Mixed state non-Abelian holonomy for subsystems

Mikael Nordling and Erik Sjöqvist*
Department of Quantum Chemistry, Uppsala University, Box 518, Se-751 20 Uppsala, Sweden.
(Dated: November 30, 2018)

Non-Abelian holonomy in dynamical systems may arise in adiabatic transport of energetically degenerate sets of states. We examine such a holonomy structure for mixtures of energetically degenerate quanital states. We demonstrate that this structure has a natural interpretation in terms of the standard Wilczek-Zee holonomy associated with a certain class of Hamiltonians that couple the system to an ancilla. The mixed state holonomy is analyzed for holonomic quantum computation using ion traps.

PACS numbers: 03.65.Vf, 03.67.Lx, 42.50.-p, 76.60.Gv

I. INTRODUCTION

Non-Abelian holonomy in simple dynamical systems [1] has found application in a variety of contexts, such as nuclear rotations of diatoms [2, 3], molecular Kramers doublets [4, 5], nuclear quadrupole resonance (NQR) [6, 7, 8], dynamics of deformable bodies [9, 10], the n-body problem [11], semiconductor heterostructures [12], and ion traps [13]. Recently, it has been pointed out [14, 15] that non-Abelian holonomy may be used in the construction of universal sets of quantum gates for the purpose of achieving fault tolerant quantum computation. Here, we wish to address this issue by putting forward an attempt to the motion of the quantal states themselves. Here, we address the appearance of non-Abelian holonomy related to the motion of the quantal states themselves. Here, we wish to address this issue by putting forward an attempt to define a concept of non-Abelian holonomy in terms of incoherent mixtures of energetically degenerate quantal states.

The main idea of this paper is to relate the mixed state time evolution of a system $S$ to the holonomic transformation of a certain class of pure state systems consisting of $S$ coupled to an ancilla system $A$. For the whole system $S+A$, this coupling creates in a purely geometric fashion an entangled state whose partial trace over $A$ corresponds in general to a nonpure state of $S$. From this perspective, we may regard the present analysis as the identification of a class of $S-A$ interaction Hamiltonians having degenerate eigenspaces that admit a non-Abelian holonomy structure for the mixed states of $S$. This structure reduces to the Wilczek-Zee pure state holonomy [1] of the system when the $S-A$ interaction vanishes.

An important issue for holonomic quantum information processing is its robustness to imperfections, such as decoherence and random unitary perturbations. The effect of imperfections on the quantum gates has been analyzed [17, 23, 30, 31], supporting the alleged robustness of holonomic quantum computation. It remains however to address the appearance of non-Abelian holonomy related to the motion of the quantal states themselves. Here, we wish to address this issue by putting forward an attempt to define a concept of non-Abelian holonomy in terms of incoherent mixtures of energetically degenerate quanital states.

An important issue for holonomic quantum information processing is its robustness to imperfections, such as decoherence and random unitary perturbations. The effect of imperfections on the quantum gates has been analyzed [17, 23, 30, 31], supporting the alleged robustness of holonomic quantum computation. It remains however to address the appearance of non-Abelian holonomy related to the motion of the quantal states themselves. Here, we wish to address this issue by putting forward an attempt to define a concept of non-Abelian holonomy in terms of incoherent mixtures of energetically degenerate quantal states.

We now apply the above Wilczek-Zee prescription in order to achieve the main idea of this paper, which is to propose a concept of mixed state non-Abelian holonomy for subsystems. To this end, we first identify a proper set of degenerate entangled energy eigensates by considering the extended Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$, where $\mathcal{H}_A$ is the Hilbert space of the ancilla.

II. MIXED STATE HOLONOMY

Let a quantal system $S$ be exposed to the Hamiltonian $H(q), q$ being some external control parameters that vary around a closed path $C : t \in [0,T] \rightarrow q(t)$ in parameter space. Let $\mathcal{H}_S$ be the Hilbert space of $S$ and let $\mathcal{M}(q)$ denote the linear span of the $N$-fold degenerate instantaneous eigensates $|\xi_a(q)\rangle, a = 0, \ldots, N - 1, N \leq \dim \mathcal{H}_S$, of $H(q)$. Under the condition that $T$ is large enough so that the adiabatic theorem holds, any initial state $|\psi(0)\rangle \in \mathcal{M}(q_0)$ is mapped to $|\psi(t)\rangle \in \mathcal{M}(q_f)$. Thus,

$$|\psi(0)\rangle = |\xi_a(q_0)\rangle \rightarrow |\psi(t)\rangle = \exp \left(-i \int_0^T E(q(t)) dt\right) \sum_b U_{ab}(t)|\xi_b(q_f)\rangle$$

