Reflected BSDEs in time-dependent convex regions
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Abstract
We prove existence and uniqueness of solutions of reflected backward stochastic differential equations in time-dependent adapted and càdlàg convex regions \( D = \{D_t; t \in [0,T]\} \). We also show that the solution may be approximated by solutions of backward equations with reflection in appropriately defined discretizations of \( D \) and by a modified penalization method. The approximation results are new even in the one-dimensional case.

1 Introduction
In the present paper we investigate the problems of existence, uniqueness and approximation of solutions to multidimensional backward stochastic differential equations (BSDEs for short) with reflection in time-dependent random convex regions.

In the one-dimensional case these problems are quite well investigated. In the pioneering paper [4] reflected BSDEs (RBSDEs) with one continuous barrier and Lipschitz continuous coefficient are thoroughly investigated. Subsequently the results of [4] were generalized to equations with possibly discontinuous barrier or coefficient satisfying less restrictive regularity or growth conditions (see, e.g., [7, 19] and the references therein). One-dimensional RBSDEs with two continuous reflecting barriers were first studied in [2]. Recent results for equations with two possibly discontinuous barriers are to be found in [3, 9, 16].

Existence, uniqueness and approximation by the penalization method of multidimensional RBSDEs were for the first time studied in [3] in the case of fixed convex domain. In [1, 15] the existence and uniqueness results of [3] were generalized to equations involving subdifferential of a fixed proper convex lower-semicontinuous function. Our main goal is to generalize the results of [3] to the case of time-dependent random regions and at the same time generalize to the multidimensional case some one-dimensional results proved in [2, 4, 7, 9, 19, 16] for continuous barriers or discontinuous barriers satisfying the so-called Mokobodzki condition.

We now describe more precisely the content of the paper. Let \( W \) be a standard \( d \)-dimensional Wiener process and let \((F_t)\) denote the standard augmentation of the natural filtration generated by \( W \). Suppose we are given a family \( D = \{D_t; t \in [0,T]\} \) of time-dependent random closed convex subsets of \( \mathbb{R}^m \) with nonempty interiors, an
and a measurable function \( f : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \) (coefficient). In the paper we consider RBSDEs in \( D \) of the form

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s + K_T - K_t, \quad t \in [0, T].
\]  

(1.1)

By a solution to (1.1) we understand a triple \((Y, Z, K)\) of \((\mathcal{F}_t)\)-adapted processes such that \(Y_t \in D_t\) for \(t \in [0, T]\), \(K\) is a process of locally bounded variation \(|K|\) increasing only when \(Y_t \in \partial D_t\) and (1.1) is satisfied.

In [3] it is proved that if \(\xi \in L^2\), \(\int_0^T |f(s, 0, 0)|^2 \, ds \in L^1\), \(f\) is Lipschitz continuous in both variables \(y, z\) and \(D_t = G\), \(t \in [0, T]\), where \(G\) is a nonrandom convex set with nonempty interior then there exists a unique solution \((Y, Z, K)\) of (1.1) such that \(Y, K\) are continuous, \(Y^*_T, K^*_T \in L^2\) and \(Z \in \mathcal{P}^2\), i.e. \(Z\) is progressively measurable and \(\left(\int_0^T \|Z_t\|^2 \, dt\right)^{1/2} \in L^2\) (Here and in the sequel we use the notation \(X^*_t = \sup_{s \leq t} |X_s|\)). In the present paper we make the same assumptions on the terminal value \(\xi\) and coefficient \(f\). Our main assumption on \(D\) says that the process \(t \mapsto D_t\) is \((\mathcal{F}_t)\)-adapted and càdlàg with respect to the Hausdorff metric, and one can find a semimartingale \(A\) of the class \(\mathcal{H}^2\) (see Section 2 for the definition) such that \(A_t \in \text{Int} D_t\) for \(t \in [0, T]\) and \(\inf_{t \leq T} \text{dist}(A_t, \partial D_t) > 0\). The last condition is an analogue of the so-called Mokobodzki condition considered up to now only in the one-dimensional case (see [9, 16] and the references therein). In our main theorem we prove that under the above assumptions on \(\xi, f, D\) there exists a unique solution \((Y, Z, K)\) of (1.1) such that \(Y, K\) are càdlàg and \(Y, Z, K\) have the same integrability properties as in [3], i.e. \(Y^*_T, K^*_T \in L^2\), \(Z \in \mathcal{P}^2\). If, in addition, \(t \mapsto D_t\) is continuous, then \(Y, K\) are continuous. Therefore our theorem generalizes the results of [3] to time-dependent regions and the same time generalizes one-dimensional results with time-dependent barriers to the multidimensional case. But let us note that in the one-dimensional case one can also prove existence and uniqueness of solutions of RBSDEs with two reflecting barriers \(L, U\) such that \(D = \{(L_t, U_t)\}, t \in [0, T]\) does not satisfy the Mokobodzki condition (see [9]) and with less restrictive assumptions on \(\xi, f\) (see [5, 9]). In general, these solutions have weaker integrability properties.

The uniqueness of solutions of (1.1) can be proved by some modification of known methods. The idea behind our proof of existence is as follows. We consider piecewise constant time-dependent processes \(\mathcal{D}^j\) such that \(\mathcal{D}^j \to \mathcal{D}\) in the Hausdorff metric uniformly in probability. Then we prove that on each random interval on which \(\mathcal{D}^j\) is a constant random set there exists a unique solution of some local RBSDE. Piecing the local solutions together we obtain a solution \((Y^j, Z^j, K^j)\) of (1.1) in \(\mathcal{D}^j\). Finally, we show that the sequence \(\{(Y^j, Z^j, K^j)\}\) converges as \(j \to \infty\) and its limit is a solution of (1.1) in \(\mathcal{D}\). The method described above is new even in the one-dimensional case. To our knowledge the results on existence and uniqueness of local solutions in random convex sets are also new.

We also consider approximation of solutions of (1.1) by the penalization method. This method proved to be useful in the case of one-dimensional RBSDEs with regular and irregular barriers (see, e.g., [3, 6, 7, 9, 11, 19]). In [11] it is observed that in the last case, i.e. if the barriers are discontinuous, the usual method provides only pointwise approximation of the first component \(Y\) and weak approximation of \(K\) and the martingale part of the solution. To generalize the penalization method to the
irregular multidimensional case and at the same time to get uniform approximation of $Y$ and strong approximation of $K$ and the martingale part we consider a modified scheme. It has the form

$$Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s) \, ds - \int_t^T Z^n_s \, dW_s + K^n_T - K^n_t, \quad t \in [0, T],$$

(1.2)

where

$$K^n_t = -n \int_0^t (Y^n_s - \Pi_{\sigma_n}(Y^n_s)) \, ds - \sum_{0 < \sigma_n, i \leq t} (Y^n_{\sigma_n, i} - \Pi_{\sigma_n, i-1}(Y^n_{\sigma_n, i}))$$

$$\equiv K^{n,c} + K^{n,d}, \quad t \in [0, T]$$

(1.3)

with $\sigma_{n,0} = 0$, $\sigma_{n,i} = \inf\{t > \sigma_{n,i-1}; \rho(D_t \cap B(0,n), D_{t-} \cap B(0,n)) > 1/n\} \wedge T$, $i = 1, \ldots, k_n$, where $k_n$ is chosen so that $P(\sigma_{n,k_n} < T) \to 0$ as $n \to \infty$ (Here $B(0,n) = \{x \in \mathbb{R}^d; |x| \leq n\}$, $n \in \mathbb{N}$). Note that $K^n$ is a càdlàg process of locally bounded variation such that

$$K^n_0 = 0, \quad \Delta K^n_{\sigma_n, i} = \Pi_{\sigma_n, i-1}(Y^n_{\sigma_n, i}) - Y^n_{\sigma_n, i} = -\Delta Y^n_{\sigma_n, i}.$$ 

In fact, on any interval $[\sigma_{n,i-1}, \sigma_{n,i})$, $i = 1, \ldots, k_n + 1$, where $\sigma_{n,k_n+1} = T$, the pair $(Y^n, Z^n)$ is a solution of the classical BSDEs with Lipschitz coefficients of the form

$$Y^n_t = \Pi_{\sigma_n, i-1}(Y^n_{\sigma_n, i}) + \int_{\sigma_n, i}^{\sigma_n, i+1} f(s, Y^n_s, Z^n_s) \, ds - \int_{\sigma_n, i}^{\sigma_n, i+1} Z^n_s \, dW_s$$

$$- n \int_{\sigma_n, i}^{\sigma_n, i+1} (Y^n_s - \Pi_{\sigma_n}(Y^n_s)) \, ds, \quad t \in [\sigma_{n,i-1}, \sigma_{n,i}).$$

Notice that as compared with the usual penalization method, in the penalization term $K^n$ the discontinuous part $K^{n,d}$ appears. If the mapping $t \mapsto D_t$ is continuous then $K^n = K^{n,c}$, so (1.2), (1.3) reduces to the usual penalization scheme. We show that under the above-mentioned assumptions under which there exists a unique solution $(Y, Z, K)$ of (1.1), it is a limit in probability of $\{(Y^n, Z^n, K^n)\}$ in the space $\mathcal{S} \times \mathcal{P} \times \mathcal{S}$ (see Section 2 for its definition). This result is new even for one-dimensional RBSEDEs with one discontinuous barrier.

It is known that one can use RBSDE to investigate viscosity solutions (see [4,12,15]) or weak solutions (see [6,8,10,18]) of variational inequalities. In fact, this work was intended as the first step to investigate by probabilistic methods this sort of problems for systems with time-dependent constraints. These problems, however, will be studied elsewhere.

2 Notation and preliminary estimates

For $x \in \mathbb{R}^m$, $z \in \mathbb{R}^{mxd}$ we set $|x|^2 = \sum_{i=1}^m |x_i|^2$, $\|z\|^2 = \text{trace}(z^*z)$. $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^m$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By $W$ we denote a standard $d$-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$ and by $(\mathcal{F}_t)$ the standard augmentation of the natural filtration generated by $W$. 

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$L^p$, $p \geq 1$, is the space of random vectors $X$ such that $\|X\|_p = E(|X|^p)^{1/p} < \infty$. $S^p$ is the space of càdlàg adapted (with respect to $(\mathcal{F}_t)$) processes $X$ such that $\|X\|_{S^p} = \|X_T\|_p < \infty$ and $P^p$ is the space of progressively measurable $m \times d$-dimensional processes $Z$ such that $\|Z\|_{P^p} = \|\int_0^T \|Z^2\| ds\|^{1/2}_p < \infty$. $S$ is the space of càdlàg adapted processes equipped with the metric $\delta(X, X') = E((X - X')^+ \wedge 1)$ and $P$ is the space of progressively measurable $m \times d$-dimensional processes $Z$ such that $\int_0^T \|Z\|^2 ds < \infty$, $P$-a.s. equipped with the metric $\delta'(Z, Z') = E(\int_0^T \|Z - Z'\|^2 ds \wedge 1)$. It is well known that $S^p$, $P^p$ are Banach spaces for $p \geq 1$ and that $S$, $P$ are complete metric spaces. By $\mathcal{H}^2$ we denote the space of $m$-dimensional special semimartingales equipped with the norm
\[
\|X\|_{\mathcal{H}^2} = \|[M]_{T}^{1/2}\|_{L^2} + \|[B]\|_{L^2},
\]
where $X = M + B$ is the canonical decomposition of $X$, $[M]_T$ is the quadratic variation of $M$ at $T$ and $[B]_T$ is the variation of $B$ on the interval $[0, T]$.

Given a process $Y$ and an $(\mathcal{F}_t)$-stopping time $\tau$ we denote by $Y^\tau$ the stopped process $\{Y_{t\land \tau}; t \in [0, T]\}$.

By $\text{Conv}$ we denote the space of all bounded closed convex subsets of $\mathbb{R}^m$ with nonempty interiors endowed with the Hausdorff metric $\rho$, i.e. any $G, G' \in \text{Conv},$
\[
\rho(G, G') = \max \left( \sup_{x \in G} \text{dist}(x, G'), \sup_{x \in G'} \text{dist}(x, G) \right),
\]
where $\text{dist}(x, G) = \inf_{y \in G} |x - y|$.

