STABILITY OF TRANSONIC CHARACTERISTIC DISCONTINUITIES IN TWO-DIMENSIONAL STEADY COMPRESSIBLE EULER FLOWS

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ABSTRACT. For a two-dimensional steady supersonic Euler flow past a convex cornered wall with right angle, a characteristic discontinuity (vortex sheet and/or entropy wave) is generated, which separates the supersonic flow from the gas at rest (hence subsonic). We proved that such a transonic characteristic discontinuity is structurally stable under small perturbations of the upstream supersonic flow in \( BV \). The existence of a weak entropy solution and Lipschitz continuous free boundary (i.e. characteristic discontinuity) is established. To achieve this, the problem is formulated as a free boundary problem for a nonstrictly hyperbolic system of conservation laws; and the free boundary problem is then solved by analyzing nonlinear wave interactions and employing the front tracking method.

1. Introduction and Main Theorem

We are concerned with the structural stability of transonic characteristic discontinuities in two-dimensional steady full compressible Euler flows, which separate supersonic flows from the static gases (that is, flows with zero-velocity, hence subsonic, cf. Figure [1]) under small perturbations in the space of functions of bounded variation of the upstream supersonic flow. The flow is governed by the two-dimensional full Euler system, consisting of the conservation laws of mass, momentum, and energy:

\[
\begin{align*}
\partial_t (\rho u) + \partial_y (\rho v) &= 0, \\
\partial_t (\rho u^2 + p) + \partial_y (\rho uv) &= 0, \\
\partial_t (\rho uv) + \partial_y (\rho v^2 + p) &= 0, \\
\partial_t (\rho u(E + \frac{\rho}{\rho})) + \partial_y (\rho v(E + \frac{\rho}{\rho})) &= 0.
\end{align*}
\]

(1.1)

As usual, the unknowns \( \mathbf{u} = (u, v), p, \) and \( \rho \) are respectively the velocity, the pressure, and the density of the flow, and

\[
E = \frac{1}{2}(u^2 + v^2) + e(p, \rho)
\]

is the total energy per unit mass with the internal energy \( e(p, \rho) \). Let \( S \) be the entropy. For polytropic gas, the constitutive relations are

\[
p = \kappa \rho^\gamma \exp\left(\frac{S}{c_p}\right), \quad e = \frac{(\gamma - 1)p}{\rho}
\]
for some positive constants $\kappa, c_\nu$, and $\gamma > 1$. The sonic speed is given by

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$  

The flow is said to be supersonic (resp. subsonic) at a state point if $u^2 + v^2 > c^2$ (resp. $u^2 + v^2 < c^2$) there. It is well-known that the Euler system (1.1) is hyperbolic for supersonic flow, and particularly hyperbolic in the positive $x$-direction if $u > c$; while it is of hyperbolic-elliptic composite-mixed type if the flow is subsonic. Hereafter, we use $U = (u, v, p, \rho)$ to represent the state of the flow under consideration.

An important physical case in which a characteristic discontinuity is generated is as follows: the characteristic discontinuity is a straight line emerging from a corner $O$ (that is the positive $x$-axis); the gas flow above (i.e., in $\{x \in \mathbb{R}, y > 0\}$) is a uniform supersonic flow with the velocity $(u, 0)$, pressure $p^+$, and density $\rho^+$ such that $u > c^+$ for the sonic speed $c^+ > 0$; below the characteristic discontinuity (i.e., in $\{x > 0, y < 0\}$), the gas is at rest with zero-velocity, pressure $p^-$, and density $\rho^-$. The question is whether such a transonic characteristic discontinuity is structurally stable under small perturbations of the upstream supersonic flow in the framework of two-dimensional steady full Euler equations, as shown in Figure 1. Notice that the characteristic discontinuity is either a combination of a vortex sheet and an entropy wave or one of them.

For related cases, when the flows on both sides of the characteristic discontinuity are supersonic, it has been shown to be structurally stable by Chen-Zhang-Zhu [3] in the framework of weak entropy solutions, and the $L^1$–stability also holds as established by Chen-Kukreja [5]; when the flow is in an infinite duct and on both sides of the characteristic discontinuity the flows are subsonic, Bae [11] proved that it is stable under small perturbations of the walls of the duct. Characteristic discontinuities appear ubiquitously in Mach reflection and refraction/reflection of shock upon an interface. For such problems, Chen [6] and Chen-Fang [7] studied the stability of subsonic characteristic discontinuities; Fang-Wang-Yuan [9] showed the local stability of supersonic characteristic discontinuity in the framework of classical solutions. Also see Zhang [15] for supersonic potential flows past a convex cornered bending wall and related geometry. As far as we know, there have been no results available so far concerning transonic characteristic discontinuities when the supersonic flows are not $C^1$ but only belong to the space of functions of bounded variation.

We remark that considerable progress has been made on the existence and stability of multidimensional transonic shocks in steady full Euler flows (see, for example, [4, 6, 12, 13, 14]; also cf. [8]). In these papers, the smooth supersonic flow is given, and the key point is to solve a one-phase elliptic free boundary problem. However, in order to solve the perturbed characteristic discontinuity in this paper, the key point is to solve a hyperbolic free boundary problem in the framework of weak entropy solutions.

In the following, we first formulate the aforementioned stability problem for the characteristic discontinuity as a free boundary problem for the Euler equations. Then, in Sections 2–5, we establish the existence and stability of the free boundary, by a front tracking method (cf. [2, 8, 11]).

To this end, we now introduce characteristic discontinuities, a kind of discontinuities that separate piecewise classical/weak solutions of (1.1). Suppose that $\Gamma$ is a Lipschitz curve with normal $n = (n_1, n_2)$ in the plane, and the flows $U = (u, v, p, \rho)$ on both sides of $\Gamma$ satisfy the Euler equations.
Supersonic flow $U = (u, v, p, \rho)$

Characteristic discontinuity

Still gas $U^− = (0, 0, p, \rho^−)$

**Figure 1.** A characteristic discontinuity emerged from the corner $O$ that separates the static gas with zero-velocity below from the supersonic flow above.

Equation (1.1) in the classical/weak sense. Then $U$ is a weak solution to (1.1) provided it satisfies (1.1) on either side of $\Gamma$ in the classical/weak sense, and the following Rankine-Hugoniot jump conditions hold along $\Gamma$:

$$
\begin{align*}
[\rho u]_1 + [\rho v]_2 &= 0, \\
[\rho u^2 + p]_1 + [\rho uv]_2 &= 0, \\
[\rho uv]_1 + [\rho v^2 + p]_2 &= 0, \\
[\rho u(E + \frac{p}{\rho})]_1 + [\rho v(E + \frac{p}{\rho})]_2 &= 0,
\end{align*}
$$

(1.2)

where $[\cdot]$ denotes the jump of the quantity across $\Gamma$. Such a discontinuity $\Gamma$ is called a characteristic discontinuity if the mass flux $m = \rho u \cdot n = (pu)_1 + (pv)_2$ through $\Gamma$ is zero. For a characteristic discontinuity, the first and fourth condition $([\rho u(E + \frac{p}{\rho})]_1 + [\rho v(E + \frac{p}{\rho})]_2 = 0)$ in (1.2) hold trivially, while the second $([pu \cdot n] + [p]_1 = 0)$ and the third $([vpu \cdot n] + [p]_2 = 0)$ imply $[p] = 0$. Thus, we see that, for a characteristic discontinuity, the only jump conditions should be

$$[p] = 0 \quad \text{and} \quad u \cdot n = 0.$$

(1.3)

This implies that there might be jumps of the tangential velocity and the entropy (i.e., the density). Therefore, in general, a characteristic discontinuity in full Euler flow is either a vortex sheet or an entropy wave. We also note that (1.3) implies (1.2).

Consider the Cauchy problem of the hyperbolic-elliptic composite-mixed system (1.1):

$$
U = \begin{cases} 
U_0, & x = 0, \ y > 0, \\
U^+, & x > 0, \ y > 0, \\
U^−, & x = 0, \ y < 0.
\end{cases}
$$

(1.4)

The discontinuous function:

$$
U = \begin{cases} 
U^+ = (u, 0, p, \rho^+), & x > 0, \ y > 0, \\
U^− = (0, 0, p, \rho^−), & x > 0, \ y < 0,
\end{cases}
$$
with $u > c^+ = \sqrt{\gamma p/\rho^+}$ is a characteristic discontinuity of (1.1), when $U_0 = U^+$, and $U^-$ is the state of the static gas below $\{x > 0, y = 0\}$.

A weak entropy solution to problem (1.4) can be defined in the standard way (cf. Definition 1.1 below): In particular, it is defined as in (1.6)–(1.10), but the domain of integration $\Omega$ is replaced by $\{x \geq 0, y \in \mathbb{R}\}$, $\Sigma$ is replaced by $\{x = 0, y \in \mathbb{R}\}$, and the right-hand sides of (1.7)–(1.8) are replaced by zero.

We note that the state of the static gas $U^-$ should be unchanged under the perturbation of the supersonic flow. This is a merit of such a transonic characteristic discontinuity, which enables us to reduce the above problem to an initial-free boundary problem of the hyperbolic Euler equations.

Suppose that the characteristic discontinuity $\Gamma$ is given by the equation:

$$y = g(x) \quad \text{for } x \geq 0,$$

with $g(0) = 0$. Then

$$n = \frac{(g'(x), -1)}{\sqrt{1 + (g'(x))^2}}.$$

The domain bounded by $\Gamma$ and $\Sigma = \{(x, y) : x = 0, y > 0\}$ is written as $\Omega$. We formulate the following free boundary problem of (1.1) in $\Omega$:

$$\begin{cases}
U = U_0 & \text{on } \Sigma, \\
p = p & \text{on } \Gamma, \\
v = g'(x)u & \text{on } \Gamma,
\end{cases}$$

(1.5)

where the first is the initial data and the last two conditions on $\Gamma$ come from (1.3).

