Model of statistically coupled chiral fields on the circle

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ABSTRACT

Starting from a field theoretical description of multicomponent anyons with mutual statistical interactions in the lowest Landau level, we construct a model of interacting chiral fields on the circle, with the energy spectrum characterized by a symmetric matrix $g_{\alpha \beta}$ with nonnegative entries. Being represented in a free form, the model provides a field theoretical realization of (ideal) fractional exclusion statistics for particles with linear dispersion, with $g_{\alpha \beta}$ as a statistics matrix. We derive the bosonized form of the model and discuss the relation to the effective low-energy description of the edge excitations for abelian fractional quantum Hall states in multilayer systems.

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1. Introduction

Recently a field theory of anyons confined to the lowest Landau level (LLL) of a strong magnetic field was formulated [1]. The authors of Ref. [1] next mapped their field theory (which is effectively one dimensional) onto a circle obtaining a theory of a chiral fermion field with linear dispersion. The problem involves a (statistics) parameter \( g \) entering the wave function of anyons in the LLL

\[
\Psi = \prod_{i<j}(z_i - z_j)^g \Phi_+(\{z_i\}) e^{-\frac{1}{2} \sum_{i} |z_i^2|},
\]

where \( \Phi_+(\{z_i\}) \) is a symmetric function of its arguments. The requirement that the function \( \Phi_+ \) should be a polynomial for all the states in the LLL implies that \( g \in [0, 2) \). Remarkably, the field theory on the circle remains consistent in a wider interval of range of the parameter \( g \), for all \( g > 0 \). The motivation to consider such an extension of the theory is the similarity of the above wave function with \( g \) odd to Laughlin wave functions of fractional quantum Hall (FQH) states [2]. It suggests that for \( g \) odd, the model of a chiral field discussed in [1] may be used to describe edge excitations for Laughlin states. Indeed, a bosonization of the theory enables one to recover correlation exponents as well as the low temperature thermodynamics [1] coinciding with those obtained with the use of the (chiral) Luttinger liquid description of edge excitations [3] (for a recent review, see [4]).

In this paper we generalize the model in [1] to the case of several species of particles. In Section 2, we discuss a field theoretical description of anyons of several species in the LLL described by the generic statistics matrix \( g_{\alpha\beta} \) (with off-diagonal elements corresponding to mutual statistical interactions [5]), and then map the system onto a circle. Upon extending the possible values of the statistics parameters, we arrive at a model of interacting chiral fields, with the energy spectrum characterized by a symmetric matrix \( g_{\alpha\beta} \) with non-negative entries. As in the case of a single species, the theory may be reformulated in a free form (in terms of field operators obeying more complicated commutation relations).

We discuss the construction of the corresponding Fock space. In Section 3 we show that the structure which is behind the above free field formulation is an ideal fractional exclusion statistics [6, 7, 8, 9, 10, 11, 12]: in the thermodynamic limit the model is equivalent to a system of free chiral particles with linear dispersion, characterized by the (bosonic) statistics matrix \( g_{\alpha\beta} \).
It was observed in Ref. [13] that as a result of the dimensional reduction, the system of two anyons in the LLL admits a description in terms of fractional statistics in one dimension. The latter statistics was originally introduced by Leinaas and Myrheim in the Heisenberg quantization of identical particles [14]. It was also argued in Ref. [14] that the system of noninteracting particles obeying fractional statistics in one dimension is equivalent to a system of particles with long range inverse square interaction (see also [15]). Using the asymptotic Bethe ansatz for the latter model [17], an equation for the single state distribution function for fractional statistics in one dimension was derived (note that in spite of being derived from the one-dimensional model, the resulting single state statistical distribution does not make any reference to the space dimension) [7]. The same distribution function was also derived [8, 9] from the generalized exclusion principle [6] and the statistics is now referred to as fractional exclusion statistics. The system of anyons in the LLL is a realization of this statistics, as can be seen from the equation of state and thermodynamic quantities [9, 10, 5].

The above explains the appearance of exclusion statistics in the model of chiral fields on the circle which has the same quantum numbers as the system of anyons in the LLL. Using the exclusion statistics representation of the model, we calculate the low temperature heat capacity. The connection with exclusion statistics enables one to describe the model in terms of equations having the form of asymptotic Bethe ansatz equations [16]. This implies a simple dynamics encoded in two-body scattering phases of a step-wise form.

We discuss some peculiarities in calculating quantities like the dressed charge matrix due to a step-wise character of the two-body scattering phases.

In Section 4 we consider the relation to the effective low-energy description of edge excitations in abelian multilayer FQH states [7, 4]. In the bosonized form, our model describes a set of chiral boson fields all propagating in the same direction with the same velocity. For the case of odd diagonal entries and integer off-diagonal entries, the statistics matrix \(g_{\alpha\beta}\) can then be identified with the topological matrix \(K_{\alpha\beta}\) describing the edge excitations. This in particular implies an interpretation of edge excitations in terms of exclusion statistics and is consistent with the results of Fukui and Kawakami who
achieved the same identification between $g_{\alpha\beta}$ and $K_{\alpha\beta}$ for edge excitations in hierarchical FQH states [12].

2. Multicomponent field theory on the circle

Consider a system of $M$ species of anyons in the LLL, described by the generic statistics matrix $g_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, M$), with $N_\alpha$ the number of particles of species $\alpha$. The wave function of the system has the form [5]

$$
\Psi = \prod_\alpha \prod_{i<j} (z_\alpha^i - z_\alpha^j)^g_{\alpha\alpha} \prod_{\alpha<\beta} \prod_{i,j} (z_\alpha^i - z_\beta^j)^g_{\alpha\beta} \Phi_+(\{z_\alpha^\alpha\}, \{z_\beta^\beta\}, \ldots) e^{-\frac{i}{2} \sum_\alpha \sum_i |z_\alpha^\alpha|^2}.
$$

Here $z_\alpha^i = \sqrt{eB/2} (x_\alpha^i + iy_\alpha^i)$ is a dimensionless complex coordinate of the $i^{th}$ particle of species $\alpha$ (the charges and masses of all the species of particles are assumed to be the same). The symmetric matrix $g_{\alpha\beta}$ is responsible for the statistics of anyons, with non-diagonal elements corresponding to mutual statistical interactions between particles of distinct species. The function $\Phi_+$ is single-valued and symmetric with respect to an interchange of coordinates of the same species, arbitrary when coordinates of distinct species are interchanged (bosonic representation).

The wave function (1) acquires a phase $\exp(i\pi g_{\alpha\alpha})$ under an (anticlockwise) interchange of two particles of species $\alpha$ and a phase $\exp(i2\pi g_{\alpha\beta})$ under winding a particle of species $\alpha$ around a particle of species $\beta$ for $\alpha \neq \beta$ (provided that each of the above closed paths encloses no other particles). In order for the function $\Phi_+$ to be a polynomial for all the states in the LLL, one should choose the intervals $g_{\alpha\alpha} \in [0, 2)$ and $g_{\alpha\beta} \in [0, 1)$ for $\alpha \neq \beta$. $g_{\alpha\beta} = 0$ and $g_{\alpha\beta} = \delta_{\alpha\beta}$ correspond to sets of bosons and fermions, respectively.

