ON THE BINOMIAL EDGE IDEALS OF PROPER INTERVAL GRAPHS

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ABSTRACT. We prove several cases of the Betti number conjecture for the binomial edge ideal \( J_G \) of a proper interval graph \( G \) (also known as closed graph). Namely, we show that this conjecture is true for the linear strand of \( J_G \), and true in general for any proper interval graph \( G \) such that the regularity of \( S/J_G \) equals two.

INTRODUCTION

The proper interval graphs are known since a while in combinatorics. They were first introduced in [13]. A finite simple undirected graph \( G \) on the vertex set \([n]\) is called a proper interval graph (in brief PI graph) if it admits a proper interval ordering. This means that there exists a labeling of the vertices of \( G \) such that for any \( 1 \leq i < j < k \leq n \), if \( \{i, k\} \) is an edge of \( G \), then \( \{i, j\} \) and \( \{j, k\} \) are edges of \( G \) as well [17, Theorem 1]. PI graphs are also known as unit interval graphs or indifference graphs. Several other properties and characterizations of PI graphs can be found in [20], [21], [10], [11], [12], [14].

Binomial edge ideals were introduced in [15] and [19]. They are defined as follows: If \( G \) is a simple graph on \([n]\), then its associated binomial ideal \( J_G \subset S=K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) is generated by the binomials \( f_{ij} = x_i y_j - x_j y_i, 1 \leq i < j \leq n \) with \( \{i, j\} \in E(G) \).

Various properties of binomial edge ideals have been studied in several papers, and some interesting still open questions in this topic exist. For example, one of the most intriguing conjectures regards binomial edge ideals associated with PI graphs. This conjecture was stated in [6] and it claims that, for any PI graph \( G \) on the vertex set \([n]\), \( J_G \) and its initial ideal with respect to the lexicographic order share the same graded Betti numbers. We shall refer to this conjecture as the Betti number conjecture for PI graphs. So far, this conjecture was proved for PI graphs whose binomial edge ideals are Cohen-Macaulay [6, Proposition 3.2].

In Theorem 3.2 we prove this conjecture for any PI graph \( G \) with \( \text{reg}(S/J_G) = 2 \). The first main step in proving Theorem 3.2 is Theorem 2.4 where we show that \( S/J_G \) and \( S/\text{in}_{<}(J_G) \) share the same linear strand in the Betti diagram for any PI graph \( G \).

The paper is organized as follows. In Section 1 we recall basic facts about binomial edge ideals of PI graphs and their initial ideals. In Section 2 we prove Theorem 2.4 which
states that, if $G$ is a PI graph, then
\[ \beta_{i,i+1}(S/J_G) = \beta_{i,i+1}(S/\text{in}_<(J_G)) = i\delta_i(\Delta(G)), \]
where $\delta_i(\Delta(G))$ denotes the number of cliques with $i+1$ vertices in the clique complex $\Delta(G)$ of $G$. Finally, we prove the Betti number conjecture for any PI graph $G$ in the case that $\text{reg}(S/J_G) = 2$.

1. Preliminaries

In this section we review fundamental results on binomial edge ideals that will be used in the next sections. To begin with, we fix some notation and present the basic notions which we use in the main sections.

Let $K$ be a field. Let $[n] = 1, 2, \ldots, n$ for $n \in \mathbb{N}$. Let $G$ be a simple graph on the vertex set $[n]$. This means that $G$ has no loops, no multiple edges, and it is undirected. We denote the edge set of $G$ by $E(G)$. For graphs, we use the standard terminology and notation. For example, if $\mathcal{S} \subset [n]$, $G_{\mathcal{S}}$ denotes the restriction of $G$ to $\mathcal{S}$ and $G^c$ denotes the complement of the graph $G$.

