Algebra properties for Besov spaces on unimodular Lie groups

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Abstract

We consider the Besov space $B^{\alpha}_{p,q}(G)$ on a unimodular Lie group $G$ equipped with a sublaplacian $\Delta$. Using estimates of the heat kernel associated with $\Delta$, we give several characterizations of Besov spaces, and show an algebra property for $B^{\alpha}_{p,q}(G) \cap L^\infty(G)$ for $\alpha > 0$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. These results hold for polynomial as well as for exponential volume growth of balls.

Keywords: Besov spaces, unimodular Lie groups, algebra property.

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1 Introduction and statement of the results

We use the following notations. $A(x) \lesssim B(x)$ means that there exists $C$ independent of $x$ such that $A(x) \leq CB(x)$ for all $x$. $A(x) \sim B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. The parameters which the constant is independent to will be either obvious from context or recalled.

1.1 Introduction

Let $d \in \mathbb{N}^*$. In $\mathbb{R}^d$, the Besov spaces $B^{p,q}_\alpha(\mathbb{R}^d)$ are obtained by real interpolation of Sobolev spaces and can be defined, for $p, q \in [1, +\infty]$ and $\alpha \in \mathbb{R}$, as the subset of distributions $\mathcal{S}'(\mathbb{R}^d)$ satisfying

$$
\|f\|_{B^{p,q}_\alpha} := \|\psi \ast f\|_{L^p} + \left( \sum_{k=1}^{\infty} \|2^{k\alpha} |\varphi_k \ast f|\|_{L^p}^q \right)^{\frac{1}{q}} < +\infty
$$

where, if $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is supported in $B(0,2) \setminus B(0,\frac{1}{2})$, $\varphi_k$ and $\psi$ are such that $\mathcal{F}\varphi_k(\xi) = \varphi(2^{-k}\xi)$ and $\mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$.

The norm of the Besov space $B^{p,q}_\alpha(\mathbb{R}^d)$ can be also written by using the heat operator. Indeed, Triebel proved in [16, 18, Section 2.12.2] that for all $p, q \in [1, +\infty]$, all $\alpha > 0$ and all integer $m > \frac{\alpha}{2}$,

$$
\|f\|_{B^{p,q}_\alpha} \simeq \|f\|_{L^p} + \left( \int_0^{\infty} t^{(m-\frac{\alpha}{2})q} \left\| \frac{\partial^M H_t}{\partial t^M} f \right\|_{L^p}^q \, dt \right)^{\frac{1}{q}}
$$

where $H_t = e^{-t\Delta}$ is the heat semigroup (generated by $-\Delta$). Note that we can give a similar characterization by using, instead of the heat semigroup, the harmonic extension or another extensions obtained by convolution (see [19, 12]).

Another characterization in term of functional using differences of functions was done. Define for $M \in \mathbb{N}^*$, $f \in L^p(\mathbb{R}^d)$, $x, h \in \mathbb{R}^d$ the term

$$
\nabla^M_h f(x) = \sum_{l=0}^{M} \binom{M}{l} (-1)^{M-l} f(x + lh)
$$

and then for $M > \alpha > 0$, $p, q \in [1, +\infty]$

$$
S^p_{\alpha,M} f = \left( \int_{\mathbb{R}^d} |h|^{-\alpha q} \|\nabla^M_h f\|_{L^p}^q \right)^{\frac{1}{q}}.
$$

We have then for all $\alpha > 0$, $p, q \in [1, +\infty]$ and $M \in \mathbb{N}$ with $M > \alpha$,

$$
\|f\|_{B^{p,q}_\alpha} \simeq \|f\|_{L^p} + S^p_{\alpha,M} f.
$$

One of the remarkable property of Besov spaces (see [7, Proposition 1.4.3], [13, Theorem 2, p. 336], [12, Proposition 6.2]) is that $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is an algebra for the pointwise product, that is for all $M > \alpha > 0$, all $p, q \in [1, +\infty]$, one has

$$
\|fg\|_{B^{p,q}_\alpha} \lesssim \|f\|_{B^{p,q}_\alpha} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^{p,q}_\alpha}.
$$

The idea of [7] consists in decomposing the product $fg$ by some paraproducts. The authors of [12] wrote $B^{p,q}_\alpha(\mathbb{R}^d)$ as a trace of some weighted (non fractional) Sobolev spaces, and thus deduced the algebra property $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ from the one of $W^{p,k}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Notice also that, when $\alpha \in (0, 1)$ and $M = 1$, the algebra property of $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a simple consequence of [4].

The property [4] have also been studied in the more general setting of Besov spaces on Lie groups. Gallagher and Sire stated in [10] an algebra property for Besov spaces on $H$-type groups, which are a subclass of Carnot groups. In order to do this, they used a some paradifferential calculus and a Fourier transform adapted to $H$-groups.

Moreover, in the more general case where $G$ is a unimodular Lie group with polynomial growth, they used the definition of Besov spaces obtained using Littlewood-Paley decomposition proved in [9]. When $\alpha \in (0, 1)$, they proved a equivalence of the Besov norms with some functionals using differences of functions, in the spirit of [9], and thus they obtained an algebra property for $B^{p,q}_\alpha(G) \cap L^\infty(G)$. They shows a recursive definition of Besov spaces because the
We will denote by $H$ the set of functions $f$ belonging to $L^p(R^n)$ for $p \in (1, +\infty)$ and $\alpha > 0$.

Note that methods used in [10] or in the present paper are similar to the ones in [5] and [6], where fractional Sobolev spaces $L^p_x(G)$ are considered on unimodular groups (and also on Riemannian manifolds). In these two last articles, the authors proved the algebra property for $L^p_x(G) \cap L^\infty(G)$ when $p \in (1, +\infty)$ and $\alpha > 0$.

1.2 Lie group structure

In this paper, $G$ is a unimodular connected Lie group endowed with its Haar measure $dx$. We recall that “unimodular” means that $dx$ is both left- and right-invariant. We denote by $\mathcal{L}$ the Lie algebra of $G$ and we consider a family $\mathcal{X} = \{X_1, \ldots, X_k\}$ of left-invariant vector fields on $G$ satisfying the Hörmander condition (which means that the Lie algebra generated by the family $\mathcal{X}$ is $\mathcal{L}$). Denote $I_\infty(N) = \bigcup_{i \in \mathbb{N}} \{1, \ldots, k\}_i$. Then if $I = (i_1, \ldots, i_n) \in I_\infty(N)$, the length of $I$ will be denoted by $|I|$ and is equal to $n$, whereas $X_I$ denotes the vector field $X_{i_1} \cdots X_{i_n}$.

A standard metric, called the Carnot-Caratheodory metric, is naturally associated with $(G, \mathcal{X})$ and is defined as follows. Let $l : [0, 1] \to G$ be an absolutely continuous path. We say that $l$ is admissible if there exist measurable functions $a_1, \ldots, a_k : [0, 1] \to \mathbb{C}$ such that

$$l'(t) = \sum_{i=1}^k a_i(t)X_i(l(t)) \quad \text{for a.e. } t \in [0, 1].$$

If $l$ is admissible, its length is defined by $|l| = \int_0^1 \left( \sum_{i=0}^k |a_i(t)|^2 \right)^{\frac{1}{2}} dt$. For any $x, y \in G$, the distance $d(x, y)$ between $x$ and $y$ is then the infimum of the lengths of all admissible curves joining $x$ to $y$ (such a curve exists thanks to the Hörmander condition). The left-invariance of the $X_i$’s implies the left-invariance of $d$. For short, $|x|$ denotes the distance between the neutral $e$ and $x$, and therefore $d(x, y) = |y^{-1}x|$ for all $x$ and $y$ in $G$.

For $r > 0$ and $x \in G$, we denote by $B(x, r)$ the open ball with respect to the Carnot-Caratheodory metric centered at $x$ and of radius $r$. Define also by $V(r)$ the Haar measure of any ball of radius $r$.

From now and abusively, we will write $G$ for $(G, \mathcal{X}, d, dx)$. Recall that $G$ has a local dimension (see [13]):

**Proposition 1.1.** Let $G$ be a unimodular Lie group and $\mathcal{X}$ be a family of left-invariant vector fields satisfying the Hörmander condition. Then $G$ has the local doubling property, that is there exists $C > 0$ such that

$$V(2r) \leq CV(r) \quad \forall 0 < r \leq 1.$$

More precisely, there exist $d \in \mathbb{N}^*$ and $c, C > 0$ such that

$$cr^d \leq V(r) \leq Cr^d \quad \forall 0 < r \leq 1.$$

For balls with radius bigger than 1, we have the result of Guivarc’h (see [11]):

**Proposition 1.2.** If $G$ is a unimodular Lie group, only two situations may occur. Either $G$ has polynomial growth and there exist $d \in \mathbb{N}^*$ and $c, C > 0$ such that

$$cr^d \leq V(r) \leq Cr^d \quad \forall r \geq 1,$$

or $G$ has exponential growth and there exist $c_1, c_2, C_1, C_2 > 0$ such that

$$c_1 e^{c_2 r} \leq V(r) \leq C_1 e^{C_2 r} \quad \forall r \geq 1.$$

We consider the positive sublaplacian $\Delta$ on $G$ defined by

$$\Delta = -\sum_{i=1}^k X_i^2.$$

We will denote by $H_t = e^{-t\Delta}$ the heat semigroup on $G$ associated with $\Delta$. 

3
1.3 Definition of Besov spaces

Definition 1.3. Let $G$ be a unimodular Lie group. We define the Schwartz space $\mathcal{S}(G)$ as the space of functions $\varphi \in C^\infty(G)$ where all the seminorms

$$N_{I,c}(\varphi) = \sup_{x \in G} e^{c|x|}|X_I \varphi(x)| \quad c \in \mathbb{N}, I \in \mathcal{I}_\infty(\mathbb{N})$$

are finite.

The space $\mathcal{S}'(G)$ is defined as the dual space of $\mathcal{S}(G)$.

Remark 1.4. Note that we have the inclusion $\mathcal{S}(G) \subset L^p(G)$ for any $p \in [1, +\infty]$. As a consequence, $L^p(G) \subset \mathcal{S}'(G)$.

Definition 1.5. Let $G$ be a unimodular Lie group and let $\alpha \geq 0$, $p, q \in [1, +\infty]$. The space $f \in B^p_q(G)$ is defined as the subspace of $\mathcal{S}'(G)$ made of distributions $f$ such that, for all $t \in (0, 1)$, $\Delta^m H_t f \in L^p(G)$ and satisfying

$$\|f\|_{B^p_q} := \Lambda^p_q f + \|H_t f\|_p < +\infty,$$

where

$$\Lambda^p_q f := \left( \int_0^1 \left( t^{q-\frac{n}{p}} \|\Delta^m H_t f\|_p^q \frac{dt}{t} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

if $q < +\infty$ (with the usual modification if $q = +\infty$) and $m$ stands for the only integer such that $\frac{n}{2} < m \leq \frac{n}{2} + 1$.

Remark 1.6. Lemma [10] provides that the heat kernel $h_t$ is in $\mathcal{S}(G)$ for all $t > 0$. Thus $H_t \varphi \in \mathcal{S}(G)$ whenever $t > 0$ and $\varphi \in \mathcal{S}(G)$. When $f \in \mathcal{S}'(G)$, the term $X_I H_t f$ denotes the distribution in $\mathcal{S}'(G)$ defined by

$$\langle X_I H_t f, \varphi \rangle = (-1)^{|I|} \langle f, H_t X_I \varphi \rangle \quad \forall \varphi \in \mathcal{S}(G).$$

1.4 Statement of the results

Proposition 1.7. Let $G$ be a unimodular Lie group. The one has for all $p \in [1, +\infty]$, all multi indexes $t \in \mathcal{I}_\infty(\mathbb{N})$ and $t \in (0, 1)$,

$$\|X_I H_t f\|_p \leq C t^{-\frac{\alpha}{p}} f \|_p \quad \forall f \in L^p(G).$$

Remark 1.8. In particular, one has that $\|t \Delta H_t\|_p \lesssim 1$ once $t \in (0, 1)$ and for all $p \in [1, +\infty]$. When $p \in (1, +\infty)$, since $\Delta$ is analytic on $L^2$ (and thus on $L^p$), we actually have $\|t \Delta H_t\|_p \lesssim 1$ for all $t > 0$. The case

The following result gives equivalent definitions of the Besov spaces $B^p_q$ only involving the Laplacian.

