A General Framework for Fair Allocation with Matroid Rank Valuations

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Abstract
We study the problem of fairly allocating a set of indivisible goods among agents with matroid rank valuations. We present a simple framework that efficiently computes any fairness objective that satisfies some mild assumptions. Along with maximizing a fairness objective, the framework is guaranteed to run in polynomial time, maximize utilitarian social welfare and ensure strategyproofness. We show how our framework can be used to achieve four different fairness objectives: (a) Prioritized Lorenz dominance, (b) Maxmin fairness, (c) Weighted leximin, and (d) Max weighted Nash welfare. In particular, our framework provides the first polynomial time algorithms to compute weighted leximin and max weighted Nash welfare allocations for matroid rank valuations.

1 Introduction
Consider the problem of assigning class seats to university students. Students have preferences over the classes they want to take; however, there is a limited number of seats available. The assignment must satisfy several additional constraints; for example, students may not take classes with conflicting schedules, and universities often impose limits on the maximal number of classes (or course credits) one can take in a semester. In addition, students often have priorities: seniors should be afforded more leeway than juniors on course selection, so that they can graduate on time; similarly, degree majors have priority in selecting classes from their home department. Course allocation can be naturally modeled as an instance of a fair allocation problem: students are agents who have a utility for receiving bundles of items (course seats). Our objective is to efficiently compute an allocation (an assignment of seats to students) that satisfies certain justice criteria. For example, one might be interested in computing an efficient allocation — in the course allocation setting, this would be an allocation that maximizes the number of course seats assigned to students who are willing and able to take them. Alternatively, one might want to find an envy-free assignment — one where every student prefers their assigned set of classes to that of any other student. Finding good allocations is computationally intractable under general agent utilities [Bouveret et al., 2016]. However, student preferences can be modeled as a well-structured utility function: student preferences over classes are submodular; in economic jargon, they have decreasing returns to scale. In addition, assuming students (or rather, university administrators) are only interested in taking the maximal number of classes relevant to them, their gain from taking an additional class is either 0 or 1. This class of preferences is known as binary submodular valuations [Benabbou et al., 2021].

Recently [Viswanathan and Zick, 2022] introduce the Yankee Swap algorithm for computing fair and efficient allocations. The algorithm starts with all items unassigned, and at every round picks an agent to play. The agent can choose to either pick an unassigned item to add to its bundle, or to steal an item from another agent. If they chose to steal, then they initiate a transfer path,
where each agent steals an item from another, until the final agent takes an unassigned item. This continues until no agents can take unassigned items they want, or no items are left.

If agents have binary submodular valuations, then Yankee Swap is guaranteed to output a Lorenz dominating allocation. As shown by Babaioff et al. [2021a], Lorenz dominating allocations satisfy a broad range of fairness and efficiency criteria: they are leximin fair, which implies that they maximize social welfare, Nash welfare, envy-freeness up to one item, and guarantee every agent at least half of their maximin share. In addition, they are truthful: no agent has an incentive to misreport their preferences over items. While Babaioff et al. [2021a] present a poly-time algorithm for computing Lorenz dominating allocations, Yankee Swap is significantly simpler to implement, and has a faster running time.

Due to its simplicity, the Yankee Swap algorithm can be easily modified to achieve other fairness objectives. In this paper, we answer the question

*What fairness objectives can be achieved by modifying Yankee Swap?*

As it turns out, the answer to this question is: surprisingly many.

### 1.1 Our Contribution

In this work, we show that a minor modification to the Yankee Swap algorithm — the order in which it lets agents play — allows it to compute allocations satisfying a broad range of justice criteria. In particular, this allows us to compute a weighted variant of leximin allocations. In addition, Yankee Swap can compute allocations maximizing the weighted Nash welfare [Chakraborty et al., 2021a]. We also show that Yankee Swap can be used to compute other fairness objectives like Lorenz dominance and maxmin fair share. Furthermore, the allocations that Yankee Swap outputs always maximize social welfare, and are always truthful.

More generally, we identify certain necessary conditions a justice criterion must satisfy in order for Yankee Swap to compute it. In more detail, a justice criterion $\psi$ describes an order over item allocations; for example, under the Nash welfare criterion an allocation $X$ is better than $Y$ if the product of agent utilities under $X$ is greater than under $Y$. In order for Yankee Swap to work, a justice criterion must satisfy Pareto dominance — if $Y$ Pareto dominates $X$ then $\psi(Y) \geq \psi(X)$ — and it must admit a gain function $\phi$. Intuitively, a gain function measures the benefit of letting an agent $i$ choose an item in a given round of Yankee Swap. When these two simple conditions are met, the justice criterion $\psi$ can be maximized by Yankee Swap.

### 1.2 Related Work

We study fair allocation when agents have matroid rank valuations. Benabbou et al. [2021] first show that it is possible to compute a welfare maximizing envy free up to one good (EF1) allocation in polynomial time. Babaioff et al. [2021a] extend this result and show that it is possible to compute Lorenz dominating allocations in polynomial time; Lorenz dominance is a stronger notion of fairness than leximin and it implies a host of other fairness properties. Recently, Viswanathan and Zick [2022] show that Lorenz dominating allocations can be computed using a simple Yankee Swap based mechanism. Barman and Verma [2021] show that it is possible to compute an allocation which guarantees each agent their maximin share. Our results use some technical lemmas from their work.

Fair allocation with asymmetric agents is a well studied problem. Several weighted fairness metrics have been proposed and studied in the literature. Farhadi et al. [2019] introduce and study the weighted maxmin share. Chakraborty et al. [2021a] introduce the notion of weighted envy freeness and propose algorithms to compute allocations which are weighted envy free up to one good. This notion was generalized by Chakraborty et al. [2022]. Aziz et al. [2020] introduce and study the notion of weighted proportionality. Babaioff et al. [2021b] propose and study additional extensions of the maximin share to the weighted setting.
Other works study the computation of existing weighted fairness measures. Chakraborty et al. [2021] study the properties of weighted fairness existing picking sequences satisfy. Garg et al. [2021] study the computation of allocations which maximize the weighted Nash welfare. Li et al. [2022] compute weighted proportional allocations for chores. Aziz et al. [2019] study the weighted maxmin bargaining solutions. Other works study the computation of existing weighted fairness measures. Chakraborty et al. [2021] study the properties of weighted fairness existing picking sequences satisfy. Garg et al. [2021] study the computation of allocations which maximize the weighted Nash welfare. Li et al. [2022] compute weighted proportional allocations for chores. Aziz et al. [2019] study the weighted maxmin bargaining solutions.

The work arguably closest to ours is Suksompong and Teh [2022]. Our result builds on that of Suksompong and Teh [2022] in two ways. First, our result propose an algorithm to compute a max weighted Nash welfare allocation when agents have binary entitlements. Second, our framework can compute several other fairness properties in addition to maximizing the weighted Nash welfare.

2 Preliminaries

We use $[t]$ to denote the set $\{1, 2, \ldots, t\}$. For ease of readability, we replace $A \cup \{g\}$ and $A \setminus \{g\}$ with $A + g$ and $A - g$ respectively.

We have a set of $n$ agents $N = [n]$ and a set of $m$ goods $G = \{g_1, g_2, \ldots, g_m\}$. Each agent $i \in N$ has a valuation function $v_i : 2^G \rightarrow \mathbb{R}^+$; $v_i(S)$ specifies the value agent $i$ has for the set of goods $S \subseteq G$. Each agent $i$ also has a positive weight $w_i \in \mathbb{R}^+$ that corresponds to their entitlement; we do not place any other constraint on the entitlement other than positivity. We use $\Delta_i(S, g) = v_i(S + g) - v_i(S)$ to denote the marginal gain of adding the good $g$ to the bundle $S$ for the agent $i$. Throughout the paper, we assume each $v_i$ is a matroid rank function (MRF). A function $v_i$ is a matroid rank function if (a) $v_i(\emptyset) = 0$, (b) for any $g \in G$ and $S \subseteq G$, we have $\Delta_i(S, g) \in \{0, 1\}$, and (c) for any $S \subseteq T \subseteq G$ and a good $g$, we have $\Delta_i(S, g) \geq \Delta_i(T, g)$. These functions are also referred to as binary submodular valuations; we use the two terms interchangeably.

An allocation $X$ is a partition of the set of goods into $n + 1$ sets $(X_0, X_1, \ldots, X_n)$ where each agent $i \in N$ gets the bundle $X_i$ and $X_0$ consists of the unallocated goods. An allocations $X$ is said to be non-redundant if for all $i \in N$, we have $v_i(X_i) = |X_i|$. For any allocation $X$, $v_i(X_i)$ is referred to as the utility or value of agent $i$ under the allocation $X$. For ease of analysis, we sometimes treat 0 as an agent with valuation function $v_0(S) = |S|$ and bundle $X_0$. None of our fairness measures take the agent 0 into account. By our choice of $v_0$, any allocation which is non-redundant for the set of agents $N$ is redundant for the set of agents $N + 0$. We have this simple useful result about non-redundant allocations. Variants of this result have been shown by Benabbou et al. [2021] and Viswanathan and Zick [2022].

**Lemma 2.1** [Benabbou et al. 2021, Viswanathan and Zick 2022]. Let $X$ be an allocation. There exists a non-redundant allocation $X'$ such that $v_i(X'_i) = v_i(X_i)$ for all $i \in N$.