It will be convenient for us to work in the fermionic representation. The latter is obtained by the change $g_{\alpha\beta} = \delta_{\alpha\beta} + \lambda_{\alpha\beta}$ and absorbing a Slater determinant corresponding to the $\delta_{\alpha\beta}$-part of $g_{\alpha\beta}$ in (1) in the definition of the function $\Phi$. This converts $\Phi_+$ into the function $\Phi_-((\{z_\alpha^\alpha\}, \{z_\beta^\beta\}, \ldots)$ which is antisymmetric with respect to an interchange of coordinates within the same species of particles.

The angular momentum for the states (1) has eigenvalues

$$
L = \sum_\alpha \left[ \sum_{j=1}^{N_\alpha} n_\alpha^j + \frac{1}{2} \lambda_{\alpha\alpha} N_\alpha (N_\alpha - 1) + \frac{1}{2} \sum_{\beta \neq \alpha} \lambda_{\alpha\beta} N_\alpha N_\beta \right],
$$

(2)
where \( n'_\alpha \) are nonnegative integers, distinct within each species, corresponding to the fermionic part of the angular momentum associated with the function \( \Phi_\ldots \).

Following Ref. [1], we introduce the fermionic field operators \( \varphi_\alpha(z) \) obeying the following anticommutation relations,

\[
\{ \varphi_\alpha(z), \varphi_\beta(z') \} = 0, \quad \{ \varphi_\alpha(z'), \varphi_\alpha^\dagger(z) \} = \frac{1}{\pi} e^{z'z}.
\] (3)

On expanding \( \varphi_\alpha(z) \) in powers of the angular momentum eigenstates

\[
\varphi_\alpha(z) = \sum_{l=0}^{\infty} \frac{1}{\sqrt{\pi l!}} \varphi_\alpha^l z^l,
\] (4)

these imply the usual fermionic anticommutation relations for the components:

\[
\{ \varphi_\alpha^l, \varphi_\beta^l \} = \{ \varphi_\alpha^\dagger l, \varphi_\beta^\dagger l \} = 0, \quad \{ \varphi_\alpha^l, \varphi_\beta^\dagger l \} = \delta_{\alpha\beta} \delta_{ll}.
\] (5)

In terms of the field operators (3), the angular momentum can be written as

\[
\hat{L} = \sum_\alpha \int d^2z e^{-\bar{z}z} \varphi_\alpha^\dagger(\bar{z}) \left[ z \partial_z + \frac{1}{2} \sum_\beta \lambda_{\alpha\beta} \hat{N}_\beta \right] \varphi_\alpha(z),
\] (6)

with the representation of the particle number operators as

\[
\hat{N}_\alpha = \int d^2z e^{-\bar{z}z} \varphi_\alpha^\dagger(\bar{z}) \varphi_\alpha(z)
\] (7)

\((d^2z \equiv dx dy)\). By using the expansion (4) and the explicit construction of the fermionic Fock space generated by the operators (5), one can verify that the formula (6) recovers the correct angular momentum eigenvalues (2) for states with fixed particle numbers.

Upon adding a harmonic potential (of frequency \( \omega \)), the Hamiltonian becomes

\[
\hat{H} = \frac{1}{2} \omega_{c}^{\text{eff}} \sum_\alpha \hat{N}_\alpha + a \hat{L},
\] (8)

where \( \omega_{c}^{\text{eff}} = \sqrt{\omega_c^2 + 4\omega^2} \), \( \omega_c = eB/m \) is the cyclotron frequency, and \( a = \omega_{c}^{\text{eff}} - \omega_c \). To leading order in \( \omega^2/\omega_c^2 \), which is assumed hereafter,

\[
\omega_{c}^{\text{eff}} \simeq \omega_c, \quad a \simeq \omega^2/\omega_c, \quad \omega \ll \omega_c.
\] (9)
The wave functions (1) for \( g_\alpha \) odd integers and \( g_{\alpha \beta} \) integers have the form of the wave functions describing FQH abelian states in multilayer systems, with \( z_i^\alpha \) playing the role of the complex coordinate of the \( i^{th} \) electron in the \( \alpha^{th} \) layer (such wave functions were first discussed by Halperin [18]). Motivated by this similarity, we consider an analytic continuation of the solutions for anyons in the LLL, allowing the parameters \( g_{\alpha \beta} \) to take any nonnegative values. For a single layer, the edge excitations are generated when the function \( \Phi^+ \) in (1) belongs to the space of symmetric polynomials in \( \{ z_i \} \) [3, 4], which is relevant to the polynomial character of the functions \( \Phi^+ \) in (1).

We now consider a mapping of the system onto a circle. Using the operators \( \varphi_i^\alpha \) from (4), we define the fields on the circle (parametrized with an angular coordinate \( \theta \)) by

\[
\chi_{\alpha}(\theta) = \sum_{n_\alpha=0}^{\infty} \frac{1}{\sqrt{2\pi}} \varphi_i^\alpha e^{in_\alpha \theta} .
\]

The commutation relations (3) then imply

\[
\{ \chi_{\alpha}(\theta), \chi_{\beta}(\theta') \} = \{ \chi_{\alpha}^+(\theta), \chi_{\beta}^+(\theta') \} = 0 ,
\]

\[
\{ \chi_{\alpha}(\theta), \chi_{\beta}^+(\theta') \} = \delta_{\alpha\beta} \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{in(\theta-\theta')} \equiv \delta_{\alpha\beta} \delta_{\text{per}}(\theta - \theta') ,
\]

where \( \delta_{\text{per}}(\theta - \theta') \) is the positive frequency part of the periodic \( \delta \)-function.

We introduce the Hamiltonian for the fields on the circle corresponding to (8) as

\[
H = \frac{1}{2} \omega_c^\text{eff} \sum_\alpha \dot{N}_\alpha + a \sum_\alpha \int_0^{2\pi} d\theta \chi_{\alpha}^+(\theta) \left( \frac{i}{2} \sum_{\beta} \lambda_{\alpha\beta} \dot{N}_\beta \right) \chi_{\alpha}(\theta) ,
\]

with the representation of the particle numbers as

\[
\dot{N}_\alpha = \int_0^{2\pi} d\theta \chi_{\alpha}^+(\theta) \chi_{\alpha}(\theta) .
\]

The equations of motion

\[
\partial_t \chi_{\alpha}(\theta) = i [H, \chi_{\alpha}(\theta)] = -a \left( \partial_\theta + \frac{1}{2} \sum_{\beta} \lambda_{\alpha\beta} \dot{N}_\beta \right) \chi_{\alpha}(\theta) ,
\]

on introducing new fields by

\[
\psi_{\alpha}(\theta) = e^{i \sum_{\beta} \lambda_{\alpha\beta} \dot{N}_\beta \theta} \chi_{\alpha}(\theta) ,
\]
take a free form, 

\[(\partial_t + a\partial_\theta)\psi_\alpha(\theta) = 0.\]  \hfill (16)

To derive the commutation relations for the \(\psi\) operators, it is convenient to use the identity

\[e^{i\sum_\beta \lambda_{\alpha\beta}\hat{N}_\beta} \chi_\gamma = \chi_\gamma e^{i\sum_\beta \lambda_{\alpha\beta}(\hat{N}_\beta - \delta_{\beta\gamma})}.\]  \hfill (17)

To prove this, we first note that the operators \(\chi_\alpha\) and \(\chi_\alpha^\dagger\) satisfy

\[[\hat{N}_\alpha, \chi_\beta] = -\delta_{\alpha\beta}\chi_\beta, \quad [\hat{N}_\alpha, \chi_\beta^\dagger] = -\delta_{\alpha\beta}\chi_\beta^\dagger.\]  \hfill (18)

The first relation in (18) written as \(\hat{N}_\alpha\chi_\beta = \chi_\beta(\hat{N}_\alpha - \delta_{\alpha\beta})\), is generalized to \(\hat{N}_\alpha^k\chi_\beta = \chi_\beta(\hat{N}_\alpha - \delta_{\alpha\beta})^k\), with \(k\) a positive integer, which straightforwardly leads to (17).

