Uplifting Amplitudes in Special Kinematics

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Abstract

We consider scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in special kinematics where all external four-dimensional momenta are restricted to a (1+1)-dimensional subspace. The amplitudes are known to satisfy non-trivial factorisation properties arising from multi-collinear limits, which we further study here. We are able to find a general solution to these multi-collinear limits. This results in a simple formula which represents an $n$-point superamplitude in terms of a linear combination of functions $S_m$ which are constrained to vanish in all appropriate multi-collinear limits. These collinear-vanishing building blocks, $S_m$, are dual-conformally-invariant functions which depend on the reduced $m$-point kinematics with $8 \leq m \leq 4\ell$. For MHV amplitudes they can be constructed directly using, for example, the approach in Ref. [1]. This procedure provides a universal uplift of lower-point collinearly vanishing building blocks $S_m$ to all higher-point amplitudes. It works at any loop-level $\ell \geq 1$ and for any MHV or $N^k$MHV amplitude. We compare this with explicit examples involving $n$-point MHV amplitudes at 2-loops and 10-point MHV amplitudes at 3-loops. Tree-level superamplitudes have different properties and are treated separately from loop-level amplitudes in our approach. To illustrate this we derive an expression for $n$-point tree-level NMHV amplitudes in special kinematics.

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1 Introduction

Scattering amplitudes in gauge theory (and gravity) are known to have a much simpler underlying structure than implied by their direct construction in terms of Feynman diagrams. One theory where these simplifications are particularly striking is the planar $\mathcal{N} = 4$ supersymmetric gauge theory. In fact, it is not unreasonable to expect that the entire S-matrix in this theory can one day be determined from methods based on integrability of planar $\mathcal{N} = 4$ SYM.

In this paper we pursue the approach of using arguments based on symmetry considerations rather than direct perturbative calculations in establishing the structure of scattering amplitudes. In addition to simplifications occurring from working in the planar limit of maximally supersymmetric Yang-Mills, we take an additional step of imposing a kinematical restriction on external momenta of scattered states. This corresponds to confining all external momenta to live in $1 + 1$ dimensions of the full $3 + 1$ dimensional Minkowski space (the loop momenta of course are not restricted). Based on experience with the currently available results at up to 2-loops, we know that their analytical form simplifies considerably when one restricts to this special external kinematics. We thus can think of this restriction as a short cut towards establishing the underlying integrable structure of the amplitude in full kinematics.

The 2d external kinematics was first introduced in [2] in the context of strong coupling where it was interpreted as the boundary of the $AdS_3$ target space of the dual string theory. The lowest-$n$ non-trivial amplitudes in this kinematics are 8-point amplitudes and Ref. [2] computed them in the strong coupling regime. At both weak and strong coupling, maximal helicity violating (MHV) amplitudes are conjectured to be dual to polygonal light-like Wilson loops [3–7] and using this duality, the 8-point MHV amplitude was computed in 2d kinematics at 2-loops in [8]. Then an infinite sequence of MHV 2-loop amplitudes was found in [9] where the 2-loop result for the 8-point MHV amplitude was extended to all $n$-points$^1$, using symmetry and collinear limits as well as an additional assumption about the analytic structure of the $n$-point answer. This construction was further sharpened in [1] where 3-loop expressions (with a few unfixed coefficients) for MHV amplitudes were obtained at 8 points and 10 points. The main motivation of the present paper is to construct a universal uplift of lower-point amplitudes to arbitrary number of points $n$. In other words, we want to upgrade the construction of 2-loop MHV $n$-point amplitudes carried out in [9] to all-loop MHV and non-MHV $n$-amplitudes in terms of the low-$n$-point building blocks.

$^1n$ must be even in the 2d kinematics.
theory, \( A_{n,k} \), are the central objects of interest. Each \( N^k \)MHV amplitude \( A_{n,k} \) is a combination of all possible physical amplitudes involving \( k + 2 \) negative helicity gluons and the rest positive helicity gluons, together with amplitudes related to these by supersymmetry. These amplitudes depend on the on-shell momenta \( p_i \) of the external particles and on Grassmann variables \( \eta_i \) necessary to specify all the particle states of the super Yang-Mills multiplet (for example, the positive helicity gluon \( g_i^+ \) is characterised by \( \eta_i \) to the power zero and, on the opposite end of the spectrum, the negative helicity gluon \( g_j^- \) corresponds to \( \eta_j^1 \eta_j^2 \eta_j^3 \eta_j^4 \), while fermions and scalars fill in powers of \( \eta \) from one to three). Each \( N^k \)MHV amplitude \( A_{n,k} \) is of degree \( (\eta)^{8+4k} \), see [10,11] for more detail.

All \( A_{n,k} \) amplitudes are naturally assembled into a single \( N = 4 \) superamplitude,

\[
A_n = \sum_{k=0}^{n-4} A_{n,k},
\]

(1.1)

and can be read back from it as \( (\eta)^{8+4k} \) coefficients in the Taylor expansion in terms of the Grassmann variables \( \eta_i^A \).

It will be convenient for us to factor out from the superamplitude \( A_n \) the tree-level contribution \( A_{n,\text{tree}} \), as well as the infrared divergences coming from loops,

\[
A_n = A_{n,\text{tree}} M_{n}^{\text{BDS}} R_n.
\]

(1.2)

Here \( M_{n}^{\text{BDS}} \) denotes the known [12] BDS-expression which contains all infrared divergences of the amplitude, and it is also known to factorise correctly under simple collinear limits, where two consecutive momenta become collinear.

Thus through (1.2) the \( N = 4 \) scattering amplitudes are determined in terms of \( R_n \) which is known as the reduced function or the remainder function. In our definition \( R_n \) itself is a superfunction, and can be Taylor-expanded in Grassmann \( \eta \)'s to give the \( N^k \)MHV remainder functions, \( R_n = \sum_{k=0}^{n-4} R_{n,k} \). Each \( R_{n,k} \) is a finite, regularisation-independent and dual-conformally invariant quantity. More precisely, for the MHV case \( k = 0 \), the amplitude/ Wilson loop duality predicts that \( R_{n,0} \) is dual conformally invariant and depends on external momenta only through conformal cross-ratios \( u \) [13]. In the general \( N^k \)MHV case, dual superconformal invariance [14], fully present at tree-level [15] but partially broken at loop level, implies that \( R_{n,k} \) depends on the external kinematics (momenta and helicities) through the cross-ratios as well as through dual superconformal invariants [14] which involve Grassmann variables. There is now a conjectured duality between the superamplitude and a super-Wilson loop [16,17] as well as with supersymmetric correlation functions [18–20], both of which explain the presence of dual superconformal symmetry.

It will be advantageous to consider the logarithm of the reduced superamplitude, \( \mathcal{R}_n = \log R_n \). In perturbation theory it can be expanded in powers of the coupling,
and independently of this also in powers of $\eta$

\[ \mathcal{R}_n := \log R_n = \sum_{\ell=1}^{\infty} a^\ell \mathcal{R}^{(\ell)}_n = \sum_{k=0}^{n-4} \sum_{\ell=1}^{\infty} a^\ell \mathcal{R}^{(\ell)}_{n,k}. \quad (1.3) \]

Note that $\mathcal{R}_{n,k}$ will thus have contributions from $R_{n,k}$ but also from products $R_{n,k'} R_{n,k-k'}$. We note that in our definition of $R_n$ in Eq. (1.2) we factored out the entire tree-level superamplitude (rather than, for example, only the MHV expression, as is sometimes done in the literature, see e.g. [21]). Thus all tree-level contributions are cleanly separated from loops and the expansion on the r.h.s. of (1.3) starts from $\ell = 1$ loop.

For MHV contributions, the expansion starts at 2-loops since $\mathcal{R}^{(1)}_{n,0} = 0$. In general four-dimensional kinematics, non-trivial two-loop contributions start at 6-points, and $\mathcal{R}^{(2)}_{6,0}$ was obtained numerically in [6,7] and later analytic expressions for $\mathcal{R}^{(2)}_{6,0}$ were derived in [22–24]. The result for general $n$, $\mathcal{R}^{(2)}_{n,0}$, can be obtained numerically from the algorithm constructed in [25]. The symbol [24] of the $n$-point amplitude, $\mathcal{R}^{(2)}_{n,0}$, is known [26] as is the symbol of the six-point 3-loop amplitude $\mathcal{R}^{(3)}_{6,0}$ [27] and the six-point two-loop NMHV amplitude $\mathcal{R}^{(2)}_{6,1}$ [28]. In special two-dimensional kinematics, remarkably compact analytic expressions for $\mathcal{R}^{(2)}_{n,0}$ were derived in [8] at $n = 8$, and in [9] for all $n$. Three-loop analytic expressions for $\mathcal{R}^{(3)}_{n,0}$, containing 7 unfixed coefficients, were obtained in [1] for $n = 8$ in special 2d kinematics and further generalised to $n = 10$.

The NMHV amplitudes in special kinematics will be addressed in [29].

The purpose of the present paper is to present a universal method for uplifting lower-point amplitudes in special kinematics, such as $\mathcal{R}^{(\ell)}_{n=8,k}$ to all higher $n$ for all classes of $N^k$MHV’s. Specifically we want to construct the uplift of the superamplitude $\mathcal{R}_n$.

We start in general 4d kinematics and our first main new result is a full explication of all $k$-preserving and $k$-decreasing multi-collinear limits. When written in terms of the full superamplitude and defined in the correct supersymmetric way, the collinear limits take on a remarkably simple form, Eqs. (2.8)-(2.10) in Section 2. Using this form for the collinear limits, we are able to completely and explicitly solve their consequences for higher point amplitudes restricted to 2d kinematics, firstly at MHV level in Section 4 and then for any superamplitude at $N^k$MHV level in Section 5. This form of the collinear uplift is the unique one with manifest collinear properties.

Tree-level amplitudes manage to possess correct collinear limits in a non-manifest way by satisfying non-trivial linear identities. We illustrate this by giving the NMHV
tree-level $n$-point amplitude in Section 6.

2 Kinematics and collinear limits

Superamplitudes are functions of bosonic variables (the light-like momenta $p_i$ of external particles) as well as fermionic variables $\eta^A_i$ [10] which take into account the different states in the super Yang-Mills multiplet which are being scattered. All $k$-components of the $N^k$MHV amplitudes $A_{n,k}$ arise from the Taylor expansion of the superamplitude in terms of the Grassmann variables $\eta^A_i$, as explained in Sec.5 of Ref. [11].

It is useful to package the external data \{${\eta}^\mu_i$, $\eta^A_i$\} in terms of the region momenta $x_i^{\alpha\dot{\alpha}}$ and their fermionic components $\theta^A_i$ defined as follows [14]

\[
\begin{align*}
    p^{\alpha\dot{\alpha}}_{x_i} &\equiv \lambda^\alpha_i \tilde{\lambda}^{\dot{\alpha}}_i = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}, \quad \alpha, \dot{\alpha} = 1, 2, \\
    \lambda^\alpha_i \eta^A_i &= \theta^A_i - \theta^A_{i+1}, \quad A = 1, \ldots, 4,
\end{align*}
\]

(2.1) (2.2)

where $\lambda^\alpha_i$ and $\tilde{\lambda}^{\dot{\alpha}}_i$ are the standard two-component helicity spinors. The chiral superspace coordinates $X_i = (x_i, \theta^A_i)$ define the vertices of the $n$-sided null polygon contour for the Wilson loop dual to the $n$-point superamplitude [3, 5, 14].

We will also use momentum supertwistors [30, 31] which transform linearly under $SU(2,2|4)$ dual superconformal transformations. The supertwistors are defined via

\[
Z_i = (Z^a_i; \chi^A_i) = (\lambda^\alpha_i, x_{\alpha\dot{\alpha}}^{i}; \theta^A_{\alpha\dot{\alpha}}^{i}).
\]

(2.3)

where $Z^a$ denote the four bosonic, and $\chi^A$ are the four fermionic components.

2.1 Multi-collinear limits

Here we will describe how the collinear limits where $m + 1$ consecutive momenta with $m \geq 1$ become collinear act on the reduced superamplitude $R_n$ and its logarithm $\mathcal{R}_n$. The full superamplitude, $A_n$, factorises in the $m + 1$ collinear limit as follows,

\[
A_n \rightarrow A_{n-m} \times \text{Split}_m,
\]

(2.4)

where $A_{n-m}$ is the superamplitude with $n - m$ external states, and the expression $\text{Split}_m$ denotes the splitting superamplitude. The latter can be expanded, $\text{Split}_m =$
\[ \sum_{p=0}^{m} \text{Split}_{m,p} \] in terms of helicity-changing (also called \( k \)-reducing) splitting functions so that the \( N^k \) MHV amplitude goes to

\[
A_{n,k} \rightarrow A_{n-m,k} \times \text{Split}_{m,0} + A_{n-m,k-1} \times \text{Split}_{m,1} + \ldots
\]

\[
= \sum_{p=0}^{k} A_{n-m,k-p} \times \text{Split}_{m,p}.
\]

(2.5)

The simplest collinear limit occurs when just 2 consecutive momenta in the colour-ordered amplitude become collinear. The amplitude \( A_n \) factorises in this limit into the amplitude with \( n-1 \) external particles times the splitting function, \( A_n \rightarrow A_{n-1} \times \text{Split}_1 \). It is well-known [12,32] that the BDS expression together with the tree-level amplitude, fully account for the splitting amplitude \( \text{Split}_1 \). As a result the reduced superamplitude has a particularly simple form under this minimal collinear limit, \( \mathcal{R}_n \rightarrow \mathcal{R}_{n-1} \), i.e. there is no splitting function entering the \( \mathcal{R} \)-equation.

