On linear wear of a layered base by a system of rough punches

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Abstract. The article is devoted to solving the problem of linear wear of a base with a nonuniform coating by a system of punches with complex base profile. Such coatings can be obtained, for example, using additive manufacturing technology. It is shown that to this end it is necessary to study and construct a solution to the system of mixed integral equations with additional conditions, which include several rapidly changing functions, since the punches can have a complex shape, and the coating nonuniformity can be described even by a discontinuous function. In the constructed solution, the functions related to the properties of the coating and the forms of the dies are explicitly identified, which allows efficient and accurate calculations.

Introduction
Earlier, we conducted a lot of research on the problems of wear of foundations with complex coatings with single hard punches in plane and axisymmetric cases [1–3]. The mathematical model of such problems was one mixed integral equation and two additional conditions. These equations could include complex, rapidly changing functions that describe the coating nonuniformity and the shapes of the contacting surfaces. Solutions for these equations constructed by special method took into account all these features.

The work [4] was devoted to solving the problem of wear of foundation with nonuniform coating by a system of flat punches. The mathematical model of such problems was system of mixed integral equation and system of additional conditions. The problem posed in this article is a development of the above problems. In this paper, we formulate and solve the multiple wear problem of a foundation with a nonuniform coating by a system of various rigid punches with complex base form. Such properties may be due to production features, for example, using additive technologies [5,6].

1. Plane multiple contact-wear problem
Elastic layer of thickness \( h_{\text{lower}} \) with nonuniform coating of thickness \( h \) which elastic properties depend on the longitudinal coordinate \( x \) (\( E = E(x) \) and \( \nu = \nu(x) \)) lies on rigid base. There are ideal contact between layers and between lower layer and underlying base. The coating is considered soft compared to the lower layer (its rigidity not exceed the rigidity of lower layer [7]). At some time moment \( \tau_0 \), a punch system begins to act on such a foundation along the \( Oz \) axis. Moreover, these punches moves along the \( Oy \) axis perpendicular to the \( Oxz \) plane. The average
magnitude of their velocity modulus is the same and equal to $V$. Due to this there is wear of the coating in the contact areas between punches and the base. The scheme for such a problem is presented in the figure 1.

Further we will assume that the contact regions are constant and bounded by the boundaries $a_i$ and $b_i$, the dimension $\min_{i=1,2,...,n}(b_i - a_i)$ is much greater than the upper layer thickness, and there is a smooth contact between punches and upper layer. Here $a_i$ and $b_i$ are left and write coordinates of $i$th punch and $n$ is number of punches.

Experiments shown that, in the linear case wear of surface is proportional to the normal load and inversely proportional to coating hardness [8–12]:

$$v_w(x,t) = -\frac{k_w}{H(x)} q(x,t)$$

where $H(x)$ is coating hardness, $q(x,t)$ is contact pressure, $k_w$ is experimentally determined constant. Then the vertical displacement of upper surface due to wear can be determined from the formula

$$u_w(x,t) = -\int_{\tau_0}^{t} v_w(x,\tau) d\tau = -\frac{k_w V}{H(x)} \int_{\tau_0}^{t} q(x,\tau) d\tau.$$  \hspace{1cm} (1)

According to [13] we can obtain vertical displacement of upper surface due to loading $-q(x,t)$:

$$u_q(x,t) = -\frac{q(x,t) h_1}{R(x)} - \frac{2(1-\nu^2)}{\pi E_{\text{lower}}} \int_{-\infty}^{\infty} k_{\text{pl}}(x-\xi) \int_{-\infty}^{\infty} k_{\text{pl}}(\xi) q(\xi,\tau) d\xi.$$  \hspace{1cm} (2)

where $\nu_{\text{lower}}$ and $E_{\text{lower}}$ are elastic moduli of lower layer, $R(x) = E(x)[1-\nu(x)]/[1-\nu(x)-2\nu^2(x)]$ is contact rigidity of upper layer [13], $k_{\text{pl}}(s) = \int_0^\infty L(u) \cos(su) du$ is kernel for plane contact problem [14], $L(u) = [2\kappa_2 \sinh(2u) - 4u]/[2\kappa_2 \cosh(2u) + \kappa_2^2 + 1 + 4u^2]$, $\kappa_2 = 3 - 4\nu_2$.

