THE AUTOMORPHISM GROUP OF A SELF-DUAL [72, 36, 16]
CODE DOES NOT CONTAIN $S_3$, $A_4$ OR $D_8$

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Abstract. A computer calculation with Magma shows that there is no extremal self-dual binary code $C$ of length 72 whose automorphism group contains the symmetric group of degree 3, the alternating group of degree 4 or the dihedral group of order 8. Combining this with the known results in the literature one obtains that $\text{Aut}(C)$ has order at most 5 or is isomorphic to the elementary abelian group of order 8.

1. Introduction

Let $C = C^\perp \subseteq \mathbb{F}_2^n$ be a binary self-dual code of length $n$. Then the weight $\text{wt}(c) := |\{i \mid c_i = 1\}|$ of every $c \in C$ is even. When in particular $\text{wt}(C) := \{\text{wt}(c) \mid c \in C\} \subseteq 4\mathbb{Z}$, the code is called doubly-even. Using invariant theory, one may show [10] that the minimum weight $d(C) := \min(\text{wt}(C \setminus \{0\}))$ of a doubly-even self-dual code is at most $4 + 4 \left\lfloor \frac{n}{24} \right\rfloor$. Self-dual codes achieving this bound are called extremal. Extremal self-dual codes of length a multiple of 24 are particularly interesting for various reasons: for example they are always doubly-even [12] and all their codewords of a given nontrivial weight support 5-designs [2]. There are unique extremal self-dual codes of length 24 (the extended binary Golay code $G_{24}$) and 48 (the extended quadratic residue code $QR_{48}$) and both have a fairly big automorphism group (namely $\text{Aut}(G_{24}) \cong M_{24}$ and $\text{Aut}(QR_{48}) \cong PSL_2(47)$). The existence of an extremal code of length 72 is a long-standing open problem [13]. A series of papers investigates the automorphism group of a putative extremal self-dual code of length 72 excluding most of the subgroups of $S_{72}$. The most recent result is contained in [3] where the first author excluded the existence of automorphisms of
order 6.
In this paper we prove that neither $S_3$ nor $A_4$ nor $D_8$ is contained in the automorphism group of such a code.

The method to exclude $S_3$ (which is isomorphic to the dihedral group of order 6) is similar to that used for the dihedral group of order 10 in [8] and based on the classification of additive trace-Hermitian self-dual codes in $F_4^2$ obtained in [7].

For the alternating group $A_4$ of degree 4 and the dihedral group $D_8$ of order 8, we use their structure as a semidirect product of an elementary abelian group of order 4 and a group of order 3 and 2 respectively. By [11] we know that the fixed code of any element of order 2 is isomorphic to a self-dual binary code $D$ of length 36 with minimum distance 8. These codes have been classified in [1]; up to equivalence there are 41 such codes $D$. For all possible lifts $\hat{D} \leq F_2^2$ that respect the given actions we compute the codes $\mathcal{E} := \hat{D}^{A_4}$ and $\mathcal{E} := \hat{D}^{D_8}$ respectively. We have respectively only three and four such codes $\mathcal{E}$ with minimum distance $\geq 16$. Running through all doubly-even $A_4$-invariant self-dual overcodes of $\mathcal{E}$ we see that no such code is extremal.

Since the group $D_8$ contains a cyclic group of order 4, say $C_4$, we use the fact [11] that $C$ is a free $F_2C_4$-module. Checking all doubly-even self-dual overcodes of $\mathcal{E}$ which are free $F_2C_4$-modules we see that, also in this case, none is extremal.

The present state of research is summarized in the following theorem.

**Theorem 1.** The automorphism group of a self-dual [72, 36, 16] code is either cyclic of order 1, 2, 3, 4, 5 or elementary abelian of order 4 or 8.

All results are obtained using extensive computations in Magma [4].

2. The Symmetric Group of Degree 3.

2.1. Preliminaries. Let $C$ be a binary self-dual code and let $g$ be an automorphism of $C$ of odd prime order $p$. Define $C(g) := \{ c \in C \mid c^g = c \}$ and $E(g)$ the set of all the codewords that have even weight on the cycles of $g$. From a module theoretical point of view, $C$ is a $F_2(g)$-module and $C(g) = C \cdot (1 + g + \ldots + g^{p-1})$ and $E(g) = C \cdot (g + \ldots + g^{p-1})$.

