MAXIMUM WEIGHT SPECTRUM CODES

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Abstract. In the recent work [9], a combinatorial problem concerning linear codes over a finite field $\mathbb{F}_q$ was introduced. In that work the authors studied the weight set of an $[n, k]_q$ linear code, that is the set of non-zero distinct Hamming weights, showing that its cardinality is bounded above by $\frac{q^k - 1}{q - 1}$. They showed that this bound was sharp in the case $q = 2$, and in the case $k = 2$. They conjectured that the bound is sharp for every prime power $q$ and every positive integer $k$. In this work we quickly establish the truth of this conjecture. We provide two proofs, each employing different construction techniques. The first relies on the geometric view of linear codes as systems of projective points. The second approach is purely algebraic. We establish some lower bounds on the length of codes that satisfy the conjecture, and the length of the new codes constructed here are discussed.

1. Introduction

In 1973 Delsarte studied the number of distinct distances for a code $C$. In the linear case, this reduces to studying the number of distinct weights of the given code [3]. In that work he underlined the importance of this parameter, analyzing its connections with the number of distinct weights of the dual code, and the minimum distance of the code and the minimum distance of the dual. These four parameters are studied in order to obtain various results on the distance properties, and, in particular, they are used to calculate the weight distributions of cosets of a code.

Discussions on set of distinct weights of a code can be traced in [6], where the author skirmished with the following question. Given a set of positive integers, $S$, is it possible to construct a code whose set of non-zero weights is $S$? Partial solutions were presented, and necessary conditions were established.

In 2015, Haily and Harzalla [5] established the existence of binary codes in which no two distinct codewords have the same Hamming weight. They refer to these codes as distinct weight (DW) codes. More recently, Shi et. al [9] studied the more general combinatorial problem concerning the number of distinct weights of linear codes over any finite field $\mathbb{F}_q$. For a code of dimension $k$ over $\mathbb{F}_q$, they showed that

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the size of the weight set is bounded above by \( \theta_q(k - 1) = \frac{q^k - 1}{q - 1} \). They proved this bound to be sharp for binary codes (independent of the results of [5]), and for all \( q \)-ary codes of dimension \( k = 2 \). They conjectured that the bound is sharp for all \( q \) and \( k \). Codes meeting this bound are called maximum weight spectrum (MWS) codes. We remark that only in the binary case it is possible to have codes with the distinct weight property, as in [5]. During the review phase of the current work, Shi et al. released [8], a companion paper to [9], focusing discussions on cyclic codes.

In this work we first quickly establish the existence of MWS codes for all \( k, q \). We provide two different constructions of \( [n, k]_q \) MWS codes. In Section 2 we give a brief recap on linear and projective codes, and define the basic tools needed for our constructions. In Section 3 we give a short proof of the existence of MWS codes via a geometric construction. The construction is pleasingly simple, but provides codes of “large” length. A different approach is taken in Section 4. The construction presented there is inductive, for dimension \( k \geq 1 \), and relies on algebraic tools. In Section 5 we investigate lower bounds on the length of MWS codes. We provide a geometric construction of a new infinite family of “shorter” MWS codes, and we determine the asymptotic length of the codes arising from both our algebraic, and our geometric construction. Finally in Section 6 we summarize our work, and discuss some remaining questions.

2. Preliminaries

2.1. Linear Codes. Let \( q \) be a prime power and \( F_q \) denote the finite field with \( q \) elements. The Hamming distance, \( d_H(a, b) \), between two elements \( a, b \in F_q^n \) is the number of coordinates in which \( a \) and \( b \) differ. The Hamming distance induces a metric on \( F_q^n \). An \([n, k]_q\) code \( C \) is a \( k \)-dimensional subspace of \( F_q^n \) equipped with the Hamming distance, the elements of \( C \) are called codewords. The Hamming weight, \( w(c) \) of a codeword, \( c \) is its distance from the all zero codeword. A generator matrix \( G \) for an \([n, k]_q\) code \( C \) is a \( k \times n \) matrix over \( F_q \) whose row vectors generate \( C \). The minimum distance \( d \) of \( C \) is the quantity \( d = \min\{d_H(u, v) \mid u, v \in C, u \neq v\} \).

Throughout, an \([n, k]_q\) code \( C \) whose minimum distance is \( d \) shall be denoted an \([n, k, d]_q\) code. For an \([n, k]_q\) code \( C \) we define the weight set of \( C \) as

\[
w(C) = \{w(c) \mid c \in C \setminus \{0\}\}.
\]

An \([n, k]_q\) code \( C \) of dimension \( k \geq 2 \) is said to be non-degenerate if no coordinate position is identically zero. Unless specified otherwise, all codes discussed here are assumed to be non-degenerate.

Given two vectors \( a \in F_q^n \), \( b \in F_q^n \) we will use the notation \((a \mid b)\) to denote the vector in \( F_q^{n_1+n_2} \) obtained by concatenating \( a \) and \( b \), i.e.

\[
(a \mid b) = (a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}),
\]

where \( a = (a_1, \ldots, a_{n_1}) \) and \( b = (b_1, \ldots, b_{n_2}) \).

Definition 2.1. Let \( G \) be the subgroup of the group of linear automorphisms of \( F_q^n \) generated by the permutations of coordinates and by the multiplication of the \( i \)-th coordinate by elements in \( F_q^* \). Two codes \( C \) and \( C' \) are said to be equivalent if there exists \( \sigma \in G \) such that \( C' = \sigma(C) \).

Recall that there always exists \( \alpha \in F_q \) that is a primitive element, i.e. \( F_q \setminus \{0\} = F_q^* = \langle \alpha \rangle \). Throughout what follows, \( \alpha \) shall denote a primitive element of \( F_q \).
Definition 2.2. Let $\beta \in \mathbb{F}_q$ and $c \in \mathbb{F}_q^n$. We define the number
\[ c[\beta] = \{i \in \{1, \ldots, n\} \mid c_i = \beta\} , \]
and the entries distribution vector for $c$ as
\[ V(c) := (c[\alpha], c[\alpha^2], \ldots, c[\alpha^{q-1}], c[0]) \in \mathbb{N}^q. \]

Some basic properties concerning the entries distribution vector $V(c)$ are presented in the following. The proofs follow readily from the respective definitions.

Proposition 2.3. Let $c \in \mathbb{F}_q^n$, $\beta \in \mathbb{F}_q^n$ and let $e \in \mathbb{F}_q^n$ be the vector whose entries are all equal to 1. The following hold:
1. $V(\beta c) = (c[\beta], c[\beta^2], \ldots, c[\beta^{q-1}], c[0])$. In particular, since $\beta = \alpha^j$ for some $j$, then the vector consisting of the first $q-1$ entries of $V(\beta c)$ is the $j$-th shift of the vector formed by the first $q-1$ entries of $V(c)$.
2. $V(c+\beta e) = (c[\alpha-\beta], c[\alpha^2-\beta], \ldots, c[\alpha^{q-1}-\beta], c[-\beta])$ and therefore $V(c+\beta e)$ is a permutation of the vector $V(c)$.
3. If $c = (a \mid b)$, then for every $\beta \in \mathbb{F}_q$, $c[\beta] = a[\beta] + b[\beta]$, i.e. $V(c) = V(a)+V(b)$.

