ELECTRIC/MAGNETIC DUALITY AND ITS STRINGY ORIGINS

M. J. Duff

Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843.

ABSTRACT

We review electric/magnetic duality in $N = 4$ (and certain $N = 2$) globally supersymmetric gauge theories and show how this duality, which relates strong to weak coupling, follows as a consequence of a string/string duality. Black holes, eleven dimensions and supermembranes also have a part to play in the big picture.

1Based on review talks delivered at the PASCOS 95 conference, Johns Hopkins University, Baltimore, March 1995 and the SUSY 95 conference, Ecole Polytechnique, Paris, June 1995. Research supported in part by NSF Grant PHY-9411543.
1 Introduction

Two of the hottest topics in theoretical high-energy physics at the moment are:

1) Electric/magnetic duality in $D = 4$ dimensional globally supersymmetric gauge theories, whereby the long distance behavior of strongly coupled electric theories are described in terms of weakly coupled magnetic theories. This sheds new light on quark confinement, the Higgs mechanism and even pure mathematics.

2) String/string duality in $D \leq 6$ dimensional superstring theory, whereby the same physics is described by two apparently different theories: one a heterotic string and the other a Type II string.

Here I wish to review both and show that the former follows as a consequence of the latter. We shall also discover that black holes, $D = 11$ dimensions and extended objects with more than one spatial dimension (the super p-branes) also make their appearance.

In discussing duality in gauge theories it is important to distinguish between exact duality and effective duality. We shall take the phrase exact duality to refer to the conjectured $SL(2, \mathbb{Z})$ symmetry that acts on the gauge coupling constant $e$ and theta angle $\theta$:

$$S \rightarrow aS + b$$

where $a, b, c, d$ are integers satisfying $ad - bc = 1$ and where

$$S = S_1 + iS_2 = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$$

This is also called electric/magnetic duality because the integers $m$ and $n$ which characterize the magnetic charges $Q_m = n/e$ and electric charges $Q_e = e(m + n\theta/2\pi)$ of the particle spectrum transform as

$$\begin{pmatrix} m \\ n \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

Such a symmetry would be inherently non-perturbative since, for $\theta = 0$ and with $a = d = 0$ and $b = c = -1$, it reduces to the strong/weak coupling duality

$$\frac{e^2}{4\pi} \rightarrow \frac{4\pi}{e^2}$$

$$n \rightarrow m, m \rightarrow -n$$

This in turn means that the coupling constant cannot get renormalized in perturbation theory and hence that the renormalization group $\beta$-function vanishes

$$\beta(e) = 0$$

This is guaranteed in $N = 4$ supersymmetric Yang-Mills and also happens in certain $N = 2$ theories. Thus the duality idea is that the theory may equivalently be described in one of two ways. In the conventional way, the W-bosons, Higgs bosons and their fermionic partners are the electrically charged elementary particles and the magnetic monopoles emerge as soliton...
solutions of the field equations. In the dual description, however, it is the monopoles which are elementary and the electrically charged particles which emerge as the solitons. Historically, the exact duality \( (1) \) was first conjectured by Montonen and Olive \([1, 2, 3, 4]\) and then generalized to include the theta angle in \([5, 6, 7, 8, 9, 10, 11, 12]\). The \( SL(2, Z) \) then means that there are, in fact, infinitely many equivalent descriptions.

We shall take the phrase effective duality to refer to the more realistic \( N = 1 \) and \( N = 2 \) gauge theories, for which \( \beta(e) \neq 0 \) and which exhibit no exact \( SL(2, Z) \) symmetry, but for which there exists two different versions, with different gauge groups and quark representations, leading to the same physics. The weak coupling region of one theory is mapped into the strong coupling region of the other. Once again there is a soliton interpretation. This effective duality has been pioneered by Seiberg and Witten \([11, 12, 17, 18]\) and it shows great promise for the understanding of quark confinement and chiral symmetry breaking. Both arise from the condensation of the magnetic monopoles. Moreover, it is these effectively dual theories which have proved most valuable in proving new results in the topology of four-manifolds \([13]\). However, my main purpose in this review is to explain how electric-magnetic duality follows by embedding these Yang-Mills theories in a superstring theory and to date most progress in this direction has been made in the context of exactly dual theories. In what follows, therefore I will focus on exact duality and confine the discussion of the stringy origins of effective duality to a few remarks at the end. This will also render possible the otherwise impossible task of reviewing the two hottest topics in just a few pages.

2 Monopoles of \( N = 4 \) and \( N = 2 \) Yang-Mills

Consider the action describing the bosonic sector of the unique \( N = 4 \) supersymmetric \( SU(2) \) Yang-Mills theory:

\[
S = \int d^4 x \left( -\frac{1}{4e^2} Tr F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} Tr D_\mu \Phi D^\mu \Phi - V(\Phi) + \frac{\theta}{16\pi^2} Tr F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \tag{6}
\]

The potential \( V(\Phi) \) admits spontaneous symmetry breaking VEVs for the scalar fields \( \langle Tr \Phi^2 \rangle = v^2 \neq 0 \) which break the \( SU(2) \) down to \( U(1) \). This means that the theory admits BPS monopoles. Both the elementary states and the monopoles belong to short 16-dimensional \( N = 4 \) supermultiplets and their masses saturate a Bogomol’nyi bound:

\[
M^2 = (4\pi)^2(m, n) \frac{v^2}{S_2} \left( \begin{array}{cc} 1 & S_1 \\ S_1 & |S|^2 \end{array} \right) \left( \begin{array}{c} m \\ n \end{array} \right) \tag{7}
\]

This universal mass formula \( (7) \) is manifestly invariant under the \( SL(2, Z) \) transformations \( (4) \) and \( (3) \). Although it was obtained by semiclassical reasoning, \( N = 4 \) non-renormalization theorems ensure that it is exact in the full quantum theory. The duality conjecture is that this \( SL(2, Z) \) is not only an exact symmetry of mass spectrum but of the entire quantum theory.

Strong evidence for this conjecture was provided by Sen \([8]\) who pointed out that given a purely electrically charged state \( (m = 1, n = 0) \), \( SL(2, Z) \) implies the existence of a state \( (p, q) \) with \( p \) and \( q \) relatively prime integers (i.e having no common divisor). Sen then went
on to construct explicitly a dyonic solution with charges \((1, 2)\) in complete agreement with the conjecture. Further evidence was supplied by Vafa and Witten [10], by studying partition functions of the twisted \(N = 4\) theory on various 4-manifolds, and also by Girardello et al [9].

Originally, it was thought that this exact duality between the elementary particles and the monopoles would work only for \(N = 4\), where both belong to the unique short supermultiplet and was shown explicitly not to work for the pure \(N = 2\) theory [4]. However, the monopole spectrum is consistent with exact duality in the \(\beta(e) = 0\) case of \(N = 2\) Yang-Mills with gauge group \(SU(2)\) coupled to four hypermultiplets in the fundamental representation, as was shown in [11, 12] for the one-monopole sector \((q = 1)\) and recently in [13, 14, 15] for the two-monopole sector \((q = 2)\). There are more subtleties in the \(N = 2\) case. For example, when the 8 real fermions in the hypermultiplets transform under \(SL(2, \mathbb{R})\), they must also transform under an \(SO(8)\) triality.

