DYNKIN GRAPHS, GABRIÉLOV GRAPHS
AND TRIANGLE SINGULARITIES

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Abstract. We consider fourteen kinds of two-dimensional triangle hypersurface singularities, and consider what kinds of combinations of rational double points can appear on small deformation fibers of these triangle singularities. We show that possible combinations can be described by Gabriélov graphs and Dynkin graphs.

1. Review of results by Russian mathematicians

In this article we assume that every variety is defined over the complex field \( \mathbb{C} \).

First I explain some results by Arnold and Gabriélov briefly.

In [1] Arnold has introduced an invariant \( m \) called modality or modules number, and has given a long classification list of hypersurface singularities. Modality \( m \) is a non-negative integer. Though we find singularities of any dimension in Arnold’s list, we consider singularities of dimension two in particular.

His class of singularities with \( m = 0 \) coincides with the class of rational double points. It is well known that each rational double point corresponds to a connected Dynkin graph of type \( A, D \) or \( E \) in the theory of Lie algebras. (Durfee [3].)

The class with \( m = 1 \) consists of three subclasses. (\( \lambda \) is a parameter.)

1. Three simple elliptic singularities: \( J_{10}, X_9, P_8 \)
2. Cusp singularities \( T_{p, q, r}, \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \right): x^p + y^q + z^r + \lambda xyz = 0 \) (\( \lambda \neq 0 \)).
3. fourteen triangle singularities (These fourteen are also called exceptional singularities.)

\[
\begin{align*}
E_{12} & : x^7 + y^3 + z^2 + \lambda x^5 y = 0 & W_{12} & : x^5 + y^4 + z^2 + \lambda x^3 y^2 = 0 \\
E_{13} & : x^7 + y^3 + z^2 + \lambda x^5 y = 0 & W_{13} & : x^7 + y^3 + z^2 + \lambda x^5 y = 0 \\
E_{14} & : x^7 + y^3 + z^2 + \lambda x^5 y = 0 & U_{12} & : x^4 + y^3 + z^3 + \lambda x^2 y z = 0.
\end{align*}
\]

(As for the other defining polynomials see Arnold [1].)

His list continues in the case \( m \geq 2 \), but we do not refer further.

We go on to Gabriélov’s results. (Gabriélov [4].)

Let \( f(x, y, z) = 0 \) be one of defining polynomials of fourteen hypersurface triangle singularities. It defines a singularity at the origin. We consider the Milnor fiber, i.e.,

\[
F = \left\{ (x, y, z) \in \mathbb{C}^3 \left| |x|^2 + |y|^2 + |z|^2 < \varepsilon^2, f(x, y, z) = t \right. \right\}
\]

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where \( \epsilon \) is a sufficiently small positive real number and \( t \) is a non-zero complex number whose absolute value is sufficiently small compared with \( \epsilon \). The pair

\[
(H_2(F, \mathbb{Z}), \text{the intersection form})
\]

is called the Milnor lattice, and \( \mu = \text{rank } H_2(F, \mathbb{Z}) \) is called the Milnor number of the singularity. Gabriélov has computed the Milnor lattice for fourteen hypersurface triangle singularities. According to him, there exists a basis \( e_1, e_2, \ldots, e_\mu \) of \( H_2(F, \mathbb{Z}) \) such that each \( e_i \) is a vanishing cycle (thus in particular \( e_i \cdot e_j = -2 \)) and the intersection form is represented by the dual graph below.

![Graph](image)

In the above graph the basis \( e_1, e_2, \ldots, e_\mu \) has one-to-one correspondence with vertices. Edges indicate intersection numbers. Two vertices corresponding to \( e_i \) and \( e_j \) are not connected, if \( e_i \cdot e_j = 0 \). They are connected by a single solid edge, if \( e_i \cdot e_j = 1 \). They are connected by a double dotted edge, if \( e_i \cdot e_j = -2 \). Three integers \( p_1, p_2, p_3 \) are the numbers of vertices in the corresponding arm. They depend on the type of the triangle singularity. The corresponding triplets \((p_1, p_2, p_3)\) to the above fourteen symbols are as follows:

\[
(2, 3, 7) \quad (2, 4, 5) \quad (3, 3, 4) \quad (2, 5, 5) \quad (3, 4, 4) \quad (4, 4, 4) \\
(2, 3, 8) \quad (2, 4, 6) \quad (3, 3, 5) \quad (2, 5, 6) \quad (3, 4, 5) \\
(2, 3, 9) \quad (2, 4, 7) \quad (3, 3, 6)
\]

(Thus the above figure is the graph for \( S_{12} \).)

