OPTIMALLY RECONNECTING WEIGHTED GRAPHS AGAINST AN EDGE-DESTROYING ADVERSARY

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Abstract

We introduce a model involving two adversaries Buster and Fixer taking turns modifying a connected graph, where each round consists of Buster deleting a subset of edges and Fixer responding by adding edges from a reserve set of weighted edges to leave the graph connected. With the weights representing the cost for Fixer to use specific reserve edges to reconnect the graph, we provide a reasonable definition for what should constitute an optimal strategy for Fixer to keep the graph connected for as long as possible as cheaply as possible, and prove that a greedy strategy for Fixer satisfies our conditions for optimality.

1. Introduction

Suppose a network must stay connected in the face of some adversary that periodically destroys subsets of its edges, with the network being reconnected after each attack by adding replacement edges, each of which has its own cost. Beyond the requirement that each individual selection of replacement edges must reconnect the network, ideally taken together these selections should, in some sense, keep the network connected for as long as possible as cheaply as possible. In Subsection 1.1 we introduce a graph-theoretic framework for formally model this situation, including a definition of optimality for selection of replacement edges, and in Subsection 1.2 we give an example of the model in action. In Subsection 1.3 we state our main theorem, that a greedy strategy for selecting replacement edges is in fact optimal, and in Subsection 1.4 we compare and contrast our model to the well-studied Maker-Breaker game as well as outline some directions for future research. Using tools developed in Section 2 we prove our main theorem in Section 3.

1. Description of Model

To initialize an instance of our model, we are given a finite multigraph $G$ (for this paper, all vertex sets of multigraphs will be unambiguous, so we view multigraphs simply as multisets of edges) and a finite multiset $R$ of weighted "reserve" edges between vertices of $G$. Each edge $r \in R$ has some nonnegative, finite weight $w(r)$; for $R' \subseteq R$, define $w(R') = \sum_{r \in R'} w(r)$.

There are two parties in this model: a positive actor Fixer and an antagonist Buster. A multigraph is to be first "busted" by Buster and then "fixed" by Fixer in each of a series of rounds. A given series will be given a name such as $S$. The $k$th round of series $S$ begins with a finite multigraph $G^S_k$ on the vertex set $V$ and a multiset $R^S_k$ of weighted "reserve" edges in the complete graph on $V$. Start with $G^S_1 = G$ and $R^S_1 = R$.

Buster begins the $k$th round by removing some nonempty multiset $B^S_k$ of edges from the current multigraph $G^S_k$. If adding all of the remaining multiset $R^S_k - B^S_k$ of reserve edges to $G^S_k - B^S_k$ does not result in a connected multigraph, then we say Buster wins in the $k$th round, and for convenience set $F^S_k = \emptyset$ (Fixer cannot reconnect the graph no matter how much she spends on reserve edges, so she might as well not spend anything at all). Otherwise, Fixer responds by creating a connected multigraph from $G^S_k - B^S_k$ by adding a (potentially empty) multiset $F^S_k$ of edges from the remaining multiset $R^S_k$ of reserve edges. In either case, set $G^S_{k+1} = (G^S_k - B^S_k) \cup F^S_k$ and $R^S_{k+1} = R^S_k - F^S_k$. If Buster does not win in the $k$th round, he has the option to quit (for instance, in a real-life scenario, external factors could prevent Buster from continuing), in which case we say Fixer wins in the $k$th round. If either Fixer or Buster wins in the $k$th round, denote it by $|S| = k$.

If $S$ is a series satisfying $|S| \geq k$, then a Fixer strategy to continue $S$ after the $k$th round is a set $\phi$ of series satisfying the following:

1. The series consisting solely of the first $k$ rounds of $S$ belongs to $\phi$.

2. If $T \in \phi$, Fixer wins $T$ in the $j$th round, and $B \subseteq G^T_j$ is nonempty, then there exists exactly one series $U \in \phi$ such that $B^U_i = B^T_i$ and $F^U_i = F^T_i$ for $1 \leq i \leq j$, $B^U_{j+1} = B$, and $|U| = j + 1$.

3. All series in $\phi$ are identical to $S$ through the first $k$ rounds.

4. If $T$ and $U$ are in $\phi$, with $B^T_i = B^U_i$ for $1 \leq i \leq j$ and $F^T_i = F^U_i$ for $1 \leq i < j$, then $F^T_j = F^U_j$.

Equivalently, $\phi$ can be defined as the set of series represented by directed paths originating from the root vertex in some decision tree $T$ with the following structure. The root vertex $r$ of $T$ represents the first $k$ rounds of $S$, and a non-root vertex $v$ of $T$ at distance $j$ from $r$ represents the $(k + j)$th round of a series whose first $k$ rounds are represented by $r$, and whose $(k + i)$th round for $1 \leq i < j$
is represented by the vertex on the path from \( r \) to \( v \) at distance \( i \) from \( r \); the path \( P \) in \( T \) from \( r \) to \( v \) represents the series \( T \) whose \( k + j \) rounds are represented by the vertices along \( P \). Given a vertex \( v \) in \( T \) and a series \( T \) whose first \( k + j \) rounds are identical to the series represented by the path from \( r \) to \( v \), for each possible Buster move \( B^j_{k+j+1} \) in the \((k+j+1)\)st round of \( T \) (i.e. each nonempty subset of a connected \( G^j_{k+j+1} \)), there is a child of \( v \) representing the \((k+j+1)\)st round of \( T \); that is, each possible Buster move \( B^j_{k+j+1} \) is assigned a single Fixer response \( F^j_{k+j+1} \). Note that each path in \( T \) from \( r \) to a leaf represents a series won by Buster, while each path in \( T \) from \( r \) to an interior vertex (including the path consisting solely of the root \( r \), if \( T \) has multiple vertices, i.e. Buster doesn’t win \( S \) in the \( k \)th round) represents a series won by Fixer.

Let \( S \) and \( S' \) be series such that \( G^j_1 = G^j_1 \) and \( R^j_1 = R^j_1 \). Say \( S \) is Fixer-superior to \( S' \) if each of the following holds:

1. Fixer wins \( S \) or Buster wins \( S' \).
2. \( \sum_{j=1}^{|S|} |B^j_S| \geq \sum_{j=1}^{|S'|} |B^j_{S'}| \)
3. \( \sum_{j=1}^{|S|} w(F^j_S) \leq \sum_{j=1}^{|S'|} w(F^j_{S'}) \)

The first requirement means that in terms of winning or losing, Fixer does at least as well in \( S \) as in \( S' \). The second requirement means that Buster works at least as hard deleting edges in \( S \) as in \( S' \). The third requirement means that Fixer spends at least as much on reserve edges in \( S \) as in \( S' \). In establishing the second and third requirements, it is often helpful to observe that \( \sum_{j=1}^{|T|} |B^j_T| = |G^j_1| + |R^j_1| - |G^j_{|T|+1}| - |R^j_{|T|+1}| \) and \( \sum_{j=1}^{|T|} w(F^j_T) = w(R^j_1) - w(R^j_{|T|+1}) \) for any series \( T \). Note that every move is Fixer-superior to itself.

Fixer move \( F^j_k \) is optimal if there exists a Fixer strategy \( \phi \) to continue \( S \) after the \( k \)th round, such that for any series \( S' \) identical to \( S \) though Buster’s move of the \( k \)th round and any Fixer strategy \( \phi' \) to continue \( S' \) after the \( k \)th round, for any \( T \in \phi \) there exists \( T' \in \phi' \) for which \( T \) is Fixer-superior to \( T' \). That is, for a series \( T \) identical to \( S \) through the first \( k \) rounds, against any Buster moves Fixer can always continue play in \( T \) with \( B^j_k, B^j_{k+1}, B^j_{k+2}, \ldots, B^j_{|T|}, B^j_{|T|+1} \), in such a way that for any series \( T' \) identical to \( S \) through Buster’s move in the \( k \)th round, against any Fixer moves Buster can always continue play in \( T' \) with \( B^j_k, B^j_{k+1}, B^j_{k+2}, \ldots, B^j_{|T'|}, B^j_{|T'|+1} \), in such a way that \( T \) is Fixer-superior to \( T' \). It is not immediately obvious that an optimal move need exist.

1.2. Example

We consider a series \( S \) starting with \( G^1_S = \{e_1, e_2, e_3\} \), where these edges form a triangle, and \( R^1_S = \{e_4, e_5\} \), where \( e_4 \) has the same endpoints as \( e_1 \) and satisfies \( w(e_4) = 1 \), while \( e_5 \) has the same endpoints as \( e_2 \) and satisfies \( w(e_5) = 2 \); see Figure 1.

![Figure 1: The solid edges form the graph \( G^j_1 \), while the dashed edges form the reserve set \( R^j_1 \), with \( w(e_4) = 1 \) and \( w(e_5) = 2 \).](image)

Suppose Buster’s first move in \( S \) is to play \( B^1_2 = \{e_1, e_2\} \), and Fixer responds with \( F^1_2 = \{e_4\} \). We detail a Fixer strategy \( \phi = \{T_1, \ldots, T_{10}\} \) to continue \( S \) after the first round below. A series \( T_i \) in the same row as the \( j \)th round means that the rounds of \( T_i \) are given by the first \( j \) rows of the given box (ignoring the “Winner” column until the \( j \)th round).

| \( T \) | \( j \) | \( G^j_1 \) | \( R^j_1 \) | \( B^j_1 \) | \( F^j_1 \) | \( \sum |B^j_1| \) | \( \sum w(F^j_1) \) | Winner |
|---|---|---|---|---|---|---|---|---|
| \( T_1 \) | 1 | \{e_1, e_2, e_3\} | \{e_4, e_5\} | \{e_1, e_2\} | \{e_4\} | 2 | 1 | Fixer |
| \( T_2 \) | 2 | \{e_3, e_4\} | \{e_5\} | \{e_3\} | \{e_5\} | 3 | 3 | Fixer |
| \( T_3 \) | 3 | \{e_4, e_5\} | {} | \{e_4\} \cup \{e_5\} \cup \{e_4, e_5\} | {} | 4 \cup 4 | 5 | Fixer |
| \( T_4 \) | 1 | \{e_1, e_2, e_3\} | \{e_4, e_5\} | \{e_1, e_2\} | \{e_4\} | 2 | 1 | Fixer |
| \( T_5 \) | 2 | \{e_3, e_4\} | \{e_5\} | \{e_4\} | \{e_5\} | 3 | 3 | Fixer |
| \( T_6 \) | 3 | \{e_4, e_5\} | {} | \{e_4\} \cup \{e_5\} \cup \{e_4, e_5\} | {} | 4 \cup 4 | 5 | Fixer |
| \( T_7 \) | 1 | \{e_1, e_2, e_3\} | \{e_4, e_5\} | \{e_1, e_2\} | \{e_4\} | 2 | 1 | Fixer |
| \( T_8 \) | 2 | \{e_3, e_4\} | \{e_5\} | \{e_3\} \cup \{e_5\} \cup \{e_3, e_5\} | {} | 4 \cup 5 | 3 | Buster |
| \( T_9 \) | 1 | \{e_1, e_2, e_3\} | \{e_4, e_5\} | \{e_1, e_2\} | \{e_4\} | 2 | 1 | Fixer |
| \( T_{10} \) | 2 | \{e_3, e_4\} | \{e_5\} | \{e_3\} \cup \{e_5\} \cup \{e_3, e_5\} | {} | 4 \cup 5 | 3 | Buster |
Now consider an alternate series $S'$ identical to $S$ up through Buster's removal of a set of edges in the first round, but suppose Fixer responds in $S'$ with $F_1^{S'} = \{e_5\}$. We detail a Fixer strategy $\phi' = \{T_1', \ldots, T_{10}'\}$ to continue $S'$ after the first round below.

| $T'$ | $j$ | $G_i^{T'}$ | $R_i^{T'}$ | $B_i^{T'}$ | $F_i^{T'}$ | $\sum |B_i^{T'}|$ | $\sum w(F_i^{T'})$ | Winner |
|------|-----|-----------|-----------|-----------|-----------|-----------------|-----------------|--------|
| $T_1'$ | 1 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_5\}$ | 2 | 2 | Fixer |
| $T_2'$ | 2 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_3'/T_4'/T_5'$ | 3 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Buster |
| $T_6'$ | 1 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_5\}$ | 2 | 2 | Fixer |
| $T_7'/T_8'/T_9'$ | 2 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_{10}'$ | 3 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Buster |
| $T_{11}'$ | 1 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_5\}$ | 2 | 2 | Fixer |
| $T_{12}'$ | 2 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_{13}'$ | 3 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Fixer |

Observe that $\phi'$ is, in fact, the only Fixer strategy to continue $S'$ after the first round. Indeed, $G_2^{S'} = \{e_3, e_5\}$, which is a path graph, and $R_2^{S'} = \{e_4\}$, where $e_4$ spans the endponts of that path. Thus Buster deleting any single edge from $G_2^{S'}$ necessitates Fixer reconnecting the graph using $e_4$, from which point on any further deletions by Buster cannot be countered by Fixer, and Buster initially deleting both edges from $G_2^{S'}$ also cannot be countered by Fixer. Furthermore, see that for $1 \leq i \leq 10$, the series $T_i$ in $\phi$ is Fixer-superior to the series $T'_i$ in $\phi'$. Hence $\phi$ is a Fixer strategy to continue $S$ after the first round, such that for any Fixer strategy $\phi'$ to continue $S'$ after the first round, for any $T \in \phi$ there exists $T' \in \phi'$ for which $T$ is Fixer-superior to $T'$.

Now consider another alternate series $S''$ identical to $S$ up through Buster's removal of a set of edges in the first round, but suppose Fixer responds in $S''$ with $F_1^{S''} = \{e_4, e_5\}$. We detail a Fixer strategy $\phi'' = \{T_1'', \ldots, T_{17}''\}$ to continue $S''$ after the first round below.

| $T''$ | $j$ | $G_i^{T''}$ | $R_i^{T''}$ | $B_i^{T''}$ | $F_i^{T''}$ | $\sum |B_i^{T''}|$ | $\sum w(F_i^{T''})$ | Winner |
|------|-----|-----------|-----------|-----------|-----------|-----------------|-----------------|--------|
| $T_1''$ | 1 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_4, e_5\}$ | 2 | 3 | Fixer |
| $T_2''$ | 2 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_3''/T_4''/T_5''$ | 3 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Buster |
| $T_6''$ | 1 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_4, e_5\}$ | 2 | 3 | Fixer |
| $T_7''/T_8''/T_9''$ | 2 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_{10}''$ | 3 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Buster |
| $T_{11}''/T_{14}''/T_{15}''/T_{17}''$ | 2 | $\{e_1, e_2, e_3\}$ | $\{e_4, e_5\}$ | $\{e_1, e_2\}$ | $\{e_4, e_5\}$ | 2 | 3 | Fixer |
| $T_{12}''$ | 3 | $\{e_3, e_5\}$ | $\{e_4\}$ | $\{e_3\}$ | $\{e_4\}$ | 3 | 3 | Fixer |
| $T_{13}''/T_{14}''/T_{15}''/T_{17}''$ | 4 | $\{e_4, e_5\}$ | $\{\}$ | $\{e_4\}/\{e_5\}/\{e_4, e_5\}$ | $\{\}$ | 4/4 | 3 | Buseter |

Again, observe that $\phi''$ is the only Fixer strategy to continue $S''$ after the first round since $R_2^{S''}$ is empty, leaving Fixer with no options besides playing the empty set when possible. See that for $1 \leq i \leq 9$, the series $T_i$ in $\phi$ is Fixer-superior to the series $T''_i$ in $\phi''$, while $T_{10}$ is Fixer-superior to $T_{11}''$. Hence $\phi$ is a Fixer strategy to continue $S$ after the first round, such that for any Fixer strategy $\phi'$ to continue $S'$ after the first round, for any $T \in \phi$ there exists $T' \in \phi'$ for which $T$ is Fixer-superior to $T'$.

