Embeddings in Non–Vacuum Spacetimes

Edward Anderson\textsuperscript{1} and James E. Lidsey\textsuperscript{2}

Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, U.K.

Abstract

A scheme is discussed for embedding $n$–dimensional, Riemannian manifolds in an $(n + 1)$–dimensional Einstein space. Criteria for embedding a given manifold in a spacetime that represents a solution to Einstein’s equations sourced by a massless scalar field are also discussed. The embedding procedures are illustrated with a number of examples.

PACS NUMBERS: 04.50.+h, 04.20.Jb, 98.80.Cq

\textsuperscript{1}Electronic address: eda@maths.qmw.ac.uk

\textsuperscript{2}Electronic address: J.E.Lidsey@qmw.ac.uk
1 Introduction

The possibility that our universe contains hidden, spatial dimensions has attracted considerable attention over recent years. In particular, advances in our understanding of the non-perturbative limits of superstring theory indicate that spacetime may be eleven-dimensional \[1\]. A further important development has been the realization that these extra dimensions need not have finite volume. Indeed, four-dimensional gravity can be recovered if the observable universe is represented by a co-dimension 1 brane embedded in a higher-dimensional space with a non-factorizable geometry \[2, 3, 4, 5, 6\].

Embedding theorems of differential geometry provide a natural framework for relating higher- and lower-dimensional theories of gravity and it is important to study such theorems in the light of the above developments. A particularly powerful theorem, due to Campbell, states that any \(n\)-dimensional, Riemannian manifold can be locally and isometrically embedded in an \((n+1)\)-dimensional, Riemannian space, where the Ricci tensor of the latter vanishes \[7\]. This theorem was discussed by Romero, Tavakol and Zalaletdinov \[8\] within the context of the non-compactified approach to Kaluza-Klein gravity \[9\]. (For earlier work, see Refs. \[10\]). Further embeddings into Ricci-flat spaces were also established for a wide class of superstring backgrounds \[11\].

The purpose of the present paper is to develop extensions of Campbell’s scheme, where the Ricci tensor of the higher-dimensional spacetime is non-trivial. Specifically, we consider the case where the embedding manifold is an Einstein space with a covariantly constant energy-momentum tensor. Such embeddings are directly relevant to the second Randall-Sundrum braneworld scenario, where the bulk corresponds to pure Einstein gravity sourced only by a negative cosmological constant \[6\]. The embedding of the four-dimensional, isotropic and homogeneous radiation universe into a Schwarzschild-Anti de Sitter space was recently investigated \[12\]. Einstein spaces with a positive cosmological constant have also become the focus of attention \[13, 14\] and cosmological solutions with such a term represent one of the simplest manifestations of the inflationary scenario \[15\].

We also consider embeddings into spacetimes sourced by a massless, minimally coupled scalar field. Such a field represents one of the simplest forms of matter and can be identified, after suitable field redefinitions, with the dilaton field that arises in the string effective action \[16\]. A massless field also parametrizes the volume of an internal, Ricci-flat space in conventional Kaluza-Klein compactification. (For a review, see, e.g., \[17\]).

The paper is organized as follows. We develop the embedding schemes in Section 2 and proceed in Section 3 to illustrate these techniques by establishing embeddings for general classes of Einstein and plane wave spacetimes. We conclude with a discussion of further applications in Section 4.
2 Embeddings in Higher Dimensions

2.1 Einstein Spaces

We consider the local and isometric embedding of the $n$-dimensional, Riemannian manifold, $(M, g_{\alpha\beta})$, with line element

$$ds^2 = g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta$$

in the $(n+1)$-dimensional manifold, $(\hat{M}, \hat{h}_{AB})$, defined by the metric

$$d\hat{s}^2 = \hat{h}_{\alpha\beta}dx^\alpha dx^\beta + \epsilon\phi^2dy^2,$$  

(2.2)

where $\hat{h}_{\alpha\beta} = h_{\alpha\beta}(x^\mu, y)$ and $\phi = \phi(x^\mu, y)$ are analytic functions of the $(n+1)$ variables $\{x^\mu, y\}$. The constant $\epsilon = \pm 1$ and we therefore allow for the possibility that the extra dimension is either spacelike or timelike. Although there are well known problems with introducing an additional timelike dimension, it has been argued that the duality symmetries of string/M theory compactified on Lorentzian tori result in extra time dimensions in the strong coupling limit \[18\]. Braneworld scenarios where the transverse dimension is timelike have also been proposed \[19\].

When evaluated on an arbitrary hypersurface, $dy = 0$, the components of the Ricci tensor calculated from the metric (2.2) take the general form

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\nabla_{\alpha}\phi}{\phi} + \frac{1}{2\epsilon\phi^2} \left( \frac{\phi}{\phi} h_{\alpha\beta} - h_{\alpha\beta} - \frac{1}{2} \hat{h}^{\gamma\delta} h_{\gamma\delta} h_{\alpha\beta} + h^{\gamma\delta} h_{\gamma\delta} h_{\alpha\beta} \right)$$

(2.3)

$$\hat{R}_{\alpha y} = \frac{\phi}{2} \nabla_{\beta} P_{\alpha}^\beta$$

(2.4)

$$\hat{R}_{yy} = -\epsilon\phi \nabla^2 \phi - \frac{1}{2} \hat{h}^{\gamma\delta} h_{\gamma\delta}^* - \frac{1}{2} \left( h^{\gamma\delta} \right)^* h_{\gamma\delta}^* + \frac{1}{2} \hat{h}^{\gamma\delta} h_{\gamma\delta}^* \frac{\phi^*}{\phi} - \frac{1}{4} \hat{h}^{\gamma\delta} h_{\gamma\delta}^* h_{\alpha\beta}^* h_{\alpha\beta},$$

(2.5)

where the $n$-dimensional Ricci tensor, $R_{\alpha\beta}$, and covariant derivative operator, $\nabla_{\alpha\beta} = \nabla_{\beta} \nabla_{\alpha}$, are calculated from $h_{\alpha\beta}$, a star denotes a partial derivative with respect to $y$, $\partial/\partial y|_{y=\text{constant}}$, evaluated on the hypersurface $y = \text{constant}$, $\nabla^2 \equiv h^{\alpha\beta} \nabla_{\alpha\beta}$ and the quantity, $P_{\alpha}^\beta$, is defined by \[20\]

$$P_{\alpha}^\beta \equiv \frac{1}{\phi} \left( h^{\gamma\delta} h_{\gamma\delta}^* - \delta_{\alpha}^{\delta} h^{\gamma\delta} h_{\gamma\delta}^* \right)$$

