Bell inequalities have arguably been regarded as “the most profound discovery in science” [2]. They provide a fundamental distinction between local hidden-variable (LHV) descriptions of physical reality and the description based on quantum mechanics wherein the concept of nonlocal entanglement is a fundamental ingredient. Violations of Bell inequalities, which reject all LHV theories and attest for the validity of quantum mechanics, have been demonstrated in numerous experiments with massless photons [3–7], but in only a handful of experiments involving massive particles [8, 9]. In addition, all massive particle experiments have so far been restricted to exploiting entanglement between internal (spin) degrees of freedom, but never between external (motional) degrees of freedom such as translational momentum. Here, we propose and simulate a matter-wave experiment which, for the first time, can demonstrate a Bell inequality violation for pairs of momentum-entangled ultracold atoms produced in a collision of two Bose-Einstein condensates [10–13]. In such a motional-state Bell inequality test, particle masses become directly relevant, thus enabling extensions of fundamental tests of quantum mechanics into regimes which may involve couplings to gravitational fields. Such regimes are relevant to theories of gravitational decoherence [14] and can therefore shed light on theoretical constructs of quantum gravity.

The original Bell inequality was formulated by John Bell [1] in response to Einstein, Podolsky, and Rosen’s (EPR) argument [15] that, under the premises of local realism, quantum mechanics appears to be incomplete and hence must be supplemented by hidden variables in order to explain the ‘spooky-action-at-a-distance’ due to entanglement between space-like separated particles. The first conclusive experimental demonstrations of Bell inequality violations with photons were reported in the early 1980s through to 1990s [4–7] and used sources of pair-correlated photons, such as from a radiative cascade or parametric downconversion. It took almost another two decades before the first massive-particle Bell violations emerged, utilising pairs of trapped ions [8] or proton pairs from the radiative decay of metastable $^{2}$He [9]. These experiments all relied on entanglement between the internal degrees of freedom—either the photon polarizations or the particle spins, with the notable exception of the Rarity-Tapster experiment [6] which explored entanglement between photons momenta (see also [16]).

In recent years, there has been an increasing number of experiments, particularly in the field of ultracold atoms [17–18] and opto-mechanics [19], generating and quantifying various forms of massive-particle entanglement (see also [20–22]). However, these should be distinguished from experiments designed to rule out LHV theories via a Bell inequality violation—the most stringent test of quantum mechanics. Ultracold atoms, nevertheless, provide a promising platform for extending these experiments towards Bell inequality tests, due to their high degree of isolation from the environment and the existing high degree of control over system parameters, including the internal and external degrees of freedom.

Here, we propose and theoretically simulate an experiment for demonstrating a motional-state Bell inequality violation for pairs of momentum-entangled atoms produced in Bose-Einstein condensate (BEC) collisions. The proposal represents an ultimate successor to recent experiments demonstrating sub-Poissonian relative atom number statistics and violation of the classical Cauchy-Schwartz inequality [12, 13], as well as to recent theoretical proposals for demonstrating the Hong-Ou-Mandel effect [23] and the EPR paradox [24], using the same collision process. In the present case, we adopt an atom-optics analog of the Rarity-Tapster optical scheme and propose to use laser-induced Bragg pulses to implement two-particle interferometry on the underlying Bell-state for two pairs of atomic scattering modes with opposite but opposite momenta. By simulating the collision dynamics and the sequence of Bragg pulses using a stochastic Bogoliubov approach [11], we predict values of the Clauser-Horne-Shimony-Holt (CHSH) parameter up to $S \approx 2.5$ for experimentally realistic parameters, showing a strong violation of the CHSH-Bell inequality bounded classically by $S \leq 2$.

The schematic diagram of the proposed experiment is shown in Fig. 1. A highly elongated (along the $x$-axis) BEC is initially split into two counterpropagating halves with momenta $\pm k_0$ along $z$ in the centre-of-mass frame [10, 11]. Constituent atoms of the condensate undergo binary $s$-wave scattering and populate a nearly spherical scattering halo (of radius $k_\sigma \approx 0.95|k_0|$) of pair-correlated atoms [11] via the process of spontaneous four-wave mixing. Previous experiments and theory [10–13, 25] have shown the existence of strong atom-atom correlation between pairs of diametrically opposite momentum modes, such as $(p, -p)$ and $(q, -q)$ (shown in Fig. 1 on the equatorial plane of the halo), similar to the correlation between twin-photons in parametric down-conversion [5–7]. After the end of the collision, we apply two separate Bragg pulses ($\pi$ and $\pi/2$) tuned to couple uncorrelated atoms from each respective pair, namely $(p, q)$ and $(-p, -q)$. The Bragg pulses replicate the atom optics analogs of a mirror and a beam splitter [see Fig. 1(b)], thus realising the two interferometer arms of the Rarity-Tapster optical setup [6] (see also Ref. [26] which proposes the same scheme for implementing phase-sensitive measurements with ultracold atoms).
variable phase shift is additionally applied before the beamsplitter (\(\pi/2\)) pulse to the two lower arms of the interferometer, corresponding to a relative phase shift of \(\phi_L\) between \(-p\) and \(-q\), and \(\phi_R = \phi_L + \phi\) between \(q\) and \(p\). This replicates the polariser angle setting or relative phase settings in the optical Bell tests of Refs. [4] [6], and can be realised by means of introducing a relative phase \(\phi_L\) between the two counterpropagating Bragg lasers that realise the \(\pi\)-pulse, combined with an additional relative phase shift \(\phi\) between the left and the right arms of the interferometer, implemented by, e.g., the well-established technique of optical phase imprinting [27].

