Extended GUP formulation with and without truncation in momentum space

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We analyze the extended GUP theory deriving from the algebra $[\hat{x}, \hat{p}] = i\hbar \sqrt{1 + 2\beta \hat{p}^2}$, which is the most general formulation satisfying the Jacobi identity. By means of functional analysis, first, we show how a natural formulation of the theory in an infinite momentum space does not lead to the emergence of a nonzero minimal uncertainty in position, then we construct a truncated formulation of the theory in momentum space, proving that only in this case we can recover the desired feature of the presence of a nonzero minimal uncertainty in position, which - as usual in these theories - can be interpreted as a phenomenological and effective manifestation of a quantum gravity effect.

Both quantization schemes are completely characterized and finally applied to study wave packets behavior and their evolution in time.

I. INTRODUCTION

Theories describing cut-off physics effects on the ultraviolet behavior of gravity, such as Loop Quantum Gravity \cite{1, 2} and Superstrings \cite{3}, can have phenomenological representations equivalent to modified formulations of non-relativistic quantum theory. On the one hand, the motion of particles in the low energy limit of string theories \cite{4-8} is well-described by the so-called Generalized Uncertainty Principle (GUP) theories, a quantum framework based on modified uncertainty relations between position and momentum operators, arising from a deformation of the Heisenberg algebra \cite{9, 10} (for minisuperspace application see \cite{11-13}. On the other, the proper application of Loop Quantum Gravity in the minisuperspace \cite{15, 16, 17} is perfectly summarized by the quantum formalism known as Polymer Quantum Mechanics (PQM), which is essentially an implementation of the quantum theory on a lattice. Nevertheless, although these theories are in principle built on two different quantization schemes, recent works \cite{18, 19} suggest that the PQM formalism can be interpreted as a GUP theory as well, i.e. it is possible to faithfully represent it as a proper deformed algebra of quantum operators.

In this paper, we concentrate our attention on the first mentioned approach to cut-off physics, in order to discuss, in some detail, its possible generalizations \cite{20-22}. More specifically, we completely re-analyze and precise some of the results discussed in \cite{23}, outlining some critical questions concerning the truncation of the momentum space of the proposed theory. As it is known, one of the main issues and at the same time one of the most interesting features of these theories is the possible existence of a nonzero minimal position uncertainty, which can be naturally interpreted as a minimal length in the theory itself. Coherently with the basic idea of a string configuration, this is exactly the case for the GUP formulation studied in \cite{10}, in which indeed the modified uncertainty principle comes directly from low energy considerations concerning the string theory (see above). In \cite{21-22}, the possibility to generalize and extend the uncertainty principle mentioned above has been inferred by some phenomenological observations, leading to the introduction of a square root term in the Heisenberg algebra, able to reproduce the original approach in \cite{10} in a proper limit.

The relevance of this generalization when ap-
plied to the minisuperspace variables is discussed in [13] and [19], where it is shown how the semi-classical model overlaps the Friedmann equation for an isotropic Universe typical of the brane cosmology [24]. In the same work, it has been also emphasized how a simple change of sign in the square root allows switching to a polymer-like formulation, associated with the Friedmann equation of Loop Quantum Cosmology [25], where the sign is semi-classically translated into the non-Einsteinian correction of the dynamics.

Many other generalizations of the analysis carried out in [10] have been considered over the years (see for example [26] for an overview), but, as discussed in [21], [22] and [23], the generalized uncertainty principle springing from the square root-modified algebra is the only one that preserves the Jacobi identity of the operators and therefore the only one providing a consistent treatment.

One of the conclusions of the analysis developed in [23], is the existence of a minimal length in the form of a nonzero minimal uncertainty in position, as in the original approach in [10]. In deriving this result a pivotal role is played by a series expansion of the square root term itself.

It is exactly on this point that our analysis is focused, aiming to precise and clarify the conditions under which the Taylor expansion is allowed and to determine the proper implications of such a procedure. Indeed, in the case under study, the series expansion is mathematically viable only if the momentum space is restricted to a compact region, needed to ensure the convergence of the series itself. It is therefore clear that the conclusions exposed in [23] are not necessarily valid nor true in a complete, non-truncated formulation of the theory, which thus asks to be studied via different methods.

On this ground, through a carefully functional analysis of the position operator and by means of the techniques first developed and discussed in [27], in particular from consideration on the divergence of the modified Lebesgue measure in the resulting Hilbert space, we preliminary arrive to show that, actually, no nonzero minimal uncertainty in the position operator exists when the momentum space is not truncated. From this result we proceed to quantize this generalized scheme, outlining its intrinsic difference from the analysis carried out in [10]. Then, by adopting a wide general functional method developed again in [27], we rigorously construct the quantum theory associated with a truncated momentum space, characterizing completely the involved operators from a functional point of view, and showing how in this case and only in this case the existence of a nonzero minimal uncertainty in the position operator emerges, although slightly different from that one proposed in [23]. This automatically allows us to construct a collection of maximally localized functions and therefore a quasi-position representation similar to that one first discussed in [10].

In this respect, this means that the only viable generalization of the GUP original formulation in [10] is the square root-modified one, but implemented through an ad hoc truncation of the momentum space, which certainly calls attention to a possible physical justification.

The paper is structured as follows: in Section II we review and re-analyze, by means of some functional analysis tools, the original GUP formulation discussed in [10], known as Kempf, Mangano, Mann (KMM) GUP, underlining some subtle aspects concerning the maximally localized states of the theory; in Section III we summarize in some detail the functional procedure introduced first in [27], which will be fully exploited to construct the physical domain of the theory in the considered Hilbert space and to determine the maximally localized functions and their minimal uncertainty in the position; in Section IV we introduce the extended GUP formulation obtained from the square root-modified Heisenberg algebra and we outline the analysis and the conclusion discussed in [23]; in Sections V and VI we carry out our complete analysis of this extended GUP formulation. In particular, in Section V we construct and study the full theory, that is the theory implemented in non-truncated momentum space, defining in a rigorous way the quantum operators and the physical domain of the the-
ory and showing how in this case the "maximally localized states" of the theory are modified plane waves which can be arbitrarily localized in space, while in Section VI through the same steps, we construct and study the truncated or compact theory, that is the theory implemented in truncated momentum space, proving this time that a nonzero minimal uncertainty in position does exist and determining the associated maximally localized functions. In the subsections VIA and VIB first we make a comparison of our results with the ones exposed in [23], then, following the arguments presented in [10] on the possibility to recover information on the position, we construct the so-called quasi-position representation within our truncated theory. Finally, in Section VII, we analyze the behavior of localized wave packets in both the truncated and non-truncated formulation, comparing the spreading properties with the standard non-relativistic quantum mechanical approach and pointing out the relevant differences. In Sections VIII we give our conclusions and summarize our results.

II. KEMPF, MANGANO, MANN GUP FRAMEWORK

One of the most studied theories descending from the deformation of the canonical commutator of Quantum Mechanics (QM) operators is the one constructed by Kempf, Mangano e Mann in their seminal paper in 1995 [10].

The modified Heisenberg algebra introduced and studied in [10] is the following:

\[ [\hat{x}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2), \]  

where \( \hat{x} \) and \( \hat{p} \) are respectively the position and momentum operator, while \( \beta \) is a positive parameter with the dimension of the inverse of a squared momentum.

The natural choice is to represent the algebra on momentum space, where the action of the operators, by means of the braket formalism, can be written as:

\[ \hat{x} |\psi\rangle \rightarrow i\hbar(1 + \beta p^2) \partial_p |\psi\rangle, \quad p \in \mathbb{R}. \]  

It is straightforward to verify that this representation satisfies the commutation relation. Nevertheless, it has to be clear that this is not the only possible choice.

What is relevant for the analysis is to understand whether there exists a minimum value in the position uncertainty \( \Delta \hat{x} \) different from zero and in that case which physical state realizes it. In order to do this in a rigorous and consistent manner, it is first strictly necessary to understand what conditions are imposed by the algebra on the operators and how these have to be properly defined.

The \( \hat{x} \) and \( \hat{p} \) operators can be densely defined on the Schwartz space \( \mathcal{S} \), i.e. the space of all the smooth functions rapidly decreasing more than any power of \( 1/x \), together with all their derivatives. The Schwartz space is a dense domain of the Hilbert space \( L^2(\mathbb{R}, dp/(1 + \beta p^2)) \), where the modified Lebesgue measure must be introduced in order to make \( \hat{x} \) symmetric on this domain. The \( \hat{p} \) operator would be symmetric in any case, with respect to any measure, acting as a multiplicative operator.

By using the tools from functional analysis it can be shown that while \( \hat{p} \) on \( \mathcal{S} \) is still an essentially self-adjoint operator, as in the ordinary quantum theory, this is not the case for \( \hat{x} \) anymore. Indeed, by appealing to the formal definition of the adjoint of an operator, it is not difficult to show that the adjoint \( \hat{p}^\dagger \) of \( \hat{p} \) is still a multiplicative operator defined on the following domain:

\[ D_{p^\dagger} = \left\{ \psi \in L^2(\mathbb{R}, dp/(1 + \beta p^2)) \mid p\psi \in L^2(\mathbb{R}, dp/(1 + \beta p^2)) \right\}. \]

This clearly shows that \( \hat{p} \) is different from \( \hat{p}^\dagger \), in particular, since \( \hat{p} \) is symmetric, it is true that \( \hat{p} \subsetneq \hat{p}^\dagger \).

At this point it is possible to calculate the deficiency indices \( (d_+, d_-) \) of \( \hat{p}^\dagger \), i.e. the di-
Since these indices are both zero this implies that \(\hat{\mathbf{p}}\) is an essentially self-adjoint operator on \(\mathcal{S}\) and in particular that the unique self-adjoint extension it admits is exactly \(\hat{\mathbf{p}}\), which hence has to be considered the "real" momentum operator of the theory [28].