Theorem 1.9. Let $G$ be a unimodular Lie group and $p, q \in [1, +\infty]$ and $\alpha \geq 0$.

If $m > \frac{n}{2}$ and $t_0$ a real in $\left\{ \begin{array}{ll} (0, 1) & \text{if } \alpha = 0 \\ [0, 1) & \text{if } \alpha > 0 \end{array} \right.$, then the following norms are equivalent to the norm of $B^p_q(G)$.

(i) $\left( \int_0^1 \left( t^{q-\frac{n}{p}} \|\Delta^m H_t f\|_p^q \frac{dt}{t} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} + \|H_{t_0} f\|_p$.

(ii) $\|H_{t_0} f\|_p + \left( \sum_{j \leq -1} \left[ 2^{j(q-\frac{n}{p})} \|\Delta^m H_{2^j t_0} f\|_p^q \right]^{\frac{1}{q}} \right)^{\frac{1}{q}}$.

(iii) $\|H_{t_0} f\|_p + \left( \sum_{j \leq -1} \left[ 2^{-j\frac{m}{2}} \left\| \int_{2^j t_0}^{2^{j+1} t_0} \|\Delta^m H_t f\|_p \frac{dt}{t} \right\|_p^q \right]^{\frac{1}{q}} \right)^{\frac{1}{q}}$.

if we assume that $\alpha > 0$.

Remark 1.10. Here and after, we say that “a norm $N$ is equivalent to the norm in $B^p_q$” if and only if the space of distributions $f \in \mathcal{S}'$ such that $\Delta^m H_t f$ is a locally integrable function in $G$ for all $t > 0$ and $N(f) < +\infty$ coincides with $B^p_q$ and the norm $N$ is equivalent to $\|\cdot\|_{B^p_q}$.

The previous theorem allows us to recover some well known facts about Besov spaces in $\mathbb{R}^d$.
\textbf{Corollary 1.11.} [Embeddings] Let $G$ be a unimodular Lie group, $p,q,r \in [1, +\infty]$ and $\alpha \geq 0$. We have the following continuous embedding

$$B^\alpha_{p,q}(G) \subset B^\alpha_{p,r}(G)$$

once $q \leq r$.

\textbf{Corollary 1.12.} [Interpolation]

Let $G$ be a unimodular Lie group. Let $s_0, s_1 \geq 0$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$.

Define

$$s^* = (1 - \theta)s_0 + \theta s_1$$

$$\frac{1}{p^*} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q^*} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$ 

The Besov spaces form a scale of interpolation for the complex method, that is, if $s_0 \neq s_1$,

$$(B^\alpha_{s_0,p_0,q_0}, B^\alpha_{s_1,p_1,q_1})[\theta] = B^\alpha_{s^*,p^*,q^*}.$$ 

The following result is another characterization of Besov spaces, using explicitly the family of vector fields $\mathbb{X}$.

\textbf{Theorem 1.13.} Let $G$ be a unimodular Lie group, $p,q \in [1, +\infty]$ and $\alpha > 0$. Let $\widehat{m}$ be an integer strictly greater than $\alpha$. Then

$$\|H_{\mathbb{X}} f\|_p + \left( \sum_{j \leq -1} \left[ 2^{j \widehat{m} \alpha} \max_{t \in [2^j, 2^{j+1}]} \sup_{|f| \leq \widehat{m}} \|X_t H_{\mathbb{X}} f\|_p \right]^q \right)^{\frac{1}{q}}$$

(6)

is an equivalent norm in $B^\alpha_{p,q}(G)$.

With the use of paraproducts, we can deduce from Corollary 1.12 and Theorem 1.13 the complete following Leibniz rule.

\textbf{Theorem 1.14.} Let $G$ be a unimodular Lie group, $0 < \alpha$ and $p, p_1, p_2, p_3, p_4, q \in [1, +\infty]$ such that

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}.$$ 

Then for all $f \in B^\alpha_{p_1,q} \cap L^{p_2}$ and all $g \in B^\alpha_{p_3,q} \cap L^{p_4}$, one has

$$\|fg\|_{B^\alpha_{p,q}} \lesssim \|f\|_{B^\alpha_{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B^\alpha_{p_4,q}}.$$ 

(7)

\textbf{Remark 1.15.} The Leibniz rule implies that $B^\alpha_{p,q}(G) \cap L^\infty(G)$ is an algebra under pointwise product, that is

$$\|fg\|_{B^\alpha_{p,q}} \lesssim \|f\|_{B^\alpha_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^\alpha_{p,q}}.$$ 

Let us state another characterization of $B^\alpha_{p,q}$ in term of functionals using differences of functions.

Define $\nabla_y f(x) = f(xy) - f(x)$ for all functions $f$ on $G$ and all $x,y \in G$. Consider the following sublinear functional

$$L^\alpha_{p,q}(f) = \left( \int_{|y| \leq 1} \left( \frac{\|\nabla_y f\|_p}{|y|^\alpha} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.$$ 

\textbf{Theorem 1.16.} Let $G$ be a unimodular Lie group. Let $p, q \in [1, +\infty]$. Then for all $f \in L^{p}(G)$,

$$L^\alpha_{p,q}(f) + \|f\|_p \simeq \Lambda^\alpha_{p}(f) + \|f\|_p$$

once $\alpha \in (0,1)$.

\textbf{Remark 1.17.} When $G$ has polynomial volume growth, Theorem 1.16 is the inhomogeneous counterpart of Theorem 2 in [13] Note that this statement is new when $G$ has exponential volume growth.

\textbf{Remark 1.18.} From Theorem 1.16, we can deduce the Leibniz rule stated in Theorem 1.13 in the case $\alpha \in (0,1)$.

As Sobolev spaces, Besov spaces can be characterized recursively.

\textbf{Theorem 1.19.} Let $G$ be a unimodular Lie group. Let $p, q \in [1, +\infty]$ and $\alpha > 0$. Then

$$f \in B^\alpha_{p,q+1}(G) \Leftrightarrow \forall i, X_if \in B^\alpha_{p,q}(G)$$

and $f \in L^{p}(G)$.

\textbf{Remark 1.20.} Note that a similar statement is established in [10]. However, we prove this fact for $p \in [1, +\infty]$ while the authors of [10] used the boundedness of the Riesz transforms and thus are restricted to $p \in (1, +\infty)$. 

5
2 Estimates of the heat semigroup

2.1 Preliminaries

The following lemma is easily checked:

**Lemma 2.1.** Let $(A, dx)$ and $(B, dy)$ be two measured spaces. Let $K(x, y) : A \times B \to \mathbb{R}_+$ be such that

$$\sup_{x \in A} \int_B K(x, y) dy \leq C_B$$

and

$$\sup_{y \in B} \int_A K(x, y) dx \leq C_A.$$

Let $q \in [1, +\infty]$. Then for all $f \in L^q(B)$

$$\left( \int_A \left| \int_B K(x, y) f(y) dy \right|^q dx \right)^{\frac{1}{q}} \leq C_B^{1-\frac{1}{q}} C_A^\frac{1}{q} \|f\|_q,$$

with obvious modifications when $q = +\infty$.

**Lemma 2.2.** Let $(a, b) \in (\mathbb{Z} \cup \{\pm \infty\})^2$ such that $a < b$, $0 < \alpha < \beta$ two real numbers and $q \in [1, +\infty]$. Then there exists $C_{\alpha, \beta} > 0$ such that for any sequence $(c_n)_{n \in \mathbb{Z}}$, one has

$$\sum_{n=a}^{b} \left[ 2^{ja} \sum_{n=a}^{b} 2^{-\max\{n,j\} \beta} c_n \right]^q \lesssim \sum_{n=a}^{b} \left[ 2^{(\alpha-\beta)n} c_n \right]^q.$$

**Proof:** We have

$$\sum_{n=a}^{b} \left[ 2^{ja} \sum_{n=a}^{b} 2^{-\max\{n,j\} \beta} c_n \right]^q = \sum_{j=a}^{b} \left[ \sum_{n=a}^{b} K(n, j) d_n \right]^q$$

with $d_n = 2^{n(\alpha-\beta)} c_n$ and $K(n, j) = 2^{(j-n)\alpha} 2^{(n-\max\{j, n\}) \beta}$.

According to Lemma 2.1 one has to check that

$$\sup_{j \in [a, b]} \sum_{n=a}^{b} K(n, j) \lesssim 1$$

and

$$\sup_{n \in [a, b]} \sum_{j=a}^{b} K(n, j) \lesssim 1.$$

For the first estimate, check that

$$\sup_{j \in [a, b]} \sum_{n=a}^{b} K(n, j) = \sup_{j \in [a, b]} \left[ 2^{j(\alpha-\beta)} \sum_{n=a}^{b} 2^{n(\beta-\alpha)} + 2^{ja} \sum_{n=j+1}^{b} 2^{-\alpha n} \right]$$

$$\leq \sup_{j \in \mathbb{Z}} \left[ 2^{j(\alpha-\beta)} \sum_{n=-\infty}^{j} 2^{n(\beta-\alpha)} + 2^{ja} \sum_{n=j+1}^{+\infty} 2^{-\alpha n} \right]$$

$$\lesssim 1,$$

since $\beta - \alpha > 0$ and $\alpha > 0$.

The second estimate can be checked similarly:

$$\sup_{n \in [a, b]} \sum_{j=a}^{b} K(n, j) = \sup_{j \in [a, b]} \left[ 2^{-\alpha n} \sum_{j=a}^{b} 2^{ja} + 2^{n(\beta-\alpha)} \sum_{j=n+1}^{b} 2^{j(\alpha-\beta)} \right]$$

$$\lesssim 1.$$
Proposition 2.3. Let $s \geq 0$ and $c > 0$. Define, for all $t \in (0, 1)$ and all $x, y \in G$,

$$K_t(x, y) = \left(\frac{|y-x|^2}{t}\right)^s \frac{1}{V(\sqrt{t})} e^{-\sqrt{t}|y-x|^2}. $$

Then, for all $q \in [1, +\infty]$,

$$\left( \int_G \left( \int_G K_t(x, y) g(y) dy \right)^q dx \right)^{\frac{1}{q}} \lesssim \|g\|_q.$$  

Proof: Let us check that the assumptions of Lemma 2.1 are satisfied. For all $x \in G$ and all $t \in (0, 1)$,

$$\int_G K_t(x, y) dy = \frac{1}{V(\sqrt{t})} \int_G \left(\frac{|y-x|^2}{t}\right)^s e^{-\sqrt{t}|y-x|^2} dy$$

$$= \frac{1}{V(\sqrt{t})} \int_{|y-x|^2 < t} \left(\frac{|y-x|^2}{t}\right)^s e^{-\sqrt{t}|y-x|^2} dy$$

$$+ \frac{1}{V(\sqrt{t})} \int_{|y-x|^2 \geq t} \left(\frac{|y-x|^2}{t}\right)^s e^{-\sqrt{t}|y-x|^2} dy$$

$$= I_1 + I_2.$$  

The term $I_1$ is easily dominated by 1. As for $I_2$, it is estimated as follows:

$$I_2 = \sum_{j=0}^{\infty} \frac{1}{V(\sqrt{t})} \int_{2^{j+1}\sqrt{t} \leq |y-x|^2 \leq 2^{j+2}\sqrt{t}} \left(\frac{|y-x|^2}{t}\right)^s e^{-\sqrt{t}|y-x|^2} dy$$

$$\lesssim \sum_{j=0}^{\infty} \frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} 4^j e^{-c4^j}.$$  