Since $F_1^{T}, F_1^{T''}$, and $F_1^{T'''}$ are Fixer's only choices for responding to Buster in the first round, we have shown that $\phi$ is a Fixer strategy to continue $S$ after the first round, such that for any Fixer strategy $\phi'$ or $\phi''$ to continue any alternate move $F_1^{T''}$ or $F_1^{T'''}$ after the first round, for any $T \in \phi$ there exists $T' \in \phi'$ and $T'' \in \phi''$ for which $T$ is Fixer-superior to $T'$ and $T''$. Hence $F_1^{T}$ is optimal. Note that because $T_1$ and $T_2$ are part of $\phi'$ and $\phi''$, respectively (the only Fixer strategies to continue $S'$ and $S''$ after the first round), but are not Fixer-superior to any series in $\phi$, by definition $F_1^{T}$ and $F_1^{T''}$ are not optimal.

### 1.3. Statement of the Main Theorem

Fixer move $F_k^S \subseteq R_k^S$ to create a connected multigraph $G_{k+1}^S$ by adding $F_k^S$ to $G_k^S - B_k^S$ is greedy if, for any other series $S'$ identical to $S$ through Buster's move of the $k$th round, Fixer move $F_k^{S'} \subseteq R_k^{S'}$ to create a connected multigraph $G_{k+1}^{S'}$ by adding $F_k^{S'}$ to $G_k^{S'} - B_k^{S'}$ satisfies $w(F_k^{S'}) \leq w(F_k^S)$. That is, Fixer plays greedily in response to a move by Buster by adding no reserve edge if the graph remains connected, and otherwise adding some cheapest set of reserve edges that connects the graph.

Note that every optimal Fixer move is necessarily greedy. Indeed, if $S$ and $S'$ are identical through Buster's move of the $k$th round, but $w(F_k^S) > w(F_k^{S'})$, then for any Fixer strategy $\phi$ to continue $S$ after the $k$th round, the series $T \in \phi$ identical to $S$ through the $k$th round but ending with $|T| = k$ cannot be Fixer-superior to any series $T'$ in a Fixer strategy $\phi'$ to continue $S'$ after the $k$th round. This is because if $|T'| = k$ then $\sum_{j=1}^{|T'|} w(F_j^T) > \sum_{j=1}^{|T'|} w(F_j^{T'})$, while if $|T'| > k$ then $\sum_{j=1}^{k} |B_j^T| < \sum_{j=1}^{k} |B_j^{T'}|$. Our main theorem states that greediness is also a sufficient condition for optimality.

**Theorem 1.1.** Every greedy move by Fixer is optimal.
After establishing some facts about spanning trees, Fixer-superiority, and optimality in Section 1.1, we use them to prove Theorem 1.1 in Section 3. Our proof will be by induction, split into cases by the number $c$ of components of $G_k^S - B_k^S$. We shall see that the case $c = 1$ is mostly trivial, the case $c = 2$ is the most difficult and requires case analysis of each move to verify that certain invariants are maintained, and the case $c \geq 3$ is better-suited for a more direct application of the inductive hypothesis.

1.4. Past and Future Work

For a family $\mathcal{F}$ of subgraphs of the complete graph $K_n$, the unbiased Maker-Breaker game on $\mathcal{F}$ consists of players Maker and Breaker taking turns claiming edges of $K_n$ (see [5] and [1] for some notable early results, and [2] for a more recent survey). Maker wins by claiming all edges of some graph in $\mathcal{F}$, while Breaker wins if all edges are claimed before Maker wins (equivalently, Breaker wins by claiming an edge from each minimal member of $\mathcal{F}$). The family $\mathcal{F}$ most relevant for comparison of the Maker-Breaker game on $\mathcal{F}$ with our game is the family of connected spanning subgraphs of $K_n$. The gameplay of Maker-Breaker differs from that of Buster-Fixer in several obvious ways, including the following:

1. Maker only needs to end up with a connected graph, while Fixer must maintain connectedness after each turn.
2. Maker cannot replace edges claimed by Breaker, whereas the reserve edges Fixer may select could include edges with the same endpoints as the edges deleted by Buster.
3. The Maker-Breaker game does not typically include weighted edges, which are is a consideration in the Buster-Fixer model.
4. Maker’s objective is to directly beat Breaker (and vice-versa), but the relationship between Fixer and Buster is less symmetric; Buster is more of an agent of chaos than a goal-oriented player (e.g. Buster can simply choose to stop participating at any point), and Fixer’s objective is to do her best to keep the graph connected for as long and cheaply as possible based on Buster’s actions, regardless of how well Buster’s edge deletions actually do to disconnect the graph in expensive ways.

We believe the last difference listed is the most important to take note of, as it provides a contrast in the fundamental structure of the models, which further dictates how results are stated for each model. A standard Maker-Breaker result (similar to many typical results in positional game theory) would be a statement of conditions on $n$ and $\mathcal{F}$ for Maker or Breaker to win, most likely constructively proven via an explicit strategy for Maker or Breaker. Our Theorem 1.1 that every greedy move by Fixer is optimal, is of a different flavor though. Since Fixer can’t even “win” if Buster plays for long enough, and has no way of forcing Buster to quit, we must compare Fixer strategies against each other, rather than use a single Fixer strategy to prove Fixer can achieve a certain goal. Hence the optimal strategy for Fixer is the statement, proven by showing its superiority to all other Fixer strategies.

Many avenues exist for future research into variations on our Buster-Fixer model. In particular, we wonder about optimal Fixer strategies for alternative games, where the condition that Fixer must maintain on the graph through each round is changed from maintaining connectedness to one of the following conditions:

1. Two given vertices $s$ and $t$ must stay in the same component.
2. The graph must stay $k$-connected for a given $k > 1$.
3. Instead of a simple graph, the graph is directed, and Fixer must maintain one of the following conditions:
   
   (a) The directed graph must stay strongly connected.
   (b) The directed graph must have directed paths from (or to) a given vertex $s$ to (or from) all other vertices.
   (c) The directed graph must have a directed path from a given vertex $s$ to a given vertex $t$.
   (d) The directed graph must have directed paths in both directions between given vertices $s$ and $t$.

2. Preliminaries

2.1. Spanning Trees and Prim’s Algorithm

A bridge in a multigraph $M$ is an edge $e$ such $M - \{e\}$ has one more component than $M$; equivalently, $e$ is part of no cycle in $M$. A spanning tree of a connected multigraph $M$ is a subgraph $T$ of $M$ such that $T$ is a tree (i.e. connected and acyclic) whose vertex set matches that of $M$. A minimum spanning tree of an edge-weighted multigraph $M$ is a spanning tree of $M$ minimizing the total weight of the edges. Minimum spanning trees are of interest to us because $F_k^S$ is greedy if and only if it is a minimum spanning tree of the multigraph whose vertices are the components of $G_k^S - B_k^S$ and whose edges are the edges of $R_k^S$ (identifying each endpoint of the edges in $R_k^S$ with the component of $G_k^S - B_k^S$ within which it lies).
Prim’s Algorithm (first discovered by Jarnik [4] and later by Prim [6] and Dijkstra [2]) finds a minimum spanning tree $T$ of a weighted connected multigraph $M$ one edge at a time by the following construction: with $T$ initialized as any vertex, iteratively add to $T$ any cheapest edge of $M$ joining a vertex in $T$ to one not yet in $T$, until all vertices of $M$ are in $T$.

We require not just the fact that Prim’s Algorithm successfully produces a minimum spanning tree, but also the fact that any minimum spanning tree can be constructed via Prim’s Algorithm.

**Proposition 2.1.** A spanning tree of a weighted connected multigraph is a minimum spanning tree if and only if it can be constructed via Prim’s Algorithm.

**Proof.** Let $M$ be a weighted connected multigraph, let $P$ be a subgraph of $M$ constructed by applying Prim’s Algorithm, and let $T$ be a minimum spanning tree of $M$. We complete the proof by showing that $P$ is in fact a minimum spanning tree of $M$, and $T$ can be constructed via Prim’s Algorithm.

If $P$ is constructed via Prim’s Algorithm, then see that $P$ is a spanning tree of $M$:

1. $P$ is connected because $P$ is initialized as a single component (its single starting vertex), and the connectedness of $P$ is maintained as each new vertex is added as the endpoint of an edge whose other endpoint was already in $P$.
2. $P$ spans $M$ because $M$ is connected, so if some vertex of $M$ is not yet in $P$, then some new vertex can always be added to $P$.
3. $P$ is acyclic because every edge added is a bridge in $P$.

If $P = T$ then $P$ is a minimum spanning tree. Otherwise, let $e$ be the first edge added during the construction of $P$ that is not in $T$, let $V$ be the set of vertices connected by the edges added before adding $e$, and let $f$ be an edge in the path through $T$ between the endpoints of $e$ such that one endpoint of $f$ is in $V$ but the other is not. Let $T'$ be the spanning tree of $M$ constructed from $T$ by replacing $f$ with $e$.

Since $e$ and $f$ are each edges with exactly one endpoint in $V$ and $e$ was added to $P$ by Prim’s Algorithm, $e$ cannot weigh more than $f$. Hence $T'$ cannot weigh more than $T$, so $T'$ must be a minimum spanning tree of $M$ as well. This process of constructing minimum spanning trees of $M$ each with one more edge in common with $P$ than the last can be continued until $P$ is the minimum spanning tree constructed.

Since $T$ and $T'$ are both minimum spanning trees of $M$, they must weigh the same. Since $T'$ was constructed from $T$ by replacing $e$ with $f$, $e$ and $f$ must weigh the same. Hence $f$ could have also been added by Prim’s Algorithm to extend the construction of a minimum spanning tree of $M$. This process of growing by an edge the subtree of $T$ that can be shown to have been created according to Prim’s Algorithm can be continued until all of $T$ is shown to have been created by Prim’s Algorithm. □

### 2.2. Facts about Fixer-superiority and Optimality

We first verify that Fixer-superiority is transitive.

**Proposition 2.2.** Suppose $S$, $S'$, and $S''$ are series such that $G_1^S = G_1^{S'} = G_1^{S''}$ and $R_1^S = R_1^{S'} = R_1^{S''}$. If $S$ is Fixer-superior to $S'$, and $S'$ is Fixer-superior to $S''$, then $S$ is Fixer-superior to $S''$.

**Proof.** We have

1. Fixer wins $S$, or Buster wins $S'$ (since $S$ is Fixer-superior to $S'$), in which case Buster wins $S''$ (since $S'$ is Fixer-superior to $S''$).
2. $\sum_{j=1}^{S} |B_j^S| \geq \sum_{j=1}^{S'} |B_j^{S'}| \geq \sum_{j=1}^{S''} |B_j^{S''}|$.
3. $\sum_{j=1}^{S} w(F_j^S) \leq \sum_{j=1}^{S'} w(F_j^{S'}) \leq \sum_{j=1}^{S''} w(F_j^{S''})$.

so $S$ is Fixer-superior to $S''$ by definition. □

We next show that Fixer playing only optimal moves past some round leads to a series that is Fixer-superior to certain other series, further justifying our definition of “optimal”.

**Lemma 2.3.** Let $S$ be identical to $S'$ up through Buster’s removal of a set of edges in the $k$th round, and let $\phi'$ be a strategy for Fixer to continue $S'$ after the $k$th round. If $F_j^S$ is optimal for $j \geq k$, then there exists $\bar{V} \in \phi'$ such that $S$ is Fixer-superior to $\bar{V}$.

**Proof.** For $k \leq j \leq |S|$, since $F_j^S$ is optimal there exists a strategy $\phi_j$ for Fixer to continue $S$ after the $j$th round, such that for any series $S_j$ identical to $S$ up through Buster’s move in the $j$th round, every series in $\phi_j$ is Fixer-superior to some series in any strategy for Fixer to continue $S_j$ after the $j$th round. For each $k \leq j < |S|$, set $\phi_j$ as the subset of $\phi_j$ consisting of its series which either end by the $j$th round or have Buster’s move in the $(j+1)$st round match $B_{j+1}^S$, and set $\phi_{k-1} = \phi'$ and $\phi_{|S|} = \phi_{|S|}$. Note that $S \in \phi_{|S|}$,
and for $k \leq j \leq |S|$ every series in $\hat{\phi}_j$ is Fixer-superior to some series in $\hat{\phi}_{j-1}$ (since $\hat{\phi}_j \subseteq \phi_j$ and for a series $S_j$ identical to $S$, up through Buster’s move in the $j$th round, $\hat{\phi}_{j-1}$ is a strategy for Fixer to continue $S_j$ after the $j$th round).

We iteratively construct a sequence $\mathcal{T}_{[S]}$, $\mathcal{T}_{[S]-1}, \ldots, \mathcal{T}_{k-1}$ of series, with $T_j \in \hat{\phi}_j$, for each $j$. First set $\mathcal{T}_{[S]} = S \in \hat{\phi}_{[S]}$. Then, having already constructed $\mathcal{T}_{[S]}, \mathcal{T}_{[S]-1}, \ldots, \mathcal{T}_j$ for some $j \geq k$, select $T_{j+1} \in \hat{\phi}_{j+1}$ so that $T_j$ is Fixer-superior to $T_{j+1}$. Hence by Proposition 2.2, $\mathcal{T}_{[S]}$ is Fixer-superior to $\mathcal{T}_{k-1}$. Since $S = \mathcal{T}_{[S]}$ and $\mathcal{T}_{k-1} \subseteq \hat{\phi}_k = \phi$, the proof is complete.

Finally, we show that to verify the optimality of some Fixer move, we need only compare it to alternate Fixer moves consisting solely of bridges.

**Proposition 2.4.** Suppose $\phi$ is a Fixer strategy to continue $S$ after the $k$th round such that for every series $S'$ identical to $S$ through Buster’s move of the $k$th round for which every edge of $F_k^{S'}$ is a bridge in $G_k^{S'}$, for any $T \in F_k^{S'}$ and any Fixer strategy $\phi'$ to continue $S'$ after the $k$th round, there exists $T' \in \phi'$ such that $\mathcal{T}$ is Fixer-superior to $T'$. Then $F_k^{S'}$ is optimal.

**Proof.** Let $T \in \phi$, let $S'$ and $S''$ be series identical to $S$ through Buster’s move of the $k$th round such that $F_k^{S'} \subseteq F_k^{S''}$ and every edge of $F_k^{S'}$ is bridge in $G_k^{S'}$, and let $\phi''$ be any Fixer strategy to continue $S''$ after the $k$th round. To complete the proof, we construct a Fixer strategy $\phi'$ to continue $S'$ after the $k$th round such that for every $T' \in \phi'$ there exists $T'' \in \phi''$ for which $T'$ is Fixer-superior to $T''$. Indeed, by the hypothesis of this proposition there would exist $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$, and if there exists $T'' \in \phi''$ such that $T'$ is Fixer-superior to $T''$, then $T$ would be Fixer-superior to $T''$, by Proposition 2.2, hence $F_k^{S'}$ would be optimal by definition since $T \in \phi$, $F_k^{S''}$, and $\phi''$ were arbitrary. To construct $\phi'$, we define an arbitrary series $T' \in \phi'$; that is, we let $T'$ be identical to $S'$ through the $k$th round, and for arbitrary plays $B_j$ from Buster in the $j$th round for $j > k$, we assign Fixer responses $F_j$ derived from $\phi''$ in such a way that $T'$ is Fixer-superior to some $T'' \in \phi''$.

First suppose $|T'| = k$, in which case let $T''$ be the lone series in $\phi''$ satisfying $|T''| = k$ (i.e. $T''$ consists of the first $k$ rounds of $S''$). Note that

1. Either Fixer wins $T'$, or Buster wins $T'$, in which case $(G_{k+1}^{T'} - B_{k+1}^{T'}) \cup R_{k+1}^{T'}$ is disconnected, leaving $(G_k^{T''} - B_k^{T''}) \cup R_k^{T''}$ also disconnected since it’s the same graph, meaning Buster also wins $T''$.

2. $\sum_{j=1}^{k} |B_j^{T'}| = \sum_{j=1}^{k} |B_j^{S'}| = \sum_{j=1}^{k} |B_j^{T''}|$

3. $\sum_{j=1}^{k} w(F_j^{T'}) = \sum_{j=1}^{k} w(F_j^{T''}) + w(F_k^{T''}) = \sum_{j=1}^{k} w(F_j^{T''})$

so $T'$ is Fixer-superior to $T''$.