Its "eigenfunctions", in momentum representation, are still Dirac deltas, that is \(\delta(p - \tilde{p})\), for which the scalar product, due to the presence of the modified measure in the Hilbert space, will be:

\[
\langle p|\tilde{p} \rangle = (1 + \beta p^2)\delta(p - \tilde{p}). \tag{5}
\]

By following the same steps for \(\hat{\mathbf{x}}\), it can be shown that whenever it is defined on \(\mathcal{S}\) it is not self-adjoint and that the adjoint operator \(\hat{\mathbf{x}}\) is defined as:

\[
\hat{\mathbf{x}} = \hat{d}^{-1} : \mathcal{D}_{\hat{\mathbf{x}}^{-1}} \rightarrow \mathcal{L}^2 \left( \mathbb{R}, \frac{dp}{1 + \beta p^2} \right)
\]

\[
\psi \mapsto i\hbar(1 + \beta p^2)\partial_p(w)\psi,
\]

where the symbol \(\partial_p\) stands for the derivative operator in the weak or distributional sense. Nevertheless, this time the deficiency \((d_+, d_-)\) are not \((0, 0)\):

\[
(\hat{\mathbf{x}} \pm \mathbf{1}) |\psi\rangle = 0,
\]

\[
\hbar(1 + \beta p^2)\partial_p(w)\psi(p) \pm \psi(p) = 0, \tag{8}
\]

\[
\partial_p \left( e^{\pm \frac{1}{1 + \beta p^2} \arctan(\sqrt{\beta p})} \psi(p) \right) = 0.
\]

Since the function on which the weak derivative acts is a locally integrable function, it is possible to conclude, by an application of the De Bois - Reymond lemma, that it is equal to a constant \(\kappa\) almost everywhere on \(\mathbb{R}\), hence:

\[
\psi(p) = \kappa e^{\pm \frac{1}{\kappa} \arctan(\sqrt{p})}. \tag{9}
\]

These functions belong to the domain of \(\hat{\mathbf{x}}\) and they span two different subspaces of dimension one, hence \((d_+, d_-) = (1, 1)\). This means that the position operator \(\hat{\mathbf{x}}\) is not essentially self-adjoint but it admits a one-parameter family of self-adjoint extensions.

A detailed and complete analysis of these self-adjoint extensions, constructed via the so-called Cayley transform, can be found in [9] and [10]. For what follows will be enough to have in mind that is possible to select one of these self-adjoint extensions for the position operator and then go ahead in the construction of the physical space of the theory with this choice.

Now, having set this precise functional framework, it is possible to proceed to the search for \(\Delta\mathbf{x}\) and the relative state to which it corresponds. In reviewing this part we will follow a slightly different approach exposed in [27].

The KMM method looks for this wave function which yields this minimum in position uncertainty among those states which are able to saturate the Heisenberg inequality:

\[
\Delta\mathbf{x}\Delta\hat{\mathbf{p}} \geq \frac{1}{2} |\langle \hat{\mathbf{x}}, \hat{\mathbf{p}} \rangle| \tag{10}
\]

These states are often called squeezed states. As it is known, relation (10) can be obtained directly from the algebra, under the minimal assumptions that the operator \(\hat{\mathbf{x}}\) and \(\hat{\mathbf{p}}\) are dense and symmetric operators on their domain of definition. Tracing back all the steps that from (1) lead to (10), it can be shown that the equality sign in (10) can be obtained from those states that are eigenstates of null eigenvalue of the following operator:

\[
\hat{A}_\Lambda |\Psi_\Lambda\rangle := (\hat{\mathbf{x}} - \xi) + i\hbar\Lambda(\hat{\mathbf{p}} - \eta) |\Psi_\Lambda\rangle = 0, \tag{11}
\]

where \(\xi, \eta, \Lambda\) are real parameters.

In p-representation, this is a first-order differential equation which solution is:

\[
\Psi_\Lambda(p) = N e^{\frac{i\Lambda(\eta - \xi)}{\kappa} z(p) - \Lambda u(p)}, \tag{12}
\]
where
\[
z(p) = \int_0^p dq (1 + \beta q^2)^{-1} = \frac{\arctan(\sqrt{\beta} p)}{\sqrt{\beta}}
\]
\[
u(p) = \int_0^p dq q(1 + \beta q^2)^{-1} = \frac{\log(1 + \beta p^2)}{2\beta}.
\]  

Clearly, the obtained wave function must be a square-integrable function, or, in other words, it has to be normalizable. Once \(\Lambda\) is fixed to be strictly positive, this condition is satisfied since:

\[
\lim_{p \to \pm\infty} z(p) = \pm \frac{\pi}{2\sqrt{\beta}}, \quad \lim_{p \to \pm\infty} u(p) = +\infty.
\]  

Furthermore, the fulfillment of the above requirements assures that the parameters \(\eta\) and \(\xi\) coincide with the expectation values of \(\hat{x}\) and \(\hat{p}\), as they should. It is relevant to notice that this last condition is related to the need of working with a symmetric position operator.

Now, keeping following the procedure in [27], it is convenient to calculate the norm of the state \(\hat{A}_l|\Psi_\Lambda\rangle = \langle \hat{A}_l | \Psi_\Lambda \rangle\), where in general \(l \neq \Lambda\):

\[
\langle \Psi_\Lambda | \hat{A}_l | \Psi_\Lambda \rangle = \langle \Psi_\Lambda | \hat{A}_l^\dagger \hat{A}_l | \Psi_\Lambda \rangle = h^2(l - \Lambda)^2 \left( \langle \Psi_\Lambda | (\hat{p} - \eta) | \Psi_\Lambda \rangle \right) = h^2(l - \Lambda)^2 \langle \Psi_\Lambda | (\hat{p} - \eta)^2 | \Psi_\Lambda \rangle,
\]  

where in the last line we have used the fact that \(\langle \Psi_\Lambda | \hat{p}^4 | \Psi_\Lambda \rangle = \langle \Psi_\Lambda | (\hat{p}^2)^2 | \Psi_\Lambda \rangle\).

We have stressed the use of parenthesis to emphasize on which side the operators act, since this subtlety it is crucial in what follows. Indeed in order to obtain from this norm the quantities \(\Delta \hat{x}_{\Psi_\Lambda}\) and \(\Delta \hat{p}_{\Psi_\Lambda}\), we need to calculate another object, namely:

\[
\langle \Psi_\Lambda | \left( \hat{A}_l^\dagger \hat{A}_l | \Psi_\Lambda \rangle \right) = \langle \Psi_\Lambda | \left( \hat{\Delta}^2 \hat{x} | \Psi_\Lambda \rangle \right) = \langle \Psi_\Lambda | \left( \hat{\Delta}^2 \hat{p} | \Psi_\Lambda \rangle \right) = \langle \Psi_\Lambda | \left( \hat{\Delta}^2 | \Psi_\Lambda \rangle \right) = \langle \Psi_\Lambda | \left( \hat{\Delta}^2 | \Psi_\Lambda \rangle \right)
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\]

The computation of this quantity instead relies basically on the fact that \(\langle \Psi_\Lambda | \langle \hat{x}^2 | \Psi_\Lambda \rangle \rangle \) = \(\langle \Psi_\Lambda | \langle \hat{x}^2 | \Psi_\Lambda \rangle \rangle \).

These two last quantities [16] and [17] are equal if and only if:

\[
\lim_{p \to \pm\infty} p|\Psi_\Lambda|^2 = 0,
\]  

which happens to be true in our case, due to the behavior of \(u(p)\). As an explicit calculation shows, this condition comes essentially from the requirement that \(\langle \Psi_\Lambda | \langle \hat{x}^2 | \Psi_\Lambda \rangle \rangle \) = \(\langle \Psi_\Lambda | \langle \hat{x}^2 | \Psi_\Lambda \rangle \rangle \).

Now we are allowed to compare [16] and [17] and to write:

\[
(\Delta \hat{x})^2_{\Psi_\Lambda} = \frac{h^2}{4} \langle \hat{f}(\hat{p}) \rangle^2_{\Psi_\Lambda},
\]  

\[
\Lambda = \frac{\langle \hat{f}(\hat{p}) \rangle_{\Psi_\Lambda}}{2(\Delta \hat{p})^2_{\Psi_\Lambda}},
\]

\[
\Delta \hat{p}_{\Psi_\Lambda} = \frac{h^2 \langle \hat{f}(\hat{p}) \rangle^2_{\Psi_\Lambda}}{4(\Delta \hat{p})^2_{\Psi_\Lambda}} = h^2 \Lambda^2 (\Delta \hat{p})^2_{\Psi_\Lambda},
\]

where the quantities \(\Delta \hat{x}_{\Psi_\Lambda}\) and \(\Delta \hat{p}_{\Psi_\Lambda}\) are defined as:

\[
\Delta \hat{x}_{\Psi_\Lambda} = \sqrt{\frac{\langle \Psi_\Lambda | \hat{x}^2 | \Psi_\Lambda \rangle}{\langle \Psi_\Lambda | \hat{p}^2 | \Psi_\Lambda \rangle} - \xi^2},
\]

\[
\Delta \hat{p}_{\Psi_\Lambda} = \sqrt{\frac{\langle \Psi_\Lambda | \hat{p}^2 | \Psi_\Lambda \rangle}{\langle \Psi_\Lambda | \hat{x}^2 | \Psi_\Lambda \rangle} - \eta^2}.
\]  

The equation [21] represents an explicit functional expression for \(\Delta \hat{x}_{\Psi_\Lambda}\):

\[
(\Delta \hat{x})^2_{\Psi_\Lambda} = \frac{h^2}{4} \int_{-\infty}^{+\infty} \frac{\exp(-2\Lambda(u(p) - \eta(p)))}{f(p)} dp,
\]

\[
(\Delta \hat{p})^2_{\Psi_\Lambda} = \frac{h^2}{4} \int_{-\infty}^{+\infty} \frac{\exp(-2\Lambda(u(p) - \eta(p)))}{f(p)} dp.
\]  

\[
(\Delta \hat{x})^2_{\Psi_\Lambda} = \frac{h^2}{4} \int_{-\infty}^{+\infty} \frac{\exp(-2\Lambda(u(p) - \eta(p)))}{f(p)} dp,
\]

\[
(\Delta \hat{p})^2_{\Psi_\Lambda} = \frac{h^2}{4} \int_{-\infty}^{+\infty} \frac{\exp(-2\Lambda(u(p) - \eta(p)))}{f(p)} dp.
\]
By minimizing this object with respect to \( \eta \) and \( \Lambda \), it is in principle possible to determine the value of \( \Delta \hat{x}^\text{min} \) and hence the wave function \( \psi \) which realizes this uncertainty in position.

