Superintegrable three-body problems

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Abstract

In this Letter superintegrable 3-body problems with $d > 1$ degrees of freedom are introduced via
superintegrability of two-dimensional Schrödinger equations. The formalism can be generalized to
many-body problems linking them to multi-dimensional Schrödinger equations.
I. INTRODUCTION

The two-dimensional Schrödinger operator

\[ \mathcal{H} = -\Delta^{(2)} + V(x, y), \quad \Delta^{(2)} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad x, y \in \mathbb{R}, \quad (1) \]

with the general potential \( V(x, y) \) is non-integrable: there exists no non-trivial differential operator that commutes with \( \mathcal{H} \). However, there exists a certain class of potentials for which the eigenvalue equation for \( \mathcal{H} \) is integrable. Typically, integrability occurs when variables can be separated: it implies the existence of a first or second order integral in the form of a first or second order differential operator. In particular, if the quantum system is \( O(2) \) symmetric, i.e. the potential is azimuthally symmetric, variables separate in polar coordinates and the angular momentum \( \hat{L} \) commutes with the Hamiltonian \( \mathcal{H} \). Moreover, among integrable potentials there exists a distinguished class of potentials, called superintegrable, for which not one but two integrals occur - the maximum possible number of integrals in two dimensions, in addition to the Hamiltonian. This class of potentials has been the subject of intense study, for review see e.g. [1]: a complete classification has been established when both integrals are of second order (see [2], [3] and references therein). A number of examples was known with higher order integrals of motion. Related to this an outstanding property was conjectured [4]: any two-dimensional (any \( n \)-dimensional in general) maximally superintegrable quantum system in flat space is exactly-solvable, no single counter-example has been found so far.

The aim of this work is to employ the above classification applying it to relative motion of the general quantum 3-body problem, giving a description of certain 3-body integrable and superintegrable potentials in \( d \) dimensions. Most of these potentials are of a unusual type: they can not be reduced to the sum of pairwise potentials.

II. FROM A \((d_1+d_2)\)-DIMENSIONAL QUANTUM SYSTEM TO A TWO-DIMENSIONAL ONE: A REDUCTION

We consider a \((d_1 + d_2)\)-dimensional quantum system in \( \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) space with \( O(d_1) \times O(d_2) \) symmetry described by the Hamiltonian

\[ \mathcal{H}_2 = -\Delta_x^{(d_1)} - \Delta_y^{(d_2)} + V_2(x, y), \quad (2) \]
where
\[ \Delta^{(d_1)}_x = \sum_{i=1}^{d_1} \frac{\partial^2}{\partial x_i \partial x_i}, \quad \Delta^{(d_2)}_y = \sum_{i=1}^{d_2} \frac{\partial^2}{\partial y_i \partial y_i} \]
are \(d_1\), \(d_2\)-dimensional Laplacians and \(x = \sqrt{x_1^2 + \ldots + x_{d_1}^2}\), \(y = \sqrt{y_1^2 + \ldots + y_{d_2}^2}\) are radial distances in \(x, y\)-spaces, respectively. This system is evidently integrable - it is characterized by two conserved angular momenta, \(\mathbf{L}_x = -i \mathbf{\omega} \wedge \frac{\partial}{\partial x}\) and \(\mathbf{L}_y = -i \mathbf{\omega} \wedge \frac{\partial}{\partial y}\) of dimensions \(d_1\) and \(d_2\), respectively, where \(\wedge\) denotes the exterior product. Indeed, one can find sets of \((d_1 - 1)\) commuting 1st and second order symmetries (integrals) in the enveloping algebra of \(so(d_1)\) to which we add the Laplacian \(\Delta^{(d_1)}_x\). Similarly, we find \((d_2 - 1)\) commuting symmetries (integrals) from the second Laplacian \(\Delta^{(d_2)}_y\), thus, the \((d_1 + d_2)\)-dimensional Hamiltonian system \(\mathbf{H}_{2,d}\) admits \((d_1 + d_2)\) commuting symmetry operators. (For explicit computation of the symmetries corresponding to any separable coordinate system on the sphere \(S^{(d_1)-1}\), see \(...\).)