The associated binomial edge ideal of $G$ is $J_G \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ which is generated by the binomials $f_{ij} = x_iy_j - x_jy_i, 1 \leq i < j \leq n$ with $\{i, j\} \in E(G)$. It is clear that we can neglect isolated vertices of $G$, hence we shall assume that our graphs have no isolated vertex throughout this paper.

In the pioneering paper [15], it is shown that the generators of $J_G$ form a (quadratic) Gröbner basis with respect to the lexicographic order induced by the natural ordering of the indeterminates if and only if $G$ is a proper interval (PI) graph. It should be noted that in [15], the authors use the term “closed graph” for the PI graph. Nevertheless, we use the notion PI graph which has been well known in combinatorics since around 60 years [13].

The equivalence of PI graphs and closed graphs is shown in [5]. Some other papers that discuss properties of PI graphs related with commutative algebra are [2], [3], [4], [6], [7], [8], [18].

For a graph $G$, its clique complex $\Delta(G)$ is the simplicial complex of all its cliques, that is, all complete subgraphs of $G$. The maximal cliques of $G$ are called facets of $\Delta(G)$. In [6, Theorem 2.2], it is shown that $G$ is a PI graph if and only if there exists a labeling of $G$ such that all the facets of the clique complex $\Delta(G)$ of $G$ are intervals $[a, b] = \{a, a+1, \ldots, b-1, b\} \subset [n]$. This means, in particular, that if $\{i, j\} \in E(G)$, then for any $i \leq k < \ell \leq j$, $\{k, \ell\} \in E(G)$. When a PI graph $G$ is given, we always assume that its vertices are labeled such that the facets of its clique complex are of the form $[a_i, b_i], 1 \leq i \leq r$, with $1 = a_1 < a_2 < \cdots < a_r < b_r = n$.

Let $G$ be a PI graph. Let $<$ be the lexicographical order on $S$, induced by $x_1 > \ldots > x_n > y_1 > \ldots > y_n$. Then the initial ideal of the binomial ideal $J_G$ is the monomial ideal, $\text{in}_<(J_G) = (x_iy_j : \{i, j\} \in E(G))$. This is the monomial edge ideal of a bipartite graph on the vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ with edge set $\{\{x_i, y_j\} : \{i, j\} \in E(G)\}$. We set $\text{in}_<(G)$ to be this bipartite graph. Therefore, we have $\text{in}_<(J_G) = I(\text{in}_<(G))$. 

2. Linear strand of a binomial edge ideal of a PI graph and of its initial ideal

We recall from [22] the formula for the linear strand of an edge ideal of a graph. Let $H$ be a graph on the vertex set $[n]$, and let $I(H) \subset R = K[x_1, \ldots, x_n]$ be its monomial edge ideal.

**Proposition 2.1.** ([22] Proposition 2.1)

\[
\beta_{i,i+1}(R/I(H)) = \sum_{\mathcal{S} \subseteq V_{H}, |\mathcal{S}| = i+1} (\#\text{comp}(H^\mathcal{S}_\mathcal{J}) - 1),
\]

where $\#\text{comp}(H^\mathcal{S}_\mathcal{J})$ is the number of the connected components of $H^\mathcal{S}_\mathcal{J}$.

From the above proposition, for bipartite graphs we can derive a more specific formula.

**Corollary 2.2.** Let $H$ be a bipartite graph on the vertex set $V(H)$. Then,

\[
\beta_{i,i+1}(R/I(H)) = \#\{\mathcal{S} \subseteq V(H) : |\mathcal{S}| = i + 1, H^\mathcal{S}_\mathcal{J} \text{ has 2 connected components}\}.
\]

**Proof.** Without lost of generality, we may take $A = \{x_1, \ldots, x_{|A|}\}$ and $B = \{y_1, \ldots, y_{|B|}\}$, the bipartition of $H$. Thus, $V(H) = A \cup B$. Let $\mathcal{S}$ be a subset of $V(H)$. If $\mathcal{S} \subseteq A$ or $\mathcal{S} \subseteq B$, then $H^\mathcal{S}_\mathcal{J}$ is a complete graph on vertex set $\mathcal{S}$ which has only one connected component.

