Relative entropy for coherent states from Araki formula

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Abstract

We make a rigorous computation of the relative entropy between the vacuum state and a coherent state for a free scalar in the framework of AQFT. We study the case of the Rindler Wedge. Previous calculations including path integral methods and computations from the lattice, give a result for such relative entropy which involves integrals of expectation values of the energy-momentum stress tensor along the considered region. However, the stress tensor is in general non unique. That means that if we start with some stress tensor, then we can “improve” it adding a conserved term without modifying the Poincaré charges. On the other hand, the presence of such improving term affects the naive expectation for the relative entropy by a non vanishing boundary contribution along the entangling surface. In other words, this means that there is an ambiguity in the usual formula for the relative entropy coming from the non uniqueness of the stress tensor. The main motivation of this work is to solve this puzzle. We first show that all choices of stress tensor except the canonical one are not allowed by positivity and monotonicity of the relative entropy. Then we fully compute the relative entropy between the vacuum and a coherent state in the framework of AQFT using the Araki formula and the techniques of Modular theory. After all, both results coincides and give the usual expression for the relative entropy calculated with the canonical stress tensor.

1 Introduction

The algebraic description of quantum field theory (AQFT) focuses on the local algebras of operators generated by fields in regions of the space rather than the field operators themselves. This gives a “basis independent” formulation which does not depend on the particular fields used for the description of the theory. Statistical properties of the state in these local algebras has been the subject of much recent interest in different areas of physics ranging from holography to condensed matter. Given one or more states and algebras, several entropic quantities can be defined which give natural measures of the statistics of fluctuations. In a certain sense, these assignments of numbers to algebras in AQFT is the analogous of the study of correlators in the approach based on field operators.

In actual computations in specific models it is customary and useful to assume a cutoff model, such as a lattice, and proceed to the computation taking the continuum limit as a final step. In general we expect the quantity computed belongs to the continuous theory as far as the result does not depend on the regularization. In the cutoff model, given a global pure state \( \Phi \in \mathcal{H} \) one can consider the reduced density matrix \( \rho_R^{\Phi} \) in a space region \( R \) of a lattice and compute its von Neumann entropy

\[
S_R^{\Phi} = -\text{tr}_{\rho_R^{\Phi}} \log \rho_R^{\Phi}.
\]  

This is divergent and not well-defined in the continuum due to the large amount of entanglement of UV modes between both sides of the region boundary. However, given two states \( \Psi \) and \( \Omega \) we can also compute the relative entropy

\[
S_R (\Phi \mid \Omega) = \text{tr}_{\rho_R^{\Phi}} (\log \rho_R^{\Phi} - \log \rho_R^{\Omega}) ,
\]
which is much better behaved than the entropy (see for example [11][14] for actual calculations). In fact, the relative entropy has an expression directly in the continuous theory for type III algebras in terms of Araki formula [15]. This shows it is free from ambiguities. Relative entropy is an important quantity in quantum information that measures distinguishability between states. It is always positive and increasing for fixed states under increasing algebras. It has recently been very useful in holography to understand the bulk-boundary map [16][19] and in the proof of the quantum null energy condition [20].

Another object that has a nice continuum limit is the following one parameter group of unitaries

\[
(\rho_\Omega^R)^{1/2} \otimes (\rho_\Omega^{R'})^{-1/2},
\]

where \( R' \) is the complement of \( R \) and we are assuming there is a decomposition of the full operator algebra as a tensor product of the algebras in \( R \) and \( R' \). This 1-parametric group is called the modular group. The generator,

\[
K_\Omega = -K_R \otimes 1 + 1 \otimes K_{R'}, \quad K_R = -\log \rho_\Omega^R,
\]

is called the modular Hamiltonian. A well-known case where the modular Hamiltonian can be computed exactly is the case when \( R \) is the Rindler wedge corresponding to a spatial slice \( x^1 > 0 \) at \( x^0 = 0 \), and the state is the vacuum. In this case \( K_\Omega = 2\pi K_1 \) with \( K_1 \) the boost generator. In terms of the energy density operator we can write

\[
K_\Omega = 2\pi \int d^{d-1} x \, x^1 \, T_{00}(x).
\]

Returning to the relative entropy, it is useful to write (1.2) as

\[
S_R (\Phi | \Omega) = \Delta \langle K_R \rangle - \Delta S_R
\]

where

\[
\Delta \langle K_R \rangle = \text{tr} \rho_\Omega^R K_R - \text{tr} \rho_\Omega^{R'} K_{R'},
\]

\[
\Delta S = S_\Phi^R - S_{\Omega}^R.
\]

Written in this way, the relative entropy is the variation in expectation value of an operator minus the variation in the entropy between the two states. The positivity of relative entropy means that \( \Delta \langle K_R \rangle \geq \Delta S_R \). In this form, when \( R \) is the Rindler wedge, this inequality has been related to Bekenstein’s bound on entropy [21].

Even if the relative entropy is well defined in the continuum, a mathematically rigorous definition of the continuum limit of the two terms in (1.3) has not been worked out in the literature yet. One difficulty is that the operator \( K_{R} \) is only half of the modular Hamiltonian (1.4). Even if the modular Hamiltonian has a good operator limit in the continuum, its half part \( K_{R} \) is at most a sesquilinear form. If we focus for simplicity on the case of the half space and where \( \Omega \) is the vacuum, we can write from (1.5) that

\[
K_{R} = 2\pi \int_{x_{1} > 0} d^{d-1} x \, x^1 \, T_{00}(x).
\]

This is not a well defined operator in Hilbert space because its fluctuation \( \langle \Omega, i \hat{K}_{R}^{2} \Omega \rangle \) diverges. However, expectation values as in \( \Delta \langle K_{R} \rangle \) can still be computed. Another more important issue is that the act of cutting the modular Hamiltonian in two pieces generate ambiguities. We are allowed for example to add field operators localized at the boundary such that \( K_{R} \) has still the same localization and commutation relations with operators inside \( R \). Another view of the same problem is that, hidden in expression (1.9), there is an ambiguity related to the non uniqueness of the stress tensor. For example, for the free hermitian scalar field, starting from the canonical stress tensor

\[
T_{\mu \nu}^{\text{can}} =: \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} \eta_{\mu \nu} (\partial_{\sigma} \phi \partial^{\sigma} \phi - m^2 \phi^2) : ,
\]

we can add an “improving term” to obtain a new stress tensor

\[
T_{\mu \nu} = T_{\mu \nu}^{\text{can}} + \frac{\lambda}{2\pi} (\partial_{\mu} \partial_{\nu} - g_{\mu \nu} \partial^2) : \phi^2 : .
\]

The Poincaré generators obtained from (1.11) equals the ones obtained from (1.10), since both expressions differs in a boundary term which vanishes when the integration region is the whole space. However, the expression (1.9) for \( K_{R} \) involves an integration in a semi-infinite region, and hence the presence of an improving term adds a non zero extra boundary term to the result,

\[
K_{R} \to K_{R} + \lambda \int_{x_{1} = 0} d^{d-2} x : \phi^2 (x) : .
\]
This is essentially the only boundary term we can add with the correct dimensions and that does not require a dimensionful coefficient with negative dimensions. This can have non zero expectation values for certain states and makes the definition of $\Delta(K_R)$ ambiguous.

Since the relative entropy is well defined, this ambiguity must be compensated by another one in the definition of $\Delta S_R$ in (1.6). This is the subtraction of two divergent quantities and again we do not have a mathematically rigorous definition in the continuum. We can make this definition unambiguous in a natural way by using a particular regularization of entropy that has been proposed in the literature [22-24]. The idea is to associate the entropy (for a pure state) with half the mutual information $I(R^+ \cap R^\prime_\epsilon^+)$ between two non intersecting regions on both sides of the boundary of $R$. The regions $R^\prime_\epsilon$ are displaced a distance $\epsilon$ from the boundary of $R$. For the case of the Rindler wedge we can take $R^\prime_\epsilon$ formed by points with $x^1 > \epsilon$ and $R^-_\epsilon$ formed by points with $x^1 < -\epsilon$. The mutual information is also a relative entropy and is well defined in the continuum. Then, a well defined $\Delta S_R$ is given by

$$\Delta S_R = \frac{1}{2} \lim_{\epsilon \to 0} \left( I(\Phi^+, R^-_\epsilon) - I_R(R^+\cap R^\prime_\epsilon) \right).$$  (1.13)

When it is computed in the lattice, it coincides with the usual $\Delta S_R$.

Defining $\Delta S_R$ rigorously through (1.13), then $\Delta(K_R)$ is also well defined through

$$\Delta(K_R) = S_R(\Phi \mid \Omega) + \Delta S_R.$$  (1.14)

Then the question that motivates this paper is whether this definition agrees with the expectation value of $\langle \phi^2 \rangle$. In such case, boundary terms in this expression should be automatically fixed, in particular we should be able to study which value of the improvement term is the correct one for a scalar field in (1.11).

In order to (partially) settle this issue, in this paper we analyze the relative entropy between a coherent state for a free scalar field and the vacuum in the Rindler wedge. Coherent states are states formed out by acting on the vacuum with a unitary operator that is the exponential of the smeared field. It is easy to see that the action of this simple type of unitaries on the algebras of two disjoint regions is equivalent to a product of unitary transformations on each of the algebras separately. Hence, these unitaries do not change the mutual information. With our definition (1.13) we automatically have $\Delta S_R = 0$ for these states. Then the question simplifies to see whether for coherent states

$$S_R(\Phi \mid \Omega) = 2\pi \int_{x^1 > 0} d^{d-1}x \ x^1 \ \langle \Phi, T_{00}(x) \Phi \rangle,$$  (1.15)

and which is the right improvement term. Notice that coherent states can change the expectation value of $\langle \phi^2 \rangle$.

In section 2 assuming that (1.15) is correct for the canonical stress tensor, we will show that the only possibility is the canonical stress tensor, i.e. $\lambda = 0$. We show this by imposing bounds which come from the positivity and monotonicity of the relative entropy.

In the rest of the paper we actually compute the relative entropy using Araki formula and show the result (1.15) is correct for the canonical stress tensor. We note that, while this paper was being prepared, a similar calculation by R. Longo has appeared in the literature [13]. A simpler case where the unitary has support inside the wedge has previously appeared in [24]. Our paper differ from the one by Longo in motivation, scope, and several details, while there is an overlap in the main technical ideas.

To make this article as much self contained as possible, in section 3 we start reviewing some basic aspects of the Wightman theory of the free scalar, to spend then some work to define the net of local algebras of the free scalar quantum field satisfying the axioms of AQFT. Because of a forthcoming necessity, we consider two different approaches. The first one is the usual approach where we define the net of algebras associated to spacetime regions. The second one consist in defining the local algebras associated to spatial sets belonging to a common Cauchy surface. We also explain how these two approaches are related. In section 4 we review the basic concepts of the Modular theory of von Neumann algebras. In particular we introduce the modular operator used to derive the modular Hamiltonian and the modular flow. We also discuss the related theorems of Tomita-Takesaki and Bisognano-Whichmann. And finally, we introduce the relative modular operator used in the definition of the relative entropy for general von Neumann algebras. Such formula will be the starting point in the computation of the proposed relative entropy, which is explicitly calculated in section 5. There, we study separately the (trivial) case when the coherent state belongs to the Wedge algebra, and the more interesting (and also more difficult) case when the coherent state has a non vanishing density along the entangling surface. We provide a complete mathematical rigorous proof of such results. For a better reading of the article, the proof of the theorems and the tedious mathematical calculations were placed into the appendices.
2 Boundary terms in the relative entropy

According to the discussion above, there is an ambiguity on the expression (1.13) for the relative entropy of a coherent state coming from the different possible choices of an improving term for the stress energy-momentum tensor. According to (1.12), the relative entropy could be written as the usual contribution with the canonical stress tensor plus a boundary term coming from the improving

\[ S_R (\Phi \mid \Omega) = \lambda \int_{x_1=0} d^{d-2}x \langle \Phi, \phi^2 (\bar{x}) \Phi \rangle + 2\pi \int_{x_1>0} d^{d-1}x x^1 \langle \Phi, T^{\alpha\beta}_{00} (\bar{x}) \Phi \rangle. \]  

(2.1)

In this section we assume this formula is correct and show that the only consistent choice is \( \lambda = 0 \).

We shall a coherent state

\( \Phi = e^{i \int x x^{-1} d^{d-1}x [\varphi(x)f_{\varphi}(x)+\pi(x)f_{\pi}(x)]\Omega}, \)

(2.2)

with \( f_{\varphi}, f_{\pi} \in \mathcal{S} (\mathbb{R}^{d-1}) \) independent and real functions, \( \varphi (\bar{x}) := \phi (0, \bar{x}) \) and \( \pi (\bar{x}) := \delta_0 \phi (0, \bar{x}) \). In this case, a straightforward computation from (2.1) gives

\[ S_R (\Phi \mid \Omega) = \lambda \int_{x_1=0} d^{d-2}x f_{\pi} (\bar{x})^2 + 2\pi \int_{x_1>0} d^{d-1}x \frac{1}{2} (f_{\varphi} (\bar{x})^2 + (\nabla f_{\pi} (\bar{x})^2)^2 + m^2 f_{\pi} (\bar{x})^2). \]  

(2.3)

Regardless what should be the true value for \( \lambda \), if we want that (2.1) and (2.3) represent real expressions for a relative entropy, they must satisfy all the properties known for a relative entropy. In particular we will concentrate on the positivity

\[ S_R (\Phi \mid \Omega) \geq 0, \]

(2.4)

and the monotonicity, that for the case of wedges implies

\[ S_R (\Phi \mid \Omega)|_{\mathcal{W}_y} \geq S_R (\Phi \mid \Omega)|_{\mathcal{W}_{y'}}, \]

(2.5)

where \( S_R (\Phi \mid \Omega)|_{\mathcal{W}_y} \) is the relative entropy for the states \( \Psi, \Omega \) but associated to the algebra of the translated Rindler wedge \( \mathcal{W}_y := \{ x \in \mathbb{R}^d : x^1 - y > |x^d| \} \). In fact, \( \mathcal{W}_y \) is obtained applying a translation of amount \( y \), in the \( x^1 \) positive direction, to the original Rindler wedge \( \mathcal{W} \). From now on, we denote \( S_R (y) := S_R (\Phi \mid \Omega)|_{\mathcal{W}_y} \).

Therefore, the strategy we adopt is to choose conveniently functions \( f_{\varphi} \) and \( f_{\pi} \) and impose (2.1) and (2.5) on (2.3) in order to bound the allowed values for \( \lambda \). In fact, we will show that from positivity we obtain \( \lambda \geq 0 \) and from the monotonicity we obtain \( \lambda \leq 0 \), an hence it must be

\[ \lambda = 0. \]  

(2.6)

Then we conclude that, if we assume that (1.13) is the correct result for the relative entropy, such expression holds for the canonical stress energy-momentum tensor (1.10).

Before we start, we make two simplifications. The first one, which is obvious, is to take \( f_{\varphi} \equiv 0 \) and denote \( f := f_{\pi} \). The second one is to work in \( d = 1 + 1 \) dimensions. The general result for any dimensions, could be obtained easily from the former.

2.1 Lower bound from positivity

We start with the expression

\[ S_R (\Phi \mid \Omega) = \lambda f (0)^2 + \pi \int_{0}^{+\infty} dx x \left( f' (x)^2 + m^2 f (x)^2 \right), \]  

(2.7)

where \( f \) is a real-valued function belonging to \( \mathcal{S} (\mathbb{R}) \). Then, the positivity of the relative entropy means that

\[ 0 \leq \lambda f (0)^2 + \pi \int_{0}^{+\infty} dx x f' (x)^2 + \pi m^2 \int_{0}^{+\infty} dx x f (x)^2. \]  

(2.8)

By scaling the function \( f(x) \to f(x)/\beta \) the first two terms of the right hand side are constant while the last one gets multiplies by \( \beta^2 \). Hence, we can make the last term as small as we want and simply take \( m = 0 \) in the following. Taking \( f \) such that \( f (0) \neq 0 \) we get

\[ 0 \leq \lambda + \pi \int_{0}^{+\infty} dx x f' (x)^2 f (0)^2. \]  

(2.9)
Now, we introduce a convenient family of real functions \((f_a)_{a>0} \in \mathcal{S}(\mathbb{R})\) given by
\[
f_a(x) := \log \left( \frac{L}{a} + x \right) e^{-\pi x}, \quad x \geq 0,
\]
and where \(L > 0\) is a dimensionful fixed constant. An straightforward computation shows that the integral in equation (2.9) behaves as
\[
\int_0^\infty dx x f'_a(x)^2 = -\log(a) + \mathcal{O}(1), \quad a \geq 0.
\]
Then replacing (2.11) into (2.9) we get
\[
0 \leq \lambda - \frac{L^2}{4} \frac{\log(a) + \mathcal{O}(1)}{\log^2(a)}.
\]
Finally taking the limit \(a \to 0^+\) we get the desired result
\[
\lambda \geq 0.
\]

