Abstract

Measures play an important role in the characterisation of various function spaces. In this paper, the structure of density measures will be investigated. These are elements of the dual of the space of essentially bounded functions. The main results presented here are a more precise representation of $L^\infty(\Omega, L^n)^*$, leading to the notion of pure measures, and the definition and analysis of density measures which constitute a large class of such measures. It is shown that density measures have applications in the context of traces. In particular, new and meaningful examples of pure measures are given on $\mathbb{R}^n$, in contrast to common examples in the literature, which are usually constructed on $\mathbb{N}$.

1 Introduction

In most mathematical texts, measures are defined to be $\sigma$-additive. For these $\sigma$-measures, a rich theory and convergence theorems hold true. Their importance follows from Riesz Representation Theorem, which characterises the dual space of the space of all compactly supported functions as the space of Radon measures. Yet, other dual spaces cannot be represented by $\sigma$-measures but by finitely additive measures, e.g. $C_b(\Omega)^*$ and $L^\infty(\Omega, L^n)^*$ (cf. [1], [5]). The reason why measures which are not $\sigma$-additive are not widely used outside of economic theory is probably due to the lack of meaningful examples in the literature. In this paper, the basic theory of finitely additive measures as used in e.g. [13] and [5] will be outlined and a new, meaningful example for such measures will be given on $\mathbb{R}^n$. Interestingly, many results on their structure are known. In particular, the dual space of the space of essentially bounded functions is known to be represented by the space of all bounded measures, which do not charge Lebesgue null sets. This
representation will be extended by the identification and definition of pure measures and later on density measures, which constitute a large class of pure measures. Some examples will show that they can be employed in the study of traces and even differential calculus.

The structure of the paper is as follows. First, some necessary notions and results from lattice theory will be recalled and useful proposition on the successive decomposition of vector lattices will be proved.

In the second section, the theory of measures and the associated integration theory will be given. The known representation of the dual space of $L^\infty (\Omega, \mathcal{L}^n)$ will be refined, using the decomposition techniques from the previous section. Pure measures will be defined and a necessary condition for a measure to be pure will be given. In plus, a first example will illustrate the new results.

The last section contains work on density measures. Following their definition, existence and some properties will be proved. In plus, the extremal points of the set of all density measures will be characterised. Exemplary applications to the traces of functions of bounded variation and differential calculus will be presented. Finally, the relation of pure measures and $\sigma$-measures which are singular with respect to Lebesgue measure will be analysed.

Concerning notation, in the following $n \in \mathbb{N}$ denotes a positive natural number and $\mathbb{R}^n$ the vector space of real $n$-tuples. For a set $\Omega \subset \mathbb{R}^n$ the set $\Omega_\delta$ denotes the open $\delta$-neighbourhood of $\Omega$. Open balls with radius $\delta > 0$ and centre $x \in \mathbb{R}^n$ are written $B_\delta (x) = \{x\}_\delta$. The Borel subsets of $\Omega$, i.e. the $\sigma$-measure generated by all relatively open sets in $\Omega$, is denoted by $\mathcal{B}(\Omega)$. $\mathcal{L}^n$ is the Lebesgue measure and $\mathcal{H}^d$ the $d$-dimensional Hausdorff measure. For set function $\mu$ on $\Omega$, $\mu | A$ denotes the restriction of $\mu$ to $A$. The Banach space of equivalence classes of $p$-integrable functions is denoted by $\mathcal{L}^p (\Omega, \mathcal{L}^n)$ and $p'$ denotes the Hölder-conjugate of $p$. Spaces of continuous functions with a support which is relatively compact in $\Omega$ will be written $C_0 (\Omega)$. (Weak) Derivates of functions $f$ are written $Df$. The divergence of a vector field $F$, be it classical or distributional, is denoted by $\text{div} F$.

## 2 Tools from Lattice Theory

First, the basic definitions for vector lattices from Rao [13, p. 24ff] (cf. [3, p. 347]) is given.

**Definition 2.1.** Let $L$ be a vector space and $\leq$ a partial order on $L$ which is compatible with $+$ and the multiplication with a scalar on $L$. If for all $l_1, l_2 \in L$ the supremum and infimum of $\{l_1, l_2\}$ exist, then $L$ is called a
vector lattice. For \( l, l_1, l_2 \in L \) write
\[
\begin{align*}
l_1 \lor l_2 & := \sup \{ l_1, l_2 \} \\
l_1 \land l_2 & := \inf \{ l_1, l_2 \} \\
l^+ & := l \lor 0 \\
l^- & := -l \lor 0 \\
|l| & := l^+ + l^-
\end{align*}
\]

\( l_1, l_2 \in L \) are called orthogonal, if \( |l_1| \land |l_2| = 0 \), written \( l_1 \perp l_2 \). If for a family \( \{ l_i \}_{i \in I} \subset L \) the supremum exists, write
\[
\bigvee_{i \in I} l_i := \sup_{i \in I} l_i .
\]

If the infimum of \( \{ l_i \}_{i \in I} \) exists, it is denoted by
\[
\bigwedge_{i \in I} l_i := \inf_{i \in I} l_i .
\]

A set \( L' \subset L \) is called bounded from above, if there exists \( l \in L \), such that \( l' \leq l \) for all \( l' \in L' \).

A vector lattice is called boundedly complete, if for every \( \{ l_i \}_{i \in I} \subset L \) which is bounded from above the supremum \( \bigvee_{i \in I} l_i \) exists.

For a vector lattice \( L \) and \( l_1, l_2 \in L \)
\[
|l_1 + l_2| \leq |l_1| + |l_2|
\]
with equality if \( l_1 \perp l_2 \) (cf. [13, p. 25]). The following example foreshadows the partial order that turns spaces of measures into vector lattices.

In order to obtain results for an orthogonal decomposition of vector lattices (and their elements), one has to define appropriate sub-structures (cf. [13, p. 28]).

Definition 2.2. A linear subspace \( L' \) of \( L \) is called a sublattice of \( L \) if \( l_1 \lor l_2 \in L' \) and \( l_1 \land l_2 \in L' \) for all \( l_1, l_2 \) in \( L' \).

A sublattice \( L' \) of \( L \) is called normal, if
1. for all \( l' \in L' \) and all \( l \in L \)
\[
|l| \leq |l'| \implies l \in L'
\]
2. if for \( \{ l_i \}_{i \in I} \subset L' \) the supremum exists in \( L \), then \( \bigvee_{i \in I} l_i \in L' \).

In order to decompose a vector lattice into normal sublattices, a notion of orthogonality is needed (cf. [13, p. 29]).
**Definition 2.3.** For a subset $L'$ of $L$, the set

$$(L')^\perp := \{ l \in L \mid \forall l' \in L' : l \perp l' \}$$

is called **orthogonal complement** of $L'$.

The following statement from [13, p. 29f] illustrates that normal sublattices and orthogonality interact in a similar way as closed linear subspaces and orthogonality in Hilbert spaces do.

**Proposition 2.4.** Let $S \subset L$, then $S^\perp$ is a normal sublattice of $L$. If $S$ is a normal sublattice, then $(S^\perp)^\perp = S$.

A useful characterisation of the orthogonal complement of a normal sublattice is the following.

**Proposition 2.5.** Let $S$ be a normal sublattice of $L$. Then $l \in S^\perp$ if and only if for every $s \in S$

$$0 \leq |s| \leq |l| \implies s = 0.$$ 

**Proof.** Assume first that $l \in S^\perp$. Then for every $s \in S$

$$0 \leq |s| \leq |l| \implies 0 = |s| \wedge |l| = |s| \implies s = 0.$$ 

Now assume for every $s \in S$

$$0 \leq |s| \leq |l| \implies s = 0.$$ 

Since $S$ is a normal sublattice

$$0 \leq |s| \wedge |l| \leq |s| \implies |s| \wedge |l| \in S.$$ 

By assumption

$$|s| \wedge |l| \leq |l| \implies |s| \wedge |l| = 0.$$ 

Thus $s \perp l$. \qed

As in the setting of Hilbert spaces, a boundedly complete vector lattice can be represented as the direct sum of a normal sublattice and its orthogonal complement (cf. [13, p. 29]).

**Proposition 2.6.** **Riesz Decomposition Theorem**

Let $S$ be a normal sublattice of $L$, then for every $l \in L$ there exist unique elements $s \in S, s^\perp \in S^\perp$ such that

$$l = s + s^\perp.$$ 

Furthermore, if $l \geq 0$, then $s = \bigvee_{s' \in S} l \wedge |s'|$. For general $l \in L$

$$s = \bigvee_{s' \in S^+} l^+ \wedge |s'| - \bigvee_{s' \in S^-} l^- \wedge |s'|.$$
The following proposition enables the successive decomposition of a lattice into sublattices. This is used in the analysis of measures. In particular, this proposition enables a better characterisation of the dual of the space of essentially bounded functions.

**Proposition 2.7.** Let $L_1, L_2$ be two normal sublattices of $L$. Then $L_1 \cap L_2$ is a normal sublattice of $L_2$. Furthermore, the orthogonal complement of $L_1 \cap L_2$ in $L_2$ is $L_1^\perp \cap L_2$.

**Proof.** Let $l_1 \in L_1 \cap L_2$ and $l_2 \in L_2$ with

$$|l_2| \leq |l_1|.$$  

Since $L_1$ is a normal sublattice of $L$,

$$l_2 \in L_1.$$  

Whence $l_2 \in L_1 \cap L_2$.

Now, let $\{l_i\}_{i \in I} \subset L_1 \cap L_2$ be such that $\bigvee_{i \in I} l_i \in L$. Since $L_1$ and $L_2$ are normal,

$$\bigvee_{i \in I} l_i \in L_1 \quad \text{and} \quad \bigvee_{i \in I} l_i \in L_2.$$  

This implies $\bigvee_{i \in I} l_i \in L_1 \cap L_2$. Thus $L_1 \cap L_2$ is a normal sublattice of $L_2$.

Let $l_2 \in L_2$ such that $l_2 \in (L_1 \cap L_2)^\perp$. Since $L_1$ is a normal sublattice of $L$, there exist $l_1 \in L_1, l_1^\perp \in L_1^\perp$ such that $l_2 = l_1 + l_1^\perp$. Now, using additivity of the total variation on orthogonal elements (cf. [13, p. 25])

$$0 \leq \sup\{|l_1|, |l_1^\perp|\} \leq |l_1| + |l_1^\perp| = |l_2|.$$  

Hence, $l_1, l_1^\perp \in L_2$ and $l_1, l_1^\perp \in (L_1 \cap L_2)^\perp$. Since $l_2 \in (L_1 \cap L_2)^\perp$,

$$0 = |l_2| \wedge |l_1| = |l_1| \wedge |l_1| + |l_1| \wedge |l_1^\perp| = |l_1| \wedge |l_1|.$$  

This implies $l_1 = 0$. Hence

$$(L_1 \cap L_2)^\perp \subset L_1^\perp \cap L_2.$$  

On the other hand, if $l_1^\perp \in L_1^\perp \cap L_2$, then for all $l_1 \in L_1 \cap L_2$

$$|l_1| \wedge |l_1^\perp| = 0,$$

whence

$$L_1^\perp \cap L_2 \subset (L_1 \cap L_2)^\perp.$$  

\[\square\]
3 A Primer On Pure Measures

In this article, set functions $\mu : \mathcal{A} \subset 2^\Omega \to \mathbb{R}$ will be called measure, if for all $m \in \mathbb{N}$ and every pairwise disjoint $\{A_k\}_{k=1}^m \subset \mathcal{A}$ with $\bigcup_{k=1}^m A_k \in \mathcal{A}$

$$\mu \left( \bigcup_{k=1}^m A_k \right) = \sum_{k=1}^m \mu(A_k).$$

If this holds with $m = \infty$, the measure is called $\sigma$-measure. A measure is called bounded if

$$\sup_{A \in \mathcal{A}} |\mu(A)| < \infty.$$

An algebra is a class of sets which is stable under union, intersection and differences and contains at least $\emptyset$. The spaces of measures considered in this paper are defined in accordance with [13].

