On restriction of roots on affine $T$-varieties

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Abstract Let $X$ be a normal affine algebraic variety with a regular action of a torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a subtorus. We prove that each root of $X$ with respect to $T$ can be obtained by restriction of some root of $X$ with respect to $\mathbb{T}$. This allows to give an elementary proof of the description of roots of the affine Cremona group. Several results on restriction of roots in the case of a subtorus action on an affine toric variety are obtained.

Keywords Locally nilpotent derivation · Torus action · Roots of an algebraic group · Affine Cremona group · Toric surface

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Introduction

Let $\mathbb{T}$ be an algebraic torus. An algebraic variety endowed with an effective action of $\mathbb{T}$ is called a $\mathbb{T}$-variety. Let $X$ be an affine $\mathbb{T}$-variety. The algebra $A = \mathbb{K}[X]$ of regular functions on $X$ is graded by the lattice $M$ of characters of the torus $\mathbb{T}$. A derivation $\partial$ on an algebra $A$ is said to be locally nilpotent (LND) if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $\partial^n(a) = 0$. We refer to Frendenburg (2006) for an algebraic theory of locally nilpotent derivations. An LND on $A$ is said to be homogeneous if it respects the $M$-grading. A homogeneous LND $\partial$ shifts the $M$-grading by some lattice vector
deg $\partial$. This vector is named the degree of $\partial$. Degrees of homogeneous LNDs are called $T$-roots of the $T$-variety $X$. This notion was introduced by Popov (2005). It imitates in some sense the notion of a root from Lie Theory. Let $G_a$ be the additive group of the ground field $K$. It is well known that LNDs on $A$ are in bijective correspondence with regular actions of $G_a$ on $X$. Namely, for a given LND $\partial$ on $A$ the map $\varphi_\partial : G_a \times A \to A$, $\varphi_\partial (t, \partial) = \exp(t\partial)(f)$, defines a $G_a$-action, and any regular $G_a$-action on $X$ arises in this way. It is easy to see that an LND on $A$ is homogeneous if and only if the corresponding $G_a$-action is normalized by the torus $T$ in the group $\text{Aut}(X)$. In these terms, a root is the character by which $T$ acts on $G_a$.

Any derivation on $K[X]$ extends to a derivation on the field of fractions $K(X)$ by the Leibniz rule. A homogeneous LND $\partial$ on $K[X]$ is said to be of fiber type if $\partial(K(X)^T) = 0$ and of horizontal type otherwise. In other words, $\partial$ is of fiber type if and only if the general orbits of the corresponding $G_a$-action on $X$ are contained in the closures of $T$-orbits. Recall that for a $T$-action on an algebraic variety, the complexity is defined as the codimension of a general orbit. Toric varieties are $T$-varieties of complexity zero. In this case roots were essentially described by Demazure (1970), see also Cox (1995, Sect. 4). There is a complete characterization of homogeneous LNDs and roots in the case of $T$-actions of complexity one due to Liendo (2010a). A description of homogeneous LNDs of fiber type in arbitrary complexity is given in Liendo (2010b). The results of Liendo (2010a, b) are based on the combinatorial description of affine $T$-varieties obtained by Altmann and Hausen (2006).

Let $T \subset \mathbb{T}$ be a subtorus. The torus $T$ also acts on $X$. Denote by $M_T$ and $M_\mathbb{T}$ the character lattices of $T$ and $\mathbb{T}$, respectively. It is obvious that if an LND respects the $M_\mathbb{T}$-grading on $K[X]$, then it respects the $M_T$-grading as well. So any $T$-root of the variety $X$ can be restricted to a $T$-root. In this paper we show that any $T$-root of $X$ arises in this way (Theorem 1). Also we study the restriction of roots in the case of affine toric varieties. For affine toric surfaces a complete description of the restriction is obtained.

Our research is motivated by the following reasons. As a rule, there are too many LNDs on an affine algebra and we have no chance to describe all of them. Often it is sufficient to study the ones that respect some grading. If all homogeneous LNDs are known, we can try to find LNDs respecting a coarser grading. Our result gives an a priori description of their degrees. This idea is fruitful for Cox rings of algebraic varieties, which have several natural gradings, see Arzhantsev et al. (2011) for the general case and Arzhantsev et al. (2014) for the case of complete varieties with torus action of complexity one. The restriction of roots is also useful, when we are interested in a subgroup $G \subset \text{Aut}(X)$ preserving some structure on an affine variety $X$. Namely, we can restrict roots of a maximal torus $\mathbb{T} \subset \text{Aut}(X)$ to a maximal torus $T \subset G$ and obtain the first approximation to the description of root subgroups in $G$. This is the case in Popov’s question concerning roots of the affine Cremona group, see Popov (2005). More precisely, the question is to find all roots and root vectors of the group of volume preserving transformations of the polynomial algebra.

In Sect. 1 we recall some basic facts on locally nilpotent derivations and roots. Section 2 contains an elementary proof of surjectivity of the restriction of roots. In Sect. 3 it is shown how the description of roots of the affine Cremona group given by Liendo (2011) can be obtained by our method. Note that Liendo’s proof is based on the description of LNDs of horizontal type in the case of $T$-actions of complexity one. We
give more direct arguments. Section 4 is devoted to the study of the restriction of roots on an affine toric variety. In Sect. 5 the case of affine toric surfaces is studied in detail.

We work over an algebraically closed field $K$ of characteristic zero.

1 Homogeneous locally nilpotent derivations

Let $X$ be an affine variety with an effective action of an algebraic torus $T$ and $A = \mathbb{K}[X]$ be the algebra of regular functions on $X$ and by $N$ the lattice of one-parameter subgroups of $T$. It is well known that there is a bijective correspondence between effective $T$-actions on $X$ and effective $M$-gradings on $A$. In fact, the algebra $A$ is graded by a semigroup of lattice points in some convex polyhedral cone $\omega \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ called the weight cone. So we have

$$A = \bigoplus_{m \in \omega_M} A_m \chi^m,$$

where $\omega_M = \omega \cap M$ and $\chi^m$ is the character corresponding to $m$. Denote by $A^+ := \bigcup_m A_m$ the set of homogeneous elements.

A derivation $\partial$ on an $M$-graded algebra $A$ is called homogeneous if it sends homogeneous elements to homogeneous ones. If $a, b \in A^+ \setminus \ker \partial$, then $\partial(ab) = a\partial(b) + \partial(a)b$ belongs to the set $A^+$ too and hence $\deg \partial(a) = \deg a = \deg \partial(b) - \deg b$. So any homogeneous derivation $\partial$ has a well defined degree given as $\deg \partial = \deg \partial(a) - \deg a$ for any $a \in A^+ \setminus \ker \partial$. Homogeneous LNDs are called root vectors with respect to the torus $T = \text{Spec} \mathbb{K}[M]$ and their degrees are said to be $T$-roots.

