TOPOLOGICAL MONOIDS OF ALMOST MONOTONE INJECTIVE CO-FINITE PARTIAL SELFMAPS OF POSITIVE INTEGERS

IVAN CHUCHMAN AND OLEG GUTIK

ABSTRACT. In this paper we study the semigroup $I^\omega_\infty(N)$ of partial co-finite almost monotone bijective transformations of the set of positive integers $N$. We show that the semigroup $I^\omega_\infty(N)$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also we prove that every Baire topology $\tau$ on $I^\omega_\infty(N)$ such that $(I^\omega_\infty(N), \tau)$ is a semitopological semigroup is discrete, describe the closure of $(I^\omega_\infty(N), \tau)$ in a topological semigroup and construct non-discrete Hausdorff semigroup topologies on $I^\omega_\infty(N)$.

1. Introduction and preliminaries

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [6, 7, 10, 27]. By $\omega$ we shall denote the first infinite cardinal.

An algebraic semigroup $S$ is called inverse if for any element $x \in S$ there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function $\text{inv}: S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called an inversion.

If $S$ is a semigroup, then by $E(S)$ we shall denote the band (i.e. the subset of idempotents) of $S$. If the band $E(S)$ is a non-empty subset of $S$, then the semigroup operation on $S$ determines the partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called natural. A semilattice is a nonempty set with an associative operation $\cdot$ and an identity $e$ such that for any $a, b \in E(S)$ we have $a \cdot b = b \cdot a$. A semilattice $E$ is called linearly ordered or chain if the semilattice operation admits a linear natural order on $E$. A maximal chain of a semilattice $E$ is a chain which is properly contained in no other chain of $E$. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [25, Definition II.5.12] chain $L$ is called $\omega$-chain if $L$ is isomorphic to $\{0, -1, -2, -3, \ldots \}$ with the usual order $\leq$. Let $E$ be a semilattice and $e \in E$. We denote $\down{e} = \{f \in E \mid f \leq e\}$ and $\up{e} = \{f \in E \mid e \leq f\}$. By $(P_{<\omega}(N), \subseteq)$ we shall denote the free semilattice with identity over the set of positive integers $N$.

If $S$ is a semigroup, then by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ the Green relations on $S$ (see [7]):

\begin{align*}
\mathcal{R}b & \text{ if and only if } aS^1 = bS^1; \\
\mathcal{L}b & \text{ if and only if } S^1a = S^1b; \\
\mathcal{D} & = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}; \\
\mathcal{H} & = \mathcal{L} \cap \mathcal{R}.
\end{align*}

A semigroup $S$ is called simple if $S$ does not contain proper two-sided ideals and bisimple if all elements of $S$ are $\mathcal{D}$-equivalent.

A semitopological (resp. topological) semigroup is a topological space together with a separately (resp. jointly) continuous semigroup operation.

Let $I_\lambda$ denote the set of all partial one-to-one transformations of a set $X$ of cardinality $\lambda$ together with the following semigroup operation: $x(\alpha \beta) = (x\alpha)\beta$ if $x \in \text{dom}(\alpha \beta) = \{y \in \text{dom} \alpha \mid y\alpha \in \text{dom} \beta\}$,
for $\alpha, \beta \in \mathcal{I}_\lambda$. The semigroup $\mathcal{I}_\lambda$ is called the symmetric inverse semigroup over the set $X$ (see [7]). The symmetric inverse semigroup was introduced by Wagner [29] and it plays a major role in the theory of semigroups.

We denote $\mathcal{I}_\lambda^n = \{ \alpha \in \mathcal{I}_\lambda \mid \text{rank } \alpha \leq n \}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathcal{I}_\lambda^n$ (or $n$) is an inverse semigroup, $\mathcal{I}_\lambda^n$ is an ideal of $\mathcal{I}_\lambda$ for each $n = 1, 2, 3, \ldots$. Further, we shall call the semigroup $\mathcal{I}_\lambda^n$ the symmetric inverse semigroup of finite transformations of the rank $n$.

Let $\mathbb{N}$ be the set of all positive integers. By $\mathcal{I}_\omega(\mathbb{N})$ we shall denote the semigroup of monotone, non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N} \setminus \text{dom } \varphi$ and $\mathbb{N} \setminus \text{rank } \varphi$ are finite for all $\varphi \in \mathcal{I}_\omega(\mathbb{N})$. Obviously, $\mathcal{I}_\omega(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathcal{I}_\omega$. The semigroup $\mathcal{I}_\omega(\mathbb{N})$ is called the semigroup of co-finite monotone partial bijections of $\mathbb{N}$ [19].

We shall denote every element $\alpha$ of the semigroup $\mathcal{I}_\omega$ by

$$\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \ldots \\ m_1 & m_2 & m_3 & m_4 & \ldots \end{pmatrix}$$

and this means that $\alpha$ maps the positive integer $n_i$ into $m_i$ for all $i = 1, 2, 3, \ldots$. We observe that an element $\alpha$ of the semigroup $\mathcal{I}_\omega$ is an element of the semigroup $\mathcal{I}_\omega(\mathbb{N})$ if and only if it satisfies the following conditions:

1. the sets $\mathbb{N} \setminus \{n_1, n_2, n_3, n_4, \ldots \}$ and $\mathbb{N} \setminus \{m_1, m_2, m_3, m_4, \ldots \}$ are finite; and

2. $n_1 < m_2 < m_3 < m_4 < \ldots$

A partial map $\alpha : \mathbb{N} \to \mathbb{N}$ is called almost monotone if there exists a finite subset $A$ of $\mathbb{N}$ such that the restriction $\alpha |_{\mathbb{N} \setminus A} : \mathbb{N} \setminus A \to \mathbb{N}$ is a monotone partial map.

