This paper investigates the method of obtaining strong coupling regime expansions for a set of Euclidean Quantum Field Theories on a general lattice, illustrated through a self-interacting scalar field $g^4\phi^4/4!$ on a cubic lattice of arbitrary dimensions. A duality is established between the strong coupling regime of this theory and the weak coupling regime of a corresponding dual theory. For Ising model, it corresponds to other choice of field-theoretical description of the same spin partition function. While the original theory is local, its dual counterpart is non-local. Using Feynman diagrammatic techniques for the dual theory, expansions for the two-point correlator and the free energy per site in the region of large and medium coupling constants $g$ are derived. These expansions appear to be regular as $g \to 0$ and they require no cumbersome calculations for derivation. Furthermore, they demonstrate sufficiently rapid numerical convergence in the regions of large and medium coupling values. Numerical analysis in dimensions $d = 2$ and $d = 3$ shows agreement with analytical results from Monte Carlo simulations. Moreover, the strong coupling regime expansions are consistent with traditional weak coupling expansions. Unfortunately, this duality does not trivially extend to continuous theories, hence they are not considered in this work.
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1 Introduction

Quantum Field Theory (QFT) stands as the bedrock of theoretical physics, providing a mathematical framework for the understanding of fundamental particles and their interactions. It combines principles of quantum mechanics and special relativity, offering a comprehensive description of the dynamics of fields permeating space and time. This intricate theoretical framework has proven to be a cornerstone in elucidating the behavior of matter at the smallest scales and has found applications in a myriad of physical phenomena. In this paper we are focusing on the scalar Euclidean Quantum Field Theory on a lattice, developing some new techniques to obtain strong coupling decompositions.

The scalar field in QFT captures our interest for several reasons. As one of the simplest and most fundamental fields, the scalar field serves as a testing ground for understanding key concepts and phenomena in QFT. Moreover, it plays a crucial role in the description of various physical systems, providing a foundational understanding of the principles that govern particle interactions. By scrutinizing the scalar field, one aims to unravel deeper insights into the broader landscape of quantum field theories.

As for the lattice field theories, we restrict our attention to them in this paper because here the notion of Path Integral is mathematically well-defined (at least for the case of finite lattices). As a result, one can perform some operations in strict rather than formal way and get more reliable results, compared to the great variety of incorrect formal transformations of Path Integral in continuous space. Besides, when speaking about lattice field theory, one can always justify the results using numerical simulations. This wouldn’t be so simple in the case of continuous theories, because we have no any direct way of numerical computation of Path Integrals in continuous space. And therefore we have to do it using some lattice simulations with the subsequent scaling of parameters or some kind of extrapolation in lattice spacing parameter. This is a powerful method, however, one faces the problem of quantum triviality during its implementation. And there is always a question whether a problem in a continuous theory we try to approximate or in the chosen way of parameters scaling. We definitely don’t want to treat these couple of complicated problems in this paper with regard to the latter obtained strong coupling decomposition and lattice theories duality. So we will consider in this paper lattice field theories and theirs alone.

Lattice field theories are the objects of independent interest, since, in fact, for the majority of condensed matter models, formulated in terms of creation and annihilation operators, can be written the explicit coordinate action, taking such form after the second quantization. For instance, the latter considered lattice \( \phi^4 \) model after second quantization becomes a well-known Hamiltonian with four-particle interaction. And the finiteness of a Brillouin zone means is equivalent to the fact that we have the sums over coordinates rather than integrals. Though when writing actions in coordinate form in condensed matter it is commonly convenient to replace sums with integrals using Euler-MacLaurin Summation formula, for our purposes it is more harmful rather than useful.

Moreover, we fix the lattice spacing and won’t speak about renormalization problems, expressing everything in via bare parameters of an action. It is natural, since all integrals are finite. And from the point of view of the authors, it is still physically interesting, because we can directly relate microscopic and macroscopic parameters of a system with no need to introduce any "dressed" qualities. The research of Renormalization Group Flow is definitely the important question in lattice theories also, but it is excessive for our problem statement.

Although this problem statement might appear overly restrictive, it is not. And we intentionally have chosen such background to testify the provided results in the environment with minimum of the additional artifacts, which could possibly influence the results in some way.

So let us discuss Lattice Euclidean QFT for scalar field with more intensity. The most widely spread technique for calculating correlation functions is Feynman Perturbation Theory, obtained by decomposition of an exponent with interaction in power series. This usually raises a series in positive powers of coupling constant, which are often diverge and appear to be only the asymptotic expansions. Commonly they diverge quite aggressively, which make us using the resummation techniques, e.g. Borel resummation. We will refer to the described series as Weak Coupling Perturbation Series in this paper.