The main part of the above graph below is called the Gabriélov graph.
The Gabriélov graph defines a lattice $P^*$ with a basis $e_1, e_2, \ldots, e_{\mu-2}$, if we apply the above mentioned rule. It is easy to check that $P^*$ has signature $(1, \mu - 3)$, and $H_2 (F, \mathbb{Z}) \cong \bigoplus \mathbb{Z}$ denotes the hyperbolic plane, i.e., a lattice of rank 2 with a basis $u, v$ satisfying $u \cdot u = v \cdot v = 0$ and $u \cdot v = v \cdot u = 1$, and $\oplus$ denotes the orthogonal direct sum.

2. Singularities on deformation fibers of triangle singularities

A finite disjoint union of connected Dynkin graphs is also called a Dynkin graph. Let $T$ denote one of the above fourteen symbols of hypersurface triangle singularities. By $PC (T)$ we denote the set of Dynkin graphs $G$ with several components such that there exists a small deformation fiber $Y$ of a singularity of type $T$ satisfying the following conditions:

1. $Y$ has only rational double points as singularities.
2. The combination of rational double points on $Y$ corresponds to graph $G$ exactly.

Here, the type of each component of $G$ corresponds to the type of a rational double point on $Y$, and the number of components of each type corresponds to the number of rational double points of each type on $Y$. If $G$ has $a_k$ of components of type $A_k$ for each $k \geq 1$, $d_\ell$ of components of type $D_\ell$ for each $\ell \geq 4$ and $e_m$ of components of type $E_m$ for $m = 6, 7, 8$, we identify $G$ with the formal sum $G = \sum a_k A_k + \sum d_\ell D_\ell + \sum e_m E_m$.

Mr. F.-J. Bilitewski informed me that he had a complete listing of Dynkin graphs of $PC (T)$ for every $T$ of the above fourteen.

Theorem 2.1. Let $T$ be one of the above fourteen symbols of hypersurface triangle singularities. Let $G$ be a Dynkin graph with only components of type $A$, $D$ or $E$. The following conditions (A) and (B) are equivalent:

(A): $G \in PC (T)$.
(B): Either (B-1) or (B-2) holds.

(B-1): $G$ can be made by an elementary transformation or a tie transformation from a Dynkin subgraph of the Gabriélov graph of type $T$.

(B-2): $G$ is one of the following exceptions:

Exceptions

$T = Z_{13} : A_7 + A_4$
$T = S_{11} : 2A_4 + A_1$
$T = U_{12} : 2D_4 + A_2, A_6 + A_4, A_5 + A_4 + A_1, 2A_4 + A_1$

The other eleven triangle singularities: None

An elementary transformation and a tie transformation in the above theorem are operations by which we can make a new Dynkin graph from a given Dynkin graph.

Definition 2.2. Elementary transformation: The following procedure is called an elementary transformation of a Dynkin graph:

1. Replace each connected component by the corresponding extended Dynkin graph.
2. Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them.
An extended Dynkin graph is a graph obtained from a connected Dynkin graph by adding one vertex and one or two edges. (Bourbaki [2]) Below we show extended Dynkin graphs. Numbers in the figures below are the coefficients of the maximal root, which will appear in the definition of a tie transformation. We can get the corresponding Dynkin graph, if we erase one vertex with the attached number 1 and edges issuing from it.

