MUCKENHOUPT-WHEEDEN CONJECTURES IN HIGHER DIMENSIONS.

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ABSTRACT. In recent work by Reguera and Thiele [24] and by Reguera and Scurry [23], two conjectures about joint weighted estimates for Calderón-Zygmund operators and the Hardy-Littlewood maximal function have been refuted in the one-dimensional case. One of the key ingredients for these results is the construction of weights for which the action of the Hilbert transform is substantially bigger than that of the maximal function. In this work, we show that a similar construction is possible for classical Calderón-Zygmund operators in higher dimensions. This allows us to fully disprove the conjectures.

1. INTRODUCTION AND STATEMENTS OF RESULTS

In this paper we will study joint weighted estimates for the Hardy-Littlewood maximal operator and classical Calderón-Zygmund operators. We consider the non-centered Hardy-Littlewood maximal operator over cubes, defined for a locally integrable function $f$ as

$$Mf(x) = \sup_{x \in Q} \int_Q |f(y)| \, dy,$$

where $Q$ denotes the family of all cubes with sides parallel to the coordinate axes in $\mathbb{R}^d$. We will also consider classical Calderón-Zygmund singular integral operators, whose action on a smooth function $f$ is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x, y) f(y) \, dy.$$

Here the kernel $K$ has the form

$$K(x, y) = \frac{\Omega(x - y)}{|x - y|^d},$$

with $\Omega$ a homogeneous function of degree 0, such that $\Omega \in C^1(S^{d-1})$ and $\int_{S^{d-1}} \Omega(x) \, d\sigma_{d-1}(x) = 0$. The Hilbert transform in one dimension and the Riesz transforms in higher dimensions are examples of such operators. We may also consider more general Calderón-Zygmund operators. In fact, our arguments work well for operators with variable kernels $K$ satisfying standard size and regularity conditions. We will not pursue here these generalizations. Instead, we will make some comments on how to extend our results to this more general setting.

In this context, a weight simply means a non-negative function $w : \mathbb{R}^n \to [0, \infty]$. Such $w$ can be interpreted as the density of an absolutely continuous measure. This measure is usually denoted by the same letter as its density. That is, if $w$ is a weight in $\mathbb{R}^n$, for a measurable $E \subset \mathbb{R}^n$ one writes $w(E) = \int_E w(x) \, dx$ and for $f$ a measurable function we say that $f \in L^p(w)$ if $\|f\|_{L^p(w)} = (\int |f|^p w)\, dx < \infty$.

In the 1970’s, B. Muckenhoupt and Wheeden among other authors began the study of weighted inequalities for maximal, Calderón-Zygmund and other operators. They defined the $A_p$ class as the
collection of weights $w$ satisfying

\begin{equation}
\sup_{Q \in \mathcal{Q}} \int_{Q} w(y) \, dy \left( \int_{Q} w(y)^{-p'/p} \, dy \right)^{p/p'} < \infty,
\end{equation}

if $1 < p < \infty$, or

\begin{equation}
MW(x) \leq CW(x) \text{ a.e. } x,
\end{equation}

with $C > 0$ independent of $x$, if $p = 1$. It is well known that $w \in A_p$ is equivalent to $M$ being bounded on $L^p(w)$, if $p > 1$, and to $M$ being weakly bounded on $L^1(w)$, if $p = 1$. It is also known that (2) and (3) are sufficient too for the same kind of estimates of a Calderón-Zygmund operator, but only necessary in the sense that if all the $d$ Riesz transforms are weakly bounded on $L^p(w)$, then $w \in A_p(w)$, for $1 \leq p < \infty$. In the one dimensional case this means in particular that the Hilbert transform is weakly bounded on $L^p(w)$ if and only if $w \in A_p$, for $1 \leq p < \infty$. For a more complete account on these facts see [13] and [14].

The situation is more complicated when one considers norm estimates with two weights. A pair of weights $(u, v)$ is in the $A_p$ class if

\begin{equation}
\sup_{Q \in \mathcal{Q}} \int_{Q} v(y) \, dy \left( \int_{Q} u(y)^{1-p'/p} \, dy \right)^{1/p'} < \infty,
\end{equation}

for $p > 1$, and in $A_1$ if

\begin{equation}
Mv(x) \leq Cu(x), \text{ a.e. } x,
\end{equation}

with $C > 0$ independent of $x$. These conditions are equivalent to the mapping $M : L^p(u) \to L^{p,\infty}(v)$ to be bounded, for $1 \leq p < \infty$, and necessary for the strong boundedness $M : L^p(u) \to L^p(v)$, if $p > 1$, but not sufficient for it. The continuity of $M$ from $L^p(u)$ to $L^p(v)$ was, nevertheless, characterized by E. Sawyer [25] to be equivalent to

\[ \int_Q M(\chi_Q u^{1-p'})^{p} u \leq C \int_Q u^{1-p'} < \infty, \]

for all $Q \in \mathcal{Q}$. In the one weight setting some of the norm estimates for Calderón-Zygmund operators were shown to be equivalent to the ones for $M$. This suggested that similar connections might be found in the two weight setting. B. Muckenhoupt, R. Wheeden and others proposed several of them. For many years they could not be confirmed or refuted and became known as Muckenhoupt-Wheeden conjectures.

Perhaps the most famous one originates in a result by C. Fefferman and E.M. Stein [12] showing that there is an absolute constant such that for any weight $w$ one has

\begin{equation}
w \left( \{ x \in \mathbb{R}^d : M f(x) > \lambda \} \right) \leq \frac{C}{\lambda} \int f(x) Mw(x) \, dx.
\end{equation}

It has been attributed to Muckenhoupt and Wheeden the conjecture that the same two weight inequality should be true for a Calderón-Zygmund operator.

**Conjecture 1.** For each classical Calderón-Zygmund operator $T$, there exists a constant $C > 0$ so that for every weight $w$ one has

\begin{equation}
w \left( \{ x \in \mathbb{R}^d : |T f(x)| > \lambda \} \right) \leq \frac{C}{\lambda} \int |f(x)| Mw(x) \, dx,
\end{equation}

for all $\lambda > 0$ and $f \in L^1(Mw)$. 


