The changing faces of the *Problem of Space* in the work of Hermann Weyl

Erhard Scholz*

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Abstract

During his life Weyl approached the *problem of space* (PoS) from various sides. Two aspects stand out as permanent features of his different approaches: the *unique determination of an affine connection* (i.e., without torsion in the terminology of Cartan) and the question *which type of group* characterizes physical space. The first feature came up in 1919 (commentaries to Riemann’s inaugural lecture) and played a crucial role in Weyl’s work on the PoS in the early 1920s. He defended the central role of affine connections even in the light of Cartan’s more general framework of connections with torsion. In later years, after the rise of the Dirac field, it could have become problematic, but Weyl saw the challenge posed to Einstein gravity by spin coupling primarily in the possibility to allow for non-metric affine connections. Only after Weyl’s death Cartan’s approach to infinitesimal homogeneity and torsion became revitalized in gravity theories.

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*University of Wuppertal, Faculty of Math./Natural Sciences, and Interdisciplinary Centre for History and Philosophy of Science, scholz@math.uni-wuppertal.de
Introduction

According to H. Weyl three aspects have to be taken into account for studying the problem of space (PoS): the extensive medium of the world (“extensives Medium der Aussenwelt”), its metrical structure, and its content by a material quality changing from place to place (“materielle Erfüllung mit einem von Stelle zu Stelle veränderlichen Quale”) (Weyl 1922a, p. 205). The problem has to be approached from two sides, by a philosophical investigation and by mathematical analysis. In the early 1920s Weyl saw the task of philosophy in clarifying the distinction and the mutual relationships between the three aspects mentioned above. In addition to this mathematics had to “search for correct knowledge of the essence of space and of the spatial structure” as far as it can be “described quantitatively, in logico-arithmetic relations”¹

Note Weyl’s double description of essence and structure; he considered them as complementary aspects of the concept of space.

Such a characterization was given by him in the phase 1921 to 1923 when Weyl developed his program of the mathematical analysis of the problem of space in a well defined, sense. In the following we shall denote it by PoS₂¹–₂³. It dealt with the question of how to generalize the Helmholtz-Lie analysis of the homogeneity conditions of classical space to the new context of relativistic physics. Weyl insisted on the necessity to reformulate homogeneity in terms of differentiable manifolds endowed with linear groups operating in the infinitesimal neighbourhoods (in modernized language, operating on the tangent spaces), or between them. He gave very general conceptually motivated conditions and analyzed their consequences. His result was an infinitesimal group structure typical for the automorphisms of his generalized differential geometry, “pure infinitesimal geometry”, developed in 1918 (Weylian metric). The contribution of Weyl to the problem of space has found much attention in the history and philosophy of mathematics. Here it will be dealt with from a specific point of view only; for more aspects and finer details the reader may consult the literature²

This was not the only situation in which Weyl addressed the problem of space. In a more general sense this problem was a recurrent theme in his thought all over his life. Weyl hit upon the PoS (in the wider sense) in different contexts and looked at it from different angles. The present paper puts Weyl’s discussion of the PoS in a wider perspective (section 1), but it would be far beyond its scope to deal with all these different facets in some

¹Für den Mathematiker handelt es sich darum, das quantitativ, in logisch-arithmetischen Relationen Erfassbare am Wesen des Raumes und der räumlichen Struktur richtig zu erkennen und mit den Hilfsmitteln der Logik, Arithmetik und Analysis auf seine einfachsten Gründe zurückzuführen” (Weyl 1922a, p. 206).
²Among many (Scheibe 1988, Sigurdsson 1991, Coleman/Korté 2001, Bernard 2013, Bernard 2015, Scholz 2004, Scholz 2016a). Weyl’s study of the PoS₂¹–₂³ was the guiding axis of the conference of which this book arose; see in particular the contributions to this volume by A. Roca-Rosell and C. Lobo.
Here we shall concentrate on selected topics which show how Weyl used context dependent relative a priori elements which he considered constitutive for determining the structure of space, or even for grasping its “essence”. Two conceptual features stand out among them: (i) the characterization of homogeneity by means of group structures and (ii) the core role assigned by Weyl to a uniquely determined affine (torsion free) connection for admissible space structures. Both features appear prominently in Weyl’s PoS but also, in different form, in other encounters of him with the problem of space. We shall discuss how Weyl dealt with the problem of homogeneity after the rise of general relativity (section 2) and contrast it with Cartan’s answer to the question (section 3). That could have given reasons for Weyl to revise the central role of uniquely determined affine connections as a kind of relative a priori for physical geometry, but it did not (section 4). The next challenge was posed by Dirac’s spinor fields in general relativity. In the light of later developments (Einstein-Cartan theory of gravity) one might expect that it could have become a problem for Weyl’s affine connection principle already in the 1930/40s. But that was not the case. Why, will be shortly discussed in section 5 before we come to a final evaluation (section 6).

Of course Weyl, like the other mathematicians of the 20th century, often used the terminology of “space” at other places in a wider sense than above, sometimes in a purely mathematical context. But in the framework of this paper the notion of space is nearly always used in the more restricted sense of a mathematical space structure which serves, or at least may serve, as a candidate for grasping physical space or space-time, the “extensive medium of the external world” in Weyl’s word. For the abbreviation PoS this is always the case.