The following well known lemma shows that any LND decomposes into a sum of homogeneous derivations, some of which are locally nilpotent, cf. Liendo (2010a, Lemma 1.10).

**Lemma 1** Let $A$ be a finitely generated $M$-graded domain. For any derivation $\partial$ on $A$ there is a decomposition

$$\partial = \sum_{e \in M} \partial_e,$$  \hfill (1)

where $\partial_e$ is a homogeneous derivation of degree $e$. Moreover, the convex hull $\Delta(\partial) \subseteq M_{\mathbb{Q}}$ of the set $\{e \in M \mid \partial_e \neq 0\}$ is a polytope and for every vertex $e$ of $\Delta(\partial)$, the derivation $\partial_e$ is locally nilpotent if $\partial$ is.

**Proof** For every homogeneous $a \in A$ there is a decomposition $\partial(a) = \sum_{e \in M} a_{\deg(a) + e}$, where the element $a_{\deg(a) + e}$ is homogeneous of degree $\deg(a) + e$. The linear map $\partial_e$ given by the rule $\partial_e(a) = a_{\deg(a) + e}$ is a homogeneous derivation of degree $e$, and $\partial = \sum_{e \in M} \partial_e$. Since $A$ is finitely generated, the set $\{e \in M \mid \partial_e \neq 0\}$ is finite and its convex hull $\Delta(\partial)$ is a polytope. Let $\hat{e}$ be a vertex of $\Delta(\partial)$ and $a \in A_m$ be some homogeneous element. Then $\partial^m(a)$ has $\partial^m_{\hat{e}}(a)$ as its homogeneous component of degree $m + n\hat{e}$. So if $\partial$ is an LND, then $\partial^m_{\hat{e}}$ is an LND as well. \hfill $\square$

Let $X$ be an affine toric variety, i.e. a normal affine variety with a generically transitive action of a torus $T$. In this case
is a semigroup algebra. We refer to Cox et al. (2011) and Fulton (1993) for more information on affine toric varieties. Recall that for a given cone \( \omega \subset \mathbb{M}_{\mathbb{Q}} \), its dual cone is defined by

\[
\sigma = \{ n \in \mathbb{N}_{\mathbb{Q}} \mid \langle n, m \rangle \geq 0 \ \forall m \in \omega \},
\]

where \( \langle , \rangle \) is the pairing between dual lattices \( \mathbb{N} \) and \( \mathbb{M} \). Let \( \sigma(1) \) be the set of rays of \( \sigma \) and \( n_\rho \) be the primitive lattice vector on the ray \( \rho \). By definition, for \( \rho \in \sigma(1) \) we put

\[
S_\rho := \{ e \in \mathbb{M} \mid \langle n_\rho, e \rangle = -1 \ \text{and} \ \langle n_\rho', e \rangle \geq 0 \ \forall \rho' \in \sigma(1), \rho' \neq \rho \}.
\]

If \( \text{rk} \mathbb{N} > 1 \) one easily checks that the set \( S_\rho \) is infinite for any \( \rho \in \sigma(1) \). The elements of the set \( \mathcal{R} := \bigsqcup_\rho S_\rho \) are called the Demazure roots of \( \sigma \), see Demazure (1970). Let \( e \in S_\rho \). One can define a homogeneous LND on the algebra \( A \) by the rule

\[
\partial_e(\chi^m) = \langle n_\rho, m \rangle \chi^m + e.
\]

It turns out that the set \( \{ \alpha \partial_e \mid \alpha \in \mathbb{K}, e \in \mathcal{R} \} \) coincides with the set of all homogeneous LNDs on \( A \), see Liendo (2010a, Theorem 2.7).

Example 1 Consider \( X = \mathbb{A}^d \) with the standard action of the torus \((\mathbb{K}^\times)^d\). It is a toric variety with the cone \( \sigma = \mathbb{Q}_{\geq 0}^d \) having rays \( \rho_1 = \langle (1, 0, \ldots, 0) \rangle_{\mathbb{Q}_{\geq 0}}, \ldots, \rho_d = \langle (0, 0, \ldots, 0, 1) \rangle_{\mathbb{Q}_{\geq 0}} \). The dual cone \( \omega \) is \( \mathbb{Q}_{\geq 0}^d \) as well.

\[
\begin{align*}
S_{\rho_1} & \quad \bullet \\
\bullet & \quad \bullet \\
\bullet & \quad \bullet \\
S_{\rho_2} & \quad \bullet \\
M_{\mathbb{Q}} & = \mathbb{Q}^2
\end{align*}
\]

In this case \( S_{\rho_1} = \{ (e_1, \ldots, e_{i-1}, -1, e_{i+1}, \ldots, e_d) \mid e_j \in \mathbb{Z}_{\geq 0} \} \). Denote \( x_1 = x^{(1,0,\ldots,0)}, \ldots, x_d = x^{(0,\ldots,0,1)} \). Then \( \mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_d] \). It is easy to see that the homogeneous LND corresponding to the root \( e = (e_1, \ldots, e_d) \in S_{\rho_1} \) is

\[
\partial_e = x_1^{e_1} \ldots x_i^{e_{i-1}} x_{i+1}^{e_{i+1}} \ldots x_d^{e_d} \frac{\partial}{\partial x_i}.
\]

2 Restriction of roots

Consider an affine \( \mathbb{T} \)-variety \( X \) and a subtorus \( T \subset \mathbb{T} \). The torus \( T \) also acts on \( X \). Evidently, every \( \mathbb{T} \)-homogeneous LND on \( \mathbb{K}[X] \) is \( T \)-homogeneous and so any \( \mathbb{T} \)-root

\[
\mathbb{T} \text{-Springer}
\]
of $X$ can be restricted to some $T$-root. The following theorem shows that every $T$-root arises in this way.

**Theorem 1** Let $X$ be an affine variety with a regular action of a torus $T$ and $T \subset T$ be a subtorus. Then the restriction of $T$-roots to $T$-roots is surjective. Moreover, if a $T$-root $e$ is the restriction of the only one $T$-root, then any $T$-homogeneous LND on $\mathbb{K}[X]$ of degree $e$ is $T$-homogeneous as well.

