Gravitation and Thermodynamics: The Einstein Equation of State Revisited

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We perform an analysis where Einstein’s field equation is derived by means of very simple thermodynamical arguments. Our derivation is based on a consideration of the properties of a very small, spacelike two-plane in a uniformly accelerating motion.

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I. INTRODUCTION

Ever since the discovery of the Bekenstein-Hawking entropy law, it has become increasingly clear that there is a deep connection between gravitation and thermodynamics (see, for instance, Refs. \cite{1, 2, 3, 4, 5}). However, even today it is not properly understood what exactly this connection may be. The most surprising point of view on these matters was probably provided by Jacobson in 1995, when he discovered that Einstein’s field equation is actually a thermodynamical equation of state of spacetime and matter fields \cite{6}. The key point in his analysis was to require that the first law of thermodynamics, which implies the fundamental thermodynamical relation

\[ \delta Q = T \, dS, \]  \hspace{1cm} (1.1)

holds for all local Rindler horizons, and that the entropy $S$ of a finite part of the Rindler horizon is one-quarter of its area. Jacobson considered an observer very close to his local Rindler horizon (which means that the proper acceleration $a$ of the observer is extremely large). For the temperature $T$ in Eq. (1.1), Jacobson took the Unruh temperature

\[ T_U = \frac{a}{2\pi} \]  \hspace{1cm} (1.2)

experienced by the observer, and the heat flow $\delta Q$ through the past Rindler horizon was defined to be the boost-energy current carried by matter. Jacobson was able to show that, under the assumptions mentioned above, the heat flow through the horizon causes a decrease in the horizon area in such a way that Einstein’s field equation is satisfied. In other words, he was able to derive Einstein’s field equation by assuming the first law of thermodynamics and the proportionality of entropy to the area of the horizon. Viewed in this way, Einstein’s field equation is nothing more than a thermodynamical equation of a state \cite{6, 7, 8}.

The purpose of this paper is to investigate whether there are some other (possibly more general) principles of nature that would imply Einstein’s field equation. Recently, it has been suggested that the concept of gravitational entropy should be extended from horizons to arbitrary spacelike two-surfaces with finite areas \cite{9, 10, 11}. In Ref. \cite{11} it was proposed that an accelerated two-plane may be associated with an entropy which is, in natural units, one-half of the area of that plane. This proposal is, in some sense, related to the well-known result that the entropy associated

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with a spacetime horizon is one-quarter of the area of the horizon. The reason for the difference in the constant of proportionality is still unclear, but it may result from the fact that a spacetime horizon is, according to observers having that surface as a horizon, only a one-sided surface, whereas an accelerated spacelike two-surface has two sides\[12\].

In this paper we shall find that Einstein’s field equation can be derived from a hypothesis which is closely related to this proposal. Our derivation will be based on a consideration of a very small, spacelike two-plane accelerating uniformly in a direction perpendicular to the plane. When the plane moves in spacetime with respect to the matter fields, matter will flow through the plane. Since the matter has, from the point of view of an observer at rest with respect to the plane, a certain non-zero temperature, it also has a certain entropy content. In other words, entropy flows through the plane. Since the plane is in an accelerating motion, the entropy flow through the plane (amount of entropy flown through the plane in unit time) is not constant, but it will change as a function of the proper time of an observer moving along with the plane.

The change in the entropy flow through the plane has two parts. One of these parts is due to the simple fact that the plane moves from one point to another in spacetime, and the entropy densities in the different points of spacetime may be different. This part has nothing to do with the acceleration of the plane. Another part in the change of the entropy flow, however, is caused by the change in the velocity of the plane with respect to the matter fields: When the velocity of the plane with respect to the matter fields changes, so does the entropy flow through the plane. This part in the change of the entropy flow is caused by the acceleration of the plane, and it is this part in the change of the entropy flow, where we shall focus our attention. For the sake of brevity and simplicity we shall call that part as the change in the acceleration entropy flow.

When the accelerating plane moves in curved spacetime, its area may change. More precisely, when the accelerating plane moves in curved spacetime, the world lines of the points of the plane may either approach to each other or recede from each other. To investigate the behaviour of those world lines, we shall consider a congruence of timelike curves with certain specific properties. The physical idea behind our consideration is that when our plane is located at a certain spacetime point \( P \), then in the local neighbourhood of that point the world lines of the points of our accelerating plane are the elements of that congruence. In broad terms, we shall define this congruence in such a way that in the immediate vicinity of the point \( P \) the tangent vectors of the world lines of the points of our plane are parallel to each other, and all points of all elements of the congruence are accelerated with a constant proper acceleration in a direction perpendicular to the plane. Such a definition is consistent with the intuitive picture of the concept of a two-plane moving in spacetime. These ideas will become more precise in the Section III of this paper, when we consider the change of the area of our plane. Because the tangent vectors of the world lines of the points of the plane are defined to be parallel to each other in the instant vicinity of the point \( P \), the proper time derivative of the area \( A \) of the plane will vanish at that point. In other words, if we parametrize the world lines of the points of our plane by means of the proper time \( \tau \) measured along those world lines such that \( \tau = 0 \), when the plane lies at the point \( P \), we must have \( \frac{dA}{d\tau} \big|_{\tau=0} = 0 \). If spacetime is curved, however, \( \frac{dA}{d\tau} \) will become non-zero, when \( \tau > 0 \). From this point on, we shall call the quantity \( -\frac{dA}{d\tau} \) as the shrinking speed of the area of the plane.

Under the assumption that the rate of change in the boost energy flow through the plane is exactly the rate of change in the heat flow, we express the following hypothesis concerning the rates of changes in the acceleration entropy flow through an accelerating plane, and in the shrinking speed of the area of the plane:

*If the temperature of the matter flowing through an accelerating, spacelike two-plane is equal to the Unruh temperature measured by an observer at rest with respect to the plane, then the rate of change in the acceleration entropy flow through the plane is, in natural units, exactly one-half of the rate of change in the shrinking speed of the area of the plane.*

Using this hypothesis, and this hypothesis only, together with Eq. (1.1), we shall obtain Einstein's field equation.
Our hypothesis may be expressed by means of a formula:

$$\left. \frac{d^2 S_a}{d\tau^2} \right|_{\tau=0} = -\frac{1}{2} \left. \frac{d^2 A}{d\tau^2} \right|_{\tau=0},$$

(1.3)

where $\frac{dS_a}{d\tau}$ denotes the acceleration entropy flow, and $-\frac{dA}{d\tau}$ the shrinking speed of the area. Because the Unruh temperature $T_U$ of Eq. (1.2) represents, in some sense, the temperature of spacetime from the point of view of an observer moving with a constant proper acceleration $a$, we may view Eq. (1.3) as an equation which holds, when matter and spacetime are, from the point of view of an accelerating observer, in a thermal equilibrium with each other. When we calculate the rate of change in the acceleration entropy flow through the plane, we must use Eq. (1.1). More precisely, we first calculate the rate of change in the flow of heat through the plane and then, using Eq. (1.1) and identifying $T$ as the Unruh temperature $T_U$ of Eq. (1.2), we calculate the rate of change in the acceleration entropy flow. We have assumed that the rate of change in the boost energy flow through our accelerating plane is exactly the rate of change in the heat flow for the simple reason that it makes the calculation of the flow of entropy very easy: We just calculate the rate of change in the boost energy flow, and then use Eq. (1.1). If there were other forms of energy, except heat, flowing through our plane, it would not be quite clear what we actually mean by the concept of entropy flow, and our analysis would become much more complicated. It is most gratifying that Einstein’s field equation follows from our hypothesis even with this rather restrictive assumption, regardless of what kind of matter we happen to have.

