On lightcone string field theory from Super Yang-Mills and holography

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Abstract

We investigate the issues of holography and string interactions in the duality between SYM and the pp wave background. We argue that the Penrose diagram of the maximally supersymmetric pp-wave has a one dimensional boundary. This fact suggests that the holographic dual of the pp-wave can be described by a quantum mechanical system. We believe this quantum mechanical system should be formulated as a matrix model. From the SYM point of view this matrix model is built out of the lowest lying KK modes of the SYM theory on an $S^3$ compactification, and it relates to a wave which has been compactified along one of the null directions. String interactions are defined by finite time amplitudes on this matrix model. For closed strings they arise as in AdS-CFT, by free SYM diagrams. For open strings, they arise from the diagonalization of the hamiltonian to first order in perturbation theory. Estimates of the leading behaviour of amplitudes in SYM and string theory agree, although they are performed in very different regimes. Corrections are organized in powers of $1/(\mu\alpha'p^+)^2$ and $g^2(\mu\alpha'p^+)^4$. 
1 Introduction

In [1], it was noticed that in a particular limit of the AdS-CFT correspondence, one can extend the duality between \( \mathcal{N} = 4 \) SYM in the \( 't \) Hooft limit [2] and string theory beyond the supergravity approximation. By taking a Penrose limit of \( AdS_5 \times S_5 \) near a null geodesic that sits in the center of AdS in global coordinates and rotates on an equator of the sphere, one obtains the maximally supersymmetric pp wave of [3, 4]:

\[
ds^2 = 2dx^+ dx^- - \mu^2 x^2 (dx^+)^2 + dx^2 \tag{1}\]

Worldsheet string theory in the pp wave was solved explicitly for the spectrum of oscillators [3, 4] and compared with a SYM planar diagram calculation finding agreement. The worldsheet time direction \( \tau \) is identified with \( x^+ \) via a lightcone gauge, and the spatial coordinate \( \sigma \) is discretized. This gives us a Hamiltonian for a discrete set of oscillators, and in the continuum limit it gives rise to a quantum system on a string. In SYM, an euclidean theory with operators defined on \( R^4 \) is related to states on \( S^3 \times R_t \), via the usual operator-state correspondence of conformal field theory. The Penrose limit on the SYM side corresponds to taking only operators with a very large number of fields, \( J \sim g_{YM} \sqrt{N} \) in the large ‘t Hooft coupling limit \( g^2 N \rightarrow \infty \). The number of fields in these operators is approximately equal with the number of lattice points on the discretized string. The Penrose limit produces the pp-wave geometry with a compact \( x^- \sim x^- + 2\pi R^2 \) (\( R \) is the radius of AdS), in such a way that the large \( N \) limit corresponds to the \( R \rightarrow \infty \) limit. However we have to be careful and remember that as far as the SYM is concerned, we are doing calculations that relate to a compactified pp-wave. This will be a very important point in the development of the present paper. In the limit it is even possible to keep \( g \) finite and small and send \( N \rightarrow \infty \), because it gives rise to a sensible perturbation expansion due to supersymmetric cancellations between diagrams. Therefore at the microscopic level one expects to have interacting strings with closed string coupling \( g^2 \). In this limit heuristic arguments [1] show that the SYM theory is dimensionally reduced on the \( S^3 \), and it is expected that the higher KK modes become infinitely massive and decouple. If this is true the SYM theory is also reduced to a quantum mechanical problem with a discrete set of oscillators that gives rise to some type of matrix model. The definition of this theory involves a scaling limit \( N \rightarrow \infty \), which is combined with a limit on the states that one needs to consider. A priori this is not a well defined theory, and we need to check that this limit can be taken systematically in the perturbation expansion we consider. Both the string states and the hamiltonian were shown to agree in between SYM and the free string on the pp wave. The string bosonic oscillators correspond to insertions of \( D_i Z \) and \( \phi^i \), with discrete phases \( e^{2\pi i n/\sqrt{N}} \) into the vacuum \( Tr(Z^J) = |0, p^+ > (i, i' = 1, ..., 4, Z, \bar{Z} \) and \( \phi^i \) are the scalars of SYM), and the field \( \bar{Z} \) dissappears from the spectrum. These calculations were performed using only planar diagrams.

In [3], the analogous treatment was done for the \( \mathcal{N} = 2 \) SYM with fundamental matter of [3, 6]. It is an \( AdS_5 \times S^5/Z_2 \) orientifold, with an O(7) plane and 4 D7-branes. It corresponds in the Penrose limit to the orientifold of the pp wave. The orientifold has both closed and open strings, and correspondingly in SYM there are operators which carry Chan-Paton representations of \( SO(8) \). The Neumann bosonic oscillators correspond to insertions of the scalars \( Z^i, \bar{Z}^i, D_i Z \) and the Dirichlet to insertions of \( W, \bar{W} \). Here \( Z, \bar{Z}^i \) are antisymmetric scalars which together form a hypermultiplet, and \( W \) is the scalar in the gauge multiplet. Open strings on pp waves arising from other branes were analyzed in [3, 7, 8] and other treatments of orbifolds appeared in [10, 11, 13] and other treatments of string interactions from the SYM theory point of view. We will begin to study splitting and joining of strings, and not just the free string spectrum. In the matrix model the number of strings is roughly the number of traces. Interactions that change the number of traces involve non-planar diagrams. Here we will begin a study of these non-planar diagrams by studying the combinatorics of these calculations in terms of powers of \( J, N \).

We find that they produce corrections that organize into powers of \( b = J^4/N^2 \), which is finite in the pp-wave limit. This serves as a new expansion parameter that controls the splitting and joining of strings. We are interested in the weak \( b \) limit, where non-planar effects are suppressed and string theory is perturbative. There is a second finite number, \( a = g^2 N/J^2 \), which controls the planar diagrams. When \( a \) is small we can trust the planar perturbation theory. For \( a \) large we need to resum the planar perturbation theory.

We also want to clarify the issue of holography in the pp wave background, since there is some confusion on the literature [21, 22, 23]. With regard to the issue of holography, we will find that the conformal boundary
of the pp wave is a one dimensional null line, parametrized by \(x^+\). In the AdS-CFT correspondence, we are focusing near a null geodesic parametrized by \(x^+\), which when projected on the boundary of \(AdS_5\) is just the time \(t\) in \(S^3 \times \mathbb{R}_t\). Therefore the holographic dual of the pp-wave is defined with the time \(t\), which is also identified with \(\tau\) on the string worldsheet. Considering that the \(S^3\) does not appear anymore on the boundary of the pp-wave, the boundary of the pp-wave is consistent with the integration of massive KK modes on \(S^3\) and the appearance of a matrix model where we only consider a truncation to the lowest lying modes on the \(S^3\). We will use the Penrose diagram as evidence in support of the existence of this matrix model.

The nature of the model suggest that observables are obtained by considering finite time transitions in the quantum mechanics on \(t\), between multistring states, of the type \(<n|e^{iHt}|m>\), with \(|n>,|m>\) multistring states, and \(H\) being given by the lightcone Hamiltonian. We could of course compute transitions depending on times \(t_1, t_2, t_3, \ldots\) by inserting operators at these times. We will argue that SYM transition amplitudes are found to have the same qualitative behavior as string amplitudes, in that they are consistent with supergravity estimates of these interactions. However, in principle, one should not compare directly these two results, since they are not valid in the same region. Instead, there should be a resummation procedure to go from the SYM calculation to the supergravity limit. In this sense our computations are limited and we can not show that they agree exactly with the flat space limit. We can expect this type of qualitative agreement based on the non-renormalization theorems for three point functions of BPS operators. This is because the states entering in the pp-wave limit are near BPS, and the deviations from the BPS results might be under control.

Since we can not yet resum the series we want to analyze, we will instead look at the \(\mathcal{N} = 2\) SYM for completeness. We want to treat a system with both open and closed strings and show that their interactions are consistent. This is, we want to show that open strings join and split with a coupling that is the square root of the closed string splitting and joining amplitude. Clearly, everything we say can be applied to the \(\mathcal{N} = 4\) system by considering just closed strings.

For closed string amplitudes, we find the usual statement of holography from AdS-CFT [24, 25, 26, 27, 28, 29, 30, 31, 33], etc., which is rather puzzling- a calculation in free SYM (which doesn’t know anything about the SYM interaction hamiltonian) finds the correct leading behaviour of the string amplitude. This comes from the overlap of a single string state with multistring states (the basis of the free hamiltonian is misidentified, and single string states are not orthogonal to multistring states).

However, for open string amplitudes, we find a more interesting behaviour: the leading behaviour of the string amplitudes is given still by free SYM (no dependence of \(g_{YM}\)), but it comes from diagonalizing the interaction hamiltonian, so it is very clear in this case that it contains information about the dynamics of the theory.

One puzzle which might be raised a priori is that the SYM transition amplitudes are of order \(1/N\) (for closed strings, and \(1/\sqrt{N}\) for open strings), so they seem to become zero in the large \(N\) limit. However, we find that this is an artifact of using the normalization appropriate for compact spaces for \(x^-\), instead of the one for the noncompact pp wave. It is consistent with the Penrose limit, where at finite \(J\) (and so \(N\)), \(x^-\) is still compact. In the noncompact normalization, the amplitudes are proportional to \(g_s\).

We analyze all amplitudes in the SYM theory with a small number of string states, namely, 3-closed string, 3-open string, open-closed string, 4-open string and 2-open-1-closed string vertices. The aim is to see that interactions are consistent with a ten dimensional picture and with basic aspects of string theory that relate the above amplitudes to the string coupling.

In order to get an understanding of what issues are involved for a complete calculation of interactions from SYM we also address the structure of corrections to the above results. We want to see that the perturbation expansion in the SYM is organized in a way which we can interpret in ten dimensions, and that the limit \(N \to \infty\) related to the pp-wave makes sense diagramatically. We are working in the \(1/(\mu \alpha' p^+)\) expansion, i.e. large RR background, which corresponds in SYM to \(g_{YM}^2N/J^2 \ll 1\). We analyze the nonplanar diagrams of \(\mathcal{N} = 4\) SYM and show on a case by case analysis that they organize into powers of \(a = g_{YM}^2N/J^2 = 1/(\mu \alpha' p^+)\) and \(b = J^4/N^2 = g^2(\mu \alpha' p^+)\), at least to first order in \(a, b, \) and \(ab^2\). This involves the study of the behavior of non-planar diagrams in the SYM, and we show that there are also cancelation of amplitudes on the non-planar diagrams taking place.
The flat space case (which is the opposite limit $\mu \alpha' p^+ \ll 1$) is hard to analyze, since one must do a resumation of diagrams. However, this is the limit where results are already available for splitting and joining amplitudes directly from string theory. One would want to compare the two amplitudes, but one needs to resum the planar diagrams before being able to do that.

The paper is organized as follows. In section 2 we review holography in $\text{AdS}_5 \times S^5$, in order to see how to proceed for the pp wave. Then in section 3 we analyze holography for the pp wave. We find the Penrose diagram of the pp wave in section 3.1, after which we derive the pp wave geometry after Tseytlin in section 3.2. We review the spectrum of closed and open strings in the $\mathcal{N} = 2$ orientifold in section 3.3. In section 3.4 we describe 4 regimes in SYM and identify the one we will be analyzing. The relation between SYM and pp wave observables follows in 3.5, and then the behaviour of SYM and supergravity observables in the Penrose limit in 3.6. In section 4 we analyze the supergravity estimates for the vertices of string field theory. In section 5 we estimate the leading behaviour of the 3-open string and open-closed amplitudes. In section 6 we treat the 3-closed string amplitude and the amplitudes related to it in string field theory. Some comments on string field theory and contact terms are given in section 7. In section 8 we treat systematically the size of corrections in SYM. It can be skipped at a first reading. We conclude in section 9. In an appendix we discuss in more detail open string 3 point function in AdS and why they disappear in the Penrose limit, and compare with closed string correlators.

## 2 Remarks on holography on $\text{AdS}_5 \times S^5$

In this section we will describe known aspects of the AdS/CFT correspondence. Part of the discussion is motivated by trying to analyze holography for the pp-wave geometry, and secondly, we need to establish the right framework to think about the calculations we will be doing.

### 2.1 Penrose diagram for $\text{AdS}_5 \times S^5$

If we chose to represent $\text{AdS}_5 \times S^5$ by the Poincare patch, then we have the metric given by

$$
ds^2 = R^2\left(\frac{1}{y^2}(dy^2 + dx_T^2) + d\Omega_5^2\right)
$$

which is valid for $y > 0$. For this metric a lightlike ray can reach $y = 0$ in finite time and return to the interior. Therefore $y = 0$ is a boundary for the spacetime. This can be written as

$$
 ds^2 = \frac{R^2}{y^2}(dy^2 + dx_T^2 + y^2d\Omega_5^2)
$$

so if we rescale the metric by $y^2$ we get a conformal embedding of the theory as Minkowski space if we combine the directions $u$ and $\Omega_5$ to obtain a flat $\mathbb{R}^6$. However, this spacetime has a boundary at $y = 0$, together with the null boundary $y = \infty$ (Killing horizon). The section $y = 0$ is flat $\mathbb{R}^{3,1}$, and the 5-sphere is of zero size. The boundary of our space is four-dimensional Minkowski space. If we think holographically, the information of $\text{AdS}_5 \times S^5$ is encoded in a four dimensional conformal field theory on the boundary, which gives us the AdS/CFT correspondence.

However, a conformal field theory does not have an S-matrix where we can identify a finite number of particles in the initial and final state. We have to take care of the infrared divergences in some manner.

Now let us consider the global coordinate system of $\text{AdS}_5 \times S^5$. Here the metric is written as

$$
 ds^2 = R^2[-dt^2(\cosh^2(\rho)) + d\rho^2 + \sinh^2(\rho)\,d\Omega_3^2 + d\Omega_2^2]
$$

which is defined for $\rho \geq 0$, and $\rho$ is a radial coordinate transverse to time. in this coordinate system $\rho = 0$ is a timelike geodesic, and the boundary of our spacetime is located at $\rho = \infty$. Notice that the warp factor for the time variable makes it so that a null ray can reach $\rho = \infty$ and back in finite time, because the integral

$$
 I = \int_0^\infty d\rho \frac{1}{\cosh \rho} < \infty
$$

4
We can now choose to rescale the metric by a factor of \( \exp(-2\rho) \) and evaluate how the metric looks at \( \rho \to \infty \). Again the term of the \( S^5 \) will have a zero coefficient, so the \( S^5 \) degenerates to a point. For the other coordinates we get the metric of \( S^3 \times R \). So the holographic dual is described by a four dimensional quantum field theory compactified on \( S^3 \). The Poincare patch described above corresponds to taking a finite interval in \( R \), of length \( I \).

In fact, one can make a coordinate transformation conformally mapping (4) to the Einstein static universe. If we write \( \tan \theta = \sinh \rho \) with \( \theta \in [0, \pi/2) \), then (4) becomes

\[
ds^2 = \frac{R^2}{\cos^2 \theta} (-dt^2 + d\theta^2 + \sin^2 \theta d\Omega_3^2 + \cos^2 \theta d\Omega_5^2)
\]

and the boundary is at \( \theta = \pi/2 \). So the Penrose diagram for \( AdS_5 \times S_5 \) is half of the Einstein universe (which has \( \theta \in (0, \pi) \)), with an extra \( S^5 \) fiber, which shrinks to zero at the boundary.