The question was extended to more general operators and the conjecture was shown to be true for some square functions in [2], but false for fractional integral operators in [1]. The closest approach, on the positive side, for Calderón-Zygmund operators is due to C. Pérez, who showed in [19] that (7) is true if $M$ is replaced by the iterated operator $M^2$ or even by the operator $M_{L(\log L)^{\varepsilon}}$, with $\varepsilon > 0$. Later, C. Pérez and D. Cruz-Uribe [7], used the extrapolation technique to show that if (7) holds for a sublinear operator $T$, then one has

$$
\hat{|Tf(x)}|_p^p w(x) \leq C \int |f(x)|^p \left( \frac{MW(x)}{w(x)} \right)^p w(x) \, dx,
$$

for all $p > 1$. This necessary condition was disproved by M.C. Reguera and C. Thiele in [24] in the case $p = 2$, thus showing the conjecture to be false. They gave a counterexample in the one-dimensional case, that is, when $T$ is the Hilbert transform. The construction was based on a simplification of the technique used by M.C. Reguera in [22] in order to refute the corresponding assertion in the dyadic setting.

Our first result shows that Conjecture 1 is false for all classical Calderón-Zygmund operators.

**Theorem 1.** Let $T$ be a Calderón-Zygmund operator with an associated kernel satisfying (7). Then, $\forall N > 0$, $\exists w$ weight, $\exists f \in L^1(Mw)$ and $\exists \lambda > 0$ so that

$$
w(\{|Tf| > \lambda\}) \geq \frac{N}{\lambda} \int |f|Mw.
$$

D. Cruz-Uribe, C. Pérez and J.M. Martell in [5] considered another conjecture relating two weight estimates for the maximal operator and Calderón-Zygmund operators. This conjecture is also attributed to Muckenhoupt and Wheeden and its precise statement is the following.

**Conjecture 2.** Let $T$ be a Calderón-Zygmund operator as above, then

$$
\begin{align*}
M : L^p(u) &\to L^p(v) \\
M : L^{p'}(u^{1-p'}) &\to L^{p'}(u^{1-p'})
\end{align*}
$$

implies $T : L^p(u) \to L^p(v)$.

**Remark.** To simplify the notation throughout this work, the symbol ‘$S : X \to Y$’ will always mean that the operator $S$ maps the elements of the space $X$ into elements of $Y$ in a continuous way. This notation has been already used in the statement of the above conjecture.

The motivation for the second condition on $M$ is the following. A simple duality argument shows that since $T$ is an essentially self-adjoint operator, $T : L^p(u) \to L^p(v)$ is equivalent to $T : L^{p'}(u^{1-p'}) \to L^{p'}(u^{1-p'})$.

This conjecture was refuted by M.C. Reguera and J. Scurry in [23] for the Hilbert transform. Their counterexample is based on the one that disproved Conjecture 1 in [24]. We show that the conjecture is false again for every classical Calderón-Zygmund operator.

**Theorem 2.** Fix $1 < p < \infty$, and let $T$ be a Calderón-Zygmund operator as in Theorem 1. Then one can construct weights $u$ and $v$ such that $M : L^p(u) \to L^p(v)$ and $M : L^{p'}(u^{1-p'}) \to L^{p'}(u^{1-p'})$ but there exists an $f \in L^p(u)$ such that $\|Tf\|_{L^p(v)} = \infty$. 
One important observation is that while an $A_p$ weight is a.e. positive, the previous results have no assumptions on the support of the weight. In order for the questions we are treating to make sense, for $w$ a weight vanishing in some set of positive Lebesgue measure, we define $L^p(w)$ as the space of the measurable functions $f$ so that $\supp f \subset \supp w$ and $\|f\|_{L^p(w)} < \infty$. Indeed, one of the key ingredients in the proofs in \cite{24} and \cite{23} is to consider weights with sparse support. In \cite{23} it is shown that in the one-dimensional setting these weights do not preserve the equivalence of the boundedness of $M$ and $H$ on weighted $L^p$. We will extend this, showing that unlike for a.e. positive weights, in this setting the boundedness of $M$ on $L^p(w)$ does not imply the same result for Calderón-Zygmund operators.

**Theorem 3.** Let $T$ be a Calderón-Zygmund operator. Then there exist a weight $u$ and a function $f \in L^p(u)$ such that $M$ is bounded on $L^p(u)$ but $\|Tf\|_{L^p(u)} = \infty$.

Although our work does not make any contribution to them, for completeness we briefly comment still other important Muckenhoupt-Wheeden conjectures. Conjecture 2 had a weak version asserting that $M : L^p(u^{1-p'}) \to L^{p'}(u^{1-p'})$ implies $T : L^p(u) \to L^{p,\infty}(u)$, for $T$ a Calderón-Zygmund operator. This has been shown to be false for the Hilbert transform by D. Cruz-Uribe, A. Reznikov and A. Volberg in \cite{10}. By duality, Conjecture 1 implied this last conjecture. Thus, the argument in \cite{10} also refutes the one-dimensional case of Conjecture 1 in an indirect way.

At last, we mention a still open conjecture. It asserts that replacing the $L^p$ or $L^{1-p'}$ integrability requirement in (4) by a slightly stronger one in the sense of Orlicz integrals will be enough to guarantee the $L^p$ boundedness of Calderón-Zygmund operators. This is known as the bump conjecture and only partial results have been obtained so far. For more details see \cite{3,10,16,21,27}.

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1, 2 and 3 assuming the existence of some weights satisfying certain specific properties. Section 3 is devoted to the construction of these weights. As usual, $C$ and $c$ will denote positive constants, that may have different values at different occurrences. Also, given two quantities $A, B > 0$, by $A \sim B$ we mean that there exist a constant $C > 0$, which may depend on the dimension but is independent otherwise of the main parameters involved, such that $A \leq CB$ and $B \leq CA$.

## 2. PROOFS OF THE THEOREMS.

The proofs of the three Theorems stated in the previous section are based on the construction of weights satisfying a local $A_1$ property but allowing large values under the action of a given Calderón-Zygmund operator.

**Proposition 4.** Let $T$ be a Calderón-Zygmund operator with an associated kernel satisfying (4). Then, for each sufficiently large $N \in \mathbb{N}$, there exists a weight $w_N$ so that if we denote $D_N := \supp w_N \subset [0,1]^d$ we have both, $w_N \geq 1$ and $Mw_N \leq Cw_N$ on $D_N$ and $|Tw_N| \geq CNw_N$ on $D_N \subset D_N$, with $|\overline{D}_N| \sim |D_N|$ and $w_N(\overline{D}_N) \sim w_N(D_N) = 1$.

The conclusion $Mw_N \leq Cw_N$ in the support of $w_N$ is what makes $w_N$ an $A_1$ weight in a local sense. We will first prove Theorems 1, 2 and 3 assuming that Proposition 4 is true, leaving its proof for the next section.