By $\mathcal{I}_\omega(\mathbb{N})$ we shall denote the semigroup of monotone, almost non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N} \setminus \text{dom } \varphi$ and $\mathbb{N} \setminus \text{rank } \varphi$ are finite for all $\varphi \in \mathcal{I}_\omega(\mathbb{N})$. Obviously, $\mathcal{I}_\omega(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathcal{I}_\omega$ and the semigroup $\mathcal{I}_\omega(\mathbb{N})$ is an inverse subsemigroup of $\mathcal{I}_\omega(\mathbb{N})$ too. The semigroup $\mathcal{I}_\omega(\mathbb{N})$ is called the semigroup of co-finite almost monotone partial bijections of $\mathbb{N}$. We observe that an element $\alpha$ of the semigroup $\mathcal{I}_\omega$ is an element of the semigroup $\mathcal{I}_\omega(\mathbb{N})$ if and only if it satisfies conditions (i) and (ii):

(iii) there exists a positive integer $i$ such that $n_i < n_{i+1} < n_{i+2} < n_{i+3} < \ldots$ and $m_i < m_{i+1} < m_{i+2} < m_{i+3} < \ldots$

Further by $\mathcal{I}$ we shall denote the identity of the semigroup $\mathcal{I}_\omega(\mathbb{N})$.

The bicyclic semigroup $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by elements $p$ and $q$ subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism $h$ of the bicyclic semigroup is either an isomorphism or the image of $\mathcal{C}(p, q)$ under $h$ is a cyclic group (see [21 Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known result of Andersen [1] states that a (0-)simple semigroup is completely (0-)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology and a topological semigroup $S$ can contain $\mathcal{C}(p, q)$ only as an open subset [9]. Neither stable nor $\Gamma$-compact topological semigroups can contain a copy of the bicyclic semigroup [21].

Also, the bicyclic semigroup does not embed into a countably compact topological inverse semigroup [18]. Moreover, in [3] and [4] the conditions were given when a countable compact or pseudocompact topological semigroup does not contain the bicyclic semigroup. However, Banakh, Dimitrova and Gutik constructed with set-theoretic assumptions (Continuum Hypothesis or Martin Axiom) an example of a Tychonoff countable compact topological semigroup which contains the bicyclic semigroup [4].

Many semigroup theorists have considered a topological semigroup of (continuous) transformations of an arbitrary topological space. Beida [5], Orlov [23] and Subbi [28] have considered semigroup and inverse semigroup topologies of semigroups of partial homeomorphisms of some classes of topological spaces.

Gutik and Pavlyk [14] considered the special case of the semigroup $\mathcal{I}_\lambda^n$: an infinite topological semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$. They showed that an infinite topological semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ does not embed into a compact topological semigroup and that $B_\lambda$ is algebraically
$h$-closed in the class of topological inverse semigroups. They also described the Bohr compactification of $B_\lambda$, minimal semigroup and minimal semigroup inverse topologies on $B_\lambda$.

Gutik, Lawson and Repovš \cite{GutikLawsonRepovs2005} introduced the notion of a semigroup with a tight ideal series and investigated their closures in semitopological semigroups, particularly inverse semigroups with continuous inversion. As a corollary they showed that the symmetric inverse semigroup of finite transformations $\mathcal{I}_\lambda^n$ of infinite cardinal $\lambda$ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion. They also derived related results about the nonexistence of (partial) compactifications of classes of considered semigroups.

Gutik and Reiter \cite{GutikReiter2004} showed that the topological inverse semigroup $\mathcal{I}_\lambda^n$ is algebraically $h$-closed in the class of topological inverse semigroups. They also proved that a topological semigroup $S$ with countably compact square $S \times S$ does not contain the semigroup $\mathcal{I}_\lambda^n$ for infinite cardinals $\lambda$ and showed that the Bohr compactification of an infinite topological semigroup $\mathcal{I}_\lambda^n$ is the trivial semigroup.

In \cite{GutikReiter2007} Gutik and Reiter showed that the symmetric inverse semigroup of finite transformations $\mathcal{I}_\lambda^n$ of infinite cardinal $\lambda$ is algebraically closed in the class of semitopological inverse semigroups with continuous inversion. There they described all congruences on the semigroup $\mathcal{I}_\lambda^n$ and all compact and countably compact topologies $\tau$ on $\mathcal{I}_\lambda^n$ such that $(\mathcal{I}_\lambda^n, \tau)$ is a semitopological semigroup.

Gutik, Pavlyk and Reiter \cite{GutikPavlykReiter2014} showed that a topological semigroup of finite partial bijections $\mathcal{S}_\lambda^n$ is algebraically isomorphic to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. They also proved that every Baire topology $\tau$ on $\mathcal{S}_\lambda^n$ such that $(\mathcal{S}_\lambda^n, \tau)$ is a topological inverse semigroup, is discrete and describe the closure of $(\mathcal{S}_\lambda^n, \tau)$ in a topological semigroup.

We remark that the bicyclic semigroup is isomorphic to the semigroup $C_\infty(\pi, \sigma)$ which is generated by partial transformations $\pi$ and $\sigma$ of the set of positive integers $\mathbb{N}$, defined as follows:

$$(n)\pi = n + 1 \quad \text{if} \quad n \geq 1, \quad \text{and} \quad (n)\sigma = n - 1 \quad \text{if} \quad n > 1.$$

Therefore the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ contains an isomorphic copy of the bicyclic semigroup $C(p, q)$.