Definition 2.4. Given an $[n,k]_q$ code $C$ with generator matrix $G$ and $r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}^{q-1}$, we define the generalized $r$-repetition code of $C$ as the code $C(r)$ whose generator matrix is
\[ \begin{bmatrix} \alpha G \mid \ldots \mid \alpha G \mid \alpha^2 G \mid \ldots \mid \alpha^{q-1} G \end{bmatrix}, \]
i.e.
\[ C(r) = \{c^r := (ac \mid \ldots \mid ac \mid \alpha^2 c \mid \ldots \mid \alpha^{q-1} c) \mid c \in C\}. \]

The next result explains some properties of the code $C(r)$.

Proposition 2.5. Let $C(r)$ be the generalized $r$-repetition code of an $[n,k]_q$ code $C$ for a non-zero vector $r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}^{q-1}$. Let moreover $R := r_1 + \ldots + r_{q-1}$. The following hold:
1. $C(r)$ is an $[Rn,k]$ linear code over $\mathbb{F}_q$.
2. $|w(C)| = |w(C(r))|$. In particular, if $C$ is an MWS code, then also $C(r)$ is an MWS code.
3. For every $i = 1, \ldots, q-1$,
\[ c^r[\alpha^i] = \sum_{j=1}^{q-1} r_j c[\alpha^{i-j}]. \]

Proof. 1-2. Follow from the definition.
3. It is an easy calculation, that follows from part 1 and 3 of Proposition 2.3.

2.2. PROJECTIVE SYSTEMS. In this section we introduce the geometric view of linear codes, as detailed in [11] (or in [1] for codes that are equivalent to linear). We start with a short overview of the fundamentals of finite projective geometry. For a detailed introduction we refer to the recent book by Ball [2]. We let $PG(k,q)$ represent the finite projective geometry of dimension $k$ and order $q$. Due to a result of Veblen and Young [12], all finite projective spaces of dimension greater than two are isomorphic up to the order $q$. The space $PG(k,q)$ can be modelled
most easily with the vector space of dimension $k + 1$ over the finite field $\mathbb{F}_q$. In this model, the one-dimensional subspaces represent the points, two-dimensional subspaces represent lines, etc. Formally, we have

$$PG(k, q) := (\mathbb{F}_q^{k+1} \setminus \{0\}) / \sim,$$

where

$$u \sim v \text{ if and only if } u = \lambda v \text{ for some } \lambda \in \mathbb{F}_q.$$  

Using this model, it is not hard to show by elementary counting that the number of points of $PG(k, q)$ is given by

$$\theta_q(k) = \frac{q^{k+1} - 1}{q - 1}.$$  

A $d$-flat $\Pi$ in $PG(k, q)$ is a subspace isomorphic to $PG(d, q)$; if $d = k - 1$, the subspace $\Pi$ is called a hyperplane. Central to the geometric view of linear codes is the idea of a projective system.

**Definition 2.6.** A projective $[n, k, d]_q$-system is a finite (multi)set $\mathcal{M}$ of points of $PG(k - 1, q)$, not all of which lie in a hyperplane, where $n = |\mathcal{M}|$, and

$$d = n - \max\{|\mathcal{M} \cap H| \mid H \subset PG(k - 1, q), \dim(H) = k - 2\}.$$  

Note that the cardinalities above are counted with multiplicities in the case of a multiset. We denote by $m(P)$ the multiplicity of the point $P$ in $\mathcal{M}$.

Two projective $[n, k, d]_q$-systems $\mathcal{M}$ and $\mathcal{M}'$ are said to be equivalent if there exists a projective isomorphism of $PG(k - 1, q)$ mapping $\mathcal{M}$ to $\mathcal{M}'$.

Let $\mathcal{C}$ be an $[n, k]_q$ code with $k \times n$ generator matrix $G$. Note that multiplying any column of $G$ by a non-zero field element yields a generator matrix for a code which is equivalent to $\mathcal{C}$. Consider the (multi)set of one-dimensional subspaces of $\mathbb{F}_q^n$ spanned by the columns of $G$. In this way the columns may be considered as a (multi)set $\mathcal{M}$ of points of $PG(k - 1, q)$.

For any non-zero vector $v = (v_1, v_2, \ldots, v_k)$ in $\mathbb{F}_q^k$, it follows that the projective hyperplane

$$v_1x_1 + v_2x_2 + \cdots + v_kx_k = 0$$

contains $|\mathcal{M}| - w$ points of $\mathcal{M}$ if and only if the codeword $vG$ has weight $w$. It follows that linear (non-degenerate) $[n, k, d]_q$ codes and projective $[n, k, d]_q$ systems are equivalent objects. That is to say, there exists a linear $[n, k, d]_q$ code if and only if there exists a projective $[n, k, d]_q$ system.

### 2.3. Maximum Weight Spectrum Codes.

The weight set of a code has been studied in many contexts of coding theory, and for different purposes. In [6], using the relation between the weight distributions of a code and its dual, the author investigated necessary conditions for the existence of a linear binary code with a given weight set. One of the first to study the cardinality of the weight set of a code was Delsarte [3]. He demonstrated its importance in computing the weight distributions of cosets of a code. Other problems concerning the weight set and its cardinality can be found in [10, 4].

Recently in [9], Shi et. al. investigated the maximum cardinality of the weight set of a code, showing the following upper bound.

**Proposition 2.7.** [9, Proposition 2] If $\mathcal{C}$ is an $[n, k]_q$ code, then

$$|w(\mathcal{C})| \leq \theta_q(k - 1),$$
Motivated by Proposition 2.7, we define a new family of codes.

**Definition 2.8.** An \([n,k]_q\) code \(C\) such that \(|w(C)| = \theta_q(k-1)\) is called a *maximum weight spectrum (MWS) code.*

**Remark 2.9.** Observe that this definition is coherent with the existing literature. If \(C\) is an \([n,k]_q\) code, then the weight spectrum of \(C\) typically denotes the vector \(A(C) = (A_1, \ldots, A_n)\), where

\[
A_i = |\{ c \in C \mid w(c) = i \}|.
\]

In this framework, the cardinality of the weight set of \(C\) coincides with the Hamming weight of the vector \(A(C)\), and \(C\) is MWS if and only if the Hamming weight of \(A(C)\) is \(\theta_q(k-1)\). Since every non-zero entry of \(A(C)\) is a multiple of \(q-1\), \(\theta_q(k-1)\) is actually the maximum possible value for the Hamming weight of \(A(C)\).

In [9] the authors conjectured, motivated by experimental results, that for every \(q\) and \(k\), MWS codes exist. In the following section, we quickly establish the truth of this conjecture.

### 3. A geometric construction of MWS codes

In this section we are going to give a geometric construction of \([n,k]_q\) MWS codes for every prime power \(q\) and every \(k \geq 2\).

Given an \([n,k,d]_q\) code \(C\), we can consider the associated projective \([n,k,d]_q\)-system \(\mathcal{M}(C)\), whose points are given by the columns of the generator matrix.

**Definition 3.1.** Let \(\mathcal{M}\) be a multiset in \(\Pi = PG(k-1,q)\). We define the character function of \(\mathcal{M}\), denoted \(\text{Char} \mathcal{M}\), mapping the power set of \(\Pi\) to the non-negative integers:

\[
\text{Char} \mathcal{M}(A) = \sum_{P \in A} m(P).
\]

So \(\text{Char} \mathcal{M}(A)\) is the number, including multiplicity, of points in \(\mathcal{M} \cap A\). With a slight abuse of notation, we will write \(m(P) = \text{Char} \mathcal{M}(P)\), for any point \(P\).

The following follows directly from the definitions.

**Lemma 3.2.** Let \(C\) be an \([n,k]_q\) code over \(\mathbb{F}_q\), and let \(\mathcal{M} := \mathcal{M}(C)\) be its associated projective system. There exists a codeword of weight \(s\) in \(C\) if and only if there exists a hyperplane \(H\) in \(\Pi\) with \(\text{Char} \mathcal{M}(H) = n-s\).

