Exactly Solvable Many-Body Systems and Pseudo-Hermitian Point Interactions

Shao-Ming Fei

Department of Mathematics, Capital Normal University, Beijing 100037
Institute of Applied Mathematics, University of Bonn, D-53115 Bonn

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We study Hamiltonian systems with point interactions and give a systematic description of the corresponding boundary conditions and the spectrum properties for self-adjoint, PT-symmetric systems and systems with real spectra. The integrability of one dimensional many body systems with these kinds of point (contact) interactions are investigated for both bosonic and fermionic statistics.

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The complex generalization of conventional quantum mechanics has been investigated extensively in recent years. In particular it is shown that the standard formulation of quantum mechanics in terms of Hermitian Hamiltonians is overly restrictive and a consistent physical theory of quantum mechanics can be built on a complex Hamiltonian that is not Hermitian but satisfies the less restrictive and more physical condition of space-time reflection symmetry (PT symmetry) [1]. It is proven that if PT symmetry is not spontaneously broken, the dynamics of a non-Hermitian Hamiltonian system is still governed by unitary time evolution. A number of models with PT-symmetric and continuous interaction potentials have been constructed and studied [2]. In this article we study Hamiltonian systems with singular interaction potentials at a point. We give a systematic and complete description of the boundary conditions and the spectra properties for self-adjoint, PT-symmetric systems and systems with real spectra. We then study the integrability of one dimensional many body systems with these kinds of point interactions.

1 Self-adjoint point interactions

Self-adjoint quantum mechanical models describing a particle moving in a local singular potential concentrated at one or a discrete number of points have been extensively discussed in the literature, see e.g. [3] and references therein. One dimensional problems with contact interactions at, say, the origin \((x = 0)\) can be characterized by separated or nonseparated boundary conditions imposed on the wave function \(\psi\) at \(x = 0\) [4]. The first model of this type with the pairwise interactions determined by \(\delta\)-functions was suggested and investigated in [7]. Intensive studies of this model applied to statistical mechanics (particles having boson or fermion statistics) are given in [8] [9].
Nonseparated boundary conditions correspond to the cases where the perturbed operator is equal to the orthogonal sum of two self-adjoint operators in \( L^2(\mathbb{R}) \) and \( L^2(0, \infty) \). The family of point interactions for the one dimensional Schrödinger operator \(-\frac{d^2}{dx^2}\) can be described by unitary \(2 \times 2\) matrices via von Neumann formulas for self-adjoint extensions of symmetric operators. The non-separated boundary conditions describing the self-adjoint extensions have the following form
\[
\left( \begin{array}{c} \psi(+0) \\ \psi'(+0) \end{array} \right) = e^{i\theta} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} \psi(-0) \\ \psi'(-0) \end{array} \right),
\]
where \( ad - bc = 1, \theta, a, b, c, d \in \mathbb{R} \). The values \( \psi(x) \) is the wave function of a spinless particle with coordinate \( x \). The values \( \theta = b = 0, a = d = 1 \) in (1) correspond to the case of a positive (resp. negative) \( \delta \)-function potential for \( c > 0 \) (resp. \( c < 0 \)). For general \( a, b, c \) and \( d \), the properties of the corresponding Hamiltonian systems have been studied in detail, see e.g. \([6, 10, 12]\).

The separated self-adjoint boundary conditions are described by
\[
\psi'(+0) = h^+ \psi(+0), \quad \psi'(-0) = h^- \psi(-0),
\]
where \( h^\pm \in \mathbb{R} \cup \{\infty\} \). \( h^+ = \infty \) or \( h^- = \infty \) correspond to Dirichlet boundary conditions and \( h^+ = 0 \) or \( h^- = 0 \) correspond to Neumann boundary conditions.

