Towards a Generalized Distribution Formalism for Gauge Quantum Fields

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ABSTRACT

We prove that the distributions defined on the Gelfand-Shilov spaces $S_\alpha^\beta$ with $\beta < 1$, and hence more singular than hyperfunctions, retain the angular localization property. Specifically, they have uniquely determined support cones. This result enables one to develop a distribution-theoretic technique suitable for the consistent treatment of quantum fields with arbitrarily singular ultraviolet and infrared behavior. The proof covering the most general and difficult case $\beta = 0$ is based on the use of the theory of plurisubharmonic functions and Hörmander’s $L^2$-estimates.

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1 Introduction

Recently the problem of an adequate choice of test functions in quantum field theory \[1\] came once again into notice with emphasis on infrared singularities which occur in gauge QFT in connection with the necessary lack of positivity \[2\]. Namely, analysis of some explicitly soluble models has shown \[3–5\] that these singularities are in general more severe than those of ultradistributions or even hyperfunctions and that appropriate test function spaces are \(S_\alpha^\beta\) by Gelfand and Shilov \[6\]. The indices \(\alpha\) and \(\beta\) control respectively the ultraviolet and infrared singularities whose order increases with decreasing their values, while the Schwartz space used in the original Wightman formalism can be identified with \(S_1^\infty\). Of great importance is the fact that the usual definition of support has sense only for generalized functions on \(S_\alpha^\beta\) with \(\beta > 1\), for otherwise the configuration-space test functions are analytic. The same is true referring to the momentum-space test functions and \(\alpha > 1\). Just then one uses the term ultradistributions and the former inequality selects in fact the Jaffe strictly localizable fields \[7\]. The space \(S_1^1\), which was put forward in \[8\] as most adequate to Meiman’s locality principle \[9\], admits a direct generalization of the notion of support and underlies the \textit{Fourier hyperfunction} QFT \[10\]. As shown in Refs. \[1–5\], the infrared behavior of gauge fields treated in generic covariant gauges is so singular that one is forced to use spaces with \(\alpha < 1\) in order to represent the fields as operator-valued distributions on a Hilbert space. This raises the problem of formulation of the spectral condition which is certainly similar to that of generalization of local commutativity beyond the localizability bound. Fortunately, the latter problem has received an extensive study which resulted particularly in new proofs of the spin-statistics and \(TCP\) theorems covering nonlocal quantum fields \[11, 12\].

In Refs. \[13–15\], we have shown that the mathematical techniques used in nonlocal QFT can essentially be improved by adoption some ideas of the theory of hyperfunctions. The principal result is evidence of the existence of uniquely defined support cones which replace supports in the nonlocalizable case. This enables one to extend a great part of the theory of distributions beyond the hyperfunctions and makes generalization of the Wightman approach quite natural. However those proofs are based on exploiting test functions of rapid decrease from the subspace \(S_1^{\alpha_{\beta}}\) and so are inapplicable to \(S_0^0\) since \(S_1^0\) is trivial. The main aim of this Letter is to fill this gap because the spaces \(S_0^0\) (and \(S_0^\beta\)) are of special interest and provide us with the widest distributional framework adequate to quantum fields with arbitrarily singular infrared (ultraviolet) behavior. There are three essential steps in proving: The extension of the scale of Gelfand-Shilov spaces to test functions defined on open and closed cones (Sec.2), the representation of these spaces in terms of complex variables (Sec.3), and the employment of Hörmander’s \(L^2\)–estimates for solutions of nonhomogeneous Cauchy-Riemann equations (Sec.5). In the last section 6, we indicate several immediate applications of the obtained results whose detailed presentation will be given in forthcoming papers.
It should be remarked that, following Hörmander, we prefer to use the term *distributions* instead of *generalized functions* under arbitrary singularity and, as usual, we call *tempered* the Schwartz distributions whose order of singularity is finite.