with $U(t)$ unitary, $E(q(t))$ the common instantaneous energy of the degenerate subspace, and we have put $\hbar = 1$ from now on. It follows that

$$U[C] = \mathbf{P} \exp \left(-i \oint_C A\right)$$

is the holonomy transformation for $C$. Here, $\mathbf{P}$ is path ordering and $A = \sum \mu \varphi_{\mu}(q) dq^\mu$ is the connection one-form with

$$A_{ab,\mu}(q) = \langle \xi_b(q)|\partial_\mu|\xi_a(q)\rangle$$

the components of the Wilczek-Zee gauge potential and $\partial_\mu = \partial/\partial q^\mu$.

*Electronic address: erik.sjoqvist@kvac.uu.se
is an ancilla Hilbert space taken to be identical to $\mathcal{H}_S$ and
the vectors in $\mathcal{H}$ being purifications of the mixed states of $S$. The essential criterion is that this set should define the standard pure state holonomy of the $S$ system in the product state limit. This may be achieved by letting $\hat{H}(q)$ be the $N$-fold degenerate Hamiltonian of the composite system $S + A$ and let the members of the instantaneous set of orthonormal eigenvectors $\{\Xi_a(q)\}$, spanning the degenerate subspace of $\hat{H}(q)$, be iso-entangled, i.e., that they correspond to purifications of mixed states $\text{Tr}_A[\Xi_a(q)]\langle \Xi_a(q)\rangle$ all having the same eigenvalues $\lambda_k$, $k = 0, \ldots, N - 1$. If $\lambda_1 \to 1$ and $\lambda_{k \neq 1} \to 0$, then it follows that $\{\nabla_a(q)\} \to \{\xi_a(q)\} \otimes |\varphi\rangle$, for some $|\varphi\rangle \in \mathcal{H}_A$, which establishes the desired connection to the holonomy in the pure state case.

Now, the iso-entangled energy eigenstates may explicitly be taken as

$$\Xi_a(q) = \sum_{k=0}^{N-1} \sqrt{\lambda_k} \xi_{[a+k]N}(q) \otimes |\varphi_k\rangle,$$

where $[a+k]N \equiv a+k \mod N$. For this orthonormal set, we obtain the gauge potential as

$$A_{ab,\mu} = \langle \xi_b(q) | \partial_\mu | \Xi_a(q) \rangle = \sum_{k=0}^{N-1} \lambda_k \xi_{[b+k]N}(q) \langle \partial_\mu | \xi_{[a+k]N}(q) \rangle. \quad (5)$$

The resulting unitarity $U[\mathcal{C}]$ for the loop $\mathcal{C}$ in parameter space is the desired mixed state non-Abelian holonomy of the set of energetically degenerate reduced density operators $\text{Tr}_A[\Xi_a(q)]\langle \Xi_a(q)\rangle$ pertaining to the subsystem $S$.

To further analyze the physical meaning of the proposed form of mixed state holonomy, let us note that the Hamiltonian corresponding to this setup may be written as

$$\hat{H}(q) = E(q) \sum_a |\Xi_a(q)\rangle\langle \Xi_a(q)| + \Delta H(q)$$

$$= H_E(q) + \Delta H(q), \quad (6)$$

where $E(q)$ is the energy of the degenerate subspace and $\Delta H(q)$ acts only on the orthogonal complement. Thus, $\hat{H}(q)$ induces interaction between $S$ and $A$ along the path $\mathcal{C}$. For example, in the $N = 2$ case, we have

$$H_E = E(q) \sigma_0^S(q) \otimes |\varphi_0\rangle\langle \varphi_0| + \lambda_1 |\varphi_1\rangle\langle \varphi_1|$$

$$+ \sqrt{\lambda_0 \lambda_1} \sigma_3^A(q) \otimes \sigma_3^A, \quad (7)$$

where

$$\sigma_0^S(q) = |\xi_0(q)\rangle\langle \xi_0(q)| + |\xi_1(q)\rangle\langle \xi_1(q)|,$$

$$\sigma_1^S(q) = |\xi_0(q)\rangle\langle \xi_1(q)| + |\xi_1(q)\rangle\langle \xi_0(q)|,$$

$$\sigma_3^A = |\varphi_0\rangle\langle \varphi_1| + |\varphi_1\rangle\langle \varphi_0|.$$

In the pure state limit $\lambda_0, \lambda_1 \to 0$ we have $|\Xi_a(q)\rangle = |\xi_a(q)\rangle \otimes |\varphi_0\rangle$, $\forall a$, which decouples the Hamiltonian and leads to the pure state Wilzcek-Zee gauge potential in Eq. (3).

