SOLVABILITY OF COUPLED SYSTEMS OF FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We present existence theorems for coupled systems of quadratic integral equations of fractional order. As applications we deduce existence theorems for two coupled systems of Cauchy problems. Also, an example illustrating the existence theorem is given.

Key words: Quadratic integral equation of fractional-order, coupled system, Cauchy problems, Schauder fixed point.

Abstrak. Makalah ini membahas teorema tentang eksistensi solusi dari sistem persamaan integral kuadratik orde fraksional. Sebagai aplikasi, teorema eksistensi untuk sistem masalah Cauchy diturunkan. Selain itu, sebuah contoh diberikan untuk menggambarkan teorema eksistensi tersebut.

Kata kunci: Persamaan integral kuadratik orde fraksional, sistem kopol, masalah Cauchy, titik tetap Schauder.

1. Introduction

Systems occur in various problems of applied nature, for instance, see (Bashir Ahmad, Juan Nieto [9]- Y. Chen, H. An [11], El-Sayed and Hashem [22], Gafiychuk, Datsko, Meleshko [24], Gejji [25] and Lazarevich [27]). Recently, X. Su [32] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [33] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space proved in El-Sayed and Hashem [18]- El-Sayed and Hashem [20]. Also, a comparison between the classical method.
of successive approximations (Picard) method and Adomian decomposition method of coupled system of quadratic integral equations proved in El-Sayed, Hashem and Ziada [21].

Let $\mathbb{R}$ be the set of real numbers whereas $I = [0, 1]$, $L_1 = L_1[I]$ be the space of Lebesgue integrable functions on $I$.

Firstly, we prove the existence of at least one continuous solution for the coupled system of quadratic functional integral equation of fractional order

$$x(t) = a_1(t) + g_1(t, y(\psi_1(t))) \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, y(\phi_1(s))) \, ds, \ t \in I, \ \alpha > 0$$

$$y(t) = a_2(t) + g_2(t, x(\psi_2(t))) \int_0^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) \, ds, \ t \in I, \ \beta > 0.$$  \hspace{1cm} (1)

Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. Argyros [1]- Banaś, Rzepka [8] and El-Sayed, Hashem[13]- El-Sayed, Rzepka[23]). However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use Tychonoff fixed point theorem. The existence of continuous solutions for some quadratic integral equations was proved by using Schauder-Tychonoff fixed point theorem Salem [31].

Also, the existence of solutions of the two Cauchy problems

$$R^\alpha D x(t) = f_1(t, y(\psi_1(t))), \ t \in (0, 1) \ and \ x(0) = 0, \ \alpha \in (0, 1)$$

$$R^\beta D y(t) = f_2(t, x(\psi_2(t))), \ t \in (0, 1) \ and \ y(0) = 0, \ \beta \in (0, 1)$$

(2)

(where $R^\alpha D$ is the Riemann-Liouville fractional order derivative) and

$$\frac{dx(t)}{dt} = f_1(t, y(\psi_1(t))), \ t \in (0, 1), \ x(0) = x_0,$$

$$\frac{dy(t)}{dt} = f_2(t, x(\psi_2(t))), \ t \in (0, 1), \ y(0) = y_0,$$

will be proved.

The proof of the main result will be based on the following fixed-point theorem.

**Theorem 1.1. (Schauder Fixed Point Theorem)Curtain and Pritchard [12].**

Let $Q$ be a nonempty, convex, compact subset of a Banach space $X$, and $T : Q \to Q$ be a continuous map. Then $T$ has at least one fixed point in $Q$. 

Definition 1.2. The fractional-order integral of order $\beta$ (positive real number) of the function $f$ is defined on $[a, b]$ by (see Kilbas, Srivastava and Trujillo [26], Podlubny [26], Miller and Ross [29] and Samko, Kilbas, Marichev [30])

$$I_0^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \, ds,$$

and when $a = 0$, we have $I_0^\beta f(t) = I_0^\alpha f(t)$.