Using the identity (17) and similar identities for the operators \(\chi_\alpha^\dagger\), we obtain from (11) the commutation relations for the \(\psi\) fields:

\[\psi_\alpha(\theta)\psi_\beta(\theta') + e^{-i\lambda_{\alpha\beta}(\theta-\theta')}\psi_\beta(\theta')\psi_\alpha(\theta) = 0 \quad (19)\]

\[\psi_\alpha^\dagger(\theta)\psi_\beta^\dagger(\theta') + e^{-i\lambda_{\alpha\beta}(\theta-\theta')}\psi_\beta^\dagger(\theta')\psi_\alpha^\dagger(\theta) = 0 \quad (19)\]

\[\psi_\alpha(\theta)\psi_\beta^\dagger(\theta') + e^{i\lambda_{\alpha\beta}(\theta-\theta')}\psi_\beta^\dagger(\theta')\psi_\alpha(\theta) = \delta_{\alpha\beta}\Delta_\alpha(\theta - \theta') ,\]

where the operators

\[\Delta_\alpha(\theta - \theta') = e^{i\sum_\beta \lambda_{\alpha\beta}\hat{N}_\beta(\theta-\theta')}\delta_{\per}(\theta - \theta')\]  \hfill (20)

behave to some extent like \(\delta\)-functions on the circle:

\[\int_0^{2\pi} d\theta'\psi_\alpha^\dagger(\theta')\Delta_\alpha(\theta - \theta') = \psi_\alpha^\dagger(\theta), \quad \int_0^{2\pi} d\theta'\Delta_\alpha(\theta - \theta')\psi_\alpha(\theta') = \psi_\alpha(\theta) .\]  \hfill (21)

Consider the “Fourier transform” of the \(\psi\) operators:

\[\psi_\alpha(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{\kappa_\alpha} a^\alpha_{\kappa_\alpha} e^{i\kappa_\alpha \theta} .\]  \hfill (22)

The (nonnegative) numbers \(\kappa_\alpha\) are not integers, in contrast to the numbers \(n_\alpha\) in the expansion (11). Allowed values of \(\kappa_\alpha\) can be obtained by calculating the matrix elements.
of the field \( \psi_\alpha \) between states with fixed particle numbers \( \{n'_\beta\} \). The only nonzero matrix elements correspond to the transition \( n'_\alpha + 1 \to n'_\alpha \), without changing the other numbers of particles. Comparing the matrix elements of \( \psi_\alpha \), in its two representations, \((22)\) and \((13)\) with \((10)\), yields the restrictions

\[
\kappa_\alpha = n_\alpha + \sum_\beta \lambda_{\alpha\beta} n'_\beta, \tag{23}
\]

where the nonnegative integers \( n_\alpha \) are the same as in \((10)\).

Let us introduce a set of basis states and label for a moment all the particles by a single sequence \( 1, \ldots, N \)

\[
|\kappa^0_N, \ldots, \kappa^0_1\rangle = \frac{1}{\sqrt{N!}} a_{\kappa N}^\dagger \cdots a_{\kappa 1}^\dagger |0\rangle. \tag{24}
\]

Then we obtain the restriction

\[
\kappa^i_\alpha = n^i_\alpha + \sum_\beta \lambda_{\alpha\beta} N^i_\beta, \quad i = 1, 2, \ldots, N_\alpha, \tag{25}
\]

where for each species \( \alpha \), \( i \) numbers particles in order of their appearance in \((24)\), \( n^i_\alpha \) correspond to the fermionic angular momentum quantum numbers (see \((2)\)). We have also denoted by \( N^i_\beta \) the number of creation operators for species \( \beta \) in the sequence \((24)\) (counted from the right to the left) standing before the operator creating the \( i^{th} \) particle of species \( \alpha \) (in other words, \( N^i_\beta \) is the number of particles of species \( \beta \) created before the \( i^{th} \) particle of species \( \alpha \)).

Inserting the expansion \((22)\) into \((19)\), we obtain the commutation relations for the operators \( a_{\kappa}^\alpha \) and \( a_{\kappa}^{\alpha\dagger} \) (here we drop the index of \( \kappa \) labelling the species of particles)

\[
a_{\kappa}^\alpha a_{\mu}^\beta + a_{\mu-\lambda_{\alpha\beta}}^\beta a_{\kappa+\lambda_{\alpha\beta}}^\alpha = 0, \\
a_{\kappa}^{\alpha\dagger} a_{\mu}^{\beta\dagger} + a_{\mu+\lambda_{\alpha\beta}}^{\beta\dagger} a_{\kappa-\lambda_{\alpha\beta}}^{\alpha\dagger} = 0, \\
a_{\kappa}^\alpha a_{\mu}^{\beta\dagger} + a_{\mu-\lambda_{\alpha\beta}}^{\beta\dagger} a_{\kappa-\lambda_{\alpha\beta}}^\alpha = \delta_{\alpha\beta} \delta_{\kappa\mu} \Pi_\kappa^\alpha. \tag{26}
\]

where \( \Pi_\kappa^\alpha \) are the projections on the subspace with the particle numbers \( \{n'_\beta\} \) determined by \((23)\). The projection operators satisfy the relations

\[
a_{\kappa}^{\alpha\dagger} \Pi_\mu^\beta = \Pi_\mu^{\beta+\lambda_{\alpha\beta}} a_{\kappa}^{\alpha\dagger}, \quad a_{\kappa}^\alpha \Pi_\mu^\beta = \Pi_\mu^{\beta-\lambda_{\alpha\beta}} a_{\kappa}^\alpha. \tag{27}
\]
For rational $\lambda_{\alpha\beta}$ with common denominator $q$, explicit expressions for these operators can be derived:

$$\Pi_\kappa^\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\theta \sum_{l=0}^\infty e^{-iq(\kappa-l-\sum_\beta \lambda_{\alpha\beta} \hat{N}_\beta)} \theta = \frac{1}{q} \sum_{n=1}^q e^{2\pi in(\kappa-\sum_\beta \lambda_{\alpha\beta} \hat{N}_\beta)} . \tag{28}$$

It is easily seen from (28) that on the states with fixed particle numbers, the operators (28) have eigenvalues 1 if the relations (23) are fulfilled and eigenvalues 0 otherwise as it should be.