As the next step, let us consider the triple collinear limit where \( m + 1 = 3 \) consecutive momenta become collinear, and furthermore we require that the helicity of the amplitude is conserved. Such limits are referred to as \( k \)-preserving collinear limits, they focus on the \( p = 0 \) term on the r.h.s of (2.5). The new feature of the triple collinear limit compared to the simple collinear limit before, is that the corresponding splitting function is no longer fully accounted for by the BDS expression \( M_{\text{BDS}} \). When interpreted in terms of the reduced amplitude, the factorisation theorem for the helicity-preserving triple collinear limit gives

\[
\lim_{k_{\text{fixed}}} R_{n,k} \rightarrow R_{n-2,k} \times \text{split}_{2,0} = R_{n-2,k} \times R_{6,0},
\]

(2.6)

where \( \text{split}_{2,0} \) is the helicity-preserving triple collinear splitting amplitude (or more precisely, the part thereof which is not accounted by the BDS expression). Importantly, this splitting amplitude agrees with the 6-point MHV reduced amplitude \( R_{6,0} \) [6,25].

Moving on to \( k \)-preserving multi-collinear limits with \( (m+1) \)-collinear momenta. Here we have

\[
\lim_{k_{\text{fixed}}} R_{n,k} \rightarrow R_{n-m,k} \times R_{m+4,0},
\]

(2.7)

where similarly to (2.6) the splitting amplitude becomes the reduced amplitude \( R_{m+4,0} \) itself [9,33].

We are now ready to consider the general multi-collinear case, where we no longer impose any restrictions on preserving the helicity degree \( k \) of the amplitude. Thus we

\footnote{Equation (2.6) was originally derived in the MHV case \( R_{n,0} \). But once we know that the \( k \)-independent splitting amplitude \( \text{split}_{2,0} \) is equal to \( R_{6,0} \), this fact can be used for general \( k \) in \( k \)-preserving collinear limits, leading to (2.6).}
write the analogue of the superamplitude factorisation (2.4) directly for the reduced superamplitude,
\[ R_n \rightarrow R_{n-m} \times R_{m+4}. \] (2.8)
This formula can also be expanded in terms of $N^{k-p}$MHV components similarly to (2.5) except that now all the splitting-function contributions are expressed in terms of $R$'s:
\[ R_{n,k} \rightarrow R_{n-m,k} \times R_{m,0} + R_{n-m,k-1} \times R_{m,1} + \ldots = \sum_{p=0}^{k} R_{n-m,k-p} \times R_{m,p}. \] (2.9)
The $k$-preserving collinear limit (2.7) is a special case of these general relations which corresponds to a single term on the r.h.s. of (2.9).

The proof of this collinear factorisation for $R_n$ in (2.8) uses known universal collinear factorisation properties of amplitudes, combined with dual superconformal symmetry of $R_n$. We know that the superamplitude $A_n$ has universal collinear factorisation limits (2.4) and so does $M_{BDS}$ being the exponent of the 1-loop MHV amplitude. Therefore the reduced amplitude $R_n$ as defined in (1.2) must also have universal collinear factorisation properties. Thus we only need to discover what the corresponding splitting superamplitude is. To do this let us focus on the maximal multi-collinear limit where $n = m + 4$. In this limit from universal factorisation we have $R_{m+4} \rightarrow R_4 \times \text{split}_m = \text{split}_m$ since $R_4$ is trivial. On the other hand, the same $(m+1)$-collinear limit on $R_{m+4}$ can be achieved via a superconformal transformation on all $m+4$ points (as we show in Appendix A). Therefore we have $R_{n+4} \rightarrow R_{m+4}$ in this case. The conclusion is that the splitting amplitude is $\text{split}_m = R_{m+4}$ and Eq. (2.8) follows.

Taking the logarithm we get the linear realisation of multi-collinear limits,
\[ \mathcal{R}_n \rightarrow \mathcal{R}_{n-m} + \mathcal{R}_{m+4}. \] (2.10)
Equations (2.10) or (2.8) constitute our main result as far as general collinear limits are concerned, and they will play a key role in in constructing the uplift to general $n$ of the amplitude in the 2d external kinematics as will be introduced below.

### 2.2 Two-dimensional kinematics

In this subsection we give details and conventions for the special kinematics, first introduced in [2], where the external momenta $p_i$ lie entirely in $1+1$ dimensions.
Figure 1: Figure illustrating the zig-zag Wilson loop contour in 2d kinematics. Vertices $x_i$ are defined in terms of light-cone co-ordinates. In 2d the contour can also be specified by giving every other vertex $x_2, x_4, x_6, \ldots$.

The corresponding contour of the dual Wilson loop has a zig-zag shape which is shown on the lightcone plane in Fig. 1. The region momenta $x_i$ (vertices of the corresponding Wilson loop contour) have the following form in light-cone coordinates $(x_+, x_-)$:

$$x_i = \begin{cases} 
    (z_{i-1}, z_i), & i \text{ even} \\
    (z_i, z_{i-1}), & i \text{ odd}
\end{cases} \quad (2.11)$$

Only an even number of vertices is possible in this 2d kinematics, and we continue denoting it as $n$ (rather than $2n$ as sometimes done in the literature). In our notation $z_i$ components with odd values of $i$ lie along the $x^+$ axis, and the even $z_i$'s are along the $x^-$ axis, as one can see instantly from Fig. 1. We will frequently refer to them as ‘odd’ and ‘even’ coordinates. All the (bosonic) functions we consider can be written in terms of Lorentz invariant intervals $z_{ij} := z_i - z_j$ where both $i, j$ are either even or odd. In this notation we must remember that the even and odd coordinates are independent of each other.

It is instructive to view the 2d kinematics from the point of view of momentum twistors (2.3). In 2d the bosonic twistors $Z_i^\alpha = (\lambda_i^\alpha, x_{\alpha i})$ reduce as follows. For all even values of $i$ we have

$$p_i^{\alpha \dot{\alpha}} = \begin{pmatrix} 0 & 0 \\ 0 & p_i^- \end{pmatrix} = \lambda_i^\alpha \bar{\lambda}_{i}^{\dot{\alpha}} \Rightarrow \lambda_i^\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{\lambda}_{i}^{\dot{\alpha}} = \begin{pmatrix} 0 \\ p_i^- \end{pmatrix}, \quad (2.12)$$
and
\[ x_\dot{\alpha} i \lambda_i^\alpha = \begin{pmatrix} x_i^+ & 0 & 0 \\ 0 & x_i^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z_i \end{pmatrix}. \tag{2.13} \]

For odd values of \( i \) the story is similar, and as a result, momentum twistors in 2d have a checkered pattern:
\[
Z_i = \begin{cases} 
(Z_i^1, 0, Z_i^3, 0) = (1, 0, z_i, 0) & \text{if } i \text{ odd} \\
(0, Z_i^2, 0, Z_i^4) = (0, 1, 0, z_i) & \text{if } i \text{ even},
\end{cases} \tag{2.14}
\]

which is a manifestation of \( SU(2, 2) \to SL(2)_+ \times SL(2)_- \) in 2d.

In 2d kinematics it is then natural to define an \( SL(2)_\pm \)-invariant two-bracket of twistors,
\[
\langle ij \rangle := \begin{cases} 
Z_i^3 Z_j^1 - Z_i^1 Z_j^3 & i \text{ and } j \text{ odd} \\
Z_i^4 Z_j^2 - Z_i^2 Z_j^4 & i \text{ and } j \text{ even} \\
0 & \text{otherwise}.
\end{cases} \tag{2.15}
\]

From (2.15) and the r.h.s. of (2.14) we have that \( \langle ij \rangle = z_{ij} \) and the Lorentz-invariant intervals \( z_{ij} \) have the standard two-bracket interpretation \( \langle ij \rangle \) (but in terms of reduced 2d twistors rather than helicity spinors).

Furthermore, the standard \( SL(2, 2) \)-invariant twistor 4-bracket contraction,
\[
\langle ijkl \rangle := \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d,
\]

reduces in 2d to a product of two-brackets if there are two even and two odd indices, or vanishes otherwise: e.g. \( \langle 1234 \rangle = \langle 13 \rangle \langle 24 \rangle \). The main point here is that lightcone coordinates are interchangeable with twistors in 2d and only two-brackets of bosonic twistors (of the same parity) can appear.

For superamplitudes in 2d, it is natural to consider a supersymmetric reduction, \( SU(2, 2|4) \to SL(2|2)_+ \times SL(2|2)_- \), under which momentum supertwistors (2.3) become [21]
\[
Z_i = (Z_i^a, \chi_i^A) = \begin{cases} 
(Z_i^1, 0, Z_i^3, 0; \chi_i^1, 0, \chi_i^3, 0) & \text{if } i \text{ odd} \\
(0, Z_i^2, 0, Z_i^4; 0, 0, \chi_i^2, \chi_i^4) & \text{if } i \text{ even},
\end{cases} \tag{2.17}
\]

and we will indeed mostly consider this additional reduction in fermionic co-ordinates also. On the other hand one should beware that while we may still compute meaningful forms for either \( R_{n,k} \) or \( R_{n,k} \) the MHV-prefactor to this from (1.2) contains
\[
\delta^{(8)} \left( \sum_{i=1}^{n} \lambda_i \eta_i \right). \tag{2.18}
\]
which under this SU(4) splitting necessarily goes to zero. For $R_{n,k}$ with $k = 0, 1$ this reduction in the fermionic superspace co-ordinates does not have a great effect. All results obtained can straightforwardly be uplifted to the case with full fermionic dependence.

The remainder function $R_n$ is dual conformally invariant $[4,13]$ and as such its lowest bosonic component, $R_{n,0}$, can be written as a function of cross-ratios. The non-MHV components, $R_{n,k>0}$, also depend on superconformal invariants involving Grassmann variables. We will first concentrate on the purely bosonic case.

We define the most general cross-ratios in special 2d kinematics as

$$u_{ij;kl} = \frac{\langle il \rangle \langle jk \rangle}{\langle ik \rangle \langle jl \rangle}.$$  \hfill (2.19)

This equation is meaningful only for $i, j, k, l$ having the same parity, in other words, all the cross-ratios fall into two separate classes with all indices being even or with all indices odd. Cross-ratios with indices of mixed parity (even and odd) don’t exist.

The general cross-ratios in 2d kinematics have to satisfy an additional constraint,

$$u_{ij;kl} = 1 - u_{il;kj}.$$  \hfill (2.20)

The set of general $u_{ij;kl}$’s can be reduced to a smaller set of cross-ratios with only two indices,

$$u_{ij} = \begin{cases} \frac{\langle i-1,j+1 \rangle \langle i+1,j-1 \rangle}{\langle i-1,j-1 \rangle \langle i+1,j+1 \rangle} = u_{i-1,i+1;j-1,j+1} & |i - j| \geq 3, \ i, j \text{ of the same parity} \\ 1 & i, j \text{ of opposite parity} \end{cases}.$$  

Indeed we have

$$u_{ij;kl} = \prod_{I=i+1}^{j-1} \prod_{K=k+1}^{l-1} u_{IK}.$$  \hfill (2.21)

This reduced set of $u_{ij}$ cross-ratios also has a 4d interpretation, $u_{ij} = \frac{x_{i,j}^1 x_{i,j+1}^2}{x_{i,j}^2 x_{i,j+1}^1 + 1}$.  

For the two lowest-$n$ cases, the octagon and the decagon, all non-trivial 2-component cross-ratios are of the form $u_{i,i+4}$, with $i = 1, \ldots, 4$ for the octagon, and $i = 1, \ldots, 10$ for the decagon. The cross-ratios $u_{ij}$ are still not all independent because of equations (2.20), leaving $n - 6$ (i.e. 2 for the octagon and 4 for the decagon) independent solutions. For the octagon (2.20) amounts to

$$n = 8 : \quad 1 - u_{i,i+4} = u_{i+2,i+6}, \quad i = 1, 2,$$  \hfill (2.22)

\(^3\text{I.e. non-vanishing } (\chi^1, \chi^2, \chi^3, \chi^4) \text{ in both equations (2.17).}\)
and for the decagon,
\[ n = 10 : \quad 1 - u_{i,i+4} = u_{i+2,i+6} u_{i-2,i+2}. \] (2.23)

To simplify notation at low \( n \), it is sometimes convenient to use
\[ u_i := u_{i,i+4}. \] (2.24)

While at \( n = 8 \) and \( n = 10 \) these are the only cross-ratios, this is no longer true at \( n \geq 12 \) where \( u_{ij} \) cross-ratios appear with \( j - i \geq 6 \). More details on the cross-ratios in the special kinematics can be found in [9].

### 2.3 Collinear limits in the 2d kinematics

From the zig-zag kinematics it is clear that the lowest collinear limit one can apply and remain within the (1+1)-dimensional kinematics is the triple collinear limit where three consecutive edges collapse into one. More precisely, this should be thought of as a collinear-soft-collinear limit, where three edges with momenta \( p_{n-2}, p_{n-1} \) and \( p_n \) collapse into a single edge \( p_n \). In practice, the middle momentum becomes soft, \( p_{n-1} \to 0 \). In terms of twistors, or the lightcone components \( z_i \)'s, we see that \( z_n \to z_{n-2} \) while the variable \( z_{n-1} \) remains free, see figure 2.

We now recall that in the 2d kinematics there are no non-trivial cross-ratios at 6-points (lowest non-trivial case being \( R_{8,0} \)) and the 6-point reduced amplitude \( R_6 \) is a (coupling dependent) constant multiplied by the tree-level amplitude, which can be reabsorbed into \( R_n \) which we now call \( \tilde{R}_n \) [1],
\[ \tilde{R}_n = R_n - \frac{n-4}{2} R_6, \] (2.25)
so that at the level of superamplitudes we have

\[ \tilde{R}_n \rightarrow \tilde{R}_{n-2} \].

