Abstract. It is shown that the operator space generated by peripheral eigenvectors of a unital completely positive map on a von Neumann algebra has a $C^*$-algebra structure. This extends the notion of non-commutative Poisson boundary by including the point spectrum of the map contained in the unit circle. The main ingredient is dilation theory. This theory provides a simple formula for the new product. The notion has implications to our understanding of quantum dynamics. For instance, it is shown that the peripheral Poisson boundary remains invariant in discrete quantum dynamics.

1. Introduction

Unital completely positive (UCP) maps on von Neumann algebras are considered as non-commutative (or quantum) Markov maps. Their fixed points are called harmonic elements. In general, this collection of harmonic elements is not an algebra. However, it is possible to introduce a new product called Choi-Effros product ([10]) which makes it a von Neumann algebra.

Inspired from the theory of classical random walks on groups Izumi [19] called this algebra (or its concrete realization) as noncommutative Poisson boundary. This concept has attracted a lot of interest in recent years (See [20], [21], [6], [26], [28], [11], [22]). The space of harmonic elements is just the eigenspaces with respect to the eigenvalue one. In this short note, we look at the operator space formed by eigenvectors corresponding to eigenvalues on the unit circle for quantum Markov maps, i.e., for UCP maps. Like in the case of harmonic elements, to begin with we have only an operator space, which may not be an algebra. But we can impose a new product to get a $C^*$-algebra. Eigenvalues on the unit circle are known as peripheral eigenvalues and the corresponding eigenvectors are called peripheral eigenvectors. Therefore we are providing an extension of Choi-Effros product to the operator space spanned by peripheral eigenvectors. To spell out our results in more detail we need some notation.

All the Hilbert spaces considered will be complex and separable whose inner product $\langle \cdot, \cdot \rangle$ is anti-linear in the first variable. We denote the algebra of all bounded operators on a Hilbert space

2020 Mathematics Subject Classification. 46L57, 47A20, 81S22.

Key words and phrases. Poisson boundary, dilation theory, peripheral spectrum, unital completely positive maps.

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\( \mathcal{H} \) by \( B(\mathcal{H}) \). In formulae, we indicate strong operator topology limit by ‘\( s - \lim \)’. The unit circle, \( \{ z \in \mathbb{C} : |z| = 1 \} \) is denoted by \( \mathbb{T} \).

Let \( \mathcal{H} \) be a Hilbert space and let \( \mathcal{A} \subseteq B(\mathcal{H}) \) be a von Neumann algebra. Suppose \( \tau : \mathcal{A} \to \mathcal{A} \) is a normal unital completely positive map. It is well-known that the fixed point space of \( \tau \) (also known as the space of harmonic elements of \( \tau \)),

\[
F(\tau) := E_1(\tau) = \{ x \in \mathcal{A} : \tau(x) = x \}
\]

admits a product ‘\( \circ \)’, called Choi-Effros product on \( F(\tau) \), which will make it a von Neumann algebra. Note that there is no modification in the norm or the involution. Concrete realization of this algebra is known as the non-commutative Poisson boundary of \( \tau \). For any \( \lambda \in \mathbb{C} \) take

\[
E_\lambda(\tau) := \{ x \in \mathcal{A} : \tau(x) = \lambda x \}.
\]

The set of all \( \lambda \in \mathbb{T} \) for which \( E_\lambda(\tau) \) is non-trivial is known as the peripheral point spectrum of \( \tau \) and the corresponding eigenvectors are known as peripheral eigenvectors. Take

\[
E(\tau) = \text{span}\{ x : x \in E_\lambda(\tau), \text{ for some } \lambda \in \mathbb{T} \}.
\]

In our main theorem (Theorem 2.3) it is shown that the norm closure of \( E(\tau) \) is a \( C^* \)-algebra with respect to a new product. Here too, the norm and the involution are not modified. Only the product is replaced with a new product. The norm is of course uniquely determined as a \( * \)-algebra can admit at most one norm which makes it a \( C^* \)-algebra. We may call this \( C^* \)-algebra as the peripheral Poisson boundary.

In fact, we have a very simple formula for the product (See Theorem 2.5). No ultra-filters are required to compute this product. It is also shown that if the UCP map leaves a faithful state invariant then the new product is same as the original product. It is not hard to see that the action of the UCP maps yields an automorphism of the peripheral Poisson boundary (Theorem 2.12).

In Section 3, we present a few examples to illustrate the theory. We see that the Poisson boundary product of Toeplitz operators, which comes from \( L^\infty(\mathbb{T}) \), has a very natural extension to the peripheral Poisson boundary by incorporating a family of unitaries parametrized by the unit circle. It is well known that the norm of a Toeplitz operator is same as the \( L^\infty \) norm of its symbol. In equation (3.1) we have an interesting extension of this result. One of the surprising features we see in this example is that the algebra we get is not a von Neumann algebra. The equation (3.1) does not extend to the strong closure.
The importance of peripheral spectrum in quantum dynamics is well known and has its origin in classical Perron-Frobenius theory (See [1], [3], [12], [13], [14], [17]). The main point is that generally the peripheral spectrum encodes the persistent/recurrent part of the dynamics and the rest has only some transient component. Below in Theorem 2.12, we answer a question raised in [15] by showing that restriction of the UCP map to the peripheral part is an automorphism in our setting.

For a UCP map \(\tau\), if we look at the Poisson boundaries of \(\tau, \tau^2, \ldots\), they keep changing, for instance if \(e^{\frac{2\pi i}{k}}\) (for some \(k > 1\)), is in the point spectrum of \(\tau\), the corresponding eigenspace gets included among harmonic elements of \(\tau^k\), but they are not part of the set of harmonic elements of \(\tau^l\) when \(l\) is not a multiple of \(k\). In other words the point spectrum keeps ‘rotating’ on the unit circle when we take powers of \(\tau\). However, as we see in Theorem 4.3, the peripheral Poisson boundary of \(\tau\) is same as that of \(\tau^k\) for all \(k \geq 1\).

Most of the papers dealing with peripheral point spectrum of UCP maps have additional conditions such as finite dimensionality of the algebra and/or existence of an invariant state. We do not impose any such restrictions.

The main technical tool we need is the following minimal dilation theorem for unital completely positive maps. In the following for Hilbert spaces \(H \subset K\), any operator \(x \in B(H)\) is identified with the operator \(pxp\) in \(B(K)\), where \(p\) is the orthogonal projection of \(K\) onto \(H\).

**Theorem 1.1.** ([4], [5], and [9]) Let \(A \subseteq B(H)\) be a von Neumann algebra and let \(\tau : A \rightarrow A\) be a normal unital completely positive map. Then there exists a triple \((K, B, \theta)\), where (i) \(K\) is a Hilbert space containing \(H\) as a closed subspace; (ii) \(B \subseteq B(K)\) is a von Neumann algebra, satisfying \(A = pBp\) where \(p\) is the orthogonal projection of \(K\) onto \(H\); (iii) \(\theta : B \rightarrow B\) is a normal, unital \(*\)-endomorphism; (iv) (dilation property): \(\tau^n(x) = p\theta^n(x)p, \forall x \in A, n \in \mathbb{Z}_+;\) (v) (minimality of the space) \(K = \text{span}\{\theta^n(x_n)\theta^{n-1}(x_{n-1})\ldots \theta(x_1)h : x_1, \ldots, x_n \in A, h \in H, n \in \mathbb{Z}_+\};\) (vi) (minimality of the algebra) \(B\) is the von Neumann algebra generated by \(\{\theta^n(x) : n \in \mathbb{Z}_+, x \in A\}\). (vii) (uniqueness) If \((K_1, B_1, \theta_1)\) is another such triple then there exists a unitary \(U : K \rightarrow K_1\) satisfying (a) \(Uh = h\) for all \(h \in H\); (b) \(B_1 = \{UbU^* : b \in B\}\); (c) \(\theta_1(x) = U\theta(x)U^*, \forall x \in A\).

