Observables

in

Topological Yang-Mills Theories

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Abstract: Using topological Yang-Mills theory as example, we discuss the definition and determination of observables in topological field theories (of Witten-type) within the superspace formulation proposed by Horne. This approach to the equivariant cohomology leads to a set of bi-descent equations involving the BRST and supersymmetry operators as well as the exterior derivative. This allows us to determine superspace expressions for all observables, and thereby to recover the Donaldson-Witten polynomials when choosing a Wess-Zumino-type gauge.

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## Contents

1 Introduction 3

2 Symmetries 4
   2.1 Superspace 4
   2.2 BRST-formalism 6
   2.3 Topological Yang-Mills theory in superspace 7
   2.4 Topological Yang-Mills in the Wess-Zumino gauge 9
      2.4.1 Symmetry transformations in the WZ-gauge 9
      2.4.2 Witten’s observables and descent equations 10
      2.4.3 Combining all symmetries 11

3 Observables in the superspace formalism 12
   3.1 Equivariant cohomology and Witten’s observables 12
   3.2 The bi-descent equations 15
   3.3 Superform solutions of the bi-descent equations 18
   3.4 General solution of the bi-descent equations for the pair \((d, D)\) 22

4 Explicit expressions 23
   4.1 An example of bi-descent and superdescent equations 23
   4.2 Some examples of observables 24
      4.2.1 Solution corresponding to the Casimir of \(U(1)\) 25
      4.2.2 Solution corresponding to the Casimir of \(SU(2)\) 26
      4.2.3 An example of “composite observables” 27

5 Concluding remarks 27

A Proofs of some propositions and lemmas 28
1 Introduction

Topological field theories have been introduced some fifteen years ago \cite{1, 2} and continue to represent a field of active interest, e.g. see ref. \cite{3, 4, 5}. The purpose of the present work is to come back to the issue of determining all of the observables for these theories (for some general reviews, see ref. \cite{6}). These observables are of a global nature, e.g. knot invariants in Chern-Simons theory \cite{2} or Donaldson invariants in topological Yang-Mills (YM) theory \cite{11} as well as the counterparts of the latter in topological gravity \cite{1}. For topological YM and gravity theories, these observables belong to the so-called equivariant cohomology as originally shown by Witten in his pioneering work on 4d topological YM theory \cite{1} and further elucidated in the sequel from the mathematical point of view \cite{8}. Equivariant cohomology amounts to computing the cohomology of a supersymmetry-like operator $\tilde{Q}$ (which is the BRST operator associated to the local shift symmetry of gauge fields) in the space of gauge invariant local functionals of the fields. A crucial point is that the cohomology of $\tilde{Q}$, although empty in the space of the unrestricted local functionals, becomes nonempty if gauge invariance is imposed on these functionals \cite{8, 9}.

As pointed out by Horne \cite{10}, the supersymmetry operator may be represented as the derivative with respect to a Grassmann-odd parameter $\theta$ within a superfield formalism in which gauge invariance is implemented as supergauge invariance following the introduction of a superconnection. Although superfield formulations of this type have been found to be quite useful for the discussion of the dynamics and symmetries of topological models of Witten-type (also termed cohomological field theories) \cite{10, 8, 11}, they have not been considered so far for the determination of observables. The present paper fills this gap and shows that one can directly apply the powerful methods and results of the BRST cohomology associated to (super)gauge invariance \cite{12}. This provides a complete basis of observables and – as expected – it allows us to recover Witten’s results which have been tackled using other approaches in the past \cite{13, 9, 14, 15}.

We shall be fairly explicit in our presentation since the present work will serve as a basis for the systematic study \cite{16, 17} of more complex models involving equivariant cohomology like topological gravity in various dimensions \cite{1} and YM theories with more than one supersymmetry generator \cite{18, 5}. We note that the techniques that
we develop for the treatment of bi-descent equations should also be useful in other contexts where equations of this type appear, e.g. see [4].

Our paper is organized as follows. In section 2 we present the general framework and, in particular, the BRST formalism for topological YM theories in the superspace associated with the shift supersymmetry. If the supergauge invariance is fixed by a Wess-Zumino type condition, we recover the field content and transformation laws that have been considered in the original literature [1, 8, 13]. In section 3 we determine the cohomology of the BRST operator in the functional space constrained by the requirements of supersymmetry invariance and zero ghost-number. We shall see that it corresponds to a certain subset of the cohomology $H(S|d)$ of the BRST operator $S$ modulo the exterior derivative $d$ in the space of differential forms whose coefficients are superfields. Some explicit examples are presented in section 4. An appendix gathers the proofs of several lemmas and propositions presented in the main body of the text.

Although the formalism is motivated by 4-dimensional topological YM theories, the value of the spacetime dimension will not be specified. In fact, we shall not consider the dynamics nor address the problem of gauge-fixing (requiring the introduction of antighosts and Lagrange multiplier fields) and thereby our results have a purely algebraic character. In particular, they are completely independent of the spacetime dimension.

2 Symmetries

Topological field theories of Witten-type can be obtained from extended supersymmetric gauge theories by performing an appropriate twist. The invariance under extended supersymmetry transformations then gives rise to a shift symmetry in the topological model. Thus, the latter invariance is often referred to as supersymmetry transformation and it can be conveniently described in a superspace [10]. The superspace formulation that we shall use is the one of Horne [10], though the latter author did not elaborate on supergauge transformations whose inclusion is essential for the discussion of observables. Let us first introduce superspace and the geometric objects that it supports.

2.1 Superspace

We extend the $n$-dimensional spacetime manifold by a single Grassmannian variable $\theta$ so as to obtain a superspace parametrized by local coordinates $(x, \theta)$. We assign a “supersymmetry-number” ($SUSY$-number or $SUSY$-charge for short) to all fields
and variables\textsuperscript{3}: for the variable \(\theta\), this number is \(-1\) and, quite generally, an upper or lower \(\theta\)-index on a field corresponds to a SUSY-number \(-1\) or \(+1\), respectively.

A superfield is a function on superspace,
\[
F(x, \theta) = f(x) + \theta f_\gamma'(x),
\tag{2.1}
\]
where \(f(x)\) has the same Grassmann parity as \(F(x, \theta)\) while its superpartner \(f_\gamma'(x)\) has the opposite parity. To be more precise, the superfield (and thereby its components) is also supposed to transform in a specific way under supersymmetry transformations, see eqs. (2.5) below.

A \(p\)-superform admits the expansion
\[
\hat{\Omega}_p(x, \theta) = \sum_{k=0}^{p} \Omega_{p-k}(x, \theta) (d\theta)^k,
\tag{2.2}
\]
where \(\Omega_{p-k}\) has \(k\) lower \(\theta\)-indices that we did not spell out. The components \(\Omega_q(x, \theta)\) of the \(p\)-superform (2.2) are \(q\)-forms whose coefficients are superfields:
\[
\Omega_q(x, \theta) = \frac{1}{q!} \Omega_{\mu_1 \ldots \mu_q}(x, \theta) \, dx^{\mu_1} \cdots dx^{\mu_q} = \omega_q(x) + \theta \omega_q'(x).
\tag{2.3}
\]
Superspace expressions of the form (2.3) will be referred to as superfield forms in the sequel. In the expansion (2.3) and in the following, the wedge product symbol is always omitted. Moreover, we shall adhere to the notational conventions used in the previous expressions: functions or forms on ordinary spacetime are denoted by small case letters (e.g. \(f, f_\gamma', \omega_q, \omega_q', \ldots\)), superfields or superfield forms by upper case letters (e.g. \(F, \Omega_q, \ldots\)) and \(p\)-superforms with \(p \geq 1\) (e.g. \(\hat{\Omega}_p\)) by upper case letters with a “hat”.

The exterior derivative in superspace is defined by
\[
\hat{d} = d + d\theta \partial_\theta \quad \text{with} \quad d = dx^\mu \partial_\mu.
\tag{2.4}
\]
We have \(0 = \hat{d}^2 = d^2 = (d \theta \partial_\theta)^2 = [d, d\theta \partial_\theta]\) where the bracket \([\cdot, \cdot]\) denotes the graded commutator.

A global, infinitesimal supersymmetry transformation is given by a translation of the \(\theta\)-variable, i.e. \(\theta \rightarrow \theta + \epsilon^\theta\). Thus, it is a supercoordinate transformation generated by the vector field \(\epsilon^\theta \partial_\theta \equiv \epsilon^\theta Q\) where \(Q = \partial_\theta\) represents the supersymmetry generator. The latter operator is nilpotent (i.e. \(Q^2 = 0\)) and it raises the SUSY-number by one unit. The supersymmetry transformations of the superfield (2.1) and of its component fields read as
\[
QF = \partial_\theta F \quad \tag{2.5}
\]
\[
Qf = f_\gamma' \quad Qf_\gamma' = 0.
\]

\textsuperscript{3}Originally this number was referred to as “ghost-number” \(U\).
Following standard practice, we use the same symbol $Q$ to denote the action of the supersymmetry generator $Q$ on either component fields or on superfields, superfield forms and superforms. On each of the latter, $Q$ acts by virtue of the $\theta$-derivative. Thus, any superfield (2.1) or superform (2.2) has the general form

$$F(x, \theta) = f(x) + \theta (Qf)(x)$$

$$\hat{\Omega}_p(x, \theta) = \Omega_p(x) + \theta (Q\hat{\Omega}_p)(x) \quad \text{with} \quad \hat{\Omega}_p(x) = \sum_{k=0}^{p} \Omega_{p-k}(x) (d\theta)^k.$$ (2.6)

While an ordinary $p$-form can be integrated over a manifold of dimension $p$, there is no directly analogous theory of integration for superforms [19]. Yet one can introduce some algebraic integration rules which are quite useful for the discussion of descent equations in the BRST formalism. To do so, we consider a collection $\mathcal{M} = (M_0, M_1, \ldots, M_p)$ of closed spacetime manifolds $M_k$ of dimension $k$ and we define the spacetime integral of a $p$-superform on this collection by the direct sum

$$\int_\mathcal{M} \hat{\Omega}_p(x, \theta) \equiv \sum_{k=0}^{p} (d\theta)^k \int_{M_{p-k}} \Omega_{p-k}(x, \theta).$$ (2.7)

This expression still depends on $\theta$. Since integration with respect to the Grassmannian variable $\theta$ means derivation with respect to $\theta$ (i.e. the operation $Q = \partial_\theta$), we set

$$\int_\theta \int_\mathcal{M} \hat{\Omega}_p(x, \theta) = Q \int_\mathcal{M} \hat{\Omega}_p(x, \theta)$$

or, more explicitly,

$$\int_\theta \int_\mathcal{M} \hat{\Omega}_p(x, \theta) = \sum_{k=0}^{p} (d\theta)^k \int_{M_{p-k}} Q\Omega_{p-k}(x, \theta) = \sum_{k=0}^{p} (d\theta)^k \int_{M_{p-k}} \omega'_{p-k}(x).$$ (2.8)

The so-defined expression can be referred to as superspace integral of a $p$-superform.

## 2.2 BRST-formalism

Within the BRST-formalism, the parameters of infinitesimal symmetry transformations are turned into ghost fields. The latter have ghost-number $g = 1$ while the fundamental fields appearing in the invariant action (i.e. the connection for topological YM theory) have a vanishing ghost-number. The Grassmann parity of an object is given by the parity of its total degree defined as the sum $p + g + s$ of its form degree $p$, its ghost-number $g$ and its SUSY-number $s$. All commutators and brackets are assumed to be graded according to this grading.

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4A manifold of dimension 0 represents a point, $M_0 = \{y\}$, and $\int_{M_0} \Omega_0(x, \theta) \equiv \Omega_0(y, \theta)$. 

6
2.3 Topological Yang-Mills theory in superspace

The basic variables are the connection 1-superform $\hat{A}(x, \theta)$ and the ghost superfield $C(x, \theta)$ which corresponds to infinitesimal supergauge transformations. These variables are Lie algebra valued, i.e.

$$\hat{A} = \hat{A}^a T_a, \quad C = C^a T_a, \quad [T_a, T_b] = f_{ab}^c T_c,$$

where the matrices $T_a$ represent the generators of the Lie group that is chosen as structure group of the theory.

The BRST transformations of $\hat{A}$ and $C$ describing the supergauge invariance of the theory read as

$$S\hat{A} = -(dC + [\hat{A}, C]), \quad SC = -C^2.$$  \hfill (2.9)

The so-defined BRST operator $S$ is nilpotent, i.e. $S^2 = 0$.