Now, since $S$ in general interacts with $A$, one should expect that the evolution of the density operator $\rho$ is nonunitary. To check this, one may assume that $\rho(0) = \text{Tr}_A[\psi(0)]\langle \psi(0)|$, with $|\psi(0)\rangle = |\Xi_a(q_0)\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A$, calculate the final density operator as

$$\rho(T) = \text{Tr}_A[\psi(T)]\langle \psi(T)| = \sum_{b,c} U^*_{ab}[\mathcal{C}]U_{ac}[\mathcal{C}]\text{Tr}_A[\Xi_b(0)]\langle \Xi_b(0)|, \quad (9)$$

and compare the initial and final degree of mixing for instance by computing $\text{Tr}\rho^2$. Due to the appearance of off-diagonal terms in $\rho(T)$, it is indeed apparent that $\text{Tr}\rho^2(T) \neq \text{Tr}\rho^2(0)$, in general.

### III. PHYSICAL EXAMPLE

A scheme to realize geometric quantum computation for ion traps has been proposed [10] (see also Ref. [13]). Here we extend the $U_2$ gate in Ref. [16], using our gauge potential for mixed states.

Suppose we have an ion with three metastable ground states denoted by $|0\rangle$, $|1\rangle$, and $|a\rangle$, and one excited state $|e\rangle$. A laser couples the ground states with the excited state, as described by the Hamiltonian

$$H = |e\rangle\langle 0|\omega_0 + |1\rangle|\omega_1 + (a|\omega_a) + \text{h.c.}, \quad (10)$$

where $\omega_0, \omega_1, \omega_a$ are complex coupling parameters that may be varied by slowly switching the laser field on and off. To realize the generalization of the $U_2$ gate of Ref. [10], restrict to paths that correspond to the parametrization $\omega_0 = \omega \sin \theta \cos \phi$, $\omega_1 = \omega \sin \theta \sin \phi$, and $\omega_a = \omega \cos \theta$. Under this condition, there is a pair of zero energy dark states of the form

$$|D_0(\theta, \phi)\rangle = \cos \theta \cos \phi |0\rangle + \cos \theta \sin \phi |1\rangle - \sin \theta |a\rangle,$$

$$|D_1(\theta, \phi)\rangle = - \sin \phi |0\rangle + \cos \phi |1\rangle.$$

For simplicity, in the adiabatic limit $\omega T \gg 1$, we may assume that the system’s density operator diagonalizes in this dark state basis, i.e.,

$$\rho(\theta, \phi) = \frac{1}{2}(1 + r)|D_0(\theta, \phi)\rangle\langle D_0(\theta, \phi)|$$

$$+ \frac{1}{2}(1 - r)|D_1(\theta, \phi)\rangle\langle D_1(\theta, \phi)| \quad (12)$$

with $-1 \leq r \leq 1$. By using Eq. (3), we obtain

$$A = ir\sigma_y \cos \theta d\phi,$$

where $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli operators in the $|0\rangle, |1\rangle$ basis. However, this connection one-form is only valid for a patch excluding the north and south pole of the sphere, where the coordinates are not uniquely defined.
This can be seen by considering an infinitesimal loop $\delta \mathcal{C}$ around the north pole, say, yielding
\[ U[\delta \mathcal{C}] = \mathbf{P} \exp \left(-\frac{\mathbf{F}}{\delta \mathcal{C}} A \right) = e^{-ir\sigma_y \delta \mathcal{C}}. \] (14)

To get around this problem we make the gauge transformation
\[ A \rightarrow A_N = V_N A V_N^T + dV_N V_N^T \] (15)
with
\[ V_N = e^{-ir\sigma_y \phi}. \] (16)

Explicitly,
\[ A_N = -ir\sigma_y (1 - \cos \theta) d\phi, \] (17)
which is a valid gauge potential for the northern hemisphere.

Now, since $[A_N, \phi(\theta), A_N, \phi(\theta')] = 0$ for any pair $\theta, \theta'$ along $\mathcal{C}$, path ordering is not needed and the mixed state holonomy becomes
\[ U[\mathcal{C}] = e^{ir\Omega}, \] (18)
where $\Omega$ is the solid angle enclosed by the path $\mathcal{C}$ in parameter space. In the pure state limit $r = 1$, $U[\mathcal{C}]$ reduces to the $U_2$ gate in Ref. 10.