1 The multiple faces of the PoS

As already mentioned we can give here only a short survey of different contexts and different forms in which Weyl met the problem of space. The following list of topics and contexts may serve as an orientation (main publications indicated in brackets):

1. Modernized presentation of the classical problem of space in the sense of Helmholtz and Lie in the first chapter of Raum - Zeit - Materie (Weyl 1923b)

2. Specification of Riemannian metrics (“Pythagorean nature” of metric) in the wider class of Finsler metrics in Riemann’s approach to geometry (Riemann/Weyl 1919)

3. Mathematical analysis of the problem of space, PoS21–23, (Weyl 1923a)
4. Characterization of $\mathbb{R}^3$ or $S^3$ (the three-dimensional sphere) by combinatorial invariants. Topological space forms and their characterization by discrete groups (operating on the universal covering space) (Weyl 1925/1988, pp. 16ff.)

5. Introduction of differential structure on continuous manifolds, in particular with regard to its justification for the concept of physical space (Weyl 1925/1988, p. 12)

6. Cartan’s general concept of infinitesimal geometries (Weyl 1925/1988, pp. 38ff.), (Weyl 1929b, Nabonnand 2005)

7. Similarities and congruences as an exemplary case for the distinction of mathematical automorphisms and physical automorphisms of space (Weyl 1949, Weyl 1948/49)

8. Specific role of Lorentz/Poincaré group and the dimension 4 of space-time (Weyl 1948/49)

9. Finally Weyl’s considerations on the possible role of non-metric affine connections for the dynamics of spinor fields in general relativity (Weyl 1950)

Under the items 1., 3., 6., 7. Weyl dealt with the question of how to characterize the homogeneity of the respective spatial structures. Here different versions of automorphism groups played a prominent role. In his discussion of the items 2., 3. and 9. Weyl’s conviction that a proper space structure in the sense of PoS carries a uniquely determined affine (i.e. torsion free linear) connection stood out. In 2., 3. the postulate of a uniquely determined affine connection was not questioned at all; in 9. he subjected it to a check whether it could be defended in the light of relativistic spinor fields. Item 7. and 8. have been discussed elsewhere. In the following sections this topic will be dealt in more detail. The topological space problem (item 4) and the problem of differentiability (item 5) have only been brought up at isolated occasions by Weyl; they cannot be discussed in this paper.

2 Homogeneity characterizations given by Weyl

In the first three editions of Raum - Zeit - Materie the classical space problem of the 19th century, posed and answered by Helmholtz, Lie, Engel (Weyl added Hilbert, Grundlagen der Geometrie, app. IV) was mentioned by Weyl only in passing (Weyl 1918b, pp. 86, 264, 1st to 3rd ed.). Even in the fourth edition published in 1921, in which he already included a first sketch

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3See the contribution of S. de Bianchi, this volume.
4(Scholz 2016b)
of his own thoughts on the PoS, Weyl did not go into more details (Weyl 1921, pp. 86, 289). Only after having finished his own analysis he gave a more extended presentation of the classical solution in the fifth edition (Weyl 1923b, p. 100). He did not use the concepts of “rigid body” and “free mobility”, which had become problematic with the advent of relativity theory, but expressed Helmholtz’ postulates abstractly in terms of group theoretical constraints for the homogeneity of classical space. He rephrased Helmholtz’ axioms of free mobility by conditions which are now called simple flag transitivity of the homogeneity group. Similar in (Weyl 1923a, 5th lecture).

Weyl’s presentation stripped Helmholtz’ analysis from the latter’s intention of founding his conditions on supposedly factual conditions (“That-sachen”) lying at the basis of any empirical measurement. He did not claim to give a historically precise account of Helmholtz’ thoughts, in fact his passage read as though Helmholtz had started from an investigation of the a priori conditions of the homogeneity of physical space. Weyl’s reconstruction of Helmholtz’s and Lie/Engels’ approach to the space problem was systematic, not historical. It assimilated the classical PoS to a perspective which prepared the way to a type of analysis which Weyl pursued in his own program between 1921 and 1923. From such a perspective the classical PoS seemed to have been the question of an a priori characterization of the homogeneity of space. It led to an answer which allowed to introduce an invariant definite quadratic differential form of constant curvature and thus to the classical spaces of Euclidean and non-Euclidean geometry.

Such a type of a priori characterization could no longer be apodictic, like Kant’s a priori judgements had been (or, at least, had been claimed to be). It no longer consisted of necessary judgements, but rather of well founded postulates, if possible of the best founded ones, which characterize the conceptually possible in a specified context. In this function it still has an a priori character in distinction from empirical determinations but only relative to the latter and to the theoretical context. With a change of context

5aMan kommt so zu der folgenden Formulierung des Homogeneitatspostulats im n-dimensionalen Raum: Es soll möglich sein, mit Hilfe einer zur Gruppe G gehörigen Abbildung ein System Σ inzidenter Richtungselemente der 0-ten bis (n − 1)-ten Stufe in ein gleichartiges, beliebig vorgegebenes System Σ’ zu überführen; aber die Identität soll unter den Abbildungen von G die einzige sein, welche ein derartiges System Σ inzidenter Richtungselemente festläßt” (Weyl 1923b, p. 100).