Nevertheless, the obtained expression has to be considered just as a formal solution, since carrying out explicitly this kind of computation is not manageable.

Fortunately, in the KMM case, there is no actual need to turn to such an expression as \( [23] \). Indeed, it is possible to express \( \Lambda \) as a function of \( \Delta \hat{p}_{\Psi_A} \) and \( \eta \), by directly computing the expectation value of the commutator, as the expression \( [20] \) suggests:

\[
\Lambda = \frac{1 + \beta \langle \hat{p} \rangle_{\Psi_A}}{2\Delta \hat{p}_{\Psi_A}^2} = \frac{1 + \beta \Delta \hat{p}_{\Psi_A}^2 + \eta^2}{2\Delta \hat{p}_{\Psi_A}^2}. \tag{24}
\]

Hence by substituting in \( [21] \):

\[
\Delta \hat{x} = \frac{1 + \beta \Delta \hat{p}^2 + \eta^2}{2\Delta \hat{p}}, \tag{25}
\]

where we have dropped the subscript \( \Psi_A \) for a cleaner notation.

The obtained function \( \Delta \hat{x}(\Delta \hat{p}) \), for \( \Delta \hat{p} = \sqrt{(1 + \eta^2)/\beta} \), reaches its minimum, which is equal to \( \Delta \hat{x}^\text{min} = \hbar \sqrt{1 + \eta^2} \). The absolute minimum value is then obtained for \( \eta = 0 \), from which follows immediately that \( \Lambda = 1 \) and that the maximally localized functions, that is the functions with the smallest possible \( \Delta \hat{x} \), can be written as:

\[
\Psi^{ml}(p) = \mathcal{N}(1 + \beta p^2)^{-\frac{1}{2}} e^{-\frac{\sqrt{\eta} \arctan(\sqrt{\eta} p)}{\sqrt{\beta}}}, \tag{26}
\]

which are exactly the states found in \( [10] \).

Nevertheless, in order to be fully consistent, this procedure has to return a function belonging to the physical space of our quantum theory. As physical space we can consider that subspace of the Hilbert space resulting from the intersection between the domains of the operators \( \hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2 \) and \( [\hat{x}, \hat{p}] \), in order to make it possible to define the uncertainty in position and momentum and so that the Heisenberg’s uncertainty principle (HUP) holds. In the end this means that the wave function \( [26] \) has to belong to the space:

\[
\mathcal{D}_{\Psi^{ml}} = \mathcal{D}_{\hat{x}} \cap \mathcal{D}_{\hat{p}^2} \cap \mathcal{D}_{[\hat{x}, \hat{p}]} \cap \mathcal{D}_{\hat{p}}. \tag{27}
\]

By constructing properly these domains by means of the \( \hat{x} \) and \( \hat{p} \) domains, it can be shown that while \( \Psi^{ml}(p) \) belongs to \( \mathcal{D}_{\hat{x}} \) and \( \mathcal{D}_{\hat{p}} \), it does not belong to \( \mathcal{D}_{\hat{p}^2} \) and \( \mathcal{D}_{[\hat{x}, \hat{p}]} \); hence it cannot belong to the whole intersection \(^1\). These considerations lead us to conclude that \( \Psi^{ml} \) is not a proper physical state.

Yet, for this state it is still possible to define an uncertainty in position and momentum such that the generalized HUP holds. Indeed, in the ordinary quantum theory, by use of the Weyl algebra formalism, it is possible to prove that Heisenberg’s inequality is still valid by taking weaker assumptions on the set to which a generic state \( \Psi \) has to belong, in particular it suffices that the state belongs just to the domain of position and momentum operator (\( [28] \), section 11.5.6). The general validity of this result can be shown to hold also in our case. Let consider the operator \( \hat{x}' = \hat{x} + a \mathbb{I} \) and \( \hat{p}' = \hat{p} + b \mathbb{I} \), where \( a \) and \( b \) are real numbers and \( \mathcal{D}_{\hat{x}'} = \mathcal{D}_{\hat{x}} \) and \( \mathcal{D}_{\hat{p}'} = \mathcal{D}_{\hat{p}} \).

By explicit computations in \( p \)-representation it is easy to show that:

\[
\langle \hat{x}' \psi | \hat{p}' \psi \rangle - \langle \hat{p}' \psi | \hat{x}' \psi \rangle = i \hbar \int_{\mathbb{R}} dp |\psi|^2 \quad \forall \psi \in \mathcal{D}_{\hat{x}} \cap \mathcal{D}_{\hat{p}}. \tag{28}
\]

First, we notice that this expression is formally equivalent to the expectation value of the commutator between position and momentum operators for those states for which it can be defined, that is:

\[
\langle \hat{x} \psi | \hat{p} \psi \rangle - \langle \hat{p} \psi | \hat{x} \psi \rangle = i \hbar \int_{\mathbb{R}} dp |\psi|^2 \quad \forall \psi \in \mathcal{D}_{[\hat{x}, \hat{p}]} \tag{29}
\]

but only under the condition:

\[
\lim_{p \to \pm \infty} p \psi(p) \frac{d}{dp} \psi(p) = 0. \tag{30}
\]

Then, by choosing \( a = \xi \) and \( b = \eta \), we can

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1 By \( \hat{x} \) operator we are referring to one of the infinite self-adjoint extensions of the operator itself, whose domain is an extension of \( \mathcal{D}_{\hat{x}} = \mathcal{S} \) and a restriction of \( \mathcal{D}_{\hat{x}^1} \).
write:

\[
\|\hat{\mathbf{x}}\psi\|^2 = \langle \hat{\mathbf{x}}\psi | \hat{\mathbf{x}}\psi \rangle = \int_{\mathbb{R}} \frac{dp}{1 + \beta p^2} (x - \xi)^2 |\psi|^2
\]

\[
\forall \psi \in D_\mathbf{x},
\]

(31)

\[
\|\hat{\mathbf{p}}\psi\|^2 = \langle \hat{\mathbf{p}}\psi | \hat{\mathbf{p}}\psi \rangle = \int_{\mathbb{R}} \frac{dp}{1 + \beta p^2} (p - \eta)^2 |\psi|^2
\]

\[
\forall \psi \in D_\mathbf{p},
\]

(32)

which are the formal expressions of the square standard deviation of the \(\hat{x}\) and \(\hat{p}\) in our framework. At this point, from (28) and (31), we can infer:

\[
\|\hat{\mathbf{x}}\psi\||\hat{\mathbf{p}}\psi|| \geq |\langle \hat{\mathbf{x}}\psi | \hat{\mathbf{p}}\psi \rangle| = \frac{\hbar}{2} \int_{\mathbb{R}} dp |\psi|^2.
\]

(33)

and, in the end, since \(\|\hat{\mathbf{x}}\psi\| = \Delta \hat{\mathbf{x}}\psi\) and \(\|\hat{\mathbf{p}}\psi\| = \Delta \hat{\mathbf{p}}\psi\):

\[
\Delta \hat{\mathbf{x}}\psi\Delta \hat{\mathbf{p}}\psi \geq \frac{\hbar}{2} \int_{\mathbb{R}} dp |\psi|^2 = \frac{\hbar}{2} |\langle 1 + \beta \hat{p}^2 \rangle\psi|.
\]

(34)

This proves exactly that the HUP is valid for all those states belonging to \(D_\mathbf{x} \cap D_\mathbf{p}\).

On behalf of this result, we can conclude that for the maximally localized states (26) it is effectively possible to define an uncertainty in position and momentum and a modified HUP which relates these two quantities, hence validating the previous procedure at a "kinematic" level; nevertheless from a "dynamic" perspective, due to the domain to which they belong, these states cannot be considered fully legitimate physical states.

III. DETOURNAY, GABRIEL, SPINDEL

FUNCTIONAL PROCEDURE

The KMM method, even if successful in the previously studied case, has some shortcomings that prevent it to represent the most general approach to the problem of finding the minimum value of \(\Delta \hat{x}\) in a GUP-modified quantum theory. Indeed, first of all, it is valid only for some class of functions \(f(p)\) associated with the operator \(f(\hat{p})\) of the deformed algebra; secondly, its more evident limit is to look for maximally localized states only among the squeezed states. For a general algebra such as \([\hat{x}, \hat{p}] = i\hbar f(\hat{p})\) it is not obvious at all the exhaustiveness of such a research. The most intuitive way to understand it is to notice that the expectation value of the commutator, in the general case, is state-dependent. Thus it is clear that it is necessary to look for these maximally localized states in a more wide domain of our Hilbert space. For this purpose, we will now review and revisit a proposal for a more general approach, which can be found again in [27] (DGS method). The Hilbert space of our theory is once again the space of all square-integrable functions with respect to the measure \(dp/f(p)\). On this space, in position-representation, the operators \(\hat{x}\) and \(\hat{p}\) can be written as:

\[
\hat{x} |\psi\rangle \rightarrow i\hbar f(p) \partial_p \psi(p) \quad (35)
\]

\[
\hat{p} |\psi\rangle \rightarrow p \psi(p), \quad p \in \mathbb{R} \quad (36)
\]

and they do satisfy the generic algebra. We can consider as the physical subspace on which looking for the minimum value of \(\Delta \hat{x}\) the domain defined in [27].

Therefore we need to properly define all the involved operators. Now, according to [27], two cases need to be distinguished: the compact and the non-compact case.

A. The compact case

Let us consider a function \(f(|p|)\) such that, for \(p \gg 1\), \(f(|p|) \approx |p|^{1+\nu}\), with \(\nu > 0\). In this case for the quantity:

\[
z(p) = \int_0^p f(q)^{-1} dq \quad (37)
\]

it is true that:

\[
z(+\infty) = \alpha_+ \quad z(-\infty) = \alpha_- \quad \alpha_{\pm} \in \mathbb{R}.
\]

(38)
It is then possible to construct a diffeomorphism between \( \mathbb{R} \) and the compact real interval \([\alpha_-, \alpha_+]\) through the map \( p \to z(p) \), moving on from the space \( L^2(\mathbb{R}, dp/f(p)) \) to the space \( L^2([\alpha_-, \alpha_+], dz) \).