This theorem was first proved for normal CP maps on \(B(H)\) in [4] and then for CP maps on general \(C^*\)-algebras in [5]. Here we have taken the setting of von Neumann algebras with normal UCP maps. This was first done in [9], using a Hilbert \(C^*\)-module approach. It is to be noted that, these articles handle discrete (where the semigroup under consideration is \(\mathbb{Z}_+\)) and the continuous case (where the semigroup is \(\mathbb{R}_+\)) at one go and the main focus was on the continuous case. Presently we need only
the much simpler discrete case. A very compact and neat exposition of this case can be seen in the Appendix of [11]. It makes use of repeated application of Stinespring’s theorem and is essentially equivalent to first constructing the weak Markov flow and then writing down the endomorphism. The increasing family of projections constructed here (which increases to identity) is the filtration of quantum Markov process theory (See [7], [8]). Some alternative versions/presentations can be seen in ([2], [23], [27]). The literature on dilation theory of UCP maps is vast and we have cited only a small sample relevant for our current purposes. Here we have confined ourselves to the von Neumann algebra version, as that is convenient and seems to be of most interest.

We will call the triple $(K, B, \theta)$ of this theorem as the minimal dilation of $\tau$. We will repeatedly use the following two facts of dilation theory:

(i) For every $n \in \mathbb{Z}_+$, $p \leq \theta^n(p)$, and $\theta^n(p)$ increases in strong operator topology to the identity of $K$. (ii) For any $z \in B$, $p\theta(z)p = \tau(pzp)$.

We have restricted our focus to dynamics in discrete time in this article. But it is clear that it is possible to develop analogous theory in continuous time.

2. Main Theorem

We assume the following setup and notation for whole of this Section. Let $A \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\tau : A \to A$ be a normal unital completely positive map. Let $(K, B, \theta)$, be the minimal dilation of $\tau$ guaranteed by Theorem [11]. Note that the orthogonal projection $p$ of $K$ onto $\mathcal{H}$ is the identity of $A$.

Our first objective is to connect the peripheral eigenvectors of $\tau$ with that of $\theta$. Then we use the $*$-endomorphism property of $\theta$ to get a $C^*$-algebra structure on the operator space generated by these eigenvectors. To begin with we have the following simple observation that peripheral eigenvectors of $\tau$ can be lifted to that of $\theta$ and peripheral eigenvectors of $\theta$ can be compressed to that of $\tau$.

**Lemma 2.1.** For every $x \in A$ satisfying $\tau(x) = \lambda x$ with $\lambda \in \mathbb{T}$, there exists unique $\hat{x} \in B$ such that $\theta(\hat{x}) = \lambda \hat{x}$ and $p \hat{x} p = x$. It is given by

$$\hat{x} = s - \lim_{n \to \infty} \lambda^{-n}\theta^n(x).$$

(2.1)

Conversely if $\hat{x} \in B$ satisfies $\theta(\hat{x}) = \lambda \hat{x}$ for some $\lambda \in \mathbb{C}$, then $\tau(x) = \lambda x$, where $x = p\hat{x} p$.

**Proof.** Let us prove the existence of $\hat{x}$ as above. Note that here, as $x \in A = pBp$, we have $x = pxp$. Take $x_0 = x$. For $n \geq 1$, set $x_n = \lambda^{-n}\theta^n(x) = \lambda^{-n}\theta^n(pxp)$. We claim that $\{x_n\}_{n \geq 0}$ forms a
martingale with respect to the filtration \( \{ \theta^m(p) : m \geq 0 \} \), that is, for \( 0 \leq m \leq n \),
\[
\theta^m(p)x_n \theta^m(p) = x_m. \tag{2.2}
\]

This can be seen easily by dilation property:
\[
\theta^m(p)x_n \theta^m(p) = \theta^m(p) \lambda^{-n} \theta^n(pxp) \theta^m(p) = \lambda^{-n} \theta^m(p) \theta^n(pxp) \theta^m(p) \\
= \lambda^{-n} \theta^m(p \theta^n-m(pxp)p) = \lambda^{-n} \theta^m(p \tau^n-m(x)p) \\
= \lambda^{-n} \theta^m(p \lambda^n-mxp) = \lambda^{-n} \lambda^n \theta^m(pxp) \\
= \lambda^{-m} \theta^m(pxp) = x_m.
\]

As \(|\lambda| = 1\) and \( \theta \) is contractive, \( \{ x_n \}_{n \geq 0} \) is a bounded sequence. Now as \( \{ \theta^m(p) \}_{m \geq 0} \) is an increasing family of projections increasing to identity, for \( h \in K \) and \( \epsilon > 0 \), we can choose \( m \) such that \( \| h - \theta^m(p)h \| < \epsilon \). Taking \( h_0 = \theta^m(p)h \), from (2.2), for \( n \geq k \geq m \), \( x_nh_0 \) and \( x_nh_0 - x_kh_0 \) are mutually orthogonal and hence \( \{ \| x_nh_0 \|^2 \}_{n \geq m} \) is an increasing sequence bounded by \( \| x \|^2 \| h_0 \|^2 \). Further, equation (2.2) implies, \( \| (x_n - x_m)h_0 \|^2 = \| x_nh_0 \|^2 - \| x_mh_0 \|^2 \), \( \forall n \geq m \). It follows that \( \{ x_nh_0 \}_{n \geq 0} \) is a Cauchy sequence. As \( \epsilon > 0 \) was arbitrary, \( \{ x_nh \}_{n \geq 0} \) is also a Cauchy sequence. Taking \( \hat{x}h = \lim_{n \to \infty} x_nh \), as \( \| x_nh \| \leq \| x \| \| h \| \), for all \( n \), we get a bounded operator \( \hat{x} = s - \lim_{n \to \infty} x_n \) in \( B \) satisfying \( \theta^m(p) \hat{x} \theta^m(p) = x_m \). Note that
\[
\theta(\hat{x}) = \theta(s - \lim_{n \to \infty} x_n) = \theta(s - \lim_{n \to \infty} \lambda^{-n} \theta^n(pxp)) = s - \lim_{n \to \infty} \lambda^{-n} \theta^{n+1}(xp) \\
= \lambda(s - \lim \lambda^{-n-1} \theta^{n+1}(xp)) = \lambda \hat{x}.
\]

Further, for \( m = 0 \), taking limit as \( n \to \infty \) in equation (2.2), \( p\hat{x}p = x \). This proves the existence of \( \hat{x} \).

Let us now prove the uniqueness. Suppose that there exists \( y' \in B \) such that \( \theta(y') = \lambda y' \) and \( py'p = x \). Then since \( \theta^n(p) \) increases to identity as \( n \) increases to infinity, we get
\[
y' = s - \lim_{n \to \infty} \theta^n(p)y' \theta^n(p) = s - \lim_{n \to \infty} \lambda^{-n} \theta^n(p) \theta^n(y') \theta^n(p) \\
= s - \lim_{n \to \infty} \lambda^{-n} \theta^n(py'p) = s - \lim_{n \to \infty} \lambda^{-n} \theta^n(x) = \hat{x}.
\]

This completes the proof of the first part. Now suppose \( \hat{x} \in B \) satisfies \( \theta(\hat{x}) = \lambda \hat{x} \) for some \( \lambda \in \mathbb{C} \). Take \( x = p\hat{x}p \). As \( \theta(p) \geq p \), we get
\[
\tau(x) = \tau(p\hat{x}p) = p\theta(p\hat{x}p)p = p\theta(p)\theta(\hat{x})\theta(p)p = p\theta(\hat{x})p = \lambda p\hat{x}p = \lambda x.
\]
\( \square \)
It is to be noted that in this Lemma, \( \hat{x} \neq 0 \) as \( p\hat{x}p = x \). So we have lifted eigenvectors of \( \tau \) to that of \( \theta \). In the computations we have crucially used the fact that we are considering peripheral eigenvalues. Later on in the Hardy space example of Section 3, we see that it is possible to have \( \lambda \) in the point spectrum of \( \tau \), which are not in that of \( \theta \) for \( |\lambda| < 1 \).