A natural consequence of Lemma 3.2 is the following result on the existence of MWS codes.

**Lemma 3.3.** There exists an \([n,k]_q\) MWS code \(C\) if and only if there exists an \([n,k,d]_q\) projective system \(\mathcal{M}\) such that \(\text{Char} \mathcal{M}\) is injective.

We now provide a construction of a projective system as required in the Lemma 3.3. Let \(\Pi = PG(k-1,q), k \geq 2\) and let the points of \(\Pi\) be denoted \(P_0, P_1, \ldots, P_{\theta_q(k-1)-1}\). For each \(i\), include \(P_i\) in \(\mathcal{M}\) with multiplicity \(2^i\).

For \(k = 2\), \(\Pi\) is the projective line, so clearly no two points will have the same character. Consider \(k \geq 3\). Each hyperplane in \(\Pi\) is incident with precisely \(\theta_q(k-2)\) distinct points, and simple counting shows that every pair of distinct hyperplanes...
are incident with precisely $\theta_q(k-3)$ distinct points. For any particular hyperplane $H$, let us suppose that $H$ is incident with $P_{i_1}, P_{i_2}, \ldots, P_{i_{\theta_q(k-3)}}$. It follows that

$$\text{Char}_M(H) = \sum_{j=1}^{\theta_q(k-2)} 2^{i_j}.$$  

It follows that no two hyperplanes have the same character (consider the binary expansion of the respective characters). We have therefore proved the following result.

**Theorem 3.4.** For each prime power $q$ and $k \geq 2$, there exists an $[n, k]_q$ MWS code, where $n = 2^{\theta_q(k-1)} - 1$.

We note that the construction used in establishing the Theorem 3.4 involves codes of considerable length (asymptotically). A natural question is whether “short” MWS codes exist. We investigate this question in the sequel.

### 4. An algebraic construction of MWS codes

In this section we give a different construction of MWS codes that relies on algebraic properties of linear codes. This construction is inductive, where the inductive step is divided in two parts.

We now define two properties playing a central role in the construction.

**(A)** There exists $\beta \in \mathbb{F}_q^*$ such that, for $a, b \in C$, $a[\beta] = b[\beta]$ only if $a = b$.

**(B)** $V(c)$ has pairwise distinct entries for every $c \in C \setminus \{0\}$.

**Proposition 4.1.** Let $q$ be a prime power, and let $C$ be an $[n, k]_q$ MWS code.

1. If $C$ satisfies (A) then there exists an $[N, k+1]_q$ MWS code $\tilde{C}$, where $N = 2n + 1$.
2. Let $q \geq 3$. If $C$ satisfies both (A) and (B), then there exists an $[N, k+1]_q$ MWS code $\tilde{C}$ which satisfies property (B), where $N = (q-1)n + (q-2) + (T + 1)\frac{(q-2)(q-3)}{2}$ and $T = \max\{c[\beta] \mid c \in C \setminus \{0\}, \beta \in \mathbb{F}_q^*\}$.

**Proof.** Let $t = \frac{q-1}{q-1}$ and let $1 \leq w_1 < w_2 < \ldots < w_t \leq n$ be the distinct weights of the code $C$. For $N > 0$ consider the embedding

$$\phi : C \longrightarrow \mathbb{F}_q^N,$$

$$c \longmapsto (c \mid 0 \ldots 0).$$

For part 1 take $N = 2n + 1$ and let $\tilde{C}$ be the $[N, k+1]_q$ code generated by $\phi(C)$ and $e$, where $e$ is the vector whose entries are all equal to 1. For every $c \in \tilde{C}$ we have $w(\phi(c)) = w(c)$, and $w(\beta e - c) = N - c[\beta]$. Since $w(c) \leq n < N - n \leq N - c[\beta]$, property (A) gives part 1.

For part 2 take $N = (q-1)n + (q-2) + (T + 1)\frac{(q-2)(q-3)}{2}$, and take the $[N, k+1]_q$ code $\tilde{C}$ generated by $\phi(C)$ and $x$, where $x$ is the vector defined as

$$x = (1, \ldots, 1, \alpha, \ldots, \alpha, \alpha^2, \ldots, \alpha^2, \ldots, \alpha^{q-2}, \ldots, \alpha^{q-2})_{n \text{ times}}, \frac{n+1 \text{ times}}{n+1+T \text{ times}}, \ldots, \frac{n+1+(q-3)(T+1) \text{ times}}{n+1+T \text{ times}}.$$

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The proof that this code is MWS is analogous to part 1, so we shall show that (B) is satisfied. For \( c \in \mathcal{C} \),
\[
V(\phi(c)) = V(c) + (0, \ldots, 0, N - n).
\]
\( \mathcal{C} \) satisfies (b), and each entry in \( V(c) \) is strictly less than \( N - n \) \((q \geq 3)\). It follows that the entries of \( V(\phi(c)) \) are pairwise distinct. Now, for some \( \alpha^j \in F_q^* \), and \( c \in \mathcal{C} \) we consider the entries of \( V(z) \), where \( z = \alpha^j x + \phi(c) \). We notice that
\[
\phi(c) + \alpha^j x = (c + \alpha^j \epsilon | \alpha^{j+1}, \ldots, \alpha^{j+1} | \ldots | \alpha^{j-1}, \ldots, \alpha^{j-1}) \text{,}
\]
Therefore, by part 3 of Proposition 2.3, \( V(z) = V(c + \alpha^j \epsilon) + V(y) \), where
\[
y[\alpha^j] = y[0] = 0, y[\alpha^{j+i}] = n + 1 + (i - 1)(T + 1).
\]
By part 2 of Proposition 2.3, \( V(c + \alpha^j \epsilon) \) is a permutation of the vector \( V(c) \) and hence the entries are pairwise distinct. This gives \( z[\alpha^{j+i}] = c[\alpha^{j+i}] + n + 1 + (i - 1)(T + 1) \), for \( i = 1, \ldots, q - 2 \), while \( z[0] = c[-\alpha^j] \) and \( z[\alpha^j] = c[0] \). The result follows, since \( c[\alpha^{j+i}] < T + 1 \) for \( i = 1, \ldots, q - 2 \), and \( c[-\alpha^j] \neq c[0] \).

We wish to use Proposition 4.1 as a basis for an inductive construction of \([n, k]_q\) MWS codes. However, starting with an \([n, k]_q\) MWS code, Proposition 4.1 gives an \([N, k + 1]_q\) MWS code \( \tilde{C} \) that does not necessarily satisfy (A). In fact, for every \( j = 1, \ldots, q - 1 \), and for any \( 1 \neq \lambda \in F_q^* \), if we take \( z_1 = \alpha^j x + \phi(c) \) and \( z_2 = \alpha^j x + \phi(\lambda c) \) as in the proof of Proposition 4.1, then \( z_1 \neq z_2 \), and \( z_1[\alpha^j] = c[0] = (\lambda c)[0] = z_2[\alpha^j] \).

Starting from this code \( \tilde{C} \), we must construct another MWS code that satisfies (A). This can be done using the generalized \( r \)-repetition code of \( C \) with a suitable vector \( r \), as we will see in the following. We first need an auxiliary result.

**Lemma 4.2.** If \( H \subseteq \mathbb{Q}^m \) is a finite union of affine hyperplanes, then there exists a non-zero vector \( z = (z_1, \ldots, z_m) \in \mathbb{N}^m \) such that \( z \notin H \).

