New perturbation theory representation of the conformal symmetry braking effects in gauge quantum field theory models

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Abstract

We propose a hypothesis on the detailed structure for the representation of the conformal symmetry breaking term in the basic Crewther relation generalized in the perturbation theory framework in QCD renormalized in the \(\overline{\text{MS}}\) scheme. We establish the validity of this representation in the \(O(\alpha^4_s)\) approximation. Using the variant of the generalized Crewther relation formulated here allows finding relations between specific contributions to the QCD perturbation series coefficients for the flavor nonsinglet part of the Adler function \(D_{ns}^A\) for the electron-positron annihilation in hadrons and to the perturbation series coefficients for the Bjorken sum rule \(S_{Bjp}\) for the polarized deep-inelastic lepton-nucleon scattering. We find new relations between the \(\alpha^4_s\) coefficients of \(D_{ns}^A\) and \(S_{Bjp}\). Satisfaction of one of them serves as an additional theoretical verification of the recent computer analytic calculations of the terms of order \(\alpha^4_s\) in the expressions for these two quantities.

Keywords: quantum field theory, conformal symmetry breaking, perturbation theory, renormalization group, relation between characteristics of inclusive processes

1. The conformal symmetry is basic for important theoretical studies in various massless quantum field models \([1, 2]\) including QED \([3]\) and QCD (see Sec. 5 in \([4]\)). Using this symmetry in studying the axialvector-vector-vector (AVV) triangle amplitude allowed establishing the fundamental relation between important characteristics of different inclusive processes \([5]\). The characteristics investigated in \([5]\) were the normalized expression for the flavor nonsinglet part \(D_{ns}^A\) of the Adler function \(D_A\) for the \(e^+e^-\) annihilation process in hadrons and the nonsinglet coefficient function \(C^{Bjp}\) of the Bjorken sum rule \(S_{Bjp}\) for the process of deep-inelastic scattering (DIS) of polarized leptons on nucleons, which also enters the nonsinglet part of the Ellis- Jaffe sum rule for the DIS of polarized leptons on nucleons.

The basic Crewther relation was soon applied in \([6]\) to the model case where diagrams with lepton insertions on the internal photon lines are not taken into account in QED. The relation is also applicable in the imaginary conformally invariant limit of QCD. In these cases, it has the form

\[
D \times C^{Bjp}|_{ci} = 1,
\]

where the quantities in the left-hand side are defined as

\[
D_{ns}^A(a_s) = \left( N_c \sum_f Q_f^2 \right) D(a_s)
\]

\[
S_{Bjp}(a_s) = \left( \frac{1}{6} \frac{g_0}{g_V} \right) C^{Bjp}(a_s)
\]

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It is known that the conformal symmetry is broken in the models of quantum field theory by renormalization of charges. These renormalizations lead to the existence of nonzero renormalization group (RG) $\beta$-functions (see \cite{7} for a detailed exposition). Moreover, the factor $\beta(a_s)/a_s$, where $a_s = \alpha_s/\pi$, appears as the result of renormalization of the trace of the energy-momentum tensor. This property was outlined theoretically in \cite{8} and demonstrated explicitly in \cite{9,12}; it is connected with the existence of the conformal anomaly. Before \cite{13} appeared, the existence of a generalization of the basic Crewther relation \cite{5} to the case of gauge theories with fermions, like QED and QCD, in higher orders of the perturbation theory (PT) with explicit manifestations of renormalizations of the coupling constants was unclear. The $SU(N_c)$ group factors were classified in \cite{13}; these factors arise when the QCD PT series for the function $D^{\text{ns}}$ in the $O(a_s^3)$ approximation (obtained analytically in \cite{14} and later in \cite{15} in the $\overline{\text{MS}}$ scheme) is multiplied by the analogous approximation for the function $C^{Bjp}$, known at that time from the calculations in \cite{16}. The studies performed in \cite{13} allowed finding an additional contribution to the right-hand side of (1):

$$D(a_s)C^{Bjp}(a_s) = 1 + \Delta_{c,sb}(a_s)$$

(4)

In the third order of the PT, the “Crewther unity” is modified by the conformal symmetry breaking term $\Delta_{c,sb}$, which is expressed as

$$\Delta_{c,sb}(a_s) = \left(\frac{\beta(a_s)}{a_s}\right) P(a_s) = \left(\frac{\beta(a_s)}{a_s}\right) \sum_{m \geq 1} K_m a_s^m.$$  

(5)

In this order of the PT, the scheme-independent two-loop RG $\beta$-function appears in the factor $\beta(a_s)/a_s$, and the coefficients $K_1$ and $K_2$ determined in \cite{13} appear in the factor $P(a_s)$. Moreover, $K_2$ depends on the quadratic Casimir operators $C_F$ and $C_A$ of the $SU(N_c)$ gauge group and on the number $n_f$ of fermion flavors.

