Based Rings for Generalized Lowest Two-Sided Cells

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Abstract

The based ring of the lowest two-sided cell of an extended affine Hecke algebras with a positive parameter map is realized by showing that the structure constants of the based ring are independent of the choice of the positive parameter map.

For any positive parameter map, we give a simple proof of a decomposition formula (due to Xi in the one parameter case, see [X90, Cor. 2.11]) by using a lemma due to Lusztig. The homomorphism from the extended affine Hecke algebra to the based ring of the lowest two-sided cell is also constructed using Xi formula in our generalized setup. And a family of irreducible representations is given using this homomorphism.

As a byproduct, Lusztig conjectures on the cells of Hecke algebras with unequal parameters are verified in the case of lowest two-sided cells of affine Hecke algebras.

Contents

1 Introduction 2

2 Preliminaries 3
2.1 Extended affine Weyl groups and parameters 3
2.2 Extended affine Hecke algebras with a parameter map 4
2.3 Lowest two-sided cells 5

3 Based rings of lowest two-sided cells 6
3.1 Lusztig Lemma 6
3.2 Boundedness of $m_{x,y,z}$ 9
3.3 Based rings of lowest two-sided cells 10

4 Homomorphisms from Hecke algebras to based rings of lowest two-sided cells 10
4.1 Xi formula 10
4.2 Homomorphism from $\mathcal{H}$ to $J_0$ 11

5 Some irreducible representations via the lowest two-sided cell 13

6 Affine case 16
6.1 Remark on the affine case 16
6.2 Lusztig conjectures 17
1 Introduction

The based rings of affine Hecke algebras are introduced in [Lus87a]. They are used to study representations of affine Hecke algebras in [Lus87b, Xi 07]. The based rings of Hecke algebras with unequal parameters (i.e. the parameters are powers of an indeterminate) are also defined similarly in [Lus03] under some conjectures which hold for Hecke algebras with one parameters but in general case are still open.

The lowest two-sided cells of extended affine Hecke algebras with two parameters are determined in [Xi94, Chapter 3]. It shows that the lowest two-sided rings are invariant under the assumption that the parameters are positive. The lowest two-sided cells of affine Hecke algebras with unequal parameters are also studied in detail. It turns out that the affine cases are similar to the extended affine case and only needing to note is that the special points of affine case may be modified. Recent work on generalized lowest two-sided cells is [Gui08, Gui14].

The based rings for lowest two-sided cells of extended affine Hecke algebras with positive parameter map are determined in this article. Our motivation is constructing based rings for lowest two-sided cells of affine Hecke algebras with unequal parameters and using it to study some representations. To avoid emphasizing the special points, we restrict ourself to the extended affine case. After constructing the based ring of affine Hecke algebras with unequal parameters we find that it is same as the case of one parameter. Even this is also can be done for the case of two free parameters, which is more useful for studying the irreducible representations of extended affine Hecke algebras over $\mathbb{C}$ with arbitrary complex parameters. So our basic object of the research is extended affine Hecke algebras with a positive parameter map.

The main structure of this article is as follows.

In Section 3, we prove that the coefficients $\gamma_{x,y,z}^{-1}$ of $q_{w_0}^{1/2}$ in $m_{x,y,z}$ for $x, y, z \in c_0$ is independent of the choice of the positive parameter map. The basic tool is the Lusztig lemma, Lemma 3.1, which have been used in the proof of the boundedness of $a$-functions in [Lus85]. In view of the fundamental role of it, we give the details of Lusztig lemma in our generalized setup.

In Section 4, we use Lusztig lemma to prove Xi formula, Theorem 4.1, in our generalized setup, which can be used to construct the homomorphism from the extended Hecke algebra with a positive parameter map to the based ring of the lowest two-sided cell. As a byproduct, we can use Xi formula (Theorem 4.1) to determine the left cells in $c_0$ (see Corollary 4.2), which have been also determined in [Gui08] in the case of affine Hecke algebras with unequal parameters. After completing our paper, we are informed that a similar decomposition formula also obtained by Guilhot in studying the cellularity of the lowest two-sided ideal of an affine Hecke algebras with parameters being powers of an indeterminate (see [Gui14, Cor. 5.4]).

In section 5, following [Xi07], we give a family of irreducible representations of extended affine Hecke algebras over $\mathbb{C}$ with complex parameters such that $C_{w_0}^2 \neq 0$. It turns out that the irreducible representations $M$ of extended affine Hecke algebras such that $C_{w_0}M \neq 0$ are the unique irreducible quotients of irreducible representations of the based ring of the lowest two-sided cell via the homomorphism in Section 4 (see Theorem 5.2).

In Section 6, we give more information on the lowest two-sided cell of the affine Hecke algebra with unequal parameters. In particular, we prove that the Lusztig conjectures in [Lus03, Chapter 14] hold in the case of the lowest two-sided cells (see Theorem 6.2).
2 Preliminaries

In this section, we recall the definition of extended affine Hecke algebras with a positive parameter map and the results on the generalized lowest two-sided cells.

2.1 Extended affine Weyl groups and parameters

Let $R$ be an irreducible reduced root system of rank $l$. Corresponding to $R$, we have Weyl group $W_0$, root lattice $Q$, and weight lattice $P$, the affine Weyl group $W' = W_0 \times Q$ and the extended affine Weyl group $W = W_0 \times P$. There is a length function $l : W \rightarrow \mathbb{Z}$ on $W$ extended the standard length function on $W'$ for Coxeter groups. Let $\Omega = \{ \pi \in W | l(\pi) = 0 \}$. Then $W = W' \rtimes \Omega$ and $\Omega \cong P/Q$. $\Omega$ can be interpreted as the group of graph automorphism of the Dynkin diagram of $W'$. Note that $W$ is not a Coxeter group.

The affine Weyl group $W'$ can be realized as a group of displacement generated by reflections of a set $\mathcal{F}$ of hyperplanes in a Euclidean space $E$ of dimension $l$. Let $X$ be the set of alcoves, i.e. the set of connected components of the set $E - \cup_{H \in \mathcal{F}} H$. $W'$ acts on $E$ and $X$ via the previous realization, which can be extended uniquely to an action of $W$ such that the subgroup $\Omega$ acts trivially. We will regard this action as right action. Another action of $W$ on $X$, regarded as a left action, was introduced in [Lus80, 1.1]. The features of this left action are that for every simple reflection $s \in S$ and every alcove $A \in X$, and $sA$ has a common face and the left action and right action commute. The common face of $A$ and $sA$ will be called a face (of $A$) of type $s$.

Let $\Gamma$ be an abelian group written multiplicatively with unit element 1. A map $q^{1/2} : S \rightarrow \Gamma$, $s \mapsto q_s^{1/2}$ is called a parameter map of $W'$ if $q_s^{1/2} = q_s^{1/2}$ holds for any two conjugate elements $s, s'$ in $S$. A parameter map $q^{1/2}$ for $W$ can be extended to a map $q^{1/2} : W \rightarrow \Gamma$ such that if, for $w \in W$, $w = \pi s_1 s_2 \cdots s_n$ is a reduced expression with $\pi \in \Omega$, $s_i \in S$, then $q^{1/2}(w) = q_{s_1}^{1/2} q_{s_2}^{1/2} \cdots q_{s_n}^{1/2}$, which will be denoted by $q_w^{1/2}$. In particular, for $\pi \in \Omega$, we have $q_\pi^{1/2} = 1_\Gamma$.

If there exists a parameter map $q^{1/2}$ for $W$ (resp. $W'$) such that $\# \text{Im} q^{1/2} = n$, then we say that $W$ (resp. $W'$) admits $n$ parameters.

Clearly, for extended affine Weyl group of type $A_l(l \geq 2)$, $D_l(l \geq 4)$, $E_6$, $E_7$, $E_8$, $q^{1/2}$ must be a constant map on $S$. For $W$ of type $B_l(l \geq 3)$, $F_4$, and $G_2$, $W$ admits 2 parameters. Also $W$ of type $C_l(l \geq 2)$ only admits 2 parameters. Indeed, noting that the nontrivial automorphism of Dynkin diagram of type $C_l(l \geq 2)$ exchanges the first node and the last one, the corresponding simple reflections are conjugate under the action of $\Omega$. Similarly $W$ of type $A_l$ admits only 1 parameter. Consequently, the extended affine Weyl group admits at most 2 parameters. In this paper, all the parameter maps are ones for extended affine Weyl groups.

We can naturally associated to each hyperplane $H \in \mathcal{F}$ an element $q_H^{1/2} \in \Gamma$ due to the following result (see [Bre97, lemma 2.1, 2.2]).

Let $W$ be an extended affine Weyl group with a parameter map $q^{1/2} : W \rightarrow \Gamma$.

(a) If $H$ supports a face of type $s \in S$ and a face of type $t \in S$, then $s$ and $t$ are conjugate in $W'$, and hence in $W$.

(b) Let $H, H'$ be two parallel hyperplanes in $\mathcal{F}$. If $H$ supports a face of type $s \in S$ and $H'$ supports a face of type $t \in S$, then $s$ and $t$ are conjugate in $W$, but maybe not conjugate in $W'$ when $W'$ is of type $C_l(l \geq 1)$.