2.2 Upper bound from monotonicity

We start with the expressions
\[
S_R(0) = \lambda f(0)^2 + \pi \int_0^\infty dx x \left( f'(x)^2 + m^2 f(x)^2 \right),
\]
\[
S_R(y) = \lambda f(y)^2 + \pi \int_y^{y+\epsilon} dx (x-y) \left( f'(x)^2 + m^2 f(x)^2 \right),
\]
where \(f\) is a real-valued function belonging to \(\mathcal{S}(\mathbb{R})\). We can eliminate the mass terms by scaling as in the previous section. The monotonicity \(S_R(0) \geq S_R(y)\) for \(y \geq 0\) reads
\[
\lambda \left( f(y)^2 - f(0)^2 \right) \leq \pi \int_0^y dx x f'(x)^2 + \pi y \int_y^{y+\epsilon} dx f'(x)^2.
\]
Now, we introduce a convenient family of functions parametrized with the constants \(\alpha \in (0, \frac{1}{2}), \delta \in (0, 1), y > 0, \epsilon > 0\) given by
\[
f_{\alpha,\delta,y,\epsilon}(x) := g_{\alpha,\delta,y}(x) \Theta_{y,\epsilon}(x), \quad \text{for } x \geq 0,
\]
where
\[
g_{\alpha,\delta,y}(x) := \left( \frac{x}{y} (1 - \delta) + \delta \right)^\alpha,
\]
and \(\Theta_{y,\epsilon}\) is a smooth step function with the condition
\[
\Theta_{y,\epsilon}(x) = \begin{cases} 1 & x \leq y, \\ 0 & x \geq y + \epsilon. \end{cases}
\]
We introduce such a step function to ensure that \(f_{\alpha,\delta,y,\epsilon} \in \mathcal{S}(\mathbb{R})\) for any values of \((\alpha, \delta, y, \epsilon)\) in the set specified above. The functions \(f_{\alpha,\delta,y,\epsilon}\) are smoothly extended to the whole real line. In particular we use
\[
\Theta_{y,\epsilon}(x) := \left[ 1 + \exp \left( -\frac{2\epsilon \left( x - y - \frac{\delta}{2} \right)}{(y - y - \frac{\delta}{2})^2 - \epsilon^2} \right) \right]^{-1}, \quad \text{if } y < x < y + \epsilon,
\]
which has the useful property \(\max_{x \in \mathbb{R}} |\Theta_{y,\epsilon}(x)| = \frac{2}{\epsilon}\). From now on, we will not write the cumbersome subindices of the above functions. For the different terms of (2.16) we have that
\[
f(y)^2 - f(0)^2 = 1 - \delta^{2\alpha},
\]
\[
\pi \int_0^y dx x f'(x)^2 = \pi \frac{\alpha}{2} \left( \frac{2\alpha \delta - \delta^{2\alpha}}{1 - 2\alpha} + 1 \right),
\]
\[
\pi y \int_y^{y+\epsilon} dx f'(x)^2 \leq \pi y \int_y^{y+\epsilon} dx \left| g'(x)^2 \Theta(x)^2 \right| + \pi y \int_y^{y+\epsilon} dx \left| g(x)^2 \Theta'(x)^2 \right|
\]
\[
+ \pi y \int_y^{y+\epsilon} dx \left| 2g'(x) g(x) \Theta(x) \Theta'(x) \right|.
\]
\(^{1}\)The functions \(f_a\) are smoothly extended to the whole real line. Such extension is guaranteed by a theorem due to Seeley \[25\].
We deal with each term of (2.24) separately
\[
\pi y \int_y^{y+\epsilon} dx \left| g'(x) \Theta(x)^2 \right| \leq \pi y \int_y^{y+\epsilon} dx \left| g'(x) \right|^2 \\
= \frac{\pi \alpha^2 (1 - \delta)}{1 - 2\alpha} \left[ 1 - \left( 1 + \frac{(1 - \delta)\epsilon}{y} \right)^{2\alpha - 1} \right] \rightarrow \frac{\pi \alpha^2 (1 - \delta)}{1 - 2\alpha}, \quad \epsilon \rightarrow +\infty, \tag{2.25}
\]
\[
\pi y \int_y^{y+\epsilon} dx \left| 2g'(x) g(x) \Theta(x) \Theta'(x) \right| \leq \pi y^2 \int_y^{y+\epsilon} dx \left| 2g'(x) g(x) \right| = 2\pi y^2 \left[ g(y + \epsilon)^2 - g(y)^2 \right] \\
= \frac{2\pi y^2}{\epsilon} \left[ 1 + \frac{(1 - \delta)\epsilon}{y} \right]^{2\alpha - 1} - 1 \rightarrow 0, \quad \epsilon \rightarrow +\infty. \tag{2.26}
\]
\[
\pi y \int_y^{y+\epsilon} dx \left| g(x)^2 \Theta'(x) \right|^2 \leq \pi y^4 \int_y^{y+\epsilon} dx \left| g(x)^2 \right|^2 \\
= \frac{4\pi y^4}{(1 + 2\alpha)(1 - 2\alpha)\epsilon^2} \left[ 1 + \frac{(1 - \delta)\epsilon}{y} \right]^{2\alpha + 1} - 1 \rightarrow 0. \tag{2.27}
\]

where in the last steps of each computation we take the limit \(\epsilon \rightarrow +\infty\). It is valid to take this limit in the inequality since it must hold for all \(\epsilon > 0\). Replacing these partial results on (2.10) we arrive at
\[
\lambda \left( 1 - \delta^{2\alpha} \right) \leq \pi \frac{\alpha}{2} \left( \frac{2\alpha \delta - \delta^{2\alpha}}{1 - 2\alpha} + 1 \right) + \pi \alpha^2 \left( 1 - \delta \right) \frac{1}{1 - 2\alpha}. \tag{2.28}
\]

Then, taking the limit \(\delta \rightarrow 0^+\) we get
\[
\lambda \leq \pi \frac{\alpha}{2} + \pi \frac{\alpha^2}{1 - 2\alpha}, \tag{2.29}
\]
and finally taking \(\alpha \rightarrow 0^+\) we arrive at the desired result
\[
\lambda \leq 0. \tag{2.30}
\]

3 The free hermitian scalar field

3.1 Preliminaries

Before we start defining the local algebras for the free hermitian scalar field, we think it is convenient to review some aspects of the Wightman theory of the free scalar field in a \(d\)-dimensional Minkowski spacetime. In such a theory, the quantum field \(\phi(x)\) is considered as an operator valued distribution acting on the (symmetric) Fock Hilbert space \(\mathcal{H}\). To describe it properly, we shall introduce the following three vector spaces

The space of test functions. The space of test functions is the Schwartz space \(\mathcal{S}(\mathbb{R}^d, \mathbb{R})\) of real, smooth and exponentially decreasing functions at infinity. This space carries naturally a representation of the restricted Poincaré group \(P^\uparrow_+\) given by \(f \mapsto f(\Lambda, a)\), with \(f(\Lambda, a)(x) := f((\Lambda (x - a))\) for any \((\Lambda, a) \in P^\uparrow_+\).

The one particle Hilbert space. The Hilbert space \(\mathcal{H}\) of one particle states of mass \(m > 0\) and zero spin is made up of the square-integrable functions on the mass shell hyperboloid \(H_m := \{p \in \mathbb{R}^d : p^2 = m^2, p^0 > 0\}\) with the Poincaré invariant measure \(d\mu(p) := \Theta(p^0) \delta (p^2 - m^2) d^d p\). It can be realized as
\[
\mathcal{H} = L^2 \left( \mathbb{R}^{d-1}, \frac{d^{d-1} p}{2\omega (\vec{p})} \right), \tag{3.1}
\]
\[
(\mathbf{f}, \mathbf{g})_{\mathcal{H}} = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1} p}{2\omega (\vec{p})} \mathbf{f}(\vec{p})^* \mathbf{g}(\vec{p}) \tag{3.2}
\]
where \(p^0 = \sqrt{\vec{p}^2 + m^2} =: \omega(\vec{p})\). Such a space carries an unitary representation of \(P^\uparrow_+\) given by
\[
(u(\Lambda, a) \mathbf{f})(p) = e^{ip.a} \mathbf{f}(\Lambda^{-1} p), \tag{3.3}
\]
for any \(\mathbf{f} \in \mathcal{H}\) and \((\Lambda, a) \in P^\uparrow_+\).

\(\footnote{We use the convention \(\eta_{\mu \nu} = \text{diag}(1, -1, \ldots, -1)\) for the spacetime metric.}\)
The Fock Hilbert space. The Fock Hilbert space $\mathcal{H}$ is the direct sum of the symmetric tensor powers of the one particle Hilbert space $\mathcal{H}_1$, i.e.

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n, \text{sym}}_1.$$  \hspace{1cm} (3.4)

For each $\mathbf{h} \in \mathcal{H}_1$, the creation and annihilation operators $A^* (\mathbf{h})$ and $A (\mathbf{h})$ act over $\mathcal{H}$ as usual. The Fock space naturally inherits from $\mathcal{H}_1$ an unitary representation of $\mathcal{P}^+_1$ which is denoted by $U (\Lambda, a)$. According to that there is a unique (up to a phase) $\mathcal{P}^+_1$-invariant vector denoted by $\Omega = 1 \in S^{\otimes 0}_v$, which is called the vacuum vector. For each $\mathbf{h} \in \mathcal{H}_1$, the normalized vector

$$e^{\mathbf{h}} := e^{-\frac{i}{2} \parallel \mathbf{h} \parallel^2} \sum_{n=0}^{\infty} \frac{\mathbf{h}^{\otimes n}}{\sqrt{n!}} \in \mathcal{H},$$

is called coherent vector, and it satisfies the relations $e^0 = \Omega$ and $\langle \Omega, e^{\mathbf{h}} \rangle_{\mathcal{H}} = e^{-\frac{i}{2} \parallel \mathbf{h} \parallel^2}$.

Now we shall consider the following real embedding $E : S (\mathbb{R}^d, \mathbb{R}) \rightarrow \mathcal{H}$

$$(Ef) (\hat{p}) := \sqrt{2\pi} \int_{\mathbb{R}^d} f (x) e^{ip \cdot x} dx \in \mathcal{H},$$

where $\hat{f} (p) := (2\pi)^{-d} \int_{\mathbb{R}^d} f (x) e^{ip \cdot x} dx$ is the usual Fourier transform. Such embedding is Poincaré covariant, i.e.

$$E (f) (\Lambda, a) = u (\Lambda, a) E (f),$$

and the set $E (S (\mathbb{R}^d, \mathbb{R})) \subset \mathcal{H}$ is dense. From now on, we naturally identified functions on $S (\mathbb{R}^d, \mathbb{R})$ with vectors on $\mathcal{H}$ through the above embedding.

For each real test function $f \in S (\mathbb{R}^d, \mathbb{R})$, the quantum field $\phi (f)$ is defined as an operator acting on $\mathcal{H}$ through the formula

$$\phi (f) := A (E (f)) + A^* (E (f)) \hspace{1cm} (3.7)$$

For complex valued functions $f = f_1 + if_2 \in S (\mathbb{R}^d)$, we define $\phi (f) := \phi (f_1) + i \phi (f_2)$ in order for $f \rightarrow \phi (f)$ to be a linear map. It’s not difficult to see that such quantum field satisfies all the Wightman axioms. Moreover, it can be shown that $\phi (f)$ is essentially self-adjoint for $f \in S (\mathbb{R}^d, \mathbb{R})$, and hence we shall denote its closure with the same symbol $\phi (f)$.

All these things are what we are going to say about the Wightman theory of the free real scalar field. Further details can be found in the literature \[26\]-\[28\].

### 3.2 Axioms of AQFT

In the algebraic approach to QFT, we associate for each region of the spacetime a $C^*$ or von Neumann algebra which encode the algebraic relations between the quantum fields. Such assignment must satisfies a set of axioms which encode the physical conditions in the algebraic framework. Unless the specific set of axioms considered could depend on the underlying theory (specially on the spacetime considered), the assumptions listed below are very standard for the treatment of QFT’s on Minkowski space time.

To start we call a double cone to any open region $\mathcal{O} \subset \mathbb{R}^d$ of Minkowski spacetime defined by the intersection of the future open null cone of some point $x \in \mathbb{R}^d$ with the past open null cone of other point $y \in \mathbb{R}^d$\footnote{In particular, if $y$ is not in the timelike future of $x$, then $\mathcal{O} = \emptyset$.} The restricted Poincaré group $\mathcal{P}^+_1$ acts in the set of double cones as

$$g \mathcal{O} := \{ \Lambda x + a : x \in \mathcal{O} \}, \hspace{1cm} g = (\Lambda, a) \in \mathcal{P}^+_1. \hspace{1cm} (3.8)$$

In the algebraic approach to quantum field theory, we assign to each double cone $\mathcal{O} \subset \mathbb{R}^d$ a $C^*$-algebra $\mathfrak{U} (\mathcal{O})$, which are called the local algebras. Then, the quasilocal algebra is defined as

$$\mathfrak{U} := \bigcup_{\mathcal{O}} \mathfrak{U} (\mathcal{O}) \hspace{1cm} (3.9)$$

The collection (net)\footnote{Mathematically, due to axiom 1, the collection of local algebras form a net indexed by the set of double cones. The set of double cones form a direct set when it is ordered by the usual set inclusion.} of local algebras must satisfy

1. Isotony: for any pair of double cones $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{U} (\mathcal{O}_1) \subset \mathfrak{U} (\mathcal{O}_2)$. 


2. Causality: if $O_1$ and $O_2$ are spacelike separated (i.e. $O_1 \sim O_2$) then $\mathcal{U}(O_1), \mathcal{U}(O_2)] = \{0\}$.

3. Poincaré covariance: there is a (norm) continuous linear representation $\alpha_0$ of $P^+_{\infty}$ in $\mathcal{U}$, such that $\alpha_0 (\mathcal{U}(O)) = \mathcal{U}(gO)$ for any double cone $O$ and all $g \in P^+_{\infty}$.

4. Vacuum: there is a pure state $\omega$ in $\mathcal{U}$ invariant under all $\alpha_0$. Then, in its GNS representation $(\pi, \mathcal{H}, \Omega)$ the linear representation $\alpha_0$ is implemented by a positive energy unitary representation of $P^+_{\infty}$ in $\mathcal{H}$ in the sense that $U(g) \pi(A) U(g)^* = \pi(\alpha_0(A))$ for all $A \in \mathcal{U}$ and all $g \in P^+_{\infty}$. Positive energy means that the representation is strongly continuous and the infinitesimal generators $P^\mu$ of the translation subgroup (i.e. $U(0, a) = e^{iP^\mu a^\mu}$) have their spectral projections on the closed forward light cone $V_+ := \{ p \in \mathbb{R}^d : p \cdot p > 0 \}$.

When we want to study states which are constructed by local perturbations around the vacuum state $\omega$, we often work directly by the collection of concrete $C^*$-algebras $\pi(\mathcal{U}(O)) \subset \mathcal{B}(\mathcal{H})$ acting on the vacuum Hilbert space $\mathcal{H}$. For technical reasons, we usually work with the net of von Neumann algebras $\mathcal{R}(O) := \pi(\mathcal{U}(O))''$, where $''$ denotes the double commutant which coincides with the weak closure. Moreover, when we want to construct a concrete example of a QFT satisfying the axioms above, it is in general easier to construct a net of von Neumann algebras $\mathcal{O} \to \mathcal{R}(O)$ acting on a given Hilbert space.

One immediate consequence of the axioms is the famous Reeh-Schlieder theorem:

**Theorem 3.1.** Reeh-Schlieder

In any QFT satisfying the axioms 1. to 4. above, the vacuum vector $\Omega$ is cyclic for any local algebra $\pi(\mathcal{U}(O))$.

In the following subsections we will concretely define the net of algebras associated to a free hermitian scalar field satisfying the axioms listed above.

### 3.3 Local algebras for spacetime regions

The theory explained in this and the following subsections is broadly discussed in [30, 32]. For our convenience, we introduce our own notation and develop some relations between the concepts which were not too much clear for us. As we anticipate before, the net of local algebras we will construct for the free hermitian scalar field is a concrete net of von Neumann algebras acting on the vacuum Hilbert space.

On the one hand, according to the results developed on section 3.1, it seems to be natural to define the local algebras simply as

$$\mathcal{R}(O) := \{ \phi(f) : f \in S(\mathbb{R}^d, \mathbb{R}) \text{ and } \text{supp}(f) \subset O \}.$$  

This gives a bad definition since the operator $\phi(f)$ is not bounded. To overcome this difficulty, we remember that such operator is selfadjoint, and hence $e^{i\phi(f)}$ are unitaries, and in particular bounded. Having this in mind, we define the embedding $W : \mathcal{S} \to \mathcal{B}(\mathcal{H})$

$$W(\mathbf{h}) := e^{i(A(\mathbf{h})+A^*(\mathbf{h}))}. \tag{3.11}$$

The operators $W(\mathbf{h})$ are called **Weyl unitaries**. These operators satisfy the canonical commutation relations (CCR) in the Segal’s form [30]

$$W(h_1) W(h_2) = e^{-i\ln(h_1, h_2)} e W(h_1 + h_2) , \tag{3.12}$$

$$W(h)^* = W(-h) . \tag{3.13}$$

A Poincaré unitary $U(\Lambda, a)$ acts covariantly on a Weyl operator according to

$$U(\Lambda, a) W(h) U(\Lambda, a)^* = W(u(\Lambda, a) h) , \tag{3.14}$$

$$W(h) \Omega = e^{ih} . \tag{3.15}$$

Furthermore, for any $f \in S(\mathbb{R}^d, \mathbb{R})$ we have that

$$W(E(f)) = e^{i\phi(f)} =: W(f) , \tag{3.16}$$

$$U(\Lambda, a) W(f) U(\Lambda, a)^* = W(f_{(\Lambda, a)}). \tag{3.17}$$

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On the other hand, it is a very well-known fact that the structure of a free QFT is completely determined by the underlying one particle quantum theory. More concretely, the assignment $\mathcal{O} \rightarrow \mathcal{R} (\mathcal{O})$ is determined by the composition of two different maps

\[
\mathcal{O} \subset \mathbb{R}^d \rightarrow K (\mathcal{O}) \subset \mathcal{S},
\]

\[
K \subset \mathcal{S} \rightarrow \mathcal{R} (K) \subset \mathcal{B} (\mathcal{H}),
\]

which are called 1st and 2nd quantization maps respectively. We will treat each map separately.

### 3.3.1 First quantization map

Given any region $\mathcal{O} \subset \mathbb{R}^d$ of Minkowski spacetime, we define its (open) spacelike complement as

\[
\mathcal{O}' := \text{Int} \left\{ x \in \mathbb{R}^d : x \sim y, \forall y \in \mathcal{O} \right\},
\]

and its causal completion as $\mathcal{O}''$. A region $\mathcal{O} \subset \mathbb{R}^d$ is called causally complete iff $\mathcal{O} \equiv \mathcal{O}''$. In particular, any double cone is causally complete. The set of causally complete regions forms an orthocomplemented lattice. For any $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^d$ causally complete regions, the lattice operations are given by

- complement: the spacelike complement, i.e. $\mathcal{O} \rightarrow \mathcal{O}'$,
- infimum: $\mathcal{O}_1 \cap \mathcal{O}_2 := \mathcal{O}_1 \cap \mathcal{O}_2$, where $\cap$ means the usual set intersection,
- and supremum: $\mathcal{O}_1 \cup \mathcal{O}_2 := (\mathcal{O}_1 \cup \mathcal{O}_2)'$, where $\cup$ means the usual set union.

On the other hand, the set of real closed subspaces of $\mathcal{S}$ defines also an orthocomplemented lattice. For $K, K_1, K_2 \subset \mathcal{S}$ real closed subspaces, the lattice operations are given by

- complement: the symplectic complement, i.e. $K' := \{ h \in \mathcal{S} : \text{Im} (h, k) = 0 \text{ for all } k \in K \}$,
- infimum: $K_1 \cap K_2 := K_1 \cap K_2$, where $\cap$ means the usual set intersection,
- and supremum: $K_1 \cup K_2 := K_1 \oplus_{\mathbb{R}} K_2$, where $\oplus_{\mathbb{R}}$ means the usual real direct sum.

The 1st quantization map is defined by

\[
\mathcal{O} \rightarrow \text{K}(\mathcal{O}) := \{ E(f) : f \in S (\mathbb{R}^d, \mathbb{R}) \text{ and } \text{supp} (f) \subset \mathcal{O} \} \subset \mathcal{S}.
\]

It is not difficult to see that such a map is a lattice homomorphism.

