Ascent sliceness

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Abstract. We introduce the notion of ascent sliceness of virtual knots. A representative of a virtual knot is an embedding $D : S^1 \hookrightarrow \Sigma_g \times I$, for $\Sigma_g$ a closed connected oriented surface of genus $g$; the virtual knot represented is slice if there exists a pair consisting of a disc $B$ and an oriented 3-manifold $M$, such that $B \hookrightarrow M \times I$, $\partial M = \Sigma_g$, and $\partial B = D(S^1)$ (the image of the embedding).

Roughly, a slice virtual knot $K$ with genus-minimal representative $S^1 \hookrightarrow \Sigma_g \times I$ is ascent slice if, given any disc and 3-manifold pair $(B, M)$ as above, and any Morse function $f : M \rightarrow I$ (whose restriction to $B$ is a Morse function also), the surface $\Sigma_{g+1}$ appears as a level set of $f$.

We use an augmented version of doubled Khovanov homology to define a property which implies ascent sliceness for slice virtual knots of minimal supporting genus 1.

1. Introduction

In this note we introduce the notion of ascent sliceness of virtual knots. Virtual links are equivalence classes of embeddings $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$ (for $\Sigma_g$ a closed oriented surface of genus $g$), up to self-diffeomorphism and certain handle stabilisations of $\Sigma_g$. A classical link (a link in $S^3$) may be represented as a link in $\Sigma_0 \times I$. As such, virtual links are equivalence classes of embeddings into equivalence classes of 3-manifolds, and we may ask a number of new types of question. The subject of this note is an example in the context of concordance.

Kuperberg showed that every virtual link has a minimal genus representative, unique up to self-diffeomorphism [7]. That is, given a virtual link $L$, there exists a $g$ such that $L$ has a representative in $\Sigma_g \times I$ but not $\Sigma_{g'} \times I$ for $g' < g$. Moreover, he verified that any two representatives of $L$ on $\Sigma_g \times I$ are related by self-diffeomorphism of $\Sigma_g$ (handle stabilisations are not required). We denote this minimal supporting genus by $m(L)$ (the quality $m(L)$ is also known as the virtual genus of $L$).

Cobordism and concordance of virtual links is defined in an analogous way to the classical case (as given in Section 2.1). The virtual case exhibits a new feature, however: a cobordism between virtual links is a pair consisting of a surface $S$ and a 3-manifold $M$ such that $S \hookrightarrow M \times I$. As a result, given a pair of virtual links, we may now ask questions of the 3-manifolds appearing in cobordisms between them, as well as of the surfaces.

For example, we define the concordance supporting genus of $L$, denoted $m'(L)$, to be the minimum $m(L')$ across all $L'$ concordant to $L$. It is a measure of how much of the knotting of $L$ about $\Sigma_g$ may be removed through a concordance.

Both $m(L)$ and $m'(L)$ are difficult to compute with currently available invariants, and their investigation is an interesting problem. However, it follows from the definition that a slice virtual knot has concordance supporting genus 0. The introduction of ascent sliceness represents an attempt to conduct a finer investigation of slice virtual knots.

Let $K$ be a slice virtual knot with representative $S^1 \hookrightarrow \Sigma_g \times I$. Roughly speaking (the full definitions are given in Section 2.2), we say that a concordance $S \hookrightarrow M \times I$ from the representative of $K$ to the unknot is an ascent concordance if, for any Morse function $f : M \rightarrow I$, the surface $\Sigma_{g+1}$ appears as a level set of $f$. We say that $K$ is ascent slice if every concordance from (a representative of) $K$ to the unknot is an ascent concordance.

It is not known whether the set of ascent slice virtual knots is nonempty; if it were, it would be a manifestation of the ubiquitous principle of "increase before decrease", as seen in the hard
unknot diagrams of Kauffman and Lambropoulou [5], and handlebody decompositions of manifolds, among many other contexts.

Whilst we do not present an ascent slice virtual knot, in Section 3 we use an augmented version of doubled Khovanov homology to define a property which implies ascent sliceness for slice virtual knots of minimal supporting genus 1.

2. Definition

In this section we give the relevant definitions regarding virtual knot theory and virtual concordance before formally defining ascent sliceness.

2.1. Virtual links and concordance

The definitions below may be expressed equivalently in terms of virtual link diagrams, but in the interest of brevity we omit those alternate definitions (they can be found in [4, 6]).

