 Cádlág curves of SLE driven by Lévy processes

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Abstract

Schramm Loewner Evolutions (SLE) are random increasing hulls defined through the Loewner equation driven by Brownian motion. It is known that the increasing hulls are generated by continuous curves. When the driving process is of the form \( \sqrt{\kappa}B + \theta^{1/\alpha}S \) for a Brownian motion \( B \) and a symmetric \( \alpha \)-stable process \( S \) with \( \kappa \) not equal to 4 and 8, we prove that the corresponding increasing hulls are generated by cádlág curves.

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1 Introduction

Let \((B_t)_{t \geq 0}\) be a standard Brownian motion and \( \mathbb{H} = \{ x + iy : y > 0 \} \). The chordal SLE is a class of random increasing hulls \((K_t)_{t \geq 0}\) in the closed upper half plane \( \mathbb{H} \), which are defined through the following stochastic Loewner equation

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \quad z \in \mathbb{H},
\]

with \( U_t = \sqrt{\kappa}B_t \) for some constant \( \kappa \geq 0 \). For \( z \in \mathbb{H} \), function \( g_t(z) \) in (1.1) can be solved when \( t \in [0, \zeta(z)] \), where \( \zeta(z) = \sup \{ t \geq 0 : g_s(z) - U_s \neq 0, \ 0 \leq s \leq t \} \) for \( z \in \mathbb{H} \). For each \( t \geq 0 \), \( K_t \) is defined by the set of ‘broken points’ before time \( t \), i.e., \( K_t = \{ z \in \mathbb{H} : \zeta(z) \leq t \} \). In Rohde, Schramm [13], Lawler, Schramm, Werner [9] for \( \kappa = 8 \), it is proved that the increasing hulls are generated by continuous curves \((\gamma(t))_{t \geq 0}\) in \( \mathbb{H} \). For \( \kappa = 2 \) and \( \kappa = 8 \) these continuous curves were originally the motivation of Schramm [17] for introducing SLE as the possible scaling limit of the two dimensional loop-erased random walk and the uniform spanning trees, respectively. By [13] we know that the increasing hulls are identity to the continuous curve for \( 0 \leq \kappa \leq 4 \), and in general \( \mathbb{H} \setminus K_t \) is the unbounded connected component of \( \mathbb{H} \setminus \gamma[0,t] \). Let \( f_t = g_t^{-1} \) and \( \hat{f}_t(z) = f_t(z + i\kappa) \). Almost surely, the curve can be defined by

\[
\gamma(t) = \lim_{y \to 0} \hat{f}_t(iy), \quad \forall t \geq 0.
\]

In [13], it is also proved that \((\gamma_t)\) is a simple curve for \( 0 \leq \kappa \leq 4 \); a self-intersecting path for \( 4 < \kappa < 8 \); and a space-filling curve for \( \kappa \geq 8 \).

The aim of this paper is to study the càdlàg structure of the stochastic Loewner evolution driven by certain Lévy processes. Let \( 0 < \alpha < 2 \) and \((S_t)_{t \geq 0}\) be a standard symmetric \( \alpha \)-stable process, i.e., a Lévy process such that \( \mathbb{E}[\exp\{\lambda S_t\}] = e^{-t|\lambda|^{\alpha}} \) (cf. Bertoin [4] and Sato [16]). In Guan, Winkel [6], for \( \kappa, \theta \geq 0 \), stochastic Loewner equation (1.1) is studied for driving processes

\[
U_t = \sqrt{\kappa}B_t + \theta^{1/\alpha}S_t
\]

on phase transition properties. See also an earlier paper Rushkin, Oikonomou, Kadanoff and Gruzberg [14] on these properties, and their recent paper [15] for some physical motivation of this model such as branching polymers. For \((U_t)\) in (1.3), we use right derivatives in (1.1) and define \( g_t, K_t, f_t, \hat{f}_t, \gamma_t \) similarly as above with \( \zeta(z) = \sup \{ t \geq 0 : g_s(z) - U_s \neq 0, \ g_s(z) - U_{s-} \neq 0 \} \).
0, 0 ≤ s ≤ t} for z ∈ ℍ \ {0}. We may also use these notations for other driving processes or functions after special remarks.

There are some theoretical motivations for the study of the stochastic Loewner equations driven by jump processes. It is easy to see that the jumps of a driven function can shift the increasing points on the boundaries of the hulls and create some tree-like clusters. Hence the jump processes may be used to construct some interesting domains with tree-like boundaries through Loewner’s equation which can not be constructed by a continuous driven function. In fact, some interesting domains in complex analysis have a tree-like boundary. In [6], Beliaev and Smirnov mentioned that SLE driven by Lévy processes may be candidates of fractal domains with high multi-fractal spectrum.

When considering a ‘nice’ càdlàg driven function (càdlàg: right continuous with left limit), we may expect that the increasing hull can be generated by a càdlàg curve. However, compared with a continuous curve, càdlàg curves can be far more complicated than it seems. To reflect their difference, we prove in the last section that a comb space, which is not a locally connected space, can be generated by a càdlàg curve. In complex analysis, locally connected is an important property for the boundary of a domain because it is equivalent to the continuous extension of a conformal map taking this domain as its image (cf. Theorem 2.1 Pommerenke [11]). This shows that we can not prove the continuous extension stated in Theorem 1.1 below as the continuous case in [13].

Due to the jumps of the (symmetric) α-stable process, the corresponding hulls have tree-like structure with infinite branching on the boundary. In [6], inspired by the results of SLE, we conjecture that the increasing hulls (Kt) are generated by càdlàg curves. As the jumps for a typical path of the α-stable process are dense on [0, ∞), this conjecture can not be deduced from the Brownian SLE curve directly. However, a function can still be defined by (1.2) provided the limit exists. Then we can study the property of (γt) and its relation to the increasing hulls. For any function f on [0, t], set f[0, t] = {f(s) : 0 ≤ s ≤ t}. A main result of this paper is the following.

**Theorem 1.1.** Let 0 < α < 2, θ ≥ 0 and κ ∈ (0, 4) ∪ (4, 8) ∪ (8, ∞). Assume that the driving process (Ut) in (1.1) is defined by (1.3). Then, almost surely, the limits in (1.2) are well defined and (γ(t)) is a càdlàg function. Moreover, almost surely, the conformal maps (ft)t≥0 extend to ℍ continuously and (Kt) is generated by (γ(t)), i.e., Ht := ℍ \ Kt is the unbounded connected component of ℍ \ γ[0, t] for t ≥ 0.

The case for κ = 0 in Theorem 1.1 is also given in Chen, Steffen [5] recently by a different method. Additionally, they prove that the Hausdorff dimension of Kt is equal to one. For general κ, the Hausdorff dimension might not depend on θ which we have not studied in this paper.

The rest part of this section are to introduce the methods in the proof of Theorem 1.1 and to prepare some other notations which will be used throughout the paper. We first consider the càdlàg property in Theorem 1.1 for the ‘truncated’ driving process (√κBt + θ1/αSδ,t), where (Sδ,t) is the truncated α-stable process for some δ > 0:

\[ S_{δ,t} := S_t - \sum_{0 < s \leq t} \Delta S_s I_{|\Delta S_s| \geq \delta}, \quad \Delta S_t = S_t - S_{t-}, \quad S_{t-} = \lim_{s \uparrow t} S_s. \]

Before introducing the truncated case in details, we see how to use it to prove Theorem 1.1.

For δ > 0, define by T1 the first time |ΔS| > δ and define by induction Tn+1 = inf{t > Tn : |ΔS_t| > δ} for n ≥ 1. By the càdlàg property of (St), almost surely we have \( \lim_{n \to \infty} T_n = \infty \) and \( T_{n+1} > T_n > 0 \) for n ≥ 1. Before stopping time T1, as processes (Sδ,t) and (St) are the same, the càdlàg property in Theorem 1.1 follows by the truncated case directly. For the time
between $T_n$ and $T_{n+1}$, we need a fact that $f_{T_n}(\cdot)$ can extend to $\mathbb{H}$ continuously (see Lemma 4.5). With the help of this fact one can check by (1.4) that
\[
\gamma_t = f_{T_n}(\beta_{t-T_n}), \quad t \in [T_n, T_{n+1}),
\]
where $(\beta_t)$ is the càdlàg curve defined by the following truncated driving process
\[
U_{T_n} + \sqrt{\kappa} B_{t+T_n} + \theta^{1/\alpha} S_{g,t+T_n} - \sqrt{\kappa} B_{T_n} - \theta^{1/\alpha} S_{\delta,T_n}, \quad t \geq 0.
\]
Thus, due to the continuity of $f_{T_n}$ on $\mathbb{H}$, the càdlàg property of $\gamma_t$ on $[T_n, T_{n+1})$ follows by that property of $\beta_t$. For the continuous extension of $f_{T_n}(\cdot)$, the proof is similar to the continuous extension of $f_t(iy)$ in [13]. Briefly, the $x$ coordinate in $f_{T_n}(x+iy)$ takes the role of the $t$ coordinate in $f_t(iy)$. See Lemma 4.4 and 4.5 for details. We can further prove that the continuous extension of $f_t$ holds for all $t \geq 0$. The proof is based on a right continuity of $\partial K_t$ followed by the càdlàg property of $(\gamma_t)$ (see [1.30]). In general this right continuous property is not true even for a continuous driving function (see a counterexample in Marshall and Rohde [10]).