Thus we can define a map $q^{1/2} : \mathcal{F} \rightarrow \Gamma$, $H \mapsto q_H^{1/2}$ such that if $H$ supports a face of type $s \in S$, then $q_H^{1/2} := q_s^{1/2}$. Moreover, $q_H^{1/2} = q_{H'}^{1/2}$ if $H$ is parallel to $H'$.
2.2 Extended affine Hecke algebras with a parameter map

Definition 2.1 Let $W$ be an extended affine Weyl group with a parameter map $q^{1/2} : W \rightarrow \Gamma$. The extended affine Hecke algebra $(H, q^{1/2})$ with the parameter map $q^{1/2}$ is defined to be an algebra over $\mathbb{Z}[\Gamma]$ with generators $\{T_w | w \in W \}$ and relations

(i) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$.

(ii) $(T_s + 1)(T_s - q_s) = 0$ if $s \in S$ and $q_s = (q_s^{1/2})^2$.

We often use that notation $\tilde{T}_w = q_w^{-1/2} T_w, w \in W$ for $w \in W$. It is easy to see that there exists an involution $\bar{\cdot}$ on $H$ such that $\bar{\gamma} = \gamma^{-1}, \gamma \in \Gamma$, $\bar{T}_s = T_s^{-1}$ and $\bar{T}_\pi = T_\pi$ where $\gamma \in \Gamma, s \in S, \pi \in \Omega$.

Let $\leq$ be a total order on $\Gamma$. Let $\Gamma^{<0} = \{ \gamma \in \Gamma | \gamma < 1_\Gamma \}, \Gamma^{>0} = \{ \gamma \in \Gamma | \gamma > 1_\Gamma \}$.

Definition 2.2 $(\Gamma, \leq)$ is called an ordered group if $\leq$ satisfies

(i) if $\gamma > 1_\Gamma, \gamma_1 > \gamma_2$, then $\gamma \gamma_1 > \gamma \gamma_2$;

(ii) if $\gamma > 1_\Gamma$, then $\gamma^{-1} < 1_\Gamma$. In other words, $(\Gamma^{>0})^{-1} = \Gamma^{<0}$.

(a) Let $H$ be an extended affine Hecke algebra with a parameter map $q^{1/2} : W \rightarrow \Gamma$ such that $\Gamma$ is an ordered group. Then there exists an unique $\mathbb{Z}[\Gamma]$ basis $\{C_w | w \in W \}$ in $H$ such that $C_w = C_w$ and $C_w \equiv \tilde{T}_w \mod \mathbb{Z}[\Gamma^{<0}] \{T_w | w \in W \}$.

The basis $\{C_w | w \in W \}$ will be referred as the KL basis of $H$ with respect to the parameter map $q^{1/2}$ and the group order $\leq$.

$C_w$ can be written as the form

$$C_w = \sum_{y \leq w} P_{y,w} \tilde{T}_y = q_w^{-1/2} \sum_{y \leq w} P_{y,w} T_y$$

where $P_{y,w}$ is a polynomial in $q_s, s \in S$, and $P_{y,w}$ is a polynomial is $q_s^{-1/2}, s \in S$. For $x, y, z \in W, m_{x,y,z}$ and $h_{x,y,z}$ are defined by the product

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z, \quad \tilde{T}_x \tilde{T}_y = \sum_{z \in W} m_{x,y,z} \tilde{T}_z.$$

The partial orders $\leq_L, \leq_R, \leq_{LR}$ on $W$ are defined using the KL basis as usual. These partial orders induces naturally equivalent relations $\sim_L, \sim_R, \sim_{LR}$. And the corresponding equivalent classes will be called left cells, right cells, and two-sided cells.

Definition 2.3 The parameter map $q^{1/2} : W \rightarrow \Gamma$ is called to be positive if $\Gamma$ is an ordered group and $q_s^{1/2} > 1_\Gamma$ for any $s \in S$.

Assumption 2.1 In the following sections, we always keep the assumption that $W$ admits two parameters, since the results in this article are not new for $W$ which only admit one parameter.

Thus we can decompose $S$ into two parts $S_1, S_2$ such that the elements in the same part are conjugate in $W$. Denote $q_s^{1/2} = q_s^{1/2}$ for some $s \in S_1$, and $q_s^{1/2} = q_s^{1/2}$ for some $s \in S_1$. And denote $\xi_1 = q_1^{-1/2} - q_1^{-1/2}, \xi_2 = q_2^{-1/2} - q_2^{-1/2}$.

Assumption 2.2 In the following sections, we always keep the assumption that $\Gamma$ is an ordered group and that $q^{1/2} : W \rightarrow \Gamma$ is a positive parameter map.
Lemma 2.1 Let \( \nu_1, \nu_2 \in \mathbb{N} \) and \( a_{i,j}, a_{i,j}' \in \mathbb{Z} \). If \( \sum_{0 \leq i \leq \nu_1} a_{i,j} \xi_1^i \xi_2^j = \sum_{0 \leq i \leq \nu_1} a_{i,j}' \xi_1^i \xi_2^j \), then \( a_{\nu_1, \nu_2} = a_{\nu_1, \nu_2}' \).

Proof. Since \( q_i^{1/2} > 1_\Gamma > q_i^{-1/2} \) for \( i = 1, 2 \), we have \( a_{\nu_1, \nu_2} q_1^{\nu_1/2} q_2^{\nu_2/2} = a_{\nu_1, \nu_2}' q_1^{\nu_1/2} q_2^{\nu_2/2} \). \( \Box \)

Example 2.1
(i) Let \( \Gamma = q^Z \). Then a positive parameter map \( q^{1/2} : W \rightarrow \Gamma \) is determined by two positive integers \( m, n \). We call \( \mathcal{H} \) in this case the extended affine Hecke algebra with integral parameters, or Hecke algebra with unequal parameters following [Lus03]. More generally, we can let \( \Gamma = q^R \) and \( m, n \) be positive real numbers.

(ii) Let \( \Gamma \) be a free abelian group of rank two with generators \( q_1^{1/2}, q_2^{1/2} \). There exists an order \( \leq \) on \( \Gamma \) such that \( \Gamma \) is an ordered group and \( q_1^{1/2}, q_2^{1/2} \) are positive (see [Xi94, 1.16]). We call the \( \mathcal{H} \) in this case the extended affine Hecke algebra with free parameters.

2.3 Lowest two-sided cells

Let \( v \) be a special point and \( W_v \) be the stabilizer of \( W \) of the set of alcoves containing \( v \) in its closure. Let \( w_v \) be the longest element of \( W_v \). The connected components of \( E - \bigcup_{v \in \mathcal{H}} \mathcal{H} \) is called a quarter with vertex \( v \). For every special point \( v \), we will fix a quarter \( \mathcal{C}^+_v \) such that for any two special point \( v, v' \), \( \mathcal{C}^+_v \) is a translate of \( \mathcal{C}^+_v \). Let \( \mathcal{C}^-_v \) be the quarter which is isometric to \( \mathcal{C}^+_v \) with respect to \( v \). And denote by \( P^-_v \) the unique alcove contained in \( \mathcal{C}^+_v \) and containing \( v \) in its closure. \( A^-_v \) is the alcove symmetric to \( A^+_v \) with respect to \( v \). Then \( A^-_v = w_v A^+_v \). \( \mathcal{F} \) is the set of hyperplanes \( H \in \mathcal{F} \) such that \( H \) is parallel to a wall of \( \mathcal{C}^+_v \). The connected components of \( E - \bigcup_{H \in \mathcal{F}} H \) is called boxes. The box containing \( A^+_v \) is unique and is denoted by \( \Pi^+_v \). For \( H \in \mathcal{F} \), \( E^+_H \) is the connected component of \( E - H \) such that \( E^+_H \) meets \( \mathcal{C}^+_v \) for any special point \( v \).

Let \( B_v = \{ w \in W | w A^+_v \subset \Pi^+_v \} \), and \( U_v = \{ w \in W | w A^+_v \subset \mathcal{C}^+_v \} \). The lowest two-sided cell of \( W \) with respect to \( \mathcal{H}_{u,v} \) has been determined in [Xi94, Theorem 3.21].

Theorem 2.1 ([Xi94, Chapter 3]) Fix a special point \( v \). Then \( e_0 = \{ w w_v w'^{-1} | w \in B_v, w' \in U_v \} \) is independent of the choice of \( v \) and is a two-sided cell in \( W \) with respect to the extended affine Hecke algebra \( \mathcal{H} \) with parameter map \( q^{1/2} \) such that \( q_s^{1/2} \) are positive for all \( s \in S \). This two-sided cell is called the (generalized) lowest two-sided cell since it is minimal under the order \( \leq_{LR} \).

Lemma 2.2 In the extended affine Hecke algebra \( \mathcal{H} \) with parameter map \( q^{1/2} : W \rightarrow \Gamma \) such that \( q_s^{1/2} \) are positive for all \( s \in S \),

(i) \( C_{w v} = q_{w v}^{-1/2} \sum_{y \leq w v} T_y = \sum_{y \leq w v} q_{y w v}^{-1/2} T_y \).

(ii) \( h_{w v, u w v} = q_{w v}^{-1/2} \sum_{y \in W_v} q_w \) is nonzero in \( \mathbb{Z}[\Gamma] \).

(iii) \( C_{w v} = E_w C_{w v} \) for \( w \in U_v \) where \( E_w = \sum_{l(w v) = l(u) + l(w v)} P_{w v, u w v} \).

(iv) \( C_{w v}^{-1} = C_{w v} F_w \) for \( w \in U_v \) where \( F_w = \sum_{l(w v) = l(u) + l(w v)} P_{w v, u w v} \).

