SUNLayer: Stable denoising with generative networks

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Abstract

It has been experimentally established that deep neural networks can be used to produce good generative models for real world data. It has also been established that such generative models can be exploited to solve classical inverse problems like compressed sensing and super resolution. In this work we focus on the classical signal processing problem of image denoising. We propose a theoretical setting that uses spherical harmonics to identify what mathematical properties of the activation functions will allow signal denoising with local methods.

1 Introduction

Deep neural networks, in particular generative adversarial networks by [Goodfellow et al., 2014] have been recently used to produce generative models for real world data that can capture very complex structures. This is especially true for natural images (see for instance [Nguyen et al., 2016]). Those generative priors have been successfully used to efficiently solve classical inverse problems in signal processing, like super resolution ([Johnson et al., 2016]) and compressed sensing ([Bora et al., 2017]). The latter numerically demonstrates that the generative prior can be exploited to solve the compressed sensing problem with ten times fewer measurements than the classic compressed sensing theory requires. Follow-up work by [Hand and Voroninski, 2017] recently explained the success of local methods (namely empirical risk minimization) in the compressed sensing task by assuming a generative model of a multi-layer neural network with random weights and ReLU activation functions.

The aim of this paper is to propose a theoretical framework that will allow us to analyze neural networks in the context of another classical inverse problem in signal processing: signal denoising. It has been experimentally established that deep neural networks can be used for image inpainting and denoising [Xie et al., 2012]. We are interested in denoising in the high-noise regime, in which modern methods that do not rely on machine learning appear less capable. In this work we propose a simple model for the generative model where linear maps are composed with non-linear activation functions, and we study what mathematical properties of the activation function will allow signal denoising with local methods. We assume our generative model can be expressed as the composition of simple neural network layers we call SUNLayer and we use tools from harmonic analysis to understand what are the good properties for activation functions for the denoising task. We perform numerical experiments to complement the theory.

1.1 Main contributions

The main contributions of this paper can be summarized in two points.

• We introduce SUNLayer, a simple model for spherical uniform neural network layers (Section 2).

• We prove performance guarantees for denoising with a generative network under the SUNLayer model. In particular, given \( y = L(x) + \text{noise} \) with SUNLayer \( L \) for some activation function, we show that all critical points of the map \( z \mapsto \|y - L(z)\|^2 \) are close to \( \{\pm x\} \) provided the activation function is well behaved and the noise is appropriately small. (Section 3).

We believe the theoretical framework we introduce in this paper could be useful to provide mathematical intuition about neural networks in a more general context. See Section 6 for a more in-depth discussion.

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2 SUNLayer: a neural network model

Let \( x \in S^n \) be an input signal, we consider the linear map \( x \mapsto f_x \in \mathcal{L}^2(S^n) \) where \( f_x(y) = x \cdot y \) the inner product in \( \mathbb{R}^{n+1} \) between \( x \) and \( y \). Let \( \theta : \mathbb{R} \to \mathbb{R} \) be an activation function. We define one layer of the SUNLayer neural network to be

\[
L_n : S^n \to \mathcal{L}^2(S^n) \\
L_n(x) = \theta \circ f_x.
\]  

Note that if instead of the linear map \( f_x \) we had considered, as one usually does in neural networks, a matrix \( M \in \mathbb{R}^{t \times n+1} \), then the analogous to \( L_n(x) \) is essentially \( \theta(Mx) \) that can be seen as a function defined in the rows of \( M \) as \( L(x) : \{1, \ldots, t\} \to \mathbb{R} \) as \( m_i \mapsto \theta(x \cdot m_i) \). The SUNLayer model is heuristically generalizing the linear step to a continuum of possible rows.

We are interested in the case where \( L_n(S^n) \subset A \subset \mathcal{L}^2(S^n) \) where \( A \) is a finite dimensional subspace of \( \mathcal{L}^2(S^n) \) (and therefore locally compact). The finite dimensionality will allow us to compose several layers of the SUNLayer model. For all \( x \in S^n \), we have that \( L_n(x) \in S^{n'} \) with \( \dim(A) = n' + 1 \). A very simple observation (see proof of Lemma 1) shows that \( \|L_n(x)\| = \|L_n(x')\| = c_{n,\theta} \) for all \( x, x' \in S^n \) where \( c_{n,\theta} \) is a constant that depends on the activation function \( \theta \) and on the dimension \( n \) of the domain. Therefore the normalization step (which a priori may have resembled practice standards like batch normalization ([Ioffe and Szegedy, 2015])) amounts to simple rescaling, and furthermore, we even have \( c_{n,\theta} = 1 \) when \( \theta \) is scaled appropriately (see Lemma 3).

We then conclude that \( L_n \circ L_n : S^n \to \mathcal{L}^2(S^n) \) is well defined as long as \( A \) is finite dimensional. In Section 4 we observe that a necessary and sufficient condition for \( A \) to be finite dimensional is that \( \theta \) is a polynomial.

2.1 Denoising

Let us assume we have a generative model \( G : S^n \to \mathbb{R}^N \) that given a parameter \( x \in S^n \) produces \( G(x) \), an element of a target space (for instance an image)\footnote{The generative model could have been produced for instance with a generative adversarial network (GAN) trained with a large set of images or more generally structured dataset (that comes from an unknown latent distribution). The GAN consists of two neural networks, one known as the generator, which aims to construct new data plausible to be coming from the latent distribution of the training set, and the other is the discriminator which aims to distinguish between instances from the true dataset and the candidates produced by the generator. Both networks get trained against each other. After training the generator produces a neural network with several layers. We assume the parameter is space normalized, so the generator finds a generative model \( G : S^n \to \mathbb{R}^N \) where \( n \ll N \). For all \( x \) we have that \( G(x) \) is an element in the target space (for instance, an image) and \( x \) is the vector of parameters that generates it.\( ^{1} \)} The question we aim to answer is when it is possible to denoise an element \( y \in \mathbb{R}^N \) to the closest element in the image of \( G \) by using local methods like gradient descent. Figure 1 shows an example of the phenomenon we aim to explain.

