On some properties and relations between restricted barred preferential arrangements, multi-poly-Bernoulli numbers and related numbers

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1. Abstract

The introduction of bars in-between blocks of an ordered set partition(preferential arrangement) results in a barred ordered set partition(barred preferential arrangement). Having the restriction that some blocks of barred preferential arrangements to have a maximum of one block results in restricted barred preferential arrangements. In this study we establish relations between number of restricted barred preferential arrangements, multi-poly-Bernoulli numbers and numbers related to multi-poly-Bernoulli numbers. We prove a periodicity property satisfied by multi-poly-Bernoulli numbers having negative index, number of restricted barred preferential arrangements and numbers related to multi-poly-Bernoulli numbers having negative index.

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2. INTRODUCTION AND PRELIMINARIES

Kaneko in [10] introduced poly-Bernoulli numbers defining them as
\[
\sum_{n=0}^{\infty} B_{n,m}^k = \frac{Li_k(1-e^{-m})}{1-e^{-m}};
\]
where \( Li_k(m) \) is a poly-logarithm defined as
\[
Li_k(m) = \sum_{s=1}^{\infty} \frac{m^s}{s^k} \text{ where } k \in \mathbb{Z}. \]
Arakawa and Kaneko in [2] further generalised poly-Bernoulli numbers to multi-poly-Bernoulli numbers with the definition
\[
\sum_{n=0}^{\infty} B_n^{(j_1,\ldots,j_b)} = \frac{Li_{j_1,\ldots,j_b}(1-e^{-m})}{(1-e^{-m})}, \text{ where } Li_{j_1,\ldots,j_b}(m) = \sum_{0<s_1<\cdots<s_b<s_{j_1/1}\ldots<s_{j_b/1}} m^{s_b}.\]

The study of preferential arrangements seems to first appear in [7], although the integer sequence itself goes far as [5]. Introducing bars in-between blocks of a preferential arrangement forms a barred preferential arrangement [1]. Recently the authors introduced the concept of restricted barred preferential arrangements by putting some restrictions on the sections of barred preferential arrangements [13].

**Barred preferential arrangements:**

The concept of preferential arrangement of an \( n \) element set was generalised by Pippenger et al in [1] by introduction of bars in-between blocks of a preferential arrangement. Examples of barred preferential arrangements of \( X_6 \) with two and three bars are respectively

a) \(| 2 \ 3 \ 64 | \ 1 \ 5 \)
b) \(| 6 \ 3 | 1 \ 24 | 5 |

With reference to the bars, the barred preferential arrangement in a) has three sections, and the barred preferential arrangement in b) has four sections (see [11]).

**Restricted barred preferential arrangements:**

In this study we view barred preferential arrangements as a result of first placing bars then distributing elements on the sections.

**Definition 1.** [13] A section of a barred preferential arrangement is a restricted section if it can only have a maximum of one block.
Definition 2. [13] A section of a barred preferential arrangement is a free section if elements distributed to the section can be preferential arranged in any possible way.

A barred preferential arrangement of an $n$-element set in which a number of fixed sections are restricted sections and other sections are free sections is referred to as a restricted barred preferential arrangement (see [13]). We denote by $p^r_j(n)$ the total number of barred preferential arrangements of an $n$-element set having $k$ bars in which $r$ fixed sections are restricted sections and the remaining $j = k+1-r$ sections are free sections. We denote the set of these barred preferential arrangements by $G^r_j(n)$, so $|G^r_j(n)| = p^r_j(n)$.

For fixed $r, j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the number $p^r_j(n)$ of restricted barred preferential arrangements for $n \geq 0$ is generated by (see [13]):

$$P^r_j(m) = \frac{e^{rm}}{(2 - e^m)^j} \quad j, r \in \mathbb{N}_0$$

For the case $j = 1$ the above family of generating functions is the following Nelsen and Schmidt family of generating functions (see [11]).

$$P^r_1(m) = \frac{e^{rm}}{2 - e^m} \quad r \in \mathbb{N}_0$$

In this study we establish relations between restricted barred preferential arrangements, multi-poly-Bernoulli numbers and some numbers related to multi-poly-Bernoulli numbers.