The spaces of measures considered in this thesis are defined in accordance with [13].

**Definition 3.1.** Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra. The set of all bounded measures $\mu : \mathcal{A} \to \mathbb{R}$ is denoted by

$$\text{ba}(\Omega, \mathcal{A}).$$

The set of all bounded $\sigma$-measures $\sigma : \mathcal{A} \to \mathbb{R}$ is denoted by

$$\text{ca}(\Omega, \mathcal{A}).$$

There is a natural partial order on $\text{ba}(\Omega, \mathcal{A})$ (cf. [13, p. 43]).

**Definition 3.2.** Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra. For $\mu, \lambda \in \text{ba}(\Omega, \mathcal{A})$ one writes

$$\mu \leq \lambda$$

if and only if for every $A \in \mathcal{A}$

$$\mu(A) \leq \lambda(A).$$

The following proposition links the theory of measures with the theory of boundedly complete vector lattices. This is essential for the subsequent results on the decomposition of measures. The proposition is taken from [13, p. 43f].

**Proposition 3.3.** Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra. Then $\text{ba}(\Omega, \mathcal{A})$ together with the partial order $\leq$ is a boundedly complete vector lattice.

The following definitions are standard in measure theory (cf. [13, p. 45]).
Definition 3.4. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. For $\mu \in \text{ba}(\Omega, \mathcal{A})$ define

$$
\mu^+ := \mu \lor 0 = \sup\{\mu, 0\} \\
\mu^- := (-\mu) \lor 0 = \sup\{-\mu, 0\} \\
|\mu| := \mu^+ + \mu^-.
$$

Call $\mu^+$ positive part of $\mu$, $\mu^-$ negative part of $\mu$ and $|\mu|$ total variation of $\mu$.

Furthermore, for $A \in \mathcal{A}$ define $\mu[A] : \mathcal{A} \to \mathbb{R}$ by

$$(\mu[A])(A') := \mu(A \cap A') \text{ for all } A' \in \mathcal{A}.$$  

The total variation and the lattice operations can be characterised in the following way (cf. [13, p. 46], [14, p. 48]).

Proposition 3.5. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. Then for every $\mu, \lambda \in \text{ba}(\Omega, \mathcal{A})$ and $A \in \mathcal{A}$

$$
|\mu|(A) = \sup \left\{ \sum_{k=1}^{m} |\mu(A_k)| \middle| \{A_k\} \subset \mathcal{A} \right\} \\
(\mu \land \lambda)(A) = \inf_{A' \subset A, A' \subset A} \mu(A') + \lambda(A \setminus A').
$$

where the supremum is taken over all finite partitions $\{A_k\}_{k=0}^{m} \subset \mathcal{A}$ of $A$.

The following proposition can be found in Rao [13, p. 44]. It states that in the space of bounded measures, the norm is compatible with the partial order.

Proposition 3.6. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. Then $\text{ba}(\Omega, \mathcal{A})$ together with $\leq$ and the norm

$$
\|\mu\| := |\mu|(\Omega) \text{ for } \mu \in \text{ba}(\Omega, \mathcal{A})
$$

is a Banach lattice, i.e. it is a Banach space and a vector lattice such that for all $\mu, \lambda \in \text{ba}(\Omega, \mathcal{A})$

$$
|\mu| \leq |\lambda| \implies \|\mu\| \leq \|\lambda\|.
$$

The following proposition is an application of Riesz’s decomposition Theorem (Proposition 2.6) (cf. [13, p. 241]). In particular, every bounded measure can be uniquely decomposed into a $\sigma$-measure and a pure measure.
Proposition 3.7. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. Then $ba(\Omega, A)$ is a boundedly complete vector lattice and $ca(\Omega, A)$ one of its normal sublattices. Hence, every $\mu \in ba(\Omega, A)$ can uniquely be decomposed into $\mu_c \in ca(\Omega, A)$ and $\mu_p \in ca(\Omega, A)^\perp$ such that

$$\mu = \mu_c + \mu_p$$

and for every $\sigma \in ca(\Omega, A)$

$$0 \leq \sigma \leq |\mu_p| \implies \sigma = 0.$$

Definition 3.8. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. Then every measure $\mu_p \in ca(\Omega, A)^\perp$ is called pure. Notice that $\mu_p$ is not $\sigma$-additive, by definition.

One important example of measures that are pure are density measures. The following new example presents a particular density measure, namely a density at zero. In the literature, examples of pure measure are only known for $\Omega = \mathbb{N}$ (cf. [13, p. 247]), they are defined on very small algebras (cf. [13, p. 246]) or they are constructed in such a way that the measure cannot be computed explicitly, even on simple sets (cf. [14, p. 57f]). The example given here is constructed on $\Omega = \mathbb{R}^n$ and lives on the Borel subsets of $\Omega$.

Example 3.9. Let $\Omega := B_1(0) \subset \mathbb{R}^n$ be open. Then there exists $\mu \in ba(\Omega, B(\Omega)), \mu \geq 0$ such that for every $B \in B(\Omega)$

$$\mu(B) = \lim_{\delta \downarrow 0} \frac{L^n(B \cap B_\delta(0))}{L^n(B_\delta(0))}$$

if this limit exists. This measure is non-unique. Its existence is shown in Proposition 5.7 (take $\lambda := L^n$ and $C = \{0\}$).

It is shown in Example 3.17 that $\mu$ is indeed pure. Figure 1 shows the family $\{A_k\}_{k \in \mathbb{N}} \subset B(\Omega)$

$$A_k := \left[\frac{1}{k + 2}, \frac{1}{k + 1}\right) \times [-1,1]^{n-1}.$$

For this family

$$\sum_{k \in \mathbb{N}} \mu(A_k \cap \Omega) = 0 \neq \mu \left(\left(0, \frac{1}{2}\right) \times [-1,1]^{n-1} \cap \Omega\right) = \mu \left(\bigcup_{k=1}^{\infty} A_k \cap \Omega\right).$$

Hence, $\mu$ is not a $\sigma$-measure.
Figure 1: A family of sets on which $\mu$ is not $\sigma$-additive

Measures that do not charge sets of Lebesgue measure zero are of special interest, because these measures lend themselves naturally to the integration of functions that are only defined outside of a set of measure zero. When treating non $\sigma$-additive measures, one carefully has to distinguish the following two notions (cf. [13, p. 159]).

**Definition 3.10.** Let $\Omega \subset \mathbb{R}^n, \mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \text{ba}(\Omega, \mathcal{A})$. Then $\mu \in \text{ba}(\Omega, \mathcal{A})$ is called

1. **absolutely continuous** with respect to $\lambda$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$

   $$|\lambda|(A) < \delta \implies |\mu(A)| < \varepsilon.$$ 

   In this case, write $\mu << \lambda$.

2. **weakly absolutely continuous** with respect to $\lambda$, if for every $A \in \mathcal{A}$

   $$|\lambda|(A) = 0 \implies \mu(A) = 0.$$ 

   In this case, write $\mu <<^w \lambda$.

The set of all weakly absolutely continuous measures in $\text{ba}(\Omega, \mathcal{A})$ is denoted by

$$\text{ba}(\Omega, \mathcal{A}, \lambda).$$

The following proposition shows that there is no pure measure which is absolutely continuous with respect to some $\sigma$-measure (cf. [13, p. 163]).

**Proposition 3.11.** Let $\Omega \subset \mathbb{R}^n, \mathcal{A} \subset 2^\Omega$ be an algebra and $\sigma \in \text{ca}(\Omega, \mathcal{A})$. Then for every $\mu \in \text{ba}(\Omega, \mathcal{A})$

$$\mu << \sigma \implies \mu \in \text{ca}(\Omega, \mathcal{A}).$$
Remark 3.12. The preceding proposition shows that one should focus on the notion of weak absolute continuity when studying measures that are continuous with respect to some $\sigma$-measure.

Example 3.13. $\mu$ from Example 3.9 is even weakly absolutely continuous with respect to $L^n$. This is evident from the construction in Proposition 5.7 (take $\lambda := L^n$ and $C := \{0\}$).

Proposition 3.14. Let $\mu_1, \mu_2 \in \text{ba}(\Omega, \mathcal{A})$ be such that $\mu_1 << \mu_2$. If $A \in \mathcal{A}$ such that $|\mu_2|(A) = 0$, then $|\mu_1|(A) = 0$.

Proof. Since $|\mu_2|$ is monotone,

$|\mu_2(A')| \leq |\mu_2|(A') \leq |\mu_2|(A) = 0$

for all $A' \in \mathcal{A}$ such that $A' \subset A$. Since

$\mu_1^+(A) = \sup_{A' \in \mathcal{A}, A' \subset A} \mu_1(A') = 0$

and a similar equation holds for $\mu_1^-$

$|\mu_1|(A) = \mu_1^+(A) + \mu_1^-(A) = 0$.

The following proposition is the key to decompose measures into $\sigma$-measures which are weakly absolutely continuous with respect to some measure and pure measures.

Proposition 3.15. Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \text{ba}(\Omega, \mathcal{A})$.

Then $\text{ba}(\Omega, \mathcal{A}, \lambda)$ is a normal sublattice of $\text{ba}(\Omega, \mathcal{A})$ and thus a boundedly complete vector lattice.

Proof. $\text{ba}(\Omega, \mathcal{A}, \lambda)$ is obviously a linear space. Let $\{\mu_i\}_{i \in \mathcal{I}} \subset \text{ba}(\Omega, \mathcal{A}, \lambda)$ be such that there exists $\mu \in \text{ba}(\Omega, \mathcal{A})$ with

$\mu_i \leq \mu$ for all $i \in \mathcal{I}$.

By Proposition 3.3 $\text{ba}(\Omega, \mathcal{A})$ is boundedly complete (cf. [13] p. 44). Hence, there exists $\mu' \in \text{ba}(\Omega, \mathcal{A})$ such that

$\mu_i \leq \mu'$ for all $i \in \mathcal{I}$

and if this holds true for another $\mu'' \in \text{ba}(\Omega, \mathcal{A})$ then $\mu' \leq \mu''$.

Assume $\mu' \notin \text{ba}(\Omega, \mathcal{A}, \lambda)$. Then there exists $A \in \mathcal{A}$ such that

$|\lambda|(A) = 0$ but $\mu'(A) \neq 0$. 

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Now, \(|\mu'|[A]| \in \text{ba}(\Omega, A)\). Whence \(\mu - |\mu'|[A]| \in \text{ba}(\Omega, A)\). Since \(\mu_i(A) = 0\)
\[
\mu_i \leq |\mu'|[A]| < \mu'
\]
in contradiction to the minimality of \(\mu'\). Hence \(\mu' \in \text{ba}(\Omega, A, \lambda)\).

Now let \(\mu' \in \text{ba}(\Omega, A)\) and \(\mu \in \text{ba}(\Omega, A, \lambda)\) such that \(|\mu'| \leq |\mu|\). Let \(A \in A\) be such that \(|\lambda|(A) = 0\). Then
\[
|\mu'|(A) \leq |\mu|(A) \leq |\mu| = 0
\]
by Proposition 3.14. Hence \(\mu' \in \text{ba}(\Omega, A, \lambda)\). Therefore, \(\text{ba}(\Omega, A, \lambda)\) is a normal sublattice and thus a boundedly complete vector lattice.