Definition 2.1. A virtual link is an equivalence class of embeddings \( \coprod S^1 \hookrightarrow \Sigma_g \times I \) up to self-diffeomorphism of \( \Sigma_g \times I \), and handle (de)stabilisations of \( \Sigma_g \) such that the product of the attaching sphere with the \( I \) summand of \( \Sigma_g \times I \) is disjoint from the image of the embedding.

A representative \( D \) of a virtual link is a particular embedding \( \coprod S^1 \hookrightarrow \Sigma_g \times I \). We abbreviate the notation to write \( D \hookrightarrow \Sigma_g \times I \).

On the level of representatives, we may generically project the image of \( D \hookrightarrow \Sigma_g \times I \) to \( \Sigma_g \) to obtain a 4-valent graph on \( \Sigma_g \). Available handle destabilisations of \( D \) are then represented by simple closed curves which do not intersect this graph.

For the remainder of this section we shall not distinguish between a virtual link \( L \) and a representative of it; we shall abuse notation and refer to \( L \) until we pass a critical point, after which they are \( \Sigma_g \times I \) are as in Definition 2.2). The 3-manifold \( \Sigma_g \) by the unknot diagram of Kauffman and Lambropoulou [5], and handlebody decompositions of manifolds, among many other contexts.

Definition 2.2. Let \( L \hookrightarrow \Sigma_g \times I \) and \( L' \hookrightarrow \Sigma_{g'} \times I \) be virtual links. We say that \( L \) and \( L' \) are cobordant if there exist a compact oriented surface \( S \) and an oriented 3-manifold \( M \), such that \( S \hookrightarrow M \times I \), \( \partial S = L \cup L' \), and \( \partial M = \Sigma_g \sqcup \Sigma_{g'} \).

Reference to \( S \) as a cobordism between \( L \) and \( L' \) (note that we do not require \( L \) and \( L' \) to have the same number of components).

Definition 2.3. Let \( S \hookrightarrow M \times I \) be a cobordism between links \( L \) and \( L' \), with \( |L| = |L'| \) (where \( |L| \) denotes the number of components of \( L \)). We say that \( L \) and \( L' \) are concordant there exists a cobordism \( S \) between them, which is a disjoint union of \( |L| \) annuli, such that each annulus has a boundary component in \( L \) and another in \( L' \). We refer to such an \( S \) as a concordance between \( L \) and \( L' \).

One notices that a cobordism between virtual links is a pair consisting of a surface and a 3-manifold; as such, we shall often denote a cobordism as a pair \((S, M)\) (where \( S \) and \( M \) are as in Definition 2.2). The 3-manifold \( M \) may be given a Morse decomposition and described in terms of level surfaces and critical points. Let \( f : M \to I \) be a Morse function: starting from \( \Sigma_g \), level surfaces of \( f \) are \( \Sigma_g \) until we pass a critical point, after which they are \( \Sigma_{g+1} \). Critical points correspond to handle (de)stabilisations. A finite number of handle (de)stabilisations are made to reach \( \Sigma_{g'} \).

Definition 2.4. Let \( K \) be a virtual knot. Define the slice genus of \( K \), denoted \( g^s(K) \), to be

\[
g^s(K) := \min \{ g(S) \mid S \text{ a cobordism between } K \text{ and the unknot} \}.
\]

If \( g^s(K) = 0 \) (so that \( K \) is concordant to the unknot) we say that \( K \) is slice.
A cobordism between virtual links $L$ and $L'$. The manifold $M \times I$ is depicted dimensionally reduced, and the blue plane depicts the submanifold $\Sigma_l \times I \subset M \times I$. The virtual link $J$ is the intersection of this submanifold with the surface $S$.

Given a cobordism from a virtual knot to the unknot we can simply cap the unknot with a disc to yield a surface whose boundary is exactly the knot. Therefore, for a virtual knot $K \hookrightarrow \Sigma_g \times I$, the question “what is the slice genus of $K$?” reads: “what is the least genus of oriented surfaces $S \hookrightarrow M \times I$ with $\partial S = K$, where $M$ is an oriented 3-manifold with $\partial M = \Sigma_g$?”.

The slice genus of a knot is a property of the surfaces which make up cobordisms from a virtual knot to the unknot; the study of ascent sliceness is an attempt to ask questions of the 3-manifolds which make up those cobordisms.

