EQUIVALENT NORMS FOR POLYNOMIALS ON THE SPHERE

JORDI MARZO AND JOAQUIM ORTEGA-CERDÀ

ABSTRACT. We find necessary and sufficient conditions for a sequence of sets $E_L \subset S^d$ in order to obtain the inequality

$$\int_{S^d} |Q_L|^p d\sigma \leq C_p \int_{E_L} |Q_L|^p d\mu, \quad \forall L \geq 0,$$

where $1 \leq p < +\infty$, $Q_L$ is any polynomial of degree smaller or equal than $L$, $\mu$ is a doubling measure and the constant $C_p$ is independent of $L$. From this description it follows an uncertainty principle for functions in $L^2(S^d)$. We consider also weighted uniform versions of this result.

1. INTRODUCTION

The classical Logvinenko-Sereda theorem describes some equivalent norms for functions in the Paley-Wiener space $PW^p_\Omega$, i.e. functions in $L^p(\mathbb{R}^d)$ whose Fourier transform is supported in a prefixed bounded set $\Omega \subset \mathbb{R}^d$.

Theorem (Logvinenko-Sereda). Let $\Omega$ be a bounded set and let $1 \leq p < +\infty$. A set $E \subset \mathbb{R}^d$ satisfies

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C_p \int_{E} |f(x)|^p dx, \quad \forall f \in PW^p_\Omega,$$

if and only if there is a cube $K \subset \mathbb{R}^d$ such that

$$\inf_{x \in \mathbb{R}^d} |(K + x) \cap E| > 0.$$

For a proof see [HI94, pp. 112–116] or the original [LS74].

Comparison norms results of this kind are known in other contexts, see [HI94] and references therein for further information. The purpose of the present paper is to prove similar comparison results for $L^p$ norms of polynomials on the unit sphere $S^d$.

In what follows $\sigma$ will denote the surface measure in $S^d$. We will prove the following theorem:

Theorem. Let $1 \leq p < +\infty$. A sequence of sets $E = \{E_L\}_{L \geq 0}$ in $S^d$ satisfies

$$\int_{S^d} |Q_L|^p d\sigma \leq C_p \int_{E_L} |Q_L|^p d\sigma, \quad \forall L \geq 0,$$

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where $Q_L$ is any polynomial of degree smaller or equal than $L$ and the constant $C_p$ is independent of $L$ if and only if
\[
\inf_{L \in \mathbb{N}, |z|=1/L} h_z(E_L) > 0,
\]
where $h_z(F)$ is the harmonic extension of $\chi_F$ to a point $z$ in the interior of the ball.

A more general (and precise) version will be stated and proved later on, see Theorem 1.5 once we have introduced some definitions and notation.

From this Theorem it follows an uncertainty principle for functions in $L^2(S^d)$. For any $f \in L^2(S^d)$ we have the spherical harmonics expansion
\[
f = \sum_{\ell \geq 0} P^\ell(f),
\]
where $P^\ell$ is the orthogonal projection from $L^2(S^d)$ to $H^\ell$.

**Corollary 1.1.** For a set $E \subset S^d$ let $\delta = \inf_{1 - |z|=1/L} h_z(E)$. There exists a constant $C > 0$ depending only on $\delta$ such that for any $f \in L^2(S^d)$
\[
\int_{S^d} |f(u)|^2 d\sigma(u) \leq C \left( \int_E |f(u)|^2 d\sigma(u) + \sum_{\ell>L} \|P^\ell(f)\|^2 \right).
\]

The proof of the Corollary amounts to show that (2) is equivalent to the inequality (1) and it can be found in [HJ94, 3.1.1.A) pp. 88–89].

1.1. **Preliminaries and Statements.** In $S^d$ we take the geodesic distance
\[
d(u,v) = \arccos \langle u,v \rangle, \quad u,v \in S^d,
\]
and let $B(\omega, \delta) \subset S^d$ denote the geodesic ball of center $\omega \in S^d$ and radius $\delta > 0$.

Let $H_\ell$ be the spherical harmonics of degree $\ell$ i.e. the restrictions to the unit sphere $S^d$ of the homogeneous harmonic polynomials in $d + 1$ variables of degree $\ell$. Let $\Pi_L = \bigcup_{\ell=0}^{L} H_\ell$ denote the spherical harmonics of degree less or equal than $L$. Observe that the restriction to $S^d$ of any polynomial in $d + 1$ variables of degree $\leq L$ belongs to $\Pi_L$.

In the Hilbert space $L^2(\sigma)$ let us denote by $Y^0_\ell, \ldots, Y^h_\ell$ an orthonormal basis of $H_\ell$. Taking all these bases for $\ell = 0, \ldots L$ together we get an orthonormal basis for $\Pi_L$.

It is well known that the reproducing kernel for $\Pi_L$ is
\[
K_L(u,v) = \sum_{\ell=0}^{L} \sum_{j=1}^{h_\ell} Y^j_\ell(u)Y^j_\ell(v), \quad u,v \in S^d,
\]
and this expression does not depend on the choice of the bases. Using the Christoffel-Darboux formula (see for instance [Jor06]) we obtain
\[
K_L(u,v) = \frac{\kappa_{d,L}}{\sigma(S^d)} P_L^{(d/2,d/2-1)}(\langle u,v \rangle),
\]
where $P_L^{(\alpha,\beta)}$ stands for the Jacobi polynomial of degree $L$ and index $(\alpha, \beta)$ and $\kappa_{d,L} \sim L^{d/2}$, as $L \to \infty$. From now on we denote $\lambda = (d-2)/2$.