(2.26)

Also, for \( R_n = \log R_n \) expressions at different order in the loop expansion do not mix,

\[
\tilde{R}^{(\ell)}_n \rightarrow \tilde{R}^{(\ell)}_{n-2}, \\
\tilde{R}^{(\ell)}_n \rightarrow \tilde{R}^{(\ell)}_{n-m} + \tilde{R}^{(\ell)}_{m+4}, \quad \text{for } m \geq 4 ,
\]

(2.27)

(2.28)

and thus \( \tilde{R}_n \) is the natural object to use for collinear uplifts of amplitudes to higher number of points.

3 n-point MHV amplitudes: Part 1

At 2-loop level, MHV amplitudes in 2-dimensional external kinematics are known for any number of external particles and, remarkably, the result for \( n \)-point amplitudes depends only on the four-logarithms structure appearing at 8-points. If we denote the 8-point 2-loop structure \( S^{(2)}_8 \), the general \( n \)-point remainder function \( \tilde{R}^{(2)}_n \) at 2 loops emerges as a linear combination of \( S^{(2)}_8 \)'s and nothing else. Can this be the case at 3 loops?

The MHV amplitudes at 3-loops were constructed for \( n = 8 \) and 10 points. In the latter case the 10-point expression contained the 8-point structures \( S^{(3)}_8 \), as well as new non-trivial contributions which did not appear at 8 points. Beyond the 10-point case, the structure of \( n \)-point 3-loop amplitude was until now an open problem.

The goal of this and the following sections is to find a decomposition of the general \( n \)-point amplitude \( \tilde{R}^{(\ell)}_n \) in terms of the lower-point building blocks, \( S_m \), with trivial multi-collinear limits. This formula will be given in the following section. Below we will briefly recall the known results about 2-loop and 3-loop amplitudes and then proceed to recast them in the form which is more appropriate to the \( \sum S_m \) generalisation.

3.1 n-points at 2 loops

In \([9]\) the \( n \)-point 2 loop MHV amplitude was obtained in 2d kinematics. The result can be written entirely in terms of logarithms of the simple cross-ratios \( u_{ij} \) and takes
where the sum runs over the set

\[ S = \left\{ i_1, \ldots, i_8 : 1 \leq i_1 < i_2 < \cdots < i_8 \leq n, \quad i_k - i_{k-1} = \text{odd} \right\}. \]  

The constant term on the r.h.s. of (3.1) arises from \( R_{n,0}^{(2)} \) and can be removed by going to \( \tilde{R}_n \) as in (2.25). Also, for ease of dealing with collinear limits it is more appropriate to work with the logarithm of the amplitude \( R_n \), though for MHV expressions the first differences would appear starting from 4-loops. We thus will use

\[ \tilde{R}_n^{(2)} = -\frac{1}{2} \left( \sum_S \log(u_{i_1i_5}) \log(u_{i_2i_6}) \log(u_{i_3i_7}) \log(u_{i_4i_8}) \right). \]  

The 2-loop \( n \)-point result was found in [9] by examining the consequences of collinear limits described above, starting with the 8-point amplitude computed (via the Wilson loop/amplitude duality) in [8] and first using an additional assumption that only logarithms of simple cross-ratios \( u_{ij} \) can appear at two loops. The general 2-loop analytic formula (and thus the logs-only assumption) was verified [9] numerically using the code developed in [25].

At higher loops, the logs-only structure no longer holds [1], and furthermore, the problem of how to obtain the all-\( n \) amplitudes starting from low-\( n \) expressions was previously an open problem.

### 3.2 8- and 10-point MHV amplitudes at 3 loops

The 8-point MHV amplitude \( \mathcal{R}_8^{(3)} \) at 3 loops was determined in Ref. [1]. This derivation was based on the fundamental assumption that the amplitude has a symbol whose entries are cross-ratios\(^5\). The reader is referred to Appendix B for more detail on this construction.

At 8-points in special kinematics, there are four non-trivial cross-ratios, \( u_1 := u_{15}, \ u_2 := u_{26}, \ u_3 := 1 - u_1, \ u_4 := 1 - u_2 \). Insisting that the 8-point function be

\(^4\)For MHV amplitudes we will suppress the \( k = 0 \) subscript to simplify notation.

\(^5\)At 1 and 2-loops this amounts to logarithms only in the amplitude [1] and starting from 3-loops the reconstructed functions involve also polylogarithms \( Li_n(u_{ij}) \).
cyclically (and parity) symmetric, and that it vanishes in the collinear limit $z_8 \rightarrow z_6$, i.e. $u_1 \rightarrow 0, u_3 \rightarrow 1$ with $u_2, u_4$ unconstrained\(^6\) leads to a 3-loop amplitude of the form:

$$\tilde{R}_8^{(3)} = \sum_{\sigma, \tau} a_{\sigma \tau} f_\sigma(u_1) f_\tau(u_2)$$

(3.4)

where $a_{\sigma \tau} = a_{\tau \sigma}$ are some rational coefficients, and the sum is over the set of functions $f_\sigma$ with the following properties:

$$f_\sigma(u) = f_\sigma(1 - u)$$

$$f_\sigma(0) = 0$$

$$f_\sigma(u) \text{ is a (generalised) polylogarithm.}$$

(3.5)

Furthermore the total polylog weight (or “degree of transcendentality”) of $\tilde{R}_8^{(3)}$ must be six due to the uniform transcendentality property of perturbative amplitudes in $\mathcal{N}=4$ SYM.

In [1] all possible functions $f_\sigma$ were listed (see also Appendix B where these functions are called $f_\sigma^+$). There is a unique weight-two function $f(u) = \log u \log(1 - u)$, 3 weight-three functions, and 7 weight-four functions, leading to a total of 13 a priori unfixed coefficients $a_{\sigma \tau}$. Further constraints arise from the OPE analysis of [34] which fix 6 of these, leaving 7 unfixed coefficients $[1]$.\(^7\)

The form of the 8-point amplitude (3.4) generalises straightforwardly beyond 3 loops by simply allowing the functions $f_\sigma$ to have more general weight, such that the total weight is $2\ell$.

$$\tilde{R}_8^{(\ell)} = \sum_{\sigma, \tau} a_{\sigma \tau} f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_2)$$

(3.6)

It is also valid at two loops where there is only one allowed function (up to a multiplicative constant), $f^{(2)}(u) = \log u \log(1 - u)$, and we reproduce the original two-loop result at 8-points found in [8]

$$\tilde{R}_8^{(2)} = -\frac{1}{2} \log(u_1) \log(u_2) \log(u_3) \log(u_4).$$

(3.7)

Let us first consider the uplift of the 8-point amplitude to 10-points following [1]. The idea is to write down all 10-point functions which reduce to the 8-point amplitude

\(^6\)Note that we need not consider cyclically equivalent collinear limits $z_i \rightarrow z_{i-2}$, since they will follow automatically from cyclic symmetry.

\(^7\) It is tempting to assume a further simplification of the structure, namely that the $f_\sigma$ are of weight 3 only. This would be consistent with all currently known facts and would leave just 3 unfixed coefficients. We will not be making this assumption in the present paper.
in the triple collinear limit, plus an additional contribution which is required to vanish in all such limits. This lead to\(^8\)

\[
\tilde{\mathcal{R}}^{(\ell)}_{10} = \frac{1}{2} \sum_{\sigma, \tau} a_{\sigma \tau} \left( f^{(\ell)}_\sigma(u_1) f^{(\ell)}_\tau(u_2) - f^{(\ell)}_\sigma(u_1) f^{(\ell)}_\tau(u_4) + \frac{1}{2} f^{(\ell)}_\sigma(u_1) f^{(\ell)}_\tau(u_6) \right) + \text{cyclic} \\
+ V^{(\ell)}_{10}.
\] (3.8)

The last term, \(V^{(\ell)}_{10}\), denotes a generic 10-point function which vanishes in all triple collinear limits. It is reproduced in Appendix B at 3-loop level from Ref. [1].

The construction of the non-vanishing contribution under triple collinear limits (everything apart from \(V^{(\ell)}_{10}\)) was specific to the case at hand where the 10-point amplitude reduces to the 8-point amplitude. If we want to uplift (3.8) to 12 and higher points, we need to come up with a more general procedure.

Note that the general 10-point expression has a more complicated structure than the result at two loops:

\[
\tilde{\mathcal{R}}^{(2)}_{10} = -\frac{1}{2} \left( \log(u_1) \log(u_2) \log(u_3) \log(u_4) + \text{cyclic} \right).
\] (3.9)

The reason for this simplification is the simple form of the 2-loop function \(f^{(2)}(u) = \log u \log(1 - u)\). Using this together with the fact that at 10 points \(1 - u_1 = u_7 u_3\) and cyclic, one can check that all the minus signs in (3.8) (for \(\ell = 2\)) disappear and the result reduces to (3.9).

### 3.3 8– and 10–points recast

We can completely and explicitly solve the constraints coming from collinear limits at 3-loops in terms of three structures, related to the 8-, 10- and 12-point amplitudes and more generally at \(\ell\)-loops in terms of the \(m\)-point functions \(S_m\) with \(m \leq 4\ell\). But first, to motivate the general formula, we will recast the 8- and 10-point amplitudes in a form more suitable for generalisation, and in the process introduce the new concepts we will need. Also for pedagogical reasons, here in this subsection, we will follow a simplified approach for relating the 8-point functions \(S_8\) directly with the amplitudes \(\mathcal{R}_8\). In the following section we will restore to the general case.

Our first step is to recast the problem back as a function of \(z\)'s, that is,

\[
\mathcal{R}_8(u_1, u_2) = \mathcal{R}_8(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8).
\] (3.10)

\(^8\)The original derivation in [1] was performed at 3-loops, but the resulting expression (in terms of the functions \(f^{(\ell)}\)) remains valid at \(\ell\) loops.
Now, on attempting to lift this to higher points, we notice that in the higher point functions the \(z\)'s always appear in consecutive pairs, but with the odd element of the pair always appearing before the even element. This is exactly what happens in the definition of \(x_i\) in terms of \(z_i\) in (2.1). It suggests that we further think of the amplitude as a function of position coordinates \(x\)'s instead of \(z\)'s so that:

\[
2S_8(x_i, x_j, x_k, x_l) := \tilde{R}_8(x_i^+, x_i^-, x_j^+, x_j^-, x_k^+, x_k^-, x_l^+, x_l^-), \tag{3.11}
\]

which implies,

\[
2S_8(x_2, x_4, x_6, x_8) = \tilde{R}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8). \tag{3.12}
\]

Here we have introduced the function \(S_8\) of \(x\)-variables which for the moment we identify with \(R_8\).\(^9\) Thus via \(S_8\) we are defining the Wilson loop zig-zag contour (see Fig. 1) by specifying every second vertex. Let us examine the symmetries of the function \(S_8(x_2, x_4, x_6, x_8)\). The symmetries of the Wilson loop, \(\tilde{R}(z_1, \ldots z_8)\) namely cyclic symmetry \(C_n\), under which each \(z_i \to z_{i+1}\), and parity of the 8-point Wilson loop, \(\tilde{R}(z_1, \ldots z_8) \to \tilde{R}(z_8, \ldots z_1)\), give the following

\[
\begin{align*}
S_8(x_2, x_4, x_6, x_8) &= S_8(x_4, x_6, x_8, x_2) = S_8(x_6, x_8, x_2, x_4) = S_8(x_8, x_2, x_4, x_6) \\
S_8(x_2, x_4, x_6, x_8) &= S_8(x_1^f, x_3^f, x_5^f, x_7^f) \\
S_8(x_2, x_4, x_6, x_8) &= S_8(x_8^f, x_6^f, x_4^f, x_2^f). \tag{3.13}
\end{align*}
\]

Here the first equation follows from cyclicity in \(z\) applied twice, i.e. \(z_i \to z_{i+2}\). The second equation is a consequence of \(z_i \to z_{i+1}\). In the last two equations we have defined the flipped \(x\) position

\[
x = (x^+, x^-) \Rightarrow x^f = (x^-, x^+) . \tag{3.14}
\]

This is necessary in order to properly define the cyclic symmetry in terms of the \(x\)-variables.

Interestingly, for the 8-point amplitude in the form (3.6) there exists an additional discrete symmetry – the flip symmetry – where each \(x\)-argument of \(S_8\) becomes flipped,

\[
S_8(x_i, x_j, x_k, x_l) = S_8(x_i^f, x_j^f, x_k^f, x_l^f) \tag{3.15}
\]

despite the fact that it is not an expected symmetry of the Wilson loop contour. To identify this symmetry consider \(S_8\) of even arguments,

\[
2S_8(x_2, x_4, x_6, x_8) = \tilde{R}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = \sum_{\sigma, \tau} a_{\sigma\tau} f_{\sigma}(u_1) f_{\tau}(u_2) \tag{3.16}
\]

\(^9\)In the following section we will in fact generalise this definition by including an additional contribution to \(S_8\) which is distinct from the 8-point amplitude \(R_8\).
and compare it with $S_8$ written in terms of the same variables being flipped,

$$2S_8(x_2^f, x_4^f, x_6^f, x_8^f) = \tilde{R}_8(z_2, z_4, z_3, z_6, z_5, z_8, z_7) = \sum_{\sigma, \tau} a_{\sigma\tau} f_{\sigma}(u_2)f_{\tau}(u_1) \tag{3.17}$$

To understand the right hand side, note that cross-ratios $u_1 = u_{15}$ and $u_2 = u_{26}$ by definition (2.21) depend only on even or on odd $z$-variables respectively,

$$u_1 = \langle 86 \rangle \langle 24 \rangle = \frac{z_{86}z_{24}}{z_{84}z_{26}}, \quad u_2 = \langle 17 \rangle \langle 35 \rangle = \frac{z_{17}z_{35}}{z_{15}z_{37}} \tag{3.18}$$

hence the distribution of $z_i$'s inside $\tilde{R}_8$ in Eqs. (3.16,3.17) implies that these two equations are related by $u_1 \leftrightarrow u_2$. The symmetry $a_{\sigma\tau} = a_{\tau\sigma}$ with the summation over all functions $f_{\sigma}$ and $f_{\tau}$ implies that the resulting expressions are symmetric under the interchange $u_1 \leftrightarrow u_2$ and equation (3.15) follows.