In [7] Huffman notes (it is a special case of Maschke’s theorem) that

$$C = C(g) \oplus E(g).$$

In particular it is easy to prove that the dimension of $E(g)$ is $\frac{(p-1)c}{2}$ where $c$ is the number of cycles of $g$. In the usual manner we can identify vectors of length $p$ with polynomials in $Q := F_2[x]/(x^p - 1)$; that is $(v_1, v_2, \ldots, v_p)$ corresponds to $v_1 + v_2x + \ldots + v_px^{p-1}$. The weight of a polynomial is the number of nonzero coefficients. Let $\mathcal{P} \subset Q$ be the set of all even weight polynomials. If $1 + x + \ldots + x^{p-1}$ is irreducible in $F_2[x]$ then $\mathcal{P}$ is a field with identity $x + x^2 + \ldots + x^{p-1}$ [9]. There is a natural map that we will describe only in our particular case in the next section, from $E(g)$ to $\mathcal{P}^e$. Let us observe here only the fact that, if $p = 3$, then $1 + x + x^2$ is irreducible in $F_2[x]$ and $\mathcal{P}$ is isomorphic to $F_4$, the field with four elements. The identification is the following:

\[
\begin{array}{cccc}
0 & 000 & \omega & 110 \\
1 & 011 & \varpi & 101 \\
\end{array}
\]

2.2. The Computations for $S_3$. Let $C$ be an extremal self-dual code of length 72 and suppose that $G \leq \text{Aut}(C)$ with $G \cong S_3$. Let $\sigma$ denote an element of order 2

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and \(g\) an element of order 3 in \(G\). By [6] and [5], \(\sigma\) and \(g\) have no fixed points. So, in particular, \(\sigma\) has 36 2-cycles and \(g\) has 24 3-cycles. Let us suppose, w.l.o.g. that
\[
\sigma = (1, 4)(2, 6)(3, 5)\ldots(67, 70)(68, 72)(69, 71)
\]
and
\[
g = (1, 2, 3)(4, 5, 6)\ldots(67, 68, 69)(70, 71, 72).
\]
As we have seen in Section 2.1
\[
C = C(\sigma) \oplus C(\sigma)
\]
where \(C(\sigma)\) is the subcode of \(C\) of all the codewords with an even weight on the cycles of \(g\), of dimension 24. We can consider a map
\[
f : \mathcal{E}(\sigma) \to \mathbb{F}_4^{24}
\]
extending the identification \(\mathcal{P} \cong \mathbb{F}_4\), stated in Section 2.1, to each cycle of \(g\).
Again by [9], \(\mathcal{E}(\sigma)' := f(\mathcal{E}(\sigma))\) is an Hermitian self-dual code over \(\mathbb{F}_4\) (that is \(\mathcal{E}(\sigma)' = \{\epsilon \in \mathbb{F}_4^{24} \mid \sum_{i=0}^{24} \epsilon_i \overline{\gamma_i} = 0 \text{ for all } \gamma \in \mathcal{E}(\sigma)'\}\), where \(\overline{\alpha} = \alpha^2\) is the conjugate of \(\alpha\) in \(\mathbb{F}_4\). Clearly the minimum distance of \(\mathcal{E}(\sigma)\)' is \(\geq 8\). So \(\mathcal{E}(\sigma)'\) is a \([24, 12, \geq 8]_4\) Hermitian self-dual code.
The action of \(\sigma\) on \(C \leq \mathbb{F}_2^{12}\) induces an action on \(\mathcal{E}(\sigma)' \leq \mathbb{F}_2^{24}\), namely
\[
(\epsilon_1, \epsilon_2, \ldots, \epsilon_{23}, \epsilon_{24})^\sigma = (\overline{\epsilon_2}, \overline{\epsilon_1}, \ldots, \overline{\epsilon_{24}}, \overline{\epsilon_{23}})
\]
Note that this action is only \(\mathbb{F}_2\)-linear. In particular, the subcode fixed by \(\sigma\), say \(\mathcal{E}(\sigma)'(\sigma)\), is
\[
\mathcal{E}(\sigma)'(\sigma) = \{(\epsilon_1, \overline{\epsilon_1}, \ldots, \epsilon_{12}, \overline{\epsilon_{12}}) \in \mathcal{E}(\sigma)'\}
\]
**Proposition 1.** (cf. [5] Cor. 5.6) The code
\[
\mathcal{X} := \pi(\mathcal{E}(\sigma)'(\sigma)) := \{(\epsilon_1, \ldots, \epsilon_{12}) \in \mathbb{F}_4^{12} \mid (\epsilon_1, \overline{\epsilon_1}, \ldots, \epsilon_{12}, \overline{\epsilon_{12}}) \in \mathcal{E}(\sigma)\}
\]
is an additive trace-Hermitian self-dual \((12, 2, 12, \geq 4)\) code such that
\[
\mathcal{E}(\sigma)' := \phi(\mathcal{X}) := \{(\epsilon_1, \overline{\epsilon_1}, \ldots, \epsilon_{12}, \overline{\epsilon_{12}}) \mid (\epsilon_1, \ldots, \epsilon_{12}) \in \mathcal{X}\} \leq \mathbb{F}_4^{12}.
\]
**Proof.** For \(\gamma, \epsilon \in \mathcal{X}\) the inner product of their preimages in \(\mathcal{E}(\sigma)'(\sigma)\) is
\[
\sum_{i=1}^{12} (\epsilon_i \overline{\gamma_i} + \epsilon_i \gamma_i)
\]
which is 0 since \(\mathcal{E}(\sigma)'(\sigma)\) is self-orthogonal. Therefore \(\mathcal{X}\) is trace-Hermitian self-orthogonal. Thus
\[
\dim_{\mathbb{F}_2}(\mathcal{X}) = \frac{1}{2} \dim_{\mathbb{F}_2}(\mathcal{E}(\sigma)'(\sigma)) = \frac{1}{2} \dim_{\mathbb{F}_2}(\mathcal{E}(\sigma)')
\]
since \(\mathcal{E}(\sigma)'\) is a projective \(\mathbb{F}_2\)-module, and so \(\mathcal{X}\) is self-dual. Since \(\dim_{\mathbb{F}_2}(\mathcal{X}) = 12 = \dim_{\mathbb{F}_4}(\mathcal{E}(\sigma)'(\sigma))\), the \(\mathbb{F}_4\)-linear code \(\mathcal{E}(\sigma)' \leq \mathbb{F}_4^{24}\) is obtained from \(\mathcal{X}\) as stated.