3 \(S\)-duality in string theory

In string theory the roles of the theta angle \(\theta\) and coupling constant \(e\) are played by the VEVs of the the four-dimensional axion field \(a\) and dilaton field \(\eta\):

\[
< a > = \frac{\theta}{2\pi} \quad (8)
\]

\[
e^2 /4\pi = < e^\eta > = 8G/\alpha'
\]

Here \(G\) is Newton’s constant and \(2\pi\alpha'\) is the inverse string tension. Hence \(S\)-duality (1) now becomes a transformation law for the axion/dilaton field \(S\):

\[
S = S_1 + iS_2 = a + ie^{-\eta} \quad (10)
\]

The \(S\)-duality conjecture in string theory has its origins in supergravity. In the late 70s and early 80s, it was realized that compactified supergravity theories exhibit non-compact global symmetries [66, 67, 68, 69] e.g. \(SL(2, R), O(22, 6), O(24, 8), E_7, E_8, E_9, E_{10}\). In 1990 it was conjectured [64, 65] that discrete subgroups of all these symmetries should be promoted to duality symmetries of either heterotic or Type \(II\) superstrings. The case for \(O(22, 6; Z)\) had already been made. This is the well-established target space duality, sometimes called \(T\)-duality [70]. Stronger evidence for a strong/weak coupling \(SL(2, Z)\) duality in string theory was subsequently provided in [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 33], stronger evidence for the combination of \(S\) and \(T\)-duality into an \(O(24, 8; Z)\) in heterotic strings was provided in [81, 34, 120, 44] and stronger evidence for their combination into a discrete \(E_7\) in Type \(II\) strings was provided in [51], where it was dubbed \(U\)-duality.

Let us first consider \(T\)-duality and focus just on the moduli fields that arise in compactification on a 2-torus of a \(D = 6\) string with dilaton \(\Phi\), metric \(G_{MN}\) and 2-form potential \(B_{MN}\) with 3-form field strength \(H_{MNP}\). Here the \(T\)-duality is just \(O(2, 2; Z)\). Let us parametrize the compactified \((m, n = 4, 5)\) components of string metric and 2-form as

\[
G_{mn} = e^{\rho - \sigma} \begin{pmatrix}
e^{-2\rho} + c^2 & -c \\
-c & 1
\end{pmatrix}
\]

(11)
and

\[ B_{mn} = b \epsilon_{mn} \]  \hspace{1cm} (12)

The four-dimensional shifted dilaton \( \eta \) is given by

\[ e^{-\eta} = e^{-\Phi} \sqrt{\text{det} G_{mn}} = e^{-\Phi - \sigma} \]  \hspace{1cm} (13)

and the axion field \( a \) is defined by

\[ e^{\mu \rho \sigma} \partial_{\sigma} a = \sqrt{-g} e^{-\eta} g^{\mu \sigma} g^{\nu \lambda} g^{\rho \tau} H_{\sigma \lambda \tau} \]  \hspace{1cm} (14)

where \( g_{\mu \nu} = G_{\mu \nu} \) and \( \mu, \nu = 0, 1, 2, 3 \). We further define the complex Kahler form field \( T \) and the complex structure field \( U \) by

\[ T = T_1 + iT_2 = b + ie^{-\sigma} \]
\[ U = U_1 + iU_2 = c + ie^{-\rho} \]  \hspace{1cm} (15)

Thus this \( T \)-duality may be written as

\[ O(2, 2; \mathbb{Z})_{TU} \sim SL(2, Z)_T \times SL(2, Z)_U \]  \hspace{1cm} (16)

where \( SL(2, Z)_T \) acts on the \( T \)-field and \( SL(2, Z)_U \) acts on the \( U \)-field in the same way that \( SL(2, Z)_S \) acts on the \( S \)-field in [1]. In contrast to \( SL(2, Z)_S \), \( SL(2, Z)_T \times SL(2, Z)_U \) is known to be not merely a symmetry of the supergravity theory but an exact string symmetry order by order in string perturbation theory. \( SL(2, Z)_T \) does, however, contain a minimum/maximum length duality mathematically similar to (4)

\[ R \rightarrow \alpha'/R \]  \hspace{1cm} (17)

where \( R \) is the compactification scale given by

\[ \alpha'/R^2 = \langle e^\sigma \rangle . \]  \hspace{1cm} (18)

Even before compactification, the Type IIB supergravity exhibits an \( SL(2, R) \) whose discrete subgroup has been conjectured to be a non-perturbative symmetry of the Type IIB string [1], [3]. We shall refer to this duality as \( SL(2, Z)_X \) to distinguish it from the others. Combining this with the known \( T \)-duality of the four dimensional theory obtained by compactification on \( T^6 \) leads to the \( E_7 \). So the explanation for \( U \)-duality devolves upon the explanation for this \( SL(2, Z)_X \). We shall return to this in section 8.

4 \hspace{1cm} S-duality from \( D = 6 \) string/string duality

Let us now investigate how both \( N = 4 \) and \( N = 2 \) exact electric/magnetic duality follows from string theory. As discussed above, there is a formal similarity between this symmetry and that of \( T \)-duality. It was argued in [32] that these mathematical similarities between \( SL(2, Z)_S \) and
$SL(2, \mathbb{Z})_T$ are not coincidental. Evidence was presented in favor of the idea that the physics of the fundamental string in six spacetime dimensions may equally well be described by a dual string and that one emerges as a soliton solution of the other [29, 30, 31, 32, 104, 99]. The string equations admit the singular elementary string solution [88]

$$ds^2 = (1 - k^2/r^2)[-d\tau^2 + d\sigma^2 + (1 - k^2/r^2)^{-2}dr^2 + r^2d\Omega_3^2]$$

$$e^\Phi = 1 - k^2/r^2$$

$$e^{-\Phi} * H_3 = 2k^2\epsilon_3$$ \hspace{1cm} (19)

where

$$k^2 = \kappa^2 T/\Omega_3$$ \hspace{1cm} (20)

$T = 1/2\pi\alpha'$ is the string tension, $\Omega_3$ is the volume of $S^3$ and $\epsilon_3$ is the volume form. It describes an infinitely long string whose worldsheet lies in the plane $X^0 = \tau, X^1 = \sigma$. Its mass per unit length is given by

$$M = T < e^{\Phi/2} >$$ \hspace{1cm} (21)

and is thus heavier for stronger string coupling, as one would expect for a fundamental string.

The string equations also admit the non-singular solitonic string solution [30, 31]

$$ds^2 = -d\tau^2 + d\sigma^2 + (1 - \tilde{k}^2/r^2)^{-2}dr^2 + r^2d\Omega_3^2$$

$$e^{-\Phi} = 1 - \tilde{k}^2/r^2$$

$$H_3 = 2\tilde{k}^2\epsilon_3$$ \hspace{1cm} (22)

whose tension $\tilde{T} = 1/2\pi\tilde{\alpha}'$ is given by

$$\tilde{k}^2 = \kappa^2 \tilde{T}/\Omega_3$$ \hspace{1cm} (23)

Its mass per unit length is given by

$$\tilde{M} = \tilde{T} < e^{-\Phi/2} >$$ \hspace{1cm} (24)

and is thus heavier for weaker string coupling, as one would expect for a solitonic string. Thus we see that the solitonic string differs from the fundamental string by the replacements

$$\Phi \rightarrow \tilde{\Phi} = -\Phi$$

$$G_{MN} \rightarrow \tilde{G}_{MN} = e^{-\Phi} G_{MN}$$

$$H \rightarrow \tilde{H} = e^{-\Phi} * H$$

$$\alpha' \rightarrow \tilde{\alpha}'$$ \hspace{1cm} (25)

The Dirac quantization rule $e\gamma = 2\pi n$ ($n=$integer) relating the Noether “electric” charge

$$e = \frac{1}{\sqrt{2\kappa}} \int_{S^3} e^{-\Phi} * H_3$$ \hspace{1cm} (26)
to the topological “magnetic” charge

\[ g = \frac{1}{\sqrt{2\kappa}} \int_{S^3} H_3 \quad (27) \]

translates into a quantization condition on the two tensions:

\[ 2\kappa^2 = n(2\pi)^3 \alpha' \tilde{\alpha}' \quad n = \text{integer} \quad (28) \]

where \(\kappa\) is the six-dimensional gravitational constant. Both the string and dual string soliton solutions break half the supersymmetries, both saturate a Bogomol’nyi bound between the mass and the charge. These solutions are the extreme mass equals charge limit of more general two-parameter black string solutions [93, 30].