Note also that $S_8$ satisfies the following properties under the collinear limit $z_8 \rightarrow z_6$ (ie $x_8 \rightarrow x_7$ as can be seen immediately from Fig. 1):

$$\lim_{x_8 \rightarrow x_7} S_8(x_2, x_4, x_6, x_8) = 0 \tag{3.19}$$

or more generally/ geometrically

$$S_8(x_i, x_j, x_k, x_l) = 0 \quad \text{if} \quad x_k, x_l \text{ are light-like separated} \tag{3.20}$$

Having defined the object $S_8$ we now re-examine the $\ell$-loop 10-point amplitude (3.8),

$$\tilde{R}_{10}^{(\ell)} = \frac{1}{4} \sum_{\sigma, \tau} a_{\sigma\tau} f_{\sigma}^{(\ell)}(u_1) \left( f_{\tau}^{(\ell)}(u_2) - f_{\tau}^{(\ell)}(u_4) + f_{\tau}^{(\ell)}(u_6) - f_{\tau}^{(\ell)}(u_8) + f_{\tau}^{(\ell)}(u_{10}) \right)$$

$$+ \text{cyclic} + V_{10}^{(\ell)} \tag{3.21}$$

This can be rewritten in terms of $S_8$ in a suggestive way which will allow generalisation to high $n$-points as

$$\tilde{R}_{10}^{(\ell)}(z_1, z_2, \ldots, z_8) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 10} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+i_2+i_3-i_4} + V_{10} \tag{3.22}$$

where, as before $V_{10}$ is an additional collinear vanishing contribution. The summation convention in this formula will be explained below (4.1), it basically states that each $i_k > i_{k-1} + 1$.

The alternating sign in the sum in this formula combined with the property (3.20) of $S_8$ are enough to show that this has the right behaviour under collinear limits and we will see this explicitly below, but more interestingly these observations lead to immediate generalisation to higher points and arbitrary loop order.
4 \ n\text{-point MHV amplitudes: Part 2}

4.1 The general formula for the \(n\)-point collinear uplift

We claim that the \(n\)-point MHV amplitude for \(\ell \geq 1\), at any loop order is given by

\[
\tilde{R}_n^{(\ell)}(z_1, z_2, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+\cdots+i_4} + \\
+ \sum_{1 \leq i_1 < \cdots < i_5 \leq n} S_{10}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1+\cdots+i_5} + \\
+ \sum_{1 \leq i_1 < \cdots < i_6 \leq n} S_{12}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1+\cdots+i_6} + \\
+ \cdots + \\
+ \sum_{1 \leq i_1 < \cdots < i_{4\ell} \leq n} S_{4\ell}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_{4\ell}})(-1)^{i_1+\cdots+i_{4\ell}} .
\tag{4.1}
\]

Here in order to simplify the notation we have defined the symbol \(\triangleright\) as follows
\(i \triangleright j \iff i < j - 1\). This operation removes terms in the sum with consecutive \(x\)'s eg \(S_m(\ldots, x_{i}, x_{i+1}, \ldots)\).

This is a deceptively simple formula. The full \(n\)-point amplitude for arbitrary \(n\), and arbitrary loop order is given explicitly in terms of just \((2\ell - 3)\) \(m\)-point functions, \(S_m\), \(m = 8, 10, 12, \ldots, 4\ell\). Let us start with the minimal case of \(n = 8\) external particles. Equation (4.1) then implies,

\[
\tilde{R}_8(z_1, z_2, \ldots, z_8) = S_8(x_2, x_4, x_6, x_8) + S_8(x_1, x_3, x_5, x_7) .
\tag{4.2}
\]

A simple possibility is that the two terms are in fact the same, \(S_8(x_2, x_4, x_6, x_8) = S_8(x_1, x_3, x_5, x_7) = \frac{1}{2}\tilde{R}_8\), and this is what we assumed previously in Section 3.3.

There is, however, a more general solution to this equation where \(S_8(x_2, x_4, x_6, x_8) \neq S_8(x_1, x_3, x_5, x_7)\). To examine it, we rewrite (4.2) in terms of \(z\)-variables,

\[
\tilde{R}_8(z_1, z_2, \ldots, z_7, z_8) = S_8(z_1, z_2, \ldots, z_7, z_8) + S_8(z_1, z_8, \ldots, z_7, z_6) .
\tag{4.3}
\]

The l.h.s. must be cyclically symmetric in \(z_i \rightarrow z_{i-1}\). To guarantee it we must impose the flip symmetry (3.15) on \(S_8\). When applied to the second term on the r.h.s. of (4.3) we find,

\[
\tilde{R}_8(z_1, z_2, \ldots, z_7, z_8) = S_8(z_1, z_2, \ldots, z_7, z_8) + S_8(z_8, z_1, \ldots, z_6, z_7) ,
\tag{4.4}
\]

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which automatically gives a cyclically symmetric combination, even though $S_8$ individually are not required to have it. We can now divide $S_8$ into two parts, so that,

\begin{align}
S_8(x_2, x_4, x_6, x_8) &= \frac{1}{2}R_8(z_1, z_2, \ldots, z_8) + T_8(x_2, x_4, x_6, x_8), \quad (4.5) \\
S_8(x_1, x_3, x_5, x_7) &= \frac{1}{2}R_8(z_1, z_2, \ldots, z_8) + T_8(x_1, x_3, x_5, x_7). \quad (4.6)
\end{align}

$T_8$ denotes an additional contribution to $S_8$, which is not determined by the amplitude $R_8$. To ensure that $T_8$’s indeed do not appear in (4.2) we require that

\begin{equation}
T_8(x_2, x_4, x_6, x_8) + T_8(x_1, x_3, x_5, x_7) = 0. \quad (4.7)
\end{equation}

This condition is guaranteed by the flip symmetry of $T_8$ together with the anti-symmetry under $z_i \rightarrow z_{i+1}$,

\begin{equation}
T_8(x_1, x_3, x_5, x_7) = T_8(x_1^f, x_3^f, x_5^f, x_7^f) = -T_8(x_2, x_4, x_6, x_8). \quad (4.8)
\end{equation}

The entire $S_8$ can be constructed using the method of [1] as we explain in Appendix B. In particular, the contributions to the amplitude $R_8$ are constructed using $f^+$ functions and the additional contributions – to $T_8$ – are constructed from $f^-$ functions, cf (B.11). In particular, $T_8^{(3)}(x_2, x_4, x_6, x_8) = b_{\sigma \tau} f_{\sigma}^-(u_1) f_{\tau}^-(u_2)$ and $T_8^{(3)}(x_1, x_3, x_5, x_7) = -b_{\sigma \tau} f_{\sigma}^-(u_1) f_{\tau}^-(u_2)$.

We now consider the next-to-minimal case $n = 10$. The first line on the r.h.s. of (4.1) gives a non-trivial sum of $S_8$ contributions. These are the contributions of $R_8$’s and contributions of $T_8$’s, the latter no longer cancel each other in the sum. Novel contributions at 10-points then arise from the second line on the r.h.s. of (4.1):

\begin{equation}
S_{10}(x_2, x_4, x_6, x_8, x_{10}) - S_{10}(x_1, x_3, x_5, x_7, x_9). \quad (4.9)
\end{equation}

To be cyclically symmetric in $z$-variables, these functions have to be anti-symmetric under the flip symmetry (due to the relative minus sign in (4.9)). Together with $T_8$’s these contributions from $S_{10}$’s will give precisely the vanishing part of the 10-point function, $V_{10}$. In a similar way there will be well-defined pieces of $S_{10}$ which do not contribute to $V_{10}$ but are instead only seen by the collinear vanishing part of the 12-point amplitude $V_{12}$. This will be explained in more detail in section 4.5.

We now return to the general expression (4.1). Interestingly the formula (4.1) is most simply written in terms of $x$ variables rather than $z$’s. To see that this is non-trivial, imagine rewriting the right-hand side back in terms of $z$ variables. We see that rather than having arbitrary $z$ dependence, the $z$’s only appear in each term as pairs of nearest neighbours, i.e. if a term depends on $z_i$, then it will necessarily depend also on either $z_{i+1}$ or $z_{i-1}$. Writing in terms of $x$’s is a short-hand way of displaying this dependence. Furthermore, the objects $S_m$ have properties which are
similar, but nicer than the corresponding low-point amplitudes $\tilde{R}_m$. We will now detail the properties of $S_m$ for general $m$ before we prove that our formula correctly solves the constraints coming from collinear limits.

The $m$-point objects $S_m$ have similar properties to $S_8$ discussed above. Firstly, they are conformally invariant functions of $m$ $z$-variables or equivalently $m/2$ $x$-variables $S_m(z_1, \ldots, z_m) = S_m(x_2, x_4, \ldots, x_m)$ where $x_2 = (z_1, z_2)$ etc. They are also symmetric under cyclic symmetry and parity up to a minus sign in $x$-variables (but not necessarily in $z$),

$$S_m(x_2, x_4, \ldots, x_m) = S_m(x_4, x_6, \ldots, x_m, x_2) = (-1)^{m/2} S_m(x_m, x_{m-2}, \ldots, x_4, x_2) .$$

(4.10)

Furthermore, we also require that they satisfy the additional flip (anti)-symmetry,

$$S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{m/2}}) = (-1)^{m/2} S_m(x_{i_1}^f, x_{i_2}^f, \ldots, x_{i_{m/2}}^f) .$$

(4.11)

The $S_m$'s must also vanish in the collinear limit $z_m \to z_{m-2}$ ie $x_m \to x_{m-1}$

$$\lim_{x_m \to x_{m-1}} S_m(x_2, \ldots, x_{m-2}, x_m) = 0 \quad \text{(collinear limits)} .$$

(4.12)

A useful and more geometrical way of saying this

$$S_m(x_i, \ldots, x_j, x_k) = 0 \quad \text{if } x_j, x_k \text{ become light-like separated} .$$

(4.13)

Finally $S_m$ must also vanish in the multi-collinear limits, where $(p+1)$ consecutive momenta become collinear, $z_m, z_{m-2}, \ldots, z_{m-p+2} \to z_{m-p}$, or $x_m \to x_{m-1}$, $x_{m-2} \to x_{m-3}$, $\ldots$, $x_{m-p+2} \to x_{m-p+1}$ for $p = 2, 4, \ldots, m-4$ ie

$$S_m(x_i, x_j \ldots, x_k) = 0 \quad \text{if any set of } 2, 3, \ldots \text{ or } (m/2 - 2) \text{ consecutive points become mutually light-like separated} .$$

(4.14)

In other words we require that the $S$-functions vanish in all allowed multi-collinear limits. By “allowed” here we mean that we can not insist that $S_m$ vanishes when too many points become collinear by conformal invariance (see Appendix A). The limit when $m/2 - 1$ points become collinear is conformally equivalent to points being in generic positions and so $S_m$ can not vanish in this limit. Similarly when all $m/2$ points become collinear.

To show that (4.1) is indeed the $n$-point function, we must first prove that this expression is cyclic, that it satisfies the correct properties under collinear limits and that it is unique. That (4.1) is cyclically symmetric in $z$-variables comes straight from its definition, the (anti)-flip symmetry (4.11) together with cyclicity of $S_m$ in its $x$-arguments. In the next subsection we argue that it behaves correctly in all collinear limits. Then we discuss the uniqueness of the structure.
4.2 \( \tilde{\mathcal{R}}_n \) has the correct collinear limits

Now consider the simplest collinear limit, \( z_n \rightarrow z_{n-2} \) i.e \( x_n \rightarrow x_{n-1} \). Using

\[
\lim_{x_n \rightarrow x_{n-1}} S_m(i, j \ldots k) = S_m(i, j \ldots k) \quad \text{for} \quad i, j, \ldots k \neq n-1, n \quad \text{and}
\]

\[
\lim_{x_n \rightarrow x_{n-1}} [S_m(i, j \ldots k, n-1) - S_m(i, j \ldots k, n)] = 0,
\]

one can see that

\[
\lim_{x_n \rightarrow x_{n-1}} \tilde{\mathcal{R}}_n(z_1, \ldots z_n) = \tilde{\mathcal{R}}_{n-2}(z_1, \ldots z_{n-2})
\]

as required under collinear limits.