(v) \( \{ C_{w v} | w \in U_v \} \) is a left cell of \( W \). Moreover, \( \mathbb{Z}[\Gamma] \{ C_{w v} | w \in U_v \} \) is a left ideal of \( \mathcal{H} \).
(vi) Let $u, u'$ both be translation in $W$ such that $uA^+ \subset C^+_v$, and $u'A^+ \subset C^+_v$ for any special point $v$. Then

$$ C_{uu'} = \sum_{w'' : w'' = A^+_v \subset C^+_v} m(u, u', u'') C_{u''u''}, $$

where $m(x, y, z) \in \mathbb{Z}_{\geq 0}$.

**Proof.** The proof of (i) is elementary (see [Xi94, Proposition 1.17(ii) and (1.14.12)]).

The Definition [2.2] implies the property of nonzero in (ii).

(iii) is first appeared in [Xi90, 2.5] and can be proved by direct calculations as follows.

$$ C_{uwv} = \sum_{z \leq uwv} P^*_z \tilde{T}_z = \sum_{y \leq wv} \left( \sum_{l(wv) = l(u) + l(wv)} q_{ywv}^{-1/2} P^*_uwv, wv \tilde{T}_u \tilde{T}_y \right) \left( \sum_{y \leq wv} q_{ywv}^{-1/2} \tilde{T}_y \right) = E_w C_{uwv}. $$

(by (i))

(iv) let $h : \mathcal{H} \rightarrow \mathcal{H}$ be the anti-involution of $\mathcal{H}$ such that $h(T_u) = \tilde{T}_{u^{-1}}$. Then $h(C_w) = C_{w^{-1}}$, $h(E_w) = F_w$. Therefore $C_{w^{-1}w} = h(C_{uwv}) = h(E_w C_{uwv}) = C_{w^{-1}w} F_w$.

(v) follows from the proof of Theorem 3.22 in [Xi90]. And (vi) is just [Xi94, Theorem 3.21(ii)].

3. Based rings of lowest two-sided cells

In this section, we will keep the Assumption [2.2] and Assumption [2.2]. We will prove that the coefficients $\gamma_{x,y,z}^{-1}$ of $q_{w_0}^{1/2}$ in $m_{x,y,z}$ is independent of the choice of the positive parameter map, and hence have the based rings of generalized lowest two-sided cells.

3.1 Lusztig Lemma

Let $C$ ba a quarter with vertex $v$, and let $A$ be an alcove cotained in $C$ and containing $v$ in its closure. There exists $w \in W_v$ such that $wA = A^+_v$. Define $q_{w_0}^{1/2}$ to be $q_{w_0}^{1/2}$. It is clear that $q_{w_0}^{1/2}$ is the product of $q_H^{1/2}$ where $H$ runs over all the hyperplanes in $\mathcal{F}$ separating $C$ from $C_v^+$. Thanks to [2.2](b), we have the facts that if $C'$ is a translate of $C$ then $q_{C'}^{1/2} = q_C^{1/2}$ and if some translate of $C$ is contained in $E_H^+$, then $q_{C_H}^{1/2} q_{C_H}^{1/2} \leq q_C^{1/2}$, where $\sigma_H$ is the reflection generated by the hyperplane $H$.

Recall that on the set $X$ of all alcoves, for $A, B \in X$, the distant function $d(A, B)$ is the number of hyperplanes separating $A$ from $B$ counting by signs [Lus80, 1.4] and there exists a partial order “$<$” on $X$ induced by the alcoves with distant 1 [Lus80, 1.5].
Lemma 3.1 (Lusztig) Let $W$ be an extended affine Weyl group with a parameter map $q^{1/2}: W \to \Gamma$ such that $q_{s}^{1/2}$ are positive for all $s \in S$. Let $v$ be a special point, $A$ an alcove containing $v$ in its closure and let $C$ be the hyperplane with respect to $v$ and containing $A$. Let $s_1, \ldots, s_k \in S$ be such that $d(A^+_v, s_k \cdots s_1A^+_v) = k$ and let $1 \leq i_1, \ldots, i_p \leq k$ such that

$$s_{i_1} \cdots s_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A) < s_{i_1} \cdots s_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A)$$

(1)

for $t = 1 \cdots p$. Then we have

(a) $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} \leq q_{C}^{1/2}$.

(b) If moreover $s_k \cdots s_1(A^+_v) \subset \Pi^+_v$ and $A \neq A^+_v$, then $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} < q_{C}^{1/2}$.

The proof of this lemma is almost the same as in [Lus85]. But we still write down the details considering its fundamental role in this article.

Proof. We prove (a) by induction on $p$. When $p = 0$ (a) is trivial. Assume now $p \geq 1$ and that the assertion holds for $p - 1$. Let $H$ be the hyperplane separating $s_{i_1} \cdots s_1(A)$ from $s_{i_1} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A)$. And let $v' = v\sigma_H, A' = A\sigma_H, C' = C\sigma_H$ where $\sigma_H$ is the reflection with respect to $H$. We have $d(A^+_v, s_k \cdots s_1A^+_v) = k$ since $A^+_v$ is a translate of $A^+_v$. Also $\hat{s}_{i_1} \cdots s_{2s_1}(A) = s_{i_1} \cdots s_{2s_1}(A')$, hence

$$s_{i_1} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A) < s_{i_1} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A')$$

(2)

for $t = 2 \cdots p$. Thus by induction hypothesis, $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} \leq q_{C}^{1/2}$ holds.

Let $H'$ be the hyperplane separating $\hat{s}_{i_1} \cdots s_1(A^+_v)$ from $s_{i_1} \cdots s_1(A^+_v)$, then $s_{i_1} \cdots s_1(A^+_v)$ and some translate of $C^+_v$ are on the same side of $H'$ since $s_{i_1} \cdots s_1(A^+_v) > \hat{s}_{i_1} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A^+_v)$. This implies that some translate of $C$ and $\hat{s}_{i_1} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A)$ are on the same side of $H$. Considering $s_{i_1} \cdots s_1(A) < \hat{s}_{i_1} \cdots s_1(A)$, some translate of $C$ is contained in $E_H$ and hence some translate of $C' = C\sigma_H$ is contained in $E'_{H'}$. Thus $q_{H}^{1/2} q_{C}^{1/2} \leq q_{C}^{1/2}$. Noting that $q_{H}^{1/2} = q_{i_1}^{1/2}$, we have $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} \leq q_{H}^{1/2} q_{C}^{1/2} \leq q_{C}^{1/2}$. This proves (a).

(b) also can be proved by inductions on $p$. When $p = 0$ the assertion holds since $A \neq A^+_v$ and $q_{C}^{1/2} > 1$ for each $s \in S$. Assume now $p \geq 1$ and that the assertion holds for $p - 1$. Let $H$ be the hyperplane separating $\hat{s}_{i_1} \cdots s_1(A)$ from $s_{i_1} \cdots s_1(A)$. And let $v' = v\sigma_H, A' = A\sigma_H, C' = C\sigma_H$ where $\sigma_H$ is the reflection with respect to $H$. We have $d(A^+_v, s_k \cdots s_1A^+_v) = k$ and $s_k \cdots s_1A^+_v \subset \Pi^+_v$, $A^+_v$ is a translate of $A^+_v$. Also $\hat{s}_{i_1} \cdots s_{2s_1}(A) = s_{i_1} \cdots s_{2s_1}(A')$, hence

$$s_{i_1} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A) < s_{i_1} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_t} \cdots \hat{s}_{i_1} \cdots s_1(A')$$

(3)

for $t = 2 \cdots p$.

If $A' \neq A^+_v$, then by induction hypothesis $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} < q_{C}^{1/2}$ and hence $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} < q_{s_{i_1}}^{1/2} q_{C}^{1/2} = q_{H}^{1/2} q_{C}^{1/2} \leq q_{C}^{1/2}$ as the argument in (a).

If $A' = A^+_v$, then $q_{C}^{1/2} = 1$. This implies that $p - 1 = 0$ using (a) to $A'$. Therefore $q_{s_{i_1}}^{1/2} \cdots q_{s_{i_p}}^{1/2} = q_{H}^{1/2} < q_{C}^{1/2}$ since some translate of $C$ is contained in $E_{H}^-$ and $H$ is not a wall of $C$ by the hypothesis in (b). This completes the proof of (b).

Let $\mathcal{M}$ be a $\mathbb{Z}[\Gamma]$ module with basis the set $X$ of all alcoves. Denote by $\tilde{A}$ the corresponding element in $\mathcal{M}$ for an alcove $A \in X$. Define $\mathcal{M}$ to be an $\mathcal{H}$ module via

$$T_s \tilde{A} = \begin{cases} \tilde{sA} & \text{if } sA > A \\ \tilde{sA} + (q_{s}^{1/2} - q_{s}^{-1/2}) \tilde{A} & \text{if } sA < A \end{cases}$$

(4)

(5)

7
and \( \tilde{T}_x \tilde{A} = \tilde{A} \) for \( \pi \in \Omega \).

Then it is easy to see that, for \( w \in W, \ A \in X, \)
\[
\tilde{T}_w \tilde{A} = \sum_{B \in \mathcal{X}} \pi_{w,A,B} \tilde{B}
\]
where \( \pi_{w,A,B} \) are polynomials in \( \xi_s = q_s^{1/2} - q_s^{-1/2}, s \in S \) with integral and positive coefficients.

We define a map \( \sigma : \mathbb{Z}[\Gamma] \to \Gamma \) through \( \sigma(\sum_{\gamma_i \in \Gamma} a_i \gamma_i) = \gamma_{i_0} \) where \( a_i \in \mathbb{Z} \) and \( \gamma_{i_0} \) is the unique maximal element in the set \( \{\gamma_i | a_i \neq 0\} \).

The following corollary is reformulated from [Lus85] 4.1.