### 2.2 Definition of observables for conformal field theories

Given a conformal field theory, there are no S-matrix observables on flat space due to infrared divergences. One needs to define the theory by imposing some sort of infrared cutoff. We can put the system in a finite box, and then the spectrum of the Hamiltonian in the theory is an observable, as well as correlation functions of insertions gauge invariant operators at finite time for some initial state of the theory. In this case it is particularly useful to write the theory compactified on an \( S^3 \) because in the AdS/CFT correspondence this choice corresponds to global coordinates on \( AdS_5 \times S^5 \).

A second possibility is to define the theory by analytic continuation to Euclidean space. The observables are the Euclidean correlation functions of the theory with insertions of local operators at various positions. Here the fact that there are finite distances between the operators imposes the infrared cutoff for theory. Since \( R^4 \) is the analytic continuation of \( R^{3,1} \), the correlation functions in this case are related to the Poincare patch of AdS, and one can use the Euclidean gravity action to define the AdS/CFT correspondence. In particular, we are interested in classifying the (super)primary operators of the theory. The correlation functions of these operators determine the full structure of the theory.

If we consider a single operator inserted at zero, \( \mathcal{O}(0) \), we can consider using radial quantization, where we use time along the radial direction. Notice that we have a punctured \( R^4, R^4/0 \). The metric of \( R^4/0 \) can be written in spherical coordinates as \( dr^2 + r^2 d\Omega_2^2 \). We can do a local conformal transformation by multiplying by \( r^{-2} \) which shows that this metric is conformally equivalent to the metric of \( S^3 \times R \). This is the analytic continuation of the boundary of \( AdS_5 \) in global coordinates. Thus, by the usual operator-state correspondence of CFT, local operators in the boundary of the Poincare patch (\( R^4 \)) are related to states in the global AdS coordinates (on \( S^3 \times R \)). Changes in the time variable for radial quantization correspond to rescalings of the punctured \( R^4 \), essentially, we have that the Euclidean time coordinate corresponds to the RG flow of the theory.

In this way we can identify that the spectrum of the Hamiltonian of the theory compactified on \( S^3 \) is the same as the spectrum of local operators of the Euclidean field theory as classified by their conformal dimension.

The observables correspond to correlations of operators inserted at various positions. Once we do the local Weyl rescaling these become questions that have to be addressed at finite (radial) time, exactly as is required for particles in a finite box.

### 3 Holography for the pp-wave

Let us now consider the pp-wave geometry and let us try to analyze how to understand holography for the pp-wave. We are interested in establishing some dictionary as was done for the AdS/CFT for this geometry. First we will analyze the Penrose diagram for the pp-wave and we will argue that it should be holographically described by some version of a matrix model. We reach this conclusion from the fact that the pp-wave has a one-dimensional boundary. Secondly we will describe how to obtain the pp-wave geometry from \( AdS_5 \times S^5 \). We will argue that the main effect of the limit is to make \( x^- \) compact (with a very large radius), and
that momentum along $x^-$ is quantized. The main point is that in order to relate an CFT calculation to a supergravity (or superstring) calculation on the pp-wave we need to take into account normalization factors that relate field theory on finite intervals with field theory in an uncompactified space.

### 3.1 Penrose diagram of the maximally supersymmetric pp-wave

A Penrose diagram is characterized by the fact that the coordinates have finite extent (infinity in the old coordinates is at a finite value in the new coordinates) and the metric is embeddable in flat space, up to a conformal factor. This requirement uniquely fixes the coordinate frame. Sometimes the requirement is even too strong, and one can’t make all coordinates be finite in extent (as in the case of the universal cover of AdS space, where as we saw in the last section the time coordinate $t$ is still infinite).

For Minkowski signature metrics, one often maps the space to a patch of the Einstein static universe

$$ds^2 = -dt^2 + d\Omega^2_{d-1}$$

which can be represented as a cylinder in $d+1$ dimensional flat spacetime. This happens for instance for flat Minkowski space, Anti-de Sitter and de Sitter spaces.

The same will happen in our case, since we can conformally map a portion of the maximally supersymmetric pp wave into flat Minkowski space.

Indeed, the metric

$$ds^2 = 2dx^+ dx^- - \vec{x}^2(dx^+)^2 + d\vec{x}^2$$

becomes

$$ds^2 = \frac{1}{1 + u^2}(2dx'^- du + (dx')^2 + x'^2d\Omega^2)$$

under the change of variables

$$u = \tan x^+$$
$$x = x' \cos x^+$$
$$x^- = x'^- + \frac{x'^2 u}{2}$$

and now we see that we have obtained flat space, up to the nontrivial conformal factor $1/(1 + u^2)$. This represents only the portion $x^+ \in (-\pi/2, \pi/2)$ (actually only $x^+ \in (0, \pi/2)$, but we have analytically extended to the mirror $u \rightarrow -u$).

One of the uses of the Penrose diagram is that it helps us decide whether the space is complete, or if we can analytically extend it, and more importantly, how to analytically extend the space. For instance, if one starts with AdS space, one could see that there are boundaries which are not singular, and hence one can analytically extend over them. Then we obtain the universal cover of AdS. If we have conformally mapped a space to a patch of the Einstein static universe, which is regular and finite, we know that all we have to analyze is the conformal factor. If it blows up, it is a genuine boundary. If it stays finite, we can analytically extend over it. In the case of AdS, the conformal factor is finite, so we can analytically extend to the universal cover. In the case of Minkowski space, the conformal factor blows up at the boundary, so there is no possible analytical continuation.

In our case however, we would have the conformal factor of Minkowski space which blows up at the boundary, together with the conformal factor relating the pp wave with Minkowski space, which factor goes to zero. So we have to analyze carefully what happens, to see what is a real boundary and what can be analytically continued over.

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1. This section was done in collaboration with Juan Maldacena
2. See also for uses of this coordinate transformation
Let us define the following consecutive coordinate transformations.

\[ u = \sigma + \tau \quad x' = \frac{\sigma - \tau}{2} \]
\[ x' = r \sin \theta \quad \sigma = r \cos \theta, \quad \theta \in (0, \pi) \]
\[ \hat{u} = r + \tau = \tan \frac{\psi + \zeta}{2} \]
\[ \hat{v} = r - \tau = - \tan \frac{\psi - \zeta}{2} \]

(11)

The first two transformations are needed to isolate the time coordinate and the overall spatial radial coordinate, in order to be able to apply the standard transformation of Minkowski space to the Einstein static universe (from \( r, \tau \) to \( \psi, \zeta \)).

Under these coordinate changes, the metric can be written as

\[ ds^2 = \frac{1}{1 + u^2} (-d\tau^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\Omega_7^2)) \]
\[ = \frac{(1 + \hat{u}^2)(1 + \hat{v}^2)}{4(1 + u^2)} (-d\psi^2 + d\zeta^2 + \sin^2 \zeta (d\theta^2 + \sin^2 \theta d\Omega_7^2)) \]
\[ = \frac{(1 + \hat{u}^2)(1 + \hat{v}^2)}{4(1 + (\hat{u} \cos^2 \theta/2 - \hat{v} \sin^2 \theta/2)^2)} (-d\psi^2 + d\zeta^2 + \sin^2 \zeta (d\theta^2 + \sin^2 \theta d\Omega_7^2)) \]
\[ = \frac{1}{4} \left( \cos \psi + \cos \zeta \right)^2 + \left( \sin \psi + \cos \theta \sin \zeta \right)^2 (-d\psi^2 + d\zeta^2 + \sin^2 \zeta (d\theta^2 + \sin^2 \theta d\Omega_7^2)) \]
\[ = \frac{1}{4} |e^{i \psi} - \cos \alpha | e^{i \beta}|^2 (-d\psi^2 + d\zeta^2 + \cos^2 \alpha d\beta^2 + \sin^2 \alpha d\Omega_7^2) \]

(12)

In the last line, we have changed between the two parametrizations of \( S^9 \) in terms of \( S^7 \) (in general \( S^{n+2} \) in terms of \( S^n \)),

\[ x_1 = - \cos \zeta = \cos \alpha \cos \beta \]
\[ x_2 = - \cos \theta \sin \zeta = \cos \alpha \sin \beta \]
\[ \vec{x} = - \sin \theta \sin \zeta \quad \vec{r} = \sin \alpha \quad \vec{F} \]

(13)

with \( \vec{x}, x_1, x_2 \) cartesian coordinates in \( R^{10} \) and \( \vec{r} \) cartesian coordinates in \( R^8 \). In the second line of (12) we have mapped to the Einstein universe, so the coordinates are finite in extent, but the conformal factor is not obvious to analyze. In the third line, we can analyze the conformal factor in terms of the usual variables \( \hat{u}, \hat{v} \), and we will do it below, but it will not be so obvious how to glue things together.

However, in the variables in the last line of (12), it is clear that the boundary is at \( \alpha = 0 \) (note that \( \alpha \in [0, \pi/2] \), so \( \cos \alpha = -1 \) is not allowed) and \( \psi = \beta \), since only then the conformal factor of the metric diverges. This is a one dimensional null line in \( S^9 \times R \) (since at \( \alpha = 0 \) the radius of \( S^7 \) shrinks to zero, as we can see) whose spatial projection lies on the maximum circle of \( S^9 \) specified by \( \alpha = 0 \).

In conclusion, we can analytically continue and cover the whole Einstein universe, except for a one dimensional null line, given by \( \alpha = 0 \) and \( \psi = \beta \), which is the real boundary of the pp wave spacetime. The Penrose diagram of the pp wave is then as represented in fig. 1e.

Note that it is not a contradiction to have a boundary which has a low dimension. In \( d \) dimensional Minkowski space, the null boundary remains \( d-1 \) dimensional, but spatial infinity and timelike infinities are mapped to points. However, in \( AdS_5 \times S^5 \) for instance, when we make the conformal transformation to map \( AdS_5 \) to a patch of Einstein universe, then the radius of \( S^5 \) shrinks to zero at the boundary. That is why the boundary of \( AdS_5 \times S^5 \) is \( S^3 \times R \).

Let us understand in a bit more detail the calculation that we did. In fig. 1a,b and c we drew the regular Penrose diagram of Minkowski space. It is a patch of the Einstein universe represented on the cylinder.
A 2-dimensional universe has a diamond as Penrose diagram (fig. 2b), since we represent a spatial coordinate living in $R$. A higher dimensional Minkowski space has a diagram which is a triangle, since we represent the radial spatial coordinate, which is positive (fig. 2c).

In our case, the conformal factor depends on the angles too, so we can’t represent the diagram in only 2 dimensions. The variables $\psi, \zeta$ vary over the usual triangle $|\psi \pm \zeta| \leq \pi, \zeta > 0$ and $\theta \in (0, \pi)$. We represented the Penrose diagram in fig. 2d, using $\psi, \zeta, \theta$ variables (suppressing the 7-sphere), $\psi$ vertically, and $\zeta, \theta$ being like polar coordinates for the perpendicular plane $x', \sigma$. This diagram suppresses a 7-sphere, but the line $\zeta = \pi, \psi = 0, \theta$ arbitrary is actually a point since the prefactor $\sin^2 \zeta$ in the metric vanishes. Likewise, the lines $\theta = 0, \pi, \psi + \zeta = \pi$ have no 7-sphere suppressed, since its radius is zero.

In the original coordinates $(x^+, x^-, \vec{x})$, the flat Einstein space described here covers only the portion $x^+ \in (-\pi/2, \pi/2)$. $(u = \infty$ is $x^+ = \pi/2)$. We notice that the correct range of $x^+$ was actually $(0, \pi/2)$, but we have extended it by symmetry in the region $(-\pi/2, 0)$ to cover the whole region of Einstein space described above. There were no obstructions to this continuation in either the original or the final picture.

We want now to see that the space should be analytically continued over the whole boundary, except at $\theta = \pi, \psi + \zeta = \pi$ and $\theta = 0, \zeta - \psi = \pi$, which are therefore the actual boundaries of this space.

We have seen this in the calculation on the last line of (12), but we want to shed some light on the fact that in the original $(x^+, x^-, \vec{x})$ coordinates there seem to be three distinct regions of infinity $x^- \rightarrow \pm \infty$ and $x \rightarrow \infty$, whereas in the final result there is only a one dimensional line. In the coordinates used in fig. 2d, the line seems to be composed of two pieces, but one has to remember that the line $\psi = 0, \zeta = \pi$ is actually just a point (io in fig. 3c), so the two lines are connected. In the coordinates on the last line of (12) it is obvious that there is only 1 continuous line, but in fig. 2d it is confusing, since we are trying to represent a 10d space in three dimensions. We would also like to understand algebraically the fact that we need to glue these Minkowski spaces (in the original coordinates it is obvious, the Minkowski space represents only the $x^+ \in (-\pi/2, \pi/2)$ patch).

The upper half of the boundary in the Penrose diagram corresponds to $\tilde{u} = \infty, \tilde{v} = \text{finite and } \theta$ arbitrary. We have

$$
x' = \frac{\tilde{u} + \tilde{v}}{2} \sin \theta
$$

$$
u = \frac{\tilde{u} \cos^2 \theta / 2 - \tilde{v} \sin^2 \theta / 2}{2}
$$

$$
x^-' = \frac{\tilde{u} \cos^2 \theta / 2 - \tilde{u} \sin^2 \theta / 2}{2}
$$

we see that in the generic case $u = \infty$. Then

$$
x = \frac{x'}{\sqrt{1 + u^2}} \approx \frac{x'}{u} \approx \tan \theta / 2
$$

$$
x^- = x^- + \frac{x^2 u}{2}
$$

(15)

and we can check that the infinite pieces cancel in $x^-$, and it becomes parametrized by $\tilde{v}$ and $\theta$. So indeed, this boundary corresponds to $u = \infty$, with $x^-$ and $x$ arbitrary, as expected. We can also check that the conformal factor is finite in this case, and so we can analytically continue through it, as expected.

The real boundary of the space is at $\theta = \pi$ in the above, because then the conformal factor of the metric blows up. Then we see from the above formulas that it depends how we approach the limits $\tilde{u} = \infty$ and $\theta = \pi$. If we choose $\lim \tilde{u}(\pi - \theta)^2 = \text{finite}$, one can check that $u$ is finite, and $x'$ (and therefore $x$) are infinite. One can moreover impose that the infinite pieces cancel in $x^-$, which implies the equation

$$
a = \lim_{\theta \rightarrow \pi} \frac{\tilde{u}(\theta - \pi)^2}{4} = \sqrt{\tilde{v}^2 + 1}
$$

(16)

This limit therefore gives the expected boundary $x^+$ arbitrary, $x^-$ finite, $x$ infinite.

A different limit, namely

$$
\lim \tilde{u}(\theta - \pi) = \text{finite}
$$

(17)
Figure 1: a) Penrose diagram of Minkowski space represented as a patch of the Einstein static universe. b) Penrose diagram of a 2d Minkowski space drawn in a plane. \( \zeta \in (-\pi, \pi) \), since the original spatial coordinate is in \((-\infty, \infty)\). c) Penrose diagram of 4d Minkowski space represented in a plane. Each point represents a 2-sphere, except for \( i^+, i^- \) and \( i_0 \). \( \zeta \in (0, \pi) \) since the spatial coordinate is radial, hence positive. d) Penrose diagram of the patch of the pp wave conformal to Minkowski space, represented in 3 dimensions. Every point is an \( S^7 \), except for the 2 thick lines which represent the real boundary of the space. All other boundary points of the diagram are analytically continued over. e) Penrose diagram of the whole pp wave spacetime. It fills the whole Einstein static universe, except for the 1 dimensional null line which is its boundary. We have also the 2 disjoint points \( i^+, i^- \) representing the timelike boundaries.
gives $x^- \to \infty$, $x$ finite, $u$ finite, as one can easily check.