Consider the compression map \( T : \mathcal{B} \to \mathcal{A} \) defined by

\[
T(z) = pzp.
\]

Clearly \( T \) is completely positive and unital. The previous Lemma shows that for \( \lambda \in \mathbb{T} \), \( T \) from \( E_\lambda(\theta) \) to \( E_\lambda(\tau) \) is a bijection. Now we wish to show that \( T \) maps \( E(\theta) \) isometrically to \( E(\tau) \). For the purpose, we need the following result on rotations of the \( d \)-torus. It is perhaps a folklore result in the theory of dynamical systems. However, we could not find a suitable reference. For readers convenience we present it in a purely Hilbert space language and provide an elementary operator theoretic proof. It maybe noted that the standard result that irrational rotations on the torus have dense orbit (See Lemma 8.1.17 of [16]) is a simple special case of this result.

**Lemma 2.2.** Suppose \( U \) is a unitary on a finite dimensional Hilbert space. Then there exists a subsequence of \( \{U^n\}_{n \in \mathbb{N}} \) converging to the identity in norm. In particular, for \( d \in \mathbb{N} \), if \( \lambda_1, \ldots, \lambda_d \) are some \( d \)-points in the unit circle \( \mathbb{T} \), then there exists a subsequence of \( \{\lambda_1^n, \ldots, \lambda_d^n\}_{n \in \mathbb{N}} \) converging to \((1, \ldots, 1)\) in \( \mathbb{C}^d \).

**Proof.** Since the set of unitaries on a finite dimensional Hilbert space is closed and bounded and hence compact, the sequence \( \{U^n\}_{n \in \mathbb{N}} \) has a convergent subsequence, say \( \{U^{nm}\}_{m \in \mathbb{N}} \). Then by Cauchy property for \( \varepsilon > 0 \), there exists \( K \), such that for \( l > m > K \), \( \|U^{nl} - U^{nm}\| < \varepsilon \). This implies,

\[
\|U^{nl} - U^{nm} - I\| \leq \|(U^{nl} - U^{nm})U^{-nm}\| \leq \|U^{nl} - U^{nm}\| < \varepsilon.
\]

Now the result is clear. The second part follows from considering the diagonal unitary with diagonal entries equal to \( \lambda_1, \ldots, \lambda_d \). \( \square \)

We remark in the passing that Arveson ([3], Lemma 2.9) has a result in the converse direction, namely that in any Banach space if identity is in the strong closure of \( \{U, U^2, \ldots\} \) for some linear contraction \( U \), then \( U \) is an automorphism. We will not need this result.

Consider the operator spaces:

\[
\mathcal{P}(\tau) := \overline{E(\tau)} = \text{span}\{x \in \mathcal{A} : \tau(x) = \lambda x, \lambda \in \mathbb{T}\};
\]

\[
\mathcal{P}(\theta) := \overline{E(\theta)} = \text{span}\{x \in \mathcal{B} : \theta(x) = \lambda x, \lambda \in \mathbb{T}\},
\]
where the closures are taken with respect to the operator norm. Clearly $\mathcal{P}(\tau), \mathcal{P}(\theta)$ are operator spaces. In fact as they are unital and $*$-closed they form operator systems. Since $\theta$ is an endomorphism, if $\theta(x) = \lambda x$ and $\theta(y) = \mu y$, then $\theta(xy) = \lambda \mu (xy)$. This shows that $\mathcal{P}(\theta)$ is a unital $C^*$-algebra. Here is our main result.

**Theorem 2.3.** Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\tau : \mathcal{A} \to \mathcal{A}$ be a normal unital completely positive map. Let $(\mathcal{K}, \mathcal{B}, \theta)$, be the minimal dilation of $\tau$. Let $T$ denote the compression map $z \mapsto pzp$ restricted to $\mathcal{P}(\theta)$. Then the completely positive map $T$ maps the $C^*$-algebra $\mathcal{P}(\theta)$ isometrically and bijectively to $\mathcal{P}(\tau)$. In particular, setting

$$x \circ y := T(T^{-1}(x)T^{-1}(y)), \quad \forall x, y \in \mathcal{P}(\tau),$$

(2.3)

makes $(\mathcal{P}(\tau), \circ)$ a unital $C^*$-algebra. Moreover, the map $T$ is a complete isometry.

**Proof.** From Lemma 2.1 we know that $T$ maps $E_\lambda(\theta)$ bijectively to $E_\lambda(\tau)$ for every $\lambda \in \mathbb{T}$. Consequently, $T$ maps $E(\theta)$ surjectively to $E(\tau)$. Recall that for any linear map eigenvectors corresponding to distinct eigenvalues are linearly independent. Hence any non-zero element $y$ in $E(\theta)$ can be written uniquely as $y = \sum_{j=1}^{d} y_j$ for some $0 \neq y_j \in E_{\lambda_j}(\theta)$ for some distinct $\lambda_1, \ldots, \lambda_d$ in $\mathbb{T}$, $d \in \mathbb{N}$. Take $x_j = p y_j p$, $1 \leq j \leq d$, and $x = pyp$. As $p$ is a projection, $\|x\| \leq \|y\|$. The construction in the proof of Lemma 2.1 tells us that

$$y_j = s - \lim_{n \to \infty} \lambda_j^{-n} \theta^n(x_j), \quad 1 \leq j \leq d.$$  

Hence,

$$y = s - \lim_{n \to \infty} \sum_{j=1}^{d} \lambda_j^{-n} \theta^n(x_j).$$  

(2.4)

Consider any $h$ in the Hilbert space $\mathcal{K}$ with $\|h\| = 1$ and let $\epsilon > 0$. By the convergence in (2.4) there exists $N$ such that for all $n \geq N$,

$$\|yh - \sum_{j=1}^{d} \lambda_j^{-n} \theta^n(x_j)h\| < \frac{\epsilon}{2}.$$  

In view of Lemma 2.2 we can find $m > N$ such that

$$|\lambda_j^m - 1| < \frac{\epsilon}{2 \sum_{k=1}^{d} \|x_k\|}, \quad \forall 1 \leq j \leq d.$$  

As $\theta$ is an endomorphism and $\|h\| = 1$, $\|\theta^m(x_j)h\| \leq \|x_j\|$ and hence

$$\|\sum_{j=1}^{d} (\lambda_j^{-m} - 1) \theta^m(x_j)h\| < \frac{\epsilon}{2}.$$
Combining with the previous inequality at $n = m$,

$$\|yh\| \leq \left\| \sum_{j=1}^{d} \theta^m(x_j) \right\| + \epsilon = \left\| \theta^m \left( \sum_{j=1}^{d} x_j \right) \right\| + \epsilon = \|\theta^m(x)\| + \epsilon \leq \|x\| + \epsilon.$$ 

As this is true for all $\epsilon > 0$, $\|y\| \leq \|x\| \leq \|y\|$. This proves that $T$ is isometric on $E(\theta)$. In particular, $T$ is injective and also it extends to an isometric bijection from $P(\theta)$ to $P(\tau)$. The second part is clear, as $P(\theta)$ is a unital $C^*$-algebra. The complete isometry property of $T$ is shown in the Appendix. □

**Definition 2.4.** Let $A \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\tau : A \to A$ be a normal unital completely positive map. Then the $C^*$-algebra $P(\tau)$ constructed above is called the peripheral Poisson boundary of $\tau$.