\[^1\]Here and in what follows $\sim$ means that the ratio of the two sides is bounded from above and from below by two positive constants.
Finally, we recall an estimate, \cite[p.~198]{Sze39}, that will be used later on:

\begin{equation}
P_L^{(1+\lambda,\lambda)}(\cos \theta) = \frac{k(\theta)}{\sqrt{L}} \left\{ \cos \left( (L + \lambda + 1)\theta - \frac{(d + 1)\pi}{4} \right) + O(1) \right\},
\end{equation}

if \( c/L \leq \theta \leq \pi - (c/L) \), where

\[ k(\theta) = \frac{\pi}{2} \left( \frac{\sin \theta}{2} \right)^{-\lambda - 3/2} \left( \frac{\cos \theta}{2} \right)^{-\lambda - 1/2}. \]

**Definition 1.2.** We say that a measure \( \mu \) is **doubling** if there exist a constant \( C > 0 \) such that for any \( u \in \mathbb{S}^d \) and any \( \delta > 0 \),

\[ \mu(B(u, 2\delta)) \leq C \mu(B(u, \delta)), \]

For such a measure \( \sup_{u, \delta} \mu(B(u, 2\delta))/\mu(B(u, \delta)) \) is called the doubling constant of \( \mu \).

It can be seen (see for instance \cite[Lemma 2.1.]{MT00}) that for \( \mu \) doubling there exist a \( \gamma > 0 \) such that for \( r > r' \)

\begin{equation}
\left( \frac{r}{r'} \right)^{1/\gamma} \leq \frac{\mu(B(u, r))}{\mu(B(u, r'))} \leq \left( \frac{r}{r'} \right)^{\gamma},
\end{equation}

with constants depending only on the doubling constant of \( \mu \).

Mimicking the Euclidean situation we define the following concept.

**Definition 1.3.** Let \( 1 \leq p < \infty \) and let \( \mu \) be a doubling measure. We say that the sequence of sets \( E = \{E_L\} \subset \mathbb{S}^d \) is \( L^p(\mu) \)-**Logvinenko-Sereda** if there exists a constant \( C_p > 0 \) such that for any \( Q \in \Pi_L \) and any \( L \)

\begin{equation}
\int_{\mathbb{S}^d} |Q(u)|^p d\mu(u) \leq C_p \int_{E_L} |Q(u)|^p d\mu(u). 
\end{equation}

**Definition 1.4.** The sequence of sets \( E = \{E_L\} \subset \mathbb{S}^d \) is \( \mu \)-**relatively dense** if there exist \( r > 0 \) and \( \varrho > 0 \) such that

\begin{equation}
\inf_{u \in \mathbb{S}^d} \frac{\mu(E_L \cap B(u, r/L))}{\mu(B(u, r/L))} \geq \varrho > 0,
\end{equation}

for all \( L \). When \( \mu \) is the Lebesgue measure we say that \( E \) is relatively dense.

Now we can state our main result.

**Theorem 1.5.** Let \( E = \{E_L\} \subset \mathbb{S}^d \) be a sequence of sets in \( \mathbb{S}^d \). \( E \) is \( L^p(\mu) \)-Logvinenko-Sereda for some \( 1 \leq p < \infty \) and \( \mu \) a doubling measure if and only if \( E \) is \( \mu \)-relatively dense.

If \( \mu \) is absolutely continuous with an \( A_\infty \) weight it is possible to reformulate the \( \mu \)-relatively density in terms of the harmonic extension.

For a weight \( \omega \geq 0 \) in \( \mathbb{S}^d \) we denote

\[ \omega(E) = \int_E \omega(u) d\sigma(u), \quad E \subset \mathbb{S}^d. \]
Definition 1.6. A weight $\omega$ belongs to $A_\infty$ if there exist constants $B, \beta > 0$ such that for any $E \subset B(u, \delta)$ measurable

$$\omega(B(u, \delta)) \leq B \left( \frac{\sigma(B(u, \delta))}{\sigma(E)} \right)^\beta \omega(E).$$

It is well known that an $A_\infty$ weight defines a doubling measure but the converse is not true, see [FM74].

Remark. Due to the reversibility of condition (7), see [Ste93, chap. V, 1.7], being relatively dense is equivalent to the same condition for the measure defined with $\omega \in A_\infty$.

We recall that for $x \in \mathbb{R}^{d+1}$ with $|x| < 1$ the harmonic measure of subset $F \subset S^d$ with respect to $x$ is

$$h_x(F) = \frac{1}{\sigma(S^d)} \int_F \frac{1 - |x|^2}{|x - u|^{d+1}} d\sigma(u) = \frac{1}{\sigma(S^d)} \int_F P(x, u) d\sigma(u),$$

and $P(x, u)$ is the Poisson kernel in $S^d$. The next result is a version for $S^d$ of the one proved in [HJ94, p. 114].

Lemma 1.7. The sequence $\{E_L\}_{L \geq 0} \subset S^d$ is relatively dense if and only if there exists $\alpha > 0$ such that

$$h_{(1 - 1/L)} u (E_L) \geq \alpha,$$

for all $x \in \mathbb{R}^{d+1}$ with $|x| = 1 - 1/L$.