All additive trace-Hermitian self-dual codes in \(\mathbb{F}_4^{12}\) are classified in [7]. There are 195,520 such codes that have minimum distance \(\geq 4\) up to monomial equivalence.

**Remark 1.** If \(\mathcal{X}\) and \(\mathcal{Y}\) are monomial equivalent, via a \(12 \times 12\) monomial matrix \(M := (m_{i,j})\), then \(\phi(\mathcal{X})\) and \(\phi(\mathcal{Y})\) are monomial equivalent too, via the \(24 \times 24\) monomial matrix \(M' := (m'_{i,j})\), where \(m'_{2i-1,2j-1} = m_{i,j}\) and \(m'_{2i,2j} = m_{i,j}\), for all \(i, j \in \{1, \ldots, 12\}\).
An exhaustive search with MAGMA (of about 7 minutes CPU on an Intel(R) Xeon(R) CPU X5460 @ 3.16GHz) shows that the minimum distance of $\phi(X)$ is $\leq 6$, for each of the 195, 520 additive trace-Hermitian self-dual $(12, 2^{12}, 4)$ codes. But $E(g)$ should have minimum distance $\geq 8$, a contradiction. So we proved the following.

**Theorem 2.** The automorphism group of a self-dual $[72, 36, 16]$ code does not contain a subgroup isomorphic to $S_3$.

3. The alternating group of degree 4 and the dihedral group of order 8.

3.1. The action of the Klein four group. For the alternating group $A_4$ of degree 4 and the dihedral group $D_8$ of order 8 we use their structure

$$A_4 \cong V_4 : C_3 \cong (C_2 \times C_2) : C_3 = \langle g, h : \langle \sigma \rangle \rangle$$

$$D_8 \cong V_4 : C_2 \cong (C_2 \times C_2) : C_2 = \langle g, h : \langle \sigma \rangle \rangle$$

as a semidirect product.

