Non-Minimal String Corrections And Supergravity

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Abstract

We reconsider the well-known issue of string corrections to Supergravity theory. Our treatment is carried out to second order in the string slope parameter. We establish a procedure for solving the Bianchi identities in the non minimal case, and we solve a long standing problem in the perturbative expansion of D=10, N=1 string corrected Supergravity, obtaining the H sector tensors, torsions and curvatures.

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1 Introduction

In view of the renewed interest in string corrected D=10, N=1 Supergravity [1], we revisit an outstanding problem concerning the case to second order in the perturbative expansion. String corrected D=10 and N=1 supergravity is believed to be the low energy limit of string theory, [1]-[4]. Some years ago a program was developed by Gates and collaborators to incorporate string corrections into the supergravity equations of motion [2]-[4]. This approach solved the problem of maintaining manifest supersymmetry. Recently the bosonic equations of motion for D=10, N=1 supergravity fields at superspace and component levels have been obtained and have been shown to be derivable from a lagrangian [1]. The authors have done this to first order in the string slope parameter perturbatively. A long standing problem has been that of obtaining the successful closure of the Bianchi identities to second order. It was stated in [5] that proceeding to higher than first order would yield interesting results. However in practice doing so was not so straightforward.\footnote{The suggestion of changing the torsion constraints was given in [6]. At the end of this paper a warning appeared: accepting that everything works well and the solution proposed by the authors is consistent, it might be that it differs from the one in ref. [5] simply by a field redefinition.}

In the present work we establish a procedure for solving the Bianchi identities to second order and we solve the long standing problem of closure in the H sector. It was suggested that a second order solution would require a modification of the torsion $T_{\alpha\beta\gamma}^{\eta}$ to second order, [1], containing the so called X tensor. We propose an Ansatz for the X tensor and show that it allows for closure in the H and Torsion sectors. We also find the relevant torsions and show mutual consistency. In the curvature sector we identify $R_{\alpha\beta\gamma\delta}^{(2)}$. Crucial to this solution are the results outlined in section (4).

The perturbative approach by Gates and coworkers is well documented and discussed in the literature, and we will not recount it here. For a recent review and for an up to date commentary see [1], and references therein. Our starting point will be the Bianchi identities as listed in [5]. The sigma matrix identities and symmetries are recorded in [3].

These geometrical methods nowadays are known as deformations [1], and the constraints have sometimes been referred to in the past as beta function favored ($\beta FF$) constraints [7] (see for related subjects e.g. [8]). In the past, such methods allowed for the determination of the most general higher derivative Yang-Mills action to order $\gamma^3$, which is globally supersymmetric and Lorentz covariant in D=10 spacetime (see e.g. [9]), a result which is important for topologically nontrivial gauge configurations of the vector field, e.g., for compactifield string theories on manifolds with topologically nontrivial properties.

This paper is organized as follows. In section (2) we set up the Bianchi identities. In section (3) we discuss some implications of the X tensor cited in [1]. In section (4) we outline key results necessary for obtaining the solutions. As the derivations are lengthy we have not included them in this letter. In section (5) we give the H sector solution through identifying $T_{\alpha\beta\gamma}^{(2)}$ and by the elimination of non linear terms. In section (6) we propose a candidate for the so called X tensor and show that it achieves closure in the H sector by a different method and that it also admits solutions to the torsion equations.

1 The suggestion of changing the torsion constraints was given in [6]. At the end of this paper a warning appeared: accepting that everything works well and the solution proposed by the authors is consistent, it might be that it differs from the one in ref. [5] simply by a field redefinition.
The remaining sections examine further checks and curvatures, finally concluding.

2 The Bianchi identities

The full set of $Q$ and $G$ Bianchi identities is listed in ref. [5]. Below we list only those which we need to consider here. The $H$ tensor is related to the $G$ tensor in superspace as follows:

$$G_{ADG} = H_{ADG} + \gamma Q_{ADG} + \beta Y_{ADG}$$  \hspace{1cm} (1)

where $Q$ is the Lorentz-Chern Simons superform, $Y_{ADG}$ is the Yang Mills Superform, and $\gamma$ is proportional to the string slope parameter. That is, the action for massless fields of heterotic or type I superstrings may be expanded, with $\beta$ set to zero, as follows [2]:

$$S_{\text{eff}} = \frac{1}{\kappa^2} \int d^{10}x (-1)[L(0) + \sum_{n=1}^{\infty} (\gamma)^n L(n)]$$  \hspace{1cm} (2)

Within the framework of the Bianchi identities we have a perturbative prescription that will allow us to incorporate string corrections into the theory, and maintain it manifestly supersymmetric. We first solve the identities satisfied by the $H$ tensor. The first three are:

$$\frac{1}{6} \nabla_{(\alpha} H_{|\beta\gamma\delta)} - \frac{1}{4} T_{(\alpha\beta|\lambda} E H_{E|\gamma\delta)} = (-\frac{\gamma}{4}) R_{(\alpha\beta|e\sigma} R_{|\gamma\delta)}^{ef}$$ \hspace{1cm} (3)

$$\frac{1}{2} \nabla_{(\alpha} H_{|\beta\gamma)} d - \nabla_d H_{\alpha\beta\gamma} - \frac{1}{2} T_{(\alpha\beta|\lambda} E H_{E|\gamma\delta)} d + \frac{1}{2} T_{d(\alpha|\lambda} E H_{E|\beta\gamma)} = (-\gamma) R_{(\alpha\beta|e\sigma} R_{|\gamma\delta)}^{ef}$$ \hspace{1cm} (4)

$$\nabla_{(\alpha} H_{|\beta\gamma)}_{cd} + \nabla_{[e} H_{|d|\alpha\beta} - T_{\alpha\beta|e\sigma} E H_{E|\sigma\delta)} - T_{e\delta} E Q_{E|\alpha\beta} - T_{(\alpha|\beta|\gamma)} E H_{E|\gamma\delta)} = -\gamma [2R_{\alpha\beta|e\sigma} R_{|\gamma\delta)}^{ef} + R_{(\alpha|\beta|e\sigma} R_{|\gamma\delta)}^{ef}$$ \hspace{1cm} (5)

We also require the torsions:

$$T_{(\alpha\beta|\lambda T_{|\gamma)} d} - T_{(\alpha\beta|g T_{|\gamma)} d} - \nabla_{(\alpha|T_{|\beta\gamma)} d} = 0$$ \hspace{1cm} (6)

and

$$T_{(\alpha\beta|\lambda T_{|\gamma)} \delta} - T_{(\alpha\beta|g T_{|\gamma)} \delta} - \nabla_{(\alpha|T_{|\beta\gamma)} \delta} - \frac{1}{4} R_{(\alpha\beta|de} \sigma^{de} |\gamma)} \delta = 0$$ \hspace{1cm} (7)

The first order solution, originally given in [5], was recently recalculated [11]. Many avenues such as null spaces were tried in order to proceed to second order, but without
success. It was suggested that a generalization of the torsion $T_{\alpha\beta}^g$ would be necessary in order to proceed to second order [1]. One of the problems at hand therefore is to find the form of this generalization, known as the X tensor. Here we do this and we also find the second order torsions $T^{(2)}_{\alpha\beta} g$, $T^{(2)}_{\alpha\beta} \lambda$, $T^{(2)}_{\alpha\beta} g$, $T^{(2)}_{\alpha\beta} \gamma$ and curvatures $R^{(2)}_{\alpha\beta\delta e}$ and $R^{(2)}_{\alpha\beta\delta e}$.

We check our results by showing mutual consistency.

