A mixed complementarity problem approach for steady-state voltage and frequency stability analysis

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Abstract—We present a mixed complementarity problem (MCP) approach for a steady-state stability analysis of voltage and frequency of electrical grids. We perform theoretical analysis providing conditions for the global convergence and local quadratic convergence of our solution procedure, enabling fast computation time. Moreover, algebraic equations for power flow, voltage control, and frequency control are compactly and incrementally formulated as a single mixed complementarity problem that subsequently is solved by a highly efficient and robust solution method. Experimental results over large grids demonstrate that our approach is as fast as the existing Newton method with heuristics while showing a more robust performance.

Index Terms—power flow, voltage and frequency regulation, mixed complementarity

I. INTRODUCTION

The growing penetration of renewable energy resources into the electrical grid and the increasing inter-dependency between the natural gas network and the grid draw attention to the impact of these components on grid stability. Unlike traditional generators, the intermittency and lack of reactive power generation capability of renewable energy resources can significantly degrade grid stability. Natural gas as the largest source of U.S. electricity generation [1] also can have a huge impact on stable operations of the grid as we have seen in the Texas power outage in its winter storm of early 2021.

One of the key measures for assessing grid stability is a steady-state power flow analysis with voltage regulation and frequency control. It basically computes a solution to alternating current power flows of the grid formulated as a nonlinear system of equations and that also satisfies several complicated conditions modeling the interactions between grid components to implement those regulations. An example of such conditions is voltage set point control via reactive power, which regulates voltage magnitude to stay at its set point by controlling its corresponding reactive power.

Computationally, these additional conditions make it difficult to directly apply the Newton-Raphson (NR) [2], which has been the norm for the conventional power flow analysis without regulations. Heuristics have been developed [3], [4] that fix some variable values and switch the bus type (e.g., PV-PQ switching) at each iteration of the NR method so that it can still be used to solve the reduced system of equations. However, they can potentially cause numerical issues [4] leading to divergence. The lack of theoretical analysis on conditions that guarantee the convergence of these heuristics further complicates the issue to avoid such divergence.

Recent works [5]–[7] introduce complementarity constraints \(0 \leq a \perp b \geq 0\) (which denotes \(ab = 0, a, b \geq 0\)) to directly incorporate regulation conditions in the model. They then transform the problem into a computationally more tractable form via equation reformulation using the Fischer-Burmeister function defined by \(\phi(a, b) = \sqrt{a^2 + b^2} - a - b\) so that \(0 \leq a \perp b \geq 0 \iff \phi(a, b) = 0\). Especially, [6] showed global convergence results that lay grounds for a numerically more robust performance than that of the NR method, as demonstrated in [6]. The computation time of this approach, however, is usually much slower than that of the NR method with heuristics, as also shown in [6], [7]. The reason is partly that the solution procedure does not exploit complementarity structure since it is obscured by reformulations into equations, resulting in no special treatment possible for it.

In this paper we present that such problems can be cast as a mixed complementarity problem (MCP) with a proper formulation as will be described in Section III. In particular, we can leverage the convergence theory and solution methods for MCPs to efficiently tackle our problem. We show sufficient conditions for obtaining global convergence as well as local quadratic convergence that enables fast computation time. Such a fast convergence is made through the solution method for MCP [8], [9] that specifically utilizes the complementarity structure. Similar to the NR method, at each iteration it linearizes the problem at the current iterate, however, instead of solving a system of equations it solves a linear complementarity problem (LCP), which provides a first-order approximation to the original MCP. Numerical experiments in Section IV demonstrate as fast computation time of our MCP approach as that of the NR method, while showing a more robust performance.

The rest of the paper is organized as follows. In Section II we briefly introduce MCP formulations, strong regularity, and their solution method. Section III introduces our MCP formulations for a steady-state power flow analysis with voltage
and frequency regulations. Numerical results are presented in Section IV, and we conclude the paper in Section V.

II. BACKGROUND OF MIXED COMPLEMENTARITY PROBLEMS

In this section we briefly introduce MCPs, its solution method based on the generalized equation [10], and conditions that guarantee local superlinear or quadratic convergence.

