \frac{1}{l_p^2} \int d^4\sigma (\sqrt{-\gamma} (\gamma^\mu\nu \partial_\mu X^I \partial_\nu X^J g_{IJ} - 1) + \partial_\mu X^I \partial_\nu X^J \partial_\rho X^K \epsilon^{\mu\nu\rho} C_{IJK}).$$

(11)

The presence of the one in the above action essentially acts like a world volume cosmological constant. Its presence is a simple indicator that the membrane unlike the string is not conformal. Dimensional reduction of the above membrane action gives the string action [27] without dilaton coupling (to obtain the dilaton coupling one has to consider the membrane partition function [28]).
Essentially one may view the action for the single membrane as a Goldstone mode action for the membrane solution that breaks eight of the eleven translation symmetries of the supergravity action. Thus there are eight physical modes on the membrane corresponding to these modes. This is easily seen once Monge gauge has been chosen and then the coordinates transverse to the brane become the physical fields on the brane world volume. Their vacuum expectation values become the classical location of the brane. Nontrivial classical solutions to the membrane equations of motion describe different membrane embeddings and have been studied at length in [29].

The five-brane is a more difficult object to describe from a world volume perspective. The field content (of a single five-brane) may be determined by again examining the Goldstone modes of the solution [30]. This gives five scalars, \( \phi^I, I = 1, \ldots, 5 \) corresponding to the five broken translations and a self-dual two form potential, \( b \) with field strength \( H = db \) corresponding to normalisable large gauge transformations of the \( C \) field in the five-brane background. There are also the Fermionic superpartners, symplectic Majoranna Spinors. Together this field content forms a \((0,2)\) tensor multiplet in six dimensions. Due to the self-duality constraint on \( H \), it is not possible to write a simple action for the tensor multiplet even at the linearised level. This maybe achieved however by introducing an auxiliary scalar that effectively allows one to gauge away the anti-self-dual degrees of freedom. This is known as the PST approach [31–33].

The PST action for the five-brane is given by:

\[
S_{PST} = \int d^6\sigma (\sqrt{-\det(g_{\mu\nu} + i\tilde{H}_{\mu\nu})} - \sqrt{-g}\tilde{H}^{\mu\nu}H_{\mu\nu\rho}\epsilon^{\rho\sigma}v_\sigma + C_6 + H \wedge C_3),
\]

(12)

where

\[
\tilde{H}_{\mu\nu} = \frac{1}{6}g_{\mu\alpha}g_{\nu\beta}\epsilon^{\alpha\beta\gamma\delta\rho\sigma}H_{\gamma\delta\rho}v_\sigma
\]

(13)

and \( v \) is an auxiliary closed one form constrained to have unit norm. \( H = db - C_3 \) is the combination of the world sheet three form field strength, with the pull back of the background three form \( C_3 \). \( g_{\mu\nu} \) is the pull back of the metric to the brane and \( C_6 \) denotes the pull back of the six form potential i.e. the dual to the usual three form potential of eleven dimension supergravity. This is similar in spirit to the actions for
D-branes with a Dirac Born-Infeld type terms followed by a Wess-Zumino type term coupling to the background fields.

Note that since the five-brane is the magnetic dual to the membrane it couples minimally to $C_6$ the dual six form potential as opposed to $C_3$. This action is useful for encoding the classical dynamics of the five-brane and determining various properties such as its relation to other branes through dimensional reduction. Many quantum aspects though may not be captured by this action. The quantisation of self-dual fields is a very subtle issue of which much has been written, see [34, 35] and references therein. We will avoid discussion of these quantum aspects of the five-brane theory but note that analysis of its anomalies and their cancellation will be crucial to understand properties of the coincident five-brane theory.

The equations of motion of the five-brane were first found in [36] using a doubly supersymmetric formalism. Soon after that the PST action was found in Green-Schwarz form and then [37] worked out the Green-Schwarz form of the equations of motion with no auxiliary fields. The relation between these results is described in [38].

The equations may be written in a useful compact form by introducing an effective metric [39, 40] defined by:

$$G_{\alpha\beta} = \frac{1 + K}{2K}(g_{\alpha\beta} + \frac{p^6}{4} \mathcal{H}_{\alpha\mu\nu} g^{\mu\sigma} g^{\nu\rho} \mathcal{H}_{\beta\sigma\rho}) , \quad (14)$$

where

$$K = \sqrt{1 + \frac{p^6}{24} \mathcal{H}^2} \quad (15)$$

Using this metric one may write the equation of motion for the scalars as [39, 40]:

$$\partial_\mu \sqrt{G} G^{\mu\nu} \partial_\nu \phi^I = 0 \quad I = 1..5. \quad (16)$$

The equation of motion for the three form field strength becomes a nonlinear self-duality equation given by:

$$\frac{1}{6} \sqrt{-\det g} \epsilon_{\mu\nu\rho\lambda\tau} \mathcal{H}^{\mu\nu\rho} = \frac{1}{2} (1 + K) (G^{-1})_{\mu}{}^{\lambda} \mathcal{H}_{\nu\rho\lambda} \quad (17)$$

In the weak field limit this is simply $H = \ast H$ but for general field configurations this is a complicated nonlinear theory.
The various non-trivial solutions to these equations yield a variety of five-brane configurations embedded in spacetime. They also can describe membranes ending on a five-brane. This is the self-dual string solution [11]. It is this that will be analysed in detail as a way of describing how membranes may end on five-branes.

In the above we have really only dealt with Bosonic aspects of the equations of motion. It is worth mentioning however that it is still not technically feasible to give the M-brane actions or equations of motion, indeed any brane action apart from the string, in component form in a generic background. That is in a form where the full theta expansion of the target space superfields are worked out. The target spaces where it is known to be possible are flat space or AdS × sphere. The reader may consult [41] for when one wishes to study more general cases.

3 Coincident Brane Degrees of freedom

Returning to the supergravity brane solutions one may attempt to analyse their properties. In particular, we are interested in knowing how many degrees of freedom will be present when we have N coincident branes; N is synonymous with the charge Q in the brane harmonic functions (8).

We must ask first of all what we mean by degrees of freedom. There will be three notions that we will use. The first will be thermal entropy of the system. Branes have horizons and so just like black holes they have thermal properties. This may be determined a la Hawking [42]. One then has a notion of entropy given by the usual laws of blackhole thermodynamics. The scaling of the entropy with N will determine how the number of degrees of freedom scale with the number of branes. We can also use the AdS/CFT correspondence to calculate thermodynamics of the decoupled brane theory using the properties of black holes in AdS. This is equivalent to the black brane picture in the decoupled limit.

Secondly, we can consider a low energy scattering calculation where one examines the infrared fluctuations of, for example, a graviton in the background of a brane solution. From this one may determine an absorption cross section for the brane solution. This
absorption cross section will scale with \( N \), the number of branes. The absorption cross section implicitly also measures the number of degrees of freedom since the more degrees of freedom an object carries the greater the absorption probability. Thus, the scaling of the absorption cross section with \( N \) measures how the degrees of freedom scale with \( N \). Of course this calculation really only measures the massless degrees of freedom since those are the only modes available to the infrared fluctuations. (It is the massless modes which are those of interest anyway). Since the absorption cross section effectively gives a transverse area of the brane it is no surprise then that this agrees with the thermodynamic calculation since the entropy of a black hole famously scales as its area. In fact, the universal nature of these calculations has been discussed in [12] and so we might wonder whether this is really an independent notion/calculation of the degrees of freedom. The view taken here is that this just confirms that they are both sensitive to the same physics.

Thirdly, for the five-brane we will have the power of anomalies at our disposal [43]. This notion is more akin to what one does in string theory where the central charge measures the number of degrees of freedom and is determined via the Weyl anomaly. For the five-brane the procedure is as follows. One first calculates the anomalies of the world volume theory for a single brane. Then one cancels the anomaly from terms in the supergravity action; this cancellation works via the so called *inflow* mechanism and with some care taken with the Chern-Simons term. One then determines how the anomaly cancellation terms in the supergravity action scale with \( N \), the number of branes. This then determines how the world volume anomaly itself scales with \( N \). (Of course here there is an assumption that the anomaly will always cancel, independently of \( N \); that is an assumption but a weak one; since if it didn’t cancel then M-theory would not be quantum mechanically consistent.) Now, once one knows how the R-symmetry anomaly scales with \( N \), then supersymmetry implies the Weyl anomaly scales the same way since they are part of the same *anomaly* multiplet. The scaling of the Weyl anomaly gives the central charge and the effective number of degrees of freedom.

Unfortunately, the membrane being a three dimensional theory is free of local anom-
lies and so we can’t use this technique in this case. It is still an open question, though, how one interprets the string Weyl anomaly from the membrane point of view. For the relationship between the Weyl symmetry of the string and the membrane see [45].

3.1 Absorption Cross Section

This was the first indication that coincident M-theory branes had an unusual scaling of the number internal degrees of freedom. One simply calculates the scattering amplitude of some field off the brane solution in supergravity and then work out the absorption cross section [8]. This will depend on the brane degrees of freedom. It is sufficient to consider the S-wave of a minimally coupled scalar in the brane background. This then reduces to a simple Coulomb like problem. In the presence of a five-brane a minimally coupled scalar field \( \phi(\rho) \) obeys:

\[
\left( \rho^{-4} \frac{d}{d\rho} \rho^4 \frac{d}{d\rho} + 1 + \frac{(\omega R_{M5})^3}{\rho^3} \right) \phi(\rho) = 0
\]

where \( \rho = \rho \omega \) the dimensionless radius, \( \omega \) is the energy and \( R_{M5} \) is defined in [8]. The solution in the inner region, \( \rho \ll (\omega R)^3 \):

\[
\phi = iy^3(J_3(y) + iN_3(y))
\]

where \( y = \frac{2(\omega R)^{3/2}}{\sqrt{\rho}} \). This can be matched onto the solution in the outer region, where \( \rho \gg (\omega R)^3 \):

\[
\phi = 24 \sqrt{\frac{2}{\pi}} \rho^{-3/2} J_{3/2}(\rho)
\]

giving an absorption probability of \( P = \frac{4}{9}(\omega R)^9 \). This finally yields an absorption cross section of

\[
\sigma_{M5} = \frac{2}{3} \pi^3 \omega^5 R_{M5}^9 \sim N_{M5}^3.
\]

A similar calculation for the membrane yields:

\[
\sigma_{M2} = \frac{2}{3} \pi^4 \omega^2 R_{M2}^9 \sim N_{M2}^{3/2}.
\]

Note that the cross section in both cases goes like \( R^9 \); it is only the dependence of \( R \) on the number of branes that changes between the membrane and five-brane.
3.2 Brane thermodynamics and AdS/CFT

The AdS/CFT correspondence [6] is an invaluable tool for examining CFT at strong coupling. It provides us with some insight into the decoupled CFTs appearing on M-theory branes. It is the shortest route to examining the number of degrees of freedom carried by a brane. Going via the AdS/CFT correspondence rather than directly looking at the brane thermodynamics is beneficial because one can also see other properties of the decoupled brane theory.

The starting point for the AdS/CFT correspondence is to take a low energy limit and simultaneously go to the near horizon of the solution such that the supergravity action remains finite in the limit. These considerations give the following limits:

\[ l_p \to 0 \quad r \to 0 \quad \frac{r^2}{l_p^3} = u \text{ fixed} \quad (23) \]

for the membrane and:

\[ l_p \to 0 \quad r \to 0 \quad \frac{r}{l_p^{3/2}} = u^2 \text{ fixed} \quad (24) \]

for the five-brane. In these limits the membrane spacetime becomes \( AdS_4 \times S^7 \) and the five-brane spacetime becomes \( AdS_7 \times S^4 \). The key information is contained in the radius of the AdS spaces. From (23) one sees that

\[ R_{AdS_4} = \frac{1}{2} (2^5 \pi^2)^{1/6} N^{1/6} l_p \quad (25) \]

with \( 2R_{AdS} = R_{S^7} \) for the membrane and

\[ R_{AdS_7} = 2\pi^{1/3} N^{1/3} l_p \quad (26) \]

with \( R_{AdS} = 2R_{S^4} \) for the five-brane.

We will follow [44] to use AdS black holes to determine thermodynamics of the dual theory. Blackhole solutions whose asymptotics are \( AdS_{n+1} \) are given by:

\[
 ds^2 = -(1 + \frac{r^2}{l^2} - \frac{Mw_n}{r^{n-2}})dt^2 + (1 + \frac{r^2}{l^2} - \frac{Mw_n}{r^{n-2}})^{-1}dr^2 + r^2 d\Omega^2 
\]
where
\[ w_n = \frac{16\pi G_N}{(n-1)Vol(S^{n-1})}, \] (28)
and \( Vol(S^{n-1}) \) refers to the volume of the unit sphere of dimension \( n-1 \). \( l^2 \) is related to the radius of the AdS space by: \( l^2 = R_{AdS} l_p \). M has the interpretation of the mass, as defined using some ADM like prescription with a subtraction scheme at infinity. The subtraction scheme is needed since the space is asymptotically AdS and a naive application of the ADM prescription would yield an infinite result.

