Magnetic Response of Disordered Ballistic Quantum Dots

Yuval Gefen\textsuperscript{1,2}, Daniel Braun\textsuperscript{1}, and Gilles Montambaux\textsuperscript{1}

\textsuperscript{1} Laboratoire de Physique des Solides, Associé au CNRS, Université Paris–Sud, 91405 Orsay, France
\textsuperscript{2} Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

(December 1993)

Abstract

The weak field average magnetic susceptibility of square shaped mesoscopic conductors is studied within a semiclassical framework. Long semiclassical trajectories are strongly affected by static disorder and differ sharply from those of clean systems. They give rise to a large linear paramagnetic susceptibility which is disorder independent and in agreement with recent experiments. The crossover field to a nonlinear susceptibility is discussed.

05.45.+b, 73.20.Dx, 03.65.Sq, 05.30.Ch
Recent developments in the theory of thermodynamics on the mesoscopic scale have brought to the focus of attention the question of the average magnetic susceptibility of an isolated mesoscopic conductor, \( \langle \chi(H) \rangle \). It has been shown that as the number of electrons in such a system is independent of the value of the applied magnetic field, one has to evaluate this quantity within the canonical ensemble. Upon energy averaging (or averaging over system size), one may discard oscillatory grandcanonical contributions, obtaining
\[
\langle \chi(H) \rangle = -\frac{\Delta}{2} \frac{\partial^2}{\partial H^2} \langle \delta N^2(H) \rangle \bigg|_{\mu},
\]
where \( \langle \delta N^2 \rangle \) is the typical (field dependent) sample to sample fluctuation in the number of levels below some effective chemical potential \( \mu \). Efforts have been concentrated on the calculation of \( \langle \chi \rangle \) in the diffusive regime, but have been extended as well to the limit of clean systems (where the only scattering is due to the sample’s boundaries) \([2–7]\). With the regime
\[
l_{el} > L
\]
made accessible experimentally \([8]\), the belief is that such theoretical studies may be employed to interpret the experimental data. Here \( l_{el} \) is the elastic mean free path, while \( L \) is the system size.

From another point of view, it has become clear that the association of all systems satisfying Eq. \((2)\) (perfectly clean or containing disorder) with a ubiquitous ballistic regime is too naive \([9,10]\). Let us consider a finite disordered system of an integrable geometry (e. g. a square). For disorder weaker than that which gives rise to diffusive behaviour, one has to compare not only \( L \) with \( l_{el} \), but also the average level spacing, \( \Delta \), with the inverse elastic mean free time \( \hbar/\tau_{el} \). We may now distinguish between two weak disorder regimes satisfying Eq. \((2)\): the **ballistic regime** ( \( \hbar/\tau_{el} > \Delta \) ) and the **clean (or perturbative) regime** ( \( \hbar/\tau_{el} < \Delta \) or, equivalently, \( l_{el} > L(k_F L)^{d-1} \). This regime is, presently, experimentally inaccessible.). For the latter, as disorder constitutes only a small perturbation on top of the clean spectrum,
one may argue that thermodynamic quantities are practically those of a perfectly clean system (of the same geometry). This is, however, not the case for the former, ballistic regime. It has been shown recently that elastic mixing of the levels give rise to non–trivial level correlations [10] (at $H = 0$). It is therefore natural to ask how thermodynamic quantities, notably the magnetic susceptibility, behave in this regime.

In the present work we address the problem of the average weak field susceptibility of (integrable) disordered ballistic quantum dots. Employing a semiclassical picture [12] we find contributions to $\langle \chi(H) \rangle$ which arise from long trajectories that are sensitive to elastic scattering. However, due to subtle cancellations that occur within this framework, contributions to the zero field susceptibility do not depend on disorder [13]. We find that this susceptibility is given by

$$\chi(H = 0) = +|\chi_L|\alpha k_F L. \tag{3}$$

It is paramagnetic and includes an enhancement factor $\alpha k_F L$ with respect to the Landau susceptibility, where the numerical factor $\alpha$ is estimated below. Our results are in qualitative agreement with the experimental data of Ref. [8]. Our approach is closely related to the de Gennes Tinkham “method of trajectories”, and some of our results bear close formal similarity to a method, discussed by Beenakker and van Houten, of treating weak localization corrections in restricted geometries [11,13].