At this stage we will draw attention to our simple second order notation. The superscript in brackets refers to the second order quantities. For brevity we will not relist the first order quantities as found in [1] and [5] unless necessary. Hence we have, for example

\[ R_{\alpha\beta\delta e} = R^{(0)}_{\alpha\beta\delta e} + R^{(1)}_{\alpha\beta\delta e} + R^{(2)}_{\alpha\beta\delta e} + \ldots \]  

(8)

where $R^{(0)}_{\alpha\beta\delta e}$ and $R^{(1)}_{\alpha\beta\delta e}$ are listed in [1] and [5]. To begin with, $H_{\beta\gamma\delta}$ is set to zero as in [1] and [5]. We have not seen that it is required to be other than zero to close the Bianchi identities. We have seen that if it is non zero the H sector Bianchi identities fail to close.

3 The Problem

Imposing the conventional constraint $T^g_{\alpha\beta} = i\sigma^g_{\alpha\beta}$ to all orders leads to failure to close the Bianchi identities to second order. In the present case that constraint is modified. From the conventional constraints listed in [1] the most general form of the zero dimensional torsion is

\[ T_{\alpha\beta}^g = i\sigma^g_{\alpha\beta} + \sigma^{pqrf}_{\alpha\beta} X_{pqrf}^g \]  

(9)

Here we absorb the coefficient $i$ used in ref. [1] into the X tensor.

Earlier, because of the existence of an apparently intractable term which arose in the H sector Bianchi identities, closure could not be obtained with $T_{\alpha\beta}^g = i\sigma^g_{\alpha\beta}$. At the time an unsolvable term could not be incorporated into the torsion $T^{(2)}_{\alpha\beta} \lambda$, which would have allowed for a solution. In the following we see that the X tensor must contribute to second order. Let us consider the Bianchi identity at dimension one-half. If the X tensor is zero, then we have, using the constraints in [5],

\[ T^{(2)}_{(\alpha\beta)} \lambda^\gamma \sigma_{(\gamma)\lambda}^d = \sigma_{(\alpha\beta)}^g T^{(2)}_{(\gamma)} g^d \]  

(10)

Therefore $T^{(2)}_{\alpha\beta} \lambda$ must have a similar sigma matrix structure to the RHS of (10). We know from the H sector Bianchi identities that $H^{(2)}_{g\gamma d}$ satisfies an equation of the form [10]

\[ \sigma^g_{(\alpha\beta)} H^{(2)}_{g(\gamma)d} = \sigma^g_{(\alpha\beta)} T^{(2)}_{(\gamma)} g d - \sigma_{d(\alpha\beta)} T^{(2)}_{(\gamma)} \lambda - 8 \gamma^2 \sigma_{e(\alpha)} \sigma_{f(\beta)} \sigma_{d(\gamma)} T^{(0)}_{kp} e T^{(0)}_{kpr} T^{(0)}_{e f p} \]  

(11)
Looking at (11) we see that we will encounter an intractable term in the $H$ sector unless we can absorb it into the $T^{(2)}_{\beta\gamma\lambda}$ term. There is no known sigma matrix identity that will allow the last term on the RHS of (11) to be written in the form necessary to solve for $H^{(2)}_{\alpha\beta\delta}$. Let us call it the $T^3$ term. We see that the $T^3$ term has the same sigma matrix structure as the $T^{(2)}_{\beta\gamma\lambda}$ term. We consider the option of equating the $T^{(2)}_{\beta\gamma\lambda}$ term with the $T^3$ term. This is possible only when $X$ is non zero, and it will constitute one of the many necessary components of the solution.

4 Results Necessary for the Solution

The following combination of results led to the solution, in combination with other observations. The first requirement will be the $X$ tensor Ansatz. We will require the zeroth order spinor derivative of $T^{(0)}_{\alpha\beta\gamma\delta}$, available from [5],

$$\nabla_\gamma T^{(0)}_{\alpha\beta\gamma\delta} = -\frac{1}{4} \sigma^{mn}_{\alpha\beta\gamma\delta} R_{efmn} + T^{(0)}_{\alpha\beta\gamma\delta} T^{(0)}_{\gamma\delta}$$

(12)

We refer to its $\chi$ part and its curvature part using a suitable subscript as in eqs. (13) and (15) below. We have the following important results, the derivations of which are lengthy and not included in this letter [10]

$$T^{(0)}_{\alpha\beta\gamma\delta} \chi = \frac{i\gamma}{6} \sigma^{pqref}_{\alpha\beta\gamma\delta} A^{(1)}_{pqrf} H^{(0)}_{def} - \frac{i\gamma}{6} \sigma^{pqref}_{\alpha\beta\gamma\delta} [H^{(0)}_{def} \nabla_\gamma A^{(1)}_{pqrf}] \chi$$

$$= + i\gamma \sigma^g_{\alpha\beta\gamma\delta} H^{(0)}_{def} \nabla_\gamma A^{(1)}_{gef} \chi$$

(13)

We also require the following:

$$+ \frac{i\gamma}{6} \sigma^{pqref}_{\alpha\beta\gamma\delta} H^{(0)}_{def} \nabla_\gamma A^{(1)}_{pqrf} \chi = \frac{\gamma^2}{12} \sigma^{pqref}_{\alpha\beta\gamma\delta} H^{(0)}_{def} \sigma_{pqrf} T^{(0)}_{kp} \epsilon^{\gamma mn} \nabla_\gamma R^{(0)}_{mn} $$

$$= \frac{\gamma^2}{2} \sigma^g_{\alpha\beta\gamma\delta} \sigma^{ef \gamma \delta} T^{(0)}_{kp} \epsilon^{\gamma mn} \nabla_\gamma R^{(0)}_{mn} + 16\gamma^2 \sigma^g_{\alpha\beta\gamma\delta} \sigma^{ef \gamma \delta} T^{(0)}_{kp} \epsilon^{ef \gamma \delta} R^{(0)}_{fg}$$

(14)

For later transparency, using the identity $\sigma^g_{\alpha\beta\gamma\delta} \sigma^{gf\gamma\delta} = 0$, this can be rewritten as

$$\frac{i\gamma}{6} \sigma^{pqref}_{\alpha\beta\gamma\delta} H^{(0)}_{def} \nabla_\gamma A^{(1)}_{pqrf} \chi$$

$$= -i\gamma \sigma^g_{\alpha\beta\gamma\delta} [\nabla_\gamma A^{(1)}_{gef}] \chi R^{(0)}_{def} + 4i\gamma \sigma^g_{\alpha\beta\gamma\delta} R^{(0)}_{gef} H^{(0)}_{def}$$

(15)

Combining (13) and (15) we get a crucial equation which will facilitate finding our solution. In a very conveniently reduced form, what once seemed apparently intractable, will become

5
\[ + \frac{i\gamma}{6} T^{(0)}_{(\alpha\beta)} \lambda \sigma^{pqref}_{[\gamma]} \lambda A^{(1)}_{pqr} H^{(0)}_{def} - \frac{i\gamma}{6} \sigma^{pqref}_{(\alpha\beta)} H^{(0)}_{def} \nabla_{[\gamma]} A^{(1)}_{pqr} \]

\[ = +i\gamma \sigma^g_{(\alpha\beta) [\nabla_{\gamma}] A^{(1)}_{gref}] H^{(0)}_{d ef} - 4i\gamma \sigma^g_{(\alpha\beta) R^{(1)}_{[\gamma]gref} H^{(0)}_{def} ef} \]  

(16)

Along with the sigma matrix identities listed in [3], we also require the important result that

\[ \sigma^{pqref}_{(\alpha\beta) [\sigma e_{[\gamma]} e_{\phi}] = - \sigma^{pqref}_{\phi(\alpha)[\sigma e_{\beta\gamma}]} \]  

(17)

The latter allows us to write

\[ \frac{i\gamma}{12} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} R^{(0)}_{[\gamma]def} \]

\[ = + \frac{\gamma}{6} \sigma^g_{(\alpha\beta) [\sigma^{pqref}_{g(\gamma)} e_{\phi}] A^{(1)}_{pqr} T^{(0)}_{def} \phi} \]  

(18)

Closure was not immediately evident in the curvature sector. To achieve this equations (58) and (72) were also required [10].

