BUNDLES WITH EVEN-DIMENSIONAL SPHERICAL SPACE FORM AS FIBERS AND FIBERWISE QUARTER PINCHED RIEMANNIAN METRICS

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Abstract. Let $E$ be a smooth bundle with fiber an $n$-dimensional real projective space $\mathbb{R}P^n$. We show that, if every fiber carries a positively curved pointwise strongly $1/4$-pinched Riemannian metric that varies continuously with respect to its base point, then the structure group of the bundle reduces to the isometry group of the standard round metric on $\mathbb{R}P^n$.

1. Main results

Recently, the Generalized Smale Conjecture has been proven for $M$ the 3-dimensional sphere, or any 3-dimensional spherical space form; that is, the diffeomorphism group of $M$ is homotopy equivalent to the isometry group of a metric of constant sectional curvature (see [2, 3, 13, 17]). Farrell, Gang, Knopf and Ontaneda in [10] proved using Ricci flow that a similar result holds for the structure group of a smooth sphere bundle over a compact manifold, when each fiber has a $1/4$-pinched Riemannian metric which depends continuously on the base point, i.e. a fiberwise metric. Henceforth, by a $1/4$-pinched metric we will always refer to a metric with positive sectional curvatures bounded between $\frac{1}{4}$ and 1.

In the present work, we study the more general case of smooth fiber bundles with an arbitrary spherical space as fiber, equipped with a fiberwise pointwise strongly $1/4$-pinched Riemannian metric.

By a smooth bundle over a space $B$, we mean a locally trivial bundle over $B$ whose structural group is $\text{Diff}(F)$, the group of self-diffeomorphisms of $F$ with the smooth topology. We assume that each fiber is equipped with a Riemannian metric such that at any point of the fiber, the ratio of the maximal to the minimal sectional curvatures is strictly less than 4.

We point out that for any $n$-dimensional spherical space form $F$, the standard round metric $\tilde{\sigma}$ of the unit round sphere $S^n \subset \mathbb{R}^{n+1}$ induces a distinguished metric $\sigma$ on $F$, which we call the standard round metric of $F$. In the case when the bundle has fibers diffeomorphic to a real projective
space and has a fiberwise pointwise strongly \( \frac{1}{4} \)-pinched Riemannian metric, we conclude that the structure group of the bundle reduces:

**Theorem A.** Let \( E \to B \) be a smooth fiber bundle over a locally compact space \( B \), whose fibers are real projective spaces. If the bundle admits pointwise strongly \( \frac{1}{4} \)-pinched fiberwise metrics, then its structure group reduces to the isometry group of the standard round metric.

Recall that an even dimensional spherical space form is either diffeomorphic to the sphere or a real projective space \( \mathbb{R}P^n \) (cf. [9, Proposition 4.4, p.166]. Thus, for the particular case of even dimensional fibers, we obtain the following corollary, which is a direct generalization of the conclusions of [10]:

**Corollary B.** Let \( E \to B \) be a smooth fiber bundle over a locally compact space \( B \), with fibers homeomorphic to an even dimensional spherical space form \( F \). If the bundle admits pointwise strongly \( \frac{1}{4} \)-pinched fiberwise metrics, then the structure group reduces to the isometry group of the standard round metric.

For the particular case of dimension 3, the work of Bamler and Kleiner [2, 3] implies that a reduction of the structure group as in Corollary B holds for any smooth bundle with fiber a 3-dimensional spherical space form, independently of whether the bundle carries a \( \frac{1}{4} \)-pinched fiberwise metric. This same observation holds for arbitrary \( \mathbb{R}P^2 \)-bundles, see [16, Proof of Theorem 3.4, p. 364].

