A WEIL REPRESENTATION OF \( \text{sp}(4) \) REALIZED BY DIFFERENTIAL OPERATORS IN THE SPACE OF SMOOTH FUNCTIONS ON \( S^2 \times S^1 \)

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In the space of complex-valued smooth functions on \( S^2 \times S^1 \), we explicitly realize a Weil representation of the real Lie algebra \( \text{sp}(4) \) by means of differential generators. This representation is a rare example of highest weight irreducible representation of \( \text{sp}(4) \) all whose weight spaces are 1-dimensional. We also show how this space splits into the direct sum of irreducible \( \text{sl}(2) \)-submodules. Selected applications: complete classification of yrast-band energies in even-even nuclei, the dynamical symmetry in some collective models of nuclear structure, the mapping methods for simplifying initial problem Hamiltonians.

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1. Introduction

In what follows, the ground field is that of real numbers, but the functions are complex-valued ones. Applications of explicit realizations of \( \text{sp}(4) \)-modules are related, e.g., to the complete classification of yrast-band energies in even-even nuclei [1], to the dynamical symmetry in some collective models of nuclear structure [2], and to the mapping methods [3] for simplifying initial problem Hamiltonians [4]. In [2], it has been shown that there are two non-equivalent Weil representations of \( \text{sp}(4k) \) in the Fock space \( \mathcal{H}_F \) constructed as the module over the Heisenberg Lie algebra \( h := h(2n) \) with generators \( a^\pm = (a^+_1, \ldots, a^+_n) \) and \( a = (a_1, \ldots, a_n) \). Monomials of degree 2 and 0 in these creation and annihilation operators, considered as elements of the enveloping algebra \( U(h) \), span the (trivial) central extension of the Lie algebra \( \text{sp}(2n) \) with respect to the bracket. Thus the representation of \( h \) in the Fock space \( \mathcal{H}_F := \mathbb{R}[a^+] \) naturally generates the representation of the Lie algebra \( \text{sp}(2n) \) of

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outer derivations of \( h \) in the same space. The authors of [2] have studied this representation for the case \( n = 2k \). This representation is reducible and decomposes into the direct sum of irreducible Weil representations: \( \mathcal{H}_F = \mathcal{H}_+ \oplus \mathcal{H}_- \) (for Weil representations, see [5, Sec. 12.3]).

For a non-negative integer \( l \) and an integer \( m \) such that \( -l \leq m \leq l \), we define the vectors

\[
[l, m] = \frac{\sqrt{2l + 1}}{2\pi \sqrt{2}} \cos(l+m) \mathcal{P}_l^m(x)
\]

in terms of the Legendre functions

\[
\mathcal{P}_l^m(x) := \frac{(-1)^m}{2^l l!} \frac{1}{(l + m + 1) \Gamma(l + m + 1)} \left(1 - x^2\right)^{l/2} \frac{d^m}{dx^m}(1 - x^2)^l.
\]

Let \( \mathcal{H}_+ := \text{span}\{[l, m] \mid l \geq 0 \text{ and } -l \leq m \leq l\} \). In what follows I realize the Weil representation of \( \mathfrak{sp}(4) \) in the space \( \mathcal{H}_+ \) by differential operators.

The starting point is a realization of the recurrence relations with respect to both parameters \( l \) and \( m \) of the Legendre functions \( \mathcal{P}_l^m(x) \) (see, e.g., [5,6]). For a fixed \( m \), set

\[
A_l^0 \mathcal{P}_l^m(x) = \sqrt{(l - m)(l + m)} \mathcal{P}_l^m(x),
\]

\[
A_l^0 \mathcal{P}_{l+1}^m(x) = \sqrt{(l - m)(l + m + 2)} \mathcal{P}_{l+1}^m(x);
\]

the operators \( A_l^0 \) can be realized as

\[
A_l^0 = \pm(1 - x^2) \frac{d}{dx} - lx.
\]

For a fixed \( l \), set

\[
B_l^m \mathcal{P}_{l-1}^m(x) = \sqrt{(l - m + 1)(l + m + 1)} \mathcal{P}_l^{m+1}(x),
\]

\[
B_l^m \mathcal{P}_l^m(x) = \sqrt{(l - m + 1)(l + m)} \mathcal{P}_l^{m+1}(x);
\]

the operators \( B_l^m \) can be realized as

\[
B_l^m = \pm \sqrt{1 - x^2} \frac{d}{dx} + \frac{(m - \frac{1}{2} \mp \frac{1}{2}) x}{\sqrt{1 - x^2}}.
\]