**Proof** Let $e$ be a $T$-root of $X$ and $\partial$ be a homogeneous LND of degree $e$. By Lemma 1, $\partial$ decomposes into a sum of $T$-homogeneous derivations, some of which are locally nilpotent. So the degree of any locally nilpotent summand is a $T$-root of $X$ and its restriction to $T$ equals $e$. If $e$ is the restriction of a unique $T$-root, then there is only one summand in the decomposition of $\partial$, and hence $\partial$ is $T$-homogeneous. \[\square\]

Now we consider an affine toric variety $X$ with an acting torus $T$ and a subtorus $T \subset T$ of codimension one. Let $M$ be the character lattice of $T$ and $N$ be the lattice of one-parameter subgroups of $T$. Denote by $\sigma_X \subset N_\mathbb{Q}$ the cone corresponding to $X$ and by $\Gamma_T \subset N_\mathbb{Q}$ the hyperplane generated by one-parameter subgroups of $T$ that are contained in the subtorus $T$.

**Proposition 1** Let $X$ be an affine toric variety with an acting torus $T$ and $T \subset T$ be a subtorus of codimension one. If $\Gamma_T \cap \sigma_X = \{0\}$, then the restriction of roots is bijective. In particular, there is only one (up to scalar) root vector for each $T$-root $e$ and any $T$-homogeneous LND on $\mathbb{K}[X]$ is $T$-homogeneous as well.

**Proof** Let $\langle \cdot, m_T \rangle = 0$ be the equation of the hyperplane $\Gamma_T$, where $m_T \in M$. The vector $m_T$ generates the kernel of the restriction of characters from $T$ to $T$. Since $\Gamma_T \cap \sigma_X = \{0\}$, we may assume that $\langle n, m_T \rangle > 0$ for all $n \in \sigma_X \setminus \{0\}$. Suppose the restrictions of roots $e_1$ and $e_2$ from $\sigma_X$ to $T$ coincide and $e_1 - e_2 = \lambda m_T$, where $\lambda > 0$. Let $\rho$ be the ray of $\sigma_X$ with $e_1 \in S_\rho$. Then $\langle n_\rho, e_1 - e_2 \rangle = -1 - \langle n_\rho, e_2 \rangle \leq 0$. On the other hand, $\langle n_\rho, e_1 - e_2 \rangle = \lambda \langle n_\rho, m_T \rangle > 0$. This contradiction implies that the restriction of roots from $T$ to $T$ is injective. By Theorem 1, any $T$-homogeneous LND on $\mathbb{K}[X]$ is $T$-homogeneous. \[\square\]

**Remark 1** In Proposition 1, we can not replace a subtorus of codimension one with a subtorus of arbitrary codimension. For example, let $X = \mathbb{A}^3$ and $T = \{(t, t, t^{-1}) \mid t \in \mathbb{T}^\times\}$. Denote by $\Gamma_T$ the line corresponding to $T$. Then $\Gamma_T = \mathbb{Q} \cdot (1, 1, -1)$ and $\Gamma_T \cap \sigma_X = \{0\}$, but the restriction of roots is not bijective. Indeed, the $T$-homogeneous LND $\partial = x_2 x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ is not $T$-homogeneous.

### 3 Roots of the affine Cremona group

Consider the polynomial algebra $\mathbb{K}[n] = \mathbb{K}[x_1, \ldots, x_n]$. The set of “volume preserving” transformations

\[\text{Aut}^*_\mathbb{K}\mathbb{K}[n] = \left\{ \gamma \in \text{Aut}_\mathbb{K}\mathbb{K}[n] \mid \det \left( \frac{\partial \gamma(x_j)}{\partial x_i} \right)_{1 \leq i,j \leq n} = 1 \right\}\]
is a closed normal subgroup of the affine Cremona group Aut\(_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\). It is an infinite dimensional simple algebraic group, see Shafarevich (1966). The maximum of dimensions of algebraic tori contained in Aut\(_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) equals \(n - 1\), and every torus in Aut\(_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) of dimension \(n - 1\) and is conjugate to

\[
T = \left\{ \gamma \in \text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]} \mid \gamma(x_i) = t_i x_i, \ t_i \in \mathbb{K}, \ \prod_{i=1}^{n} t_i = 1 \right\}.
\]

If \(\partial\) is an LND on \(\mathbb{K}^{[n]}\), then \(\exp t \partial \in \text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) for any \(t \in \mathbb{K}\), and so \(\partial \in \text{Lie}(\text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}).\) A nonzero LND \(\partial\) of \(\mathbb{K}^{[n]}\) is called a root vector of \(\text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) with respect to \(T\) if there exists a nontrivial character \(\chi : T \rightarrow \mathbb{K}^{\times}\) such that

\[
\gamma \circ \partial \circ \gamma^{-1} = \chi(\gamma) \partial \quad \forall \gamma \in T.
\]

The character \(\chi\) is said to be the root of \(\text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) with respect to \(T\) corresponding to \(\partial\). It is easy to see that two last definitions are equivalent to the definitions of root vectors and roots given in Sect. 2 in the case \(X = \mathbb{A}^n\). The first Popov’s question in Popov (2005) was to find all roots and root vectors of \(\text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) with respect to \(T\). The answer is given in Liendo (2011). It claims that the root vectors are exactly the LNDs \(x^\alpha \frac{\partial}{\partial x_i}\), where \(x^\alpha\) is any monomial not depending on \(x_i\). Let us show that this statement immediately follows from Proposition 1. We have the affine toric variety \(\mathbb{A}^n\) with the standard action of the torus \(\mathbb{T} = (\mathbb{K}^{\times})^n\) and the subtorus \(T \subset \mathbb{T}\) given by equation \(\prod_{i=1}^{n} t_i = 1\). The cone corresponding to \(\mathbb{A}^n\) is just \(\sigma_X = \mathbb{Q}^n_{\geq 0}\). The hyperplane \(\Gamma_T\) is given by equation \(z_1 + \cdots + z_n = 0\) and we get \(\Gamma_T \cap \sigma_X = \{0\}\). Thus all root vectors are homogeneous with respect to \(\mathbb{T}\) and hence, as it is shown in Example 1, have the form \(\lambda x^\alpha \frac{\partial}{\partial x_i}\), where \(\lambda \in \mathbb{K}^{\times}\). So we obtain the following theorem.

**Theorem 2** The set of root vectors of \(\text{Aut}_{\mathbb{K}}^{*}\mathbb{K}^{[n]}\) with respect to \(T\) coincides with the set

\[
\left\{ \lambda x^\alpha \frac{\partial}{\partial x_i} \mid \lambda \in \mathbb{K}^{\times}, \ i \in \{1, \ldots, n\}, \alpha \in \mathbb{Z}^n_{\geq 0}, \alpha_i = 0 \right\}.
\]

The root corresponding to \(\lambda x^\alpha \frac{\partial}{\partial x_i}\) is the character \(\chi_{i, \alpha} : T \rightarrow \mathbb{K}^{\times}\) given by \(\chi_{i, \alpha}(\gamma) = t_i^{-1} \prod_{j=1}^{n} t_j^{\alpha_j}\).