To prove the correct property under multi-collinear limits we need to work a little harder. The multi-collinear limit, where \( p+1 \) edges become collinear is defined for even \( p \) as \( z_n, z_{n-2}, \ldots, z_{n-p+2} \rightarrow z_{n-p} \). This is the same as pairwise limits on consecutive \( x \)-variables, \( x_n \rightarrow x_{n-1}, x_{n-2} \rightarrow x_{n-3}, \ldots, x_{n-p+2} \rightarrow x_{n-p+1} \) as can be easily seen from Fig. 2. More geometrically, we can separate all the \( n \) \( x \)-variables into two sets:

\[
\underbrace{x_{n-p}, x_{n-p+1}, \ldots, x_{n-1}}_{S_{p+2}} \leftarrow \underbrace{x_{n-p+2}, \ldots, x_n, x_1, x_2, \ldots, x_{n-p-1}}_{S_{n-p-2}}
\]

In this limit all points in the set \( S_{p+2} = \{x_{n-p}, x_{n-p+1}, \ldots, x_1\} \) are becoming mutually light-like separated (i.e. collinear), whereas the points in the set \( S_{n-p-2} = \{x_2, \ldots, x_{n-p-1}\} \) remain unchanged. Now the \( S \)'s vanish whenever \( r \) consecutive points become light-like separated for \( r = 2, 3, \ldots, \frac{m}{2} - 2 \) as discussed in (4.14). Since all the points in \( S_{p+2} \) become light-like separated from each other, this means that \( S_m \) vanishes unless all, or all but one of the points are in \( S_{p+2} \) or unless, all, or all but one of the points are in \( S_{n-p-2} \), i.e

\[
S_m(x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_{m-r}}) \rightarrow 0 \quad \text{for} \quad r = 2, \ldots, \bar{m} - 2 \quad \text{and where}
\]

\[
\{i_1, \ldots, i_r\} \in S_{m-p-2} \quad \text{and} \quad \{j_1, \ldots, j_{m-r}\} \in S_{p+2},
\]

where we have defined \( \bar{m} = m/2 \). So the sum of \( S \)'s appearing in \( \mathcal{R} \) reduces to

\[
\sum_{2 \leq i_1 < \cdots < i_{\bar{m}} \leq n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1+\cdots+i_{\bar{m}}} \quad \rightarrow
\]

\[
\sum_{2 \leq i_1 < \cdots < i_{\bar{m}} \leq n-p-1} \sum_{j=i_{\bar{m}}+1}^{n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}-1}, x_j)(-1)^{i_1+\cdots+i_{\bar{m}-1}-1}j
\]

\[
+ \sum_{n-p \leq j_1 < \cdots < j_{\bar{m}-1} \leq n+1} \sum_{i=2}^{j_1-2} S_m(x_i, x_{j_1}, x_{j_2}, \ldots, x_{j_{\bar{m}-1}})(-1)^{j_1+\cdots+j_{\bar{m}-1}-1}i.
\]
Now consider the first term of this last expression, and in particular focus on the sum over $j$. We have that

$$\sum_{j=i_{\bar{m}}-1+2}^{n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}-1}, x_{j})(-1)^j$$

$$= \sum_{i_{\bar{m}}=i_{\bar{m}}-1+2}^{n-p-1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_{\bar{m}}} + \sum_{j=n-p}^{n+1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}-1}, x_{j})(-1)^j$$

$$= \sum_{i_{\bar{m}}=i_{\bar{m}}-1+2}^{n-p-1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_{\bar{m}}} + S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}-1}, x_{n-p}) - S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}-1}, x_{n+1})$$

(4.20)

where in the last equality we have used the fact that $x_j$ is one of the vertices becoming collinear, and in the limit $x_n \to x_{n-1}$, $x_{n-2} \to x_{n-3}$, $\ldots$, $x_{n-p+2} \to x_{n-p+1}$, thus the alternating sum collapses to the two boundary cases. Inserting this back into (4.19) and using cyclicity, we can include this most succinctly by including the end-points $n-p$ and $n+1=1$ in the sum to rewrite the first term on the right-hand side of (4.19) in the suggestive form

$$\sum_{1 \leq i_1 < \ldots < i_{\bar{m}} \leq n-p} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1+\ldots+i_{\bar{m}}}.$$  (4.21)

So we have massaged the first term on the r.h.s. of (4.19) into a nice form. The second term in (4.19), despite its similarity to the first term looks trickier to manipulate into something pleasant, since instead of 1 point becoming collinear $m/2 - 1$ of the points are becoming collinear. However at this point we can make use of the fact (used in [35]) that the collinear limit we are performing is conformally equivalent to a different multi-collinear limit, in which instead of the points in $S_{p+2}$ becoming collinear and the points in $S_{n-p-2}$ remaining unchanged, we have the converse: the points in $S_{p+2}$ remain unchanged and the points in $S_{n-p-2}$ become collinear. With this observation we see that in this conformally equivalent setting, only the point $x_i$ is becoming collinear and the points $x_j$ remain unchanged. We can then perform similar manipulations to those leading to (4.21) on the second term on the r.h.s. of (4.19) to obtain the final result

$$\sum_{2 \leq i_1 < \ldots < i_{\bar{m}} \leq n+1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1+\ldots+i_{\bar{m}}}$$

$$\to \sum_{1 \leq i_1 < \ldots < i_{\bar{m}} \leq n-p} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1+\ldots+i_{\bar{m}}} + \sum_{n-p-1 \leq j_1 < \ldots < j_{\bar{m}} \leq n+2} S_m(x_{j_1}, \ldots, x_{j_{\bar{m}}})(-1)^{j_1+\ldots+j_{\bar{m}}}.$$  (4.22)
Now this is true for any value of $m$ and since our general formula for the amplitude (4.1) is made from such structures as these we conclude that in the multi-collinear limit

$$\tilde{R}_n \to \tilde{R}_{n-p} + \tilde{R}_{p+4},$$

precisely as we expect.

### 4.3 Discussion of the result

So (4.1) gives a solution of the constraints from collinear limits. How unique is this solution? To examine this question, imagine that the formula (4.1) failed to give the correct result for $\tilde{R}_n$ at $n$-points (but succeeded below this point). Then consider the difference between the prediction from (4.1) and $\tilde{R}_n$, $\tilde{R}_n - \tilde{R}_n^{(4.1)}$. Since both obey the same collinear limits, this is an $n$-point function which vanishes in all allowed collinear limits. So one might expect that we can always absorb this into the collinear vanishing object $S_n$. However this not quite as straightforward as it first appears. In the following subsections we will argue, first by considering explicit special cases, and then outlining the general case, that we can always absorb the collinear vanishing piece into $S_n$ and $S_{n-2}$. This means that (4.1) would indeed give the unique result if we allowed $S_n$ for all $n$.

We have, however also restricted the number of collinear vanishing functions so that in particular $S_m^{(\ell)} = 0$ for $m > 4\ell$. So we now focus on the question of why $S_n$, and hence the collinear vanishing part of $\tilde{R}_n$ should be restricted in this way. The point is simply that one can not write down a collinear vanishing, conformally invariant $\ell$-loop function beyond $4\ell$-points. This was argued in [1] and for completeness we briefly recast the argument here. It is based on examining the symbol of $S_m$. The central assumption of [1] was that the basis of the symbol (in 2d kinematics) is made out of simple cross-ratios $u_{ij}$. These cross-ratios have a clear and simple behaviour in two collinear limits, those associated with the edges $i$ or $j$. Specifically, $\log u_{ij} \to 0$ when either $z_{i+1} \to z_{i-1}$, or $z_{j+1} \to z_{j-1}$. Thus, the presence of $u_{ij}$ in the symbol of $S_m$ makes it vanish in the collinear limits associated with the edges $i$ or $j$. To make sure that $S_m$ vanishes in all possible collinear limits, its symbol must contain $u_{ij}$’s for all pairs of edges. At $\ell$-loops, there are $2\ell$ tensor products of $u_{ij}$’s in the symbol, and they can connect maximally $4\ell$ different edges. This means that collinearly vanishing functions exist only up to $m = 4\ell$ points. Thus the formula (4.1) gives a unique uplift.

In the next section we will extend this analysis to non-MHV amplitudes and obtain similar conclusions. In Section 6 we will consider the tree-level NMHV amplitudes.
They evade our conclusions by not manifestly having the correct collinear behaviour (and indeed they are not manifestly cyclic either). They only have these properties after taking into account special linear identities. We believe this is special to tree-level and that at loop level the only solution is (4.1).

4.4 Special cases

We first look again at the $n$-point 2-loop result of [9]. At 2-loops, inserting the 8-point result for $S^{(2)}$

$$S^{(2)}_8(z_1, \ldots, z_8) = -\frac{1}{4} \log(u_{17,53}) \log(u_{31,75}) \log(u_{28,64}) \log(u_{42,86})$$

into (4.1)

$$\tilde{R}_n^{(2)}(z_1, z_2, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} S^{(2)}_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+\ldots+i_4},$$

and rewriting in terms of the basis $u_{ij}$s correctly reproduces the form of the two-loop result in (3.1).

Next, at 3 loops the formula (4.1) for any number of points contains essentially only three independent terms. It reduces to

$$\tilde{R}_n^{(3)}(z_1, z_2, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} S^{(3)}_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+\ldots+i_4}$$

$$+ \sum_{1 \leq i_1 < \ldots < i_5 \leq n} S^{(3)}_{10}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1+\ldots+i_5}$$

$$+ \sum_{1 \leq i_1 < \ldots < i_6 \leq n} S^{(3)}_{12}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1+\ldots+i_6},$$

where the multi-collinearly-vanishing function $S_m$ are constructable with methods of [1] as we will now demonstrate. We will show that the general formula (4.1) correctly reproduces the 10-point result of (3.22) and gives the entire collinear vanishing term $V_{10}$.

4.4.1 $S_{10}$ contribution to $V_{10}$

We first consider the $S_{10}$ collinear vanishing contribution to $R_{10}$. The building blocks are collinear vanishing functions $(f_1, f_2, f_3)$ of even and odd cross-ratios, derived in [1]
and listed in (B.21). Since \( f_1 \) and \( f_2 \) give 5 independent functions via cyclic permutations of their arguments, whereas \( f_3 \) is already cyclically symmetric, giving only 1 independent function, we have in total 11 functions.

Let us now rewrite these functions in a basis which diagonalizes the action of the cyclic group\(^{10}\),

\[
\begin{align*}
    f_1^{(k)}(z_1, z_3, z_5, z_7, z_9) &= \sum_{j=1}^{5} f_1(u_{2j}, u_{2j+2}, u_{2j+4})e^{2\pi ijk/5} & k = 0 \ldots 4 \\
    f_2^{(k)}(z_1, z_3, z_5, z_7, z_9) &= \sum_{j=1}^{5} f_2(u_{2j}, u_{2j+2}, u_{2j+4})e^{2\pi ijk/5} & k = 0 \ldots 4 \\
    f_3^{(0)}(z_1, z_3, z_5, z_7, z_9) &= f_3(u_2, u_4, u_6, u_8, u_{10})
\end{align*}
\]

These new functions lie in irreducible representations of the cyclic group, in fact they are eigenstates of the cyclic group,

\[
    f_a^{(k)}(z_3, z_5, z_7, z_9, z_1) = e^{2\pi ik/5} f_a^{(k)}(z_1, z_3, z_5, z_7, z_9) .
\]

We also have that under parity \( f^{(k)} \to f^{(5-k)} \).

Then by construction both \( V_{10} \) appearing in (3.8) and \( S_{10} \) appearing in (4.1) are \( C_5 \) and parity invariant combinations of these functions. To obtain cyclic \((C_5)\) invariant combinations, a function carrying cyclic representation \( k \) must multiply a function carrying cyclic representation \(-k\).

Let us first construct \( V_{10} \). It is given by a linear combination of the 12 collinear vanishing contributions to the remainder function listed in (B.22). These are now written as

\[
\begin{align*}
    f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) + e^{-2\pi ik/5} f_b^{(-k)}(z_{\text{odd}}) f_a^{(k)}(z_{\text{even}}) &+ a \leftrightarrow b , \\
    f_3^{(0)}(z_{\text{odd}}) f_a^{(0)}(z_{\text{even}}) + f_a^{(0)}(z_{\text{odd}}) f_3^{(0)}(z_{\text{even}}) &+ a = 1, 2, 3
\end{align*}
\]

where \( z_{\text{odd}} := z_1, z_3, z_5, z_7, z_9 \) and \( z_{\text{even}} := z_2, z_4, z_6, z_8, z_{10} \). In the first equation we have \( a, b = 1, 2 \) and \( k = 0, 1, 2 \), thus it gives 9 independent functions, in the second equation \( a = 1, 2, 3 \) giving 3 more. Clearly these 12 functions are simple recombinations of the 12 functions in (B.22). This is what we have for \( V_{10} \).

Let us compare this with the construction of \( S_{10} \). These are constructed from the same building block functions, with an additional constraint that \( S_{10} \) must be

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\(^{10}\)Here we mean the cyclic group which acts separately on the even and odd variables, so in this case it is \( C_5 \).
antisymmetric under flip symmetry. They are given as

\[ S_{10}(x_2, x_4, x_6, x_8, x_{10}) \ni f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) - f_b^{(k)}(z_{\text{odd}}) f_a^{(-k)}(z_{\text{even}}) \]

\[ + f_b^{(k)}(z_{\text{odd}}) f_a^{(-k)}(z_{\text{even}}) - f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) \] \quad (4.31)

Non vanishing contributions arise from \( k = 1, 2 \) and \( a, b = 1, 2 \), so we have 6 contributions in total. Note in particular that the invariant representation \( k = 0 \) drops out here. Now, the contribution from \( S_{10} \)'s to \( R_{10} \), dictated by the \( S \)-formula (4.1),

\[ \tilde{R}_{10} \ni S_{10}(x_2, x_4, x_6, x_8, x_{10}) - S_{10}(x_1, x_3, x_7, x_9) \] \quad (4.32)

is

\[ (1 - e^{2\pi i k/5}) \left( f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) + e^{-2\pi i k/5} f_b^{(k)}(z_{\text{odd}}) f_a^{(-k)}(z_{\text{even}}) \right) \]

\[ + a \leftrightarrow b \] \quad (4.33)

We can now see that the contribution of \( S_{10} \)'s to the 10-point amplitude (4.29) gives a clearly identifiable subset of the most general collinearly vanishing contribution \( V_{10} \) in (4.30). They are the same functions, simply multiplied by a constant factor \((1 - e^{2\pi i k/5})\) which plays no role, except in the case \( k = 0 \) where it vanishes.

Thus we see clearly that the contribution of \( S_{10} \) yields the entire collinear vanishing part of \( R_{10} \) except the pieces constructed from the cyclically invariant functions \( f_a^{(0)} \). We will now see how these missing building blocks are correctly filled in by contributions from \( S_8 \) or more precisely \( T_8 \).

### 4.4.2 \( S_8 \) contribution to \( V_{10} \)

Now consider the contribution of \( S_8 \) to \( R_{10} \). Following Section 4.1 we split \( S_8 \) into \( R_8 \) and \( T_8 \) parts (4.5)-(4.6). The role of \( R_8 \) is completely clear. It is the 8-point amplitude and furthermore it contributes to the collinearly non-vanishing part of all higher point amplitudes. The first contribution of \( T_8 \) however arises only at 10-points where it contributes to the collinearly vanishing part of the answer. Here we wish to trace through the \( T_8 \) contribution to \( V_{10} \).

