On the singular two-parameter eigenvalue problem II*

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Abstract

In the 1960s, Atkinson introduced an abstract algebraic setting for multiparameter eigenvalue problems. He showed that a nonsingular multiparameter eigenvalue problem is equivalent to the associated system of generalized eigenvalue problems, which is a key relation for many theoretical results and numerical methods for nonsingular multiparameter eigenvalue problems. In 2009, Muhič and Plestenjak extended the above relation to a class of singular two-parameter eigenvalue problems with coprime characteristic polynomials and such that all finite eigenvalues are algebraically simple. They introduced a way to solve a singular two-parameter eigenvalue problem by computing the common regular eigenvalues of the associated system of two singular generalized eigenvalue problems. Using new tools, in particular the stratification theory, we extend this connection to singular two-parameter eigenvalue problems with possibly multiple eigenvalues and such that characteristic polynomials can have a nontrivial common factor.

Keywords: singular two-parameter eigenvalue problem, Kronecker canonical form, minimal reducing subspace, stratification

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1. Introduction

We consider the algebraic two-parameter eigenvalue problem

\begin{align*}
W_1(λ, µ)x_1 & := (A_1 + λB_1 + µC_1)x_1 = 0, \\
W_2(λ, µ)x_2 & := (A_2 + λB_2 + µC_2)x_2 = 0,
\end{align*}

(W)

where $A_i$, $B_i$, and $C_i$ are $n_i \times n_i$ matrices over $\mathbb{C}$, $λ, µ \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$. If $(λ, µ)$ and nonzero $x_1, x_2$ satisfy (W), then the tensor product $x_1 \otimes x_2$ is the corresponding \textit{(right) eigenvector}. Similarly, $y_1 \otimes y_2$ is the corresponding \textit{left eigenvector}.

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eigenvector if \( y_i \neq 0 \) and \( y_i^* W_i(\lambda, \mu) = 0 \) for \( i = 1, 2 \). In the generic case, problem (W) has \( n_1 n_2 \) eigenvalues counting multiplicities \((\lambda, \mu)\) that are roots of the system of two bivariate characteristic polynomials

\[
\begin{align*}
p_1(\lambda, \mu) &:= \det(W_1(\lambda, \mu)) = 0, \\
p_2(\lambda, \mu) &:= \det(W_2(\lambda, \mu)) = 0.
\end{align*}
\] (1)

A two-parameter eigenvalue problem (W) is related to a coupled pair of generalized eigenvalue problems

\[
\begin{align*}
\Delta_1 z &= \lambda \Delta_0 z, \\
\Delta_2 z &= \mu \Delta_0 z,
\end{align*}
\] (\(\Delta\))

where \( n_1 n_2 \times n_1 n_2 \) matrices

\[
\begin{align*}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\
\Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2, \\
\Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2
\end{align*}
\] (2)

are called operator determinants and \( z = x_1 \otimes x_2 \) is a decomposable vector, for details see, e.g., [1]. A generic two-parameter eigenvalue problem (W) is nonsingular, which means that the corresponding operator determinant \( \Delta_0 \) is nonsingular. In this case (see, e.g., [1]), the matrices \( \Delta_0^{-1} \Delta_1 \) and \( \Delta_0^{-1} \Delta_2 \) commute and the eigenvalues of (W) agree with the eigenvalues of (\(\Delta\)). This relation enables us to solve a nonsingular two-parameter eigenvalue problem numerically using standard tools for generalized eigenvalue problems, see, e.g., [8].

Several applications lead to singular two-parameter eigenvalue problems, where \( \Delta_0 \) is singular and (\(\Delta\)) is a pair of singular generalized eigenvalue problems, see, e.g., the computation of the signed distance between two overlapping ellipsoids [9], solving a system of two bivariate polynomials [13], an application in stochastic games [2], aerelastic flutter problem [14], and applications in [12]. We assume that (W) is singular in the homogeneous setting, i.e., for all linear combinations \( \alpha_0 \Delta_0 + \alpha_1 \Delta_1 + \alpha_2 \Delta_2 \) are singular, as otherwise the problems can be transformed into a nonsingular one by a linear substitution of parameters \( \lambda \) and \( \mu \). In the singular case, the relation between problems (W) and (\(\Delta\)) is much less understood. For a start, we need to define what is an eigenvalue of (W) and (\(\Delta\)) in the singular case.

We assume that bivariate pencils \( W_1(\lambda, \mu) \) and \( W_2(\lambda, \mu) \) are both regular, i.e., \( p_i(\lambda, \mu) \neq 0 \) for \( i = 1, 2 \). The problem (W) is then singular if either

a) degree of \( p_i \) is less than \( n_i \) for \( i = 1, 2 \), or
b) polynomials \( p_1 \) and \( p_2 \) have a nontrivial common factor.

**Definition 1.** Let \((\lambda_0, \mu_0) \in \mathbb{C}^2\) be a point that does not belong to a nontrivial common factor of characteristic polynomials \( p_1 \) and \( p_2 \). We say that \((\lambda_0, \mu_0)\) is a finite eigenvalue of (W) if there exist nonzero vectors \( x_1 \) and \( x_2 \) that satisfy (W).
It is easy to see that all isolated solutions of the system of characteristic polynomials \( \Pi \) are finite eigenvalues of \( W \).

A definition of eigenvalue that covers both regular and singular pencils is that \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of a matrix pencil \( A - \lambda B \) if \( \text{rank}(A - \lambda_0 B) < \text{nrank}(A - \lambda B) \), where \( \text{nrank}(A - \lambda B) := \max_{\xi \in \mathbb{C}} \text{rank}(A - \xi B) \) is the normal rank of a matrix pencil. In a similar way, \( \infty \) is an eigenvalue if \( \text{rank}(B) < \text{nrank}(A - \lambda B) \).

We define a common finite eigenvalue of a pair of matrix pencils \( \Delta \) as follows.

**Definition 2.** A point \( (\lambda_0, \mu_0) \in \mathbb{C}^2 \) is a (finite) eigenvalue of matrix pencils \( \Delta_1 - \lambda \Delta_0 \) and \( \Delta_2 - \mu \Delta_0 \) if the following is true:

a) \( \text{rank}(\Delta_1 - \lambda_0 \Delta_0) < \text{nrank}(\Delta_1 - \lambda \Delta_0) \),

b) \( \text{rank}(\Delta_2 - \mu_0 \Delta_0) < \text{nrank}(\Delta_2 - \mu \Delta_0) \),

c) there exists a common regular eigenvector \( z \), i.e., a nonzero \( z \) such that

\[
(\Delta_1 - \lambda_0 \Delta_0)z = 0, \quad (\Delta_2 - \mu_0 \Delta_0)z = 0, \quad \text{and} \quad z \notin R(\Delta_i, \Delta_0) \text{ for } i = 1, 2,
\]

where \( R(\Delta_i, \Delta_0) \) is the minimal reducing subspace (for a definition, see Section 2) for \( \Delta_i \) and \( \Delta_0 \).

Under the assumption that \( p_1 \) and \( p_2 \) are coprime, i.e., they do not have a nontrivial common factor, it is shown in [12] that if all eigenvalues of \( W \) are algebraically simple, then eigenvalues of \( W \) and \( \Delta \) agree. Also, if \( (\lambda_0, \mu_0) \in \mathbb{C}^2 \) is an algebraically simple eigenvalue of \( W \), then \( (\lambda_0, \mu_0) \) is an eigenvalue of \( \Delta \). Several numerical examples in [12] indicate that the relation between the eigenvalues of \( W \) and \( \Delta \) could be extended to multiple eigenvalues and also to the case where \( p_1 \) and \( p_2 \) have a nontrivial common factor. In this paper we confirm these findings by providing the supporting theory.