The proposition above enables the decomposition of measures into pure parts and \(\sigma\)-measures, analogously to Proposition 3.7.

**Theorem 3.16.** Let \(\Omega \subset \mathbb{R}^n, A \subset 2^\Omega\) be an algebra and \(\lambda \in \text{ba}(\Omega, A)\).

Then for every \(\mu \in \text{ba}(\Omega, A, \lambda)\) there exist unique \(\mu_c \in \text{ca}(\Omega, A) \cap \text{ba}(\Omega, A, \lambda)\), \(\mu_p \in \text{ca}(\Omega, A)^\perp \cap \text{ba}(\Omega, A, \lambda)\) such that
\[
\mu = \mu_c + \mu_p.
\]

**Proof.** Since \(\text{ba}(\Omega, A, \lambda)\) and \(\text{ca}(\Omega, A)\) are normal sublattices of \(\text{ba}(\Omega, A)\), Proposition 2.7 yields that
\[
\text{ca}(\Omega, A) \cap \text{ba}(\Omega, A, \lambda)
\]
is a normal sublattice of \(\text{ba}(\Omega, A, \lambda)\) whose orthogonal complement is
\[
\text{ca}(\Omega, A)^\perp \cap \text{ba}(\Omega, A, \lambda).
\]
This, together with Riesz’s decomposition Proposition 2.6 yields the statement of the theorem. 

**Example 3.17.** Since the measure \(\mu\) from Example 3.9 is positive and \(\mu_c \perp \mu_p\), using the additivity of the total variation on orthogonal element (cf. [13, p. 25]) yields
\[
0 \leq |\mu_c| \leq |\mu_c| + |\mu_p| = |\mu| = \mu.
\]
Hence, for every \(\delta > 0\)
\[
|\mu_c|(B_\delta(0)^c) = 0.
\]
Thus
\[
|\mu_c| (\Omega \setminus \{0\}) = \lim_{\delta \downarrow 0} |\mu_c|(B_\delta(0)^c) = 0.
\]
But \(|\mu_c|(\{0\}) \leq \mu(\{0\}) = 0\). Hence
\[
|\mu_c| (\Omega) = 0
\]
and \(\mu = \mu_p\) is pure.
When $\lambda$ is a $\sigma$-measure, the structure of $\mu_c$ is well known by the Radon Nikodym theorem (cf. [9, p. 128ff]).

**Proposition 3.18.** Radon-Nikodym Theorem

Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra. Furthermore, let $\sigma \in \text{ca}(\Omega, \Sigma)$ and $\mu \in \text{ca}(\Omega, \Sigma)$ be such that $\mu \ll^w \sigma$. Then there exists $f \in L^1(\Omega, \Sigma, \sigma)$ such that

$$
\mu(A) = \int_A f \, d\sigma
$$

for every $A \in \Sigma$.

The structure of $\mu_p$ is described by the following proposition taken from [13, p. 244] (cf. [14, p. 56]).

**Remark 3.19.** The following results are stated for $\sigma$-measures $\sigma \geq 0$. They also hold for arbitrary $\sigma$-measures $\sigma$ when using $|\sigma|$.

**Proposition 3.20.** Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra and $\sigma \in \text{ca}(\Omega, \Sigma)$, $\sigma \geq 0$. Then $\mu \in \text{ba}(\Omega, \Sigma, \sigma)$ is pure if and only if there exists a decreasing sequence $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$ such that

$$
\sigma(A_k) \xrightarrow{k \to \infty} 0
$$

and for all $k \in \mathbb{N}$

$$
|\mu_p|(A_k^c) = 0.
$$

Intuitively speaking, weakly absolutely continuous measures are pure if and only if they concentrate in the vicinity of a set of measure zero. Reviewing Example [5.3] the support (cf. [2, p.30]) of the measure can be seen to lie outside of $\Omega \setminus \{0\}$. Yet the construction of the measure would still work on this set. Hence, it is possible for a pure measure to have support outside of its domain of definition. This necessitates the following definition of core.

**Definition 3.21.** Let $\Omega \subset \mathbb{R}^n$, $A \subset 2^\Omega$ be an algebra containing every relatively open set in $\Omega$. Furthermore let $\mu \in \text{ba}(\Omega, A)$. Then the set

$$
core \mu := \{x \in \mathbb{R}^n \mid |\mu|(V \cap \Omega) > 0, \forall V \subset \mathbb{R}^n, V \text{ open}, x \in V\}
$$

is called core of $\mu$.

Let $d \in [0, n]$ be the Hausdorff dimension of core $\mu$. Then $d$ is called core dimension of $\mu$ and $\mu$ is called $d$-dimensional.

**Remark 3.22.** Note that there is a slight difference to the notion of support of a measure as defined in classic measure theory (cf. [3, p. 60]). The core of a measure is not necessarily contained in $\Omega$, the support of a $\sigma$-measure is.
Example 3.23. The measure $\mu$ from Example 3.9 has

$$\text{core } \mu = \{0\}$$

and is thus 0-dimensional.

Now, an example for a density measure with a larger core is given. Note that in this thesis

$$C_\delta := \text{dist}^{-1}_\Omega((\infty, \delta)) \text{ for } C \subset \mathbb{R}^n.$$  

Example 3.24. Let $\Omega \subset \mathbb{R}^n$ be open, $d \in [0, n)$ and $C \subset \Omega$ be closed with Hausdorff dimension $d$. Then there exists a pure measure $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$, $\mu \geq 0$ such that for every $B \in \mathcal{B}(\Omega)$

$$\mu(B) = \lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B \cap C_\delta \cap \Omega)}{\mathcal{L}^n(C_\delta \cap \Omega)} =: \text{dens}_C(B),$$

if this limit exists. Here, $C_\delta$ is the open $\delta$-neighbourhood of $C$. Furthermore

$$\text{core } \mu = C$$

and $\mu$ is thus $d$-dimensional.

The existence of this measure is evident by Proposition 5.7 (take $\lambda := \mathcal{L}^n$).

Proposition 3.25. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra containing every relatively open set and $\mu \in \text{ba}(\Omega, A)$. Then core $\mu$ is a closed set in $\mathbb{R}^n$.

Proof. Set $B := \text{core } \mu$ and let $x \in B^c$. Then there is an open neighbourhood $V \subset \mathbb{R}^n$ of $x$ such that

$$|\mu|(V \cap \Omega) = 0.$$  

Now let $x' \in V$ and $V' \subset \mathbb{R}^n$ be an open neighbourhood of $x'$. Then

$$|\mu|(V \cap V' \cap \Omega) \leq |\mu|(V \cap \Omega) = 0.$$  

Thus, $x' \in B^c$. Since $x$ was arbitrary, it follows that for every $x \in B^c$ there exists an open neighbourhood $V \subset \mathbb{R}^n$ of $x$ such that $V \subset B^c$, whence $B^c$ is open and $B$ closed. 

On bounded domains, the core is non-empty.

Proposition 3.26. Let $\Omega \subset \mathbb{R}^n$ be bounded, $A \subset 2^\Omega$ be an algebra containing every relatively open set in $\Omega$ and $\mu \in \text{ba}(\Omega, A)$, $\mu \neq 0$. Then core $\mu$ is non-empty and for every $\delta > 0$

$$|\mu|(\Omega \cap ((\text{core } \mu)_\delta)^c) = 0.$$
Proof. Set $B := \text{core } \mu$. Assume core $\mu$ was empty. Then, by compactness of $\overline{\Omega}$ there exists an open covering $\{V_k\}_{k=0}^m$ of $\overline{\Omega}$ such that for $k = 0, \ldots, m$

$$|\mu|(V_k \cap \Omega) = 0.$$ 

But then

$$|\mu|(\Omega) \leq \sum_{k=0}^m |\mu|(V_k \cap \Omega) = 0$$

in contradiction to $\mu \neq 0$.

Now, let $\delta > 0$. For every $x \in (B_\delta)^c$ there is a $0 < \delta_x < \frac{\delta}{2}$ such that

$$|\mu|(B(x, \delta_x) \cap \Omega) = 0.$$ 

Otherwise, $x \in \text{core } \mu$. Now

$$\{B(x, \delta_x)\}_{x \in (B_\delta)^c}$$

is an open covering of

$$(B_\delta)^c \cap \Omega.$$ 

Since $\Omega$ is relatively compact in $\mathbb{R}^n$, there exists a finite open sub-covering

$$\{B(x_l, \delta_{x_l})\}_{l=0}^m$$

of

$$(B_\delta)^c \cap \Omega.$$ 

Hence

$$|\mu|((B_\delta)^c \cap \Omega) \leq \sum_{l=0}^m |\mu|(B(x_l, \delta_{x_l}) \cap \Omega) = 0.$$ 

Remark 3.27. If $\Omega$ is unbounded, the statement of the preceding proposition need not be true. The measures in Example 10.4.1 in [13, p. 245] can be shown to have empty core, since they concentrate near infinity.

The core itself does not give all information on the way in which a pure measure concentrates. Hence, the sequences from Proposition 3.20 are investigated further.

Definition 3.28. Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra, $\sigma \in \text{ca}(\Omega, \Sigma)$, $\sigma \geq 0$ and $\mu_p \in \text{ba}(\Omega, \Sigma, \sigma)$ be pure. Then every $A \in \Sigma$ such that

$$|\mu_p|(A^c) = 0$$

is called aura of $\mu_p$.

Any decreasing sequence $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$ of auras for $\mu_p$ such that

$$\sigma(A_k) \xrightarrow{k \to \infty} 0$$

is called aura sequence.
Now, it is shown that any aura sequence can be restricted to neighbourhoods of the core.

**Proposition 3.29.** Let \( \Omega \subset \mathbb{R}^n \) be bounded and \( \Sigma \subset 2^\Omega \) be a \( \sigma \)-algebra containing every relatively open set in \( \Omega \). Furthermore, let \( \sigma \in \text{ca}(\Omega, \Sigma) \) with \( \sigma \geq 0 \) and \( \mu_p \in \text{ba}(\Omega, \Sigma, \sigma) \) be pure. Then for every aura sequence \( \{A_k\}_{k \in \mathbb{N}} \subset \Sigma \) of \( \mu_p \) the sequence

\[
\{A'_k\}_{k \in \mathbb{N}} := \left\{ A_k \cap (\text{core } \mu_p) \right\} \subset \Sigma
\]

is an aura sequence of \( \mu_p \) with

\[
\text{core } \mu_p = \bigcap_{k \in \mathbb{N}} A'_k \mathbb{R}^n.
\]

**Proof.** Let \( C := \text{core } \mu_p \). Note that \( |\mu_p| \) is pure and let \( \{A_k\}_{k \in \mathbb{N}} \subset \Sigma \) be any aura sequence of \( \mu_p \). Then for every \( k \in \mathbb{N} \), \( x \in \left( \overline{A_k} \mathbb{R}^n \right)^c \) and any open neighbourhood \( V \subset \left( \overline{A_k} \mathbb{R}^n \right)^c \) of \( x \)

\[
|\mu_p|(V \cap \Omega) \leq |\mu_p| \left( \left( \overline{A_k} \mathbb{R}^n \right)^c \cap \Omega \right) \leq |\mu_p|(A'_k \cap \Omega) = 0.
\]

Hence

\[
C \subset \overline{A_k} \mathbb{R}^n \quad \text{for every } k \in \mathbb{N}.
\]

Thus,

\[
C \subset \bigcap_{k \in \mathbb{N}} \overline{A_k} \mathbb{R}^n.
\]

For \( k \in \mathbb{N} \) set

\[
A'_k := A_k \cap C_1^\mathbb{R}_k.
\]

Then for every \( k \in \mathbb{N} \)

\[
|\mu_p|(A'_k) \leq |\mu_p|(A_k) + |\mu_p| \left( \left( C_1^\mathbb{R} \right)^c \cap \Omega \right) = 0,
\]

by Proposition 3.26.