4 Subtorus actions on affine toric varieties

Let us start with three questions concerning roots on affine \(T\)-varieties. The first one was communicated to us by Liendo.

**Question 1** Let \(e\) be a root of an affine \(T\)-variety. How many root vectors correspond to the root \(e\)?
As we have seen, in the toric case there is only one (up to scalar) root vector for each root. It follows from Liendo (2010b, Theorem 2.4) that root vectors of fiber type having degree \( e \) form a vector space (of finite or infinite dimension). For root vectors of horizontal type there is no complete answer even for \( T \)-action of complexity one.

**Question 2** Let \( X \) be an affine \( \mathbb{T} \)-variety and \( T \subset \mathbb{T} \) be a subtorus. Consider a \( T \)-root \( e \) of \( X \). How many \( \mathbb{T} \)-roots have \( e \) as their restrictions to \( T \)?

**Question 3** Are all \( T \)-homogeneous LNDs of degree \( e \) on \( \mathbb{K}[X] \) homogeneous with respect to \( \mathbb{T} \)?

Theorem 1 shows that if a \( T \)-root \( e \) is the restriction of only one \( \mathbb{T} \)-root \( \hat{e} \), then any \( T \)-homogeneous LND of degree \( e \) is \( \mathbb{T} \)-homogeneous and there are as many root vectors corresponding to \( e \) as root vectors corresponding to \( \hat{e} \).

In the present paper we investigate these questions only in the toric case. Consider an affine toric variety \( X \) with the acting torus \( \mathbb{T} \) and a subtorus \( T \subset \mathbb{T} \) of codimension one. As above, denote by \( N \) the lattice of one-parameter subgroups, by \( M \) the character lattice of \( \mathbb{T} \), by \( \sigma_X \subset N_\mathbb{Q} \) the cone corresponding to \( X \), and by \( \Gamma_T \) the hyperplane corresponding to subtorus \( T \). Let \( \langle \cdot, m_T \rangle = 0 \) be the equation of the hyperplane \( \Gamma_T \), where \( m_T \in M \). The set of \( \mathbb{T} \)-roots of the variety \( X \) is the set \( \mathcal{R}_T = \bigcup_{\rho \in \sigma_X(1)} S_\rho \) of Demazure roots of the cone \( \sigma_X \). Let \( \mathcal{R}_T \) be the set of \( T \)-roots and \( \pi : \mathcal{R}_T \to \mathcal{R}_T \) be the restriction of roots.

The answers to Questions 1, 2 and 3 depend on a relative position of the cone \( \sigma_X \) and the hyperplane \( \Gamma_T \). We already have studied the case \( \sigma_X \cap \Gamma_T = \{0\} \), see Proposition 1. The following three propositions describe the restriction of roots in other cases.

**Proposition 2** If \( \Gamma_T \) intersects the interior of the cone \( \sigma_X \) and does not contain rays of \( \sigma_X \), then any \( T \)-root is the restriction of at most two \( \mathbb{T} \)-roots.

**Proof** Suppose that the restrictions of pairwise distinct \( \mathbb{T} \)-roots \( \hat{e}_1 \in S_{\rho_1}, \hat{e}_2 \in S_{\rho_2} \) and \( \hat{e}_3 \in S_{\rho_3} \) coincide. If \( \rho_i = \rho_j \) for \( i \neq j \), then \( \langle n_{\rho_i}, e_i - e_j \rangle = 0 \) and thus the ray \( \rho_i \) is contained in \( \Gamma_T \), a contradiction. So \( \rho_i \neq \rho_j \) whenever \( i \neq j \). We may assume without loss of generality, that \( \hat{e}_1 - \hat{e}_2 = \lambda m_T \) and \( \hat{e}_2 - \hat{e}_3 = \mu m_T \), where \( \lambda > 0 \) and \( \mu > 0 \). Then \( \lambda \langle n_{\rho_2}, m_T \rangle = 1 + \langle n_{\rho_2}, \hat{e}_1 \rangle > 0 \) and \( \mu \langle n_{\rho_2}, m_T \rangle = -1 + \langle n_{\rho_2}, \hat{e}_3 \rangle < 0 \), a contradiction. \( \square \)

**Proposition 3** Suppose that \( \Gamma_T \cap \sigma_X \) is a face of the cone \( \sigma_X \) of a positive dimension. Then

(a) \( \pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) = \emptyset \) whenever \( \rho_1 \neq \rho_2 \).
(b) If a ray \( \rho \) is not contained in \( \Gamma_T \cap \sigma_X \), then \( \pi|_{S_{\rho}} : S_{\rho} \to \pi(S_{\rho}) \) is bijective.
(c) If \( \rho \subset \Gamma_T \cap \sigma_X \), then for any \( e \in \pi(S_{\rho}) \) there are infinitely many elements of \( S_{\rho} \), having \( e \) as their restriction, and the root vectors corresponding to \( e \) form an infinite dimensional vector space.

**Proof** (a), (b) If \( \Gamma_T \cap \sigma_X \) is a face of the cone \( \sigma_X \), then \( \langle n, m_T \rangle \geq 0 \) for all \( n \in \sigma_X \). Suppose \( \hat{e}_1 \in S_{\rho_1}, \hat{e}_2 \in S_{\rho_2} \), and \( \hat{e}_1 - \hat{e}_2 = \lambda m_T \), where \( \lambda > 0 \). Then \( \lambda \langle n_{\rho_1}, m_T \rangle = -1 + \langle n_{\rho_1}, \hat{e}_2 \rangle \leq 0 \) and \( \lambda \langle n_{\rho_2}, m_T \rangle = 1 + \langle n_{\rho_2}, \hat{e}_1 \rangle \geq 0 \). It is possible if and only if \( \rho_1 = \rho_2 \subset \Gamma_T \cap \sigma_X \).
(c) Let \( \hat{e} \in S_\rho \), where \( \rho \subseteq \Gamma_T \cap \sigma_X \). The vector \( \hat{e} + \lambda m_T \) is a root if and only if it belongs to the lattice and \( \langle n_{\rho'}, \hat{e} \rangle + \lambda \langle n_{\rho'}, m_T \rangle \geq 0 \) for each \( \rho' \neq \rho \). So there are infinite many roots, whose restrictions are equal to restriction of \( \hat{e} \). It remains to note that if \( \hat{e}_1, \hat{e}_2 \in S_\rho \), then corresponding LNDs commute and hence their sum is again an LND.

\[ \square \]

**Proposition 4** Suppose \( \Gamma_T \) intersects the interior of the cone \( \sigma_X \) and contains some rays of \( \sigma_X \). Then

(a) If \( \rho \not\subseteq \Gamma_T \cap \sigma_X \), then any \( e \in \pi(S_\rho) \) is the restriction of at most two \( \mathbb{T} \)-roots.