**Proof.** For each agent $i$, create $X'_i$ starting at 0 and add goods one by one from $X_i$ as long as the good added provides a marginal gain of 1; stop when no good in $X_i \setminus X'_i$ can provide a marginal gain of 1 to $X'_i$. Allocate all leftover goods to $X_0$. It is easy to see that $X'$ is non-redundant; we only added goods when they had a marginal gain of 1. It remains to show that $v_i(X'_i) = v_i(X_i)$.

Let $X_i \setminus X'_i = \{g_1, g_2, \ldots, g_k\}$. Let $Z^j = \{g_1, g_2, \ldots, g_{j-1}\}$ ($Z^1 = \emptyset$). By the definition of $X'_i$, we have $\Delta_i(X'_i, g_j) = 0$ for all $j \in [k]$. By the submodular property, this implies that $\Delta_i(X'_i \cup Z^j, g_j) = 0$ as well for all $j \in [k]$. We have

$$v_i(X_i) = v_i(X'_i) + \sum_{j=1}^k \Delta_i(X'_i \cup Z^j, g_j) = v_i(X'_i)$$

This completes the proof. \qed

This completes the proof.

3
We define the exchange graph of a non-redundant allocation \( X \) (denoted by \( G(X) \)) as a directed graph defined over the set of goods \( G \). An edge exists from good \( g \in X_i \) to another good \( g' \) if \( v_i(X_i-g+g') = v_i(X_i) \). Intuitively, this means that from the perspective of agent \( i \) (who owns \( g \)), \( g \) can be replaced with \( g' \) with no decrease to agent \( i \)'s utility. The exchange graph is a useful representation since it can be used to compute valid transfers of goods between agents.

Let \( P = (g_1, g_2, \ldots, g_t) \) be a path in the exchange graph for the allocation \( X \). We define a transfer of goods along the path \( P \) in the allocation \( X \) as the operation where \( g_t \) is given to the agent who has \( g_{t-1} \), \( g_{t-1} \) is given to the agent who has \( g_{t-2} \) and so on till finally \( g_1 \) is discarded.

This transfer is called path augmentation; the bundle \( X_i \) after path augmentation with the path \( P \) is denoted by \( X_i \Delta P \) and defined as \( X_i \Delta P = (X_i - g_t) \oplus \{g_j, g_{j+1} : g_j \in X_i \} \) where \( \oplus \) denotes the symmetric set difference operation. While the conventional notation for path augmentation uses \( \Delta \) [Barman and Verma 2021, Schrijver 2003], we replace it with \( \Lambda \) to avoid confusion with the other definition of \( \Delta \) as the marginal gain of adding an item to a bundle.

For any non-redundant allocation \( X \) and agent \( i \), we define \( F_i(X) = \{ g \in G : \Delta_i(X_i, g) = 1 \} \) as the set of goods which give agent \( i \) a marginal gain of 1. For any agent \( i \), let \( P = (g_1, \ldots, g_t) \) be the shortest path from \( F_i(X) \) to some \( X_{j} \neq i \). Then path augmentation with the path \( P \) and giving \( g_t \) to \( i \) results in an allocation where \( i \)'s value for their bundle goes up by 1, \( j \)'s value for their bundle goes down by 1 and all the other agents do not see any change in value. This is formalized below and exists both in [Viswanathan and Zick 2022, Lemma 5] and [Barman and Verma 2021, Lemma 1].

**Lemma 2.2** [Viswanathan and Zick 2022, Barman and Verma 2021]. Let \( X \) be a non-redundant allocation. Let \( P = (g_1, \ldots, g_t) \) be the shortest path from \( F_i(X) \) to \( X_j \) for some \( i \in N + 0 \) and \( j \in N + n - i \). Then, we have for all \( k \in N - i - j \), \( v_k(X_k \Delta P) = v_k(X_k) \), \( v_i(X_i \Delta P + g_t) = v_i(X_i) + 1 \) and \( v_j(X_j \Delta P) = v_j(X_j) - 1 \). Furthermore, the new allocation is non-redundant.

This lemma is particularly useful when the path ends at some good in \( X_0 \); then transferring goods along the path creates an allocation where no agent loses any value but one agent strictly gains in value. We say there is a path from some agent \( i \) to some agent \( j \) in an allocation \( X \) is there is a path from \( F_i(X) \) to \( X_j \) in the exchange graph \( G(X) \). [Viswanathan and Zick 2022, Theorem 3] establish a sufficient condition for a path to exist; we present it below.

**Lemma 2.3** [Viswanathan and Zick 2022]. Let \( X \) and \( Y \) be two non-redundant allocations. Let \( S^- \) be the set of agents \( k \in N + 0 \) such that \( |X_k| < |Y_k| \), \( S^+ \) be the set of agents \( k \in N + 0 \) such that \( |X_k| > |Y_k| \). Then, for any \( i \in S^- \), there is a path from \( F_i(X) \) to \( X_j \) in the exchange graph \( G(X) \) where \( j \in S^+ \).

We define the utility vector of an allocation \( X \) as \( \vec{u}^X = (v_1(X_1), v_2(X_2), \ldots, v_n(X_n)) \). In general, a fairness objective (denoted by \( \psi \)) can be viewed as a function that maps the utility vector of an allocation to a totally ordered space \( \mathcal{Y} \). The goal is to compute an allocation with a utility vector that maximizes \( \psi \). We sometimes abuse notation and use \( \psi(X) \) to denote \( \psi(\vec{u}^X) \).

For example, a popular fairness objective is the Nash welfare defined as \( \prod_{i \in N} (v_i(X_i)) \). In this case, the output of \( \psi \) is a real valued number equal to \( \prod_{i \in N} (v_i(X_i)) \). The commonly associated goal with Nash welfare is to maximize it i.e. to maximize \( \psi \).

For ease of readability, in this section, we only define a few necessary terms from the fair division literature. We define specific terms like specific fairness objectives as and when required.

**Definition 2.4.** An allocation \( X \) is said to maximize utilitarian social welfare (denoted by \( \text{MAX-USW} \)) if it maximizes \( \sum_{i \in N} v_i(X_i) \).

**Definition 2.5.** Let \( \vec{x}, \vec{y} \in \mathbb{R}^c \) be two vectors for some positive integer \( c \). \( \vec{x} \) is said to lexicographically dominate \( \vec{y} \) if there exists a \( k \in [c] \) such that for all \( j \in [k - 1] \), we have \( x_j = y_j \) and we have \( x_k > y_k \). A real valued vector \( \vec{x} \) is lexicographically dominating with respect to a set of vectors \( V \) if there
exists no \( \vec{y} \in V \) which lexicographically dominates \( \vec{x} \). This definition can be extended to allocations as well. An allocation \( X \) is said to lexicographically dominate an allocation \( Y \) if the utility vector of \( X \) lexicographically dominates the utility vector of the allocation \( Y \). Similarly, an allocation \( X \) is lexicographically dominating with respect to a set of allocations \( V \) if there exists no \( Y \in V \) which lexicographically dominates \( X \).

**Definition 2.6.** An allocation \( X \) is said to pareto dominate another allocation \( Y \) for all \( h \in N \), we have \( v_h(X_h) \geq v_h(Y_h) \) with the inequality being strict for at least one \( h \in N \).

### 3 General Yankee Swap

In this section, we present our framework to compute fair allocations. Our framework is a generalization of the Yankee Swap algorithm by Viswanathan and Zick [2022].

In the Yankee Swap algorithm, all goods are initially unallocated. At every round, the agent with the least utility picks a good they like from the unallocated pile or initiate a transfer path where they steal a good they like from another agent who then steals a good they like from another agent and so on until an agent finally takes a good they like from the unallocated pile of goods. If there is no such path to the unallocated pile of goods, the agent is removed from the game (denoted by their removal from the set \( P \)). We terminate once there are no agents playing.

We now present General Yankee Swap (Algorithm 1). We make only one change in the implementation: instead of picking the least utility agent at every round, we pick an agent that maximizes a general gain function \( \phi \). \( \phi \) takes as input a non-redundant allocation (the partial allocation we have so far) and an index \( i \in N \); its output is a \( b \)-dimensional vector. When \( b > 1 \), \( \phi(X,i) > \phi(X,j) \) if \( \phi(X,i) \) lexicographically dominates \( \phi(X,j) \). We break ties by choosing the the agent with the least index. The gain function \( \phi \) depends on the fairness objective we optimize. The original Yankee Swap is a specific case of the general Yankee Swap where \( \phi(X,i) = \frac{1}{n} \).

We make an additional change in notation to the Yankee Swap in Viswanathan and Zick [2022]. We replace transfer path with the shortest path in the exchange graph. Viswanathan and Zick [2022] show that a transfer path is equivalent to the shortest path in the exchange graph, so this does not affect the final output of the algorithm.