It is important to note that our hypothesis contained in Eq. (1.3) involves second proper time derivatives only. The reason for this is easy to understand: If the initial velocity of an accelerated plane with respect to the matter fields is undetermined, the acceleration of the plane contributes to the rate of change of the entropy flow through the plane only, and not to the entropy flow itself, which depends on the velocity of the plane only. Hence we must consider the second, instead of the first proper time derivative of the amount of entropy carried through the plane. For similar reasons we must consider the rate of change of the shrinking speed of the plane, instead of the shrinking speed itself: In a given point of spacetime the shrinking speed of the plane depends on the initial conditions given for the tangent vectors of the world lines of the plane only, and therefore it has no direct connection with the geometric properties of spacetime. In contrast, the rate of change in the shrinking speed, or the negative of the second proper time derivative of the area, does indeed depend on the spacetime geometry. Hence we may conclude that if we want to find the relationship between acceleration, entropy and spacetime geometry, we must consider the second, instead of the first proper time derivatives of the area and entropy.

We begin our investigations in Sec. II by considering the trajectory of our plane. We shall assume that at a certain point $P$ of spacetime we have an orthonormal geodesic frame of reference, where all components of the energy momentum stress tensor $T_{\mu\nu}$ of matter are fixed and finite. In this frame of reference we shall then introduce a very small spacelike two-plane, which moves, at the point $P$, with a velocity very close to the speed of light to the direction of its normal and, at the same time, accelerates with a constant proper acceleration to the opposite direction. In order to make our analysis sufficiently local, the proper acceleration of the plane is taken to be very large. For sufficiently large values of the proper acceleration, one may view the local neighbourhood of the point $P$ as a region of spacetime which possesses the ordinary properties of the Rindler spacetime, including the Unruh temperature $T_U$ of Eq. (1.2).

In Sec. III we shall focus our attention to the change in the area of our accelerating two-plane. As the final result of Sec. III we shall get the rate of change in the shrinking speed of the plane.

The motivation for our decision to consider a plane moving with a very high speed in a chosen frame of reference becomes obvious in Sec. IV where we consider the flow of heat through our accelerating plane. It is fairly easy to show that if matter consists of a gas of non-interacting massless particles, i.e., of massless radiation, then the flow of boost energy through the plane is exactly the heat flow through the plane. Unfortunately, if the particles of the matter fields are massive, the situation becomes more complicated, because in that case other forms of energy, except heat, (mass-energy, for instance) are carried through the plane. However, if the plane moves with an enormous velocity
with respect to the matter fields, then the kinetic energies of the particles of the fields vastly exceed, in the rest frame of the plane, all the other forms of energy. In this limit we may consider matter, in effect, as a gas of non-interacting massless particles, and the rate of change in the boost energy flow is exactly the rate of change in the heat flow. We identify that part in the rate of change in the heat flow, which is due to the mere acceleration of the plane, and using Eq. (1.1) we calculate the rate of change in the acceleration entropy flow.

After obtaining an expression for the rate of change in the acceleration entropy flow in Sec. IV and for the rate of change in the shrinking speed in Sec. III we are finally able to obtain, in Sec. V, Einstein’s field equation by means of our hypothesis in the special case, where matter consists of massless, non-interacting radiation fields (electromagnetic field, for example), which are initially in a thermal equilibrium in the rest frame of our plane. In Sec. VI it is found that Einstein’s field equation for general matter fields is a straightforward consequence of our hypothesis in the limit, where the plane moves with a velocity very close to the speed of light with respect to the matter fields.

We close our discussion in Sec. VII with some concluding remarks.

II. TRAJECTORY OF THE PLANE

It is now time to specify our thermodynamical system in detail. Take a spacetime point \( P \) and define an orthonormal geodesic system of coordinates \( t, x, y, z \) at the local neighbourhood of that point. The origin of the coordinates is taken to lie at \( P \). Consider then a uniformly accelerated observer with a proper acceleration \( a \) travelling through \( P \) in the direction of the positive \( z \)-axis. We denote the velocity of that observer at \( P \) by \( v > 0 \). Furthermore, we assume that the acceleration of that observer is directed (in space) towards the negative \( z \)-axis. With the accelerating observer we shall now associate a small accelerated two-plane in the following way: In the local neighbourhood surrounding the observer, it is possible to define the concept of a two-plane. We consider a small two-plane which always remains at rest with respect to the observer. This means that at every point of the world line of the observer, we visualize a certain spacelike two-plane, constantly moving along with the observer. We assume that this two-plane is perpendicular to the \( z \)-axis, which means that the acceleration is directed perpendicular to the plane.

There are obvious physical reasons to require that the proper acceleration of the plane must be very large. When spacetime is curved, one may associate the ordinary Rindler wedge of the accelerating observer with the local neighbourhood of the point \( P \) only. Hence, if we want to employ the properties of Rindler spacetime in our calculations, we must analyze the thermodynamics of the plane in the limit where the proper acceleration \( a \) becomes very large. However, we shall not specify the actual magnitude of the proper acceleration in more detail. When the curvature of spacetime is reasonable large, one may always make the analysis sufficiently local by increasing the value of \( a \). Only in very special circumstances, that is, when the effects of the curvature on the metric of spacetime become significant at the Planck scale of distances, our arguments probably fail to hold. In all what follows, we shall therefore always assume that the proper acceleration \( a \) is sufficiently large.