3. ON SOME PROPERTIES OF RESTRICTED BARRED PREFERENTIAL ARRANGEMENTS

Theorem 1. For $j \in \mathbb{N}_0$ and $n, r \in \mathbb{N}$

$$p^r_j(n) = \sum_{s=0}^n \binom{n}{s} r^s p^0_j(n-s)$$

Proof. On an element $\mathbf{w} \in G^r_j(n)$ we assume there are $s$ elements which are distributed among the $r$ restricted sections. The $s$ elements can be selected in $\binom{n}{s}$ ways. We can preferentially arrange the $s$ elements among the $r$ restricted sections in $r^s$ ways. We can then preferentially arrange remaining $n-s$ elements among the $j$ free sections in $p^0_j(n-s)$ ways. Taking the product and summing over $s$ we obtain the result. \qed
Lemma 1. \[7\] For a fixed \( s \in \mathbb{N}_0 \) and \( n \geq 1 \) the following congruence holds:
\[
 s^{n+4} - s^n \equiv 0 \mod 10
\]

Lemma 2. \[7\] For \( n \geq 1 \) the last digit of the sequence \( p^j_1(n) \) has a four cycle.

Theorem 2. For fixed \( r, j \geq 0 \) such that \( r > 0 \) or \( j > 0 \) the last digit of the sequence \( p^j_r(n) \) has a four cycle for \( n \geq 1 \).

Proof. \[
P^r_j(m) = \frac{e^m}{(2-e^m)^r} = \frac{1}{2} \sum_{s=0}^{\infty} (-1)^s e^{s+1} m^s
\]
Hence \[
p^j_r(n) = \left[ \frac{m^n}{n!} \right] P^r_j(m) = \frac{1}{2} \sum_{s=r}^{\infty} (-1)^s \left( \frac{n}{2^s} \right) e^{s+1} m^s \]
Letting \( u = r + s \) we have
\[
p^j_r(n + 4) - p^j_r(n) = \frac{1}{2j} \sum_{u=r}^{\infty} (-1)^s e^{s+1} \left[ 2^{n+4} - u^n \right]
\]
Applying lemma 1 on (3) we obtain the result. \( \square \)

Lemma 3. For \( n \geq 1, r \geq 0 \)
\[
p^j_r(n) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} (-1)^s (k + r - s)^n
\]

Theorem 3. For \( n, j \geq 1 \) and \( r \geq 0 \)
\[
p^j_r(n) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} (-1)^s p^{r+k-s}_{j-1}(n)
\]
Proof. The theorem is a generalisation of an un-labelled equation in \[7\].
\[
P^r_j(m) = \frac{e^m}{(2-e^m)^r} = \frac{e^m}{(2-e^m)^r} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \binom{k}{s} (-1)^s e^{s+1} m^s
\]
Hence \[
p^j_r(n) = \left[ \frac{m^n}{n!} \right] P^r_j(m) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} (-1)^s P^{r+k-s}_{j-1}(n).
\]
(3)

Theorem 4. For \( r, j \geq 1 \) such that \( r \leq j \) we have
\[
p^{j-r}_{j-r}(n) = \sum_{s=1}^{r} \binom{r}{s} (-1)^{s+1} \times p^s_{j-s}(n)
\]
Proof. On barred preferential arrangements having \( j \) free sections, we fix \( r \geq 1 \) sections. By the inclusion/exclusion principle the number of those barred preferential arrangements from \( G^0_j(n) \) such that all the \( r \) fixed sections have more than one block is
\[
p^0_j(n) - \binom{j}{r} p^1_{j-1}(n) + \binom{j}{r} p^2_{j-2}(n) + \cdots + \binom{j}{r} p^r_{j-r}(n)(-1)^r = \sum_{s=0}^{r} \binom{j}{s} p^s_{j-s}(n)(-1)^s.
\]
Hence the number of those barred preferential arrangements such that all the \( r \) fixed sections have a maximum of one block is 
\[
\sum_{s=1}^{r} \binom{r}{s} p_{j-s}^s(n)(-1)^{s+1} = p_{j-r}^r(n). 
\]

\[\square\]