**Algorithm 1** Weighted Yankee Swap

\[
\begin{align*}
X &= (X_0, X_1, \ldots, X_n) \leftarrow (G, \emptyset, \ldots, \emptyset) \\
P &\leftarrow N \\
\text{while } P \neq \emptyset \text{ do} \\
&\quad S \leftarrow \text{arg max}_{k \in P} \phi(X,k) \\
&\quad i \leftarrow \min\{j : j \in S\} \\
&\quad \text{Find the shortest path in the exchange graph } G(X) \text{ from } F_i(X) \text{ to } X_0 \\
&\quad \text{if a path } P = (g_{i_1}, g_{i_2}, \ldots, g_{i_k}) \text{ exists then} \\
&\quad &\quad X_k \leftarrow X_k \text{AP for all } k \in N - i \\
&\quad &\quad X_i \leftarrow X_i \text{AP} + g_{i_1} \\
&\quad &\quad X_0 \leftarrow X_0 \text{AP} \\
&\quad \text{else} \\
&\quad &\quad P \leftarrow P - i \\
\text{end if} \\
\text{end while}
\]

General Yankee Swap works when the fairness objective \( \psi : \mathbb{R}^n \rightarrow \mathcal{Y} \) has the following properties:

(C1) – Pareto Dominance: \( \psi(X) \leq \psi(Y) \) whenever \( Y \) Pareto dominates \( X \).
(C2) – Gain Function: \( \psi \) admits a gain function \( \phi \) that maps each non-redundant allocation to a real-valued \( b \)-dimensional vector with the following properties:

(G1) Let \( X \) be a non-redundant allocation. Let \( Y_1 \) be the non-redundant allocation resulting from giving the good \( g \) to \( i \) under \( X \), such that \( \Delta(X, i) = 1 \). Similarly, let \( Y_2 \) be the non-redundant allocation resulting from giving the good \( g \) to \( j \) under \( X \) such that \( \Delta(X, j) = 1 \). Then, if \( \phi(X, i) \geq \phi(X, j), \psi(Y_1) \geq \psi(Y_2) \). Equality holds if and only if \( \phi(X, j) = \phi(X, i) \).

(G2) For any non-redundant allocation \( X, \phi(X, i) \) is a function of \( |X_i| \) such that for any two non-redundant allocations \( X \) and \( Y \), if \( |X_i| \leq |Y_i| \), then \( \phi(X, i) \geq \phi(Y, i) \) with equality holding if \( |X_i| = |Y_i| \).

Intuitively, \( \phi \) can be thought of as a function describing the marginal ‘gain’ of giving a good to agent \( i \) given some allocation \( X \). The higher the value of \( \phi(X, i) \), the more valuable it is to add a good to the bundle \( X_i \) for some allocation \( X \).

General Yankee Swap outputs a non-redundant \( \psi \) maximizing allocation which is also welfare maximizing. Moreover, among all allocations that maximize \( \psi \), the output of General Yankee Swap is lexicographically dominating. This is a stronger statement and is required to show strategyproofness — a technique used by Halpern et al. [2020] and Suksompong and Teh [2022]. While we use the same idea and notation of Viswanathan and Zick [2022, Theorem 2] in our proof, our arguments are more general.

**Theorem 3.1.** Let \( \psi \) be a fairness objective that satisfies (C1) and (C2) with a gain function \( \phi \). When agents have matroid rank valuations, General Yankee Swap with input \( \phi \) maximizes \( \psi \). Moreover, among all the allocations which maximize \( \psi \), the allocation output by General Yankee Swap is lexicographically dominating.

**Proof.** It is easy to show that Algorithm 1 always terminates: at every iteration, we either reduce the number of unallocated goods or remove some agent from \( P \) while not changing the number of unallocated goods.

Let \( X \) be the output of Algorithm 1. Note that \( X \) is guaranteed to be non-redundant at every iteration. We start with the non-redundant empty allocation, and using Lemma 2.2, non-redundancy is maintained after every transfer. We next show that \( X \) maximizes \( \psi \) and among all the allocations which maximize \( \psi \), \( X \) is lexicographically dominating.

Let \( Y \) be a non-redundant allocation that maximizes \( \psi \) — such an allocation is guaranteed to exist since \( Y \) induces a total order over allocations, and every allocation can be made non-redundant (Lemma 2.1). If there are multiple such \( Y \), pick one that lexicographically dominates all others. If for all \( h \in N \), \( \upsilon_h(X_h) \geq \upsilon_h(Y_h) \), then \( X \) maximizes \( \psi \) (using (C1)) and is lexicographically dominating — we are done. Assume for contradiction that this does not hold.

Let \( i \in N \) be the agent with highest \( \phi(X, i) \) in \( X \) such that \( \upsilon_i(X_i) < \upsilon_i(Y_i) \); if there are multiple, break ties in favor of the lowest index agent. Let \( W \) be the non-redundant allocation at the start of the iteration where \( i \) was removed from \( P \). Note that in this iteration, \( i \) was the agent with highest \( \phi(W, i) \) among all the agents in \( P \). We use \( t \) to denote this iteration. We have the following lemma.

**Lemma 3.2.** For all \( h \in N \), \( |Y_h| \geq |W_h| \)

**Proof.** Assume for contradiction that this is not true. Let \( j \in N \) be the agent with highest \( \phi(Y, j) \) such that \( |Y_j| < |W_j| \); if there are multiple, break ties in favor of the one with the least index.

Consider the bundle \( W_j \). Let \( W' \) be the allocation when \( j \) moved from a bundle of size \( |W_j| - 1 \) to \( |W_j| \). When \( j \) moved from a bundle of size \( |W_j| - 1 \) to \( |W_j| \), it was the agent with maximum \( \phi(W', j) \) within \( P \). Combining this with the fact that bundle sizes for each agent monotonically increase and that agents never return to \( P \) once moved, we have \( \phi(Y, j) \geq \phi(W', j) \geq \phi(W', i) \geq \phi(W, i) \). The first and third inequality hold due to (G2). If \( \phi(Y, j) = \phi(W', j) = \phi(W', i) = \phi(W, i) \), then \( j < i \).
since the General Yankee Swap breaks final ties using the index of the agent and \( j \) was chosen at \( W' \) instead of \( i \).

Therefore, we have

**Observation 3.3.** \( \phi(Y,j) \geq \phi(W,i) \). If equality holds, then \( j < i \).

Invoking Lemma 2.2 with allocations \( Y \) and \( W \) and agent \( j \), there must be a path from \( F_j(Y) \) to \( Y_k \) for some \( k \in N + 0 \) where \( |Y_k| > |W_k| \) in the exchange graph of \( Y \). Invoking Lemma 2.2 transferring goods along the shortest path from \( F_j(Y) \) to \( Y_k \) results in a non-redundant allocation \( Z \) where \(|Z_j| = |Y_j| + 1\), \(|Z_k| = |Y_k| - 1\) and for all other agents, the bundle size remains the same.

If \( k = 0 \), we are done since \( Z \) pareto dominates \( Y \). Therefore using (C1), \( \psi(Y) \leq \psi(Z) \). Combining this with the fact that pareto dominance implies lexicographic dominance, we contradict our assumption on \( Y \). We therefore assume \( k \neq 0 \).

Consider \( \phi(W,k) \). If \( \phi(W,k) > \phi(W,i) \), since \( i \) was chosen as the agent with highest \( \phi(W,i) \) among the agents in \( P \) at iteration \( t \), we must have that \( k \notin P \) at iteration \( t \). This gives us \( v_k(X_k) = v_k(W_k) \leq v_k(Y_k) \). Combining this with our initial assumption that \( \phi(W,k) > \phi(W,i) \), we get \( \phi(X,k) = \phi(W,k) > \phi(W,i) \geq \phi(X,i) \) using (G2). This contradicts our choice of \( i; \ i \) is not the agent with highest \( \phi(X,i) \) utility such that \(|X_i| < |Y_i|\). \( \phi(X,k) > \phi(X,i) \) and \( k \) satisfies the condition. This gives us the following observation (combined with Observation 3.3).

**Observation 3.4.** \( \phi(W,k) \leq \phi(W,i) \leq \phi(Y,j) \)

Let \( Y' \) be a non-redundant allocation starting at \( Y \) and removing any good from \( k \). Note that, to show \( \psi(Z) \geq \psi(Y) \), it suffices to show that \( \phi(Y',j) \geq \phi(Y',k) \) (using (G1)). From Observation 3.3 and (G2), we have \( \phi(Y',k) \leq \phi(W,k) \leq \phi(W,i) \leq \phi(Y,j) = \phi(Y',j) \). If any of these inequalities are strict, we have \( \phi(Y',j) > \phi(Y',k) \) which implies \( \psi(Z) > \psi(Y) \) — a contradiction.

This gives us the following observation.

**Observation 3.5.** \( \phi(Y',k) = \phi(W,k) = \phi(W,i) = \phi(Y,j) = \phi(Y',j) \)

Since \( \phi(W,k) = \phi(W,i) \) and the algorithm picked \( i \) at iteration \( t \), we must have \( i < k \). Assume for contradiction that this is not true. If \( k \in P \) at iteration \( t \), the algorithm would have picked \( k \) instead of \( i \), a contradiction. If \( k \notin P \), then \( \phi(X,k) = \phi(W,k) = \phi(W,i) \geq \phi(X,i) \) using (G2). Combining this with \( i > k \) contradicts our choice of \( i \). Therefore we have, \( i < k \). Combining this with Observation 3.3 we have:

**Observation 3.6.** \( j < i < k \)

Combining Observations 3.5 and 3.6, we get that \( \phi(Y',k) = \phi(Y,j) = \phi(Y',j) \) using (G2). Therefore \( Z \) maximizes \( \psi \) using (G1).

