Cyclographic Modeling of Surface Forms of Highways

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Abstract. A cyclographic approach towards the formation of a highway surface is suggested. The approach is based on mapping a point \((x, y, z)\) of the space \(\mathbb{R}^3\) onto a cycle (a circle with direction) in the plane \(\Pi_{xy}\). The centre of the cycle is a point \((x, y)\) and its radius is defined as \(R = \pm |z|\). In case \(z > 0\), the cycle radius \(R = z\) and the cycle has one direction. In case \(z < 0\), the cycle radius \(R = -|z|\) and the cycle has the opposite direction. Thus, a one-to-one correspondence is established between the set of \(\infty^3\) points of the space \(\mathbb{R}^3\) and the set of \(\infty^3\) cycles of the plane \((xy)\).

The main linear element of a road is its axis which in general is a spatial curve. The spatial forms of road elements are a roadway, road shoulders, slopes; they are determined by the main linear element. Proceeding from this principle of formation of geometric and mathematical models of road elements spatial forms, the use of road axis cyclographic projection is proposed. The parametric equations of the envelope of cycles modeling the axis point in the plane are obtained for the general case of cyclographic mapping. In a general case, unlike the classic one, the use of regular angle variation at the vertex of mapping cones is suggested. Taking the envelope of cycles and the axis of the road as the guidelines for the surface, the surface forms of road elements are obtained in the form of developable surfaces. The presence of two modifications of road elements’ surface forms distinguishes the cyclographic approach from existing ones and gives additional possibilities for the choice of optimal surface forms in the design of highways.

1. Introduction
Modern level of development of industrial and civil construction objects, roads and railways is impossible without mathematical modeling, using modern achievements in the field of classical and applied mathematics, system analysis, modern information technologies and computer facilities. If it is possible to build a visual and figurative representation of initial data as a set of geometrical objects and connections between them at the stage of research and design objectives definition, then the opportunity to obtain a geometric model for solving the problem arises. The greatest effect of the application of geometric modeling in solving practical problems can be achieved by using the internal capabilities of the geometric model due to its mathematical nature. This nature makes it possible to carry out within the model a transition to operating methods of classical and applied mathematics and obtain the results of calculations and their subsequent geometric interpretation within the framework of the geometric model domain of definition. The combination of geometric and mathematical
methods within the applied geometric model is objectively necessary since a relatively small number of applied problems, including tasks in the field of road construction, can be effectively solved purely geometrically. In most cases it is necessary to apply geometric and quantitative abstractions with different degrees of their combination, which depends to a large extent on the content of the tasks and the level of geometric and mathematical knowledge needed to solve them.

As good examples of the application of geometric-quantitative algorithms of geometric modeling serve solutions of various problems in computational geometry, in which mathematical representation and computer visualization of various geometric objects is constructed throughout the representation of their simpler components.

The application of geometric-quantitative algorithms in geometric models in highway design and construction makes it possible at the modern level of science and engineering to achieve optimal performance and structural characteristics of these objects.

2. Relevance
In Russian modern design and construction of highways different approaches to mathematical modelling of road surface are used [1-11]. All of them are oriented at state requirements [12], according to which a straight section of the road must contain a discrete skeleton of typical cross-sectional profiles, while the bent part of the road must contain a discrete skeleton of transverse profiles in the form of rectilinear segments of constant slope. The variety of applicable road surface design mathematical models is reduced to mathematical algorithms for the organization of continuous skeletons of transverse profiles from given discrete standard skeletons. Herewith the junction of compartments of the resulting continuous frames on the boundaries of road sections with differing transverse profiles is hardly ever mathematically justified, which in practice leads to a dynamic impact on the car in motion. Special mention should be made of the papers [8-10], in which geometrical and mathematical models of linear and surface forms of roads are proposed on the basis of analysis and systematic consideration of existing problems in the design of highways. These models are based on the skeleton-kinematic method known in applied geometry. Currently these are the only models in which the road is viewed as an integral object, the components of which, differing in performance and structural characteristics, are joined together in a certain order of smoothness.