We assume our generative model is the composition of layers from the SUNLayer model defined in (1). We solve the denoising problem one layer at a time. Fix \( x^2 \in S^n \). Given \( y = \theta \circ f_{x^2} + \eta \) for some \( \theta : \mathbb{R} \to \mathbb{R} \) and noise \( \eta \in \mathcal{L}^2(S^n) \), then denoising for one SUNLayer corresponds with the least squares problem

\[
\min_{x \in S^n} \|\theta \circ f_x - y\|^2_{\mathcal{L}^2(S^n)}.
\]

There exists at least one minimizer for (2) due to compactness.

3 Preliminaries: spherical harmonics

To analyze denoising under the SUNLayer model, we leverage ideas from spherical harmonics. In this section we summarize some classical results about spherical harmonics that can be found on Chapter 2 of [Morimoto, 1998], focusing on theorems and definitions we use in this paper. We refer the reader to [Morimoto, 1998] for a comprehensive review.

Let \( P_k(\mathbb{R}^{n+1}) \) the space of homogeneous polynomials of degree \( k \) in \( n+1 \) variables (we could have also considered real or complex coefficients but real is enough for the scope of this paper).
Figure 1: Denoising with generative priors

(First line) Digits from the MNIST test set ([LeCun, 1998]). (Second line) Random noise is added to the digits. (Third line) Denoising of images by shrinkage in wavelet domain ([Donoho and Johnstone, 1994]). (Fourth line) Denoising by minimizing total variation ([Rudin et al., 1992]). (Fifth line) We train a GAN using the training set of MNIST to obtain a generative model $G$. We denoise by finding the closest element in the image of $G$ using stochastic grading descent.
**Definition 1** (Spherical harmonics). The Laplacian is the differential operator defined as

\[ \Delta_x = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n+1}^2}, \]

and the space of spherical harmonics is defined as:

\[ \mathcal{H}_k(S^n) = \{ H_k \in \mathcal{P}_k(S^n) : \Delta H_k = 0 \} \subset \mathcal{L}^2(S^n). \] \hspace{1cm} (3)

In other words, \( \mathcal{H}_k(S^n) \) is the restriction of the polynomials with Laplacian 0 to \( S^n \).

**Proposition 1.** \( \mathcal{H}_k(S^n) \) is a finite dimensional space and

\[ \mathcal{L}^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^n). \] \hspace{1cm} (4)

In the sequel, we let \( \alpha_{n,k} \) denote the dimension of \( \mathcal{H}_k(S^n) \).

**Definition 2.** For fixed \( k \) and \( n \) let \( \{ Y_{1}^{\alpha_{n,k}} \ldots Y_{k}^{\alpha_{n,k}} \} \) an orthonormal basis of \( \mathcal{H}_k(S^n) \). Define the bilinear form

\[ F_k(\sigma, \tau) = \sum_{i=1}^{\alpha_{n,k}} Y_{i}^{\sigma}(\sigma) Y_{i}^{\tau}(\tau). \]

A simple computation shows that \( F_k \) is independent of the choice of the orthonormal basis. The bilinear forms \( F_k(\cdot, \cdot) \) will be very useful in the analysis of the SUNLayer model. Some of their relevant properties are summarized in the following lemma.

**Proposition 2.** The following statements hold.

1. Reproducing property: \( \langle H, F_k(\cdot, \cdot) \rangle = H(\sigma) \) for all \( H \in \mathcal{H}_k(S^n) \).

2. Zonal property: there exists \( \varphi_{n,k} : \mathbb{R} \to \mathbb{R} \) so that \( \langle F_k(\sigma_1, \cdot), F_k(\sigma_2, \cdot) \rangle = F_k(\sigma_1, \sigma_2) = \varphi_{n,k}(\sigma_1 \cdot \sigma_2) \). In particular \( F_k(\sigma_1, \sigma_2) \) only depends on \( \sigma_1 \cdot \sigma_2 \).

3. The function \( \varphi_{n,k} : \mathbb{R} \to \mathbb{R} \) is the Gegenbauer polynomial of degree \( k \) and dimension \( n+1 \). The set \( \{ \varphi_{n,k} \}_{k=0}^{\infty} \) is an orthogonal basis of polynomials over \([-1, 1]\) with respect to the measure

\[ d\mu_n = (1 - t^2)^{(n-2)/2} dt \] \hspace{1cm} (5)

(here \( dt \) is the standard Borel measure in \( \mathbb{R} \)). Note that this is not a standard normalization for the Gegenbauer polynomials but we use it to simplify the results of this paper. In fact Chapter 2 of [Morimoto, 1998] considers the Legendre polynomials to be \( P_{n,k}(t) = \frac{\text{vol}(S^n)}{\alpha_{n,k}} \varphi_{n,k}(t) \) (the term \( \text{vol}(S^n) \) is the \( n \)-dimensional volume of the sphere and it does not show up in Morimoto’s analysis since he uses the normalized measure in the spheres). In Chapter 5 Morimoto considers the Gegenbauer polynomials as a generalization of the Legendre polynomials where \( n > 0 \) can be any real number, with a different normalization.