Let $\mathcal{C}$ be some extremal $[72, 36, 16]$ code such that $\mathcal{H} \leq \text{Aut}(\mathcal{C})$ where $\mathcal{H} \cong A_4$ or $\mathcal{H} \cong D_8$. Then by [6] and [5] all non-trivial elements in $\mathcal{H}$ act without fixed points and we may replace $\mathcal{C}$ by some equivalent code so that

$$g = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \ldots (71, 72)$$

$$h = (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12) \ldots (70, 72)$$

$$\sigma = (1, 5, 9)(2, 7, 12)(3, 8, 10)(4, 6, 11) \ldots (64, 66, 71) \quad (\text{for } A_4)$$

$$\sigma = (1, 5)(2, 8)(3, 7)(4, 6) \ldots (68, 70) \quad (\text{for } D_8)$$

Let

$$\mathcal{G} := C_{S_2}(\mathcal{H}) := \{ t \in S_2 \mid tg = gt, th = ht, t\sigma = \sigma t \}$$

denote the centralizer of this subgroup $\mathcal{H}$ in $S_2$. Then $\mathcal{G}$ acts on the set of extremal $\mathcal{H}$-invariant self-dual codes and we aim to find a system of orbit representatives for this action.

**Definition 1.** Let

$$\pi_1 : \{ v \in F_2^36 \mid v^g = v \} \to F_2^36$$

$$(v_1, v_1, v_2, v_2, \ldots, v_{36}, v_{36}) \mapsto (v_1, v_2, \ldots, v_{36})$$

denote the bijection between the fixed space of $g$ and $F_2^{36}$ and

$$\pi_2 : \{ v \in F_2^36 \mid v^g = v \text{ and } v^h = v \} \to F_2^{18}$$

$$(v_1, v_1, v_1, v_2, v_2, \ldots, v_{18}) \mapsto (v_1, v_2, \ldots, v_{18})$$

denote the bijection between the fixed space of $\langle g, h \rangle \leq A_4$ and $F_2^{18}$. Then $h$ acts on the image of $F_2^{18}$ as

$$(1, 2)(3, 4) \ldots (35, 36).$$

Let

$$\pi_3 : \{ v \in F_2^{36} \mid v^{\pi_1(h)} = v \} \to F_2^{18},$$

$$(v_1, v_1, v_2, v_2, \ldots, v_{18}, v_{18}) \mapsto (v_1, v_2, \ldots, v_{18}),$$

so that $\pi_2 = \pi_3 \circ \pi_1$.

**Remark 2.** The centraliser $C_{S_2}(g) \cong C_2 \wr S_{36}$ of $g$ acts on the set of fixed points of $g$. Using the isomorphism $\pi_1$ we obtain a group epimorphism which we again denote by $\pi_1$

$$\pi_1 : C_{S_2}(g) \to S_{36}$$
Remark 4. For \( H \sim A_1 \) we obtain the epimorphism
\[
\pi_3 : C_{S_{36}}(\pi_1(h)) \rightarrow S_{18}.
\]
The normalizer \( N_{S_{36}}((g, h)) \) acts on the set of \( (g, h) \)-orbits which defines a homomorphism
\[
\pi_2 : N_{S_{36}}((g, h)) \rightarrow S_{18}.
\]

Let us consider the fixed code \( C(g) \) which is isomorphic to
\[
\pi_1(C(g)) = \{(c_1, c_2, \ldots, c_{36}) \mid (c_1, c_1, c_2, c_2, \ldots c_{36}, c_{36}) \in C\}.
\]
By \([1]^{\text{II}}\), the code \( \pi_1(C(g)) \) is some self-dual code of length 36 and minimum distance 8. These codes have been classified in \([1]^{\text{II}}\) up to equivalence (under the action of the full symmetric group \( S_{36} \)) there are 41 such codes. Let
\[
Y_1, \ldots, Y_{41}
\]
be a system of representatives of these extremal self-dual codes of length 36.

Remark 3. \( C(g) \in D \) where
\[
D := \left\{ D \leq \mathbb{F}_{36}^2 \mid D = D^\perp, d(D) = 8, \pi_1(h) \in \text{Aut}(D) \right. \text{ and } \left. \pi_2(\sigma) \in \text{Aut}(\pi_3(D(\pi_1(h)))) \right\}.
\]
For \( 1 \leq k \leq 41 \) let \( D_k := \{ D \in D \mid D \cong Y_k \} \).

Let \( G_{36} := \{ \tau \in C_{S_{36}}(\pi_1(h)) \mid \pi_3(\tau))\pi_2(\sigma) = \pi_2(\sigma)\pi_3(\tau)) \} \).

Remark 4. For \( H \cong A_4 \) the group \( G_{36} \) is isomorphic to \( C_2 \wr C_3 \wr S_6 \). It contains \( \pi_1(G) \cong A_4 \wr S_6 \) of index 64.