(2.6)

$^1$Greek and Latin indices run from $(0, 1, \ldots, n-1)$ and $(0, 1, \ldots, n)$, respectively, and the coordinate of the $(n+1)$th dimension is denoted by $y$. All curvature tensors relevant to the $(n+1)$-dimensional metric, $\hat{h}_{AB}$, are represented with a circumflex accent and those constructed from the hypersurface metric, $h_{\alpha\beta}$, have no accent. We employ Wald’s conventions with signature $(-, +, +, \ldots)$ for the $n$-dimensional spacetime, $g_{\alpha\beta}$ \[21\]. In all cases, the embeddings considered in this paper are local and isometric and do not refer to any aspects of the global topology of the spaces.
If we now define the functions $\Omega_{\alpha\beta}(x^\mu, y)$ \cite{7, 8}: 

$$\frac{\partial h_{\alpha\beta}}{\partial y} \equiv -2\phi \Omega_{\alpha\beta},$$  

(2.7)

it follows that Eqs. (2.3)–(2.5) simplify to

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \nabla_\alpha \phi \nabla_\beta \phi + \epsilon \left[ \frac{1}{\phi} \delta_\alpha^\gamma h_{\gamma\delta} \Omega_{\delta\beta} - h_{\alpha\beta} \right],$$

(2.8)

$$\hat{R}_{\alpha y} = \phi \nabla_\beta \left[ \delta_\alpha^\gamma h_{\gamma\delta} \Omega_{\delta\alpha} - h_{\gamma\alpha} \right],$$

(2.9)

$$\hat{R}_{yy} = -\phi h_{\gamma\beta} \left[ \epsilon \nabla_{\gamma\beta} \phi - \Omega_{\gamma\beta} - \phi h_{\alpha\beta} \Omega_{\alpha\gamma} \Omega_{\beta\delta} \right].$$

(2.10)

In this subsection, we show that the metric (2.1) can be embedded in an Einstein space of the form (2.2), where the constraint equations

$$\hat{R}_{AB} = 2\Lambda - \frac{n}{1 - n} \hat{h}_{AB},$$

(2.11)

are satisfied and $\Lambda$ is a spacetime constant. That such an embedding is possible was stated by Campbell \cite{7}, but the proof was not given. The proof proceeds iteratively by first assuming that the equations (2.11) are valid on a specific hypersurface $y = y_0$, where $y_0$ is arbitrary, and then verifying that they are also valid for any $y$ in the neighbourhood of this hypersurface.

To proceed, we substitute Eq. (2.11) into Eqs. (2.8)–(2.10):

$$\Omega_{\alpha\beta}^* = h_{\lambda\mu} (\Omega_{\alpha\beta} \Omega_{\lambda\mu} - 2\Omega_{\alpha\lambda} \Omega_{\beta\mu}) + \epsilon \left[ \frac{1}{\phi} \delta_\alpha^\gamma h_{\gamma\delta} \Omega_{\delta\beta} - h_{\alpha\beta} \right]$$

(2.12)

$$h_{\mu\nu} (\nabla_\mu \Omega_{\alpha\nu} - \nabla_\alpha \Omega_{\mu\nu}) = 0$$

(2.13)

$$h_{\lambda\beta} (\epsilon \nabla_{\lambda\beta} \phi - \Omega_{\lambda\beta}^* - \phi h_{\alpha\beta} \Omega_{\alpha\gamma} \Omega_{\beta\delta}) - \frac{2\epsilon \Lambda \phi}{n - 1} = 0.$$  

(2.14)

Subtracting the trace of Eq. (2.12) from Eq. (2.14) then results in the contracted ‘Gauss’ equation:

$$\Omega^2 - \Omega_{\alpha\beta} \Omega_{\alpha\beta} = \epsilon (R + 2\Lambda),$$

(2.15)

where $\Omega \equiv h_{\alpha\beta} \Omega_{\alpha\beta}$ and the covariant constancy of the metric in Eq. (2.13) yields the ‘Codazzi’ equation:

$$\nabla_\nu \Omega_{\alpha\nu} = \nabla_\alpha \Omega.$$  

(2.16)

A crucial property of the higher–dimensional metric (2.2) is that it must simplify to the embedded metric (2.1) when on the hypersurface, $y = y_0$:

$$h_{\alpha\beta}(x^\mu, y_0) = g_{\alpha\beta}(x^\mu).$$  

(2.17)

We then assume that the symmetric functions

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$$

(2.18)
can be found that satisfy the constraints (2.15) and (2.16) on this ‘initial’ hypersurface. Moreover, it is also assumed that these functions evolve according to Eq. (2.7) and the set of differential equations\footnote{The metric $h_{\alpha\beta}(x^\mu, y)$ is employed in Eq. (2.19) to raise and lower indices and in calculating the curvature tensors and covariant derivatives.}

$$\frac{\partial \Omega^\gamma_\beta}{\partial y} = \epsilon \nabla^\gamma_\beta \phi + \phi \left( \Omega \Omega^\gamma_\beta - \epsilon R^\gamma_\beta - \frac{2\epsilon \Lambda}{n-1} \delta^\gamma_\beta \right),$$

(2.19)

where the boundary conditions

$$h^*_{\alpha\beta} = -2\phi(x^\mu, y_0)\Omega_{\alpha\beta}(x^\mu, y_0)$$

(2.20)

are satisfied.