In Section 2 we introduce the Kronecker canonical form and other tools. We introduce stratification together with some new related results in Section 3. In Section 4 we show that if \( p_1 \) and \( p_2 \) are coprime, then each eigenvalue of \( W \) is an eigenvalue of \( \Delta \) and vice versa. For problems such that \( p_1 \) and \( p_2 \) do have a nontrivial common factor we extend the relation to points that do not belong to a common factor. In Section 5 we discuss conditions for a point \( (\lambda_0, \mu_0) \) on a common curve that would make it an eigenvalue and give several illuminating numerical examples.

### 2. Auxiliary results

The Kronecker canonical form (KCF) of a matrix pencil is one of the main tools in this paper. For more details on the KCF and its numerical computation see, e.g., [4], [5], [7], and [12].

**Definition 3.** Let \( A - \lambda B \in \mathbb{C}^{m \times n} \) be a matrix pencil. Then there exist non-singular matrices \( P \in \mathbb{C}^{m \times m} \) and \( Q \in \mathbb{C}^{n \times n} \) such that

\[
P^{-1}(A - \lambda B)Q = \bar{A} - \lambda \bar{B} = \text{diag}(A_1 - \lambda B_1, \ldots, A_k - \lambda B_k)
\]
is in **Kronecker canonical form.** Each block $A_i - \lambda B_i, \; i = 1, \ldots, k$, has one of the following forms: $J_d(\alpha), N_d, L_d$, or $L_d^T$, where the matrices

$$J_d(\alpha) = \begin{bmatrix}
\alpha - \lambda & 1 \\
\vdots & \ddots \\
0 & \ddots & 1 \\
\alpha - \lambda & & \cdots & 1
\end{bmatrix} \in \mathbb{C}^{d \times d}, \quad N_d = \begin{bmatrix}
1 & -\lambda \\
\vdots & \ddots & \ddots \\
0 & \ddots & \ddots & -\lambda \\
0 & \cdots & \cdots & 1
\end{bmatrix} \in \mathbb{C}^{d \times d},$$

$$L_d = \begin{bmatrix}
-\lambda & 1 \\
\vdots & \ddots \\
0 & \cdots & -\lambda & 1
\end{bmatrix} \in \mathbb{C}^{d \times (d+1)}, \quad L_d^T = \begin{bmatrix}
-\lambda & \cdots & \cdots & -\lambda \\
1 & \ddots & \ddots & \cdots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1
\end{bmatrix} \in \mathbb{C}^{(d+1) \times d},$$

represent a finite regular block, an infinite regular block, a right singular block, and a left singular block, respectively. To each Kronecker block we associate a **Kronecker chain** of linearly independent vectors as follows:

a) A finite regular block $J_d(\alpha)$ is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$(A - \alpha B)u_1 = 0,$$

$$(A - \alpha B)u_{i+1} = Bu_i, \quad i = 1, \ldots, d - 1.$$

b) An infinite regular block $N_d$ is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$Bu_1 = 0,$$

$$Bu_{i+1} = Au_i, \quad i = 1, \ldots, d - 1.$$

c) A right singular block $L_d$ is associated with vectors $u_1, \ldots, u_{d+1}$ that satisfy

$$Bu_1 = 0,$$

$$Bu_{i+1} = Au_i, \quad i = 1, \ldots, d,$$

$$0 = Au_{d+1}.$$

d) For $d \geq 1$, a left singular block $L_d^T$ is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$Bu_i = Au_{i+1}, \quad i = 1, \ldots, d - 1.$$

The union of Kronecker chains for all Kronecker blocks is a basis for $\mathbb{C}^n$.

One can see that $L_d \begin{bmatrix} 1 & \lambda & \cdots & \lambda^d \end{bmatrix}^T = 0$ and to each right singular block there corresponds one of linearly independent vectors from the kernel of $A - \lambda B$. If vectors $u_1, \ldots, u_{d+1}$ form a Kronecker chain for a right singular block $L_d$, then

$$(A - \lambda B)(\lambda^d u_1 + \lambda^{d-1} u_2 + \cdots + \lambda u_d + u_{d+1}) = 0$$
for all \( \lambda \). Thus, to each right singular block \( L_d \) there corresponds a polynomial vector \( p(\lambda) \) of degree \( d \) such that \( (A - \lambda B)p(\lambda) = 0 \) for all \( \lambda \).

We know that \( \text{nrank}(A - \lambda B) = n - s \), where \( s \geq 0 \) is the number of the right singular blocks \( L_d \) in the KCF of \( A - \lambda B \). If \( A - \lambda B \) is a singular pencil then \( s \geq 1 \) and there exist \( s \) linearly independent polynomial vectors \( p_1(\lambda), \ldots, p_s(\lambda) \), each corresponding to a different right singular block, such that for a generic \( \lambda_0 \in \mathbb{C} \) vectors \( p_1(\lambda_0), \ldots, p_s(\lambda_0) \) form a basis for \( \text{Ker}(A - \lambda_0 B) \). We say that span\( (p_1(\lambda), \ldots, p_s(\lambda)) \) is the \textit{generic kernel} of \( A - \lambda B \) and denote it by \( \text{GKer}(A - \lambda B) \). For each \( \lambda \in \mathbb{C} \), \( \dim(\text{GKer}(A - \lambda B)) = s \) and \( \text{GKer}(A - \lambda B) \subset \text{Ker}(A - \lambda B) \).

A subspace \( M \subset \mathbb{C}^n \) is a \textit{reducing subspace} for \( A - \lambda B \) if \( \dim(AM + BM) = \dim(M) - s \). The vectors from the Kronecker chains of all right singular blocks \( L_d \) form a basis for the \textit{minimal reducing subspace} \( R(A, B) \), which is a subset of all reducing subspaces. \( R(A, B) \) is unique and can be computed from the generalized upper triangular form (GUPTRI), see, e.g., [4, 5, 10]. One can also compute \( R(A, B) \) numerically by exploiting the relation

\[
R(A, B) = \bigcup_{\xi \in \mathbb{C}} \text{GKer}(A - \xi B).
\]

If \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of a singular matrix pencil \( A - \lambda B \), then the eigenvector cannot be defined in a usual way, but the following is true.

**Lemma 4.** For a (singular) matrix pencil \( A - \lambda B \) the following is equivalent:

a) \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of \( A - \lambda B \),

b) \( \text{rank}(A - \lambda_0 B) < \text{nrank}(A - \lambda B) \),

c) \( \dim(\text{GKer}(A - \lambda_0 B)) < \dim(\text{Ker}(A - \lambda_0 B)) \),

d) \( \) there exists a nonzero \( v \) such that \( v \notin \text{GKer}(A - \lambda_0 B) \) and \( (A - \lambda_0 B)v = 0 \),

e) there exists a nonzero \( v \) such that \( v \notin R(A, B) \) and \( (A - \lambda_0 B)v = 0 \).

If \( \lambda_0 \) is an eigenvalue of \( A - \lambda B \), then we say that each nonzero vector \( v \) such that \( v \notin R(A, B) \) and \( (A - \lambda_0 B)v = 0 \) is a corresponding (right) regular eigenvector. Clearly, if we pick a random vector \( v \) from \( \text{Ker}(A - \lambda_0 B) \), then \( v \) is a regular eigenvector with probability one.

For the study of \( \Delta \), we will apply the following result from [11] that shows that the kernel of an operator determinant \( \Delta = A \otimes D - B \otimes C \) can be constructed from the Kronecker chains of matrix pencils \( A - \lambda B \) and \( C - \mu D \).

**Theorem 5 ([11] Theorem 4).** A basis for the kernel of \( \Delta = A \otimes D - B \otimes C \) is the union of sets of linearly independent vectors associated with the pairs of Kronecker blocks of the following types:

a) \( (J_{d_1}(\alpha_1), J_{d_2}(\alpha_2)) \), where \( \alpha_1 = \alpha_2 \),

b) \( (N_{d_1}, N_{d_2}) \),

c) \( (N_{d_1}, L_{d_2}) \),

d) \( (L_{d_1}, N_{d_2}) \).
e) \((L_{d_1}, J_{d_2}(\alpha))\),
f) \((J_{d_1}(\alpha), L_{d_2})\),
g) \((L_{d_1}, L_{d_2})\),
h) \((L_{d_1}, L_{d_2}^T)\), where \(d_1 < d_2\),
i) \((L_{d_1}^T, L_{d_2})\), where \(d_1 > d_2\),

where the left block of each pair belongs to the pencil \(A - \lambda B\) and the right block belongs to the pencil \(C - \mu D\).