The discovery of this QCD generalization of the Crewther relation in the third-order PT \cite{13} in the $\overline{\text{MS}}$ scheme with the factor $\beta(a_s)/a_s$ was the first independent theoretical indication of the validity of computer analytic calculations of the $O(a_s^3)$ corrections in the PT series for the $D$ functions \cite{14,15} and of the analogous calculations of the third term in the PT series for $C^{Bjp}$ \cite{16}. We note that this “cross-checking” theoretical indication was later confirmed by independent computer calculations of the contributions of the order $a_s^3$ to the $D$ function in \cite{17} using a different theoretical approach. The next-to-leading PT corrections to the $D$ function were previously evaluated analytically in \cite{18} and numerically in \cite{19}. These results were soon confirmed analytically \cite{20}. In the case of $C^{Bjp}$, corrections of the same PT order were obtained in \cite{21} and later confirmed in \cite{22} using a different symbolic computational technique.

To explain the origin of the effect of the factorization $\beta(a_s)/a_s$ in Eq. (5), the operator product expansion method in the momentum space was applied in \cite{23} to the triangle diagram of the AVV currents (see \cite{24} for a more detailed discussion). Also in \cite{23}, arguments were presented for the absence of an inconsistency between the one-loop nature of the axial anomaly \cite{25} and the multiloop structure of the QCD generalization of the Crewther relation, based on their relation to different form factors in the AVV triangle diagram \cite{23}. The possibility that the multiloop factor $\beta(a_s)/a_s$ in the conformal symmetry breaking term in Eq. (5) is factorable in all PT orders \cite{23} with the coefficients $K_m$ of the polynomial $P(a_s)$ unfixed in the presented considerations was also indicated. The considerations imply the application of the $\overline{\text{MS}}$ scheme in which the coefficient functions of the leading operators in the operator product expansion method can be explicitly defined. The second conclusion in \cite{23} was proved in the coordinate space in \cite{26} and was previously discussed in \cite{27} but was published only recently \cite{4}. This variant of the QCD generalization of the Crewther relation in the $\overline{\text{MS}}$ scheme \cite{13} was considered from a more phenomenological standpoint in \cite{28} and \cite{29}, where the characteristic energy scales were fixed by applying the multiloop version (developed in \cite{30}) of the Brodsky- Lepage-Mackenzie approach \cite{31} supplemented by the procedure for constructing the “commensurate
scale relations” in [32]. As a result, the “Crewther unity” was restored in the right-hand side of [33] at the $O(a_s^4)$ level by absorbing the conformal symmetry breaking term into the energy scale of the effective charge for the $D$ function, which is equivalent to choosing a certain scale in the invariant charge for the $D$ function and including the conformal symmetry breaking term in the energy scale of the effective charge for $C^{BP}$ [29].

In the case of the $SU(N_c)$ group, the $O(a_s^4)$ corrections to the functions $D^{as}$ and $C^{BP}(a_s)$ are known thanks to the recent analytic calculation in the $\overline{MS}$ scheme [33]. These calculations with the three-loop analytic contributions to the QCD $\beta$-function in this scheme taken into account [34], [35] allowed fixing the coefficient $K_3$ of the polynomial $P(a_s)$ in expression (5) and demonstrating the validity of the results in [28] with the $O(a_s^4)$ PT contributions taken into account. We note that the explicit results for $C^{BP}$ confirmed the expressions for the $\zeta_3$-containing QED contributions to the Bjorken sum rule that appeared first in fourth-order QED PT corrections to the function $D^{as}$ [32]. This term in the Bjorken sum rule was previously obtained in [37] from the results in [36], arguments based on the conformal symmetry and the basic Crewther relation. The found agreement was the first confirmation of the validity of the calculations in [33].

Our main purpose here is to justify the detailed representation of the generalized Crewther relation at the $a_s^4$ level previously proposed in [39]. Its new feature is writing the right-hand side of Eq. (5) in the form of a double power expansion in which the first expansion “parameter” is the function $\beta(a_s)/a_s$ and the second is the coupling constant $a_s$, namely,

$$\Delta_{csb}(a_s) = \sum_{n \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n(a_s) = \sum_{n \geq 1} \sum_{r \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n^{(r)} a_s^r = \sum_{n \geq 1} \sum_{r \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n^{(r)} [k, m] C^k F C^m a_s^r,$$

where $k + m = r$ and the coefficients $P_n^{(r)} [k, m]$ contain rational fractions and Riemann $\zeta$-functions of odd arguments. In contrast to the coefficients of the polynomial $P(a_s)$ in Eq. (5), the coefficients of $P_n(a_s)$ in Eq. (6) are independent of the number $n_f$ of quark flavors.