**Remark 2.1** (see Protter [17]). (a) For a special semimartingale $X$,
\[
\|X\|_{\mathcal{H}^2} \leq 3 \sup_{H \text{ predictable}, \|H\| \leq 1} \left( \int_0^T H_s dX_s \right)^* \|L^2 \leq 9 \|X\|_{\mathcal{H}^2}.
\]
(b) $\|X\|_{S^2} \leq c\|X\|_{\mathcal{H}^2}$ and $\|[X]_{T}^{1/2}\|_{L^2} \leq \|X\|_{\mathcal{H}^2}$. Moreover, for any predictable and locally bounded $H$,
\[
\left\| \int_0^T H_s dX_s \right\|_{\mathcal{H}^2} \leq \|H^*\|_{L^2} \|X\|_{\mathcal{H}^2}.
\]

**Remark 2.2** (Menaldi [13]). (a) Let $G$ be a closed convex domain with nonempty interior and let $\mathcal{N}_y$ denote the set of inward normal unit vectors at $y \in \partial G$. It is well known that $n \in \mathcal{N}_y$ iff $\langle y - x, n \rangle \leq 0$. for every $x \in G$ (Here $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^d$).
(b) If moreover $a \in \text{Int}G$ then for every $n \in \mathcal{N}_y$,
\[
\langle y - a, n \rangle \leq -\text{dist}(a, \partial G).
\]
(c) If $\text{dist}(x, G) > 0$ then there exists a unique $y = \Pi_G(x) \in \partial G$ such that $|y - x| = \text{dist}(x, G)$. One can observe that $(y - x)/|y - x| \in \mathcal{N}_y$. Moreover, for every $a \in \text{Int}G$,
\[
\langle x - a, y - x \rangle \leq -\text{dist}(a, \partial G)|y - x|.
\]
(d) For all $x, x' \in \mathbb{R}^m$,
\[
\langle x - x', (x - \Pi_G(x)) - (x' - \Pi_G(x')) \rangle \geq 0.
\]
In the paper we will assume that we are given an $\mathcal{F}_T$-measurable $m$-dimensional random vector $\xi$, a generator $f : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$, which is measurable with respect to $\text{Prog} \otimes B(\mathbb{R}^m) \otimes B(\mathbb{R}^{m \times d})$, where $\text{Prog}$ denotes the $\sigma$-field of all progressive subsets of $[0, T] \times \Omega$ and a family $\mathcal{D} = \{D_t; t \in [0, T]\}$ of random closed convex sets in $\mathbb{R}^m$ with nonempty interiors such that the process $[0, T] \ni t \mapsto D_t \in \text{Conv}$ is $(\mathcal{F}_t)$-adapted. Moreover, we will assume that

(H1) $\xi \in D_T$, $\xi \in L^2$,

(H2) $E \int_0^T |f(s, 0, 0)|^2 ds < \infty$,

(H3) There are $\mu, \lambda \geq 0$ such that for any $t \in [0, T]$,

$$|f(t, y, z) - f(t, y', z')| \leq \mu|y - y'| + \lambda||z - z'||, \quad y, y' \in \mathbb{R}^m, z, z' \in \mathbb{R}^{m \times d},$$

(H4) For each $N \in \mathbb{N}$ the mapping $t \mapsto D_t \cap B(0, N) \in \text{Conv}$ is càdlàg $P$-a.s. (with the convention that $D_T = D_{T-}$) and there is a semimartingale $A \in \mathcal{H}^2$ such that $A_t \in \text{Int}D_t$ for $t \in [0, T]$ and

$$\inf_{t \leq T} \text{dist}(A_t, \partial D_t) > 0.$$

In (H1), (H3), (H4) and in the sequel we understand that the equalities and inequalities hold true $P$-a.s.

**Definition.** We say that a triple $(Y, Z, K)$ of $(\mathcal{F}_t)$-progressively measurable processes is a solution of the RBSDE (1.1) if

(a) $Y_t \in D_t$, $t \in [0, T]$,

(b) $K$ is a càdlàg process of locally bounded variation such that $K_0 = 0$ and for every $(\mathcal{F}_t)$ adapted càdlàg process $X$ such that $X_t \in D_t$, $t \in [0, T]$, we have

$$\int_0^T (Y_{s-} - Y_{s-}, dK_s) \leq 0,$$

(c) Eq. (1.1) is satisfied.

**Proposition 2.3.** Assume (H1)–(H4). If $(Y, Z, K)$ is a solution of (1.1) such that $Y \in \mathcal{S}^2$ then there exists $C > 0$ depending only on $\mu, \lambda, T$ such that

$$E((Y_T^2 + \int_0^T \|Z_s\|^2 ds + \sum_{0 < s \leq T} |\Delta K_s|^2 + (K_T^+)^2 + \int_0^T \text{dist}(A_{s-}, \partial D_{s-}) d|K|_s))$$

$$\leq C\left(E(|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds) + \|A\|^2_{\mathcal{H}^2}\right)$$

**Proof.** We first show that

$$E\left(\int_0^T \|Z_s\|^2 ds + \sum_{0 < s \leq T} |\Delta K_s|^2 + (K_T^+)^2 + \int_0^T \text{dist}(A_{s-}, \partial D_{s-}) d|K|_s)\right)$$

$$\leq C\left(E((Y_T^2 + \int_0^T |f(s, 0, 0)|^2 ds) + \|A\|^2_{\mathcal{H}^2}\right). \quad (2.1)$$
Let $\tau_n = \inf\{t > 0; \int_0^t \|Z_s\|^2 ds > n\} \land T$, $n \in \mathbb{N}$. By Itô’s formula,
\[
|Y_0|^2 + \int_0^{\tau_n} \|Z_s\|^2 ds + \sum_{s \leq \tau_n} |\Delta K_s|^2 = |Y_{\tau_n}|^2 + 2 \int_0^{\tau_n} \langle Y_s, f(s, Y_s, Z_s) \rangle ds
\]
\[
- 2 \int_0^{\tau_n} \langle Y_s, Z_s dW_s \rangle + 2 \int_0^{\tau_n} \langle Y_{s-}, dK_s \rangle.
\]
By Remark 2.2(b), the integration by parts formula and the fact that $dK_t = n_Y d|K|_t$,
\[
\int_0^{\tau_n} \langle Y_{s-}, dK_s \rangle = \int_0^{\tau_n} \langle Y_{s-} - A_{s-}, dK_s \rangle + \int_0^{\tau_n} \langle A_{s-}, dK_s \rangle
\]
\[
\leq - \int_0^{\tau_n} \text{dist}(A_{s-}, \partial D_{s-}) d|K|_s
\]
\[
+ \langle K_{\tau_n}, A_{\tau_n} \rangle - \int_0^{\tau_n} \langle K_{s-}, dA_s \rangle - \sum_{s \leq \tau_n} \langle \Delta K_s, \Delta A_s \rangle,
\]
wheras by (H3),
\[
\int_0^{\tau_n} \langle Y_s, f(s, Y_s, Z_s) \rangle ds \leq C_1 \left( (Y_T^*)^2 + \int_0^T |f(s, 0, 0)|^2 ds \right) + \frac{1}{4} \int_0^{\tau_n} \|Z_s\|^2 ds
\]
for some $C_1 > 0$. Putting together the above inequalities and using the fact that
\[
-2 \sum_{s \leq \tau_n} \langle \Delta K_s, \Delta A_s \rangle \leq 2 \left( \sum_{s \leq \tau_n} |\Delta K_s|^2 \right)^{1/2} \left( \sum_{s \leq \tau_n} |\Delta A_s|^2 \right)^{1/2} \leq \frac{1}{2} \sum_{s \leq \tau_n} |\Delta K_s|^2 + 2[A]_T
\]
we obtain
\[
\int_0^{\tau_n} \|Z_s\|^2 ds + \sum_{s \leq \tau_n} |\Delta K_s|^2 + 4 \int_0^{\tau_n} \text{dist}(A_{s-}, \partial D_{s-}) d|K|_s
\]
\[
\leq 2(Y^*)^2 + 2C_1 \left( (Y_T^*)^2 + \int_0^T |f(s, 0, 0)|^2 ds \right) + 4K_{\tau_n}^* A_T^* + 4[A]_T
\]
\[
- 4 \int_0^{\tau_n} \langle K_{s-}, dA_s \rangle - 4 \int_0^{\tau_n} \langle Y_s, Z_s dW_s \rangle.
\]
Since $K_t = Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s dW_s$, $t \in [0, T]$, we have
\[
K_{\tau_n}^* \leq (2 + \mu T)Y_T^* + \int_0^T |f(s, 0, 0)| ds + \lambda \int_0^{\tau_n} \|Z_s\| ds + \sup_{t \leq \tau_n} \int_0^t Z_s dW_s.
\]
Therefore there is $C_2 > 0$ such that
\[
E(K_{\tau_n}^*)^2 \leq C_2 E((Y_T^*)^2 + \int_0^T |f(s, 0, 0)|^2 ds + \int_0^{\tau_n} \|Z_s\|^2 ds).
\]
Since $E[\int_0^{\tau_n} \langle K_{s-}, dA_s \rangle] \leq c \|K_{\tau_n}^*\|_{L^2} \|A\|_{\mathcal{H}^2}$ and $\int_0^{\tau_n} \langle Y_s, Z_s dW_s \rangle$ is a uniformly integrable martingale, from (2.3) it follows that there is $C_3 > 0$ such that
\[
E\left( \int_0^{\tau_n} \|Z_s\|^2 ds + \sum_{s \leq \tau_n} |\Delta K_s|^2 + 4 \int_0^{\tau_n} \text{dist}(A_{s-}, \partial D_{s-}) d|K|_s \right)
\]
\[
\leq C_3 \left( E((Y^*)^2 + \int_0^T |f(s, 0, 0)|^2 ds + (A_T^*)^2 + [A]_T + \|A\|_{\mathcal{H}^2}^2) \right)
\]
\[
+ (2C_2)^{-1} E(K_{\tau_n}^*)^2.
\]
Combining (2.3) with (2.4), using the fact that \( E(A_T^+)^2, E([A]_T) \leq \|A\|^2_{H^2} \) and then letting \( n \to \infty \) we obtain (2.1).

In the second part of the proof we will estimate \( E(Y_T^*)^2 \). Using Itô’s formula gives
\[
|Y_t|^2 + \int_t^T \|Z_s\|^2 \, ds + \sum_{t < s \leq T} |\Delta K_s|^2 \\
= |\xi|^2 + 2 \int_t^T \langle Y_s, f(s, Y_s, Z_s) \rangle \, ds - 2 \int_t^T \langle Y_s, Z_s \, dW_s \rangle + 2 \int_t^T \langle Y_{s-}, dK_s \rangle \\
\leq |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 \, ds + (2\mu + 2\lambda^2 + 1) \int_t^T |Y_s|^2 \, ds \\
+ \frac{1}{2} \int_t^T \|Z_s\|^2 \, ds - 2 \int_t^T \langle Y_s, Z_s \, dW_s \rangle + 2 \int_t^T \langle A_{s-}, dK_s \rangle.
\] (2.6)

From this we deduce that there is \( C_4 > 0 \) such that
\[
|Y_t|^2 + \frac{1}{2} \int_t^T \|Z_s\|^2 \, ds \leq X + C_4 \int_t^T |Y_s|^2 \, ds - 2 \int_t^T \langle Y_s, Z_s \, dW_s \rangle, \quad t \in [0, T], \quad (2.7)
\]
where \( X = |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 \, ds + \sup_{t \leq T} |\int_0^t \langle A_{s-}, dK_s \rangle| \). Note that from (H1), (H2) and earlier considerations it follows that \( X \) is integrable. Since \( \int_0^t \langle Y_s, Z_s \, dW_s \rangle \) is a uniformly integrable martingale,
\[
E \int_t^T \|Z_s\|^2 \, ds \leq 2EX + 2C_4 E \int_t^T |Y_s|^2 \, ds, \quad t \in [0, T].
\]
Consequently,
\[
E \sup_{s \in [t,T]} |Y_s|^2 \leq EX + C_4 E \int_t^T |Y_s|^2 \, ds + 2cE(\int_t^T |Y_s|^2 \|Z_s\|^2 \, ds)^{1/2} \\
\leq EX + C_4 E \int_t^T |Y_s|^2 \, ds + 2cE(\sup_{s \in [t,T]} |Y_s|)(\int_t^T |Y_s|^2 \|Z_s\|^2 \, ds)^{1/2} \\
\leq EX + C_4 E \int_t^T |Y_s|^2 \, ds + \frac{1}{2} E(\sup_{s \in [t,T]} |Y_s|^2) + 2c^2 E(\int_t^T \|Z_s\|^2 \, ds)^{1/2} \\
\leq (4c^2 + 1)EX + C_4(4c^2 + 1) E \int_t^T |Y_s|^2 \, ds + \frac{1}{2} E(\sup_{s \in [t,T]} |Y_s|^2). \quad (2.8)
\]

Therefore the are \( C_5, C_6 > 0 \) such that
\[
E \sup_{s \in [t,T]} |Y_s|^2 \leq C_5 EX + C_6 \int_t^T E(\sup_{u \in [s,T]} |Y_u|^2) \, ds, \quad t \in [0, T].
\]
Hence, by Gronwall’s lemma, \( E(Y_T^*)^2 \leq C_5 EX e^{C_6 T} \). Since by the integration by part formula and the previously used arguments there is \( C_7 > 0 \) such that
\[
E \sup_{t \leq T} |\int_0^t \langle A_{s-}, dK_s \rangle| \leq (2C_1C_5)^{-1} e^{-C_6 T} E(\sum_{0 < s \leq T} |\Delta K_s|^2 + (K_T^*)^2) + C_7 \|A\|^2_{H^2},
\]
it follows from (2.1) that
\[
E(Y_T^+)^2 \leq C_5 e^{C_6 T} E(|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds) + \frac{1}{2} E(Y_T^-)^2 \\
+ \frac{1}{2} \left( E \int_0^T |f(s, 0, 0)|^2 ds + \|A\|_{\mathcal{H}^2}^2 \right) + C_5 C_7 e^{C_6 T} \|A\|_{\mathcal{H}^2}^2,
\]
which completes the proof. \(\square\)

