On the Three-dimensional Lattice Model

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Abstract

Using the restricted star-triangle relation, it is shown that the N-state spin integrable model on a three-dimensional lattice with spins interacting round each elementary cube of the lattice proposed by Mangazeev, Sergeev and Stroganov is a particular case of the Bazhanov-Baxter model.

Keywords: Three-dimensional lattice model, Interaction-round-cube (IRC) model, Bazhanov-Baxter model, Restricted star-triangle relation, Tetrahedron equation, Three-dimensional star-star relation, Weight function

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1 Introduction

As a generalization of Zamolodchikov model with the $Z_2$ symmetry, the interaction-round-cube (IRC) model with the $Z_N$ symmetry was first proposed by Bazhanov and Baxter [1]. In that model the tetrahedron equation [2, 3] plays an important role, by which the commutativity of layer-to-layer transfer matrices is ensured in the three-dimensional lattice model. Later the three-dimensional star-star relation and the tetrahedron equation were discussed in Refs. [4, 5, 6, 7, 8].

Korepanov investigated the solution of the vertex tetrahedron equation in Ref. [9]. The elliptic solutions for modified tetrahedron equations related to the 3D integrable models were studied by Boos et al [10]. Bellon et al discussed the cubic lattice model by imposing some restricted conditions on the entries of the $R$-matrix [11].

Very recently in Ref. [12], another IRC model has proposed in which the weight function is a vertex solution of the tetrahedron equation with the arbitrary number $N$ of the spin variables as a generalization of Hietarinta’s result. This solution has the multiplicative form and can be written as

$$R_{i_1j_1}^{j_2j_3j_4} = \omega^{j_1(i_3-j_3)} w(p_2, p_{12}, p_1|i_3-j_3) w(p_4, p_{34}, p_3|i_1-j_1) / w(p_6, p_{56}, p_5|i_1-j_1)$$

(1)

where the spin variables $i_1, i_2, i_3, j_1, j_2, j_3$ satisfy the conditions $j_2 = i_1 + i_3$, $i_2 = j_1 + j_3$ and take their values in $Z_N$. And the functions $w(p_2, p_{12}, p_1|i_3-j_3), \ldots$ have the following form:

$$w(x, y, z|l) = \prod_{j=1}^{l} \frac{y}{z-x\omega^j}, \quad x^N + y^N = z^N,$$

(2)

with the notations

$$\omega = \exp(2\pi i/N), \quad \omega^{1/2} = \exp(\pi i/N).$$

(3)

The weight function $R$ satisfies the vertex type tetrahedron equation [13]

$$\sum_{k_1,k_2,k_3, k_4,k_5,k_6} R_{i_1,i_2,i_3}^{k_1,k_2,k_3} R_{j_1,j_2,j_3}^{k_4,k_5,k_6} R_{k_2k_4k_6}^{j_2j_4j_6} R_{k_4k_5k_6}^{i_2i_4i_6} = \sum_{k_1,k_2,k_3, k_4,k_5,k_6} R_{i_3,i_5,i_6}^{m,k_5,k_6} R_{i_2i_4i_6}^{p,k_2k_4k_6} R_{i_2i_4i_6}^{k_1j_3j_4} R_{k_1j_2j_4}^{j_1j_2j_3}.$$

(4)

It ensures the commutativity of layer-to-layer transfer matrices constructed from weight functions $R$ and $R'$. In view of this, a natural question may be raised. What is the difference between the above model and the Bazhanov-Baxter model? Since the latter has the more parameters in the weight function, we may ask if the former one is a special case of the latter model. So far this question has only been answered in an affirmative way for the case of $N = 2$ (See Ref. [12]). The aim of
this letter is to give an affirmative answer for the generic case of arbitrary number of spin variables.

The outline of this letter is as follows. The Bazhanov-Baxter model is described in §2. In section 3, the weight function denoted by Eq. (1) is derived from the Bazhanov-Baxter model. This demonstrates that the former model is the particular case of that of latter. Finally, some remarks are given.

2 The Bazhanov-Baxter Model

Following Ref. [1], consider a simple cubic lattice $\mathcal{L}$. At each site of it has a spin $\sigma \in \mathbb{Z}_N$ so that all possible interaction is allowed within each elementary cube (See Fig.1). The partition function of the Bazhanov-Baxter model can be given from

\begin{equation}
V(a|efg|bcd|h) = w_{pq}^{-1}(a-g-f+b)s(c-h,d-h) \\
\times s(g,a-g-f+b)\sum_{\sigma=0}^{N-1} w_{pq}^{-1}(e-c-\sigma)w_{pq}(d-h-\sigma) \\
\times w_{pq}(\sigma-f+b)\tilde{w}_{pq}(a-g-\sigma)s(\sigma,a-c-f+h),
\end{equation}

where $w_{pq}^{-1}(l)$ denotes $1/w_{pq}(l)$, and

\begin{equation}
w_{pq}(k) = w(p/q,k), \quad \tilde{w}_{pq}(k) = \Phi(k)w(p/q,k), \\
\Phi(k) = (\omega^{1/2})^{k(N+k)}, \quad s(k,l) = \omega^{kl},
\end{equation}

\begin{equation}
\frac{w(p/q,l)}{w(p/q,0)} = [\Delta(p/q)]^{l} \prod_{k=1}^{l} (1 - \omega^{kp/q})^{-1}, \quad \Delta(p/q) = (1 - p^{N}/q^{N})^{1/N},
\end{equation}