4. The discussion in pages 26–27 of [Morimoto, 1998] shows that \( P_{n,1}(1) = 1 \). This together with the facts \( \alpha_{n,0} = 1, \alpha_{n,1} = n + 1 \),

\[ \alpha_{n,k} = \binom{n+k}{k} - \binom{n+k-2}{k-2} = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} = O(k^{n-1}) \quad \text{for } k \geq 2, \]

and \( \text{vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \) allow us to identify the correct normalization for the Gegenbauer polynomials.
5. Using that \( \varphi_{n,k}(t) = \frac{\alpha_{n,k}}{\text{vol}(S^n)} P_{n,k}(t) \) and Theorems 2.29 and 2.34 of [Morimoto, 1998] one obtains the following identities:

\[
\|\varphi_{n,k}(t)\|_\infty = \varphi_{n,k}(1) = \frac{\alpha_{n,k}}{\text{vol}(S^n)},
\]

\[
\|\varphi_{n,k}(t)\|^2_{L^2(\mu_n)} = \int_{-1}^{1} \varphi_{n,k}(t)^2(1-t^2)^{(n-2)/2}dt = \frac{\alpha_{n,k}}{\text{vol}(S^n) \text{vol}(S^{n-1})}.
\]

6. Using (5.1) and (5.3) of [Morimoto, 1998] (pages 97–98) one can express a relationship between \( \varphi_{n,k}(t) \) and its derivative \( \varphi'_{n,k}(t) := \frac{d}{dt} \varphi_{n,k}(t) \), namely

\[
\varphi'_{n,k}(t) = \frac{(n+1) \text{vol}(S^n)}{\text{vol}(S^{n+2})} \varphi_{n+2,k-1}(t).
\]

7. Let \( h \in L^2(S^n) \) a \( C^{2r} \) function, then one can decompose \( h \) in the spherical harmonics as \( h(\tau) = \sum_{k=0}^{\infty} h_k(\tau) \) where \( h_k \in \mathcal{H}_k(S^n) \). Theorem 2.45 of [Morimoto, 1998] in particular shows that for all \( k \geq 0 \) one has

\[
k^{2r} \|h_k(\tau)\|_{L^2(S^n)} \leq \|((\Delta_{S^n})^r h)\|_{L^2(S^n)} \tag{9}
\]

where \( \Delta_{S^n} \) is the spherical Laplacian. In particular, if there exists an axis under which \( h \) is rotationally invariant (i.e. \( h(\tau) = \theta(\omega \cdot \tau) \) for some fixed \( \omega \) and some \( \theta : [-1, 1] \rightarrow \mathbb{R} \)) then if \( \omega \cdot \tau = t \)

\[
\Delta_{S^n}(h) = \theta''(t)(1-t^2) - n t \theta'(t) \tag{10}
\]

(see for instance (2.9)).

Note that \( F_k(\sigma, \cdot) \in \mathcal{H}_k(S^n) \) for all \( \sigma \in S^n \), thus \( \text{span}(\{F_k(\sigma, \cdot)\}_{\sigma \in S^n}) \subseteq \mathcal{H}_k(S^n) \). The reproducing property says that for all \( H \in \mathcal{H}_k(S^n) \)

\[
(H, F_k(\sigma, \cdot)) = H(\sigma). \tag{11}
\]

Observe that for all \( H \neq 0 \) there exists \( \sigma \in S^{n-1} \) such that \( H(\sigma) \neq 0 \). Then \( H \not\in \mathcal{H}_k(\sigma, \cdot) \) which implies that

\[
\text{span}(\{F_k(\sigma, \cdot)\}_{\sigma \in S^n}) = \mathcal{H}_k(S^n) \subset L^2(S^n).
\]

4 Analysis

Given an activation function \( \theta : \mathbb{R} \rightarrow \mathbb{R} \), then since \( \{\varphi_{n,k}\}_{k=0}^{\infty} \) form an orthogonal basis of polynomials over \([-1, 1] \) with respect to some measure, we can decompose \( \theta \) as

\[
\theta(t) = \sum_{k=0}^{\infty} a_k \varphi_{n,k}(t),
\]

for some \( a_0, \ldots, a_k, \ldots \in \mathbb{R} \). Then

\[
(\theta \circ f_x)(y) = \theta(x \cdot y) = \sum_{k=0}^{\infty} a_k \varphi_{n,k}(x \cdot y) = \sum_{k=0}^{\infty} a_k F_k(x, y). \tag{12}
\]

In other words one layer of the SUNLayer neural network model \( [1] \) can be expressed as

\[
L_n(x) = \theta \circ f_x = \sum_k a_k F_k(x, \cdot).
\]

Note that if \( \theta \) is a polynomial of degree \( K \), then \( L_n(S^n) = \{\theta \circ f_x : x \in S^n\} \subset \oplus_{k=1}^{K} \mathcal{H}_k(S^n) \) which is finite dimensional. Reciprocally, finite dimensional subspaces of \( L^2(S^n) \) are included in \( \oplus_{k=1}^{K} \mathcal{H}_k(S^n) \) for some finite \( K \). This observation, combined with the remark from Section \( [2] \) suggest that polynomial activation functions are a useful model for studying the composition of multiple layers.

Lemma \( [2] \) shows an alternative expression for the least squares problem \( [2] \).
Lemma 1. For all \( y \in \mathcal{L}^2(S^n) \) we have
\[
\arg \min_{x \in S^n} \| \theta \circ f_x - y \|_{\mathcal{L}^2(S^n)}^2 = \arg \max_{x \in S^n} (\theta \circ f_x, y)_{\mathcal{L}^2(S^n)}.
\]

Proof. Note that for all rotations \( Q \in O(n) \) we have
\[
\theta \circ f_{Qx}(z) = \theta(x^T Q^T z) = (\theta \circ f_x)(Q^T z),
\]
and so
\[
\| \theta \circ f_{Qx} \|_{\mathcal{L}^2(S^n)}^2 = \int_{z \in S^n} |(\theta \circ f_{Qx})(z)|^2 \, dz = \int_{z \in S^n} |(\theta \circ f_x)(Q^T z)|^2 \, dz
\]
\[
= \int_{z \in S^n} |\theta \circ f_x(z)|^2 \, dz = \| \theta \circ f_x \|^2.
\]
Therefore \( \| \theta \circ f_x \| \) is constant for all \( x \in S^n \), which implies the lemma since
\[
\| \theta \circ f_x - y \|^2 = \| \theta \circ f_x \|^2 + \| y \|^2 - 2(\theta \circ f_x, y) = \text{constant} - 2(\theta \circ f_x, y).
\]