**Corollary 3.1** Let \( v \) be a special point, \( u \in W \) be such that \( uA_v^+ \subset \Pi_v^+ \) and \( y \in W_v \) such that \( A = yA_v^+ \neq A_v^+ \). Then
\[
\sigma(\pi_{u,A,B}) < q_y^{1/2}
\]
for any \( B \in X \) where \( \tilde{T}_u \tilde{A} = \sum_{B \in \mathcal{X}} \pi_{u,A,B} \tilde{B} \).

**Proof.** Let \( u = \pi s_k \cdots s_1 \) be a reduced expression with \( \pi \in \Omega, s_i \in S, 1 \leq i \leq k \). Then \( uA_v^+ \subset \Pi_v^+ \) implies that \( d(A_v^+, s_k \cdots s_1 A_v^+) = k \).

By definition, we have
\[
\tilde{T}_u \tilde{A} = \tilde{T}_{s_k} \cdots \tilde{T}_{s_1}(A) = \sum_{I \in \mathcal{I}} \prod_{k=1}^{p_I} (q_{s_{i_k}}^{1/2} - q_{s_{i_k}}^{-1/2}) s_k \cdots \tilde{s}_{i_{p_I}} \cdots \tilde{s}_{i_2} \cdots \tilde{s}_{i_1} \cdots s_1(A)
\]
where \( \mathcal{I} \) is the set of all the sequence such that
\[
s_t \cdots \tilde{s}_{i_{t-1}} \cdots \tilde{s}_{i_2} \cdots \tilde{s}_{i_1} \cdots s_1(A) < \tilde{s}_t \cdots \tilde{s}_{i_{t-1}} \cdots \tilde{s}_{i_2} \cdots \tilde{s}_{i_1} \cdots s_1(A)
\]
for \( t = 1, \ldots, p_I \). Now using lemma 3.1 (ii), we obtain the conclusion. \( \square \)

**Corollary 3.2** Let \( v \) be a special point, \( u \in W \) be such that \( uA_v^+ \subset \Pi_v^+, u' \in W \) be such that \( u' A_v^+ \subset C_v^+ \) and \( y \in W_v \) be such that \( y < w_v \). Write the product \( \tilde{T}_u \tilde{T}_y \tilde{T}_{u^{-1}} \) as \( \sum_z a_{u,y,w}^z \tilde{T}_z \) with \( a_{u,y,w}^z \in \mathbb{Z}[\Gamma] \). Then we have \( \sigma(a_{u,y,w}^z) < q_y^{1/2} \) in \( \Gamma \).

**Proof.** The strategy of this proof is same as that in [Lus85] 7.9 whose aim is to prove the boundness of \( \sigma \)-function for affine Weyl group.

Let \( A = u' A_v^+ \). Then \( A \subset C_v^+ \) and hence \( \tilde{T}_{u'} \tilde{A} = \tilde{A} = w_v A_v^+ \). This implies that \( \tilde{T}_y \tilde{T}_{u^{-1}} \tilde{A} = yw_v A_v^+ \).

On one hand, \( \tilde{T}_u \tilde{T}_y \tilde{T}_{u^{-1}} \tilde{A} = \tilde{T}_u yw_v A_v^+ = \sum_B \pi_{u,y,w,A_v^+} \tilde{B} \). On the other hand, \( \sum_z a_{u,y,w,u}^z T_z \tilde{A} = \sum_z a_{u,y,w,u}^z \pi_{z,A,B} \tilde{B} \). Thus we have \( \sum_z a_{u,y,w,u}^z \pi_{z,A,B} = \pi_{u,y,w,A_v^+} \). Since \( a_{u,y,w,u}^z, \pi_{z,A,B} \) and \( \pi_{u,y,w,A_v^+} \) are all polynomials in \( \xi_s \) with positive integral coefficients, we have \( \sigma(a_{u,y,w,u}^z) \leq \sigma(\pi_{u,y,w,A_v^+}) = \sigma(\pi_{u,y,w,A_v^+}) < q_y^{1/2} \). It is easy to see that \( \pi_{z,A,B} \) has constant term 1. Therefore \( \sigma(a_{u,y,w,u}^z) \leq \sigma(\sum_z a_{u,y,w,u}^z \pi_{z,A,B} \leq \sigma(\pi_{u,y,w,A_v^+}) < q_y^{1/2} \).

The last strict inequality is exactly the Corollary 3.1. This completes the proof. \( \square \)
3.2 Boundedness of \( m_{x,y,z} \)

**Lemma 3.2** Let \( C = yA_v^+, \; y \in W_v, \; w' A_v^+ \subset C_v^+ \), \( w' = \pi s_k \cdots s_1 \) is a reduced expression with \( \pi \in \Omega, \; s_i \in S, \; 1 \leq i \leq k \). As before we can write
\[
\tilde{T}_w \tilde{C} = \sum_{l \in \mathcal{I}} (q_{1/2}^l - q_{1/2}^{-l})^t (q_{2/2}^l - q_{2/2}^{-l})^t p_{s_k}^l \cdots \tilde{s}_i \cdots s_1 (C)
\]
where \( \mathcal{I} \) is the set of all the sequence \( i_1 < i_2 < \cdots < i_{p_l} \) such that
\[
s_1 \cdots \tilde{s}_{i_{t-1}} \cdots \tilde{s}_{i_t} \cdots s_1 (C) < \tilde{s}_t \cdots \tilde{s}_{i_{t-1}} \cdots \tilde{s}_{i_t} \cdots s_1 (C)
\]
for \( t = 1, \ldots, p_l \). Then
\[
p_{l_1}^t \leq l'(y), \; p_{l_2}^t \leq l'(y)
\]
where \( p_{l_1}^t = \# \{ s_i q_{l_i} = q_{1/2}^l, \; t = 1 \ldots p_l \} \), \( p_{l_2}^t = \# \{ s_i q_{l_i} = q_{2/2}^l, \; t = 1 \ldots p_l \} \) and \( l'(y) \) (resp. \( l''(y) \)) is the number of simple reflections in the reduced expression of \( y \) whose parameters are \( q_{1/2}^l \) (resp. \( q_{2/2}^l \)).

It is not difficult to see that this is just the lemma 3.1(a) when one of the parameters is taken to be 1, whose proof is completely same as that of lemma 3.1(a). \( \square \)

**Lemma 3.3** For any \( x, \; y, \; z \in W \), \( m_{x,y,z} \) is a polynomial in \( \xi_i = q_{1/2}^l - q_{1/2}^{-l}, \; i = 1, 2 \) with the degree of \( \xi_1 \) (resp. \( \xi_2 \) bounded by \( \min \{ l'(x), l'(y), l'(z) \} \) (resp. \( \min \{ l''(x), l''(y), l''(z) \} \)).

The proof of this lemma is same as lemma 7.4 in [Lus85]. \( \square \)

**Proposition 3.1** The degree of \( \xi_1 \) (resp. \( \xi_2 \)) in \( \pi_{w,A,B} \) is bounded by \( \nu_1 := l'(w_v) \) (resp. \( \nu_2 = l''(w_v) \)).

**Proof.** Choose a special point \( v \) in the closure of \( A \). We can uniquely write \( w \) as \( w' w_1 \) with \( w_1 \in W_v \) and \( w' \) satisfying \( w' A_v^+ \subset C_v^+ \). We can find \( w_2 \in W_v \) such that \( A = w_2 A_v^- \).

Thus \( \tilde{T}_{w_1} \tilde{A} = \tilde{T}_{w_1} \tilde{T}_{w_2} \tilde{A}_v = \sum_{w_3 \in W_v} m_{w_1,w_2,w_3} \tilde{T}_{w_3} (\tilde{A}_v) = \sum_{w_3 \in W_v} m_{w_1,w_2,w_3} \tilde{T}_{w_3} (\tilde{A}_v) \). So \( \tilde{T}_w \tilde{A} = \sum_{w_3 \in W_v} m_{w_1,w_2,w_3} \tilde{T}_{w_3} (\tilde{w}^3 A_v) \). So \( \pi_{w,A,B} = \sum_{w_3 \in W_v} m_{w_1,w_2,w_3} \xi_1^p s_2^q \), where \( I \) runs through \( \mathcal{I} \) such that
\[
s_k \cdots \tilde{s}_i \cdots \tilde{s}_1 \cdots s_1 (w_3 A_v^-) = B.
\]
By the above two lemmas, \( p_{l_1}^t \leq l'(w_3 w_v), \; p_{l_2}^l \leq l'(w_3 w_v), \; \deg_\xi m_{w_1,w_2,w_3} \leq l'(w_3), \; \deg_\xi m_{w_1,w_2,w_3} \leq l''(w_v) \). Therefore \( \deg_\xi \pi_{w,A,B} \leq l'(w_v) \), \( \deg_\xi \pi_{w,A,B} \leq l''(w_v) \). \( \square \)

**Theorem 3.1** Let \( m_{x,y,z} \in \mathbb{Z}[\Gamma] \) be defined by \( \tilde{T}_x \tilde{T}_y = \sum_{z \in W} m_{x,y,z} \tilde{T}_z \) for \( x, \; y \in W \). \( \xi_1 = q_{1/2}^l - q_{1/2}^{-l}, \; \xi_2 = q_{2/2}^l - q_{2/2}^{-l} \). Then we can write \( m_{x,y,z} \) as the form
\[
m_{x,y,z} = \sum_{0 \leq i \leq \nu_1, \; 0 \leq j \leq \nu_2} a_{i,j} \xi_1^i \xi_2^j \tag{6}
\]
with \( a_{i,j} \in \mathbb{Z} \) and \( \nu_1 = l'(w_v), \; \nu_2 = l''(w_v) \).