6bVon einem tieferen, gruppentheoretischen Gesichtspunkt aus hat Helmholtz zuerst die Homogeneitatsfrage gestellt. Helmholtz setzt nicht die Gültigkeit des Pythagoreischen Lehrsatzes im Unendlichkleinen, ja nicht einmal die Meßbarkeit der Linienlänge vor, er spricht allein von dem wahren Grundbegriff der Geometrie, der Gruppe G der kongruenten Abbildungen des Raumes.” (Weyl 1923b, p. 100).

7bEs ist eine wunderbare gruppentheoretische Tatsache, die von Helmholtz, strenger und allgemeiner von S. Lie bewiesen wurde, daß die einzigen dieser Bedingung genügenden Gruppen G die Gruppen Gλ ... [sind]” ibid. By Gλ Weyl denoted the congruence groups of hyperbolic, parabolic, or elliptic geometry (in the terminology introduced by F. Klein).
and/or more refined empirical knowledge, formerly well established a priori conditions can become obsolete and may have to be revised, in agreement with the analysis given by Friedman (1999, 59ff.).

Such a revision was the goal of Weyl’s analysis of the problem of space (1921–1923). By several reasons simple flag transitivity could no longer be upheld as a feature characterizing the homogeneity of space. Firstly special relativity had integrated space proper and time to spacetime as the “extensive medium of the world”. That destroyed flag transitivity because it now became necessary to account for the qualitative difference of timelike and spacelike directions. Moreover general relativity, Einstein’s theory of gravity, broke with the paradigm of constant curvature and made curvature dependent on the distribution of matter and energy, thus giving it an a posteriori character. Therefore Weyl, and a little later Cartan, posed the question of homogeneity of spacetime anew, in forms adapted to the context of general relativity. Both came to different, only partially overlapping answers which we shall discuss in the following.

Weyl started from a conceptual analysis of what he considered the most general, minimal conditions for congruence geometry founded on infinitesimal structures like those he had proposed in his purely infinitesimal geometry of 1918. In his investigations 1921–1923 he wanted to dig deeper and to motivate, or even derive, a generalized metrical structure from congruence and similarity concepts in the infinitesimal and a generalized homogeneity principle. In a move which he presented as a conceptual analysis of the idea of congruence in the infinitesimal he established conditions for (linear) groups characterizing congruence by generalized “rotations” in the infinitesimal neighbourhoods of each point of spacetime. An intuitive idea of homogeneity then demanded that the type of group (more precisely the conjugation class in the general linear group) was equal for any two points. But Weyl did not use the terminology “homogeneous/homogeneity” in this context; he rather spoke of the fixed nature of the group (the conjugation class) which could be expressed by pointwise changing “orientations” of the group (members of the class)(Weyl 1923a, p. 48).

Moreover a transition between neighbouring points had to be specified by a linear connection which is compatible with the similarities with regard to the “rotations” (more technically with the normalizer of the rotations). All this was justified by by what Weyl considered an a priori analysis of the concepts involved (Weyl 1923a, p. 49).

But his was not all; Weyl added:

8Similarly in the 5th edition of Raum - Zeit - Materie, where he characterized the “nature” of a Riemannian metric by its signature and its “orientation” by the point dependent value of the respective quadratic differential form (Weyl 1923b, p. 102). In the 4th edition (translated into English and French) he still fought more indirectly with the problem that space as “a form of phenomena ... is necessarily homogeneous”, while the Riemannian metric is not (Weyl 1922b, pp. 96ff.).
I now come to the synthetic part in the Kantian sense. The task is now to formulate precisely the postulate, up to now only indicated, which finally determines the type of rotation group which is characteristic for the real world. (ibid.)

The synthetic component of his a priori justification of infinitesimal congruence structures consisted of a two-part postulate, the first of which demanded a kind of wide adaptability to a posteriori distribution structure of matter (postulate of “free deformability”), the second one was the postulate of unique determination of a compatible affine connection. Later the first part turned out to be mathematically redundant, leaving the second part as the mathematical and philosophical core of Weyl’s synthetic a priori of the PoS.

Mathematically it was crucial for constraining the groups which could serve as candidates for infinitesimal congruences so strongly that in the end Weyl could show that only the generalized orthogonal groups (of arbitrary signature) satisfy the constraints (main theorem of Weyl’s PoS). A philosophical analogy of this principle with intersubjectivity relations in practical philosophy was expressed and emphasized by Weyl in his Barcelona lectures (Weyl 1923a, p. 46). The nature of this analogy is being analyzed by N. Sieroka (this volume) and related to Weyl’s exchange with F. Medicus and his reading of Fichte.

The existence of a uniquely determined affine connection remained a stable feature in Weyl’s understanding of a good geometric structure designed for representing space from 1919 onward, although the mathematical feasibility of it might have become doubtful after Cartan’s answer to the homogeneity challenge of general relativity and even stronger after the advent of relativistic spinor fields. Only much later, in the years between 1948 and 1950, Weyl subjected it to an investigation of its empirical acceptability. His result was that this part of his a priori may be sustained even relative to a context including general relativistic spinor fields (see section 5).