However, since \( j < k \), \( Z \) lexicographically dominates \( Y \). This is because, all agents \( h < j \) receive the same value in both allocations and \( v_j(Z_j) > v_j(Y_j) \). This contradicts our assumption on \( Y \) and completes the proof.

Using Lemma 2.2 we construct a path from \( i \) to 0 in \( W \). We create a new non-redundant allocation \( Z \) where each \( Z_h \) is an arbitrary \( |W_h| \) sized subset of \( Y_h \) for all \( h \in N - i \). This can be done thanks to Lemma 3.2, \( Z_i = Y_i \) and \( Z_0 \) is defined as all the goods not in any \( Z_h \) for \( h \in N \).

Note that \(|W'_i| < |Z_i|\), \(|W'_0| > |Z_0|\) and \(|W'_h| = |Z_h|\) for all \( h \in N - i \). Using Lemma 2.2 with the allocations \( W' \) and \( Z \) and the agent \( i \), we get that there is a path from \( i \) to 0 in \( W' \) — 0 is the only agent in \( S^+ \). This is a contradiction since we chose the start of the iteration where \( i \) was removed from \( P \), indicating that there was no path from \( i \) to 0 in \( W \).

Due to the similarity of our algorithm with that of Viswanathan and Zick [2022], their time complexity result applies to our algorithm as well.
We define the steps of the Yankee Swap mechanism as follows:

1. Elicit the valuation function $v_i$ of each agent $i \in N$. If an agent’s valuation function is not an MRF, set the agent’s valuation of every bundle to 0.

2. Use General Yankee Swap to compute a non-redundant allocation that maximizes $\psi$ for the valuation profile $\{v_i\}_{i \in N}$.

Remark 3.8. As we shall see in the coming sections, $b + T_\phi(n, m)$ is usually $O(1)$.

We also show that, for any $\phi$, the output of the General Yankee Swap is always $\text{MAX-USW}$.

Proposition 3.9. For any $\phi$, the output of the General Yankee Swap is $\text{MAX-USW}$.

4 The Yankee Swap Mechanism

In this section, we show that if preferences are elicited before running the General Yankee Swap, being truthful is the dominant strategy.

A mechanism is said to be strategyproof if no agent can get a better outcome by not reporting a false valuation function. We define the steps of the Yankee Swap Mechanism as follows:

1. Elicit the valuation function $v_i$ of each agent $i \in N$. If an agent’s valuation function is not an MRF, set the agent’s valuation of every bundle to 0.

2. Use General Yankee Swap to compute a non-redundant allocation that maximizes $\psi$ for the valuation profile $\{v_i\}_{i \in N}$. 

Proof. Our proof uses the arguments of Viswanathan and Zick [2022] who show that the Yankee Swap algorithm runs in $O(m^2(n+\tau)(m+n))$ time.

We make three observations. First, the algorithm runs for at most $(m+n)$ iterations. At each round either $|X_h|$ reduces by 1 or an agent is removed from $P$. $X_0$ monotonically decreases and agents do not return to $P$; therefore we can only have at most $m+n$ iterations.

Second, constructing the exchange graph, checking for a path and transferring goods along the path can be done in $O(m^2(n+\tau))$ time. This is shown by Viswanathan and Zick [2022, Section 3.4].

Finally, finding $i$ involves computing $\phi$ for each agent and then comparing them. Since the output of $\phi$ has $b$ components, each comparison takes at most $O(b)$ time, finding $i$ takes $n(b + T_\phi(n,m))$ time. Combining these three observations we get the required time complexity.

\[\text{Remark 3.8. As we shall see in the coming sections, } b + T_\phi(n,m) \text{ is usually } O(1).\]

We also show that, for any $\phi$, the output of the General Yankee Swap is always $\text{MAX-USW}$.

Proposition 3.9. For any $\phi$, the output of the General Yankee Swap is $\text{MAX-USW}$.

Proof. Let $X$ be the non-redundant allocation output by General Yankee Swap. Assume for contradiction that $X$ is not $\text{MAX-USW}$. Let $Y$ be a non-redundant $\text{MAX-USW}$ allocation which minimizes $\sum_{h \in N} |v_h(X_h) - v_h(Y_h)|$.

If $|X_h| \leq |Y_h|$ for all $h \in N$, there must be at least one agent $i$ such that $|X_i| < |Y_i|$. Consider the allocation $W$ at the start of the iteration where $i$ was removed from $P$. Create a new allocation $Z$ where each $Z_h$ is an arbitrary $|W_h|$ sized subset of $Y_h$ for all $h \in N - i$. This can be done since $|W_h| \leq |X_h| \leq |Y_h|$ by assumption. $Z_i = Y_i$ and $Z_0$ is defined as all the goods not in any $Z_h$ for $h \in N$.

Note that $|W_i| < |Z_i|$, $|W_0| > |Z_0|$ and $|W_h| = |Z_h|$ for all $h \in N - i$. Using Lemma 2.3 with the allocations $W$ and $Z$ and the agent $i$, we get that there is a path from $i$ to 0 in $W - 0$ is the only agent in $S^+$. This is a contradiction since we chose the start of the iteration where $i$ was removed from $P$ indicating that there was no path from $i$ to 0 in $W$.

Therefore, we must have at least one agent $j$ such that $|Y_j| < |X_j|$. Applying Lemma 2.3 with allocations $X$, $Y$ and the agent $j$, we get that there is a path from $j$ to some agent $k \in N + 0$ in the exchange graph of $Y$ such that $|X_k| < |Y_k|$. Transferring goods along the shortest path from $j$ to $k$, using Lemma 2.2 leads to a non-redundant allocation $Z$ where $|Z_j| = |Y_j| + 1$ and $|Z_k| = |Y_k| - 1$.

If $k = 0$, $Z$ has a higher USW than $Y$ contradicting our assumption on $Y$.

If $k \neq 0$, then $\sum_{h \in N} |v_h(X_h) - v_h(Y_h)| > \sum_{h \in N} |v_h(X_h) - v_h(Z_h)|$ and $Z$ is $\text{MAX-USW}$; again contradicting our assumption on $Y$. Therefore, $X$ is $\text{MAX-USW}$. 

Before we show the final result, we prove some useful lemmas. Our proof uses the same ideas as the strategyproofness result in [Babaioff et al. 2021a] Theorem 5. Given a set \( T \subseteq G \), we define the function \( f_T : 2^G \to \mathbb{R}^+ \) as \( f_T(S) = |S \cap T| \). Note that for any \( T \), \( f_T \) is an MRF.

**Lemma 4.1.** Let \( X \) be the output allocation of the Yankee Swap mechanism with valid input gain function \( \phi \) and valuation profile \( \{v_i\}_{i \in N} \). For some agent \( i \in N \), replace \( v_i \) with some \( f_T \) such that \( T \subseteq X_i \) and run the mechanism again to get an allocation \( Y \). We must have \( Y_i = T \).

**Proof.** Since the allocation \( Y \) is non-redundant, we have that \( Y_i \subseteq T \). Assume for contradiction that \( Y_i \neq T \). Define an allocation \( Z \) as \( Z_h = X_h \) for all \( h \in N - i \) and \( Z_i = T \); allocate the remaining goods in \( Z \) to \( Z_0 \). Note that both \( Y \) and \( Z \) are non-redundant under both valuation profiles (with the old \( v_i \) and the new valuation function \( f_T \)).

Let us compare \( Y \) and \( Z \). Let \( p \in N \) be the agent with highest \( \phi(Y,p) \) such that \( |Y_p| < |Z_p| \); choose the agent with the least \( p \) if there are ties. Such an element is guaranteed to exist since \( |Y_i| < |Z_i| \).

If there exists no \( q \in N \) such that \( |Y_q| > |Z_q| \), we must have \( |Y_0| > |Z_0| \) (since \( |Y_i| < |Z_i| \)). Using Lemma 2.3 with agent \( p \) and the allocations \( Y \) and \( Z \), we get that there is a path from \( p \) to \( 0 \) in the exchange graph of \( Y \). Transferring goods along the shortest path results in an allocation with a higher USW than \( Y \) under the new valuation profile contradicting the fact that \( Y \) is \textsc{MAX-USW}.

Let \( q \in N \) be the agent with highest \( \phi(Z,q) \) such that \( |Y_q| > |Z_q| \); break ties by choosing the least \( q \). Further, note that since \( Z \) and \( X \) only differ in \( i \)’s bundle and \( |Y_i| < |Z_i| \), we must have \( |X_p| = |Z_q| \).

Consider two cases:

(i) \( \phi(X,q) > \phi(Y,p) \),

(ii) \( \phi(X,q) = \phi(Y,p) \) and \( q < p \)

If any of the above two conditions occur, then invoking Lemma 2.3 with allocations \( X \), \( Y \) and the agent \( q \), there exists a transfer path from \( q \) to some agent \( k \) in the exchange graph of \( X \) (w.r.t. the old valuations) where \( |Y_k| < |X_k| \). Transferring along the shortest such path gives us a non-redundant allocation \( X' \) where \( |X'_q| = |X_q| + 1 \) and \( |X'_k| = |X_k| - 1 \) (Lemma 2.2). Let \( X'' \) be an allocation starting at \( X' \) and removing one good from \( X'_q \). If \( \phi(X'',q) > \phi(X'',k) \), then \( \psi(X') > \psi(X) \) (using (G1)) contradicting our assumption on \( X \).

