A NOTE ON THE DUAL COMPLEX ASSOCIATED TO A RESOLUTION OF SINGULARITIES

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1. Construction

The dual complex associated to a resolution of singularities generalizes the notion of a resolution graph of a surface singularity to any dimension. Let $X$ be an algebraic variety over a field of characteristic 0 or an analytic space and let $S \subset X$ be the singular locus of $X$. Usually we consider the germ of singularity $(X, S)$. Take a resolution $\pi: (Y, E) \to (X, S)$ such that the exceptional set $E$ is a divisor with simple normal crossings. This resolution exists by Hironaka theorem (see [1], [2]); following terminology of the minimal model theory, we say that $\pi$ is a log-resolution.

The dual complex $\Gamma(E)$ associated to the resolution $\pi$ is a dual $CW$-complex associated to the divisor $E$ as to a reducible variety. More precisely, decompose the divisor $E$ to its prime components $E_i$: $E = \sum_{i=1}^{N} E_i$. Then $k$-dimensional cells $\Delta^j_{i_0...i_k}$ of the complex $\Gamma(E)$ are in one-to-one correspondence with irreducible components $E^j_{i_0...i_k}$ of the intersections

$$E_{i_0} \cap \cdots \cap E_{i_k} = \cup_j E^j_{i_0...i_k},$$

$k = 0, \ldots, n-1$, \quad $1 \leq i_0 < i_1 < \cdots < i_k \leq N$.

Every cell $\Delta^j_{i_0...i_k}$ is a standard $k$-dimensional simplex with vertices marked by numbers $i_0, \ldots, i_k$. For instance, the 0-dimensional skeleton of the complex $\Gamma(E)$ is a union of the points $\Delta_i$ corresponding to the prime divisors $E_i$. The patching map from the boundary $\partial \Delta^j_{i_0...i_k}$ of a $k$-dimensional cell to the $k-1$-dimensional skeleton is simplicial. It maps the face $\Delta^j_{i_0...\hat{i}_s...i_k}$ onto simplex $\Delta^{j'}_{i_0...\hat{i}_s...i_k}$ such that

$$E^j_{i_0...i_k} \cap E^{j'}_{i_0...\hat{i}_s...i_k} \neq \emptyset,$$

and sends the vertex $i_r$ of the face $E^j_{i_0...\hat{i}_s...i_k}$ to the vertex $i_r$ of the simplex $\Delta^{j'}_{i_0...\hat{i}_s...i_k}$.

The complex $\Gamma(E)$ was considered by G. L. Gordon in the paper [3].

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2. Homotopy type of the dual complex is an invariant of a singularity

**Theorem 1.** Let \( \pi': (Y', E') \to (X, o) \) and \( \pi'': (Y'', E'') \to (X, o) \) be two log-resolutions of an isolated singularity \((X, o)\). Then the topological spaces \( \Gamma(E') \) and \( \Gamma(E'') \) have the same homotopy type.

This result is based on the following theorem by Abramovich-Karu-Matsuki-Wlodarczyk on factorization of birational maps (see [4]).

**Theorem 2 (Weak Factorization Theorem in the Logarithmic Category).** Let \((U_{X_1}, X_1)\) and \((U_{X_2}, X_2)\) be complete nonsingular toroidal embeddings (this means that \(U_{X_i}\) is an open set in a nonsingular variety \(X_i\) and the boundary \(X_i \setminus U_{X_i}\) is a divisor with simple normal crossings, \(i = 1, 2\)) over an algebraically closed field of characteristic zero. Let \(\varphi: (U_{X_1}, X_1) \dashrightarrow (U_{X_2}, X_2)\) be a birational map which is an isomorphism over \(U_{X_1} = U_{X_2}\). Then the map \(\varphi\) can be factored into a sequence of blowups and blowdowns with smooth admissible and irreducible centers disjoint from \(U_{X_1} = U_{X_2}\). That is to say, there exists a sequence of birational maps between complete nonsingular toroidal embeddings

\[
(U_{X_1}, X_1) = (U_{V_1}, V_1) \dashrightarrow (U_{V_2}, V_2) \dashrightarrow \ldots \\
\vdots \dashrightarrow (U_{V_i}, V_i) \dashrightarrow (U_{V_{i+1}}, V_{i+1}) \dashrightarrow \ldots \\
\vdots \dashrightarrow (U_{V_{i-1}}, V_{i-1}) \dashrightarrow (U_{V_i}, V_i) = (U_{X_2}, X_2),
\]