With regard to homogeneity in the modern (general relativistic) context Weyl performed a tight-rope walk. Clearly, space “as a form of phenomena . . . is necessarily homogeneous” (Weyl 1922b, p. 96), but Riemannian spaces are not. He solved, or circumvented, the problem by arguing that (simply connected) neighbourhoods of any two points are diffeomorphic and the “nature” of the metric remains the same for all points. Both belong to the a priori determinations of space, while the metrical structure is fixed by a posteriori determinations.
ori, empirically given factual conditions (or contingent model assumption, we might add). For him the manifold and the nature of the metric, both invariant under diffeomorphisms, expressed the generalized idea of homogeneity in the relativistic context. But he did not speak of “generalized homogeneity”, he rather reserved the terminology of homogeneity for the classical situation of metrically homogeneous spaces (Weyl 1918b, Weyl 1923a, Weyl 1949).

In spite of this terminological decision, the view that the diffeomorphisms of the spacetime manifold are part of the physical automorphisms of general relativistic field theories persisted in Weyl’s thought. At the time of preparing the different editions of Raum - Zeit - Materie (1918–1923) Weyl argued for such an understanding by means of the plasticine analogy discussed in J. Bernard’s contribution, this volume. This was an intuitive, “didactical”, way for expressing the more general postulate that under a physical automorphism the field structures are “dragged along” with the diffeomorphisms which can thus be considered as dynamical symmetries in present physicists’ terminology. Otherwise they would not be able to preserve the (a posteriori) field structures and the metric induced by them. Weyl insisted on such a generalized understanding of homogeneity in a talk on physical and mathematical automorphisms given in the late 1940s (Weyl 1948/49).

3 Cartan’s concept of infinitesimal homogeneity

Elie Cartan approached the problem of homogeneity in a different way. In 1922 he presented a new type of infinitesimal geometry to the public Sur une généralisation de la notion de courbure ... (Cartan 1922). He had developed the necessary tools (differential forms and Lie group theory) over a long time and elaborated the basic ideas for his new geometry during the year 1921 in an interplay of differential geometry, Einstein’s gravity theory, and the brothers Cosserat’s generalized theory of elasticity. In the years to come he would expand his approach to a broad program of generalized infinitesimal geometries later called Cartan spaces. In his survey talk at the International Congress of Mathematicians in Toronto, 1924, Cartan motivated the approach by indicating that general relativity was confronted with

...the paradoxical task of interpreting in a non-homogeneous universe ...the multiple experiences made by observers who believed in the homogeneity of this universe (Cartan 1924, p. 86).

Although general relativity had helped to induce a first step towards bridging the gap between (non-homogeneous) Riemannian geometry and Euclidean

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13 Similar in (Weyl 1949, p. 87).
14 Cf. (Scholz 2016b)
15 (Cogliati 2015, Nabonnand 2016, Scholz 2016a); for a modern mathematical introduction to Cartan geometry see (Sharpe 1997).
geometry (homogeneous in the sense of F. Klein) by motivating T. Levi-Civita’s concept of infinitesimal parallelism, he did not yet see the gap closed.\(^{16}\)

Cartan alluded to the Kleinean understanding of homogeneity and indicated the idea underlying his approach:

\[\ldots\text{while a Riemannian space does not possess absolute homogeneity, it possesses a kind of infinitesimal homogeneity; in the immediate neighbourhood of a point it can be assimilated to a Euclidean space. (ibid.)}\(^{17}\)\]

That is, Cartan wanted to implement \textit{infinitesimal homogeneity} in his new generalized spaces, in addition to their infinitesimal (generalized) rotational symmetries. Weyl, as we have seen, translated homogeneity in the new, relativistic context to the possibility of comparing the neighbourhoods of any two points of spacetime (not infinitesimally close ones) with each other, which boiled down to considering the diffeomorphisms of the underlying manifold as part of the automorphisms of the geometric structure.

Cartan’s program thus consisted in an infinitesimalization of the Kleinean concept of geometry as a homogeneous space \(S\) in the sense of \(S \cong G/H\) with a main group \(G\) and generalized rotations (isotropy group) \(H \subset G\)\(^{18}\). Cartan considered the infinitesimal version of the groups, i.e. the corresponding Lie algebras \(\mathfrak{g} = \text{Lie } G, \mathfrak{h} = \text{Lie } H\) and \(\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{h}\), and assimilated the quotient of the two with the infinitesimal neighbourhoods in the manifold \(M\), such that \(T_p M \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{i}\), for any point \(p\) of \(M\). His crucial symbolic tools were ensembles of differential forms, which can be read as differential forms with values in the respective Lie algebras. With their help he introduced a generalized type of connection, now called \textit{Cartan connection}, which led to two kinds of curvature effects\(^{19}\).

The curvature with respect to the generalized rotations \(\mathfrak{h}\) corresponded to the Riemann curvature (in slightly different form) which was well known at the time, while the curvature with respect to the generalized translations \(\mathfrak{i}\) was a new effect. Cartan called it \textit{torsion} because in the context of the generalized elasticity theory in the sense of the Cosserats it could be interpreted as a rotational momentum in the medium. If translated to the

\(^{16}\) Or, c’est le développement même de la théorie de la relativité, liée par l’obligation paradoxale d’interpréter dans et par un Univers non homogène les résultats de nombreuses expériences faites par des observateurs qui croyaient à l’homogénéité de cet Univers, qui permet de combler en partie le fossé qui séparait les espaces de Riemann de l’espace euclidien. Le premier pas dans cette voie fut l’œuvre de M. Levi-Civita, par l’introduction de sa notion de parallélisme.” (Cartan 1924, p. 86))

\(^{17}\) ...si un espace de Riemann ne possède pas une homogénéité absolue, il possède cependant une sorte d’homogénéité infinitésimale; au voisinage immédiat d’un point donné il est assimilable à une espace euclidien” (Cartan 1924, p. 85).