In the low-gain regime of atomic four-wave mixing, this process realises a prototypical Bell state of the form \(|\Psi\rangle = \frac{1}{\sqrt{2}} (|1_p, 1_q\rangle + |1_q, 1_p\rangle)|\), which corresponds to a pair of atoms in a quantum superposition of belonging to either the momentum modes \(p\) and \(-p\), or \(q\) and \(-q\). By measuring appropriate second-order correlation functions using atom-atom coincidences between certain pairs of atom detectors \(D_i\) (\(i = 1, 2, 3, 4\)), for a chosen set of applied phases \(\phi_L\) and \(\phi_R\), one can construct (see below) the CHSH-Bell parameter \(S\) for the Clauser-Horne-Shimony-Holt (CHSH) version of the Bell inequality [4] [28]. The choice of phase settings \(\phi_R\) and \(\phi_L\) gives rise to non-locality in the vein of the original EPR paradox as atom-atom coincidences are intrinsically dependent on both phase settings, analogous to choosing polarization directions in archetypal optics experiments [3] [4]. Indeed, the Rarity-Tapster interferometric scheme can be mapped to a spin-1/2 or polarization-entangled system [3], wherein choosing the phases \(\phi_L\) and \(\phi_R\) directly controls the polarization basis in which each measurement is made.

Apart from coupling two pairs of momentum modes, \((p, q)\) and \((-q, -p)\), shown on the equatorial plane of Fig. 1(a), the Bragg pulses couple many other pairs of scattering modes that have the same wave-vector difference of \(2k_r\). \(|p - q|\) = \(|(p - q)|\). Quartets of such modes, forming independent Bell states, can be identified on any other plane obtained from the equatorial plane by rotating it by an angle \(\theta\) around the \(y\)-axis. Atom-atom coincidences between these modes can therefore be used as independent measurements for evaluating the respective CHSH-Bell parameter \(S\). Averaging over many coincidence counts obtained in this way on a single scattering halo (in addition to averaging over many experimental runs) can be used to increase the signal-to-noise ratio and ultimately help the acquisition of a statistically significant result for \(S\).

To simulate the generation and detection of Bell states via the proposed scheme we use the stochastic Bogoliubov approach in the positive \(P\)-representation [11]. This approach has previously been used to accurately model a number of condensate collision experiments, including the measurement and characterisation of atom-atom correlations via sub-Poissonian relative number statistics [12] and violation of the classical Cauchy-Schwarz inequality [13]. The positive \(P\)-representation has also been used in Ref. [29] for direct probabilistic sampling of an idealised, polarisation-entangled Bell state to show how a Bell inequality violation can be simulated using the respective phase-space distribution function. Complementary to Ref. [29], we do not assume any pre-existing Bell state in our analysis, but adopt an operational approach of calculating a set of pair-correlation functions \(C_{ij}\) that define the CHSH-Bell parameter \(S\), after real-time simulations of the collision dynamics and the application of Bragg pulses. The parameter \(S\) corresponding to our measurement protocol, performed for four pairs of phase settings, is defined as [6] [28]

\[
S = |E(\phi_L, \phi_R) - E(\phi_L, \phi'_R) + E(\phi'_L, \phi_R) + E(\phi'_L, \phi'_R)|
\]

where

\[
E(\phi_L, \phi_R) = \frac{C_{14} + C_{23} - C_{12} - C_{34}}{C_{14} + C_{23} + C_{12} + C_{34}} \phi_L \phi_R
\]
The CHSH-Bell inequality states that any local hidden variable theory satisfies an upper bound given by $S \leq 2$, irrespective of the phase settings $\phi_L, \phi_R, \phi'_L$, and $\phi'_R$. The atom number correlation functions $C_{ij}$ are given by $C_{ij} = \langle N_i N_j \rangle$, where the number operator $N_i$ ($i = 1, 2, 3, 4$) corresponds to the number of atoms detected in a detection bin centered around the vectors $\{k_1, k_2, k_3, k_4\} \equiv \{p, -p, q, -q\}$.