For bundles with spherical space form as fibers of odd dimension we give some sufficient conditions to obtain the same conclusions of Theorem A. In [10], Farrell, Gang, Knopf and Ontaneda show that for any trivialization \( \alpha_i: p^{-1}(U_i) \to U_i \times F \) of a smooth fiber bundle \( S^n \to \tilde{E} \overset{\beta}{\to} B \) equipped with a fiberwise metric \( \{g_b\}_{b \in B} \) of constant sectional curvature 1 there is a continuous map \( \phi_i: U_i \to \text{Diff}(S^n) \), which for each \( b \in U_i \) takes the pushforward metric \( (\alpha_i)_*g_b \) on \( S^n \) and then sends \( b \) to an isometry \( \phi_i(b) : (S^n, (\alpha_i)_*g_b) \to (S^n, \sigma) \). Denote by \( \Gamma \) any finite subgroup of \( O(n+1) \) acting freely by isometries on \( S^n \) with respect to the standard round metric, and let \( F \) be the spherical space form \( S^n/\Gamma \). Given a smooth fiber bundle \( p: E \to B \) with fiber \( F \), we say that an \( n \)-sphere bundle \( \tilde{p}: \tilde{E} \to B \) is a covering bundle of \( p: E \to B \), if there exist a map \( \pi: \tilde{E} \to E \) which is an extension of the covering \( S^n \to F \) of the fibers. Observe that in general \( \tilde{E} \) is not the universal covering of \( E \), since for any sphere bundle over \( B \) the fundamental group of the total space is isomorphic to the fundamental space of the base.

In the next theorem we give a sufficient condition for the structure group of a smooth fiber bundle, with fibers an \( n \)-dimensional space form to reduce to the isometry group of the standard round metric on the fiber.

**Theorem C.** Let \( F \) be an \( n \)-dimensional spherical space form with fundamental group \( \Gamma \). Denote by \( I = \text{Isom}(F, \sigma) \) the group of isometries of \( F \) with respect to the standard round metric \( \sigma \). Let \( F \to E \overset{p}{\to} B \) be a smooth fiber bundle over a locally compact space \( B \), with pointwise strongly
quarter pinched fiberwise metrics. Assume that there exist a covering bundle $S^n \to \tilde{E} \to B$ extending $\pi: S^n \to F$. Consider $\phi_i: U_i \to \text{Diff}(S^n)$ the continuous maps described above. When the images $\phi_i(U_i)$ are contained in the normalizer of $\Gamma$ in $\text{Diff}(S^n)$, i.e. $\phi_i(U_i) \subset N_{\text{Diff}(S^n)}(\Gamma)$, for all indices $i$, then the structure group of $p: E \to B$ reduces to $I$.

We point out that in general, for an arbitrary smooth fiber bundle $F \to E \to B$ with spherical space form as fibers, an extension $\tilde{E} \to E$ of the covering $S^n \to F$ may not exist. In Section 3 we exhibit a family of smooth $\mathbb{R}P^3$-bundles over closed simply-connected 4-manifolds which do not admit an extension of the covering $S^3 \to \mathbb{R}P^3$.

The proofs of both Theorems A and C rely on the fact that, locally, the spherical space form fiber bundle under consideration is finitely covered by a trivial sphere bundle. With this observation, several results presented in [10] can be generalized. The main difference between Theorem A and Theorem C is an equivariance problem. Namely, when the fibers are projective spaces, we can do all constructions in a $\mathbb{Z}_2$ equivariant fashion. It is not clear whether the analogous constructions can be done in a continuous fashion for an arbitrary finite group $\Gamma$ of $O(n + 1)$ acting freely and by isometries on the unit round $n$-sphere, and thus we impose the additional, sufficient conditions in Theorem C.

The present article is organized as follows: First we give the preliminaries needed for our discussion and recall several facts from [10], for the sake of completeness. In Section 3, we prove Theorem C, and in the last section we prove Theorem A.

Acknowledgements. We thank Fernando Galaz-Garcia, for comments on the first versions of the present manuscript, and Wilderich Tuschmann for useful conversations.

2. Preliminaries

For a closed smooth manifold $M$, we denote by $\text{Met}(M)$ the space of smooth Riemannian metrics on $M$ equipped with the $C^\infty$ Whitney topology. For a survey on the properties of this space, the interested reader can consult, for example, [7], [14], [15], [18]. A Riemannian metric on $M$ is pointwise strongly $1/4$-pinched if for any point $p \in M$, the ratio of the maximal to the minimal sectional curvatures at $p$ is strictly less than 4. We denote by $\text{Met}^{1/4}(M)$ the space of pointwise strongly $1/4$-pinched metrics with the subspace topology induced by $\text{Met}^{1/4}(M) \subseteq \text{Met}(M)$. In an analogous fashion we define $\text{Met}^1(M)$ to be the space of round metrics, i.e. metrics with constant sectional curvature 1, with the induced topology given by $\text{Met}^1(M) \subset \text{Met}(M)$.