The equation for the world line of our plane may be now written as
\[
(z - z_0)^2 - (t - t_0)^2 = \frac{1}{a^2},
\]
where \( z_0 \) and \( t_0 \) are constants depending on the values of \( a \) and \( v \) at the point \( P \). In the (flat) tangent space of the point \( P \), these constants have solid geometrical interpretations (see Fig. 1). Equation (2.1) gives the equation of the world line of the plane in an immediate vicinity of the point \( P \) with respect to the orthonormal geodesic coordinates \( t, x, y, \) and \( z \). If we solve \( z \) from Eq. (2.1) and differentiate \( z \) with respect to the time coordinate \( t \), we find that the velocity of the plane is, as a function of the time \( t \),
\[
\frac{dz}{dt} = \frac{a(t_0 - t)}{\sqrt{1 + a^2(t_0 - t)^2}}.
\]
Hence, at the point \( P \), the velocity of the plane is

\[
v = \frac{at_0}{\sqrt{1 + a^2t_0^2}}.
\]

(2.3)

It is convenient to write the velocity \( v \) by means of a new parameter \( \epsilon \in (0, 1) \) such that

\[
v = \frac{1 - \epsilon}{1 + \epsilon},
\]

(2.4)

and it follows from Eq. (2.3) that the constant \( t_0 \) may be expressed in terms of \( \epsilon \) and \( a \) as

\[
t_0 = \frac{1 - \epsilon}{a\sqrt{2\epsilon}}.
\]

(2.5)

As one may observe, for fixed \( a \) the quantity \( t_0 \) goes to infinity when \( \epsilon \) goes to zero.

![Diagram](image)

FIG. 1: Geometrical interpretations of the constants \( t_0 \) and \( z_0 \). In this figure, the world line of the accelerated two-plane (or, equivalently, the world line of the accelerated observer) going through \( P \) is drawn in the frame of reference equipped with the geodesic coordinates \( t \) and \( z \). The origin of the coordinates \( t \) and \( z \) should lie at the point \( P \). The past and the future Rindler horizons of the plane are the thick lines which intersect at the point \( P_0 \). The constant \( t_0 \) is then the value of the coordinate \( t \) at the point \( P_0 \), whereas the constant \( z_0 \) is the value of the coordinate \( z \) at \( P_0 \).

Now, what shall be the role of the parameter \( \epsilon \) in our analysis? We see from Eq. (2.4) that \( \epsilon \) describes the velocity of our plane at \( P \) with respect to the given system of coordinates. In the limit, where \( \epsilon = 1 \), the plane is at rest at the point \( P \). On the other hand, when \( \epsilon \) takes its values within the interval \((0, 1)\), the plane has initially a certain velocity relative to the positive \( z \)-axis such that in the limit where \( \epsilon \to 0 \), the velocity becomes close to 1, the speed of light in the natural units. Obviously, for sufficiently small \( \epsilon \), the plane moves with relativistic speeds with respect to all matter fields, regardless of the properties of matter at \( P \). Similar results hold also vice versa: As \( \epsilon \) approaches zero, the velocity of the flow of the matter fields across the plane approaches the speed of light. We have previously argued that under these circumstances the flow of heat vastly dominates other forms of energy transfer (the demonstration of this claim will be given in Sec. IV). Therefore, we shall henceforth always require that the parameter \( \epsilon \) becomes very small. Only in this limit, we may always interpret the energy flow through the plane as heat. As we shall soon see, in this limit the calculations also turn out relatively simple.
So far we have managed to find an appropriate parameter which determines the velocity of the matter flux across the accelerating two-plane. It is now time to formulate our ideas by using this parameter. We denote the future pointing unit tangent vector of the observer’s world line by \( \xi^\mu \) and a spacelike unit normal vector of the plane by \( \eta^\mu \). Because the observer, together with the plane, is assumed to move in the direction perpendicular to the plane, the vectors \( \xi^\mu \) and \( \eta^\mu \) are orthogonal. Moreover, we choose \( \eta^\mu \) in such a way that the observer is accelerated in the direction of the vector \( -\eta^\mu \). Since the observer is assumed to move, at the point \( P \), with the velocity \( v \) to the direction of the positive \( z \)-axis, the non-zero components of the vectors \( \xi^\mu \) and \( \eta^\mu \) are

\[
\begin{align*}
\xi^0 &= \cosh \phi, \\
\xi^3 &= \sinh \phi, \\
\eta^0 &= \sinh \phi, \\
\eta^3 &= \cosh \phi,
\end{align*}
\]

where

\[
\phi := \text{arsinh} \left( \frac{v}{\sqrt{1 - v^2}} \right)
\]

is the boost angle, or rapidity. Using Eq. (2.4) we find:

\[
\begin{align*}
\xi^\mu &= \frac{1}{2} \left( \frac{k^\mu}{\sqrt{\epsilon}} + \sqrt{\epsilon} l^\mu \right), \\
\eta^\mu &= \frac{1}{2} \left( \frac{k^\mu}{\sqrt{\epsilon}} - \sqrt{\epsilon} l^\mu \right),
\end{align*}
\]

where \( k^\mu := (1, 0, 0, 1) \) and \( l^\mu := (1, 0, 0, -1) \) are null vectors. This means that when the parameter \( \epsilon \) becomes small, the world line of the observer seems to lie very close to the null geodesic generated by the null vector \( k^\mu \). In the limit, where the proper acceleration \( a \) goes to infinity, the null vector \( k^\mu \) becomes a generator of the past Rindler horizon of the observer moving with the plane, whereas the null vector \( l^\mu \) becomes a generator of the future Rindler horizon (see Fig. 2).

### III. CHANGE OF AREA

In the previous Section we considered the trajectory of an observer at rest with respect to our accelerating plane in an infinitesimal neighborhood of an arbitrary point \( P \) of spacetime. The neighborhood in question was assumed be equipped with an orthonormal geodesic system of coordinates. Our main result was Eq. (2.8). That equation told in which way the future directed tangent vector \( \xi^\mu \) of the observer’s world line, as well as the spacelike unit normal vector \( \eta^\mu \) of the plane may be expressed, at the point \( P \), by means of the null vectors \( k^\mu \) and \( l^\mu \), provided that the velocity of the plane is known. It should be emphasized that Eq. (2.8) is always valid, no matter whether spacetime at the point \( P \) is flat or curved.

In this Section we proceed to calculate the area change of our plane. In contrast to the previous Section, where we considered the world line of an observer at rest with respect to the plane we shall, in this Section, consider the world lines of all the points of the plane. In other words, we shall consider the congruence of the world lines of the points of our plane. We shall denote the future directed unit tangent vector field of the smooth congruence under question by \( \xi^\mu \), and the spacelike unit vector field orthogonal to the plane by \( \eta^\mu \), because we shall assume that at the point \( P \) the vector fields \( \xi^\mu \) and \( \eta^\mu \) will coincide with the vectors \( \xi^\mu \) and \( \eta^\mu \) of Sec. II. At the point \( P \) we shall use exactly the same geodesic system of coordinates as we did in Sec. II. This implies that when our plane lies at the point \( P \), it is parallel to the \( xy \)-plane. We shall parametrize the world lines of the points of the plane by the proper time \( \tau \).
FIG. 2: The world line of the accelerated spacelike two-plane. \( \xi^\mu \) is the future pointing unit tangent vector of the world line, and \( \eta^\mu \) is the spacelike unit normal vector of the plane. As one may observe, the world line of the plane lies close to its past Rindler horizon \( \mathcal{H}^- \), which is generated by the null vector \( k^\mu \), whereas its future Rindler horizon \( \mathcal{H}^+ \) is generated by the null vector \( l^\mu \). The large arrow represents the heat that flows through the past horizon.

measured along these world lines. When the plane lies at the point \( \mathcal{P} \), we have \( \tau = 0 \) for all of the points of the plane. When the plane moves away from the point \( \mathcal{P} \) and \( \tau > 0 \), the plane is represented by the set of points, where \( \tau = \text{constant} \ (> 0) \) along the world lines.