Notice that Propositions 1.1 and 1.2 imply that $\frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} \lesssim 2^{jd}$ if $2^j \sqrt{t} \leq 1$ and

$$\frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} = \frac{V(2^{j+1}\sqrt{t})}{V(1)} \frac{V(1)}{V(\sqrt{t})} \lesssim e^{C2^j} 2^{jd}$$

if $2^j \sqrt{t} \geq 1$. Hence,

$$\sum_{j=0}^{\infty} \frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} 4^j e^{-c4^j} \lesssim \sum_{j=0}^{\infty} e^{-c4^j} \lesssim 1,$$

which yields with the uniform estimate

$$\int_G K_t(x, y) dy \lesssim 1.$$  

In the same way, one has

$$\int_G K_t(x, y) dx \lesssim 1.$$  

Lemma 2.1 provides then the desired result. \qed

Proposition 2.4. Let $s \geq 0$ and $c > 0$. Define

$$K(t, y) = \left(\frac{|y|^2}{t}\right)^s \frac{V(|y|)}{V(\sqrt{t})} e^{-c|y|^2}.$$  

Then, for all $q \in [1, +\infty]$,

$$\left( \int_0^1 \left( \int_G K(t, y) g(y) dy \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \left( \int_G |g(y)| \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.$$  

\[7\]
Proof: Let us check again that the assumptions of Lemma 2.1 are satisfied, that are in our case
\[
\sup_{t \in (0, 1)} \int_G K(t, y) dy \leq C_B
\]
and
\[
\sup_{y \in G} \int_0^1 K(t, y) \frac{dt}{t} \leq C_A.
\]
The first one is exactly as the estimate (2). For the second one, check that
\[
\int_0^1 K(t, y) \frac{dt}{t} = \int_0^1 \frac{V(|y|)}{V(\sqrt{t})} \left( \frac{|y|^2}{t} \right)^s e^{-c \frac{|y|^2}{t}} \frac{dt}{t} + \int_0^\infty \frac{V(|y|^2)}{V(\sqrt{t})} \left( \frac{|y|^2}{t} \right)^s e^{-c \frac{|y|^2}{t}} \frac{dt}{t}
\]
\[
\leq \int_0^\infty \frac{V(|y|)}{V(\sqrt{t})} \left( \frac{|y|^2}{t} \right)^s e^{-c \frac{|y|^2}{t}} dt + 1
\]
\[
\leq \sum_{j=0}^{+\infty} 2^{(d+2)s} e^{C2^j} e^{-c4^j} + 1
\]
\[
\leq 1,
\]
where the last but one line is obtained with the estimate (2).

\[\square\]

2.2 Estimates for the semigroup

Because of left-invariance of \(\Delta\) and hypoellipticity of \(\frac{D}{dt} + \Delta\), \(H_t = e^{-t\Delta}\) has a convolution kernel \(h_t \in C^\infty(G)\) satisfying, for all \(f \in L^1(G)\) and all \(x \in G\),
\[
H_t f(x) = \int_G h_t(y^{-1} x) f(y) dy = \int_G h_t(y) f(xy) dy = \int_G h_t(y) f(xy^{-1}) dy.
\]
The kernel \(h_t\) satisfies the following pointwise estimates.

Proposition 2.5. Let \(G\) be a unimodular Lie group. For all \(I \in I_{\infty}(\mathbb{N})\), there exist \(C_1, c_1 > 0\) such that for all \(x \in G\), all \(t \in (0, 1]\), one has
\[
|X_I h_t(x)| \leq \frac{C_I}{t^{\frac{d}{4}}} V(\sqrt{t}) \exp\left(-c_I \frac{|x|^2}{t}\right).
\]

Proof: It is a straightforward consequence of Theorems VIII.2.4, VIII.4.3 and V.4.2. in [20].

\[\square\]

Lemma 2.6. Let \(G\) be a unimodular group. Then \(h_t \in S(G)\) for all \(t > 0\).

Proof: The case \(t < 1\) is a consequence of the estimates on \(h_t\). For \(t \geq 1\), just notice that \(S(G) * S(G) \subset S(G)\).

\[\square\]

Proposition 2.7. For all \(I \in I_{\infty}(\mathbb{N})\) and all \(p \in [1, +\infty]\), one has
\[
\|X_I H_t f\|_p \leq t^{-\frac{d}{4p}} \|f\|_p \quad \forall t \in (0, 1], \forall f \in L^p(G).
\]

Proof: Proposition 2.8 yields for any \(t \in (0, 1]\)
\[
\|X_I H_t f\|_p \leq t^{-\frac{d}{4p}} \left( \int_G \left| \int_G K_t(x, y) f(y) dy \right|^p dx \right)^{\frac{1}{p}}
\]
where \(K_t(x, y) = \frac{1}{V(\sqrt{t})} \exp\left(-c_1 \frac{|y|^2}{t}\right)\).
The conclusion of Proposition 2.11 is an immediate consequence of Proposition 2.8.

\[\square\]
3  Littlewood-Paley decomposition

We need a Littlewood-Paley decomposition adapted to this context. In [10], the authors used the Littlewood-Paley decomposition proven in [9, Proposition 4.1], only established in the case of polynomial volume growth. We state here a slightly different version of the Littlewood-Paley decomposition, also valid for the case of exponential volume growth.

**Lemma 3.1.** Let $G$ be a unimodular group and let $m \in \mathbb{N}^*$. For any $\varphi \in \mathcal{S}(G)$ and any $f \in \mathcal{S}'(G)$, one has the identities

$$\varphi = \frac{1}{(m-1)!} \int_0^1 (t \Delta)^m H_t \varphi \frac{dt}{t} + \sum_{k=0}^{m-1} \frac{1}{k!} \Delta^k H_1 \varphi,$$

where the integral converges in $\mathcal{S}(G)$, and

$$f = \frac{1}{(m-1)!} \int_0^1 (t \Delta)^m H_t f \frac{dt}{t} + \sum_{k=0}^{m-1} \frac{1}{k!} \Delta^k H_1 f,$$

where the integral converges in $\mathcal{S}'(G)$.

**Proof:** We only have to prove the first identity since the second one can be obtained by duality.

Let $\varphi \in \mathcal{S}(G)$. Check first the formula

$$(m-1)! = \int_0^{+\infty} (tu)^m e^{-tu} \frac{dt}{t} = \int_0^1 (tu)^m e^{-tu} \frac{dt}{t} + \sum_{k=0}^{m-1} \frac{1}{k!} (m-1)! u^k e^{-u}.$$

Thus by functional calculus, since $\varphi \in L^2(G)$, one has

$$\varphi = \frac{1}{(m-1)!} \int_0^1 (t \Delta)^m H_t \varphi \frac{dt}{t} + \sum_{k=0}^{m-1} \frac{1}{k!} \Delta^k H_1 \varphi,$$

where the integral converges in $L^2(G)$. Since the kernel $h_t$ of $H_t$ is in $\mathcal{S}(G)$ for any $t > 0$ (see Lemma 2.6), the formula (10) will be proven if we have for any $c \in \mathbb{N}$ and any $I \in \mathcal{I}_\infty(\mathbb{N})$,

$$\lim_{u \to 0} N_{I,c} \left( \int_0^u (t \Delta)^m H_t \varphi \frac{dt}{t} \right) = 0. \quad (11)$$

Let $n > \frac{d}{1-t}$ be an integer. Similarly to (10), one has for all $x \in G$ and all $t \in (0, 1)$,

$$H_t \varphi(x) = \frac{1}{(n-1)!} \int_0^1 (v-t)^{n-1} \Delta^n H_v \varphi(x) dv + \sum_{k=0}^{n-1} \frac{1}{k!} (1-t)^k \Delta^k H_1 \varphi(x).$$

Hence, for all $x \in G$ and all $u \in (0, 1)$, we have the identity

$$\int_0^u (t \Delta)^m H_t \varphi(x) \frac{dt}{t} = \frac{1}{(n-1)!} \int_0^1 \Delta^{n+m} H_v \varphi(x) \left( \int_0^{\min(u,v)} t^{m-1} (v-t)^{n-1} dt \right) dv$$

$$+ \sum_{k=0}^{n-1} \frac{1}{k!} \Delta^{k+m} H_1 \varphi(x) \int_0^u t^{m-1} (1-t)^k dt.$$

Note that

$$\int_0^{\min(u,v)} t^{m-1} (v-t)^{n-1} dt \lesssim u^m v^{n-1}$$

and

$$\int_0^u t^{m-1} (1-t)^k dt \lesssim u^m.$$

Therefore, the Schwartz seminorms of $\int_0^u (t \Delta)^m H_t \varphi \frac{dt}{t}$ can be estimated by

$$N_{I,c} \left( \int_0^u (t \Delta)^m H_t \varphi \frac{dt}{t} \right) \lesssim u^m \int_0^1 v^{n-1} \sup_{x \in G} e^{c|v|} |X_I \Delta^{n+m} H_v \varphi(x)| dv$$

$$+ u^m \sum_{k=0}^{n-1} \sup_{x \in G} e^{c|k|} |X_I \Delta^{k+m} H_1 \varphi(x)|. \quad (12)$$
Check then that for all \( w \in (0, 1] \) and all \( l \in \mathbb{N} \), we have
\[
\sup_{x \in G} e^{c|x|}|X_l \Delta^l H_w \varphi(x)| = \sup_{x \in G} e^{c|x|}|X_l H_w \Delta^l \varphi(x)| \\
\leq \sup_{x \in G} e^{c|x|} \int_G |X_l h_w(y^{-1} x)| |\Delta^l \varphi(y)| dy \\
\lesssim \sup_{x \in G} \int_G e^{c|y^{-1} x|} |X_l h_w(y^{-1} x)| e^{c|\Delta^l \varphi(y)|} dy \\
\lesssim \left( \sup_{x \in G} \int_G e^{c|y^{-1} x|} |X_l h_w(y^{-1} x)| dy \right) \sum_{|I|=2l} N_{l,c}(\varphi)
\tag{13}
\]
where the third line holds because \( |x| \leq |y^{-1} x| + |x| \).
However, for all \( x \in G \) and all \( w \in (0, 1] \), Proposition \ref{prop:blowup} yields that, for all \( x \in G \),
\[
\int_G e^{c|y^{-1} x|} |X_l h_w(y^{-1} x)| dy \lesssim w^{-\frac{m}{n}} \frac{1}{V(\sqrt{w})} \int_G e^{c|y^{-1} x|} e^{-c'(x^{-1} w^{2})} dy \\
\lesssim w^{-\frac{m}{n}} \frac{1}{V(\sqrt{w})} \int_G e^{-c'(x^{-1} w^{2})} dy \\
\lesssim w^{-\frac{m}{n}}.
\tag{14}
\]
By gathering the estimates \ref{eq:bound_x}, \ref{eq:bound_y} and \ref{eq:bound_z}, we obtain
\[
N_{l,c} \left( \int_0^1 (t \Delta)^m H_t \varphi \frac{dt}{t} \right) \lesssim u^m \left[ \sum_{|I| \leq 2(m+n)} N_{l,c}(\varphi) \right] \left[ \int_0^1 v^{n-1} v^{-\frac{m}{n}} dv + \sum_{k=0}^{n-1} 1 \right] \\
\lesssim u^m \sum_{|I| \leq 2(m+n)} N_{l,c}(\varphi) \\
\xrightarrow{u \to 0} 0,
\]
which proves \ref{eq:main_bound} and finishes the proof. \[\square\]

4 Proof of Theorem \ref{thm:main} and of its corollaries

4.1 Proof of Theorem \ref{thm:main}

In this section, we will always assume that \( \alpha \geq 0 \), \( p, q \in [1, +\infty] \).

Proposition 4.1. For all \( t_1, t_0 \in (0, 1) \) and all integers \( m > \frac{n}{2} \),
\[
\|f\|_p \lesssim \|H_{t_0} f\|_p + \left( \int_0^1 \left( t^{m-\frac{n}{2}} \|\Delta^m H_t f\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
\forall f \in S'(G)
\]
when \( \alpha > 0 \) and
\[
\|H_{t_1} f\|_p \lesssim \|H_{t_0} f\|_p + \left( \int_0^1 \left( t^{m-\frac{n}{2}} \|\Delta^m H_t f\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
\forall f \in S'(G)
\]
when \( \alpha \geq 0 \) and \( q < +\infty \), with the usual modification when \( q = +\infty \).