Let \(D' = \{D'_t; t \in [0, T]\}\) be another family of random closed sets satisfying (H4) with some semimartingale \(A'\) and let \(\xi'\) be an \(\mathcal{F}_T\)-measurable random variable such that \(\xi' \in D'_T\) and \(\xi' \in L^2\). In the following proposition together with (1.1) we consider RBSDE with terminal condition \(\xi'\), coefficient \(f\) and family \(D'\), i.e. equation of the form
\[
Y'_t = \xi' + \int_t^T f(s, Y'_s, Z'_s) ds - \int_t^T \langle Z'_s, dW_s \rangle + K'_T - K'_t, \quad t \in [0, T]. \tag{2.9}
\]
In its proof we will use the following notation
\[
\text{sgn}(x) = \frac{x}{|x|} 1(x \neq 0), \quad x \in \mathbb{R}^d.
\]

**Proposition 2.4.** Let \((Y, Z, K)\) and \((Y', Z', K')\) be solutions of (1.1) and (2.9), respectively, and let \(Y = Y - Y', Z = Z - Z', K = K - K'\). If \(f\) satisfies (H3) and \(Y, Y' \in S^2\) then for every \(p \in (1, 2]\) there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that for any stopping time \(\sigma\) such that \(0 \leq \sigma \leq T\) we have
\[
E\left(\sup_{t < \sigma} |\tilde{Y}_t|^p + \int_0^\sigma |\tilde{Y}_s|^{p-2} \|\tilde{Z}_s\|^2 ds + I_{\sigma^-}\right) \\
\leq C \left( E(|\tilde{Y}_{\sigma^-}|^p + \int_0^{\sigma^-} |\tilde{Y}_{s^-}|^{p-2} |\Pi_{D_{s^-}}(Y'_{s^-}) - Y'_{s^-}| 1_{\{Y'_{s^-} \notin D_{s^-}\}} d|K|_{s^-}) \\
+ E(\int_0^{\sigma^-} |\tilde{Y}_{s^-}|^{p-2} |\Pi_{D'_{s^-}}(Y'_{s^-}) - Y_{s^-}| 1_{\{Y_{s^-} \notin D'_{s^-}\}} d|K'|_{s^-})\right),
\]
where \(I_t = \sum_{s \leq t} \langle |\tilde{Y}_s|^p - |\tilde{Y}_{s^-}|^p - p|\tilde{Y}_{s^-}|^{p-1} \langle \text{sgn}(\tilde{Y}_{s^-}), \Delta \tilde{Y}_s\rangle\rangle, t \geq 0\).

**Proof.** By Itô’s formula for the convex function \(x \rightarrow |x|^p\) (see Klionsky [9]), for any \(t < \sigma\) we have
\[
|\tilde{Y}_t|^p + \frac{p(p-1)}{2} \int_t^\sigma |\tilde{Y}_s|^{p-2} \|\tilde{Z}_s\|^2 ds + I_{\sigma^-} - I_t \\
= |\tilde{Y}_{\sigma^-}|^p + p \int_t^\sigma |\tilde{Y}_s|^{p-1} \langle \text{sgn}(\tilde{Y}_s), f(s, Y_s, Z_s - f(s, Y'_s, Z'_s) \rangle ds \\
- p \int_t^\sigma |\tilde{Y}_s|^{p-1} \langle \text{sgn}(\tilde{Y}_s), \tilde{Z}_s dW_s \rangle + p \int_t^\sigma |\tilde{Y}_s|^{p-1} \langle \text{sgn}(\tilde{Y}_s), dK_s\rangle.
\]
Since
\[
p \int_t^\sigma |\tilde{Y}_s|^{p-1} \langle \text{sgn}(\tilde{Y}_s), f(s, Y_s, Z_s - f(s, Y'_s, Z'_s) \rangle ds \\
\leq C_1 \int_t^\sigma |\tilde{Y}_s|^{p} ds + \frac{p(p-1)}{4} \int_t^\sigma |\tilde{Y}_s|^{p-2} 1_{\{\tilde{Y}_s \neq 0\}} \|\tilde{Z}_s\|^2 ds
\]

for some $C_1 > 0$, it follows that

$$
|\bar{Y}_t|^{p} + \frac{p(p-1)}{4} \int_t^{t\wedge \sigma} |\bar{Y}_s|^{p-2}1_{\{\bar{Y}_s \neq 0\}} \|\bar{Z}_s\|^2 \, ds + I_{t\wedge \sigma} - I_t
$$

$$
\leq |\bar{Y}_\sigma|^{p} + C_1p \int_t^T |\bar{Y}_s|^{p-1} \, ds - p \int_t^{t\wedge \sigma} |\bar{Y}_s|^{p-1} \langle \text{sgn}(\bar{Y}_s), \bar{Z}_s \rangle \, dW_s
$$

$$
+ p \int_t^{t\wedge \sigma} |\bar{Y}_s|^{p-1} \langle \text{sgn}(\bar{Y}_s), d\bar{K}_s \rangle
$$

(2.10)

for $t \in [0, T]$. Since \( \int_t^\sigma |\bar{Y}_s|^{p-1} \langle \text{sgn}(\bar{Y}_s), d\bar{K}_s \rangle = \int_t^\sigma |\bar{Y}_s|^{p-2}1_{\{\bar{Y}_s \neq 0\}} \langle \bar{Y}_s \rangle, d\bar{K}_s \rangle \) and

$$
\langle \bar{Y}_s, d\bar{K}_s \rangle = \langle Y_{\sigma} - \Pi_{D_{\sigma}}(Y_{\sigma}'), d\bar{K}_s \rangle + (\Pi_{D_{\sigma}}(Y_{\sigma}') - Y_{\sigma}', d\bar{K}_s) + \langle \Pi_{D_{\sigma}}(Y_{\sigma}') - Y_{\sigma}, d\bar{K}_s \rangle
$$

$$
\leq |\Pi_{D_{\sigma}}(Y_{\sigma}') - Y_{\sigma}| d\bar{K}_s + |\Pi_{D_{\sigma}}(Y_{\sigma}) - Y_{\sigma}| dK'_s,
$$

(2.11)

we see that \( \sup_{t \in [0, T]} |\bar{Y}_s|^{p-1} \langle \text{sgn}(\bar{Y}_s), d\bar{K}_s \rangle \leq X_1 \), where

$$
X_1 = \int_0^\sigma |\bar{Y}_s|^{p-2} \Pi_{D_{\sigma}}(Y_{\sigma}') - Y_{\sigma}'| 1_{\{Y_{\sigma} \notin D_{\sigma}\}} d|\Pi_{D_{\sigma}}(Y_{\sigma}') - Y_{\sigma}| d|K|_s
$$

$$
+ \int_0^\sigma |\bar{Y}_s|^{p-2} \Pi_{D_{\sigma}'}(Y_{\sigma}) - Y_{\sigma}| 1_{\{Y_{\sigma} \notin D_{\sigma}'\}} d|K'|_s.
$$

Since \( \int_0^t |\bar{Y}_s|^{p-1} \langle \text{sgn}(\bar{Y}_s), \bar{Z}_s \rangle \, dW_s \) is a uniformly integrable martingale it follows from (2.10) and (2.11) that

$$
\frac{p(p-1)}{4} E \int_t^{t\wedge \sigma} |\bar{Y}_s|^{p-2}1_{\{\bar{Y}_s \neq 0\}} \|\bar{Z}_s\|^2 \, ds + E(I_{t\wedge \sigma} - I_t)
$$

$$
\leq EX + EpC_1 \int_t^T |\bar{Y}_s|^{p} \, ds, \quad t \in [0, T],
$$

(2.12)

where \( X = |\bar{Y}_\sigma|^{p} + X_1 \). Arguing as in the proof of (2.8) we deduce from the above that there exist constants \( C_5, C_6 > 0 \) such that

$$
E \sup_{s \in [t, T]} |\bar{Y}_s|^{p} \leq C_5 EX + C_6 \int_t^T E( \sup_{u \in [s, T]} |\bar{Y}_u|^{p}) \, ds, \quad t \in [0, T].
$$

By Gronwall’s lemma, \( E( \sup_{t < \sigma} |\bar{Y}_t|^{p} ) = E \sup_{t \in [0, T]} |\bar{Y}_t|^{p} \leq C_5 EX e^{C_6 T} \). Putting \( t = 0 \) in (2.12) completes the proof.

\( \square \)

**Corollary 2.5.** Under the assumptions of Proposition 2.4 if moreover \( D' = D'^{-} \) then

$$
E \left( \sup_{t < \sigma} |\bar{Y}_t|^{p} + \int_0^\sigma |\bar{Y}_s|^{p-2}1_{\{\bar{Y}_s \neq 0\}} \|\bar{Z}_s\|^2 \, ds + I_{\sigma} \right) \leq CE( |\bar{Y}_\sigma|^{p} ).
$$

**Remark 2.6.** Since \( \bar{Y} = -\Delta \bar{K} \), in the case \( p = 2 \) we have

$$
I_t = \sum_{s \leq t} |\Delta K_s|^2, \quad t \geq 0.
$$
3 Existence and uniqueness of solutions of RBSDEs

Our main goal is to prove that under (H1)–(H4) there exists a unique solution \((Y,Z,K)\) of (1.1) such that \(Y,K \in \mathcal{S}^2\) and \(Z \in \mathcal{P}^2\). The uniqueness follows easily from Corollary 2.5. In the proof of the existence we will use the method of approximation of \(D\) by discrete time-dependent process described in the following proposition.

**Proposition 3.1.** Let \(\sigma_0 = 0 \leq \sigma_1 \leq \ldots \leq \sigma_{k+1} = T\) be stopping times and let \(D^0, D^1, \ldots, D^k\) be random closed convex subsets of \(\mathbb{R}^m\) with nonempty interiors such that \(D^i\) is \(\mathcal{F}_{\sigma_i}\)-measurable. Let \((Y,Z,K)\) be a triple of \((\mathcal{F}_t)\)-progressively measurable processes such that

\[
\begin{align*}
(a) & \quad \xi = Y_T \in D^k, \quad K \text{ is a càdlàg process of locally bounded variation such that } K_0 = 0, \\
& \quad \Delta K_{\sigma_i} = \Pi_{D^{i-1}}(Y_{\sigma_i}) - Y_{\sigma_i} = -\Delta Y_{\sigma_i}, \\
(b) & \quad \text{on each interval } [\sigma_{i-1}, \sigma_i], \ i = 1, \ldots, k + 1, \text{ we have}
\end{align*}
\]

\[
Y_t = \Pi_{D^{i-1}}(Y_{\sigma_i}) + \int_t^{\sigma_i} f(s,Y_s,Z_s) \, ds - \int_t^{\sigma_i} Z_s \, dW_s + K_{\sigma_i} - K_t, \quad (3.1)
\]

where \(Y_t \in D^{i-1}\),

\[
(c) \quad \int_{(\sigma_{i-1},\sigma_i)} (Y_{s^-} - X_{s^-} \, dK_s) \leq 0 \quad \text{for every } (\mathcal{F}_t)\text{-adapted càdlàg process } X \text{ such that } X_t \in D^{i-1} \text{ for } t \in [\sigma_{i-1}, \sigma_i].
\]

Then \((Y,Z,K)\) is a unique solution of (1.1) with terminal value \(\xi\) and \(\{D_t; t \in [0,T]\}\) such that \(D_t = D^i\), \(t \in [\sigma_{i-1}, \sigma_i], \ i = 1, \ldots, k + 1, \ D_T = D_{T^-}\).

**Proof.** Note that \(Y_{\sigma_i} + K_{\sigma_i} = \Pi_{D^{i-1}}(Y_{\sigma_i}) + K_{\sigma_i-}, \ i = 1, \ldots, k, \) so \((Y,Z,K)\) satisfies (1.1). Let \(X\) be an \((\mathcal{F}_t)\)-adapted càdlàg process such that \(X_t \in D_t, t \in [0,T]\). Clearly

\[
\sum_{i=1}^{k+1} \int_{(\sigma_{i-1},\sigma_i)} (Y_{s^-} - X_{s^-} \, dK_s) \leq 0.
\]

On the other hand,

\[
Y_{\sigma_i-} = Y_{\sigma_i} - \Delta Y_{\sigma_i} = Y_{\sigma_i} + \Delta K_{\sigma_i} = \Pi_{D^{i-1}}(Y_{\sigma_i}), \quad i = 1, \ldots, k.
\]

Since \(X_{\sigma_i-} \in D_{\sigma_i-}\), from Remark 2.1(a) it follows that

\[
\langle Y_{\sigma_i-} - X_{\sigma_i-}, \Delta K_{\sigma_i} \rangle = \langle Y_{\sigma_i-} - X_{\sigma_i-}, \Pi_{D_{\sigma_i-}}(Y_{\sigma_i}) - Y_{\sigma_i} \rangle
\]

\[
= \langle \Pi_{D_{\sigma_i-}}(Y_{\sigma_i}) - X_{\sigma_i-}, \Pi_{D_{\sigma_i-}}(Y_{\sigma_i}) - Y_{\sigma_i} \rangle \leq 0,
\]

which completes the proof. \(\square\)

Now we are going to study the problem of existence of solutions of (3.1). To this end, we first consider local RBSDEs on closed random intervals.

