Spin squeezing via One- and two-axis twisting induced by a single off-resonance stimulated Raman scattering in a cavity

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Squeezed spin states have an important application in quantum metrology and sensing. It has been shown by Sørensen and Mølmer (2002 Phys. Rev. A 66 022314) that an effective one-axis-twisting interaction can be realized in a cavity setup via a double off-resonance stimulated Raman scattering, resulting in a noise reduction scaling $\propto 1/N^{2/3}$ with $N$ the atom number. Here, we show that, by making an appropriate change of the initial input spin state, it is possible to produce one-axis-twisting spin squeezing via a single off-resonance stimulated Raman scattering, which thus can greatly simplify the realistic implementation. We also show that the one-axis-twisting interaction can be transformed into more efficient two-axis-twisting interaction by rotating the collective spin while coupling to the cavity, yielding a Heisenberg limited noise reduction $\propto 1/N$. Considering the noise effects due to atomic decoherence and cavity decay, we find that substantial squeezing is still attainable with current laboratory technique.

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I. INTRODUCTION

Squeezed spin states (SSS) [1] play an essential role in quantum information processing [2,4,41] and precision measurement [5,7]. They have been shown to have many applications such as to detect quantum entanglement, to improve the precision in Ramsey spectroscopy, as well as to make more precise atomic clocks. Recently, various methods have been proposed [8–17] to create such states, including quantum non-demolition (QND) measurements of collective spins [18–24], nonlinear interaction between spins based on either one-axis twisting (OAT) [1,8] or two-axis twisting (TAT) [12,13]. Among them, the QND-based methods have the advantage of simple implementation, while, on the other hand, it also suffers the drawback of being difficult to produce highly spin squeezed state due to its inefficient noise-reduction scaling $\propto N^{-1/2}$ with $N$ the total number of atoms. On the contrary, the nonlinear-interaction-based methods have been proved to work much more efficiently than the QND scheme [1], as the theoretical limit of spin squeezing for OAT scales as $\propto N^{-2/3}$, and the noise reduction for TAT can even reach the Heisenberg limit $\propto N^{-1}$.

To date, much attention has been drawn to realize both OAT and TAT evolution in atomic systems. In atomic Bose-Einstein condensates, the OAT Hamiltonian arises from binary atomic collisions [25,27]. In free-space atomic samples, the interference of multiple atom-light QND interactions can induce atomic OAT and even TAT interactions [13,28,29]. The most studied systems, however, for realizing nonlinear interaction between individual spins, are atomic systems in cavities [8,12,30–36]. Among them, an impressive work by Sørensen and Mølmer proposed to realize OAT evolution via a double off-resonance stimulated Raman scattering (SRS) [8]. They showed that two classical driving fields (which have different central frequency and resonant Rabi frequency) together with one vacuum cavity mode can simultaneously flip a pair of atoms in a way that is analogous to the emission of correlated photon pair in optical parametric amplification, which creates the entanglement between individual spins and this is the essence of OAT squeezing. One of the advantages of this method is that it can produce unitary OAT spin squeezing, which, as shown in [37], can yield better clock stability than nonunitary squeezing. Another advantage is that, due to the collective enhancement effect of the atoms-light coupling [38,39], this method is in principle possible to work in any optical cavity, such as the bad cavity. Recently, this method has also been extended to the case of TAT spin squeezing by Borregaard et al. [12] via adding another two classical driving fields.

In this paper, we show that, by making an appropriate change of the initial input atomic state, it is possible to realize unitary OAT spin squeezing by using a single SRS interaction between atoms and light. Our approach inherits all the advantages of [8]. Meanwhile, in contrast to the mechanism in [8], our approach requires only a single classical driving field and can thereby significantly simplify the possible experimental realizations. By an appropriate coherent control of the collective spin by means of adding a rotation to the spin-polarized direction during the OAT interaction, we also show that the OAT can be converted into the TAT, leading to faster and stronger squeezing. Compared to the TAT method of [12] that uses four classical laser fields, our TAT protocol has the advantages of less experimental resource costs and simpler realization. Further investigation indicates that the present schemes can be made robust with respect to realistic imperfections, including the atomic decoherence and
the cavity decay. During the preparation of this work, we became aware of the proposal [10] that is similar to ours. Compared to that work whose pseudo angular moment operators connect to the ground and excited states of the atom, ours connect to the two ground-state sublevels, which therefore has the advantage of long coherence time.

The rest of the paper is structured as follows. In Section II we first review some basic concepts and then give details analysis of the processes of spin squeezing in an atomic ensemble. In Section III, we will consider the noise effects. After that, the experimental feasibility of the scheme is also discussed. Finally, Sec. IV contains brief conclusions.