The occupation number description determined by the creation and annihilation operators $a^{\alpha\dagger}_\kappa$ and $a^\alpha_\kappa$ is not unique: as can be seen from (26), the interchange of any two of the creation operators leads to an equivalent state corresponding to the same fermionic quantum numbers \( \{n^i_\alpha\} \). Rearranging an $N$-particle state does not change the total energy of the state, which is given by the sum over single-particle energies,

$$E = \sum_{\alpha} \sum_{j=1}^{N_\alpha} \left( \frac{1}{2} \omega^{\text{eff}}_\alpha + a\kappa^j_\alpha \right) . \tag{29}$$

An ordering procedure may be introduced by demanding that each particle added into the system has to have a higher (or equal) value of $\kappa$ than the largest $\kappa$ already present in the system. This means that only the states (24) corresponding to the ordering $\kappa_1 \leq \cdots \leq \kappa_N$ are regarded as physical states, which makes the occupation number picture unique. Any of such normal ordered states can also be constructed in a unique way starting from a given set of the fermionic angular momentum quantum numbers \( \{n^i_\alpha\} \) by putting particles into the system one by one and always making sure to put in the particle which gets the lowest possible value of $\kappa$.

Eq. (25) then admits the interpretation that the spectrum of available single particle energy levels for a particle of a given species, say $\alpha$, is shifted upwards by the presence of all particles below it. Each particle, say of species $\beta$, causes a shift $\lambda_{\alpha\beta}$. By convention, particles of distinct species in the same level shift each other by $\frac{1}{2} \lambda_{\alpha\beta} (= \frac{1}{2} \lambda_{\beta\alpha})$. This means that the order of filling of levels of the same energy with particles of distinct species is unessential.

Figure 1 illustrates the above normal ordering procedure in the case of a simple two-component system with integer statistics parameters.
3. Relation to exclusion statistics

3.1. Thermodynamic limit of the free particle formulation

With the ordering procedure described at the end of the previous section, Eq. (25) can be written in a compact form as

$$\kappa_{\alpha j} = n_{\alpha j} + \sum_\beta \sum_\lambda \lambda_{\alpha\beta} h(\kappa_{\alpha j} - \kappa_{\beta l}),$$  \hspace{1cm} (30)

where \(h(x)\) is the step function, which is defined here as \(h(x) = 1\) for \(x > 0\), \(h(x) = \frac{1}{2}\) for \(x = 0\), and \(h(x) = 0\) for \(x < 0\). We assume the ordering \(\kappa_{1\alpha} < \kappa_{2\alpha} < \cdots < \kappa_{N_\alpha}\) within each species (the resulting numbers \(\kappa_{\alpha j}\) within the same species are distinct). In the form (30), the numbers \(\kappa_{\alpha j}\) are similar to “renormalized” quantum numbers introduced for integrable models of a Calogero type [19].

We now specify the length of the circle as \(L\) and use the identification \[1\]

$$a = \frac{2\pi}{L} v.$$  \hspace{1cm} (31)

If one assumes that the harmonic potential is created in an anyon droplet of radius \(L/2\pi\) by an electric field, then the velocity \(v\) can be interpreted as the drift velocity on the edge \(E/B\) (the velocity of the edge excitations) where \(E = (m/e)\omega^2 R\) is the electric field on the edge. The thermodynamic limit is understood as \(\omega \to 0\) and \(L \to \infty\) while \(v\) is kept fixed. We also introduce the momenta and pseudomomenta

$$p_{\alpha j} = \frac{2\pi}{L} n_{\alpha j}, \quad k_{\alpha j} = \frac{2\pi}{L} \kappa_{\alpha j},$$  \hspace{1cm} (32)

distributed in the thermodynamic limit with the densities \(\nu_{\alpha}(p^\alpha)\) and \(\rho_{\alpha}(k^\alpha)\), respectively, so that the number of particles of species \(\alpha\) in the interval \((p^\alpha, p^\alpha + dp^\alpha)\) or in the corresponding interval \((k^\alpha, k^\alpha + dk^\alpha)\) is\[1\]

$$\nu_{\alpha}(p^\alpha) dp^\alpha = \rho_{\alpha}(k^\alpha) dk^\alpha$$  \hspace{1cm} (33)

\[1\]The transition to the thermodynamic limit below is similar to that given in Ref. [20]. We stress however that our starting point is the free field occupation number picture rather than the spectrum of an integrable model as in the above reference.
In terms of (32), Eq. (30) reads
\[ p^\alpha_j = k^\alpha_j - \frac{2\pi}{L} \sum_\beta \sum_l \lambda_{\alpha\beta} h(k^\alpha_j - k^\beta_l), \] (34)
or, in the thermodynamic limit,
\[ p^\alpha = k^\alpha - \sum_\beta \int_0^\infty \lambda_{\alpha\beta} h(k^\alpha - k^\beta) \rho_\beta(k^\beta) dk^\beta. \] (35)

From this equation, we get
\[ \partial p^\alpha / \partial k^\alpha = 1 - \sum_\beta \lambda_{\alpha\beta} \rho_\beta(k^\alpha) \] (36)
and with the use of (33),
\[ \nu^\alpha = \frac{\rho^\alpha}{1 - \sum_\beta \lambda_{\alpha\beta} \rho_\beta}. \] (37)

The fermionic description of the energy levels implies that the (non-equilibrium) entropy has the form
\[ S = \frac{L}{2\pi} \sum_\alpha \int_0^\infty \left[ -\nu^\alpha \ln \nu^\alpha - (1 - \nu^\alpha) \ln(1 - \nu^\alpha) \right] dp^\alpha \] (38)
or, in terms of \( \rho_\alpha \),
\[ S = \frac{L}{2\pi} \sum_\alpha \int_0^\infty \left\{ -\rho_\alpha \ln \rho_\alpha + \left[ 1 - \sum_\beta \lambda_{\alpha\beta} \rho_\beta \right] \ln \left[ 1 - \sum_\beta \lambda_{\alpha\beta} \rho_\beta \right] \\
- \left[ 1 - \sum_\beta (\delta_{\alpha\beta} + \lambda_{\alpha\beta}) \rho_\beta \right] \ln \left[ 1 - \sum_\beta (\delta_{\alpha\beta} + \lambda_{\alpha\beta}) \rho_\beta \right] \right\} dk^\alpha. \] (39)

The latter expression, along with the expression for the energy
\[ E = \frac{L}{2\pi} \sum_\alpha \int_0^\infty (\frac{1}{2} \omega_c + \nu^\alpha) \rho_\alpha(k^\alpha) dk^\alpha. \] (40)
shows that in the thermodynamic limit the model on the circle is equivalent to a gas of free chiral particles with linear dispersion (with a gap), the same for all the species, obeying _ideal_ fractional exclusion statistics \[8, 9\] with the fermionic statistics parameters \( \lambda_{\alpha\beta} \). We refer to the _bosonic_ and _fermionic_ statistics parameters \[21\] \( g_{\alpha\beta} \) and \( \lambda_{\alpha\beta} \) as those appearing in the bosonic and fermionic counting of the statistical weight
\[ W = \prod_\alpha \frac{(D^b_\alpha + N_\alpha - 1)!}{N_\alpha!(D^b_\alpha - 1)!} = \prod_\alpha \frac{D^f_\alpha!}{N_\alpha!(D^f_\alpha - N_\alpha)!}. \] (41)
with
\[ D^b_\alpha = G_\alpha - \sum_\beta g_{\alpha\beta} N_\beta, \quad D^f_\alpha = G_\alpha - \sum_\beta \lambda_{\alpha\beta} N_\beta, \] (42)
resulting in the expression for the entropy (39) upon taking the thermodynamic limit \( N_\alpha \to \infty, G_\alpha \to \infty, \) with \( \rho_\alpha = N_\alpha/G_\alpha \) kept constant. Eq. (41) implies the relation between the bosonic and fermionic statistics parameters
\[ g_{\alpha\beta} = \delta_{\alpha\beta} + \lambda_{\alpha\beta}. \] (43)

