Scalar-tensor cosmologies with a potential in the general relativity limit

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Abstract. We consider Friedmann-Lemaître-Robertson-Walker flat cosmological models in the framework of general Jordan frame scalar-tensor theories of gravity with arbitrary coupling functions, in the era when the energy density of the scalar potential dominates over the energy density of ordinary matter. To study the regime suggested by the local weak field tests (i.e. close to the so-called limit of general relativity) we propose a nonlinear approximation scheme, solve for the phase trajectories, and provide a complete classification of possible solutions. We argue that the topology of phase trajectories in the nonlinear approximation is representative of those of the full system, and thus can tell for which scalar-tensor models general relativity functions as an attractor. To the classes of models which asymptotically approach general relativity we give the solutions also in cosmological time and conclude with some observational implications.

1. Introduction
One of the most outstanding problems for contemporary physics is how to explain the dynamics behind the current accelerated expansion of the universe. A plethora of various approaches have been proposed, most of which can be subsumed under the categories of dark energy (extraordinary matter) or dark gravity (modifications to Einstein’s general relativity). As new astrophysical and cosmological precision data is continuously becoming available, it remains a cumbersome task to systematically sift out the models which can meet all observational requirements. What one would like to have are reliable model independent tools for testing gravity at large scales, perhaps something akin to the parametrized post-Newtonian (PPN) formalism for weak fields.

While there has been some progress in designing such tools (the post-Friedmannian formalism), here we would like to introduce a bit more modest approach in scope, yet with a congruent aim. First we note that many proposed modified gravity theories can be cast in the form of scalar-tensor gravity (STG) – higher dimensional theories, braneworld models, $f(R)$ type gravities, varying speed of light theories, etc. Second, wide classes of STG cosmologies dynamically converge to the limit of general relativity favored by local weak field experiments, the so-called “attractor mechanism” [1]. Therefore we can use generic STG as a template or paradigm which implicitly incorporates many different theories, in order to confront them with various observations and see how good they are in comparison with each other. In principle this should provide an instrument of selection within the zoo of modified gravities.

In the present paper we review our recent work towards this direction [2, 3, 4, 5]. After setting the notation and describing the local weak field constraints in Section 2, we write the
scalar-tensor cosmology as a dynamical system and briefly outline the features of the phase space in Section 3, followed by the discussion of the fixed points in Section 4. The fixed point corresponding to limit of general relativity needs a more careful consideration, facilitated by the approximation scheme of Section 5. This leads to approximate equations, which are used to find and classify the phase portraits in Section 6 and to find the solutions in time in Section 7. Finally, Section 8 provides a brief summary and outlook.

2. Scalar-tensor gravity and local weak field constraints

We start with a generic Jordan frame STG using the “Brans-Dicke” parametrization,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \Psi R[g_{\mu\nu}] - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \nabla^\rho \Psi - 2\kappa^2 V(\Psi) \right] + S_m. \quad (1)$$

In fact, the action above describes a family of theories, each pair of the functions $\omega(\Psi)$ and $V(\Psi)$ specifies a particular STG. The non-minimal coupling between curvature and the scalar field $\Psi$ means variable gravitational “constant”, $8\pi G = \frac{\kappa^2}{\Psi}$, and it makes sense to assume $0 < \Psi < \infty$. To ensure that the energy density of the scalar field is positive, one also needs $2\omega(\Psi) + 3 \geq 0$ and $V(\Psi) \geq 0$. The term $S_m$ gives the usual matter part of the action.

Local weak field experiments (e.g. observations in the Solar System) reckoned in terms of the PPN formalism, impose rather severe constraints on the theory. If $V(\Psi)$ can be neglected, then

$$8\pi G = \frac{\kappa^2}{\Psi} \frac{2\omega + 4}{2\omega + 3}, \quad (2)$$

$$\beta - 1 = \frac{\kappa^2}{G} \frac{d\omega}{d\Psi} \frac{2\omega + 3}{(2\omega + 4)} \lesssim 10^{-4}, \quad (3)$$

$$\gamma - 1 = -\frac{1}{\omega + 2} \lesssim 10^{-5}, \quad (4)$$

$$\frac{\dot{G}}{G} = -\frac{2\omega + 3}{2\omega + 4} \left( G + \frac{2}{(2\omega + 3)^2} \frac{d\omega}{d\Psi} \right) \lesssim 10^{-13} \text{yr}^{-1}. \quad (5)$$