For the proof of the truncated case, the main steps is similar to the continuous case in Rohde and Schramm [13] and Lawler [7]. However, due to the jumps we need to overcome some different difficulties. In [13], to prove the convergence in (1.2), a key step is to obtain some estimates for the complex derivative of $f_t(\cdot)$, denoted by $f_t'(\cdot)$. By Lemma 3.1 [13], this reduces to the derivative estimates for the backward flow of (1.1). Let $(\tilde{B}_t)_{t \geq 0}$ and $(\tilde{S}_{\delta,t})_{t \geq 0}$ be another Brownian motion and truncated symmetric $\alpha$-stable process, respectively. We assume that $(B_t)_{t \geq 0}$, $(\tilde{S}_{\delta,t})_{t \geq 0}$ and $(\tilde{S}_{\delta,t})_{t \geq 0}$ are independent and define
\[
V_t := \sqrt{\kappa} B_t + \theta^{1/\alpha} S_{\delta,t}, \quad V_{-t} := \sqrt{\kappa} \tilde{B}_t + \theta^{1/\alpha} \tilde{S}_{\delta,t}, \quad \forall \ t \geq 0.
\]
Next we take $(V_t)_{t \in \mathbb{R}}$ as the driving process in (1.1) and extend the solution of (1.1) to the negative times. The notations $g_t, K_t, f_t, \tilde{f}_t$ will be also used in this setting. For $t < 0$, the derivative in (1.1) is defined by the left derivative and $g_t$ is a conformal map from $\mathbb{H}$ onto a subset of $\mathbb{H}$. The following lemma is a straightforward extension of Lemma 3.1 in [13] which can be proved by the stochastic Loewner equation.

**Lemma 1.2.** For driving process $(V_t)$ and for all fixed $t \geq 0$, the map $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - V_t$.

In [13], the estimates of $|g_{-t}(z)|$ for SLE are obtained by moment estimates which in turn are given by constructing martingales. Next we introduce this method in the context of this paper. Let $\tilde{z} = \hat{x} + iy \in \mathbb{H}$ and set for $t \geq 0$
\[
z(t) = (x(t), y(t)) := g_{-t}(\tilde{z}) - V_{-t}, \quad \psi(t) = \frac{\tilde{y}}{y(t)} |g'_{-t}(\tilde{z})|.
\]
Direct calculations show,
\[
dx(t) = \frac{-2ydt}{x^2 + y^2} - dV_{-t}, \quad dy(t) = \frac{2ydt}{x^2 + y^2}, \quad d\log \psi(t) = \frac{-4y^2dt}{(x^2 + y^2)^2}.
\]
In [13], some function $F$ and constant $\mu$ are constructed so that the process
\[
M_t = \psi(t)^\mu F(z(t)), \quad t \geq 0
\]
is a martingale with respect to the natural filtration. By Itô’s formula and (1.6), the drift of the process $(M_t)$ is
\[
\int_0^t \psi(s)^\mu A F(z(s)) ds,
\]
where $\Lambda$ is the following operator

$$
\Lambda := \frac{-4\mu y^2}{(x^2 + y^2)^2} - \frac{2x}{x^2 + y^2} \partial_x + \frac{2y}{x^2 + y^2} \partial_y + \frac{\kappa}{2} \partial^2_x + \theta \Delta_{x/y}^{\alpha/2}.
$$
(1.8)

Here $\Delta_{x/y}^{\alpha/2}$ is the generator of $(S_{\delta,t})$ given in (2.11) below. Thus a martingale $(M_t)$ can be constructed by choosing a harmonic function $F$ of $\Lambda$, i.e., $\Lambda F = 0$. Let $b \in \mathbb{R}$. When $\theta = 0$, it is proved in [13] that function

$$
F(x, y) := (1 + (x/y)^2)^b y^n
$$
(1.9)

is harmonic with respect to $\Lambda$ by setting

$$
\mu := 2b + \kappa b(1 - b)/2, \quad \nu := 4b + \kappa b(1 - 2b)/2.
$$
(1.10)

For general Lévy processes, it may not be possible to find explicit nontrivial harmonic functions. Instead, we look for super-harmonic functions which is enough for our purpose. To this end we consider the function $F$ in (1.9) with some different parameters (see (2.17)). To prove the super harmonic property of $F$, we control $\Delta_{x/y}^{\alpha/2} F$ by $\partial^2_y F$. This is achievable when $0 \leq \kappa < 8$ with $b \in (1/2, 1]$. When $\kappa > 8$, this can be done only for $|x|/y$ sufficiently large. For smaller $|x|/y$, we obtain a weaker estimate of $\Delta_{x/y}^{\alpha/2} F$ which can be controlled by the first term in (1.8) when choosing a bigger $\mu$. See more details in Section 2. For $\kappa = 4$, the estimates in Proposition 2.2 is not strong enough to prove the continuous extension property (cf. Remark 4.1). It is interesting to know whether Theorem 1.1 holds or not for $\kappa = 4, 8$ as the curves in these two cases are in a critical situation.

Another crucial ingredient in proving the SLE curves in [13] is the pathwise increment estimates of the Brownian motion. In [7], the role of this property is distilled into a determined version. For a determined continuous driving function $(u_t)$, a condition in Lemma 4.32 [7] is

$$
sup_{k/2^n < t \leq (k+1)/2^n} |u_t - u_{k/2^n}| \leq c \sqrt{t}/2^n, \quad 0 \leq k \leq 2^n - 1, \quad n \geq 1
$$
(1.11)

for some constant $c > 0$. Due to the jumps, one can not expect a property like (1.11) for the $\alpha$-stable process or the truncated $\alpha$-stable process. In fact, it might not be easy to prove whether the following weak local increment property is true or not:

$$
\sup_{0 \leq t \leq 1} \limsup_{h \downarrow 0} |S_{t+h}(\omega) - S_t(\omega)|/\sqrt{h \ln h} < c(\omega), \quad a.s..
$$
(1.12)

Here, we give another form of regularity for the $\alpha$-stable process which may have independent interest. This includes a kind of uniform increment estimate on each interval $[k/2^n, (k+1)/2^n)$ for a truncated stable process with jumps less than $2^{-\beta n}$, where $\beta > 1$. On the other hand, for $\beta < 2/\alpha$, the number of the jumps bigger than $2^{-\beta n}$ in each interval $[k/2^n, (k+1)/2^n)$ can be controlled by a finite number. See Lemma 3.1 and 3.2 for details. According to these two properties, Lemma 4.3 gives some sufficient conditions on the existence of a càdlàg curve for the Loewner evolution.

The structure of this paper is the following. We give the moments estimates of $\hat{f}'_t$ in Section 2 and prove the path increments properties of the symmetric $\alpha$-stable process in Section 3. In Section 4 we give a determined version for the existence of a càdlàg curve and apply it to prove Theorem 1.1. The proof of Theorem 1.2 and an example are given in the last section.
2 Derivative estimates for the truncated case

Let $0 < \alpha < 2$. The generator of the truncated $\alpha$-stable process $(S_{\delta,t})$ is the truncated fractional Laplacian

$$\Delta_{x|\delta}^{\alpha/2} f(x) := \lim_{\varepsilon \downarrow 0} A(1,-\alpha) \int_{\{y:|y-x|<\varepsilon\}} \frac{f(y) - f(x)}{|x-y|^{1+\alpha}} dy,$$

(2.1)

provided the limit exists, where $A(1,-\alpha) = \alpha 2^{\alpha-1} \pi^{-1/2} \Gamma((1+\alpha)/2)/\Gamma(1-\alpha/2)$ and $f$ is a real function on $\mathbb{R}$ (cf. [6]). The following lemma is to compare operator $\Delta_{x|\delta}^{\alpha/2}$ with Laplacian on some functions which will be used later.

**Lemma 2.1.** Let $0 < \alpha < 2$ and $a > 0$. Set for $b \in (0,1]$ 

$$f(x) = (1 + ax^2)^b.$$ 

Then there exists a positive constant $C_1 = C_1(\alpha, b)$ such that for $c > 0$

$$|\Delta_{x|c}^{\alpha/2} f(x)| < C_1 c^{2-\alpha} f''(x), \quad b \in (1/2, 1], \ x \in \mathbb{R};$$

(2.2)

$$\Delta_{x|c}^{\alpha/2} f(x) < C_1 c^{2-\alpha} |f''(x)|, \quad b \in (0,1/2), \ |x| > 2^2((1 - 2b)a)^{-1/2}, \ \alpha \neq 2b;$$

(2.3)

$$\Delta_{x|c}^{\alpha/2} f(x) < C_1 c^{2-\alpha} (1 + |\log c|) |f''(x)|, \quad b \in (0,1/2), \ |x| > 2^2((1 - 2b)a)^{-1/2}, \ \alpha = 2b;$$

(2.4)

$$|\Delta_{x|c}^{\alpha/2} f(x)| < C_1 ac^{2-\alpha}, \quad b \in (0,1], \ x \in \mathbb{R}.$$ 

(2.5)

**Proof** Direct calculation shows that

$$f''(x) = 2ab(1 + a(2b - 1)x^2)(1 + ax^2)^{b-2}.$$ 

(2.6)

First we prove (2.2). Let $b \in (1/2, 1]$. By (2.6),

$$2ab(2b - 1)(1 + ax^2)^{b-1} \leq f''(x) \leq 2ab(1 + ax^2)^{b-1}.$$ 

(2.7)