Consider a possible multiply reflected semiclassical trajectory within our square geometry. The same trajectory may be described within an extended zone scheme (Fig. 1a), where boundary scattering does not occur. We next consider a uniform perpendicular magnetic field in the $z$–direction. Within the Landau gauge the vector potential is $A = -Hy\hat{x}$. The vector potential in the extended zone scheme is shown schematically in Fig. 1b. According to this scheme, the electron is moving in a staggered field [14], which may be written as sum of its Fourier components.
\[
A = -\frac{16HL\hat{x}}{\pi^3} \sum_{m,n=0}^{\infty} \frac{(-1)^n}{(2m+1)(2n+1)^2} \sin((2m+1)\frac{\pi x}{L}) \sin((2n+1)\frac{\pi y}{L}).
\]  

(4)

Imagine now a random walk trajectory (in the extended zone scheme) of total length \(L\), starting from point \((x_0, y_0)\), and consisting of \(L/l_{el}\) uncorrelated segments of length \(l_{el}\) each. The end point of this trajectory may be mapped back onto the reduced zone scheme. For long trajectories \((L > l_{el})\) we assume that ergodicity holds and that the image of the end point in the reduced zone scheme is uniformly distributed. In the absence of an applied field, the probability of returning to the origin at time \(t\) is given by (see e. g. \[12\])

\[
p(t) \equiv p_0 = 2 \frac{1}{(2\pi \hbar)^2},
\]

where the factor 2 in front represents an enhancement due to an interference of a trajectory with its time reversed image (a “Cooperon contribution”). The two–level spectral correlation function, \(Y_2(\Delta E)\), is given then by the fourier transform of \(|t|p(t)\) \[12\], and finally the level fluctuation is given by

\[
\langle \delta N^2(\Delta E) \rangle = 2 \int_0^{\Delta E} (\Delta E - s)Y_2(s) ds
\]
\[
= 4 \int_0^{\infty} dt \frac{\hbar^2}{t} p(t) (1 - \cos(\frac{\Delta E t}{\hbar})).
\]

(6)

Our goal is to find the field dependence of \(p(t)\), hence \(\langle \delta N^2 \rangle\), then employ Eq. (1). We first note that considering long trajectories, i. e. time scales larger than the elastic mean free time \(\tau_{el}\), allows us to evaluate reliably energy windows such that \(\Delta E < \frac{\hbar}{\tau_{el}}\). This, in turn, may be employed to evaluate the weak field susceptibility. (In other words, long trajectories affect the small energy contribution of the level–level correlation function to \(\langle \delta N^2 \rangle\) \[13,14\].

Estimates of the field scales for which the present treatment is valid are given below. Upon application of a magnetic field there is an extra phase that may be associated with the amplitude of the \(j\)th returning trajectory, \(\phi_j = \frac{2\pi}{\phi_0} \int_{(j)} A \cdot dx\), where the integral is taken
along the \( j \)th trajectory. To calculate \( p(t) \) one has to take the square modulus of the sum over all trajectories corresponding to time \( t \). Upon disorder averaging, only terms that may be represented as a product of an amplitude with its complex conjugate or with the complex conjugate of its reversed survive [12,17]. The factor 2 (Eq.(5)) is now to be replaced by a factor \((1 + \cos 2\phi_j)\) to be averaged over all trajectories \( j \) [18]. We note that following the disorder averaging, the dependence on random (trajectory specific) phases disappears. Our remaining task is to evaluate the distribution function of \( 2\phi_j \). We first assume that the magnetic phases (contributions from line integrals) accumulated at different steps of size \( l_{el} \) of a trajectory are independent and identically distributed, with a distribution function \( P_{l_{el}}(\tilde{\phi}) \). The average over \( \cos 2\phi_j \) is then given by [14]

\[
\text{Re} \left[ \int_{-\infty}^{\infty} P_{l_{el}}(\tilde{\phi})e^{i2\tilde{\phi}} d(\tilde{\phi}) \right]^{L/l_{el}} = \text{Re} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} (\delta(\tilde{\phi} - \phi(\theta)))e^{i2\tilde{\phi}} d\theta d(\tilde{\phi}) \right]^{L/l_{el}} = \text{Re} \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i2\phi(\theta)} \right]^{L/l_{el}} \equiv \text{Re} \left( \zeta(H) \right)^{L/l_{el}},
\]