Fig.1. Arrangement of the spins $a, b, c, d, e, f, g, h$ on the elementary cube

the following weight function $V$ [1]:

\begin{align*}
V(a|efg|bcd|h) &= w_{pq}^{-1}(a-g-f+b)s(c-h,d-h) \\
&\times s(g,a-g-f+b)\sum_{\sigma=0}^{N-1} w_{pq}^{-1}(e-c-\sigma)w_{pq}(d-h-\sigma) \\
&\times w_{pq}(\sigma-f+b)\tilde{w}_{pq}(a-g-\sigma)s(\sigma,a-c-f+h),
\end{align*}

where $w_{pq}^{-1}(l)$ denotes $1/w_{pq}(l)$, and

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\end{align*}

\begin{align*}
\frac{w(p/q,l)}{w(p/q,0)} &= [\Delta(p/q)]^{l} \prod_{k=1}^{l} (1 - \omega^{kp/q})^{-1}, \quad \Delta(p/q) = (1 - p^{N}/q^{N})^{1/N},
\end{align*}

3
where \( w(p/q, 0) \) is yet arbitrary. The weight function \( V \) satisfies the three-dimensional star-star relation [1, 7]

\[
\bar{V}(a|efg|bcd|h) = \frac{w(z, c-h-g+b)s(g+h, g-b)}{w(z, e-a-a+f)s(a+d, a-f)},
\]

where

\[
\bar{V}(a|efg|bcd|h) = w_{q'q}(d-h-f+b)w_{q'q}^{-1}(e-c-a+g)s(c, g-a)
\]

\[
\times s(h, f-b) \sum_{\sigma=0}^{N-1} w_{p'q}^{-1}(\sigma-f+b)w_{pq}(\sigma-a+g)
\]

\[
\times w_{q'p}(e-c-\sigma)\tilde{w}_{p'q}(\sigma-d+h)s(-\sigma, a-c-f+h),
\]

(8)

\[
z = e^{-i\pi/N}(\Gamma(p, p', q, q'))^{1/N}, \quad \Gamma(p, p', q, q') = \frac{(p^N - q^N)(p'^N - q'^N)}{(p^N - q'^N)(p'^N - q^N)}.
\]

3 The derived of the weight function \( R \) from Bazhanov-Baxter model

Set

\[
x_1 = q, \; x_2 = q', \; x_3 = p, \; x_4 = p', \;
\]

\[
x_i^N - x_j^N = x_{ij}^N, \; i < j, \; i, j = 1, 2, 3, 4.
\]

(9)

Following the notations in (2), the weight function \( V \) of the Baxter-Bazhanov model can be written into the form

\[
V(a|efg|bcd|h)
\]

\[
= \frac{w(x_4, x_{34}, x_3|e-c-d+h)s(g, a-g-f+b)\Phi(a-g)}{w(x_4, x_{34}, x_3|a-g-f+b)s(c-h, h-d)\Phi(b-f)}
\]

\[
\times \left\{ \sum_{\sigma=0}^{N-1} \frac{w(3, x_{13}, x_1|\sigma+d-h)w(x_4, x_{24}, x_2|\sigma+a-g)s(\sigma+b+c-g-h)}{w(x_4, x_{14}, x_1|\sigma+e-c)w(x_3, x_{23}, \omega x_2|\sigma+f-b)} \right\}_0
\]

(10)

where the subscript "0" after the curly brackets indicates that the expression in the braces is divided by itself with the zero exterior spins and we have used the property

\[
w(x, y, z|l)w(z, \omega^{1/2}y, \omega x|-l)\Phi(l) = 1, \quad l \in \mathbb{Z}_N,
\]

(11)

and \( \Phi(l) \) is given by (3). Let

\[
c = h, \; d = e.
\]

(12)
Then the above weight function $V$ can be viewed as a Boltzmann weight function with the spins interacting round the triangular prism as in Fig. 2.

![Fig. 2. Arrangement of the spins $a, b, e, f, g, h$ on the elementary triangular prism]

Furthermore, when we make the choice of

$$a + b = f + g,$$

this figure can be denoted by Fig. 3.