2.2. Ascent sliceness

We give the formal definition of ascent concordance, which specialises to the case of ascent sliceness.

**Definition 2.5.** Let $(S, M)$ be a cobordism. Fix a Morse function $f : M \to I$ such that the restriction of $f$ to $S$ is a Morse function also. We say that a virtual link $J \hookrightarrow \Sigma_l \times I$ appears in $S$ if $S \cap (f^{-1}(t) \times I) = J$, for some $t \in I$ with $f^{-1}(t) = \Sigma_l$. The situation is depicted in Figure 1. 

**Definition 2.6.** Let $L$ and $L'$ be concordant virtual links, and $S$ a concordance between them. We say that $S$ is ascent if a (representative of a) virtual link, $J \hookrightarrow \Sigma_g \times I$, appears in $S$ such that $g > m(L), m(L')$. If every concordance between $L$ and $L'$ is ascent we say that $L$ and $L'$ are ascent concordant.

That is, a concordance $S \hookrightarrow M \times I$ is ascent if the genus of surfaces appearing in $M$ is at some point greater than the minimal supporting genera of both $L$ and $L'$.

**Definition 2.7.** Let $K$ be a slice virtual knot. If every concordance from $K$ to the unknot is ascent, then $K$ is ascent slice.
Of course, the unknot has minimal supporting genus 0, so that a slice virtual knot $K$ is ascent slice if, in every concordance from $K$ to the unknot, a virtual link appears whose minimal supporting genus is greater than that of $K$.

Boden and Nagel showed that if a classical knot is slice when treated as a virtual knot, then it was already slice as a classical knot [1]. Therefore, there are no ascent slice classical knots (the two statements are equivalent, in fact).

3. A source of potential examples

A virtual knot is classical if and only if it has minimal supporting genus 0, and as a consequence of the aforementioned work of Boden and Nagel it is reasonable to expect that there do not exist ascent slice virtual knots of minimal supporting genus 1. In other words, one may suspect that more complicated virtual knots must be considered in order to find examples of ascent sliceness. Nevertheless, in this section we define a property which implies ascent sliceness for slice virtual knot of minimal supporting genus 1.

The property is defined using an augmented version of doubled Khovanov homology [8, Section 3]; if there are indeed no ascent slice virtual knots of minimal supporting genus 1, this would feed back to a structural result regarding the augmented homology theory.

The augmented homology theory takes as its argument links in thickened surfaces. Such links are closely related to virtual links, but there are important distinctions between the two theories. We outline these differences before describing the homology theory.

For the remainder of this note we shall need to distinguish between representatives of virtual links and virtual links themselves. Therefore, by $D$ a representative of a virtual link $L$ we mean a particular embedding $D \hookrightarrow \Sigma_g \times I$ representing the equivalence class $L$.

3.1. Links in thickened surfaces

Here we give the essential details of links in thickened surfaces and their relationship to virtual links (for full details see, for example, [2]). To differentiate between the two types of objects, we denote links in thickened surfaces with “mathfrak” letters, while virtual links remain uppercase Roman letters.

A link in a thickened surface is an isotopy class of embeddings $\sqcup \Sigma_g \times I$. Let $D$ be a link in a thickened surface, and consider the 4-valent graph obtained from a regular projection to $\Sigma_g$ of a particular embedding $\sqcup \Sigma_g \times I$ representing $D$. A diagram of $D$, denoted $\Sigma$, is obtained by decorating the vertices of this graph with the appropriate overcrossing and undercrossing information. Two diagrams represent the same link in a thickened surface if they are related by a finite sequence of Reidemeister moves (those familiar from classical knot theory).

By the Isotopy Extension Theorem an isotopy of an embedding $\sqcup \Sigma_g \times I$ extends to a self-diffeomorphism of $\Sigma_g \times I$. Of course, there may be many self-diffeomorphisms which cannot be realised as extensions of isotopies of embeddings; for example, Dehn twists. Therefore, one can obtain a virtual link from a link in a thickened surface by simply considering the latter up to Dehn twists and the permitted handle stabilisations (as given in Definition 2.1). If we can obtain a virtual link $L$ from a link in a thickened surface $\Sigma$ in this manner we say that $\Sigma$ projects to $L$.