**Definition 1.1.** A pair $(g, U)$ with $y = g(x) \in \text{Lip}([0, \infty); \mathbb{R})$ and $U = (u, v, p, \rho) \in L^\infty(\Omega; \mathbb{R}^4)$ is called a weak entropy solution to problem (1.5) provided the following hold:

\(\blacklozenge\) $U$ is a weak solution to (1.1) in $\Omega$ and satisfies the initial-boundary conditions in the trace sense: For any $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_\Omega (pu \partial_x \phi + pv \partial_y \phi) \, dx \, dy + \int_\Sigma pu \phi \, dy = 0,$$  (1.6)

$$\int_\Omega ((p u^2 + p) \partial_x \phi + puv \partial_y \phi) \, dx \, dy + \int_\Sigma (p u^2 + p) \phi \, dy = p \int_\Gamma \phi n_1 \, ds,$$  (1.7)

$$\int_\Omega ((puv) \partial_x \phi + (pv^2 + p) \partial_y \phi) \, dx \, dy + \int_\Sigma (puv) \phi \, dy = p \int_\Gamma \phi n_2 \, ds,$$  (1.8)

$$\int_\Omega (pu(E + \frac{p}{\rho}) \partial_x \phi + pv(E + \frac{p}{\rho}) \partial_y \phi) \, dx \, dy + \int_\Sigma pu(E + \frac{p}{\rho}) \phi \, dy = 0;$$  (1.9)

\(\blacklozenge\) $U$ satisfies the entropy inequality, i.e., the steady Clausius inequality:

$$\partial_x (puS) + \partial_y (pvS) \geq 0$$

in the sense of distribution in $\Omega$: For any $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\phi \geq 0$:

$$\int_\Omega (puS \partial_x \phi + pvS \partial_y \phi) \, dx \, dy + \int_\Sigma puS \phi \, dy \leq 0.$$  (1.10)
We remark that, if \((g,U)\) is a weak entropy solution to problem (1.5), then

\[
\tilde{U} = \begin{cases} 
U & \text{in } \{ y > g(x), \ x \geq 0 \}, \\
U^- & \text{in } \{ y < g(x), \ x \geq 0 \}
\end{cases}
\]

is a weak entropy solution to problem (1.4). This can be checked by integration by parts in \(\{ x \geq 0, y < g(x) \} \); thus, we omit the details. From now on, we focus on the solution of problem (1.5). The main result of this paper is the following.

**Theorem 1.1.** There exists positive constants \(\varepsilon\) and \(C\) depending only on \(U^\pm\) so that, if

\[
\| U_0 - U^+ \|_{\text{BV}(\Gamma)} \leq \varepsilon,
\]

then problem (1.5) has a weak entropy solution \((g,U)\). Moreover, the solution satisfies

(i) \( g \in \text{Lip}([0,\infty); \mathbb{R}) \) with \(g(0) = 0\) and \(\|g'\|_{L^\infty([0,\infty))} \leq C\varepsilon\);

(ii) There exists \(U_0 \in \mathbb{R}^4\) so that

\[
U - U_0 \in C([0,\infty); L^1(g(x), \infty)), \quad \|(U - U^+)(x,\cdot)\|_{\text{BV}(\int_{\{g(x),\infty\})}) \leq C\varepsilon.
\]

**Remark 1.1.** We note that \(\| U_0 - U^+ \|_{\text{BV}(\Sigma)} \leq \varepsilon\) implies that \(\lim_{y \to \infty}(U_0 - U^+)(y)\) exists. Then there exists \(U_0 \in \mathbb{R}^4\) as claimed in Theorem 1.1 so that

\[
\lim_{y \to \infty} U_0(y) = \overline{U}_0,
\]

and

\[
|U_0 - U^+| \leq \varepsilon.
\]

To prove Theorem 1.1, we establish the compactness and convergence of approximate free boundaries to the free boundary of the exact solution in supersonic-subsonic flows in the framework of front tracking method, while some other essential tools/notions of the front tracking method are extended, modified, and further clarified working in the presence of the free boundary such as a generation of fronts to control the finiteness of physical fronts and the errors from approximate Riemann solvers for the nonstrictly hyperbolic free boundary problem. For this, two new nonlinear Riemann problems are involved: One is the Riemann problem at the convex corner connected with the still gas state (subsonic state); and the other is the Riemann problem determining the evolution of the free boundary, for which we establish the boundedness of the key reflection coefficient of the reflected wave into the supersonic region after the interaction of the incident wave with the free boundary. To achieve the compactness, we have to identify the right scales and global weights to control the Glimm functional to make it monotonically decrease in the flow direction, while preserving the overall structural stability of the characteristic boundary as the hyperbolic region evolves in complicated ways under any small BV perturbation yet the subsonic state remains stable beneath the free boundary.

We also remark in passing that, as an example of one-phase hyperbolic free boundary problems for nonstrictly hyperbolic systems, we deal with the problem in the physical space, the Euler coordinates throughout this paper. This represents a first example of an approach to apply the front-tracking method to study structural stability of interfaces between different mediums, one of them is subsonic. Our approach offers further opportunities to initiate the study of vortex
sheets/entropy waves in the space of bounded variation in nozzles, jets, etc. for mixed-type flows, transonic flows. In a forthcoming paper, we will deal with this problem and related $L^1$-stability in a different approach.

The rest of this paper is devoted to establishing Theorem 1.1. We will mainly employ a version of the front tracking method introduced in Holden-Risebro [11] for convenience to deal with the problem. Thus, in Section 2, we review some facts concerning the solvability of various Riemann problems for the steady Euler equations, and present some essential interaction estimates. It manifests clearly in the simplest case how such a hyperbolic free boundary problem can be solved. Then, in Section 3, we construct approximate solutions by the front tracking algorithm. The key point is to show such an approximate solution can be established for $x \in [0, \infty)$ by constructing a Glimm functional. Then, in Section 4, with the uniform $BV$ estimate of approximate solutions obtained from the Glimm functional, we establish the compactness of the family of approximate solutions and that the limit is actually an entropy solution. Finally, we discuss the asymptotic behavior of the weak entropy solutions as $x \to \infty$ in Section 5.

2. Riemann problems and interaction estimates

In this section we first review certain basic properties of the steady hyperbolic Euler equations (2.1) that are used later for self-containedness (cf. Chen-Zhang-Zhu [3, pp.1665-1670]). Then we show the solvability of “free boundary” Riemann problem and interaction estimate between weak waves and the free boundary, which are the new ingredients in this paper.

2.1. Euler Equations. As in [3], we write the Euler equations (1.1) in the form

$$ \partial_x W(U) + \partial_y H(U) = 0, \quad U = (u, v, p, \rho), \tag{2.1} $$

where

$$ W(U) = (\rho u, \rho u^2 + p, \rho uv, \rho u(\frac{\gamma p}{\gamma - 1} + \frac{u^2 + v^2}{2}))^\top, $$

and

$$ H(U) = (\rho v, \rho uv, \rho v^2 + p, \rho v(\frac{\gamma p}{\gamma - 1} + \frac{u^2 + v^2}{2}))^\top. $$

The eigenvalues $\lambda$ of this system are determined by $\det(\lambda \nabla_U W(U) - \nabla_U H(U)) = 0$, or explicitly,

$$ (v - \lambda u)^2 ((v - \lambda u)^2 - c^2(1 + \lambda^2)) = 0. $$

Thus, if $u > c$, we have four real eigenvalues:

$$ \lambda_j = \frac{uv + (-1)^j \sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 4; \quad \lambda_k = \frac{v}{u}, \quad k = 2, 3. \tag{2.2} $$

The associated linearly independent right-eigenvectors are

$$ r_j = \kappa_j (-\lambda_j, 1, \rho(\lambda_j u - v), \frac{\rho(\lambda_j u - v)}{c^2})^\top, \quad j = 1, 4; \tag{2.3} $$

$$ r_2 = (u, v, 0, 0)^\top, \quad r_3 = (0, 0, 0, \rho)^\top, \tag{2.4} $$

where $\kappa_j$ are renormalized factors so that $r_j \cdot \nabla_U \lambda_j(U) \equiv 1$ since the $j$-th characteristic fields are genuinely nonlinear, $j = 1, 4$. While the second and third characteristic fields are linearly degenerate: $r_j \cdot \nabla_U \lambda_j(U) \equiv 0, j = 2, 3$. Although the steady Euler system is not strictly hyperbolic, we can still employ the general ideas presented in [8, 11] to treat related Riemann and Cauchy
problems. The only difference is that, although the characteristic discontinuity has only one front in physical space (since two of the four characteristic eigenvalues coincide), we need two independent parameters (one corresponds to $\lambda_2$ for the vortex sheet, and the other to $\lambda_3$ for the entropy wave) to represent its strength.

At the unperturbed reference state $U^+ = (u, 0, p, \rho^+)$, we easily see that

$$\lambda_1(U^+) < \lambda_2(U^+) = 0 = \lambda_3(U^+) < \lambda_4(U^+) = -\lambda_1(U^+).$$

Also, Lemma 2.3 in [3] indicates that the re-normalization factors $\kappa_j(U), j = 1, 4$, are positive in a small neighborhood of $U^+$.

2.2. Wave Curves in the Phase Space. As shown in [3], at each state $U_0 = (u_0, v_0, p_0, \rho_0)$ with $u_0 > c_0$ in the phase space, there are four curves in a neighborhood of $U_0$:

- Vortex sheet curve $C_2(U_0) : U = (u_0 e^{\alpha_2}, v_0 e^{\alpha_2}, p_0, \rho_0)$.
  
  These are the states $U$ that can be connected to $U_0$ by a vortex sheet with slope $\frac{u_0}{u_0}$ and strength $\alpha_2 \in \mathbb{R}$;

- Entropy wave curve $C_3(U_0) : U = (u_0, v_0, p_0, \rho_0 e^{\alpha_3})$.
  
  These are the states $U$ that can be connected to $U_0$ by an entropy wave with slope $\frac{u_0}{u_0}$ and strength $\alpha_3 \in \mathbb{R}$;

- Rarefaction wave curve $R_j(U_0)$:
  
  $$dp = c^2 d\rho, du = -\lambda_j d\nu, \rho(\lambda_j u - v) d\nu = dp \quad \text{for } \rho < \rho_0, u > c, \quad j = 1, 4.$$

  These are the states $U$ that can be connected to $U_0$ from the lower by a rarefaction wave of the $j$-th family;

- Shock wave curve $S_j(U_0)$:
  
  $$[p] = \frac{c_0^2}{b} [\rho], [u] = -s_j [v], p_0(s_j u_0 - v_0) [v] = [p] \quad \text{for } \rho > \rho_0, u > c, \quad j = 1, 4.$$

  These are the states $U$ that can be connected to $U_0$ from the lower by a shock wave of the $j$-th family, with the slope of the discontinuity to be

  $$s_j = \frac{u_0 v_0 + (-1)^j \tilde{c} \sqrt{u_0^2 + v_0^2 - \tilde{c}^2}}{u_0^2 - \tilde{c}^2}, \quad j = 1, 4,$$

  where $\tilde{c} = \frac{\rho_0^{\alpha_2}}{\rho_0}$ and $b = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \frac{\rho}{\rho_0}$.

One can also parameterize $R_j(U_0)$ and $S_j(U_0) \ (j = 1, 4)$ so that there is a curve given by a $C^2$ map $\alpha_j \mapsto \Phi_j(\alpha_j; U_0)$ in a neighborhood of $U_0$, with $\alpha_j \geq 0$ being the part of $R_j(U_0)$, and $\alpha_j < 0$ the part of $S_j(U_0)$, and

$$\Phi_j(0; U_0) = U_0, \quad \partial_{\alpha_j} \Phi_j(0; U_0) = r_j(U_0). \quad (2.5)$$

We can also write the curve $C_j(U_0) \ (j = 2, 3)$ as $\alpha_j \mapsto \Phi_j(\alpha_j; U_0)$ which is still $C^2$ so that $(2.5)$ hold for $j = 2, 3$. Since $\{r_j(U_0)\}_{j=1}^4$ are linearly independent, such curves consist locally a (curved) coordinate system in a neighborhood of $U_0$. This guarantees the solvability of the Riemann problems stated below.

