I. INTRODUCTION

Relativistic hydrodynamical models have been applied to studies of matter that is produced in high-energy hadron or nuclear collisions. Fluid dynamical descriptions, in particular, provide a simple picture of the space-time evolution of the hot/dense matter produced by ultra-relativistic heavy-ion collisions at RHIC and LHC [1, 2]. It is expected that this simple picture makes it possible to investigate the strongly interacting quark and gluon matter present at the initial stage of the collisions. The fluid model assumes that there exist local thermal quantities of the matter, and the pressure gradients of the matter cause collective phenomena [3]. These expected phenomena have been successfully observed as an elliptic flow coefficient $v_2$ in the CERN SPS experiment NA49 [4], at RHIC experiment [5, 6] and recent ALICE experiments in LHC [7] including the higher order flow harmonic $v_3$ for $(n = 3, 4)$. These have been observed as a function of various characteristics, including the transverse momentum $p_T$ or rapidity $y$ and so on. Hence, the hydrodynamical model has been widely accepted because of such experimental evidences. To investigate the properties of the quark and gluon matter created during such ultra-relativistic heavy-ion collisions more precisely, it is necessary to consider the effects of the viscosities and corresponding dissipation [8]. These effects are introduced into the hydrodynamic simulations and a detailed comparison between simulation and experimental data is made (see, for example, Ref. [9]). However, as several authors have noted, dissipative hydrodynamics is not yet completely understood and there are issues associated to the determination of the hydrodynamical flow [10–12] (see also, Ref. [13]). In this article, the Landau matching (fitting) condition that is necessary to specify the initial conditions with dissipative fluid dynamics is discussed. This may be related to the issue of defining the local rest frame [14].

The fundamental equations of relativistic fluid dynamics are defined by the conservation laws of energy-momentum and the charge current vector (in this paper, I assume the net baryon density as the conserved charge),

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\mu N^\nu(x) = 0.$$  

(1a) \hspace{1cm} (1b)

Here, $T^{\mu\nu}$ and $N^\nu$ are respectively the energy-momentum tensor and the conserved charge current at a given point in space-time $x$, which can be obtained by a coarse-graining procedure [10] with some finite size (fluid cell size), $l_{\text{fluid}}$. Hence, the fluid dynamical model expressed as a coarse-graining theory describing macroscopic phenomena can be derived from the underlying kinetic theory. In the case of a perfect fluid limit, the microscopic collision time scale $\tau_{\text{micro}}$ is much shorter than the macroscopic evolution time scale $\tau_{\text{macro}}$, thus

$$\tau_{\text{macro}} \gg \tau_{\text{micro}}.$$  

(2)

If the condition in eq. (2) is satisfied, the distribution function instantaneously relaxes to its local equilibrium form. In the local rest frame of the fluid, i.e., the frame in which the fluid velocity is given by $u^\mu(x) = (1, 0, 0, 0)$, the local equilibrium distribution functions for particles and for anti-particles are respectively given (within the
Boltzmann approximation) as

$$f_{k0}(x) = \exp \left[ \frac{-k_\mu u^\mu(x) + \mu_0(x)}{T_0(x)} \right], \quad (3a)$$

$$\bar{f}_{k0}(x) = \exp \left[ \frac{-k_\mu u^\mu(x) - \mu_0(x)}{T_0(x)} \right], \quad (3b)$$

where $k^\mu$ is a four momentum vector of a particle or an anti-particle within a cube of the coarse-graining scale $l_{\text{fluid}}$ on a side (fluid cell). I decompose $k^\mu$ by using the local flow vector $u^\mu$, i.e., $k^\mu = (k^\mu u^\mu) + k^{(\mu)}$ where $k^\mu u^\mu \equiv E_k$ is a scalar and $k^{(\mu)} \equiv \Delta_k^\mu k^\nu$ is a vector, which is orthogonal to $u^\mu$. In the local rest frame, the scalar $E_k$ coincides with the zero-th component of the four vector energy of the classical particle, $k^0$. The projection tensor onto the 3-space that is orthogonal to the flow velocity is defined by $\Delta k^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ with $g^{\mu\nu}$ being the metric tensor, $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. $T_0(x)$ and $\mu_0(x)$ are the equilibrium temperature and chemical potential respectively.

When the distribution function is given by eq. (3), the energy-momentum tensor and the net baryon number current vector are defined as

$$T_{\text{eq}}^{\mu\nu} = \frac{d^4k}{(2\pi)^3k^0} \left[ f_{k0}(x) + \bar{f}_{k0}(x) \right] = \varepsilon_{\text{eq}} u^\mu u^\nu - P_{\text{eq}}(\varepsilon_{\text{eq}}, n_{\text{eq}}) \Delta k^{\mu\nu}, \quad (4a)$$

$$N_{\mu}^{\text{eq}} = \frac{d^4k}{(2\pi)^3k^0} \left[ f_{k0}(x) - \bar{f}_{k0}(x) \right] = n_{\text{eq}} u^\mu, \quad (4b)$$

respectively, where $\varepsilon_{\text{eq}}$ is the energy density in the local rest frame, $n_{\text{eq}}$ is the net baryon density in the local rest frame, and $P_{\text{eq}}$ is the pressure in the equilibrium state.

If the fluid expands very rapidly (i.e., the macroscopic evolution time scale $\tau_{\text{macro}}$ becomes shorter), especially for fluids produced by ultra-relativistic heavy-ion collisions, equation (2) may not be satisfied everywhere in the fluid during the early stages.

In such cases, microscopic processes cannot keep pace with the changes in local energy and baryon density; therefore, I can write the following equation:

$$\tau_{\text{micro}} \gtrsim \tau_{\text{macro}}. \quad (5)$$

Under the condition eq. (6), the distribution function in the local rest frame does not obey the local equilibrium form eq. (3) and hence I have

$$f_k(x) = f_{k0}[T(x), \mu(x)] + \delta f_k(x), \quad (6a)$$

$$\bar{f}_k(x) = \bar{f}_{k0}[T(x), \mu(x)] + \delta \bar{f}_k(x), \quad (6b)$$

where $\delta f_k$ and $\delta \bar{f}_k$ are the deviations from the corresponding equilibrium distribution functions. I define a temperature $T(x)$ and chemical potential $\mu(x)$ (hereafter the separation temperature and separation chemical potential, respectively) that is distinguished from the equilibrium temperature $T_0(x)$ and chemical potential $\mu_0(x)$ with the assumption that $\delta f_k$ and $\delta \bar{f}_k$ also contribute to the energy and baryon number density of the system. In the inside of the fluid cell, within scales less than the coarse-graining scale $l_{\text{fluid}}$, the extremely rapid expansion of matter causes tiny disturbances to the flow in addition to the local flow velocity field $u^\mu(x)$. Although such tiny disturbance in the flow vectors may be cancelled out by defining a new local rest frame (because of the randomness of such tiny disturbances), it also causes a disturbance in the distribution function $\delta f_k(x)$ and $\delta \bar{f}_k(x)$. For simplicity, I assume that the fluid considered is stable against such disturbances; otherwise, the disturbance would escalate and the fluid would eventually become turbulent. The space-time evolution of the disturbance flow is independent from that of main background flow even if heat is supplied by the main background flow. Since the disturbance flow has no mechanism for obtaining energy other than heat originated from the shear viscosity of the main background flow, it finally disappears within a finite time scale by converting kinetic energy to heat which is then dissipated by the fluid. Thus, the disturbances, $\delta f_k$ and $\delta \bar{f}_k$, do not belong to any equilibrium state. As the macroscopic evolution time scale $\tau_{\text{macro}}$ grows longer due to the expansion of matter and the pressure gradients of the fluid decrease, $\delta f_k$ and $\delta \bar{f}_k$ approaches to zero (an assumption of hydrodynamic stability), and the condition eq. (2) is restored and local equilibrium is achieved.

The question I now consider is how to find local separation temperature $T(x)$ and net chemical potential $\mu(x)$ in eq. (6) from a given off-equilibrium state characterized by

$$T^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^3k^0} \left[ f_k(x) + \bar{f}_k(x) \right], \quad (7a)$$

$$N^{\mu}(x) = \int \frac{d^4k}{(2\pi)^3k^0} \left[ f_k(x) - \bar{f}_k(x) \right]. \quad (7b)$$

Usually the local separation temperature $T(x)$ and chemical potential $\mu(x)$ in eq. (6) are determined by imposing the so-called Landau matching conditions [17 21]

$$\delta T^{\mu\nu} u_\mu u_\nu = 0, \quad (8a)$$

$$\delta N^{\mu} u_\mu = 0, \quad (8b)$$

where $\delta T^{\mu\nu} \equiv T^{\mu\nu} - T_{\text{eq}}^{\mu\nu}$ and $\delta N^{\mu} \equiv N^{\mu} - N_{\text{eq}}^{\mu}$ are the deviation of the energy-momentum tensor and the baryon charge current from the matching equilibrium state, respectively. In this procedure, it is necessary to select a Lorentz frame. There are usually two natural choices for the Lorentz frame, namely the Landau frame [18] and the Eckart frame [21]. If the Landau frame is employed, for example, the local flow velocity $u^\mu$ is determined by the eigenvector of the following eigenvalue equation,

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu, \quad (9a)$$

where the eigenvalue $\varepsilon$ is the energy density measured in the rest frame. In the Landau matching condition, shown in eq. (5a), the energy density should be matched to an energy density in an equilibrium state parameterized by a temperature $T$ and a chemical potential $\mu$, i.e., $\varepsilon =
\[ T = T(T_0, \mu_0; \delta f_k, \delta \bar{f}_k), \]

\[ \mu = \mu(T_0, \mu_0; \delta f_k, \delta \bar{f}_k), \]

with boundary conditions \( T \rightarrow T_0 \) and \( \mu \rightarrow \mu_0 \) when the disturbance flow disappears, \( \delta f_k, \delta \bar{f}_k \rightarrow 0 \). The boundary condition is the Landau matching condition.