As shown in the appendix, if conditions (2.7), (2.15)–(2.20) are satisfied, it follows that

$$\left( \nabla^\nu \Omega_{\alpha\nu} - \nabla_\alpha \Omega \right)^* = 0$$

(2.21)

$$\left( \Omega^2 - \Omega^\gamma_\alpha \Omega^\gamma_\beta - \epsilon (R + 2\Lambda) \right)^* = 0$$

(2.22)

and Eqs. (2.21) and (2.22) then imply that Eqs. (2.15), (2.16) and (2.18) are valid for all hypersurfaces, $dy = 0$, in the \textit{neighbourhood} of $y = y_0$. Given the validity of Eq. (2.19), therefore, we may further deduce that the Einstein conditions (2.11) are satisfied for all $y$ in this neighbourhood. Consequently, the $(\alpha\beta)$–components of the higher–dimensional metric $\hat{h}_{AB}$ can be expanded as a Taylor series in $y$ to first–order:

$$\hat{h}_{\alpha\beta} = g_{\alpha\beta} - 2\phi(x^\mu, y_0)\Omega_{\alpha\beta}(x^\mu, y_0)y,$$

(2.23)

where Eqs. (2.7) and (2.17) have been employed. Likewise, the value of $\Omega_{\alpha\beta}$ in this vicinity can be determined from Eq. (2.19). Since the analysis is valid for an arbitrary hypersurface, this local extension can be repeated recursively and this establishes the embedding of the metric (2.1) in the Einstein space (2.2). The embedding is not unique since more than one choice of $\Omega_{\alpha\beta}$ may be possible for a given embedded metric [11]. When $n = 3$ and $\epsilon = -1$, the determination of $\Omega_{\alpha\beta}$ from the point of view of the boundary value problem follows by identifying $y$ with the timelike coordinate, $h_{\alpha\beta}$ with the spatial three–metric and $\Omega_{\alpha\beta}$ with the extrinsic curvature [21]. The conditions that these functions must satisfy are the five relations (2.15)–(2.16) and (2.18)–(2.20). We may conclude, therefore, that any $n$–dimensional, Riemannian manifold may be locally and isometrically embedded in an $(n + 1)$–dimensional Einstein space when Eqs. (2.15)–(2.20) are satisfied.

### 2.2 Massless Scalar Fields

It is also of interest to consider whether Campbell’s technique can be extended to include embeddings of the metric (2.1) in non–vacuum spacetimes, $(\hat{M}, \hat{h}_{AB})$. One
possible source of matter is a massless, minimally coupled scalar field, \( \chi \), that satisfies the Einstein field equations

\[
\hat{R}_{AB} = \frac{1}{2} \hat{\nabla} A \hat{\nabla} B \chi \tag{2.24}
\]

\[
\hat{h}^{AB} \hat{\nabla}_{AB} \chi = 0 \tag{2.25}
\]

For the metric ansatz (2.2) the components of the \((n+1)\)-dimensional Ricci tensor (2.24) reduce to

\[
R_{\alpha\beta} - \frac{\nabla_{\alpha\beta} \phi}{\phi} + \epsilon \left[ \frac{1}{\phi^2} \Omega^{*} - \Omega_{\alpha\beta} \right] = \frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \chi \tag{2.26}
\]

\[
\phi \nabla_{\beta} \left[ \delta_{\alpha}^{\gamma} h_{\gamma\delta} \Omega_{\alpha\beta} - h_{\alpha\delta} \Omega_{\gamma} \right] = \frac{1}{2} \chi \nabla_{\alpha} \chi \tag{2.27}
\]

\[
\phi h^{\gamma\beta} \left[ \epsilon \nabla_{\gamma\beta} \phi - \Omega^{*}_{\gamma\beta} - \phi h^{\alpha\delta} \Omega_{\alpha\gamma} \Omega_{\beta\delta} \right] = -\frac{1}{2} (\chi^*)^2 \tag{2.28}
\]

on the hypersurface, \( y = y_0 \), where the symmetric functions, \( \Omega_{\alpha\beta} = \Omega_{\beta\alpha} \), are defined, as before, in Eq. (2.7) and the boundary conditions (2.17) and (2.20) are also assumed to be valid. The left–hand side of the scalar field equation (2.25) takes the form

\[
h^{\alpha\beta} \nabla_{\alpha\beta} \chi + \frac{1}{\phi} h^{\alpha\beta} \nabla_{\alpha} \phi \nabla_{\beta} \chi + \epsilon \left[ \frac{1}{\phi^2} \chi^{**} - \frac{1}{\phi^2} \phi^* \chi^* - \Omega \chi^* \right] = 0 \tag{2.29}
\]

on this hypersurface and taking the trace of Eq. (2.26) and subtracting Eq. (2.28) implies that

\[
\Omega_{\alpha\beta} \Omega^{\alpha\beta} - \Omega^2 = \epsilon \left[ \frac{1}{2} \nabla_{\alpha} \nabla^{\alpha} \chi - R - \frac{\epsilon}{2\phi^2} (\chi^*)^2 \right]. \tag{2.30}
\]

We now consider the scalar function, \( \chi \), and the symmetric functions, \( \Omega_{\alpha\beta} \), that satisfy the conditions (2.27), (2.29) and (2.30) on the hypersurface, \( y = y_0 \), and evolve according to

\[
\frac{\partial \Omega^{\gamma}_{\beta}}{\partial y} = \epsilon \nabla_{\gamma} \phi - \epsilon \phi \left[ R_{\beta}^{\gamma} - \frac{1}{2} \nabla_{\gamma} \chi \nabla_{\beta} \chi \right] + \Omega_{\gamma} \Omega^{\gamma}_{\beta}. \tag{2.31}
\]

It can then be shown, following an argument similar to that presented in the Appendix, that

\[
\left( \nabla_{\lambda} \Omega_{\mu} \right)^* = \Omega_{\kappa}^{\mu} \left[ \Omega_{\eta}^{\kappa} \nabla_{\mu} \phi + \phi \nabla_{\mu} \Omega_{\eta}^{\kappa} \right] - \Omega_{\mu}^{\eta} \left[ \Omega_{\eta} \nabla_{\mu} \phi + \phi \nabla_{\mu} \Omega \right]
\]

\[
+ \epsilon \nabla_{\mu} \nabla^2 \phi + \Omega \nabla_{\mu} \nabla_{\lambda} \phi + \frac{\epsilon}{2} \nabla_{\lambda} \phi \nabla_{\mu} \chi \nabla^{\lambda} \chi + \frac{\epsilon}{2} \nabla_{\mu} \nabla^{\lambda} \chi \nabla_{\lambda} \chi + \epsilon \frac{\chi}{2} \nabla_{\mu} \nabla^{\lambda} \chi \nabla_{\lambda} \chi \tag{2.32}
\]