Let us now introduce the components of the 1-superform $\hat{A}$,

$$\hat{A} = A(x, \theta) + d\theta A_\theta(x, \theta),$$  \hfill (2.10)

as well as the spacetime components of all superfield forms:

$$A(x, \theta) = a(x) + \theta \psi_\theta(x), \quad C(x, \theta) = c(x) + \theta c'_\theta(x)$$

$$A_\theta(x, \theta) = \chi_\theta(x) + \theta \phi_\theta(x).$$  \hfill (2.11)

Here, $a$ denotes the connection 1-form associated to ordinary gauge transformations and $c$ the corresponding ghost. In the sequel, the covariant derivative with respect to $a$ will be denoted by $D_a c = dc + [a, c]$ and the $\theta$-indices labeling spacetime fields will be omitted in order to simplify the notation.

Substitution of (2.10) into (2.9) yields the BRST transformations of $A$ and $A_\theta$,

$$SA = -(dC + [A, C]) \equiv -D_A C, \quad SA_\theta = -(\partial_\theta C + [A_\theta, C])$$  \hfill (2.12)

and the expansions (2.11) provide the BRST transformations of the spacetime fields:

$$Sa = -D_a c, \quad Sc = -c^2$$

$$S\psi = -[c, \psi] - D_a c', \quad S\psi' = -[c, c']$$  \hfill (2.13)

$$S\phi = -[c, \phi] - [X, c'], \quad S\chi = -[c, \chi] - c'.$$

The supersymmetry transformations of all component fields appearing in (2.11) follow from (2.3):

$$Qa = \psi, \quad Q\chi = \phi, \quad Qc = c'$$

$$Q\psi = 0, \quad Q\phi = 0, \quad Qc' = 0.$$  \hfill (2.14)
We have the graded commutation relations
\[ [\mathcal{S}, Q] = [\mathcal{S}, d] = [d, Q] = 0. \]

It is quite useful to consider the following redefinitions of superfield \( S \):
\[ \Psi \equiv \partial_\theta A + D_A A_\theta = \psi + D_A A_\theta \]
\[ \Phi \equiv \partial_\theta A_\theta + A_\theta^2 = \phi + A_\theta^2 \]
\[ K \equiv -(\partial_\theta C + [A_\theta, C]) = -c' - [A_\theta, C]. \] \tag{2.15}

In fact, in terms of these expressions, the \( d\theta \)-expansion of the supercurvature form \( \hat{F} = \hat{d}\hat{A} + \hat{A}^2 \) reads as
\[ \hat{F} = F_A + \Psi d\theta + \Phi (d\theta)^2, \quad \text{with} \quad F_A = dA + A^2, \] \tag{2.16}
while the BRST transformations read as
\[ SA = -D_A C, \quad SC = -C^2 \]
\[ S\Psi = -[C, \Psi], \quad S\Phi = -[C, \Phi] \]
\[ SA_\theta = K, \quad SK = 0, \] \tag{2.17}
and the supersymmetry transformations are given by
\[ QA = \Psi - D_A A_\theta, \quad Q\Psi = -D_A \Phi - [A_\theta, \Psi] \]
\[ QF_A = -D_A \Psi - [A_\theta, F_A], \quad Q\Phi = -[A_\theta, \Phi]. \] \tag{2.18}

We note that \( Q \) acts on \( A, \Psi, \Phi \) and the curvature \( F_A \) according to
\[ Q = Q_0 + S|_{C=A_\theta}, \]
with
\[ Q_0 A = \Psi, \quad Q_0 \Psi = -D_A \Phi, \quad Q_0 \Phi = 0 \]
\[ (Q_0)^2 = \text{infinitesimal supergauge transformation with parameter } \Phi. \] \tag{2.19}

Thus, the operator \( Q_0 \) is nilpotent when acting on an invariant polynomial depending on the variables \( F_A, \Psi, \Phi, D_A \Psi, D_A \Phi \).

In this paper, all space-time forms will be taken as polynomials of the basic forms \( a, \psi, \chi, \phi, c, c' \) and their \( d \)-derivatives. Superfield forms and superforms will be taken as polynomials of the basic superfield forms \( A, A_\theta, C \) and their \( \partial_\theta \)- and \( d \)-derivatives. Since we only discuss the kinematics, we do not fix \( a \) priori the space-time dimension. The respective functional spaces will be denoted by
\[ E : \text{space-time forms}, \quad E_S : \text{superfield forms}, \quad \hat{E}_S : \text{superforms}. \] \tag{2.20}

We conclude this section with two results which will be important for our investigations:
Proposition 2.1 The cohomology $H(Q)$ of the supersymmetry operator $Q$ in the space $\mathcal{E}$, $\mathcal{E}_S$ or $\hat{\mathcal{E}}_S$ is trivial, i.e.

$$\text{If } Q\varphi = 0 , \text{ then } \varphi = Q\varphi' ,$$

with both $\varphi$ and $\varphi'$ belonging to either $\mathcal{E}$, $\mathcal{E}_S$ or $\hat{\mathcal{E}}_S$.

(2.21)

Proof: For the functional space $\mathcal{E}$, the proof follows from the fact that all fields represent $Q$-doublets $\{f, f'\}$ with $Qf = f'$ and $Qf' = 0$, and from a well-known result according to which such doublets do not contribute to the cohomology (e.g. see proposition 5.8. of reference [21]). The extension of this result to the spaces $\mathcal{E}_S$ and $\hat{\mathcal{E}}_S$ is straightforward, the action of the operator $Q$ being given on these spaces by the derivative $\partial_\theta$.

$q.e.d.$

Proposition 2.2 ("Algebraic Poincaré Lemma"). The cohomology $H(d)$ of the exterior derivative $d$ in the space $\mathcal{E}$, $\mathcal{E}_S$ or $\hat{\mathcal{E}}_S$ is trivial.

Proof: The result for the space $\mathcal{E}$ is well known [12] within the present context where the space-time dimension is not fixed a priori. The extension to the spaces $\mathcal{E}_S$ or $\hat{\mathcal{E}}_S$ follows by considering an expansion in $\theta$ or in $\theta$ and $d\theta$, respectively, and by using the linearity of $d$.

$q.e.d.$

2.4 Topological Yang-Mills in the Wess-Zumino gauge

2.4.1 Symmetry transformations in the WZ-gauge

The supergauge freedom can be reduced to the ordinary gauge freedom by imposing the Wess-Zumino (WZ) supergauge condition

$$\chi = 0 .$$

(2.22)

By virtue of eqs. (2.13), the $S$-invariance of this condition requires $c' = 0$. The $S$-variations (2.13) then reduce to the BRST transformations in the WZ-gauge which describe ordinary gauge transformations:

$$Sa = -D_a c , \quad S\psi = -[c, \psi] , \quad S\phi = -[c, \phi] , \quad Sc = -c^2 .$$

(2.23)

Condition (2.22) is not invariant under the SUSY generator $Q$, i.e. under the variations (2.14). A modified SUSY operator $\tilde{Q}$ which leaves this condition stable is obtained by combining $Q$ with a compensating BRST transformation (2.13) according to

$$\tilde{Q} = Q + S|_{\chi = c = 0, c' = 0} \text{ on } a, \psi, \phi .$$

(2.24)
Thus, we get the *supersymmetry transformations in the WZ-gauge*,

\[ \tilde{Q} a = \psi \ , \quad \tilde{Q} \psi = -D_a \phi \ , \quad \tilde{Q} \phi = 0 \ , \]

(2.25)

which satisfy

\[ \tilde{Q}^2 = \text{infinitesimal gauge transformation with parameter } \phi \ . \]

(2.26)

A crucial point of the theory is the fact that the operator \( \tilde{Q} \) is nilpotent when acting on invariant polynomials, very much like the operator \( Q_0 \) defined in (2.19). We also note that the algebra generated by the forms \( a, \psi \) and \( \phi \) and their exterior derivatives, together with the action of the operators \( S \) and \( \tilde{Q} \) given by (2.23) and (2.25) is isomorphic to the algebra generated by the superfield forms \( A, \Psi \) and \( \Phi \) (defined in eq. (2.15)) and their exterior derivatives, together with the action of the operators \( S \) and \( Q_0 \) given by (2.17) and (2.19).

The supersymmetry-BRST formalism defined by eqs. (2.25) and (2.23) is the one used by Witten in his pioneering work on four-dimensional topological YM theory \[1\]. We emphasize that the only ghost field in this approach is \( c \) (as well as \( c' \) in a general supergauge). This fact is in contrast to some other approaches where \( \psi \) and \( \phi \) have ghost-numbers 1 and 2, respectively (e.g. see \[8, 13\]).

For later reference, we display the \( \theta \)-expansion of the superconnection \( \hat{A} \) and of the associated curvature \( \hat{F} = \hat{d} \hat{A} + \hat{A}^2 \) in the WZ-gauge (cf. (2.16)):

\[ \hat{A} = a + \theta \psi + \theta d\theta \phi \]
\[ \hat{F} = F_a + \psi d\theta + \phi (d\theta)^2 - \theta D_a \psi - \theta d\theta D_a \phi \ . \]

(2.27)

### 2.4.2 Witten’s observables and descent equations

The expression (2.27) of \( \hat{F} \) has the form

\[ \hat{F} = \mathcal{F} + \theta \tilde{Q} \mathcal{F} \quad \text{with} \quad \mathcal{F} \equiv F_a + \psi d\theta + \phi (d\theta)^2 , \]

(2.28)

i.e. it is of the same form as a generic superform in a general gauge, cf. eq. (2.6). More specifically, one can check that we have

\[ \tilde{Q} \mathcal{F} = -(D_a \mathcal{F}) (d\theta)^{-1} \ , \]

(2.29)

where the notation \((d\theta)^{-1}\) is symbolic, though it can be further justified.

The quantity \( \mathcal{F} \) represents the curvature of the universal bundle considered by Baulieu and Singer \[13\] in their derivation of Witten’s observables. (Actually, these

\[^5\text{In fact, in these approaches our ghost- and SUSY-numbers are added together so as to yield a single BRST ghost-number.}\]
authors did not introduce the monomials $d\theta$, rather they associated ghost-numbers 1 and 2 to $\psi$ and $\phi$, respectively.) For the derivation of observables, we can argue as follows. For $m = 1, 2, \ldots$, we have

$$\text{Tr} \hat{F}^m = \text{Tr} F^m + \theta \hat{Q} \text{Tr} F^m,$$

where the first term yields the Donaldson-Witten polynomials,

$$\text{Tr} F^m = \text{Tr} F^m + \text{Tr} (mF_a (m-1) \psi) d\theta + \cdots + \text{Tr} (m\psi (m-1) (d\theta)^2 + \text{Tr} \phi^m (d\theta)^2$$

$$\equiv \sum_{p=0}^{2m} \omega_p (d\theta)^{2m-p}$$

(2.30)

and where the second term represents a total derivative by virtue of eq.(2.29):

$$\hat{Q} \text{Tr} F^m = -d \text{Tr} F^m (d\theta)^{-1}.$$

By substituting the expansion (2.30) into the last relation, we obtain Witten’s descent equations for the polynomials $\omega_p$:

$$\hat{Q} \omega_p + d\omega_{p-1} = 0 \quad (p = 0, 1, \ldots, 2m).$$

(2.31)

Here and in the following, the forms of negative form degree are assumed to vanish by convention.

### 2.4.3 Combining all symmetries

It is possible to incorporate the transformations (2.25) into the BRST algebra by introducing a constant commuting ghost $\varepsilon$: the BRST operator then acts on $a, \psi, \phi$ according to

$$s_{\text{tot}} = s + \varepsilon \hat{Q}$$

(2.32)

and on $c, \varepsilon$ according to

$$s_{\text{tot}} c = -c^2 + \varepsilon^2 \phi, \quad s_{\text{tot}} \varepsilon = 0,$$

(2.33)

which ensures the nilpotency of the operator $s_{\text{tot}}$. More explicitly, we have the expansion [9]

$$s_{\text{tot}} = s_0 + \varepsilon s_1 + \varepsilon^2 s_2,$$

(2.34)

with

$$s_0 a = -D_a c, \quad s_1 a = \psi, \quad s_2 a = 0$$

$$s_0 \psi = -[c, \psi], \quad s_1 \psi = -D_a \phi, \quad s_2 \psi = 0$$

$$s_0 \phi = -[c, \phi], \quad s_1 \phi = 0, \quad s_2 \phi = 0$$

(2.35)

$$s_0 c = -c^2, \quad s_1 c = 0, \quad s_2 c = \phi.$$
where

\[(S_0)^2 = 0, \quad [S_0, S_1] = 0, \quad (S_1)^2 + [S_0, S_2] = 0. \tag{2.36}\]

In terms of the notation introduced above, we have \(S_0 = S\) and \(S_1 = \tilde{Q}\) on \(a, \psi, \phi\). If we only consider functionals \(\Delta\) depending on \(a, \psi, \phi\) and not on \(c\) (i.e. functionals of zero ghost-number), then the last relation of (2.36) is nothing but (2.26). If these functionals are, in addition, gauge invariant, then the operator \(S_1\) is nilpotent:

\[S_0\Delta = 0 = S_2\Delta \implies (S_1)^2 = 0. \tag{2.37}\]

Its cohomology is referred to as equivariant cohomology and will be further discussed in the next section. (Thus, equivariant cohomology is the cohomology of the operator \(S_1\) in the space of local functionals of \(a, \psi, \phi\) and \(c\) restricted by \(S_2\) - and \(S_0\) - invariance.)