Proof: Lemma \ref{lem:apriori} (recall that \( L^p(G) \subset S'(G) \)) yields the estimate
\[
\|f\|_p \lesssim \int_0^1 t^m \|\Delta^m H_t f\|_p \frac{dt}{t} + \sum_{k=0}^{m-1} \|\Delta^k H_1 f\|_p.
\]
However, for all \(k \in \mathbb{N}\), \(\|\Delta^k H_t f\|_p \leq \frac{C}{(t-t_0)^{\delta}} \|H_{t_0} f\|_p\). Then, when \(\alpha > 0\),
\[
\|f\|_p \lesssim \int_0^1 t^m \|\Delta^m H_t f\|_p \frac{dt}{t} + \|H_{t_0} f\|_p \\
\lesssim \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} \left( \int_0^1 \left( \frac{\alpha}{2} \frac{dt}{t} \right)^{q'} \frac{dt}{t} \right)^{\frac{1}{q'}} + \|H_{t_0} f\|_p \\
\lesssim \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} + \|H_{t_0} f\|_p,
\]
which proves the case \(\alpha > 0\).

If \(\alpha = 0\), Lemma 3.31 for the integer \(m + 1\) implies
\[
\|H_t f\|_p \lesssim \int_0^1 \frac{t^{m+1}}{t_1} \|\Delta^{m+1} H_{t+t_1} f\|_p \frac{dt}{t} + \sum_{k=0}^{m} \|\Delta^k H_{t+t_1} f\|_p \\
\lesssim \int_0^1 \frac{t^{m+1}}{t_1} \|\Delta^m H_t f\|_p \frac{dt}{t} + \|H_{t_0} f\|_p \\
\lesssim \left( \int_0^1 \left( t^{m} \|\Delta^m H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} \left( \int_0^1 \left( \frac{1}{t_1} \right)^{q'} \frac{dt}{t} \right)^{\frac{1}{q'}} + \|H_{t_0} f\|_p \\
\lesssim \left( \int_0^1 \left( t^{m} \|\Delta^m H_{t_0} f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} + \|H_{t_0} f\|_p.
\]

\(~\square\)

**Proposition 4.2.** For all integers \(m > \frac{\alpha}{2}\),
\[
\left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \|f\|_p + \left( \int_0^1 \left( t^{m+1-\frac{\alpha}{2}} \|\Delta^{m+1} H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q'}}.
\]

**Proof:** We use Lemma 3.31 and get
\[
\Delta^m H_t f = \int_0^1 s \Delta^m H_s H_t f \frac{ds}{s} + H_t \Delta^m H_t f.
\]

Thus,
\[
\left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} \\
\leq \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \int_0^1 \|\Delta^{m+1} H_{t+s} f\|_p ds \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \|\Delta^{m+1} H_{t+s} f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q'}} \\
:= I_1 + I_2.
\]

We start with the estimate of \(I_1\). One has \(\Delta^{m+1} H_{t+s} f = H_s \Delta^{m+1} H_t f = H_t \Delta^{m+1} H_s f\). Then
\[
I_1 \leq \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \int_0^1 \|\Delta^{m+1} H_t f\|_p ds \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^1 \left( t^{m-\frac{\alpha}{2}} \int_0^1 \|\Delta^{m+1} H_{s} f\|_p ds \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q'}} \\
:= II_1 + II_2.
\]

Notice
\[
II_1 = \left( \int_0^1 \left( t^{m+1-\frac{\alpha}{2}} \|\Delta^{m+1} H_t f\|_p \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q'}}
\]
which is the desired estimate. As far as \(II_2\) is concerned,
\[
II_2 = \left( \int_0^1 \left( \int_0^1 K(s,t) g(s) ds \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q'}}
\]

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with \( g(s) = s^{m+1-\frac{\alpha}{2}} \| \Delta^{m+1} H_s f \|_p \) and \( K(s, t) = \left( \frac{t}{s} \right)^{m-\frac{\alpha}{2}} \). Since

\[
\int_0^1 K(s, t) \frac{ds}{s} \lesssim 1 \quad \text{and} \quad \int_0^1 K(s, t) \frac{dt}{t} \lesssim 1,
\]

Lemma 2.1 yields then

\[
II_2 \lesssim \left( \int_0^1 g(s) \frac{ds}{s} \right)^{\frac{1}{q}}
\]

which is also the desired estimate.

It remains to estimate \( I_2 \). First, verify that Proposition 2.7 of \( H_t \) implies \( \| \Delta^m H_{t+1} f \|_p \lesssim \| f \|_p \). Then we obtain

\[
I_2 \lesssim \| f \|_p
\]

since \( \int_0^1 t^{q(m-\frac{\alpha}{2})} dt < +\infty. \)

**Proposition 4.3.** For all integers \( \beta \geq \gamma > \frac{\alpha}{2} \),

\[
\left( \int_0^1 \left( t^{\beta - \frac{\alpha}{2}} \| \Delta^\beta H_t f \|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \left( \int_0^1 \left( t^{\gamma - \frac{\alpha}{2}} \| \Delta^\gamma H_t f \|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
\]

**Proof:** Proposition 2.7 implies \( \| \Delta^\beta H_t f \|_p \lesssim t^{\gamma - \gamma} \| f \|_p \). Then

\[
\left( \int_0^1 \left( t^{\beta - \frac{\alpha}{2}} \| \Delta^\beta H_t f \|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \left( \int_0^1 \left( t^{\gamma - \frac{\alpha}{2}} \| \Delta^\gamma H_t f \|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_0^1 \left( u^{\gamma - \frac{\alpha}{2}} \| \Delta^\gamma H_u f \|_p \right)^q \frac{du}{u} \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_0^1 \left( t^{\gamma - \frac{\alpha}{2}} \| \Delta^\gamma H_t f \|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
\]

**Remark 4.4.** Propositions 4.1, 4.2 and 4.3 imply (i) of Theorem 1.9.

**Proposition 4.5.** Let \( m > \frac{\alpha}{2} \). Then

\[
\| f \|_p + \left( \sum_{j \leq -1} \left[ 2^{j(m-\frac{\alpha}{2})} \| \Delta^m H_{2^j f} \|_p \right]^q \right)^{\frac{1}{q}}
\]

is an equivalent norm in \( B^p_q(G) \).

**Proof:** Assertion (i) in Theorem 1.9 and the following calculus prove the equivalence of norms:

\[
\left( \sum_{j \leq -1} \left[ 2^{j(m-\frac{\alpha}{2})} \| \Delta^m H_{2^j f} \|_p \right]^q \right)^{\frac{1}{q}} \lesssim \left( \sum_{j \leq -1} \int_{2^j}^{2^{j+1}} \left[ 2^{j(m-\frac{\alpha}{2})} \| \Delta^m H_{t f} \|_p \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_0^1 \left[ t^{m-\frac{\alpha}{2}} \| \Delta^m H_t f \|_p \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_0^1 \left[ t^{m-\frac{\alpha}{2}} \| \Delta^m H_t f \|_p \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
= \left( \sum_{j \leq -1} \int_{2^j}^{2^{j+1}} \left[ t^{m-\frac{\alpha}{2}} \| \Delta^m H_t f \|_p \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\leq \left( \sum_{j \leq -1} \left[ 2^{j(m-\frac{\alpha}{2})} \| \Delta^m H_{2^j f} \|_p \right]^q \right)^{\frac{1}{q}}.
\]

This proves item (ii) in Theorem 1.9.

\[\square\]
\textbf{Proposition 4.6.} Let \( \alpha > 0 \) and \( l > \frac{\alpha}{2} \). Then

\[
\| H^j f \|_p + \left( \sum_{j=-1}^{l} \left[ 2^{-j/2} \left\| \int_{2^j}^{2^{j+1}} (t\Delta)^j H_1 f \frac{dt}{t} \right\|_p \right]^q \right)^{\frac{1}{q}}
\]

is an equivalent norm in \( B^{\alpha,q}_p(G) \).

\textbf{Proof:} We denote by \( \| \cdot \|_{B^{\alpha,q}_p} \) the norm defined in (15). It is easy to check, using assertion (i) in Theorem 1.9 the Hölder inequality and the triangle inequality, that

\[
\| f \|_{B^{\alpha,q}_p} \lesssim \| f \|_{B^{\alpha,q}_p}.
\]

For the converse inequality, we proceed as follows. Fix an integer \( m > \frac{\alpha}{2} \).

1. **Decomposition of \( f \):**

   The first step is to decompose \( f \) as in Lemma 3.1

\[
f = \frac{1}{(l-1)!} \int_0^1 (t\Delta)^l H_1 f \frac{dt}{t} + \sum_{k=0}^{l-1} \frac{1}{k!} \Delta^k H_1 f \text{ in } S'(G).
\]

   We introduce

\[
f_n = - \int_{2^n}^{2^{n+1}} (t\Delta)^l H_1 f \frac{dt}{t} dt
\]

   and

\[c_n = \left\| \int_{2^n}^{2^{n+1}} (t\Delta)^l H_1 f \frac{dt}{t} \right\|_p .
\]

Remark then that

\[
f = \frac{1}{(l-1)!} \sum_{n=-\infty}^{\infty} f_n + \sum_{k=0}^{l-1} \frac{1}{k!} \Delta^k H_1 f \text{ in } S'(G).
\]

2. **Estimates of \( \Delta^m H^j f_n \)**

   Note that

\[
\Delta^m H^j f_n
\]

\[
= -\Delta^m H_{2n-1+2j} \int_{2^n}^{2^{n+1}} (t\Delta)^l H_{1-2n-1} f \frac{dt}{t} dt - \Delta^m H_{2n+2j} \int_{3\cdot2^n}^{2^{n+1}} (t\Delta)^l H_{1-2n} f \frac{dt}{t} dt
\]

Then Proposition 2.7 implies,

\[
\| \Delta^m H^j f_n \|_p \lesssim \left[ (2^{(n-1)} + 2^j)^{-m} \left\| \int_{2^n}^{2^{n+1}} (t + 2^{n-1}) \Delta^l H_1 f \frac{dt}{t} \right\|_p + [2^n + 2^j]^{-m} \left\| \int_{2^n}^{2^{n+1}} (t + 2^n) \Delta^l H_1 f \frac{dt}{t} \right\|_p \right.\]

\[
\left. \lesssim [2^n + 2^j]^{-m} \left\| \int_{2^n}^{2^n} (t\Delta)^l H_1 f \frac{dt}{t} \right\|_p .
\]

In other words,

\[
\| \Delta^m H^j f_n \|_p \lesssim \begin{cases} 2^{-jn} c_n & \text{if } j \leq n \quad (16) \\ 2^{-jm} c_n & \text{if } j > n \quad (17) \end{cases}
\]

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3. Estimate of $\Lambda^{p,q}_\alpha(\sum f_n)$

As a consequence,

$$\sum_{j \leq -1} \left[ 2^{j(m-\frac{q}{2})} \left\| \Delta^m H_{2j} \sum_{n \leq -1} f_n \right\|_p \right]^q \lesssim \sum_{j \leq -1} \left[ 2^{j(m-\frac{q}{2})} \sum_{n \leq -1} 2^{-m \max\{j,n\} c_n} \right]^q.$$ 

According to Lemma 2.2 since $0 < m - \frac{q}{2} < m$, one has

$$\sum_{j \leq -1} \left[ 2^{j(m-\frac{q}{2})} \left\| \Delta^m H_{2j} \sum_{n \leq -1} f_n \right\|_p \right]^q \lesssim \sum_{n = -\infty}^{-1} 2^{-n \frac{q}{2} c_n}.$$ 

4. Estimate of the remaining term

Remark that

$$\|f\|_{B^{p,q}_\alpha} \lesssim \|H_t f\|_p + \Lambda^{p,q}_\alpha\left( \sum f_n \right) + \sum_{k=0}^{l-1} \Lambda^{p,q}_\alpha(\Delta^k H_1 f).$$ 

From the previous step and Proposition 4.5, we proved that

$$\Lambda^{p,q}_\alpha\left( \sum f_n \right) \lesssim \|f\|_{B^{p,q}_\alpha}.$$ 

In order to conclude the proof of Proposition 4.6, it suffices then to check that for all $k \in [0, l - 1]$, one has

$$\|\Delta^k H_1 f\|_{B^{p,q}_\alpha} \lesssim \|f\|_{L^p}. \quad (18)$$

Indeed, one has for all $j \leq -1$

$$\|\Delta^m H_{2j} \Delta^k H_1 f\|_p = \|\Delta^{m+k} H_{1+2j} f\|_p \lesssim (1 + 2^j)^{(m+k)} \|f\|_p \lesssim \|f\|_p.$$ 