2. We consider the PT series for the nonsinglet part of the Adler function $D$ and the coefficient function $C^{BP}$ for the Bjorken sum rule respectively defined in (2) and (3) and normalized to unity:

$$D = 1 + \sum_{n=1} d_n a_s^n, \quad C^{BP} = 1 + \sum_{l=1} c_l a_s^l.$$

Explicit expressions for $d_1$, $d_2$, $d_3$ and $c_1$, $c_2$, $c_3$ in terms of the $SU(N_c)$ group factors are well known (see, e.g., [14], [16]). In the concrete case of the $SU(3)$ group, the fourth coefficient $d_4$ of the $D$ function was evaluated analytically in [40]. This result was recently generalized to the case of an arbitrary color group $SU(N_c)$ in [33]. The analogous coefficient $c_4$ for $C^{BP}$, also

\footnote{The arguments for the possibility of explaining the appearance at this PT level of $\zeta_3$ term, untypical for previously known diagram contributions characterizing the photon function of the QED vacuum polarization without fermion loop insertions into internal photon lines, were presented in [35].}
calculated in [32] (see the supplemental file to the electronic preprint version of [33]), is

\[
c_4 = \left[ \frac{3}{16} + \frac{1}{4} \zeta_3 + \frac{5}{4} \zeta_5 \right] \frac{d_{F}^{abcd} d_{A}^{abcd}}{d_R} + \left[ \frac{13}{16} + \frac{\zeta_3}{3} - \frac{5}{2} \zeta_5 \right] \frac{d_{F}^{abcd} d_{F}^{abcd}}{d_R} - \frac{4823}{2048} + \frac{3}{8} \zeta_3 \right] C_F^4 + \]

\[
+ \frac{839}{2304} + \frac{451}{96} \zeta_3 - \frac{145}{24} \zeta_5 \right] C_F^2 T_F n_f^2 + \left[ -\frac{265}{576} + \frac{29}{24} \zeta_3 \right] C_F^2 T_F n_f^2 + \left[ \frac{605}{648} \right] C_F T_F^3 n_f^2 \]

\[
+ \frac{3707}{4608} - \frac{971}{96} \zeta_3 + \frac{1045}{48} \zeta_5 \right] C_F^2 T_F n_f^2 + \left[ \frac{87403}{13824} - \frac{1289}{144} \zeta_3 - \frac{275}{144} \zeta_5 + \frac{35}{4} \zeta_7 \right] C_F^2 C_A T_F n_f^2 \]

\[
+ \frac{165283}{20736} - \frac{43}{144} \zeta_3 + \frac{5}{12} \zeta_5 + \frac{1}{6} \zeta_3 \right] C_F C_A T_F n_f^2 \]

\[
+ \frac{1071641}{55296} + \frac{1591}{144} \zeta_3 - \frac{1375}{144} \zeta_5 - \frac{385}{16} \zeta_7 \right] C_F^2 C_A \]

\[
+ \frac{1238827}{41472} + \frac{59}{64} \zeta_3 - \frac{1855}{288} \zeta_5 + \frac{11}{12} \zeta_3 - \frac{35}{16} \zeta_7 \right] C_F C_A^2 T_F n_f \]

\[
+ \frac{3004727}{248832} + \frac{6159}{576} \zeta_3 + \frac{12545}{1152} \zeta_5 - \frac{121}{96} \zeta_3 + \frac{385}{64} \zeta_7 \right] C_F C_A^3 \]  

(8)

where \( \zeta_{2q+1} = \sum_{k=1}^{\infty} (1/k)^{2q+1} \) is the Riemann function of an odd argument. In the fundamental representation of \( SU(N_c) \), the group factors are defined as \( C_F = (N_c^2 - 1)/(2N_c) \), \( C_A = N_c \), \( T_F = 1/2 \), \( d_{F}^{abcd} d_{A}^{abcd}/d_R = N_c(N_c^2 - 6)/18 \), and \( d_{F}^{abcd} d_{F}^{abcd}/d_R = (N_c^2 - 6N_c^2 + 18)/(36N_c^2) \). For the \( SU(3) \) group, which corresponds to the QCD case, we have \( C_F = 4/3 \), \( C_A = 3 \), \( d_R = 3 \) and \( d_{F}^{abcd} d_{A}^{abcd} = 15/2 \), \( d_{F}^{abcd} d_{F}^{abcd} = 5/12 \).