For simplicity, we set

$$\Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_0) = \Phi_4(\alpha_4; \Phi_3(\alpha_3; \Phi_2(\alpha_2; \Phi_1(\alpha_1; U_0)))) \quad (2.6)$$
Then
\[ \Phi(0,0,0,0;U_0) = U_0, \quad \partial_{\alpha_j} \Phi(0,0,0,0;U_0) = r_j(U_0), \quad j = 1, 2, 3, 4. \] (2.7)

2.3. Standard Riemann Problem. We now consider the standard Riemann problem, that is, system (1.1) with the piecewise constant (supersonic) initial data
\[ U|_{x=x_0} = \begin{cases} U^+, & y > y_0; \\ U^-, & y < y_0, \end{cases} \] (2.8)
where \( U^+ \) and \( U^- \) are the constant states which are regarded as the above state and below state with respect to the line \( y = y_0 \), respectively.

**Lemma 2.1** (Lemma 2.2 in [3]). There exists \( \epsilon > 0 \) such that, for any states \( U^- \) and \( U^+ \) lie in the ball \( O_\epsilon(U_0) \subset \mathbb{R}^4 \) with radius \( \epsilon \) and center \( U_0 \), the above Riemann problem admits a unique admissible solution consisting of four elementary waves. In addition, the state \( U^+ \) can be represented by
\[ U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-). \] (2.9)

It is noted (cf. Lemma 4.1 in [3]) that one can use the parameters \( \alpha_j, j = 1, \ldots, 4 \), to bound \( |U^+ - U^-| \): There is a constant \( B \) depending continuously on \( U_0 \) and \( \epsilon \) so that, for \( U^\pm \) connected by (2.9),
\[ \frac{1}{B} \sum_{j=1}^4 |\alpha_j| \leq |U^+ - U^-| \leq B \sum_{j=1}^4 |\alpha_j|. \]

For later applications, it is also important to express the Riemann solver from the upper state \( U^+ \) to the lower state \( U^- \), rather than the usual way given above. For \( U^+ = \Phi_j(\alpha_j; U^-) \), we may have a \( C^2 \)-map \( U^- = \Psi_j(\alpha_j; U^+) \) with \( \Psi_j(0;U) = U \) and \( \partial_{\alpha_j} \Psi_j(0;U) = -r_j(U) \). Thus, for \( U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-) \), we may express \( U^- \) in terms of \( U^+ \) by
\[ U^- = \Psi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U^+) = \Psi_1(\alpha_1; \Psi_2(\alpha_2; \Psi_3(\alpha_3; \Psi_4(\alpha_4; U^+)))). \]
Then \( \Psi(0,0,0,0;U) = U \) and \( \partial_{\alpha_j} \Psi(0,0,0,0;U) = -r_j(U) \).

2.4. Free Boundary Riemann Problem. We now consider the following Riemann problem of (1.1) involving a free boundary—a characteristic discontinuity. The initial data is a constant state \( U = U^+ \) given on the positive \( y \)-axis, and the free boundary is a straight line \( y = kx \) with \( k \in \mathbb{R} \) to be solved. The boundary conditions on the free boundary are \( p = \frac{\alpha_2}{\alpha_1} \) and \( k = \frac{\alpha_1}{\alpha_4} \). Since the free boundary—characteristic discontinuity — is of the second/third characteristic family, the Riemann solver should contain only one 4-wave with parameter \( \alpha_4 \) and a middle constant state \( U^* \); see Figure 2 below.

**Lemma 2.2.** There exists \( \epsilon > 0 \) so that, for \( U^+ \in O_\epsilon(U^+) \), there is only one admissible solution consisting of a 4-wave that solves the above free boundary Riemann problem. The middle state \( U^* \) can be represented by \( U^* = \Psi_4(\alpha_4; U^+) \), and the free boundary is determined by \( k = \frac{n^*}{\alpha_4} \). There also holds
\[ \alpha_4 = K_1(p^+ - p) + M_1|U^+ - U^+|^2, \quad |k| \leq K_1'|U^+ - U^+|, \] (2.10)
with the constants $K_1, K'_1 > 0$ and a bounded quantity $M_1$ only depending continuously on $\underline{U}^+$ and $\epsilon$.

Proof. 1. We write $U^{(k)}$ to denote the $k$-th argument of the vector $U$, $k = 1, \ldots, 4$. Consider the function:

$$L(\alpha, U^+) = (\Psi_4(\alpha; U^+))^3 - p = (\Psi_4(\alpha; U^+) - \Psi_4(0; \underline{U}^+))^3,$$

for which $L(0; \underline{U}^+) = 0$. Then

$$\partial_\alpha L(0; \underline{U}^+) = -(r_4(\underline{U}^+))^3 = -(\kappa_4 \rho u \lambda_4)|_{\underline{U}^+} < 0.$$

From the implicit function theorem, we infer that $\alpha$ can be viewed as a function of $U^+ \in O_\epsilon(\underline{U}^+)$ for suitably small $\epsilon > 0$. In particular, $\alpha(\underline{U}^+) = 0$. This completes the existence proof.

2. Since $\nabla_U \Psi_4(0; U) = I_4$, $\partial_U L(0; \underline{U}^+) = (0, 0, 1, 0)$. Then

$$\nabla_U \alpha(\underline{U}^+) = \frac{(0, 0, 1, 0)}{(\kappa_4 \rho u |\lambda_4)|_{\underline{U}^+}}.$$

Thus, by the Taylor expansion, we conclude

$$\alpha = K_1(p^+ - \underline{p}) + M_1|U^+ - \underline{U}^+|^2,$$

where $K_1 = \frac{1}{(\kappa_4 \rho u |\lambda_4)|_{\underline{U}^+}} > 0$, and $M_1$ is a constant depending continuously and only on $\underline{U}^+$ and $\epsilon$.

3. From the above, we have

$$|U^* - U^+| \leq B|\alpha| \leq B'|U^+ - \underline{U}^+|.$$

Then we have

$$|U^* - \underline{U}^+| \leq B''|U^+ - \underline{U}^+|$$

for some constant $B'' > 0$. Hence, regarding $\frac{\psi}{u}$ as a function of $U$ and by the mean value theorem, we have

$$|\frac{\psi}{U^*}| \leq C|U^* - \underline{U}^+| \leq K_1' |U^+ - \underline{U}^+|.$$
as desired. \qed

2.5. **Approximate Riemann Solver.** The front tracking method involves approximating the rarefaction waves appeared in the Riemann problems or (free) boundary Riemann problems by several artificial discontinuities separating piecewise constant states.

Suppose that $U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-)$ gives the solution to the standard Riemann problem (2.8), with middle states $U^1 = \Phi_1(\alpha_1; U^-)$ and $U^2 = \Psi_4(\alpha_4; U^+)$. For any $\delta > 0$, we define a $\delta$-approximate solution $U^\delta$ to the Riemann problem as follows:

- If $\alpha_1 > 0$, then the 1-wave is a rarefaction wave that requires modification as follows. Set $\nu$ be the closest integer to $\frac{\alpha_1}{\nu}$ (that is, $\nu \in \mathbb{Z}$ and $\frac{\alpha_1}{\nu} - \frac{1}{2} \leq \nu < \frac{\alpha_1}{\nu} + \frac{1}{2}$), as well as $U_{1,0} = U^-$, $U_{1,\nu} = U^1$, and $U_{1,k} = \Phi_1(\frac{1}{\nu} \alpha_1; U_{1,k-1})$ for $k \in \{1, \ldots, \nu - 1\}$. Then, in the wedge $\{(x, y) : x > 0, y < \lambda_\nu x\}$, we define

$$U^\delta = \begin{cases} U^-, & y < \lambda_1(U^-)x, \\ U_{1,k}, & \lambda_1(U_{1,k-1})x < y < \lambda_1(U_{1,k})x, & k = 1, \ldots, \nu - 1, \\ U^1, & \lambda_1(U_{1,\nu-1})x < y < \lambda_\nu x. \end{cases} \quad (2.11)$$

Here $\lambda_\nu$ is a constant chosen so that $\sup_{U \in O_i(U^+)} \lambda_1 < \lambda_\nu < \inf_{U \in O_i(U^+)} \lambda_2$, which exists when $\epsilon$ is small.

Then the rarefaction wave is replaced by “step” functions with width (strength) $\frac{\alpha_1}{\nu}$, and the discontinuity between two steps moves with the characteristic speed of the lower state.

- If $\alpha_1 < 0$, then the 1-wave is a shock, and no change is necessary. In the wedge $\{(x, y) : x > 0, y < \lambda_\nu x\}$, we define

$$U^\delta = \begin{cases} U^-, & y < s_1x, \\ U^1, & s_1x < y < \lambda_\nu x, \end{cases}$$

where $s_1$ is the speed of the shock front.

- For $\alpha_2, \alpha_3$, there is always no change.

- Similar to the case of the 1-wave, we can define $U^\delta$ in $\{x > 0, y > -\lambda_\nu x\}$ by considering whether the 4-waves is a rarefaction wave (with modification) or a shock (without modification).

2.6. **Interaction of Weak Waves.** The following weak wave interaction estimate is classical; see Lemma 3.2 in [3, p.1670].

**Lemma 2.3.** Suppose that $U^+, U^m,$ and $U^-$ are three states in a small neighborhood of $U_0$ with $U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-)$, $U^m = \Phi(\beta_4, \beta_3, \beta_2, \beta_1; U^-)$, and $U^+ = \Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; U^-)$. Then

$$\gamma_j = \alpha_j + \beta_j + O(1)\Delta(\alpha, \beta)$$

where $\Delta(\alpha, \beta) = |\alpha_4|(|\beta_1| + |\beta_2| + |\beta_3|) + (|\alpha_2| + |\alpha_3|)|\beta_1| + \sum_{j=1,4} \Delta_j(\alpha, \beta)$, with

$$\Delta_j(\alpha, \beta) = \begin{cases} 0, & \alpha_j \geq 0, \beta_j \geq 0, \\ |\alpha_j||\beta_j|, & \text{otherwise}. \end{cases}$$
2.7. **Interaction of Weak Wave and Free Boundary.** We now consider the change of strength when a weak wave interacts with the free boundary (see Figure 3). It is only possible that a weak 1-wave $\alpha_1$ impinges on the characteristic discontinuity $S^l$, and resulting a reflected 4-wave with parameter $\alpha_4$, and the characteristic discontinuity itself is also deflected to a new direction, denoted to be $S^r$. We note that both $U^r$ and $S^r$ can be solved by the free boundary Riemann problem with initial data $U^m$.

\[ \text{Figure 3. A 1-wave } \alpha_1 \text{ is reflected by the characteristic discontinuity } S^l, \text{ resulting in a reflected 4-wave } \alpha_4 \text{ and deflected characteristic discontinuity } S^r. \]

**Lemma 2.4.** Suppose that $U^l, U^m,$ and $U^r$ are three states in $O_{\epsilon}(U^+)\text{ for sufficiently small } \epsilon,$ with $U^m = \Phi_1(\alpha_1; U^l) = \Phi_4(\alpha_4; U^r).$ Then

\[ \alpha_4 = -K_2\alpha_1 + M_2|\alpha_1|^2, \tag{2.13} \]

with the constant $K_2 > 0$ and the quantity $M_2$ bounded in $O_{\epsilon}(U^+)$. Furthermore, for $U^l = (u_l, v_l, p_l, \rho_l), |K_2| > 1, |K_2| < 1,$ and $|K_2| = 1$ when $v_l < 0$, $v_l > 0$, and $v_l = 0$, respectively.

