Ordered Linear Spaces and Categories as Frameworks for Information-Processing Characterizations of Quantum and Classical Theory

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Abstract

The advent of quantum computation and quantum information science has been accompanied by a revival of the project of characterizing quantum and classical theory within a setting significantly more general than both. Part of the motivation is to obtain a clear conceptual understanding of the sources of quantum theory’s greater-than-classical power in areas like cryptography and computation, as well as of the limits it appears to share with classical theory. This line of work suggests supplementing traditional approaches to the axiomatic characterization of quantum mechanics within broader classes of theories, with an approach in which some or all of the axioms concern the information-processing power of the theory.

In this paper, we review some of our recent results (with collaborators) on information processing in an ordered linear spaces framework for probabilistic theories. These include demonstrations that many “inherently quantum” phenomena are in reality quite general characteristics of non-classical theories, quantum or otherwise. As an example, a set of states in such a theory is broadcastable if, and only if, it is contained in a simplex whose vertices are cloneable, and therefore distinguishable by a single measurement. As another example, information that can be obtained about a system in this framework without causing disturbance to the system state, must be inherently classical. We also review results on teleportation protocols in the framework, and the fact that any non-classical theory without entanglement allows exponentially secure bit commitment in this framework. Finally, we sketch some ways of formulating our framework in terms of categories, and in this light consider the relation of our work to that of Abramsky, Coecke, Selinger, Baez and others on information processing and other aspects of theories formulated categorically.
1. Introduction

The advent of quantum information theory has been accompanied by a resurgence of interest in the convex, or ordered linear spaces, framework for operational theories, as researchers seek to understand the nature of information processing in increasingly abstract terms, both in order to illuminate the sources of the difference between the information processing power of quantum versus classical theory, and because quantum information has occasioned renewed interest in foundational aspects of quantum theory, often with the new twist that axioms or principles concerning information processing are considered. A representative (but by no means exhaustive) sample might include the work of Hardy [18, 19], D'Ariano [13], Barrett [8], and others. The general drift of this work is to suggest that it may be possible to characterize quantum mechanics largely or entirely in terms of informational properties, and especially its information processing capabilities. A particularly sharp conjecture advocated by Brassard [12] and Fuchs [16, 17] suggests that QM may be uniquely characterized by three information-theoretic constraints: a no-signaling constraint on compound systems, the impossibility of bit commitment, and the possibility of tamper-evident key distribution using a small amount of authenticated public classical communication.

In order to address such questions, it is necessary to have a clear picture of the information-theoretic properties of probabilistic theories more general than quantum mechanics. Working in a well-established mathematical framework for such theories, in which compact convex sets serve as state spaces, we and our coauthors (Jonathan Barrett and Matthew Leifer) have shown that many “characteristically quantum” phenomena—in particular, many aspects of entanglement, as well as no-cloning and no-broadcasting theorems—are quite generic features of a wide class of non-classical probabilistic theories in which systems are coupled subject to no-signaling and local observability requirements. In [6], Barnum, Dahlsten, Leifer and Toner have shown that the impossibility of a bit-commitment protocol for non-classical state spaces (in the same broad framework) implies the existence of entangled states. Since most ways of combining nonclassical state spaces yield entanglement, this is only a weak constraint on theories. Of course, the impossibility of bit commitment might imply much more about theories, but we do not yet have such results.

This suggests that the impossibility of bit-commitment is in fact not a particularly strong constraint on probabilistic theories. On the other hand, as discussed in [5], the existence of a teleportation protocol is known to be a strong constraint, moving one somewhat closer to quantum theory. Even so, one can construct many models of teleportation—and even of deterministic teleportation—that are neither classical nor quantum. The first part of this contribution gives a concise overview of this work.

In a different direction, with impetus from linear logic and topological quantum field theory, various authors—in particular Abramsky and Coecke [1], Baez [2],
and Selinger [33, 34]—have established that many of the most striking phenomena associated with quantum information processing notably, various forms of teleportation, as well as restrictions on cloning—arise much more generally in any compact closed category, including, for instance, the category of sets and relations. In the second part of this paper, we make a preliminary attempt to relate this approach to ours.

2. General Probabilistic Theories

There is a well-established mathematical framework for generalized probability theory, based on ordered linear spaces, deriving from the work of Mackey [30] in the late 1950s and further developed by Ludwig [26, 27, 28, 29], Davies and Lewis [14], Edwards [15], Holevo [20] and others in the 1960s and 70s. This section gives a whirlwind overview of this framework, focusing on finite-dimensional systems.

2.1 Abstract state spaces

By an abstract state space, we mean a pair \((A, u_A)\) where \(A\) is a finite-dimensional ordered real vector space, with closed, generating positive cone \(A^+_+\), and where \(u_A : A \to \mathbb{R}\) is a distinguished linear functional, called the order unit, that is strictly positive on \(A^+_+ \setminus \{0\}\). A state is normalized iff \(u_A(\alpha) = 1\). (Henceforth, when we say just “cone”, we will mean a positive, generating cone.) We write \(\Omega_A\) for the convex set of normalized states in \(A^+_+\). By way of illustration, if \(A\) is the space \(\mathbb{R}^X\) of real-valued functions on a set \(X\), ordered pointwise on \(X\), with \(u_A(f) = \sum_{x \in X} f(x)\), then \(\Omega_A = \Delta(X)\), the simplex of probability weights on \(X\). If \(A\) is the space \(L(H)\) of hermitian operators on a (finite-dimensional) complex Hilbert space \(H\), with the usual operator ordering, and if the unit is defined by \(u_A(a) = \text{Tr}(a)\), then \(\Omega_A\) is the set of density operators on \(H\). On any abstract state space \(A\), there is a canonical norm (the base norm) such that for \(\alpha \in A^+_+\), \(\|\alpha\| = u_A(\alpha)\). For \(\mathbb{R}^X\), this is the \(\ell^1\)-norm; for \(L(H)\), it is the trace norm \((\|L\| := \text{Tr}\sqrt{L^*L})\).