A representative of a virtual link $D \hookrightarrow \Sigma_g \times I$ defines a link in a thickened surface simply by considering $D$ up to isotopy. It is important to note that two representatives of the same virtual link may define non-equivalent links in thickened surfaces, even if the representatives are in the same thickened surface. For example, given $D$ and $D'$, two representatives of a virtual link $L$, we may need to apply handle stabilisations and Dehn twists to obtain $D$ from $D'$. Kuperberg’s Theorem (as given on section 1), however, ensures that if $D$ and $D'$ are representatives in $\Sigma_g(L) \times I$, then we need only apply self-diffeomorphisms i.e. isotopies and Dehn twists.
The relationship between virtual links and links in thickened surfaces is depicted in Figure 2. The notions of cobordism and concordance of links in thickened surfaces are defined essentially identically to those of virtual links (see [10]). In what follows we shall use a restricted class of cobordisms of links in thickened surfaces: we shall only be interested in cobordisms embedded in 4-manifolds of the form $\Sigma_g \times I \times I$. This is contrast to cobordisms of virtual knots, which are embedded into thickened oriented 3-manifolds with appropriate boundary.

### 3.2. The homology theory

Doubled Khovanov homology is an extension of Khovanov homology to virtual links [9]. In [8] an augmented version of it is defined for links in thickened surfaces. We shall give an outline of the construction of this augmentation, focussing only on what is relevant for this note (for full details consult [8, Section 3]).

The construction is familiar from other Khovanov-style theories: a cube of smoothings is associated to a diagram, which is then turned into an algebraic chain complex. The homology of this chain complex is an invariant of the link (in a thickened surface) represented by the diagram.

First, we use auxiliary information to add extra decorations to the cube of smoothings.

**Definition 3.1.** Let $\mathcal{D} \hookrightarrow \Sigma_g \times I$ be a diagram of an oriented link in a thickened surface $\Sigma$. Form the cube of smoothings of $D$ in the same way as for a virtual link diagram: smoothings are embeddings of disjoint unions of $S^1$ into $\Sigma$. An example is given in Figure 4.

Pick an element $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. Decorate the cube of smoothings as follows: a circle within a smoothing is decorated with a dot if it has non-zero image under $c$. The assignment of dots to all circles of all smoothings within the cube is referred to as the dotting associated to $c$.

Two examples of dottings are given in Figure 4: green dots represent the dotting associated to the element of $H^1(\Sigma_g; \mathbb{Z}_2)$ coloured green, and the element coloured red does not produce any dots. The fully decorated cube is referred to as the dotted cube of smoothings of $\mathcal{D}$ with respect to $c$, denoted $[\mathcal{D}, c]$.

Next, we form a chain complex in the usual way (vertices are sent to modules, and edges are sent to module maps), keeping track of the extra decoration we have applied to the cube.
Definition 3.2. Let $\cD \hookrightarrow \Sigma_g \times I$ be a diagram of an oriented link in a thickened surface $\cU$. Pick an element $c \in H^1(\Sigma_g; \mathbb{Z}_2)$ and form the dotted cube $[\cD, c]$ as in Definition 3.1. We form the \textit{doubled Khovanov complex of $\cD$ with respect to $c$} in the standard way, but augmented by adding dots above modules assigned to circles which are dotted. These dots persist to elements of the dotted module; that is, we denote the elements of $\mathcal{A}$ as $\nu_+^d$ and $\nu_-$.

As in unaugmented Khovanov homology, the components of the differential are matrices of the appropriate maps, which are assigned signs in the standard way. The resulting chain complex is denoted $\text{CDKh}(\cD, c)$, and an example of such a complex is given in Figure 5.

We can use this extra decoration to define a third grading on the chain complex.

Definition 3.3. Let $\cD \hookrightarrow \Sigma_g \times I$ be a diagram of an oriented link in a thickened surface $\cU$ and $\text{CDKh}(\cD, c)$ its dotted doubled Khovanov complex with respect to $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. By an abuse of notation we denote by $c$ both the cohomology class and a $\mathbb{Z}[\frac{1}{2}]$-grading on $\text{CDKh}(\cD, c)$ defined in the following manner. Given $x \in \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (where the copies of $\mathcal{A}$ may or may not be dotted) define

$$c(x) := \#(\nu_+^d) - \#(\nu_-^d) + \frac{1}{2}j(x)$$
where \( \#(v^*) \) denotes the number of \( v^* \) in \( x \) (likewise \( \#(v^\circ) \) the number of \( v^\circ \)), and \( j \) the standard quantum degree. We refer to this grading as the \( c \)-grading.