**Proof.** 1. We have $U^m = \Phi_1(\alpha; U^l)$ and $U^r = \Psi_4(\beta; U^m)$. Consider the following function:

\[ L(\beta, \alpha) := (\Psi_4(\beta; \Phi_1(\alpha; U^l)) - U^l)^{(3)}. \]

Then $L(0, 0) = 0$, and $\partial_\beta L(0, 0) = -(r_4(U^l))^{(3)} < 0$. By the implicit function theorem, there exists a function $\beta = \beta(\alpha)$ so that $L(\beta(\alpha), \alpha) = 0$ for small $\alpha$. We see $\beta(0) = 0$.

2. We calculate $\partial_\alpha L(0, 0) = (r_1(U^l))^{(3)} < 0$. Thus, $\frac{d\beta(0)}{d\alpha} = -K_2 := \frac{\lambda_1(U^l)}{(r_4(U^l))^{(3)}} < 0$. Therefore the equality in (2.13) follows from Taylor expansion.

3. The coefficient

\[ K_2 := \frac{(r_1(U^l))^{(3)}}{(r_4(U^l))^{(3)}} = \frac{\lambda_1(U^l)}{\lambda_4(U^l)} > 0 \]

and, for any state $U = (u, v, p, \rho) \in O_{\epsilon}(U^+)$, there holds $\lambda_1(U) < \lambda_2(U) = \frac{\lambda_3(U)}{\lambda_4(U)} < \lambda_4(U)$. Using these two facts with the expressions for $\lambda_1(U)$ and $\lambda_4(U)$ given in (2.2), it follows that $|K_2| < 1, |K_2| > 1$, and $|K_2| = 1$ when $v_l > 0, v_l < 0$, and $v_l = 0$, respectively. \hfill \Box
3. Construction of Approximate Solutions and Uniform Estimates

In this section we adopt the front tracking method in Holden-Risebro [11] to construct a family of approximate solutions \( \{ (g^\delta, U^\delta) \}_{\delta > 0} \) of the problem (1.5) and present some uniform estimates independent of \( \delta \), which is necessary for a compactness argument in §4 to show the existence of a weak entropy solution to (1.5).

3.1. Construction of Approximate Solutions. For any given \( \delta > 0 \), we now describe the construction of an approximate solution \( (g^\delta, U^\delta) \) to the free boundary problem (1.5).

We first approximate the initial data \( U^0(y) \) by a piecewise constant function \( U^\delta_0(y) \) as done in the study of the Cauchy problem. We require that

\[
\lim_{\delta \to 0} \| U^0 - U^\delta_0 \|_{L^1([0, \infty))} = 0.
\]

(3.1)

By Remark 1.1, we may also assume that, for each \( \delta > 0 \), there holds \( U^\delta_0(y) = U^0 \) for large \( y \).

We solve the Riemann problems with initial data on \( \{ x = 0, y > 0 \} \) and a free boundary Riemann problem at the corner \((0,0)\), and then approximate rarefaction waves as carried out in §2.5 with parameter \( \delta \) to obtain new discontinuities. Note the resulting (approximate) solution is piecewise constant.

Then we need do nothing until as \( x \) increases to some value \( x = \tau \), where

(i) either two fronts interact;

(ii) or there is a weak 1-wave that interacts the free boundary (it is obtained by solving the free boundary Riemann problem before) from above.

As noted in [2], by adjusting the slopes of the discontinuities, we can assume that, at each \( \{ x = \tau \} \), only one of the two cases above happens. This is harmless since the error can be made to be arbitrarily small.

For case (i), as mentioned above, by adjusting the slopes of these discontinuities (with arbitrarily small error), we may assume that only two discontinuities collide. Suppose that the lower discontinuity is of \( r \)-family and has a parameter \( \alpha \) with the lower (constant) state \( U^l \) and upper (constant) state \( U^m \), the upper discontinuity is of \( s \)-family and has a parameter \( \beta \) with the lower (constant) state \( U^m \) and upper (constant) state \( U^r \), and they collide at the point \((\tau, \eta)\). Then, as before, we solve a Riemann problem at \((\tau, \eta)\) with the lower state \( U^l \) and upper state \( U^r \), by applying the approximate Riemann solver to obtain new discontinuities.

For case (ii), we may still assume only one discontinuity collides with the free boundary (transonic characteristic discontinuity). Then we solve a wave reflection-deflection problem with a 1-wave reflected by the free boundary, obtaining a reflected 4-wave and a deflected characteristic discontinuity (see Figure 2). If the reflected 4-wave is a rarefaction wave, by approximating the rarefaction wave, we obtain again the approximate solver containing new discontinuities.

Continuing this procedure and, in some cases, removing certain quite weak fronts (cf. §3.4.2 below for details), we obtain an approximate solution \( (g^\delta, U^\delta) \).

Remark 3.1. To ensure that the above procedure works to construct an approximate solution for all \( x \in [0, \infty) \), we need to show that, for any \( 0 < x < \infty \),

- The total variation is small: \( \text{T.V.}(U^\delta(x, \cdot)) \leq C\varepsilon \);
• An \( L^\infty \)-bound: The solution still lies in a small neighborhood of \( U^+ \);
• Given any finite \( T > 0 \), there happens only a finite number of collisions/reflections for \( \{0 < x < T\} \).

The first two are necessary so that we can actually solve the standard or free boundary Riemann problem. Here \( C \) is a universal constant independent of \( \varepsilon \) and \( x > 0 \). The third one guarantees that the global approximate solutions defined up to any \( x > 0 \) can be actually obtained.

In the following three subsections, we deal with these three issues.

3.2. Bounds of Total Variation. We now establish the bounds of total variation of the approximate solutions \( U^\delta(x,y) \).

3.2.1. Glimm Functional. We introduce the following version of Glimm functional

\[
G(x) = V(x) + \kappa Q(x),
\]

where \( \kappa > 0 \) is a large constant to be chosen. The terms \( V \) and \( Q \) are explained below. By the properties of the approximate Riemann solver, \( T.V.(U^\delta(x,\cdot)) \) is equivalent to \( V(x) \). Then it suffices to prove

\[
V(x) \leq C_0 \varepsilon
\]

for a constant \( C_0 \) depending only on \( U^+ \). Recall here \( \varepsilon = \| U_0 - U^+ \|_{BV([0,\infty))} \) measures the strength of the perturbation of initial data.

For a weak wave/discontinuity \( \alpha \) of \( i_\alpha \)-family, we define its weighted strength as

\[
b_\alpha = \begin{cases} 
  k_+ \alpha & \text{if } \alpha \in \Upsilon_t \text{ and } i_\alpha = 1, \\
  \alpha & \text{if } \alpha \in \Upsilon_t \text{ and } i_\alpha = 2,3,4,
\end{cases}
\]

where \( k_+ > |K_2| \) for the coefficient \( K_2 \) appeared in Lemma 2.4 and we use \( \Upsilon_t \) to denote the set of weak waves/discontinuities (not including the free boundary) that cross the line \( \{x = t\} \).

- The weighted strength term \( V(t) \). We define the total (weighted) strengths of weak waves/discontinuities at \( x = t \) as

\[
V(t) = \sum_{\alpha \in \Upsilon_t} |b_\alpha|.
\]

- The interaction potential term \( Q(t) \). The interaction potential term we use here is the same one as introduced by Glimm [10], that is:

\[
Q(t) = \sum_{(b_\alpha,b_\beta) \in \mathcal{A}(t)} |b_\alpha b_\beta|,
\]

where \( \mathcal{A}(t) \) is the approaching set defined by pairs \( (b_\alpha,b_\beta) \) so that, for \( x = t \), the waves/discontinuity with strength \( b_\alpha \) lies in the lower side of the waves/discontinuity with strength \( b_\beta \), and \( b_\alpha \) is of family \( i_\alpha \) and \( b_\beta \) is of family \( i_\beta \), where \( i_\alpha > i_\beta \), or both are of the same family but at least one of them is a shock. Note we do not consider the free boundary as a wave/discontinuity in this paper.

As shown by Lemma 6.2 in [11], at \( x = \tau \), if two discontinuities of strengths \( b_\alpha \) and \( b_\beta \) collide, then we have

\[
Q(\tau^+) - Q(\tau^-) = -\frac{1}{2} |b_\alpha b_\beta|,
\]
provided that
\[ V(\tau) \leq \mu := \frac{1}{2}O(1). \] (3.8)

It is here one needs Lemma 2.3. If no discontinuities collide at \( x = \tau \), then \( Q(\tau^+) = Q(\tau^-) \).

3.2.2. Non-increasing of the Glimm Functional. We now show the bounds of total variation by proving that the Glimm functional \( G(x) \) is non-increasing for \( x \). There are the following three cases.

(i) Collision of discontinuities. For \( x = \tau \) where two discontinuities \( b_\alpha \) and \( b_\beta \) collide, there is no other wave interaction and reflection upon the free boundary as we assumed. Therefore, the decreasing of \( G(\tau) \) is classical. By Lemma 2.3 we have
\[
G(\tau^+) - G(\tau^-) = (V(\tau^+) - V(\tau^-)) + \kappa(Q(\tau^+) - Q(\tau^-))
\leq M|b_\alpha b_\beta| + \kappa(\frac{1}{2}|b_\alpha b_\beta|) \leq 0,
\]
if we choose \( \kappa \geq 2M \) sufficiently large. Note that \( O(1) \) does not depend on the approximation parameter \( \delta \).

(ii) Weak 1-wave interacts with the free boundary. For \( x = \tau \), a weak wave \( \alpha_1 \) of 1-family interacts with the free boundary from above, resulting in a reflected 4-wave \( \alpha_4 \). By Lemma 2.4, we have
\[
G(\tau^+) - G(\tau^-) = (V(\tau^+) - V(\tau^-)) + \kappa(Q(\tau^+) - Q(\tau^-))
\leq |b_{\alpha_4}| - |b_{\alpha_1}| + \kappa\mu|b_{\alpha_4}|
\leq ((\kappa\mu + 1)(-K_2 + M_2\mu) - k_+)|\alpha_1| \leq 0
\]
if we choose \( k_+ \) sufficiently large (independent of \( \delta \)).

(iii) Other situation. If, for \( x = \tau \), no collision or reflection upon the free boundary happens, then we still have \( G(\tau^+) = G(\tau^-) \).

In the above, we have determined \( \kappa \) and \( k_+ \), independent of \( \delta \), and proved that, for any \( x = \tau > 0 \), there holds \( G(\tau^+) \leq G(\tau^-) \), provided (3.8) holds.