In the present paper we study the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. We show that the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. They proved that every locally compact topology $\tau$ on $\mathcal{I}_\infty^\nu(\mathbb{N})$ such that $(\mathcal{I}_\infty^\nu(\mathbb{N}), \tau)$ is a topological inverse semigroup, is discrete and describe the closure of $(\mathcal{I}_\infty^\nu(\mathbb{N}), \tau)$ in a topological semigroup.

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In the present paper we study the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. We show that the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also we prove that every Baire topology $\tau$ on $\mathcal{I}_\infty^\nu(\mathbb{N})$ such that $(\mathcal{I}_\infty^\nu(\mathbb{N}), \tau)$ is a semitopological semigroup is discrete, describe the closure of $(\mathcal{I}_\infty^\nu(\mathbb{N}), \tau)$ in a topological semigroup and construct non-discrete Hausdorff semigroup topologies on $\mathcal{I}_\infty^\nu(\mathbb{N})$.

2. Algebraic properties of the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$

Proposition 2.1. \(i\) An element $\alpha$ of the semigroup $\mathcal{I}_\infty^\nu(\mathbb{N})$ is an idempotent if and only if \((x)\alpha = x\) for every $x \in \text{dom}\, \alpha$, and hence $E(\mathcal{I}_\infty^\nu(\mathbb{N})) = E(\mathcal{I}_\infty^\nu(\mathbb{N}))$.

\(ii\) If $\varepsilon, \iota \in E(\mathcal{I}_\infty^\nu(\mathbb{N}))$, then $\varepsilon \preceq \iota$ if and only if $\text{dom}\, \varepsilon \subseteq \text{dom}\, \iota$.

\(iii\) The semilattice $E(\mathcal{I}_\infty^\nu(\mathbb{N}))$ is isomorphic to $(\mathcal{P}_{<\omega}(\mathbb{N}), \subseteq)$ under the mapping $(\varepsilon)h = \mathbb{N}\setminus\text{dom}\, \varepsilon$.

\(iv\) Every maximal chain in $E(\mathcal{I}_\infty^\nu(\mathbb{N}))$ is an $\omega$-chain.

\(v\) For every $\varepsilon, \iota \in E(\mathcal{I}_\infty^\nu(\mathbb{N}))$ there exists $\alpha \in \mathcal{I}_\infty^\nu(\mathbb{N})$ such that $\alpha\varepsilon = \varepsilon$ and $\alpha^{-1}\iota = \iota$.

\(vi\) $\mathcal{I}_\infty^\nu(\mathbb{N})$ is a simple semigroup.
(vii) $αBβ$ in $I^\infty_\nu(N)$ if and only if $\text{dom} \ α = \text{dom} \ β$.
(viii) $αLβ$ in $I^\infty_\nu(N)$ if and only if $\text{rank} \ α = \text{rank} \ β$.
(ix) $αHβ$ in $I^\infty_\nu(N)$ if and only if $\text{dom} \ α = \text{dom} \ β$ and $\text{rank} \ α = \text{rank} \ β$.
(x) $I^\nu_\nu(N)$ is a bisimple semigroup.

Proof. Statements (i) – (iv) are trivial and their proofs follow from the definition of the semigroup $I^\nu_\nu(N)$.

For the idempotents $ε = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $t = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ we put

$α = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then $αα^{-1} = ε$ and $α^{-1}α = t$.

Let $α = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $β = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ be any elements of the semigroup $I^\nu_\nu(N)$, where $n_i, m_i, k_i, l_i ∈ N$ for $i = 1, 2, 3, \ldots$. We put $γ = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ n_1 & n_2 & n_3 & n_4 & \cdots \end{pmatrix}$

and $δ = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then we have that $γαδ = β$. Therefore $I^\nu_\nu(N) ∗ α ∗ I^\nu_\nu(N) = I^\nu_\nu(N)$ for any $α ∈ I^\nu_\nu(N)$ and hence $I^\nu_\nu(N)$ is a simple semigroup.

Let $α, β ∈ I^\nu_\nu(N)$ be such that $αBβ$. Since $αI^\nu_\nu(N) = βI^\nu_\nu(N)$ and $I^\nu_\nu(N)$ is an inverse semigroup, Theorem 1.17 \textsuperscript{[7]} implies that $αI^\nu_\nu(N) = αα^{-1}I^\nu_\nu(N)$, $βI^\nu_\nu(N) = ββ^{-1}I^\nu_\nu(N)$ and $αα^{-1} = ββ^{-1}$. Hence $\text{dom} \ α = \text{dom} \ β$.

Conversely, let $α, β ∈ I^\nu_\nu(N)$ be such that $\text{dom} \ α = \text{dom} \ β$. Then $αα^{-1} = ββ^{-1}$. Since $I^\nu_\nu(N)$ is an inverse semigroup, Theorem 1.17 \textsuperscript{[7]} implies that $αI^\nu_\nu(N) = αα^{-1}I^\nu_\nu(N) = βI^\nu_\nu(N)$ and hence $αI^\nu_\nu(N) = βI^\nu_\nu(N)$.

The proof of statement (viii) is similar to (vii).

Statement (ix) follows from (vii) and (viii).