To conclude, we note that the algebra (2.32)–(2.33) can also be obtained along a slightly different, though equivalent line of reasoning. In fact, we could include the supersymmetry variations generated by \(\varepsilon Q\) right away into the BRST transformations (2.13): the stability of the WZ-condition (2.22) then restricts the ghost field \(c'\) to be equal to \(\varepsilon \phi\) and readily yields the results (2.32)–(2.33). The decoupling (2.23)–(2.25) is realized \([9]\) by considering the filtration \(N = \varepsilon \partial / \partial \varepsilon\) and the expansion (2.34).

\section{Observables in the superspace formalism}

In the following, the expression \(*\varphi^g_p\) denotes a \(p\)-form \(\varphi\) of ghost-number \(g\) and SUSY-number \(s\).

\subsection{Equivariant cohomology and Witten’s observables}

Let us first consider the WZ-gauge setting described in the preceding section since the latter has been chosen in all former discussions of observables. The representation (2.34)–(2.36) of the complete set of symmetry transformations is quite useful for specifying the cohomological characterization of observables. As is well known, the cohomology of the operator \(S_{\text{tot}}\) is empty \([5]\). Not so the equivariant cohomology which can be described in several different ways \([11]\). As mentioned in the last section, it can be characterized as the cohomology of the operator \(\tilde{Q}\) (defined by (2.25)) in the space of the gauge invariant local functionals of \(a, \psi, \phi\) \([11]\). Thus, at form degree zero, one looks for a local functional \(*\Delta(0) = \int_{M_0} *\omega^0_0(x)\) which solves the \(\tilde{Q}\)-cyclicity condition, i.e.

\[\tilde{Q} *\Delta(0) = 0, \tag{3.1}\]
and which is constrained by gauge invariance, i.e.

$$S^s \Delta_{(0)} = 0 .$$  (3.2)

This cocycle is required to be non-trivial, i.e.

$$s \Delta_{(0)} \neq \tilde{Q}^{s-1} \Delta'_{(0)} \quad \text{with} \quad S^{s-1} \Delta'_{(0)} = 0 ,$$  (3.3)

where $s^{-1} \Delta'_{(0)} = \int_{M_0} s^{-1} \omega^0_0(x)$. From the $\tilde{Q}$-transformation laws (2.25), it follows that zero-forms cannot be written as a $\tilde{Q}$-variation. Thus, the non-triviality condition (3.3) is automatically satisfied at form degree zero. (Note that this is no longer true at higher form degree: an expression of the form

$$\int \tilde{Q} P_{\text{inv}}(F_a, \psi, \phi, D_a \psi, D_a \phi) ,$$  (3.4)

where $P_{\text{inv}}$ is a \(S\)-invariant polynomial, is $\tilde{Q}$- and $S$-invariant, but $\tilde{Q}$-trivial.) As pointed out by Witten [1], the equations for the integrand of $s \Delta_{(0)}$, i.e. $\tilde{Q}^{s} \omega^0_0 = 0 = S^s \omega^0_0$, are solved by the gauge invariant polynomials $P(\phi)$. Thereby, the equivariant cohomology is given by the differential forms generated from these polynomials by virtue of the descent equations of $\tilde{Q}$ modulo $d$, i.e. eqs. (2.31). After integrating each of these forms over closed cycles, one obtains global observables which only depend on the homology class of these cycles. These observables will be referred to as Witten’s observables.

Equivalently, the equivariant cohomology can be defined as the cohomology of the BRST operator $S_{\text{tot}}$ restricted to the space of local functionals of $a, \psi, \phi, c$ which are independent of $c$ and gauge invariant [8]. The mathematical techniques of equivariant cohomology [20, 15] then allow to construct some cohomology representatives which turn out to coincide with Witten’s observables. However, a complete determination of the cohomology classes along these lines seems to be a difficult task.

Yet, one can also apply standard cohomological techniques while working in a restricted functional space. Using this approach, the authors of reference [9] found that the solution of the cohomological problem is given by certain $S$-cohomology classes of ghost-number zero (reproducing again Witten’s observables). This result suggests to look for representatives of the equivariant cohomology within the cohomology of the operator $S$ (describing gauge transformations, see (2.34)–(2.35)) in the space of local functionals of $a, \psi, \phi, c$ which are of ghost-number zero and invariant under the supersymmetry operator $\tilde{Q}$. From this viewpoint, one looks for a non-trivial solution of the $S$-cocycle condition $S^s \Delta_{(d)} = 0$ which satisfies the constraint $\tilde{Q}^s \Delta_{(d)} = 0$, where the non-triviality requirement now concerns the $S$-operator, i.e.

$$s \Delta_{(d)} \neq S^s \Delta'_{(d)} \quad \text{with} \quad \tilde{Q}^s \Delta'_{(d)} = 0 .$$  (3.5)

However, $\Delta_{(d)}$ is of ghost-number zero and we do not have any fields of ghost-number minus one, therefore the non-triviality condition (3.5) is automatically satisfied whatever the form degree $d$. Thus, at form degree zero, this approach also
reduces to the cohomology problem (3.1) (3.2) without any further requirements. At higher form degree, it regards as non-trivial the solutions of the form (3.4) which are trivial representatives of equivariant cohomology.

The latter approach can easily be extended beyond the WZ-gauge: in a general supergauge, the equivariant cohomology can be determined by looking for the ghost-number zero cohomology classes of the BRST operator (2.9) or (2.13) in the space of the supersymmetric local functionals (the supersymmetry transformations being defined by means of the operator $Q$ according to (2.14)). In the following, we shall completely determine this cohomology while working within the superspace formalism, only specifying to the WZ-gauge ($\chi = 0$) towards the end.

Thus, let us consider a fixed SUSY-number $s \geq 0$ and a fixed degree $d \geq 0$. The task is to find a solution of the cocycle condition

$$\mathcal{S}^s \Delta_{(d)} = 0$$

satisfying the SUSY constraint

$$Q^s \Delta_{(d)} = 0$$

Here,

$$\Delta_{(d)} = \int_{M_d} s \omega_0(x)$$

denotes a local functional of SUSY-number $s$ which depends on the components of the superfield forms $A, A_\theta, C$ and their exterior derivatives. Since the solution of the problem (3.6) (3.7) proceeds in several steps, we shall present a summary of results at the end of each of the following sections.

Our discussion will be purely algebraic and does not assume a specification of the spacetime dimension $n$. If the latter is specified, all forms of degree greater than $n$ vanish identically. Those of degree $d$ smaller than $n$ can be integrated over oriented submanifolds $M_d$. The latter manifolds are assumed to be closed which implies the absence of boundary terms upon integration over $M_d$. Thus, we exclude from our discussion the “trivial” solution of (3.6) (3.7) which exists for $d = 2m$,

$$0 \Delta_{(2m)} = \int_{M_{2m}} 0 \omega_{2m} \equiv \int_{M_{2m}} \text{Tr} F_a^m$$

since the Pontrjagin density $\text{Tr} F_a^m$ is locally given by the exterior derivative of the Chern-Simons form of degree $2m - 1$.

Before tackling the cohomological problem in full generality, we already note that the determination of observables that we presented for the WZ-gauge in subsection 2.4.2 can be generalized to a general supergauge as follows.

According to equation (2.6), the curvature 2-superform has the general form

$$\hat{F}(x, \theta) = \hat{F}(x) + \theta (Q \hat{F})(x),$$
where the first term of this expansion can also be written as \( \hat{F}\big|_{\theta=0} \equiv \hat{F} \). (In the WZ-gauge, the latter expression reduces to the form \( \mathcal{F} \) introduced in eq. (2.28).) For \( m = 1, 2, \ldots \), the \( 2m \)-superform \( \text{Tr} \, \hat{F}^{2m}(x, \theta) \) admits an analogous expression:

\[
\text{Tr} \, \hat{F}^{2m} = \text{Tr} \, \hat{F}\big|^{2m} + \theta QT \text{Tr} \, \hat{F}^{2m} \\
\text{with} \quad \text{Tr} \, \hat{F}\big|^{m} = \sum_{p=0}^{2m} 2^{m-p} w^{0}_p \, (d\theta)^{2m-p}.
\]

(3.10)

Since \( \text{Tr} \, \hat{F}^{2m} \) is a closed superform,

\[
0 = \hat{d} \, \text{Tr} \, \hat{F}^{2m} = (d\theta \partial_{\theta} + d) \, \text{Tr} \, \hat{F}^{2m} = (Q \, \text{Tr} \, \hat{F}^{2m}) \, d\theta + d \, \text{Tr} \, \hat{F}^{2m},
\]

it follows by projection onto the \( \theta = 0 \) component that

\[
Q \, \text{Tr} \, \hat{F}\big|^{m} = -(\text{Tr} \, \hat{F}\big|^{m}) \, (d\theta)^{-1}.
\]

By substituting the expansion (3.10) into this relation, we get *Witten’s descent equations in a general supergauge*:

\[
Q^{2m-p} w^{0}_p + d^{2m-p+1} w^{0}_{p-1} = 0 \quad (p = 0, 1, \ldots, 2m).
\]

(3.11)

Explicit expressions for the polynomials \( w_p \) for \( m = 1 \) and \( m = 2 \) will be given in section 4 below and here we only note that \( ^0 w^{0}_{2m} = \text{Tr} \, F^{m} \) whatever the value of \( m \).

The task of the next subsections is to determine if other solutions can be obtained by virtue of a systematic study in superspace.

### 3.2 The bi-descent equations

In this section, we shall show that the cohomological problem (3.6)-(3.7) leads to a set of bi-descent equations involving superfield forms. Let us first solve the SUSY constraint (3.7) for \( s \Delta_{(d)} \) given by (3.8). For the integrand \( s \omega_0^d(x) \), it implies

\[
Q \, s \omega_0^d + d \, s+1 \omega_0^{d-1} = 0.
\]

(3.12)

In view of this relation, we shall prove the following proposition:

**Proposition 3.1** Let \( p \) and \( s \) be non-negative integers. (Here, we do not refer to the ghost-number which only represents a passive label in this proposition.)

(i) The cocycle condition

\[
Q \, s \omega_p + d \, s+1 \omega_{p-1} = 0
\]

(3.13)

implies the \( Q \) modulo \( d \) triviality of the space-time form \( s \omega_p \) and the \( d \) modulo \( Q \) triviality of the space-time form \( s+1 \omega_{p-1} \):

\[
s \omega_p = Q \, s-1 \varphi_p + d \, s \varphi_{p-1}
\]

(3.14)

\[
s+1 \omega_{p-1} = Q \, s \varphi_{p-1} + d \, s+1 \varphi_{p-2},
\]

(3.15)
with the same space-time form \( *\varphi_{p-1} \) appearing in both equations.