Now suppose $|T'| > k$. Note that $F_k^{S''} \subseteq F_k^{S''}$ implies for any $T'' \in \phi''$ both that $R_k^{T''} \subseteq R_{k+1}^{T'}$ (since $R_k^{T''} \subseteq R_{k+1}^{T'}$ were constructed from $R_k^{S''}$ by removing $F_k^{S''}$ and $F_k^{S''}$, respectively) as well as that $G_k^{T''}$ is a subgraph of $G_{k+1}^{T'}$ (since $G_k^{T''}$ and $G_{k+1}^{T'}$ were constructed from $G_k^{S''} - B_k^{S''}$ by adding $F_k^{S''}$ and $F_k^{S''}$, respectively). By the latter of these observations, there exists $T'' \in \phi''$ such that $B_{k+1}^{T''} = B_{k+1}^{T'}$; we shall choose our $T'' \in \phi''$ for which $T'$ is Fixer-superior to $T''$ to satisfy $B_{k+1}^{T''} = B_{k+1}^{T'}$. Note that for $D = F_k^{T''} - F_k^{S'}$,

$$(G_{k+1}^{T'} - B_{k+1}^{T'}) \cup R_{k+1}^{T'} = (((G_k^{T''} - B_k^{T''}) \cup (F_k^{T''} - B_k^{T''}) - B_k^{T''}) \cup (R_k^{T''} - F_k^{T''}))$$

$$= (((G_k^{T''} - B_k^{T''}) \cup (F_k^{T''} - D)) - B_k^{T''}) \cup (R_k^{T''} - F_k^{T''}) \cup D$$

$$= (((G_k^{T''} - B_k^{T''}) \cup (F_k^{T''} - B_k^{T''}) - B_k^{T''}) \cup (R_k^{T''} - F_k^{T''}))$$

$$= (G_{k+1}^{T''} - B_{k+1}^{T''}) \cup R_{k+1}^{T''}$$

so $(G_{k+1}^{T''} - B_{k+1}^{T''}) \cup R_{k+1}^{T''}$ is connected if and only if $(G_{k+1}^{T''} - B_{k+1}^{T'}) \cup R_{k+1}^{T'}$ is connected.

If Fixer wins $T'$ in the $(k + 1)$st round, then

1. Buster also wins $T''$ in the $(k + 1)$st round, as $(G_{k+1}^{T''} - B_{k+1}^{T''}) \cup R_{k+1}^{T''}$ is disconnected since $(G_{k+1}^{T''} - B_{k+1}^{T'}) \cup R_{k+1}^{T'}$ is disconnected due to Buster winning $T''$ in the $(k + 1)$st round.

2. $\sum_{j=1}^{k} |B_j^{T'}| = \sum_{j=1}^{k} |B_j^{S'}| + |B_{k+1}^{T'}| = \sum_{j=1}^{k} |B_j^{S'}| + |B_{k+1}^{T'}| = \sum_{j=1}^{k} |B_j^{T''}|$

3. $\sum_{j=1}^{k} w(F_j^{T'}) = \sum_{j=1}^{k} w(F_j^{T''}) + w(F_k^{T''}) + 0 \leq \sum_{j=1}^{k} w(F_j^{T''}) + w(F_k^{T''}) + 0 = \sum_{j=1}^{k} w(F_j^{T''})$ since $F_{k+1}^{T'} = F_k^{T''} = \emptyset$

so $T'$ is Fixer-superior to $T''$. Therefore, we may suppose either Fixer wins $T'$ in the $(k + 1)$st round or $|T'| \geq k + 2$. Then $(G_{k+1}^{T''} - B_{k+1}^{T''}) \cup R_{k+1}^{T''}$ is connected, so $(G_{k+1}^{T''} - B_{k+1}^{T'}) \cup R_{k+1}^{T''}$ is also connected, meaning either Fixer wins $T''$ in the $(k + 1)$st round or $|T''| \geq k + 2$. Suppose according to $\phi''$ that Fixer repairs $G_k^{T''} - B_{k+1}^{T''}$ with the set $F_{k+1}^{T''} \subseteq R_{k+1}^{T''}$ to create the connected graph $G_{k+2}^{T''}$. Define $\phi'$ so that Fixer repairs $G_{k+1}^{T''} - B_{k+1}^{T'}$ with the set $F_{k+1}^{T'} = F_{k+1}^{T''} \cup ((F_k^{T''} - F_k^{T'}) - B_{k+1}^{T'})$ to create the graph $G_{k+2}^{T'}$. 


First, note that \( F_{k+1}^T \subseteq R_{k+1}^T \) since \( F_{k+1}^S \subseteq R_{k+1}^T \) and \( F_{k+1}^T - F_k^T = R_{k+1}^T - R_{k+1}^T \subseteq R_{k+1}^T \). Hence Fixer can play \( F_{k+1}^T \) as long as it makes \( G_{k+2}^T \) connected, which is the case since \( G_{k+2}^T \) is connected and \( G_{k+2}^T = G_{k+2}^T \):

\[
G_{k+2}^T = (G_{k+1}^T - B_{k+1}^T) \cup F_{k+1}^T \\
= (G_{k+1}^T - (F_k^T - F_k^T) - B_{k+1}^T) \cup F_{k+1}^T \\
= (G_{k+1}^T - B_{k+1}^T) \cup F_{k+1}^T \\
= (G_{k+1}^T - B_{k+1}^T) \cup F_{k+1}^T \\
= G_{k+2}^T
\]

Next, note that \( R_{k+2}^T = R_{k+2}^T \) since

\[
R_{k+2}^T = R_k^T - (F_k^T \cup F_{k+1}^T) \\
= R_k^T - (F_k^T \cup F_{k+1}^T \cup ((F_k^T - F_k^T) - B_{k+1}^T)) \\
= R_k^T - ((F_k^T - B_{k+1}^T) \cup F_{k+1}^T) \\
= R_k^T - (F_k^T \cup F_{k+1}^T) \\
= R_{k+2}^T
\]

(using the facts that \( R_k^T = R_{k+1}^T \), that \( F_{k+1}^T = F_{k+1}^T \cup ((F_k^T - F_k^T) - B_{k+1}^T) \), and that \( F_k^T \cap (B_{k+1}^T - F_k^T) = (F_k^T - F_k^T) \cap B_{k+1}^T = \emptyset \) since \( F_k^T - F_k^T \subseteq R_{k+1}^T \) and \( R_{k+1}^T \cap B_{k+1}^T = \emptyset \).

Finally, see that since \( G_{k+2}^T = G_{k+2}^T \) and \( R_{k+2}^T = R_{k+2}^T \), \( \phi' \) can continue to be defined by copying \( \phi'' \) in the following way. Assuming \( T' \) has been defined up to the start of the \( j \)th round for some \( j \geq k + 2 \) in such a way that \( G_j^T = G_j^T \) and \( R_j^T = R_j^T \) for some \( T'' \in \phi'' \), and Buster removes some set \( B_j^T \) of edges from \( G_j^T \) in \( T' \), set \( B_j'' = B_j'' \) and let \( F_j'' \) be Fixer’s response in \( T'' \) prescribed by \( \phi'' \). Then set \( F_j'' = F_j'' \), leaving \( G_{j+1}^T = G_{j+1}^T \) and \( R_{j+1}^T = R_{j+1}^T \). Continuing this process up through the final round \( \ell \) of \( T' \), which we also let be the final round of \( T'' \) (either automatically if Buster wins, or by letting Buster quit if Fixer wins), we see that \( T' \) is Fixer-superior to \( T'' \) because

1. Either Fixer wins \( T' \), or Buster wins \( T' \), in which case \((G_j^T - B_j^T) \cup R_j^T \) is disconnected, leaving \((G_j'' - B_j^T') \cup R_j^T'' \) also disconnected since it’s the same graph, meaning Buster also wins \( T'' \).

2. \( \sum_{j=1}^{T'} |B_j^T| = |G_1^T| + |G_1^T| - |G_{T+1}^T| = |G_1^T| + |R_1^T| - |G_{T+1}^T| = \sum_{j=1}^{T''} |B_j^T| \)

3. \( \sum_{j=1}^{T'} w(F_j^T) = w(F_1^T) - w(F_{T+1}^T) = w(F_1^T) - w(R_1^T) = \sum_{j=1}^{T''} w(F_j^T) \)

and thus we have constructed \( \phi' \) so that for every \( T' \in \phi' \) there exists \( T'' \in \phi'' \) for which \( T' \) is Fixer-superior to \( T'' \). Hence \( F_k^S \) is optimal.

\( \square \)

3. Proof of Main Theorem

To prove our main theorem, that during any series \( S \), any greedy Fixer move \( F_k^S \) is optimal, we perform induction on \( |G_k^S| + |R_k^S| \). To help with the base case, we use the following proposition.

**Proposition 3.1.** If Buster wins \( S \) in the \( k \)th round, then \( F_k^S \) is greedy and optimal.

**Proof.** If Buster wins \( S \) in the \( k \)th round, then \((G_k^S - B_k^S) \cup R_k^S \) is disconnected, and by convention \( F_1^S = \emptyset \), which is greedy. Clearly the only series identical to \( S \) through Buster’s move in the \( k \)th round is \( S \) itself, and the only strategy for Fixer to continue \( S \) after the \( k \)th round is \( \{ \} \), so \( F_k^S \) is optimal because \( S \) is Fixer-superior to itself.

\( \square \)

Let \( V \) be the vertex set of \( G_1^S \), with \( |V| = n \), and without loss of generality assume \( k = 1 \). Note that \( |G_1^S| \geq n - 1 \) since \( G_1^S \) is connected, so for our base case we consider \( |G_1^S| + |R_1^S| = n - 1 \). In this case, \( |G_1^S| = n - 1 \), \( R_1^S = \emptyset \), and \( B_1^S \subseteq G_1^S \) is nonempty; then \((G_1^S - B_1^S) \cup R_1^S \) is disconnected because it has at most \( n - 2 \) edges, in which case Buster wins \( S \) during the first round, and Proposition 3.1 applies.

Hence we may suppose \( |G_1^S| + |R_1^S| \geq n \), and inductively assume during any series \( T \) such that \( G_1^T \) is a connected graph on \( V \) and \( |G_1^T| + |R_1^T| < |G_1^S| + |R_1^S| \), any greedy Fixer move \( F_1^T \) is optimal. Furthermore, by Proposition 3.1 we may assume that Buster does not win \( S \) during the first round. Let \( \phi \) be a greedy Fixer strategy to continue \( S \) after Fixer’s greedy move \( F_1^S \) of the first round; by
the inductive hypothesis, all Fixer moves in $\phi$ past the first round are optimal. Let $S'$ be an arbitrary series identical to $S$ through Buster’s move of the first round such that every edge of $F_{k+1}^{S'}$ is a bridge in $G_{k+1}^{S'}$, and let $\phi'$ be an arbitrary strategy for Fixer to continue $S'$ after the first round. By Proposition 2.1 in order to show $F_{k+1}^{S'}$ is optimal, it suffices to show that for any $T \in \phi$ there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$. Note that if Fixer wins $T$ in the first round then $T$ is clearly Fixer-superior to the only series $T' \in \phi'$ satisfying $|T'| = 1$, so we may assume $|T| > 1$.

For the rest of this section, fix any greedy Fixer strategy $\phi$ to continue $S$ after Fixer’s greedy move $F_{1}^{S}$ of the first round, fix any series $T \in \phi$ satisfying $|T| > 1$, and fix any Fixer strategy $\phi'$ of $\phi'$ to continue $S'$ after the first round. Let $c$ equal the number of components of $G_{1}^{S} - B_{1}^{S}$. Let $M$ be the multigraph whose vertices are the components of $G_{1}^{S} - B_{1}^{S}$ and whose edges are the edges of $R_{1}^{S}$ (identifying each endpoint of the edges in $R_{1}^{S}$ with the component of $G_{1}^{S} - B_{1}^{S}$ within which it lies), so $F_{1}^{S}$ is a minimum spanning tree of $M$, and $F_{1}^{S'}$ is a spanning tree of $M$. We complete the proof by showing for each value of $c$ that there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$, handling separately the cases $c = 1$, $c = 2$, and $c \geq 3$ in Subsections 3.1, 3.2, and 3.3 respectively.

### 3.1. The case $c = 1$

If $c = 1$, then the only spanning tree of $M$ is edgeless, so $F_{1}^{S} = F_{1}^{S'} = \emptyset$. For our fixed series $T \in \phi$, let $T'$ be any series in $\phi'$ identical to $T$ through Buster’s move of the second round. Since $F_{2}^{S}$ is optimal for $j \geq 2$, and the subset $\phi''$ of $\phi'$ consisting of all series in $\phi'$ identical to $T'$ through the second round forms a strategy for Fixer to continue $T'$ after the second round, by Lemma 2.3 there exists $T'' \in \phi'' \subseteq \phi'$ such that $T$ is Fixer-superior to $T''$.

### 3.2. The case $c = 2$

If $c = 2$, then the spanning trees of $M$ are the individual edges in $R_{1}^{S}$ joining the two components of $G_{1}^{S} - B_{1}^{S}$. Hence for two such edges $s$ and $s'$, where no such edge is cheaper than $s$, we have $F_{1}^{S} = \{s\}$ and $F_{1}^{S'} = \{s'\}$. We establish Lemmas 3.2 and 3.3 in order to prove Proposition 3.3 which provides a strategy for proving that $F_{1}^{S} = \{s\}$ is optimal.

**Lemma 3.2.** Suppose $U$ is a series identical to $S$ through the first round, with $F_{j}^{U}$ greedy for $j > k$ for some $k$, which satisfies $1 \leq k \leq |U| - 2$ if Buster wins $U$ and $1 \leq k \leq |U| - 1$ if Fixer wins $U$. Then there exists a series $U(k + 1)$ identical to $U$ through the $k$th round such that $|B_{k+1}^{U(k+1)}| = 1$, $F_{j}^{U(k+1)}$ is greedy for $j > k$, and $U$ is Fixer-superior to $U(k + 1)$.

**Proof.** If $|B_{k+1}^{U}| = 1$, set $U(k + 1) = U$, so $U(k + 1)$ is identical to $U$ through the $k$th round, $|B_{k+1}^{U(k+1)}| = |B_{k+1}^{U}| = 1$, $F_{j}^{U(k+1)} = F_{j}^{U}$ is greedy for $j > k$, and $U$ is Fixer-superior to $U(k + 1)$ since every series is Fixer-superior to itself.