It is clear from the definition that if \( y \) is a simple closed curve on \( \Sigma_g \) which is disjoint from \( \Sigma \), then the \([y]\)-grading of \( CDKh(\Sigma, [y]) \) will contain no new information (every smoothing will also be disjoint to \( y \), so that the cube of smoothings will acquire no dots). That is, it can be obtained from the quantum grading by multiplication by \( \frac{1}{2} \). This observation (made explicit in Theorem 3.4) is a key point of our argument in Section 3.3.

The components of the differential split with respect to the \( c \)-grading (the splitting is given in Equations 3.2 to 3.6 of [8]), so that the differential itself splits into a part which preserves the \( c \)-grading, and a part which increases it. Denote by \( DKh(\Sigma, c) \) the homology of \( CDKh(\Sigma, c) \) with respect to the \( c \)-degree preserving part. It is shown in [8, Theorem 3.4] that \( DKh(\Sigma, c) \) is an invariant of \( \Sigma \), and we write \( DKh(\Psi, c) \) for the \textit{doubled Khovanov homology of} \( \Psi \) \textit{with respect to} \( c \).

We only need a perturbed version of this theory, however. Just as Lee homology is defined by adding a term to the differential which increases the \( j \)-grading, we can define a perturbed version of \( DKh(\Psi, c) \) by adding a term which increases the \( j \)-grading, and another which increases the \( c \)-grading.

Taking a filtration in both the \( j \)- and \( c \)-gradings, and adding these terms, we arrive at a homology theory known as the \textit{totally reduced homology of} \( \Psi \) \textit{with respect to} \( c \), denoted \( DKh''(\Psi, c) \). (In direct analogy to Lee homology, the chain complex whose homology is \( DKh''(\Psi, c) \), denoted \( CDKh''(\Psi, c) \), has chain spaces identical to those of \( CDKh(\Psi, c) \), and an altered differential.)

To conclude this section we list some properties of the totally reduced homology, which we shall rely on later.

\textbf{Theorem 3.4.} Let \( \Psi \) be a link in the thickened surface \( \Sigma_g \times I \). Given an element \( c \in H^1(\Sigma_g; \mathbb{Z}_2) \), the totally reduced homology \( DKh''(\Psi, c) \) has the following properties:

(i) \[ 8, \text{Corollary 3.9} \]: Ignoring the \( c \)-grading, \( DKh''(\Psi, c) \cong DKh'(L) \), where \( DKh'(L) \) is the standard doubled Lee homology of the virtual link \( L \) projected to by \( \Psi \). (Doubled Lee homology is defined in [9, Section 3].)

(ii) \[ 8, \text{Contrapositive to Proposition 3.5} \]: If there are elements of \( DKh''(\Psi, c) \) with non-trivial \( c \)-grading, then there does not exist \( y \) a simple closed curve on \( \Sigma_g \) with \([y] = c\), such that \( \Sigma \cap y = \emptyset \), for any diagram \( \Sigma \) of \( \Psi \).

(iii) \[ 8, \text{Proposition 3.15} \]: Let \( \mathcal{R}, \mathcal{R}' \) be knots in the thickened surface \( \Sigma_g \times I \), and \( S \hookrightarrow \Sigma_g \times I \times I \) be a concordance between them. Then \( S \) defines a triply-graded isomorphism \( \phi_S : DKh''(\mathcal{R}, c) \rightarrow DKh''(\mathcal{R}', c) \), for all \( c \in H^1(\Sigma_g; \mathbb{Z}_2) \).
Alternatively, if $S$ is a genus 0 cobordism between $\mathcal{R}$ and a link $\mathcal{L}$ in $\Sigma_g \times I$, then $\phi_S : \text{DKh}''(\mathcal{R}, c) \to \text{DKh}''(\mathcal{L}, c)$ is injective and graded of degree 0 in all three gradings.

(iv) [8, Theorem 3.16]: Let $\mathcal{R}$ a knot in the thickened surface $\Sigma_g \times I$. If there exists a $c$ such that $\text{DKh}''(\mathcal{R}, c) \neq \text{DKh}''(\mathcal{O}, c)$ (where $\mathcal{O}$ denotes the unique knot in $\Sigma_g \times I$ which bounds a disc) then $\mathcal{R}$ does not bound a disc in $\Sigma_g \times I \times I$.

