HARBINGERS OF ARTIN’S RECIPROCITY LAW.
II. IRREDUCIBILITY OF CYCLOMATIC POLYNOMIALS

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In [8], we have presented the history of auxiliary primes from Legendre’s proof of the quadratic reciprocity law up to Artin’s reciprocity law. We have also seen that the proof of Artin’s reciprocity law consists of several steps, the first of which is the verification of the reciprocity law for cyclotomic extensions of \( \mathbb{Q} \). In this article we will show that this step can be identified with one of Dedekind’s proofs of the irreducibility of the cyclotomic polynomial.

7. Irreducibility of Cyclotomic Polynomials

Let \( \zeta \) be a primitive \( m \)-th root of unity, and let \( \mu_m = \langle \zeta \rangle \) denote the group of \( m \)-th roots of unity. The polynomial \( \Phi_m(X) = \prod_{(r,m) = 1} (X - \zeta^r) \) is called the \( m \)-th cyclotomic polynomial. Since every automorphism of \( \mathbb{Q}/\mathbb{Q} \) maps a primitive \( m \)-th root of unity to another primitive \( m \)-th root of unity, the coefficients of \( \Phi_m(X) \) must be rational numbers, and actually are integers since \( \zeta \) is an algebraic integer.

The first proofs of the irreducibility of the cyclotomic polynomial were obtained by Gauss, Eisenstein, Arndt, and Kronecker. Dedekind gave three proofs: [3] was published in 1857, his second proof was contained in his review of Bachmann’s [2], and the last one [4] in 1894. If Dedekind published a new proof of a classical result then he did so because he thought that the new proof was superior to the known proofs. In fact, his third proof will turn out to be a small first step in Artin’s and Tate’s proof of Artin’s reciprocity law. In this section we will go through Dedekind’s third proof; we have updated the form, but not the content, of Dedekind’s results.

Now fix a primitive \( m \)-th root of unity \( \zeta \), and assume that \( \Phi_m(X) \) is reducible over \( \mathbb{Q} \). Then \( \zeta \) is a root of one of the factors of \( \Phi_m(X) \), and if we let \( f \) denote the minimal polynomial of \( \zeta \) we can write \( \Phi_m(X) = f(X)g(X) \) for polynomials \( f, g \in \mathbb{Q}[X] \) with \( f(\zeta) = 0 \).

Let \( L = \mathbb{Q}(\mu_m) \) denote the splitting field of \( \Phi_m(X) \) and \( K = \mathbb{Q}(\zeta) \) the splitting field of \( f \); observe that \( L \) contains all \( m \)-th roots of unity, and \( K \) only those conjugate to \( \zeta \). In particular, we have \( K \subseteq L, (K : \mathbb{Q}) = \deg f \) and \( (L : \mathbb{Q}) = \phi(m) \).

Thus \( \Phi_m(X) \) is irreducible if and only if \( (K : \mathbb{Q}) \geq \phi(m) \). The following lemma collects other ways of expressing this fact:

**Lemma 7.1.** The following assertions are equivalent:

1. \( \Phi_m(X) \) is irreducible over \( \mathbb{Q} \);
2. \( (K : \mathbb{Q}) \geq \phi(m) \) for \( K = \mathbb{Q}(\zeta) \);
3. \( K = L \), where \( L \) is the splitting field of \( \Phi_m(X) \);
4. for every \( r \) coprime to \( m \), the element \( \zeta^r \) is a conjugate of \( \zeta \);
5. We have \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \{ \sigma_r : \gcd(r,m) = 1 \} \), where \( \sigma_r(\zeta) = \zeta^r \).

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In both of Dedekind’s proofs of the irreducibility of $\Phi_m(X)$ it is shown that (4) holds. The two main ingredients of Dedekind’s third proof are the following two lemmas:

**Lemma 7.2.** Let $F$ be a number field containing $\mu_m$, and let $p$ be a prime ideal in $\mathcal{O}_F$ coprime to $m$. Then the natural projection $\pi : \mu_m \rightarrow (\mathcal{O}_F/p)^\times$ is injective.