Furthermore

\[
0 \leq \sigma(A'_k) \leq \sigma(A_k) \xrightarrow{k \to \infty} 0.
\]

Obviously

\[
\bigcap_{k \in \mathbb{N}} A_k \cap C_1^\mathbb{R}_k \subset \bigcap_{k \in \mathbb{N}} C_1^\mathbb{R}_k = C.
\]

It remains to show that

\[
C \subset \bigcap_{k \in \mathbb{N}} A_k \cap C_1^\mathbb{R}_k.
\]
Let \( x \in C \). Then \( x \in \overline{A_k}^{\mathbb{R}^n} \) for every \( k \). Hence, for every \( k \) there is a sequence \( \{x_l^k\}_{l \in \mathbb{N}} \subseteq A_k \) such that
\[
x_l^k \xrightarrow{l \to \infty} x. \]
In particular, there is an \( t_0^k \in \mathbb{N} \) such that
\[
\|x_l^k - x\| < \frac{1}{k} \quad \text{for } l \geq t_0^k.
\]
Hence, for every \( k \in \mathbb{N} \),
\[
x \in \overline{A_k} \cap C^{\mathbb{R}^n}. \]
Since \( x \in C \) was arbitrary, this finally implies
\[
C \subseteq \bigcap_{k \in \mathbb{N}} \overline{A_k} \cap C^{\mathbb{R}^n}. \]

Figure 2: An aura sequence \( \{A_k\}_{k \in \mathbb{N}} \) of a 1-dimensional measure with core \( C = \bigcap_{k \in \mathbb{N}} A_k \)

The following lemma identifies a big class of pure measures. In particular, if the core of a measure is a Lebesgue null set, the measure is necessarily pure.

**Proposition 3.30.** Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \) and \( \mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \).

If core \( \mu \cap \Omega \) is a \( \mathcal{L}^n \)-null set then \( \mu \) is pure.

**Proof.** Let \( B := \text{core} \mu \). Then by the definition of the core, for every \( \delta > 0 \)
\[
|\mu| (B^c_{\delta} \cap \Omega) = 0.
\]

Now let $B_k := B_1 \cap \Omega$ for $k \in \mathbb{N}$ and $\sigma \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$, $\sigma \geq 0$ be a $\sigma$-measure such that 

$$0 \leq \sigma \leq |\mu|.$$ 

Then for every $k \in \mathbb{N}$ 

$$0 \leq \sigma((B_k)^c) \leq |\mu|(B_k^c) = 0.$$ 

On the other hand, since core $\mu \cap \Omega$ is a $\mathcal{L}^n$-null set, 

$$\sigma(\Omega \cap B) = 0.$$ 

Hence 

$$\sigma(\Omega) = \sigma(\Omega \cap B) + \sigma \left( \bigcup_{k \in \mathbb{N}} B_k^c \right) = \lim_{k \to \infty} \sigma(B_k^c) = 0.$$ 

This implies $\sigma = 0$.

Since $\sigma$ was arbitrary, $\mu$ is pure by Proposition 2.5 and Proposition 3.7.

**Remark 3.31.** Note that core $\mu \subset \Omega$. If $\Omega \subset \mathbb{R}^n$ is open such that $\mathcal{L}^n(\partial\Omega) > 0$, then there is $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that core $\mu = \partial\Omega$. Hence core $\mu$ is not a null set, but core $\mu \cap \Omega = \emptyset$. Thus, $\mu$ is necessarily pure.

The following proposition is taken from [13, p. 70]. It shows that there are many degrees of freedom when choosing an extension of a measure to a larger class of sets. Since all pure measures used below are constructed using an extension argument, they are in general not unique.

**Proposition 3.32.** Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra on $\Omega$. Let $\mu \in \text{ba}(\Omega, A), \mu \geq 0$. Let $A \in 2^\Omega \setminus A$ and $A' \subset 2^\Omega$ the smallest algebra such that $A, \{A\} \subset A'$. Then for any $c \in [0, \infty)$ such that 

$$\sup\{\mu(A') \mid A' \in \mathcal{A}, A' \subset A\} \leq c \leq \inf\{\mu(A') \mid A' \in \mathcal{A}, A' \subset A\}$$

there exists an extension $\mu' \in \text{ba}(\Omega, A')$, $\mu' \geq 0$ of $\mu$ to all of $A'$ such that 

$$\mu'(A) = c.$$ 

## 4 Integration Theory and $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)^*$

Now, integration with respect to measure which are not necessarily $\sigma$-additive is outlined. Measurability of functions is not defined through the regularity of preimages but by approximability by simple functions in measure. In this definition, the measure is needed on possibly non-measurable sets. Hence, an outer measure has to be used. This outer measure is defined as in the case of $\sigma$-measures (cf. [13, p. 86], [9, p. 42]).
Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra. For $\mu \in \text{ba}(\Omega, A)$, $\mu \geq 0$ the outer measure of $\mu$ is defined for $B \in 2^\Omega$ by
$$
\mu^*(B) := \inf_{A \in A, B \subset A} \mu(A).
$$

Now, convergence in measure can be defined. The definition is taken from [13, p. 92] (cf. [9, p. 91]).

Definition 4.2. Let $\Omega \subset \mathbb{R}^n$ and $A \subset 2^\Omega$ be an algebra and $\mu : A \to \mathbb{R}$ be a measure. A sequence $\{f_k\}_{k \in \mathbb{N}}$ of functions $f_k : \Omega \to \mathbb{R}$ is said to converge in measure to a function $f : \Omega \to \mathbb{R}$ if for every $\varepsilon > 0$
$$
\lim_{k \to \infty} |\mu|^*\{x \in \Omega \mid |f_k(x) - f(x)| > \varepsilon\} = 0.
$$
In this case, write $f_k \mu \nrightarrow f$.

Note that the limit in measure is not unique, yet. Therefore, the following notion of equality almost everywhere is needed. The definition is taken from [13, p. 88].

Definition 4.3. Let $\Omega \subset \mathbb{R}^n$, $A \subset 2^\Omega$ and $\mu : A \to \mathbb{R}$ be a measure.

Then $f : \Omega \to \mathbb{R}$ is called null function, if for every $\varepsilon > 0$
$$
|\mu|^*\{x \in \Omega \mid |f(x)| > \varepsilon\} = 0.
$$

Two functions $f_1 : \Omega \to \mathbb{R}$, $f_2 : \Omega \to \mathbb{R}$ are called equal almost everywhere (a.e.) with respect to $\mu$, if $f_1 - f_2$ is a null function.

In this case, write $f_1 = f_2 \mu$-a.e.

Remark 4.4. If $f : \Omega \to \mathbb{R}$ is a null function, then it need not be true that
$$
|\mu|^*\{x \in \Omega \mid f(x) \neq 0\} = 0. \tag{1}
$$
Take e.g. the density measure $\mu$ introduced in Example 3.9 and $f(x) := |x|$. Then $f$ is a null function but
$$
|\mu|^*\{x \in \mathbb{R}^n \mid f(x) \neq 0\} = \mu(B_1(0) \setminus \{0\}) = 1 > 0.
$$

This entails that the notion of equality almost everywhere that was defined above does not imply the existence of a null set such that $f_1 = f_2$ outside of that set. Take e.g. the density measure introduced in Example 3.9 $f_1(x) := |x|$ and $f_2(x) := 2f_1(x)$.

On the other hand, if $\mu$ is a $\sigma$-measure and $A$ a $\sigma$-algebra, then Equation (1) is equivalent to $f$ being a null function (cf. [13, p. 89]).
The limit in measure turns out to be unique in the sense of almost equality. This is stated in the following proposition taken from [13, p. 92].

**Proposition 4.5.** Let \( \Omega \subset \mathbb{R}^n \), \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. Furthermore let \( \{f_k\}_{k \in \mathbb{N}} \) be a sequence of functions \( f_k : \Omega \to \mathbb{R} \) and \( f, \tilde{f} : \Omega \to \mathbb{R} \) be functions such that

\[
f_k \xrightarrow{\mu} f.
\]

Then

\[
f_k \xrightarrow{\mu} \tilde{f} \iff f = \tilde{f} \text{ } \mu \text{-a.e.}
\]

Now, the notion of measurability is introduced. The definition is similar to the definition of \( T_1 \)-measurability in [13, p. 101].

**Definition 4.6.** Let \( \Omega \subset \mathbb{R}^n \) and \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. A function \( f : \Omega \to \mathbb{R} \) is called **measurable** if there exists a sequence \( \{h_k\}_{k \in \mathbb{N}} \) of simple functions \( h_k : \Omega \to \mathbb{R} \) such that

\[
h_k \xrightarrow{\mu} f.
\]

The integral for measurable functions can now be defined via \( L^1 \)-Chauchy sequences. This is of course well-defined (cf. [13, p. 102]).

**Definition 4.7.** Let \( \Omega \subset \mathbb{R}^n \), \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. A function \( f : \Omega \to \mathbb{R} \) is said to be **integrable** if there exists a sequence \( \{h_k\}_{k \in \mathbb{N}} \) of integrable simple functions \( h_k : \Omega \to \mathbb{R} \) such that

1. \( h_k \xrightarrow{\mu} f \).
2. \( \lim_{k,l \to \infty} \int_{\Omega} |h_k - h_l| \, d|\mu| = 0 \).

In this case, denote

\[
\int_{\Omega} f \, d\mu := \lim_{k \to \infty} \int_{\Omega} h_k \, d\mu.
\]

The sequence \( \{h_k\}_{k \in \mathbb{N}} \) is called **determining sequence** for the integral of \( f \).

**Remark 4.8.** In particular, integrable functions are measurable. This notion of integral is also called Daniell-Integral in the literature (cf. [13]).

The \( L^p \)-spaces are defined in the usual way (cf. [13, p. 121]).

**Definition 4.9.** Let \( \Omega \subset \mathbb{R}^n \), \( A \subset 2^\Omega \) be an algebra, \( \mu : A \to \mathbb{R} \) be a measure and \( p \in [1, \infty) \). Then the set of all measurable functions \( f : \Omega \to \mathbb{R} \) such that \( |f|^p \) is \( |\mu| \)-integrable is denoted by

\[
L^p(\Omega, A, \mu).
\]

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If $\mathcal{A} = \mathcal{B}(\Omega)$, write $L^p(\Omega, \mu)$.

For $f_1, f_2 \in L^p(\Omega, \mathcal{A}, \mu)$

$$f_1 = f_2 \text{ } \mu\text{-a.e.}$$

defines an equivalence relation. The set of all equivalence classes of this relation is denoted by $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$.

If $\mathcal{A} = \mathcal{B}(\Omega)$, write $L^p(\Omega, \mu)$.

Definition 4.10. Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \rightarrow \mathbb{R}$ a measure. Then for every $p \in [1, \infty)$ and $f \in L^p(\Omega, \mathcal{A}, \mu)$ write

$$\|f\|_p := \left(\int_{\Omega} |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

Furthermore, for measurable $f : \Omega \rightarrow \mathbb{R}$ define

$$\text{esssup } f := \inf \{ K \in \mathbb{R} \mid |\mu|^{*}(\{x \in \Omega | f(x) > K\}) = 0 \}$$

and

$$\|f\|_\infty := \text{esssup } |f|.$$ 

The set of all measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that

$$\|f\|_\infty < \infty$$

is denoted by $L^\infty(\Omega, \mathcal{A}, \mu)$.