We now make the major assumption of string/string duality: the dual string may be regarded as a fundamental string in its own right with a worldsheet action that couples to the dual variables and has the dual tension given in (27). It follows that the dual string equations admit the dual string (27) as the fundamental solution and the fundamental string (19) as the dual solution. When expressed in terms of the dual metric, however, the former is singular and the latter non-singular. It also follows from (28) that in going from the fundamental string to the dual string and interchanging \(\alpha'\) with \(\tilde{\alpha}' = 2\kappa^2/(2\pi)^3 \alpha'\), one also interchanges the roles of worldsheet and spacetime loop expansions. Moreover, since the dilaton enters the dual string equations with the opposite sign to the fundamental string, it was argued in [29, 30, 31] that in \(D = 6\) the strong coupling regime of the string should correspond to the weak coupling regime of the dual string:

\[ g_6^2/(2\pi)^3 = \langle e^\Phi \rangle = (2\pi)^3/\tilde{g}_6^2 \quad (29) \]

where \(g_6\) and \(\tilde{g}_6\) are the six-dimensional string and dual string loop expansion parameters.

On compactification to four spacetime dimensions, the two theories appear very similar, each acquiring an \(O(2,2;Z)\) target space duality. One’s first guess might be to assume that the strongly coupled four-dimensional fundamental string corresponds to the weakly coupled dual string, but in fact something more subtle and interesting happens: the roles of the \(S\) and \(T\) fields are interchanged [79] so that the strong/weak coupling \(SL(2,Z)_S\) of the fundamental string emerges as a subgroup of the target space duality of the dual string:

\[ O(2,2;Z)_{SU} \sim SL(2,Z)_S \times SL(2,Z)_U \quad (30) \]

This duality of dualities is summarized in Table (1). As a consistency check, we note that since \((2\pi R)^2/2\kappa^2 = 1/16\pi G\) the Dirac quantization rule (28) becomes (choosing \(n=1\))

\[ 8GR^2 = \alpha' \tilde{\alpha}' \quad (31) \]

Invariance of this rule now requires that a strong/weak coupling transformation on the fundamental string \((8G/\alpha' \rightarrow \alpha'/8G)\) must be accompanied by a minimum/maximum length transformation of the dual string \((\tilde{\alpha}'/R^2 \rightarrow R^2/\tilde{\alpha}')\), and vice versa.
Turning now to the electric and magnetic fields, these fall naturally into two categories: (1) the gauge fields already present in the $D = 6$ string theory and whose details will depend on how we arrived at this theory; (2) the $U(1)^4$ fields which arise in going from 6 to 4 dimensions on a generic $T^2$ and which appear in $G_{\mu n}$ (the Kaluza-Klein gauge fields) and $B_{\mu n}$ (the winding gauge fields). We begin with (2) which are easier to discuss. The target space $T$-duality and $U$-duality and the strong-weak coupling $S$-duality transform the four field strengths $F_{\mu \nu}$ and their duals $\tilde{F}_{\mu \nu}$ as

$$F_{\mu \nu} \rightarrow (\omega_T^{-1} \times \omega_U^{-1}) F_{\mu \nu}$$

$$\left( \begin{array}{c} F_{\mu \nu} \\ \tilde{F}_{\mu \nu} \end{array} \right) \rightarrow \omega_S^{-1} \left( \begin{array}{c} F_{\mu \nu} \\ \tilde{F}_{\mu \nu} \end{array} \right)$$

where $\omega_S$, $\omega_T$ and $\omega_U$ are the respective $SL(2, Z)$ matrices. Thus $T$-duality transforms Kaluza-Klein electric charges ($F^3$, $F^4$) into winding electric charges ($F^1$, $F^2$) (and Kaluza-Klein magnetic charges into winding magnetic charges), $U$-duality transforms the Kaluza-Klein and winding electric charge of one circle ($F^3$, $F^2$) into those of the other ($F^4$, $F^1$) (and similarly for the magnetic charges) but $S$-duality transforms Kaluza-Klein electric charge ($F^1$, $F^2$) into winding magnetic charge ($\tilde{F}_3$, $\tilde{F}_4$) (and winding electric charge into Kaluza-Klein magnetic charge). In a way which should now be obvious, an entirely similar story applies to the dual theory with $T$ and $S$ exchanging roles. Note that the solitonic magnetic $H$-monopoles [118, 119] of the fundamental string are the fundamental electric winding states of the dual string [15, 34]. The Kaluza-Klein states are common to both.

It is here that the black hole connection enters. In [15] it was shown that these $H$-monopoles are in fact magnetically charged black holes in the limiting case where the charge equals the...
mass. By $T$-duality, they are related to the Kaluza-Klein monopoles which are also extreme (mass=magnetic charge) black holes. But by $S$-duality if these solitonic magnetically charged string states are black holes, then the elementary electrically charged string states must also be extreme (mass=electric charge) black holes \[81\] because they belong to the same $SL(2, \mathbb{Z})$ doublet! To be precise, if we denote by $N_L$ and $N_R$ the number of left and right oscillators of the heterotic string, then the extreme black holes correspond to $N_R = 1/2$. In the literature, black holes are frequently labeled by a parameter $\alpha$ which describes how the Maxwell field couples to a single scalar field formed from a combination of dilaton and moduli. The $(N_R = 1/2, N_L = 1)$ states yield $\alpha = \sqrt{3}$ and the $(N_R = 1/2, N_L > 1)$ states (with vanishing left-moving internal momentum) yield $\alpha = 1$. One might worry that this identification of Bogomol’nyi string states with extreme black holes works only at the level of the charge and mass spectrum, so it is worth emphasizing that it extends also to the gyromagnetic ratios \[81, 82\]. Moreover, further dynamical evidence has been supplied in \[116\], where it was shown (in the limit of low velocities) that the $\alpha = \sqrt{3}$ and $\alpha = 1$ extreme black holes have the same scattering amplitudes as the $(N_R = 1/2, N_L = 1)$ and $(N_R = 1/2, N_L > 1)$ string states. It has recently been shown that there are also massless charged black holes corresponding to $(N_R = 1/2, N_L = 0)$ states \[35, 36, 109\].

6 Symmetry enhancement

All of our discussions of the compactifying torus $T^2$ have so far assumed that we are at a generic point in the moduli space of vacuum configurations and that the unbroken gauge symmetry in going from $D = 6$ to $D = 4$ is the abelian $U(1)^4$. However, we know that at special points in moduli space two of the four $U(1)$s may be enhanced \[25\] to a simply laced rank 2 non-abelian group. For $T = U; T = U = i; T = U = exp(2\pi i/3)$, the enhanced symmetries are $SU(2) \times U(1), SU(2) \times SU(2)$ and $SU(3)$, respectively. String/string duality now suggests a new (non-perturbative) phenomenon, however. In theories with an $S \leftrightarrow T$ symmetry, such as the one discussed in section (9) below, a similar enhancement of the dual gauge symmetry also occurs in the dual theory when $S = U$ and $S = T$ \[32, 19\].

In order to discuss the gauge fields present already in $D = 6$ and the questions of symmetry enhancement there, it is first necessary to be more specific about the nature of the dual string.