By (2.7) we have

$$|\Delta_{x|c}^{\alpha/2} f(x)|$$

$$= A(1,-\alpha) \int_{x-c}^{x+c} f(y) - f(x) - f'(x)(y-x) dy$$

$$= A(1,-\alpha) \left| \int_{x-c}^{x+c} \frac{f(y) - f(x)}{|x-y|^{1+\alpha}} dy \right|$$

$$\leq A(1,-\alpha) \left| \int_{x-c}^{x+c} \frac{f''(u)(y-u)}{|x-y|^{1+\alpha}} dy \right|$$

$$\leq A(1,-\alpha)(2b - 1)^{-1} f''(x) \int_{-1}^{1} \frac{|t|^{-\alpha}}{|t|^{1-a}} \left( \frac{1 + ax^2}{1 + a(x + tu)^2} \right)^{1-b} dt$$

$$\leq 2A(1,-\alpha)(2b - 1)^{-1} (2 - \alpha)^{-1} f''(x) c^{2-\alpha} \sup_{t>0} \int_{-1}^{1} \left( \frac{1 + ax^2}{1 + a(x + tu)^2} \right)^{1-b} du.$$ 

We claim that

$$M := \sup_{t>0, a>0, x\in\mathbb{R}} \int_{-1}^{1} \left( \frac{1 + ax^2}{1 + a(x + tu)^2} \right)^{1-b} du < \infty.$$ 

(2.9)
When $|x| \geq 2t$, we have $M < \int_{-1}^{1} \left( \frac{1}{1 + ax^2} \right)^{1-b} du < 2^{3-2b}$. When $|x| < 2t$ and $\sqrt{1/a} < 2t$, by the assumption that $b \in (1/2, 1]$, we have for $t > 0$

$$\int_{-1}^{1} \left( \frac{1 + ax^2}{1 + ax^2} \right)^{1-b} du = 1 \int_{-t}^{t} \left( \frac{1}{1 + a(x + tu)^2} \right)^{1-b} du < 1 \int_{-t}^{t} 2(2t)^{2-2b} du \leq \frac{2^{3}}{2b - 1}.$$ 

Similarly, we have for $|x| < 2t$ and $\sqrt{1/a} > 2t$

$$\int_{-1}^{1} \left( \frac{1 + ax^2}{1 + ax^2} \right)^{1-b} du \leq 2^{2-b}, \quad t > 0.$$ 

Combing the facts above, we get (2.9) and hence (2.2) is true.

Next we assume that $b \in (0, 1/2)$ and $|x| > 2^2((1 - 2b)a)^{-1/2}$. Assume also that $x > 0$ by symmetry. By (2.6), we have for $|y| \geq 2((1 - 2b)a)^{-1/2}$

$$ab(1 - 2b)(1 + ay^2)^{b-1} \leq |f''(y)| \leq 2ab(1 - 2b)(1 + ay^2)^{b-1}. \quad (2.10)$$

By (2.10) we have for $|y| \leq x/2$

$$|f''(x + y)| \leq 2^{2(2-b)} |f''(x)|. \quad (2.11)$$

Therefore, if in addition that $x > 2c$, we have by (2.8)

$$|\Delta_{x|c}^{\alpha/2} f(x)| \leq A(1, -\alpha)(2 - \alpha)^{-1} 2^{2(3-b)} c^{2-\alpha} |f''(x)|. \quad (2.12)$$

Noticing that for $y \in \{z : |z - x| \geq x/2\}$ we have $|y| \leq 3|x - y|$ and

$$1 + 9a|x - y|^2 \leq (a|x - y|^2) \frac{4}{ax^2} + 9a|x - y|^2 \leq 10a|x - y|^2.$$ 

Hence for $2^2((1 - 2b)a)^{-1/2} < x \leq 2c$, we have by (2.11) and $f > 0$

$$A(1, -\alpha)^{-1} \Delta_{x|c}^{\alpha/2} f(x)$$

$$\leq \int_{x-c}^{x+c} \frac{f(y) - f(x)}{|x - y|^{1+\alpha}} I_{[y - x] \geq x/2} dy + \int_{x-c}^{x+c} \int_{x-y} f''(u)(y - u) du I_{|y - x| < x/2} dy$$

$$\leq \int_{x-c}^{x+c} \left( 1 + 9a|x - y|^2 \right)^b \frac{1}{|x - y|^{1+\alpha}} I_{|y - x| \geq x/2} dy + (2 - \alpha)^{-1} 2^{2(3-b)} c^{2-\alpha} |f''(x)|$$

$$\leq 2(10a)^b \int_{0}^{x+c} \frac{1}{|y - x|^{1+\alpha - 2b}} I_{|y - x| \geq x/2} dy + (2 - \alpha)^{-1} 2^{2(3-b)} c^{2-\alpha} |f''(x)|. \quad (2.13)$$

If we further assume that $\alpha > 2b$, (2.10) and (2.13) give

$$A(1, -\alpha)^{-1} \Delta_{x|c}^{\alpha/2} f(x)$$

$$\leq \frac{2 \cdot 10^b}{\alpha - 2b} 2^{2(3-b)} c^{2-\alpha} |f''(x)|$$

$$\leq c_1 c^{2-\alpha} |f''(x)|$$

for some constant $c_1 = c_1(\alpha, b)$. If $\alpha < 2b$, (2.10) and (2.13) give

$$A(1, -\alpha)^{-1} \Delta_{x|c}^{\alpha/2} f(x)$$
We first prove the case $0 < \kappa < \theta$ for $t \in (0, 1]$. We refer to Applebaum [1] and [4] for the details.

The construction of Lévy processes by Poisson random measures and the Itô’s formula of Lévy processes is similar to (2.3). By (2.6), we see that $|f''(x)| \leq 2ab$ for $b \in (0, 1]$ and hence (2.5) follows by (2.8).

Next we turn to the derivative estimates for the truncated case. Recall that for $t \in \mathbb{R}, V_t$ is defined by (1.4). In this section, we consider the stochastic Loewner equation (1.1) with $(U_t)$ replaced by $(V_t)_{t \in \mathbb{R}}$ and adopt the previous notations $g_t, K_t, f_t, \hat{f}_t$ in this setting. Let $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$. Define $z(t) = (x(t), y(t))$ and $\psi(t)$ for $t \geq 0$ by (1.5). For $u \geq \log \hat{y}$, set

$$T_u = -\sup\{t \in \mathbb{R} : \text{Im}(g_t(z)) \geq e^u\}. \quad (2.16)$$

Notice that $T_u$ is less than infinity almost surely which can be proved by the recurrence and the Markov property of $t(V_t)$. Let $b, \kappa_1, \kappa_2, \alpha' \geq 0$ and define

$$\begin{cases} a := 2b + \kappa_1 b(1 - b)/2, & \lambda := 4b + \kappa_1 b(1 - 2b)/2, \quad 0 \leq \kappa < 8, \quad b \in (1/2, 1]; \\ a := 2b + \kappa_2 b(1 - b)/2 + \alpha', & \lambda := 4b + \kappa_2 b(1 - 2b)/2, \quad \kappa > 8, \quad b \in (0, 1/2]. \end{cases} \quad (2.17)$$

In the following proposition and Section 3, we need some results of Lévy processes, such as the construction of Lévy processes by Poisson random measures and the Itô’s formula of Lévy processes. We refer to Applebaum [1] and [4] for the details.

**Proposition 2.2.** Let $b \in (0, 1/2] \cup (1/2, 1], \kappa_1, \kappa_2, \alpha' \geq 0$ and define $a, \lambda$ by (2.17). Let $0 < \hat{y} < 1$ and set for $u > \log \hat{y}$

$$F(\hat{z}) = \hat{y}^a e^{-au} e^{\lambda u} \mathbb{E}[(1 + x(u)^2/y(u)^2)^b |\hat{f}_{T_u}(\hat{z})|^a]. \quad (2.18)$$

1) If $0 \leq \kappa < 8, \theta \geq 0, b \in (1/2, 1]$ and $\kappa_1 > \kappa$, then there exists a constant $\delta_0 = \delta_0(\theta, \kappa_1 - \kappa, \alpha, b) \in (0, 1)$ such that for any $0 < \delta < \delta_0$,

$$F(\hat{z}) \leq (1 + (x/y)^2)^b \hat{y}^\lambda, \quad 0 < \delta < \delta_0, \quad (2.19)$$

where $\delta$ is the parameter in (1.4).

2) If $\kappa > 8$ and $b \in (0, 1/2)$, then we have the same inequality (2.19) for some constant $\delta_0 = \delta_0(\theta, \kappa - \kappa_2, \alpha, b) \in (0, 1)$ provided $\kappa - \kappa_2 > 0$ small enough.