The proof of Lemma 7.2 will be given below; Dedekind’s formulation of this result was the following:

Let $p$ be a prime ideal in a number field $\Omega$, let $p'$ be the prime number divisible by $p$, and set $N_p = p f$, where $f$ is the inertia degree of $p$. If $\alpha$ is a primitive $m$-th root of unity in $\Omega$, and if we set $m' = m p'$, where $p'$ is the highest power of $p$ dividing $m$, then $\alpha$ belongs to the exponent $m' \text{ mod } p$. In the case we are interested in we have $p \nmid m$, hence $m' = m$; the exponent to which a root of unity $\zeta$ belongs mod $p$ is the smallest positive integer $e$ such that $\zeta^e \equiv 1 \text{ mod } p$. Thus Dedekind claims that the order of $\zeta$ and that of $\zeta \text{ mod } p$ coincide, which is exactly the content of Lemma 7.2 above.

The next lemma provides us with the existence of the Frobenius automorphism $^1$, which is a well known result from the Galois theory of normal extensions of number fields:

**Lemma 7.3.** Let $L/K$ be a normal extension of number fields, $p$ a nonzero prime ideal in $\mathcal{O}_K$ that does not ramify in $L/K$, and $\mathfrak{P}$ a prime ideal in $\mathcal{O}_L$ above $p$. The residue class fields $\mathbb{F} = \mathcal{O}_K/p$ and $\mathbb{L} = \mathcal{O}_L/\mathfrak{P}$ are finite fields, and the natural projection

$Z(\mathfrak{P}|p) \rightarrow \text{Gal}(\mathbb{L}/\mathbb{F})$

from the decomposition group $Z(\mathfrak{P}|p)$ to the Galois group of the (cyclic) extension $\mathbb{L}/\mathbb{F}$ of finite fields is an isomorphism. The unique preimage of the Frobenius automorphism of $\mathbb{F}$ is called the Frobenius automorphism of $p$. This Frobenius automorphism $\sigma$ is characterized by the property that $\sigma(\alpha) \equiv \alpha^p \text{ mod } p$ for every $\alpha \in \mathcal{O}_K$.

Now we claim

**Proposition 7.4.** The natural homomorphism $\phi : \text{Gal}(K/Q) \rightarrow (\mathbb{Z}/m\mathbb{Z})^\times$ sending $\sigma \in \text{Gal}(K/Q)$ to the residue class of $r \text{ mod } m$ determined by $\sigma(\zeta) = \zeta^r$ is surjective.

Since $\text{Gal}(K/Q)$ has order $(K : Q)$ (recall that $K$ is the splitting field of the factor $f$ of $\Phi_m$, hence normal), this implies that $\Phi_m(X)$ is irreducible over $Q$.

**Proof of Prop. 7.4.** Since the coprime residue classes modulo $m$ are generated by classes represented by primes $p$ not dividing $m$, it is more than enough to show that the residue classes generated by these $p$ are in the image (this reduction can

$^1$In a letter to Dedekind, probably written on the first days of June 1882, Frobenius asked whether Dedekind knew a certain result concerning the Galois group of an extension of number fields $K/Q$ and the decomposition of prime ideals in $K$. Dedekind replied on June 8 that he knew this result, and Frobenius gave his proof of the existence of the Frobenius substitution in his answer. Dedekind explained his own proof in his letter dated June 14, and this version was used by Frobenius in his publication [7]. For the relevant parts of these letters see [9].
already be found in Dedekind’s first proof in [3], as well as in van der Waerden’s Algebra).

Let \( p \) denote a prime number coprime to \( m \). We have to show that there is a \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) with \( \sigma(\zeta) = \zeta^p \). Let \( \sigma \) be the Frobenius automorphism for \( p \); then \( \sigma(\alpha) \equiv \alpha^p \mod p \). Applying this to \( \zeta \) we find \( \sigma(\zeta) \equiv \zeta^p \mod p \). Since \( \sigma(\zeta) \) must be a root of \( \Phi_m(X) \), we can write \( \sigma(\zeta) = \zeta^s \) for some integer \( s \) with \( 1 \leq s < m \); then \( \zeta_s \equiv \zeta^p \mod p \), and Lemma 7.5 implies that \( s = p \), that is, \( \sigma(\zeta) = \zeta^p \). \( \square \)

Observe that we did not assume the existence of any primes lying in certain residue classes modulo \( m \), but rather showed that if there is a prime in some residue class, then it is the image of its Frobenius automorphism.