5 The H Sector Solution

It is a straightforward to solve for \( H^{(2)}_{\gamma\delta} \). To do so we use the constraints in ref. [5] and substitute them into equation (3). We then extract a sigma matrix coefficient from each term and solve. After some algebra we obtain

\[ H^{(2)}_{\gamma\delta} = [-4\gamma H^{(0)}_{gref R^{(1)}_{\gamma\delta} ef} - \frac{1}{2} T^{(2)}_{\gamma\delta}] \]  

(19)

We can also write the latter expression as follows:

\[ H^{(2)}_{\alpha\beta d} = \sigma_{\alpha\beta g} [8i\gamma H^{(0)}_{def R^{(1)}_\gamma ef} - i\gamma H^{(0)}_{def A^{(1)}_g ef}] + \sigma^{pqref}_{(\alpha\beta) \frac{-i\gamma}{6} H^{(0)}_{def A^{(1)}_{pqr} - \frac{1}{2} X_{pqrdefd}} \]  

(20)

For the next Bianchi identity, applying the constraints of ref. [5] to (4) reduces it to

\[ + \frac{i}{2} \sigma^g_{(\alpha\beta)} H^{(2)}_{g\gamma\delta d} = \frac{1}{2} \nabla_{(\alpha} H_{\beta]\gamma d} \text{Order} \gamma^2 - \frac{1}{2} T^{(0)}_{(\alpha\beta) \lambda H^{(2)}_{\lambda\gamma d} - \frac{i}{2} \gamma R^{(0)}_{gref R^{(1)}_{[\gamma]def} + H^{(0)}_{gref R^{(1)}_{[\gamma]def} ef}] + \frac{i\gamma}{24} \sigma^{pqref}_{(\alpha\beta) R^{(0)}_{[\gamma]def A^{(1)}_{pqr} - \frac{i}{4} \sigma^g_{d(\alpha)[\lambda T^{(2)}_{\beta\gamma} + \frac{i}{4} \sigma^g_{(\alpha\beta) T^{(2)}_{d\gamma\delta}] \]  

(21)
where
\[
\Pi^{(1)}_{\ g\ ef} = L^{(1)}_{\ g\ ef} - \frac{1}{8} A^{(1)}_{\ g\ ef} \quad (22)
\]

We write it in the above form for transparency and for future reference. In particular we note that the last two terms in (21) can be obtained using equation (6), and that they can be written in terms of the X tensor. Equation (21) contains a proliferation of non solvable \(\chi\) terms. The X tensor is also contained within this equation. It is lengthy and does not advance our argument to write it out in full here.

We will solve this Bianchi identity by two routes. Firstly we solve it by eliminating the X tensor, and by identifying \(T^{(2)}_{\alpha\beta\gamma}\). We need to consider the spinor derivative of \(H^{\beta\gamma d}_{\alpha\beta\gamma}^{Order \gamma^2}\). We do not yet know the form of the X tensor, so we employ a torsion and a curvature to eliminate \(\chi\) and X tensor terms. We will later calculate it directly and show agreement. We must remember to include the second order derivative contributions which come from the first order result. To first order, [5], we have

\[
H_{\beta\gamma d} = \frac{i}{2} \sigma_{d\beta\gamma} + i \sigma_{\alpha\beta} g_{ef} H^{(0)}_{gef} H^{(0)}_{d\ ef} \quad (23)
\]

As pointed out in [7], the field independence of the leading term in the \(H_{\alpha\beta\gamma}^{Order \gamma^2}\) component of the 3-form field strength is indicative of the correspondence of these equations in the limit \(\gamma = 0\), to the superspace geometry in a string-frame description of pure supergravity. Taking the spinor derivative of this first order term will generate second order contributions. We have from first order, which we leave as is,

\[
\frac{1}{2} \nabla_{(\alpha} H^{(1)}_{\beta|\gamma) d}^{Order \gamma^2} = \sigma^9_{(\alpha\beta}(2i\gamma \nabla_{[\gamma)} [H^{(0)}_{gef} H^{(0)}_{d\ ef}]^{Order (1)}) \quad (24)
\]

We seek to solve for \(H^{(2)}_{\gamma\beta\gamma} d\) in an expression of the form

\[
\sigma^9_{(\alpha\beta} H^{(2)}_{|\gamma) d} = \sigma^9_{(\alpha\beta} M^{(2)}_{|\gamma) d} \quad (25)
\]

Finally we extract the correct expression for \(H^{(2)}_{\gamma\beta\gamma} d\) with an appropriate symmetrization operator, (77).

We need to take the derivative of equation (19). To do this we use a first order curvature and the dimension one-half torsion to second order. This will eliminate the \(\chi\) terms in (21), and it will also isolate the torsion \(T^{(2)}_{\alpha\beta\gamma}\).

The first order curvature required is

\[
\nabla_{(\alpha} R^{(1)}_{|\beta}\lambda\gamma e f} = T^{(0)}_{(\alpha\beta|} \lambda R^{(1)}_{\gamma\lambda e f} + T^{(1)}_{(\alpha\beta|} \lambda R^{(0)}_{\gamma\lambda e f} - T^{(0)}_{(\alpha\beta|} g R^{(1)}_{\gamma e f} - T^{(1)}_{(\alpha\beta|} g R^{(0)}_{\gamma e f} \quad (26)
\]

The required torsion is given in equation (6). This gives the complete expression for the derivative

\[
\frac{1}{2} \nabla_{(\alpha} H_{\beta|\gamma) d}^{Order 2} = \sigma^9_{(\alpha\beta} [2i\gamma \nabla_{[\gamma)} (H^{(0)}_{gef} H^{(0)}_{d\ ef})^{Order 2} + 4i\gamma [\nabla_{[\gamma)} H^{(0)}_{gef}]^{Order (1)} e f]}
\]
\[-\frac{i\gamma}{12} \sigma^{pqre}_{(\alpha\beta)}(\nabla_{|\gamma|} H^{(0)}_{def}) A^{(1)}_{pqr} + \frac{1}{2} \chi(\alpha\gamma) T^{(2)}_{|\beta\gamma|d} - \frac{i}{4} \sigma^{d(\alpha\lambda)} T^{(2)}_{|\beta\gamma|\lambda} \]

\[
+ \sigma^{g}_{(\alpha\beta)} \left[ \frac{1}{4} \sigma^{\phi\lambda} X_{\phi} T^{(2)}_{|\gamma|\lambda d} + \frac{i}{4} T^{(2)}_{|\gamma|\gg d} \right]
+ \sigma^{g}_{(\alpha\beta)} \left[ 2 \gamma \sigma^{\phi\lambda} X_{\phi} R^{(1)}_{|\gamma|\gamma e f} H^{(0)}_{def} + 2 i \gamma H^{(0)}_{def} R^{(1)}_{|\gamma|e f} \right]
+ 4 \gamma H^{(0)}_{def} \chi(\alpha\gamma) R^{(1)}_{|\beta\gamma|e f} \]

(27)