Proof. Observe that both conditions are rotation invariant. For $u$ such that $d(u, N) < r/L$ we have $C L^d \leq P(|x|N, u) \leq 2L^d$, where $C > 0$ is a constant depending on $r$ and $d$. For $\theta = d(u, N) > r/L$

$$P(|x|N, u) \lesssim \frac{2 - \frac{1}{L}}{\sin^{d+1} \frac{\theta}{2}} \lesssim \frac{L^d}{r^{d+1}}.$$

These bounds are all we need to prove the result. In one direction

$$h_{|x|N} (E_L) \gtrsim L^d \sigma(E_L \cap B(N, r/L)) \gtrsim \varrho > 0.$$

Conversely

$$\sigma(S^d) \alpha \leq \int_{E_L} P(|x|N, u) d\sigma(u) \leq \int_{E_L} (\chi_{B(N,R/L)}(t) + \chi_{B(N,R/L)^c}(t)) P(|x|N, u) d\sigma(u) \leq 2L^d \sigma(E_L \cap B(N, r/L)) + \sum_{j \geq 0} \int_0^{2^j r < d(u, N) < 2^{j+1} r} \chi_{E_L}(u) P(|x|N, u) d\sigma(u) \leq C_r \frac{\sigma(E_L \cap B(N, r/L))}{\sigma(B(N, r/L))} + C \sum_{j \geq 0} \frac{1}{2^{dj}}.$$

Taking $r > 0$ big enough we get the result. □

Remark. We have proved that there exist $r, \varrho$ such that $\sigma(E_L \cap B(u, r/L)) \geq \varrho \sigma(B(u, r/L))$ for $L$ big enough if and only if there exists $\alpha$ such that $h_{(1 - 1/L)u} (E_L) \geq \alpha$. This new formulation depends only on one parameter.
In Theorem 1.5 when the dimension $d = 1$ there are already some results known. In this case it is possible to replace polynomials by holomorphic polynomials. If moreover $\mu$ is the Lebeug measure the space of holomorphic polynomials can be seen as a model space, so Volberg result \[Vol81\] extending the original theorem of Logvinenko and Sereda to model spaces apply. Also when $d = 1$ and the measure $\mu$ is an $A_\infty$ weight, the sufficiency of condition (6) was proved in \[MT00, Theorem 5.4\].

Condition (6) is true for some $\omega \in A_\infty$ if and only if it is true for the Lebesgue measure $\sigma$. So we have comparison of norms for any $\omega \in A_\infty$ if and only if we have (6) for the Lebesgue measure $\sigma$. The discussion following \[MT00, Theorem 5.4\] shows that this is not true for arbitrary doubling measures.

To the best of our knowledge, for dimensions greater than one Theorem 1.5 is new, even in the case of Lebesgue measure.

The outline of this paper is as follows. In Section 2 we will prove Theorem 1.5.

In Section 3 we deal with the uniform norm case. In this setting we have an analogous result to Proposition 2.2, namely Theorem 3.1. To consider weighted versions of this result an obvious requirement is to take weights bounded above. We take the reverse Hölder class $RH_\infty$ of those weights satisfying reverse Hölder inequalities in a uniform way. This class that was also introduced in \[MT00\] for the one-dimensional case, is shown to be optimal in a certain sense.

2. Main Results

Proposition 2.1. Let $1 \leq p < \infty$, $\mu$ be a doubling measure and let $E = \{E_L\}_{L \geq 0}$ be a sequence of sets in $\mathbb{S}^d$. If $E$ is $L^p(\mu)$-Logvinenko-Sereda, then it is $\mu$-relatively dense.

Proof. We focus on $d \geq 2$ but only minor changes will prove the one-dimensional case. The strategy is to apply the $L^p(\mu)$-comparison of norms to a power of the reproducing kernel and to use classical estimates on the Jacobi polynomials.

Let $Q(v) = (P_L^{1+\lambda,\lambda}(\langle v, N \rangle))^{t} \in \Pi_{tL}$ and let $0 < r << R$. We have by hypothesis,

\[
\int_{B(N, r/L)} |Q(v)|^p d\mu(v) \leq \int_{\mathbb{S}^d} |Q(v)|^p d\mu(v) \leq C \int_{E_{tL}} |Q(v)|^p d\mu(v)
\]

\[
\lesssim \int_{E_{tL} \cap B(N, R/L)} |Q(v)|^p d\mu(v) + \int_{\mathbb{S}^d \setminus B(N, R/L)} |Q(v)|^p d\mu(v).
\]

Observe that $Q$ reach its maximum in $N$, \[Sze39\] so applying Bernstein’s inequality to the polynomial restricted to a great circle we get for any $v$ such that $d(v, N) < r/L$

\[
|Q(v) - Q(N)| \leq |Q(N)| \ell r.
\]

Therefore for $r$ small enough we have $|Q(v)|^p \sim |Q(N)|^p$ if $d(v, N) < r/L$. We can bound the integral in the left hand side of (8) as

\[
\int_{B(N, r/L)} |Q(v)|^p d\mu(v) \gtrsim (P_L^{1+\lambda,\lambda}(1))^{tL^d \mu(B(N, r/L))} \sim L^{tL^d} \mu(B(N, r/L)).
\]
Since \(|P_L^{1+\lambda \mu}(\cos \theta)| \leq L^\lambda\) for \(\pi - \frac{R}{L} \leq \theta \leq \pi\),

\[ L^{\frac{pl+1}{2}} \mu(B(N, r/L)) \lesssim L^{\frac{pl+2}{2}} \mu(E_{\ell L} \cap B(N, R/L)) + L^{\frac{pl(d-2)}{2}} \mu(B(S, R/L)) + \int_{R/L < d(v, N) < \pi - R/L} |Q(v)|^p d\mu(v). \]

To control the last integral we may use Szegö estimate (3)

\[ \int_{R/L < d(v, N) < \pi - R/L} |Q(v)|^p d\mu(v) \lesssim L^{-\frac{pl}{2}} \int_{R/L < d(v, N) < \pi/2} \left| \sin^{d+1} \frac{d(v, N)}{2} \right|^{-\ell p/2} d\mu(v) + L^{-\frac{pl}{2}} \int_{\pi/2 < d(v, N) < \pi - R/L} \left| \cos^{d+1} \frac{d(v, N)}{2} \right|^{-\ell p/2} d\mu(v) = I + II. \]