The other possible choice is when $\mathcal{S}$ consists of a subset of $A$ and a subset of $B$. That is, $\mathcal{S} = \{x_{k_1}, \ldots, x_{k_a} : 1 \leq k_1 < \cdots < k_a \leq |A|\} \cup \{y_{l_1}, \ldots, y_{l_b} : 1 \leq l_1 < \cdots < l_b \leq |B|\}$.

Consider the case when $H^\mathcal{S}_\mathcal{J}$ is a complete bipartite graph, i.e., $\{x_{k_i}, y_{l_j}\} \in E(H)$ for all $i \in [a], j \in [b]$. Then, $H^\mathcal{S}_\mathcal{J}$ has 2 connected components, since it does not have any edge between $A \cap \mathcal{S}$ and $B \cap \mathcal{S}$.

The last case is when $H^\mathcal{S}_\mathcal{J}$ is a non-complete bipartite graph. Then there exist $i \in [a], j \in [b]$ such that $\{x_{k_i}, y_{l_j}\} \notin E(H)$. Then $\{x_{k_i}, y_{l_j}\} \in E(H^\mathcal{S}_\mathcal{J})$. Since in $H^\mathcal{S}_\mathcal{J}$ the vertex $x_{k_i}$ is connected to all other vertices in $A \cap \mathcal{S}$ and the vertex $y_{l_j}$ is connected to all other vertices in $B \cap \mathcal{S}$, then $H^\mathcal{S}_\mathcal{J}$ has only one connected component.

Therefore, for a bipartite graph $H$ and any $\mathcal{S} \subseteq V(H)$, $H^\mathcal{S}_\mathcal{J}$ has either one or two connected components, which means that $\#\text{comp}((H^\mathcal{S}_\mathcal{J}) - 1)$ is either 0 or 1, respectively. Hence, by Proposition 2.1 we have $\beta_{i,i+1}(R/I(H))$ equals the number of subset $\mathcal{S}$ with $i + 1$ vertices such that $H^\mathcal{S}_\mathcal{J}$ has two connected components. \hfill $\Box$

The following proposition is a particular case of Corollary 4.3 in [16].

**Proposition 2.3.** Let $G$ be a graph on the vertex set $[n]$ and $J_G$ be its binomial edge ideal. Let $\Delta(G)$ be the clique complex of $G$ and let $f_i(\Delta(G))$ denote the number of the cliques of $G$ with $i + 1$ vertices. Then

\[
(2) \quad \beta_{i,i+1}(S/J_G) = if_i(\Delta(G)).
\]

**Proof.** cf. [16] Corollary 4.3. \hfill $\Box$

Now we are ready to prove our first main theorem.
Theorem 2.4. Let $G$ be a PI graph over the vertex set $[n]$. Let $<$ be the lexicographical order on $S$, induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Then we have
\begin{equation}
\beta_{i,i+1}(S/J_G) = \beta_{i,i+1}(S/\text{in}_<(J_G)) = i_f(\Delta(G)).
\end{equation}

Proof. By Proposition 2.3, we know that $\beta_{i,i+1}(S/J_G) = i_f(\Delta(G))$. Hence, we need to show that $\beta_{i,i+1}(S/\text{in}_<(J_G)) = i_f(\Delta(G))$.

As we have stated in the introduction, we may assume that the graph $G$ is labeled such that the facets of $\Delta(G)$ are the intervals $[a_1,b_1],\ldots,[a_r,b_r]$ where $1 = a_1 < a_2 < \cdots < a_r < b_r = n$. Since $G$ is a PI graph, one can consider the bipartite graph $H = \text{in}_<(G)$ on the vertex set $V(H) = \{x_1,\ldots,x_n\} \cup \{y_1,\ldots,y_n\}$ with the edge set $E(H) = \{(x_r,y_s) : r < s \text{ and } \{r,s\} \in E(G)\}$. Observe that $I(H)$ is an ideal in $K[x_1,\ldots,x_n,y_1,\ldots,y_n] = S$.

