Auxiliary Field Formulation of Supersymmetric Nonlinear Sigma Models

Kiyoshi Higashijima\textsuperscript{a} and Muneto Nitta\textsuperscript{b}

\textsuperscript{a}Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
\textsuperscript{b}Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan

Abstract

Two dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on hermitian symmetric spaces are formulated in terms of the auxiliary superfields. If we eliminate auxiliary vector and chiral superfields, they give D- and F-term constraints to define the target manifolds. The integration over auxiliary vector superfields, which can be performed exactly, is equivalent to the elimination of the auxiliary fields by the use of the classical equations of motion.

\textsuperscript{1}Talk given by M. N. at XXXth International Conference on High Energy Physics (ICHEP 2000), July 27-August 2, 2000, Osaka, Japan

\textsuperscript{*}E-mail: higashij@phys.sci.osaka-u.ac.jp.

\textsuperscript{†}E-mail: nitta@th.phys.titech.ac.jp
1 Introduction

Two dimensional nonlinear sigma models (NLσM) have been interested in, since they have many similarities to four dimensional QCD such as asymptotic freedom, the mass gap, instantons and so on. Non-perturbative analyses of NLσM can be easily done by the large-$N$ method. In the $O(N)$ model, the mass gap appears as non-perturbative effect. In the $CP^{N-1}$ model, a gauge boson is dynamically generated. $\mathcal{N} = 1$ SUSY NLσM (SNLσM) have been also investigated. The $\mathcal{N} = 1$ $O(N)$ model is simply a combination of the bosonic $O(N)$ NLσM and the Gross-Neveu model which shows dynamical chiral symmetry breaking [1].

Along this line it is interesting to discuss non-perturbative analyses of $\mathcal{N} = 2$ SNLσM, since they may have similarities to four dimensional $\mathcal{N} = 1$ QCD. To discuss non-perturbative effects of NLσM by the large-$N$ method, it is necessary to reformulate them by the auxiliary field method. However there was no auxiliary field formulation of $\mathcal{N} = 2$ SNLσM except for the $CP^{N-1}$ and the Grassmann models [2, 3]. In this talk, we formulate $\mathcal{N} = 2$ SNLσM on hermitian symmetric spaces (HSS) $G/H$ (see Table 1) by the auxiliary field method [4, 5, 6]. Since $\mathcal{N} = 2$ SUSY in two dimensions is equivalent to $\mathcal{N} = 1$ SUSY in four dimensions, we use the notation of four dimensions.

Table 1: Hermitian symmetric spaces (HSS).

| Type  | $G/H$                                      | dim$_{\mathbb{C}}(G/H)$ |
|-------|--------------------------------------------|--------------------------|
| AIII1 | $CP^{N-1} = SU(N)/SU(N-1) \times U(1)$     | $N - 1$                  |
| AIII2 | $G_{N,M}(C) = U(N)/U(N - M) \times U(M)$   | $M(N - M)$               |
| BDI   | $Q^{N-2}(C) = SO(N)/SO(N - 2) \times U(1)$| $N - 2$                  |
| CI    | $Sp(N)/U(N)$                               | $\frac{1}{2}N(N + 1)$   |
| DIII  | $SO(2N)/U(N)$                              | $\frac{1}{2}N(N - 1)$   |
| EIII  | $E_6/SO(10) \times U(1)$                   | 16                       |
| EVII  | $E_7/E_6 \times U(1)$                      | 27                       |
2 Auxiliary Field Formulation

2.1 SNLσM without F-term constraints

It was recognized in Ref. [2] that the $\mathbb{C}P^{N-1}$ model can be constructed by an auxiliary vector superfield $V$: Let $\phi$ be dynamical chiral superfields belonging to $N$ of $SU(N)$. Then its Kähler potential can be written as

$$K(\phi, \phi^\dagger, V) = e^V \phi^\dagger \phi - cV,$$

where $cV$ is a Fayet-Iliopoulos (FI) D-term. ($c$ is a positive constant called an FI-parameter.)