II. GENERATION OF SPIN SQUEEZING

A. Ideal case

Let us first review the definition of spin squeezing. Consider an atomic system consisting of $N$ two-level atoms, which can be described by the pseudo angular momentum operators $\hat{S}_x = \sum_k |1\rangle_k \langle 1| - |2\rangle_k \langle 2|)/2$ and $\hat{S}_+ = \sum_k |1\rangle_k \langle 2|$, where the sum is over all the individual atoms, and $|1\rangle$, $|2\rangle$ are the two internal states of the atoms. The spin components in three orthogonal directions satisfy the commutation relations $[\hat{S}_x, \hat{S}_y] = iS_z$, where $\hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2$ and $\hat{S}_y = (\hat{S}_+ - \hat{S}_-)/2i$, resulting in the Heisenberg uncertainty relation $(\Delta \hat{S}_x)^2 (\Delta \hat{S}_y)^2 \geq (|\langle \Delta \hat{S}_z \rangle|^2)/4$. A commonly used measure for the degree of squeezing in an atomic ensemble is Wineland criterion [11], which is defined as

$$\xi^2 = \min_\theta \left[ \frac{N(\Delta \hat{S}_y)^2}{(\hat{S}_x)^2} \right],$$  

(1)

where $\hat{S}_y = \cos(\theta) \hat{S}_x + \sin(\theta) \hat{S}_z$ is perpendicular to $\hat{S}_x$, with $\theta \in [0, 2\pi]$. If the squeezing parameter $\xi^2 < 1$, the collective atomic state is said to be spin squeezed.

Our scheme relies on a system of $N$ three-level atoms interacting with one classical driving field (with Rabi frequency $\Omega$ and frequency $\omega$) and one quantized cavity mode $\hat{c}$ with frequency $\omega_0$ (see Fig. 1). The cavity mode is initially in a vacuum state, and the atoms each are initially prepared in the equal superposition of their ground states, forming the coherent spin state (CSS) $|CSS\rangle = 2^{N/2}(|1\rangle + |2\rangle)\otimes \hat{c}$, which is an eigenstate of the $\hat{S}_x$ operator with eigenvalue $N/2$. The classical field is detuned from $1 \rightarrow 3$ resonance [with an energy difference $\omega_{13}$ (hereafter we use the unit $\hbar = 1$)] by an amount $\Delta$, while the quantized field involves in the off-resonant atomic transition $2 \rightarrow 3$ (with an energy difference $\omega_{23} \equiv \omega_{13}$) with detuning $\Delta + \delta$, where $\delta = \omega - \omega_0$ is the two-photon detuning [see Fig. 1(a)]. For such an atoms-light system, the interaction Hamiltonian can be written as

$$\hat{H} = \omega_0 \hat{c}^\dagger \hat{c} + \sum_k \omega_{13} |3\rangle_k \langle 3| + \sum_k \left( \frac{\Omega}{2} e^{-i\omega t} |3\rangle_k \langle 1| + g \hat{c} |3\rangle_k \langle 2| + \text{h.c.} \right),$$  

(2)

where the first two terms describe the energy of the cavity field and the atoms, and the last term accounts for the atoms-light interaction, with the coupling $g = d \sqrt{\omega_0/(2\pi \epsilon_0 V_0)}$, where $d$ is the dipole moment of the $|2\rangle \rightarrow |3\rangle$ transition, $\epsilon_0$ is the vacuum permittivity, $V_0$ is the mode volume. Changing to a rotating frame with respect to $\omega_0 \hat{c}^\dagger \hat{c} + \sum_k \omega |3\rangle_k \langle 3|$, the interaction Hamiltonian of (2) is changed into

$$\hat{H} = \Delta \hat{\sigma}_{33} + \frac{\Omega}{2} \hat{\sigma}_{31} + g \hat{\epsilon} \hat{\sigma}_{32} + \text{h.c.},$$  