From Corollary 2.2 we have
\begin{equation}
\beta_{i,i+1}(S/\text{in}_<(J_G)) = \beta_{i,i+1}(S/I(H)) = 
#\{\mathcal{S} \subseteq V(H) : |\mathcal{S}| = i+1, H_{\mathcal{S}} \text{ has 2 connected components}\}.
\end{equation}

Let $X = \{x_1,\ldots,x_n\}$, $Y = \{y_1,\ldots,y_n\}$, and $\mathcal{S} \subset V(H)$ be a set with $|\mathcal{S}| = i + 1$. By the proof of Corollary 2.2 we know that $H_{\mathcal{S}}$ has two connected components if and only if $\mathcal{S} \cap X$ and $\mathcal{S} \cap Y$ are nonempty and $\{x_r,y_s\} \in E(H)$ for all $x_r \in \mathcal{S} \cap X$ and $y_s \in \mathcal{S} \cap Y$.

Let $\mathcal{S} \cap X = \{x_{k_1},\ldots,x_{k_j}\}$, $k_1 < \cdots < k_j$, and $\mathcal{S} \cap Y = \{y_{k_{j+1}},\ldots,y_{k_{i+1}}\}$, $k_{j+1} < \cdots < k_{i+1}$, for an integer $1 \leq j \leq i$. Then, $H_{\mathcal{S}}$ has two connected components if and only if $\{x_r,y_{k_r}\} \in E(H)$ for all $1 \leq r \leq j$ and $j+1 \leq s \leq i+1$, which is equivalent to saying that $\{k_r,k_s\} \in E(G)$ for all $1 \leq r \leq j$ and $j+1 \leq s \leq i+1$. This implies that the clique $\{k_1,\ldots,k_j,k_{j+1},\ldots,k_{i+1}\}$ is a clique of $G$.

This shows that, for any $\mathcal{S} \subset V(H)$ such that $|\mathcal{S}| = i + 1$ and $H_{\mathcal{S}}$ has two connected components, we may associate it to a pair $(j,C_{i+1})$ where $1 \leq j \leq i$ is an integer and $C_{i+1}$ is a clique of the graph $G$ with $i + 1$ vertices.

Conversely, let $1 \leq j \leq i$ be an integer and let $C_{i+1} = \{k_1,\ldots,k_{i+1}\}$ be a clique of $G$, with $k_1 < \cdots < k_{i+1}$. Then, if $\mathcal{S} = \{x_{k_1},\ldots,x_{k_j}\} \cup \{y_{k_{j+1}},\ldots,y_{k_{i+1}}\}$, we have $|\mathcal{S}| = i + 1$ and $\{k_r,k_s\} \in E(G)$ for all $1 \leq r \leq j$ and $j+1 \leq s \leq i+1$, hence $\{x_r,y_{k_r}\} \in E(H)$. Thus, $H_{\mathcal{S}}$ has two connected components, namely the complete graphs on the vertex sets $\{x_{k_1},\ldots,x_{k_j}\}$ and $\{y_{k_{j+1}},\ldots,y_{k_{i+1}}\}$, respectively. Obviously, the above maps $\mathcal{S} \rightarrow (j,C_{i+1})$ and $(j,C_{i+1}) \rightarrow \mathcal{S}$ are inverse.

Therefore, we have obtained the following equality:
\begin{align*}
#\{\mathcal{S} \subseteq V(H) : |\mathcal{S}| = i + 1, H_{\mathcal{S}} \text{ has 2 connected components}\} &= 
#\{(j,C_{i+1}) : 1 \leq j \leq i \text{ and } C_{i+1} \text{ is a clique of } G \text{ with } i + 1 \text{ vertices}\},
\end{align*}
which implies that $\beta_{i,i+1}(S/I(H)) = i_f(\Delta(G))$. 