Given a group $G$, a $G$-covariant lattice $L$ is a lattice equipped with a family of lattice automorphisms $\{ \alpha_g \}_{g \in G}$ such that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $g, h \in G$. Then, the lattices described above are $\mathcal{P}_+^1$-covariant lattices if we consider the following lattices automorphisms

\[
\mathcal{O} \rightarrow g\mathcal{O} := \{ \lambda x + a : x \in \mathcal{O} \} \subset \mathbb{R}^d,
\]

\[
K \rightarrow gK := u (\lambda, a) K,
\]

where $g = (\Lambda, a) \in \mathcal{P}_+^1$. As a consequence, the 1st quantization map is a $\mathcal{P}_+^1$-covariant lattice homomorphism, i.e. $g\mathcal{K}(\mathcal{O}) = \mathcal{K}(g\mathcal{O})$.

### 3.3.2 Second quantization map

Similarly, the set of von Neumann subalgebras of $\mathcal{B} (\mathcal{H})$ forms an orthocomplemented lattice. For any $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{B} (\mathcal{H})$ von Neumann subalgebras, the lattice operations are given by

- complement: the algebraic commutant, i.e. $\mathcal{R}' := \{ A : [A, B] = 0 \text{ for all } B \in \mathcal{R} \} \subset \mathcal{B} (\mathcal{H})$,
- infimum: $\mathcal{R}_1 \cap \mathcal{R}_2 := \mathcal{R}_1 \cap \mathcal{R}_2$, where $\cap$ means the usual set intersection,
- and supremum: $\mathcal{R}_1 \vee \mathcal{R}_2 := (\mathcal{R}_1 \cup \mathcal{R}_2)'$, where $\cup$ means the usual set union.

This lattice is also a $\mathcal{P}_+^1$-covariant lattice according to

\[
\mathcal{R} \rightarrow g\mathcal{R} := U (\Lambda, a) \mathcal{R} U (\Lambda, a)^*.
\]

The 2nd quantization map is defined through

\[
K \in \mathcal{S} \rightarrow \mathcal{R} (K) := \{ W (k) : k \in K \}' \subset \mathcal{B} (\mathcal{H}).
\]

It is not difficult to see that such map is a lattice $\mathcal{P}_+^1$-covariant homomorphism between $\mathcal{P}_+^1$-covariant lattices, i.e. $g\mathcal{R}(K) = \mathcal{R}(gK)$ \[\[\].

---

5We always consider open regions.

6It is always true that $\mathcal{O} \subset \mathcal{O}''$.
3.3.3 Net of local algebras for spacetime regions

According to the above discussion, the net of local algebras \( \mathcal{O} \subset \mathbb{R}^d \to \mathcal{R}(\mathcal{O}) \subset B(\mathcal{H}) \) of the free hermitian scalar field is defined as the composition of the 1st and 2nd quantization maps, i.e.

\[
\mathcal{R}(\mathcal{O}) := \mathcal{R}(K(\mathcal{O})) .
\]

The above assignment is a \( \mathcal{P}^+ \)-covariant lattice homomorphism, which means the following axioms hold [31]

\[
\begin{align*}
\mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2), & \quad & \text{if } \mathcal{O}_1 \subset \mathcal{O}_2, \\
\mathcal{R}(\mathcal{O}') = \mathcal{R}(\mathcal{O})', & \quad & (3.27) \\
\mathcal{R}(\mathcal{O}_1 \land \mathcal{O}_2) = \mathcal{R}(\mathcal{O}_1) \land \mathcal{R}(\mathcal{O}_1), & \quad & (3.28) \\
\mathcal{R}(\mathcal{O}_1 \lor \mathcal{O}_2) = \mathcal{R}(\mathcal{O}_1) \lor \mathcal{R}(\mathcal{O}_1), & \quad & (3.29) \\
\mathcal{R}(\mathbb{R}^d) = B(\mathcal{H}), & \quad & (3.30) \\
U(\Lambda, a) \mathcal{R}(\mathcal{O}) U(\Lambda, a)^{-1} = \mathcal{R}((\Lambda, a)\mathcal{O}), & \quad & (3.31)
\end{align*}
\]

where \( \mathcal{O}, \mathcal{O}_1, \mathcal{O}_2, \subset \mathbb{R}^d \) are causally complete regions. In particular, the weaker set of axioms stated on section [32] are satisfied.

3.4 Local algebras at fixed time

In this subsection, we discuss the theory of the von Neumann algebras for the real scalar free field at fixed time. Naively speaking, they are the local algebras generated by the field operator at fixed time \( \varphi(\hat{x}) \) and its canonical conjugate momentum field \( \pi(\hat{x}) \). This theory will be very useful for the computation of the relative entropy in section [5].

For that, we start decomposing \( \mathcal{H} \) into two \( \mathbb{R} \)-linear closed subspaces

\[
\begin{align*}
\mathcal{H}_\varphi := \{ h \in \mathcal{H} : h(\hat{p}) = h(\hat{p})^* \text{ a.e.} \}, \\
\mathcal{H}_\pi := \{ h \in \mathcal{H} : h(\hat{p}) = -h(\hat{p})^* \text{ a.e.} \},
\end{align*}
\]

where \( \mathcal{H}_\varphi = \mathcal{H}_\varphi' \oplus \mathcal{H}_\pi \). Each \( h \in \mathcal{H} \) is uniquely decomposed into \( h = h_\varphi + h_\pi \) with

\[
h_\varphi(\hat{p}) = \frac{h(\hat{p}) + h(\hat{p})^*}{2} \text{ and } h_\pi(\hat{p}) = \frac{h(\hat{p}) - h(\hat{p})^*}{2},
\]

and we have the useful relations

\[
\text{Im} \langle h_\varphi, h'_\varphi \rangle = \text{Im} \langle h_\pi, h'_\pi \rangle = \text{Re} \langle h_\varphi, h'_\varphi \rangle = 0,
\]

for all \( h_\varphi, h'_\varphi \in \mathcal{H}_\varphi \) and \( h_\pi, h'_\pi \in \mathcal{H}_\pi \). The free field theories allow a correct definition for the quantum field at fixed time \( x^0 = 0 \). For that we shall consider the following real embeddings \( E_{\varphi, \pi} : \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathcal{H}_\varphi, \mathcal{H}_\pi \)

\[
\begin{align*}
(E_\varphi f)(\hat{p}) := \hat{f}(\hat{p}), \\
(E_\pi f)(\hat{p}) := \hat{\omega}(\hat{p}) \hat{f}(\hat{p}),
\end{align*}
\]

where \( \hat{f}(\hat{p}) := (2\pi)^{-d-1} \int_{\mathbb{R}^{d-1}} f(\hat{x}) e^{-i\hat{p} \cdot \hat{x}} d^{d-1}x \). The sets \( E_{\varphi, \pi}(\mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})) \subset \mathcal{H}_\varphi, \mathcal{H}_\pi \) are dense. From now on, we naturally identify functions on \( \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \) with vectors on \( \mathcal{H}_\varphi, \mathcal{H}_\pi \) through these embeddings.

The field operator at fixed time \( \varphi(f) \) and its canonical conjugate momentum field \( \pi(f) \) are defined as operator valued distributions over \( \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \) by the formulae

\[
\begin{align*}
\varphi(f) := A(E_\varphi(f)) + A^*(E_\pi(f)), \\
\pi(f) := A(E_\pi(f)) + A^*(E_\varphi(f)).
\end{align*}
\]

The map \( E_\varphi \) (resp. \( E_\pi \)) is actually defined on a bigger class of test functions, namely \( H_{\frac{1}{2}}^{-} \) (resp. \( H_{\frac{1}{2}}^{+} \)) (\( \mathbb{R}^{d-1}, \mathbb{R}) \), i.e.

\[
E_\varphi : H_{\frac{1}{2}}^{-} \to \mathcal{H}_\varphi \quad \text{and} \quad E_\pi : H_{\frac{1}{2}}^{+} \to \mathcal{H}_\pi ,
\]

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where $H^\alpha (\mathbb{R}^{d-1}, \mathbb{R})$ is the real Sobolev space of order $\alpha$ (see appendix A.1). Then, we have that $E_\phi \left( H^{\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \right) = \delta_\phi$ and $E_\pi \left( H^{-\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \right) = \delta_\pi$. Similarly as we did in the last subsection, for each $h_\phi \in \delta_\phi$ and $h_\pi \in \delta_\pi$ we define the Weyl unitaries

$$W_\phi (h_\phi) := W (h_\phi) \quad \text{and} \quad W_\pi (h_\pi) := W (h_\pi),$$

which satisfy the CCR in the Weyl form

$$W_\phi (h_\phi) W_\pi (h_\pi) W_\phi ^* (h_\phi^*) W_\pi ^* (h_\pi^*) = W_\phi (h_\phi + h_\phi^*) W_\pi (h_\pi + h_\pi^*) e^{2i \text{Im}(h_\phi h_\pi^*)},$$

and in particular for $f_\phi \in H^{\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R})$ and $f_\pi \in H^{-\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R})$

$$W_\phi (E_\phi (f_\phi)) = e^{i \phi (f)} := W_\phi (f_\phi),$$

$$W_\pi (E_\pi (f_\pi)) = e^{i \pi (f)} := W_\pi (f_\pi).$$

As before, the local algebras at fixed time are also defined through the 1st and 2nd quantization map. They are defined as follows.

**First quantization map.** We say that $\mathcal{C} \subset \mathbb{R}^{d-1}$ is a spatially complete region iff it is open, regular \footnote{An open set $U \subset \mathbb{R}^n$ is regular iff $U \equiv \text{Int} (U)$.} and with regular boundary \footnote{The boundary $\partial \mathcal{C} \subset \mathbb{R}^{d-1}$ is a smooth submanifold of dimension $d - 2$, or several manifolds joined together along smooth manifolds of dimension $d - 3$. [31]}. The set of all spatially complete regions forms a lattice according to the following operations

- complement: the (open) space complement, i.e. $\mathcal{C}' := \mathbb{R}^3 - \overline{\mathcal{C}}$,
- infimum: $\mathcal{C}_1 \cap \mathcal{C}_2 := \mathcal{C}_1 \cap \overline{\mathcal{C}_2}$, where the $\cap$ means the usual set intersection,
- and supremum: $\mathcal{C}_1 \vee \mathcal{C}_2 := \text{Int} (\overline{\mathcal{C}_1 \cup \mathcal{C}_2})$, where $\cup$ means the usual set union.

Then the 1st quantization map are defined by the following two lattice homomorphism

$$\mathcal{C} \to \mathcal{K}_\phi (\mathcal{C}) := \{ E_\phi (f) : f \in S(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and supp} (f) \subset \mathcal{C} \} \subset \delta_\phi,$$

$$\mathcal{C} \to \mathcal{K}_\pi (\mathcal{C}) := \{ E_\pi (f) : f \in S(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and supp} (f) \subset \mathcal{C} \} \subset \delta_\pi.$$ (3.46) (3.47)

It can be shown that

$$\mathcal{K}_\phi (\mathcal{C}) = \left\{ E_\phi (f) : f \in H^{\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \text{ and supp} (f) \subset \overline{\mathcal{C}} \text{ a.e.} \right\},$$

$$\mathcal{K}_\pi (\mathcal{C}) = \left\{ E_\pi (f) : f \in H^{-\frac{\alpha}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \text{ and supp} (f) \subset \overline{\mathcal{C}} \text{ a.e.} \right\}.$$ (3.48) (3.49)

**Second quantization map.** For each pair $\mathcal{K}_\phi \subset \delta_\phi$ and $\mathcal{K}_\pi \subset \delta_\pi$ of $\mathbb{R}$-linear closed subspaces, we define the von Neumann algebra

$$(\mathcal{K}_\phi, \mathcal{K}_\pi) \to \mathcal{R}_0 (\mathcal{K}_\phi, \mathcal{K}_\pi) := \{ W_\phi (k_\phi) W_\pi (k_\pi) : k_\phi \in \mathcal{K}_\phi, k_\pi \in \mathcal{K}_\pi \}'' \subset \mathcal{B} (\mathcal{H}).$$ (3.50)

**Net of local algebras at fixed time.** The net of local algebras $\mathcal{C} \subset \mathbb{R}^{d-1} \to \mathcal{R}_0 (\mathcal{C}) \subset \mathcal{B} (\mathcal{H})$ of the free hermitian scalar field at fixed time is then defined as the composition of the 1st and 2nd quantization maps, i.e.

$$\mathcal{R}_0 (\mathcal{C}) := \mathcal{R}_0 (\mathcal{K}_\phi (\mathcal{C}), \mathcal{K}_\pi (\mathcal{C})).$$ (3.51)

The above map is a lattice homomorphism, which means the following axioms hold

$$\mathcal{R}_0 (\mathcal{C}_1) \subset \mathcal{R}_0 (\mathcal{C}_2) \quad \text{if} \quad \mathcal{C}_1 \subset \mathcal{C}_2,$$

$$\mathcal{R}_0 (\mathcal{C}') = \mathcal{R}_0 (\mathcal{C})',$$

$$\mathcal{R}_0 (\mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{R}_0 (\mathcal{C}_1) \cap \mathcal{R}_0 (\mathcal{C}_2),$$

$$\mathcal{R}_0 (\mathcal{C}_1 \vee \mathcal{C}_2) = \mathcal{R}_0 (\mathcal{C}_1) \vee \mathcal{R}_0 (\mathcal{C}_2),$$

$$\mathcal{R}_0 (\mathbb{R}^{d-1}) = \mathcal{B} (\mathcal{H}).$$ (3.52) (3.53) (3.54) (3.55) (3.56)
3.5 Relation between the two approaches

In this subsection we study the close relation existing between the two approaches of sections 3.3 and 3.4.

Relation between nets. Given any spatially complete region $\mathcal{C} \subset \mathbb{R}^{d-1}$, we define its domain of dependence $\mathcal{O}_C \subset \mathbb{R}^d$ as

$$\mathcal{O}_C := \{ x \in \mathbb{R}^d : x \sim (0, \vec{y}) \text{ and } \vec{y} \in \mathcal{C}' \}. \quad (3.57)$$

Then the following key relation holds \[33\]

$$K (\mathcal{O}_C) = K_\phi (\mathcal{C}) \oplus \mathbb{R} K_\pi (\mathcal{C}) , \quad (3.58)$$

and hence we have the equality between the von Neumann algebras

$$\mathcal{R}_0 (\mathcal{C}) = \mathcal{R} (\mathcal{O}_C) . \quad (3.59)$$

The relations developed along the above subsections can be summarized in the following schematic diagram

$$\begin{array}{ccc}
\mathcal{O}_C \subset \mathbb{R}^d & \xrightarrow{E} & K \subset \mathcal{H} \\
\uparrow & \oplus_{\mathbb{R}} & \uparrow \oplus_{\mathbb{R}} \\
\mathcal{C} \subset \mathbb{R}^{d-1} & \xrightarrow{E \times \pi} & (K_\phi , K_\pi) \subset \mathcal{N}_\phi \oplus \mathcal{N}_\pi \\
& \xrightarrow{W \times \pi} & \mathcal{R} \subset \mathcal{B} (\mathcal{H}) \\
& \xrightarrow{\mathbb{R} \times \pi} & \mathcal{R}_0 \subset \mathcal{B} (\mathcal{H}).
\end{array} \quad (3.60)$$

Relation between test functions. More explicitly, given $f \in \mathcal{S} (\mathbb{R}^d, \mathbb{R})$ there exist functions $f_\phi , f_\pi \in \mathcal{S} (\mathbb{R}^{d-1}, \mathbb{R})$ such that

$$E (f) = E_\phi (f_\phi) + E_\pi (f_\pi). \quad (3.61)$$

Indeed, given $f \in \mathcal{S} (\mathbb{R}^d, \mathbb{R})$, if we define

$$F (x) := \int_{\mathbb{R}^d} \Delta (x - y) f (y) \, dy , \quad (3.62)$$

where $\Delta (x) := -i (2\pi)^{-(d-1)} \int_{\mathbb{R}^d} e^{-ip \cdot x} \delta (p^2 - m^2) \, dp$, and then we find that

$$f_\phi (\vec{x}) = - \frac{\partial F}{\partial x^\mu} (0, \vec{x}) \quad \text{and} \quad f_\pi (\vec{x}) = F (0, \vec{x}). \quad (3.63)$$

Moreover, since $F (x) = 0$ if $x \in \text{supp} (f)'$, then we have

$$\text{supp} (f) \subset \mathcal{O}_C \implies \text{supp} (f_\phi) , \text{supp} (f_\pi) \subset \mathcal{C}. \quad (3.64)$$

In other words, $f$ generates the solution of the equation of motion (3.62), in the sense that $f (p)$ is the positive energy density of $F$. The functions (3.63) are simply the initial data at time $x^0 = 0$ of such wave solution.

Relation between Weyl unitaries. For the particular case of Weyl unitaries, it follows that

$$W (f) = e^{i \text{Im} \langle f_\phi , f_\pi \rangle} W_\phi (f_\phi) W_\pi (f_\pi), \quad (3.65)$$

where

$$\text{Im} \langle f_\phi , f_\pi \rangle \delta = \frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\phi (\vec{x}) f_\pi (\vec{x}) \, d^{d-1} \vec{x}. \quad (3.66)$$

4 Modular theory

In this section we discuss the key points of the modular theory in the framework of AQFT. We start with the definition of the modular Hamiltonian for a generic von Neumann algebra and the statement of Tomita-Takesaki theorem. After that we discuss the canonical example when the underlying algebra corresponds to the algebra of observables associated to the Rindler wedge on Minkowski spacetime (Bisognano-Whichmann theorem), and a second example of the modular operator for a state constructed by applying a unitary operator to the vacuum. Then we introduce the relative modular Hamiltonian which is needed for the definition of the Araki formula for the relative entropy. For a detailed reading, see [34][36] for (relative) modular theory, [34][37] for Bisognano-Whichmann theorem, and [15] for the Araki formula for the relative entropy.
4.1 Modular Hamiltonian and modular flow

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra in a standard form, and $\Omega \in \mathcal{H}$ a cyclic and separating vector then there exist two unique (generally unbounded) antilinear operators $S_\Omega$ and $F_\Omega$ such that

$$S_\Omega A \Omega = A' \Omega, \quad (4.1)$$

$$F_\Omega A' \Omega = A'' \Omega, \quad (4.2)$$

for all $A \in \mathcal{R}$ and $A' \in \mathcal{R}'$. Both of them are well defined on dense domains $\mathcal{D}(S_\Omega) = \mathcal{R} \Omega$ and $\mathcal{D}(F_\Omega) = \mathcal{R}' \Omega$. From (4.2) it follows that these operators are closable and

$$S_\Omega^* = \overline{F}_\Omega, \quad F_\Omega^* = \overline{S}_\Omega. \quad (4.3)$$

From now on we denote $\overline{S}_\Omega$ and $\overline{F}_\Omega$ simply as $S_\Omega$ and $F_\Omega$. Let

$$S_\Omega = J_\Omega \Delta_\Omega^\frac{1}{2} \quad (4.4)$$

be the polar decomposition of $S_\Omega$. Then $\Delta_\Omega$ (positive selfadjoint and generally unbounded) is called the modular operator associated with the pair $\{ \mathcal{R}, \Omega \}$ and $J_\Omega$ (antiunitary) is called the modular conjugation. Finally the modular Hamiltonian is defined by

$$K_\Omega := - \log (\Delta_\Omega). \quad (4.5)$$

4.1.1 Tomita-Takesaki theorem

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $\Omega \in \mathcal{H}$ a cyclic and separating vector and $S_\Omega = J_\Omega \Delta_\Omega^\frac{1}{2}$ the operator defined above. The 1-parameter (strongly continuous) group of unitaries $\Delta_\Omega^t$ is called modular group or modular flow. The Tomita-Takesaki theorem states

$$J_\Omega \mathcal{R} J_\Omega = \mathcal{R}', \quad \Delta_\Omega^t \mathcal{R} \Delta_\Omega^{-it} = \mathcal{R} \quad \text{and} \quad \Delta_\Omega^t \mathcal{R}' \Delta_\Omega^{-it} = \mathcal{R}', \quad (4.6)$$

for all $t \in \mathbb{R}$. Despite of that, $\Delta_\Omega^t \notin \mathcal{R} \cup \mathcal{R}'$ generally.