where we have employed Eq. (2.31), (A.1) and (A.3)–(A.4). Evaluating the derivative with respect to \( x^{\mu} \) of the trace of Eq. (2.31) and combining the result with Eq. (2.32)
then implies that
\begin{equation}
\left( \nabla_\lambda \Omega_\mu^\lambda - \nabla_\mu \Omega + \frac{1}{2\phi} \chi^* \nabla_\mu \chi \right)^* = \frac{\epsilon}{2} \phi \nabla_\mu \chi \left( \hat{h}^{AB} \hat{\nabla}_{AB} \chi \right) \bigg|_{y=y_0},
\end{equation}
where we have substituted for the scalar field equation (2.29), employed Eqs. (2.27) and Eqs. (2.30) and the right–hand side is evaluated at $y = y_0$. It follows from Eq. (2.29), therefore, that the right–hand side of Eq. (2.33) vanishes. Consequently, Eq. (2.27) is also valid for all hypersurfaces in the neighbourhood of $y = y_0$.

The validity of Eq. (2.30) in this neighbourhood is established by determining the derivative of the Ricci scalar, $R(h)$, with respect to the variable, $y$. We find that
\begin{equation}
R^* = 2\phi \Omega_\mu^\mu R_\mu - 2\Omega \nabla^2 \phi - 2\Omega_\tau \nabla^\tau \phi
+ \chi^* \nabla^2 \chi + \nabla^\mu \nabla_\mu (\chi^*) + \frac{1}{\phi} \chi^* \nabla_\mu \chi \nabla^\mu \phi.
\end{equation}
Moreover, it follows from Eq. (2.31) and its trace that
\begin{equation}
\left( \Omega_\mu^\mu \Omega_\mu^\mu - \Omega^2 \right)^* = 2\epsilon \Omega_\alpha^\mu \nabla_\beta \phi - 2\epsilon \Omega \nabla^2 \phi - 2\epsilon \phi \Omega_\alpha \Omega_\beta
+ \epsilon \phi \Omega_\alpha \nabla_\beta \chi \nabla^\alpha \chi - \frac{\Omega^2}{\phi} (\chi^*)^2
\end{equation}
and combining Eqs. (2.34) and (2.35) implies that
\begin{equation}
\left( \Omega_{\alpha\beta} \Omega_{\alpha\beta} - \Omega^2 + \epsilon R - \frac{\epsilon}{2} \nabla^\alpha \chi \nabla_\alpha \chi + \frac{1}{2\phi^2} (\chi^*)^2 \right)^* = \epsilon \chi^* \hat{h}^{AB} \hat{\nabla}_{AB} \chi \bigg|_{y=y_0}.
\end{equation}

The right–hand side of Eq. (2.36) also vanishes when Eq. (2.29) is satisfied and this implies that Eq. (2.30) is valid in the neighbourhood near $y = y_0$. We may conclude, therefore, that the field equations (2.24) are valid for all $y$ and Eq. (2.25) then follows from the contracted Bianchi identity.

To summarize, for the functions $\Omega_{\alpha\beta} = \Omega_{\alpha\beta}(x^\mu, y)$, $\phi = \phi(x^\mu, y)$ and $\chi = \chi(x^\mu, y)$ that satisfy Eqs. (2.27), (2.29) and (2.30) on the hypersurface $y = y_0$, and evolve according to Eqs. (2.7), (2.17), (2.20) and (2.31), the metric (2.1) can be embedded in the manifold (2.2), where the latter is a solution to the Einstein field equations (2.24) and (2.25) for a massless, minimally coupled scalar field.

This concludes our discussion of the embedding schemes. In the following Section, we employ the procedures to embed Einstein and plane wave spacetimes in non–vacuum, higher–dimensional manifolds.

### 3 Applications of the Embedding Schemes
3.1 Embedding Einstein Spaces in Einstein Spaces

We first consider the embedding of an \( n \)-dimensional Einstein space

\[
R_{\alpha\beta}(g) = \frac{2\lambda}{2-n}g_{\alpha\beta}
\]  

(3.1)

in the \((n + 1)\)-dimensional Einstein space (2.11) for arbitrary constants \( \{\lambda, \Lambda\} \). The embedding is achieved by invoking the ansatz

\[
\Omega_{\alpha\beta} \equiv Ch_{\alpha\beta}, \quad \phi = 1,
\]

(3.2)

where \( C = C(x^{\mu}, y) \) is a scalar function. Eq. (2.16) immediately implies that the prefactor, \( C \), may be a function only of \( y \). On the other hand, Eq. (2.15) implies that

\[
C^2 = \frac{\epsilon(R + 2\Lambda)}{n(n-1)}
\]

(3.3)

and, since \( \phi = 1 \), it follows that Eq. (2.7) may be formally integrated to yield

\[
h_{\alpha\beta} = a^2(y)g_{\alpha\beta},
\]

(3.4)

where the ‘warp factor’, \( a \), is defined by \( a \equiv \exp \left[ - \int y dy' C(y') \right] \). If the constant of integration is chosen such that \( a(y_0) = 1 \), the \( n \)-dimensional metric \( g_{\alpha\beta} \) may be interpreted as the embedded Einstein space satisfying Eq. (3.1). Indeed, only Eqs. (2.19) and (3.3) remain to be solved for the embedding to be determined and these equations reduce to

\[
\frac{1}{a} \frac{d^2a}{dy^2} = \frac{2\epsilon\Lambda}{n(n-1)}
\]

(3.5)

\[
\left( \frac{da}{dy} \right)^2 = \frac{2\epsilon\Lambda}{n(n-1)}a^2 - \frac{2\epsilon\lambda}{(n-1)(n-2)},
\]

(3.6)

respectively. The general solution satisfying the boundary condition (2.20) is then given by

\[
a = \cosh \left( \sqrt{\frac{2\epsilon\Lambda}{n(n-1)}} (y - y_0) \right) + B \sinh \left( \sqrt{\frac{2\epsilon\Lambda}{n(n-1)}} (y - y_0) \right),
\]

(3.7)

where

\[
B^2 = 1 - \frac{n\lambda}{(n-2)\Lambda}
\]

(3.8)

and the embedding of the Einstein space (3.1) is therefore given by

\[
ds^2 = a^2(y)g_{\alpha\beta}dx^\alpha dx^\beta + dy^2,
\]

(3.9)
where Eqs. (3.7) and (3.8) are satisfied.