For \( H \cong D_8 \) we get \( G_{36} = \pi_1(G) \cong C_2 \wr C_2 \wr S_6 \).

Lemma 1. A set of representatives of the \( G_{36} \) orbits on \( D_k \) can be computed by performing the following computations:

- Let \( h_1, \ldots, h_s \) represent the conjugacy classes of fixed point free elements of order 2 in Aut(\( Y_k \)).
- Compute elements \( \tau_1, \ldots, \tau_s \in S_{36} \) such that \( \tau_i^{-1}h_i\tau_i = \pi_1(h) \) and put \( D_i := Y_k^{\tau_i} \) so that \( \pi_1(h) \in \text{Aut}(D_i) \).
- For all \( D_i \) let \( \sigma_1, \ldots, \sigma_{t_i} \) a set of representatives of the action by conjugation by the subgroup \( \pi_3(C_{\text{Aut}(D)}(\pi_1(h))) \) on fixed point free elements of order 3 (for \( H \cong A_4 \)) respectively 2 (for \( H \cong D_8 \)) in Aut(\( \pi_3(D_i(\pi_1(h))) \)).
- Compute elements \( \rho_1, \ldots, \rho_{t_i} \in S_{36} \) such that \( \rho_j^{-1}\sigma_j\rho_j = \pi_3(\sigma) \), lift \( \rho_j \) naturally to a permutation \( \tilde{\rho}_j \in S_{36} \) commuting with \( \pi_1(h) \) (defined by \( \tilde{\rho}_j(2a - 1) = 2\rho_j(a) - 1, \tilde{\rho}_j(2a) = 2\rho_j(a) \)) and put
\[
D_{i,j} := (D_i)^{\tilde{\rho}_j} = Y_k^{\tau_i\tilde{\rho}_j}
\]
so that \( \pi_3(\sigma) \in \text{Aut}(\pi_2(D_{i,j}(\pi_1(h)))) \).

Then \( \{ D_{i,j} \mid 1 \leq i \leq s, 1 \leq j \leq t_i \} \) represent the \( G_{36} \)-orbits on \( D_k \).

Proof. Clearly these codes lie in \( D_k \).
Now assume that there is some \( \tau \in G_{36} \) such that
\[
Y_k^{\tau_i\tilde{\rho}_j} = D_{i,j} = Y_k^{\tau_i\tilde{\rho}_j}.
\]
Then
\[
\epsilon := \pi_i\tilde{\rho}_j^{-1}\tau_i^{-1} \in \text{Aut}(Y_k)
\]
satisfies \( e \theta_i e^{-1} = h \varphi, \) so \( h \) and \( h \varphi \) are conjugate in \( \text{Aut}(Y_k) \), which implies \( i = i' \) (and so \( \pi_i = \pi_{i'} \)). Now, 
\[
Y_k^{\pi_i \hat{\rho}_i \tau} = D_i^{\hat{\rho}_i \tau} = D_i^{\hat{\rho}_j} = Y_k^{\pi_i \hat{\rho}_j}.
\]
Then
\[
\epsilon' := \hat{\rho}_j \tau \hat{\rho}_j^{-1} \in \text{Aut}(D_i)
\]
commutes with \( \pi_1(h) \). We compute that \( \pi_3(\epsilon')\sigma_j \pi_3(\epsilon'^{-1}) = \sigma_j' \) and hence \( j = j' \).

Now let \( D \in D_k \) and choose some \( \xi \in S_{36} \) such that \( D^\xi = Y_k \). Then \( \pi_1(h)^\xi \) is conjugate to some of the chosen representatives \( h_i \in \text{Aut}(Y_k) \) \( (i = 1, \ldots, s) \) and we may multiply \( \xi \) by some automorphism of \( Y_k \) so that \( \pi_1(h)^\xi = h_i = \pi_1(h)^{\tau_i} \). So \( \xi \tau_i \in C_{S_{36}}(\pi_1(h)) \) and \( D^\xi \tau_i = Y_k^{\tau_i} = D_i \). Since \( \pi_3(\sigma) \in \text{Aut}(\pi_3(D(\pi_1(h)))) \) we get 
\[
\pi_3(\sigma)^{\pi_3(\xi \tau_i)} \in \text{Aut}(\pi_3(D_i(\pi_1(h))))
\]
and so there is some automorphism \( \alpha \in \pi_3(C_{\text{Aut}(D_i)}(\pi_1(h))) \) and some \( j \in \{1, \ldots, t_i\} \) such that \( (\pi_3(\sigma)^{\pi_3(\xi \tau_i)})^\alpha = \sigma_j \). Then 
\[
D^\xi \tau_i \alpha \hat{\rho}_j = D_{i,j}
\]
where \( \xi \tau_i \alpha \hat{\rho}_j \in G_{36} \).