For the cases relevant to M-theory we have \( n=3 \) and \( 6 \) corresponding to the membrane and five-brane respectively. There is obviously a horizon whose radius is given by the largest root of the equation:
\[ 1 + \frac{r^2}{l^2} - \frac{M w_n}{r^{n-2}} = 0. \] (29)
This is quite complicated for general \( n \) however in the large M limit the horizon radius, \( r_h \) will scale as:
\[ r_h \sim (w_n l^2)^{\frac{\frac{1}{n}}{\frac{1}{n}}}. \] (30)
The temperature is given by:
\[ T_{bh} = \frac{nr_h^2 + ((n - 2)l^2)}{4\pi l^2 r_h}. \] (31)
We will find it useful to consider the large mass or equivalently the large horizon limit limit where:
\[ T_{bh} \sim \frac{r_h}{l^2}. \] (32)
This is true independent of the dimension (the constant of proportionality will change however). The entropy is given by the usual Hawking area formula:
\[ S = \frac{\text{Area}}{4 G_N}. \] (33)
Now we must be careful with what we mean by \( G_N \). This will be \( G_N \) in the \( AdS_{n+1} \) space where the black hole is. The relation to the eleven dimensional \( G_N^{(11)} \) is given by dividing by the volume of the \( 11 - (n + 1) \) dimensional sphere upon which eleven
dimensional supergravity has been reduced. Since the radius of the sphere is proportional to $R_{AdS} \sim l^2$ we have:

$$G_N \sim \frac{G_N^{(11)}}{(l)^{2(10-n)}}. \quad (34)$$

The area of the black hole will scale as $r_h^{n-1}$ which using (32) implies the area scales as $l^{2(n-1)}T^{n-1}$. Combining this expression with (34) gives the entropy in the canonical ensemble as:

$$S \sim l^{18}T^{n-1} \sim R^9T^{n-1}. \quad (35)$$

Note, as with the absorption cross section, the scaling with $l$ or equivalently $R$ is independent of the dimension. We now use the relationship between the $AdS$ radius and the number of branes (25,26); this will be dimensionally dependent to give the entropies for the membrane and five-brane respectively:

$$S_{M2} \sim N^2T^2 \quad S_{M5} \sim N^3T^5. \quad (36)$$

These thermodynamic relations relating the temperature to entropy are inevitable for any conformal theory since there is no scale and the entropy is expected to be extensive; this fixes the temperature dependence. The $N$ dependence though is not determined by dimensional arguments. This factor is governed by the number of degrees of freedom present in the system. One might wonder what would happen if we moved away from the large black hole limit. These corrections may be calculated and are associated to finite size effects in the conformal field theory [46].

### 3.3 The five-brane anomaly and its cancellation

The (0,2) tensor multiplet world volume theory of the five-brane is anomalous. There are two sources of anomalies. The chiral two form is anomalous under world volume diffeomorphisms and the Fermions have an SO(5) R-symmetry anomaly. (In more formal terms, the SO(5) R-symmetry acts as SO(5) gauge transformations on the normal bundle $N$ to the brane and the diffeomorphisms act as local SO(1,5) transformations on the tangent bundle). The total anomaly may be determined through descent by an
eight form characteristic class, which for the world volume theory we denote as, $I_8^{WV}$. (For a relevant review of anomalies see [47]). That is $I_8^{WV}$ determines the anomaly $I_6^1$, via

$$I_8^{WV} = dI_7^0 \quad \delta I_7^0 = dI_6^1,$$

(37)

where $\delta$ indicates the variation under which the action is anomalous. The specific eight form may be read off using [47] and is described in [34]. The anomaly is cancelled by a combination of an inflow mechanism and a careful treatment of the Chern-Simons term. The inflow mechanism is as follows. There is an interaction term present in the supergravity action at one loop in $l_p$.

$$S_{\text{inflow}} = F_4 \wedge I_7^{\text{inflow}} = C_3 \wedge I_8^{\text{inflow}},$$

(38)

with $I_8^{\text{inflow}} = dI_7^{\text{inflow}}$. Taking the variation of this implies:

$$\delta S_{\text{inflow}} = F_4 \wedge \delta I_7^0 = F_4 \wedge dI_6^1 = -dF_4 \wedge I_6^1,$$

(39)

In the presence of a five-brane

$$dF_4 = Q_5 \delta^6$$

(40)

where $\delta^6$ is essentially a Dirac delta like object restricting the form to lie on the five-brane. It is a representative of the five-brane Thom class [48]. One then sees that after inserting (40) into (39) and integrating one produces a term on the five-brane world volume. The sum of the inflow term and the world volume anomaly is given by:

$$I_8^{WV} + I_8^{\text{inflow}} = \frac{p_2(N)}{24},$$

(41)

where $p_2(N)$ denotes the second Pontryagin class of the normal bundle $N$. The tangent bundle anomaly has vanished but the normal bundle anomaly remains [34]. The remaining contribution comes from a careful treatment of the M-theory Chern-Simons term when there are five-branes present [49, 50].

This is quite subtle, the essential idea is that in the presence of five-branes the Chern-Simons term is modified and becomes anomalous. We introduce $\rho$, the integral of a
bump form $d\rho$, and $e_3^0$ related to the global angular form $\frac{1}{2}e_4$ via $de_3^0 = e_4$. We define $\sigma_3 = \frac{1}{2}d\rho e_3^0$. Then the Chern-Simons term becomes

$$S_{CS} = \text{Lim}_{\epsilon \to 0} - \frac{3}{\pi} \int_{M^{11} - D_\epsilon(M^6)} (C_3 - \sigma_3) \wedge d(C_3 - \sigma_3) \wedge d(C_3 - \sigma_3)$$

(42)

where $D_\epsilon(M^6)$ denotes the excision of a neighbourhood of radius $\epsilon$ around the five-brane world volume $M^6$. The variation of this term under SO(5) normal bundle gauge transformations may be calculated using a result of Bott and Cattaneo to yield (again expressed through descent) [49]:

$$I_{CS}^8 = -\frac{p_2(N)}{24}$$

(43)

which cancels the remaining normal bundle anomaly.

All this has been carried out for a single five-brane. The interesting thing happens if we consider the case of $Q_5$ coincident branes and see how these terms scale with $Q_5$. The key point is that although the inflow term is linear in $C_3$ the Chern-Simons term is cubic and thus will scale as $Q_5^3$. We can now determine the anomaly of the general five-brane world volume theory by insisting that the total theory is anomaly free ie. the anomaly cancellation persists for any number of coincident branes. Since we know how the terms that cancel the world volume anomaly scale with $Q_5$ we can infer that the world volume anomaly, $I_{WV}(Q_5)$ for $Q_5$ five-branes must be:

$$I_{WV}(Q_5) = Q_5I_{WV}|_{Q_5=1} + \frac{1}{24}(Q_5^3 - Q_5)p_2(N)$$

(44)

This is an example of the extraordinary power of anomalies. By demanding anomaly cancellation we see that the mysterious unknown world volume theory has a normal bundle anomaly scaling as $Q_5(Q_5^2 - 1)$ To leading order in $Q_5$, this is the usual $Q_5^3$ term. The R-symmetry anomaly is in the same supermultiplet as the Weyl anomaly and so the Weyl anomaly will scale the same way. The Weyl anomaly provides the central charge of theory and is an independent measure of the number of degrees of

\footnote{Here we use the conventions of [49, 50] for the Chern-Simons term since the application of the Bott and Cattaneo formula is more immediate.}
freedom. It is satisfying that the various different methods agree to leading order in $Q_5$. The gravity description is not valid away from that limit though the anomaly calculation is valid for any $Q_5$.

The membrane theory is non-anomalous and so we can’t use anomaly arguments there. The Weyl anomaly has also been calculated via the AdS/CFT correspondence by [51]. This again yields the $Q_5^3$ dependence but now to get the next to leading term one must go to next order in the $l_p$ expansion [52].

3.4 Anomalies in the Coulomb branch

A more sophisticated five-brane set up may be considered where the five-branes are in the Coulomb branch and the different five-branes knit together. That is they may wind around each other with $|\phi| = c$, fixed but $\phi^a(\sigma^\mu)$ a function of the five-brane world volume. Also the possibility of a more general gauge group may be considered along with different breakings of this group. This was studied and the anomaly cancellation analysed by [53, 54]. Here we will follow [54].

We consider the case where the mysterious interacting five-brane theory associated to some group G is broken to a subgroup H by the vacuum expectation values of the scalars as follows: $\phi^a_i = \phi^a(T)_{ii}$ where the $(T)_{ii}$ are the diagonal components of a generator of the Cartan of G whose little group is $H \times U(1)$. The massless spectrum of the world volume theory when $<\phi^a> \neq 0$ is the (0,2) theory now with group H and the tensor multiplet associated to the U(1). The $\phi^a$s are the scalars in the tensor multiplet. One would then expect that for energies much less than the vev ie. $E << (\phi^a\phi^a)^{1/4}$ these theories would decouple and the U(1) multiplet become free.

Importantly, in order to ensure ’t Hooft anomaly matching between theory with group G and the broken theory with group $H \times U(1)$ there is a Wess-Zumino interaction term on the five-brane world volume. This term is independent of the value of the

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So far we have not really mentioned a specific group associated to the brane theory but since theory under a toroidal reduction goes to Yang-Mills theory with gauge group G, it is natural to associate a gauge group to the unknown five-brane theory; the default being, as with D-branes, SU(N).
scalar vev and persists at all energy scales. Since it is a topological term and its coefficient is quantised, it won’t be renormalised.

For \(|\phi| \neq 0\) the configuration space of the scalar fields will be \(SO(5)/SO(4) = S^4\) with coordinates given by \(\phi^a = \frac{\phi^a}{|\phi|}\). The required Wess-Zumino term is calculated to be [54]:

\[
S_{WZ} = \frac{1}{6} (c(G) - c(H)) \int_{M^7} \Omega_3(\hat{\phi}, A) \wedge d\Omega_3(\hat{\phi}, A). \tag{45}
\]

\(M^7\) is the seven dimension space whose boundary is the five-brane world volume. \(\Omega_3(\hat{\phi}, A)\) is defined via the equation \(d\Omega_3(\hat{\phi}, A) = \eta_4 = \frac{1}{2} c_4^M\) where \(c_4^M\) is the pull back to \(M^7\) via \(\hat{\phi} : M^7 \to S^4\) of the global angular, Euler class 4-form \(\eta_4\) which enters in the usual anomaly cancellation mechanism. (\(A\) denotes the \(SO(5)\) connection). The coefficient \(c(G)\) in (45) is the term in front of the normal bundle cancellation term ie. the coefficient of the \(\frac{p_1(N)}{24}\) term in (44). For the case considered above, corresponding to the \(SU(N)\) theory, it is given by, \(c(G) = Q_5(Q_5^2 - 1)\). Intriligator conjectures that for a general group:

\[
c(G) = |G| C_2(G) \tag{46}
\]

where \(C_2(G)\) is the dual Coxeter number, normalised to \(N\) for \(SU(N)\). It would be good to have an independent check of this conjecture but so far there are none.

There is also a Wess-Zumino term coupling to the three form field strength, \(H\) given by [54]:

\[
S_{WZ} = \alpha(Q_5) \int_{M^7} H_3 \wedge d\Omega_3(\hat{\phi}, A), \tag{47}
\]

with

\[
\alpha(Q_5) = \frac{1}{4} (|G| - |H| - 1). \tag{48}
\]

where \(|G|\) denotes the dimension of the group, \(G\), which depends on \(Q_5\) Again, this is derived somewhat indirectly. We will see later that this term (47) is crucial for anomaly cancellation of the self-dual string and without it the self-dual string would have a normal bundle anomaly.

Other anomaly considerations such as for five-branes at fixed points of orbifolds have been studied in [55].
4 Brane Interactions

So far we have explored the properties of membranes and five-branes on their own. We now wish to examine how they may interact. The fundamental interaction is via the membrane ending on the five-brane. From this point of view the five-brane is the Dirichlet brane for the membrane; a sort of D-brane for M-theory.

4.1 How can branes end on branes

The realisation that the membrane may end on a five-brane is due to Townsend [10,56] and Strominger [9]. This would also be expected from dimensional reduction arguments. Since the membrane reduces to a fundamental string and the five-brane to a D4-brane, the fact that a fundamental string may end on a D4 implies a membrane should be able to end on a five-brane once theory is decompactified to eleven dimensions.

To see directly how the membrane may end on a five-brane one examines the Chern-Simons term in the supergravity action.

\[ S_{CS} = \int C_3 \wedge F_4 \wedge F_4 \]  

Even though the potential, \( C_3 \), appears in the action, this term is gauge invariant in the absence of boundaries via the usual Chern-Simons argument.

\[ \delta \lambda S_{CS} = \int d\lambda \wedge F_4 \wedge F_4 = \int d(\lambda \wedge F_4 \wedge F_4) \]  

Thus the variation is a simple boundary term. Now because of this term the equations of motion for \( C_3 \) are:

\[ d^* F_4 + F_4 \wedge F_4 = 0 \]  

which one may write as

\[ d(\wedge F_4 + F_4 \wedge C_3) = 0 . \]  

Thus the charge of the membrane is actually calculated by:

\[ Q_2 = \int_{M^7} * F_4 + F_4 \wedge C_3 . \]
For an infinite membrane $M^7$ would be some seven cycle enclosing the membrane capturing all the flux. However if the membrane has a boundary then the seven cycle may be slipped off the end of the membrane and contracted. One needs to consider the presence of a five-brane at the boundary of the membrane. The seven cycle $M^7$ would now decompose into a product $M^4 \times M^3$ where the $M^4$ is the four cycle enclosing the five-brane (which we take to be infinite) and the $M^3$ is a three cycle enclosing the boundary of the membrane inside the five-brane world volume. Let us examine the second part of this integral (53). The integral becomes:

$$Q_2 = \int_{M^4} F_4 \int_{M^3} C_3 = Q_5 \int_{M^3} C_3$$

(54)

For now take $Q_5 = 1$ corresponding to a membrane ending on a single five-brane. We see that the membrane charge must be given by

$$Q_2 = \int_{M^3} C_3$$

(55)

where $M^3$ is a cycle surrounding the string that is the membrane boundary inside the five-brane world volume. Thus, for a membrane to end on a five-brane requires the five-brane to carry a nontrivial $C_3$ field. Note that since $M^3$ is closed this expression for the charge is gauge invariant as it must be.