We introduce double spherical coordinates
\[ \mathbf{x} = \{x, \Omega_x\}, \quad \mathbf{y} = \{y, \Omega_y\}, \]
where \(\Omega_x, \Omega_y\) are Euler angles on spheres \(S^{(d_1)-1}, S^{(d_2)-1}\), respectively, so that the Hamiltonian \(\mathbf{H}_{2,d}\) is reduced to
\[ \mathbf{H}_{2,r} = -\frac{\partial^2}{\partial x^2} - \frac{d_1 - 1}{x} \frac{\partial}{\partial x} \Delta^{S^{(d_1)-1}}_x - \frac{\partial^2}{\partial y^2} - \frac{d_2 - 1}{y} \frac{\partial}{\partial y} \Delta^{S^{(d_2)-1}}_y + V_2(x, y). \]
Here \(\Delta^{S^{(d_1)-1}}_x, \Delta^{S^{(d_2)-1}}_y\) are Laplacians on spheres \(S^{(d_1)-1}, S^{(d_2)-1}\), respectively. Since the eigenfunctions of the Laplacian on the sphere are spherical harmonics, each Laplacian decomposes its domain into a direct sum of eigenspaces, so each can be replaced by its eigenvalue:
\[ \mathbf{H}_{2,radial}(x, y) = -\frac{\partial^2}{\partial x^2} - \frac{d_1 - 1}{x} \frac{\partial}{\partial x} \frac{L_x(L_x + d_1 - 2)}{x^2} - \frac{\partial^2}{\partial y^2} - \frac{d_2 - 1}{y} \frac{\partial}{\partial y} \frac{L_y(L_y + d_2 - 2)}{y^2} + V_2(x, y), \]
where \(L_x, L_y\) index the angular momentum eigenvalues in \(x, y\) spaces, respectively. The configuration space is the quadrant \(\mathbb{R}_+(x) \times \mathbb{R}_+(y)\) in the \(E_2\) plane. Through the gauge rotation of the Hamiltonian \(\mathbf{H}_{2,r}\),
\[ x^{d_1-1} y^{d_2-1} \mathbf{H}_{2,radial} x^{-d_1-1} y^{-d_2-1} \equiv \mathbf{H}_2, \]
we arrive at a two-dimensional Hamiltonian,
\[ \mathbf{H}_2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + W_2(x, y), \]
where the new potential $W_2$ has absorbed the singular terms $\sim 1/x^2$ and $\sim 1/y^2$.

Recently, a new method [6] based on operator algebras was introduced to construct higher order superintegrable potentials separating in Cartesian coordinates $W_2(x, y) = V_1(x) + V_2(y)$. For these potentials, one (polynomial) integral is of second order in momenta while the other one is of order $N \geq 2$. For $N = 2$, the simplest example is given by the two-dimensional isotropic harmonic oscillator

$$W_2 = \omega^2 (x^2 + y^2),$$

thus with two 2nd order integrals, whereas for the caged (anisotropic) harmonic oscillator potential

$$\tilde{W}_N = \omega^2 (N^2 x^2 + y^2),$$

one of the integrals is of $N$th order. It is worth mentioning that superintegrable potentials in classical and quantum mechanics can be different - in quantum case $\hbar$ dependence can occur [7, 8]. In particular, for potentials possessing a higher order integral (higher than two), terms depending on $\hbar$ typically appear. This is true for any Hamiltonian in $E_2$ with an integral of order $N > 2$, independently on the separation of variables. However, in the present study we focus on $\hbar$-independent potentials.

Also, there is a recent classification [9] of higher order superintegrable potentials separating in polar coordinates $V(\rho, \theta) = R(\rho) + S(\theta)/\rho$. Related to this, it is worth mentioning the TTW potential [10]

$$V^{(k)}_{\text{TTW}} = \omega^2 \rho^2 + \frac{1}{\rho^2} \left[ \frac{\alpha}{\cos^2(k \theta)} + \frac{\beta}{\sin^2(k \theta)} \right],$$

written naturally in polar coordinates $(\rho, \theta)$, instead of Cartesian ones $(x, y)$ with $R(\rho) = \omega^2 \rho^2$. Here $k$ is rational, $k = m/n$ with $m$ and $n$ integers (with no common divisors). For $m = n = 1$, the TTW system possesses two 2nd order integrals (as the consequence of multiseparability of variables in Cartesian and polar coordinates). In general, it admits an additional $N = (2k)$-th order integral [13, 14]

$$N = 2(m + n - 1).$$

Another higher order superintegrable system separating in polar coordinates is the PW potential [11]

$$V_{\text{PW}}^{(k)} = -\frac{a}{\rho} + \frac{1}{\rho^2} \left[ \frac{\mu}{\cos^2(\frac{k}{2} \theta)} + \frac{\nu}{\sin^2(\frac{k}{2} \theta)} \right],$$
where again $k = m/n$ is rational. Note that the PW potential is related to TTW by coupling constant metamorphosis [12].

Our approach differs from the above two in that it is focused on solutions of the 3-body problem in $d$ dimensions, not on a particular coordinate system.