(b) If \( \rho \subseteq \Gamma_T \cap \sigma_X \), then any \( e \in \pi(S_\rho) \) is the restriction of a finite number \( k_e \) of \( \mathbb{T} \)-roots, there exists \( e \) with \( k_e \geq 2 \), and root vectors corresponding to \( e \) form a \( k_e \)-dimensional vector space.

**Proof** (a) Suppose that \( e = \pi(\hat{e}_1) = \pi(\hat{e}_2) \), where \( \hat{e}_1 \in S_\rho, \hat{e}_2 \in S_{\rho'}, \rho \not\subseteq \Gamma_T \cap \sigma_X \), and \( \rho' \subseteq \Gamma_T \cap \sigma_X \). Then \( \hat{e}_1 - \hat{e}_2 = \lambda m_T \) and hence \( \langle n_{\rho'}, \hat{e}_1 \rangle = -1 \). It is a contradiction. If \( e \in \pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) \cap \pi(S_{\rho_3}) \), where \( \rho_1 \not\subseteq \Gamma_T \cap \sigma_X \), we can use the same arguments as in Proposition 2.

(b) Let us show that any \( e = \pi(\hat{e}) \), where \( \hat{e} \in S_\rho \) and \( \rho \subseteq \Gamma_T \cap \sigma_X \), is the restriction of a finite number of elements. As above, the vector \( \hat{e} + \lambda m_T \) is a root if and only if it belongs to the lattice \( N \) and \( \langle n_{\rho'}, \hat{e} \rangle + \lambda \langle n_{\rho'}, m_T \rangle \geq 0 \) for any \( \rho' \neq \rho \). In this case there are rays \( \rho' \) with both positive and negative values \( \langle n_{\rho'}, m_T \rangle \). Hence only a finite number of \( \lambda \) satisfies these conditions. In conclusion, let \( \hat{e} \in S_\rho \) be a root such that \( \langle n_{\rho'}, \hat{e} \rangle \geq -\langle n_{\rho'}, m_T \rangle \) whenever \( \langle n_{\rho'}, m_T \rangle < 0 \). Then \( \hat{e} + m_T \) is \( \mathbb{T} \)-root and \( k_e \geq 2 \) for \( e = \pi(\hat{e}) \).

\[ \square \]

**Example 2** Consider \( X = \mathbb{A}^3 \) with the standard action of the torus \( \mathbb{T} = (\mathbb{K}^\times)^3 \). Let \( T = \{(s_1, s_1, s_2) \mid s_1, s_2 \in \mathbb{K}^\times \} \) be a subtorus of codimension one. In this case the hyperplane \( \Gamma_T \) is generated by vectors \((1, 1, 0)\) and \((0, 1, 1)\). The restriction of a \( \mathbb{T} \)-root \( e = (e_1, e_2, e_3) \) equals \((e_1 + e_2, e_3)\). Using Example 1, we obtain that

\[ \mathcal{R}_T = \{(a, b), (c, -1) \mid a \in \mathbb{Z}_{\geq -1}, b, c \in \mathbb{Z}_{\geq 0}\}, \]

each root \((a, b)\) is the restriction of two \( \mathbb{T} \)-roots \((a + 1, -1, b)\) and \((-1, a + 1, b)\), and each root \((c, -1)\) is the restriction of \( c + 1 \) \( \mathbb{T} \)-roots of the form \((i, c - i, -1)\), where \( 0 \leq i \leq c \).

Thus we have a complete description of the restriction of roots in the case, when \( \sigma_X \cap \Gamma_T \) is a face of the cone \( \sigma_X \). If the hyperplane \( \Gamma_T \) intersect the interior of \( \sigma_X \), the answers to Questions 1 and 3 are remain unknown for \( \mathbb{T} \)-roots \( e \in \pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) \), where \( \rho_1, \rho_2 \not\subseteq \Gamma_T \). The following proposition gives some sufficient condition on \( \Gamma_T \), under which there are infinitely many root vectors corresponding to \( e \). In Sect. 5 we show that this condition is also necessary in the case of affine toric surfaces.

**Proposition 5** Suppose that \( e_1 \in S_{\rho_1}, e_2 \in S_{\rho_2}, \rho_1 \neq \rho_2, \pi(e_1) = \pi(e_2) \), and \( \langle n_{\rho_1}, m_T \rangle = -1 \). Then the linear map \( \partial : \mathbb{K}[X] \rightarrow \mathbb{K}[X] \) given by

\[ \partial \chi^m = \chi^{m+e_2}(\alpha \langle n_{\rho_1}, m \rangle \chi^{m_T} + \beta \langle n_{\rho_2}, m \rangle)(\alpha \chi^{m_T} - \beta \langle n_{\rho_2}, m_T \rangle)^{(n_{\rho_1}, e_2)}, \]

for all \( m \in \omega_M \),

\[ (2) \]

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is a $T$-homogeneous LND of degree $\pi(e_1)$ for every $\alpha, \beta \in K$.

Proof By assumptions, $e_1 - e_2 = (\langle n_{\rho_1}, e_2 \rangle + 1)m_T$ and $\partial$ is a $T$-homogeneous derivation on $A$. By direct calculation we get

$$\partial^{k+1} \chi^m = \chi^{m+(k+1)e_2}(\alpha \chi^{m_T} - \beta \langle n_{\rho_2}, m_T \rangle) \sum_{j=0}^{k+1} \alpha^j \beta^{k-j+1} l_j^{(k+1)} \chi^{jm_T},$$

where $l_j^{(k+1)} = \langle n_{\rho_1}, m \rangle - j + 1 l_j^{(k)} + \langle n_{\rho_2}, m \rangle + j \langle n_{\rho_2}, m_T \rangle - k l_j^{(k)}$, $l_1^{(k)} = \langle n_{\rho_1}, m \rangle$, and $l_0^{(1)} = \langle n_{\rho_2}, m \rangle$. We have $l_0^{(k+1)} = (\langle n_{\rho_2}, m \rangle - k) l_0^{(k)}$. If $k = \langle n_{\rho_2}, m \rangle$, then $l_0^{(k+1)} = 0$ and thus $l_0^{(k)} = 0$ for all $k \geq \langle n_{\rho_2}, m \rangle + 1$. Let us prove by induction that $l_j^{(k)} = 0$ whenever $k \geq \langle n_{\rho_2}, m \rangle + d \langle n_{\rho_2}, m_T \rangle + 1$ and $j \leq d$. Suppose that the assumption holds for some $d$. Then $l_j^{(k+1)} = 0$ for $k \geq \langle n_{\rho_2}, m \rangle + d \langle n_{\rho_2}, m_T \rangle + 1$ and $j \leq d$. Also $l_{d+1}^{(k+1)} = \langle n_{\rho_2}, m \rangle + (d + 1) \langle n_{\rho_2}, m_T \rangle - k l_{d+1}^{(k)}$ and it equals zero whenever $k \geq \langle n_{\rho_2}, m \rangle + (d + 1) \langle n_{\rho_2}, m_T \rangle$.