This is entirely in terms of the supergravity fields. As we have seen the five-brane has a world volume two form field corresponding to the Goldstone modes of $C_3$. The above integral in terms of the world volume field would yield,

$$Q_2 = \int_{M^3} H$$

(56)

where $H$ is the field strength of the Goldstone field in the five-brane world volume.

Powerfully, one may use the fact that the membrane may end on a five-brane to derive the five-brane equations of motion. Just as $\kappa$-symmetry of a closed membrane allows one to determine the supergravity equations of motion, the requirements of $\kappa$-symmetry for an open membrane allows one to determine the five-brane equations of motion [57] (the five-brane field being the background fields for the membrane boundary).
4.2 World Volume solitons

The above argument demonstrates how a world volume field of the five-brane with nontrivial flux gives rise to membrane charge. This argument is somewhat cohomological and does not actually yield a membrane description. To do this we must solve the five-brane equations of motion and find a solution with a charge (56). We begin with the equations of motion for the five-brane. We will look for a solution that corresponds to a membrane ending on a five-brane and so we expect this to be a string solution from the five-brane perspective. Thus we make an ansatz where the five-brane world volume fields are independent of time and $x^1$ the string direction. The remaining rotational symmetry implies that the fields may only depend on $r$, where $r$ is the radial coordinate of the space transverse to the string i.e. $r^2 = x_2^2 + x_3^2 + x_4^2 + x_5^2$.

The solution we are looking for will be somewhat like a string monopole since it will be charged magnetically as given by (56). The simplest monopole-like solutions will be supersymmetric $\frac{1}{2}$ BPS states of the five-brane world volume theory. Thus the simplest approach is to take the five-brane supersymmetry transformations and search for a half supersymmetric solution. This is just like looking for a brane solution in supergravity, one makes an ansatz for the fields and then searches for solutions that preserve half the supersymmetry variations.

Following the intuition gained from brane solutions in supergravity and monopole solutions in gauge theories one picks the supersymmetry projector to be:

$$\pi_{\text{string}} = \frac{1}{2} (1 + \gamma^7 \gamma^{01})$$  \hspace{1cm} (57)

where $\gamma^i$ are five-brane world volume gamma matrices and $\gamma^7$ is a gamma matrix in the transverse space; that is it acts on the spin cover of the SO(5) R-symmetry of the five-brane world volume theory. Note, that it is essentially the same as the membrane projector for a membrane whose world volume lies along the 017 directions. The 01 directions being common to the five-brane and the 7 direction orthogonal.

Using this projector and the field ansatz where all field are only functions of $r$ reveals the following from the demanding the supersymmetry variation on the five-brane
vanishes:

\[ H = *_4 d\phi \]  \hspace{1cm} (58)

where \( *_4 \) denotes Hodge duality in the four transverse directions to the string. The SO(5) R-symmetry has been used to fix a single scale to be excited. Using the Bianchi identity for H then implies:

\[ dH = d* d\phi(r) = 0 \]  \hspace{1cm} (59)

which implies \( \phi \) is a harmonic function. Its solution is given by:

\[ \phi(r) = \frac{2Q_{SD} l_p^2}{r^2} \equiv \frac{R^2_{SD}}{r^2} \]  \hspace{1cm} (60)

with the field strength H related to \( \phi \) by (58) and H obeying the self duality relation. Note this solution obeys, \( H = *H \) even though the field equation is nonlinear. This is typical of a BPS solution where the nonlinearity becomes linearised in some sense. \( Q_{SD} \) refers to the self-dual string charge and via (56) also gives the numbers of membranes ending on the five-brane world volume.

This solution will obviously be a brane in its own right and as such there will be Goldstone modes associated to it and one might imagine constructing an effective world volume theory describing its low energy dynamics. The obvious Bosonic modes will be the four scalars coming from the four broken translation symmetries of the string solution in the six dimensional five-brane world volume. The Fermionic superpartners of these Bosons will be charged under

\[ SO(1, 1) \times SU(2) \times SU(2) \times SU(2) \times SU(2) \]  \hspace{1cm} (61)

the first two SU(2) form an SO(4) which is the symmetry of the space transverse to the string but parallel to the five-brane worldvolume. The second two SU(2)s form the SO(4) that is the symmetry of the space transverse to both the membrane and five-brane. From analysing the projector (57) one may determine the supersymmetry of the Goldstone modes. There will be (4,4) supersymmetry in two dimensions with the Fermions in the following representations of (61):

\[ +\frac{1}{2}, 1, 2, 2, 1 \]  \hspace{1cm} (62)

\[-\frac{1}{2}, 2, 1, 1, 2. \]

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We thus now have a description of the field content of the self-dual string. In what follows we will analyse the self-dual string and its properties just as we have analysed the properties of the membrane and five-brane itself.

First a word of warning, the self-dual string will couple to the self-dual two form potential, $b_2$ in the five-brane world volume. The coupling constant in a self-dual theory must be fixed to be of order one and so it is never in a perturbative regime. This means one must be very careful in determining when a classical approximation is valid and a low energy effective description can make sense.

The self-dual string is thus a mini-version of an M-theory brane. We know the field content from analysing Goldstone modes and can as we will see determine many properties indirectly but we do not have a fundamental version of theory. How self-dual strings interact and how to describe coincident self-dual strings will be yet another M-theory mystery.

There have also been solutions found that are non-BPS by brute force approaches to solving the nonlinear equations [58]. This solution involves the two form potential only and does not excite any of the world volume scalars. In fact a whole family of non-BPS solutions were found using a solution generating symmetry of the five-brane equations of motion [59]. This family of solutions interpolates between the solution found in [58] where no scalar field was excited and the BPS solution where the scalar field becomes singular at the origin. These are bump like solutions which being non-BPS will undoubtedly decay to the BPS state with the same membrane charge. This process has so far not been studied but would be an interesting topic for future research.

5 The self-dual string and its properties

We will follow our intuition of how we explored the five-brane and membrane by using techniques such as: scattering cross sections, anomalies and various limits. First, let us state our goals. We have a solution of the five-brane equations of motion and we would like to know how many degrees of freedom that object carries as a function
of the five-brane charge and the self-dual string charge. This is analogous to our original questions concerning the membrane and five-brane given in previous sections. Since ultimately the self-dual string describes open membranes from the five-brane perspective it is hoped that this will throw light on our original question on the degrees of freedom in M-theory.

5.1 Absorption Cross section

The absorption cross section for s-wave scalar fluctuations is calculated at low energy by expanding the five-brane equations of motion around the self-dual string background. This gives the following Coulomb type equation for the fluctuations, $\phi(\rho)$, where we have introduced the dimensionless radius $\rho = \omega r$ as before:

$$\left(\rho^{-3} \frac{d}{d\rho}\rho^3 \frac{d}{d\rho} + \frac{(R_{SD} \omega)^6}{\rho^6}\right) \phi(\rho) = 0.$$  \hfill (63)

Solving this equation in the exterior region where $(R \omega)^6 << \rho^4$ gives:

$$\phi(\rho) = \rho^{-1}(AJ_1(\rho) + BN_1(\rho)).$$  \hfill (64)

The interior region ($\rho << R \omega$) has solution:

$$\phi(\rho) = A' \cos\left(\frac{(R \omega)^3}{2\rho^2}\right) + B' \sin\left(\frac{(R \omega)^3}{2\rho^2}\right).$$  \hfill (65)

In the overlap region $R \omega >> \rho >> (R \omega)^{3/2}$ we can match the solution and so calculate the transmission coefficient.

This gives an absorption cross section for the self-dual string to be [60]:

$$\sigma \sim R_{SD}^3 \omega^3 \sim Q_5^3 \omega^3.$$  \hfill (66)

This indicates that theory of $Q_2$ coincident self-dual strings has a $Q_2$ scaling of degrees of freedom. This calculation however does not reveal anything about the dependence of self-dual string on the five-brane charge $Q_5$ since we could only calculate the cross section using theory of a single five-brane. To do this we resort to an anomaly calculation analogous to that of the five-brane itself.
This is a half BPS state. One can also consider a system with less supersymmetry. One suggestion is the emergence self-dual string webs, [61]. This is when the strings end on each other forming a net or web like structure. This would correspond to the situation of intersecting membranes ending on the five-brane. The supersymmetry restricts the possible angles allowed for membrane intersections and equivalently, for the tension of the string web to balance, the string vertices must intersect at particular angles.

5.2 Self-dual string anomalies

As can be seen from the table of the representations of the Fermions of the self-dual string world sheet (63), theory has chiral Fermions and so possess anomalies. Using the results of [47] we may read off the overall resulting normal bundle anomaly. (Also see [43] for a good review of these issues). The normal bundle of the string splits up into two SO(4) bundles, one that is tangent to the five-brane and one that is normal to the five-brane. The normal bundle tangent to the five-brane we denote by T and the normal bundle that is normal to the five-brane we denote by N.

The anomalies are given through descent [47] by:

$$I_4 = \pi (\chi(T) + \chi(N))$$

(67)

where $\chi(T/N)$ denotes the Euler character of the SO(4) T/N bundles respectively. Given theory is anomalous we must introduce local terms in the five-brane world volume and the self-dual string that will cancel this anomaly.

To cancel the T bundle anomaly one has a term:

$$I_{mc} = Q_2 \int_{M^6} b_2 \wedge \delta(\Sigma_2 \hookrightarrow M^6)$$

(68)

on the five-brane world volume. $\delta(\Sigma_2 \hookrightarrow M^6)$ denotes the Poincare dual of the string world volume $\Sigma_2$ in the five-brane world volume $M^6$. This is just the minimal coupling of the string to the two form on the five-brane under which is it charged. Its variation
under SO(4) T-bundle transformations is:

\[ \delta I_{mc} = Q_{SD} \pi \int_{\Sigma_2} \chi_2^{(1)} \]

where we have used the representation of the Poincare dual to be:

\[ \delta_4(\Sigma_2 \hookrightarrow M^6) = d\rho(r) \wedge e_3/2. \]

with \( \rho(r) \) the bump form and \( e_3 \) the global angular form over the \( S^3 \) fibres transverse to \( \Sigma_2 \) in \( M^6 \) [64]. Note the minimal coupling is proportional to \( Q_{SD} \) and so this indicates the world volume theory for an arbitrary number of strings has an anomaly that scales linearly with \( Q_{SD} \). This agrees with the absorption cross section described in the previous section.

The anomaly in the N bundle may be cancelled by a term that originates from the term introduced by Ganor and Motl [53] and Intriligator [54] to cancel anomalies of five-branes in the Coulomb branch. That is the Wess-Zumino described in (47). Their term, for the case of a self-dual string, where one excites a single scalar reduces to:

\[ I = -\frac{1}{2} \alpha(Q_5) \int_{M^6} H_3 \wedge \chi^{(0)}(A_N) \]

which is the term required to cancel the N bundle anomaly of the self-dual string. The crucial part is to examine the charge dependence of this term. It is linear in \( H_3 \) which means it is linear in \( Q_{SD} \). The \( Q_5 \) dependence is given by (48). For the case \( G = SU(Q_5 + 1) \), \( H = SU(Q_5) \) then \( \alpha = \frac{1}{2} Q_5 \) which gives a linear dependence. The more surprising result is if \( G = SU(Q_5 + 1) \), \( H = U(1)^{Q_5} \) then \( \alpha = \frac{1}{8} (Q_5^2 + Q_5 - 1) \). This implies the anomaly is proportional to

\[ Q_{SD} Q_5^2, \]

in the large \( Q_5 \) limit. This \( Q_5^2 \) dependence is something that cannot be seen at this point by any other means. [47]) Here we have discovered a further M-theory mystery, what are these \( Q_5^2 \) degrees of freedom on the self-dual string?

How might this be related to the degrees of freedom on the five-brane? Well if one considers the case of maximal breaking where \( H = U(1)^{Q_5-1} \) then we are left with \( Q_5 - \)

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1 tensor multiplets and each tensor multiplet may have a self-dual string associated with it. Each string will carry $Q_5^2$ degrees of freedom so that in total all the self-dual strings will carry $Q_5^3$ degrees of freedom which agrees with the five-brane in the large $Q_5$ limit. Note, however the next to leading order in $Q_5$ does not agree; for the five-brane there is no term quadratic in $Q_5$ whereas there is such a dependence if one counts self-dual string degrees of freedom. Work on anomalies of the self-dual string has also been discussed in [62, 63].

5.3 AdS limits of the self-dual string

One may try to construct a decoupling limit in which the low energy dynamics of the self-dual string decouple from the five-brane bulk [60]. This will be analogous to the Maldacena near horizon limits for branes.

One takes a low energy limit while keeping fixed some energy scale:

$$l_p \rightarrow 0 \quad u = \frac{r^2}{l_p^3} \text{ fixed}.$$  \hspace{1cm} (73)

This limit is determined simply by the dimensions of the two dimensional string scalars or via appealing to the description in terms of membranes. This is identical to the Maldacena limit for membranes (as one might expect). Taking this limit for $Q_{SD}$ fixed, gives the following. (Note, $Q_{SD} \gg 1$ so that the five-brane equations of motion are valid, i.e. there are no derivative corrections). The effective metric \(\text{AdS}_3 \times S^3\) which governs the dynamics of the brane becomes conformal to: \(\text{AdS}_3 \times S^3\). The radius of the \(\text{AdS}_3\) is given by:

$$R_{\text{AdS}_3} = R_{S^3} = (2Q_{SD}) \frac{1}{3} l_p.$$ \hspace{1cm} (74)

This is of course for $Q_5 = 1$. The area of the sphere is therefore proportional to $Q_{SD}$, in keeping with our intuition that the cross section goes linearly with $Q_{SD}$. We cannot repeat the calculation for the case where the $Q_5 > 1$ since we don’t have a description of the five-brane theory in this case.