Eq. (3.9) generalizes the embedding of maximally symmetric, four-dimensional Einstein spaces in five dimensions [22] as well as the embedding found in Ref. [23] for \( \lambda < 0 \). When the embedded manifold is Ricci–flat (\( \lambda = 0 \)), the warp factor (3.7) is exponential and Eq. (3.9) reduces to the metric considered in Ref. [24].

One interesting consequence of the embedding (3.9) is that it provides the bulk solution for non–fine–tuned versions of the Randall–Sundrum–type braneworld scenarios, where the co–dimension 1 branes have a non–vanishing cosmological constant [22, 25, 26, 27]. Since the embedded metric is arbitrary in our analysis, it may be viewed as a non–linear generalization of the graviton zero mode on the brane. Within this context, a specific example is given by the Siklos class of solutions to Eq. (3.1) representing gravitational waves propagating in anti–de Sitter spacetime [28, 29].

3.2 Plane Waves

We now consider the embedding of the plane wave backgrounds [30]

\[
    ds^2 = -dudv + du^2 + f_{ij}dx^idx^j
\]

(3.10)
in a manifold sourced by a massless scalar field, \( \chi \), following the approach outlined in Section 2.2 for \( \epsilon = 1 \). The arbitrary function \( f_{ij} = f_{ij}(u) \) is symmetric and depends only on the light–cone coordinate, \( u \). The metric (3.10) admits a covariantly constant, null Killing vector field, \( \partial/\partial v \), that is orthogonal to the Riemann curvature tensor. Consequently, all curvature invariants vanish and this implies that metrics of this form can represent perturbatively exact solutions to the string equations of motion when the dilaton and antisymmetric form fields satisfy appropriate conditions [31, 32]. The only non–trivial component of the Ricci tensor is \( R_{uu} \) and is also a function only of \( u \).

To proceed with the embedding, we assume that the scalar field is independent of the coordinate, \( y \), and further invoke the ansatz

\[
    \Omega_{\alpha\beta} = \begin{cases} 
    y/y_0^2 & \text{if } \alpha = \beta = u \\
    0 & \text{otherwise}
    \end{cases}
\]

(3.11)
together with the condition

\[
    \phi = -1. \quad (3.12)
\]

On the hypersurface \( y = y_0 \), where indices are raised with \( g^{\alpha\beta} \), the only non–trivial components of \( \Omega_{\alpha\beta} \) and \( \Omega^\alpha_{\beta} \) are \( \Omega^{uu} = 2\Omega^u_u = 4\Omega_{uu} \). Thus, Eq. (2.27) is solved since the embedded metric (3.10) and \( \Omega_{\alpha\beta} \) are both independent of \( v \). Eq. (2.30) is also satisfied if the scalar field is a function only of \( u \), \( \chi = \chi(u) \), and this latter condition also ensures that Eq. (2.29) holds when \( y = y_0 \). We may then solve the set of equations (2.7) to deduce that

\[
    h_{\alpha\beta} = \begin{cases} 
    (y/y_0)^2 & \text{if } \alpha = \beta = u \\
    g_{\alpha\beta} & \text{otherwise}
    \end{cases}
\]

(3.13)
This implies that in the neighbourhood of the hypersurface, only $\Omega^u$ and $\Omega^{uv}$ are non-trivial and, consequently, Eqs. (2.27) and (2.30) are solved for arbitrary $y$. Moreover, Eq. (2.25) is trivially satisfied, since the scalar field is null. Thus, only Eq. (2.31) is yet to be solved and this set of conditions reduces to the single constraint:

$$\left(\frac{d\chi}{du}\right)^2 = 2 \left[ R_{uu} - \frac{1}{y_0^2} \right], \quad (3.14)$$

where $R_{uu}$ is the $(uu)$-component of the Ricci tensor calculated from the embedded metric (3.10). We may conclude, therefore, that the $(n+1)$-dimensional embedding metric is given by

$$ds^2 = -dudv + \left(\frac{y}{y_0}\right)^2 du^2 + f_{ij}dx^idx^j + dy^2, \quad (3.15)$$

where the scalar field is determined by the quadrature

$$\chi = \sqrt{2} \int^u du' \left[ R_{uu}(u') - \frac{1}{y_0^2} \right]^{1/2}. \quad (3.16)$$

An interesting example of this embedding arises for the four-dimensional backgrounds defined by $f_{ij} = f^2(u)\delta_{ij}$, where $\delta_{ij}$ is the two-dimensional Kronecker delta and $f = f(u)$ is an arbitrary function that parametrizes the amplitude of the plane wave. The Ricci tensor for such a metric is given by $R_{uu} = -2f^{-1}(d^2f/du^2)$ and Eq. (3.14) therefore has the form of a one-dimensional Helmholtz equation:

$$\frac{d^2}{du^2} + V(u) f = 0, \quad (3.17)$$

where the effective potential, $V(u)$, is determined by the kinetic energy of the scalar field:

$$V \equiv \frac{1}{2} \left[ \frac{1}{2} \left( \frac{d\chi}{du} \right)^2 + \frac{1}{y_0^2} \right]. \quad (3.18)$$

It follows that if a particular solution, $f_1(u)$, to Eq. (3.14) can be found for a given choice of $\chi(u)$, the general solution can be expressed directly in terms of this solution such that

$$f_{\text{gen}} = [\kappa + \int^u \frac{du'}{f_1^2(u')} \big] f_1(u), \quad (3.19)$$

where $\kappa$ is an arbitrary constant. In general, this implies that there is not a one-to-one correspondence between the amplitude of the embedded metric and the functional form of the scalar field that generates the Ricci curvature of the embedding metric.

Finally, a second metric of interest is the Nappi–Witten WZW model

$$ds^2 = -dudv + du^2 + dx^2 + 2 \cos u dx dy + dy^2 \quad (3.20)$$

that corresponds to a conformal field theory describing a homogeneous, monochromatic plane wave [33]. The Ricci tensor of this background is $R_{uu} = 1/2$, implying that the scalar field takes the particularly simple form $\chi = \left[ 1 - (2/y_0^2) \right]^{1/2}u$. 