Given \( y = \theta \circ f_{x^t} \), according to Lemma 1 and equation (12) we need to find \( x \in S^n \) that maximizes
\[
(\theta \circ f_x, \theta \circ f_{x^t}) = \sum_{k=0}^{\infty} (a_k F_k(x, \cdot), a_k F_k(x^t, \cdot)) = \sum_{k=0}^{\infty} a_k^2 F_k(x, x^t)
\]
\[
= \sum_{k=0}^{\infty} a_k^2 \varphi_{n,k}(x \cdot x^t) =: g_\theta(x \cdot x^t).
\]
(13)

Note that the second equality is a consequence of the reproducing property (11). The function \( g_\theta \) will be particularly useful in our analysis.

Definition 3. Let \( \theta : \mathbb{R} \to \mathbb{R} \) be an activation function, with Gegenbauer decomposition \( \theta(t) = \sum_{k=0}^{\infty} a_k \varphi_{n,k}(t) \). Then we define \( g_\theta : \mathbb{R} \to \mathbb{R} \) as \( g_\theta(t) = \sum_{k=0}^{\infty} a_k^2 \varphi_{n,k}(t) \).

Lemma 2. If \( \theta(t) : [-1, 1] \to \mathbb{R} \) is \( C^2 \) and \( \theta(t) = \lim_{K \to \infty} \sum_{k=0}^{K} a_k \varphi_{n,k}(t) \) (convergence in \( \mathcal{L}^2(\mu_n) \)) then the functions \( g_\theta(t) = \lim_{K \to \infty} \sum_{k=0}^{K} a_k^2 \varphi_{n,k}(t) \) and \( h_\theta = \lim_{K \to \infty} \sum_{k=0}^{K} a_k^2 \varphi'_{n,k}(t) \) are well-defined (and the convergence is also point-wise and absolute). Furthermore, if \( \theta \) is \( C^4 \) we also have that \( g_\theta'(t) = h_\theta(t) \) for all \( t \in [-1, 1] \).

Proof. See Appendix 7

Lemma 3. If \( \theta(t) = \sum_{k=0}^{\infty} a_k \varphi_{n,k}(t) \) then
\[
c_{n,\theta}^2 = \| \theta \circ f_x \|_{\mathcal{L}^2(S^n)}^2 = \text{vol}(S^{n-1}) \| \theta \|_{\mathcal{L}^2(\mu_n)}^2 = g_\theta(1)
\]
Theorem 1. Let $\theta$ be a measurable function. The following Theorem provides a sufficient condition that makes recovery possible in the noiseless case.

4.1 Noiseless case

Theorem 1. Suppose $g'_\theta(t) > 0$ for all $t \in [-1,1]$. Then for each $x^x \in S^n$, the only critical points of

$$x \mapsto \|\theta \circ f_x - \theta \circ f_{x^x}\|^2$$

are $\pm x^x$, with $x^x$ being the unique local minimizer.

4.2 Denoising

The following Theorem is the main result of this paper.

Theorem 2. Let $\theta(t) = \sum_{k=0}^{K} a_k \varphi_{n,k}(t)$ and $y = \theta \circ f_{x^x} + \eta$. We decompose $\eta$ as follows:

$$\eta = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} e_{k,i} F_k(\sigma_{k,i}, \cdot) =: \sum_{k=0}^{\infty} \eta_k \in \mathcal{H}_k(S^n).$$

Let $\epsilon = \left\| \sum_{k=0}^{K} \sum_{i=1}^{d_k} e_{k,i} \varphi_{n,k}(x \cdot \sigma_{k,i}) \sigma_{k,i} \right\|$ and let $T = \inf_{t \in [-1,1]} |g'_\theta(t)|$. Then

(a) Every critical point $\hat{x}$ of $x \mapsto \|\theta \circ f_x - y\|_2^2 \mathcal{F}^2(S^n)$ satisfies that $|\hat{x} \cdot x^x| > 1 - \frac{2\epsilon}{T+\epsilon}$.
(b) Define $M_k := \max_{t \in [-1,1]} |\varphi'_{n,k}(t)|$, then $\epsilon \leq \sum_{k=1}^{K} M_k |a_k| \|\eta_k\|$.

**Proof. of Theorem 2(a)** According to Lemma 1 we need to solve
\[
\max_{x \in S^n} (\theta \circ f_x, \theta \circ f_x + \eta) = \max_{x \in S^n} g_\theta(x \cdot x^2) + \langle \theta \circ f_x, \eta \rangle.
\]
The reproducing property implies
\[
\langle \theta \circ f_x, \eta \rangle = \sum_{k=0}^{K} \langle \epsilon F_k(x, \cdot), \sum_{i=1}^{d_k} e_{k,i} F_k(\sigma_{k,i}, \cdot) \rangle = \sum_{k=0}^{K} a_k \sum_{i=1}^{d_k} e_{k,i} \varphi_{n,k}(x \cdot \sigma_{k,i}).
\]
Therefore the denoising objective is
\[
\max_{x \in S^n} g_\theta(x \cdot x^2) + \sum_{k=0}^{K} a_k \sum_{i=1}^{d_k} e_{k,i} \varphi_{n,k}(x \cdot \sigma_{k,i}) \tag{14}
\]
For $x$ critical point of (14) Lagrange multipliers give us
\[
\mathcal{L}_n(x, \lambda) = g_\theta(x \cdot x^2) + \sum_{k=0}^{K} a_k \sum_{i=1}^{d_k} e_{k,i} \varphi_{n,k}(x \cdot \sigma_{k,i}) + \lambda (\|x\|^2 - 1)
\]
and $\frac{\partial}{\partial x} \mathcal{L}_n = 0$ implies
\[
0 = \frac{\partial}{\partial x} g_\theta(x \cdot x^2) x^2 + A \sum_{k=0}^{K} a_k \sum_{i=1}^{d_k} e_{k,i} \varphi'_{n,k}(x \cdot \sigma_{k,i}) \lambda + 2\lambda x
\]
By hypothesis we have $\|B\| < \|A\|$ then $\|2\lambda x\| = \|A + B\| \geq \|A\| - \|B\| > 0$ which implies $\lambda \neq 0$, therefore
\[
x = -\frac{1}{2\lambda} (g_\theta(x \cdot x^2) x^2 + B)
\]
and
\[
2|\lambda| = \|g_\theta' (x \cdot x^2) x^2 + B\| \leq |g_\theta'(x \cdot x^2)| + \epsilon
\]
therefore
\[
|x \cdot x^2| = \frac{1}{2|\lambda|} |g_\theta(x \cdot x^2) + B x^2| \geq \frac{g_\theta'(x \cdot x^2)}{2|\lambda|} \geq \frac{g_\theta'(x \cdot x^2) - \epsilon}{g_\theta'(x \cdot x^2) + \epsilon} \geq 1 - \frac{2\epsilon}{T + \epsilon}.
\]