Definition 1.3. The Riemann-Liouville fractional-order derivative of order $\alpha \in (0, 1)$ of the function $f$ is given by (see Kilbas, Srivastava and Trujillo [26], Podlubny [26], Miller and Ross [29] and Samko, Kilbas, Marichev [30])

$$D^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t).$$

For the properties of fractional calculus (see Kilbas, Srivastava and Trujillo [26], Podlubny [26], Miller and Ross [29] and Samko, Kilbas, Marichev [30] for example).

2. Existence of Continuous Solutions

Now, the coupled system (1) will be investigated under the assumptions:

(i) $a_i : I \rightarrow \mathbb{R}$, $i = 1, 2$ are continuous and bounded with $M_i = \sup_{t \in I} |a_i(t)|$.

(ii) $g_i : I \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$ are continuous and bounded with

$$N_i = \sup_{(t, x) \in I \times \mathbb{R}} |g_i(t, x)|, \quad i = 1, 2.$$

(iii) There exist constants $h_i$, $l_i$, $i = 1, 2$ respectively satisfying

$$|g_i(t, x) - g_i(s, y)| \leq l_i |t - s| + h_i |x - y|$$

for all $t, s \in I$ and $x, y \in \mathbb{R}$.

(iv) $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$ satisfy Carathéodory condition (i.e. measurable in $t$ for all $x : I \rightarrow \mathbb{R}$ and continuous in $x$ for all $t \in I$).

(v) There exist two functions $m_i \in L_1$ and positive constants $b_i$ such that

$$|f_i(t, x)| \leq m_i(t) + b_i |x| \quad (\forall (t, x) \in I \times \mathbb{R})$$

and $k_i = \sup_{t \in I} m_i(t), \quad i = 1, 2$

for any $\gamma_1 \leq \alpha, \gamma_2 \leq \beta$.

(vi) $\psi, \phi_i : I \rightarrow I$ are continuous.

Let $C(I)$ be the class of all real functions defined and continuous on $I$ with the norm

$$|| x || = \sup_{} \{ | x(t) | : t \in I \}.$$

Now, we define the Banach space $X = \{ x(t) | x(t) \in C(I) \}$ endowed with the norm $||x||_X = \sup_{t \in I} |x(t)|$, $Y = \{ y(t) | y(t) \in C(I) \}$ endowed with the norm $||y||_Y = \sup_{t \in I} |y(t)|$. For $(x, y) \in X \times Y$, let $||(x, y)||_{X \times Y} = \sup_{t \in I} \{ ||x||_X, ||y||_Y \}$.

Clearly, $(X \times Y, ||(x, y)||_{X \times Y})$ is a Banach space.
Define the operator $T$ by 
\[ T(x, y)(t) = (T_1 y(t), T_2 x(t)), \]
where 
\[ T_1 y(t) = a_1(t) + g_1(t, y(\psi_1(t))) \int_0^t (t-s)^{\alpha-1} \Gamma(\alpha) f_1(s, y(\phi_1(s))) \, ds, \]
\[ T_2 x(t) = a_2(t) + g_2(t, x(\psi_2(t))) \int_0^t (t-s)^{\beta-1} \Gamma(\beta) f_2(s, x(\phi_2(s))) \, ds. \]

**Theorem 2.1.** Let the assumptions (i)-(vi) be satisfied, then the coupled system of quadratic integral equations of fractional order (1) has at least one solution in $X \times Y$.

**Proof.** Define 
\[ U = \{ u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : ||(x, y)||_{X \times Y} \leq r \}. \]
For $(x, y) \in U$, we have 
\[ |T_1 y(t)| \leq |a_1(t)| + |g_1(t, y(\psi_1(t)))| \int_0^t (t-s)^{\alpha-1} \Gamma(\alpha) |f_1(s, y(\phi_1(s)))| \, ds \]
\[ |T_1 y(t)| \leq M_1 + N_1 \int_0^t (t-s)^{\alpha-1} \Gamma(\alpha) |g_1(t, y(\psi_1(t)))| \, ds. \]