2.2 Events and Processes

Events (e.g., measurement outcomes) associated with an abstract state space \(A\) are represented by effects, i.e., positive linear functionals \(a \in A^*\), with \(0 \leq a \leq u_A\) in the dual ordering. Note that 0 and \(u_A\) are, by definition, the least and greatest effects. If \(\alpha\) is a normalized state in \(A\)—that is, if \(u_A(\alpha) = 1\)—then we interpret \(a(\alpha)\) as the probability that the event represented by the effect \(f\) will occur if measured. Accordingly, a discrete observable on \(A\) is a list \((a_1, ..., a_n)\) of effects with \(a_1 + a_2 + \cdots + a_n = u_A\), representing a collection of events—the possible outcomes of an experiment, for instance—one of which must certainly occur. We represent a physical process with initial state space \(A\) and final state space \(B\) by a positive mapping \(\tau : A \to B\) such that, for all \(\alpha \in A^+_+\), \(u_B(\tau(\alpha)) \leq u_A(\alpha)\)—equivalently, \(\tau\) is norm-contractive. We can regard \(\|\tau(\alpha)\| = u_B(\tau(\alpha))\) as the probability that the process represented by \(\tau\) takes place in initial state \(\alpha\); this event is represented by the effect \(u_B \circ \tau\) on \(A\).
It is important to note that, in the framework just outlined, the state space $A$ and its dual space $A^*$ have (in general) quite different structures: $A$ is a cone-base space, i.e., an ordered space with a preferred base, $\Omega_A$, for $A_+$, while $A^*$ is an order-unit space, i.e., an ordered space with a preferred element in the interior of its positive cone. Indeed, the spaces $A$ and $A^*$ are generally not even isomorphic as ordered spaces. Where there exists a linear order-isomorphism (that is, a positive linear mapping with positive inverse) between $A$ and $A^*$, we shall say that $A$ is weakly self-dual. Where this isomorphism induces an inner product on $A$ such that $A_+ = \{ b \in A | \langle b, a \rangle \geq 0 \forall a \in A_+ \}$, we say that $A$ is self-dual. Finite dimensional quantum and classical state spaces are self-dual in this sense. A celebrated theorem of Koecher [22] and of Vinberg [37] tells us that if $A$ is an irreducible, finite-dimensional self-dual state space, and if the group of affine automorphisms of $A_+$ acts transitively on the interior of $A_+$, then the space $\Omega_A$ of normalized states is affinely isomorphic to the set of density operators on an $n$-dimensional real, complex or quaternionic Hilbert space, or to a ball, or to the set of $3 \times 3$ trace-one positive semidefinite matrices over the octonions.

2.3 Information and disturbance

With Barrett and Leifer, we have shown (as described in [8]) that in nonclassical theories, the only information that can be obtained about the state without disturbing it is inherently classical information—information about which of a set of irreducible direct summands of the state cone the state lies in. Call a positive map $T : A \rightarrow A$ nondisturbing on state $\omega$ if $T(\omega) = c_\omega \omega$ for some nonnegative constant $c_\omega$ that in principle could depend on the state. Say such a map is nondisturbing if it is nondisturbing on all pure states. A norm-nonincreasing map nondisturbing in this sense is precisely the type of map that can appear associated with some measurement outcome in an operation that, averaged over measurement outcomes, leaves the state (pure or not) unchanged.

A cone $C$ in a vector space $V$ is a direct sum of cones $D$ and $E$ if $D$ and $E$ span disjoint (except for 0) subspaces of $V$, and every element of $C$ is a positive combination of vectors in $D$ and $E$. A cone is irreducible if it is not a nontrivial direct sum of cones. Every finite-dimensional cone is uniquely expressible as a direct sum $C = \oplus_i C_i$ of irreducible cones $C_i$. Information about which of the summands a state is in should be thought of as “inherently classical” information about the state.

**Theorem 1.** The nondisturbing maps on a cone that is a sum $C = \oplus_i C_i$ of irreducible $C_i$, are precisely the maps $M = \sum_i c_i \text{id}_i$, where $\text{id}_i$ is the identity operator on the summand $V_i$ and the zero operator elsewhere, and $c_i$ are arbitrary nonnegative constants.

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1Often termed a base-norm space, when considered as a normed ordered space with the base norm discussed above.

2Of course, this definition permits mixed states to be disturbed by a nondisturbing map—that can be viewed as something like an inevitable “epistemic” disturbance associated with obtaining information.
For a nondisturbing map, $c_\omega$ can depend only on the irreducible component a state is in; thus, the fact that a nondisturbing map has occurred can give us no information about the state within an irreducible component. In other words, as claimed, only inherently classical information is contained in the fact that a nondisturbing map has occurred.

The existence of information that cannot be obtained without disturbance is often taken to be the principle underlying the possibility of quantum key distribution, so the fact that it is generic in nonclassical theories in the framework leads us (with Barrett and Leifer) to conjecture that secure key distribution, given an authenticated public channel, is possible in all nonclassical models.