\qed

The key parameter $\epsilon$ in Theorem 2 depends on both the noise $\eta$ and the activation function $\theta$. In order to understand the behavior of $\epsilon$ in terms of the noise $\eta$, and prove Theorem 2(b), we choose $\{\sigma_{k,i}\}$, $(i = 1, \ldots, N_t)$ so that $\{F_k(\sigma_{k,i}, \cdot)\}$ forms a tight frame. To this end it suffices for $\{\sigma_{k,i}\}$ to form a spherical $t$-design for $t = 2k$.

**Definition 4 (Spherical $t$-design).** A spherical $t$-design is a sequence of $N_t$ points $\{x_1, \ldots, x_{N_t}\} \subset S^n$ such that for every polynomial $p$ of degree at most $2t$ we have
\[
\frac{1}{N_t} \sum_{i=1}^{N_t} p(x_i) = \int_{S^n} p(x) dx.
\]
**Definition 5** (Tight frame). Let \((V, \langle \cdot, \cdot \rangle)\) be vector space with an inner product. A tight frame is a sequence \(\{v_k\}_{k \in I \subseteq \mathbb{N}} \subset V\) such that there exists a constant \(c\) so that for all \(v \in V\)
\[
\sum_{k \in I} |\langle v, v_k \rangle|^2 = c \|v\|^2.
\]

**Lemma 4.** If \(\{\sigma_{k,i}\}_{i=1}^{N_k}\) form a spherical \(t\)-design with \(t = 2k\) then \(\{F_k(\sigma_{k,i}, \cdot)\}_i\) is a tight frame for \(\mathcal{H}_k(S^n)\) with constant \(c = N_k\).

**Proof.** Let \(\{Y_j\}\) be an orthonormal basis for \(\mathcal{H}_k(S^n)\). Consider \(\delta_{a,b} = 1\) if \(a = b\) and 0 otherwise. It suffices to show
\[
N_k \delta_{j,j'} = \sum_{i=1}^{N_k} \langle F_k(\sigma_{k,i}, \cdot), Y_j \rangle \langle F_k(\sigma_{k,i}, \cdot), Y_{j'} \rangle
\]
\[
= \sum_{i=1}^{N_k} \left( \int_{\mathbb{S}^n} F_k(\sigma_{k,i}, \tau) Y_j(\tau) \right) \left( \int_{\mathbb{S}^n} F_k(\sigma_{k,i}, \tau) Y_{j'}(\tau) \right)
\]
\[
= \sum_{i=1}^{N_k} \left( \sum_{j''} Y_{j''}(\sigma_{k,i}) \int_{\mathbb{S}^n} Y_{j''}(\tau) Y_j(\tau) \right) \sum_{j'''} Y_{j'''}(\sigma_{k,i}) \int_{\mathbb{S}^n} Y_{j'''}(\tau) Y_{j'}(\tau)
\]
\[
= \sum_{i=1}^{N_k} Y_j(\sigma_{k,i}) Y_{j'}(\sigma_{k,i})
\]
Observe that if \(Y_j, Y_{j'} \in \mathcal{H}_k(S^n)\) then \(p(x) = Y_j(x)Y_{j'}(x)\) is a polynomial of degree \(2k\). Then using the \(t\)-design property we get
\[
\sum_{i=1}^{N_k} Y_j(\sigma_{k,i}) Y_{j'}(\sigma_{k,i}) = N_k \frac{1}{N_k} \sum_{i=1}^{N_k} Y_j(\sigma_{k,i}) Y_{j'}(\sigma_{k,i})
\]
\[
= N_k \int_{\mathbb{S}^n} Y_j(\tau) Y_{j'}(\tau)
\]
\[
= N_k \langle Y_j, Y_{j'} \rangle_{\mathcal{H}_k(S^n)}
\]
which proves the theorem. \(\square\)