In the above limit the effective tension of open membranes is constant in units of $l_p$ and so the open membranes do not decouple [66]. The bulk five-brane theory is therefore
described by open membranes on an $AdS_3 \times S^3$ background with a decoupling of the closed membrane sector. Therefore the low energy dynamics of the decoupled self-dual string (in the large $Q_{SD}$ limit) appears to be described by open membranes on $AdS_3 \times S^3$. Unfortunately we do not have sufficient control of either side of this correspondence to make further statements.

### 5.4 Self-dual string effective actions

A conjectured action for a single self-dual string has been given by [67–70]. Of course the self-dual string as described above has infinite tension. To get a finite tension one imagines the open membrane ending on another five-brane so the string has a finite effective tension. The low energy effective action of the Bosonic sector is given by (where low energy means small with respect to the energy scale given by the inverse of the five-brane separation):

$$S = \frac{1}{l_p^3} \int d^2 \sigma |\phi| \left( \sqrt{|\det g_{\mu\nu}} + dX^M \wedge dX^N b_{MN} \right),$$

where $g_{\mu\nu}$ is the induced world sheet metric. The first term is essentially a Nambu-Goto term but with a tension set by $|\phi|$. The second term is just the minimal coupling to the five-brane world volume two form $b$. One may check that this captures the dynamics described above. The scattering amplitude of waves off the string has been calculated in [67]. Comparison with [60] agrees with the scattering off the self-dual string soliton.

Further work has been done with this action. However, it has a restricted range of validity and is probably not valid when there are more than one coincident strings or five-branes. One further approach may be to calculate the moduli space of two (or more) strings and determine their effective action as a sigma model on that moduli space. This is just what one would do for the effective action of two monopoles for example.
6 The membrane boundary theory

In the preceding sections the membrane has been described as a five-brane soliton - the self dual string and its dynamics analysed as a low energy effective description for the soliton. We now move to describe the open membrane directly by using the action for the single membrane and paying special attention to the minimal coupling of the C field to the membrane.

6.1 The boundary of the membrane

\[ \int_{M^3} C_{IJK} \partial_\mu X^I \partial_\nu X^J \partial_\rho X^K \epsilon^{\mu\nu\rho} \]  

(76)

is the term that dictates the boundary membrane dynamics when there is a large C field. As such, the large C field limit provides an expansion for the membrane dynamics. It is instructive to see how the open membranes behave in this limit. We begin by choosing C to be constant on the five-brane. Since the pullback of C is also governed by the nonlinear self-duality relation one needs to take the limit in a way that preserves this equation of motion. A description of the details of this limit are given in [65]. Here we will note that the limit exists and analyse the boundary of the membrane in the presence of a constant C field given by \( C_{IJK} = \epsilon_{IJK} C \) where I,J,K are indices labelling a three dimensional subspace of the five-brane.

For such a constant C field the above minimal coupling term becomes:

\[ S = \int_{\partial M^3} C \epsilon_{IJK} X^I \partial_\sigma X^J \partial_\tau X^K \]  

(77)

This is a first order action. Its canonical momentum will be:

\[ P_t = C \epsilon_{IJK} X^J \partial_\sigma X^K. \]  

(78)

This momentum relation must then be imposed as a constraint because the action is first order and so the time derivatives will not be related to the momentum. Calculating the brackets of these constraints implies that they are second class which in turn
implies one should replace Poisson brackets with Dirac brackets. We then calculate the Dirac bracket for the $X^I(\sigma)$ [65]:

$$[X^I(\sigma), X^J(\sigma')]_D = \delta(\sigma - \sigma') \epsilon_{IJK} \frac{\partial X^K(\sigma)}{|\partial X|^C}$$

(79)

Now we see that the $X^I(\sigma)$ do not commute. This is the M-theory analogue of noncommutativity on a D-brane in the presence of a Neveu Schwarz two form. In that case, the boundary of a string was a point and the noncommutativity was an ordinary sort. For example, for a constant Neveu-Schwarz two from in a two dimensional subspace of the D-brane $B_{IJ} = \epsilon_{IJ}B$ a similar analysis for the open string in the large B limit yields:

$$[X^I, X^J] = \frac{\epsilon^{IJ}}{B}.$$ 

(80)

The M-theory relation (79) is therefore interpreted as defining a noncommutative loop space in comparison to the simple noncommutative space described by (80). Physically this occurs is because the membrane boundary is a loop as opposed to the case of the open string which has a point-like boundary.

How one interprets this from the five-brane perspective is difficult and we will return to this later.

### 6.2 From strings to ribbons

We will now consider the case where the membrane has two boundaries on the same 5-brane in the presence of non-vanishing flux.

First we observe that [73] for a single boundary the equations of motion of the boundary string

$$C_{ijk} \partial_\sigma X^j \wedge \partial_\tau X^k = 0$$

(81)

imply that $X^i$ has to depend on the worldsheet coordinates through only one function $f(\tau, \sigma)$. The boundary of the membrane therefore is a static string in the $(X^1, X^2, X^3)$ plane. A useful analogy is that of a vortex line in a three-dimensional fluid: just as in two-dimensions, the transverse motion of a vortex line is effectively confined by
the rotational motion of the fluid itself, on Landau-like orbits. In addition, there exist soft modes propagating along the vortex line known as Kelvin modes. In fact, as noticed in [72, 73], the boundary coupling (77) is precisely the one describing the Magnus effect in fluid hydrodynamics.

Membranes with two boundaries, however, have no overall charge and therefore can propagate freely just as vortex anti-vortex lines may propagate in hydrodynamics. In the absence of a $C$ field, the two boundaries lie on top of each other, leading to an effectively tensionless string, the tentative fundamental degrees of freedom of the (0,2) theory. In the presence of a $C$ field however, these tensionless strings polarise into thin ribbons, whose width is proportional to the local momentum density. Indeed, the canonical momentum on the membrane, neglecting the contribution of the Nambu-Goto term, is

$$P^i = C_{ijk} \partial_\sigma X^j \partial_\rho X^k$$

(82)

where $\sigma$ is the coordinate along the boundary string, and $\rho$ the coordinate normal to it. The ribbon thus grows as

$$\Delta^i \sim \partial_\rho X^i = \frac{1}{C|\partial_\sigma X|^2} \epsilon_{ijk} P^j \partial_\sigma X^k$$

(83)

where we retain in $\Delta$ only the component orthogonal to $\sigma$ (the parallel component could be reabsorbed by a diffeomorphism on the membrane worldvolume).

Let us consider a simple classical solution corresponding to an infinite strip of width $\Delta$ moving at a constant velocity $v$ transverse to it: We thus consider the classical solution

$$X^i = p^i_0 \tau + u^i \sigma + \Delta^i \rho .$$

(84)

The boundary condition

$$\sqrt{\gamma} \gamma^{\rho\rho} \partial_\rho X^i - C_{ijk} \partial_\sigma X^j \partial_\tau X^k = 0$$

(85)

with induced metric $\gamma = \text{diag}(m^2, |u|^2, |\Delta|^2)$, implies that the direction of the polarisation vector is orthogonal to the plane formed by the tangent vector to the string.
\( \vec{u} = \partial_x \vec{X} \) and the local velocity \( \vec{p}_0 = \partial_t \vec{X} \),

\[
\frac{\Delta \vec{X}}{|\Delta|} = C \frac{\vec{u}}{|u|} \wedge \frac{\vec{p}_0}{m}.
\]  

Calculating the local canonical momentum

\[
P^i = \sqrt{\gamma^{\tau\tau}} \partial_\tau X^i - C_{ijk} \partial_\sigma X^j \partial_\rho X^k
\]

\[
= \frac{|\vec{u}| |\vec{\Delta}|}{m(1 + C^2)} \left[ (1 + C^2) p_0^i - C^2 \frac{\vec{u} \cdot \vec{p}_0}{|u|^2} u^i \right]
\]

one may express the local velocity in terms of \( P_i \),

\[
p_0^i = \frac{m}{|\vec{u}| |\vec{\Delta}|(1 + C^2)} \left[ P_i + C^2 \frac{\vec{u} \cdot \vec{P}}{|u|^2} u^i \right]
\]

and obtain the relationship between the membrane polarisation and canonical momentum,

\[
\frac{\Delta}{|\Delta|} = \Theta \frac{\vec{u}}{|u|^2} \wedge \vec{P}, \quad \Theta = \frac{C}{1 + C^2}.
\]

For convenience, we will use the gauge \( m = |\vec{u}| |\vec{\Delta}| \) from now on.

We thus recover the “open membrane non-commutativity parameter” \( \Theta \), defined in [74]. In this work, this parameter was determined by studying the physics of five-branes probing supergravity duals with \( C \)-flux longitudinal to the probe brane world volume. We now understand this result classically as the polarisability of open membranes in a \( C \)-field. Of course in the large \( C \) limit it becomes \( \frac{1}{C} \) as we expect from (79) and is the obvious generalisation to the open string non-commutativity parameter [76].

### 6.3 An effective Schild action for string ribbons

We will now describe an effective string theory that describes the ribbons as strings. This is a good approximation for scales less than the ribbon width [76]. Let us start with the light-cone formulation [77] of the membrane, with Hamiltonian

\[
P^- = \int d\sigma d\rho \frac{1}{2P^+} \left[ (p_0^i)^2 + g \right]
\]
where $g$ is the determinant of the spatial metric, hence the square of the area element (and the membrane tension is set to 1). In this gauge, one should enforce the constraint

$$
\partial_\sigma X^i \partial_\rho \partial_\tau X^i - \partial_\rho X^i \partial_\sigma \partial_\tau X^i = 0 ,
$$

which is trivially satisfied on zero-mode configurations \[84\]. For a thin ribbon of width $\vec{\Delta}$ given by \[90\], the square of the area element is

$$
g = |\vec{u} \wedge \vec{\Delta}|^2 = \frac{C^2}{(1 + C^2)^2} \left[ \vec{P}^2 - \frac{(\vec{u} \cdot \vec{P})^2}{|u|^2} \right].
$$

On the other hand, using \[89\], the kinetic energy may be written as

$$
(\vec{p}_0)^2 = \frac{1}{(1 + C^2)^2} \left[ \vec{P}^2 + C^2(C^2 + 2) \frac{(\vec{u} \cdot \vec{P})^2}{|u|^2} \right].
$$

Finally, the total Hamiltonian thus takes the form

$$
P^- = \int d\sigma \frac{1}{2P^+(1 + C^2)} \left[ P^2 + C^2 \frac{(\vec{P} \cdot \partial_\sigma \vec{X})^2}{|\partial_\sigma \vec{X}|^2} \right].
$$

From this expression, specifying to a gauge choice where $\vec{P}$ and $\partial_\sigma \vec{X}$ are orthogonal, we see that the effective metric in the transverse directions is rescaled by a factor of $(1 + C^2)$,

$$
G_{ij} = \left[ 1 + C^2 \right] \delta_{ij}.
$$

This agrees with the membrane metric found from very different considerations in [74, 75], up to the conformal factor $Z = (1 - \sqrt{1 - 1/K^2})^{1/3}$ with $K = \sqrt{1 + C^2}$.

Finally, we may perform a Legendre transform on $P_i$ to find the Lagrangian density of the ribbon,

$$
\mathcal{L} = \int d\sigma \left\{ \frac{1}{2} (\partial_\tau X^i)^2 + \frac{C^2}{2|\partial_\sigma X|^2} \sum_{i,j} \{X^i, X^j\}^2 \right\}
$$

where we have defined the Poisson bracket on the Lorentzian string worldsheet\[7\] as

$$
\{A, B\} = \partial_\sigma A \partial_\tau B - \partial_\sigma B \partial_\tau A.
$$

Note that the relative sign between the two terms

\[7\]This should not be confused with the Poisson bracket formulation of the membrane, which refers to the two \textit{spatial} directions of the membrane world-volume.
in (97) is consistent with the fact that they both contribute to kinetic energy. For vanishing $C$, (97) reduces to the Lagrangian for a tensionless string, as expected. While we have mostly worked at the level of zero-modes, it is easy to see that (97) remains correct for arbitrary profiles $X^i(\tau, \sigma)$, as long as the dependence on membrane coordinate $\rho$ is fixed by Eqs. (84), (90).

After fixing the invariance of the Lagrangian (97) under general reparameterizations of $\sigma$ by choosing $|\partial_\sigma X^i| = 1$, we recognise in the second term the Schild action, which provides (in the case of a Lorentzian target-space) a unified description of both tensile and tensionless strings, depending on the chosen value for the conserved quantity $\omega = \{X^i, X^j\}^2$, as long as the dependence on membrane coordinate $\rho$ is fixed by Eqs. (84), (90).

As usual, it is possible to give a regularisation of this membrane action, by replacing the Poisson bracket (now in light-cone directions on the worldsheet) by commutators in a large $N$ matrix model. One thus obtains a lower-dimensional analogue of the type IIB IKKT matrix model [79],

$$P^- = \frac{1}{2P^+} \left( [A_0, X^i]^2 + C^2 \sum_{i < j} [X^i, X^j]^2 \right).$$

(98)

The analysis of this matrix model has not been carried out. It would be interesting to see what one could learn about membrane ribbons from quantising this action.