3. Composite systems and Entanglement

Given two physical systems, represented by abstract state spaces $A$ and $B$, we naturally want to describe composite systems having $A$ and $B$ as subsystems. We make the (non-trivial) assumption that a bipartite state $\omega$ on systems $A$ and $B$ is defined by a joint probability weight $\omega: [0, u_A] \times [0, u_B] \rightarrow \mathbb{R}$.

In other words, we assume that the joint state of two systems be determined by the probabilities assigned local measurements, i.e., measurements pertaining to the two systems separately. Barrett [8] calls this the global state assumption; we follow [5] in calling it the local observability condition. Such a bipartite state is non-signaling iff, for all observables $E$ on $A$,

$$\omega_E(a,b) := \sum_{a \in E} \omega(a,b)$$

is independent of $E$, and similarly for the other component. One can show (see [21, 38]) that $\omega$ is non-signaling iff it extends to a bilinear form on $A_1^* \times A_2^*$.

It is clear that, conversely, any bilinear form $\omega$ that is positive in the sense that $\omega(a,b) \geq 0$ for all $(a,b) \in A_1^* \times B_1^*$, and normalized by $\omega(u_A, u_B)$, defines a state. Thus, we can identify the set of possible bipartite states with the space $B(A^*, B^*)$ of bilinear forms on $A^* \times B^*$, ordered by the cone of positive forms.

For our purposes, it will be convenient to identify the space $B(A^*, B^*)$ with the tensor product $A \otimes B$, interpreting the pure tensor $\alpha \otimes \beta$ of states $\alpha \in A, \beta \in B$ as the form given by $(\alpha \otimes \beta)(f,g) = f(\alpha)g(\beta)$ where $f \in A^*, g \in B^*$. We call a form $\omega \in A \otimes B$ positive iff $\omega(a,b) \geq 0$ for all $(a,b) \in A_1^* \times B_1^*$. If $\omega$ is positive and $\omega(u_A, u_B) = 1$, then $\omega(a,b)$ can be interpreted as a joint probability for effects $a \in A^*$ and $b \in B^*$. Thus, the most general model of a composite of $A$ and $B$ consistent with our definition of a joint state and the no-signaling requirement, is the space $A \otimes B$, ordered by the cone of all positive forms, and with order unit given by $u_A \otimes u_B : \omega \mapsto \omega(u_A, u_B)$. This gives us an abstract state space, which

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3The condition is violated by both real and quaternionic quantum mechanics [21].
we term the maximal tensor product of $A$ and $B$, and denote $A \otimes_{\text{max}} B$. At the
other extreme, we might wish to allow only product states $\alpha \otimes \beta$, and mixtures of
these, to count as bipartite (normalized) states. This gives us the minimal tensor product,
$A \otimes_{\text{min}} B$. These coincide if either factor is classical, that is, if $\Omega_A$ or $\Omega_B$
is a simplex \cite{32}; in general, however, the maximal tensor product allows many
more states than the minimal. A state in $\Omega_{A \otimes_{\text{max}} B}$ not belonging to $\Omega_{A \otimes_{\text{min}} B}$ is
entangled.

More generally, we define a composite of $A$ and $B$ to be any state space $AB$
consisting of bilinear forms on $A^* \times B^*$, ordered by a cone $AB_+$ of positive forms
containing every product state $\alpha \otimes \beta$, where $\alpha \in \Omega_A$ and $\beta \in \Omega_B$. Equivalently,
$AB$ is a composite iff $A \otimes_{\text{min}} B \leq AB \leq A \otimes_{\text{max}} B$ (where, for abstract state spaces
$A$ and $B$, $A \leq B$ means that $A$ is a subspace of $B$, that $A_+ \subseteq B_+$, and that $u_A$
is the restriction of $u_B$ to $A$). More generally still, a composite of $n$ state spaces
$A_1, ..., A_n$ is a state space $A$ of $n$-linear forms on $A_1^* \times \cdots \times A_n^*$, ordered by any
cone of positive forms containing all product states. In such a composite, define
the conditional state space of a subset $J$ of the parts as the set of states obtainable
by conditioning on all product effects of parts not in $J$. Following \cite{3} we call such
a composite regular if all of its conditional state spaces are also composites.

We’ll call the formalism outlined above, in which the unnormalized effects on a
system are the full dual cone of the state cone, composites are non-signalling,
contain all product states, and satisfy local observability, simply “the framework”
in order to avoid repetition of this list of assumptions. It is the framework used,
for the most part, in \cite{1, 3, 6}, which are the sources of most of the results
we outline below. Barrett’s framework \cite{8} is essentially the same except it does
not require the effects to be the full dual cone.

4. Cloning, Broadcasting, Bit Commitment and Teleportation

Many of the most celebrated results of quantum information theory turn out to
have much more general formulations in terms of abstract state spaces. An example,
first pointed out by Barrett \cite{8}, is that the impossibility of universal cloning,
far from being a specifically quantum phenomenon, is generically non-classical,
in the sense that it is a feature of all probabilistic theories involving non-classical
state spaces. In this section, we survey recent work in this direction.