In any off-equilibrium state, most literature using the Landau matching condition published to date assumes that \( T \equiv T_0 \) and \( \mu \equiv \mu_0 \). However, in this paper, I consider the problem where \( T \) and \( \mu \) are dependent on the strength of the off-equilibrium state, i.e., \( T \) and \( \mu \) are functions of \( \delta f_k \) and \( \delta \bar{f}_k \), as seen in eq. (12).

This article is organized as follows. In Section II, I obtain an expression for eq. (6) using the irreducible tensor expansion technique for the off-equilibrium distribution function, as recently employed by G.S. Denicol and his collaborators \[24\] under the general matching condition of eq. (11). I also apply the irreducible tensor expansion technique to an off-equilibrium fluid in which the Landau matching condition published to date assumes that \( T \equiv T_0 \) and \( \mu \equiv \mu_0 \). However, in this paper, I consider the problem where \( T \) and \( \mu \) are dependent on the strength of the off-equilibrium state, i.e., \( T \) and \( \mu \) are functions of \( \delta f_k \) and \( \delta \bar{f}_k \), as seen in eq. (12).

In the kinetic approach, since the energy-momentum tensor and the conserved charge current are defined by the second and first moment of the single particle distribution function, the Landau matching conditions eq. (8) are equivalent to the following expressions \[22\]

\[ \int \frac{d^3k}{(2\pi)^3k^0} \left[ \delta f_k(x) + \delta \bar{f}_k(x) \right] = 0, \]  

\[ \int \frac{d^3k}{(2\pi)^3k^0} \left[ \delta f_k(x) - \delta \bar{f}_k(x) \right] = 0. \]

These conditions strongly constrain the disturbances \( \delta f_k \) and \( \delta \bar{f}_k \). In fact, the separation temperature \( T \) and chemical potential \( \mu \) defined in eq. (6) under the condition eq. (10) are unrelated to the equilibrium values after relaxation. In order to restore the physical meaning of \( \delta f_k \) and \( \delta \bar{f}_k \), it is necessary to exclude the restrictions imposed by eq. (8) or eq. (10) and to generalize the equations \( \delta f_k \) and \( \delta \bar{f}_k \) the following form.

\[ \delta T^{\mu\nu}u_\mu u_\nu = \Lambda, \quad \delta n^{\mu}u_\mu = \delta n. \]

Here, \( \Lambda \) and \( \delta n \) can be considered as the energy density and net baryon number density of the disturbance (tiny turbulent) flow caused by rapid expansion. Except for particles constituting the disturbance flow, it is assumed that the remaining particles in the fluid cell approximately obey the local thermal distribution form \( f_{k0} \) or \( \bar{f}_{k0} \). The corresponding temperature and chemical potential are different from those obtained at the equilibrium limit, i.e., \( T(x) \neq T_0(x) \) and \( \mu(x) \neq \mu_0(x) \).

Therefore, to find the separate temperature \( T \) and chemical potential \( \mu \) in eq. (10), it is required to find \( \mu \) as functions of \( \delta f_k \) and \( \delta \bar{f}_k \).

II. SEPARATION EQUILIBRIUM PART FROM THE OFF-EQUILIBRIUM DISTRIBUTION FUNCTION

A. Expansion of the single particle distribution function by irreducible tensors

Considering a relativistically expanding fluid in which microscopic processes cannot keep pace with the quick macroscopic changes, as characterized by eq. (5) in Section I. To generalize eq. (6) further, I assume that such an off-equilibrium state can be expressed by a function of the deviations of \( \phi_k(x) \) and \( \bar{\phi}_k(x) \) as

\[ f_k(x) = f_{k0}(T[\phi_k, \bar{\phi}_k], \mu[\phi_k, \bar{\phi}_k]) (1 + \phi_k), \]  

\[ \bar{f}_k(x) = \bar{f}_{k0}(T[\phi_k, \bar{\phi}_k], \mu[\phi_k, \bar{\phi}_k]) (1 + \bar{\phi}_k), \]

where \( \phi_k(x) \equiv \delta f_k(x)/f_{k0}(x) \) and \( \bar{\phi}_k(x) \equiv \delta \bar{f}_k(x)/\bar{f}_{k0}(x) \) are deviations from the local thermal equilibrium function eq. (5). The deviations \( \phi_k(x) \) and \( \bar{\phi}_k(x) \) involve information about a given off-equilibrium state characterized not only by scalars such as \( \Lambda, \delta n, \)
and $\gamma \equiv \Pi/P_{eq}$ but also by vectors and tensors, for example, the heat flow vector $W^\mu$, shear tensor $\pi^{\mu\nu}$ of the dissipative fluid and so on. In order to expand $\phi_k$ and $\hat{\phi}_k$, it is necessary to use the orthogonal base of irreducible tensors [23, 24], \{1, $k_{(\mu)}$, $k_{(1)}$, $k_{(2)}$, $k_{(3)}$, \},

\[ F(E_k) = \sum_{l=0}^{\infty} a_{\nu}^{(l)}(E_k) \gamma, \quad (17) \]

where $F(E_k)$ is an arbitrary function of the energy $E_k$. Since the tensors defined in eq. (14) are orthogonal, I may expand the deviation $\phi_k$ and $\hat{\phi}_k$ as follows:

\[ \phi_k(x) = \sum_{l=0}^{\infty} \lambda^{(\mu_1\cdots\mu_l)} k_{(\mu_1)} \cdots k_{(\mu_l)}, \quad (16a) \]

\[ \hat{\phi}_k(x) = \sum_{l=0}^{\infty} \tilde{\lambda}^{(\mu_1\cdots\mu_l)} k_{(\mu_1)} \cdots k_{(\mu_l)}, \quad (16b) \]

respectively, where $\lambda^{(\mu_1\cdots\mu_l)}$ and $\tilde{\lambda}^{(\mu_1\cdots\mu_l)}$ are coefficient tensors of l-rank in the above expansions. Since the tensors in the above expanded equations have momentum dependence (denoted by the subscript $k$), I further expand these coefficients tensors according to a set of polynomial functions of the energy $E_k$ having a maximum order of $n$, \[ P_{kn}^{(l)} = \sum_{r=0}^{n} a_{\nu r}^{(l)}(E_k) \gamma. \quad (17) \]

Here, the coefficients $a_{\nu r}^{(l)}$ ($r = 0, 1, \ldots, n$) satisfy the following orthogonality relation:

\[ \int \frac{d^3k}{(2\pi)^3 k_0} \omega_k^{(l)} P_{km}^{(l)} \delta_{mn} = 0, \quad (18) \]

where $\omega_k^{(l)}$ is an $l$-dependent weight factor defined by

\[ \omega_k^{(l)} = \frac{\cal V}{(2l + 1)!!} [\Delta_{\alpha\beta} k_{(\alpha)} k_{(\beta)}] f_{uk}, \quad (19) \]

and $\cal V$ in eq. (19) is a normalization factor (See Appendix [E]). The orthogonal condition of eq. (18) gives a relation between the coefficients $a_{\nu r}$ and the integral as follows:

\[ I_r^{(l)} \equiv \int \frac{d^3k}{(2\pi)^3 k_0} \omega_k^{(l)}(E_k) \gamma, \quad (20) \]

where the irreducible tensor of the l-rank is defined by

\[ k_{(\mu_1) \cdots (\mu_l)} \equiv \Delta_{\mu_1 \cdots \mu_l}^{\nu_1 \cdots \nu_l} k_{(\nu_1)} \cdots k_{(\nu_l)} \gamma, \quad (14) \]

and the projection tensor $\Delta_{\mu_1 \cdots \mu_l}^{\nu_1 \cdots \nu_l}$ used in eq. (14) are defined in Appendix [A] (see also Appendix [B, C] and [D, 24]). These irreducible tensors $k_{(\mu_1) \cdots (\mu_l)}$ satisfy the following orthogonal condition (for derivation, see Appendix [D]),

\[ \int \frac{d^3k}{(2\pi)^3 k_0} F(E_k) \Delta_{\mu_1 \cdots \mu_l}^{\nu_1 \cdots \nu_l} \delta_{mn} = \frac{(2m + 1)!!}{2^m m!} \int \frac{d^3k}{(2\pi)^3 k_0} [\Delta_{\alpha\beta} k_{(\alpha)} k_{(\beta)}] m, \quad (15) \]

that is,

\[ \left( \begin{array}{cccc} 1 & I_1^{(l)} & \cdots & I_{n}^{(l)} \\ I_1^{(l)} & I_2^{(l)} & \cdots & I_{n+1}^{(l)} \\ \cdots & \cdots & \cdots & \cdots \\ I_{n-1}^{(l)} & I_n^{(l)} & \cdots & I_{2n-1}^{(l)} \\ I_n^{(l)} & I_{n+1}^{(l)} & \cdots & I_{2n}^{(l)} \end{array} \right) = \left( \begin{array}{cccc} a_{00}^{(l)} & a_{01}^{(l)} & \cdots & a_{0n}^{(l)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n0}^{(l)} & a_{n1}^{(l)} & \cdots & a_{nn}^{(l)} \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{a_{nn}} \end{array} \right). \quad (21) \]

