Dynamical spin-spin susceptibility of Silicene

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We present a detailed study of the imaginary and real parts of the spin-susceptibility of silicene which can be generalized to other buckled honeycomb structure. We find that while the off-diagonal components are non-zero in individual valleys, they add up to zero upon including contributions from both the valleys. We investigate the interplay of the spin-orbit interaction and an external electric field applied perpendicular to the substrate and find that although the $xx$ and $yy$ components of the susceptibility are identical, they differ from the $zz$-component. The external electric field plays an important role in modifying the allowed inter-subband regions. In the dynamic limit, the real part of the susceptibility exhibits log-divergence, position of which can be tuned by the electric field and therefore has implications for spin-collective excitations. The effect of the electric field on the static part of the susceptibility and its consequence for the long distance decay of the spin-susceptibility have been explored.

I. INTRODUCTION

Spin-orbit (SO) interaction is one of the key ingredients in a spintronics device required for controlling and manipulating the spin degrees of freedom of an electron via the electric field$^{12}$. In this regard, enormous progress has already been made in the study of semiconductor based spintronics device$^{14}$. Recently the possibility of graphene and other 2D materials, in particular, silicene and germanene$^4$, topological insulators$^{5,6}$, Weyl semimetals$^{7-9}$, along with monolayer transition metal dichalcogenides such as MoS$_2$$^{10}$ with intrinsic and extrinsic SO coupling, have garnered wide attention from the fundamental physics point of view as well as for their potential for spintronics applications.

The low energy effective theory of many of these new materials is governed by the Dirac physics. In graphene, due to the relatively small mass of carbon atoms the SO coupling is very weak therefore the physics is effectively described by the massless Dirac theory. The conduction and valance bands meet at the two inequivalent Dirac points, called the $K$ and $K'$ points, which is where the Fermi energy also lies. On the other hand, due to the higher mass of silicon atoms SO coupling in silicene is appreciable ($\approx 3.9$ meV)$^{11-13}$. Unlike graphene which is completely planar, silicene has a buckled honeycomb sublattice structure resulting in the explicit breaking of inversion symmetry$^{14}$. An electric field applied perpendicular to the silicene surface leads to a staggered potential which in combination with the SO term determines the gap in the energy spectrum. Consequently, electric field can be used as a control parameter to drive silicene from a trivial band insulator phase to symmetry protected topological phase (e.g spin Hall insulator$^{15,16}$). At the critical point the band-gap closes$^{17,18}$ and silicene enters into a valley-spin polarized metallic state$^{19-21}$. These features in the energy spectrum provide the possibility for detecting quantum, anomalous and valley hall effects in silicene$^{19,22,23}$.

Useful insights into the electronic properties of materials are obtained by studying their charge response function or the charge polarization operator. It yields information regarding the single particle and collective excitations which are crucial for understanding the static and dynamical properties of many body systems$^{24,25}$. While the modifications to the response function due to the SO coupling in 2DEG with parabolic dispersion have been investigated in great detail$^{26,27}$, it is only recently that similar studies on the charge response function of materials with Dirac like dispersion have been made$^{5-9,28-35}$. There have also been studies on the spin response of SO coupled 2D electron system and of the helical surface states of a 3D topological insulator$^{5,36-39}$. By considering the dynamical spin-susceptibility of SO coupled 2D electron system the existence of spin-collective excitations was established$^{39}$, moreover, the surface states of a 3D topological insulator, described by the Dirac spectrum, have been predicted to host hybridized spin-charge coupled plasmons$^{5}$. Recently, Raman spectroscopy was used to reveal the collective spin-excitations of the chiral surface states of the three dimensional topological insulator Bi$_2$Se$_3$$^{40}$. On the other hand, the modifications to the static spin-susceptibility due to the SO terms yield additional interaction terms like Dzyaloshinskii-Moriya and Ising terms besides the usual isotropic Rudermann-Kittel-Kasuya-Yosida (RKKY) interaction term$^{41-44}$.

Recent studies of the charge polarization function of silicene have predicted the existence of charge collective excitation with a $\sqrt{q}$ dispersion at small $q$$^{20,45,46}$. However, the study of spin collective modes in silicene and other buckled honeycomb lattice is an ongoing and challenging work. As a first step towards the better understanding of the role of spin-orbit interaction in silicene we study in detail the spin-susceptibility in the non-interacting limit. The imaginary part of the spin-susceptibility, which yields information regarding the single particle spin decay channel, are identical for the $xx$ and $yy$ components while the $zz$-components are different. We discuss in detail the allowed single-particle transitions and the regions in the $(q, \omega)$ plane where the imaginary part of the susceptibility is non-zero. The role of electric field in extending the allowed regions for
particle-hole excitations is examined. We calculate the real part of susceptibility, with particular emphasis on the dynamic and static limits. We show that the real part of spin-susceptibility exhibits log-divergence in the dynamic limit (in the $xx$ and $yy$ channels) and discuss its significance with regard to the spin-collective modes. The static part of the spin-susceptibility exhibits Kohn-anomaly, interestingly nature of this anomaly and the momentum at which this happens can be controlled by electric field. The consequence of it for the long distance decay behavior of the spin-susceptibility have been studied.

Our paper is organized as follows: In sec.II we provide a general description of our model along with the low energy effective Hamiltonian of silicene. In sec. III we define the spin-susceptibility operator and discuss the contributions to the imaginary part of the spin-susceptibility arising from different transition scenarios. In sec. IV the real part of the spin-susceptibility in the dynamical and statical limits have been calculated. Summary of the results are provided in sec. V.

II. MODEL

The tight binding Hamiltonian of 2D silicene is given by

$$H = -t \sum_{\langle i,j \rangle, \alpha} \hat{c}_{i \alpha}^\dagger \hat{c}_{j \alpha} + \frac{\lambda_{SO}}{3\sqrt{3}} \sum_{\langle i,j \rangle, \alpha \beta} \nu_{ij} \hat{c}_{i \alpha}^\dagger \sigma_{\alpha \beta}^z \hat{c}_{j \beta} + l \sum_{\langle i \rangle, \alpha} \zeta_i E_z \hat{c}_{i \alpha}^\dagger \hat{c}_{i \alpha} - \mu \sum_{\langle i \rangle, \alpha} \hat{c}_{i \alpha}^\dagger \hat{c}_{i \alpha} + \hat{\mu} \sum_{i, \alpha} \hat{c}_{i \alpha}^\dagger \hat{c}_{i \alpha} ,$$

where the first term represents the nearest-neighbor hopping on the honeycomb lattice, the second term represents the effective SO term which couples next nearest-neighbor sites. The coupling parameter is denoted by $\lambda_{SO}$, $\sigma^z$ the pauli spin matrix, $\nu_{ij} = \hat{z} \cdot (\hat{d}_{i} \times \hat{d}_{j})/|\hat{d}_{i} \times \hat{d}_{j}|$ with $\hat{d}_{i}$ and $\hat{d}_{j}$ being the bonds between the two next nearest-neighbor sites. The third term represents the staggered sublattice potential, where $\zeta_\| = \pm 1$ for the A(B) sites and $2l$ is the separation between the A and B sublattices in the z-direction, $E_z$ is an applied electric field perpendicular to the plane and $\mu$ is the chemical potential. For silicene $t = 1.6$ eV, $\lambda_{SO} = 3.9$ meV and $l = 0.23\AA^{14,19,47}$. The Hamiltonian receives an additional contribution due to the Rashba SO-term, however, the magnitude of this term ($\lambda_R = 0.7$ meV) is almost an order of magnitude less than $\lambda_{SO}$. Moreover, near the Dirac points the rashba term is given by the linear $\sim \lambda_R k$ term which can be neglected when describing the low-energy physics$^{19,47}$. We note that germanene which has a buckled structure is also described by the Hamiltonian given in Eq. 1, with $t = 1.3$ eV, $\lambda_{SO} = 43$ meV and $l = 0.33\AA^{4,14,19,47}$, here also the Rashba term can be neglected when describing the low energy physics.

The low-energy effective Hamiltonian about the two inequivalent Dirac points $K, K'$ (where $\eta = \pm 1$) in the basis

$$H_\eta = \hbar v_F \left( \hat{k}_z (\hat{I} \otimes \hat{\tau}_I) - \eta k_y (\hat{I} \otimes \hat{\tau}_J) \right) + \eta l E_z (\hat{I} \otimes \hat{\tau}_3)$$

where the Pauli-matrix $\hat{\tau}$ acts on the sublattice basis and $\eta$ is the valley index. Henceforth, we will set $v_F = 1$ and $\hbar = 1$. In the presence of both the electric field and the spin-orbit term the spectrum is given by $E_{\eta \beta} = \alpha \sqrt{k^2 + \Delta_{\eta \beta}^2}$, where $\alpha = \pm 1$ and the inequivalent gaps for spins $\beta = \pm 1$ are given by $\Delta_{\eta \beta} = |E_z - \eta \beta \lambda_{SO}|$. In Fig. 1 we plot the energy spectrum near the K, K' points, where the energy gaps are $\Delta_{1/2} = |E_z \mp \lambda_{SO}|$. We note that the strength of the gap can be tuned by external electric fields, in particular, for the critical field $E_z = \lambda_{SO}/l$ the Hamiltonian exhibits gapless modes.

III. POLARIZATION FUNCTION

The non-interacting generalized susceptibility in the Matsubara formalism is given by

$$\chi_{ij}(q, \omega_n) = -\int_P \text{Tr} \left[ \hat{\sigma}_i \hat{G}_P \hat{\sigma}_j \hat{G}_P^+ Q \right] ,$$

where $\text{Tr}$ denotes trace over spin and sublattice degrees of freedom, $i, j = 0, x, y, z$, $P = (\vec{p}, \Omega_m)$ and $Q = (\vec{q}, \omega_n)$. Note that the polarization function/operator is related to the susceptibility via the relation, $\Pi_{ij}(q, \omega_n) = -\chi_{ij}(q, \omega_n)$. In the rest of the text we will be using the two terms interchangeably.

The corresponding zero temperature Matsubara Green’s function used in the above equation has the following form

$$\hat{G}_P = \frac{1}{4} \sum_{\beta, \alpha = \pm 1} \left[ (\hat{I} + \beta \hat{\sigma}_3) \otimes (\hat{I} - \alpha (\hat{p}_3 \cdot \hat{\tau})/E_{P \beta}) \right]$$

$$\left( i\Omega_n + \alpha E_{P \beta} \right) \right] ,$$

![FIG. 1. (Color online) Energy spectrum near the K and K’ points. The arrows indicate the orientation of the spin in the respective band.](image-url)
where $\alpha = \pm 1$ represents lower and upper bands respectively, $\vec{p}_\eta$ is the momentum for one of the valleys, the other valley yields identical contribution to $\Pi$ respectively. In the next two sub-sections, we will separate terms in the expression of $\Pi$ of the polarization operator, in particular, those arising from both the valleys. On the other hand, the diagonalization function due to the $\Delta_2 \to \Delta_1$ transition. 1A and 2A regions denote contributions from the transitions $u_{\Delta_2} \to u_{\Delta_1}$, whereas 1B, 2B and 3B denote contributions from $l_{\Delta_2}$ to $u_{\Delta_1}$. Here $\mu' = \sqrt{k_{F_1}^2 + \Delta_2^2}$, $k_{F_1} = \sqrt{\mu^2 - \Delta_1^2}$ and $k_{F_2} = \sqrt{\mu^2 - \Delta_2^2}$.