As mentioned above, we expect the mixing of the state to change along $\mathcal{C}$. To check this, let $\rho(0) = \frac{1}{2} (1 + r) |0\rangle \langle 0| + \frac{1}{2} (1 - r) |1\rangle \langle 1|$, with purification $|\psi(0)\rangle = \sqrt{\frac{1}{2}} (1 + r) |0\rangle \otimes |\varphi_0\rangle + \sqrt{\frac{1}{2}} (1 - r) |1\rangle \otimes |\varphi_1\rangle \equiv |\psi_0\rangle$. After having traversed the loop, we obtain
\[ |\psi(T)\rangle = U_{00}[\mathcal{C}] |\psi_0\rangle + U_{01}[\mathcal{C}] |\psi_1\rangle = \cos(\Omega) |\psi_0\rangle + \sin(\Omega) |\psi_1\rangle \] (19)
with $|\psi_1\rangle = \sqrt{\frac{1}{2}} (1 + r) |1\rangle \otimes |\varphi_0\rangle + \sqrt{\frac{1}{2}} (1 - r) |0\rangle \otimes |\varphi_1\rangle$, which is a purification of
\[ \rho(T) = \frac{1}{2} \left(I + \sin(2r\Omega) \sigma_z + r \cos(2r\Omega) \sigma_x \right), \] (20)
whose Bloch vector traces out an ellipse in the $(x, z)$ plane with half axes $(1, |r|)$ when $\Omega$ varies. We obtain
\[ \text{Tr} \rho^2(T) = \frac{1}{2} \left(1 + r^2 + (1 - r^2) \sin^2(2r\Omega) \right), \] (21)
which equals $\text{Tr} \rho^2(0) = \frac{1}{2} (1 + r^2)$ if and only if $|r| = 1$ (pure states) or $\Omega = n\pi/(2r)$, $n$ integer.

For $\Omega = (n + \frac{1}{2}) \pi/(2r)$, we obtain that the initial mixed state $\rho(0)$ polarized along the $z$ axis has transformed into the pure state $\rho(T) = \frac{1}{2} (I \pm \sigma_z)$. This may be understood by looking at the final purification, which for $n = 0$ reads
\[ |\tilde{\psi}(T)\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \left(\sqrt{\frac{1}{2}} (1 + r) |\varphi_1\rangle + \sqrt{\frac{1}{2}} (1 - r) |\varphi_2\rangle \right) \] (22)
corresponding to a product state in $\mathcal{H}_S \otimes \mathcal{H}_A$. Conversely, one may transform a pure input state into a mixed one using the mixed state holonomy $U[\mathcal{C}]$. To illustrate this latter point, let $|\psi_0^\prime\rangle = \frac{1}{2} \sqrt{(1 + r)} |0\rangle \otimes |\varphi_0\rangle + \frac{1}{2} \sqrt{(1 - r)} |1\rangle \otimes |\varphi_1\rangle$ and $|\psi_1^\prime\rangle = \frac{1}{2} \sqrt{(1 + r)} (|0\rangle - |1\rangle) \otimes |\varphi_2\rangle$. After having traversed the loop $\mathcal{C}$, we obtain
\[ |\tilde{\psi}(T)\rangle = \frac{1}{\sqrt{2}} \left(U_{00}[\mathcal{C}] + U_{01}[\mathcal{C}] \right) |\psi_0^\prime\rangle \] (23)
which is a purification of the final density operator
\[ \rho^\prime(T) = \frac{1}{2} \left(I - r \sin(2r\Omega) \sigma_x + r \cos(2r\Omega) \sigma_z \right) \] (24)
that is nonpure for $|r| < 1$ unless $\Omega = n\pi/(2r)$, $n$ integer. Note that the accessible final states again constitute an ellipse in the $(x, z)$ plane, now with half axes $(|r|, 1)$.

More general manifolds of final states are obtained if the initial $S + A$ state is mixed. As an illustration of this, consider the input state
\[ \rho(0) = \frac{1 + R}{2} |\psi_0\rangle \langle \psi_0| + \frac{1 - R}{2} |\psi_1\rangle \langle \psi_1| \] (25)
with $-1 \leq R \leq 1$ and $|\psi_0\rangle, |\psi_1\rangle$ defined above. This state evolves into $\rho(T) = U[\mathcal{C}] \rho(0) U^\dagger[\mathcal{C}]$, yielding the final density matrix of $S$ as
\[ \rho(T) = \frac{1}{2} \left(I + R \sin(2r\Omega) \sigma_x + r R \cos(2r\Omega) \sigma_z \right) \] (26)
which corresponds to a shrinking of the ellipse in the $(x, z)$ plane by a factor $|R|$.