We shall require that the vector fields \( \xi^\mu \) and \( \eta^\mu \) have the following properties at the point \( \mathcal{P} \):

\[
\begin{align*}
\xi^\mu &= \frac{1}{2} \left( \frac{k^\mu}{\sqrt{\epsilon}} + \sqrt{\epsilon} l^\mu \right), \\
\eta^\mu &= \frac{1}{2} \left( \frac{k^\mu}{\sqrt{\epsilon}} - \sqrt{\epsilon} l^\mu \right), \\
\xi_{;1}^\mu &= \xi_{;2}^\mu = 0, \\
a_{;1}^\mu &= a_{;2}^\mu = 0, \\
(a_{\mu} a_{\mu})^{1/2} &= -a_\mu \eta_\mu = a,
\end{align*}
\]

where

\[
a_\mu := \xi^\alpha \xi_{;\alpha}^\mu
\]

is the proper acceleration vector field of our congruence. Equations \( (3.1a) \) and \( (3.1b) \), respectively, are identical to Eqs. \( (2.8a) \) and \( (2.8b) \), and they state that our plane moves, at the point \( \mathcal{P} \), with a certain velocity \( v \) determined by the parameter \( \epsilon \) with respect to the geodesic coordinates associated with the point \( \mathcal{P} \). Eqs. \( (3.1c) \) and \( (3.1d) \) involve derivatives with respect to the coordinates \( x \) and \( y \) only. Eq. \( (3.1c) \) states that the tangent vectors of the world lines of the points of our plane are parallel to each other, whereas Eq. \( (3.1d) \) states that the proper acceleration vectors of the points of the plane are parallel. Finally, we have Eq. \( (3.1e) \). That equation states that our plane accelerates, at the point \( \mathcal{P} \), with a proper acceleration \( a \) to the direction opposite to that of the vector \( \eta^\mu \).

As a whole our assumptions contained in Eq. \( (3.1) \) correspond to our intuitive picture of a material, accelerating plane: When a plane is in an accelerating motion, all its points are, initially, moved and accelerated to the same
direction, and the possible changes in its area result from the curvature of spacetime, rather than from the initial conditions posed for the trajectories of its points. It should be emphasized, however, that all of the assumptions mentioned in Eq. (3.1) are just technical assumptions posed for the world lines of the points of our plane. The only really physical assumption of our paper is the hypothesis expressed in Eq. (1.3).

Since our plane is parallel to the \(xy\)-plane, when \(\tau = 0\) and the plane lies at the point \(P\), the points of the plane have \(z = 0\), when \(\tau = 0\). At the point \(P\) the change of the area \(A\) of our very small plane during an infinitesimal proper time interval \(d\tau\) is therefore:

\[
dA = A(\xi^1_{;1} + \xi^2_{;2}) d\tau. \tag{3.3}
\]

Hence we find, using Eq. (3.1c):

\[
\frac{dA}{d\tau} = 0, \tag{3.4}
\]

when \(\tau = 0\) and the plane lies at the point \(P\). In other words, the first proper time derivative of the area vanishes, when \(\tau = 0\). The second proper time derivative of the area, however, does not necessarily vanish, when \(\tau = 0\). We have:

\[
\left. \frac{d^2 A}{d\tau^2} \right|_{\tau=0} = A(\dot{\xi}^1_{;1} + \dot{\xi}^2_{;2}), \tag{3.5}
\]

where the dot means proper time derivative such that \(\dot{\xi}^\mu_{;\nu} := \frac{d}{d\tau}(\xi^\mu_{\nu})\). It is shown in the Appendix that when Eq. (3.1) holds, we have:

\[
\dot{\xi}^1_{;1} = R^1_{\mu\nu\eta} \xi^\mu_{\nu}, \tag{3.6a}
\]

\[
\dot{\xi}^2_{;2} = R^2_{\mu\nu\eta} \xi^\mu_{\nu}, \tag{3.6b}
\]

\[
R_{\mu\nu} \xi^\mu_{\nu} = \dot{\xi}^1_{;1} + \dot{\xi}^2_{;2} + R^\alpha_{\mu\nu\beta} \eta^\alpha \eta^\beta \xi^\mu_{\nu}. \tag{3.6c}
\]

where \(R^\alpha_{\mu\nu\beta}\) and \(R_{\mu\nu}\), respectively, are the Riemann and the Ricci tensors. So we see that we may write, in general:

\[
\left. \frac{d^2 A}{d\tau^2} \right|_{\tau=0} = AR_{\mu\nu} \xi^\mu_{\nu} - AR^\alpha_{\mu\nu\beta} \eta^\alpha \eta^\beta \xi^\mu_{\nu}. \tag{3.7}
\]

Equation (3.7) tells in which way the second proper time derivative of the area of our accelerating plane depends on the geometry of spacetime. The first special case of interest is the one, where the plane is at rest at the point \(P\), and spacetime is isotropic in the neighborhood of the point \(P\), i.e. it expands and contracts in exactly the same ways in all spatial directions. In that case we have

\[
\eta^\mu = \delta^\mu_3, \tag{3.8}
\]

which implies that

\[
R^\alpha_{\mu\nu\beta} \eta^\alpha \eta^\beta \xi^\mu_{\nu} = R^3_{\mu\nu\beta} \xi^\mu_{\nu}, \tag{3.9}
\]

and because spacetime is assumed to be isotropic, we have:

\[
R^1_{\mu\nu\lambda} \xi^\mu_{\nu} = R^2_{\mu\nu\lambda} \xi^\mu_{\nu} = R^3_{\mu\nu\lambda} \xi^\mu_{\nu}. \tag{3.10}
\]

Hence Eqs. (3.6) and (3.7) imply:

\[
\left. \frac{d^2 A}{d\tau^2} \right|_{\tau=0} = \frac{2}{3} AR_{\mu\nu} \xi^\mu_{\nu}. \tag{3.11}
\]
Another special case of interest is the one, where the parameter \( \epsilon \) gets close to zero, and our plane moves with an enormous velocity with respect to the orthonormal geodesic system of coordinates associated with the point \( P \). Using Eq. (2.8) and the symmetry properties of the Riemann and the Ricci tensors we find that we may write for general \( \epsilon > 0 \):

\[
\frac{d^2 A}{d\tau^2} \bigg|_{\tau=0} = \frac{1}{4\epsilon} AR_{\mu\nu} k^\mu k^\nu + \frac{1}{2} AR_{\mu\nu} k^\mu l^\nu - \frac{1}{4} AR_{\alpha\mu\nu\beta} k^\alpha l^\mu l^\nu k^\beta + \frac{\epsilon}{4} AR_{\mu\nu} l^\mu l^\nu. \tag{3.12}
\]