Let $F \to E \to B$, $M$ be a smooth fiber bundle with compact fiber $F$. A fiberwise metric on the bundle is a continuous family of Riemannian metrics $g_x$ on every fiber $F_x$. Here, by “continuous” we mean that for any local trivialization $\alpha: p^{-1}(U) \to U \times F$ of $p: E \to B$, the induced map $U \to \text{Met}(F)$ given by $x \mapsto \alpha_x g_x$ is continuous. Observe that for two local trivializations, $\alpha_i: p^{-1}(U_i) \to U_i \times F$ and $\alpha_j: p^{-1}(U_j) \to U_j \times F$, the transition map
\[ \alpha_{ij}(b) = \alpha_j \circ \alpha_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \] is of the form \[ \alpha_{ij}(b, v) = (b, \alpha_{ij}(b)(v)). \]

The definition of fiberwise Riemannian metric implies that for any \( b \in U_i \cap U_j \) the map \( \alpha_{ij}(b) : (F, (\alpha_j)_* g_b) \to (F, (\alpha_j)_* g_b) \) is an isometry.

The normalized Ricci flow on a smooth compact \( n \)-dimensional manifold \( M \) is the solution of the following differential equation:

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t)) + \frac{2}{n} \left( \frac{\int_M \text{Scal}(g(t)) d\text{vol}(g(t))}{\int_M d\text{vol}(g(t))} \right) g(t),
\]

where \( g(t) \) is a time-dependent family of Riemannian metrics on \( M \). In [6], Brendle and Schoen proved that for a closed manifold \( M \) of dimension \( n \geq 4 \), the normalized Ricci flow starting at a pointwise strongly \( 1/4 \)-pinched metric converges to a metric with constant sectional curvature 1. In dimension 3 this same result holds by the work of Hamilton in [12].

Consider a finite subgroup \( \Gamma < \text{O}(n+1) \) acting freely on the unit round sphere. A spherical space form is the quotient manifold \( S^n/\Gamma \). Since \( \Gamma \) is a subgroup of isometries of the standard round metric \( \tilde{\sigma} \) of \( S^n \), the standard round metric on \( S^n \) induces a distinguished round metric \( \sigma \) on \( S^n/\Gamma \). We refer to this metric as the standard round metric on \( S^n/\Gamma \).

As in the case of sphere bundles, for a smooth fiber bundle with fiber a spherical space form, if the fibers are pointwise strongly \( 1/4 \)-pinched, then the normalized Ricci flow gives a continuous fiberwise deformation of the fiberwise metrics to a round metric (not necessarily the standard one). This leads to the following fiberwise version of [6, Theorem 1]:

**Proposition 2.1.** Let \( F = S^n/\Gamma \) be an \( n \)-dimensional spherical space form, and \( \pi : E \to B \) a smooth \( F \)-fiber bundle over a locally compact space \( B \). Assume that the bundle admits strongly \( 1/4 \)-pinched fiberwise metrics, then the bundle admits fiberwise round metrics.

**Proof.** Locally, the continuous fiberwise family of Riemannian metrics of \( \pi : E \to B \) is given by a continuous map \( F : U \to \text{Met}^{1/4}(S^n/\Gamma) \), where \( U \subset B \) is a open set over which the fiber bundle is trivial. Since \( B \) is locally compact, we can assume w.l.o.g. that \( \tilde{U} \) is compact and the bundle is also trivial over \( \tilde{U} \). Via the universal cover \( \rho : S^n \to S^n/\Gamma \) we obtain a continuous map \( \tilde{F} : \tilde{U} \to \text{Met}^{1/4}(S^n) \). The metric \( \tilde{F}(b) \) on \( S^n \) is the pull-back of \( F(b) \) along \( \rho \). Consider the map \( F^* : \tilde{U} \to \text{Met}^1(S^n) \) that sends \( \tilde{F}(b) \) to its limit \( \tilde{F}^*(b) \) under the normalized Ricci flow. Theorem 1 in [10] implies that this map is continuous on the compact set \( \tilde{U} \), hence on \( U \). Furthermore, for each fixed \( b \in B \), by construction the action of \( \Gamma < \text{O}(n+1) \) on \( S^n \) is by isometries with respect to the metric \( \tilde{F}(b) \). Since the normalized Ricci flow preserves isometries (see [1]), this implies that the action of \( \Gamma \) on \( S^n \) is by isometries with respect to \( \tilde{F}^*(b) \). Thus we obtain a new Riemannian metric \( F^*(b) \) on \( S^n/\Gamma \).