After substitution of the derivative term, (27), into (21) we eliminate the non-linear terms and after many cancelations we arrive at

\[
\frac{i}{2} \sigma^{g}_{(\alpha\beta)} H^{(2)}_{g|\gamma|d} = \sigma^{g}_{(\alpha\beta)} \left[ + 2 i \gamma \nabla_{|\gamma|} (H^{(0)}_{def} H^{(0)}_{g ef}) - 2 i \gamma R^{(1)}_{|\gamma|d e f} e f H^{(0)}_{g ef} \right.
- 2 i \gamma \Gamma^{(1)}_{g ef} H^{(0)}_{def} - 2 \nabla_{|\gamma|} H^{(0)}_{def} \bigg]
+ \frac{i \gamma}{24} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} [R^{(0)}_{|\gamma|def} - 2 \nabla_{|\gamma|} H^{(0)}_{def}]
- \frac{i}{2} \sigma^{d(\alpha\lambda)} T^{(2)}_{|\beta\gamma| \lambda} + \sigma^{g}_{(\alpha\beta)} \frac{i}{2} T^{(2)}_{|\beta\gamma| g} \]

(28)

We write the expression this way for later convenience and transparency. Now we consider the sigma five part. Although we do not do it now, we note that the term with R^{(0)}_{\gamma def} allows it to be written as a solvable term because of the identity, (18). After a lengthy calculation we find the following (see [10] for the detailed calculation):

\[
+ \frac{i \gamma}{24} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} R^{(0)}_{|\gamma|def} - 2 \nabla_{|\gamma|} H^{(0)}_{def} = - \frac{\gamma}{4} \sigma^{g}_{(\alpha\beta)} A^{(1)}_{g e f \sigma^{d|\gamma| \lambda} T^{(0)} e f \lambda}
+ 4 i \gamma^{(2)} e f \sigma^{(\alpha|e| \sigma_{f|\beta|} \sigma^{d|\gamma| \lambda} T^{(0)} e f \lambda}^{(0)} T^{(0)} k p r = - \frac{\gamma}{24} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{p q \sigma^{d|\gamma| \phi} T^{(2)}_{e f \phi} \}

(29)

We now make the following identification:

\[
i \sigma^{d(\alpha\lambda)} T^{(2)}_{|\beta\gamma| \lambda} = + \frac{i \gamma}{12} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} R^{(0)}_{|\gamma|def} - 2 \nabla_{|\gamma|} H^{(0)}_{def} \]

(30)

or

\[
T^{(2)}_{\alpha\beta \gamma} = - \frac{i \gamma}{12} \sigma^{pqref}_{\alpha\beta} A^{(1)}_{p q r} T^{(2)}_{e f \lambda} \]

(31)

This scenario will give for H^{(2)}_{g|\gamma|d}

\[
\frac{i}{2} \sigma^{g}_{(\alpha\beta)} H^{(2)}_{g|\gamma|d} = \sigma^{g}_{(\alpha\beta)} \left[ + 2 i \gamma \nabla_{|\gamma|} (H^{(0)}_{def} H^{(0)}_{g ef}) - 2 i \gamma R^{(1)}_{|\gamma|d e f} e f H^{(0)}_{g ef} \right.
- 2 i \gamma \Gamma^{(1)}_{g ef} H^{(0)}_{def} - 2 \nabla_{|\gamma|} H^{(0)}_{def} \bigg]
+ \sigma^{g}_{(\alpha\beta)} \frac{i}{2} T^{(2)}_{|\beta\gamma| g} \]

(32)
This is now in the solvable form and the result can be extracted using the operator (77). It is written in terms of the torsion $T^{(2)}_{d\gamma g}$, which is given later in (81).

## 6 Closure Via the X tensor

We now propose an Ansatz for the X tensor and show that, in conjunction with the results (16) and (30), it succeeds in closing the H sector by a different route. Furthermore we can add support to this Ansatz by deriving the torsion $T^{(2)}_{\beta\gamma \lambda}$ through solving the dimension one-half torsion. Let us consider the last two terms in (28), consisting of torsions.

Using the dimension one-half torsion (6), we can write the last two terms of (28) as

$$-\frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}_{|\beta\gamma)} + \sigma^g_{(\alpha\beta]}\frac{i}{2}T^{(2)}_{d|\gamma)g}$$

$$= \frac{1}{2}T^{(0)}_{(\alpha\beta]}\lambda T^{(2)}_{|\gamma)\lambda d} - \frac{1}{2}\nabla_{(\alpha}T^{(2)}_{|\beta\gamma)}d$$  \hspace{1cm} (33)

Let the X tensor be given by the following:

$$X_{pqrefd} = -\frac{i\gamma}{6}H^{(0)}_{defA^{(1)}_{pqr}} + Y_{pqrefd}$$  \hspace{1cm} (34)

Our procedure is successful with $Y_{pqrefd} = 0$, hence we use

$$T^{(2)}_{\alpha\beta}d = -\frac{i\gamma}{6}\sigma_{pqref\alpha\beta}H^{(0)}_{defA^{(1)}_{pqr}}$$  \hspace{1cm} (35)

We obtain

$$-\frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}_{|\beta\gamma)} + \sigma^g_{(\alpha\beta]}\frac{i}{2}T^{(2)}_{d|\gamma)g}$$

$$= -\frac{i\gamma}{12}T^{(0)}_{(\alpha\beta]}\lambda \sigma_{pqref\gamma)\lambda}H^{(0)}_{defA^{(1)}_{pqr}} + \frac{i\gamma}{12}\sigma_{pqref(\alpha\beta]}H^{(0)}_{def[\nabla_{\gamma)}A^{(1)}_{pqr}]}
+ \frac{i\gamma}{12}\sigma_{pqref(\alpha\beta][\nabla_{\gamma)}H^{(0)}_{def}]A^{(1)}_{pqr}}$$  \hspace{1cm} (36)

Now apply the result (16) to (36) to obtain

$$-\frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}_{|\beta\gamma)} + \sigma^g_{(\alpha\beta]}\frac{i}{2}T^{(2)}_{d|\gamma)g}$$

$$= -\frac{i\gamma}{2}\sigma^g_{(\alpha\beta][\nabla_{\gamma)}A^{(1)}_{gef]}H^{(0)}_{def}$$
$$+ 2i\gamma\sigma^g_{(\alpha\beta]|R^{(1)}_{|\gamma)gef}H^{(0)}_{def}$$  \hspace{1cm} (37)
Hence also using (18) we find

\[
\frac{i}{2} \sigma^g_{(\alpha\beta)} H^{(2)\alpha\beta|\gamma} g_{\gamma} d = \sigma^g_{(\alpha\beta)} \left\{ +2i\gamma \nabla_{|\gamma} (H^{(0)}_{def} H^{(0)}_{g} e^f) - 2i\gamma R^{(1)}_{|\gamma} d e^f H^{(0)}_{g} e^f \right. \\
- 2i\gamma \left[ \Pi^{(1)}_{g} e^f \right] R^{(0)}_{|\gamma} d e^f - 2\nabla_{|\gamma} H^{(0)}_{d e^f} \} + \frac{\gamma}{12} \sigma^g_{(\alpha\beta)} \sigma^{pqref} g_{|\gamma} \phi A^{(1)}_{pqr} T^{(0)}_{de^f} - \frac{i\gamma}{2} \sigma^g_{(\alpha\beta)} \left[ \nabla_{|\gamma} A^{(1)}_{g e^f} H^{(0)}_{d e^f} + 2i\gamma \sigma^g_{(\alpha\beta)} R^{(1)}_{|\gamma} g e^f H^{(0)}_{d e^f} \right]
\]

(38)

Thus we see that we can solve for $H^{(2)\alpha\beta|\gamma} g_{\gamma} d$ using this $X$ tensor in conjunction with our result for $T^{(2)\beta\gamma^\lambda}$.