It is easy to see that when \( \epsilon \to 0 \), the first term on the right hand side of Eq. (3.12) will dominate. Hence we may write, for very small \( \epsilon \):

\[
\frac{d^2 A}{d\tau^2} \bigg|_{\tau=0} = \frac{1}{4\epsilon} AR_{\mu\nu} k^\mu k^\nu + \mathcal{O}(1), \tag{3.13}
\]

where \( \mathcal{O}(1) \) denotes the terms, which are of the order \( \epsilon^0 \), or higher. The negative of the right hand side of Eq. (3.13) gives the rate of change in the shrinking speed of the plane.

**IV. FLOW OF HEAT**

So far we have considered the properties of our accelerating plane only. We shall now turn our attention to the matter fields. In general, the boost energy flow of matter, or the boost energy flown per unit time through a very small plane with area \( A \) is, from the point of view of an observer at rest with respect to the plane:

\[
\frac{dE_b}{d\tau} = -AT_{\mu\nu}\xi^\mu\eta^\nu, \tag{4.1}
\]

where, as in the previous Sections, \( \xi^\mu \) is the timelike unit tangent vector of the worldline of the observer, and \( \eta^\mu \) is a spacelike unit normal vector of the plane, orthogonal to \( \xi^\mu \). The negative sign comes from the fact that our plane is assumed to move, with respect to the matter fields, to the direction of the vector \( \eta^\mu \), and that vector also determines the direction of the boost energy flow through our plane.

When the plane is in an accelerating motion, the boost energy flow through the plane is not constant, but it will change in time. The rate of change in the boost energy flow is:

\[
\frac{d^2 E_b}{d\tau^2} = -\dot{AT}_{\mu\nu}\xi^\mu\eta^\nu - \dot{AT}'_{\mu\nu}\xi^\mu\eta^\nu - AT_{\mu\nu}\dot{\xi}^\mu\eta^\nu - AT_{\mu\nu}\dot{\eta}^\mu\xi^\nu, \tag{4.2}
\]

where the dot means the proper time derivative. It follows from Eq. (3.13) that when \( \tau = 0 \), which means that our plane lies at the point \( P \), the first term on the right hand side of Eq. (4.2) vanishes. Because, in general,

\[
\dot{\xi}^\mu = -a\eta^\mu, \tag{4.3a}
\]

\[
\dot{\eta}^\mu = -a\xi^\mu \tag{4.3b}
\]

for an observer moving with a proper acceleration \( a \) on the left hand side of the Rindler wedge, we may write Eq. (4.2), by means of the chain rule, as:

\[
\frac{d^2 E_b}{d\tau^2} = \frac{d^2 E_{b,t}}{d\tau^2} + \frac{d^2 E_{b,a}}{d\tau^2}, \tag{4.4}
\]

where we have denoted:

\[
\frac{d^2 E_{b,t}}{d\tau^2} := -AT_{\mu\nu,\alpha}\xi^\mu\eta^\nu \xi^\alpha, \tag{4.5a}
\]

\[
\frac{d^2 E_{b,a}}{d\tau^2} := aAT_{\mu\nu}(\xi^\mu\xi^\nu + \eta^\mu\eta^\nu). \tag{4.5b}
\]
All quantities have been calculated at the point $P$.

The first term on the right hand side of Eq. (4.4) is now due to the simple fact that the energy momentum stress tensor $T_{\mu\nu}$ of the matter fields may be different in different points of spacetime. That term has nothing to do with the acceleration of the plane. The second term, in turn, is due to the mere acceleration of the plane: When the plane is in an accelerating motion, the velocity of the plane with respect to the matter fields changes as a function of the proper time $\tau$. In what follows, we shall focus our attention to the second term.

The question is now: In which cases will the second term on the right hand side of Eq. (4.4) give the rate of change in the flow of heat, caused by the mere acceleration of the plane? After all, we assumed in our hypothesis that the rate of change in the boost energy flow is exactly the rate of change in its heat flow. We shall see in the next Section that at least in the special case, where the matter consists of massless, non-interacting radiation (electromagnetic radiation, for instance) in thermal equilibrium, the second term on the right hand side of Eq. (4.4) does indeed give the rate of change in the heat energy flow through our plane. Another special case is the one, where there is, instead of massless non-interacting radiation in thermal equilibrium, a steady flow of thermal non-interacting massless particles, all propagating to the one and the same direction. For instance, we may put a source of light to the focus of a parabolic mirror. In that case the photons reflected from the mirror will all propagate to the one and the same direction. The photons come out from the source of light with all the possible wave lengths, and the energy density

$$\rho = T_{\mu\nu} \xi^\mu \xi^\nu \quad (4.6)$$

of the photon gas from the point of view of an observer at rest with respect to our accelerating plane depends on the absolute temperature $T$ of the gas only. If the photons propagate to the direction orthogonal to the plane, the pressure exerted by the photons against the plane is, in the rest frame of the plane:

$$P = T_{\mu\nu} \eta^\mu \eta^\nu = \rho. \quad (4.7)$$

In other words, the energy density and the pressure of the photon gas under consideration are equals.

It is easy to see that the second term on the right hand side of Eq. (4.4) really gives the rate of change in the flow of heat through an accelerating plane for the photon gas described above. According to the first law of thermodynamics the change in the heat content of a system with total energy $E$ and pressure $P$ is:

$$\delta Q = dE + P dV, \quad (4.8)$$

where $V$ is the volume of the system. Because the energy density $\rho$ of our photon gas is independent of its volume $V$, its total energy is

$$E = \rho V, \quad (4.9)$$

and therefore it follows from Eqs. (4.6) – (4.8) that the heat energy possessed by our photon gas per unit volume is

$$\frac{\delta Q}{dV} = T_{\mu\nu} (\xi^\mu \xi^\nu + \eta^\mu \eta^\nu). \quad (4.10)$$

Hence we find that the rate of change in the heat energy flow through our accelerating plane is, from the point of view of an observer at rest with respect to our plane:

$$\frac{\delta^2 Q}{d\tau^2} = a A T_{\mu\nu} (\xi^\mu \xi^\nu + \eta^\mu \eta^\nu), \quad (4.11)$$

which is exactly the second term on the right hand side of Eq. (4.4). It should be noted that the same result holds whenever the matter consists of massless, non-interacting particles only (not just photons), all propagating to the one and the same direction.
It is remarkable that when our accelerating plane moves with respect to the matter fields with a velocity very close to that of light, then the components of the energy-momentum stress tensor $T^{\mu\nu}$ of the matter fields behave, from the point of view of an observer at rest with respect to the plane, in exactly the same way as do the components of $T^{\mu\nu}$ for matter consisting solely of massless, non-interacting particles, all moving in a direction perpendicular to the plane. In other words, in the rest frame of a plane moving with an enormous velocity we may consider arbitrary matter, in effect, as a gas of non-interacting massless particles.