(ii) The same result holds for superfield forms, i.e.

\[
Q \, s\Omega_p + d \, s^{+1}\Omega_{p-1} = 0
\]  

implies

\[
\begin{align*}
*\Omega_p &= Q \, s^{-1}\Phi_p + d \, *\Phi_{p-1} \\
&s^{+1}\Omega_{p-1} = Q \, s\Phi_{p-1} + d \, s^{+1}\Phi_{p-2} .
\end{align*}
\]  

**Proof:** See appendix A.1.

With the help of this proposition, we deduce from (3.12) that

\[
s\omega_0^0 = Q \, s^{-1}\omega_0^0 .
\]  

Here and in the following, the total derivative term is suppressed without loss of generality, since it does not contribute to the integrated cocycle \( s\Delta_d \). Furthermore, without loss of generality, we can assume \( s^{-1}\omega_0^0 \) to be a superfield form

\[
s^{-1}\Omega_0^0 = s^{-1}\omega_0^0 + \theta \, Q \, s^{-1}\omega_0^0 ,
\]  

so that (3.19) reads as

\[
s\omega_0^0 = Q \, s^{-1}\Omega_0^0 .
\]

Since the operator \( Q \) acts on superfield forms by the \( \theta \)-derivative, this shows that \( s\Delta_d \) is the superspace integral of \( s^{-1}\Omega_0^0 \):

\[
Q \, s\Delta_d = 0 \implies s\Delta_d = \int_{M_d} Q \, s^{-1}\Omega_0^0 .
\]  

Next, we turn to the cocycle condition (3.6). Since the cohomology of \( d \) in the space of local field polynomials is trivial \[12\], this condition implies the descent equations\[6\]

\[
S \, s\omega_{d-p}^0 + d \, s\omega_{d-p-1}^0 = 0 \quad (p = 0, \ldots, d) .
\]  

**Lemma 1** The SUSY constraint implies that every form in (3.22) (and not just the one of highest form degree, i.e. \( s\omega_0^0 \)) can be written as a SUSY variation,

\[
s\omega_{d-p}^0 = Q \, s^{-1}\Omega_{d-p}^0 \quad (p = 0, \ldots, d) ,
\]  

where \( s^{-1}\Omega_{d-p}^0 \) is a superfield form.

\[6\]Every form of negative form degree, ghost-number or SUSY-number is assumed to vanish by convention.
The proof of this statement proceeds by induction, see appendix A.2.

We note that in eqs. (3.22) and thus in (3.23) and in the equations to follow, the array of descent equations may terminate at some positive form degree that we denote by \( p > 0 \). (A simple illustration of such a “termination of descent” within abelian gauge theory (with field strength \( F = da \)) is given by the cocycles \( \omega_0^0 = aF \) and \( \omega_2^1 = cF \), which satisfy \( S\omega_0^0 + d\omega_1^1 = 0 \) and \( S\omega_2^1 = 0 \), so that \( p = 2 \) in this case.) In such a case, we use the convention that every form of form degree less than \( p \) is vanishing.

We are now going to prove:

**Proposition 3.3** For given values of \( s \) and \( d \), the descent equations (3.22) together with the SUSY constraint (3.23) imply a set of descent equations involving superfield forms and all of the three indices:

\[
S^{s-r-1} \Omega^{d-p+r}_p + d^{s-r-1} \Omega^{d-p+r+1}_{p-1} + Q^{s-r-2} \Omega^{d-p+r+1}_p = 0
\]

\( (r = 0, \ldots, s-1 ; \quad p = 0, \ldots, d) \).  \( (3.24) \)

We shall call this set of equations the **bi-descent equations for the pair \((d, s)\)**.

**Proof:** In order to derive this result, we first rewrite (3.22), using the result (3.23), as

\[
Q \left( S^{s-1} \Omega^{d-p}_p + d^{s-1} \Omega^{d-p+1}_{p-1} \right) = 0 .
\]

The triviality of the \( Q \)-cohomology then implies

\[
S^{s-1} \Omega^{d-p}_p + d^{s-1} \Omega^{d-p+1}_{p-1} + Q^{s-2} \Omega^{d-p+1}_p = 0 \quad (p = 0, \ldots, d) , \quad (3.25)
\]

where \( s-2 \Omega^{d-p+1}_p \) is again taken to be a superfield form. The latter equation is nothing but (3.24) with \( r = 0 \). The validity of the bi-descent equations for all values of \( r \) is shown by induction, see appendix A.3. \( q.e.d. \)

The bi-descent equations for the pair \((d, s)\) as given by eqs. (3.24) involve the forms \( s-r-1 \Omega^{d-p+r}_p \) which all have the same total degree

\[
D = d + s - 1 . \quad (3.26)
\]

In the following, we shall consider this number to be fixed to some arbitrary value \( D \geq 0 \). For given values of \( D, d \) and \( s \) related by (3.26), the bi-descent equations for the pair \((d, s)\) then become the **bi-descent equations for the pair \((d, D)\)**, i.e. the same set of equations with a different labeling of ghost- and SUSY-indices:

\[
S^{D-p-g} \Omega^{g}_p + d^{D-p-g} \Omega^{g+1}_{p-1} + Q^{D-p-g-1} \Omega^{g+1}_p = 0
\]

\( (p = 0, \ldots, d ; \quad g = d - p, \ldots, D - p) \).  \( (3.27) \)
We note that the domain of variation of the indices \( p \) and \( g \) in this set of equations is a parallelogram \( \text{Par}(d, D) \) in the \((p, g)\) plane which is bounded by the straight lines \( p = 0, \ p = d, \ p + g = d \) and \( p + g = D \). More precisely, each point of \( \text{Par}(d, D) \) represents exactly one of the bi-descent equations \( \text{BDE} \), these equations being parametrized by the form degree and ghost-number of the \( S \)-term. The example \( D = 3 \) is presented in detail in section 4.

**Summary:** By definition, the observables of the theory are the integrated local functionals \( \Delta_{(d)} \) of the form \( \text{Eq.}(3.8) \) satisfying the cocycle condition \( \text{Eq.}(3.6) \) and the supersymmetry constraint \( \text{Eq.}(3.7) \). For a fixed maximal degree \( D \equiv d + s - 1 \geq 0 \), they are given by superspace integrals of superfield \( d \)-forms of ghost-number 0, i.e.

\[
D - d + 1 \Delta_{(d)} = \int_{M_d} Q^{D - d} \Omega_0^d \quad (d = 0, \ldots, D),
\]

where \( D - d \Omega_0^d \) is a non-trivial solution of the bi-descent equations for the pair \((d, D)\), i.e. eqs. \( \text{BDE} \).

### 3.3 Superform solutions of the bi-descent equations

When varying \( d \) from 0 to its maximum value \( D \), the parallelograms \( \text{Par}(d, D) \) fill up the triangle \( \text{Tri}(D) \) of vertices \((0, 0), (D, 0)\) and \((0, D)\). The points of this triangle describe in a one-to-one fashion the bi-descent equations for the forms of total degree \( D \), i.e. the bi-descent equations for the pairs \((d, D)\) with \( d = 0, \ldots, D \):

\[
S^{D - p - g} \Omega_p^g + d^{D - p - g} \Omega_{p-1}^{g+1} + Q^{D - p - g - 1} \Omega_{p}^{g+1} = 0 \quad (p \geq 0, \ g \geq 0, \ p + g \leq D).
\]

The bi-descent equations \( \text{BDE} \) represent subsets of the latter equations which are closed in the sense that each equation of \( \text{BDE} \) corresponding to a point \((p, g) \in \text{Par}(d, D)\) only involves forms corresponding to points of \( \text{Par}(d, D) \), i.e. it represents an equation of the set \( \text{BDE} \). Hence a non-trivial solution of \( \text{BDE} \) also represents a solution of \( \text{BDE} \). However, the converse is not necessarily true. Indeed, two forms \( D^{D - p - g} \Omega_p^g \) and \( D^{D - p - g'} \Omega_p^{g'} \) belonging to the intersection of two parallelograms \( \text{Par}(d, D) \) and \( \text{Par}(d', D) \) might represent different solutions of the two corresponding sets of bi-descent equations.

In this subsection, we shall look for solutions of the system of equations \( \text{BDE} \), thereby providing a special set of solutions of the bi-descent equations \( \text{BDE} \). The search of the general solution of eqs. \( \text{BDE} \) is postponed to section 3.4.

We first introduce the set of \( q \)-superforms (cf. eq. \( \text{Eq.}(2.2) \))

\[
\hat{\Omega}_q^{D - q} = \sum_{p=0}^{q} q^{D - q} \Omega_p^{D - q} \Omega^{q - p} \quad (q = 0, \ldots, D),
\]
which contains all of the superfield forms appearing in equations (3.29). In fact, the superform $\hat{\Omega}^{D-q}$ contains the superfield forms of ghost-number $D-q$, i.e. those located on the horizontal line $g = D - q$ of the triangle $Tri(D)$. It is easy to check that the equations (3.29) are equivalent to the superdescent equations

$$S\hat{\Omega}^{D-q} + \hat{d}\hat{\Omega}^{D-q+1} = 0 \quad (q = 0, \ldots, D),$$

(3.31)

where $\hat{d}$ denotes the exterior derivative in superspace, see eq.(2.4). This equivalence allows us to solve the bi-descent equations (3.29) in superspace in terms of superforms. In fact, in this subsection, we shall only be interested in superforms which are polynomials of the basic superforms $\hat{A}(x, \theta), C(x, \theta)$ and their exterior superderivatives.

The corresponding observables (3.28) are then obtained by integrating a non-trivial solution $\hat{\Omega}^0_D$ of the superdescent equations (3.31) over the collection $\mathcal{M} = (M_0, M_1, \ldots, M_D)$ of manifolds according to (2.8):

$$\hat{\Delta}^{(D)} = \int_{\mathcal{M}} \hat{\Omega}^0_D = \sum_{p=0}^{D} D^{-p+1} \Delta(p) (d\theta)^D - p,$$

(3.32)

where each of the expressions $D^{-p+1} \Delta(p)$ involves another component of the superform $\hat{\Omega}^0_D$ (cf.(3.30)):

$$D^{-p+1} \Delta(p) = \int_{M_p} Q D^{-p} \Omega^0_p = \int_{M_p} D^{-p+1} \omega^0_p \quad (p = 0, \ldots, D).$$

(3.33)

Here, a solution $\hat{\Omega}$ of (3.31) is considered to be non-trivial if it cannot be written as $S\hat{\Omega}' + d\hat{\Omega}'$.

Let us now determine the non-trivial solutions of the superdescent equations (3.31), i.e. the elements of the cohomology $H(\mathcal{S}|\hat{d})$ of the BRST operator $\mathcal{S}$ modulo the superderivative $\hat{d}$, in the space $\hat{\mathcal{E}}_S$ of the local polynomials of the superconnection $\hat{A}$, the superghost $C$ and their $\hat{d}$-derivatives. Since the BRST transformations (2.9) for $\hat{A}(x, \theta)$ and $C(x, \theta)$ have exactly the same structure as in ordinary pure YM theory, the well known results valid in the latter theory (see reference [22] and the reviews [21, 12]) can directly be applied after putting “hats” on all quantities. We will use the notation of reference [12].

First, we introduce the following supercocycles:

$$\theta_r(C) = (-1)^{m_r-1} \frac{m_r!(m_r-1)!}{g_r!} \text{Tr} C^{g_r} \quad (g_r = 2m_r - 1)$$

$$f_r(\hat{F}) = \text{Tr} \hat{F}^{m_r} \quad (r = 1, \ldots, \text{rank } G).$$

(3.34)

Here, $\hat{F} = d\hat{A} + \hat{A}^2$ is the curvature of the superconnection $\hat{A}$ and the index $r$ labels the $r^{th}$ Casimir operator of the structure group (gauge group) $G$, whose degree is.
The cocycles (3.34) are related by superdescent equations involving superforms \([\hat{\theta}_r]^{g_r-p}_p\) of form degree \(p \geq 0\) and ghost-number \(g_r - p\):

\[
S [\hat{\theta}_r]^{g_r-p}_p + \hat{d} [\hat{\theta}_r]^{g_r-p+1}_p = 0 \quad (p = 0, \ldots, g_r),
\]

with \([\hat{\theta}_r]^0_{g_r} = \theta_r(C)\) and \(\hat{d} [\hat{\theta}_r]^0_{g_r} = f_r(\hat{F})\).

According to the last equation, the “top” superform \([\hat{\theta}_r]^{0}_{g_r}\) is the Chern-Simons superform of degree \(g_r\) associated to the \(r\)th Casimir operator.

Obviously, (3.35) corresponds to the superdescent equations (3.31) and thus yields a solution of the latter equations. More general solutions are found by multiplying the cocycle \(\theta_r(C)\) by a certain number of factors \(f_r(\hat{F})\) since the latter are both \(S\)- and \(\hat{d}\)-invariant. Thus, we introduce the following supercocycle (belonging to the \(S\)-cohomology in the space \(\hat{E}_S\)):

\[
\hat{H}_R \equiv \theta_{r_1}(C)f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}),
\]

with \(L \geq 1\), \(r_i \leq r_{i+1}\).

This cocycle is of ghost-number \(g_{r_1}\) and superform degree \(D_R = \sum_{i=2}^{L} 2m_{r_i}\) (cf.(3.34)).