(3)

where we have defined the new operator $\hat{\epsilon} = \hat{c} e^{i\delta t}$ and the collective atomic operators $\hat{\sigma}_{uv} = \sum_k |u\rangle_k \langle v|$ with $u, v \in \{1, 2, 3\}$. Corresponding to this Hamiltonian, one may evaluate the Heisenberg equations for light and atoms,
yielding the following Maxwell-Bloch equations:

\[
\begin{align*}
\dot{\sigma}_{12} &= i\Omega^* \sigma_{13} - i\sigma_{23} \sigma_{13}^\dagger, \\
\dot{\sigma}_{22} &= ig\varepsilon \sigma_{32} - ig\sigma_{23} \sigma_{13}^\dagger, \\
\dot{\sigma}_{13} &= \frac{i}{2} \sigma_{32} - ig\sigma_{13} \sigma_{12}^\dagger, \\
\dot{\varepsilon} &= -ig\sigma_{23} + i\delta \varepsilon, \\
\dot{\sigma}_{13} &= -i\Delta \sigma_{13} - \frac{i}{2} (\sigma_{11} - \sigma_{33}) - ig\varepsilon\sigma_{12}, \\
\dot{\sigma}_{23} &= -i\Delta \sigma_{23} - \frac{i}{2} \sigma_{21} - ig\varepsilon(\sigma_{22} - \sigma_{33}), \\
\dot{\sigma}_{33} &= \frac{i}{2} \sigma_{13} + i\sigma_{31}^\dagger + ig\sigma_{23} \sigma_{13}^\dagger - ig\sigma_{32}.
\end{align*}
\]

Next, we assume that (i) the power of the classical driving field is sufficiently weak and (ii) the detuning is very large \(\Delta \gg 1\). With such assumptions, it is reasonable to suppose that the population of the excited state is very small, and thus one may adiabatically eliminate the excited state, leading to \(\sigma_{33} \approx 0\). Large detuning also makes it possible to quickly drive the coherences \(\sigma_{13}, \sigma_{23}\) into steady states, resulting in \(\sigma_{13} \approx -(\Omega_{12}/2 + g\varepsilon\sigma_{12})/\Delta, \sigma_{23} \approx -(\Omega_{21}/2 + g\varepsilon\sigma_{21})/\Delta\). Furthermore, we also assume that the two-photon detuning is large enough \(\delta \gg 1\) such that there is no significant photon excitation created in the cavity during interaction, which enables the adiabatical elimination of the cavity field, leading to \(\varepsilon = -g^2/\Omega^2 \Delta^2 \delta\). With these assumptions, the equations for the atomic ground states can be written as

\[
\begin{align*}
\dot{\sigma}_{12} &= -i\kappa_0 \hat{S}_z \sigma_{12} - i\chi_0 \sigma_{12}, \\
\dot{\sigma}_{11} &= \dot{\sigma}_{22} = 0,
\end{align*}
\]

where we have defined \(\kappa_0 = |\Omega|^2 |g|^2 / 4\delta \Delta^2\) and \(\chi_0 = |\Omega|^2 / 4\Delta\). In the language of pseudothermal operators, Eqs. (11) can be expressed as

\[
\begin{align*}
\dot{\hat{S}}_x &= \kappa_0 \left( \hat{S}_y \hat{S}_z + \hat{S}_z \hat{S}_y + \hat{S}_y^\dagger \hat{S}_y \right) + \chi_0 \hat{S}_y, \\
\dot{\hat{S}}_y &= -\kappa_0 \left( \hat{S}_x \hat{S}_z + \hat{S}_z \hat{S}_x + \hat{S}_x^\dagger \hat{S}_x \right) - \chi_0 \hat{S}_x, \\
\dot{\hat{S}}_z &= 0.
\end{align*}
\]

From these equations, one may infer that the atomic dynamics are produced by the effective Hamiltonian

\[
\hat{H}_{eff} = -\chi_0 \hat{S}_z - \kappa_0 (\hat{S}_x + \hat{S}_z^\dagger),
\]

which is exactly the OAT-type interaction [11]. The first term in [15] arises because of ac-Stark shifts of the ground states, while the rest terms stem from the two-photon off-resonance Raman transition. The origin of the non-linear term in [15] can be understood by considering a double Raman process as shown in Fig. 1(b). Assuming that an atom in the ground state [1] absorbs a photon from the classical driving field and emits a photon to the cavity field that is absorbed by another atom in the ground state [2], which then emits back into the classical driving field again, resulting in the effective transitions of the form \(|12\rangle \rightarrow |21\rangle\). It is this two atom process that is kept on resonance and responsible for the spin-spin entanglement (and thus the spin squeezing) generation. The dynamics of such two-atom flipping can be described by an effective Hamiltonian \(\hat{H}_{eff} = \hat{H}_S + \hat{H}_R = N/2(N/2 + 1) - \hat{S}_z^2 \propto \hat{S}_z^2\). It should be mentioned that there exists a probability that the cavity photon emitted by an atom is absorbed by the atom itself, which suppresses the two atom process. This is why we here take the equal-superposition spin state as the input state, as the number of atoms in such a state that participate in cavity-photon reabsorption is around \(N/2\), which can greatly suppress the effect of self-reabsorption and therefore makes the two atom process dominant.