We will assume that hardness of coating an its contact rigidity are proportional to each other (see examples in [15] and [16])

$$H(x) = k_H R(x).$$  \hspace{1cm} (3)
From formulas (1)–(3) and the condition that the displacement of the upper face of the foundation is equal to the displacements of the punch bases, we can obtain the following system of mixed integral equations

\[
\frac{k_W q_i(x,t)}{H(x)} + \frac{k_W V}{H(x)} \int_{\tau_0}^{t} q_i(x,\tau) \, d\tau + \frac{2(1 - \nu^2_{\text{lower}})}{\pi E_{\text{lower}}} \sum_{j=1}^{n} \int_{a_j}^{b_j} \frac{k_{\text{pl}}(x - \xi)}{h_{\text{lower}}} q_j(\xi, t) \, d\xi
\]

\[= \delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x), \quad (4)\]

where \(\delta_i(t)\) and \(\alpha_i(t)\) are settlements and tilt angles of \(i\)th punch, \(\eta_i = (a_i + b_i)/2\) is midpoint of \(i\)th punch, \(g_i(x)\) is base form of \(i\)th punch.

These equations should be supplemented by system of equilibrium conditions on each punch:

\[
\int_{a_i}^{b_i} q_i(\xi, t) \, d\xi = P_i(t), \quad \int_{a_i}^{b_i} (\xi - \eta_i)q_i(\xi, t) \, d\xi = P_i(t)e_i(t), \quad t \geq \tau_0, \quad (5)
\]

where \(P_i(t)\) and \(e_i(t)\) are applied forces and its eccentricities.

2. Dimensionless form

We introduce into (4) and (5) new variables and functions by the formulas

\[
x^* = \frac{2(x - \eta_i)}{\bar{a}_i}, \quad \xi^* = \frac{2(\xi - \eta_j)}{\bar{a}_j}, \quad t^* = \frac{t}{\tau_0}, \quad \lambda = \frac{2h_{\text{lower}}}{\bar{a}}, \quad \xi^{**} = \frac{\bar{a}_i}{\bar{a}}, \quad \eta^{**} = \frac{2\eta_i}{\bar{a}},
\]

\[
\delta^{**}(t^*) = \frac{2\delta_i(t)}{\bar{a}_i}, \quad \alpha^{**}(t^*) = \frac{\xi^{**} \alpha_i(t)}{\bar{a}}, \quad g^{**}(x^*) = \frac{2g_i(x)}{\bar{a}}, \quad m^{**}(x^*) = \frac{k_{H}hE_{\text{lower}}}{\bar{a}_iH(x)(1 - \nu^2_{\text{lower}})},
\]

\[
q^{**}(x^*, t^*) = \frac{2\xi^{**} q_i(x, t)(1 - \nu^2_{\text{lower}})}{E_{\text{lower}}}, \quad k^{ij}(x^*, \xi^*) = \frac{1}{\pi} k_{\text{pl}} \left( \frac{\xi^{**} x^* + \eta^{**} - \xi^{**} - \eta^{**}}{\lambda} \right),
\]

\[
P^{**}(t^*) = \frac{4P_i(t)(1 - \nu^2_{\text{lower}})}{\bar{a}_i E_{\text{lower}}}, \quad M^{**}(t^*) = \frac{8P_i(t)e_i(t)(1 - \nu^2_{\text{lower}})}{\bar{a}_i a_{\text{lower}} E_{\text{lower}}}, \quad V^* = \frac{k_W V\tau_0}{k_{H}h},
\]

\[x \in [a_i, b_i], \quad \xi \in [a_j, b_j], \quad t \geq \tau_0, \quad i, j = 1, 2, \ldots, n.
\]

