A COMBINATORIAL PROOF OF STRICT UNIMODALITY FOR
\(q\)-BINOMIAL COEFFICIENTS

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Abstract. I. Pak and G. Panova recently proved that the \(q\)-binomial coefficient \(\binom{m+n}{m}_q\) is a strictly unimodal polynomial in \(q\) for \(m, n \geq 8\), via the representation theory of the symmetric group. We give a direct combinatorial proof of their result by characterizing when a product of chains is strictly unimodal and then applying O’Hara’s structure theorem for the partition lattice \(L(m,n)\). In fact, we prove a stronger result: if \(m, n \geq 8d\), and \(2d \leq r \leq mn/2\), then the \(r\)-th rank of \(L(m,n)\) has at least \(d\) more elements that the next lower rank.

1. Introduction

Recall that the lattice \(L(m,n)\) consists of integer partitions whose Young diagrams fit inside an \((m \times n)\)-rectangle, ordered by inclusion:
\[
L(m,n) = \{ (\lambda_1, \ldots, \lambda_n) \mid n \geq \lambda_1 \geq \ldots \lambda_m \geq 0 \}.
\]
The rank generating function for \(L(m,n)\) is the \(q\)-binomial coefficient:
\[
\binom{m+n}{m}_q = \prod_{i=1}^{n} \frac{1-q^{m+i}}{1-q^i} = \sum_{r=0}^{mn} p_r(m,n)q^r.
\]
It is easy see that complementary ranks have the same size: \(p_r(m,n) = p_{mn-r}(m,n)\). Sylvester was the first to prove the unimodality property [7]:
\[
p_0(m,n) \leq \cdots \leq p_{\lfloor mn/2 \rfloor}(m,n).
\]
Several other proofs have been discovered over the years using many different techniques, e.g. [5, 6, 9]. In particular, O’Hara gave a purely combinatorial proof by decomposing the underlying ranked set of \(L(m,n)\) into a centered disjoint union of products of chains [3]. We use a slightly modified version of O’Hara’s theorem to prove the following:

1.1. Theorem. If \(m, n \geq 8d\) and \(2d \leq r \leq mn/2\), then:
\[
p_r(m,n) - p_{r-1}(m,n) \geq d.
\]
If \(d = 1\), we recover the strict unimodality theorem of Pak-Panova [4]:
1.2. **Theorem.** If \( m, n \geq 8 \) and \( 2 \leq r \leq mn/2 \), then:

\[
p_r(m, n) - p_{r-1}(m, n) > 0.
\]

In what follows we will ignore the partial order on \( L(m, n) \) and simply work with the underlying ranked set. The following is a restatement of O’Hara’s structure theorem [2, 3]:

1.3. **Theorem.** There is a decomposition of \( L(m, n) \) into centered rank-symmetric unimodal subsets \( Q_m(d_0, \ldots, d_k) \), where:

\[
k = \lfloor m/2 \rfloor \quad \text{and} \quad n = d_0 + 2d_1 + \cdots + (k+1)d_k,
\]

and there exist isomorphisms of ranked sets:

\[
Q_m(d_0, \ldots, d_k) \cong L(r, \ell_m(d_0, \ldots, d_k)) \times Q_{m-2r}(d_r, \ldots, d_k)
\]

where:

\[
r = 1 + \min\{j \mid d_j > 0\} \quad \text{and} \quad \ell_m(d_0, \ldots, d_k) = \sum_{j=0}^{k} (m-2j)d_j.
\]

1.4. **Remark.** There are several equivalent ways to define the sets \( Q_m(d_0, \ldots, d_k) \). For example, they are the level sets of certain tropical polynomials [1]:

\[
f_{m,r}(\lambda) = \min_{0 \leq t_0 \leq \cdots \leq t_{m-2r} \leq r} \sum_{j=0}^{m-2r} a_{2t_j+j},
\]

where \( \lambda_0 = n, \lambda_{m+1} = 0 \), and \( a_i = \lambda_i - \lambda_{i+1} \). We have the formula:

\[
Q_m(d_0, \ldots, d_k) = \{ \lambda \in L(m, n) \mid f_{m,r}(\lambda) = \sum_{j=r}^{k} (j+1-r)d_j \text{ for } 1 \leq r \leq k \}.
\]