The results of our numerical simulations of the collision dynamics and ensuing Bragg pulses are shown in Fig. 2 (a) shows the equatorial slice of the momentum distribution $n(k, \tau)$ of the scattering halo at the end of the collision; (b) and (c) show the same slice after the application of the $\pi$ and $\pi/2$ pulses, respectively. The upper and lower semicircles in (b) correspond to Bragg-kicked populations between the targeted momenta around $p$ and $q$, and between $-q$ and $-p$, while (c) shows the final distribution after mixing. The density modulation in (c) (in parts of the halo lying outside the vicinity of the targeted momentum modes, where the transfer of population during the $\pi$ pulse is not 100% efficient) is simply the result of interference between the residual and transferred atomic populations upon their recombination on the beamsplitter [25].

We next use the stochastic Bogoliubov simulations to calculate the atom-atom correlations $C_{ij}$, for the optimal choice of phase angles $\phi_L = 0$, $\phi'_L = \pi/2$, $\phi_R = \pi/4$, and $\phi'_R = 3\pi/4$ [6]. The dependence of the resulting correlation coefficient $E$ on the relative phase $\phi \equiv \phi_L - \phi_R$ is shown in Fig. 2 (d); it displays a sinusoidal dependence $E_0 \cos \phi$ which can also be predicted from a simple Gaussian-fit analytic model:

$$E(\phi_L, \phi_R) = \frac{h \prod_d \alpha_d}{h \prod_d \alpha_d + 2 \prod_d (\lambda_d)^2} \cos(\phi_L - \phi_R).$$

Here, $h$ is the height (above the background level of $n^2$) of the density-density correlation function $C(k, k', t_1)$ after the collision, $\sigma_d$ is the rms width, $\lambda_d \equiv \Delta k_d/2\sigma_d$ is the relative bin size, and $\alpha_d \equiv (e^{-2\lambda_d^2} - 1) + \sqrt{2\pi} \lambda_d \text{erf}(\sqrt{2}\lambda_d)$ (see Methods). The particular form of $E$ in Eq. (3) is obtained from this model (see Supplementary Material) by assuming the subsequent ‘mirror’ and ‘beam-splitter’ mix the coupled modes exactly. The visibility of the correlation coefficient $E$ bounds the maximum attainable violation of the CHSH-Bell inequality for a specific set of phase settings, with a lower-limit of $E_0 = 1/\sqrt{2}$ required for $S > 2$, and a maximum value of $E_0 = 1$ corresponding to $S = 2\sqrt{2}$.

The results of calculations of the CHSH-Bell parameter $S$ are shown in Fig. 3 where we explore its dependence on the strength of atom-atom correlations and the detection bin size. The dependence on the correlation strength, for a fixed collision velocity and trap frequencies, reflects essentially the dependence on the peak density of the initial BEC, which itself depends on the total number of atoms loaded in the trap [25]. The results of stochastic simulations in Fig. 3 are plotted alongside the predictions of the Gaussian-fit analytic model, which from Eq. (3) gives

$$S = \frac{h \prod_d \alpha_d}{h \prod_d \alpha_d + 2 \prod_d (\lambda_d)^2}. \quad \text{(4)}$$

As we see, the analytic prediction agrees reasonably well with the numerical results, both showing that strong Bell violations are favoured for: (i) smaller condensates, leading to lower mode population in the scattering halo and thus higher correlation strength, and (ii) smaller bin sizes, for which the strength of atom number correlations does not get diluted due to the
finite detection resolution. The discrepancies between the numerical and analytical results are due the fact that the analytic model assumes uniform halo density across the integration bin and perfect Bragg pulses, both in terms of the intended transfer efficiency and its insensitivity to the momentum offsets within the integration bin, whereas the numerical simulations are performed with realistic Bragg pulses acting on the actual inhomogeneous scattering halo. Nevertheless, an important conclusion that we reach here is that the Bell violation in our scheme can tolerate experimentally relevant imperfections that are often ignored in oversimplified models.

The general form of Eq. (4) displays similar behavior to that obtained in a simple model of four-mode parametric down-conversion in the undepleted pump approximation (see the Supplementary Material). In this model, the maximum value of $S$ is given by $S = 2\sqrt{2}/(1 + n)/(1 + 3n)$, where $n = n_k$, $i = 1, 2, 3, 4$ is the mean occupation number of the downconverted modes, which are all equal to each other. This is an insightful result from the simplest analytic treatment as it shows the scaling of $S$ with the mode population of the scattering halo: for $n \ll 1$ we see a strong violation of the CHSH-Bell inequality, tending to the maximal value of $S = 2\sqrt{2}$ as $n \to 0$, whilst we find an upper bound of $n \approx 0.26$ above which the violation is no longer observed. Comparing to Eq. (4), we note that a strong violation requires large $h$ which in general corresponds $n \ll 1$ as the relevant normal-

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