I assume that the energy dependent tensors $\lambda^{(\mu_1\cdots\mu_l)}_k$ and $\tilde{\lambda}^{(\mu_1\cdots\mu_l)}_k$ can be written as products of the energy-independent tensors $c_{\nu_1 \cdots \nu_l}$, $\tilde{c}_{\nu_1 \cdots \nu_l}$, and a polynomial function of $E_k$, $P_{kn}^{(l)}$, as the following equation:

\[ \lambda^{(\mu_1\cdots\mu_l)}_k = \sum_{n=0}^{N_1} c_{\nu_1 \cdots \nu_l} P_{kn}^{(l)}, \quad (22a) \]

\[ \tilde{\lambda}^{(\mu_1\cdots\mu_l)}_k = \sum_{n=0}^{N_1} \tilde{c}_{\nu_1 \cdots \nu_l} P_{kn}^{(l)}. \quad (22b) \]

Although the order of expansion $N_1$ should, in principle, be equal to infinity, in practice one must truncate the expansion of the polynomial functions $P_{kn}^{(l)}$ with some finite number $N_1$. Note that, without loss of generality, one can set $P_{k0}^{(l)} = 1$ for arbitrary $l$, which is equivalent to $a_{00}^{(l)} = 1$. For anti-particles, the normalization factor $\tilde{\cal V}^{(l)}$ can be similarly defined (i.e., by replacement of $\mu \rightarrow -\mu$). Note that, the weight factor expressed in eq. (19) has no chemical potential dependence and I do not therefore need to introduce a factor $\tilde{\omega}_k^{(l)}$. Applying the two orthogonalities of eqs. (15) and (18) to the expression given in eqs. (16a) and (16b), I obtain

\[ c_{\nu_1 \cdots \nu_l}^{(\mu_1 \cdots \mu_l)} = \frac{\cal V}{l!} \int \frac{d^3k}{(2\pi)^3 k_0} f_{uk} P_{kn}^{(l)} k_{(\nu_1) \cdots k_{(\nu_l)}}, \quad (23a) \]

\[ \tilde{c}_{\nu_1 \cdots \nu_l}^{(\mu_1 \cdots \mu_l)} = \frac{\tilde{\cal V}}{l!} \int \frac{d^3k}{(2\pi)^3 k_0} f_{uk} P_{kn}^{(l)} \tilde{k}_{(\nu_1) \cdots \nu_l}, \quad (23b) \]
Substituting the definition of $P^{(l)}_{kn}$ given by eq.(17) into eq. (23) and denoting
\[
\rho^{\mu_1\ldots\mu_l} = \int \frac{d^3k}{(2\pi)^3} \phi_k f_{0k} (E_k)^r (k^{\mu_1} \ldots k^{\mu_l}),
\]
\[
\bar{\rho}^{\mu_1\ldots\mu_l} = \int \frac{d^3k}{(2\pi)^3} \phi_k f_{0k} (E_k)^r (k^{\nu_1} \ldots k^{\nu_l}),
\]
the following expressions are obtained
\[
c^{(\mu_1\ldots\mu_l)}_n = \sum_{m=0}^{n} a_{nm} \rho_m^{\mu_1\ldots\mu_l},
\]
\[
\bar{c}^{(\mu_1\ldots\mu_l)}_n = \sum_{m=0}^{n} b_{nm} \bar{\rho}_m^{\mu_1\ldots\mu_l},
\]
where $b_{nm}$ is a coefficient series expansion of $c^{(\mu_1\ldots\mu_l)}_n$. Note that the linear combination of eqs. (23a) and (24b)
yields $\rho_2 + \bar{\rho}_2 = u_\mu \delta^{\mu
u} u_\nu = \Lambda$, $\rho^I_1 + \bar{\rho}^I_1 = \Delta^I_\mu \delta^{\mu\nu} u_\nu = W_\mu$, $\rho_0^I + \bar{\rho}^I_0 = \delta T^{(\mu\nu)} = \pi^{\mu\nu}$, $\rho_1 - \bar{\rho}_1 = u_\mu \delta N^{\mu} = \delta n$ and $\rho_0^I - \bar{\rho}^I_0 = N^{(\mu)} = V^\mu$, where $W_\mu$, $\pi^{\mu\nu}$, and $V^\mu$ are the energy flow, shear tensor, and baryon number flow, respectively. As seen in eq. (24a), the $l$-rank coefficient tensors $c^{(\mu_1\ldots\mu_l)}_n$ are linear combinations of the $m$-th moment ($m \leq n$) of the energy $E_k$ with the irreducible tensor $\rho_m^{\mu_1\ldots\mu_l}$ ($\bar{\rho}_m^{\mu_1\ldots\mu_l}$). Hence, $\lambda^{(\mu_1\ldots\mu_l)}_k$ and $\bar{\lambda}^{(\mu_1\ldots\mu_l)}_k$ must be rewritten as linear combinations of $\rho_m^{\mu_1\ldots\mu_l}$ or $\bar{\rho}_m^{\mu_1\ldots\mu_l}$ using the expressions given in eq. (24b). Thus, I resubstitute eq. (24a) into the definition of $\lambda_k$ obtained from eq. (22a) to yield the following expression
\[
\lambda^{(\mu_1\ldots\mu_l)}_k = \frac{W^{(l)}}{l!} \left( \rho_0^{\mu_1\ldots\mu_l}, \rho_1^{\mu_1\ldots\mu_l}, \ldots, \rho_N^{\mu_1\ldots\mu_l} \right) \times \left( \begin{array}{ccc} a_{00} & \cdots & a_{0l} \\ a_{10} & 0 & \cdots \\ \vdots & \vdots & \vdots \\ a_{Nl} & \cdots & 0 \end{array} \right) \left( \begin{array}{ccc} a_{00} & 0 & \cdots \\ a_{l0} & a_{l1} & \cdots \\ \vdots & \vdots & \vdots \\ a_{lN} & \cdots & a_{lN} \end{array} \right) = \begin{pmatrix} E_k^0 \\ E_k^1 \\ \vdots \\ E_k^N \end{pmatrix},
\]
where I use eq. (17) in the following matrix form
\[
\begin{pmatrix} P^{(l)}_{k0} \\ P^{(l)}_{k1} \\ \vdots \\ P^{(l)}_{kn} \end{pmatrix} = \begin{pmatrix} a_{00} & 0 & \cdots \\ a_{l0} & a_{l1} & \cdots \\ \vdots & \vdots & \vdots \\ a_{lN} & \cdots & a_{lN} \end{pmatrix} \begin{pmatrix} E_k^0 \\ E_k^1 \\ \vdots \\ E_k^N \end{pmatrix}.
\]
These expressions given by eqs. (31a) and (31b) are consistent with the results obtained directly from eqs. (24a) and (24b) using the orthogonality condition for the irreducible tensors of eq. (15).

In principle, all ranks of tensors should be added \(k_{(\mu_1 \cdots k_{\mu_l})}\) for \((l = 1, 2, \cdots \infty)\) as defined in the irreducible expansion of eq. (14). However, I assume that the deviations \(\phi\) and \(\bar{\phi}\) can be adequately represented by including terms up to the second rank tensor, that is the heat and baryon number flow \((l = 1)\) and the shear tensor \((l = 2)\) added to the scalar component \(\Lambda\) and \(\delta n\) \((l = 0)\). Furthermore, I also assume that the energy dependence of the coefficient tensor \(\lambda_{(\mu_1 \cdots \mu_l)}\) and \(\bar{\lambda}_{(\mu_1 \cdots \mu_l)}\) can also be adequately expressed up to the first \((n = 1)\) and 0-th order \((n=0)\) polynomial function \(P^{(l)}(E_k)\) for \(l = 1\) and \(l = 2\), respectively. For \(l = 0\), however, the coefficient irreducible tensor corresponding to \(\Lambda\) and \(\delta n\) originate from the disturbance (tiny turbulent) flow and their energy dependence may be complicated. Let me assume that one can terminate at the second order of the energy \(E_k\) for \(l = 0\), and later (during discussion of the stability of the entropy current in off-equilibrium states) I will discuss this issue in detail.

Thus, if the expansion of the deviation of the distribution function is truncated at \(t = 2\), with \(N_0 = 2, N_1 = 1, N_2 = 0\), also known as the possible lowest scheme [24], then linear combinations of the r.h.s of eq. (31a) and eq. (31b), at \(l = 0\), gives

\[
\Lambda = \int \frac{d^3k}{(2\pi)^3 k_0} \left[ \lambda_k f_{k0}(x) + \bar{\lambda}_k \bar{f}_{k0}(x) \right], \tag{32a}
\]

\[
\delta n = \int \frac{d^3k}{(2\pi)^3 k_0} \left[ \lambda_k f_{k0}(x) - \bar{\lambda}_k \bar{f}_{k0}(x) \right], \tag{32b}
\]

and the bulk pressure is given by

\[
\Pi = \frac{1}{3} \int \frac{d^3k}{(2\pi)^3 k_0} \left[ \lambda_k f_{k0}(x) + \bar{\lambda}_k \bar{f}_{k0}(x) \right]. \tag{32c}
\]