**Proof.** Choose an alcove \( A \in X \) such that \( \tilde{T}_y \tilde{A} = y \tilde{A} \). \( \tilde{T}_x \tilde{T}_y = \sum_{z \in W} m_{x,y,z} \tilde{T}_z \) implies that \( \tilde{T}_x y A = \sum_{z} m_{x,y,z} \tilde{T}_z \tilde{A} \). Thus \( \pi_{x,y,A,B} = \sum_{z} m_{x,y,z} \pi_{z,A,B} \). Since \( \pi_{x,y,A,B}, \; m_{x,y,z}, \; \pi_{z,A,B} \) are all polynomial in \( \xi_1, \xi_2, \) with positive integral coefficients and \( \xi_1, \xi_2 \neq 0 \), we have \( \deg_\xi m_{x,y,z} \leq \deg_\xi \pi_{z,A,B} \leq \nu_1 \) for \( i = 1, 2 \) where the last inequality follows from Proposition 3.1. \( \square \)

**Corollary 3.3** For any \( x, \; y, \; z \in W \), we have \( \sigma (m_{x,y,z}) \leq q_{w_v}^{1/2} \) in \( \Gamma \). \( \square \)

**Definition 3.1** Let \( x, \; y, \; z \in W \) such that one of them is in \( c_0 \), \( \gamma_{x,y,z}^{-1} \) is defined to be the coefficient of \( \xi_1^3 \xi_2^2 \) in \( m_{x,y,z} \).
3.3 Based rings of lowest two-sided cells

As the one parameter case, it is easy to see that \( h_{x,y,z} \) is also bounded by \( q^{1/2}_w \) and the coefficient of \( q^{1/2}_w \) in \( h_{x,y,z} \) is \( \gamma_{x,y,z}^{-1} \) for \( x, y, z \in c_0 \).

Let \( J_0 \) be a ring generated by \( \{ t_w \mid w \in c_0 \} \) as a free \( \mathbb{Z} \) module and with multiplication \( t_xt_y = \sum_{z \in c_0} \gamma_{x,y,z}^{-1}t_z \) for \( x, y \in c_0 \). Viewing \( \gamma_{x,y,z}^{-1} \) as the coefficient of \( q^{1/2}_w \) in \( h_{x,y,z} \), it is easy to see that \( J_0 \) is an associative ring. But it is not easy to determine the unit of \( J_0 \). Our strategy is to show that \( \gamma_{x,y,z} \) is actually independent of the choice of the parameter map.

**Theorem 3.2** For \( x, y, z \in W \), \( \gamma_{x,y,z}^{-1} \) is independent of the choice of \( \Gamma \) under the assumption that \( q^{1/2}_s \) > 1\( \Gamma \) for \( s \in S \).

**Proof.** Let \( \Phi \) be the free abelian group with free generators \( \tilde{q}^{1/2}_1, \tilde{q}^{1/2}_2 \). Then there exists a unique homomorphism \( \phi : \Phi \rightarrow \Gamma \) such that \( \phi(q^{1/2}_1) = q^{1/2}_1 \), \( \phi(q^{1/2}_2) = q^{1/2}_2 \). Denote by \( m'_{x,y,z} \) (resp. \( \tilde{\xi}_1, \tilde{\xi}_2 \)) the counterpart of \( m_{x,y,z} \) (resp. \( \xi_1, \xi_2 \)) in the case when the parameter map is \( \tilde{q} : W \rightarrow \Phi \). Then, by Theorem 3.1 we can write \( m'_{x,y,z} = \sum_{0 \leq i \leq v_1} a'_{i,j} \xi_1^i \xi_2^j \). It is easy to see that \( \phi(m'(x,y,z)) = m_{x,y,z} \). That is \( \sum_{0 \leq i \leq v_1} a'_{i,j} \xi_1^i \xi_2^j = \sum_{0 \leq j \leq v_2} a_{i,j} \xi_1^i \xi_2^j \). Therefore \( a_{v_1,v_2} = a'_{v_1,v_2} \) by Lemma 2.1. This completes the proof.

From the proof we see that this theorem depends largely on Theorem 3.1.

**Theorem 3.3** \( J_0 \) is an associative ring with unit \( \sum_{d \in D_0} t_d \) where \( D_0 \) is the set of distinguished involutions \( \{ ww^{-1} | w \in W, wA_+^0 \subset \Pi_+^0 \} \). The ring \( J_0 \) will be called the based ring of (generalized) lowest two-sided cell \( c_0 \) for the Hecke algebra \( H \) with parameter map \( q^{1/2} : W \rightarrow \Gamma \).

**Proof.** In the case of one parameter, the assertion is already known [Lus87a, Lus87b]. Then the above Theorem 3.2 implies this theorem.

Now we are in the place that the lowest two-sided cell \( c_0 \) and the based ring \( J_0 \) of \( c_0 \) are both independent of the choice of the positive parameter map.

4 Homomorphisms from Hecke algebras to based rings of lowest two-sided cells

In this section, we will keep the Assumption 2.1 and Assumption 2.2 we will use the Lusztig lemma to prove Xi formula, which make the structure of lowest two-sided rings very clear. Using this formula we can construct the homomorphisms from Hecke algebras to based rings of lowest two-sided cells.

4.1 Xi formula

The following formula plays an important role in understanding the lowest two-sided cell \( c_0 \). It is first appeared in [Xi90] in the setup of affine Hecke algebra with one parameter.
Theorem 4.1 Let \( v \) be a special point, \( w \in W \) be such that \( wA_v^+ \subset \Pi_v^+ \), \( w' \in W \) be such that \( w'A_v^+ \subset C_v^+ \). Then

\[
C_{wwv}w'^{-1} = E_wC_{ww}F_{w'}.
\]

Proof. On one hand,

\[
E_wC_{ww}F_{w'} = \sum_{u,y,u'} P^u_{wwv,vuv}a^{-1/2}_u P^u_{u'w,vwv}z \tilde{T}_u \tilde{T}_y \tilde{T}_{u'}^{-1} \\
= \sum_{u,y,u,z} P^u_{wwv,vuv}P^{u'}_{u'w,vwv}((a^{-1/2}_u a^z_{u,y,u'})z) \tilde{T}_z \\
\equiv \tilde{T}_v \tilde{T}_w \tilde{T}_{w'}^{-1} \mod \mathbb{Z}[\Gamma<0] \quad \text{(using Corollary 3.2)}
\]

On the other hand, \( E_wC_{ww}F_{w'} \) is \( \tilde{\gamma} \)-invariant due to lemma 2.2(ii)(iii)(iv). Then the theorem follows from the \( \tilde{\gamma} \)(a). \( \square \)

Corollary 4.1 Let \( w \in B_v \), \( w' \in B_v \), and \( u \in W \) such that \( uA_v^+ = A_v^+ \subset C_v^+ \). Then

\[
C_{wwuv}u'^{-1} = E_wC_{ww}F_{w'}.
\]

Proof. It suffices to prove \( E_wC_{ww} = E_wC_{ww} \). Write \( uwv = u^{-1}wv \) and \( u'wv = u^{-1}wv \) and \( u'v = A_v^+ \subset C_v^+ \), which implies that \( u'v = A_v^+ \subset C_v^+ \). Therefore \( E_wC_{wwv}w' = E_wC_{wwv}w' = E_wC_{wwv}F_{w'} = C_{wwv}w' = C_{wwv} = E_wC_{ww} \). This completes the proof. \( \square \)

Corollary 4.2 Fix a special point \( v \). Let \( \Sigma_w = \{ w'w_v^{-1} \mid w' \in U_v \} \) for \( w \in B_v \). Then \( \Sigma_w \) is all the left cells in \( c_0 \). The number of left cells in \( c_0 \) is \( |W_0| \).

Proof. This follows from lemma 2.2(iv) and Theorem 4.1. \( \square \)

Remark 4.1 (i) Note that \( \Sigma_w \) depends on the choice of \( v \) while \( c_0 \) does not.

(ii) In the case of affine Hecke algebras with unequal parameters, the lowest two-sided cells have been largely studied in [Gui08, Gui14]. For \( \mathfrak{sl}_2 \), the left cells in the lowest two-sided cells of affine Hecke algebras with unequal parameters have been determined in [Gui08]. And a formula similar to Corollary 4.1 are also obtained in [Gui14, Cor. 5.4]. In some sense, Corollary 4.1 is more precise considering the result in [Xi94, Ch. 3] which connects \( C_{ww} \) with the elements in the center of \( \mathcal{H} \).

4.2 Homomorphism from \( \mathcal{H} \) to \( J_0 \)

Let \( \mathcal{H}' \) be another Hecke algebra with parameter map \( q' : W \rightarrow \Gamma' \), which is completely same as \( \mathcal{H} \). The counterparts of \( h_{x,y,z}, c_x \) in \( \mathcal{H}' \) will be denoted by \( c'_x, h'_{x,y,z} \).