Recall that two metrics in \( \text{Met}(S^n/\Gamma) \) are close in the Whitney topology if for any atlas, the matrix coefficients are uniformly close. Since \( F^* \) depends continuously on \( b \), and \( \rho : (S^n, \tilde{F}(b)) \to (S^n/\Gamma, F^*(b)) \) is a local isometry, this implies that \( F^* : \tilde{U} \to \text{Met}^1(S^n/\Gamma) \) is a continuous function.
For the sake of completeness, we recall [10, Lemma 1], which will be used in Section 3.

**Lemma 2.2 (Lemma 1 in [10]).** Let \( \overline{\sigma} \) denote the standard round metric on \( S^n \). There is a continuous map

\[
\phi: \text{Met}^1(S^n) \rightarrow \text{Diff}(S^n),
\]

such that \( \phi(g): (S^n, g) \rightarrow (S^n, \overline{\sigma}) \) is an isometry, for any round metric \( g \in \text{Met}^1(S^n) \).

We observe that, for a round metric \( g \) with isometry group \( \text{Isom}(g) \), the map \( \phi(g) \) is in general not \( \text{Isom}(g) \)-equivariant. Given any finite subgroup \( \Gamma \) of \( O(n+1) \), acting freely and isometrically on the unit round sphere, and \( f \in \text{Diff}(S^n) \), we obtain a new effective representation \( \rho_f: \Gamma \hookrightarrow \text{Diff}(S^n) \), defined as follows: For any \( \gamma \in \Gamma \) and \( v \in S^n \) we set

\[
\rho_f(\gamma)(v) = f(\gamma(f^{-1}(v))).
\]

**Remark 2.3.** In the case that such representation \( \rho_f \) is again a subgroup of \( O(n+1) \), de Rahm showed in [8] that in odd dimensions \( \rho_f \) has the same character as the inclusion \( \Gamma \hookrightarrow O(n+1) \). This implies that there is some \( h_f \in O(n+1) \), such that for all \( \gamma \in \Gamma \) we have

\[
h_f^{-1} \gamma h_f = \rho_f(\gamma).
\]

In other words, the map \( h_f^{-1} \circ f \in \text{Diff}(S^n) \) is \( \Gamma \)-equivariant. When we consider the maps \( h_f^{-1} \circ \phi(g) \), we get an equivariant map. It is not clear, however, whether this map is continuous with respect to the metric \( g \).

### 3. General spherical space form case

Let \( p: E \rightarrow B \) and \( \tilde{p}: \tilde{E} \rightarrow B \) be two smooth fiber bundles, \( \tilde{F} \) and \( F \) their fibers over a specific base point \( b \in B \), and \( \pi: \tilde{F} \rightarrow F \) a covering.

We call a covering \( \pi: \tilde{E} \rightarrow E \) a covering of the bundle \( E \rightarrow B \) extending the covering \( \pi: \tilde{F} \rightarrow F \) if the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{i} & \tilde{E} \\
\downarrow{\pi} & & \downarrow{\pi} \\
F & \xrightarrow{i} & E \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \tilde{p} \\
& & \downarrow{p} \\
& B
\end{array}
\]

**Remark 3.1.** We will only consider the universal cover \( \pi: \tilde{F} \rightarrow F \), and given a fiber bundle \( p: E \rightarrow B \) with fiber \( F \), we denote a covering bundle of \( \tilde{F} \rightarrow F \) by \( \tilde{p}: \tilde{E} \rightarrow B \). Note that \( \tilde{E} \) is, in general, not the universal cover of \( E \), since the fundamental group of \( \tilde{E} \) is isomorphic to the fundamental group of \( B \).

Let \( F \) be an \( n \)-dimensional spherical space form. Denote by \( I \) the isometry group \( \text{Isom}(F, \sigma) \) of \( F \) with respect to the standard round metric \( \sigma \), and by \( G \) the structure group of a smooth fiber bundle \( F \rightarrow E \xrightarrow{p} B \). For the universal cover \( \pi: S^n \rightarrow F \), let \( G^\pi \) denote the group of diffeomorphisms of
\( S^n \) which consists of all the lifts of elements in \( G \). Observe that there is a natural continuous group homomorphism \( \psi : G^* \to G \).