For a given \( m \), the Legendre functions are orthogonal on the interval \(-1 \leq x \leq 1\) with respect to the inner product with the measure \( dx \):

\[
\int_{-1}^{1} \mathcal{P}_l^m(x) \mathcal{P}_l^{m'}(x) dx = \frac{2}{2l + 1} \delta_{m m'}.
\]

2. Irreducible \( \mathfrak{sl}(2) \)-submodules of \( \mathcal{H}_+ \)

One can show that in the overlapping region of both north and south coordinate patches \((\theta, \phi)\) of the sphere we have (here \( Y_l^m(\theta, \phi) \) with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \) are the
spherical harmonics, see [5]), so the vectors \(|l, m\rangle\), where \(x = -\cos \theta\), are smooth functions on \(S^2 \times S^1\):

\[|l, m\rangle_{x = -\cos \theta} = \frac{e^{i\theta}}{\sqrt{2\pi}} Y^m_l(\theta, \phi).\]

By choosing an appropriate decomposition \(sp(4) = g^+ \oplus h \oplus g^-\), where \(h\) is the Cartan subalgebra, we show that \(\mathcal{H}_{+}\) is an \(sp(4)\)-module with highest weight. We realize the root vectors of \(sp(4)\) depicted by the diagram

\[
\begin{array}{c c c c c}
J_{-} & J_{10} & J_{+} \\
\downarrow & \downarrow & \downarrow \\
J_{0} & J_{-} & J_{+}
\end{array}
\]

by the following differential operators obtained from (1.5) and (1.8) under the change of variable \(x = -\cos \theta\):

\[
\begin{align*}
J_{0+} &= e^{i\phi} \left(\sqrt{1-x^2} \frac{\partial}{\partial x} - i x \frac{\partial}{\partial \theta}\right), \\
J_{0-} &= e^{-i\phi} \left(-\sqrt{1-x^2} \frac{\partial}{\partial x} + i x \frac{\partial}{\partial \theta}\right), \\
J_{+0} &= e^{i\phi} \left(1-x^2 \frac{\partial}{\partial x} + i x \frac{\partial}{\partial \theta}\right), \\
J_{-0} &= e^{-i\phi} \left(-1-x^2 \frac{\partial}{\partial x} + i x \frac{\partial}{\partial \theta}\right).
\end{align*}
\]

The space \(\mathcal{H}_{+}\) can be split into the direct sums of the \(sl(2)\)-submodules in four different ways in accordance with the four ways select the \(sl(2)\)-subalgebra in \(sp(4)\): the horizontal, vertical and two diagonal ones, see (2.1) (here the symbol \(\lfloor \cdot \rfloor\) means the integer part of a given number):

\[
\begin{align*}
\mathcal{H}_{+} &= \bigoplus_{l \in \mathbb{Z}} \mathcal{H}_l, & \mathcal{H}_l &= \text{span}\left\{|l, m\rangle_{x = \cos \theta}, \right\}_{-l \leq m \leq l}, \\
\mathcal{H}_{+} &= \bigoplus_{m = -\infty}^{m = \infty} \mathcal{H}_m, & \mathcal{H}_m &= \text{span}\left\{|l, m\rangle_{x = \cos \theta}\right\}_{|l| \geq |m|}, \\
\mathcal{H}_{+} &= \bigoplus_{k = 0}^{\infty} \mathcal{H}_k, & \mathcal{H}_k &= \text{span}\left\{|l, l - k\rangle_{x = \cos \theta}\right\}_{l \geq |k|}, \\
\mathcal{H}_{+} &= \bigoplus_{k = 0}^{\infty} \mathcal{H}_{-k}, & \mathcal{H}_{-k} &= \text{span}\left\{|l, -l + k\rangle_{x = \cos \theta}\right\}_{l \geq |k|}.
\end{align*}
\]