**Proof.** Let \( s \in \mathbb{N} \) and \( H = \bigcup_{j=1}^{s} H_j \), where
\[
H_j = \left\{ x \in \mathbb{Q}^m \left| \sum_{i=1}^{m} f^{(j)}_i x_i = 0 \right. \right\},
\]
with \( f^{(j)} \)'s not all zeros. We take a vector of the form \( v = (1, t, \ldots, t^{m-1}) \) and show that it cannot be in \( H \) for every \( t \in \mathbb{N} \). In fact, \( v \in H \) if and only if there exists an \( \ell \) such that \( f^{(j)}(t) := \sum_{i=1}^{m} f^{(j)}_i t^{i-1} = 0 \), if and only if
\[
F(t) := \prod_{j=1}^{s} f^{(j)}(t) = 0.
\]
Since \( F(T) \) is a non-zero polynomial in \( \mathbb{Q}[T] \) having degree at most \( s(m - 1) \), it cannot vanish on the whole \( \mathbb{N} \).
or, equivalently,
\[ \sum_{j=1}^{q-1} r_j (a[\alpha^{i-j}] - b[\alpha^{i-j}]) \neq 0. \]

This is equivalent to the condition
\[ r = (r_1, \ldots, r_{q-1}) \notin \bigcup_{a,b \in C} H^a_b, \]
where
\[ H^a_b := (a[\alpha^{i-1}] - b[\alpha^{i-1}], a[\alpha^i] - b[\alpha^i])^\perp. \]

**Remark 4.3.** Observe that here the choice of \( i \) does not really matter. In fact, if for all \( a, b \in C \) with \( a \neq b \) we have that \( a[\alpha^j] \neq b[\alpha^j] \) for some \( i \), then for any \( j = 1, \ldots, q-1 \) we have
\[ a[\alpha^j] = (\alpha^{i-j} a)[\alpha^i] \neq (\alpha^{i-j} b)[\alpha^i] = b[\alpha^i]. \]

The following lemma gives necessary and sufficient conditions for the existence of a vector \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}^{q-1} \) that satisfies (\( \ast \)).

**Lemma 4.4.** A vector \( r = (r_1, \ldots, r_{q-1}) \) that satisfies (\( \ast \)) exists if and only if for every \( a, b \in C \) with \( a \neq b \) we have
\[ V(a) \neq V(b). \]
In particular, if \( C \) is an \([n, k]_q\) MWS code that satisfies (B), then (\( \ast \)) has solution.

**Proof.** Suppose \( V(a) = V(b) \) for some \( a \neq b \). Then the vector \( (a[\alpha^{i-1}] - b[\alpha^{i-1}], \ldots, a[\alpha^i] - b[\alpha^i]) \) is the zero vector. This implies that
\[ H^a_b = (a[\alpha^{i-1}] - b[\alpha^{i-1}], a[\alpha^i] - b[\alpha^i])^\perp = \mathbb{Q}^{q-1} \]
and in (\( \ast \)), no such \( r \) exists.

On the other hand, if \( V(a) \neq V(b) \), then \( V(a) \) and \( V(b) \) differ in at least two entries. Hence there exists at least one \( 1 \leq i \leq q-1 \) such that \( a[\alpha^i] - b[\alpha^i] \neq 0 \), and the vector \( (a[\alpha^{i-1}] - b[\alpha^{i-1}], a[\alpha^i] - b[\alpha^i]) \) is non-zero. It follows that
\[ \bigcup_{a,b \in C \atop a \neq b} H^a_b \]
is a finite union of hyperplanes. By Lemma 4.2 there exists a solution for (\( \ast \)) in \( \mathbb{N}^{q-1} \).

For the second part, suppose \( C \) is an MWS code that satisfies (B), and let \( a, b \in C \) with \( V(a) = V(b) \). Since \( C \) is MWS, \( w(a) \neq w(b) \) whenever \( a \) is not of the form \( \lambda b \) for some \( \lambda \in \mathbb{F}_q^* \). Moreover, by part 1 of Proposition 2.3, the first \( q-1 \) entries \( V(\lambda b) \) are a shift of the first \( q-1 \) entries of \( V(b) \). Since \( V(b) \) has all distinct entries, this implies \( \lambda = 1 \), i.e. \( a = b \). This concludes the proof.

We wish to ensure that property (B) is preserved when we extend the code to the generalized \( r \)-repetition code \( \mathcal{C}(r) \). As such, we shall require for every \( c^r \in \mathcal{C}(r) \setminus \{0\} \), that \( c^r[\alpha^i] \neq c^r[\alpha^\ell] \) for every \( i < \ell \), and moreover \( c^r[\alpha^\ell] \neq c^r[0] \) i.e.
\[ \sum_{j=1}^{q-1} r_j (c[\alpha^{i-j}] - c[\alpha^{\ell-j}]) \neq 0 \quad \text{and} \quad \sum_{j=1}^{q-1} r_j (c[\alpha^{\ell-j}] - c[0]) \neq 0. \]
This condition can be reformulated as

\[(r_1, \ldots, r_{q-1}) \notin \bigcup_{i=0}^{q-1} \bigcup_{\ell=i+1}^{q-1} H^c_{i,\ell},\]

where

\[H^c_{i,\ell} := (c[\alpha^{i-1}] - c[\alpha^{\ell-1}], \ldots, c[\alpha^i] - c[\alpha^\ell])^\perp\]

for \(1 \leq i < \ell \leq q - 1\), and

\[H^c_{0,\ell} := (c[\alpha^{\ell-1}] - c[0], \ldots, c[\alpha^\ell] - c[0])^\perp.\]

Imposing this for every \(c \in C \setminus \{0\}\), we get

\[(\star\star) \quad r = (r_1, \ldots, r_{q-1}) \notin \bigcup_{c \in C \setminus \{0\}} \bigcup_{i=0}^{q-1} \bigcup_{\ell=i+1}^{q-1} H^c_{i,\ell}.\]

As we did for (\star), we now find conditions such that (\star\star) has solutions.

**Lemma 4.5.** If (B) holds, then there exists a vector \(r\) satisfying (\star\star).

*Proof.* If \(V(c)\) has all distinct elements it is clear that \(H^c_{i,\ell}\) is an hyperplane for every \(i < \ell\). Therefore, we conclude using Lemma 4.2. \(\square\)

The following theorem summarizes the properties of our generalized \(r\)-repetition code \(C(r)\), and is fundamental to our algebraic construction of MWS codes.

**Theorem 4.6.** If \(C\) is an \([n,k]_q\) MWS code that satisfies property (B), then there exists \(r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}^{q-1}\) such that \(C(r)\) is an \([Rn,k]_q\) MWS code that satisfies (A) and (B).

*Proof.* Consider \(r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}^{q-1}\) not identically zero such that both (\star) and (\star\star) are satisfied. Such a vector \(r\) exists, since we can always find, by Lemmas 4.2, 4.4 and 4.5, a non-zero vector in \(\mathbb{N}^{q-1}\) that is not in \(X_1 \cup X_2\), where

\[X_1 = \bigcup_{a,b \in C} H^{a,b}_{i,\ell}\] and \(X_2 = \bigcup_{c \in C \setminus \{0\}} \bigcup_{i=0}^{q-1} \bigcup_{\ell=i+1}^{q-1} H^c_{i,\ell}.\]

With this choice of \(r\) the code \(C(r)\) satisfies (A) and (B) by construction (also see Remark 4.3). Moreover, by part 1 and 2 of Proposition 2.5, it is an \([Rn,k]_q\) MWS code. \(\square\)

The following construction gives an alternative proof of the main conjecture.