An MCP \((B, F)\) is defined by a vector-valued continuous function \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) and a box constraint \(B = [l, u] = \prod_{i=1}^n [l_i, u_i]\) with \(l_i \leq u_i\) and \(l_i, u_i \in \mathbb{R} \cup \{-\infty, +\infty\}\) for \(i = 1, \ldots, n\). We note that \(F\) is a square system of equations, that is, it has the same number of variables and equations. We say that \(x\) is a solution to the MCP \((B, F)\) if it satisfies one of the following three conditions for each \(i = 1, \ldots, n\):

\[
\begin{align*}
  x_i & = l_i \quad & F_i(x) & \geq 0 \\
  l_i & \leq x_i \leq u_i \quad & F_i(x) & = 0 \\
  x_i & = u_i \quad & F_i(x) & \leq 0
\end{align*}
\]

We use a complementarity notation \(l_i \leq x_i \leq u_i \perp F_i(x)\) to denote (1). Its vector form \(l \leq x \leq u \perp F(x)\) implies that (1) holds component-wise.\(^1\)

MCPs subsume many different problem classes. When \(B = \mathbb{R}^n\), the MCP \((B, F)\) becomes a square system of nonlinear equations seeking \(x \in \mathbb{R}^n\) that satisfies \(F(x) = 0\). The Karush-Kuhn-Tucker (KKT) conditions of an optimization problem \(\min_{x \in \mathbb{R}^n} f(x)\) s.t. \(c(x) = 0\) can be formulated as an MCP \((B, F)\) with \(B = [l, u] \times \mathbb{R}^m\) and \(F(x, \lambda) = ((\nabla f(x) + \nabla c(x)\lambda)^T, (c(x))^T)^T \in \mathbb{R}^{n+m}\) where \(c(x), \lambda \in \mathbb{R}^m\). Many other different problem classes including generalized Nash equilibrium problems and quasi-variational inequalities can be also formulated as MCPs. We refer to [11], [12] for more details.

The generalized equation (GE) provides a machinery to solve MCPs [8], [13]. It is defined by

\[
0 \in F(x) + N_B(x)
\]

where \(N_B(x)\) is a normal map to \(B\) at \(x \in B\) defined by \(N_B(x) := \{v \mid \langle v, y - x \rangle \leq 0, \forall y \in B\}\) and \(N_B(x) := \emptyset\) when \(x \notin B\). One can easily verify that \(x\) satisfies (2) if and only if it satisfies (1). A Newton’s method [8], [13] finds a solution to the GE by iteratively linearizing and solving the linearized GE, where it becomes an LCP in this case with \(F\) being an affine function. This method is similar to the NR method for a nonlinear system of equations, however, it takes into account complementarity at its linearization, and its complementary pivoting solution procedure effectively exploits the complementarity structure in finding a solution. We will see its fast and robust computational performance in Section IV.

\(^1\)As a variation, we may use the notation \(l_i \leq x_i \perp F_i(x)\geq 0\) when \(u_i = \infty\). This emphasizes that \(F_i(x)\) should be nonnegative at a solution since the third condition of (1) cannot hold in this case. Similarly, we may denote \(x_i \leq u_i \perp F_i(x)\leq 0\) when \(l_i = -\infty\). If \(l_i = -\infty\) and \(u_i = \infty\), we may omit both bounds and simply denote \(x_i \perp F_i(x)\).

Table I: Variables fixed/free at each bus \(i\) according to its type

| Bus type | Variables fixed | Variables free |
|----------|----------------|----------------|
| Slack    | \(v_i, \delta_i\) | \(p_{gi}, q_{gi}\) |
| PQ bus   | \(p_{gi}, q_{gi}\) | \(v_i, \delta_i\) |
| PV bus   | \(p_{gi}, q_{gi}\) | \(v_i, \delta_i\) |

A key condition to define the domain of attraction for local convergence to a solution \(x^*\) of (2) is strong regularity defined below:

**Definition 1** (Strong regularity [14]). Let \(x^*\) be a solution to (2). Let \(T := LF_{x^*} + L_N\), where \(LF_{x^*} := F(x^*) + \nabla F(x^*)(x - x^*)\). We say that \(x^*\) is strongly regular if and only if there exist neighborhoods \(U\) of the origin and \(V\) of \(x^*\) such that the restriction to \(U\) of \(T^{-1} \cap V\) is a single-valued Lipschitzian function from \(U\) to \(V\).