The condition that $G$ is a PI graph in Theorem 2.4 cannot be omitted. For example, consider the graph $G$ on the vertex set $\{1,2,3,4\}$ and with edge set $\{\{1,2\},\{1,3\},\{1,4\}\}$. Then $\beta_{2,3}(S/J_G) = \beta_{3,4}(S/J_G) = 0$, while $\beta_{2,3}(S/\text{in}_<(J_G)) = 3$ and $\beta_{3,4}(S/\text{in}_<(J_G)) = 1$. 

\[\square\]
3. The Betti number theorem for PI graphs associated with binomial edge ideals with small regularity

Before we discuss our main theorem about the graded Betti numbers for the binomial edge ideals of a PI graph $G$ with $\text{reg}(S/J_G) = 2$, first we show when one gets $\text{reg}(S/(J_G)) = 2$.

**Proposition 3.1.** Let $G$ be a PI graph on the vertex set $[n]$. Then $\text{reg}(S/J_G) = 2$ if and only if $G$ is in one of the following forms:

1. $G = K_m \cup K_p$ with $m + p = n$,
2. $G$ is connected and $\Delta(G)$ is generated by two maximal cliques of the form $[1, b], [a, n]$ where $1 < a \leq b < n$.

**Proof.** First consider the case when $G$ is not connected, that is, $G = G_1 \cup \cdots \cup G_c$ with $c \geq 2$, where $G_1 \cup \cdots \cup G_c$ are the connected components of $G$. Then

$$\text{reg}(S/J_G) = \text{reg}(S/J_1) + \cdots + \text{reg}(S/J_c).$$

From [9, Theorem 3.2], $\text{reg}(S/J_i) = \ell_1 + \cdots + \ell_c$, where $\ell_i$ denotes the length of a longest induced path in $G_i$, $1 \leq i \leq c$. If $\text{reg}(S/J_G) = 2$, the equality $\ell_1 + \cdots + \ell_c = \text{reg}(S/J_G) = 2$ occurs if and only if $G$ has only two connected components and $\ell_1 = \ell_2 = 1$, which means that $G_1, G_2$ are complete graphs.

The other case is when $G$ is connected. By [9, Theorem 3.2], the length of the longest induced path in $G$ is equal to the regularity, which is 2. This holds if and only if $\Delta(G)$ has two maximal cliques of the form $[1, b], [a, n]$ where $1 < a \leq b < n$. Indeed, suppose that $\Delta(G)$ has at least 3 facets, that is, the facets are $[1, b_1], [a_2, b_2], \ldots, [a_r, b_r = n], r \geq 3$. Then there exists an induced path in $G$ of length at least 3 which contains the vertices $1, a_2, b_2, b_3$. \hfill $\square$

Now we are ready to prove our main result, which shows that the Betti number conjecture for PI graphs is true in the case where $\text{reg}(S/J_G) = 2$.

**Theorem 3.2.** Let $G$ be a PI graph over vertex the set $[n]$. If $\text{reg}(S/J_G) = 2$, then

$$\beta_{ij}(S/J_G) = \beta_{ij}(S/\text{in}_<(J_G))$$

for all $i, j$.

**Proof.** Notice that since $G$ is a PI graph, then, from [9, Theorem 3.2] we have $\text{reg}(S/\text{in}_<(J_G)) = \text{reg}(S/J_G) = 2$. Therefore, $\beta_{ij}(S/\text{in}_<(J_G)) = 0 = \beta_{ij}(S/J_G)$ for all $j \geq i + 3$. We also have Theorem 2.4 for $j = i + 1$. Therefore, we need to prove the equality only for $j = i + 2$.