**Construction 4.7.** Let \(q \geq 3\) be a prime power, and \(k > 2\) be a positive integer.

1. For \(\ell = 1\), we consider the \([\frac{q(q-1)}{2}]_q\) MWS code \(C\) generated by the codeword

\[c = (1, \alpha, \alpha^2, \alpha^2, \alpha^2, \ldots, \alpha^{q-2}, \ldots, \alpha^{q-2})^{q-1 \text{ times}}\]

where \(\alpha\) is a primitive element of \(\mathbb{F}^*_q\). It obviously satisfies (B).
2. Let $2 \leq \ell \leq k$ be a positive integer, and let $C_{\ell-1}$ be an $[n^{(\ell-1)}, \ell-1]_q$ MWS code satisfying (B). By Theorem 4.6, we take $r^{(\ell-1)} \in \mathbb{N}^{\ell-1}$ such that $C_{\ell-1}(r^{(\ell-1)})$ is an $[R^{(\ell-1)}]_q^{(\ell-1)}, \ell-1]_q$ MWS code satisfying properties (A) and (B), where $R^{(\ell-1)} = r^{(\ell-1)}_1 + \ldots + r^{(\ell-1)}_{q-1}$.

Let $C_{\ell-1}(r^{(\ell-1)})$ be an MWS code satisfying (A) and (B).

(i) If $\ell < k$, by part 2 of Proposition 4.1 we get an $[n^{(\ell)}, \ell]_q$ MWS code $C_\ell$ with property (B).

(ii) If $\ell = k$, by part 1 of Proposition 4.1 we get an $[n^{(k)}, k]_q$ MWS code $C_k$.

The following example demonstrates how Construction 4.7 may be applied.

**Example 4.8.** Here we see what happens in the easiest case that was not covered in [9], i.e. when $q = k = 3$. We consider the finite field $\mathbb{F}_3 = \{0, 1, 2\}$. The code $C_1 \subseteq \mathbb{F}_3^3$ is $\{(0, 0, 0), (1, 2, 2), (2, 1, 4), (3, 4, 0), (4, 3, 0), (5, 0, 2), (5, 0, 2), (6, 0, 1), (0, 6, 1)\}$, and the inductive construction gives a $[7, 2]_3$ code $C_2$ whose generator matrix is

$$G_2 = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ \end{pmatrix}.$$  

Here, we can first compute the vectors $V(c)$ that are

$$\{(1, 2, 4), (2, 1, 4), (3, 4, 0), (4, 3, 0), (5, 0, 2), (5, 0, 2), (6, 0, 1), (0, 6, 1)\},$$  

and we compute the hyperplanes $H_i^{a, b}$ that are the orthogonal complements of the vectors

$$\{(1, -1), (1, 1), (1, 3), (3, 1), (1, -3), (1, -2), (2, -1), (3, -1), (5, -2), (4, -1), (3, -4), (2, -3), (6, -5), (1, 0), (1, -4), (2, -5), (3, -2), (4, -3), (0, 1), (5 - 6)\}.$$  

Furthermore, the hyperplanes $H_i^{c, \ell}$ are the orthogonal complements of

$$\{(3, 2), (2, 3), (3, 4), (4, 3), (2, -3), (3, -2), (5, -1), (1, -5)\}.$$  

At this point we can choose $r^{(2)} = (1, 6)$ that is not contained in any of those hyperplanes, and consider the code $C_2(r^{(2)})$. Such a code is a $[49, 2]_3$ code over $\mathbb{F}_3$ whose generator matrix is given by

$$\begin{pmatrix} 2G_2 | G_2 | G_2 | G_2 | G_2 | G_2 | G_2 \end{pmatrix}.$$  

Here the vectors $V(c)$, for $0 \neq c \in C_2(r^{(2)})$, are given by

$$\{(8, 13, 28), (13, 8, 28), (22, 27, 0), (27, 22, 0), (5, 30, 14), (30, 5, 14), (36, 6, 7), (6, 36, 7)\}.$$  

The code $C_2(r^{(2)})$ satisfies (B). Moreover, $a[1] \neq b[1]$ for all $a, b \in C_2(r^{(2)})$ with $a \neq b$. Hence it also satisfies (A), and we may therefore use part 1 of Proposition 4.1 to construct a $[99, 3]_q$ code. According to the proof of Proposition 4.1, we first embed the code $C_2(r^{(2)})$ into $\mathbb{F}_3^{99}$, by concatenating every codeword with the 0 vector of length 50. We then add to the resulting code the codeword whose entries are all equal to 1, and consider the subspace generated. The obtained code $C_3$ will have 13 distinct non-zero weights, given by

$$w(C_3) = \{21, 49, 35, 42, 99, 91, 86, 77, 72, 94, 69, 63, 93\}.$$  

(i.e. $C_3$ is a $[99, 3]_3$ MWS code).
5. LENGTH OF MWS CODES

In this section we investigate lower bounds on the lengths of $[n,k]_q$ MWS codes. A trivial lower bound is of course given by

$$n \geq \theta_q(k-1).$$

In the case $q = 2$ this bound is sharp, in the sense that it is possible to construct a $[\theta_2(k-1),k]_2$ MWS code for each $k \geq 1$ [9, Theorem 1]. This lower bound is not optimal when $q \geq 3$. Indeed, we have the following result.

**Lemma 5.1.** If $C$ is an $[n,k]_q$ MWS code with $k \geq 2$, then

$$n \geq \left\lceil \frac{q \cdot \theta_q(k-1)}{2} \right\rceil = \left\lceil \frac{q^{k+1} - q}{2(q-1)} \right\rceil = \left\lceil \frac{1}{2} \left[ q^k + q^{k-1} + \cdots + q \right] \right\rceil.$$

**Proof.** Let $C$ be an $[n,k]_q$ MWS code with $k \geq 2$. Let the columns of a generator matrix correspond to the $[n,k,d]_q$ projective system $M$ in $\Pi = PG(k-1,q)$. Consider the set $S$ of incident point-hyperplane pairs $(P, \Lambda)$, where $P \in M$. Summing over all members of $M$ we obtain

$$(1) \quad |S| = \sum_{P \in M} \theta_q(k-2) = n \cdot \theta_q(k-2).$$

On the other hand, summing over all hyperplanes of $\Pi$ we obtain

$$(2) \quad |S| = \sum_{H \in \Pi} \text{Char}_M(H) \geq \sum_{i=0}^{\theta_q(k-1)-1} i = \theta_q(k-1)(\theta_q(k-1) - 1) \cdot \frac{\theta_q(k-1) \cdot q \cdot \theta_q(k-2)}{2},$$

where the inequality is a consequence of Lemma 3.3. The result follows from (1) and (2).

As we have seen in Theorem 3.4, there exist $[n,k]_q$ MWS codes of length $n = O(2^{q^{k-1}})$. We are therefore motivated to determine values of $n$ for which $[n,k]_q$ MWS codes exist with

$$\left\lceil \frac{q \cdot \theta_q(k-1)}{2} \right\rceil \leq n < 2^{\theta_q(k-1)} - 1.$$

**Corollary 5.2.** If $C$ is an $[n,k]_q$ MWS code satisfying property (A), then

$$n \geq \left\lceil \frac{q^{k+2} - 3q + 2}{4(q-1)} \right\rceil.$$

**Proof.** By Proposition 4.1, $C$ gives rise to an $[2n+1,k+1]_q$ MWS code $\tilde{C}$. The result follows by applying the bound in Lemma 5.1 to $\tilde{C}$. \qed

**Remark 5.3.** Observe that in the case $q = 2$, the bound (3) becomes

$$n \geq \left\lceil \frac{2^{k+2} - 6 + 2}{4} \right\rceil = 2^k - 1 = \theta_2(k-1),$$

which coincides with the bound in Lemma 5.1. For $q \geq 3$ the two bounds diverge, so the question remains as to whether the bound in Lemma 5.1 may be sharp for some $q \geq 3$. For $k = 2$ we have an answer.