Let \( \mathcal{E} \) be the free \( \mathbb{Z}[\Gamma] \otimes \mathbb{Z}[\Gamma'] \) module generated by \( \{ \mathcal{E}_w | w \in c_0 \} \). We can define a left (resp. right) module structure of \( \mathcal{H} \) (resp. \( \mathcal{H}' \)) on \( \mathcal{E} \) by defining

\[
C_x \mathcal{E}_w = \sum_{z \in W} h_{x,w,z} \mathcal{E}_z \quad \text{for} \ x \in W \text{ and } w \in c_0
\]

\[
\mathcal{E}_w C_y = \sum_{z \in W} h'_{w,y,z} \mathcal{E}_z \quad \text{for} \ y \in W \text{ and } w \in c_0
\]

Lemma 4.1 The actions of \( \mathcal{H} \) and \( \mathcal{H}' \) on \( \mathcal{E}_0 \) are commutative.
Proof. The following claim is needed.
(a) Let \( u, u' \) be two elements in \( W \) such that \( uA_+ \subset C_+ \), \( u'A_+ \subset C_+ \), then \( (C_{uw}, E_{wu})C'_{w,u'} = C_{uw}(E_{wu}C'_{w,u'}) \).

\[
(C_{uw}, E_{wu})C'_{w,u'} - 1 = h_{wu, w'u'} C'_{w,u'} - 1
= h_{wu, w'u'} \sum_{z \in G} h'_{wu, w'w'z} E_z
= h_{wu, w'u'} \gamma'_{wu, w'w'z} E_z
\]

The last equality is from Lemma 2.2 (v) and Corollary 4.1. Then the similar computation for \( C_{uw}, E_{wu}C'_{w,u'} \) implies the claim (a).

Now we prove the lemma. For any \( x, y \in W, w \in B_v, w' \in U_v \), we have

\[
(C_x E_{wu}C'_{w,u'}) C_y = \frac{1}{h_{wu, w'u'}} (C_x E_{wu}C'_{w,u'}) C_y
= \frac{1}{h_{wu, w'u'}} (C_x E_{wu}C'_{w,u'}) C_y
\]

by Lemma 2.2 (iv)

\[
= \frac{1}{h_{wu, w'u'}} \left( \sum_{u \in U_v} h_{x, wu, w'u} C_{uw} E_{wu} C'_{w,u'} \right) C_y
= \frac{1}{h_{wu, w'u'}} \left( \sum_{u \in U_v} h_{x, wu, w'u} C_{uw} E_{wu} C'_{w,u'} \right) C_y
\]

by claim (a)

\[
= \frac{1}{h_{wu, w'u'}} \left( \sum_{u \in U_v} h_{x, wu, w'u} C_{uw} E_{wu} C'_{w,u'} \right) C_y
= \frac{1}{h_{wu, w'u'}} \left( \sum_{u \in U_v} h_{x, wu, w'u} C_{uw} E_{wu} C'_{w,u'} \right) C_y
\]

Similarly we can get

\[
C_x (E_{wu}C'_{w,u'}) C_y = \frac{1}{h_{wu, w'u'}} \sum_{u \in U_v} h_{x, wu, w'u} h'_{w'u, w'u'} (C_{uw} E_{wu} C'_{w,u'})
\]

Using claim (a) again, we see that \( (C_x E_{wu}C'_{w,u'}) C_y = C_x (E_{wu}C'_{w,u'}) C_y \), \( \square \).
Corollary 4.3  For any \( x, z \in W \), \( y \in c_0 \), we have
\[
\sum_{w \in c_0, v \in c_0} h_{x,y,w} \gamma_{w,z,v}^{-1} = \sum_{w \in c_0, v \in c_0} h_{w,z,v}^{-1} \gamma_{y,z,w}^{-1}.
\]

Proof. By Lemma \[\ref{lem:main} \] we have \((C_x E_y) C_z' = C_x(E_y C_z')\) in \( \mathcal{E} \). Thus \( \sum_{w \in c_0, v \in c_0} h_{x,y,w} h_{w,z,v}^{-1} = \sum_{w \in c_0, v \in c_0} h_{w,z,v}^{-1} h_{y,z,w}^{-1} \). Taking the coefficients of \( q_0^{1/2} \) in this equation, we then get the corollary. □

Theorem 4.2  There is a homomorphism of rings preserving units
\[
\phi : \mathcal{H} \rightarrow Z[\Gamma] \otimes Z J_0, \quad C_x \mapsto \sum_{d \in D_0, z \in c_0} h_{x,d,z} t_z
\]
for the extended affine Hecke algebra with parameter map \( q^{1/2} : W \rightarrow \Gamma \).

Proof. To prove \( \phi(C_x C_y) = \phi(C_x) \phi(C_y) \) for \( x, y \in W \), we have to prove
\[
\sum_{w \in C_0, d \in D_0} h_{x,y,w} h_{w,d,z} = \sum_{d', d'' \in D_0, u,v \in c_0} h_{x,d',u} h_{d'',v} \gamma_{u,v,z}^{-1}
\]
for \( z \in c_0 \).

By the Corollary \[\ref{cor:main} \] we have \( \sum_{w \in c_0} h_{x,d,u} \gamma_{u,v,z} = \sum_{u \in c_0} h_{x,u,z} \gamma_{d,v,u}^{-1} \). Recall that for \( d \in D_0 \), \( \gamma_{d,v,u}^{-1} \neq 0 \) implies that \( u = v \) and for any \( y \in c_0 \), there exists a unique \( d \in D_0 \) such that \( \gamma_{d,y,y}^{-1} \neq 0 \). Hence
\[
\sum_{u \in c_0, d \in D_0} h_{x,d,u} \gamma_{u,v,z}^{-1} = h_{x,v,z}, \quad (10)
\]
Thus the right hand side of equation \[(10)\] is equal to \( \sum_{d \in D_0} h_{y,d,v} h_{x,v,z} \). And this is equal to the left hand side of \( (9) \) by taking the coefficients of \( C_z \) in \( \sum_{d \in D_0} C_z(C_y C_d') = \sum_{d \in D_0}(C_x C_y) C_d' \). Now we have proved \( \phi \) preserves the multiplications.

It is easy to see that \( \phi(1) = \sum_{d \in D_0, z \in c_0} h_{e,d,z} t_z = \sum_{d \in D_0} t_d \), which is the unit of \( Z[\Gamma] \otimes Z J_0 \). This completes the proof.

5 Some irreducible representations via the lowest two-sided cell

The based ring has been used to study efficiently the irreducible representation of affine Hecke algebra with one parameter \[\Xi07\]. Now that the based rings of lowest two-sided cells of extended affine Hecke algebras with two parameters have been determined in the previous sections, so we can obtain a class of irreducible representations of extended affine Hecke algebras over \( \mathbb{C} \) with parameters being complex numbers. The method comes from the one in \[\Xi07\].

In this section, \( \mathcal{H} \) represents an extended affine Hecke algebras with free parameters (see Example \[2.1(ii)\]).

After specializing to \( \mathbb{C} \), the homomorphism \( \phi : \mathcal{H} \rightarrow J_0 \otimes \mathbb{Z}[\Gamma] \) gives a homomorphism \( \phi : \mathcal{H}_\mathbb{C} \rightarrow J_{0,\mathbb{C}} := J_0 \otimes \mathbb{C} \). We can associate a \( J_{0,\mathbb{C}} \) module \( J_{0,\mathbb{C}} N \) to an \( \mathcal{H}_\mathbb{C} \) module \( \mathcal{H}_\mathbb{C} N \) through the homomorphism \( \phi : \mathcal{H}_\mathbb{C} \rightarrow J_{0,\mathbb{C}} \). This module also is denoted by \( N_\phi \) or
\( \phi_*(N) \). Let \( Z_0 \) be vector space over \( \mathbb{C} \) with basis \( \{ Z_w | w \in c_0 \} \). Define a left \( J_{0,\mathcal{C}} \) module structure on \( Z_0 \) by

\[
t_u Z_w = \sum_{v \in c_0} \gamma_{u,w,v}^{-1} Z_v \quad \text{for } u \in c_0, w \in c_0
\]

and a right \( \mathcal{H}_\mathcal{C} \) module structure on \( Z_0 \) by

\[
Z_w C_x = \sum_{v \in c_0} h_{w,x,v} Z_v \quad \text{for } w \in c_0, x \in W.
\]

Then by Corollary 4.3 \( Z_0 \) is a \( J_{0,\mathcal{C}}\mathcal{H}_\mathcal{C} \) bimodule. Via the homomorphism \( \phi : \mathcal{H}_\mathcal{C} \to J_{0,\mathcal{C}} \), \( Z_0 \) becomes an \( \mathcal{H}_\mathcal{C} \) bimodule. Then we claim that the left \( \mathcal{H}_\mathcal{C} \) action on \( Z_0 \) is just the action \( C_x Z_w = \sum_{v \in c_0} h_{x,w,v} Z_v \) for \( x \in W, w \in c_0 \), which can be verified using the equation (10).

Let \( M \) be an \( \mathcal{H}_\mathcal{C} \) module. Then \( \hat{M} := J_{0,\mathcal{C}} Z_0 \mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{C} \) \( M \) becomes a \( J_{0,\mathcal{C}} \) module and \( \hat{M} := \mathcal{H}_\mathcal{C} Z_0 \mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{C} \) \( M \) becomes a \( \mathcal{H}_\mathcal{C} \) module. Note that \( \hat{M} = \phi_*(\hat{M}) \). It is easy to verify that \( \hat{M} \mapsto M, Z_w \otimes m \mapsto C_u m \) is a homomorphism of \( \mathcal{H}_\mathcal{C} \) modules.

Let \( E \) be a \( J_{0,\mathcal{C}} \) module and \( N \) be a \( \mathcal{H}_\mathcal{C} \) submodule of \( E_\phi \). Then it can be verified that \( \hat{N} \mapsto E, Z_w \otimes n \mapsto \phi(C_u) n \) is a homomorphism of \( J_{0,\mathcal{C}} \) modules using equation (10). The image of this map is \( I_0 N \), where \( I_0 \) is the two-sided ideal of \( \mathcal{H}_\mathcal{C} \) corresponding to the lowest two-sided cell \( c_0 \).