Equations for the distribution functions \( \rho_\alpha \) in the equilibrium can be derived from Eqs. (39,40) in the usual way. In terms of the single state grand partition functions \( \xi_\alpha, \) related to the \( \rho_\alpha \)'s by [5]
\[ \xi_\alpha = 1 + \frac{\rho_\alpha}{1 - \sum_\beta (\delta_{\alpha\beta} + \lambda_{\alpha\beta}) \rho_\beta}, \] (44)
these read
\[ (\xi_\alpha - 1) \prod_\beta \xi_\beta^{\lambda_{\alpha\beta}} = e^{(\mu_\alpha - \frac{1}{2} \omega c - v k^\alpha)/T}. \] (45)

3.2. Heat capacity

We now assume that all the species are kept at the same chemical potential, \( \mu_\alpha = \mu. \) We also restrict to the case where all the distributions \( \rho_\alpha \) (and, correspondingly, \( \xi_\alpha \)) are equal to each other. It then follows from (45) that
\[ \lambda M \equiv \sum_{\beta=1}^{M} \lambda_{\alpha\beta} \] (46)
is independent of \( \alpha, \) and (45) coincides with the equation for the partition function for a single species of particles with the statistics parameter (46):
\[ (\xi_\alpha - 1) \xi_\alpha^{\lambda M} = e^{(\mu_\alpha - \frac{1}{2} \omega c - v k^\alpha)/T}. \] (47)

We calculate the low temperature heat capacity for this case following essentially Ref. [22] for the case of a constant density of states, which holds for a linear dispersion in
one dimension (the difference from Ref. [22] is that in the case at hand the particles are chiral and have a gap in the dispersion law). Introducing the variable $w$ by

$$\xi_\alpha = \frac{1}{1 - w} \quad (48)$$

we represent the particle density $d = N/L = \sum_\alpha \int_0^\infty \rho_\alpha(k) \frac{dk}{2\pi}$ and the energy (40) as integrals over the variable $w$:

$$d = \frac{MT}{2\pi v} \int_0^{w(0)} \frac{dw}{1 - w},$$

$$\frac{E}{L} = \frac{MT}{2\pi v} \int_0^{w(0)} \frac{dw}{1 - w} \left[ \mu_\alpha + (1 + \lambda M) T \ln(1 - w) - T \ln w \right] \quad (49)$$

where $w(0)$ is the solution of Eqs. (47) and (48) for $k = 0$. From the first equation in (49) we find

$$w(0) = 1 - e^{-2\pi vd/MT}. \quad (50)$$

From Eqs. (47), (48) we then have

$$\mu_\alpha = \frac{1}{2} \omega_c + 2\pi (\lambda M + 1) v \frac{d}{M} + T \ln \left( 1 - e^{-2\pi vd/MT} \right). \quad (51)$$

Finally, combining this with the second equation in (49), we obtain

$$\frac{E}{L} = d \left[ \frac{1}{2} \omega_c + (\lambda M + 1) \pi v \frac{d}{M} \right] + T d \ln w(0) - \frac{MT^2}{2\pi v} \int_0^{w(0)} dw \frac{\ln w}{1 - w}. \quad (52)$$

At low temperatures $T \ll 2\pi vd/M$, up to non-perturbative corrections, containing the exponential factor $e^{-2\pi vd/MT}$, $w(0)$ can be set to one. The integral in (52) then equals $-\frac{1}{6} \pi^2$, which results in the heat capacity

$$\frac{C_V}{L} = M \frac{\pi}{6v} T. \quad (53)$$

In the single species case this result reduces to that obtained in Ref.[1] in a different way. The present derivation shows that (53) exhausts all the perturbative (behaving as powers of the temperature) terms in the heat capacity.

3.3. Asymptotic Bethe ansatz equations
The connection with ideal exclusions statistics suggests that the equations of the model can be represented in a form similar to asymptotic Bethe ansatz equations \[16\] (the latter have the form of the thermodynamic Bethe ansatz equations \[23\], with two-body scattering phases (matrices) having a step-wise form). Indeed, introducing the “dressed” energies \(\varepsilon_\alpha\) by

\[
\nu_\alpha = \frac{1}{e^{\varepsilon_\alpha/T} + 1}
\]

so that

\[
\xi_\alpha = 1 + e^{-\varepsilon_\alpha/T},
\]

we find that Eqs. \[15\] become

\[
\varepsilon_\alpha = -\mu_\alpha + \varepsilon_\alpha^0(k_\alpha) + T \sum_\beta \lambda_{\alpha\beta} \ln \left(1 + e^{(\mu_\beta - \varepsilon_\beta)/T}\right),
\]

where

\[
\varepsilon_\alpha^0(k_\alpha) = \frac{1}{2} \omega_c + v k_\alpha.
\]

Equations \[56\], along with \[35\], have the form of the asymptotic Bethe ansatz equations for a system of particles with the “bare” energy \[57\] and the step-wise “two-body scattering phases”

\[
\theta(k_\alpha^\alpha - k_\beta^\beta) = \lambda_{\alpha\beta} h(k_\alpha^\alpha - k_\beta^\beta).
\]

One can in addition introduce the density of holes \(\rho_\alpha^h(k)\), related to the interval \(dk^\alpha\), by

\[
(1 - \nu_\alpha(p^\alpha))dp^\alpha = \rho_\alpha^h(k^\alpha)dk^\alpha.
\]

With the use of \[33\] and \[56\], we obtain

\[
\rho_\alpha^h = 1 - \sum_\beta (\lambda_{\alpha\beta} + \delta_{\alpha\beta}) \rho_\beta.
\]

For \(T = 0\), in the case of equal boundary pseudomomenta, \(k_0^\alpha = k_0\), we have

\[
\rho_\alpha = \sum_\beta (\lambda_{\alpha\beta} + \delta_{\alpha\beta})^{-1}, \quad \rho_\alpha^h = 0; \quad k < k_0;
\]

\[
\rho_\alpha = 0, \quad \rho_\alpha^h = 1; \quad k > k_0.
\]
We now turn again to the above special case of equal $\rho_\alpha$ for all species and evaluate the Fermi velocities of quasiparticles (the same for all species) in the picture of interacting fermions for $T = 0$

$$v_F^\alpha \equiv \frac{\partial \varepsilon_\alpha}{\partial p} \bigg|_{p=p_F} = \left( \frac{\partial k}{\partial p} \frac{\partial \varepsilon_\alpha}{\partial k} \right) \bigg|_{k=k_0},$$

(62)

where $p_F$ is the Fermi momentum for the distributions $\nu_\alpha$ and $k_0$ is the boundary (pseudo Fermi) momentum for the distributions $\rho_\alpha$.