If \( k = 0 \), we improve USW contradicting the fact that \( X \) is \textsc{MAX-USW} with respect to the original valuations \( \{v_i\}_{i \in N} \).

**For case (i):** If \( k \neq 0 \), we have \( \phi(X'',q) = \phi(X,q) > \phi(Y,p) \geq \phi(Y,k) \geq \phi(X'',k) \) (using (G1) and (G2)). Therefore \( \phi(X'',q) > \phi(X'',k) \) and \( X \) does not maximize \( \psi \) — a contradiction.

**For case (ii):** If \( k \neq 0 \), we have \( \phi(X'',q) = \phi(X,q) = \phi(Y,p) \geq \phi(Y,k) \geq \phi(X'',k) \). If any of these weak inequalities are strict, we can use analysis similar to that of case (i) to show that \( X \) does not maximize \( \psi \). Therefore, all the weak inequalities must be equalities and we must have \( \phi(X'',q) = \phi(X,q) = \phi(Y,p) = \phi(Y,k) = \phi(X'',k) \). This implies that \( \psi(X') = \psi(X) \) using (G1).

Moreover, by our choice of \( p \) we have \( p < k \) and by assumption, we have \( q < p \). Combining the two, this gives us \( q < k \). Therefore, \( X' \) lexicographically dominates \( X \) — a contradiction to Theorem 3.1.

Let us now move on to the remaining two possible cases

(iii) \( \phi(Z,q) < \phi(Y,p) \),

(iv) \( \phi(Z,q) = \phi(Y,p) \) and \( q > p \)

Recall that both \( Y \) and \( Z \) are non-redundant with respect to the new valuation profile (with \( f_T \)). If any of the above two conditions occur, then invoking Lemma 2.3 with allocations \( Y \), \( Z \) and the agent \( p \), there exists a transfer path from \( p \) to some agent \( l \) in the exchange graph of \( Y \) where
\(|Y_i| > |Z_i|\). Transferring along the shortest such path gives us a non-redundant allocation \(Y'\) where \(|Y'_p| = |Y_p| + 1\) and \(|Y'_q| = |Y_q| - 1\) (Lemma \textit{2.2}).

Let \(Y''\) be an allocation starting at \(Y'\) and removing one good from \(Y'_p\). If \(\phi(Y'',p) > \phi(Y'',l)\), then \(\psi(Y') > \psi(Y)\) contradicting our assumption on \(Y\).

If \(l = 0\), we improve USW contradicting the fact that \(Y\) is MAX-USW with respect to the new valuation \(f_Y\).

\textbf{For case (iii):} If \(l \neq 0\), we have \(\phi(Y'',p) = \phi(Y,p) > \phi(Z,q) \geq \phi(Z,l) \geq \phi(Y'',l)\) (using (G1) and (G2)). Therefore \(\phi(Y'',p) > \phi(Y'',l)\) and \(Y\) does not maximize \(\psi\) — a contradiction.

\textbf{For case (iv):} If \(l \neq 0\), we have \(\phi(Y'',p) = \phi(Y,p) = \phi(Z,q) \geq \phi(Z,l) \geq \phi(Y'',l)\). If any of the weak inequalities are strict, similar to that of case (iii) to show that \(Y\) does not maximize \(\psi\).

Therefore, all the weak inequalities must be equalities and we must have \(\phi(Y'',p) = \phi(Y,p) = \phi(Z,q) = \phi(Z,l) = \phi(Y'',l)\). This implies that \(\psi(X) = \psi(X')\).

Moreover, by our choice of \(q\) we have \(q < l\) and by assumption, we have \(p < q\). Combining the two, this gives us \(p < l\). Therefore, \(X'\) lexicographically dominates \(X\) — a contradiction to Theorem \textit{3.1}.

Since \(|Z_q| = |X_q|\), \(\phi(Z,q) = \phi(X,q)\). Therefore, cases (i)–(iv) cover all possible cases. Each of the cases lead to a contradiction. Therefore, our proof is complete and \(Y_i = T\).

\textbf{Lemma 4.2.} Let \(X\) be the output allocation of the weighted leximin mechanism with input valuation profile \(\{v_i\}_{i \in N}\). For some agent \(i \in N\), replace \(v_i\) with some \(v'_i\) such that \(v'_i(S) \geq v_i(S)\) for all \(S \subseteq G\) and run the mechanism again to get an allocation \(Y\). We must have \(|Y_i| \geq |X_i|\).

\textbf{Proof.} This proof is very similar to that of Lemma \textit{4.1}.

Define the new valuation functions by \(\{v'_j\}_{j \in N}\) where \(v'_j = v_j\) for all \(j \in N - i\). To prevent any ambiguity, whenever we discuss the \(\psi\) value of \(X\), we implicitly discuss it with respect to the valuations \(v\). Similarly, whenever we discuss the \(\psi\) value of \(Y\) we implicitly discuss it with respect to the valuations \(v'\).

Assume for contradiction that \(|Y_i| < |X_i|\). Let \(T\) be a subset of \(Y_i\) such that \(|T| = v_i(T) = v_i(Y_i)\) (such a set exists using an argument similar to Lemma \textit{2.4}). Define an allocation \(Z\) as \(Z_h = Y_h\) for all \(h \in N - i\) and \(Z_i = T\); allocate the remaining goods in \(Z\) to \(Z_0\). Note that both \(X\) and \(Z\) are non-redundant under both valuation profiles. This implies for all \(j \in N\), \(v'_j(X_j) = v_j(X_j) = |X_j|\) and \(v'_j(Z_j) = v_j(Z_j) = |Z_j|\).

Let us compare \(X\) and \(Y\). Let \(p \in N\) be the agent with highest \(\phi(Y,p)\) such that \(|Y_p| < |X_p|\); break ties by choosing the least \(p\). Such an element is guaranteed to exist since \(|Y_i| < |X_i|\).

If there exists no \(q \in N\) such that \(|Y_q| > |X_q|\), we must have \(|Y_0| > |X_0|\) (since \(|X_i| > |Y_i|\)). Using Lemma \textit{2.3} with the allocations \(X\) and \(Y\) with \(p\), we get that there is a path from \(p\) to 0 in the exchange graph of \(Y\). Transferring goods along the shortest such path results in an allocation with a higher USW than \(Y\) contradicting the fact that \(Y\) is MAX-USW.

Let \(q \in N\) be the agent with highest \(\phi(X,q)\) such that \(|Y_q| > |X_q|\). Break ties by choosing the least \(q\). Note that since \(Z\) and \(Y\) only differ in \(i\)'s bundle and \(|Y_i| < |X_i|\), we must have \(|Y_q| = |Z_q|\).

Therefore, \(|Z_q| > |X_q|\) as well.

Consider two cases:

(i) \(\phi(X,q) > \phi(Y,p)\),

(ii) \(\phi(X,q) = \phi(Y,p)\) and \(q < p\)

If any of the above two conditions occur, then invoking Lemma \textit{2.3} with allocations \(X\), \(Z\) and the agent \(q\), there exists a transfer path from \(q\) to some agent \(k\) where \(|Z_k| < |X_k|\). Transferring along the shortest such path gives us a non-redundant (w.r.t. the valuation profile \(v\)) allocation \(X'\) where \(|X'_q| = |X_q| + 1\) and \(|X'_k| = |X_k| - 1\) (Lemma \textit{2.4}). Since \(v'_j(X'_j) \geq v_j(X'_j)\) for all \(j \in N\), we must have that \(X'\) is non-redundant with respect to both valuation profiles \(v\) and \(v'\). By our definition
of Z, if \( k \in N - i \), we have \( |X_k| > |Z_k| = |Y_k| \) and if \( k = i \), we have \( |X_i| > |Y_i| \). Therefore, we have \( |X_k| > |Y_k| \).

Let \( X'' \) be an allocation obtained from starting at \( X \) and removing a good from \( k \). From (G1), we have that if \( \phi(X'', k) \leq \phi(X'', q) \), we have \( \psi(X) \leq \psi(X') \) (with respect to the valuations \( v \) with equality holding if and only if \( \phi(X'', k) = \phi(X'', q) \).

\( X, X', X'' \) and \( Y \) are non-redundant with respect to the new valuation function profile \( v' \). We can therefore, compare their \( \phi \) values directly.

If \( k = 0 \), we improve USW (w.r.t. the valuation profile \( v \)) contradicting the fact that \( X \) is weighted leximin.

**For case (i):** If \( k \neq 0 \), we have \( \phi(X'', k) = \phi(X', k) \leq \phi(Y, k) \leq \phi(Y, p) < \phi(X, q) = \phi(X'', q) \). This gives us \( \phi(X'', k) < \phi(X'', q) \); a contradiction.

**For case (ii):** If \( k \neq 0 \), we have \( \phi(X'', k) \leq \phi(Y, k) \leq \phi(Y, p) = \phi(X, q) = \phi(X'', q) \). If any of these weak inequalities are strict, we can use analysis similar to that of case (i) to show that \( X \) does not maximize \( \psi \). Therefore, all the weak inequalities must be equalities and we must have \( \phi(X'', k) = \phi(Y, k) = \phi(Y, p) = \phi(X, q) = \phi(X'', q) \).

Since \( \phi(X', k) = \phi(X'', p) \), we have \( \psi(X') = \psi(X) \). Moreover, by our choice of \( p \) we have \( p < k \) and by assumption, we have \( q < p \). Combining the two, this gives us \( q < k \). Therefore, \( X' \) lexicographically dominates \( X \) — a contradiction to Theorem 3.1.