In opposite to Weak Coupling Perturbation Series, there were some attempts to build strong coupling decompositions, or, at least, to extend the weak series to the region of "middle" values of coupling constant. They include (we also mention the methods applied to continuous theories, since it is usually more general case):

1. The various types of Borel resummations, for instance \([16, 15, 15, 17, 8, 9]\). However, Borel resummations necessitate extensive technical computations and rigorous analytical scrutiny. Sometimes these resummations result in series in inverse coupling or its roots, which exhibit aggressive behavior at low coupling strengths. Besides, it is usually not clear how to get the final answer in terms of bare theory parameters. which could be also interesting sometimes;
2. The method of "taking out" the kinetic part (rather than the potential, like in standard Feynman Technique) outside the exponent \[7\], which is the method the most close to the one provided in this paper. Some decompositions for finite momentum cut-off obtained then and some attempts to approach the continuum limit are made. However, the terms of the obtained series grow rapidly when the coupling constant goes to zero;

3. The previous method, combined with the use of some instantonic solutions \[20, 21, 19\]. This instantonic solution, though, is selected somewhat arbitrarily, raising legitimate concerns about how the results might vary if other potential solutions were considered. An additional drawback of this method is the complexity involved in deriving higher-order terms of decomposition in inverse powers of the coupling constant. Yet, despite these challenges, this approach is viable, yielding results that are both qualitatively and, following some numerical tuning of renormalized parameters, quantitatively accurate. Finally, the author has succeeded in passing to the continuous limit, which is a remarkable result;

4. Ising-like decompositions in spirit of Kramers-Wannier duality \[3, 12, 1, 6, 2\]. It is a common and acceptable approach, however, Ising-like decompositions demand a sophisticated spatial comprehension in higher dimensions. Moreover, they are not applicable to the models with other form of partition function or demand essential modification;

5. Some Pade-Approximant techniques, giving the expressions, suitable for larger couplings \[16, 5\]. Though these expansions can obey some nice properties and give reasonable results, this solution is more computational than physical and also based on extrapolation techniques that are inherently unreliable;

and so on.

In contrast, the methodology presented in this paper amalgamates the advantages of the aforementioned techniques while circumventing their principal shortcomings. It is straightforward and easy to implement, it can be established in a 'canonical' manner, and delivers effective approximations for both middle and strong coupling regimes, maintaining finite values even at lower coupling levels. Importantly, this method eliminates the necessity to select any particular classical solution for the instantonic approach, obviating the need to aggregate all instantonic contributions. The terms of our derived decompositions are regularized for small couplings, enhancing their convergence within the medium coupling range. Furthermore, our numerical experiments confirm robust convergence across an expansive interval of coupling constants. Ultimately, this paper offers not merely a perturbation series but establishes a duality between lattice theories — a significantly potent outcome. Hence, it paves the way for the potential development of saddle-point techniques in the dual theory, alongside the exploration of other established methods. As for the weaknesses of the provided approach, they contain include the following points:

1. The physical sense of the obtained duality is not very clear, though one can try to make some speculations on it, looking at Ising model case (Appendix \[A\]). Notably, for Ising model it corresponds to some other choice of field-theoretical description of spin partition function.

2. For any finite number of terms, it could give finite, but irrelevant results for couplings small enough (reflecting, for instance, in the negative free energy predicted value). However, it still converges better than the other approaches mentioned above;

3. This duality should be treated currently only as a duality of lattice rather than continuous theories. And for the authors it is still an open question if it is possible to extend this duality to the continuous way and obtain results consistent with Renormalization Group Theory results, such as quantum triviality and existence of fixed points in different dimensions.

The primary objective of this paper is to present a novel perspective on an already well-examined theory, and the authors hope that the approach discussed may prove beneficial to some applications.

This paper is structured as follows. In this section, we have given the brief introduction into the field of research. In the second section we will give the necessary definitions and introduce the concepts we will use throughout the paper. Then we will describe the general construction of the duality between lattice field theories, which will be used for the obtaining of strong coupling decomposition. Finally, we apply it to the case of power potentials. In section 3 we will derive strong and weak coupling decompositions for a free energy per cite \[28\]. Obviously, the weak coupling decomposition is not a new result, but we will need it further for verification of numerical simulations and strong coupling decompositions. Section 4 will be devoted to the numerical simulations. We start from describing the essence of the Hamiltonian Monte Carlo method we use to compute correlators numerically. Then we present the results of simulations and their comparison with the obtained weak and strong coupling decompositions. In the fifth section we will provide analytical forms of strong and weak coupling decompositions, as well as compare them with the numerical results. Finally, in section 6 we will make a discussion of the obtained results and give some prospects in section 7.
2 General description of duality between lattice theories

2.1 The concepts and definitions used throughout the paper

In this paper we are going to consider only lattice field theories with finite cell size \( l \) and with the total number of cites \( N \). We won’t treat the problem of renormalization of the considered theories as well as a problem of passing to the continuous limit. All the provided results refer only to lattice field theories, being a separate object of interest, as it was explained in the previous section. However, in the sections 6 and 7 we will give some prospects and criticism of the application of the presented approach to the continuous field theories. However, we found it useful and convenient to consider a thermodynamical limit where it is possible, which resides in passing to the infinite number of cites \( N \to \infty \) keeping \( l = \text{const} \).

Moreover, during this paper free energy per site will be our main object of interest. We made this choice since it is a relatively simple quantity for both analytical and numerical calculations. However, the presented method can be also applied to correlators of any order straightforwardly. We will formulate the provided approach in the most general form, and then apply it to the simple case of lattice \( \phi^4 \), defined in the following.