Thus we may assume $|B_{k+1}^{U}| > 1$. Let $U(k + 1)$ be the series identical to $U$ through the $k$th round, with the rest of $U(k + 1)$ constructed as follows. We show that there exists an edge $b \in B_{k+1}^{U}$ such that if Buster plays $B_{k+1}^{U(k+1)} = \{b\}$ in $U(k + 1)$, then Fixer can respond with some greedy $F_{k+1}^{U(k+1)} \subseteq F_{k+1}^{U}$. If this is the case, then for the $(k + 2)$nd round in $U(k + 1)$ Buster could play $B_{k+2}^{U(k+1)} = B_{k+1}^{U} - \{b\}$ since

\[
B_{k+2}^{U(k+1)} = B_{k+1}^{U} - \{b\} \\
\subseteq G_{k+1}^{U} - \{b\} \\
= \left(G_{k+1}^{U} - B_{k+1}^{U}\right) \\
\subseteq \left(G_{k+1}^{U} - B_{k+1}^{U(k+1)}\right) \cup F_{k+1}^{U(k+1)} \\
= G_{k+2}^{U(k+1)}
\]

and Fixer could respond with $F_{k+2}^{U(k+1)} = F_{k+1}^{U} - F_{k+1}^{U(k+1)}$ since

\[
F_{k+2}^{U(k+1)} = F_{k+1}^{U} - F_{k+1}^{U(k+1)} \\
\subseteq R_{k+1}^{U} - R_{k+1}^{U(k+1)} \\
= R_{k+1}^{U(k+1)} - F_{k+1}^{U(k+1)} \\
= R_{k+1}^{U(k+1)}
\]

and we are done.
and

\[ G_{k+3}^{U(k+1)} = (G_{k+2}^{U(k+1)} - B_{k+1}^{U(k+1)}) \cup F_{k+2}^{U(k+1)} \]

\[ = (((G_{k+1}^{U(k+1)} - \{b\}) \cup U_{k+1}^{U(k+1)}) - (B_{k+1}^{U(k+1)} - \{b\})) \cup (F_{k+1}^{U(k+1)} - F_{k+1}^{U(k+1)}) \]

\[ = (((G_{k+1}^{U(k+1)} - \{b\}) \cup U_{k+1}^{U(k+1)}) - (B_{k+1}^{U(k+1)} - \{b\})) \cup (F_{k+1}^{U(k+1)} - F_{k+1}^{U(k+1)}) \]

\[ = (G_{k+1}^{U(k+1)} - B_{k+1}^{U(k+1)}) \cup F_{k+1}^{U(k+1)} \]

\[ = G_{k+2}^{U(k+1)} \]

which is connected since Buster doesn’t win \( U \) in the \((k+1)\)st round. Furthermore, Fixer’s move \( F_{k+1}^{U(k+1)} \) in \( U(k+1) \) would be greedy, since otherwise Fixer’s move \( F_{k+1}^{U(k+1)} = B_{k+1}^{U(k+1)} \cup F_{k+1}^{U(k+1)} \) in \( U \) would not have been greedy, contradicting the hypotheses of this lemma. Then \( G_{k+3}^{U(k+1)} = G_{k+2}^{U(k+1)} \) and \( F_{k+3}^{U(k+2)} = R_{k+1}^{U(k+1)} - (F_{k+1}^{U(k+1)} \cup U_{k+1}^{U(k+1)}) = R_{k+1}^{U(k+1)} - F_{k+1}^{U(k+1)} = R_{k+1}^{U(k+1)} - B_{k+1}^{U(k+1)} \), so setting \( B_{k+2}^{U(k+1)} = B_{j}^{U(k+1)} \) and \( F_{j+1}^{U(k+1)} = F_{j}^{U(k+1)} \) for \( j \geq k+2 \) would be valid plays by Buster and Fixer in \( U(k+1) \), with Fixer’s moves being greedy because they were greedy in \( U \). Thus we’d have

1. Fixer wins \( U \), or Buster wins \( U \) and thus also wins \( U(k+1) \).
2. \( \sum_{j=1}^{k} |B_{j}^{U(k+1)}| = \sum_{j=1}^{k} |U_{j}^{U(k+1)}| \)
3. \( \sum_{j=1}^{k} w(F_{j}^{U(k+1)}) = \sum_{j=1}^{k} w(U_{j}^{U(k+1)}) \)

so \( U \) would be Fixer-superior to \( U(k+1) \). We complete the proof by showing there exists \( b \in B_{k+1}^{U(k+1)} \) such that Fixer can respond to \( B_{k+1}^{U(k+1)} = \{b\} \) with some greedy \( F_{k+1}^{U(k+1)} \subseteq F_{k+1}^{U(k+1)} \).

Finally, suppose \( G_{k+1}^{U(k+1)} - \{b\} \) is disconnected for every \( b \in B_{k+1}^{U(k+1)} \). Since \( F_{k+1}^{U(k+1)} \) is greedy, it is therefore a minimum spanning tree of the multigraph \( H \) whose vertices are the components of \( G_{k+1}^{U(k+1)} - B_{k+1}^{U(k+1)} \) and whose edges are the edges of \( R_{k+1}^{U(k+1)} \), identifying each endpoint of the edges in \( R_{k+1}^{U(k+1)} \) with the component of \( G_{k+1}^{U(k+1)} - B_{k+1}^{U(k+1)} \) within which it lies. By Proposition 2.1, there exists an ordering of \( F_{k+1}^{U(k+1)} \) over these edges in the order they were added by Prim’s Algorithm; let \( f \) be the first edge in this ordering. Let \( b \) be an edge in \( B_{k+1}^{U(k+1)} \) such that \( (G_{k+1}^{U(k+1)} - \{b\}) \cup \{f\} \) is connected; note that such a \( b \) exists because otherwise \( f \) would be a loop in \( H \) and therefore couldn’t be part of any minimum spanning tree. Let Buster play \( B_{k+1}^{U(k+1)} = \{b\} \) in \( U(k+1) \), and have Fixer respond with \( F_{k+1}^{U(k+1)} = \{f\} \). Fixer’s move is greedy because if there were some edge \( r \in R_{k+1}^{U(k+1)} \) such that \( w(r) < w(f) \) and \( (G_{k+1}^{U(k+1)} - \{b\}) \cup \{r\} \) was connected, then \( r \) would’ve been chosen before \( f \) by Prim’s Algorithm in constructing a minimum spanning tree of \( H \), contradicting \( f \) being the first edge chosen.

**Lemma 3.3.** Suppose \( U(k) \) is a series identical to \( S \) through the first round, with \( F_{j}^{U(k)} \) greedy for \( j \geq k \) for some \( 1 < k < |U(k)| \). If \( F \) is a subset of \( R_{k}^{U(k)} \) such that \( (G_{k}^{U(k)} - B_{k}^{U(k)}) \cup F \) is connected, then there exists a series \( U(k+1) \) identical to \( U(k) \) through Buster’s move in the \( k \)-th round such that \( F_{k}^{U(k+1)} = F \) if \( |B_{k+1}^{U(k+1)}| = 1 \) if Buster doesn’t win \( U(k+1) \) in the \((k+1)\)st round, \( F_{j}^{U(k+1)} \) is greedy for \( j > k \), and \( U(k) \) is Fixer-superior to \( U(k+1) \).

**Proof.** By the inductive hypothesis of this section, for \( j \geq k \), \( F_{j}^{U(k)} \) is optimal since \( F_{j}^{U(k)} \) is greedy. Let \( \hat{\phi} \) be a greedy strategy for Fixer to continue the series \( U \) after the \( k \)-th round, where \( \hat{U} \) is identical to \( U(k) \) through Buster’s move in the \( k \)-th round, and \( \hat{F}^{U(k)} = F^{U(k)} \). Since \( U(k) \) is part of some Fixer strategy to continue \( U(k) \) after the \( k \)-th round where Fixer only makes optimal moves after the \( k \)-th round, and \( \hat{U} \) is a series identical to \( U(k) \) through Buster’s move in the \( k \)-th round, by Lemma 2.3 \( U(k) \) is Fixer-superior to some series \( \hat{U} \subseteq \hat{\phi} \). Note that \( k < |U| \), since otherwise

\[ \sum_{j=1}^{k} |B_{j}^{U(k)}| > \sum_{j=1}^{k} |B_{j}^{U(k)}| \]

\[ = \sum_{j=1}^{k} |B_{j}^{U(k)}| \]

contradicting \( U(k) \) being Fixer-superior to \( U \). If Buster wins \( U \) in the \((k+1)\)st round, then \( F_{k+1}^{U(k+1)} \) is empty and greedy by convention, so we can set \( U(k+1) = U \). If Buster doesn’t win \( U \) in the \((k+1)\)st round, then by Lemma 2.2 there exists a series \( U(k+1) \) identical to \( U \) through the \( k \)-th round such that \( |B_{k+1}^{U(k+1)}| = 1 \), \( F_{j}^{U(k+1)} \) is greedy for \( j > k \), and \( U \) is Fixer-superior to \( U(k+1) \). Since \( U(k) \) is Fixer-superior to \( U \), and \( U \) is Fixer-superior to \( U(k+1) \), by Proposition 2.2 \( U(k) \) is Fixer-superior to \( U(k+1) \).
Recall that in order to show $F^S_1$ is optimal, we fixed a series $T \in \phi$ that we must show is Fixer-superior to some $T' \in \phi'$, where $\phi$ is a greedy Fixer strategy to continue $S$ after the first round, and $\phi'$ is any Fixer strategy to continue $S'$ after the first round.

**Proposition 3.4.** Suppose there exists a Fixer strategy to continue $S$ after the first round such that for its subset $\phi_1$ consisting of each of its series $T_S$ satisfying $|B^T_{j+1}| = 1$ for $1 < j < |T_S|$, and also $|B^T_{j+1}| = 1$ for $j = |T_S|$ if Fixer wins $T_S$ (i.e., Buster is restricted to removing singletons after the first round, except for the final round if Buster wins), for every $T_1 \in \phi_1$ there exists a series $T'_1$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $j > 1$ and $T_1$ is Fixer-superior to $T'_1$. Then there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$.

**Proof.** We first show that there exists a series $T'_g$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $j > 1$ and $T$ is Fixer-superior to $T'_g$.

If Buster wins $T$ in the second round, then $T \in \phi_1$, so by hypothesis there exists a series $T'_g$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $j > 1$ and $T$ is Fixer-superior to $T'_g$.

If Buster doesn’t win $T$ in the second round, then by Lemma 3.3 there exists a series $U(2)$ such that $U(2)$ is identical to $T$ through the first round, $|B^U_{2j+1}| = 1$, $|B^U_{2j+2}|$ is greedy for $j > 1$, and $T$ is Fixer-superior to $U(2)$. We iteratively apply Lemma 3.3 to construct a sequence $U(2), U(3), \ldots, U(\ell)$ such that for $1 < k < \ell$, $U(k+1)$ is a series identical to $U(k)$ through Buster’s move in the $k$th round such that $|B^U_{2j+1}| = 1$ is the set $F$ prescribed by $\phi_1$, $|B^U_{2j+2}| = 1$ if Buster doesn’t win $U(k+1)$ in the $(k+1)$st round, $F^U_{2j+2}$ is greedy for $j > k$, and $U(k)$ is Fixer-superior to $U(k+1)$; the sequence terminates after we reach a series $U(\ell)$ in $\phi_1$ (that is, either Buster wins $U(\ell)$ in round $\ell$, or $|B^U_{2j+1}| = 1$ and Fixer wins $U(\ell)$ in round $\ell$, or $|B^U_{2j+2}| = 1$ and Buster wins $U(\ell)$ in round $\ell+1$). Since $U(\ell) \in \phi_1$, by the hypothesis of this proposition there exists a series $T'_g$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $j > 1$ and $U(\ell)$ is Fixer-superior to $T'_g$. Since $T$ is Fixer-superior to $U(2), U(k)$ is Fixer-superior to $U(k+1)$ for $1 < k < \ell$, and $U(\ell)$ is Fixer-superior to $T'_g$, by Proposition 2.2 $T$ is Fixer-superior to $T'_g$.

Thus regardless of whether Buster wins $T$ in the second round, there exists a series $T'_g$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $j > 1$ and $T$ is Fixer-superior to $T'_g$. We now show that $T'_g$ is Fixer-superior to some $T' \in \phi'$. Since all Fixer moves after the first round of $T'_g$ are greedy, by the inductive hypothesis of the section they are optimal. Let $\phi''$ be the subset of $\phi'$ consisting of its series for which Buster’s move in the second round makes $B_{1'}^T$, and let $T''$ be any element of $\phi''$, so $\phi''$ is a strategy for Fixer to continue $T''$ after the second round. Since $F^T_{1'}$ is optimal for $j > 2$, by Lemma 3.3 there exists $T' \in \phi'' \subseteq \phi'$ such that $T'_g$ is Fixer-superior to $T''$.

Hence $T$ is Fixer-superior to $T'_g$, which is Fixer-superior to $T'$. By Proposition 2.2 $T$ is Fixer-superior to $T'$, as desired since $T' \in \phi'$.

We use Proposition 3.4 to complete the proof for the case $c = 2$ by showing there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$. We define a subset $\phi_1$ of a Fixer strategy to continue $S$ after the first round consisting of each of its series $T_1$ satisfying $|B^T_{k+1}| = 1$ for $1 < k < |T_1|$, and also $|B^T_{k+1}| = 1$ for $k = |T_1|$ if Fixer wins $T_1$, by constructing an arbitrary member $T_1 \in \phi_1$. $T_1$ will be constructed simultaneously alongside some series $T'_g$ identical to $S'$ through the first round such that $F^T_{1'}$ is greedy for $k > 1$, in the following way. Let $T_1$ be identical to $S$ through the first round, and let $T'_g$ be identical to $S'$ through the first round. For a given round $k > 1$, Buster will remove an arbitrary singleton set $B^T_{k+1}$ of edges from $G^T_k$ in $T_1$ (unless Buster wins $T_1$ in the $k$th round, in which case the singleton requirement is dropped for $B^T_{k+1}$), then based on that move in $T_1$ Buster will remove a set $B^T_{k+1}$ of edges from $G^T_k$ in $T'_g$ (or, in a particular case, have $T'_g$ skip a round with respect to $T_1$, only to make it up later). Fixer will then respond in $T'_g$ with some greedy set $F^T_{k+1}$ of edges from $B^T_{k+1}$ to connect $G^T_k - B^T_{k+1}$, then based on that response in $T'_g$ Fixer will add a set $F^T_{k+1}$ of edges from $B^T_{k+1}$ to connect $G^T_k - B^T_{k+1}$ in $T_1$. Since each Buster move in $T_1$ is an arbitrary singleton after the first round and before the final round, and in the final round is an arbitrary singleton if Fixer wins an arbitrary set and Fixer wins, $T_1$ is an arbitrary member of $\phi_1$, so our procedure for defining $T_1$ fully defines $\phi_1$. Since $T_1$ is an arbitrary member of $\phi_1$, if we show $T_1$ is Fixer-superior to $T'_g$, then by Proposition 3.4 there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$.

In order to analyze $T_1$ and $T'_g$, we categorize the corresponding rounds of each series into Scenarios 3.2.1, 3.2.2, and 3.2.3. Each scenario will include a list of conditions that must be satisfied by $T_1$ and $T'_g$, plus round-by-round instructions for both Buster to make moves in $T_1$ based on his moves in $T_1$ as well as for Fixer to respond in $T_1$ based on her responses in $T'_g$. After each round we shall show either that $T_1$ and $T'_g$ are complete, with $T_1$ Fixer-superior to $T'_g$, or that $T_1$ and $T'_g$ still satisfy the conditions of the current scenario, or that $T_1$ and $T'_g$ have advanced to a new scenario.

**3.2.1. Scenario where Buster has not used $s$ in $T_1$ and Fixer has not used $s$ in $T'_g$**

This scenario involves $T_1$ and $T'_g$ starting the $k$th round with the following properties:

1. $s \in G^T_k$, $s' \in G^T_k$, and $G^T_k \cup \{s' - \{s\}\}$ (i.e., the only difference between graphs is $s$ in $G^T_k$ being replaced by $s'$ in $G^T_k$)
2. \( s' \in R_k^T, \ s \in R_k^T, \) and \( R_k^T - \{s'\} = R_k^T - \{s\} \) (i.e. the only difference between reserve sets is \( s' \) in \( R_k^T \) being replaced by \( s \) in \( R_k^T \)).

3. \( s \) and \( s' \) are bridges in \( G_k^T \) and \( G_k^T \), respectively, between the same subgraphs \( X_k \) and \( Y_k \), but perhaps in different spots (i.e. removing both edges from their respective graphs leaves the same graphs, each with two components); see Figure 2

4. for every \( r \in R_k^T \cup R_k^T \) such that \( r \) joins \( X_k \) to \( Y_k \), \( w(r) \geq w(s) \) (i.e. in either series, no reserve edge going between subgraphs \( X_k \) and \( Y_k \) can be cheaper than \( s \))

![Figure 2: Graphs \( G_k^T \) and \( G_k^T \) from Scenario 3.2.1](image)

We divide our analysis of this scenario in the following way. Proposition 3.5 deals with the case that Fixer wins \( T_g \) in the \((k-1)st\) round (i.e. Buster decides to quit before the \( k \)th round of \( T_k \)). Propositions 3.7 and 3.8 deal with the case that Buster wins \( T_g \) in the \( k \)th round, each dealing with a subcase of whether \( s \in B_k^T \). The remaining propositions in our analysis of this scenario deal with the remaining case that Buster makes a move in the \( k \)th round, and Fixer is able to reconnect the graph in response. Proposition 3.9 deals with the subcase where \( B_k^T - B_k^T \) is connected, while Proposition 3.10 deals with the subcase where \( B_k^T = \{s\} \) (which would result in \( G_k^T - B_k^T \) being disconnected, since \( s \) is a bridge in \( G_k^T \)). Propositions 3.11, 3.12, and 3.13 deal with the remaining subcases in a manner described later on.