4.1 No-cloning and no-broadcasting

Let $AA$ be any composite of $A$ with (a copy of) itself, and let $\varphi : A \to AA$ be any
positive, norm-preserving mapping. We say that $\varphi$ clones a set $\Gamma$ of (normalized)
states iff $\varphi(\alpha) = \alpha \otimes \alpha$ for all $\alpha \in \Gamma$, and that $\varphi$ broadcasts $\Gamma$
iff $\varphi(\alpha)_A = \varphi(\alpha)_B = \alpha$ for all $\alpha \in \Gamma$. In \cite{1, 3}, it is shown that (i) $\Gamma$ is broadcastable iff it
is contained in the convex hull of a set of clonable states, and (ii) $\Gamma$ is clonable
iff jointly distinguishable. The standard quantum no-broadcasting theorem is an
easy corollary (and, indeed, this provides the easiest known proof of the latter). In
fact, \cite{3} shows slightly more: for any positive map, the set of states it broadcasts
is precisely such a simplex generated by distinguishable states.

4.2 Bit commitment

Quantum theory has mixed states whose representation as a convex combination of pure states is not unique. So do all nonclassical theories: uniqueness of the decomposition of mixed states into pure states is an easy characterization—sometimes used as a definition—of simplices (see, for example, the proof in [8]). While we are not aware of any quantum information processing task whose possibility is directly traced to the non-unique decomposability of mixed states into pure, this was certainly proposed as a possible basis for quantum bit commitment schemes, though (as shown in [10] for their proposed scheme, and in [31, 25] for more elaborate schemes) these schemes do not work because of entanglement.

Bit commitment is an important cryptographic primitive in which one party (“Alice”) can perform an act that commits her, vis-a-vis a partner (“Bob”) to the value of a bit in such a way that she is able, at will, to reveal the committed bit value to Bob and have him accept it (perhaps on the basis of some tests he performs) as genuine. The protocol is binding: once committed to a bit value, Alice will not be able to get Bob to accept the other value as the revealed bit. It is also hiding: once Alice has committed, Bob knows she has committed, but knows nothing about the value of the bit. Information-theoretically secure bit commitment (applicable even to parties with unlimited computational power) is impossible in classical probability theory. In [6] it is shown that the existence of bit commitment protocols is universal in nonclassical theories in the framework, provided that the tensor products used do not permit entanglement. Consider theories generated by a finite set $\Sigma$ of “elementary” systems modeled by convex sets $\Omega, \Gamma \ldots$ in finite dimensions, containing at least one nonsimplex. Let it be closed under the minimal, or separable, tensor product, which we write with the ordinary tensor product symbol $\otimes$.

The protocol. Let a system have a non-simplicial, convex, compact state space $\Omega$ of dimension $d$, embedded as the base of a cone of unnormalized states in a vector space $V$ of dimension $d + 1$. The protocol uses a state $\mu$ that has two distinct decompositions into finite disjoint sets $\{\mu_i^0\}, \{\mu_j^1\}$ of exposed states, that is,

$$\omega = \sum_{i=1}^{N_0} p^0_i \mu_i^0 = \sum_{j=1}^{N_1} p^1_j \mu_j^1,$$

(1)

A state $\mu_i^b$ is exposed if there is a measurement outcome $a^b_i$ that has probability 1 when, and only when, the state is $\mu_i^b$; the protocol exists for all nonclassical systems because, as we show, any non-simplicial convex set of affine dimension $d$ always has a state $\omega$ with two decompositions (as above), into disjoint set of states whose total number $N_0 + N_1$ is $d + 1$ The disjointness and the bound on cardinality are used in the proof of exponential security given in [6].

In the honest protocol, Alice first decides on a bit $b \in \{0,1\}$ to commit to. She
then draws \(n\) samples from \(p^b\), obtaining a string \(\mathbf{x} = (x_1, x_2, \ldots, x_n)\). To commit, she sends the state \(\mu^b_x = \mu^{b}_{x_1} \otimes \mu^{b}_{x_2} \otimes \ldots \otimes \mu^{b}_{x_n}\) to Bob. To reveal the bit, she sends \(b\) and \(\mathbf{x}\) to Bob. Bob measures each subsystem of the state he has. On the \(k\)-th subsystem, he performs a measurement, (which will depend on \(b\)) containing the distinguishing effect for \(\mu^b_x\) and rejects if the result is not the distinguishing effect. If he obtains the appropriate distinguishing effect for every system, he accepts. The protocol is perfectly sound (if Alice and Bob are honest, Bob never accuses her of cheating and always obtains the correct bit), perfectly hiding (if Alice is honest, Bob cannot gain any information about the bit until Alice reveals it), and has an exponentially low probability, in \(n\), of Alice’s successfully cheating (is exponentially binding).

### 4.3 Conditioning and teleportation protocols

In the standard quantum teleportation protocol [11], Alice and Bob share an entangled state; Alice possesses an ancillary system in an unknown state. By making a suitable entangled measurement on her total system, and instructing Bob to make suitable unitary corrections on his wing of the shared system, Alice can guarantee that the final state of Bob’s wing is identical to the unknown initial state of her ancilla (which, in compliance with the no-cloning theorem, is irrevocably altered).

In recent work [5] with Jonathan Barrett and Matthew Leifer, we consider what such a protocol looks like in the setting of abstract state spaces. We find that teleportation is possible in a much broader class of probability theories than just quantum and classical theory. However, unlike the no-cloning and no-broadcasting theorems, which are generically non-classical (at least in finite dimensions), the existence of a teleportation protocol imposes a real constraint on nonclassical theories.