**Proof** 1) We first prove the case $0 \leq \kappa < 8, b \in (1/2, 1]$. Let $F_1(z) = (1 + (x/y)^2)^b y^\lambda$ and set for $t \geq 0$

$$M_t = \psi(t)^a F_1(z(t)).$$

Let $\tilde{N}(dt, dx)$ be the Poisson random martingale measure on $[0, \infty) \times (-\theta^{1/\alpha} \delta, \theta^{1/\alpha} \delta)$ for the truncated stable process $(\theta^{1/\alpha} \hat{S}_{bt})$. The Poisson intensity function of $\tilde{N}(dt, dx)$ is

$$(\mathcal{A}(1, -\alpha) \theta/|x|^{1+\alpha}) I_{|x| < \theta^{1/\alpha} \delta} I_{t \geq 0}. \quad (2.20)$$

By (1.6) and Itô’s formula (cf. [11]) we have for $t > 0$

$$M_t = M_0 - \sqrt{\kappa} \int_0^t \psi(s)^a \sigma_x F_1(z(s)) \, d\hat{B}_s,$$
\[ -\int_0^t \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \psi(s)^a (F_1(z(s^+) + x) - F_1(z(s^-))) \tilde{N}(ds, dx) + \int_0^t \psi(s)^a \Lambda F_1(z(s)) \, ds, \quad (2.21) \]

where \( \Lambda \) is defined by (1.8) with \( \mu \) replaced by \( a \). Direct calculation shows

\[ H(z) := \Lambda F_1(z) = \frac{\kappa - \kappa_1}{2} \partial_z^2 F_1(z) + \theta \Delta_x^{\alpha/2} F_1(z), \quad y > 0. \quad (2.22) \]

By Lemma 2.1 \( \kappa < \kappa_1 \) and the fact that \( \partial_z^2 F_1(z) > 0 \), we can choose \( \delta_0 = \delta_0(\theta, \kappa_1 - \kappa, \alpha, b) \in (0, 1 \land \theta^{-1/\alpha}) \) small enough such that \( H(z) \leq 0 \) for \( 0 < \delta < \delta_0 \). Thus by (2.21), we have

\[ M_t \leq M_0 - \sqrt{2} \int_0^t M_s \frac{2b\alpha(s)}{x(s)^2 + y(s)^2} \, dB_s \]
\[ -\int_0^t \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} M_s (F_1(z(s^+) + x) - F_1(z(s^-))) F_1(z(s))^{-1} \tilde{N}(ds, dx). \quad (2.23) \]

Define the right hand side of (2.23) by \( \tilde{M}_t \) which is a local martingale. Let \( u = u(\tilde{z}, t) := \log|\text{Im}g-t(\tilde{z})| \). By (1.1), \( \partial_t u = 2(x(t)^2 + y(t)^2)^{-1} = 2(\tilde{x}(u)^2 + \tilde{y}(u)^2)^{-1} \). For \( M > 0 \), set \( T = \inf\{t \geq 0 : |M_t| > M\} \). By \( b > 1/2 \) and the mean value principle, for \( |x| < 1 \) we have

\[ |((\tilde{x}(s) + x)^2 + \tilde{y}(s)^2) - (\tilde{x}(s) + \tilde{y}(s)^2)| \leq 2b|x|(|\tilde{x}(s)| + 1)^2 + \tilde{y}(s)^2|^{b-1/2}. \]

Hence by (2.20) and Itô's isometry, we have for \( 0 < \delta < \delta_0 \)

\[
\mathbb{E} \tilde{M}^2_{T_u \wedge T} \leq 3M_0^2 + 3\kappa \mathbb{E} \left[ \int_0^{T_u \wedge T} M_s^2 \frac{(2bx(s))^2}{(x(s)^2 + y(s)^2)^2} \, ds \right] \\
+ 3\mathcal{A}(1, -\alpha) \theta \mathbb{E} \left[ \int_0^{T_u \wedge T} M_s^2 \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \frac{(F_1(z(s^+) + x) - F_1(z(s^-))) F_1(z(s))^{-2}}{|x|^{1+\alpha}} \, dxds \right] \\
\leq 3M_0^2 + 3\kappa \mathbb{E} \left[ \int_0^u M_s^2 \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \frac{2b^2 \tilde{x}(s)^2}{x(s)^2 + y(s)^2} \, ds \right] + 3\frac{1}{2} \mathcal{A}(1, -\alpha) \theta. \\
\mathbb{E} \left[ \int_0^u M_s^2 \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \frac{((\tilde{x}(s) + x)^2 + \tilde{y}(s)^2)^b - (\tilde{x}(s)^2 + \tilde{y}(s)^2)^b}{|x|^{1+\alpha}} \, dxds \right] \\
\leq 3M_0^2 + 6b^2 \kappa \mathbb{E} \left[ \int_0^u M_s^2 \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \frac{d(\tilde{x}(s) + 1)^2 + \tilde{y}(s)^2)^b}{|x|^{b-1}} \, dxds \right] + 6\mathcal{A}(1, -\alpha) b^2 \theta. \\
\mathbb{E} \left[ \int_0^u M_s^2 \int_{-\theta^{1/\alpha} \delta}^{\theta^{1/\alpha} \delta} \frac{(|\tilde{x}(s)| + 1)^2 + \tilde{y}(s)^2)^b}{|x|^{b-1}} \, dxds \right]. \quad (2.24) \]

Noticing that \( \theta^{1/\alpha} < 1 \) and considering \(|\tilde{x}(s)| < 1 \) and \(|\tilde{x}(s)| \geq 1 \) respectively, the term on line of (2.24) is less than

\[
\frac{12b^2 \theta \mathcal{A}(1, -\alpha)}{2 - \alpha} \left( (4/\tilde{y}^2 + 1)^{2b-1} + 2b^2 \right) \mathbb{E} \left[ \int_0^u M_s^2 \, ds \right] \leq c \mathbb{E} \left[ \int_0^u \tilde{M}^2_{T_u \wedge T} \, ds \right] \quad (2.25) \]

for some constant \( c = c(\kappa, \theta, b, \alpha, \tilde{y}) \). By (2.24) and (2.25), we have for \( u \geq \log \tilde{y} \)

\[ \mathbb{E} \tilde{M}^2_{T_u \wedge T} \leq 3M_0^2 + c \mathbb{E} \left[ \int_0^u \tilde{M}^2_{T_u \wedge T} \, ds \right]. \quad (2.26) \]
Hence by Gronwall’s Lemma (cf. [12]),
\[ \mathbb{E}\left[ \tilde{M}_{T_u \wedge T}^2 \right] \leq 3M_0^2 e^{c(u-\log \hat{y})}, \]
which gives the finiteness of \( \mathbb{E}\left[ \tilde{M}_{T_u}^2 \right] \) by taking \( M \to \infty \). This proves that \( \tilde{M}_{T_u} \) is a martingale. By optional stopping theorem and (2.23) we complete the proof of this case.

2) Assume that \( \kappa > 8, b \in (0, 1/2) \) and \( \alpha' > 0 \). We only prove the case \( \alpha = 2b \) as the proof for the case \( \alpha = 2b \) is similar. By (2.3), (2.5) and (2.6) we have for \( \theta^{1/\alpha} \delta < 1 \)
\[
\Lambda F_1(z) = -\frac{4\alpha'y^2}{(x^2+y^2)^2}(1+(x/y)^2)b\lambda + \frac{\kappa - \kappa_2}{2} \partial_x^2 F_1(z) + \theta \Delta_{x/\theta^{1/\alpha}, \delta} F_1(z)
\]
\[
\leq -a'(\frac{2y^2}{x^2+y^2})^2(1+(x/y)^2)b\lambda^2 + b(\kappa - \kappa_2) (1+ (2b-1)(x/y)^2) (1+(x/y)^2)^{b-2}y^2 \lambda^2
\]
\[
+ C_1 \theta (\theta^{1/\alpha} \delta)^2 -\alpha |1 + (2b-1)(x/y)^2| (1+(x/y)^2)^{b-2}y^2 \lambda^2 I_{|x| \geq 2^2(1-2b)^{-1/2}y}
\]
\[
+ C_1 \theta (\theta^{1/\alpha} \delta)^2 -\alpha y^{2-2} \lambda^2 I_{|x| \leq 2^2(1-2b)^{-1/2}y}.
\] (2.27)

We claim that there exist \( \delta_0 \in (0, 1 \wedge \theta^{-1/\alpha}) \) and \( \kappa - \kappa_2 > 0 \) small enough such that \( \Lambda F_1(z) \leq 0 \) for \( \delta \in (0, \delta_0) \). By (2.27) and \( \alpha' > 0 \), we can first choose \( \delta \) and \( \kappa - \kappa_2 \) small enough so that \( \Lambda F_1(z) \) is negative for \( |x| \leq 2^2(1-2b)^{-1/2}y \). By \( \kappa > \kappa_2 \) and \( b \in (0, 1/2) \), we can further decrease \( \delta \) so that \( \Lambda F_1(z) \) is negative for \( |x| > 2^2(1-2b)^{-1/2}y \).