In particular, it follows that \( Q_m(d_0, \ldots, d_k) \) has a unique minimal element, and its rank is equal to:

\[
\sum_{j=0}^{k} j(j+1)d_j.
\]

1.5. **Remark.** Our proof involves choosing subsets \( Q_m(d_0, \ldots, d_k) \subset L(m, n) \) whose strictly unimodal ranges provide a covering of the desired interval. F. Zanello has given a similar combinatorial proof [8].
2. Strict unimodality for chain products

Let $P$ be a ranked poset $P$ of length $n$. Let $p_i$ denote the size of the $i$-th rank of $P$. We say that $P$ is rank-symmetric if $p_i = p_{n-i}$ for all $0 \leq i \leq n$. We say that a rank-symmetric poset $P$ is strictly unimodal if:

$$p_0 < \cdots < p_{\lfloor n/2 \rfloor}.$$

In this section, we determine when a product of chains is strictly unimodal. We will apply this result in the next section to prove the theorem.

Given a non-negative integer $a$, let $[a] = \{0 < \cdots < a\}$ denote a chain of length $a$.

2.1. Lemma. Let $P = [a_1] \times \cdots \times [a_n]$, where $a_1 \geq \cdots \geq a_n \geq 0$. Let $C$ be a shortest chain in a symmetric chain decomposition of $P$.

1. $P$ is strictly unimodal from rank 0 up to the lowest rank of $C$.

2. The length of $C$ is equal to:

$$\ell(C) = \max(a_1 - (a_2 + \cdots + a_n), \epsilon)$$

where $\epsilon = 0$ (resp. $\epsilon = 1$) if $a_1 + \cdots + a_n$ is even (resp. odd).

3. $P$ is strictly unimodal if and only if:

$$a_1 \leq a_2 + \cdots + a_n + 1.$$

Proof. We will simultaneously prove these statements by induction on $n$. Consider the base case $n = 2$. There is a well-known symmetric chain decomposition:

$$[a_1] \times [a_2] \simeq \bigsqcup_{i=0}^{a_2} [a_1 + a_2 - 2i]$$

where the lowest rank of the $i$-th chain is $i$. Since there is a new symmetric chain starting at each rank from 0 to $a_2$, we see that $[a_1] \times [a_2]$ is strictly unimodal from rank 0 up to $a_2$, which is equal to the lowest rank of the shortest chain. The length of the shortest chain is:

$$a_1 + a_2 - 2a_2 = a_1 - a_2,$$

so $[a_1] \times [a_2]$ is strictly unimodal if and only if $a_1 - a_2 \leq 1$.

For $n \geq 3$, let $R = [a_2] \times \cdots \times [a_n]$ so:

$$P = [a_1] \times R.$$

Let $D$ be a shortest chain in a symmetric chain decomposition of $R$, and let $r$ denote the lowest rank of $D$. By induction, $R$ is strictly unimodal from rank 0 up $r$ and the length of $D$ is equal to:

$$\ell(D) = \max(a_2 - (a_3 + \cdots + a_n), \epsilon')$$

where $\epsilon' = 0$ (resp. $\epsilon' = 1$) if $a_2 + \cdots + a_n$ is even (resp. odd). In any case, we have:

$$a_1 \geq a_2 \geq \ell(D).$$
Now, the length of the longest chain in $R$ is $a_2 + \cdots + a_n$, and the lengths of all the symmetric chains in $R$ have the same parity.