4.1.2 Example 1: Bisognano-Wichmann theorem

Let $\mathcal{W} := \{ x \in \mathbb{R}^d : x^1 > |x^0| \}$ the right Rindler wedge and $\Sigma := \{ \bar{x} \in \mathbb{R}^{d-1} : x^1 \geq 0 \}$. Then by (3.57) we have that $\mathcal{O}_\Sigma = \mathcal{W}$. From now on, we will denote the orthogonal coordinates to the Rindler Wedge as $\bar{x}_\perp := (x^2, \ldots, x^{d-1})$ and hence any spacetime point can be expressed as $x = (x^0, x^1, \bar{x}_\perp)$.

Using the definitions (3.29), (3.51) and the relation (3.59) we can define the von Neumann algebras

$$\mathcal{R}_\mathcal{W} := \mathcal{R}(\mathcal{W}) = \mathcal{R}_0(\Sigma), \quad (4.8)$$

$$\mathcal{R}_\mathcal{W}' := \mathcal{R}(\mathcal{W}') = \mathcal{R}_0(\Sigma'), \quad (4.9)$$

acting on the Fock Hilbert space $\mathcal{H}$. From relations (3.32), (3.56) we have

$$\mathcal{R}_\mathcal{W}' = \mathcal{R}_\mathcal{W}, \quad \mathcal{R}_\mathcal{W} \lor \mathcal{R}_\mathcal{W}' = \mathcal{B}(\mathcal{H}). \quad (4.10)$$

From Reeh-Schlieder theorem we know that the vacuum $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathcal{R}_\mathcal{W}$. The Bisognano-Wichmann theorem [37] states that

$$J_\Omega = \Theta U(\Lambda_1(\pi)) \quad \text{and} \quad \Delta_\Omega = e^{-2\pi K_1}, \quad (4.12)$$

where $K_1$ is the infinitesimal generator of the 1-parameter group of boost symmetries in the plane $(x^0, x^1)$, i.e.

$$U(\Lambda_1^s, 0) = e^{i K_1 s} \quad \text{with} \quad \Lambda_1^s := \begin{pmatrix} \cosh(s) & \sinh(s) & 0 \\ \sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \quad (4.13)$$

$\Theta$ is the CPT operator and $U(\Lambda_1(\pi))$ is the Lorentz unitary operator representing a space rotation of angle $\pi$ along the $x^1$ axes. Despite we are working with the net of local algebras for the real scalar field, the above result holds for any relativistic QFT which satisfies the Wightman axioms.

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Footnote: Then $\Omega$ is cyclic and separating for $\mathcal{R}'$. 

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13
4.1.3 Example 2: coherent vector

Inspired by our main task of computing the relative entropy between a coherent state and the vacuum, we now obtain an expression for the modular operator for a vector $\Psi = U\Omega$ in terms of the modular operator of $\Omega$ when $U$ is an unitary operator.\(^{11}\) For a general $U$ the computation is not easy, but in the case when such unitary satisfies\(^{11}\)

\[ U^* AU \in \mathcal{R} \quad \text{for all} \ A \in \mathcal{R}, \]  

we have

\[ S_\Psi = US_\Omega U^*, \]

since

\[ S_\Psi A\Psi = US_\Omega U^* AU\Omega = U(U^* AU)^* \Omega = UU^* A^* U\Omega = A^* \Psi. \]  

Then

\[ J_\Psi = UJ_\Omega U^* \quad \text{and} \quad \Delta_\Psi = U\Delta_\Omega U^*. \]  

For the free hermitean scalar field theory discussed above, it is easy to see that any Weyl unitary $U = W(h)$, with $h \in \mathfrak{s}_1$ satisfies \(^{11}\) when $\mathcal{R} = \mathcal{R}(\mathcal{O})$ is any local algebra. This is because for any local operator $A = W(f) \in \mathcal{R}(\mathcal{O})$ with $supp(f) \subset \mathcal{O}$ we have that

\[ W(h)W(f)W(h)' = e^{-2i\text{Im}(f,h)}W(f) \in \mathcal{R}(\mathcal{O}), \]

and an approximation argument shows that equation \(^{4.17}\) holds for a general operator $A \in \mathcal{R}(\mathcal{O})$.

4.2 Relative modular Hamiltonian and relative modular flow

Now we consider $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra in a standard form and two cyclic and separating vectors $\Omega, \Psi \in \mathcal{H}$. Then there exists a unique (generally unbounded) antilinear operator such that

\[ S_{\Psi,\Omega} A\Omega = A^* \Psi \]  

for all $A \in \mathcal{R}$. It’s well defined on a dense domain $\mathcal{D}(S_{\Psi,\Omega}) = \mathcal{R}\Omega$. It can be shown that $S_{\Psi,\Omega}$ is closable, and hence we name $S_{\Psi,\Omega}$ simply as $S_{\Psi,\Omega}$. From its polar decomposition

\[ S_{\Psi,\Omega} = J_{\Psi,\Omega}\Delta_{\Psi,\Omega}^{\frac{1}{2}}, \]  

we name $\Delta_{\Psi,\Omega}$ the relative modular operator associated with the pair $\{\mathcal{R},\Omega,\Psi\}$. Then the relative modular Hamiltonian is defined by

\[ K_{\Psi,\Omega} := -\log(\Delta_{\Psi,\Omega}) \]  

The relative modular flow $\Delta_{\Psi,\Omega}^t$ acts as the modular flow $\Delta_{\Psi}^t$ for the algebra $\mathcal{R}$ and as $\Delta_{\Omega}^t$ for the algebra $\mathcal{R}'$, i.e.

\[ \Delta_{\Psi,\Omega}^t A \Delta_{\Psi,\Omega}^{-t} = \Delta_{\Psi}^t A \Delta_{\Psi}^{-t} \quad A \in \mathcal{R}, \]  

\[ \Delta_{\Omega,\Omega}^t A' \Delta_{\Omega,\Omega}^{-t} = \Delta_{\Omega}^t A' \Delta_{\Omega}^{-t} \quad A' \in \mathcal{R}'. \]

As it happens for the modular flow, $\Delta_{\Psi,\Omega}^t \notin \mathcal{R} \cup \mathcal{R}'$ generally. However, we can define the unitaries\(^{12}\)

\[ u_{\Psi,\Omega}(t) = \Delta_{\Psi,\Omega}^t \Delta_{\Omega}^{-t}, \]  

\[ u_{\Psi,\Omega}'(t) = \Delta_{\Psi,\Omega}^t \Delta_{\Omega}^{-t}, \]

and for them we have

\[ u_{\Psi,\Omega}(t) \in \mathcal{R} \quad \text{and} \quad u_{\Psi,\Omega}'(t) \in \mathcal{R}'. \]

\(^{10}\)The relation between this and the calculation of the relative entropy will be clear in future sections.

\(^{11}\)In particular it holds when $U = VV'$ with unitaries $V \in \mathcal{R}$ and $V' \in \mathcal{R}'$.

\(^{12}\)These operators, which are not 1-parameter groups, are best known as Radon-Nikodym cocycle.
4.3 Araki formula for relative entropy

The definition of the relative entropy for a general von Neumann algebra is due to Araki. Let be $\mathcal{R} \subset \mathcal{B} (\mathcal{H})$ a von Neumann algebra in standard form. For any given two $\omega, \phi$ two faithful normal states, there exists cyclic and separating vectors representatives $\Omega, \Phi \in \mathcal{H}$ (in particular choose them in the natural cone of the standard vector of $\mathcal{R}$). Then the relative entropy $S_R (\phi \mid \omega)$ is defined through the relative modular Hamiltonian $K_{\Omega, \Phi}$ as

$$ S_R (\phi \mid \omega) := \langle \Phi, K_{\Omega, \Phi} \rangle_{\mathcal{H}}, $$

which is independent of the vector representatives chosen for the given states. Indeed, any other vector representatives for the same state must be related as

$$ \Omega' = V' \Omega \quad \text{and} \quad \Phi' = V' \Phi, $$

where $V' \Omega, V' \Phi \in \mathcal{R}'$ are a partial isometries. In this case, since all states in (4.27) are cyclic for $\mathcal{R}$, then $V' \Omega, V' \Phi$ must be unitaries. Then a straightforward computation using (4.18) and (4.27) shows that

$$ S_{\Omega', \Phi'} = V_2 S_{\Omega, \Phi} V_1^* \Rightarrow K_{\Omega', \Phi'} = V'_2 K_{\Omega, \Phi} V'_1^*, $$

which implies that (4.26) is independent of the chosen vector representatives. The relative entropy satisfies the well-known properties of strict-positivity, lower semi-continuity, convexity and monotonicity, which are all well discussed in Araki's original work [15]. When the relative entropy is finite (in particular when $\Omega$ belongs to the domain of $K_{\Omega, \Phi}$) the following useful expression holds

$$ S_R (\phi \mid \omega) = \lim_{t \to 0} \left( \langle \Phi, \Delta^{(t)}_{\Omega, \Phi} \Phi \rangle_{\mathcal{H}} - 1 \right). $$

4.3.1 Symmetric relative entropy for coherent states

For any unitary operator $V \in \mathcal{B} (\mathcal{H})$ satisfying the property (4.14), we have the following symmetry property for the modular operator $\Delta V_{\Omega, \Phi}$

$$ S_{V \Omega, V \Phi} = V S_{\Omega, \Phi} V^*. $$

This implies that

$$ \Delta_{V \Omega, V \Phi} = V \Delta_{\Omega, \Phi} V^* \Rightarrow K_{V \Omega, V \Phi} = VK_{\Omega, \Phi} V^*. $$

Then

$$ S_R (\phi \mid \omega) = \langle \Phi, K_{\Omega, \Phi} \rangle_{\mathcal{H}} = \langle \Phi, V^* K_{V \Omega, V \Phi} V \Phi \rangle_{\mathcal{H}}. $$

The above expression is useful when we want to compute the relative entropy between the vacuum state $\omega$ and a coherent state $\phi$ with vector representatives $\Omega$ (vacuum vector) and a coherent vector $\Phi := U \Omega$, where $U = W (f)$ is a Weyl unitary. In such case, we have that $U^* = W (-f)$ satisfies relation (4.14) because of (4.17). Now using $V := U^*$ in (4.32) and defining $\Psi := U^* \Omega$, we obtain

$$ S_R (\phi \mid \omega) = \langle \Omega, K_{\Psi, \Omega} \rangle_{\mathcal{H}} = S_R (\omega \mid \psi), $$

where $\psi$ is the coherent state which vector representative $\Psi = U^* \Omega$. In the rest of the paper we will show that the relative entropy for a coherent state is

$$ S_R (\phi \mid \omega) = 2 \pi \int_{\Sigma} d^{d-1} x x^1 \left( \frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right|_{x^0 = 0}, $$

where $F (x) = \int_{\mathbb{R}^d} \Delta (x - y) f (y) dy$. We want to remark two features of the above expression. On one hand, as we have anticipated, expression (4.13) holds for the canonical stress tensor. On the other hand, combining (4.33) and the fact that (4.34) is quadratic in the generating function $f$, we get that the relative entropy for a coherent state is symmetric, i.e.

$$ S_R (\phi \mid \omega) = S_R (\omega \mid \phi), \quad \text{with } \phi \text{ coherent state.} $$

We expect that this property should hold for any (reasonable) region, beyond the Rindler wedge.

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13 Contrary to the notation employed in sections 11 and 12 on the l.h.s. expression (4.26) we emphasize that the relative entropy depends on the states rather than the vector representatives used to define it. We use this new notation in the rest of the paper.

14 At first glance, we can ask ourselves if the new states $V \Omega$ and $V \Phi$ are cyclic and separating for the underlying algebra $\mathcal{R}$. In general, we can not give an affirmative answer. However, in the next section, we prove this issue for the case we are interested in: $\mathcal{R} \equiv \mathcal{R}_V$ is the wedge algebra of a QFT, $\Omega$ is the vacuum state vector and $V$ is a local unitary operator.


5  Relative entropy for a coherent state

In this section we compute, for the theory of the free hermitian scalar field, the relative entropy for the Rindler wedge algebra \( \mathcal{R}_W \) between the vacuum vector \( \Omega \) and an excitation from the vacuum of the form \( \Phi := U\Omega \), with \( U \in \mathcal{B}(\mathcal{H}) \) unitary. Hence, the aim of the work is to calculate the relative modular Hamiltonian \( K_{\Omega,\Phi} \) (or \( K_{\Phi,\Omega} \) according to (4.33)) for such local algebra and the corresponding states. We distinguish between 2 cases

\[
\text{easy case : } U = U_R U_L, \\
\text{hard case : } U \neq U_R U_L,
\]

where \( U_R \in \mathcal{R}_W \) and \( U_L \in \mathcal{R}_{W'} \) are unitaries.\(^{13}\)

First of all, we need to show that \( \Omega, \Psi \) are cyclic and separating for \( \mathcal{R}_W \). From Reeh-Schlieder theorem we know that such statement holds for the vacuum \( \Omega \). Relying on that it is not difficult to show that \( \Phi = U\Omega \) is also cyclic and separating for \( \mathcal{R}_W \). In the case where the unitary is a product as in (5.1), we have that

\[
\mathcal{H} = \overline{\mathcal{R}_W\Omega} = U_L \overline{\mathcal{R}_W\Omega} = U_L \overline{\mathcal{R}_W U_R \Omega} = \overline{\mathcal{R}_W U_R \Omega} = \overline{\mathcal{R}_W \Phi},
\]

and hence \( \Phi \) is cyclic for \( \mathcal{R}_W \). The same argument holds for \( \mathcal{R}_{W'} \), and finally we get that \( \Psi \) is separating for \( \mathcal{R}_W \). In the case that \( U \) is not a product as in (5.2), but it is localized in the algebra of some bounded region, we can always take a “bigger” right wedge \( \mathcal{W}_R \supset \mathcal{W} \) such that \( U \in \mathcal{R}(\mathcal{W}_R) \). Then, using the above result, we have that \( \Phi \) is separating for \( \mathcal{R}(\mathcal{W}_R) \) and hence \( \Phi \) is separating for \( \mathcal{R}_W \). Applying the same reasoning, we can also construct a left wedge \( \mathcal{W}_L \supset \mathcal{W} \) such that \( U \in \mathcal{R}(\mathcal{W}_L) \). Then \( \Phi \) is separating for \( \mathcal{R}(\mathcal{W}_L) \) implies \( \Phi \) is separating for \( \mathcal{R}_W \), and hence \( \Phi \) is cyclic for \( \mathcal{R}_W \).

In the following subsections, we deal with each case (5.1) and (5.2) separately.

5.1 Easy case: \( U = U_R U_L \)

In this case, we simply have that

\[
S_{\Omega,\Phi} = U_R S_{\Omega} U_L^*.
\]

To see this, for \( A \in \mathcal{R}_W \)

\[
S_{\Omega,\Phi} A \Phi = U_R S_{\Omega} U_L^* A U_L U_R \Phi = U_R S_{\Omega} A U_R \Omega = U_R (A U_R)^* \Omega = A^* \Omega.
\]

Then the polar decomposition for \( S_{\Omega,\Phi} \) comes from

\[
S_{\Omega,\Phi} = U_R S_{\Omega} U_L^* = U_R J_{\Omega} \Delta_{\Omega}^{1/2} U_L^* = U_R J_{\Omega} U_L^* U_L \Delta_{\Omega}^{1/2} U_L^* \\
\Rightarrow J_{\Omega,\Phi} = U_R J_{\Omega} U_L^* \quad \text{and} \quad \Delta_{\Omega,\Phi} = U_L \Delta_{\Omega} U_L^* \\
\Rightarrow K_{\Omega,\Phi} = U_L K_{\Omega} U_L^*.
\]

Despite the notation used, the above expressions are valid for any von Neumann algebra \( \mathcal{A} \) having a cyclic and separating vector \( \Omega \), and unitaries \( U_R \in \mathcal{A} \) and \( U_L \in \mathcal{A}' \). Then the relative entropy is

\[
S_R (\phi | \omega) = \langle \Phi, U_L K_{\Omega} U_L^* \Phi \rangle_{\mathcal{H}} = \langle \Omega, U_R^* K_{\Omega} U_R \Omega \rangle_{\mathcal{H}}.
\]

**Coherent state.** Given the expression (5.7) for the relative modular Hamiltonian, we can calculate the relative entropy using this and formula (4.26) in the case when the unitary is given by a coherent state

\[
U_R = W(f) \quad f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}), \ \text{supp}(f) \subset \mathcal{W}.
\]

Remembering that the second quantized Poincaré unitary operator \( U (A_1^t, 0) = e^{i K_1 s} \), acting on the the Fock space \( \mathcal{H} \), is constructed from the Poincaré unitary operator \( u (A_1^t, 0) = e^{i k_1 s} \), acting on the 1-particle Hilbert space \( \mathcal{H}_1 \), then we have that

\[
S_R (\phi | \omega) = \langle \Omega, U_R K_{\Omega} U_R^* \Omega \rangle_{\mathcal{H}} = 2\pi \langle \Omega, W(-f) K_1 W(f) \Omega \rangle_{\mathcal{H}} = 2\pi \langle f, k_1 f \rangle_{\mathcal{H}}.
\]

where the last equality is fully calculated in appendix A.2. Thus, the relative entropy between the coherent state and the vacuum, can be expressed, in the 1-particle Hilbert space \( \mathcal{H}_1 \), in terms of the expectation value

\(^{13}\)In particular, the easy case includes the cases when \( U \in \mathcal{R}_W \) or \( U \in \mathcal{R}_{W'} \).
of the boost operator $k_1$ in the vector $E(f)$ which generates the coherent state. In addition, a straightforward calculation explained in appendix A.3 allows us to rewrite the expression in terms of the generating function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ as

$$S_R(\phi \mid \omega) = 2\pi \int_{\mathbb{R}^d} d^{d-1}x x^1 \frac{1}{2} \left( \frac{\partial F}{\partial x^0} \right)^2 \left| \nabla F \right|^2 + m^2 F^2 \right|_{x^0=0},$$

where $F(x) = \int_{\mathbb{R}^d} \Delta(x-y) f(y) \, dy$. Since $\text{supp}(f) \subset \mathcal{W}_R \Rightarrow \text{supp}(F|_{x^0=0}) \subset \Sigma$, we should rewrite this last equation as

$$S_R(\phi \mid \omega) = 2\pi \int_{\Sigma} d^{d-1}x x^1 \frac{1}{2} \left( \frac{\partial F}{\partial x^0} \right)^2 \left| \nabla F \right|^2 + m^2 F^2 \right|_{x^0=0}. \quad (5.10)$$

This result coincides with (1.15) for the canonical stress tensor (1.10).