\footnote{In a similar fashion, the expression $w^{\alpha}(x)$ for negative $\alpha$ is set to be zero at the points $x$ where $w(x) = 0$.}
Proof of Theorem 7 Consider $T^*$ the adjoint operator of $T$. Note that $T$ is an essentially self-adjoint operator, indeed we have $T^* f(x) = T f(-x)$. Given $N > 0$ consider the weight $w_N$ associated to $T^*$ from Proposition 4. Taking $f = w_N T^* w_N / (M w_N)^2$, we have

$$\int T f \, w_N = \int f \, T^* w_N = \int \left| \frac{T^* w_N}{M w_N} \right|^2 w_N \geq C N^2 w_N (D_N) \geq C N^2 > 0.$$  \hspace{1cm} (10)

Considering $F$ to be the non-increasing rearrangement of $|T f|$ with respect to $w_N$ in $\mathbb{R}^d$, we also have

$$\int \left| \frac{T^* w_N}{M w_N} \right|^2 w_N = \int T f \, w_N \leq \int |T f| \, w_N = \int w_N(\mathbb{R}^d) F(t) \, dt \leq \int \frac{1}{t^{1/2}} \sup_{s > 0} s^{-1/2} F(s)$$

$$= 2 \sup_{\lambda > 0} \lambda w_N(\{|T f| > \lambda\})^{1/2} \hspace{1cm} (11)$$

for some $\lambda_0$. Combined with (10), this yields

$$\left( \int \left| \frac{T^* w_N}{M w_N} \right|^2 w_N \right)^{1/2} \leq C \frac{1}{N} \int \left| \frac{T^* w_N}{M w_N} \right|^2 w_N \leq C \frac{1}{N} \lambda_0 w_N(\{|T f| > \lambda_0\})^{1/2}. \hspace{1cm} (12)$$

Now we define $E = \{|T f| > \lambda_0\}$ and $\omega = \chi_E w_N$. Using Hölder’s inequality and (12) we have

$$\int |f| M w = \int w_N \frac{T^* w_N}{(M w_N)^2} M w \leq \left( \int \left| \frac{T^* w_N}{M w_N} \right|^2 w_N \right)^{1/2} \left( \int \left| \frac{M w}{M w_N} \right|^2 w_N \right)^{1/2} \leq \frac{C}{N} \lambda_0 w_N(E),$$

the last inequality provided we show the following Lemma.

**Lemma 5.** There exists a constant $C > 0$ so that for all weights $v$ and all measurable sets $E \subset \mathbb{R}^n$ one has

$$\left( \int \frac{M(\chi_E v)^2}{M v} \right)^{1/2} \leq C v(E)^{1/2}. \hspace{1cm} (13)$$

**Proof.** Given a weight $v$ we define the operator $S_v$ for $f \in L^1_{\text{loc}}(v)$ as

$$S_v f(x) = \frac{M(fv)(x)}{M v(x)}. \hspace{1cm}$$

We will prove indeed a stronger result, that for all $p > 1$ one has

$$\int |S_v f|^p v \leq C \int |f|^p v.$$ 

Since $M(fv) \leq \|f\|_{L^\infty(v)} M v$, one has that $S_v$ is bounded on $L^\infty(v)$ with operator norm 1. By interpolation, the result is proved if we show that $S_v$ is of weak type $L^1(v)$ with a constant independent of $v$. Since it makes no essential difference, we will see it for $\tilde{S}_v f = \tilde{M} (fv) / \tilde{M} v$, where $\tilde{M}$ denotes the centered maximal operator. Let $f \in L^1(v)$ and $0 < \lambda < 1$. If $\tilde{S}_v f(x) > \lambda$, there exists $R_x > 0$ so that

$$\int_{Q(x, R_x)} |f| v > \lambda \tilde{M} v(x) \geq \lambda \int_{Q(x, R_x)} v > 0,$$

where by $Q(x, R)$ we mean the cube in $Q$ of edge length $R$ and centered at $x$. This implies that

$$v(Q(x, R_x)) \leq \frac{1}{\lambda} \int_{Q(x, R_x)} |f| v.$$
Observe that the cubes \( Q(x, R_x) \) with \( x \in A_\lambda := \{ x \in \mathbb{R}^d : \cdot \mathcal{F}(x) > \lambda \} \) are a Besicovitch cover of \( A_\lambda \). By Besicovitch Covering Theorem (see \cite{15}) there is a subcover by cubes \( Q(x, R_x) \), with \( x \in A_\epsilon \subset A_\lambda \), such that each \( x \in \mathbb{R}^d \) belongs to at most \( b_d \) cubes of the subcover, where \( b_d \) is a number that only depends on the dimension. Then we have

\[
v(A_\lambda) \leq v \left( \bigcup_{x \in A_\epsilon} Q(x, R_x) \right) \leq \sum_{x \in A_\epsilon} v(Q(x, R_x)) \leq \frac{1}{\lambda} \sum_{x \in A_\epsilon} \int_{Q(x, R_x)} |f| v \leq \frac{b_d}{\lambda} \int |f| v.
\]

This proves the lemma and, hence, Theorem 1 too.

Let us now prove Theorem 3.

**Proof of Theorem 3** We use the same ‘hump gliding’ argument as in \cite{23}. Let \( z \in \mathbb{R}^d \) be a unitary vector. We define \( w := \sum_{N=N_0}^{\infty} \tilde{w}_N \), where \( \tilde{w}_N(x) = w_N(x - 3^N z) \) and \( w_N \) are the weights described in Proposition 4 starting at some \( N_0 \) large. We also define \( g := \sum_{N=N_0}^{\infty} \frac{1}{N^s} \chi_{Q_N} \) with \( Q_N = [0, 1]^d + 3^N z \) and \( 1/p < \varepsilon < 1 \). Finally, we take \( u = w^{1-p} \) and \( f = gw \).