Let \(\tau, \sigma\) be stopping times such that \(0 \leq \tau \leq \sigma \leq T, \ D\) be an \(\mathcal{F}_\tau\)-measurable random convex set with nonempty interior and let \(\zeta \in L^2\) be an \(\mathcal{F}_\sigma\)-measurable random variable. We consider equations of the form

\[
Y_t = \zeta + \int_t^\sigma f(s,Y_s,Z_s) \, ds - \int_t^\sigma Z_s \, dW_s + K_\sigma - K_t, \quad t \in [\tau, \sigma]. \quad (3.2)
\]
**Definition.** We say that a triple \((Y, Z, K - K_\tau)\) of \((\mathcal{F}_t)\)-progressively measurable processes on \([\tau, \sigma]\) is a solution of the local RBSDE on \([\tau, \sigma]\) if it satisfies (3.2) and

(a) \(Y_t \in D, t \in [\tau, \sigma]\),

(b) \(K_\tau = 0, K\) is a càdlàg process of locally bounded variation on the interval \([\tau, \sigma]\) such that \(\int_\tau^\sigma \langle Y_{s-} - X_{s-}, dK_s \rangle \leq 0\) for every \((\mathcal{F}_t)\)-adapted càdlàg process \(X\) with values in \(D\).

We will assume that

\(\text{(H1*)} \, \zeta \in D, \zeta \in L^2\),

\(\text{(H2*)} \, E \int_\tau^\sigma |f(s, 0, 0)|^2 \, ds < \infty\),

\(\text{(H3*)} \, \text{There are } \mu, \lambda \geq 0 \text{ such that for any } t \in [\tau, \sigma], \)

\[|f(t, y, z) - f(t, y', z')| \leq \mu |y - y'| + \lambda |z - z'|, \quad y, y', z, z' \in \mathbb{R}^m, \]

\(\text{(H4*)} \, \text{There is an } \mathcal{F}_\tau\text{-measurable random variable } A \in L^2 \text{ such that } A \in \text{Int}\, D\).

**Proposition 3.2.** Assume (H1*)–(H4*). If \((Y, Z, K)\) is a solution of (3.2) such that \(\sup_{\tau \leq t \leq \sigma} |Y_t| \in L^2\) then there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that

\[E \left( \sup_{\tau \leq t \leq \sigma} |Y_t|^2 + \int_\tau^\sigma \|Z_s\|^2 \, ds \mid \mathcal{F}_\tau \right) \leq CE \left( \|\zeta\|^2 + |A|^2 + \int_\tau^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau \right)\]

and

\[E(\|K\|_\tau^\sigma \mid \mathcal{F}_\tau) \leq C(\text{dist}(A, \partial D))^{-1} E \left( \|\zeta\|^2 + |A|^2 + \int_\tau^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau \right).

**Proof.** It is sufficient to apply arguments from the proof of Proposition 2.3 and use the fact that dist\((A, \partial D)\) is a strictly positive \(\mathcal{F}_\tau\)-measurable random variable.

Let \(D'\) be an \(\mathcal{F}_\tau\)-measurable random convex set with nonempty interior, \(\zeta' \in L^2\) be an \(\mathcal{F}_\sigma\)-measurable random variable such that \(\zeta' \in D'\) P-a.s. and there is an \(\mathcal{F}_\tau\)-measurable random variable \(A' \in L^2\) such that \(A' \in \text{Int}\, D'\). Consider the local RBSDE on \([\tau, \sigma]\) of the form

\[Y'_t = \zeta' + \int_\tau^\sigma f(s, Y'_{s-}, Z'_{s-}) \, ds - \int_\tau^\sigma Z'_{s-} \, dW_s + K'_\sigma - K'_\tau, \quad t \in [\tau, \sigma]. \quad (3.3)\]

**Proposition 3.3.** Let \((Y, Z, K)\) and \((Y', Z', K')\) be solutions of (3.2) and (3.3), respectively. If \(f\) satisfies (H3*) and \(\sup_{\tau \leq t \leq \sigma} |Y_t|, \sup_{\tau \leq t \leq \sigma} |Y'_t| \in L^2\) then there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that

\[E \left( \sup_{\tau \leq t \leq \sigma} |Y_t - Y'_t|^2 + \int_\tau^\sigma \|Z_s - Z'_s\|^2 \, ds \mid \mathcal{F}_\tau \right) \]

\[\leq C \left( E(\|\zeta - \zeta'\|^2 \mid \mathcal{F}_\tau) + E \left( \sup_{\tau \leq t \leq \sigma} |\Pi_D(Y_{t-}) - Y_{t-}||K'_\tau\| \mid \mathcal{F}_\tau \right) \]

\[+ E \left( \sup_{\tau \leq t \leq \sigma} |\Pi_{D'}(Y_{t-}) - Y_{t-}||K'_\tau\| \mid \mathcal{F}_\tau \right) \right).

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Proof. We apply the arguments from the proof of Proposition 2.4 with $p = 2$. 

We will need the following assumption: there exists $N \in \mathbb{N}$ such that

$$D \subset B(0, N). \quad (3.4)$$

**Proposition 3.4.** Assume (H1*)–(H4*) and (3.4). Then there exists a unique solution $(Y, Z, K)$ of the local RBSDE (3.2) such that

$$\sup_{\tau \leq t \leq \sigma} |Y_t| \in L^2, \quad \int_{\tau}^{\sigma} \|Z_s\|^2 \, ds \in L^1, \quad \sup_{\tau \leq t \leq \sigma} |K_t - K_\tau| \in L^2. \quad (3.5)$$

**Proof.** The uniqueness follows from Proposition 3.3. To prove the existence we first assume that $D$ is nonrandom, i.e. $D = G$, where $G$ is some fixed convex set with nonempty interior. Set $g(s, \cdot, \cdot) = f(s, \cdot, \cdot)1_{[0, \sigma]}(s)$. By [3] Theorem 5.9 there exists a solution $(Y, Z, K)$ of the following RBSDE in $G$

$$Y_t = \zeta + \int_t^{T} g(s, Y_s, Z_s) \, ds - \int_t^{T} Z_s \, dW_s + K_T - K_t, \quad t \in [0, T]. \quad (3.6)$$

Since $Y_t = \zeta$, $Z_t = 0$ and $K_T = K_t$ for $t \geq \sigma$, it is clear that for any $\tau \leq \sigma$ the triple $(Y, Z, K - K_\tau)$ is also a solution of the local RBSDE on $[\tau, \sigma]$. It is well known that in the space $\text{Conv} \cap B(0, N)$ there exists a countable dense set $\{G_1, G_2, \ldots\}$ of convex polyhedrons such that $G_i \subset B(0, N)$, $i \in \mathbb{N}$. By what has already been proved for each $i \in \mathbb{N}$ there exists a solution $(Y_i, Z_i, K_i)$ of the local RBSDE in $G_i$ with terminal value $\zeta_i = \Pi_{G_i}(\zeta)$. Set $C_i^j = \{\rho(G_i, D) \leq 1/j\}$ and

$$C_i^j = \{\rho(G_i, D) \leq 1/j, \rho(G_1, D) > 1/j, \ldots, \rho(G_{i-1}, D) > 1/j\}, \quad i = 2, 3, \ldots$$

Furthermore, for $j \in \mathbb{N}$ set

$$\zeta^j = \sum_{i=1}^{\infty} \Pi_{G_i}(\zeta) \mathbf{1}_{C_i^j}, \quad D^j = \sum_{i=1}^{\infty} G_i \mathbf{1}_{C_i^j}.$$

Since $C_i^j \in \mathcal{F}_\tau$ for $i \in \mathbb{N}$, $(Y^j, Z^j, K^j) = \sum_{i=1}^{\infty} (Y_i, Z_i, K_i) \mathbf{1}_{C_i^j}$ is a solution of the local RBSDE in $D^j$ and terminal value $\zeta^j$. Set

$$A^j = \begin{cases} A, & \text{if dist}(A, \partial D) > 1/j, \\ a_i \in \text{Int}G_i, & \text{if dist}(A, \partial D) \leq 1/j \text{ and } D^j = G_i, \quad i \in \mathbb{N}, \end{cases}$$

and observe that $|\zeta^j| \leq N$ and $|A^j| \leq N$, $j \in \mathbb{N}$. Therefore, by Proposition 3.2 for any $j \in \mathbb{N}$,

$$E \left( \sup_{\tau \leq t \leq \sigma} |Y^j_t|^2 + \int_{\tau}^{\sigma} \|Z^j_s\|^2 \, ds \right) \leq C \left( N^2 + \int_{\tau}^{\sigma} |f(s, 0, 0)|^2 \, ds \right),$$

and

$$E( |K^j|^{\sigma} | \mathcal{F}_\tau ) \leq C(\text{dist}(A^j, \partial D^j))^{-1} \left( N^2 + E \int_{\tau}^{\sigma} |f(s, 0, 0)|^2 \, ds \right).$$

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Since \( P(\text{dist}(A, \partial D) > 1/j) \uparrow 1 \) and \( \text{dist}(A^j, \partial D^j) > \text{dist}(A, \partial D) - 1/j \) if \( \text{dist}(A, \partial D) > 1/j \), it follows that

\[
\{ E(|K^j|^\sigma \mid \mathcal{F}_\tau); \, j \in \mathbb{N} \}
\]

is bounded in probability. (3.7)

By Proposition 3.3 for any \( j, k \in \mathbb{N} \) we have

\[
E\left( \sup_{\tau \leq t \leq \sigma} |Y^j_t - Y^j_t^{\xi+k}|^2 + \int_\tau^\sigma \| Z^j_s - Z^{j+k}_s \|^2 \, ds \mid \mathcal{F}_\tau \right)
\leq C \left( E(|\zeta^j - \zeta^{j+k}|^2 \mid \mathcal{F}_\tau) + \rho(D^j, D^{j+k}) E(|K^j|^\sigma + |K^{j+k}|^\sigma \mid \mathcal{F}_\tau) \right).
\]

(3.8)

By the construction, \( |\zeta^j - \zeta^{j+k}| \leq 2/j \) and \( \rho(D^j, D^{j+k}) \leq 2/j \) for \( k \in \mathbb{N} \). Therefore from (3.7) and (3.8) it follows that \( \{Y^j, Z^j, K^j\} \) is a Cauchy sequence on \( [\tau, \sigma] \) in the space \( \mathcal{S} \times \mathcal{P} \times \mathcal{S} \). By using standard methods we show that its limit \( (Y, Z, K) \) is a solution of the local RBSDE (3.2).

\[ \Box \]

Let us remark that in fact assumption (3.4) in Proposition 3.4 is superfluous (see Remark 3.8).

**Lemma 3.5.** Let \( \{G^j; t \in [0,T]\} \) be a family of bounded closed subsets of \( \mathbb{R}^m \) such that \( t \mapsto G^j_t \) is càdlàg with respect to the Hausdorff metric \( \rho \) and \( G_T = G^j_T \). Define its discretization \( \{G^j_t; t \in [0,T]\} \) by putting \( G^j_t = G^j_{t^{-1}} \), \( t \in [t^{-1}, t^j], \ G^j_T = G^j_T \), where \( t^{-1}_0 = 0, \ t^j_i = (t^j_{i-1} + 1/j) \land \inf\{t > t^j_{i-1}; \rho(G^j_{t^j_{i-1}}, G^j_t) > 1/j\} \land T, \ i, j \in \mathbb{N} \). Then \( \sup_{t \leq T} \rho(G^j_t, G^j_t) \to 0 \) as \( j \to \infty \).

**Proof.** Suppose the assertion of the lemma is false. Then there exists \( t \in [0,T] \) and a sequence \( \{t_j\} \) such that \( t_j \to t \) and

\[
\rho(G^j_{t_j}, G^j_{t_j}) \neq 0.
\]

(3.9)

Observe that for each \( j \in \mathbb{N} \), \( G^j_{t_j} = G^j_s \), where \( s_j = \max\{t^j_i; t^j_i \leq t_j\} \). Since \( t_j - 1/j \leq s_j \leq t_j, s_j \to t \). If \( \rho(G^j_{t^-}, G^j_t) = 0 \) then

\[
0 \leq \rho(G^j_{t^-}, G^j_{t^-}) \leq \rho(G^j_{t^-}, G^j_{t_j}) + \rho(G^j_{t_j}, G^j_{t^-}) \to 0,
\]

which contradicts (3.9). If \( \rho(G^j_{t^-}, G^j_t) > 0 \) then \( t \in \{t^j_i; i \in \mathbb{N} \cup \{0\}\} \) for sufficiently large \( j \) (such that \( \rho(G^j_{t^-}, G^j_t) > 1/j \)). Set \( J^+ = \{j; t_j \geq t\}, \ J^- = \{j; t_j < t\} \) and assume that both sets are infinite. If \( t \leq t_j \) then for sufficiently large \( j \), \( t = \max\{t^j_i; t^j_i \leq t\} \leq s_j \). Consequently, \( \lim_{j \in J^+} \rho(G_{t_j}, G_t) = 0 \) and \( \lim_{j \in J^-} \rho(G_{s_j}, G_t) = 0 \). Since \( s_j \leq t_j \), \( \lim_{j \in J^-} \rho(G_{t_j}, G_{t^-}) = 0 \) and \( \lim_{j \in J^-} \rho(G_{s_j}, G_{t^-}) = 0 \). Hence

\[
0 \leq \rho(G^j_{t^-}, G^j_{t^-}) \leq (\rho(G_{t^-}, G_t) + \rho(G_t, G_{s_j}))1_{\{j \in J^+\}}
+ (\rho(G_{t^-}, G_{t^-}) + \rho(G_{t^-}, G_{s_j}))1_{\{j \in J^-\}} \to 0,
\]

which also contradicts (3.9). \[ \Box \]

We are now ready to prove our main theorem of this section.
**Theorem 3.6.** Assume (H1)–(H4). Then there exists a unique solution \((Y, Z, K)\) of the \(\text{RBSDE}~ \{1.1\}\) such that \(Y, K \in \mathcal{S}^2\) and \(Z \in \mathcal{P}^2\).