For part I we take \(\ell\) big enough to get \(C(\mu) < 2^{d(d+1)/4}\), where \(C(\mu)\) is the doubling constant of \(\mu\). We split the sphere in diadic “bands” around the north pole and using the doubling property for \(\mu\) we get

\[ L^{pl/2} I \lesssim \int_{R/L < d(v, N) < \pi - R/L} \frac{1}{d(v, N)^{(d+1)p/2}} d\mu(v) \leq \sum_{j \geq 0} \int_{2^{j}R/L < d(v, N) < 2^{j+1}R/L} \frac{d\mu(v)}{(2^jR/L)^{(d+1)p/2}} \leq \sum_{j \geq 0} \frac{\mu(B(N, 2^j R/L))}{(2^j R/L)^{(d+1)p/2}} \leq \mu(B(N, R/L)) \sum_{j \geq 0} \left( \frac{C(\mu)}{2^{\lambda+\lambda}} \right)^j \lesssim \frac{\mu(B(N, R/L))}{(R/L)^{(d+1)p/2}}. \]

where \(\mathbb{N} \ni J \geq \log_2(\pi L/R).\)

For part II the same computation taking dyadic “bands” around the south pole shows that

\[ L^{pl/2} II \lesssim \frac{\mu(B(S, R/L))}{(R/L)^{(d-1)p/2}} \lesssim \mu(B(S, R/L)) L^{(d-1)p/2}. \]

We use now property (4) and the \(\gamma\) given there for \(\mu\) to estimate \(\mu(B(S, R/L))\). If we put all estimates together and for \(\ell\) big enough we get

\[ \mu(B(N, r/L)) \lesssim \mu(E_{\ell L} \cap B(N, R/L)) + L^{-\ell} \mu(B(S, R/L)) + \mu(B(N, R/L)) \frac{R^\gamma}{L^{\gamma+pl}} + \frac{R^\gamma}{r} \mu(B(N, r/L)) \frac{R}{R^{(d+1)p/2}}. \]

As \(\mu(B(N, r/L)) \geq (r/L)^{1/\gamma}\) the second term is \(o(\mu(B(N, r/L)))\) when \(L \to \infty\) for \(\ell\) big enough. For the third term we choose \(\ell\) such that \((R/r)^\gamma \leq R^{(d+1)p/2}/2.\) Thus picking \(\ell\) big enough we have proved that

\[ \mu(B(N, r/L)) \lesssim \mu(E_{\ell L} \cap B(N, R/L)), \quad \text{if } L \geq L_0 \]

Of course, the constants do not depend of the center of the balls being the north pole. Moreover by the doubling property \(\mu(B(N, R/L)) \simeq \mu(B(N, r/L))\). By choosing a bigger \(R\) we get

\[ \mu(B(z, R/L)) \lesssim \mu(E_{\ell L} \cap B(z, R/L)), \quad \forall z \in \mathbb{S}^d, \quad L \geq 0. \]
Finally we have only controlled the density of the sequence of sets \( \{ E_{IL} \}_{L \geq 0} \). But we could have used the same argument to the sequence \( \mathcal{E}' = \{ E_{L+1} \}_{L \geq 0} \) from the very beginning and we will obtain then a control of the density the sets \( \{ E_{IL+1} \}_{L \geq 0} \). By repeating the argument \( l \) times we get the desired result. \( \square \)

Remark. The somehow simpler polynomials

\[
\left( \frac{1 + \langle v, N \rangle}{2} \right)^L \text{ or } \left( \frac{1 - \langle v, N \rangle^{L+1}}{(L+1)(1 - \langle v, N \rangle)} \right)^L
\]

that peak at \( N \) and have been considered in other contexts do not decrease fast enough near the pole north to be chosen as test functions for the comparison of norms as we did with the polynomial \( Q \) above.

**Proposition 2.2.** If \( \{ E_L \}_{L \geq 0} \) is \( \mu \)-relatively dense for some doubling measure \( \mu \) then it is \( L^p(\mu) \)-Logvinenko-Sereda for any \( 1 \leq p < \infty \).

**Proof.** We consider a regularized version of \( \mu \)

\[
\mu_L(u) = \frac{\mu(B(u, 1/L))}{\sigma(B(u, 1/L))}, \quad L \geq 0.
\]

By Corollary 3.4. in [Dai06] we have

\[
\int_{\mathbb{S}^d} |Q_L(u)|^p d\mu(u) \sim \int_{\mathbb{S}^d} |Q_L(u)|^p \mu_L(u) d\sigma(u), \quad Q_L \in \Pi_L.
\]

The regularization of \( \mu \) is pointwise equivalent to a polynomial. Indeed, there exists \( R_L \in \Pi_L \) nonnegative such that for any \( u \in \mathbb{S}^d \)

\[
\mu_L(u) \sim R_L(u)^p,
\]

with constant depending only on \( d \), the doubling constant for \( \mu \) and \( p \), see [Dai06, Lemma 4.6]. Given \( Q_L \in \Pi_L \) let \( M_{2L} \in \Pi_{2L} \) such that \( M_{2L} = Q_L R_L \) in \( \mathbb{S}^d \). Following an idea of D. H. Luecking [Lue83] we consider, for \( \epsilon > 0 \) and \( r > 0 \), the set of points \( z \in \mathbb{S}^d \) such that \( M_{2L}(z) \) has the same size as its average, i.e.

\[
A = A_{\epsilon, r, M_{2L}} = \left\{ z \in \mathbb{S}^d : |M_{2L}(z)|^p \geq \epsilon \int_{B(z, r/L)} |M_{2L}(u)|^p dm(u) \right\}.
\]