Consequently,

$$\sum_{j \leq -1} \left[ 2^{j(m-\frac{q}{2})} \|\Delta^m H_{2j} \Delta^k H_1 f\|_p \right]^q \lesssim \|f\|_p^q \sum_{j \leq -1} 2^{j(m-\frac{q}{2})} \lesssim \|f\|_p^q.$$ 

\[\Box\]

4.2 Proof of Theorem 1.13

Proof: (Theorem 1.13)

We denote by $\|\cdot\|_{B^{p,q}_{\alpha,X\sup}}$ the norm defined in (13). Since

$$\|\Delta^m H_{2j} f\|_p \leq \max_{t \in [2^j, 2^{j+1}]} \sup_{|I| \leq 2m} \|X_I H_t f\|_p,$$

it is easy to check that

$$\|f\|_{B^{p,q}_{\alpha,X\sup}} \lesssim \|f\|_{B^{p,q}_{\alpha}}.$$ 

For the converse inequality, it is enough to check that

$$\|f\|_{B^{p,q}_{\alpha}} \lesssim \|f\|_{B^{p,q}_{\alpha,X\sup}}.$$ 

We proceed then as the proof of Proposition 4.6 since Proposition 2.7 yields

$$\max_{t \in [2^j, 2^{j+1}]} \sup_{|I| \leq 2m} \|X_I H_t f\|_p \lesssim \begin{cases} 2^{-n \frac{q}{2} c_n} & \text{if } j \leq n \\ 2^{-j \frac{q}{2} c_n} & \text{if } j > n \end{cases}$$

with a proof analogous to the one of (17). \[\Box\]
4.3 Embeddings and interpolation

Proof: (of Corollary 1.11) The proof is analogous to the one of Proposition 2.3.2/2 in [17] using Proposition 4.5. It relies on the monotonicity of $l_q$ spaces, see [17] 1.2.2/4. □

Let us turn to interpolation properties of Besov spaces, that implies in particular Corollary 1.12.

Corollary 4.7. Let $s_0, s_1, s \geq 0$, $1 \leq p_0, p_1, p, q_0, q_1, r \leq \infty$ and $\theta \in (0,1)$.

Define

\[ s^* = (1 - \theta)s_0 + \theta s_1, \]
\[ \frac{1}{p^*} = \frac{1 - \theta}{p_0} + \theta \frac{1}{p_1}, \]
\[ \frac{1}{q^*} = \frac{1 - \theta}{q_0} + \theta \frac{1}{q_1}. \]

i. If $s_0 \neq s_1$ then

\[ (B^{p_0, q_0}_{s_0}, B^{p_1, q_1}_{s_1})_{\theta, r} = B^{p^*, q^*}_{s^*}. \]

ii. In the case where $s_0 = s_1$, we have

\[ (B^{p_0, q_0}_{s}, B^{p_1, q_1}_{s})_{\theta, r} = B^{p^*, q^*}_{s}. \]

iii. If $p^* = q^* := r$,

\[ (B^{p_0, q_0}_{s}, B^{p_1, q_1}_{s})_{\theta, r} = B^{p^*, q^*}_{s}. \]

iv. If $s_0 \neq s_1$,

\[ (B^{p_0, q_0}_{s_0}, B^{p_1, q_1}_{s_1})_{[r]} = B^{p^*, q^*}_{s}. \]

Proof: The proof is inspired by [2, Theorem 6.4.3].

Recall (see Definition 6.4.1 in [2]) that a space $B$ is called a retract of $A$ if there exists two bounded linear operators $J : B \to A$ and $P : A \to B$ such that $P \circ J$ is the identity on $B$.

Therefore, we just need to prove that the spaces $B^{p^*, q^*}_{s^*}$ are retracts of $l^p_q(L^p)$ where, for any Banach space $A$ (see paragraph 5.6 in [2]),

\[ l^p_q(A) = \left\{ u \in A^{\mathbb{Z}^-}, \|u\|_{l^p_q(A)} := \left( \sum_{j \leq 0} [2^{-j \frac{q}{p}} \|u_j\|_A]^q \right)^{\frac{1}{q}} < +\infty \right\}. \]

Then interpolation on the spaces $l^p_q(L^p)$ (see [2], Theorems 5.6.1, 5.6.2 and 5.6.3) provides the result. Note the weight appearing $l^q_q(A)$ is $2^{-j \frac{q}{p}}$ (and not $2^j \frac{q}{p}$) because we sum on negative integers.

Fix $m > \frac{q}{2}$. Define the functional $J$ by $Jf = ((Jf)_j)_{j \leq 0}$ where

\[ (Jf)_j = 2^{jm} \Delta^m H_{2^j - 1} f \]

if $j \leq -1$ and

\[ (Jf)_0 = H_{\frac{1}{2}} f. \]

Moreover, define $P$ on $l^p_q(L^p)$ by

\[ Pu = \sum_{k=0}^{2m-1} \frac{1}{k!} \triangle^k H_{\frac{1}{2}} u_0 + \frac{1}{(2m-1)!} \sum_{j \leq -1} 2^{-jm} \int_{2^{j+1}}^{2^{j+1}} 2^{2m} \Delta^m H_{2^{j+1} - 2^j - 1} u_j \frac{dt}{t}. \]

We will see below that $P$ is well-defined on $l^p_q(L^p)$. Proposition 4.3 implies immediately that $J$ is bounded from $B^{p^*, q^*}_{s^*}$ to $l^p_q(L^p)$. Moreover, Lemma 3.1 easily provides that

\[ P \circ J = \text{Id}_{B^{p^*, q^*}_{s^*}}. \]
It remains to verify that $\mathcal{P}$ is a bounded linear operator from $l^p_n(L^p)$ to $B^p,q$. The proof is similar to the one of Proposition 1.13. Indeed, proceeding as the fourth step of Proposition 1.13 one gets

$$\left\| \sum_{k=0}^{2m-1} \frac{1}{k!} \Delta^k u_0 \right\|_{B^p,q} \lesssim \|u_0\|_p.$$  

It is plain to see that

$$\left\| H_+ \sum_{j \leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \Delta^m H_{t-2^{j-1}u_j} \frac{dt}{t} \right\|_p \leq \sum_{j \leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} \left| t^{2m} \Delta^m H_{t-2^{j-1}u_j} \right| \frac{dt}{t} \lesssim \sum_{j \leq -1} 2^{jm} \left\| u_j \right\|_p \lesssim \|u\|_{l^p_q(L^p)}.$$  

Then the proof of the boundedness of $\mathcal{P}$ is reduced to the one of

$$I := \left( \sum_{k \leq -1} \left( 2^{k(m-\frac{m}{2})} \sum_{j \leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} (t\Delta)^{2m} H_{t-2^{j-1}+2^k u_j} \frac{dt}{t} \right) \right)^{\frac{q}{p}} \lesssim \|u\|_{l^p_q(L^p)}. \quad (19)$$  

Indeed,

$$I^q \lesssim \sum_{k \leq -1} \left( 2^{k(m-\frac{m}{2})} \sum_{j \leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} (t\Delta)^{2m} H_{t-2^{j-1}+2^k u_j} \frac{dt}{t} \right)^q \lesssim \sum_{k \leq -1} \left( 2^{k(m-\frac{m}{2})} \sum_{j \leq -1} 2^{jm} \left\| \Delta^m H_{2^{j-1}u_j} \right\|_p \right)^q \lesssim \sum_{k \leq -1} \left( 2^{k(m-\frac{m}{2})} \sum_{j \leq -1} 2^{jm} \left\| u_j \right\|_p \right)^q \lesssim \sum_{k \leq -1} \left( 2^{k(m-\frac{m}{2})} \sum_{j \leq -1} 2^{-2m(j+k)} \left\| u_j \right\|_p \right)^q.$$  

Check that $0 < m - \frac{m}{2} < 2m$. Thus, Lemma 2.2 yields

$$I^q \lesssim \sum_{j \leq -1} \left[ 2^{-j(\frac{m}{2}+m)} 2^{jm} \left\| u_j \right\|_p \right]^q \lesssim \|u\|_{l^p_q(L^p)}^q,$$

which proves (19) and thus concludes the proof.

5 Algebra under pointwise product - Theorem 1.14

We want to introduce some paraproducts. The idea of paraproducts goes back to \cite{[4]}. The term “paraproducts” is used to denote some non-commutative bilinear forms $\Lambda_i$ such that $fg = \sum \Lambda_i(f,g)$. They are introduced in some cases, where the bilinear forms $\Lambda_i$ are easier to handle than the pointwise product.

In the context of doubling spaces, a definition of paraproducts is given in \cite{[3],[8]}. We need to slightly modify the definition in \cite{[3]} to adapt them to non-doubling spaces.
For all $t > 0$, define
\[
\phi_t(\Delta) = -\sum_{k=0}^{m-1} \frac{1}{k!} (\Delta)^k H_t,
\]
and observe that the derivative of $t \mapsto \phi_t(\Delta)$ is given by
\[
\phi'_t(\Delta) = \frac{1}{(m-1)!} (\Delta)^m H_t := \frac{1}{t} \phi_t(\Delta).
\]

**Remark 5.1.** Even if $\phi_t$ actually depends on $m$, we do not indicate this dependence explicitly.

Recall that Lemma 5.1 provides the identity
\[
f = \int_0^1 \psi_t(\Delta) f \frac{dt}{t} - \phi_1(\Delta) f \quad \text{in } S'(G).
\]  

**Proposition 5.2.** Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} := \frac{1}{r} + \frac{1}{q} \leq 1$. Let $(f, g) \in L^p(G) \times L^q(G)$. One has the formula
\[
fg = \Pi_f(g) + \Pi_g(f) + \phi_1(\Delta) [\phi_1(\Delta)f \cdot \phi_1(\Delta)g] \quad \text{in } S'(G),
\]

where
\[
\Pi_f(g) = \int_0^1 \phi_t(\Delta) \left[ \int_0^1 \psi_t(\Delta)\frac{dt}{t} - \phi_t(\Delta) \right] = \frac{1}{t} \phi_t(\Delta) f \quad \text{in } S'(G),
\]
and
\[
\Pi(f, g) = \int_0^1 \psi_t(\Delta) \left[ \phi_t(\Delta) f \cdot \phi_t(\Delta) g \right] \frac{dt}{t}.
\]

**Proof:** Since $fg \in L^r \subset S'(G)$, the formula (20) provides in $S'(G)$
\[
[f \cdot g] = \int_0^1 \psi_t(\Delta) [f \cdot g] \frac{dt}{t} - \phi_1(\Delta) [f \cdot g].
\]

We can use again twice (one for $f$ and one for $g$) the identity (20) to get
\[
[f \cdot g] = \int_0^1 \psi_t(\Delta) \left[ \left\{ \int_0^1 \psi_u(\Delta) f \left( \frac{du}{u} - \phi_1(\Delta) g \right) \right\} \frac{dt}{t} \right.
\]
\[
- \phi_1(\Delta) \left[ \left\{ \int_0^1 \psi_u(\Delta) f \left( \frac{du}{u} - \phi_1(\Delta) g \right) \right\} \frac{dt}{t} \right] \]
\[
= \int_0^1 \int_0^1 \psi_t(\Delta) \left[ \psi_u(\Delta) f \cdot \psi_v(\Delta) g \right] \frac{dt}{t} \frac{du}{u} \frac{dv}{v} - \int_0^1 \int_0^1 \psi_t(\Delta) \left[ \phi_1(\Delta) f \cdot \psi_v(\Delta) g \right] \frac{dt}{t} \frac{dv}{v} - \int_0^1 \int_0^1 \psi_t(\Delta) \left[ \psi_v(\Delta) f \cdot \phi_1(\Delta) g \right] \frac{dt}{t} \frac{du}{u}
\]
\[
- \int_0^1 \int_0^1 \phi_1(\Delta) \left[ \psi_u(\Delta) f \cdot \psi_v(\Delta) g \right] \frac{du}{uv} - \int_0^1 \int_0^1 \phi_1(\Delta) \left[ \phi_1(\Delta) f \cdot \psi_v(\Delta) g \right] \frac{du}{uv}
\]
\[
+ \int_0^1 \phi_1(\Delta) \left[ \phi_1(\Delta) f \cdot \phi_1(\Delta) g \right] \frac{du}{uv} + \int_0^1 \phi_1(\Delta) \left[ \phi_1(\Delta) f \cdot \phi_1(\Delta) g \right] \frac{du}{uv}
\]
\[
- \phi_1(\Delta) \left[ \phi_1(\Delta) \cdot \phi_1(\Delta) \right] \]
\[
:= R(f, g) + \int_0^1 \int_0^1 \int_0^1 \psi_t(\Delta) \left[ \psi_u(\Delta) f \cdot \psi_v(\Delta) g \right] \frac{dt}{t} \frac{du}{u} \frac{dv}{v} - \phi_1(\Delta) \left[ \phi_1(\Delta) \cdot \phi_1(\Delta) \right] .
\]