Also, from assumption (v) we obtain 
\[ |T_1 y(t)| \leq M_1 + N_1 \int_0^t (t-s)^{\alpha-1} \Gamma(\alpha) |g_1(t, y(\psi_1(t)))| \, ds \]
\[ + N_1 b_1 r_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, ds, \]
\[ ||y||_Y = \sup_{t \in I} |y(t)| \leq r_1. \]

Then 
\[ ||T_1 y(t)|| \leq M_1 + \frac{N_1 k_1}{\Gamma(\alpha - \gamma_1 + 1)} + \frac{N_1 b_1 r_1}{\Gamma(1 + \alpha)}. \]

From the last estimate we deduce that 
\[ r_1 = (M_1 + \frac{N_1 k_1}{\Gamma(\alpha - \gamma_1 + 1)} + \frac{N_1 b_1 r_1}{\Gamma(1 + \alpha)})^{-1}. \]

By a similar way as done above we have 
\[ ||T_2 x(t)|| \leq M_2 + \frac{N_2 k_2}{\Gamma(\beta - \gamma_2 + 1)} + \frac{N_2 b_2 r_2}{\Gamma(1 + \beta)}. \]

and 
\[ r_2 = (M_2 + \frac{N_2 k_2}{\Gamma(\beta - \gamma_2 + 1)} + \frac{N_2 b_2 r_2}{\Gamma(1 + \beta)})^{-1}. \]

Therefore, 
\[ ||T u(t)|| = ||T(x, y)(t)|| = \max_{t \in I} \{ ||T_1 y(t), T_2 x(t)|| \} = r. \]

From the last estimate we can choose 
\[ r = \max_{t \in I} \{ r_1, r_2 \}. \]
then, for every $u = (x, y) \in U$ we have $Tu \in U$ and hence $TU \subseteq U$.

It is clear that the set $U$ is closed and convex. Assumptions (ii) and (iv) imply that $T : U \to C(I) \times C(I)$ is a continuous operator. Now, for $u = (x, y) \in U$, and for each $t_1, t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we get

$$|(T_2 x)(t_2) - (T_2 x)(t_1)| = |a_2(t_2) - a_2(t_1) + g_2(t_2, x(\psi_2(t_2))) I^\beta f_2(t_2, x(\phi_2(t_2))) - g_2(t_1, x(\psi_2(t_1))) I^\beta f_2(t_1, x(\phi_2(t_1))) + g_2(t_1, x(\psi_2(t_1))) I^\beta f_2(t_2, x(\phi_2(t_2))) - g_2(t_1, x(\psi_2(t_1))) I^\beta f_2(t_2, x(\phi_2(t_2)))|$$

$$\leq |a_2(t_2) - a_2(t_1)| + |g_2(t_2, x(\psi_2(t_2))) - g_2(t_1, x(\psi_2(t_1)))| I^\beta |f_2(t_2, x(\phi_2(t_2))) - f_2(t_1, x(\phi_2(t_1)))|$$

but

$$|I^\beta f_2(t_2, x(\phi_2(t_2))) - I^\beta f_2(t_1, x(\phi_2(t_1)))| = \left| \int_{0}^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds - \int_{0}^{t_1} \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds - \int_{t_1}^{t_2} \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds \right|$$

$$\leq \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds + \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) ds$$

Then

$$|I^\beta f_2(t_2, x(\phi_2(t_2))) - I^\beta f_2(t_1, x(\phi_2(t_1)))|$$

$$\leq I_1^\beta |f_2(t_2, x(\phi_2(t_2)))| + b_2 I_1^\beta \sum_{i=1}^{m_2(t_2)} |x(\phi_2(t_2))|$$

$$\leq I_1^\beta \sum_{i=1}^{m_2(t_2)} |x(\phi_2(t_2))| + b_2 I_1^\beta \sum_{i=1}^{m_2(t_2)} |x(\phi_2(t_2))|$$

$$\leq k_2 \frac{(t_2 - t_1)^{\beta - \gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + b_2 r_2 \frac{(t_2 - t_1)^{\beta}}{\Gamma(\beta + 1)}.$$

