GAUGE INVARIANT SYSTEMS OF A GENERAL FORM: COUNT OF THE PHYSICAL DEGREES OF FREEDOM

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Abstract

The relationship between various methods to calculate the physical degrees of freedom for gauge invariant systems of a general form is established. The set of hidden parameters caused for the superfluous degrees of freedom is revealed.
1. In spite of more than 40 years of history of the constrained dynamics and vast list of publications appeared from the first paper by P. Dirac \[1\] there still are lacunas in this theory even on the classical level. First of all, it concerns the correspondence between Lagrangian and Hamiltonian formulations of the theory and geometrical meaning of the Legendre transformation for the degenerate case (see \[4, 2, 3\] and refs. therein). Another question deals with the general form of gauge symmetry transformations and corresponding calculation of physical degrees of freedom \[5, 6, 7\]. At the quantum level of the theory we get the problem of consistent description of the state space of gauge-invariant systems within the framework of any quantization scheme, e. g. BRST-quantization. This problem is still far from completion \[8\]. In this report we will deal with the count of the physical (dynamical) degrees of freedom. We suppose the summation over repeated indexes.

2. Let us consider the mechanical system given by the Lagrangian

$$L(q, \dot{q}) = \sum_{k=0}^{N} \varepsilon_{\alpha}^{(k)} \psi^{r}_{\alpha}(q, \dot{q}), \quad r = 1, \ldots, R,$$

where \(\varepsilon_{\alpha}(t), \alpha = 1, \ldots, A\), are infinitesimal arbitrary functions of time. The velocity phase space \(V\) is described by the set of generalized coordinates \(q^r\) and generalized velocities \(\dot{q}^r\). To calculate correctly the dynamical (physical) degrees of freedom of the system, it is necessary to consider the gauge transformations of all coordinates of the velocity phase space point \(\{\delta V\} = \{\delta q^r; \delta \dot{q}^r\}\) on the trajectory of the system, which is defined by the Lagrange equations

$$L_r(q, \dot{q}, \ddot{q}) = W_{rs}(q, \dot{q}) \ddot{q}^s - R_r(q, \dot{q}) := 0,$$

where

$$R_r(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^r} - \dot{q}^s \frac{\partial^2 L(q, \dot{q})}{\partial q^s \partial \dot{q}^r}, \quad W_{rs}(q, \dot{q}) = \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^r \dot{q}^s},$$

and the symbol := means that the corresponding equation is valid on the trajectory satisfying the relations \(\dot{q}^r(t) = d/dt q^r(t)\). Because of the gauge invariance the Hessian of the system \(W_{rs}\) is a singular matrix, and we get that the vectors \(\psi^{r}_{\alpha}\) are null–vectors of this matrix. Suppose that any null–vector of the Hessian is a linear combination of \(\hat{\psi}^{r}_{\alpha}\). Hence, we can express the generalized accelerators \(\ddot{q}^r\) from the Lagrange equations (2) only in the following form

$$\ddot{q}^r := W^{rs} R_s + x^r \hat{\psi}^{r}_{\alpha},$$

where \(W^{rs}\) is some pseudo–inverse matrix \[9\] for the Hessian \(R_{rs}\), and \(x^r\) are undefined arbitrary parameters. Further, it is easy to show that the vectors entering the gauge symmetry transformations satisfy the relations

$$\hat{\psi}^{r}_{\alpha} \frac{\partial \psi^{s}_{\beta}}{\partial \dot{q}^r} = A^{[k] \gamma}_{\alpha \beta} \hat{\psi}^{r}_{\gamma}, \quad k = 0, 1, \ldots, N,$$
where \( [k] A_{\gamma}^\alpha \) are some functions of the velocity phase space coordinates. Using Eqs. (1)-(5) we get

\[
\delta \varepsilon_r = \sum_{k=0}^{N} (k) \varepsilon^{\alpha} \left( \left[ (N-k+1) \right] A_{\gamma}^\alpha + T \left[ (N-k) \right] \psi^r_{\alpha} \right) + \left( (N+1) \varepsilon^{\alpha} + x^\beta \sum_{k=0}^{N} (k) A_{[k+1] \beta}^\gamma \right) \psi^r_{\alpha},
\]

where \( T \) is the differential operator of the form

\[
T = \dot{q}^s \frac{\partial}{\partial q^s} + R_s W^s_t \frac{\partial}{\partial \dot{q}^s}.
\]

On the trajectory of the system we have that

\[
T := \frac{d}{dt},
\]

hence, the operator \( T \) has the sense of the time evolution operator of the gauge invariant systems.

In order to calculate the physical degrees of freedom we must reveal all arbitrary parameters of the system and fix them for some initial value of time \( t = t_0 \). Upon this, the gauge transformations are treated as the transformations of the initial data, where \( (k) \varepsilon^{\alpha} \) form the set of independent arbitrary parameters.