9
3.3 Embeddings of Ricci–flat Spaces

We conclude this Section by establishing a class of embeddings where the scalar field is independent of the coordinates of the embedded metric (2.1), i.e., $\chi$ is a function only of $y$. We employ the ansatz (3.2) for the functions $\{\Omega_{\alpha\beta}, \phi\}$. In this case, Eq. (2.27) implies that $\nabla_{\alpha}C = 0$ and Eq. (2.30) implies that

$$n(n - 1)C^2 = \epsilon R + \frac{1}{2} \left( \frac{d\chi}{dy} \right)^2.$$  \hspace{1cm} (3.21)

The scalar field equation (2.29) simplifies to

$$\frac{d^2\chi}{dy^2} = nC \frac{d\chi}{dy}$$ \hspace{1cm} (3.22)

and the evolution equation (2.31) takes the form

$$R^\gamma_{\beta} = - \epsilon \left( \frac{dC}{dy} - nC^2 \right) \delta^\gamma_{\beta}.$$ \hspace{1cm} (3.23)

Eq. (3.22) admits the first integral

$$\left( \frac{d\chi}{dy} \right)^2 = \frac{m^2}{a^{2n}},$$ \hspace{1cm} (3.24)

where $m$ is an arbitrary constant and we have defined the function $C \equiv -d\ln a/dy$. Taking the trace of (3.23) and combining Eqs. (3.21) and (3.24) then implies that

$$\frac{d^2a}{dy^2} = - \frac{m^2}{2n} a^{1-2n}$$ \hspace{1cm} (3.25)

and Eq. (3.25) may be integrated to yield the solution

$$a = \left[ 1 + b(y - y_0) \right]^{1/n},$$ \hspace{1cm} (3.26)

where $b^2 \equiv nm^2/[2(n - 1)]$. By substituting Eq. (3.26) into Eq. (3.23), we then deduce that the Ricci tensor of the embedded metric must vanish. Finally, the form of the embedded metric follows by integrating Eq. (2.7):

$$d\hat{s}^2 = \left[ 1 + b(y - y_0) \right]^{2/n} g_{\mu\nu} dx^\mu dx^\nu + \epsilon dy^2$$ \hspace{1cm} (3.27)

and this satisfies the boundary conditions (2.17) and (2.20).

Thus, Eq. (3.27) represents the embedding of an arbitrary, $n$–dimensional, Ricci–flat spacetime into a manifold sourced by a massless scalar field, where the latter is given by $\chi = (m/b) \ln[1 + b(y - y_0)]$. In the special case where the embedded metric is four–dimensional, flat Minkowski space, Eq. (3.27) represents the bulk metric for a class of braneworld models, where the cosmological constant on the brane is arbitrary but has no influence on the brane dynamics [34]. These models are interesting because they provide a new perspective on solving the cosmological constant problem. The embedding considered in this subsection indicates that this problem may also discussed within a wider context [35].
4 Discussion

In this paper, we have developed a procedure, introduced by Campbell, to embed a given Riemannian manifold into an Einstein space with a non–trivial cosmological constant. Such an embedding has a number of applications.

Firstly, the scheme is iterative and does not depend on the dimensionality of the embedded space. Thus, if the embedding of a particular $n$–dimensional space, $M$, in an $(n + 1)$–dimensional Einstein space, $M_{\text{Ein}}$, can be determined, an embedding of the space $M$ into an $(n + 2)$–dimensional Einstein space follows immediately by embedding $M_{\text{Ein}}$ along the lines outlined in Section 3.1.

This provides a method for generating and classifying exact solutions to higher–dimensional theories of gravity. For example, the infra–red limit of M–theory is eleven–dimensional supergravity, with a bosonic sector consisting of the graviton and a three–form antisymmetric potential \[1\]. Recently, it was shown that the field equations for this theory can be written in such a way that only ten–dimensional Poincare invariance is manifest \[10\]. This is equivalent to performing a generalized Scherk–Schwarz dimensional reduction to ten dimensions, where the fields are allowed to depend specifically on the compactifying coordinate \[37\]. The resulting ten–dimensional theory represents a ‘massive’ extension of type IIA supergravity. It was further shown that if the gauge fields are then frozen out, the ten–dimensional equations of motion reduce to the single equation \[36\]

\[
\hat{R}_{AB} = m^2 \hat{h}_{AB},
\]

(4.28)

where $m^2$ represents a cosmological constant. Thus, the embeddings that we have discussed in this paper may be employed to generate solutions to the massive type IIA supergravity and eleven–dimensional supergravity theories.

Embeddings in Einstein spaces are also relevant to Wesson’s ‘spacetime–matter’ (STM) theory, where the matter on any $(3 + 1)$–dimensional hypersurface is encoded at a classical level purely in terms of five-dimensional vacuum geometries \[1, 38\]. As discussed in Ref. \[8\], this interpretation is closely linked to that of Campbell’s theorem \[7\]. Thus, embeddings in Einstein spaces would be related to a generalisation of the STM theory, although such a generalization could only be achieved at the price of introducing a curvature length scale. It would be of interest to investigate the relationship between four–dimensional matter and the geometry of five–dimensional Einstein spaces further. Moreover, such a generalization would enable direct comparisons to be made between the STM theory and braneworld models. In particular, both approaches attempt to attach physical significance to the fifth coordinate \[38, 39\] and these attempts should share some common obstacles and insights.

In establishing the embedding of Einstein spaces we invoked the ansatz \[32\]. This restriction could be relaxed by allowing $\Omega_{\alpha\beta}$ to have more degrees of freedom. One possibility is to specify $\Omega_{\alpha\beta} = Q^\gamma_\alpha h_{\gamma\beta}$, where $Q^\gamma_\gamma$ has the block–diagonal form

\[
Q^\gamma_\gamma = \text{diag} \left[ C(x^A), \ldots, C(x^A), D(x^A), \ldots, D(x^A) \right]
\]

(4.29)
for some scalar functions \( \{C, D\} \). It would be natural to consider such an ansatz when embedding an Einstein space that itself is the product of two or more lower-dimensional Einstein spaces. An alternative approach—relevant to spatially homogeneous cosmologies—is to first embed the \((n-1)\)-dimensional spacelike hypersurface in a space with an extra spatial dimension and to then view the embedding to \((n+1)\) dimensions as an initial value problem \[21\].