Let us now move on to the remaining two possible cases

(iii) \( \phi(X, q) < \phi(Y, p) \),

(iv) \( \phi(X, q) = \phi(Y, p) \) and \( q > p \)

Recall that \( X \) and \( Y \) are non-redundant with respect to the new valuation functions \( v' \). If any of the above two conditions occur, then invoking Lemma 2.3 with allocations \( Y, X \) and the agent \( p \), there exists a transfer path from \( p \) to some agent \( l \) in the exchange graph of \( Y \) where \( |Y_l| > |X_l| \). Transferring along the shortest such path gives us a non-redundant allocation \( Y' \) (according to the valuation \( v' \)) where \( |Y'_p| = |Y_p| + 1 \) and \( |Y'_l| = |Y_l| - 1 \) (Lemma 2.3).

Let \( Y'' \) be an allocation obtained from starting at \( Y \) and removing a good from \( l \). From (G1), we have that if \( \phi(Y'', l) \leq \phi(Y'', p) \), we have \( \psi(Y) \leq \psi(Y') \) (with respect to the valuations \( v' \)) with equality holding if and only if \( \phi(Y'', l) = \phi(Y'', p) \).

If \( l = 0 \), we improve USW (according to the valuations \( v' \)) contradicting the fact that \( Y \) is leximin.

**For case (iii):** If \( l \neq 0 \), we have \( \phi(Y'', l) \leq \phi(X, l) \leq \phi(X, q) < \phi(Y, p) = \phi(Y'', p) \). Compressing the inequality, we get that \( \phi(Y'', l) < \phi(Y'', p) \); a contradiction.

**For case (iv):** If \( l \neq 0 \), we have \( \phi(Y'', l) \leq \phi(X, l) \leq \phi(X, q) = \phi(Y, p) = \phi(Y'', p) \). If any of these weak inequalities are strict, we can use analysis similar to that of case (iii) to show that \( Y \) does not maximize \( \psi \). Therefore, all the weak inequalities must be equalities and we must have \( \phi(Y'', l) = \phi(X, l) = \phi(X, q) = \phi(Y, p) = \phi(Y'', p) \). This implies that \( \psi(Y') = \psi(Y) \).

Moreover, by our choice of \( q \) we have \( q < l \) and by assumption, we have \( p < q \). Combining the two, this gives us \( p < l \). Therefore, \( X' \) lexicographically dominates \( X \) — a contradiction to Theorem 3.1.

Since cases (i)–(iv) cover all possible cases, our proof is complete.

We are now ready to show strategyproofness.

**Theorem 4.3.** When agents have matroid rank valuations, the Yankee Swap mechanism is strategyproof.

**Proof.** Assume an agent \( i \) reports \( v_i' \) instead of their true valuation \( v_i \) to generate the allocation \( X' \) via the Yankee Swap mechanism. Let \( X \) be the allocation generated by the mechanism had they
reported their true valuation \( v_i \). We need to show that \( v_i(X_i) \geq v_i(X'_i) \). We can assume w.l.o.g. that \( v'_i \) is an MRF; otherwise, \( i \) gets nothing and \( v_i(X'_i) = 0 \).

Let \( B \) be a subset of \( X'_i \) such that \( |B| = v_i(B) = v_i(X'_i) \). Let us also find a set \( S \) such that by construction we have \( v_i(S) \geq v_i(S \cap B) = |S \cap B| = f_B(S) \) for all \( S \) such that \( S \cap B \neq \emptyset \) and \( v_i(S) \geq 0 = f_B(S) \) otherwise. Since \( v_i(B) = v_i(X'_i) \), the proof is complete.

5 Applying General Yankee Swap

In this section, we show how Yankee Swap can be applied to optimize commonly used fairness metrics. The key challenge here is to encode fairness properties using a valid fairness metric \( \psi \) such that a straightforward gain function \( \phi \) exists. This section also showcases how simple the problem of optimizing fairness objectives becomes when using Yankee Swap.

While we do not prove this explicitly, note that in each case \( b + T_\phi(n, m) = O(1) \).

5.1 Prioritized Lorenz Dominating Allocations

As a sanity check, we first show how General Yankee Swap computes prioritized Lorenz dominating allocations.

An allocation \( X \) is Lorenz dominating if for all allocations \( Y \) and for any \( k \in [n] \), the sum of the utilities of the \( k \) agents with least utility in \( X \) is greater than the sum of the utilities of the \( k \) agents with least utility in \( Y \). An allocation \( X \) is leximin if it maximizes the lowest utility and subject to that; maximizes the second lowest utility and so on.

Both these metrics can be formalized using the sorted utility vector. The sorted utility vector of an allocation \( X \) (denoted by \( s^X \)) is defined as the utility vector \( u^X \) sorted in ascending order. An allocation \( X \) is Lorenz dominating if for all allocations \( Y \) and all \( k \in [n] \), we have \( \sum_{j \in [k]} s_j^X \geq \sum_{j \in [k]} s_j^Y \). An allocation \( X \) is leximin if the sorted utility vector of \( X \) lexicographically dominates all other sorted utility vectors. A Lorenz dominating allocation is not guaranteed to exist, but when it does, it is equivalent to a leximin allocation (which is guaranteed to exist). This result holds for arbitrary valuation functions.

**Lemma 5.1.** When a Lorenz dominating allocation exists, an allocation is leximin if and only if it is Lorenz dominating.

**Proof.** Let \( Y \) be any Lorenz dominating allocation and let \( X \) be any leximin allocation. Assume for contradiction that they do not have the same sorted utility vector. Let \( k \) be the lowest index such that \( s_k^X \neq s_k^Y \). If \( s_k^X < s_k^Y \), then \( Y \) lexicographically dominates \( X \) contradicting the fact that \( X \) is leximin. If \( s_k^X > s_k^Y \), then \( Y \) is not Lorenz dominating. Since \( s_k^X = s_k^Y \) for all \( k \), both allocations have the same sorted utility vector. This implies that \( X \) is Lorenz dominating and \( Y \) is leximin. \( \square \)

Babaioff et al. [2021a] introduce and study the concept of prioritized Lorenz dominating allocations. Each agent is given a priority which is represented using a permutation \( \pi : [n] \rightarrow [n] \). When agents have MRF valuations \( \{v_i\}_{i \in N} \), prioritized Lorenz dominating allocations are defined as Lorenz dominating allocations for the fair allocation instance where valuations are defined as \( v'_i(S) = v_i(S) + \frac{\pi(i)}{m} \). Babaioff et al. [2021a] show that when agents have MRF valuations, a prioritized Lorenz dominating allocation is guaranteed to exist and satisfies several desirable fairness properties such as leximin, envy freeness up to any good (EFX) and maximizing Nash welfare.

Using Lemma 5.1 these prioritized Lorenz dominating allocations are equivalent to leximin allocations with respect to the valuations \( v' \). Given this, we have an obvious choice of fairness
objective $\psi$. We define $\psi(X)$ as the sorted utility vector of $X$ with respect to the valuations $v'$. $\psi(X) > \psi(Y)$ if $\psi(X)$ lexicographically dominates $\psi(Y)$. Any leximin allocation with respect to the valuations $v'$ maximizes $\psi(X)$. As Benabou et al. [2021] show, leximin allocations are also Pareto optimal, which implies that $\psi$ satisfies (C1) (Pareto Dominance).

For (C2), we define the gain function $\phi(X, i) = (-|X_i|, -\pi(i))$ for any non-redundant allocation $X$ and agent $i$.

**Lemma 5.2.** The function $\phi(X, i) = (-|X_i|, -\pi(i))$ is a valid gain function for the fairness objective $\psi(X) = \bar{s}^X$ with respect to the valuations $v'$.

**Proof.** This function trivially satisfies (G2), as the term $-|X_i|$ is a decreasing function of $|X_i|$. Let us next show that $\phi$ satisfies (G1). Let $X$ be a non-redundant allocation. Let $Y$ be the non-redundant allocation resulting from giving a good $g$ to $i$ under $X$, such that $\Delta(X, i) = 1$. Let $Z$ be the non-redundant allocation resulting from giving a good $g$ to $j$ under $X$ such that $\Delta(X, j) = 1$. We assume that $\phi(X, i) > \phi(X, j)$, and show that $\psi(Y) \geq \psi(Z)$. Note that since $\pi(i) \neq \pi(j)$, $\phi(X, i)$ can never equal $\phi(X, j)$.

To show that $\phi$ satisfies (G1), we need the following Lemma.

**Lemma 5.3.** Let $r$ be any positive real valued number. Let $Y$ and $Z$ be two non-redundant allocations such that at most two agents $(j, k \in N)$ receive different utilities in the two allocations. If $\min\{v_j(Z_j), v_k(Z_k)\} > r$ and $\min\{v_j(Y_j), v_k(Y_k)\} \leq r$, then $\bar{s}^Z$ lexicographically dominates $\bar{s}^Y$.