First, we check that \( f \in L^p(u) \):

\[
\int |f|^p u = \int g^p w = \sum_{N=N_0}^{\infty} \int_{Q_N} \frac{1}{N^p} w_N(x - 3^N z) \, dx = \sum_{N=N_0}^{\infty} \frac{1}{N^p} < \infty.
\]

Next, we see that \( Tf \notin L^p(u) \). In order to do so, we write \( \|Tf\|_{L^p(u)} \) as

\[
\left( \sum_{N=N_0}^{\infty} \int_{Q_N} \left( \frac{1}{N^\varepsilon} T \tilde{w}_N(x) + \sum_{J \neq N} \frac{1}{J^\varepsilon} T \tilde{w}_J(x) \right)^p \tilde{w}_N(x)^{1-p} \, dx \right)^{1/p}.
\]

By the triangle inequality this is greater than or equal to \( A - B \), where

\[
A = \left( \sum_{N=N_0}^{\infty} \int_{Q_N} \frac{1}{N^\varepsilon} T \tilde{w}_N(x) \left| \tilde{w}_N(x)^{1-p} \right| dx \right)^{1/p},
\]

\[
B = \left( \sum_{N=N_0}^{\infty} \int_{Q_N} \sum_{J \neq N} \frac{1}{J^\varepsilon} T \tilde{w}_J(x) \left| \tilde{w}_N(x)^{1-p} \right| dx \right)^{1/p}.
\]

We will see that \( A = \infty \) and \( B < \infty \). We begin with \( B \). If \( x \in Q_N \) and \( J \neq N \) we have

\[
|T \tilde{w}_J(x)| \leq \int_{Q_J} |K(x, y)w_J(y) - 3^J z| \, dy \leq \int_{R_J} \frac{C}{|3^N - 3^J|} \, w_J(y - 3^J z) \, dy \leq \frac{C}{\max\{3^N, 3^J\}^d} \sum_{y \in [0, 1]^d} w_J(y - 3^J z) \leq \frac{C}{3^{dN/2}3^dJ/2}.
\]

Here we have used that for \( y \in Q_J \) and \( J \neq N \) one has \( |x - y| \sim |3^N - 3^J| \sim 3^N + 3^J \). Hence,

\[
B^p \leq C \sum_{N=N_0}^{\infty} \int_{Q_N} \left( \sum_{J \neq N} \frac{1}{J^\varepsilon} \frac{1}{3^{dN/2}3^dJ/2} \right)^p \tilde{w}_N(x)^{1-p} \, dx
\]

\[
\leq C \sum_{N=N_0}^{\infty} \left( \sum_{J \neq N} \frac{1}{J^\varepsilon} \frac{1}{3^{dN/2}3^dJ/2} \right)^p < \infty.
\]
Now we proceed with $A$. Using an obvious change of variables in the integration and the property that $|T w_N| \geq C N w_N$ in $D_N$ we have

$$A^p = \sum_{N=N_0}^{\infty} \frac{1}{N N^p} \int_{[0,1]^d} |T w_N(x)|^p w(x)\, dx \geq \sum_{N=N_0}^{\infty} \frac{1}{N N^p} \int_{D_N} |T w_N(x)|^p w(x)\, dx$$

$$\geq C \sum_{N=N_0}^{\infty} N^p \frac{N^p}{N N^p} \int_{D_N} w(x)\, dx \geq C \sum_{N=N_0}^{\infty} N^p (1 - \varepsilon) = \infty.$$

It remains to prove that $M$ is bounded on $L^p(w)$. Since it makes no essential difference we will prove it for the centered maximal operator $\tilde{M}$ again. We define $Q_w = \{ Q \in Q : w(Q) > 0 \}$. For $f \in L^p(u)$ and $Q \in Q_w$ we have

$$\frac{1}{|Q|} \int_{Q} |f| = \frac{w(Q)}{|Q|} \frac{1}{w(1)} \int_{Q} |f w^{-1}| w.$$

This implies that

$$\tilde{M} f \leq \tilde{M} w \tilde{w} (f w^{-1}),$$

where $\tilde{M} w$ is the centered maximal operator associated to $w$ defined by

$$\tilde{M} w \ g(x) = \sup_{R>0, w(Q(x,R))>0} \frac{1}{w(Q(x,R))} \int_{Q(x,R)} |g| w.$$

It is easy to check that for $x \in Q_N$ one has $M w(x) \sim M w_N(x - 3N z) \leq C w_N(x - 3N z) = C w(x)$, that is

$$M w \sim w \quad \text{in supp } w.$$ (14)

Hence, since the same is true for $\tilde{M}$, we have

$$\int |\tilde{M} f|^p w^{1-p} \leq \int |\tilde{M} w|^p \tilde{w} (f w^{-1})^p w^{1-p} \leq C \int |\tilde{M} w (f w^{-1})|^p.$$ A well-known consequence of Besicovitch Covering Theorem is that $\tilde{M} w$ is bounded on $L^p(w)$. This, together with the observation that $f \in L^p(u^{1-p})$ if and only if $f w^{-1} \in L^p(w)$, finishes the proof.

\[\square\]

We now present the proof of Theorem\textsuperscript{2}. As we will see, everything reduces to the same arguments used in the proof of Theorem\textsuperscript{3}.

**Proof of Theorem\textsuperscript{2}** At this point we assume that the reader is familiar with the notation and the circle of ideas surrounding the proof of Theorem\textsuperscript{3}. Taking again $u(x) = \sum_{N=N_0}^{\infty} w_N(x - 3N z)$ we consider the weights $u = (M w/w)^p w$ and $w$. In view of (14), we have $u \sim w$ in $W = \text{supp } w$, which reduces the problem to the one weight setting.

It is easy to see that for an essentially self-adjoint operator $T$, the following inequalities are equivalent

$$\| T f \|_{L^p(u)} \leq C_u \| f \|_{L^p(u)},$$

$$\| T (f u^{1-p'}) \|_{L^p(w)} \leq C_u \| f \|_{L^p(u^{1-p'})},$$

$$\| T (f w) \|_{L^{p'}(u^{1-p'})} \leq C_u \| f \|_{L^{p'}(w)}.$$ (15)
Instead of (15) we will disprove (16). Taking again \( g = \sum_{N=N_0}^{\infty} \frac{1}{N^2} \chi_{Q_N} \), with \( 1/p < \varepsilon < 1 \), we have that \( g \in L^p(w) \). On the other hand,

\[
\|T(gw)\|^p_{L^p(w^1 - p')} = \int |T(gw)|^p \frac{w}{(Mw)^p} \geq C \int |T(gw)|^p w^{1-p'},
\]

and this last quantity was shown to be infinite in the proof of Theorem 3, except that the roles of \( p \) and \( p' \) were interchanged.