2.1.1 Lattice description and introduction of the notations

As for the Lattice Field Theory, the monograph [14] can be a bright introduction. However, for our problem statement we will use slightly different notation from the traditional ones in Lattice Field theory, which are more close to the continuous form. Let us consider the lattice of general form, which will be some connected graph \( G \) (which we usually consider to be a lattice) with the set of vertices \( V \) and the set of edges \( E \). In this paper, we restrict ourselves to the case of cubic \( d \)-dimensional lattice. However, the described approach could be directly extended to the majority of lattices types, so we will describe it in this section in a more general form.

For the case of a cubic \( d \)-dimensional lattice with \( M \) cites along each dimension, we are able to write the set of vertices as:

\[
x \in V = \{1, \ldots, M\}^d,
\]

We will denote \( N = |V| \), which is \( M^d \) for the case of \( d \)-dimensional cubic lattice. In all summations and products we will use the following conventions:

\[
\sum_x := \sum_{x \in V}, \quad \prod_x := \prod_{x \in V}.
\]

Let \( J = (J_{x,x'}) \) be the adjacency matrix of the considering graph, and \( L = (L_{x,x'}) \) be some matrix, defining the interactions of sites with each other. We will consider \( J \) and \( L \) as operators on a space of functions defined on graph vertices. Under such a consideration, \( (J_{x,x'}) \) and \( (L_{x,x'}) \) are nothing else but the matrices of these operators in a frame of delta-functions, located in one cite. We also impose \( L \) to be a positive operator, i. e. all its eigenvalues have to be greater than zero, which is necessary for the convergence of the integrals we will write.

When \( G \) is a lattice, we will require \( L \) to be translational-invariant and symmetric, which means that:

\[
L_{x+a,x'+a} = L_{x,x'}, \quad L_{x,x'} = L_{x',x},
\]

for any \( x \) and \( x' \) from \( V \) and any \( a \) from the space of admissible translations. Let us note that this causes the imposition of periodic boundary conditions, i. e. identifying the opposite sites. After that, both these conditions are also satisfied automatically for adjacency matrix \( J \).

Let us also introduce the notion of real scalar product between two lattice functions. Namely, here we understand angle brackets as:

\[
\langle \psi, \phi \rangle = \sum_x \phi(x)\psi(x),
\]

which makes \( L \) and \( J \) operators both self-adjoint. Under the imposed conditions, operator \( L \) is diagonalizable. We will denote its set of eigenvalues and corresponding orthonormal frame of eigenvectors as \( \{\lambda_k\}_{k \in \mathbb{P}} \) and \( \{\epsilon_k(x)\}_{k \in \mathbb{P}} \) for some set of indices \( \mathbb{P} \), enumerating all the states. We will usually mark these eigenvalues and corresponding eigenvectors with indices \( k \), \( p \), and \( q \), motivated by the fact that they are enumerated with the admissible momenta in the case of lattices. In this way, we will denote the set of admissible momenta as \( \mathbb{P} \) for the case of lattices and will refer
to it as a Brillouin zone. Besides, due to the translational and reflection symmetries of the lattice, we consider $\mathbb{P}$ to be inversion-symmetric, which means the parity of the spectrum of $L$.

Similarly to the case of coordinates, we will follow the conventions:

$$\sum_k := \sum_{k \in \mathbb{P}}, \quad \prod_k := \prod_{k \in \mathbb{P}},$$

(5)

and the same for indices $p$ and $q$. Summarizing, we have for the eigenvectors:

$$(L e_k)(x) = \lambda_k e_k(x), \quad k \in \mathbb{P}.\tag{6}$$

Moreover, they satisfy the completeness relation:

$$\sum_k e_k(x)e_k(y) = \delta_{x,y},$$

(7)

as well as the orthogonality and normalization condition:

$$\sum_x e_k(x)e_p(x) = \delta_{k,p}.\tag{8}$$

Let us underline that the real orthonormal frame of eigenvectors has to exist since $L$ is symmetric and corresponds to sine and cosine solutions in terms of Fourier transform. This frame will be convenient for deriving the duality in the section 2.2, but in all calculations we will use the frame with different normalization, namely:

$$\sum_x e_k(x)e_p(x) = \delta_{k+p,0}.\tag{9}$$

This normalization is possible due to the parity of the Brillouin zone $\mathbb{P}$ and in terms of Fourier transform corresponds to the frame of complex exponents. In all places where it is not clear, we will specify explicitly what normalization we are considering in this place.

Along with a frame of $L$ eigenvectors, there is another natural orthonormal frame in a space of lattice functions, which is the set of all cite delta-functions:

$$\{\delta_y(x) := \delta_{x,y} \}_{y \in \mathbb{V}}.\tag{10}$$

Let us also note that a transirion matrix $U$ from frame of eigenvectors to a frame of delta-functions is orthogonal, since they both are orthonormal. This means, in particular, that:

$$|\det U| = 1.\tag{11}$$

Together with the interaction operator $L$ it is useful to consider its inverse $G$, which is natural to refer as a Green function:

$$G = L^{-1},\tag{12}$$

and in terms of the introduced eigenvalues and eigenfunctions they have the representations:

$$G_{x,y} = \sum_k \frac{e_k(x)e_k(y)}{\lambda_k}, \quad L_{x,y} = \sum_k \lambda_k e_k(x)e_k(y).\tag{13}$$

Let us note that in quantum-mechanical notations, Green function can be rewritten as:

$$G = \sum_{|\vec{p}|} \frac{|\vec{p}| \langle \vec{p} |}{\lambda_{\vec{p}}},\tag{14}$$

where we have denoted $|\vec{p}| = \frac{1}{\sqrt{N}} e^{i(p_k \cdot x)}$. 

