HOMOCLINIC ORBITS AND ENTROPY FOR THREE-DIMENSIONAL FLOWS

A.M. LOPEZ, R.J. METZGER, C.A. MORALES

Abstract. We prove that every $C^1$ three-dimensional flow with positive topological entropy can be $C^1$ approximated by flows with homoclinic orbits. This extends a previous result for $C^1$ surface diffeomorphisms [7].

1. Introduction

In his classical paper [6] Katok proved that every $C^{1+\alpha}$ surface diffeomorphism with positive topological entropy has a homoclinic orbit. In [7] Gan asked if this result is true for $C^1$ surface diffeomorphisms too. He didn’t answer this question but managed to prove that every $C^1$ surface diffeomorphism with positive entropy can be $C^1$ approximated by diffeomorphisms with homoclinic orbits. More recently, the authors [4] proved that every three-dimensional flow can be $C^1$ approximated by Morse-Smale flows or by flows with a homoclinic orbit (this entails the weak Palis conjecture for three-dimensional flows). From this they deduced that there is an open and dense subset of three-dimensional flows where the property of having zero topological entropy is invariant under topological equivalence. Moreover, the $C^1$ approximation by three-dimensional flows with robustly zero topological entropy is equivalent to the $C^1$ approximation by Morse-Smale ones.

In this paper we will extend [7] from surface diffeomorphisms to three-dimensional flows. In other words, we will prove that every $C^1$ three-dimensional flow with positive topological entropy can be $C^1$ approximated by flows with homoclinic orbits. Let us state our result in a precise way.

The term flow will be referred to $C^1$ vector fields $X$ defined on a compact connected boundaryless Riemannian manifold $M$. To emphasize its differentiability we say that $X$ is a $C^r$ flow, $r \in \mathbb{N}^+$. When $\dim(M) = 3$ we say that $X$ is a three-dimensional flow. The flow of $X$ will be denoted by $\phi_t$ (or $\phi_t^X$ to emphasize $X$), $t \in \mathbb{R}$. We denote by $\Phi_t = \Phi_t^X$ the derivative of $\phi_t$. The space of $C^r$ flows $\mathcal{X}^r$ is endowed with the standard $C^r$ topology. We say that $x \in M$ is a periodic point of a flow $X$ if there is a minimal positive number $\pi(x)$ (called period) such that $\phi_{\pi(x)}(x) = x$. Notice that 1 is always an eigenvalue of the derivative $DX(\pi(x))$ with eigenvector $X(x)$. The remainder eigenvalues will be referred to as the eigenvalues of $x$. We say that the orbit $O(x) = \{X_t(x) : t \in \mathbb{R}\}$ of a periodic point $x$ (or the periodic point $x$) is hyperbolic if it has no eigenvalue of modulus 1. In case there are eigenvalues of modulus less and bigger than 1 we say that the hyperbolic periodic point is a saddle.

2010 Mathematics Subject Classification. Primary 37D25; Secondary 37C40.
Key words and phrases. Hyperbolic ergodic measure, Lyapunov exponents, Flow.
Partially supported by MATHAMSUB 15 MATH05-ERGOPTIM, Ergodic Optimization of Lyapunov Exponents.
The Invariant Manifold Theory [5] asserts that through any periodic saddle \( x \) it passes that through any periodic saddle \( x \) it
passes through invariant manifolds, the so-called strong stable and unstable manifolds \( W^{ss}(x) \) and \( W^{uu}(x) \), tangent at \( x \) to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating them with the flow we obtain the stable and unstable manifolds \( W^s(x) \) and \( W^u(x) \) respectively. We say that \( O \) is a homoclinic orbit (associated to a periodic saddle \( x \)) if \( O \in W^s(x) \cap W^u(x) \setminus O(x) \).

If, additionally, \( \dim(T_q W^s(x) \cap T_q W^u(x)) \neq 1 \) then we say that \( O \) is a homoclinic tangency.

We say that \( E \subset X \) is \((T, \epsilon)-\text{separated}\) for some \( T, \epsilon > 0 \) if for any distinct point \( x, y \in E \) there exists \( 0 \leq t \leq T \) such that \( d(X_t(x), X_t(y)) > \epsilon \). The number

\[
    h(X) = \lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \log \sup \{ \text{Car}(E) : E \text{ is } (T, \epsilon)\text{-separated} \}
\]

is the so-called topological entropy of \( X \). With these definitions we can state our result.

**Theorem 1.** Every \( C^1 \) three-dimensional flow with positive topological entropy can be \( C^1 \) approximated by flows with homoclinic orbits.