The bundle \( p: E \to B \) admits a smooth covering fiber bundle \( \tilde{p}: \tilde{E} \to B \) extending the universal cover \( \pi : S^n \to F \) if there exists a continuous group homomorphism \( c: G \to G^* \) such that \( \psi \circ c = Id_G \) (see [4, Section 3]). In particular, if \( p: E \to B \) admits a pointwise strongly \( 1/4 \)-pinched fiberwise metric, so does \( \tilde{p}: \tilde{E} \to B \). From the Main Theorem in [10], the structure group of \( \tilde{p}: \tilde{E} \to B \) is \( O(n + 1) \). We give sufficient conditions to guarantee that this reduction descends to \( p: E \to B \).

**Remark 3.2.** Recall that for a local trivialization \( \tilde{\alpha}_i : \tilde{p}^{-1}(U_i) \to U_i \times S^n \) of the bundle \( \tilde{p}: \tilde{E} \to B \) with round fiberwise metrics, the map \( \phi \) given by Lemma 2.2 induces a map \( \phi_i : U_i \to \text{Diff}(S^n) \) as follows:

\[
\phi_i(b) = \phi((\tilde{\alpha}_i)_*, g_b),
\]

where \( g_b \) is the metric on the fiber \( \tilde{p}^{-1}(b) \).

**Lemma 3.3.** Let \( F \) be an \( n \)-dimensional spherical space form with fundamental group \( \Gamma \). Let \( F \to E \xrightarrow{\tilde{p}} B \) be a smooth fiber bundle over a locally compact space \( B \), with fiberwise round metrics. Assume that there exists a covering bundle \( S^n \to \tilde{E} \xrightarrow{\tilde{p}} B \) extending \( \pi : S^n \to F \). Consider the continuous maps \( \phi_i : U_i \to \text{Diff}(S^n) \) from Remark 3.2. If \( \phi_i(U_i) \subset N_{\text{Diff}(S^n)}(\Gamma) \) for all indices \( i \), then the structure group of \( p: E \to B \) reduces to the isometry group of \( F \) with respect to the standard round metric.

**Proof.** Since \( p: E \to B \) has a family of fiberwise round metrics Riemannian metrics, so does \( \tilde{p}: \tilde{E} \to B \). By [10], the structure group of \( \tilde{p}: \tilde{E} \to B \) reduces to \( O(n + 1) \). We recall how such a reduction is obtained. Fix a local trivialization \( \{(U_i, \alpha_i)\} \) of \( p: E \to B \). Denote by \( \sigma \) the standard round metric on \( S^n \), and consider the trivialization \( \tilde{\alpha}_i : \tilde{p}^{-1}(U_i) \to U_i \times S^n \) induced by \( \alpha_i : p^{-1}(U_i) \to U_i \times F \). Fix \( x \in U_i \cap U_j \) and consider

\[
\phi_i(x) : (S^n, (\tilde{\alpha}_i)_*, \tilde{\sigma}_x) \to (S^n, \sigma).
\]

This is the isometry described in Lemma 2.2. Since \( \phi_i(x) \) and \( \alpha_{ij}(x) \) are isometries, then by definition the map \( \tilde{\beta}_{ij} : U_i \cap U_j \to \text{Diff}(S^n) \) given by

\[
\tilde{\beta}_{ij}(x) = \phi_j(x) \circ (\tilde{\alpha}_{ij}(x)) \circ \phi_i(x)^{-1},
\]

is an element of \( O(n + 1) \). Observe that the hypothesis \( \phi_i(x) \in N_{\text{Diff}(S^n)}(\Gamma) \) implies that both maps \( \phi_i(x) \) and \( \phi_j(x) \) leave the orbits invariant, i.e. a point \( q \) in the orbit \( \Gamma(p) \) is mapped to another point in \( \Gamma(p) \). This is also true for \( \tilde{\alpha}_{ij}(x) \) by construction. Thus, \( \tilde{\beta}_{ij}(x) \) maps orbits to orbits. This implies that for any \( x \in U_i \cap U_j \), the map \( \tilde{\beta}_{ij}(x) \) induces an isometry \( \beta_{ij}(x) : (F, \sigma) \to (F, \sigma) \), i.e. we have a reduction to the isometry group of \( (F, \sigma) \). \( \square \)

**Proof of Theorem C.** By Proposition 2.1, the fiber bundle \( E \to B \) admits round fiberwise metrics. The result follows from Lemma 3.3. \( \square \)
3.1. Obstructions to lifting spherical space form bundles to sphere bundles.