We have the following cases as in Proposition 3.1:

**Case 1:** Let $G = G_1 \cup G_2$ where $G_1 = K_m, G_2 = K_p, n = m + p$. Let $S_1 = K[x_1, \ldots, x_m, y_1, \ldots, y_m]$ and $S_2 = K[x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n]$ be the polynomial rings with variables related to the vertices in $G_1$ and $G_2$, respectively. As $G_1, G_2$ are complete graphs, it is well known that,
since the binomial edge ideal of a complete graph and its initial ideal have a linear resolution,

\[
\beta_{ij}(S_k / (J_{G_k})) = \beta_{ij}(S_k / \text{in}_<(J_{G_k})), k = 1, 2.
\]

The resolution of \( S/J_G \) is obtained by tensoring the resolution of \( S_1/J_{G_1} \) with the resolution of \( S_2/J_{G_2} \), and the resolution of \( S/(\text{in}_<(J_G)) \) is obtained by tensoring the resolution of \( S_1/(\text{in}_<(J_{G_1})) \) with the resolution of \( S_2/(\text{in}_<(J_{G_2})) \). Hence, the Betti numbers of \( S/J_G \) and \( S/(\text{in}_<(J_G)) \) are equal.

**Case 2:** Let \( G \) be a connected graph with \( \Delta(G) = \langle 1, b \rangle, [a, n] \rangle, \) with \( 1 < a \leq b < n \). Then, by [15, Section 3], obviously \( J_G \) has two minimal primes, namely \( P = J_{K_a} \) and \( (\{x_i, y_i\}_{a \leq i \leq b}, J_{K[a,a-1]}, J_{K[b+1,n]}) = Q \). We denote by \( K_{[a,b]} \) the complete graph on the vertex set \([a, b]\).

This implies, by [15, Theorem 3.2], that \( J_G = P \cap Q \). We have the following exact sequence:

\[
0 \to S/J_G \to S/P \bigoplus S/Q \to S/(P + Q) \to 0.
\]

Observe that
\[
P + Q = J_{K_a} + (\{x_i, y_i\}_{a \leq i \leq b}, J_{K[a,a-1]}, J_{K[b+1,n]}) = J_{K_n} + (\{x_i, y_i\}_{a \leq i \leq b}) = J_{K_n[a,b]} + (\{x_i, y_i\}_{a \leq i \leq b}).
\]

To simplify the notation, we set \( \text{Tor}_k(M) =: \text{Tor}_k(M, K)_{[a,b]} \) for any \( k, \ell \in \mathbb{N} \). We consider the following long exact sequence of Tor which follows from (5):

\[
\cdots \to \text{Tor}_{i+2}(S/J_G)_{i+2} \to (\text{Tor}_{i+2}(S/P) \oplus \text{Tor}_{i+2}(S/Q))_{i+2} \to \text{Tor}_{i+2}(S/(P + Q))_{i+2} \to \text{Tor}_{i+1}(S/J_G)_{i+2} \to (\text{Tor}_{i+1}(S/P) \oplus \text{Tor}_{i+1}(S/Q))_{i+2} \to \text{Tor}_{i+1}(S/(P + Q))_{i+2} \to \cdots
\]

Obviously, we have have \( \text{Tor}_{i+2}(S/J_G)_{i+2} = 0 \) and \( \text{Tor}_{i+2}(S/P)_{i+2} = \text{Tor}_{i+2}(S/Q)_{i+2} = 0 \).

As
\[
\text{reg}(S/(P + Q)) = \text{reg}(\frac{K[\{x_i, y_i\}_{i \in [n]\setminus[a,b]}]}{K_{n[a,b]}}, a \leq i \leq b) = 1 + 0 = 1,
\]
we get \( \beta_{ij}(S/(P + Q)) = 0 \) for \( j \geq i + 2 \), which implies that \( \text{Tor}_j(S/(P + Q))_{i+2} = 0 \).