### IV. CONCLUSIONS

Holonomy effects for mixed states, as first conceived by Uhlmann [32] and later reconsidered in Ref. 33, have recently attracted interest due to its potential importance to fault tolerant quantum information processing. These previous analyzes have mainly been concerned with the Abelian case, while, on the other hand, universal sets of quantum gates must involve noncommuting one- and two-qubit gates. In other words, to examine the effect of nonpure input states in all-geometric quantum information processing requires a concept of non-Abelian mixed state holonomy.

In this paper, we have introduced such a concept of non-Abelian holonomy for adiabatic transport of energetically degenerate mixed quantal states. The holonomy effect may be understood as arising in certain types
of slowly changing Hamiltonians that couple the considered system to an ancilla.

The present type of holonomy effect has an inherent additional richness compared to the standard Wilczek-Zee pure state holonomy in that it may change the mixedness of the input state. We have explicitly demonstrated that this feature may transform a mixed state into a pure one and vice versa by purely geometric means. This provides a method for robust coherent control of quantal states, which may be of use for quantum information processing and communication purposes.

Acknowledgments

The work by E.S. was supported in part by the Swedish Research Council.

[1] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
[2] J. Moody, A. Shapere, and F. Wilczek, Phys. Rev. Lett. 56, 893 (1986).
[3] R. Jackiw, Phys. Rev. Lett. 56, 2779 (1986).
[4] C.A. Mead, Phys. Rev. Lett. 59, 161 (1987).
[5] C.A. Mead, Rev. Mod. Phys. 64, 51 (1992).
[6] R. Tyecko, Phys. Rev. Lett. 58, 2281 (1987).
[7] A. Zee, Phys. Rev. A 38, 1 (1988).
[8] J.W. Zwaniger, M. Koenig, and A. Pines, Phys. Rev. A 42, 3107 (1990).
[9] A. Shapere and F. Wilczek, J. Fluid Mech. 198, 557 (1989).
[10] A. Shapere and F. Wilczek, Am. J. Phys. 57, 514 (1989).
[11] R.G. Littlejohn and M. Reinsch, Rev. Mod. Phys. 69, 213 (1997).
[12] D.P. Arovas and Y. Lyanda-Geller, Phys. Rev. B 57, 12302 (1998).
[13] R.G. Unanyan, B.W. Shore, and K. Bergmann, Phys. Rev. A 59, 2910 (1999).
[14] P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999).
[15] J. Pachos, P. Zanardi, and M. Rasetti, Phys. Rev. A 61, 010305(R) (2000).
[16] L.-M. Duan, J. I. Cirac, and P. Zoller, Science 292, 1695 (2001).
[17] J. Pachos, Phys. Rev. A 66, 042318 (2002).
[18] L. Faoro, J. Siewert, and R. Fazio, Phys. Rev. Lett. 90, 023013 (2003).
[19] M.-S. Choi, J. Phys.: Condens. Matter 15, 7823 (2003).
[20] M. Cholascinski, Phys. Rev. B 69, 134516 (2004).
[21] S.-L. Zhu, Z.D. Wang, and P. Zanardi, e-print: quant-ph/0403004.
[22] L.X. Cen, X.Q. Li, Y.J. Yan, H.Z. Zheng, and S.J. Wang, Phys. Rev. Lett. 90, 147902 (2003).
[23] A.O. Niskanen, M. Nakahara, and M.M. Salomaa, Phys. Rev. A 67, 012319 (2003).
[24] S. Tanimura, D. Hayashi, and M. Nakahara, Phys. Lett. A 325, 199 (2004).
[25] P. Zanardi, Phys. Rev. Lett. 87, 077901 (2001).
[26] I. Fuentes-Guridi, J. Pachos, S. Bose, V. Vedral, and S. Choi, Phys. Rev. A 66, 022102 (2002).
[27] K.-P. Marzlin, S.D. Bartlett, and B.C. Sanders, Phys. Rev. A 67, 022316 (2003).
[28] Y. Li, P. Zhang, P. Zanardi, and C.P. Sun, Phys. Rev. A 70, 032330 (2004).
[29] I. Fuentes-Guridi, F. Girelli, and E. Livine, Phys. Rev. Lett. 94, 020503 (2005).
[30] P. Solinas, P. Zanardi, and N. Zanghi, Phys. Rev. A 70, 042316 (2004).
[31] L.-X. Cen and P. Zanardi, Phys. Rev. A 70, 052323 (2004).
[32] A. Uhlmann, Rep. Math. Phys. 24, 229 (1986).
[33] E. Sjöqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. 85, 2845 (2000).