### 6.4 Non-commutative string field theory

Just like open strings, open membranes interact only when their ends coincide. Since their boundaries are tensionless closed strings which polarise into thin ribbons in the presence of a strong $C$ field, one may expect that the effect of the $C$ field can be encoded by a deformation of the string field theory describing the membranes boundaries. Despite the fact that string field theory of closed strings, not to mention tensionless ones, is a rather ill-defined subject, it is natural to represent the string
field as a functional in the space of loops. The effect of the polarisation of the ribbons can thus be represented by

\[ V \sim \int [DX(i)] \Phi \left[ X^i - \frac{1}{2} \frac{\Theta}{|\partial_\sigma X|^2} \epsilon_{ijk} \partial_\sigma X^j \frac{\delta}{\delta X^k} \right] \times \Phi \left[ X^i + \frac{1}{2} \frac{\Theta}{|\partial_\sigma X|^2} \epsilon_{ijk} \partial_\sigma X^j \frac{\delta}{\delta X^k} \right] \tag{99} \]

where we represented the momentum density \( P_i \), canonically conjugate to \( X^i(\sigma) \), as a derivative operator in the space of loops. Defining the operators

\[ \tilde{X}^i(\sigma) = X^i - \frac{1}{2} \frac{\Theta}{|\partial_\sigma X|^2} \epsilon_{ijk} \partial_\sigma X^j \frac{\delta}{\delta X^k} \tag{100} \]

reproduces the non-commutative loop space in the “static” gauge \( X^3(\tau,\sigma) = \sigma \),

\[ [\tilde{X}^1(\sigma), \tilde{X}^2(\sigma')] = \Theta \delta(\sigma - \sigma') \tag{101} \]

as proposed in [65,80]. The fact that the transverse fluctuations of a vortex line are effectively confined by an harmonic potential is well known in fluid dynamics. In the more covariant gauge \( |\partial_\sigma \vec{X}| = 1 \), one obtains a tensionless limit of the \( SU(2) \) current algebra,

\[ [\tilde{X}^i(\sigma), \tilde{X}^j(\sigma')] = \Theta \epsilon_{ijk} \partial_\sigma X^k \delta(\sigma - \sigma') . \tag{102} \]

The same relations may be directly obtained by Dirac quantisation of the topological open membrane Lagrangian (77).

More generally, much as in the non-commutative case, this deformation amounts to multiplying the closed string scattering amplitudes by a phase factor proportional to the volume enclosed by the ribbons as they interact.

In this section we have seen how the presence of a C-field on the five-brane deforms the membrane boundary and its canonical quantisation leads to a noncommutative loop space on the five-brane. This also could be viewed as a thickening of the membrane boundary into ribbons. Recently there was a proposal for an effective description in terms of a field theory with nonassociative algebra [81]. The nonassociativity arises from a product made with the ribbons just as a product of open strings leads to a noncommutative field theory.
There are many open questions on key aspects of the quantum nature of this nonassociative theory. One hint, however, that this may be in the right direction is the appearance of nonassociativity in the membrane five-brane system from other considerations which will become apparent later.

7 Five-Branes from membranes

We now move to describing the membrane ending on the five-brane from the membrane perspective. The idea is that this will be a BPS solution of some multiple membrane effective action. This is in analogy with the description of the D1 brane ending on a D3 brane where, from the D1 brane perspective, the system is described as a fuzzy funnel solution of the D1 brane $\frac{1}{2}$ BPS equation [82]. Essentially the D1 has a (fuzzy) two sphere cross section whose radius diverges to give an additional three dimensions forming the D3 brane. The diverging fuzzy two sphere is known as a fuzzy funnel.

The goal will be to reverse engineer the membrane action. First we decide on the properties of the required solution needed to describe a membrane ending on a five-brane. Then we construct an equation whose solutions have such properties and finally we will be able to deduce the action from which that equation is a Bogmolnyi equation. It should be stressed that this is all conjectural since the starting point cannot be derived from some fundamental action as the non-Abelian D-brane action can. Instead the justification will that the equation has the right properties required to describe the membrane five-brane system. This is essentially M-theory phenomenology. We will follow thesis by Neil Copland [83] and the original work of Basu and Harvey, [14].

7.1 Expected properties for the membrane fuzzy funnel

We know that the five-brane picture of the membrane ending is in terms of the self-dual string. Examining the self-dual solution [60] we see the relationship between the radial direction, $R$, tangent to the five-brane (i.e. $R = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2}$),
and $s$, the membrane worldvolume direction away from the five-brane, will be:

$$s \approx \frac{Q}{R^2},$$  \hspace{1cm} (103)

(Q is the membrane charge). As the self-dual string is a static solution with no dependence on $\sigma$, the coordinate along the string, the active scalars should only depend on $s.$

In the D1-D3 system there were three active scalars transverse to the string in the directions of the D3-brane worldvolume. This meant the cross section was a (fuzzy) two-sphere, as expected for a Blon spike. In the membrane five-brane system we have an extra transverse dimension so the cross section should be a (fuzzy) three-sphere, associated to the $SO(4)_T$ R-symmetry of the self-dual string.

Coordinates on a fuzzy three sphere, $G^i$, obey the equation

$$G^i + \frac{1}{2(n + 2)} \epsilon_{ijkl} G_5 G^j G^k G^l = 0,$$  \hspace{1cm} (104)

where $G_5$ is a constant matrix obeying $\{G_5, G^i\} = 0$ and $(G_5)^2 = 1,$ [14]. $n$ is associated to an $SO(4)$ representation as follows. Matrices in the fuzzy sphere algebra may be thought of in terms of $SO(4)$ representations; $SO(4)$ is decomposed into $SU(2) \oplus SU(2)$ and under this decomposition the fuzzy three sphere matrices are restricted to be in $(\frac{n+1}{4}, \frac{n-1}{4})$ with $n$ labelling the representation. (In what follows we move between using $N$ as referring to $N \times N$ matrices and $n$ as referring to the above $SO(4)$ representation associated to the fuzzy sphere algebra.) This equation is a quantised version of a higher Poisson bracket equation for a three sphere and was first derived in [14].

7.2 The Basu-Harvey Equation

By analogy with the D1-D3 system the Basu-Harvey equation is conjectured to be a Bogomol’nyi equation for minimising the energy. It should also, as usual, follow from the vanishing of the supersymmetry variation of the Fermions on the membrane.
The Basu-Harvey equation is given by
\[ \frac{dX^i}{ds} + \frac{M_{11}^3}{8\pi\sqrt{2N}} \frac{1}{4!} \varepsilon_{ijkl}[G_5, X^j, X^k, X^l] = 0. \] (105)

The anti-symmetric 4-bracket is a sum over permutations with sign, e.g.
\[ [X^1, X^2, X^3, X^4] = \sum_{\text{perms } \sigma} \text{sign}(\sigma) X^{\sigma(1)} X^{\sigma(2)} X^{\sigma(3)} X^{\sigma(4)}, \] (106)

and it can be thought of as a quantum Nambu bracket \[103, 104\]. We could have used such a bracket for the second term of (104) once the correct combinatorial factor is included.

\(G_5\) and the scalars will belong to an algebra containing the fuzzy three-sphere. There are three main possibilities for what this algebra is. The first is \(\text{Mat}_N(\mathbb{C})\), the algebra of \(N \times N\) matrices, where \(N\) is the dimension of the representation of \(SO(4)\). For the fuzzy three-sphere only, this dimension coincides with the square of the radius in terms of the \(G^i\) (i.e. \(\sum_i trG^iG^i = N\)). \(N\) is what we will identify with the number of membranes, and this is the \(N\) that appears in the Basu-Harvey equation (105). A second possibility for the algebra is that generated by the \(\{G^i\}\), which is a sub-algebra of \(\text{Mat}_N(\mathbb{C})\). The third option is the algebra which in the large-\(N\) limit agrees with classical algebra of functions on \(S^3\), namely the spherical harmonics. For now, we will assume our fields are in \(\text{Mat}_N(\mathbb{C})\).

### 7.3 The Membrane Fuzzy Funnel Solution

We expect a static solution with scalars proportional to the fuzzy three sphere coordinates \(G^i\) and only depending on \(s\). An ansatz
\[ X^i(s) = f(s)G^i \] (107)
leads quickly to the solution
\[ X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{\frac{3}{2}} \sqrt{s}} G^i. \] (108)
The physical radius is given by

\[ R = \sqrt{\frac{\text{Tr} \sum (X^i)^2}{\text{Tr} \frac{1}{2}}} ] \]  

(109)

This implies,

\[ s \sim \frac{N}{R^2}, \]  

(110)

which is the self dual string behaviour we expect when we identify \( N \) with the number of membranes. Other solutions to the Basu-Harvey equation have been found that describe multiple parallel five-branes [105]. Later we will describe generalisations of the equation to allow the description of arbitrary five-brane calibrated geometries.

8 An Action for Multiple Membranes

Given that the Basu-Harvey equation should arise as a Bogomol’nyi equation, we define the energy of our static configuration with four non-zero scalars to be [14]:

\[
E = T_2 \int d^2\sigma \text{Tr} \left[ \left( \frac{dX^i}{ds} + \frac{M_{ii}^3}{8\pi \sqrt{2N}} \frac{1}{4!} \epsilon_{ijkl} [G^5, X^j, X^k, X^l] \right)^2 \right. \\
\left. + \left( 1 - \frac{M_{ii}^3}{16\pi \sqrt{2N}} \frac{1}{4!} \epsilon_{ijkl} \left\{ \frac{dX^i}{ds}, [G^5, X^j, X^k, X^l] \right\} \right)^2 \right]^{1/2}. 
\]

(111)

The membrane tension \( T_2 \) is given by \( T_2 = M_{ii}^3/(2\pi)^2 \) and the integral is over the two spatial worldvolume directions \( \sigma \) and \( s \). In what follows we will consider the \( X^i \) to obey \( \{G^5, X^i\} = 0 \). In terms of the fuzzy three-sphere algebra this means restricting the \( X^i \) to lie in \( \text{Hom}(R^+, R^-) \) or \( \text{Hom}(R^-, R^+) \). This is of course obeyed by \( \{G^i\} \).

It means that by multiplying out the squares in (111) \( G^5 \) can be eliminated.

If the scalars obey the Basu-Harvey equation then Bogomol’nyi bound is satisfied and the energy density linearises:

\[
E = T_2 \int d^2\sigma \text{Tr} \left( 1 - \frac{M_{ii}^3}{8\pi \sqrt{2N}} \epsilon_{ijkl} \frac{dX^i}{ds} G^5 X^j X^k X^l \right). 
\]

(112)

The energy can be written for large \( N \) as

\[
E = NT_2L \int ds + T_5L \int 2\pi^2 dRR^3, 
\]

(113)
where we have used \( T_5 = M_{11}^6/(2\pi)^5 \) and \( L \) is the length of the string. These two terms have the energy densities you would expect for \( N \) membranes and a single five-brane respectively.

We would expect this analysis to be only valid at the core \((s \to \infty)\), though for large \( N \) it agrees with the M5-brane picture, a description which should only be valid in the opposite limit. Examining (111) we can see that a Taylor expansion in terms of powers of \( X^i \) is valid when \( M_{11}^6 R^6 N^3 \ll 1 \), that is \( R \ll \sqrt{N} M_{11}^{-1} \). Thus if \( N \) is large, \( R \) can be large as well.

Given the expression for the energy (111) we can expand and deduce terms in the associated action. Also the action should be a function of the eight transverse scalars and allow \( \sigma \) dependence. This reasoning yields:

\[
S = -T_2 \int d^3 \sigma \text{Tr} \left[ 1 + (\partial_a X^M)^2 - \frac{1}{2N.3!} [X^M, X^N, X^P][X^M, X^N, X^P] \\
+ \frac{1}{2N.4.3!} [\partial_a X^L, [X^M, X^N, X^P]] \times \left( [\partial^a X^L, [X^M, X^N, X^P]] \\
+ [\partial^a X^M, [X^L, X^P, X^N]] + [\partial^a X^N, [X^L, X^M, X^P]] \\
+ [\partial^a X^P, [X^L, X^N, X^M]] \right) \right]^{1/2}
\]

(114)

for a multiple membrane action. \( L, M, \ldots \) labels the 8 transverse directions and \( a, b, \ldots \) the 3 worldvolume directions. The three-bracket used is defined analogously to the quantum Nambu 4-bracket (106) as a sum over the six permutation of the entries, with sign.

Just as in the case of a single membrane one may use supersymmetry to limit the form of the action. Significant progress in this direction has made in [15].