**Lemma 4.** \[6\]
For \( n \geq 0 \)
\[
p_1^0(n) = \sum_{s=0}^{\infty} \frac{s^n}{2^s} 
\]

**Lemma 5.** \[11\]
For \( n \geq 0 \)
\[
p_1^2(n) = 2 \sum_{s=2}^{\infty} \frac{s^n}{2^s} 
\]

**Theorem 5.** For \( j \geq 1 \) and \( n, r \geq 0 \)
\[
p_{r}^j(n) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{r}^{r+s}(n)}{2^s} 
\]

**Proof.** \( P_{r}^j(m) = \frac{e^{em}}{(2-e)^{2r}} \)
\[
= \frac{1}{2} \sum_{s=0}^{\infty} \frac{e^{e^{s+m}}}{(2-e)^{2r-s}} 
\]
Hence \( p_{r}^j(n) = \left( \frac{\binom{n}{m}}{m!} \right) P_{j}^r(m) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{r}^{r+s}(n)}{2^s} \)
\[\square\]

**Lemma 6.** \[7\]
For \( j, n \geq 1 \)
\[
p_1^0(n) = \sum_{s=0}^{n-1} \binom{n}{s} p_1^0(n-s) + 1 
\]

**Lemma 7.** \[12\]
For \( j, n \geq 1 \)
\[
p_1^2(n + 1) = \sum_{s=0}^{n} \binom{n+1}{s} p_1^2(s) + 2^{n+1} 
\]

**Theorem 6.** For \( j, n \geq 1 \)
\[
p_{r}^j(n) = p_{r-1}^j(n) + \sum_{s=0}^{n-1} \binom{n}{s} p_{r}^j(s) 
\]

**Proof.** The theorem is a generalisation of (9) of \[7\].

By theorem \[5\] we have

\[
(4) \quad p_{r}^j(n) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{r-1}^{r+s}(n)}{2^s} 
\]

This implies that

\[
(5) \quad \sum_{m=0}^{n-1} \binom{n}{m} p_{r}^j(n-m) = \frac{1}{2} \sum_{s=0}^{\infty} \left[ \sum_{m=0}^{n} \binom{n}{m} p_{r-1}^{r+s}(n-m) \times 1^s - 1 \right] \frac{1}{2^s} 
\]

So
\[
\sum_{m=0}^{n-1} \binom{n}{m} p_{r}^j(n-m) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{r-1}^{r+s+1}(n)}{2^s} - 1 
\]
Letting $s + 1 = k$ and applying (4) we obtain

\[ p_j^r(n) = \sum_{m=1}^{n} \binom{n}{m} p_j^r(n - m) + p_{j-1}^r(n) \]

\[ \square \]

4. Poly-Bernoulli numbers

**Proposition 1.** [14] A formal power series $p(m) = \sum_{n=0}^{\infty} c_n \times m^n$ has a reciprocal if $c_0 \neq 0$.

The $n^{th}$ term of the reciprocal $\frac{1}{p(m)} = \sum_{n=0}^{\infty} c_n \times m^n$ when it exists is given by (see [14])

\[ c_n^* = -\frac{1}{c_0} \sum_{s=1}^{n} c_s c_{n-s}^* \quad \text{where} \quad c_0^* = \frac{1}{c_0} \]

A closed form for the poly-Bernoulli numbers $B_n^{-2}$ (see [9])

\[ B_n^{-2} = 2 \times 3^n - 2^n \quad \text{where} \quad n \in \mathbb{N}_0 \]

**Lemma 8.** For $n \geq 0$ and fixed $j \in \mathbb{Z}$

\[ B_n^j = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{1}{(s+1)!} \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} (i-s)^n \]

**Proof.** By definition $\sum_{n=0}^{\infty} B_n^j \frac{m^n}{n!} = \frac{L_{L_j}(1-e^{-m})}{(1-e^{-m})^j}$ \[ \Rightarrow \sum_{n=0}^{\infty} B_n^j \frac{m^n}{n!} = \sum_{s=0}^{\infty} \frac{(1-e^{-m})^s}{(s+1)!}. \]

From this it follows that $B_n^j = \sum_{s=0}^{\infty} \frac{1}{(s+1)!} \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} \sum_{n=0}^{\infty} (i-s)^n \]

We recall the family of generating functions for number of restricted barred preferential arrangements for $j = 1$ is $P_1^r(m) = \frac{1-e^{-m}}{2e^m}$ (where $r \in \mathbb{N}_0$). We denote by $P_1^r(m)^*$ the reciprocal of the generating function $P_1^r(m)$; we denote as $P_1^r(m)^* = \sum_{n=0}^{\infty} \frac{a_1^r(n) \times m^n}{n!}$.