\(^{18}\) In the Euclidean case \(G \cong \mathbb{R}^n \times SO(n, \mathbb{R})\) with \(H = SO(n, \mathbb{R})\), \(S = G/H \cong \mathbb{R}^n\).

\(^{19}\) Cf. (Sharpe 1997).
coordinate notation of differential geometry (the calculus of Ricci and Levi-Civita) Cartan’s connection could, in many cases, be expressed in the form of a linear connection $\Gamma$ with coefficients $\Gamma^\lambda_{\mu\nu}$ which are no longer symmetric in the lower indices. In fact the condition $\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\nu\mu}$ is equivalent to non-vanishing torsion.\footnote{20 $\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = T^\lambda_{\nu\mu}$ is the torsion tensor, i.e., the translational curvature expressed in coordinate coefficients.}

Cartan did not try to argue on a philosophical level as explicitly as Weyl did, but his conceptual analysis of the “paradoxical task” posed by general relativity may be read as establishing a new type of a priori framework for physical geometry, different from Weyl’s in PoS\textsuperscript{21–23}. Cartan’s new relativ a priori was wider than Weyl’s in two respects. His framework allowed a larger variety of infinitesimal isotropy and homogeneity groups than Weyl’s. Moreover, in his view it would not appear natural to consider the existence and uniqueness of a compatible affine connection (torsion zero) as a “synthetic” a priori of the physical space concept. Cartan reformulated Weyl’s PoS in his framework, but he dealt with it from a mathematical point of view rather then of a philosophical one, and with a slightly different outcome.\footnote{21 Cf. (Scholz 2016a).}

Weyl responded to Cartan’s proposal of a large class of infinitesimal geometries, but at the beginning he was not convinced that the wider perspective was helpful for extending the a priori framework of physical geometry. In the correspondence between him and Cartan he expressed doubts even with regard to the specific geometrical achievements of Cartan’s generalization, although at the end both authors came to a basic acceptance of the other’s viewpoint (Nabonnand 2005).\footnote{22 For a survey see (Scholz 2016a); a more refined evaluation of the correspondence is being prepared by C. Eckes and P. Nabonnand.}

In his contribution to the Lobachevsky anniversary volume written in 1925, but published only posthumously, Weyl discussed Cartan’s approach and acknowledged that it allowed a “far-reaching generalization of infinitesimal geometry” (Weyl 1925/1988, p. 38); in particular:

\ldots it achieves the natural widest possible range of concept formation which still allows to establish a theory of curvature in analogy to Riemann’s.\footnote{23 Ünd darauf beruht wohl überhaupt die mathematische Bedeutung seines allgemeinen Schemas: es erreicht den natürlichen weitesten Umfang der Begriffsbildung, welche die Aufstellung einer Krümmungstheorie analog der Riemannschen noch ermöglicht” (Weyl 1925/1988, p. 39).}

Thus Weyl acknowledged Cartan’s generalization of the curvature concept, but without mentioning that it carries the potential to undermine the central role of the affine connection, which he considered as the most important part
of the “synthetic” a priori of the space concept.

4 Affine connection, synthetic a priori or just a special condition among others?

Shortly after Levi-Civita’s invention of the infinitesimal parallel displacement in Riemannian geometry Weyl introduced affine connections as a concept of its own for differential geometry, which allowed to talk about *infinitesimal parallel displacements* in allusion to Levi-Civita’s terminology but independent of the structure of a Riemannian metric and without reference to an embedding into a higher dimensional Euclidean space (Weyl 1918c). He simply demanded that (1) for any vector $\xi$ attached to a point $p$ of the manifold the change induced by parallel displacement from $p$ to an infinitesimal close point $p'$ depends linearly on the vector $\xi p'$, and (2) if for two points $p_1, p_2$, both infinitesimally close to $p$, the parallel displacement of $p p_1$ along $p p_2$ leads to the end point $p_{21}$ and the displacement of $p p_2$ along $p p_1$ to $p_{12}$, then $p_{12} = p_{21}$: “The result is an infinitesimally small parallelogram.”

In the paper directed to mathematicians Weyl argued conceptually, sometimes even in a philosophical style. He presented the task of geometry being “to fathom out the essence of the metrical concepts” He thus understood the conditions (1) and (2) as postulates which arise from an *analysis of the concept* of infinitesimal parallel displacement. In the philosophical language used in (Weyl 1923a) the generalized affine connection resulted from an a priori conceptual analysis.

A little later, in his commentaries to Riemann’s inaugural lecture (Riemann/Weyl 1919), he pondered on the question of how the Riemannian metric could be specified among the wider class of Finsler metrics (which had a striking a priori justification in being homogeneous with regard to rescaling). He conjectured that the Riemannian metrics are just those which are compatible with a uniquely determined affine connection. D. Laugwitz

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24In the same article he reiterated that he still stood to the content of his PoS: “Das neue gruppentheoretische Raumproblem, das vom Standpunkt der Relativitätstheorie an Stelle des Helmholtz-Lie’schen tritt, glaube ich in meiner Schrift "Mathematische Analyse des Raumproblems" (1923, Vorlesung 7 und 8) formuliert und gelöst zu haben.” (Weyl 1925/1988, p. 37)

25Es entsteht eine unendlich kleine Parallelogrammfigur” (Weyl 1918c, p. 7).