To prove \( M : L^p(u) \to L^p(v) \) is easy. For \( f \in L^p(u) \), using Fefferman-Stein inequality (6) and (14), we have

\[
\|Mf\|^p_{L^p(u)} = \int |Mf|^p w \leq C \int |f|^p Mw \leq C \int |f|^p w \leq C \int |f|^p \left( \frac{Mw}{w} \right)^p w = C\|f\|^p_{L^p(u)}.
\]

We finish showing that \( M : L^{p'}(u^{1-p'}) \to L^{p'}(u^{1-p'}) \). Similarly as before, for \( f \in L^{p'}(u^{1-p'}) \) we have

\[
\|Mf\|^p_{L^{p'}(u^{1-p'})} = \int |Mf|^p \frac{w}{(Mw)^p} \leq \int |Mf|^p w^{1-p'} \leq C \int |f|^p w^{1-p'},
\]

where the last inequality was obtained in the proof of Theorem 3 for \( p \) instead of \( p' \).

3. The construction of the weights

The construction of the weights \( w_N \) in Proposition 4 is an extension to higher dimension of the one by M.C. Reguera and C. Thiele in [24], which in turn was a simplification of the construction by M.C. Reguera in [22]. The argument is long and involves some technicalities.

Proof of Proposition 4 First we will give the basics of the construction of the weight \( w_N \) and of the sets \( D_N \) and \( \overline{D}_N \). Then we will proceed to estimate \( Mw_N \) on \( D_N \) and \( Tw_N \) on \( \overline{D}_N \), and we will complete the details of the construction of \( w_N \) so that the conclusion is reached.

The triadic decomposition. For \( k \in \mathbb{Z} \), we say that \( Q \) is a triadic cube of the \( k \)-th generation in \( \mathbb{R}^n \), if \( Q \) has edge length \( 3^{-k} \) and its vertices are points of the grid \( 3^{-k}\mathbb{Z}^n \). For any cube \( Q = Q(x, R) \) we define its triadic middle child as \( \hat{Q} = Q(x, R/3) \). For \( k = 0, 1, 2, \ldots \) we will consider \( T_k \) as a family of triadic cubes of the \( (Nk) \)-th generation, with \( N \in \mathbb{N} \) fixed. We define these families inductively. We begin with \( T_0 = \{[0, 1]^d\} \). Once \( T_k \) is determined, for each \( Q \in T_k \) we will select a family \( T_{k+1}(Q) \) of triadic subcubes so that \( T_{k+1}(Q) \subset \{ \text{triadic } Q' \subset \hat{Q}, \ |Q'| = 3^{-N_k} \} \) and \( \#T_{k+1}(Q) = A \sim 3^{(N_k - 1)d} \), with \( A \in \mathbb{N} \) a fixed number depending neither on \( Q \) nor on \( k \). The exact way of selecting these cubes will be explained later. Then we take \( T_{k+1} = \bigcup_{Q \in T_k} T_{k+1}(Q) \).

Contained in each \( Q \in T_k \) we consider a triadic cube \( J(Q) \) such that \( |J(Q)| = |Q'| = 3^{-N_k} \) for any \( Q' \in T_{k+1} \). We will place \( J(Q) \) having disjoint interior with respect to \( \hat{Q} \) but contiguous to it, in the sense that their boundaries intersect. In particular, the elements of the family \( \{ J(Q) \} \) are all disjoint. Moreover, if \( N \geq 3 \) and \( Q_0 \in T_k \), for some \( k_0 \),

(17) \[
\text{dist} \left( J(Q_0), \bigcup_{k=0}^{\infty} \bigcup_{Q \in T_k} J(Q) \right) \geq \frac{\ell}{3} - \frac{\ell}{3^N} \geq \frac{\ell}{4},
\]

with \( \ell = |J(Q)|^{1/d} \).
The construction of the weight. We define a weight $w_N$ supported in

$$D_N = \bigcup_{k=0}^{\infty} \bigcup_{Q \in T_k} J(Q),$$

so that $w_N$ is constant over each $J(Q)$ and if $x \in J(Q)$ with $Q \in T_k$ one has

$$\alpha_k = w_N(x) = \frac{w_N(J(Q))}{|J(Q)|} = \frac{w_N(Q')}{|Q'|},$$

for any $Q' \in T_{k+1}$. In this way

$$w_N(x) = \sum_{k=0}^{\infty} \alpha_k \sum_{Q \in T_k} \chi_{J(Q)}.$$

Observe that for $Q \in T_k$

$$w_N(Q) = w_N(J(Q)) + w_N(\overline{Q}) = w_N(J(Q)) + \sum_{Q' \in T_{k+1}(Q)} w_N(Q').$$

Using (18), the previous formula can be rewritten as

$$\alpha_k|Q| = \alpha_k|J(Q)| + \alpha_k \sharp T_{k+1}(Q) |J(Q)| = \alpha_k|J(Q)| + \alpha_k A |J(Q)|,$$

obtaining that

$$\frac{\alpha_k}{\alpha_{k-1}} = \frac{3^{Nd}}{1 + A} = a.$$

Hence, $\alpha_k = a^k \alpha_0$, for certain $\alpha_0$ and

$$w_N([0, 1]^d) = \sum_{k=0}^{\infty} \sum_{Q \in T_k} w_N(J(Q)) = \alpha_0 \sum_{k=0}^{\infty} \sharp T_k |J(Q)| a^k = \alpha_0 \sum_{k=0}^{\infty} A^{k-1} 3^{-Nd(k+1)} a^k

= \alpha_0 3^{-Nd} \sum_{k=0}^{\infty} \left( \frac{A}{1 + A} \right)^k = \alpha_0 3^{-Nd(1 + A)} = \alpha_0/a.$$

We take $\alpha_0 = a$ so that $w_N$ is a probability measure and $w_N \geq a > 1$ in $D_N$, as stated.