\[
\begin{align*}
A_k & \\
\begin{figure}
\end{figure}
\end{align*}
\]

\[
\begin{align*}
D_l & \\
\begin{figure}
\end{figure}
\end{align*}
\]

\[
\begin{align*}
E_8 & \\
\begin{figure}
\end{figure}
\end{align*}
\]

\[
\begin{align*}
E_7 & \\
\begin{figure}
\end{figure}
\end{align*}
\]

\[
\begin{align*}
E_6 & \\
\begin{figure}
\end{figure}
\end{align*}
\]
Definition 2.3. Tie transformation: Assume that by applying the following procedure to a Dynkin graph $G$ we have obtained the Dynkin graph $\bar{G}$. Then, we call the following procedure a tie transformation of a Dynkin graph:

1. Replace each component of $G$ by the extended Dynkin graph of the same type. Attach the corresponding coefficient of the maximal root to each vertex of the resulting extended graph $\tilde{G}$.

2. Choose, in an arbitrary manner, subsets $A, B$ of the set of vertices of the extended graph $\tilde{G}$ satisfying the following conditions:

   (a): $A \cap B = \emptyset$

   (b): Choose arbitrarily a component $\tilde{G}''$ of $\tilde{G}$ and let $V$ be the set of vertices in $\tilde{G}''$. Let $\ell$ be the number of elements in $A \cap V$. Let $n_1, n_2, \ldots, n_\ell$ be the numbers attached to $A \cap V$. Also, let $N$ be the sum of the numbers attached to elements in $B \cap V$. (If $B \cap V = \emptyset$, $N = 0$.) Then, the greatest common divisor of the $\ell + 1$ numbers $N, n_1, n_2, \ldots, n_\ell$ is $1$.

3. Erase all attached integers.

4. Remove vertices belonging to $A$ together with the edges issuing from them.

5. Draw another new vertex called $\theta$. Connect $\theta$ and each vertex in $B$ by a single edge.

Remark. After following the above procedure 1–5, the resulting graph $\bar{G}$ is often not a Dynkin graph. We consider only the cases where the resulting graph $\bar{G}$ is a Dynkin graph, and then we call the above procedure a tie transformation. The number $\#(B)$ of elements in the set $B$ satisfies $0 \leq \#(B) \leq 3$. $\ell = \#(A \cap V) \geq 1$.

Example 2.4. We consider the case $T = W_{13}$. The Gabriélov graph in this case is the following, and it has a Dynkin subgraph of type $E_8 + A_2$:

![Diagram of the Gabriélov graph]

First we apply a tie transformation to $E_8 + A_2$. In the second step of the transformation we can choose subsets $A$ and $B$ as follows:

![Diagram of the tie transformation]
For the component of type $E_8$, $\ell = 1$, $n_1 = 4$, $N = 1$ and thus $G.C.D. (n_1, N) = 1$. For the component $A_2$, $\ell = 1$, $n_1 = 1$, $N = 1$ and thus $G.C.D. (n_1, N) = 1$. One sees that the condition (b) is satisfied. As the result of the transformation one gets a graph of type $A_6 + D_5$. By our theorem one can conclude $A_6 + D_5 \in PC (W_{13})$.

Second we apply an elementary transformation to $E_8 + A_2$.

As in the above figure we can get $E_6 + 2A_2$. Thus $E_6 + 2A_2 \in PC (W_{13})$.

3. K3 surfaces and lattice theory

It is known that fourteen hypersurface triangle singularities have interesting property called the strange duality. (Pinkham [6].) Let $T$ be one of the above fourteen symbols of hypersurface triangle singularities. Associated with $T$, we have another symbol $T^*$ also in the above fourteen symbols of hypersurface triangle singularities. This $T^*$ is called the dual of $T$. The dual of the dual coincides with the original one, i.e., $(T^*)^* = T$.

For the following six singularities the dual coincides with itself, i.e., $T^* = T$: $E_{12}$, $Z_{12}$, $Q_{12}$, $W_{12}$, $S_{12}$, $U_{12}$. For the following four pairs the dual is another member of the pair: $\{ E_{13}, Z_{11} \}$, $\{ E_{14}, Q_{10} \}$, $\{ Z_{13}, Q_{11} \}$, $\{ W_{13}, S_{11} \}$.