Note that $l_{k+1}^{(k+1)} = (\langle n_{\rho_1}, m \rangle - k) l_{k}^{(k)}$. Hence $l_{k}^{(k)} = 0$ for $k \geq \langle n_{\rho_1}, m \rangle + 1$. It can be proven by induction that $l_i^{(k)} = 0$ whenever $k \geq \langle n_{\rho_1}, m \rangle + 1$ and $i \geq \langle n_{\rho_1}, m \rangle + 1$. Thus if $k = \langle n_{\rho_2}, m \rangle + \langle n_{\rho_2}, m_T \rangle \langle n_{\rho_1}, m \rangle$, then $\partial^{k+1} \chi^m = 0$, and $\partial$ is locally nilpotent. □

5 An example: affine toric surfaces

Normal affine surfaces with $\mathbb{C}^\times$-actions and LNDs on them were studied in Flenner and Zaidenberg (2003, 2005). Here we consider a particular case. Namely, let $X$ be an affine toric surface with the acting torus $T$ and $T \subset \mathbb{T}$ be a one-dimensional subtorus. We give complete answers to Questions 1, 2 and 3 for $X$. Let $N_T$ and $N_\mathbb{T}$ be the lattices of one parameter subgroups of tori $T$ and $\mathbb{T}$ respectively, and $\sigma_X \subset (N_T)_Q$ be the cone corresponding to $X$. Up to automorphism of the lattice $N_T$ we may assume that the cone $\sigma_X$ is generated by the vectors $n_{\rho_1} = (1, 0)$ and $n_{\rho_2} = (a, b)$, where $a, b \in \mathbb{Z}^\geq$, $a < b$, and $\text{gcd}(a, b) = 1$.

The set of Demazure roots of $\sigma_X$ has the form:

$$S_{\rho_1} = \left\{(-1, m) \mid m \in \mathbb{Z}, m \geq \frac{a}{b} \right\},$$
$$S_{\rho_2} = \left\{(m_1, m_2) \mid m_1, m_2 \in \mathbb{Z}, m_1 \geq 0, am_1 + bm_2 = -1 \right\}.$$
Let $\Gamma_T$ be the line corresponding to the subtorus $T$. There are three alternatives for a relative position of the cone $\hat{\sigma}_X$ and the line $\Gamma_T$:

1. $\hat{\sigma}_X \cap \Gamma_T = \{0\}$;
2. $\hat{\sigma}_X \cap \Gamma_T$ is a ray of $\hat{\sigma}_X$;
3. $\Gamma_T$ intersects the interior of $\hat{\sigma}_X$.

**Case 1** Suppose $\hat{\sigma}_X \cap \Gamma_T = \{0\}$. It follows from Proposition 1 that there is only one (up to scalar) root vector for each $T$-root $e$, only one $T$-root has $e$ as its restriction, and all $T$-homogeneous LNDs are $T$-homogeneous as well. In the following figure we illustrate the $T$-roots of $X$. The restrictions of the elements of $S_{\rho_1}$ and $S_{\rho_2}$ are denoted by “•” and “◦” respectively.

![Diagram](image)

**Case 2** Assume that the intersection of the cone $\hat{\sigma}_X$ with $\Gamma_T$ is the ray $\rho_1$. In this case the restriction of the set $S_{\rho_1}$ is just the point $-1$. By Proposition 3 the root vectors corresponding to $e = -1$ form an infinite dimensional vector space. For each root $e \neq -1$ there is only one $T$-root, whose restriction equals $e$. The set of the $T$-roots looks like:

![Diagram](image)

**Case 3** Suppose $\Gamma_T$ intersects the interior of the cone $\hat{\sigma}_X$. Then the restriction $\pi$ on each set $S_{\rho_1}$ is injective and the number $D = rb - qa$ is positive for one of primitive lattice vectors $(r, q)$ of the line $\Gamma_T$. Let us prove the following lemma.

**Lemma 2** Suppose the line $\Gamma_T$ intersects the interior of the cone $\hat{\sigma}_X$. Then the intersection $\pi(S_{\rho_1}) \cap \pi(S_{\rho_2})$ is not empty if and only if $a - 1$ is divisible by $\gcd(q, rb - qa)$.

**Proof** The restrictions of $S_{\rho_1}$ and $S_{\rho_2}$ have the form $\{-r + mq \mid m \in \mathbb{Z}_{\geq 0}\}$ and $\{rm_1^0 + qm_2^0 + kD \mid k \in \mathbb{Z}_{\geq 0}\}$, where $(m_1^0, m_2^0) \in S_{\rho_2}$ is the root with the minimal first coordinate. The intersection of these two sets is non-empty if and only if there exist numbers $m_0$ and $k_0$ such that

$$r + rm_1^0 + qm_2^0 = m_0q - k_0D. \tag{3}$$

Multiplying this equation by $a$ and using the fact that $(m_1^0, m_2^0) \in S_{\rho_2}$, we get

$$r(a - 1) - Dm_2^0 = m_0qa - k_0Da. \tag{4}$$

So if $a - 1$ is not divisible by $\gcd(q, D)$, we obtain a contradiction. Conversely, suppose that the left part of (4) is divisible by $\gcd(q, D)$. Since $b$ is divisible by $\gcd(q, D)$ and $\gcd(a, b) = 1$, the left part of (3) is also divisible by $\gcd(q, D)$. Hence there exist $m_0$ and $k_0$ such that (3) holds. \qed
Let us draw the set of the \( T \)-roots as above.

\[
\begin{align*}
\text{a) } & \quad \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \quad \gcd(q, rb - qa) \nmid a - 1 \\
\text{b) } & \quad \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \quad \gcd(q, rb - qa) \mid a - 1
\end{align*}
\]

In case (a) the restriction of roots from \( \mathbb{T} \) to \( T \) is bijective and any \( T \)-homogeneous LND is \( \mathbb{T} \)-homogeneous. In case (b) denote by \( \Lambda \) the intersection \( \pi(S_{\rho_3}) \cap \pi(S_{\rho_2}) \). For the roots \( e \notin \Lambda \) we can use Theorem 1. It remains to answer Questions 1 and 3 for \( e \in \Lambda \).