2 PT-symmetric point interactions

An operator is said to be PT-symmetric if it commutes with the product operator of the parity operator \( P \) and the time reversal operator \( T \). It can be shown that the family of PT-symmetric second derivative operators with point interactions at the origin coincides with the set of restrictions of the second derivative operator to the domain of functions satisfying the boundary conditions at the origin \([11]\):
\[
\left( \begin{array}{c} \psi(+0) \\ \psi'(+0) \end{array} \right) = B \left( \begin{array}{c} \psi(-0) \\ \psi'(-0) \end{array} \right),
\]
for non-separated type, where
\[
B = e^{i\theta} \left( \begin{array}{cc} \sqrt{1 + bc} e^{i\phi} \\ b \sqrt{1 + bc} e^{-i\phi} \end{array} \right),
\]
the real parameters \( b \geq 0, c \geq -1/b^2 \), \( \theta, \phi \in [0, 2\pi] \); or corresponding to the separated type
\[
h_0 \psi'(+0) = h_1 e^{i\theta} \psi(+0), \quad h_0 \psi'(-0) = -h_1 e^{-i\theta} \psi(-0)
\]
with the real phase parameter \( \theta \in [0, 2\pi] \) and the parameter \( h = (h_0, h_1) \) taken from the (real) projective space \( \mathbb{P}^1 \).

1) If the parameter \( b \) is equal to zero, then the second inequality should be neglected.
The spectrum of any PT-symmetric second derivative operator with point interactions at the origin consists of the branch \([0, \infty)\) of the absolutely continuous spectrum and at most two (counting multiplicity) eigenvalues, which are real negative or are (complex) conjugated to each other. The eigenvalues corresponding to PT-symmetric eigenfunctions are real and negative. Every eigenfunction corresponding to any real eigenvalue can be chosen either PT-symmetric or -antisymmetric.

The spectrum of the PT-symmetric second derivative operator with non-separated type point interaction at the origin is pure real if and only if the parameters appearing in (3) satisfy in addition at least one of the following conditions:

\[
bc \sin ^2 \phi \leq \cos ^2 \phi; \tag{5}
\]

\[
bc \sin ^2 \phi \geq \cos ^2 \phi \quad \text{and} \quad \cos \phi \geq 0. \tag{6}
\]

### 3 Point interactions with real spectra

We consider further non-separated type boundary conditions at the origin leading to second derivative operators with real spectrum. A general form of the boundary condition can be written as

\[
\begin{pmatrix}
\psi(0) \\
\psi'(0)
\end{pmatrix} =
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
\psi(-0) \\
\psi'(-0)
\end{pmatrix}, \tag{7}
\]

where \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\). We suppose that the matrix \(B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) appearing in the boundary conditions (7) is non degenerate (from \(GL(2, \mathbb{C})\)). Again it is easy to prove that the operator has branch of absolutely continuous spectrum \([0, \infty)\). To study its discrete spectrum we use the following Ansatz for the eigenfunction

\[
\psi(x) = \begin{cases} 
c_1 e^{-ikx}, & x < 0 \\
c_2 e^{ikx}, & x > 0
\end{cases}, \quad \Im k > 0,
\]

corresponding to the energy \(\lambda = k^2\). Substituting this function into the boundary conditions (7) we get the dispersion equation \(k^2 \beta + ik(\alpha + \delta) - \gamma = 0\).

The set of coefficients \(\alpha, \beta, \gamma, \delta\) satisfying the condition \(\exists k_{1,2} \leq 0\) can be parameterized by 8 real parameters and leads to operators with pure absolutely continuous spectrum \([0, \infty)\). Pure imaginary solutions to the dispersion equation leads to nontrivial discrete spectrum. Set \(\tau = \alpha + \delta\). The solutions are pure imaginary if and only if the following conditions are satisfied:

\[
\tau = te^{i\theta}, \quad \beta = be^{i\theta}, \quad \gamma = ce^{i\theta}, \quad 4b^2 \leq \frac{t^2}{\beta^2}, \tag{8}
\]

for \(\beta \neq 0\), where \(t, b, c\) are real numbers. If \(\beta = 0\), the spectrum is guaranteed to be real when \(\alpha + \delta\) and \(\gamma\) have the same phases, i.e.,

\[
\tau = te^{i\theta}, \quad \gamma = ce^{i\theta}. \tag{9}
\]
The real spectrum point interaction (8) is parameterized by 6 real parameters. The four-parameter family of self-adjoint (non-separated) boundary conditions (1) is contained in this 6-parameter family. The family of PT-symmetric (non-separated) boundary conditions leading to operators with real spectrum is also included in the family of boundary conditions (8) or (9).