(7)

where \( \theta \), the scattering angle at a given step (cf. Fig. 1a), is assumed to be uniformly distributed (a consequence of isotropic scattering). The function \( 2\phi(\theta) \) (associated with a segment of length \( l_{el} \)) is evaluated by inserting Eq. (4) into the expression for the line integral. Approximating \( A_x(x, y) \) by the \( m = n = 0 \) terms in Eq. (4), we obtain

\[
2\phi(\theta) = -2 \cdot 2\pi \frac{8HL}{\pi^3\phi_0} \int_0^{l_{el}} ds \cos \theta \left[ \cos \left( \frac{\pi}{L}(x_0 - y_0) + \frac{\pi s}{L}(\cos \theta - \sin \theta) \right) - \cos \left( \frac{\pi}{L}(x_0 + y_0) + \frac{\pi s}{L}(\cos \theta + \sin \theta) \right) \right],
\]

(8)

where \((x_0, y_0), (x_0 + l_{el} \cos \theta, y_0 + l_{el} \sin \theta)\) are the endpoints of the segment at hand. We next write

\[
p(t) = \frac{1}{2} p_0 (1 + \text{Re} \zeta(H)^{L/l_{el}})
\]

(9)

Accounting only for the field sensitive (Cooperon like) contribution \( \Delta p(t) \), writing \( \mathcal{L} = v_F t \), \( l_{el} = v_F \tau \) and introducing a cutoff factor [19], \( \gamma \), we obtain
\[ \Delta p(t) = \frac{1}{2} p_0 \xi \frac{t}{\tau} e^{-\frac{t^2}{\tau}}. \] (10)

Substituting this into Eq. (6), we obtain that the flux sensitive part of \( \delta N^2(\Delta E) \) is given by

\[ \delta N^2(\Delta E) = p_0 h^2 \ln \left( \frac{(\Delta E)^2 + (\frac{\hbar}{\tau_{el}} \ln \frac{1}{\xi} + \gamma)^2}{(\frac{\hbar}{\tau_{el}} \ln \frac{1}{\xi} + \gamma)^2} \right). \] (11)

To evaluate the very weak field behaviour we may expand

\[ \zeta(H) \simeq 1 - \int_{-\pi}^{\pi} \frac{d\theta}{4\pi} (2\phi(\theta))^2 \equiv 1 - \frac{H^2 L^4}{\phi_0^2}. \] (12)

Averaging over \( x_0, y_0 \) we obtain

\[ c = \frac{32}{\pi^5} \int_{-l_{el}/L}^{l_{el}/L} ds \int_{-l_{el}/L}^{l_{el}/L} ds' \int_{-\pi}^{\pi} d\theta \cos(\pi \sqrt{2}(s' - s) \sin \theta), \] (13)

which, following a few straightforward steps, yields

\[ c = \frac{512}{\pi^4} \int_0^{l_{el}/L} du J_0(\sqrt{2}\pi u) \left( \frac{l_{el}}{L} - u \right), \] (14)

where \( J_0 \) is a Bessel function. As \( \frac{l_{el}}{L} \gg 1 \), this integral may be approximated by

\[ c \simeq \frac{256\sqrt{2} l_{el}}{\pi^5}. \] (15)

The most remarkable feature of Eq. (15) is the proportionality of \( c \) to \( l_{el} \). The main contribution to the quadratic correction \( cH^2 \) comes from segments oriented at angles \( \theta \simeq \pm \pi/4 \), for which \( \langle \phi^2 \rangle \propto l_{el}^2 \). However, the angular width around \( \theta = \pm \pi/4 \) corresponding to such exceedingly large contributions is \( \sim \frac{L}{l_{el}} \), rendering \( c \sim l_{el} \). It follows then that \( \zeta_{el} \simeq 1 - \frac{c}{l_{el}^2} \frac{H^2 L^4}{\phi_0^2} \), rendering the quadratic term, hence the zero field susceptibility, disorder independent. Employing Eqs. (1) and (11), and considering an energy window \( \Delta E \simeq \frac{h}{\tau_{el}} \gg \Delta \), we obtain

\[ \langle \chi(H = 0) \rangle = \frac{\Delta}{\gamma} p_0 h^3 \frac{2c}{\tau_{el} \phi_0^2} L^4 = \frac{1536\sqrt{2} \Delta}{\pi^8} \langle \chi_L \rangle L^4 \] (16)

\[ \simeq 0.23 \frac{\Delta}{\gamma} \langle \chi_L \rangle L^4, \] (16)
where for spinless electrons \( \chi_L = \frac{-e^2 L^2}{24 \pi mc} \). For \( \frac{1}{2} \) spin electrons \( \chi \), as well as \( |\chi_L| \), should be multiplied by 2 \[23\]. The range for which this very weak field analysis may be employed (i.e. the regime of linear susceptibility, eq. (16)) is found by (i) examining the range of validity of Eq. (12), i.e., by requiring that \( \zeta \) may be expanded to quadratic order in \( H^2 \) and (ii) by requiring that the field dependent correction in Eq. (11) be smaller than \( \gamma \)!