Next we prove that \( \tilde{M}_{T_u} \) defined above is still a martingale in this case. By \( 2b-1 < 0 \), we have for \( |\bar{x}(s)| < 2 \)
\[
N_1 := \sup_{|x|<1} (|\bar{x}(s) + x|^2 + \bar{y}(s)^2)^{2b-1} (\bar{x}(s)^2 + \bar{y}(s)^2)^{1-2b} \leq (1 + 2^2 \bar{y}^{-2})^{1-2b},
\]
and for \( |\bar{x}(s)| \geq 2 \)
\[
N_2 := \sup_{|x|<1} (|\bar{x}(s) + x|^2 + \bar{y}(s)^2)^{2b-1} (\bar{x}(s)^2 + \bar{y}(s)^2)^{1-2b} \leq 2^{4b}.
\]
By these two estimates, (2.21) is replaced by
\[
\mathbb{E}\tilde{M}_{T_u \wedge T}^2 \leq 2M_0^2 + (2b)^2 \kappa \mathbb{E}\left[ \int_{\log \hat{y}}^u M_{T_u \wedge T}^2 \ ds \right] + A(1, -\alpha)(2b)^2 \theta.
\]
\[
\mathbb{E}\left[ \int_{\log \hat{y}}^u M_{T_u \wedge T}^2 \ ds \right] \sup_{|x| < \theta^{1/\alpha}, \delta} \mathbb{E}\left[ \int_{\theta^{1/\alpha}, \delta} |\bar{x}(s) + x|^2 + \bar{y}(s)^2)^{2b-1} (\bar{x}(s)^2 + \bar{y}(s)^2)^{1-2b} \right] dxds
\]
\[
\leq 2M_0^2 + (2b)^2 \kappa \mathbb{E}\left[ \int_{\log \hat{y}}^u M_{T_u \wedge T}^2 \ ds \right] + 2A(1, -\alpha)(2b)^2 \theta (N_1 + N_2) (2-\alpha)^{-1} \mathbb{E}\left[ \int_{\log \hat{y}}^u M_{T_u \wedge T}^2 \ ds \right],
\]
which gives the martingale property of \( (\tilde{M}_{T_u \wedge T})_{u \geq \log \hat{y}} \). Therefore we can prove the conclusion by following the arguments in case 1. □

3 Path increments of the symmetric \( \alpha \)-stable process

For any càdlàg function \( (u_t)_{t \geq 0} \), denote \( u_{t\downarrow} = \lim_{s \downarrow t} u_s \) and \( \Delta u_t = u_t - u_{t\downarrow} \). For \( \delta > 0 \), write \( S_{\delta, t} = S_t - \sum_{0 < s \leq t} \Delta S_s I_{\Delta S_s \geq \delta} \) and call \( (S_{\delta, t})_{t \geq 0} \) the truncated symmetric \( \alpha \)-stable process with jumps less than \( \delta \).
Lemma 3.1. Let $0 < \alpha < 2$ and $(S_t)_{t \geq 0}$ be the symmetric $\alpha$-stable process. Then for $\beta < 2/\alpha$ and $L = [3(2 - \alpha\beta)^{-1}] + 1$,

$$
\lim_{m \to \infty} P\{ \bigcup_{n \geq m} \bigcup_{0 \leq j < 2^{2n} - 1} A_{n,j} \} = 0,
$$

where

$$A_{n,j} = \{ \text{there exist } (t_i)_{i=1}^L \in [j/2^{2n}, (j + 1)/2^{2n}) \text{ such that } |\Delta S_{t_i}| > 2^{-\beta n}, \text{ for } i = 1, 2, \cdots, L \}.$$

Proof. We know that the jumps of the symmetric $\alpha$-stable process is a Poisson point process with intensity $(A(1, -\alpha)/|x|^{1+\alpha})I_{t \geq 0}$ on $\mathbb{R} \times \mathbb{R}_+$ (cf. [3]). Thus for $n$ big enough

$$
P(A_{n,j}) = \exp(-A(1, -\alpha)\alpha^{-1}2^{1-(2-\alpha\beta)n}) \sum_{k=L}^{\infty} \frac{(A(1, -\alpha)\alpha^{-1}2^{1-(2-\alpha\beta)n})^k}{k!},$$

which leads to (3.1). \hfill \Box

Lemma 3.2. Let $(S_t)_{t \geq 0}$ be the symmetric $\alpha$-stable process and let $\beta > 1$. Then

$$
\lim_{m \to \infty} P\{ \bigcup_{n \geq m} \bigcup_{0 \leq j < 2^{2n} - 1} B_{n,j} \} = 0,
$$

where

$$B_{n,j} = \{ \sup_{j/2^{2n} < t < (j+1)/2^{2n}} |S_{2^{-\beta n}, t} - S_{2^{-\beta n}, j/2^{2n}}| > 2^{-n} \}.$$ 

Proof. Let $c > 0$ and let $S_{c,t}$ be the truncated symmetric $\alpha$-stable process with jumps less than $c$. For $k \geq 1$, we denote

$$\mathcal{P}_k = \{(2k_1, 2k_2, \cdots, 2k_l) : 1 \leq k_1 \leq k_2 \leq \cdots \leq k_l, \sum_{s=1}^{l} k_s = k \text{ for some positive integer } l \geq 2\}.$$

For each $(2k_1, 2k_2, \cdots, 2k_l) \in \mathcal{P}_k$, we say that a partition $(A_k)_{k=1}^l$ for $\{1, 2, \cdots, 2k\}$ is of type $(2k_1, 2k_2, \cdots, 2k_l)$ if the cardinal numbers of $A_k, 1 \leq k \leq l$, are $(2k_1, 2k_2, \cdots, 2k_l)$ without considering the order. Denote by $C(2k_1, 2k_2, \cdots, 2k_l)$ the cardinal number of the type $(2k_1, 2k_2, \cdots, 2k_l)$ partitions for the set $\{1, 2, \cdots, 2k\}$. Set $(\Delta_t)_{t \geq 0} = (\Delta S_{c,t})_{t \geq 0}$. Since $(\Delta_t)_{t \geq 0}$ is the Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $(A(1, -\alpha)/|x|^{1+\alpha})I_{0 < |x| < c} I_{t \geq 0}$, by expansion and taking limit we have for integer $k \geq 1$

$$
E|S_{c,t}|^{2k} = m \sum_{0 < s \leq t} \Delta s I_{|\Delta s| > \varepsilon}^{2k} = \lim_{m \to \infty} \sum_{s=0}^{m} \sum_{0 < s \leq t} \Delta s I_{|\Delta s| > \varepsilon}^{2k} = \mathcal{A}(1, -\alpha) \int_0^t \int_{-\varepsilon}^{c} \frac{|x|^{2k}}{|x|^{1+\alpha}} dx ds
$$
with respect to \( t \) integer \( U \) with determined setting. Set \( V \) for any conformal map curve. Here we divide each rectangle in a lattice decomposition of \( L \) lattice decomposition of time-space is important in proving the existence of the continuous jumps and fluctuations of a càdlàg driven function to give an extension of Lemma 4.32 [7].

Thus, we put some conditions on the regularity of the stable processes studied in the last section, we put some conditions on the of the Brownian SLE curve can be proved. Here we adopt this strategy. Motivated by the Lemma 4.32 in [7] gives some sufficient conditions on the existence of the continuous curve for some \( c = 2^{-\beta n} \) in (3.3), we obtain

\[
P\{B_{n,j}\} = \sup_{0 < t \leq 1} |S_{1/2^{3n}, 1/2^{2n}}| > 2^{-n}
\]

\[
\leq 2P\{|S_{1/2^{3n}, 1/2^{2n}}| > 2^{-n}\}
\]

\[
\leq 2^{1 + 2kn} \mathbb{E}|S_{1/2^{3n}, 1/2^{2n}}|^{2k}
\]

\[
\leq c_1 2^{1 + 2kn} \max_{1 \leq l \leq k} 2^{-(2\beta l + (2 - \alpha)l)n},
\]

Noticing that \( \beta > 1 \), we can choose \( k \) big enough so that \( P\{B_{n,j}\} \leq 2^{-3n} \), which gives the conclusion.

4 Càdlàg curves

Lemma 4.32 in [7] gives some sufficient conditions on the existence of the continuous curve for the Loewner evolution. By checking these conditions holding almost surely, the existence of the Brownian SLE curve can be proved. Here we adopt this strategy. Motivated by the regularity of the stable processes studied in the last section, we put some conditions on the jumps and fluctuations of a càdlàg driven function to give an extension of Lemma 4.32 [7].

Lattice decomposition of time-space is important in proving the existence of the continuous curve. Here we divide each rectangle in a lattice decomposition of \( \{(t, y) \in [0, N] \times [0, 1]\} \) according to the jumps and fluctuations of a driven function. The similar method was also used in [13] for the continuous case.

Let \( (u_t)_{t \geq 0} \) be a real càdlàg function. Suppose that \( (g_t)_{t \geq 0} \) is the Loewner chain in (1.1) with \( U_t = u_t \). For convenience, the previous notations \( g_t, f_t, f_t \) and \( \gamma_t \) will be also used for this determined setting. Set \( V(t, y) = f_t(iy) \) for \( t \geq 0 \) and \( y > 0 \). Let \( \beta > 0 \) and denote for each integer \( j \geq 1 \), \( u_j = u_t - \sum_{0 \leq s < t} \Delta u_s I_{|\Delta u_s| > 2^{-\beta}} \). By differentiating both sides of \( f_t(g_t(z)) = z \) with respect to \( t \), we have (cf. [7])

\[
\partial_t f_t(z) = -\frac{2f''(z)}{z - u_t},
\]

\[
\partial_t f'_t(z) = -\frac{2f''(z)}{z - u_t} + \frac{2f'(z)}{(z - u_t)^2}.
\]

For any conformal map \( f \) defined on the unite disc to itself with \( f(0) = 0 \) and \( |f'(0)| = 1 \), we know that \( |f''(0)| \leq 2 \) by Bieberbach. This implies that for any conformal map \( g \) from \( \mathbb{H} \) to \( \mathbb{H} \),

\[
|g''(z)| \leq 2|g'(z)|/|Im(z)|, \quad z \in \mathbb{H}.
\]
The following lemma is a càdlàg version of Lemma 4.32 in [7] and the proof is similar.