A strong verification of the self-consistency of the results obtained in [10] and [33] follows from the validity of QCD-generalized Crewther relation [13] after the \( O(a_s^4) \) contributions to the left-hand side of (11) evaluated in the \( \overline{\text{MS}} \)-scheme are taken into account. We recall that the existence of this generalization with the factored multiplier \( \beta(a_s)/a_s \) is not accidental. It was discovered in the preceding PT order [13] and proved in all orders in [26]. It was shown in [13] that the coefficients of the polynomial \( P(a_s) \) in (11) in the third order of the PT can be expressed as

\[
K_1 = K_1[1, 0, 0]C_F, \]

\[
K_2 = K_2[2, 0, 0]C_F^2 + K_2[1, 1, 0]C_F C_A + K_2[1, 0, 1]C_F T_F n_f, \]  

(9)

The fourth-order PT calculations in [33] lead to the fixation of the third term in the polynomial \( P(a_s) \) in the form of a sum of six terms proportional to the Casimir operators of the \( SU(N_c) \) group times the number \( n_f \) of fermion flavors:

\[
K_3 = K_3[3, 0, 0]C_F^3 + K_3[2, 1, 0]C_F^2 C_A + K_3[1, 2, 0]C_F C_A^2 + K_3[2, 0, 1]C_A^2 T_F n_f + K_3[1, 1, 1]C_F C_A T_F n_f + K_3[1, 0, 2]C_F (T_F n_f)^2. \]  

(10)

The analytic expression for the last coefficient \( K_3[1, 0, 2] \) in (10) coincides with the result in [13] obtained when calculating analogous coefficients generated in higher PT orders by multiplying the contributions to the functions \( D(a_s) \) and \( C^{Bjp} \) of the diagrams with a large number of one-loop fermion insertions into the internal gluon lines.

In correspondence with the structure of the term \( \Delta_{\mu
u}^{ab}(a_s) \) in (5) and (6), we need concrete values of the coefficients of the RG \( \beta \)-function in the \( \overline{\text{MS}} \)-scheme

\[
\mu^2 \frac{d}{d\mu^2} a_s = \beta(a_s) = -a_s^2 \left( \beta_0 + \beta_1 a_s + \beta_2 a_s^2 \right), \]  

(11)

We recovered this expression, which agrees with the result contained in the electronic supplement to the preprint of [33], from its text in which the result for \( 1/C^{Bjp} \) was presented.
found in the three-loop approximation in [34] and confirmed in [35]. The coefficients $\beta_i$ can be expressed in the forms

$$\begin{align*}
\beta_0 &= \beta_0[0, 1, 0]C_A + \beta_0[0, 0, 1]T_F n_f, \\
\beta_1 &= \beta_1[0, 2, 0]C_A^2 + \beta_1[0, 1, 1]C_A T_F n_f + \beta_1[1, 0, 1]C_F T_F n_f, \\
\beta_2 &= \beta_2[0, 3, 0]C_A + \beta_2[0, 2, 1]C_A^2 T_F n_f + \beta_2[1, 1, 1]C_F C_A T_F n_f + \beta_2[0, 1, 2]C_F T_F^2 n_f^2, \\
\end{align*}$$

(12)

with the elements $\beta_i[...]:$

$$\begin{align*}
\beta_0[0, 1, 0] &= \frac{11}{12}, \quad \beta_0[0, 0, 1] = -\frac{1}{3}, \\
\beta_1[0, 2, 0] &= \frac{2857}{3456}, \quad \beta_1[0, 1, 1] = -\frac{5}{12}, \quad \beta_1[1, 0, 1] = -\frac{1}{4}, \\
\beta_2[0, 3, 0] &= \frac{79}{864}, \quad \beta_2[0, 2, 1] = -\frac{1415}{1728}, \quad \beta_2[1, 1, 1] = -\frac{205}{576}, \\
\beta_2[0, 1, 2] &= \frac{1}{32}, \quad \beta_2[1, 0, 2] = \frac{11}{144}. \\
\end{align*}$$

(13)

3. We now consider the issue of the uniqueness of a detailed generalization of Crewther relation [5] in powers of the $\beta$-function. We here present additional arguments for our assumption that such a generalization exists (see [32]) and justify it using the results of the fourth-order PT approximation for [4] and [5] obtained in [33]. The derivation of the detailed generalization of the Crewther relation in the $\overline{\text{MS}}$ scheme is based on the requirement that the coefficients of the polynomials $P_n$ in (6) should be independent of the $\beta$-function coefficients and consequently independent of the number $n_f$ of fermion flavors. This property can be realized by passing from representation [5] with the single factored $\beta$-function in the expression for the conformal symmetry breaking term $\Delta_{cab}(a_s)$ in (1) to representation [6] in the form of an expansion in powers of $\beta(a_s)/a_s$. The validity of this form of writing $\Delta_{cab}(a_s)$ in the fourth PT order was assumed in [39] before the publication of the analytic results of calculations of the $D$-function and $C_{B_j p}(a_s)$ in the $a_s^4$ order [33]. To derive it explicitly, we should equate the right-hand sides of the two representations for $\Delta_{cab}(a_s)$ from [4] and [6] at each order of the expansion in the coupling constant $a_s$. In the PT approximations we are interested in, the coefficients in the right-hand side of (6) are related to the analogous contributions to (6) by the system of linear equations