### 5.2 Hard case: $U \neq U_R U_L$

For the computation in the previous section, it was essential that the unitary, which defines the state $\Phi = U \Omega$, was a product of the form $U = U_R U_L$ with $U_R \in \mathcal{R}_W, U_L \in \mathcal{R}_W$. Our next step is to get the result for the relative entropy when $U$ is not such a product. For that we will work with the concrete case when $U = W(f)$, $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, $\text{supp}(f) \not\subset \mathcal{W}, \mathcal{W}'$, $\text{supp}(f)$ compact. \quad (5.11)

Before we continue, we want to remark that, in this case, the relative entropy must be finite. The proof is as follows. Since $\text{supp}(f)$ is compact, there exists a “bigger” right wedge $\mathcal{W}_R \supset W$ such $U \in \mathcal{W}_R$. Then the relative entropy for those both states and for the algebra $\mathcal{R}(\mathcal{W}_R)$ is as the one computed in the previous section, which is finite because the generating function $f$ is smooth. Then by monotonicity, the relative entropy for the original wedge $\mathcal{W}$ must be finite. In particular, we are allowed to use expression (4.29).

The first question which comes up is if we could split the unitary into two unitaries, one belonging to the right wedge $\mathcal{W}$ and the other to the left wedge $\mathcal{W}'$. In other words, if there exists $U_R \in \mathcal{R}_W$ and $U_L \in \mathcal{R}_W$ unitaries such that $U = U_L U_R$. Unfortunately the answer is no, almost for the most general interesting case. This fact arises when we try to explicitly split $U$. To begin, it seems natural to split the function $f$ simply as

$$f_R(x) := \Theta_W(x) f(x), \quad f_L(x) := \Theta_{W'}(x) f(x), \quad (5.12)$$

where $\Theta_W$ is the characteristic function of the right Rindler wedge (equivalently for $\Theta_{W'}$). However, it will lead to a wrong result, since $f_R + f_L \neq f$. Moreover, if we take for example we shall start with a function $f$ supported in the upper light cone $\mathcal{V}^+ := \{ x \in \mathbb{R}^d : x^0 > |x| \}$, then equation (5.12) implies that $f_R \equiv 0$ and hence we obtain $S_R(\phi \mid \omega) = 0$, which is obviously the wrong result. To make a consistent splitting, we must use the relations explained in subsection 5.2.1. Given the spacetime function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ we can construct $f_\nu, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ satisfying the relation (3.65). The correct result is to split these functions $f_\nu, f_\pi$, which are the initial data at $x^0 = 0$ of the Klein-Gordon solution generated by $f$. The property (5.11) about the $\text{supp}(f)$ implies that an open neighborhood of the origin $x = 0$ is included in the supports of $f_\nu$ and $f_\pi$. Now, we would like to write

$$f_\nu = f_{\nu,L} + f_{\nu,R} \quad \text{and} \quad f_\pi = f_{\pi,L} + f_{\pi,R}, \quad (5.14)$$

with $\text{supp}(f_{\nu,L}) \subset \Sigma'$ and $\text{supp}(f_{\nu,R}) \subset \Sigma (\nu = \varphi, \pi)$. The right way to do this is taking

$$f_{\nu,L}(x) := f_\nu(x) \cdot \Theta(-x^1) \quad \text{and} \quad f_{\nu,R}(x) := f_\nu(x) \cdot \Theta(x^1), \quad (5.15)$$

where $\Theta$ is the usual step Heaviside function. The problem is that $f_{\nu,L}$ and $f_{\nu,R}$ are no longer smooth, and nothing guarantee that $E_{\nu}(f_{\nu,L}) \in \mathcal{S}_\nu$ (the same problem occurs for $f_{\nu,L}$). More precisely, since $f_{\nu,R} \in L^2(\mathbb{R}^{d-1}, \mathbb{R}) = H^0(\mathbb{R}^{d-1}, \mathbb{R})$, and because of the inclusions (see appendix A.1)

$$H^+ \left( \mathbb{R}^{d-1}, \mathbb{R} \right) \subset H^0(\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}), \quad (5.16)$$

we have that $f_{\nu,R} \not\in \mathcal{S}_\phi$ but $f_{\pi,R} \not\in \mathcal{S}_\pi$. In other words, $f_{\pi,R}$ is not an appropriate smear function for the canonical conjugate field $\pi(x)$. This problem does not arise because the test function is no longer smooth, it is just because $f_{\pi,R}$ is no longer continuous. On the other hand, if $f_{\pi,R}$ is continuous, the problem can be solved due to the following lemma.

---

14The compactly supported condition is needed in our proof that $\Psi$ is cyclic and separating for $\mathcal{R}_W$. 
Theorem 5.1.
Let \( f \in L^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \cap C^1_t(\mathbb{R}^n) \) and \( \partial_j f \in L^2(\mathbb{R}^n) \) for \( j = 1, \ldots, n \). Then \( f \in H^1(\mathbb{R}^n) \).

Proof. See appendix \[\text{A.1}\].

Then, having this in mind, the strategy we adopt below is to make a splitting for some other smear function which, by construction, we know it is continuous.

5.2.1 Splitting the unitary

In this subsection we will obtain a general expression for the relative modular flow, assuming that some non local operator can be written as a product of two new operators, one belonging to \( \mathcal{R}_W \) and the other to \( \mathcal{R}_{W'} \). The possibility of doing such splitting will be unclear at this stage, and it will be justified in the next subsection when we apply the results derived here to the theory of the free hermitian scalar field. For simplicity, throughout this section and below, we make use of the symmetry relation \[\text{[A.3]}\]. In other words, we will compute the relative modular Hamiltonian \( K_{\Psi,\Omega} \).

The strategy here is to obtain an expression for the operator \( u_{\Psi,\Omega}(t) \) defined by \[\text{[B.4]}\], and then we obtain the equality \( \Delta_{\Psi,\Omega}^{i\alpha} = u_{\Psi,\Omega}(t) \Delta_{\Omega}^{i\alpha} \). We expect that such computation involves the splitting of some test function, but since \( u_{\Psi,\Omega}(t) \in \mathcal{R}_W \), we expect that the splitting will be indeed well defined. To get some intuition, from \[\text{[5.7]}\] we know that

\[
u_{\Psi,\Omega}(t) = U_R^* \Delta_{\Omega}^{i\alpha} U_R \Delta_{\Omega}^{it} \quad \text{when } \Psi = U^* \Omega = U_R^* U_L^* \Omega \quad \text{with } U_R \in \mathcal{R}_W, \ U_L \in \mathcal{R}_{W'}.
\]

By analogy, for the general case, we start with the combination \( U^* \Delta_{\Omega}^{i\alpha} U \Delta_{\Omega}^{it} \), and suppose that exist \[\text{[5.7]}\] unitaries \( V_R(t) \in \mathcal{R}_W \), \( V_L(t) \in \mathcal{R}_{W'} \) such that \[\text{[5.7]}\] we know that

\[
u_{\Psi,\Omega}(t) = U_R^* \Delta_{\Omega}^{i\alpha} U_R \Delta_{\Omega}^{it} \quad \text{when } \Psi = U^* \Omega = U_R^* U_L^* \Omega \quad \text{with } U_R \in \mathcal{R}_W, \ U_L \in \mathcal{R}_{W'}.
\]

Now we prove that \( V_R(t) \simeq u_{\Psi,\Omega}(t) \), in the sense that \( V_R(t) \Delta_{\Omega}^{it} \) has the same action than \( \Delta_{\Psi,\Omega}^{i\alpha} \) over every \( A \in \mathcal{R}_W \) and \( A' \in \mathcal{R}_{W'} \) , indeed

\[
\mathcal{R}_W \ni V_R(t) \Delta_{\Omega}^{it} A \Delta_{\Omega}^{it} V_R(t) = V_R(t) V_L(t) \Delta_{\Omega}^{it} A \Delta_{\Omega}^{it} V_R(t) V_L(t) = U_R^* U_L^* \Delta_{\Omega}^{it} A \Delta_{\Omega}^{it} U_R^* U_L^* \Delta_{\Omega}^{it} U_R^* \Delta_{\Omega}^{it} U^* = U_R^* U^* A \Delta_{\Omega}^{it} U_R^* \Delta_{\Omega}^{it} U^* = \Delta_{\Psi,\Omega}^{i\alpha} A \Delta_{\Omega}^{it} = \Delta_{\Psi,\Omega}^{i\alpha} A \Delta_{\Omega}^{it} ,
\]

and

\[
V_R(t) \Delta_{\Omega}^{it} A' \Delta_{\Omega}^{it} V_R(t) = V_R(t) V_R(t) \Delta_{\Omega}^{it} A' \Delta_{\Omega}^{it} = \Delta_{\Psi,\Omega}^{i\alpha} A' \Delta_{\Omega}^{it} = \Delta_{\Psi,\Omega}^{i\alpha} A' \Delta_{\Omega}^{it} .
\]

Hence, for all \( B \in \mathcal{R}_W \cup \mathcal{R}_{W'} \) we have

\[
(V_R(t) \Delta_{\Omega}^{it}) B (V_R(t) \Delta_{\Omega}^{it})^* = \Delta_{\Psi,\Omega}^{i\alpha} B \Delta_{\Psi,\Omega}^{it} , \quad \Rightarrow \quad B (V_R(t) \Delta_{\Omega}^{it})^* \Delta_{\Psi,\Omega}^{i\alpha} = (V_R(t) \Delta_{\Omega}^{it})^* \Delta_{\Psi,\Omega}^{i\alpha} B ,
\]

and then \( (V_R(t) \Delta_{\Omega}^{it})^* \Delta_{\Psi,\Omega}^{i\alpha} \) belongs to \( (\mathcal{R}_W \cup \mathcal{R}_{W'})' = \{ \lambda \cdot 1 \} \), which means that

\[
\Delta_{\Psi,\Omega}^{i\alpha} = \lambda(t) V_R(t) \Delta_{\Omega}^{it} , \quad \text{with } \lambda : \mathbb{R} \to C .
\]

Moreover, since all operators in \[\text{[5.22]}\] are unitaries, then

\[
\Delta_{\Psi,\Omega}^{i\alpha} = e^{-i \alpha(t)} V_R(t) \Delta_{\Omega}^{it} , \quad \text{with } \alpha : \mathbb{R} \to \mathbb{R} \text{ and } \alpha(0) = 0 .
\]

Finally the relative modular Hamiltonian \[\text{[5.23]}\]

\[
K_{\Psi,\Omega} = i \left. \frac{d}{dt} \right|_{t=0} \Delta_{\Psi,\Omega}^{i\alpha} = \left. \frac{d}{dt} \right|_{t=0} e^{-i \alpha(t)} V_R(t) \Delta_{\Omega}^{it} = \alpha'(0) 1 + i \dot{V}_R(0) + K_{\Omega}.
\]

\[\text{[A.1]}\] is the set of piecewise differentiable functions. See appendix \[\text{[A.1]}\] for a proper definition.

\[\text{[B.4]}\] This happens in the theory of the free scalar field as we explain in the next subsection.

\[\text{[5.7]}\] They are not necessarily 1-parameter groups for \( t \in \mathbb{R} \).

\[\text{[5.22]}\] The derivative in \[\text{[5.24]}\] has to be understood as a limit in the strong operator topology of \( \mathcal{H} \).
This formula gives a well defined expression for the relative modular Hamiltonian up to a constant. One way to determine such constant is using that \( \Delta^u_{\psi,\Omega} \) is a 1-parameter group of unitaries and must fulfill the concatenation equation
\[
\Delta^u_{\psi,\Omega} \Delta^u_{\psi,\Omega} = \Delta^u_{\psi,\Omega}^{(t_1 + t_2)} , \quad \forall t_1, t_2 \in \mathbb{R} .
\] (5.25)
We will discuss the computation to determine \( \alpha' (0) \) in subsection 5.2.3.

### 5.2.2 The splitting for a coherent state

In the last subsection we obtain a way to split the unitary (5.13) in order to obtain the expression (5.24) for the modular relative Hamiltonian. Our aim now is to justify why this splitting works for the free hermitian scalar field.

From now on we set \( s = -2\pi t \). Replacing \( U = W (f) \) in (5.18) we obtain that
\[
W (f)^* \Delta^u_{\Omega} W (-f) \Delta^u_{\Omega}^{-1} = W (-f) e^{i s K^1} W (f) e^{-i s K^1} = W (-f) W \left( f (\Lambda^1,0) \right) = e^{i \text{Im}(f,f')_\psi} W \left( f (\Lambda^1,0) - f \right) = e^{i \text{Im}(f,f')_\psi} W (f^s - f) ,
\] (5.26)
where we have defined \( f^s := f (\Lambda^1,0) \). Applying the decomposition (3.61) to \( g^s := f^s - f \) we have
\[
W (f)^* \Delta^u_{\Omega} W (f) \Delta^u_{\Omega}^{-1} = e^{i \text{Im}(f,f')_\psi} W (g^s) = e^{i \text{Im}(f,f')_\psi} e^{i \text{Im}(\tilde{g}_s^\ast g^s)} W (g^s) W (g^s) = (5.27)
\]
with
\[
g^s_x (\bar{x}) = \frac{\partial g^s_x (0, \bar{x})}{\partial x^i} = - \cosh (s) \frac{\partial F}{\partial x^i} (\bar{x})^s + \sinh (s) \frac{\partial F}{\partial \bar{x}^i} (\bar{x})^s + \frac{\partial F}{\partial x^i} (0, \bar{x}) ,
g^s_x (\bar{x}) = G^s (0, \bar{x}) = F (\bar{x}) - F (0, \bar{x}) ,
\] (5.28)
where \( \bar{x} = (-x^1 \sinh (s), x^1 \cosh (s), \bar{x}_\perp) = (\Lambda^1 x)_\perp \) and
\[
G^s (x) = F (\Lambda^1 x) - F (x) ,
F (x) = \int_{\mathbb{R}^d} \Delta (x-y) f (y) dy = (5.30)
\]
Now, we explicitly split the unitaries \( W_\varphi (g^s_\varphi) \) and \( W_\pi (g^s_\pi) \) in equation (5.24) defining
\[
g^s_{\varphi, R} (\bar{x}) := g^s_x (\bar{x}) \Theta (x^1) \quad \text{and} \quad g^s_{\varphi, L} (\bar{x}) := g^s_x (\bar{x}) \Theta (-x^1) ,
g^s_{\varphi, R} (\bar{x}) := g^s_x (\bar{x}) \Theta (x^1) \quad \text{and} \quad g^s_{\varphi, L} (\bar{x}) := g^s_x (\bar{x}) \Theta (-x^1) ,
\] (5.32)
which clearly implies that \( g^s_{\varphi, R} + g^s_{\varphi, L} = g^s_\varphi \) and \( g^s_{\pi, R} + g^s_{\pi, L} = g^s_\pi \). Moreover
\[
g^s_{\varphi, R}, g^s_{\varphi, L} \in L^2 (\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{-\frac{1}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \text{supp} (g^s_{\varphi, R}) \subset \Sigma \text{ and supp} (g^s_{\varphi, L}) \subset \Sigma' \] \Rightarrow g^s_{\varphi, R} \in K_\varphi (\Sigma) \quad \text{and} \quad g^s_{\varphi, L} \in K_\varphi (\Sigma') .
\] (5.34)
Furthermore, we have that \( g^s_{\pi, R}, g^s_{\pi, L} \) are real-valued functions and they satisfy the hypothesis of lemma (5.1). Then
\[
g^s_{\pi, R}, g^s_{\pi, L} \in H^1 (\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{\frac{1}{2}} (\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \text{supp} (g^s_{\pi, R}) \subset \Sigma \quad \text{and supp} (g^s_{\pi, L}) \subset \Sigma' \] \Rightarrow g^s_{\pi, R} \in K_\pi (\Sigma) \quad \text{and} \quad g^s_{\pi, L} \in K_\pi (\Sigma') ,
\] (5.35)
which means that the splits (5.32-5.33) work. Coming back to (5.26),
\[
W (f) \Delta^u_{\Omega} W (f) \Delta^u_{\Omega}^{-1} = e^{i \text{Im}(f,f')_\psi} e^{i \text{Im}(\tilde{g}_s^\ast g^s)} W (g^s_{\varphi, L} + g^s_{\varphi, R}) W (g^s_{\pi, L} + g^s_{\pi, R}) = e^{i \text{Im}(f,f')_\psi} e^{i \text{Im}(\tilde{g}_s^\ast g^s)} W (g^s_{\varphi, L}) W (g^s_{\varphi, R}) W (g^s_{\pi, L}) W (g^s_{\pi, R})
\] (5.36)
\[= e^{i \text{Im}(f,f')_\psi} e^{i \text{Im}(\tilde{g}_s^\ast g^s)} \frac{W (g^s_{\varphi, L}) W (g^s_{\varphi, R}) W (g^s_{\pi, L}) W (g^s_{\pi, R})}{e^{K_{W_\varphi}} e^{K_{W_\pi'}}} .
\]
any vector of finite number of particles belong to the domain of

\[ \Delta_{\Psi, \Omega}^{it} = e^{i\alpha(s) W_{\Phi}} (g_{\pi, R}^s) W_{\pi} (g_{\pi, R}^s) \Delta_{\Omega}^{it} \]

Using the fact that the relative modular flow \( \Delta_{\Psi, \Omega}^{it} \) is strongly continuous and that the relative entropy \( S_R (\phi \mid \omega) \) is finite (see discussion at the beginning of section 5.2) and hence the expression (4.29) holds, it is not difficult to show that the function \( t \mapsto \langle \Omega, \Delta_{\Psi, \Omega}^{it} \Omega \rangle_H \) is continuous differentiable. Furthermore, taking the vacuum expectation value on the r.h.s of (5.37), it can be proven that the function \( \alpha (s) \in C^1 (\mathbb{R}) \). Finally, from (5.24) we get the following expression for the relative modular Hamiltonian

\[ K_{\Psi, \Omega} = 2\pi (\alpha' (0)) \{ 1 + \varphi (h_{\pi, R}) + \pi (h_{\pi, R} + K_1) \} , \]

where

\[ h_{\varphi, R} (\bar{x}) := \frac{d}{ds} \big|_{s=0} g_{\pi, R}^s (\bar{x}) = \left( x^1 \frac{\partial^2 F}{(\partial x^0)^2} (0, \bar{x}) + \frac{\partial F}{\partial x^1} (0, \bar{x}) \right) \cdot \Theta (x^1) , \]

\[ h_{\pi, R} (\bar{x}) := \frac{d}{ds} \big|_{s=0} g_{\pi, R}^s (\bar{x}) = \left( -x^1 \frac{\partial F}{\partial x^0} (0, \bar{x}) \right) \cdot \Theta (x^1) . \]

With similar arguments used above, we have that \( h_{\varphi, R} \in K_{\Psi} (\Sigma) \) and \( h_{\pi, R} \in K_{\pi} (\Sigma) \).