Following Looijenga [3], we explain the relation between triangle singularities and K3 surfaces below. Let $T$ be one of the above fourteen symbols of hypersurface triangle singularities. Let $\Gamma^*$ be the Gabriëlov graph of the dual $T^*$. We can define a reducible curve $IF$ on a surface whose dual graph coincides with $\Gamma^*$. The curve $IF = IF (T)$ is called the curve at infinity of type $T$. The irreducible components are all smooth rational curves $C$ with $C \cdot C = -2$ and have one-to-one correspondence with vertices of $\Gamma^*$. For two components $C$, $C'$ of $IF$ the intersection number $C \cdot C'$ is equal to one or zero, according as the corresponding vertices in $\Gamma^*$ are connected in $\Gamma^*$ or not.

Let $G$ be a Dynkin graph with components of type $A$, $D$ or $E$ only. Assume that there exists a smooth K3 surface $Z$ satisfying the following conditions (a) and (b):

(a): $Z$ contains the curve at infinity $IF = IF (T)$ of type $T$ as a subvariety.
(b): Let $E$ be the union of all smooth rational curves on $Z$ disjoint from $IF$.

The dual graph of the components of $E$ coincides with graph $G$.

(Note that an irreducible curve $C$ on a K3 surface is a smooth rational curve if, and only if, $C \cdot C = -2$.)

Contracting every connected component of $E$ to a rational double point and then removing the image of $IF$, we obtain an open variety $\tilde{Y}$. 
Proposition 3.1 (Looijenga [5]).

1. Under the above assumption there exists a small deformation fiber \( Y \) of a singularity of type \( T \) homeomorphic to \( \hat{Y} \).

2. Let \( Y \) be a small deformation fiber of a singularity of type \( T \). Assume that \( Y \) has only rational double points as singularities, and the combination of rational double points on \( Y \) corresponds to a Dynkin graph \( G \). Then, there exists a K3 surface satisfying (a) and (b), and the corresponding \( \hat{Y} \) is homeomorphic to \( Y \).

By the above proposition our study is reduced to the study of K3 surfaces containing the curve \( IF = IF(T) \). K3 surfaces are complicated objects, but it is known that by the theory of periods we can reduce the study of K3 surfaces to the study of lattices.

Below we explain several terminologies in the lattice theory. (Urabe [9].) A free module over \( \mathbb{Z} \) of finite rank equipped with an integral symmetric bilinear form \((\ ,\ )\) is called a lattice. Besides, if a free module \( L \) over \( \mathbb{Z} \) of finite rank has a symmetric bilinear form \((\ ,\ )\) with values in rational numbers, then \( L \) is called a quasi-lattice. For simplicity we write \( x^2 = (x, x) \).

Let \( L \) be a quasi-lattice and \( M \) be a submodule. The submodule \( \tilde{M} = \{ x \in L \mid mx \in M \text{ for some non-zero integer } m \} \)

is called the primitive hull of \( M \) in \( L \). We say that \( M \) is primitive, if \( M = \tilde{M} \), and an element \( x \in L \) is primitive, if \( M = \mathbb{Z} x \) is primitive. We say that an embedding \( M \to L \) of quasi-lattices is a primitive embedding, if the image is primitive. If \( M \) is non-degenerate and primitive as a sub-quasi-lattice, we can define the canonical induced bilinear form on the quotient module \( L/M \).

Let \( L \) be a quasi-lattice, and \( FL \) be a submodule of \( L \) such that the index \(#(L/FL)\) is finite. Set

\[
R = \{ \alpha \in FL \mid \alpha^2 = -2 \} \cup \{ \beta \in L \mid \beta^2 = -1 \text{ or } -2/3 \}
\]

\[
\cup \{ \gamma \in L \mid \gamma^2 = -1/2, 2\gamma \in FL \}
\]

The set \( R = R(L, FL) \) is called the root system of \((L, FL)\), and every element \( \alpha \in R \) is called a root. If the pair \((L, FL)\) satisfies the following conditions (R1) and (R2), then \((L, FL)\) is called a root module:

(R1): \( 2(x, \alpha) / \alpha^2 \) is an integer for every \( x \in L \) and \( \alpha \in R \).