The embedding of manifolds in Einstein gravity with a massless scalar field can also provide the seed for generating new, higher-dimensional solutions to the string equations of motion. In the case where the embedded metric admits an Abelian isometry associated with a Killing vector, \( \partial/\partial z \), a conformal transformation to the string frame, followed by a T–duality transformation, may be performed. This symmetry transformation inverts the string–frame metric coefficient associated with \( z \) and results in a new dilaton field. Solutions with non-trivial form fields may also be found by employing further duality transformations \[17, 32\].

Finally, we remark that since the cosmological constant and scalar field considered in Section 2 were uncoupled, it follows that embeddings in manifolds sourced by both degrees of freedom can in principle be found by extending the above analyses. In particular, we may deduce immediately that if a solution, \( \{g_{\alpha\beta}, \chi\} \), to the \( n \)-dimensional field equations

\[
R_{\alpha\beta}(g) = \frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \chi + \frac{2\lambda}{2-n} g_{\alpha\beta}
\]

\[
g^{\alpha\beta} \nabla_{\alpha} \chi = 0
\]

is known, the metric

\[
ds^2 = a^2(y) g_{\alpha\beta} dx^\alpha dx^\beta + dy^2
\]

represents a solution of the equations of motion derived from the \((n+1)\)-dimensional action

\[
S = \int d^{n+1}x \sqrt{-h} \left[ \hat{R} - \frac{1}{2} \left( \hat{\nabla} \chi \right)^2 - 2\Lambda \right],
\]

where the warp factor, \( a = a(y) \), is given by Eqs. (3.7) and (3.8), and the functional form of the scalar field is unaltered. Scalar field spacetimes satisfying Eqs. (4.30) and (4.31) were recently studied within the context of the AdS/CFT correspondence \[10\]. This embedding generalizes the embedding for \( \lambda = 0 \) found recently by Feinstein, Kunze and Vazquez-Mozo \[33\].

**Acknowledgements** EA is supported by the Particle Physics and Astronomy Research Council (PPARC) and JEL by the Royal Society. We thank J. Barbour and R. Tavakol for discussions.
References

[1] Witten E 1995 *Nucl. Phys.* B **443** 85

[2] Akama K 2000 [hep-th/0001113](https://arxiv.org/abs/hep-th/0001113)
   Rubakov V A and Shaposhnikov M E 1985 *Phys. Lett.* B **159** 22
   Arkani–Hamed N, Dimopoulos S and Dvali G 1998 *Phys. Lett.* B **429** 263
   Gogberashvili M 2000 *Europhys. Lett.* **49** 396

[3] Hořava P and Witten E 1996 *Nucl. Phys.* B **460** 506
   Hořava P and Witten E 1996 *Nucl. Phys.* B **475** 94

[4] Lukas A, Ovrut B A, Stelle K S and Waldram D 1999 *Phys. Rev. D* **59** 086001
   Lukas A, Ovrut B A and Waldram D 1999 *Phys. Rev. D* **60** 086001

[5] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370

[6] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690

[7] Campbell J E 1926 *A Course of Differential Geometry* (Oxford: Clarendon Press)

[8] Romero C, Tavakol R and Zalaletdinov R 1996 *Gen. Rel. Grav.* **28** 365

[9] Overduin J M and Wesson P S 1997 *Phys. Rep.* **283** 303

[10] Magaard L 1963 *Zur einbettung riemannscher raume in Einstein–raume und konforeuclidische raume* (Kiel: PhD thesis)
    Goenner H 1980 In *General Relativity and Gravitation One Hundred Years after the Birth of Albert Einstein* Vol I p. 444 (New York: Plenum Press)

[11] Lidsey J E, Romero C, Tavakol R and Rippl S 1997 *Class. Quantum Grav.* **14** 865
    Lidsey J E 1997 *Phys. Lett.* B **417** 33

[12] Savonije I and Verlinde E 2001 *Phys. Lett.* B **507** 305

[13] Hawking S W, Maldacena J and Strominger A 2001 *JHEP* **0105** 001
    Banks T 2000 [hep-th/0007146](https://arxiv.org/abs/hep-th/0007146)
    Maldacena J and Nunez C 2001 *Int. J. Mod. Phys.* A **16** 822

[14] Chamblin A and Lambert N D 2001 [hep-th/0102159](https://arxiv.org/abs/hep-th/0102159)

[15] Lyth D H and Liddle A R 2000 *Cosmological Inflation and Large–Scale Structure* (Cambridge: Cambridge University Press)

[16] Polchinski J 1998 *String Theory* Vols. I and II (Cambridge: Cambridge University Press)
[17] Lidsey J E, Wands D and Copeland E W 2000 Phys. Rep. 337 343

[18] Bars I, Deliduman C and Andreev O 1998 Phys. Rev. D 58 066004
Hull C M 1998 JHEP 9811 017
Hull C M and Khuri R R 1998 Nucl. Phys. B 536 219
Pope C N, Sadrzadeh A and Scuro S R 2000 Class. Quantum Grav. 17 623

[19] Gogberashvili M and Midodashvili P 2000 hep-ph/0005298
Gogberashvili M 2000 Phys. Lett. B 484 124
Dvali G, Gabadadze G and Senjanovic G 1999 hep-ph/9910207
Chaichian M and Kobakhidze A B 2000 Phys. Lett. B 488 117
Berezhiani Z, Chaichian M, Kobakhidze A B and Yu Z H 2001 hep-th/0102207

[20] Wesson P S and Ponce de Leon J 1992 J. Math. Phys. 33 3883

[21] Wald R M 1984 General Relativity (Chicago: Chicago University Press)

[22] Kaloper N 1999 Phys. Rev. D 60 123506
Nihei T 1999 Phys. Lett. B 465 81

[23] Garriga J and Sasaki M 2000 Phys. Rev. D 62 043523

[24] Cvetiˇc M, Lü H and Pope C N hep-th/0009188

[25] Karch A and Randall L 2001 Int. J. Mod. Phys. A 16 780

[26] Kim H B and Kim H D 2000 Phys. Rev. D 61 064003
Kogan I I, Mouslopoulos S and Papazoglou A 2001 Phys. Lett. B 501 140