For each pair of Kronecker blocks that satisfies a) or b) we can construct an associated set of linearly independent vectors \(z_1, \ldots, z_d\) in the kernel of \(\Delta\) as follows. Let the vectors \(u_1, \ldots, u_d\) form a Kronecker chain associated with the block \(J_{d_1}(\alpha_1)\) (or \(N_{d_1}\)) of the pencil \(A - \lambda B\) and let the vectors \(v_1, \ldots, v_d\) form a Kronecker chain associated with the block \(J_{d_2}(\alpha_2)\) (or \(N_{d_2}\)) of the pencil \(C - \mu D\). Then \(d = \min(d_1, d_2)\) and

\[
z_j = \sum_{i=1}^{d_j} u_i \otimes v_{j+1-i}, \quad j = 1, \ldots, d.
\]

In the above theorem we omit the constructions of vectors in the kernel for all pairs of Kronecker blocks that are irrelevant to our case. For a complete description, see [11]. A similar technique is used to describe the kernels of generalized Sylvester operators in [3].

**Definition 6.** For a pair of matrix pencils \(A - \lambda B\) and \(C - \mu D\) we denote by \(T(A - \lambda B, C - \mu D)\) the size of the kernel of \(A \otimes D - B \otimes C\).

### 3. Stratification

If we perturb a regular matrix pencil \(A - \lambda B\) into \(\tilde{A} - \lambda \tilde{B}\), where \(\tilde{A}\) and \(\tilde{B}\) are arbitrarily close to \(A\) and \(B\), respectively, then the stratification theory [3, 10] explains which are possible structures of the KCF of the perturbed pencil, based on the KCF of the initial pencil.

Let \(\alpha_1, \ldots, \alpha_k\) be different eigenvalues (including a possible eigenvalue \(\infty\)) of \(A - \lambda B\). For an eigenvalue \(\alpha_j\) let \(d_{j1} \geq d_{j2} \geq \cdots \geq d_{jm_j}\) be sizes of the corresponding Jordan blocks (or infinite blocks for eigenvalue \(\infty\)) for \(j = 1, \ldots, k\).

The KCF of \(\tilde{A} - \lambda \tilde{B}\) can be obtained from the KCF of the initial pencil \(A - \lambda B\) using a finite sequence of the following two moves:

- **MLW (minimum leftward move):** For one eigenvalue \(\alpha\) we get from blocks of sizes \(d_1 \geq d_2 \geq \cdots \geq d_m\) blocks of sizes \(d'_1 \geq d'_2 \geq \cdots \geq d'_m\), where \(d'_p = d_p + 1\) and \(d'_q = d_q - 1\) for \(p < q\) and \(d_p = d_{p+1} = \cdots = d_{q-1} = d_q\), for an eigenvalue \(\alpha'\) close to \(\alpha\) (it can remain unchanged). The number of blocks corresponding to \(\alpha'\) equals the number of blocks for \(\alpha\), except when \(q = m\) and \(d_q = 1\) in which case the perturbed pencil has \(m - 1\) blocks.
• **HC (horizontal cut):** From one eigenvalue $\alpha$ and its blocks of sizes $d_1 \geq d_2 \geq \cdots \geq d_m$ we get two eigenvalues $\beta \neq \gamma$, both close to $\alpha$ (one of them can remain $\alpha$), such that the sizes of blocks for $\beta$ are $b_1 \geq b_2 \geq \cdots \geq b_m$, where $b_j = \min(d_j, C)$, and the sizes of blocks for $\gamma$ are $c_1 \geq c_2 \geq \cdots \geq c_m$, where $c_j = d_j - b_j$ for $j = 1, \ldots, m$ and $1 \leq C < d_1$ is the size of the cut. If $c_j = 0$ for some $j \in \{1, \ldots, m\}$, then there are less than $m$ blocks for the eigenvalue $\gamma$.

In the above two moves, whenever $\alpha = \infty$, it can perturb into finite eigenvalues, while a finite eigenvalue cannot perturb into an infinite one. In addition to the above two moves, each eigenvalue can perturb into a nearby one without any change in the structure of Jordan (or infinite) blocks.

We say that matrix pencils $A - \lambda B$ and $C - \lambda D$ are equivalent if there exist nonsingular matrices $P$ and $Q$ such that $C - \lambda D = P(A - \lambda B)Q$. The orbit of a matrix pencil is the manifold of all equivalent pencils, which means that all pencils in the orbit have equal KCF. A bundle is the union of all orbits with the same canonical structure (number of blocks and their sizes) but with the eigenvalues unspecified. We denote the bundle that corresponds to the canonical structure of the pencil $A - \lambda B$ by $\mathcal{B}(A - \lambda B)$. We say that bundle $\mathcal{B}'$ covers bundle $\mathcal{B}$ if we get $\mathcal{B}'$ by applying a MLW or HC move on $\mathcal{B}$. More precisely, we say that $\mathcal{B}'$ is a MLW cover of $\mathcal{B}$ or a HC cover of $\mathcal{B}$.

In the following section we will study the relation between the eigenvalues of $\mathcal{W}$ and $\mathcal{D}$. One condition for $(\lambda_0, \mu_0)$ to be an eigenvalue of $\mathcal{D}$ is that $\operatorname{rank}(\Delta_1 - \lambda_0 \Delta_0) < \operatorname{rank}(\Delta_1 - \xi \Delta_0)$ for a generic $\xi \neq \lambda_0$. It follows from

$$\Delta_1 - \lambda \Delta_0 = W_1(\lambda, 0) \otimes C_2 - C_1 \otimes W_2(\lambda, 0)$$

that a perturbation from $\lambda = \lambda_0$ to $\lambda = \xi$ changes the structure of the KCF in Theorem 5. To understand how this affects $\operatorname{rank}(\Delta_1 - \lambda_0 \Delta_0)$ it is enough to study what happens when we apply a MLW or HC move to the corresponding bundles. The following two lemmas provide the appropriate theory.

**Lemma 7.** Let $A - \lambda B$ and $C - \mu D$ be a pair of regular matrix pencils and let $\tilde{A} - \lambda \tilde{B}$ and $\tilde{C} - \mu \tilde{D}$ be their arbitrarily close perturbations, such that $\mathcal{B}(\tilde{A} - \lambda \tilde{B})$ is a MLW cover of $\mathcal{B}(A - \lambda B)$ and $\mathcal{B}(\tilde{C} - \mu \tilde{D}) = \mathcal{B}(C - \mu D)$. More precisely, let the MLW move be applied to an eigenvalue $\alpha$ of $A - \lambda B$ and let $d_1 \geq d_2 \geq \cdots \geq d_m$ be sizes of the corresponding blocks in the KCF. If the MLW move affects blocks with indices $p$ and $q$, where $p < q$ and $d_p = d_{p+1} = \cdots = d_{q-1} \geq d_q$, then

1. $\tilde{T} \leq T$, where $T = \dim(\ker(\tilde{A} \otimes D - B \otimes C))$ and $\tilde{T} = \dim(\ker(\tilde{A} \otimes \tilde{D} - \tilde{B} \otimes \tilde{C}))$.
2. If the KCF of $C - \mu D$ contains a block $J_e(\alpha)$ such that $d_p \geq e \geq d_q$, then $\tilde{T} \leq T$.