The proof follows Gan’s arguments [7] using the variational principle (e.g. [2]) and Ruelle’s inequality. But we simplify such arguments by using recent tools as a flow-version of a result by Crovisier [3] and Gan-Yang [4].

Denote by \( \text{Cl}(\cdot) \) and \( \text{int}(\cdot) \) the closure and interior operations. As in [7] we get from Theorem 1 the following corollary.

**Corollary 2.** If \( H_+ = \{ X \in X^1 : h(X) > 0 \} \), then \( \text{Cl}(\text{int}(H_+)) = H_+ \) and so \( H_+ \) has no isolated points.

**2. Proof of Theorem 1**

Denote by \( \text{Sing}(X) \) the set of singularities of a flow \( X \). Given \( \Lambda \subset M \), we denote by \( \Lambda^* = \Lambda \setminus \text{Sing}(X) \) the set of regular points of a flow \( X \) in \( \Lambda \). Define by \( E^X \) the map assigning to \( p \in M \) the subspace of \( T_p M \) generated by \( X(p) \). It turns out to be a one-dimensional subbundle of \( TM \) when restricted to \( M^* \). Define also the normal subbundle \( N \) over \( M^* \) whose fiber \( N_p \) at \( p \in M^* \) is the orthogonal complement of \( E^X_p \) in \( T_p M \). Denoting by \( \pi = \pi_p : T_p M \to N_p \) the orthogonal projection we obtain the linear Poincaré flow \( \psi_t : N \to N \) defined by \( \psi_t(p) = \pi_{\phi_t(p)} \circ \Phi_t(p) \). When necessary we will use the notation \( N^X \) and \( \psi_t^X \) to indicate the dependence on \( X \).

For a (nonnecessarily compact) invariant set \( \Omega \subset M^* \), one says that \( \Omega \) has a dominated splitting with respect to the Poincaré flow if there are a continuous splitting \( N_\Omega = N^- \oplus N^+ \) into \( \psi_t \)-invariant subbundles \( N^- \), \( N^+ \) and positive numbers \( K, \lambda \) such that

\[
    \| \psi_t \|_{N^-} \cdot \| \psi_t \|_{N^+} \leq K e^{-\lambda t}, \quad \forall x \in \Omega, t \geq 0.
\]

Let \( \mu \) be a Borel probability measure of \( M \). We say that \( \mu \) is nonatomic if it has no points with positive mass. We say that \( \mu \) is supported on \( H \subset M \) if \( \text{supp}(\mu) \subset H \), where \( \text{supp}(\mu) \) denotes the support of \( \mu \). We say that \( \mu \) is invariant if \( \mu(X_t(A)) = \mu(A) \) for every borelian \( A \) and every \( t \in \mathbb{R} \). Moreover, \( \mu \) is ergodic if it is invariant and every measurable invariant set has measure 0 or 1.

Oseledets’s Theorem [10] ensures that every ergodic measure \( \mu \) is equipped with an invariant set of full measure \( R \), a positive integer \( k \), real numbers \( \chi_1 < \chi_2 < \cdots < \chi_k \)
Proposition 4. For every flow, every hyperbolic ergodic measure whose Oseledets decomposition is dominated with respect to the Poincaré flow is supported on a homoclinic class.

Proof. Let $\mu$ be a hyperbolic ergodic measure of a flow $X$. Suppose that the Oseledets decomposition of $\mu$ is dominated with respect to the linear Poincaré flow. By Lemma 3, there are $\eta, T > 0$ such that $\mu$ is ergodic for $\phi_T^X$, 
\[
\int \log \|\psi_T|_{N^s}\| d\mu \leq -\eta \quad \text{and} \quad \int \log \|\psi_{-T}|_{N^u}\| d\mu \leq -\eta.
\]

We shall use the following lemma.

Lemma 3. Let $\mu$ be a hyperbolic ergodic measure of a flow $X$ whose Oseledets decomposition is dominated with respect to the Poincaré flow. Then, there are $\eta, T > 0$ such that $\mu$ is ergodic for $\phi_T^X$, 
\[
\int \log \|\psi_T|_{N^s}\| d\mu \leq -\eta \quad \text{and} \quad \int \log \|\psi_{-T}|_{N^s}\| d\mu \leq -\eta.
\]