$$\begin{align*}
K_1[1, 0, 0] &= P_1^{(1)}[1, 0], \\
K_2[2, 0, 0] &= P_1^{(2)}[2, 0], \\
K_2[1, 1, 0] &= P_1^{(2)}[1, 1] - \beta_0[0, 1, 0] P_2^{(1)}[1, 0], \\
K_2[1, 0, 1] &= -\beta_0[0, 0, 1] P_2^{(1)}[1, 0], \\
K_3[3, 0, 0] &= P_1^{(3)}[3, 0], \\
K_3[2, 1, 0] &= P_1^{(3)}[2, 1] - \beta_0[0, 1, 0] P_2^{(2)}[2, 0], \\
K_3[1, 2, 0] &= P_1^{(3)}[1, 2] - \beta_0[0, 1, 0] P_2^{(2)}[1, 1] - \beta_1[0, 2, 0] P_1^{(1)}[1, 0] + (\beta_0[0, 1, 0])^2 P_3^{(1)}[1, 0], \\
K_3[2, 0, 1] &= -\beta_1[1, 0, 1] P_2^{(1)}[1, 0] - \beta_0[0, 0, 1] P_2^{(2)}[2, 0], \\
K_3[1, 1, 1] &= -\beta_1[0, 1, 1] P_2^{(1)}[1, 0] - \beta_0[0, 0, 1] P_2^{(2)}[1, 1] + 2\beta_0[0, 1, 0] \beta_0[0, 0, 1] P_3^{(1)}[1, 0], \\
K_3[1, 0, 2] &= (\beta_0[0, 0, 1])^2 P_3^{(1)}[1, 0]. \\
\end{align*}$$

(14)

The unique solution of this system determines the explicit expressions for the three polynomials
$\mathcal{P}_n(a_s)$ with coefficients $P^{(r)}_{n}[k,m]$ independent of the number of flavours:

\[
\mathcal{P}_1(a_s) = \left( -\frac{21}{8} + 3\zeta_3 \right) C_F a_s + \left[ \left( \frac{397}{96} + \frac{17}{2} \zeta_3 - 15\zeta_5 \right) C_F^2 + \left( -\frac{47}{48} + \zeta_3 \right) C_F C_A \right] a_s^2
\]
\[
+ \left[ \left( \frac{2471}{768} + \frac{61}{8} \zeta_3 - \frac{715}{8} \zeta_5 + \frac{315}{4} \zeta_7 \right) C_F^3
+ \left( \frac{16649}{1536} - \frac{1113}{192} \zeta_3 + \frac{1015}{24} \zeta_5 - \frac{105}{8} \zeta_7 + \frac{99}{4} \zeta_9 \right) C_F^2 C_A
+ \left( \frac{2107}{192} + \frac{2503}{72} \zeta_3 - \frac{355}{18} \zeta_5 - 33\zeta_7 \right) C_F C_A^2 \right] a_s^3 + O(a_s^4);
\]

\[
\mathcal{P}_2(a_s) = \left( \frac{163}{8} - 19\zeta_3 \right) C_F a_s + \left[ \left( -\frac{13597}{384} - \frac{2523}{16} \zeta_3 + \frac{375}{2} \zeta_5 + 27\zeta_7 \right) C_F^2
\]
\[
+ \left( \frac{1433}{32} - \frac{1}{4} \zeta_3 - \frac{85}{2} \zeta_5 - 6\zeta_7 \right) C_F C_A \right] a_s^2 + O(a_s^3);
\]

\[
\mathcal{P}_3(a_s) = \left( -\frac{307}{2} + \frac{203}{2} \zeta_3 + 45\zeta_5 \right) C_F a_s + O(a_s^2).
\]

We note that the four-loop term $\beta_3$ of the RG $\beta$-function, evaluated analytically in the case of $SU(N_c)$ in [41] and confirmed in [42], contains three new group structures $d_{A}^{abcd}d_{A}^{abcd}$, $d_{F}^{abcd}d_{A}^{abcd}n_{I}$ and $c_{F}^{abcd}d_{F}^{abcd}n_{I}$. In view of the factorization of the $\beta$-function in (5) in all PT orders (see the proofs in [29, 42]), we conclude that the appearance of these extra group terms does not spoil the $\beta$-function factorability in (5) and also in the first term of the sum in (5).

One more conclusion follows from higher contributions in powers of $n_{I}$ calculated in [13], equivalent to calculating the corrections proportional to higher powers of the first coefficient $\beta_{0}$ of the RG $\beta$-function. These corrections determine the leading contributions to the polynomials $\mathcal{P}_n(a_s)$ of the new representation for $\Delta_{csb}$ in (4), which have the form

\[
\mathcal{P}_n(a_s) = \frac{S_n 2^{(n-1)} C_F a_s}{4^n} + O(a_s^2).
\]

The first nine coefficients $S_{n}$, $1 \leq n \leq 9$, were calculated analytically in [12].