By virtue of equations (3.35), the superforms

\[
\hat{\Omega}_{D_R+p}^{g_{r_1}-p} = [\hat{\theta}_{r_1}]^{g_{r_1}-p}_p f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}) \quad (p = 0, \ldots, g_{r_1}; \quad L \geq 1)
\]

obey the superdescent equations

\[
S \hat{\Omega}_{D_R+p}^{g_{r_1}-p} + \hat{d} \hat{\Omega}_{D_R+p+1}^{g_{r_1}-p} = 0 \quad (p = 0, \ldots, g_{r_1}).
\]

The most general solution of the superdescent equations (3.31) in the space \(\hat{E}_S\) is obtained by considering a supercocycle of the form (3.36) which is non-linear in the monomials \(\theta_r(C)\). However, in view of the construction of observables (which have zero ghost-number by definition), we are only interested in the most general solution containing superforms of ghost-number 0 and the latter are given by (3.37) according to the results of section 10.7 of reference [12], adapted to the present superspace formalism.

The corresponding observables are now constructed according to (3.32) (3.33), by using (3.37) for \(p = g_{r_1}\), i.e.

\[
\hat{\Omega}^0_D = [\hat{\theta}_{r_1}]^{0}_{g_{r_1}} f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}) \equiv \sum_{p=0}^{D} \Omega^0_p (d\theta)^{D-p},
\]

with \(D = D_R + g_{r_1} = 2 \sum_{i=1}^{L} m_{r_i} - 1\), \(L \geq 1\).
Note that $D$ is necessarily odd.

There is an alternative way of writing the observables which amounts to a simpler manner of extracting the polynomials $D^{-p+1}\omega^0_p(x)$ from the superform $\hat{\Omega}^0_D$. This procedure is suggested by the fact that the exterior derivative $\hat{d} = d\theta \partial_\theta + d$ differs from the superspace SUSY operator $\partial_\theta$ by a factor $d\theta$ and the addition of a spacetime derivative. Indeed, let us consider the exterior derivative of $\hat{\Omega}^0_D$ and write its expansion with respect to $d\theta$ (see (2.16) (2.30)):

$$\hat{d}\hat{\Omega}^0_D = f_{r_1}(\hat{F}) \cdots f_{r_L}(\hat{F}) = f_{r_1}(F_A) \cdots f_{r_L}(F_A) + \sum_{p=0}^D D^{+1-p}W^0_p (d\theta)^{D+1-p},$$

(3.40)

with

$$D^{-p+1}W^0_p = Q D^{-p}\Omega^0_p + dD^{-p+1}\Omega^0_{p-1} = D^{-p+1}\omega^0_p(x) + dD^{-p+1}\Omega^0_{p-1}(x,\theta).$$

Henceforth, in the integral (3.33) which yields the observables, we can substitute the form $D^{-p+1}\omega^0_p$ by the value of $D^{-p+1}W^0_p$ at $\theta = 0$:

$$D^{-p+1}\Delta(p) = \int_{M_p} D^{-p+1}w^0_p \quad (p = 0, \ldots, D),$$

with $D^{-p+1}w^0_p = D^{-p+1}W^0_p|_{\theta=0}$.

(3.41)

Before concluding, we note that application of the superderivative $\hat{d}$ to (3.40) and use of its nilpotency, leads to

$$Q D^{-p+1}W^0_p + dD^{-p+2}W^0_{p-1} = 0 \quad (p = 0, \ldots, D),$$

which, taken at $\theta = 0$, yields

$$Q D^{-p+1}w^0_p + dD^{-p+2}w^0_{p-1} = 0 \quad (p = 0, \ldots, D).$$

(3.42)

These relations for the integrands $D^{+1-p}w^0_p$ of the observables (3.41) are nothing but Witten’s descent equations in a general supergauge (generalizing eqs. (2.31) which hold in the WZ-gauge and involve $Q$ rather than $\hat{Q}$). In the WZ-gauge, the polynomials $D^{+1-p}w^0_p$ reduce - by construction - to the Donaldson-Witten polynomials discussed in subsection 2.4.2. In particular, in the WZ-gauge, we obtain

$$D^{+1}\omega^0_0 = f_{r_1}(\phi) \cdots f_{r_L}(\phi) \quad \text{with} \quad L \geq 1, \quad r_i \leq r_{i+1},$$

(3.43)

i.e. Witten’s well-known result [1] that the algebra of observables is generated, at form degree zero, by the invariant monomials $f_s(\phi)$. The examples $D = 1$ and $D = 3$ will be presented in more detail in section [4].
Anticipating the discussion of the next subsection, which shows that there are no other non-trivial observables, we can summarize our results as follows.

Summary: Apart from the ‘trivial’ observables (3.9), there exist further ones. All of these observables, as defined by the conditions (3.6) and (3.7), are given by eqs. (3.11). In the latter expressions, the superfield forms \( D^{-p+1} W^0_p \) are the coefficients appearing in the expansion (3.40), the superform \( \hat{\Omega}^0_D \) being the non-trivial solution (3.39) of the superdescent equations (3.31) in the space \( \hat{E}_S \) of polynomials in the basic superforms \( \hat{A}, C \) and their exterior superderivatives. The forms \( D^{-p+1} w^0_p \) satisfy the generalization of Witten’s descent equations to a general supergauge, i.e. eqs. (3.42).

3.4 General solution of the bi-descent equations for the pair \((d, D)\)

As noted at the beginning of the last section, the solutions of the superdescent equations (3.31) in the space \( \hat{E}_S \) (which is generated by the superforms \( \hat{A}, C \) and the operators \( S, d \)) represent a priori only a special set of solutions of the bi-descent equations for the pair \((d, D)\), i.e. eqs. (3.27). Henceforth, we have to determine the general non-trivial solution of the latter equations in order to obtain the general set of observables. At this point, we only state and comment on the main result, leaving the proof for appendix A.4.

Proposition 3.4 The general solution of the bi-descent equations (3.27) for the pair \((d, D)\) is generated, at ghost-number zero, by two classes of solutions. The first one is given by the superfield forms

\[
D^{-d} \Omega^0_d (d \theta)^{D-d} = \left[ \hat{\theta}_{r_1}^0 f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}) \right]_{s=D-d, p=d}
\]

with \( D = 2 \sum_{i=1}^{L} m_{r_i} - 1 \), \( L \geq 1 \),

where the Chern-Simons superform \( \left[ \hat{\theta}_{r}^0 \right]_{g_r} \) and supercurvature invariant \( f_{r}(\hat{F}) \) are defined by (3.35) (3.34).

The second class of solutions depends on the superfield forms \( F_A, \Psi \) and \( \Phi \) defined in eqs. (2.16) (2.15) and it is given by

\[
D^{-d} \Omega^0_d = D^{-d} Z^0_d (F_A, \Psi, \Phi, D_A \Psi, D_A \Phi) .
\]

Here, \( D^{-d} Z^0_d \) is an arbitrary invariant polynomial of its arguments, which has a form degree \( d \) and SUSY-number \( D - d \) and which is non-trivial in the sense that \( D^{-d} Z^0_d \neq d D^{-d} \Phi^0_{d-1} + Q D^{-d-1} \Phi^0_d \).
**Proof:** See appendix A.4.

Concerning the invariant forms (3.45), we note that they represent the general cohomology classes of the BRST operator $S$ in the space $\mathcal{E}_S$ by virtue of a mere adaptation of the results of section 8 of reference [12].

According to (3.33), the observables are obtained as integrals of the $Q$-variation of the solutions (3.44) over a $d$-dimensional manifold. The ones corresponding to (3.44) coincide with the corresponding expressions calculated in section 3.3 (i.e. (3.41) with $p = d$) since the corresponding integrands only differ by a total derivative. In fact, for $d = 0, 1, \ldots, D$, the superfield forms $D^{-d}\Omega^0_d$ given in (3.44) are nothing but those introduced in (3.39). Hence, the solutions (3.44) provide the same observables as the superform solutions (3.39).

On the other hand, for the solutions (3.45), one gets the integrals

$$D^{-d+1} \Delta^{(d)} = \int_{M_d} \mathcal{Q} D^{-d} \Omega^0_d (F_A, \Psi, \Phi, D_A \Psi, D_A \Phi).$$

(3.46)

As pointed out after equation (2.18), the operator $Q$ simply reduces to $Q_0$ when acting on an invariant polynomial $D^{-d} \Omega^0_d$. Yet, as noted after equation (2.26), the action of $Q_0$ is isomorphic to the one of the operator $\tilde{Q}$ describing supersymmetry transformations in the WZ-gauge. This means that the solution (3.46), if written out in the WZ-gauge, is simply the $\tilde{Q}$-variation of a gauge invariant polynomial. Thus, it is trivial in the sense of equivariant cohomology, see eqs. (3.1)-(3.3). This explicitly shows that the cohomology defined by equations (3.6)-(3.7) is not equivalent to the equivariant cohomology: the difference is precisely given by the expressions of the form (3.46), which are manifestly $\tilde{Q}$-trivial.

We conclude that, apart from the solutions (3.45) which are “equivariantly” trivial, the general solution of the bi-descent equations that we described in this section does not yield any more solutions than those obtained in terms of superforms in section 3.3. In other words, the solution constructed by using superforms represents the most general, equivariantly non-trivial expression for the observables.

### 4 Explicit expressions

#### 4.1 An example of bi-descent and superdescent equations

There is a graphical way of representing the sets of bi-descent equations which allows us to exhibit explicitly the combinatorics leading to the superdescent equations (3.31).
By way of illustration, let us consider the case of total degree $D = 3$. We have $10$ superfield forms $3-p-g\Omega_p^g$ in the bi-descent equations for total degree $D = 3$, i.e. eqs.(3.29), which can be represented in the $(p, g)$ diagram:

$$
\begin{array}{cccc}
& & & \\
& & 3 & \\
& 2 & & \\
1 & & & \\
0 & & 1 & 2 & 3 & p \\
\end{array}
$$

E.g. the point on the outer right represents the form $^0\Omega_3^0$. The forms listed in the previous diagram appear in different sets of bi-descent equations (3.27): for $d = 0, 1, 2, 3$ and $D = 3$, the latter bi-descent equations correspond to the following sub-diagrams of the previous diagram:

$$
\begin{array}{cccc}
\text{d} = 0 & & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\quad
\begin{array}{cccc}
\text{d} = 1 & & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\quad
\begin{array}{cccc}
\text{d} = 2 & & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\quad
\begin{array}{cccc}
\text{d} = 3 & & & \\
& & & \\
& & & \\
& & & \\
\end{array}
$$

Thus, one clearly sees how the parallelograms representing equations (3.27) overlap for the various values of $d$ (and a fixed value $D$) finally covering the full triangle in the $(p, g)$ plane which represents the bi-descent equations (3.29). Obviously, this triangle also represents the superdescent equations (3.31). The latter equations presently take the same form as the descent equations in 3-dimensional Chern-Simons field theory for which the solution is well known, e.g. see reference [21]. Thus, for $D = 3$, the non-trivial solution of the superdescent equations (3.31) is given by

$$
\hat{\Omega}_3^0 = \text{Tr} (\hat{A}\hat{d}\hat{A} + \frac{2}{3}\hat{A}^3) , \quad \hat{\Omega}_3^1 = \text{Tr} (\hat{A}\hat{d}\hat{C}) , \quad \hat{\Omega}_3^2 = \text{Tr} (\hat{C}\hat{d}\hat{C}) , \quad \hat{\Omega}_3^3 = -\frac{1}{3} \text{Tr} C^3 .
$$

(4.1)

4.2 Some examples of observables

In this section, we consider the structure group $U(1) \times SU(2)$ to illustrate the conclusions of section 3. For this group, there are two Casimir operators: the $U(1)$ generator itself (the charge) and the quadratic Casimir of $SU(2)$. Their degrees are respectively $m_1 = 1$ and $m_2 = 2$.

