ON THE UNIQUENESS OF CLASSICAL SOLUTIONS OF CAUCHY PROBLEMS

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Abstract. Given that the terminal condition is of at most linear growth, it is well known that a Cauchy problem admits a unique classical solution when the coefficient multiplying the second derivative is also a function of at most linear growth. In this note, we give a condition on the volatility that is necessary and sufficient for a Cauchy problem to admit a unique solution.

Key Words and Phrases: Cauchy problem, a necessary and sufficient condition for uniqueness, European call-type options, Strict local martingales.

1. Main Result

Given a terminal-boundary data $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $g(x) \leq C(1 + x)$ for some constant $C > 0$, we consider the following Cauchy problem

$$
\begin{align*}
    u_t + \frac{1}{2} \sigma^2(x) u_{xx} &= 0, \quad (x, t) \in (0, \infty) \times [0, T), \\
    u(0, t) &= g(0), \quad t \leq T, \\
    u(x, T) &= g(x),
\end{align*}
$$

(1)

where $\sigma \neq 0$ on $(0, \infty)$, $\sigma^{-2} \in L^1_{loc}(0, \infty)$ (i.e., $\int_a^b \sigma^{-2}(x)dx < \infty$ for any $[a, b] \subset (0, \infty)$), and $\sigma = 0$ on $(-\infty, 0]$.

A solution $u : \mathbb{R}_+ \times [0, T] \mapsto \mathbb{R}$ of (1) is said to be a classical solution if $u \in C^{2,1}((0, \infty) \times [0, T))$. A function $f : \mathbb{R}_+ \times [0, T] \mapsto \mathbb{R}$ is said to be of at most linear growth if there exists a constant $C > 0$ such that $|f(x, t)| \leq C(1 + x)$ for any $(x, t) \in \mathbb{R}_+ \times [0, T]$.

A well-known sufficient condition for (1) to have a unique classical solution among the functions with at most linear growth is that $\sigma$ itself is of at most linear growth; see e.g. Chapter 6 of [8] and Theorem 7.6 on page 366 of [10]. On the other hand, consider the SDE

$$
    dX^{t,x}_s = \sigma(X^{t,x}_s) dW_s, \quad X^{t,x}_t = x > 0.
$$

(2)

The assumptions on $\sigma$ we made below (1) ensure that (2) has a unique weak solution which is absorbed at zero. (See [6].) The solution $X^{t,x}$ is clearly a local martingale. Delbaen and Shirakawa

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shows in [4] that $X^{t,x}$ is a martingale if and only if
\[ \int_1^\infty \frac{x}{\sigma^2(x)} \, dx = \infty. \] (3)

Also see [2].

Below, in Theorem 2, we prove that (3) is also necessary and sufficient for the existence of a unique classical solution of (1). First, in the next theorem, we show that (3), which is weaker than the linear growth condition on $\sigma$, is a sufficient condition for the uniqueness.

**Theorem 1.** The Cauchy problem (1) has a unique classical solution (if any) in the class of functions with at most linear growth if (3) is satisfied.

**Proof.** It suffices to show that $u \equiv 0$ is the unique solution of the Cauchy problem in (1) with $g \equiv 0$. Let us define a sequence of stopping times $\tau_n \triangleq \inf\{s \geq t : X^{t,x}_s \geq n \text{ or } X^{t,x}_s \leq 1/n\} \wedge T$ for each $n \in \mathbb{N}_+$ and $\tau_0 \triangleq \{s \geq t : X^{t,x}_s = 0\} \wedge T$. Then as in the proof of Theorem 1.6 in [4] we can show that the function defined by
\[
\Psi(x) = \begin{cases} 
    x, & x \leq 1; \\
    x + \int_1^x \frac{n}{\sigma^2(u)}(x - u) \, du, & x \geq 1.
\end{cases}
\]
satisfies $\mathbb{E}[\Psi(X^{t,x}_{\tau_n})] \leq \Psi(x) + xT/2$. Since $\Psi$ is convex, (3) implies that $\lim_{x \to \infty} \Psi(x)/x = \infty$. Then the criterion of de la Vallée Poussin implies that $\{X^{t,x}_{\tau_n} : n \in \mathbb{N}\}$ is a uniformly integrable family.