For energy and net baryon number flow \((l = 1)\) the relevant expressions are

\[
W^{\mu} = -\frac{1}{3} \int \frac{d^3k}{(2\pi)^3 k_0} \left[ \lambda^{(\mu)}_k f_{k0}(x) + \bar{\lambda}^{(\mu)}_k \bar{f}_{k0}(x) \right], \tag{32d}
\]

\[
V^{\mu} = -\frac{1}{3} \int \frac{d^3k}{(2\pi)^3 k_0} \left[ \lambda^{(\mu)}_k f_{k0}(x) - \bar{\lambda}^{(\mu)}_k \bar{f}_{k0}(x) \right], \tag{32e}
\]

and the shear viscosity (for \(l = 2\) is)

\[
\pi^{\mu\nu} = \frac{2}{15} \int \frac{d^3k}{(2\pi)^3 k_0} \left( \lambda^{(\mu\nu)}_k f_{k0}(x) + \bar{\lambda}^{(\mu\nu)}_k \bar{f}_{k0}(x) \right), \tag{32f}
\]

where \(k^2 = k \cdot k = -k_a k_b \Delta^{ab}\). Eqs. (22a) and (22b) offer the key to understand the separation temperature \(T\) and the separation chemical potential \(\mu\) thus, I have

\[
\varepsilon = \varepsilon(T, \mu, \lambda_k, \bar{\lambda}_k) = \varepsilon_{eq}(T, \mu) + \Lambda, \tag{33a}
\]

\[
n = n(T, \mu, \lambda_k, \bar{\lambda}_k) = n_{eq}(T, \mu) + \delta n. \tag{33b}
\]

\(T\) and \(\mu\) are functions of \(\phi\) and \(\bar{\phi}\), respectively, as shown by eq. (13). However, \(T\) and \(\mu\) can only be expressed by the \(l = 0\) component, i.e., \(\lambda_k\) and \(\bar{\lambda}_k\). Hence the separation temperature \(T\) and the separation chemical potential \(\mu\) can be expressed as a function of \(\lambda_k(x)\) and \(\bar{\lambda}_k(x)\), respectively:

\[
T(x) = T[\lambda_k(x), \bar{\lambda}_k(x); T_0], \tag{34a}
\]

\[
\mu(x) = \mu[\lambda_k(x), \bar{\lambda}_k(x); \mu_0]. \tag{34b}
\]

### B. Entropy current of the non-equilibrium state and thermodynamic stability

When the microscopic phase-space distribution function \(f_k(x)\) is expressed by eq. (13), the local entropy current is divided into three parts as follows

\[
s_{\mu}^{eq}(x) = -\frac{1}{(2\pi)^3 k_0} \left[ f_k \ln f_k - f_k \right] = s_{\mu}^{eq}(x) + \delta s_{1}^{\mu}(x) + \delta s_{2}^{\mu}(x), \tag{35}
\]

where \(s_{\mu}^{eq}\) represents the equilibrium part given by the separation temperature \(T\) and chemical potential \(\mu\). The remaining two terms, \(\delta s_{1}^{\mu}\) and \(\delta s_{2}^{\mu}\), are given by

\[
\delta s_{1}^{\mu} \equiv -\int \frac{d^3k}{(2\pi)^3 k_0} \left[ [\phi_k \ln f_{k0}] f_{k0} + [\bar{\phi}_k \ln \bar{f}_{k0}] \bar{f}_{k0} \right], \tag{36a}
\]

\[
\delta s_{2}^{\mu} \equiv -\int \frac{d^3k}{(2\pi)^3 k_0} \left[ [(1 + \phi_k) \ln (1 + \phi_k) - \phi_k] f_{k0} + [(1 + \bar{\phi}_k) \ln (1 + \bar{\phi}_k) - \bar{\phi}_k] \bar{f}_{k0} \right], \tag{36b}
\]

respectively. Since the entropy density should be maximum and stable in the limit of equilibrium, it must not include any linear terms of scalar off-equilibrium quanti-
ties such as $\delta n$, $\Lambda$, and $\Pi$:
\[
\frac{\partial (\delta s_1^\mu u_\mu)}{\partial \delta n} \bigg|_{\delta n=0} = \frac{\partial (\delta s_1^\mu u_\mu)}{\partial \Lambda} \bigg|_{\Lambda=0} = \frac{\partial (\delta s_1^\mu u_\mu)}{\partial \Pi} \bigg|_{\Pi=0} = 0.
\]

(37)

Note that, the thermodynamic stability condition defined in eq.(37) must be satisfied not only approximately but exactly. Therefore, one must add all terms for $\lambda_k$ and $\lambda_k$ without termination in the energy polynomial function $P_n(\lambda_k)$ at a finite $n = N_l$ and for $l = 0$. Thus the first order correction of the off-equilibrium entropy current should be given as the following:
\[
\delta s_1^\mu = [-\alpha \delta n + \beta \Lambda + \xi]u^\mu + [\alpha V^\mu + \beta W^\mu + \xi^\mu].
\]

(38)

Here, $\alpha \equiv \mu/T$, $\beta \equiv 1/T$ and the terms $\xi$ and $\xi^\mu$ are residual terms which were ignored in the truncation of the polynomial $P_n(\lambda_k)$ at $n = N_l$;
\[
\xi^{\mu_1\cdots\mu_l} \equiv -\int d^3k E_k \left[ \alpha \delta \Psi^{(\mu_1\cdots\mu_l)} - \beta \delta \Psi^{(\mu_1\cdots\mu_l)} \right].
\]

(39)

where I denote
\[
\delta \Phi_k^{(\mu_1\cdots\mu_l)} \equiv \delta \chi_k^{(\mu_1\cdots\mu_l)} - f_{k0} - \delta \chi_k^{(\mu_1\cdots\mu_l)} - f_{k0}, \quad (40a)
\]
\[
\delta \Psi_k^{(\mu_1\cdots\mu_l)} \equiv \lambda_k^{(\mu_1\cdots\mu_l)} - \bar{f}_{k0} + \bar{\lambda}_k^{(\mu_1\cdots\mu_l)} - \bar{f}_{k0}, \quad (40b)
\]

and
\[
\delta \lambda_k^{(\nu_1\cdots\nu_l)} \equiv \sum_{n=N_l+1}^{\infty} e_n^{(\nu_1\cdots\nu_l)} P_{kn}, \quad (41a)
\]
\[
\delta \bar{\lambda}_k^{(\nu_1\cdots\nu_l)} \equiv \sum_{n=N_l+1}^{\infty} \bar{e}_n^{(\nu_1\cdots\nu_l)} P_{kn}, \quad (41b)
\]

In the usual formulation using the Landau matching conditions, the factor $[-\alpha \delta n + \beta \Lambda + \xi]u^\mu \equiv 0$ because $\delta n = 0$, $\Lambda = 0$, and $\xi = 0$. Therefore, in this case I can write
\[
\delta s_1^\mu u_\mu = 0,
\]

(42)

which prevents the occurrence of instability in the entropy current. (38) (See also Ref.[27]). However, in the case of eq.(11), the condition of eq.(42) is
\[
-\alpha \delta n + \beta \Lambda + \beta \chi \Pi \equiv 0,
\]

(43)

where $\chi \equiv \xi/(\beta \Pi)$ for the off-equilibrium state to be thermodynamically stable. I require that the condition defined by eq.(43) is satisfied for both particles and antiparticles because each of the subsystems should be independently stable;
\[
-\alpha \delta n_+ + \beta \Lambda_+ + \beta \chi_+ \Pi_+ = 0, \quad (44a)
\]
\[
+\alpha \delta n_- + \beta \Lambda_- + \beta \chi_- \Pi_- = 0. \quad (44b)
\]

Here suffixes $+$ and $-$ in eq.(43) denote the contributions from particles ($\lambda_k f_{k0}$ in eq.(32)) and antiparticles ($\bar{\lambda}_k \bar{f}_{k0}$ in eq.(32)). Then $\chi$ in the eq.(43) as
\[
\chi = \chi_+ \Pi_+ + \chi_- \Pi_-. \quad (44c)
\]

In the case of $\chi_+ > -3$ and $\chi_- < -3$, integration by parts for the first and the second terms in eq.(43), yield
\[
\delta n_+ = -\left( E_k f_\lambda \frac{\partial \lambda_k}{\partial k} \right)_0 + \frac{\Pi_+}{T}, \quad (45a)
\]
\[
\delta n_- = -\left( E_k f_\lambda \frac{\partial \bar{\lambda}_k}{\partial k} \right)_0 + \frac{\Pi_-}{T}, \quad (45b)
\]

respectively. Note that, eqs.(45a) and (45b) can be regarded as “Equations of State” for the off-equilibrium condition that correspond to $P_{eq} = n_{eq} T$ in the equilibrium state. Combining eq.(45) and eq.(43) gives a differential equation for $\lambda_k$ as
\[
\frac{\alpha E_k}{3 \beta} \left( \frac{d \lambda_k}{d k} \right) + \left( E_k^2 \frac{\alpha}{3} \frac{\alpha + \chi_+}{k} \frac{\lambda_k}{k} \right) = 0. \quad (46)
\]

The above differential equation can be solved by the following integration form
\[
\int \frac{d \lambda_k}{\lambda_k} = \beta [1 - \frac{\chi_+}{\alpha} \int \frac{k dk}{\sqrt{k^2 + m^2}} - \frac{3}{\alpha} \frac{\sqrt{k^2 + m^2}}{k} dk], \quad (47)
\]

thus I obtain
\[
\lambda_k = C_\gamma \left[ \frac{E^2_k + m^2}{k} \right]^{\frac{\alpha}{m}} \exp \left[ \left(1 - \frac{3 + \chi_+}{\alpha} \right) \beta E_k \right]. \quad (48a)
\]

For $\bar{\lambda}_k$, a similar expression can be obtained;
\[
\bar{\lambda}_k = C_\gamma \left[ \frac{E^2_k + m^2}{k} \right]^{-\frac{\alpha}{m}} \exp \left[ \left(1 + \frac{3 + \chi_-}{\alpha} \right) \beta E_k \right]. \quad (48b)
\]

The constant $C_\gamma$ in eq.(48) is an arbitrary integration constant. Physically, it determines the absolute values of off-equilibrium quantities such as $\Pi$. The value of $C_\gamma$ satisfies
\[
\Pi = \gamma \ P_{eq}(T, \mu). \quad (49)
\]

$\gamma$ indicates the strength of the off-equilibrium state. Hence, a given off equilibrium state can be specified by $T$, $\mu$, and $\gamma$. It is possible to write $\chi \equiv -3 + \delta \chi$ because an exchange $\alpha \leftrightarrow -\alpha$ in eq.(48) gives $\lambda_k \leftrightarrow \bar{\lambda}_k$. Then, using eqs.(44a) and (44b), I can express $\chi$ in eq.(43) in a more simple form
\[
\chi = -3 + \delta \chi \frac{\delta \Pi}{\Pi}, \quad (50)
\]

where $\delta \Pi \equiv (\Pi_+ - \Pi_-)$.