**Proposition 3.5.** If Fixer wins \( T_g \) in the \((k-1)st\) round, then Buster can quit after the \((k-1)st\) round of \( T_g \), resulting in Fixer winning \( T_g \) in the \((k-1)st\) round, and \( T_g \) being Fixer-superior to \( T_g \).

**Proof.** We have

1. Fixer wins \( T_g \)
2. \( \sum_{j=1}^{|T_g|} |B_j^T| = |G_1^T| + |R_1^T| - |G_1^T| = |G_1^T| + |R_1^T| - |G_1^T| = \sum_{j=1}^{|T_g|} |B_j^T| \)
3. \( \sum_{j=1}^{|T_g|} w(F_j^T) = w(R_1^T) - w(R_k^T) = (w(R_k^T) - w(s) + w(s')) \leq w(R_1^T) - w(R_k^T) = \sum_{j=1}^{|T_g|} w(F_j^T) \)

and thus \( T_g \) is Fixer-superior to \( T_g \).

**Lemma 3.6.** If Buster wins \( T_g \) in the \( k \)th round, and there exists \( \ell \) such that \( T_g \) satisfies either \( \bigcup_{j=0}^{\ell-1} B_j^T \subseteq B_k^T \) with \( (G_1^T - B_j^T) \cup R_j^T \) disconnected, or \( \bigcup_{j=0}^{\ell} B_j^T = B_k^T \), then Fixer wins \( T_g \) in the \( \ell \)th round and \( T_g \) is Fixer-superior to \( T_g \).

**Proof.** To begin, note that \( G_1^T \subseteq R_k^T \) (i.e. \( G_1^T \subseteq \{s'\} \cup \{s\} \cup \{s'\} \cup \{s\} \subseteq G_k^T \subseteq R_k^T \)). If \( (G_1^T - B_j^T) \cup R_j^T \) is disconnected, then Fixer wins \( T_g \) in the \( \ell \)th round by definition, so to show Fixer wins \( T_g \) in the \( \ell \)th round in the case \( \bigcup_{j=0}^{\ell} B_j^T = B_k^T \) we show \( (G_1^T - B_j^T) \cup R_j^T \) is a spanning subgraph of a disconnected graph and thus disconnected itself. Indeed

\[
(G_1^T - B_j^T) \cup R_j^T = (G_k^T \cup \bigcup_{j=0}^{\ell-1} F_j^T) \cup (B_k^T - B_j^T) = (G_j^T - B_j^T) \cup (R_j^T - B_j^T) = (G_k^T \cup B_k^T) - B_k^T = (G_k^T \cup R_k^T) - B_k^T = (G_k^T - B_k^T) \cup R_k^T
\]

which is disconnected since Buster wins \( T_g \) in the \( k \)th round. Furthermore, noting that the convention \( F_j^T = \emptyset \) implies \( R_{k+1}^T = R_k^T \), we have
1. Buster wins $T'_{g}$ in the $\ell$th round because $(G_{T'_{g}}^{T'_{g}} - B_{T'_{g}}^{T'_{g}}) \cup R_{T'_{g}}^{T'_{g}}$ is disconnected

2. $\sum_{j=1}^{T'_{g}} |B_{j}^{T'_{g}}| = |G_{1}^{T'_{g}}| + |R_{1}^{T'_{g}}| - |G_{k}^{T'_{g}}| - |R_{k}^{T'_{g}}| \geq |G_{1}^{T'_{g}}| + |R_{1}^{T'_{g}}| - |G_{k}^{T'_{g}}| - |R_{k}^{T'_{g}}| + \sum_{j=k}^{T'_{g}} |B_{j}^{T'_{g}}| = \sum_{j=1}^{T'_{g}} |B_{j}^{T'_{g}}|$

3. $\sum_{j=1}^{T'_{g}} w(F_{j}^{T'_{g}}) = w(R_{1}^{T'_{g}}) - w(R_{1}^{T'_{g}}) = w(R_{1}^{T'_{g}}) - (w(R_{1}^{T'_{g}}) - w(s) + w(s')) \leq w(R_{1}^{T'_{g}}) - w(R_{1}^{T'_{g}}) = \sum_{j=1}^{T'_{g}} w(F_{j}^{T'_{g}}) \leq \sum_{j=1}^{T'_{g}} w(F_{j}^{T'_{g}})$

so $T_1$ is Fixer-superior to $T'_{g}$, as desired.

**Proposition 3.7.** If Buster wins $T_1$ in the $k$th round and $s \notin B_{k}^{T_1}$, then Buster can play $B_{k}^{T_1' = B_{k}^{T_1}}$ in $T'_g$, resulting in Buster winning $T'_g$ in the $k$th round and $T_1$ Fixer-superior to $T'_g$.

**Proof.** Buster can play $B_{k}^{T_1'} = B_{k}^{T_1}$ in $T'_g$ because $B_{k}^{T_1'} \subseteq G_{k}^{T_1} - \{s\} \subseteq G_{k}^{T_1'}$, so Buster wins $T'_g$ in the $k$th round and $T_1$ is Fixer-superior to $T'_g$ by Lemma 3.6.

**Proposition 3.8.** If Buster wins $T_1$ in the $k$th round and $s \in B_{k+1}^{T_1}$, then in $T'_g$ Buster can play $B_{k}^{T_1'} = B_{k}^{T_1} - \{s\}$, as well as $B_{k+1}^{T_1'} = \{s\}$ following any Fixer response if $(G_{k}^{T_1'} - B_{k}^{T_1'}) \cup R_{k}^{T_1'}$ is connected, resulting in Buster winning $T'_g$ in the $k$th or $(k+1)$st round and $T_1$ Fixer-superior to $T'_g$.

**Proof.** Note that $B_{k}^{T_1} \neq \{s\}$, since otherwise setting $F_{k}^{T_1'} = \{s'\}$ would contradict Buster winning $T_1$ in the $k$th round since $G_{k+1}^{T_1} = (G_{k}^{T_1} - \{s\}) \cup \{s\} = G_{k}^{T_1}$, which is connected. Hence $\emptyset \neq B_{k}^{T_1} - \{s\} \subseteq G_{k}^{T_1}$, so Buster can play $B_{k}^{T_1'} = B_{k}^{T_1} - \{s\}$.

If $(G_{k}^{T_1'} - B_{k}^{T_1'}) \cup R_{k}^{T_1'}$ is disconnected, then Buster wins $T'_g$ in the $k$th round and $T_1$ is Fixer-superior to $T'_g$ by Lemma 3.6.

If $(G_{k}^{T_1'} - B_{k}^{T_1'}) \cup R_{k}^{T_1'}$ is connected, then Fixer will respond with some greedy $F_{k}^{T_1'}$ to create a connected graph $G_{k+1}^{T_1'}$. Note that

$s \in F_{k}^{T_1'}$, since otherwise $G_{k+1}^{T_1} = (G_{k}^{T_1'} - B_{k}^{T_1'}) \cup F_{k}^{T_1'} \subseteq (G_{k}^{T_1'} - B_{k}^{T_1'}) \cup R_{k}^{T_1'}$, meaning $G_{k+1}^{T_1}$ is a spanning subgraph of a disconnected graph and thus disconnected itself, a contradiction. Hence Buster can play $B_{k+1}^{T_1'} = \{s\}$, so $B_{k}^{T_1'} \cup B_{k+1}^{T_1'} = B_{k+1}^{T_1'}$, so Buster wins $T'_g$ in the $(k+1)$st round and $T_1$ is Fixer-superior to $T'_g$ by Lemma 3.6.

**Proposition 3.9.** If Buster plays in the $k$th round of $T_2$ and $G_{k}^{T_2} - B_{k}^{T_2}$ is connected, then Buster can copy her move from $T_2$ to $T'_g$ by playing $B_{k}^{T_2'} = B_{k}^{T_2} = \{b\}$, and Fixer can respond greedily in both $T'_g$ and $T_1$ with $F_{k}^{T_2'} = F_{k}^{T_2} = \emptyset$ to maintain the conditions of this scenario.

**Proof.** Note that $b \neq s$, since $s$ is a bridge and $G_{k}^{T_2} - \{b\}$ is connected. Hence Buster can play $B_{k}^{T_2'} = \{b\} \subseteq G_{k}^{T_2} - \{s\} \subseteq G_{k}^{T_2}$. Both $X_b - \{b\}$ and $Y_b - \{b\}$ are connected (since the only edge in $G_{k}^{T_2}$ with one endpoint in $X_b$ and the other in $Y_b$ is $s$, which is a bridge), and thus $G_{k}^{T_2} - B_{k}^{T_2} = G_{k}^{T_2}$ is connected, as the only edge in $G_{k}^{T_2}$ with one endpoint in $X_b$ and the other in $Y_b$ is $s'$, which is a bridge such that $b \neq s'$ (since $b \in G_{k}^{T_2}$ but $s' \notin B_{k}^{T_2}$). Hence the only greedy response for Fixer is $F_{k}^{T_2'} = \emptyset$, which Fixer can copy in $T_2$ with $F_{k}^{T_2'} = \emptyset$. Setting $X_{k+1} = X_b - \{b\}$ and $Y_{k+1} = Y_b - \{b\}$, we have

1. $s \in G_{k}^{T_2} - \{b\} = G_{k+1}^{T_2} - \{s\} = G_{k+1}^{T_2}$, and $G_{k+1}^{T_2} - \{s\} = G_{k+1}^{T_2} - \{b, s\} = G_{k+1}^{T_2} - \{s\}$

2. $s' \in R_{k}^{T_2} = R_{k+1}^{T_2}$, and $R_{k+1}^{T_2} - \{s\} = R_{k+1}^{T_2} - \{s\} = R_{k+1}^{T_2} - \{s\}$

3. $s$ and $s'$ are bridges in $G_{k+1}^{T_2}$ and $G_{k+1}^{T_2}$, respectively, between $X_{k+1}$ and $Y_{k+1}$

4. for every $r \in R_{k+1}^{T_2} \cup R_{k+1}^{T_2}$ such that $r$ joins $X_{k+1}$ to $Y_{k+1}$, $w(r) \geq w(s)$, since $r \in R_{k+1}^{T_2} \cup R_{k+1}^{T_2} = R_{k+1}^{T_2} \cup R_{k+1}^{T_2}$ and $r$ would have also joined $X_{k}$ to $Y_{k}$ (because $X_b$ and $X_{k+1}$ share the same set of vertices, as do $Y_{b}$ and $Y_{k+1}$)

and thus the conditions of this scenario are maintained.

**Proposition 3.10.** If $B_{k}^{T_2} = \{s\}$, then Fixer can play $F_{k}^{T_2} = \{s'\}$ to advance to Scenario 3.2.2

**Proof.** Letting $T'_g$ fall a round behind $T_1$ and setting $X_{k+1} = X_b$ and $Y_{k+1} = Y_b$, we have

1. $s' \in (G_{k+1}^{T_2} - \{s\}) \cup \{s'\} = G_{k+1}^{T_2}$

2. $R_{k+1}^{T_2} = R_{k+1}^{T_2}$ and $s \in R_{k+1}^{T_2}$

3. $s'$ and $s$ are bridges in $G_{k+1}^{T_2}$ and $(G_{k+1}^{T_2} - \{s\}) \cup \{s\}$, respectively, between $X_{k+1}$ and $Y_{k+1}$
4. for every \( r \in R^k_{T^0} \) such that \( r \) joins \( X_{k+1} \) to \( Y_{k+1} \), \( w(r) \geq w(s) \), since \( r \in R^k_{T^0} \cup R^k_{B^0} \) and \( r \) would have also joined \( X_k \) to \( Y_k \) and thus the conditions of Scenario 3.2.2. are satisfied.

Now suppose Fixer doesn’t win \( T_k \) in the \((k-1)\)st round, Buster doesn’t win \( T_{k+1} \) in the \( k \)th round, \( G^k_{T^0} - B^k_{T^0} \) is disconnected, and \( B^k_{T^0} = \{b\} \neq \{s\} \). Since \( s \) bridges \( X_k \) and \( Y_k \) in \( G^k_{T^0} \), either \( b \in X_k \) and \( X_k - \{b\} \) is disconnected with two components, or \( b \in Y_k \) and \( Y_k - \{b\} \) is disconnected with two components. Without loss of generality, assume \( b \in X_k \) and \( X_k - \{b\} \) is disconnected with two components \( X^1_k \) and \( X^2_k \), with \( s \) bridging \( X^1_k \) and \( Y_k \); see Figure 3a. Note that Buster can copy his move from \( T_1 \) in \( T^0_k \) by playing \( B^k_{T^0} = B^k_{T^0} = \{b\} \), since \( b \in G^k_{T^0} - \{s\} \subseteq G^k_{T^0} \); the two possibilities for \( G^k_{T^0} \) are shown in Figures 3b and 3c. Furthermore, if \( s \notin F^k_{T^0} \) and \((G^k_{T^0} - B^k_{T^0}) \cup F^k_{T^0}\) is connected, then Fixer can copy her move from \( T'_1 \) in \( T^0_k \) by playing \( F^k_{T^0} = R^k_{T^0} \) since \( F^k_{T^0} \subseteq R^k_{T^0} \) and \( s \subseteq R^k_{T^0} \).

Figure 3: \( G^k_{T^0} \) and the two possibilities for \( G^k_{T^0} \)

We separate into the following cases that together comprise every remaining possibility. Proposition 3.11 deals with the case that \( s' \) bridges \( X^1_k \) and \( Y_k \) in \( G^k_{T^0} \) (see Figure 4a). Propositions 3.12 and 3.13 deal with the case that \( s' \) bridges \( X^2_k \) and \( Y_k \) in \( G^k_{T^0} \), with the former dealing with the subcase that the cheapest connecting edge \( f \) in \( R^k_{T^0} \) is between \( X^1_k \) and \( X^2_k \) (see Figure 4b) and the latter dealing with the subcase that the cheapest connecting edge \( f \) in \( R^k_{T^0} \) is between \( X^1_k \) and \( Y_k \) (see Figure 4c).

Figure 4: The three remaining possibilities for \( G^k_{T^0} - B^k_{T^0} \), where in the latter two \( f \) is the cheapest edge in \( R^k_{T^0} \) such that \( (G^k_{T^0} - B^k_{T^0}) \cup \{f\} \) is connected.

**Proposition 3.11.** If \( s' \) bridges \( X^1_k \) and \( Y_k \) in \( G^k_{T^0} \), then for \( B^k_{T^0} = B^k_{T^0} = \{b\} \) Fixer can copy her greedy response from \( T'_1 \) in \( T^0_k \) by playing \( F^k_{T^0} = F^k_{T^0} = \{f\} \) in order to maintain the conditions of this scenario.

**Proof.** If \( f \) bridges \( X^1_k \) and \( X^2_k \), set \( X_{k+1} = (X_k - \{b\}) \cup \{f\} \) and \( Y_{k+1} = Y_k \). Then \( s \) and \( s' \) are bridges in \( G^k_{T^0} \) and \( G^k_{T^0} \), respectively, between \( X_{k+1} \) and \( Y_{k+1} \), as \( s \) and \( s' \) each bridged \( X^1_k \) and \( Y_k \), and the only new edge \( f \) resides entirely inside \( X_{k+1} \). For every \( r \in R^k_{T^0} \cup R^k_{T^0} \) such that \( r \) joins \( X_{k+1} \) to \( Y_{k+1} \), \( w(r) \geq w(s) \), since \( r \in R^k_{T^0} \cup R^k_{T^0} \subseteq R^k_{T^0} \cup R^k_{T^0} \) and \( r \) would have also joined \( X_k \) and \( Y_k \) (because \( X_k \) and \( X_{k+1} \) share the same set of vertices, as do \( Y_k \) and \( Y_{k+1} \)).