3.2. The Computations for \( A_4 \). We now deal with the case \( H \cong A_4 \).

**Remark 5.** With MAGMA we use the algorithm given in Lemma 1 to compute that there are exactly 25, 299 \( G_{36} \)-orbits on \( D \), represented by, say, \( X_1, \ldots, X_{25,299} \).

As \( G \) is the centraliser of \( A_4 \) in \( S_{72} \) the image \( \pi_1(G) \) commutes with \( \pi_1(h) \) and \( \pi_2(G) \) centralizes \( \pi_2(\sigma) \). In particular the group \( G_{36} \) contains \( \pi_1(G) \) as a subgroup. With MAGMA we compute that \([G_{36} : \pi_1(G)] = 64\). Let \( g_1, \ldots, g_{64} \in G_{36} \) be a left transversal of \( \pi_1(G) \) in \( G_{36} \).

**Remark 6.** The set \( \{X_i^{g_j} \mid 1 \leq i \leq 25, 299, 1 \leq j \leq 64\} \) contains a set of representatives of \( \pi_1(G) \)-orbits on \( D \).

**Remark 7.** For all \( 1 \leq i \leq 25, 299, 1 \leq j \leq 64 \) we compute the code 
\[
E := E(X_i^{g_j}, \sigma) := \tilde{D} + \tilde{D}^\sigma + \tilde{D}^{\sigma^2}, \text{ where } \tilde{D} = \pi_1^{-1}(X_i^{g_j})
\]
For three \( X_i \) there are two codes \( \tilde{D}_{i,1} = \pi_1^{-1}(X_i^{g_{j_1}}) \) and \( \tilde{D}_{i,2} = \pi_1^{-1}(X_i^{g_{j_2}}) \) such that \( E(X_i^{g_{j_1}}, \sigma) \) and \( E(X_i^{g_{j_2}}, \sigma) \) are doubly even and of minimum distance 16. In all three cases, the two codes are equivalent. Let us call the inequivalent codes \( E_1, E_2 \) and \( E_3 \), respectively. They have dimension 26, 26, and 25, respectively, minimum distance 16 and their automorphism groups are 
\[
\text{Aut}(E_1) \cong S_4, \text{ Aut}(E_2) \text{ of order } 432, \text{ Aut}(E_3) \cong (A_4 \times A_5) : 2
\]
All three groups contain a unique conjugacy class of subgroups conjugate in \( S_{72} \) to \( A_4 \) (which is normal for \( E_1 \) and \( E_3 \)).

These computations took about 26 hours CPU, using an Intel(R) Xeon(R) CPU X5460 @ 3.16GHz.

**Corollary 1.** The code \( C(g) + C(h) + C(gh) \) is equivalent under the action of \( G \) to one of the three codes \( E_1, E_2 \) or \( E_3 \).
Let \( \mathcal{E} \) be one of these three codes. The group \( \mathcal{A}_4 \) acts on \( \mathcal{V} := \mathcal{E}^\perp/\mathcal{E} \) with kernel \( \langle g, h \rangle \). The space \( \mathcal{V} \) is hence an \( \mathbb{F}_2(\sigma) \)-module supporting a \( \sigma \)-invariant form such that \( \mathcal{C} \) is a self-dual submodule of \( \mathcal{V} \). As in Section 2.1 we obtain a canonical decomposition
\[
\mathcal{V} = \mathcal{V}(\sigma) \perp \mathcal{W}
\]
where \( \mathcal{V}(\sigma) \) is the fixed space of \( \sigma \) and \( \sigma \) acts as a primitive third root of unity on \( \mathcal{W} \).

For \( \mathcal{E} = \mathcal{E}_1 \) or \( \mathcal{E} = \mathcal{E}_2 \) we compute that \( \mathcal{V}(\sigma) \cong \mathbb{F}_2^4 \) and \( \mathcal{W} \cong \mathbb{F}_4^8 \). For both codes the full preimage of every self-dual submodule of \( \mathcal{V}(\sigma) \) is a code of minimum distance < 16.