Equation (12) can be readily solved to yield [1]:

\[
\hat{S}_x(t) = \hat{S}_x(0) \exp[i(\mu \hat{S}_z + 1/2 + \varphi/\mu)] + \exp[-i(\mu \hat{S}_z + 1/2 + \varphi/\mu)] \hat{S}_z(0)/2, \quad \mu = -2\kappa_0 t \quad \text{and} \quad \varphi = -\alpha t.
\]

The mean value, after writing the CSS in the basis of Dicke states \(|CSS\rangle = \sum_m = -S 2^{-S} S^m \langle S + m| S - m \rangle |m\rangle\), can be calculated as:

\[
\langle CSS|\hat{S}_z|CSS\rangle = S \cos^{2S-1} \frac{\alpha}{\mu} \sin \varphi.
\]

For \(S \gg 1\) and \(|\mu|, |\varphi| \ll 1\), one approximately has \(\langle \hat{S}_x \rangle \approx S\), which means that almost all the atoms are still polarized along the \(x\) direction after the OAT interaction. One thus can use the Holstein-Primakoff approximation [12] to define new atomic quantum variables \(X_a = \hat{S}_y/\sqrt{\hat{S}_z}, \hat{P}_a = \hat{S}_z/\sqrt{\hat{S}_z}\), which satisfy \([X_a, \hat{P}_a] = i\) and have zero mean \(\langle X_a \rangle = \langle \hat{P}_a \rangle = 0\) and a normalized variance \(\langle \Delta X_a^2 \rangle = \langle \Delta \hat{P}_a^2 \rangle = 1/2\) for the initial CSS. In this language, the solutions to Eqs. (13) and (14) can be expressed as

\[
\begin{align*}
\hat{X}_a^{\text{out}} &= \hat{X}_a^{\text{in}} + \alpha \hat{P}_a^{\text{in}} + \beta, \quad \hat{P}_a^{\text{out}} = \hat{P}_a^{\text{in}},
\end{align*}
\]

where “in” and “out” refers to the atoms before and after the interaction and we have defined the coupling constant \(\alpha = S\mu\) and the displacement parameter \(\beta = \sqrt{\hat{S}_z}\varphi\) that arises because of the linear term of the Hamiltonian [15]. Apparently, the spin state of [16] is produced by first squeezing the spin state via a Hamiltonian quadratic in \(\hat{P}_a^2\), and then displacing the SSS in the phase space along the \(\hat{P}_a\) direction by an amount \(\beta\). Since the displacement operation (linear operation) in phase space does not reduce the atom-atom entanglement created by the OAT evolution [13], the displacement \(\beta\) can then be neglected when we estimate the amount of squeezing of the atomic system. To see how much squeezing is created, we rotate the spin state around the \(x\)-axis by the unitary transformation \(\hat{X}_a^{\text{out}} = \exp(i\theta \hat{H}_{SR}) \hat{X}_a^{\text{out}} \exp(-i\theta \hat{H}_{SR}) = \cos \theta \hat{X}_a^{\text{in}} + (\alpha \cos \theta + \sin \theta) \hat{P}_a^{\text{in}}\), where \(\hat{H}_{SR} = -2\hat{S}_z \simeq \hat{X}_a^2 + \hat{P}_a^2\) is the spin-rotation Hamiltonian [14]. Optimizing the variance \(\langle \Delta X_a^{\text{out}} \rangle^2\) with respect to \(\theta\), we finally get \(\xi_{\text{OAT}} = 2\langle \Delta X_a^{\text{out}} \rangle^2 = 1 + \alpha^2/2 - (\alpha^4/4 + \alpha^2)^{1/2} \Rightarrow \lim_{\alpha \to \infty} \xi_{\text{OAT}} = 1/2\).

The amount of squeezing can be dramatically increased if one can transform the OAT into the TAT [23]. To do
so, we add a rotation about the $x$-direction during OAT interaction with an angular frequency $\Omega_0$ [43, 47] (which can be realized by applying a homogeneous magnetic field along the $x$ axis as shown in Fig. 1), resulting in the Hamiltonian $\hat{H}_{TAT} = \Omega_0 \hat{H}_{SR}/2 + \hat{H}_{eff} = \Omega_0 (\hat{X}_a^2 + \hat{P}_a^2)/2 - \kappa_0 S \hat{P}_a^2 - \sqrt{\kappa_0 S} \hat{P}_a = \kappa_0 S (\hat{X}_a^2 - \hat{P}_a^2)/2 - \sqrt{\kappa_0 S} \hat{P}_a$ for $\Omega_0 = \kappa_0 S$, which is exactly the TAT-type interaction [1] and squeezes the spin fluctuations at a rate that scales exponentially with coupling constant, that is $c^2 T_{AT} = \exp(-\alpha)$. In contrast to the OAT that creates squeezing polynomially, the exponential scaling of the TAT method will greatly enhance the entanglement between individual atoms and thus enable us to perform nontrivial control of collective spin.