3.2.3. Boundedness of Total Variation. The bound \( V(\tau) \leq C_0\varepsilon \) then follows from an induction argument as shown in [11] p.217] for the proof of Lemma 6.3 there, provided that \( \varepsilon \) is small.

We first set \( 0 < \tau_1 < \ldots < \tau_k < \ldots \) as the sequence so that, for \( x = \tau_k \), either collision or reflection upon the free boundary occurs, and set \( V_k, G_k \) the value of \( V(\tau_k^-), G(\tau_k^-) \) respectively.

We know that there exists a constant \( C_1 \) independent of \( \delta > 0 \) so that \( V(\tau) \leq C_1 T.V.(U^\delta(\tau, \cdot)) \) for all \( x \geq 0 \). Note here the choice of weight \( k_+ \) is in essence only determined by \( U^+ \). Define
\[ C_0 = C_1 + \kappa C_1^2. \]

We choose positive \( \varepsilon < 1 \) small so that
\[ C_1\varepsilon + \kappa(C_1\varepsilon)^2 \leq \mu, \quad C_2C_0\varepsilon \leq \varepsilon. \]

Here \( \varepsilon \) is the value so that the Riemann problems or the free boundary Riemann problems can be solved when the Riemann data are in \( O(U^+) \), and \( C_2 \) is the constant depending only on \( U^+ \) so that \( T.V.(U^\delta(x, \cdot)) \leq C_2V(x) \) for any \( x > 0 \).
By assumption on the initial data, we have $T.V.(U^\delta_0) \leq \varepsilon$. Thus, by a property of the Riemann problem, we may have

$$V_1 \leq C_1 \varepsilon \leq \min\{C_0 \varepsilon, \mu\},$$

and furthermore,

$$G_1 \leq V_1 + \kappa V_1^2 \leq C_1 \varepsilon + \kappa(C_1 \varepsilon)^2 \leq \min\{C_0 \varepsilon, \mu\}. $$

Suppose that, for $n \leq k$, we have proved

$$V_n \leq \min\{C_0 \varepsilon, \mu\}. $$

Then, by decreasing of the Glimm functional, we have proved that there holds

$$V_{k+1} \leq G_{k+1} \leq G_k \leq \ldots \leq G_1. $$

This shows

$$V_n \leq \min\{C_0 \varepsilon, \mu\} \quad \text{for all } n.$$

If we further choose $\varepsilon$ small so that $C_0 \varepsilon \leq \mu$, we obtain the bound $V(\tau) \leq C_0 \varepsilon$ as desired. This again implies the uniform estimate:

$$T.V.(U^\delta(x, \cdot)) \leq C_2 C_0 \varepsilon. \quad (3.9)$$

### 3.3. $L^\infty$-Estimate of $\{U^\delta\}$ and Lipschitz Estimate of $\{g^\delta\}$.

The fact that $\{U^\delta\}_{\delta>0}$ is uniformly bounded follows directly. For each $x$, the solution $U^\delta(x, y)$ is just the constant state $U_0$ for sufficiently large $y$, by the finiteness of propagation speed and the fact that the initial data $U^\delta_0(y) \rightarrow U_0$ for $y \rightarrow \infty$. Since we have proved $T.V.(U^\delta(t, \cdot)) \leq C_2 C_0 \varepsilon$ for any $t > 0$, then, by definition of the total variation, we conclude

$$\|U^\delta(x, \cdot) - U^+\|_{L^\infty} \leq C_2 C_0 \varepsilon \quad (3.10)$$

for some new constant $C_2$.

Estimate (3.10) implies the following uniform estimate on the free boundary that is given by the equation $y = g^\delta(x)$:

$$\|(g^\delta)'\|_{L^\infty} \leq C_3 \varepsilon. \quad (3.11)$$

with a constant $C_3$ depending only on $U^+$. In particular, by construction, for fixed $\delta > 0$, $g^\delta$ is a piecewise linear (affine) function, and except for countable points $\{\tau_k\}$, it is differentiable, with

$$(g^\delta)'(x) = \frac{\nu^\delta(x,g^\delta(x))}{\nu^\delta(x,g^\delta(x))}. $$

Thus, by the mean value theorem,

$$| (g^\delta)'(x) | \leq C' |U^\delta(x, g^\delta(x)) - U^+| \leq C' \|U^\delta - U^+\|_{L^\infty} \leq C' C_2 C_0 \varepsilon,$$

where the constant $C'$ depends only on $U^+$. 

### 3.4. Finiteness of Collisions and Reflections.

To show that the numbers of fronts/discontinuities and collisions/reflections do not approach infinity in $\{0 < x < \tau\}$ for any finite $\tau > 0$, the basic idea presented in [11] for the Cauchy problem works well, but we have to consider additional issues such as the reflections off the free boundary and the fact that the Euler system is not strictly hyperbolic in the argument. For completeness, we give the proof below, which closely follows that in [11].
3.4.1. *Generation of Fronts and Modified Construction of Approximate Solutions.* Firstly, we define the notion of *generation* of a front. We set that each initial front starting at \( x = 0 \) belongs to the first generation. Take two first-generation fronts of families \( d \) and \( h \), respectively, that collide. The resulting fronts of families \( d \) and \( h \) belong to the first generation, while all the remaining fronts resulting from the collision are called second-generation fronts. Generally, if a front of family \( d \) and generation \( m \) interacts with a front of family \( h \) and generation \( n \), the resulting front of families \( d \) and \( h \) are still of generation \( m \) and \( n \), respectively, while the remaining fronts resulting from this collision are given generation \( n + m \). The fronts of 4-family resulting from reflection of a front \( \alpha \) of 1-family off the free boundary has the same generation of the front \( \alpha \). The point of this notion is that the fronts of high generation are quite weak.

Given the approximation parameter \( \delta > 0 \), we remove all fronts with generation higher than \( N \), with

\[
N = \left\lfloor \ln_{4KT}(\delta) \right\rfloor
\]  

(3.12)

in our construction of approximate solution \((g^\delta, U^\delta)\). Here \( \left\lfloor z \right\rfloor \) denotes the integer larger than but closest to \( z \) and, following the notations in [11, p.218], we set

\[
T = T(x) = \sum_{\alpha \in \Upsilon_x} |\alpha| \leq V(x), \quad K = \frac{1}{4C_0\varepsilon_0},
\]

with \( \varepsilon_0 \) sufficiently small and fixed, and taking later \( \varepsilon < \varepsilon_0 \), so that \( T < \frac{1}{4K} \).

More precisely, if two fronts of generation \( n \) and \( m \) collide, at most two waves will retain their generation. If \( n + m > N \), then the remaining waves will be removed; however, if \( n + m \leq N \), we use the original (approximate) solution. When we remove the fronts, we let the function \( U^\delta \) be equal to the value that has to be the lower of the removed fronts, provided that the removed fronts are not the upmost fronts in the solution of the Riemann problem. If the upmost are removed, then \( U^\delta \) is set equal to the value immediately to the upper of the removed fronts.

We remark that this process of removing (very) weak waves in approximate Riemann solver in our construction of approximate solutions will not influence the uniform estimates we obtained in §3.2–3.3. In particular, we still have \( T < \frac{1}{4K} \).

3.4.2. *Finiteness of Fronts and Collisions.* We will show that there exists only a finite number of fronts of generation less than or equal to \( N \) and that, for a fixed \( \delta \), there is only a finite number of collisions/reflections.

For this, as we know that \( T < \frac{1}{4K} \), then the strength of each individual front is bounded by \( \frac{1}{4K} \). For later reference, we also note that, by (3.12),

\[
(4KT)^{N+1} \leq \delta.
\]  

(3.13)

First we consider the number of fronts of first generation. This number can increase when the first-generation rarefaction fronts split into several rarefaction fronts. By the term *rarefaction front* we mean a front approximating a rarefaction wave. Note that, by the construction of the approximate Riemann problem, the strength of each split rarefaction front is at least \( \frac{3}{4} \delta \). Given that \( T \) is
uniformly bounded, we find

\[(\text{Number of first \(-\) generation fronts}) \leq (\text{Number of initial fronts}) + \frac{4T}{3\delta}. \quad (3.14)\]

Thus, the number of first-generation fronts is finite. This also means that there will be only a finite number of collisions/grounds between first-generation fronts and free boundary. To see this, note first having the assumption of strict hyperbolicity would have implied that each wave family will have speeds that are distinct. However, we see that, although the Euler system is not strictly hyperbolic, the multiplicity of the eigenvalues is constant for the states \(U\) near the background state \(U^+\). That is, \(\lambda_1(U) < \lambda_2(U) = \lambda_3(U) < \lambda_4(U)\) for any state \(U \in O_r(U^+)\), and hence the eigenvalues are separable in the same way for any state \(U\).

Hence, we can still conclude that each first-generation front will remain in a wedge in the \((x, y)\)-plane determined by the slowest and fastest speeds of that family. Eventually, all first-generation fronts will have interacted at most finite times, and we can also conclude that there can be only a finite number of collisions between first-generation fronts and free boundary globally, since once a front is reflected, it will never meet the free boundary again.

Assuming now that, for some \(m \geq 1\), there will be only a finite number of fronts of generation \(i\), for all \(i < m\), and that there will only be a finite number of interactions between the fronts and fronts reflection off free boundary of generation less than \(m\). Then, in analogy to (3.14), we find

\[
\text{Number of } m\text{-th generation fronts} \\
\leq 2 \times (\text{Number of } j\text{-th and } i\text{-th-generation fronts; } i + j = m) + \frac{4T}{3\delta} < \infty. \quad (3.15)
\]

Consequently, the number of fronts of generation less than or equal to \(m\) is finite. We can now repeat the arguments above showing that there is only a finite number of collisions between the first-generation fronts (and reflections off free boundary), just replacing “first generation” by “of generation less than or equal to \(m\)” and show that there is only a finite number of collisions producing the fronts of generation of \(m + 1\). Thus, we can conclude that there is only a finite number of fronts of generation less than \(N + 1\), and that these interact (reflect off free boundary) only a finite number of times.

4. Convergence and Existence of Weak Entropy Solutions

In this section we show the strong convergence of a subsequence of the approximate solutions to a weak entropy solution of problem (1.5).

4.1. Compactness. We first show there exists a subsequence of approximate solutions \(\{(g^\delta, U^\delta)\}_{\delta > 0}\) that converges to some \((g, U)\) almost everywhere. In §4.2, we show that \((g, U)\) is actually a weak entropy solution to problem (1.5).