So in conclusion, both boundaries are mapped to (different limits of) the same boundary of the Einstein universe, namely the one dimensional lightlike line $\zeta + \psi = \pi$, $\theta = \pi$. Over the rest, one could analytically continue.

We have discussed the $x^+ \in (-\pi/2, +\pi/2)$ patch, but of course we have the boundaries $x^+ = \pm \infty$ too, which correspond to the usual boundaries $\psi = \pm \infty$ of the Einstein universe. These are 2 disjoint points by the usual argument (e.g. for Anti-de Sitter): if we bring timelike infinity to a finite distance, the Einstein universe shrinks to the time line, hence timelike infinity is a point.

So what happens to geodesics in the pp wave? A timelike geodesic ends at $i^+$ or $i^-$. A null geodesic in the $x^-$ direction ends on the null line. Since the null line is parametrized by $x^+$, and $x^+$ is the pp wave time, a null geodesic in the $x^+$ direction will go parallel to the null boundary and end up at $i^+$ or $i^-$ too. A spacelike geodesic will end on the null boundary too.

Naively, one should have a null infinity for geodesics propagating in the transverse $x^i$ directions too. But in the transverse directions, particles behave as harmonic oscillators, so a null geodesic will be periodic in $x^i$, and it will not reach spatial infinity at $x^+ = \infty$, but it will still reach $i^+$ (or $i^-$).

For completeness, let us also see what happens for the 11 dimensional maximally supersymmetric wave. Its metric is

$$ds^2 = 2dx^+ dx^- - \left( \frac{x^2}{9} + \frac{y^2}{36} \right) (dx^+)^2 + dx^2 + dy^2$$  \hfill (18)

Under the transformation

$$x = x' \cos(x^+/3)$$
$$y = y' \cos(x^+/6)$$
$$x^- = x^- + \frac{1}{12} x^2 \sin 2x^+/3 + \frac{1}{24} y^2 \sin x^+ / 3$$  \hfill (19)

the metric becomes

$$ds^2 = 2dx^+ dx^- + (dx')^2 \cos^2(x^+/3) + (dy')^2 \cos^2(x^+/6)$$  \hfill (20)

Then after rescaling $x^+ \to 3x^+$, $x^- \to x^- / 3$ and then $u = \tan x^+$, we get

$$ds^2 = \frac{1}{1 + u^2} \left( 2dx^+ du + dx^2 + x^2 d\Omega_2^2 + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 + u^2}} \right) (dy'^2 + y'^2 d\Omega_5^2) \right)$$  \hfill (21)

and here we have $u \in (0, \infty)$. Notice that if we first rescaled $x^+ \to 6x^+$, $x^- \to x^- / 6$ and then $u = \tan x^+$, we would have gotten the metric

$$ds^2 = \frac{1}{1 + u^2} \left( 2dx^+ du + dy'^2 + y'^2 d\Omega_5^2 + \frac{(1 - u^2)^2}{1 + u^2} (dx^2 + x^2 d\Omega_2^2) \right)$$  \hfill (22)

with $u$ varying from 0 to 1.

Instead, we keep the first transformation of coordinates, and now make a similar transformation as for the 10d wave case:

$$u = \sigma + \tau \quad x'^- = \frac{\sigma - \tau}{2}$$
$$\sigma = r \cos \theta \quad \theta \in (0, \pi)$$
$$x' = r \sin \theta \cos \phi \quad \frac{y'}{\sqrt{2}} = r \sin \theta \sin \phi$$
$$\tilde{u} = r + \tau \quad \tilde{v} = r - \tau$$
$$\tilde{u} = \tan \frac{\psi + \zeta}{2} \quad \tilde{v} = -\tan \frac{\psi - \zeta}{2}$$  \hfill (23)
and get a metric which near \( u = \infty \) looks like
\[
\frac{(1 + \tilde{u}^2)(1 + \tilde{v}^2)}{4(1 + u^2)} \left( -d\psi^2 + d\zeta^2 + \sin^2 \zeta (d\theta^2 + \sin^2 \theta (d\phi^2 + \cos^2 \phi d\Omega_5^2 + \sin^2 \phi d\Omega_3^2)) + o(1/u^2) \right) = \frac{(1 + \tilde{u}^2)(1 + \tilde{v}^2)}{4(1 + u^2)} \left( -d\psi^2 + d\zeta^2 + \sin^2 \zeta d\Omega_5^2 + o(1/u^2) \right)
\]
and where as before
\[
u = \tilde{u}\cos^2 \theta / 2 - \tilde{v}\sin^2 \theta / 2
\]

If it wouldn’t be for the \( u \)-dependent factor multiplying \( d\Omega_5^2 \), we would have the same metric as before, just in 11 dimensions. But at \( u = \infty \), the analysis is the same, since then the factor is 1. Moreover, for \( u \) finite, the factor varies from 1/2 to 1. So we can analytically continue this metric in exactly the same way over the \( u = \infty \) regions.

Then the pp wave will be conformally equivalent to slices of slightly deformed Einstein space glued together, but it will be exactly Einstein space over the gluing region. There would be a difference on the real boundary, at \( \theta = \pi \), except that there the nontrivial piece is multiplied by \( \sin^2 \theta \), which is zero. So again the boundary is a one dimensional line plus 2 disjoint points \( i^+ \) and \( i^- \), and the Penrose diagram will look more complicated then fig. 1e, since what used to be a Minkowski patch will be now deformed as in (21), but the general structure is the same.

Finally, it is not very obvious that one can generalize this analysis to other pp waves, since it is not obvious whether one can map a patch of a general pp wave into anything that looks like a Minkowski space at infinity, as it happened in (21).

Now, let us comment on three papers on holography in the pp wave which have previously appeared. In [21], the authors analyzed the pp wave duality, and from the fact that the string transverse oscillators have \( SO(4) \) invariance, the same as the invariance of the \( D_i Z \) insertions in euclidean SYM, concluded that the SYM must be defined on an 4 dimensional euclidean space corresponding to 4 of the transverse directions of the pp wave. But in this way one forgets that the pp wave duality is a limit of the AdS-CFT correspondence. In AdS-CFT, the euclidian theory comes from Wick rotation of the SYM defined on \( S^3 \times R \) at the boundary of global AdS. The \( SO(4) \) invariance is just the invariance of the sphere. And what used to be AdS time is not in the transverse direction of the pp wave. In [23], the authors tried to implement a procedure similar to the AdS-CFT, and argue that the radial direction is a holographic direction. The pp wave spacetime appears by focusing in on a geodesic in the middle of AdS, and the boundary of AdS lies well outside the pp wave spacetime. The wave has an entirely new structure. Finally, the authors of [22] proposed that there is a 9 dimensional holographic screen at fixed \( x^+ \). They analyzed the Minkowski slice of the pp wave \((x^+ \in (0, \pi/2))\), and decided that its boundary is the holographic screen. But that is related to a misleading analogy to AdS-CFT. In AdS, the spatial boundary of the Poincare patch is \( S^3 \times I \) \((I=\text{interval})\), as part of the whole boundary \( S^3 \times R \). The euclidean theory one is working with is defined on the euclideanized Poincare patch. However, the full boundary of Minkowskian AdS is \( S^3 \times R \), on which the states of SYM are defined. Since for the pp wave there is no “euclidianized version”, one has to consider the whole pp wave, not just the Minkowski patch.

3.2 Derivation of the pp-wave geometry after Tseytlin

In [3, 1], it was shown that the pp-wave geometry can be obtained from that of \( AdS_5 \times S^5 \) by taking a Penrose limit. Here we will describe a slightly modified version of the Penrose limit which makes certain conceptual issues about perturbation theory in the Yang-Mills theory more clear.

If we consider \( AdS_5 \times S^5 \) as a geometry, and we want to take the Penrose limit, we are interested in considering a null geodesic whose time direction flows along the global time coordinate in AdS, and that winds around a great circle of \( S^5 \).

---

3 We thank Arkady Tseytlin for communicating this modified Penrose limit to us. See also [2].
This is, we single two coordinates $t, \psi$, and we write a metric adapted to these geodesics
\begin{equation}
  ds^2 = R^2(-\cosh^2(\rho)dt^2 + dp^2 + \sinh^2(\phi)d\Omega_3^2 + \cos^2(\theta)d\psi^2 + d\theta^2 + \sin^2(\theta)d\Omega_{4}^2)\tag{26}
\end{equation}

Now, we want to take lightcone coordinates adapted to the geodesic, so a naive guess would be to take $x^\pm = t \pm \psi$. Notice however that the coordinate $\psi$ is periodic with period $2\pi$, so in the above equation the coordinates $x^\pm$ are identified with a periodicity, and in particular there are shifts in the time coordinate of the lightcone time $x^+$. This is not a good choice of coordinates if we want to use $x^+$ as a time variable. We want a time variable that does not get shifted when we rotate $\psi$. Therefore let us consider the following modified version of the lightcone variables
\begin{equation}
  x^+ = t, x^- = R^2(t - \psi)\tag{27}
\end{equation}
and rescale the fields $\rho = Rx, \theta = Ry$, and then take the limit metric when $R \to \infty$. The $x, y$ combined with the spherical angles $\Omega_3, \Omega_4$ give rise to a flat metric, and the only interesting piece of the metric is the one involving $t, \psi$. Let us concentrate on these terms, and remember that $dt = dx^+, d\psi = dx^+ - dx^- R^{-2}$. Substituting these expressions we find that
\begin{equation}
  ds_{LC}^2 \sim R^2(-(1 + \rho^2/R^2)(dx^+)^2 + (1 - y^2/R^2)((dx^+)^2 + 2dx^+ dx^- R^{-2})
  \rightarrow 2dx^+ dx^- - (p^2 + y^2)(dx^+)^2 \tag{28}
\end{equation}
so this choice of coordinates leads again to the pp-wave geometry. There is only one noncompact pp-wave geometry, the leading effect of $R$ is to make the variable $x^-$ periodic, with period $2\pi R^2$, so it compactifies one of the lightcone variables.

The momentum associated to this compactification becomes discrete, and is naturally identified with $p^+ = JR^{-2} = -iR^{-2}\partial_\phi$. Therefore, when we are doing an AdS/CFT calculation, we need to think about supergravity in a box of size $2\pi R^2$ for $x^-$. As is natural in lightcone gauge, we will find that in the expression of the Hamiltonian there are powers of $p^+$ in denominators. In the paper [1] the identification with $p^+ = (\Delta + J)/2R^2 \sim J/R^2$ does not naturally suggest that it is quantized and that it can be described in terms of partons, although in the limit they behave in essentially the same way.

Also, the lightcone energy is as before $p^- = i\partial_t + i\partial_\phi = \Delta - J$.

The directions transverse to the lightcone are also in a box, which is given by the quadratic gravitational potential, but the size of that box depends on the amount of $p^+$ momentum carried by a state. The more $p^+$ a state has, the more focused it is.

A supergravity type calculation will be reliable if the transverse box is large, this is when $p^+$ is small. At large $p^+$ we need to use the full string theory to make predictions.

Now, when we consider the AdS/CFT correspondence, the Hamiltonian associated to the motion in the $x^+$ direction is given by $\Delta - J$. Since $J$ is a conserved quantity that arises from a global symmetry, when we deform the theory by changing the gauge coupling we are only doing a perturbation theory for $\Delta$. Indeed there is a non-renormalization theorem for $J$, so we are interested in finding the values of the dimension of local operators in the SYM theory, and we will be effectively computing the anomalous dimension of the operators.

3.3 The spectrum of open and closed strings in the pp-wave limit of the orientifold

In this section we are going to describe the spectrum of open and closed strings in the orientifolded pp-wave limit from the SYM perspective. The main idea is to set up some notation which we will be using throughout the paper.

The orientifold field theory is an $N = 2$ supersymmetric conformal field theory in four dimensions, which is realized as the low energy effective field theory of D3 branes parallel to an $O(7)$ plane with D7 branes cancelling the RR charge. It contains a vector multiplet for the gauge group $Sp(N)$, a hypermultiplet in the antisymmetric representation of $Sp(N)$, and four hypermultiplets in the fundamental. Naively the global symmetry of the flavors is $SU(4)$, but it is enhanced to an $SO(8)$ symmetry that mixes the flavor
hypermultiplets with their complex conjugates. This is the gauge group associated to 4 D7 branes at the orientifold, which in the infrared of the D3-brane theory becomes a global symmetry of the four dimensional low energy effective field theory.

The global symmetries of the theory are \( SO(2) \times SU(2)_L \times SU(2)_R \times SO(8) \). The \( SO(2) \) are the rotations transverse to the D7 brane, and the \( SU(2) \times SU(2) \) arises from the rotations transverse to the D3 brane that are parallel to the D7 brane.

Their quantum numbers under the \( SU(2) \) appear in the following table, together with \( \Delta - J_3^L - J_3^R \), which is the quantity relevant for the lightcone hamiltonian.

| Field | \( J_3^L \) | \( J_3^R \) | \( J^L \) | \( J^R \) | \( \Delta - J \) |
|-------|-------------|-------------|---------|---------|------------|
| \( W \) | 0           | 0           | 0       | 0       | 1          |
| \( \bar{W} \) | 0           | 0           | 0       | 0       | 1          |
| \( Z \) | 1/2         | 1/2         | 1/2     | 1/2     | 0          |
| \( \bar{Z} \) | -1/2        | -1/2        | 1/2     | 1/2     | 2          |
| \( Z' \) | 1/2         | -1/2        | 1/2     | 1/2     | 1          |
| \( \bar{Z}' \) | -1/2        | 1/2         | 1/2     | 1/2     | 1          |
| \( Q_i \) | 0           | 1/2         | 0       | 1/2     | 1/2        |
| \( \bar{Q}_i \) | 0           | -1/2        | 0       | 1/2     | 3/2        |
| \( D_i \) | 0           | 0           | 0       | 1       | 1          |

Here we are just describing the bosonic states. The quantum numbers of the fermions are obtained by applying supersymmetry transformations. The general closed string state of the pp-wave of momentum \( J/R^2 \) is built of (cyclic) words that are a trace where there are \( J \) fields \( Z \) and a finite number of the fields with two gauge indices and dimension less than or equal to 1. This is, of the form

\[
\text{tr}(S_0 \Omega(Z \Omega)^{n_1} S_1 \Omega(Z \Omega)^{n_2} S_2 \ldots)
\]  

where \( S_i \) is any of the fields with \( \Delta - J = 1 \), \( \Omega \) is the antisymmetric invariant tensor of \( Sp(N) \) which is used to raise indices so that we can multiply the fields as matrices and \( \sum n_i = J \). As written above, we have ordered defects in the word made out of the \( Z \), and the values of \( n_i \) are the distances between the defects in a lattice whose sites are made out of the fields \( Z \). This a description of the string with the defects written in position space. It is convenient to go to a basis which diagonalizes the leading planar diagram interactions to first order, and these are furnished by discrete fourier transforms of the above operators with respect to the positions of the defects inside the trace. One can argue that this is the right basis because the trace is invariant under rotations (cyclicity of the trace).

The vacuum state of the string is identified with the operator \( |0, p^+> = A \text{tr}(Z^J) \) up to a normalization constant \( A \). In the planar limit \( A = \frac{1}{\sqrt{4\pi^2 J^2}} \). A string state with two oscillators of momenta \( n \) and \( -m \) is written as

\[
a^+_n a^+_{-m} |0, p^+> = A \sum_{l, l_2} \text{tr}(Z^l S_1 Z^{l_2} S_2 Z^{J-l-l_2}) \exp(2\pi i n l/J + 2\pi i (-m)(l + l_2)/J) 
\]

which will vanish by ciclicity of the trace unless \( n = m \), and this corresponds to the level matching condition. Generically in the large \( J \) limit the locus where the defects coincide is supressed by powers of \( 1/J \), so one can ignore the exact details of the operators when the defects coincide, at least at the free string theory level. The orientifold projection is imposed automatically by the symmetries of the operators with respect to transposition, which reverses the order of elements in the trace.