7 A concrete $N = 4$ example

When we say that one string is dual another, exactly which strings are we talking about? After all, any string in $D = 6$ will exhibit the fundamental and solitonic string solutions of section \[4\] because all strings couple to the metric, 2-form and dilaton. The solitonic zero-modes that describe the field content of the dual string worldsheet will, however, depend crucially on the nature of the fundamental string. In the author’s opinion, one of the most important unsolved problems in string/string duality is that there is as yet no well-defined criterion for deciding when a solitonic string should be promoted to the status of being fundamental in its own right. Must it be critical, for example? At the moment, therefore, the game is to play safe and focus
only on those dual solitonic strings that correspond to fundamental critical string theories that we already know. The example that has attracted the most interest is the conjecture put forward by Hull and Townsend [51] and by Witten [28] which states that the $\mathcal{D}=10$ heterotic string compactified to $\mathcal{D}=6$ on $T^4$ is dual to the $\mathcal{D}=10$ Type IIA string compactified to $\mathcal{D}=6$ on $K3$. $K3$ is a four-dimensional compact closed simply-connected manifold. It is equipped with a self-dual metric and hence its holonomy group is $SU(2)$. It was first invoked in a Kaluza-Klein context in [57, 98] where it was used as a way of compactifying $\mathcal{D}=11$ supergravity to $\mathcal{D}=7$ and $\mathcal{D}=10$ supergravity to $\mathcal{D}=6$. Half the spacetime supersymmetry remains unbroken as a consequence of the $SU(2)$ holonomy, and hence the Type IIA theory gives rise to an $N=2$ string in $\mathcal{D}=6$. In fact, in 1986, it was pointed out [56] that $\mathcal{D}=11$ supergravity compactified on $K3 \times T^n$ [57] and the $\mathcal{D}=10$ heterotic string compactified on $T^n$ [58] have the same moduli spaces of vacua, namely

$$\mathcal{M} = \frac{SO(16+n,n)}{SO(16+n) \times SO(n)}$$

(34)

It was subsequently confirmed [59, 60], in the context of the $\mathcal{D}=10$ Type IIA theory compactified on $K3 \times T^{n-4}$, that this equivalence holds globally as well as locally. These “coincidences” lend further credence to the conjecture. We shall have more to say on the role of $\mathcal{D}=11$ supergravity in section (10).

Further $T^2$ compactification yields a dual pair of $\mathcal{D}=4$, $N=4$ strings that are related by the interchange of $S$ and $T$ discussed in section [4]. In particular, the $S$-duality of the Type IIA string follows automatically from the $T$-duality of the heterotic string. Moreover, just as special points in moduli space lead to enhanced gauge symmetries on the heterotic side, string/string duality implies that symmetry enhancement must occur at special $K3$ points on the IIA side [51, 28, 102]. If we now take the global limit of this theory, we find $N=4$ Yang-Mills theories with the desired $SL(2,Z)$. Of course, this global limit involves starting with a theory of gravity and then switching the gravity off. Witten [84] has suggested that a more direct way of embedding the Yang-Mills theory in string theory in order to derive the $SL(2,Z)$ would be via the anti-self-dual string [44] in the sense that this may be the minimal manifestly $S$-dual extension of the $N=4$ super Yang-Mills theory.

8 Four-dimensional string/string/string triality

We have seen that in six spacetime dimensions, the heterotic string is dual to a Type IIA string. On further toroidal compactification to four spacetime dimensions, the heterotic string acquires an $SL(2,Z)_S$ strong/weak coupling duality and an $SL(2,Z)_T \times SL(2,Z)_U$ target space duality acting on the dilaton/axion, complex Kahler form and the complex structure fields $S, T, U$ respectively. Strong/weak duality in $\mathcal{D}=6$ interchanges the roles of $S$ and $T$ in $\mathcal{D}=4$ yielding a Type IIA string with fields $T, S, U$.

However, as discussed in section (3), the target space symmetry of the heterotic theory also contains an $SL(2,Z)_U$ that acts on $U$, the complex structure of the torus. This suggests that, in addition to these $S$ and $T$ strings there ought to be a third $U$-string whose axion/dilaton
field is $U$ and whose strong/weak coupling duality is $SL(2, Z)_U$. From a $D = 6$ perspective, this seems strange since, instead of $(2^3)$, we now interchange $G_{15}$ and $B_{15}$. Moreover, of the two electric field strengths which become magnetic, one is a winding gauge field and the other is Kaluza-Klein! So such a duality has no $D = 6$ Lorentz invariant meaning. In fact, this $U$ string is a Type IIB string, a result which may also be understood from the point of view of mirror symmetry: interchanging the roles of Kahler form and complex structure (which is equivalent to inverting the radius of one of the two circles) is a symmetry of the heterotic string but takes Type IIA into Type IIB \[23, 24\]. In summary, if we denote the heterotic, IIA and IIB strings by $H, A, B$ respectively and the axion/dilaton, complex Kahler form and complex structure by the triple $X Y Z$ then we have a triality between the $S$-string ($H_{STU} = H_{SUT}$), the $T$-string ($B_{TUS} = A_{TUS}$) and the $U$-string ($A_{UST} = B_{UTS}$) \[19\]. Related results have been obtained independently in \[101\] and \[113\]. Each string in $D = 4$ will then exhibit the same total symmetry

$$SL(2, Z)_S \times O(6, 22; Z)_{TU} \supset SL(2, Z)_S \times SL(2, Z)_T \times SL(2, Z)_U$$

with the 28 gauge field strengths and their duals transforming as a $(2, 28)$. Of course, there will be different interpretations for the three $SL(2, Z)$ factors. So although there is a discrete symmetry under $T \leftrightarrow U$ interchange, there is no such $U \leftrightarrow S$ or $S \leftrightarrow T$ symmetry. As discussed in \[32\], it is the degrees of freedom associated with going from 10 to 6 which are responsible for this lack of $S$–$T$–$U$ democracy. This will also be reflected in the Bogomol’nyi spectrum of electric and magnetic states that belong to the short and intermediate $N = 4$ supermultiplets. The three strings also admit a soliton interpretation: one may identify the $S$-string with the elementary string solution of \[88\], the $T$-string with the dual solitonic string solution of \[73\] and the $U$-string with (a limit of) the stringy cosmic string solution of \[19\]. In $D = 3$ dimensions, all three strings are related by $O(8, 24; Z)$ transformations.

The compactification to $N = 4$, $D = 4$ reveals one or two surprises: although the $S$-string supergravity action has an off-shell $O(6, 22; Z)$ which continues to contain $SL(2, Z)_T \times SL(2, Z)_U$, the $T$-string action has only an off-shell $SL(2, Z)_U \times O(3, 19; Z)$ which does not contain $SL(2, Z)_S$. Similarly, the $U$-string action has only an $SL(2, Z)_T \times O(3, 19; Z)$ which does not contain $SL(2, Z)_S$. In short, none of the actions is $SL(2, Z)_S$ invariant! This lack of off-shell $SL(2, Z)_S$ in the Type II actions can be traced to the presence of the extra 24 gauge fields which arise from the Ramond-Ramond (R-R) sector of Type II strings: $S$-duality in the heterotic picture acts as an on-shell electric/magnetic transformation on all 28 gauge fields and continues to be an on-shell transformation on the 24 which remain unchanged under the string/string/string triality. At first sight, this seems disastrous for deriving the strong/weak coupling duality of the heterotic string from target space duality of the Type II string. The whole point was to explain a non-perturbative symmetry of one string as a perturbative symmetry of another \[32\]. Fortunately, all is not lost: although $SL(2, Z)_S$ is not an off-shell symmetry of the Type II supergravity actions, it is still a symmetry of the Type II string theories. To see this we first note that $D = 6$ general covariance is a perturbative symmetry of the Type IIB string and therefore that the $D = 4$ Type IIB strings must have a perturbative $SL(2, Z)$ acting on the complex structure of the compactifying torus. Secondly we note that for both Type IIB theories, $B_{TUS}$ and $B_{UTS}$, $S$ is the complex structure field. Thus the $T$
string has $SL(2, Z)_U \times SL(2, Z)_S$ and the $U$ string has $SL(2, Z)_S \times SL(2, Z)_T$ as required. In this sense, four-dimensional string/string/string triality fills a gap left by six-dimensional string/string duality: although duality satisfactorily explains the strong/weak coupling duality of the $D = 4$ Type IIA string in terms of the target space duality of the heterotic string, the converse requires the Type IIB ingredient.

Note that all of the three $SL(2, Z)_{(S,T,U)}$ take Neveu-Schwarz (NS)-NS states into NS-NS states and that none can be identified with the conjectured [91, 51, 28] non-perturbative $SL(2, Z)_X$ of section 8, where $X$ is the complex scalar of the Type IIB theory in $D = 10$, which transforms NS-NS into R-R. However, this $SL(2, Z)_X$ is a subgroup of $O(6, 22; Z)$. Since this is a perturbative target space symmetry of the heterotic string, the conjecture follows automatically from the $D = 4$ string/string/string triality hypothesis. Thus we can say that evidence for this triality is evidence not only for the electric/magnetic duality of all three $D = 4$ strings but also for the $SL(2, Z)_X$ of the $D = 10$ Type IIB string and hence for all the conjectured non-perturbative symmetries of string theory.