We note that we have also found the torsion $T^{(2)\beta\gamma^\lambda}$, by comparing (32) to (38). However for clarity we will solve the dimension one-half torsion directly. Looking at the dimension one-half torsion (6), and substituting (30) and (35) into it, yields

\[
- \frac{i\gamma}{6} T^{(0)}_{|\lambda} \lambda \sigma^{pqref} g_{|\gamma} A^{(1)}_{pqr} H^{(0)}_{def} + \frac{i\gamma}{12} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} R^{(0)}_{|\gamma} H^{(0)}_{def} \\
- \frac{i\gamma}{6} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} \nabla_{|\gamma} H^{(0)}_{def} + \frac{i\gamma}{6} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} \nabla_{|\gamma} H^{(0)}_{def} + \frac{i\gamma}{12} \sigma^{pqref}_{(\alpha\beta)} A^{(1)}_{pqr} \nabla_{|\gamma} A^{(1)}_{pqr} - i\sigma^g_{(\alpha\beta)} T^{(2)\alpha\beta|\gamma} g_{\gamma} d = 0
\]

(39)

We now follow the same procedure as before and use (16) and (18). We obtain convenient cancelations of non solvable terms to get

\[
+ i\sigma^g_{(\alpha\beta)} T^{(2)\alpha\beta|\gamma} g_{\gamma} d = -i\gamma \sigma^g_{(\alpha\beta)} \left[ \nabla_{|\gamma} A^{(1)}_{g e^f} H^{(0)}_{d e^f} + 4i\gamma \sigma^g_{(\alpha\beta)} R^{(1)}_{|\gamma} g e^f H^{(0)}_{d e^f} + \frac{\gamma}{6} \sigma^{pqref}_{(\alpha\beta)} g_{|\gamma} \phi A^{(1)}_{pqr} T^{(0)}_{de^f} \right]
\]

(40)

To find $T^{(2)\alpha\beta|\gamma} g_{\gamma} d$ the above must be symmetrized. The expression is listed in the conclusions, see eq. (81), and so is the spinor derivative $\nabla_{|\gamma} A^{(1)}_{g e^f}$.

7 Direct Method

In the following we show agreement and consistency with the previous results, by choosing to take the direct derivative of $H^{(2)\alpha\beta|\gamma} g_{\gamma} d$ (equation (20)), instead of eliminating non linear terms using a torsion and curvature. This will involve the derivative of $L^{(1)_{abc}}$. We begin with equation (21). We require the following:

\[
\nabla_{|\gamma} \Pi^{(1)}_{g} e^f = \nabla_{|\gamma} \left[ L^{(1)}_{g} e^f - \frac{1}{8} A^{(1)}_{g} e^f \right]
\]

(41)
We also have the important observation that, using\( \sigma^g_{(\alpha\beta|\sigma|\gamma)\lambda} = 0 \), we can make the identification

\[
\nabla_\alpha L^{(1)}_{abc} = R^{(1)}_{\gamma abc}
\]

(42)

With this we can compare the two results. Taking the derivative of (20) with the use of this result yields

\[
\frac{1}{2} \nabla_\alpha \left[ H^{(2)}_{(0)} |_{[\beta\gamma]} \right] \sigma_\alpha^{\beta\gamma} = \sigma^g_{(\alpha\beta|\sigma|\gamma)\lambda} \left( - \frac{i}{2} \nabla_\gamma \left[ H^{(0)}_{(0) de} f H^{(0)}_{(0) g e f} \right] \sigma_\gamma^{\lambda\delta} + 4i \nabla_\gamma \left[ H^{(0)}_{(0) de} f \right] \Pi^{(1)}_{\gamma} g e f 
\]

\[
+ 4i \nabla_\gamma \left[ H^{(0)}_{(0) de} f R^{(1)} |_{[\gamma]} \right] g e f - \frac{i}{2} \nabla_\gamma \left[ A^{(1)} g e f \right] - \frac{i}{24} \sigma^{pqref} \sigma_{(\alpha\beta|\sigma|\gamma)\lambda} \left( \nabla_\gamma \left[ H^{(0)}_{(0) de} f \right] \right) A^{(1)}_{pqr}
\]

\[
- \frac{i}{24} \sigma^{pqref} \sigma_{(\alpha\beta|\sigma|\gamma)\lambda} \left( \nabla_\gamma \left[ H^{(0)}_{(0) de} f \right] \right) A^{(1)}_{pqr}
\]

(43)

Substituting (43) as well as (42) into (21) yields exactly (32) with the expression\( \sigma^g_{(\alpha\beta|T^{(2)}_{(\gamma)|d} g} \) as found in (40). We regard this as an important check for consistency.

\section{Torsion Equation for \( T^{(2)}_{\alpha d \delta} \)}

To second order we have the following torsion equation:

\[
T^{(0)}_{(\alpha\beta|}\lambda T^{(2)}_{(\gamma)|}\delta + T^{(2)}_{(\alpha\beta|}\lambda T^{(0)}_{(\gamma)|}\delta - i \sigma^g_{(\alpha\beta|T^{(2)}_{(\gamma)|}\delta - \nabla_\alpha T^{(2)}_{(\beta\gamma)} |\delta
\]

\[
- \frac{1}{4} R^{(2)}_{(\alpha\beta|de} \sigma_{d e |(\gamma) \delta} = 0
\]

(44)

We must take care not to neglect second order contributions from the derivative\( \nabla_\alpha T^{(0)}_{\beta\gamma |\delta} \). We find

\[
- \nabla_\alpha T^{(2)}_{(\beta\gamma) |\delta} = [2 \delta_{(\alpha\delta |\beta|\gamma) \lambda + \sigma^g_{(\alpha\beta|\sigma|\delta \lambda}] \nabla_\gamma \chi_\lambda =
\]

\[
- \frac{i}{2} \sigma^g_{(\alpha\beta|\sigma|\gamma) \lambda} \left[ L^{(2) mn} + \frac{1}{4} A^{(2) mn} \right]
\]

(45)

Once again using (12), (16) and (31), reduces (44) to
\[-\frac{i\gamma}{2}\sigma^{\mu\nu}_{(\alpha\beta)}[\nabla_\gamma]A^{(1)}_{\mu\nu} T^{(0)}_{\mu\nu} \delta + 2i\gamma\sigma^{\mu\nu}_{(\alpha\beta)} R^{(1)}_{\mu\nu} T^{(0)}_{\mu\nu} \delta \]
\[-\frac{i\gamma}{12}\sigma^{pqref}_{(\alpha\beta)}A^{(1)}_{pqr} T^{(0)}_{\mu\nu} \lambda T^{(0)}_{\gamma\lambda} \delta - i\gamma\sigma^{\mu\nu}_{(\alpha\beta)} T^{(2)}_{\mu\nu} \delta \]
\[+ \frac{i\gamma}{12}\sigma^{pqref}_{(\alpha\beta)}A^{(1)}_{pqr} \left[-\frac{1}{4}\sigma^{mn}_{|\gamma}\delta R_{efmn}\right] \]
\[+ T^{(0)}_{\mu\nu} \lambda T^{(0)}_{\gamma\lambda} \delta - \frac{1}{4} R^{(2)}_{(\alpha\beta)de\delta\gamma} \delta \]
\[-\frac{i}{2}\sigma^{\mu\nu}_{(\alpha\beta)}\sigma^{mn}_{|\gamma}\delta \left[L^{(2)}_{gmn} + \frac{1}{4} A^{(2)}_{gmn}\right] = 0 \] (46)