A single string state is not of trace type, but instead it is capped with quarks at the ends. The ground states are of the form

\[
|0, p^+>^{ij} = Q^i Z^J Q^j
\]

they have \( p^− = 1 \), and are antisymmetric in \( i, j \). From the quantization of the superstring in the pp-wave this non-zero expression of the Hamiltonian results from a mismatch between fermionic and bosonic zero modes. However, the state as written above is still BPS (the operator is chiral). Again, one can introduce defects
in the above states. The diagonalization of the interactions to first order planar approximation produces Dirichlet boundary conditions for \( W, W, \) and Neumann boundary conditions for all the other defects. One can perform \( SU(2) R \times SU(2) L \) rotations of the above states and produce BPS states. These will insert \( Z' \) and \( \bar{Z}' \) at general positions with no phases. There are \( 1/J \) effects that can turn a quark \( Q_i \) into \( \bar{Q}_i \), but these can be ignored in general, as they are subleading effects in \( 1/J \).

Notice that the combinations \( ZZ' + Z'Z \) and \( Q_i Q_i \), with no trace over the gauge indices, but traced over the flavor indices have the same \( SO(8) \), \( J_3 \) and \( J_\perp \) quantum numbers. However, the operators have different \( J_\perp \) quantum number, as \( ZZ' + Z'Z \) is related by \( SU(2) L \) to \( ZZ \), which has spin 1, whereas the state \( QQ \) is singlet.

On the other hand, the combination \( ZZ' - Z'Z \) does have the same quantum numbers as \( Q_i \bar{Q}_i \), and they can mix. This is how strings will be able to split and join. The word combination \( ZZ' - Z'Z \) can not appear in BPS operators which are fully symmetrized in the \( Z, Z' \), but it can appear in non-BPS operators.

### 3.4 Comments on the regimes of SYM

Let us analyze in a bit of detail the various regimes that are a priori possible in the SYM theory. In SYM, the size of the operators \( J \) can vary, and there are 3 scales it can be compared with: \( g_{YM} \sqrt{N}, \sqrt{N} \) and \( \sqrt{N}/g_{YM} \).

- The regime \( J \ll g_{YM} \sqrt{N} \) (but still parametrically comparable) is the regime of flat space string theory. Indeed, in \( \Box \) we saw that in this limit, the string spectrum in the pp-wave background reduced to the flat space spectrum, so in this limit SYM describes strings in flat space. In SYM, this limit corresponds to large (nonperturbative) \( g_{YM}^2 N/J^2 \) corrections. The usual diagrammatic expansion has to be resummed. In pp wave variables, the limit corresponds to \( \alpha' \mu p^+ \ll 1 \). Of course, if one goes to still lower \( J \) (parametrically lower), we are back to the usual AdS-CFT case, of operators with a small number of fields.

- The next possible regime is \( g_{YM} \sqrt{N} \ll J \ll \sqrt{N} \). In pp-wave variables, this corresponds to \( (\alpha' \mu p^+)^2 \gg 1 \), but \( (\alpha' \mu p^+)^2 g_s \ll 1 \). This is the regime of strings in large RR background, where the string oscillators have almost the same energy. From the SYM point of view, this is the regime where one can trust perturbative computations. Now \( J^4/N^2 = (\alpha' \mu p^+)^4 g_s^2 \ll 1 \), and we will see that this corresponds to string loop corrections i.e. nonplanar SYM diagrams being negligible.

This is the regime we are working with mostly in this paper. Here as we will see, three-point functions are nonzero (tree-level interactions), but (string) quantum corrections are negligible. It is an extension of the same situation one encounters in the usual AdS-CFT correspondence. Note that since we are looking at near-BPS operators and \( g_{YM}^2 N/J^2 \ll 1 \), perturbative SYM is a good approximation, and it should correspond to string theory in the pp wave background. However, it is not obvious that string theory and sugra should give the same result, since \( \alpha' \to 0 \) (or rather \( \alpha' \mu p^+ \to 0 \)) corresponds to \( g_{YM}^2 N/J^2 \to \infty \), which is the opposite limit to the one considered. Also note that therefore it is essential to look at the \( 1/(\alpha' \mu p^+)^2 \) corrections, which we will analyze later.

Also note that in the previous regime \( g_{YM}^2 N/J^2 \gg 1 \), so it is not clear whether non-BPS operators correspond to the same (sugra) behaviour of the n-point functions. Of course, the BPS operators would not have any corrections due to the nonrenormalization theorems at work for the usual AdS-CFT correspondence, so for them the free SYM result should still be the sugra result.

- If we go still higher in \( J \) we encounter the most intriguing regime: \( \sqrt{N} \ll J \ll \sqrt{N}/g_{YM} \). In pp wave variables this is \( (\alpha' \mu p^+)^2 g_s \gg 1 \), but \( (\alpha' \mu p^+)^2 g_s^2 \ll 1 \). This is a strongly coupled string theory, but in SYM this is an apparently very simple theory: perturbative corrections are negligible \( (g_{YM}^2 N/J^2 \ll 1) \), but nonplanar diagrams dominate \( (J^4/N^2 \gg 1) \). It would be very interesting to describe this theory.

- Finally, if \( J \gg \sqrt{N}/g_{YM} \), the theory should describe giant gravitons. Indeed, as we saw in \( \Box \), then the scale of the giant gravitons in the pp wave background becomes bigger than the string scale (giant gravitons are really giant). In pp wave variables, the limit is \( (\alpha' \mu p^+) g_s \gg 1 \). In SYM, in this limit
not only \( J^4/N^2 \gg 1 \), but also \((J^4/N^2)(g_{YM}^2N/J^2) \gg 1\), so both free nonplanar contributions are important, as well as interacting nonplanar contributions (although planar interacting contributions are all the more negligible). We can understand why this is a giant graviton theory from the following. As explained in [37, 38], a giant graviton is a subdeterminant operator in SYM, which means it is a sum of all possible multitraces. Since in this limit, free nonplanar diagrams dominate, it is clear that the mixing of multitrace operators becomes maximal (string interactions dominate). However, unlike the previous regime, now also interacting nonplanar diagrams dominate over free diagrams, and as such they will create an interacting hamiltonian which presumably should be diagonalized exactly by giant graviton states (subdeterminants). It would be very interesting to find that behavior explicitly. However, this is beyond the scope of the present paper.

3.5 Relations between observables in SYM and the pp-wave

We have seen already that the limit of the \( AdS \times S \) geometry that gives rise to a pp-wave produces a wave where \( x^- \) is compact. Traditionally we think of the pp-wave as a noncompact spacetime along \( x^- \), and then it is appropriate to think of calculations on the pp-wave as describing some form of S-matrix. Notice however that since we have finite box in \( x^- \), we can not talk about S-matrix calculations because there is no way to form asymptotic states that are separated asymptotically at large times from each other. Instead we need to do amplitudes at finite time, and in particular we are interested in computing the spectrum of the Hamiltonian for single and multiparticle states. Also, since we are in a box, there is going to be mixing between single and multiparticle states, and in the CFT calculations these mixing amplitudes will be related to coefficients in the operator product expansion.

Notice, on the other hand, that a calculation for compact \( x^- \) at finite times is essentially the same as a calculation with \( x^- \) non compact, except that the momenta are quantized, and one has to use slightly different normalizations for the states. In the noncompact case we normalize one particle states so that they give rise to a delta function in momentum. Namely

\[
\phi_{nc} \sim \frac{1}{\sqrt{p^+}} \exp^{ip^+ x^-} \psi_T
\]

where \( \psi_T \) is the transverse wave function to the lightcone directions. When we put the particles in a box we normalize the wave function so that it has norm one in the box. This means that we need to normalize

\[
\phi_c \sim \frac{1}{\sqrt{p^+ \text{vol}(x^-)}} \exp^{ip^+ x^-} \psi_T
\]

and we have an extra factor of the square root of the volume appearing in the calculation. When we compute a Feynman diagram we need to take into account the factors of the volume. If there are \( m \) particles between the in and out state, the relation between the respective amplitudes (taking away the term that corresponds to conservation of momentum) will be

\[
A_{NC} \sim A_{C \text{vol}(x^-)^{(m-2)/2}}
\]

Now, since we have that \( \text{vol}(x^-) \sim R^2 \), the relation between amplitudes will involve factors of \( R \). Also, there are relations between the normalizations that enforce momentum conservation

\[
\delta(p_{in} - p_{out}) \sim \text{vol}(x^-) \delta_{p_{in} p_{out}}
\]

For a three particle function the relation between the amplitudes will involve a factor of \( R \) difference in the normalization.

In conclusion, the noncompact normalization [32] is used in the gravity and string theory calculations in the pp wave background (which is the exact limit of \( AdS_5 \times S^5 \)), and the compact normalization [33] corresponds to the SYM computation. Indeed, the SYM amplitudes are expressed in terms of \( J = p^+ R^2 = p^+ \text{vol}(x^-) \), and the gravity amplitudes in terms of \( p^+ \). For SYM, \( J \) is one of the R charges, but via AdS-CFT
is related in supergravity to an SO(6) charge on $S^5$, that is compact momentum. Then in (14) and (15), the l.h.s is expressed in terms of $p^+$, and the r.h.s. in terms of $J$.

Another issue we might consider is that the line which is the boundary of the pp wave is null, while in SYM, the line is the time direction. But we have to remember that the null geodesic we are focusing on in the pp wave limit is moving on the $S^5$, so its projection on the $S^3 \times R$ boundary of $AdS_5$ is the time direction. So the time direction is the correct coordinate to define SYM on.

Finally, what are the observables in the pp wave duality? On the SYM side, we saw that the theory reduced to a one dimensional quantum mechanics of states acted on by a hamiltonian. Correspondingly, in the $(\sigma$ discretized) worldsheet string theory, we have a hamiltonian acting on string states. The hamiltonian defines a time dependent problem, with finite time $t = t_0$ transition amplitudes formally of the type $< n | e^{iHt} | m >$, with $| n >, | m >$ eigenvectors of the free hamiltonian. To first order, the transition amplitudes are given by terms linear in $H$, and then higher orders in $H$ give corrections. In particular we have $H_{\text{free}}$ and we are interested in computing the full Hamiltonian. Time flow in the Feynman diagrams we draw will corresponds to the time in the matrix model. This is, when considering the spectrum of operators in $R^4$ the time will flow along the radial direction via $t = \log(r)$. The amplitudes will then be measured by OPE coefficients. For example $< O_1 (0) O_2 (z) O_3 (\bar{z}) >$ can be interpreted as an amplitude $< O_1 | e^{iH(t-z)} | O_2 e^{iH(t-\bar{z})} | O_3 >$ in the matrix model, if we smear the operators on a sphere around the origin. From here a power law behavior in $r$ translates to an exponential behavior in $t$, and therefore one can read the terms of the Hamiltonian by analyzing the situation where $t_i - t_j$ is small. This is exactly what is measured by the OPE coefficients, as one has to let two of the operators become very close to each other. Evaluation of amplitudes at finite time will give terms linear in $t = \log(r)$ to first order, so in general they will appear as power series in terms of $\log(x - x')$ and these correspond to the calculation of the anomalous dimensions of operators in the SYM by using standard Feynman diagrams (in position space). Notice that $t$ has to be sufficiently small to apply perturbation theory. In supergravity, the corresponding calculation is to integrate the action over solutions of the equations of motion, over the transverse coordinates (in which the wavefunction goes to zero at infinity), over $x^-$, which is compact, and over $x^+$ from 0 to $t$, with boundary conditions on the null boundary line.

3.6 Behaviour of SYM and supergravity observables in the Penrose limit

There has been a lot of work in the AdS-CFT correspondence calculating SYM correlators from supergravity, so we have learned a lot from them. We have learned that in N=4 SYM there should be nonrenormalization theorems which guarantee that at least the 3-point functions (24, 26, 27, 29), etc. at strong ’t Hooft coupling are given by the free diagrams. They are loop diagrams due to the fact that the operators are composite (for 3 R currents for instance, the free diagram is the usual triangle graph, with both anomalous and nonanomalous pieces). It is unclear whether 4 point functions are not renormalized.

However, in the case of “extremal” correlators, when the number of fields of one operator matches the sum of the others, $k_1 = k_2 + \ldots + k_n$, or more generally $k_1 + \ldots + k_n = k_{n+1} + \ldots + k_{n+m}$, it was conjectured in (33) that there are also nonrenormalization theorems at work.

For “extremal” correlators, the AdS calculation comes with a coefficient (coefficient of the n-point supergravity coupling) which is zero: $\alpha_3 = (k_1 + k_2 - k_3)$ for the 3 point functions, etc. (in the Penrose limit this is approximately equal to $(J_1 + J_2 - J_3)$). However, in (27) it was found that after calculating the AdS 3-point function the result is not singular anymore, since it gets multiplied with factors of $1/(k_1 + k_2 - k_3)$. This fact for 3-point functions was noticed in (34), where it was argued that the procedure of analytic continuation in $\alpha_3$ is the right one. In order to get the 3-point supergravity couplings with $\alpha_3$ coefficient, one has to perform nonlinear redefinitions of fields (24), or use the equations of motion and partial integration (31), which however generates boundary terms. But these boundary terms give extra contributions to 3-point functions (31). It was argued in (32, 33) that the correct procedure is to perform nonlinear redefinitions, since they correspond to consistent truncations for the massless fields. For the massive fields, nonlinear redefinitions are induced by consistency. Further checks of the fact that one gets a consistent truncation after a field redefinition were found in (34), and a different interpretation provided in (35).

When we take the Penrose limit, the $J$ momenta become continuous ($p^+$), as we saw in the last subsection, and the delicate problem of analytic continuation dissappears.
Let us now see how the natural observables in the AdS-CFT correspondence get restricted to the pp wave duality observables in the limit.

The first observation to be made is that only SYM transitions between states at time 0 and t (on S\(^3 \times R_t\)) remain in the limit, and in terms of the correlation functions that corresponds to correlators with a set of operators at zero (we cut a small hole around the operator, so we start with log(r) = t_0 finite), and another at \(\vec{x} = <O_1...O_n(0)O_{n+1}...O_{n+m}(x) >\). Of course that really means correlators between 2 multi-trace operators (multi-string states). We could have operators at different times, e.g. \(<O_1(0)O_2(x)O_3(y)>\), but the simplest (giving the OPE and characterizing string splittings), is for \(x=0\). Let us take for example a 3-point function of operators with sizes \(k_1, k_2, k_3\),

\[
< O_{k_1}(0)O_{k_2}(x)O_{k_3}(y) > \sim \frac{1}{x^{\alpha_1}} \frac{1}{y^{\alpha_2}} \frac{1}{(x - y)^{\alpha_1}},
\]

where the \(\alpha_i\)’s were defined above. The \(k_i\)’s are approximately equal to the \(J_i\)’s in the pp wave limit, so one of the \(\alpha\) is approximately zero, while the other two are large. These large factors come mainly from contractions of \(Z\) with \(Z\) in the operators. Only the oscillator terms can be contracted between \(O_{k_1}\) and \(O_{k_2}\), since neither of them contains \(\bar{Z}\). However, planarity would require to put the contracted \(\phi\)’s together at the end of the operators. For example \(Tr(Z^{J_1}\phi\bar{\phi}) Tr(Z^{J_1+J_2}\phi\bar{\phi}) Tr(Z^{J_1+J_2})\), with obvious contractions. This is a term subleading in \(J\) (there’s a probability of \(1/J\) to find the \(\phi\) at the end, since they are distributed uniformly in the trace), so these correlators disappear in the limit. This is, the correlation functions scale in such a way that in the limit \(J \to \infty\) they go to zero, even after taking into account the normalization issues discussed in the previous subsection. This gives us that one of the \(\alpha\)’s is strictly zero, so there is no pole in the OPE of the multi-string states and this fixes the possible normal ordering problem of the operators. This is essentially saying that the operators are mutually BPS.