We should emphasize, of course, that string/string duality and string/string/string triality are themselves still only conjectures, albeit very plausible ones, so checks on $S$-duality in string theory are still useful. Trying to find all the $S$-dual magnetic partners of the elementary string states is not an easy task, however, and seems to require a better understanding of the role of $K3$ [83, 114].

9 More dual string pairs

The next conjecture to attract attention was the one put forward by Ceresole et al [50], Kachru and Vafa [47] and Ferrara et al [45] that the $D = 10$ heterotic string compactified to $D = 4$ on $K3 \times T^2$ is dual to the $D = 10$ Type IIA string compactified on a Calabi-Yau manifold. This yields $D = 4, N = 2$ dual pairs with similar symmetry enhancement properties. Taking the global limit of this theory, we find $N = 2$ Yang-Mills theories which also have the $SL(2, Z)$ and hence vanishing $\beta$ function.

Prior to this recent surge of interest in a duality between heterotic and Type IIA strings [51], [28], [104, 99, 19], however, it was conjectured, on the basis of $D = 10$ heterotic string/fivebrane duality [53, 92] discussed in section 10 below, that in $D \leq 6$ dimensions there might exist a duality between one heterotic string and another [23, 79, 50, 31, 34, 33, 32]. In particular, if one could find a compactification to $D = 6$ for which the heterotic string was dual to itself, then this would automatically guarantee: (1) the quantum consistency of the dual string [31, 32], (2) an exact symmetry between the spacetime and worldsheet loop expansions [79, 34, 33, 32], (3) that on further $T^2$ compactification the resulting $D = 4, N = 2$ theory would exhibit an exact symmetry under interchange of the dilaton $S$ and the complex Kahler form $T$, and hence (4) enhanced (non-perturbative) gauge symmetries for special values of $S$ in addition to enhanced (perturbative) gauge symmetries for special values of $T$ [32] as described in section 8; a phenomenon that does not occur in the $N = 4$ theories.

The comparative lack of interest in heterotic/heterotic duality is presumably due to the lack of a convincing compactification. The example of [31], where the $D = 10$ $SO(32)$ heterotic
string compactified to $D = 6$ on $K3$ was conjectured to be dual to the $D = 10$ heterotic fivebrane wrapped around $K3$, seemed like a possible candidate, but encountered problems with the wrong sign for some of the gauge kinetic terms. In [21] this idea is re-examined but instead one considers the $E_8 \times E_8$ string with symmetric embedding of the anomaly in the two $E_8$’s. It is conjectured that this heterotic string is indeed dual to itself and that the resulting $D = 4, N = 2$ theory has the above $S - T$ interchange symmetry. This self-duality of the heterotic string in $D = 6$ does not rule out the possibility that in $D = 4$ it is also dual to a Type II string. In fact, as discussed in [17], when the gauge group is completely Higgsed, an obvious candidate is provided by the Type $IIA$ string compactified on a Calabi-Yau manifold with hodge numbers $h_{11} = 3$ and $h_{21} = 243$, since this has the same massless field content $(M, N) = (244, 4)$ where $M = h_{21} + 1$ counts the number of hypermultiplets and $N = h_{11} + 1$ counts the vectors (including the graviphoton). Such a manifold does indeed exist and is given by the degree 24 hypersurface in $WP^{4,1,1,2,8,12}$ which has recently been studied in [112, 46].

More recently, attention has turned to dual $N = 2$ and $N = 1$ dual string pairs in $D = 4$ whose global limit would correspond to the effective duality mentioned in the Introduction [12, 15, 17, 18, 14, 13, 74, 77, 107, 106]. We will not go into the details here but simply note that although the field theory duality may be only effective, the underlying string/string duality is exact. Interestingly enough, all these constructions seem to involve the ubiquitous $K3$ in one way or another.

10 Web of interconnections

The 1984 superstring revolution [23] answered many puzzles about the existence of a consistent perturbatively finite quantum theory of gravity and its unification with the other forces. However, it also raised some new and important questions:

i) The maximum spacetime dimension permitting a classically consistent supersymmetric field theory is $D = 11$ [103, 106]. Yet superstrings demand $D \leq 10$. If superstrings really are the theory of everything, does that mean that all the previous work on $D = 11$ Kaluza-Klein supergravity [26] (including its compactification on K3 [57, 56]) is of no consequence?

ii) Ideally, one would like the ultimate theory to be unique, yet already in $D = 10$ there were five consistent string theories: Type $IIA$, Type $IIB$, Heterotic $E_8 \times E_8$, Heterotic $SO(32)$ and Type 1 $SO(32)$. Although Heterotic $E_8 \times E_8$ compactified to $D = 4$ on a Calabi-Yau manifold seemed most promising phenomenologically, the vacuum degeneracy problem then raised its ugly head: there were literally billions of such manifolds each yielding different gauge groups, different fermion representations and different numbers of families. How do we choose the right one?

iii) The $D = 11$ question became more acute in 1987 when the $D = 11$ supermembrane was discovered [61, 62] and when it was pointed out [38] that the $(d = 2, D = 10)$ Green-Schwarz action of the Type $IIA$ superstring follows by simultaneous worldvolume/spacetime dimensional reduction of the $(d = 3, D = 11)$ Green-Schwarz action of the supermembrane. Could we really afford to ignore supermembranes and other higher dimensional extended objects? For example, the $d = 6$ worldvolume of the $D = 10$ fivebrane couples to a rank six antisymmetric tensor
potential $\tilde{B}_{MNPQRS}$ just as the $d = 2$ worldsheet of the string couples to the rank two potential $B_{MN}$. Since the $H_3 = dB_2$ version of supergravity corresponds to the field theory limit of a superstring, it was conjectured \[55\] that there exists a fivebrane which may be regarded as fundamental in its own right and whose field theory limit is the dual $H_7 = d\tilde{B}_6$ version of supergravity. Further evidence was provided by the discovery of fivebrane soliton solutions of string theory \[22, 33, 30, 41\] and evidence of a strong/weak coupling duality between the string and the fivebrane. Indeed it was this $D = 10$ string/fivebrane duality conjecture that inspired the $D = 6$ string/string duality conjecture of section (4).

Recent events have dramatically thrust these issues back into the limelight:

i) In a startling paper, Witten \[28\] has shown that $D = 11$ supergravity emerges as the strong coupling limit of the Type IIA superstring and has conjectured a whole web of interconnections between Type IIA and the remaining four string theories. For example, the strong coupling limit of the $SO(32)$ heterotic string is the $SO(32)$ Type I string. Further evidence for this conjecture came from the discovery \[22, 21\] that one is the soliton of the other. The strong coupling limit of the $D = 7$ heterotic string is $D = 11$ supergravity on $K3$, in accordance with the “coincidence” of the moduli in (34). The strong coupling limit of the $D = 6$ heterotic string is the Type IIA string on $K3$. This is subsumed by the string/string duality example of section (7). The strong coupling limit of the $D = 5$ heterotic string is the Type IIB string on $K3$. Following Witten’s paper \[28\] it was furthermore proposed \[100\] that the combination of perturbative and non-perturbative states of the $D = 10$ Type IIA string could be assembled into $D = 11$ supermultiplets.

ii) Townsend \[94\] has suggested that the $D = 10$ Type IIA superstring should be identified with the $D = 11$ supermembrane compactified on $S^1$, with the charged extreme black holes of the former interpreted as the Kaluza-Klein modes of the latter.