### 8.1 Fluctuations on the Funnel

We now analyse the fluctuations analysis on the membrane fuzzy funnel in the four directions transverse to both the membrane and the five-brane. We take a general kinetic term and the sextic coupling given in the last section and carry out a linear analysis of the membrane fluctuations. In flat space and static gauge, the pull back
of the metric is given by $P[G]_{ab} = \eta_{ab} + \partial_a X^M \partial_b X^M$, taking the determinant will lead to the first two terms of (114). This gives the action used for fluctuation analysis,

$$S = -T_2 \int d^3 \sigma \text{Tr} \sqrt{- \text{det}(P[G]_{ab}) - \frac{1}{2N} \frac{1}{3!} [X^M, X^N, X^P][X^M, X^N, X^P]}.$$  

(115)

The fluctuations may depend on all three worldvolume coordinates and are proportional to the identity in the fuzzy sphere algebra, $\delta X^m(t,s,\sigma) = f^m(t,s,\sigma) \mathbb{1}_N$. Keeping terms up to quadratic order in the fluctuations, gives

$$[X^M, X^N, X^P][X^M, X^N, X^P] = 3(f^m)^2[X^i,X^j]^2,$$  

(116)

where $M, N, P$ run over all indices, $m$ runs over the directions transverse to the both branes and $i, j$ run over the non-zero scalars of the solution. Evaluation of the commutator squared on the right-hand side proceeds via

$$[G^i, G^j]^2 = 2G^i G^j G^i G^j - 2N^2 \mathbb{1}$$  

(117)

and

$$G^i G^j G^i G^j \mathcal{P}_{R^+} = -(n+1)(n+3) \mathcal{P}_{R^+} = -2N \mathcal{P}_{R^+}.$$  

(118)

Finally this leads to

$$[G^i, G^j]^2 = -2N(N+2).$$  

(119)

Returning to the action (115) we now have

$$S = -T_2 \int d^3 \sigma \text{Tr} \sqrt{H - H(\partial_t f^m)^2 + (\partial_s f^m)^2 + H(\partial_\sigma f^m)^2 + \frac{N+2}{2s^2}(f^m)^2},$$  

(120)

where

$$H = 1 + \frac{\pi N}{2M_{11}^3 s^3}.$$  

(121)

The equation of motion for the linearised fluctuations becomes

$$(H \partial_t^2 - \partial_s^2 - H \partial_\sigma^2) f^m(t,s,\sigma) + \frac{N+2}{2s^2} f^m(t,s,\sigma) = 0.$$  

(122)

In the $s \to \infty$ limit (where we have a flat membrane) the equation of motion reduces to

$$(-\partial_t^2 + \partial_s^2 + \partial_\sigma^2) f^m = 0.$$  

(123)
The solutions to this equation are plane waves with $SO(2, 1)$ symmetry in the world-volume directions, as one would expect for a membrane. Although in the opposite limit the analysis should not be valid, as per the earlier discussion we keep $N$ large and find agreement with what we would expect. As $s \to 0$, $H \sim s^{-3}$ and the equation of motion gives

$$(-\partial_t^2 + \partial_R^2)f^m + R^{-3} \frac{\partial}{\partial R} \left( R^3 \frac{\partial f^m}{\partial R} \right) = 0.$$  \hspace{1cm} (124)

This has exactly the $SO(2, 1) \times SO(4)$ symmetry that we would expect for the M5 worldvolume with string soliton.

9 Calibrations and a Generalised Basu-Harvey Equation

While the Basu-Harvey equation is successful in reproducing many of the properties desired for the M2-M5 system, it was not derived from any fundamental principle. Given this somewhat ad-hoc construction, more detailed and involved checks are desirable to establish its validity. The Basu-Harvey equation only excited four of the eight scalars on the membrane. It is natural to consider whether the Basu-Harvey equation may be generalised to include the other scalars on the membrane. This will give rise to a membrane description of five-brane calibrated geometries. Instead of the membranes blowing up into a single five-brane we will describe membranes blowing up into five-brane intersections or equivalently five-branes on calibrated cycles. (Recall that a calibrated cycle is a minimal volume surface that possesses a calibration form defining the cycle, see [91] for a description of calibrations.) This is essentially an M-theory version of [84] where D1 branes end on calibrated three brane geometries. This constitutes a good check on the validity of the Basu-Harvey equation; the membranes have to be able to describe any allowed supersymmetric five-brane configuration.

As in [84] we work with a ‘linearised’ action to describe BPS solutions of the coincident membrane theory. We then present the generalised Basu-Harvey equation, with the conditions necessary for it to appear as a Bogomol’nyi equation. This is then encoded as a supersymmetry variation (though not actually with a supersymmetric action). This generalised Basu-Harvey equation will have solutions that describe co-
incident membranes ending on intersections of five-branes corresponding to calibrated geometries. We will then list these possible configurations and some simple solutions to give a flavour of the intricacies involved. Along the way, we must solve the generalised form of the BPS equation and some algebraic conditions on the brackets. This leads to additional algebraic conditions whose two-bracket equivalents were identically solved in the simpler D1-D3 case. These algebraic conditions hint at the possibility of a new algebraic structure for the membrane fields.

9.1 A Linear Action for Coincident Membranes

The generalised version of the Nahm equation was found as the Bogomol’nyi equation of the linear action in [84]. The linear action is simply dimensionally reduced super Yang-Mills. For BPS states, the linear action is in fact equal to the full action (see [97] for a fuller discussion of this point) this allows us to work with the linear action rather than the much more complicated nonlinear Born-Infeld type action when dealing with BPS states. There is an analogous situation for the membrane theory. The starting point is the energy for coincident membranes ending on a single five-brane (111). If the Basu-Harvey equation (105) holds and \( \{ G_5, X^i \} = 0 \), then the remaining terms under the square-root can be rewritten as the perfect square

\[
E = T_2 \int d^2 \sigma \text{Tr} \left[ \left( 1 + \frac{1}{2} (\partial_a X^i)^2 - \frac{1}{2N} \frac{1}{2.3!} [X^i, X^j, X^k]^2 \right)^2 \right]^{1/2}.
\]

(125)

This is the energy that one would get from an action

\[
S = -T_2 \int d^3 \sigma \text{Tr} \left( 1 + \frac{1}{2} (\partial_a X^i)^2 - \frac{1}{2N} \frac{1}{2.3!} [X^j, X^k, X^l]^2 \right),
\]

(126)

which is the linearised form of the membrane action (114) for three non-zero scalars. This is the action we will use when looking for the generalised Basu-Harvey equation but with the indices \( i, j, k \) running over all eight scalars. Since we are dealing with static configurations it is sufficient to consider the energy functional ie. the Hamiltonian.
As is usual in M-theory, we do not have a coupling constant in which to expand. We can however expand in powers of $X^i$ and it is in this expansion that we are working to leading order.

### 9.2 A Generalised Basu-Harvey Equation

Consider the Hamiltonian,

$$E = \frac{T_2}{2} \int d^2 \sigma \text{Tr} \left( X^{i'} X^{i'} - \frac{1}{3!} [X^j, X^k, X^l] [X^j, X^k, X^l] \right)$$  

(a trivial constant piece corresponding to the flat brane has been subtracted). The indices $i, j, \ldots$ run from 2 to 9 and $X^{10}$ is identified with $\sigma$. The factor of $1/(2N)$, has been scaled out as was previously done with the numerical factors to simplify the presentation. Reintroduction of this factor just involves inserting a factor of $1/\sqrt{2N}$ with each three- or four-bracket. We proceed by using the usual Bogomol’nyi construction to write

$$E = \frac{T_2}{2} \int d^2 \sigma \left\{ \text{Tr} \left( X^{i'} + g_{ijkl} \frac{1}{4!} [H^*, X^j, X^k, X^l] \right)^2 + T \right\},$$

where $T$ is a topological piece given by

$$T = -T_2 \int d^2 \sigma \text{Tr} \left( g_{ijkl} X^{i'} \frac{1}{4!} [H^*, X^j, X^k, X^l] \right).$$

When there are only four non-zero scalars ($X^2, \ldots, X^5$) and $g_{ijkl} = \epsilon_{ijkl}$ this gives the Basu-Harvey equation and the topological piece gives the energy of the five-brane on which the membranes end. For more scalars $g_{ijkl}$ is essentially the calibration form of the five-brane on which the membrane ends.

If more than four scars are non-zero, then we must impose

$$\frac{1}{3!} g_{ijkl} g_{ipqr} \text{Tr} \left( [H^*, X^j, X^k, X^l][H^*, X^p, X^q, X^r] \right)$$

in order to be able to rewrite the action as in (128). $H^*$ is a more general form of $G_5$, chosen to have the analogous properties $\{H^*, X^i\} = 0$ and $(H^*)^2 = 1$. In fact using
these properties (130) reduces to the simpler form
\[ \frac{1}{3!} g_{ijkl} g_{ipqr} \text{Tr} \left( [X^j, X^k, X^l] [X^p, X^q, X^r] \right) = \text{Tr} \left( [X^i, X^j, X^k] [X^i, X^j, X^k] \right), \quad (131) \]
which is the M-theory version of the constraints given in [84] for the D1-D3 brane system.

Once we have written the energy in the form (128) using (131) then we can clearly minimise it by imposing the generalisation of the Basu-Harvey equation
\[ \frac{\partial X^i}{\partial s} + \frac{M_3^3}{\sqrt{2N}8\pi} g_{ijkl} \frac{1}{4!} [H^*, X^j, X^k, X^l] = 0. \quad (132) \]
This equation has factors restored, and \( g \) is a general anti-symmetric four-tensor. When \( g_{ijkl} = \epsilon_{ijkl} \) we recover the Basu-Harvey equation and (131) is an identity.

9.3 The Equation of Motion

The equation of motion following from the action (126) is given by
\[ X^{i''} = -\frac{1}{2} [X^j, X^k, [X^i, X^j, X^k]] \quad (133) \]
where the three bracket \( [A, B, C] \) is the sum of the six permutations of the three entries, but with the sign of the permutation determined only by the order of the first two entries; i.e. \( ABC, ACB \) and \( CAB \) are the positive permutations. By using the Bogomol’nyi equation (132) twice on the left-hand side it is equivalent to:
\[ \frac{1}{3!} g_{ijkl} g_{ipqr} [X^k, X^l, [X^p, X^q, X^r]] = -[X^j, X^k, [X^i, X^j, X^k]]. \quad (134) \]
After multiplying by \( X^i \) and taking the trace, we recover the constraint equation (131). Thus in summary, the solutions of the generalised Basu-Harvey equation (132) that obey the algebraic equation of motion (134) are minimal energy solutions to the equations of motion of the proposed membrane action (126).
9.4 Supersymmetry

In the D1-D3 system the Nahm equation could be derived either as the Bogomol’nyi equation for minimising the energy, or as a requirement for preserving half the supersymmetry.

Here we do not have a supersymmetry variation or indeed an action from which to start. What we can do is to impose by fiat a simple generalisation of the linearised supersymmetry variation for the D1-strings and determine whether it leads to a consistent picture of membranes ending on five-branes.

We find that if the generalised form of the Bogomol’nyi equation is satisfied then it leads to a simplified form of the supersymmetry variation, where the route to further simplification is the imposition of a set of projectors corresponding to the non-zero components of $g_{ijkl}$. Compatible sets of these projectors are in correspondence with the known calibrated five-brane intersections and the imposition of these projectors leads to the preservation of a certain fraction of supersymmetry, the fraction being that preserved by the corresponding five-brane intersection with a membrane attached. This is of course provided we satisfy the algebraic conditions on the brackets left over in the supersymmetry condition after imposition of the projectors. Similar to the case of D3-brane intersections, the total space of all the intersecting five-branes in the submanifold calibrated by $g = \frac{1}{4!}g_{ijkl}dx^i \wedge dx^j \wedge dx^k \wedge dx^l$.

It remains to check if the algebraic conditions on the brackets are enough to satisfy the constraint (131) and thus the equations of motion. It turns out that it is not quite enough, there are additional algebraic conditions, a set of equations of similar form for all configurations, which must be satisfied to solve the constraint. For the D1-D3 case these had a simpler form and were satisfied identically.

The most obvious suggestion for the supersymmetry variation is

$$\delta \lambda = \left( \frac{1}{2} \partial_\mu X^i \Gamma^\mu_\mu - \frac{1}{2.4!} [H^s, X^i, X^j, X^k] \Gamma^{ijk} \right) \epsilon.$$  

We substitute the generalised Basu-Harvey equation in the first term and rearrange.
The requirement that the supersymmetry variation vanishes becomes that

\[
\sum_{i<j<k} [X^i, X^j, X^k] \Gamma^{ijk}(1 - g_{ijkl} \Gamma^{ijkl}) \epsilon = 0,
\]

where we have removed an overall factor of $H^*$ from the left-hand side since, like $G_5$, it has trivial kernel. $\epsilon$ is the preserved supersymmetry on the membrane world volume and we have $\Gamma^{01} \epsilon = \epsilon$, where the membrane’s worldvolume is in the 0, 1 and 10 = $\#$ directions. We can then solve the supersymmetry condition (136) by defining projectors

\[
P_{ijkl} = \frac{1}{2} (1 - g_{ijkl} \Gamma^{ijkl}),
\]

where there is no sum over $i, j, k$ or $l$.

We normalise $g_{ijkl} = \pm 1$ so they obey $P_{ijkl} P_{ijkl} = P_{ijkl}$. (Note, in all the cases that we will consider, for each triplet $i, j, k$, $g_{ijkl}$ is only non-zero for at most one value of $l$).

We impose $P_{ijkl} \epsilon = 0$ for each $i, j, k, l$ such that $g_{ijkl} \neq 0$. Then by using the membrane projection ($\Gamma^{01} \epsilon = \epsilon$) we can see that each projector $P_{ijkl}$ corresponds to a five-brane in the $0, 1, i, j, k, l$ directions. To apply the projectors simultaneously, the matrices $\Gamma_{ijkl}$ need to commute with each other. $[\Gamma_{ijkl}, \Gamma_{i'j'k'l'}] = 0$ if and only if the sets $\{i, j, k, l\}$ and $\{i', j', k', l'\}$ have two or zero elements in common, corresponding to five-branes intersecting over a three-brane soliton or a string soliton.