In this paper another approach to road surface modelling is proposed. It differs from the carcass-parametric approach in that the basis of geometric modelling is the cyclographic modelling of the first and main element of the road - its axis. Surface forms of roadway, road shoulders, and slopes, obtained on the basis of the cyclographic model of the axis of the road, are simple in mathematical representation and in geometric interpretation of their formation. They are developable or non-developable ruled surfaces. The presence of two types of surface forms and their mathematical representation within the framework of one geometric model of highway provides additional opportunities for efficient and high-quality design.

3. Formulation of the problem
The highway is a complex extensive construction that consists of rectilinear and cornering sections. Each of the cornering sections includes a transitional subsection as well as the corner itself. The main stage of highway design, which determines the subsequent stages, is the design of horizontal projection of the road axis (the route plan) and its longitudinal profile [9]. This stage of highway design, including road surface mathematical modelling, is thoroughly considered in [9]. A distinctive feature of this model is the parametric form of equations describing road axis.

In this paper the axis of the road is assumed to be a given, predesigned element, and is a smooth space curve with smoothness class no lower than $C^4$. If initially the axis of the road is set discretely in the form of a spatial row of points, with subsequent projective interpolation of this row using third-order spline functions, the spatial curve $\vec{P}(t)$ formed as a result of interpolation becomes smooth $C^2$
[9]. This circumstance does not affect the domain of determination of geometric model, which is based on the proposed cyclographic projection of the curve $\tilde{P}(t)$.

The problem of road surface modelling can be formulated as follows: given the axis of the road in the form of a smooth (class of smoothness no lower than $C^4$) the spatial curve $\tilde{P}(t)$ described by the vector equation:

$$\tilde{P}(t) = (x(t), y(t), z(t)), \quad \tilde{P}'(t) \neq 0, t \in R : T_0 \leq t \leq T$$  

it is required to construct a theoretical surface of the roadway on the basis of the cyclographic projection of the curve $\tilde{P}(t)$.

4. Theoretical part (solution of the problem)

A cyclographic projection of a point $(x, y, z)$ of space $R^3$ is a directed circle (a cycle) in a plane $\Pi_{1}(xy)$. The cycle is centred at the point $(x, y)$ and has radius $R = \pm |z|$, where $R = z$ in case $z > 0$ and $R = -|z|$ in case $z < 0$. Therefore, a set $\infty^3$ of points of the space $R^3$ can be put in a one-to-one correspondence with a set $\infty^3$ of cycles of the plane $(xy)$.

Mapping of points of space onto cycles of the plane $(xy)$ is called cyclographic mapping. It was proposed and developed by German geometers W. Fiedler [13], E. Muller and J. Krames [14]. The current level of development of the cyclographic mapping and its numerous theoretical applications and practical implementations are considered in [15-21].

Consider the formation and generation of cyclographic projection of a curve(1). According to the classical scheme of cyclographic projection generation of a curve(1), each point $(x(t), y(t), z(t))$ of the curve $\tilde{P}(t)$ is mapped onto the $(xy)$ plane by a cone of revolution with a vertex at a given point and a half-angle at the vertex $\alpha = 45° = \text{const}$. A certain law of variation of the half-angle at the vertex is adopted as a function:

$$\beta = f(t), f(t) \subset C^1, 0 < \beta(t) < 90°$$

Basing on the scheme of the cyclographic mapping of a curve (Figure 1), the equation of a one-parameter set of cycles - images of its points and the condition of envelope formation of this set – is formed:

$$\left\{ \tilde{P}_{\beta} - \tilde{P}_{1}(t), \tilde{P}_{\beta} - \tilde{P}_{1}(t) \right\} - \bar{r}_{\beta}(t) = 0,$$

$$\left\{ \tilde{P}_{\beta} - \tilde{P}_{1}(t), \tilde{P}_{1}(t) \right\} + \bar{r}_{\beta}(t) \cdot \bar{r}_{\beta}(t) = 0$$

In these equations the following is accepted: $\tilde{P}_{\beta} = (x_{\beta}, y_{\beta})$ is the radius vector of a point of the envelope of the cycles, $\tilde{P}_{1}(t)$ is the radius vector of the point of the orthogonal projection of the curve $\tilde{P}(t)$ onto the plane $\Pi_{1}(xy)$, $\bar{r}_{\beta}$ is the vector of the cycle point.