4.1.1. Compactness of \(\{g^\delta\}\). We first show the compactness of the approximate free boundary \(\{g^\delta\}_{\delta > 0}\). More explicitly, we have

**Lemma 4.1.** Let \(g^\delta(x)\) be the free boundary for the approximate solution \(U^\delta(x)\). Then there is a subsequence \(\delta_j \to 0\) so that \(g^{\delta_j}(x) \to g(x)\) uniformly in any compact set. Furthermore, the limit \(g(x)\) is Lipschitz continuous: \(|g(x_1) - g(x_2)| \leq C_3\varepsilon|x_1 - x_2|\) for some constant \(C_3\).
Proof. By (3.11), that is, \( \|(f^\delta)\|_{L^\infty([0,\infty))} \leq C_3 \varepsilon \) and \( g^\delta(0) = 0 \), we see that, for fixed \( T > 0 \), the family \( \{g^\delta\} \) is uniformly bounded and equicontinuous on \([0, T]\). Then, by the Arzela-Ascoli compactness criterion, there is a subsequence \( \delta_j \to 0 \) so that \( g^{\delta_j} \to g \) uniformly for some \( g \) in \([0, T]\) and one easily proves that \( |g(x_1) - g(x_2)| \leq C_3 \varepsilon |x_1 - x_2| \) for \( x_1, x_2 \in [0, T] \). By taking a diagonal subsequence for \( 2T, 3T, \ldots \), we can prove that \( g \) is defined for \( x \in [0, \infty) \) and \( g^{\delta_j} \to g \) uniformly in any compact subset of \([0, \infty)\), and \( |g(x_1) - g(x_2)| \leq C_3 \varepsilon |x_1 - x_2| \) for any finite \( x_1 \) and \( x_2 \). \( \square \)

4.1.2. Compactness of \( \{U^\delta\} \). We use the following compactness lemma, which is a modification of Theorem A.8 in [11].

**Lemma 4.2.** Let \( \{u_\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}^d\}_\eta \) be a family of functions such that, for each positive \( T \),

(a) \( |u_\eta(x, \theta)| \leq C_T \) for \( (x, \theta) \in [0, T] \times [0, \infty) \) with a constant \( C_T \) independent of \( \eta \);

(b) For all \( t \in [0, T] \), there holds

\[
\sup_{\|\xi\| \leq \rho} \int_B |u_\eta(x, \theta + \xi) - u_\eta(x, \theta)| \, d\theta \leq \nu_B(|\rho|),
\]

for a modulus of continuity \( \nu \) and all compact \( B \subset [0, \infty) \) (here \( u_\eta(x, t) \) is extended to be zero for \( x \notin [0, \infty) \));

(c) Furthermore, for any \( R > 0 \), for \( s \) and \( t \) in \([0, T]\), there holds

\[
\int_0^R |u_\eta(t, \theta) - u_\eta(s, \theta)| \, d\theta \leq \omega_T(|t - s|) \quad \text{as} \ \eta \to 0,
\]

for some modulus of continuity \( \omega_T \).

Then there exists a sequence \( \eta_j \to 0 \) such that, for each \( x \in [0, T] \), the function \( u_{\eta_j}(x) \) converges to a function \( u(x) \) in \( L^1([0, \infty)) \). The convergence is in the topology of \( C([0, T]; L^1[0, \infty)) \).

For any \( T > 0 \), note that \( U^\delta(x, y) \) is defined for \( 0 < x < T \) and \( g^\delta(x) < y < \infty \). By introducing \( \theta = y - g^\delta(x) \), we may regard \( \bar{U}^\delta \) as a function of \( \theta \in [0, \infty) \) and \( x \in [0, T] \) by defining

\[
\bar{U}^\delta(x, \theta) = U^\delta(x, \theta + g^\delta(x))
\]

to apply Lemma 4.2. Obviously \( \|\bar{U}^\delta\|_{L^\infty} = \|U^\delta\|_{L^\infty} \), and \( T.V.(\bar{U}^\delta)(x, \cdot) = T.V.(U^\delta)(x, \cdot) \). Then, by (3.10), we see immediately that (a) is valid for \( \{\bar{U}^\delta\}_{\delta > 0} \).

Using the boundedness of \( L^\infty \) norm and total variation of \( \bar{U}^\delta \) (cf. (3.9)), the verification of (b) is elementary. Without loss of generality, we assume \( \xi > 0 \). Then by monotone convergence theorem,

\[
\int_{\mathbb{R}^+} |\bar{U}^\delta(x, \theta + \xi) - \bar{U}^\delta(x, \theta)| \, d\theta = \sum_{k=0}^{\infty} \int_{k\xi}^{(k+1)\xi} |\bar{U}^\delta(x, \theta + \xi) - \bar{U}^\delta(x, \theta)| \, d\theta \\
= \int_0^\xi \sum_{k=0}^{\infty} |\bar{U}^\delta(x, z + (k + 1)\xi) - \bar{U}^\delta(x, z + k\xi)| \, dz \\
\leq (T.V.\bar{U}^\delta(x, \cdot)) \xi \leq (C_2 C_0 \varepsilon ) \xi.
\]

The verification of (c) is also not difficult. For \( 0 < s < t < T \), we will prove that

\[
\int_0^R |\bar{U}^\delta(t, \theta) - \bar{U}^\delta(s, \theta)| \, d\theta \leq C(t - s),
\]

for any \( R > 0 \) and a constant \( C \) independent of \( \delta, t, \) and \( s \).
To this end, for given approximate solution $U^\delta$, suppose the “collision times” are

$$0 < \tau_1 < \ldots < \tau_k < \ldots$$

Then, for $x \in (\tau_i, \tau_{i+1})$, nothing happens on the (approximate) free boundary, and then we may ignore the free boundary and write $U^\delta(x,y)$ in the form

$$U^\delta(x,y) = \sum_{k=1}^{N_i} (U^i_{k+1} - U^i_k) H(y - y^i_k(x)) + U^i_1,$$  \hspace{1cm} (4.2)

with $H(\cdot)$ the Heaviside step function (whose value is 0 for the negative argument and 1 for the positive argument). Here we have assumed that, for $x \in (\tau_i, \tau_{i+1})$, there are $N_i$ discontinuities with equation $y = x^i_k(x)$ (from the lower to upper as $k = 1, \ldots, N_i$), and the state in the lower side of \{y = x^i_k\} is $U^i_k$. From (3.42) we know that $N_i < \infty$.

With the above expression, for $\tau_i < s < t < \tau_{i+1}$, we have

$$\int_{R^+} |\tilde{U}^\delta(t, \theta) - \tilde{U}^\delta(s, \theta)| d\theta = \int_{R^+} \left| \int_s^t \frac{d}{d\tau} \tilde{U}^\delta(\tau, \theta) d\tau \right| d\theta$$

$$\leq \int_{R^+} \int_s^t \sum_{k=1}^{N_i} |U^i_{k+1} - U^i_k| \left| H'( (g^\delta(\tau) + \theta) - y^i_k(\tau)) \right| \left( \left| \frac{dg^\delta(\tau)}{d\tau} \right| + \left| \frac{dy^i_k(\tau)}{d\tau} \right| \right) d\tau d\theta$$

$$\leq (L + C_3 \varepsilon) \sum_{k=1}^{N_i} \left| U^i_{k+1} - U^i_k \right| \int_s^t \left| H'( (g^\delta(\tau) + \theta) - y^i_k(\tau)) \right| d\theta d\tau$$

$$= (L + C_3 \varepsilon) \sum_{k=1}^{N_i} \left| U^i_{k+1} - U^i_k \right| d\tau$$

$$\leq (L + C_3 \varepsilon) C.T.V. (U^\delta(\tau_{i+1}, \cdot))(t - s) \leq (L + C_3 \varepsilon) C_2 C_0 \varepsilon (t - s).$$  \hspace{1cm} (4.3)

Here we have set

$$L = \sup_{U \in \mathcal{D}(\mathbb{R}^+)} (|\lambda_1(U)|, |\lambda_2(U)|, |\lambda_4(U)|)$$  \hspace{1cm} (4.4)

to be the maximal characteristic speed, and used the fact that $\left| \frac{dg^\delta(x)}{dx} \right| \leq L$. Estimate (3.11) is also used to control $\left| \frac{dg^\delta}{d\tau} \right|$.

We note (4.3) also holds for $s = \tau_i$ and/or $t = \tau_{i+1}$. Then, for $s \in (\tau_i, \tau_{i+1})$ and $t \in (\tau_j, \tau_{j+1})$ with $i < j$, using (4.3) repeatedly in the intervals $(s, \tau_{i+1})$, $(\tau_{i+1}, \tau_{i+2})$, $\ldots$, $(\tau_{j-1}, \tau_j)$, and $(\tau_j, t)$, we obtain (4.1) with $C = (L + C_3 \varepsilon) C_2 C_0 \varepsilon$.

Therefore, by Lemma 4.2 we can find a subsequence $\{\tilde{U}^\delta_j\}$ that converges to some $\tilde{U}$ under the metric of $C([0,T]; L^1([0, \infty)))$. In addition, upon at most a further subsequence, $g^\delta_j \to g$. Now set $U(x,y) = \tilde{U}(x,y - g(x))$, which is defined in the domain $\Omega = \{x > 0, y > g(x)\}$, with $D = \{y = g(x)\}$ being the lateral (free) boundary. In §4.2, we show that $(g, U)$ is actually a weak entropy solution of problem (1.5). In the following, for simplification, we also write $\delta_j$ as $\delta$.

4.2. Existence of a Weak Entropy Solution. For $0 \leq s \leq t \leq T_0$, define $\Omega_{s,t} := \Omega \cap \{x \in [s,t]\}$, $\Sigma_s = \Omega \cap \{x = s\}$, and $\Gamma_{s,t} := \partial \Omega \cap \{s \leq x \leq t\}$. By the definition of weak entropy solutions (Definition 1.1), a pair of bounded measurable functions $(g, U) = (g(x), U(x,y))$ is a weak entropy solution of problem (1.5) provided that
We note by our construction of approximate solutions that $U_\delta$ of the Euler equations (1.1) in $\Omega$ ≤ 0. For $0 \leq s \leq t \leq T_0$, define

$$\Omega^\delta := \{x > 0, y > g^\delta\}$$

and

$$\Gamma^\delta := \{y = g^\delta(x)\}.$$ 

For any $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$(\rho u + p)\partial_x \psi + \rho v \partial_y \psi \psi) dy dx + \int_{\Sigma_s} \rho u \psi_x dy - \int_{\Sigma_t} \rho u \psi dy = 0; \quad (4.5)$$

$$(\rho u + p)\partial_x \psi + \rho v \partial_y \psi \psi) dy dx + \int_{\Sigma_s} \rho u \psi_x dy - \int_{\Sigma_t} \rho u \psi dy = 0; \quad (4.6)$$

$$(\rho u + p)\partial_x \psi + \rho v \partial_y \psi \psi) dy dx + \int_{\Sigma_s} \rho u \psi_x dy - \int_{\Sigma_t} \rho u \psi dy = 0; \quad (4.7)$$

$$(\rho u + p)\partial_x \psi + \rho v \partial_y \psi \psi) dy dx + \int_{\Sigma_s} \rho u \psi_x dy - \int_{\Sigma_t} \rho u \psi dy = 0; \quad (4.8)$$

$$E^\delta_s := \int_{\Omega_{s,t}} (\rho u \partial_x \psi + \rho v \partial_y \psi \psi) dy dx + \int_{\Sigma_s} \rho u \psi_x dy - \int_{\Sigma_t} \rho u \psi dy \leq 0. \quad (4.9)$$

4.2.1. Estimate on the Total Strength of the Removed Fronts. For any approximate solution $(g^\delta, U^\delta)$, we set

$$\Omega^\delta := \{x > 0, y > g^\delta\}$$

and

$$\Gamma^\delta := \{y = g^\delta(x)\}.$$ 

We note by our construction of approximate solutions that $U^\delta$ may not be a weak entropy solution of the Euler equations (1.1) in $\Omega^\delta$, since there are possible errors introduced by the approximating rarefaction wave via several fronts, and removing weak fronts of higher generations. In the following we will estimate these errors and show that they actually vanish as $\delta \to 0$. The analysis is again quite similar to [11]. We first list below Lemma 6.5 in [11] for later reference.