By statements (vii) and (viii) it is sufficient to show that every distinct idempotents of the semigroup $I^\nu_\nu(N)$ are $D$-equivalent. For the idempotents $ε = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $t = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ we put $α = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then by statements (vii) and (viii) we have that $εRα$ and $αLε$, and hence $εDε$. □

Proposition 2.2. For every $α, β ∈ I^\nu_\nu(N)$, both sets $\{ χ ∈ I^\nu_\nu(N) | α ∗ χ = β \}$ and $\{ χ ∈ I^\nu_\nu(N) | χ ∗ α = β \}$ are finite. Consequently, every right translation and every left translation by an element of the semigroup $I^\nu_\nu(N)$ is a finite-to-one map.

Proof. We denote $A = \{ χ ∈ I^\nu_\nu(N) | α ∗ χ = β \}$ and $B = \{ χ ∈ I^\nu_\nu(N) | α^{-1} ∗ χ = α^{-1} ∗ β \}$. Then $A ⊆ B$ and the restriction of any partial map $χ ∈ B$ to $\text{dom} \ (α^{-1} ∗ α)$ coincides with the partial map $α^{-1} ∗ β$. Since every partial map from the semigroup $I^\nu_\nu(N)$ is almost monotone (i.e., almost non-decreasing) and co-finite, the set $B$ is finite and hence so is $A$. □

For an arbitrary non-empty set $X$ we denote by $S_\infty(X)$ the group of all bijective transformations of $X$ with finite supports (i.e., $α ∈ S_\infty(X)$ if and only if the set $\{ x ∈ X | (x)α ≠ x \}$ is finite).

The definition of the semigroup $I^\nu_\nu(N)$ implies the following proposition:

Proposition 2.3. Every maximal subgroup of the semigroup $I^\nu_\nu(N)$ is isomorphic to $S_\infty(N)$.

The semigroup $I^\nu_\nu(N)$ contains $I^\nu_\nu(N)$ as a subsemigroup and Theorem 2.9 of \textsuperscript{[19]} states that if $S$ is a semigroup and $h: I^\nu_\nu(N) \to S$ is a non-annihilating homomorphism, then either $h$ is a monomorphism or $(I^\nu_\nu(N))h$ is a cyclic subgroup of $S$. This arises the following problem: To describe all homomorphisms of the semigroup $I^\nu_\nu(N)$.

The definition of the semigroup $I^\nu_\nu(N)$ implies the following proposition:
Proposition 2.4. For every $γ ∈ 𝕀₈^∞(N)$ there exists $n_γ ∈ N$ such that $i - n_γ = (i)α - (n_γ)α$ for all $i ≥ n_γ$, $i ∈ N$.

Lemma 2.5. For every $γ ∈ 𝕀₈^∞(N)$ there exists an idempotent $ε ∈ ℋ_n(π, σ)$ such that $γ · ε, ε · γ ∈ ℋ_n(π, σ)$. Consequently, for every idempotent $i ∈ 𝕀₈^∞(N)$ there exists $ε_0 ∈ E(ℋ_n(π, σ))$ such that $i · ε_0 = ε_0 · i = ε_0$.

Proof. Let $n_γ ∈ N$ be such as in the statement of Proposition 2.4. We put $m_γ = \max\{n_γ, (n_γ)γ\}$ and define

$$
ε = \begin{pmatrix}
m_γ & m_γ + 1 & m_γ + 2 & \cdots \\
m_γ & m_γ + 1 & m_γ + 2 & \cdots
\end{pmatrix}.
$$

Then we have that $γ · ε, ε · γ ∈ ℋ_n(π, σ)$.

Let $i$ be an arbitrary idempotent of the semigroup $ℋ₈^∞(N)$. By the first assertion of the lemma there exists $ε ∈ E(ℋ_n(π, σ))$ such that $i · ε = ε · i ∈ ℋ_n(π, σ)$. Since the semigroup $ℋ₈^∞(N)$ is inverse Theorem 1.17 [7] implies that $ε_0 = i · ε = ε · i$ is an idempotent of $ℋ_n(π, σ)$. Hence we have that $i · ε_0 = ε_0 · i = ε_0$.

Lemma 2.6. Let $S$ be a semigroup and $h: ℋ₈^∞(N) → S$ be a non-annihilating homomorphism such that the set $(E(ℋ_n(π, σ)))h$ is singleton. Then $(ℋ₈^∞(N))h = (ℋ_n(π, σ))h$.

Proof. Suppose that $(E(ℋ_n(π, σ)))h = \{e\}$. Since $ℋ₈^∞(N)$ is an inverse semigroup and $E(ℋ₈^∞(N)) = E(ℋ_n(π, σ))$ we conclude that $e$ is a unique idempotent in $(ℋ₈^∞(N))h$. Fix an arbitrary element $γ$ of $ℋ₈^∞(N)$. Let $ε$ be such as in Lemma 2.5. Then we have

$$(γ)h = (γ · γ⁻¹ · γ)h = (γ)h · (γ⁻¹ · γ)h = (γ)h · (ε)h = (γ · ε)h ∈ (ℋ_n(π, σ))h,$$

the assertion of the lemma holds.

We need the following theorem from [19]:

Theorem 2.7 ([19] Theorem 2.9]). Let $S$ be a semigroup and $h: ℋ₈^∞(N) → S$ a non-annihilating homomorphism. Then either $h$ is a monomorphism or $(ℋ₈^∞(N))h$ is a cyclic subgroup of $S$.

Lemma 2.8. Let $S$ be a semigroup and $h: ℋ₈^∞(N) → S$ be a homomorphism such that the restriction $h|_{ℋ₈^∞(N)}: ℋ₈^∞(N) → (ℋ₈^∞(N))h ⊆ S$ is an isomorphism. Then $h$ is an isomorphism.