From equations (2) and (3) follow the parametric equations of the envelope $\tilde{P}_{\beta}$ of cycles:

$$x_{\beta}(t) = x(t) + z(t) \cdot \varepsilon(t) \cdot \frac{-x'(t) \cdot \mu(t) \mp x'(t) \cdot \sqrt{2} \lambda(t) - \mu^2(t)}{\lambda(t)}$$

$$y_{\beta}(t) = y(t) + z(t) \cdot \varepsilon(t) \cdot \frac{-y'(t) \cdot \mu(t) \mp y'(t) \cdot \sqrt{2} \lambda(t) - \mu^2(t)}{\lambda(t)}$$

(4)
\[ \mu(t) = e(t) \cdot z'(t) + z(t) \cdot e'(t) \]
\[ \lambda(t) = x'(t)^2 + y'(t)^2 \]
\[ e(t) = \tan \beta(t) \]

The envelope equations follow from the equations (4): in case \( e = const \), they are the envelope equations for cycles (\( \beta = const \)) and in case \( e = 1 \), they are the known envelope equations for the classical scheme of the cyclographic projection (\( \alpha = 45^\circ \)) [15, 19, 20]. Therefore, the equations (4) should be regarded as generalized equations of cyclographic projection of a spatial curve.

Equations (4) allow us to obtain equations for the ruled surface \( \Phi_\beta \) enveloping a one-parameter set of cones that map the points of the line (1) in the set of cycles (2) in the plane \( \Pi_1(xy) \). The generators of this ruled surface pass through the corresponding points \( A(A_1, A_2) \), \( B(B_1, B_2) \) and \( C(C_1, C_2) \), forming pairs respectively: \( A - B \) and \( A - C \) (see figure 1). Equations (4) describe an envelope that can be a single object \( \tilde{P}_\beta(t) \) or consist of two branches \( \tilde{P}_{\beta(1)}(t) \) and \( \tilde{P}_{\beta(2)}(t) \), which is determined by the geometry of the original curve \( \tilde{P}(t) \). The envelope equations of the ruled surface \( \Phi_\beta \) have the form:

\[ X(t,l) = x(t) + l \left[ x_\beta(t) - x(t) \right] \]
\[ Y(t,l) = y(t) + l \left[ y_\beta(t) - y(t) \right] \]
\[ Z(t,l) = z(t) \cdot (1 - l) \]
\[ T_0 \leq t \leq T, L_0 \leq l \leq L \]  

As follows from the geometric formation scheme of the surface \( \Phi_\beta, \Phi_\beta \) is a developable surface, the generators of which belong to the normal planes of the envelope \( \tilde{P}_\beta(t) \). Therefore, when the surface \( \Phi_\beta \) intersects the normal planes of the curve \( \tilde{P}_1(t) \) in the section, we obtain a curve with a vertex on the original curve \( \tilde{P}(t) \). It differs from the typical transverse profile, which is generally accepted for a straight-line road section [1,9,12]. In this connection it becomes necessary to transform the developable surface \( \Phi_\beta(5) \) into a ruled surface \( \Phi_{\beta(1,2)} \), which satisfies the accepted norms. The transformation is based on "twisting" a pair of straight lines forming each cone of the map into the normal plane of the curve \( \tilde{P}_1 \) (Figure 2). As a result of continuous execution of such transformations for each point \( A \in \tilde{P}_1(t) \) we obtain a new curve \( \tilde{P}(t) \) of equation (6):

\[ x_{1,2}(t) = x(t) \mp v_x(t) \cdot R_x(t) \]
\[ y_{1,2}(t) = y(t) \pm v_y(t) \cdot R_y(t) \]  

where

\[ v_x(t) = -y'(t) \cdot \frac{1}{\sqrt{\lambda(t)}}; v_y(t) = x'(t) \cdot \frac{1}{\sqrt{\lambda(t)}}; R_x(t) = \sqrt{(x(t) - x_\beta(t))^2 + (y(t) - y_\beta(t))^2} \]

\[ \vec{v}_1 = (v_x(t), v_y(t)) \]