**Proof.** A result of a generic perturbation would be that $\tilde{A} - \lambda \tilde{B}$ and $\tilde{C} - \mu \tilde{D}$ do not have any common eigenvalues and thus $\tilde{T} = 0$. In our analysis we therefore assume that the eigenvalues of perturbed pencils $\tilde{A} - \lambda \tilde{B}$ and $\tilde{C} - \mu \tilde{D}$ match in a way that gives the maximum possible $T$. 

7
Let \( e_1 \geq e_2 \geq \cdots \geq e_\ell \) be the sizes of blocks corresponding to \( \alpha \) in the KCF of the matrix pencil \( C - \mu D \). From Theorem 5 it follows that these blocks contribute

\[
T(\alpha) = \sum_{i=1}^{m} \sum_{j=1}^{\ell} \min(d_i, e_j)
\]

linearly independent vectors to \( \text{Ker}(A \otimes D - B \otimes C) \). If \( \alpha \) perturbs into \( \alpha' \) in the MLW move, then we assume that \( \alpha' \) is also an eigenvalue of \( \tilde{C} - \mu \tilde{D} \) and that the KCF of \( \tilde{C} - \mu \tilde{D} \) contains blocks \( J_{e_i}(\alpha') \) for \( q = 1, \ldots, \ell \).

What changes in the MLW move are sizes of the blocks. After the MLW move the blocks that correspond to \( \alpha' \) in \( \tilde{A} - \lambda \tilde{B} \) have sizes \( d'_1 \geq d'_2 \geq \cdots \geq d'_m \), where \( d'_p = d_p + 1 \), \( d'_q = d_q - 1 \) and \( d'_s = d_s \) for \( s \neq p, q \). It follows that

\[
T - \tilde{T} = \sum_{j=1}^{\ell} (\min(d_p, e_j) - \min(d_p + 1, e_j)) + \sum_{j=1}^{\ell} (\min(d_q, e_j) - \min(d_q - 1, e_j)).
\]

The difference in the first sum is 1 for indices where \( d_p < e_j \) and the difference in the second sum is -1 for indices where \( e_j > d_q \). Since \( d_p \geq d_q \), we clearly have \( \tilde{T} \leq T \), and if there exists \( e_j \) such that \( d_p \geq e_j \geq d_q \), then \( \tilde{T} < T \).

**Lemma 8.** Let \( A - \lambda B \) and \( C - \mu D \) be a pair of regular matrix pencils and let \( \tilde{A} - \tilde{\lambda} \tilde{B} \) and \( \tilde{C} - \tilde{\mu} \tilde{D} \) be their arbitrarily close perturbations such that \( B(\tilde{A} - \tilde{\lambda} \tilde{B}) \) is a HC cover of \( B(A - \lambda B) \) and \( B(\tilde{C} - \tilde{\mu} \tilde{D}) \) is a HC cover of \( B(C - \mu D) \). Then \( \tilde{T} \leq T \), where \( T = \dim(\text{Ker}(A \otimes D - B \otimes C)) \) and \( \tilde{T} = \dim(\text{Ker}(\tilde{A} \otimes \tilde{D} - \tilde{B} \otimes \tilde{C})) \).

**Proof.** A HC move splits an eigenvalue into two nearby eigenvalues. As only the eigenvalues that pencils \( A - \lambda B \) and \( C - \mu D \) have in common (including the infinite ones) contribute to \( T \), we only have to consider a situation when we apply a HC move to a common eigenvalue \( \alpha \).

If \( d_1 \geq \cdots \geq d_m \) and \( e_1 \geq \cdots \geq e_\ell \) are sizes of the blocks corresponding to \( \alpha \) in the KCF of \( A - \lambda B \) and \( C - \mu D \), respectively, then these blocks contribute \( T(\alpha) \) from (3) to \( T \). If we perform the HC move only to one of the pencils, then clearly \( \tilde{T} \leq T \). It is easy to see that we get maximum possible \( \tilde{T} \) when we apply the HC move to both pencils in such way that eigenvalue \( \alpha \) simultaneously splits into two common eigenvalues \( \alpha_1 \) and \( \alpha_2 \) of \( \tilde{A} - \tilde{\lambda} \tilde{B} \) and \( \tilde{C} - \tilde{\mu} \tilde{D} \).

It is enough to study the situation for just one pair of blocks. Suppose that the block \( J_d(\alpha) \) in \( A - \lambda B \) splits into blocks \( J_{d_1}(\alpha_1) \) and \( J_{d_2}(\alpha_2) \), and similarly, the block \( J_e(\alpha) \) in \( C - \mu D \) splits into blocks \( J_{e_1}(\alpha_1) \) and \( J_{e_2}(\alpha_2) \), such that \( d = d_1 + d_2 \) and \( e = e_1 + e_2 \). Without loss of generality we can assume that \( d \leq e \). Then

\[
\min(d_1, e_1) + \min(d_2, e_2) \leq d_1 + d_2 = d = \min(d, e).
\]

As the above inequality holds for all possible pairs of one block from \( A - \lambda B \) and the other one from \( C - \mu D \), it follows that \( \tilde{T} \leq T \).
4. Relation between the eigenvalues of \((W)\) and \((\Delta)\)

In the nonsingular case the eigenvalues of \((W)\) agree with the eigenvalues of the associated pair of generalized eigenvalue problems \((\Delta)\), see, e.g., [1]. In this section we show that in the singular case in a similar way the finite regular eigenvalues of \((W)\) are related to the finite regular eigenvalues of \((\Delta)\). While this was established for algebraically simple eigenvalues of \((W)\) with coprime polynomials \(p_1\) and \(p_2\) in [12], we extend the relation in this paper to all eigenvalues.

First we show that we can assume that the configuration of eigenvalues is as general as possible.

**Lemma 9.** It is always possible to apply a linear substitution of parameters \(\lambda\) and \(\mu\) in \((W)\) in such way that if \((\lambda_0, \mu_0) \in \mathbb{C}^2\) is an eigenvalue of \((W)\) as in Definition 1 then the following is true:

1. Matrix pencil \(W_i(\lambda_0, 0) - \gamma C_i\) is regular for \(i = 1, 2\).
2. Matrix pencil \(W_i(0, \mu_0) - \beta B_i\) is regular for \(i = 1, 2\).
3. In a small neighbourhood of \(\lambda_0\) it holds that \(\text{rank}(W_i(\eta, \mu_0)) = n_i\) for all \(\eta \neq \lambda_0\) and \(i = 1, 2\).
4. In a small neighbourhood of \(\mu_0\) if holds that \(\text{rank}(W_i(\lambda_0, \theta)) = n_i\) for all \(\theta \neq \mu_0\) and \(i = 1, 2\).
5. If \((\lambda_1, \mu_1) \neq (\lambda_0, \mu_0)\) is another eigenvalue of \((W)\) then \(\lambda_1 \neq \lambda_0\) and \(\mu_1 \neq \mu_0\).
6. In a small neighbourhood of \((\lambda_0, \mu_0)\) it holds for all \((\eta, \theta) \neq (\lambda_0, \mu_0)\) that

\[
\sum_{i=1}^{2} \text{rank}(W_i(\lambda_0, \mu_0)) < \sum_{i=1}^{2} \text{rank}(W_i(\eta, \theta)).
\]

**Proof.** We can apply a substitution in the form of an orthogonal rotation

\[
\begin{bmatrix}
\lambda \\
\mu
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
\tilde{\lambda} \\
\tilde{\mu}
\end{bmatrix} =: R(\phi)\begin{bmatrix}
\tilde{\lambda} \\
\tilde{\mu}
\end{bmatrix},
\]

where \(\phi \in [0, 2\pi)\).