To see how this important result comes out, let us fix the rest frame of our plane at the point $P$ such that its $z$-axis coincides with the vector $\eta^\mu$, and its $t$-axis with the vector $\xi^\mu$. In this frame the relevant components of the energy momentum stress tensor of arbitrary matter are:

$$T_{00}^{'00} = T_{\mu\nu} \xi^\mu \xi^\nu,$$
$$T_{33}^{'33} = T_{\mu\nu} \eta^\mu \eta^\nu,$$
$$T_{03}^{'03} = T_{30}^{'30} = -T_{\mu\nu} \xi^\mu \eta^\nu,$$

where $T_{\mu\nu}$ denotes the components of the energy momentum stress tensor in the orthonormal geodesic system of coordinates used in Secs. 11 and 111. For matter consisting solely of massless, non-interacting particles carrying no angular momentum and propagating to the direction orthogonal to the plane these are the only non-zero components of $T_{\mu\nu}^{'}$, and we have, in the natural units:

$$T_{00}^{'00} = T_{33}^{'33} = -T_{03}^{'03} = -T_{30}^{'30} = \rho,$$

where $\rho$ is the energy density of the matter in the rest frame of the plane.

Now, suppose that instead of having a gas of non-interacting massless particles, the matter fields are arbitrary. In that case it follows from Eqs. 2.8 and 4.12 that we have, in the rest frame of the plane:

$$T_{00}^{'00} = \frac{1}{4\epsilon} T_{\mu\nu} k^\mu k^\nu + O(1),$$
$$T_{33}^{'33} = \frac{1}{4\epsilon} T_{\mu\nu} k^\mu k^\nu + O(1),$$
$$T_{03}^{'03} = T_{30}^{'30} = -\frac{1}{4\epsilon} T_{\mu\nu} k^\mu \eta^\nu + O(1),$$

where $O(1)$ denotes the terms, which are of the order $\epsilon^0$, or higher. So we see that, in the limit, where $\epsilon \to 0$, which means that the plane moves with a velocity very close to the speed of light with respect to the matter fields, we have:

$$T_{00}^{'00} = T_{33}^{'33} = -T_{03}^{'03} = -T_{30}^{'30} = \rho,$$

which is exactly Eq. 4.13. In other words, in the rest frame of a plane moving with an enormous velocity with respect to the matter fields, the components of the energy momentum stress tensor are exactly the same as are its components for massless, non-interacting particles moving in a direction perpendicular to the plane, independently of the kind of matter we happen to have. This means that we may consider arbitrary matter, from the point of view of an observer moving with respect to the matter fields with a very great speed, as a gas of non-interacting massless particles. Actually, this is something one might expect: When an observer moves with a very high speed with respect to the matter fields, the particles of the matter fields move with respect to the observer with velocities close to that of light, and their kinetic energies vastly exceed all the other forms of energy (mass-energy, for instance). As a result the observer will see, in effect, a gas of non-interacting massless particles.

Our investigations imply that in the limit where the velocity of an accelerating plane with respect to the matter fields gets close to the speed of light, the boost energy flow becomes to the heat flow, independently of the kind of matter we happen to have. In this limit the second term on the right hand side of Eq. 4.11 gives that part of the
rate of change in the heat flow which is caused by the mere acceleration of the plane. Using Eq. (2.8) we find that in the high speed limit the second term on the right hand side of Eq. (4.4) may be written as:

\[ aAT_{\mu\nu}(\xi^\mu \xi^\nu + \eta^\mu \eta^\nu) = \frac{1}{2\epsilon}aAT_{\mu\nu}k^\mu k^\nu + \mathcal{O}(\epsilon), \quad (4.16) \]

where \( \mathcal{O}(\epsilon) \) denotes the terms, which are of the order \( \epsilon \), or higher. It is obvious that in the high speed limit, where \( \epsilon \to 0 \), the terms proportional to \( 1/\epsilon \) will dominate. When the second term on the right hand side of Eq. (4.4) gives that part of the rate of change in the heat flow which is caused by the mere acceleration, we define the rate of change in the acceleration entropy flow as:

\[ \frac{d^2S_a}{d\tau^2} \bigg|_{\tau=0} := \frac{1}{T}aAT_{\mu\nu}(\xi^\mu \xi^\nu + \eta^\mu \eta^\nu), \quad (4.17) \]

where \( T \) is the absolute temperature, in the rest frame of the plane, of the matter flowing through the plane. Using Eq. (4.16) we find that in the high speed limit Eq. (4.17) takes the form:

\[ \frac{d^2S_a}{d\tau^2} \bigg|_{\tau=0} = \frac{1}{2\epsilon}aAT_{\mu\nu}k^\mu k^\nu + \mathcal{O}(\epsilon). \quad (4.18) \]

V. MASSLESS, NON-INTERACTING RADIATION FIELDS

So far we have managed to find explicit expressions for the both sides of Eq. (1.3). Its right hand side was obtained in two important cases in Sect. III whereas its left hand side was obtained in Sect. IV. We shall now equate these both sides and see, whether Einstein’s field equation really follows from the thermodynamical hypothesis of Eq. (1.3).

In this Section we shall derive Einstein’s field equation in the special case, where matter consists solely of massless, non-interacting radiation fields, which are in thermal equilibrium in the rest frame of our plane. A typical example of a massless, non-interacting radiation field is, of course, the electromagnetic field. Whatever massless non-interacting radiation field in thermal equilibrium in the rest frame of our plane we may have, its energy density is always, in the rest frame of the plane,

\[ \rho = T_{\mu\nu}\xi^\mu \xi^\nu, \quad (5.1) \]

and its pressure

\[ P = T_{\mu\nu}\eta^\mu \eta^\nu \quad (5.2) \]

has the property:

\[ P = \frac{1}{3}\rho. \quad (5.3) \]

It is an important property of the radiation field described above that its energy momentum stress tensor \( T_{\mu\nu} \) is traceless, i.e.,

\[ T^\alpha_\alpha = 0. \quad (5.4) \]

This property will play an important role in our derivation of Einstein’s field equation in the special case considered in this Section.