Most of the norm of \( M_{2L} \) is concentrated on \( A \),

\[
\int_{\mathbb{S}^d \setminus A} |M_{2L}(z)|^p d\sigma(z) \leq \epsilon \int_{\mathbb{S}^d \setminus A} \left( \int_{B(z, r/L)} |M_{2L}(u)|^p dm(u) \right) d\sigma(z)
\]

\[
\leq \epsilon \int_{|z| < r/L} |M_{2L}(u)|^p \left( \int_{\mathbb{S}^d \setminus A} \frac{\chi_B(z, r/L)(u)}{m(B(z, r/L))} dm(u) \right) d\sigma(z)
\]

\[
\leq \epsilon L \int_{|z| < r/L} |M_{2L}(u)|^p dm(u) \sim \epsilon \int_{\mathbb{S}^d} |M_{2L}(z)|^p d\sigma(z),
\]

using [Jor06, Corollary 4.3] in the last estimate, the constants are independent of \( L \).
Thus it is enough to show that
\[
\int_{A} |M_{2L}(u)|^p d\sigma(u) \lesssim \int_{E_L} |Q_L(u)|^p d\mu(u).
\]

All we need to prove is the existence of a constant \( C > 0 \) such that for all \( \omega \in A \)
\[
|Q_L(\omega)|^p \leq \frac{C}{\mu(B(\omega, r/L))} \int_{B(\omega, r/L) \cap E_L} |Q_L(u)|^p d\mu(u).
\]

Indeed, if this is the case then
\[
\int_{A} |M_{2L}(\omega)|^p d\sigma(\omega) \leq C \int_{E_L} |Q_L(u)|^p \int_{S^d} \frac{\chi_{B(\omega, r/L)}(u)}{\mu(B(\omega, r/L))} \mu_L(\omega) d\sigma(\omega) d\mu(u)
\]
\[
\lesssim \int_{E_L} |Q(u)|^p d\mu(u).
\]

To prove (9) we argue by contradiction. If (9) is false there are for any \( n \in \mathbb{N} \) polynomials \( Q_n \in \Pi_{L_n} \) and \( \omega_n \in A \) such that
\[
|Q_n(\omega_n)|^p > \frac{n}{\mu(B(\omega_n, r/L_n))} \int_{B(\omega_n, r/L_n) \cap E_{L_n}} |Q_n(u)|^p d\mu(u).
\]

Since \( \mu \) is doubling then \( R_{L_n}(\omega_n) \sim R_{L_n}(u) \) for any \( u \in B(\omega_n, r/L_n) \). Let \( M_n \in \Pi_{2L_n} \) be such that \( M_n = Q_n R_{L_n} \in S^d \)
\[
|M_n(\omega_n)|^p \gtrsim \frac{n}{\mu(B(\omega_n, r/L_n))} \int_{B(\omega_n, r/L_n) \cap E_{L_n}} |M_n(u)|^p d\mu(u).
\]

By means of a rotation, a dilation and a translation we send \( \omega_n \) to the origin in \( \mathbb{R}^{d+1} \), the ball \( B(\omega_n, r/L_n) \) to \( B(0, 1) \subset \mathbb{R}^{d+1} \) and the set \( E_{L_n} \) to
\[
E_n \subset \partial B(-(L_n/r)N, L_n/r) \cap B(0, 1).
\]

The composition of these applications with our harmonic polynomials \( M_n \) are harmonic functions \( f_n \) that, after normalization, we can assume that satisfy
\[
\int_{B(0,1)} |f_n|^p dm = 1.
\]

The subharmonicity of \( |f_n|^p \) and the fact that \( \omega_n \in A \) tells us that
\[
\epsilon \lesssim |f_n(0)|^p \lesssim 1,
\]
and this property together with (10) yields
\[
\frac{1}{n} \gtrsim \int_{B(0,1) \cap E_n} |f_n(u)|^p d\mu_n(u),
\]
where \( \mu_n \) is the push forward of the measure \( \mu \), supported in \( \partial B(-(L_n/r)N, L_n/r) \cap B(0, 1) \), and normalized in such a way that \( \mu_n(B(0, 1)) = 1 \).

We have that \( \{f_n\} \) is a normal family in \( B(0, 1) \) and therefore there exist a subsequence that converges locally uniformly on \( B \) to an harmonic function that we call \( f \).
We observe that the relative density hypothesis yields
\[ \inf \mu_n(E_n \cap B(0,1)) > 0. \]

Let \( \tau \) be a weak-* limit of a subsequence of \( \tau_n = \mu_n \chi_{E_n} \), having \( \text{supp} \tau \subset \mathbb{R}^d \times \{0\} \) and \( \tau \neq 0 \). We will consider the measure \( \tau_n \) that has support in \( \partial B(-L_n/r, L_n/r) \cap B(0,1) \) as having support in \( \mathbb{R}^d \times \{0\} \). To do so we define the measure \( \tilde{\tau}_n \) as the “projection” of the measure \( \tau_n \) to \( \mathbb{R}^d \times \{0\} \), i.e. \( \tilde{\tau}_n(A) = \tau_n(A \times [-1,1]) \), for \( A \subset \mathbb{R}^d \).

We observe that \( f \) restricted to \( \mathbb{R}^d \times \{0\} \) is real analytic. Condition (11) implies that \( f = 0 \) \( \tau \)-a.e. and therefore \( \text{supp} \tau \subset \{ f = 0 \} \).

We want to show that \( \text{supp} \tau \subset \mathbb{R}^d \) cannot lie on a real analytic \((d-1)\)-dimensional submanifold \( S \subset \mathbb{R}^d \) (the worst case). We argue by contradiction. Let \( x \in \text{supp} \tau \subset S \) and \( \delta > 0 \) be such that \( \tau(B(x,\delta)) = \epsilon > 0 \).