Then we get

$$|(T_2 x)(t_2) - (T_2 x)(t_1)|$$

$$\leq |a_2(t_2) - a_2(t_1)| + |l_2| |t_2 - t_1| + h_2 |x(\psi_2(t_2)) - x(\psi_2(t_1))| + I^\beta |f_2(t_2, x(\phi_2(t_2))) - f_2(t_1, x(\phi_2(t_1)))|$$

$$+ |g_2(t_1, x(\psi_2(t_1)))| I^\beta \left( k_2 \frac{(t_2 - t_1)^{\beta - \gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + b_2 r_2 \frac{(t_2 - t_1)^{\beta}}{\Gamma(\beta + 1)} \right).$$
i.e.,
\[ |(T_2 x)(t_2) - (T_2 x)(t_1)| \]
\[ \leq |a_2(t_2) - a_2(t_1)| \]
\[ + |l_2|t_2 - t_1| + h_2|x(t_2) - x(t_1)| \]
\[ + b_2 N_2 \frac{(t_2 - t_1)^{\beta - \gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + N_2 b_2 r_2 \frac{\alpha}{\Gamma(\beta + 1)} (t_2 - t_1)^\beta. \]

As done above we can obtain
\[ |(T_1 y)(t_2) - (T_1 y)(t_1)| \]
\[ \leq |a_1(t_2) - a_1(t_1)| \]
\[ + \frac{k_1}{\Gamma(\alpha - \gamma_1 + 1)} |l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \]
\[ + \frac{b_1 r_1}{\Gamma(\alpha + 1)} |l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \]
\[ + \frac{k_1 N_1}{\Gamma(\alpha - \gamma_1 + 1)} (t_2 - t_1)^{\alpha - \gamma_1} + N_1 b_1 r_1 \frac{\alpha}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha. \]

Now, from the definition of the operator \( T \), we get
\[ Tu(t_2) - Tu(t_1) = T(x, y)(t_2) - T(x, y)(t_1) \]
\[ = (T_1 y(t_2), T_2 x(t_2)) - (T_1 y(t_1), T_2 x(t_1)) \]
\[ = (T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1)), \]

and
\[ ||Tu(t_2) - Tu(t_1)|| = \max_{t_1, t_2 \in I} {||T_1 y(t_2) - T_1 y(t_1)|| + ||T_2 x(t_2) - T_2 x(t_1)||} \]
\[ \leq ||a_1(t_2) - a_1(t_1)|| + \frac{k_1}{\Gamma(\alpha - \gamma_1 + 1)} |l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \]
\[ + \frac{b_1 r_1}{\Gamma(\alpha + 1)} |l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \]
\[ + \frac{k_2}{\Gamma(\beta - \gamma_2 + 1)} |l_2|t_2 - t_1| + h_2|x(t_2) - x(t_1)| \]
\[ + \frac{k_2}{\Gamma(\beta - \gamma_2 + 1)} |l_2|t_2 - t_1| + h_2|x(t_2) - x(t_1)| \]
\[ + \frac{k_2}{\Gamma(\beta + 1)} (t_2 - t_1)^\beta \]
Hence
\[ |t_2 - t_1| < \delta \implies \|Tu(t_2) - Tu(t_1)\| < \varepsilon(\delta). \]
This means that the functions of \( TU \) are equi-continuous on \( I \). Then by the Arzela-Ascoli Theorem Curtain and Pritchard [12] the closure of \( TU \) is compact. Since all conditions of the Schauder Fixed-point Theorem hold, then \( T \) has a fixed point in \( U \) which completes the proof.

**Example 2.2.** Consider the following coupled system of functional equations for \( t \in I \)
\[
\begin{align*}
x(t) &= 1 + \sqrt{t^2 + 5} + t(|\log(|y(t)| + 3)| + 1) t^{2/3} \left[ 2t + \frac{1}{3-t} y(\sin(t^2 + 3)) \right], \\
y(t) &= 1 + \left[ \frac{1+2t}{10} + \frac{x^2(t)}{30} e^{-t} \right] t^{2/3} \left[ -\ln(1-t) + \frac{1}{3-t} x(\sin(t^2 + 4t)) \right].
\end{align*}
\]