**FIG. 2.** (Color online) Shaded regions in the figure indicate non-zero contributions to the imaginary part of the polarization function due to the $\Delta_2 \to \Delta_1$ transition. 1A and 2A regions denote contributions from the transitions $u_{\Delta_2}$ to $u_{\Delta_1}$, whereas 1B, 2B and 3B denote contributions from $l_{\Delta_2}$ to $u_{\Delta_1}$. Here $\mu' = \sqrt{k_{F_1}^2 + \Delta_2^2}$, $k_{F_1} = \sqrt{\mu^2 - \Delta_1^2}$ and $k_{F_2} = \sqrt{\mu^2 - \Delta_2^2}$.

**FIG. 3.** (Color online) Plotted are $\text{Im}\Pi_{21}$ vs $\omega$ for $k_{F_1} - k_{F_2} < q = 1.25\mu < k_{F_1} + k_{F_2}$ and $k_{F_1} + k_{F_2} < q = 2.25\mu$. Here and in subsequent plots $\omega$ and $\Pi$ are in units of $\mu$.

**A. ($\Delta_2 \to \Delta_1$) Transition**

The transition from $u_{\Delta_2}$ to $u_{\Delta_1}$ is allowed for particles with energy $\epsilon$ in the range: $\max[|\mu - \omega, \Delta_2|] < \epsilon < \mu$. The angular integration of Eq. 5 (with $\alpha = \alpha' = -1$) yields,

$$\text{Im}\Pi_{21}^\alpha(q, \omega) = -\text{Re} \left[ \frac{1}{\sqrt{q^2 - \omega^2}} \int_{L_\perp} dx \frac{(x - \omega_1)^2 - \gamma_0}{8\pi} \right],$$

where $\gamma_0 = q^2 + \Delta_1^2$, $\omega_1 = \omega(\gamma_21 - 1)$, $\xi_21 = \sqrt{q^2 \gamma_21 + 4q^2 \Delta_2^2/(q^2 - \omega^2)}$, $\gamma_21 = 1 - \Delta_2 \Delta_d/(q^2 - \omega^2)$, along with the redefine parameter $\Delta_s = \Delta_2 + \Delta_1$ and $\Delta_d = \Delta_2 - \Delta_1$. Performing the integration by tak-
where the limits of integration to be $U_x = 2\mu + \omega \gamma_{21}$ and $L_x = 2\max[\mu - \omega, \Delta_2] + \omega \gamma_{21}$, we obtain

$$\text{Im} \Pi_{21}^{uu}(q, \omega) = -\frac{1}{4\pi} \frac{1}{\sqrt{\omega^2 - q^2}} \times \left\{ \begin{array}{ll}
G_{21}^{uu}(2\mu + \omega \gamma_{21}) - G_{21}^{uu}(2\max[\mu - \omega, \Delta_2] + \omega \gamma_{21}) & :1A \\
G_{21}^{uu}(\xi_{21}) - G_{21}^{uu}(\xi_{21}) & :2A
\end{array} \right\},$$

where

$$G_{21}^{uu}(x) = \frac{1}{4} \left\{ -2q^2 - 2\Delta_d^2 + \xi_{21}^2 + 2(\omega \gamma_{21} - \omega)^2 \right\} \log \left( \sqrt{x^2 - \xi_{21}^2} + x \right) + \left[ x - 4(\omega \gamma_{21} - \omega) \right] \sqrt{x^2 - \xi_{21}^2}. \quad (7)$$

The regions in the $(q, \omega)$ plane where $\text{Im} \Pi_{21}^{uu}(q, \omega)$ is non-zero are [see Fig. (2)]:

1A : $\omega < \mu - \mathcal{F}(k_{F_1}, \Delta_2)$
2A : $\pm \mu \mp \mathcal{F}(k_{F_{12}}, \Delta_{2(1)}) < \omega < -\mu + \mathcal{F}(-k_{F_2}, \Delta_1)$,

where $\mathcal{F}(x, y) = \sqrt{(q - x)^2 + y^2}$. The allowed regions for particle-hole (p-h) excitation in the $(q, \omega)$ plane can be obtained via kinematic consideration (see $\omega < q$ region in Fig. 2). For example, in the scenario being discussed, the minimum momentum required for p-h generation is $k_{F_1} - k_{F_2}$, this involves the collinear transition of a particle from the Fermi level of $u_{\Delta_2}$ to the Fermi level of $u_{\Delta_1}$ without a change in energy. Indeed, the particle’s energy need not change for the transition from the Fermi-level of one band to the Fermi-level of the other band, thus the maximum momentum change for such a process is $k_{F_2} + k_{F_1}$. For a given momentum $q > k_{F_1} - k_{F_2}$, the energy upper bound for a transition from $u_{\Delta_2}$ to $u_{\Delta_1}$ is $\omega_{\text{max}} = \sqrt{(k_{F_2} - q)^2 + \Delta_1^2} - \mu$. The process involves a particle getting excited from the Fermi level of $u_{\Delta_2}$ to an unoccupied level of $u_{\Delta_1}$, with the final direction being the same as the initial one. On the other hand the lower boundary (for $q > k_{F_1} + k_{F_2}$) is set by transition involving back-scattering of particle from the Fermi-level of $u_{\Delta_2}$ to $u_{\Delta_1}$ (with momentum change $q - k_{F_2}$) which requires

$$\omega_{\text{min}} = \sqrt{(k_{F_2} - q)^2 + \Delta_1^2} - \mu.$$

A lower, $L_x$, to upper band $u_{\Delta_1}$ transition requires the particle to have energy $\epsilon$ in the range: $\mu - \omega < \epsilon < -\Delta_2$. Performing the angular integration of Eq. 5 yields

$$\text{Im} \Pi_{21}^{uu}(q, \omega) = -\text{Re} \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \int_{L_x}^{U_x} \frac{dx \gamma_0 - (x + w_1)^2}{8\pi \sqrt{\xi_{21}^2 - x^2}} \right],$$

where the limits of integration are $U_x = 2(\omega - \mu) - \omega \gamma_{21}$ and $L_x = 2\Delta_2 - \omega \gamma_{21}$. Integrating the above equation we obtain the following result:

$$\text{Im} \Pi_{21}^{uu}(q, \omega) = -\frac{1}{4\pi} \frac{1}{\sqrt{\omega^2 - q^2}} \times \left\{ \begin{array}{ll}
G_{21}^{uu}(2\omega - \mu - \omega \gamma_{21}) - G_{21}^{uu}(\xi_{21}) & :1B \\
G_{21}^{uu}(\xi_{21}) - G_{21}^{uu}(\xi_{21}) & :2B \\
G_{21}^{uu}(\xi_{21}) - G_{21}^{uu}(\xi_{21}) & :3B
\end{array} \right\},$$

where,

$$G_{21}^{uu}(x) = \frac{1}{4} \left\{ 2q^2 + 2\Delta_d^2 - \xi_{21}^2 - 2(\omega \gamma_{21} - \omega)^2 \right\} \tan^{-1} \left( \frac{x}{\sqrt{\xi_{21}^2 - x^2}} \right) + \left[ x - 4(\omega \gamma_{21} - \omega) \right] \sqrt{\xi_{21}^2 - x^2}. \quad (8)$$

The non-zero regions in the $(q, \omega)$ plane are described by the following equations

1B : $\mu + \mathcal{F}(k_{F_1}, \Delta_2) < \omega < \mu + \mathcal{F}(-k_{F_1}, \Delta_2)$
2B : $\omega > \mu + \mathcal{F}(-k_{F_1}, \Delta_2)$
3B : $\sqrt{q^2 + \Delta_2^2} < \omega < \mu + \mathcal{F}(k_{F_1}, \Delta_2)$

Unlike the transitions involving only the upper bands, $q = 0$ particle-hole transitions are now allowed for all frequencies $\omega > \mu + \sqrt{k_{F_1}^2 + \Delta_2^2}$ (see $\omega > q$ region in Fig. 2). As $q$ is increased, the threshold frequency given by $\omega = \mu + \sqrt{(k_{F_1} - q)^2 + \Delta_1^2}$ exhibits a downturn, these are realized by processes involving particle with momentum $p < k_{F_1}$ moving to the upper Fermi level while maintaining its initial direction. For the above process, the minimum allowed frequency $\omega = \mu + \Delta_2$ is reached for $q = k_{F_1}$, where the transitioning particle had originally momentum $p = 0$. Increasing $q$ further, the threshold frequency exhibits an upturn. The process now involves particle from $l_{\Delta_2}$ moving to the upper Fermi level by
changing its initial direction. A further increase in $q$ changes the threshold frequency to $\omega = \sqrt{q^2 + \Delta_q^2}$ and is obtained by minimizing $\sqrt{(p-q)^2 + \Delta_q^2 + \sqrt{p^2 + \Delta_q^2}}$ with respect to $p$.

Combining $\Im\Pi_{12}^{\omega}$ and $\Im\Pi_{21}^{\omega}$ yields the contribution to the imaginary part of the polarization operator from the $2 \rightarrow 1$ processes represented as $\Im\Pi_{21}^{\omega}$. In Fig. 3 we have plotted $\Im\Pi_{21}^{\omega}$ as a function of $\omega$ for two values of $q$. The frequencies for which $\Im\Pi_{21}^{\omega}$ vanishes represent regions for which single p-h excitations are forbidden. For $l_{\Delta_2} \rightarrow u_{\Delta_1}$ transition (right most curves of Fig. 3), the threshold behavior exhibits contrasting features depending on whether $q$ is lesser or greater than $\sqrt{(\Delta_s + \sqrt{\mu^2 - \Delta_s^2})^2/\Delta_1}$ (the value at which $\omega = \sqrt{q^2 + \Delta_q^2}$ and $\omega = \mu + \sqrt{(q - k_{F1})^2 + \Delta_q^2}$ curves intersect). For $q$ values greater than $q^* = (\Delta_s + \sqrt{\mu^2 - \Delta_s^2})/\Delta_1$ the threshold behavior exhibits a step jump (shown by the black curve) to a finite value given by $q^2\Delta_1\Delta_2/\Delta_3^3$, whereas for lesser values of $q$ it vanishes with the derivative acquiring a square-root singularity at $\omega = \mu + \sqrt{(q - k_{F1})^2 + \Delta_q^2}$ (shown by the red curve). On the other hand, for $u_{\Delta_2}$ to $u_{\Delta_1}$ transition, the threshold behavior at the upper edge of region $2A$ vanishes, while the derivative diverges again with square-root singularity. Moreover, inside the allowed regions the plot exhibits a weak kink at various boundaries.