We note that for a system of nonlinear equations \(F(x) = 0\) strong regularity reduces to the condition that \(\nabla F(x^*)\) has a continuous linear inverse map. This condition is a typical assumption for the local convergence of the NR method. In Section III we present conditions for local convergence of our power flow analysis with voltage and frequency regulations via MCPs.

The following shows an asymptotic convergence rate of the Newton’s method [8], [13] for the GE under strong regularity.

**Theorem 1** (Local Q-quadratic convergence [13]). For a given MCP \((B, F)\) with \(B \neq \emptyset\), suppose that \(x^*\) is a strongly regular solution of it, and \(\nabla F\) is locally Lipschitz continuous. Then for \(x^*\) near \(x^*\), the sequence generated by solving LCPs is locally Q-quadratically convergent to \(x^*\).

III. MIXED COMPLEMENTARITY FORMULATIONS FOR A UNIFIED POWER FLOW ANALYSIS

This section presents MCP formulations that encapsulate power flow equations, voltage regulation, and frequency control in an incremental fashion. Starting with power flow equations in Section III-A, we incrementally build MCP formulations in Sections III-B–III-D, each of which models a specific regulation and control feature. At the end of this section, we describe the global and local convergence results. We note that complementarity constraints for voltage and frequency regulations are known in the literature, but our work is the first showing these also can be compactly formulated as a single MCP, enabling us to utilize its convergence theory and solution method.

A. Modeling power flow equations

In the conventional power flow analysis, we are interested in finding voltage magnitudes and angles of the grid for a given set of parameters such as load, real power generation, voltage magnitudes at regulated buses, and network coefficients. To
achieve this, we solve the following system of nonlinear equations defined at each bus $i = 1, \ldots, N$:

$$
P_i(x) = p_i^{\text{ij}} - \sum_{k=1}^{N} v_i v_k (G_{ik} \cos(\delta_{ik}) + B_{ik} \sin(\delta_{ik}))$$

$$
Q_i(x) = q_i^{\text{ij}} - \sum_{k=1}^{N} v_i v_k (G_{ik} \sin(\delta_{ik}) - B_{ik} \cos(\delta_{ik}))
$$

(3)

where $p_i^{\text{ij}} := p_{gi} - P_{di}$ and $q_i^{\text{ij}} := q_{gi} - Q_{di}$ are the net real and reactive power injections at bus $i$ with $p_{gi}$ and $q_{gi}$ being real and reactive power generated at the bus and $P_{di}$ and $Q_{di}$ denoting its real and reactive loads. Voltage magnitude and angle at bus $i$ are denoted by $V_i$ and $\delta_i$, respectively, and the angle difference is defined by $\delta_i := \delta_i - \delta_k$. We denote by $x := ([p_{gi}, q_{gi}, v_i, \delta_i]_{i=1}^{N})$ the encapsulation of all variables. $G_{ik}$ and $B_{ik}$ are network coefficients computed from a nodal admittance matrix and bus shunt values. Later in Section III-C they may be changed to variables for voltage regulation.