Here \(\bar{a}_i = b_i - a_i\) is width of this punch and \(\bar{a}_{\text{bar}}\) is characteristic dimension of punches, for example \(\bar{a} = \min_{i=1,2,\ldots,n} \bar{a}_i\). Having omitted asterisks in the relations obtained, we get dimensionless system of equations and additional conditions

\[
m^i(x) \left[ q^i(x, t) + V \int_{-1}^{1} q^i(x, \tau) \, d\tau \right] + \sum_{j=1}^{n} \int_{-1}^{1} k^{ij}(x, \xi) q^j(\xi, t) \xi \, d\xi = \delta^i(t) + \alpha^i(t)x - g^i(x),
\]

\[
\int_{-1}^{1} q^i(\xi, t) \, d\xi = P^i(t), \quad \int_{-1}^{1} \xi q^i(\xi, t) \, d\xi = M^i(t), \quad x \in [-1,1], \quad t \geq 1.
\]

If we introduce the Fredholm operator for the first term in the obtained mixed integral equations, then this system become as follow

\[
m^i(x)(I - V)q^i(x, t) + \sum_{j=1}^{n} \int_{-1}^{1} k^{ij}(x, \xi) q^j(\xi, t) \xi \, d\xi = \delta^i(t) + \alpha^i(t)x - g^i(x),
\]

\[
Vf(t) = - \int_{1}^{t} V f(\tau) \, d\tau.
\]
It is easy to show that if \((I + R)f(t) = (I - V)^{-1}f(t)\) then

\[
R f(t) = -V \int_1^t e^{-V(t-\tau)} f(\tau) \, d\tau, \quad (I + R)1 = e^{-V(t-1)}.
\]

Note that functions \(m^i(x)\) and \(q^i(x)\) in (8) relate with hardness of upper layer and punch base forms; these parameters can be described by a rapidly changing function.

### 3. Solution of the problem

So we should solve system of mixed integral equations [17] with set of rapidly changing functions supplemented by additional conditions. We will present solution for the case with known \(P^0(t)\) and \(M^0(t)\), and unknown \(\delta^i(t)\), \(\alpha^i(t)\), and \(q^i(x, t)\). In this case system of integral equations (8) will have unknown terms both on the left and on the right sides.

Based on studies conducted with similar systems in the works [18–20] contained our investigation we can obtain following representation for contact stresses, settlements, and tilt angles:

\[
q^i(x, t) = \frac{1}{m^i(x)} \left[ z_0^i(t) p_0^i(x) + z_1(t) p_1^i(x) + \sum_{k=2}^{\infty} z_k(t) \sum_{m=2}^{\infty} \psi_{km}^i \mu_m(x) \right] - e^{-V(t-1)} \frac{g^i(x)}{m^i(x)},
\]

\[
\delta^i(t) = \frac{1}{\sqrt{J_{0,i}}} \left\{ z_0^j(t) + V \int_0^t z_1(t) \, d\tau - g_0^j e^{-V(t-1)}
\right. \\
+ \left. \sum_{j=1}^n \left[ K_{0,j}^i z_0^j(t) + K_{0,j}^i z_1^j(t) + \sum_{k=2}^{\infty} \left( \sum_{l=2}^{\infty} K_{0,k}^j \psi_{kl}^j \right) z_k(t) \right] \right\} - \frac{\alpha^i(t)}{J_{0,i}},
\]

\[
\alpha^i(t) = \sqrt{\frac{J_{0,i}}{J_{0,i} J_{2,i} - J_{1,i}^2}} \left\{ \frac{z_1^i(t)}{z_0^i(t)} + V \int_0^t z_1^i(\tau) \, d\tau - g_1^i e^{-V(t-1)}
\right. \\
+ \left. \sum_{j=1}^n \left[ K_{10,j}^i z_0^j(t) + K_{11,j}^i z_1^j(t) + \sum_{k=2}^{\infty} \left( \sum_{l=2}^{\infty} K_{1,k}^j \psi_{kl}^j \right) z_k(t) \right] \right\},
\]

where the functions included in these equations are determined analytically when constructing the solution. They are not given here due to cumbersomeness. It is only necessary to note that the solution was constructed using special representation, basis functions, and the generalized projection method [17].

### Conclusions and remarks

The multiple contact-wear plane problem for regular system of rigid punches with complex base forms and elastic layer with nonuniform coating is formulated. The solution for one version is constructed in analytical form. It should be noted that functions associated with the forms of punches and coating properties highlighted by separated factors and terms in the expression for contact pressures. This allows one to provide effective calculations for the case when shapes and properties described by complex functions. Other well-known methods will lead to a significant error, since the solution features do not take into account the features of the problem.

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