Given a fibration \( p: E \to B \) with fiber \( F = p^{-1}(b) \), for some fixed \( b \in B \), and a covering \( \pi: \tilde{E} \to E \), the map \( \tilde{p} := p \circ \pi: \tilde{E} \to B \) is also a fibration, with fiber \( \tilde{F} = \tilde{p}^{-1}(b) \). If \( \tilde{F} \) is connected, the restriction \( \pi: \tilde{F} \to F \) is then also a covering, and we call \( \pi: \tilde{E} \to E \) a covering of the fibration \( E \to B \) extending the covering \( \tilde{F} \to F \). In this case, diagram (3.1) commutes.

A smooth covering fiber bundle is a particular example of a covering fibration, but in general not every covering fibration is a smooth fiber bundle. The following theorem gives sufficient and necessary conditions for the existence of a covering fibration for the universal cover \( \pi: \tilde{F} \to F \).

Theorem 3.4 (Theorem 1 in [4]). Let \( p: E \to B \) be a fibration with fiber \( F \) and \( \pi: \tilde{F} \to F \) be the universal cover. Then, a covering \( \tilde{E} \to E \) of the fibration \( E \to B \) extending \( \tilde{F} \to F \) exists if and only if the following two conditions are satisfied:

1. \( i_*: \pi_1(F) \to \pi_1(E) \) is injective and,
2. \( p_*: \pi_1(E) \to \pi_1(B) \) has a right inverse homomorphism.

With this characterization, we show that there are smooth fiber bundles with spherical space form as fibers which do not satisfy the hypothesis of Theorem C relating to the existence of a covering bundle \( S^n \to \tilde{E} \to B \).

Example 3.5. Consider \( SO(3) \to ESO(3) \to BSO(3) \) the universal bundle of \( SO(3) \approx \mathbb{R}P^3 \). From the long exact sequence of the universal bundle, we see that \( \pi_k(BSO(3)) \cong \pi_k(SO(3)) \), and in particular \( \pi_3(BSO(3)) = \pi_2(SO(3)) = 0 \) and \( \pi_2(BSO(3)) = \pi_1(SO(3)) = \mathbb{Z}_2 \). Since \( \pi_1(ESO(3)) \) is the trivial group, then the map \( i^*: \pi_1(SO(3)) = \mathbb{Z}_2 \to \pi_1(ESO(3)) \) is not injective. Thus by Theorem 3.4 there does not exist a covering fibration extending the universal cover \( S^3 \to \mathbb{R}P^3 \approx SO(3) \).

From this previous example we can give more examples of spherical space form bundles which do not admit a covering fibration.

Example 3.6. For any simply-connected smooth 4-dimensional manifold \( M \), with \( H_2(M, \mathbb{Z}) = \mathbb{Z}^n \), we construct a smooth fiber bundle \( \mathbb{R}P^3 \to E \to M \) which does not have a covering fibration \( S^3 \to \tilde{E} \to M \) extending the universal cover \( S^3 \to \mathbb{R}P^3 \), and hence it does not admit a covering bundle.

Recall from the proof of Theorem 1.2.25 in [11], that every simply-connected smooth 4-manifold \( M \) with \( H_2(M, \mathbb{Z}) \cong \mathbb{Z}^n \) is homotopy equivalent to a CW-complex of the form

\[
M \simeq \left( \bigvee_{i=1}^n S^2 \right) \cup_{S^3} D^4,
\]

where the boundary of \( D^4 \) is attached to the wedge of spheres via a gluing map \( F: S^3 \to \bigvee_{i=1}^n S^2 \). Since \( \mathbb{Z}_2 \cong \pi_2(BSO(3)) \), we can choose non-null homotopic continuous maps \( g_i: S^2 \to BSO(3) \). We define a map \( G: \bigvee_{i=1}^n S^2 \to BSO(3) \) by \( G = \bigvee_{i=1}^n g_i \). Composing the attaching map \( F \) with \( G \) we get \( [G \circ F] \in \pi_3(BSO(3)) = 0 \). Thus we can extend \( G \) to
3.4 and we get a continuous map $M \to B\text{SO}(3)$, which induces a principal SO(3)-bundle $E \to M$.