As in the case $p \in [1, \infty)$,

$$L^\infty(\Omega, \mathcal{A}, \mu)$$

denotes the set of all equivalence classes in $L^\infty(\Omega, \mathcal{A}, \mu)$ with respect to equality almost everywhere.

In the case $\mathcal{A} = \mathcal{B}(\Omega)$, only write $L^\infty(\Omega, \mu)$ and $L^\infty(\Omega, \mu)$ respectively.

The integral defined in this way shares many properties of the Lebesgue-integral. The H"older and Minkwoski inequality hold true. Furthermore, dominated convergence is available when using convergence in measure instead of pointwise convergence (cf. [13, p. 105ff]).

Before proceeding to the characterisation of the dual of $L^\infty$, a new integral symbol is introduced, which gives formulas for traces and integrals over pure measures a more pleasing shape.
**Definition 4.11.** Let $\Omega \subset \mathbb{R}^n$ be bounded and $C \subset \overline{\Omega}$ be closed. Then for every $\mu \in ba(\Omega, \mathcal{B}(\Omega), L^n)$ such that $\text{core } \mu \subset C$, every $f \in L^1(\Omega, \mu)$ and $\delta > 0$ write

$$\int_C f \, d\mu := \int_{C \cap \Omega} f \, d\mu.$$ 

**Remark 4.12.** This notion of integral is well-defined since the definition of $\text{core } \mu$ yields $|\mu|(C_\delta^c) = 0$ for any $\delta > 0$.

The following proposition is a specialised version of the proposition from [13, p. 139] (cf. [14, p. 53]).

**Proposition 4.13.** Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra and $\sigma : \Sigma \to \mathbb{R}$ be a $\sigma$-measure. Then for every $u^* \in (L^\infty(\Omega, \Sigma, \sigma))^*$ there exists a unique $\mu \in ba(\Omega, \Sigma, \sigma)$ such that

$$\langle u^*, f \rangle = \int_\Omega f \, d\mu$$

for every $f \in L^\infty(\Omega, \Sigma, \sigma)$ and

$$\|u^*\| = \|\mu\| = |\mu|(\Omega).$$

On the other hand, every $\mu \in ba(\Omega, \Sigma, \sigma)$ defines $u^* \in L^\infty(\Omega, \Sigma, \sigma)^*$. Hence, $L^\infty(\Omega, \Sigma, \sigma)^*$ and $ba(\Omega, \Sigma, \sigma)$ can be identified.

Using the decomposition Theorem 3.16 that was proved earlier, one obtains a more refined characterisation of the dual of $L^\infty(\Omega, \Sigma, \sigma)$. In particular, every element of the dual space is the sum of a $\sigma$-measure with $L^n$-density and a pure measure. In contrast to the literature, this makes the intuitive idea of the dual of $L^\infty$ being $L^1$ plus something which is not weakly absolutely continuous with respect to Lebesgue measure precise.

**Theorem 4.14.** Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra and $\sigma : \Sigma \to \mathbb{R}$ be a $\sigma$-measure. Then for every $u^* \in L^\infty(\Omega, \Sigma, \sigma)^*$ there exists a unique pure $\mu_p \in ba(\Omega, \Sigma, \sigma)$ and a unique $h \in L^1(\Omega, \Sigma, \sigma)$ such that

$$\langle u^*, f \rangle = \int_\Omega fh \, dL^n + \int_\Omega f \, d\mu_p$$

for every $f \in L^\infty(\Omega, \Sigma, \sigma)$. 

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Proof. Let \( u^* \in L^\infty(\Omega, \Sigma, \sigma)^* \). Then by Proposition 4.13 there exists \( \mu \in ba(\Omega, \Sigma, \sigma) \) such that for all \( f \in L^\infty(\Omega, \Sigma, \sigma) \)

\[
\langle u^*, f \rangle = \int_\Omega f \, d\mu.
\]

Now, by proposition 3.16 there exist unique \( \mu_c, \mu_p \in ba(\Omega, \Sigma, \sigma) \) such that

\[
\mu = \mu_c + \mu_p
\]

and \( \mu_c \) is a \( \sigma \)-measure and \( \mu_p \) is pure. By the Radon-Nikodym Theorem (Proposition 3.18) there is \( h \in L^1(\Omega, \Sigma, \sigma) \) such that

\[
\mu_c(A) = \int_A h \, d\sigma
\]

for every \( A \in \Sigma \). Since the integral is obviously linear in \( \mu \)

\[
\int_\Omega f \, d\mu = \int_\Omega f \, d\mu_c + \int_\Omega f \, d\mu_p = \int_\Omega fh \, d\sigma + \int_\Omega f \, d\mu_p
\]

for every \( f \in L^\infty(\Omega, \Sigma, \sigma) \), whence the statement of the proposition follows.

Remark 4.15. Note that the \( L \)-space over a measure \( \mu \geq 0 \) is in general not complete. Nevertheless, the completion is known to be the set of all absolutely continuous measures whose \( p \)-norm is finite, i.e. all bounded measures \( \lambda \) with \( \lambda \ll \mu \) and

\[
\lim_{P \in \mathcal{P}} \sum_{A \in P} \frac{|\lambda(A)|^p}{\mu(A)} \mu(A) < \infty.
\]

Here, the limit is taken over the directed set \( \mathcal{P} \) of all partitions \( P \) of \( \Omega \). See [13, p. 185ff] for reference. Using the convention \( \frac{0}{0} = 0 \), this limit is the same as the refinement integral

\[
\int_\Omega \left| \frac{\lambda}{\mu} \right|^p \, d\mu
\]

as defined by Kolmogoroff in [10].

5 Density Measures

This section will present the new class of measures, called density measures. These measures extend on Example 3.9. It turns out that the signed distance function plays an important role.
Definition 5.1. Let $\Omega \subseteq \mathbb{R}^n$ be non-empty. The function
\[
\text{dist}_\Omega : \mathbb{R}^n \to (-\infty, \infty)
\]
defined by
\[
\text{dist}_\Omega(x) := \begin{cases} 
\inf_{y \in \Omega} |x - y| & \text{if } x \notin \Omega \\
-\inf_{y \in \mathbb{R}^n \setminus \Omega} |x - y| & \text{if } x \in \Omega.
\end{cases}
\]
is called signed distance function.

For sets $B \subset \mathbb{R}^n$ write
\[
\text{dist}_\Omega(B) := \inf_{x \in B} \text{dist}_\Omega(x).
\]
Furthermore, neighbourhoods of sets prove useful. Therefore, set
\[
\Omega_\delta := \text{dist}^{-1}_\Omega((-\infty, \delta))
\]
for $\delta \in \mathbb{R}$.

Remark 5.2. Note that $\text{dist}_\Omega$ is Lipschitz continuous, since it is the sum of two Lipschitz continuous functions. If $\Omega \subset \mathbb{R}^n$ is bounded, then by [11, p. 2788]
\[
\mathcal{H}^{n-1}(\partial(\Omega_\delta)) < \infty
\]
for every $\delta \in \mathcal{R}(\text{dist}_\partial \Omega)$, the range of $\text{dist}_\partial \Omega$. Note that there exist $\Omega \subset \mathbb{R}^n$ having finite perimeter with
\[
\lim_{\delta \downarrow 0} \mathcal{H}^{n-1}(\partial(\Omega_\delta)) = \infty.
\]
See Kraft [11, p. 2781] for reference.

Now, density measures can be defined. The basic definition essentially demands the measure to be a probability measure whose core is a Lebesgue null set. By scaling, any bounded positive measure whose support has no volume can be seen as a density measure.

Definition 5.3. Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be closed and $\mathcal{L}^n(C \cap \Omega) = 0$. A measure $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is called a density measure for $C$, if $\mu \geq 0$ and for all $\delta > 0$
\[
\mu(C_\delta \cap \Omega) = \mu(\Omega) = 1.
\]
The set of all density measures for $C$ is denoted by
\[
\text{Dens}(C).
\]

Remark 5.4. If $\mathcal{L}^n(\Omega \cap C_\delta) = 0$ for some $\delta > 0$ or $C = \emptyset$, then
\[
\text{Dens}(C) = \emptyset.
\]
The following proposition shows that density measures indeed have core on $C$ and that they are pure.

**Proposition 5.5.** Let $\Omega \in B(\mathbb{R}^n)$ and $C \subset \Omega$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. Then for every $\mu \in \text{Dens}(C)$

$$\text{core} \mu \subset C$$

and $\mu$ is pure.

**Proof.** Let $x \in \mathbb{R}^n \setminus C$. Let

$$\delta := \frac{1}{2} \text{dist}_C(x).$$

Then for every $0 < \tilde{\delta} < \delta$

$$\mu(B_{\tilde{\delta}}(x)) \leq \mu(\Omega \setminus C_{\tilde{\delta}}) = 0.$$

Hence

$$x \notin \text{core} \mu,$$

and thus

$$\text{core} \mu \subset C.$$

Finally

$$\mathcal{L}^n(\text{core} \mu \cap \Omega) \leq \mathcal{L}^n(C \cap \Omega) = 0.$$

By Proposition 3.30, $\mu$ is pure.

Density measures can be characterised in a way that justifies their name. In essence, they are densities of other measures on their core.

**Proposition 5.6.** Let $\Omega \in B(\mathbb{R}^n)$ and $C \subset \Omega$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. A measure $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is a density measure for $C$ if and only if there exists a measure $\lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ with $\lambda \geq 0$ satisfying

$$\lambda(C_{\delta} \cap \Omega) > 0 \text{ for all } \delta > 0,$$

such that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\int_{\Omega} f \, d\mu \leq \limsup_{\delta \downarrow 0} \int_{C_{\delta} \cap \Omega} f \, d\lambda.$$

Then for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\int_{C} f \, d\mu = \lim_{\delta \downarrow 0} \int_{C_{\delta} \cap \Omega} f \, d\lambda \quad (2)$$

if this limit exists.
Proof. Let \( \mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \).

Assume there exists \( \lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) with \( \lambda \geq 0 \) satisfying

\[
\lambda(C_\delta \cap \Omega) > 0 \text{ for all } \delta > 0
\]
such that for \( f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \)

\[
\int_{\Omega} f \, d\mu \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda.
\]

Note that since

\[
\int_{\Omega} -f \, d\mu \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} -f \, d\lambda
\]

for \( f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \),

\[
\liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda \leq \int_{\Omega} f \, d\mu.
\]

Then for \( \delta > 0 \)

\[
\mu(\Omega) = \mu(C_\delta \cap \Omega) = \lim_{\delta \downarrow 0} \frac{\lambda(C_\delta \cap \Omega)}{\lambda(C_\delta \cap \Omega)} = 1.
\]

Furthermore, for every \( B \in \mathcal{B}(\Omega) \)

\[
\mu(B) \geq \liminf_{\delta \downarrow 0} \frac{\lambda(B \cap C_\delta)}{\lambda(C_\delta \cap \Omega)} \geq 0.
\]

Thus, \( \mu \) is a density measure for \( C \). Equation (2) follows with Proposition 5.5 and the previous estimates.

Now assume \( \mu \) to be a density measure for \( C \). Set \( \lambda = \mu \). Note that \( \lambda(C_\delta \cap \Omega) > 0 \) for every \( \delta > 0 \). Then for all \( f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \)

\[
\int_{\Omega} f \, d\mu = \lim_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\mu \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda.
\]

Now, existence is proved. It turns out that every measure \( \lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \), which does not vanish near \( C \), induces a density measure.