**Lemma 4.1.** Let $\beta > 1$ and $L, N$ be positive integers. Let $(r_j)_{j \geq 1}$ be a sequence of increasing positive numbers with $\sum_{j=1}^{\infty} 1/r_j < \infty$. Suppose there exist constants $c, c_0 > 0$ such that, for each integer $j \geq j_0$ and $0 \leq k, l \leq N^{2^j} - 1$, there exist two increasing sequences $(t(j, k, l))_{l=1}^{L_j}$ and $(s(j, k, l))_{l=1}^{L_j}$ both belonging to $[k/2^{2j}, (k + 1)/2^{2j})$ and satisfying

$$\{ t : |\Delta u_t| > 2^{-3j}, \ t \in [k/2^{2j}, (k + 1)/2^{2j}) \} \subseteq \{ t(j, k, l) : l = 1, 2, \ldots, L \},$$

$$\sup_{0 \leq t \leq r_j} \sup_{s(j, k, l) \leq s, t < s(j, k, l + 1)} |u_{j,t} - u_{j,s}| \leq c2^{-j},$$

where $s(j, k, 0) = k/2^{2j}$ and $s(j, k, r_j + 1) = (k + 1)/2^{2j}$. Suppose that for $j \geq j_0$ and $0 \leq k \leq N^{2^j} - 1$ we also have

$$|f'_{(j, k, l)}(i2^{-j} - \Delta u_{(j, k, l)})| \leq 2^{j}/r_j^2, \ l = 1, \ldots, L;$$

$$|\hat{f}'_{s(j, k, m)}(i2^{-j})| \leq 2^{j}/r_j^2, \ m = 0, 1, \ldots, r_j + 1.$$  

Then $\gamma(t)$ in (4.14) is well defined and is a càdlàg function on $[0, N]$.

**Proof** Let $j \geq j_0, 0 \leq k \leq N^{2^j} - 1, 1 \leq l \leq L$ and define $t(j, k, L + 1) = t(j, k + 1, l)$ with convention that $t(j, N^{2^j} + 1) = N$. We see that

$$[t(j, k, l), t(j, k, l + 1)] \subseteq [k/2^{2j}, N \wedge (k + 2)/2^{2j}).$$

Hence there are at most $2(r_j + 1)$ numbers from $\cup_{0 \leq k \leq N^{2^j} - 1} \{ (s(j, k, l))_{l=0}^{r_j} \}$ belonging to the interval $[t(j, k, l), t(j, k, l + 1))$. Denote them by $a_1 \leq a_2 \cdots \leq a_{r_j}$ and define $a_0 = t(j, k, l), a_{r_j + 1} = t(j, k, l + 1)$. By (4.12) and (4.13) we have $|\partial_t f'(z)| \leq 6|f'(z)|/(Im(z))^2$ and hence

$$|f'_{t-s}(z)| \leq \exp\{6s/(Im(z))^2\}|f'(z)|, \ 0 < s < t.$$  

By (4.6) and (4.8), we have

$$|f'(2^{-j}i + u_{(j, k, l+1)} - z)| \leq e^{12^{j} / r_j^2}, \ \text{for} \ t \in [a_j, a_{r_j + 1}).$$

By Distortion Theorem (see e.g. [7] [11]), there exists a constant $K$ such that for any conformal map $f$ from $H$ to $H$,

$$|f'(w)| \leq K(|z-w|/y) + 1 |f'(z)|, \ \text{Im}(z), \ \text{Im}(w) \geq y > 0.$$

Combining (4.3), (4.9) and (4.10), we have for $t \in [a_j, a_{r_j + 1})$

$$|\hat{f}'(i2^{-j})| = |f'(i2^{-j} + u_{(j, k, l+1)} - (u_t - u_{(j, k, l+1)} - z)| \leq K^{c+1}e^{12^{j} / r_j^2}.$$  

By (4.10) and (4.11) we get

$$|\hat{f}'(iy)| \leq K^{c+1}e^{12^{j} / r_j^2}, \ 2^{-j-1} \leq y \leq 2^{-j}, \ t \in [a_j, a_{r_j + 1}).$$

Similarly, by (4.17) we can also prove

$$|\hat{f}'(iy)| \leq K^{c+1}e^{12^{j} / r_j^2}, \ 2^{-j-1} \leq y \leq 2^{-j}, \ t \in [a_m, a_{m+1}), \ 0 \leq m \leq J - 1.$$  

Noticing that the estimates in (4.13) only depend on $j$, we have for any $t \in [0, N]$

$$|\hat{f}'(iy) - \hat{f}'(i2^{-j})| \leq K^{c+1}e^{12} \sum_{i=j}^{\infty} r_i^{-2}, \ 0 < y < 2^{-j}.$$
which converges to zero as $j \to \infty$ by the condition $\sum_{j=1}^{\infty} 1/r_j < \infty$. This implies that $\gamma(t)$ in \(1.2\) is well defined for $t \in [0, N]$.

By \(4.1\) we have $|\partial_t f_i(z)| \leq 2f_i(z)|y^{-1}$. Applying \(4.10\) one can also check that $|\hat{f}_i(i2^{-j} + x)| \leq K^2(c+1) e^{12j^2/r_i^2}$ if $|x| < c2^{-j}$ and $t \in [0, N]$. Therefore, by \(4.5\) and \(4.11\)-(4.14), we have for $0 < y, y_1 < 2^{-j}$ and $t_1, t_2 \in [a_0, a_{j+1})$,

$$\begin{align*}
|\hat{f}_i(iy) - \hat{f}_t(iy_1)| &\leq |\hat{f}_i(iy) - \hat{f}_t(i2^{-j})| + |\hat{f}_t(i2^{-j}) - \hat{f}_t(i2^{-j} + ut) + \hat{f}_t(i2^{-j} + ut - u_t)| \\
&\leq 2 \cdot K^{c+3} e^{12j} \sum_{i=j}^{\infty} r_i^{-2} + |\hat{f}_t(i2^{-j}) - \hat{f}_t(i2^{-j} + ut - u_t)| \\
&\leq 5 \cdot K^{2c+3} e^{12j} \sum_{i=j}^{\infty} r_i^{-2}.
\end{align*}$$

As $J \leq 2(r_j + 1)$ and $r_i$ is increasing, this gives that for $0 < y, y_1 < 2^{-j}$ and $t_1, t_2 \in [a_0, a_{J+1})$

$$\begin{align*}
|\hat{f}_i(iy) - \hat{f}_t(iy_1)| &\leq 5 \cdot K^{2c+3} e^{12j} \sum_{i=j}^{\infty} 2(r_j + 1)r_i^{-2} \\
&\leq 10 \cdot K^{2c+3} e^{12j} \sum_{i=j}^{\infty} r_i^{-1},
\end{align*}$$

which gives the conclusion by $\sum_{j=1}^{\infty} 1/r_j < \infty$. \qed

From the proof above we see that $(\gamma_t)$ is continuous on the continuous point of $(u_t)$. Next we denote by $(\Omega, \mathcal{F}, P)$ the probability space that all the stochastic processes are considered. In the following proposition the driving process is $(V_t)_{t \in \mathbb{R}}$ defined by \(1.4\) with $\delta \in (0, \delta_0)$, where $\delta_0$ is the positive constant specified by Proposition \(2.2\).

**Proposition 4.2.** Let $\kappa \in [0, 8) \cup (8, \infty)$. Let $b \in (0, 1/2)$ for $\kappa > 8$ and $b \in (1/2, 1]$ for $\kappa < 8$. Define $a, \lambda$ by \(2.17\) and suppose that $\kappa_1, \kappa_2, a', \delta, \delta_0$ satisfies all the assumptions in Proposition \(2.2\). For $T, t_1, t_2, a_1, a_2 > 0$ with $0 < t_2 < t_1 < T$, define stopping times $(T_k)_{k \geq 0}$ and $(S_k)_{k \geq 0}$ by induction

$$\begin{align*}
T_0 = t_1, & \quad T_k = \inf \{t < T_{k-1} : |\Delta V_t| > a_1 \} \lor t_2, \quad k \geq 1, \\
S_0 = t_1, & \quad S_k = \inf \{t < S_{k-1} : |B_t - B_{S_{k-1}}| > a_2 \} \lor t_2, \quad k \geq 1.
\end{align*}$$

Then there exist constants $C_2(\kappa_1, \kappa_2, \alpha, \theta, b), C_3(\kappa_1, \kappa_2, \alpha, \theta, b)$ such that the following estimates hold for all $y, \rho \in (0, 1], x \in \mathbb{R}$ and $k = 0, 1, 2, \ldots$,

\begin{align*}
P(|\hat{f}_i(T_k(x - \Delta V_{T_k} + iy)) > \rho y^{-1}| &\leq C_2(1 + (x/y)^2 b(y/\rho)|y|^3 \vartheta(\rho, a - \lambda), \quad (4.15) \\
P(|\hat{f}_i(S_k(x + iy)) > \rho y^{-1}| &\leq C_3(1 + (x/y)^2 b(y/\rho)|y|^3 \vartheta(\rho, a - \lambda), \quad (4.16)
\end{align*}

where

$$\vartheta(\rho, s) = \begin{cases} \rho^{-s}, & s > 0, \\
1 + |\log \rho|, & s = 0, \\
1, & s < 0.\end{cases}$$
Proof. By Lemma 1.2, Proposition 2.2 and the methods in Corollary 3.5 [13], we can prove (4.15) for \( k = 0 \). The only difference is that we apply the estimate (2.19) directly to replace the scale invariant property of the Brownian SLE used in [13]. Notice that for \( k \geq 1 \) and \( t > 0 \), the process \((V_{t-u} - V_{t-})_{0 < u < t}\) has the same distribution with \((V_u)_{0 < u \leq t}\) conditional on \( T_k = t \). From this fact and \( V_0 = V_{t_2} = V_{t_2} \) almost surely, we have

\[
\begin{align*}
P\{|\hat{f}_{tk}^*(x - \Delta V_{tk} + iy)| > \delta y^{-1}\} = & \int_{[t_1, t_2]} P\{|\hat{f}_{tk}^*(x - \Delta V_{tk} + iy)| > \delta y^{-1} | T_k = t\} P\{T_k \in dt\} \\
& \leq \int_{[t_1, t_2]} P\{|\hat{f}_{tk}^*(x + V_{t-} + iy)| > \delta y^{-1} | T_k = t\} P\{T_k \in dt\} \\
& \leq \sup_{t_1 \leq t \leq t_2} P\{|\hat{f}_{tk}^*(x + iy)| > \delta y^{-1}\}.
\end{align*}
\]

(4.17)

By the result for \( k = 0 \), (4.15) follows by (4.17). The proof of (4.16) is similar to (4.15). See also Theorem 3.6 [13].