**Lemma 5.1** Let \( E \) be a simple \( J_{0,\mathcal{C}} \) module such that \( C_{w_0} E_\phi \neq 0 \) where \( w_0 \) is the longest element of \( W_0 \). Then \( K = \{ b \in E_\phi | C_u b = 0 \text{ for all } u \in c_0 \} \) is the unique maximal submodule of \( E_\phi \).

In particular, \( E_\phi \) has only one composition factor \( M' \) such that \( C_{w_0} M' \neq 0 \). And \( M' \) is the unique simple quotient of \( E_\phi \).

**Proof.** It is easy to see that \( K \) is an \( \mathcal{H}_\mathcal{C} \) submodule of \( E_\phi \).

Let \( v \in E_\phi, v \notin K \) and \( N = \mathcal{H}_\mathcal{C} v \). Then the image \( I_0 N \) of the homomorphism \( \hat{N} \mapsto E \) is nonzero. Since \( E \) is a simple \( J_{0,\mathcal{C}} \) module, we have \( E = I_0 N \). Thus \( N = E_\phi \). Therefore \( K \) is the unique maximal submodule of \( E_\phi \). This completes the proof. \( \square \)

**Theorem 5.1** Let \( S \) be the set of simple \( J_{0,\mathcal{C}} \) modules \( E \) such that \( C_{w_0} E_\phi \neq 0 \) and \( T \) be the set of simple \( \mathcal{H}_\mathcal{C} \) modules \( M \) such that \( C_{w_0} M \neq 0 \). Then there is a well-defined map \( \rho : S \to T, E \mapsto \rho(E) := E_\phi / K \) where \( K \) is the unique maximal submodule of \( E_\phi \). Furthermore, \( \rho \) is bijective.

**Proof.** Lemma 5.1 implies that \( \rho \) is well-defined.

Now we prove \( \rho \) is surjective. Let \( M \in T \). Then the map \( \hat{M} \mapsto M \) is nonzero and hence surjective. Since \( \hat{M} = \phi_*(\hat{M}) \), \( \hat{M} \) must have a composition factor \( E \) such that \( E_\phi \) has a composition factor \( M \). Since \( C_{w_0} M \neq 0 \), by Lemma 5.1 \( M \) is the unique simple quotient of \( E \), i.e. \( \rho(E) = M \). Therefore \( \rho \) is surjective.

Now we prove \( \rho \) is injective. Let \( E \in S \) and \( \pi : E_\phi \to M \) is the quotient map. Let \( \rho' : \hat{E} \to \hat{M}, Z_u \otimes e \mapsto Z_u \otimes \pi(e) \) be the homomorphism of \( J_{0,\mathcal{C}} \) modules induced from \( \pi \). Then we have commutative diagram

\[
\begin{array}{ccc}
\hat{E} & \xrightarrow{\rho'} & \hat{M} \\
\downarrow{\theta} & & \downarrow{\rho} \\
E & \xrightarrow{\pi} & M
\end{array}
\]
where $\theta$ and $p$ are the natural maps as previous.

$p'$ induces a surjective homomorphism $p' : \hat{E}/\ker \theta \to \hat{M}/p'(\ker \theta)$ of $J_{0,\mathbb{C}}$ modules. On one hand, $C_{w_0}M \neq 0$ implies that $p$ is surjective and hence $\ker p \neq \hat{M}$. On the other hand, the commutative diagram implies that $p'(\ker \theta) \subset \ker p$. Then we have $\hat{M}/p'(\ker \theta) \neq 0$. Since $\hat{E}/\ker \theta \simeq E$ is simple, $\hat{p}'$ is an isomorphism. Therefore $E$ is a composition factor of $\hat{M}$. Using Lemma 5.1, it is easy to see that $\hat{M}$ admits one and only one composition factor $E'$ such that $C_{w_0}E'_\phi \neq 0$. Since $C_{w_0}E_\phi \neq 0$, $E$ is unique determined by $\hat{M}$. Thus $\rho$ is injective. This completes the proof. \qed

We now recall the irreducible representations of $J_{0,\mathbb{C}}$, which are given explicitly in [Xi90].

Let $M_{B_\sigma \times B_\sigma}(\mathbb{C})$ be the set of matrices with rows and columns are indexed by the set $B_v = \{w \in W|wA^+_\sigma \subset \Pi^+_\sigma\}$. Denote by $e_{w,w'}$ the matrix in $M_{B_\sigma \times B_\sigma}(\mathbb{C})$ with 1 at place $(w, w')$ and 0 elsewhere.

Let $G$ be the simply connected simple algebraic groups over $\mathbb{C}$ corresponding to the root system $\Pi$. Let $u \in W$ be such that $uA_\sigma^+ = A_\sigma^+ \subset C_\sigma^+$. Then $u$ corresponds to a dominant weight and hence corresponds to a simple module of $G$, which is denoted by $V(u)$.

Let $s$ be a semisimple element of $G$. Then the map $\psi_s : J_{0,\mathbb{C}} \to M_{B_\sigma \times B_\sigma}(\mathbb{C})$, $t_{wuw,w'-1} \mapsto \text{tr}(s, V(u))e_{w,w'}$ gives an irreducible representations of $J_{0,\mathbb{C}}$. Let $SS$ be the set of representatives of semisimple conjugacy classes of the algebraic groups $G$, and let $\text{Irr} J_{0,\mathbb{C}}$ be the set of all irreducible representations of $J_{0,\mathbb{C}}$. Then $s \mapsto \psi_s$ gives a bijective map from $SS$ to $\text{Irr} J_{0,\mathbb{C}}$, see [Xi90]. Denote by $E_s$ the underlying vector space of the representation $\psi_s$. Then $\text{Irr} J_{0,\mathbb{C}} = \{E_s|s \in SS\}$.

**Theorem 5.2** When $C^2_{w_0} \neq 0$, then $C_{w_0}E_\phi \neq 0$ for all $E \in \text{Irr} J_{0,\mathbb{C}}$. Therefore, $s \mapsto \rho(E_s)$ is a bijective map from $SS \to \mathcal{T}$.

**Proof.** $\phi(C_{w_0})$ acts on $E_s$ by matrix $\psi_s(\phi(C_{w_0})) = \psi_s(\sum_{d \in D_\sigma, v \in c_0} h_{w_0,d,v}t_v)$, whose entry at place $(w_0,w_0)$ is $h_{w_0,w_0,w_0}$. Since $C^2_{w_0} = h_{w_0,w_0,w_0}C_{w_0} \neq 0$, $h_{w_0,w_0,w_0} \neq 0$. Hence the action of $\phi(C_{w_0})$ on $E_s$ is nonzero for all $s \in SS$. \qed

**Corollary 5.1** Assume $C^2_{w_0} \neq 0$. Then $\hat{M}$ is simple for any $M \in \mathcal{T}$. Hence $M \mapsto \hat{M}$ gives an inverse map of $\rho : S \to \mathcal{T}$.

**Proof.** We have know from the proof of Theorem 5.1 that $\hat{M}$ has only one composition factor $E$ such that $C_{w_0}E_\phi \neq 0$. By Theorem 5.2 all the simple $J_{0,\mathbb{C}}$ modules $E'$ satisfy $C_{w_0}E'_\phi \neq 0$. Thus $\hat{M}$ is simple.

At last, we give a formula for the dimension of the simple $H_{\mathbb{C}}$ modules $M$ such that $C_{w_0}M \neq 0$.

**Proposition 5.1** Let $s$ be a semisimple element in $G$ and $C_{w_0}(E_s)_\phi \neq 0$. Then the dimension of the irreducible $H_{\mathbb{C}}$ module $\rho(E_s)$ is the rank of the matrix $(m_{w,w'}(s))_{w,w' \in B_\sigma}$, where

$$m_{w,w'}(s) = \sum_{u \in W} h_{w,u^{-1},w',uw} \text{tr}(s, V(u)).$$

**Proof.** By Lemma 5.1 the $(E_s)_\phi$ has a unique maximal submodule $K = \{v \in (E_s)_\phi|Cu v = 0 \text{ for all } u \in c_0\}$. Furthermore, by Theorem 4.1 we have $K = \{v \in (E_s)_\phi|C_{w_0,u^{-1}}v = 0 \text{ for all } w \in B_\sigma\}$. 

15
Noting that
\[
\psi_\alpha(\phi(C_{w,w^{-1}})) = \sum_{w' \in B_v} h_{w,w^{-1},w,w',w^{-1}} \text{tr}(s,V(u))e_{1,w'}
\]
\[
= \sum_{w' \in B_v} h_{w,w^{-1},w,w',w} \text{tr}(s,V(u))e_{1,w'}
\]
\[
= \sum_{w' \in B_v} m_{w,w'}e_{1,w'},
\]
we have \(\dim(K) = \# B_v - \text{rank}(m_{w,w'}(s))_{w,w' \in B_v}\). Therefore
\[
\dim(\rho(E_s)) = \text{rank}(m_{w,w'}(s))_{w,w' \in B_v}.
\]

6 Affine case

In this section, we will extend our previous results about the lowest two-sided cells of extended affine Hecke algebras with positive parameter maps to the case of affine Hecke algebras with positive parameter maps. And we will see that the Lusztig conjectures in [Lus03, Chapter 14] hold in the case of lowest two-sided cells.