Controlling the maximal function. We prove here that $Mw_N \leq Cw_N$ in $D_N$, with a constant $C$ independent of $N$. Fix $x \in J(Q)$ with $Q \in T_k$ and take an arbitrary cube $R$ containing $x$. We want to show that

$$\frac{w_N(R)}{|R|} \leq C w(x).$$

If $|R|^{1/d} < 1/4|J(Q)|^{1/d}$, then $R \cap D_N = R \cap J(Q)$ from (17). This says that $w$ is constant in $R \cap J(Q)$ and the result is obvious. If, on the contrary, $|R|^{1/d} \geq 1/4|J(Q)|^{1/d}$ and we consider

$$A = \{ \text{triadic } Q', Q' \cap R \neq \emptyset, |Q'| = |J(Q)| \},$$

then $\sum_{Q' \in A} |Q'| \leq 9^d |R|$. We claim that if $L \subset [0, 1]^d$ is a triadic cube with size $|L| = |J(Q)|$ then

$$w_N(L) \leq \alpha_k |L|.$$

Using this, one has

$$\frac{w_N(R)}{|R|} \leq \frac{1}{|R|} \sum_{Q' \in A} w_N(Q') \leq \frac{\alpha_k}{|R|} \sum_{Q' \in A} |Q'| \leq g^d w_N(x).$$

The proof of (19) is easy. We have three possible situations:

i) $L \cap D_N = \emptyset$, and there is nothing to show.
ii) $L \subset J(Q_0)$, for some $Q_0 \in \mathcal{T}_j$ and $j \leq k$. In this case $w_N$ is constant in $L$ with value $\alpha_j$. Since $\alpha_j \leq \alpha_k$, the result follows immediately.

iii) $L = Q'$ for some $Q' \in \mathcal{T}_{k+1}$. Here we have directly $w_N(L) = \alpha_k |L|$ by definition.

**Splitting $T w_N$ into ‘continuous’ and ‘discrete’ pieces.** Taking, by a slight abuse of notation, $\widehat{D_N} := \bigcup_{k=0}^{\infty} \bigcup_{Q \in \mathcal{T}_k} J(Q)$, we want to prove that $|T w_N| \geq C N w_N$ in $\widehat{D_N}$.

Let $x \in \widehat{J(Q)}$, with $Q \in \mathcal{T}_k$. Then we have

$$T w_N(x) = \int_{Q^c} K(x, y) w_N(y) \, dy + \int_{Q \setminus J(Q)} K(x, y) w_N(y) \, dy + \text{p.v.} \int_{J(Q)} K(x, y) w_N(y) \, dy$$

$$= I + II + III.$$ We further split $I$ and $II$ into a ‘continuous’ and a ‘discrete’ part. Denoting by $c_R$ the center of a cube $R$, we have

$$I = \sum_{L \text{ triadic}, |L| = |Q|, L \neq Q} \int_L K(x, y) w_N(y) \, dy$$

$$= \sum_{L \text{ triadic}, |L| = |Q|, L \neq Q} K(c_Q, c_L) w_N(L) \, dy + \sum_{L \text{ triadic}, \mathcal{L} \neq Q, L \mathcal{L} \mathcal{Q}} \int_L (K(x, y) - K(c_Q, c_L)) \, w_N(y) \, dy$$

$$= I_1 + I_2,$$

and

$$II = \sum_{L \in \mathcal{T}_{k+1}(Q)} \int_L K(x, y) w_N(y) \, dy$$

$$= \sum_{L \in \mathcal{T}_{k+1}(Q)} K(c_{J(Q)}, c_L) w_N(L) \, dy + \sum_{L \in \mathcal{T}_{k+1}(Q)} \int_L (K(x, y) - K(c_{J(Q)}, c_L)) \, w_N(y) \, dy$$

$$= II_1 + II_2.$$ First, we will show that the ‘continuous’ parts $I_2$, $II_2$ and $III$ are ‘small’ in the sense that $|I_2| + |II_2| + |III| \lesssim w_N(x)$. Then we will show that $I_1$ is much bigger than $w_N$ by showing that $|II_1| \gtrsim N w_N(x)$. Although we will not have any control on $I_1$, we will construct $J(Q)$ and $\mathcal{T}_{k+1}(Q)$ so that $II_1$ has the same sign as $I_1$. In this way, we will have $|I_1 + II_1| \geq |II_1| \gtrsim N w_N(x)$. At that point we will get

$$|T w_N(x)| \geq |I_1 + II_1| - |I_2 + II_2 + III| \geq (cN - C) w_N(x) \geq C N w_N(x),$$

for sufficiently large $N$. This would prove the result.

**The ‘continuous’ pieces.** We recall the well-known fact (see [26] for instance) that our hypotheses on $K$ imply the following estimates: there exist $\delta, \eta > 0$ so that

$$|K(x, y) - K(x, \bar{y})| \leq C \frac{|y - \bar{y}|^\delta}{|x - y|^{d+\delta}},$$

if $|x - y| > (1 + \eta)|y - \bar{y}|$, and

$$|K(x, y) - K(\bar{x}, y)| \leq C \frac{|x - \bar{x}|^\delta}{|x - y|^{d+\delta}},$$

if $|x - \bar{x}| > (1 + \eta)|y - \bar{y}|$. Thus, we have

$$|I_1| \gtrsim \frac{1}{N} \int_{\widehat{D_N}} K(x, y) w_N(y) \, dy,$$

and

$$|II_1| \lesssim \frac{1}{N} \int_{\widehat{D_N}} K(x, y) w_N(y) \, dy,$$

where $\frac{1}{N}$ comes from the fact that $w_N$ has an average value of $N$. Since $N w_N(x) \gtrsim \frac{1}{N} \int_{\widehat{D_N}} K(x, y) w_N(y) \, dy$, we have shown that $|T w_N| \gtrsim C N w_N$. This completes the proof.
if $|x - y| > (1 + \eta)|x - \bar{x}|$. These estimates give rise to the so called $\delta$–Calderón-Zygmund kernels. Although in our case we have $\delta = 1$, it is worth observing that this part of the construction works for these more general kernels too.

When estimating $I_2$, first we use that $x \in \tilde{Q}$ and $y \in \tilde{L}$ to deduce

$$|x - y| \sim |x - c_L| \sim |c_Q - c_L|,$$

and as a consequence

$$|K(x, y) - K(c_Q, c_L)| \leq |K(x, y) - K(x, c_L)| + |K(x, c_L) - K(c_Q, c_L)| \lesssim \frac{|y - c_L|^\delta}{|x - y|^{d+\delta}} \lesssim Q^\delta/d$$

(22)

Hence,

$$|I_2| \lesssim Q^\delta/d \sum_{|L| = |Q| \in [0,1]^n \cap L \neq Q} \left| w_N(y) \int_L \frac{1}{|x - y|^{d+\delta}} dy \right| \leq Q^\delta/d \int_{|x - y| > |Q|^{1/\delta}} \frac{w_N(y)}{|x - y|^{d+\delta}} dy \lesssim M w_N(x).$$

The last inequality follows from the fact that $x \mapsto |x|^{-d-\delta}$ is a radially decreasing function and

$$\int_{|x - y| > |Q|^{1/\delta}} \frac{1}{|x - y|^{d+\delta}} dy = \left| Q^{d-1} \right| \int_{|Q|^{1/\delta}} \frac{t}{t^{d+\delta}} dt \sim \frac{1}{|Q|^{\delta/d}}.$$

(See [26].)