**Proof of Theorem \([2](b)\).** We choose \(\{\sigma_{k,i}\}_{i=1}^{N_k}\) so that \(\{F_k(\sigma_{k,i}, \cdot)\}_i\) is a tight frame for \(\mathcal{H}_k(S^n)\) with constant \(N_k\). We write \(\eta = \sum_{k=0}^{\infty} \sum_{\delta_{k,i}} e_{k,i} F_k(\sigma_{k,i}, \cdot)\). One can uniquely decompose \(\eta = \sum_{k=0}^{\infty} \eta_k \in \mathcal{H}_k(S^n)\) and we have \(\sum_i |e_{k,i}|^2 = \frac{1}{N_k} \|\eta_k\|^2\). In fact \(e_{k,i}\) can be chosen so that \(e_{k,i} = \frac{1}{N_k} \langle \eta, F_k(\sigma_{k,i}, \cdot) \rangle\). Following the notation in the proof of Theorem \([2](a)\) we have:
\[
B = \sum_{k=1}^{K} a_k \sum_{i=1}^{N_k} e_{k,i} \varphi'_{n,k}(x \cdot \sigma_{k,i}) \sigma_{k,i}
\]
and
\[
\epsilon = \|B\| \leq \sum_{k=1}^{K} |a_k| \left\| \sum_{i=1}^{N_k} e_{k,i} \varphi'_{n,k}(x \cdot \sigma_{k,i}) \sigma_{k,i} \right\|
\]
Let \(G_{k,x} : \mathcal{L}^2(S^n) \rightarrow S^n\) such that \(G_{k,x}(\eta) = \sum_{i=1}^{N_k} \frac{1}{N_k} \langle e, F_k(\sigma_{k,i}, \cdot) \rangle \varphi'_{n,k}(x \cdot \sigma_{k,i}) \sigma_{k,i}\). Let
\[
\|G_{k,x}\|_{2 \rightarrow 2} = \sup_{\|\nu\|_{\mathcal{L}^2(S^n)} = 1} \|G_{k,x}(\nu)\|_{S^n}
\]
then for all $x \in S^n$ we have
\[ \epsilon \leq \left( \sum_{k=0}^{K} |a_k| ||G_{k,x}||_{2 \to 2} \right) ||\eta||. \]

Since $M_k = \max_{t \in [-1,1]} |\phi'_{n,k}(t)|$ we bound
\[ \max_{x} ||G_{k,x}||_{2 \to 2} \leq \frac{M_k}{N_k} \sum_{i=1}^{N_k} \langle \eta, F(\sigma_{k,i}, \cdot) \rangle ||\sigma_{k,i}|| = M_k \sum_{i=1}^{N_k} |e_{k,i}|, \]

obtaining the bound
\[ \epsilon \leq \sum_{k=1}^{K} M_k |a_k| \sum_{i=1}^{N_k} |e_{k,i}|. \]

Using Theorem 2(a) we conclude that denoising is possible provided that
\[ \sum_{k=1}^{K} M_k |a_k| \sum_{i=1}^{N_k} |e_{k,i}| \leq \inf_{t \in [-1,1]} \sum_{k=1}^{K} a_k^2 \phi'_{n,k}(t). \]

Note that the left hand side depends on the activation function $\theta$ and the noise $\eta$ whereas the right hand side depends only on $\theta$. Using the frame properties and Cauchy-Schwarz inequality one can write
\[ \epsilon \leq \sum_{k=1}^{K} M_k |a_k| ||\eta_k|| \]

and prove Theorem 2(b). Note that this implies a sufficient condition for denosing
\[ \sum_{k=1}^{K} M_k |a_k| ||\eta_k|| < \inf_{t \in [-1,1]} \sum_{k=1}^{K} a_k^2 \phi'_{n,k}(t), \]

that does not depend on the frame choice.

\section{Numerical experiments}

\subsection{Denoising a generative model for MNIST}

Figure 1 shows denosing of simple images using a generative model in comparison with classical denoising algorithms. In order to produce the generative model we use a 4-layer convolutional neural network with ELU as the activation function. We train it in the entire MNIST training set using stochastic gradient descent.

\subsection{Gegenbauer approximations of common activation functions}

Figures 3 and 4 consider the most common activation functions. The first column plots the activation function, the second column shows the Gegenbauer approximation truncating to $K = 30$. The third column plots $g_\theta(t)$ for the approximation of $\theta$ from the second column and the fourth column plots $g'_\theta(t)$. The difference between Figures 3 and 4 is the space considered (in Figure 3 $n = 2$ whereas in Figure 4 $n = 10$). According to Theorem 2 the best activation functions for denoising will be the ones where $g'_\theta(t)$ is bounded away from zero, in particular we observe that the performance for all nonlinearities seem to deteriorate by increasing $n$. We also observe that ELU [Clevert et al., 2015] and GELU (Gaussian Error Linear Unit) [Hendrycks and Gimpel, 2016] theoretically have better denoising properties than Softplus [Nair and Hinton, 2010], LeakyReLU [Maas et al., 2013], Swish [Ramachandran et al., 2017] or ReLU. Note that GELU is not monotonous but $g'_\theta(t) > 0$ for all $[-1,1]$.

Table 1 shows lower bounds for $\inf_{t \in [-1,1]} |g'_\theta(t)|$ for popular activation functions. The strategy to produce such lower bounds mainly uses the bound (9) and it is explained in Appendix 8. Note that we do not produce a provable bound for ELU or ReLU because they are not smooth enough.
Figure 2: Denoising performance of the SUNLayer for different activation functions

We consider ReLU, ELU and softplus. The denoising performance of ELU is superior to ReLU and softplus, which is consistent with the theory from Section 4 and the properties shown in Figure 4. Note that softplus satisfies that $g_\theta'(t) > 0$ for all $t \in [-1, 1]$ but the values of $g_\theta(t)$ are close to zero.
The estimation of the signal is not possible unless the number of samples exceeds \( p \) in priors. Such a result would be particularly significant for the multi-reference alignment problem (on some regimes it is reasonable to suspect that the sample complexity of this problem may significantly decrease by the use of generative priors). The theoretical framework we propose potentially applies to other inverse problems for which deep generative priors are used.