Set
\[
\begin{align*}
f_1(t,x) &= 2t + \frac{1}{3-t} x, \quad g_1(t,y) = \sqrt{t^2 + 5} + t(|\log(|y(t)| + 3)| + 1), \quad t \in I \\
f_2(t,x) &= -\ln(1-t) + \frac{1}{3-t} x, \quad g_2(t,x) = \frac{1+2t}{10} + \frac{x^2}{30} e^{-t}, \quad t \in I.
\end{align*}
\]
Then easily we can deduce that:

- \( M_1 = M_2 = 1 \).
- \( |f_2(t,x)| \leq \frac{1}{2} x \) and \( |f_1(t,x)| \leq 2t + 1/2 x. \)
- \( \phi_1(t) = \sin(t^2 + 3t), \quad \phi_2(t) = \sin(t^2 + 4t), \quad \psi_1(t) = \psi_2(t) = t, \)
- \( |g_1(t,z) - g_1(s,y)| = |\sqrt{t^2 + 5} + t(|\log(|z(t)| + 3)| + 1) - \sqrt{s^2 + 5} - s(|\log(|y(s)| + 3)| + 1)| \)
- \( -\sqrt{s^2 + 5} - s(|\log(|y(s)| + 3)| + 1) + |t - s| \)
- \( + |t \log(|y(s)| + 3) - s \log(|y(s)| + 3)| \)
- \( \leq \frac{6}{5} |t - s| + |z - y| + |t - s| + r|t - s| \)
- \( \leq (2 + r) |t - s| + |z - y|, \quad r > 0 \)
and
\[ |g_2(t, z) - g_2(s, x)| = \left| \frac{1 + 2t}{10} + \frac{z^2}{30} e^{-t} - \frac{1 + 2s}{10} - \frac{x^2}{30} e^{-s} \right| \]
\[ \leq \frac{2}{10} |t - s| + \frac{1}{30} |e^{-t}z^2 - e^{-s}x^2| + \frac{1}{30} |e^{-t}x^2 - e^{-s}x^2| \]
\[ \leq \frac{1}{5} |t - s| + \frac{1}{30} |x + z| |x - z| + \frac{r^2}{30} |e^{-t} - e^{-s}| \]
\[ \leq \frac{1}{5} |t - s| + \frac{2r}{30} |x - z| + \frac{r^2}{30} |t - s| \]
\[ \leq \frac{6 + r^2}{30} |t - s| + \frac{r}{15} |x - z|, \ r > 0. \]

Then all the assumptions of Theorem 2.1 are satisfied so the coupled system of the functional equations (5) possesses at least one solution in \( X \times Y \).

3. Spacial Cases

Corollary 3.1. Let the assumptions of Theorem 2.1 be satisfied (with \( \psi_i(t) = \phi_i(t) = t, \ i = 1, 2 \)), then the coupled system of the fractional-order quadratic integral equations
\[
x(t) = a_1(t) + g_1(t, y(t)) \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, y(s)) \, ds, \ t \in I, \ \alpha > 0
\]
\[
y(t) = a_2(t) + g_2(t, x(t)) \int_0^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(s)) \, ds, \ t \in I, \ \beta > 0
\]
has at least one solution in \( X \times Y \).

Corollary 3.2. Let the assumptions of Theorem 2.1 be satisfied (with \( g_i(t, x) = 1, \ i = 1, 2 \)), then the coupled system of the fractional-order integral equations
\[
x(t) = a_1(t) + \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, y(\phi_1(s))) \, ds, \ t \in I, \ \alpha > 0
\]
\[
y(t) = a_2(t) + \int_0^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s))) \, ds, \ t \in I, \ \beta > 0
\]
has at least one solution in \( X \times Y \).

Now, letting \( \alpha, \beta \to 1 \), we obtain
Corollary 3.3. Let the assumptions of Theorem 2.1 be satisfied (with \( g_i(t,x) = 1 \), \( a_1(t) = x_0 \), \( a_2(t) = y_0 \) and letting \( \alpha, \beta \to 1 \)), then the coupled system of the integral equations
\[
\begin{align*}
x(t) &= x_0 + \int_0^t f_1(s,y(\phi_1(s))) \, ds, \quad t \in I, \alpha > 0 \\
y(t) &= y_0 + \int_0^t f_2(s,x(\phi_2(s))) \, ds, \quad t \in I, \beta > 0
\end{align*}
\]
has at least one solution in \( X \times Y \) which is equivalent to the coupled system of the initial value problems (3).