We note that the spinor derivative of $T^{(0)}_{\mu\nu} \delta$ results in a convenient cancelation of otherwise unsolvable terms, and so finally we obtain

\[+ i\gamma\sigma^{\mu\nu}_{(\alpha\beta)} T^{(2)}_{\mu\nu} \delta \]
\[+ \frac{i\gamma}{2}\sigma^{\mu\nu}_{(\alpha\beta)}[\nabla_\gamma]A^{(1)}_{\mu\nu} T^{(0)}_{\mu\nu} \delta - 2i\gamma\sigma^{\mu\nu}_{(\alpha\beta)} R^{(1)}_{\mu\nu} T^{(0)}_{\mu\nu} \delta \]
\[+ \frac{1}{4} R^{(2)}_{(\alpha\beta)de\delta\gamma} \delta + \frac{i\gamma}{12}\sigma^{pqref}_{(\alpha\beta)}A^{(1)}_{pqr} \left[+\frac{1}{4}\sigma^{mn}_{|\gamma}\delta R_{efmn}\right] \]
\[+ \frac{i}{2}\sigma^{\mu\nu}_{(\alpha\beta)}\sigma^{mn}_{|\gamma}\delta \left[L^{(2)}_{gmn} + \frac{1}{4} A^{(2)}_{gmn}\right] = 0 \] (47)

From the above we extract the candidates

\[T^{(2)}_{\gamma\delta} = -\frac{\gamma}{2}[\nabla_\gamma A^{(1)}_{gef} T^{(0)}_{ef} \delta + 2\gamma R^{(1)}_{gef} T^{(0)}_{ef} \delta] \] (48)

and

\[R^{(2)}_{\alpha\beta de} = -\frac{i\gamma}{12}\sigma^{pqrab}_{\alpha\beta} A^{(1)}_{pqr} R_{deab} - 2i\gamma\sigma^{\mu\nu}_{\alpha\beta} \left[L^{(2)}_{gde} + \frac{1}{4} A^{(2)}_{gde}\right] \] (49)

### 9 Curvature Equation for $R^{(2)}_{\lambda gde}$

We need to solve the curvature that will give $R^{(2)}_{\lambda gde}$. We have the curvature Bianchi identity which we need to solve at second order as follows:

\[T_{(\alpha\beta)}^{\lambda} R_{\gamma\lambda de} - T_{(\alpha\beta)}^{g} R_{\gamma gde} - \nabla_{(\alpha} R_{\beta\gamma) de} = 0 \] (50)

Here we must consider second order contributions from the spinor derivative $\nabla_{(\alpha} R_{|\beta\gamma) de}$. We write the full curvature to second order for clarity.

12
\[ R_{\beta\gamma de} = -2i\sigma^g_{\alpha\beta}\Pi'_{gde} + \frac{i}{24}\sigma^{pqr}_{d\alpha\beta}A^{(1)}_{pqr} - \frac{i\gamma}{12}\sigma^{pqrab}_{\alpha\beta}A^{(1)}_{pqr}R_{deab} \quad (51) \]

Where \( \Pi' \) is the modified \( \Pi \), but is any case is of the solvable form.

\[ \Pi' = [L^{(0)}_{gde} + L^{(1)}_{gde} + L^{(2)}_{gde} - \frac{1}{4}A^{(1)}_{gde} + \frac{1}{4}A^{(2)}_{gde}] \quad (52) \]

With hindsight and in order to eliminate an apparently intractable term we begin by making the following observations. In we found \( T^{(2)}_{\alpha\beta\lambda} \), hence we also encountered the quantity

\[ \sigma_d(\alpha|\lambda T^{(2)}_{\alpha\beta\lambda} = -\frac{i\gamma}{12}\sigma_d(\alpha|\lambda \sigma^{pqref}_{\beta\gamma})A^{(1)}_{pqr}T^{(0)}_{ef}\lambda \quad (53) \]

Using the torsion, equation (6) we can write

\[ \sigma_d(\alpha|\lambda T^{(2)}_{\beta\gamma} = -\sigma^g_{\alpha\beta}T^{(2)}_{d|\gamma}g + iT^{(0)}_{\alpha\beta\lambda}T^{(2)}_{\gamma|\lambda}d - i\nabla_{(\alpha}T^{(2)}_{\beta\gamma)}d \quad (54) \]

Using the second order torsion results which we found then gives

\[ +\sigma_d(\alpha|\lambda T^{(2)}_{\beta\gamma} = +\sigma^g_{\alpha\beta}T^{(2)}_{d|\gamma}g + \frac{\gamma}{6}T^{(0)}_{\alpha\beta|\lambda}\lambda H^{(0)}_{def}A^{(1)}_{pqr} - \frac{\gamma}{6}\sigma^{pqref}_{\alpha\beta}H^{(0)}_{def}[\nabla_{|\gamma}]A^{(1)}_{pqr} \]

\[ -\frac{\gamma}{6}\sigma^{pqref}_{\alpha\beta}[\nabla_{|\gamma}]H^{(0)}_{def}]A^{(1)}_{pqr} \quad (55) \]

Now applying our key equation, (16) to (55) gives

\[ +\sigma_d(\alpha|\lambda T^{(2)}_{\beta\gamma} = +\sigma^g_{\alpha\beta}T^{(2)}_{d|\gamma}g + \gamma\sigma^g_{\alpha\beta}[\nabla_{|\gamma}]A^{(1)}_{gdf}H^{(0)}_{d}ef - 4\gamma\sigma^g_{\alpha\beta}R^{(1)}_{|\gamma}gH^{(0)}_{d}ef \quad (56) \]

We now substitute in out result for \( \sigma^g_{\alpha\beta}T^{(2)}_{d|\gamma}g \), (40) into (56) to obtain cancelations and the simple result

\[ \sigma_d(\alpha|\lambda T^{(2)}_{\beta\gamma} = -\frac{i\gamma}{6}\sigma^g_{\alpha\beta}[\nabla_{|\gamma}]\phi A^{(1)}_{pqr}T^{(0)}_{de}\phi = -\frac{i\gamma}{12}\sigma_d(\alpha|\lambda \sigma^{pqref}_{\beta\gamma})A^{(1)}_{pqr}T^{(0)}_{ef}\lambda \quad (57) \]
From which we extract the following result \[10\]

\[
\sigma_{pqref}^{\alpha\beta|\sigma_{d|\gamma}\phi} A_{pqr}^{(1)} T_{ef}^{(0)} = 2\sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} T_{ef}^{(0)} \phi
\]

(58)

From this we deduce a hitherto unknown identity, albeit indirectly. Using our second order torsion and curvature results, (31), (35), and (51), the full curvature at second order becomes

\[
T_{(\alpha\beta)}^{(0)} \lambda [- \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda A_{pqr}^{(1)} R_{(0)abde}] + [- \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde}] + \frac{i\gamma}{6} \sigma_{pqab|\gamma}\lambda R_{(0)abde}
\]

\[
+ \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda A_{pqr}^{(1)} R_{(0)abde} + \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde} + \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde}
\]

\[
- \frac{i\gamma}{24} \sigma_{pqab|\gamma}\lambda A_{pqr}^{(1)} R_{(0)abde} + 2i\gamma \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} T_{ef}^{(0)} \phi
\]

(59)

Using (16) again gives two more solvable terms

\[
+ \frac{i\gamma}{12} T_{(\alpha\beta)}^{(0)} \lambda \sigma_{pqab|\gamma}\lambda A_{pqr}^{(1)} R_{(0)abde} - \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde} + \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde}
\]

\[
= + \frac{i\gamma}{2} \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} R_{(0)abde} - 2i\gamma \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} R_{(0)abde}
\]

(60)