We first consider $P_1^3(m)^* = \frac{2-e^{-m}}{e^m} = \sum_{n=0}^{\infty} \frac{a_1^3(n) \times m^n}{n!}$. So $a_1^3(n) = (-1)^n(2 \times 3^n - 2^n)$.

\[ \Rightarrow |a_1^3(n)| = 2 \times 3^n - 2^n = B_n^{-2} \quad \text{(by [14])}. \]

By (6) we have;

\[ p_1^3(n) = \sum_{s=1}^{n} \binom{n}{s} (-1)^{s+1} B_s^{-2} \times p_1^3(n-s) \quad \text{for} \quad n \geq 1 \]
From the generating functions we deduce that for $r \geq 3, \ j \geq 1$

\[
p_j^{-3}(n) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s B_s^{-2} \times p_j^r(n-s)
\]

**Multi-poly-Bernoulli numbers:**

**Theorem 7.** \[ \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_b=0}^{\infty} B_n^{(-j_1,\ldots,-j_b)} \frac{r_1^{j_1} r_2^{j_2} \cdots r_b^{j_b} m^n}{j_1! \cdots j_b! n!} = \frac{e^{(b-1)m}}{e^{-r} + e^{-m} - 1} \]

On theorem 7 when we let $r_2 = r_3 = \cdots = r_b = 0$ we obtain

\[
\sum_{j=0}^{\infty} B_n^{(j,\ldots,\ldots,0)} \frac{r^j m^n}{j! n!} = \frac{e^{(b-1)m}}{e^{-r} + e^{-m} - 1} = \left( \frac{e^{b-1}}{e^{-r} + e^{-m} - 1} \right)
\]

**Corollary 1.** For fixed $b \in \mathbb{N}$ and $j \in \mathbb{N}_0$

\[
B_n^{(j,0,\ldots,\ldots,0)} = \sum_{s=0}^{n} \binom{n}{s} B_s^{(0,\ldots,\ldots,0)} B_{n-s}^{-j}
\]

**Theorem 8.** \[ \text{For a fixed } b \in \mathbb{N} \text{ and } j_1, j_2, \ldots, j_b \in \mathbb{N}_0 \text{ such that } (j_1, j_2, \ldots, j_b) \neq (0,0,\ldots,0). \text{ Let } j = j_1 + j_2 + \cdots + j_b. \text{ Then the following identity holds}
\]

\[
B_n^{(-j_1,\ldots,-j_b)} = \sum_{s=1}^{j} \mu_s^{(j_1,\ldots,-j_b)} (s+b)^n
\]

Where $\mu_s^{(j_1,\ldots,-j_b)}$ are integers recursively defined in the following way

1. $\mu_s^{(j_1)} = (-1)^{s+j_1} s! \binom{j_1}{s}$
2. $\mu_s^{(j_1,\ldots,j_b-1,0)} = \mu_s^{(j_1,\ldots,j_b-1,0)}$
3. $\mu_s^{(j_1,\ldots,j_b-1,j_b+1)} = (s+b-1) \mu_s^{(j_1,\ldots,j_b-1,j_b)} - s \times \mu_s^{(j_1,\ldots,j_b-1,j_b)}$

Where $\mu_0^{(j_1,\ldots,j_b-1,j_b)} = 1$ if $(j_1,\ldots,j_b-1,j_b) = (0,0,\ldots,0)$

\[
B_n^{(-2)} = 2 \times 3^n - 2^n
\]

\[
B_n^{(-2,0)} = 2 \times 4^n - 3^n
\]
\[ B_n^{(-2,0,0)} = 2 \times 5^n - 4^n \]
\[ B_n^{(-2,0,0,0)} = 2 \times 6^n - 5^n \]

Inductively,

\[ B_n^{(-2,\cdots,0)} = 2 \times (3 + b)^n - (3 + b - 1)^n \quad \text{where } b \in \mathbb{N}_0 \]