4. The coupled system of the fractional order functional differential equations

For the coupled system of the initial value problems of the nonlinear fractional-order differential equations (2) we have the following theorem.

Theorem 4.1. Let the assumptions of Theorem 2.1 be satisfied (with \( a_i(t) = 0 \) and \( g_i(t,x(t)) = 1 \), \( i = 1,2 \)), then the coupled system of the Cauchy problems (2) has at least one solution in \( X \times Y \).

Proof. Integrating (2) we obtain the coupled system of the integral equations
\[
\begin{align*}
x(t) &= \int_0^t \frac{(t-s)^{a_i-1}}{\Gamma(a_i)} f_i(s,y(\phi_i(s))) \, ds, \quad t \in I, \alpha > 0 \\
y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,x(\phi_2(s))) \, ds, \quad t \in I, \beta > 0
\end{align*}
\]
(8)
which by Theorem 2.1 has the desired solution.

Operating with \( RD^\alpha \) on the first equation of the coupled system (8) and with \( RD^{\beta} \) on the second equation of the coupled system (8) we obtain the coupled system of the initial value problems (2). So the equivalence between the coupled system of the initial value problems (2) and the coupled system of the integral equations (8) is proved and then the results follow from Theorem 2.1.

References
[1] Argyros, I.K., "Quadratic Equations and Applications to Chandrasekhar’s and Related Equations", Bull. Austral. Math. Soc., 32 (1985), 275-292.
[2] Argyros, I.K., "On a Class of Quadratic Integral Equations with Perturbations", Funct. Approx. 20 (1992), 51-63.
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[3] Banaś, J., Lecko, M., and El-Sayed, W.G., "Existence Theorems for Some Quadratic Integral Equation", J. Math. Anal. Appl., 222 (1998), 276 - 285.

[4] Banaś, J., and Martinon, A., "Monotonic Solutions of a Quadratic Integral Equation of Volterra Type", Comput. Math. Appl., 47 (2004), 271 - 279.

[5] Banaś, J., Caballero, J., Rocha, J., and Sadarangani, K., "Monotonic Solutions of a Class of Quadratic Integral Equations of Volterra Type", Computers and Mathematics with Applications, 49 (2005), 943-952.

[6] Banaś, J., Rocha Martin, J., and Sadarangani, K., "On the Solutions of a Quadratic Integral Equation of Hammerstein Type", Mathematical and Computer Modelling, 43 (2006), 97-104.

[7] Banaś, J., and Rzepka, B., "Nondecreasing Solutions of a Quadratic Singular Volterra Integral Equation", Math. Comput. Modelling, 49 (2009), 488-496.

[8] Banaś, J., and Rzepka, B., "Monotonic Solutions of a Quadratic Integral Equations of Fractional Order", J. Math. Anal. Appl., 332 (2007), 1370 -1379.

[9] Ahmad, B., and Nieto, J.J., "Existence Results for a Coupled System of Nonlinear Fractional Differential Equations with Three-Point Boundary Conditions", Computers and Mathematics with Applications, 58 (2009,) 1838-1843.

[10] Bai, C., and Fang, J., "The existence of a Positive Solution for a Singular Coupled System of Nonlinear Fractional Differential Equations", Appl. Math. Comput., 150 (2004), 611-621.

[11] Chen, Y., and An, H., "Numerical Solutions of Coupled Burgers Equations with Time and Space Fractional Derivatives", Appl. Math. Comput., 200 (2008), 87-95.

[12] Curtain, R.F., and Pritchard, A.J., Functional Analysis in Modern Applied Mathematics, Academic Press, 1977.

[13] El-Sayed, A.M.A., and Hashem, H.H.G., "Carathéodory Type Theorem for a Nonlinear Quadratic Integral Equation", Math. Sci. Res. J., 12(4) (2008), 71-95.