For zero temperature, because of the jumps in the distributions of pseudoparticles and holes, the expression (62) is not well defined for $k = k_0$. It is well defined however for $k = k_0^-(\equiv k_0 - 0)$ and for $k = k_0^+(\equiv k_0 + 0)$. In the above special case of equal distribution functions for all the species, we obtain from (56), $\varepsilon_\alpha'(k_-) = v(\lambda M + 1)^{-1}$ and $\varepsilon_\alpha'(k_+^+) = v$. With $\partial p/\partial k = (\lambda M + 1)^{-1}$ for $k = k_0^-$ and $\partial p/\partial k = v$ for $k = k_0^+$ following from (36), we conclude that

$$v_F^\alpha(k_-) = v_F^\alpha(k_0^+) = v, \quad \text{for} \quad T = 0.$$  

(63)

Representing then the Fermi velocities (62) as

$$v_F^\alpha = \begin{cases} 
\frac{\varepsilon_\alpha'(k)}{\rho(k)}, & k = k_0^-; \\
\varepsilon_\alpha'(k), & k = k_0^+,
\end{cases}$$

(64)

we remark that the first line in (64), which is normally used to evaluate the Fermi velocities in the thermodynamic Bethe ansatz method (see e. g. [24]), in the case of step-wise two body phase shifts (asymptotic Bethe ansatz) holds only for $k < k_0$.

Similar peculiarities due to a step-wise form of the two-body scattering phase also arise for other quantities determined by the asymptotic Bethe ansatz equations. We discuss here the dressed charge matrix (which is related to Friedel oscillations, conductivity etc., see [25]). With the two-body scattering phases (58), the dressed charge matrix is determined by the equations [24]

$$Z_{\alpha\beta}(k_\beta) = \delta_{\alpha\beta} - \sum_\gamma \int_0^{k_0} Z_{\alpha\gamma}(k'_\gamma) \lambda_{\gamma\beta} \delta(k'_\gamma - k_\beta) dk'_\gamma.$$  

(65)
The function $Z_{\alpha\beta}(k_\beta)$ thus has a jump at the boundary pseudomomenta $k_0$: below the pseudo Fermi level

$$Z_{\alpha\beta}(k_0^-) = (g^{-1})_{\alpha\beta}, \quad (66)$$

while above the pseudo Fermi level

$$Z_{\alpha\beta}(k_0^+) = \delta_{\alpha\beta}. \quad (67)$$

The relation between the dressed charge matrix and exclusion statistics matrix can also be written in a matrix form as

$$Z(k_0^-)Z(k_0^+) = g^{-1}. \quad (68)$$

which is different from the result $Z(k_0^-)Z(k_0^+) = g^{-1}$ obtained in Ref. [26].

Note that the representation of the Fermi velocity (64) and the relation of the dressed charge matrix to the statistics matrix (68) are valid for an arbitrary “bare” energy of particles $\varepsilon_\alpha^0(k)$ (which in the case at hand is (57)). For the case of a single species, the relation similar to (68), between the dressed charge function and the fractional exclusion statistics parameter, was discussed in Ref. [27].

3.4. Thermal excitations

Because of the nontrivial temperature dependence of the dressed energy $\varepsilon(k)$ determined by Eq.(56), it is not obvious that the above zero temperature quasiparticle excitations survive at finite temperatures. In this subsection, using the exclusion statistics representation of the model, we calculate the Fermi velocities (62) at low but nonzero temperatures. The derivatives with respect to $k$ in (62) then have to be evaluated at the point $\mu_\alpha = \varepsilon_\alpha^0(k)$.

The derivative $\partial p/\partial k$ is given by (36), which in the case at hand reads

$$\frac{\partial p}{\partial k} = 1 - \lambda M \rho_\alpha. \quad (69)$$

To calculate the derivative $\partial \varepsilon_\alpha/\partial k$, we note that, according to (13) and (53), $\varepsilon_\alpha$ depend on $k$ only via $x \equiv e(\mu_\alpha - \varepsilon_\alpha^0(k))/T$, which yields

$$\frac{\partial \varepsilon_\alpha}{\partial k} = \frac{\partial x}{\partial k} \frac{\partial \varepsilon_\alpha}{\partial x} = v_x \frac{\partial \varepsilon_\alpha}{\partial x} \frac{1}{x - 1}. \quad (70)$$
Taking into account the relation between the single state distribution functions and partition functions \((\xi = \Pi_\alpha \xi_\alpha)\),

\[
\rho_\alpha = x \frac{1}{\xi} \frac{\partial \xi}{\partial x} = M x \frac{1}{\xi_\alpha} \frac{\partial \xi_\alpha}{\partial x},
\]

and expressing \(\xi_\alpha\) in terms of \(\rho_\alpha\) with the use of (44), we get

\[
\frac{\partial \varepsilon_\alpha}{\partial k} = v(1 - \lambda M \rho_\alpha).
\]

Note that both the derivatives (69) and (72) are well defined at the point \(\mu_\alpha = \varepsilon^0_\alpha(k)\).

Using these, we finally obtain the Fermi velocities of quasiparticles for \(T \neq 0\)

\[
v^\alpha_F = v.
\]

This calculation proves that the Fermi velocities of the quasiparticles at nonzero temperatures remain the same as at \(T = 0\). As the derivation shows, the conclusion holds for an arbitrary bare energy \(\varepsilon^0_\alpha(k)\) in the asymptotic Bethe ansatz equations (56).

4. Bosonization and relation to edge excitations in multilayer FHQ states

In this section we bosonize the model governed by the Hamiltonian (12) generalizing the procedure given in Ref. [1] to the case of several chiral fields. We then discuss the relation of the bosonized form of the model to the effective low energy description of edge excitations in multilayer FQH states [17, 4].

Consider the Hamiltonian (12) of chiral fields on the circle of length \(L\) specified by (31). The fact that the model is described with only one velocity \(v\) means that in the picture of the anyon droplet, all the species of particles have the same boundary radius. We assume that the boundary (pseudo Fermi) values of \(\kappa\) in the ground state are the same for all species. It then follows from the occupation rules and the ordering procedure described in Section 2 (or, equivalently, from Eq. (30)) that the pseudo Fermi angular momentum is

\[
\kappa^\alpha_0 = (\lambda_{\alpha\alpha} + 1)(N_{0\alpha} - 1) + \sum_{\beta(\neq \alpha)} \lambda_{\alpha\beta} N_{0\beta},
\]
where \( N_{0\alpha} \) is the number of particles of species \( \alpha \) in the ground state. We introduce the chemical potentials for zero temperature by

\[
\mu^0_\alpha = \frac{1}{2} \omega_{\text{eff}} + a \kappa^0_F ,
\]

with

\[
\kappa^0_F = (\lambda_{\alpha\alpha} + 1) \left( N_{0\alpha} - \frac{1}{2} \right) + \sum_{\beta(\neq \alpha)} \lambda_{\alpha\beta} N_{0\beta} .
\]

This choice corresponds to the chemical potential lying on the midway between the highest occupied and the lowest unoccupied energy levels.

The pseudo Fermi levels in the model on the circle are directly related to the physical size of the anyon droplet: for large \( N_\alpha \) one has

\[
R^2 \simeq r_0^2 \kappa^0_F \simeq r_0^2 \sum_\beta g_{\alpha\beta} N_{0\beta} ,
\]

where \( r_0 = \sqrt{2/eB_{\text{eff}}} \) is the effective magnetic length. This means that all \( \kappa^0_F \) have the same value, \( \kappa_F \), from which follows that \( \sum_\beta g_{\alpha\beta} N_{0\beta} \) should not depend on \( \alpha \). This also implies that \( \mu^0_\alpha = \mu_0 \).