This reduces (59) to

\[
+ \frac{i\gamma}{2} \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} R_{(0)abde} - 2i\gamma \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} R_{(0)abde}
\]

\[
- \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde} + \frac{i\gamma}{6} \sigma_{pqab|\gamma}\lambda R_{(0)abde} + \frac{i\gamma}{12} \sigma_{pqab|\gamma}\lambda R_{(0)abde}
\]

\[
- \frac{i\gamma}{24} \sigma_{pqab|\gamma}\lambda A_{pqr}^{(1)} R_{(0)abde} + 2i\gamma \sigma^{g}_{(\alpha|\sigma_{pqre|\gamma}\phi} A_{pqr}^{(1)} T_{ef}^{(0)} \phi
\]

(61)

We now list the sigma five terms separately.
\[ + \frac{i\gamma}{12} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ - T^{(0)}_{ab} \lambda R^{(0)}_{\lambda\gamma} ]_{de} + 2 H^{(0)}_{ab} g R^{(0)}_{\gamma gde} + \nabla_{\gamma} R^{(0)}_{abde} \] (62)

\[ = + \frac{i\gamma}{12} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ - T^{(0)}_{ab} \lambda R^{(0)}_{\lambda\gamma} ]_{de} - T^{(0)}_{ab} g R^{(0)}_{\gamma gde} + \nabla_{\gamma} R^{(0)}_{abde} \] (63)

We have the Bianchi Identity

\[ \nabla_a R_{abde} - T_{[a} X R_{b]de} - T_{ab} X R_{Xade} + \nabla_{[a} R_{b]ade} = 0 \] (64)

The second term on the LHS of (64) is zero at zeroth order. Hence we have as follows:

\[ + \frac{i\gamma}{12} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ \nabla_{\gamma} R^{(0)}_{abde} ] = \]

\[ + \frac{i\gamma}{12} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ + T^{(0)}_{ab} \lambda R^{(0)}_{\lambda\gamma} ]_{de} - T^{(0)}_{ab} g R^{(0)}_{\gamma gde} - 2 \nabla_a R^{(0)}_{b\gamma de} \] (65)

Substituting (65) into (61) gives

\[ + \frac{i\gamma}{2} \sigma_{(\alpha\beta)} [ \nabla_{\gamma} A^{(1)}_{gab} ] R^{(0)}_{abde} - 2 i\gamma \sigma_{(\alpha\beta)} R^{(1)}_{\lambda\gamma gde} R^{(0)}_{abde} \]

\[ - i\sigma_{(\alpha\beta)} R^{(2)}_{\lambda\gamma gde} + \frac{i\gamma}{6} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ \nabla_a R^{(0)}_{\gamma gde} ] + 2 H^{(0)}_{ab} g R^{(0)}_{\gamma gde} \]

\[ - \frac{i}{24} \sigma_{pqr}^{ab} (\alpha\beta) [ \nabla_{\gamma} A^{(1)}_{pqr} ] + 2 i\gamma \sigma_{(\alpha\beta)} \nabla_{\gamma} \Pi_{gde} = 0 \] (66)

Now consider the sigma five terms in (66). Using our new result result (58) allows for solving these terms, and we obtain

\[ + \frac{\gamma}{6} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ \nabla_a R^{(0)}_{\gamma gde} ] + 2 H^{(0)}_{ab} g R^{(0)}_{\gamma gde} \]

\[ = \frac{\gamma}{6} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ \nabla_a T^{(0)}_{b\gamma} ] + 2 H^{(0)}_{ab} g T^{(0)}_{\gamma gde} \]

\[ = \frac{\gamma}{3} \sigma_{(\alpha\beta)} [ \sigma_{pqr}^{ab} g_{\gamma\delta} ] A^{(1)}_{pqr} [ \nabla_{\gamma} T^{(0)}_{a\gamma} ] + 2 H^{(0)}_{ab} g T^{(0)}_{\gamma gde} \] (67)

Hence we obtain

\[ i\sigma_{(\alpha\beta)} R^{(2)}_{\lambda\gamma gde} = + \frac{i\gamma}{2} \sigma_{(\alpha\beta)} [ \nabla_{\gamma} A^{(1)}_{gab} ] R^{(0)}_{abde} - 2 i\gamma \sigma_{(\alpha\beta)} R^{(1)}_{\lambda\gamma gde} R^{(0)}_{abde} \]

\[ + \frac{\gamma}{6} \sigma_{pqr}^{ab} (\alpha\beta) A^{(1)}_{pqr} [ \nabla_a T^{(0)}_{b\gamma} ] + 2 H^{(0)}_{ab} g T^{(0)}_{\gamma gde} \]

\[ - \frac{i}{24} \sigma_{pqr}^{ab} (\alpha\beta) [ \nabla_{\gamma} A^{(1)}_{pqr} ] + 2 i\gamma \sigma_{(\alpha\beta)} \nabla_{\gamma} \Pi_{gde} = 0 \] (68)
We now look at the remaining unsolved term $-\frac{i}{24}(\sigma_{de(\alpha\beta)}\lbrack[\nabla_{\gamma}]A^{(1)}_{pqr}\rbrack$. This term cannot be manipulated into a solvable term because of the placement of the free indices. Using the results found in [5] we have

\[
-\frac{i}{24}(\sigma_{de(\alpha\beta)}\lbrack[\nabla_{\gamma}]A^{(1)}_{pqr}\rbrack = \gamma_{(12)(24)}(\sigma_{pqr}\delta_{gde}(\alpha\beta|\nabla|\gamma)A^{(1)}_{pqr})
\]

Using the sigma matrix identities as given in (3) it can be shown that these two terms cannot be written in the solvable form, that is with the same structure as $\sigma_g(\alpha\beta|R(2)|\gamma)gde$. Hence we look at the origin of these terms. For the derivative of $T_{kl}\tau$ we have the following Bianchi identity.

\[
\nabla_\gamma T_{kl}\tau = T_{\gamma[k|}T_{\lambda|l]}\tau + T_{\gamma[k}gT_{\lambda|l]}\tau + T_{kl}\lambda T_{\gamma}\tau + T_{kl}gT_{\gamma}\tau - \nabla_{[k|}T_{|l]}\gamma\tau - R_{kl}\gamma\tau
\]

At first order this reduces to

\[
\nabla_\gamma T_{kl}\tau^{\text{Order}(1)} = T^{(1)}_{\gamma[k|}T^{(0)}_{\lambda|l]}\tau\]

In references [1] and [2] it appears that $R^{(1)}_{kl}\gamma\tau$ was set to zero. With the form of the curvature $R_{\alpha\beta gde}$ and this choice of super current supertensor $A_{abc}$ we will always be led to the term $\frac{i}{24}(\sigma_{de(\alpha\beta)}\lbrack[\nabla_{\gamma}]A^{(1)}_{pqr}\rbrack^{(\text{order2})}$ because of the spinor derivative in the Bianchi identity (32) as given in (45). This term is not reducible as we require so it must be incorporated into this curvature. Hence we must identify the following curvature at first order:

\[
R^{(1)}_{kl}\gamma\tau = \frac{1}{48}\lbrack2H\delta_{kl}g\sigma^{g}\gamma\lambda\sigma_{pqr}\lambda\tau A^{(1)}_{pqr} - \sigma_{[k|\gamma\lambda\sigma_{pqr}\lambda\tau}(\nabla_{|l]}A^{(1)}_{pqr})\rbrack
\]

The second order form of this curvature is already solved in the Bianchi identity (71). All the quantities in this Bianchi identity are known. Hence it can be written in full in a later review. It is the role of this paper simply to arrive at the second order solution, and to overcome obstacles to obtaining this solution.