In the sequel, we shall use an index ‘(a)’ for ‘abelian’. The ghosts, connections
and curvatures are, respectively, given by the following superfields and -forms:

\[ U(1) : \quad C_{(a)} , \quad \hat{A}_{(a)} , \quad \hat{F}_{(a)} = \hat{d}\hat{A}_{(a)} \]

\[ SU(2) : \quad C , \quad \hat{A} , \quad \hat{F} = \hat{d}\hat{A} + \hat{A}^2 . \]

The “canonical” basis (3.34) of the cohomology \( H(S) \) reads as

\[
\begin{align*}
\theta_1 &= \theta_1(C) = C_{(a)} , \quad f_1 = f_1(\hat{F}_{(a)}) = \hat{F}_{(a)} \\
\theta_2 &= \theta_2(C) = -\frac{1}{3} \text{Tr} C^3 , \quad f_2 = f_2(\hat{F}) = \text{Tr} \hat{F}^2
\end{align*}
\]

and the canonical descent equations (3.35) involve the forms

\[
\begin{align*}
\hat{\theta}_1{}^0 &= \theta_1 , & \hat{\theta}_1{}^1 &= \hat{A}_{(a)} \\
\hat{\theta}_2{}^3 &= \theta_2 , & \hat{\theta}_2{}^2 &= \text{Tr} (C\hat{d}C) \\
\hat{\theta}_2{}^1 &= \text{Tr} (\hat{A}\hat{d}C) , & \hat{\theta}_2{}^0 &= \text{Tr} (\hat{A}\hat{d}\hat{A} + \frac{2}{3} \hat{A}^3) .
\end{align*}
\]

The latter coincide with the “canonical” superforms (4.3) for \( U(1) \).

According to the results of section 3.3 (see eqs. (3.40), (3.41)), the observables are obtained from the superspace exterior derivative of the top superform \( \hat{\Omega}_0{}^1 \) (which has ghost-number zero) and given by the expansion at \( \theta = 0 \):

\[
\begin{align*}
\hat{\Omega}_0{}^1 &= 0 , \quad S\hat{\Omega}_0{}^0 + \hat{d}\hat{\Omega}_0{}^1 = 0 ,
\end{align*}
\]

are solved by the superforms

\[
\begin{align*}
\hat{\Omega}_0{}^1 &= 0 , \quad \theta_1 = C_{(a)} \\
\hat{\Omega}_0{}^0 + \hat{\Omega}_0{}^1 d\theta &= \hat{A}_{(a)} .
\end{align*}
\]

The latter coincide with the “canonical” superforms (4.3) for \( U(1) \).

According to the results of section 3.3 (see eqs. (3.40), (3.41)), the observables are obtained from the superspace exterior derivative of the top superform \( \hat{\Omega}_1{}^0 \) (which has ghost-number zero) and given by the expansion at \( \theta = 0 \):

\[
\begin{align*}
\hat{\Omega}_1{}^0 |_{\theta=0} &= \hat{F}_{(a)} |_{\theta=0} = F_{(a)} + w_{(a)}^0 (d\theta)^2 , \\
\text{with} \quad F_{(a)} &= da_{(a)} , \quad w_{(a)}^0 = \psi_{(a)} + d\chi_{(a)} , \quad \hat{A}_{(a)} = \hat{F}_{(a)} + \hat{A}_{(a)}^2 .
\end{align*}
\]
Apart from the ‘trivial’ observable \( \int_{M_2} F(a) \), the observables are the integrals of the forms \( 1^w_1, 2^w_0 \) on closed submanifolds \( M_1 \) and \( M_0 \), respectively. The polynomials \( w_p \) satisfy Witten’s descent equations in a general supergauge, i.e. eqs. (3.42).

Equivalently – cf. (3.32)-(3.33) – the superspace integral of the superform \( \hat{\Omega}_1^0 \) over a collection \( \mathcal{M} = (M_0, M_1) \) of closed submanifolds is a direct sum of two integrals,

\[
\hat{\Delta}_{(1)} \equiv \int_\theta \int_\mathcal{M} \hat{\Omega}_1^0 = \int_\mathcal{M} \partial_\theta \hat{\Omega}_1^0 = \sum_{p=0}^1 (d\theta)^{1-p} \int_{M_p} 2^{-p} \omega_p^0 ,
\]

where the \( p \)-forms \( 2^{-p} \omega_p^0 \) are the coefficients of the expansion of \( \partial_\theta \hat{\Omega}_1^0 \). Each integral in (4.5) defines an expression belonging to the SUSY-constrained cohomology of \( \mathcal{S} \). These integrals coincide with those of the forms \( w_p \) defined in (4.4).

In the WZ-gauge \( \chi = 0 \), the expressions (4.4) reduce to the Donaldson-Witten polynomials generated from the invariant \( \phi(a) \) using the supersymmetry operator \( \hat{Q} \), see eqs. (2.30) with \( m = 1 \). In our approach, these polynomials have been generated for \( D = 1 \) from the bottom superform \( \hat{\Omega}_3^0 = C(a) \) which solves superdescent equations.

### 4.2.2 Solution corresponding to the Casimir of \( SU(2) \)

The bottom form \( \hat{\Omega}_3^0 = \theta_2 \) has total degree \( D = 3 \) and the solution of the superdescent equations is given by the superforms (4.3). These expressions coincide with the canonical superforms (4.3) for \( SU(2) \). Applying again proposition 4, we obtain the observables from the expansion

\[
\hat{d} \hat{\Omega}_3^0 \big|_{\theta=0} = \text{Tr} \hat{F}^2 \big|_{\theta=0} = \text{Tr} F^2 + \sum_{p=0}^3 4^{-p} \omega_p^0 (d\theta)^{4-p} ,
\]

the last term being an exterior derivative. By substituting the component field expansions (2.10) (2.11) of \( \hat{A} \) into \( \hat{\Omega}_3^0 \), we obtain the following explicit expressions for the spacetime forms:

\[
\begin{align*}
4^w_0 &= \text{Tr} (\phi^2 + 2\phi \chi^2) \\
3^w_1 &= \text{Tr} 2(\psi \phi + \psi \chi^2 + \phi D_a \chi) + d \text{Tr} \left( \frac{2}{3} \chi^3 \right) \\
2^w_2 &= \text{Tr} (\psi^2 + 2\phi F_a + 2\psi D_a \chi) + d \text{Tr} (\chi D_a \chi) \\
1^w_3 &= \text{Tr} (2\psi F_a) + d \text{Tr} (2\chi F_a) .
\end{align*}
\]

The observables are the integrals of these forms (and of \( \text{Tr} F_a^3 \)) on closed submanifolds of appropriate dimension.

In the WZ-gauge \( \chi = 0 \), the expressions for the observables again reduce to Witten’s result (generated from the quadratic invariant \( \text{Tr} \phi^2 \)), i.e. eqs. (2.30) with \( m = 2 \).
4.2.3 An example of “composite observables”

As stated at the end of section 3.3, all other observables are integrals whose integrands are polynomials of the forms \( w_p \) that we constructed in the last two subsections (i.e. of the forms associated to the Casimir operators). Let us illustrate this with the simplest example, generated by the bottom form \( \hat{\Omega}_0^1 = \theta f_1 \) which is of total degree 3. The corresponding top superform is given by \( \hat{\Omega}_0^3 = \hat{A}(a) \hat{F}(a) \) and the expansion of its superderivative \( \hat{d} \hat{\Omega}_0^3 = (\hat{F}(a))^2 \) yields the following integrands for the observables:

\[
\begin{align*}
4 \tilde{w}_0^0 &= (2w_0^0)^2 = \phi_{(a)}^2 \\
3 \tilde{w}_1^1 &= 2(1w_0^0)(2w_0^0) \\
2 \tilde{w}_2^2 &= 2^2w_0^0 F(a) + (1w_0^0)^2 \tag{4.8} \\
1 \tilde{w}_3^3 &= 2^1w_1^1 F(a) \\
0 \tilde{w}_4^4 &= F^2(a). 
\end{align*}
\]

Obviously, these forms are polynomials in the basic forms given in eqs. (4.4) and in the abelian curvature invariant \( f_1 = F(a) \).

5 Concluding remarks

We have shown that the problem of determining the equivariant cohomology of topological Yang-Mills theories can be reduced to that of computing the Yang-Mills BRST cohomology (modulo \( d \)) in the space of polynomials depending on the components of the Yang-Mills superconnection \( \hat{A} \), its superghost \( C \) and their exterior derivatives – all these components being superfields. The determination of this cohomology relies on different extensions of well-known techniques [1 2], on one hand to superspace, and on the other hand to the case where one has two BRST-like operators, namely \( S \) and \( Q \). This leads to the consideration of “bi-descent equations” generalizing the usual descent equations.

Our main result is the following one. Apart from solutions of the bi-descent equations that are trivial in the sense of equivariant cohomology (i.e. the trivial observables determined by (3.45)), the general non-trivial solution (3.44) of these equations (describing an observable \( ^s\Delta^{(d)} \) of dimension \( d \) and SUSY-number \( s \)) is given as the superspace integral \( \int_{M_d} d\theta ^{s-1}\Omega_d^0 \), where \( ^{s-1}\Omega_d^0 \) is a coefficient of some superform which has total degree \( D = d + s - 1 \), this superform being a solution of a set of “super-descent equations”. In other words, the observables are determined by the cohomology of the BRST operator (modulo the exterior superderivative \( d \)) in the space of superforms which are polynomials in the superconnection \( \hat{A} \), the Faddeev-Popov ghost-superfield \( C \) and their exterior superderivatives. When specialized
to the Wess-Zumino gauge, our result reproduces Witten’s observables [1]. The generalization of our approach to more complex models is currently under study and will be reported upon elsewhere [16, 17].

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A Proofs of some propositions and lemmas

A.1 Proof of Proposition 3.1

The proof of the results (3.14)(3.15) and (3.17)(3.18) is based on the triviality of the cohomologies \( H(Q) \) and \( H(d) \) for the functional spaces \( E \) and \( E_s \), respectively, see propositions 2.1 and 2.2. Here, we outline the proof of (3.14)(3.15), the one of (3.17)(3.18) being analogous.

Equation (3.14) follows from the cocycle condition (3.13) by virtue of a corollary of theorem 9.2 of ref. [12]. In the present context, this corollary states that, if the cohomologies \( H(Q) \) and \( H(d) \) are both trivial, and if a form \( s_\omega \) is \( Q \)-invariant modulo \( d \) (i.e. condition (3.13) holds), then \( s_\omega \) is \( Q \)-exact modulo \( d \), i.e. (3.14) holds. The same corollary, with the roles of \( Q \) and \( d \) interchanged, implies that \( s_\omega \) is \( d \)-exact modulo \( Q \), i.e.

\[
s_{p-1} + 1 \omega p - 1 = d s_{p-1} + 1 \chi_{p-2} + Q s_{p-1} - 1 \chi_{p-1} . \tag{A.1}
\]

By substituting the expressions (3.14) and (A.1) into (3.13), we obtain the equation

\[
dQ ( s_\chi_{p-1} - s_\varphi_{p-1} ) = 0 ,
\]

which, due to the triviality of the cohomology \( H(d) \), implies

\[
Q ( s_\chi_{p-1} - s_\varphi_{p-1} ) + d s_{p-1} \chi_{p-2} = 0 .
\]

This \( Q \) modulo \( d \) invariance condition is solved by

\[
s_\chi_{p-1} = s_\varphi_{p-1} + Q s_{p-1} - 1 \xi_{p-1} + d s_{p-2} .
\]

Introducing this result into eq. (A.1) and defining

\[
s_{p-2} + 1 \varphi_{p-2} = s_{p-2} + 1 \chi_{p-2} - Q s_{p-2}
\]

finally yields the result (3.15).
A.2 Proof of Lemma 1

The proof proceeds by induction. Equation (3.23) already holds at form degree $d$. Let us assume relation (3.23) to be true at form degree $p + 1$ and show its validity at degree $p$. By applying $Q$ to the descent equation (3.22) at degree $p + 1$ and using the induction hypothesis, we find

$$dQ s \omega_p^{d-p} = 0.$$  

Due to the triviality of the $d$-cohomology, this equation implies the $Q$ modulo $d$ cocycle condition

$$Q s \omega_p^{d-p} + d s^{p+1} \omega_p^{d-p} = 0.$$  

According to Proposition 3.1, the general solution of the latter is

$$s \omega_p^{d-p} = Q s^{-1} \omega_p^{d-p} + d s \omega_p^{d-p}.$$  

Discarding the derivative term since it does not contribute to the descent equation (3.22) at degree $p + 1$, we thus obtain the result (3.23) after replacing the spacetime form $s^{-1} \omega_p^{d-p}$ by a superfield form $s^{-1} \Omega_p^{d-p}$ thanks to the argument leading from (3.19) to (3.20).

A.3 Proof of Proposition 3.3

The first part of the proof was already presented after Proposition 3.3.