6
One also can perform the decomposition of any lattice functions in terms of $L$ operator eigenvectors, namely:

$$\phi(x) = \sum_k \phi_k e_k(x), \quad \phi_k = \sum_x \phi(x) e_k(x),$$  \hspace{1cm} (15)

and we fix the notation $\phi_k$ for the component of field $\phi(x)$ along the eigenvector $e_k(x)$. Clearly, it is nothing else but the Discrete Fourier Transform on the Lattice. Though, in the following part of the paper under the Discrete Fourier Transform we will mean the similar relation but with other normalization, since the chosen modes $\phi_p$ die out in the thermodynamic limit $N \to \infty$. This other definition of Discrete Fourier Transform will be discussed further.

### 2.1.2 Classical field theory description

Now let us consider the action of the following form:

$$S[\phi] = \frac{1}{2} \sum_{x,x'} L_{x,x'} \phi(x)\phi(x') + \sum_x V(\phi(x)), \quad V(\phi) = \frac{1}{(2\pi)^d} \int^2 \phi^{2n} \phi^{2n},$$  \hspace{1cm} (16)

for some integer $n > 1$. Such action is exactly the sum of «free» quadratic part, defined through the operator $L$, and some interaction, defined as a sum of the potential values over all sites. Let us note that, unlike usual continuous theories, we define action with summations rather than integrations. Such models do have a sense, as an example of Ising’s model shows. Let us also note that the free part of this action can be rewritten in terms of the introduced scalar product:

$$\frac{1}{2} \sum_{a,b} L_{ab} \phi(x_a)\phi(x_b) = \frac{1}{2} \langle L\phi, \phi \rangle.$$  \hspace{1cm} (17)

For the practical purposes, it is useful to consider the operator $L$ in the form of lattice Laplacian:

$$L = -\alpha \Delta + \gamma,$$  \hspace{1cm} (18)

with some coefficients $\alpha$ and $\gamma$, which we impose to be such that $L$ was a positive operator. Usually, it means $\alpha > 0$. Also, the values $\gamma < 0$ can lead to infrared divergence. In this paper, we will consider the cases $\alpha < 0$ or $\gamma < 0$ only from the point of view of analytical continuation.

Finally, for the considering case of cubic $d$-dimensional lattice, Laplacian has the form:

$$(\Delta f)(\vec{r}) = \sum_{j=1}^d \left( f(\vec{r} + \vec{e}_j) - 2f(\vec{r}) + f(\vec{r} - \vec{e}_j) \right),$$  \hspace{1cm} (19)

where $\vec{e}_j$ is $j$th vector of an orthonormal basis, corresponding to the considering cubic lattice with unit size of lattice cell in each direction. This formula is a nothing else but a finite difference approximation of continuous Laplacian.

We intentionally omit the lattice scale $l$ since in this paper we are not going to scale lattice size, considering only thermodynamic limit, and also would like to simplify the formulas. If needed, this length parameter can be recovered with little efforts.

For periodic boundary conditions on a cubic lattice with $M$ cites in each dimension, the discrete Laplacian \[a\] eigenvalues $\{\lambda_k(\Delta)\}$ and eigenvectors $\{e_k(x)\}$ has the following form \[b\]:

$$\lambda_k(\Delta) = -4 \sum_{j=1}^d \sin^2 \left( \frac{\pi j}{M} \right), \quad \vec{p}_k = \frac{2\pi \vec{k}}{M}, \quad \vec{k} \in \{0, \ldots, M-1\}^d, \quad e_k(x) = \frac{1}{\sqrt{N}} e^{i(p_k \cdot x)},$$  \hspace{1cm} (20)

which is derived in the Appendix\[x\]. Moreover, in the same Appendix, there are given some other motivation to consider an interaction in the form of Laplacian (or inverse Laplacian).

Summarizing, the eigenvalues and eigenvectors of the considering $L$ operator have the form:

$$\lambda_k = \gamma + 4\alpha \sum_{j=1}^d \sin^2 \left( \frac{\pi j}{M} \right), \quad \vec{p}_k = \frac{2\pi \vec{k}}{M}, \quad \vec{k} \in \{0, \ldots, M-1\}^d, \quad e_k(x) = \frac{1}{\sqrt{N}} e^{i(p_k \cdot x)},$$
with the same eigenvectors as for lattice Laplacian.

2.1.3 Statistical field theory description

Now, let us write the definition of partition function in external field \( j(x) \) we are going to consider with action \( 16 \):

\[
Z[j] = \int_{\mathbb{R}^N} \frac{\prod_k d\phi_k}{\sqrt{\pi^N} \det G} \exp \{-S[\phi] + \langle j, \phi \rangle\},
\]

(21)

which is inspired by Ising or XY-models [23] or just common physical sense, because we should integrate over all system’s states \( \mathbb{R}^V \). In this paper we will concentrate majorly on free energy density, which means \( j = 0 \), but for generality we provide a form of duality for general \( j \).