Before we proceed to obtain the constant \( \alpha' (0) \), we want to emphasize the importance of it,

\[ S_R (\phi \mid \omega) = \langle \Omega, K_{\Psi, \Omega} \Omega \rangle_H = 2\pi \alpha' (0) . \]

Thus, it is the desired result for the relative entropy. Regardless of this problem, expressions (5.38-5.40) gives us an explicit exact expression for the relative modular Hamiltonian \( K_{\Psi, \Omega} \) up to a constant. It is interesting to notice that the difference \( K_{\Psi, \Omega} - K_{\Omega} \) is just a linear term on the fields operators plus a constant term. We expect that this structure holds not just for the Rindler wedge, but for any kind of region as long as \( \Psi \) is a coherent vector.

### 5.2.3 Determination of \( \alpha' (0) \)

In this subsection we determine the constant \( \alpha' (0) \) and hence the value for the relative entropy. Most of the calculation is straightforward and it is carefully done in the appendix A.3. We take the vacuum expectation value on both sides in expression (5.37),

\[ \langle \Omega, \Delta_{\Psi, \Omega}^{it_1} \Delta_{\Psi, \Omega}^{it_2} \Omega \rangle_H = \langle \Omega, \Delta_{\Psi, \Omega}^{it_1 + t_2} \Omega \rangle_H , \]

and we replace the expression (5.37) obtained for the relative modular flow. Applying \( \frac{d}{ds} \big|_{s=0} = -\frac{1}{2\pi} \frac{d}{dt} \big|_{t=0} \) on both sides of (5.41) \(^{22}\) and matching real and imaginary parts separately we get \(^{23}\)

\[ \alpha' (s_2) - \frac{d}{ds_2} \text{Im} \langle g_{\pi, R}^{s_2} g_{\pi, R}^{s_2} \rangle_{\bar{\theta}} = \alpha' (0) - \frac{d}{ds_1} \big|_{s_1=0} \text{Im} \langle g_{\pi, R}^{s_1} g_{\pi, R}^{s_2} \rangle_{\bar{\theta}} , \]

\[ \frac{d}{ds_1} \big|_{s_1=0} \| g_{\pi, R}^{s_1 + s_2} \|^2_{\bar{\theta}} = \frac{d}{ds_1} \big|_{s_1=0} \| g_{\pi, R}^{s_1} + u (A_{\pi}^{s_1}, 0) g_{\pi, R}^{s_2} \|^2_{\bar{\theta}} , \]

where \( g_{\pi, R}^{s} = E_{\pi} (g_{\pi, R}^{s}) + E_{\pi} (g_{\pi, R}^{s}) \). The second equation is useless to determine \( \alpha' (0) \), then we concentrate in the first one which is a differential equation for \( \alpha' (s) \), with the particularity that \( \alpha' (0) \) appears on it. To solve

\(^{22}\) An explicit computation of the strong derivative in equation (5.38) shows that the vacuum vector \( \Omega \), any coherent vector and any vector of finite number of particles belong to the domain of \( K_{\Psi, \Omega} \).

\(^{23}\) Analytic properties of the relative modular flow ensures that both sides of (5.41) are continuous differentiable functions on \( t_1 \) and \( t_2 \).

\(^{23}\) The \( \frac{d}{ds_2} \) in (5.42) appears because in some terms the dependence on \( s_1 \) of the expression is through \( s_1 + s_2 \).
it, let us analyze the second term on the right hand side of equation (5.42). In appendix A.4 we compute

\[ 2 \text{Im} \langle g_R^{s_1}, g_R^{s_2} \rangle_B = 2 \text{Im} \left\langle g_{φ,R}^{s_1} + g_{ζ,R}^{s_1}, g_{φ,R}^{s_2} + g_{ζ,R}^{s_2} \right\rangle_B \]

\[ = \int_\Sigma f_φ(ξ) f_ζ(ξ) d^{d-1}ξ - \int_\Sigma f_φ(ξ) f_ζ(ξ) d^{d-1}ξ + \gamma(s_2), \tag{5.44} \]

The function \( \gamma \) includes all the \( s_1 \)-independent terms, which they do not contribute to (5.42). In the same appendix we analyze \( P, Q, R \) carefully and we get

\[ \frac{dP}{ds_1} \bigg|_{s_1=0} = \int_\Sigma x^1 \left( \frac{∂F}{∂x} \right)^2 + (\nabla F)^2 + m^2 F^2 \right|_{x^0=0} =: S, \tag{5.45} \]

\[ \frac{d}{ds_2} (Q - R) \bigg|_{s_1=0} = -\frac{d}{ds_2} (Q - R) \bigg|_{s_1=0}. \tag{5.46} \]

Coming back to (5.42), we have that

\[ α'(s_2) - \frac{d}{ds_2} \text{Im} \left\langle g_{φ,R}^{s_2}, g_{ζ,R}^{s_2} \right\rangle_B = α'(0) - \frac{d}{ds_1} \bigg|_{s_1=0} \text{Im} \left\langle g_R^{s_1}, g_R^{s_2} \right\rangle_B \]

\[ = α'(0) - \frac{1}{2} \frac{d}{ds_1} \bigg|_{s_1=0} \left( P(s_1) + Q(s_1, s_2) - R(s_1, s_2) \right) \]

\[ = α'(0) - \frac{1}{2} S + \frac{1}{2} \frac{d}{ds_2} (Q(s_0, s_2) - R(s_0, s_2)). \tag{5.47} \]

Then, integrating this last equation respect to \( s_2 \) we have\(^{24}\)

\[ α(s_2) - \text{Im} \left\langle g_{φ,R}^{s_2}, g_{ζ,R}^{s_2} \right\rangle_B = α'(0) s_2 - \frac{1}{2} S s_2 + \frac{1}{2} (Q(s_0, s_2) - R(s_0, s_2)). \tag{5.48} \]

To determine \( α'(0) \), we will use the analyticity property of the relative modular flow \([33]\), which reflects the KMS-condition. It states that for any \( A, B \in R \) there exits a unique continuos function \( G_{A,B} : \mathbb{R} + i [-1,0] \rightarrow \mathbb{C} \), analytic on \( \mathbb{R} + i (-1,0) \) such that

\[ G_{A,B}(t) = \langle Ω, Δ^A_{Ω} Ω, Δ^B_{Ω} Ω \rangle_H, \tag{5.49} \]

\[ G_{A,B}(-i) = \langle Ψ, Δ^B_{Ψ} Ω, Δ_{Ψ} Ω \rangle_H, \tag{5.50} \]

for all \( t \in \mathbb{R} \). Moreover, the function above is uniquely determined by one of its boundary values. For our purpose, it’s enough to take \( A = B = 1 \) and we simply call \( G(z) \) to the underlying function. In such case,

\[ G(t) = \langle Ω, Δ^t_{Ω} Ω \rangle_H \quad \rightarrow \quad G(-i) = \langle Ψ, Ψ \rangle_H = 1. \tag{5.51} \]

In terms of the real variable \( s = -2πt \), the function \( G(s) \) is in analytic on \( \mathbb{R} + i (0, 2π) \), and relation (5.51) must hold for \( s \rightarrow 2πi \). Using (5.37), we have that

\[ G(s) = e^{iα(s)} \left( Ω, e^{iφ(s)} g_{φ,R}^{s}, e^{iπ(s)} g_{ζ,R}^{s} \right) \]

\[ = e^{iα(s)} - i \text{Im} \left\langle g_{φ,R}^{s}, g_{ζ,R}^{s} \right\rangle_B - \frac{1}{2} ||g_R^{s}||^2, \tag{5.52} \]

and hence

\[ iα(s) - i \text{Im} \left\langle g_{φ,R}^{s}, g_{ζ,R}^{s} \right\rangle_B - \frac{1}{2} ||g_R^{s}||^2 \quad \rightarrow_{s \rightarrow 2πi} \quad i 2π, \quad n \in \mathbb{Z}. \tag{5.53} \]

Having this into account, we come back to (5.48) and write

\[ iα(s) - i \text{Im} \left\langle g_{φ,R}^{s}, g_{ζ,R}^{s} \right\rangle_B - \frac{1}{2} ||g_R^{s}||^2 = iα'(0) s - \frac{1}{2} S s + \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} ||g_R^{s}||^2. \tag{5.54} \]

\(^{24}\)Since \( g_R^{s=0} = g_{ζ,R}^{s=0} = 0 \Rightarrow \text{Im} \left\langle g_{φ,R}^{s=0}, g_{ζ,R}^{s=0} \right\rangle_B = 0 \). Independently, from the definition of \( Q \) and \( R \) we have: \( Q(0, 0) - R(0, 0) = 0. \)
Before we take limit $s \rightarrow 2\pi i$, we may notice that $\bar{x}^s = (x^1 \sinh(s), x^1 \cosh(s), \bar{x}_\perp) \longrightarrow (0, \bar{x})$, which informally suggests that

$$
\begin{align*}
g_R^s \xrightarrow{s \rightarrow 2\pi i} 0 & \quad \Rightarrow \quad \|g_R^s\|_H^2 \xrightarrow{s \rightarrow 2\pi i} 0, \\
\hat{f}_\nu \xrightarrow{s \rightarrow 2\pi i} f_\nu & \quad \Rightarrow \quad Q(0, s) - R(0, s) \xrightarrow{s \rightarrow 2\pi i} 0, \quad \text{where } \nu = \varphi, \pi.
\end{align*}
$$

We prove on appendix A.5 that the function

$$
N(s) := \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_H^2,
$$

of the variable $s \in \mathbb{R}$, can be analytically extended on the strip $\mathbb{R} + i(0, 2\pi)$ and that $\lim_{s \to 2\pi i} N(s) = 0$. Then, taking the limit $s \to 2\pi i$ on (5.4) we get

$$
i2\pi \alpha = -\alpha'(0) \cdot 2\pi + \frac{1}{2} S 2\pi,
$$

but since $\alpha'(0)$ and $S$ are real numbers, then it must be $n = 0$. Finally we get

$$
S_R(\phi | \omega) = 2\pi \alpha'(0) = 2\pi \int_{\Sigma} d^{d-1}x x^1 \frac{1}{2} \left( \left( \frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right) |_{x^0 = 0},
$$

where we recall that $F(x) = \int_{\mathbb{R}^d} \Delta(x - y) f(y) d^d y$. Indeed, this expression coincides with (1.15) for the canonical stress tensor (1.10).

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A Appendix

A.1 Sobolev spaces

For the definition and properties of Sobolev spaces we follow [39]. Here we adapt the notation to our convenience.

Consider the test function space $\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ of smooth and compactly supported functions, with its usual topology. The $n$-dimensional complex Sobolev space of order $\alpha \in \mathbb{R}$ is defined as

$$
H^\alpha(\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \hat{f}(\hat{\rho}) \omega^\alpha P_\hat{\rho} \in L^2(\mathbb{R}^n) \right\},
$$

where $\omega^\rho = \sqrt{\rho^2 + 1}$ and $\hat{f}(\hat{\rho}) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\rho \cdot x} d^d x$ is the usual Fourier transform. From the definition follows that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n) \subset H^{\alpha'}(\mathbb{R}^n)$ if $\alpha > \alpha'$.

The Sobolev space $H^\alpha(\mathbb{R}^n)$ is a Hilbert space under the inner product

$$
\langle f, g \rangle_{H^\alpha} := \int_{\mathbb{R}^n} \hat{f}(\hat{\rho})^{\alpha} \hat{g}(\hat{\rho})^{\alpha} P_\hat{\rho} |_{L^2} = \int_{\mathbb{R}^n} d^dp \hat{f}(\hat{\rho})^{\alpha} \hat{g}(\hat{\rho})^{\alpha} P_\hat{\rho}.
$$

Furthermore, for $f \in H^\alpha(\mathbb{R}^n)$ we have that $\|f\|_{H^\alpha} \leq \|f\|_{H^{\alpha'}}$ if $\alpha > \alpha'$, and hence the natural injections $H^\alpha(\mathbb{R}^n) \hookrightarrow H^{\alpha'}(\mathbb{R}^n)$ for $\alpha > \alpha'$ are continuous. We also have that the set $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ is dense in $H^\alpha(\mathbb{R}^n)$.

When $\alpha = k \in \mathbb{N}_0$, there is also another useful equivalent characterization of the Sobolev spaces in term of weak derivatives

$$
H^k(\mathbb{R}^n) = \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : D^\mu f \in L^2(\mathbb{R}^n), \text{ for all } |\mu| \leq k \right\},
$$

\[25\] The weak derivative of an element of $\mathcal{D}'(\mathbb{R}^n)$ is its usual derivative in the distributional sense.
It is useful to introduce a new norm in $H^k(\mathbb{R}^n)$ as

$$
\|f\|_{H^k} := \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} d^n x |D^\alpha f(x)|^2 \right)^{\frac{1}{2}},
$$

which is equivalent to the former norm $\|\cdot\|_{H^s}$.

The real Sobolev spaces $H^s(\mathbb{R}^n, \mathbb{R})$ are defined in a similar manner as above, but restricting to real valued functions.

In general, it is easier to calculate the usual pointwise derivatives rather than the weak derivatives. Then, the following lemma states sufficient conditions for both notions of derivatives coincide. Before we formulate it, we need to introduce the notions of $C^k$-piecewise function.

**Definition A.1.**

Let $U \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(U)$ and $k \in \mathbb{N}_0$. We say that $f$ is a $C^k$-piecewise function iff there exists a finite family of pairwise disjoint open sets $\{\Omega_j\}_{j=1,\ldots,J} \subset U$ such that

1. $\bigcup_{j=1}^J \Omega_j = U$.
2. $f \in C^k(\Omega_j)$ for all $j = 1,\ldots,J$.
3. For all $j = 1,\ldots,J$, $\forall x_0 \in \partial\Omega_j$ and for all multi-index $|\alpha| \leq k$, the $\lim_{x \to x_0} D^\alpha f(x)|_{\Omega_j}$ exist and are finite (where $D^\alpha$ is the usual multi-order pointwise derivative).

We denote $C^k_{\text{piece}}(U)$ the set of $C^k$-piecewise functions on $U$.

Now, we formulate the lemma that ensures that weak derivatives and pointwise derivatives coincide.

**Lemma A.2.** Let $U \subset \mathbb{R}^n$ be open and $f \in C^0(U) \cap C^1_{\text{piece}}(U)$. Then the (first order) weak derivatives of $f$ coincides with the usual pointwise derivatives.

**Proof.** Since $f \in C^0(U) \cap C^1_{\text{piece}}(U)$ we have that $f$ is locally Lipschitz continuous on $U$ (see Corollary 4.1.1 on [10]). Then we have that $f$ is locally absolute continuous on $U$, and of course $f \in L^1_{loc}(U)$. Then $f$ is weakly differentiable and the (first order) weak and pointwise derivatives of $f$ coincide a.e. \qed

Now, using the above lemma and the alternative definition (eq. (A.3) for the Sobolev space $H^1(\mathbb{R}^n)$, the proof of lemma 5.1 is trivial.