10 The Super-current

The starting point in references [1] and [2] were the conventional constraints as listed in [1]. Among these constraints we have

\[
\]

16
\[ T_{ab}^\delta = \frac{1}{48} \sigma_{ba\lambda} \sigma^{pqr\lambda} \delta A_{pqr} \]  

(73)

The choice of

\[ A_{pqr} = -i \gamma \sigma_{pqr\alpha} T_{kp} \epsilon^{kpr} \]  

(74)

was made for on shell conditions, [1]. This conventional constraint can be imposed to all orders. We have found \( T_{ab}^\delta \) and it is given in in equation (48). Hence we can find \( A_{pqr}^{(2)} \), by solving the above [10]. No modification to this super-current was required to close the identities other than this. Hence we use a suitable inverting operator along with our results (42) and (48) to obtain

\[ A_{(2)}^{(2)} gef = -\frac{1}{20} \sigma_{gef\gamma} \sigma^{b\lambda\phi} \rho^{(2)}_{\phi b} \]  

(75)

\[ = \frac{\gamma}{20} \sigma_{gef\gamma} \sigma^{b\lambda\phi} \left[ \nabla_\phi \left( \frac{1}{4} A^{(1)}_{bmn} - 2 L^{(1)}_{bmn} \right) \right] T^{(0)mn\gamma} \]  

(76)

11 Conclusions

We have analyzed the second order non minimal case of string corrected supergravity. We found a procedure for solving the Bianchi identities to this order. This involved the equations (16), (31), (35), (42), (58), and (72), which we used in conjunction with several other key observations. We found a mechanism which allows for closure of the H sector Bianchi identities and also related torsions and curvatures.

We have seen how the X tensor is necessary for achieving closure of these identities and we have proposed a candidate for this tensor which succeeds. Adding the second part, \( Y_{pqrdef} \), to the X tensor appears to result in failure to close in the H sector. With \( H^{(2)}_{\alpha\beta\delta} \) set to zero we obtained \( H^{(2)}_{\alpha\beta\delta} \), given in equation (20).

\( H^{(2)}_{\alpha bg} \) must be extracted from (32). We use the following operator, \( \hat{O} \), to obtain the symmetrized \( H^{(2)}_{\alpha bg} \):

\[ \hat{O} = \left[ \frac{1}{2} \delta^a_{[a} \delta^g_{b]} \sigma_{\alpha\beta} \phi + \frac{1}{12} \sigma_{ab\alpha} \phi + \frac{1}{24} \delta^a_{[a} \sigma_{b\alpha]} \phi \right] \]  

(77)

We find after a long calculation we obtain the result

\[
H^{(2)}_{\alpha ab} = 2 \gamma [\nabla_\alpha (H^{(0)}_{[aef} H^{(0)}_{b]} e f) - \sigma_{aba} \phi \nabla_\phi (H^{(0)}_{gef} H^{gef})] \\
+ 2i \gamma \sigma_{[a\alpha} T_{ef} \phi^{(1)}_{b]} e f - 2i \gamma \sigma_{aba} \lambda \sigma_{g\lambda\phi} T_{ef} \phi^{(1)gef} \\
- \frac{\gamma}{6} \sigma_{[a\alpha} \phi \sigma_{b\lambda\phi} T_{ef} \lambda \phi^{(1)}_{g]} e f - \frac{\gamma}{6} \sigma_{[a\alpha} \phi \sigma_{g\lambda\phi} T_{ef} \lambda \phi^{(1)}_{g]} e f \\
- 4 \gamma R^{(1)}_{\alpha[a} \phi^{(1)}_{b]} e f H^{(0)}_{e f} + T^{(2)}_{\alpha ab}
\]  

(78)
where \( T^{(2)}_{aab} \) is given in equation (81). In the case of \( H^{(2)}_{abc} \), the Bianchi identity has already given us the result. From the term \( T_{\alpha\beta}E_{Ecd} \) in equation (5) we isolate an expression of the form

\[
T^{(0)}_{\alpha\beta}H^{(2)}_{gcd} = i\sigma_{\alpha\beta}^{\psi}H^{(2)}_{gcd} = M_{\alpha\beta cd}
\]

(79)

The right hand side contains now known torsions and curvatures. However they need only be substituted into (79) generating a long expression. We then use the fact that

\[
\sigma_{a\alpha\beta}\sigma_{b\alpha\beta} = -16\delta^b_a
\]

(80)

and solve for \( H^{(2)}_{gcd} \).

This is left for the next stage of work. We obtained the full set or torsions and curvatures, (31), (35), (48) and (49).

Extracting the symmetrized torsion \( T^{(2)}_{\gamma gd} \) from (40) gives

\[
T^{(2)}_{\gamma ab} = -\frac{\gamma}{2}[\nabla_{\gamma}A^{(1)}_{[a|ef]}H^{(0)}_{|b]} ef + 2\gamma R^{(1)}_{|a|efH^{(0)}[b]} ef - \frac{i\gamma}{12}\sigma^{pqrg}_{|a|\lambda T^{(0)}_{|b]} g^{\lambda A^{(1)}_{pqr}}}
\]

\[
+ \sigma_{a\gamma} \phi [+ \frac{\gamma}{12}(\nabla_{\phi}A^{(1)}_{g|ef})H^{(0)}_{g|ef} + \gamma R^{(1)}_{\phi g|efH^{(0)}[g]} ef - \frac{i\gamma}{72}\sigma^{pqrg}_{\phi\lambda A^{(1)}_{pqr} T^{(0)}_{eg\lambda}}]
\]

\[
+ \sigma_{[a\gamma} \phi [- \frac{\gamma}{2}(\nabla_{\phi}A^{(1)}_{b|ef})H^{(0)}_{g|ef} ef - \frac{\gamma}{2}(\nabla_{\phi}A^{(1)}_{g|ef})H^{(0)}_{|b]} ef - \frac{\gamma}{6}R^{(1)}_{\phi [b|efH^{(0)}[g]} ef + \frac{\gamma}{6}R^{(1)}_{\phi g|efH^{(0)}[b]} ef]
\]

\[
+ \frac{i\gamma}{144}A^{(1)}_{pqr}[\sigma^{pqre}_{|b|\phi\lambda T^{(0)}_{eg\lambda} + \sigma^{pqre}_{g\phi\lambda T^{(0)}_{e|b]}}]
\]

(81)

where

\[
\nabla_{\gamma}A^{(1)}_{gef} = i\gamma\sigma_{gef\epsilon\tau}T^{(0)}_{kp}\epsilon[2T^{(0)kp\lambda T^{(0)}_{\gamma\lambda}} - \frac{1}{2}\sigma^{mn}_{\gamma\tau R^{(0)kp}}_{mn}]
\]

(82)

We also find the adjusted curvature \( R_{k\lambda\gamma} \). For \( R^{(2)}_{\alpha\beta de} \) we have reduced it to solvable form. After imposing condition (72) we have

\[
i\sigma^{g}_{(\alpha\beta)}R^{(2)}_{(\alpha\beta)g} = \sigma^{g}_{(\alpha\beta)}[- \frac{\gamma}{2}[\nabla_{\gamma}A^{(1)}_{g|ab]}R^{(0)ab}_{de} - 2i\gamma R^{(1)}_{|gab}R^{(0)_{abde}}
\]

\[
+ \frac{\gamma}{3}\sigma^{pqra}_{g|\alpha\beta}A^{(1)}_{pqr}\nabla_{[d\gamma}T^{(0)}_{a|e]} + 2H^{(0)}_{[d|a}T^{(0)}_{|c|e]} + 2i\gamma\nabla_{|\gamma}\gamma')gde = 0
\]

(83)

\( R^{(2)}_{\alpha\beta de} \) can be extracted from the above result. Finally we have found the supercurrent \( A^{(2)}_{abc} \) as given in equation (76).
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