If \( f \) bridges \( X^2_k \) and \( Y_k \), set \( X_{k+1} = X^1_k \) and \( Y_{k+1} = X^2_k \) \( \cup \{f\} \cup Y_k \). Then \( s \) and \( s' \) are bridges in \( G^k_{T^0} \) and \( G^k_{T^0} \), respectively, between \( X_{k+1} \) and \( Y_{k+1} \), as \( s \) and \( s' \) each bridged \( X^1_k \) and \( Y_k \), and the only new edge \( f \) resides entirely inside \( X_{k+1} \). For every \( r \in R^k_{T^0} \cup R^k_{T^0} \) such that \( r \) joins \( X_{k+1} \) to \( Y_{k+1} \), \( w(r) \geq w(s) \), since \( r \in R^k_{T^0} \cup R^k_{T^0} \subseteq R^k_{T^0} \cup R^k_{T^0} \) and either \( r \) joined \( X^1_k \) to \( Y_k \) (so \( w(r) \geq w(s) \) by hypothesis of this scenario), or \( r \) joined \( X^1_k \) to \( X^2_k \) (so \( w(r) \geq w(f) \) because \( F^k_{T^0} = \{f\} \) was greedy, and \( w(f) \geq w(s) \) by hypothesis of this scenario, since \( f \in R^k_{T^0} \cup R^k_{T^0} \) and \( f \) joined \( X_k \) to \( Y_k \)).

Thus
1. $s \in G_{k}^{t_s} - \{b\} \subset G_{k+1}^{t_s} - \{s\} \subset G_{k+1}^{t_s}$, and $G_{k+1}^{t_s} - \{s\} = (G_{k}^{t_s} - \{b, s\}) \cup \{f\} = G_{k+1}^{t_s} - \{s'\}$

2. $s' \in R_{k}^{t_s} - \{f\} = R_{k+1}^{t_s} - \{f\} = R_{k+1}^{t_s} - \{s\} = R_{k+1}^{t_s} - \{s, f\} = R_{k+1}^{t_s} - \{f, s\} = R_{k+1}^{t_s} - \{s\}$

3. $s$ and $s'$ are bridges in $G_{k+1}^{t_s}$ and $G_{k+1}^{t_s}$, respectively, between $X_{k+1}$ and $Y_{k+1}$

4. for every $r \in R_{k+1}^{t_s} \cup R_{k+1}^{t_s}$ such that $r$ joins $X_{k+1}$ to $Y_{k+1}$, $w(r) \geq w(s)$

so the conditions of this scenario are maintained.

Proposition 3.12. If $s'$ bridges $X_{k}^{2}$ and $Y_{k}$ in $G_{k}^{t_s}$, and the cheapest edge $f \in R_{k}^{t_s}$ such that $(G_{k}^{t_s} - \{b\}) \cup \{f\}$ is connected has one endpoint in $X_{k}^{1}$ and the other in $X_{k}^{2}$, then for $B_{k}^{t_s} = B_{k}^{t_s} = \{b\}$ Fixer can copy her greedy response from $T_{y}^{t_s}$ in $T_{1}$ by playing $F_{k}^{t_s} = F_{k}^{t_s} = \{f\}$ in order to maintain the invariants in this scenario.

Proof. By hypothesis of this proposition, Fixer can greedily play $F_{k}^{t_s} = \{f\}$ in $T_{y}^{t_s}$ to create a connected graph $G_{k+1}^{t_s}$. Note that $f \in R_{k}^{t_s}$, since $f \in R_{k}^{t_s}$ and $R_{k}^{t_s} - R_{k}^{t_s} = \{s\} \neq \{f\}$, as $f$ has both endpoints in $X_{k}$ whereas $s$ has one in $Y_{k}$. Furthermore, $G_{k}^{t_s} - B_{k}^{t_s}$ consists of two components, one $X_{k}^{2}$ and the other the featuring $X_{k}^{1}$ and $Y_{k}$ being bridged by $s$, so Fixer can also play $F_{k}^{t_s} = \{f\}$ in $T_{1}$ to create a connected graph $G_{k+1}^{t_s}$. Set $X_{k+1} = (X_{k} - \{b\}) \cup \{f\}$ and $Y_{k+1} = Y_{k}$, so

1. $s \in G_{k}^{t_s} - \{b\} \subset G_{k+1}^{t_s} - \{s\} \subset G_{k+1}^{t_s}$, and $G_{k+1}^{t_s} - \{s\} = (G_{k}^{t_s} - \{b, s\}) \cup \{f\} = (G_{k}^{t_s} - \{b, s'\}) \cup \{f\} = G_{k+1}^{t_s} - \{s'\}$

2. $s' \in R_{k}^{t_s} - \{f\} = R_{k+1}^{t_s} - \{f\} = R_{k+1}^{t_s} - \{s\} = R_{k+1}^{t_s} - \{s, f\} = R_{k+1}^{t_s} - \{f, s\} = R_{k+1}^{t_s} - \{s\}$

3. $s$ and $s'$ are bridges in $G_{k+1}^{t_s}$ and $G_{k+1}^{t_s}$, respectively, between $X_{k+1}$ and $Y_{k+1}$

4. for every $r \in R_{k+1}^{t_s} \cup R_{k+1}^{t_s}$ such that $r$ joins $X_{k+1}$ to $Y_{k+1}$, $w(r) \geq w(s)$, since $r \in R_{k+1}^{t_s} \cup R_{k+1}^{t_s} \subset R_{k+1}^{t_s} \cup R_{k+1}^{t_s}$ and $r$ would have also joined $X_{k}$ to $Y_{k}$ (because $X_{k}$ and $X_{k+1}$ share the same set of vertices, as do $Y_{k}$ and $Y_{k+1}$)

so the conditions of this scenario are maintained.

Proposition 3.13. If $s'$ bridges $X_{k}^{2}$ and $Y_{k}$ in $G_{k}^{t_s}$, and the cheapest edge $f \in R_{k}^{t_s}$ such that $(G_{k}^{t_s} - \{b\}) \cup \{f\}$ is connected has one endpoint in $X_{k}^{1}$ and the other $Y_{k}$, then for $B_{k}^{t_s} = B_{k}^{t_s} = \{b\}$ Fixer can greedily play $F_{k}^{t_s} = \{s\}$ in $T_{y}^{t_s}$, as well as play $F_{k}^{t_s} = \{s'\}$ in $T_{1}$, to advance to Scenario 3.2.2.

Proof. Since the cheapest connecting reserve edge in $T_{y}^{t_s}$ is between $X_{k}^{1}$ and $Y_{k}$, $s$ is such an edge by hypothesis of this scenario, so Fixer can greedily play $F_{k}^{t_s} = \{s\}$ in $T_{y}^{t_s}$. Since $G_{k}^{t_s} - B_{k}^{t_s}$ consists of two components, one $X_{k}^{2}$ and the other the featuring $X_{k}^{1}$ and $Y_{k}$ being bridged by $s$, and $s' \in R_{k}^{t_s}$ bridges $X_{k}^{2}$ and $Y_{k}$, Fixer can play $F_{k}^{t_s} = \{s'\}$ in $T_{1}$ to leave $T_{1}$ and $T_{y}^{t_s}$ so that

1. $G_{k+1}^{t_s} = (G_{k}^{t_s} - \{b\}) \cup \{s'\} = (G_{k+1}^{t_s} - \{b\}) \cup \{s\} = G_{k+1}^{t_s}$

2. $R_{k+1}^{t_s} = R_{k+1}^{t_s} - \{s\} = R_{k+1}^{t_s} - \{s\} = R_{k+1}^{t_s}$

which are the conditions of Scenario 3.2.2.

3.2.2. Scenario where Fixer has used $s$ in $T_{1}$ but Buster has not used $s$ in $T_{y}^{t_s}$, which is a round behind $T_{1}$

This scenario involves $T_{1}$ starting the $k$th round and $T_{y}^{t_s}$ starting the $(k-1)$st round with the following properties:

1. $s' \in G_{k}^{t_s} = G_{k-1}^{t_s}$ (i.e. the graphs are identical and contain $s'$)

2. $R_{k}^{t_s} = R_{k-1}^{t_s} - \{s\}$ and $s \in R_{k-1}^{t_s}$ (i.e. the only difference between reserve sets is $s$ being in $R_{k-1}^{t_s}$ but not in $R_{k}^{t_s}$)

3. $s'$ and $s$ are bridges in $G_{k}^{t_s}$ and $(G_{k-1}^{t_s} - \{s'\}) \cup \{s\}$, respectively, between the same connected subgraphs $X_{k}$ and $Y_{k}$, but perhaps in different spots (i.e. removing both edges from their respective graphs leaves the same graphs, each with two components)

4. for every $r \in R_{k-1}^{t_s}$ such that $r$ bridges $X_{k}$ and $Y_{k}$, $w(r) \geq w(s)$ (i.e. in either series, no reserve edge bridging subgraphs $X_{k}$ and $Y_{k}$ can be cheaper than $s$)
Note that in this scenario Buster can always copy his move from \( T_k \) with \( B^\gamma_{k-1} = B^\gamma_k \) since \( B^\gamma_k \subseteq G^\gamma_k = G^\gamma_{k-1} \). Furthermore, if Buster doesn’t win \( T_k \) in the \( k \)th round and \( B^\nu_{k-1} = B^\nu_k \), then Buster doesn’t win \( T'_g \) in the \((k-1)\)st round, since if \((G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \) is connected, then so must be \((G^\nu_k - B^\nu_k) \cup R^\nu_k \) because \( G^\gamma_k = G^\gamma_{k-1} \), \( B^\gamma_k = B^\gamma_{k-1} \), and \( R^\gamma_k \subseteq R^\nu_k \).

We divide our analysis of this scenario in the following way. Proposition 3.14 deals with the case that Fixer wins \( T_k \) in the \((k-1)\)st round (i.e. Buster decides to quit before the \( k \)th round of \( T_k \)). Propositions 3.15 and 3.16 deal with the case that Buster wins \( T_g \) in the \( k \)th round, each dealing with a subcase of whether \((G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \cup \{s\} \) is connected. Propositions 3.17 and 3.18 deal with the remaining case that Buster makes a move in the \( k \)th round, and Fixer is able to reconnect the graph in response; each deals with a subcase of whether Fixer can respond greedily in \( T'_g \) to \( B^\gamma_{k-1} = B^\gamma_k \) with \( F^\gamma_k \) containing \( s \).

**Proposition 3.14.** Suppose Fixer wins \( T_k \) in the \((k-1)\)st round. Then for \( B^\nu_{k-1} = \{s'\} \) Fixer can play \( F^\gamma_k \) as a greedy response in \( T'_g \), and if Buster subsequently quits then Fixer wins \( T'_g \) in the \((k-1)\)st round and \( T_k \) is Fixer-superior to \( T'_g \).

**Proof.** Since \( G^\gamma_k - \{s'\} \) is the graph consisting of the components \( X_k \) and \( Y_k \), and \( s \in R^\nu_{k-1} \) is a bridge between \( X_k \) and \( Y_k \), \( F^\gamma_{k-1} = \{s\} \) is a valid move by Fixer in \( T'_g \). Furthermore, \( F^\gamma_{k-1} = \{s\} \) is a greedy move because for every \( r \in R^\nu_{k-1} \) such that \( r \) bridges \( X_k \) and \( Y_k \), \( w(r) \geq w(s) \). Note that this leaves \( s' \in G^\gamma_k - s \in G^\gamma_k \), and \( G^\gamma_k - \{s\} = G^\gamma_{k-1} - \{s'\} \) (i.e. the only difference between graphs is \( s' \) in \( G^\gamma_k \) being replaced by \( s \) in \( G^\gamma_k \)) as well as \( R^\nu_k = R^\nu_{k-1} - \{s\} \). Hence

1. Fixer wins \( T_k \)
2. \[ \sum_{j=1}^{T_k} |B^\gamma_{j} \cap T_k| = |G^\gamma_k| + |R^\nu_k| - |R^\nu_k| = |G^\gamma_k| + |R^\nu_k| - |R^\nu_k| = |T_k| \]
3. \[ \sum_{j=1}^{T_k} w(F^\gamma_k) = w(R^\nu_k) - w(R^\nu_k) = w(R^\nu_k) - w(R^\nu_k) = \sum_{j=1}^{T_k} w(F^\gamma_k) \]

so \( T_k \) is Fixer-superior to \( T'_g \).

**Proposition 3.15.** Suppose Buster wins \( T_k \) in the \( k \)th round and \((G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \cup \{s\} \) is disconnected. Then for \( B^\nu_{k-1} = \{s'\} \) Fixer can play \( F^\gamma_k \) as a greedy response in \( T'_g \). Furthermore, Buster playing \( B^\nu_k = B^\nu_k \) in \( T'_g \) if \( s' \notin B^\nu_k \), or Buster playing \( B^\nu_k = (B^\nu_k - \{s'\}) \cup \{s\} \) in \( T'_g \) if \( s' \in B^\nu_k \), both result in Buster winning \( T'_g \) in the \( k \)th round and \( T_k \) Fixer-superior to \( T'_g \).

**Proof.** Since \( G^\gamma_k - \{s'\} \) is the graph consisting of the components \( X_k \) and \( Y_k \), and \( s \in R^\nu_{k-1} \) is a bridge between \( X_k \) and \( Y_k \), \( F^\nu_{k-1} = \{s\} \) is a valid move by Fixer in \( T'_g \). Furthermore, \( F^\nu_{k-1} = \{s\} \) is a greedy move because for every \( r \in R^\nu_{k-1} \) such that \( r \) bridges \( X_k \) and \( Y_k \), \( w(r) \geq w(s) \). Note that this leaves \( s' \in G^\gamma_k - s \in G^\gamma_k \), and \( G^\gamma_k - \{s\} = G^\gamma_{k-1} - \{s'\} \) (i.e. the only difference between graphs is \( s' \) in \( G^\gamma_k \) being replaced by \( s \) in \( G^\gamma_k \)) as well as \( R^\nu_k = R^\nu_{k-1} - \{s\} \).

Let Buster play \( B^\nu_k = B^\nu_k \) in \( T'_g \) if \( s' \notin B^\nu_k \), or play \( B^\nu_k = (B^\nu_k - \{s'\}) \cup \{s\} \) in \( T'_g \) if \( s' \in B^\nu_k \). We first show in either case that Buster wins \( T'_g \) in the \( k \)th round by showing that \((G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \) is a spanning subgraph of \((G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \cup \{s\} \) and thus disconnected as well. If \( s' \notin B^\nu_k \), then \( B^\nu_k = B^\nu_k \) and

\[
(G^\gamma_k - B^\gamma_k) \cup R^\nu_k = ((G^\gamma_k - \{s'\}) \cup \{s\}) - B^\nu_k \\
\subseteq (G^\gamma_k - B^\gamma_k) \cup R^\gamma_k \cup \{s\}
\]

since \( s \notin B^\nu_k \), as \( s \notin G^\gamma_k \) and \( B^\nu_k \subseteq G^\gamma_k \). If \( s' \in B^\nu_k \), then \( B^\nu_k = (B^\nu_k - \{s'\}) \cup \{s\} \) and

\[
(G^\gamma_k - B^\gamma_k) \cup R^\nu_k = (((G^\gamma_k - \{s'\}) \cup \{s\}) - (B^\nu_k - \{s'\})) \cup R^\gamma_k \\
= (G^\gamma_k - B^\gamma_k) \cup R^\gamma_k
\]

since \( s' \in G^\gamma_k \cap B^\gamma_k \) and \( s \notin G^\gamma_k \cup B^\gamma_k \).