26Die Geometrie “ergründet, was im Wesen der metrischen Begriffe liegt” (Weyl 1918c, p. 2). In the paper presenting his purely infinitesimal geometry to physicists (as the geometrical background for his unified field theory) Weyl introduced the affine connection in more concrete form and axiomatically (Weyl 1918a, p. 32). (Weyl 1918/1997, p. 26).

27Bei der fundamentalen Bedeutung, die nach den neueren Untersuchungen (…) dem affinen Grundbegriff der infinitesimalen Parallelverschiebung eines Vektors für den Aufbau der Geometrie zukommt, erhebt sich insbesondere die Frage, ob die Mannigfaltigkeiten der Pythagoreischen Raumklasse die einzigen sind, welche die Aufstellung dieses Begriffs ermöglichen und welche dementsprechend nicht bloß eine Metrik, sondern auch affinen Zusammenhang besitzen. Die Antwort lautet wahrscheinlich bejahend, ein Beweis dafür
would later call this conjecture *Weyl’s first problem of space* – and answered it positively (Laugwitz 1958).

Weyl was convinced of the fundamental import of the principle of a uniquely determined affine connection already in 1919; in the following we shall speak about it as *Weyl’s affine connection principle*. Its crucial role for deriving the main theorem of the PoS made it so convincing for Weyl that in 1922/23 he even raised it to the status of a “synthetic” a priori.

Considered from a wider perspective this was neither self-evident nor imperative (in distinction to a Kantian understanding of synthetic a priori). Only a few weeks after Weyl’s Barcelona lectures in February 1922, É. Cartan presented his first public note on generalized spaces to the Paris Académie des sciences (Cartan 1922). Of course Weyl could not know about it at the time of his lectures, nor apparently while preparing them for publication, but in hindsight it could have become clear to him that Cartan’s generalized spaces also opened the pathway towards a different relative a priori for relativistic spacetime.

For Cartan the difference was not so much of a philosophical nature, but mathematically it was clear to him from the outset. If a parallel displacement, \( \Gamma(\xi, p p') = \xi' - \xi \), is expressed by connection coefficients \( \Gamma^\lambda_{\mu
u} \) with regard to a coordinate basis of the infinitesimal neighbourhoods (the tangent spaces), the closing condition (2) boils down to the symmetry of the coefficients, \( \Gamma^\lambda_{\mu
u} = \Gamma^\lambda_{\nu\mu} \). Cartan’s torsion tensor expressed in coordinate coefficients, on the other hand, becomes \( T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \); Weyl’s condition (2) thus amounts to vanishing torsion.\(^{28}\)

It was not easy for Weyl and Cartan to disentangle their different points of view on their differences with regard to generalized spaces, although Cartan could treat the mathematical aspects of Weyl’s PoS quite easily as a special case of his methods and he acknowledged Weyl’s deep philosophical analysis, but without discussing it from his side (Cartan 1923b). Weyl, on the other hand, found it difficult to grasp the subtleties of Cartan’s approach, while he soon understood the wider mathematical generality of the latter’s approach and acknowledged it (see above). In 1929 he even adapted certain aspects of Cartan’s approach for his proposal to of a general relativistic version of Dirac’s electron theory (Weyl 1929a). In the same year he gave a survey talk on Cartan’s theory to the Princeton group of differential geometry and mathematical physics. There he argued that for a proper geometric usage one has to add certain restrictions to Cartan’s scheme, among them the exclusion of torsion (Weyl 1929b, p. 210).

Cartan had developed and extended his methodology in the course of the 1920s in very general terms, with infinitesimal Klein spaces of many

\(^{28}\)Cartan discussed this point in his investigation of Weyl’s PoS in slightly different terms (Cartan 1923b, §3).
different types and even without assuming that, pointwise, the infinitesimal quotient group (Lie algebra) $l \cong g/\mathfrak{h}$ can be identified with the tangent spaces of the manifold (not even the dimension needed to be the same). For Weyl it seemed indispensable that for a proper geometrical usage of Cartan’s general scheme one had to assume pointwise identifications of $l$, which he called the “tangent plane” denoted by “$T_P$” (sic!), with the infinitesimal neighbourhoods of the manifold, the tangent spaces in the later terminology\footnote{In the modern understanding of Cartan spaces this is an indispensable property inbuilt in the definition of a Cartan gauge, e.g. (Sharpe 1997, p. 174). Note that Weyl’s “$T_P$” stood for for $l$ in the function of what would now be considered the tangent space of the transitive subgroup in the fibre direction.}. He called this an “embedding” of “$T_P$” into the manifold and added additional restrictions motivated by the specific geometrical structure considered. He spoke of “special manifolds”, in particular with regard to projective and conformal structures\footnote{Weyl drew upon the results of the Princeton school of differential geometry, A.L. Eisenhart, O. Veblen, T.Y Thomas, intending to build bridges between the French (E. Cartan) and the US (Princeton school) traditions in differential geometry.}. The specialization conditions contained, in particular, the “invariantive restriction to require that our manifold ...is without torsion” (Weyl 1929b, p. 210).