On our new Hilbert space the \( \hat{x} \) operator is a symmetric multiplicative operator defined on the whole Hilbert space, hence, it is automatically self-adjoint and, due to the Hellinger-Toeplitz theorem, it is bounded.

The same can be stated for \( \hat{z}^2 \).

With regards to the \( \hat{x} \) operator, the most natural choice is to define it as follows:

\[
\hat{x} : D_\hat{x} \to L^2([\alpha_-, \alpha_+], dz) \quad (39)
\]

\[
\psi \mapsto i\hbar \partial_z \psi,
\]

where \( D_\hat{x} = \{ \psi(z) \in H^{1,2}([\alpha_-, \alpha_+], dz) \mid \psi(\alpha_-) = \psi(\alpha_+) = 0 \} \). (41)

This operator is not self-adjoint as it can be seen through a direct construction of its adjoint, which turns out to be a true extension, that is \( \hat{x} \subseteq \hat{x}^\dagger \).

In particular:

\[
\hat{x}^\dagger : D_{\hat{x}^\dagger} \to L^2([\alpha_-, \alpha_+], dz) \quad (42)
\]

\[
\psi \mapsto i\hbar \partial_z^2 \psi,
\]

where:

\[
D_{\hat{x}^\dagger} = \{ \psi(z) \in H^{1,2}([\alpha_-, \alpha_+], dz) \} \quad (44)
\]

Being the adjoint operator well-defined, it is now straightforward to obtain the deficiency indices of \( \hat{x} \). By following the same procedure of the previous section, it turns out that \( (d_+, d_-) = (1, 1) \), showing therefore that \( \hat{x} \) is not essentially self-adjoint, but rather it admits a one-parameter family of self-adjoint extensions. We notice that this kind of result is common whenever the theory is implemented in a compact space. These self-adjoint extensions \( \hat{x}_\lambda \) has the same action of \( \hat{x} \) but are defined on the domain:

\[
D_{\hat{x}_\lambda} = \{ \psi(z) \in H^{1,2}([\alpha_-, \alpha_+], dz) \mid \psi(\alpha_+) = e^{-i\lambda} \psi(\alpha_-) \} \quad (45)
\]

Not surprisingly, neither the squared position operator \( \hat{x}^2 \) is essentially self-adjoint, but admits a one-parameter family of self-adjoint extensions given by \( \hat{x}_\lambda^2 \). Nevertheless, as it is extensively discussed in [27], the more convenient choice for the construction of the squared position operator is the operator \( \hat{x}_\lambda \hat{x} \), which domain is exactly \( D_{\hat{x}} \).

Clearly, since \( \hat{x}^2 \subseteq \hat{x} \), \( \hat{x}_\lambda^2 \) is an extension of \( \hat{x}^2 \) and it results to be self-adjoint.

Our final list of operators will be then represented by \( \hat{x}_\lambda, \hat{x}_\lambda^\dagger, \hat{x}, \hat{x}^2 \).

We can now define the domain of the commutator \( [\hat{x}_\lambda, \hat{z}] \), which turns to be:

\[
D_{[\hat{x}_\lambda, \hat{z}]} = D_{\hat{x}_\lambda, \hat{z}} \cap D_{\hat{x}_\lambda \hat{z}} = D_{\hat{x}_\lambda \hat{z}} \cap D_{\hat{x}_\lambda} = D_{\hat{x}}. \quad (46)
\]

Finally the physical space on which to look for the maximally localized states will be:

\[
D_{\hat{z}_\lambda} \cap D_{\hat{x}_\lambda \hat{z}} \cap D_{\hat{z}_\lambda} \cap D_{\hat{x}_\lambda} \cap D_{[\hat{x}_\lambda, \hat{z}]} = D_{\hat{z}}. \quad (47)
\]

At this point it is possible to write down \( \Delta \hat{x} \) as a functional object and define the maximally localized states as those states which minimize this expression in \( D_{\hat{x}} \):

\[
(\Delta \hat{x})^2 = \frac{\langle \Psi | (\hat{x}^\dagger \hat{x} - \xi) | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad | \Psi \rangle \in D_{\hat{x}}, \quad (48)
\]

\[
(\Delta \hat{x}^{\min})^2 = \min \frac{\langle \Psi | (\hat{x}^\dagger \hat{x} - \xi) | \Psi \rangle}{\langle \Psi | \Psi \rangle} := \mu^2. \quad (49)
\]

Two constraints must be imposed for a consistent procedure:

\[
\xi = \frac{\langle \Psi | \hat{x}_\lambda | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad \xi \in \mathbb{R}, \quad (50)
\]

\[
\gamma = \frac{\langle \Psi | \hat{p} | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad \gamma \in \mathbb{R}. \quad (51)
\]
The first one concerns the existence of the expectation value of $\hat{x}_{\lambda}$, while the second one, which plays a crucial role in the whole method, is an extra condition to select only those states which admit a finite expectation value of some function of the momentum operator, e.g. the energy, from a physical point of view. The constrained variational principle gives back, in $p$-representation, the Euler-Lagrange equations for the system:

$$
\left[ -\left(f(p)\partial_p^{(w)}\right)^2 - \xi^2 + 2a\left(if(p)\partial_p^{(w)} - \xi\right) + 2b(v(p) - \gamma) - \mu^2 \right] \Psi(p) = 0.
$$

(52)

Since $v(p)$ is an arbitrary function, it is impossible to write down a general solution of (52), expect for the case $b=0$. For this particular choice of the Lagrange multiplier, taken into account the first constraint and the boundary conditions of the domain, the solution eventually will be:

$$
\Psi(p) = C\exp[-i\xi z(p)] \sin\{\mu [z(p) - \alpha_-]\},
$$

(53)

$$
|C| = \sqrt{\frac{2}{\alpha_+ - \alpha_-}}, \quad \mu = \frac{n\pi}{\alpha_+ - \alpha_-}, \ n \in \mathbb{N}_0.
$$

(54)

This function correctly belongs to $\mathcal{D}_{\hat{x}}$ and it has a non-vanishing uncertainty in position, obtained for $n = 1$, which corresponds to the minimum of the functional [18], for $b = 0$. Of course, the value $b = 0$ is a special case. Yet, it can be shown (see [27]) that $(\Delta x_{\text{min}}^n)^2 \mid_{\nu = 0}$ is a local minimum with respect to $\gamma$, hence signaling the importance of such a solution. On the other hand, by finding a solution for the case $b = 0$, the method is assuming a priori the existence of a finite expectation value of the function $v(p)$ on the solution of (52). This means that this condition must be checked a posteriori. As explained in [27], if this is not the case, fixed a particular $v(p)$, the eq. (52) must be resolved for arbitrary values of $b$, check what solutions are compatible with the $\gamma$-constraint and then find the minimum of $\Delta \hat{x}$ with respect to $b$.

**B. The non-compact case**

If the quantity $z(p)$ diverges for $p \to \infty$, the previous procedure, which is essentially based on the mapping between $\mathbb{R}$ and a real compact interval, is not available. In general this happens when $f(p) = |p|^{1+\nu}$, for $|p| \to \infty$, with $\nu < 0$. Indeed in this case the squeezed states are physical states, in the sense fixed by the constrained variational principle of the previous section, but their uncertainty in position can be made arbitrarily small, hence there is no non-vanishing minimum in the quantity $\Delta \hat{x}$ (see [27] for a complete discussion).

**IV. EXTENDED GUP FORMULATION**

The literature is plenty of GUP-modified frameworks built as an extension or a generalization of the KMM one. Indeed, it can be noticed at first glance that the KMM-modified commutator can be regarded as a perturbative expansion in $\beta$ (at the first order) of some more general operator-valued function.

One of the most interesting cases is doubtless represented by the GUP theory studied in [22], [20] and [23], which deformed algebra can be written as:

$$
[\hat{x}, \hat{p}] = i\hbar \sqrt{1 + 2\beta \hat{p}^2}.
$$

(55)

This specific modification of the canonical commutation relation (CCR) has a high degree of generality. The reason for that lies in the fact that, in a 3-dimensional setting, the most general modified algebra that can be written, asking that the groups of translations and the group of rotations be undeformed, is:

$$
[\hat{x}_i, \hat{x}_j] = \frac{\hbar}{\kappa e^2} a(\hat{p})_{ij} \hat{j}_k,
$$

(56)

$$
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} f(\hat{p}).
$$

(57)
where $a(p)$ and $f(p)$ are sufficiently regular functions of the momentum operator, $\kappa$ is a parameter with the dimension of a mass and $c$ is the speed of light in vacuum. By imposing, as it should be, that the constructed algebra satisfies the Jacobi identities, a differential equations system is obtained for the form of $a(p)$ and $f(p)$, which solution states that $a(p) = \pm 1$, previous a rescaling of $\kappa$, and consequently $f(p) = \sqrt{\alpha \pm p^2/\kappa c^2}$, where $\alpha$ is an integration constant. By choosing $\alpha = 1$ in order to recover in the proper limit the standard HUP and rewriting $1/(\kappa^2 c^2) = 2\beta_0/\mathcal{M}_{pl}^2 c^2 = 2\beta$ to make contact with the KMM notation - where $\beta_0$ is a dimensionless constant and $\mathcal{M}_{pl}$ is the Planck mass - in the end we can write $f(p) = \sqrt{1 \pm 2\beta p^2}$. Therefore, the modified algebra (55), is one of the two general solutions obtained by imposing that the fundamental requirement of Jacobi identities is fulfilled by the general commutators (56).