Note that in (iii) the 4-manifold $\Sigma_g \times I \times I$ into which $S$ embeds has a Morse decomposition with no critical points i.e. no handle stabilisations. This is because there are difficulties in incorporating the handle stabilisations of virtual knot theory into the totally reduced homology, and thus extending it to an invariant of virtual links. In the next section, however, we show that the theory can be used to investigate virtual knots of minimal supporting genus 1.

3.3. The obstruction

We can now define the obstruction advertised above. For ease, we shall fix a basis of $H^1(\Sigma_1; \mathbb{Z}_2)$. Let $\alpha$ be the Poincaré dual of the homology class represented by the curve of that label in Figure 6, and likewise $\beta$, so that $\{\alpha, \beta\}$ forms a basis. We shall abuse notation and denote by $\alpha$ both the curve and the cohomology class associated to it (likewise $\beta$).

For the remainder of this section all virtual knots and links have minimal supporting genus equal to 1 unless otherwise stated. Recall that $\mathcal{O}$ denotes the unique knot in $\Sigma_1 \times I$ which bounds a disc, whose totally reduced homology $\text{DKh}''(\mathcal{O}, \gamma)$ is as follows: it is of rank 4, supported in homological grading 0, quantum gradings $\{1, 0, -1, -2\}$, and $\gamma$-gradings $\{\frac{1}{2}, 0, -\frac{1}{2}, -1\}$, for all $\gamma \in \{\alpha, \beta, \alpha + \beta\}$.

**Definition 3.5.** Let $\mathcal{R}$ be a knot in $\Sigma_1 \times I$. We say that $\mathcal{R}$ is totally non-trivial if $\text{DKh}''(\mathcal{R}, \gamma) \neq \text{DKh}''(\mathcal{O}, \gamma)$, for all $\gamma \in \{\alpha, \beta, \alpha + \beta\}$.

A non-trivial example of the totally reduced homology of a knot in $\Sigma_1 \times I$ is given in Figure 7 (this knot projects to the virtual knot $4.12$ in Green’s table [3]). The totally reduced homology with respect to the red curve is equal to that of $\mathcal{O}$, however, so the knot depicted is not totally non-trivial.

**Proposition 3.6.** Let $D \leftrightarrow \Sigma_1 \times I$ and $D' \leftrightarrow \Sigma_1 \times I$ be representatives of a virtual knot $K$. As described in Section 3.1 $D$ and $D'$ define knots in $\Sigma_1 \times I$, denoted $\mathcal{R}_D$ and $\mathcal{R}_{D'}$, respectively. If $\mathcal{R}_D$ is totally non-trivial, then $\mathcal{R}_{D'}$ is also.

**Proof.** As $D$ and $D'$ are genus-minimal representatives of the same virtual knot, Kuperberg’s Theorem implies that $\mathcal{R}_D$ and $\mathcal{R}_{D'}$ are related by finite sequence of isotopies and Dehn twists i.e. there is the sequence of diagrams

$$\mathcal{D}_1 \to \mathcal{D}_2 \to \cdots \to \mathcal{D}_n$$
Figure 7. The totally reduced homology of the knot in \( \Sigma_1 \times I \) depicted with respect to the curve depicted in green. All of the generators are at homological grading 0, and the horizontal (vertical) axis denotes the \( \bigcirc \)-grading (quantum grading).

where \( \mathcal{D}_i \) is a diagram of \( \mathcal{R}_i, \mathcal{R}_i = \mathcal{R}_D, \mathcal{R}_n = \mathcal{R}_{D'} \), and \( \mathcal{D}_{i+1} \) is obtained from \( \mathcal{D}_i \) by a Reidemeister move or Dehn twist. We may also assume that if \( \mathcal{D}_i \rightarrow \mathcal{D}_{i+1} \) is a Dehn twist, it is a twist about \( \alpha \) or \( \beta \) (all twists may be written as a composition of these elementary twists).

If \( \mathcal{D}_i \rightarrow \mathcal{D}_{i+1} \) is a Reidemeister move, then the proposition holds as the totally reduced homology is an invariant of links in thickened surfaces.