Due to analyticity of the Legendre functions \(P^m_l(x)\), the infinite-dimensional representations of \(sl(2)\) can be realized in the space of functions on the sphere instead of those on the hyperbolic plane [7]. The well-known generators of rotation about the \(x\)- and \(y\)-axes can be described as \(J_x = \frac{\partial}{\partial x}\) and \(J_y = \frac{\partial}{\partial y}\), respectively. Clearly, \(J_z = J_0\) is the generator of rotation about the \(z\)-axis. In this case, the matrix elements of the rotation operator \(D(R)\) are given by 

\[2\pi \delta_{m'M} \langle l, m | \exp(-i\alpha \cdot \vec{J}) | l', m'\rangle,\]

where the unit vector \(\vec{n}\) and the angle \(\alpha\) describe the rotation \(R\) (see [8,9]).
Proposition 2.1. The differential operators \( J_{0+}, J_{0-} \) and \( J_{03} = -i \frac{\partial}{\partial \phi} \) span \( \mathfrak{sl}(2) \):
\[
[J_{0+}, J_{0-}] = 2J_{03}, \quad [J_{03}, J_{0\pm}] = \pm J_{0\pm}.
\]

Each of the finite-dimensional submodules \( \mathcal{H}_l \), realizes an irreducible representation of the horizontal algebra \( \mathfrak{sl}(2) \):
\[
J_{0+}[l, m - 1] = \sqrt{(l - m + 1)(l + m)}[l, m],
\]
\[
J_{0-}[l, m] = \sqrt{(l - m + 1)(l + m)}[l, m - 1],
\]
\[
J_{03}[l, m] = m[l, m].
\]

Proposition 2.2. The differential operators \( J_{\pm 0}, J_{0-} \) and \( J_{00} = -i \frac{\partial}{\partial \psi} + \frac{1}{2} \) span a copy of \( \mathfrak{sl}(2) \):
\[
[J_{\pm 0}, J_{0-}] = -2J_{00}, \quad [J_{00}, J_{0\pm}] = \pm J_{0\pm}.
\]

Each of the infinite-dimensional submodules \( \mathcal{H}_m \), realizes an irreducible representation of the vertical algebra \( \mathfrak{sl}(2) \):
\[
J_{\pm 0}[l, m - 1] = \frac{\sqrt{2l + 1}}{2}(l - m)(l + m)[l, m],
\]
\[
J_{0-}[l, m] = \frac{\sqrt{2l + 1}}{2}(l - m)(l + m)[l, m - 1],
\]
\[
J_{00}[l, m] = \left( l + \frac{1}{2} \right)[l, m].
\]

The generators \( J_{\pm 0} \) and \( J_{0-} \) are deformations of certain left-invariant vector fields on the homogeneous manifold \( AdS_2 \) and, together with \( J_{00} \), they realize a representation of the Lie algebra \( \mathfrak{sl}(2) \) in the space of hyperbolic harmonics. The following two propositions are immediate.

Proposition 2.3. Set
\[
J_{\pm \pm} := [J_{0+}, J_{0\pm}] = e^{i\phi} \left( -x \sqrt{1 - x^2} \frac{\partial}{\partial x} + \frac{i}{\sqrt{1 - x^2}} \frac{\partial}{\partial \phi} + i \sqrt{1 - x^2} \frac{\partial}{\partial \psi} - \sqrt{1 - x^2} \right),
\]
\[
J_{\pm -} := [J_{0-}, J_{0\pm}] = e^{-i\phi} \left( x \sqrt{1 - x^2} \frac{\partial}{\partial x} + \frac{i}{\sqrt{1 - x^2}} \frac{\partial}{\partial \phi} + i \sqrt{1 - x^2} \frac{\partial}{\partial \psi} \right),
\]
\[
J_{00} + J_{0\pm} = -i \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \psi} \right) \frac{1}{2}.
\]

These operators span a copy of \( \mathfrak{sl}(2) \):
\[
[J_{\pm \pm}, J_{\pm -}] = -4(J_{00} + J_{0\pm}), \quad [J_{00} + J_{0\pm}, J_{\pm \pm}] = \pm 2J_{\pm \pm}.
\]

Each of the infinite-dimensional submodules \( \mathcal{H}_k \), where \( k \) is an arbitrary nonnegative integer, realizes an irreducible representation of the right diagonal algebra \( \mathfrak{sl}(2) \).
irreducible representations for the diagonal subalgebras $\mathfrak{sl}_v$ vectors. Set $\mathfrak{sl}$ horizontal and the second labels of the vectors, respectively. Moreover, the submodules $H$ completely reducible over each of the four 