Proof. It follows from the hypothesis that $\mu(\text{Sing}(X)) = 0$. On the other hand, $\mu$ is ergodic for $X$ so there is $T_1 > 0$ such that $\mu$ is totally ergodic for $\phi_{T_1}$ (c.f. [9]). Since $\mu$ is hyperbolic, there is $\eta_0 > 0$ such that any Lyapunov exponent off the flow direction belongs to $\mathbb{R} \setminus [-\eta_0, \eta_0]$. From this and the Furstenberg-Kesten Theorem (see also p. 150 in [12]) we obtain 
\[
\lim_{n \to \infty} \frac{1}{n} \int \log \|\psi_{nT_1}|_{N^s}\| d\mu \leq -\eta_0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \int \log \|\psi_{-nT_1}|_{N^s}\| d\mu \leq -\eta_0,
\]

for $\mu$-a.e. $x \in M$. Hence
\[
\lim_{n \to \infty} \frac{1}{n} \int \log \|\psi_{nT_1}|_{N^s}\| d\mu \leq -\eta_0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \int \log \|\psi_{-nT_1}|_{N^s}\| d\mu \leq -\eta_0
\]

by the Dominated Convergence Theorem. Now take $T = nT_1$ and $\eta = n \frac{\eta_0}{2}$ with $n$ large.

Denote by $\text{Cl}(\cdot)$ the closure operation. We say that $H \subset M$ is a homoclinic class of $X$ if there is a periodic saddle $x$ such that
\[
H = \text{Cl}(\{q \in W^s(x) \cap W^u(x) : \dim(T_qW^s(x) \cap T_qW^u(x)) = 1\}).
\]

A homoclinic class is nontrivial if it does not reduce to a single periodic orbit.

The following is the flow-version of Proposition 1.4 in [3].

Proposition 4. For every flow, every hyperbolic ergodic measure whose Oseledets decomposition is dominated with respect to the Poincaré flow is supported on a homoclinic class.

Proof. Let $\mu$ be a hyperbolic ergodic measure of a flow $X$. Suppose that the Oseledets decomposition of $\mu$ is dominated with respect to the linear Poincaré flow. By Lemma 3 there are $\eta, T > 0$ such that $\mu$ is ergodic for $\phi_T^X$, 
\[
\int \log \|\psi_T|_{N^s}\| d\mu \leq -\eta \quad \text{and} \quad \int \log \|\psi_{-T}|_{N^u}\| d\mu \leq -\eta.
\]
It follows from the hypothesis that \( \mu(\text{Sing}(X)) = 0 \). Since \( \mu \) is ergodic, we obtain
\[
\int \log \| \Phi_T \|_{E^X} \, d\mu = 0.
\]
Replacing in the two previous inequalities we obtain
\[
\int \log \| \psi^s_T \|_{N^s} \, d\mu \leq -\eta \quad \text{and} \quad \int \log \| \psi^u_T \|_{N^u} \, d\mu \leq -\eta,
\]
where
\[
\psi^s_t = \frac{\psi_t}{\| \Phi_t(x) \|_{E^X}}, \quad x \in M^s, t \in \mathbb{R}
\]
is the scaled linear Poincaré flow (c.f. [11]).

On the other hand, standard arguments (c.f. [5]) imply that the decomposition \( N_R = N^s \oplus N^u \) (which is dominated for the Poincaré flow by hypothesis) extends continuously to a dominated splitting \( N_{\text{supp}(\mu)} = N^s \oplus N^u \) with respect to the linear Poincaré flow. By Lemma 2.29 in [11] there are a neighborhood \( U \) of \( \text{supp}(\mu) \) and a splitting \( N\Lambda = N^s \oplus N^u \) extending \( N_{\text{supp}(\mu)} = N^s \oplus N^u \) where \( \Lambda = \bigcap_{t \in \mathbb{R}} X_t(U) \).

From this point forward we can reproduce the arguments on p. 214 of [11] to conclude the proof.

Proof of Theorem 4. Let \( X \) be a three-dimensional flow with positive topological entropy. By the variational principle (e.g. [2]) there is an invariant measure \( \mu \) of \( X \) such that \( h_\mu(X_1) > 0 \), where \( h_\mu \) is the metric entropy operation. By the ergodic decomposition theorem we can assume that \( \mu \) is ergodic.

By Ruelle’s inequality (e.g. Theorem 5.1 in [2]) we get that \( \mu \) has at least one positive Lyapunov exponent. By applying this inequality to the reversed flow we obtain that \( \mu \) has also a negative exponent. Since \( \dim(M) = 3 \), we conclude that \( \mu \) is hyperbolic of saddle-type (i.e. with positive and negative exponents).

By the Ergodic Closing Lemma for flows (c.f. Theorem 5.5 in [11]) there are a sequence of flows \( X^n \) and a sequence of hyperbolic periodic orbits \( \gamma_n \) of \( X_n \) such that \( X_n \to X \) and \( \gamma_n \to \text{supp}(\mu) \) as \( n \to \infty \) where the latter convergence is with respect to the Hausdorff topology of compact subsets of \( M \). By passing to a subsequence if necessary we can assume that the index (stable manifold dimension) of these periodic orbits is constant (i.e. say).