We estimate $II_2$ in a similar way. Since $J(Q)$ is not contained in $\tilde{Q}$, for $x \in J(Q)$, $y \in L \in T_{k+1}(Q)$ and $v_{J(Q)} \in J(Q)$ to be determined later, one has

$$|x - y| \sim |x - c_L| \sim |v_{J(Q)} - c_L|,$$

and

$$|K(x, y) - K(v_{J(Q)}, c_L)| \lesssim |K(x, y) - K(x, c_L)| + |K(x, c_L) - K(v_{J(Q)}, c_L)| \lesssim \frac{|J(Q)|^{\delta/d}}{|y - x|^{d+\delta}}.$$

Then, reasoning as before we obtain again

$$|II_2| \lesssim |J(Q)|^{\delta/d} \sum_{L \in T_{k+1}(Q)} \int_L \frac{w_N(y)}{|x - y|^{d+\delta}} dy \leq |J(Q)|^{\delta/d} \int_{|x - y| > |J(Q)|^{1/3}} \frac{w_N(y)}{|x - y|^{d+\delta}} dy \lesssim M w_N(x).$$

In order to bound $III$, we use that $w_N$ is constant over $J(Q)$ and the cancellation property of $K$ on $E = \{ y : |x - y| < |J(Q)|^{1/3} \}$ to obtain

$$III = w_N(x) \int_{J(Q)} \frac{\Omega(x - y)}{|x - y|^d} dy$$

$$= w_N(x) \int_{J(Q) \setminus E} \frac{\Omega(x - y)}{|x - y|^d} dy.$$

(23)
Hence
\[ |III| \leq w_N(x) \int_{J(Q) \setminus E} \frac{\|\Omega\|_{L^\infty}}{|J(Q)|} dy \lesssim w_N(x). \]

**Remark.** Observe that in all the above estimates we have not needed a precise description of the construction of the families \( \mathcal{T}_{k+1}(Q) \) and the cubes \( J(Q) \). The only information we have used so far is that each \( Q' \in \mathcal{T}_{k+1}(Q) \) is a triadic subcube of \( \hat{Q} \) of size \( 3^{-N(k+1)} \) and that \( J(Q) \) is of the same size and ‘touch’ \( \hat{Q} \) from the outside.

Another important observation is that for \( Q \in \mathcal{T}_k \), the term \( I_1 = I_1(Q) \) does not depend on the triadic cubes of the next generation. In particular, \( I_1(Q) \) is independent of \( \mathcal{T}_i \), for all \( i > k \). This is consistent with the inductive process that we use in order to define our weights \( w_N \).

The ‘discrete’ pieces in a simpler case: Riesz Transforms. To get some intuition of the construction, we will first consider a concrete example. Assume that \( T \) is a Riesz Transform, that is \( T = R_j \) for some \( j \in \{1, 2, \ldots, d\} \), where
\[
R_j f(x) = c_d \text{ p.v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) \, dy,
\]
and \( c_d \) is a normalizing constant depending on the dimension. In this case, given \( Q \in \mathcal{T}_k \), we choose \( \mathcal{T}_{k+1}(Q) \) to consist of all the triadic subcubes of \( \hat{Q} \) of size \( 3^{-N(k+1)} \). We take \( J(Q) \) to be a triadic cube of size \( 3^{-N(k+1)} \) contiguous to \( \hat{Q} \) so that their boundaries only share a point, hence a vertex. For \( x \in \mathbb{R}^d \), we denote by \( x_j \) its \( j \)-th coordinate. Now if \( I_1 \geq 0 \) we place \( J(Q) \) so that \( \min_{x \in J(Q)} x_j \geq \max_{x \in \hat{Q}} x_j \) and if \( I_1 \leq 0 \) we require instead \( \max_{x \in J(Q)} x_j \geq \min_{x \in \hat{Q}} x_j \). This makes the signs of \( I_1 \) and \( II_1 \) coincide. Calling
\[
\mathcal{T}_{k+1}(Q) = \{ L \in \mathcal{T}_{k+1}(Q) : |(c_L)_j - (c_{J(Q)})_j| = |c_L - c_{J(Q)}|_\infty = 3^{-N(k+1)} \},
\]
and taking \( v_{J(Q)} = c_{J(Q)} \) we have
\[
|II_1| \gtrsim \sum_{L \in \mathcal{T}_{k+1}(Q)} \frac{|(c_L)_j - (c_{J(Q)})_j|}{|c_L - c_{J(Q)}|_\infty} w_N(L) = a^{k+1} |J(Q)| \sum_{i=1}^{3^{N-1}} \frac{1}{|c_L - c_{J(Q)}|_\infty} = w_N(x) |J(Q)| \sum_{i=1}^{3^{N-1}} \frac{1}{i} \gtrsim N w_N(x).
\]

Observe also that in this case \( A = 3^{(N-1)d} \) and, therefore, \( a = \frac{3^{Nd}}{1 + A} \sim 3^d \).

**Finishing the construction of the measure for a general operator.** We will now explain how we chose \( J(Q) \) and \( \mathcal{T}_{k+1}(Q) \) so that \( II_1 \) behaves the way we need when \( T \) is a general Calderón-Zygmund operator. This choice will depend on \( T \).

Since \( \Omega \) is a continuous function over the sphere with null integral mean, there exist \( \lambda > 0, r > 0 \) and two points in the sphere \( z_0 \) and \( z_\infty \) so that for any \( y \in B^+ = B(z_+, r) \cap \mathbb{S}^{d-1} \) one has \( \Omega(y) > \lambda \) and for any \( y \in B^- = B(z_-, r) \cap \mathbb{S}^{d-1} \) one has \( \Omega(y) < -\lambda \). We have the same bounds for \( \Omega \) all over the cones \( U^+ = \{ tx : t > 0, x \in B^+ \} \) and \( U^- = \{ tx : t > 0, x \in B^- \} \). Using a rotation if necessary, we can assume that \( z_+ \) and \( z_- \) are symmetric with respect to all the coordinate axis and
that none of their coordinates are zero. This can be expressed in terms of coordinates with the relation

\[ |(z_+)_i| = |(z_-)_i| \neq 0 \text{ for all } i = 1, \ldots, d, \text{ or } z_+ = \tau z_- \text{ with} \]

\[
\tau = \begin{pmatrix}
\delta_{i,j} \frac{\text{sign}(z_-)_i}{\text{sign}(z_+)_j} & i,j=1,\ldots,d
\end{pmatrix} = \begin{pmatrix}
\pm 1 & 0 & \cdots & 0 \\
0 & \pm 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \pm 1
\end{pmatrix}.
\]

Note that also \( U^- = \tau U^+ \).