Lemma 4.3. Let $\mathcal{G}_m$ denote the set of all fronts of generation $m$, and let $T_m$ denote the sum of the strengths of fronts of generation $m$: $T_m = \sum_{\alpha_j \in \mathcal{G}_m} |\alpha_j|$. Then $T = \sum_{m=1}^N T_m$, and

$$T_m \leq C(4KT)^m$$

for some constant $C$. In particular, for $m = N + 1$, we have $T_{N+1} \leq C\delta$ (cf. (3.13)).
4.2.2. Exact Riemann Solutions. For a given approximate solution \((g^\delta, U^\delta)\), suppose as before that the collision/reflection “times” are \(x = \tau_1 < \tau_2 < \ldots\). For a fixed interval \([\tau_j, \tau_{j+1}]\), set \(s_1 = \tau_j\). We solve the following initial–free boundary problem with \(i = 1\) (cf. (2.1)): 

\[
\begin{aligned}
\partial_x W(\bar{U}) + \partial_y H(\bar{U}) &= 0, \quad x > s_i, \; y > \bar{g}(x), \\
\bar{U} &= U^\delta, \quad x = s_i, \; y > \bar{g}(s_i) := g^\delta(s_i), \\
p &= \bar{p}, \quad x > s_i, \; \text{on } \Gamma := \{y = \bar{g}(x)\}, \\
\bar{v} &= \bar{g}'\bar{u}, \quad x > s_i, \; \text{on } \bar{\Gamma}.
\end{aligned}
\] (4.10)

Since the “initial data” \(U^\delta(s_1, \cdot)\) is piecewise constant, the solution \((\bar{g}_1, \bar{U}_1)\) is obtained by solving the Riemann problems. It can be solved up to \(x = s_2\) when two waves interaction or reflection off the free boundary occurs (if \(s_2 > \tau_{j+1}\), we set \(s_2 = \tau_{j+1}\)). Then we solve \((\bar{g}_2, \bar{U}_2)\) from problem (4.10) with \(i = 2\) (note that the initial data is \(U^\delta(s_2, \cdot)\)), up to some \(s_3\). Repeat this process, we obtain

\[(\bar{g}_i, \bar{U}_i) \quad \text{in } [s_i, s_{i+1})
\]

with \(\cup_{i=1}^{\infty}[s_i, s_{i+1}) = [\tau_j, \tau_{j+1})\). We can then define \((\bar{g}, \bar{U})\) piecewise in \(x \in [\tau_j, \tau_{j+1})\) by \((\bar{g}, \bar{U}) = (\bar{g}_i, \bar{U}_i)\) for \(x \in [s_i, s_{i+1})\).

4.2.3. Error of Splitting Rarefaction Waves. Let \(\bar{U}^\delta\) be the approximate solution obtained from problem (4.10) in \([s_i, s_{i+1})\), with the approximating parameter \(\delta\). This means that the rarefaction waves in \(\bar{U}\) are separated into many discontinuities; while there is no front to be removed since each front in \(\bar{U}\) is of generation one. Also, by our rule of splitting rarefaction waves, the lowermost state of \(\bar{U}^\delta\) is the same as \(\bar{U}\). This implies that the corresponding free boundaries are the same, and both \(\bar{U}\), \(\bar{U}^\delta\) are defined in the same domain. The analysis below is similar to [11]. We present details here to show the ideas there still work for our free boundary problem.

Suppose that there is a rarefaction wave in \(\bar{U}\) with the lower state \(\bar{U}_l\) and upper state \(\bar{U}_r\). Then this rarefaction wave is replaced by a step function \(\bar{U}^\delta\). There also holds

\[|\bar{U}^\delta(x, y) - \bar{U}(x, y)| \leq O(\delta)
\]

by our splitting process (it is zero for the points not in rarefaction wave fan). We also want to find the error in the \(L^1\)-space. To this end, we note that there are at most \(\frac{\bar{U}_r - \bar{U}_l}{O(\delta)}\) steps, and the width of each step is at most \((x - s_i)\triangle\lambda\), with \(\triangle\lambda\) the difference of characteristic speeds of two adjacent approximate fronts of each step — it is less than \(O(\delta)\) (cf. (2.11)). Using the mean value theorem (since we know uniform \(L^\infty\) bounds of \(\bar{U}\) and \(\bar{U}^\delta\)), and summing up for all rarefaction wave fans across \(x\), we find

\[
\int_{y > \bar{g}(x)} |W(\bar{U}^\delta)(x, y) - W(\bar{U})(x, y)\ dy \leq C \int_{y > \bar{g}(x)} |\bar{U}^\delta(x, y) - \bar{U}(x, y)\ dy \\
\leq O(\delta) \sum_k |\bar{U}_r^k - \bar{U}_l^k| |x - s_i| \leq O(\delta) T.V.(U^\delta)|x - s_i| = O(\delta)|x - s_i|.
\] (4.11)

We note here that \(\sum_k |\bar{U}_r^k - \bar{U}_l^k|\) is actually controlled by the total variation of the initial data by using the property of the Riemann solution. Similar inequality also holds when \(W(U)\) is replaced by \(H(U)\).
4.2.4. Error of the Removing Weak Fronts. We then compare $\tilde{U}^\delta$ and $U^\delta$ in $x \in [s_i, s_{i+1}]$. We note that both $\tilde{U}^\delta$ and $U^\delta$ satisfy the same initial data. The only difference between them is that some fronts in $\tilde{U}^\delta$ of generation $N + 1$ are ignored to obtain $U^\delta$. Note that, by the removing fronts of generation $N + 1$, we always keep the lowermost state the same as before. This means that the free boundary of $U^\delta$ is the same as $\tilde{U}^\delta$, hence still to be $y = \tilde{g}(x) = g^\delta(x)$. Consequently, $\tilde{U}^\delta$ is different from $U^\delta$ in $x \in (s_i, s_{i+1})$ only in a number of wedges emanating from the discontinuities in $U^\delta(s_i, \cdot)$, and in each wedge, the difference is bounded by the strength of the removing fronts $\alpha$ that are of generation $N + 1$. We also note the width of each wedge is controlled by $O(x - s_i)$. By Lemma 4.3, we then find

$$
\int_{y > \tilde{g}(x)} |W(\tilde{U}^\delta)(x, y) - W(U^\delta)(x, y)| \, dy \leq C \int_{y > \tilde{g}(x)} |\tilde{U}^\delta(x, y) - U^\delta(x, y)| \, dy
$$

$$
\leq O(|x - s_i|) \sum_{\alpha \in \mathcal{G}_{N+1}} |\alpha| \leq O(\delta)|x - s_i|.
$$

(4.12)

Similar inequality is also true for $H(U)$.

4.2.5. Total Error of Approximate Solutions. Since $\tilde{U}$ is obtained by the exact Riemann solvers for $x \in [s_i, s_{i+1}]$, there must hold (with $\Omega_{s,t}$ and $\Sigma_s$, and $\Gamma_{s,t}$ in the integrals replaced by $\mathcal{Q}_{s,t}^\delta$, $\Sigma_s^\delta$, and $\Gamma_{s,t}^\delta$ respectively, since we have shown that the free boundary of $\tilde{U}$ is the same as $U^\delta$):

$$
F_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad G_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad I_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad J_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad E_{s_i}^{s_{i+1}}(\tilde{U}) \leq 0.
$$

From (4.11) and (4.12), we also obtain that, for any $x \in [s_i, s_{i+1}]$,

$$
\int_{\Sigma^\delta_s} |W(U^\delta)(x, y) - W(\tilde{U})(x, y)| \, dy \leq O(\delta)|x - s_i|,
$$

(4.13)

$$
\int_{\Sigma^\delta_s} |H(U^\delta)(x, y) - H(\tilde{U})(x, y)| \, dy \leq O(\delta)|x - s_i|.
$$

(4.14)

Therefore, as an example, we find (note that the boundary term involving the pressure $\int_{\Gamma} \psi_1 \, ds$ canceled because the boundary is the same):

$$
|G_{s_i}^{s_{i+1}}(U^\delta)| = |G_{s_i}^{s_{i+1}}(U^\delta) - G_{s_i}^{s_{i+1}}(\tilde{U})| \leq \int_{s_i}^{s_{i+1}} \int_{\Sigma^\delta_s} |(W_2(U^\delta) - W_2(\tilde{U})) \partial_x \phi + (H_2(U^\delta) - H_2(\tilde{U})) \partial_y \phi| \, dy \, dx + \int_{\Sigma^\delta_{s_{i+1}}} |(W_2(U^\delta) - W_2(\tilde{U})) \phi| \, dy \leq M \int_{s_i}^{s_{i+1}} \int_{\Sigma^\delta_s} |W_2(U^\delta) - W_2(\tilde{U})| \, dy \, dx + M \int_{s_i}^{s_{i+1}} \int_{\Sigma^\delta_{s_{i+1}}} |H_2(U^\delta) - H_2(\tilde{U})| \, dy \, dx + M \int_{\Sigma^\delta_{s_{i+1}}} |W_2(U^\delta) - W_2(\tilde{U})| \, dy + M \int_{\Sigma^\delta_{s_{i+1}}} |W_2(U^\delta) - W_2(\tilde{U})| \, dy \leq O(\delta)(s_{i+1} - s_i) + O(\delta)(s_{i+1} - s_i),
$$

where $M := \|\phi\|_{W^{1,\infty}}$. Then we find

$$
|G_{r_j}^{r_{j+1}}(U^\delta)| \leq O(\delta) \sum_{i=1}^\infty ((s_{i+1} - s_i)^2 + (s_{i+1} - s_i)) \leq O(\delta)((\tau_{j+1} - \tau_j)^2 + (\tau_{j+1} - \tau_j)).
$$
Thus it is clear that
\[ |G_s^t(U^\delta)| \leq O(\delta)(|t-s|^2 + |t-s|) \quad \text{for any } 0 \leq s < t \leq \infty, \]
and
\[ \lim_{\delta \to 0} G_s^t(U^\delta) = 0. \]

We now need to prove
\[ \lim_{\delta \to 0} G_s^t(U^\delta) = G_s^t(U). \]