Let \( \mathcal{D}_i \rightarrow \mathcal{D}_{i+1} \) be a Dehn twist. This twist sends \( \mathcal{D}_i \) to \( \mathcal{D}_{i+1} \), and as it is a self-diffeomorphism of \( \Sigma_i \) it must permute the elements of the set \( \{ \alpha, \beta, \alpha + \beta \} \), from which the proposition follows.

We may then define a virtual knot to be totally non-trivial if it has a totally non-trivial representative, and state the following theorem.

**Theorem 3.7.** Let \( K \) be a slice virtual knot. If \( K \) is totally non-trivial then \( K \) is ascent slice.

A slice virtual knot \( K \) is, of course, concordant to a classical knot. We prove Theorem 3.7 by focussing on cobordisms from \( K \) embedded into \( \Sigma_1 \times I \times I \), in order to demonstrate that if a link appears in such a cobordism, then it cannot be destabilised, so that a stabilisation to higher genus surface must be made before a destabilisation can occur (which must occur as one end of the cobordism is to be the unknot).

**Proof of Theorem 3.7.** As \( K \) is a slice virtual knot, \((i)\) of Theorem 3.4 shows that the homological and quantum gradings of \( \text{DKh}''(\mathcal{R}_D, \gamma) \) are those of \( \bigcirc \) (for any representative \( D \) of \( K \))\(^1\). Therefore, the totally non-trivial condition is equivalent to the homology groups \( \text{DKh}''(\mathcal{R}_D, \alpha) \), \( \text{DKh}''(\mathcal{R}_D, \beta) \), and \( \text{DKh}''(\mathcal{R}_D, \alpha + \beta) \) having non-trivial \( \alpha \), \( \beta \), and \( (\alpha + \beta) \) gradings, respectively.

By \((ii)\) of Theorem 3.4, \( K \) does not possess a representative \( D \) such that \( \mathcal{R}_D \) bounds a disc in \( \Sigma_1 \times I \times I \). To show that \( K \) is ascent slice, therefore, we must show that it does not possess a representative which is concordant (in \( \Sigma_1 \times I \times I \)) to a link which can be destabilised. That is, we must show that if there exists a link, \( \mathcal{L} \), in \( \Sigma_1 \times I \) and a genus 0 cobordism, \( S \hookrightarrow \Sigma_1 \times I \times I \), with \( \partial S = \mathcal{R}_D \cup \mathcal{L} \), then for \( \gamma \) a simple closed curve on \( \Sigma_1 \) we have \( \gamma \cap \mathcal{L} \neq \emptyset \).

Assume towards a contradiction that there exists such a link and genus 0 cobordism pair, \( \mathcal{L} \) and \( S \), and a simple closed curve \( \gamma \) on \( \Sigma_1 \) such that \( \gamma \cap \mathcal{L} = \emptyset \). Then by \((ii)\) of Theorem 3.4

\(^1\)both the homological and quantum gradings of doubled Lee homology are slice obstructions, so that the homology of a slice virtual knot must be trivial in both.
we have that $DKh''(\xi, [y])$ possesses no non-trivial $[y]$-gradings. Further, $[y] \in \{\alpha, \beta, \alpha + \beta\}$, and by assumption $DKh''(K_D, [y])$ has non-trivial $[y]$-gradings. But by (iii) of Theorem 3.4 the map $\varphi_S$ is an isomorphism onto its image in $DKh''(\xi, [y])$. As $DKh''((K)_D, [y])$ has non-trivial $[y]$-gradings, while $DKh''(\xi, [y])$ does not, the existence of this graded isomorphism yields the desired contradiction.

Therefore, there does not exist a representative of $K$ which is concordant (in $\Sigma_1 \times I \times I$) to a link which can be destabilised, and any concordance (of virtual knots) from $K$ to the unknot must exhibit a handle stabilisation to at least $\Sigma_2 \times I$.

We conclude this note by remarking that totally non-trivial property can be defined for virtual knots of higher minimal supporting genus than 1, and that it can be demonstrated that it implies ascent sliceness for such virtual knots in essentially identical fashion to the genus 1 case. However, for a virtual knot of minimal supporting genus $g$, determining if it is totally non-trivial requires the computation of $\sum_{i=0}^{2g} (2g - i) = g(2g + 1)$ homology groups, rendering the technique impractical without computer assistance.

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