**Proposition 5.4.** For $k = 2$, the bound in Lemma 5.1 is sharp for all prime powers $q$. 

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Proof. Denote the points of $\ell = PG(1, q) = \{P_0, P_1, \ldots, P_q\}$, and consider the projective system $\mathcal{M}$ in which every point $P_i$ appears with multiplicity $i$. The corresponding linear code is MWS and of length $n = \frac{q(q+1)}{2}$.

**Proposition 5.5.** There exists an $[7, 3]_2$ MWS code, and there exists an $[32, 3]_3$ MWS code.

*Proof.* For the first part, pick a triangle of points $P_0, P_1, P_2$ in the (Fano) plane $\Pi = PG(2, 2)$. Construct a projective system $\mathcal{M}$, whereby $m(P_i) = 2^i$. One easily verifies that the characters of the seven lines in $\Pi$ are $0, 1, 2, \ldots, 6$ respectively. Hence, the corresponding code is MWS.

For the second part, let $\Pi = PG(2, 3)$, and define the projective system $\mathcal{M}$ with point multiplicities as indicated in the diagram.

![Diagram](image)

**Figure 1.**

The characters of the lines of $\Pi$ are $1, 2, 4, 5, 6, 8, 10, 11, 12, 13, 16, 18$, and $22$ respectively.

Note that the existence of a $[7, 3]_2$-MWS code is also shown in [9].
Proposition 5.6. For each $q$, there exists an $[n,3]_q$ MWS code with $n \leq \frac{q-1}{2}(q^3 + q^2 + q)$.

Proof. For $q = 2, 3$, the result follows from Proposition 5.5, so assume $q > 3$. Let $\Pi = PG(2,q)$ and consider a set $T = \{P, Q, R\}$ of three non-collinear points, and the three lines that are their joins, $\ell_0 = \langle P, R \rangle$, $\ell_1 = \langle P, Q \rangle$, and $\ell_2 = \langle Q, R \rangle$. Let $\ell \setminus T = \{P_1, P_2, \ldots, P_{q-1}\}$, $\ell_1 \setminus T = \{Q_1, Q_2, \ldots, Q_{q-1}\}$, and $\ell_2 \setminus T = \{R_1, R_2, \ldots, R_{q-1}\}$.

Next, we assign multiplicities to points to construct the projective system $\mathcal{M}$.

Consider first the case that $q > 2$ is even. Let $m(P_i) = i$, $m(Q_i) = iq$, and $m(R_i) = iq^2$, $1 \leq i \leq q - 1$. We claim that no two lines of $\Pi$ have the same character. For each line, $\ell$, $\text{Char}_\mathcal{M}(\ell)$ may be written uniquely as $a + bq + cq^2 + dq^3$, where $0 \leq a, b, c, d \leq q - 1$. Moreover, $|\ell \cap T| = t$, where $0 \leq t \leq 2$. Consider the following cases.

$t=0$: In this case, $1 \leq a, b, c \leq q - 1$, and $d = 0$. Moreover, as two points uniquely determine a line, no two such lines agree in as many as two of $a, b, c$.

$t=1$: In this case, exactly one of $a, b, c$ are non-zero, and $d = 0$. That no two such lines have the same character follows from the fact that two points determine an unique line, and that no two (distinct) points in $\{\ell \cup \ell_1 \cup \ell_2\} \setminus T$ have the same multiplicity.

$t=2$: For $\ell_0$ we have $a = \frac{q}{2}$, $b = \frac{q-2}{2}$, $c = d = 0$; for $\ell_1$ we have $b = \frac{q}{2}$, $c = \frac{q-2}{2}$, $a = d = 0$; and for $\ell_2$ we have $c = \frac{q}{2}$, $d = \frac{q-2}{2}$, and $a = b = 0$.

Consider now the case that $q > 2$ is odd. In this case, we assign all multiplicities as in the case $q$ is even, with the following exceptions: $m(R_{q-1}) = 0$, and $m(P) = \frac{q+1}{2}q^2$.

For each line, $\ell$, $\text{Char}_\mathcal{M}(\ell)$ may be written uniquely as $a + bq + cq^2 + dq^3$, where $0 \leq a, b, c, d \leq q - 1$. Moreover, $|\ell \cap T| = t$, where $0 \leq t \leq 2$. Consider the following cases.

$t=0$: In this case, $1 \leq a, b \leq q - 1$, $0 \leq c \leq q - 1$, $c \neq \frac{q-1}{2}$, and $d = 0$. As two points uniquely determine a line, no two such lines agree in both $a$ and $b$.

$t=1$: If $P \in \ell$ then $a = b = 0$, $0 \leq c \leq q - 1$, and $0 \leq d \leq 1$. If $Q \in \ell$ then $b = c = 0$, $1 \leq a \leq q - 1$, and $d = 0$. If $R \in \ell$ then $a = c = 0$, $1 \leq b \leq q - 1$, and $d = 0$. No two such lines have the same character follows from the fact that two points determine an unique line, and that no two (distinct) points in $\{\ell_0 \cup \ell_1 \cup \ell_2\} \setminus T$ have the same multiplicity.
If \( \ell = \ell_0 \) then \( b = c = \frac{q-1}{2} \), and \( a = d = 0 \). If \( \ell = \ell_1 \) then \( a = b = d = 0 \), and \( c = q - 1 \). If \( \ell = \ell_2 \) then \( a = b = 0 \), \( c = \frac{q+1}{2} \), and \( d = \frac{q-3}{2} \).

Consequently, for \( q > 3 \) we arrive at an \([n,3]_q\) MWS code \( C \), where

\[
n = \sum_{j=0}^{2} \sum_{i=1}^{q-1} iq^j = \frac{q-1}{2} (q^3 + q^2 + q).
\]

\[\square\]

**Lemma 5.7.** If \( C \) is an \([n,k]_q\)-MWS code with projective system \( M \) in \( \Pi = PG(k-1,q) \), then \( M \) spans \( \Pi \).

**Proof.** The result clearly holds for \( k = 2 \), where \( M \) must contain at least \( q > 2 \) distinct points. Consider \( k > 2 \), and suppose \( \lambda \) is an hyperplane of \( \Pi \) with \( M \subset \Pi \).

Let \( \sigma \) be any \((k-2)\)-flat within \( \lambda \). Through \( \sigma \) there are \( q \) hyperplanes distinct from \( \lambda \), each of which has character \( \text{Char}_M(\sigma) \). Since \( q > 2 \) we have a contradiction.

\[\square\]

**Theorem 5.8.** Suppose there exists an \([n,k]_q\)-MWS code with \( n < q^t \). Then there exists an \([N,k+1]_q\)-MWS code with \( N < q^{t+k+1} \).

**Proof.** Let \( M \) be the projective system in \( \Pi = PG(k-1,q) \) associated with the \([n,k]_q\)-MWS code, \( C \). Let \( T = \{T_0, T_1, \ldots, T_{k-1}\} \) be a set of \( k \) points of \( \Pi \) that are in general position (\( T \) spans \( \Pi \)). Note that such a set may always be chosen.

Embed \( \Pi \) in \( \Sigma = PG(k+1,q) \), and distinguish a point \( P \in \Sigma^* = \Sigma \setminus \Pi \).