We write the generating functions for number of restricted barred preferential arrangements for \( j = 1, r \geq 3 \) as

\[ P_{r}^{+b}(m) = \frac{e^{(3+b)m}}{2 - e^m} \quad \text{where } b \in \mathbb{N}_0 \]

So \( P_{1}^{+b}(m)^* = \sum_{n=0}^{\infty} \frac{P_{1}^{+b}(m)^*}{n!} \) and

\[ P_{1}^{+b}(m)^* = (-1)^n \left[ 2 \times (3 + b)^n - (3 + b - 1)^n \right] \]

\[ \Rightarrow \left[ \frac{m^n}{n!} \right] P_{1}^{+b}(m)^* = B_n^{(-2,0,0,\cdots,0)} \quad \text{(by (11))}. \]

So by (6) we have

\[ p_{1}^{3+b}(n) = \sum_{s=1}^{n} \binom{n}{s} (-1)^s + 1 B_s^{(-2,0,0,\cdots,0)} \times p_{1}^{3+b}(n-s) \quad \text{where } b \in \mathbb{N}_0 \]

Where \( p_{1}^{3+b}(n) \) denotes number of restricted barred preferential arrangements. The result in (13) can equivalently be written as;

\[ B_n^{(-2,0,0,\cdots,0)} = \sum_{s=1}^{n} \binom{n}{s} p_{1}^{3+b}(s) \times (-1)^{n-s+1} B_{n-s}^{(-2,0,0,\cdots,0)} \]

On a convolution of \( P_{j}^{r}(m) = \sum_{n=0}^{\infty} \frac{P_{j}^{r}(n) \times m^n}{n!} \) and

\[ P_{1}^{r}(m)^* = \sum_{n=0}^{\infty} (-1)^n B_n^{(-2,0,0,\cdots,0)} \frac{m^n}{n!} \]

we obtain;

\[ p_{j-1}^{r-(3+b)}(n) = \sum_{s=0}^{n} \binom{n}{s} p_{j}^{r}(s) \times (-1)^{n-s} \times B_{n-s}^{(-2,0,0,\cdots,0)} \quad \text{for } r \geq 3+b, j \geq 1 \]

**Lemma 9.** For \( n \geq 1 \) and fixed \( b \in \mathbb{N}_0 \) the last digit of the sequence \( B_n^{(-2,0,0,\cdots,0)} \) has a four cycle.
Proof. By (11) we have $B_n^{(−2,0,\cdots,0)} = 2(3+b)^n − (3+b−1)^n$. So $B_{n+4}^{(−2,0,\cdots,0)} − B_n^{(−2,0,\cdots,0)} = 2[(3+b)^{n+4} − (3+b)^n] − [(2+b)^{n+4} − (2+b)^n]$. By lemma 4 both $[(3+b)^{n+4} − (3+b)^n]$ and $[(2+b)^{n+4} − (2+b)^n]$ are divisible by 10. □

**Theorem 9.** For fixed $j_1, j_2, \ldots, j_b \in \mathbb{N}_0$ the last digit of the sequence $B_n^{(−j_1,\ldots,−j_b)}$ for $n \geq 1$ has a four cycle.

Proof. By definition $\sum_{n=0}^{\infty} B_n^{(−j_1,\ldots,−j_b)} \frac{m^n}{n!} = \frac{L_{−j_1,\ldots,−j_b}(1−e^{−m})}{(1−e^{−m})^b} = \sum_{0<s_1<s_2<\cdots<s_b} s_1^{j_1} \times s_2^{j_2} \times \cdots \times s_b^{j_b} (1−e^{−m})^{s_b−b}$

$\implies B_n^{(−j_1,\ldots,−j_b)} = \sum_{0<s_1<s_2<\cdots<s_b} s_1^{j_1} \times \cdots \times s_b^{j_b} \times (−1)^{s_b−b} \sum_{i=0}^{s_b−b} \binom{s_b−b}{i} (−1)^{s_b−b−1} (−1)^i n^i$.

$\implies B_{n+4}^{(−j_1,\ldots,−j_b)} − B_n^{(−j_1,\ldots,−j_b)} = \sum_{0<s_1<s_2<\cdots<s_b} s_1^{j_1} \times s_2^{j_2} \times \cdots \times s_b^{j_b} (−1)^{s_b−b} \sum_{i=0}^{s_b−b} \binom{s_b−b}{i} (−1)^{s_b−b−1} (−1)^i [n^{i+4} − i^n]$.