Once we impose the set of mutually commuting projectors, the supersymmetry transformation (136) reduces to

\[
\sum_{g_{ijkl}=0} [X^i, X^j, X^k] \Gamma^{ijk} \epsilon = 0.
\]

Here we sum over triplets $i, j, k$, such that $g_{ijkl} = 0$ for all $l$. Using the projectors allows us to express these as a set of conditions on the 3-brackets alone.

10 **Five-Brane Configurations**

We will now describe the specific equations that correspond to the various possible intersecting five-brane configurations.
The five-branes must always have at least one spatial direction in common, corresponding to the direction in which the membrane ends. These configurations of five-branes can also be thought of as a single five-brane stretched over a calibrated manifold [91]. These five-brane intersections can be found in [92, 93, 109]. We list the conditions following from the modified Basu-Harvey equation, those following from the supersymmetry conditions (138) (with $\nu$ the fraction of preserved supersymmetry) and then discuss any remaining conditions required to satisfy the constraint (134).

### 10.1 Calibrated five-branes

The first configuration is the original set up of a membrane ending on a single five-brane.

$$g_{2345} = 1 \quad \nu = 1/2$$

(139)

\[
X^{2'} = -H^*[X^3, X^4, X^5] \quad , \quad X^{3'} = H^*[X^4, X^5, X^2] ,
\]

\[
X^{4'} = -H^*[X^5, X^2, X^3] \quad , \quad X^{5'} = H^*[X^2, X^3, X^4] .
\]

The next case is two five-branes intersecting over a three-brane corresponding to an SU(2) Kähler calibration of a two-surface embedded in four dimensions. In terms of the first five-brane’s worldvolume theory the condition for preserved supersymmetry is the Cauchy-Riemann equations for the the complex scalar $Z = X^6 + iX^7$. That is $Z$ must be a holomorphic function of the complex worldvolume coordinate $z = x^4 + ix^5$. The calibration form for this intersection is the Kähler form. The activated scalars in this case are $X^2$ to $X^7$. 

50
\[ g_{2345} = g_{2367} = 1 \quad \nu = 1/4 \] (140)

\[
X^{2'} = -H^*[X^3, X^4, X^5] - H^*[X^3, X^6, X^7] \quad X^{3'} = H^*[X^4, X^5, X^2] + H^*[X^6, X^7, X^2]
\]
\[
X^{4'} = -H^*[X^5, X^2, X^3] \quad X^{5'} = H^*[X^2, X^3, X^4]
\]
\[
X^{6'} = -H^*[X^7, X^2, X^3] \quad X^{7'} = H^*[X^2, X^3, X^6]
\]
\[
[X^2, X^4, X^6] = [X^2, X^5, X^7] \quad [X^2, X^5, X^6] = -[X^2, X^4, X^7]
\]
\[
[X^3, X^4, X^6] = [X^3, X^5, X^7] \quad [X^3, X^5, X^6] = -[X^3, X^4, X^7]
\]
\[
[X^4, X^5, X^6] = [X^4, X^5, X^7] = [X^4, X^6, X^7] = [X^5, X^6, X^7] = 0.
\]

In order to satisfy the constraint we need the \( X^i \)'s to satisfy the following equations:

Choose \( m \in \{2, 3\} \), \( i, j, k, l \in \{4, 5, 6, 7\} \), then

\[
\epsilon_{ijk} [X^i, X^m, [X^m, X^j, X^k]] = 0, \quad \text{(no sum over} \ m) \]
\[
\epsilon_{ijkl} [X^i, X^j, [X^m, X^k, X^l]] = 0. \quad (141)
\]

In the string theory case there were no additional equations, as apart from the Nahm like equations and algebraic conditions on the brackets all that was needed to solve the constraint was the Jacobi identity, \( \epsilon_{ijk} [\Phi^i, [\Phi^j, \Phi^k]] = 0 \). If \( X^m \) anti-commutes with \( X^i, X^j, X^k \) then the first equation of (141) reduces to the Jacobi identity. Similarly if \( X^m \) anti-commutes with \( X^i, X^j, X^k, X^l \) the second equation reduces to

\[
\epsilon_{ijkl} X^i X^j X^k X^l = 0. \quad (142)
\]

For all the following configurations the additional algebraic conditions take the same form as (141). Note, that although equations of this form are not satisfied for general matrices in \( \text{Mat}_N(\mathbb{C}) \), it is possible that if the \( X^i \) are restricted to a particular algebra then these equations could become identities. This suggests it is necessary to perhaps restrict the algebra of fields on the membrane. We will return to this idea later.
One can also have the following configurations as described in [16] with more and more complicated sets of equations: three five-branes intersecting on a three-brane corresponding to an SU(3) Kähler calibration of a two-surface embedded in six dimensions; three five-branes intersecting over a string which corresponds to an SU(3) Kähler calibration of a four-surface in six dimensions; and three five-branes and an anti-five-brane intersecting over a membrane which corresponds to the SU(3) special Lagrangian calibration of a three-surface embedded in six dimensions. These are all of the configurations preserving $1/8$ of the membrane supersymmetry. There exist additional calibrations preserving less supersymmetry with more five-branes, which can be treated in the same manner.

10.2 Solutions

One can solve the cases of intersecting five-branes by using multiple copies of the Basu-Harvey solution. The first multi-five-brane case is solved by setting

\[ X^i(s) = i\sqrt{\frac{2}{\pi}} \frac{1}{M_{11}^{3/2}} s H^i, \]  

where the $H^i$ are given by the block-diagonal $2N \times 2N$ matrices

\[
\begin{align*}
H^2 &= \text{diag} (G^1, G^1) \\
H^3 &= \text{diag} (G^2, G^2) \\
H^4 &= \text{diag} (G^3, 0) \\
H^5 &= \text{diag} (G^4, 0) \\
H^6 &= \text{diag} (0, G^3) \\
H^7 &= \text{diag} (0, G^4) \\
H^* &= \text{diag} (G^5, G^5),
\end{align*}
\]

which are such that

\[ H^i + \frac{1}{2(n+2)} g_{ijkl} \frac{1}{4!} [H^*, H^j, H^k, H^l] = 0. \]
This makes sure that the conditions following from the generalised Basu-Harvey equation vanish. The remaining conditions, following from the supersymmetry transformation, are satisfied trivially as all terms in the three brackets involved vanish for this solution. The first additional algebraic equation of (141) is satisfied for this solution and the second additional algebraic equation is trivially satisfied as there are no non-zero products of five different $X^i$'s.

The more complicated cases follow easily for example the third configuration is also given by the block-diagonal $3N \times 3N$ matrices

$$
H^2 = \text{diag} (G^1, G^1, G^1) \\
H^3 = \text{diag} (G^2, G^2, G^2) \\
H^4 = \text{diag} (G^3, 0, 0) \\
H^5 = \text{diag} (G^4, 0, 0) \\
H^6 = \text{diag} (0, G^3, 0) \\
H^7 = \text{diag} (0, G^4, 0) \\
H^8 = \text{diag} (0, 0, G^3) \\
H^9 = \text{diag} (0, 0, G^4) \\
H^* = \text{diag} (G^5, G^5, G^5). \quad (146)
$$

The other configurations have similar block diagonal solutions.

There will exist many solutions containing off-diagonal terms. These will describe configurations when the branes are no longer flat. It would be fascinating to find and analyse such solutions since then one would be describing curved five-branes using non-Abelian membranes.

11 Fuzzy Spheres, membranes and $Q^{3/2}_2$

So far, the membrane fields are taken to lie in the algebra of complex $N \times N$ matrices, $Mat_N(\mathbb{C})$. This choice is not unique as discussed in [17]. An alternative that we shall now consider is $A_n(S^3)$, the algebra that reduces to the classical algebra of functions
on the sphere in the large-$N$ limit. It is significantly more technically involved than $Mat_N(\mathbb{C})$, since it is not closed under multiplication. The lack of closure implies it is necessary to project back into the algebra after multiplying. The projection then leads to the algebra to become nonassociative, though obviously the nonassociativity disappears in the large-$N$ limit as it must to reproduce the classical algebra of functions.

To see how the projection works, decompose the full $Mat_N(\mathbb{C})$ fuzzy three-sphere basis into a basis of $SO(4)$ Young tableau. The projection is a restriction to allowing only completely symmetric diagrams, that is those with only one row. This is described in detail in [17, 83].

The essential point is that the algebra $\mathcal{A}_n(S^3)$ has a tantalising property. $n$ refers to representation of $SO(4)$ given by $(\frac{n+1}{4}, \frac{n-1}{4})$ which is a representation of the fuzzy sphere algebra. The number of degrees of freedom is no longer given by $N^2$ as it is for $Mat_N(\mathbb{C})$, but it is now given by $D = (n+1)(n+2)(2n+3)/6$ [17]. Thus for large $N$ where $n \sim \sqrt{N}$ we have that

$$D \sim N^{\frac{3}{2}},$$

exactly as expected for $N$ coincident membranes in the large $N$ limit.

We can see intuitively why a fuzzy three sphere should have such a scaling. Recall that the dependence of the radius of the fuzzy sphere on the number of membranes is given by:

$$R \sim \sqrt{Q_2}.$$  \hspace{1cm} (148)

A noncommutative space is equipped with a natural ultraviolet cut off given by the noncommutative scale. There is also an infrared cutoff given by the sphere size. Simply summing over all the spherical harmonics, ie. the modes on the sphere, with the cut-offs prescribed as above yields the number of degrees of freedom. In units where the noncommutativity scale is one (and the radius is large) this sum yields $R^3$. Therefore, the number of modes on a fuzzy three sphere whose radius scales as $\sqrt{Q_2}$ is:

$$D = Q_2^{\frac{3}{2}}.$$  \hspace{1cm} (149)
This means that one can interpret the $Q^{3/2}$ degrees of freedom corresponding to the non-Abelian membrane theory as coming from modes on the fuzzy sphere.

The appearance of a nonassociative algebra on five-branes in the presence of background flux is described in [110]. Here there is flux due to the ending of the membrane on the five-brane and so it would be natural for nonassociativity to appear as the membranes expand out to form a five-brane. Later we will see nonassociativity arising from another perspective when trying to supersymmetrise the Basu-Harvey equation.

Since the anti-symmetric bracket vanishes when we project onto symmetric representations, one may ask how the Basu-Harvey equation may still hold? One possible prescription is that the projection does not act inside the bracket, which instead is thought of as an operator $[G_5, \cdot, \cdot, \cdot] : (A_n(S^3))^3 \rightarrow A_n(S^3)$. To understand this we should remember that the Basu-Harvey equation (or rather the fuzzy sphere equation (104) on which it is based) is a quantised version of a higher Poisson bracket equation. If we were to fix one of the coordinates in a Poisson Bracket, the derivatives acting on that coordinate would give zero and the bracket would not hold. However, if we evaluate the derivatives and then fix the coordinate, the bracket equation still holds. Here the anti-symmetric bracket essentially encodes the derivatives, which is why the projection is not made inside the bracket.

12 A Nonassociative Membrane theory?

In the previous sections we have seen the appearance of nonassociativity resulting from the algebra of the fuzzy three sphere. Also there has emerged a sextic coupling of the proposed non-Abelian membrane theory and a proposed supersymmetric variation of the membrane theory. What was missing was a proper supersymmetric action including Fermions containing the sextic coupling described above and an appropriate supersymmetry variation. Supersymmetric theories have been studied for many years and theories with the sort of coupling in (114) were known not to be supersymmetrisable. The membrane must be a supersymmetric theory and so we must resolve the puzzle of how to make this theory supersymmetric.
Bagger and Lambert [108] had the novel idea to consider the membrane fields to be part of some nonassociative algebra. As we will see below, the presence of the nonassociativity allows theory to be made supersymmetric. This is remarkable since this now opens up the possibility of a whole new class of supersymmetric theories that are only supersymmetric if the fields take values in some nonassociative algebra.

The goal will be to have a supersymmetric theory with eight scalars for which the condition for half the supersymmetry transformations to vanish will be the Basu-Harvey equation. That is, the BPS equation of theory, as defined by configurations that preserve half supersymmetry, will be the Basu-Harvey equation.

We will consider the eight scalars labelled by $X^I$, $I = 1..8$ with Fermions, $\Psi$. The supersymmetry transformations will be:

\[
\delta X^I = i \bar{\epsilon} \Gamma^I \Psi \\
\delta \Psi = \partial_\mu X^I \Gamma^\mu \Gamma^I \epsilon + i \kappa [X^I, X^J, X^K] \Gamma^{IJ} \epsilon
\]

Then the vanishing of the supersymmetry transformation on Fermions obeying:

\[
P_{M2} \Psi = \Psi \quad \text{and} \quad P_{M5} \Psi = \Psi
\]

implies the Basu Harvey equation.

Now if $X^I$ took values in a Lie Algebra then the triple bracket would be equivalent to the Jacobi identity and so would vanish. We thus take $X^I$ to be values in a nonassociative algebra where the Jacobi identity is not obeyed. Given the product on the algebra $(.)$, then the associator is defined as:

\[
<X^I, X^J, X^K> = (X^I . X^J) . X^K - X^I . (X^K . X^K)
\]

The triple bracket is then defined as:

\[
[X^I, X^J, X^K] = \frac{1}{12} < X^{[I}, X^{J}, X^{K]} >
\]

The closure of this algebra is interesting since it only closes up to translations and a conjectured local symmetry transformation. This is in analogy with the D2 brane
where the supersymmetry algebra only closes up to translations and gauge transformations.