For each \( i \), \( 0 \leq i \leq k-1 \), let \( \ell_i = \langle P, T_i \rangle = \{P, T_i, Q_{i,0}, Q_{i,1}, \ldots, Q_{i,q-1}\} \). Define the projective system \( M' \) by letting \( \text{Char}_{M'}(R) = \text{Char}_M(R) \) for each point \( R \in \Pi \), and by letting \( \text{Char}_{M'}(Q_{i,j}) = jq^{t+1} \). We claim that \( M' \) is a projective system of an MWS code.

Suppose \( \lambda \) is an hyperplane, with \( \text{Char}_{M'}(\lambda) = \text{Char}_{M'}(\Pi) \). Each hyperplane must meet each \( \ell_i \) in at least one point. For each \( i, j \), we have \( m(Q_{i,j}) > n = \text{Char}_{M'}(\Pi) \). If \( \lambda \neq \Pi \), then it must be the case that \( \lambda \cap \ell_i = \{P\} \) for \( i = 0, 1, \ldots, k-1 \).
1, and therefore $\text{Char}_{\mathcal{M}'}(\lambda) = \text{Char}_{\mathcal{M}}(\lambda) = n$, contradicting Lemma 5.7. It follows that if $\lambda \neq \Pi$ is a hyperplane of $\Sigma$, then $\text{Char}_{\mathcal{M}'}(\lambda) \neq \text{Char}_{\mathcal{M}}(\Pi)$. It therefore suffices to consider hyperplanes distinct from $\Pi$.

Let $\lambda_1, \lambda_2$ be hyperplanes with $\text{Char}_{\mathcal{M}'}(\lambda_1) = a_0 + a_1 q + a_2 q^2 + \cdots + a_{k+1} q^{k+\ell}$, and $\text{Char}_{\mathcal{M}'}(\lambda_2) = b_0 + b_1 q + b_2 q^2 + \cdots + b_{t+1} q^{t+\ell}$. If $\lambda_1 \cap \Pi \neq \lambda_2 \cap \Pi$, then $(a_0, a_1, a_2, \ldots, a_{t-1}) \neq (b_0, b_1, b_2, \ldots, b_{t-1})$ (Lemma 5.6). If $\lambda_1 \cap \Pi \neq \lambda_2 \cap \Pi$, then $\text{Char}_{\mathcal{M}'}(\lambda_1) \neq \text{Char}_{\mathcal{M}'}(\lambda_2 \cap \Pi)$ since $C$ is MWS). It therefore suffices to consider the case that $\lambda_1 \cap \Pi = \lambda_2 \cap \Pi$.

Let $\sigma$ be a $(k - 2)$-flat in $\Pi$. Since the points of $T$ are in general position, there exists at least one $j$ with $T_j \notin \sigma$. A dimension argument shows the hyperplanes containing $\sigma$ must meet $\ell_j$ in mutually distinct points. By considering $\sigma = \lambda_1 \cap \lambda_2$, it follows that $b_j \neq a_j$, and whence $\text{Char}_{\mathcal{M}'}(\lambda_1) \neq \text{Char}_{\mathcal{M}'}(\lambda_2)$.

Therefore, $\mathcal{M}'$ is a projective system of an $[n, k+1]_q$-MWS code, where

$$N = n + \sum_{j=0}^{k-1} \sum_{i=1}^{q-1} i q^{t+j} = n + \frac{q-1}{2} [q^{t+1} + q^{t+2} + \cdots + q^{t+k}] < q^{t+k+1}.$$

$\square$

**Corollary 5.9.** For each $k \geq 3$, there exists an $[n, k]_q$-MWS code with $n < q^{4+\binom{k+1}{2} - 2}$.

**Proof.** For $k = 3$ the result follows from Proposition 5.6. From the Theorem 5.8, and Proposition 5.6 we have for $k > 3$ there exists an $[n, k]_q$-MWS code with

$$n < q^{4+\binom{k}{2} + \cdots + k} = q^{\frac{k(k+1)}{2} - 2}.$$

$\square$

**Remark 5.10.** The bound in Lemma 5.1 is rather optimistic in the following sense: The only way a code can achieve this bound is if there is an hyperplane of every character, from 0 to $\theta_q(k - 1) - 1$. For $k > 2$, constructions for codes meeting the bounds in Lemma 5.1, even asymptotically, seem quite elusive.

### 5.1. Length of the Codes Arising from the Algebraic Construction

In order to give a partial answer to the question above about the length of MWS codes, we will try to estimate the length $n^{(k)}$ of the code $C_k$ obtained using Construction 4.7. We will show that the algebraic construction provides codes that asymptotically are considerably smaller in length to those provided in Theorem 3.4. For this purpose we need to examine how the length increases after each of the two partial steps.

We will denote by $C_k$ the $k$-dimensional MWS code, by $n^{(k)}$ its length and by $r^{(k)} \in \mathbb{N}^{r-1}$ the vector that minimize the quantity $R^{(k)} = r_1^{(k)} + \cdots + r_{q-1}^{(k)}$, obtained in Theorem 4.6. We seek a recurrence relation or upper bound for $n^{(k)}$. Any such bound will certainly involve $R^{(k)}$ and $T^{(k)} = \max\{c'[\beta] | c' \in C_k(r^{(k)}) \setminus \{0\}, \beta \in F_q^*\}$. Since we wish to minimize the length of our MWS codes, at each step the vector $r^{(k)}$ is chosen as a vector that minimizes $R^{(k)}$ among all the vectors that satisfy both ($\ast$) and ($\ast\ast$).

Observe that we start with $C_1 = F_q$ of length $n^{(1)} = 1$, $r^{(1)} = (2, 3, \ldots, q - 1, 1)$, $R^{(1)} = \frac{q(q-1)}{2}$ and $C_1(r^{(1)})$ is the code generated by the codeword

$$c = (1, \alpha, \alpha, \alpha^2, \alpha^2, \alpha^2, \ldots, \alpha^{q-2}, \ldots, \alpha^{q-2}).$$
Proposition 5.11. The sequences of integers \( n^{(k)} \) and \( T^{(k)} \) satisfy the following recurrence relations.

\[
T^{(k+1)}(4) = R^{(k)} n^{(k)} + (T^{(k)} + 1) \sum_{j=0}^{q-2} r_{ij},
\]

where we have reordered the \( r_i \)'s such that \( r_{i_0} \leq r_{i_1} \leq \ldots \leq r_{i_{q-2}} \), and

\[
n^{(k+1)} = (q - 1) R^{(k)} n^{(k)} + 1 + \frac{(q - 3)(q - 2)}{2} (T^{(k)} + 1),
\]

with initial conditions \( n^{(1)} = 1 \) and \( T^{(1)} = q - 1 \).

Proof. By Theorem 4.6, \( C_k(r^{(k)}) \) has length \( R^{(k)} n^{(k)} \), and then by part 2 of Proposition 4.1 we obtain \( n^{(k+1)} = (q - 1) R^{(k)} n^{(k)} + 1 + \frac{(q - 3)(q - 2)}{2} (T^{(k)} + 1) \).