**Proof.** Step 1. We begin by proving the theorem under the additional assumption that there exists \(N \in \mathbb{N}\) such that

\[
D_t \subset B(0, N), \quad t \in [0, T].
\]

(3.10)

For \(j \in \mathbb{N}\) set \(\sigma_j^0 = 0\) and

\[
\sigma_i^j = (\sigma_{i-1}^j + 1/j) \land \inf\{t > \sigma_{i-1}^j; \rho(D_{t-}, D_t) > 1/j\} \land T, \quad i \in \mathbb{N}.
\]

Since \(t \to D_t\) is càdlàg, for every \(j \in \mathbb{N}\) there is \(k_j\) such that \(P(\sigma_{k_j}^j < T) \leq 1/j\). Set

\[
D_t^j = \begin{cases} 
D_{\sigma_{i-1}^j}, & t \in [\sigma_{i-1}^j, \sigma_i^j], \ i = 1, \ldots, k_j - 1, \\
D_{\sigma_{k_j}^j}, & t \in [\sigma_{k_j}^j, T].
\end{cases}
\]

Then

\[
\sup_{t \leq T} \rho(D_t^j, D_t) \to 0
\]

(3.11)

as \(j \to \infty\). Indeed, by Lemma \ref{lem_5.5} \(\sup_{t \leq \sigma_{k_j}^j} \rho(D_t^j, D_t) \to 0\) \(P\)-a.s. Therefore for every \(\varepsilon > 0\),

\[
P(\sup_{t \leq T} \rho(D_t^j, D_t) > \varepsilon) \leq P(\sup_{t \leq \sigma_{k_j}^j} \rho(D_t^j, D_t) > \varepsilon) + 1/j \to 0.
\]

By the above and (H4) one can find a sufficiently slowly decreasing sequence \(\delta_j \downarrow 0\) such that the sequence \(\{\gamma_j\}\) defined as

\[
\gamma_j = \inf\{t; \text{dist}(A_t, \partial D_t^j) < \delta_j\} \land T, \quad j \in \mathbb{N}
\]

has the property that \(P(\gamma_j < T) \to 0\). By Propositions \ref{prop_3.1} and \ref{prop_3.4} for each \(j \in \mathbb{N}\) there exists a solution \((Y^j, Z^j, K^j)\) of RBSDE in the stopped time-dependent region \(D_t^{\gamma_j -} = D_t^{\gamma_j -} = D_t^{\gamma_j -} \cap (0, T]\) with terminal value \(\xi^j = \Pi_{D_t^{\gamma_j -}}(\xi)\). Set \(A_t^j = A_{\gamma_j -}^j, t \in [0, T]\), and observe that \(\inf_{t \leq T} \text{dist}(A_t^j, \partial D_t^j) \geq \delta_j > 0\). Since for any predictable locally bounded process \(H\),

\[
\left( \int_0^T (H_s, dA_t^j) \right)^*_T = \left( \int_0^T (H_s, dA_s) \right)^*_T
\]

it follows from Remark \ref{rem_2.4} that there is \(c > 0\) such that \(\|A^j\|_{\mathcal{H}^2} \leq c\|A\|_{\mathcal{H}^2}, j \in \mathbb{N}\). Hence, by Proposition \ref{prop_2.3} there exists \(C > 0\) such that for every \(j \in \mathbb{N}\),

\[
E\left( \sup_{t \leq T} |Y_t^j|^2 + \int_0^T \|Z_t^j\|^2 \, ds + \sum_{s \leq T} |\Delta K_s^j|^2 + \int_0^T \text{dist}(A_{s-}^j, \partial D_{s-}^j) \, d|K^j|_s \right)
\]

\[
\leq C \left( N^2 + E \int_0^T |f(s, 0, 0)|^2 \, ds + \|A\|_{\mathcal{H}^2}^2 \right).
\]

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For every $\varepsilon > 0$ there is $M > 0$, a stopping time $\sigma_j \leq T$ and $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$,

$$P(\sigma_j < T) \leq \varepsilon, \quad |K^j|_{\sigma_j} \leq M. \quad (3.12)$$

Indeed, by (H4) there is $\delta > 0$ such that $P(\inf_{t \leq T} \text{dist}(A_t, \partial D_t) \leq \delta) \leq \varepsilon/4$. On the other hand, by (3.11), there is $j_0$ such that for $j \geq j_0$, $P(\sup_{t \leq T} \rho(D^j_t, D_t) > \delta) \leq \varepsilon/4$. Therefore for every $j \geq j_0$,

$$P(\inf_{t \leq T} \text{dist}(A^j_{t}, \partial D^j_t) \leq \delta) \leq P(\inf_{t \leq T} \text{dist}(A_t, \partial D_t) \leq 2\delta) + P(\sup_{t \leq T} \rho(D^j_t, D_t) > \delta) \leq \frac{\varepsilon}{2}.$$ 

Set $c = C(E(N^2 + \int_0^T \|f(s, 0, 0)\|^2 ds) + \|A\|_{\mathcal{H}_2}^2)$, $M = (2c)/(\varepsilon \delta)$ and $\sigma_j = \inf\{t; |K^j|_t > M\} \wedge T$. By Proposition 2.3 and Tchebyshev’s inequality,

$$P(\sigma_j < T) \leq P(|K^j|_T > M) \leq P(|K^j|_T > M, \inf_{t \leq T} \text{dist}(A^j_t, \partial D^j_t) > \delta) + \frac{\varepsilon}{2}$$

$$\leq P(\inf_{t \leq T} \text{dist}(A^j_t, \partial D^j_t)|K^j|_T > M\delta) + \frac{\varepsilon}{2}$$

$$\leq P(\int_0^T \text{dist}(A^j_{s-}, \partial D^j_{s-}) \, d|K^j|_s > M\delta) + \frac{\varepsilon}{2}$$

$$\leq \frac{c}{M\delta} + \frac{\varepsilon}{2} \leq \varepsilon.$$

If we set $\sigma = \sigma_j \wedge \sigma_{j+k} \wedge \gamma_j$ then by Proposition 2.3,

$$E\left(\left(\sup_{t \leq \sigma} |Y^j_t - Y^{j+k}_t|^2 + \int_0^\sigma \|Z^j_s - Z^{j+k}_s\|^2 \, ds\right)\right)$$

$$\leq C\left(E\left(\left|Y^j_{\sigma} - Y^{j+k}_{\sigma}\right|^2 + \int_0^{\sigma-} \rho(D^j_{s-}, D^{j+k}_{s-}) \, d(|K^j|_s + |K^{j+k}|_s)\right)\right)$$

$$\leq C\left(E\left(\left|\xi^j - \xi^{j+k}\right|^2 + 2\varepsilon N^2 + 2M \min_{s \leq T} \rho(D^j_{s-}, D^{j+k}_{s-}), N\right)\right).$$

Since $\lim_{j \to \infty} \sup_k E|\xi^j - \xi^{j+k}|^2 = 0$ and by (3.11),

$$\lim_{j \to \infty} \sup_k E \min_{s \leq T} \rho(D^j_{s-}, D^{j+k}_{s-}), N = 0,$$

it follows that $\{(Y^j, Z^j, K^j)\}$ is a Cauchy sequence in $\mathcal{S} \times \mathcal{P} \times \mathcal{S}$. Its limit $(X, Z, K)$ is a solution of RBSDE (1.1).

Step 2. We will show how to dispense with assumption (3.10). Set $\gamma_j = \inf\{t \geq 0 : \sup_{s \leq t} |A_s| > N_j\} \wedge T$, $j \in \mathbb{N}$, where $N_j \uparrow \infty$ and

$$D^j_t = D^j_t \cap B(A^j_t, N_j), \quad t \in [0, T].$$

Clearly $D^j_t \subset B(0, 2N_j)$ and $P(\gamma_j < T) \leq P(\sup_{t \leq T} |A_t| > N_j) \downarrow 0$. By Step 1 for each $j \in \mathbb{N}$ there exists a solution $(Y^j, Z^j, K^j)$ of RBSDE in $\{D^j_t; t \in [0, T]\}$ with terminal value $\xi^j = 0$. Set $A^j_t = A^j_t \wedge (0, T]$. Since by Remark 2.1 there is $c > 0$ such that $\|A^j\|_{\mathcal{H}_2} \leq c\|A\|_{\mathcal{H}_2}$ for $j \in \mathbb{N}$, using Proposition 2.3 we obtain

$$E\left(\left(\sup_{t \leq T} |Y^j_t|^2 + \int_0^T \|Z^j_s\|^2 \, ds + \sum_{s \leq T} |\Delta K^j_s|^2 + \int_0^T \text{dist}(A^j_{s-}, \partial D^j_{s-}) \, d|K^j|_s\right)\right)$$

$$\leq C\left(E\left(\xi^2 + \int_0^T |f(s, 0, 0)|^2 \, ds + \|A\|_{\mathcal{H}_2}\right)\right).$$
Set $\tau_{j,k} = \inf \{ t; \sup_{s \leq t} |Y^{j+k}_{\sigma} | > 2N_j \} \wedge T$ for $j, k \in \mathbb{N}$ and observe that by Tschebyshev's inequality,

$$P(\tau_{j,k} < T) \leq P(\sup_{t \leq T} |Y^{j+k}_{t} | > 2N_j) \leq (2N_j)^{-2} C \left( E(\xi^2 + \int_0^T |f(s, 0, 0)|^2 ds) + ||A||_{H^2} \right),$$

which implies that $\lim_{j \to \infty} \sup_k P(\tau_{j,k} < T) = 0$. Let $\sigma = \tau_{j,k}$. Since $Y^{j}_{t} \in D^{j+k}_{t}$ for $t \in [0, T]$ and $Y^{j+k} \in D^{j}_{t}$ for $t < \sigma$, from Proposition 2.4 it follows that for $p < 2$,

$$E(\sup_{t < \sigma} |Y^{j}_{t} - Y^{j+k}_{t}|^p) \leq CE(|Y^{j}_{\sigma} - Y^{j+k}_{\sigma}|^p)$$

$$\leq E(|\xi^{j} - \xi^{j+k}|^p + E|Y^{j}_{\sigma} - Y^{j+k}_{\sigma}|^p 1_{\{\sigma < T\}})$$

$$\leq E(|\xi^{j} - \xi^{j+k}|^p + (E|Y^{j}_{\sigma} - Y^{j+k}_{\sigma}|^2)^{p/2}(P(\sigma < T))^{(2-p)/2}.$$

Hence $\lim_{j \to \infty} \sup_k E(\sup_{t < \sigma} |Y^{j}_{t} - Y^{j+k}_{t}|^p) = 0$ for $p < 2$ from which we deduce that $\{(Y^{j}, Z^{j}, K^{j})\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $S \times \mathcal{P} \times \mathcal{S}$. Using standard arguments one can show that its limit $(X, Z, K)$ is a solution of the approximated RBSDEs.

**Remark 3.7.** Arguing as in Step 2 of the above proof one can dispense with assumption (3.3) in Proposition 3.4. Therefore under (H1$^+$)--(H4$^+$) there exists a unique solution $(Y, Z, K - K_{\tau})$ of (3.2) such that (3.5) is satisfied.

**Remark 3.8.** The assumption that $D_T = D_{T-}$ in Theorem 3.6 is superfluous, because if $D_T \neq D_{T-}$ then from Theorem 3.6 it follows that there exists a unique solution of the RBSDE

$$Y_t = \Pi_{D_{T-}}(\xi) + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + K_T - K_t, \quad t \in [0, T]$$

in $\{D^{T-}_{t}; t \in [0, T]\}$. Set

$$Y'_t = \begin{cases} Y_t, & \text{if } t < T, \\ \xi, & \text{if } t = T, \end{cases} \quad Z = Z', \quad K'_t = \begin{cases} K_t, & \text{if } t < T, \\ K_T + \Pi_{D_{T-}}(\xi) - \xi, & \text{if } t = T. \end{cases}$$

Then $(Y', Z', K')$ is a unique solution of RBSDEs in $\{D_t; t \in [0, T]\}$.

**Remark 3.9.** In Step 1 of the proof of Theorem 3.6 and in Lemma 3.5 one can use stopping times $\sigma_i^j$ defined as follows: $\sigma_0^j = 0$ and

$$\sigma_i^j = (\sigma_{i-1}^j + a_i^j) \wedge \in \{ t > \sigma_{i-1}^j; \rho(D_{i-}, D_i) > 1/j \} \wedge T, \quad i, j \in \mathbb{N},$$

where $a_i^j$ is an arbitrary constant such that $1/j \leq a_i^j \leq 2/j$. This follows from the fact that if we use the modified stopping times $\sigma_i^j$ to define the process $D^j$ then (3.11) still holds true. We will use this simple observation in the next section.