### B. Noise effect

So far, we have neglected the noise effects. As in reality, the photons leak out from the cavity into the environment at a rate $\kappa$ and the excited state decays to the ground state with a radiative decay rate $\gamma_1 = \gamma_3 \equiv \gamma = \omega_0^2 2/\sqrt{8} \pi c^3$ [8]. In the presence of decays as well as spin rotation about the $x$ direction, the time evolution of the atomic operators (see Appendix A for more details) can be written as

\[
\frac{d}{dt} \begin{pmatrix} \hat{X}_a \\ \hat{P}_a \end{pmatrix} = \mathcal{G} \begin{pmatrix} \hat{X}_a \\ \hat{P}_a \end{pmatrix} - \sqrt{\kappa} \begin{pmatrix} \varphi_0 \\ \eta \end{pmatrix} + \sqrt{2\eta} \begin{pmatrix} \hat{F}_y \\ \hat{F}_z \end{pmatrix}
\]

for the case of $r_0 = \kappa/2\delta \ll 1$, with

\[
\mathcal{G} = \Omega_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 2S\kappa_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \eta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]

where $\eta = \chi_0^0 \gamma / \Delta$ is the atomic decay parameter and $\hat{F}_y, \hat{F}_z$ are Langevin operators that have zero means and satisfy $\langle \hat{F}_y (t) \hat{F}_y (t') \rangle = i\delta(t - t')/2$, and $\langle \hat{F}_z (t) \hat{F}_z (t') \rangle = \delta(t - t')/2$. The first term of $\mathcal{G}$ represents atoms turning around with $\Omega_0$ around the $x$ axis. The second term in $\mathcal{G}$ denotes the coherent OAT interaction induced by light, while the third term stands for the transverse decay of atoms caused by optical pumping. Note that $\hat{P}_a$ is now also displaced at a rate proportional to $\eta$, which is due to the ground-state population transfer from 1 to 2 induced by the strong light field. The solution to this differential equation is

\[
\begin{pmatrix} \hat{X}_a^{out} \\ \hat{P}_a^{out} \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} \hat{X}_a^{in} \\ \hat{P}_a^{in} \end{pmatrix} - \mathcal{A}(t) \int_0^t d\tau A^{-1}(\tau) \times \begin{pmatrix} \varphi_0 \\ \eta \end{pmatrix} - \sqrt{2\eta} \begin{pmatrix} \hat{F}_y(\tau) \\ \hat{F}_z(\tau) \end{pmatrix},
\]

with the homogeneous solution $\mathcal{A}(t) = \exp(\mathcal{G}t)$. For the case of OAT squeezing (that is, $\Omega_0 = 0$), we have $\mathcal{A}(t) = e^{-\eta t} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and obtain directly from (17)

\[
\begin{pmatrix} \hat{X}_a^{out} \\ \hat{P}_a^{out} \end{pmatrix} = \sqrt{1 - \eta_0} \begin{pmatrix} \hat{X}_a^{in} + \alpha \hat{P}_a^{in} + \beta' \\ + \sqrt{\eta_0} \hat{F}_X \end{pmatrix}, \tag{18}
\]

\[
\begin{pmatrix} \hat{P}_a^{out} \\ \hat{F}_p^{out} \end{pmatrix} = \sqrt{1 - \eta_0} \begin{pmatrix} \hat{P}_a^{in} + \beta'' \\ + \sqrt{\eta_0} \hat{F}_p \end{pmatrix}, \tag{19}
\]

where we have defined the parameters $\eta_0 = 2\eta$, $\beta' = \beta + \eta_0 \alpha / 4$, and $\beta'' = \sqrt{\eta_0} / 2$, and used the condition $\eta_0 \ll 1$. The modified noise operators are of the form: $\hat{F}_X = \frac{1}{\sqrt{\eta}} e^{-\eta t} \int_0^t d\tau e^{\eta \tau} [\hat{F}_y (\tau) - \alpha (\tau - t) / \sqrt{\eta} \hat{F}_X (\tau)]$, $\hat{F}_p = \frac{1}{\sqrt{\eta}} e^{-\eta t} \int_0^t d\tau e^{\eta \tau} \hat{F}_p (\tau)$, which can be easily checked to have $\langle \hat{F}_X \rangle = \langle \hat{F}_p \rangle = 0$, $\langle \hat{F}_X \hat{F}_p \rangle \simeq i(1 - \alpha / 2) / 2$, $\langle \hat{F}_X^2 \rangle \simeq (1 + \alpha^2 / 2) / 2$, and $\langle \hat{F}_p^2 \rangle \simeq 1 / 2$. With help of the atomic input-output relations [18] and [19], one may calculate the variance $(\Delta \hat{X}_\theta^{out})^2$, and thus obtain the optimized variance