In order to prove the validity of the bi-descent equations (3.24) for all values of $r$ by induction, it is convenient to use the formalism of “extended forms” [23]. The latter involve superfield forms of the same total degree, but of different form degrees and ghost-numbers. In general, an extended form $\tilde{\Omega}_{d+r}$ is supposed to be of the form

$$\tilde{\Omega}_{d+r} = \Omega_{d+r}^0 + \Omega_{d+r-1}^1 + \cdots + \Omega_0^{k+r},$$

but, for the present application, we truncate the expansion so as to have $d$ as highest form degree, i.e. we consider

$$s^{-1-r} \tilde{\Omega}_{d+r} = \sum_{p=0}^d s^{-1-r} \Omega_p^{d-p+r} \quad (r = 0, \ldots, s - 1).$$  

The “extended differential” acting on these extended forms is defined by

$$\tilde{d} = S + d$$  

and it is nilpotent ($\tilde{d}^2 = 0$).

The set of bi-descent equations (3.24) may then be rewritten in terms of extended forms as

$$\tilde{d} s^{-r-1} \tilde{\Omega}_{d+r} + Q s^{-r-2} \tilde{\Omega}_{d+r+1} = d s^{-r-1} \Omega_d^r \quad (r = 0, \ldots, s - 1),$$  

(A.4)
where the \((d+1)\)-form on the right-hand side cancels the spurious \((d+1)\)-form which is present on the left-hand side.

Knowing that (A.4) is true for \(r = 0\) (i.e. eq.(3.25)) and assuming that it holds for \(r\), let us prove it for \(r+1\). Application of the nilpotent operator \(\tilde{d}\) to (A.4) yields

\[
Q \tilde{d} \bar{s}^{-2}\bar{\Omega}_{d+r+1} = d\bar{s}^{-1}\Omega_d^r .
\]

We now use the \(d\)-form component of equation (A.4), which is nothing but the bi-descent equation (3.24) for \(p = d\) and a fixed index \(r\), to get

\[
Q \tilde{d} \bar{s}^{-2}\bar{\Omega}_{d+r+1} = -dQ \bar{s}^{-2}\Omega_{d+1}^{r+1} = Q d \bar{s}^{-2}\Omega_{d+1}^{r+1} .
\]

Due to the triviality of the \(Q\)-cohomology, this relation implies the existence of an extended form \(s^{-3}\bar{\Omega}_{d+r+2}\) and thus leads to equation (A.4) with \(r+1\).

**A.4 Proof of Proposition 3.4**

Since we are specifically interested in solving, for some fixed values of \(d\) and \(D\), the bi-descent equations (3.27) which correspond to the parallelogram \(\text{Par}(d, D)\) defined thereafter, we have to restrict the functional space to that of superfield forms having SUSY-number \(s\) and form degree \(p\) constrained by

\[
s \leq D - d , \quad p \leq d .
\]

(A.5)

Thus, we introduce truncated \(q\)-superforms (more simply referred to as truncated forms in the following) of ghost-number \(g\):

\[
\tilde{\Omega}_q^g = \left[ \hat{\Omega}_q^g \right]^{\text{tr}} \equiv \sum_{p=q-D+q}^{d} \bar{q}^{-p}\bar{Q}_{p}^g (d\theta)^{q-p} .
\]

(A.6)

Here, the coefficients \(\bar{q}^{-p}\bar{Q}_{p}^g\) are superfield forms. In the special case where \(q + g = D\), the truncated form (A.6) contains superfield forms of ghost-number \(g\) belonging to the parallelogram \(\text{Par}(d, D)\). Depending on the relative values of \(g\), \(D\), and \(d\), the expansion (A.6) may involve terms of negative SUSY-number or negative form degree, but all of these terms vanish by virtue of our conventions.

Moreover, we define the (nilpotent) truncated differential \(\tilde{d}\) which has the property of mapping truncated forms to truncated forms:

\[
\tilde{d}\tilde{\Omega} = \left[ d\tilde{\Omega} \right]^{\text{tr}} ,
\]

(A.7)

We note that the arguments in brackets in the definitions (A.6) (A.7) are superforms. Truncation simply means cutting down all of their components which do not satisfy the condition (A.5).
The space of truncated forms for the fixed pair \((d, D)\) will be denoted by \(\mathcal{E}_{(d,D)}\). It is generated by the basic superfield forms \(A, A_\theta, C\) and the action of the operators \(\mathcal{S}, d\) and \(Q\), the latter operator giving rise to the expressions

\[
QA = \psi, \quad QA_\theta = \phi, \quad QC = c' .
\]

The obvious relations

\[
\left[\hat{\Omega} \hat{\Phi}\right]^{tr} = \left[\hat{\Omega} \hat{\Phi}\right]^{tr} \\
\hat{d} \left[\hat{\Omega} \hat{\Phi}\right]^{tr} = \left[(\hat{d} \hat{\Omega}) \hat{\Phi} \pm \hat{\Omega} \hat{d} \hat{\Phi}\right]^{tr} ,
\]

which hold for arbitrary truncated superforms \(\hat{\Omega}\) and \(\hat{\Phi}\), show that the projection from the algebra of superforms to the algebra of truncated superforms represents an homomorphism.

In terms of truncated superforms, the bi-descent equations for the pair \((d, D)\), i.e. eqs.(3.27), read as

\[
S^{g}_{D-g} + \hat{d} \hat{\Omega}^{g+1}_{D-g-1} = 0 \quad (g = 0, \ldots, D) .
\]

These *truncated superdescent equations* define the cohomology \(H(S|\hat{d})\) of \(S\) modulo \(\hat{d}\) in the functional space \(\mathcal{E}_{(d,D)}\). This cohomological problem can be solved using the algebraic techniques of reference [12].

Thus, as before, we do not fix the form degree and we assume that the forms of negative form degree, ghost-number or SUSY-number vanish. The *first step* consists of determining the cohomology \(H(S)\) in the functional space \(\mathcal{E}_S\) of superfield forms introduced in eqs.(2.20) (and subsequently in the functional space \(\mathcal{E}_{(d,D)}\)). In this respect, it is convenient to consider the superfield variables \(\{A, A_\theta, C, \Psi, \Phi, K\}\) where \(\Psi,\Phi\) and \(K\) have been defined in eqs.(2.15). By virtue of the BRST transformations (2.17), the fields \(A_\theta\) and \(K\) form a BRST doublet and therefore they are absent from the cohomology [21, 12]. The remaining fields consist of the gauge superfield form \(A\) and its ghost \(C\), as well as the two “matter superfields” \(\Psi\) and \(\Phi\). From this fact, we conclude [12] that the cohomology \(H(S)\) is algebraically generated by the invariant polynomials depending on \(C\), the supercurvature \(F_A\) and the matter superfields \(\Psi, \Phi\) – all of which fields transform covariantly – as well as their covariant exterior derivatives. More precisely, the cohomology \(H(S)\) in the space \(\mathcal{E}_S\) is generated by the cocycles

\[
\theta_r(C) \quad (r = 1, \ldots, \text{rank } G) \quad \text{and} \quad P^{\text{inv}}(F_A, \Psi, \Phi, D_A \Psi, D_A \Phi) ,
\]

where \(\theta_r\) is the cocycle (3.34) associated to the \(r\)th Casimir operator of the structure group \(G\) and where \(P^{\text{inv}}(\cdots)\) is any invariant polynomial of its arguments. A straightforward generalization of this result from superfield forms to truncated superforms yields the following lemma.
Lemma A.1 The cohomology \( H(S) \) in the functional space \( \mathcal{E}_{(d,D)} \) is given by the truncated forms whose non-vanishing coefficients are polynomials in the superfield forms given in (A.10).

Let us now determine the cohomology \( H(S|\hat{d}) \) in the space \( \mathcal{E}_{(d,D)} \) by starting from the bottom equation of (A.9), i.e. the equation for \( g = D \): \( S\hat{\Omega}_0^D = 0 \). According to lemma A.1, the general non-trivial solution of the latter equation is given by a truncated superform \( \hat{\Omega}_0^D = \theta_r(C) \Omega_D^0 \) which is a polynomial in the \( \theta_r \)'s. However, just as in the case of complete superform solutions discussed in section 3.3, only a linear term in \( \theta_r(C) \) allows us to work our way up to ghost-number zero, i.e. to construct observables. Therefore, we assume that \( \hat{\Omega}_0^D = \theta_r(C) \) for some value of \( r \). The total degree \( D \) will then be odd and given by

\[
D = g_r = 2m_r - 1, \tag{A.11}
\]

where \( m_r \) is the degree of the \( r \)th Casimir operator.

The form \( \hat{\Omega}_0^D = \theta_r(C) \) generates a special solution of the truncated superdescent equations (A.9), namely the truncation to the parallelogram \( \text{Par}(d,D) \) of the superforms \( [\hat{\theta}_r]_{D-g}^g \) \( (g = 0, \ldots, D) \) obeying the superdescent equations (3.35). For the top form, this yields

\[
\hat{\Omega}_D = D - d \Omega_D^0 (d\theta)^{D-d} = \left[ [\hat{\theta}_r]^0 \right]_{D}^\text{tr}. 
\]

The general solution corresponding to the same bottom form \( \hat{\Omega}_0^D \) is obtained by adding to it the general solution of the truncated superdescent equations beginning with \( \hat{\Omega}_0^D = 0 \). Since we are only interested in cohomology classes, we shall consider the slightly more general, though equivalent form

\[
\hat{\Omega}_0^D = S\hat{\Phi}_0^{D-1}, \tag{A.12}
\]

where \( \hat{\Phi}_0^{D-1} \) is a truncated \( (D - 1 - g) \)-superform of ghost-number \( g \). Solving this problem for a value of \( D \) which is not necessarily equal to \( g_r \) as in (A.11) will give us the general solution of (A.9) corresponding to a bottom form \( \hat{\Omega}_0^D \) which is vanishing or trivial in the sense of eq.(A.12).

The procedure is iterative. Let us assume that we have arrived, at the stage of ghost-number \( g + 1 \), to the trivial solution

\[
\hat{\Omega}_{D-g-1}^{g+1} = S\hat{\Phi}_{D-g-1}^g + \hat{d}\hat{\Phi}_{D-g-2}^{g+1}
\]

for some truncated superforms \( \hat{\Phi}_{D-g-1}^g \) and \( \hat{\Phi}_{D-g-2}^{g+1} \). By inserting this expression into the descent equation (A.9) for ghost-number \( g \) and using the nilpotency of \( \hat{d} \), we obtain

\[
S \left( \hat{\Omega}_{D-g}^g - \hat{d}\hat{\Phi}_{D-g-1}^g \right) = 0.
\]
From lemma A.1, it follows that the solution of this relation is given by
\[
\tilde{\Omega}_{D-g}^g = S\tilde{\Phi}_{D-g}^{g-1} + \tilde{d}\tilde{\Phi}_{D-g-1}^g + \tilde{Z}_{D-g}^g, (A.13)
\]
where the truncated superform \(\tilde{Z}_{D-g}^g\) belongs to the cohomology \(H(S)\) as given by lemma A.1, if there exists a representative of the latter with the right ghost-number and form degree. The supercocycle \(\tilde{Z}_{D-g}^g\) has to satisfy a consistency condition ensuring that the next descent equation in (A.9) is integrable. Indeed, let us substitute (A.13) into the descent equation for ghost-number \(g - 1\), thus obtaining
\[
S \left( \tilde{\Omega}_{D-g+1}^{g-1} - \tilde{d}\tilde{\Phi}_{D-g}^{g-1} \right) + \tilde{d}\tilde{Z}_{D-g}^g = 0. (A.14)
\]
In order for this equation to admit a solution \(\tilde{\Omega}_{D-g+1}^{g-1}\), there must exist a truncated superform \(\tilde{H}_{D-g+1}^{g-1}\) such that
\[
S \tilde{H}_{D-g+1}^{g-1} + \tilde{d}\tilde{Z}_{D-g}^g = 0. (A.15)
\]
Then, the solution of (A.14) is given by
\[
\tilde{\Omega}_{D-g+1}^{g-1} = S\tilde{\Phi}_{D-g+1}^{g-2} + \tilde{d}\tilde{\Phi}_{D-g}^{g-1} + \tilde{H}_{D-g+1}^{g-1} + \tilde{Z}_{D-g+1}^{g-1}, (A.16)
\]
where \(\tilde{Z}_{D-g+1}^{g-1}\) is again an element of \(H(S)\) which, in turn, has to obey a consistency condition like (A.15).