For proving Lemma 7.2, Dedekind uses the prime ideal factorization of the element \( 1 - \zeta \) in \( \mathbb{Q}(\mu_m) \):

Lemma 7.5. In the splitting field \( \mathbb{Q}(\mu_m) \) of \( \Phi_m(X) \), we have

1. \( \zeta - 1 = 0 \) if \( m = 1 \).
2. \( (\zeta - 1)^{\phi(m)} = \varepsilon^p \) if \( m \) is a power of \( p \), where \( \varepsilon \) is a unit; moreover, if \( \gcd(r, p) = \gcd(s, p) = 1 \), then \( \frac{\zeta^r - 1}{\zeta - 1} \) is a unit.
3. \( \zeta - 1 = \varepsilon \) if \( m \) is divisible by at least two distinct primes.

These are simple properties of cyclotomic fields whose proofs can be found in most textbooks on algebraic number theory. In order to convince ourselves that Dedekind is not using the irreducibility of \( \Phi_m(X) \) along the way, let us derive these results here:

Proof of Lemma 7.5. Property (1) is trivial. For a proof of (2), let \( r \) run through all \( \phi(m) \) coprime residue classes and set \( rr' \equiv 1 \mod m \). The elements

\[
\frac{\zeta^r - 1}{\zeta - 1} \quad \text{and} \quad \frac{\zeta - 1}{\zeta' - 1} = \frac{\zeta^{rr'} - 1}{\zeta^r - 1}
\]

are clearly integral, hence are units in the splitting field \( L \) of \( \Phi_m(X) \). Moreover, we have

\[
\Phi_m(X) = \frac{X^m - 1}{X^{m/p} - 1} = \prod(X - \zeta^r),
\]

where the product is over all \( 1 \leq r < m = p^n \) with \( \gcd(r, m) = 1 \). Writing \( X^m - 1 = \prod_{r=0}^{m-1} (X - \zeta^r) \), hence \( (\zeta - 1)^{\phi(m)} = \varepsilon^p \) for some unit \( \varepsilon \in \mathcal{O}_L \).

Next, since both \( \frac{\zeta^r - 1}{\zeta - 1} \) and \( \frac{\zeta - 1}{\zeta' - 1} \) are units whenever \( r \) and \( s \) are integers coprime to \( m \), so is their quotient \( \frac{\zeta^r - 1}{\zeta - 1} \).

If finally \( m = pqn \) is divisible by two distinct primes \( p \) and \( q \), then \( \zeta - 1 \) is a common divisor of \( \zeta^p - 1 = \zeta_p - 1 \) and \( \zeta^q - 1 = \zeta_q - 1 \), hence a common divisor of \( p = N(\zeta_p - 1) \) and \( q = N(\zeta_q - 1) \), and thus a unit. \( \square \)

Dedekind uses Lemma 7.5 to give the following

Proof of Lemma 7.2. Let \( f \geq 1 \) be the minimal integer with \( \zeta^f \in \ker \pi \), that is, with \( \zeta^f - 1 \in \mathfrak{p} \). Then \( \zeta^f \) is a primitive \( n \)-th root of unity for some \( n \mid m \), and there are several cases:

1. \( n \) is divisible by two distinct primes; then the prime ideal \( \mathfrak{p} \) contains the unit \( \zeta^f - 1 \): contradiction.
(2) $n$ is a power of a prime $p$; then $\zeta^f - 1 \in p$, where $p$ is a prime ideal above $p \mid m$ contradicting our assumption.

(3) $n = 1$: then $f = m$, hence $\ker \pi = 1$, and this is what we wanted to prove.

\[\square\]

An Irreducibility Proof based on Lemma 7.5. The irreducibility of the cyclotomic polynomial $\Phi_m(X)$ for prime powers $m$ follows from part (2) of Lemma 7.5: the equation $p = \varepsilon^{-1}(\zeta - 1)^{\phi(m)}$ is valid in $L$, but since $p$ and $\zeta$ are in $K$, so is $\varepsilon$; thus $(p)$ is the $\phi(m)$-th power of the ideal $(\zeta - 1)$ in $K$, and we must have $(K : Q) \geq \phi(m)$.