From (B.11) We have

\[ T_8(x_2, x_4, x_6, x_8) = \sum_{\sigma, \tau} b_{\sigma \tau} f_{\sigma}^-(u_1) f_{\tau}^-(u_2) \] \quad (4.34)

where \( b_{\sigma \tau} = b_{\tau \sigma} \) and the functions \( f_{\sigma}^- \), \( \sigma = 1, 2, 3 \), are listed in (B.18). These functions \( f_{\sigma}^- \) are all weight 3. It turns out that contributions of \( T_8 \) of the form \((\text{weight 2}) \times \)
vanish at all points. We will discuss this point further at the end of this subsection. In terms of the \(z\)-variables these functions satisfy the following property (cf (B.12)),
\[
f_{\sigma}^{-}(z_3, z_5, z_7, z_9) = -f_{\sigma}^{-}(z_1, z_3, z_5, z_7),
\]
ie they are invariant with an alternating sign under cyclic symmetry.

Inserting \(T_8\) into the S-formula
\[
\mathcal{R}_{10} \ni \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 10} S_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+i_2+i_3+i_4}
\]
\[
\ni \mathcal{R}_{10} \ni \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 10} T_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+i_2+i_3+i_4} \tag{4.36}
\]
and performing the sum we have
\[
\mathcal{R}_{10} \ni \sum_{\sigma, \tau} b_{\sigma, \tau} F_{\sigma}(z_1, z_3, z_5, z_7, z_9) F_{\tau}(z_2, z_4, z_6, z_8, z_{10}) \tag{4.37}
\]
where
\[
F_{\sigma}(z_1, z_3, z_5, z_7, z_9) \tag{4.38}
\]
\[
= f_{\sigma}^{-}(z_1, z_3, z_5, z_7) - f_{\sigma}^{-}(z_1, z_3, z_5, z_9) + f_{\sigma}^{-}(z_1, z_3, z_7, z_9) - f_{\sigma}^{-}(z_1, z_5, z_7, z_9) + f_{\sigma}^{-}(z_3, z_5, z_7, z_9)
\]
\[
= f_{\sigma}^{-}(z_1, z_3, z_5, z_7) + f_{\sigma}^{-}(z_9, z_1, z_3, z_5) + f_{\sigma}^{-}(z_7, z_9, z_1, z_3) + f_{\sigma}^{-}(z_5, z_7, z_9, z_1) + f_{\sigma}^{-}(z_3, z_5, z_7, z_9)
\]

Note that the functions \(F_{\sigma}\), although constructed from four-point building blocks, are in fact cyclically invariant 5-point functions. Furthermore, inspection of the r.h.s. of (4.38) shows that they also vanish in collinear limits. Thus we see that the r.h.s of (4.37) corresponds precisely to \(k = 0\) contributions to \(V_{10}\) in (4.29),(4.30). We have six contributions
\[
F_1 F_1, \ F_1 F_2 + F_2 F_1, \ F_1 F_3 + F_3 F_1, \ F_2 F_2, \ F_2 F_3 + F_3 F_2, \ F_3 F_3 \tag{4.39}
\]
and these are the six previously missing contributions in \(V_{10}\), not accounted by \(S_{10}\), of the previous subsection.

One obvious question is what happens if we use (weight 2) \(\times\) (weight 4) functions, \(f^{-}\), to construct \(T_8\). According to the above discussion this should produce a (weight 2) \(\times\) (weight 4) collinear vanishing contribution to \(S_{10}\) which we know is not present in \(V_{10}\) giving an apparent contradiction. In reality it is easy to see that all such contributions vanish. There is a unique weight 2 function \(f^{-}(u) = Li_{2}(u) - Li_{2}(1 - u)\) and when we plug it into (4.38) we see that the corresponding function \(F\) vanishes. When
written in terms of the symbol this identity is manifest; in terms of the polylogarithms this becomes the equation

\[
\text{Li}_2(u_1) + \text{Li}_2(u_3) + \text{Li}_2(u_5) + \text{Li}_2(u_7) + \text{Li}_2(u_9) - (u_i \leftrightarrow 1 - u_i) = \text{constant}
\]

(4.40)

which, when writing in terms of \(u_1, u_5\) via the relation (2.23)

\[
u_3 = 1 - u_1 u_5 \quad u_7 = \frac{1 - u_5}{1 - u_1 u_5} \quad u_9 = \frac{1 - u_1}{1 - u_1 u_5}
\]

(4.41)
is equivalent to the famous non-trivial five-term identity for the dilogarithm, first discovered by Spence in 1809. Thus as mentioned earlier no (weight 2) \times (weight 4) contributions survive in \(V_{10}\) while weight 3 functions have already been accounted above.

We also note that the contributions involving weight 2 functions \(f^\pm\) also disappear from \(V_n\) at all higher \(n\).

To summarise we have demonstrated that \(T_8\) and \(S_{10}\) together generate all possible collinear vanishing 10-point functions. And this confirms that the \(S\)-formula does not miss anything.

4.5 Higher points

This general pattern continues in a similar way to higher points. We construct \(S_{2m}\)'s from the product of collinear vanishing building block functions of even and odd \(z\)’s. We choose a basis of these which diagonalise the cyclic group, and call them \(f_a^{(k)}\) where \(k\) is the representation of the cyclic group \(C_m\) and \(a\) labels the inequivalent functions. Then the \(S\)-formula gives the contribution

\[
S_{2m} = \sum_{a,b} \alpha_{ab;k} \left( f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) + (-1)^m f_b^{(-k)}(z_{\text{odd}}) f_a^{(k)}(z_{\text{even}}) \right) + \text{parity}
\]

(4.42)

where \(k = 0, 1, \ldots, m\), giving the contribution to (the collinear vanishing part of) \(R_{2m}\) of

\[
\sum_{a,b} \alpha_{ab;k} \left( 1 + (-1)^m e^{2\pi i k/m} \right) \left( f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) + e^{-2\pi i k/m} f_b^{(-k)}(z_{\text{odd}}) f_a^{(k)}(z_{\text{even}}) \right) + \text{parity}.
\]

(4.43)

This yields all possible collinear vanishing \(m\)-point amplitudes, simply multiplied by an irrelevant overall factor \((1 + (-1)^m e^{2\pi i k/m})\), except those built from \(k = 0\) \((m\ \text{odd})\)
or \( k = m/2 \) \((m \text{ even})\) where the factor vanishes. In other words the \( S_{2m} \) contribution to \( R_{2m} \) misses out the cyclically invariant symmetric building block functions if \( m \) is odd, or in the \( m \) even case it misses out the functions cyclically invariant up to a sign.

However, just as in Section 4.4.2, these missing contributions at \( 2m \)-points will arise from \( 2m - 2 \) points and together should fill the full space of collinearly vanishing functions \( V_{2m} \).

Thus the \( S \)-formula (4.1) gives an explicit formula for the \( n \)-point amplitude, in terms of collinearly vanishing objects \( S_m \) once they have been constructed, and in this paper we have shown how to do that using the method of [1].

## 5 Collinear uplift of \( n \)-point \( N^k \)MHV amplitudes

The general formula for lifting MHV amplitudes to higher points looks very general and immediately suggests generalisation to \( N^k \)MHV superamplitudes. To do so we will need to examine odd superspace variables in 2d and the form of the collinear limit.

As discussed in Section 2, superamplitudes can be written in chiral superspace depending on superspace coordinates \( X_i = (x_i, \theta_i^A) \) where the bosonic components \( x_i \) are given in terms of 2d lightcone coordinates by Eq. (2.11). Examining the implications of the light-like condition for the \( \theta \)’s (2.2) in 2d kinematics, we find that the condition can be solved in an analogous manner to the way we write \( x \)’s in terms of \( z \)’s, namely for the Grassmann coordinates, \( \theta \)’s and \( \chi \)’s

\[
\theta_i^{\alpha A} = \begin{cases} 
(\chi_i^{A}, \chi_i^{\bar{A}}), & i \text{ even} \\
(\chi_i^{A}, \chi_{i-1}^{\bar{A}}), & i \text{ odd} 
\end{cases}
\]  

(5.1)

Indeed, comparing with the supertwistor in the form of Eq. (2.17) we find that the \( \chi \)’s are precisely the odd supertwistor variables just as the \( z \) were the bosonic twistors.

The general formula (4.1) giving all \( n \)-point MHV amplitudes in terms of a finite number of collinear vanishing functions generalises immediately now to the non-MHV case. Indeed the collinear limits \( z_n \to z_{n-2} \) must be accompanied by identical limits for the Grassmann coordinates \( \chi_n \to \chi_{n-2} \). Indeed, one has to be careful about the relevant speed at which we take the limit. We here take the collinear limit in a supersymmetric way. More precisely the collinear limit can be taken as a particular superconformal transformation on the relevant vertices. The details of this limit are given in Appendix A.
So precisely as for the MHV case we have collinear vanishing functions, this time of the super-co-ordinates \( S_m(X_2, X_4, \ldots, X_m) \) which satisfy cyclicity and (anti-)parity in their \((X)\) arguments, flip (anti-)symmetry and collinear vanishing property in all (allowed) collinear limits, so
\[
S_m(X_2, X_4, \ldots, X_m) = S_m(X_4, X_6, \ldots, X_m, X_2) = (-1)^{m/2} S_m(X_m, X_{m-2}, \ldots, X_2) = (-1)^{m/2} S_m(X_2^f, X_4^f, \ldots, X_m^f)
\]
\[
\lim_{x_m \rightarrow X_{m-1}} S_m(X_2, X_4, \ldots, X_m) = 0 .
\] (5.2)

Or more generally \( S_m \) vanishes whenever any (allowed) number of consecutive \( X \)'s become light-like separated (in the supersymmetric sense: \( X_1 = (x_1, \theta_1) \) and \( X_2 = (x_2, \theta_2) \) are light-like separated if \( x_{12}^2 = 0 \) and \( \theta_{12a} x_{12}^{\alpha a} = 0 \)) i.e.
\[
S_m(X_i, X_j, \ldots, X_k) = 0 \quad \text{if any set of 2, 3, \ldots or } m/2 - 2 \text{ consecutive points}
\]
become mutually light-like separated.

(5.3)

We note that \( S_m(X_1, X_3, \ldots, X_{m-1}) \) is a function of superspace variables, and is not the same object as the MHV function \( S_m(x_1, x_3, \ldots, x_{m-1}) \) with purely bosonic-variables from the previous section. The latter however is given by the zero-th order in \( \theta \) expansion of the former. As we are talking here about superamplitudes, the \( N^k \)MHV label \( k \) does not appear in \( \tilde{R} \) and \( S \) in formulae below, but the \( N^k \)MHV amplitudes, \( \tilde{R}_{n,k} \), will arise as \( \theta^{4k} \) components of \( \tilde{R}_n(X) \).

The general formula for the \( n \)-point amplitude is given, in exact analogy with the MHV case, by
\[
\tilde{R}^{(l)}_n(Z_1, Z_2, \ldots, Z_n) = \sum_{1 \leq i_1 < \cdots < i_4 \leq n} S_8^{(l)}(X_{i_1}, X_{i_2}, \ldots, X_{i_4}) (-1)^{i_1 + \cdots + i_4}
\]
\[
+ \sum_{1 \leq i_1 < \cdots < i_5 \leq n} S_{10}^{(l)}(X_{i_1}, X_{i_2}, \ldots, X_{i_5}) (-1)^{i_1 + \cdots + i_5}
\]
\[
+ \sum_{1 \leq i_1 < \cdots < i_6 \leq n} S_{12}^{(l)}(X_{i_1}, X_{i_2}, \ldots, X_{i_6}) (-1)^{i_1 + \cdots + i_6}
\]
\[
+ \cdots
\]
\[
+ \sum_{1 \leq i_1 < \cdots < i_{m_{\max}/2} \leq n} S_{m_{\max}}^{(l)}(X_{i_1}, X_{i_2}, \ldots, X_{i_{m_{\max}/2}}) (-1)^{i_1 + \cdots + i_{m_{\max}/2}} .
\] (5.4)

One can easily verify that (5.4) gives a formula with the correct properties under collinear limits. Indeed for the multi-collinear limit in which superspace points in the set \( S_{p+2} = \{ X_{n-p}, X_{n-p+1}, \ldots, X_1 \} \) become light-like separated (in the supersymmetric
sense) from all other points in $S_{p+2}$ (i.e. collinear) whereas the points in the set $S_{n-p-2} = \{ x_2, \ldots, x_{n-p-1} \}$ remain unchanged. Importantly this limit can be described by performing a conformal transformation on the points in $S_{p+2}$ (see Appendix A). In this limit one can see that

$$\tilde{R}_n \to \tilde{R}_{n-p} + \tilde{R}_{p+4},$$

exactly as required (2.10). The proof follows by direct analogy to the arguments in the MHV case around (4.15).

Thus the only question is how many $S$’s are there, i.e. what is $m_{\text{max}}$. This will depend on the loop level $\ell$ and the order in $\chi$-expansion, i.e. the value of $k$. Based on the MHV bound, $m_{\text{MHV}} \leq 4\ell$ and the $\overline{Q}$-equation of Ref. [21, 36] which related $N^k$MHV amplitudes at $\ell$-loops to $\overline{Q}$-$N^{k-1}$MHV amplitudes at $(\ell+1)$-loops, one could expect that $m_{\text{max}} = 4(\ell + k)$.

6 Tree-level NMHV amplitude

In this section we reduce the known $n$-point tree-level NMHV superamplitudes down to 2d kinematics. This is a non-trivial procedure, since each term diverges in 2d kinematics and only certain combinations are finite.