As I shall see in the Section III (Fig. 4), when $\gamma$ is small and the temperature $T$ is high enough, $\phi_k$ and $\phi_k$ satisfy $\phi_k \ll 1$ and $\phi_k \ll 1$, respectively. In this case, I can ignore the higher order $(n > 2)$ contributions of $\phi_n^a$ and $\phi_n^b$ to the entropy current, and thus I can approximate $\delta s_2^\mu$ as the following:
\[ \delta s_2^\mu \approx - \int \frac{d^3k}{(2\pi)^3k^0} \left[ 1 - \frac{1}{2} u_0 \frac{1}{k^0} \right] \left[ \frac{1}{2} \dot{\psi}_k \right] \left[ \frac{1}{2} \dot{\psi}_k \right] \]

where for \( n \geq m \)

\[ \psi_{\mu_1 \cdots \mu_m} \equiv \lambda_{k_1}^{(\mu_1 \cdots \mu_m)} \lambda_{k_2}^{(\mu_1 \cdots \mu_m)} \dot{f}_{k_0} \]

Here, for the second term \( \delta s_2 \), I also ignore any contribution of \( \delta \lambda_{k_1}^{(\mu_1 \cdots \mu_1)} \). In current literature, the second term in the entropy current is parametrized according to

\[ \delta s_2^\mu = \zeta_\Pi \Pi^\mu + \zeta_W W^\mu W_\mu + \zeta_\gamma \Pi^\mu \Pi^\nu \]

III. NUMERICAL RESULTS AND DISCUSSION

In this section, I demonstrate the separation of the corresponding equilibrium energy density \( \varepsilon_{eq}(T, \mu) \) and net baryon density \( n_{eq}(T, \mu) \) from those in an off-equilibrium state \( (\varepsilon, n) \) provided that the strength of the off-equilibrium state \( \gamma \) is defined. The boundary conditions for the corresponding equilibrium temperature \( T_0 \) and chemical potential \( \mu_0 \) are determined by the Landau matching condition. For all numerical results shown, the classical particle mass \( m \) is 5 MeV.

The \( \Lambda \) and \( \delta n \) are given by eqs. (32a) and (32b), respectively, and the scalars \( \lambda \) and \( \gamma \) are given by eqs. (48) and (49). Hence, I have

\[ \varepsilon - \varepsilon_{eq}(\alpha, \beta) = \Lambda(\alpha, \beta, \gamma) \]

\[ n - n_{eq}(\alpha, \beta) = \delta n(\alpha, \beta, \gamma) \]

Note that, in eq. (56) I can define \( \Lambda \) and \( \delta n \) as an explicit function of \( \alpha, \beta \) and \( \gamma \). Therefore, determination of the separation energy density and net baryon density is equivalent to solving the nonlinear simultaneous equations concerning \( \alpha, \beta, \gamma \) for given \( \varepsilon, n, \) and \( \gamma \). The off-equilibrium \( \varepsilon \) and \( n \) are determined by the boundary conditions,

\[ \varepsilon = \varepsilon_{eq}(\alpha_0, \beta_0), \quad n = n_{eq}(\alpha_0, \beta_0) \]

where \( \alpha_0 \equiv \mu_0/T_0 \) and \( \beta_0 \equiv 1/T_0 \). Note that, eq. (57) appears to be the same expression as the Landau matching condition, but eq. (57) is satisfied only for the case of \( \gamma \to 0 \). Eq. (56) may be regarded as expressions for the conservation of energy and net baryon density within the coarse-grain scale of a fluid \( l_{fluid} \) (the fluid cell size). I assume that the non-thermal part of the energy and baryon number dissipates within the spatial scale \( l_{fluid} \) and ultimately obeys a thermal equilibrium distribution during the relaxation time scale.

Figures 1 and 2 show the separation temperature \( T \) and separation chemical potential \( \mu \) as a function of \( \gamma = \Pi/P_{eq} \), \( \mu_0=5.0 \) MeV (Fig.1) and \( \mu_0=25 \) MeV (Fig.2).
Figure 3 shows the $\gamma$ decreases due to the decrease in temperature the total number of particles and anti-particles. Generally, $b$ increases as $T$ decreases, i.e., the difference between the number of primary $n_+$ and anti-particles $n_-$ must be fixed while the total number of particles and anti-particles $n_+ + n_-$ decreases due to the decrease in temperature $T$.

$T_0$ is fixed at 200 MeV [panels (a) and (d) of Figs. 1 and 2], 400 MeV [panels (b) and (e) of Figs. 1 and 2], and 600 MeV [panels (c) and (f) of Figs. 1 and 2]. For very small $\gamma$ region ($\gamma \lesssim 10^{-4}$) in both Figs. 1 and 2, one can observe that $T$ and $\mu$ are almost constant with $T$, $\mu \approx T_0$ and $\mu \approx \mu_0$. In this region, the Landau matching condition works well. However, in the region $\gamma \gg 10^{-3}$, the separation temperature $T$ decreases with increasing $\gamma$ and the separation chemical potential increases. The reason for the decrease of the separation temperature $T$ is that the total energy of the fluid is partially used in non-thermal motion such as tiny scale turbulent flow as discussed in Section II. However, the separation chemical potential increases as $\gamma$ increases for an off-equilibrium system. This is because of the constraint of conservation of the net-baryon number, i.e., the difference between the number of particles $n_+$ and anti-particles $n_-$ must be fixed while the total number of particles and anti-particles $n_+ + n_-$ decreases due to the decrease in temperature $T$.

Figure 3 shows the $\gamma$ dependence of the $\chi$ parameter for different $\mu_0$ and $T_0$. In the limit $\mu_0 \to 0$ $\chi$ approaches $-3$ because $\delta \chi \to 0$.

I have already documented the result $\Lambda = 3\Pi$ for the net baryon free case in the NeXDC correspondence [28], as well as the hydrodynamical model in the presence of a long-range correlation [29]. When the strength of the off-equilibrium state $\gamma \equiv \Pi/P_{eq}$ is larger than around $\sim 1/100$, the value of $\chi$ increases rapidly. This is because the bulk pressure gap between $\Pi_+$ and $\Pi_-$ becomes large in the region of $\gamma > 1/100$. Since the parameter $\chi$ is derived from $\xi$ ($\xi = \chi_0 \Pi$), which is the residual (originally neglected) term obtained by truncation of the energy polynomial function $P_{\lambda_0}$. This means that not only quadratic, but also higher order energy dependences of both $\lambda_k$ and $\lambda_0$ play an important role in the thermodynamical stability of the off-equilibrium entropy current. These higher order contributions in the energy polynomial function may result in the existence of the tiny turbulent flow in the fluid. For the finite net baryon number case, I have also obtained the same expression as given in eq. (43) [25, 26]. I found there that the value of $\chi$ plays an important role to restore not only the thermodynamical stability but also the causality of the solution obtained from the relativistic dissipative fluid dynamical equations. In ref. [25, 26], the value of $\chi$ is introduced phenomenologically into the expression for an off-equilibrium entropy current, however, in this article, I illustrate the result numerically using kinetic theory as shown in Fig. 3.

In figures 4 and 5, I show the off-equilibrium distribution $\mu$ [GeV] (upper panels (a), (b), and (c)) and separation temperature $T$ [GeV] (lower panels (d), (e) and (f)) for an off-equilibrium fluid as functions of $\gamma = \Pi/P_{eq}$. The $T_0=0.2$ GeV for figure (a) and (d), $T_0=0.4$ GeV, for (b) and (e), and $T_0=0.6$ GeV for (c) and (f), while the $\mu_0=5.0$ MeV is fixed for all panels.
FIG. 2. The same as Fig.1 but $\mu_0=25$ MeV.

Note that, the condition $\delta f_k \ll f_{k0}$ (or $\phi_k \ll 1$) is not assumed for the entire momentum region. The deviation of $\phi_k$ is expanded by an orthogonal set of irreducible tensors $k^{(\mu_1 \mu_2 \cdots \mu_m)}$ and the orthogonal polynomial function $P_{k(n)}$. These expansions are truncated in a finite rank of tensor and finite order of the polynomials of the energy $E_k$, respectively, and do not require the condition $\phi_k \ll 1$. Hence, formulations based on an irreducible tensor expansion are able to deal with an off-equilibrium distribution function with a long tail momentum (a distribution function having a large momentum dispersion).