\[
2(\Delta \hat{X}_\theta^{out})^2 = 1 + \frac{\alpha^2}{3} \left( 1 - \frac{2}{3} \eta_0 \right)
\]

\[
- \sqrt{\frac{1 - \frac{2}{3} \eta_0}{2}} \frac{\alpha^4}{4} + \left( 1 - \frac{1}{2} \eta_0 \right)^2 \alpha^2
\]

\[
\Rightarrow \frac{1}{\alpha^2} + \frac{\eta_0}{3}, \quad \alpha \gg 1, \tag{20}
\]

for $\theta = \arctan[(2 - \eta_0)/(\alpha - 2\eta_0 \alpha / 3)] / 2 + \pi / 2$. Eq. (20) shows that the atomic decay set a limit, that is, $\eta_0 / 3$, to the highest degree of squeezing that can be achieved. For atoms situated on the cavity antinode, the coupling $g$ can be conveniently expressed in terms of the excited-state linewidth $\eta \omega_0^2 / 3 \pi c^3$, with the cavity optical depth (OD) $d_c = 2\pi / \eta_0 / 2 = 2\pi / N_{\alpha_0} / \eta_0$, where $F$ is the cavity finesse, $d_0 = N_{\alpha_0} / \eta_0$ is the sample’s OD in free space, $\sigma_0$ is the photon-absorption cross section of an atom, $A_0$ is the effective cross sectional area of the antinode. The coupling constant $\alpha$ can then be reexpressed as $\alpha = r d_c / \eta_0 / 2$. Consequently, the amount of squeezing for large $\alpha$ may be written as

\[
2(\Delta \hat{X}_\theta^{out})^2 = 1/(r d_c / \eta_0 / 2 + \eta_0 / 3) \gg 3 \alpha^2 / (r d_c / \eta_0 / 2 + \eta_0 / 3), \quad \alpha \gg 1,
\]

which is exactly the OAT squeezing as mentioned above.

For the case of TAT squeezing (that is, $\Omega_0 = S\kappa_0$), we have $\mathcal{A}(t) = e^{-\eta t} \left( \begin{pmatrix} \cosh \frac{\alpha}{2} & -\sinh \frac{\alpha}{2} \\ -\sinh \frac{\alpha}{2} & \cosh \frac{\alpha}{2} \end{pmatrix} \right)$, and thus one may derive the input-output relations for the atomic quadrature $\hat{X}_\pi / 4$ from Eq. (17)

\[
\hat{X}_\pi^{out} = \sqrt{1 - \eta_0} \left[ e^{-\eta_0 / 4} \hat{X}_\pi^{in} + \frac{\beta - \beta''}{\sqrt{2}} \right] + \sqrt{\eta_0} \hat{F}_\pi^{out}, \tag{21}
\]

with $\hat{F}_\pi^{out} = \frac{1}{\sqrt{2\eta}} e^{-[(\eta_0 + \alpha) / 2]} \int_0^t d\tau e^{(\eta_0 + \alpha) / 2} \hat{F}_\pi (\tau)$.