### III. FROM THE 3-BODY $d$-DIMENSIONAL QUANTUM SYSTEM TO A 2-DIMENSIONAL ONE

Now let us consider the 3-body quantum system of $d$-dimensional particles of masses $m_1, m_2, m_3$. Its kinetic energy is of the form,

$$\mathcal{T} = -\sum_{i=1}^{3} \frac{1}{2m_i} \Delta_i^{(d)},$$

with coordinate vector of $i$th particle $\mathbf{r}_i \equiv \mathbf{r}_i^{(d)} = (x_{i,1}, \cdots, x_{i,d})$. Here, $\Delta_i^{(d)}$ is the $d$-dimensional Laplacian,

$$\Delta_i^{(d)} = \frac{\partial^2}{\partial r_i \partial r_i},$$

associated with the $i$th particle. Evidently, its 3$d$-dimensional configuration space is $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ space and (9) is $O(d) \times O(d) \times O(d)$ symmetric. This system is evidently integrable - it is characterized by three conserved angular momenta, $\mathbf{\hat{L}}_{1,2,3}$, respectively.

We introduce the center-of-mass, $d$-dimensional vectorial coordinate

$$\mathbf{R}_0 = \frac{1}{\sqrt{M}} \sum_{k=1}^{3} m_k \mathbf{r}_k , \quad M = m_1 + m_2 + m_3 ,$$

where $M$ is the total mass, and two, $d$-dimensional, vectorial Jacobi coordinates

$$\mathbf{r}_1^{(J)} = \sqrt{\frac{m_1 m_2}{m_1 + m_2}} (\mathbf{r}_2 - \mathbf{r}_1) , \quad \mathbf{r}_2^{(J)} = \sqrt{\frac{m_3 (m_1 + m_2)}{M}} \left( \mathbf{r}_3 - \frac{m_1}{m_1 + m_2} \mathbf{r}_1 - \frac{m_2}{m_1 + m_2} \mathbf{r}_2 \right) ,$$

see e.g. [15] and also [16] for discussion. In Jacobi coordinates the Laplacian (9) becomes diagonal,

$$\mathcal{T} = -\frac{\partial^2}{\partial \mathbf{R}_0 \partial \mathbf{R}_0} - \frac{\partial^2}{\partial \mathbf{r}_1^{(J)} \partial \mathbf{r}_1^{(J)}} - \frac{\partial^2}{\partial \mathbf{r}_2^{(J)} \partial \mathbf{r}_2^{(J)}} \equiv \mathcal{T}_0 + \mathcal{T}_r ,$$

where $\mathcal{T}_0 = -\Delta_{\mathbf{R}_0}$ is the kinetic energy of the center-of-mass motion, while $\mathcal{T}_r$ is the kinetic energy of relative motion. Assuming the 3-body potential is translation-invariant, hence,
no dependence on the center-of-mass coordinate, and separating out the $R_0$-coordinate, one can define the Hamiltonian of the relative motion,

$$\mathcal{H}_r = -\Delta_1^{(d)} - \Delta_2^{(d)} + V_r(r_1^{(J)}, r_2^{(J)}) .$$

Here $\Delta_i^{(d)} = \frac{\partial^2}{\partial r_i^{(j)} \partial r_i^{(j)}}$, $i = 1, 2$ is the $i$th Laplacian, and $r_i^{(j)} = |r_i^{(j)}|$ is the $i$th Jacobi distance, where we assume that the potential $V_r$ depends on Jacobi distances only. This $(2d)$-dimensional system is evidently integrable - it is characterized by two conserved angular momenta, $\hat{\mathbf{L}}_1 = -i r_1^{(j)} \wedge \frac{\partial}{\partial r_1^{(j)}}$ and $\hat{\mathbf{L}}_2 = -i r_2^{(j)} \wedge \frac{\partial}{\partial r_2^{(j)}}$, so is $O(d) \times O(d)$ symmetric.