Also, as it has already been noted, the jacobian of transition from a frame of eigenvectors to a frame of cite delta functions equals to one, so:

\[
\prod_{k \in \mathcal{P}} d\phi_k = \prod_{x \in \mathcal{V}} d\phi(x),
\]

(22)

and both these products are finite. As a result, we can write the path integral both in terms of integration over function values at cites or over all possible values of its modes. All these expressions are well-defined rather than formal, since we will conduct all the transformations for the case of finite number of nodes, and will pass to thermodynamic limit only in correlators or free energy density.

This partition function can also be written in usual path-integrals notations, where the rigorous definition of ”measure” was given above:

\[
Z[j(x)] = \int \mathcal{D}\phi(x) e^{-S[\phi] + i\langle j, \phi \rangle},
\]

(23)

For the given partition function, we understand the correlation functions or, equivalently, correlators as:

\[
\langle \phi(x_1) \cdot \cdot \cdot \phi(x_k) \rangle = \frac{1}{Z[0]} \int_{\mathbb{R}^N} \frac{\prod_k d\phi_k}{\sqrt{\pi^N} \det G} \phi(x_1) \cdot \cdot \cdot \phi(x_k) \exp \{-S[\phi]\},
\]

(24)

which is a common definition for Quantum Field Theory [23, 13].

Let us underline, that we have chosen the special normalization, such that \( Z = 1 \) for \( j = 0 \) and \( g = 0 \). It is not only a convenience, but a necessary step to consider further the thermodynamic limit. Also, for the case of free theory, when \( V(\phi) = 0 \), the partition function can be easily calculated:

\[
Z_0[j(x)] = \exp \left( -\frac{1}{2} \sum_{x,x'} j(x)G_{x,x'} j(x') \right)
\]

(25)

Besides, the kinetic part in \( Z[j(x)] \) can be also rewritten in terms of modes of \( \phi(x) \) in a frame of \( L \) eigenvectors:

\[
Z[j(x)] = \int_{\mathbb{R}^N} \frac{\prod_k d\phi_k}{\sqrt{(2\pi)^N} \det G} e^{-\frac{1}{2} \sum_k \lambda_k \phi_k^2 + i \sum_k j_k \phi_k - \sum_x V(\phi(x))}
\]

(26)

Let us underline that we have expressed the kinetic and potential parts of the action in terms of different coordinates (values in cites and modes) deliberately rather than accidentally. Namely, this form will clarify the way we construct the duality between lattice field theories in section 2.2. Even more precisely, it hints that both kinetic and potential part can be made ”diagonal”, but in different frames, the frames of \( L \) eigenvectors and cite delta functions correspondingly.

We will also assume that the integrals \( \int_{\mathbb{R}} d\phi e^{-V(\phi)} \) converge.

Moreover, throughout the paper we will need the notion of a free energy, which we will identify up to a sign with a Generating Functional of Connected Green Functions. Namely, we define Free Energy Functional \( \mathcal{F}[j(x)] \) as:
Duality between lattice theories

\[ Z[j(x)] = \exp(-\mathcal{F}[j(x)]). \]  

(27)

Obviously, if one will use Feynman diagrams for calculating \( Z[j(x)] \), then the value of \( \mathcal{F}[j(x)] \) will be given by contribution of connected diagrams only. We are also interested in free energy density per cite in thermodynamical limit, which we define as:

\[ f = -\lim_{N \to \infty} \frac{\mathcal{F}[0]}{N}. \]  

(28)

We won’t prove analytically that this limit exists and will be satisfied with numerical substantiation of its finiteness. As an addition verification, let us note the fact that every term in both strong and weak perturbation theory we deduce later gives the finite contribution to \( f \).

### 2.1.4 Free theory Green function

During calculation, we will restrict our attention to the translational invariant case and will denote \( G_{x,x} = G_0 \) and \( G_{x,0} = G_x \). Generally, due to the translational invariance of the considered type of propagators, one can write that \( G_{x+a,y+a} = G_{x,y} \). So we can introduce the propagator as a function of one variable as:

\[ G(x-y) = G_{x-y} = G_{x,y}. \]  

(29)

We can write down a more convenient expression for Green function, assuming that cite number \( N \gg 1 \), as it is usually done in lattice field theories [14]:

\[ G(x-x') = (L^{-1})_{x,x'} = \frac{1}{(2\pi)^d} \left( \prod_{i=1}^d \int_0^{2\pi} dq_i \right) \left( 4\alpha \sum_{j=1}^d \sin^2 \left( \frac{q_j}{2} \right) + \gamma \right)^{-1} e^{-i(q \cdot (x-x'))}. \]  

(30)

This equality can be obtained from [13] with a substitution of explicit eigenvalues and eigenvectors of lattice Laplacian [20] with subsequent application of Euler-MacLaurin summation formula. For the sake of convenience, we will also use the following notation for the Fourier Transform of \( G_{x,x'} \) in sense of (37), which we will denote as \( G(q) \):

\[ G(q) = \frac{1}{(2\pi)^d} \left( 4\alpha \sum_{j=1}^d \sin^2 \left( \frac{q_j}{2} \right) + \gamma \right)^{-1}, \]  

(31)

which will be distinguished from \( G(x-y) \) by the notation of its argument. Namely, we will continue denoting the coordinates with Latin letters \( x, y, z \), and will use letters \( p, q, k \) for the lattice momenta.