## 2 Nonlocalizable distributions versus hyperfunctions

### Definition 1.

Let $O$ be an open set in $\mathbb{R}^n$. The space $S^\beta_\alpha(O)$ consists of all complex valued infinitely differentiable functions on $O$ with the property that the norm

$$
\| \varphi \|_{O,a,b} \overset{\text{def}}{=} \sup_{x \in O, k, q} x^{k} \partial^{q} \varphi(x) \left[ \frac{1}{a^{|k|} \cdot b^{q}} \cdot k^{\alpha} \cdot q^{\beta} \right] ^{1}
$$

is finite for some positive $a, b$ dependent on $\varphi$.

Here $k$ and $q$ are multi-indices and the standard notation relating to functions of several variables is used. The set of norms (1) determines naturally an inductive limit topology on $S^\beta_\alpha(O)$ whose strong dual space is denoted by $S^\prime_\alpha(O)$. By making certain regularity assumptions regarding $O$, it is easy to show that the former is a DFS space and the latter is an FS space, see [14] for details. Hence they have nice topological properties in perfect analogy to the original spaces $S^\beta_\alpha = S^\beta_\alpha(\mathbb{R}^n)$ and $S^\prime_\alpha$. When dealing with hyperfunctions, it is useful to bear in mind the following

### Proposition 1.

The inductive limit of the spaces $S^1_\beta(|x|>R) \ (R \to \infty)$ is a Hausdorff space wherein $S^1_\alpha$ is embedded.

This is the case due to analyticity of these test functions. It implies that the mappings $S^1_\alpha(|x|>R_1) \to S^1_\alpha(|x|>R_2) \ (R_1 < R_2)$ are injective and hence the limit is also DFS. Clearly every element of $S^\prime_\alpha$ which is continuous under the topology induced on $S^1_\alpha$ by that of the inductive limit should be considered as attached to infinity, if a localization is possible at all. By contrast, in the case $\beta > 1$ the kernel of the canonical mapping from $S^\beta_\alpha$ into the corresponding inductive limit is everywhere dense and the only ultradistribution continuous under the induced topology is zero. Therefore it is reasonable to adapt the definition of the presheaf $S^1_\alpha(O)$ to a compactification of $\mathbb{R}^n$. Following Kawai [16], we use the radial compactification $\mathcal{R}^n$ and identify the space $S^1_\alpha(\mathcal{O})$, where $\mathcal{O}$ is an open subset of $\mathcal{R}^n$, with $S^1_\alpha(\mathcal{O} \cap \mathcal{R}^n)$. Then the space $S^1_\alpha(\mathcal{K})$ corresponding to a compact set $\mathcal{K} \subset \mathcal{R}^n$ is defined by $S^1_\alpha(\mathcal{K}) = \text{inj lim} S^1_\alpha(\mathcal{O})$, where $\mathcal{O}$ runs over neighborhoods of $\mathcal{K}$ in $\mathcal{R}^n$, and a compactum is said to be a carrier of $f \in S^\prime_\alpha$ if $f$ has a continuous extension to the space assigned to it.

### Theorem 1.

For any pair of compact sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{R}^n$, the sequence

$$
0 \to S^\prime_\alpha(\mathcal{K}_1 \cap \mathcal{K}_2) \to S^\prime_\alpha(\mathcal{K}_1) \oplus S^\prime_\alpha(\mathcal{K}_2) \to S^\prime_\alpha(\mathcal{K}_1 \cup \mathcal{K}_2) \to 0
$$

is exact.
For a proof, see [14]. Its essential points are the same as those in the paper of Kawai who treated the case of Fourier hyperfunctions $\alpha = 1$ and used, however, much more involved cohomology techniques. The mappings in (2) are naturally defined via restrictions and the next to last arrow maps a pair of hyperfunctions into the difference of their restrictions. This theorem expresses two simple but fundamental facts:

(i) Every hyperfunction carried by $\mathcal{K}_1 \cup \mathcal{K}_2$ can be decomposed into a sum of two hyperfunctions with carriers $\mathcal{K}_1$ and $\mathcal{K}_2$.

(ii) If both $\mathcal{K}_1$ and $\mathcal{K}_2$ are carriers of a hyperfunction, then so is $\mathcal{K}_1 \cap \mathcal{K}_2$.

It is worthwhile to recall that in the case of tempered distributions and ultradistributions the property (i) holds only for closed sets subject to some regularity conditions. Furthermore from the statement (ii), by standard compactness arguments, it follows that every hyperfunction has in $\mathbb{R}^n$ a unique minimal carrier, the support.