4. Representation (4) can be obtained differently by using the $\beta$-expansion formalism of the coefficients of the PT series (in the $\overline{MS}$ scheme) developed in [13]. In this approach, instead of the commonly used expansions of the coefficients in powers of the flavor-dependent factor $T_{FNI}$, the quadratic Casimir operators $C_F$ and $C_A$, and the structure constants of the color group $SU(N_c)$, it is proposed to consider expansions of the coefficients $d_{n}$ and $c_{n}$ in powers of the coefficients $\beta_{0}$, $\beta_{1}, \ldots$ of the $\beta$-function with the weight elements $d_{n}[n_0, n_1, \ldots]$ and $c_{n}[n_0, n_1, \ldots]$. Their first arguments $(n_0)$ determine the powers of the coefficients $\beta_{0}$ of the elements $d_{n}[^{\ldots}, 0, \ldots]$ and $c_{n}[0, 0, \ldots]$, the second arguments $(n_1)$ give the powers of the coefficients $\beta_{1}$, and so on. The elements $d_{n}[0, 0, \ldots, 0]$ and $c_{n}[0, 0, \ldots, 0]$ are the contributions “cleaned” of the charge renormalizations and the factors $\beta_{i}$, whose powers are here equal to zero ($n_{i} = 0$). These elements coincide with the values of the coefficients $d_{n}$ and $c_{n}$ in the hypothetical limit with the $\beta$-function identically equal to zero in all PT orders in QCD. This limit corresponds to restoring the conformal symmetry in the effective quantum field model. We regard the transition to this model as a technical trick here. If all arguments $n_{i}$ of the elements $d_{n}[^{\ldots}, m, 0, \ldots, 0]$ and $c_{n}[^{\ldots}, m, 0, \ldots, 0]$ after some index $m$ are zero, then we simplify the notation as $d_{n}[^{\ldots}, m, 0, \ldots, 0] = d_{n}[^{\ldots}, m]$ and $c_{n}[^{\ldots}, m, 0, \ldots, 0] = c_{n}[^{\ldots}, m]$. The corresponding $\beta$-representations for the first few coefficients of (7) are

\[
d_2 = \beta_0 d_2[1] + d_2[0],
\]

\[
d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0, 1] + \beta_0 d_3[1] + d_3[0],
\]

\[
d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1, 1] + \beta_2 d_4[0, 0, 1] + \beta_0^2 d_4[2] + \beta_1 d_4[0, 1] + \beta_0 d_4[1] + d_4[0].
\]
Analogous representations also hold for higher coefficients \( d_n \) in the PT series for the \( D \) function and for the coefficients \( c_l \) of the PT series for \( C_{Bj} \) given by (7), and so on. We stress that the representations like (17)-(19) are unique. The coefficients \( d_n[n-1] \) and \( c_n[n-1] \) are identical to the terms generated by the chains of one-particle-reducible one-loop fermion insertions into the gluon propagators and can be found, for example, in [13]. Determining the explicit forms of the other elements is a separate and not simple task. Their diagram representation was discussed in [33]. Below, we consider a way to obtain concrete analytic expressions for the elements of the coefficients \( d_n \) and \( c_l \) up to corrections of the order \( a_s^4 \). Expansion (6) together with (17)-(19) allows finding the relation between the unknown elements of the fourth-order PT coefficients \( d_4 \) and \( c_4 \) and the elements in the expressions for the third order of the PT series (which are presented explicitly below).

By virtue of relation (1) following from the unbroken conformal symmetry restored in the hypothetical case at \( \beta_1 = 0 \), we find an explicit relation between the contributions “cleaned” from the charge renormalizations:

\[
c_n[0] + d_n[0] + \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0] = 0. \tag{20}
\]

The special feature of this recurrence relation is the possibility to express the sum of the \( n \)-th order PT elements in terms of the analogous elements in the coefficients of lower PT approximations. The relation for the “cleaned” elements \( c_4[0] \) and \( d_4[0] \) of the coefficients of the fourth-order PT hence follows:

\[
c_4[0] + d_4[0] = 2d_1d_3[0] - 3d_1^2d_2[0] + (d_2[0])^2 + d_4^1. \tag{21}
\]

We note that this equation contains contributions proportional not only to the Casimir operators \( C_F \), \( C_F^4 \), is equivalent to the relation previously used in [37] to formulate the proposed verification of the QED result for an analogue of \( d_4 \) first published in [50]. The explicit expression for \( d_3 \) in the \( \beta \)-expansion was obtained in [33] thanks to using the analytic result evaluated in [17] for the contribution to the third coefficient of the PT series for the Adler function \( D(\alpha_s, \nu_t, \nu_g) \) with \( \nu_g \) gluino multiplets when the contributions from scalar quarks (squarks) are neglected in the supersymmetric variant of QCD. At the level of \( a_s^3 \) corrections, the analytic result for the gluino contributions in [17] coincides with the numerical result in [44], and the gluino correction of the order \( a_s^3 \) evaluated analytically in the \( \overline{MS} \) scheme in [17] was confirmed in [45]. It is easy to obtain the element \( d_3[2] \) in [18]. Its value can also be extracted from the results in [13]. We should then separate the contributions from the terms \( \beta_1d_3[0,1] \) and \( \beta_0d_3[1] \) in the expression for \( d_3 \). They are both linear in the number \( \nu_t \) of quark flavors. They separate if we use additional degrees of freedom, the abovementioned gluino contributions labeled by the number \( \nu_g \) of gluino multiplets.