By applying lemma 4 we obtain the result. □

**Theorem 10.** For fixed $b \in \mathbb{N}_0$ we consider barred preferential arrangements of $X_n$ having $3+b$ bars where all the sections are restricted sections. For fixed sections the $i^{th}$ and the $j^{th}$ the poly-Bernoulli number $B_n^{(−2,0,\cdots,0)}$ is the number of restricted barred preferential arrangements such that the $i^{th}$ or $j^{th}$ section is empty.

Proof. We consider restricted barred preferential arrangements of $X_n$ having $3+b$ bars where all the sections are restricted sections. We fix two sections (the $i^{th}$ and the $j^{th}$ sections). The number of those restricted barred preferential arrangements whose $i^{th}$ section is empty is $(3+b)^n$. The number of those restricted barred preferential arrangements whose $j^{th}$ section is empty is also $(3+b)^n$. The number of those restricted barred preferential arrangements whose $i^{th}$ and $j^{th}$ section are empty is $(3+b−1)^n$. By the inclusion/exclusion principle the number of restricted barred preferential arrangements whose $i^{th}$ or $j^{th}$ sections is empty, is $2 \times (3+b)^n − (3+b−1)^n$ where $b \geq −2$. Hence on (11) the number $2 \times (3+b)^n − (3+b−1)^n = B_n^{(−2,0,\cdots,0)}$ is the number of restricted barred preferential arrangements of $X_n$. 


having \(3+b\) bars; where all the sections are restricted sections such that the \(i^{th}\) or \(j^{th}\) section is empty. \(\square\)

5. ON SOME RELATED NUMBERS

We define numbers \(U_n^{(j_1,\ldots,j_b)}\) by the generating function

\[
\sum_{n=0}^{\infty} U_n^{(j_1,\ldots,j_b)} \frac{m^n}{n!} = \frac{L_{j_1,\ldots,j_b}(1-e^{-m})}{(1-e^{-m})^b} e^{-m}
\]

(For the case \(b = 1\) the numbers appears in [2])

By section 4 we have

\[
\sum_{n=0}^{\infty} B_n(-2,0,\ldots,0) \frac{m^n}{n!} = 2 - \frac{e^m}{e^{(3+b)m}}
\]

(15)

By definition of \(U_n^{(j_1,\ldots,j_b)}\) we have

\[
\sum_{n=0}^{\infty} U_n^{(-2,0,\ldots,0)} \frac{m^n}{n!} = \frac{2 - e^m}{e^{(3+b+1)m}}
\]

(16)

By [12] and (10) we have

\[
p^{3+b+1}(n) = \sum_{s=1}^{n} \binom{n}{s} (-1)^{s+1} U_s^{(-2,0,\ldots,0)} \times p^{3+b+1}(n-s)
\]

(17)

From (16) we deduce that \(U_n^{(-2,0,\ldots,0)} = B_n^{(-2,0,\ldots,0)}\). Hence for fixed \(b \in \mathbb{N}_0\) the sequence \(U_n^{(-2,0,\ldots,0)}\) for \(n \geq 1\) has a four cycle (by lemma 9).

**Theorem 11.** For fixed \(j_1, j_2, \ldots, j_b \in \mathbb{N}_0\) the last digit of the sequence

\(U_n^{(-j_1,\ldots,-j_b)}\) for \(n \geq 1\) has a four cycle.

Proof. By definition

\[
\sum_{n=0}^{\infty} U_n^{(-j_1,\ldots,-j_b)} \frac{m^n}{n!} = \frac{L_{-j_1,\ldots,-j_b}(1-e^{-m})}{(1-e^{-m})^b} e^{-m}
\]