**4 Approximation of solutions of RBSDEs by the modified penalization method**

We start with a priori estimates for solutions of the penalized BSDEs and their local versions.
**Proposition 4.1.** Assume (H1)–(H4). If \((Y^n, Z^n, K^n)\) is a solution of (1.2) such that \(Y^n \in \mathcal{S}^2\) then there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that

\[
E\left( (Y_T^{n})^* \right)^2 + \int_0^T \|Z^n\|^2 \, ds + \sum_{0 \le s \le T} |\Delta K^n_s|^2 + (K_T^{n})^2 + \int_0^T \text{dist}(A_{s-}, \partial D_{s-}) \, d|K^n_s|_s
\]

\[
\le C \left( E(\|\xi\|^2 + \int_0^T |f(s, 0, 0)|^2 \, ds) + \|A\|_{lip}^2 \right).
\]

**Proof.** The proof is similar to that of Proposition 2.3. To get the desired estimate it suffices to repeat step by step arguments from the proof of Proposition 2.3, the only difference being in the fact that to obtain an analogue of (2.2) we have to prove that

\[
\int_0^t \langle Y^n_s - A_{s-}, dK^n_s \rangle \le - \int_0^t \text{dist}(A_{s-}, \partial D_{s-}) \, d|K^n_s|_s, \quad t \in [0, T].
\]

To prove (4.1) let us define \(K^{n, d}, K^{n, c}\) by (1.3). Observe that by Remark 2.2(c),

\[
\int_0^t \langle Y^n_s - A_{s-}, dK^{n, c}_s \rangle = n \int_0^t \langle Y^n_s - A_{s-}, \Pi D_s(Y^n_s) - Y^n_s \rangle \, ds
\]

\[
\le -n \int_0^t \text{dist}(A_{s-}, \partial D_{s-}) \|\Pi D_s(Y^n_s) - Y^n_s\| \, ds
\]

\[
= - \int_0^t \text{dist}(A_{s-}, \partial D_{s-}) \, d|K^{n, c}|_s.
\]

By Remark 2.2(b), for \(i = 1, 2, \ldots, k_n\) we have

\[
\langle Y^n_{\sigma_{n,i}-} - A_{\sigma_{n,i}-}, \Delta K^n_{\sigma_{n,i}} \rangle = \langle Y^n_{\sigma_{n,i}-} - A_{\sigma_{n,i}-}, \Pi D_{\sigma_{n,i}-}(Y^n_{\sigma_{n,i}}) - Y^n_{\sigma_{n,i}} \rangle
\]

\[
\le -\text{dist}(A_{\sigma_{n,i}-}, \partial D_{\sigma_{n,i}-}) \|\Pi D_{\sigma_{n,i}-}(Y^n_{\sigma_{n,i}}) - Y^n_{\sigma_{n,i}}\|
\]

\[
= -\text{dist}(A_{\sigma_{n,i}-}, \partial D_{\sigma_{n,i}-}) |\Delta K^n_{\sigma_{n,i}}|.
\]

Putting together the above two estimates we get (4.1). \(\square\)

Let \(\xi' \in L^2\) and let \(\mathcal{D}' = \{D'_t, t \in [0, T]\}\) be a family satisfying (H4) with some semimartingale \(A'\). In the next proposition we consider RBSDE in \(\mathcal{D}'\) of the form

\[
Y_t^{m} = \xi' + \int_t^T f(s, Y_s^{m}, Z_s^{m}) \, ds - \int_t^T Z_s^{m} \, dW_s + K_T^{m} - K_t^{m}, \quad t \in [0, T],
\]

where

\[
K_t^{m} = -n \int_0^t \langle Y_s^{m} - \Pi D'_s(Y_s^{m}) \rangle \, ds - \sum_{\sigma'_{n,k} \le t} \langle Y_{\sigma'_{n,k}-}^{m} - \Pi D'_{\sigma'_{n,k}-}(Y_{\sigma'_{n,k}-}^{m}) \rangle, \quad t \in [0, T]
\]

and \(\sigma'_n, 0 = 0, \sigma'_{n,i} = \inf\{t > \sigma'_{n,i-1}: \rho(D'_t \cap B(0,n), D'_{t-} \cap B(0,n)) > 1/n\} \land T, \; i = 1, \ldots, k'_n\) for some \(k'_n \in \mathbb{N}\).
Proposition 4.2. Let \( (Y^n, Z^n, K^n), (Y^m, Z^m, K^m) \) be solutions of (12) and (12), respectively, such that \( Y^n, Y^m \in S^2 \). Set \( \bar{Y}^n = Y^n - Y^m \), \( Z^n = Z^n - Z^m \), \( K^n = K^n - K^m \). If \( f \) satisfies (H3) then for every \( p \in (1, 2] \) there exists \( C > 0 \) depending only on \( \mu, \lambda, T \) such that for every stopping time \( \sigma \) such that \( 0 \leq \sigma \leq T \),

\[
E \left( \sup_{t \leq \sigma} |\bar{Y}^n_t|^p + \int_0^\sigma |\bar{Y}^n_s|^p |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s)| d|K^{n,c}|_s \right) \\
\leq CE \left( |\bar{Y}^n_0|^p + \int_0^\sigma |\bar{Y}^n_s|^p |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s)| d|K^{n,c}|_s \right) \\
+ \int_0^\sigma |\bar{Y}^n_s|^p |\Pi D^\prime_{s-}(\Pi D_{s-}(Y^m_s)) - \Pi D_{s-}(Y^m_s)| d|K^{m,d}|_s \\
+ \int_0^\sigma |\bar{Y}^n_s|^p |\Pi D_{s-}(Y^m_s) - Y^m_s| d|K^{n,d}|_s + |\Pi D^\prime_{s-}(Y^m_s) - Y^m_s| d|K^{m,d}|_s, \right),
\]

where \( P^n = \sum_{s \leq t} (|\bar{Y}^n_s|^p - |\bar{Y}^n_s|^p - p)|\bar{Y}^n_s|^p - 1 \langle \text{sgn}(\bar{Y}^n_s, \Delta \bar{Y}^n_s) \rangle, t \geq 0. \)

Proof. The proof is similar to that of Proposition 2.4. We first apply Itô’s formula to the function \( x \rightarrow |x|^p \) and the semimartingale \( Y^n \) to get an analogue of (2.10). Then we estimate the terms of the right-hand side of the equality thus obtained in much the same way as in the proof of Proposition 2.4, except for an analogue of (2.11). Now

\[
\int_t^\sigma |\bar{Y}^n_s|^{p-1} \langle \text{sgn}(\bar{Y}^n_s, d\bar{K}^n_s) = \int_t^\sigma |\bar{Y}^n_s|^{p-1} \langle \text{sgn}(Y^m_s, d\bar{K}^n_s) \right)
\]

and instead of (2.11) we have to show that

\[
\langle \bar{Y}^n_s, d\bar{K}^n_s \rangle \leq |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s)| d|K^{n,c}|_s \\
+ |\Pi D^\prime_{s-}(\Pi D_{s-}(Y^m_s)) - \Pi D_{s-}(Y^m_s)| d|K^{m,d}|_s \\
+ |\Pi D_{s-}(Y^m_s) - Y^m_s| d|K^{n,d}|_s + |\Pi D^\prime_{s-}(Y^m_s) - Y^m_s| d|K^{m,d}|_s. \tag{4.3}
\]

To see this, we first observe that

\[
\langle \bar{Y}^n_s, d\bar{K}^n_s \rangle = \langle \bar{Y}^n_s - \Pi D_{s-}(Y^m_s), d\bar{K}^n_s \rangle + |\Pi D_{s-}(Y^m_s) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s| \\
\leq |\Pi D_{s-}(Y^m_s) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s|,
\]

because

\[
\langle \bar{Y}^n_s - \Pi D_{s-}(Y^m_s), d\bar{K}^n_s \rangle = -n(\bar{Y}^n_s - \Pi D_{s-}(Y^m_s)) + \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s \\
\leq |\Pi D_{s-}(Y^m_s) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s|,
\]

so

\[
|\Pi D_{s-}(Y^m_s) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s| = \frac{1}{2} \left( \langle \bar{Y}^n_s - \Pi D_{s-}(Y^m_s), d\bar{K}^n_s \rangle + \langle \bar{Y}^n_s - \Pi D_{s-}(Y^m_s), d\bar{K}^n_s \rangle \right) \\
= -n(\bar{Y}^n_s - \Pi D_{s-}(Y^m_s) - \Pi D^\prime_{s-}(Y^m_s), Y^m_s - \Pi D_s(Y^m_s)) d\bar{K}^n_s \\
\leq |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s|.
\]

By Remark 2.2(b),

\[
|\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s| \\
= -n(\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D_{s-}(Y^m_s), Y^m_s - \Pi D_s(Y^m_s)) d\bar{K}^n_s \\
+ |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s| \\
\leq |\Pi D_{s-}(\Pi D^\prime_{s-}(Y^m_s)) - \Pi D^\prime_{s-}(Y^m_s), d\bar{K}^n_s|.
\]
Using similar estimate for $\langle \Pi_{D_1^n} (Y^n_{s-}) - \Pi_{D_1^n} (Y^n_m) , -dK^n_{s-} \rangle$ we obtain

$$
\langle \bar{Y}_{s-}^n , d\bar{K}^n_{s-} \rangle \leq |\Pi_{D_1^n} (\Pi_{D_1^n}(Y^n_{s-})) - \Pi_{D_1^n} (Y^n_{s-})| d|K^n_{s-}|_s
$$

$$
+ |\Pi_{D_1^n} (\Pi_{D_1^n}(Y^n_{s-})) - \Pi_{D_1^n} (Y^n_{s-})| d|K^n_{s-}|_s. \quad (4.4)
$$

On the other hand, by Remark 2.2(a), for $i = 1, 2, \ldots, k$ we have

$$
\langle Y^m_{\sigma_{i,i}}, - \Pi_{D_{\sigma_{i,i}}} (Y^m_{\sigma_{i,i}}), \Delta K^m_{\sigma_{i,i}} \rangle = \langle Y^m_{\sigma_{i,i}} - \Pi_{D_{\sigma_{i,i}}} (Y^m_{\sigma_{i,i}}), \Pi_{D_{\sigma_{i,i}}} (Y^m_{\sigma_{i,i}}) - Y^m_{\sigma_{i,i}} \rangle \leq 0
$$

and

$$
\langle Y^m_{\sigma_{i,i}} - \Pi_{D_{\sigma_{i,i}}} (Y^m_{\sigma_{i,i}}), \Delta K^m_{\sigma_{i,i}} \rangle \leq 0,
$$

which implies that

$$
\langle \bar{Y}_{s-}^n, \Delta \bar{K}^n \rangle \leq \langle \Pi_{D_1^n} (Y^n_{s-}) - Y^n_{s-}, \Delta K^n_{s-} \rangle + \langle \Pi_{D_1^n} (Y^n_{s-}) - Y^n_{s-}, \Delta K^n_{s-} \rangle. \quad (4.5)
$$

Combining (4.4) with (4.5) yields (4.3). We leave the details of the rest of the proof to the reader.

**Corollary 4.3.** Under the assumptions of Proposition 4.2, if moreover $D^{\sigma -} = D^{\sigma -}$, then

$$
E \left( \left( \sup_{\tau \leq \sigma} |\bar{Y}_{t}^m|_s \right) \int_{0}^{\sigma} |\bar{Y}_{s}^m|_s^{-2} 1_{\{Y_s \neq 0\}} \|Z^m_s\|^2 ds + I^m_{\sigma} \right) \leq CE|\bar{Y}_{\sigma -}^m|^p.
$$

**Proof.** Follows immediately from Proposition 4.2.

Note that in Propositions 4.1, 4.2 and Corollary 4.3 we do not assume that $\xi \in D_T$, $\xi' \in D'_T$.