where

(i) \(\varphi = \psi_{i-1} \circ \psi_{i-2} \circ \cdots \circ \psi_1\),

(ii) \(\psi_i\) are isomorphisms over \(U_{V_i}\), and

(iii) either \(\psi_i\) or \(\psi_i^{-1}\) is a morphism obtained by blowing up a smooth irreducible center \(C_i\) (or \(C_{i+1}\)) disjoint from \(U_{V_i} = U_{V_{i+1}}\) and transversal to the boundary \(D_{V_i} = V_i \setminus U_{V_i}\) (or \(D_{V_{i+1}} = V_{i+1} \setminus U_{V_{i+1}}\)), i. e., at each point \(p \in V_i\) (or \(p \in V_{i+1}\)) there exists a regular coordinate system \(\{x_1, \ldots, x_n\}\) in a neighborhood \(p \in U_p\) such that

\[
D_{V_i} \cap U_p \ (\text{or} \ D_{V_{i+1}} \cap U_p) = \left\{ \prod_{j \in J} x_j = 0 \right\}
\]

and

\[
C_i \cap U_p \ (\text{or} \ C_{i+1} \cap U_p) = \left\{ \prod_{j \in J} x_j = 0, \ x_{j'} = 0 \ \forall j' \in J' \right\},
\]

\(J, J' \subseteq \{1, \ldots, n\}\).
Remark. The same holds if $X_i$ are complex manifolds and $\varphi, \psi_i$ are bimeromorphic maps.

To apply Theorem 2 in our case, take the resolutions $(Y' \setminus E', Y')$ and $(Y'' \setminus E'', Y'')$ as toroidal embeddings and compactify $Y'$ and $Y''$ to smooth varieties (here we use the fact that the given singularity $(X, o)$ is isolated). Now Theorem 1 follows from the

Lemma. Let $\sigma: (X' \setminus E', X') \to (X \setminus E, X)$ be a blowup of an admissible center $C \subset E$ in a nonsingular toroidal embedding $(X \setminus E, X)$, $X' \setminus E' \simeq X \setminus E$. Then the topological spaces $\Gamma(E')$ and $\Gamma(E)$ have the same homotopy type.

Proof. Let $E = \sum_{i=1}^{N} E_i$ be the decomposition of $E$ into its prime components, and let $C \subset E_i$ for $1 \leq i \leq l$ and $C \not\subset E_i$ for $l < i \leq N$. Assume that $C$ has nonempty intersections also with $E_{l+1}, \ldots, E_r$, $l < r \leq N$. There are two possibilities.

1) $\dim C = n - l$ ($n = \dim X$), i. e., $C$ coincides with one of the irreducible components of the intersection $E_1 \cap \cdots \cap E_l$: $C = E^1_{1,\ldots,l}$. Then after the blowup the intersection of the proper transforms $E'_1, \ldots, E'_l$ of the divisors $E_1, \ldots, E_l$ has $J - 1$ irreducible components (if $J$ is the number of components of $E_1 \cap \cdots \cap E_l$), but all these proper transforms intersect the exceptional divisor $F$ of the blowup $\sigma$. Futhermore, $F$ intersects proper transforms $E'_{l+1}, \ldots, E'_r$ of the divisors $E_{l+1}, \ldots, E_r$. Now it is clear that the complex $\Gamma(E')$ is obtained from $\Gamma(E)$ by the barycentric subdivision of the simplex $\Delta^1_{1,\ldots,l}$ with the center at the point corresponding to the divisor $F$. Thus the complexes $\Gamma(E')$ and $\Gamma(E)$ are even homeomorphic.

2) $\dim C < n - l$, let $C \subset E^1_{1,\ldots,l}$. In this case divisors $E_{i_1}, \ldots, E_{i_s}$ have nonempty intersection if and only if their proper transforms $E'_{i_1}, \ldots, E'_{i_s}$ have nonempty intersection. Therefore the complex $\Gamma(E')$ is obtained from the complex $\Gamma(E)$ in the following way. Add to $\Gamma(E)$ a new vertex corresponding to the exceptional divisor $F$ of the blowup $\sigma$ and construct cones with vertex at $F$ over all the maximal cells $\Delta^j_{i_1,\ldots,i_s}$ of the complex $\Gamma(E)$ possessing the property

$$E^j_{i_1,\ldots,i_s} \cap C \neq \emptyset.$$

Note that the simplex $\Delta^j_{F,1,\ldots,l}$ corresponding to the intersection $F \cap E^1_{1,\ldots,l}$ is regarded as a common simplex for all constructed cones. Now we can define the homotopy equivalence between $\Gamma(E')$ and $\Gamma(E)$ as a contraction of the constructed cones: it sends the vertex $F$ of the complex $\Gamma(E')$ to any of the vertices $E_1, \ldots, E_l$ of the cell $\Delta^1_{1,\ldots,l}$ of the complex $\Gamma(E)$ and it is identity on other vertices of $\Gamma(E')$ ($\Gamma(E)$). Then the induced simplicial map is our homotopy equivalence. \qed
3. Toric singularities