[27] Alonso–Alberca N, Meessen P and Ortin T 2000 Phys. Lett. B 482 400
Chamseddine A H and Sabra W A 2001 hep-th/0105207
Chamseddine A H and Sabra W A 2001 hep-th/0106092

[28] Siklos S T C 1985 Galaxies, Axially Symmetric Systems and Relativity ed M A H MacCallum (Cambridge: Cambridge University Press)

[29] Kaigorodov V R 1963 Sov. Phys. Doklady 7 893
Podolsky J 1998 Class. Quantum Grav. 15 719
Chamblin A and Gibbons G W 2000 Phys. Rev. Lett. 84 1090
Banados M, Chamblin A and Gibbons G W 2000 Phys. Rev. D 61 081901

[30] Brinkmann H 1923 Proc. Natl. Acad. Sci. USA 9 1

[31] Horowitz G T and Steif A R 1990 Phys. Rev. Lett. 64 260

[32] Tseytlin A A 1995 Class. Quantum Grav. 12 2896

[33] Nappi C R and Witten E 1993 Phys. Rev. Lett. 71 3751
A Appendix

In this appendix we derive the conditions (2.21) and (2.22). In doing so, we employ the expressions

$$\nabla_\mu \nabla^2 \phi = \nabla_\lambda \left( \nabla^\lambda \phi \right) - R^\lambda_\mu \nabla_\lambda \phi$$  \hspace{1cm} (A.1)

$$(\nabla_\beta - \nabla_\alpha) T^\gamma_{\delta} = -R^\gamma_{\alpha\beta\delta} T^\varepsilon_{\delta} + R^\gamma_{\alpha\beta\delta} T^\varepsilon_{\varepsilon}$$  \hspace{1cm} (A.2)

for a scalar field and a tensor field $T^\gamma_{\delta}$ derived from the Ricci lemma [21]. We also require expressions for the $y$–derivative of the Christoffel matrices [7]:

$$\Omega^\eta_{\rho\kappa} \left( \Gamma^\eta_{\rho\kappa} \right)^* = - (\phi \nabla_\rho \Omega^\eta_{\kappa} + \Omega^\eta_{\kappa} \nabla_\rho \phi) \Omega^\xi_{\eta}$$  \hspace{1cm} (A.3)

$$\left( \Gamma^\eta_{\rho\kappa} \right)^* = - \nabla_\eta (\phi \Omega).$$  \hspace{1cm} (A.4)

We first consider Eq. (2.21). Differentiating Eq. (2.19) with respect to $x^\gamma$, and employing Eqs. (A.1), (A.3) and (A.4) implies that

$$(\nabla_\kappa \Omega^\kappa_{\mu})^* = \Omega^\kappa_{\kappa}(\phi \Omega^\kappa_{\eta\mu} + \Omega^\eta_{\kappa} \nabla_\mu \phi) - \Omega^\kappa_{\mu}(\Omega \nabla_\kappa \phi + \phi \nabla_\kappa \Omega) + \nabla_\mu (\nabla^2 \phi)$$

$$+ \Omega^\kappa_{\mu} \nabla_\kappa \phi - \frac{2\Lambda}{n-1} \nabla_\mu \phi + \phi (\Omega^\kappa_{\mu} \nabla_\kappa \Omega + \Omega^\kappa_{\kappa} \Omega^\kappa_{\mu} - \epsilon \nabla_\lambda R^\lambda_{\mu}).$$  \hspace{1cm} (A.5)

The trace of Eq. (2.19), on the other hand, is given by

$$\Omega^* = \epsilon \nabla^2 \phi - \epsilon \phi \left( R + \frac{2n\Lambda}{n-1} \right) + \phi \Omega^2$$  \hspace{1cm} (A.6)
and combining Eq. (A.3) with the covariant derivative of Eq. (A.6) with respect to $x^\mu$ then implies that
\[
\left( \nabla_\lambda \Omega^\lambda_\mu - \nabla_\mu \Omega \right)^* = \nabla_\mu \phi \left( \epsilon (R + 2\Lambda) + \Omega^\mu_\kappa \Omega^\kappa_\eta - \Omega^2 \right) \\
+ \phi \left( \Omega^\mu_\eta \nabla_\mu \Omega^\kappa + \Omega \nabla_\lambda \Omega^\lambda_\mu - 2\Omega \nabla_\mu \Omega + \epsilon \nabla_\mu R - \epsilon \nabla_\lambda R^\lambda_\mu \right).
\] (A.7)

The first bracketed term on the right hand side of Eq. (A.7) vanishes due to Eq. (2.15). The second bracketed term vanishes due to the covariant derivative of Eq. (2.15) with respect to $x^\mu$ and the contracted Bianchi identity
\[
\nabla_\lambda R^\lambda_\mu = \frac{1}{2} \nabla_\mu R.
\] (A.8)

Hence, Eq. (2.21) is valid and Eq. (2.16) propagates.

In establishing the validity of Eq. (2.22), it is necessary to calculate $R^*$. Since $R$ is a scalar, its derivative can be evaluated in normal coordinates \[.\] Employing Eqs. (2.16), (A.2) and (A.4) implies that
\[
R^* = 2\Omega \nabla^2 \phi + 2R_{\alpha\beta} \Omega^\alpha_\phi - 2\Omega^\beta_\nu \nabla^\beta_\nu \phi.
\] (A.9)

However, it follows from Eq. (2.19) and (A.6) that
\[
(\Omega^\mu_\phi \Omega^\lambda_\mu - \Omega^2)^* = 2\Omega \phi \left( \Omega^\mu_\phi \Omega^\lambda_\mu - \Omega^2 + \epsilon (R + 2\Lambda) \right) \\
- 2\epsilon \left( \phi \Omega^\mu_\phi R^\lambda_\mu + \Omega \nabla^2 \phi - \Omega^\mu_\phi \nabla^\lambda_\mu \phi \right).
\] (A.10)

Thus, substitution of Eqs. (2.15) and (A.9) into Eq. (A.10) implies that Eq. (2.22) is valid.