Its variance can be directly calculated to yield

\[
2(\Delta \hat{X}_\pi^{out})^2 = (1 - \eta_0) e^{-\alpha} + \eta_0 \frac{1 - (1 - \eta_0) e^{-\alpha}}{\eta_0 + \alpha} \Rightarrow \frac{\eta_0}{\alpha} \propto \frac{1}{N}, \quad \alpha \gg 1. \tag{22}
\]
which indicates that the TAT scheme produces a Heisenberg-scaling squeezing. Furthermore, it is also required to take into account the effect of the $x$-component decay according to the definition given in Eq. (3), which is $(\tilde{S}_x)^2 \approx (1 - \eta_0)N/2$ as can be derived from Eq. (A2). Finally, we are able to calculate the squeezing parameter $\xi^2 \approx 2(\Delta \hat{x}_g^2)/(1 - \eta_0)$ and plot in Fig. 2(a) the amount of squeezing in their dependence on coupling strength $\alpha$ for various values of atomic decay. As can be seen from the figure that, if the atomic decay is small than 10%, a high degree of squeezing larger than 10 dB would be obtainable for the interaction parameter $\alpha = 5$. Besides, as expected, the TAT scheme works much more efficiently than the OAT scheme even in the presence of noises. While a further investigation of the performance of the protocols versus atomic decay [as shown in Fig. 2(b)] indicates that the TAT protocol is sensitive to noises, the OAT protocol, on the contrary, is much more robust to the atomic decay. Consequently, an extremely low decay rate is required to fully preserve the advantages of the TAT protocol. In a realistic implementation, it is quite convenient to use the accessible experimental parameter OD to assess the performance of the proposed protocols. In Fig. 4(c), the best achievable squeezing (optimized with respect to $\eta_0$) of the two proposed protocols versus cavity OD $d_c$ is also plotted. For room-temperature vapors whose OD in free space is around 30 [2], if one set the parameter $r_0 = 0.1$ and the finesse $F \simeq 100$, degrees of squeezing created by OAT and TAT should be as high as 13.4 dB and 19.6 dB, respectively.

III. CONCLUSION

In conclusion, we have presented a realistic scheme for generating highly spin-squeezed state of an atomic ensemble in an optical cavity. The process is based on off-resonance SRS interaction between light and spin-polarized atomic ensembles. By sending a strong pulse through a polarized atomic vapors placed in an optical cavity that is initially in a vacuum state, we find that unitary OAT squeezing can be realized. The stronger the interaction between cavity field and atoms, the higher the degree of squeezing. We also show that the OAT protocol can be transformed into the more efficient TAT protocol by just adding a homogeneous magnetic field along the spin-polarized direction. The proposed schemes are also tested by adding different noise effects, finding that (i) substantial squeezing off more than 10 dB is still obtainable even in the presence of 10% atomic decay, (ii) although the performance of the TAT protocol is, in general, superior to the OAT protocol, it is much more sensitive to the atomic decay, and (iii) the OAT protocol is quite robust against noises. We thus believe that, although the OAT protocol is not good in squeezing scaling (compared to the TAT protocol), its good characteristics of “easy surviving” in a noisy environment, as well as simpler experimental setup make it applicable to a wide range of atomic systems. We expect that the proposed protocols can be beneficial in the context of quantum information processing and quantum metrology.

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Appendix A: Details of the derivation of the equations of motion with noises

In this appendix, we analyze the performance of our proposal in the presence of spontaneous emission and cavity decay. By taking into account the noise effects, the Maxwell-Bloch equations of (5)-(10) are then changed into

\[\dot{\hat{\sigma}}_{11} = -\Omega_0 \hat{\sigma}_y + i \frac{\Omega}{2} \hat{\sigma}_{31} - i \frac{\Omega^*}{2} \hat{\sigma}_{13} + \gamma \hat{\sigma}_{33} + F_{11},\]

\[\dot{\hat{\sigma}}_{22} = \Omega_0 \hat{\sigma}_y + ig \hat{\sigma}_{32} - ig^* \hat{\sigma}_{23}^\dagger + \gamma \hat{\sigma}_{33} + F_{22},\]

\[\dot{\hat{\sigma}}_{12} = i \Omega_0 \hat{\sigma}_z + i \frac{\Omega}{2} \hat{\sigma}_{32} - ig^* \hat{\sigma}_{13}^\dagger,\]

\[\dot{\hat{\sigma}}_{13} = -\frac{\Omega_0}{2} \hat{\sigma}_{23} - (i\Delta + \gamma) \hat{\sigma}_{13} - i \frac{\Omega}{2} (\hat{\sigma}_{11} - \hat{\sigma}_{33}) - i \gamma \hat{\sigma}_{12} + F_{13},\]

\[\dot{\hat{\sigma}}_{23} = -\frac{\Omega_0}{2} \hat{\sigma}_{13} - (i\Delta + \gamma) \hat{\sigma}_{23} - i \gamma \hat{\sigma}_{22} + \hat{\sigma}_{33} - \frac{\Omega}{2} \hat{\sigma}_{21} + F_{23},\]

\[\dot{\hat{\sigma}}_{33} = i \frac{\Omega^*}{2} \hat{\sigma}_{13} - i \frac{\Omega}{2} \hat{\sigma}_{31} + ig \hat{\sigma}_{23}^\dagger - i \gamma \hat{\sigma}_{32} - 2\gamma \hat{\sigma}_{33} + F_{33}.\]  