We could have called this as ‘non-commutative peripheral Poisson boundary’. The word ‘non-commutative’ has been dropped for brevity as it is clear from the context. We have defined the peripheral Poisson boundary using the minimal dilation. It is possible to write down the algebra structure without referring to the minimal dilation as below. However we do not know as to how to prove the existence of strong operator topology limit in this theorem without using dilation. Initial formulae for Choi-Effros product included limits over ultra-filters. It was W. Arveson and M. Izumi who first saw that a much simpler formula can be provided using dilation theory (See the Appendix of [21]). Here we have similar formula for peripheral Poisson boundary.

**Theorem 2.5.** Under the setting of Theorem 2.3, for $x, y \in A$ with $\tau(x) = \lambda x, \tau(y) = \mu y$, $\lambda, \mu \in \mathbb{T}$,

$$x \circ y = s - \lim_{n \to \infty} (\lambda \mu)^{-n} \tau^n(xy).$$

(2.5)

If there exists a faithful state left invariant by $\tau$, then $x \circ y = xy$.

**Proof.** Take $\hat{x} = T^{-1}(x)$ and $\hat{y} = T^{-1}(y)$ in $B$. Then from Lemma 2.1 we know that $\hat{x} = s - \lim_{n \to \infty} \lambda^{-n} \theta^n(x)$ and $\hat{y} = s - \lim_{n \to \infty} \mu^{-n} \theta^n(y)$. Since strong operator topology convergence respects products of sequences, $\hat{x}\hat{y} = s - \lim_{n \to \infty} (\lambda \mu)^{-n} \theta^n(xy)$. Therefore,

$$x \circ y = p\hat{x} \hat{y}.$$

To see the second part let $\phi$ be a faithful state left invariant by $\tau$. Consider $x \in E_\lambda(\tau), y \in E_\mu(\tau)$ with $\lambda, \mu \in \mathbb{T}$. As $\tau$ is UCP, for any $y \in A$, $\tau(y^*y) \geq \tau(y)^*\tau(y)$. Hence

$$\phi(\tau(y^*y) - \tau(y)^*\tau(y)) = \phi(\tau(y^*y) - |\mu|^2 y^*y) = \phi(y^*y) - \phi(y^*y) = 0.$$
The faithfulness of $\phi$ implies $\tau(y^*y) - \tau(y)^*\tau(y) = 0$. Then by dilation property $p\theta(y)^*[p + (1 - p)]\theta(y)p - p\theta(y)^*p\theta(y)p = 0$ or $(1 - p)\theta(y)p = 0$. Therefore, $\tau(xy) = p\theta(x)[p + (1 - p)]\theta(y)p = \tau(x)\tau(y) = \lambda\mu xy$ and the result follows from the first part. 

This result about UCP maps preserving faithful states has applications to the theory of quantum channels. For instance, it shows that in such cases products of two peripheral eigenvectors is again a peripheral eigenvector. We may also note the reverse implication that if the product in the peripheral Poisson boundary does not match with the ordinary product then no faithful state is left invariant by the UCP map.

**Corollary 2.6.** If the von Neumann algebra $\mathcal{A}$ is abelian then the peripheral Poisson boundary of $\tau : \mathcal{A} \to \mathcal{A}$ is also abelian.

**Proof.** If the algebra is abelian we get $x \circ y = y \circ x$ from equation (2.5). □

**Corollary 2.7.** If $x \in E_\lambda(\tau)$ and $y \in E_\mu(\tau)$ then $x \circ y \in E_{\lambda,\mu}(\tau)$. In particular, if $\lambda,\mu$ is not in the point spectrum of $\tau$, then $x \circ y = 0$.

**Proof.** This follows as we have lifts $\hat{x} \in E_\lambda(\theta), \hat{y} \in E_\mu(\theta)$ and consequently $\hat{x} \circ \hat{y} \in E_{\lambda,\mu}(\theta)$. □

We remark that instead of considering the whole group $\mathbb{T}$, we may consider any subgroup $G$ of $\mathbb{T}$ and define $E_G(\tau) = \text{span}\{x \in E_\lambda(\tau) : \lambda \in G\}$. Like in the main theorem, $T$ maps $E_G(\theta)$ isometrically and bijectively to $E_G(\tau)$. Following the same procedure $\mathcal{P}_G(\tau) := E_G(\tau)$ becomes a unital $C^*$-algebra, which is in fact the $C^*$-subalgebra of $\mathcal{P}(\tau)$ generated by $E_G(\tau)$. The following case is of special interest.

**Definition 2.8.** For $k \in \mathbb{N}$, $k$-cyclic Poisson boundary of UCP map $\tau : \mathcal{A} \to \mathcal{A}$ is defined as the $C^*$-algebra $\mathcal{P}_{G_k}(\tau)$, where $G_k$ is the group $\{\omega^j : 0 \leq j \leq k - 1\}$ with $\omega = e^{2\pi i/k}$.

By considering the trivial group $\{1\}$, we get the usual noncommutative Poisson boundary as a unital $C^*$-subalgebra of $\mathcal{P}(\tau)$. Here are some additional properties of peripheral eigenvectors as elements of the peripheral Poisson boundary.

**Theorem 2.9.** Let $E_\lambda(\tau)$ for $\lambda \in \mathbb{T}$ be the spaces as above.

(i) Define $I_\lambda(\tau) = \text{span}\{x^* \circ y : x, y \in E_\lambda(\tau)\}$. Then $(I_\lambda(\tau), \circ)$ is a closed two sided ideal of the Poisson boundary $C^*$-algebra, $(F(\tau), \circ)$.

(ii) $E_\lambda(\tau)$ with $\langle x, y \rangle := x^* \circ y$ (and natural left and right actions) is a two-sided Hilbert $C^*$-module over the $C^*$-algebra $(F(\tau), \circ)$, with inner products taking values in the ideal $(I_\lambda(\tau), \circ)$. 

(iii) The mapping \( x \mapsto x^* \) from \( E_\lambda(\tau) \) to \( E_\lambda(\tau) \) is an anti-linear isomorphism.

Proof. All the computations can be done using the minimal dilation \( \theta \) as \( E_\lambda(\theta) \) and \( E_\lambda(\tau) \) are isomorphic and the product operation on \( \mathcal{P}(\tau) \) is borrowed from \( \mathcal{P}(\theta) \). Now the results (i)-(iii) follow from the \(*\)-homomorphism property of \( \theta \).

Corollary 2.10. For \( \lambda \in \mathbb{T} \) if there exist \( v_1, v_2 \in E_\lambda(\tau) \), such that \( v_1^* \circ v_2 = 1 \), then

\[
E_\lambda(\tau) = \{ x \circ v_2 : x \in F(\tau) \} = \{ x \circ v_1 : x \in F(\tau) \}.
\]

Similarly, if there exist \( v_1, v_2 \in E_\lambda(\tau) \) such that \( v_1 \circ v_2^* = 1 \), then

\[
E_\lambda(\tau) = \{ v_2 \circ x : x \in F(\tau) \} = \{ v_1 \circ x : x \in F(\tau) \}.
\]

In either case, \( \dim \left( E_\lambda(\tau) \right) = \dim \left( F(\tau) \right) \).