In [13–14], we have argued that there is a reminiscence of these properties in the nonlocalizable case. Namely, an analogue of Theorem 1 is valid for the closed cones with vertex at the origin if their associated spaces are defined similarly, through the use of a filter of neighborhoods in $\mathbb{R}^n$. We refer to [14, 17] for terminology and notation concerning cones, though these are quite customary.

**Definition 2.** Let $\beta < 1$ and let $K$ be a closed cone in $\mathbb{R}^n$. The space $S^\beta_\alpha(K)$ is defined to be the inductive limit of the spaces $S^\beta_\alpha(U_\star)$, where $U_\star = U \cup B$, $U$ runs over open cones such that $K \subset \subset U$, and $B$ is the unit ball centered about the origin.

Strictly speaking, the ball must shrink to the origin, however this does not change the space as may easily be verified by using the Taylor formula and as follows, in particular, from Theorem 3 below. The only role of $B$ is to provide connectedness.

**Definition 3.** A closed cone $K \subset \mathbb{R}^n$ is said to be a carrier of $f \in S^\beta_\alpha$ if $f$ is continuous under the topology induced on $S^\beta_\alpha$ by that of $S^\beta_\alpha(K)$ or, equivalently, if $f$ has a continuous extension to $S^\beta_\alpha(K)$.

**Theorem 2.** For any $\beta < 1$ and for any pair of closed cones $K_1, K_2 \subset \mathbb{R}^n$, the sequence

$$0 \to S^\beta_\alpha(K_1 \cap K_2) \to S^\beta_\alpha(K_1) \oplus S^\beta_\alpha(K_2) \to S^\beta_\alpha(K_1 \cup K_2) \to 0$$

is exact. Moreover it is topologically exact since all the involved spaces are FS.

This theorem formalizes the property that we call angular localizability. We notice that, for a given $f \in S^\beta_\alpha$, it ensures the existence of a smallest carrier among the closed cones which can be called the support cone of $f$. For $0 < \beta < 1$, Theorem 1 can be derived in an elementary way described in Sec.4. A more general, but more complicated proof covering the borderline case $\beta = 0$ is presented in Sec.5. We shall also show that $S^\beta_\alpha$ is dense in every
space \( S^\beta_\alpha(K) \) and so the linear extension referred to in Definition 3 is unique. Actually, we shall prove the exactness of the dual sequence

\[
0 \leftarrow S^\beta_\alpha(K_1 \cap K_2) \leftarrow S^\beta_\alpha(K_1) \oplus S^\beta_\alpha(K_2) \leftarrow S^\beta_\alpha(K_1 \cup K_2) \leftarrow 0.
\] (4)

This is completely equivalent to the initial problem since all the spaces here are reflexive and their strong dual spaces are Fréchet, so the mappings in (4) are of closed range if and only if the dual mappings possess this property (see [18], Sec.IV.7.7) and our claim is proved by using the formulae

\[
\text{Ker} \ u' = (\text{Im} \ u)^\perp, \quad \overline{\text{Im} \ u'} = (\text{Ker} \ u)^\perp
\]

which is valid for any continuous linear mapping \( u \) with dual \( u' \), and wherein the bar stands for the closure under the weak topology coinciding with that under the strong topology for the reflexive spaces. Certainly the exactness of (4) in the term \( S^\beta_\alpha(K_1 \cap K_2) \) is the only point that need be argued, everything else being evident. In other words, we have to prove that each element of \( S^\beta_\alpha(K_1 \cap K_2) \) can be decomposed into a sum of two functions belonging to \( S^\beta_\alpha(K_1) \) and \( S^\beta_\alpha(K_2) \). As a first step, we shall represent the test function spaces under consideration in another, more convenient form.