### Discussion and open problems

A classification problem (with \( n \) classes) can be thought as a function \( c : D \rightarrow [n] \) where \( D \subset S \). An interesting question is what classification functions \( c \) can be approximated by using functions \( L^{(f)}(x) := L_{n_1} \circ \ldots \circ L_{n_2} \circ L_n(x) \in \mathcal{L}^2(S^n) \) where \( \ell \) is the number of layers. In this framework the classifier would be approximated by a function \( \hat{c}(x) = \arg \max_{y \in [n]} \{(L^{(f)}(x), y_i)\}_{i \in [n]} \) for some \( y_1, \ldots, y_n \in \mathcal{L}^2(S^n) \) and \( y_i \) are the objects we may want to find using local methods. We believe an answer to a problem of this form may involve the study of the geometric or topological properties of \( L^{(f)}(S^n) \subset \mathcal{L}^2(S^n) \).

Finally, an intriguing question that arises from this analysis is what the condition \( g_\theta(t) > 0 \) means for the activation function \( \theta \). For instance, squaring the coefficients of the Fourier decomposition of a function corresponds with convolving the function with itself in the time domain. Is there an interesting interpretation of squaring the coefficients of the Gegenbauer decomposition?

### Table 1

| Activation function | \( n \) | \( T := \inf_{t \in [-1, 1]} |g_\theta(t)| \) | \( c_{\theta,n} \) | \( T/c_{\theta,n} \) |
|---------------------|-------|---------------------------------|----------------|--------------------|
| \( \text{id}(x) = x \) | 2     | 4.189                           | 4.189          | 1.000              |
|                     | 10    | 1.884                           | 1.884          | 1.000              |
| \( \text{softplus}(x) = \log(1 + e^x) \) | 2     | 0.998                           | 7.83           | 0.127              |
|                     | 10    | 0.462                           | 10.759         | 0.043              |
| \( \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \) | 2     | 2.959                           | 2.996          | 0.988              |
|                     | 10    | 1.635                           | 1.639          | 0.997              |
| \( \text{sigmoid}(x) = \frac{1}{1 + e^{-x}} \) | 2     | 0.238                           | 3.380          | 0.071              |
|                     | 10    | 0.113                           | 5.295          | 0.021              |
| \( \text{swish}(x) = \frac{x}{1 + e^{-x}} \) | 2     | 0.864                           | 1.187          | 0.728              |
|                     | 10    | 0.437                           | 0.497          | 0.880              |
| \( \text{gelu}(x) = xe^{-x^2} \) | 2     | 1.208                           | 1.454          | 0.831              |
|                     | 10    | 1.207                           | 1.240          | 0.973              |

We report \( \|G(\hat{x}) - G(x^\star)\| \) and \( \langle \hat{x}, x^\star \rangle \) for different noise levels and activation functions. The solid curve corresponds to the mean over the 10 experiments, whereas the shaded area shows the standard deviation. The activation functions we consider are ReLU, softplus and ELU. The denoising performance of ELU is empirically superior to ReLU and softplus. This observation is consistent with the theory from Section 4 and the properties shown in Figure 4. Note that softplus satisfies that \( g_\theta(t) > 0 \) for all \( t \in [-1, 1] \) but the values of \( g_\theta(t) \) are close to zero.

### 5.3 Denoising in a synthetic framework

In Figure 2 we perform a numerical experiment to illustrate the theory developed in Section 4. We consider a random instance of one layer of the SUNLayer model. Here \( G(x) = \theta(Bx) \) where \( x \in S \) for \( n = 9 \) and \( B \in \mathbb{R}^{100 \times 10} \) is a fixed random Gaussian matrix with normalized rows. We perform 10 independent experiments where we draw random \( x^\star \in S \) and we let \( y = G(x^\star) + \eta \) where \( \eta \) is Gaussian in \( \mathbb{R}^{100} \). For each \( y \) we use stochastic gradient descent to find \( \hat{x} \), a local minimizer of \( \|G(x) - y\| \).

We report \( \|G(\hat{x}) - G(x^\star)\| \) and \( \langle \hat{x}, x^\star \rangle \) for different noise levels and activation functions. The solid curve corresponds to the mean over the 10 experiments, whereas the shaded area shows the standard deviation. The activation functions we consider are ReLU, softplus and ELU. The denoising performance of ELU is empirically superior to ReLU and softplus. This observation is consistent with the theory from Section 4 and the properties shown in Figure 4. Note that softplus satisfies that \( g_\theta(t) > 0 \) for all \( t \in [-1, 1] \) but the values of \( g_\theta(t) \) are close to zero.

The theoretical framework we propose potentially applies to other inverse problems for which deep generative priors may be obtained, like phase retrieval ([Candes et al., 2015]) or multi-reference alignment ([Bandeira et al., 2014]). It is reasonable to suspect that the sample complexity of these problems may significantly decrease by the use of generative priors. Such a result would be particularly significant for the multi-reference alignment problem (on some regimes estimation of the signal is not possible unless the number of samples exceeds \( 1/\text{SNR}^3 \) ([Perry et al., 2017])).

A different direction to explore is whether it is possible to use this framework to study classification problems. A classification problem with \( n \) classes can be thought as a function \( c : D \rightarrow [n] \) where \( D \subset S \). An interesting question is what classification functions \( c \) can be approximated by using functions \( L^{(f)}(x) := L_{n_1} \circ \ldots \circ L_{n_2} \circ L_n(x) \in \mathcal{L}^2(S^n) \) where \( \ell \) is the number of layers. In this framework the classifier would be approximated by a function \( \hat{c}(x) = \arg \max_{y \in [n]} \{(L^{(f)}(x), y_i)\}_{i \in [n]} \) for some \( y_1, \ldots, y_n \in \mathcal{L}^2(S^n) \) and \( y_i \) are the objects we may want to find using local methods. We believe an answer to a problem of this form may involve the study of the geometric or topological properties of \( L^{(f)}(S^n) \subset \mathcal{L}^2(S^n) \).