Proposition 2.4. Each of the infinite-dimensional submodules $H_{(k)}$ are separately spanned by the basis elements $\{l,m\}$ with the given values for $l = m + k$:

$$J_{\pm}|l - 1, m - 1\rangle = \sqrt{\frac{2l - 1}{2l + 1}}(l + m - 1)(l + m)|l, m\rangle,$$

$$J_{\mp}|l, m\rangle = \sqrt{\frac{2l + 1}{2l - 1}}(l + m - 1)(l + m)|l - 1, m - 1\rangle,$$

$$(J_{0h} + J_{0v})|l, m\rangle = \left(l + m + \frac{1}{2}\right)|l, m\rangle.$$

These operators span a copy of $\mathfrak{sl}(2)$:

$$[J_{\pm}, J_{\mp}] = -4(J_{0h} - J_{0v}),$$

$$[J_{0h} - J_{0v}, J_{\pm}] = \pm 2J_{\pm}.$$

Each of the infinite-dimensional submodules $H_{(k)}$ where $k$ is an arbitrary nonnegative integer; realizes an irreducible representation of the left diagonal algebra $\mathfrak{sl}(2)$ with $l = m + k$:

$$J_{\pm}|l - 1, m\rangle = \sqrt{\frac{2l - 1}{2l + 1}}(l - m)(l + m + 1)|l, m - 1\rangle,$$

$$J_{\mp}|l, m - 1\rangle = \sqrt{\frac{2l + 1}{2l - 1}}(l - m)(l + m + 1)|l - 1, m\rangle,$$

$$(J_{0h} + J_{0v})|l, m\rangle = \left(l - m + \frac{1}{2}\right)|l, m\rangle.$$

The irreducible submodules $H_{(l)}$ and $H_{(m)}$ of the horizontal and vertical subalgebras $\mathfrak{sl}(2)$ are separately spanned by the basis elements $\{|l, m\rangle$ with the given values for the first and the second labels of the vectors, respectively. Moreover, the submodules $H_{(l)}$ and $H_{(m)}$ constructed by all basis vectors $\{|l, m\rangle$ with given values for $l - m$ and $l + m$, constitute irreducible representations for the diagonal subalgebras $\mathfrak{sl}(2)$. Therefore, the module $H_{(l)}$ is completely reducible over each of the four $\mathfrak{sl}(2)$-subalgebras.

Let us now fix bases in each of the four $\mathfrak{sl}(2)$-subalgebras. Since the submodules over the horizontal $\mathfrak{sl}(2)$-algebra are finite-dimensional, they have both highest and lowest weight vectors. Set

$$H^h := 2J_{0h}, \quad X^h_{\pm} := J_{0h} \pm 1.$$
Modules over the other three copies of $\mathfrak{sl}(2)$ (vertical, left and right diagonal) are infinite-dimensional. Set:

$$H^r := -2J_{30}, \quad X^+ \defeq \pm J_{90}, \quad (2.29)$$
$$H^{rd} := -(J_{30} + J_{90}), \quad X^{rd} \defeq \frac{1}{2}J_{\pm \pm}, \quad (2.30)$$
$$H^{id} := J_{30} - J_{90}, \quad X^{id} \defeq \frac{1}{2}J_{\pm \pm}. \quad (2.31)$$

It remains only to identify the $\mathfrak{sl}(2)$-submodules $\mathcal{H}_1$, $\mathcal{H}_m$, $\mathcal{H}^+_1$ and $\mathcal{H}^-_1$ with the modules $T_r$ with highest weight $\nu$:

$$\mathcal{H}_1 \cong T_{2l}, \quad \mathcal{H}_m \cong T_{-2m} \cdot \frac{1}{2}, \quad \mathcal{H}^+_{2 j + 1} \cong T_{-\frac{1}{2}}, \quad \mathcal{H}^-_{2 j} \cong T_{-\frac{1}{2}}. \quad (2.32)$$

The isomorphisms $\mathcal{H}_j \cong \mathcal{H}^+_{2 j}$ (for any $j = 0, 1, 2, \ldots$) immediately follow from (2.32). Indeed,

$$H^{rd}(m + 2 j + 1, m) = -\left(2m + 2j + \frac{3}{2}\right)(m + 2 j + 1, m). \quad (2.33)$$

So, for the smallest $m = -j$, the highest weight (independent of $j$) is equal to $-\frac{3}{2}$. The same applies to the last two isomorphisms in (2.32).