**B. $(\Delta_1 \rightarrow \Delta_2)$ Transition**

Similar to the earlier discussed upper band transitions, the transition from $u_{\Delta_1}$ to $u_{\Delta_2}$ are allowed for particles with energy $\epsilon$ in the range: $\max[\mu - \omega, \Delta_1] < \epsilon < \mu$. The major difference is that now the particle-hole transitions are allowed even for $\omega > q$ regions, albeit the phase-space is much smaller than the phase space for the dominant $\omega < q$ regions [see the lower part of the $(q, \omega)$ plane in Fig. 4].

The maximum allowed frequency for such a transition is given by $\omega_{\text{max}} = \max[\mu - \sqrt{(k_{F2} - q)^2 + \Delta_q^2}, \sqrt{(k_{F1} + q)^2 + \Delta_q^2} - \mu]$. The first term in the square bracket is the energy $\mu - \sqrt{(k_{F2} - q)^2 + \Delta_1^2}$ required for a collinear transition of a particle from $u_{\Delta_1}$ to the Fermi level of $u_{\Delta_2}$. These transitions serve as the upper bound for frequency at small momentum transfer. The second frequency term $\sqrt{(k_{F1} + q)^2 + \Delta_1^2} - \mu$ is due to the collinear transition of a particle to $u_{\Delta_2}$ originating from the Fermi-level of $u_{\Delta_1}$. The lower bound of frequency for the $u_{\Delta_1}$ to $u_{\Delta_2}$ transition include $\sqrt{(k_{F1} - q)^2 + \Delta_2^2} - \mu$ (collinear transition from the Fermi level of the first band to the second band with the reduced momentum of the final particle) for momentum exchanges which lie between $0 < q < k_{F1} - k_{F2}$. In the range $k_{F1} - k_{F2} < q < k_{F1} + k_{F2}$ the transition can take place without change in the energy of the particle. While in the range $k_{F1} + k_{F2} < q$ the minimum energy required is $\sqrt{(k_{F1} - q)^2 + \Delta_2^2} - \mu$, which involves a transition from the Fermi level of the first band to a higher energy level of the second band with the final momentum reversing its direction.

The contribution to the imaginary part of the polarization function are as follows:
\[\text{Im} \Pi_{12}^{\text{uu}}(q, \omega) = - \frac{1}{4\pi} \frac{1}{\sqrt{|q^2 - \omega^2|}} \times \left\{ \begin{array}{ll}
G_{u_1}^{\text{uu}}(2\mu + \omega \gamma_{12}) - G_{u_2}^{\text{uu}}(2\max[\mu - \omega, \Delta_1] + \omega \gamma_{12}) & : 1A \\
G_{u_1}^{\text{uu}}(2\mu + \omega \gamma_{12}) - G_{u_2}^{\text{uu}}(\xi_{12}) & : 2A \\
G_{u_1}^{\text{uu}}(2\mu + \omega \gamma_{12}) - G_{u_2}^{\text{uu}}(2\max[\mu - \omega, \Delta_1] + \omega \gamma_{12}) & : 3A \\
G_{u_1}^{\text{uu}}(\xi_{12}) - G_{u_2}^{\text{uu}}(2\max[\mu - \omega, \Delta_1] + \omega \gamma_{12}) & : 4A \\
G_{u_1}^{\text{uu}}(2\mu + \omega \gamma_{12}) - G_{u_2}^{\text{uu}}(-\xi_{12}) & : 5A \\
G_{u_1}^{\text{uu}}(\xi_{12}) - G_{u_2}^{\text{uu}}(-\xi_{12}) & : 6A \\
G_{u_1}^{\text{uu}}(2\mu + \omega \gamma_{12}) - G_{u_2}^{\text{uu}}(-\xi_{12}) & : 7A
\end{array} \right\},\]

where \(\gamma_{12} = 1 + \Delta_2 \Delta_4/(q^2 - \omega^2), \xi_{12} = \sqrt{q^2 \gamma_{12}^2 + 4q^2 \Delta_2^2/(q^2 - \omega^2)}\) and

\[\begin{aligned}
G_{u_1}^{\text{uu}}(x) &= \frac{1}{4} \left\{ -2q^2 - 2\Delta_4^2 + \xi_{12}^2 + 2(\omega \gamma_{12} - \omega)^2 \right\} \log \left( x \frac{\sqrt{x^2 - \xi_{12}^2} + x}{x - 4(\omega \gamma_{12} - \omega)\sqrt{x^2 - \xi_{12}^2}} \right), \quad (9) \\
G_{u_1}^{\text{uu}}(x) &= \frac{1}{4} \left\{ -2q^2 - 2\Delta_4^2 + \xi_{12}^2 + 2(\omega \gamma_{12} - \omega)^2 \right\} \tan^{-1} \left( \frac{x}{\sqrt{x^2 - \xi_{12}^2}} \right) - \left[ x - 4(\omega \gamma_{12} - \omega)\sqrt{x^2 - \xi_{12}^2} \right] \quad (10)
\end{aligned}\]

The different allowed regions in the \((q, \omega)\) plane for the \(u_{\Delta_1}\) to \(u_{\Delta_2}\) transition (Fig. 4) are as follows,

1A: \(\omega < \mu - \mathcal{F}(k_{F_2}, \Delta_1)\),
2A: \(\pm \mu + \mathcal{F}(\pm k_{F_2}, \Delta_1) < \omega < -\mu + \mathcal{F}(-k_{F_2}, \Delta_2)\),
3A: \(\omega > q; \omega < \mu + \mathcal{F}(k_{F_2}, \Delta_1); \omega > \mu - \mathcal{F}(-k_{F_2}, \Delta_1)\) or \(\omega < -\mu + \mathcal{F}(-k_{F_2}, \Delta_2)\),
4A: \(\omega > q; \omega < \mu + \mathcal{F}(k_{F_2}, \Delta_1); \omega > \mu + \mathcal{F}(-k_{F_2}, \Delta_1)\) or \(\omega < -\mu + \mathcal{F}(-k_{F_2}, \Delta_2)\),
5A: \(\omega > q; \omega < -\mu + \mathcal{F}(-k_{F_2}, \Delta_2)\) or \(\omega > -\mu + \mathcal{F}(k_{F_2}, \Delta_2)\) or \(\omega < \mu - \mathcal{F}(k_{F_2}, \Delta_1)\),
6A: \(\omega > q; \omega < -\mu + \mathcal{F}(k_{F_2}, \Delta_1)\) or \(\omega > -\mu + \mathcal{F}(-k_{F_2}, \Delta_2)\) or \(\omega < -\mu + \mathcal{F}(k_{F_2}, \Delta_1)\),
7A: \(\omega > q; \omega > \mu + \mathcal{F}(k_{F_2}, \Delta_1)\) or \(\omega < -\mu + \mathcal{F}(k_{F_2}, \Delta_1)\),

A lower band \(l_{\Delta_1}\) to upper band \(u_{\Delta_2}\) transition requires the particle to have energy \(\epsilon\) in the range: \(\mu - \omega < \epsilon < -\Delta_1\). The derivation of the threshold frequencies are very similar as for the case of \(l_{\Delta_2}\) to \(u_{\Delta_1}\) transition and are obtained by simply exchanging the indices \(1 \leftrightarrow 2\). The threshold frequency for small \(q\) has the form \(\omega = \mu + \sqrt{(k_{F_2} - q)^2 + \Delta_2^2}\) which changes to \(\omega = \sqrt{q^2 + \Delta_2^2}\) at the point of intersection of the two curves. The contribution to the imaginary part of the polarization function are obtained to be:

\[\text{Im} \Pi_{12}^{lu}(q, \omega) = - \frac{1}{4\pi} \frac{1}{\sqrt{\omega^2 - q^2}} \times \left\{ \begin{array}{ll}
G_{l_2}^{lu}(2(\omega - \mu) - \omega \gamma_{12}) - G_{l_1}^{lu}(\xi_{12}) & : 1B \\
G_{l_2}^{lu}(\xi_{12}) - G_{l_1}^{lu}(\xi_{12}) & : 2B \\
G_{l_2}^{lu}(\xi_{12}) - G_{l_1}^{lu}(-\xi_{12}) & : 3B
\end{array} \right\},\]

where,

\[\begin{aligned}
G_{l_1}^{lu}(x) &= \frac{1}{4} \left\{ 2q^2 + 2\Delta_4^2 - \xi_{12}^2 - 2(\omega \gamma_{12} - \omega)^2 \right\} \tan^{-1} \left( \frac{x}{\sqrt{x^2 - \xi_{12}^2}} \right) + \left[ x - 4(\omega \gamma_{12} - \omega)\sqrt{x^2 - \xi_{12}^2} \right] \quad (11)
\end{aligned}\]

The non-zero regions in the \((q, \omega)\) plane (Fig. 4) are,

1B: \(\mu + \mathcal{F}(k_{F_2}, \Delta_1) < \omega < \mu + \mathcal{F}(-k_{F_2}, \Delta_1)\)
2B: \(\omega > \mu + \mathcal{F}(-k_{F_2}, \Delta_1)\)
3B: \(\sqrt{q^2 + (\Delta_2 + \Delta_1)^2} < \omega < \mu + \mathcal{F}(k_{F_2}, \Delta_1)\).

Fig. 5 shows \(\text{Im} \Pi_{12} = \text{Im} \Pi_{12}^{lu} + \text{Im} \Pi_{12}^{uu}\) plotted as a function of \(\omega\) for three different values of \(q\). The behavior for \(l_{\Delta_1} \rightarrow u_{\Delta_2}\) (right most curves of Fig. 5) transition is similar to those considered in Fig. 3. In this case, the main change is in the position of \(q\) value given by \(q^* = (\Delta_2 + \sqrt{\mu^2 - \Delta_1^2})/\Delta_2\) which separates the two
threshold behaviors. As before, for $q$ values greater than it, the threshold behavior exhibits a step jump to the same finite value $q^2 \Delta_1 \Delta_2 / \Delta_3^2$ (shown by the black curve), whereas for lesser $q$ values the derivative at the threshold diverges (shown by the red curve). Also, for $u_{\Delta_1}$ to $u_{\Delta_2}$ transition, the threshold behavior at the upper edge vanishes everywhere, while the derivative diverges with square-root singularity. For the additional region shown in the inset, at small $\omega$ and $q < k_{F_i} - k_{F_{\Delta_i}}$, the threshold behavior at both the edges exhibits square-root divergence of the derivatives. It turns out that in this region the real part of the polarization operator exhibits singular features, details of which are provided in sec. IV.

Finally to conclude this section, $\Im \Pi_{xx/yy}$ is given by $\Im \Pi_{xx/yy} = \Im \Pi_{21} + \Im \Pi_{12}$. It is worth mentioning that in the absence of electric-field, $\Delta_1 = \Delta_2$, therefore $\Im \Pi_{12}$ and $\Im \Pi_{21}$ will be identical.