Proof. We may replace \( \tau \) by \( \theta \) and consider the ordinary product. Clearly if \( x \in F(\theta) \) then \( xv_2 \in E_\lambda(\theta) \) as \( \theta(xv_2) = x.(\lambda v_2) \). Now if \( y \in E_\lambda(\theta) \), \( y = (yv_1^*) \cdot v_2 \) and \( yv_1^* \in F(\theta) \). This proves the first part. Now if there exist \( v_1, v_2 \in E_\lambda(\theta) \) such that \( v_1^* v_2 = 1 \). If \( x_1, \ldots, x_n \) are linearly independent in \( F(\theta) \), then \( v_2 x_1, v_2 x_2, \ldots, v_2 x_n \) are linearly independent in \( E_\lambda(\theta) \) \( (\sum_i c_i v_2 x_i = 0 \) implies \( \sum_i c_i x_i = \sum_i c_i v_1^* v_2 x_i = 0) \) and conversely if \( y_1, y_2, \ldots, y_n \) are linearly independent in \( E_\lambda(\theta) \), then \( y_1 v_1^*, y_2 v_1^*, \ldots, y_n v_1^* \) linearly independent in \( F(\theta) \). This shows \( \dim \left( E_\lambda(\tau) \right) = \dim \left( F(\tau) \right) \). The proofs of other parts are similar.

Corollary 2.11. If \( F(\tau) \) is one dimensional and \( E_\lambda(\tau) \) is non-trivial, then \( E_\lambda(\tau) \) is also one dimensional.

Proof. If \( v_1, v_2 \) are in \( E_\lambda(\theta) \), \( v_1^* v_2 \in F(\theta) \) and hence it is a scalar. Now the result is clear from previous corollary.

The following answers the main question of [15] for general von Neumann algebras.

Theorem 2.12. Let \( \tau : A \to A \) be UCP map as above. Then \( x \mapsto \tau(x) \) is an automorphism of the peripheral Poisson boundary \( \mathcal{P}(\tau) \). The peripheral Poisson boundary of this automorphism is \( \mathcal{P}(\tau) \).

Proof. Consider \( x \mapsto \theta(x) \) in the peripheral Poisson boundary of \( \theta \). Since \( \theta \) maps \( E(\theta) \) to itself, \( \theta \) restricted to the peripheral Poisson boundary is a \(*\)-endomorphism. Since \( \theta(x) = \lambda.x \) for \( x \in E_\lambda(\theta) \) it is surjective.

Now we prove that \( \theta \) is isometric on \( E(\theta) \), and consequently it is isometric on \( \mathcal{P}(\theta) \). That will prove in particular that \( \theta \) is injective. As \( \theta \) is a \(*\)-endomorphism \( \|x\| \geq \|\theta(x)\| \geq \|\theta^n(x)\| \) for all
Consider $x = \sum_{j=1}^{d} x_j$, with $x_j \in E_{\lambda_j}(\theta)$ for distinct $\lambda_1, \ldots, \lambda_d$ in $\mathbb{T}$ and some $d \geq 1$. We have $\theta^n(x) = \sum_{j=1}^{d} \lambda_j^n x_j$ for all $n \geq 1$. For $\epsilon > 0$, by Lemma 2.2 there exists $n$ such that

$$|\lambda_j^n - 1| < \frac{\epsilon}{2 \sum_{k=1}^{d} \|x_k\|}, \quad 1 \leq j \leq d.$$ 

Then $\|\theta^n(x) - x\| = \|\sum_{j=1}^{d} (\lambda_j^n - 1)x_j\| \leq \frac{\epsilon}{2 \sum_{k=1}^{d} \|x_k\|} \sum_{j=1}^{d} \|x_j\| < \epsilon$. As this is true for all $\epsilon > 0$, we get $\|\theta(x)\| = \|x\|$. This proves the result for $\theta$. Now consider the compression map $T$ on $E(\theta)$ and we are done. The proof of the second part is straightforward. □

### 3. Examples

In general it is difficult to explicitly determine Poisson boundaries of UCP maps. For instance, these von Neumann algebras can be factors of different types. There are large number of papers devoted to these computations. In this Section we present some examples, deliberately chosen to be simple, to show that the peripheral Poisson boundary can be very distinct from the Poisson boundary. To begin with here is a simple finite dimensional example.

**Example 3.1.** Fix some $d \geq 2$ and let $U$ be a unitary in $M_d(\mathbb{C})$. Let $\tau: M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be the automorphism

$$\tau(X) = U^* X U, \quad X \in M_d(\mathbb{C}).$$

The minimal dilation of $\tau$ is $\tau$ itself. For $\mu \in \mathbb{C}$, let $\mathcal{H}_\mu := \{h \in \mathbb{C}^d : Uh = \mu h\}$ be the corresponding eigenspace. Then the fixed point space of $\tau$ is given by

$$F(\tau) = \bigoplus_{\mu \in \sigma(U)} B(\mathcal{H}_\mu)$$

and for any $\lambda \in \mathbb{T}$,

$$E_\lambda(\tau) = \{X \in M_d(\mathbb{C}) : X(\mathcal{H}_\mu) \subseteq \mathcal{H}_{\lambda \mu}, \forall \mu\}.$$ 

The Poisson boundary of $\tau$ is the commutant of $U$ and the peripheral Poisson boundary of $\tau$ is whole of $M_d(\mathbb{C})$. These results can be seen easily by taking $U$ as a diagonal matrix and observing that all matrix units of the standard basis become eigenvectors of $\tau$.

Here is a class of examples which are not $*$-homomorphisms.

**Example 3.2.** Fix some $d \geq 2$ and let $U, V$ be unitaries in $M_d(\mathbb{C})$ such that $VU = e^{\frac{2\pi i}{n}} UV$. (Such unitaries are known as Weyl unitaries.) Take $\mathcal{H} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$ ($n$-times, for some $n \in \mathbb{N}$). Let $p_j, 1 \leq j \leq n$ be positive scalars such that $\sum_{j} p_j = 1$. Let $U_j = I^\otimes_{j-1} \otimes U \otimes I^\otimes_{n-j}$. Define
\[ \tau : B(\mathcal{H}) \to B(\mathcal{H}) \text{ by } \]
\[ \tau(X) = \sum_j p_j U_j^* X U_j. \]

Then we can see that \( \tau(V^\otimes n) = e^{2\pi i} V^\otimes n \). So the peripheral Poisson boundary is distinct from the Poisson boundary.

Perhaps the most well-known example of a noncommutative Poisson boundary comes from the theory of Toeplitz operators. Understanding the peripheral Poisson boundary of this example is very instructive. Let us recall the setup.

Let \( \mathcal{K} = L^2(\mathbb{T}) \), with the inner product coming from the normalized Haar measure on the unit circle. It has the standard orthonormal basis \( \{ e_n : n \in \mathbb{Z} \} \) where
\[ e_n(e^{2\pi it}) = e^{2\pi int}, \quad 0 \leq t < 2\pi. \]

Now consider the Hardy space \( \mathcal{H} := H^2(\mathbb{T}) \subset L^2(\mathbb{T}) \) with its standard orthonormal basis \( \{ e_n : n \in \mathbb{Z}_+ \} \). Let \( \mathcal{A} \) be the von Neumann algebra \( B(\mathcal{H}) \). For \( f \in L^\infty(\mathbb{T}) \) we have the multiplication operator on \( \mathcal{K} \) defined by
\[ M_fh(z) = f(z)h(z), \quad z \in \mathbb{T}. \]

Then the Toeplitz operator with symbol \( f \in L^\infty(\mathbb{T}) \) is the operator
\[ T_f = PM_f|_\mathcal{H}. \]

where \( P \) is the projection of \( \mathcal{K} \) onto \( \mathcal{H} \). Let \( S \) denote the Toeplitz operator with symbol “\( z \)”, which is of course the unilateral shift on \( \mathcal{H} \) mapping \( e_n \) to \( e_{n+1} \) for \( n \in \mathbb{Z}_+ \). Consider the UCP map \( \tau : B(\mathcal{H}) \to B(\mathcal{H}) \) defined by
\[ \tau(X) = S^* XS, \quad X \in B(\mathcal{H}). \]

By a well known result of Brown and Halmos the space \( F(\tau) \) of fixed points of \( \tau \) is precisely the space of Toeplitz operators \( \{ T_f : f \in L^\infty(\mathbb{T}) \} \). Furthermore, the Poisson boundary structure on this space is given by
\[ T_f \circ T_g = T_{fg}, \quad f, g \in L^\infty(\mathbb{T}). \]

In particular, it is a commutative von Neumann algebra. Now to describe the peripheral Poisson boundary, let us look at the minimal dilation.