**Proof.** Since all the other elements of the sorted utility vector have the same value, it suffices to only compare the sorted utility vectors restricted to the agents $i$ and $j$. In other words, to decide lexicographic dominance from giving a good $g$ to $i$ under $X$, such that $\Delta(X, i) = 1$. Let $Z$ be the non-redundant allocation resulting from giving a good $g$ to $j$ under $X$ such that $\Delta(X, j) = 1$. We assume that $\phi(X, i) > \phi(X, j)$, and show that $\psi(Y) \geq \psi(Z)$. Note that since $\pi(i) \neq \pi(j)$, $\phi(X, i)$ can never equal $\phi(X, j)$.

If $\phi(X, i) > \phi(X, j)$ one of the following two cases must be true.

**Case 1:** $|X_i| < |X_j|$. If this is true, we have $v_j'(Y_j) = |X_j| + \frac{\pi(j)}{n} > |X_i| + \frac{\pi(i)}{n} = v_i'(X_i) = v_i'(Z_i)$ since $\frac{\pi(i)}{n} - \frac{\pi(j)}{n} < 1 \leq |X_j| - |X_i|$. Therefore, invoking Lemma 5.2 with allocations $Y$ and $Z$ and $r = v_i'(X_i)$, we get that $\bar{s}^Y$ lexicographically dominates $\bar{s}^X$ i.e. $\psi(Y) > \psi(Z)$.

**Case 2:** $|X_i| = |X_j|$ and $\pi(i) < \pi(j)$. If this is true, we have $v_j'(Y_j) = |X_j| + \frac{\pi(i)}{n} > |X_i| + \frac{\pi(i)}{n} = v_i'(X_i) = v_i'(Z_i)$ by assumption. Therefore, invoking Lemma 5.3 with allocations $Y$ and $Z$ and $r = v_i'(X_i)$, we get that $\bar{s}^X$ lexicographically dominates $\bar{s}^Y$ i.e. $\psi(Y) > \psi(Z)$.

Invoking Lemma 5.2 we immediately obtain the following claim.

**Theorem 5.4.** When agents have MRF valuations, General Yankee Swap with $\phi(X, i) = (-|X_i|, -\pi(i))$ computes prioritized Lorenz dominating allocations with respect to priority $\pi$.

As a sanity check, this is the exact tie breaking scheme used by Viswanathan and Zick [2022] in the original Yankee Swap to compute prioritized Lorenz dominating allocations.

### 5.2 Weighted Leximin Allocations

Let us next consider the case where agents have entitlements. When each agent $i$ has a positive weight $w_i$, the weighted utility of an agent $i$ is defined as $\frac{v_i(X_i)}{w_i}$. A weighted leximin allocation maximizes the least weighted utility and subject to that, maximizes the second least weighted utility and so on.

More formally, we define the weighted sorted utility vector of an allocation $X$ (denoted by $\bar{e}^X$) as $\left(\frac{v_1(X_1)}{w_1}, \frac{v_2(X_2)}{w_2}, \ldots, \frac{v_n(X_n)}{w_n}\right)$ sorted in ascending order. An allocation $X$ is weighted leximin if
for no other allocation $Y$, $\vec{e}^X$ lexicographically dominates $\vec{e}^Y$. For the weighted leximin objective, we use $\psi(X) = e^X$, $\psi(X) > \psi(Y)$ if $e^X$ lexicographically dominates $e^Y$. It is easy to see that the allocation that maximizes $\psi$ is weighted leximin. It is also easy to see that weighted leximin trivially satisfies Pareto dominance (C1): any Pareto improvement increases the utility of one agent, while not decreasing the utility of any other agent, resulting in a strictly dominating sorted weighted utility vector.

For (C2), we define the gain function as $\phi(X, i) = (\frac{|X_i|}{w_i}, -w_i)$ for any non-redundant allocation $X$.

**Lemma 5.5.** The function $\phi(X, i) = (\frac{|X_i|}{w_i}, -w_i)$ is a valid gain function for the weighted leximin fairness objective.

**Proof.** $\phi(X, i)$ clearly satisfies (G2). We use a proof technique similar to Lemma 5.2 to show that $\phi$ is a valid gain function.

**Lemma 5.6.** Let $r$ be any positive real valued number. Let $Y$ and $Z$ be two non-redundant allocations such that at most two agents $j, k \in N$ receive different utilities in the two allocations. If $\min\{\frac{v_j(Z_j)}{w_j}, \frac{v_k(Z_k)}{w_k}\} > r$ and $\min\{\frac{v_j(Y_j)}{w_j}, \frac{v_k(Y_k)}{w_k}\} \leq r$, then $e^Z$ lexicographically dominates $e^Y$.

**Proof.** Since all the other elements of the weighted sorted utility vector have the same value, it suffices to only compare the sorted utility vectors restricted to the agents $i$ and $j$. In other words, to decide lexicographic dominance, we need to only compare the sorted version of $\{\frac{v_j(Z_j)}{w_j}, \frac{v_k(Z_k)}{w_k}\}$ and $\{\frac{v_j(Y_j)}{w_j}, \frac{v_k(Y_k)}{w_k}\}$. By assumption, we have $\min\{\frac{v_j(Z_j)}{w_j}, \frac{v_k(Z_k)}{w_k}\} > \min\{\frac{v_j(Y_j)}{w_j}, \frac{v_k(Y_k)}{w_k}\}$. Therefore $e^Z$ lexicographically dominates $e^Y$.

Assume $\phi(X, i) > \phi(X, j)$ for some agents $i, j \in N$ and non-redundant allocation $X$. Let $Y$ be the allocation that results from adding one unit of utility to $i$ in $X$ and let $Z$ be the allocation that results from adding one unit of utility to $j$ in $X$.

If $\phi(X, i) > \phi(X, j)$, then one of the following two cases must be true.

**Case 1:** $\frac{|X_i|}{w_i} < \frac{|X_j|}{w_j}$. Invoking Lemma 5.3 with allocations $Z$ and $Y$ and $r = \frac{|X_i|}{w_i}$, we get that $\psi(Y) > \psi(Z)$.

**Case 2:** $\frac{|X_i|}{w_i} = \frac{|X_j|}{w_j}$ and $w_i < w_j$. If this is true, we have $\frac{|Y_j|}{w_j} = \frac{|Z_j|}{w_j}$ by assumption. However, $\frac{|Y_i|}{w_i} = \frac{|X_i| + 1}{w_i} > \frac{|X_j| + 1}{w_j} = \frac{|Z_j|}{w_j}$. Since the two allocations differ only in the utilities allocated to $j$ and $i$, this implies $\psi(Y) > \psi(Z)$.

This implies that when $\phi(X, i) > \phi(X, j)$, we have $\psi(Y) > \psi(Z)$ as required. When $\phi(X, i) = \phi(X, j)$, we must have $\frac{|X_i|}{w_i} = \frac{|X_j|}{w_j}$ and $w_i = w_j$. This gives us $\frac{|Y_j|}{w_j} = \frac{|Z_j|}{w_j}$ and $\frac{|Y_i|}{w_i} = \frac{|X_i| + 1}{w_i}$ which implies that both $Y$ and $Z$ have the same weighted sorted utility vector. This implies that $\psi(Y) = \psi(Z)$.

We conclude that $\phi$ satisfies (G1) as well.

Since the weighted leximin justice criterion admits a valid gain function, we obtain the following theorem.

**Theorem 5.7.** When agents have MRF valuations and each agent $i$ has a weight $w_i$, General Yankee Swap with $\phi(X, i) = (\frac{|X_i|}{w_i}, -w_i)$ computes a weighted leximin allocation.

### 5.3 Individual Fair Share Allocations

We now turn to justice criteria that guarantee each agent a minimum fair share amount. One such popular notion is the maxmin share. An agent’s maxmin share [Budish, 2011] is defined as the utility an agent would receive if they divided the set of goods into $n$ bundles themselves and picked
the worst bundle. More formally, the maxmin share of an agent $i$ (denoted by $\text{MMS}_i$) is defined as $\text{MMS}_i = \max_{X=(X_1, X_2, \ldots, X_n)} \min_{e \in [n]} c_i(X_e)$. There are several other generalizations of the maxmin share popular in the literature [Farhadi et al., 2019, Babaioff et al., 2021b]. All of these metrics have the same objective — each agent $i$ has an instance dependent fair share $c_i \geq 0$ and the goal is to compute allocations that guarantee each agent a high fraction of their share [Procaccia and Wang, 2014, Ghodsi et al., 2018].

We define the fair share fraction of an agent $i$ in an allocation $X$ as $\frac{v_i(X_i)}{c_i}$ when $c_i > 0$ and 0 when $c_i = 0$. When $c_i = 0$, any bundle of goods (even the empty bundle) will provide $i$ its fair share; therefore, agents $i$ with $c_i = 0$ can be ignored when allocating bundles. When agents have matroid rank valuations, Yankee Swap can be used to maximize the lowest fair share fraction received by an agent and subject to that, maximize the second lowest fair share fraction and so on. Using a proof very similar to Theorem 5.7, we can show that the appropriate $\phi(X,i)$ to achieve such a fairness objective is defined as follows:

$$
\phi(X,i) = \begin{cases} 
\left(-\frac{|X_i|}{c_i}, -c_i\right) & c_i > 0 \\
(-M, -c_i) & c_i = 0 
\end{cases} 
$$

(1)

where $M$ is a large positive number greater than any possible $|X_i|/c_i$. This can be seen as setting the weight of each agent $i$ to their share $c_i$ and computing a weighted leximin allocation. The only minor change we make is take into account cases where $c_i$ could potentially be 0: in such a case we make $\phi(X, i)$ the lowest possible value it can take so we do not allocate any goods to these agents. The only time Yankee Swap will allocate goods to these agents is when all the other agents with positive shares do not derive a positive marginal gain from any of the remaining unallocated goods. More formally, we have the following Theorem whose proof is omitted due its similarity to Theorem 5.7.