In the following correspondence Cartan insisted that his research program did not need such restrictions and deplored that it was not fairly represented by Weyl’s survey. The ensuing exchange of letters centered on the role of “embedding” of $l$ (Weyl’s “$T_P$”) and the specialization conditions. Although torsion played only a subordinate role, the correspondence shows once again that Cartan considered torsion zero as a technical specialization condition among others without particular conceptual import (Nabonnand 2005).

5 The challenge of spinor fields in the 1930-40s

In the light of later developments in Einstein-Cartan theory of gravity (see final section) it ought to be added that, to my knowledge, in the 1930s neither Cartan nor Weyl considered the question whether the general relativistic Dirac electron field with non-vanishing spin ($\text{spin} = \frac{1}{2}$) might undermine Weyl’s affine principle from the physical, perhaps even from an empirical side. Even in the late 1940s, when Weyl started to analyze the consequences of an independent spin coupling of the Dirac field to gravity, he did not pose the question whether the affine principle had to be given up in favour of Cartan’s view ($\text{torsion} \neq 0$). He rather chose an approach which Einstein had studied in the middle of the 1920s, in which an affine connection and the metric of a generalized Lagrangian were varied independently (Einstein 1925). He thus relaxed the condition of metricity of the connection (calling it a “mixed” theory) rather than that of vanishing torsion. For a Lagrangian of the electron field with a Dirac term, a spin term and a mass term Weyl...
found that “by the influence of matter a slight discordance between affine connection and metric is created” (Weyl 1950, p. 288 and equs. (2), (3)).

This interesting observation clearly posed a challenge to Weyl’s affine connection principle. But “by somewhat laborious calculations” (not presented in the paper) he was able to show that by adding a small term of the form \(-12\pi G l_2\) to the Lagrangian (\(l_2\) a quartic scalar invariant in the 4-component spinor field \(\psi\)) the metric theory became equivalent to the mixed theory (without the additional term). He concluded:

To this extent then there is a complete equivalence between the mixed metric-affine and the purely metric conception of gravity.

(Weyl 1950, p. 288)

The metric theory of gravity could be upheld by a small addition to the Lagrangian even in the light of an electron field’s spin coupling to gravity. In this way the challenge posed by Dirac spinor fields to Weyl’s affine connection principle was neutralized and the relative a priori of the uniquely determined metric affine connection successfully defended.

6 Late endorsement for Cartan’s infinitesimal homogeneity principle and general discussion

In his PoS\textsuperscript{21–23} Weyl tried to found a new conceptual framework for space and time that lived up to the challenge of the theories of relativity, like the homogeneous spaces of the late 19th century had done with regard to classical physics. The challenge arose from physical theories in which Einstein had evaluated both, empirical and theoretical knowledge in a quite specific sense. Regarding physical concepts Einstein was an ingenious innovator (role of time, space, simultaneity, equivalence principle, metric as gravitational field etc.), but with regard to the mathematical theories, he had built with conceptual material inherited from contemporaneous mathematics (Riemannian geometry, Ricci-Levi Civita’s absolute calculus). Weyl intended to go beyond the constraints of the inherited and to “fathom out” (as he said in the above quote) the minimal ingredients of spacetime concepts, necessary for obtaining infinitesimal congruence structures which were able to build a bridge between the general notion of a differentiable manifold and specific metrical determinations. The latter ought to be able to adapt as flexible as possible to contingent distributions of matter and fields. He did so in what he considered an a priori move of conceptual analysis, as he said in open allusion to the Kantian terminology, and found that he had to add the

\[ l_2 = (\bar{\psi}_3 \psi_1 + \bar{\psi}_4 \psi_2)(\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4). \]

The “laborious calculations” seem to have been presented in the manuscript (not preserved) for the publication (Weyl 1948). K. Chandrasekharan, the editor of Weyl’s Gesammelte Abhandlungen remarks about this paper that “due to typographical errors, it is incomprehensible” (Weyl 1968, vol. 4, p. 285, footnote). It was therefore not reprinted in the edition.
“synthetic” affine connection principle. Although he tried to motivate the synthetic principle by analogy with considerations of practical philosophy (respect for a “common good”) the finally decisive motivation of the principle lay in its success for deriving the main theorem of the PoS, not different from what one would ordinarily expect from the axioms of a mathematical theory.

Weyl was well aware that his a priori was different from Kant’s; in particular it was no longer apodictic and made sense only in the context of relativistic physics and the horizon of new differential geometric structures on manifolds. It thus was relative with regard to the theoretical context and open for potential revisions, like the classical understanding of homogeneity had been. In Weyl’s view this did not make the striving for a well understood a priori obsolete. In his view the role of a priori statements had changed from being necessary judgements to well motivated possible concepts and structures. But their function with regard to more specific theoretical and empirical knowledge remained. In his view “physics projects what is given onto the background of the possible” (Weyl 1949, p. 220) and mathematics explores the conceptually possible.