Comparing the long exact sequence of homotopy groups of $\text{SO}(3) \to E \to M$ with the one of the classifying bundle $\text{SO}(3) \to E\text{SO}(3) \to B\text{SO}(3)$, via the map $G \circ F$, we get the following commutative diagram:

$$
\begin{array}{c}
\pi_2(E) \xrightarrow{\delta_2} \pi_2(M) \xrightarrow{i_*} \pi_1(\text{SO}(3)) \xrightarrow{\pi} \pi_1(E) \\
0 \xrightarrow{G_*} \pi_2(B\text{SO}(3)) \xrightarrow{\cong} \pi_1(\text{SO}(3)) \xrightarrow{i_*} 1
\end{array}
$$

Since the map $G_*: \pi_2(M) \to \pi_2(B\text{SO}(3))$ is surjective by construction, we see that the connecting homomorphism $\delta_2: \pi_2(M) \to \pi_1(\text{SO}(3))$ is surjective. This implies that the induced map $i_*: \pi_1(\text{SO}(3)) \to \pi_1(E)$ is the trivial group homomorphism. Again, by Theorem 3.4 this principal SO(3)-bundle cannot admit a covering fibration extending the cover $S^3 \to \mathbb{R}P^3$.

4. $\mathbb{R}P^n$-Bundles

In light of Example 3.6, we show that the Main Result in [10] still holds for smooth $\mathbb{R}P^n$-bundles with fiberwise pointwise strongly 1/4-pinched metric, i.e. we prove Theorem A. We begin by proving an analogous statement to Lemma 2.2

**Theorem 4.1.** Let $g$ be a round metric on $\mathbb{R}P^n$ and let $\sigma$ denote the metric induced by the Riemannian covering $(S^n, \tilde{\sigma}) \to (\mathbb{R}P^n, \sigma)$, where $\tilde{\sigma}$ denotes the standard round metric on $S^n$. Then there is an isometry $\tilde{\phi}_g: (\mathbb{R}P^n, g) \to (\mathbb{R}P^n, \sigma)$. Moreover, $\tilde{\phi}_g$ depends on $g$ in a continuous way.

**Proof.** Denote by $\tilde{g}$ the lift of $g$ to $S^n$. Consider $N$ to be the “north pole” of $S^n$, and let $\{e_i\}$ be the standard orthonormal basis of $(T_N(S^n), \tilde{\sigma})$ given by the inclusion $S^n \subset \mathbb{R}^{n+1}$. Via the Gram-Schmidt algorithm, we obtain an orthonormal basis $\{\tilde{e}_i\}$ on $(T_N(S^n), \tilde{g})$. Denote by $i: T_N(S^n) \to T_N(S^n)$ the change of basis. Then, by Cartan’s Theorem (see [9, Chapter 8]), there exists a unique isometry $\tilde{\phi}: (S^n, \tilde{g}) \to (S^n, \tilde{\sigma})$ such that $\tilde{\phi}(N) = N$ and $D_N\tilde{\phi}(\tilde{e}_i) = e_i$ for $i = 1, 2, \ldots, n$.

We now show that the isometry $\tilde{\phi}$ is equivariant under the action of $Z_2 \cong \{1d, -1d\}$. To see this, we show that the cut locus of any point $p \in S^n$ consists of its antipodal point $-p$, with respect to the metric $\tilde{g}$ as well as $\tilde{\sigma}$. Fix $p \in M$ and consider a shortest geodesic $c$ from $p$ to $-p$ with respect to the metric $\tilde{g}$. Then the curve $(-1d) \circ c$ is, after reversing its direction, another shortest geodesic from $p$ to $-p$ with the same length. We observe that these two geodesics cannot be identical since otherwise the middle point of $c$ would be fixed by $-1d$. The images of these curves via the isometry $\tilde{\phi}$ give two different shortest geodesics from $\tilde{\phi}(p)$ to $\tilde{\phi}(-p)$ with respect to the standard round metric $\tilde{\sigma}$. Clearly, this is only possible if $\tilde{\phi}(-p) = -\tilde{\phi}(p)$.

This implies $\tilde{\phi}(-1d) = (-1d) \circ \tilde{\phi}$, so the isometry $\tilde{\phi}$ is equivariant under the group action of $Z_2$ and therefore induces an isometry $\phi: \mathbb{R}P^n \to \mathbb{R}P^n$. 
We end the proof by showing that \( \phi_y \) depends continuously on the metric \( g \). Consider two metrics \( g_1 \) and \( g_2 \) on \( \mathbb{R}^P \) close to each other with respect to \( C^\infty \) Whitney-topology. Since \( \phi_{g_1} \) and \( \phi_{g_2} \) are given by the exponential maps, as well as the Gram-Schmidt orthonormalization process with respect to \( g_1 \) and \( g_2 \), then the isometry \( \phi_{g_1} \) is close to the isometry \( \phi_{g_2} \) in \( \text{Diff}(\mathbb{R}^P) \) with respect to the \( C^\infty \) Whitney topology (see [7, Section 2.6]).

\[
\square
\]

With this we are able to give the proof of Theorem A. Note that the proof is analogous to the proof of the Main Theorem in [10].