**Theorem 5.8.** When agents have MRF valuations and every agent $i$ has a fair share $c_i$, then General Yankee Swap run with $\phi(X, i)$ given by (1) computes an allocation which maximizes the lowest fair share fraction received by an agent and subject to that, the second lowest fair share fraction and so on.

With Theorem 5.8, it is no longer necessary to find a fair share fraction that can be guaranteed to all agents and then design an algorithm which allocates each agent at least this fraction of their fair share. General Yankee Swap will automatically compute an allocation which maximizes the lowest fair share fraction received by any agent. A straightforward corollary is that, when there exists an allocation that guarantees each agent their fair share, Yankee Swap outputs one such allocation. Barman and Verma [2021] show that when agents have MRF valuations, an allocation which guarantees each agent their maxmin share always exists. Using their result with Theorem 5.8, we have the following Corollary.

**Corollary 5.9.** When agents have MRF valuations and every agent has $c_i = \text{MMS}_i$, General Yankee Swap run with $\phi(X, i)$ given by (1) computes a MAX-USW allocation which gives each agent their maxmin share.

Barman and Verma [2021] also present a polynomial time algorithm to compute the maxmin share of each agent. Since the procedure to compute maxmin shares is not necessarily strategyproof, the overall procedure of computing a maxmin share allocation using Yankee Swap may not be strategyproof either.

### 5.4 Max Weighted Nash Welfare Allocations

We continue to assume that each agent $i$ has a weight $w_i > 0$. For any allocation $X$, let $P_X$ be the set of agents who receive a positive utility under $X$. An allocation $X$ is said to be max weighted Nash welfare if
welfare (denoted by $\mathbb{WW}$) if it first minimizes the number of agents who receive a utility of zero; subject to this, $X$ maximizes $\prod_{i \in P_X} v_i(X_i)^{w_i}$, Chakraborty et al. [2021a] Suksompong and Teh [2022].

Note that the exponent is necessary since simply using the product of the weighted utilities of each agent is equivalent to the Nash welfare with equal entitlements when assuming that all agents receive a positive utility. This is because any allocation that maximizes $\prod_{i \in N} v_i(X_i)$ also maximizes $\prod_{i \in N} \frac{v_i(X_i)}{w_i}$.

The function trivially satisfies (G2).

This implies that they are equivalent notions of fairness.

Example 5.12. Consider an example with two agents $N = \{1, 2\}$ and six goods: $w_1 = 2$ and $w_2 = 8$. Both agents have additive valuations, and value all items at 1. Any weighted leximin allocation allocates two goods to agent 1 and four goods to agent 2, with a weighted sorted utility vector of $(\frac{1}{2}, 1)$. However, any max weighted Nash welfare allocation allocates one good to agent 1 and five goods to agent 2. This allocation has a worse weighted sorted utility vector $(\frac{4}{2}, \frac{5}{8})$, but a higher weighted Nash welfare.

Lemma 5.10. $\phi(X, i)$ given by (2) is a valid gain function when $\psi(X) = (|P_X|, \prod_{i \in P_X} v_i(X_i)^{w_i})$.

Proof. This function trivially satisfies (G2).

To show that $\phi$ satisfies (G1), some minor case work is required. Let $X$ be a non-redundant allocation and $i, j \in N$ be two agents. Let $Y$ be the allocation resulting from adding one unit of utility to $i$ in $X$ and similarly, let $Z$ be the allocation resulting from adding one unit of utility to $j$ in $Y$. We need to show that if $\phi(X, i) < \phi(X, j)$ then $\psi(Y) < \psi(Z)$ and if $\phi(X, i) = \phi(X, j)$, then $\psi(Y) = \psi(Z)$.

Case 1: $|X_i| = |X_j| = 0$. In this case, it is easy to see that both $\phi(X, i) = \phi(X, j)$ and $\psi(Y) = \psi(Z)$.

Case 2: $|X_i| > |X_j| = 0$. Then, by construction $\phi(X, j) > \phi(X, i)$. We also have $\psi(Z) > \psi(Y)$ since $|P_Z| > |P_Y|$. Note that this argument also covers the case where $|X_j| > |X_i| = 0$.

Case 3: $|X_i| > 0$ and $|X_j| > 0$. In this case, note that

$$\phi(X, i) = \frac{|X_i| + 1}{|X_i|^{w_i}} = \frac{|Y_i|^{w_i}}{|X_i|^{w_i}} = \prod_{h \in P_X} |Y_h|^{w_h} \prod_{h \in P_X} |X_h|^{w_h}$$

since $|P_Y| = |P_X| > 0$. Similarly, we have $\phi(X, j) = \prod_{h \in P_X} |Z_h|^{w_h}$. Therefore, since $|P_Y| = |P_Z| > 0$, we have $\phi(X, i) > \phi(Y, i)$ if and only if $\psi(Y) > \psi(Z)$. Similarly, we have $\phi(X, i) = \phi(Y, i)$ if and only if $\psi(Y) = \psi(Z)$.

The above analysis shows that $\phi$ satisfies (G1) as well.

Invoking Lemma 5.10 we have the following Theorem.

Theorem 5.11. When agents have MRF valuations and each agent $i$ has a weight $w_i$, General Yankee Swap with $\phi$ given by (2) computes a max weighted Nash welfare allocation.
6 Limitations

The previous section describes several fairness objectives for which Yankee Swap works. This raises the natural question: *is there any reasonably fairness notion where Yankee Swap does not work?*

The major limitation of Yankee Swap is that it cannot be used to achieve envy based fairness properties. An allocation \( X \) is said to be envy free if \( v_i(X_i) > v_i(X_j) \) for all \( i, j \in N \). Indeed, this is not always possible to achieve. This impossibility has resulted in several relaxations like envy free up to one good (EF1) \cite{Lipton2004, Budish2011} and envy free up to any good (EFX) \cite{Caragiannis2016, Plaut2017}. However, at its core, these envy relaxations are still fairness objectives that violate the pareto dominance property (C1): by increasing the utility of an agent currently being envied by other agents, we decrease the fairness of the allocation while pareto dominating the allocation. One work around for this is using Yankee Swap to compute leximin allocations and hope that leximin allocations have good envy guarantees. This works when all the agents have equal weights — prioritized Lorenz dominating allocations are guaranteed to be EFX. However, when agents have different weights, Yankee Swap fails to compute weighted envy free up to one good (WEF1) allocations \cite{Chakraborty2021}. For clarity, an allocation \( X \) is WEF1 when for all \( i, j \in N \) \( \frac{v_i(X_i)}{w_i} > \frac{v_i(X_j - g)}{w_j} \) for some \( g \in X_j \). Yankee Swap fails mainly due to the fact that when agents have MRF valuations, it may be the case that no \text{MAX-USW} allocation is WEF1. Therefore, irrespective of the choice of \( \phi \), Yankee Swap cannot always compute a WEF1 allocation (Proposition 3.9). This is illustrated in the following example.

**Example 6.1.** Consider an example with two agents \( \{1, 2\} \) and four goods \( \{g_1, g_2, g_3, g_4\} \). We have \( w_1 = 10 \) and \( w_2 = 1 \). The valuation function of every agent \( i \) is the MRF \( v_i(S) = \min\{|S|, 2\} \). Any \text{MAX-USW} allocation will assign two goods to both agents. However, no such allocation is WEF1: agent 1 (weighted) envies agent 2. This is because agent 1 receives a weighted utility of \( \frac{2}{10} \) but agent 2 receives a weighted utility of \( \frac{2}{1} \). Even after dropping any good, agent 2’s weighted utility (as seen by both agents) will be 1.

There does exist a WEF1 allocation in this example but achieving WEF1 comes at an unreasonable cost of welfare. To achieve WEF1, we must allocate 3 goods to agent 1. However, the third good offers no value to agent 1. In effect, we are obligated to “burn” an item in order to satisfy WEF1.

This example seems to suggest that WEF1 is not a suitable envy relaxation for MRF valuations. We leave the question of creating suitable envy relaxations to future work.

7 Conclusion and Future Work

In this work, we study the fair division of goods when agents have MRF valuations. Our main contribution is a flexible framework that can be used to optimize several fairness objectives. The General Yankee Swap framework is fast, strategyproof and always maximizes utilitarian social welfare.

The General Yankee Swap framework is a strong theoretical tool. We believe it has several applications outside the ones described in Section 5. This is a very promising direction for future work. One specific example is the computation of weighted maxmin share allocations. Theorem 5.8 shows that to compute an allocation which gives each agent the maximum fraction of their fair share possible, it suffices to simply input these fair shares into Yankee Swap. Therefore, the problem of computing fair share allocations have effectively been reduced to the problem of computing fair shares when agents have MRF valuations. It would be very interesting to see the different kinds of weighted maxmin shares that can be computed and used by Yankee Swap.

Another interesting direction is to show the tightness of this framework. It is unclear whether the conditions (C1) and (C2) are absolutely necessary for Yankee Swap to work. If they are not, it would be interesting to see how they can be relaxed to allow for the optimization of a larger class of fairness properties.
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