For a \( Q \in T_k \) we denote by \( v_+ \) (respectively, \( v_- \)) the only vertex of \( \hat{Q} \) such that the half-line \( s_+ \equiv v_+ + tz_+ \) (respectively, \( s_- \equiv v_- + tz_- \)), for \( t > 0 \), intersects the interior of \( \hat{Q} \). If \( I_1 \geq 0 \) we will choose \( v = v_{J(Q)} := v_+, z = z_+ \) and \( U = U^+ \). On the other hand, if \( I_1 \leq 0 \) we choose \( v = v_{J(Q)} := v_-, z = z_- \) and \( U = U^- \). Now we take \( J(Q) \) to be the only triadic cube of size \( 3^{-Nd} |Q| \) so that the boundaries of \( J(Q) \) and of \( \hat{Q} \) intersect only at \( v \). Once this is done we take

\[ T_{k+1}(Q) = \{ \text{triadic } R \subset \hat{Q} : |R| = 3^{-Nd} |Q|, c_R \in \hat{v} + U \}. \]

The construction guarantees that \( A = \sharp T_{k+1}(Q) \sim 3^{(N-1)d} \) is independent of \( k \) and \( Q \), as required before.

\[ Q \in T_k \]
We want to find a lower estimate for the last sum. We could use an argument similar to the one for the Riesz transforms but we will use a more direct one. For a positive integer \( i \) we define
\[
\Gamma_i = \{ x \in v + U \cap \hat{Q} : 3^{i-1} 3^{-N(k+1)} < |x - v| \leq 3^i 3^{-N(k+1)} \}.
\]
We also define
\[
\mathcal{T}^i_{k+1}(Q) = \{ R \in \mathcal{T}_{k+1}(Q) : c_R \in \Gamma_i \}.
\]

Now we choose \( N \) large enough to make \( |J(Q)| \) very small compared to \( \Gamma_{\lfloor N/2 \rfloor} \), so that the measure of \( \Gamma_i \) is comparable to the sum of the measures of the cubes in \( \mathcal{T}^i_{k+1} \) for \( \lfloor N/2 \rfloor \leq i \leq N - 1 \), that is
\[
|\Gamma_i| \sim \sum_{R \in \mathcal{T}^i_{k+1}(Q)} |R| = \#T^i_{k+1}(Q) |J(Q)|.
\]

Note that \( |J(Q)| = 3^{-Nd(k+1)} \) and \( |\Gamma_i| = \beta (3^{d(i-N(k+1))} - 3^{d(i-1-N(k+1))}) = 2\beta 3^{d(i-1-N(k+1))} \) for certain \( \beta > 0 \) that depends on the opening of the cube \( U \). The previous choice of \( N \) is possible since the quotient of the measures of \( \Gamma_{\lfloor N/2 \rfloor} \) and \( J(Q) \) is of the order of \( 3^{dN/2} \). The conclusion is that for \( \lfloor N/2 \rfloor \leq i \leq N - 1 \) one has \( \#T^i_{k+1}(Q) \sim 3^{di} \) and as a consequence
\[
|II_1| \geq \lambda w_N(x) |J(Q)| \sum_{i=\lfloor N/2 \rfloor}^{N-1} \sum_{L \in \mathcal{T}^i_{k+1}(Q)} \frac{1}{|c_L - v|^d}
\]
\[
\geq \lambda w_N(x) |J(Q)| \sum_{i=\lfloor N/2 \rfloor}^{N-1} \frac{\#T^i_{k+1}(Q)}{|3^i 3^{-N(k+1)}|^d} \geq \lambda N w_N(x).
\]

This finishes the proof of Proposition [4] \( \square \)
4. Final Remarks.

Variable Kernels. We point out that most of the arguments of the previous proof also work if \( K \) is a variable Calderón-Zygmund kernel with the standard size conditions. Thus, a similar construction is possible for such kernels, if in addition they have an adequate distribution of signs so that one can find cones defining \( T_{k+1}(Q) \) as before.

Counterexamples for condition (8). It is implicit in the proof of Theorem 1 that the weights \( w_N \) together with the functions \( f_N = w_N T^* w_N / (Mw_N)^2 \) give counterexamples for the condition (8) established by C. Pérez and D. Cruz-Uribe. As already pointed out in [23] the election of \( u \) and \( v \) in the proof of Theorem 2 gives again counterexamples for (8). The point in the given proof of Theorem 1 is to produce an explicit counterexample for Conjecture 1. An interesting observation is that the weights \( w_N \) do satisfy Conjecture 1. To see this, recall that (7) is true for \( M \) replaced by \( M^2 \) and then apply the ‘local’ \( A_1 \) condition \( Mw_N \lesssim w_N \) in \( D_N \).

‘Local \( A_p \)’ weights. It is clear that ‘local’ \( A_p \) weights share some of the properties of the usual Muckenhoupt \( A_p \) weights. For example, it is easy to see that conditions (3) and (2), satisfied on the support of the weight, are equivalent to the weak boundedness of \( M \) on weighted \( L^p \). However, there are some important differences too. One of them is the non existence of a reverse Hölder inequality for local weights. In fact, we have the following

Lemma 6. Let \( w \) be the local \( A_1 \) weight defined in the proofs of Theorems 2 and 3. Then, for all \( \varepsilon > 0 \), \( w^{1+\varepsilon} \) is not even a locally integrable function.

Proof. Observe that for each \( N \)

\[
\int_{Q_N} w^{1+\varepsilon} = \int_{D_N} w^{1+\varepsilon} = \sum_{k=0}^{\infty} \left( \frac{3^N k!}{(1+A)^k} \right)^{(k+1)(1+\varepsilon)} A^k 3^{-N(k+1)}
\]

\[
= \frac{1}{A} \sum_{k=0}^{\infty} \left( \frac{3^N k!}{(1+A)^{k+1+\varepsilon}} \right)^{k+1}.
\]

Since \( A \leq 3^{(N-1)d} \), by taking \( N \) large so that \( \frac{3^N k!}{(1+A)^{k+1+\varepsilon}} > 1 \), we see that the series diverges to \( \infty \). This shows that \( w^{1+\varepsilon} \notin L^1_{\text{loc}}(\mathbb{R}^d) \).

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