It is not hard to see that the minimal dilation of \( \tau \) is given by \( (\mathcal{K}, B(\mathcal{K}), \theta) \) where \( \theta \) is the automorphism of \( B(\mathcal{K}) \) given by
\[ \theta(Z) = U^* Z U, \quad Z \in B(\mathcal{K}), \]
with $U = M_\mathcal{K}$. The space of fixed points of $\theta$ is given by $\{M_f : f \in L^\infty(\mathbb{T})\}$. For $\lambda \in \mathbb{T}$, consider $V_\lambda \in B(\mathcal{K})$ defined by

$$V_\lambda f(z) = f(\lambda z), \ f \in \mathcal{K}.$$ 

Clearly $V_\lambda$ is a unitary and $V_\lambda e_n = \lambda^n e_n, \ \forall n \in \mathbb{Z}$. It is easy to see that $\theta(V_\lambda) = \lambda V_\lambda$, that is, $V_\lambda \in E_\lambda(\theta)$. Then by Corollary 2.10

$$E_\lambda(\theta) = \{M_f V_\lambda : f \in L^\infty(\mathbb{T})\} = \{V_\lambda M_f : f \in L^\infty(\mathbb{T})\}.$$ 

Then by the main theorem,

$$E_\lambda(\tau) = \{T_{\lambda,f} : f \in L^\infty(\mathbb{T})\},$$

where $T_{\lambda,f} = P_H M_f V_\lambda |_{H}$. The operators $T_{\lambda,f}$ can be called $\lambda$-Toeplitz operators, with $\lambda = 1$ corresponding to usual Toeplitz operators. Such operators have been studied before (See [18]). Since $M_f V_\lambda M_g V_\mu = M_f(\cdot)g(\cdot \cdot) V_\lambda V_\mu$, the product in the peripheral Poisson boundary is given by

$$T_{\lambda,f} \circ T_{\mu,g} = T_{\lambda\mu,(\cdot)g(\cdot \cdot)}.$$ 

Making use of the notion of covariant representations we may express the observations made above as follows.

**Theorem 3.3.** Let $\tau : B(L^2(\mathbb{T})) \to B(L^2(\mathbb{T}))$ be the map $X \mapsto S^* XS$ as above. Consider the $C^*$-dynamical system $(\mathbb{T}, L^\infty(\mathbb{T}), \alpha)$, where $\mathbb{T} \overset{\alpha}{\curvearrowright} L^\infty(\mathbb{T})$ is the natural action given by $\alpha(\lambda)f(z) = f(\lambda^{-1}z)$ for $f \in L^\infty(\mathbb{T}), \lambda \in \mathbb{T}$. Then the peripheral Poisson boundary of $\tau$ is equal to the $C^*$-algebra generated by the images of covariant representation $(\pi, u)$ on $L^2(\mathbb{T})$ given by

$$\pi(f) = M_f, \ u(\lambda) = V_\lambda, \ f \in L^\infty(\mathbb{T}), \lambda \in \mathbb{T}.$$ 

**Proof.** It is easy to see that these representations are covariant as

$$(V_\lambda M_f V_\lambda^*)(h)(z) = (M_f V_\lambda^*)(h(\bar{\lambda}z)) = f(\bar{\lambda}z)(V_\lambda)^*(h)(\bar{\lambda}z) = f(\bar{\lambda}z)h(z).$$ 

Hence $V_\lambda M_f V_\lambda^* = M_{\alpha(\lambda)(f)}$. □

In the theory of Toeplitz operators the observation that

$$\|T_f\| = \|M_f\| = \|f\|_\infty,$$

is considered as a very significant and useful result. From Theorem 2.3, we have an extension of this result to have

$$\|PXP\| = \|X\|$$
whenever $X$ is in $\text{span}\{M_fV_\lambda : f \in L^\infty(\mathbb{T}), \lambda \in \mathbb{T}\}$. In particular,

$$\| \sum_{j=1}^n c_j T_{\lambda_j, f_j} \| = \| \sum_{j=1}^n c_j M_{f_j} V_{\lambda_j} \| \quad (3.1)$$

for $c_j \in \mathbb{C}, f_j \in L^\infty(\mathbb{T}), \lambda_j \in \mathbb{T}, 1 \leq j \leq n, n \in \mathbb{N}$.

Surprisingly we are unable to take closure in the strong operator topology as the map $X \mapsto PXP$ won’t remain isometric if we do that. This can be seen as follows. In the example above, $F(\theta) = \{M_f : f \in L^\infty(\mathbb{T})\}$ is a von Neumann algebra whose commutant is itself. Therefore the commutant of $F(\theta) \cup \{V_\lambda\}$ is contained in $L^\infty(\mathbb{T})$ for any $\lambda$. Then it is easy to see that if $\lambda = e^{2\pi i \gamma}$ with $\gamma$ irrational, then this commutant consists of just scalars. In particular, the von Neumann algebra generated by $E(\theta)$ is whole of $B(K)$. Obviously the compression map $X \mapsto PXP$ is not isometric on $B(K)$. In other words, we are not able to take the peripheral Poisson boundary as a von Neumann algebra.

The previous example shows another feature of dilation. If $|\lambda| < 1$ and $X \in E_\lambda(\tau)$, it is possible that there is no $X' \in E_\lambda(\theta)$ such that $PXP = X$. This follows trivially for this example as an automorphism being isometric can’t have points outside of the unit circle in its point spectrum.

Now we consider the non-commutative extension of random walks on discrete groups studied by M. Izumi [20].

**Example 3.4.** Let $G$ be a countable discrete group and let $\mu$ be a probability measure on $G$. The Hilbert space under consideration is $l^2(G)$ and let $L$, $R$ be respectively the left and right regular representations of $G$, defined by $L(g)(\delta_h) = \delta_{gh}$ and $R(g)(\delta_h) = \delta_{h^{-1}g}$ on standard basis vectors and extended as unitaries. Let $\tau : B(l^2(G)) \to B(l^2(G))$ be defined by

$$\tau(X) = \sum_{g \in G} \mu(g) R(g) X R(g)^*, X \in B(l^2(G)).$$

Izumi [20] has identified the fixed point space of $\tau$ as a crossed product of classical harmonic elements with $G$. Let $S$ be the support of $\mu$, $S := \{g \in G : \mu(g) > 0\}$. Let $\hat{G}_S$ be the group of all characters on $G$ which are constant on $S$. Now consider any $\lambda \in \mathbb{T}$. Suppose there is a character $\chi \in \hat{G}_S$ such that

$$\chi(g) = \lambda, \ \forall g \in S. \quad (3.2)$$

Let $V_\chi$ be the unitary defined on $l^2(G)$ by setting $V_\chi \delta_h = \chi(h) \delta_h$ on basis vectors. Then $R(g)V_\chi R(g)^* \delta_h = R(g)V_\chi \delta_{h^{-1}g} = R(g) \chi(hg) \delta_{h^{-1}g} = \chi(hg) \delta_h = \lambda V_\chi \delta_h$ for every $g \in S$. Therefore $V_\chi$ is in $E_\lambda(\tau)$. Then by Lemma 2.10, we have

$$E_\lambda(\tau) = \{ A \circ V_\chi : A \in E_1(\tau) \} = \{ V_\chi \circ A : A \in E_1(\tau) \}.$$
We believe that if there is no character $\chi$ satisfying $\chi(0) \neq 0$, then $E_\lambda(\tau) = \{0\}$.