iii) Hull and Townsend showed that the (extreme electric and magnetic black hole \[25, 81\]) Bogomol’nyi spectrum necessary for the $E_7 U$-duality of the $D = 10$ Type IIA string compactified to $D = 4$ on $T^6$ can be given an explanation in terms of the wrapping of either the elementary $D = 11$ supermembrane solution \[87\] or the $D = 11$ solitonic superfivebrane solution \[117\] around the extra dimensions \[51\].

iv) Strominger \[38\] has shown that the R-R black holes of the Type II strings can become massless and in doing so resolve the so-called conifold singularities in the moduli space of Calabi-Yau vacua. In the Type IIA theory, these black holes are nothing but membranes which wrap around two-surfaces of the Calabi-Yau manifold. In the Type IIB theory, they are threebranes which wrap around three-surfaces. Moreover, Greene, Morrison and Strominger \[39\] then showed that such black-hole condensation signals a smooth transition to a new Calabi-Yau space with different topology. Thus string theory unifies the moduli space of many or possibly all Calabi-Yau vacua!

v) The conjectured equivalence of the $D = 10$ heterotic string compactified on $T^4$ and the $D = 10$ Type IIA string compactified on $K3$ \[51, 28\], combined with the above conjectures implies that the $d = 2$ worldsheet action of the $D = 6$ ($D = 7$) heterotic string may be obtained by $K3$ compactification of the $d = 6$ worldvolume action of the $D = 10$ Type IIA fivebrane ($D = 11$ fivebrane) \[27, 13\].

vi) Putting all this together, one may thus conjecture that membrane/fivebrane duality
in $D=11$ implies Type IIA string/Type IIA fivebrane duality in $D=10$, which in turn implies Type IIA string/heterotic string duality in $D=6$. To test this conjecture, Duff, Liu and Minasian [96] correctly reproduced the corrections to the 3-form field equations of the $D=10$ Type IIA string (a mixture of tree-level and one-loop effects [52]) starting from the Chern-Simons corrections to the 7-form Bianchi identities of the $D=11$ fivebrane (a purely tree-level effect). $K3$ compactification of the latter then yields the familiar gauge and Lorentz Chern-Simons corrections to 3-form Bianchi identities of the heterotic string.

Further evidence for the importance of $D=11$ dimensions, supermembranes and super-fivebranes in superstring theory continues to appear daily on the internet: Schwarz [110], and independently Aspinwall [103], have identified the $SL(2,\mathbb{Z})$ of the Type IIB string with the modular group of the torus appearing in the compactification of $D=11$ supergravity down to $D=9$. Cadavid et al [41], Papadopoulos and Townsend [40], Schwarz and Sen [83], Harvey, Lowe and Strominger [113], Chaudhuri and Lowe [111], Acharya [108] and Aspinwall [103] have noted that $N=1, D=4$ heterotic strings can be dual to $D=11$ supergravity compactified on seven-dimensional spaces of $G_2$ holonomy which also yield $N=1$ in $D=4$ [27, 26]. Becker, Becker and Strominger [13] have shown that membranes and fivebranes of the Type IIA theory, obtained from the $D=11$ supermembrane by compactification on $S^1$, yield $e^{-1/g_s}$ effects, where $g_s$ is the string coupling. Polchinski [121] has proposed that the Type II $p$-branes carrying Ramond-Ramond charges can be given an exact conformal field theory description via open strings with Dirichlet boundary conditions, thus heralding the era of D-branes.

It has even been shown by Horava and Witten [122] that the $E_8 \times E_8$ heterotic string in $D=10$ may be obtained by compactifying the $D=11$ theory on $S^1/Z_2$ just as the Type IIA string may be obtained from $S^1$. As shown by Duff, Minasian and Witten [20], the $D=6$ heterotic/heterotic duality discussed in the previous section may then be deduced by looking in two different ways at the $D=11$ theory compactified on $K3 \times S^1/Z_2$, just as heterotic/Type II duality may be deduced by looking in two different ways at the $D=11$ theory compactified on $K3 \times S^1$ [119].

The rehabilitation of $D=11$ and the recognition of the importance of supermembranes should come as no surprise to those who believe in supersymmetry: what is not forbidden must be allowed. The picture that seems to be emerging from all this, however, is that the underlying theory is very different from the traditional theory of superstrings. It is as though the theory has some enormous moduli space: in one corner of moduli space it looks like a Type I $SO(32)$ string, in another corner like an $E_8 \times E_8$ heterotic fivebrane, in another like a $D=11$ supermembrane and so on. In his book Infinite in All Directions, Freeman Dyson [123] divides theoretical physicists into unifiers and diversifiers. The current developments in duality might be described as unification via diversification!

11 Acknowledgements

I have enjoyed useful conversations with the participants of the Aspen workshop on duality, July 1995, and would like to thank the organizers and the Aspen Center for their hospitality. Thanks are also due Ramzi Khuri, Jim Liu, Jian Xin Lu, Ruben Minasian, Joachim Rahmfeld
References

[1] P. Goddard, J. Nuyts and D. Olive, *Gauge theories and magnetic charge*, Nucl.Phys. B 125 (1977) 1.

[2] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. B 72 (1977) 117.

[3] D. Olive and E. Witten, *Supersymmetry algebras that include topological charges*, Phys. Lett. B 78 (1978) 97.

[4] H. Osborn, *Topological charges for N = 4 supersymmetric gauge theories and monopoles of spin 1*, Phys. Lett. B 83 (1979) 321.

[5] J. Cardy and E. Rabinovici, *Phase structure of Z(P) models in the presence of a theta parameter*, Nucl. Phys. B 205 (1982) 1.

[6] J. Cardy, *Duality and the theta parameter in abelian lattice models*, Nucl. Phys. B 205 (1982) 17.

[7] A. Shapere and F. Wilczek, *Selfdual models with theta terms*, Nucl. Phys. B 320 (1989) 669.

[8] A. Sen, *Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space and SL(2,Z) invariance in string theory*, TIFR/TH/94-08, hep-th/9402032.

[9] L. Girardello, A. Giveon, M. Porratti, and A. Zaffaroni, *S-duality in N = 4 Yang-Mills theories with general gauge groups* Nucl. Phys. B 448 (1995) 127.

[10] C. Vafa and E. Witten, *A strong coupling test of S-duality*, HUTP-94/A017, IASSNS-HEP-94-54, hep-th/9408074.

[11] N. Seiberg and E. Witten, *Electric/magnetic duality, monopole condensation and confinement in N=2 supersymmetric Yang-Mills theory*, RU-94-52, IAS-94-43, hep-th/9407087.

[12] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, RU-94-60, IASSNS-HEP-94-55, hep-th/9408099.

[13] E. Witten, *Monopoles and four-manifolds*, hep-th/9411102.

[14] M. Cederwall, G. Ferretti, B. E. W. Nilsson and Per Salomonson, *Low energy dynamics of monopoles in N = 2 SYM with matter*, Goteborg-ITP-95-16 hep-th/9508124.

[15] J. P. Gauntlett and J. A. Harvey, *S-duality and the dyon spectrum in N = 2 super Yang-Mills theory* CALT-68-2017, EFI-95-56, hep-th/9508156.
[16] S. Sethi, M. Stern and E. Zaslow, *Monopole and dyon bound states in $N = 2$ supersymmetric Yang-Mills* theories, HUTP-95/A031, DUK-M-95-12, hep-th/9508117.

[17] N. Seiberg, *The power of duality–exact results in 4D susy field theory*, IASSNS-HEP-95/46, hep-th/9506077.

[18] N. Seiberg, *Electric-magnetic duality in supersymmetric non-abelian gauge theories*, RU-94-82, IASSNS-HEP-94/98, hep-th/9411143.

[19] M. J. Duff, J. T. Liu and J. Rahmfeld, *Four-dimensional string/string/string triality*, CTP-TAMU-27/95, hep-th/9508094.

[20] M. J. Duff, R. Minasian and E. Witten, *Evidence for heterotic/heterotic duality*, CTP-TAMU-54/95, hep-th/9601036.

[21] C. M. Hull, *String/string duality in ten dimensions*, QMW-95-25, hep-th/9506124.