1. We know that problem \((W)\) has less than \(n_1n_2\) eigenvalues. Therefore, for a generic \(\phi\) the rotation (4) leads to eigenvalues \((\tilde{\lambda}, \tilde{\mu})\) such that different eigenvalues differ in both components.
2. Matrix pencil \(W_i(\lambda_0, 0) - \gamma C_i\) is singular if \(p_i(\lambda_0, \gamma) \equiv 0\), which means that \(p_i(\lambda, \mu)\) is divisible by \(\lambda - \lambda_0\). As \(p_i\) for \(i = 1, 2\) can have only finitely many such divisors, none of them is present after a substitution (4) for a generic \(\phi\).
3. Similar as above.
4. The assumption is that \(W_i(\lambda, \mu)\) is regular, i.e., \(\text{nrank}(W_i(\lambda, \mu)) = n_i\) for \(i = 1, 2\). Eigenvalue \((\lambda_0, \mu)\) lies on one or more curves \(q_i(\lambda, \mu) = 0\), where \(q_i(\lambda, \mu)\) divides \(p_i(\lambda, \mu)\). If we use a generic rotation \(\phi\) than none of these curves has a vertical tangent at \((\lambda_0, \mu_0)\) and point 4. holds.
5. Similar as above.
6. This property does not depend on the substitution, but follows from the assumption that \((\lambda_0, \mu_0)\) is an isolated solution of \(p_1(\lambda, \mu) = p_2(\lambda, \mu) = 0\). Therefore, there exists a neighbourhood of \((\lambda_0, \mu_0)\) that does not include other eigenvalues and does not intersect with any curve \(q(\lambda, \mu) = 0\), where \(q(\lambda, \mu)\) is a common factor of \(p_1(\lambda, \mu)\) and \(p_2(\lambda, \mu)\).

**Theorem 10.** Let \((\lambda_0, \mu_0) \in \mathbb{C}^2\) be an eigenvalue of \((\text{I})\) and let all assumptions from Lemma 9 be satisfied. Then \((\lambda_0, \mu_0)\) is an eigenvalue of the associated problem \((\Delta)\).

**Proof.** In order to obtain a basis for \(\text{Ker}(\Delta_1 - \eta\Delta_0)\) from Theorem 5 we apply the equality

\[
\Delta_1 - \eta\Delta_0 = W_1(\eta, 0) \otimes C_2 - C_1 \otimes W_2(\eta, 0).
\]

It follows from item 3 in Lemma 9 that \(W_i(\eta, 0) - \gamma C_i\) is regular for \(\eta\) sufficiently close to \(\lambda_0\) and its KCF contains only blocks of types \(J_d\) and \(N_d\) for \(i = 1, 2\). We will show that rank\((\Delta_1 - \lambda_0\Delta_0)\) < rank\((\Delta_1 - \eta\Delta_0)\) for all \(\eta \neq \lambda_0\) sufficiently close to \(\lambda_0\).

We know from Theorem 5 that a basis for \(\text{Ker}(\Delta_1 - \lambda_0\Delta_0)\) is the union of sets of linearly independent vectors associated with the pairs of Kronecker blocks of the type \((J_d(\gamma_1), J_d(\gamma_2))\), where \(\gamma_1 = \gamma_2\), or \((N_d, N_d)\). If we change \(\lambda_0\) to \(\eta\) by an infinitesimally small perturbation, then \(W_i(\eta, 0) - \gamma C_i\) is still regular for \(i = 1, 2\). A basis for \(\text{Ker}(\Delta_1 - \eta\Delta_0)\) thus depends on pairs of Kronecker blocks of the type \((J_d(\gamma_1), J_d(\gamma_2))\), where \(\gamma_1 = \gamma_2\), or \((N_d, N_d')\), where \(J_d'\) and \(N_d'\) are blocks from the KCF of \(W_i(\eta, 0) - \gamma C_i\) for \(i = 1, 2\).

We now apply the stratification theory from Section 3. It follows that one gets from the bundle of \(W_i(\lambda_0, 0) - \gamma C_i\) to the bundle of \(W_i(\eta, 0) - \gamma C_i\) by a finite sequence of the moves MLW and HC introduced in Section 3. There exists a finite sequence of pairs of bundles \((B_{i,j}, B_{i,j+1})\), \(j = 0, \ldots, k\), such that \(B_{0,0} = B(W_i(\lambda_0, 0) - \gamma C_i), B_{0,k} = B(W_i(\eta, 0) - \gamma C_i),\) and either \(B_{i,j+1} = B_{i,j}\) or \(B_{i,j+1}\) covers \(B_{i,j}\) for \(i = 1, 2\) and \(j = 0, \ldots, k - 1\). In addition, the eigenvalues in \(B_{i,j+1}\) are arbitrarily close to the eigenvalues in \(B_{i,j}\), which means that, due to the continuity of the eigenvalues, \(B_{i,j+1}\) cannot have less distinct eigenvalues as \(B_{i,j}\).

Let us denote \(T_0 = \text{dim}(\text{Ker}(\Delta_1 - \lambda_0\Delta_0))\) and \(T_k = \text{dim}(\text{Ker}(\Delta_1 - \eta\Delta_0))\). We know from Theorem 5 that we can compute \(T_0\) and \(T_k\) from the canonical structures of \(W_i(\lambda_0, 0) - \gamma C_i\) and \(W_i(\eta, 0) - \gamma C_i\) for \(i = 1, 2\). In a similar way we define \(T_j\) for \(j = 1, \ldots, k - 1\) as the number of linearly independent vectors that we get for the pair of bundles \((B_{i,j}, B_{i,j+1})\) from matching blocks in \(B_{i,j}\) and \(B_{i,j+1}\) as in Theorem 5 where we assume that the eigenvalues of \(B_{i,j}\) match the eigenvalues \(B_{i,j+1}\) in a way that gives the maximum possible \(T_j\).

Lemmas 7 and 8 show that \(T_{j+1} \leq T_j\) for \(j = 0, \ldots, k - 1\). For a MLW move it is enough to consider the situation when we apply MLW only to one bundle in a pair as we can combine a sequence of two individual changes into one step. This is done in Lemma 7. On the other hand, if we apply a HC move then we assume that a HC move was applied to both bundles in such way that we get
the same way we can show that being an isolated solution of (1). Therefore, $T_k \leq T_0$.

We can also show that $T_k < T_0$. In the beginning, pencils $W_i(\lambda_0, 0) - \gamma C_i$, $i = 1, 2$, have a common eigenvalue $-\mu_0$ and combinations of the form $(J_{d_1}(-\mu_0), J_{d_2}(-\mu_0))$ contribute at least one linearly independent vector to $\ker(\Delta_1 - \lambda_0\Delta_0)$. In the end pencils $W_i(\eta_0, 0) - \gamma C_i$, $i = 1, 2$, do not have a common eigenvalue $\theta$ close to $-\mu$ as this would mean that $(\eta, -\theta)$ close to $(\lambda_0, \mu_0)$ and $\eta \neq \lambda_0$ is an eigenvalue of $(W)$ which is in contradiction with $(\lambda_0, \eta_0)$ being an isolated solution of (1). Therefore, $T_k < T_0$ and $\text{rank}(\Delta_1 - \lambda_0\Delta_0) < \text{rank}(\Delta_1 - \eta\Delta_0)$. This means that $\lambda_0$ is an eigenvalue of $\Delta_1 - \lambda\Delta_0$. In the same way we can show that $\mu_0$ is an eigenvalue of $\Delta_2 - \mu\Delta_0$.

What remains to show is that there exists a common regular eigenvector that satisfies Definition 2. If we take a nonzero $z \in \ker(W_1(\lambda_0, \mu_0)) \otimes \ker(W_2(\lambda_0, \mu_0))$, then it is clear from the above that $z \in \ker(\Delta_1 - \lambda_0\Delta_0) \cap \ker(\Delta_2 - \mu_0\Delta_0)$ but $z \notin \ker(\Delta_1 - \lambda_0\Delta_0)$ and $z \notin \ker(\Delta_2 - \mu_0\Delta_0)$. This shows that $z$ is a common regular eigenvector and $(\lambda_0, \mu_0)$ is an eigenvalue of $\Delta_0$.