If $a_1 \leq a_2 + \cdots + a_n$, then the smallest difference between $a_1$ and the lengths of all the symmetric chains in $R$ is equal to 0 or 1, depending on the parity of $a_1 + \cdots + a_n$. In other words, the length of a shortest chain $C$ in any SCD of $P$ will be 0 or 1. On the other hand, if $a_1 > a_2 + \cdots + a_n$, then:

$$\ell(C) = a_1 - (a_2 + \cdots + a_n)$$

by our formula from the base case. Therefore, we obtain:

$$\ell(C) = \max(a_1 - (a_2 + \cdots + a_n), \epsilon).$$

Note that $P$ is strictly unimodal from rank 0 up to the lowest rank of $C$ because $P$ is a disjoint union over products of the form $[a_1] \times D'$, where $D'$ runs over the symmetric chains in a fixed SCD of $R$. Finally, $P$ is strictly unimodal if and only if $\ell(C) \leq 1$, and the result follows. 

3. Proof of the theorem

3.1. Theorem. If $m, n \geq 8d$ and $2d \leq r \leq mn/2$, then:

$$p_r(m, n) - p_{r-1}(m, n) \geq d.$$ 

Proof. Step 1: Consider $Q_m(d_0, d_1, 0, \ldots, 0) \subset L(m, n)$ where $d_0, d_1 > 0$. This ranked set is isomorphic to a product of two chains:

$$[md_0 + (m-2)d_1] \times [(m-2)d_1].$$

The lowest rank element of this set has rank $2d_1$. Note that $n = d_0 + 2d_1$ so $n > 2d_1$. We know that a product of two chains of the form $[a+b] \times [b]$ is strictly unimodal from rank 0 to rank $b$, so $Q_m(d_0, d_1, 0, \ldots, 0)$ is strictly unimodal from rank $2d_1$ up to rank:

$$2d_1 + (m-2)d_1 = md_1.$$ 

Therefore, we see that:

$$p_r(m, n) - p_{r-1}(m, n) \geq d$$

for all $2d \leq r \leq mn/2 - dm$ and $n > 2d$.

Step 2: Consider $Q_m(d_0, d_1, d_2, 0, \ldots, 0) \subset L(m, n)$ where $d_0, d_1, d_2 > 0$. This ranked set is isomorphic to a product of three chains:

$$[md_0 + (m-2)d_1 + (m-4)d_2] \times [(m-2)d_1 + (m-4)d_2] \times [(m-4)d_2].$$

The lowest rank element of this set has rank $2d_1 + 6d_2$. We know that the product of three chains of the form $[a+b+c] \times [b+c] \times [c]$ is strictly unimodal if and only if $a \leq c + 1$, so $Q_m(d_0, d_1, d_2, 0, \ldots, 0)$ is strictly unimodal if and only if:

$$md_0 \leq (m-4)d_2 + 1.$$
Let \( N_3(m, n) \) denote the number of strictly unimodal subsets in \( L(m, n) \) of the above type:
\[
N_3(m, n) = \{(d_0, d_1, d_2) \in \mathbb{Z}_{\geq 0}^2 \mid d_0 + 2d_1 + 3d_2 = n, \; md_0 \leq (m - 4)d_2 + 1\}.
\]

Now let us calculate the largest possible value of \( d_2 \), depending on \( n \) (mod 3):
- \( n = 0 \) (mod 3) \( \Rightarrow \) \( (d_0, d_1, d_2) = (1, 1, (n - 3)/3) \).
- \( n = 1 \) (mod 3) \( \Rightarrow \) \( (d_0, d_1, d_2) = (2, 1, (n - 4)/3) \).
- \( n = 2 \) (mod 3) \( \Rightarrow \) \( (d_0, d_1, d_2) = (1, 2, (n - 5)/3) \).

Therefore, the largest possible value of \( 2d_1 + 6d_2 \) is equal to \( 2n - 4 \) if \( n = 0 \) (mod 3) and \( 2n - 6 \) otherwise.