Under (R1), for every \( \alpha \in R \) we can define an isomorphism \( s_\alpha : L \to L \) preserving the bilinear form, by setting for \( x \in L \) \( s_\alpha(x) = x - 2(x, \alpha) \alpha / \alpha^2 \).

(R2): \( s_\alpha(FL) = FL \) for every \( \alpha \in R \).

Let \((L, FL)\) be a root module. If \( L = FL \), we say that it is regular and abbreviate \( FL \). Let \( M \) be a submodule of \( L \). It is easy to check that the pair \((M, FL \cap M)\) is again a root module. Below we identify \( M \) with the pair \((M, FL \cap M)\). If the root system of \( M \) and the root system of \( \tilde{M} \) coincide, then we say that \( M \) is full. An embedding \( M \to L \) of quasi-lattices is a full embedding, if the image is full.

Let \( G \) be a Dynkin graph with several components of type \( A, D \) or \( E \) only. We can define a lattice and its basis such that the corresponding dual graph coincides with \( G \). This lattice is called the root lattice of type \( G \) and is denoted by \( Q(G) \). \( Q(G) \) is a regular root module with a basis \( \alpha_1, \alpha_2, \ldots, \alpha_r \) with \( \alpha_i^2 = -2 \) for every \( i \). Let \( \Lambda_N \) denote the even unimodular lattice with signature \((N, 16 + N)\) for \( N \geq 0 \). The isomorphism class of \( \Lambda_N \) is unique if \( N \geq 1 \) and thus \( \Lambda_N \cong \Lambda_{N-1} \oplus H \).

For a K3 surface \( Z \) the second cohomology group \( H^2(Z, \mathbb{Z}) \) with the intersection
form is a lattice isomorphic to $\Lambda_3$. Let $P = P(T)$ be the lattice whose dual graph is the Gabrielov graph $\Gamma^*$ of the dual $T^*$. Assume that there exists a K3 surface $Z$ satisfying the above condition (a). The classes of the components of $IF$ generate a primitive sublattice in $H^2(Z, \mathbb{Z})$, which is isomorphic to $P$.

**Proposition 3.2.**
1. If $N \geq 1$, there is a primitive embedding $P \to \Lambda_N$.
2. If $N \geq 2$, a primitive embedding $P \to \Lambda_N$ is unique up to automorphisms of $\Lambda_N$.
3. If $N \geq 1$, for any embedding $P \to \Lambda_N$, the pair $\left(\Lambda_N/\hat{P}, F_N\right)$ is a root module, where $F_N$ is the image of the orthogonal complement of $P$ in $\Lambda_N$ by the canonical surjective homomorphism $\Lambda_N \to \Lambda_N / \hat{P}$.
4. For any primitive embedding $P = P(T) \to \Lambda_2$ the orthogonal complement $F_2$ of $P$ in $\Lambda_2$ has a basis whose dual graph coincides with the Gabrielov graph of type $T$.

With aid of Looijenga’s results in [3] we can show the following:

**Proposition 3.3.** We fix a primitive embedding $P \to \Lambda_3$. There exists a K3 surface $Z$ satisfying the above conditions (a) and (b) if, and only if, there is a full embedding $Q(G) \to \Lambda_3/P$.

**Corollary 3.4.** $G \in PC(T)$ if, and only if, there is a full embedding $Q(G) \to \Lambda_3/P(T)$.

By Proposition 3.3 our study has been reduced to the lattice theory. Next, we have to consider properties of the lattice $P = P(T)$ depending on $T$ closely. Let $T$ be one of fourteen symbols of hypersurface triangle singularities.