Substituting Kant’s necessary a priori judgements by the “background of the possible” points also towards another shift: the relative a priori need no longer be uniquely determined. In our case study we have come across a possible loss of uniqueness, the underdetermination of the relative a priori, by comparing Weyl’s analysis of the PoS with Cartan’s generalized spaces. The latter’s conceptual motivation was the implementation of infinitesimal homogeneity in addition to the global homogeneity achieved by structure dragging diffeomorphisms. That gave them the potential for becoming a competing a priori structure for relativistic space concepts. In our discussion of Weyl’s different approaches to the space problem this potentiality did not materialize, apparently because Cartan did not like to argue too much on the philosophical level and Weyl did not see imperative reasons to give up his affine connection principle. It needed a change of generations and deep conceptual as well as technical studies of physicists before the a priori potential of Cartan spaces became apparent. A detailed account of this next shift would need a publication of its own (or more). Here we have to content ourselves with an outline.

The success of conceiving electromagnetism as a $U(1)$-gauge field induced physicists to study field theoretic consequences of point dependent infinitesimal symmetries of other groups. In the terminology of physics the

\[32\] The dual nature of reality accounts for the fact that we cannot design a theoretical image of being except upon the background of the possible. Thus the four-dimensional continuum of space and time is the field of the a priori existing possibilities of coincidences.” (Weyl 1949, p. 231)

\[33\] For a rich collection of sources with detailed commentaries from the theoretical physics side see (Blagojević/Hehl 2013).
Symmetries were "localized". Most striking and best known is the case of at first strong, later weak isospin $SU(2)$ (Yang/Mills) and the later generalizations to gauge field theories in elementary particle physics. But also the Poincaré group, the symmetry group of special relativity, was "localized" independently and nearly simultaneously by T. Kibble and D. Sciama (Sciama 1962, Kibble 1961). From the point of view of physics the conserved currents of infinitesimal symmetries supplied by the Noether theorems played a crucial heuristic role for this research. For the Poincaré group $\mathbb{R}^4 \cong SO(3,1)$ that led to considering the spin current, the Noether current with regard to the Lorentz rotations, as an additional source for the gravitational field, supplementing energy-momentum, the current with regard to the translation group.

The physical idea of "localizing" the translations of the Poincaré group together with the Lorentz rotations was very close to Cartan’s idea to implement infinitesimal homogeneity in addition to infinitesimal isotropy in his concept of generalized spaces, although neither Kibble nor Sciama noticed the kinship of their studies with Cartan’s geometrical framework. This was brought into the open by the work of F. Hehl and A. Trautman. Then it also became clear that the simplest Lagrangian in Kibble’s approach, as well as in Sciama’s, is equivalent to the one discussed by Cartan in passing, when he showed what his approach could contribute to understand and to extend Einstein’s theory of gravity (Cartan 1923a, §83). It is now being called Einstein-Cartan gravity (EC).

Einstein-Cartan gravity modifies Einstein’s general relativity only to a tiny degree; for vanishing spin it reduces to the latter. Moreover, outside of spinning matter field the torsion is zero and the influence of spin on the metric can be taken into account by a small modification of the energy-momentum source of the Einstein equation, similar to what Weyl had found for the non-metricity induced by spin in the “mixed” theory. Only for mass densities more than $10^{38}$ times the one of a neutron star, respectively a nucleon mass compressed to $10^6$ Planck lengths, which signifies energy densities at the hypothetical grand unification scale of elementary particle interactions, the experts expect EC “to overtake” Einstein’s general relativity.

These seemingly technical results of modified gravity are important in our context, because they show that Cartan’s principle of infinitesimal ho-

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34 See A. Afriat’s paper, this volume.
35 To be more precise: Sciama presupposed an Einsteinean background and gained spin as an additional current, modifying Einstein gravity to what was later called Einstein-Cartan gravity. Kibble, on the other hand, started from localizing the symmetries of Minkowski space and considered different Lagrangians, the simplest of which led to Einstein-Cartan theory (Blagojević/Hehl 2013, p. 106).
36 (Hehl 1970, Trautman 1973, Hehl e.a. 1976) and others.
37 (Trautman 2006, Hehl 2016).
38 (Hehl e.a. 1976, p. 406), (Trautman 2006, p. 194).
39 (Trautman 2006, p. 194) (Blagojević/Hehl 2013, p. 108).
Mogeneity has finally become important in foundational studies of gravity. In the last third of the 20th century it has turned into a relative apriori for relativistic spacetime theories with, at least, the same right as Weyl’s affine connection principle and alternative to the latter. It even would have the advantage of being closer to what Weyl called the “physical automorphisms” of modern physics by fitting well to the Noether current paradigm for infinitesimal symmetries, prominent in contemporary physics (Weyl 1948/49).

On the other hand, there is (still?) no actual empirical evidence which would force us to revise Weyl’s analysis of the PoS21–23 and to relegate his affine connection principle from the status of a relative apriori to an empirical principle, valid only in “weak” field constellations. In this sense, we seem to be here in the situation of a presumably temporal underdetermination of the relative a priori principles of Weyl and Cartan. This seems to be another feature of present a priori structures, at least as important as their being established in the context of wider scientific results and being open to revision with them.

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40 Cf. (Scholz 2016b).

41 The so-called teleparallel version of Cartan geometric gravity rearranges the coordination between Noether currents and dynamical equations: Energy-momentum becomes the source of translational curvature and the spin current of the rotational curvature, rather than the other way round as in EC gravity. In oral communications D. Lehmkuhl has indicated that teleparallel gravity may be an interesting example of a possibly principled (in contrast to temporal) underdetermination of empirically equivalent gravity theories.
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