If \(AB\) is a composite of state spaces \(A\) and \(B\), we can define for any normalized state \(\omega \in AB_+\) and any effect \(a \in A\), both a marginal state \(\omega_A(\cdot) = \omega(\cdot, u_B)\) and a conditional state \(\omega^B_a\) defined by the condition \(\omega^B_a(b) = \omega(a,b)/\omega_A(a)\) (with the usual proviso that if \(\omega_A(a) = 0\), the conditional state is also 0). We shall also refer to the partially evaluated state \(\omega_B(a) := \omega(a,\cdot)\) as an un-normalized conditional state.

Note that any state \(\omega \in AB\) gives rise to a positive operator \(\hat{\omega} : A^* \to B\), given by \(\hat{\omega}(a)(b) = \omega(a,b)\). That is, \(\hat{\omega}(a)\) is the “un-normalized” conditional state corresponding to conditioning on \(a\). As a partial converse, any positive operator \(\psi : A^* \to B\) with \(\psi(u_A) \in \Omega_B\)—that is, with \(\psi^*(u_B) := u_B \circ \psi = u_A\)—corresponds to a state in the maximal tensor product \(A \otimes_{\text{max}} B\). Dually, any effect \(f \in (AB)^*\) yields an operator \(\hat{f} : A \to B^*\), given by \(\hat{f}(\alpha)(\beta) = f(\alpha \otimes _B \beta)\); and any positive operator \(\varphi : A \to B^*\) with \(\varphi(\alpha) \leq u_B\) for all \(\alpha \in \Omega_A\)—that is, with \(\|\varphi\| \leq 1\)—corresponds to an effect in \((A \otimes_{\text{min}} B)^*\).

Suppose now that \(f\) is an effect in \((A \otimes_{\text{min}} B)^*\) and \(\omega\) is a state in \(B \otimes_{\text{max}} C\). Then,
for any $\alpha \in A$, it is not difficult to check that
\[
(\alpha \otimes \omega)^C_f = \hat{\omega}(\hat{f}(\alpha))\|\hat{\omega} \circ \hat{f}(\alpha)\|.
\] (2)

Notice that, in consequence, if $c$ is any effect in $C^*$, $f \otimes c$ is positive on product states of the form $\alpha \otimes \omega$, with $\alpha \in A$ and $\omega \in B \otimes_{\max} C$, and hence, defines an effect in $A \otimes_{\min} (B \otimes_{\max} C)$.

Put differently, this shows that $(A^* \otimes_{\max} B^*) \otimes_{\min} C^*$ is an ordered subspace of $(A \otimes_{\min} (B \otimes_{\max} C))^*$. This allows us to interpret equation (2) as follows: if the tripartite system $ABC = A \otimes_{\min} (B \otimes_{\max} C)$ is in a state $\alpha \otimes \omega$, with $\alpha$ unknown, then conditional on securing measurement outcome $f$ in a measurement on $A \otimes_{\max} B$, the state of $C$ is, up to normalization, a known function of $\alpha$. This is very like a teleportation protocol. Indeed, suppose that $C$ is a copy of $A$, and that $\eta : A \rightarrow C$ is a specified isomorphism allowing us to match up states in the former with those in the latter:

**Definition 2.** With notation as above, $(f, \omega)$ is a (one-outcome, post-selected) teleportation protocol iff there exists a positive, norm-contractive correction map $\tau : C \rightarrow C$ such that, for all $\alpha \in A$, $\tau(\alpha \otimes \omega)^C_f = \eta(\alpha)$\(^4\)

If $(f, \omega)$ is a teleportation protocol, the un-normalized conditional state of $\alpha \otimes \omega$ is exactly $\hat{\omega}(\hat{f}(\alpha))$. If we let $\mu := \hat{\omega} \circ \hat{f}$, the normalized conditional state can be written as $\hat{\mu}(\alpha)/u(\hat{\mu}(\alpha))$. Thus, $(f, \omega)$ is a teleportation protocol iff there exists a norm-contractive mapping $\tau$ with $(\tau \circ \mu)(\alpha) = \|\mu(\alpha)\| \eta$ for all $\alpha \in \Omega_A$. $\|\mu(\alpha)\| > 0$ (which is the probability of getting measurement outcome $f \otimes u_C$ on state $\alpha \otimes \omega$) is the probability that the teleportation protocol succeeds on state $\alpha$.

Henceforth, we simply identify $C$ with $A$, suppressing $\eta$. We say a composite $ABA$ supports a conclusive teleportation protocol if such a protocol $(f, \omega)$ exists with $f, \omega$ allowed states and effects of the composite. Note that if $(f, \omega)$ is a teleportation protocol on a regular composite [AW: “regular composite” not yet defined...?] $ABA$ of $A$, $B$ and (a copy of) $A$, then, as $f$ lives in $(AB)^* \leq (A \otimes_{\min} B)^*$ and $\omega$ lives in $BA \leq B \otimes_{\max} C$, one can also regard $(f, \omega)$ as a teleportation protocol on $A \otimes_{\min} (B \otimes_{\max} A)$.

**Theorem 3 (5).** $A \otimes_{\min} (B \otimes_{\max} A)$ supports a conclusive teleportation protocol iff $A$ is order-isomorphic to the range of a compression (a positive idempotent mapping) $P : B^* \rightarrow B^*$.

**Corollary 4.** If $B$ is order-isomorphic to $A^*$ then $A \otimes_{\min} (B \otimes_{\max} A)$ supports a conclusive teleportation protocol.

**Corollary 5.** If $A$ can be teleported through a copy of itself, then $A$ is weakly self-dual.

\(^4\)One could also allow protocols in which the correction has a nonzero probability to fail. For details, see [5].
In order to deterministically teleport an unknown state $\alpha \in A$ through $B$, we need not just one entangled effect $f$, but an entire observable’s worth.