Proof. Suppose to the contrary that the map $h: ℋ₈^∞(N) → S$ is not an isomorphism. Then by Theorem 2.7 we have that the restriction $h|_{ℋ₈^∞(N)}: ℋ₈^∞(N) → (ℋ₈^∞(N))h ⊆ S$ is an isomorphism. Since $ℋ₈^∞(N)$ is an inverse semigroup we conclude that if $(α)h = (β)h$ for some $α, β ∈ ℋ₈^∞(N)$ then $α ℋ β$. Otherwise if $α$ and $β$ are not $ℋ$-equivalent and $(α)h ≠ (β)h$ then $(α⁻¹)h ≠ (β⁻¹)h$ and therefore either $(αα⁻¹)h ≠ (ββ⁻¹)h$ or $(α⁻¹α)h ≠ (β⁻¹β)h$, a contradiction to the assumption that the restriction $h|_{ℋ₈^∞(N)}: ℋ₈^∞(N) → (ℋ₈^∞(N))h ⊆ S$ is an isomorphism. Thus by Green Theorem (see [7] Theorem 2.20) without loss of generality we can assume that $(I)h = (α)h$ for some $α ∈ H(I)$. Since the group $S₈(N)$ has only one proper normal subgroup and such subgroup is the group $A₈(N)$ of even permutations of $N$ (see [22] and [12] pp. 313–314, Example) we conclude that $(A₈(N))h = (I)h$. We denote

$$
β = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n & \cdots \\ 2 & 3 & 1 & 4 & 5 & \cdots & n & \cdots
\end{pmatrix}
$$

and

$$
ε_{1,2} = \begin{pmatrix} 3 & 4 & 5 & \cdots & n & \cdots \\ 1 & 4 & 5 & \cdots & n & \cdots
\end{pmatrix}.
$$

Then $β ∈ A₈(N)$. Therefore we have that

$$(ε_{1,2})h = (ε_{1,2} · I)h = (ε_{1,2})h · (I)h = (ε_{1,2})h · (β)h = (ε_{1,2} · β)h$$

and similarly $(ε_{1,2})h = (β · ε_{1,2})h$. Since

$$
β · ε_{1,2} = \begin{pmatrix} 2 & 4 & 5 & \cdots & n & \cdots \\ 3 & 4 & 5 & \cdots & n & \cdots
\end{pmatrix}
$$

and

$$
ε_{1,2} · β = \begin{pmatrix} 3 & 4 & 5 & \cdots & n & \cdots \\ 1 & 4 & 5 & \cdots & n & \cdots
\end{pmatrix}.$$
we conclude that \( \beta \cdot \varepsilon_{1,2} \cdot \varepsilon_{1,2} \cdot \beta \in \mathcal{I}_\infty(N) \). Hence by Theorem 2.7 the set \( \mathcal{I}_\infty(N)h \) contains only one idempotent and therefore the assertions of Lemma 2.6 hold. This completes the proof of the lemma.

\[ \blacksquare \]

Theorem 2.7 and Lemmas 2.6 and 2.8 imply the following theorem:

**Theorem 2.9.** Let \( S \) be a semigroup and \( h : \mathcal{I}_\infty(N) \rightarrow S \) a non-annihilating homomorphism. Then either \( h \) is a monomorphism or \( (\mathcal{I}_\infty(N))h \) is a cyclic subgroup of \( S \).

### 3. Topologizations of some classes of countable semigroups

**Definition 3.1.** We shall say that a semigroup \( S \) has:

- an **S-property** if for every \( a, b \in S \) there exist \( c, d \in S^1 \) such that \( c \cdot a \cdot d = b \);
- an **F-property** if for every \( a, b, c, d \in S^1 \) the sets \( \{ x \in S \mid a \cdot x = b \} \) and \( \{ x \in S \mid x \cdot c = d \} \) are finite or empty;
- an **FS-property** if \( S \) has F- and S-properties.

**Remark 3.2.** We observe that

1) every simple (resp., left simple, right simple) semigroup has S-property;
2) every free (Abelian) semigroup has F-property;
3) \( \mathcal{I}_\infty(N) \), \( \mathcal{I}_\infty(N) \) and the bicyclic semigroup have FS-property.

**Lemma 3.3.** Let \( S \) be a Hausdorff semitopological semigroup with FS-property. If \( S \) has an isolated point then \( S \) is the discrete topological space.

**Proof.** Let \( t \) be an isolated point in \( S \). Since the semigroup \( S \) has the FS-property we conclude that for every \( s \in S \) there exist \( a, b \in S^1 \) such that \( a \cdot s \cdot b = t \) and the equation \( a \cdot x \cdot b = t \) has a finite set of solutions. Therefore the continuity of translations in \( (S, \tau) \) implies that the element \( s \) has a finite open neighbourhood, and hence Hausdorffness of \( (S, \tau) \) implies that \( s \) is an isolated point of \( (S, \tau) \). This completes the proof of the lemma.

A topological space \( X \) is called **Baire** if for each sequence \( A_1, A_2, \ldots, A_i, \ldots \) of nowhere dense subsets of \( X \) the union \( \bigcup_{i=1}^{\infty} A_i \) is a co-dense subset of \( X \) [10].

**Theorem 3.4.** Let \( S \) be a countable semigroup with FS-property. Then every Baire topology \( \tau \) on \( S \) such that \( (S, \tau) \) is a Hausdorff semitopological semigroup is discrete.

**Proof.** We consider countable cover \( \Gamma = \{ s \mid s \in S \} \) of the Baire space \( (S, \tau) \). Then there exists an isolated point \( t \) in \( S \). By Lemma 3.3 the topological space is discrete.