The new *gauge* symmetry on the membrane, required to close its supersymmetry algebra, is given by:

$$
\delta X^I = 6i\kappa v_{JK}[X^I, X^J, X^K]
$$

and

$$
\delta \Psi = \frac{3i\kappa}{16} v_{KL} \Gamma^K \Gamma^{IJ}[X^I, X^J, \Psi] - \frac{9i\kappa}{8} v_{IJ}[X^I, X^J, \Psi] - 6i\kappa v_{IK} \Gamma^{JK}[X^I, X^J, \Psi] + \frac{3i\kappa}{192} v_{JKN} \Gamma^{J} \Gamma^{KN} \Gamma^P[X^I, X^P, \Psi],
$$

where

$$
v_{IJ} = i\bar{\epsilon}_1 \Gamma_{IJ} \epsilon_2, \quad v_{JKN} = -i\bar{\epsilon}_1 \Gamma_J \Gamma_{KN} \epsilon_2.
$$

The meaning of these transformations is still uncertain. Obviously one would require some sort of connection to make these transformations local but there cannot be any more local degrees of freedom so the gauge field would need to be be purely topological. This is still an open question.\(^8\)

The action which possess this supersymmetry is most easily expressed in superspace formalism. In what follows we construct a Lagrangian that is invariant under four of the sixteen supersymmetries. Only the $SU(4) \times U(1)$ subgroup of the $SO(8)$ $R$-symmetry will be manifest. This is the best that one can do (as yet), the full $SO(8)$ invariant version is still not known. It is conjectured that this is because the above unknown gauge symmetry is not understood and to get the full $SO(8)$ invariant theory will require the gauge field. This is analogous to $\mathcal{N} = 4$ Yang-Mills where in the purely scalar-spinor sector one can only see a $SU(3) \times U(1)$ subgroup of the $SO(6)$ $R$-symmetry. The fields transform under the $SU(4) \times U(1)$ symmetry as follows:

$$
X^I \rightarrow Z^A \oplus \bar{Z}_{\dot{A}} \in 4(1) \oplus \bar{4}(-1)
$$

\(^8\)Current as yet unpublished work by Niel Lambert may throw further light on this question.
\[ \Psi \rightarrow \psi^A \oplus \bar{\psi}_A \in \mathbf{4}(-1) \oplus \mathbf{4}(1) \]
\[ \epsilon \rightarrow \epsilon \oplus \epsilon^* \oplus \epsilon^{AB} \in \mathbf{1}(-2) \oplus \mathbf{1}(2) \oplus \mathbf{6}(0). \]

Restricting to supersymmetries generated by \( \epsilon \) i.e. the N=2 algebra defined by \( \epsilon^{AB} = 0 \), gives:
\[ \delta Z^A = i\bar{\epsilon}\psi^A, \quad \delta \psi^A = \gamma^\mu \partial_\mu Z^A \epsilon + i\kappa_1 \epsilon^{ABCD}[Z_B, Z_{\bar{C}}, Z_{\bar{D}}] \epsilon^* + 3i\kappa_3[Z^A, Z^B, Z_{\bar{B}}] \epsilon \]

This corresponds to imposing:
\[ \Gamma_{5678} \epsilon = \Gamma_{56910} \epsilon = -\epsilon. \]  

We can then interpret these projections as that corresponding to the SU(3) Kahler calibrated geometry described in [16].
Thus this set up seems to be related to a particular calibrated five-brane configuration. How one may relate this to other five-brane calibrations is still not known. One must also note that the above supersymmetries do not close and so will require an additional constraint. This is most easily written in the superspace formalism which we now come to. Using the chiral superfield,
\[ Z^A = Z^A(y) + \bar{\theta}^* \psi^A(y) + \bar{\theta}^* \theta F^A(y) \]
where \( y^\mu = x^\mu + i\bar{\theta}\gamma^\mu \theta \), the action may be expressed as:
\[ S = \int D^4\theta tr(Z^A \bar{Z}_A) + \int D^2\theta W(Z) + \int D^2\theta^* \bar{W}(Z_A) \]
where the superpotential \( W(Z) \) is given by:
\[ W(Z) = -\frac{\kappa}{8} \epsilon_{ABCD} Tr(Z^A, [Z^B, Z^C, Z^D]). \]
To make this work we need a nonassociative algebra since if \( Z^A \) were Lie algebra valued then this superpotential would vanish. This is described in the next section.
We must also supplement this with the superspace constraint
\[ [Z^A, Z^B, Z_{\bar{B}}] = 0 , \]
which ensures the closure of the supersymmetry algebra.
12.1 The nonassociative algebra

We assume that the algebra is equipped with a bilinear trace form:

\[ Tr : A \times A \rightarrow C \]  

which obeys:

\[ Tr(A, B) = Tr(B, A) \quad Tr(A.B, C) = Tr(A, B.C) \]  

There is also a complex involution of the algebra,

\[ # : A \rightarrow A \]  

such that \( #^2 = 1 \) and

\[ Tr(A, A#) \geq 0 \]  

with equality only when \( A = 0 \). Complex conjugation in the algebra is then given by \( # \).

For a Lie algebra such as SU(N) there is an associative anti-symmetric product \( i[,] : A \times A \rightarrow A \) which preserves Hermiticity. What is required here is a trilinear map that preserves the Hermitian condition. That is a trilinear map \( [,,,] : A \times A \times A \rightarrow A \) that obeys:

\[ (i[X^I, X^J, X^K])# = i[X^I, X^J, X^K] . \]  

We will now produce an explicit example of a nonassociative algebra. Take \( N \times N \) matrices but with a product defined as follows:

\[ A.B = QABQ \]  

Where \( Q \) is a constant invertible matrix and the right handside is the usual matrix product. The trace is defined by:

\[ Tr(A, B) = tr(Q^{-1}AQ^{-1}B) \]  

where \( tr \) is the usual matrix trace. The generalised conjugation is given by:

\[ A# = QA^{\dagger}(Q^{-1})^{\dagger} \]
The associator of this algebra is:

\[ <A, B, B, C> = Q^2ABQCQ - QAQBCQ^2 \]  

To eventually yield the Basu-Harvey equation we take:

\[ Q = \frac{1 + iG_5}{\sqrt{2}} \]  

It is curious to see how \( G_5 \) now plays a key role in the nonassociative algebra. Using the above we may write the superspace action as:

\[ S = \frac{1}{2} \int d^4\theta Tr(Z^A Z_A) + -\frac{\kappa}{8} \int d^2\theta \epsilon_{ABCD} Tr(G_5 Z^A, Z^B Z^C Z^D) + h.c. \]  

and then the BPS equation becomes:

\[ \partial_\sigma Z_A + \frac{\kappa}{4!} \epsilon_{ABCD}[G_5, Z^B, Z^C, Z^D] = 0 \]  

which we recognise as the Basu-Harvey equation with the parameter \( \kappa \) identified appropriately.

The suggestive punch line, however, is the counting of degrees of freedom. Although the above representation uses \( N \times N \) matrices for which we would expect \( N^2 \) degrees of freedom, the algebra allows us to truncate the matrices to only \( N^{3/2} \) degrees of freedom. To see this, consider \( N \times N \) matrices with \( N = n^2 \) of the form:

\[ X = \sum_{k}^{n-1} X_k \otimes \Omega^k \]  

where \( \Omega \) is an \( n \times n \) matrix, such that \( \Omega^n = 1 \). Thus \( X \) has \( n^3 = N^{3/2} \) degrees of freedom. Using the multiplication rule defined above with \( Q = \Omega^q \otimes 1 \) for some integer \( q \) we obtain a nonassociative algebra with the appropriate degrees of freedom. Taking \( q = n/4 \) and \( G = \Omega^{n^2/2} \otimes 1 \) we reproduce the Basu-Harvey equation.

This nonassociativity is of course different fundamentally to that discussed previously in the context of fuzzy three spheres. There the nonassociativity and \( N^{3/2} \) arose naturally but the system was not supersymmetric. Here the supersymmetry is manifest and naturally requires the nonassociativity but the \( N^{3/2} \) is a possibility that is imposed ad hoc.
There are numerous open questions concerning this formalism. Perhaps the most important is the aforementioned local symmetry required by the closure of the supersymmetry algebra. Understanding the role of this symmetry is crucial in determining whether Bagger and Lambert’s conjectured membrane action has a chance of being related to the non-Abelian membrane.

One crucial observation is the appearance of $\frac{1}{\sqrt{N}}$ in the Basu-Harvey equation and by extension also to $\kappa$ in the Bagger Lambert formulation of the nonassociative membrane. This indicates that theory we are discussing is likely to be a theory in the large $N$ expansion. Of course in the previous discussion of validity of the Basu Harvey equation we saw how this was also the realm of validity of the fuzzy funnel solution. Given that the membrane is a theory whose coupling is outside the quantum perturbative regime this explains why one can have some sort of simple semiclassical formalism. Carrying out a $\frac{1}{N}$ expansion provides a possible perturbative parameter for theory. More work on the $\frac{1}{N}$ expansion for the D2 brane seems to provide a hope to understand to how the Basu-Harvey equation from the string theory point of view.

13 Other ideas

In the previous sections we have described the basic problems of constructing theories with the appropriate degrees of freedom that satisfy the properties which we determined via various circuititious means. In this final section we will describe some other ideas and issues without entering into the details.

One might wish to take the view that the five-brane is simply the strong coupling limit of D4 brane and so should have a description in terms of a 5 dimensional Yang-Mills theory or even, after further compactification, a four dimensional gauge theory. From examining the supergravity description one can determine when the cross over from the D4 to the M-theory five-brane should occur. This scale when interpreted from the Yang-Mills point of view may be associated to fractional instantons.\footnote{This idea was expressed to me by Tong and attributed to Arkani-Hamed.} Thus the extra degrees of freedom in the five-brane theory may have an interpretation in terms...
of fractional instantons that have become in some sense light. A similar idea to this is the so called deconstruction approach where one has the $(0,2)$ theory arise out of a limit of a lower dimensional quiver gauge theory [129]. Both these ideas provide interesting connections to gauge theories but don’t reveal much more of the strict M-theoretic description.

One missing piece of information, that would throw light onto the self-dual string, would be a supergravity description of the membrane ending on the five-brane. There have been many attempts at this problem. Yet still a solution is far from being constructed. The reason for this is as follows. One would be tempted to impose the supersymmetry projector for the five-brane and membrane simultaneously and with a $SO(1,1) \times SO(4) \times SO(4)$ metric ansatz, look for brane solutions. This does not lead to the membrane ending on a five-brane but actually to solutions of smeared insecting five-branes all be it carrying membrane charge [130]. The reason for this is that imposing a projector in supergravity essentially imposes an isometry in those directions; this prevents a priori the solution from having the properties we are looking for. There cannot be any metric dependence on either brane world volume. This is in clear contradiction to what we would expect from the self-dual string. One possible solution to this is to consider deforming the known supersymmetry brane projectors so as to remove the presence of some Killing directions. This is essentially the type of solution constructing technique developed in [131]. There has, however, been recent progress in finding brane solutions in M-theory $AdS$ spaces, [132]. In this work, solutions are found with $SO(2,2) \times SO(4) \times SO(4)$ symmetry. This is precisely the symmetries one would expect for the supergravity dual of the decoupled self-dual string; the $SO(2,2)$ being the isometry group of $AdS_3$ and the $SO(4) \times SO(4)$ being the R-symmetry group of the string. Following this there has been important work in determining supergravity solutions of strings ending on branes and indeed the M-theory analogue of the membrane ending on the five-brane [133]. There have also been recent studies of the five-brane wilson surface operators using $AdS/CFT$ [136] aswell as further studies of the self dual string soliton in $AdS_4$ [137].

\[10^{10}\]The author is greatful to Chethan Gowdigere for discussion on this question. 62
One approach to the five-brane theory is simply to take it as a theory of a tensor multiplet and see how to construct a non-Abelian version. There are well known obstructions to this program. A clear no go theorem is expressed in [138] which details the inconsistencies of making a two form connection non-Abelian. However one may take the following view. A two form potential may be used to form a connection on loop space. Explicitly, for coordinates on the space of loops $X^\mu(\sigma)$, the loop space covariant derivative is:

$$D(\sigma)_\mu = \frac{\delta}{\delta X^\mu(\sigma)} + B_{\mu\nu}(X^\mu(\sigma))\partial_\sigma X^\nu(\sigma).$$  \hspace{1cm} (177)

The commutator of covariant derivatives yields a field strength:

$$[D(\sigma)_\mu, D(\sigma')_\nu] = H_{\mu\nu\rho}\partial X^\mu(\sigma)\delta(\sigma - \sigma')$$  \hspace{1cm} (178)

where $H = dB$, is the field strength of $B$, the Abelian two form connection. So one idea has been to generalise, not the two form connection, but the loop space connection. The no go theorem then states that this loop space connection cannot be expressed as an ultra local pull back of some target space field. The appearance of the deformation of the loop space by the presence of the background $C$ field on the brane indicated that this may well be a way to view the five-brane. Ideas of this sort have been explored in [134, 135] though as yet there is no clear theory of the five-brane.

This article on M-theory is missing a key section on the Matrix models of M-theory [18]. This is because in matrix theory the hardest things to explain seem to be the M-theory branes themselves, never mind their interactions. There has been some clear progress [139] on this issue, but gaining an insight into the $N^3$ or $N^{3/2}$ degrees of freedom or the membrane five-brane interaction seems to be out of the realm of matrix theory at the moment.

And so where are we at? This review has described the evidence for new light degrees of freedom in M-theory. There are hints that coincident membranes may be described (perhaps in the large $N$ limit) by some nonassociative field theory. However, many mysteries remain to be solved concerning aspects of M-theory brane interactions.
There are clues to new and exciting theories with novel properties; to solve M-theory questions we must move beyond examining the usual field theory suspects.

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