Let us emphasize that Theorem 10 also covers the case when $p_1$ and $p_2$ have a nontrivial common factor. We showed that, regardless to the existence of a common factor, each eigenvalue of $(W)$ as defined in Definition 2 is an eigenvalue of $\Delta_0$. To show the implication in the other direction, we require that $p_1$ and $p_2$ are coprime. The following theorem is a generalization of Theorem 3.7 in [12] where it is in addition required that all eigenvalues of $(W)$ are algebraically simple.

**Theorem 11.** Let $(\lambda_0, \mu_0) \in \mathbb{C}^2$ be an eigenvalue of a pair of matrix pencils $(\Delta)$ associated to a (singular) two-parameter eigenvalue problem $(W)$ that satisfies all assumptions in Lemma 8. If the characteristic polynomials of $(W)$ are coprime, then $(\lambda_0, \mu_0)$ is an eigenvalue of $(\Delta)$.

**Proof.** As the proof is a slight modification of the proof of Theorem 3.7 in [12], we will omit most of the details. We start with a nonzero $z \in \ker(\Delta_1 - \lambda_0\Delta_0) \cap \ker(\Delta_2 - \mu_0\Delta_0)$, which is a regular eigenvector for the eigenvalue $(\lambda_0, \mu_0)$ and apply the relation

$$\Delta_1 - \lambda\Delta_0 = W_1(\lambda, 0) \otimes C_2 - C_1 \otimes W_2(\lambda, 0). \quad (6)$$

If follows from Theorem 5 that $z$ is a linear combination of vectors associated with pairs $(J_{d_1}(\alpha), J_{d_2}(\alpha))$ or $(N_{d_1}, N_{d_2})$ of pencils $W_1(\lambda, 0) - \gamma C_1$ and $W_2(\lambda, 0) - \gamma C_2$. All other combinations from Theorem 5 are not possible due to Lemma 9.

In a similar way as in [12] we can show that $z$ cannot be a linear combination of vectors solely from the combinations of type $(N_{d_1}, N_{d_2})$. Namely, if this would be true, then $z$ would belong to the minimal reducing subspace $R(\Delta_2, \Delta_0)$ which is in contradiction with $z$ being a regular eigenvector for the eigenvalue $\mu_0$ of $\Delta_2 - \mu\Delta_0$. 

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Therefore, \( z \) must include a contribution from at least one Kronecker pair of type \((J_{d_1}(\gamma_0), J_{d_2}(\gamma_0))\), where \( \gamma_0 \) is a common eigenvalue of \( W_1(\lambda, 0) - \gamma C_1 \) and \( W_2(\lambda, 0) - \gamma C_2 \). As the characteristic polynomials of \( (W) \) are coprime, it follows that \((\lambda_0, -\gamma_0)\) is an eigenvalue of \( (W) \) and then by Theorem \[ \text{Theorem 10} \] \((\lambda_0, -\gamma_0)\) is also an eigenvalue of \((\Delta)\). In a similar way we can show that there exists an eigenvalue \((-\beta_0, \mu_0)\) of \( (W) \) and then \((-\beta_0, \mu_0)\) is an eigenvalue of \((\Delta)\) as well. Since \( (W) \) has only finitely many eigenvalues, it follows that \( (\Delta) \) has finitely many eigenvalue as well.

A generic substitution \[(4)\] used in the proof of Lemma \[ \text{Lemma 9} \] affects the problem \((\Delta)\) and its eigenvalues as well. Thus we can assume that item 5 from Lemma \[ \text{Lemma 9} \] holds for the eigenvalues of \((\Delta)\) as well. But, then the only option is that \( \gamma_0 = -\mu_0 \) and \( \beta_0 = -\lambda_0 \) which means that \((\lambda_0, \mu_0)\) is an eigenvalue of \((W)\).

In numerical experiments we often noticed that pencils \( \Delta_1 - \lambda \Delta_0 \) and \( \Delta_2 - \mu \Delta_0 \) belong to the same bundle, i.e., they have the same canonical structure regarding blocks and their sizes. The following lemma explains the observation.

**Lemma 12.** Let us assume that we applied a generic orthogonal rotation \[(4)\] to a two-parameter eigenvalue problem \((W)\). Then the associated matrix pencils \( \Delta_1 - \lambda \Delta_0 \) and \( \Delta_2 - \mu \Delta_0 \) belong to the same bundle.

**Proof.** We consider a family of pencils

\[
\cos \varphi \Delta_1 + \sin \varphi \Delta_2 - \lambda \Delta_0.
\]

As \( \varphi \) goes from 0 to \( 2\pi \), pencil \[(7)\] moves between bundles and returns back to the initial bundle at \( \varphi = 2\pi \). For all except finitely many values \( \varphi \) pencil \[(7)\] belongs to the same bundle, which we can call the generic bundle for \[(7)\]. Since pencils \( \Delta_1 - \lambda \Delta_0 \) and \( \Delta_2 - \mu \Delta_0 \) are just special cases of \[(7)\] for \( \varphi = 0 \) and \( \varphi = \pi/2 \), respectively, they generically belong to the same bundle.

**Example 13.** We take example 4.4 from \[12\], where

\[
W_1(\lambda, \mu) = \begin{bmatrix} 2 + \lambda & 1 + 2\lambda & \lambda \\ \lambda & 2 + 2\lambda + 2\mu & \mu \\ \mu & 1 + 2\mu & 2 + \mu \end{bmatrix}
\]

and

\[
W_2(\lambda, \mu) = \begin{bmatrix} 1 + \lambda & 1 + 2\lambda & \lambda \\ \lambda & 1 + 2\lambda + 2\mu & \mu \\ \mu & 1 + 2\mu & 1 + \mu \end{bmatrix}.
\]

It turns out (for details see \[12\]) that \((\lambda_1, \mu_1) = (-\frac{1}{2}, -\frac{1}{2})\) is a quadruple eigenvalue of both \((W)\) and \((\Delta)\), but due to the multiplicity an implication in either side could not be deduced from the theory available at the time. The extended theory in this paper covers the example. As the characteristic polynomials of \( W_1 \) and \( W_2 \) are coprime, we can apply Theorems \[10\] and \[11\] to establish that the eigenvalues of \((W)\) and \((\Delta)\) do agree.
Example 14. If we apply linearization 1 from [13] to bivariate polynomials

\[ p_1(\lambda, \mu) = (x^2 + y^2 - 2)(x + y - 1), \]
\[ p_2(\lambda, \mu) = (x^2 + y^2 - 2)(x - y + 1), \]  
we get determinantal representations

\[ W_1(\lambda, \mu) = \begin{bmatrix} 2 - 2\lambda - 2\mu & -\lambda & \lambda + \mu & 0 & \lambda + \mu - 1 \\ \lambda & -1 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 \\ \mu & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mu & -1 \end{bmatrix} \]

and

\[ W_2(\lambda, \mu) = \begin{bmatrix} -2 - 2\lambda + 2\mu & \lambda & \lambda - \mu & 0 & \lambda - \mu + 1 \\ \lambda & -1 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 \\ \mu & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mu & -1 \end{bmatrix} \]

such that \( \det(W_i(\lambda, \mu)) = p_i(\lambda, \mu) \) for \( i = 1, 2 \). The obtained two-parameter eigenvalue problem is singular by construction and, in addition, \( p_1 \) and \( p_2 \) have a common nontrivial factor. Theorem 10 still holds for the eigenvalue \( (\lambda_1, \mu_1) = (0, 1) \) of the corresponding problem \((W)\), which is an isolated solution of \( p_1(\lambda, \mu) = p_2(\lambda, \mu) = 0 \). Point \( (\lambda_1, \mu_1) \) is the only eigenvalue of the corresponding problem \((\Delta)\) and in this case eigenvalues of \((W)\) and \((\Delta)\) agree as well.

In the next section we explore the situation where \((\Delta)\) has an eigenvalue on the common nontrivial factor of \( \det(W_1(\lambda, \mu)) \) and \( \det(W_2(\lambda, \mu)) \).