We can then find the explicit forms of the functions \( \nu_f = n_f(\beta_0, \beta_1) \) and \( \nu_g = n_g(\beta_0, \beta_1) \). These expressions can be obtained after taking the gluino contributions to the first two coefficients of the \( \beta \)-functions for this type of extension of QCD into account. These two-loop results are found from the calculations in [48]. The coefficients of the \( \beta \)-expansions of the terms \( d_2 \) and \( d_3 \) defined in [17] and (18) were obtained just this way in [33]. We here present the results

\[ \footnote{We note that the possible existence of a gluino with a mass in the region \( m_g \geq 195 \text{ GeV}, \) lighter than the squark in the minimal supersymmetric standard model (MSSM), is not excluded by the existing Tevatron data [45] but was recently excluded by LHC data. Nevertheless, the joint detailed analysis of the available LHC data by the CMS and ATLAS collaborations still does not exclude the possible existence of a gluino with a mass in the region \( m_g \geq 400 \text{ GeV} \) heavier than squarks [47].} \]
in [43] only slightly changing the normalization coefficients:

\[ d_1 = \frac{3}{4}C_F, \quad d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)C_F, \quad d_2[0] = \frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A, \quad (22) \]

\[ d_3[2] = \left(\frac{151}{6} - 19\zeta_3\right)C_F, \quad d_3[1] = \left(\frac{-27}{8} - \frac{39}{4}\zeta_3 + 15\zeta_5\right)C_F^2 - \left(\frac{9}{64} - 5\zeta_3 + \frac{5}{2}\zeta_5\right)C_F C_A, \quad (23) \]

\[ d_3[0, 1] = \left(\frac{101}{16} - 6\zeta_3\right)C_F, \quad d_3[0] = -\frac{69}{128}C_F^3 + \frac{71}{64}C_F^2 C_A + \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)C_F C_A^2. \quad (24) \]

We now express the elements \(c_3[\ldots]\) in an analogous form. Using (20) to determine \(c_3[0]\) and taking the analytic expression for \(d_3[0]\) in (24) into account, we obtain

\[ c_1 = -\frac{3}{4}C_F, \quad c_2[1] = -\frac{3}{2}C_F, \quad c_2[0] = \frac{21}{32}C_F^2 - \frac{1}{16}C_F C_A, \quad (25) \]

Expansions similar to (17), (18) were previously used in [49] both for the Adler function and for the Bjorken sum rule. But only the terms proportional to powers of \(\beta_0\) (including its zeroth power) were then taken into account. In general, it is more consistent to use the approach in [43], which prescribes also taking the contribution of the two-loop coefficients \(\beta_1\) of the RG \(\beta\)-function into account. Now substituting the corresponding forms (17)-[19] for \(d_i\) and \(c_i\) in our proposed representation (6), we obtain the expressions

\[ P_1(a_s) = a_s \left\{ P_1^{(1)} + a_s P_1^{(2)} + a_s^2 P_1^{(3)} \right\} \]
\[ = -a_s \left\{ c_2[1] + d_2[1] + a_s \left( c_3[1] + d_3[1] + d_1 \left( c_2[1] - d_2[1] \right) \right) \right. \]
\[ +a_s^2 \left( c_4[1] + d_4[1] + d_1 \left( c_3[1] - d_3[1] \right) + d_2[0] c_2[1] + d_1[2] c_2[0] \right) \} \quad (26) \]

\[ P_2(a_s) = a_s \left\{ P_2^{(1)} + a_s P_2^{(2)} \right\} \]
\[ = a_s \left( c_3[2] + d_3[2] + a_s \left( c_4[2] + d_4[2] - d_1 \left( c_3[2] - d_3[2] \right) \right) \right) \quad (27) \]

\[ P_3(a_s) = a_s P_3^{(1)} = -a_s \left( c_4[3] + d_4[3] \right) = a_s C_F \left( \frac{307}{2} - \frac{203}{2} \zeta_3 - 45\zeta_5 \right) \quad (28) \]

\[ P_n(a_s) = a_s P_n^{(1)} = (-1)^n - 1 a_s \left\{ c_n[n - 1] + d_n[n - 1] \right\} \quad (29) \]