**Proposition 5.7.** Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \) and \( C \subset \overline{\Omega} \) be closed with \( \mathcal{L}^n(C \cap \Omega) = 0 \).

Furthermore, let \( \lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) with \( \lambda \geq 0 \) be such that for all \( \delta > 0 \)

\[
\lambda(C_\delta \cap \Omega) > 0.
\]

Then there exists a density measure \( \mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) such that for every \( f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \)

\[
\liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda \leq \int_{C} f \, d\mu \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda.
\]

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Remark 5.8. In particular, if $\mathcal{L}^n(C_\delta \cap \Omega) > 0$ for every $\delta > 0$, then $\text{Dens}(C) \neq \emptyset$. In order to see this, note that $\lambda = \mathcal{L}^n|\Omega$ satisfies the assumptions of the preceding proposition. Furthermore, every density measure arises in this way (cf. Proposition 5.6).

Proof. Let $\lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be such that for every $\delta > 0$

$$\lambda(C_\delta \cap \Omega) > 0.$$ 

Then

$$p : \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \to \mathbb{R} : f \mapsto \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda$$

is a positively homogeneous, subadditive functional. Set $X := \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$ and

$$X_0 := \left\{ f \in X \mid \lim_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda \text{ exists} \right\}.$$ 

Then $X_0$ is a linear subspace of $X$ and

$$u^*_0 : X_0 \to \mathbb{R} : f \mapsto \lim_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda$$

is a continuous linear functional which is bounded by $p$. The subadditive version of the Hahn-Banach theorem [5, p. 62] yields the existence of a linear extension $u^*$ of $u^*_0$ to all of $X$ which is bounded by $p$. Note that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\langle u^*, f \rangle \leq p(f) \leq \|f\|_{\infty}$$

since $\lambda \ll^w \mathcal{L}^n$. Hence, $u^*$ is a continuous linear functional on $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)$. By Proposition 4.13 there exists $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\langle u^*, f \rangle = \int_{\Omega} f \, d\mu.$$ 

Note that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\int_{\Omega} -f \, d\mu \leq p(-f) = \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} -f \, d\lambda$$

which implies

$$\liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\lambda \leq \int_{\Omega} f \, d\mu.$$ 

Now it is easy to see that for every $B \in \mathcal{B}(\Omega)$

$$0 \leq \liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} \chi_B \, d\lambda \leq \mu(B).$$
Hence, $\mu \geq 0$. Furthermore,

$$1 = \liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} \chi \Omega \, d\lambda \leq \mu(\Omega) \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} \chi \Omega \, d\lambda = 1.$$ 

Finally, let $\delta > 0$. Then

$$1 = \liminf_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} \chi_{C_\delta \cap \Omega} \, d\lambda \leq \mu(C_\delta \cap \Omega) \leq \limsup_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} \chi_{C_\delta \cap \Omega} \, d\lambda = 1.$$ 

Thus, $\mu$ is a density measure of $C$.

**Example 5.9.** Let $\Omega \subset \mathbb{R}^2$ be a cusped set as in Figure 3 below and $C = \{x\}$, where $x \in \mathbb{R}^2$ is the point at the cusp. Then for every $\delta > 0$

$$\mathcal{L}^n(C_\delta \cap \Omega) > 0.$$ 

Hence there exists a density measure $\mu \in \text{Dens}(C)$ such that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\int_C f \, d\mu = \lim_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \, d\mathcal{L}^n,$$

if this limit exists. This example is in essence identical to Example 3.9.

![Figure 3: Existence of a density measure at a cusp](image)

The integral with respect to a density measure can be estimated by the essential supremum and the essential infimum of the integrand near the core.

**Proposition 5.10.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. Furthermore, let $\mu \in \text{ba}((\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be a density measure of $C$. Then for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\lim \text{ess inf}_f \leq \int_C f \, d\mu \leq \lim \text{ess sup}_f.$$
Proof. It suffices to prove the right-hand side of the inequality.

Let \( f \in L^\infty(\Omega, \mathcal{L}^n) \). Since \( \mu \geq 0 \), for every \( \delta > 0 \)
\[
\hat{\int}_{C_\delta \cap \Omega} f \, d\mu = \sup_{C_\delta \cap \Omega} \int_{C_\delta \cap \Omega} f \, d\mu = \sup_{C_\delta \cap \Omega} \sup_{C_\delta \cap \Omega} f \, d\mu = \sup_{C_\delta \cap \Omega} f .
\]
\( \sup f \) is increasing in \( \delta > 0 \) and bounded. Passing to the limit yields the statement.

If \( \text{Dens}(C) \neq \emptyset \) is ensured, then the inequalities in the preceding proposition are sharp.

**Proposition 5.11.** Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \) and \( C \subset \overline{\Omega} \) be non-empty, closed with \( \mathcal{L}^n(C \cap \Omega) = 0 \) such that for every \( \delta > 0 \)
\[
\mathcal{L}^n(C_\delta \cap \Omega) > 0 .
\]
Furthermore, let \( f \in L^\infty(\Omega, \mathcal{L}^n) \). Then
\[
\sup_{\mu \in \text{Dens}(C)} \int_{C} f \, d\mu = \lim_{\delta \downarrow 0} \sup_{C_\delta \cap \Omega} f .
\]

Proof. Let \( f \in L^\infty(\Omega, \mathcal{L}^n) \) and \( \varepsilon > 0 \). Set
\[
M_\varepsilon := \{ x \in \Omega \mid f(x) \geq \lim_{\delta \downarrow 0} \sup_{C_\delta \cap \Omega} f - \varepsilon \}
\]
and
\[
\lambda_\varepsilon := \mathcal{L}^n[M_\varepsilon] .
\]
Then \( \lambda_\varepsilon \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) is positive and such that for every \( \delta > 0 \)
\[
\lambda_\varepsilon(C_\delta \cap \Omega) > 0 .
\]
Hence by Proposition 4.13, there exists a density measure \( \mu_\varepsilon \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) of \( \Omega \) such that
\[
\int_{\Omega} f \, d\mu_\varepsilon \geq \lim_{\delta \downarrow 0} \inf_{C_\delta \cap \Omega} \int_{C_\delta \cap \Omega} f \, d\lambda_\varepsilon \geq \lim_{\delta \downarrow 0} \sup_{C_\delta \cap \Omega} f - \varepsilon .
\]
Hence
\[
\sup_{\mu \in \text{Dens}(C)} \int_{\Omega} f \, d\mu \geq \sup_{\varepsilon > 0} \int_{\Omega} f \, d\mu_\varepsilon \geq \lim_{\delta \downarrow 0} \sup_{C_\delta \cap \Omega} f .
\]
On the other hand, Proposition 5.10 yields
\[
\sup_{\mu \in \text{Dens}(C)} \int_{\Omega} f \, d\mu \leq \lim_{\delta \downarrow 0} \sup_{C_\delta \cap \Omega} f .
\]
The statement for \( \text{essinf} \) follows analogously.

\( \square \)
The set of all density measures is a weak* compact convex set, as the following proposition shows.

**Proposition 5.12.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be non-empty, closed such that $\mathcal{L}^n(C \cap \Omega) = 0$. Then $\text{Dens}(C)$ is a convex weak* compact subset of $\text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ as the dual of $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)$.

**Proof.** W.l.o.g. $\text{Dens}(C) \neq \emptyset$.

Let $\mu_1, \mu_2 \in \text{Dens}(C)$ and $a_1, a_2 \in [0, 1]$ such that $a_1 + a_2 = 1$. Then for every $\delta > 0$

$$a_1\mu_1(C_\delta \cap \Omega) + a_2\mu_2(C_\delta \cap \Omega) = a_1\mu_1(\Omega) + a_2\mu_2(\Omega) = a_1 + a_2 = 1,$$

and

$$a_1\mu_1 + a_2\mu_2 \geq a_1\mu_1 \geq 0.$$

Hence, $\text{Dens}(C)$ is a convex set.

For $\mu \in \text{Dens}(C)$$$
\|\mu\| = |\mu(\Omega)| = \mu(\Omega) = 1.$$

Hence, $\text{Dens}(C)$ is a bounded set.

Now let $\lambda \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \setminus \text{Dens}(C)$. Then either $\lambda(\Omega) \neq 1$ or there is a $\delta > 0$ such that $\lambda(C_\delta \cap \Omega) \neq 1$ or there is $B \in \mathcal{B}(\Omega)$ such that $\lambda(B) < 0$.

Consider the first case. Set $\varepsilon := \frac{1}{2}|\lambda(\Omega) - 1|$. Then

$$V(\lambda) := \{\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) | |\mu(\Omega) - \lambda(\Omega)| < \varepsilon\}$$

is a weak* open set such that

$$V(\lambda) \cap \text{Dens}(C) = \emptyset.$$

In the second case set $\varepsilon := \frac{1}{2}|\lambda(C_\delta \cap \Omega) - 1|$ and

$$V(\lambda) := \{\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) | |\mu(C_\delta \cap \Omega) - \lambda(C_\delta \cap \Omega)| < \varepsilon\}$$

is a weak* open set and

$$V(\lambda) \cap \text{Dens}(C) = \emptyset.$$

In the third and final case set $\varepsilon := \frac{1}{2}|\lambda(B)|$ and

$$V(\lambda) := \{\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) | |\mu(B) - \lambda(B)| < \varepsilon\}.$$ 

Also in this case

$$V(\lambda) \cap \text{Dens}(C) = \emptyset.$$

Since $\lambda$ was arbitrary, the complement of $\text{Dens}(C)$ is weak* open and thus, $\text{Dens}(C)$ is weak* closed. The statement of the proposition follows by the Banach-Alaoglu/Alaoglu-Bourbaki Theorem (cf. [15, p. 777]). □

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Now, the action of $\text{Dens}(C)$ on a fixed essentially bounded function can be characterised.

**Corollary 5.13.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be non-empty, closed such that $\mathcal{L}^n(C \cap \Omega) = 0$ and for every $\delta > 0\,$

$$\mathcal{L}^n(C_\delta \cap \Omega) > 0.$$  

Furthermore, let $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$. Then

$$\langle \text{Dens}(C), f \rangle = \left[ \lim_{\delta \downarrow 0} \text{essinf}_{C_\delta \cap \Omega} f, \lim_{\delta \downarrow 0} \text{esssup}_{C_\delta \cap \Omega} f \right].$$

**Proof.** Since $\text{Dens}(C)$ is a weak* compact convex subset of $\text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$, $\langle \text{Dens}(C), f \rangle$ is a convex compact subset of $\mathbb{R}$. In order to see this, note that

$$f \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)^*.$$  

Since continuous images of compact sets are again compact, $\langle \text{Dens}(C), f \rangle$ is compact. The convexity follows from the convexity of $\text{Dens}(C)$. By Proposition 5.10 and Proposition 5.11

$$\left[ \lim_{\delta \downarrow 0} \text{essinf}_{C_\delta \cap \Omega} f, \lim_{\delta \downarrow 0} \text{esssup}_{C_\delta \cap \Omega} f \right] \subset \langle \text{Dens}(C), f \rangle \subset \left[ \lim_{\delta \downarrow 0} \text{essinf}_{C_\delta \cap \Omega} f, \lim_{\delta \downarrow 0} \text{esssup}_{C_\delta \cap \Omega} f \right].$$  

This, together with the fact that $\langle \text{Dens}(C), f \rangle$ is closed, implies the statement. \hfill \Box

Recall that for a convex set $M$ in a locally convex topological vector space $m \in M$ is an extremal point if for every $m_1, m_2 \in M$ with $m_1 \neq m_2$ and $a_1, a_2 \in [0, 1]$ with $a_1 + a_2 = 1$

$$m = a_1 m_1 + a_2 m_2 \implies a_1 = 1 - a_2 \in \{0, 1\}.$$  

The importance of extremal points follows from the theorem of Krein-Milman (cf. [8, p. 154], [10, p. 157]). In particular, every compact convex set is the closure of the convex hull of its extremal points. Note that the theorem also implies that the set of extremal points is non-empty. Hence, the extremal points of $\text{Dens}(C)$ can be regarded as spanning $\text{Dens}(C)$. The following proposition gives a sufficient and necessary condition for a density measure to be an extremal point.
Proposition 5.14. Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \), \( C \subset \overline{\Omega} \) be non-empty, closed such that \( \mathcal{L}^n(C \cap \Omega) = 0 \) and \( \mu \in \text{Dens}(C) \).