A Tychonoff space \( X \) is called **Čech complete** if for every compactification \( cX \) of \( X \) the remainder \( cX \setminus c(X) \) is an \( F_\sigma \)-set in \( cX \) [10].

Since every Čech complete space (and hence every locally compact space) is Baire Theorem 3.4 implies the following:

**Corollary 3.5.** Every Hausdorff Čech complete (locally compact) countable semitopological semigroup with FS-property is discrete.

A topological space \( X \) is called **hereditary Baire** if every closed subset of \( X \) is a Baire space [10]. Every Čech complete (and hence locally compact) space is hereditary Baire (see [10, Theorem 3.9.6]). We shall say that a Hausdorff semitopological semigroup \( S \) is an I-Baire space if either \( sS \) or \( Ss \) is a Baire space for every \( s \in S \).

**Remark 3.6.** We observe that every left ideal \( Ss \) and every right ideal \( sS \) of a regular semigroup \( S \) are generated by some idempotents of \( S \). Therefore every principal left or right ideal of a regular Hausdorff semitopological semigroup \( S \) is a closed subset of \( S \). Hence every regular Hausdorff hereditary Baire semitopological semigroup is the I-Baire space.
Theorem 3.7. Let $S$ be a countable semilattice with $F$-property. Then every $I$-Baire topology $\tau$ on $S$ such that $(S, \tau)$ is a Hausdorff semitopological semilattice is discrete.

Proof. Let $s$ be an arbitrary element of the semilattice $S$. We consider a countable cover $\Gamma = \{e \mid e \in sS\}$ of $sS$. Since $(S, \tau)$ is an $I$-Baire space we conclude that there exists an isolated point $t$ in $sS$. Since $S$ is a semilattice we have that $s \cdot t = t$. Then $\uparrow_s t = \{x \in sS \mid x \cdot t = t\}$ is a finite subset of $S$ which contains $s$ and by Proposition VI.1.13 we get that $\uparrow_s t$ is an open subset of $sS$. Hence there exists an open neighbourhood $U(s)$ of $s$ in $S$ such that $U(s) \cap sS = \{s\}$. The continuity of translations in $S$ implies that there exists an open neighbourhood $V(s) \subseteq U(s)$ such that $V(s) \subseteq \{x \in S \mid x \cdot s = s\}$. Since the semilattice $S$ is Hausdorff and has $F$-property we have that $s$ is an isolated point of $S$. □

Theorem 3.7 implies the following:

Corollary 3.8. Every $I$-Baire topology $\tau$ on the countable free semilattice $\text{FSL}_\omega$ such that $(\text{FSL}_\omega, \tau)$ is a Hausdorff semitopological semilattice is discrete.

4. On topologizations and closures of the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$

Theorem 3.3 implies the following two corollaries:

Corollary 4.1. Every Baire topology $\tau$ on $\mathcal{I}_\infty^\omega(\mathbb{N})$ such that $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau)$ is a Hausdorff semitopological semigroup is discrete.

Corollary 4.2. Every Baire topology $\tau$ on $\mathcal{I}_\infty^\omega(\mathbb{N})$ such that $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau)$ is a Hausdorff semitopological semigroup is discrete.

We observe that Corollary 4.2 generalizes Theorem 3.3 from [10].

The following example shows that there exists a non-discrete topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$ such that $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$ is a Tychonoff topological inverse semigroup.

Example 4.3. We define a topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$ as follows. For every $\alpha \in \mathcal{I}_\infty^\omega(\mathbb{N})$ we define a family

$$B_F(\alpha) = \{U_\alpha(F) \mid F \text{ is a finite subset of } \text{dom } \alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathcal{I}_\infty^\omega(\mathbb{N}) \mid \text{dom } \alpha = \text{dom } \beta, \text{ran } \alpha = \text{ran } \beta \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

Since conditions (BP1)–(BP3) [10] hold for the family $\{B_F(\alpha)\}_{\alpha \in \mathcal{I}_\infty^\omega(\mathbb{N})}$ we conclude that the family $\{B_F(\alpha)\}_{\alpha \in \mathcal{I}_\infty^\omega(\mathbb{N})}$ is the base of the topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$.

Proposition 4.4. $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$ is a Tychonoff topological inverse semigroup.

Proof. Let $\alpha$ and $\beta$ be arbitrary elements of the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$. We put $\gamma = \alpha \beta$ and let $F = \{n_1, \ldots, n_i\}$ be a finite subset of $\text{dom } \gamma$. We denote $m_i = (n_i)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_i = (n_i)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_i)\beta = k_i, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1, \ldots, n_i\}) \cdot U_\beta(\{m_1, \ldots, m_i\}) \subseteq U_\gamma(\{n_1, \ldots, n_i\})$$

and

$$(U_\gamma(\{n_1, \ldots, n_i\}))^{-1} \subseteq U_{\gamma^{-1}}(\{k_1, \ldots, k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$.

We observe that the group of units $\text{H}(\mathbb{I})$ of the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$ with the induced topology $\tau_F(\text{H}(\mathbb{I}))$ from $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$ is a topological group (see [12] pp. 313–314, Example] and [22]) and the definition of the topology $\tau_F$ implies that every $\mathcal{H}$-class of the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$ is an open-and-closed subset of the topological space $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$. Therefore Theorem 2.20 [1] implies that the topological space $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$ is homeomorphic to a countable topological sum of topological copies of $(\text{H}(\mathbb{I}), \tau_F(\text{H}(\mathbb{I})))$. Since every $T_0$-topological group is a Tychonoff topological space (see [20] Theorem 3.10] or [20] Theorem 8.4]) we conclude that the topological space $(\mathcal{I}_\infty^\omega(\mathbb{N}), \tau_F)$ is Tychonoff too. This completes the proof of the proposition. □
Remark 4.5. We observe that the topology $\tau_F$ on $\mathcal{I}_\infty^\forall(\mathbb{N})$ induces discrete topologies on the sub-semigroups $\mathcal{I}_\infty^\forall(\mathbb{N})$ and $E(\mathcal{I}_\infty^\forall(\mathbb{N}))$.