Now we describe a recall of \( T \)-varieties of complexity one. Let \( N \) and \( M \) be two mutually dual lattices with the pairing denoted by \( \langle \cdot, \cdot \rangle \). Let \( \sigma \) be a cone in \( N_\mathbb{Q} \), \( \omega \subset M_\mathbb{Q} \) be its dual cone, and \( C \) be a smooth curve. Consider a divisor \( \mathcal{D} = \sum_{z \in C} \Delta_z \cdot z \), whose coefficients \( \Delta_z \) are \( \sigma \)-tailed polyhedra in \( N_\mathbb{Q} \). Such objects are called \( \sigma \)-polyhedral divisors, see Altmann and Hausen (2006). For every \( m \in \omega_M \) we can get the \( \mathbb{Q} \)-divisor \( \mathcal{D}(m) = \sum_{z \in C} \min \langle p, m \rangle \cdot z \). One can define the \( M \)-graded algebra

\[
A[C, \mathcal{D}] = \bigoplus_{m \in \omega_M} A_m x^m, \quad \text{where} \quad A_m = H^0(C, \mathcal{D}(m))
\]

and the multiplication is determined in the natural way. It follows from Altmann and Hausen (2006) that \( A[C, \mathcal{D}] \) is a normal affine domain and that every normal affine \( M \)-graded domain \( A \) with \( \text{tr. deg.} \ A = \text{rk} \ M + 1 \) is equivariantly isomorphic to \( A[C, \mathcal{D}] \) for some \( C \) and \( \mathcal{D} \).

Also we need a description of homogeneous LNDs of horizontal type for \( \mathbb{T} \)-variety \( X \) of complexity one. Below we follow the approach given in Arzhantsev and Liendo (2012). We have \( \mathbb{K}[X] = A[C, \mathcal{D}] \) for some \( C \) and \( \mathcal{D} \). It turns out that \( C \) is isomorphic to \( \mathbb{A}^1 \) or \( \mathbb{P}^1 \) whenever there exists a homogeneous LND of horizontal type on \( A[C, \mathcal{D}] \), see Liendo (2010a, Lemma 3.15). Let \( C \) be \( \mathbb{A}^1 \) or \( \mathbb{P}^1 \), \( \mathcal{D} = \sum_{z \in C} \Delta_z \cdot z \) a \( \sigma \)-polyhedral divisor on \( C, z_0 \in C, z_\infty \in C \{z_0\} \), and \( v_z \) a vertex of \( \Delta_z \) for every \( z \in C \). Put \( C' = C \) if \( C = \mathbb{A}^1 \) and \( C' = C \{z_\infty\} \) if \( C = \mathbb{P}^1 \). A collection \( \widehat{\mathcal{D}} = \{\mathcal{D}, z_0, z_\infty; v_z, \forall z \in C\} \) if \( C = \mathbb{P}^1 \) is called a colored \( \sigma \)-polyhedral divisor on \( C \) if the following conditions hold:

\[
(1) v_{\text{deg}} := \sum_{z \in C'} v_z \text{ is a vertex of deg } \mathcal{D} \ \text{on } C', \Delta_z ; \quad (2) \text{ every } z \in N \ \text{for } z \neq z_0.
\]

Let \( \widehat{\mathcal{D}} \) be a colored \( \sigma \)-polyhedral divisor on \( C \) and \( \delta \subset N_\mathbb{Q} \) be the cone generated by \( \text{deg } \mathcal{D} \ \text{on } C', \delta = 0 \) and \( (v_{z_0}, 1) \) if \( C = \mathbb{A}^1 \), and by \( (\delta, 0), (v_{z_0}, 1) \) and \( (\Delta_{\infty} + v_{\text{deg}} - v_{z_0} + \delta, -1) \) if \( C = \mathbb{P}^1 \). By definition, put \( d \) the minimal positive integer such that \( d \cdot v_{z_0} \in N \). A pair \( \langle \mathcal{D}, v \rangle \), where \( v \in M \), is said to be coherent if

\[
(1) \text{ There exists } s \in \mathbb{Z} \text{ such that } e = (e, s) \in M \oplus \mathbb{Z} \text{ is a Demazure root of the cone } \delta \text{ with distinguished ray } \rho = (d \cdot v_{z_0}, d).
(2) \langle v, e \rangle \geq 1 + \langle v, e \rangle \text{ for every } z \in C' \{z_0\} \text{ and every vertex } v \neq v_z \text{ of the polyhedron } \Delta_z.
(3) d \cdot \langle v, e \rangle \geq 1 + \langle v_{z_0}, e \rangle \text{ for every vertex } v \neq v_{z_0} \text{ of the polyhedron } \Delta_{z_0}.
(4) \text{ If } Y = \mathbb{P}^1, \text{ then } d \cdot \langle v, e \rangle \geq -1 - d \cdot \sum_{z \in Y} \langle v_z, e \rangle \text{ for every vertex } v \text{ of the polyhedron } \Delta_{z_\infty}.
\]
It follows from Arzhantsev and Liendo (2012, Theorem 1.10) that homogeneous LNDs of horizontal type on $A[C, \mathfrak{D}]$ are in bijection with coherent pairs $(\tilde{\mathfrak{D}}, e)$.

Let us return to our case. Following Altmann and Hausen (2006, Sect. 11) we will show how to determine $C$ and $\mathfrak{D}$ such that $X \cong \text{Spec } A[C, \mathfrak{D}]$. Since $T \subset \mathbb{T}$, we have an exact sequence

$$0 \longrightarrow N_T \xrightarrow{F} N_T \xrightarrow{p} N_T / N_T \longrightarrow 0.$$ 