4 Integrable many-body systems

The self-adjoint boundary conditions (1) and (2), PT-symmetric boundary conditions (3) and (4), and the real spectrum boundary conditions (7) with the parameters satisfying (8) or (9) also describe two spinless particles moving in one dimension with contact interaction when they meet (i.e. the relative coordinate \( x = 0 \)). When the particles have spin \( s \) but without any spin coupling among the particles, \( \psi \) represents any one of the components of the wave function. In the following we study the integrability of one dimensional systems of \( N \)-identical particles with general contact interactions described by the non-separated boundary conditions that are imposed on the relative coordinates of the particles. We first consider the case of two particles (\( N = 2 \)) with coordinates \( x_1, x_2 \) and momenta \( k_1, k_2 \) respectively. Each particle has \( n \)-‘spin’ states designated by \( s_1 \) and \( s_2 \), \( 1 \leq s_i \leq n \). For \( x_1 \neq x_2 \) these two particles are free. The wave functions \( \varphi \) are symmetric (resp. antisymmetric) with respect to the interchange \( (x_1, s_1) \leftrightarrow (x_2, s_2) \) for bosons (resp. fermions).

In the region \( x_1 < x_2 \), from the Bethe ansatz the wave function is of the form,

\[
\varphi = \alpha_{12} e^{i(k_1 x_1 + k_2 x_2)} + \alpha_{21} e^{i(k_2 x_1 + k_1 x_2)},
\]

where \( \alpha_{12} \) and \( \alpha_{21} \) are \( n^2 \times 1 \) column matrices. In the region \( x_1 > x_2 \),

\[
\varphi = (P_{12} \alpha_{12}) e^{i(k_1 x_2 + k_2 x_1)} + (P_{12} \alpha_{21}) e^{i(k_2 x_2 + k_1 x_1)},
\]

where according to the symmetry or antisymmetry conditions, \( P_{12} = P^{12} \) for bosons and \( P^{12} = -P^{12} \) for fermions, \( P^{12} \) being the operator on the \( n^2 \times 1 \) column that interchanges \( s_1 \leftrightarrow s_2 \).

Set \( k_{12} = (k_1 - k_2)/2 \). In the center of mass coordinate \( X = (x_1 + x_2)/2 \) and the relative coordinate \( x = x_2 - x_1 \), we get, by substituting (10) and (11) into the boundary conditions (7) at \( x = 0 \),

\[
\begin{align*}
\alpha_{12} + \alpha_{21} &= \alpha P^{12} (\alpha_{12} + \alpha_{21}) + i \beta k_{12} P^{12} (\alpha_{12} - \alpha_{21}), \\
\delta k_{12} (\alpha_{21} - \alpha_{12}) &= \gamma P^{12} (\alpha_{12} + \alpha_{21}) + i \delta k_{12} P^{12} (\alpha_{12} - \alpha_{21}).
\end{align*}
\]

Eliminating the term \( P^{12} \alpha_{12} \) from (12) we obtain the relation

\[
\alpha_{21} = Y_{21}^{12} \alpha_{12},
\]

where

\[
Y_{21}^{12} = \frac{2 i k_{12} (\alpha \delta - \beta \gamma) P^{12} + i k_{12} (\alpha - \delta) + (k_{12})^2 \beta + \gamma}{i k_{12} (\alpha + \delta) + (k_{12})^2 \beta - \gamma}.
\]
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For $N \geq 3$ and $x_1 < x_2 < ... < x_N$, the wave function is given by

$$\varphi = \alpha_{12...N}e^{i(k_1x_1+k_2x_2+...+k_Nx_N)} + \alpha_{12...N}e^{i(k_2x_1+k_1x_2+...+k_Nx_N)} + \sum_{p=1}^{N!-2} \text{other terms}. \tag{15}$$