As a further example, all 3-point functions of open string massless operators are zero, since they must involve a contraction of the \(q\)’s in between operators 1 and 2, and this is a term subleading in \(J\). More details of the disappearance of correlators in the pp wave limit are given in the Appendix.

So we have established that \(k_3 = k_1 + k_2\), and that there are no contractions between operators 1 and 2. The interactions are such that for the pp-wave states at the free field theory level they only mix operators with the same conformal dimension but with different numbers of strings. This is, the string splitting and joining problem is given also by degenerate perturbation theory, in the same limit that the string spectrum is generated by degenerate perturbation theory of planar diagrams.

### 4 Supergravity estimates for the n-point functions of string field theory: 3-open, 3-closed, open-closed, 4-open and 2-open-1-closed

In this section we will make some predictions for n-point functions of open and closed string fields based on the supergravity limit. This is the regime where strings have low values of \(p^+\) in \(\alpha'\) units, and it is reasonable to use a supergravity approximation where the string behaves almost as in flat space. The idea is to give us a feel for how these amplitudes should behave. In particular, for some BPS states there are non-renormalization theorems that predict that certain n-point functions are protected, so the behavior of these amplitudes at small values of \(p^+\) can give us a reasonable idea of what happens at large values of \(p^+\). In the SYM theory, where perturbation theory is valid at high values of \(p^+\), one expects that the level of the string oscillators give subleading effects and the essential features of the interactions are fairly independent of the states one considers, which are all almost BPS. It is because of this property that it is sensible to compare the transition amplitudes in these two very different regimes. In this section we will just describe the supergravity calculations.

Consider a closed string field \(\phi\) propagating on the plane wave background. Let us say a mode of the graviton. It has an interaction term in the action of the form

\[
g \int d^8 r dx^+ dx^- \phi^2 \Box \phi \quad (37)
\]
The laplacian in the pp wave background is
\[ \Box = 2\partial_+ \partial_- - \mu^2 r^2 \partial_+^2 + (\partial_+)^2, \quad i = 1, ..., 8 \] (38)

The solution of the wave equation for a massless scalar in the background is
\[ \phi = e^{ip^+ x^+} e^{ip^- x^-} H_{m_i}(\sqrt{p^+} x^i) \] (39)

It is essentially that of harmonic oscillators in the (massive) transverse directions and free fields in the \( x^+ \) and \( x^- \) directions. We normalize it in the \( x^- \) direction as in (32), and the transverse wavefunction is normalized as
\[ \psi_T \sim (p^+)^2 e^{-x^2 T / p^+} \] (40)

In supergravity we compute amplitudes of normalizable modes, but one should be able to understand them in terms of non-normalizable modes giving boundary values on the null line boundary, as in \([29]\). This type of prescription, where one has to include non-normalizable modes has also been considered for the pp-wave geometry in \([23]\).

Then the three point functions of closed string fields (gravitons) behaves as
\[ A \sim g \frac{p \times p^2}{p_1 p_2 p_3} \delta(p_3 - p_1 - p_2) \] (41)

where the first factor of \( p \) comes from the \( \partial^2 \) term in the lagrangian and the second factor of \( p^2 \) comes from \( \psi_T \). Notice that (40) also implies that \( \partial_+ \partial_- \sim \partial_+ \partial_i \sim r^2 \partial_+^2 \sim p_- \) for the first factor.

By going to the SYM variables (including the factor of \( \sqrt{\text{vol}} \sim R \)), we see that the corresponding SYM result should give us something that behaves as
\[ A \sim 1 N^{J^3 / J^{3/2}} J_{J_1 + J_2, J_3} \] (42)

independent of the 't Hooft coupling and therefore should be obtainable from free field theory results.

We also notice that the SYM result seems to go to zero for \( N \to \infty \), but that is just because of normalization. If \( g \) is finite, (41) doesn’t go to zero.

The open strings correspond to supergravity modes stuck on the D7 branes situated at the O(7) plane in the pp-wave metric. If we would write a 3-point interaction it would be given by the term
\[ \sqrt{g} \int d^6 x' d^4 x^- A_M A_N \partial_M A_N \quad M, N = \mu, +, - \quad \mu = 1, ..., 6 \] (43)

We normalize the fields in a canonical way (as in (32)) in the \( x^- \) direction, and again the wavefunction in the transverse directions is one for harmonic oscillators of amplitude \( A = 1 / \sqrt{p^-} \), just that now there are only 6 transverse directions for the open string, so we have
\[ A_\mu(x_i) \sim k_\mu p^{3/2} e^{-r^2 / A^2} \quad \mu = 1, ..., 6 \] (44)

where \( r^2 = \sum_{\mu=1}^6 (x^\mu)^2 \) and \( k_\mu \) is a constant polarization vector. The gaussian function takes into account the harmonic potential well transverse to the lightcone directions, and \( A = 1 / \sqrt{p^-} \). Notice that there are only six of those directions which are also along the brane. Hence our wave functions can only depend on these transverse coordinates. The full solution is like in (29), except that we have a polarization vector \( k_\mu \) and six transverse directions only.

Let us understand the Lorentz structure. The equation for the \( A_\mu \) components is
\[ \Box A_\mu + \partial_\mu (\partial \cdot A) - (\partial_\mu g^-) \partial_- A_\mu = 0 \] (45)
If we would choose the usual Lorentz gauge $\partial \cdot A = 0$, then the equation would become $\Box A_\mu = r'_\mu \partial_\mu A_\mu$, and we would have a nontrivial Lorentz structure. However, because $g^{--} = r^2$, we can write (43) as

$$\Box A_\mu + \partial_\mu(\partial \cdot A - \int d(r'^2)\partial_\mu A_\mu) = 0$$  \hspace{1cm} (46)$$
and therefore choose the gauge $\partial \cdot A - \int d(r'^2)\partial_\mu A_\mu = 0$. Then we have the solution in (44).

The effective action on the D7 brane worldvolume will be essentially the one of $N = 4$SYM in four dimensions, but with an extra WZW term from the RR potential that is important for the determination of the conformal dimensions of the primary operators. Namely

$$S \sim \int \frac{1}{4} F_{MN}^2 + \frac{1}{2} \sum_{p=1,2} (D_M X_p)^2 + \frac{1}{2} [X_1, X_2]^2 + \bar{\psi} D_M \Gamma^M \psi + WZW$$  \hspace{1cm} (47)$$
It can be seen from replacing the above solution that the answer is going to give a polarization structure of the form $\int d^3r'(k \cdot r')(k \cdot r') e^{-r'^2/r_+} = 0$, so the three point functions of massless vectors will vanish. This is also true for the scalars $X$ as there are no cubic couplings among the $X$. Also one can see that scalar scalar vector interactions are zero again because we will get a polarization of the form $k \cdot r'$ to integrate.

So there are no massless (sugra) bosonic 3 point functions. Similarly there are no 3-point functions for fermions. One might be a little bit concerned about this, but we have to consider that in flat space there are no on-shell three point functions of massless open strings as well. It is not clear from the SYM point of view whether the splitting and joining of strings is happening on-shell or off-shell. We believe that in the end the holographic model is calculating on-shell bulk physics, so we are finding no apparent inconsistency. Here we will report that we have consistent amplitudes between the SYM and the string theory. Certainly in the AdS/CFT dictionary, the ten dimensional (euclideanized) supergravity is on-shell.

However, there will be 3 point functions of massive fields (corresponding to the decay of massive string modes). These are also allowed off-shell in flat space. Indeed, for instance for massive vectors there is no gauge invariance, so one has the full equation (45) to satisfy with the addition of a mass term. Then, the solution will be behaving like (44), where now $k_\mu$ will depend on $r'_\mu$.

So then the open string vertex for three vectors will be

$$\sqrt{g} p^{3/2} / p^{1/2} \frac{1}{\sqrt{p'}} \delta(p_1 + p_2 - p_3)$$  \hspace{1cm} (48)$$
where the factor of $p^{3/2}$ comes from the wavefunction in the transverse directions (44), the factor of $p^{1/2}$ comes from the derivative in (43), and the last factor comes from factors of $p^{-1/2}$ in the expansion of the field. We don’t need to know the exact behaviour of $A_\mu$, only the scaling with $x$ and the exponential decay, to determine the $p$ dependence of the amplitude.

Then (48) would imply the following result in terms of gauge theory variables, by including the factor of $\sqrt{\text{vol}} \sim R$ difference in normalization of the amplitudes

$$\frac{1}{N} \int f^2 \frac{1}{f^{3/2}} \delta(J_i + J_{i+1}, J_{i+2})$$  \hspace{1cm} (49)$$
so the result should be independent of $g$ and be visible directly in the free field limit.

The supergravity 4 point functions of open strings are now nonzero, unlike the 3 point function case. Indeed, for instance the interaction term

$$\int d^6r'dx^+ dx^- A_\mu A_\nu A_\mu A_\nu$$  \hspace{1cm} (50)$$
has a Lorentz structure given by $(k^2)^2 = 1$, and then the 4 point function is

$$g_s (p^{3/2})^2 (p^{1/2})^4 \delta(p_1 + p_2 - p_3 - p_4)$$  \hspace{1cm} (51)$$
where \((p^{3/2})^2\) comes from the wavefunction in the transverse directions and \(1/(p^{1/2})^4\) from the normalization in \(x^-\), as usual. In SYM variables this is

\[
\frac{1}{N} J\delta_{J_1+J_2,J_3+J_4}
\]  

(52)

Let us turn to the open-closed transition. The supergravity result would come now from an interaction vertex of the type

\[
\sqrt{g} \int d^6r' dx^+ dx^- \text{tr}((\partial_M A_N)) \partial_M \partial_N \phi
\]  

(53)

That is so because the gauge invariant open string field \(\text{tr}(\partial_M A_N)\) could only couple to a bulk scalar field as shown. The transverse wavefunctions for the closed string field \(\phi\) and the open string field \(A_{\mu}\) have been defined in (40) and (44). However now we see that even if such a coupling would exist for massless fields, it would be zero, since we would have again a term \(k \cdot r'\) averaged over the whole space, giving zero.

Moreover, there is no such coupling between open and closed massless strings in supergravity (the trace is zero for the gauge field terms). But there could be in string theory, but then the vector would be massive, and just as for the open string 3 point function, we would get a nonzero result, since \(k_{\mu}\) would depend on \(r'_\mu\).

Then it follows that the result of the open string -closed string transition is of order

\[
\sqrt{g} p^{3/2} p^{1/2} \frac{1}{p} \delta(p_1 - p_2)
\]  

(54)

where the \(p^{3/2}\) factor comes from the derivatives in the action, the \(p^{1/2}\) from the difference in normalization in the transverse directions (the closed string field wavefunction extends in directions 7, 8 too, it is not localized to the D7-brane worldvolume, even though the integration is only over 6d). The \(1/p\) comes from the norm in the \(x^-\) directions, as above.

In terms of the SYM variables, (54) becomes

\[
\frac{1}{\sqrt{N}} J\delta_{J_1,J_2}
\]  

(55)

Finally, let us look at the open string to open plus closed string amplitude. In supergravity, there is a coupling

\[
\int d^6r' dx^+ dx^- \phi \partial_\mu A_\nu \partial_\mu A_\nu
\]  

(56)

(for instance the coupling of the dilaton to the \(F^2\) term). It gives the 3 point function

\[
g \frac{p^2 p}{(p^{1/2})^3} \delta(p_1 + p_2 - p_3)
\]  

(57)

(i.e. the same as the 3-closed string amplitude) where \(p^2\) comes from the wavefunction normalization in the transverse directions, \(p\) from the two derivatives in the action, and the denominators are again due to the expansion in \(x^-\). In terms of the SYM variables, this becomes (42). In the SYM side, we will perform the calculations in the opposite limit of large curvature. There the oscillators have a behavior which is very insensitive to the momenta on the worldsheet, and all of them should be treated on the same footing, as opposed to just looking at the zero modes and treating the other oscillators as if we are in flat space.

These are our expectations for the three point functions in the supergravity limit where \(p^+\) is small. The stringy oscillators should behave pretty much as they do in flat space in this limit, a fact which has appeared in the computation in [40]. We will compare these expectations with the results of free SYM theory, and we will find agreement.

However, for the 3-closed string amplitude and the ones in the same string field theory category (open string to open plus closed string amplitude and 4-open string amplitude), we will find the same puzzling
behaviour as in AdS-CFT: the SYM interactions don’t seem to play a role in holography [24, 26], etc. The correct supergravity results are obtained from free SYM correlators without any input from the interactions.

For the 3-open string amplitude and the open-closed amplitude something more interesting happens: we need to diagonalize the SYM hamiltonian, so they contain some information about the dynamics of SYM.

We will start therefore with the latter.

5 Leading SYM calculations for splitting and joining of open strings and open to closed amplitude

In the description of open strings at the orientifold given in [6] open strings were constructed with quarks at their ends, and the subleading corrections to the form of the operators were ignored. This was possible because of the nature of the limits taken. However, when we address the question of interactions, we need to be more precise. It will turn out that the subleading terms that were ignored in that calculation are necessary in order to get three point functions with a leading dependence which is independent of the ’t Hooft coupling.

If we consider operators which are holomorphic, then a two open string state built out of two BPS open strings can be given as

\[
(S^1)^\dagger(S^2)^\dagger|0\rangle \sim q_a^\dagger(Z^1)^{J_1}q_b^\dagger(Z^2)^{J_2}q_d^\dagger|0\rangle
\]

where the operators above describe constant modes of the corresponding fields on $S^3$. The normalization factor for such an operator is $1/(N(J_1+1)/2N(J_2+1)/2)$.

We expect to be able to glue the strings at their ends when $b = c$ to make a single string state, just by looking at their $SO(8)$ quantum numbers. However, a naïve single string state will have a quark occupation number of 2 instead of four. If we consider the operators above in the SYM what we find is that the operators $QZ^J Q$ and $QZ^J Q^\dagger$ do not have any free field contractions when both are holomorphic, so they should naturally produce what would be considered a two string state, with no mixing with one string states. If one of the $Q$ where instead $\overline{Q}$, there would be an allowed contraction between the quark indices that might produce a single string state in their OPE. However, a state with $\overline{Q}$ at the end in the pp-wave limit will appear in operators of the form $\sum qZ_1^l\bar{Z}^lZ_{\bar{Z}}^lq$ when we look at their end points. The coefficient for such a term in a general operator will be suppressed by factors of $1/\sqrt{J}$, so the three point function would vanish in the limit appropriate for the string theory. This is as it should be, since we expect that three point functions of members of the vector multiplet vanish. However, we need to examine these single string operators with more care, especially for the strings with oscillators excited.

Consider the state

\[
\sum Q_aZ^lZ'\bar{Z}'Z'^{-1}Q_d\cos(\pi inl/J)
\]

in [6] it was argued that this state represents an open string with an oscillator with momentum $n$ on the open string. In the free field theory, the states with all possible values of $n$ are degenerate in their conformal dimension, and the above state results from diagonalizing the planar diagrams that ignore quarks in the center. States with quarks in the center are also degenerate with these states. For example, we can consider a state

\[
\sum Q_aZ^{l-1}Q_b\bar{Z}^{l-1}\cos(2\pi inl/J)
\]

which is also degenerate with (59). When we diagonalize the interactions, there are field theory diagrams that mix these two kinds of operators coming from the potential in the theory.