The domain $[0,1]^3$ can be divided in the subsets $D(t,u,v), D(u,t,v)$ and $D(v,u,t)$ where $D(a,b,c) = \{(a,b,c) \in$
Proof: Let $\alpha > 0$ and $p,p_1,p_2,q \in [1, +\infty]$ such that
\[
\frac{1}{p} + \frac{1}{p_2} = \frac{1}{p}.
\]
Then for all $f \in B^\alpha_{p_1,q}$ and all $g \in L^{p_2}$, one has
\[
\Lambda^p_{\alpha,q}[\Pi_f(g)] \lesssim \|f\|_{B^\alpha_{p_1,q}} \|g\|_{L^{p_2}}.
\]

Proof: Let $m > \frac{q}{p}$ and $j \leq -1$. Notice that, for all $u \in (0,1)$,
\[
\|\Delta^m H_u \Pi_f(g)\|_p \leq \int_0^1 \|\Delta^m H_u \phi(t)\|_p \frac{dt}{t}.
\]
Remind that
\[
\|\phi(t)\|_r \lesssim \|H^{\frac{r}{2}}_r\|_r
\]
for all $r \in [1, +\infty]$ and all $h \in L^r$. As a consequence,
\[
\|\Delta^m H_u \phi(t)\|_p = \|\phi(t)\Delta^m H_u \phi(t)\|_p \lesssim \left(\frac{t}{2} + u\right)^{-m} \|\phi(t)\|_p \lesssim \min \{t^{-m}, u^{-m}\} \|\phi(t)\|_p.
\]

We deduce then
\[
\Lambda^p_{\alpha,q}[\Pi_f(g)]^q \lesssim \|g\|_{L^{p_2}}^q \sum_{j=-1}^{\infty} \left(2^{j(m - \frac{q}{p})} \sum_{n=-\infty}^{-1} 2^{-m \max\{j,n\}} \|\Delta^m H_{2^n} f\|_{p_2} \right)^q.
\]
Proof: Notice first that
\[ \Lambda_{\alpha}^{p,q}[\Pi f(g)] \lesssim \|g\|_{L^{p_2}} \left( \sum_{n \leq -1} 2^{nq(m-\frac{m}{p_2})} \|\Delta^m H_{2^n f}\|_{p_1} \right)^{\frac{1}{q}} \]
where we used Lemma 2.2 for the last line. As a consequence, we obtain if
\[ \Lambda_{\alpha}^{p,q}[\Pi f(g)] \lesssim \|g\|_{L^{p_2}} \|f\|_{B^{p,q}_g} \]
where we used Proposition 4.5 for the last line. □

Proposition 5.4. Let G be a unimodular Lie group. Let \( \alpha > 0 \) and \( p, p_1, p_2, p_3, p_4, q \in [1, +\infty] \) such that
\[ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p} \]
Then for all \( f \in B^{p,q}_g \cap L^{p_3} \) and all \( g \in B^{p,q}_g \cap L^{p_2} \), one has
\[ \Lambda_{\alpha}^{p,q}[\Pi(f,g)] \lesssim \|f\|_{B^{p,q}_g} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B^{p,q}_g}. \]

Proof: Notice first that
\[ \|\Delta^m H_u \Pi(f,g)\|_p \leq \int_0^1 \|\Delta^m H_u H_t(\Delta)^m [\phi_t(\Delta)f \cdot \phi_t(\Delta)g]\|_p \frac{dt}{t}. \]
Let us recall then that \( X_i(f \cdot g) = f \cdot X_i g + X_i f \cdot g \). Consequently, since \( \Delta = \sum_{i=1}^k X_i^2 \), one has
\[ \|\Delta^m [f \cdot g]\|_p \lesssim \|\Delta^m f \cdot g\|_p + \|f \cdot \Delta^m g\|_p + \sum_{k=1}^{2m-1} \sup_{|I_1|=k, |I_2|=m-k} \|X_{I_1} f \cdot X_{I_2} g\|_p. \]
In the following computations, \((Y_{I_1}, Z_{I_2})\) denotes the couple \((X_{I_1}, X_{I_2})\) if \( |I_1| \neq 0 \) and \( |I_2| \neq 0 \), \((\Delta^{|I_1|/2}, I)\) if \( |I_2| = 0 \) and \((I, \Delta^{|I_2|/2})\) if \( |I_1| = 0 \). With these notations, one has
\[ \|\Delta^m H_{u+t}(\Delta)^m[\phi_t(\Delta)f \cdot \phi_t(\Delta)g]\|_p \]
\[ \lesssim \min \{ t^{-m}, u^{-m} \} \left\| (t\Delta)^m[\phi_t(\Delta)f \cdot \phi_t(\Delta)g] \right\|_p \]
\[ \lesssim \min \{ t^{-m}, u^{-m} \} \sum_{k=0}^{m-1} \left\| (t\Delta)^m[(t\Delta)^k H_t f \cdot (t\Delta)^l H_t g] \right\|_p \]
\[ \lesssim \min \{ t^{-m}, u^{-m} \} \sum_{k=0}^{m-1} 2^m \sup_{|I_1|=k, |I_2|=2m-k} \left\| Y_{I_1} (t\Delta)^k H_t f \cdot Z_{I_2} (t\Delta)^l H_t g \right\|_p \]
\[ = \min \{ t^{-m}, u^{-m} \} \sum_{k=0}^{m-1} 2^m \sum_{i=0}^{m+k+l} \sup_{|I_1|=i, |I_2|=2m-k} \left\| Y_{I_1} H_{t+k+l} f \cdot Z_{I_2} H_{t+l} g \right\|_p \]
\[ \lesssim \min \{ t^{-m}, u^{-m} \} \sum_{k=0}^{m-1} 2^m \sum_{i=0}^{m+k+l} t^{k+i} \sup_{|I_1|=i, |I_2|=l} \left\| Y_{I_1} H_t f \cdot Z_{I_2} H_t g \right\|_p \].
Setting \( c_n = \sum_{2m \leq k+l \leq 6m-4} \int_0^{2^{n+1}} \int_0^{t^{k+l}} l^{\frac{1}{k+l}} \sup_{|I_1|=k \ |I_2|=l} ||Y_{I_1} H_t f \cdot Z_{I_2} H_t g||_p \frac{dt}{l} \), one has

\[
A_{\alpha,q}^p \Gamma(f,g)^q \lesssim \int_0^1 \left( \int_0^1 \| \Delta^m H_{t+1} (t \Delta)^m [\phi_1(t) \Delta] f \cdot [\phi_1(t) \Delta] g \|_p \frac{dt}{u} \right)^q du \\lesssim \int_0^1 \left( \int_0^1 \min \left( t^{m}, t^{-m} \right) \sum_{2m \leq k+l \leq 6m-4} l^{\frac{k+l}{k+l}} \sup_{|I_1|=k \ |I_2|=l} ||Y_{I_1} H_t f \cdot Z_{I_2} H_t g||_p \frac{dt}{l} \right)^q du \\lesssim \sum_{n \leq -1} 2^{-nq} c_n \]

where the last line is a consequence of Lemma 2.2 since \( 0 < m - \frac{q}{2} < m \).

It remains to prove that for any couple \((k,l) \in \mathbb{N}^2\) satisfying \( 6m-4 \geq k+l \geq 2m \) and \( k+l \) even, we have

\[
T := \left( \sum_{n \leq -1} 2^{-nq} \left( \int_0^{2^{n+1}} l^{\frac{1}{k+l}} \sup_{|I_1|=k \ |I_2|=l} ||Y_{I_1} H_t f \cdot Z_{I_2} H_t g||_p \frac{dt}{l} \right)^q \right)^{\frac{1}{q}} \lesssim \| f \|_{B^p_{\infty,q}} \| g \|_{L^q_2} + \| f \|_{L^{q_2}} \| g \|_{B^p_{\infty,q}}.\]

1. If \( k = 0 \) or \( l = 0 \):

Since \( k \) and \( l \) play symmetric roles, we can assume without loss of generality that \( l = 0 \). In this case, \( k \) is even and if \( k = 2k' \),

\[
\sup_{|I_1|=k \ |I_2|=0} ||Y_{I_1} H_t f \cdot Z_{I_2} H_t g||_p = \| \Delta^{k'} H_t f \cdot H_t g \|_p \leq \| \Delta^{k'} H_t f \|_{p_1} \| H_t g \|_{p_2} \leq \| \Delta^{k'} H_t f \|_{p_1} \| g \|_{p_2}.\]

Therefore,

\[
T \leq \| g \|_{L^{q_2}} \left( \sum_{n \leq -1} 2^{-nq} \left( \int_0^{2^{n+1}} l^{k'} \| \Delta^{k'} H_t f \|_{p_1} \frac{dt}{l} \right)^q \right)^{\frac{1}{q}} \lesssim \| g \|_{L^{q_2}} \| f \|_{B^p_{\infty,q}}.\]

where the second line is due to the fact that \( k' \geq m > \frac{q}{2} \).

2. If \( k \geq 1 \) and \( l \geq 1 \):

Define \( \alpha_1, \alpha_2, r_1, r_2, q_1 \) and \( q_2 \) by

\[
\alpha_1 = \frac{k}{k+l} \alpha, \quad \alpha_2 = \frac{l}{k+l} \alpha,
\]

\[
\frac{k+l}{r_1} = \frac{k}{p_1} + \frac{l}{p_3}, \quad \frac{k+l}{r_2} = \frac{k}{p_2} + \frac{l}{p_4},
\]

\[
\frac{k+l}{q_1} = \frac{k}{q}, \quad \frac{k+l}{q_2} = \frac{l}{q}.\]

In this case, notice that \( k > \alpha_1 \) and \( l > \alpha_2 \). One has then

\[
\sup_{|I_1|=k \ |I_2|=l} ||X_{I_1} H_t f \cdot X_{I_2} H_t g||_p \leq \sup_{|I_1|=k} ||X_{I_1} H_t f||_{r_1} \sup_{|I_2|=l} ||X_{I_2} H_t g||_{r_2}\]

20
Proof: With the use of Propositions 5.2, 5.3 and 5.4, it remains to check that Lemma 6.1.

Let \( \theta = \frac{k}{\nu+1} \). Complex interpolation (Corollary 1.12) provides

\[
(B_0^{p_1, \infty}, B_\alpha^{p_1, q})[\theta] = B_\alpha^{r_1, q_1}
\]

and

\[
(B_\alpha^{p_1, q}, B_0^{p_1, \infty})[\theta] = B_\alpha^{r_2, q_2}.
\]

Remark also that \( L^s(G) \) is continuously embedded in \( B_0^{p_1, \infty}(G) \) (this can be easily seen from the definition of Besov spaces). As a consequence,

\[
T \lesssim \|f\|_{B_\alpha^{r_1, q_1}} \|g\|_{B_\alpha^{r_2, q_2}}
\]

\[
\lesssim \|f\|_{L^{p_1}} \|f\|_{B_\alpha^{p_1, q_1}} \|g\|_{B_\alpha^{p_1, q_1}} \|g\|_{L^{p_2}}
\]

\[
\lesssim \|f\|_{L^{p_1}} \|g\|_{B_\alpha^{p_1, q_1}} + \|f\|_{B_\alpha^{p_1, q_1}} \|g\|_{L^{p_2}}
\]

which is the desired conclusion.