As an example let us find the homotopy type of the dual complex associated to a resolution of a toric singularity.

**Theorem 3.** Let \( o \in X \) be an isolated singularity of a toric variety \( X \). Then for any log-resolution \( \pi: (Y, E) \to (X, o) \) the dual complex \( \Gamma(E) \) is homotopy equivalent to a point.

*Proof.* We can restrict ourselves to the case when the variety \( X \) is affine. Then it can be represented as a toric variety \( X = U_\sigma = \text{Spec} \mathbb{C}[\sigma] \) corresponding to a convex polyhedral cone \( \sigma \subset \mathbb{R}^n \). According to Theorem 1, it is enough to determine the homotopy type for one arbitrary log-resolution of the singularity \((X, o)\).

There exists a fan \( \Sigma \) such that the support \( \text{supp} \Sigma = \sigma \) and the birational morphism \( \pi: Y = X(\Sigma) \to X \) is a log-resolution for the variety \( X \) (see [5]). First consider the dual complex for the divisor \( E' = \sum T_i + E \), where \( T_i \) are invariant divisors corresponding to the edges \( \tau_i \) of the cone \( \sigma \) and \( E \) is the exceptional divisor of the morphism \( \pi \). Recall that the prime exceptional divisors \( E_i, E = \sum E_i \), correspond to 1-dimensional cones of the fan \( \Sigma \) different from \( \tau_i \); the divisors \( E_i \) and \( E_j \) have a nonempty intersection if and only if the corresponding 1-dimensional cones span a 2-dimensional cone belonging to the fan \( \Sigma \), and so on.

Let \( L \) be a hyperplane in the space \( \mathbb{R}^n \) such that \( L \cap \sigma = K \) is a compact polyhedron. Then the fan \( \Sigma \) determines some triangulation of the polyhedron \( K \). We denote by the same letter \( K \) the corresponding simplicial complex. Now it is clear that the dual complex \( \Gamma(E') \) is homeomorphic to the complex \( K \) and it obviously has the homotopy type of a point.

The dual complex \( \Gamma(E) \) is obtained from \( K \) by deleting simplexes which contain at least one of divisors \( T_i \) as a vertex. The triangulation \( \Sigma \) can be chosen sufficiently small, so it is evident that after such deleting we obtain a complex which is homotopy equivalent to the original complex \( K \), i. e., which has the homotopy type of a point. \( \square \)

4. Some remarks

The surface cusp

\[
X = \{x^4 + y^4 + z^4 + xyz = 0\} \subset \mathbb{C}^3;
\]

gives an example of singularity with homotopy nontrivial dual complex associated to a resolution. It is easy to see that if \( \pi: Y \to X \) is a
log-resolution, $E$ is its exceptional divisor, then the complex $\Gamma(E)$ is homotopy equivalent to the circle $S^1$.

The singularity
$$\{x^8 + y^8 + z^8 + x^2y^2z^2 = 0\} \subset \mathbb{C}^3$$
gives a more complicated example (see [3]). If we blow up the origin, the exceptional divisor $E'|_X$, consists of 3 lines $E_i$, $i = 1, 2, 3$; every 2 of them intersect at a single point. The proper transform $X'$ is singular along these lines. A resolution of $X$ can be obtained by blowing up the variety $X'$ along $E_1 \cup E_2 \cup E_3$. The dual complex associated to this resolution has a nontrivial first homology group; in particular, $\dim H_1(\Gamma(E), \mathbb{Q}) = 4$.

On the other hand, it is known (see [6]) that the resolution graphs of rational surface singularities are trees, thus they have homotopy type of a point. It would be interesting to verify if this fact generalizes to higher dimensions. More precisely, if $(X, o)$ is an isolated rational singularity and $\pi: (Y, E) \to (X, o)$ is a log-resolution, then is it right that the complex $\Gamma(E)$ has the homotopy type of a point?

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