We can consider for any \( y \in B(x,\delta) \cap S \) the unitary vector \( \nu_y \) in \( \mathbb{R}^d \) normal to \( S \) in the point \( y \) (see figure 1) and define the “square” \( B(x,\delta) \subset R_x \)
\[ R_x = \{ y + \eta \nu_y : y \in B(x,\delta) \cap S, |\eta| < \delta \}. \]

Now we can define measures \( \nu_n \) in \( S_x^\perp = \{ x + \eta \nu_x : |\eta| < \delta \} \) just by taking for \( A \subset S_x^\perp \) the set \( \tilde{\alpha} \subset (-\delta,\delta) \) such that \( x + \tilde{\alpha} \nu_x = A \) and defining
\[ \nu_n(A) = \tilde{\tau}_n(\{ y + \eta \nu_y \in R_x : \eta \in \tilde{\alpha} \}). \]

By hypothesis \( \nu_n \) converges vaguely to some nonzero measure \( \nu \) with support in \( \{ x \} \), because \( \nu_n(S_x^\perp) = \tilde{\tau}_n(R_x) \geq \tilde{\tau}_n(B(x,\delta)) \geq \epsilon > 0 \). To get a contradiction it is enough to show that \( \nu \) is dominated by a doubling measure in \( S_x^\perp \).

We define
\[ \gamma_n(A) = \tilde{\mu}_n(\{ y + \eta \nu_y \in R_x : \eta \in \tilde{\alpha} \}), \quad A \subset S_x^\perp, \]
where as before \( \tilde{\mu}_n \) is the “projection” of \( \mu_n \) to \( \mathbb{R}^d \). Observe that \( \nu_n(A) \leq \gamma_n(A) \) and that \( \gamma_n \) are doubling measures all with the same doubling constant. Indeed, for any \( \delta > \alpha > 0 \) there exist
\[ y_1, \ldots, y_N \in B(x, \delta) \cap S \text{ such that} \]
\[
\{ y + \eta y_j \in R_x : |\eta| < 2\alpha \} \subset \bigcup_{j=1}^{N} B(y_j, 5\alpha/2), \quad \sum_{j=1}^{N} \chi_{B(y_j, 5\alpha/4)} \leq C.
\]

The “projection” of the \( \mu_n \) to \( \mathbb{R}^d \) are doubling measures all with the same doubling constant so
\[
\gamma_n(x + (-2\alpha, 2\alpha) \nu_x) \leq \sum_{j=1}^{N} \tilde{\mu}_n(B(y_j, 5\alpha/2)) \leq C \sum_{j=1}^{N} \tilde{\mu}_n(B(y_j, 5\alpha/4)) \leq C \gamma_n(x + (\alpha, \alpha) \nu_x).
\]

Therefore by (4) we have \( C, \gamma > 0 \) constants such that \( \nu_n(x + (-r, r) \nu_x) \leq Cr^\gamma \) and the same holds for \( \nu \). Observe that \( \text{supp} \nu \) has to be of Hausdorff dimension \( \geq \gamma > 0 \) and this would contradict \( \text{supp} \nu = \{ x \} \).

\[ \Box \]

3. Uniform Norm Case

In this section we want to find sufficient conditions in the sequence \( \mathcal{E} = \{ E_L \}_{L \geq 0} \) in order to get the \( L^\infty \)-Logvinenko-Sereda property, i.e.
\[
\sup_{u \in S^d} |Q_L(u)| \leq C \sup_{u \in E_L} |Q_L(u)|, \quad \text{for any } Q_L \in \Pi_L,
\]
with \( C \) a constant that does not depend on \( L \).

Our main result is the following:

**Theorem 3.1.** If \( \mathcal{E} \) is relatively dense, then it is \( L^\infty \)-Logvinenko-Sereda.

**Remark.** The converse is false because there exist discrete sets (so with zero Lebesgue measure) with comparison property (12).

In [MT00] the authors deal with the weighted one-dimensional case of Theorem 3.1. In this uniform case it is a natural assumption to consider only bounded weights. They considered the family of weights \( \omega \geq 0 \) such that
\[
\omega(u) \leq \frac{C}{\sigma(B)} \int_B \omega(v)d\sigma(v),
\]
for any spherical \( B \subset S^d \) and \( u \in B \). Following [CN95] we call \( RH_\infty \) this family.

**Definition 3.2.** Let \( \omega \geq 0 \) be a function such that property (13) holds for almost every \( u \in S^d \), we say that \( \omega \) is in the reverse Hölder class \( RH_\infty \).

To justify the name of this class observe that for \( \omega \in RH_\infty \) the reverse Hölder inequality
\[
\left( \frac{1}{\sigma(B)} \int_B \omega^s(u)d\sigma(u) \right)^{s} \leq \frac{C}{\sigma(B)} \int_B \omega(u)d\sigma(u), \quad B \subset S^d \text{ spherical cap}
\]
holds for each $s > 1$, (i.e. $\omega \in RH_s$) and the best constant $C$ is bounded by the constant appearing in (13). And conversely, if the reverse Hölder inequality holds for each $s > 1$ with a constant independent of $s$, then $\omega \in RH_\infty$, see [CN95].

Observe that $RH_\infty \subset A_\infty$. Roughly speaking $\omega$ belongs to $A_1$ if and only if $1/\omega \in RH_\infty$. These weights can have high order zeros in $\mathbb{S}^d$.

In this section we will prove the one-dimensional unweighted result first and then extend it to $\mathbb{S}^d$. Using this unweighted case and adapting some results from [Dai06, MT00] we will prove the weighted result.