An analysis carried out in [23], shows how this formulation of the quantum theory leads to the existence of a minimal length in the theory, namely a nonzero minimal uncertainty in position. By following the reasoning path of the authors and restricting for simplicity to one-dimensional case, from (55) the uncertainty principle can be derived:

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + 2\beta \hat{p}^2). \quad (58)$$

In order to evaluate explicitly the expectation value of the commutator, a series expansion is performed:

$$1 + 2\beta \hat{p}^2 = \sum_n c_n (2\beta \hat{p}^2)^n, \quad (59)$$

where $c_n$ is the generalized binomial coefficient:

$$c_n = \binom{1/2}{n} = \frac{(-1)^2 (2n)!}{2^{2n} (1-2n)(n)!^2}. \quad (60)$$

Then, after a brief chain of inequalities:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \sqrt{1 + 2\beta (\Delta \hat{p})^2}. \quad (61)$$

From this it is immediate to see that there exists indeed a nonzero minimum value for $\Delta x$, which is reached asymptotically, that is for $\Delta \hat{p}$ approaching infinity. In particular:

$$\Delta x_{\text{min}} = l_P \sqrt{\frac{\beta_0}{2}}, \quad (62)$$

where $l_P$ is the Planck length. So according to [23], this framework provides the theory with a "natural" minimal length and it is capable of doing so in a way that resembles much closer to the ordinary quantum theory with respect to the KMM GUP, due to the asymptotic behavior discussed above.

Nevertheless this procedure suffers from a subtle problem. The operator-valued function $1 + 2\beta \hat{p}^2$ admits a series expansion which is convergent if and only if the $\hat{p}$ operator has finite norm, in particular if and only if $\|\hat{p}\| \leq 1/\sqrt{2\beta}$. This means that the procedure discussed in [23] cannot be valid for the "usual" momentum operator, which is an unbounded operator, but it holds only for a theory where the momentum operator is properly bounded. If this is the case, it is clear that also $\Delta \hat{p}$ will be a bounded quantity, therefore it does not make any sense to explore indefinitely the $\Delta \hat{p}$-region, since it will be accessible only up to a certain finite positive value. From these considerations, it is clear that other paths are needed to properly explore this GUP theory and thus address these problems which make the conclusion unreliable.

Two frameworks will be developed and studied: the full or non-truncated theory and the compact or truncated theory. The motivations of such a structure will be naturally clarified in the analysis itself.

V. FULL GUP THEORY

As a first thing, we shall follow a functional analysis analog to the one we carried out in revisiting the KMM theory.

Once again the natural choice is to represent the algebra (55) on momentum space. The position and momentum operators that we are going to construct, first of all, must be densely defined
and symmetric operators. For these reasons we define them as:

\[
\hat{p} : D_\hat{p} \rightarrow L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right)
\]

\[
\psi \mapsto p\psi,
\]

(63)

\[
\hat{x} : D_\hat{x} \rightarrow L^2 \left( \mathbb{R}, \frac{d\beta p}{\sqrt{1 + 2\beta p^2}} \right)
\]

\[
\psi \mapsto i\hbar \sqrt{1 + 2\beta p^2} \partial_p \psi,
\]

(64)

where \( D_\hat{p} \equiv D_\hat{x} \equiv S \) are dense subset of our Hilbert space \( H \equiv L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right) \), with a deformed Lebesgue measure, introduced as usual for the symmetry of the position operator. Also for this case it is easy to prove that the chosen representation satisfies the commutator (55), even if - we again stress it - it is not the only possible one. As a first thing, we notice, simply by following the same procedure of the second section, that the \( \hat{p} \) operator on \( S \) is essentially self-adjoint and that the unique self-adjoint extension is given by the adjoint of \( \hat{p} \), which is a multiplicative operator defined on the following domain:

\[
D_\hat{p}^\dagger = \left\{ \psi \in L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right) \left| p\psi \in L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right) \right. \right\}.
\]

(65)

(66)

Therefore from now on \( \hat{p}^\dagger := \hat{p} \) will be our "true" momentum operator. Also in this framework the momentum "eigenfunctions", in momentum representation, are Dirac deltas and their scalar product will be defined as:

\[
\langle p|\bar{p} \rangle = \sqrt{1 + 2\beta p^2} \delta(p - \bar{p}).
\]

(67)

For the position operator \( \hat{x} \), as expected, the analysis is more subtle. As the explicit construction of its adjoint shows, it is not self-adjoint on \( S \):

\[
D_\hat{x}^\dagger = \left\{ \psi \in L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right) \left| \exists \sqrt{1 + 2\beta p^2} \partial_p \psi \in L^2 \left( \mathbb{R}, \frac{dp}{\sqrt{1 + 2\beta p^2}} \right) \right. \right\},
\]

(68)

thus \( \hat{x} \subset \hat{x}^\dagger \).

In order to understand if this position operator \( \hat{x} \) is essentially self-adjoint once again we need to calculate the dimension of the kernel of the two operators \( \hat{x}^\dagger \pm i\mathbf{I} \), which results in finding the solution of the following differential equations:

\[
\hbar \sqrt{1 + 2\beta p^2} \partial_p \psi(p) = \mp \psi(p).
\]

(69)

(70)

By the same distributional analysis considerations that we have mentioned for the KMM case, we can write down the solutions:

\[
\psi(p) = \kappa e^{\pm \sinh^{-1}(\sqrt{\beta}p)}.
\]

(71)

Nevertheless these functions are not square-integrable functions, unless \( \kappa = 0 \), this means that \( \text{Ker}(\hat{x}^\dagger \pm i\mathbf{I}) = 0 \) and that \( (d_+, d_-) = (0, 0) \). Therefore we can conclude that, differently to the KMM case, in this framework the position operator (on \( S \)) is essentially self-adjoint and its unique self-adjoint extension is exactly \( \hat{x}^\dagger \), defined above. This difference between the two formulations is non-trivial and extremely relevant. Indeed, as pointed out by Kempf et al. in \cite{9}, \cite{10}, giving up the self-adjointness of the position operator is the mathematical feature that allows the theory to host a nonzero minimal uncertainty in position, thus a "natural" minimal length. To prove that indeed this is the case, we now turn to the instrument provided by the DGS procedure. Our function \( f(p) = \sqrt{1 + 2\beta p^2} \) goes as \( f(p) \approx |p| \) for \( |p| \gg 1 \), hence the exponent \( \nu \) is equal to zero. As pointed out in \cite{27}, in this case it is not possible to say anything about the integral function \( z(p) \) a priori, but everything will depend on the precise behavior of \( f(p) \).
In the case under study, the quantity \( z(p) \) is:

\[
\begin{align*}
  z(p) &= \int_0^p dq \frac{1}{\sqrt{1 + 2\beta q^2}} = \frac{\sinh^{-1}(\sqrt{2\beta}q)}{\sqrt{2\beta}} \\
  \text{and it is divergent for } |p| \to \infty.
\end{align*}
\]

According to the DGS scheme, we are in the non-compact case, therefore the whole procedure discussed above in the compact case, which eventually leads to finding a nonzero minimal uncertainty in position, is not available.

This is still not enough to conclude that in this framework the minimal value of \( \Delta \hat{x} \) is zero. It is indeed necessary to prove that also in this specific case with \( \nu = 0 \), the squeezed states are physical states, which uncertainty in position can be made arbitrarily small. As a first thing we need the general form of the squeezed states of the theory, which turns out to be:

\[
\Phi_\Lambda(p) = Ne^{(h\lambda \eta - \xi)} \frac{\sinh^{-1}(\sqrt{2\beta}p)}{h\sqrt{\beta}} - \Lambda \frac{\sqrt{1 + 2\beta p^2 - 1}}{2\beta},
\]

where we recall that \( \eta \in \mathbb{R} \) and \( \Lambda \in \mathbb{R}^+ \). Now we should evaluate explicitly the quantity \( \Delta \hat{x} \). Unfortunately an analytical resolution seems not viable, thus we need to estimate numerically the value of this integral as function of the couple \( (\Lambda, \eta) \). For a more accurate calculation we will use the following general formula to express an integral over the whole real axis as an integral over a finite interval:

\[
\int_{-\infty}^{\infty} h(x) dx = \int_0^1 \left[ h \left( \frac{1}{t} - 1 \right) + h \left( -\frac{1}{t} + 1 \right) \right] t^{-2} dt.
\]

The numerical integration gives back the 2D-surface shown in Fig. 1, which represents the changing of the value of \( \Delta \hat{x}_{\Phi_\Lambda} \) with respect to \( \Lambda \) and \( \eta \), in an exemplifying region delimited by some chosen value of the two independent variables.

We are now interested in finding - if they exist at all - the values of \( \Lambda \) and \( \eta \) which minimizes \( \Delta \hat{x}_{\Phi_\Lambda} \). By visually inspecting the plot in Fig. 1, it appears clear that, with respect to \( \eta \), the surface reaches its minimum for \( \eta = 0 \), hence the minimum with respect to \( \Lambda \) - if it exists - will lie along this specific \( \eta \)-curve.

Setting \( \eta = 0 \), we are now able to represent the changing of the value of \( \Delta \hat{x}_{\Phi_\Lambda} \) only with respect to \( \Lambda \), as showed in Fig. 2.

The graphic shows distinctly what we have anticipated: by making the value of \( \Lambda \) arbitrarily small, it is possible to obtain states with an arbitrarily small uncertainty in posi-
expression (73): normalization constant of a plane wave and where we have set η = 0, hence obtained there no exist a nonzero minimal uncertainty. Therefore we can conclude that in this theory there no exist a nonzero minimal uncertainty in position. (76), is:

\[ \psi(p) = \langle p | \psi \rangle = \frac{1}{\sqrt{2\pi \hbar}} \int dx e^{\frac{i x \sinh^{-1}(\sqrt{\bar{\beta} p})}{\sqrt{2\beta}}} \psi(x). \tag{77} \]

For a free particle of fixed momentum \( \bar{p} \) we have that \( \langle \bar{p} | \psi \rangle = \sqrt{1 + 2\bar{\beta} p^2 \delta(\bar{p} - \bar{p})} \) and coherently its generalized Fourier transform, through (76), is:

\[ \psi^\dagger(x) = \frac{1}{\sqrt{2\pi \hbar}} e^{\frac{i x \sinh^{-1}(\sqrt{\bar{\beta} p})}{\sqrt{2\beta}}} \tag{78} \]

that is the same expression (75) (except for a sign), where now \( p = \bar{p} \) is fixed and \( x \) is the independent variable.

Finally we make two observations:

- even if we have chosen the η-curve of the surface obtained for η = 0, which can be considered the most natural choice, we would have come to the same conclusion for any other value of η. Indeed, as the Fig. 1 shows, every η-curve goes asymptotically to zero, for Λ → 0.