The columns $\alpha$ have $n^N \times 1$ dimensions. The wave functions in the other regions are determined from (15) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane $x_i = x_{i+1}$, $i = 1, 2, ..., N-1$, from similar considerations we have

$$\alpha_{l_1l_2...l_{i-1}l_{i+1}...l_N} = Y_{i+l_1l_1...l_{i-1}l_{i+1}...l_N}^{i+l_1l_1...l_{i-1}l_{i+1}...l_N}, \tag{16}$$

where

$$Y_{i+l_1l_1...l_{i-1}l_{i+1}...l_N}^{i+l_1l_1...l_{i-1}l_{i+1}...l_N} = \frac{2ik_{l_1l_1}((\alpha - \beta)\gamma)p_{i+l_1l_1} + ik_{l_1l_1}(\alpha - \beta) + (k_{l_1l_1})^2\beta + \gamma}{ik_{l_1l_1}(\alpha + \beta) + (k_{l_1l_1})^2\beta - \gamma}. \tag{17}$$

Here $k_{l_1l_1} = (k_i - k_{i+1})/2$ play the role of spectral parameters. $p_{i+l_1l_1}$ for bosons and $p_{i+l_1l_1}$ for fermions, with $p_{i+l_1l_1}$ the operator on the $n^N \times 1$ column that interchanges $s_i \leftrightarrow s_{i+1}$.

For consistency $Y$ must satisfy the Yang-Baxter equation with spectral parameters [8, 14], i.e.,

$$Y_{ij}^{m,m+1}Y_{kj}^{m+1,m+2}Y_{ki}^{m,m+1} = Y_{ji}^{m,m+1}Y_{kj}^{m+1,m+2}Y_{ki}^{m,m+1},$$

or

$$Y_{ij}^{mr}Y_{kj}^{rs}Y_{ki}^{mr} = Y_{ji}^{rs}Y_{kj}^{mr}Y_{ki}^{rs} \tag{18}$$

if $m, r, s$ are all unequal, and

$$Y_{ij}^{mr}Y_{ji}^{mr} = 1, \quad Y_{ij}^{mr}Y_{kl}^{sq} = Y_{kl}^{sq}Y_{ij}^{mr} \tag{19}$$

if $m, r, s, q$ are all unequal. These Yang-Baxter relations are satisfied when

$$\alpha \delta - \beta \gamma = 1, \quad \beta = 0, \quad \alpha = \delta, \quad \tag{20}$$

i.e., $\beta = 0, \alpha = \delta = \pm 1$.

Therefore for self-adjoint contact interactions [1], the $N$-body system is integrable when $\theta = 0$, $a = d = \pm 1$, $b = 0$, $c$ arbitrary. The case $a = d = 1$, $\theta = b = 0$ corresponds to the usual $\delta$-function interactions, which has been investigated in [5] [9]. The case $a = d = -1$, $\theta = b = 0$, is related to a kind of anti-$\delta$ interaction [2].

For $N$-body systems with PT-symmetric contact interactions [9], the integrable condition [20] implies that $\theta = \phi = b = 0$, which is just the usual self-adjoint $\delta$-interaction.

From [9] and [20], we also have that an many-body system with contact interaction and real spectra is integrable only when $\alpha = \delta = \pm 1, \gamma = c$ for some $c \in \mathbb{R}$,

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Fig. 1. Relations among point interactions according to integrability.

which is again the self-adjoint $\delta$-type interactions, see Fig.1 for the relations among point interactions according to integrability.

We have presented a complete picture of self-adjoint, PT-symmetric and real spectrum point interactions, and their corresponding integrability. What we concerned here are just the case of particles with only pure contact interactions, and the possible contact coupling of the spins of two particles are not taken into account [15]. A further study along this direction would possibly give rise to more interesting integrable quantum many-body systems with various symmetries and spectrum properties.

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