Then \( \mu \) is an extremal point of \( \text{Dens}(C) \) if and only if for every \( B \in \mathcal{B}(\Omega) \) either \( \mu(B) = 0 \) or \( \mu(B^c) = 0 \).

Proof. Let \( \mu \in \text{Dens}(C) \) be such that for every \( B \in \mathcal{B}(\Omega) \) either \( \mu(B) = 0 \) or \( \mu(B^c) = 0 \). Assume \( \mu = a_1 \mu_1 + a_2 \mu_2 \) for \( \mu_1, \mu_2 \in \text{Dens}(C) \) and \( a_1, a_2 \in (0, 1) \) such that \( a_1 + a_2 = 1 \) and \( \mu_1, \mu_2 \neq \mu \). Then there is \( B \in \mathcal{B}(\Omega) \) such that

\[
\mu_1(B) \neq \mu_2(B).
\]

Suppose \( \mu(B) = 0 \). Then \( \mu_1(B) = \mu_2(B) = 0 \), a contradiction to the assumption.

Hence \( \mu(B) = 1 \) and \( \mu(B^c) = 0 \).

This implies

\[
\mu_1(B^c) = \mu_2(B^c) = 0
\]

and thus

\[
\mu_1(B) = 1 = \mu_2(B),
\]

a contradiction to the assumption.

Hence \( \mu_1 = \mu_2 = \mu \) and \( \mu \) is an extremal point of \( \text{Dens}(C) \).

Now, assume \( \mu \) to be an extremal point of \( \text{Dens}(C) \) and assume, there exists \( B \in \mathcal{B}(\Omega) \) such that \( \mu(B), \mu(B^c) > 0 \). Set

\[
\mu_1 := \frac{1}{\mu(B)} \mu|B
\]

and

\[
\mu_2 := \frac{1}{\mu(B^c)} \mu|B^c.
\]

Then \( \mu_1 \) and \( \mu_2 \) are density measures and

\[
\mu = \mu(B) \mu_1 + \mu(B^c) \mu_2,
\]

and \( \mu \) is not an extremal point of \( \text{Dens}(C) \) in contradiction to the assumption. \( \square \)

A simple consequence is that the core of extremal points contains exactly one point. This is the same in the case of Radon measure, where the Dirac-measures are the extremal points of the unit ball (cf. [6, p. 156]).

Corollary 5.15. Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \), \( C \subset \overline{\Omega} \) be non-empty, closed, \( \mathcal{L}^n(C \cap \Omega) = 0 \) and \( \mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) be an extremal point of \( \text{Dens}(C) \).

Then \( \text{core } \mu \) is a singleton.

Proof. Assume there were \( x, y \in \text{core } \mu \) such that \( x \neq y \). Let \( \delta > 0 \) be such that \( \delta < \frac{1}{2} |x - y| \). Then either

\[
\mu(B_\delta(x)) = 0 \quad \text{or} \quad \mu(B_\delta(y)^c) = 0
\]

in contradiction to \( x, y \in \text{core } \mu \). \( \square \)
Another obvious corollary gives the values of extremal points on sets $B$ whose boundary does not meet the core of the extremal point.

**Corollary 5.16.** Let $\Omega \subset \mathbb{R}^n$, $C \subset \overline{\Omega}$ be non-empty, closed, $\mathcal{L}^n(C \cap \Omega) = 0$ and $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be an extremal point of $\text{Dens}(C)$ with core $\mu = \{x\}$ for some $x \in \overline{\Omega}$.

Then for every $B \in \mathcal{B}(\Omega)$

$$
\mu(B) = \begin{cases} 
1 & \text{if } x \in \text{int } B, \\
0 & \text{if } x \notin \overline{B}.
\end{cases}
$$

The question arises, what happens on sets whose boundary meets the core. The following proposition gives a partial answer to this. It states that extremal points concentrate along one-dimensional directions.

**Proposition 5.17.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$ and $\mu \in \text{Dens}(C)$ be an extremal point. Then there exist unique $x \in C$ and $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that for every $\alpha \in (0, \frac{\pi}{2})$

$$
\mu(K(x, v, \alpha) \cap \Omega) = 1,
$$

where

$$
K(x, v, \alpha) := \{y \in \mathbb{R}^n | y \neq x, \angle(y - x, v) < \alpha\}.
$$

**Proof.** By Corollary 5.15 there is a unique $x \in C$ such that

$$
\text{core } \mu = \{x\}.
$$

Let $\{\alpha_k\}_{k \in \mathbb{N}} \subset (0, \frac{\pi}{2})$ be such that

$$
\lim_{k \to \infty} \alpha_k = 0.
$$

Let $S^n := \partial B_1(0)$ and for every $k \in \mathbb{N}$ and $v \in S^n$

$$
V^k_v := \{v' \in S^n | \angle(v, v') < \alpha_k\}.
$$

Then for each $k \in \mathbb{N}$

$$
\left\{V^k_v\right\}_{v \in S^n}
$$

is an open covering of $S^n$. Assume that for every $v \in S^n$

$$
\mu(K(x, v, \alpha_k) \cap \Omega) = 0.
$$

Since $S^n$ is compact, there exists a finite set $M \subset S^n$ such that

$$
S^n \subset \bigcup_{v \in M} V^k_v.
$$
But then
\[ B_1(x) \cap \Omega \subset \left( \{x\} \cup \bigcup_{v \in M} K(x, v, \alpha_k) \right) \cap \Omega. \]
Hence
\[ \mu(\Omega) = \mu(B_1(x) \cap \Omega) \leq \mu(\{x\} \cap \Omega) + \sum_{v \in M} \mu(K(x, v, \alpha_k) \cap \Omega) = 0, \]
in contradiction to
\[ \mu(\Omega) = 1. \]
Hence, for every \( k \in \mathbb{N} \), there exists \( v_k \in S^n \) such that
\[ \mu(K(x, v_k, \alpha_k) \cap \Omega) = 1. \]
Since \( S^n \) is compact, up to a subsequence
\[ v_k \xrightarrow{k \to \infty} v \in S^n. \]
Now let \( \alpha > 0 \) and \( k_0 \in \mathbb{N} \) be such that for every \( k \in \mathbb{N}, k \geq k_0 \)
\[ \angle(v_k, v) < \frac{\alpha}{2} \quad \text{and} \quad \alpha_k < \frac{\alpha}{2}. \]
Then
\[ K(x, v, \alpha) \supseteq K(x, v_k, \alpha_k) \]
for every \( k \geq k_0 \) and thus
\[ \mu(K(x, v, \alpha) \cap \Omega) \geq \mu(K(x, v_k, \alpha_k) \cap \Omega) = 1. \]
In order to prove that \( v \) is unique, assume there exists \( v' \in \mathbb{R}^n, v' \neq v \) such that the statement of the proposition holds. Set
\[ \alpha := \frac{1}{3} \angle(v, v') \]
and note that
\[ K(x, v, \alpha) \cap K(x, v', \alpha) = \emptyset. \]
But then
\[ \mu(\Omega \cap (K(x, v, \alpha) \cup K(x, v', \alpha))) = \mu(\Omega \cap K(x, v, \alpha)) + \mu(\Omega \cap K(x, v', \alpha)) = 2 \]
a contradiction to \( \mu(\Omega) = 1. \)

**Remark 5.18.** The proposition above shows that extremal points in \( \text{Dens}(C) \) concentrate around one dimensional directions. Figure \( \textbf{1} \) illustrates this. Note that it is only necessary for an extremal point of \( \text{Dens}(C) \) to concentrate in this way. A sufficient condition might be that it concentrates on a cusp but this is still an open problem.
Remark 5.19. The extremal points of $\text{Dens}(C)$ are called directionally concentrated density measures.

Integration with respect to bounded density measures that was laid out is well-suited for essentially bounded functions $f \in L^\infty(\Omega, \mathcal{L}^n)$ but in general it is not suited for unbounded functions. The following example illustrates this.

Example 5.20. Let $n = 1$, $\Omega = B_1(0) \subset \mathbb{R}$ and $C := \{0\}$. Let

$$f(x) := \frac{1}{\sqrt{|x|}} (\chi_{(-\infty,0)}(x) - \chi_{[0,\infty)}(x))$$

for $x \in \mathbb{R}$. Then

$$\lim_{\delta \downarrow 0} \int_{\hat{B}_\delta(0)} f \, d\mathcal{L}^n = 0 .$$

Let $\mu \in \text{Dens}(C)$ be a density measure of $C$. Then for every $\varepsilon > 0$ and every simple $h \in L^\infty(\Omega, \mathcal{L}^n)$

$$|\mu|(\{|f-h| > \varepsilon\}) \geq |\mu|(\{|f| > \|h\|_\infty + \varepsilon\}) = 1. \quad (3)$$

Hence there is no sequence of simple function that converge in measure to $f$ and thus $f$ is not $\mu$-integrable.

This chapter is closed with some suggestions of further uses for density measures. For example, the trace of a function of bounded variation can be computed using density measures.

Example 5.21. Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. For $x \in \partial \Omega$ let $\mu_x \in \text{Dens}(\{x\})$ be such that

$$\int_{\{x\}} f \, d\mu_x \leq \limsup_{\delta \downarrow 0} \int_{\hat{B}_\delta(x) \cap \Omega} f \, d\mathcal{L}^n$$
for every \( f \in L^\infty(\Omega, \mathcal{L}^n) \). Then for every \( f \in BV(\Omega) \cap L^\infty(\Omega, \mathcal{L}^n) \) and \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial \Omega \)

\[
T^\Omega(f)(x) = \int_{\{x\}} f \, d\mu_x ,
\]

where \( T^\Omega \) is the usual trace operator for functions of bounded variation (cf. [7, p. 181]). For fixed \( f \in BV(\Omega) \) it even holds true that for a.e. \( x \in \partial \Omega \)

\[
f \in L^1(\Omega, \mu_x) .
\]

In order to see this, note that by Evans [7, p. 181]

\[
\lim_{\delta \downarrow 0} \int_{B_\delta(x) \cap \Omega} |f - T^\Omega(f)(x)| \, d\mathcal{L}^n = 0 .
\]

For every \( k \in \mathbb{N} \) set

\[
h_k := T^\Omega(f)(x) \chi_\Omega .
\]

For \( \varepsilon > 0 \) set

\[
B_\varepsilon := \{ y \in \Omega \mid |f(y) - h_k(y)| \geq \varepsilon \} .
\]

Then

\[
\mu_x(B_\varepsilon) \leq \limsup_{\delta \downarrow 0} \frac{\mathcal{L}^n(B_\varepsilon \cap \Omega \cap B_\delta(x))}{\mathcal{L}^n(B_\delta(x))} \leq \limsup_{\delta \downarrow 0} \frac{1}{\varepsilon} \int_{\Omega \cap B_\delta(x)} |f(y) - h_k(y)| \, d\mathcal{L}^n = 0 .
\]

Hence,

\[
h_k \overset{\mu_x}{\longrightarrow} f .
\]

Furthermore, the sequence is constant and thus \( L^1 \)-Cauchy. Thus \( f \in L^1(\Omega, \mu_x) \) and

\[
\int_{\{x\}} f \, d\mu_x = T^\Omega(f)(x) .
\]

This shows, that even the trace of unbounded functions of bounded variation can be expressed this way.