## 3 Representations of the test function spaces in terms of complex variables

**Theorem 3.** Let \( U \) be a nonempty open cone in \( \mathbb{R}^n \) and let \( U_* \) be as in Definition 2. Denote by \( d(x, U) \) the distance of \( x \) from \( U_* \). If \( \beta < 1 \), the space \( S^\beta_\alpha(U_*) \) is identical to the inductive limit of Banach spaces \( E^{\beta,b}_{\alpha,a}(U) \) consisting of entire analytic functions on \( \mathbb{C}^n \) with the finite norms

\[
\| \varphi \|_{U,a,b} = \sup_z |\varphi(x)| \exp\{-\rho_{U,a,b}(z)\},
\] (5)

where

\[
\rho_{U,a,b}(z) = -|x/a|^{1/\alpha} + d(bx, U)^{1/(1-\beta)} + |by|^{1/(1-\beta)}.
\] (6)

The same is true for empty \( U \) if we set \( d(x, U) = |x| \) and drop the first term in (6).

We remark that \( d(bx, U) = bd(x, U) \) and that in view of the inductive limit procedure the choice of the norm in \( \mathbb{R}^n \) is unessential here since all these norms are equivalent. It should also be noted that for the particular case \( U = \mathbb{R}^n \) this reformulation is essentially due to Gelfand and Shilov, see [18], Sec. IV.7.5.

**Proof.** We shall proceed along the same lines as in [13], where an analogous representation has been obtained for \( S^\beta_{\infty}(U_*) \). Due to the condition \( \beta < 1 \) the Taylor series expansion of
\( \varphi \in S^{\beta,b}_{a,a}(U_\star) \) is convergent for all \( z \in \mathbb{C}^n \) and since any point \( \xi \in U_\star \) can be taken as its center, the analytic continuation \( \hat{\varphi}(z) \) is bounded by

\[
\|\varphi\|_{U,a,b} \inf_{\xi \in U_\star} \inf_k \frac{a^{|k|} \kappa^{|k|}}{|\xi^k|} \sum_q b^{|q|} q^\beta q! |(z - \xi)^q|.
\]

The infimum over \( k \) and sum over \( q \) can be estimated in the same manner as in [6] and are dominated by

\[
C \exp\{-|\xi'|^{1/\alpha} + |b'(x - \xi)|^{1/1-\beta} + |b'y|^{1/1-\beta}\}
\]

with new constants \( a', b' \). We see that, for any nonempty \( U \), the space \( S^{\beta}_{a,a}(U_\star) \) is trivial if \( \alpha < 1 - \beta \), since then the infimum over \( \xi \) is zero. Recall incidentally that \( S^{\beta}_{1-\beta} \) is nontrivial [3] and this fact will be exploited in the next section. Taking \( \xi \in U \) to be a point with minimal distance to \( x \) and using the inequality

\[
-|\xi|^{1/\alpha} \leq -|x|/2^{1/\alpha} + |x - \xi|^{1/\alpha},
\]

we infer that \( S^{\beta,b}_{a,a}(U_\star) \) is continuously embedded into a space \( E^{\beta,b}_{a,a}(U) \). In the degenerate case \( U = \emptyset \), the subscript is certainly inessential and we obtain the same result with the above stipulation concerning \( \rho \).

To prove the converse embedding, let \( \varphi \in E^{\beta,b}_{a,a}(U) \) and let \( \xi \in U_\star \). We employ Cauchy's inequality

\[
|\partial^q \varphi(\xi)| \leq q! r^{-q} \sup_{z \in D} |\varphi(\xi - z)|,
\]

where \( D = \{ z \in \mathbb{C}^n : |z_j| \leq r_j, \quad j = 1, ..., n \} \), and use (5) to estimate \( \varphi(\xi - z) \). Next we apply (8) with the interchanged position of \( \xi \) and \( x \) and notice that the term \( |x|^{1/\alpha} \) can be replaced by \( |x|^{1/(1-\beta)} \) if \( \alpha \geq 1 - \beta \). Then, since \( d(\xi - x, U) \leq d(\xi, U) + |x| \leq 1 + |x| \) and since

\[
|x|^{1/(1-\beta)} + |y|^{1/(1-\beta)} \leq h \sum_j |z_j|^{1/(1-\beta)}
\]

for some \( h > 0 \), we obtain

\[
|\partial^q \varphi(\xi)| \exp\{|\xi|/2 \alpha\}^{1/\alpha} \leq C\|\varphi\|_{U,a,b} r^{-q} \exp\{\sum_j (b' r_j)^{1/(1-\beta)}\}.
\]