### C. $(\Delta_{1(2)} \rightarrow \Delta_{1(2)})$ Transition

For completeness we will enumerate the known result corresponding to the case of intra and inter-band transitions within the same gap, i.e., $\Delta_i \rightarrow \Delta_i$ where $i \in (1, 2)$. These give contributions to only $\Im \Pi_{zz}$ and as before they arise due to $u_{\Delta_i} \rightarrow u_{\Delta_i}$ and $l_{\Delta_i} \rightarrow u_{\Delta_i}$ transitions. The contribution to the imaginary part of the polarization function from the $u_{\Delta_i} \rightarrow u_{\Delta_i}$ transitions are as follows,

$$\Im \Pi_{uu}(q, \omega) = -\frac{1}{4\pi} \frac{1}{\sqrt{q^2 - \omega^2}} \times \left\{ G^{uu}(2 \mu + \omega) - G^{uu}(2 \max[\mu - \omega, \Delta_i] + \omega) : 1A' \right\},$$

where $\xi = \sqrt{q^2 + 4q^2 \Delta_i^2 / (q^2 - \omega^2)}$, and $G^{uu}(x) = \frac{1}{4} \left\{ [x^2 - 2q^2] \log \left( \sqrt{x^2 - \xi^2} + x \right) + x \sqrt{x^2 - \xi^2} \right\}$. (12)

The allowed regions for the transitions are (see Fig. 6)

1A': $\omega < \mu - F(k_{F_i}, \Delta_i)$

2A': $\pm \mu \mp F(k_{F_i}, \Delta_i) < \omega < -\mu + F(-k_{F_i}, \Delta_i)$.

Unlike the earlier two cases, the transitions within the same band allows the creation of particle-hole pairs having $\omega = 0$ and infintesimally small momentum $q$.

The contribution from $l_{\Delta_i} \rightarrow u_{\Delta_i}$ transitions are,

$$\Im \Pi_{uu}(q, \omega) = -\frac{1}{4\pi} \frac{1}{\sqrt{q^2 - \omega^2}} \times \begin{cases} G^{lu}(\omega - 2\mu) - G^{lu}(-\xi) : 1B' \\ G^{lu}(\xi) - G^{lu}(-\xi) : 2B' \\ G^{lu}(\xi) - G^{lu}(-\xi) : 3B' \end{cases},$$

where

$$G^{lu}(x) = \frac{1}{4} \left( 2q^2 - \xi^2 \right) \tan^{-1} \left( \frac{x}{\sqrt{\xi^2 - x^2}} \right) + x \sqrt{\xi^2 - x^2}$$

(13)

and the allowed regions in the $(q, \omega)$ plane are (see Fig. 6)

1B': $\mu + F(k_{F_i}, \Delta_i) < \omega < \mu + F(-k_{F_i}, \Delta_i)$

2B': $\omega > \mu + F(-k_{F_i}, \Delta_i)$

3B': $\omega > (2k_{F_i}); \sqrt{q^2 + (2\Delta_i)^2} < \omega < \mu + F(k_{F_i}, \Delta_i)$. 

---

**FIG. 6.** (Color online) The $A'$ regions denote contributions from $u_{\Delta_i} \rightarrow u_{\Delta_i}$, whereas $B'$ denote those from $l_{\Delta_i} \rightarrow u_{\Delta_i}$. Note $k_{F_i} = \sqrt{\mu^2 - \Delta_i^2}$. 

---
We note that the qualitative behavior of this region is similar to the earlier two studied cases. As an additional remark, we would like to point out that in the scenario of vanishing electric field, the \(zz\) component obtains identical contribution to \(xx/yy\) components.

IV. REAL PART OF SPIN-SUSCEPTIBILITY

The real part of spin-susceptibility is evaluated from Eq. 5, where some of the parts have been calculated with the help of Kramers-Kronig technique and the rest via direct integration. The \(\text{Re} \chi_{xx}\) and \(\text{Re} \chi_{yy}\) are identical and obtain contributions from transitions involving \(\Delta_1 \rightarrow \Delta_2\) and vice versa, while \(\Delta_i \rightarrow \Delta_i\) \((i = 1, 2)\) transitions yield contributions to \(\text{Re} \chi_{zz}\). Details of the calculation are provided in appendix VII. In the following two subsections we will limit our discussion to the case of dynamic and static susceptibility.

A. Dynamic limit: \(q = 0\)

It is easy to show that for finite frequencies and \(q = 0\), \(\text{Re} \chi_{xx}^0(q = 0, \omega)\) vanishes identically due to the Fermi-distribution terms in (5) \((\alpha = \alpha')\) and form factor (6) \((\alpha = -\alpha')\). In contrast, \(\text{Re} \chi_{xx}^0(0, \omega)\) and \(\text{Re} \chi_{yy}^0(0, \omega)\) are in general non-zero and exhibit interesting behavior in regions where the corresponding imaginary part vanishes. In the following, we will take a closer look into the different contributions to the real part of the susceptibility. As before, we will discuss the susceptibility in terms of the polarization operator which differs by a sign.

The non-interacting real part of the polarization operator \((xx\) and \(yy)\) contributions is split in to three parts labelled as \(\text{Re} \Pi_a\), \(\text{Re} \Pi_b\) and \(\text{Re} \Pi_c\) (details of the decomposition and their derivation are given in the appendix VII C). The first part, \(\text{Re} \Pi_a\), is independent of \(\mu\) and takes on the value,

\[
\text{Re} \Pi_a(\omega) = -\frac{\Delta_0^2}{4\pi\omega} \left\{ \log \left[ \frac{\Delta_s + \omega}{\Delta_s - \omega} \right] \left( 1 - \frac{\Delta_s^2}{\omega^2} + \frac{2\Delta_s}{\omega} \right) \right\}
\]

where \(\Delta_d = \Delta_2 - \Delta_1\) and \(\Delta_\alpha = \Delta_1 + \Delta_2\). The second term, \(\text{Re} \Pi_b\), is non-zero for \(\mu > \Delta_1\) and obtains contribution from the integrals containing \(n_F(\sqrt{p^2 + \Delta^2_f} - \mu)\) and \(n_F((p + q)^2 + \Delta^2_f - \mu)\) terms and is given by,

\[
\text{Re} \Pi_b(\omega) = -\frac{\Delta_0^2}{4\pi\omega} \left\{ \log \left[ \frac{\Delta_s + \omega}{\Delta_s - \omega} \right] \left( 1 - \frac{\Delta_s^2}{\omega^2} + \frac{2\Delta_s}{\omega} \right) \right\}
\]

The third term denoted as \(\text{Re} \Pi_c\) obtains contribution from the integrals containing \(n_F(\sqrt{p^2 + \Delta^2_f} - \mu)\) and \(n_F((p + q)^2 + \Delta^2_f - \mu)\) terms and exhibits log divergence. It has the following form,

\[
\text{Re} \Pi_c(\omega) = -\frac{\Delta_0^2}{4\pi\omega} \left\{ \log \left[ \frac{\Delta_s + \omega}{\Delta_s - \omega} \right] \left( 1 - \frac{\Delta_s^2}{\omega^2} + \frac{2\Delta_s}{\omega} \right) \right\}
\]

Combining all the contributions, \(\text{Re} \Pi_0^0(\omega) = \text{Re} \Pi_a(\omega) + \text{Re} \Pi_b(\omega) + \text{Re} \Pi_c(\omega)\), we obtain the following compact expression,

\[
\text{Re} \Pi_0^0(\omega) = \frac{\Delta_0^2}{8\pi\omega^3} L(w),
\]

where

\[
L(w) = \log \left[ \frac{(\omega^2 - 2\mu^2)^2 - \Delta_0^2\Delta_s^2}{(\omega^2 + 2\mu^2)^2 - \Delta_0^2\Delta_s^2} \right].
\]

Let us next consider the possibility of spin collective excitations occurring in the \(xx\) and \(yy\) channels when coupled with interactions. The ladder diagrams yield an equation for spin collective excitations which typically has the form, \(\text{Re} \Pi_{xx/yy}^0(\omega) = -1/u^*\) \((u^*\) is the screened interaction). It is clear that this equation is satisfied,
One can deduce from the corresponding imaginary part that the processes responsible for the contribution involve upper-band transitions from $\Delta_1 \to \Delta_2$ as shown in Fig. 7.

Interestingly, the real part of the polarization operator is also negative for frequencies $\omega$ close to and less than $\omega_U = \mu + \sqrt{\mu^2 + \Delta_1^2 - \Delta_2^2}$ (the upper threshold for the single particle excitation) and is logarithmically divergent right at $\omega = \omega_U$. The log-divergence in this scenario is due to the vanishing of the first term in the numerator of the log-term (corresponding to Re$\Pi$, Eq. 15) at the frequency $\omega_U$. Once again we can pin-point the specific integral causing the divergence and it is due to

\[
I \propto \int \frac{dp}{8\pi} \left[ \left( \frac{p^2 + \Delta_1 \Delta_2}{\sqrt{p^2 + \Delta_1^2} \sqrt{p^2 + \Delta_2^2}} \right) - \frac{\Theta \left( \sqrt{\mu^2 - \Delta_1^2} - p \right)}{w - \sqrt{p^2 + \Delta_1^2 - \sqrt{p^2 + \Delta_2^2}}} \right],
\]

where the contributions again arise from $\Delta_1 \to \Delta_2$ transition but now $\Delta_1$ and $\Delta_2$ corresponds to the lower and upper bands respectively.

Solving the pole equations yield two solutions close to the threshold frequencies (see Fig. 7 for solution near the lower threshold) given by

\[
\omega_1 = \omega_L - \frac{2\mu(\mu' - \mu)(\mu' - \mu'' - 2\mu)(\mu' + \mu'' - 2\mu)}{\mu'(\mu' - \mu'')(\mu' + \mu'')} \times \exp \left[ \frac{8\pi(\mu' - \mu)^3}{u^* [(\mu' - \mu)^2 - \Delta_2^2] \Delta_d} \right],
\]

and a solution just below the upper threshold,

\[
\omega_2 = \omega_U - \frac{2\mu(\mu + \mu'')(\mu' - \mu'' - 2\mu)(\mu' + \mu'' + 2\mu)}{\mu''(\mu' - \mu'')(\mu' + \mu'')} \times \exp \left[ \frac{8\pi(\mu' - \mu)^3}{u^* [(\mu' - \mu)^2 - \Delta_2^2] \Delta_d} \right],
\]

where $\mu' = \sqrt{\mu^2 + \Delta_1 \Delta_2}$ and $\mu'' = \sqrt{\mu^2 - \Delta_1^2 + \Delta_2^2}$. We note that in the absence of external electric field the two gaps $\Delta_1$ and $\Delta_2$ are identical, therefore the Re$\Pi^0(\omega)$ vanishes identically and no pole solutions are possible. In Fig. (8) we show the explicit dependence of the threshold frequencies $\omega_L$ and $\omega_L$ and the lower pole position on the perpendicular electric field $E_z$. For non-zero electric field, the slope of $\omega_L$ and $\omega_L$ are $2E_z/\sqrt{\mu^2 + 4E_z^2 \lambda_{SO}}$, respectively, therefore the width of the real region given by $\omega_L - \omega_L$ grows wider. At the same time the slope of pole position for a fixed screened interaction $u^*$ is even lesser than the slope of $\omega_L$ therefore the width between the pole position and $\omega_L$ also increases.