These result from integrating out the F-terms of the superpartner of the vector multiplet, and are of the form

\[
g^2\text{tr}([\bar{Z}, Z]q_aq_b)
\]
The first state has a normalization factor that goes like \((\sqrt{JN^{(J+1)/2}})^{-1}\), while the ‘two string’ state has a normalization factor that goes like \((\sqrt{JN^{J/2}})^{-1}\). When we evaluate the planar diagrams involving the above interaction terms between these two states we are going to get a matrix of the form

\[
\begin{pmatrix}
g^2 \sqrt{Nn^2} & g^2 \sqrt{Nn} \\
g^2 \sqrt{Nn} & g^2 
\end{pmatrix}
\]  

where we ignore constants of order one. The different \(N\) dependence comes about because of the normalizations, and because quarks do not have a double line propagator. The \(N\) dependence can be pictorially represented by the figure 2

Figure 2: Diagrams for splitting and joining amplitude

The fact that the off-diagonal terms are proportional to \(n/J\) is because we have only one commutator in the potential term, so we only pick the difference between two consecutive phases. This results from a mixing of \([Z, Z']\) with \(QQ\) via the F-term for the field \(W\). If we consider the ‘t Hooft limit \(\lambda = g^2 N\) fixed with \(N \to \infty\) and \(J\) finite but very large, then we see that the components of the matrix scale very differently. In this case, the largest term in the matrix is the one that corresponds to the one string state alone with the oscillator, and the other terms correspond to very small numbers. This is the type of matrices that appear in See-Saw models for mass generation. Notice also that the off diagonal terms are determined exclusively from the place where the strings join and split, since this is the only place where we can insert the Feynman rule associated to the potential (61) in the leading planar approximation.

When we are writing the string state with oscillators it is the large term that dominates, and the two string state can be ignored. However, when we consider interactions, the true primary operator will be of the form

\[|S_1 > + \epsilon |S_2 S_3 >\]  

and it is of interest to calculate \(\epsilon\). The state we need is an eigenvector of (62), with eigenvalue approximately equal to \(g^2 Nn^2/J^2\). This fixes \(\epsilon \sim J/(n\sqrt{N})\) for \(n \neq 0\). For \(n = 0\) the above argument gives many zeros, and the one string state is an eigenvector of the matrix which is orthogonal to the two string state.

At high values of \(J\), the overlap between the two open string state of momentum \(J_1\) and \(J_2\) and the one string state primary is of order \(\frac{\lambda}{n\sqrt{N} J} \sim \epsilon_n/\sqrt{J}\), times the amplitude for the wave at the splitting point \(\cos(\pi n J_1 / J)\) which is of order one, and it is zero for \(n = 0\). The two string state with momenta \(J_1\) and \(J_2\) is of the form

\[QQZ^{J_1}QZ^{J_2}Q\]  

From here we can see that the mixing parameter is independent of the ‘t Hooft coupling, as expected from section 2, and it agrees with the general expectations we had from supergravity (49) in their \(J\) dependence as well. For \(J\) small we have to be careful since the higher order terms in the perturbation expansion will play a role in the infinite ‘t Hooft coupling limit \(g^2 N \to \infty\), with \(g^2\) fixed; which is appropriate to obtain interacting strings in the pp-wave. hence this calculation is not appropriate to understand the weakly curved case of small \(p^+\).
Notice that the way the calculation worked made very little use of the quarks at the ends $Q_a, Q_d$, except to fix the boundary conditions for the amplitudes of states with the defects at different positions. Instead we could have written
\[
Q_a Z^{J_1} Z' Z^{J_2} Q_d = \text{tr}(Q_d Q_a Z^{J_1} Z' Z^{J_2})
\]
so that it is clear that we can replace $q_d q_a$ with an operator which is not of the form $Z^m$. The essential step in the calculation remains identical, and this shows that open string to closed string mixing is essentially the same as mixing due to splitting and joining of open strings. The only difference is in the normalization. Once it is taken into account we get (65).

Above we have written some very particular amplitudes and we have shown that they roughly agree with our expectations from supergravity. However we can consider general mixing amplitudes for all the open strings. Here it is convenient to isolate the one $Z'$ that is important for the calculation and write it in momentum space, whereas for all the other defects we can write them in position space. The reason for this is that leading planar diagrams require contractions of words that are ordered in the same manner. If we consider two words $W_1$ and $W_2$ made by multiplying mostly $Z$ matrices, and the gauge invariant operators $Q W_1 Q$ and $Q W_2 Q$, these will mix with $Q W_1 (Z Z') W_2 Q$ with $Z'$ inserted everywhere with phases in such a way that only the behavior of these amplitudes near the interaction point matter to determine the off-diagonal terms that change the number of strings. Therefore when the strings split and join, they keep their shape away from the splitting point, and they add a defect at the splitting point. This is exactly what we expect of string theory splitting and joining amplitudes, they should affect the strings only locally on the worldsheet. They might affect slightly the behavior of the strings at the endpoints where they meet because one hopes that the way strings split and join is local on the worldsheet and that it does not reorganize the string completely.

6 Leading SYM calculation of 3 point functions involving at least one closed string and open string 4 point function

Now let us turn to the problem of closed string calculations. For example, we can consider the overlap amplitudes between two string states
\[
\text{tr}(Z' Z^{J_1}) \text{tr}(Z' Z^{J_2})
\]
and
\[
\sum_l \text{tr}(Z' Z^{J_l} Z' Z^{J_{-l}})
\]
We have chosen to include some pertubation states which are not ground states to provide an origin for the closed strings and make it easier to visualize. We could have considered ground states as well.

The strings in equation (66) are normalized with a factor of $(N^{(J_i+1)/2})^{-1}$, and the one in (67) is normalized with a factor of $(\sqrt{J} N^{(J+2)/2})^{-1}$. The non-planar overlap is of the order
\[
\frac{1}{N \sqrt{J}} J^2
\]
The first term comes from normalization, and the extra factors of $J^2$ come from a choice of where to break the two strings to be glued with respect to their origin. Again we see agreement between the overlaps and the expectations of supergravity. Notice however that here the overlaps seem not to care at all about the interactions. This is exactly the same as in the AdS-CFT case, since this calculation is just a limit of the AdS-CFT one, as we saw in section 3.6. We do not need to do any diagonalization of the interacting Hamiltonian to get the mixing terms to be right. This is as puzzling as it is in AdS-CFT, but nonrenormalization theorems ensure the free theory gives the right result. The appropriate diagram to calculate is given by figure 3.
Our intuition is that in truncating from infinite $N$ to finite $N$ we lose the concept of number of closed strings, to a notion that is only approximate. This is roughly the stringy exclusion principle, and is also in accordance with the fact that in Einstein units the gravitational constant of the dimensionally reduced supergravity to $AdS_5$ is $N$. So in some sense the theory is secretly implementing aspects of the interactions without ever giving us the details of those interactions. This is consistent with our calculations of splitting and joining of open strings. We see that we can effectively say that the closed string coupling is the square of the open string coupling because we matched the expectations from supergravity.

Similarly we can do an open plus closed to open calculation. The result will be very similar to the one above (same estimate) (68). The extra factor of $J^2$ in the numerator comes from choosing how far from the end we break the open string, and do we choose to break the closed string. Also the factor of $1/\sqrt{J}$ will come from the normalization of one of the states. This means that the open strings interact with gravity and with closed strings just as expected. The diagram we calculate is given in figure 4b.

Finally, the open string 4-point function is of the same string field theory type as the closed string 3-point function, and should behave similarly. The appropriate diagram is given in figure 4a. There are $J_1$ ways to
break the first open string, and for each we get exactly one SYM diagram satisfying $\delta_{J_1+J_2,J_3+J_4}$. The open string SYM operators are normalized to 1, without factors of $J$, so we get \((52)\).

7 Comments on string field theory and contact terms

String field theory in the light-cone gauge in flat space has vertices that involve more then three strings (see e.g. [41, 42, 43, 44]): 3-open, open-closed; 3-closed, 4-open, 2-open-1-closed. Schematically, the light-cone field action is

\[ L = \Phi^3 + \Phi\Psi + \Psi^3 + \Phi^4 + \Phi^3\Psi + \ldots \] (69)

Although we are in the pp wave background, the general structure of interactions should be the same as in flat space. These vertices are represented pictorially in figure 5. One can see that the vertices are locally of two types: the first two are of string breaking type, and the last three of string exchange type. Putting together all these 5 vertices, one can generate the whole integration region for the S matrix elements (see e.g. [41, 42, 43, 44]), except for singular points (of measure zero). Indeed, Greensite and Klinkhamer [47] and Green and Seiberg [46] have found that one needs to add at least quartic contact terms to the hamiltonian.

They arise either from imposing the susy algebra \(\{Q, Q\} \sim H\), or by cancelling divergencies in S matrices, due to 2 vertices colliding. In the susy algebra, one obtains

\[ \{Q^4_2, Q^4_4\} + \{Q^4_4, Q^4_2\} + \{Q^5_3, Q^5_5\} = 2\delta^{ab}H_4 \] (70)

where the subscript denotes the number of fields. In the S matrix, we get divergencies when two vertex operators collide, and these divergencies can be cancelled by contact terms.

In the string field theory language, the distinction between real vertices and contact vertices is somewhat moot, since they all are geometric in nature. For instance, the 3-open string field vertex in bosonic string
field theory is

$$S_3 = \int d^4 p \delta \left( \sum_{r=1}^{3} p^+ r \right) \int D X_{123} \Phi(3) \Phi(1) \Phi(2) \delta_{123} + h.c. \quad (71)$$

where

$$\delta_{123} = \prod_{r=1}^{3} \delta \left[ X_r(3) - \theta(\pi \alpha_1 - \sigma) X_r(1) - \theta(\pi \alpha_2) X_r(2) \right]$$

$$D X_{123} = D X_1 D X_2 D X_3 \quad (72)$$

and $0 \leq \sigma \leq \pi (\alpha_1 + \alpha_2)$, $\sigma_1 = \sigma$ for $0 \leq \sigma \leq \pi \alpha_1$, $\sigma_2 = \sigma - \pi \alpha_1$ for $\pi \alpha_1 \leq \sigma \leq \pi (\alpha_1 + \alpha_2)$, $\sigma_3 = \pi (\alpha_1 + \alpha_2) - \sigma$ for $0 \leq \sigma \leq \pi (\alpha_1 + \alpha_2)$, $\sum_{i=1}^{3} \alpha_i = 0$.

This vertex can be depicted as in figure 6. Using it one can build more complicated Mandelstam interacting string diagrams [48]. Contact interaction arise when 2 (or more) vertex operators collide. In figure 7

$$\sigma = \pi (\alpha_1 + \alpha_2)$$

$$\sigma = \pi \alpha_1$$

$$\sigma = 0$$

Figure 6: Parametrization of the three-string vertex.

we have depicted the contact terms found in [47, 46]. The terms in fig. 7a and 7b: arise when a string (open or closed) splits and then reconnects with another one at the same time. The corresponding time separated processes shown in (7b) and (7d) are made of usual three-string vertices.

These contact terms were found by requiring the closure of the susy algebra $\{Q, Q\} \sim H$, or in other words boundedness of the energy ($E \geq 0$). It is evident that for a closed string theory, where only the cubic vertex exists, the energy can’t be positive definite, so we can understand in this way the need for a quartic interaction as in (7c). The exception to this is Witten’s cubic open string field theory [49].

We have identified in SYM the regular string field theory vertices. Indeed, the 3-open string and the 3-closed string amplitudes, and the open-closed amplitude will just give the corresponding string field theory vertices. For the open string to open plus closed string amplitude, we could have a contribution from the open string breaking string vertex, followed by an open-closed vertex. But the latter contribution vanishes for supergravity external states, as we saw, whereas the open string to open plus closed string vertex occurs even for external supergravity states. So we can identify it. A similar story holds for the 4 open string amplitude. It could have a contribution from 2 3-open string vertices, but that vanishes for external supergravity states.

The contact terms are also easily found in SYM. They appear when two vertices collide. The SYM free diagrams correspond roughly to the Mandelstam diagrams formed by things like fig. 6 except that we can separate vertices only in $\sigma$, not in $\tau$.

The 4-open-string contact term is shown in fig. 17a and b.

One can see that the diagram can’t be separated into 2 3-open string vertices- they occur at the same point, and moreover this diagram does’t occur because of $ZZ' \rightarrow qq$ mixing in the SYM operator, as for the 3-open string vertices (see section 3). It occurs even for supergravity states (corresponding to BPS operators), which don’t have 3-open string vertex. It is qualitatively different and we need to add it to the string field theory. We notice however by comparing figs. 3a and 17b that the SYM diagram can actually be interpreted as one of the diagrams contributing to the regular 4-point open string interaction. So by this, we
can see already that the contact term is $1/J$ down from a normal 4-point function, and since there is nothing singular happening in SYM. This result comes solely from free field theory. Essentially the contact term diagram will disappear in the limit we are working, namely $\alpha' \mu p \rightarrow \infty$, or in SYM variables, $g_{YM}^2 N/J^2 \rightarrow 0$, interactions become negligible, so there will be no string field theory contact term in this limit. However, we can also see that in the flat space limit ($g_{YM}^2 N/J^2 \rightarrow \infty$), SYM interactions become dominant and a contact term can be generated from summing these diagrams. For the 4-closed contact term, things are similar. In fig. 7c we drew the contact term diagram, to be contrasted with a diagram composed of 2 3-vertices, shown in fig. 7d, where the interactions occur at two different $\sigma$’s. But again the two diagrams are qualitatively different. In the regime we are working, when SYM interactions are negligible, the contact diagram is again subleading in $1/J$ and there is nothing singular about it, so it disappears in the pp wave limit. But again, for the flat space regime $g_{YM}^2 N/J^2 \rightarrow \infty$, there will be new SYM interactions appearing. So then, the contact diagram would be genuinely different, and inclusion of (17d) will probably not imply inclusion of interaction-corrected (17c).

We can also easily see that there will be more contact terms appearing when 3 or more vertex operators collide. For instance, we have shown a 6-closed-string contact term in fig. (17e). Their existence was only conjectured in lightcone string field theory, but from SYM point of view, if there is a 4-string contact term, there will be others as well.

In conclusion, we have seen that all the regular string field theory interactions are obtained, and that the contact terms present in flat space are not present in the pp wave, but could appear in the (hard to analyze) flat space limit.

8 Estimates of corrections

Let us analyze now the non-planar corrections to the results we have already presented. We will look only at closed string corrections (they can appear in the open string amplitudes as well, just that they will not involve the quarks). Therefore, we will use the language of the $N = 4$ SYM theory in this section. We want to show that these corrections are consistent with a spacetime effective action which is a power series in $1/\alpha'$ and in $g$ with appropriate powers of the derivatives of the fields. In other words we want to see that the SYM nonplanar corrections organize in powers of $J^4/N^2 = g^2 (\mu \alpha' \mu p)^4$ and $g_{YM}^2 N/J^2 = 1/(\mu \alpha' \mu p)^2$, since these are the only expansion parameters in the string theory. We will treat them systematically to first nontrivial order ($g_{YM}^2 N/J^2, J^4/N^2$ and $(J^4/N^2)(g_{YM}^2 N/J^2)^2$).