The exponential on the left side can be replaced by \( \sup_k |\xi^k|/\alpha^{|k|} \kappa^{|k|} \), and evaluation of the lower bound over \( r \) yields the factor \( b^{|q|} q^{-\beta} q^{-\beta} q! \). Thus we arrive at the inequality

\[
\|\varphi\|_{U,a',b'} \leq C\|\varphi\|_{U,a,b} \cdot
\]

The case of empty \( U \) is treated with obvious simplifications and it remains to say a little about the spaces \( E^{\alpha}_{\beta}(U) = \text{inj lim } E^{\beta,b}_{a,a}(U) \) with \( \alpha < 1 - \beta, U \neq \emptyset \). Let
\( \varphi \) be an element of such a space. Evidently the second term in (6) is negligible now and the function \( \varphi(z)\varphi(iz) \) tends to zero as \( |z| \to \infty \). Hence by Liouville’s theorem, it is identically zero and the proof is finished.

**Theorem 4.** For the entire analytic functions, the family of norms (5) is equivalent to that determined by the scalar products

\[
\langle \varphi, \psi \rangle_{U,a,b} = \int \varphi(z)\psi(z) \exp\{-2\rho_{U,a,b}(z)\} \, dv,
\]

with \( dv \) denoting the Lebesgue measure on \( \mathbb{C}^n \), and hence

\[
S_\beta^\alpha(U_\bullet) = \text{inj lim } H_{\alpha,a}^{\beta,b}(U) \quad (a, b \to \infty),
\]

where \( H_{\alpha,a}^{\beta,b}(U) \) are the corresponding Hilbert spaces.

**Proof.** Let \( \| \cdot \|_{U,a,b}'' \) be the norm determined by (9). Clearly \( \| \varphi \|_{U,a,b}'' \leq C \| \varphi \|_{U,a',b'}'' \) for \( a > a', b > b' \). Conversely, by Cauchy’s integral formula,

\[
|\varphi(\zeta)| \leq C \| \varphi \|_{L^2(D)}
\]

for any polydisk \( D \) centered about the point \( \zeta \). One can multiply (11) by \( \exp\{-\rho_{U,a,b}(\zeta)\} \), write \( \zeta = z + (\zeta - z) \) on the right side and then apply the triangle inequality to every term of the exponent assuming that \( |\zeta - z| \leq 1 \). On doing so, we obtain

\[
|\varphi(\zeta)|^2 \exp\{-2\rho_{U,a,b}(\zeta)\} \leq C' \int_D |\varphi(z)|^2 \exp\{-2\rho_{U,a',b'}(z)\} \, dv
\]

with \( a' < a, b' < b \) and hence \( \| \varphi \|_{U,a,b}'' \leq C' \| \varphi \|_{U,a',b'}'' \), which proves the desired identification (10).

## 4 An elementary derivation of the density and decompositions theorems in the case \( \beta > 0 \).

**Theorem 5.** The space \( S_\alpha^\beta, 0 < \beta < 1 \), is dense in \( S_\alpha^\beta(U_\bullet) \) for any open cone \( U \subset \mathbb{R}^n \), and all the more in each space \( S_\alpha^\beta(K) \), where \( K \) is a closed cone.

In [14], this result was obtained as a by-product of a theorem proven in terms of real variables and expressing the angular-support property as a fall-off property in the complementary directions. The representation \( S_\alpha^\beta(K) = \text{inj lim } E_{\alpha,a}^{\beta,b}(U) \) enables one to present a simple direct proof. Namely, let \( \varphi \in E_{\alpha,a}^{\beta,b}(U) \). We choose a function \( \chi_0 \in E_{1-\beta,a_0}^{\beta,b_0}(U) \) so that \( \int \chi_0(\xi) \, d\xi = 1 \) and define an approximating sequence \( \varphi_\nu \) by \( \varphi_\nu(z) = \chi_\nu(z)\varphi(z) \), where \( \chi_\nu \) is a sequence of Riemann sums for the integral \( \int \chi_0(z - \xi) \, d\xi \) or, more explicitly,