While the above discussion hints at the possibility of collective excitations it turns out that the presence of the sub-lattice degrees of freedom complicates the analysis. The pole equation has its structure modified due to the presence of $\tau_i$ type of terms in the Green’s function.

\[
\Pi_{xx} = \int_p \text{Tr} [\sigma_x G_P \Lambda^0_L \tilde{G}_{P + Q}],
\]

where $\Lambda$ satisfies the equation:

\[
\Lambda^\beta_j = \sigma_j \tau_\beta - u \int \tilde{G}_P \Lambda^\beta_j \tilde{G}_{P + Q}.
\]
Multiplying both sides of Eq. 23 with \( \sigma \) the trace yields,

\[
\sigma_k \tau_\gamma + \int \hat{G}_P \sigma_k \tau_\gamma \hat{G}_{P+Q} M^{[4k+\gamma]}_{[4j+\beta]} = \sigma_j \tau_\beta. \tag{23}
\]

Multiplying both sides of Eq. 23 with \( \sigma_m \tau_\nu \) and taking the trace yields,

\[
(\delta_{m,k} \delta_\nu,\gamma + \frac{u}{4} \tilde{\Pi}^{[4m+\nu]}_{[4k+\gamma]} M^{[4k+\gamma]}_{[4j+\beta]} = \delta_{m,j} \delta_\nu,\beta. \tag{24}
\]

where \( \tilde{\Pi} \) is the generalized susceptibility whose elements are defined as \( \tilde{\Pi}^{[4m+\nu]}_{[4j+\beta]} = \text{Tr}[\int d\tau \sigma_m \tau_\nu \hat{G}_P \sigma_j \tau_\beta \hat{G}_{P+Q}] \).

The matrix \( M \) is thus given by \( M = (I + u\tilde{\Pi}/4)^{-1} \).

It turns out that many of the elements of \( \tilde{\Pi}(\omega) \) matrix exhibit ultra-violet divergence. We will illustrate one such example, consider the \( \Pi_{55} \) element given by \( \Pi_{55}(\omega) = \text{Tr}[\int d\tau \sigma_3 \tau_3 \hat{G}_{P+Q}] \). Here the terms independent of the chemical potential, i.e.,

\[
I_\pm \propto \int dp d\nu \left( 1 + \frac{\Delta_1 \Delta_2}{E_{P_1} E_{P_2}} \right) \frac{\Theta(\nu - 2k_{F_1})}{E_{P_1} + E_{P_2} + \omega}, \tag{25}
\]

obtain divergent contributions from the upper limit due to the Dirac spectrum and necessitates one to consider non-linear terms arising from the exact energy spectrum.

The divergence of Eq. 16 is expected to be altered in the interacting version \( \Pi_{55} \equiv \int d\tau [\sigma_3 \tau_3 \hat{G}_{P+Q}] \), however, the fate of collective excitations is not apriori clear, i.e., whether it survives at all or survives with its peak position and peak width renormalized.

### B. Static limit: \( \omega = 0 \)

Following earlier discussion, the components of spin-susceptibility that yield non-vanishing contributions are \( \Pi_{zz} \) and \( \Pi_{xx/yy} \). \( \Pi_{zz} \) can be conveniently decomposed into the sum of \( \Pi_{zz} = \Pi_{zz-1} + \Pi_{zz-2} \) which are the contributions from transitions involving \( \Delta_1 \rightarrow \Delta_i (i = 1, 2) \). For \( q < 2k_{F_1} \), \( \Pi_{zz-1} \) is a constant. Subtracting the constant part we obtain

\[
\delta \text{Re} \Pi_{zz-1} = \left[ \frac{\mu \sqrt{q^2 - (2k_{F_1})^2}}{4\pi q} - \frac{q^2 - 4\Delta_1^2}{8\pi q} \right] \tan^{-1} \left( \frac{\sqrt{q^2 - (2k_{F_1})^2}}{2\mu} \right) \Theta(q - 2k_{F_1}). \tag{26}
\]

Thus one can deduce from simple power counting arguments that at large distances the \( zz \) component of the spin-susceptibility decays as \( 1/r^2 \) and the contribution to exchange interaction is oscillatory with two wavelengths given by \( \pi/k_{F_1} \) and \( \pi/k_{F_2} \). For electric field strength equal to \( E_c = \lambda_{SO}/l \), \( \Delta_1 = 0 \) and therefore the first derivative of \( \text{Re} \chi_{zz} \) vanishes. It is the second derivative which diverges at \( 2k_{F_1} \) as \( \text{Re} \chi_{zz} \approx - (1/8\pi \sqrt{k_{F_1}})/\sqrt{q - 2k_{F_1}} \) that determines the long distance behavior of \( \text{Re} \chi_{zz} \). The susceptibility now acquires a faster \( 1/r^3 \) decay. For \( \mu > \Delta_2 \) this behavior will be masked by the slower \( 1/r^2 \) decay arising due to \( \text{Re} \chi_{zz} \), however for \( \mu < \Delta_2 \), only the \( 1/r^3 \) term will survive.

Next consider the behavior of \( \text{Re} \Pi_{xx/yy} \) (details of the
derivation are given in the appendix VII B). The terms which are independent of the chemical potential yield regular contributions for all values of $q$ given by

$$\text{Re} \Pi_s(q) = -\frac{\Delta_s^2 + q^2}{4\pi q^3} \left\{ [q^2 - \Delta_s^2] \tan^{-1} \left( \frac{q}{\Delta_s} \right) + q\Delta_s \right\}. \tag{30}$$

While from the integrals containing $n_F(\sqrt{p^2 + \Delta_s^2} - \mu)$ we obtain

$$\text{Re} \Pi_0(q) = \begin{cases} -\frac{\mu - \Delta_1}{2\pi} - \frac{\text{sgn}(q^2 + \Delta_1^2 - \Delta_2^2)}{4\pi q}[Y(\mu) - Y(\Delta_1)], & \text{for } q < k_{F_1} - k_{F_2}, \text{ or } q > k_{F_1} + k_{F_2}, \\ -\frac{\mu - \Delta_1}{2\pi} - \frac{\text{sgn}(q^2 + \Delta_2^2 - \Delta_1^2)}{4\pi q}[Y(\xi) - Y(\Delta_1)], & \text{for } k_{F_1} - k_{F_2} < q < k_{F_1} + k_{F_2}, \end{cases} \tag{31}$$

where

$$Y(x) = \left\{-2x\sqrt{\xi^2 - x^2} - \tan^{-1} \left( \frac{x}{\sqrt{\xi^2 - x^2}} \right) \left[ q^2 - 2\xi^2 + \Delta_s^2 \right] \right\}. \tag{32}$$

The remaining term arising from the integrals containing $n_F(\sqrt{p^2 + \Delta_s^2} - \mu)$ denoted by $\text{Re} \Pi_c(q)$ is obtained by simply changing $\Delta_1$ to $\Delta_2$ and vice-versa in Eq. 31. The derivatives of both $\text{Re} \Pi_0(q)$ and $\text{Re} \Pi_c(q)$ diverge at $q_d = k_{F_1} - k_{F_2}$ and $q_s = k_{F_1} + k_{F_2}$ (see Fig. 9). However, combining them together we find that the divergence at $q_d$ is cancelled and that $\text{Re} \Pi_{xx/yy}$ is constant for $q < q_s$, while the divergence at $q_s$ remains. Removing the constant part, the full expression for the static-susceptibility is given by

$$\delta \text{Re} \Pi_{xx/yy}(q) = \left[ \frac{\mu(\sqrt{q^2 - q_s^2})(q^2 - q_d^2)}{2\pi q^2} - \frac{(q^2 + \Delta_s^2)(q^2 - \Delta_s^2)}{4\pi q^3} \right] \tan^{-1} \left( \frac{\sqrt{(q^2 - q_s^2)(q^2 - q_d^2)}}{2\mu q} \right) \Theta(q - k_{F_1} - k_{F_2}).$$

The derivative of the polarization operator has a square-root singularity at $q = q_s$ given by,

$$\delta \text{Re} \Pi'_{xx/yy}(q) \approx -\frac{\sqrt{k_{F_1}k_{F_2}}}{4\sqrt{2}\pi q^{3/2}} \left( \frac{q^2 + \Delta_1^2}{q^2 - \Delta_s^2} \right) \frac{\mu^2 q^2}{q - q_s},$$

therefore the real space decay exhibits $1/r^2$ power-law dependence at large distances while the oscillatory wavelength is now given by $2\pi/q_s = 2\pi/(k_{F_1} + k_{F_2})$. Rather interestingly, for the $xx$ and $yy$ parts of the spin-susceptibility, unless both the gaps are equal ($\Delta_1 = \Delta_2$) closing of one of the gaps does not lead to vanishing of the singular behavior of the derivative at $k_{F_1} + k_{F_2}$. Thus the $1/r^2$ power-law dependence at large distances is maintained irrespective of the tuning of the gaps by the electric field.

The real space analysis thus far yields the behavior of spin-spin correlation function between spins that are widely separated from each other and are delocalized on few sites. The calculation of spin-correlations thus entails disregarding intervalley scattering and taking the trace of the sub-lattice degrees of freedom. In contrast, the behavior of spin-correlations between two impurity spins that are localized on specific sites of the lattice is given by a different version of static spin-susceptibility that also yields the Rudermann-Kittel-Kasuya-Yosida (RKKY) interaction between the two localized spins (see33,44 for a detailed analysis for the case of silicene). Due to the short-range nature of interactions between the localized impurities and itinerant electrons, an intervalley scattering of the electrons via large $2K$ momentum exchange is allowed leading to additional contributions to the spin-susceptibility. Moreover, the position of the spin-impurities (whether the two spins are on $A$-$A$/$B$-$B$ sites or $A$-$B$ sites) also crucially determines the behavior of spin-correlations. In what follows, we will briefly discuss the differences and similarities between the results arising from these two different scenarios.