Let $\Sigma$ be the coarsest quasifan in $(N_T / N_T)_Q$ refining all cones $P(\tau)$, where $\tau$ runs over all faces of $\hat{\sigma}_X$. It follows from Altmann and Hausen (2006, Sect. 6) that the desired curve $C$ is the toric variety corresponding to $\Sigma$. Let us choose the projection $s : N_T \rightarrow N_T$ which satisfies $s \circ F = \text{id}$. The divisor $\mathfrak{D}$ on $C$ such that $X \cong \text{Spec } A[Y, \mathfrak{D}]$ is $\mathfrak{D} = \sum_{\rho \in \Sigma(1)} \Delta_{\rho} \cdot D_{\rho}$, where $\Sigma(1) \subset \Sigma$ is the set of one-dimensional cones, $D_{\rho}$ is the prime divisor corresponding to $\rho$, $\Delta_{\rho} = s(\hat{\sigma}_X \cap P^{-1}(n_{\rho})) \subset (N_T)_Q$, and $n_{\rho}$ is the primitive lattice vector on $\rho$. Here all $\Delta_{\rho}$ are $\sigma$-tailed polyhedra with $\sigma = s(\hat{\sigma}_X \cap (F(N_T))_Q)$. In our case

$$F = \begin{pmatrix} r \\ q \end{pmatrix}, \quad s = (u, v) \quad \text{and} \quad P = (q, -r),$$

where $u, v \in \mathbb{Z}, ru + qv = 1$. By direct calculation we obtain $Y = \mathbb{P}^1, \sigma = \mathbb{Q}_{\geq 0}$, and

$$\mathfrak{D} = (p_1 + \sigma) \cdot [0] + (p_2 + \sigma) \cdot [\infty],$$

where $p_1 = \frac{u}{q}$ and $p_2 = \frac{au + bv}{rb - qa}$. In this case there is no homogeneous LND of fiber type on $A[C, \mathfrak{D}]$. Homogeneous LNDs of horizontal type of degree $e$ are in bijection with coherent pairs $(\tilde{\mathfrak{D}}, e)$. If neither $p_1$ nor $p_2$ belong to $\mathbb{Z}$, then we should put either $z_0 = 0$ and $z_\infty = \infty$, or $z_0 = \infty$ and $z_\infty = 0$ in the definition of a colored $\sigma$-polyhedral divisor $\tilde{\mathfrak{D}}$. It means that there exist at most two LNDs of degree $e$. So for a root $e \in \Lambda$ there are two root vectors and they are $\mathbb{T}$-homogeneous. It is clear that $p_1$ is integer if and only if $q$ equals 1. Let us find out when $p_2$ is integer.

**Lemma 3** The number $p_2 = \frac{au + bv}{rb - qa}$ is integer if and only if $rb - qa = 1$.

**Proof** Denote $D = rb - qa$. Suppose $au + bv = kD$, where $k \in \mathbb{Z}$. We also know that $ru + qv = 1$. Using these equations, we get

$$a = aru + aqv = D(rk - v), \quad b = bru + bqv = D(qk - u).$$

So $a$ and $b$ are divisible by $D$. This holds if and only if $D = 1$. 

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Now suppose \( p_1 \notin \mathbb{Z} \) and \( p_2 \in \mathbb{Z} \). Then for any colored divisor either \( z_0 = 0 \) or \( z_\infty = 0 \) holds. It is not hard to check that in the former case the pair \((\widetilde{D}, e)\) is coherent if and only if \( \frac{1}{q} + ep_1 \in \mathbb{Z} \) and \( qep_2 + 1 \geq 0 \). These conditions do not depend on the choice of \( z_\infty \). So if there is one coherent pair \((\widetilde{D}, e)\) with \( z_0 = 0 \), then any pair \((\widetilde{D}', e)\) with \( z_0 = 0 \) and \( z_\infty \in \mathbb{P}^1 \setminus \{z_0\} \) is coherent. Since \( p_2 \in \mathbb{Z} \), we consider all \( \widetilde{D} \) with \( z_\infty = 0 \) as the same colored \( \sigma \)-polyhedral divisor. Let \( e \in \Lambda \). We already know two root vectors of degree \( e \), which are \( \mathbb{T} \)-homogeneous LNDs, whose degrees belong to different \( S_{\rho_i} \). Hence the inequality \( z_0 \neq z'_0 \) holds for corresponding coherent pairs \((\widetilde{D}, e)\) and \((\widetilde{D}', e)\). Thus there are as many root vectors corresponding to \( e \) as points on \( \mathbb{P}^1 \). All these root vectors are described by Proposition 5. The cases \( p_1 \in \mathbb{Z}, p_2 \notin \mathbb{Z} \) and \( p_1, p_2 \in \mathbb{Z} \) are analyzed similarly.

The following figure illustrates the lines \( \Gamma_T \) with \( p_1 \in \mathbb{Z} \) or \( p_2 \in \mathbb{Z} \).

Summarizing, we obtain the following statement.

**Proposition 6** Let \( X \) be an affine toric surface with the acting torus \( \mathbb{T} \) and \( T \subset \mathbb{T} \) be a one-dimensional subtorus. Suppose that the cone \( \hat{\sigma}_X \) corresponding to \( X \) is generated by \( n_{\rho_1} = (1, 0) \) and \( n_{\rho_2} = (a, b) \), where \( a, b \in \mathbb{Z} \geq 0, a < b, \) and \( \gcd(a, b) = 1 \), and the line \( \Gamma_T \) is generated by \((r, q)\), where \( \gcd(r, q) = 1 \). Let \( \pi \) be the restriction of roots from \( \mathbb{T} \) to \( T \), \( e \) be a \( T \)-root, \( \Lambda = \pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) \), and \( D = rb - qa \). Then the following table describes the restriction of roots.

| Condition | The number of root vectors corresponding to \( e \) | The number of \( \mathbb{T} \)-roots having \( e \) as their restriction | Are all \( \mathbb{T} \)-homogeneous LNDs on \( X \) \( \mathbb{T} \)-homogeneous? |
|-----------|-----------------------------------------------|-------------------------------------------------|-----------------------------------------------|
| 1. \( \hat{\sigma}_X \cap \Gamma_T = \{0\} \) | 1 | 1 | Yes |
| 2. \( \hat{\sigma}_X \cap \Gamma_T = \mathbb{Q} \geq 0 \cdot \rho_i \) \( e = -1 \) | An infinite dimensional space | \( \infty \) | No |
| 3. \( \hat{\sigma}_X \cap \Gamma_T \neq \emptyset \) \( e \neq -1 \) | 1 | 1 | Yes |
| 3.1. \( \gcd(q, D) \nmid a - 1 \) | 1 | 1 | Yes |
| 3.2. \( q \neq 1, D \neq 1 \) and \( \gcd(q, D) \mid a - 1 \) | 1 | 1 | Yes |
| 4. \( e \notin \Lambda \) \( e \in \Lambda \) | 1 | 1 | Yes |
| 5. \( e \notin \Lambda \) or \( D = 1 \) | 2 | 2 | Yes |
| 6. \( e \notin \Lambda \) \( e \in \Lambda \) \( \text{As many as points on } \mathbb{P}^1 \) | 1 | 1 | Yes |

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