**Definition 6.** A deterministic teleportation protocol for $A$ through $B$ consists of an observable $E = (f_1, \ldots, f_n)$ on $A \otimes B$ and a state $\omega$ in $B \otimes A$, such that for all $i = 1, \ldots, n$, the operator $\hat{f}_i \circ \hat{\omega}$ is physically invertible.

By physically invertible, we mean its inverse is a norm-contractive positive map.

The following result provides a sufficient condition (satisfied, e.g., by any state space $A$ with $\Omega_A$ a regular polygon) for such a protocol to exist.

**Theorem 7** ([5]). Let $A = B$. Suppose that $G$ is a finite group acting transitively on the pure states of $A$, and let $\omega$ be a state such that $\hat{\omega}$ is a $G$-equivariant isomorphism. For all $g \in G$, let $f_g \in (A \otimes_{\text{max}} A)^*$ correspond to the operator

$$\hat{f}_g = \frac{1}{|G|} \hat{\omega}^{-1} \circ g.$$ 

Then $E = \{f_g | g \in G\}$ is an observable, and $(E, \omega)$ is a deterministic teleportation protocol.

**5. Categorical considerations**

In this section we briefly sketch one way of relating the above-described convex framework to the category-theoretic framework for theories developed by Abramsky and Coecke and by Selinger. If an abstract state space and its dual “effect space” provide an abstract probabilistic model, one would like to say that a probabilistic theory is a class of such models, perhaps closed under appropriate operations producing models of composite systems from models of their components. Each such model should be equipped with a dynamical semigroup of allowed evolutions, which must be positive maps. A natural way of formalizing such an approach is to say that a theory is a category whose objects are abstract state spaces and whose morphisms are positive linear maps between these spaces, composed as usual.

Such a category will in addition have a “unit object”, referred to as $I$, consisting of $\mathbb{R}$ understood as a one-dimensional vector space over itself, ordered by its usual (and in fact unique) positive cone $\mathbb{R}_+$, with order unit equal to the identity function from $\mathbb{R}$ to $\mathbb{R}$. Since the morphisms from, say, $A$ to $B$ live in the real vector space $\mathcal{L}(A, B)$ of linear maps from $A$ to $B$, we can and will require $\text{Hom}(A, B)$ to itself be a (pointed, generating) positive cone. We will require that $\text{Hom}(I, A)$ be isomorphic to $A_+$. The isomorphism $\eta : A_+ \to \text{Hom}(I, A); \omega \mapsto f_\omega$ is taken to satisfy $f_\omega(1) = \omega$. That is, the states of $A$ can be viewed as morphisms from the unit to $A$. Since (not-necessarily normalized) effects are positive maps from $A$ to $I$, all effects are potentially elements of $\text{Hom}(I, A)$. We will require $\text{Hom}(I, A)$ to contain $u_A$ in its interior; thus, the order-unit structure for each object specifies a distinguished morphism $u_A \in \text{intHom}(A, I)$. This formalism easily accommodates one
small but important divergence from the framework described above: by requiring
all Hom-sets to be positive cones, but not necessarily requiring every positive linear
functional to belong to Hom(A, I), we have not enforced the assumption we made
above, that the cone of unnormalized effects was the full dual cone of the cone of
states. (Rather, our formalism implies only that it is a subcone of the dual cone.)
We’ll call an object for which Hom(A, I) is the cone dual to Hom(I, A) saturated.
We will call a theory in which all objects are saturated locally saturated. The ability
to formulate theories that are not locally saturated is crucial for dealing with,
for example, convex versions of Rob Spekkens’ theory of toy bits [36]. Relaxing
the assumption of local saturation likely has nontrivial implications for informatic
phenomena.

The interpretation of such a category is somewhat loose, but we may think of
the morphisms as processes that a system may undergo. This is essentially the
Abramsky-Coecke point of view on categorically formulated theories. Since mor-
phisms may connect objects with different objects, this allows a process to change
the nature of a system, to one described by a different object. Whether this is an
actual physical change in the nature of a system, or whether we want to regard
it as a change in point of view, is a somewhat delicate point of interpretation.
It could represent, for example, discarding or disregarding part of a system, or
cobbling up a new system and combining it with the old one, or receiving a new
system delivered from some other agent. We’ll remain fairly agnostic on this point,
as various interpretations may be useful for various applications. In our setting,
the norm-contractive morphisms are the operationally relevant ones, corresponding
to something that can actually be realized in the system being modelled by the
formalism; the other ones are merely mathematically convenient to include in the
formalism.

In particular, the morphisms \( f_\omega : I \to A \) may be viewed as “bringing up a new
system, \( A \), prepared in state \( \omega \)”\). The morphisms \( A \to I \) are effects. Let \( g \) be such
an effect; \( g(\omega) \), the probability of observing effect \( g \) when \( A \) is prepared in state
\( \omega \), may be represented as \( g(f_\omega(1)) \), or if one prefers, \( g \circ f_\omega(1) \). More generally,
for any state \( \omega \) of \( A \), and process \( \varphi : A \to B \), the probability of the state undergoing
the process is calculated by applying the order unit, i.e. as \( u_B \circ \varphi \circ f_\omega(1) \). (Strictly
speaking, these are not guaranteed to be probabilities unless all morphisms involved
are norm-contractive.) The structure of a category does several useful things for
us. Since morphisms are positive, and the composition of morphisms must be a
morphism, it enforces that we will never get a negative number out of any chain
of morphisms \( I \to I \). In other words, no process constructed (in any way) from
other processes will ever have negative probability. Similarly, processes composed
out of norm-contractive processes will have probabilities bounded above by 1.