The effective interaction between two magnetic impurities $\vec{S}_i$ and $\vec{S}_j$ (localized at sites $\vec{R}_i$ and $\vec{R}_j$, respectively) is given by $H_{\text{RKKY}} = -J^2 \chi_{\alpha\beta}^{cd} S_i^\alpha S_j^\beta$, where there is a repeated summation on only the spin indices $\alpha, \beta = x, y, z$; the indices $c, d$ refer to the $A$ or $B$ sites and $J$ is the interaction term between the magnetic impurity and itinerant electrons. The spin-susceptibility matrix
FIG. 9. (Color online) Here we consider the contributions to Re\(\Pi_{xx/yy}(q, \omega = 0)\). As in the appendix VII B, we split the full integral into Re\(\Pi_a\), Re\(\Pi_b\) and Re\(\Pi_c\) and examine their behavior. The contribution from Re\(\Pi_a\) is smooth and continuous. The sharp features of Re\(\Pi_b\) and Re\(\Pi_c\) at \(k_F_1 - k_F_2\) come with opposite sign, however, the kink like features at \(k_F_1 + k_F_2\) have same sign and when all three contributions are combined the features at \(k_F_1 + k_F_2\) are enhanced while those at \(k_F_1 - k_F_2\) cancel exactly.

has the form,

\[
\chi_{\alpha,\beta}(R_{ij}) = \frac{1}{\hbar} \int_0^\infty \text{Tr}[\sigma_\alpha G(i, j; \tau)\sigma_\beta G(j, i; -\tau)]d\tau \tag{33}
\]

where the trace is only over the spin degrees of freedom.\(^{12,40}\) The Green’s function is a \(4 \times 4\) matrix,

\[
G(i, j; \tau) = \mp \sum_n \psi_n(j)e^{i\epsilon_n\tau}\Theta(\tau - \epsilon_n), \tag{34}
\]

where \(n \in \{\eta, p, s\}\) is a summation on valley, momentum and spin degrees of freedom, \(\epsilon_n = \epsilon_n - \mu\) and the wavefunctions in the basis \(\psi_n = (\psi_{A\uparrow}, \psi_{B\uparrow}, \psi_{A\downarrow}, \psi_{B\downarrow})^T\) are given by,

\[
\psi_{(\eta,p,\uparrow)} = \frac{e^{i(\eta k + p)\tilde{R}_i}}{\sqrt{2\epsilon_{\eta\uparrow}(\epsilon_{\eta\uparrow} + \nu\Delta_{\eta\uparrow})}} \begin{bmatrix}
\nu e^{-\nu\theta} \\
\nu \epsilon_{\eta\uparrow} + \Delta_{\eta\uparrow} \\
0 \\
0
\end{bmatrix}, \tag{35}
\]

and

\[
\psi_{(\eta,p,\downarrow)} = \frac{e^{i(\eta k + p)\tilde{R}_i}}{\sqrt{2\epsilon_{\eta\downarrow}(\epsilon_{\eta\downarrow} + \nu\Delta_{\eta\downarrow})}} \begin{bmatrix}
0 \\
0 \\
\nu e^{-\nu\theta} \\
\nu \epsilon_{\eta\downarrow} + \Delta_{\eta\downarrow}
\end{bmatrix}, \tag{36}
\]

where \(\nu = \pm 1\) represents conduction/valence band respectively.

Let us for example consider \(\chi^{AA}_{xx}\) and \(\chi^{AA}_{xy}\) which are obtained from the following integrals

\[
\chi^{AA}_{xx} = \int_0^\infty (g^{AA}_{\uparrow\uparrow} g^{AA}_{\downarrow\downarrow} + g^{AA}_{\uparrow\downarrow} g^{AA}_{\downarrow\uparrow})d\tau/\hbar \tag{37}
\]

and

\[
\chi^{AA}_{xy} = -i \int_0^\infty (g^{AA}_{\uparrow\uparrow} g^{AA}_{\downarrow\downarrow} - g^{AA}_{\uparrow\downarrow} g^{AA}_{\downarrow\uparrow})d\tau/\hbar \tag{38}
\]

where \(g^{AA}_{\eta,s} = e^{i\mu\tau} \sum_n e^{i\eta\tilde{R}_i} \lambda^{\eta,s}_{\eta,s} \) and

\[
\lambda^{\eta,s}_{\eta,s} = -\frac{a^2}{4\pi} \int p^3 dp \Theta(\epsilon_{\eta,s} - \mu) f_0(\mu | \tilde{R}_{ij}|) e^{-\epsilon_{\eta,s} \tau}. \tag{39}
\]

Similarly \(\tilde{\lambda}^{AA}_{\eta,s} = e^{i\mu\tau} \sum_n e^{-i\eta\tilde{R}_i} \lambda^{\eta,s}_{\eta,s}\) where

\[
\tilde{\lambda}^{AA}_{\eta,s} = \frac{a^2}{4\pi} \int p^3 dp \Theta(\mu - \nu\epsilon_{\eta,s}) J_0(p | \tilde{R}_{ij}|) e^{\nu\epsilon_{\eta,s} \tau}.
\]

Taking the product

\[
g^{AA}_{\uparrow\uparrow} g^{AA}_{\downarrow\downarrow} = \sum_{\eta,\eta'} e^{i(\eta - \eta')\tilde{R}_i} \lambda^{\eta,s}_{\eta,s}, \tag{40}
\]

and

\[
g^{AA}_{\uparrow\downarrow} g^{AA}_{\downarrow\uparrow} = \sum_{\eta,\eta'} e^{i(\eta - \eta')\tilde{R}_i} \lambda^{\eta,s}_{\eta,s}. \tag{41}
\]

we identify that the classifications can be classified into intra \((\eta = \eta')\) and inter-valley \((\eta = -\eta')\) terms. While for \(\chi^{xx}_{xy}\) the intra terms add-up, they cancel identically for \(\chi^{AA}_{xy}\). Similar cancellation holds for \(\chi^{AB}_{xy}\). This result is consistent with our earlier result (which takes into consideration only the intra terms) regarding the vanishing of \(\chi_{xy}\) term when contributions from the valleys are added together. However due to the inter-valley scattering processes, \(\chi^{AB}_{xy}\) and \(\chi^{AA}_{xy}\), obtain additional non-vanishing contributions. Another important difference is that, besides the oscillatory dependence with wave-number \(2\pi/|k_{F_1} + k_{F_2}|\) due to the intravalley process, the interinterval processes yield additional oscillatory dependence on \(\tilde{R}\) arising from terms of the type \(e^{2i\eta\tilde{R}_i} A^{\eta}_{\eta,s}\) (see Eq. 40).

V. SUMMARY

To summarize, in this article, we have presented a detailed study of the spin-susceptibility for silicene, that can be generalized to other buckled honeycomb structured materials e.g., germanene and stanene which also exhibit an electric field tunable band gap. We find that while the \(xx\) and \(yy\) components of the spin-susceptibility are identical, the \(zz\) component is different. The \(xx\) and \(yy\) components obtain contributions from only those electronic transitions for which the spins are flipped, while the \(zz\) component obtain contributions from spin conserving processes. Although the off-diagonal components of the spin-susceptibility, \(0z\) and \(xy\), are non-zero in individual valleys, adding the contributions from the valleys leads to cancellation. The study of the imaginary
part of spin-susceptibility reveals regions in the \((q, \omega)\) plane where the single-particle excitations are allowed. We find that the threshold behavior for the lower to upper-band transition is especially interesting since its behavior changes upon increasing the value of \(q\). For \(q\) values smaller than \(q^*\) the threshold behavior exhibits a square-root singularity in its derivative, whereas for \(q > q^*\) the susceptibility acquires a finite jump. We have investigated the role of electric field \(E_Z\) in extending the allowed regions for particle-hole transitions. Electric field is also responsible for yielding differing contributions for the \(\Delta_1 \rightarrow \Delta_2\) transitions as compared to those from the \(\Delta_2 \rightarrow \Delta_1\) transitions. Moreover, the magnitude of the \(xx/yy\) components and \(zz\) component also differ due to non-zero electric field.

We have studied the real part of spin-susceptibility, with particular emphasis on the dynamic and static limits. In the dynamic limit, we show that the real part of spin-susceptibility exhibits log-divergence. The origin of divergence at low frequencies can be traced to the \(u_{\Delta_1} \rightarrow u_{\Delta_2}\) transitions, whereas those at high frequencies can be attributed to \(l_{\Delta_1} \rightarrow u_{\Delta_2}\) transitions. We explore the significance of the divergence for spin-collective excitations and the dependence of the excitations on external electric field. The study of the static part of the spin-susceptibility reveals Kohn-anomaly at \(k_F + k_F\) for the \(xx/yy\) components of the spin-susceptibility, whereas for the \(zz\) component the anomaly is present at \(2k_F\) and \(2k_F\). Tuning the electric field effects the behavior of the \(x\) and \(y\) components at \(2k_F\). We have explored the consequence of the Kohn-anomaly on the long distance behavior of the spin-susceptibility.

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VII. APPENDIX

A. Derivation of \(\text{Re} \Pi_{xx/yy}(q, \omega)\)

We will integrate the terms of Eq. 5 by first obtaining the contribution from \(\Delta_1 \rightarrow \Delta_2\) transition by taking \(\beta = +1\) and \(\beta' = -1\) (for all possible values of \(\alpha, \alpha'\) for the K-valley). We divide the real part of polarization operator \(\Pi(q, w)\) as follows,

\[
A_1 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left(1 + \frac{\tilde{p}_1 \cdot (\tilde{p} + \tilde{q})_2}{E_1(p) - E_2(p + q)} \right) \left[ \frac{n_F[E_1(p)]}{-E_1(p) + E_2(p + q) - \omega} - \frac{n_F[E_2(p + q)]}{-E_1(p) + E_2(p + q) - \omega} \right]
\]

\[
A_2 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left(1 - \frac{\tilde{p}_1 \cdot (\tilde{p} + \tilde{q})_2}{E_1(p) - E_2(p + q)} \right) \left[ \frac{1}{+E_1(p) + E_2(p + q) - \omega} - \frac{n_F[E_2(p + q)]}{+E_1(p) + E_2(p + q) - \omega} \right]
\]

\[
A_3 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left(1 - \frac{\tilde{p}_1 \cdot (\tilde{p} + \tilde{q})_2}{E_1(p) - E_2(p + q)} \right) \left[ \frac{n_F[E_1(p)]}{-E_1(p) - E_2(p + q) - \omega} - \frac{1}{-E_1(p) - E_2(p + q) - \omega} \right],
\]

where \(F^{-1}_{xx/yy} = 2\), \(\tilde{p}_{1/2} = p_x \hat{e}_1 + \eta p_y \hat{e}_2 + \Delta_{1/2} \hat{e}_3\), \(E_1(p) = \sqrt{\vec{p}^2 + \Delta_1^2}\), and \(E_2(p) = \sqrt{\vec{p}^2 + \Delta_2^2}\).

The first term of \(A_2\) and second term of \(A_3\) yield terms that are independent of \(\mu\), we combine them together and represent it as \(\Pi_{12-a}\). \(\text{Im} \Pi_{12-a}\) is given by

\[
\text{Im} \Pi_{12-a}(q, \omega) = -\frac{1}{16} \Theta \left(\omega^2 - q^2 - \Delta_2^2\right) Y(q, \omega),
\]

where

\[
Y(q, \omega) = \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ \left[ q^2 + 2\Delta_2^2 \right] + \frac{2q^2 \left( \Delta_2^2 + \Delta_3^2 - 2(\Delta_1 \Delta_2) \right)}{2 \omega^2} \right\}.
\]

We use the above result to calculate \(\text{Re} \Pi_{12-a}\) via the Kramers-Kronig relation:

\[
\text{Re} \Pi_{12-a}(q, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im} \Pi_{12-a}(q, \omega')}{(\omega' - \omega)} \text{sgn}(\omega') = -\frac{1}{16\pi} P \left( \int_{\gamma}^{\infty} d\omega' \frac{Y(q, \omega')}{(\omega' - \omega)} - \int_{-\infty}^{-\gamma} d\omega' \frac{Y(q, \omega')}{(\omega' - \omega)} \right)
\]

\[
= -\frac{1}{16\pi} \left( \Theta(q - \omega) f(q, \omega) + \Theta(\omega - q) g(q, \omega) \right).
\]
The first integral is performed with the aid of the following variable change \( \omega' = q(1 + x^2)/(1 - x^2) \). Similar transformation is used for the second integral.