When solving (3), some components of variable $x$ are fixed depending on the bus type as described in Table I. Since we are interested mainly in voltage magnitudes and angles, we solve for equations that involve these voltage values as variables. These equations correspond to the real and reactive power flow equations at PQ buses and real power flow equations at PV buses. Therefore the conventional power flow analysis solves the following square system of nonlinear equations:

$$
\begin{bmatrix}
P_{\text{PQ}}(x) \\
Q_{\text{PQ}}(x) \\
P_{\text{PV}}(x)
\end{bmatrix} = 0
$$

(4)

where we use the notation PQ or PV in the subscript to indicate the set of bus indices belonging to the indicated bus type. We note that variables in (4) are $\delta_{\text{PQ}}, \nu_{\text{PQ}},$ and $\delta_{\text{PV}}$. Using the fact that variables are set to be free variables in (4), we can formulate (4) as an MCP as follows:

$$
\begin{align*}
\delta_{\text{PQ}} & \perp P_{\text{PQ}}(x) \\
\nu_{\text{PQ}} & \perp Q_{\text{PQ}}(x) \\
\delta_{\text{PV}} & \perp P_{\text{PV}}(x)
\end{align*}
$$

(5)

From (1), one can easily verify that $x$ solves (4) if and only if it solves (5).

B. Modeling generator voltage control

Generator voltage control can be compactly formulated by using the following MCP formulation:

$$
g_{\text{g PV}}^{\min} \leq q_{\text{g PV}} \leq q_{\text{g PV}}^{\max} \perp v_{\text{PV}} - v_{\text{PV}}^{\text{sp}}
$$

(6)

where $v_{i}^{\text{sp}}$ denotes the set point of the voltage magnitude for bus $i \in \text{PV}$. From (1), we see that $v_i = v_i^{\text{sp}}$ when $q_{\text{g min}} < q_{gi} < q_{\text{g max}}$, $v_i \geq v_i^{\text{sp}}$ for $q_{gi} = q_{\text{g min}}^{\text{min}}$, and $v_i \leq v_i^{\text{sp}}$ for $q_{gi} = q_{\text{g max}}^{\text{max}}$. Therefore (6) formulates the desired behavior for generator voltage control. In contrast to the conventional power flow analysis, we note that $v_i, i \in \text{PV}$ in (6) is not fixed to its set point.

Since both $v_i$ and $q_{gi}$ are variables for $i \in \text{PV}$, the reactive power flow equations at PV buses should be added in addition to the complementarity constraints (6). Together with (5), the following incrementally formulates a power flow analysis with generator voltage control as an MCP:

$$
v_{\text{PV}} \perp Q_{\text{PV}}(x)
$$

(7)

C. Modeling tap-changing transformer and switched shunt voltage control

These additional controllers are used to regulate the bounds on voltage magnitudes as defined below using MCP formulations. We note that similar formulation has been given in [2].

$$
\begin{align*}
0 & \leq u_{\text{PV}}^{+} \perp u_{\text{PV}} + \nu_{\text{PV}} - v_{\text{PV}}^{\min} \geq 0 \\
0 & \leq v_{\text{PV}}^{+} \perp v_{\text{PV}} + u_{\text{PV}} + v_{\text{PV}}^{+} \geq 0 \\
0 & \leq u_{\text{PV}}^{-} \perp u_{\text{PV}} - v_{\text{PV}}^{\min} \geq 0 \\
0 & \leq v_{\text{PV}}^{-} \perp v_{\text{PV}}^{\max} - u_{\text{PV}} \geq 0
\end{align*}
$$

(8)

The first two conditions allow an increase (or a decrease) of a tap ratio $u_{\text{PV}}$ from its set point $v_{\text{PV}}^{sp}$, when a voltage magnitude reaches its limit. The third and fourth conditions together with the first two conditions capture a further drift from voltage magnitude limits when the corresponding tap ratio reaches its limit. For switched shunt devices, we can employ similar MCP formulations.

We note that tap ratios and susceptances in general take discrete values—choosing among finite values—but we instead treat them as continuous variables in (8). This is a common approach as in [3] and still provides useful values for finding a discrete solution [3], [7], [15].

In addition to the generator voltage control by (7), the formulation (8) can be easily concatenated to (7) to include in the model the effect of tap-changing transformer and switched shunt devices for voltage control as well. This provides a great flexibility in modeling the impact of various regulation components in power flow analysis via MCPs. In Section 4.3, we present a modeling example of incrementally including voltage regulating components to regulate the voltage magnitude.