4.2.6. Verification of Weak Entropy Solutions. Set
\[ \bar{\phi}(x, \theta) = \phi(x, \theta + g^\delta(x)), \quad \bar{W}_2(x, \theta) = W_2(U^\delta)(x, \theta + g^\delta(x)), \quad \bar{H}_2(x, \theta) = H_2(U^\delta)(x, \theta + g^\delta(x)), \]
where \( W_2^\delta = W_2(U^\delta) \) and \( H_2^\delta = H_2(U^\delta) \). Then we have
\[ G_s^t(U^\delta) = \int_s^t \int_{y > g^\delta(x)} (W_2^\delta \partial_x \phi + H_2^\delta \partial_y \phi) \, dy \, dx \]
\[ + \int_{y > g^\delta(s)} (W_2^\delta \phi)_{|x=s} \, dy - \int_{y > g^\delta(t)} (W_2^\delta \phi)_{|x=t} \, dy - \int_{\Gamma_{x,t}} \phi n_1 \, ds \]
\[ = \int_s^t \int_{\mathbb{R}^+} \left( \bar{W}_2^\delta \partial_x \bar{\phi} - \partial_y \bar{\phi} \right) + \bar{H}_2^\delta \partial_\theta \bar{\phi} \right) \, d\theta \, dx \]
\[ + \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=s} \, d\theta - \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=t} \, d\theta \]
\[ - 2\int_s^t \phi(x, g^\delta(x)) \frac{(g^\delta)'(x)}{\sqrt{1 + ((g^\delta)'(x))^2}} \, dx. \]

By Lemma 4.1, we know that \( g^\delta \to g \) uniformly for \( x \in [s, t] \). Since \( \bar{U}^\delta \) is uniformly bounded and converges to \( \bar{U} \) under the metric of \( C([s, t]; L^1(\mathbb{R}^+)) \), then \( \bar{W}^\delta \) and \( \bar{H}^\delta \) are also uniformly bounded and converges to \( \bar{W} \) and \( \bar{H} \) respectively in the topology of \( C([s, t]; L^1(\mathbb{R}^+)) \). From these facts, one can easily use the Lebesgue dominant convergence theorem to show (with \( \bar{\phi} = \phi(x, \theta + g(x)) \)) that, as \( \delta \to 0 \),
\[ \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=s} \, d\theta - \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=t} \, d\theta \to \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=s} \, d\theta - \int_{\mathbb{R}^+} \left( \bar{W}_2 \bar{\phi} \right)_{|x=t} \, d\theta, \]
\[ \int_s^t \int_{\mathbb{R}^+} \left( \bar{W}_2 \partial_x \bar{\phi} + \bar{H}_2 \partial_\theta \bar{\phi} \right) \, d\theta \, dx \to \int_s^t \int_{\mathbb{R}^+} \left( \bar{W}_2 \partial_x \bar{\phi} + \bar{H}_2 \partial_\theta \bar{\phi} \right) \, d\theta \, dx. \]

Since \( \{(g^\delta)\}' \) is uniformly bounded, we may assume that \( (g^\delta) \to h \) in the weak* topology of \( L^\infty(\mathbb{R}^+) \). Since \( g^\delta \to g \) uniformly in \([s, t]\), we find that \( (g^\delta) \to g' \) in the sense of distributions. Thus, we must have \( h = g' \). Therefore, as \( \phi(x, g^\delta(x)) \to \phi(x, g(x)) \) uniformly in \([s, t]\), and \( (g^\delta) \to g' \) in the weak* \( L^\infty \), we have
\[ \int_s^t \phi(x, g^\delta)(g^\delta)'(x) \, dx \to \int_s^t \phi(x, g)g'(x) \, dx. \]
We also find
\[
\int_s^t \int_{\mathbb{R}^+} (\tilde{W}^2_2 \partial_t \tilde{\phi} (g^\delta)' - \tilde{W}_2 \partial_t \tilde{\phi} ) \, d\theta \, dx
\]
\[= \int_s^t \int_{\mathbb{R}^+} \tilde{W}^2_2 (\partial_t \tilde{\phi} - \partial \tilde{\phi} ) (g^\delta)' \, d\theta \, dx + \int_s^t \int_{\mathbb{R}^+} \tilde{W}^2_2 \partial_t \tilde{\phi} ((g^\delta)' - g') \, d\theta \, dx
\]
\[+ \int_s^t \int_{\mathbb{R}^+} \tilde{W}^2_2 - \tilde{W}_2 \right) \partial_t \tilde{\phi} (x) \, d\theta \, dx.
\]
Using the boundedness of \( \{ \tilde{W}^2_2 \} \) and \( \{(g^\delta)\}' \), and the uniform convergence \( \partial_t \tilde{\phi} \to \partial \tilde{\phi} \), the first integral in the right-hand side goes to zero as \( \delta \to 0 \). The third one converges to zero directly from \( \tilde{W}^2_2 \to \tilde{W}_2 \) in \( C([s,t]; L^1(\mathbb{R}^+)) \). For the second integral, it can be written as
\[
\int_s^t \int_{\mathbb{R}^+} (\tilde{W}^2_2 - \tilde{W}_2) \partial_t \tilde{\phi} ((g^\delta)' - g') \, d\theta \, dx + \int_s^t \int_{\mathbb{R}^+} \tilde{W}_2 \partial_t \tilde{\phi} ((g^\delta)' - g') \, d\theta \, dx.
\]
By the boundedness of \( (g^\delta)' - g' \), the first one then converges to zero; for the second one, using again \( (g^\delta)' \to g' \) in the weak* topology of \( L^\infty \). Then we have also proved
\[
\int_s^t \int_{\mathbb{R}^+} \tilde{W}^2_2 \partial_t \tilde{\phi} (g^\delta)' \, d\theta \, dx \to \int_s^t \int_{\mathbb{R}^+} \tilde{W}_2 \partial_t \tilde{\phi} g' \, d\theta \, dx.
\]
Hence, we have
\[
\lim_{\delta \to 0} G_s^t (U^\delta) = \int_s^t \int_{\mathbb{R}^+} (\tilde{W}_2 (\partial_t \tilde{\phi} - \partial \tilde{\phi} g') + \tilde{H}_2 \partial_t \tilde{\phi} ) \, d\theta \, dx
\]
\[+ \int_{\mathbb{R}^+} (\tilde{W}_2 \tilde{\phi} ) |_{x=s} \, d\theta - \int_{\mathbb{R}^+} (\tilde{W}_2 \tilde{\phi} ) |_{x=t} \, d\theta - \int_s^t \phi (x, g(x)) g' (x) \, dx
\]
\[= G_s^t (U)
\]
by a change of variables \( (x, y) = (x, \theta + g(x)) \).

Therefore, we have proved \( G_s^t (U) = 0 \) as desired. Similarly, we can conclude
\[
E_s^t (U) \leq 0, \quad F_s^t (U) = 0, \quad I_s^t (U) = 0, \quad J_s^t (U) = 0,
\]
and hence the limit \( (g, U) \) obtained from the approximate solutions \( (g^\delta, U^\delta) \) is actually a weak entropy solution to problem \([1.5]\).

It is clear that \( g \) should satisfy the estimate listed in Theorem 1.1 as guaranteed by Lemma 4.1. To show \( \| (U - U^+) (x, \cdot) \|_{BV} \leq C \varepsilon \), we note that we have proved
\[
\| (U^\delta - U^+) (x, \cdot) \|_{BV} \leq C \varepsilon.
\]
Then, by Helly’s theorem, without loss of generality, we may assume
\[
(U^\delta - U^+) (x, \cdot) \to (\tilde{U} - U^+) (x, \cdot) \quad \text{pointwise}
\]
for some \( (\tilde{U} - U^+) (x, \cdot) \) so that
\[
\| (\tilde{U} - U^+) (x, \cdot) \|_{BV} \leq C \varepsilon \quad \text{as } \delta \to 0.
\]

However, by uniqueness of the pointwise limit, we must have \( \tilde{U} = U \). This completes the proof.
5. Asymptotic Behavior of Weak Entropy Solutions

Finally we discuss the asymptotic behavior of the weak entropy solution \((g, U)\) as \(x \to \infty\).

For any given \(\delta > 0\) and the corresponding approximate solution \((g^\delta, U^\delta)\), we know that there are a finite number of fronts and collisions/reflections. Thus, there exists \(x_\delta > 0\) so that, for \(x > x_\delta\), there is no collisions and reflections. Suppose then that there are \(m + 1\) different states \(\{U_j^\delta\}_{j=0}^m\) from the upper to lower. It is obvious that \(U_0^\delta = U_0\) (cf. Remark 1.1), and there is \(m_0\) with \(1 \leq m_0 < m\) so that each pair \((U_{j-1}^\delta, U_j^\delta)\) \((j = 1, \ldots, m_0)\) is connected by a discontinuity of the first characteristic family, while each \((U_{j-1}^\delta, U_j^\delta)\), \(j = m_0 + 1, \ldots, m\), is connected by a characteristic discontinuity (of the second and/or third characteristic family). Since no fronts interact, it is only possible that, for \(m_0 > 1\), all the discontinuities of the first family must be rarefaction waves; for \(m_0 = 1\), this discontinuity might be a shock or a rarefaction wave. For states \(U_j^\delta\) \((j = m_0 + 1, \ldots, m)\), the pressure must be \(p\) by the boundary condition and the Rankine–Hugoniot jump conditions of characteristic discontinuities.

Now we solve the free boundary Riemann problem of the Euler equations (1.1) with the initial data \(U = U_0\) and the boundary condition \(p = p_0\) on the free boundary \(y = kx\). Suppose that the solution is given by \(U_\infty = \Psi_4(\alpha; U_0)\). Then \(k = \frac{u_\infty}{u_\infty}\), and the resulting 4-wave is a shock if \(p > p_0\), and a rarefaction wave if \(p < p_0\). It is clear that both \(\frac{v_j^\delta}{u_j^\delta}\) \((j = m_0 + 1, \ldots, m)\) and \((g^\delta)'\) should be \(\frac{v_\infty}{u_\infty}\) for all \(x > x_\delta\).

We now define \(p_\delta(\xi, \theta) = p^\delta(\xi + x_\delta + 1, \theta + g^\delta(\xi + x_\delta))\) for \(\xi \geq 0\) and \(\theta \geq 0\). It is easy to see that \(|p_\delta(\xi, \cdot)|\) and \(T.V.(p_\delta)(\xi, \cdot) = [p_0 - p]\) are bounded for all \(\delta > 0\) and given \(\xi\). Thus, by Helly’s theorem, there is a subsequence (still denoted as \(\delta\)) so that \(\lim_{\delta \to 0} p_\delta(\xi, \theta) = p(\theta) = p\) for \(\theta > 0\) pointwise. This should imply that, for a.e. \(\theta \geq 0\) and the weak entropy solution \(U = (u, v, p, \rho)\),

\[
\lim_{x \to \infty} p(x, \theta + g(x)) = p,
\]

Similarly, we have

\[
\lim_{x \to \infty} \frac{v}{u}(x, \theta + g(x)) = \frac{v_\infty}{u_\infty}.
\]

It then follows from \(g' = \frac{v}{u}\) that

\[
\lim_{x \to \infty} g'(x) = \frac{v_\infty}{u_\infty}.
\]

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