It follows that:
\[
p_r(m, n) - p_{r-1}(m, n) \geq N_3(m, n)
\]
for all \( 2n - 4 \leq r \leq mn/2 \).

**Step 3:** We see that we are reduced to satisfying the following inequalities:
\[
mn/2 - dm \geq 2n - 4 \quad \text{and} \quad N_3(m, n) \geq d.
\]
The first inequality is equivalent to:
\[
m \geq \frac{4n - 8}{n - 2d}
\]
If \( d \geq 1 \) and \( m, n \geq 8d \), then the above inequality does hold:
\[
m(n - 2d) \geq 8d(n - 2d) = d(8n - 16d) \geq 8n - 16d \geq 4n + 32d - 16d = 4n + 16d \geq 4n - 8.
\]

Let us now estimate a lower bound for \( N_3(m, n) \). For a fixed value of \( d_0 \), we find that:
\[
3d_2 = n - d_0 - 2d_1 \leq n - 2 - d_0.
\]
Given a possible solution \( (d_1, d_2) \), note that the next solution is \( (d_1 + 3, d_2 - 2) \), so all the possible values of \( d_2 \) must have the same parity mod 2. Therefore:
\[
\frac{md_0 - 1}{m - 4} \leq d_2 \leq \frac{n - 2 - d_0}{3}
\]
where all the values of \( d_2 \) must have the same parity mod 2. So the number of solutions is at least:
\[
\frac{1}{2} \sum_{d_0} \left( \frac{n - 2 - d_0}{3} - \frac{md_0 - 1}{m - 4} \right).
\]
Summing over the allowed values of \( d_0 \), we obtain:
\[
N_3(m, n) \geq \frac{1}{2} \sum_{d_0} \left( \frac{n - 2 - d_0}{3} - \frac{md_0 - 1}{m - 4} \right).
\]
Note that, if \( m \geq 8 \), then:
\[
md_0 - 1 < 2d_0(m - 4) = 2md_0 - 8d_0
\]
because:
\[
md_0 \geq 8d_0 > 8d_0 - 1.
\]
Therefore:

$$\frac{1}{2} \left( \frac{n - 2 - d_0}{3} - \frac{md_0 - 1}{m - 4} \right) > \frac{1}{2} \left( \frac{n - 2 - d_0}{3} - 2d_0 \right) = \frac{n - 2 - 7d_0}{6}.$$ 

If \(d \geq 3\), then \(m, n \geq 24\) and by considering the first three terms we obtain:

$$N_3(m, n) \geq \frac{n - 9}{6} + \frac{n - 16}{6} + \frac{n - 23}{6} = \frac{n - 16}{2} \geq 4d - 8 \geq d$$

for all \(d \geq 3\). It remains to prove the theorem for \(1 \leq d \leq 2\).

**Step 4:** Consider \(Q_m(d_0, d_1, d_2, d_3, 0, \ldots, 0) \subset L(m, n)\) where \(d_0, d_1, d_2, d_3 > 0\). This ranked set is isomorphic to a product of four chains:

\([\ell_0] \times [\ell_1] \times [\ell_2] \times [\ell_3],\)

where \(\ell_3 = (m - 6)d_3, \ell_2 = \ell_3 + (m - 4)d_2, \ell_1 = \ell_2 + (m - 2)d_1,\) and \(\ell_0 = \ell_1 + md_0\). It is strictly unimodal if and only if:

$$\ell_0 - \ell_1 \leq \ell_2 + \ell_3 + 1,$$

which translates to the condition that:

$$md_0 \leq (m - 4)d_2 + 2(m - 6)d_3.$$ 

If \(d_2, d_3 \geq d_0\) we get:

$$(m - 4)d_2 + 2(m - 6)d_3 \geq d_0(3m - 16) \geq md_0$$

for all \(m \geq 8\). In other words, \(Q_m(1, d_1, d_2, d_3, 0, \ldots, 0) \subset L(m, n)\) is strictly unimodal if there exist \(x, y, z \geq 0\) such that \(2x + 3y + 4z = n - 10\). The standard generating function argument shows that the existence of at least one solution for all \(n \geq 12\) and at least two solutions for all \(n \geq 16\). So the \(d = 2\) case is finished, and for \(d = 1\) we can check the remaining cases \(8 \leq m, n \leq 11\) by inspection.