This implies that

\[
U_n^{(-j_1,\ldots,-j_b)} = \sum_{0<s_1<s_2<\ldots<s_b} s_1^{j_1} \times \cdots \times s_b^{j_b} \frac{(-1)^{s_b-b+1} (s_b-b+1)}{s_b-b+1} \sum_{i=0}^{s_b-b+1} \frac{(s_b-b+1)}{i} \times
\]

\[
\sum_{n=0}^{\infty} (-i)^{n+1} = \sum_{n=0}^{\infty} \sum_{0<s_1<s_2<\ldots<s_b} s_1^{j_1} \times \cdots \times s_b^{j_b} \frac{(-1)^{s_b-b+1} (s_b-b+1)}{s_b-b+1} \sum_{i=0}^{s_b-b+1} \frac{(s_b-b+1)}{i} \times
\]

\(-1)^{s_b-b+1-i} (-i)^{n+1}.\) This implies that

\[
U_{n+4}^{(-j_1,\ldots,-j_b)} = U_n^{(-j_1,\ldots,-j_b)} = \sum_{n=0}^{\infty} \sum_{0<s_1<s_2<\ldots<s_b} s_1^{j_1} \times \cdots \times s_b^{j_b} \frac{(-1)^{s_b-b+1} (s_b-b+1)}{s_b-b+1} \times
\]

\[
\sum_{n=0}^{\infty} (-i)^{n+1}
\]

\[
\sum_{n=0}^{\infty} (-i)^{n+1} = \sum_{n=0}^{\infty} \sum_{0<s_1<s_2<\ldots<s_b} s_1^{j_1} \times \cdots \times s_b^{j_b} \frac{(-1)^{s_b-b+1} (s_b-b+1)}{s_b-b+1} \sum_{i=0}^{s_b-b+1} \frac{(s_b-b+1)}{i} \times
\]

\(-1)^{s_b-b+1-i} (-i)^{n+1}.\) This implies that
\[ \sum_{i=0}^{s_b-1} (s_i - b + 1)(-1)^{s_i - b + 1 + (-i)^{n+1} - (-i)^{n+1}}. \] By lemma 1, the sequence \( U_n(-j_1, \ldots, -j_b) \) for \( n \geq 1 \) has a four cycle. \( \square \)

**Theorem 12.** For fixed \( j_1, j_2, \ldots, j_b \in \mathbb{Z} \)

\[ U_n^{(j_1, j_2, \ldots, j_b)} = (-1)^{n+1} \sum_{s_b=b}^{s-b+1} \sum_{0<s_1<\cdots<s_b} \frac{1}{s_1! \cdots s_b!} (-1)^{s_b-b+1} \times (s_b - b)! \left\{ \frac{n+1}{s_b-b+1} \right\} \]

**Proof.** By definition \( \sum_{n=0}^{\infty} \frac{U_n^{(j_1, \ldots, j_b)} m^n}{n!} = \frac{Li_{j_1, \ldots, j_b}(1-e^{-m})}{(1-e^{-m})^b} e^{-m} \)

\[ \Rightarrow \sum_{n=0}^{\infty} \frac{U_n^{(j_1, \ldots, j_b)} m^n}{n!} = \sum_{0<s_1<\cdots<s_b} \frac{(1-e^{-m})^{s_b-b} e^{-m}}{s_1! s_2! \cdots s_b!} \]

\[ = \sum_{0<s_1<\cdots<s_b} \frac{1}{s_1! s_2! \cdots s_b!} d \left( \frac{(1-e^{-m})^{s_b-b+1}}{s_b-b+1} \right). \]

Now applying the identity \( \sum_{n=0}^{\infty} \frac{n^s}{s!} = \frac{(e^m-1)^s}{s!} \) (see [3], pp 32)

We have \( \sum_{n=0}^{\infty} \frac{U_n^{(j_1, \ldots, j_b)} m^n}{n!} \)

\[ = \sum_{0<s_1<\cdots<s_b} \frac{(-1)^{s_b-b+1}}{s_1! s_2! \cdots s_b!} \sum_{n=s_b-b}^{n+1} (s_b - b)! \left\{ \frac{n+1}{s_b-b+1} \right\} \]

\[ = \sum_{n=0}^{\infty} \left[ (-1)^{n+1} \sum_{s_b=b}^{s-b+1} \frac{(-1)^{s_b-b+1}}{s_1! s_2! \cdots s_b!} \left\{ \frac{n+1}{s_b-b+1} \right\} \right] \frac{m^n}{n!}. \]

\( \square \)

The theorem is an analogue of theorem 7 of [8]

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