**Proposition 3.5.** We fix $N \geq 1$.
1. For any $T$ and for any embedding $P(t) \to \Lambda_N$ the quasi-lattice $\Lambda_N / \hat{P}(T)$ does not contain an element $\beta$ with $\beta^2 = -1$.
2. The root module $\left(\Lambda_N / \hat{P}(T), F_N\right)$ contains a root $\gamma$ with $\gamma^2 = -1/2$ for some embedding $P(T) \to \Lambda_N$ if, and only if, $T = E_{13}$, $Z_{12}$, $Q_{11}$, $W_{13}$ or $U_{12}$. It contains a root $\gamma$ with $\gamma^2 = -1/2$ for some primitive embedding $P(T) \to \Lambda_N$ if, and only if, $T = E_{13}, Z_{12}$ or $Q_{11}$.
3. The root module $\left(\Lambda_N / \hat{P}(T), F_N\right)$ contains a root $\beta$ with $\beta^2 = -2/3$ for some embedding $P(T) \to \Lambda_N$ if, and only if, $T = E_{14}, Z_{13}$ or $Q_{12}$.

Consider the case where $(L, FL)$ is a root module such that the bilinear form on $L$ has signature $(1, \text{rank } L - 1)$. In this case we can apply the hyperbolic geometry, and we can give the generalization of the theory in the negative definite case such as the Weyl chamber and the Dynkin graph. The generalized Dynkin graph in this case is called the Coxeter-Vinberg graph. (Vinberg [1].)

We need consider the Coxeter-Vinberg graph of $(\Lambda_2/P(T), F_2)$. By Proposition 3.2.4 we can expect that it is related to the Gabrielov graph. We fix a primitive embedding $P(T) \to \Lambda_2$.

**Proposition 3.6.** Let $\tilde{\Gamma}$ denote the Coxeter-Vinberg graph of $(\Lambda_2/P(T), F_2)$.
1. We can draw $\tilde{\Gamma}$ in finite steps if, and only if, $T \neq S_{11}, S_{12}$.
2. If $T \neq W_{12}, W_{13}, S_{11}, S_{12}, U_{12}$, every vertex in $\tilde{\Gamma}$ corresponds to a root.
3. If $T = W_{12}$, every vertex in $\tilde{\Gamma}$ corresponds to either a root $\alpha$ with $\alpha^2 = -2$ or an element $\delta$ with $\delta^2 = -2/5$.

4. If $T = W_{13}$ or $U_{12}$, every vertex in $\tilde{\Gamma}$ corresponds to either a root $\alpha$ with $\alpha^2 = -2$ or an element $\delta$ with $\delta^2 = -1/2$ and $2\delta \notin F_2$.

5. If $T = E_{12}$, $Z_{11}$, or $Q_{10}$, the Gabriélov graph coincides with $\tilde{\Gamma}$.

6. If $T = E_{13}$, $E_{14}$, $Z_{12}$, $Z_{13}$, $Q_{11}$, $Q_{12}$, $W_{12}$, or $U_{12}$, the Gabriélov graph is the subgraph of $\tilde{\Gamma}$ consisting of all vertices corresponding to a root $\alpha$ with $\alpha^2 = -2$.

7. If $T = W_{13}$, the Gabriélov graph is the maximal subgraph of $\tilde{\Gamma}$ such that every vertex corresponds to a root $\alpha$ with $\alpha^2 = -2$, and if $\alpha$, $\beta$ are roots corresponding to two vertices, then $(\alpha, \beta) \neq -2$.

We can explain main ideas in the verification of our Theorem 2.1 here. Let $PC(T)$ denote the set of all Dynkin graphs made from a Dynkin subgraph of the Gabriélov graph of type $T$ by an elementary transformation or a tie transformation. We assume that $G \in PC(T)$ was made from a Dynkin subgraph $G'$ of the Gabriélov graph. Besides, we fix a primitive embedding $P \to \Lambda_N$ for $N = 2, 3$. By Proposition 8.2 there is a primitive embedding $Q(G) \to F_3$. By the theory of elementary and tie transformations (Urabe [1, 2]), we can conclude that there is a full embedding $Q(G) \to F_3 \cong F_2 \oplus H$ into the regular root module $F_3$. Assume $T \neq E_{13}, Z_{12}, Q_{11}, E_{14}, Z_{13}, Q_{12}$ here. By Proposition 8.3, the composition $Q(G) \to F_3 \subset \Lambda_3/P$ defines a full embedding into the root module $(\Lambda_3/P, F_3)$ in these cases. By Corollary 8.4 we have $G \in PC(T)$. Thus $PC(T) \subset PC(T)$.