Let us now prove Theorem 1.14

Proof: With the use of Propositions 5.2, 5.3 and 5.4, it remains to check that

\[
\|H_{\Delta} f \cdot g\|_{L^p} \lesssim \|f\|_{B_\alpha^{p_1, q_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|g\|_{B_\alpha^{p_1, q_1}}
\]  

(25)

and

\[
\|\phi_{\Delta} f \cdot \phi_{\Delta} g\|_{B_\alpha^{p_1, q_1}} \lesssim \|f\|_{B_\alpha^{p_1, q_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|g\|_{B_\alpha^{p_1, q_1}}.
\]  

(26)

The inequality (25) is easy to check. By Proposition 1.11 one has

\[
\|H_{\Delta} f \cdot g\|_{L^p} \leq \|f \cdot g\|_p \leq \|f\|_{p_1} \|g\|_{p_2} \leq \|f\|_{B_\alpha^{p_1, q_1}} \|g\|_{L^{p_2}}.
\]

For (26), recall that (13) implies

\[
\|\phi_{\Delta} f \cdot \phi_{\Delta} g\|_{B_{\alpha}^{p_1, q_1}} \lesssim \|\phi_{\Delta} f \cdot \phi_{\Delta} g\|_{L^p}
\]

\[
\lesssim \|\phi_{\Delta} f\|_{p_1} \|\phi_{\Delta} g\|_{p_2}
\]

\[
\lesssim \|f\|_{B_{\alpha}^{p_1, q_1}} \|g\|_{L^{p_2}}.
\]

\[\square\]

6 Other characterizations of Besov spaces

6.1 Characterization by differences of functions - Theorem 1.16

Lemma 6.1. Let \( p, q \in [1, +\infty] \) and \( \alpha > 0 \). There exists \( c > 0 \) such that, for all \( f \in L^p(G) \),

\[
\Lambda_{\alpha}^p (f) \lesssim \left( \int_{G} \left( \frac{\|\nabla y f\|_{p, \infty}}{|y|^{\alpha}} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.
\]
Proof: Since \( \int_G \frac{\partial}{\partial t} h(y) dy = 0 \),
\[
\frac{\partial H}{\partial t} f(x) = \int_G \frac{\partial}{\partial t} h(y) f(xy) dy
\]
\[
= \int_G \frac{\partial}{\partial t} h(y) (f(xy) - f(x)) dy
\]
\[
= \int_G \frac{\partial}{\partial t} h(y) \nabla_y f(x) dy.
\]
Consequently,
\[
\left\| \frac{\partial H}{\partial t} f \right\|_p \leq \int_G \left\| \frac{\partial h}{\partial t}(y) \right\| \nabla_y f \|_p dy.
\]
Proposition 2.5 provides
\[
\Lambda_\alpha^p(f) \lesssim \left( \int_0^1 \left( t^{1-\frac{3}{4}} \int_G \frac{1}{tV(\sqrt{t})} e^{-c' \frac{|y|^2}{t}} \| \nabla_y f \|_p dy \right)^{\frac{q}{4}} \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
\lesssim \left( \int_0^1 \left( t^{1-\frac{3}{4}} \int_G \frac{1}{tV(\sqrt{t})} e^{-c' \frac{|y|^2}{t}} \| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}} dy \right)^{\frac{q}{4}} \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
\lesssim \left( \int_0^1 \left( t^{1-\frac{3}{4}} \int_G \frac{1}{tV(\sqrt{t})} e^{-c' \frac{|y|^2}{t}} \| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}} dy \right)^{\frac{q}{4}} \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
= \left( \int_0^1 \left( \int_G K(t,y) g(y) \frac{dy}{V(|y|)} \right)^{\frac{q}{4}} \frac{dt}{t} \right)^{\frac{1}{q}}
\]
with \( c' = \frac{c}{2} \), \( g(y) = \frac{\| \nabla f \|_p e^{-c' \frac{|y|^2}{t}}} {V(|y|)} \) and \( K(t,y) = \frac{V(|y|)}{t^{1/2}} \) \( e^{-c' \frac{|y|^2}{t}} \) (note that we used the fact that \( t \in (0, 1) \) in the third line). Lemma 2.11 and Proposition 2.4 imply then
\[
\Lambda_\alpha^p(f) \lesssim \left( \int_G |g(y)|^q dy \right)^{\frac{1}{q}} \frac{dy}{V(|y|)}
\]
\[
= \left( \int_G \left( \frac{\| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}}} {V(|y|)} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.
\]
\[\square\]

**Proposition 6.2.** Let \( p, q \in [1, +\infty] \) and \( \alpha > 0 \), then
\[
\Lambda_\alpha^p(f) \lesssim L_\alpha^p(f) + \| f \|_p.
\]

**Proof:** According to Lemma 6.1 it is sufficient to check that
\[
\left( \int_G \left( \frac{\| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}}} {V(|y|)} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}} \lesssim L_\alpha^p(f) + \| f \|_p.
\]
Since we obviously have
\[
\left( \int_{|y| \leq 1} \left( \frac{\| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}}} {V(|y|)} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}} \leq L_\alpha^p(f),
\]
all we need to prove is
\[
T = \left( \int_{|y| \geq 1} \left( \frac{\| \nabla_y f \|_p e^{-c' \frac{|y|^2}{t}}} {V(|y|)} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}} \lesssim \| f \|_p.
\]
Indeed, $\|\nabla_y f\|_p \leq 2\|f\|_p$ and thus

$$T \lesssim \|f\|_p \left( \int_{|b| \geq 1} \left( e^{-c|b|^2} \right)^q dy \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_p \left( \sum_{j=0}^{\infty} e^{-cq^j V(2^{j+1})} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_p,$$

where the last line holds because $V(r)$ have at most exponential growth. □

**Proposition 6.3.** Let $p, q \in [1, +\infty]$ and $\alpha \in (0, 1)$. Then

$$L_{\alpha}^{p,q}(f) \lesssim \Lambda_{\alpha}^{p,q}(f) + \|f\|_p \quad \forall f \in B_{\alpha}^{p,q}(G).$$

**Proof:**

1. **Decomposition of $f$:**
   The first step is to decompose $f$ as
   $$f = (f - H_1 f) + H_1 f.$$
   We introduce
   $$f_n = - \int_{2^n}^{2^{n+1}} \frac{\partial H_1 f}{\partial t} dt = - \int_{2^n}^{2^{n+1}} \Delta H_1 f dt$$
   and
   $$c_n = \int_{2^n}^{2^{n+1}} \left\| \frac{\partial H_1 f}{\partial t} \right\|_p dt.$$
   Remark then that
   $$\|f_n\|_p \leq c_{n+1}$$
   and Lemma 3.1 provides
   $$f - H_1 f = \sum_{n=-\infty}^{-1} f_n \quad \text{in } S'(G).$$

2. **Estimate of $X_i f_n$:**
   Let us prove that if $n \leq -1$, one has for all $i \in [1, k]$
   $$\|X_i f_n\|_p \lesssim 2^{-\frac{n}{4}} c_n \quad (28)$$
   Indeed, notice first
   $$f_n = -2 \int_{2^n}^{2^{n+1}} \Delta H_2 f dt$$
   $$= -2H_{2^{n-1}} \int_{2^n}^{2^{n+1}} H_{t-2^{n-1}} \Delta H_1 f dt$$
   $$:= H_{2^{n-1}} g_n.$$ 
   Proposition 2.7 implies then
   $$\|X_i f_n\|_p \lesssim 2^{-\frac{n}{4}} \|g_n\|_p$$
   $$\lesssim 2^{-\frac{n}{4}} \int_{2^n}^{2^{n+1}} \left\| H_{t-2^{n-1}} \frac{\partial H_1 f}{\partial t} \right\|_p dt$$
   $$\lesssim 2^{-\frac{n}{4}} \int_{2^n}^{2^{n+1}} \left\| \frac{\partial H_1 f}{\partial t} \right\|_p dt$$
   $$= 2^{-\frac{n}{4}} c_n.$$ 

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If $\varphi : [0, 1] \to G$ is an admissible path linking $e$ to $y$ with $l(\varphi) \leq 2|y|$, 
\[
\nabla_y f_n(x) = \int_0^1 \frac{d}{ds} f_n(x \varphi(s)) ds 
= \int_0^1 \sum_{i=1}^k c_i(s) X_i f_n(x \varphi(s)) ds.
\]

Hence, (28) implies
\[
\|\nabla_y f_n\|_p \leq \int_0^1 \sum_{i=1}^k |c_i(s)||X_i f_n(x \varphi(s))| ds
= \|X_i f_n\|_p \int_0^1 \sum_{i=1}^k |c_i(s)| ds
\lesssim 2^{-\frac{n}{2}} c_n \int_0^1 \sum_{i=1}^k |c_i(s)| ds
\lesssim |y| 2^{-\frac{n}{2}} c_n
\]
where the second line is a consequence of the right-invariance of the measure and the last one follows from the definition of $l(\varphi)$. Thus, one has
\[
\|\nabla_y f_n\|_p \lesssim \begin{cases} 
|y| 2^{-\frac{n}{2}} c_n & \text{if } |y|^2 < 2^n 
\cr 
c_{n+1} & \text{if } |y|^2 \geq 2^n 
\end{cases} \tag{29}
\]

3. Estimate of $L^{p,q}_\alpha(f - H_1 f)$

As a consequence of (29),
\[
[L^{p,q}_\alpha(f - H_1 f)]^q = \sum_{j=-\infty}^{-1} \int_{2^j < |y|^2 \leq 2^{j+1}} \left( \frac{\|\nabla_y f\|_p}{|y|^{\alpha}} \right)^q \frac{dy}{V(|y|)} 
\lesssim \sum_{j=-\infty}^{-1} \int_{2^j < |y|^2 \leq 2^{j+1}} \left( \sum_{n=-\infty}^{-1} \frac{\|\nabla_y f_n\|_p}{|y|^{\alpha}} \right)^q \frac{dy}{V(|y|)} 
\lesssim \sum_{j=-\infty}^{-1} 2^{-\frac{n}{2} q} \left( \sum_{n=-\infty}^{j} c_{n+1} + \sum_{n=j+1}^{-1} 2^{-\frac{n}{2}} c_n \right)^q 
\lesssim \sum_{j=-\infty}^{-1} 2^{j q(1-\frac{q}{2})} \left( \sum_{n=-\infty}^{-1} 2^{-\frac{\alpha}{2} n} [c_{n+1} + c_n] \right)^q 
\lesssim \sum_{n=-\infty}^{-1} (2^{-\frac{\alpha}{2} \frac{n}{2}} [c_{n+1} + c_n])^q 
\lesssim \sum_{n=-\infty}^{0} (2^{-\frac{\alpha}{2} \frac{n}{2}} c_n)^q
\]
Note that the third line holds since $2^j \leq 1$, so that $V(2^{j+1}) \lesssim V(2^j)$ and the fifth one is obtained with Lemma 272 since $\alpha \in (0, 1)$. 