\( \vec{v}_1 \) is the unit vector of the normal of the curve \( \tilde{P}_1(t) \).
The pairs of corresponding points - one on the original curve $\overrightarrow{P}(t)$ and one on the resulting curve $\overrightarrow{P}_{(1,2)}(t)$ - determine the generator of the new ruled surface $\Phi_{\beta(1,2)}$. The equations of this surface have the form:

$$X_{(1,2)}(t) = x(t) + l \times \left[ x_{(1,2)}(t) - x(t) \right]$$

$$Y_{(1,2)}(t) = y(t) + l \times \left[ y_{(1,2)}(t) - y(t) \right]$$

$$Z_{(1,2)}(t) = z(t) \times (1 - l)$$

$$T_0 \leq t \leq T, L_0 \leq l \leq L$$

**Figure 1.** Cyclographic projection formation scheme of the curve $\overrightarrow{P}(t)$.

**Figure 2.** Formation scheme of the guide curve $\overrightarrow{P}_{(1,2)}$ of the ruled surface $\Phi_{\beta(1,2)}$.

5. Results of numerical experiments

The equation of the curve $\overrightarrow{P}(t)$ taken as the axis of a road is given in parametric form:

$$x = A \cdot \cos t, y = B \cdot \sin t, z = 2t, \; A = 1000, B = 200, \; 0.2\pi \leq t \leq 0.25\pi, \beta = 1.75t$$

For experimental roadside surface formation, the half-angle at the vertex of the mapping cone for the points of the roadside edge is accepted constant and equal to $\beta = 1.1$ rad. Using the equations (5) of the developable surface $\Phi_{\beta}$ and performing the necessary calculations in accordance with the initial data, the calculation result visualization is obtained as shown in Figure 3a. It depicts the surfaces of the roadway, roadsides and cycle envelopes of the $\overrightarrow{P}(t)$ road axis and the edge of the roadway in the $(xy)$ plane. Calculations with the same initial data and on the basis of equations (6) of the ruled non-developable surface $\Phi_{\beta(1,2)}$ were performed, the respective result visualization is shown in Figure 4a. Figures 3b and 4b extensively show top views of formatted surfaces with position of normal to curve $\overrightarrow{P}(t)$ sectional plane $\Delta$ indicated. In the first case, the plane $\Delta$ defines spatial curvilinear profile of normal section, in the second case – linear profile of normal section.

6. Conclusions

A geometrical model of road surface formation is proposed in two modifications:
1. Cyclographic modification, based on the construction of a cyclographic projection of the main element of the road - its axis. This modification allows the shaping of the roadway in the form of a developable surface. The straight lines forming this surface are perpendicular to the cyclographic projection of the road axis, and its transverse profile has a curved shape with a vertex point on the axis of the road. The curvilinear form approximates the shape of a typical transverse profile in the form of a broken line.

2. Non-cyclographic modification based on the cyclographic modification and allowing the shaping of the roadway in the form of a non-developable ruled surface to obtain a typical transverse profile. Both modifications can be used to form the roadsides and slopes of the highway. They are mathematically formalizable and allow analytic solutions of the systems of equations of their mathematical models. This makes the modifications available for the use with modern information technologies and computer facilities.

Figure 3. The result of roadway and the roadside cyclographic modelling at $\beta = \text{var}$: a) spatial visualization; b) top view.

Figure 4. The result of non-cyclographic roadway and roadside modelling at $\beta = \text{var}$: a) spatial visualization; b) top view.
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