FIG. 3. The $\gamma$ dependence of the factor $\chi$ eq. with the equilibrium temperature $T_0=200, 400, \text{and} 600$ MeV, and the equilibrium chemical potential $\mu_0=5$ MeV (panel (a)) and for $\mu_0=25$ MeV (panel (b)). In the limit of $\gamma \rightarrow 0$ and $\mu_0 \rightarrow 0$, the $\chi$ approaches to $-3$. 

IV. SUMMARY AND CONCLUDING REMARKS

I have proposed a novel way to specify the initial conditions for a dissipative fluid dynamical model as an alternative to the so-called Landau matching condition eq. employed in most of the literature published so far. The large expansion rate of matter produced in ultrarelativistic heavy-ion collisions may prevent the local equilibrium condition eq. (2) from holding true. Specifically, the microscopic collision time scale is not much shorter than any macroscopic evolution time scale because the macroscopic evolution time scale is consider-
For a given off-equilibrium state specified by the energy density \(\varepsilon = \varepsilon_{\text{eq}}(T, \mu)\) and net baryon density \(n = n_{\text{eq}}(T, \mu)\). When the matching conditions are imposed, the \(\delta f_k\) and \(\delta \bar{f}_k\) are not physically relevant. Note that, other off-equilibrium quantities such as the bulk pressure \(\Pi\) are also distorted by the Landau matching condition. \(\Lambda = \delta n = 0\). Therefore, I assert that the Landau matching condition be relaxed as shown in eq. (11),
\[
\delta T^{\mu\nu} u_{\mu} u_{\nu} = \Lambda \neq 0, \quad \delta n^{\mu} u_{\mu} = \delta n \neq 0.
\]

Then, \(\Lambda\) and \(\delta n\) obtain their entities of physical existence, i.e., they are the energy density and baryon number density of the disturbance flow caused by the rapid expansion of the viscous fluid.

An important consequence of the irreducible tensor expansion for the off-equilibrium distribution function is that scalar quantities such as \(\Lambda\), \(\delta n\), and \(\Pi\) may be written as functions of the separation temperature \(T\), chemical potential \(\mu\), and the 0-th order (scalar) expansion coefficients \(\lambda_k\) and \(\chi_k\) (see eqs. (32a)-(32c)). In addition to this, the thermodynamical stability condition requires a constraint of the scalar off-equilibrium thermodynamical quantities \(\Lambda\), \(\delta n\), and \(\Pi\) (eq. (44), or eq. (44a) and (44b)). These conditions practically determine both \(\lambda_+\) and \(\chi_-\) (and thus also \(\chi\)). It also defines the differential equations for \(\lambda_k\) and \(\chi_k\) (see eq. (11)) that can be solved

FIG. 4. The off-equilibrium distribution function for \(T_0 = 200\) MeV, \(\mu_0 = 5.0\) MeV, \(T = 193\) MeV, \(\mu = 5.4\) MeV (panel (a)), \(400\) MeV (panel (b)), and \(600\) MeV (panel (c)) in the case of \(\gamma = 10^{-3}\). The straight line is the Boltzmann distribution \(f_{\text{eq}}\) with the separation temperature \(T\), and the separation chemical potential \(\mu\) are shown in each panel. The open circles shows the off-equilibrium distribution function \(f_{\text{eq}} + \delta f_k\).

FIG. 5. The same as Fig. 4 but \(\gamma = 10^{-2}\).
easily, excluding uncertainty in the integration constant $C_\gamma$ (see eq. (18a) and (18b)). As seen in the solutions to the differential equations, one can observe that $\chi_+$ and $\chi_-$, which are determined by the conditions of thermodynamical stability, play an important role in the energy dependence of $\lambda_k$ and $\lambda_k$. Note that, the integration constant $C_\gamma$ defines a scale of $\lambda_k$ and $\lambda_k$, or in other words, the constant $C_\gamma$ regulates the ‘strength’ of the off-equilibrium state. I fix this constant by introducing an index $\gamma \equiv \Pi/P_{eq}$. Then for a given $\gamma$ I obtain the non-linear simultaneous equations, eq. (58a) and eq. (58b), concerning the separation temperature $T$ and chemical potential $\mu$ (or $\alpha = \mu/T$ and $\beta = 1/T$) that can be solved numerically. The separation temperature $T$ and chemical potential $\mu$ obtained show (see Fig. 1 and 2) that $\epsilon < T_{\approx}$. In this region, the Landau matching condition approximately holds true. However, it must be done carefully to obtain some physical results (see eq.(48a) and (48b)). As seen in the solutions to the differential equations, one can observe that $\chi_+$ and $\chi_-$, which are determined by the conditions of thermodynamical stability, play an important role in the energy dependence of $\lambda_k$ and $\lambda_k$. Note that, the integration constant $C_\gamma$ defines a scale of $\lambda_k$ and $\lambda_k$, or in other words, the constant $C_\gamma$ regulates the ‘strength’ of the off-equilibrium state. I fix this constant by introducing an index $\gamma \equiv \Pi/P_{eq}$. Then for a given $\gamma$ I obtain the non-linear simultaneous equations, eq. (58a) and eq. (58b), concerning the separation temperature $T$ and chemical potential $\mu$ (or $\alpha = \mu/T$ and $\beta = 1/T$) that can be solved numerically. The separation temperature $T$ and chemical potential $\mu$ obtained show (see Fig. 1 and 2) that $\epsilon < T_{\approx}$. In this region, the Landau matching condition approximately holds true. However, it must be done carefully to obtain some physical results (see eq.(48a) and (48b)).

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### Appendix A: Projection tensor

The deviation from equilibrium $\phi(k, x)$ can be expanded by a series of irreducible tensors $1, k, k^2, k^3, \cdots$ forming a complete and orthogonal set. A component of the set of irreducible tensors is given by

$$k^{[\mu_1 \cdots \mu_m]} \equiv \Delta^{[\mu_1 \mu_2 \cdots \mu_m]} k^{\mu_1} k^{\mu_2} \cdots k^{\mu_m}, \quad (A1)$$

where a projection tensor, $\Delta^{[\mu_1 \mu_2 \cdots \mu_m]}$, is given by (see Appendix A). See also Appendix F of ref. [24]

$$\Delta^{[\mu_1 \cdots \mu_m]} = g_{\mu_1 \sigma_1} \cdots g_{\mu_m \sigma_m} \Delta^{(\mu_1 \cdots \mu_m)(\sigma_1 \cdots \sigma_m)} \quad (A2a)$$

and

$$\Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = \sum_{k=0}^{[m/2]} C_{mk} \Phi^{(\mu_1 \cdots \mu_m)(\sigma_1 \cdots \sigma_m)} \quad (A2b)$$

The parentheses in the indexes of $\Phi_{mk}^{(\mu_1 \cdots \mu_m)(\sigma_1 \cdots \sigma_m)}$ denotes symmetrization under the exchange of indexes within $\mu_1 \cdots \mu_m$ and $\sigma_1 \cdots \sigma_m$ of the tensor $\Phi_{mk}$. The symmetric tensor is written by a sum of all possible permutations of $\mu$-type and $\nu$-type indexes as follows:

$$\Phi_{mk}^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = \frac{1}{\mathcal{N}_{m,k}} \sum_{\gamma, \nu_{(\nu)}} \Delta^{(\mu_1 \mu_2 \cdots \mu_m)} \mu_1 \mu_2 \cdots \mu_m \nu_1 \nu_2 \cdots \nu_m \Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)}$$

where $\sum_{\gamma, \nu_{(\nu)}}$ represents the summation of all distinct permutations of the $\mu$-type and $\nu$-type indexes. The coefficient $C_{mk}$ in eq. (A2b) is introduced to satisfy the following conditions

$$g_{\mu_1 \nu_1} \Delta^{(\mu_1 \mu_2 \cdots \mu_m)} = 0, \quad g_{\mu_1 \nu_1} \Delta^{(\mu_1 \mu_2 \cdots \mu_m)} = 0, \quad (A4)$$

for $1 \leq i, j \leq m$. The symbol $[m/2]$ denotes the largest integer not exceeding $m/2$, and the factor $\mathcal{N}_{m,k}$ is the total number of distinct permutations in the indexes $\mu$ and $\nu$ to be summed. In order to consider the symmetric tensor, let me introduce the following notations for the $\mu$-type rank-2 and rank-4 tensors

$$\{\Delta^{(\mu_1 \mu_2)}\} = \frac{1}{2} \sum_{\gamma, \gamma', \gamma''} \Delta^{(\mu_1 \mu_2)} = \frac{1}{2} \sum_{\gamma, \gamma'} \Delta^{(\mu_1 \mu_2)} \Delta^{(\mu_1 \mu_2)}, \quad (A5a)$$

$$\{\Delta^{(\mu_1 \mu_2)}\}^2 = \frac{1}{2^2 2!} \sum_{\gamma, \gamma', \gamma''} \Delta^{(\mu_1 \mu_2)} = \frac{1}{2^2 2!} \sum_{\gamma, \gamma'} \Delta^{(\mu_1 \mu_2)} \Delta^{(\mu_1 \mu_2)} = \frac{1}{2^2 2!} \sum_{\gamma, \gamma'} \Delta^{(\mu_1 \mu_2)} \Delta^{(\mu_1 \mu_2)}$$

respectively, where the factors $\frac{1}{2}$ and $\frac{1}{2^2 2!}$ in eq. (A5a) and eq. (A5b) are introduced so as to exclude duplication caused by the possible exchange of suffixes. For example,
\{(\Delta_{\mu_1\mu_2})^k\}^k with m = 4 and k = 2 means the following rank-2k tensor,
\{(\Delta_{\mu_1\mu_2})^2\}^2 = \Delta_{\mu_1\mu_2}^2 \Delta_{\mu_3\mu_4} + \Delta_{\mu_1\mu_3} \Delta_{\mu_2\mu_4} + \Delta_{\mu_1\mu_4} \Delta_{\mu_2\mu_3}.