**A.2 Calculation of $\langle \Omega, W(f) K_1 W(-f) \Omega \rangle$**

Given $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$,

$$
\langle \Omega, W(-f) K_1 W(f) \Omega \rangle_H = \left< e^{-\|f\|_H^2} \sum_{n=0}^{\infty} \frac{i^n f^{\otimes n}}{n!} K_1 e^{\|f\|_H^2} \sum_{n=0}^{\infty} \frac{i^n f^{\otimes n}}{n!} \right>_{H} = e^{-\|f\|_H^2} \sum_{n=0}^{\infty} \frac{(-i)^n (i)^n}{n!} \langle f^{\otimes n}, K_1 f^{\otimes n} \rangle_{B^\otimes n} = e^{-\|f\|_H^2} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle f, k_1 f \rangle_{B} \langle f, f \rangle_{B}^{n-1} = e^{-\|f\|_H^2} \langle f, k_1 f \rangle_{B} e^{\|f\|_H^2} \langle f, f \rangle_{B} = \langle f, k_1 f \rangle_{B}. (A.5)
$$

**A.3 Calculation of $\langle f, k_1 f \rangle_{B}$**

Given $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$,

$$
\langle f, k_1 f \rangle_{B} = \text{Re} \langle f, k_1 f \rangle_{B} = \text{Re} \left( -i \left< \frac{d}{ds} \right>_{s=0} \langle f, e^{i k_1 s} f \rangle_{B} \right) = \left. \frac{d}{ds} \right|_{s=0} \text{Im} \langle f, u(\Lambda^t_1, 0) f \rangle_{B} = \left. \frac{d}{ds} \right|_{s=0} \text{Im} \langle f, f^t \rangle_{B} = \left. \frac{d}{ds} \right|_{s=0} \text{Im} \langle f, f^t \rangle_{B} \quad (A.6)
$$
where we have defined \( f^s = f(\Lambda^s_{x^0}) \). As we explained in section \( \S 5.3 \) there exist functions \( \varphi, f, \varphi^s, \varphi^s_\pi \in \mathcal{S}(\mathbb{R}^{d-1},\mathbb{R}) \) such that

\[
E(f) = E_\varphi(f_\varphi) + E_\pi(f_\pi) \quad \text{and} \quad E(f^s) = E_\varphi(f^s_\varphi) + E_\pi(f^s_\pi).
\]

Replacing these into (A.6) we get

\[
\langle f, k_1 f \rangle_B = \frac{d}{ds} \bigg|_{s=0} \text{Im} \langle f_\varphi + f_\pi, f_\varphi^s + f_\pi^s \rangle_B = \frac{d}{ds} \bigg|_{s=0} \left( \text{Im} \langle f_\varphi, f_\varphi^s \rangle_B + \text{Im} \langle f_\pi, f_\varphi^s \rangle_B \right)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\varphi(x) f_\varphi^s(x) d^{d-1}x - \frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\pi(x) f_\pi^s(x) d^{d-1}x,
\]

where we have used the relations (3.34) in second line and (3.66) in last line. From the Poincaré invariance of the distribution \( \Delta(x) \) we have that

\[
F^s(x) = \int_{\mathbb{R}^d} \Delta(x-y) f^s(x) d^d y,
\]

where we have defined \( F^s = F(\Lambda^s_{x^0}) \). Then

\[
\begin{align*}
\varphi_\varphi(x) &:= -\frac{\partial F}{\partial x^0}(0,x), \\
\varphi_\pi(x) &:= F(0,x), \\
\varphi^s_\varphi(x) &:= -cosh(s) \frac{\partial F}{\partial x^0}(x^s) + sinh(s) \frac{\partial F}{\partial x^1}(x^s), \\
\varphi^s_\pi(x) &:= F(x^s),
\end{align*}
\]

being \( x^s := (-x^1 \sinh(s), x^1 \cosh(s), x_\perp) \), and hence

\[
\begin{align*}
\frac{d}{ds} \bigg|_{s=0} f^s_\varphi(x) &:= x^1 \frac{\partial^2 F}{(\partial x^0)^2}(0,x) + \frac{\partial F}{\partial x^1}(0,x), \\
\frac{d}{ds} \bigg|_{s=0} f^s_\pi(x) &:= -x^1 \frac{\partial F}{\partial x^0}(0,x).
\end{align*}
\]

Replacing such expressions into (A.8) we finally get\(^{26}\)

\[
\langle f, k_1 f \rangle_B = \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}x \left( \frac{\partial F}{\partial x^0} \right)^2 - x^1 \frac{\partial F}{\partial x^0} \right) d^{d-1}x - \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}x \left( x^1 \frac{\partial^2 F}{(\partial x^0)^2} + \frac{\partial F}{\partial x^1} \right) F
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}x \left[ x^1 \left( \frac{\partial F}{\partial x^0} \right)^2 - \left( x^1 (\nabla^2 - m^2) + F \right) + \frac{\partial F}{\partial x^1} \right]
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}x \left[ x^1 \left( \frac{\partial F}{\partial x^0} \right)^2 - x^1 (\nabla^2 F) - \frac{\partial F}{\partial x^1} F + m^2 F^2 \right]
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}x \left[ \left( \frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right].
\]

### A.4 Explicit computations of section \( \S 5.2.3 \)

We replace the expression (5.37) obtained for \( \Delta^d_{\varphi,\Omega} \) on both sides of (5.41) and we set \( s = -2\pi t \). Working with each side of (5.41) separately

\(^{26}\) All functions involved in the following expressions are evaluated at \( x^0 = 0 \).
\[
\langle \Omega, \Delta_{\psi,\Omega}^{(t_1)}, \Delta^{(t_2)} \Omega \rangle_{\mathcal{H}} = \langle \Omega, e^\iota (s_1) W_{\psi} \left( g_{\overline{s},R}^{s_1} \right) W_{\pi} \left( g_{\pi,R}^{s_1} \right) \Delta^\dagger_{\Omega} e^\iota (s_2) W_{\psi} \left( g_{\overline{s},R}^{s_2} \right) W_{\pi} \left( g_{\pi,R}^{s_2} \right) \Delta_{\Omega} \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1) + (s_2) \langle \Omega, W_{\psi} \left( g_{\overline{s},R}^{s_1} \right) W_{\pi} \left( g_{\pi,R}^{s_1} \right) e^\iota (s_1) K W_{\psi} \left( g_{\overline{s},R}^{s_2} \right) W_{\pi} \left( g_{\pi,R}^{s_2} \right) \Omega \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1) + (s_2) - \text{Im} \left( g_{\overline{s},R}^{s_1} g_{\pi}^{s_1} \right) - \text{Im} \left( g_{\overline{s},R}^{s_2} g_{\pi}^{s_2} \right) \langle \Omega, W \left( g_{\overline{s},R}^{s_1} \right) W (u (\Lambda_{s_1}^1) g_{\overline{s},R}^{s_2} \Omega) \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1) + (s_2) - \text{Im} \left( g_{\overline{s},R}^{s_1} g_{\pi}^{s_1} \right) - \text{Im} \left( g_{\overline{s},R}^{s_2} g_{\pi}^{s_2} \right) \langle \Omega, W \left( g_{\overline{s},R}^{s_1} \right) W (u (\Lambda_{s_1}^1) g_{\overline{s},R}^{s_2} \Omega) \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1) + (s_2) - \text{Im} \left( g_{\overline{s},R}^{s_1} g_{\pi}^{s_1} \right) - \text{Im} \left( g_{\overline{s},R}^{s_2} g_{\pi}^{s_2} \right) \langle \Omega, W \left( g_{\overline{s},R}^{s_1} \right) W (u (\Lambda_{s_1}^1) g_{\overline{s},R}^{s_2} \Omega) \rangle_{\mathcal{H}}
\]  \hspace{1cm} (A.17)

\[
\langle \Omega, \Delta_{\psi,\Omega}^{(t_1 + s_2)} \rangle_{\mathcal{H}} = \langle \Omega, e^\iota (s_1 + s_2) W_{\psi} \left( g_{\overline{s},R}^{s_1 + s_2} \right) W_{\pi} \left( g_{\pi,R}^{s_1 + s_2} \right) \Delta_{\Omega} \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1 + s_2) \langle \Omega, W_{\psi} \left( g_{\overline{s},R}^{s_1 + s_2} \right) W_{\pi} \left( g_{\pi,R}^{s_1 + s_2} \right) \Omega \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1 + s_2) - \text{Im} \left( g_{\overline{s},R}^{s_1 + s_2} g_{\pi}^{s_1 + s_2} \right) \langle \Omega, W \left( g_{\overline{s},R}^{s_1 + s_2} \right) \Omega \rangle_{\mathcal{H}}
\]

\[
eq e^\iota (s_1 + s_2) - \text{Im} \left( g_{\overline{s},R}^{s_1 + s_2} g_{\pi}^{s_1 + s_2} \right) \langle \Omega, W \left( g_{\overline{s},R}^{s_1 + s_2} \right) \Omega \rangle_{\mathcal{H}}
\]  \hspace{1cm} (A.18)

where we remember that $g_{\overline{s},R} = E_{\pi} \left( g_{\pi,R} \right) + E_{\pi} \left( g_{\overline{s},R} \right) \in \mathcal{H}$. Taking $\frac{d}{ds_1} |_{s_1=0}$ on both expressions above we obtain

\[
\frac{d}{ds_1} |_{s_1=0} \langle \Omega, \Delta_{\psi,\Omega}^{(t_1)}, \Delta^{(t_2)} \Omega \rangle_{\mathcal{H}} = i \alpha' (0) - i \frac{d}{ds_1} |_{s_1=0} \text{Im} \langle g_{\overline{s},R}^{s_1} \Omega \rangle_{\mathcal{H}} - i \frac{d}{ds_1} |_{s_1=0} \text{Im} \langle g_{\pi,R}^{s_1} \Omega \rangle_{\mathcal{H}}
\]

\[
eq \frac{1}{2} \frac{d}{ds_1} |_{s_1=0} \left\| g_{\overline{s},R}^{s_1} + u (\Lambda_{s_1}^1) g_{\overline{s},R}^{s_1} \right\|_{\mathcal{H}}^2
\]

\[
\langle \Omega, \Delta_{\psi,\Omega}^{(t_1 + s_2)} \rangle_{\mathcal{H}} = i \alpha' (s_2) - i \frac{d}{ds_1} |_{s_1=0} \text{Im} \langle g_{\pi,R}^{s_1 + s_2} \Omega \rangle_{\mathcal{H}} - \frac{1}{2} \frac{d}{ds_1} |_{s_1=0} \left\| g_{\pi,R}^{s_1 + s_2} \right\|_{\mathcal{H}}^2
\]

\[
\neq i \alpha' (s_2) - i \frac{d}{ds_1} |_{s_1=0} \text{Im} \langle g_{\pi,R}^{s_1 + s_2} \Omega \rangle_{\mathcal{H}} - \frac{1}{2} \frac{d}{ds_1} |_{s_1=0} \left\| g_{\pi,R}^{s_1 + s_2} \right\|_{\mathcal{H}}^2
\]  \hspace{1cm} (A.19)

Matching real and imaginary parts of these two last expressions, we arrive to formulae (B.42 - B.43).

Expressions (7.44) follows from

\[
2 \text{Im} \langle g_{\pi,R}^{s_1} \rangle_{\mathcal{H}} = 2 \text{Im} \langle g_{\pi,R}^{s_1} \rangle_{\mathcal{H}}
\]

\[
= \int_{\mathcal{S}} d^{d+1-1} x g_{\pi}^{s_1} (x) g_{\pi}^{s_1} (x) - \int_{\mathcal{S}} d^{d+1-1} x f_{\pi}^{s_1} (x) f_{\pi}^{s_1} (x)
\]

\[
= \int_{\mathcal{S}} d^{d+1-1} x \left( f_{\pi}^{s_1} (x) - f_{\pi} (x) \right) \left( f_{\pi}^{s_1} (x) - f_{\pi} (x) \right) - \int_{\mathcal{S}} d^{d+1-1} x \left( f_{\pi}^{s_1} (x) - f_{\pi} (x) \right) \left( f_{\pi}^{s_1} (x) - f_{\pi} (x) \right)
\]

\[
= \int_{\mathcal{S}} d^{d+1-1} x f_{\pi}^{s_1} (x) f_{\pi} (x) - \int_{\mathcal{S}} d^{d+1-1} x f_{\pi}^{s_1} (x) f_{\pi} (x)
\]

\[
= \int_{\mathcal{S}} d^{d+1-1} x f_{\pi}^{s_1} (x) f_{\pi} (x) - \int_{\mathcal{S}} d^{d+1-1} x f_{\pi}^{s_1} (x) f_{\pi} (x) \hspace{1cm} (A.21)
\]
where the function $\gamma$ includes all the $s_1$-independent terms.

The function $P(s_1)$ is essentially the same as (A.3) in appendix A.3, with the difference that now the integration is over the region $\Sigma$ instead of the whole $\mathbb{R}^{d-1}$. Despite this, the final result is the same and hence we get\footnote{Following the computation of (A.3) in appendix A.3, now appears a boundary term after the integration by parts. Fortunately, this term vanishes since the integrand is zero at the boundary of $\Sigma.$}

$$
\left. \frac{dP}{ds_1} \right|_{s_1=0} = \int_\Sigma d^{d-1}x \left[ \left( \frac{\partial F}{\partial x^0} \right)^2 + (\nabla F)^2 + m^2 F^2 \right] \bigg|_{x^0=0} =: S. \tag{A.22}
$$

Now we explicitly obtain the relations (5.46). Indeed

$$
\frac{dR}{ds_1} \bigg|_{s_1=0} = \frac{d}{ds_1} \bigg|_{s_1=0} \int_\Sigma d^{d-1}x \left( - \cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_1}) \right) F(\bar{x}^{s_1})
= \int_\Sigma d^{d-1}x \left( - \cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_1}) \right) \left( -x^1 \frac{\partial F}{\partial x^0}(\bar{x}) \right)
= \frac{d}{ds_2} \int_\Sigma d^{d-1}x \left( - \frac{\partial F}{\partial x^0}(\bar{x}) \right) F(\bar{x}^{s_2})
= \frac{d}{ds_2} \bigg|_{s_1=0} \int_\Sigma d^{d-1}x f_{\varphi^1}^1(x) f_{\varphi^2}^2(x) = \frac{dQ}{ds_2} \bigg|_{s_1=0}. \tag{A.23}
$$

Similarly we start with

$$
\frac{dQ}{ds_1} \bigg|_{s_1=0} = \frac{d}{ds_1} \bigg|_{s_1=0} \int_\Sigma d^{d-1}x f_{\varphi^1}^1(x) f_{\varphi^2}^2(x)
= \frac{d}{ds_1} \bigg|_{s_1=0} \int \Sigma d^{d-1}x \left( - \cosh(s_1) \frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_1) \frac{\partial F}{\partial x^1}(\bar{x}^{s_1}) \right) F(\bar{x}^{s_2})
= \int \Sigma d^{d-1}x \left( x^1 \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x}) \right) F(\bar{x}^{s_2})
= \int \Sigma d^{d-1}x \left( x^1 (\nabla^2 - m^2) F(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x}) \right) F(\bar{x}^{s_2}) .
$$

First we integrate the Laplacian term by parts

$$
\frac{dQ}{ds_1} \bigg|_{s_1=0} = - \int \Sigma d^{d-1}x x^1 m^2 F(\bar{x}) F(\bar{x}^{s_2}) - \int \Sigma d^{d-1}x x^1 \nabla \cdot \nabla F(\bar{x}) \cdot \nabla F(\bar{x}^{s_2})
- \int \Sigma d^{d-1}x x^1 \frac{\partial F}{\partial x^0}(\bar{x}) \left( -\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) .
$$

After a second integration by parts we get

$$
\frac{dQ}{ds_1} \bigg|_{s_1=0} = \int \Sigma d^{d-1}x x^1 \nabla F(\bar{x}) (\nabla^2 - m^2) F(\bar{x}^{s_2}) + \int \Sigma d^{d-1}x F(\bar{x}) \left( -\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right)
+ \int \Sigma d^{d-1}x x^1 F(\bar{x}) \left( \sinh^2(s_2) \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) - 2 \sinh(s_2) \cosh(s_2) \frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \cosh^2(s_2) \frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2}) \right) .
$$

Now we form a Laplacian term in the first line and we use the equation of motion for $F$,

$$
\frac{dQ}{ds_1} \bigg|_{s_1=0} = \int \Sigma d^{d-1}x x^1 F(\bar{x}) \left( \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) \right) + \int \Sigma d^{d-1}x F(\bar{x}) \left( -\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right)
+ \int \Sigma d^{d-1}x x^1 F(\bar{x}) \left( \sinh^2(s_2) \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) - 2 \sinh(s_2) \cosh(s_2) \frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \sinh^2(s_2) \frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2}) \right) .
$$
Finally, a straightforward computation shows that
\[
\left. \frac{dQ}{ds_1} \right|_{s_1=0} = \int \frac{d^{d-1}x}{\Sigma} \frac{d}{ds_2} \left( - \cosh (s_2) \frac{\partial F}{\partial x^2} (\bar{x}^{2^2}) + \sinh (s_2) \frac{\partial F}{\partial x^1} (\bar{x}^{2^2}) \right) F(\bar{x})
\]
\[
= \frac{d}{ds_2} \left| \int \frac{d^{d-1}x}{\Sigma} \left( - \cosh (s_2) \frac{\partial F}{\partial x^2} (\bar{x}^{2^2}) + \sinh (s_2) \frac{\partial F}{\partial x^1} (\bar{x}^{2^2}) \right) F(\bar{x}^{2^1}) \right|_{s_1=0}.
\]

Using (A.23) and (A.24) we arrive to (5.46).

A.5 Analytic continuation for \( N(s) \)

In order to show that formulae (5.56) hold, we need to explicitly show the analytic continuation for the function
\[
N(s) = \frac{i}{2} (Q(0,s) - R(0,s)) - \frac{1}{2} \| g_R \|_B^2,
\]
(A.25)

or more specifically, we need to show that there exists a continuous function \( \tilde{N} : \mathbb{R} + i[0,2\pi] \to \mathbb{C} \), analytic on \( \mathbb{R} + i(0,2\pi) \) such that
\[
\tilde{N}(s + i0) = N(s).
\]

To begin with, we notice that
\[
\frac{i}{2} Q(0,s) = \frac{i}{2} \int \frac{d^{d-1}x}{\Sigma} f_\varphi(\bar{x}) f_\varphi^* (\bar{x}) = i \text{Im} \left\langle f_\varphi, R, f_\varphi^*, R \right\rangle_B,
\]
(A.27)

\[
\frac{i}{2} R(0,s) = \frac{1}{2} \int \frac{d^{d-1}x}{\Sigma} f_\varphi(\bar{x}) f_\varphi(\bar{x}) = i \text{Im} \left\langle f_\varphi^*, R, f_\varphi, R \right\rangle_B,
\]
(A.28)

where the above expressions make sense regardless \( f_\varphi^*, R \notin \mathcal{S} \). This is because
\[
\left\langle f_\varphi^*, R, f_\varphi^*, R \right\rangle_B = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_p} \hat{f}_\varphi(R)(\hat{p})^* i\omega_p \hat{f}_\varphi(R)(\hat{p}) = \frac{i}{2} \left( \hat{f}_\varphi(R), \hat{f}_\varphi(R) \right)_{L^2},
\]
(A.29)

which is convergent. The problem involving scalar products of split functions \( f_\varphi^*, R \) and \( f_\varphi^*, R \) happens only when we try to compute the scalar product of two cut test functions of the momentum operator, e.g.
\[
\left\langle f_\pi, R, f_\pi^*, R \right\rangle_B = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_p} \left( i\omega_p f_\pi(R)(\hat{p}) \right)^* i\omega_p \hat{f}_\pi(R)(\hat{p}) = \frac{1}{2} \left( \hat{f}_\pi, \hat{f}_\pi \right)_{H^1_B},
\]
(A.30)

which is in general divergent. Such divergency comes from the non-continuity of the function \( f_\pi, R (\bar{x}) = f_\pi, R (\bar{x}) \Theta (\bar{x})^{\pm} \) at \( x^1 = 0 \). To overcome this difficulty we introduce a family of smooth functions (for \( \epsilon > 0 \))
\[
f_\varphi^\varepsilon, R(\bar{x}) := f_\varphi(\bar{x}) \Theta_\varepsilon \left( x^1 \right),
\]
(A.31)

\[
f_\pi^\varepsilon, R(\bar{x}) := f_\pi(\bar{x}) \Theta_\varepsilon \left( x^1 \right),
\]
(A.32)

where \( \Theta_\varepsilon \in C^\infty (\mathbb{R}) \) is a regularized Heaviside function such that
\[
\Theta_\varepsilon (t) = \begin{cases} 
0 & \text{if } t \leq \frac{\varepsilon}{2} \\
1 & \text{if } t \geq \varepsilon
\end{cases}
\]
(A.33)