where we have introduced the radiative decay rate of the excited state \(|3\rangle\), \(\gamma_3 = \gamma_{13} + \gamma_{23} = 2\gamma\) (we assume \(\gamma_{13} = \gamma_{23} \equiv \gamma\)), the Langevin noise operators \(F_{uv}\) for the atomic operators, the cavity decay rate \(\kappa\), and the input field \(\hat{\xi}_{in}\) for the cavity mode. The correlation functions of Langevin noise operators can be derived by using the the generalized Einstein relation \(\langle \hat{F}_{uv}(t)\hat{F}_{uv}(t') \rangle = -\langle D(\sigma_{uv})\hat{F}_{uv}(t')\hat{D}(\sigma_{uv})\rangle \delta(t-t')\), where \(D(\sigma_{uv})\) denotes the evolution for \(\sigma_{uv}\) obtained from the Heisenberg-Langevin equation but with the Langevin noise omitted. The input cavity satisfies \(\hat{\xi}_{in}(t), \hat{\xi}_{in}^\dagger(t') = \delta(t-t')\). Here we assume without loss of generality that there is no decay between 1 and 2 (as the coherence time of the ground state is normally much longer than the interaction time \(t\) in a realistic implementation). Besides, we also introduced a spin rotation to the system about the \(x\) direction, which is obtained by adding to the Hamiltonian of Eq. (5) a time-independent term \(\Omega_0 \hat{F}_x\), resulting in the terms in Eqs. (A1) that are proportional to \(\Omega_0\). Corresponding to this set of coupled equations, the evolution for the ground states can be derived along the lines outlined in the main context above to give

\[\dot{\hat{\sigma}}_{11} = -\Omega_0 \hat{\sigma}_y - \eta \hat{\sigma}_{11} + \hat{F}_{11},\]

\[\dot{\hat{\sigma}}_{22} = \Omega_0 \hat{\sigma}_y + \eta \hat{\sigma}_{11} + \hat{F}_{22},\]

\[\dot{\hat{\sigma}}_{12} = i \Omega_0 \hat{\sigma}_z - \left(\frac{\Omega^2}{4\Delta^2} + \frac{i\Omega^2|y|^2}{2\kappa/\Delta \Delta^*} \right) \hat{\sigma}_{12} + \hat{F}_{12},\]

where \(\eta = \chi_0 \gamma/\Delta\) is an optical pumping rate and we have defined \(\delta_{x/2} = \frac{\kappa}{2} - i\delta, \Delta_\gamma = \gamma + i\Delta,\) and \(\hat{F}_{11}, \hat{F}_{22}, \hat{F}_{12}\) are modified Langevin noise operators. In deriving Eqs. (A2) we have assumed the angular frequency \(\Omega_0 \ll \Delta\) and thus neglected its influence on the adiabatic-elimination procedure. From Eqs. (A2) one may directly deduce the time evolution of the collective spin operators

\[\dot{\hat{S}}_y = \Omega_0 \hat{S}_z - \frac{\kappa_0}{1 + \kappa_0} \left(\hat{S}_z \hat{S}_z + \hat{S}_x \hat{S}_x + \hat{S}_z \hat{S}_x \right) + \frac{r_0\kappa_0}{1 + r_0} \left(\hat{S}_y \hat{S}_z + \hat{S}_x \hat{S}_y + \hat{S}_y \right) - \chi_0 \hat{S}_x - \hat{S}_y + \sqrt{2S\eta}\hat{F}_y,\]

\[\dot{\hat{S}}_z = -\Omega_0 \hat{S}_y - \hat{S}_y - \hat{S}_x + \sqrt{2S\eta}\hat{F}_z,\]

where \(r_0 = \kappa/2\delta\) and we have neglected the ac-Stark shifts of the ground states induced by the cavity mode, and we also used the relation \(\hat{\sigma}_{11} \simeq \hat{S}_z + \hat{S}\) in the right-hand side of Eq. (A4). and defined the new vacuum noise operators \(\hat{F}_y = (\hat{F}_{12} - \hat{F}_{12})/2\sqrt{S\eta}, \hat{F}_z = (\hat{F}_{11} - \hat{F}_{22})/2\sqrt{S\eta}\), which, according to the Einstein relations, have the correlations \(\langle \hat{F}_y(t)\hat{F}_y(t') \rangle \simeq \delta(t-t')/2\) and \(\langle \hat{F}_z(t)\hat{F}_y(t') \rangle \simeq \delta(t-t')/2\). The above equations indicate that the noises cause a decay of the transverse spin components and a redistribution of the populations of the ground states since \(\langle \hat{S}_x \rangle \propto S\eta\), as can be seen from Eq. (A4). Note that the second line of equation (A3) arising because of cavity decay is negligible in the limit of \(r_0 \ll 1\), which is the case in the main context.

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