So, let us analyze Eq. (5). It can be shown that

\[
[k] A_{\gamma}^\alpha = 0 \quad \text{for} \quad k > 1.
\]

Hence, from Eq. (5) we obtain the differential equations of the form

\[
[k] \psi^r_{\alpha} \frac{\partial \psi^r_{\alpha}}{\partial q^s} = 0, \quad k = 0, 1, \ldots, N - 2.
\]

It means that the functions \( \psi^r_{\alpha}(q, \dot{q}) \), \( k = 0, 1, \ldots, N - 2 \), take constant values on the surfaces \( S_0 \) having parametric representation of the form

\[
q^r(\lambda) = \dot{q}^r, \quad \dot{q}^r(\lambda) = \dot{q}^r + [k] \lambda \psi^r_{\alpha}(q, \dot{q}).
\]

One can show that the arbitrariness in definition of nonzero functions \( [0] A_{\gamma}^\alpha \), \( [1] A_{\gamma}^\alpha \) is fixed by fixing of \( [x] - \) parameters, whereas the arbitrariness of the vectors \( \psi^r_{\alpha} \), \( k = 0, 1, \ldots, N - 2 \), is fixed by fixing of \( [\lambda] - \) parameters in Eqs. (11), (12).

Now taking into account the relations (1), (4), (6), (11), (12) we see that to determine the physical degrees of freedom we must fix \((N+2) \times A [\varepsilon]-\)parameters, \( A [x]-\)parameters, \((N-1) \times A [\lambda]-\)parameters. Hence, we get the number of the physical degrees of freedom of the system to be equal

\[
2F_L = 2R - 2(N + 1) \times A.
\]
3. Another way to obtain the number of the physical degrees of freedom is the following. Consider the space of trajectories of the system, where $\dot{q}^r(t) = dq^r(t)/dt$. In this scheme $\varepsilon^\alpha$ are not independent parameters. They satisfy the differential equations $\varepsilon^\alpha(t) = d^k\varepsilon^\alpha(t)/dt^k$. It can be shown that the system under consideration has the following hierarchy of the Lagrangian constraints

$$\Lambda^\alpha_k(q, \dot{q}) := 0, \quad k = 1, \ldots, N,$$

where the functions $\Lambda^\alpha_k(q, \dot{q})$ obey the Noether identities of the form

$$\Lambda^\alpha_k = \psi^r_k \Lambda^\alpha_r - \dot{q}^s \partial \Lambda^\alpha_s,$$

$$\psi^r_k \Lambda^\alpha_r = - \partial \Lambda^\alpha_s \dot{q}^s,$$

where $k = 0, 1, \ldots, N$ and $\Lambda^\alpha_0 \equiv 0$. Now to have the correct number of the physical degrees of freedom we must choose $A$ functions $\varepsilon^\alpha(t)$ and impose $N \times A$ conditions (14) on the trajectories of the system. Hence, we get the formula for the physical degrees of freedom

$$F_L = R - (N + 1) \times A,$$

that gives the result coinciding with (13).

4. It is desirable to obtain the same result from the Hamiltonian formulation of the theory. Indeed, it can be shown that in the Hamiltonian description of our system there appears a set of $A$ primary constraints $\Phi^\alpha_0$ and a set of $N \times A$ secondary constraints $\Phi^\alpha_k$, $k = 1, \ldots, N$, of $N$ stages. These functions $\Phi^\alpha_0$, $\Phi^\alpha_k$ are functionally independent and fulfil the relations

$$\Phi^\alpha_0(V) = 0,$$

$$\Phi^\alpha_k(V) = \Lambda^\alpha_k, \quad k = 1, \ldots, N,$$

in the terms of the velocity phase space coordinates. One can show that the Hamiltonian constraints of the system form the constraint algebra of the first class. The explicit form of the constraint algebra has been found in ref. within the framework of the so-called standard extension procedure presented first in ref. and developed in refs. It is worth to note that the scheme of standard extension allows us to intersect the "gauge orbits" – surfaces $S_0$ – in such a way that, in particular, one can put the hidden $[x]$- and $[\lambda]$-parameters, considered above, to be:

$$\lambda^\alpha_k = 0, \quad x^\alpha = \dot{q}^r \chi^\alpha_r(q, \dot{q}),$$

where the vectors $\chi^\alpha_r$ are dual to the null–vectors $\psi^r_\alpha$

$$\psi^r_\alpha \chi^\alpha_r = \delta^\alpha_\beta.$$
From the other side such a choice of the hidden parameters (20), (21) provides us with the following correspondence between the differential operators in $V$ and $\Gamma$:

$$\frac{\partial}{\partial q^r} \longleftrightarrow \frac{\partial}{\partial q^r}; \quad \frac{\partial}{\partial p^r} \longleftrightarrow W^{rs} \frac{\partial}{\partial q^s},$$