Proof. We start with the one-dimensional case. Using the Lemma 1.7 we get $h_x(E_n) \geq \alpha$, for any $|x| = 1 - 1/L$. Let $p$ be a polynomial of degree $L$, there exists an holomorphic polynomial $q$ of degree $2L$ such that $|p| = |q|$ in $\mathbb{S}^1$. So for any $x \in \mathbb{R}^2$ with $|x| = 1 - 1/L$,

$$\log |q(x)| \leq h_x(E_L) \log(\max_{E_L} |q|) + h_x(S^1 \setminus E_L) \log(\max_{S^1 \setminus E_L} |q|)$$

$$= \log \|q\|_{S^1} + h_x(E_L) \log \|q\|_{E_L} \|q\|_{S^1} \leq \log \|q\|_{S^1} + \alpha \log \|q\|_{E_L},$$

because $\|q\|_{E_L}/\|q\|_{S^1} \leq 1$ and so $|q(x)| \leq \|p\|_{E_L}^{\alpha} |p|_{S^1}^{1-\alpha}$. Finally on can see that

$$\max_{x \in \mathbb{S}^1} |q(x)| \leq C \max_{|x| = 1 - 1/L} |q(x)|,$$

where $C$ is independent of $L$, see [OS05, Lemma 2].

Now we consider the case $d > 1$. Let $Q \in \Pi_L$ and suppose that $\max_{\mathbb{S}^d} |Q| = |Q(N)| = 1$. We have that

$$\sigma(E_L \cap B(N, r/L)) \geq \epsilon > 0.$$  

Denoting $\tilde{\omega} = (\omega, 0) \in \mathbb{R}^{d+1}$ for $\omega \in \mathbb{S}^{d-1}$ we have that $G_\omega(\theta) = N \cos \theta + \tilde{\omega} \sin \theta$ is a geodesic in $\mathbb{S}^d$ if $\theta \in [-\pi, \pi]$. Therefore denoting $S_+^{d-1} = \{\omega \in \mathbb{S}^{d-1} : \omega_d > 0\}$

$$\sigma(E_L \cap B(N, r/L)) = \int_{S_+^{d-1}} \int_{r/L}^{r/L} \chi_{E_L}(G_\omega(\theta)) \sin^{d-1} \theta d\theta d\omega.$$

Now

$$\epsilon \sigma(B(N, r/L)) \leq \int_{S_+^{d-1}} \int_{r/L}^{r/L} \chi_{E_L}(G_\omega(\theta)) \sin^{d-1} \theta d\theta d\omega$$

$$\leq \int_{S_+^{d-1}} \int_{r/L}^{r/L} \chi_{E_L}(G_\omega(\theta)) \left(\frac{r}{L}\right)^{d-1} d\theta d\omega$$

$$\leq \left(\frac{r}{L}\right)^{d-1} \int_{S_+^{d-1}} \sigma(E_L \cap B_\omega(N, r/L)) d\omega$$

where $B_\omega(N, r/L) = \{G_\omega(\theta) : |\theta| \leq r/L\}$. We get

$$\int_{S_+^{d-1}} \frac{\sigma(E_L \cap B_\omega(N, r/L))}{\sigma(B_\omega(N, r/L))} d\omega \geq C_d \epsilon,$$
and therefore there exists a direction \( \omega \in S^d_{+} \) such that

\[
\sigma(E_L \cap B_\omega(N, r/L)) \geq C_d \epsilon > 0.
\]

We get the result applying the one-dimensional case to the trigonometric polynomial \( Q(G_\omega(\theta)) \).

Using Theorem 3.1 we prove the following weighted version.

**Corollary 3.3.** If \( E \) is relatively dense and \( \omega \in RH_\infty \), then

\[
\sup_{u \in S^d} |Q_L(u)|\omega(u) \leq C \sup_{u \in E_L} |Q_L(u)|\omega(u), \text{ for any } Q_L \in \Pi_L,
\]

with \( C \) a constant that does not depend on \( L \).

**Remark.** This result is optimal in some sense because there are unbounded weights belonging to all reverse Hölder classes i.e. in particular \( RH_\infty \subset \cap_{s>1} RH_s \), see [CN95, p. 2948].

**Proof.** By definition of \( RH_\infty \) weight

\[
\omega(u) \leq C \omega_L(u) = \frac{1}{\sigma(B(u, 1/L))} \int_{B(u, 1/L)} \omega(v) d\sigma(v).
\]

Now [Dai06, Lemma 4.6.] provide us with \( R_L \in \Pi_L \) nonnegative such that for any \( u \in S^d \)

\[
\omega_L(u) \sim R_L(u), \text{ with constant depending only on the doubling constant for } \omega_L.
\]

Now we want to construct a relatively dense regularization of \( E_L \) that we will denote \( E^*_L \).

Given \( \epsilon > 0 \) let \( V = V_{\epsilon, L} \subset S^d \) discrete and such that

\[
S^d \subset \bigcup_{v \in V} B(v, \epsilon/L), \text{ and } \sum_{v \in V} \chi_{B(v, \epsilon/L)}(u) \leq C_d, \quad u \in S^d.
\]

For \( \delta > 0 \), that we will determine afterwards, let

\[
V_g = \{v \in V : \sigma(B(v, \epsilon/L) \cap E_L) \geq \delta \sigma(B(v, \epsilon/L))\}, \text{ and } E^*_L = \bigcup_{v \in V_g} B(v, \epsilon/L).
\]

We denote \( V_b = V \setminus V_g \). Let \( V(u) \) be the set of those \( v \in V \) such that \( B(v, \epsilon/L) \cap B(u, r/L) \neq \emptyset \) and likewise we split \( V(u) = V_g(u) \cup V_b(u) \)

\[
\sigma(B(u, r/2L)) \leq \sigma\left( \bigcup_{v \in V_g(u)} B(v, \epsilon/L) \right) + \sigma\left( \bigcup_{v \in V_b(u)} B(v, \epsilon/L) \right)
\]

\[
\leq \sigma\left(E^*_L \cap B(u, r/L) \right) + \sigma\left( \bigcup_{v \in V_b(u)} B(v, \epsilon/L) \right).
\]