- once we have set η = 0, it is not difficult to verify through numerical integration that the resulting squeezed states are real physical states, in the sense that they belong to the domain \( D_\xi \cap D_\bar{\xi} \cap D_\bar{p} \cap D_{\bar{p}^2} \cap D_{[\bar{\xi}, \bar{p}]}. \)

VI. TRUNCATED OR COMPACT GUP THEORY

What we have just learned is that the full theory based on the algebra (55) does not seem to lead to the existence of a nonzero minimal uncertainty, asymptotically going to zero for Λ → 0. Therefore we can conclude that in this theory there no exist a nonzero minimal uncertainty in position. The corresponding asymptotically "maximally localized" states are hence obtained for η = 0, by making the limit for Λ → 0 of the expression (73).

\[ \Phi_\Lambda(p) = \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{i x \sinh^{-1}(\sqrt{\bar{\beta} p})}{\sqrt{2\beta}}} , \tag{75} \]

where we have set \( \mathcal{N} = 1/\sqrt{2\pi \hbar} \) as usual for the normalization constant of a plane wave and \( \xi = x \) because, being these states perfectly localized in position, the expectation value of \( \hat{x} \) coincides with the exact position \( x \) of the particle itself.

These states are modified plane waves and coherently they are the eigenfunctions of the position operator of the theory in momentum representation. Of course they are not physical states, but they can play exactly the same role as the plane waves in the ordinary quantum theory and they can be approximated with arbitrary precision by sequences of physical states of increasing localization. This means that within this framework the position representation is available and it has the usual physical interpretation. It is immediate to write down the map from momentum space to position space:

\[ \psi(x) = \langle x | \psi \rangle = \frac{1}{\sqrt{2\pi \hbar}} \int dp e^{\frac{i x \sinh^{-1}(\sqrt{\bar{\beta} p})}{\sqrt{2\beta}}} \psi(p) , \tag{76} \]

FIG. 2. Plot of the functional Δx as a function only of Λ, being η = 0, in the same units of the previous plot. Here it can be appreciated how the minimum of the functional along the η-curve for η = 0, is reached for Λ = 0. Since physically we have that Λ ≠ 0, this means that the value of Δx can be made arbitrarily small for Λ → 0 and hence that the theory does not contain a natural minimum length as minimal nonzero uncertainty in position.

\[ \Phi_\Lambda(x) = \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{i x \sinh^{-1}(\sqrt{\bar{\beta} p})}{\sqrt{2\beta}}} , \tag{78} \]
uncertainty in position, which would play the role of a "natural" minimal length. In light of this, it makes sense to ask whether or not some modifications of the previous framework which can account for such a desired feature are possible. One of the possible paths could be the implementation of the GUP theory on a one-dimensional truncated or compact momentum space. The arguments for such a choice are essentially two:

- the DGS functional procedure clearly shows that whenever it is possible to recast the theory, through a proper diffeomorphism, in a compact momentum space, a nonzero minimal uncertainty in position appears. Hence it should be automatic for a theory that is built on a compact momentum space in order to preserve the obvious symmetry under parity. It is trivial to notice that the quantity \( \Delta \) is a fully legitimate physical state. It is worth posing by the variational method, therefore it automatic for a theory that is built on a compact momentum space in order to preserve the obvious symmetry under parity.

- an adequate truncation of momenta could allow us to recover the series expansion method used in \([23]\) - even if the following analysis will have to be handled differently - and it could be then compared to the DGS scheme.

Our new Hilbert space \( \mathcal{H} \) will now be \( \mathcal{L}^2 ([-p_0, p_0], dp/\sqrt{1 + 2\beta p^2}) \), where \( p_0 \) is a generic point of the real line and we have chosen a symmetric interval with respect to zero as compact momentum space in order to preserve the obvious symmetry under parity. It is trivial to notice that the quantity \( z(p) \) does not diverge anymore towards the endpoints of the interval:

\[
\lim_{p \to \pm p_0} \frac{\sin^{-1}\left(\sqrt{2\beta} p\right)}{\sqrt{2\beta}} = \pm \frac{\sin^{-1}\left(\sqrt{2\beta} p_0\right)}{\sqrt{2\beta}}.
\]

(79)

Since we are working on compact momentum space for construction, there is no real need to make use of the map \( p \to z(p) \), which in this case is just a diffeomorphism between two compact intervals. The functional analysis and all the considerations about the momentum operator \( \hat{p} \), the position operator \( \hat{x} \), their squares and the commutator are exactly the same discussed in the compact DGS case, with the only difference that now everything is written with respect to the \( p \)-variable, thus it will be necessary to take into account in every step the measure \( \sqrt{1 + 2\beta p^2} \).

Hence the physical domain of the theory will be:

\[
\mathcal{D}_x = \{ \psi(p) \in \mathcal{H}^{1,2}([-p_0, p_0], dp/\sqrt{1 + 2\beta p^2}) \mid \psi(-p_0) = \psi(p_0) = 0 \}. \tag{80}
\]

By applying the DGS method we are now able to find in our physical domain \( \mathcal{D}_x \) the maximally localized states of the truncated theory and the corresponding nonzero uncertainty in position, which will be the minimum length of the theory:

\[
\Psi^\xi(p) = \mathcal{K} e^{-i \xi \sinh^{-1}(\sqrt{2\beta} p)} \cos \left( \frac{\pi}{2} \sinh^{-1}(\sqrt{2\beta} p_0) \right), \quad \xi \in \mathbb{R}, \quad p_0 \in [0, \infty).
\]

(81)

\[
\Delta x_{\text{min}} = \sqrt{\frac{\beta}{2} \sinh^{-1}(\sqrt{2\beta} p_0)}, \quad (82)
\]

where \( \mathcal{K} = \sqrt{\frac{\sqrt{2\beta}}{\sinh^{-1}(\sqrt{2\beta} p_0)}} \).

Of course, by construction, the state \( \Psi^\xi \) belongs to \( \mathcal{D}_x \) and respects all the constraints imposed by the variational method, therefore it is a fully legitimate physical state. It is worth noticing that the quantity \( \Delta x_{\text{min}} \) is inversely proportional, through the hyperbolic arcsine function, to \( p_0 \), which can be read as the half-length of the symmetric closed real interval we have chosen as momentum space. This implies that the larger this interval, the smaller this length will be and in the limit for \( p_0 \to \infty \) we obtain \( \Delta x_{\text{min}} = 0 \).

Since the limit \( p_0 \to \infty \) restores the real line as momentum space, marking the transition from the compact formulation to the non-compact one, this result is perfectly coherent with our conclusions about the full theory and can be interpreted as a further confirmation of what we have discussed previously. Furthermore, once the normalization condition is relaxed, also the
maximally localized states, for \( p_0 \to \infty \), are reduced to the modified plane waves \( (75) \), which, as stated before, are the "maximally localized states" of the full theory, even if they are not proper physical states.

Regarding the relation with ordinary quantum theory, it is straightforward to see that if, once we have taken the limit for \( p_0 \to \infty \) - i.e. once we are dealing with the full theory - we take the limit for \( \beta \to 0 \), from \( (75) \), we re-obtain the plane waves of the standard quantum mechanics as "maximally localized states", with zero uncertainty in position. Furthermore, the two limits commute. Indeed if we first take the limit for \( \beta \to 0 \), the expression \( (81) \) and \( (82) \) become:

\[
\lim_{\beta \to 0} K e^{-i \frac{\pi}{2} \frac{\sinh^{-1}(\sqrt{2} \beta p)}{\pi} \cos \left( \frac{\pi}{2} \frac{\sinh^{-1}(\sqrt{2} \beta p)}{\pi} \right)} = \sqrt{\frac{\pi}{p_0}} e^{-i \frac{\pi}{2} \frac{p}{p_0}} \cos \left( \frac{\pi}{2} \frac{p}{p_0} \right).
\]

\[
\lim_{\beta \to 0} \sqrt{\frac{\beta}{2} \frac{\pi}{\sinh^{-1}(\sqrt{2} \beta p_0)}} = \frac{\pi}{2} \frac{p_0}{p_0}.
\]

These are respectively the maximally localized states and the minimal uncertainty in position, obtained through a DGS scheme, of the ordinary quantum theory implemented in a one-dimensional compact momentum space, as it can be directly verified. At this point, once the consistency of the first limit is accepted, by relaxing again the normalization condition, it is possible to make the limit for \( p_0 \to \infty \) and once again we find the plane waves of the standard theory, with zero uncertainty in position.

A. Comparison with the series expansion procedure

In the previous analysis, which led us to explicitly find the maximally localized states and the corresponding uncertainty in position in the truncated theory, we never specified the norm of \( \hat{p} \) and consequently neither the set of its possible eigenvalues, since the procedure does not require it and holds in general. Nevertheless, if we want to make contact with the analysis carried out in [23] and if we want to use correctly a series expansion, some conditions must be imposed. Indeed the series \( (59) \) converges to \( \sqrt{1 + 2 \beta \hat{p}^2} \) if and only if \( \lVert \hat{p} \rVert \leq 1/(2 \beta) \) or equivalently if \( \lVert \hat{p} \rVert \leq \sqrt{1/(2 \beta)} \). Without loss of generality we made the maximal choice and set \( \lVert \hat{p} \rVert = \sqrt{1/2 \beta} \), meaning that the eigenvalues of \( \hat{p} \) belong to the set \( [-\sqrt{1/(2 \beta)}, \sqrt{1/(2 \beta)}] \), which represents our compact momentum space. Under these assumptions it is now possible to write:

\[
\Delta \hat{x} \Delta \hat{p} \geq \frac{\hbar}{2} \sqrt{1 + 2 \beta (\Delta \hat{p})^2}.
\]

Yet, two fundamental facts must be taken into account to interpret correctly the above expression:

- since our momentum space is a compact space, \( \Delta \hat{p} \) cannot take arbitrary values up to infinity, but only in the set \( [0, \sqrt{2/\beta}] \);
- there are no physical states which are able to saturate the inequality, hence the equal sign must be removed. This is because the squeezed states, which general form has been obtained in the first section, have no place in the truncated formulation because they do not belong to the physical domain \( D_\beta \) from expression \( (80) \).