The first task is to check, whether the radiation fields under consideration really satisfy the assumptions of our hypothesis. In other words, we must check, whether the rate of change in the boost energy flow of our radiation is really the rate of change in the heat flow. Obviously, this is the case: It is a well-known result of elementary
thermodynamics that the entropy density (entropy per unit volume) of massless, non-interacting radiation in thermal equilibrium is \[ s = \frac{4}{T^3} \rho, \] (5.5)

where \( T \) is the absolute temperature of the radiation. So we find that the rate of change in the flow of entropy carried by radiation through our plane is

\[ \frac{d^2 S_a}{d\tau^2} \bigg|_{\tau=0} = \frac{4}{3} \frac{a A \rho}{T^3}. \] (5.6)

It is easy to see that this is exactly the same result as the one given by Eq. (4.17), when we use Eqs. (5.1)–(5.3).

Hence we may conclude that Eq. (4.17) really gives the rate of change in the acceleration entropy flow through our plane for our radiation fields. Reasoning backwards and using Eq. (1.1) then implies that the rate of change in the boost energy flow is the rate of change in the heat flow, as required. In other words, the assumptions of our hypothesis are satisfied. Eqs. (5.1) and (5.6) imply:

\[ \frac{d^2 S_a}{d\tau^2} \bigg|_{\tau=0} = \frac{4}{3} \frac{a A T \xi^\mu \xi^\nu}{T^3}. \] (5.7)

If the absolute temperature \( T \) of the radiation agrees with the Unruh temperature \( T_U \) of Eq. (1.2), our final expression for the acceleration entropy flow takes the form:

\[ \frac{d^2 S_a}{d\tau^2} \bigg|_{\tau=0} = \frac{8\pi}{3} A T \xi^\mu \xi^\nu. \] (5.8)

It should be emphasized, however, that although we have taken the temperature of the radiation to agree with the Unruh temperature measured by an observer at rest with respect to our accelerating plane, Eq. (5.8) does not give the acceleration entropy flow of the Unruh radiation. Our radiation field is just an ordinary, massless, non-interacting radiation field, which has no connection whatsoever with the Unruh radiation. We may either heat up or cool down the radiation until its temperature agrees with the Unruh temperature. Our claim is that after the Unruh temperature has been reached, the hypothesis of Eq. (1.3) holds, and it implies Einstein’s field equation.

After finding an expression for the acceleration entropy flow we shall now turn our attention to the rate of change in the shrinking speed of the plane. Since the radiation is assumed to be in thermal equilibrium with respect to the plane, when the plane lies at the point \( \mathcal{P} \), it is isotropic in the neighborhood of the point \( \mathcal{P} \), and spacetime expands and contracts in exactly the same ways in all spatial directions at the point \( \mathcal{P} \). This means that \( R_{\mu \nu 1} \xi^\mu \xi^\nu \), \( R_{\mu \nu 2} \xi^\mu \xi^\nu \) and \( R_{\mu \nu 3} \xi^\mu \xi^\nu \) are equals, and we may use Eq. (3.11). Using Eq. (5.8) for the left hand side, and the negative of the right hand side of Eq. (3.11) for the right hand side of Eq. (1.3), we find that Eq. (1.3) implies:

\[ \frac{8\pi}{3} A T \xi^\mu \xi^\nu = -\frac{1}{3} A R_{\mu \nu} \xi^\mu \xi^\nu, \] (5.9)

or:

\[ R_{\mu \nu} \xi^\mu \xi^\nu = -8\pi T_{\mu \nu} \xi^\mu \xi^\nu. \] (5.10)

Since \( \xi^\mu \) is an arbitrary timelike unit vector field, we must have:

\[ R_{\mu \nu} = -8\pi T_{\mu \nu}, \] (5.11)

which is exactly Einstein’s field equation

\[ R_{\mu \nu} = -8\pi(T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T^a_a), \] (5.12)
or
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \quad (5.13) \]
in the special case, where the tensor \( T_{\mu\nu} \) is traceless, i.e., Eq. (5.4) holds. Since \( T_{\mu\nu} \) is indeed traceless for massless, non-interacting radiation fields in thermal equilibrium, we have managed to obtain Einstein’s field equation from our hypothesis for such radiation fields.

VI. GENERAL MATTER FIELDS

We saw in the previous Section how Einstein’s field equation follows from our hypothesis concerning the thermodynamical properties of spacetime and matter fields at least in the special case, where matter consists of massless, non-interacting radiation in thermal equilibrium only. An advantage of such radiation is that the rate of change in its boost energy flow through our accelerating plane is exactly the rate of change in the heat flow, and therefore the assumptions of our hypothesis are automatically satisfied. When attempting to generalize the thermodynamical derivation of Einstein’s field equation of the Sec. V for general matter fields, however, one meets with difficulties, because for general matter fields the boost energy flow may include other forms of energy, except heat as well (mass energy, for instance), and hence the assumptions of our hypothesis are not necessarily satisfied for general matter fields.

Fortunately, we managed to show in Sec. IV that there is a way out of this problem: We take our accelerating plane to move, at the point \( \mathcal{P} \) under consideration, with a velocity \( v \) very close to the speed of light with respect to the matter fields. From the point of view of an observer moving with an enormous velocity with respect to the matter fields all matter behaves, in effect, as a gas of non-interacting massless particles, and the rate of change in the boost energy flow equals with the rate of change in the heat flow. Hence we may apply our hypothesis in the limit, where \( v \) gets close to 1, the speed of light in the natural units. In this limit the rate of change in the shrinking speed of the plane is given by Eq. (3.13) and the rate of change in the acceleration entropy flow is given by Eq. (4.18). Using Eq. (3.13) for the left hand side, and the negative of the right hand side of Eq. (4.13) for the right hand side of Eq. (1.3) we find:

\[ \frac{1}{T} \frac{\alpha}{2\epsilon} A T_{\mu\nu} k^\mu k^\nu + \mathcal{O}(\epsilon) = -\frac{1}{8\epsilon} AR_{\mu\nu} k^\mu k^\nu + \mathcal{O}(1). \quad (6.1) \]

Again, if the absolute temperature \( T \) of the matter agrees with the Unruh temperature \( T_U \) of Eq. (1.2) we get, in the high speed limit, where \( \epsilon \to 0 \):

\[ R_{\mu\nu} k^\mu k^\nu = -8\pi T_{\mu\nu} k^\mu k^\nu. \quad (6.2) \]

Since \( k^\mu \) is an arbitrary, future directed null vector field, we have:

\[ R_{\mu\nu} + fg_{\mu\nu} = -8\pi T_{\mu\nu}, \quad (6.3) \]

where \( f \) is some function of the spacetime coordinates. It follows from the Bianchi identity

\[ (R^\mu_\nu - \frac{1}{2} R g^\mu_\nu)_{;\nu} = 0, \quad (6.4) \]

that \( f = -\frac{1}{2} R + \Lambda \) for some constant \( \Lambda \), and hence we arrive at the equation

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}, \quad (6.5) \]

which is Einstein’s field equation with the cosmological constant \( \Lambda \).
VII. CONCLUDING REMARKS

In this paper we have obtained Einstein’s field equation by means of very simple thermodynamical arguments concerning the properties of a very small spacelike two-plane in a uniformly accelerating motion. Our derivation was based on a hypothesis that when matter flows through the plane, and the temperature of the matter is the same as the Unruh temperature measured by an observer at rest with respect to the plane, then the rate of change in the flow of entropy caused by the mere acceleration of the plane is, in natural units, exactly one-half of the rate of change in the shrinking speed of the area of the plane. From this hypothesis we obtained, by means of the fundamental thermodynamical relation \( \delta Q = T \, dS \), Einstein’s field equation.