The interpretation also explains the meaning of our assumption that all Hom-sets
are cones: operationally, it means that for any two processes, there is another
process consisting of doing one or the other of the first two processes, conditional
on the outcome of a “coin-flip” (where the coin may be chosen to have arbitrary
We call such a category of positive maps *saturated* if there is no way to enlarge it by adding positive maps to some $\text{Hom}(A, B)$ (while keeping all the morphisms we started with). Note that a category may be saturated without being locally saturated, and vice versa. We call a category *locally Hom-saturated at $(A, B)$* if the subcategory whose objects are $A, B, I$ and whose Hom-sets are those of the original category, is saturated, and *locally Hom-saturated* if it is Hom-saturated at every pair of objects. The category whose objects are finite-dimensional mixed quantum state spaces and whose morphisms are completely positive maps, for example, is locally saturated, but neither locally Hom-saturated nor saturated. There are examples of categories that are locally saturated, Hom-saturated, and saturated: for example, the categories generated by taking a fixed set of objects and closing under coupling the state spaces $\text{Hom}(I, A)$ by the minimal tensor product, and determining the remaining Hom-sets by local saturation and local Hom-saturation. A similar construction, but closing under the maximal tensor product, also exhibits all three properties.

Various kinds of structure may be added to our categories. Here, we consider structures representing one object’s being a composite of other objects. In [5] we introduced a very general notion of multipartite composite, though not in an explicitly category-theoretic framework. We now briefly consider a specific way of modeling compositeness in a category of positive maps, with a view to bringing out connections with the work of Abramsky and Coecke and of Selinger. Namely, we consider such categories that are, in addition, *monoidal*, with monoidal tensor written $\otimes$. Our first order of business is to compare $A \otimes B$ with the notion of composite introduced above. The bifunctoriality of $\otimes$ implies that $\text{Hom}(A \otimes B, C \otimes D)$ contains $\text{Hom}(A, C) \times \text{Hom}(B, D)$ where elements $(\alpha, \beta)$ of this Cartesian product are written $\alpha \otimes \beta$, and turn out to be just the usual tensored pairs of linear maps. This implies that the composite state space includes all product states, and the composite effect space $\text{Hom}(A \otimes B, I)$ contains all product effects. Hence the states of the composite must be positive on all product effects, so we have two of the properties we required of a bipartite composite in the framework of earlier sections. More precisely, we have them for saturated objects, hence for all objects in a locally saturated theory; the statement in the previous sentence is the natural generalization of the two requirements to theories whose objects are not necessarily saturated.

However, the third condition on composites in the above framework, that of *local observability*, does not necessarily hold in such a category. Local observability, though extremely natural and still permitting an extraordinarily wide range of composite systems and of informatic phenomena, is an assumption with relatively substantive implications. For example, it makes bit commitment significantly harder, by ruling out “intrinsically nonlocal” degrees of freedom. Allowing the latter permits theories [7] which effectively have bipartite “lockbox-key pairs” along the lines suggested by John Smolin [35] in which a bit may be put by Alice
and “stranded” between Alice and Bob when she sends Bob one of the systems as her commitment.

As remarked above for the case of quantum systems, the category, which we’ll call $C^*\text{CPOS}$, of $C^*$-algebra state spaces and completely positive maps, is not saturated. We can enlarge it by adding maps $\varphi : A \to B$ that are positive but not completely positive, as long as we do not add the maps $\varphi \otimes \text{id} : A \otimes C \to B \otimes C$ except in the trivial case $C = I$. In the category $C^*\text{CPOS}$, one can introduce a natural symmetric monoidal structure, which in the case of quantum systems just gives us the standard tensor product. If we attempt to add positive but not completely positive maps while maintaining monoidality, we fail, for we are forced to include all maps $f \otimes \text{id}$, and these can fail to be positive. We call a monoidal category of positive maps saturated if one cannot enlarge it by adding positive maps, while maintaining monoidality; we’ll usually say it’s monoidally saturated to avoid confusion in cases, such as $C^*\text{CPOS}$, which are not saturated as categories, but are saturated as monoidal categories.

In the case where local observability happens to hold, the associativity and bifunctoriality of $\otimes$ together imply that multipartite composites are regular in the sense of [5] and the preceding section. It should not be hard to generalize the notion of regularity to systems lacking local observability, and we expect the appropriate generalization to hold for all monoidal categories of positive maps as defined above. There are very natural monoidal categories of abstract state spaces and positive maps for which local observability fails. The category whose objects are spaces of real symmetric $n \times n$ matrices ordered by their positive semidefinite cones and with order unit given by the trace, and whose morphisms are completely positive maps between such spaces, is a case in point. This theory is, of course, just the finite-dimensional mixed-state version of real (as opposed to complex) quantum mechanics. Here if $A_+$ is the unnormalized density matrices (i.e. the positive semidefinite matrices) on real $n$-dimensional Hilbert space $H_A$, and $B$ is the same for $H_B$, then $A \otimes B$ is the vector space of real symmetric matrices on $H_A \otimes H_B$, which is higher dimensional than the tensor product of $A$ and $B$ as vector spaces. Nevertheless, even in this case, the projection onto the tensor product of the underlying vector spaces is a composite in the sense of our earlier framework. We suspect that using this composite as $A \otimes B$ would still give a symmetric monoidal category. However, in this case there is a compelling reason for going beyond it: to preserve the representation of the state space as the set of density matrices on a real Hilbert space. This has nice properties like self-duality and homogeneity of the state space; preserving such properties is a more abstract motivation, probably applicable to this case, for sometimes going beyond local observability.