[22] A. Dabholkar, *Ten dimensional heterotic string as a soliton*, CALT-68-2002, hep-th/9506160.

[23] J. Dai, R. Leigh and J. Polchinski, *New connections between string theories* Mod. Phys. Lett. A4 (1989) 2073-2083.

[24] M. Dine, P. Huet and N. Seiberg, *Large and small radius in string theory* Nucl. Phys. B322 (1989) 2073.

[25] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, 1987).

[26] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. 130 (1986) 1.

[27] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *$N = 8$ supergravity breaks down to $N = 1$*, Phys. Rev. Lett. 50 (1983) 294.

[28] E. Witten, *String theory dynamics in various dimensions*, IASSNS-HEP-95-18, hep-th/9503124.

[29] M. J. Duff and J. X. Lu, *Loop expansions and string/five-brane duality*, Nucl. Phys. B 357 (1991) 534.

[30] M. J. Duff and J. X. Lu, *Black and super $p$-branes in diverse dimensions*, Nucl. Phys. B 416 (1994) 301.

[31] M. J. Duff and R. Minasian, *Putting string/string duality to the test*, Nucl. Phys. B 436 (1995) 507.

[32] M. J. Duff, *Strong/weak coupling duality from the dual string*, Nucl. Phys. B 442 (1995) 47.
[33] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rep. 259 (1995) 213.

[34] M. J. Duff, *Classical/quantum duality*, in *Proceedings of the International High Energy Physics Conference*, Glasgow, July 1994 (Eds. P. J. Bussey and I. G. Knowles, I. O. P.,1995).

[35] K. Behrdt, *About a class of exact string backgrounds*, HUB-EP-95/6, [hep-th/9506106](http://arxiv.org/abs/hep-th/9506106).

[36] R. Kallosh and A. Linde, *Exact supersymmetric massive and massless black holes*, SUTP-95-14, [hep-th/9507022](http://arxiv.org/abs/hep-th/9507022).

[37] G. Lopez Cardoso, D. Lust and T. Mohaupt, *Non-perturbative monodromies in N=2 heterotic string vacua*, [hep-th/9504090](http://arxiv.org/abs/hep-th/9504090).

[38] A. Strominger, *Massless black hole and conifolds in string theory*, [hep-th/9504090](http://arxiv.org/abs/hep-th/9504090).

[39] B. R. Greene, D. R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, CLNS-95/1335, [hep-th/9504145](http://arxiv.org/abs/hep-th/9504145).

[40] G. Papadopoulos and P. K. Townsend, *Compactification of D=11 Supergravity on spaces with exceptional holonomy*, R/95/31, [hep-th/9506150](http://arxiv.org/abs/hep-th/9506150).

[41] A. C. Cadavid, A. Ceresole, R. D’Auria and S. Ferrara, *11-dimensional supergravity compactified on Calabi-Yau threefolds*, CERN-TH/95-166, POLFIS-TH.08/95, UCLA/95/TEP/23, [hep-th/9506144](http://arxiv.org/abs/hep-th/9506144).

[42] M. Billo, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre, T. Regge, P. Soriani and A. Van Proeyen, *A search for non-perturbative dualities of local N = 2 Yang-Mills theories from Calabi-Yau threefolds*, SISSA 64/95/EP, POLFIS-TH 07/95, CERN-TH/95-140, UCLA/95/TEP/19, IFUM 508FT, KUL-TF-95/18, [hep-th/9506144](http://arxiv.org/abs/hep-th/9506144).

[43] V. Kaplunovsky, J. Louis and S. Thiesen, *Aspects of duality in N=2 string vacua*, LMU-TPW 95-9, UTTG-12-95, [hep-th/9506110](http://arxiv.org/abs/hep-th/9506110).

[44] M. J. Duff, S. Ferrara, R. R. Khuri and J. Rahmfeld, *Supersymmetry and dual string solitons*, Phys. Lett. B 356 (1995) 479.

[45] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, EFI-95-26, HUTP-95/A019, CERN-TH 95/131, UCLA/95/TEP/17, [hep-th/9505162](http://arxiv.org/abs/hep-th/9505162).

[46] A. Klemm, W. Lerche and P. Mayr, *K3-fibrations and heterotic-Type-II duality*, CERN-TH/95-165, [hep-th/9506112](http://arxiv.org/abs/hep-th/9506112).

[47] S. Kachru and C. Vafa, *Exact results for N=2 compactifications of heterotic strings*, HUTP-95/A016, [hep-th/9505105](http://arxiv.org/abs/hep-th/9505105).

[48] K. Becker, M. Becker and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, NSF-ITP-95-62, [hep-th/9507153](http://arxiv.org/abs/hep-th/9507153).
[49] B. R. Greene, A. Shapere, C. Vafa, and S. T. Yau, *Stringy cosmic strings*, Nucl. Phys. B 340 (1990) 33.

[50] A. Ceresole, R. D’Auria, S. Ferrara and A. van Proyen, *On electromagnetic duality in locally supersymmetric $N = 2$ Yang-Mills Theory*, CERN-TH.7510/94, POLFIS-th.08/94, UCLA 94/TEP/45, KUL-TF-94/44, hep-th/9412200.

[51] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B 438 (1995) 109.

[52] C. Vafa and E. Witten, *A one-loop test of string duality*, HUTP-95-A015, IASSNS-HEP-95-33, hep-th/9505053.

[53] C. Vafa and E. Witten, *Dual string pairs with $N = 1$ and $N = 2$ supersymmetry in four dimensions*, HUTP-95-A028, TIFR/TH/95-41, hep-th/9507050.

[54] A. Sen and C. Vafa, *Dual pairs of Type II string compactification*, HUTP-95-A023, IASSNS-HEP-95-58, hep-th/9507050.

[55] M. J. Duff, *Supermembranes: The first fifteen weeks*, Class. Quantum Grav. 5 (1988) 189.

[56] M. J. Duff and B. E. W. Nilsson, *Four-dimensional string theory from the K3 lattice*, Phys. Lett. B 175 (1986) 417.

[57] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Compactification of $D = 11$ supergravity on $K3 \times T^3$*, Phys. Lett. B 129 (1983) 39.

[58] K. S. Narain, *New heterotic string theories in uncompactified dimensions $< 10$*, Phys. Lett. B 169 (1986) 41.

[59] N. Seiberg, *Observations on the moduli space of superconformal field theories*, Nucl. Phys. B 303 (1988) 286.

[60] P. S. Aspinwall and D. R. Morrison, *String theory on K3 surfaces*, DUK-TH-94-68, IASSNS-HEP-94/23, hep-th/9404151.

[61] E. Bergshoeff, E. Sezgin and P. K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. B 189 (1987) 75.

[62] E. Bergshoeff, E. Sezgin and P. K. Townsend, *Properties of the eleven-dimensional supermembrane theory*, Ann. Phys. 185 (1988) 330.

[63] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, *Superstrings in $D = 10$ from supermembranes in $D = 11$*, Phys. Lett. B 191 (1987) 70.

[64] M. J. Duff and J. X. Lu, *Duality rotations in membrane theory*, Nucl. Phys. B 347 (1990) 394.
[65] M. J. Duff and J. X. Lu, *Duality for strings and membranes*, in Duff, Sezgin and Pope, editors, *Supermembranes and Physics in 2 + 1 Dimensions* (World Scientific, 1990); also published in Arnowitt, Bryan, Duff, Nanopoulos, Pope and Sezgin, editors, *Strings 90* (World Scientific, 1991).

[66] E. Cremmer, J. Scherk and S. Ferrara, *SU(4) invariant supergravity theory*, Phys. Lett. B 74 (1978) 61.

[67] E. Cremmer and B. Julia, *The SO(8) supergravity*, Nucl. Phys. B 159 (1979) 141.

[68] N. Marcus and J. H. Schwarz, *Three-dimensional supergravity theories*, Nucl. Phys. B 228 (1983) 145.