For \( q > \omega \), the result of the integration is \( f(q, \omega) \) which is expressed as a sum of three parts, \( f(q, \omega) = f_1(q, \omega) + f_2(q, \omega) + f_3(q, \omega) \) (corresponding to the square brackets of \( Y(q, \omega) \)) and they are given by,

\[
f_1(q, \omega) = \left( q^2 + 2\Delta^2 \right) \left\{ \frac{2}{q + \omega} \left[ \frac{1}{\beta_1} \tan^{-1} \left( \frac{x}{\beta_1} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

\[
f_2(q, \omega) = \left[ 2q^2(\Delta_1^2 + \Delta_2^2) - 2(\Delta_1^4 + \Delta_2^4) \right] \left\{ \frac{1}{2(q + \omega)q^2} \left[ x - \frac{1}{\beta_1} x - \frac{(\beta_1^2 + 1)^2}{\beta_1^3} \tan^{-1} \left( \frac{x}{\beta_1} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

\[
f_3(q, \omega) = \left[ \frac{-3(\Delta_1^2 + \Delta_2^2)}{8(q + \omega)q^2} \right] \left[ \frac{x^3}{3} - x(\beta_2^2 + 4) + \frac{4\beta_2^2}{\beta_2^4} - \frac{1}{3\beta_2^4} x^3 + \left( \frac{\beta_2^2 - 1}{\beta_2^3} \right)^4 \tan^{-1} \left( \frac{x}{\beta_2} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

While for \( \omega > q \) regions, the result is expressed in terms of \( g(q, \omega) \), where as before it is expressed as sum of three parts, \( g(q, \omega) = g_1(q, \omega) + g_2(q, \omega) + g_3(q, \omega) \), which are given by

\[
g_1(q, \omega) = \left( q^2 + 2\Delta^2 \right) \left\{ \frac{-2}{q + \omega} \left[ \frac{1}{\beta_2} \ln \left( \frac{x + \beta_2}{|x - \beta_2|} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

\[
g_2(q, \omega) = \left[ 2q^2(\Delta_1^2 + \Delta_2^2) - 2(\Delta_1^4 + \Delta_2^4) \right] \left\{ \frac{-1}{2(q + \omega)q^2} \left[ -x - \frac{1}{\beta_2} x + \left( \frac{\beta_2^2 - 1}{\beta_2^3} \right) \ln \left( \frac{x + \beta_2}{|x - \beta_2|} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

\[
g_3(q, \omega) = \left[ \frac{3(\Delta_1^2 + \Delta_2^2)}{8(q + \omega)q^2} \right] \left[ \frac{-x^3}{3} - x(\beta_2^2 - 4) + \frac{4\beta_2^2 - 1}{\beta_2^4} - \frac{1}{3\beta_2^4} x^3 + \left( \frac{\beta_2^2 - 1}{\beta_2^3} \right)^4 \ln \left( \frac{x + \beta_2}{|x - \beta_2|} \right) \right] + (\omega \rightarrow -\omega) \right\}_{\tan[\gamma'/2]}^{1} \]

where \( \gamma = \sqrt{q^2 + \Delta^2} \), \( \gamma' = \cos^{-1}[q/\Delta] \), \( \beta_1^2 = (q - \omega)/(q + \omega) \), \( \beta_2^2 = (\omega - q)/(q + \omega) \) and \( (\omega \rightarrow -\omega) \) represents similar terms with sign of \( \omega \) changed.

As a next step, \( n_F[E_1(p)] \) terms from \( A_1 \) and \( A_3 \) are combined together and labelled as \( \text{Re} \Pi_{12-b} \):

\[
\text{Re} \Pi_{12-b} = -\int \frac{d^dp}{(2\pi)^2} n_F[E_1(p)] \left\{ \frac{E_1(p) + \omega}{[E_2(p + q)]^2 - [E_1(p) + \omega]^2} + \frac{\beta_1 \cdot (\vec{p} + \vec{q})}{E_1(p)} \right\} \frac{1}{[E_2(p + q)]^2 - [E_1(p) + \omega]^2} \left[ \frac{1}{4\pi} \right] \left[ \int_{\Delta_1} dE_1 \sqrt{\omega^2 - q^2} \left\{ \left( 2E_1(\omega + \omega^2) - q^2 - \Delta^2 \right) \text{sgn} \left[ \alpha_b - E_1 \right] \left( \mu - \Delta_1 \right) \right\} \right] \]

where \( \gamma_b = \left( \frac{-x^2 - q^2 - \Delta_1^2}{\omega - q^2} \right) \), \( \alpha_b = \left( \frac{x^2 + \Delta_1^2 - q^2}{2\omega} \right) \) and we have used, \( \int_0^{2\pi} d\phi/(a + b \cos \phi) = 2\pi \text{ Sgn}[a]/\sqrt{a^2 - b^2} \) to perform the angular integration. Due to the \( \text{sgn} \) function the result of the integration depends on the value of \( \alpha_b \) with respect to the upper and lower limits, we obtain:

\[
(i). \quad \alpha_b > \mu \quad \Rightarrow \quad \text{Re} \Pi_{12-b} = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_b(\mu + \omega \gamma_b) - F_b(2\Delta_1 + \omega \gamma_b) \right\} + (\mu - \Delta_1) \right] \]

\[
(ii). \quad \mu > \alpha_b > \Delta_1 \quad \Rightarrow \quad \text{Re} \Pi_{12-b} = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_b(\mu + \omega \gamma_b) + F_b(2\Delta_1 + \omega \gamma_b) - 2F_b(2\alpha_b + \omega \gamma_b) \right\} \right] \]

\[
(iii). \quad \alpha_b < \Delta_1 \quad \Rightarrow \quad \text{Re} \Pi_{12-b} = +\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_b(\mu + \omega \gamma_b) - F_b(2\Delta_1 + \omega \gamma_b) \right\} + (\mu - \Delta_1) \right] \]

where \( \Re \) represents the real part of the corresponding function and

\[
F_b(x) = \frac{1}{2} \left[ \left( \xi_b^2 - 2\Delta_1^2 - 2q^2 + 2(\omega \gamma_b - \omega) \right) \log \left( \sqrt{x^2 - \xi_b^2} + x \right) + \left( x - 4(\omega \gamma_b - \omega) \right) \sqrt{x^2 - \xi_b^2} \right]
\]
and $\xi_b = \sqrt{q^2\gamma_b^2 - \frac{4q^2\Delta_2^2}{\omega^2 - q^2}}$.

Finally the terms corresponding to $n_F[E_2(p + q)]$ from $A_1$ and $A_2$ are combined together into $\text{Re} \Pi_{12-c}$:

$$
\text{Re} \Pi_{12-c} = -\frac{1}{4\pi} \int_{\Delta_2} dE_2 \frac{dE_2}{\sqrt{\omega^2 - q^2}} \left[ \left( E_2 - \omega - \Delta_2 \right)^2 \text{sgn} \left[ E_2 - \alpha_c \right] \right] + (\mu - \Delta_2),
$$

where $\gamma_c = \frac{(\omega^2 - q^2 + \Delta_2^2 - \omega q)^2}{2\omega}$ and $\alpha_c = \frac{(\omega^2 - q^2 + \Delta_2^2 - \omega q)^2}{2\omega}$.

As before, due to the sgn function, the integral yields three different results depending on the value of $\alpha_c$. They are

(i). $\alpha_c > \mu \Rightarrow \text{Re} \Pi_{12-c} = +\frac{1}{4\pi} \int \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_c(2\mu - \omega\gamma_c) - F_c(2\Delta_2 - \omega\gamma_c) \right\} \right] + (\mu - \Delta_2),$

(ii). $\mu > \alpha_c > \Delta_2 \Rightarrow \text{Re} \Pi_{12-c} = -\frac{1}{4\pi} \int \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_c(2\mu - \omega\gamma_c) + F_c(2\Delta_2 - \omega\gamma_c) - 2F_c(2\alpha_c - \omega\gamma_c) \right\} \right] + (\mu - \Delta_2),$

(iii). $\alpha_c < \Delta_2 \Rightarrow \text{Re} \Pi_{12-c} = -\frac{1}{4\pi} \int \left[ \frac{1}{\sqrt{\omega^2 - q^2}} \left\{ F_c(2\mu - \omega\gamma_c) - F_c(2\Delta_2 - \omega\gamma_c) \right\} \right] + (\mu - \Delta_2),$

where

$$F_c(x) = \frac{1}{2} \left[ \left( \xi_c^2 - 2\Delta_2^2 - 2q^2 + 2(\omega\gamma_c - \omega)^2 \right) \log \left( \sqrt{x^2 - \xi_c^2} + x \right) + \left( x + 4(\omega\gamma_c - \omega) \right) \sqrt{x^2 - \xi_c^2} \right],$$

and $\xi_c = \sqrt{q^2\gamma_c^2 - \frac{4q^2\Delta_2^2}{\omega^2 - q^2}}$.

Similar to the earlier derivation we will next integrate the terms of Eq. 5 by considering the contributions from $\Delta_2 \rightarrow \Delta_1$ transition by considering $\beta = -1$ and $\beta' = +1$ (in the case of K-valley), for all possible values of $\alpha$ and $\alpha'$.