We now turn to the approximation of local RBSDEs. Let $\tau, \sigma$ be stopping times such that $0 \leq \tau \leq \sigma \leq T$, Let $D, D'$ be $\mathcal{F}_\tau$-measurable random convex sets with nonempty interiors and let $\zeta, \zeta' \in L^2$ be $\mathcal{F}_\sigma$-measurable random variables (we do not assume neither that $\zeta \in D P$-a.s. nor that $\zeta' \in D' P$-a.s.). We consider approximations of the form

$$
Y_{t}^n = \zeta + \int_{t}^{\sigma} f(s, Y^n_{s}, Z^n_{s}) ds - \int_{t}^{\sigma} Z^n_{s} dW_s + K^n_{s} - K^n_{t}, \quad t \in [\tau, \sigma] \quad (4.6)
$$

and

$$
Y'^{mn}_{t} = \zeta' + \int_{t}^{\sigma} f(s, Y'^{mn}_{s}, Z'^{mn}_{s}) ds - \int_{t}^{\sigma} Z'^{mn}_{s} dW_s + K'^{mn}_{s} - K'^{mn}_{t}, \quad t \in [\tau, \sigma], \quad (4.7)
$$

where

$$
K^n_{t} = -n \int_{\tau}^{t} \langle Y^n_{s} - \Pi_D(Y^n_{s}) \rangle ds, \quad K'^{mn}_{t} = -n \int_{\tau}^{t} \langle Y'^{mn}_{s} - \Pi_{D'}(Y'^{mn}_{s}) \rangle ds, \quad t \in [\tau, \sigma]. \quad (4.8)
$$
Corollary 4.4. Assume (H1*)–(H4*). Let \((Y^n, Z^n, K^n)\) be a solution of (4.6) such that \(\sup_{\tau \leq t \leq \sigma} |Y^n_t| \in L^2\). Then there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that for any \(n \in \mathbb{N}\)

\[
E\left( \sup_{\tau \leq t \leq \sigma} |Y^n_t|^2 + \int_\tau^\sigma \|Z^n_s - Z^n_s\|^2 \, ds \mid \mathcal{F}_\tau \right) \leq CE\left( |\zeta|^2 + |A|^2 + \int_\tau^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau \right)
\]

and

\[
E(|K^n|^2 \mid \mathcal{F}_\tau) \leq C(\text{dist}(A, \partial D))^{-1}E\left( |\zeta|^2 + |A|^2 + \int_\tau^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau \right).
\]

Proof. Follows from the proof of Proposition 4.2. \(\square\)

Corollary 4.5. Let \((Y^n, Z^n, K^n)\), \((Y^n, Z^n, K^n)\) be solutions of (4.6) and (4.7), respectively, such that \(\sup_{\tau \leq t \leq \sigma} |Y^n_t|, \sup_{\tau \leq t \leq \sigma} |Y'^n_t| \in L^2\). If \(f\) satisfies (H2) then there exists \(C > 0\) depending only on \(\mu, \lambda, T\) such that for any \(n \in \mathbb{N}\),

\[
E\left( \sup_{\tau \leq t \leq \sigma} |Y^n_t - Y'^n_t|^2 + \int_\tau^\sigma \|Z^n_s - Z^n_s\|^2 \, ds \mid \mathcal{F}_\tau \right)
\]

\[
\leq C\left( E(|\zeta|^2 \mid \mathcal{F}_\tau) + E\left( \sup_{\tau \leq t \leq \sigma} |\Pi_\mathcal{D}(\Pi_\mathcal{D}')(Y^n_t) - \Pi_\mathcal{D}(\Pi_\mathcal{D}')(Y'^n_t)|^2 \mid \mathcal{F}_\tau \right)
\right)
\]

\[
+ E\left( \sup_{\tau \leq t \leq \sigma} |\Pi_\mathcal{D}(\Pi_\mathcal{D}'(Y^n_t)) - \Pi_\mathcal{D}(Y'^n_t)|^2 \mid \mathcal{F}_\tau \right).
\]

Proof. Follows from the proof of Proposition 4.2. \(\square\)

Proposition 4.6. Assume (H1*)–(H4*) and (3.4). Then

\[
\sup_{\tau \leq t \leq \sigma} |Y^n_t - Y_t| \to 0, \quad \int_\tau^\sigma \|Z^n_s - Z_s\|^2 \, ds \to 0, \quad \sup_{\tau \leq t \leq \sigma} |K^n_t - K_t| \to 0,
\]

where \((Y, Z, K)\) is a unique solution of the local RBSDE (3.2).

Proof. First set \(D = G\) for some fixed convex set with nonempty interior. Consider approximations of the form

\[
Y^n_t = \zeta + \int_t^T g(s, Y^n_s, Z^n_s) \, ds - \int_t^T Z^n_s \, dW_s + K^n_T - K^n_t, \quad t \in [0, T]
\]

where \(g(s, \cdot, \cdot) = f(s, \cdot, \cdot)1_{[0, \sigma]}(s)\). By [14, Theorem 5.9], \((Y^n, Z^n, K^n) \to (Y, Z, K)\) in \(S^2 \times \mathcal{P}^2 \times S^2\), where \((Y, Z, K)\) is a solution of RBSDEs of the form (3.6) in \(G\). Since \(Y^n_t = Y_t = \zeta, Z^n_t = Z_t = 0\) and \(K^n_T = K^n_T\) for \(t \geq \sigma\), it is clear that for any \(\tau \leq \sigma\), \((Y^n, Z^n, K^n)\) converges in \(S^2 \times \mathcal{P}^2 \times S^2\) to the solution of our local RBSDE on \([\tau, \sigma]\).

Now let us define \(D^j, \zeta^j, A^j, j \in \mathbb{N}\) as in the proof of Proposition 3.4 (observe that \(|\zeta^j| \leq N\) and \(|A^j| \leq N\, j \in \mathbb{N}\)) and by \((Y^{j,n}, Z^{j,n}, K^{j,n})\) denote a solution of the local BSDE

\[
Y^{j,n}_t = \zeta^j + \int_t^\sigma f(s, Y^{j,n}_s, Z^{j,n}_s) \, ds - \int_t^\sigma Z^{j,n}_s \, dW_s + \int_t^\sigma (Y^{j,n}_s - \Pi_\mathcal{D}')(Y^{j,n}_s) \, ds, \quad t \in [\tau, \sigma].
\]
Using the first part of the proof and arguments from the proof of Proposition \ref{th4.3} one can show that for every \( j \in \mathbb{N} \),

\[
(Y^{j,n}, Z^{j,n}, K^{j,n}) \to (Y^{(j)}, Z^{(j)}, K^{(j)}) \quad \text{in } \mathcal{S} \times \mathcal{P} \times \mathcal{S},
\]

(4.9)

where \((Y^{(j)}, Z^{(j)}, K^{(j)})\) is a solution of the local RBSDE in \( D^j \) with terminal value \( \zeta^j \). Since \(|\zeta^j - \zeta| \leq 2/j\) and \( \rho(D^j, D) \leq 2/j \), from Corollary \ref{cor4.5} it follows that

\[
E(\sup_{\tau \leq t \leq \sigma} |Y^\tau_t - Y^j_{\tau,t}|^2 + \int_{\tau}^\sigma \|Z^\tau_s - Z^j_{\tau,s}\|^2 \, ds \mid \mathcal{F}_\tau) \\
\leq C(E(|\zeta - \zeta|^2 \mid \mathcal{F}_\tau) + \rho(D, D)E(|K^\tau|^2 + |K^j| \mid \mathcal{F}_\tau)) \\
\leq C\left(\frac{4}{j^2} + \frac{2}{j}E(|K^\tau|^2 + |K^j| \mid \mathcal{F}_\tau)\right).
\]

(4.10)

By Corollary \ref{cor4.3}

\[
E(|K^\tau|^2 \mid \mathcal{F}_\tau) \leq C(\text{dist}(A, \partial D))^{-1}E\left(N^2 + \int_{\tau}^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau\right).
\]

Using once again Corollary \ref{cor4.3} and the fact that on the set \{dist(A, \partial D) > 1/j\} we have dist(A, \partial D) - 1/j, we conclude that

\[
E(|K^\tau|^2 \mid \mathcal{F}_\tau) \leq C(\text{dist}(A, \partial D) - \frac{1}{j})^{-1}E\left(N^2 + \int_{\tau}^\sigma |f(s, 0, 0)|^2 \, ds \mid \mathcal{F}_\tau\right)
\]

on \{dist(A, \partial D) > 1/j\}. Since \( P(\text{dist}(A, \partial D) > 1/j) \uparrow 1 \), it follows from (4.11) that for every \( \varepsilon > 0 \),

\[
\lim_{j \to \infty} \limsup_{n \to \infty} P\left(\sup_{\tau \leq t \leq \sigma} |Y^\tau_t - Y^j_{\tau,t}|^2 + \int_{\tau}^\sigma \|Z^\tau_s - Z^j_{\tau,s}\|^2 \, ds \mid \mathcal{F}_\tau\right) \geq \varepsilon = 0.
\]

(4.11)

Combining (4.9) with (4.11) and the fact that \{(Y^{(j)}, Z^{(j)}, K^{(j)})\} converges in \( \mathcal{S} \times \mathcal{P} \times \mathcal{S} \) to the solution \((Y, Z, K)\) of the local RBSDE in \( D \) we get the desired convergence results.

\[ \square \]

**Remark 4.7.** Proposition \ref{th4.6} may be slightly generalized to encompass different terminal values in the approximation sequence. More precisely, let \( \zeta_n \in L^2 \) be a sequence of \( \mathcal{F}_\tau \)-measurable random variables such that \( \zeta_n \to \zeta \) in \( L^2 \), where \( \zeta \in D \) \( P \)-a.s. Let us define \((\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n)\) by (4.7), (4.8) but with \( \zeta' \) replaced by \( \zeta_n \). Then Proposition \ref{th4.6} holds true with \((Y^n, Z^n, K^n)\) replaced by \((\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n)\). To see this it suffices to observe in Corollary \ref{cor4.5} we do not assume that \( \zeta \in D, \zeta' \in D' \). Therefore for any \( n \in \mathbb{N} \),

\[
E\left(\sup_{\tau \leq t \leq \sigma} |Y^\tau_t - \tilde{Y}^\tau_t|^2 + \int_{\tau}^\sigma \|Z^\tau_s - \tilde{Z}^\tau_s\|^2 \, ds \mid \mathcal{F}_\tau\right) \leq CE(|\zeta - \zeta_n|^2 \mid \mathcal{F}_\tau),
\]

which leads to the desired conclusion.

**Proposition 4.8.** Assume (H1*)–(H4*). Let \( 0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{k+1} = T \) be stopping times and let \( D^0, D^1, \ldots, D^k \) be random closed convex subsets in \( \mathbb{R}^m \) such that \( D^j \) is \( \mathcal{F}_{\sigma_j} \)-measurable and there is \( m \in \mathbb{N} \) such that \( D^j \subset B(0, N), \) \( i = 1, \ldots, k \). Let
(Y, Z, K) be a unique solution of RBSDE (1.1) in \{D_t; t \in [0, T]\} such that \(D_t = D_{t-1}\), \(t \in [\sigma_{i-1}, \sigma_i], i = 1, \ldots, k + 1, D_T = D_{T-1}\), and let \((Y^n, Z^n, K^n)\) be a solution of (1.2). Then
\[
(Y^n, Z^n, K^n) \rightarrow_p (Y, Z, K) \quad \text{in } S \times P \times S.
\]

**Proof.** By Proposition 4.6
\[
\sup_{\sigma_k \leq t \leq T} \left| Y^n_t - Y_t \right| \to 0, \quad \int_{\sigma_k}^T \| Z^n_s - Z_s \|^2 ds \to 0
\]
and
\[
\sup_{\sigma_k \leq t \leq T} \left| n \int_{\sigma_k}^t (Y^n_s - \Pi_{D_k}(Y^n_s)) ds - (K_t - K_{\sigma_k}) \right| \to 0.
\]
Since
\[
Y^n_{\sigma_{k-1}} = \left\{ \begin{array}{ll}
\Pi_{D_{k-1}}(Y^n_{\sigma_k}), & \text{if } \text{dist}(D_k; D_{k-1}) > 1/n, n \geq N, \\
Y^n_{\sigma_k}, & \text{otherwise},
\end{array} \right.
\]
it is clear that \(Y^n_{\sigma_{k-1}} \rightarrow_p \Pi_{D_{k-1}}(Y^n_{\sigma_k}) = Y_{\sigma_{k-1}}\). Similarly, if \(Y^n_{\sigma_{i-1}} \rightarrow_p \Pi_{D_{i-1}}(Y^n_{\sigma_{i-1}}) = Y_{\sigma_{i-1}}\) for \(i = k, k - 1, \ldots, 1\) then by Proposition 4.6
\[
\sup_{\sigma_{i-1} \leq t \leq \sigma_i} \left| Y^n_t - Y_t \right| \to 0, \quad \int_{\sigma_{i-1}}^{\sigma_i} \| Z^n_s - Z_s \|^2 ds \to 0
\]
and
\[
\sup_{\sigma_{i-1} \leq t \leq \sigma_i} \left| n \int_{\sigma_{i-1}}^t (Y^n_s - \Pi_{D_k}(Y^n_s)) ds - (K_t - K_{\sigma_{i-1}}) \right| \to 0.
\]
Consequently, \(Y^n_{\sigma_{i-1}} \rightarrow_p \Pi_{D_{i-1}}(Y^n_{\sigma_{i-1}}) = Y_{\sigma_{i-1}}\) for \(i \geq 2\). Using backward induction completes the proof. \(\square\)

**Theorem 4.9.** Assume (H1)–(H4). Let \(\{(Y^n, Z^n, K^n)\}\) be a sequence of solutions of (1.2). Then
\[
(Y^n, Z^n, K^n) \rightarrow_p (Y, Z, K) \quad \text{in } S \times P \times S,
\]
where \((Y, Z, K)\) is a unique solution of (1.1).