\[
\chi_\nu(z) = \sum_{k \in \mathbb{Z}^n, |k| < \nu^2} \chi_0(z - k/\nu) \nu^{-n}.
\]
Clearly $\varphi_\nu \in S^{\beta}_{1-\beta} \subset S^{\beta}_a$ if $a_0 < 1/b$. The series (12) converges uniformly on compact sets because it can be dominated there by a convergent number series. The limit function is analytic and equals 1 at real points. Hence so does it on the whole of $\mathbb{C}^n$. Furthermore the sequence $\varphi_\nu$ is bounded in the norm $\| \cdot \|_{U,a,b'}$ with $b' = b + b_0$. By standard arguments, it follows that $\| \varphi - \varphi_\nu \|_{U,a',b''} \to 0$ for any $a' > a, b'' > b$ since
\[
\lim_{|z| \to \infty} \exp\{\rho_{U,a,b'}(z) - \rho_{U,a',b''}(z)\} = 0.
\]

**Theorem 6.** Let $K_1$ and $K_2$ be closed cones in $\mathbb{R}^n$ and let $K = K_1 \cap K_2$. For every $\varphi \in S^{\beta}_a(K)$, where $0 < \beta < 1$, one can find a pair of functions $\varphi_j \in S^{\beta}_a(K_j)$ such that $\varphi = \varphi_1 + \varphi_2$.

**Proof.** We may assume that $\varphi \in E_{a,a}^{\beta}(U)$, where $U$ is a cone-shaped neighborhood of $K$, and choose another open cone $W$ so that $K \subset U \subset W$. Since the closed cones $K_j \setminus W$ have disjoint projections, there are open cones $W_j \supset K_j \setminus W_j$ with nonzero angular separation, that is,
\[
|x - \xi| \geq \theta |x| \text{ and } |x - \xi| \geq \theta |\xi| \text{ for all } x \in W_1, \xi \in W_2,
\]
where $\theta$ is a positive constant. Let us take $\chi_0 \in E_{1-\beta, a_0}^{\beta,b_0}$ as before and set
\[
\chi(z) = \int_{W_2} \chi_0(z - \xi) d\xi.
\]
This function is entire analytic and we claim that one can define $\varphi_1$ to be $\chi \varphi$. More specifically, if $a_0 < \theta/b$, the product belongs to the space $E_{a,a}^{\beta,b_1}(U_1)$, where $K_1 \subset U_1 \subset W_1 \cup U$ and $b_1$ is large enough. In fact, after taking $b_1 \geq b_0 + b$, we need only to inspect the dependence on $x$. The inequalities (13) show that, for $x \in W_1$, the bounded function $\chi(x)$ satisfies the estimate
\[
|\chi(x)| \leq C_a \exp\{-|\theta x/a'|^{1/(1-\beta)}\}
\]
with any $a' > a$. Since $d(bx, U) \leq b|x|$, we see that $\chi \varphi$ decreases inside $W_1 \cup U$ no slower than $\varphi$ providing $a_0 < \theta/b$. Outside this cone $d(bx, U_1) \geq \theta_1 |x|$ and therefore the additional restriction $b_1 > b/\theta_1$ ensures the bound
\[
|\varphi_1(z)| \leq C_1 \exp\rho_{U_1,a,b_1}(z),
\]
which proves the claim. On the other hand, $1 - \chi$ satisfies an estimate similar to (15) for $x$ in any $W'_2 \subset W_2$. Hence, by the same arguments and under condition $a_0 < \theta'/b$, the function $(1 - \chi)\varphi$ belongs to $E_{a}^{\beta}(U_2)$, where $K_2 \subset U_2 \subset W'_2 \cup U$. Thus
\[ \varphi = \chi \varphi + (1 - \chi) \varphi \]  

is the desired decomposition. This completes the proof of Theorem 5 and thereby that of Theorem 2 for \( \beta = 0 \).