We fix the ground state subtracting \( \mu_0 \hat{N} \) from the Hamiltonian (12),

\[
H' \equiv H - \mu_0 \sum_\alpha \hat{N}_\alpha = a \sum_\alpha \int_0^{2\pi} d\theta \chi_\alpha^\dagger(\theta) \left[ (-i\partial_\theta - \kappa_F) + \frac{1}{2} \sum_\beta \lambda_{\alpha\beta} \hat{N}_\beta \right] \chi_\alpha(\theta) .
\]

By normal ordering the Hamiltonian with respect to the Fermi levels, subtracting a constant and redefining the field operators \( \chi_\alpha \) for the rest of this section by \( \chi_\alpha \rightarrow e^{-iN_{0\alpha}\theta} \chi_\alpha \), we obtain the Hamiltonian describing the excitations

\[
H' = a \sum_\alpha \left[ \int_0^{2\pi} d\theta : \chi_\alpha^\dagger(\theta) \left( -i\partial_\theta + \frac{1}{2} \right) \chi_\alpha(\theta) : + \frac{1}{2} \sum_\beta \lambda_{\alpha\beta} Q_\alpha Q_\beta \right] ,
\]

where \( Q_\alpha = \hat{N}_\alpha - N_{0\alpha} \) are the charge operators.

The fermion fields \( \chi_\alpha \) admit the representation (for details of the bosonization procedure, see Ref. [28])

\[
\chi_\alpha^\dagger(\theta) = \frac{e^{i\theta}}{\sqrt{2\pi}} e^{i\frac{\pi}{2} \sum_{\beta(\neq \alpha)} \text{sgn}(\alpha-\beta) Q_\beta} : e^{i\phi_\alpha(\theta)} : ,
\]
where the chiral boson fields are given by

$$\phi_\alpha(\theta) = \phi_\alpha^0 - Q_\alpha \theta + i \sum_{n > 0} \frac{1}{\sqrt{n}} \left[ a_{\alpha n} e^{i n \theta} - a_{\alpha n}^\dagger e^{-i n \theta} \right] ,$$

with $[\phi_\alpha^0, Q_\beta] = i \delta_{\alpha\beta}$ and $[a_{\alpha m}, a_{\beta n}^\dagger] = \delta_{\alpha\beta} \delta_{mn}$. The normal ordering in (80) refers only to $a_{\alpha n}$ and $a_{\alpha n}^\dagger$. The fields $\phi_\alpha$ satisfy the commutation relations

$$[\phi_\alpha(\theta), \phi_\beta(\theta')] = i \pi \delta_{\alpha\beta} \text{sgn}_{\text{per}}(\theta - \theta') ,$$

where $\text{sgn}_{\text{per}}(\theta)$ is the periodic sign function: $(\partial/\partial \theta) \text{sgn}_{\text{per}}(\theta) = 2 \delta_{\text{per}}(\theta)$. The extra phase in (80) containing the charge operators ensures that the fermion operators for distinct species anticommute (rather than commute) with each other.

The bosonized form of the Hamiltonian (79) reads

$$H' = \sum_\alpha \left[ \frac{1}{4\pi} \int_0^{2\pi} d\theta : (\partial_\theta \phi_\alpha)^2 : + \frac{1}{2} \sum_\beta \lambda_{\alpha\beta} Q_\alpha Q_\beta \right]$$

$$= \sum_\alpha \left[ \sum_n n a_{\alpha n}^\dagger a_{\alpha n} + \frac{1}{2} \sum_\beta g_{\alpha\beta} Q_\alpha Q_\beta \right] ,$$

where $g_{\alpha\beta}$ are the bosonic statistics parameters (see (43)).

Since $g_{\alpha\beta}$ is a symmetric matrix, it can be diagonalized by an orthogonal transformation, $(O^{-1} g O)_{\alpha\beta} = \Lambda_\alpha \delta_{\alpha\beta}$. The Hamiltonian then becomes that of a set of uncoupled components with (bosonic) statistics parameters $\Lambda_\alpha$,

$$H' = \sum_\alpha \left[ \sum_n n \tilde{a}_{\alpha n}^\dagger \tilde{a}_{\alpha n} + \frac{1}{2} \Lambda_\alpha \tilde{Q}_\alpha^2 \right] ,$$

where $\tilde{a}_{\alpha n} = O_{\alpha\beta}^{-1} a_{\beta n}$, $\tilde{Q}_\alpha = O_{\alpha\beta}^{-1} Q_\beta$. The transformed operators $\tilde{a}_{\alpha m}$ and $\tilde{a}_{\beta n}^\dagger$ still obey the bosonic commutation relations.

At this stage one has to demand that all the eigenvalues $\Lambda_\alpha$ of the matrix $g_{\alpha\beta}$ should be positive, in order for the Hamiltonian (85) to be positive definite. This means, as we will see below, that all the edge modes in our model propagate in the same direction. Note that the diagonal form of the Hamiltonian (85) implies that the low-temperature heat capacity is just a sum of heat capacities from all the species, $\pi T/6\nu$ from each, in agreement with (53).
The Hamiltonian (85) can be transformed to a free form

\[ H' = \frac{a}{4\pi} \sum_\alpha \int_0^{2\pi} d\theta : (\partial_\theta \bar{\phi}_\alpha)^2 : , \]  

where the new Bose fields are defined by

\[ \bar{\phi}_\alpha(\theta) = \frac{1}{\sqrt{\Lambda_\alpha}} \tilde{\phi}_0^\alpha - \sqrt{\Lambda_\alpha} \tilde{Q}_\alpha \theta + i \sum_{n>0} \frac{1}{\sqrt{n}} \left[ \tilde{a}_{\alpha n} e^{in\theta} - \tilde{a}_{\alpha n}^\dagger e^{-in\theta} \right], \]  

with \([\tilde{\phi}_0^\alpha, \tilde{Q}_\beta] = i\delta_{\alpha\beta}\).

One can define the charged operators corresponding to the fields \(\bar{\phi}_\alpha\):

\[ \tilde{\chi}_\alpha^\dagger(\theta) = \frac{e^{i\theta\gamma^2_\alpha}}{\sqrt{2\pi}} \sum_{\beta\neq\alpha} \frac{\text{sgn}(\alpha-\beta) \tilde{Q}_\beta}{\sqrt{\Lambda_\beta}} e^{i\gamma_\alpha \tilde{\phi}_\alpha(\theta)} : . \]  

These operators have charges \(\bar{q}_\alpha = \gamma_\alpha/\sqrt{\Lambda_\alpha}\) (determined by the commutator \([\tilde{Q}_\alpha, \tilde{\chi}_\alpha^\dagger]\)) as well as “statistics” \(\gamma^2_\alpha\) in the sense that

\[ \tilde{\chi}_\alpha^\dagger(\theta) \tilde{\chi}_\alpha^\dagger(\theta') = e^{i\pi \gamma^2_\alpha \text{sgn}_{\text{per}}(\theta-\theta')} \tilde{\chi}_\alpha^\dagger(\theta') \tilde{\chi}_\alpha^\dagger(\theta) . \]  

In particular, if one chooses \(\gamma_\alpha = \sqrt{\Lambda_\alpha}\) and if \(\Lambda_\alpha = 2m_\alpha + 1\), with \(m_\alpha\) a positive integer, the operator (88) is an “electron” operator of the kind discussed by Wen [3]. Another special case \(\gamma_\alpha = 1/\sqrt{\Lambda_\alpha} = 1/\sqrt{2m_\alpha + 1}\) corresponds to a particle with fractional charge and statistics equal to \(1/(2m_\alpha + 1)\) like a fundamental quasiparticle in a \(\nu = 1/(2m_\alpha + 1)\) FQH layer. The fields corresponding to distinct species anticommute with each other, \(\{\tilde{\chi}_\alpha^\dagger(\theta), \tilde{\chi}_\beta^\dagger(\theta')\} = 0\), etc.