5. Points on a nontrivial common curve

Suppose that we have a singular two-parameter eigenvalue problem where pencils \( W_1(\lambda, \mu) \) and \( W_2(\lambda, \mu) \) are both regular and characteristic polynomials \( p_1 \) and \( p_2 \) have a common nontrivial factor. For such problems we know from Theorem 10 that if \( (\lambda_0, \mu_0) \) is an eigenvalue of \((W)\) then \( (\lambda_0, \mu_0) \) is also an eigenvalue of \((\Delta)\). In the other direction, an eigenvalue of \((\Delta)\) is just a candidate for an eigenvalue of \((W)\). We will show that a point \( (\lambda_0, \mu_0) \) that belongs to a common factor of \( p_1 \) and \( p_2 \) can be an eigenvalue of \((\Delta)\), but it is not clear if it is possible to extend Definition 1 to include points on the common factors as well.

Example 15. We take a two-parameter eigenvalue problem

\[ W_1(\lambda, \mu) = \begin{bmatrix} \lambda + \mu \\ \lambda - \mu \end{bmatrix}, \quad W_2(\lambda, \mu) = \begin{bmatrix} \lambda + \mu - 2 \\ \lambda - \mu \end{bmatrix}. \]
The problem is clearly singular since $\lambda - \mu$ is a common factor of characteristic polynomials $p_1$ and $p_2$. Matrices of the corresponding problem $\Delta$ are

$$
\Delta_0 = \begin{bmatrix}
0 & -2 \\
-2 & 2 \\
0 & 2
\end{bmatrix}, \quad \Delta_1 = \begin{bmatrix}
-2 & 0 \\
0 & 2 \\
0 & 0
\end{bmatrix}, \quad \Delta_2 = \begin{bmatrix}
2 & 0 \\
0 & 2 \\
0 & 0
\end{bmatrix}.
$$

The canonical forms of $\Delta_1 - \lambda \Delta_0$ and $\Delta_2 - \mu \Delta_0$ are both equal to $L_0 \oplus L_0^T \oplus J_1(1) \oplus J_1(0) \oplus N_1$ and thus have the same structure as predicted by Lemma 12. The minimal reducing subspace, which is the same for both pencils, is $\text{Lin}(e_4)$.

Problem $\Delta$ has two eigenvalues $(0, 0)$ and $(1, 1)$ with regular eigenvectors $e_2$ and $e_3$, respectively.

Both points $(0, 0)$ and $(1, 1)$ lie on the zero set of the common factor curve $\lambda - \mu = 0$. If we look at a situation at the point $(0, 0)$, we can see that the KCF of $W_1(0, 0) - \gamma C_1$ and $W_2(0, 0) - \gamma C_2$ is $2 J_1(0)$ and $J_1(-2) \oplus J_1(0)$ respectively. This gives $\dim(\ker(\Delta_1)) = 2$ by Theorem 5. If we perturb pencils into $W_1(\varepsilon, 0) - \gamma C_1$ and $W_2(\varepsilon, 0) - \gamma C_2$, then the KCF changes to $J_1(\varepsilon) \oplus J_1(-\varepsilon)$ and $J_1(-\varepsilon) \oplus J_1(-2 + \varepsilon)$, respectively, which gives $\dim(\ker(\Delta_1 - \varepsilon \Delta_0)) = 1$. This means that $\lambda_0 = 0$ is an eigenvalue of $\Delta_1 - \lambda \Delta_0$. In a similar way we could show that $\mu_0 = 0$ is an eigenvalue of $\Delta_2 - \mu \Delta_0$ and then finally check that $e_2$ is a regular eigenvector for $(0, 0)$.

Using the same approach we could show that $(1, 1)$ is an eigenvalue of $(\Delta)$. Both points $(0, 0)$ and $(1, 1)$ are intersections of a common curve and another curve from $p_1$ or $p_2$. This might look like a good condition that makes a point an eigenvalue of $(\Delta)$, but the next example immediately gives a counterexample for this assumption.

**Example 16.** We slightly change the first pencil from Example 15 and use

$$
W_1(\lambda, \mu) = \begin{bmatrix}
\lambda + \mu & 1 \\
\lambda - \mu & 1
\end{bmatrix}, \quad W_2(\lambda, \mu) = \begin{bmatrix}
\lambda + \mu - 2 & -1 \\
\lambda - \mu & 1
\end{bmatrix}.
$$

Point $(0, 0)$ is still an intersection of the common curve $\lambda = \mu$ and curve $\lambda = -\mu$ from $W_1$, but now $(0, 0)$ is not an eigenvalue of the corresponding problem $\Delta$, whose matrices are

$$
\Delta_0 = \begin{bmatrix}
0 & -2 \\
-2 & 2 \\
0 & 0
\end{bmatrix}, \quad \Delta_1 = \begin{bmatrix}
-2 & 0 \\
0 & 2 \\
0 & 0
\end{bmatrix}, \quad \Delta_2 = \begin{bmatrix}
2 & 1 \\
0 & 2 \\
0 & 0
\end{bmatrix}.
$$

The KCF of $\Delta_1 - \lambda \Delta_0$ is $L_0 \oplus L_0^T \oplus J_1(1) \oplus N_1$ for $i = 1, 2$. If we look at a situation at $(0, 0)$, we see that the KCF of $W_1(0, 0) - \gamma C_1$ is $J_2(0)$ and the KCF of $W_1(\varepsilon, 0) - \gamma C_1$ is $J_1(\varepsilon) \oplus J_1(-\varepsilon)$, while canonical forms related to $W_2$ are same as in Example 15. We get $\dim(\ker(\Delta_1)) = \dim(\ker(\Delta_1 - \varepsilon \Delta_0)) = 1$, which confirms that 0 is not an eigenvalue of $\Delta_1 - \lambda \Delta_0$. 

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In the above two examples we were investigating an intersection point between the common factor and a curve belonging to one of the pencils. The examples show that this alone is not enough for a point to be an eigenvalue of \( \Delta \). In the following example situation is even more complicated as \( p_1 = p_2 \).

**Example 17.** We take

\[
p(x) = x^2 + 2x + 1.
\]

Now \( p(x) = p(y) = (x+y)^2 \). It turns out that \( (0, 0) \) is a double eigenvalue of \( \Delta \), whose matrices are

\[
\Delta_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta_1 = 0, \quad \Delta_2 = 0.
\]

The KCF of \( \Delta - \lambda \Delta_0 \) is \( 2L_0 \oplus 2L_0^T \oplus 2J_1(0) \) for both \( i = 1, 2 \). The minimal reducing subspace, which is the same for both pencils, has dimension 2 and is spanned by \( e_1 \) and \( e_2 \), while \( e_3 \) and \( e_4 \) are regular eigenvectors for the eigenvalue \( (0, 0) \).

If we look at a situation at \( (0, 0) \), we see that the KCF of \( W_i(0, 0) - \gamma C_i \) is \( 2J_1(0) \) for \( i = 1, 2 \). The perturbed pencils \( W_i(\varepsilon, 0) - \gamma C_i \) for \( i = 1, 2 \) have canonical forms \( J_2(\varepsilon) \) and \( 2J_1(\varepsilon) \), respectively. This shows that \( 4 = \dim(\ker(\Delta_1)) > \dim(\ker(\Delta_1 - \varepsilon \Delta_0)) = 2 \) and confirms that \( 0 \) is a double eigenvalue of \( \Delta_1 - \lambda \Delta_0 \).

In a similar way we show that \( 0 \) is a double eigenvalue of \( \Delta_2 - \mu \Delta_0 \). As \( e_3 \) and \( e_4 \) are regular eigenvectors for both pencils, \( (0, 0) \) is a double eigenvalue of \( \Delta \).