We will see that nonplanar corrections to the 3-string vertex are of the type $g_{YM}^2 N/J^2 = 1/(\mu \alpha' \mu p)^2$. String loop corrections $J^4/N^2 = g^2 (\mu \alpha' \mu p)^4$ will appear first in the normalization of a closed string field,
but they appear as subdiagrams in any string diagram.

Consider first the nonplanar corrections to the 3-string vertex. We want to look at genuine 
\[ g^2_{YM} N/J \]
(1/N^2) nonplanarity, we must use the “nonplanarity” in the free theory (see fig. [3]), which is situated at the string vertex.

So we must put the 
\[ g^2_{YM} \]
interaction (the interaction hamiltonian is of order \( g^2_{YM} \)) at the string breaking point.

If it is on only one side, it gives the usual planar 
\[ g^2_{YM} N/J \]
corrections, not at subdiagrams giving string loop corrections. In order not to create an extra 
\( 1/N^2 \) nonplanarity, we must use the "nonplanarity" in the free theory (see fig. [3]), which is situated at the string vertex.

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\( 1/N^2 \) nonplanarity, we must use the "nonplanarity" in the free theory (see fig. [3]), which is situated at the string vertex.

It is of the type we are looking for.

Let us now study the case where the SYM interaction 4-vertex (\( g^2_{YM} \)) is situated at the string interaction point, 2 neighbouring lines on the \( J_3 \) string interacting and going into \( J_1 \) and \( J_2 \), as in fig. [8]. They cannot be only \( z \) lines, since we know these corrections vanish. Indeed, in the computation of the anomalous dimension, we found that the \( z^4 \) interaction, together with photon exchange and self energy corrections, are all present for BPS operators too, therefore vanish.

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So the vertex has to have at least a \( \phi \) line. Let’s take for concreteness a state with 2 \( \phi \) lines going to 2 states with 1 \( \phi \) line. If the interaction is a \( \phi \) line interacting with a \( z \) line, there are \( J \) possible diagrams, because the \( \phi \) line is fixed to be at the end of the \( J_1 \) string, for instance, but the noninteracting \( \phi \) line can be at \( J \) positions on the \( J_2 \) string (see figure [8b]). However, the vertex \( (\phi z)^2 \) is really a commutator, and the order of the \( \phi \) and \( z \) legs on the \( J_3 \) string can be interchanged, therefore there will be a \( e^{2\pi i j/J_3} - 1 \sim n/J \) suppression (\( n=\)momentum on the \( \phi \) line). This will give a factor of

\[ \frac{g^2_{YM} N J^{3/2}}{N J_{J_3-J_1+J_2}} \]

that is, a \( 1/(\mu \alpha' p)^2 \) correction to the string interaction, as expected.

We can convince ourselves (by hopping the legs of the SYM interaction in all possible ways) that there are no further diagrams at order \( 1/N \). A priori one could have thought that we could have used the 3 independent trace cyclicities of the operators to make further diagrams, but it is not possible.

The only case which remains is when the interaction vertex is a \( \phi^4 \) interaction (see figure [8a]). For concreteness, let’s say there’s a \( \phi^3_n, \phi^3_m \) pair on the \( J_1 \) string going to a \( \phi^4_p, \phi^4_q \) pair (\( n,m,p,q=\)momenta, with \( n+m = p+q \)). We can write down only one such diagram as before, we can convince ourselves that we can’t hop any more lines without reducing further the power of \( N \) (see figure [9]). So this diagram is of order \( 1/J^2 \) different from the free theory. So again we have an \( 1/(\mu \alpha' p)^2 \) correction to the string interaction. But why does this correction vanish for a BPS operator (zero momentum insertions)?

The point is that we could have such an interaction as a subdiagram in the 1 string going to 1 string amplitude, and there we know that the combination of this diagram with photon exchange should cancel for zero momentum (BPS) operators (see figure [10]). In the nonzero momentum, the correction is therefore

\[ e^{2\pi i (n+m)/J} e^{-2\pi i (p+q)/J} - 1 = 0 \] (j= the site where the interaction occurs).
Figure 9: The $J_3$ string has insertions $\phi^3_m, \phi^3_n$ which turn into $\phi^4_p$ on $J_1$ and $\phi^4_q$ on $J_2$.

However, as a subdiagram inside the 3 string amplitude, it is now proportional to $e^{2\pi i (n+m)/J_1} e^{-2\pi i (p+q)/J_3} - 1$ which is now of order 1 for nonzero momentum (not suppressed by $1/J^2$), but it is equal to zero at zero momentum.

![Diagram](image)

Figure 10: In the corresponding 1 string to 1 string amplitude, we have the sum of the diagram (a) and photon exchange (b) and self energy corrections (c) gives a $1/J^2$ suppression.

So indeed we have found that all the order $g^2_{YM}$ corrections to the 3-string vertex sum up to order $g^2_{YM} N/J^2 = 1/(\mu \alpha' p^+)^2$.

Let us now turn to string loop corrections. As we mentioned, the first place they appear is in the normalization of a closed string field, and as such are subdiagrams of all string amplitudes. One could take the point of view that since the 3-string vertex was obtained, the string loop corrections should follow. But it seems a priori not obvious how nonplanar diagrams sum up to the right result, so let us see this explicitly.

When we talk about normalizations, for definiteness we always assume that there is only one extra $\phi$ line situated at site $j=0$, fixing the origin.
The free theory contribution comes from diagrams where we hop m bits situated at site j+l, over k other sites, and then the l bits at site j over k +m sites, as in figure 11. We can check by explicitly (straightforwardly, case by case drawing double line diagrams) that these are the only diagrams at order 1/N^2 (first nonplanarity). Any other thing will introduce extra nonplanarities. By summing over j, k, l, m, we get a total of order J^4 diagrams, so the contribution of these nonplanar diagrams to the amplitude is of order J^4/N^2 = g^2(α′µp^+)^4, as advertised.

![Figure 11: The diagram which gives the first string loop correction to the Yang Mills amplitude. It comes from the first nonplanarity (1/N^2). The diagram is a worldsheet with a handle made of m+l bits (double ribbon). The figure on the right shows that it gives only a factor of 1/N^2, the same contribution as a one ribbon graph.](image)

When we write the action in terms of momenta, we see that this loop correction gives a term in the string effective action of the form

\[ g^2(α′µp^+)^4\phi\phi \]

which is polynomial in derivatives of the fields. According to this calculation the non-planar diagrams will begin to be very important when J^4/N^2 ∼ 1. We have here a normalization in p^+ space, as usual.

Let us now see what happens at the first interaction order, g^2_YM. Let us take a 4-vertex and see what kind of nonplanar generalizations we can make. First, we notice that again we can’t have a 4-z vertex, since together with the other corrections this is zero as a subdiagram in the 2-point function of a BPS operator, therefore is zero as a subdiagram of the 2-point function of any other operator. So we only look at vertices involving a φ at least.

We can hop one leg over k sites. Again, we can also hop m bits over the same k sites, and another l bits before the m can be hopped over k+m sites, as in figure 12. So for each planar diagram, there are of order J^3 nonplanar ones, coming with a factor of 1/N^2. However, now since we hop many sites (of order J), not just 1 as in the planar case, for non-BPS operators we don’t expect a n^2/J^2 suppression anymore. Then we would have a problem, since we seem to get a divergent contribution, of order g^2_YM N(1/N^2)J^3. However, first we still have a [φ, z] commutator on the planar side of the diagram, giving a e^{2πip/J} − 1 ∼ 1/J suppression, (p=momentum of the φ line) while on the nonplanar side we will have the φ line hopping either -l+k or -l-m sites (see figures 13a and b). So when we sum over m,l, k, we will have a sum over (almost) all phases, which will give zero except for a few (of order 1) terms. So we get something like

\[ (\sum_{k,l,m} e^{2πip(k+m)/J})(e^{2πip/J} − 1) ∼ J^3p^2/J^2 \]

which means that in the end the g^2_YM nonplanar contribution is subleading in 1/J:

\[ g^2_YM N^2 = \frac{g^2_YM N J^4}{J^3 N^2} \]

Note however that if there is a different momentum on the top and the bottom of the diagram, there is no 1/J suppression due to the sum over phases, hence these diagrams will contribute to mixing between operators of different momentum.
Figure 12: Maximal nonplanar generalization of a 4 point interaction. The diagram is a worldsheet with a handle made of \(m+l+1\) bits, one of them being a leg of the 4-vertex.

Figure 13: The \(\phi\) line can hop an effective number of sites equal to \(+k+1-l\) (a) or \(-m-l-1\) (b). In both cases, on the planar (lower) side of the diagram, one can still commute the \(\phi\) and \(z\) lines, giving a \(1/J\) suppression.

There are however genuine contributions even to the anomalous dimension (two point function of the same operator), which come from diagrams with \(m+k\) bits situated in between the left 2 legs and the right 2 legs of the 4 point interaction. The \(m+k\) bits are crossed as the corresponding ones in fig. 11. For each planar diagram there are of order \(J^2\) nonplanar diagrams (sum over \(m\) and \(k\)), but there is no extra suppression factor: exactly as in fig. 11, the factor for fixed \(m\) and \(k\) is \(2(1 - \cos(2\pi n(m + k)/J))\), with \(n=\)momentum. The sum over \(m\) and \(k\) doesn’t produce any suppression. So these corrections come with a factor

\[
\frac{g_{YM}^2 N}{J^2} \frac{J^4}{N^2} \tag{77}
\]

Let us look at nonplanar Yang Mills interactions at order \(g_{YM}^4\), and see that they are proportional to \((g_{YM}^4 N^2/J^4)(J^4/N^2) = g_{YM}^4\).

If we first look at the planar diagrams with 2 4-vertices separated by a large number \(k\) of sites, which is the generic situation, and we make one of the vertices nonplanar, for instance as in fig. 12, it is easy to see that the picture carries through. That is, the planar 4-vertex subdiagram is a spectator. For the anomalous dimension, the correction will be of order

\[
\frac{g_{YM}^2 N}{J^3} \frac{J^4}{N^2} \frac{g^2 N}{J^2} \tag{78}
\]

---

In the first version of this paper, we had missed these diagrams, as well as the fact that the diagrams in fig. 12 can contribute to mixing of operators. After that, the paper [50] appeared, which had a correct treatment. The missed diagrams appear in their fig.9
therefore again subleading, as it should, since it is not a true $g_{YM}^4$ contribution, but merely an iteration of the subleading $g_{YM}^2$ contribution. The leading terms are similar.

So let us start with a genuinely $g_{YM}^4$ planar diagram. 2 neighbouring lines interact in a 4 vertex, and one of the resulting lines interacts again with a neighbouring line in another 4-vertex, as shown in figure 14.

![Figure 14: Planar diagram which is genuinely of order $g_{YM}^4$. We will look at its nonplanar generalizations.](image)

We can make the biggest number of nonplanar diagrams by the following: The first interaction occurs at site $j+r$, and the second after another $l+k$ sites. $r$ bits at site $j$ jump over $k+l$ sites, and the $l$ bits at site $j+r+2$ jump over the $k$ sites, as drawn in figure 15. The sum over $j,k,l,r$ gives of the order of $J^4$ diagrams for one planar diagram. Now let’s analyze what kind of lines can be in the original planar diagram.

![Figure 15: The nonplanar generalizations of the planar diagram. It is a worldsheet with a handle made of $r+l$ bits. The Yang-Mills interactions are situated on the handle.](image)

We agreed before that we can’t have $z$ lines only, since then the other contribution (photon corrections and self-energy corrections) would cancel it (and the cancellations remain in effect for the nonplanar generalizations).

The first case we can have is when $z$ line hops over 2 consecutive $\phi$ lines (figure 16a). This diagram has $1/J^2$ from the normalization of the states, $J^4$ diagrams, $1/N^2$ from nonplanarity, $g_{YM}^4 N^2$ from the vertices. There are however also suppression factors from the fact that the $z$ line can hop the $\phi$ line or be on the same side, giving a factor of $1/J$ at each interaction, and again a $1/J$ suppression from the sum over phases, in
\[
\frac{1}{J^2} \frac{J^4}{N^2} g_{YM}^4 N^2 \frac{1}{J^2} \frac{1}{J} = \left( \frac{g_{YM}^4 N^2}{J^4} \right) \frac{J^4}{N^2} \]

therefore is subleading.

\[\text{(a)} \quad \phi_m^3 \phi_n^4 \]
\[\phi \]
\[\phi_p^4 \phi_q^4 \]

Figure 16: In the planar diagram, we can have a z line hopping two \(\phi\) lines (a), a \(\phi\) line hopping two z lines, or a \(\phi^3_n\) interacting with a \(\phi^4_m\) to give two z’s, one of which interacts again to give a \(\phi^4_p, \phi^4_q\) pair (\(m+n=p+q\)).

The next case is when a \(\phi\) line hops 2 z lines (figure 16b). This gives the same contributions, except that the norm is now \(1/J\) instead of \(1/J^2\), and so gives a finite \(1/(\mu \alpha' p)^4\) correction

\[\left( \frac{g_{YM}^4 N^2}{J^4} \right) \frac{J^4}{N^2} \]

The last case is when 2 \(\phi\) lines (say \(\phi^3_m\) and \(\phi^4_n\)) decay into 2 z lines, and one of the z lines interacts with another z line and decays into another 2 \(\phi\) lines (say \(\phi^4_p\) and \(\phi^4_q\), \(n+m=p+q\)). This diagram (figure 16c) has again a norm factor of \(1/J^2\), \(1/N^2\) nonplanarity and a \(g_{YM}^4 N^2\) vertex factor.

However, the corresponding planar diagram would come with a \(1/J^2\) norm, \(J^4\) diagrams, and a \(g_{YM}^4 N^2\) vertex factor. It is conjectured that string interactions do not make this contribution subleading, that is suppress it with \(1/J^2\). We can see that by noticing that the planar diagram comes with a factor of \(e^{2\pi i(n+m)/J} + e^{-2\pi i(n+m)/J}\), and so for the BPS operators (\(n=m=0\)), the momentum independent contributions should cancel it. So in conclusion, these nonplanar contributions should come with

\[\frac{1}{J^2} \frac{J^4}{N^2} g_{YM}^4 N^2 \frac{1}{J^2} = \frac{g_{YM}^4 N^2}{J^4} \frac{J^4}{N^2} \]

and are again \(1/(\mu \alpha' p)^4\) corrections.

In conclusion, in this section we have seen that the SYM interactions organize to first nontrivial order into \(1/(\mu \alpha' p)^2\) corrections for the 3-string vertex, and \(g^2(\alpha' \mu p^+)^4\) and \(g^2(\alpha' \mu p^+)^2\) for the string loop corrections.

9 Discussion and conclusions

In this paper we have analyzed holography and string interactions in the pp wave duality.
We have seen that the Penrose diagram of the maximally supersymmetric pp wave is an Einstein universe with a 1d null line removed, corresponding to the boundary. That boundary is projected in SYM to the time $t$ in the $S^3 \times R$ boundary of global $AdS_5 \times S^5$. Correspondingly, the SYM dimensionally reduced to quantum mechanics was related to a discretized worldsheet string theory.

We have shown also that the 11d wave has a similar (yet more complicated) Penrose diagram, with a 1 dimensional null line (parametrized by $x^+$) as a boundary. That means that the matrix model of [1], with worldline time $t = x^+$, describing the 11d M theory in the pp wave background, should be thought of in some sense as living on the boundary. And correspondingly, the M2 brane and M5 brane theories which are dual to the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds, should in the Penrose limit reduce to the matrix model quantum mechanics, with $t= \text{time in radial quantization}.$

This version of holography suggests that the natural holographic dual of the pp-wave is a matrix model, and differs markedly from other approaches to this question [21, 22, 23], as explained in detail at the end of subsection 3.1. The main insight of our construction is that the pp-wave is focusing on a null geodesic deep inside AdS space, and that the boundary of the pp-wave geometry has no relation to the boundary of AdS.