5 General proof based on Hörmander’s \( L^2 \)-estimate

Let us now turn to the case \( \beta = 0 \). Because of triviality of \( S_0 \), the above considerations fail to show the desired result. However a method used for resolving the Cousin problem in the theory of analytic functions of several complex variables points the way to get over this difficulty. Let \( \chi_0(|x|) \) be any positive, smooth bump function with compact support in the unit ball. Then functions constructed as above have the wanted behavior at infinity but are nonanalytic. However one can put the decomposition (16) in order by writing \( \varphi_1 = \chi \varphi - \psi, \varphi_2 = (1 - \chi) \varphi + \psi \), where \( \psi \) must obey the nonhomogeneous Cauchy-Riemann equations

\[ \frac{\partial \psi}{\partial \bar{z}_j} = \eta_j, \quad j = 1, \ldots, n \]  

with \( \eta_j = \varphi \partial \chi/\partial \bar{z}_j \). We notice that the latter functions satisfy the compatibility conditions

\[ \frac{\partial \eta_j}{\partial \bar{z}_k} = \frac{\partial \eta_k}{\partial \bar{z}_j}, \quad j, k = 1, \ldots, n \]  

and have support in the 1-neighborhood of the boundary of \( W_2 \). By this reason, they vanish in the cone \( W_1 \cup W_2' \cup U \) except for this neighborhood, where an decrease like \( \exp\{ -|x/a|^{1/\alpha} \} \) takes place. Therefore

\[ |\eta_j| \leq C_j \| \varphi \|_{U,a,b} \exp \rho_{U_1 \cup U_2', a, b'}, \]  

where \( U_1 \) and \( U_2 \) are chosen as before and \( b' \) is sufficiently large. In order to extend Theorem 6 to \( \beta = 0 \), we only need to show that the system (17) has a solution with the same growth and fall-off properties as those of \( \eta_j \)’s. For this purpose, we shall take advantage of the representation (10) and the following result of Hörmander [13].

Suppose that \( \varrho \in C^2(\mathbb{C}^n) \) is a strictly plurisubharmonic function, i.e.,

\[ \kappa(z) = \inf_{\zeta} \sum_{j,k} \frac{\partial \varrho}{\partial z_j \partial \bar{z}_k} \hat{\zeta}_j \hat{\zeta}_k \left( \sum_j |\zeta_j|^2 \right)^{-1} > 0. \]

Then for every collection of functions \( \eta_j \) satisfying the condition (18), one can find a solution \( \psi \) of the system of equations (17) such that \( \psi \in L^2(\mathbb{C}^n, e^{-\varrho} dv) \) and

\[ \int |\psi|^2 e^{-\varrho} dv \leq \int |\eta_j|^2 e^{-\varrho_1} \kappa^{-1} dv. \]
Lemma 1. There is a smooth plurisubharmonic function $\rho^*_{U,a,b}$ such that the replacement $\rho_{U,a,b} \to \rho^*_{U,a,b}$ in the definition (5) gives an equivalent family of norms.

Proof. Let us denote by $\rho_{\min}$ the greatest plurisubharmonic minorant of $\rho$. It exists for any continuous and even for upper semicontinuous function \([17]\). If $\varphi(z)$ is entire, then, modulus of an analytic function being logarithmically plurisubharmonic, the inequality $|\varphi| \leq C \exp \rho$ is equivalent to $|\varphi| \leq C \exp \rho_{\min}$. The function $\rho_{\min}$ is locally integrable but is not necessarily smooth. In order to eliminate this trouble, one can exploit the regularization $\rho^*_{U,a,b}(z) = \int \rho_{\min}(z - \zeta) \chi_0(|\zeta|)dv$ which is $C^\infty$ and plurisubharmonic providing $\chi_0$ is nonnegative. As before, by making use of the triangle inequality, we obtain

$$\rho^*_{U,a,b}(z) \leq \sup_{|\zeta| < 1} \rho_{U,a,b}(z - \zeta) \leq \rho_{U,a',b'}(z) + C,$$

where $a' > a$, $b' > b$ and can be taken arbitrarily close to $a, b$. It follows that

$$\rho^*_{U,a,b}(z) \leq \rho_{\min}^*_{U,a',b'}(z) + C.$$

Conversely, for any $\zeta$ in the unit ball, we have

$$\rho_{\min}^*_{U,a,b}(z) \leq \rho_{U,a',b'}(z - \zeta) + C.$$

Since the left side of this inequality is plurisubharmonic, one can replace the right side by its greatest minorant and after that perform the multiplication by $\chi_0(\zeta)$ and the integration over $\zeta$, which gives