(22)

where $W^{rs}(q, \dot{q})$ is the pseudo-inverse matrix \[9\] for the Hessian $W_{rs}(q, \dot{q})$. $W^{rs}$ is uniquely defined by the relations

$$W^{rs} \chi_s^\alpha = 0, \quad W^{rt} W_{ts} = \delta^r_s - \chi_s^\alpha \psi_s^\alpha.$$ 

(23)

Thus, we get that the following $(N + 1) \times A$ relations

$$\phi^{[k]}_\alpha \approx 0, \quad k = 0, 1, \ldots, N$$

(24)

restrict the possible values of the canonical phase space coordinates $\Gamma$.

Besides, the gauge transformations are mapped to the (canonical) phase space as follows

$$\delta_\varepsilon q^r := \{ q^r, G_\varepsilon \}, \quad (25)$$

$$\delta_\varepsilon p^r := \{ p^r, G_\varepsilon \}, \quad (26)$$

where $G_\varepsilon$ is the linear combination of the constraints

$$G_\varepsilon = \sum_{k=0}^N g^{[k]}_\alpha(\varepsilon) \phi^{[k]}_\alpha.$$ 

(27)

The functions $g^{[k]}_\alpha$ depend on the gauge parameters $\varepsilon^\alpha$ and their derivatives $^{(k)} \varepsilon^\alpha$ up to $N$-th order. Hence, fixing the initial value of time $t = t_0$ and treating again Eqs.(25), (26) as the transformations of the initial data of the system we see that it is necessary to fix $(N + 1) \times A$ parameters $^{(k)} \varepsilon^\alpha$, $k = 0, 1, \ldots, N$, and to impose $(N + 1) \times A$ constraints (24) on the phase space coordinates. This procedure gives the number of the dynamical degrees of freedom to be equal

$$2F_H = 2R - 2(N + 1) \times A.$$ 

(28)

The same result can be obtained according to the Dirac scheme \[1\]. Actually, we have the set of $(N + 1) \times A$ constraints of the first class. Thus, to remove the degeneracy one should impose $(N + 1) \times A$ gauge conditions on the canonical variables. This way also leads to Eq.(28) for calculation of the physical degrees of freedom.

Thus, we have obtained the number of the physical degrees of freedom for the system under consideration using different approaches and established the mechanism of annihilation of the superfluous (unphysical) degrees of freedom.

5. Let now the vectors $^{[N-k]} \psi^r_\alpha$ depend on higher order derivatives of the generalized coordinates up to $M$-th order. Hence, instead of (24) we have the gauge transformations

$$\delta_\varepsilon q^r = \sum_{k=0}^N ^{(k)} \varepsilon^\alpha \psi^{[N-k]}_\alpha(q, \dot{q}, \ldots, \dddot{q})$$

(29)
to be the local symmetry transformations of the Lagrangian $L(q, \dot{q})$. It follows from the Noether identities

$$\sum_{k=0}^{N} (-1)^k \frac{d^k}{dt^k} \left( \psi \left[ N-k \right] r \alpha L_r \right) = 0$$

that the vectors $\psi \left[ N-k \right] r \alpha$ are determined up to combinations of the form

$$\theta \left[ N-k \right] rs \alpha L_s, \quad \theta \left[ N-k \right] rs \alpha = - \theta \left[ N-k \right] sr \alpha.$$

Thus, from the Noether identities, gauge algebras and Jacobi identities, using this arbitrariness, one can conclude that the dependence of the vectors $\psi \left[ N-k \right] r \alpha$ on higher order derivatives $\dot{q}^r$ is effectively governed by the number $N$ — maximal order of time derivatives of the gauge parameters $\varepsilon^\alpha(t)$. Namely, one can get $M = N$. Moreover, the vectors $\psi \left[ N \right] r \alpha$ are linear in the variables $\dot{q}^r$, quadratic in $\ddot{q}^r$, and so on, whereas the null–vectors $\psi \left[ 0 \right] r \alpha$ depend only on the velocity phase space coordinates $\dot{q}^r, \ddot{q}^r$. Such a dependence on higher order derivatives gives rise to the additional superfluous arbitrary parameters of the type discussed above. These additional parameters, as it turns out, may lead, in general, to violation of the above correspondence between Lagrangian and Hamiltonian approaches to calculation of the physical degrees of freedom. Detailed discussion of this question (see e. g. [7] and refs. therein) is beyond the limits of our report.

An analysis of the papers cited here in addition to our talk allows us to point out one more evidence that the classical aspects of gauge invariant systems are deeply elaborated, whereas the problems of quantization are up to now solved only for the simplest (but actually nontrivial [8]) case of abelian gauge symmetry group.

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