4. Discrete quantum dynamics

In quantum theory of open systems dynamics is described through completely positive maps. In discrete time it just means that we are studying powers $\tau, \tau^2, \ldots$ of a single completely positive maps. Here we study how peripheral eigenspaces move when we consider powers of a UCP map. Here our focus will be on quantum channels on the algebra $M_d(\mathbb{C})$ of $d \times d$ matrices and more generally CP maps which preserve a faithful normal state. However, initial few results are valid for CP maps on general $C^*$-algebras.

We find the following result from linear algebra very useful. It is valid even for infinite dimensional vector spaces. Recall that if $V_1, \ldots, V_n$ are subspaces of a vector space $V$, the subspace $W := \text{span}(\bigcup_{j=1}^n V_j)$ is a vector space direct sum if every vector $w$ in $W$ decomposes uniquely as $w = v_1 + \cdots + v_n$, with $v_j \in V_j$. We will denote this by writing $W = \bigvee_{j=1}^n V_j$.

**Lemma 4.1.** Let $V$ be a complex vector space and let $\tau : V \to V$ be a linear map. Let $p$ be a polynomial of degree $n$ with distinct roots $\lambda_1, \ldots, \lambda_n$ for some $n \geq 2$. Let $E_\lambda(\tau)$ be the kernel of $\tau - \lambda I$. Then
\[
\ker(p(\tau)) = \bigvee_{j=1}^n E_{\lambda_j}(\tau).
\]

**Proof.** Without loss of generality we may assume that $p$ is monic, so that $p(x) = (x - \lambda_1) \cdots (x - \lambda_n)$. For computational convenience we wish to assume $\lambda_i \neq 0$ for every $i$. If not, choose $\mu \in \mathbb{C}$ such that $\lambda_i := \lambda_i + \mu \neq 0$ for every $i$. Take $\tilde{p}(x) = \Pi_i(x - \lambda_i)$ and $\tilde{\tau} = \tau + \mu I$. Then $\tilde{p}(\tilde{\tau}) = p(\tau)$ and $E_{\lambda_i}(\tilde{\tau}) = E_{\lambda_i}(\tau)$ and so proving the result for $\tilde{p}(\tilde{\tau})$ will suffice. Therefore we may assume $\lambda_i \neq 0$ for every $i$. Now taking $d = \Pi_{k=1}^n (-\lambda_k)$, we have the algebraic identity
\[
p(x) = d[1 - \sum_{i=1}^n \frac{x \Pi_{j\neq i}(x - \lambda_j)}{\lambda_i \Pi_{j\neq i}(\lambda_i - \lambda_j)}],
\]
which can be verified easily by evaluating both sides at $0, \lambda_1, \ldots, \lambda_n$. Now if $y \in \ker(p(\tau))$, $p(\tau)y = 0$ implies $d[1 - \sum_{i=1}^n \frac{x \Pi_{j\neq i}(\tau - \lambda_j)}{\lambda_i \Pi_{j\neq i}(\lambda_i - \lambda_j)}]y = 0$. Hence $y = \sum_{i=1}^n \frac{x \Pi_{j\neq i}(\tau - \lambda_j)}{\lambda_i \Pi_{j\neq i}(\lambda_i - \lambda_j)}y = \sum_{i=1}^n y_i$, where $y_i = \frac{x \Pi_{j\neq i}(\tau - \lambda_j)}{\lambda_i \Pi_{j\neq i}(\lambda_i - \lambda_j)}y$. Note that $(\tau - \lambda_i)\Pi_{j\neq i}(\tau - \lambda_j) = p(\tau)$. Hence $(\tau - \lambda_i)y_i = 0$. So we get $y = \sum_i y_i$ with $y_i \in E_{\lambda_i}(\tau)$. The converse part namely if $y = \sum_i y_i$ with $y_i \in E_{\lambda_i}(\tau)$, then $y \in \ker(p(\tau))$ is obvious, and also in such a case $y_i$ can be recovered as $\frac{\Pi_{j\neq i}(\tau - \lambda_j)y}{\Pi_{j\neq i}(\lambda_i - \lambda_j)}$, which proves uniqueness of the decomposition.  

In the following special case of our current interest on peripheral spectrum we can do a kind of discrete Fourier inversion.
Lemma 4.2. Let $V$ be a complex vector space and let $\tau : V \to V$ be a linear map. Fix $k \geq 2$ and let $\omega = e^{\frac{2\pi i}{k}}$. Then

$$F(\tau^k) := E_1(\tau^k) = \bigvee_{j=0}^{k-1} E_{\omega^j}(\tau).$$

Furthermore, $x \in F(\tau^k)$ decomposes uniquely as $x = \sum_{j=0}^{k-1} x_j$ with $x_j \in E_{\omega^j}(\tau)$ where

$$x_j = \frac{1}{k} \sum_{l=0}^{k-1} \omega^{-lj} \tau^l(x). \quad (4.1)$$

Proof. The first part is an immediate consequence of the previous lemma, by considering the polynomial $p(x) = x^k - 1 = \prod_{j=0}^{k-1} (x - \omega^j)$. For the second part, as $\tau^k(x) = x$,

$$\tau(x_j) = \sum_{l=0}^{k-1} \omega^{-lj} \tau^{l+1}(x) = \omega^j \sum_{l=0}^{k-1} \omega^{-(l+1)j} \tau^{l+1}(x) = \omega^j \sum_{l=0}^{k-1} \omega^{-lj} \tau^l(x) = \omega^j x_j.$$  

Hence $x_j \in E_{\omega^j}(\tau)$. Further,

$$\sum_{j=0}^{k-1} x_j = \frac{1}{k} \sum_{j,l=0}^{k-1} \omega^{-lj} \tau^l(x) = \frac{1}{k} \sum_{j=0}^{k-1} \tau^j(x) \sum_{l=0}^{k-1} \omega^{-lj} = \frac{1}{k} x.k = x,$$

as $\sum_{l=0}^{k-1} \omega^{-lj} = \frac{\omega^{kj} - 1}{\omega - 1} = 0$ for $j \neq 0$. \qed

Fix $k \in \mathbb{N}$. Recall the $k$-cyclic Poisson boundary $P_{G_k}(\tau)$ defined in Definition, using the subgroup $G_k = \{1, \omega, \ldots, w^{k-1}\}$ of $\mathbb{T}$ where $\omega = e^{\frac{2\pi i}{k}}$.

Theorem 4.3. Let $A \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\tau : A \to A$ be a normal unital completely positive map. Then for every natural number $k \geq 1$, the peripheral Poisson boundary,

$$P(\tau^k) = P(\tau), \quad (4.2)$$

and the Poisson boundary

$$P_{\{1\}}(\tau^k) = P_{G_k}(\tau). \quad (4.3)$$

In particular, $k$-cyclic Poisson boundary is a von Neumann algebra for every $k$.

Proof. Let $(\mathcal{K}, \mathcal{B}, \theta)$, be the minimal dilation of $\tau$. It suffices to show these equalities for $\theta$ instead of $\tau$. Then the product $\circ$ is same as the usual product.