$$\rho_{\min}^*_{U,a,b}(z) \leq \rho^*_{U,a',b'}(z) + C$$

and ends the proof of the lemma. Note that when $\beta = 0$ and consequently $\alpha > 1$, the above inequalities are valid with $a' = a$, $b' = b$, that is, in this case not only the whole family but also every norm is equivalent to that obtained by the replacement $\rho \to \rho^*$. Now we set

$$\varrho(z) = 2\rho^*_{U_1 \cup U_2, a,b}(z) + 2\ln(1 + |z|^2) \quad (21)$$

with $a$ and $b$ slightly greater than those in (19). Then $\kappa \geq 2(1 + |z|^2)^{-2}$ and we are in a position to apply Hörmander’s estimate, which shows that the functions $\varphi_j$ corrected by adding $\psi$ have finite $\| \cdot \|_{U,a_j,b_j}$ norm, where $a_j$ is arbitrarily close to the $a$ characterizing fall-off of the initial function $\varphi$ and $b_j$ is large enough. On the other hand, it is well known that each distribution and all the more any integrable function satisfying the homogeneous
Cauchy-Riemann equations is actually a function analytic in the usual sense. Thus we have proved that Theorem 6 holds true for $\beta = 0$.

In exactly the same manner one can extend to $\beta = 0$ the density theorem. Here again we take $\chi_0$ with compact support and apply the former approximate procedure (12). The sequence $\chi_\nu \varphi$ tends to $\varphi$ in the norm $\| \cdot \|_{U,a,b}''$ with $a,b$ properly chosen but it consists of nonanalytic functions. However $\partial \chi_\nu / \partial \bar{z}_j$ is uniformly convergent to zero and hence $\| \varphi \partial \chi_\nu / \partial \bar{z}_j \|_{U,a,b}'' \to 0$. This norm squared and doubled exceeds the integral on the right side of (20) if we set $\eta_j = \varphi \partial \chi_\nu / \partial \bar{z}_j$ and define $\varrho(z)$ in the manner of (21). Thus there exist functions $\psi_\nu$ such that $\chi_\nu \varphi - \psi_\nu$ are analytic and compose a sequence convergent to $\varphi$ in $S^\beta_\alpha(U,\bullet)$.

6  Concluding remarks and outlook

We consider the present work as a step towards a distribution-theoretic formalism for consistent treatment of gauge field models with arbitrarily singular infrared behavior. We hope it may be of use in constructing nontrivial field models on $S_0^0(R^n)$ fulfilling all the Wightman axioms and perhaps divergence-free at the cost of replacement of local commutativity by an asymptotical commutativity condition. Besides this ultimate goal, we would like to point out a few immediate applications of the obtained results. Firstly, these lead to a natural generalization of the Paley-Wiener-Schwartz theorem to the distributions defined on the space $S_0^0$ and carried by a closed cone. This in turn enables one to reformulate the generalized spectral condition proposed by Moschella and Strocchi for infrared singular quantum fields [3] as a support property of Wightman functions and then to present a more general formulation. Another possible application is the theory of Lorentz invariant and Lorentz covariant distributions of arbitrarily high singularity. In particular, this includes extension of the theorems [20, 21] concerning the structure of covariant tempered distributions, an invariant splitting of distributions carried by the closed light cone, some theorems on odd distributions, etc.

We conclude by noting that results similar to those listed above can be derived for distributions on the space $S_\infty^0$ which is used in the nonlocal QFT [11, 12]. In this case, a part of the derivations is even simpler since $S_\infty^0$ is none other than the Fourier transform of Schwartz’s space $D$, i.e., such distributions are tempered in momentum space. However the topological structure of $S_\infty^0(K)$ is more complicated compared to $S_\alpha^0(K)$, which gives rise to an additional trouble in proving an analogue of the key theorem 6. One may overcome this difficulty in the same manner as in showing the existence of support for $f \in S_\infty^1$ in Refs. [14,15].

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