Example 4.6. We define a topology $\tau_{WF}$ on the semigroup $\mathcal{I}_\infty^\forall(\mathbb{N})$ as follows. For every $\alpha \in \mathcal{I}_\infty^\forall(\mathbb{N})$ we define a family

$$\mathcal{B}_{WF}(\alpha) = \{U_\alpha(F) \mid F \text{ is a finite subset of } \text{dom } \alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathcal{I}_\infty^\forall(\mathbb{N}) \mid \text{dom } \beta \subseteq \text{dom } \alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

Since conditions (BP1)–(BP3) hold for the family $\{\mathcal{B}_{WF}(\alpha)\}_{\alpha \in \mathcal{I}_\infty^\forall(\mathbb{N})}$ we conclude that the family $\{\mathcal{B}_{WF}(\alpha)\}_{\alpha \in \mathcal{I}_\infty^\forall(\mathbb{N})}$ is the base of the topology $\tau_{WF}$ on the semigroup $\mathcal{I}_\infty^\forall(\mathbb{N})$.

Proposition 4.7. $(\mathcal{I}_\infty^\forall(\mathbb{N}), \tau_{WF})$ is a Hausdorff topological inverse semigroup.

Proof. Let $\alpha$ and $\beta$ be arbitrary elements of the semigroup $\mathcal{I}_\infty^\forall(\mathbb{N})$. We put $\gamma = \alpha\beta$ and let $F = \{n_1, \ldots, n_i\}$ be a finite subset of dom $\gamma$. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1, \ldots, n_i\}) \cdot U_\beta(\{m_1, \ldots, m_i\}) \subseteq U_\gamma(\{n_1, \ldots, n_i\})$$

and

$$(U_\gamma(\{n_1, \ldots, n_i\}))^{-1} \subseteq U_{\gamma^{-1}}(\{k_1, \ldots, k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathcal{I}_\infty^\forall(\mathbb{N}), \tau_{WF})$.

Later we shall show that the topology $\tau_{WF}$ is Hausdorff. Let $\alpha$ and $\beta$ be arbitrary distinct points of the space $(\mathcal{I}_\infty^\forall(\mathbb{N}), \tau_{WF})$. Then only one of the following conditions holds:

(i) dom $\alpha$ = dom $\beta$;

(ii) dom $\alpha$ $\neq$ dom $\beta$.

In case dom $\alpha$ = dom $\beta$ we have that there exists $x \in$ dom $\alpha$ such that $(x)\alpha$ $\neq$ $(x)\beta$. The definition of the topology $\tau_{WF}$ implies that $U_\alpha(\{x\}) \cap U_\beta(\{x\}) = \emptyset$.

If dom $\alpha$ $\neq$ dom $\beta$, then only one of the following conditions holds:

(a) dom $\alpha$ $\subseteq$ dom $\beta$;

(b) dom $\beta$ $\supseteq$ dom $\alpha$;

(c) dom $\alpha$ \ dom $\beta$ $\neq$ $\emptyset$ and dom $\beta$ \ dom $\alpha$ $\neq$ $\emptyset$.

Suppose that case (a) holds. Let $x \in$ dom $\beta$ \ dom $\alpha$ and $y \in$ dom $\alpha$. The definition of the topology $\tau_{WF}$ implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \emptyset$.

Case (b) is similar to (a).

Suppose that case (c) holds. Let $x \in$ dom $\beta$ \ dom $\alpha$ and $y \in$ dom $\alpha$ \ dom $\beta$. The definition of the topology $\tau_{WF}$ implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \emptyset$.

This completes the proof of the proposition. □

Remark 4.8. We observe that the topology $\tau_{WF}$ on $\mathcal{I}_\infty^\forall(\mathbb{N})$ induces non-discrete topologies on the semigroup $\mathcal{I}_\infty^\forall(\mathbb{N})$ and the semilattice $E(\mathcal{I}_\infty^\forall(\mathbb{N}))$. Moreover, every $\mathcal{H}$-class of the semigroup $(\mathcal{I}_\infty^\forall(\mathbb{N}), \tau_{WF})$ is homeomorphic to every $\mathcal{H}$-class of the semigroup $(\mathcal{I}_\infty^\forall(\mathbb{N}), \tau_W)$

The proof of the following proposition is similar to Theorem 3.4.

Proposition 4.9. Every Hausdorff Baire topology $\tau$ on a countable group $G$ such that left (right) translations in $(G, \tau)$ are continuous is discrete.

Theorem 4.10. Let $S$ be a topological semigroup which contains a dense discrete subspace $A$ such that every equations $a \cdot x = b$ and $y \cdot c = d$ have finitely many solutions in $A$. Then $I = S \setminus A$ is an ideal of $S$. 
Proof. Suppose that \( I \) is not an ideal of \( S \). Then at least one of the following conditions holds:

1) \( IA \not\subseteq I \), 2) \( AI \not\subseteq I \), or 3) \( II \not\subseteq I \).