In either case of final Buster moves, noting that the convention of \( F^\nu_k = F^\nu_k = \emptyset \) implies \( |G^\gamma_{k+1}| = |G^\gamma_k| - |B^\gamma_k| = |G^\gamma_{k-1}| - |B^\gamma_k| = \\
|G^\gamma_{k-1} - \{s'\}) \cup \{s\}| - |B^\gamma_k| = |G^\gamma_{k+1}| + R^\nu_k = R^\nu_k = R^\nu_k \) as well as \( R^\nu_{k+1} = R^\nu_k = R^\nu_k \),

1. Buster wins \( T'_g \)
2. \[ \sum_{j=1}^{T_k} |B^\gamma_{j} \cap T_k| = |G^\gamma_k| + |R^\nu_k| - |R^\nu_k| = |G^\gamma_k| + |R^\nu_k| - |R^\nu_k| = \sum_{j=1}^{T_k} |B^\gamma_{j} \cap T_k| \]
3. \( \sum_{j=1}^{|T_1|} w(F_j^{T_1}) = w(R_1^{T_1}) - w(R_{k+1}^{T_1}) = w(R_1^{T_1}) - w(R_{k+1}^{T_1}) = \sum_{j=1}^{|T_1|} w(F_j^{T_1}) \)

   so \( T_1 \) is Fixer-superior to \( T_1' \).

\[ \square \]

**Proposition 3.16.** Suppose Buster wins \( T_1 \) in the \( k \)th round and \( (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1} \cup \{s\} \) is connected. Then for \( B_k^{T_1} = B_k^{T_2} \), Buster does not win \( T_1 \) in \( (k - 1) \)st round, and any valid Fixer response in \( T_1' \) must satisfy \( s \in F_k^{T_1} \). Furthermore, this allows Buster to play \( B_k^{T_1} = \{s\} \) in \( T_1' \), resulting in Buster winning \( T_1' \) in the \( k \)th round and \( T_2 \) Fixer-superior to \( T_1' \).

**Proof.** First see that Buster does not win \( T_1' \) in \( (k - 1) \)st round since \( (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1} \cup \{s\} \), which is connected by hypothesis of this proposition. Next, note that any valid Fixer response in \( T_1' \) must satisfy \( s \in F_k^{T_1} \), since otherwise \( (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1} \subseteq (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1} \), which is disconnected because Buster wins \( T_1 \) in the \( k \)th round.

With \( B_k^{T_1} = \{s\} \), Buster wins \( T_1' \) in the \( k \)th round since

\[ (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1} = ((G_k^{T_1} - B_k^{T_1}) \cup F_k^{T_1} - \{s\}) \cup (R_k^{T_1} - F_k^{T_1}) \]

so \( T_1 \) is Fixer-superior to \( T_1' \).

\[ \square \]

**Proposition 3.17.** Suppose Buster plays some set \( B_k^{T_1} = \{s\} \) that does not win \( T_2 \) for Buster in the \( k \)th round, and when Buster copies that move in \( T_2 \), with \( B_k^{T_1} = \{s\} \), Fixer responds greedily in \( T_2' \) with a set \( F_k^{T_1} \), but no possible greedy response for Fixer in \( T_2' \) contains \( s \). Then Fixer can copy that move in \( T_2 \) with \( F_k^{T_1} = F_k^{T_1} \) to stay in this scenario.

**Proof.** Fixer can validly play \( F_k^{T_1} = F_k^{T_1} \) in \( T_2 \) because \( R_k^{T_1} \subseteq R_k^{T_1} - \{s\} = R_k^{T_1} \) and

\[ G_k^{T_1} = (G_k^{T_1} - B_k^{T_1}) \cup F_k^{T_1} \]

\[ = (G_k^{T_1} - B_k^{T_1}) \cup F_k^{T_1} \]

which is connected. In addition to \( G_k^{T_1} = G_k^{T_1} \), we also have

\[ R_k^{T_1} = R_k^{T_1} - F_k^{T_1} \]

\[ = (R_k^{T_1} - \{s\}) - F_k^{T_1} \]

\[ = R_k^{T_1} - \{s\} \]

and \( s \in R_k^{T_1} - F_k^{T_1} = R_k^{T_1} \).

We show that \( e \neq s' \) by showing that if \( e = s' \) then Fixer could have contradicted the assumption that no possible greedy response in \( T_2' \) contained \( s \) by playing \( F_k^{T_1} = \{s\} \). Indeed, \( F_k^{T_1} = \{s\} \) would have been a valid Fixer response since \( s \) is a bridge in \( (G_k^{T_1} - \{s\}) \) between connected subgraphs \( X_k \) and \( Y_k \). Furthermore, it would have been a greedy response since \( s' \) being a bridge between \( X_k \) and \( Y_k \) implies that any greedy response would have to be a single edge in \( R_k^{T_1} \) bridging \( X_k \) and \( Y_k \), and for every \( r \in R_k^{T_1} \) such that \( r \) bridges \( X_k \) and \( Y_k \), \( w(r) \geq w(s) \) by assumption of this scenario.
To complete the proof that Fixer playing $F^{T_k}_{k+1} = F^{T_k}_{k+1}$ in $T_1$ maintains the conditions of this scenario, we show that $G^{T_k}_{k+1} - \{s'\}$ consists of two components $X_{k+1}$ and $Y_{k+1}$ such that $s'$ and $s$ are bridges in $G^{T_k}_{k+1}$ and $(G^{T_k}_{k+1} - \{s'\}) \cup \{s\}$, respectively, between $X_{k+1}$ and $Y_{k+1}$, with every $r \in R^{T_k}_{k}$ bridging $X_{k+1}$ and $Y_{k+1}$ also satisfying $w(r) \geq w(s)$. If $e$ is not a bridge in $X_k$ or $Y_k$, then both $X_k - \{e\}$ and $Y_k - \{e\}$ are connected, and furthermore $G^{T_k}_{k+1} - \{e\}$ is connected because $s'$ is a bridge between $X_k - \{e\}$ and $Y_k - \{e\}$ and $e \neq s'$; hence $F^{T_k}_{k} = F^{T_k}_{k-1} = \emptyset$ since that would be the only greedy move by Fixer in $T'_g$, so we can set $X_{k+1} = X_k - \{e\}$ and $Y_{k+1} = Y_k - \{e\}$. Thus without loss of generality we may assume $e$ is a bridge in $X_k$ between the two components $X_k^1$ and $X_k^2$ of $X_k - \{e\}$, and $F^{T_k}_{k} = F^{T_k}_{k-1} = \{f\}$ for some edge $f \in R^{T_k}_{k}$ either bridging $X_k^1$ and $X_k^2$, or bridging $Y_k$ and one of $X_k^1$ or $X_k^2$. If $f$ bridges $X_k^1$ and $X_k^2$, then we may set $X_{k+1} = (X_k - \{e\}) \cup \{f\}$ and $Y_{k+1} = Y_k$. Thus without loss of generality we may assume $f$ bridges $X_k^1$ and $Y_k$, so $w(f) \geq w(s)$ by the assumptions of this scenario, and $w(f) \leq w(r)$ for any $r \in R^{T_k}_{k-1}$ bridging $X_k^1$ and $X_k^2$, since otherwise $F^{T_k}_{k-1} = \{r\}$ would have been a cheaper valid response for Fixer in $T'_g$, contradicting $F^{T_k}_{k-1} = \{f\}$ being greedy. Hence we can set $X_{k+1} = X_k^1$ and $Y_{k+1} = Y_k \cup X_k^2 \cup \{f\}$, since for every $r \in R^{T_k}_{k+1}$ such that $r$ bridges $X_k^1$ and $Y_k$, or $r$ bridged $X_k$ and $Y_k$ in which case $w(r) \geq w(s)$ by the assumptions of this scenario, or $r$ bridged $X_k^1$ and $X_k^2$, in which case we’ve already shown $w(r) \geq w(f) \geq w(s)$. \[ \square \]

**Proposition 3.18.** Suppose Buster plays some set $B^{T_k}_{k}$ that does not win $T_1$ for Buster in the $k$th round, and when Buster copies that move in $T'_g$ with $B^{T_k}_{k-1} = B^{T_k}_{k}$, Fixer can respond greedily in $T'_g$ with a set $F^{T_k}_{k-1}$ containing $s$. Then Buster can play $B^{T_k}_{k} = \{s\}$ and Fixer can create a connected graph $G^{T_k}_{k+1}$ with some greedy $F^{T_k}_{k}$ in $T'_g$, and Fixer can play $F^{T_k}_{k} = (F^{T_k}_{k-1} - \{s\}) \cup F^{T_k}_{k}$ in $T_1$, to trigger Scenario 32.3 for the $(k+1)$st round.

**Proof.** Buster can play $B^{T_k}_{k} = \{s\}$ in $T'_g$ because $s \in F^{T_k}_{k-1} \subseteq G^{T_k}_{k}$. Fixer can respond with some $F^{T_k}_{k} \subseteq R^{T_k}_{k}$ such that $G^{T_k}_{k} = (G^{T_k}_{k} - B^{T_k}_{k}) \cup F^{T_k}_{k}$ is connected (so Buster doesn’t win $T'_g$ in the $k$th round), because Buster’s failure to win $T_1$ in the $k$th round implies $(G^{T_k}_{k} - B^{T_k}_{k}) \cup R^{T_k}_{k}$ is connected, and

$$
(G^{T_k}_{k} - B^{T_k}_{k}) \cup R^{T_k}_{k} = (G^{T_k}_{k} \cup R^{T_k}_{k}) - \{s\}
= ((G^{T_k}_{k-1} \cup R^{T_k}_{k-1}) - B^{T_k}_{k-1}) - \{s\}
= (G^{T_k}_{k-1} - B^{T_k}_{k-1}) \cup (R^{T_k}_{k-1} - \{s\})
= (G^{T_k}_{k-1} - B^{T_k}_{k}) \cup R^{T_k}_{k}
$$

so $(G^{T_k}_{k} - B^{T_k}_{k}) \cup R^{T_k}_{k}$ is connected as well. Fixer can play $F^{T_k}_{k} = (F^{T_k}_{k-1} - \{s\}) \cup F^{T_k}_{k}$ in $T_1$ because

$$
F^{T_k}_{k} = (F^{T_k}_{k-1} - \{s\}) \cup F^{T_k}_{k}
\subseteq R^{T_k}_{k-1} - \{s\}
= R^{T_k}_{k}
$$

and

$$
G^{T_k}_{k+1} = (G^{T_k}_{k} - B^{T_k}_{k}) \cup F^{T_k}_{k}
= (G^{T_k}_{k-1} - B^{T_k}_{k-1}) \cup (F^{T_k}_{k-1} - \{s\}) \cup F^{T_k}_{k}
= ((G^{T_k}_{k-1} - B^{T_k}_{k-1}) \cup F^{T_k}_{k-1} - \{s\}) \cup F^{T_k}_{k}
= (G^{T_k}_{k} - B^{T_k}_{k}) \cup F^{T_k}_{k}
= G^{T_k}_{k+1}
$$

which we already showed was connected because there existed a valid Fixer move $F^{T_k}_{k}$ to prevent Buster from winning $T'_g$ in the $k$th
round. Since we just showed $G_{k+1}^{T_1} = G_{k+1}^{T_6}$, and

\begin{align*}
R_k^{T_1} &= R_k^{g} - F_k^{T_1} \\
&= (R_k^{g} - \{s\}) - ((F_{k-1}^{T_1} - \{s\}) \cup F_k^{T_1}) \\
&= (R_k^{g} - F_k^{T_1}) - F_k^{T_1} \\
&= R_k^{g} - F_k^{T_1} \\
&= R_{k+1}^{T_6}
\end{align*}

Scenario 3.2.3 is triggered for the $(k+1)$st round. □

### 3.2.3. Scenario where $T_1$ and $T_6$ are in the same state

This scenario involves $T_1$ and $T_6$ each starting the $k$th round with the following properties:

1. $G_k^{T_1} = G_k^{T_6}$
2. $R_k^{T_1} = R_k^{T_6}$

Note that in this scenario, after Buster plays some $B_k^{T_1}$ in $T_1$, Buster can copy that move in $T_6$ with $B_k^{T_6} = B_k^{T_1}$ since $G_k^{T_1} = G_k^{T_6}$.

We divide our analysis of this scenario in the following way. Proposition 3.19 deals with the case that Fixer wins $T_1$ in the $(k-1)$st round (i.e. Buster decides to quit before the $k$th round of $T_1$). Proposition 3.20 deals with the case that Buster wins $T_1$ in the $k$th round. Proposition 3.21 deals with the remaining case that Buster makes a move in the $k$th round, and Fixer is able to reconnect the graph in response.

**Proposition 3.19.** If Fixer wins $T_1$ in the $(k-1)$st round, then Buster can quit after the $(k-1)$st round of $T_6$, resulting in Fixer winning $T_6$ in the $(k-1)$st round, and $T_1$ being Fixer-superior to $T_6$.

**Proof.** We have

1. Fixer wins $T_1$
2. $\sum_{j=1}^{T_1}|B_j^{T_1}| = |G^{T_1}_1| + |R_1^{T_1}| - |G^{T_6}_1| - |R_1^{T_6}| = |G^{T_6}_1| + |R_1^{T_6}| - |G^{T_6}_1| - |R_1^{T_6}| = \sum_{j=1}^{T_6}|B_j^{T_6}|$
3. $\sum_{j=1}^{T_1} w(F_j^{T_1}) = w(R_1^{T_1}) - w(R_1^{T_6}) = w(R_1^{T_6}) - w(R_1^{T_6}) = \sum_{j=1}^{T_6} w(F_j^{T_6})$

and thus $T_1$ is Fixer-superior to $T_6$. □

**Proposition 3.20.** If Buster wins $T_1$ in the $k$th round, then Buster playing $B_k^{T_6} = B_k^{T_1}$ results in Buster winning $T_6$ in the $k$th round and $T_1$ Fixer-superior to $T_6$.

**Proof.** We have $(G^{T_6}_k - B_k^{T_6}) \cup R_k^{T_1} = (G^{T_1}_k - B_k^{T_1}) \cup R_k^{T_1}$, which is disconnected since Buster wins $T_1$ in the $k$th round, so Buster wins $T_6$ in the $k$th round. Hence $F_1^{T_6} = F_1^{T_1} = \emptyset$ by convention, implying $G_{k+1}^{T_1} = G_k^{T_1} - B_k^{T_1} = G_k^{T_6} - B_k^{T_6} = G_k^{T_6}$ and $R_{k+1}^{T_1} = R_k^{T_1} = R_k^{T_6} = R_{k+1}^{T_6}$, so

1. Buster wins $T_6$
2. $\sum_{j=1}^{T_1}|B_j^{T_1}| = |G^{T_1}_1| + |R_1^{T_1}| - |G^{T_6}_1| - |R_1^{T_6}| = |G^{T_6}_1| + |R_1^{T_6}| - |G^{T_6}_1| - |R_1^{T_6}| = \sum_{j=1}^{T_6}|B_j^{T_6}|$
3. $\sum_{j=1}^{T_1} w(F_j^{T_1}) = w(R_1^{T_1}) - w(R_1^{T_6}) = w(R_1^{T_6}) - w(R_1^{T_6}) = \sum_{j=1}^{T_6} w(F_j^{T_6})$

and thus $T_1$ is Fixer-superior to $T_6$. □

**Proposition 3.21.** If Buster plays some set $B_k^{T_1}$ that does not win $T_1$ for Buster in the $k$th round, then after Buster plays $B_k^{T_6} = B_k^{T_1}$, Fixer can copy her greedy move from $T_6$ in $T_1$ by playing $F_1^{T_6} = F_1^{T_1}$ to stay in this scenario.

**Proof.** Since Buster does not win $T_1$ in the $k$th round and $(G_k^{T_6} - B_k^{T_6}) \cup R_k^{T_1} = (G_k^{T_1} - B_k^{T_1}) \cup R_k^{T_1}$, Buster also doesn’t win $T_6$ in the $k$th round, so Fixer can respond greedily in $T_6$ with some $F_1^{T_6}$. Since $R_k^{T_6} = R_k^{T_1}$, Fixer can copy that move in $T_1$ with $F_1^{T_6} = F_1^{T_1}$. Hence $G_{k+1}^{T_1} = G_{k+1}^{T_6}$ and $R_{k+1}^{T_1} = R_{k+1}^{T_6}$, leaving us in the same scenario. □
3.3. The case \( c \geq 3 \)

If \( c \geq 3 \), then \( F_1^S \) and \( F_1^{S'} \) each have multiple edges. Let \( S'' \) be a series such that \( G_1^S = G_1^{S'} = G_1^{S''} \), \( R_1^S = R_1^{S'} = R_1^{S''} \), and \( B_1^S = B_1^{S'} = B_1^{S''} \). To show that for our fixed series \( \bar{S} \in \phi \), there exists \( \bar{T} \in \phi' \) such that \( \bar{S} \) is Fixer-superior to \( \bar{T} \), we first show (via Proposition 3.23) that there exists a Fixer move \( F_1^{S''} \) and strategy \( \phi'' \) for Fixer to continue \( S'' \) after the first round such that \( F_1^{S''} \) contains an edge \( e \in F_1^S \) and for every \( \bar{T} \in \phi' \) there exists \( \bar{U} \in \phi' \) such that \( \bar{U} \) is Fixer-superior to \( \bar{T} \). We then show (via Proposition 3.24) that for every \( \bar{T} \in \phi' \) there exists \( \bar{U} \in \phi' \) such that \( \bar{T} \) is Fixer-superior to \( \bar{U} \). Then for every \( \bar{T} \in \phi' \), we would have \( \bar{T} \in \phi'' \) and \( \bar{T} \in \phi' \) such that \( \bar{T} \) is Fixer-superior to \( \bar{T}'' \), and \( \bar{T}'' \) is Fixer-superior to \( \bar{T}' \). Since Fixer-superiority is transitive, \( \bar{T} \) would be Fixer-superior to \( \bar{T}' \).