Using the relative density of \( E_L \) and the property of being in \( V_b \) we get

\[
\rho \sigma(B(u, r/2L)) \leq \sigma\left(E_L \cap B(u, r/L) \right)
\]

\[
\leq C_d \delta \sigma(B(u, r/L)) + \sigma\left(E_L \cap (B(u, r/L) \setminus \bigcup_{v \in V_b(u)} B(v, \epsilon/L)) \right).
\]
so for \( \delta \) small enough
\[
\sigma(B(u, r/L)) - \sigma\left( \bigcup_{v \in V_b(u)} B(v, \epsilon/L) \right) \geq \frac{\delta}{2} \sigma(B(u, r/L)),
\]
so using (16) and (15) we get
\[
\frac{\delta}{2} \sigma(B(u, r/2L)) \leq \sigma(E_L^* \cap B(u, r/L)),
\]
and thus \( E_L^* \) is relatively dense.

Applying our unweighted result Theorem 3.1 to \( E_L^* \) and to \( M_{2L} \in \Pi_{2L} \), such that \( M_{2L} = Q_L R_L \) in \( S^d \), we get
\[
\sup_{u \in S^d} |Q_L(u)| \omega_L(u) \lesssim \sup_{u \in E_L^*} |Q_L(u)| \omega_L(u), \quad Q_L \in \Pi_L.
\]

We can take \( \epsilon > 0 \) small enough so that spherical harmonics of degree \( \leq L \) are pointwise equivalents in spherical caps of radius \( \epsilon/L \) where they reach their maximum. Indeed all we have to do is to apply Bernstein’s inequality as we did in proving Proposition 2.1.

Let \( w \in B(v, \epsilon/L) \) with \( v \) the center of a cap in \( E_L^* \). We apply the \( A_\infty \) condition getting
\[
\omega_L(w) = \frac{1}{\sigma(B(w, 1/L))} \int_{B(w, 1/L)} \omega(u) d\sigma(u)
\]
\[
\leq \frac{K}{\sigma(B(w, 1/L))} \left( \frac{\sigma(B(w, 1/L))}{\sigma(B(v, \epsilon/L) \cap E_L)} \right)^{s} \int_{B(v, \epsilon/L) \cap E_L} \omega(u) d\sigma(u)
\]
\[
\leq \frac{C}{\sigma(B(w, 1/L))} \left( \frac{\sigma(B(w, 1/L))}{\delta \sigma(B(v, \epsilon/L))} \right)^{s} \int_{B(v, \epsilon/L) \cap E_L} \omega(u) d\sigma(u)
\]
\[
= C \varepsilon \delta L^d \int_{B(v, \epsilon/L) \cap E_L} \omega(u) d\sigma(u).
\]

Finally, there exists \( u \in V_g \) such that
\[
\sup_{u \in E_L^*} |Q_L(u)| \omega_L(u) = \sup_{u \in B(v, \epsilon/L)} |Q_L(u)| \omega_L(u),
\]
for any \( w \in B(v, \epsilon/L) \)
\[
\inf_{u \in B(v, \epsilon/L)} |Q_L(u)| \omega_L(w) \leq L^d \inf_{u \in B(v, \epsilon/L)} |Q_L(u)| \int_{B(v, \epsilon/L) \cap E_L} \omega(z) d\sigma(z)
\]
\[
\leq \sup_{u \in B(v, \epsilon/L) \cap E_L} |Q_L(u)| \omega(u),
\]
and the result follows easily.

\[\square\]

\textbf{REFERENCES}

[BD05] G. Brown, F. Dai, \textit{Approximation of smooth functions on compact two-point homogeneous spaces}, J. Funct. Anal. 220, no. 2, 401–423, 2005.

[CN95] D. Cruz-Uribe, C. J. Neugebauer, \textit{The structure of the reverse Hölder classes}, Trans. Amer. Math. Soc. 347, no. 8, 2941–2960, 1995.
[Dai06] F. Dai, *Multivariate polynomial inequalities with respect to doubling weights and $A_\infty$ weights*, J. Funct. Anal. 235, no. 1, 137–170, 2006.

[FM74] C. Fefferman, B. Muckenhoupt, *Two nonequivalent conditions for weight functions*, Proc. Amer. Math. Soc. 45, 99–104, 1974.

[HJ94] V. Havin, B. Joricke, *The uncertainty principle in harmonic analysis*, Springer-Verlag, Berlin, 1994.

[Jor06] J. Marzo, *Marcinikiewicz-Zygmund inequalities and interpolation by spherical harmonics*, preprint, 2006. [http://arxiv.org/abs/math.FA/0611503](http://arxiv.org/abs/math.FA/0611503)

[LS74] V. N. Logvinenko, Ju. F. Sereda, *Equivalent norms in spaces of entire functions of exponential type*, Teor. funktsii, funk. analiz i ich prilozhenia 20, 102–111, 175, 1974.

[Lue83] D. H. Luecking, *Equivalent norms on $L^p$ spaces of harmonic functions*, Monatsh. Math. 96, no. 2, 133-141, 1983.

[MT00] G. Mastroianni, V. Totik, *Weighted polynomial inequalities with doubling and $A_\infty$ weights*, Constr. Approx. 16, no. 1, 37–71, 2000.

[OS05] J. Ortega-Cerdà, J. Saludes, *Marcinikiewicz-Zygmund inequalities*, J. Approx. Theory, doi: 10.1016/j.jat.2006.09.001

[Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[Sze39] G. Szegö, *Orthogonal polynomials*, American Mathematical Society, Colloquium Publications, vol. 23, 1991.

[Vol81] A. L. Volberg, *Thin and thick families of rational fractions*, Complex analysis and spectral theory (Leningrad, 1979/1980), 440–480, LNM 864, Springer, Berlin-New York, 1981.