6.1 Remark on the affine case

Let \(H'\) be an affine Hecke algebra corresponding to an affine Weyl group \(W'\) and a positive parameter map \(q^{1/2} : W' \rightarrow \Gamma\). Now the affine Weyl group \(W'\) of type \(\widetilde{A}_1\) admits 3 parameters, \(W'\) of type \(\widetilde{A}_1\) admits 2 parameters, and \(q_H^{1/2}\) may be not equal to \(q_H^{1/2}\) when \(H\) is parallel to \(H'\) for \(W'\) (see [Bre97]). But the work of [Bre97] suggests that all the things will be almost the same to the extended affine case if we modified the concept of special points.

Let \(T\) be the set of \(v \in E\) such that \(\prod_{v \in \Gamma, w \in F} q_{\Gamma}^{1/2}\) is maximal, which generalizes the concept of special points.

Theorem 6.1 Let \(H'\) be an affine Hecke algebra with a positive parameter map \(q^{1/2} : W' \rightarrow \Gamma\).

(i) The subset \(c'_0 = \{ww,w'w'^{-1} | w \in U'_v, w' \in B'_v, v \in T\}\) of \(W'\) is the lowest two-sided cell of \(W'\) with respect to \(H'\) and the positive parameter map \(q^{1/2} : W' \rightarrow \Gamma\), where \(U'_v = \{w \in W' | wA^+_w \subset C^+_w\}\) and \(B'_v = \{w \in W' | wA^+_w \subset \Pi^+_w\}\).

(ii) We still have Xi formula
\[
C_{ww,ww'^{-1}} = E_wC_{ww'}F_{ww'^{-1}}
\]
for \(v \in T, w \in U'_v, w' \in B'_v\) where \(E_w = \sum_{u \leq w} P^*_{ww,ww} \tilde{T}_u\) and \(F_w = \sum_{l(wu_v) = l(u) + l(w_v)} P^*_{ww,ww} \tilde{T}_u^{-1}\). This formula also implies that \(N_{\lambda,z} = \{C_{z'w',w'^{-1}} \mid z' \in U'_v\}, z \in B'_v\) form a \(\mathbb{Z}[\Gamma]\) basis of a left ideal of \(H'\) (see also [Gui08]).
(iii) Define a multiplication on $J'_0 = \{t_w | w \in W'\}$ by $t_xt_y = \sum_{z \in c'_0} t_{x'y}^{-1}$ for $x, y \in c'_0$, where $\gamma_{x,y,z}^{-1}$ is the coefficients of $q_{w_0}^{1/2}$ in $m_{x,y,z}$. Then $J'_0$ becomes an associative ring with the unit $\sum_{d \in D'_0} t_d$, where $D'_0 = \{ww_0w^{-1} | w \in B'_v, v \in T\} = D_0$ as a subset of $W$. In fact, $\gamma_{x,y,z}$ is independent of the choice of the positive parameter maps.

(iv) We have a homomorphism $\phi' : H' \rightarrow \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} J'_0$, $C_w \mapsto \sum_{d \in D'_0, z \in c'_0} h_{w,d,z}t_z$ of rings preserving units.

6.2 Lusztig conjectures

Assume in this section that $\Gamma = v\mathbb{Z}$. Then the positive parameter map $q^{1/2} : W' \rightarrow \Gamma$ is same as the function $L : W' \rightarrow \mathbb{Z}^+$ such that $q_{w_0}^{1/2} = vL(w)$. Then we call $(H', L)$ the affine Hecke algebra with unequal parameters following [Lus03].

In [Lus03] Chapter 14, a numerous conjectures are formulated. We can now deal with these conjectures in the situation of lowest two-sided cells using the previous observations in the process of constructing the based ring of $c_0$.

For any $x, y, z \in W'$, $h_{x,y,z}$ can be written as $\gamma_{x,y,z}v^a(z)$ + lower terms, and $P_{e,z}^* = n_zv^{-\Delta(z)} +$ lower terms, where $a(z)$ is the a-function and $n_z \neq 0$. If $x$ or $y \in c'_0$, $\gamma_{x,y,z}$ coincides with Definition 3.1 since $a(z) = L(w_u)$ for $z \in c'_0$.

**Proposition 6.1**

(i) For any $z \in c'_0$, $a(z) = L(w_u)$.

(ii) For any $z \in c'_0$, $\Delta(z) \geq L(w_0)$.

(iii) $D_0 = \{z \in c'_0 | a(z) = \Delta(z)\}$.

(iv) For $d \in D_0, n_d = 1$.

**Proof.** Using Xi formula, it is easy to see that $\gamma_{ww_0w,w,ww_0w^{-1}} = 1$ for $w \in B_v, w' \in U_v$, and (i) follows.

Define the homomorphism $\tau : H \rightarrow \mathbb{Z}[\Gamma]$ by $\tau(\tilde{T}_x) = \delta_{x,e}$. For $w \in B_v, w' \in U_v$,

$$\tau(C_{ww_0w^{-1}}) = \tau(E_{w,c}F_{w'}) = \sum_{w \leq w_0} P_{ww_0w,w_0}^* P_{w_0w_0}^* P_{w'w_0,w}^* \tau(\tilde{T}_u \tilde{T}_y \tilde{T}_u^{-1})$$

It is easy to see that $\tau(\tilde{T}_u \tilde{T}_y \tilde{T}_u^{-1}) \neq 0$ implies that $y = 1, u = u'$. Therefore

$$P_{e,ww_0w^{-1}}^* = \tau(C_{ww_0w^{-1}})$$

$$= \sum_{w \leq w_0} P_{ww_0w,w_0}^* v^{-L(w_u)} P_{ww_0w,w_0}^*$$

$$\begin{cases} v^{L(w_u)} + \text{lower terms} & \text{if } w = w' \\ \text{every term has degrees less than } v^{L(w_0)} & \text{if } w \neq w'. \end{cases}$$

Then (ii), (iii), (iv) follow. 

The following facts have been essentially proved dispersively in the previous sections. We collect them here just for comparing with the conjectures in [Lus03] Chapter 14, P1–15.
Theorem 6.2 Let $H'$ be an affine Hecke algebra with unequal parameter map $L : W' \to \mathbb{Z}^{>0}$. Then the following statements hold.

$P'1$. For any $z \in c'_0$, we have $a(z) = L(w_0) \leq \Delta(z)$.

$P'2$. If $d \in D_0$, $x, y \in c'_0$ satisfying $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.

$P'3$. For any $y \in c'_0$, there exists a unique $d \in D_0$ such that $\gamma_{y^{-1},y,d} \neq 0$.

$P'5$. If $d \in D_0$, $y \in c'_0$ and $\gamma_{y^{-1},y,d} \neq 0$ then $\gamma_{y^{-1},y,d} = n_d = 1$.

$P'6$. If $d \in D_0$, then $d^2 = e$.

$P'7$. If $x, y$ or $z \in c'_0$, then $\gamma_{x,y,z} = \gamma_{y,z,x}$.

$P'8$. If $x, y, z \in c'_0$, then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$, $z \sim_L x^{-1}$.

$P'9$. If $z, z' \in c'_0$ such that $z \leq_L z'$, then $z \sim_L z'$.

$P'10$. If $z, z' \in c'_0$ such that $z \leq_R z'$, then $z \sim_L z'$.

$P'13$. For any left cell $\Theta$ in $c'_0$, $\Theta$ contains a unique $d \in D_0$, and $\gamma_{x^{-1},x,d} \neq 0$ for any $x \in \Theta$.

$P'15$. $\sum_{y' \in c'_0} h'_{w,x',y'} h_{x,y',y} = \sum_{y' \in c'_0} h_{x,w,y'} h'_{x',y',y}$ for $w \in c'_0$ and $x, x' \in W'$.

Proof. $P'1, P'2, P'3, P'5, P'6$ follow from Proposition 6.1 and Theorem 3.2 in the affine case. $P'7,P'8,P'9,P'10, P'13$ also follow from Theorem 3.2 in the affine case. $P'15$ is just Lemma 4.1 in the affine case. □
References

[Bre97] Kirsten Bremke. On generalized cells in affine Weyl groups. *Journal of Algebra*, 191(1):149–173, 1997.

[Gui08] Jeremie Guilhot. On the lowest two-sided cell of an affine Weyl groups. *Representation Theory*, 12:327–345, 2008.

[Gui14] Jeremie Guilhot. Cellularity of the lowest two-sided ideal of an affine Hecke algebra. *Advances in Mathematics*, 255:525–561, 2014.

[Lus80] George Lusztig. Hecke algebras and Jantzens generic decomposition patterns. *Advances in Mathematics*, 37:121–164, 1980.

[Lus85] George Lusztig. Cells in affine Weyl group. In *Algebraic groups and related topics*, volume 6 of *Advanced Studies in Pure Math*, pages 255–287. Kinokunia and North Holland, 1985.

[Lus87a] George Lusztig. Cells in affine Weyl groups, II. *Journal of Algebra*, 109(2):536–548, 1987.

[Lus87b] George Lusztig. Cells in affine Weyl groups, III. *Journal of The Faculty of Science, The University of Tokyo, Section IA, Mathematics*, 34:223–243, 1987.

[Lus03] George Lusztig. *Hecke algebras with unequal parameters*, volume 18. American Mathematical Soc., 2003.

[Xi90] Nanhua Xi. The based ring of the lowest two-sided cell of an affine Weyl group. *Journal of Algebra*, 134(2):356–368, 1990.

[Xi94] Nanhua Xi. *Representations of affine Hecke algebras*, volume 1587. Springer-Verlag Berlin-Heidelberg, 1994.

[Xi07] Nanhua Xi. Representations of affine Hecke algebras and based rings of affine Weyl groups. *Journal of the American Mathematical Society*, 20(1):211–217, 2007.