Now we assume by contradiction that \( X \) cannot be approximated by flows with homoclinic orbits. Hence \( X \) cannot be approximated by flows with homoclinic tangencies either.

Since \( \dim(M) = 3 \), \( i \) can take the values 0, 1, 2 only. If \( i = 2 \) then each \( \gamma_n \) is an attracting periodic orbit of \( X^n \). Since \( X \) cannot be approximated by flows with homoclinic tangencies, Lemma 2.9 in [11] implies that there is \( T > 0 \) such that
\[
\| \psi_T \|_{N X^n} \leq \frac{1}{2}
\]
for all \( n \in \mathbb{N} \) and all \( x \in \gamma_n \). Letting \( n \to \infty \) we get
\[
\| \psi_T \|_{N x} \leq \frac{1}{2}
\]
for all \( x \in \text{supp}(\mu) \). This would imply that the Lyapunov exponents of \( \mu \) off the flow direction are all negative. Since \( \mu \) is saddle-type, we obtain a contradiction proving \( i \neq 2 \). Similarly, \( i \neq 0 \) and so \( i = 1 \). This allows us to apply Corollary 2.10 in [11] to obtain a dominated splitting \( N_{\text{supp}(\mu)} = N^- \oplus N^+ \) of index 1 (i.e. \( \dim(N^-) = 1 \)) with respect to the Poincaré flow.

Next we observe that both the Oseledets splitting \( N^s \oplus N^u \) for the linear Poincaré flow and the splitting \( N^- \oplus N^+ \) obtained above are pre-dominated of index 1 in the sense of Definition 2.1 in [5]. Since pre-dominated splittings of prescribed index are unique (c.f. Lemma 2.3 in [5]), we get \( N^s \oplus N^u = N^- \oplus N^+ \).
Since $N^- \oplus N^+$ is dominated with respect to the Poincaré flow, the Oseledets decomposition $N^s \oplus N^u$ of $\mu$ is dominated with respect to the linear Poincaré flow either. We conclude that $\mu$ is supported on a homoclinic class by Proposition 4.

Since $\mu$ has positive metric entropy, such a homoclinic class is nontrivial and so $X$ has a homoclinic orbit against the assumption. This contradiction completes the proof of the theorem. □

References

[1] Araújo, V., Pacífico, M.J., Three-dimensional flows. With a foreword by Marcelo Viana. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 53. Springer, Heidelberg, 2010.

[2] Bowen, R., Ruelle, D., The ergodic theory of Axiom A flows, Invent. Math. 29 (1975), no. 3, 181–202.

[3] Crovisier, S., Partial hyperbolicity far from homoclinic bifurcations, Adv. Math. 226 (2011), no. 1, 673–726.

[4] Gan, S., Yang, D., Morse-Smale systems and horseshoes for three dimensional singular flows, Preprint arXiv:1302.0946v1 [math.DS] 5 Feb 2013.

[5] Hirsch, M.W., Pugh, C.C., Shub, M., Invariant manifolds, Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[6] Katok, A., Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Inst. Hautes Études Sci. Publ. Math. No. 51 (1980), 137–173.

[7] Gan, S., Horseshoe and entropy for $C^1$ surface diffeomorphisms, Nonlinearity 15 (2002), no. 3, 841–848.

[8] Li, M., Gan, S., Wen, L., Robustly transitive singular sets via approach of an extended linear Poincaré flow, Discrete Contin. Dyn. Syst. 13 (2005), no. 2, 239–269.

[9] Pugh, C., Shub, M., Ergodic elements of ergodic actions, Compositio Math. 23 (1971), 115–122.

[10] Simić, S.N., Oseledets regularity functions for Anosov flows, Comm. Math. Phys. 305 (2011), no. 1, 1–21.

[11] Shi, Y., Gan, S., Wen, L., On the singular-hyperbolicity of star flows, J. Mod. Dyn. 8 (2014), no. 2, 191–219.

[12] Wójcikowski, M., Invariant families of cones and Lyapunov exponents, Ergodic Theory Dynam. Systems 5 (1985), no. 1, 145–161.

Institute of Exact Sciences (ICE), Universidade Federal Rural do Rio de Janeiro, 23890-000 Seropédica, Brazil.

E-mail address: barragan@im.ufrj.br.

Instituto de Matemática y Ciencias Afines (IMCA), Universidad Nacional de Ingeniería, Calle Los Biólogos 245, Urb. San César La Molina Lima 12, Lima, Peru.

E-mail address: metzger@imca.edu.pe.

Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970 Rio de Janeiro, Brazil.

E-mail address: morales@impa.br.