In full 4d kinematics, the tree-level NMHV amplitude is [14, 37]

$$R_{n,1}^{\text{tree}} = \frac{1}{2} \sum_{i,j} [1, i-1, i, j-1, j].$$

where the 5-brackets (which are totally anti-symmetric in their arguments) can be written in momentum supertwistors (2.3) as [31]

$$[i, j, k, l, m] = \frac{\delta^{0|4}(\chi^i<jklm> + \text{cyclic})}{\langle i j k l \rangle \langle j k l m \rangle \langle k l m i \rangle \langle l m i j \rangle \langle m i j k \rangle}.$$  

The 4-brackets $\langle i j k l \rangle$ are defined in (2.16) and as mentioned there, in 2d kinematics these vanish unless there are precisely 2 even particles and 2 odd particles. This clearly can not be the case for all five 4-brackets in the denominator, and so we conclude that the 5-bracket inevitably diverges in 2d. None-the-less it must be that the 2d kinematics should lead to sensible amplitudes, so (6.2) should be rewritable in terms of some finite combinations of 5-brackets. At this point we notice that in fact the poles which diverge in 2d are in fact spurious poles which can not be present in the amplitude itself. Guided by this insight and the analysis of spurious poles given in [32] we are able to find simple combinations of 5-brackets which are finite in
Furthermore, these combinations of 5-brackets factorise into two 3-brackets in 2d kinematics:

\[
\tilde{R}(i, j, k) := [i, j - 1, j, k - 1, k] + [j, k - 1, k, i - 1, i] + [k, i - 1, i, j - 1, j] \\
= [i, j, k][i - 1, j - 1, k - 1] \quad (i, j, k \text{ all even/odd})
\]

\[
\tilde{R}(i, j, k) := -[i, j - 1, j, k - 1, k] - [j, k - 1, k, i - 1, i] + [k - 1, i - 1, i, j - 1, j] \\
= [i, j, k - 1][i - 1, j - 1, k] \quad (i \text{ even/odd}; k \text{ odd/even})
\]

\[
\tilde{R}(i, j, k) := [i, j - 1, j, k - 1, k] - [j - 1, k - 1, k, i - 1, i] - [k - 1, i - 1, i, j - 1, j] \\
= [i, j - 1, j, k - 1][i - 1, j, k] \quad (i \text{ even/odd}; k, j \text{ odd/even}) \quad . \tag{6.3}
\]

Here on the right-hand side we have used 3-brackets, the natural analogue of the 5-bracket in 2d kinematics, an invariant of \( SL(2|2) \), defined as

\[
[ijk] = \delta^{012} (\chi^i \langle jk \rangle + \chi^j \langle ki \rangle + \chi^k \langle ij \rangle) \\
\langle ij \rangle \langle jk \rangle \langle ki \rangle . \quad \tag{6.4}
\]

In actual fact, as mentioned earlier, at least at NMHV level, we do not need to reduce the internal \( SU(4) \) group, but can perfectly well keep the full \( \chi \) structure. I.e, we do not need to reduce the 4-component \( \chi \)'s to 2-component \( \chi \)'s as in (2.17). In this case the above 3-brackets would simply have two antisymmetric \( SU(4) \) indices \( \langle ijk \rangle ^{AB} \) which would be contracted with an \( \epsilon_{ABCD} \) in (6.3). Nevertheless for simplicity we stick to the reduced version.

Notice that just as the 5-brackets satisfy a 6-term identity

\[
[ijklm] + [jklmn] + [klmni] + [lnmij] + [mnijk] + [nijkl] = 0 , \quad \tag{6.5}
\]

(and indeed one can use this identity to rewrite (6.3) in an alternative way) so the 3-brackets satisfy a simple 4-term identity:

\[
[ijk] - [jkl] + [kli] - [lij] = 0 . \quad \tag{6.6}
\]

This can be quickly checked by considering \( \chi \)-components and using Schouten identities.

Now let us reduce the NMHV tree-level amplitudes to 2d kinematics. Consider first of all the first non-trivial case, the 6-point amplitude. This is

\[
\mathcal{R}^{tree}_{0;1} = \frac{1}{2} \left( [13456] + [12356] + [12345] \right) = \frac{1}{2} \tilde{R}(1, 3, 5) = \frac{1}{2} [135][246] . \quad \tag{6.7}
\]

Here the first equality comes directly from the general formula (6.1), and the second and third from (6.3).
By considering higher points, in particular we looked at 8- and 10-points in great
detail. Due to the large number of identities, it is not clear which is the best way of
representing any amplitude at low points. However gradually a general picture begins
to emerge and we obtain a simple formula for the $n$-point NMHV tree-level amplitude
in 2d kinematics in terms of 3-brackets. The result can be written

$$ R_{\text{tree}}^{n;1} = \sum_{4 \leq j < k \leq n} \frac{1}{2} \tilde{R}(2, j, k) (-1)^{j+k} . $$

which at 6-points correctly reproduces (6.7).

Let us then compare this NMHV tree-level result with our general result for loop
level superamplitudes, given in formula (5.4). First of all we see that the formulae
are strikingly similar with the same type of alternating sum. The tree-level formula
starts at 6 points however whereas (5.4) starts at 8-points. Looking closer we see that
the main difference is that in the tree-level formula (6.8) only two out of the three
indices are summed over, the first index remaining fixed. This does not look cyclically
invariant and indeed verification of cyclic invariance requires the implementation of
non-trivial linear identities between the six-point $\tilde{R}(i, j, k)$ at different points. We
can of course make it manifestly cyclically symmetric by adding together cyclic terms
to give

$$ R_{\text{tree}}^{n;1} = \sum_{i < j < k < n} \frac{1}{2^n} \tilde{R}(i, j, k) (-1)^{i+j+k} . $$

This has a form very similar to the general $S$-formula (5.4), the difference is the
appearance of the rather asymmetric looking $(-1)^{i+j+k}$ instead of the more symmetric
$(-1)^{i+j+k}$ one would expect. Indeed, imagine extending the $S$-formula to $m = 6$ to
give

$$ \sum_{1 \leq i < j < k \leq n} S_6(i, j, k)(-1)^{i+j+k} $$

(6.10)

with a $(-1)^{i+j+k}$ factor.

So the question remains, how does the tree-level NMHV formula get round this
obstacle? The answer is that the $S$-formula is derived to obey manifest cyclicity
and manifest collinear limits. The NMHV tree-level formula does not satisfy these
requirements, but instead only satisfies cyclicity after taking into account non-trivial
linear identities.

For example, first consider taking the triple/soft collinear limit $Z_n \rightarrow Z_{n-1}$ (ie
$X_n \rightarrow X_{n-1}$) on the tree-level NMHV expression (6.8). This gives

$$ \sum_{4 \leq j < k \leq n} \frac{1}{2} \tilde{R}(2, j, k) (-1)^{j+k} \xrightarrow{X_n \rightarrow X_{n-1}} \sum_{4 \leq j < k \leq n-2} \frac{1}{2} \tilde{R}(2, j, k) (-1)^{j+k} + \frac{1}{2} \tilde{R}(2, n-2, n) $$

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correctly reproducing the collinear limit \( R_{n;1} \to R_{n-2;1} + R_{6;1} \) manifestly. On the other hand if we instead perform the limit \( Z_{n-1} \to Z_{n-3} \) (ie \( X_{n-1} \to X_{n-2} \)) on the tree-level NMHV expression (6.8) we get

\[
\sum_{4 \leq j < k \leq n} \frac{1}{2} \tilde{R}(2, j, k) (-1)^{j+k}
\]

\[
\to_{X_{n-1} \to X_{n-2}} \sum_{4 \leq j < k \leq n} \frac{1}{2} \tilde{R}(2, j, k) (-1)^{j+k}
\]

\[
+ \frac{1}{2} \left( \tilde{R}(2, n - 3, n - 1) - \tilde{R}(2, n - 3, n) + \tilde{R}(2, n - 2, n) \right)
\]

This also correctly reproduces the collinear limit \( R_{n;1} \to R_{n-2;1} + R_{6;1} \) but only after taking into account the linear identity (coming from (6.6))

\[
\tilde{R}(2, n - 3, n - 1) - \tilde{R}(2, n - 3, n) + \tilde{R}(2, n - 2, n) = \tilde{R}(1, n - 3, n - 1) .
\]

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**A Collinear limits and (super)conformal transformations**

The reason for the very simple form of the collinear factorisation of reduced amplitudes under the \( m+1 \) collinear limit comes from universal collinear factorisation of superamplitudes, combined with (dual) superconformal symmetry. Applying the \( m+1 \) collinear limit on a \((m+4)\)-point reduced amplitude gives the 4-point superamplitude (which is simply 1 for the reduced superamplitude) multiplied by the splitting superamplitude. On the other hand as we shall show now, performing the \( m+1 \) collinear limit on the \( m+4 \) point superamplitude can be achieved via a superconformal transformation. Indeed this superconformal transformation will become the
definition of the collinear limit, defining precisely the relative speed with which the fermionic coordinates approach collinearity compared to the bosonic variables. We will give collinear limits in terms of superconformal transformations for the case of interest in this paper only, namely in 2d kinematics, since the discussion is particularly simple here: we discuss the superconformal group $SL(2|2)$ acting on unconstrained variables $(z, \chi)$. The bosonic case is simply the well-known Möbius transformation. The general 4d bosonic case was discussed in [35] where it was related to the family of conformal transformations preserving a light-like square and the generalisation of this to the superspace case should follow.

So we start with an $(m+4)$-point reduced superamplitude $\mathcal{R}_{m+4}(Z_1, \ldots, Z_{m+4})$ and wish to perform the $m+1$ collinear limit on this. To this effect we want to send $z_{m+4}, z_{m+2}, \ldots, z_6 \to z_4$ and similarly $\chi_{m+4}, \chi_{m+2}, \ldots, \chi_6 \to \chi_4$. In particular all odd-point variables are unchanged and we do not act on them (in 2d kinematics they are acted on via a separate $SL(2|2)_+$ which we can choose to be the identity) but more importantly $z_2$ and $\chi_2$ are also unchanged. In other words we wish to find an $SL(2|2)_-$ transformation (or more precisely family of transformations) which keeps $z_2, \chi_2$ fixed whilst all other $z \to z_4$ and all other $\chi \to \chi_4$.

We can find precisely such a transformation. We use standard coset techniques to implement the $SL(2|2)$ transformations. For example, the conformal part of $SL(2|2)$ acts as follows

$$z \to \frac{az + b}{cz + d}, \quad \chi \to \frac{\chi}{cz + d}. \quad (A.1)$$

We first use this to send $z_2 \to 0$, $z_4 \to \infty$ and $\chi_2, \chi_4 \to 0$. At this point there is a simple family of transformations keeping these points fixed ($b = c = 0$, $d = 1/a$), so that $z \to a^2 z$, $\chi \to a\chi$ with $a$ parametrising a family of conformal transformations, and $a \to 0$ corresponding to the collinear limit. Finally, transforming back to the original coordinates we thus construct the explicit conformal transformation implementing our collinear limit as

$$z \to \frac{z_2 a^2(z - z_4) - z_4(z - z_2)}{a^2(z - z_4) - (z - z_2)} \quad \chi \to \frac{a\chi (z_4 - z_2) + (1 - a)\left[a\chi_2(z - z_4) + \chi_4(z - z_2)\right]}{(z - z_2) - a^2(z - z_4)}. \quad (A.2)$$

Notice that the $z$ transformation is simply a Möbius transformation as expected. The points $(z_2, \chi_2)$ and $(z_4, \chi_4)$ are fixed, but in the limit $a \to 0$ all other points approach $(z_4, \chi_4)$ corresponding to the collinear limit.

In particular When $z$ is close to $z_4$ the transformation simplifies to

$$z - z_4 \to a^2(z - z_4) + O(z - z_4)^2 \quad \chi - \chi_4 \to a(\chi - \chi_4) + O(z - z_4). \quad (A.3)$$
We see that we are taking a very specific collinear limit, where the \( \chi \)'s approach the limit at half the speed that the \( z \)'s do.

Thus we have shown that the \((m + 1)\)-collinear limit \( z_{m+4}, z_{m+2}, \ldots, z_6 \to z_4 \) and similarly \( \chi_{m+4}, \chi_{m+2}, \ldots, \chi_6 \to \chi_4 \) can be implemented (and indeed explicitly defined) via a family of superconformal transformations. Since \( R_{m+4} \) is superconformally invariant, the function is unchanged by the collinear limit, in particular it is finite and we have \( R_{m+4} \to R_{m+4} \). Thus \( R_{m+4} \) is the \((m + 1)\)-collinear splitting amplitude.

B Symbols and functions at 3-loops

The conjecture at the centre of the method outlined in [1] for constructing MHV amplitudes in special kinematics states that (the logarithms of) the fundamental cross-ratios \( u_{ij} \) form the basis for the vector space on which the symbol of the amplitude is defined.