By the similar arguments as above, we can prove the following result.

Corollary 4.3. Suppose that all the assumptions in Proposition 2.2 hold. Let \( \sigma \) be a bounded nonnegative random variable on \((\Omega, \mathcal{F}, P)\) which is independent with process \((V_t)_{t \geq 0}\). Then the estimate in (4.16) still holds with \( S_k \) replaced by \( \sigma \).

Lemma 4.4. Let \( f \) be an analytic function on \( \mathbb{H} \). Let \( (r_j)_{j \geq 1} \) be a sequence of increasing positive numbers with \( \sum_{j=1}^{\infty} 1/r_j < \infty \). If for each \( N \geq 1 \), there exists a constant \( C_4 = C_4(N) \) such that

\[
|f'(\frac{k}{2^j} + i2^{-j})| \leq C_42^j/r_j, \quad k = 0, \pm 1, \cdots, \pm N 2^j, \quad \text{for each } x \in [-N, N] \text{ and } y \in (0, 2^{-j}].
\]

then \( f \) can extend to \( \mathbb{H} \) continuously.

Proof. By distortion estimate (4.10) and the condition (4.18),

\[
|f'(x + iy)| \leq C_4K^32^j/r_j, \quad |x| \leq N, \quad 2^{-j-1} < y \leq 2^{-j}.
\]

(4.19)

Therefore for each \( x \in [-N, N] \) and \( y \in (0, 2^{-j}] \), we have

\[
|f(x + iy) - f(x + i2^{-j})| \leq C_4K^3\sum_{n \geq j} 1/r_n.
\]

(4.20)

Hence for \( x_1, x_2 \in [-N, N] \) with \( |x_1 - x_2| \leq 1/2^j \) and \( y_1, y_2 \in (0, 2^{-j}] \),

\[
|f(x_1 + iy_1) - f(x_2 + iy_2)| \leq |f(x_1 + iy_1) - f(x_1 + i2^{-j})| + |f(x_2 + i2^{-j}) - f(x_1 + i2^{-j})| + |f(x_2 + iy_2) - f(x_2 + i2^{-j})|
\]

\[
\leq 3C_4 \cdot K^3 \sum_{n \geq j} 1/r_n,
\]

which leads to the conclusion by \( \sum_{j=1}^{\infty} 1/r_j < \infty \).

Lemma 4.5. Let \((V_t)_{t \geq 0}\) be the driving process in (1.1) and let \( \sigma \) be a nonnegative random variable on \((\Omega, \mathcal{F}, P)\). Suppose that \( \kappa \neq 4 \) and all the parameters, such as \( \kappa_1, \kappa_2, \delta, b, a, \lambda, a' \), satisfy the assumptions in Proposition 2.2. Suppose also that the estimate (4.16) holds with \( S_k \) replaced by \( \sigma \wedge M \) for any \( M > 0 \) (\( C_3 \) in (4.16) may depend on \( M \)). Then, almost surely, the conformal map \( f_\sigma(\cdot) \) extends to \( \mathbb{H} \) continuously.
Proof Case 1. $\kappa \in (0, 4) \cup (4, 8)$. We assume that $\sigma \equiv t \geq 0$ because the proof is similar. Let $b = 1 \land [(4 + \kappa)/4\kappa]$. By definition (2.17) we can check that for $\kappa_1 - \kappa > 0$ small enough
\[
\lambda - 2b = 2b + \kappa_1 b(1 - 2b)/2 > 1.
\] (4.21)
For any $\varepsilon > 0$, we can find $M' > 0$ such that
\[
P\{A\} \geq 1 - \varepsilon, \quad \text{for } A := \{\sup_{0 \leq u \leq t} |V_u| \leq M'\}.
\] (4.22)
Let $N \geq 1$. For each $j \geq 1$, by estimate (4.10) with $S_k$ replaced by $t$ and (4.21), we can choose $\sigma > 0$ small enough such that for some $\varepsilon > 0$
\[
P\{|f_t^j(k2^{-j} + i2^{-j})| > 2^j2^{-\sigma j}\} \leq C_3(\kappa, \theta, b)(1 + k^2)2^{-j}2^{3\kappa} \rho(2^{-\sigma j}, a - \lambda) \leq C_3(\kappa, \theta, b)(1 + (N + M')^2)2^{-j}2^{-\varepsilon j},
\] (4.23)
for $k = 0, \pm 1, \cdots, \pm 2^j(N + M')$. By (4.10) and (4.23), almost surely for $\omega \in A$ we can find $J(\omega)$ such that
\[
|f_t^j(k2^{-j} + i2^{-j})| \leq c2^j2^{-\sigma j}, \quad k = 0, \pm 1, \cdots, \pm 2^jN, \quad j \geq J(\omega).
\]
By Lemma 4.4 this implies that, almost surely for $\omega \in A$, $f_t(\cdot)$ extends to $\mathbb{H}$ continuously. Hence the proof for this case is completed by taking $\varepsilon \to 0$.

Case 2. $\kappa \geq 8$. By taking $b = (4 + \kappa)/(4\kappa)$ in (2.17), we can also check that $\lambda - 2b = 2b - \kappa_2 b(1 - 2b)/2 > 1$ for $\kappa - \kappa_2 > 0$ small enough. Therefore we can prove the assertion as case 1.  

Remark 4.1. For $\kappa = 4$, we have $\sup_{b>0}(\lambda - 2b) = 1$ and the maximum is equal to 1 at $b = 1/2$, which is not enough to apply Lemma 4.2.

Proof of Theorem 1.1 First we consider the truncated case. Let $\delta_0$ be the positive constant in Proposition 2.2 and let $(V_t)$ be the stochastic process defined by (1.4) with $\delta \in (0, \delta_0)$. Choose $\beta$ such that $1 < \beta < 2/\alpha$ and set $L = [(2 - \alpha\beta)^{-1}] + 1$. For $N \geq 1$, $j \geq 1$ and $0 \leq k \leq N2^j - 1$, define stopping times $(H_{j,k,l})_{1 \leq l \leq L}$ by
\[
H_{j,k,1} = \inf\{t \leq (k + 1)2^{-2j} : \theta^{1/\alpha}|\Delta S_t| > 2^{-\beta j}\} \lor k2^{-2j},
\]
\[
H_{j,k,l} = \inf\{t < H_{j,k,l-1} : \theta^{1/\alpha}|\Delta S_t| > 2^{-\beta j}\} \lor k2^{-2j}, \quad 2 \leq l \leq L,
\]
and define stopping times $(R_{j,k,m})_{0 \leq m \leq j^2}$ by $R_{j,k,0} = (k + 1)/2^{2j}$ and
\[
R_{j,k,1} = \inf\{t < (k + 1)2^{-2j} : |B_t - B_{(k+1)2^{-2j}}| > 2^{-j}\} \lor k2^{-2j},
\]
\[
R_{j,k,m} = \inf\{t < R_{j,k,m-1} : |B_t - B_{R_{j,k,m-1}}| > 2^{-j}\} \lor k2^{-2j}, \quad 2 \leq m \leq j^2.
\]
By Lemma 3.1, Lemma 3.2 and the distribution of the Brownian motion, we see that for a.s. $\omega \in \Omega$, there exists an integer $J_0 = J_0(\omega)$ such that for $j \geq J_0(\omega)$
\[
H_{j,k,L}(\omega) = R_{j,k,j^2}(\omega) = k2^{-2j}, \quad 0 \leq k \leq N2^{2j} - 1,
\] (4.24)
\[
\sup_{k/2^{2j} \leq t < (k+1)/2^{2j}} |S_{2^{-\beta j}t} - S_{2^{-\beta j}k/2^{2j}}| \leq 2^{-j}, \quad 0 \leq k \leq N2^{2j} - 1,
\] (4.25)
\[
\sup_{0 \leq m \leq j^2 - 1} \sup_{R_{j,k,m+1} \leq t < R_{j,k,m}} |B_t - B_{R_{j,k,m}}| \leq 2^{-j}, \quad 0 \leq k \leq N2^{2j} - 1.
\] (4.26)
Let $\kappa \neq 8$, $b = ((8 + k)/4k) \land 1$ and assume that all the assumptions in Proposition 2.2 hold. By choosing $\kappa_1 - \kappa$, $\kappa - \kappa_2$ small enough and applying Proposition 4.2, we can find $\varepsilon$ small enough such that
\[
P\{|f_{H_{j,k,l}}^j(i2^{-j}) - \Delta V_{H_{j,k,l}}| > 2^j/j^4\} \leq O(1)2^{-2j}2^{-\varepsilon j},
\]
for any $\omega \in A$.
\[ P\{1 \leq m \leq j^2 : |f'_{R_{j,k,m}}(i2^{-j})| > 2^j/j^4\} \leq O(1)2^{-2j2^{-\varepsilon j}}. \]  