Note that, possible permutations of the suffixes include
\Delta_{\mu_1\mu_2} \Delta_{\mu_3\mu_4}, \Delta_{\mu_2\mu_3} \Delta_{\mu_1\mu_4}, \Delta_{\mu_2\mu_1} \Delta_{\mu_3\mu_4}, which gives exactly the same contribution as the above first term \Delta_{\mu_1\mu_2} \Delta_{\mu_3\mu_4}. In addition to this, the permutation also includes terms such as \Delta_{\mu_1\mu_4} \Delta_{\mu_2\mu_3}, which is also gives the same contribution. In order to exclude these terms, the factor \(\frac{2!}{2!}\) is needed. Thus, for the general 2k-rank tensor, \{(\Delta_{\mu_1\mu_2})^k\}^k (k \leq m/2), it is defined by
\{(\Delta_{\mu_1\mu_2})^k\}^k = \frac{1}{k!} \sum_{\nu_1, \nu_2, \ldots, \nu_k} \Delta_{\mu_1\mu_2} \Delta_{\mu_1\mu_3} \Delta_{\mu_2\mu_4} \ldots \Delta_{\mu_{2k-1}\mu_{2k}}, (A6a)

where m denotes a permissible maximum number of the suffix. Similarly for \(\nu\)-type tensors, I define
\{(\Delta_{\nu_1\nu_2})^k\}^k = \frac{1}{k!} \sum_{\nu_1, \nu_2, \ldots, \nu_k} \Delta_{\nu_1\nu_2} \Delta_{\nu_1\nu_3} \Delta_{\nu_2\nu_4} \ldots \Delta_{\nu_{2k-1}\nu_{2k}}. (A6b)

Considering linear combinations for the projection tensor \(\Phi^{(\mu_1\mu_2\ldots\mu_m)(\nu_1\nu_2\ldots\nu_m)}_{mk}\) with different k values we obtain
\[(\Delta_{\mu_1\mu_2\ldots\mu_m}(\nu_1\nu_2\ldots\nu_m)) = \sum_{k=0}^{[m/2]} C_{m,k} \left(\Delta_{\mu_1\mu_2}\right)^{k-1}\left(\Delta_{\nu_1\nu_2}\right)^k\left(\Delta_{\mu_3\mu_4}\right)^{m-2k}. (A11)

Note that, the value of \(C_{m,0}\) can be set arbitrarily so that it may be set to a value of 1 without loss of generality. Thus, \(C_{m,k}\) can then be determined as the following:
\[C_{m,k} = (-1)^k \frac{(m)!^2}{(2m)!} \frac{(2m-2k)!}{k!(m-k)!(m-2k)!}. (A12)\]

The derivation of eq. (A12) is given in Appendix B.

Appendix B: Derivation of the \(C_{m,k}\)

When the indices \(\mu_1\) and \(\mu_2\) are expressed explicitly, the projection tensor is written as follows:
\[\Delta^{\mu_1\mu_2\ldots\mu_m}(\nu_1\nu_2\ldots\nu_m) = \sum_{k=0}^{[m/2]} C_{m,k} \left[\Delta_{\mu_1\mu_2}^{k-1}\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right] + \left[\Delta_{\mu_1\mu_2}^{k-2}\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right] + \left[\Delta_{\mu_1\mu_2}^{k-3}\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right] + \left[\Delta_{\mu_1\mu_2}^{k-1}\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right] + \left[\Delta_{\mu_1\mu_2}^1\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right] + \left[\Delta_{\mu_1\mu_2}^1\Delta_{\nu_1\nu_2}^k\Delta_{\mu_3\mu_4}^{m-2k}\right].\]
Here, I take contraction of two indices \( \mu_1 \) and \( \mu_2 \) and obtain

\[
g_{\mu_1\mu_2} \Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = \sum_{k=0}^{[m/2]} \frac{C_{m,k}}{N_{m,k}} \left[ x \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} k \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \right. \\
+ c^{(1)}_{m,k} \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} k \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \\
+ 2 c^{(2)}_{m,k} \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} k \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \\
+ c^{(3)}_{m,k} \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \left] = 0, \quad \text{(B1)} \right.
\]

where \( x = g_{\mu_1 \mu_2} \Delta^{\mu_1 \mu_2} \) and \( c^{(1)}_{m,k}, c^{(2)}_{m,k}, c^{(3)}_{m,k} \) are given by

\[
c^{(1)}_{m,k} = \frac{\binom{\mu_1 \mu_2}{\nu_1 \nu_2}}{n_{m-2,1} n_{m-3,1} n_{m-4,1}} = 2k - 2, \quad \text{(B2a)}
\]
\[
c^{(2)}_{m,k} = \frac{\binom{\mu_1 \mu_2}{\nu_1 \nu_2}}{n_{m-2,1} n_{m-3,1} n_{m-4,1}} = m - 2k, \quad \text{(B2b)}
\]
\[
c^{(3)}_{m,k} = \frac{\binom{\mu_1 \mu_2}{\nu_1 \nu_2}}{n_{m-2,1} n_{m-3,1} n_{m-4,1}} = 2k + 2. \quad \text{(B2c)}
\]

Since the contraction \( g_{\mu_1 \mu_2} \Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = 0 \), the coefficients \( c^{(1)}_{m,k}, c^{(2)}_{m,k} \) and \( c^{(3)}_{m,k} \), need to satisfy the following equation:

\[
\frac{C_{m,k-1}}{N_{m,k-1}} c^{(3)}_{m,k-1} + \frac{C_{m,k}}{N_{m,k}} (x + c^{(2)}_{m,k}) = 0. \quad \text{(B3)}
\]

From the above equation, I obtain the following recursion formula for \( C_{m,k} \):

\[
C_{m,k} = - \frac{(m - 2k + 2)(m - 2k + 1)}{2k(2m - 2k + 1)} C_{m,k-1}. \quad \text{(B4)}
\]

Without loss of generality, the coefficient \( C_{m,0} \) is set to 1. Then, the other \( C_{m,k} \) to general \( k \) can be determined according to

\[
C_{m,k} = (-1)^k \frac{(m!)}{(2m)!} \frac{(2m - 2k)!}{k!(m - k)!(m - 2k)!}, \quad \text{(B5)}
\]

which is eq. (A12).

**Appendix C: Total contraction of the projection tensor**

Defining an identity

\[
g_{\mu_1 \nu_1} \cdots g_{\mu_m \nu_m} \Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = 2m + 1,
\]

which is used in the derivation of the orthogonality condition for the irreducible projection tensors in Appendix D.

When the indices \( \mu_1 \) and \( \nu_1 \) are expressed explicitly, the projection tensor is written as follows:

\[
\Delta^{(\mu_1 \cdots \mu_m)(\nu_1 \cdots \nu_m)} = \sum_{k=0}^{[m/2]} \frac{C_{m,k}}{N_{m,k}} \left[ \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} m_{m-2k} \right. \\
+ \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \\
+ \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \\
+ \{ \Delta^{(\mu_1 \mu_2)} m_{m-2} \} \{ \Delta^{(\nu_1 \nu_2)} m_{m-2} \} m_{m-2k} \\
\left. (m - 2k) \right].
\]
Contracting the indices $\mu_1$ and $\nu_2$, I get
\[
g_{\mu_1 \nu_1} \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m) = \sum_{k=0}^{[m/2]} \frac{c_{m,k}}{\mathcal{N}_{m,k}} \left[ d_{m,k}^{(1)} \Delta_{m-1}^{(\nu_1 \nu_2)} \Delta_{m-2k+1}^{(\mu_1 \mu_2)} - d_{m,k}^{(2)} \Delta_{m-1}^{(\nu_1 \nu_2)} \Delta_{m-2k}^{(\mu_1 \mu_2)} \right], \tag{C1}
\]
where
\[
d_{m,k}^{(1)} = \frac{\gamma_{m-1,k} \gamma_{m-2,k-1} \gamma_{m-2k+1,m-2k-1} \gamma_{m-2k+1,m-2k} \gamma_{m-2k+2,m-2k-2}}{m - 2k + 1}, \tag{C2a}
\]
\[
d_{m,k}^{(2)} = \frac{\gamma_{m-1,k} \gamma_{m-2,k-1} \gamma_{m-2k+1,m-2k-1} \gamma_{m-2k+2,m-2k-2}}{2k}, \tag{C2b}
\]
\[
d_{m,k}^{(3)} = \frac{\gamma_{m-1,k} \gamma_{m-2,k-1} \gamma_{m-2k+1,m-2k-1} \gamma_{m-2k+2,m-2k-2}}{2k}, \tag{C2c}
\]
\[
d_{m,k}^{(4)} = \frac{\gamma_{m-1,k} \gamma_{m-2,k-1} \gamma_{m-2k+1,m-2k-1} \gamma_{m-2k+2,m-2k-2}}{1}, \tag{C2d}
\]
\[
d_{m,k}^{(5)} = \frac{\gamma_{m-1,k} \gamma_{m-2,k-1} \gamma_{m-2k+1,m-2k-1} \gamma_{m-2k+2,m-2k-2}}{m - 2k + 1}. \tag{C2e}
\]