When spacetime is filled with isotropic, massless, non-self-interacting radiation field (electromagnetic field, for instance) in thermal equilibrium, it is very easy to obtain Einstein’s field equation from our hypothesis, because it turns out that in this case the rate of change in the boost energy flow through the plane is exactly the rate of change in the heat flow of the radiation. However, if the fields are massive, or self-interacting, the situation becomes more complicated, because the boost energy flow involves other forms of energy, except heat, as well (mass-energy, for instance). In that case we may consider the situation, where the plane moves with respect to the matter fields with a velocity very close to that of light. When the plane moves with respect to the matter fields with an enormous velocity, it turns out that the amount of heat vastly exceeds the amounts of other forms of energy carried by matter through the plane, and Einstein’s field equation for general matter fields follows from our hypothesis.

Our derivation of Einstein’s field equation by means of purely thermodynamical arguments provides support for the idea, earlier expressed by Jacobson, that Einstein’s field equation may actually be understood as a thermodynamical equation of state of spacetime and matter fields. Although our thermodynamical derivation of Einstein’s field equation bears a lot of similarities with Jacobson’s derivation, it should be strongly emphasized the radical difference between these two derivations: Jacobson considered the boost energy flow through a horizon of spacetime, whereas we considered the boost energy flow through an accelerating, spacelike two-plane. Horizons of spacetime are certain null hypersurfaces of spacetime, and therefore they are created, when all points of a spacelike two-surface move along certain null curves of spacetime. In contrast, our spacelike two-plane was assumed to move in spacetime with a speed less than that of light, and therefore all of its points move along timelike curves of spacetime. Because of that our two-plane should not be considered as a part of any horizon of spacetime. Nevertheless, we found that if the entropy carried by matter through the plane is connected with the change in its area in a certain manner, then Einstein’s field equation follows. The fact that an assumption of a simple proportionality between the rates of changes in the entropy flow and in the shrinking speed of the plane yields Einstein’s field equation even when that two-plane is not a part of any horizon of spacetime strongly suggests that one may associate meaningfully the concept of gravitational entropy not only with horizons, but also with arbitrary spacelike two-surfaces of spacetime. It is still uncertain what the consequences of such a possibility may be, but they will most likely have some influence on our views of the nature of gravitational entropy.

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APPENDIX: PROOF OF EQ. (3.6)

In this Appendix we shall show that Eq. (3.1) implies Eq. (3.6). It follows from the chain rule that, in geodesic coordinates,

\[ \dot{\xi}^\mu := \frac{d}{d\tau}(\xi^\mu) = \xi^\alpha \xi^\mu_{,\alpha}. \]  

(A.1)

Using the trivial identity

\[ \xi^\alpha \xi^\mu_{,\alpha} = \xi^\alpha \xi^\mu_{,\beta} - \xi^\alpha R^\mu_{\alpha\beta\gamma} \xi^\gamma, \]  

(A.2)

the product rule of covariant differentiation, and the basic properties of the Riemann tensor we get:

\[ \dot{\xi}^\mu = (\xi^\alpha \xi^\mu_{,\alpha})_{,\mu} - \xi^\alpha \xi^\mu_{,\alpha;\mu} - \xi^\alpha R^\mu_{\alpha\beta\gamma} \xi^\gamma. \]  

(A.3)

Eq. (3.2) and the symmetry properties of the Riemann tensor imply:

\[ \dot{\xi}^\mu = a^\mu_{,\mu} - \xi^\alpha \xi^\mu_{,\alpha} + R^\mu_{\alpha\beta\gamma} \xi^\gamma. \]  

(A.4)

So we have:

\[ \dot{\xi}_1 = a^1_{,1} - \xi^\alpha \xi^1_{,1} + R^1_{\alpha1\beta} \eta^\beta, \quad \dot{\xi}_2 = a^2_{,2} - \xi^\alpha \xi^2_{,2} + R^2_{\alpha2\beta} \eta^\beta. \]  

(A.5a, A.5b)

Eqs. (3.1c) and (3.1d) imply that the first two terms on the right hand sides of Eqs. (A.5a)–(A.5b) will vanish. So we get:

\[ \dot{\xi}_1 = R^1_{\mu1\nu} \xi^\mu \xi^\nu, \quad \dot{\xi}_2 = R^2_{\mu2\nu} \xi^\mu \xi^\nu. \]  

(A.6a, A.6b)

which are Eqs. (3.6a)–(3.6b).

It only remains to prove Eq. (3.6c). To this end we note first that the vectors $\xi^\mu$, $\eta^\mu$, $\epsilon^\mu_{(1)}$, and $\epsilon^\mu_{(2)}$, respectively, are the spacelike unit vectors parallel to the $x$- and the $y$-axes of the orthonormal geodesic system of coordinates at the point $P$, constitute an orthonormal set of vectors. In other words, the vectors $\xi^\alpha$, $\eta^\alpha$, $\epsilon^\alpha_{(1)}$ and $\epsilon^\alpha_{(2)}$ may be taken to be the base vectors of a new orthonormal geodesic system of coordinates at the point $P$. In this system of coordinates our plane is at rest at the point $P$, and it has been obtained from the original orthonormal geodesic system of coordinates by means of the Lorentz boost. Hence we find, by means of the antisymmetry properties

\[ R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu}, \]  

(A.7)

of the Riemann tensor that

\[ R^1_{\mu1\nu} \xi^\mu \xi^\nu + R^2_{\mu2\nu} \xi^\mu \xi^\nu + R_{\alpha\mu\beta} \eta^\alpha \eta^\beta \xi^\mu \xi^\nu = R_{\mu\nu} \xi^\mu \xi^\nu, \]  

(A.8)

where $R_{\mu\nu}^\sigma$ is the Ricci tensor. Using Eqs. (A.6a)–(A.6b) we therefore get:

\[ R_{\mu\nu} \xi^\mu \xi^\nu = \dot{\xi}_1 + \dot{\xi}_2 + R^\alpha_{\mu\nu} \eta^\alpha \xi^\mu \xi^\nu, \]  

(A.9)

which is Eq. (3.6c).

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[13] Actually, the parameter $\epsilon$ is the square of the so-called Doppler shift factor.

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