Finally, we check that the lowest rank of \(Q_m(1, d_1, d_2, d_3, 0, \ldots, 0)\) is at most \(mn/2 - dm\) for \(d = 1, 2\) and \(m, n \geq 8d\). The largest possible value of \(d_3\), depending on \(n \pmod{4}\), is given by:

$$n = 0 \pmod{4} \implies (d_0, d_1, d_2, d_3) = (1, 2, 1, (n - 8)/4).$$

$$n = 1 \pmod{4} \implies (d_0, d_1, d_2, d_3) = (1, 1, 2, (n - 9)/4).$$

$$n = 2 \pmod{4} \implies (d_0, d_1, d_2, d_3) = (1, 1, 1, (n - 6)/4).$$

$$n = 3 \pmod{4} \implies (d_0, d_1, d_2, d_3) = (1, 2, 2, (n - 11)/4).$$

Therefore, the largest possible value for the lowest rank \(2d_1 + 6d_2 + 12d_3\) is \(3n - 10\). Now:

$$mn/2 - dm \geq 4n - 8d \geq 3n + n - 8d \geq 3n \geq 3n - 10,$$

so \(mn/2 - dm \geq 3n - 10\) for \(m, n \geq 8d\) as desired. \(\square\)
3.2. **Remark.** The lower bounds on \( m, n \), and \( r \) given above are certainly not the best possible. One could improve the bounds on \( m \) and \( n \) by calculating the number of strictly unimodal products of five or more chains. One could also improve the bound on \( r \) by finding the overlaps among the strictly unimodal ranges for all products of three or more chains. The lower bound on \( r \) given in [8] is quadratic in \( d \), while our lower bound is linear in \( d \). The best possible lower bound for \( r \) is logarithmic in \( d \), since it involves the inverse of the number of partitions of \( r \).

**References**

[1] V. Dhand, Tropical decomposition of Young’s partition lattice. *J. Alg. Comb.* (2013) DOI 10.1007/s10801-013-0467-4.

[2] V. Dhand, Rank unimodality of Young’s lattice via explicit chain decomposition. [http://arxiv.org/abs/1303.2536](http://arxiv.org/abs/1303.2536).

[3] K. O’Hara, Unimodality of Gaussian coefficients: a constructive proof, J. Comb. Theory Ser. A. 53, 29-52 (1990).

[4] I. Pak and G. Panova, Strict unimodality of \( q \)-binomial coefficients, *C. R. Acad. Sci. Paris*, Ser. I (2013)

[5] R. A. Proctor, Solution of two difficult combinatorial problems with linear algebra. *Amer. Math. Monthly* 89, (1982) 721–734.

[6] R. P. Stanley, Weyl Groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Alg. and Disc. Methods* 1, (1980) 168–184.

[7] J. J. Sylvester, Proof of the hitherto undemonstrated fundamental theorem of invariants. In *The collected mathematical papers of James Joseph Sylvester*, Vol. 3 Cambridge University Press, Chelsea, New York (1973), 117–126.

[8] F. Zanello, Zeilberger’s KOH theorem and the strict unimodality of \( q \)-binomial coefficients, [http://arxiv.org/abs/1311.4480](http://arxiv.org/abs/1311.4480).

[9] D. Zeilberger, Kathy O’Hara’s constructive proof of the unimodality of the Gaussian polynomials. *Amer. Math. Monthly* 96 (1989), 590–602.