Hence, I obtain a formula
\[
g_{\mu_1 \nu_1} \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m) = \sum_{k=0}^{[m/2]} \gamma_{m,k} \left[ (m - 2k + 1) \mathcal{M}_{m-1,k} + (m + 2k + 2) \mathcal{M}_{m-1,k} \right], \tag{C3}
\]
where
\[
\gamma_{m,k} \equiv \frac{c_{m,k}}{\mathcal{N}_{m,k}}, \tag{C4}
\]
\[
\mathcal{M}_{m,k} \equiv \{ \Delta_{m}^{(\mu_1 \mu_2)} \} \{ \Delta_{m-2k}^{(\nu_1 \nu_2)} \} \{ \Delta_{m-2k+1}^{(\mu_1 \mu_2)} \}. \tag{C5}
\]
This means
\[
g_{\mu_1 \nu_1} \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m) = \sum_{k=0}^{[m/2]-1} \gamma_{m-1,k} \mathcal{M}_{m-1,k}
\]
\[
= \frac{2m+1}{2m-1} \sum_{k=0}^{[m/2]-1} \gamma_{m-1,k} \mathcal{M}_{m-1,k}
\]
\[
= \frac{2m+1}{2m-1} \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m), \tag{C6}
\]
Using the above formula iteratively, I obtain
\[
g_{\mu_1 \nu_1} \cdots g_{\mu_m \nu_m} \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m) = 2m+1. \tag{C7}
\]

**Appendix D: Derivation of the orthogonality condition**

In order to discuss the orthogonality of the irreducible projection tensor, consider the following integral, $I_m^{(l)}$ for the case of $F_k$ which is an arbitrary function of $E_k \equiv g_{\alpha_1 \beta_1} \cdots g_{\alpha_m \beta_m} \Delta^{(\alpha_1 \cdots \alpha_m)}(\alpha_1 \cdots \alpha_m) \Delta^{(\beta_1 \cdots \beta_m)}(\nu_1 \cdots \nu_m)$
\[
= \Delta^{(\mu_1 \cdots \mu_m)}(\nu_1 \cdots \nu_m), \tag{D2}
\]
and eq. (17). The integral part in eq. (22), can be evaluated as

\[
\frac{d^3 k F_k}{(2\pi)^3 k_0} k^{(\alpha_1 \ldots \kappa_{\alpha_m})} k_{(\alpha_1 \ldots \kappa_{\alpha_m})} = \sum_{j=0}^{[m/2]} \gamma_{m,j} \times N_{m,j} \int \frac{d^3 k F_k}{(2\pi)^3 k_0} [\Delta_{\alpha\beta} k^\alpha k^\beta]^m
\]

where \( P_m(x) \) is a Legendre function of the order of \( m \). Hence, I can write

\[
\frac{d^3 k F_k}{(2\pi)^3 k_0} k^{(\mu_1 \kappa_2 \ldots \kappa_{\mu_m})} k_{(\nu_1 \kappa_2 \ldots \kappa_{\mu_m})} = \frac{m!}{(2m + 1)!!} \int \frac{d^3 k F_k}{(2\pi)^3 k_0} [\Delta_{\alpha\beta} k^\alpha k^\beta]^m.
\] (D3)

**Appendix E: Evaluation of the \( I_r^{(l)} \)**

I require to calculate the following integrals also denoting \( I_r^{(l)} \)

\[
I_r^{(l)} = \int \frac{d^3 k \omega_k^{(l)}}{(2\pi)^3 k_0} (E_k)^r
\]

\[
= \frac{(-1)^r}{2\pi^2 (2l + 1)!!} \frac{W(l)}{E_k} \left[ \sqrt{E_k^2 - m^2} \right]^{2l} [E_k]^r
\times e^{-(E_k - \mu_0)/T}.
\] (E1)

This may be expressed in the following form,

\[
I_r^{(l)} = \frac{(-1)^r}{2\pi^2 (2l + 1)!!} \frac{W(l)}{E_k} \left[ \sqrt{E_k^2 - m^2} \right]^{2l} (aT)^{2l+2+r}
\times \left( \frac{d}{da} \right) \int_0^{\infty} dy \sinh^2 y \left[ \frac{1}{E_k} \right] \frac{1}{E_k - \mu_0} (aT)^{2l+2+r}
\times \frac{d}{da} \frac{1}{a^{l+1}} K_{l+1}(a),
\] (E2)

where \( a = m/T \). Using the identity about the following differential operators I have

\[
\left[ \frac{d}{dz} \right]^r = \sum_{k=0}^{[r/2]} \frac{r!}{2^k k!(r - 2k)!} z^{r-2k} \left[ \frac{1}{z} \frac{d}{dz} \right]^{r-k}.
\] (E3)

Then I obtain

\[
I_r^{(l)} = \frac{1}{2\pi^2} W(l) \frac{e^{\mu_0/T} T^{2l+1+r}}{(2\pi)^3 k_0}
\times \sum_{k=0}^{[r/2]} \frac{(-1)^{l-k} r!}{2^k k!(r - 2k)!} a^{l+1+r-k} K_{l+1+r-k}(a).
\] (E4)

By requiring \( I_0^{(l)} = 1 \), I can determine the normalization factor,

\[
\frac{1}{W(l)} = \frac{(-1)^l e^{\mu_0/T} T^{2l+1}}{(2\pi)^3 k_0} \left( \frac{m}{T} \right)^{l+1} K_{l+1}(m/T).
\] (E5)

Moreover, substituting the result obtained previously in the normalization factor \( W(l) \), I finally obtain

\[
I_r^{(l)} = m^r r! \sum_{k=0}^{[r/2]} \frac{(-1)^k}{2^k k!(r - 2k)!} \frac{m}{T}^{-k} K_{l+1+r-k}(m/T) K_{l+1}(m/T).
\] (E6)

**Appendix F: Polynomial \( P_{kn}^{(m)} \) and its coefficients \( a_{n_l}^{(l)} \)**

The orthogonal condition of the polynomial \( P_{kn}^{(m)} \) in eq. (18) can be written in the following form

\[
\begin{pmatrix}
1 & I_1^{(l)} & \ldots & I_n^{(l)} \\
I_1^{(l)} & I_2^{(l)} & \ldots & I_n^{(l)} \\
\vdots & \vdots & \ddots & \vdots \\
I_n^{(l-1)} & I_n^{(l-1)} & \ldots & I_{n+2n}^{(l)}
\end{pmatrix}
\begin{pmatrix}
a_{n0}^{(l)} \\
a_{n1}^{(l)} \\
\vdots \\
a_{nn}^{(l)}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
1/a_{nn}^{(l)}
\end{pmatrix}
\] (F1)

and

\[
a_{nk}^{(l)} = \frac{\mathcal{T}_{nk}^{(l)}}{\sqrt{\det I_n^{(l)} I_{nn}}},
\] (F2)

where \( \det I_n^{(l)} \) is the determinant of the matrix

\[
\{I_n^{(l)}\}_{ij} = \begin{pmatrix}
1 & I_1^{(l)} & \ldots & I_n^{(l)} \\
I_1^{(l)} & I_2^{(l)} & \ldots & I_n^{(l)} \\
\vdots & \vdots & \ddots & \vdots \\
I_n^{(l-1)} & I_n^{(l-1)} & \ldots & I_{n+2n}^{(l)}
\end{pmatrix},
\] (F3)

and \( \mathcal{T}_{nk}^{(l)} \) is a co-factor (cross out the entries that lie in the corresponding row \( n \) and column \( k \) of the matrix \( I_n^{(l)} \). Thus, I can express the orthogonal function as the following

\[
P_{kn}^{(l)} = \sum_{r=0}^{N_l} \frac{\mathcal{T}_{nk}^{(l)} E_r^{(l)}}{\sqrt{\det I_n^{(l)} / \mathcal{T}_{nn}^{(l)}}}
\] (F4)

Then for eq. (28) I obtain, for example, in the \( N_l = 2 \) case

\[
\begin{pmatrix}
a_{00}^{(l)} & a_{10}^{(l)} \\
0 & a_{11}^{(l)}
\end{pmatrix}
\begin{pmatrix}
a_{00}^{(l)} \\
a_{10}^{(l)}
\end{pmatrix}
= \begin{pmatrix}
I_0^{(l)} & I_1^{(l)} \\
I_1^{(l)} & I_2^{(l)}
\end{pmatrix}^{-1}.
\] (F5)

Therefore,

\[
\begin{pmatrix}
a_{00}^{(l)} & a_{10}^{(l)} \\
0 & a_{11}^{(l)}
\end{pmatrix}
\begin{pmatrix}
I_0^{(l)} & I_1^{(l)} \\
I_1^{(l)} & I_2^{(l)}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\] (F6)
This is satisfied if eq. (29) holds true; i.e., for the orthogonal 0, 0 and 1, 1 components, respectively

\[
(a_{00}^2 + a_{10}^2)I_0 + a_{10}a_{11}I_1 = 1 + a_{10}(I_0a_{10} + I_1a_{11}) = 1,
\]

(F7a)

\[
a_{10}a_{11}I_1 + a_{11}^2I_2 = a_{11}(I_0a_{10} + I_1a_{11}) = 1,
\]

(F7b)

where I use \(a_{00} = 1\) and \(I_0 = 1\). However, the off-orthogonal 0, 1 and 1, 0 components are

\[
(a_{00}^2 + a_{10}^2)I_1 + a_{10}a_{11}I_2 = I_1 + a_{10}(I_0a_{10} + I_2a_{11}) = I_1 + a_{10}/a_{11}
\]

(F7c)

\[
a_{10}a_{11}I_2 + a_{11}^2I_1 = a_{11}(I_0a_{10} + I_1a_{11}) = 0,
\]

(F7d)