A general charged operator can be composed of operators of the above type:

\[ \chi^\dagger(\theta) = \frac{e^{i\theta \sum_\alpha \gamma^2_\alpha}}{(2\pi)^{M/2}} : e^{i\sum_\alpha \gamma_\alpha \tilde{\phi}_\alpha(\theta)} : , \]  

which corresponds to the total charge \(\sum_{\alpha\beta} O_{\alpha\beta} \gamma_\beta/\sqrt{\Lambda_\beta}\) associated with the operator \(\sum_\alpha Q_\alpha\).

To discuss the relation to the edge excitations in multilayer FQH systems [17, 4], we rescale the Bose fields \(\tilde{\phi}_\alpha\) and transform back to the non-diagonal picture, using the inverse of the above orthogonal transformation. This defines new fields

\[ \bar{\phi}_\alpha \equiv \frac{1}{\sqrt{\Lambda_\beta}} \bar{\phi}_\beta , \]  

(91)
obeying the commutation relations (cf. (92))

\[
\left[ \bar{\phi}_\alpha(\theta), \bar{\phi}_\beta(\theta') \right] = i\pi g^{-1}_{\alpha\beta} \text{sgn}_{\text{per}}(\theta - \theta') .
\] (92)

The Hamiltonian (79) takes the form

\[
H' = \frac{a}{4\pi} \sum_{\alpha\beta} g_{\alpha\beta} \int_0^{2\pi} d\theta : (\partial_\theta \bar{\phi}_\alpha)(\partial_\theta \bar{\phi}_\beta) : .
\] (93)

For the case where the diagonal elements of the matrix \( g_{\alpha\beta} \) are odd numbers and the off-diagonal elements are integers, Eqs. (92), (93) reduce to those describing edge excitations in multilayer abelian FQH states [4] if one identifies

\[
g_{\alpha\beta} = K_{\alpha\beta}, \quad v g_{\alpha\beta} = V_{\alpha\beta},
\] (94)

where \( K_{\alpha\beta} \) is the topological matrix (in the symmetric basis) and \( V_{\alpha\beta} \) is a positive definite matrix describing the interaction between the chiral bosonic modes [4]. Thus, our model with only one velocity \( v \), corresponds to the special case where the matrices \( K \) and \( V \) are proportional to each other. In the more general case of the effective Luttinger liquid description of the edge excitations, when different edge modes may have different velocities, the matrices \( V \) and \( K \) can be simultaneously diagonalized, and the velocities are determined by the ratios of their respective eigenvalues [29].

Note that the conclusion about the identification of the exclusion statistics matrix with the topological matrix (the first relation in (94)) was previously obtained by Fukui and Kawakami for hierarchical FQH states (by comparing the excitation spectrum for the edge excitations and chiral particles obeying ideal fractional exclusion statistics) [12].

In addition, expressing the general charged operator (90) in terms of the fields \( \bar{\phi}_\alpha \) as

\[
\chi^\dagger(\theta) = \frac{e^{i\theta} \sum_{\alpha\beta} l_{\alpha} g_{\alpha\beta}^{-1} l_{\beta}}{(2\pi)^{M/2}} : e^{i \sum_{\alpha} l_{\alpha} \bar{\phi}_\alpha(\theta)} : ,
\] (95)

with \( l_{\alpha} = O_{\alpha\beta} \sqrt{A_{\gamma\beta}} \), we find that this operator has the total charge

\[
Q_1 = \sum_{\alpha\beta} l_{\beta} g_{\alpha\beta}^{-1}
\] (96)
and statistics $\sum_{\alpha\beta} l_\alpha g_{\alpha\beta}^{-1} l_\beta$. For integer $l_\alpha$, this agrees with the expressions obtained by Wen and Zee in the symmetric basis [17]. Finally, the filling factor is

$$\nu = \sum_{\alpha\beta} g_{\alpha\beta}^{-1} = \sum_\alpha \tilde{t}_\alpha \frac{1}{\Lambda_\alpha} \tilde{t}_\alpha$$

(97)

where $\tilde{t}_\alpha = \sum_\beta O_{\alpha\beta}^{-1}$. Figure 1 illustrates the first equality in (97) for a simple example of two components.

5. Concluding Remarks

We have constructed a model describing several chiral fields on the circle which has the same quantum numbers as a system of anyons of several species in the lowest Landau level. The parameters of the model range in wider intervals than the original (statistics) parameters of anyons; this is achieved by the analytic continuation of the solutions corresponding to anyons in the LLL.

The model incorporates some features of the physics described by the electron wave functions corresponding to abelian multilayer FQH states. The harmonic potential added into the system of anyons plays the role of the confining potential of electrons in the FQH states. The model on the circle was found to recover the effective low energy description of edge excitations in the special case of equal velocities of all edge modes. In this sense, the model obtained can be considered as a possible dynamic theory underlying the effective chiral Luttinger liquid description of edge excitations.

In this context we note that some of the correlation functions as well as low temperature thermodynamics of edge excitations have been obtained using only the effective Luttinger liquid description of the edge excitations (see e. g. the review [4]). On the other hand, recent calculations of nonequilibrium transport properties through a point contact in a Luttinger liquid (which is related to tunneling transport between edge states in FQH devices) are based on a particular (integrable) dynamic model of the Luttinger liquid [30].

In this connection the question arises to what extent these nonequilibrium properties depend on a particular dynamic model of edge excitations. The model we have discussed has a simple dynamics encoded in step-wise two-body scattering phases when it is for-
mulated in the form of the asymptotic Bethe ansatz equations (see Sect. 3). It seems therefore to be interesting to use this model to investigate the above issue.

Another remark concerns the edge excitations for hierarchical FQH states. Integrable models with long range interactions can be constructed which are described by the same matrices as the topological matrices corresponding to hierarchical FQH states (see [31]). A simple link between the integrable models with long range interactions and edge excitations for FQH states is given by the (ideal) fractional exclusion statistics. Exclusion statistics provides, in addition, a connection with anyons in the LLL [9, 10, 5]. This suggests that an explicit dynamic model of edge excitations for the hierarchical FQH states can also be constructed, along the lines of the present paper, starting from the appropriate picture of anyons in the LLL.

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Fig. 1: Ground state of a two-component system with \( g_{11} = 3 \), \( g_{22} = 5 \) and \( g_{12} = g_{21} = 1 \) (bosonic representation). The white and black circles denote component 1 and component 2, respectively. The state has been built up according to the normal ordering convention, with increasing energy (\( \kappa \)) values. The pattern repeats periodically. The filling factor is \( 3/7 = \sum_{\alpha \beta} g_{\alpha \beta}^{-1} \) (if the number of particles in the ground state is such that the Fermi level is the same for both species).