These conditions read

\[
H < \frac{\phi_0}{L_{el}} \quad \text{(flux in the sample \( < \frac{L}{l_{el}}\phi_0 \)) [24]} \quad (i)
\]

\[
H < \phi_0 \sqrt{\frac{\pi L^3}{l_{el}}} \quad \text{(ii)}
\]

Typically, inequality (17 (ii)) is stricter, and yields the dependence of the crossover field on both the elastic and inelastic scattering. Beyond this range the susceptibility is not constant, and may even change sign (cf. Fig. 2). We next recall that our semiclassical considerations apply to energy intervals (close to \( \epsilon_F \)) smaller than \( \frac{\hbar}{r_{el}} \). We now find the largest field for which this picture may be employed to evaluate the total \( H \)-dependence of \( \langle \delta N^2 \rangle \) (hence \( \chi \)). The condition is that \( \langle (2\phi(\theta))^2 \rangle < 1 \), yielding

\[
H < \frac{\phi_0}{\sqrt{l_{el}L^3}}.
\] (18)

To obtain contributions from long \( (L > l_{el}) \) trajectories, we require that the dephasing length \( (l_{\phi}) \) satisfies \( l_{\phi} > l_{el} \). In this case \( \gamma \simeq \frac{\hbar D}{l_{\phi}} \), hence \( \chi / |\chi_L| \) scales as \( l_{ph}^2 / L_{el} \). The magnetic susceptibility of two-dimensional ballistic dots (squares) has recently been measured by Levy et al. \[8\]. They have obtained \( \chi_{(H=0)} / |\chi_L| \simeq k_F L \simeq 100 \). In their experiments \( l_{el} \simeq L - 2L, l_{\phi} \simeq 3L - 10L \). Eq.(16), pushed to the limit of its validity, yields \( (l_{el} = 1.5L, l_{\phi} = 8L) \chi \simeq 250|\chi_L| \), in rough agreement with the experiment. Evidently, a more detailed comparison with our theory (e. g., measurements of the crossover field from linear susceptibility, which is sensitive to disorder) is needed before one may confirm the validity of our picture.
To summarize, the central point of our analysis was based on the observation that when the condition $l_φ \gg l_{el}$ is met, the weak field behaviour carries the signature of the long, disorder dependent trajectories. It is possible that the contribution calculated in Refs. [1, 6] should be added on top of the present result [25]. It is the dependence on $\ell$ that cancels out which gives rise to the disorder independent result, Eq. (10), in sharp contrast with, e.g., the results expected in the Aharonov–Bohm case. The scale of the field for which the susceptibility decreases may be disorder dependent (Eq. (17)) [26].

Acknowledgments: We have benefitted from useful discussions with A. Altland, N. Arga-man, R. A. Jalabert, K. Richter, and D. Ullmo. In particular we are grateful to D. Mukamel for pointing out to us the usefulness of the expansion Eq. (4) in the present context. Y. G. acknowledges the hospitality of H. Bouchiat and G. Montambaux in Orsay. This research was supported in part by the German–Israel Foundation (GIF), the U. S. –Israel Binational Science Foundation (BSF), the Claussen Stiftung, and EC Science program no. SCC–CT90–0020.
REFERENCES

[1] B. L. Altshuler, Y. Gefen, Y. Imry, Phys. Rev. Lett. **66**, 88 (1991).

[2] K. Nakamura, H. Thomas, Phys. Rev. Lett. **61**, 247 (1988).

[3] J. M. van Ruitenbeck, D. A. van Leuwen, Phys. Rev. Lett. **67**, 640 (1991).

[4] F. von Oppen, E. K. Riedel, Phys. Rev. **B 48**, 9170 (1993).

[5] F. von Oppen, thesis, unpublished.

[6] D. Ullmo, K. Richter, R. Jalabert, preprint (Orsay).

[7] Y. Ovchinikov, A. Schmid, unpublished.