(4.27)

For the proof of (4.27), see (3.20) in [13] for details. This implies that almost surely, there exists an integer \( J_1 = J_1(\omega) \) such that for \( j \geq J_1(\omega) \) and \( 0 \leq k \leq N2^{-j} - 1 \),

\[
|f'_{H_{j,k,l}}(i2^{-j}) - \Delta V_{H_{j,k,l}}| \leq 2^j/j^4, \quad 1 \leq l \leq L; 
\]

(4.28) 
\[
|f'_{R_{j,k,m}}(i2^{-j})| \leq 2^j/j^4, \quad 1 \leq m \leq j^2. 
\]

(4.29)

Combining (4.28)-(4.29) and applying Lemma 4.1 by setting \( r_j = j^2 \) and \( c = 2(\sqrt{K} + \theta^{1/\alpha}) \), we can prove that \((\gamma_t)\) is a càdlàg curve almost surely, where \( \gamma_t \) is defined by (1.2) for the driving process \( (V_t) \).

From now on we further assume that \( \kappa \neq 4 \) and use the notations \( \gamma_t, f_t, \zeta, K_t \) for the driving process \((U_t)\) in (1.3). As in Section 1, define \( T_1 \) to be the first time \( |\Delta S_t| > \delta \) and define by induction \( T_{n+1} = \inf\{t \geq T_n : |\Delta S_t| > \delta\} \) for \( n \geq 1 \). By the arguments in Section 1, to obtain the càdlàg property of \( \gamma_t \) from that of \((\gamma_t)\), we only need to prove that \( f_{T_{k}}, k \geq 1 \), can extend to \( \mathbb{R} \) continuously. For \( k = 1 \), noticing that \( U_t = V_t \) for \( t \in [0, T_1) \) and \( T_1 \) is independent with \((V_t)\), this follows by Proposition 4.2, Corollary 4.3 and Lemma 4.5. Suppose by induction that it holds for \( k = n \). For \( k = n + 1 \), by relation \( f_{T_{n+1}} = f_{T_n}(g_{T_n} \circ f_{T_{n+1}}) \), the assertion for \( f_{T_{n+1}} \) can reduce to the continuous extension of conformal map \( g_{T_n} \circ f_{T_{n+1}} \). By considering the truncated process

\[ U_{T_n} + \sqrt{\kappa}B_{t+T_n} + \theta^{1/\alpha}S_{h,t+T_n} - \sqrt{\kappa}B_{t+n} - \theta^{1/\alpha}S_{h,T_n}, \quad t \geq 0, \]

the continuous extension of \( g_{T_n} \circ f_{T_{n+1}} \) can be proved as the case \( k = 1 \). Hence we obtain the first assertion of the theorem.

Next we turn to the relation between \((K_t)_{t \geq 0}\) and \((\gamma_t)_{t \geq 0}\). Denote

\[ A = \{w : (\gamma_t)_{t \geq 0} \text{ is a càdlàg curve}\}. \]

We have \( P\{A\} = 1 \). Let \( B_t \) be the unbounded component of \( \mathbb{R} \setminus [0, t] \). We want to show that \( B_t = H_t := \mathbb{R} \setminus \partial K_t \) for \( \omega \in A \). If this is not true, then there exist some \( \omega \in A \) and \( t > 0 \) such that \( H_t \neq B_t \). By that \( H_t \subseteq B_t \) and that \( \gamma(0, t] \) is closed, we can find \( z_0 \in \partial K_t \setminus \gamma(0, t] \) with \( \text{dist}(z_0, \gamma(0, t]) > 0 \). Choose \( z_1 \in H_{\zeta(z_0)} \) with \( |z_1 - z_0| < \text{dist}(z_0, \gamma(0, t]) \) and set \( z_2 = z_1 + s_0(z_0 - z_1) \) with \( s_0 = \inf\{s > 0 : z_1 + s(z_0 - z_1) \in \partial K_{\zeta(z_0)}\} \). By the definition of \( \zeta(z_2) \) and Proposition 2.14 [11], we have \( \lim_{t \uparrow 1} g_{\zeta(z_2)}(\zeta_t) = U_{\zeta(z_2)} \), where \( \zeta(t) = (1 - t)z_1 + tz_2 \). Set \( \xi_t = g_{\zeta(z_2)}(i(1-t)) \) for \( 0 < t < 1 \), we also have \( \lim_{t \uparrow 1} g_{\zeta(z_2)}(\xi_t) = U_{\zeta(z_2)} \). Therefore we have \( z_2 = \gamma(\zeta(z_2)) \) by Proposition 2.14 [11], which contradicts with \( \text{dist}(z_2, \gamma(0, t]) > 0 \).

To prove the second part of Theorem 1.1 for \( \kappa \neq 4, 8 \), we only need to prove that almost surely, \( f_t(\cdot) \) extends to \( \mathbb{R} \) continuously for all \( t \in [0, \infty) \). By Theorem 2.1 [11], for any \( t \geq 0 \), the continuous extension of \( f_t \) is equivalent to the local connected property of \( \partial K_t \). Thus by Proposition 4.2 and Lemma 4.5 \( \partial K_t \) is local connected for \( t \in Q_+ \) almost surely, where \( Q_+ \) is the set of positive rational numbers. Since \( \partial K_t \subseteq \gamma(0, t]) \) by \( B_t = H_t \) and \( \gamma(t) \) is right continuous, we have the following right continuity of \( \partial K_t \)

\[
\limsup_{h \downarrow 0} \{|x - y] : x, y \in \partial K_{t+h} \setminus \partial K_t\} = 0, \quad t \geq 0.
\]

(4.30)

This can extend the locally connected property of \( \partial K_t \) from \( t \in Q_+ \) to \( t \in [0, \infty) \). Thus by the equivalent conditions in Theorem 2.1 [11] once again, the continuous extension of \( f_t \) holds for all \( t \in [0, \infty) \) almost surely.

At last we give an example to show that some non locally connected sets can be generated by a càdlàg curve. Define a comb space by

\[ D = \cup_{n=1}^{\infty} \{(x, y) : x = 1/n, \ 0 \leq y \leq 1\} \cup \{(x, y) : x = 0, \ 0 \leq y \leq 1\}. \]
Next we show that there exists a càdlàg curve $(\xi(t))_{0 \leq t \leq 2}$ in $\mathbb{R}^2$ such that
\[
D = \xi[0,2] := \{\xi(t) : 0 \leq t \leq 2\},
\] (4.31)
and $\xi[0,t]$ is a connected set in $\mathbb{R}^2$ for each $t \in [0,2]$. For two subsets $A$ and $B$ of $\mathbb{R}$, we write $A + B = \{x + y : x \in A, y \in B\}$. Set
\[
R_1 = \{1\} + [1/4,1/2) \cup [3/4,1) := I_{1,1} \cup I_{1,2},
\]
\[
R_2 = \{1\} + [1/16,2/16) \cup [3/16,4/16) \cup [9/16,10/16) \cup [11/16,12/16) := I_{2,1} \cup I_{2,2} \cup I_{2,3} \cup I_{2,4}.
\]
Inductively, define for $n \geq 2$
\[
R_{n+1} = \{1\} + ([0,1) \setminus R_n) \setminus \bigcup_{k=0}^{2^{n+1}} [(2k)2^{-2(n+1)}, (2k+1)2^{-2(n+1)}] := \bigcup_{k=1}^{2n+1} I_{n+1,k},
\] (4.32)
where $(I_{n+1,k})$ are intervals of length $2^{-2(n+1)}$ arranged by the increasing order. Set $\xi(t) = (t,0)$ for $0 \leq t < 1$ and $\xi(2) = (0,0)$. Define for $n \geq 1$ and $1 \leq k \leq 2^n$
\[
\xi_t = (1/n, (k-1)2^{-n} + (t - t_{n,k})2^n), \quad t \in I_{n,k},
\] (4.33)
where $t_{n,k} = \inf I_{n,k}$. For each point $t \in [1,2) \setminus \cup_{n \geq 1} R_n$, we can find a sequence of integers $(k_n)$ such that $(t_{n,k_n})$ is a decreasing sequence in $\cup_{n \geq 1} R_n$ and $\lim_{n \to \infty} t_{n,k_n} = t$. By definition, it is easy to check that $\xi(t_{n,k_n})$ converges to a number which does not depend on the choice of $(t_{n,k_n})$. Define
\[
\xi(t) = (0, \lim_{n \to \infty} \xi(t_{n,k_n})).
\] (4.34)
By definition we can check directly that $\xi(t)$ is a right continuous and left limit curve satisfying (4.31). Notice that for each $n = 1, 2, \cdots, \infty$ and $0 \leq y_1 < y_2 \leq 1$, we have $\inf\{s : 1/n, y_1 \in \xi[0,s]) < \inf\{s : (1/n, y_2) \in \xi[0,s]\}$, where $1/\infty = 0$. This implies that $\xi[0,t]$ is connected for each $t \in [0,2]$.

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