These constraints are satisfied by STG in the “Nordtvedt limit” [6]

$$\frac{1}{2\omega + 3} \to 0, \quad \frac{d\omega}{d\Psi} \frac{(2\omega + 3)^3}{(2\omega + 3)^3} \to 0, \quad (6)$$

in which case the predictions of STG and GR coincide and we may call (6) also “the limit of general relativity”. (If $V(\Psi)$ gives a contribution, then the PPN parameters get a correction [7, 8].)
3. Scalar-tensor cosmology as a dynamical system

The field equations for flat \((k = 0)\) Friedmann-Lemaître-Robertson-Walker spacetime filled with barotropic matter fluid \(p = w \rho\) read

\[
H^2 = -H \frac{\dot{\Psi}}{\Psi} + \frac{1}{6} \frac{\dot{\Psi}^2}{\Psi^2} \omega(\Psi) + \frac{\kappa^2}{3} \frac{\rho}{\Psi} + \frac{\kappa^2}{3} V(\Psi),
\]

(7)

\[
2 \dot{H} + 3 H^2 = -2H \frac{\dot{\Psi}}{\Psi} - \frac{1}{2} \frac{\dot{\Psi}^2}{\Psi^2} \omega(\Psi) - \frac{\ddot{\Psi}}{\Psi} + \frac{\kappa^2}{\Psi} \rho + \frac{\kappa^2}{\Psi} V(\Psi),
\]

(8)

\[
\dot{\Psi} = -3H \dot{\Psi} - \frac{1}{2\omega(\Psi) + 3} \frac{d\omega(\Psi)}{d\Psi} \dot{\Psi}^2 + \frac{\kappa^2}{2\omega(\Psi) + 3} (1 - 3w) \rho
\]

\[+ \frac{2\kappa^2}{2\omega(\Psi) + 3} \left[ 2V(\Psi) - \Psi \frac{dV(\Psi)}{d\Psi} \right],
\]

(9)

\[
\dot{\rho} = -3H (w + 1) \rho,
\]

(10)

where \(H\) is the Hubble parameter. Denoting \(\Pi = \dot{\Psi}\) the equations (7)-(10) can be rewritten as a dynamical system

\[
\dot{\Psi} = \Pi,
\]

(11)

\[
\dot{\Pi} = -\frac{1}{2\omega(\Psi) + 3} \left[ \frac{d\omega(\Psi)}{d\Psi} \Pi^2 - \kappa^2 (1 - 3w) \rho
\]

\[+ 2\kappa^2 \left( \frac{dV(\Psi)}{d\Psi} \Psi - 2V(\Psi) \right) \right] - 3H \Pi,
\]

(12)

\[
\dot{H} = \frac{1}{2\Psi(2\omega(\Psi) + 3)} \left[ \frac{d\omega(\Psi)}{d\Psi} \Pi^2 - \kappa^2 (1 - 3w) \rho
\]

\[+ 2\kappa^2 \left( \frac{dV(\Psi)}{d\Psi} \Psi - 2V(\Psi) \right) \right]
\]

\[\frac{1}{2\Psi} \left[ 6H^2 \Psi + 2H \Pi - \kappa^2 (1 - w) \rho - 2\kappa^2 V(\Psi) \right],
\]

(13)

\[
\dot{\rho} = -3H (1 + w) \rho,
\]

(14)

while the vector \((\dot{\Psi}, \dot{\Pi}, \dot{H}, \dot{\rho})\) gives a tangent to the trajectories (solutions) in the phase space \(\{\Psi, \Pi, H, \rho\}\). The trajectories lie on the 3-surface

\[
H = -\frac{\Pi}{2\Psi} \pm \sqrt{(2\omega(\Psi) + 3) \frac{\Pi^2}{12\Psi^2} + \frac{\kappa^2 (\rho + V(\Psi))}{3\Psi}}.
\]

(15)

By carefully inspecting Eq. (11)-(14) we can elicit the following generic information about the boundaries in the phase space [2, 3]:

- \(|H| \to \infty, |\rho| \to \infty, \) or \(|\dot{\Psi}| \to \infty\) imply a spacetime curvature singularity;
- \(\dot{\Psi} \to 0\) generally also leads to a singularity (solutions can not slip from “attractive” to “repulsive” gravity);
- \(\Psi \to \infty\) is not a singularity, but gravitational “constant” \(\frac{\kappa^2}{\Psi}\) vanishes;
- \(V \to \infty\) or \(2\omega + 3 \to 0\) is again a singularity, solutions can not safely pass to the region where \(\Psi\) turns to a ghost \((2\omega + 3 < 0)\);
- \(\frac{1}{\omega + 3} \to 0\) turns out to be a singularity as well, unless \(\dot{\Psi} = \Pi \to 0\).
4. Fixed points
If on cosmological scales the potential dominates over matter density \((V \neq 0, \rho \equiv 0)\) we can eliminate \(H\) using (15) and obtain a 2-dimensional system:

\[
\begin{align*}
\dot{\Psi} &= \Pi \\
\dot{\Pi} &= \left( \frac{3}{2\Psi} - \frac{1}{2\omega(\Psi) + 3} \frac{d\omega}{d\Psi} \right) \Pi^2 + \frac{2\kappa^2}{2\omega(\Psi) + 3} \left( 2V(\Psi) - \Psi \frac{dV}{d\Psi} \right) \\
&\quad + \frac{1}{2\Psi} \sqrt{3(2\omega(\Psi) + 3)\Pi^2 + 12\kappa^2\Psi V(\Psi)} \Pi.
\end{align*}
\]

(17)

The fixed points \((\dot{\Psi} = 0, \dot{\Pi} = 0)\) are of two types, given by [2]:

\[
\begin{align*}
\Psi_\ast : & \quad \left. \frac{dV}{d\Psi} \right|_{\Psi_\ast} = 2V(\Psi_\ast) = 0, \\
\Psi_* : & \quad \frac{1}{2\omega(\Psi_\ast) + 3} = 0, \quad \frac{1}{(2\omega(\Psi_\ast) + 3)^2} \left. \frac{d\omega}{d\Psi} \right|_{\Psi = \Psi_\ast} \neq 0.
\end{align*}
\]

(Strictly speaking, the latter case assumes \((2\omega(\Psi_\ast) + 3)\Pi_*^2 = 0\), see below Section 5 for an improved approach.) The properties of the fixed points (stable, unstable) and the form of the solutions around the fixed points are determined by the eigenvalues, and these by the values \(\omega(\Psi_\ast), \frac{d\omega}{d\Psi}|_{\Psi = \Psi_\ast}, V(\Psi_\ast), \frac{dV}{d\Psi}|_{\Psi = \Psi_\ast}, \frac{d^2V}{d\Psi^2}|_{\Psi = \Psi_\ast}\). Notice that \(\Psi_\ast\) is compatible with the “limit of general relativity”, i.e. the local weak field experiments (the matter density in the Solar System is still much higher than the energy density of the potential, even if on the cosmological scales the potential may dominate). On the other hand, the \(\Psi_*\) point is generally not in accord with the weak field observations, although in some special situations it may perhaps be possible to salvage it by taking into account the contribution of the potential to the PPN parameters.

If matter density dominates over potential \((\rho \neq 0, V \equiv 0)\) we can use a new time variable \(dp \equiv \left| H + \frac{\Psi}{2\Psi} \right| dt\) to eliminate \(H\) and obtain

\[
\begin{align*}
\Psi' &= \Upsilon \\
\Upsilon' &= \frac{2\omega(\Psi) + 3}{8\Psi^2} \Upsilon^3 + \frac{6\omega(\Psi) + 9 - 4\Psi \frac{d\omega}{d\Psi}}{4\Psi(2\omega(\Psi) + 3)} \Upsilon^2 + \frac{3}{2} \Upsilon + \frac{3\Psi}{2\omega(\Psi) + 3}.
\end{align*}
\]

(21)

The fixed point \((\Psi' = 0, \Upsilon' = 0)\) in \(p\)-time corresponds to a fixed point \((\dot{\Psi} = 0, \dot{\Pi} = 0)\) in \(t\)-time, and is given by [2]:

\[
\begin{align*}
\Psi_\ast : & \quad \frac{1}{2\omega(\Psi_\ast) + 3} = 0, \quad \frac{1}{(2\omega(\Psi_\ast) + 3)^2} \left. \frac{d\omega}{d\Psi} \right|_{\Psi = \Psi_\ast} \neq 0.
\end{align*}
\]

(22)

Again it is good to see that the fixed point turns out to be compatible with the “limit of general relativity”.

5. Approximation scheme for the “limit of general relativity”
In the GR limit \((\Psi_{\ast}, \Pi_\ast)\), i.e. \(a\) \(\frac{1}{2\omega(\Psi_\ast) + 3} \to 0\), \(b\) \(\Psi \equiv \Pi \to 0\) the equations contain a \(0\) type indeterminacy (like \(\frac{2}{(\Pi + 3)}\) at the origin), thus demanding a more careful analysis. Let us focus around the vicinity of this point, \(\Psi = \Psi_\ast + x, \Pi = \Pi_\ast + y = y\), and expand in series \((x \text{ and } y \text{ being first order small})\):

\[
\begin{align*}
\frac{1}{2\omega(\Psi) + 3} &= \frac{1}{2\omega(\Psi_\ast) + 3} + A_\ast x + \ldots \approx A_\ast x, \\
(2\omega(\Psi) + 3)\Pi^2 &= \frac{y^2}{0 + A_\ast x + \ldots} = \frac{y^2}{A_\ast x} (1 + O(x)) \approx \frac{y^2}{A_\ast x}.
\end{align*}
\]