Suppose $\tilde{u}$ is another classical solution of at most linear growth. Applying the Itô’s lemma, we obtain
\[
\tilde{u}(X^{t,x}_{\tau_n}, \tau_n) = \tilde{u}(x, t) + \int_t^{\tau_n} \left[ \tilde{u}_s(X^{t,x}_s, s) + \frac{1}{2} \sigma^2(X^{t,x}_s) \tilde{u}_{xx}(X^{t,x}_s, s) \right] \, ds + \int_t^{\tau_n} \tilde{u}_x(X^{t,x}_s, s) \sigma(X^{t,x}_s) \, dW_s.
\]
Thanks to our choice of $\tau_n$, the expectation of the stochastic integral is zero. Therefore, taking the expectation of both sides of the above identity, we get $\tilde{u}(x, t) = \mathbb{E}[\tilde{u}(X^{t,x}_{\tau_n}, \tau_n)]$ for each $n \in \mathbb{N}$.

On the other hand, since $\tilde{u}$ is of at most linear growth, there exists a constant $C$ such that $|\tilde{u}(x, t)| \leq C(1 + x)$. Therefore $\{\tilde{u}(X^{t,x}_{\tau_n}, \tau_n) : n \in \mathbb{N}_+\}$ is a uniformly integrable family. This is because it is bounded from above by the uniformly integrable family $\{C(1 + X^{t,x}_{\tau_n}) : n \in \mathbb{N}_+\}$. As a result,
\[
\tilde{u}(x, t) = \lim_{n \to \infty} \mathbb{E}[\tilde{u}(X^{t,x}_{\tau_n}, \tau_n)] = \mathbb{E}[\lim_{n \to \infty} \tilde{u}(X^{t,x}_{\tau_n}, \tau_n)] = \mathbb{E}[\tilde{u}(X^{t,x}_{\tau_0}, \tau_0)]
\] (4)
\[ = \mathbb{E}[g(X^{t,x}_T)1_{\{\tau_0 = T\}}] + \mathbb{E}[\tilde{u}(X^{t,x}_{\tau_0}, \tau_0)1_{\{\tau_0 < T\}}]
\] (5)
\[ = \mathbb{E}[g(X^{t,x}_T)1_{\{\tau_0 = T\}}] + \mathbb{E}[g(0)1_{\{\tau_0 < T\}}] = 0.
\]
Here the third equality holds since $X^{t,x}$ does not explode (i.e., $\inf\{s \geq t : X^{t,x}_s = +\infty\} = \infty$, see Problem 5.3 in page 332 of [11]) and one before the last equality follows since $X^{t,x}_{\tau_0} = 0$ on the set $\{\tau_0 < T\}$. \qed
Theorem 2. If we further assume that \( \sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is locally Hölder continuous with exponent 1/2 and \( g \) is of linear growth, then the Cauchy problem in (1) has a unique classical solution if and only if (3) is satisfied.

Proof. First let us prove the existence of a solution. Let \( u(x, t) \triangleq \mathbb{E} g(X^{x,t}_T) \) (the value of a call-type European option). Thanks to the Hölder continuity of \( \sigma \), it follows from Theorem 3.2 in [5] that \( u \) is a classical solution of (1). Moreover, it is of at most linear growth due to the assumption that \( g \) is of at most linear growth.

Proof of sufficiency. This follows from Theorem 1.

Proof of necessity. If (3) is violated, then \( X \) is a strict local martingale (see [4] and [2]). Using Theorem 3.2 in [5], it can be seen that \( u^*(x, t) \triangleq x - \mathbb{E}[X^{x,t}_T] > 0 \) is a classical solution of (1) with zero boundary and terminal conditions. (Note that the Hölder continuity assumption on \( \sigma \) is used in this step as well.) This function clearly has at most linear growth. Therefore \( u + \lambda u^* \), for any \( \lambda \in \mathbb{R} \), is also a classical solution of (1) which is of at most linear growth.

A related result is given by Theorem 4.3 of [5] on put-type European options: When \( g \) is of strictly sublinear growth (i.e., \( \lim_{x \to \infty} g(x)/x = 0 \)) then (1) has a unique solution among the functions with strictly sublinear growth (without assuming (3)).

Our result in Theorem 2 complements Theorem 3.2 of [5], which shows that the call-type European option price is a classical solution of (1) of at most linear growth. We prove that (3) is necessary and sufficient to guarantee that the European option price is the only classical solution (of at most linear growth) to this Cauchy problem. [3] and [9] had already observed that the Cauchy problem corresponding to European call options have multiple solutions. (Also see [7] and [1], which consider super hedging prices of call-type options when there are no equivalent local martingale measures.) However, a necessary and sufficient condition under which there is uniqueness/nonuniqueness remained unknown.

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