Since \( A \) is a discrete dense subspace of \( S \), Theorem 3.5.8 \cite{10} implies that \( A \) is an open subspace of \( S \). Suppose there exist \( a \in A \) and \( b \in I \) such that \( b \cdot a = c \not\in I \). Since \( A \) is a dense open discrete subspace of \( S \) the continuity of the semigroup operation in \( S \) implies that there exists an open neighbourhood \( U(b) \) of \( b \) in \( S \) such that \( U(b) \cdot \{a\} = \{c\} \). But the equation \( a \cdot x = b \) and \( y \cdot c = d \) have finitely many solutions in \( A \). This contradicts to the assumption that \( a, b \in S \setminus A \). Therefore \( b \cdot a = c \in I \) and hence \( IA \subseteq I \). The proof of the inclusion \( AI \subseteq I \) is similar.

Suppose there exist \( a, b \in I \) such that \( a \cdot b = c \not\in I \). Since \( A \) is a dense open discrete subspace of \( S \) the continuity of the semigroup operation in \( S \) implies that there exist open neighbourhoods \( U(a) \) and \( U(b) \) of \( a \) and \( b \) in \( S \), respectively, such that \( U(a) \cdot U(b) = \{c\} \). But the equations \( x \cdot a = c \) and \( y \cdot c = d \) have finitely many solutions in \( A \). This contradicts to the assumption that \( a, b \in S \setminus A \). Therefore \( a \cdot b = c \in I \) and hence \( II \subseteq I \). \( \Box \)

Theorem 4.10 implies Corollaries 4.11 and 4.12.

Corollary 4.11. Let \( S \) be a topological semigroup which contains a dense discrete subsemigroup \( \mathcal{I}_\infty^\nu(N) \). If \( I = S \setminus \mathcal{I}_\infty^\nu(N) \neq \emptyset \) then \( I \) is an ideal of \( S \).

Corollary 4.12 \((19)\). Let \( S \) be a topological semigroup which contains a dense discrete subsemigroup \( \mathcal{I}_\infty^\nu(N) \). If \( I = S \setminus \mathcal{I}_\infty^\nu(N) \neq \emptyset \) then \( I \) is an ideal of \( S \).

Proposition 4.13. Let \( S \) be a topological semigroup which contains a dense discrete subsemigroup \( \mathcal{I}_\infty^\nu(N) \). Then for every \( c \in \mathcal{I}_\infty^\nu(N) \) the set

\[ D_c(A) = \{(x, y) \in \mathcal{I}_\infty^\nu(N) \times \mathcal{I}_\infty^\nu(N) | x \cdot y = c\} \]

is a closed-and-open subset of \( S \times S \).

Proof. Since \( \mathcal{I}_\infty^\nu(N) \) is a discrete subspace of \( S \) we have that \( D_c(A) \) is an open subset of \( S \times S \).

Suppose that there exists \( c \in \mathcal{I}_\infty^\nu(N) \) such that \( D_c(A) \) is a non-closed subset of \( S \times S \). Then there exists an accumulation point \((a, b) \in S \times S\) of the set \( D_c(A) \). The continuity of the semigroup operation in \( S \) implies that \( a \cdot b = c \). But \( \mathcal{I}_\infty^\nu(N) \times \mathcal{I}_\infty^\nu(N) \) is a discrete subspace of \( S \times S \) and hence by Corollary 4.11 the points \( a \) and \( b \) belong to the ideal \( I = S \setminus \mathcal{I}_\infty^\nu(N) \) and hence \( p \cdot q \in S \setminus \mathcal{I}_\infty^\nu(N) \) cannot be equal to \( c \). \( \Box \)

A topological space \( X \) is defined to be pseudocompact if each locally finite open cover of \( X \) is finite. According to [10, Theorem 3.10.22] a Tychonoff topological space \( X \) is pseudocompact if and only if each continuous real-valued function on \( X \) is bounded.

Theorem 4.14. If a topological semigroup \( S \) contains \( \mathcal{I}_\infty^\nu(N) \) as a dense discrete subsemigroup then the square \( S \times S \) is not pseudocompact.

Proof. Since the square \( S \times S \) contains an infinite closed-and-open discrete subspace \( D_c(A) \), we conclude that \( S \times S \) fails to be pseudocompact (see [10, Ex. 3.10.F(d)] or [5]). \( \Box \)

Remark 4.15. Recall that, a topological semigroup \( S \) is called \( \Gamma \)-compact if for every \( x \in S \) the closure of the set \( \{x, x^2, x^3, \ldots\} \) is a compactum in \( S \) (see [21]). Since the semigroup \( \mathcal{I}_\infty^\nu(N) \) contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [3], [4], [18], [21] imply that if a topological semigroup \( S \) satisfies one of the following conditions: (i) \( S \) is compact; (ii) \( S \) is \( \Gamma \)-compact; (iii) the square \( S \times S \) is countably compact; (iv) \( S \) is a countably compact topological inverse semigroup; or (v) the square \( S \times S \) is a Tychonoff pseudocompact space, then \( S \) does not contain the semigroup \( \mathcal{I}_\infty^\nu(N) \) and hence the semigroup \( \mathcal{I}_\infty^\nu(N) \).

The proof of the following theorem is similar to Theorem 4.14.

Theorem 4.16. If a topological semigroup \( S \) contains \( \mathcal{I}_\infty^\nu(N) \) as a dense discrete subsemigroup then the square \( S \times S \) is not pseudocompact.
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Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine

*E-mail address: chuchman_i@mail.ru*

Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine

*E-mail address: o_gutik@franko.lviv.ua, ovgutik@yahoo.com*