3. The Well Representation $\mathcal{H}_+$ of the Lie Algebra $\mathfrak{sp}(4)$

Between the 10 generators of $\mathfrak{sp}(4)$ there can be $(10 \times 9)/2 = 45$ commutation relations. In addition to the commutation relations obtained in (2.8), (2.12), (2.16), (2.17), (2.22) and (2.23), there are only the following nonzero commutation relations:

$$[J_{30}, J_{30}] = \pm J_{30}, \quad [J_{30}, J_{90}] = \pm J_{90}, \quad [J_{90}, J_{30}] = \mp J_{30}, \quad [J_{30}, J_{90}] = \mp J_{90}, \quad (3.1)$$

Equation (3.1) completes the proof of the following proposition:

**Proposition 3.1.** The ten differential operators $J_{01}$, $J_{0-}$, $J_{03}$, $J_{00}$, $J_{00}$, $J_{30}$, $J_{30}$, $J_{30}$, $J_{03}$, $J_{03}$ and $J_{03}$, in the space $\mathcal{H}_+$ of smooth functions on $S^2 \times S^1$ satisfy the same commutation relations of the basis elements of the Lie algebra $\mathfrak{sp}(4)$ as in Eqs. (2.8), (2.12), (2.16), (2.17), (2.22), (2.23) and (3.1). The generators $\{J_{30}, J_{90}\}$ constitute the Cartan subalgebra. The $\mathfrak{sp}(4)$-module $\mathcal{H}_+$ is irreducible with highest weight $(0, -\frac{3}{2})$ with respect to $H_1 := 2J_{30}$ and $H_2 := -(J_{30} + J_{90})$.

This representation is a rare example of highest weight irreducible representation of $\mathfrak{sp}(4)$ all whose weight spaces are 1-dimensional.

**Proof.** Take $\alpha_1$ and $\alpha_2$ for simple roots. Therefore the subalgebra $\mathfrak{g}^+$ is generated by $X^+_{\pm} := J_{90}$ and $X^-_{\pm} := J_{30}$, and $\mathfrak{g}^-$ is generated by $X^-_{\pm} := J_{90}$ and $X^+_{\pm} := J_{30}$. Now the weight $\omega([l, m])$ of the vector $[l, m]$ with respect to $H_1$ and $H_2$ is equal to $(2m, -m - l - \frac{3}{2})$.

So the weight of the highest weight vector $(0, 0)$ of the representation is $\omega((0, 0)) = (0, -\frac{3}{2})$. 

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Irreducibility of this representation follows easily from the fact that for any given weight \( \omega \in \mathfrak{h} \), the space \( \mathcal{H}^\omega \) of vectors of weight \( \omega \) in \( \mathcal{H}_+ \) is just 1-dimensional, and derived from the theory of Verma modules. So, on an abstract level, the representation is the irreducible quotient-module of the Verma module \( M(0, -1/2) \).

Let us equip the space \( \mathcal{H}_+ \) with the inner product (bar is for the complex conjugation)

\[
(l, m|) = \int_{x=-1}^{1} \int_{\psi=0}^{2\pi} \int_{\phi=0}^{2\pi} \left( \sqrt{2l+1} \sqrt{2l'+1} \right) \left( e^{il\psi} \psi + e^{im\phi} \right) \left( e^{il'\psi} \psi + e^{im\phi} \right) d\phi d\psi dx.
\]

Using the orthogonality relation (1.9), one can conclude that the basis of \( \mathcal{H}_+ \) constitutes an orthonormal set with respect to both indices \( l \) and \( m \):

\[
(l, m|l', m') = \delta_{ll'} \delta_{mm'}.
\]

4. Concluding Remarks

Propositions 2.1 and 2.2 unite representations of the horizontal and vertical subalgebras \( sl(2) \) in the spaces of functions on the sphere and hyperbolic plane into the representation on functions on the space \( S^2 \times S^1 \). The Lie algebra of differential operators generated by both horizontal and vertical subalgebras lead to a larger Lie algebra with a representation in an infinite-dimensional space. Propositions 2.3 and 2.4 together with the commutation relations (3.1) show that this larger algebra is isomorphic to \( sp(4) \). From Fig. 1 on [2, p. 1186], representing the reduction of the representations \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) with respect to the maximal subalgebra \( su(2) \times u(1) \), and by comparing Proposition 3.1 with the Bose representations embedded in the \( \mathcal{H}_F \), one sees immediately that the representation (2.9), (2.10), (2.11), (2.13), (2.14), (2.15), (2.19), (2.20), (2.25) and (2.26) for \( sp(4) \) is equivalent to the irreducible Weil representation \( \mathcal{H}_+ \) realized in the space of smooth functions on \( S^2 \times S^1 \).

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