Next, we determine the difference $PC(T) - PC(T)$. Let $r$ be the number of vertices of a graph $G$. It is easy to see that if $G \in PC(T)$, then $r \leq \mu - 2$. In case $T \neq S_{11}, S_{12}$, using Proposition 8.6, we can show that conditions $G \in PC(T)$ and $G \in PC(T)$ are equivalent if $r \leq \mu - 5$. Thus we can assume $r = \mu - 2, \mu - 3$ or $\mu - 4$. For triangle singularities the Milnor number $\mu$ is relatively small, and it is easy to check whether a Dynkin graph $G$ belongs to $PC(T) - PC(T)$ case-by-case. To tell the truth, we could not succeed in finding any effective method except case-by-case checking. This is a weak point of our theory. I regret this fact and hope that somebody can improve it. If $T = S_{11}$ or $S_{12}$, the checking becomes more complicated since we have no Coxeter-Vinberg graph.

Now, if $T \neq W_{12}, W_{11}, S_{11}, S_{12}, U_{12}$, then because of Proposition 8.6, we can formulate another theorem. (Urabe [10].) In this another theorem we start from not a Gabriélov graph but a Dynkin graph possibly with a component of type $BC_1$ or $G_2$, and the number of transformations is not one but two. There, no exception appears even in the case $Z_{13}$. (We can make $A_7 + A_4$ from $E_7 + G_2$ by two tie transformations.) For $T = E_{13}, Z_{12}, Q_{11}, E_{14}, Z_{13}$ or $Q_{12}$ our theorem in this article follows from this theorem in another formulation. Also for $T = W_{12}, W_{13}, S_{11}, S_{12}, U_{12}$ the theorem in another formulation is possible, but becomes very complicated, because Proposition 8.6 does not hold for them. It is not worth mentioning.

Details of the verification will appear elsewhere.
Now, it is very strange that our Theorem 2.1 has a few exceptions in a few cases. Perhaps this is because our theory has a missing part.

**Problem.** Find the missing part of our theory and give a simple characterization of the set $P C (T)$ without exceptions.

This problem may be very difficult, but I believe that there exists a solution.

**References**

[1] V. I Arnold, *Local normal forms of functions*, Invent. Math. 35 (1976), 87–109.
[2] N. Bourbaki, *Groupes et algèbre de Lie*, Hermann, Paris, 1968.
[3] A. H. Durfee, *Fifteen characterization of rational double points and simple critical points*, Enseign. Math. II 25 (1979), 131–163.
[4] A. M. Gabriélov, *Dynkin diagrams for unimodular singularities*, Functional Anal. Appl. 8 (1974), 192–196.
[5] E. Looijenga, *The smoothing components of a triangle singularity II*, Math. Ann. 269 (1984), 357–387.
[6] H. C. Pinkham, *Singularités exceptionelles, la dualité étrange d’Arnold et les surfaces K-3*, C. R. Acad. Sci. Paris Sér. A 284 (1977), 615–618.
[7] T. Urabe, *Elementary transformations of Dynkin graphs and singularities on quartic surfaces*, Invent. Math. 87 (1987), 549–572.
[8] , *The transformations of Dynkin graphs and singularities on quartic surfaces*, Invent. Math. 100 (1990), 207–230.
[9] , *Dynkin graphs and quadrilateral singularities*, Springer-Verlag, Berlin Heidelberg New-York, 1993.
[10] , *Dynkin graphs and triangle singularities*, Kodai Math. J. 17 (1994), 395–401.
[11] E. B. Vinberg, *On groups of unit elements of certain quadratic forms*, Math. USSR Sbornik 16 (1972), 17–35.

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