Finally, an intriguing question that arises from this analysis is what the condition \( g_\theta(t) > 0 \) means for the activation function \( \theta \). For instance, squaring the coefficients of the Fourier decomposition of a function corresponds with convolving the function with itself in the time domain. Is there an interesting interpretation of squaring the coefficients of the Gegenbauer decomposition?
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Proposition 3. If for all $t \in I \subset \mathbb{R}$ we have
(a) $\lim_{n \to \infty} S_n(t) = f(t)$
(b) $\lim_{n \to \infty} S'_n(t) = g(t)$
(c) $|S'_n(t)| < C$ for some $C$ independent of $n$
then $g'(t) = f(t)$ for all $t \in I \subset \mathbb{R}$.

Proof. For $h > 0$ we have
$$|f(x + h) - f(x) - hg(x)| \leq |S_n(x + h) - f(x + h)| + |S_n(x) - f(x)| + h|S'_n - g(x)| + |S_n(x + h) - S_n(x) - hS'_n(x)|$$

Proof of Lemma 2 Let $\omega \in S^n$ and define the function $h : S^n \to \mathbb{R}$ as $h(\tau) = \theta(\omega \cdot \tau)$. Then $h \in L^2(S^n)$ and $h$ is $C^2$. We have $h(\tau) = \sum_{k=0}^{\infty} a_k \varphi_n,k(\omega \cdot \tau)$, in particular $h_k = a_k F_k(\omega, \cdot) \in H_k(S^n)$. Then (9) implies $k^2 \|a_k \varphi_n,k(t)\|_{L^2(\mu_n)} < A_n$ for some constant $A_n$ that depends on $n$ but it does not depend on $k$. Using (7) we obtain
$$a_k^2 < \frac{B_n}{k^{n+3}},$$
which implies
$$\sum_{k=0}^{\infty} a_k^2 \varphi_n,k(t) < \sum_{k=0}^{\infty} \frac{1}{k^{n+3}} \varphi_n,k(t) \leq B_n \sum_{k=0}^{\infty} \frac{k^{n-1}}{k^{n+3}} < \infty$$
which establishes the pointwise convergence of $\sum_{k=0}^{\infty} a_k^2 \varphi_n,k(t)$ to a function $g_0(t)$ for $n \geq 1$. Now we consider the derivatives and we use the identity (8) and we get
$$\sum_{k=0}^{\infty} a_k^2 \varphi_n,k'(t) = C_n \sum_{k=0}^{\infty} a_k^2 \varphi_{n+2,k-1}(t) \leq D_n \sum_{k=0}^{\infty} a_k^2 k^{n+1}$$
where $C_n, D_n$ are constants depending only on $n$. Note that this argument guarantees the pointwise convergence of $\sum_{k=0}^{\infty} a_k^2 \varphi_n,k(t) < \infty$.

Using (8) again we get
$$\sum_{k=0}^{\infty} a_k^2 \varphi_n''(t) = F_n \sum_{k=0}^{\infty} a_k^2 \varphi_{n+2,k-1}(t) = F_n \sum_{k=0}^{\infty} a_k^2 \varphi_{n+4,k-2}(t) \leq G_n \sum_{k=0}^{\infty} a_k^2 k^{n+3}.$$
Now the bound (15) is not good enough to bound the second derivative, but if we have that $\theta$ is $C^4$ we can use (9) with $r = 2$ obtaining a bound that allows us to use Proposition 3 and complete the proof of Lemma 2.
8 Derivation of bounds in Table 1

Let $\theta : [-1, 1] \rightarrow \mathbb{R}$ be a $C^4$ activation function. Let $h : S^n \rightarrow \mathbb{R}$ be a function defined as $h(\tau) = \theta(\tau \cdot \omega)$ for some $\omega \in S^n$ fixed. Then bound 9 says that $k^4\|h_k\| \leq \|\Delta^2 h\|$. Similar computations than the ones in the proof of Lemma 3 show that $\|h_k\|^2 = a^2_k \frac{\alpha_{n,k}}{\text{vol}(S^n)}$ and the norm of the laplacian can be computed as

$$\|\Delta^2 h\|^2 = \text{vol}(S^{n-1}) \int_{-1}^{1} (\Delta^2_{S^n}(\theta(t)))^2 (1 - t^2)^{\frac{n-2}{2}} dt =: \text{vol}(S^{n-1})\Delta_{\theta,n},$$

where $\Delta_{\theta,n}$ is a constant depending only on $\theta$ and $n$ that we can compute for each activation function. We have

$$a^2_n \leq \frac{\Delta_{\theta,n} \text{vol}(S^{n-1})}{\text{vol}(S^n)\alpha_{n,k}k^8}.$$

Then using (8) and (6) we have

$$|a^2_k \varphi'_{n,k}(t)| \leq \frac{\Delta_{\theta,n} \text{vol}(S^{n-1}) (n + 1) \text{vol}(S^n)}{\text{vol}(S^n)\alpha_{n,k}k^8} |\varphi_{n+2,k-1}(t)| = \frac{\Delta_{\theta,n} \text{vol}(S^{n-1})}{\alpha_{n,k}k^8} \frac{(n + 1) \alpha_{n+2,k-1}}{\text{vol}(S^n+2) \text{vol}(S^{n+2})}.$$

Note that $\frac{\alpha_{n+2,k-1}}{\alpha_{n,k}} = \frac{(k+n-1)k}{n+2}$, obtaining

$$|a^2_k \varphi'_{n,k}(t)| < A_{\theta,n} \frac{1}{k^6}.$$

One can uniformly bound the tail $\sum_{k=K+1}^{\infty} |a^2_k \varphi'_{n,k}(t)|$ by observing that $\sum_{k=K+1}^{\infty} \frac{1}{k^6} \leq \int_{K}^{\infty} \frac{1}{t^6} dt = \frac{K^{-5}}{5}.$
Figure 3: Activation functions and their Gegenbauer approximations for $K = 30$ and $n = 2$. 
Figure 4: Activation functions and their Gegenbauer approximations for $K = 30$ and $n = 10$. 