In terms of double spherical coordinates

$$r_i^{(j)} = \{r_i^{(j)}, \Omega_i\}, \quad r_i^{(j)} = \{r_i^{(j)}, \Omega_i\} ,$$

where $\Omega_1, \Omega_2$ are Euler angles on spheres $S^{(d-1)}$, the Hamiltonian (13) is reduced to the double radial-spherical operator

$$\mathcal{H}_r = -\frac{\partial^2}{(\partial r_1^{(j)})^2} - \frac{d - 1}{r_1^{(j)}} \frac{\partial}{\partial r_1^{(j)}} - \frac{\Delta s_1^{(d-1)}}{(r_1^{(j)})^2} - \frac{\partial^2}{(\partial r_2^{(j)})^2} - \frac{d - 1}{r_2^{(j)}} \frac{\partial}{\partial r_2^{(j)}} - \frac{\Delta s_2^{(d-1)}}{(r_2^{(j)})^2} + V_r(r_1^{(j)}, r_2^{(j)}) .$$

Here $\Delta s_{1,2}^{(d-1)}$ are Laplacians on spheres $S_{1,2}^{(d-1)}$, respectively. Since the eigenfunctions of the Laplacian on the sphere are spherical harmonics, each Laplacian can be replaced by an eigenvalue. We arrive at

$$\mathcal{H}_{r,\text{radial}}(x, y) = -\frac{\partial^2}{(\partial r_1^{(j)})^2} - \frac{d - 1}{r_1^{(j)}} \frac{\partial}{\partial r_1^{(j)}} - \frac{L_1(L_1 + d - 2)}{(r_1^{(j)})^2} -$$

$$\frac{\partial^2}{(\partial r_2^{(j)})^2} - \frac{d - 1}{r_2^{(j)}} \frac{\partial}{\partial r_2^{(j)}} - \frac{L_2(L_2 + d - 2)}{(r_2^{(j)})^2} + V_r(r_1^{(j)}, r_2^{(j)}) ,$$

where $L_1, L_2$ index the angular momenta in 1, 2 spaces, respectively. The configuration space is the quadrant $R_+(1) \times R_+(2)$ in the $E_2$ plane. Through the gauge rotation of the Hamiltonian (15),

$$(r_1^{(j)}, r_2^{(j)})^{d-1 \over 2} \mathcal{H}_{r,\text{radial}}(r_1^{(j)}, r_2^{(j)})^{-d-1 \over 2} \equiv H_r ,$$

we arrive at the two-dimensional Hamiltonian,

$$H_r = -\frac{\partial^2}{(\partial r_1^{(j)})^2} - \frac{\partial^2}{(\partial r_2^{(j)})^2} + W_r(r_1^{(j)}, r_2^{(j)}) ,$$

where the new potential $W_r$ has absorbed the singular terms $\sim 1/(r_1^{(j)})^2$ and $\sim 1/(r_2^{(j)})^2$. It can be immediately seen that by identifying $(r_1^{(j)}, r_2^{(j)})$ with $(x, y)$, the Hamiltonian $H_r$
becomes the Hamiltonian $H_2$. This identification allows us to transform the theory of superintegrable systems in the $(x, y)$-plane to the theory of superintegrable systems in the space of relative motion of the 3-body problem parameterized by Jacobi distances $(r_1^{(J)}), (r_2^{(J)})$. In particular, the 3-body analogue of the TTW potential (6) is of the form

$$W_r^{(k)}(\rho_r, \theta_r; \omega, k, \alpha, \beta) = \omega^2 \rho_r^2 + \frac{\alpha k^2}{\rho_r^2 \cos^2 k \theta_r} + \frac{\beta k^2}{\rho_r^2 \sin^2 k \theta_r}, \quad (17)$$

being written in polar coordinates $(\rho_r, \theta_r)$ instead of Jacobi distances $(r_1^{(J)}, r_2^{(J)})$, where $\rho_r = \sqrt{(r_1^{(J)})^2 + (r_2^{(J)})^2}$ and $\theta_r$ is polar angle. In these coordinates the 3-body TTW Hamiltonian admits separation of variables. Here $\alpha, \beta > -\frac{1}{4k^2}$, $\omega$ and $k \neq 0$ are parameters. Correspondingly, the Hamiltonian,

$$\mathcal{H}_r = \mathcal{T} + W_r^{(k)},$$

see (9), presents one of the most striking examples of a 3-body superintegrable system with $(2k)$ extra integrals of motions.

IV. CONCLUSIONS AND DISCUSSION

We have established a quantum mechanical construction linking planar dynamics to one related to 3-body relative motion. It is evident that this construction can be extended even further by making a connection of the $n$-dimensional Schrödinger equation to an $(n+1)$-body quantum system by identifying the Cartesian coordinates with Jacobi distances. This allows us to connect integrality/solvability properties of quantum dynamics in Cartesian coordinates with the quantum dynamics of relative motion in the many-body problem. It has an obvious classical mechanical analog for all the potentials that we have presented. In particular, in the paper [17] it is shown how to construct additional classical superintegrable 2D Euclidean systems with higher order integrals, following the TTW method [18]. It has not yet been established which of these new systems is also quantum superintegrable. This will be done elsewhere.

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