Discarding for the moment all of the supercocycles \(\tilde{Z}\) which have appeared or may still appear during this process, we finally arrive, at ghost-number zero, to the trivial solution
\[
\tilde{\Omega}_D^0 = \tilde{d}\tilde{\Phi}_D^0,
\]
which, according to definition (A.6), reads explicitly as
\[
D-d\Omega_d^0 = d D-d\Phi_d^0 + Q D-d\Phi_d^0. (A.17)
\]
By virtue of equation (3.28), this solution corresponds to a vanishing observable:
\[
D-d+1\Delta(d) \equiv \int_M Q D-d\Omega_d^0 = 0.
\]

Let us now go back to one of the steps where a cohomological term \(\tilde{Z}\) appears. Since this term belongs to the cohomology \(H(S)\), it is a polynomial in the cocycles (A.10) – again to be taken as (at most) linear in the \(\theta_r(C)\). Thus, we consider a generic term of one of the two following forms, which generalizes the superform expression (3.36):
\[
\tilde{Z}_D^0 = \tilde{P}_D^0(F_A, \Psi, \Phi, D_A\Psi, D_A\Phi) (A.18)
\]
or
\[
\tilde{Z}_{D-g_r}^{g_r} = \theta_{r_1}(C) \tilde{P}_{D-g_r}^0 (F_A, \Psi, \Phi, D_A\Psi, D_A\Phi). (A.19)
\]
Here, \(\theta_{r_1}(C)\) denotes the cocycle (3.34) of ghost-number \(g_{r_1}\), while the truncated superforms \(\tilde{P}_D^0\) and \(\tilde{P}_{D-g_r}^0\) are invariant polynomials of their arguments.
Let us begin with the solution (A.18) which may be encountered in the last step of the process described above, namely at ghost-number zero. In this case, the coefficient $\frac{D-d}{D-d}Z^0_d$ of the truncated form $\tilde{Z}^0_d = (d\theta)^{D-d}$ is a BRST invariant polynomial in the superfield forms $F_A$, $\Psi$, $\Phi$ and their covariant derivatives as in equation (A.10). By virtue of (A.13) with $g = 0$, the expression $\tilde{\Omega}^0_d = \tilde{Z}^0_d$ solves the truncated superdescent equations (A.9). This solution is cohomologically non-trivial if

$$D-d\tilde{Z}^0_d \neq d^{D-d-1}\tilde{\Phi}^0_{d-1} + Q^{D-d-1}\tilde{\Phi}^0_d.$$  \hspace{1cm} (A.20)

This result yields the second class of solutions announced in proposition 3.4, i.e. the one given in equation (3.45).

Next, we turn to the case given by the solution (A.19), which case may be encountered at a ghost-number $g_{r_1} > 0$. We now have to solve the consistency condition (A.15). We recall that the cocycle $\tilde{\theta}^{r_1}$ generates a set of (complete) superforms $[\tilde{\theta}^{r_1}]^{g_{r_1}-p}$ ($p = 0, \ldots, g_{r_1}$) obeying the superdescent equations (3.35). Substituting the expression (A.19) into (A.15) and using the superdescent equation (3.35) for the ghost cocycle $[\tilde{\theta}^{r_1}]^{g_{r_1}-1}$, we obtain the following relation with the help of the properties (A.8):

$$S\tilde{H}^{g_{r_1}-1}_{D-g_{r_1}+1} - \left( S\left[\tilde{\theta}^{r_1}\right]^{g_{r_1}-1}_{1} \right)^{tr} + (-1)^{g_{r_1}}\tilde{\theta}^{r_1}(C) \tilde{\theta}^{r_1} = 0 \left( g = 0, \ldots, g_{r_1} \right).$$  \hspace{1cm} (A.21)

Here, the exponent ‘tr’ of the second term means truncation according to the definition (A.6). The last term in (A.21) is a non-trivial $S$-cohomology class, whereas the first two terms are $S$-exact. Therefore, both expressions must vanish separately. This implies the following consistency condition for $\tilde{P}$:

$$\tilde{d}\tilde{P}^0_{D-g_{r_1}} = 0.$$  \hspace{1cm} (A.22)

Assuming this relation to hold, condition (A.21) can now be solved by

$$\tilde{H}^{g_{r_1}-1}_{D-g_{r_1}+1} = \left[\tilde{\theta}^{r_1}\right]^{g_{r_1}-1}_{1} \tilde{P}^0_{D-g_{r_1}} \right]^{tr},$$  \hspace{1cm} (A.23)

where we discarded possible $S$-exact terms as well as terms belonging to $H(S)$ which, for their part, would generate further solutions. As a matter of fact, the solution (A.23) is the first of a chain of truncated supercocycles

$$\tilde{H}^g_{D-g} \equiv \left[\tilde{\theta}^{r_1}\right]^{g}_{g_{r_1}-g} \tilde{P}^0_{D-g_{r_1}} \right]^{tr} \quad (g = 0, \ldots, g_{r_1} - 1),$$  \hspace{1cm} (A.24)

which obey the following truncated superdescent equations by virtue of eqs. (3.35) and (A.22):

$$S\tilde{H}^g_{D-g} + \tilde{d}\tilde{H}^{g+1}_{D-g-1} = 0 \quad (g = 0, \ldots, g_{r_1} - 1).$$  \hspace{1cm} (A.25)

We still have to solve condition (A.22). This requires the determination of the cohomology $H(\tilde{d})$ in the space $\tilde{E}_{(d,D)}$, the result being expressed by the following lemma:
Lemma A.2 The cohomology $H(\hat{d})$ in the space $\hat{\mathcal{E}}_{(d,D)}$ is given by the truncated superforms of ghost-number 0 and total degree $D$:

$$\hat{\Omega}_D^0 = (d^0 (d\theta)^{D-d}) .$$  \hspace{1cm} (A.26)

Here, $D^0 \Omega_d^0$ is a BRST invariant polynomial in the superfield forms $F_A$, $\Psi$, $\Phi$ and their covariant derivatives as in eq. (A.10), of degree $d$ and ghost-number 0, but subject to the non-triviality condition

$$D^0 \Omega_d^0 \neq d D^0 \Phi_{d-1}^0 + Q D^{D-d} \Phi_d^0 ,$$

where the superfield forms $D^0 \Phi_{d-1}^0$ and $D^{D-d} \Phi_d^0$ are the components of a truncated superform belonging to $\hat{\mathcal{E}}_{(d,D)}$. In particular, the cohomology $H(\hat{d})$ is trivial for truncated superforms of degree strictly smaller than $D$.

Proof: In this proof, we do not specify the ghost-number which is irrelevant for the present discussion. We have to solve the equation $d \hat{\Omega}_q = 0$ for the truncated form (A.6). Let us begin with the generic case, i.e. $q < D$. The condition $d \hat{\Omega}_q = 0$ can then be written as a set of equations, one for each form degree:

$$Q^{q-p-1} \Omega_{p+1}^q + d^{q-p} \Omega_p^q = 0 \quad (p = q - D + d, \ldots, d - 1) .$$  \hspace{1cm} (A.27)

The first of these equations, namely the one for $p = q - D + d$, may be solved by using proposition 3.1 (see (3.13)-(3.18)), which yields

$$D^d \Omega_{q-D+d}^q = Q D^{d-1} \Phi_{q-D+d}^q + d D^d \Phi_{q-D+d-1}^q ,$$

$$D^{d-1} \Omega_{q-D+d+1}^q = Q D^{d-2} \Phi_{q-D+d+1}^q + d D^{d-1} \Phi_{q-D+d}^q .$$  \hspace{1cm} (A.28)

Substituting this result into the second equation of the set (A.27), we get

$$Q \left( D^{d-2} \Omega_{q-D+d+2}^q - d D^{d-2} \Phi_{q-D+d+1}^q \right) = 0 ,$$

whose solution, due to the triviality of $H(Q)$, is given by

$$D^{d-2} \Omega_{q-D+d+2}^q = Q D^{d-3} \Phi_{q-D+d+2}^q + d D^{d-2} \Phi_{q-D+d+1}^q .$$  \hspace{1cm} (A.29)

The procedure continues along these lines until the solution of the last equation of the set (A.27):

$$q^d \Omega_d^q = Q q^{d-1} \Phi_d^q + d q^d \Phi_{d-1}^q .$$  \hspace{1cm} (A.30)

By combining these results, we conclude that, for $q < D$, we have

$$\hat{\Omega}_q = \hat{d} \Phi_{q-1}^q , \quad \text{with} \quad \Phi_{q-1}^q = \sum_{p=q-D+d-1}^d q^{1-p} \Phi_p^q (d\theta)^{q-1-p} ,$$  \hspace{1cm} (A.31)

i.e. the general solution is trivial. Thus, we are left with the case $q = D$, i.e. the cocycle condition $d \hat{\Omega}_D = 0$, which is identically satisfied, whence the result (A.26).

q.e.d.
Let us now solve equation (A.22) with the help of this lemma. Since $D - g_r < D$, it follows from lemma A.2 that the solution $\tilde{P}^0_{D - g_r}$ of (A.22) is a $\tilde{d}$-variation, i.e. $\tilde{P}^0_{D - g_r} = \tilde{d} \tilde{R}^0_{D - g_r - 1}$. Thereby, equation (A.24) for $g = 0$ can be written as

$$\tilde{H}_D^0 = \left[ \left[ \hat{\theta}_r \right]_{g_r}^0 \tilde{d} \tilde{R}^0_{D - g_r - 1} \right]^\text{tr}. \quad \text{(A.32)}$$

Since $\tilde{P}^0_{D - g_r}$ is $\mathcal{S}$-invariant, the polynomial $\tilde{R}^0_{D - g_r - 1}$ again has to be a solution of a system of truncated superdescent equations in the functional space $\mathcal{E}_{(d,D)}$:

$$\mathcal{S} \tilde{R}^g_{D - g_r - g - 1} + \tilde{d} \tilde{R}^{g+1}_{D - g_r - g - 2} = 0 \quad (g = 0, \ldots, D - g_r - 1). \quad \text{(A.33)}$$

These are solved in the same way as we did in the first step above\(^7\). The non-trivial solution of the bottom equation is a supercocycle

$$\tilde{R}^{g_r}_0 = \theta_{r_2}(C) \quad \text{with} \quad g_r \equiv D - g_r - 1, \quad \text{(A.34)}$$

if any such exists with this ghost-number. Otherwise $\tilde{R}^{g_r}_0 = 0$ and one has to climb up equations (A.33) until meeting a non-trivial cohomology – as we did in the first step – and then continue, starting from this cohomology. But let us consider the case where (A.34) holds. Then, we get the following result by using (3.35) and by discarding possible new cohomology that may appear in the process of climbing up:

$$\tilde{H}^0_{D - g_r - 1} = \left[ \left[ \hat{\theta}_r \right]_{g_r}^0 \tilde{d}_r \tilde{R}^{g_r}_0 \right]^\text{tr}. \quad \text{(A.35)}$$

Substitution of this expression into (A.32) and application of the rules (A.7)-(A.8) then yields

$$\tilde{H}^0_D = \left[ \left[ \hat{\theta}_r \right]_{g_r}^0 \tilde{d}_r \left[ \hat{\theta}_{r_2} \right]_{g_{r_2}}^0 \right]^\text{tr} = \left[ \left[ \hat{\theta}_r \right]_{g_r}^0 f_{r_2}(\hat{F}) \right]^\text{tr}, \quad \text{with} \quad g_r + g_{r_2} + 1 = D, \quad \text{(A.35)}$$

where we have used the last of equations (3.35).

Going back to the previous step where new cohomology might have been encountered, we may repeat the whole argument, producing in this way solutions involving more and more factors $\tilde{d}_r \left[ \hat{\theta}_r \right]_{g_r}^0 = f_r(\hat{F})$. Thus, the generic solution of the truncated superdescent equations reads as $\tilde{H}_D^0 = \tilde{H}_D^0 \equiv D - d \tilde{H}_D^0 (d\theta)^{D-d}$ where

$$\tilde{H}_D^0 = \left[ \left[ \hat{\theta}_r \right]_{g_r}^0 f_{r_2}(\hat{F}) \cdots f_{r_L}(\hat{F}) \right]^\text{tr}, \quad \text{with} \quad \sum_{k=1}^L g_{r_k} + L - 1 = D \quad (L \geq 1). \quad \text{(A.36)}$$

This conclusion is precisely the result (3.44) stated in proposition 3.14.

\(^7\)However, there is a difference here: since the total degree is less than $D$ and the truncation is made relative to the pair $(d, D)$, the parallelogram is replaced by a pentagram defined by the lines $p = 0$, $p = d$, $s = 0$, $s = D - d$, $g = 0$. 36
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