For the parameter \( T^{(k+1)} \) we consider what happens when we go from \( C_k(r^{(k)}) \) to \( C_{k+1}(r^{(k+1)}) \). Let \( T^{(k)} = \max \{ c^r \mid c^r \in C_k(r^{(k)}) \setminus \{0\}, r \in \mathbb{F}_q^* \} \). After applying the construction of part 2 of Proposition 4.1, we get a code \( C_{k+1} \) and now consider a codeword \( c \in C_{k+1} \). If we sort in increasing order the values \( c[\alpha^{f_j}] \), we get

\[
c[\alpha^{f_0}] < c[\alpha^{f_1}] < \ldots < c[\alpha^{f_{q-1}}],
\]

and it is easy to check that for every \( j = 1, \ldots, q - 1 \), we have that

\[
c[\alpha^{f_j}] \leq n + 1 + (j - 1)(T^{(k)} + 1) + T^{(k)}.
\]

At this point we construct the code \( C_{k+1}(r^{(k+1)}) \). In this code, by part 3 of Proposition 2.5, the entries of \( V(c^r) \) for \( c^r \in C_{k+1}(r^{(k+1)}) \) are given by

\[
c^r[\alpha^i] = \sum_{j=1}^{q-1} r_{ij} c[\alpha^{i-j}].
\]

Reordering the \( r_i \)'s such that \( r_{i_0} \leq r_{i_1} \leq \ldots \leq r_{i_{q-2}} \), we get

\[
c^r[\alpha^i] = \sum_{j=1}^{q-1} r_{ij} c[\alpha^{i-j}] \leq \sum_{j=1}^{q-1} r_{ij} c[\alpha^{f_j}].
\]

Combining this inequality with (6), we get the desired result. \( \square \)

Now it only remains to determine an upper bound for \( R^{(k)} \). One possible strategy to find an estimate on \( R^{(k)} \) is the following and relies on Schwartz-Zippel Lemma [7]. Let \( H \) be the union of \( D^{(k)} \) distinct linear hyperplanes obtained after the \( k \)-th step of the algebraic construction. A translation of the Schwartz Zippel Lemma [7, Corollary 1] states that

\[
| \{ x = (x_1, \ldots, x_{q-1}) \mid x \in H, x_i \in \{0, \ldots, m - 1\}, x \neq 0 \} | \leq D^{(k)}(m^{q-2} - 1).
\]

We can also find an upper bound on \( D^{(k)} \).

Lemma 5.12. Let \( D^{(k)} \) denote the number of distinct linear hyperplanes obtained at the \( k \)-th step of the iterative construction.

\[
D^{(k)} \leq \left( \frac{q^k}{2} \right) + (q^k - 1)(q - 1).
\]
Corollary 5.13. For $k \geq 2$, there exists an $r^{(k)} \in \mathbb{N}^{q-1}$ satisfying Theorem 4.6 such that

$$R^{(k)} \leq (q-1)\left(\frac{q^k}{2}\right) + (q^k - 1)(q-1)^2 = O(q^{2k+1}).$$

Proof. Equation (7) with $m = D^{(k)}$ implies that there exists a non-zero $r^{(k)} = (r_1^{(k)}, \ldots, r_{q-1}^{(k)}) \in \mathbb{N}^{q-1}$ that is not in the union of the $D^{(k)}$ hyperplanes, such that each of the $r_i^{(k)}$ belongs to \{0, 1, \ldots, D^{(k)} - 1\}. This implies that

$$R^{(k)} \leq (q-1)D^{(k)} \leq (q-1)\left(\frac{q^k}{2}\right) + (q^k - 1)(q-1)^2,$$

where the last inequality follows from Lemma 5.12. \qed

Now we estimate the asymptotic order of the parameters $n^{(k)}$ and $T^{(k)}$.

Proposition 5.14. For $k \geq 1$ we have

$$n^{(k+1)} = O(q^{(k+1)(k+2)-3}) \quad \text{and} \quad T^{(k+1)} = O(q^{k(k+3)}).$$

Proof. We prove it by induction. Recall that $n^{(1)} = 1$, $T^{(1)} = q - 1$ and $R^{(1)} = \frac{q(q-1)}{2}$. For $k = 1$, we can compute

$$T^{(2)} \leq R^{(1)}n^{(1)} + (T^{(1)} + 1)\sum_{j=0}^{q-2} jr_{ij} = O(q^2) + O(q^3) = O(q^4),$$

$$n^{(2)} = (q-1)R^{(1)}n^{(1)} + 1 + \frac{(q-3)(q-2)}{2}(T^{(1)} + 1) = O(q^3) + O(q^4) = O(q^5).$$

Suppose that the result holds for $n^{(k)}$ and $T^{(k)}$. By (4) we have

$$T^{(k+1)} \leq R^{(k)}n^{(k)} + (T^{(k)} + 1)\sum_{j=0}^{q-2} jr_{ij} = O(q^{2k+1+k(k+1)-3}) + O(q^{k(k+1)+2k+2}) = O(q^{k(k+3)-2}) + O(q^{k(k+3)}) = O(q^{k(k+3)}).$$

Moreover, by (5), it holds

$$n^{(k+1)} = (q-1)R^{(k)}n^{(k)} + 1 + \frac{(q-3)(q-2)}{2}(T^{(k)} + 1) = O(q^{2k+2+k(k+1)-3}) + O(q^{k(k+1)+2}) = O(q^{k+1}(k+2)-3).$$

\qed
At this point we can conclude our estimate of the length of MWS codes obtained via the algebraic construction.

**Theorem 5.15.** Using Construction 4.7, we can obtain an \([n,k]_q\) MWS codes with length
\[n = \mathcal{O}(q^{k(k+1)-4}).\]

**Proof.** Using Construction 4.7 we produce an \([n^{(k-1)},k-1]_q\) MWS code \(C_{k-1}\) satisfying (B), with \(n^{(k-1)} = \mathcal{O}(q^{(k-1)k-3})\) by Proposition 5.14. By Theorem 4.6 there exists \(r^{(k-1)} \in \mathbb{N}^{q-1}\) such that \(C_{k-1}(r^{(k-1)})\) is an \([R^{(k-1)}n^{(k-1)}]_q\) MWS code satisfying (A) and (B). Therefore, by part 1 of Proposition 4.1, there exists a \([2R^{(k-1)}n^{(k-1)}+1,k]_q\) MWS code. Since by Corollary 5.13 \(R^{(k-1)} = \mathcal{O}(q^{2k-1})\), we finally obtain
\[2R^{(k-1)}n^{(k-1)}+1 = \mathcal{O}(q^{2k-1+k(k-1)-3}) = \mathcal{O}(q^{k(k+1)-4}).\]

\[\square\]

**Remark 5.16.** Observe that the asymptotic estimate of the length of an MWS given in Theorem 5.15 is done considering the worst-case scenario, and therefore the algebraic construction could give a shorter code in practice. Indeed it could be possible to improve the asymptotic estimate of \(R^{(k)}\) that we computed with a worst-case argument.

### 6. Conclusions and open questions

In this paper we have introduced the concept of \([n,k]_q\) maximum weight spectrum (MWS) codes. We provided two different constructions for MWS codes, showing that they exist for all dimensions, and over every finite field. In Lemma 5.1 we provide a lower bound on the length of MWS codes, which is shown to be sharp for \(k = 2\). The infinite families of \([n,k]_q\)-MWS codes provided here have lengths
\[n \geq \mathcal{O}(q^{2k+4-k^2}).\] This raises the natural question: How short may an MWS code be? In other words, the general problem for fixed \(k\) and \(q\), is to determine the least value \(n\) such that an \([n,k]_q\)-MWS code exists. It is with some trepidation that we refer to such codes as optimal MWS codes (the term optimal is over used, but seems most appropriate here). Rather than pose a conjecture on the length of optimal MWS codes, the authors invite investigation into the existence of infinite families of \([n,k]_q\)-MWS codes with \(\mathcal{O}(q^k) \leq n \leq \mathcal{O}(q^{2k+4-k^2})\).

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