From (6), we derive the following equality:

\[
\beta_{i+2,i+2}(S/G) - \beta_{i+2,i+2}(S/(P + Q)) = \beta_{i+1,i+2}(S/P) - (\beta_{i+1,i+2}(S/Q) + \beta_{i+1,i+2}(S/Q))
\]

\[
+ \beta_{i+1,i+1}(S/(P + Q)) - \beta_{i+1,i+2}(S/P) = 0.
\]

Now consider the ideal \( \text{in}_<(J_G) \). From [11, Lemma 1.3], we know that \( \text{in}_<(J_G) = \text{in}_<(P) \cap \text{in}_<(Q) \) if and only if \( \text{in}_<(P) + \text{in}_<(Q) = \text{in}_<(P + Q) \). The latter equality is equivalent to

\[
\text{in}_<(J_{K_n}) + (\{x_i, y_i\}_{a \leq i \leq b}, \text{in}_<(J_{K[a,a-1]}), \text{in}_<(J_{K[b+1,n]})) = (\{x_i, y_i\}_{a \leq i \leq b}) + \text{in}_<(J_{[n]\setminus[a,b]}).
\]

But this is obviously true.
Hence, we also have the following exact sequence:

\[(8) \quad 0 \to S/\text{in}_<(J_G) \to S/\text{in}_<(P) \bigoplus S/\text{in}_<(Q) \to S/\text{in}_<(P+Q) \to 0.\]

As in the case of \(J_G\), we can consider the following exact sequence of Tor for \(\text{in}_<(J_G)\) which follows from (8):

\[\cdots \to \text{Tor}_{i+2}(\frac{S}{\text{in}_<(J_G)}) \to (\text{Tor}_{i+2}(\frac{S}{\text{in}_<(P)}) \bigoplus \text{Tor}_{i+2}(\frac{S}{\text{in}_<(Q)})) \to \text{Tor}_{i+2}(\frac{S}{\text{in}_<(P+Q)}) \to \cdots\]

With the same arguments as in the case of \(J_G\), we obtained the equality:

\[(9) \quad -\beta_{i+1,i+2}(\frac{S}{\text{in}_<(Q)}) + \beta_{i+1,i+2}(\frac{S}{\text{in}_<(P)}) - \beta_{i+1,i+2}(\frac{S}{\text{in}_<(J_G)}) = 0.\]

Compare equations (7) and (9). We know, from Theorem 2.4, that

\[\beta_{i+1,i+2}(\frac{S}{J_G}) = \beta_{i+1,i+2}(\frac{S}{\text{in}_<(J_G)}).\]

We also know that

\[\beta_{i+1,i+2}(\frac{S}{P}) = \beta_{i+1,i+2}(\frac{S}{\text{in}_<(P)}),\]

since \(P = J_{\mathbb{K}_n}\). In addition, we have

\[\beta_{i+1,i+2}(\frac{S}{Q}) = \beta_{i+1,i+2}(\frac{S}{\text{in}_<(Q)}),\]

\[\beta_{i+1,i+2}(\frac{S}{Q}) = \beta_{i+1,i+2}(\frac{S}{\text{in}_<(Q)}),\]

and

\[\beta_{i,i+2}(\frac{S}{Q}) = \beta_{i,i+2}(\frac{S}{\text{in}_<(Q)}),\]

due the particular form of the ideal \(Q\).

Finally, we also have the following equalities:

\[\beta_{i+2,i+2}(\frac{S}{P+Q}) = \beta_{i+2,i+2}(\frac{S}{\text{in}_<(P+Q)})\]

and

\[\beta_{i+1,i+2}(\frac{S}{P+Q}) = \beta_{i+1,i+2}(\frac{S}{\text{in}_<(P+Q)})\]

due the particular form of the ideal \(P+Q\).

Therefore, \(\beta_{i,i+2}(\frac{S}{J_G}) = \beta_{i,i+2}(\frac{S}{\text{in}_<(J_G)})\).\]
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