[14] El-Sayed, A.M.A., and Hashem, H.H.G., "Monotonic Positive Solution of Nonlinear Quadratic Hammerstein and Urysohn Functional Integral Equations", Commentationes Mathematicae, 48(2) (2008), 199-207.

[15] El-Sayed, A.M.A., and Hashem, H.H.G., "Monotonic Solutions of Functional Integral and Differential Equations of Fractional Order", EJQTDE, 7 (2009), 1-8.

[16] El-Sayed, A.M.A., and Hashem, H.H.G., "Monotonic Positive Solution of a Nonlinear Quadratic Functional Integral Equation", Appl. Math. and Comput., 216 (2010), 2576-2580.

[17] El-Sayed, A.M.A., Hashem H.H.G., and Ziada, E.A.A., "Picard and Adomian Methods for Quadratic Integral Equation", Comp. Appl. Math., 29(3) (2010), 447-463.

[18] El-Sayed, A.M.A., and Hashem, H.H.G., "Coupled Systems of Hammerstein and Urysohn Integral Equations in Reflexive Banach Spaces", Differential Equations And Control Processes, 1 (2012), 1-12.

[19] El-Sayed, A.M.A., and Hashem, H.H.G., "Coupled Systems of Integral Equations in Reflexive Banach Spaces", Acta Mathematica Scientia, 32B(5), (2012), 1-8.

[20] El-Sayed, A.M.A., and Hashem, H.H.G., "A coupled System of Fractional Order Integral Equations in Reflexive Banach Spaces", Commentationes Mathematicae, 52(1) (2012), 21-28.

[21] El-Sayed, A.M.A., Hashem, H.H.G., and Ziada, E.A.A., "Picard and Adomian Methods for Coupled Systems of Quadratic Integral Equations of Fractional Order", Journal of Nonlinear Analysis and Optimization: Theory & Applications, 3(2), (2012), 171-183.

[22] El-Sayed, A.M.A., and Hashem, H.H.G., "Existence Results for Coupled Systems of Quadratic Integral Equations of Fractional Orders", Optimization Letters, 7 (2013), 1251-1260.

[23] El-Sayed, W.G., and Rzepka, B., "Nondecreasing Solutions of a Quadratic Integral Equation of Urysohn Type", Comput. Math. Appl., 51 (2006), 1065-1074.

[24] Gafiychuk, V., Datsko, B., and Meleshko, V., "Mathematical Modeling of Time Fractional Reaction-Diffusion Systems", J. Comput. Appl. Math., 220 (2008), 215-225.

[25] Gejji, V. D., "Positive Solutions of a System of Non-Autonomous Fractional Differential Equations", J. Math. Anal. Appl., 302 (2005), 56-64.
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[26] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland, 2006.

[27] Lazarevich, M.P., "Finite Time Stability Analysis of $PD^\alpha$ Fractional Control of Robotic Time-Delay Systems", *Mech. Res. Comm.* 33 (2006), 269-279.

[28] Podlubny, I., *Fractional Differential Equations*, San Diego-NewYork-London, 1999.

[29] Miller, K.S., and Ross, B., *An Introduction to Fractional Calculus and Fractional Differential Equations*, John Wiley, New York, 1993.

[30] Samko, S.G., Kilbas, A.A., and Marichev, O., *Integrals and Derivatives of Fractional Orders and Some of Their Applications*, Nauka i Teknika, Minsk, 1987.

[31] Salem, H.A.H., "On the Quadratic Integral Equations and Their Applications", *Computers and Mathematics with Applications*, 62 (2011), 2931-2943.

[32] Su, X., "Boundary Value Problem for a Coupled System of Nonlinear Fractional Differential Equations", *Appl. Math. Lett.*, 22 (2009), 64-69.

[33] Gafiychuk, V., Datsko, B., Meleshko, V., and Blackmore, D., "Analysis of the Solutions of Coupled Nonlinear Fractional Reaction-Diffusion Equations", *Chaos Solitons Fractals*, 41 (2009), 1095-1104.