In order to prove Propositions 3.23 and 3.24 we make use of the following observation. Suppose \( P \) and \( U \) are series such that for some subset \( F \) of \( F_1^P \), the situation facing Fixer during her first move in \( U \) is the same situation she faced in \( P \) after having partially fixed \( G_1^P = B_1^P \) with \( F \) from \( R_1^U \) (i.e. \( G_1^U = B_1^U = (G_1^P - B_1^P) \cup F \) and \( R_1^U = R_1^P - F \)). Further suppose \( Q \) is a series identical to \( P \) up through Fixer partially fixing each graph with \( F \) at the start of her first move, but Fixer finishes her first move in \( Q \) by copying her entire first move in \( U \) (i.e. \( G_1^Q = G_1^P \), \( R_1^Q = R_1^P \), \( B_1^Q = B_1^P \), and \( F_1^Q = F \)). Further suppose \( F_1^U \) is optimal, \( \phi_U \) is a strategy of optimal moves for Fixer to continue \( U \) after the first round constructed by replacing the first round of each series in \( \phi_U \) with the first round of \( Q \), and \( \phi_P \) is a strategy for Fixer to continue \( P \) after the first round. Then for every \( Q' \in \phi_Q \), we should expect by dint of Fixer copying the first part of her move from \( P \) into \( Q' \) before (in a sense) finishing that move optimally, and playing all subsequent moves optimally, that there exists \( P' \in \phi_P \) such that \( Q' \) is Fixer-superior to \( P' \). We formally verify below that this indeed holds true.

**Lemma 3.22.** Let \( P \) and \( U \) be series such that for some subset \( F \) of \( F_1^P \), \( G_1^U = B_1^U = (G_1^P - B_1^P) \cup F \) and \( R_1^U = R_1^P - F \). Let \( Q \) be a series such that \( G_1^Q = G_1^P \), \( R_1^Q = R_1^P \), \( B_1^Q = B_1^P \), and \( F_1^Q = F \). If \( F_1^Q \) is optimal, \( \phi_Q \) is a strategy of optimal moves for Fixer to continue \( U \) after the first round, \( \phi_U \) is the strategy for Fixer to continue \( U \) after the first round constructed by replacing the first round of each series in \( \phi_U \) with the first round of \( Q \), and \( \phi_P \) is a strategy for Fixer to continue \( P \) after the first round, then for every \( Q' \in \phi_Q \) there exists \( P' \in \phi_P \) such that \( Q' \) is Fixer-superior to \( P' \).

**Proof.** Fixer’s strategy \( \phi_Q \) against any Buster strategy will be a translation of Fixer’s strategy \( \phi_U \) against the same Buster strategy. Note that

\[
G_2^Q = (G_1^Q - B_1^Q) \cup F_1^Q = (G_1^P - B_1^P) \cup F \cup F_1^U = (G_1^U - B_1^U) \cup F_1^U = G_2^U
\]

and

\[
R_2^Q = R_1^Q - F_1^Q = R_1^P - (F \cup F_1^U) = (R_1^P - F) - F_1^U = R_1^U - F_1^U = R_2^U
\]

so \( Q \) and \( U \) are equivalent starting in the second round.

Since \( F_1^U \) is optimal, and \( \phi_U \) is a strategy of optimal moves for Fixer to continue \( U \) after the first round, by Lemma 2.23 for any series \( V \) identical to \( U \) through Buster’s move of the first round, for any \( U' \in \phi_U \) and any strategy \( \phi_V \) for Fixer to continue \( V \) after the first round, there exists \( U' \in \phi_V \) such that \( U' \) is Fixer-superior to \( V \). Let \( V \) be identical to \( U \) through Buster’s move of the first round, but set \( F_1^V = F_1^P - F \). Note that

\[
G_2^P = (G_1^P - B_1^P) \cup F_1^P = (G_1^P - B_1^P) \cup F \cup (F_1^P - F) = (G_1^U - B_1^U) \cup (F_1^P - F) = (G_1^V - B_1^V) \cup F_1^V = G_2^V
\]
and

\[ R_2^p = R_1^p - F_1^p \]

\[ = (R_1^U - F) - (F_1^T - F) \]

\[ = R_1^U - F_1^V \]

\[ = R_1^T - F_1^V \]

\[ = R_2^U \]

so \( P \) and \( V \) are equivalent starting in the second round.

Let \( \phi_Q \) be the strategy for Fixer to continue \( Q \) after the first round constructed by replacing the first round of each series in \( \phi_U \) with the first round of \( Q \), let \( \phi_P \) be a strategy for Fixer to continue \( P \) after the first round, and let \( \phi_V \) be the strategy for Fixer to continue \( V \) after the first round constructed by replacing the first round of each series in \( \phi_P \) with the first round of \( V \). Let \( Q' \in \phi_Q \), and let \( U' \in \phi_U \) be the series from which \( Q' \) was constructed by replacing the first round with the first round of \( Q \). Let \( V' \in \phi_V \) be a series such that \( U' \) is Fixer-superior to \( V' \), and let \( P' \in \phi_P \) be the series from which \( V' \) was constructed by replacing the first round of \( P' \) with the first round of \( V \). Then

1. Fixer wins \( Q' \), or Buster wins \( Q' \), in which case \((G_{U'|Q'} - B_{U'|Q'}) \cup R_{U'|Q'}^e\) is disconnected, implying Buster wins \( U' \) since \((G_{V'|Q'} - B_{V'|Q'}) \cup R_{V'|Q'}^e\) is disconnected, implying Buster wins \( V' \) since \( U' \) is Fixer-superior to \( V' \), implying \((G_{V'|Q'} - B_{V'|Q'}) \cup R_{V'|Q'}^e\) disconnected, implying Buster wins \( P' \) since \((G_{P'|Q'} - B_{P'|Q'}) \cup R_{P'|Q'}^e\) = \((G_{V'|Q'} - B_{V'|Q'}) \cup R_{V'|Q'}^e\).

2. \[ \sum_{j=1}^{U'} |B_j^U| = |B_1^U| + \sum_{j=1}^{V'} |B_j^V| \geq |B_1^P| - |B_1^V| + \sum_{j=1}^{V'} |B_j^V| = \sum_{j=1}^{P'} |B_j^P| \]

3. \[ \sum_{j=1}^{U'} w(F_j^U) = w(F) + \sum_{j=1}^{V'} w(F_j^V) \leq w(F) + \sum_{j=1}^{V'} w(F_j^V) = \sum_{j=1}^{P'} w(F_j^P) \]

so \( Q' \) is Fixer-superior to \( P' \).

**Proposition 3.23.** There exists a move \( F_1^{S''} \) and strategy \( \phi'' \) for Fixer to continue \( S'' \) after the first round such that \( F_1^{S''} \) contains an edge \( e \in F_1^{S} \) and for every \( U'' \in \phi'' \) there exists \( V'' \in \phi' \) such that \( U'' \) is Fixer-superior to \( V'' \).

**Proof.** Since every series is Fixer-superior to itself, if \( F_1^{S} \cap F_1^{S'} \neq \emptyset \), then we could set \( F_1^{S''} = F_1^{S'} \) and \( \phi'' = \phi' \). Hence we may assume \( F_1^{S} \cap F_1^{S'} = \emptyset \).

Since \( c \geq 3, F_1^{S} \) has multiple edges. Let \( e \) be the cheapest edge of \( F_1^{S} \), and let \( e' \) be in the path through \( F_1^{S'} \) between the endpoints of \( e \); see Figure 66. Recalling that \( F_1^{S} \) is a minimum spanning tree of the multigraph \( M \) whose vertices are the components of \( G_1^{S} - B_1^{S} \) and whose edges are the edges of \( R_1^{S} \) (identifying each endpoint of the edges in \( R_1^{S} \) with the component of \( G_1^{S} - B_1^{S} \) within which it lies), note that no non-loop edge in \( M \) (i.e. edge in \( R_1^{S} \) joining two components of \( G_1^{S} - B_1^{S} \)) can be cheaper than \( e \), or else by Proposition 2.1 it would have been added to \( F_1^{S} \) by Prim’s algorithm as the edge immediately after the first of its endpoints joined the tree.

Consider the series \( U \), initialized by the following constructions of \( G_1^{U} \) and \( R_1^{U} \). Construct \( G_1^{U} \) by adding \( F_1^{S'} - \{e'\} \) to \( G_1^{S'} \) and then deleting from that graph all the edges in \( B_1^{S'} \) except for an edge \( g \) joining the two components of \( G_2^{S'} - e' \); see Figure 65. Note that \( g \) must exist, or else \( F_1^{S''} \) would not be a tree. Construct \( R_1^{U} \) by deleting \( F_1^{S'} - \{e'\} \) from \( R_1^{S} \).

Note that \( G_1^{U} \) is connected. Indeed, \( G_2^{U} \) is connected, and \( G_1^{U} \) is \( G_2^{S} \) with \( e' \) replaced by \( g \), where \( e' \) and \( g \) connect the same two components of \( G_2^{S} - e' \).

Furthermore, \( |G_1^{U} \cup R_1^{U}| < |G_1^{S} \cup R_1^{S}| \). The graph \( G_1^{U} \) and reserve edge set \( R_1^{U} \) are obtained from the graph \( G_1^{S} \) and reserve edge set \( R_1^{S} \) by transferring the edges \( F_1^{S'} - \{e'\} \) from \( R_1^{S} \) to \( G_1^{U} \), and then deleting a positive number of edges from the graph, since \( |B_1^{S''}| > 1 \) or else \( c < 3 \).

Hence by the inductive hypothesis, against any set \( B_1^{U} \) of edges removed by Buster, any greedy choice of \( F_1^{U} \) by Fixer is optimal. Set \( B_1^{U} = \{g\} \).

We claim that \( F_1^{U} = \{e\} \) is optimal. First, see that \( e \in R_1^{U} \):

\[ e \in F_1^{S} - F_1^{S'} \]

\[ \subseteq R_1^{S} - F_1^{S'} \]

\[ = R_1^{S} - F_1^{S'} \]

\[ \subseteq R_1^{S} - (F_1^{S'} - \{e'\}) \]

\[ = R_1^{U} \]

Next, see that adding \( e \) would connect \( G_1^{U} - B_1^{U} \), since \( G_1^{U} - B_1^{U} = G_2^{S'} - \{e'\} \), and \( G_2^{S'} \) is connected, with \( e \) and \( e' \) both joining the two components of \( G_2^{S} - \{e'\} \). Finally, see that no edge \( h \) in \( R_1^{U} \) that would connect \( G_1^{U} - B_1^{U} \) can be cheaper than \( e \), since
Replacing the first round of each series in $\phi$ and $\psi$ by replacing the first round of each series in $\phi$ by $G_w$ and $\phi^*$ respectively then $h$ must join two components of $G^e_w - \{e\}$ and $e' \in F^e_w$ then $h$ must join two components of $G^e_w - F^e_w = G^e_w - B^e_w = G^e_w - B^e_w$, so $h$ being cheaper than $e$ would contradict $e$ being the cheapest edge in $M$. Hence $F^e_w = \{e\}$ is optimal, by the inductive hypothesis.

Let $\phi_U$ be a strategy of greedy moves for Fixer to continue $\bigcup$ after the first round; by the inductive hypothesis, these greedy moves are optimal. Let $S''$ be a series such that $G^e_w = G^e_w$, $R^e_w = R^e_w$, $B^e_w = B^e_w$, and $F^e_w = (G^e_w - \{e\}) \cup \{e\}$. Note that for $F = F^e_w - \{e\}$ we have $F \subseteq F^e_w$, $G^e_w = (G^e_w \cup F) - (B^e_w - \{g\}) - \{g\} = (G^e_w \cup F)$ (since $F \subseteq F^e_w$ and $F^e_w \cap B^e_w = \emptyset$ imply $F \cap B^e_w = \emptyset$), $R^e_w = R^e_w - F$, and $F^e_w = F \cup F^e_w$. Let $\phi''$ be the strategy for Fixer to continue $S''$ after the first round constructed by replacing the first round of each series in $\phi_U$ with the first round of $S''$. By Lemma 3.22 for every $T'' \in \phi''$ there exists $T' \in \phi'$ such that $T''$ is Fixer-superior to $T'$.

Figure 5: Two graphs from the proof of Proposition 3.23. The blobs are the components of $G^w - B^w$.

**Proposition 3.24.** If $F^e_w$ contains an edge $e \in F^e_w$ and $\phi''$ is a strategy for Fixer to continue $S''$ after the first round, then for every $T \in \phi$ there exists $T'' \in \phi''$ such that $T$ is Fixer-superior to $T''$.

Proof. Pick $h \in B^e_w$ that joins the two components of $G^e_w - \{e\}$, and consider the series $U$ satisfying $G^U_w = (G^e_w - \{h\}) \cup \{e\}$, $R^U_w = R^e_w - \{e\}$, and $B^U_w = B^e_w - \{h\}$. Then $G^U_w - B^U_w = (G^e_w - B^w) \cup \{e\}$ and $R^U_w = R^e_w - \{e\}$. Since $G^U_w \cup R^U_w = (G^e_w \cup R^e_w) - \{h\}$ where $h \in B^e_w \subseteq G^w$, by the inductive hypothesis any greedy play by Fixer is optimal, including $F^U_w = F^e_w - \{e\}$.

Let $\phi_U$ be the strategy for Fixer to continue $\bigcup$ after the first round constructed by replacing the first round of each series in $\phi$ with the first round of $\bigcup$. Since all Fixer moves in $\phi$ are greedy, all Fixer moves in $\phi_U$ are also greedy, so all Fixer moves in $\phi_U$ are optimal by the inductive hypothesis. Furthermore, $\phi$ is the strategy for Fixer to continue $\bigcup$ after the first round constructed by replacing the first round of each series in $\phi_U$ with the first round of $\bigcup$. Since $\bigcup$ is a series such that $G^w = G^e_w$, $R^w = R^e_w$, $B^w = B^e_w$, and $F^w = \{e\} \cup F^e_w$, where $\{e\} \subseteq F^e_w$ satisfies $G^U_w - B^U_w = (G^e_w - B^e_w) \cup \{e\}$ and $R^U_w = R^e_w - \{e\}$, by Lemma 3.22 for every $T \in \phi$ there exists $T'' \in \phi''$ such that $T$ is Fixer-superior to $T''$.

Combining the previous two propositions with the transitivity of Fixer-superiority yields the following conclusion to this subsection.

**Corollary 3.25.** For every $T \in \phi$, there exists $T' \in \phi'$ such that $T$ is Fixer-superior to $T'$.

Proof. By Proposition 3.23 there exists a move $F^e_w$ and strategy $\phi''$ for Fixer to continue $S''$ after the first round such that $F^e_w$ contains an edge $e \in F^e_w$ and for every $T'' \in \phi''$ there exists $T' \in \phi'$ such that $T''$ is Fixer-superior to $T'$. By Proposition 3.24 for every $T \in \phi$ there exists $T'' \in \phi''$ such that $T$ is Fixer-superior to $T''$. Hence for every $T \in \phi$, there exists $T'' \in \phi''$ and $T' \in \phi'$ such that $T$ is Fixer-superior to $T''$, and $T''$ is Fixer-superior to $T'$. By Proposition 2.22 $T$ is Fixer-superior to $T'$.

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