The concrete expression for (29) is defined in (16). We stress that the analytic form of formulae (15) obtained previously acquires a concrete relation to the \(\beta\)-expansion method (see [26,28]). The elements \(d_n[n - 1](c_n[n - 1])\) are defined by the diagrams containing a single gluon propagator with a chain of one-particle-reducible one-loop fermion insertions (so-called leading renormalon contributions) and can be determined from the results obtained in [13]. The elements \(d_n[l], \ l < n - 1\), are defined by the diagrams with at least two gluon propagators with both containing the one-particle-reducible one-loop fermion insertions (so-called subleading renormalon contributions). The similar classes of diagrams have not yet been evaluated explicitly. Using main theoretical result (4), which we have explicitly verified in the fourth PT order, we can obtain the relations between the elements of the \(\beta\)-expansion coefficients \(d_1(d_n)\) and \(c_4(c_n)\). Thus, the first term of the polynomial \(P_1(a_s)\) in (6) is determined by the chain of
The second term $P_1^{(2)}$ of the same polynomial, analytically fixed in (15), also relates different elements of the $\beta$-expansion approach:

\[
P_1^{(2)} = -c_3[1] - d_3[1] - d_1(c_2[1] - d_2[1]) = -c_4[0, 1] - d_4[0, 1] - d_1(c_3[0, 1] - d_3[0, 1]) = \ldots = -c_n[0, \ldots, 1] - n \cdot d_n[0, \ldots, 1] - d_1\left(c_{n-1}[0, \ldots, 1] - d_{n-1}[0, \ldots, 1]\right) = \left(\frac{397}{96} + \frac{17}{2} \zeta_4 - 15 \zeta_5\right) C_F^2 \]

But to obtain analytic expressions for $P_1^{(3)}$ and $P_1^{(4)}$ using the same $\beta$-expansion method, we must find the $\beta$-expansion representations for the fourth-order coefficients in the PT series for the functions $D(a_s)$ and $C^{Bijp}(a_s)$.

This can be done after additionally evaluating the gluino contributions to these important quantities analytically in the fourth PT order and taking the three-loop gluon effects in the QCD RG $\beta$-function evaluated in the MS- scheme in [50] into account. The relations obtained above allow deriving a new theoretical expression for the sum $d_4 + c_4$ of the fourth-order coefficients of the PT series. For this, we fix the number $n_f$ of fermion flavors from the condition $\beta_0(n_f = n_0) = 0$ which corresponds to the Banks-Zaks ansatz [51] and leads to the value $T_F n_0 = (11/4) C_A$. In this case, we obtain

\[
c_4(n_0) + d_4(n_0) = c_4[0] + d_4[0] + \beta_2(n_0) (c_4[0, 0, 1] + d_4[0, 0, 1]) + \beta_1(n_0) (c_4[0, 1] + d_4[0, 1]) \]

The terms in the right-hand side of (32) are known from (21) and (30) (i.e., $-c_4[0, 0, 1] - d_4[0, 0, 1]$) and from (31) (i.e., $-c_3[0, 1] - d_3[0, 1]$). Substituting the value $n_0$ fixed above in $\beta_1$ and $\beta_2$ and using (32), we obtain

\[
d_4(n_0) + c_4(n_0) = \left(\frac{333}{1024} C_F^4 + C_A C_F^3 \left(\frac{-1661}{3072} + \frac{1309}{128} \zeta_3 - \frac{165}{16} \zeta_5\right) + C_A^3 C_F^2 \left(\frac{3337}{1536} + \frac{7}{2} \zeta_3 - \frac{105}{16} \zeta_5\right) + C_A^3 C_F \left(\frac{-28931}{12288} + \frac{1351}{512} \zeta_3\right)\right). \]

Fixing the number $n_f = n_0$ of quark flavors in the concrete analytic expression $d_4(n_0) + c_4(n_0)$ following from the calculations in [33], we find agreement with the right-hand side of (33).

In summary, using the new representation of the generalized Crewther relation derived here (see Eq. (13)) and also the $\beta$-expansion method in [43] and the Banks-Zaks ansatz [51] allowed obtaining an additional argument for the correctness of the results of complicated and lengthy computer analytic calculations performed by a group from the Institute for Nuclear Research, the Institut fur Theoretische Teilchenphysik (Karlsruhe), and the Skobeltsyn Institute of Nuclear Physics (Moscow State University) [33]. Moreover, the absence of transcendental terms proportional to $\zeta_7$ and $\zeta_5^2$ from the right-hand side of (33) after the $\beta_0$ coefficient vanishes confirms the observation made in [33] that such contributions to the coefficients $d_4$ and $c_4$ determined in the MS scheme are proportional to the first coefficient $\beta_0$ of the QCD RG $\beta$-function (see the results in [33] and expression (3)).
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