String interactions in 10 dimensions were introduced by finite time transition amplitudes in the quantum mechanics, connecting two multistring states. In the closed string case, the holography functions in the same way as for AdS-CFT; free SYM amplitudes give the strong coupling string result, without any reference to the SYM interactions. This is as puzzling now as was for AdS-CFT. For the open strings, a more natural situation happens: the string result is reproduced by diagonalizing the SYM interaction hamiltonian.

We have analyzed the action of the Penrose limit on SYM (and supergravity) correlators, and found that only “extremal” correlators survive, and seem to be better defined in the limit. It would be interesting to go back and reanalyze them more carefully from the perspective of the large J limit.

The transition amplitudes analyzed here in principle cover all the vertices of string field theory in flat space, but the calculation is done in the large RR background. One should more properly say that the knowledge of these vertices we calculated should determine all of the higher order contact terms which are probably necessary for a supersymmetric string theory. Here all of the string oscillators are essentially degenerate, and they can be treated at the same level as the zero modes. Namely, most of the physics is related to the one associated to the supergravity states. Along these lines of thought, one can also imagine doing the same type of calculations we did for the case of orbifolds, and to show that the glueing of twisted sectors is consistent with the spacetime expectations.

Contact terms seem to be unimportant in SYM, but they should become important in the flat space limit. Notice that if one is able to completely define string field theory in flat space by this limit, one has a nonperturbative definition of string theory based on SYM, so it would be very interesting to explore the string field theory and the flat space limit further. The nonperturbative definition was already implicit in the usual AdS-CFT correspondence, but now there is a chance to make that statement explicitly calculable.

We have identified 4 regimes in SYM in the Penrose limit, as a function of increasing J. The first corresponds to the flat space limit, the second is the large RR background analyzed in this paper. The third regime is the most puzzling: it corresponds in string theory to a strongly coupled phase, but in SYM it is described by just free nonplanar diagrams, which dominate. Finally, there is the regime of giant gravitons in the pp wave. In SYM, it is described by the domination of free and interacting nonplanar diagrams. The diagonalization of the interacting nonplanar contributions should select giant graviton operators in SYM. It would be very interesting to study these last two regimes further.

In [10] the splitting and joining of string amplitudes were calculated up to a state-independent function of $p^+$, determined at small values of $p^+$ (the supergravity regime where the effective curvature for the string is small), without analyzing the SYM side. It would be very interesting to be able to connect our calculation to the one appearing in [10] by resumming the planar diagrams to all orders, as well as their dressing of the non-planar diagrams to leading order.

While this paper was being written, we learned of the work [51] which has some partial overlap with the present work. They calculate the $J^1/N^2$ free field perturbation series to all orders for two and three point functions.

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Appendix

In this appendix we will look a bit more closely at the case of 3-point functions of open string operators in SYM, and see why they disappear in the Penrose limit. We will compare them with the disappearance of various correlators of closed string operators in the limit.

We will first review some things about the AdS-CFT duality in the \( N = 2 \) orientifold, from [3].

The open string supergravity action in \( AdS_5 \times S_5/Z_2 \) lives in a \( AdS_5 \times S_5 \) subspace (the orientifold 7-plane). In 8 dimensions, we have a vector \( A_M \) and a complex scalar \( \sigma \). By dimensionally reducing on \( AdS_5 \times S_5 \), we get the fields

\[
Z(x, y) = \sum_k Z_k(x)Y^k(y) \\
A_\mu(x, y) = \sum_k A_\mu^k(x)Y^k(y) \\
A_a(x, y) = \sum_k A_a^k(x)Y^k(y) 
\]

where the fields fall into representations of the symmetry group \( SO(4) \times SO(2) \simeq SU(2)_R \times SU(2)_L \times U(1)_R \).

The supermultiplet structure is as follows. The supercharges are in the \( (2, 1, 0) \) representation, coming from \( A_k \).

- the lowest state \( |0> \) is a real scalar in the \((k + 2, k)_0\) representation, coming from \( A_k \).
- at the next level, coming from \( Q^0|0> + \bar{Q}^0|0> \), we have a complex scalar in the \((k, k)_2\) representation, coming from \( Z_k \).
- at the same level, coming from \( QQ|0> \), we have a vector in the \((k, k)_0\) representation, coming from \( A_k^\mu \).
- at the last level, coming from \( Q^2\bar{Q}^2|0> \), we have a real scalar in the \((k - 2, k)_0\) representation, coming from \( A_k \).

In the above \( k \geq 1 \), so the \( k=1 \) representation is shorter. In SYM, the scalars are the fundamental hypermultiplet \( q^A = \sum_{i=1}^4 q_i^{A_i} \) in the \((2, 1)_0\) (and also in the bifundamental of \( SO(8) \times Sp(2N) \)), and the antisymmetric multiplet \( Z^{AB}_k \) in the \((2, 2)_0\), as well as the gauge scalar \( W_{ab} \). They satisfy reality conditions \( q^{A_i} = \epsilon^{AB} \Omega_{ab}(q^i)_{B}^A \)

and \( Z^{AB}_k = \epsilon^{AB} \Omega_{ab} Z^{a}^{\ d}_{b} \). Here \( A, B \) are \( SU(2)_R \) indices, \( A', B' \) are \( SU(2)_L \) indices, \( a, b \) are gauge indices, \( i, j \) are \( SO(8) \) indices. The superpartners of \( q's, \psi_i \) are singlets (in the \((1, 1)_1\) representation), and the superpartners of the \( Z's, \psi_Z \), are in the \((1, 2)_1\) representation.

In terms of the split used in the CFT-pp-wave correspondence, \( (q^i, \bar{q}^i) \) (with \( i=1, \ldots, 4 \)) form an \( SO(8) \) vector multiplet, whereas \( (q, \bar{q}) \) form a \( SU(2)_R \) doublet. So \( q \) and \( \bar{q} \) are related by an \( SO(8) \) rotation together with an \( SU(2)_R \) rotation. Then \( (Z, Z') \) form an \( SU(2)_L \) doublet, whereas \( (\bar{Z}, \bar{Z}') \) form an \( SU(2)_R \) doublet, and \( Z \) and \( \bar{Z} \) are related by a \( SU(2)_L \) rotation followed by an \( SU(2)_R \) rotation.

Let us look at SYM operators and their 3 point functions.

The short representation comes from the lowest state \( qq \) in the antisymmetric representation of \( SO(8) \) (the \( 28 \)) and in the \((3, 1)_0\) of the global symmetry. The vector singlet \((1, 1)_0\) at the next level is the \( SO(8) \) current \( \sim \bar{q}\partial_\mu q + \bar{q}_\mu \gamma_\mu \psi_q \). The 3-point function of the \( SO(8) \) currents is nonzero by the same computation as in [4], since the calculation is in the same \( AdS_5 \) space. And by the AdS-CFT correspondence (and a completely analogous computation as in [4], only the group indices differ), one should get the same result in SYM, already at the one loop (free) level. However, note that the fields in that SYM diagram contract pairwise (there is no contraction where 2 of the 3 operators don’t have contractions among each other).

At the next levels \( k \geq 2 \), the lowest member \(|0>\) is in the \( SU(2)_R \times SU(2)_L \times SO(8) \) multiplet which starts with \( q^i \bar{Z}^{k-1}q'' \). As we mentioned, the \( SO(8) \) rotations turn \( q \) into \( \bar{q} \), \( SU(2)_L \) rotations turn \( Z \) into \( Z' \), and \( SU(2)_R \) rotations turn \( q \) into \( \bar{q} \) and \( Z \) into \( Z' \). The vectors will be obtained by acting with \( QQ \) on the above operator, etc.
If we want to compute 3-point functions of the lowest members, we need to contract pairwise all the 3 operators. We can see that already at the group theory level, since from representations (k+2,k) and (l+2,l) we can form a (k+l+3,k+1+1) representation without contracting the \( SU(2)_R \times SU(2)_L \) indices, but this is not a representation of a lowest member. Let us see this at the level of fields. In the planar limit, the only free diagram comes from contracting \( <qZ^{k-1}\tilde{q}(x)\tilde{q}Z^{l-1}q(y)(qZ^{k+l-2}q)^*(0)> \) in the obvious way (i.e. contract \( \tilde{q}\tilde{q} \) between operators 1 and 2 and the rest planarly between operators 1 and 3 and also between 2 and 3). The piece \( \tilde{q}Z^{l-1}q \) is only one term in the corresponding operator, the other terms being \( \sum qZ^rZ^{l-r-2}q \), but we have emphasized the term which gives a nonzero diagram. Here we have chosen the \( SU(2)_R \times SU(2)_L \) indices in such a way as to be relevant to the pp wave limit, i.e. 2 of the operators are in the pp wave vacuum.

If we would want to compute 3 point functions of vectors, they would have each 2 fermions (and a subleading piece when \( Q \) and \( Q \) act on the same field to give a \( D_\mu \), antisymmetrized with the rest of the fields). We would need to contract these 6 fermions, again pairwise, in the same way as for the \( k=1 \) case (the \( SO(8) \) current). But on top of that, we would have to contract the \( q \)'s as for the scalars above. Note now that on purely group theoretic grounds, one could have contracted the indices only between operators and 3 and also between 2 and 3, since from the representations (k,k) and (l,l) we can construct \( (k+l-1,k+l-1) \) without contracting indices. But the point is that one of the indices in (k,k) is split into two in the operator: there is a fermion \( q_2 \) in the (1,2) and a \( q \) in the (2,1), which together make a (2,2) representation, but which can't be contracted at the free level with a \( Z \) in the (2,2).

So all the nonzero 3-point functions of open string operators involve at least one contraction between operators 1 and 2, namely between the \( q \)'s. Also note that for the 3 point function of vector closed string operators, we have again 2 fermions per operator, so one has again to consider pairwise contractions of the fermions.

What happens when we take the pp wave limit? For the 3 point function of the lowest members of supermultiplets, the correlator \( <qZ^{l_1-1}\tilde{q}(x)\tilde{q}Z^{l_2-1}q(y)(qZ^{l_1+l_2-2}q)^*(0)> \) (here again we have written only the term in operator 2 which contributes to the free diagram) is representative. Extra oscillator insertions will not change the result. Operators 1 and 3 have no factors of \( J \), whereas operator 2 has a \( 1/J \) from the norm (since there are \( J \) terms in the operator), so the correlator is proportional to \( 1/(\sqrt{N}\sqrt{J}) \), that is \( 1/J \) down from the expected sugra result. That means that the correlator dissappears from sugra in the pp wave limit. Similarly, all 3-point correlators of lowest members will become zero.

Moreover, this argument only relied on the necessity of contracting the \( q \)'s, and the fact that a \( \tilde{q} \) insertion is a \( 1/\sqrt{J} \) correction of a \( Z \) insertion. That means that all open string 3-point correlators will be down by at least a \( 1/J \) from the expected sugra result.

How can this be? To gain some insight, let us look at a case where some correlators dissappear, and some remain, namely closed string 3-point functions. Why do correlators where we contract fields between operators 1 and 2 dissapear in the pp wave limit? The simplest example is the correlator \( <Tr(Z'^nZ')\times Tr(Z'^mZ')(y)(Tr(Z'^nZ'^m)^*)> \), where again the contractions are made in the order suggested in the expression, with the neighbouring \( Z' \) and \( Z' \) contracted. The contracted \( Z' \) and \( Z' \) insertions which contribute to this diagram are only 1 term out of \( J \) on each side (because of the the planar limit), so we see that because of the norm factors this correlator is \( 1/J \) down with respect to \( <Tr(Z'^nZ')(y)(Tr(Z'^mZ'^m)^*)> \) (where now there are no contractions between operators 1 and 2). Notice that we couldn't have had a \( Z \) \( - \) \( Z \) contractions between 1 and 2, since \( Z \) insertions are not allowed, hence we were forced to introduce a \( Z' \) impurity on the first operator.

So the first correlator dissappears in the pp wave limit, while the second remains. Notice that this is not because we are contracting only 1 line in between operators 1 and 2. If we contract 2 lines, as in \( <Tr(Z'^2Z'^2)\times Tr(Z'^2Z'^2)(y)(Tr(Z'^nZ'^m)^*)> \) we get a \( 1/J^2 \) suppression, etc. The point is that because of the planarity, we only get one contributing diagram, and that produces the suppression factors. However, this was all in the hypothesis of small number of \( Z' \) insertions at the end of operator 1 ("dilute gas approximation"). If the number of \( Z' \) insertions is comparable with \( J \), we don't get any suppression anymore!

So now we understand in SYM why contractions between operators 1 and 2 are not allowed in the pp wave limit. But how do we see the correlators dissappearing in the pp wave limit on the sugra side? The AdS-CFT correspondence equates the SYM result with the sugra result, so the corresponding sugra amplitudes should dissappear in the pp wave limit. The suppression comes because there are much more \( Z \) fields than \( Z' \) fields.
In other words, the suppression is due to powers of $K/J$, where $K$ is a charge on the sphere other than $J$ ($K$ can be $J'$, for instance). This translates in sugra in powers of $p^i R/p^+ R^2$. The point is that before the pp wave limit, correlators have low $SO(4) = SU(2)_R \times SU(2)_L$ quantum numbers (momenta on the sphere). After the boost, the momentum in the boost direction ($p^+ R^2$) becomes much bigger than the momenta in the other sphere directions (because $J$ goes to infinity, whereas other charges, like $J'$, remain finite). And what used to be dependence of the 3 point functions on the quantum numbers of the sphere now becomes suppression in $p^i R/p^+ R^2$.

Let us examine one more closed string example before going to the open string: the 3 point function of closed string vectors. We saw in SYM that we need to contract operators 1 and 2 to get a nonzero result. But there is a group theoretic argument. In supergravity, the spherical harmonics on the sphere are $V_{\mu}^{[AB]A_1...A_n} = D_\mu Y^{[B} Y^A]Y^{A_1}...Y^{A_n}$, with 2 antisymmetric indices $(A,B)$ and $n+1$ symmetric ones $(A, A_1, ..., A_n)$, and where $Y^A$ are cartesian coordinates on the sphere. We can easily see that we need to contract pairwise at least the $A,B$ indices between the 3 spherical harmonics. We can understand that the sugra 3 point function dissapears in the pp wave limit because this is the KK tower of the vectors associated with the $SO(4)$ symmetry of the sphere. By taking the limit, those symmetries lose their nonabelian nature, only $U(1)^2$ symmetry surviving.

So what happens for the open string? The answer is the same as for the closed string. The $1/J$ suppression is really a $K/J$ suppression. Indeed, the q’s carry different charges than the Z’s, and the suppression appears because there is only one such field ($q$) in the operators. For the 3 point function of lowest component scalars, we saw that we couldn’t match the group theory factors without contracting operators 1 and 2, but for the 3 point function of vectors we could a priori, yet the only nonzero result happens when we contract them. Therefore the sugra 3 point functions vanish in the pp wave limit as $p^i R/p^+ R^2$ also.
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Figure 17: a) SYM diagram corresponding to the 4-open-string contact term drawn in a suggestive way. b) The same diagram drawn in the usual way. Thin lines are adjoint, thick lines are quarks. c) The 4-closed-string contact term. Both interactions occur at the same site (worldsheet $\sigma$ coordinate). We have drawn adjoints as double lines for clarity. d) Corresponding diagram made of two 3-closed-string vertices. e) New 6-closed-string contact term in SYM.