6. Conclusion

We have reviewed some of our recent work, with various sets of collaborators, on information processing and informatic phenomena in the convex operational frame-

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5One can add all the positive maps if one likes, obtaining a category we’ll call $C^*\text{POS}$. 
work for theories, and outlined a way of connecting it to the category-theoretic framework. The information-processing results fall into two basic classes.

First, there are results which elucidate the distinction between classical and nonclassical information, thereby elucidating the distinction between classical and nonclassical theories, as we can characterize classical theories as ones in which all information is classical. From these results, two types of classical information emerge within general theories in the framework. First, there is *intrinsically* classical information. Emerging as the condition for information to be extractable from a system without disturbing it, this is information that tells us which summand in a direct-sum decomposition of the cone of states the state is in. It is intrinsically classical in the sense that when a cone has a direct sum decomposition, by definition every extremal ray is in one summand or the other; this is to say that no pure state of the theory is a *superposition* of, and no mixed state exhibits *coherence* between, states in different summands. The notion of superposition in the convex framework was investigated in \cite{9, 23, 24}, and we will explore it and our mixed-state extension of the notion to that of *coherence*, in more depth elsewhere. Second, there is a notion of information that might be called *facultatively classical*, corresponding to a *classical substructure* in a model. This is information about which of a set of perfectly distinguishable states of the theory the system are in. Intrinsically classical information is of course also facultatively classical, but not vice versa: the theory may allow coherence between perfectly distinguishable states, and then distinguishing the states will disturb other states, including some pure ones, that have coherence between the distinguishable states. But the information about which state we have within a facultatively classical set of states—a simplex within the convex set of states, whose vertices are perfectly distinguishable states—*can* be gathered without disturbing *those states* (i.e., the vertices will not be disturbed), and similarly this kind of information can be broadcast.

Second, there are results that establish informatic properties of broad classes of theories, but do not just provide a way of demarcating classical from non-classical theories. These include very general and simple results linking to well-known properties of composites such as entanglement (which generalizes easily to this framework), such as the result that nonclassical theories without entanglement support exponentially secure bit commitment. They also include results that bring out the importance of new properties of theories, or at least ones that have been less stressed in the quantum literature, such as weak self-duality of state spaces, and the existence of states and maps that implement that self-duality, which is sufficient for teleportation. Our study of teleportation, besides yielding interesting necessary and sufficient conditions for conclusive teleportation in a three-part composite, has also turned up nice sufficient conditions for *deterministic teleportation*, relating the existence of a high degree of symmetry of the state space—a transitive group of automorphisms—and, again, appropriate bipartite states—to teleportation. We expect that further investigation of which classes of theories permit bit commitment, will bear the same sort of conceptual fruit.
Finally, we related this convex framework to the categorial framework in which much recent work has explored quantum theory and protocols—and increasingly, non-quantum foil theories as well. As we have implemented it, the framework treats systems that may be built up from multiple subsystems, or at any rate analyzable in terms of subsystem structure—using notions of \textit{composite} or \textit{regular composite} that are somewhat looser, structurally, than the symmetric monoidal structure usually used in the category-theoretic approach. But formulating convex theories as categories of positive maps, as we do in the last section, is very natural and provides a natural way to relax two assumptions that are usually present in our use of the convex framework but which are substantive and which, as we have noted, it will be sometimes desirable to relax. These are that for each system the effect cone is the full dual of the state cone (“local saturation”), and that the probabilities of product effects determine the state (“local observability”). It also shows how the convex approach can provide a wealth of concrete examples of theories that may or may not have a monoidal structure, but that are in either case much more general than the strongly (a.k.a. dagger) compact closed ones that have formed the main object of study in the categorial approach. Using these concrete examples, the extensive results that exist concerning ordered linear spaces, their face lattices, and their symmetries, and the currently developing, and potentially very rich, theory of how this order structure behaves in composite systems, especially how it interacts with monoidality, we believe the convex operational approach and the categorial approach will continue to fruitfully interact in the project of characterizing quantum theory by its informatic properties and its information-processing powers, not just in contradistinction to classical theory, but in contrast to the panoply of other theories that both approaches provide as foils. A promising avenue along which progress will likely be made is to understand the implications of strong compact closure in the convex setting, and the implications of natural properties of ordered linear spaces, notably homogeneity and self-duality, for categorially structured theories. Both strong compact closure and homogeneous self-duality single out significantly restricted classes of theories, classes that include quantum theory. We venture to guess that these classes may be somewhat similar, and that combining concepts of categorial and of convex origin may help us, both in narrowing down the class of theories and thus obtaining formal results characterizing quantum theory within a broad space of glistening foils set off to th’quantum world, and in understanding the operational content, and implications for information processing, of the principles used in such characterization results. This will give us a fascinating, exciting, and useful perspective on the essence of quantum theory, as the unique theory, within a broad framework for theories, in which information, and the ways in which it can be processed, has a specific set of fundamental properties. We expect this understanding to have pragmatic applications in the development of new quantum information processing protocols, and in understanding the limits on such protocols. But we also expect it may contribute to progress on the vexing questions of how quantum theory is to be interpreted, what it implies for the way in which our physics relates to the
world, and how it fits with the rest of physical theory.

7. * References

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