**Proof of Theorem A.** Let \( p : E \to B \) be a smooth fiber bundle whose fibers are diffeomorphic to \( \mathbb{R}^P \). Assume that \( E \) is equipped with a pointwise strongly 1/4-pinched fiberwise Riemannian metric \( \{g_b\}_{b \in B} \). We will show that given any trivializing bundle charts, we can give transition maps which are isometries of the standard round metric of \( \mathbb{R}^P \).

Take an atlas of trivializing bundle charts \( \{U_i, \alpha_i\}_{i \in J} \) of \( E \), such that the closure \( U_i \) is compact (since \( B \) is locally compact, such an atlas exists). Taking \( U_{ij} = U_i \cap U_j \) and \( \alpha_{ij} = \alpha_j \circ \alpha_i^{-1} : U_{ij} \times \mathbb{R}^P \to U_{ij} \times \mathbb{R}^P \), we have

\[
\alpha_{ij}(b, v) = (b, \alpha_{ij}(b)(v)),
\]

and by definition of a fiberwise Riemannian metric, for any \( b \in U_{ij} \) the map \( \alpha_{ij}(b) : (\mathbb{R}^P, (\alpha_i)_* g_b) \to (\mathbb{R}^P, (\alpha_j)_* g_b) \) is an isometry.

By Proposition 2.1 evolving the fiberwise metric via the normalized Ricci flow we obtain a fiberwise Riemannian metric \( \{g_b^\bullet\}_{b \in B} \) in \( \text{Diff}(\mathbb{R}^P) \) with respect to each Riemannian metric \( g_b^\bullet \) is a round metric. Thus we obtain continuous maps \( G_i : U_i \to \text{Met}^1(\mathbb{R}^P) \) given by

\[
G_i(b) = (\alpha_i)_* g_b^\bullet.
\]

We now consider the continuous map \( \phi : \text{Met}^1(\mathbb{R}^P) \to \text{Diff}(\mathbb{R}^P) \) given by Theorem 4.1. Composing with \( G_i \) we obtain a continuous map \( \phi \circ G_i : U_i \to \text{Diff}(\mathbb{R}^P) \). We take the diffeomorphism \( f_i : U_{ij} \times \mathbb{R}^P \to U_{ij} \times \mathbb{R}^P \) defined as

\[
f_i(b, v) = (b, \phi \circ G_i(b)(v)).
\]

For any pair of indices \( i, j \in J \) we define the diffeomorphism \( \beta_{ij} : U_{ij} \times \mathbb{R}^P \to U_{ij} \times \mathbb{R}^P \) as

\[
\beta_{ij}(b, v) = f_j \circ \alpha_{ij} \circ f_i^{-1}(b, v).
\]

Thus by construction the following diagram commutes:

\[
\begin{array}{ccc}
U_{ij} \times \mathbb{R}^P & \xrightarrow{f_i} & U_{ij} \times \mathbb{R}^P \\
\downarrow{\alpha_{ij}} & & \downarrow{\beta_{ij}} \\
U_{ij} \times \mathbb{R}^P & \xrightarrow{f_j} & U_{ij} \times \mathbb{R}^P
\end{array}
\]

This implies that \( p : E \to B \) is isomorphic as a fiber bundle to the smooth fiber bundle \( p' : E' \to B \) associated to the transition maps \( \beta_{ij} \).

Fix \( b \in U_{ij} \subset B \). Then

\[
\beta_{ij}(b) = \phi_{(\alpha_{ij})_* g_b^\bullet} \circ \alpha_{ij}(b) \circ \left( \phi_{(\alpha_i)_* g_b} \right)^{-1} : (\mathbb{R}^P, \sigma) \to (\mathbb{R}^P, \sigma)
\]
is by construction an isometry of the standard round metric $\sigma$. Thus the structure group of $p': E' \to B$ is contained in the isometry group of $\sigma$, as claimed. □

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EVEN-DIM. SPHER. SPACE FORM BUNDLES AND 1/4-PINCHED RIEM. METRICS

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