If we look back to the pencils \( W_1 \) and \( W_2 \) from [W] and try to figure out what makes point \( (0, 0) \) an eigenvalue, we can observe that \( 0 = \text{rank}(W_1(0, 0)) < \text{rank}(W_1) = 2 \), \( 0 = \text{rank}(W_2(0, 0)) < \text{rank}(W_2) = 2 \), and \( \text{rank}(W_1(0, 0)) + \text{rank}(W_2(0, 0)) < \text{rank}(W_1(0, 0)) + \text{rank}(W_2(0, 0)) + \text{rank}(W_2(0, 0)) \) for all \( (\theta, \eta) \neq (0, 0) \). All other points \( (\lambda_0, -\lambda_0) \), where \( \lambda_0 \neq 0 \), on the common factor satisfy \( \text{rank}(W_1(\lambda_0, -\lambda_0)) < \text{rank}(W_1) \) and \( \text{rank}(W_2(\lambda_0, -\lambda_0)) < \text{rank}(W_2) \), but fail to satisfy the condition \( \text{rank}(W_1(\lambda_0, -\lambda_0)) + \text{rank}(W_2(\lambda_0, -\lambda_0)) < \text{rank}(W_1(\lambda_0, -\lambda_0)) + \text{rank}(W_2(\lambda_0, -\lambda_0)) \) for all \( (\theta, \eta) \neq (\lambda_0, -\lambda_0) \) close to \( (\lambda_0, -\lambda_0) \). But, as we will see in the next example, this alone is not enough for a point to be an eigenvalue.

**Example 18.** We slightly change the second pencil from Example [17] into

\[
W_1(\lambda, \mu) = \begin{bmatrix} \lambda + \mu & \lambda \\ \lambda & \lambda + \mu \end{bmatrix}, \quad W_2(\lambda, \mu) = \begin{bmatrix} \lambda + \mu & 1 \\ \lambda & \lambda + \mu \end{bmatrix}.
\]

The result of the change is that \( \Delta \) has no eigenvalues. The corresponding \( \Delta \) matrices are

\[
\Delta_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.
\]
The KCF of \(\Delta_1 - \lambda \Delta_0\) is \(L_0 \oplus L_1 \oplus L_0^T \oplus L_1^T\) for \(i = 1, 2\).

We can observe that \(\text{rank}(W_1(0, 0)) < \text{rank}(W_1), \text{rank}(W_2(0, 0)) < \text{rank}(W_2),\)
and \(\text{rank}(W_1(0, 0)) + \text{rank}(W_2(0, 0)) < \text{rank}(W_1(\theta, \eta)) + \text{rank}(W_2(\theta, \eta))\) for all \((\theta, \eta) \neq (0, 0)\), which makes a point \((0, 0)\) special, but not special enough to be an eigenvalue.

**Example 19.** In the last example \(0\) is an eigenvalue of \(\Delta_1 - \lambda \Delta_0\) for \(i = 1, 2\), but there does not exist a common regular eigenvector that would make \((0, 0)\) an eigenvalue of \(\Delta\). We take

\[
W_1(\lambda, \mu) = \begin{bmatrix}
\lambda + \mu & 1 & \lambda \\
\lambda + \mu & 1 & \lambda + \mu \\
\lambda + \mu & 1 & \lambda + \mu
\end{bmatrix},
\]

\[
W_2(\lambda, \mu) = \begin{bmatrix}
\lambda + \mu & 1 & \lambda + \mu \\
\lambda + \mu & 1 & \lambda + \mu \\
\lambda + \mu & 1 & \lambda + \mu
\end{bmatrix}.
\]

The KCF of \(\Delta_1 - \lambda \Delta_0\) (we leave the computation of the corresponding \(\Delta\) matrices of size \(16 \times 16\) to the reader) is \(6L_0^T \oplus 2L_1 \oplus 4L_0 \oplus 2J_1(0) \oplus 6N_1\) for \(i = 1, 2\).

To obtain the generic kernel and the minimal reducing subspace for the pencil \(\Delta_1 - \lambda \Delta_0\) we study a pair of matrix pencils \(W_i(\lambda, 0) - \gamma C_i\) such that \(\lambda \neq 0\) for \(i = 1, 2\). The KCF of \(W_1(\lambda, 0) - \gamma C_1\) has a block \(J_1(\lambda)\) with a Kronecker chain \(\lambda e_1, e_3, e_4\) and a block \(J_1(\lambda)\) with eigenvector \(-\lambda e_2 + e_3\). The KCF of \(W_2(\lambda, 0) - \gamma C_2\) has two blocks \(J_2(\lambda)\) with chains \(e_1, e_2\) and \(e_3, e_4\). It follows from Theorem 5 that the generic kernel of \(\Delta_1 - \lambda \Delta_0\) has dimension 6 and \(\text{GKer}(\Delta_1 - \lambda \Delta_0) = \text{Lin}(e_1 \otimes e_1, e_1 \otimes e_3, e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_4 + e_2 \otimes e_3, \lambda e_1 \otimes e_2 + e_3 \otimes e_1, \lambda e_1 \otimes e_4 + e_3 \otimes e_3)\).

In a similar way we get that the generic kernel of \(\Delta_2 - \mu \Delta_0\) has dimension 6 as well and \(\text{GKer}(\Delta_2 - \mu \Delta_0) = \text{Lin}(e_1 \otimes e_1, e_1 \otimes e_3, e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_4 + e_2 \otimes e_3, -\mu e_1 \otimes e_2 + e_3 \otimes e_1, -\mu e_1 \otimes e_4 + e_3 \otimes e_3)\). This shows that both pencils have the same minimal reducing subspace. \(R(\Delta_1, \Delta_0) = R(\Delta_2, \Delta_0) = \text{Lin}(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_1 \otimes e_4, e_2 \otimes e_1, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_3)\).

In point \(\lambda = 0\) and \(\mu = 0\) both pencils have the same generic kernel \(G := \text{Lin}(e_1 \otimes e_1, e_1 \otimes e_3, e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_4 + e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_3)\). But, although nullspaces \(\text{Ker}(\Delta_1)\) and \(\text{Ker}(\Delta_2)\) both have dimension 8, they intersect in the common generic kernel and do not have a common regular vector. A detailed analysis shows that \(\text{Ker}(\Delta_1) = G \oplus \text{Lin}(e_3 \otimes e_2 + e_4 \otimes e_1, e_3 \otimes e_4 + e_4 \otimes e_3)\) and \(\text{Ker}(\Delta_2) = G \oplus \text{Lin}(-e_1 + e_3) \otimes e_2 + e_4 \otimes e_1, (-e_1 + e_3) \otimes e_4 + e_4 \otimes e_3)\), therefore \(\text{Ker}(\Delta_1) \cap \text{Ker}(\Delta_2) = G\) and \((0, 0)\) is not an eigenvalue of \(\Delta\).

### 6. Conclusion

We extend the equivalence of eigenvalues of a two-parameter eigenvalue problem \((W)\) and the eigenvalues of the associated pair of generalized eigenvalue
problem $\Delta$ to all singular problems where pencils $W_1(\lambda, \mu)$ and $W_2(\lambda, \mu)$ are regular and $\det(W_1(\lambda, \mu))$ and $\det(W_2(\lambda, \mu))$ do not have a nontrivial common factor. This was first established for nonsingular problems by Atkinson in [1] and then generalized in [12] for singular problems such that all eigenvalues are algebraically simple.

For the case where $\det(W_1(\lambda, \mu))$ and $\det(W_2(\lambda, \mu))$ do have a nontrivial common factor, we show the implication in one direction, which makes the eigenvalues of $\Delta$ candidates for the eigenvalues of $W$. We give several examples in Section 5 and, while the eigenvalues of $\Delta$ are well defined regardless to the presence of common factors, it is still open how to extend the definition of eigenvalues of $W$ to cover the points from the common factors so that they would agree to the eigenvalues of $\Delta$.

Based on numerical experiments we believe that it should be possible to generalize Theorems 10 and 11 to singular multiparameter eigenvalue problems with more than two parameters as well. However, the proof would have to be based on a different approach as Theorem 5 which is our key tool, cannot be generalized to three or more parameters in an elegant way.

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