This model can be immediately generalized to the Grassmann model, $G_{N,M}(\mathbb{C})$ ($N > M$) by replacing $\phi$ by an $N \times M$ matrix chiral superfield $\Phi$ and $V$ by an $M \times M$ matrix vector superfield $V = V^A T_A$, where $T_A$ are generators of $U(M)$ gauge group:

$$K(\Phi, \Phi^\dagger, V) = \text{tr}(\Phi^\dagger \Phi e^V) - c \text{tr} V.$$

After we integrate out $V$ and fix a gauge in these models, we obtain Kähler potentials of the Fubini-Study metric of $\mathbb{C}P^{N-1}$ and its generalization to the Grassmann manifold.

2.2 SNLσM with F-term constraints

In this section, to obtain the rest of HSS, we introduce auxiliary chiral superfields $\phi_0$ or $\Phi_0$ besides auxiliary vector superfields and construct $G$-invariant superpotentials as summarized in Table 2 [4]. Integration over the auxiliary chiral superfields gives F-term constraints, which are holomorphic.

The simplest example is $Q^{N-2}(\mathbb{C})$, where dynamical fields constitute an $SO(N)$ vector $\phi$. It can be embedded into $\mathbb{C}P^{N-1}$ by an F-term constraint $\phi^2 = 0$. Hence an auxiliary chiral superfield is taken to be a singlet $\phi_0$ and a superpotential to be $W = \phi_0 \phi^2$.

Both $SO(2N)/U(N)$ and $Sp(N)/U(N)$ can be embedded into $G_{2N,N}$ by F-term constraints $\Phi^T J \Phi = 0$, where $\Phi$ is a $2N \times N$ matrix chiral superfield. Here
Table 2: F-term constraints and embedding.

| $G/H$ | $G$-invariants | superpotentials | constraints | embedding |
|-------|----------------|-----------------|-------------|-----------|
| $SO(N)$ | $I_2 = \phi^2$ | $\phi_0 I_2$ | $I_2 = 0$ | $CP^{N-1}$ |
| $SO(N-2) \times U(1)$ | $I_2' = \Phi^T J \Phi$ | $\text{tr} (\Phi_0 I_2')$ | $I_2' = 0$ | $G_{2N,N}$ |
| $SO(2N)$, $Sp(N)$ | $I_3 = \Gamma_{ijk} \phi^i \phi^j \phi^k$ | $\Gamma_{ijk} \phi_0^i \phi^j \phi^k$ | $\partial I_3 = 0$ | $CP^{26}$ |
| $U(N)$ | $I_3 = \Gamma_{ijk} \phi^i \phi^j \phi^k$ | $d_{\alpha \beta \gamma \delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta$ | $\partial I_4 = 0$ | $CP^{55}$ |
| $E_6$ | $I_4 = d_{\alpha \beta \gamma \delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta$ | $d_{\alpha \beta \gamma \delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta$ | | |
| $SO(10) \times U(1)$ | | | | |
| $E_6$ | | | | |

$J = \left( \begin{array}{cc} 0 & 1_N \\ \epsilon 1_N & 0 \end{array} \right)$, where $\epsilon = +1$ (or $-1$) for $SO(2N)$ (or $Sp(N)$). Then auxiliary chiral superfields constitute an $N \times N$ matrix $\Phi_0$ belonging to the symmetric (anti-symmetric) tensor representation of $SO(2N)$ ($Sp(N)$), and the superpotential is $W = \text{tr} (\Phi_0 \Phi^T J \Phi)$.

It is less trivial to find F-term constraints for $E_6$ and $E_7$ models. The F-term constraints are $G$-invariants ($I_2$ or $I_2'$ in Table 2) for the classical groups; on the other hand they are not $G$-invariants but the differentiation of $G$-invariants ($\partial I_3$ or $\partial I_4$ in Table 2) for the exceptional groups. We must introduce auxiliary chiral superfields $\phi_0^i$ and $\phi_0^\alpha$ ($i = 1, \cdots, 27; \alpha = 1, \cdots, 56$) belonging to the fundamental representations of $E_6$ and $E_7$. Superpotentials can be written as $W = \Gamma_{ijk} \phi_0^i \phi^j \phi^k$ and $W = d_{\alpha \beta \gamma \delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta$ for $E_6$ and $E_7$ models, where $\Gamma$ and $d$ are rank-3 and rank-4 symmetric tensors of $E_6$ and $E_7$, respectively. At first sight one might consider the number of the F-term constraints are too large. However some of them are not independent due to the identities of invariant tensors $\Gamma$ and $d$. (Only 10 of 27 equations and 28 of 56 equations are independent for $E_6$ and $E_7$ cases, respectively.)