This can easily be extended to a proof of the irreducibility of $\Phi_m(X)$ in the general case: we have to show that $(Q(\zeta_m) : Q) \geq \phi(m)$. The idea of the proof becomes clear enough by treating the case where $m = PQ$ is a product of two prime powers $P$ and $Q$. Then $Q(\zeta_m)$ is the compositum of the fields $Q(\zeta_P)$ and $Q(\zeta_Q)$, which have degrees $\phi(P)$ and $\phi(Q)$, respectively. Since the fields $Q(\zeta_P)$ and $Q(\zeta_Q)$ are independent ($p$ is fully ramified in $Q(\zeta_P)$ and unramified in $Q(\zeta_Q)$), we must have $(Q(\zeta_m) : Q) = (Q(\zeta_P) : Q)(Q(\zeta_Q) : Q) = \phi(P)\phi(Q) = \phi(m)$.

The fact that Dedekind chose not to give this proof does not mean that he did not see it: Dedekind avoided artificial tricks and shortcuts whenever he thought that this would make the proof less conceptional. In the case of the irreducibility of $\Phi_m(X)$ he also wanted to avoid arguments that do not carry over to the theory of complex multiplication: there one had to prove the irreducibility of certain polynomials (coming from the theory of elliptic curves and modular functions) whose roots generate abelian extensions of complex quadratic number fields.

8. The Proof of Artin’s Reciprocity Law.

The proof of Artin’s reciprocity law is quite involved; the first (essential) step is verifying the reciprocity law for cyclotomic extensions of $Q$; this step (see e.g. Artin & Tate [1, Lemma 1, p. 42]) is nothing but Dedekind’s proof of the irreducibility of the cyclotomic polynomial discussed above:

Lemma 8.1. Let $\zeta$ be a primitive $m$-th root of unity. An automorphism of $Q(\zeta)/Q$ sends $\zeta$ into a power $\zeta^n$ where $n$ is coprime to $m$. Conversely, to any given $n$ prime to $m$ there is an automorphism $\sigma$ such that $\zeta^n = \zeta^m$.

Proof. The first part of the lemma is trivial. As for the second part, it suffices to prove the statement if $n$ is a prime $p$ that does not divide $m$. Recall that $\zeta$ satisfies the equation $x^m - 1 = f(x) = 0$, and $f'(\zeta) = m\zeta^{m-1}$ is prime to $p$. The local field $Q_p(\zeta)/Q_p$ is therefore unramified. The Frobenius substitution sends $\zeta$ into an $m$-th root of unity that is congruent to $\zeta^p$. Since $f'(\zeta) = \prod_{\nu}(\zeta - \zeta^\nu)$ is prime to $p$ it follows that no two $m$-th roots of unity are in the same residue class; $\zeta^p$ is therefore the image of $\zeta$ under the Frobenius substitution. This automorphism of the local field is induced by an automorphism of the global field $Q(\zeta)/Q$ and this proves the lemma. \[\square\]

In this proof, Dedekind’s Lemma 7.2 is condensed into the observation that, for $f(x) = x^m - 1$, the element $f'(\zeta) = \prod_{\nu \neq 1}(\zeta - \zeta^\nu)$ is coprime to $p$. 
Emmy Noether’s Comments. The fact that Dedekind’s proof of the irreducibility of the cyclotomic equation is connected with Artin’s reciprocity law was first noticed by Emmy Noether. In her comments on Dedekind’s proof in his Collected Works she writes:

The proof of the irreducibility of the cyclotomic equation given here is based on the following two facts:

1. A primitive $m$-th root of unity remains primitive modulo each prime ideal $p$ not dividing $m$ in the field of $m$-th roots of unity.

2. There is a substitution $\psi_0$ of the decomposition group of $p$ for which $\psi_0(\omega) \equiv \omega^p \mod p$ for every integral $\omega$ in $K$.

The map given in (2) from the Galois group and the class group – from which irreducibility follows immediately – is the map given by Artin’s reciprocity law, but in a very weak and therefore elementary form.

In fact, Prop. 7.4 gives an isomorphism $\text{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. Composed with the isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \simeq \text{Cl}_\mathbb{Q}\{m\infty\}$ this gives a canonical isomorphism between $\text{Cl}_\mathbb{Q}\{m\infty\}$ and $\text{Gal}(K/\mathbb{Q})$.

Finally let us remark that the “small piece of Artin’s Reciprocity Law for cyclotomic extensions”, which, according to [6, p. 54], is contained in Chebotarev’s article [9, p. 200], is based on the irreducibility of the cyclotomic equation, which Tschebotareff takes for granted.