For \( \mathcal{E} = \mathcal{E}_3 \) the dimension of \( \mathcal{V}(\sigma) \) is 2 and there is a unique self-dual submodule of \( \mathcal{V}(\sigma) \) so that the full preimage \( \mathcal{E}_3^\perp \) is doubly-even and of minimum distance \( \geq 16 \). The element \( \sigma \) acts on \( \mathcal{E}_3^\perp/\mathcal{E}_3 \cong \mathcal{W} \) with irreducible minimal polynomial, so \( \mathcal{E}_3^\perp/\mathcal{E}_3 \cong \mathbb{F}_4^{10} \). The code \( \mathcal{C} \) is a preimage of one of the 58, 963, 707 maximal isotropic \( \mathbb{F}_4 \)-subspaces of the Hermitian \( \mathbb{F}_4 \)-space \( \mathcal{E}_3^\perp/\mathcal{E}_3 \).

The unitary group \( GU(10, 2) \) of \( \mathcal{E}_3^\perp/\mathcal{E}_3 \cong \mathbb{F}_4^{10} \) acts transitively on the maximal isotropic subspaces. So a quite convenient way to enumerate all these spaces is to compute an isometry of \( \mathcal{E}_3 \) with the standard model used in MAGMA and then compute the \( GU(10, 2) \)-orbit of one maximal isotropic space (e.g. the one spanned by the first 5 basis vectors in the standard model). The problem here is that the orbit becomes too long to be stored in the available memory (4GB). So we first compute all 142, 855 one dimensional isotropic subspaces \( \mathcal{E}_3/\mathcal{E}_3 \cong \mathbb{F}_4 \) for which the code \( \mathcal{E}_3 \) has minimum distance \( \geq 16 \). The automorphism group \( \text{Aut}(\mathcal{E}_3) = \text{Aut}(\mathcal{E}_3) \) acts on these codes with 1, 264 orbits. For all these 1, 264 orbit representatives \( \mathcal{E}_3 \) we compute the 114, 939 maximal isotropic subspaces of \( \mathcal{E}_3 \) (as the orbits of one given subspace under the unitary group \( GU(8, 2) \) in MAGMA) and check whether the corresponding doubly-even self-dual code has minimum distance 16. No such code is found.

Note that the latter computation can be parallelised easily as all 1, 264 computations are independent of each other. We split it into 10 jobs. To deal with 120 representatives \( \mathcal{E}_3 \) took between 5 and 10 hours on a Core i7 870 (2.93GHz) personal computer.

This computation shows the following.

**Theorem 3.** The automorphism group of a self-dual [72, 36, 16] code does not contain a subgroup isomorphic to \( \mathcal{A}_4 \).

### 3.3. The Computations for \( \mathcal{D}_8 \)

For this section we assume that \( \mathcal{H} \cong \mathcal{D}_8 \). Then \( \pi_1(\mathcal{G}) = \mathcal{G}_{36} \) and we may use Lemma 11 to compute a system of representatives of the \( \pi_1(\mathcal{G}) \)-orbits on the set \( \mathcal{D} \).

**Remark 8.** \( \pi_1(\mathcal{G}) \) acts on \( \mathcal{D} \) with exactly 9, 590 orbits represented by, say, \( X_1, \ldots, X_{9,590} \). For all \( 1 \leq i \leq 9,590 \) we compute the code
\[
\mathcal{E} := \mathcal{E}(X_i, \sigma) := \tilde{\mathcal{D}} + \tilde{\mathcal{D}}^\sigma, \text{ where } \tilde{\mathcal{D}} = \pi_1^{-1}(X_i).
\]

For four \( X_i \) the code \( \mathcal{E}(X_i, \sigma) \) is doubly even and of minimum distance 16. Let us call the inequivalent codes \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) and \( \mathcal{E}_4 \), respectively. All have dimension 26 and minimum distance 16.

**Corollary 2.** The code \( \mathcal{C}(g) + \mathcal{C}(h) + \mathcal{C}(gh) \) is equivalent under the action of \( \mathcal{G} \) to one of the four codes \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) or \( \mathcal{E}_4 \).
This computation is very fast (it is due mainly to the fact that $G_{36} = \pi(G)$). It took about 5 minutes CPU on an Intel(R) Xeon(R) CPU X5460 @ 3.16GHz.