Then
\[
f_\varphi^\varepsilon, R(\bar{x}) \xrightarrow{\epsilon \to 0^+} f_\varphi, R(\bar{x}) \quad \text{and} \quad f_\pi^\varepsilon, R(\bar{x}) \xrightarrow{\epsilon \to 0^+} f_\pi, R(\bar{x}),
\]
(A.34)

where the above convergence must be in a sense that we will specify opportunistically below. Before we get into such convergence issues, we notice that \( f_\varphi^\varepsilon, R, f_\pi^\varepsilon, R \in \mathcal{S} (\mathbb{R}^{d-1}, \mathbb{R}) \) and hence the scalar product (A.30) is now well defined. Then we define the function
\[
N^\varepsilon (s) := i \text{Im} \left\langle f_\varphi^\varepsilon, R, f_\varphi^\varepsilon, R \right\rangle_B - i \text{Im} \left\langle f_\pi^\varepsilon, R, f_\pi^\varepsilon, R \right\rangle_B - \frac{1}{2} \| g_R \|_B^2,
\]
(A.35)
which is just the regularized version of (A.25). It can be rewritten as

\[
N^\epsilon (s) = i \Im \left( f_{\varphi,R}^{\epsilon} \right)_R - i \Im \left( f_{\varphi,R} \right)_R - \frac{1}{2} \left\| f_{\varphi,R}^{\epsilon} \right\|_R^2
\]

\[
= i \Im \left( f_{\varphi,R}^{\epsilon} \right)_R + i \Im \left( f_{\varphi,R} \right)_R - \frac{1}{2} \left\langle f_{\varphi,R}^{\epsilon}, f_{\varphi,R} \right\rangle_R
\]

\[
= \left( f_{\varphi,R}^{\epsilon} \right)_R - \frac{1}{2} \left( f_{\varphi,R} \right)_R - \frac{1}{2} \left( f_{\varphi,R}^{\epsilon} \right)_R
\]

\[
= \int R^{d-1} p \left\{ \left( f_{\varphi,R}^{\epsilon} + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} \right)^* \left( f_{\varphi,R} + i \omega_p \hat{f}_{\varphi,R} \right) - \left( f_{\varphi,R} + i \omega_p \hat{f}_{\varphi,R} \right)^* \left( f_{\varphi,R}^{\epsilon} + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} \right) \right\} .
\]

(A.36)

where in the penultimate line we have used that \( f_{\varphi,R}^{\epsilon + s_{2} \epsilon} = u \left( A_{\varphi}^\epsilon \right) f_{\varphi,R}^{\epsilon} \) for all \( s_1, s_2 \in \mathbb{R} \). For a moment, let assume that this last expression converges to

\[
N (s) = \int R^{d-1} p \left\{ \left( f_{\varphi,R}^{\epsilon} + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} \right)^* \left( f_{\varphi,R} + i \omega_p \hat{f}_{\varphi,R} \right) - \left( f_{\varphi,R} + i \omega_p \hat{f}_{\varphi,R} \right)^* \left( f_{\varphi,R}^{\epsilon} + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} \right) \right\} ,
\]

(A.37)

when \( \epsilon \to 0^+ \). We will prove this in the next subsection. The second term of the above integrand is independent on \( s \) and hence its analytic continuation is trivial. Let then focus on the first term. Using the covariance and causality of the Klein-Gordon equation, it is not difficult to show that

\[
\hat{f}_{\varphi,R}^{\epsilon} (p) + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} (p) = \hat{f}_{\varphi,R} (A_{\varphi}^\epsilon \hat{p}) + i A_{\varphi}^\epsilon \omega_p \hat{f}_{\varphi,R} (A_{\varphi}^\epsilon \hat{p}) ,
\]

(A.38)

where \( A_{\varphi}^\epsilon \hat{p} = \left( p, \cos (h) - \omega_p \sin (h), \hat{p}_\perp, \hat{\gamma}_\perp \right) \) and \( A_{\varphi}^\epsilon \omega_p = \omega_p \cos (h) - p^1 \sinh (h) \). Then, the first integrand term of (A.37) becomes

\[
\left( f_{\varphi,R}^{\epsilon} + i \omega_p \hat{f}_{\varphi,R}^{\epsilon} \right)^* \left( f_{\varphi,R} + i \omega_p \hat{f}_{\varphi,R} \right)
\]

\[
= \int R^{d-1} x d^{d-1} y \left( f_{\varphi,R} (x) - i \omega_p f_{\varphi,R} (x) \right) \left( f_{\varphi,R} (y) + i \omega_p f_{\varphi,R} (y) \right) e^{i A_{\varphi}^\epsilon \hat{p} (x) \cdot \hat{\gamma} (y)} ,
\]

(A.39)

where \( -i A_{\varphi}^\epsilon \hat{p} \cdot \hat{y} = -i \left( -\sinh \left( \frac{s}{2} \right) \omega_p + \cosh \left( \frac{s}{2} \right) \right) y^1 - i \hat{p}_\perp \cdot \hat{y}_\perp \), and equivalently for \( i A_{\varphi}^\epsilon \hat{p} \cdot \hat{x} \). Then

\[
\left( -i \left( -\sinh \left( \frac{s}{2} \right) \omega_p + \cosh \left( \frac{s}{2} \right) \right) y^1 \right) \eta^1
\]

\[
\longrightarrow_{s \to s + i \sigma} \left( -i \left( -\sinh \left( \frac{s + i \sigma}{2} \right) \omega_p + \cosh \left( \frac{s + i \sigma}{2} \right) \right) y^1 \right)
\]

\[
= \left( -i \left( -\sinh \left( \frac{s}{2} \right) \omega_p + \cosh \left( \frac{s}{2} \right) \right) y^1 \cos \left( \frac{s}{2} \right) - \cosh \left( \frac{s}{2} \right) \omega_p \sin \left( \frac{s}{2} \right) y^1 \right) \eta^1
\]
To do this, we must precise in which sense the functions $f^{s, R}_{\varphi, R}, f^{s, R}_{\pi, R}$ converge in (A.34). To begin the choice we make the following smooth step function

$$\Theta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq \frac{\varepsilon}{2}, \\ \exp\left(\frac{\varepsilon(t - \frac{\varepsilon}{2})}{(t - \frac{\varepsilon}{2})^2 - (\frac{\varepsilon}{2})^2}\right) + 1 & \text{if } \frac{\varepsilon}{2} < t < \varepsilon, \\ 1 & \text{if } t \geq \varepsilon. \end{cases} \tag{A.46}$$

Let first we focus in the limit (A.45). Looking back to (A.36), we can rewrite the l.h.s. of that expression as

$$N^s(s) = \left\langle f^{s, R}_{\varphi, R}, f^{s, R}_{\pi, R} \right\rangle_{B_0} - \left\langle f^{s, R}_{\pi, R} \right\rangle_{B_0} = \left\langle f^{s, R}_{\varphi, R}, f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \right\rangle_{B_0} - \left\langle f^{s, R}_{\pi, R} \right\rangle_{B_0}$$

$$= \left\langle f^{s, R}_{\varphi, R}, f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \right\rangle_{B_0} + \left\langle f^{s, R}_{\pi, R}, f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \right\rangle_{B_0}$$

$$+ \left\langle f^{s, R}_{\pi, R}, f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \right\rangle_{B_0} + \left\langle f^{s, R}_{\pi, R} \right\rangle_{B_0}$$

$$\otimes \left\langle f^{s, R}_{\pi, R}, f^{s, R}_{\pi, R} \right\rangle_{B_0} \otimes \left\langle f^{s, R}_{\pi, R}, f^{s, R}_{\pi, R} \right\rangle_{B_0}. \tag{A.47}$$

It is not difficult to see that

$$f^{s, R}_{\varphi, R} \to f^{s, R}_{\varphi, R} \quad \text{and} \quad f^{s, R}_{\pi, R} \to f^{s, R}_{\pi, R}, \quad \text{in } L^2(\mathbb{R}^{d-1}), \tag{A.49}$$

which implies that all terms in (A.48) are convergent, except perhaps those pointed by $\otimes$. Now we concentrate in those remaining terms, e.g.

$$\left\langle f^{s, R}_{\pi, R}, f^{s, R}_{\pi, R} \right\rangle_{B_0} = \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}p \int f^{s, R}_{\pi, R} (\hat{p}) \left(\hat{f}^{s, R}_{\pi, R} (\hat{p}) - \hat{f}^{s, R}_{\pi, R} (\hat{p})\right) \omega_\rho. \tag{A.50}$$

The convergence of (A.50) is guaranteed by the fact that

$$f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \to f^{s, R}_{\pi, R} - f^{s, R}_{\pi, R} \quad \text{in } H^1(\mathbb{R}^{d-1}), \tag{A.51}$$

$$\left(\hat{f}^{s, R}_{\pi, R} - \hat{f}^{s, R}_{\pi, R}\right) \omega_\rho \to \left(\hat{f}^{s, R}_{\pi, R} - \hat{f}^{s, R}_{\pi, R}\right) \omega_\rho \quad \text{in } L^2(\mathbb{R}^{d-1}). \tag{A.52}$$

In order to probe (A.51) we remember that $f^{s, R}_{\pi, R}(\hat{x}) - f^{s, R}_{\pi, R}(\hat{x}) = g^s_{\pi}(\hat{x}) \Theta (x^1)$ with $g^s_{\pi} \in S(\mathbb{R}^{d-1}, \mathbb{R})$ and $g^s_{\pi} \mid_{x^1=0} = 0$. Then the following lemma ensures (A.51).

**Lemma A.3.**

Let $g \in S(\mathbb{R}^n)$ with $g_{x^1=0} = 0$, $g_R(\hat{x}) = g(\hat{x}) \Theta (x^1)$ and $g^s_R(\hat{x}) = g(\hat{x}) \Theta (x^1)$ with $\Theta$ as (A.46). Then $g^s_R \in H^1(\mathbb{R}^n)$ and $g^s_R \to g^s_R$ in $H^1(\mathbb{R}^n)$.

**Proof.** The fact that $g^s_R \in H^1(\mathbb{R}^n)$ is guaranteed by lemma [5.1] Then we prove the convergence for $n = 1$. The generalization to $n > 1$ is straightforward. Since $g_R, g^s_R$ satisfy the hypothesis of the lemma, their weak derivatives coincide with their pointwise derivatives and hence

$$\|g_R - g_R\|_{H^1}^2 = \int_{-\infty}^{+\infty} dx |g(x)\Theta(x) - g(x)\Theta(x)|^2 + \int_{-\infty}^{+\infty} dx |\partial_x [g(x)\Theta(x) - g(x)\Theta(x)]|^2$$

$$\leq \int_{-\infty}^{+\infty} dx |g(x)|^2 |\Theta(x) - \Theta(x)|^2 + \int_{-\infty}^{+\infty} dx |g'(x)|^2 |\Theta'(x) - \Theta(x)|^2$$

$$+ 2 \int_{-\infty}^{+\infty} dx |g(x)| |g'(x)| |\Theta(x) - \Theta(x)| |\Theta'(x)|$$

$$\leq \int_{\frac{1}{2}} dx \left( |g(x)|^2 + |g'(x)|^2 \right) + \int_{\frac{1}{2}} dx |g(x)|^2 |\Theta'(x)|^2 + 2 \int_{\frac{1}{2}} dx |g(x)| |g'(x)| |\Theta'(x)|. \tag{A.53}$$

We notice that since $g \in C^\infty(\mathbb{R})$ and $g(0) = 0$, by the Taylor theorem we have that $g(x) = g'(0)x + r(x)x$ with $r(x) \to 0$ and $r \in C^\infty(\mathbb{R})$. We also have that $\max_{x \in \mathbb{R}} |\Theta'(x)| = \frac{1}{4}$ which follows from the definition of
that function. Then using the above properties and assuming $0 < \epsilon \leq 1$
\[
\|g_R - g\|^2_{H^1} \leq \max_{x \in [0, 1]} \left( |g(x)|^2 + |g'(x)|^2 \right) \int_0^\epsilon dx + \max_{x \in [0, 1]} |g'(0) + r(x)|^2 \frac{16}{\epsilon^2} \int_0^\epsilon dx x^2
+ \max_{x \in [0, 1]} |g'(x)| \max_{x \in [0, 1]} |g'(0) + r(x)| \frac{2}{3} \epsilon \int_0^\epsilon dx x
\leq \max_{x \in [0, 1]} \left( |g(x)|^2 + |g'(x)|^2 \right) \frac{\epsilon}{2} + \max_{x \in [0, 1]} |g'(0) + r(x)|^2 \frac{14}{3} \epsilon
+ \max_{x \in [0, 1]} |g'(x)| \max_{x \in [0, 1]} |g'(0) + r(x)| 3 \epsilon \xrightarrow{\epsilon \to 0^+} 0 .
\]

Then we have that all terms in (A.48) converge. By continuity of the scalar product, the limit of (A.48) is just this same expression but evaluated at $\epsilon = 0$, which coincides with the l.h.s of (A.49).

We use the same arguments to prove the limit (A.44). The first two terms of (A.44) are convergent due to (A.49), and the remaining term is also convergent due to (A.51) and (A.52). Then by continuity of the scalar product we have that
\[
N^{\epsilon}(s) \xrightarrow{\epsilon \to 0^+} N(s) ,
\]
where $N(s)$ is given by (A.25). Finally, expression (A.37) holds.

References

[1] David D. Blanco, Horacio Casini, Ling-Yan Hung, and Robert C. Myers. Relative Entropy and Holography. JHEP, 08:060, 2013.
[2] Nima Lashkari. Relative Entropies in Conformal Field Theory. Phys. Rev. Lett., 113:051602, 2014.
[3] Gábor Sárosi and Tomonori Ugajin. Relative entropy of excited states in two dimensional conformal field theories. JHEP, 07:114, 2016.
[4] Gábor Sárosi and Tomonori Ugajin. Relative entropy of excited states in conformal field theories of arbitrary dimensions. JHEP, 02:060, 2017.
[5] Paola Ruggiero and Pasquale Calabrese. Relative Entanglement Entropies in 1+1-dimensional conformal field theories. JHEP, 02:060, 2017.
[6] Stefan Hollands, Onirban Islam, and Ko Sanders. Relative entanglement entropy for widely separated regions in curved spacetime. J. Math. Phys., 59(6):062301, 2018.
[7] Roberto Longo and Feng Xu. Relative Entropy in CFT. Adv. Math., 337:139–170, 2018.
[8] Stefan Hollands. Relative entropy close to the edge. 2018.
[9] Raúl E. Arias, Horacio Casini, Marina Huerta, and Diego Pontello. Entropy and modular Hamiltonian for a free chiral scalar in two intervals. Phys. Rev., D98(12):125008, 2018.
[10] Sara Murciano, Paola Ruggiero, and Pasquale Calabrese. Entanglement and relative entropies for low-lying excited states in inhomogeneous one-dimensional quantum systems. 2018.
[11] Feng Xu. On Relative Entropy and Global Index. 2018.
[12] Roberto Longo. Entropy distribution of localised states. 2018.
[13] Roberto Longo. Entropy of Coherent Excitations. 2019.
[14] Stefan Hollands. Relative entropy for coherent states in chiral CFT. 2019.
[15] H. Araki. Relative Entropy of States of von Neumann Algebras. 1974.
[16] Daniel L. Jafferis, Aitor Lewkowycz, Juan Maldacena, and S. Josephine Suh. Relative entropy equals bulk relative entropy. JHEP, 06:004, 2016.
[17] Xi Dong, Daniel Harlow, and Aron C. Wall. Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality. Phys. Rev. Lett., 117(2):021601, 2016.

[18] Thomas Faulkner and Aitor Lewkowycz. Bulk locality from modular flow. JHEP, 07:151, 2017.

[19] Thomas Faulkner, Min Li, and Huajia Wang. A modular toolkit for bulk reconstruction. 2018.

[20] Fikret Ceyhan and Thomas Faulkner. Recovering the QNEC from the ANEC. 2018.

[21] H. Casini. Relative entropy and the Bekenstein bound. Class. Quant. Grav., 25:205021, 2008.

[22] H. Casini. Mutual information challenges entropy bounds. Class. Quant. Grav., 24:1293–1302, 2007.

[23] Horacio Casini, Marina Huerta, Robert C. Myers, and Alexandre Yale. Mutual information and the F-theorem. JHEP, 10:003, 2015.

[24] Nima Lashkari, Hong Liu, and Srivatsan Rajagopal. Modular Flow of Excited States. 2018.

[25] R. T. Seeley. Extension of $c^\infty$ functions defined in a half space. Proc. Amer. Math. Soc., 15:625–626, 1964.

[26] M. Reed and B. Simon. Methods of Modern Mathematical Physics: Functional analysis, volume 2. Academic Press, 1980, 1975. Page 210.

[27] R. F. Streater and A. S. Wightman. PCT, spin and statistics, and all that. 1989.

[28] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov. General principles of quantum field theory. 1990.

[29] S. Schlieder H. Reeh. Bemerkungen zur unitäräquivalenz von lorentzinvarianten feldern. Il Nuovo Cimento (1955-1965), Società Italiana di Fisica, 22:1051–1068, 1961.

[30] H. Araki. A lattice of von neumann algebras associated with the quantum theory of a free bose field. Journal of Mathematical Physics, 4(11):1343–1362, 1963.

[31] H. Araki. Von neumann algebras of local observables for free scalar field. Journal of Mathematical Physics, 5(1):1–13, 1964.

[32] S.S. Horuzhy. Introduction to algebraic quantum field theory. 1990.

[33] Paolo Camassa. Relative Haag duality for the free field in Fock representation. Annales Henri Poincare, 8:1433–1459, 2007.

[34] R. Haag. Local Quantum physics: fields, particles and algebras. 1993.

[35] M. Takesaki. Tomita’s Theory of Modular Hilbert Algebras and its Applications. 1970.

[36] O. Bratteli and D. W. Robinson. Operator Algebras and Quantum Statistical Mechanics 1. C* and W* Algebras, Symmetry Groups, Decomposition of States. 1979.

[37] J. J Bisognano and E. H. Wichmann. On the Duality Condition for a Hermitian Scalar Field. J. Math. Phys., 16:985–1007, 1975.

[38] Paolo Bertozzini, Roberto Conti, and Wicharn Lewkeeratiyutkul. Modular Theory, Non-commutative Geometry and Quantum Gravity. SIGMA, 6:067, 2010.

[39] L. C. Evans. Partial Differential Equations. 2010.

[40] S. Scholtes. Introduction to Piecewise Differentiable Equations. 2012.