At this point we can write:

\[
\Delta \hat{x} > \frac{\hbar}{2} \sqrt{\frac{1}{\Delta \hat{p}^2} + 2 \beta}
\]

and, by inserting the value of \( \Delta \hat{p} \) which minimizes the left-hand side, we obtain a lower bound for \( \Delta \hat{x} \), namely:

\[
\Delta \hat{x} > \frac{1}{2} \sqrt{\frac{5}{2} \frac{\hbar}{\sqrt{\beta}} = \frac{1}{2} \sqrt{\frac{5}{2} \frac{\hbar}{\sqrt{\beta}} \approx 0.79 \frac{\hbar}{\sqrt{\beta}}}.}
\]

If we now calculate the value of \( \Delta \hat{x}^{\text{min}} \) of the
expression (82) for \( p_0 = \sqrt{1/(2\beta)} \) we obtain:

\[
\Delta x_{\text{min}}^x = \sqrt{\frac{\beta}{2}} \frac{\pi \hbar}{\sinh^{-1}(1)} = \frac{\pi}{\sqrt{2} \sinh^{-1}(1)} l_p \sqrt{\beta_0} \approx 2.52 l_p \sqrt{\beta_0},
\]

which is in agreement with the constraint represented by the modified HUP.

The values of \( \Delta x \) between the lower bound just defined in (87) and the minimal uncertainty of the maximally localized states reported in (88) are evidently ruled out from the theory, since there no exist physical states, i.e. states \((88)\) are evidently ruled out from the theory, since there no exist physical states, i.e. states \( \Psi(x) \) which respect the constraints of the DGS variational principle, which can realize them.

### B. Quasi-position representation

Even if a position representation formally still exists and can be constructed, its physical meaning in some respects is lost due to the presence of a limit in localizing physical objects. Indeed, while in the ordinary quantum theory the position eigenbasis, even if it is made up of non-physical states, can be approximated by a sequence of physical states of uncertainty in position decreasing to zero, this is no longer possible in our framework for the formal position eigenbasis of the \( \hat{x} \) operator, hence the usual interpretation of the position representation and the density probability amplitude \( \Psi(x) \) is not valid anymore. Nevertheless, as pointed out first in [10], information on position can still be recovered by exploiting the maximally localized states. In particular it is possible to project any arbitrary physical state \( |\psi\rangle \) onto the maximally localized state \( |\xi\rangle \), defining in this way the probability amplitude of finding the particle maximally localized around the position \( \xi \). In this way the maximally localized states of the theory can be interpreted as constituting a basis for a new representation, namely the quasi-position representation:

\[
\psi(\xi) := \langle \xi | \psi \rangle = \int_{-p_0}^{p_0} dp \sqrt{\frac{1}{1 + 2\beta p^2}} \frac{Ke^{i\xi \sinh^{-1}(\sqrt{2\beta p})}}{\sqrt{2\beta}} \langle \xi | \psi \rangle.
\]

These wave functions are consequently called quasi-position wave functions. We notice that the basis made by the maximally localized states is not orthogonal (see Fig. 3):

\[
\langle \psi_{\xi'} | \psi_{\xi} \rangle = \int_{-p_0}^{p_0} dp \sqrt{\frac{1}{1 + 2\beta p^2}} \left( K e^{i \xi' \sinh^{-1}(\sqrt{2\beta p})} \right)^* \langle \xi' | \psi \rangle = \int_{-p_0}^{p_0} dp \sqrt{\frac{1}{1 + 2\beta p^2}} \left( K e^{i \xi \sinh^{-1}(\sqrt{2\beta p})} \right) \langle \xi' | \psi \rangle.
\]

This is analog to what happens in the original GUP formulation in [10], where the lack of the orthogonality property of the quasi-position basis is attributed to the "fuzziness" of the space.

The map (89) from momentum space to quasi-position space is clearly a generalization of the Fourier transformation. In order to see that this object is well defined, expression (89) can be rewritten as:

\[
\psi(\xi) = \int_{-q_0}^{q_0} dq \sqrt{\frac{1}{q_0}} e^{i \xi \frac{q}{2q_0}} \psi(q) \cos \left( \frac{\pi}{2} \frac{q}{q_0} \right) \]

where \( q := \sinh^{-1}(\sqrt{2\beta p}) / \sqrt{2\beta} \).

What we have obtained now is a standard Fourier transform of the compactly supported function \( \tilde{\psi}(q) \cos \left( \frac{\pi}{2} \frac{q}{q_0} \right) \).
By momentarily promoting $\xi$ to be a complex variable, we can make use of the Paley-Wiener theorem that assures us that the function $\psi(\xi)$ exists and in particular that it is an entire complex function, which is square-integrable over horizontal lines in the complex plane and therefore also for real values of $\xi$, the only ones in which we are interested, being $\xi$ the position expectation value of an arbitrary state. It is interesting to notice that, since we are dealing with compactly supported functions, it is possible to express $\psi(\xi)$ as a power series of $\xi$:

$$
\psi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n
$$

$$
a_n = \frac{1}{n!} \int_{-\infty}^{\infty} dq \frac{1}{q_0} \phi(q) \cos \left( \frac{\pi}{2} \frac{q}{q_0} \right) \left( \frac{iq}{\hbar} \right).
$$

(93)

It can be shown that this series is absolutely convergent and this implies that the $\psi(\xi)$ functions are $C^\infty$-smooth, as they should. If we now choose as $\psi(p)$ the momentum "eigenfunction" $\sqrt{1+2\beta p^2} \delta(p-\bar{p})$, through the map (89) we obtain:

$$
\psi(\xi) = K \cos \left( \frac{\pi}{2} \operatorname{sinh}^{-1} \left( \sqrt{2\beta p_0} \right) \right) e^{i \frac{\pi}{2} \operatorname{sinh}^{-1} \left( \sqrt{2\beta p_0} \right)}
$$

(94)

The function (94), which is a modified plane wave, represents of course a free particle in quasi-position representation, with fixed momentum $\bar{p}$ and fully delocalized in the $\xi$-space. The obtained Fourier map is invertible and the inverse transformation can be obtained by starting from (91):

$$
\tilde{\psi}(q) \Theta^{q+q_0}_{-q+q_0} = \frac{1}{2\pi \hbar} \frac{\sqrt{q_0}}{\cos \left( \frac{\pi}{2} \frac{q}{q_0} \right)} \int_{-\infty}^{\infty} d\xi \psi(\xi) e^{-i\xi q / \hbar},
$$

(95)

where $\Theta^{q+q_0}_{-q+q_0} := \Theta(q+q_0) + \Theta(-q+q_0)$ is the sum of two Heaviside functions, which natural presence signals that the inverse map is correctly giving back compactly supported functions in the interval $]-q_0, q_0[$. By making use of the relation $q(p)$ we can of course come back to the $p$ variable and rewrite (95) as a function of $p$. It is natural at this point to ask which is the action of the momentum operator and position operator in the quasi-position representation. By carefully using the definition of the generalized Fourier transform (89) it is possible to show what follows:

$$
\langle \xi | \hat{p} | \psi \rangle = \frac{1}{\sqrt{2\beta}} \sinh \left( -i\hbar \sqrt{2\beta} \frac{d}{d\xi} \right) \psi(\xi)
$$

(96)

$$
\langle \xi | \hat{x} | \psi \rangle = \xi \psi(\xi) + \frac{\pi}{2} \frac{i\hbar \sqrt{2\beta}}{\sinh^{-1} \left( \sqrt{2\beta p_0} \right)} \tan \left( \frac{\pi}{2} \frac{i\hbar \sqrt{2\beta}}{\sinh^{-1} \left( \sqrt{2\beta p_0} \right)} \frac{d}{d\xi} \right) \psi(\xi).
$$

(97)

As expected, they are non-local differential operators and their action can be made explicit by a series expansion in the derivative operator itself. Nevertheless for a generic function $\psi(\xi)$ the series is in general not convergent.

VII. WAVE PACKETS

We want now to explore some physical consequences of the theory within both formulations by studying one of the simplest physical systems, that is a free wave packet. Exactly as in the ordinary quantum theory, we can construct
a wave packet evolving in time as a superposition of time-dependent plane waves:

\[ \Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \tilde{\Psi}(p)e^{ipx/\hbar - itE(p)/\hbar} dp, \tag{98} \]

where \( \tilde{\Psi}(p) \) is the ordinary Fourier transform of the function \( \Psi(x, t) \) at \( t = 0 \) and \( E(p) \) is the dispersion relation between energy and momentum. On this ground, in the full GUP theory we will use as infinite basis for the wave packet the modified plane waves \( \tilde{\Psi}(p) \), which correctly are the eigenfunctions of the position operator in momentum representation:

\[ \Xi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \left\{ \frac{dp}{\sqrt{1 + 2\beta p^2}} \tilde{\Xi}(p) e^{ipx/\hbar - itE(p)/\hbar} \right\}, \tag{99} \]

where \( \tilde{\Xi}(p) \) is obtained via the generalized Fourier transform of \( \Xi(x, t) \) at \( t = 0 \). On the other hand, in the compact theory, to recover physical information on position we need to rely on quasi-position representation. It is therefore natural to use maximally localized states as infinite basis for the construction of the wave packet. Coherently we notice that in the ordinary quantum theory and in the full GUP theory the plane waves and the modified plane waves used as basis for the wave packet can be obtained respectively as the Fourier transform and generalized Fourier transform of a Dirac Delta \( \delta(p - \bar{p}) \). This holds true also in the compact theory, where via the generalized Fourier transform of a Dirac Delta \( \delta(p - \bar{p}) \) we obtain the states \( \tilde{\Psi}(p) \). In light of this we can write:

\[ \Phi(\xi, t) = \int_{-p_0}^{+p_0} \left\{ \frac{dp}{\sqrt{1 + 2\beta p^2}} \tilde{\Phi}(p) e^{ip\xi/\hbar - itE(p)/\hbar} \cos \left( \frac{\pi}{2} \frac{\sinh^{-1}(\sqrt{2\beta p}/\sqrt{2\beta})}{\sinh^{-1}(\sqrt{2\beta})} \right) \right\}. \tag{100} \]

Since we are interested in free motion, the dispersion relation in both the GUP theories will be \( E(p) = p^2/2m \). This is the same one of the ordinary theory since the free particle Hamiltonian is left untouched by the modification of the commutator. Nevertheless, if we express the dispersion relation in terms of the frequency \( \omega \) and the wave number \( k \) we are able to appreciate the deep difference between the GUP theories and the standard one:

\[ \omega(k) = \frac{\hbar k^2}{2m} \quad \text{Standard theory}, \]

\[ \omega(k) = \frac{\sinh^2(\sqrt{2\beta}\hbar k)}{4m\hbar\beta} \quad \text{GUP theories.} \tag{102} \]

FIG. 4. Plot of the GUP-modified dispersion relation for a free particle together with its first derivative, which represents the group velocity of the wave packet and its second derivative, which instead is responsible for the dispersion of the wave packet.