[69] M. J. Duff, *E_8 \times SO(16) symmetry of D = 11 supergravity?*, in Batalin, Isham and Vilkovisky, editors, *Quantum Field Theory and Quantum Statistics, Essays in Honour of E. S. Fradkin* (Adam Hilger, 1986).

[70] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rep. 244 (1994) 77.

[71] A. Font, L. Ibanez, D. Lust and F. Quevedo, *Strong-weak coupling duality and nonperturbative effects in string theory*, Phys. Lett. B 249 (1990) 35.

[72] S.-J. Rey, *The confining phase of superstrings and axionic strings*, Phys. Rev. D 43 (1991) 526.

[73] S. Kalara and D. V. Nanopoulos, *String duality and black holes*, Phys. Lett. B 267 (1991) 343.

[74] A. Sen, *Electric magnetic duality in string theory*, Nucl. Phys. B 404 (1993) 109.

[75] A. Sen, *Quantization of dyon charge and electric magnetic duality in string theory*, Phys. Lett. B 303 (1993) 22.

[76] J. H. Schwarz and A. Sen, *Duality symmetries of 4-D heterotic strings*, Phys. Lett. B 312 (1993) 105.

[77] J. H. Schwarz and A. Sen, *Duality symmetric actions*, Nucl. Phys. B 411 (1994) 35.

[78] P. Binetruy, *Dilaton, moduli and string/five-brane duality as seen from four-dimensions*, Phys. Lett. B 315 (1993) 80.

[79] M. J. Duff and R. R. Khuri, *Four-dimensional string/string duality*, Nucl. Phys. B 411 (1994) 473.

[80] A. Sen, *SL(2, Z) duality and magnetically charged strings*, Mod. Phys. Lett. A 8 (1993) 5079.
[81] M. J. Duff and J. Rahmfeld, Massive string states as extreme black holes, Phys. Lett. B 345 (1995) 441.

[82] A. Sen, Black hole solutions in heterotic string theory on a torus, TIFR-TH-94-47, hep-th/9411187.

[83] J. P. Gauntlett and J. A. Harvey, S-duality and the spectrum of magnetic monopoles in heterotic string theory, EFI-94-11, hep-th/9407111.

[84] E. Witten, Some Comments on String Dynamics, IASSNS-HEP-95-63, hep-th/9507121.

[85] J. H. Schwarz and A. Sen, Type IIA dual of six-dimensional CHL compactifications, CALT-68-2007, hep-th/9507027.

[86] M. J. Duff and J. X. Lu, The selfdual Type IIB superthreebrane, Phys. Lett. B 273 (1991) 409.

[87] M. J. Duff and K. S. Stelle, Multimembrane solutions of $D = 11$ supergravity, Phys. Lett. B 253 (1991) 113.

[88] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz-Ruiz, Superstrings and solitons, Nucl. Phys. B 340 (1990) 33.

[89] M. J. Duff and J. X. Lu, Elementary fivebrane solutions of $D = 10$ supergravity, Nucl. Phys. B 354 (1991) 141.

[90] C. G. Callan, J. A. Harvey and A. Strominger, World sheet approach to heterotic instantons and solitons, Nucl. Phys. B 359 (1991) 611.

[91] C. G. Callan, J. A. Harvey and A. Strominger, Worldbrane actions for string solitons, Nucl. Phys. B 367 (1991) 60.

[92] A. Strominger, Heterotic solitons, Nucl. Phys. B 343 (1990) 167; E: Nucl. Phys. B 353 (1991) 565.

[93] G. T. Horowitz and A. Strominger, Black strings and $p$-branes, Nucl. Phys. B 360 (1991) 197.

[94] P. K. Townsend, The eleven-dimensional supermembrane revisited, hep-th/9501068.

[95] M. J. Duff, R.R Khuri, R. Minasian and J. Rahmfeld, New black hole, string and membrane solutions of the four dimensional heterotic string, Nucl. Phys. B 418 (1994) 195.

[96] M. J. Duff, J.T. Liu, R. Minasian, Eleven dimensional origin of string-string duality: a one-loop test, CTP-TAMU-26/95, hep-th/9506126.

[97] P. K. Townsend, String-membrane duality in seven dimensions, DAMTP-R/95/12, hep-th/9504095.
[98] P. K. Townsend, A new anomaly-free chiral supergravity theory from compactification on K3, Phys. Lett. B139 (1984) 283.

[99] J. A. Harvey and A. Strominger, The heterotic string is a soliton, EFI-95-16, hep-th/9504047.

[100] I. Bars, Stringy evidence for D = 11 structure in strongly coupled Type II-A superstring, USC-95/HEP-B2, hep-th/9503228.

[101] P. S. Aspinwall and D. R. Morrison, U-duality and integral structures, CLNS-95/1334, hep-th/9505025.

[102] P. S. Aspinwall, Enhanced gauge symmetries and K3 surfaces, CLNS-95/1334, hep-th/9505025.

[103] P. S. Aspinwall, Some relationships between dualities in string theory, CLNS-95/1359, hep-th/9508154.

[104] A. Sen, String string duality conjecture in six dimensions and charged solitonic strings, TIFR-TH-95-16, hep-th/9504027 (April 1995).

[105] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B 135 (1978) 409.

[106] E. Cremmer, B. Julia and J. Scherk, Supergravity in theory in 11 dimensions, Phys. Lett. 76B (1978) 409.

[107] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, N = 2 Type II-heterotic duality and higher derivative F-terms, IC/95/177-NUB-3122, CPTH-RR368.0795, hep-th/9507115.

[108] B. S. Acharya, N = 1 Heterotic-supergauge duality and Joyce manifolds, QMW/th-95-28.

[109] M. Cvetic and D. Youm, BPS Saturated and non-extreme states in abelian Kauza-Klein theory and effective N = 4 supersymmetric string vacua, UPR-675-T, NSF-ITP-74.

[110] J. H. Schwarz, An SL(2, Z) multiplet of Type IIB superstrings, CALT-68-2013, hep-th/9508143.

[111] S. Chaudhuri and D. A. Lowe, Type IIA-Heterotic duals with maximal supersymmetry, NSF-ITP-95-76, UCSBTH-95-23, hep-th/9508144.

[112] S. Hosono, A. Klemm, S. Thiesen and S. T. Yau, Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces, Comm. Math. Phys. 167 (1995) 301, hep-th/93008122.

[113] N. Kaloper, Hidden finite symmetries in string theory and duality of dualities, McGill 95-42 hep-th/9508132.
[114] L. Girardello, M. Porrati and A. Zaffaroni, *Heterotic-Type II string duality and the H-monopole problem*, CERN-TH/95-217, NYU-TH-95/07/02, IFUM/515/FT, *hep-th/9508056*.

[115] J. A. Harvey, D. A. Lowe and A. Strominger, *N = 1 string duality*, EFI-95-46, UCSBTH-95-22, *hep-th/9507168*.

[116] R. R. Khuri and R. C. Myers, *Dynamics of Extreme Black Holes and Massive String States*, McGill/95-38, CERN-TH/95-213, *hep-th/9508045*.

[117] R. Guven, *Black p-brane solutions of d = 11 supergravity theory*, Phys. Lett. B 276 (1992) 49.

[118] R. R. Khuri, *A heterotic multimonopole solution*, Nucl. Phys. B387 (1992) 315.

[119] J. Gauntlett, J. H. Harvey and J. Liu, *Magnetic monopoles in string theory*, Nucl. Phys. B 409 (1993) 363.

[120] A. Sen, *Strong-weak coupling duality in three-dimensional string theory* Nucl. Phys. B228 (1995) 179.

[121] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *hep-th/9510017*.

[122] P. Horava and E. Witten, *Heterotic and Type I string dynamics from eleven dimensions*, IASSNS-HEP-95-86, PUPT-1571, *hep-th/9510209*.

[123] F. J. Dyson, *Infinite in All Directions: Gifford lectures given at Aberdeen, Scotland* (Harper & Row, New York, 1988).