(23) (24)
where (c) $A_* \equiv \left. \frac{d}{d\Psi} \left( \frac{1}{2\omega(\Psi)+3} \right) \right|_{\Psi_*} \neq 0$ ja (d) $\frac{1}{2\omega(\Psi)+3}$ is differentiable at $\Psi_*$. In the following we will focus on the case when the matter density can be neglected in favor of the potential (i.e. take $V \neq 0, \rho \equiv 0$). Keeping only the terms which are of first order in $x$ and $y$, the dynamical system (16), (17) becomes [4]

\begin{align*}
\dot{x} & = y, \\
\dot{y} & = \frac{y^2}{2x} - C_1 y + C_2 x, 
\end{align*}

where the constants

\begin{align*}
C_1 & \equiv \pm \sqrt{\frac{3\kappa^2 V(1)}{\Psi_*}}, \\
C_2 & \equiv 2\kappa^2 A_* \left( 2V(\Psi) - \frac{dV(\Psi)}{d\Psi} \right) \bigg|_{\Psi_*},
\end{align*}

encode the behavior of the functions $\omega$ and $V$ near the GR point. In a similar manner the PPN parameters (3), (4) and $\frac{\dot{G}}{G} (5)$ are approximated by

\begin{align*}
\beta - 1 & = -\frac{1}{2} A^2 \Psi_*, \\
\gamma - 1 & = -2A_* x, \\
\frac{\dot{G}}{G} & = -\frac{1}{1 - A_* \Psi_*} \dot{x}.
\end{align*}

Via the Friedmann equation (15) we can also express $H(x(t)), \dot{H}(x(t))$ and get the equation of state parameter as

\begin{align*}
w & = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{1}{C^2 \Psi_*} \left[ \frac{3}{2} \left( 1 + \frac{1}{\Psi_* A_*} \right) \frac{x^2}{x} - 4C_1 \dot{x} + 3C_2 x \right].
\end{align*}

6. Phase space trajectories

The phase trajectories for the nonlinear approximate system (25), (26) are determined by the equation

\begin{align*}
\frac{dy}{dx} = \frac{y^2}{2x} - C_1 + \frac{x}{y} C_2.
\end{align*}

Its solutions depend on the sign of $C^2 + 2C \equiv C$ [4]:

\begin{align*}
|x| K & = \left| \frac{1}{2} y^2 + C_1 y x - C_2 x^2 \right| \exp(-C_1 f(u)) , \quad u \equiv \frac{y}{x}, \\
f(u) & = \frac{1}{\sqrt{C}} \ln \left| \frac{u + C_1 - \sqrt{C}}{u + C_1 + \sqrt{C}} \right| \quad \text{if } C > 0, \\
& = -\frac{2}{u + C_1} \quad \text{if } C = 0, \\
& = \frac{2}{\sqrt{|C|}} \left( \arctan \frac{u + C_1}{\sqrt{|C|}} + n\pi \right) \quad \text{if } C < 0.
\end{align*}

The phase portraits which emerge for the various values of the constants are classified in Table 1 and depicted on Figure 1. As the approximate system is non-linear the phase portraits split
Table 1. The topology of trajectories for the approximate nonlinear system (32).

| No. | Parameters | Topology of trajectories |
|-----|------------|--------------------------|
| 1.a | \( C_1 > 0 \) \( C_2 > 0 \) | 2 hyperb., 2 st. & 2 unst. parab. sectors |
| 1.b | \( C_1 > 0 \) \( C_2 = 0 \) | 1 stable & 1 unstable parabolic sector, 2 stable sectors of degenerate fixed points |
| 1.c | \( C_1 > 0 \) \( -\frac{C_2^2}{2} < C_2 < 0 \) | 2 elliptic, 4 stable parabolic sectors |
| 1.d | \( C_1 = 0 \) \( C_2 > 0 \) | 2 hyperb., 2 st. & 2 unst. parab. sectors |
| 1.e | \( C_1 < 0 \) \( C_2 > 0 \) | 2 hyperb., 2 st. & 2 unst. parab. sectors |
| 1.f | \( C_1 < 0 \) \( C_2 = 0 \) | 1 stable & 1 unstable parabolic sector, 2 unst. sectors of degenerate fixed points |
| 1.g | \( C_1 < 0 \) \( -\frac{C_2^2}{2} < C_2 < 0 \) | 2 elliptic, 4 unstable parabolic sectors |
| 2.a | \( C_1 > 0 \) \( C_2 = -\frac{C_2^2}{2} \) | 2 elliptic, 2 stable parabolic sectors |
| 2.b | \( C_1 = 0 \) \( C_2 = 0 \) | 2 stable & 2 unstable parabolic sectors |
| 2.c | \( C_1 < 0 \) \( C_2 = -\frac{C_2^2}{2} \) | 2 elliptic, 2 unstable parabolic sectors |
| 3.a | \( C_1 > 0 \) \( C_2 < -\frac{C_2^2}{2} \) | 2 elliptic sectors |
| 3.b | \( C_1 = 0 \) \( C_2 < 0 \) | 2 elliptic sectors |
| 3.c | \( C_1 < 0 \) \( C_2 < -\frac{C_2^2}{2} \) | 2 elliptic sectors |