**Remark 5.22.** Slightly adapting the technique from the previous example, one can show that all unbounded functions are integrable with respect to density measures whose core is one of the Lebesgue points of the function. This way, traces for Sobolev functions and functions of bounded variation can also be computed on the interior of the domain.

For functions of bounded variation, this technique also works at jump points, i.e. points where the precise representative is the mean of the one-sided traces.
It is also possible to use density measures to define a set-valued gradient for Lipschitz continuous functions.

**Example 5.23.** Let \( C = \{ x \} \subset \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be Lipschitz continuous. Note that by Rademachers Theorem (cf. [7, p. 81]), \( Df \) exists almost everywhere and is essentially bounded. Set

\[
\partial_d f(x) := \langle \text{Dens}(\{ x \}), Df \rangle.
\]

Then \( \partial_d f(x) \) is a weak* compact, convex set which is contained in \( B_L(0) \), where \( L \) is the Lipschitz constant of \( f \). In plus, the linearity of the integral implies that for every \( f_1, f_2 \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}) \)

\[
\partial_d(f_1 + f_2)(x) \subset \partial_d f_1(x) + \partial_d f_2(x).
\]

and

\[
\partial_d(f_1 f_2)(x) \subset f_1(x)\partial_d(f_2)(x) + f_2(x)\partial_d(f_1)(x).
\]

Note that the definition of \( \partial_d \) hints at similarities to a characterisation of Clarkes Generalised Gradient in [4, p. 63].

The following proposition states that every pure measure induces a Radon measure on its core.

**Proposition 5.24.** Let \( \Omega \in B(\mathbb{R}^n) \) be bounded and \( \mu \in \text{ba}(\Omega, B(\Omega), L^n) \).

Then there exists a Radon measure \( \sigma \) supported on \( \text{core} \mu \subset \Omega \) such that for every \( \phi \in C(\Omega) \)

\[
\int_{\Omega} \phi \, d\mu = \int_{\text{core} \mu} \phi \, d\sigma.
\]

**Proof.** First, note that for every \( \phi \in C(\Omega) \)

\[
\left| \int_{\Omega} \phi \, d\mu \right| \leq \| \phi \|_C \cdot |\mu|(\Omega)
\]

Furthermore, note that every \( \phi \in C(\Omega) \) can be extended to a function \( \tilde{\phi} \in C_0(\overline{\Omega}) \) and every element of \( C_0(\overline{\Omega}) \) can be restricted to \( \Omega \) to obtain an element of \( C(\Omega) \). Hence

\[
\begin{align*}
\phi^* : C_0(\overline{\Omega}) \rightarrow \mathbb{R} : \phi & \mapsto \int_{\Omega} \phi \, d\mu \\
\end{align*}
\]

is a continuous linear operator and by the Riesz Representation Theorem (cf. [8, p. 106]) there is a Radon measure \( \sigma \) on \( \overline{\Omega} \) such that for every \( \phi \in C(\overline{\Omega}) \)

\[
\int_{\Omega} \phi \, d\mu = \int_{\overline{\Omega}} \phi \, d\sigma.
\]

Now let \( x \in \overline{\Omega} \setminus \text{core} \mu \). Then there exists a \( \delta > 0 \) such that \( B_\delta(x) \cap \text{core} \mu = \emptyset \).
Then for every $\phi \in C_0 (B_\delta (x) \cap \Omega)$
\[
\int_{\Omega} \phi \, d\sigma = \int_{\Omega} \phi \, d\mu = 0.
\]
Hence
\[
|\sigma| (B_\delta (x)) = 0
\]
and thus $x$ is not in the support of the $\sigma$-measure $\sigma$. Since $x \in \overline{\Omega} \setminus \text{core } \mu$ was arbitrary, it is proved that the support of $\sigma$ is indeed a subset of $\text{core } \mu$.
This proves the statement of the proposition.

**Remark 5.25.** In the setting of the proposition above, $\sigma$ is said to be a representation of $\mu$ on $\text{core } \mu$.

The next proposition gives a partial inverse to the statement of the proposition above. In particular, any Radon measure can be extended to a measure on all of its domain.

**Proposition 5.26.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and every $\delta > 0$
\[
\mathcal{L}^n (B_\delta (x) \cap \Omega) > 0.
\]
Furthermore, let $\sigma$ be a Radon measure on $C$. Then there exists $\mu \in \text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C (\overline{\Omega})$
\[
\int_{\Omega} \phi \, d\mu = \int_{C} \phi \, d\sigma.
\]
In particular,
\[
\text{core } \mu \subset C
\]
and
\[
|\mu| (\Omega) = |\sigma| (C).
\]

**Remark 5.27.** The conditions of the statement are satisfied if, for example, $C \subset \partial_* \Omega \cup \Omega_{\text{int}}$.

**Proof.** Let $\phi \in C (\overline{\Omega})$. Then
\[
\|\phi|_C \|_C \leq \|\phi\|_\infty.
\]
In order to see this, let $\varepsilon > 0$ and $x \in C$ be such that
\[
|\phi(x) - \|\phi|_C \|_C| < \frac{\varepsilon}{2}.
\]
Let $\delta > 0$ be such that for all $y \in B_\delta (x) \cap \Omega$
\[
|\phi(x) - \phi(y)| < \frac{\varepsilon}{2}.
\]
By assumption

\[ L^n(B_\delta(x) \cap \Omega) > 0 \]

whence

\[ \|\phi\|_\infty \geq |\phi(x)| - \frac{\varepsilon}{2} \geq \|\phi\|_C - \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, the statement follows.

Set

\[ u^*_0 : C(\overline{\Omega}) \subset L^\infty(\Omega, L^n) \to \mathbb{R} : \phi \mapsto \int_C \phi \, d\sigma \]

and note that for every \( \phi \in C(\overline{\Omega}) \)

\[ |(u^*_0, \phi)| \leq \|\phi\|_C |\sigma|(C) \leq \|\phi\|_\infty |\sigma|(C). \]

By the Hahn-Banach theorem (cf. [5, p. 63]) there exists a continuous extension \( u^* \) of \( u^*_0 \) to all of \( L^\infty(\Omega, L^n) \) such that

\[ \|u^*\| = \|u^*_0\|. \]

But \( L^\infty(\Omega, L^n)^* = \text{ba}(\Omega, B(\Omega), L^n) \) by Proposition [4.13]. Hence, there exists \( \mu \in \text{ba}(\Omega, B(\Omega), L^n) \) such that for every \( \phi \in C(\overline{\Omega}) \)

\[ \int_\Omega \phi \, d\mu = \int_C \phi \, d\sigma. \]

Let \( \phi \in C(\overline{\Omega}) \) such that \( \|\phi\|_\infty \leq 1 \). Then

\[ \|\phi\|_C \leq \|\phi\|_\infty \leq 1. \]

Hence,

\[ |\mu|(\Omega) = \|u^*_0\| = \sup_{\phi \in C(\overline{\Omega})} \int_\Omega \phi \, d\mu \leq \sup_{\|\phi\|_\infty \leq 1} \int_\Omega \phi \, d\mu \leq \sup_{\|\phi\|_C \leq 1} \int_C \phi \, d\sigma \leq |\sigma|(C). \]

Note that for every \( \phi \in C(\overline{\Omega}) \)

\[ \max(\min(\phi, 1), -1) \in C(\overline{\Omega}) \]

and that every \( \phi \in C_0(C) \) can be extended to all of \( \overline{\Omega} \), preserving the norm (cf. [12, p. 25]). Hence, every \( \phi \in C_0(C) \) can be extended to \( \overline{\phi} \in C_0(\overline{\Omega}) \) such that

\[ \|\phi\|_C = \|\overline{\phi}\|_C. \]

Thus

\[ |\sigma|(C) = \sup_{\|\phi\|_C \leq 1} \int_C \phi \, d\sigma = \sup_{\|\phi\|_C \leq 1} \int_\Omega \overline{\phi} \, d\mu \leq \sup_{\|\overline{\phi}\|_C \leq 1} \int_\Omega \overline{\phi} \, d\mu \leq |\mu|(\Omega). \]

Since changing \( \phi \) outside of \( C \) does not change the integral, core \( \mu \subset C \).

This finishes the proof. \( \square \)
The measure from the preceding proposition is pure if the Radon measure is singular with respect to Lebesgue measure.

**Corollary 5.28.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and $\delta > 0$

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0$$

and

$$\mathcal{L}^n(C \cap \Omega) = 0.$$  

Furthermore, let $\sigma$ be a Radon measure on $C$.

Then there exists $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for all $\phi \in C_0(\Omega)$

$$\int_\Omega \phi \, d\mu = \int_C \phi \, d\sigma.$$  

Furthermore,

$$|\mu|(\Omega) = |\sigma|(C)$$

and $\mu$ is pure.

**Proof.** The preceding proposition and Proposition 3.30 yield the statement. □

The following example presents another way to construct a density at zero.

**Example 5.29.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $x \in \overline{\Omega}$ such that for every $\delta > 0$

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0.$$  

Then there exists a pure $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_{\partial \Omega} \phi \, d\mu = \phi(x).$$  

The next example shows an extension for $\mathcal{H}^{n-1}$.

**Example 5.30.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be open, bounded and have smooth boundary. Then $\mathcal{L}^n(\partial \Omega) = 0$ and $C = \partial \Omega$ satisfies the assumptions of Proposition 5.26. Hence, there exists $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for all $\phi \in C(\overline{\Omega})$

$$\int_{\partial \Omega} \phi \, d\mathcal{H}^{n-1} = \int_{\Omega} \phi \, d\mu.$$  

The following example shows, that the surface part of a Gauß formula can be expressed as an integral with respect to a pure measure.
Example 5.31. Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be a bounded set with smooth boundary. Then $C = \partial \Omega \subset \overline{\Omega}$ is a closed set and for every $k \in \mathbb{N}$ such that $1 \leq k \leq n$

$$\nu^k \cdot \mathcal{H}^{n-1}|_{\partial \Omega}$$

is a Radon measure on $C$. By Proposition 5.26 there exists $\mu_k \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_{\partial \Omega} \phi \cdot \nu^k \, d\mathcal{H}^{n-1} = \int_{\Omega} \phi \, d\mu_k = \int_{\partial \Omega} \phi \, d\mu_k$$

and 

$$\text{core } \mu_k \subset \partial \Omega.$$ 

Hence, there exists $\mu \in (\text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^\mathbb{N}$ such that for all $\phi \in C^1(\Omega, \mathbb{R}^n)$

$$\int_{\partial \Omega} \phi \, d\mu = \int_{\Omega} \phi \, d\mu = \int_{\partial \Omega} \phi \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\Omega} \text{div } \phi \, d\mathcal{L}^n,$$

where the Gauß formula for sets with finite perimeter from Evans [7, p. 209] was used. Furthermore,

$$\text{core } \mu \subset \partial \Omega$$

and $\mu$ is pure by Proposition 3.30.

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