D. Modeling primary frequency control

Primary frequency control is governor’s reaction to real power imbalance and can be formulated using an MCP formulation as described in (9). When $P_{\text{g min}} < P_{gi} < P_{\text{g max}}$, we have $P_{gi} = P_{gi}^{\text{min}} + \nu_{gi} \Delta f$ following the linear generation characteristic. If $P_{gi} = P_{gi}^{\text{max}}$, then it stays at its limit, but frequency could further drop resulting in $P_{gi} < P_{gi}^{\text{min}} + \nu_{gi} \Delta f$. Similarly, we could have $P_{gi} > P_{gi}^{\text{min}} + \nu_{gi} \Delta f$ when $P_{gi} = P_{gi}^{\text{min}}$. In (9), we note that the real power flow equation of the slack bus is added to the formulation for the synchronized generation.

$$
\begin{align*}
\Delta f & \perp P_{\text{slack}}(x) \\
P_{gi}^{\text{min}} \leq P_{gi} \leq P_{gi}^{\text{max}} \perp P_{gi} + \nu_{gi} \Delta f \\
\forall i \in \text{PV} \cup \text{Slack}
\end{align*}
$$

(9)
By combining (9) with voltage regulation formulations in Sections III-B and III-C, we can perform a power flow analysis with voltage and frequency regulations simultaneously using a single MCP model.

E. Convergence results

The following proposition shows sufficient conditions for strong regularity of a solution to an MCP. The conditions can be applied to any MCPs defined in the previous sections. By applying Theorem 1 with strong regularity, we obtain a local $Q$-quadratic convergence result.

**Proposition 1** (Theorem 3.1 in [14]). Let $x^*$ be a solution to an MCP $(B, F)$. Let $\alpha := \{i \mid x_i^{\min} < x_i^* < x_i^{\max}, F_i(x^*) = 0\}$, $\beta := \{i \mid x_i^* \in \{x_i^{\min}, x_i^{\max}\}, F_i(x^*) = 0\}$, and $\gamma := \{i \mid x_i^* \in \{x_i^{\min}, x_i^{\max}\}, F_i(x^*) \neq 0\}$. If the submatrix $\nabla F(x^*)_{\alpha,\alpha}$ is nonsingular and the Schur-complement $\nabla F(x^*)_{\alpha,\alpha}^{-1}$ is nonsingular, then $x^*$ is a strongly regular solution.

We note that when $\beta = \emptyset$ in Proposition 1 the conditions correspond to having a continuous linear inverse mapping for the reduced space defined by $\alpha$.

Global convergence results are obtained by assuming a sequence having an accumulation point that converges to a strongly regular solution. See [9, Theorem 5] for details.

IV. EXPERIMENTAL RESULTS

In this section we present numerical results of our MCP approach and compare its performance with those of the NR method with switching heuristics and the FB-based optimization approach [6]. Experiments were performed over large grids included in the MATPOWER package [16] on a Mac machine having Intel 6-Core i7@2.6 GHz and 32 GB of memory. The PATH and Ipopt were used for solving MCP and the FB-based problem, respectively.

A. Generator voltage control

In Table II we demonstrate the computational performance of each method for voltage control via reactive power. For MCP we solve the MCP defined by (5) and (7) in this case. Our MCP method showed the most robust performance as solving all of the problem, whereas the NR method failed the convergence on 3120sp, and the FB method could not find a solution for ACTIVSg25k. Also, the MCP method demonstrated as fast computation time as the NR method, while the FB method showed a much slower computation time.

We found for the 3120sp data that the solutions from the FB method and our MCP method violate some of the voltage magnitude bounds. Since reactive power cannot control such violations once it reaches its limit, we need another controller such as tap ratios and switched shunt devices, as described in Section III-C, in order to guarantee the feasibility. This will be addressed in the following section.

B. Voltage control using transformer tap ratios and switched shunt devices

In order to control the problematic voltage magnitudes of the 3120sp data to be within their bounds, we incorporated (8) into our MCP problem (so we solved (5)+(7)+(8)) by allowing some buses to change their tap ratios and switched shunt devices.