Here it is important to pay attention to the fact that while in the full GUP theory the dispersion relation \( \omega(k) \) does not contain boundaries on the possible values of \( k \) and \( \omega \), in the compact theory, since the momentum is constrained in some interval \([-p_0, p_0] \), \( k \) will be automatically limited and this leads to the existence of a minimum wavelength \( \lambda \), as it would be expected from the presence of a minimum length in the theory. It is also interesting to notice that the compactness of the momentum space implies an upper bound on the angular frequency \( \omega \), which could be interpreted as a lower bound on the possible time interval. In
order to make a comparison between the time-evolution of a wave packet in the three different frameworks we will analyze wave packets built up by fixing a gaussian-like wave function in momentum space, being careful that the chosen states belong to the physical domain of the different theories.

Formally, we will have:

Standard theory:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} A e^{-\frac{(p-p_0)^2}{2\beta}} e^{i\frac{p^2}{\hbar} - it\frac{p^2}{2m\hbar}} dp, \quad (103)$$

Full GUP theory:

$$\Xi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \beta e^{-\beta(p-\nu)^2} e^{i\frac{\nu}{\hbar} \sinh^{-1}(\sqrt{\pi p})} - it\frac{p^2}{2m\hbar}, \quad (104)$$

Compact GUP theory:

$$\Phi(\xi, t) = \int_{-p_0}^{+p_0} dp \beta e^{-\beta(p-\nu)^2} \cos \left( \frac{\pi}{2\hbar} \sinh^{-1}(\sqrt{2\beta p}) \right) e^{i\xi \sinh^{-1}(\sqrt{\pi p})} - it\frac{p^2}{2m\hbar}, \quad (105)$$

where $A, B, C$ are the normalization constants and $\gamma, \nu, \kappa$ are real parameters.

The quantity $\gamma$ in (103) represents, in the ordinary theory, the (initial) expectation value of the momentum operator $\hat{p}$ for the considered state, but this is not true for the parameters $\nu$ and $\kappa$ in (104) and (105) in the GUP theories. Since we want to compare states with the same initial conditions the parameters $\nu$ and $\kappa$ will hence be fixed in order to have $\langle \hat{p} \rangle = p_0$ also for the wave packets in the two modified theories. On the other hand, the initial $\langle t = 0 \rangle$ expectation value $\langle \hat{x} \rangle$ for the position operator is automatically zero for all the wave packets.

Numerical evaluation of these integrals - for which analytical solutions seem not available - are shown in the graphics below in Fig. 5 where the probability density at different times for the three wave packets is plotted, for an arbitrary yet proper choice of the free parameters.

A more quantitative picture of the situation can be obtained by inspecting the plot in Fig. 6 of the expectation value of position $\langle \hat{x} \rangle$ and the plot in Fig. 7 of the relative uncertainty in position $\Delta \hat{x} / \Delta \hat{x}_0$ as a function of time for the three cases, where $\Delta \hat{x}_0$ is the initial uncertainty.

From the first plot we notice that, even if $\langle \hat{p} \rangle$ is the same for all the wave packets, the time-evolution law for $\langle \hat{x} \rangle$ is different. This can be easily understood by looking at the evolution of the $\hat{x}$ operator in the Heisenberg picture in the

![FIG. 6. Plot of the expectation value of the position operator in units of $\hbar \sqrt{\beta}$ as a function of time (measured in units of $\hbar \beta$) for the different wave packets in the three quantum framework here considered. We can notice how the difference in the relations (106) here results in wave packets with an expectation value of the position changing more rapidly in the GUP theories with respect to the ordinary quantum theory.](image)
FIG. 5. Plots of the spreading in time of the wave packets $\{103\} - \{105\}$ in the different frameworks here discussed. Space is naturally measured in units of $\hbar \sqrt{\beta}$, time in units of $m \hbar \beta$ and momentum in units of $1/\sqrt{\beta}$. In figure (a) is shown the spreading of the gaussian wave packet $\{103\}$ in the ordinary quantum theory, in figure (b) is possible to appreciate the spreading of the wave packet $\{104\}$ in the full GUP theory, while in figures (c) and (d) is exhibited the spreading of the wave packet $\{105\}$ in the compact GUP theory for two different choices of the closed interval of the momentum space, respectively $[-p_0, p_0] = [-5, 5]$ and $[-p_0, p_0] = [-3, 3]$, in units of $1/\sqrt{\beta}$.

different frameworks:
\[
\frac{d\hat{x}}{dt} = \frac{i}{\hbar} \left[ \frac{\hat{p}^2}{2m}, \hat{x}^2 \right] \Rightarrow \hat{x}(t) = \hat{x}(0) + \frac{\hat{p}}{m} t, \quad (106)
\]
\[
\frac{d\hat{x}}{dt} = \frac{i}{\hbar} \left[ \frac{\hat{p}^2}{2m}, \hat{x}^2 \right] \Rightarrow \hat{x}(t) = \hat{x}(0) + \sqrt{1 + 2\beta \hat{p}^2} \frac{\hat{p}}{m} t. \quad (107)
\]

From here it is clear why the expectation values of the position operator are different. By looking at the second plot in Fig. 5 instead, it is evident how differently the wave packets spread in the different frameworks. In particular we see that the wave packet of the full GUP theory spreads more rapidly than the wave packet of the ordinary theory, while the spreading of the wave packet in the truncated GUP theory really depends on how the compact interval of momentum is fixed. Thus, according to the chosen interval, we can have wave packets spreading more or less rapidly with respect to the ordinary theory, but always more slowly than the full GUP theory, which spreading curve represents an upper limit for the region that the position uncertainty of these wave packets can explore.
FIG. 7. Plot of relative uncertainty in position as a function of time for the different wave packets studied in the three quantum frameworks, in the same units of the previous plot. We are able to see how the wave packet spread in the full GUP theory is always more rapid with respect to the one in the ordinary theory, while the wave packet spread in the compact GUP theory strongly depends on the real interval chosen as a momentum space, producing thus physical objects which can spread more or less rapidly with respect to the one in the standard theory.

We stress the fact that within the compact GUP theory is then possible to obtain wave packets that spread really slowly in time and in the end this is due to the truncation process which cuts out all the modified plane waves with higher momentum.

VIII. CONCLUDING REMARKS

We have analyzed in detail the extended formulation of the GUP theory, in one dimension, deriving from a square root-modified algebra, which lower order Taylor expansion reproduces the original formulation and which verify the Jacobi identity, as proposed in [23].

The main merit of our study is to have proved that, differently from what was stated in [23] and from the original approach in [10], the considered formulation, without any truncation of the momentum space, is not associated with a minimal uncertainty for the position operator. This result was first of all signaled by the functional analysis of the position operator, which has resulted to be essentially self-adjoint, exactly as in the ordinary quantum theory and differently from the KMM GUP theory, and it was then supported by considerations regarding the modified Lebesgue measure of the theory, which integral is divergent, and consequently by an explicit calculation carried out according to the functional methods presented in [23]. To obtain a minimal uncertainty for the position operator different from zero and, in this sense, to extend the original formulation, we have shown that a truncation (by hand) of the momentum space is necessary. Then, we constructed the so-called quasi-position representation and, by following a similar scheme to that one presented in [10], we arrived at a complete characterization of the modified quantum theory. From a physical point of view, a significant difference with respect to the original analysis consists in having obtained a minimal uncertainty in position realized by states that do not belong to the boundary of the uncertainty relation (i.e. when the equality sign holds). These states are indeed ruled out from the theory since they cannot satisfy the boundary conditions of the obtained physical domain [30].

In particular, whenever the truncation is chosen in such a way that is possible to apply the series expansion method discussed in [23], the uncertainty relation, which can be explicitly found in this case, is strictly an inequality, setting a lower bound $\Delta \hat{x}_{GUP}$ for the value of $\Delta \hat{x}$, compatible with the value of $\Delta \hat{x}_{min}$ obtained in our analysis through the methods discussed above. In this specific case, this fact suggests that all the states which have minimal uncertainty in position in the range $[\Delta \hat{x}_{GUP}, \Delta \hat{x}_{min}]$ must correspond to non-physical states as well, for which, for instance, the energy is diverging or not well-defined. Finally, we have analyzed the spreading of localized wave packets both in the truncated and non-truncated theory, comparing them with the standard quantum mechanics spread in equivalent initial conditions. We have shown that the non-truncated or full theory outlines wave packets that spread more rapidly than the ordinary quantum theory. Instead, the truncated or compact theory exhibits spreading features faster or slower than ordinary quantum mechanics, depending on the width of the real closed interval chosen as mo-
momentum space. This could have some interesting implications for possible minisuperspace implementation of the GUP theory, concerning the possibility to deal, in the truncated formulation, with wave packets which, differently from the Wheeler-De Witt dynamics [29], are slowly spreading even close to the initial singularity or the Big-Bounce, allowing there for a quasi-classical approximation of the quantum dynamics.

We conclude by observing that in [13] and [19] it has been clarified that the classical GUP dynamics (that is a modification of the Poisson brackets in place of the commutators), naturally leads, in the case of an extended formulation with the square root, to the same Friedmann equation for the isotropic Universe, emerging in brane cosmology. Our analysis then opens the interesting questions about which of the two proposed extended approaches really corresponds to this singular brane cosmology. Since the Poisson brackets have been studied without any restrictions on the momentum space, we are led to argue that the above correspondence should be valid for the non-truncated theory. Nevertheless, an intriguing question still would remain on the ground: which kind of cosmological behavior is predicted by the quasi-classical limit of the isotropic Universe in the truncated scenario?

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