[8] L. P. Levy, D. H. Reich, L. Pfeiffer, K. West, Physica **B 189**, 204 (1993).

[9] U. Sivan, Y. Imry, Phys. Rev. **B 33**, 551, (1986).

[10] A. Altland, Y. Gefen, Phys. Rev. Lett., **71**, 3339 (1993).

[11] P. G. de Gennes and M. Tinkham, Physics **1**, 107 (1964). See also C. W. J. Beenakker and H. van Houten, Phys. Rev. **B 38**, 3232 (1988).

[12] S. Chakravarty, A. Schmid, Phys. Rep. **140**, 193 (1986); N. Argaman, Y. Imry, U. Smilansky, Phys. Rev. **B 47**, 4440 (1993).

[13] This is by no means obvious. Indeed, it has been recently shown that the magnetization of a ballistic cylinder threaded by an Aharonov–Bohm flux does depend on disorder, hence is significantly different from that of a clean cylinder [Y. Gefen, G. Montambaux, A. Altland, to be published]. A similar cancellation (involving magnetotransport of infinite systems) has been found in Ref. [11].

[14] We treat these trajectories as consisting of an integer number of equal length steps. This “discretization” of the paths introduces small oscillatory terms on top of our results (e.g. Eq.(14)), which will be smeared out upon assigning a continuous distribution to
$l_{el}$.

[15] B. L. Altshuler, Y. Gefen, Y. Imry, G. Montambaux, Phys. Rev. B 47, 10335 (1993).

[16] Y. Imry, Proceedings on the 1991 Tanuguchi Symposium on Mesoscopic Physics, H. Fukuyama and T. Ando Editors, Springer 1992.

[17] G. Bergmann, Phys. Rep. 107, 1 (1984).

[18] As we consider bilinear products of the amplitudes the dependence on Maslov indices disappears.

[19] The factor $\gamma$ may represent inelastic scattering. However, even when the inelastic time is large, $\gamma$ should be of the order of $\Delta$, accounting for the fact that for $t > \hbar/\Delta$ our semiclassical picture ceases to be valid [20,21].

[20] M. V. Berry, Proc. Roy. Soc. London, A 423, 219 (1989).

[21] B. L. Al’tshuler, B. I. Shklovskii, Sov. Phys. JETP 64, 127 (1986).

[22] Strictly speaking Eq. (I0) holds only for $t \gg \tau_{el}$. The error in substituting Eq. (I0) into Eq. (I) can be shown to be small.

[23] In the limit of vanishing inelastic broadening one may substitute $\gamma = 0.392\Delta$, cf. N. Dupuis, G. Montambaux, Phys. Rev. B 43, 14390 (1991).

[24] The analogue of this condition in the diffusive regime is $\phi < \frac{1}{\sqrt{g}}\phi_0$, where $g$ is the dimensionless conductance (cf. [15]).

[25] When dealing with two–dimensional electron gas systems, one may encounter situations where the transport mean free path is larger than $L$ (i. e., the system is ballistic), while $l_{el} \ll L$. This may imply that invoking a semiclassical description applied to a clean system (employing continuous families of periodic trajectories, cf. Refs. [39]) may not be adequate.
We propose the following conjecture concerning the generalization of this result to clean systems of non-integrable geometry. In such geometries ergodicity is achieved for trajectories $\mathcal{L} > L$. One may be tempted then to employ the results of the preceding analysis, with the identification $L \sim l_{el}$. That would imply that on the axis $\frac{L}{l_{el}}$, generic quantum chaotic systems are basically represented by one point. Evidently one may produce non-integrable systems for which the effective $\frac{L}{L_{el}} \ll 1$ (e.g. a sequence with one of its corners a bit rounded).
FIGURES

FIG. 1.a. A multiply reflected semiclassical trajectory (thick line), and its image within the extended zone scheme. A dot denotes the location of a scatterer.

FIG. 1.b. Vector potential within the extended zone scheme (arrow lengths are proportional to $|A_x|$). Alternating directions of the staggered field are indicated.

FIG. 2. The field dependent susceptibility calculated numerically (Eqs. (1) and (11)) with $x_0 = y_0 = 0$ for $H < \frac{\phi_0}{\sqrt{l_{el}L^2}}$. Here $l_{el}/L = 50.3$. Note that $\langle \chi(H) \rangle$ changes sign for $H > \frac{\phi_0}{L l_{el}}$. These oscillations will be partially smeared out due to fluctuations in the value of $l_{el}$.