### 2.1.5 Conventions about Fourier Transform

Finally, let us specify the notion of Fourier Transform we are going to use in this paper. Namely, we will follow the normalization:

\[ F[f(x)](p) := \int_{\mathbb{R}} dx \, f(x) e^{-ipx}, \]  

(32)

for the direct Fourier Transform, and, consequently:

\[ F^{-1}[f(p)](x) := \int_{\mathbb{R}} \frac{dp}{2\pi} f(p) e^{ipx}, \]  

(33)

for the inverse Fourier Transform. We will also need Plancherel identity. Let us write it down explicitly for the chosen normalization. Let us take two functions: \( f(x) \in L_2(\mathbb{R}) \) and \( g(x) \in L_2(\mathbb{R}) \). One can define their scalar product in the following way:

\[ \langle f, g \rangle = \int_{\mathbb{R}} dx \, f(x) g^*(x). \]  

(34)
Then according to the fact that Fourier transform is the unitary (up to constant multiplier) map, we result in Plancherel Identity:

\[ \langle f, g \rangle = \int_{\mathbb{R}} \frac{dp}{2\pi} F[f(x)](p) F^*[g(x)](p), \]  

(35)

and similarly for multidimensional integrals:

\[ \langle f, g \rangle = \int_{\mathbb{R}^N} \frac{d^Np}{(2\pi)^N} F[f(x)](p) F^*[g(x)](p). \]  

(36)

We would like to emphasize that an application of this equality is the simplest known to authors way of deriving the desired strong-weak duality, so we apply it shortly.

Now we are going to make a definition of a Discrete Fourier Transform. Let us note once again that we don’t satisfied with the definition (15), because the coefficients \( \phi_k \) of a function \( \phi(x) \) die out in the thermodynamic limit \( N \to \infty \). So, for any lattice function \( h(x) \), we can write down the expression for its Fourier Transform \( h(p) \) in the following form:

\[ h(x) = \frac{1}{(2\pi)^dN} \sum_p h(p)e^{ipx}. \]  

(37)

This formula is equivalent to:

\[ h(p) = \sum_x h(x)e^{ipx}, \]  

(38)

which is a direct recipe of a calculation of the discrete Fourier Transform \( h(p) \) from the given set of coordinate values \( h(x) \). In the thermodynamic limit one can also write:

\[ h(x) = \int_{[0;2\pi]^d} \frac{d^dp}{(2\pi)^d} h(p)e^{-ipx}, \quad N \to \infty, \]  

(39)

which is a desired convenient formula, where all the terms stay finite in the prescribed limit. Let us mark that we can always distinguish the function \( h(x) \) from its Discrete Fourier Transform \( h(p) \) by the notion of their arguments. For the functions themselves we will use the Latin “coordinate” letters, such as \( x, y, z \), etc., and the letters \( p, k, q \), etc. for the lattice momenta. This notation also differs from the notation of Fourier Modes (15) \( h_p \), linked to the new definition as \( h_p = \frac{1}{\sqrt{N}} h(p) \). So, since it is also commonly accepted in physical literature, we will use it throughout the paper.

Finally, let us underline that the all above about Fourier Transform can be also applied to correlation functions (24) themselves. For our purposes in would be sufficient to note that the correlation function \( \langle \phi(x)\phi(y) \rangle \) is translational invariant for the considering theories, which means:

\[ \langle \phi(x)\phi(y) \rangle = \langle \phi(0)\phi(y-x) \rangle. \]  

(40)

So we will denote the Fourier transform of a correlator as:

\[ \langle \phi(p)\phi(-p) \rangle = \sum_x \langle \phi(0)\phi(x) \rangle e^{ipx}, \]  

(41)

in alignment with (38). The arguments in the left-hand side \( p \) and \( -p \) are opposite because of the translational invariance of a correlation function (24). So if one decides to find its two-dimensional Fourier Transform, both in \( x \) and \( y \), unlike we did, the only nonzero values will be the ones from (41).

2.2 Construction of duality between general lattice theories

The core idea of our discussion is the duality in general lattice theories, which can be derived in two different ways. One method (see Appendix) involves modifying the kinetic operator and using certain transformations to achieve a dual action. This approach is advantageous because it provides a clear physical understanding of the theory’s construction.
However, it does complicate the calculations. Alternatively, using the Plancherel Identity (36), we can streamline the calculations and clarify the mathematical framework of the dual theory, though its physical significance becomes less clear.

We are going to look to the exponent as a product of exponentials of potential and kinetic part of action. More precisely, let us write:

\[ \exp (-S[\phi] + i \langle j, \phi \rangle) = \exp \left( -\frac{1}{2} \langle L \phi, \phi \rangle + i \langle j, \phi \rangle \right) \exp \left( -\sum_x V(\phi(x)) \right), \]

(42)

And now we are going to apply Plancherel identity (36) to partition function (21). Then we will use the fact that Fourier Transforms of exponents of kinetic and potential parts separately both could be computed without devoting much efforts. It is so, since kinetic part is diagonalizable in a basis of Fourier modes of quadratic part operator, as well as a potential part is “diagonal” in basis of coordinate delta-functions (10). Obviously, these both parts are not likely to be diagonalized simultaneously, but since one has to compute their Fourier Transforms independently, that’s doesn’t matter.