As it seems to be quite hard to compute all $D_8$-invariant self-dual overcodes of $\mathcal{E}_i$ for these four codes $\mathcal{E}_i$ we apply a different strategy which is based on the fact that $h = (g\sigma)^2$ is the square of an element of order 4. So let

$$k := g\sigma = (1, 8, 3, 6)(2, 5, 4, 7) \ldots (66, 69, 68, 71) \in D_8.$$

By [11], $C$ is a free $\mathbb{F}_2(k)$-module (of rank 9). Since $\langle k \rangle$ is abelian, the module is both left and right; here we use the right notation. The regular module $\mathbb{F}_2(k)$ has a unique irreducible module, 1-dimensional, called the socle, that is $\langle (1+k+k^2+k^3) \rangle$. So $C$, as a free $\mathbb{F}_2(k)$-module, has socle $C(k) = C \cdot (1+k+k^2+k^3)$. This implies that, for every basis $b_1, \ldots, b_9$ of $C(k)$, there exist $w_1, \ldots, w_9 \in C$ such that $w_i \cdot (1+k+k^2+k^3) = b_i$ and

$$C = w_1 \cdot \mathbb{F}_2(k) + \ldots + w_9 \cdot \mathbb{F}_2(k).$$

To get all the possible overcodes of $\mathcal{E}_i$, we choose a basis of the socle $\mathcal{E}_i(k)$, say $b_1, \ldots, b_9$, and look at the sets

$$W_{i,j} = \{ w + \mathcal{E}_i \in \mathcal{E}_i^\perp / \mathcal{E}_i \mid w \cdot (1+k+k^2+k^3) = b_j \text{ and } d(\mathcal{E}_i + w \cdot \mathbb{F}_2(k)) \geq 16 \}.$$

For every $i$ we have at least one $j$ for which the set $W_{i,j}$ is empty. This computation (of about 4 minutes CPU on the same computer) shows the following.

**Theorem 4.** The automorphism group of a self-dual [72, 36, 16] code does not contain a subgroup isomorphic to $D_8$.

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**References**

[1] C. Aguilar Melchor and P. Gaborit, *On the classification of extremal [36, 18, 8] binary self-dual codes*, IEEE Trans. Inform. Theory 54 (2008) 4743-4750.

[2] E. F. Assmuss and H.F. Mattson, *New 5-designs*, J. Combin. Theory 6 (1969) 122–151.

[3] M. Borello, *The automorphism group of a self-dual [72, 36, 16] binary code does not contain elements of order 6*, IEEE Trans. Inform. Theory 58, No. 12 (2012), 7240–7245.

[4] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system I: The user language*, J. Symbol. Comput. 24 (1997) 235–265.

[5] S. Bouyuklieva, *On the automorphism group of a doubly even [72, 36, 16] code*, IEEE Trans. Inform. Theory 50 (2004) 544–547.

[6] S. Bouyuklieva, *On the automorphisms of order 2 with fixed points for the extremal self-dual codes of length 24m*, Des. Codes Cryptogr. 25 (2002) 5–13.

[7] L.E. Danielsen and M.G. Parker, *On the Classification of All Self-Dual Additive Codes over $GF(4)$ of length up to 12*, Journal of Combinatorial Theory, Series A 112(7), pp. 1351–1367, October 2006.

[8] T. Feulner and G. Nebe, *The automorphism group of an extremal [72, 36, 16] code does not contain $\mathbb{Z}_7$, $\mathbb{Z}_9 \times \mathbb{Z}_9$, or $D_{10}$*, IEEE Trans. Inform. Theory 58 (11) 2012, 6916–6924.

[9] W.C. Huffman, *Automorphisms of codes with application to extremal doubly even codes of length 48*, IEEE Trans. Inform. Theory IT-28 (1982) 511–521.

[10] C.L. Mallows and N.J.A. Sloane, *An upper bound for self-dual codes*, Information and Control 22 (1973) 188–200.

[11] G. Nebe, *An extremal [72, 36, 16] binary code has no automorphism group containing $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Q}_8$, or $D_{10}$, Finite Fields and their applications 18 (2012) 563–566.*
[12] E.M. Rains, *Shadow bounds for self-dual codes*, IEEE Trans. Inform. Theory 44 (1998), 134–139.

[13] N.J.A. Sloane, *Is there a (72; 36) d = 16 self-dual code?*, IEEE Trans. Inform. Theory 2 (1973) 251.

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