As before, we divide the real part of polarization operator $\Pi(q, w)$ as follows,

$$B_1 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ 1 + \frac{\tilde{p}_2 \cdot (\tilde{p} + \tilde{q})}{E_2(p) \cdot E_1(p + q)} \right] \frac{n_F[E_2(p)]}{-E_2(p) + E_1(p + q) - \omega} - \frac{n_F[E_1(p + q)]}{-E_2(p) + E_1(p + q) - \omega},$$

$$B_2 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ 1 - \frac{\tilde{p}_2 \cdot (\tilde{p} + \tilde{q})}{E_2(p) \cdot E_1(p + q)} \right] \frac{n_F[E_2(p)]}{-E_2(p) + E_1(p + q) + \omega} - \frac{n_F[E_1(p + q)]}{-E_2(p) + E_1(p + q) + \omega},$$

$$B_3 = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ 1 - \frac{\tilde{p}_2 \cdot (\tilde{p} + \tilde{q})}{E_2(p) \cdot E_1(p + q)} \right] \frac{n_F[E_2(p)]}{-E_2(p) - E_1(p + q) - \omega} \frac{1}{-E_2(p) - E_1(p + q) + \omega}.$$

The first term of $B_2$ and the second term of $B_3$ yield terms that are independent of $\mu$, we combine them together and represent it as $\text{Re} \Pi_{21-a}$. Performing the following change of variables $p + q \rightarrow p$ and $p \rightarrow -p$ it is easy to show that $\text{Re} \Pi_{21-a}(q, \omega) = \text{Re} \Pi_{12-a}(q, \omega)$. The combined contribution represented as $\text{Re} \Pi_a$ is thus given by

$$\text{Re} \Pi_a = \text{Re} \Pi_{21-a}(q, \omega) + \text{Re} \Pi_{12-a}(q, \omega).$$

Similar to the evaluation of $\text{Re} \Pi_{12-b}$, we combine terms corresponding to $n_F[E_1(p + q)]$ from $B_1$ and $B_2$ and denote the contributions as $\text{Re} \Pi_{21-b}$. Change of variables as above yields,

$$\text{Re} \Pi_{21-b}(q, \omega) = -\int \frac{d^2 p}{(2\pi)^2} \frac{n_F[E_1(p)]}{[E_2(p + q)]^2 - [E_1(p) - \omega]^2} \left[ \frac{E_1(p) - \omega}{[E_2(p + q)]^2 - [E_1(p) - \omega]^2} + \frac{\tilde{p}_1 \cdot (\tilde{p} + \tilde{q})}{E_1(p)} \right] \frac{1}{[E_2(p + q)]^2 - [E_1(p) - \omega]^2},$$

thus $\text{Re} \Pi_{21-b}(q, \omega) = \text{Re} \Pi_{12-b}(q, -\omega)$. The total contribution is, $\text{Re} \Pi_b(q, \omega) = \text{Re} \Pi_{21-b}(q, \omega) + \text{Re} \Pi_{12-b}(q, \omega)$. Following essentially same arguments we obtain $\text{Re} \Pi_{12-c}(q, \omega) = \text{Re} \Pi_{21-c}(q, -\omega)$, thus $\text{Re} \Pi_c(q, \omega) = \text{Re} \Pi_{12-c}(q, \omega) + \text{Re} \Pi_{21-c}(q, \omega)$. Therefore, the full result for $\text{Re} \Pi_{xx/yy}$ is

$$\text{Re} \Pi_{xx/yy} = \text{Re} \Pi_a(q, \omega) + \text{Re} \Pi_b(q, \omega) + \text{Re} \Pi_c(q, \omega).$$
B. \( \text{Re} \Pi_{\text{xx},\text{yy}}(q, \omega = 0) \)

We will use the expression of \( \text{Re} \Pi_{\text{xx},\text{yy}}(q, \omega) \) as given in Appendix VII A to obtain the \( \omega = 0 \) limit. As before, \( \text{Re} \Pi_{\text{xx},\text{yy}} \) can be expressed as sum of three components, \( \text{Re} \Pi_{\text{xx},\text{yy}}(q) = \text{Re} \Pi_a(q) + \text{Re} \Pi_b(q) + \text{Re} \Pi_c(q) \). The results of the calculations for the individual terms are as follows. The integral without the chemical potential term is given by

\[
\text{Re} \Pi_a(q) = -\frac{1}{8\pi} \left( f_1 + f_2 + f_3 \right), \tag{49}
\]

where the lower limit on all three integrals are \( l = \tan \left[ \frac{1}{2} \cos^{-1} \left( \frac{q}{\sqrt{q^2 + \Delta^2_s}} \right) \right] \).

\[
f_1(q, 0) = \frac{4}{q} \left[ q^2 + 2\Delta^2_s \right] \tan^{-1} \left( \frac{q}{\sqrt{q^2 + \Delta^2_s}} \right), \tag{50}
\]

\[
f_2(q, 0) = \frac{1}{q^2} \left[ 2q^2(\Delta_1^2 + \Delta_2^2) - 2(\Delta_1 \Delta_2)^2 \right] \left[ x - \frac{1}{x} - 4 \tan^{-1}(x) \right], \tag{51}
\]

\[
f_3(q, 0) = -\frac{3(\Delta_1 \Delta_2)^2}{4q^4} \left[ \frac{x^3}{3} - 5x + \frac{5}{x} - \frac{1}{3x^3} + 2^4 \tan^{-1}(x) \right]. \tag{52}
\]

Combining them together we obtain, \( \text{Re} \Pi_a(q) = -\frac{\Delta^2_s + q^2}{4\pi q^2} \left[ q^2 - \Delta^2_s \right] \tan^{-1} \left( \frac{q}{\Delta_s} \right) + q \Delta_s \). \( \text{Re} \Pi_b(q) \) includes contributions from all integrals that have \( n_F[E_1(p)] \) and \( n_F[E_1(p + q)] \) terms:

\[
\text{Re} \Pi_b(q) = -\frac{1}{2\pi} \left\{ \mu - \Delta_1 + \int_{\Delta_1}^{\mu} \frac{dx}{2q} \left[ 4x^2 - q^2 - \Delta^2_s \right] \frac{\text{sgn}(q^2 + \Delta_1 \Delta_2)}{\sqrt{\xi^2 - x^2}} \right\}, \tag{53}
\]

where \( \xi = \sqrt{\frac{(q^2 + \Delta_1 \Delta_2)^2}{q^2} + 4q^2 \Delta^2_s} \). The result of the integration is given in Eq. 31 of the main text. The last term, \( \text{Re} \Pi_c(q) \), includes contributions from all integrals containing \( n_F[E_2(p)] \) and \( n_F[E_2(p + q)] \) terms and is given by

\[
\text{Re} \Pi_c(q) = -\frac{1}{2\pi} \left\{ \mu - \Delta_2 + \int_{\Delta_2}^{\mu} \frac{dx}{2q} \left[ 4x^2 - q^2 - \Delta^2_s \right] \frac{\text{sgn}(q^2 + \Delta_1 \Delta_2)}{\sqrt{\xi^2 - x^2}} \right\}, \tag{55}
\]

Final result for \( \text{Re} \Pi_c(q) \) is obtained from \( \text{Re} \Pi_b(q) \) by exchanging \( \Delta_1 \) with \( \Delta_2 \) and vice-versa. Adding together the three terms we find that the static part of the polarization function has a constant value for \( q < k_{F_1} + k_{F_2} \), while the change in polarization function from the constant value for \( q > k_{F_1} + k_{F_2} \) is given by

\[
\delta \left| \text{Re} \Pi_{\text{xx},\text{yy}}(q) \right| = \frac{1}{2\pi} \left[ \frac{\mu \sqrt{(q^2 - q_d^2)(q^2 - q_s^2)}}{q^2} - \frac{(q^2 + \Delta^2_s)(q^2 - \Delta^2_s)}{2q^4} \right] \tan^{-1} \left( \frac{\sqrt{(q^2 - q_d^2)(q^2 - q_s^2)}}{2\mu q} \right), \tag{54}
\]

where \( q_s = k_{F_1} + k_{F_2} \) and \( q_d = k_{F_1} - k_{F_2} \).

C. Derivation of \( \text{Re} \Pi_{\text{xx},\text{yy}}(q = 0, \omega) \)

The \( \text{Im} \Pi_a(\omega) \) term with contributions from both \( \Delta_1 \rightarrow \Delta_2 \) and \( \Delta_2 \rightarrow \Delta_1 \) transitions is given by,

\[
\text{Im} \Pi_a(\omega) = \frac{1}{8} \Theta(\omega^2 - \Delta^2_s) Y(\omega); \quad Y(\omega) = \frac{2\Delta^2_s}{\omega} \left[ 1 - \frac{\Delta^2_s}{\omega^2} \right]. \tag{55}
\]

Utilizing the Kramers-Kronig relation we obtain for \( \text{Re} \Pi_a(\omega) \)

\[
\text{Re} \Pi_a(\omega) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \text{Im} \Pi_a(\omega') \text{sgn}(\omega') = -\frac{\Delta^2_s}{4\pi \omega} \left\{ \log \left[ \frac{\Delta_s + \omega}{|\Delta_s - \omega|} \right] \left( 1 - \frac{\Delta^2_s}{\omega^2} \right) + \frac{2\Delta_s}{\omega} \right\}. \tag{56}
\]
A direct integration by considering contributions from the integrals containing $n_F(E_1)$ term yields

$$\text{Re} \Pi_b = - \left\{ \int_{\Delta_1} \frac{d^2 p}{(2\pi)^2} n_F(E_1(p)) \left[ \frac{E_1(p) + \omega}{[E_2(p) - E_1(p) - \omega]^2} + \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1(p)} \right] \frac{1}{[E_2(p) - E_1(p) - \omega]^2} \right\} [\omega \to -\omega]$$

$$= \frac{1}{4\pi} \left\{ \int_{\Delta_1} dE_1 \left[ \frac{(2E_1 + \omega)^2 - \Delta_0^2}{\Delta_s \Delta_d - \omega^2 - 2E_2 \omega} \right] \right\} [\omega \to -\omega] = \frac{1}{4\pi} \left\{ \frac{\Delta_0^2 (\Delta_s^2 - \omega^2)}{2\omega^3} \right\} \left( \log \left[ \frac{(-\Delta_d \Delta_s - 2\mu \omega + \omega^2) (-\Delta_d \Delta_s + 2\omega \Delta_1 + \omega^2)}{(-\Delta_d \Delta_s + 2\omega \Delta_1 + \omega^2) (-\Delta_d \Delta_s - 2\omega \Delta_1 + \omega^2)} \right] - \frac{2\Delta_d \Delta_s (\mu - \Delta_1)}{\omega^2} \right\}. \quad (56)$$

Similarly, $n_F(E_2)$ term yields contribution to $\text{Re} \Pi_c$ given by

$$\text{Re} \Pi_c = - \left\{ \int_{\Delta_2} \frac{d^2 p}{(2\pi)^2} n_F(E_2(p)) \left[ \frac{E_2(p) - \omega}{[E_1(p) + \omega]^2 - [E_2(p) - \omega]^2} + \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1(p)} \right] \frac{1}{[E_2(p) + \omega]^2 - [E_2(p) - \omega]^2} \right\} [\omega \to -\omega]$$

$$= \frac{1}{4\pi} \left\{ \int_{\Delta_2} dE_2 \left[ \frac{(2E_2 - \omega)^2 - \Delta_0^2}{-\Delta_s \Delta_d - \omega^2 + 2E_2 \omega} \right] \right\} [\omega \to -\omega] = \frac{1}{4\pi} \left\{ \frac{\Delta_0^2 (\Delta_s^2 - \omega^2)}{2\omega^3} \right\} \left( \log \left[ \frac{(-\Delta_d \Delta_s - 2\mu \omega + \omega^2) (-\Delta_d \Delta_s + 2\omega \Delta_2 + \omega^2)}{(-\Delta_d \Delta_s + 2\mu \omega + \omega^2) (-\Delta_d \Delta_s - 2\omega \Delta_2 + \omega^2)} \right] + \frac{2\Delta_d \Delta_s (\mu - \Delta_2)}{\omega^2} \right\}. \quad (57)$$
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