So in terms of functions from (36), we have:

\[ f(\{\phi_k\}) = e^{-\frac{1}{2} \sum_k \lambda_k \phi_k^2 + i \sum_k j_k \phi_k} \quad \text{and} \quad g^\ast(\{\phi_k\}) = e^{-\sum_x V(\phi(x))}, \]

where \( \{\phi_k\} \) is the set of all coordinates of a field \( x \) in frame of eigenvectors \( \{e_k(x)\}_{k \in \mathbb{P}} \) of the operator \( L \). Let us denote the set of Fourier variables, corresponding to \( \{\phi_k\} \) as \( \{\psi_k\} \) Now we can apply Plancherel Identity for each integration over \( \phi_k^\ast \):

\[ Z[j(x)] = \int_{\mathbb{R}^N} \frac{\prod_k d\psi_k}{(2\pi)^{\frac{3N}{2}} \det G} F\left[ f(\{\phi_p\}) \right]\left( \{\psi_p\} \right) F\left[ g(\{\phi_p\}) \right]\left( \{-\psi_p\} \right). \]

One can calculate Fourier transformations separately. For the kinetic part we receive:

\[ F\left[ f(\{\phi_p\}) \right]\left( \{\psi_p\} \right) = (2\pi)^{N/2} \sqrt{\det G} e^{-\frac{1}{2} \sum_k (\mu_k - \psi_k)^2 \psi_k}, \]

(43)

taking Gaussian integral. As for the potential part, to calculate its Fourier Transform, one should firstly change coordinates from frame of Fourier modes \( \{e_k(x)\}_{k \in \mathbb{P}} \) to frame of coordinate “delta-functions” \( \{\psi_k\} \). Then the coordinates become \( \{\phi(x)\}_{x \in \mathbb{V}} \) instead of \( \{\phi_k\}_{k \in \mathbb{P}} \). So we obtain:

\[ F\left[ g(\{\phi_p\}) \right]\left( \{\psi_p\} \right) = |\det U| \prod_x \int_{\mathbb{R}} d\phi(x) e^{-V(\phi(x))-i \sum_x \psi(x)\phi(x)}, \]

(44)

where \( U \) is a transition matrix. But it is orthogonal, according to (11), so \( \det U = \pm 1 \). Hence, introducing dual potential \( \tilde{V}(\phi) \) by:

\[ \tilde{V}(\psi) := \int_{\mathbb{R}} d\phi e^{-V(\phi)-i\phi\psi}, \]

(45)

we finish at:

\[ F\left[ g(\{\phi_p\}) \right]\left( \{\psi_p\} \right) = \prod_x \tilde{V}(\psi(x)) = e^{\sum_x \ln[\tilde{V}(\psi(x))]}, \]

(46)

Here we understand logarithm of possibly negative function in sense of taking some branch, coinciding with real logarithm on positive real axis. Since we write a branch in the exponent, its choice does affect anything. So we would like to choose the main branch with zero phase on positive real axis. The position of cutoff is not essential, one may think that it is a ray \([0; -i\infty]\). Let us note that the Taylor series of \( \ln[\tilde{V}(\psi)] \) can have a finite radius of convergence.

However, after exponentiation one should obtain the series with infinite convergence radius, since \( \tilde{V}(\psi) \) is an entire function.

Finally we obtain a dual form of a partition function:

\[ Z[j(x)] = \int_{\mathbb{R}^N} \frac{\prod_k d\psi_k}{2\pi} e^{-\frac{1}{2} \sum_k (\mu_k - \psi_k)^2 \psi_k + \sum_x \ln[\tilde{V}(\psi(x))]}, \]

(47)

11
or, equivalently:

\[ Z[j(x)] = \int_{\mathbb{R}^n} \prod_k \frac{d\psi_k}{2\pi} \exp \left\{ -\frac{1}{2} \langle G \cdot (\psi - j), (\psi - j) \rangle + \sum_x \ln \left[ \tilde{V} (\psi(x)) \right] \right\}, \quad (48) \]

where, recall, \( G = L^{-1} \) and \( \cdot \) has been inserted to underline that \( G \) is an operator acting on fields rather than some their function. In the sake of additional verification of the obtained formula, in Appendix B we propose another way of derivation with the same result. Let us denote the expression in the exponent without minus by \( S^* [\psi] \) and call it the dual action:

\[ S^* [\psi] = \frac{1}{2} \langle G \cdot (\psi - j), (\psi - j) \rangle - \sum_x \ln \left[ \tilde{V} (\psi(x)) \right] \]  

(49)

Let us note that the dual action \( S^* \) for local action \( S \) is essentially non-local: the operator \( G \) is an integral (lattice) operator unlike \( L \), which is a finite difference approximation of a derivative. The first and main question that arise immediately after justification of this duality - does it admit continuous limit, or one also see the kismet of quantum triviality in this approach. The answer is yes, at least for the naive approach, its continuous limit is trivial in some sense, but we won’t discuss it in this paper in more details.

### 2.3 Application of duality to the case of a scalar field with an even power-law self-action

Now we are going to consider interactions of the form:

\[ V(\phi) = \frac{1}{(2n)!} g^{2n} \phi^{2n}, \]

(50)

as the simplest non-trivial and quite widely-spread. In this section we present the results for the general power, but in the following we will focus on \( \phi^4 \) theory.