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However
\[
\sum_{n=-\infty}^{0} \left[ 2^{-n} \sum_{n=-\infty}^{0} \left[ 2^{-n} \int_{2^{n-1}}^{2^n} \left| \frac{\partial H_1 f}{\partial t} \right|_p \right]^q \right]^q \\
\leq \sum_{n=-\infty}^{0} 2^{-nq} 2^{n(q-1)} \int_{2^{n-1}}^{2^n} \left| \frac{\partial H_1 f}{\partial t} \right|_p^q \, dt \\
\leq \sum_{n=-\infty}^{0} \int_{2^{n-1}}^{2^n} \left( t^{-\frac{n}{2}} \left| \frac{\partial H_1 f}{\partial t} \right|_p \right)^q \, dt \\
= \int_{0}^{1} \left( t^{-\frac{n}{2}} \left| \frac{\partial H_1 f}{\partial t} \right|_p \right)^q \, dt \\
= (L^p_{\alpha,q}(f))^q.
\]

4. **Estimate of** $L^p_{\alpha,q}(H_1f)$

With computations similar to those of the second step of this proof, we find that
\[
\| \nabla_y H_1 f \|_p \lesssim |g| \| f \|_p.
\]
Consequently,
\[
L^p_{\alpha,q}(H_1f) \leq \| f \|_p \left( \int_{|y| \leq 1} |g|^{\frac{(1-\alpha)}{q}} \frac{dy}{V(|y|)} \right)^\frac{1}{q} \\
\leq \| f \|_p \left( \sum_{n \leq -1} \int_{2^n < |y| \leq 2^{n+1}} |g|^{\frac{(1-\alpha)}{q}} \frac{dy}{V(|y|)} \right)^\frac{1}{q} \\
\lesssim \| f \|_p \left( \sum_{n \leq -1} 2^q(1-\alpha) \right)^\frac{1}{q} \\
\lesssim \| f \|_p
\]
where the third line is a consequence of the local doubling property.

\[\square\]

**Theorem 6.4.** Let $G$ be a unimodular Lie group and $\alpha \in (0,1)$, then we have the following Leibniz rule. If $p_1, p_2, p_3, p_4, p, q \in [1, +\infty]$ are such that
\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}
\]
then for all $f \in B^{p,q}_{\alpha}(G) \cap L^p(G)$ and all $g \in B^{p,q}_{\alpha}(G) \cap L^p(G)$, one has
\[
\|fg\|_{B^{p,q}_{\alpha}} \lesssim \|f\|_{B^{p,q}_{\alpha}} \|g\|_{L^p} + \|f\|_{L^p} \|g\|_{B^{p,q}_{\alpha}}.
\]

**Proof:** Check that
\[
\nabla_y (f \cdot g)(x) = g(xy) \cdot \nabla_y f(x) + f(x) \cdot \nabla_y g(x).
\]
Thus, with Hölder inequality,
\[
\|fg\|_{B^{p,q}_{\alpha}} \approx \|f \cdot g\|_{p} + L^{p,q}_{\alpha}(f \cdot g) \\
\lesssim \|f\|_{p_1} \|g\|_{p_2} + L^{p,q}_{\alpha}(f) \cdot \|g\|_{L^p} + \|f\|_{L^p} \cdot L^{p,q}_{\alpha}(g) \\
\lesssim \|f\|_{B^{p,q}_{\alpha}} \|g\|_{p_2} + \|f\|_{L^p} \|g\|_{B^{p,q}_{\alpha}}.
\]
\[\square\]
6.2 Characterization by induction - Theorem 1.19

Proposition 6.5. Let $p, q \in [1, +\infty]$ and $\alpha > -1$. Let $m > \frac{q}{2}$. One has for all $i \in \mathbb{I}$,
\[ A^{p,q}_\alpha(X,f) \lesssim A^{p,q}_{\alpha+1}f + \|f\|_p = \|f\|_{B^{p,q}_\alpha}. \]

Proof: The scheme of the proof is similar to Proposition 4.6.

1. Decomposition of $f$:
   Let $M$ be an integer with $M > \frac{m+1}{2}$. We decompose $f$ as in Lemma 5.1,
   \[ f = \frac{1}{(M-1)!} \int_0^1 (t\Delta)^M H_t f \frac{dt}{t} + \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f \]
   and we introduce
   \[ f_n = -\int_{2^n}^{2^{n+1}} (t\Delta)^M H_t f \frac{dt}{t} \]
   and
   \[ c_n = \int_{2^{n-1}}^{2^n} t^M \|\Delta^M H_t f\|_p \frac{dt}{t}. \]
   Remark then that
   \[ \|f_n\|_p \leq c_{n+1} \]
   and
   \[ f = \frac{1}{(M-1)!} \sum_{n=-\infty}^{-1} f_n + \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f. \]

2. A first estimate of $\Delta^m H_t X_i f_n$:
   Let us prove that if $n \leq -1$, one has for all $i \in \mathbb{I}$,
   \[ \|\Delta^m X_i f_n\|_p \lesssim 2^{-n(m+\frac{1}{2})} c_n. \]
   Indeed, notice first
   \[ f_n = -2^M \int_{2^n}^{2^{n+1}} (t\Delta)^M H_{2^t} f \frac{dt}{t} \]
   \[ = -2^M H_{2^n-1} \int_{2^{n-1}}^{2^n} H_{t-2^{n-1}} (t\Delta)^M H_t f \frac{dt}{t} \]
   \[ := H_{2^{n-1}} g_n. \]
   Thus, since $\Delta = -\sum_{i=1}^\infty X_i^2$ can be written as a polynomial in the $X_i$'s, we obtain with the upper estimate of the heat kernel (Proposition 2.5),
   \[ \|\Delta^m X_i f_n\|_p \lesssim \left( \int_G \left| \int_G |\Delta^m X_i h_{2^n-1}(z^{-1}x)|g_n(z)dz \right|^p dx \right)^{\frac{1}{p}} \]
   \[ \lesssim 2^{-n(m+\frac{1}{2})} \left( \int_G \left| \int_G \exp \left( -c\frac{|z^{-1}x|^2}{2^n} \right) g_n(z)dz \right|^p dx \right)^{\frac{1}{p}} \]
   \[ \lesssim 2^{-n(m+\frac{1}{2})} \|g_n\|_p \]
   \[ \lesssim 2^{-n(m+\frac{1}{2})} c_n \]
   where the second line is due to the fact that $V(2^\frac{q}{2}) \lesssim V(2^{-m+\frac{1}{2}})$ and the last two lines are obtained by an argument analogous to the one for (28).

   \[ \|\Delta^m H_t X_i f_n\|_p \lesssim \left( \frac{1}{p} \right) \|X_i f_n\|_p \]
   \[ \lesssim \left( \frac{1}{p} \right) \|X_i f_n\|_p \]
   As a consequence, one has for all $t \in (0, 1]$,
   \[ \|\Delta^m H_t X_i f_n\|_p = \|H_t \Delta^m X_i f_n\|_p \lesssim 2^{-n(m+\frac{1}{2})} c_n, \]
   since $H_t$ is uniformly bounded.
3. A second estimate of $\Delta^m H_t X_i f_n$:

Let us prove that for all $f \in L^p(G)$ and for all $i \in [1, k]$, one has

$$\|\Delta^m H_t X_i f\|_p \lesssim t^{-m-\frac{1}{2}} \|f\|_p. \tag{32}$$

First, notice that

$$\Delta^m H_t X_i f(x) = \int_G \frac{\partial^m}{\partial t^m} h_t(y)(X_i f)(xy) dy$$

$$= \int_G \frac{\partial^m}{\partial t^m} h_t(y)[X_i f(x)](y) dy$$

$$= - \int G X_i \frac{\partial^m}{\partial t^m} h_t(y) f(xy) dy$$

$$= - \int G X_i \Delta^m h_t(x^{-1} y) f(y) dy.$$

Then, using the estimates on the heat kernel (Proposition 2.5) and the fact that $\Delta = -\sum X_i^2$, we obtain

$$\|\Delta^m H_t X_i f\|_p \lesssim \left( \int_G \left| \frac{t^{-m-\frac{1}{2}}}{V(\sqrt{t})} \int G \exp \left( -c \frac{|x^{-1} y|^2}{t} \right) |f(y)| dy \right|^p dx \right)^{\frac{1}{p}}$$

$$:= t^{-m-\frac{1}{2}} \left( \int G \left| \int G K(x, y) |f(y)| dy \right|^p dx \right)^{\frac{1}{p}}$$

with $K(x, y) = \frac{1}{V(\sqrt{t})} \exp \left( -c \frac{|x^{-1} y|^2}{t} \right)$. Proposition 2.3 yields the estimate (32).

4. Estimate of $A_n^{p, q} (\sum f_n)$

The two previous steps imply

$$\|\Delta^m H_t X_i f_n\|_p \lesssim \begin{cases} 2^{-n(m+\frac{1}{2})} c_n & \text{if } t < 2^n \\ t^{-m-\frac{1}{2}} c_{n+1} & \text{if } t \geq 2^n \end{cases}.$$

As a consequence,

$$\int_0^1 \left( t^{-m-\frac{1}{2}} \|\Delta^m H_t X_i \sum_{n=-\infty}^{-1} f_n\|_p \right)^q \frac{dt}{t} \lesssim \sum_{j=-\infty}^{-1} \int_{2^j t \leq 2^{j+1}} \left( t^{-m-\frac{1}{2}} \sum_{n=-\infty}^{-1} \|\Delta^m H_t X_i f_n\|_p \right)^q \frac{dt}{t}$$

$$\lesssim \sum_{j=-\infty}^{-1} \left( 2^{j(m-\frac{1}{2})} \sum_{n=-\infty}^{-1} 2^{-j(m+\frac{1}{2})} c_{n+1} + \sum_{n=j+1}^{-1} 2^{-n(m+\frac{1}{2})} c_n \right)^q$$

$$\lesssim \sum_{j=-\infty}^{-1} \left( 2^{j(m-\frac{1}{2})} \sum_{n=-\infty}^{-1} 2^{-\max(j, n)(m+\frac{1}{2})} c_{n+1} \right)^q$$

$$\lesssim \sum_{n=-\infty}^{-1} \left[ 2^{-n(m+\frac{1}{2})} c_n + c_{n+1} \right]^q$$

$$\lesssim \sum_{n=-\infty}^{0} \left[ 2^{-n(m+\frac{1}{2})} c_n \right]^q$$

where we used Lemma 2.2 for the fourth estimate, relevant since $-1 < \frac{m}{2} < m$ by assumption. We get then the domination

$$\int_0^1 \left( t^{-m-\frac{1}{2}} \|\Delta^m H_t X_i \sum_{n} f_n\|_p \right)^q \frac{dt}{t} \lesssim \sum_{n=-\infty}^{0} \left[ 2^{-n(m+\frac{1}{2})} c_n \right]^q. \tag{33}$$
However computations analogous to those leading to (30) prove that
\[
\sum_{n=-\infty}^{0} \left[ 2^{-n^{\alpha+1}} c_n \right]^q \lesssim \int_0^1 \left( t^{M-\alpha+1} \| \Delta^M f \|_p \right)^q \frac{dt}{t} \lesssim (\Lambda_{\alpha+1}^p f)^q.
\]

5. **Estimate of the remaining term.**

Recall that
\[
f = \frac{1}{(M-1)!} \sum_{n=-\infty}^{-1} f_n + \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f.
\]
We already estimated \( \Lambda_{\alpha}^p(\sum f_n) \). What remains to be estimated is \( \Lambda_{\alpha}^p(\sum 1 \Delta^k H_1 X_i f) \).
Proposition 2.7 provides as well
\[
\| \Delta^m H_1 X_i \Delta^k H_1 f \|_p \lesssim \| \Delta^m X_i \Delta^k H_1 f \|_p \lesssim \| f \|_p.
\]
As a consequence, we get,
\[
\int_0^1 \left( t^{m-\alpha} \left\| \Delta^m H_1 X_i \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f \right\|_p \right)^q \frac{dt}{t} \lesssim \left( \int_0^1 t^{(m-\alpha)q} \frac{dt}{t} \right)^q \lesssim \| f \|_p^q.
\]

\[\square\]

**Corollary 6.6.** Let \( p, q \in [1, +\infty] \) and \( \alpha > 0 \).
\[
\| f \|_{B_{\alpha+1}^{p,q}} \simeq \| f \|_{L^p} + \sum_{i=1}^{k} \| X_i f \|_{B_{\alpha}^{p,q}}.
\]

**Proof:** The main work was done in the previous proposition. Indeed, notice that Proposition 6.3 implies
\[
\Lambda_{\alpha+1}^p f = \Lambda_{\alpha-1}^p f(X_i f)
\]
\[
\lesssim \sum_{i=1}^{k} \Lambda_{\alpha}^p f(X_i f)
\]
\[
\lesssim \sum_{i=1}^{k} \| X_i f \|_{B_{\alpha}^{p,q}},
\]
which provides the domination of the first term by the second one.

The converse inequality splits into two parts. The first one is the domination of \( \Lambda_{\alpha}^p f(X_i f) \) by \( \| f \|_{B_{\alpha+1}^{p,q}} \), which is an immediate application of Proposition 6.5. The second one is the domination of \( \| X_i f \|_p \). But recall that Theorem 1.10 states that we can replace \( \| X_i f \|_p \) by \( H_{\frac{1}{2}} X_i f \|_p \) in the Besov norm, and (32) provides that
\[
\| H_{\frac{1}{2}} X_i f \|_p \lesssim \| f \|_p \leq \| f \|_{B_{\alpha}^{p,q}}.
\]

\[\square\]

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