![Fig. 3. Arrangement of the spins $b, e, f, g, h$ corresponding to Eq. (13)]

From the three-dimensional star-star relation we know that the weight function $V(a|efg|bcd|h)$ is the same with $\tilde{V}(a|efg|bcd|h)$ modulo the factor $\omega^{(g-b)(b-f+h-e)}$ in this case. And the function $V$ can be written as

$$V(a|efg|bcd|h) = \left\{ \sum_{\sigma=0}^{N-1} \frac{w(x_3, x_{13}, x_1|\sigma + e - h)w(x_4, x_{24}, x_2|\sigma + f - b)s(\sigma, b - g)}{w(x_4, x_{14}, x_1|\sigma + e - h)w(x_3, x_{23}, \omega x_2|\sigma + f - b)} \right\}_0,$$  \hspace{1cm} (14)

Now we make the transformation:

$$b \rightarrow -h, \; e \rightarrow c, \; f \rightarrow -f,$$
$$g \rightarrow a - b - e, \; h \rightarrow a,$$  \hspace{1cm} (15)
V(a|efg|bcd|h) \rightarrow w(a|efg|bcd|h),

which means that the plane (bgh) in Fig.3 is substituted by the plane (achb) in Fig.4. Setting

\[ x_4 = 0, \ x_1 = x_{14}, \ x_2 = x_{24}, \] (16)

we have that

\[ w(a|efg|bcd|h) \]

\[ = \omega^{(h-f)(a-b-e+h)} \left\{ \sum_{\sigma=0}^{N-1} \frac{w(x_3,x_{13},x_1| - \sigma - a + c + f - h)}{w(x_3,x_{23},\omega x_2| - \sigma)s(\sigma,-a+b+e-h)} \right\}. \] (17)

![Fig.4. Arrangement of the spins after transforming (13)](image)

The restricted star-triangle relation of the Bazhanov-Baxter model has the form [4, 8, 14, 15, 16]

\[ \sum_{i=1}^{N} \frac{w_{pR(q)}(n-l)}{w_{qR}(l-m)w_{pq}(k-l)} = \rho(pqr) \frac{w_{pR(q)}(n-m)w_{R^{-1}(q)r}(k-n)}{w_{pr}(k-m)} \] (18)

where \( \rho(pqr) \) is a scalar factor and

\[ R(a_p, b_p, c_p, d_p) = (b_p, \omega a_p, d_p, c_p), \ w_{pq}(n) = w(\omega^{-1}c_p b_q, d_p a_q, b_p c_q|n) \] (19)

with \( a_p = d_r = 0 \). The vectors \( (a_p, b_p, c_p, d_p) \) etc satisfy

\[ a_p^N + k'b_p^N = kd_p^N, \ k'a_p^N + b_p^N = kc_p^N \] (20)

with \( k^2 + k'^2 = 1 \). By using the above restricted star-triangle relation, Eq. (17) can be transformed into the form:

\[ w(a|efg|bcd|h) = \omega^{(h-f)(a-b-e+h)} \times \]
$$\times \frac{w(p_2, p_{12}, p_1|a - b - e + h)w(p_4, p_{34}, p_3| - a + c + f - h)}{w(p_6, p_{56}, p_5| - b + c - e + f)} \tag{21}$$

where

$$p_1 = \omega x_{13}x_2, \quad p_2 = x_1x_{23}, \quad p_3 = \omega x_1x_3,$$

$$p_4 = \omega x_2x_3, \quad p_5 = \omega x_{13}x_3, \quad p_6 = x_{23}x_3 \tag{22}$$

with $p_{ij}^N = p_i^N - p_j^N$ for $i < j$. This is just the relation (2.9) of the Ref. [12]. It can be written as the form of the equation (1) by making a proper choice of the spin variables. So we get the connection of the Bazhanov-Baxter model and the one proposed by Mangazeev et al.

4 Conclusion and Remarks

As the above discussion, we have obtained the $N$-state spin integrable model on a three-dimensional lattice with the spins interacting round each elementary cube of the lattice, proposed by Mangazeev, Sergeev and Stroganov, from the Bazhanov-Baxter model. The key point in the above derivation is that the spectrums in Eq. (1) have the relation $p_6/p_5 = p_2p_4/(p_1p_3)$ and the spin variable appeared in the denominator of the right hand side term of Eq. (1) is the sum of that appeared in the numerator, which is similar to the property of the restricted star-triangle relation of the Bazhanov-Baxter model. We know that the tetrahedron equation plays an important role in the IRC model. And it is proved that the weight function $V(a|efg|bcd|h)$ of the Eq. (4) satisfies the tetrahedron equation by using the method of the spherical trigonometry parametrization where some additional multipliers was introduced [13]. But the weight function (1) cannot be derived from the result of this parametrization when $N > 2$ [12]. It would be an interesting question to find a new parametrization for weight function $V(a|efg|bcd|h)$ of the Bazhanov-Baxter model which satisfies the tetrahedron equation and contains the relation (21) as a limited case.

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