Fix $k \geq 2$. For any $\lambda \in \mathbb{T}$, the polynomial $p(x) = x^k - \lambda$ has distinct roots, $\{\mu \omega^j : 0 \leq j \leq (k-1)\}$ for some $\mu$ satisfying $\mu^k = \lambda$. Therefore, from Lemma 4.1

$$E_\lambda(\theta^k) = \bigvee_{j=0}^{k-1} E_{\mu \omega^j}(\theta).$$
As this is true for every $\lambda$ in the circle, we get $E(\theta^k) = E(\theta)$. Taking norm closures we get the first equality. The second part follows in a similar fashion from Lemma 4.2.

It may be noted that equation (4.3) along with Lemma 4.2, tells us as to how to decompose a vector in the Poisson boundary of $\tau^k$ in terms of elements of $k$-cyclic peripheral Poisson boundary of $\tau$.

**Corollary 4.4.** Let $A$ be a von Neumann algebra and let $\tau : A \to A$ be a normal UCP map. Suppose the group generated by the peripheral point spectrum of $\tau$ is finite. Then the peripheral Poisson boundary of $\tau$ is same as the Poisson boundary of $\tau^k$ for some $k$. In particular, it is a von Neumann algebra.

**Proof.** The hypothesis implies that there exists $k \in \mathbb{N}$ such that every $\lambda \in \mathbb{T}$ with $E_\lambda(\tau)$ non-trivial is of the form $e^{\frac{2\pi ij}{k}}$ for some $j$. Then by the previous theorem $\mathcal{P}(\tau) = \mathcal{P}_{G_k}(\tau) = \mathcal{P}_{\{1\}}(\tau^k)$. □

**Appendix**

Here we show that the compression map $T : \mathcal{P}(\theta) \to \mathcal{P}(\tau)$ of Theorem 2.3 is in fact, a complete isometry. To see this, let us consider the $m^{th}$ ampliation map

$$T^{(m)} := I_m \otimes T : M_m \otimes \mathcal{P}(\theta) \to M_m \otimes \mathcal{P}(\tau).$$

Since $M_m \otimes E(\theta)$ is dense in $M_m \otimes \mathcal{P}(\theta)$, it suffices to show that $\|T^{(m)}(Y)\| = \|Y\|$ for every $Y \in M_m \otimes E(\theta)$.

Let $Y = \sum_{i,j=1}^{m} E_{ij} \otimes y_{ij} \in M_m \otimes E(\theta)$, with $y_{ij} \in E(\theta)$, where $\{E_{ij} : i, j \leq m\}$ are the matrix units of $M_m$. For each $1 \leq i, j \leq m$, set $x_{ij} := py_{ij}p = T(y_{ij})$ and $X = T^{(m)}(Y) = \sum_{i,j=1}^{m} E_{ij} \otimes x_{ij}$.

As $\|I_m \otimes p\| = \|I_m\| \|p\| = \|p\| \leq 1$, we have

$$\|X\| = \left\| \sum_{i,j=1}^{m} E_{ij} \otimes (py_{ij}p) \right\| = \left\| \left( \sum_{i=1}^{m} E_{ii} \otimes p \right) \left( \sum_{i,j=1}^{m} E_{ij} \otimes y_{ij} \right) \left( \sum_{i=1}^{m} E_{ii} \otimes p \right) \right\| \leq \|Y\|.$$

Suppose $y_{ij} = \sum_{k_{ij}=1}^{d_{ij}} y_{ijk_{ij}}$ where $y_{ijk_{ij}} \in E_{\lambda_{ijk_{ij}}}(\theta)$. So from Lemma 2.1 we have

$$y_{ijk_{ij}} = s - \lim_{n \to \infty} \frac{1}{(\lambda_{ijk_{ij}})^n} \theta^n(x_{ijk_{ij}}),$$

where $s = \lim_{n \to \infty} \frac{1}{(\lambda_{ijk_{ij}})^n} \theta^n(x_{ijk_{ij}})$.
where \( x_{ijk} = py_{ijk}p \). Therefore

\[
Y = \sum_{i,j=1}^{m} E_{ij} \otimes y_{ij} = \sum_{i,j=1}^{m} E_{ij} \otimes \left( \sum_{k_{ij}=1}^{d_{ij}} \lambda_{ijk} \right) y_{ijk}
\]

\[
= s - \lim_{n \to \infty} \sum_{i,j=1}^{m} E_{ij} \otimes \left( \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} \theta^n(x_{ijk}) \right)
\]

\[
= s - \lim_{n \to \infty} \sum_{i,j=1}^{m} E_{ij} \otimes \theta^n \left( \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} \right)
\]

\[
= s - \lim_{n \to \infty} \left( \theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} \right) \right),
\]

where \( \theta^n(m) = I_m \otimes \theta^n \) is the \( m \)th ampliation of \( \theta^n \).

Let \( h \in \oplus_{i,j=1}^{m} K \) with \( ||h|| = 1 \) and let \( \epsilon > 0 \) be arbitrary. We claim that \( ||Yh|| \leq ||X|| + \epsilon \). From triangle inequality it follows that

\[
||Yh|| \leq \left| |Yh - \theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} \right) \right| h \right|
\]

\[
+ \left| |\theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} \right) h - \theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} x_{ijk} \right) h \right| h \right|
\]

\[
+ \left| |\theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} x_{ijk} \right) h \right| h \right|
\]

Using the expression of \( Y \) as a certain strong limit, one may find \( N \in \mathbb{N} \) such that for every \( n > N \) the first one of these three terms is less than \( \epsilon/2 \). The third term is less than or equal to \( ||X|| \) as \( \theta^n(m) \) is a contraction. The following inequalities explain how one can control the middle term by repeated application of Lemma [22] to the \( m^2 \) sequences determined by each fixed \( i \) and \( j \).

\[
\left| |\theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} \right) h - \theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} x_{ijk} \right) h \right| h \right|
\]

\[
= \left| |\theta^n(m) \left( \sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} - x_{ijk} \right) h \right| h \right|
\]

\[
\leq \left| |\sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k_{ij}=1}^{d_{ij}} \frac{1}{(\lambda_{ijk})^n} x_{ijk} - x_{ijk} ||h|| \right|
\]
\[ \leq \sum_{i,j=1}^{m} \left\| E_{ij} \right\| \left\| \left( \sum_{k_{ij}=1} \frac{1}{\lambda_{ij k_{ij}}} x_{ij k_{ij}} - x_{ij k_{ij}} \right) \right\| \leq \sum_{i,j=1}^{m} \left\| \left( \sum_{k_{ij}=1} \frac{1}{\lambda_{ij k_{ij}}} x_{ij k_{ij}} - x_{ij k_{ij}} \right) \right\| \leq \epsilon/2. \]

This shows that \( \|Y h\| \leq \|X\| + \epsilon \) and hence \( \|Y\| \leq \|X\| + \epsilon \). As \( \epsilon \) was arbitrary, we obtain \( \|X\| = \|Y\| \), which proves that \( T \) is a complete isometry.

Acknowledgments

Bhat gratefully acknowledges funding from SERB(India) through JC Bose Fellowship No. JBR/2021/000024. Talwar thanks ISI Bangalore: TSMD/PnC/2021-2022/RA/TSMU-Bangalore/009 for financial support through Research Associate scheme. A part of this work was completed during his research assistantship at Nazarbayev University and he appreciates their support. Kar is thankful to NBHM (India): 0204/21/2021/R&D-II/9988 for funding. We thank J. Sarkar for informing us about the literature on \( \lambda \)-Toeplitz operators. We extend our gratitude to the reviewer for carefully reading the manuscript and noticing that the map \( T \) is a complete isometry.

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