Figure 1 demonstrates how voltage bound violations were decreasing and eventually removed as we increased the allowed bounds on the tap ratios and switched shunt devices. These results demonstrate the capability of our method that can easily simulate the effect of several voltage controllers in an incremental fashion.

C. Frequency and voltage regulations

In Table III, we report the numerical results from our method for the MCP problem (5)+(7)+(9) with frequency and voltage controls on ACTIVSg25k data. For frequency control we gradually increased real power generation loss by turning off up to 5 generators that had the most real power generation among other generators (i.e., 1557, 2345, 2346, 235, and 2015). Those generators were chosen from the solution of generator voltage control we performed in Section IV-A, and we warm-started from that solution to solve the MCP. To the best of our knowledge, there exists only one recent paper [17] that uses an NR-based heuristics for solving the problem without any convergence guarantee. However, we do not consider NR method due to the lack of convergence guarantee.

In all cases, the MCP was able to quickly find a solution containing frequency changes while controlling the voltage magnitudes to stay at their set points as much as possible and be within their bounds as well.

V. CONCLUSION

We presented a mixed complementarity problem approach for a steady-state power flow analysis with voltage regulation and frequency control. The key advantage of our approach is leveraging the existing algorithms with the theoretical support for the global convergence with quadratic local convergence rate, which guarantees the numerical stability and fast computation. We demonstrated the greater computational
performance of our approach, as compared with the existing NR method and FB method, by using large MATPOWER test instances.

**TABLE II: Generator voltage control**

| Data         | NR method | MCP | FB method |
|--------------|-----------|-----|-----------|
|              | Iter      | Time (secs) | max $|v - v^p|$ | Iter | Time (secs) | max $|v - v^p|$ | Iter | Time (secs) | max $|v - v^p|$ |
| 1354pegase   | 4         | 0.06 | 2.64e-02 | 4    | 0.05 | 2.64e-02 | 22   | 0.82 | 2.64e-02 |
| 2869pegase   | 6         | 0.18 | 1.82e-02 | 6    | 0.29 | 1.64e-02 | 24   | 2.21 | 1.65e-02 |
| 3120sp       | f         | n/a  | n/a      | 6    | 0.15 | 6.95e-02 | 925  | 82.15 | 6.95e-02 |
| 6468tne      | 5         | 0.26 | 2.79e-02 | 4    | 0.28 | 2.78e-02 | 301  | 66.83 | 2.79e-02 |
| 9241pegase   | 6         | 0.52 | 2.47e-02 | 7    | 0.92 | 2.46e-02 | 74   | 36.95 | 2.47e-02 |
| 13659pegase  | 5         | 0.63 | 5.66e-02 | 5    | 0.93 | 5.02e-03 | 35   | 23.59 | 5.03e-03 |
| ACTIVSg10k   | 5         | 0.36 | 4.07e-05 | 3    | 1.40 | 4.06e-05 | 37   | 10.95 | 1.67e-04 |
| ACTIVSg25k   | 4         | 0.80 | 5.83e-04 | 5    | 1.37 | 5.82e-04 | 253  | 263.05 | 7.31e-04 |
| ACTIVSg70k   | 5         | 3.12 | 1.03e-03 | f    | n/a  | n/a      | f    | n/a  | n/a      |

**TABLE III: Frequency and voltage regulations over AC- TIVSg25k**

| Generation loss | Frequency | max $|v - v^p|$ | Time (secs) |
|-----------------|-----------|--------------|-------------|
| 1.299 MW        | 59.92 Hz  | 3.41e-03     | 1.27        |
| 2.597 MW        | 59.84 Hz  | 9.50e-03     | 1.28        |
| 3.895 MW        | 59.76 Hz  | 1.96e-02     | 1.56        |
| 5.185 MW        | 59.68 Hz  | 2.80e-02     | 1.77        |
| 6.455 MW        | 59.62 Hz  | 2.90e-02     | 1.77        |

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