Fundamental cross-ratios are given by, (cf. Eq. (2.21))

\[
  u_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2} = \frac{\langle i-1, j+1 \rangle \langle i+1, j-1 \rangle}{\langle i-1, j-1 \rangle \langle i+1, j+1 \rangle} = u_{i-1,i+1;j-1,j+1}. \tag{B.1}
\]

For the lowest in \( n \) cases, \( n = 8 \) and \( n = 10 \), all non-trivial 2-component cross-ratios are of the form \( u_{i,i+4} \), with \( i = 1, \ldots, 4 \) for the octagon, and \( i = 1, \ldots, 10 \) for the decagon with the additional constraint:

\[
  n = 8 : \quad 1 - u_{i,i+4} = u_{i+2,i+6}, \quad i = 1, 2 \tag{B.2}
\]

\[
  n = 10 : \quad 1 - u_{i,i+4} = u_{i+2,i+6}, \quad u_{i-2,i+2}, \quad i = 1, \ldots, 10. \tag{B.3}
\]

At \( n = 8 \) points there are just four fundamental cross-ratios, \( u_1, u_2, v_1 \) and \( v_2 \):

\[
  u_1 := u_{1,5} , \quad u_2 := u_{2,6} , \quad u_3 := 1 - u_1 := v_1 , \quad u_4 := 1 - u_2 := v_2 . \tag{B.4}
\]

The symbol [24] associates to any (generalised) polylogarithm, a tensor whose entries are rational functions of the arguments. The rank of the tensor is equal to the weight of the polylogarithm. For example \( \log x \) has weight 1 and gives rise to a 1-tensor

\[
  \text{Symb} \left( \log x \right) = x \tag{B.5}
\]

whereas the symbol of the classical polylogarithm of weight \( n \) is

\[
  \text{Symb} \left( \text{Li}_n(x) \right) = -(1 - x) \otimes x \otimes \ldots \otimes x . \tag{B.6}
\]
The symbol has the properties inherited from the logarithm
\[
\ldots \otimes x \, y \otimes \ldots = \ldots \otimes x \otimes \ldots + \ldots \otimes y \otimes \ldots \quad (B.7)
\]
\[
\ldots \otimes 1/x \otimes \ldots = -\ldots \otimes x \otimes \ldots
\]
For the product of functions the symbol is given by taking the shuffle product of the symbol of each function
\[
\text{Symb}(fg) = \text{Symb}(f) \text{III Symb}(g) . \quad (B.8)
\]
For example
\[
\text{Symb}(\text{Li}_2(x) \log y) = \left( - (1 - x) \otimes x \right) \text{III } y
\]
\[
= -(1 - x) \otimes x \otimes y - (1 - x) \otimes y \otimes x - y \otimes (1 - x) \otimes x . \quad (B.9)
\]
In the formalism of [1] 8-point MHV 3-loop amplitudes have the following structure:
\[
\tilde{\mathcal{R}}_8^{(3)} = \sum_{\sigma, \tau} a_{\sigma \tau} f^+_\sigma(u_1) f^+_\tau(u_2) \quad (B.10)
\]
where \(a_{\sigma \tau} = a_{\tau \sigma}\) are rational coefficients, and the sum is over the set of functions \(f^+_\sigma\) with the properties given in (3.5). The total polylog weight of \(\tilde{\mathcal{R}}_8^{(3)}\) must be six which implies that the transcendental weights of individual functions \(f^+_\sigma\) can be 2, 4 and 3.
We can now similarly write down the expression for \(S_8\),
\[
S_8^{(3)}(x_2, x_4, x_6, x_8) = \sum_{\sigma, \tau} a_{\sigma \tau} f^+_\sigma(u_1) f^+_\tau(u_2) + b_{\sigma \tau} f^-_\sigma(u_1) f^-_\tau(u_2)
\]
\[
= \frac{1}{2} \tilde{\mathcal{R}}_8^{(3)} + T_8^{(3)}(x_2, x_4, x_6, x_8) \quad (B.11)
\]
with \(b_{\sigma \tau} = b_{\tau \sigma}\) and which utilize functions \(f^\pm_\sigma\) with the property
\[
f^\pm_\sigma(u) = \pm f^\pm_\sigma(v) , \quad v = 1 - u . \quad (B.12)
\]
It can be checked that \(T_8\) with correct properties (4.8) indeed arises from \(f^-_\sigma(u_1) f^-_\tau(u_2)\) combination. In particular, the transformation of the arguments \((x_2, x_4, x_6, x_8) \leftrightarrow (x_1, x_3, x_5, x_7)\) corresponds in terms of the cross-ratios to \(u_2 \leftrightarrow 1 - u_2\) with \(u_1\) (and \(1 - u_1\)) unchanged. Thus for \(T_8^{(3)}(x_2, x_4, x_6, x_8) = b_{\sigma \tau} f^-_\sigma(u_1) f^-_\tau(u_2)\) , for the alternative selection of arguments in \(T_8\) we have \(T_8^{(3)}(x_1, x_3, x_5, x_7) = b_{\sigma \tau} f^-_\sigma(u_1) f^-_\tau(1 - u_2) = -b_{\sigma \tau} f^-_\sigma(u_1) f^-_\tau(u_2)\). This is in agreement with (4.8).

In [1] all possible (symbols and) functions \(f^+_\sigma(u)\) were listed. It is straightforward to generalise this construction to functions \(f^\pm_\sigma\). For weight-2 there is only one function \(f^-\) and one function \(f^+\) with properties (3.5) or (B.12),
\[
\text{weight } 2 : \quad f^+_{\text{weight } 2}(u) = \log(u) \log(v) \quad f^-_{\text{weight } 2}(u) = \text{Li}_2(u) - \text{Li}_2(v) . \quad (B.13)
\]
These weight-2 functions are accompanied in (B.11) by functions $f^\pm_\sigma(u)$ of weight-4. For completeness we list below symbols for all functions $f^\pm_\sigma(u)$. They come in two types, type-a and type-b:

**weight 4 a**

\[
\text{Symb}[f^+_a] := u \otimes u \otimes u \otimes v \pm v \otimes v \otimes v \otimes u
\]
\[
\text{Symb}[f^+_a] := u \otimes u \otimes v \otimes u \pm v \otimes v \otimes u \otimes v
\]
\[
\text{Symb}[f^+_a] := u \otimes v \otimes u \otimes u \pm u \otimes v \otimes v \otimes v
\]
\[
\text{Symb}[f^+_a] := v \otimes u \otimes u \otimes u \pm u \otimes v \otimes v \otimes v
\]
\[
\text{Symb}[f^+_a] := u \otimes u \otimes v \otimes u \pm v \otimes v \otimes u \otimes u
\]

**weight 4 b**

\[
\text{Symb}[f^+_b] := u \otimes u \otimes v \otimes v \pm v \otimes v \otimes u \otimes u
\]
\[
\text{Symb}[f^+_b] := u \otimes v \otimes u \otimes v \pm u \otimes v \otimes u \otimes u
\]
\[
\text{Symb}[f^+_b] := u \otimes v \otimes v \otimes u \pm v \otimes u \otimes u \otimes v
\]

At the end of Section 4.4.2 we explain that there are no contributions to $\mathcal{R}_n$ from weight 2 functions $f^-$. Thus there are also no contributions from weight 4 functions $f^-$ as they would have had to be accompanied by weight 2 functions.

What remains is to examine the weight-3 functions, known as type-c. Here we have (cf. [1]),

\[
\text{Symb}[f^+_c] := u \otimes u \otimes v \pm v \otimes v \otimes u
\]
\[
\text{Symb}[f^+_c] := u \otimes v \otimes u \pm v \otimes u \otimes v
\]
\[
\text{Symb}[f^+_c] := u \otimes v \otimes v \pm v \otimes u \otimes u
\]

For the 8-point 3-loop amplitude itself, only the functions $f^+$ appear in Eq. (B.10). After imposing the constraint arising from the near-collinear OPE of [34] the final result of Ref. [1] for the octagon at 3-loops is given by

\[
\tilde{\mathcal{R}}^{(3)}_8 = \log u_1 \log(1 - u_1) \left[ \alpha_1 f^+_{a3}(u_2) + \alpha_2 f^+_{a4}(u_2) + \alpha_3 f^+_{b3}(u_2) + \alpha_4 f^+_{b3}(u_2) \right] + \alpha_5 f^+_{c2}(u_1)f_c(u_2) + \alpha_6 f^+_{c2}(u_1)f^+_{c3}(u_2) + \alpha_7 f^+_{c3}(u_1)f^+_{c3}(u_2) + f^+_{c1}(u_1) \left[ \frac{1}{2} f^+_{c1}(u_2) + 2 f^+_{c3}(u_2) + f^+_{c3}(u_2) \right] + (u_1 \leftrightarrow u_2)
\]

with the $f^+_a$, $f^+_b$ and $f^+_c$ functions are straightforwardly reconstructed from their symbols in (B.14)-(B.16) and are listed in Eqs. (5.15) of Ref. [1].

To fully determine $S_8$ at 3 loops, in addition to $\tilde{\mathcal{R}}^{(3)}_8$ we need the contribution $T^{(3)}_8$ in (B.11) which comes solely from the $f^-$ functions. Since ultimately there will be no $f^-$ contributions at weight 2 (as shown in Section 4.4.2), the contributions to $T^{(3)}_8$ relevant for $V_n$ can arise only from the weight-3 times weight-3 functions $f^-$ in
(B.16). The $f^-$ functions are of the form\footnote{We should note that the third function does not vanish in the collinear limits, but goes to a constant, $f_{c3}^{-}(u,v) \to \pm \zeta_3$ when $u$ or $v$ got to 1. This is not a problem, as these constant terms cancel in the $S$-formula.}:

$$
- f_{c1}^{-}(u,v) = \text{Li}_3(u) + \left(\text{Li}_2(v) + \frac{\pi^2}{6}\right) \log(u) + \frac{1}{2} \log(v) \log^2(u) - (u \leftrightarrow v)
$$

$$
f_{c2}(u,v) = 2\text{Li}_3(u) + \left(\text{Li}_2(v) - \frac{\pi^2}{6}\right) \log(u) + \log(v) \log^2(u) - (u \leftrightarrow v)
$$

$$
f_{c3}(u,v) = \text{Li}_3(u) - \text{Li}_3(v),
$$

They give 6 possible combinations,

$$
T_{8}^{(3)} \supset \begin{align*}
&f_{c1}^{-}(u_1, u_3)f_{c1}^{-}(u_2, u_4) \\
&f_{c1}^{-}(u_1, u_3)f_{c2}^{-}(u_2, u_4) + f_{c2}^{-}(u_1, u_3)f_{c1}^{-}(u_2, u_4) \\
&f_{c1}^{-}(u_1, u_3)f_{c3}^{-}(u_2, u_4) + f_{c3}^{-}(u_1, u_3)f_{c1}^{-}(u_2, u_4) \\
&f_{c2}^{-}(u_1, u_3)f_{c2}^{-}(u_2, u_4) \\
&f_{c2}^{-}(u_1, u_3)f_{c3}^{-}(u_2, u_4) + f_{c3}^{-}(u_1, u_3)f_{c2}^{-}(u_2, u_4) \\
&f_{c3}^{-}(u_1, u_3)f_{c3}^{-}(u_2, u_4)
\end{align*}
$$

We now turn our attention to the 10-point amplitude, which was originally obtained in [1] in the form given by Eq. (3.8). The first term on the $r.h.s.$ gives a particular solution to the multi-collinear constraints. It is reproduced by the $S_8$ contributions (specifically by the $f^+f^+$ terms in (B.11). On the other hand, the second term, $V_{10}$ denotes a generic 10-point function which is constrained to vanish in all triple collinear limits. This collinearly vanishing contribution was constructed in [1].

Here, for convenience of the reader, we reproduce the form of $V_{10}$ from [1]. In order to be able to uplift the 10-point result to 12 points and all higher points using our general $S$-formula we do not need $S_{12}$ but we need to know that it can be deconstructed in terms of collinearly vanishing $T_8$ and collinearly vanishing $S_{10}$ contributions.

At 10-points there are 10 fundamental cross-ratios

$$
u_i := u_{i,i+4}, \quad i = 1, \ldots, 10
$$

(B.19)

which can be divided into 5 parity-even $(u_1, u_3, \ldots, u_9)$, and 5 parity-odd cross-ratios $(u_2, u_4, \ldots, u_{10})$. It was argued in [1] that $V_{10}$ is assembled from functions of even cross-ratios (times functions of odd $u$’s as follows:

$$
f_i(u_{\text{even}})f_j(u_{\text{odd}}) + \text{cyclic} + \text{parity}.
$$

(B.20)

These functions $f_i$ must themselves vanish in any collinear limit. To do this they must have weight-3 and each term must contain 3 consecutive cross-ratios of given
parity, e.g. \( u_2, u_4, u_6 \). They are not difficult to find analytically [1]:

\[
\begin{align*}
  f_1(u_2, u_4, u_6) &= \log(u_2) \log(u_4) \log(u_6) \\
  f_2(u_2, u_4, u_6) &= \log(u_4) \left( \text{Li}_2(u_2) - \text{Li}_2(1 - u_4) + \text{Li}_2(u_6) - \pi^2/6 \right) \\
  f_3(u_2, u_4, u_6, u_8, u_{10}) &= \sum_{i=2,4,6,8,10} \left( \text{Li}_3(u_i) - \text{Li}_3(1 - u_i) \right) - \zeta_3. 
\end{align*}
\]

(B.21)

Here \( f_1 \) and \( f_2 \) give 5 independent functions via cyclic permutations of the arguments, whereas \( f_3 \) is cyclically symmetric giving only 1 independent function, thus we have 11 functions in total. These functions are combined together to give a total of 12 independent weight-6 collinear vanishing contributions to \( V_{10} \):

\[
\begin{align*}
  f_1(u_1, u_3, u_5) f_1(u_2, u_4, u_6) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_1(u_4, u_6, u_8) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_1(u_6, u_8, u_{10}) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_2(u_2, u_4, u_6) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_2(u_4, u_6, u_8) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_2(u_6, u_8, u_{10}) &+ \text{cyclic + parity} \\
  f_2(u_1, u_3, u_5) f_2(u_2, u_4, u_6) &+ \text{cyclic + parity} \\
  f_2(u_1, u_3, u_5) f_2(u_4, u_6, u_8) &+ \text{cyclic + parity} \\
  f_2(u_1, u_3, u_5) f_2(u_6, u_8, u_{10}) &+ \text{cyclic + parity} \\
  f_1(u_1, u_3, u_5) f_3(u_i^-) &+ \text{cyclic + parity} \\
  f_2(u_1, u_3, u_5) f_3(u_i^-) &+ \text{cyclic + parity} \\
  f_3(u_1, u_3, u_5) f_3(u_i^-) &+ \text{cyclic + parity} 
\end{align*}
\]

(B.22)

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