3 Integration over Auxiliary Fields

The path integration over auxiliary chiral superfields $\phi_0$ or $\Phi_0$ is easy since they are linear in the lagrangian; on the other hand, the path integration over auxiliary vector superfields $V$ is nontrivial. However we can perform integration over $V$ exactly although they are not quadratic [5]. The result for Abelian $V$ is

$$
\int [dV] \exp \left[ i \int d^P x d^4 \theta \left( \phi^i \phi e^V - c V \right) \right] = \exp \left[ i \int d^P x d^4 \theta c \log(\phi^i \phi) \right].
$$

(3)
This coincides with the result obtained by the equation of motion of \( V \). Its coincidence is highly nontrivial since there are infinite number of corrections in bosonic cases. Eq. (3) can be proved by the following theorem.

**Theorem.** Let \( \sigma(x, \theta, \bar{\theta}) \) and \( \Phi(x, \theta, \bar{\theta}) \) be vector superfields and \( W \) be a function of \( \sigma \). Then,

\[
\int [d\sigma] \exp \left[ i \int d^D x d^A \theta \left( \sigma \Phi - W(\sigma) \right) \right] = \exp \left[ i \int d^D x d^A \theta U(\Phi) \right].
\]

Here \( U(\Phi) \) is defined as \( U(\Phi) = \hat{\sigma}(\Phi)\Phi - W(\hat{\sigma}(\Phi)) \), where \( \hat{\sigma} \) is a solution of the stationary equation \( \frac{\partial}{\partial \sigma} (\sigma \Phi - W(\sigma))|_{\sigma=\hat{\sigma}} = \Phi - W'(\hat{\sigma}) = 0 \).

To prove Eq. (3), put \( \Phi = \phi \dagger \phi, \sigma = e^V \) and \( W = c \log \sigma \), and note that \([d\sigma] = [dV]\).

This theorem can be generalized to many variables and to matrix variables, which can be applied to integration over non-Abelian vector superfields. As an application of the theorem, we can show that

\[
\int [dV] \exp \left[ i \int d^D x d^A \theta \left( f(\phi \dagger \phi e^V) - c V \right) \right] = \int [dV] \exp \left[ i \int d^D x d^A \theta \left( \phi \dagger \phi e^V - c V \right) \right],
\]

where \( f \) is an arbitrary function.

### 4 Discussion

Non-perturbative analyses of \( \mathcal{N} = 2 \) SNLSM on HSS are possible, which are in progress [8].

Let us discuss a generalization to an arbitrary Kähler \( G/H \). It is known that \( H \) must be of the form \( H = H_{s,s} \times U(1)^n \), where \( H_{s,s} \) is the semi-simple subgroup of \( H \) and \( n = \text{rank} G - \text{rank} H_{s,s} \), and a Kähler potential has \( n \) free parameters [7].

As seen in Eq. (3), the FI-parameter \( c \) represents a size of \( G/H \) after integration over \( V \). Hence to obtain a Kähler \( G/H \), it is needed to introduce \( n \) FI-terms by considering a gauge group including \( n \) Abelian factors.
Acknowledgements

The work of M. N. is supported in part by JSPS Research Fellowships.

References

[1] E. Witten, Phys. Rev. D16 (1977) 2991; O. Alvarez, Phys. Rev. D17 (1978) 1123.

[2] E. Witten, Nucl. Phys. B149 (1979) 285; A. D’adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B152 (1979) 125.

[3] S. Aoyama, Nuovo Cim. 57A (1980) 176.

[4] K. Higashijima and M. Nitta, Prog. Theor. Phys. 103 (2000) 635, hep-th/9911139.

[5] K. Higashijima and M. Nitta, Prog. Theor. Phys. 103 (2000) 833, hep-th/9911225.

[6] K. Higashijima and M. Nitta, hep-th/0006025, to appear in Proceedings of Confinement 2000.

[7] K. Itoh, T. Kugo and H. Kunitomo, Nucl. Phys. B263 (1986) 295.

[8] K. Higashijima, T. Kimura, M. Nitta and M. Tsuzuki, in preparation.