**Proof.** Step 1. As in the proof of Theorem 3.6 we first assume additionally that (3.10) is satisfied. For \(j \in \mathbb{N}\) set \(\sigma_{j, 0} = 0\) and
\[
\sigma^j_i = (\sigma^j_{i-1} + a^j_i) \wedge \inf\{t > \sigma^j_{i-1}; \rho(D_{\tau-}, D_t) > 1/j \} \wedge T, \quad i \in \mathbb{N},
\]
where \(a^j_i \in [1/j, 2/j]\) is a constant chosen via the following procedure. Suppose that \(\tau \equiv \sigma^j_{i-1}\) is such that \(\tau + 1/j < T\). By Proposition 4.1 there is \(c > 0\) such that
\[
\int_0^{T-j} E(\text{dist}(A_{\tau+s}, D_{\tau+s}) | Y^n_{\tau+s} - \Pi_{D_{\tau+s}}(Y^n_{\tau+s})| ds \leq E \int_0^T \text{dist}(A_s, D_s) | Y^n_s - \Pi_{D_s}(Y^n_s)| ds \leq cn^{-1}
\]
for every \(n \in \mathbb{N}\). Therefore we can find \(s \in [1/j, 2/j]\), which we denote by \(a^j_i\), such that \(E(\text{dist}(A_{\tau+s}, D_{\tau+s}) | Y^n_{\tau+s} - \Pi_{D_{\tau+s}}(Y^n_{\tau+s})|) \to 0\) as \(n \to \infty\). Since \(\text{dist}(A_{\tau+s}, D_{\tau+s}) > 0\),
\[ Y^\sigma_{\tau+s} - \Pi_{D \gamma}(Y^\sigma_{\tau+s}) \to 0 \text{ for } s = a_i^j. \]

It follows that the stopping times \( \sigma^j_i \) have the property that

\[ Y^{\sigma_{j,i-1} + a_{i-1}^j} - \Pi_{D \gamma}(Y^{\sigma_{j,i-1} + a_{i-1}^j}) \to 0, \quad j, i \in \mathbb{N}. \]  

(4.12)

Now let us define \( D^j = \{ D^j_t; t \in [0, T] \}, \xi^j, A^j = \{ A^j_t; t \in [0, T] \} \) as in Step 1 of the proof Theorem 3.6, (observe that \( |\xi^j| \leq N \) and \( |A^j_t| \leq N, j \in \mathbb{N} \)). Let \( (Y^j, Z^j, K^j) \) denote the solution of the BSDE

\[ Y_{t}^j = \xi^j + \int_{t}^{T} f(s, Y_{s}^j, Z_{s}^j) \, ds - \int_{t}^{T} Z_{s}^j \, dW_{s} + K_{T}^j, \quad t \in [0, T], \]  

(4.13)

where

\[ K_{t}^j = -n \int_{0}^{t} (Y_{s}^j - \Pi_{D^j}(Y_{s}^j)) \, ds - \sum_{\sigma^j_{n,i} \leq t} (Y_{s}^j - \Pi_{D^j}(Y_{s}^j)) \]  

(4.14)

with \( \sigma^j_{n,0} = 0, \sigma^j_{n,i} = \inf\{ t > \sigma^j_{n,i-1}; \rho(D^j_t, D^j_{t-}) > 1/n \} \land T, i = 1, \ldots, k^j_n \) with \( k^j_n \) chosen so that \( P(\sigma^j_{n,k^j_n} < T) \to 0 \) as \( n \to \infty \). From Proposition 4.8 we deduce that for each \( j \in \mathbb{N} \),

\[ (Y^j, Z^j, K^j) \to (\xi^j, Z^j, K^j) \]  

\[ \text{in } \mathcal{S} \times \mathcal{P} \times \mathcal{S}, \]  

(4.15)

where \( (\xi^j, Z^j, K^j) \) is a solution of RBSDE in \( D^j \) with terminal value \( \xi^j \). Moreover, by Proposition 4.1 there exists \( C > 0 \) such that for \( j, n \in \mathbb{N} \),

\[ E\left( \sup_{t \leq T} |Y_t^j|^2 + \int_{0}^{T} ||Z_t^j||^2 \, ds + \sum_{s \leq T} |\Delta K_{s}^j|^2 + \int_{0}^{T} \text{dist}(A_{s}^j, \partial D_{s}^j) \, d||K_{s}^j|| \right) \]

\[ \leq C \left( N^2 + E \int_{0}^{T} |f(s, 0, 0)|^2 \, ds + ||A||^2_{H^2} \right). \]

Consequently, by the same arguments as in the proof of Theorem 3.6 we deduce that for every \( \varepsilon > 0 \) there exist \( M > 0 \), stopping times \( \sigma^n_j \leq T \) and \( j_0 \in \mathbb{N} \) such that for every \( j \geq j_0 \) and \( n \in \mathbb{N} \),

\[ P(\sigma^n_j < T) \leq \varepsilon, \quad |K_{\sigma^n_j}|_{\sigma^n_j} \leq M. \]  

(4.16)

Similarly, by Proposition 4.1 we show that for every \( \varepsilon > 0 \) there exist \( M > 0 \) and stopping times \( \tau^n \leq T \) such that for every \( n \in \mathbb{N} \),

\[ P(\tau^n < T) \leq \varepsilon, \quad |K^n|_{\tau^n} \leq M. \]

Putting \( p = 2 \) and \( \sigma = \sigma^n \land \tau^n \) in Proposition 4.2 we obtain

\[ E(\sup_{t < \sigma} |Y_t^j - Y_t^n|^2 + \int_{0}^{\sigma} ||Z_s^j - Z_s^n||^2 \, ds) \]

\[ \leq C \left( E(|Y_{\sigma^n}^j - Y_{\sigma^n}^n|^2 + \int_{0}^{\sigma^n} \rho(D_{s}^j, D_{s}^n) \, d(||K_{s}^j, c|| + ||K_{s}^n, c||) \right) \]

\[ + \sum_{\sigma^j_{n,i} < \sigma} ||\Pi_{D_{\sigma^j_{n,i}}^j} (Y_{\sigma^j_{n,i}}^n - Y_{\sigma^j_{n,i}}^n) || \cdot |\Delta K_{\sigma^j_{n,i}}^j| \]

\[ + \sum_{\sigma_{n,i} < \sigma} ||\Pi_{D_{\sigma_{n,i}}^j} (Y_{\sigma_{n,i}}^n - Y_{\sigma_{n,i}}^n) || \cdot |\Delta K_{\sigma_{n,i}}^n| \right). \]
If \( \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}) > 0 \) then \( Y_{\sigma_n}^{j+1} \in D_{\sigma_n}^{j+1} \) and
\[
|\Pi_{D_{\sigma_n}^{j}} D_{\sigma_n}^{j+1} - Y_{\sigma_n}^{j+1}| \leq \rho(D_{\sigma_n}^{j+1}, D_{\sigma_n}^{j+1})
\]
for \( n \geq \max(j, N) \). Using the above estimate if \( \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}) > 0 \) and (4.12) if \( \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}) = 0 \) we obtain
\[
\limsup_{n \to \infty} E \left( \sum_{\sigma_n < \sigma} |\Pi_{D_{\sigma_n}^{j}} (Y_{\sigma_n}^{j+1}) - Y_{\sigma_n}^{j+1}| \cdot |\Delta K_{\sigma_n}^{j+1}| \right)
\]
\[
\leq \limsup_{n \to \infty} E \left( \int_{0}^{\sigma} \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}) d(|K_{\sigma_n}^{j+1}|) \right).
\]

Similarly,
\[
\limsup_{n \to \infty} E \left( \sum_{\sigma_n < \sigma} |\Pi_{D_{\sigma_n}^{j+1}} (Y_{\sigma_n}^{j+1}) - Y_{\sigma_n}^{j+1}| \cdot |\Delta K_{\sigma_n}^{j+1}| \right)
\]
\[
\leq \limsup_{n \to \infty} E \left( \int_{0}^{\sigma} \rho(D_{\sigma_n}^{j+1}, D_{\sigma_n}^{j+1}) d(|K_{\sigma_n}^{j+1}|) \right).
\]

Hence
\[
\limsup_{n \to \infty} E \left( (\sup_{t < \sigma} |Y_{t}^{j,n} - Y_{t}^{n}|^2 + \int_{0}^{\sigma} \|Z_{s}^{j,n} - Z_{s}^{n}\|^2 ds \right)
\]
\[
\leq CE((\xi^j - \xi^2 + 2\varepsilon N^2 + 2M \min(\sup_{t} \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}), N)).
\]

Consequently,
\[
\lim_{j \to \infty} \limsup_{n \to \infty} E \left( (\sup_{t < \sigma} |Y_{t}^{j,n} - Y_{t}^{n}|^2 + \int_{0}^{\sigma} \|Z_{s}^{j,n} - Z_{s}^{n}\|^2 ds \right) \leq 2C\varepsilon N^2,
\]
(4.17)

because \( \lim_{j \to \infty} \sup_{k} E|\xi^j - \xi|^2 = 0 \) and \( E \sup_{t \leq T} \rho(D_{\sigma_n}^{j}, D_{\sigma_n}^{j+1}) \to 0 \) by (5.10), (3.11) and Remark 3.9. Furthermore, by the Step 1 of the proof of Theorem 3.9 and Remark 3.9
\[
(Y^{(j)}, Z^{(j)}, K^{(j)}) \to_{P} (Y, Z, K) \text{ in } S \times P \times S,
\]
(4.18)

where \((Y, Z, K)\) is a unique solution of (1.1). Combining (4.14) with (4.17) and (4.18) we conclude that \((Y_n, Z_n, K_n) \to_{P} (Y, Z, K) \) in \( S \times P \times S \) under the additional assumption (3.10).

Step 2. In the general case we will use arguments from Step 2 of the proof of Theorem 3.6. Let \( N_j, \gamma_j, D^j = \{D_{\gamma_j}^j; t \in [0, T]\}, \xi^j, A^j = \{A_{\gamma_j}^j; t \in [0, T]\} \) be defined as in that step. Note that \( D_{\gamma_j}^j \subset B(0, 2N_j) \). Let \((Y_{t}^{j,n}, Z_{t}^{j,n}, K_{t}^{j,n})\) be a solution of (4.13) with \( \sigma_{n,0}^j = 0, \sigma_{n,i}^j = \inf\{t > \sigma_{n,i-1}^j; \rho(D_{\sigma_{n,i}^j} \cap B(0, n), D_{\sigma_{n,i}^j} \cap B(0, n)) > 1/n\} \wedge T, i = 1, \ldots, k_n^j \), and \( k_n^j \) is chosen so that \( P(\sigma_{n,k_n^j}^j < T) \to 0 \) as \( n \to \infty \). From the first part of the proof we know that for each \( j \in \mathbb{N} \),
\[
(Y_{t}^{j,n}, Z_{t}^{j,n}, K_{t}^{j,n}) \to (\tilde{Y}_t^{(j)}, \tilde{Z}_t^{(j)}, \tilde{K}_t^{(j)}) \text{ in } S \times P \times S,
\]
(4.19)
where \( (\tilde{Y}(j), \tilde{Z}(j), \tilde{K}(j)) \) is a solution of RBSDE in \( \mathcal{D}^j \) with terminal value \( \xi^j \). Moreover, by Proposition 4.1 there exists \( C > 0 \) such that for \( j, n \in \mathbb{N} \),

\[
E\left( \sup_{t \leq T} |Y_t^{j,n}|^2 + \int_0^T |Z_{s}^{j,n}|^2 \, ds + \sum_{s \leq T} |\Delta K_{s}^{j,n}|^2 + \int_0^T \text{dist}(A_{s-}, \partial D_{s-}) \, d|K_{s}^{j,n}|_s \right) \\
\leq C \left( E(\xi^2) + \int_0^T |f(s, 0, 0)|^2 \, ds \right) + \|A\|^2_{\mathcal{H}^2}.
\]

Set \( \tau_{j,n} = \inf\{t; \sup_{s \leq t} |Y_s^n| > 2N_j \} \) and \( T \) and observe that by Tschebyshev’s inequality and Proposition 4.1

\[
P(\tau_{j,n} < T) \leq P(\sup_{t \leq T} |Y_t^n| > 2N_j) \leq (2N_j)^{-2} C \left( E(\xi^2) + \int_0^T |f(s, 0, 0)|^2 \, ds \right) + \|A\|^2_{\mathcal{H}^2},
\]

which implies that \( \lim_{j \to \infty} \sup_n P(\tau_{j,n} < T) = 0 \). Let \( \sigma = \tau_{j,k} \wedge \gamma_j \). Since \( Y_t^n \in B(0, 2N_j) \) for \( t < \sigma \), we may and will assume that \( \mathcal{D}^{j-} = \mathcal{D}^{\sigma-} \). By Corollary 4.3

\[
E(\sup_{t < \sigma} |Y_t^n - Y_t^{j,n}|^p) \leq CE(|Y_{\sigma}^n - Y_{\sigma}^{j,n}|^p) \\
\leq E|\xi - \xi^j|^p + E|Y_{\sigma}^n - Y_{\sigma}^{j,n}|^p 1_{\{\sigma < T\}} \\
\leq E|\xi - \xi^j|^p + (E|Y_{\sigma}^n - Y_{\sigma}^{j,n}|^2)^{p/2}(P(\sigma < T))^{(2-p)/2}.
\]

Hence \( \lim_{j \to \infty} \limsup_{n \to \infty} E(\sup_{t < \sigma} |Y_t^n - Y_t^{j,n}|^p) = 0 \). Consequently, for every \( \varepsilon > 0 \),

\[
\lim_{j \to \infty} \limsup_{n \to \infty} P\left( \int_0^T \|Z_{s}^n - Z_{s}^{j,n}\|^2 \, ds \right) + \sup_{t \leq T} |Y_t^n - Y_t^{j,n}| > \varepsilon) = 0. \tag{4.20}
\]

The desired convergence follows from (4.19), (4.20), because from Step 2 of the proof of Theorem 3.6 it follows that that \( (\tilde{Y}(j), \tilde{Z}(j), \tilde{K}(j)) \to P(Y, Z, K) \), where \( (Y, Z, K) \) is a unique solution of (4.1).

\[\square\]

**Remark 4.10.** Using arguments from the proof of Step 2 of Theorem 4.9 one can show that in fact assumption (3.4) in Proposition 4.6 (and consequently in the convergence statement in Remark 4.7) is superfluous.

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