into sectors, which can be hyperbolic, elliptic, or parabolic; stable or unstable. Typically there are many trajectories passing through the GR point either once or multiple times.

By comparing the tangents of the trajectories in full system (16)-(17) with the ones of the nonlinear approximate system (25)-(26), we can argue [4] that the topology of trajectories in the nonlinear approximation is representative of those of the full system. Therefore one should take the results of the nonlinear approximation seriously. In the end, the classification of the phase portraits reveals that only if \( C_1 > 0 \) and \( C_2 < 0 \), i.e.

\[
\left. \frac{d}{d\Psi} \left( \frac{1}{2\omega(\Psi) + 3} \right) \right|_{\Psi_*} \left( 2V(\Psi) - \frac{dV(\Psi)}{d\Psi} \Psi \right)_{\Psi_*} < 0
\] (35)

does the GR point function as an asymptotic attractor for the flow of all trajectories in the vicinity.
Figure 1. Phase portraits of the nonlinear approximation (32) near the GR point. (Axes: $x = \Psi - \Psi^*$, horizontal and $y = \dot{\Psi}$, vertical.)
7. Time evolution

The approximate system (25)-(26) can be also combined into a second order equation

$$\ddot{x} + C_1 \dot{x} - C_2 x = \frac{\dot{x}^2}{2x}, \quad (36)$$

yielding solutions in terms of the cosmological time [5]:

$$\pm x = e^{-C_1 t} \left[ M_1 e^{\frac{1}{2} t v C} - M_2 e^{-\frac{1}{2} t v C} \right]^2, \quad \text{if } C > 0, \quad (37)$$

$$= e^{-C_1 t} \left[ e^{\frac{1}{2} C t - M_2} \right]^2, \quad \text{if } C = 0, \quad (38)$$

$$= e^{-C_1 t} \left[ N_1 \sin\left(\frac{1}{2} t \sqrt{|C|}\right) - N_2 \cos\left(\frac{1}{2} t \sqrt{|C|}\right) \right]^2, \quad \text{if } C < 0, \quad (39)$$

where $M_1, M_2, t_1, N_1, N_2$ are constants of integration (determined by the initial conditions). In particular, the behavior of solutions which approach general relativity can be classified under two characteristic types: (i) exponential or linear exponential convergence, and (ii) damped oscillations around general relativity.

Given the solutions above, one can now plug $x(t)$ into the expressions for the PPN parameters (28), (29), or $\frac{\dot{G}}{G}$ (30), or $w$ (31) to find how they should evolve in time. The solutions for which GR is an attractor eventually converge to de Sitter, of course. But as they do it, some of them have oscillating, some monotonically evolving $w$. There are solutions which do not cross the phantom divide ($w = -1$), solutions which cross it only once, and solutions which cross it periodically. The particular behavior is determined by the constants $C_1$, $C_2$ and $A_*$ (and to certain extent also by the initial conditions).

8. Summary and outlook

We have found and characterized the fixed points of STG cosmology in the case when potential dominates over cosmological matter density. In particular we have also found the general analytic form of solutions around the fixed points. This can be applied to cosmological expansion: our results inform whether the solutions of any particular theory have oscillating, phantom crossing etc behavior.

The analysis in the case of matter domination should be refined by carefully dealing with the indeterminacy in the equations, it’s a work in progress. Next step, if possible, would be to study the cross-over phase from matter domination to potential domination.

The idea to be pursued is to rely upon the attractor mechanism – instead of scanning the full phase space range of all theories, it makes sense to focus upon the vicinity of certain points which are favored by the cosmological dynamics. Hence arises a selection principle: only those theories and models are viable which possess attractive fixed points around where the solutions satisfy observational constraints.

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