So the dual potential (45) has the form:

\[ \tilde{V}_g(\psi) := \int_{\mathbb{R}} d\phi \, e^{-g^{2n} \phi^{2n} / (2n)! - i\phi\psi}. \]

(51)

Here we renamed the potential (45) from \( \tilde{V} \) to \( \tilde{V}_g \) to underline that it has nontrivial dependence on the coupling constant, and since it would be more convenient to reserve this notation for slightly another function for the case of poser potentials. This integral can be calculated exactly in terms of Generalized Hypergeometric functions. Though, we won’t need the exact expression, so we restrict ourselves to the needed Taylor coefficients. Let us also make the potential dimensionless, namely:

\[ \tilde{V}_g(\psi) := \frac{1}{g} \tilde{V} \left( \frac{\psi}{g} \right), \quad \tilde{V}(\psi) = \int_{\mathbb{R}} d\phi \, e^{-\phi^{2n} / (2n)! - i\phi\psi}, \]

(52)

and further in this paper we will understand \( \tilde{V} \) exactly in the sense on (52). This dual potential \( \tilde{V} \) can be expressed explicitly in terms of generalised hypergeometric functions, but we won’t need any of these formulas, so we won’t present them here. Besides, it is instructive to plot the graphs of \( \tilde{V} \) and \( \ln |\tilde{V}| \). Both these plots can be found in the figure 2.3.3.

So the partition function now takes the form:

\[ Z = \frac{1}{g^N} \int_{\mathbb{R}^n} \prod_x \frac{d\psi_x}{(2\pi)^N} \exp \left\{ \frac{1}{2} \sum_{x,x'} \psi_x G_{x,x'} \psi_{x'} + \sum_x \ln \left[ \tilde{V} (\psi_x) \right] \right\}. \]

(53)

Finally, let us note that the application of the considered duality to Ising Model (viewed as lattice field theory using the Hubbard-Stratonovich Transform [23]) is equivalent to some other chose of the quadratic operator. This example is considered in more details in the Appendix C.
2.3.1 Dual Potential power series expansion

Let’s expand the dual action $S^*$ in a few first orders near zero:

\[
S^*[\psi] = \frac{1}{2} \sum_k (\psi_k - j_k)^2 - \ln \tilde{V}(0) + \frac{a}{2} \sum_x \psi^2(x) + \frac{b}{24} \sum_x \psi^4(x) + \frac{c}{6!} \sum_x \psi^6(x) + \ldots, \tag{54}
\]

where:

\[
a := -\frac{\tilde{V}''(0)}{\tilde{V}(0)} > 0, \quad b = -\frac{\tilde{V}(0)\tilde{V}'(0) - 3\tilde{V}''(0)^2}{\tilde{V}'(0)}, \quad c = -\frac{30\tilde{V}'(0)^2 - 15\tilde{V}(0)\tilde{V}''(0) + \tilde{V}'(0)^2\tilde{V}(0)}{\tilde{V}'(0)^2},
\]

and so on. Here we assume that $1/g$ is a small parameter and expand on it. However, the terms Perturbation Series we obtain in the following sections will be regular at zero, since we will include the quadratic part of $\ln \tilde{V}$ into the Green function of the obtained dual theory. Also, all $\tilde{V}^{(2k+1)}(0) = 0$ due to the evenness of the original potential $V$. For power potentials, one can write down explicit formulas:

\[
\tilde{V}^{(2m)}(0) = 2(-1)^m \int_0^\infty d\phi \phi^{2m} \exp \left[ -\frac{1}{(2n)!} \phi^{2n} \right] = (-1)^m \frac{\Gamma \left( \frac{2m+1}{2n} \right) \Gamma(2n+1) \Gamma \left( \frac{2n+2m+1}{2n} \right)}{n}, \tag{56}
\]

Therefore:

\[
a = \frac{\Gamma \left( \frac{3}{2n} \right) \Gamma(2n+1)^{1/n}}{\Gamma \left( \frac{1}{2n} \right)} > 0, \quad b = \frac{3\Gamma \left( \frac{3}{2n} \right)^2 - \Gamma \left( \frac{1}{2n} \right) \Gamma \left( \frac{5}{2n} \right) \Gamma(2n+1)^{2/n}}{\Gamma \left( \frac{1}{2n} \right)^2} \geq 0, \tag{57}
\]

\[
c = \frac{30\Gamma \left( \frac{3}{2n} \right)^3 - 15\Gamma \left( \frac{1}{2n} \right) \Gamma \left( \frac{5}{2n} \right) \Gamma \left( \frac{3}{2n} \right) + \Gamma \left( \frac{1}{2n} \right) \Gamma \left( \frac{7}{2n} \right) \Gamma(2n+1)^{3/n}}{\Gamma \left( \frac{1}{2n} \right)^3} \geq 0, \tag{58}
\]

\[
d = \frac{\Gamma(2n+1)^{4/n}}{\Gamma \left( \frac{3}{2n} \right)^4} \left( \Gamma \left( \frac{1}{2n} \right) ^3 \Gamma \left( \frac{9}{2n} \right) - 7 \left( 90\Gamma \left( \frac{3}{2n} \right) ^4 - 60\Gamma \left( \frac{1}{2n} \right) \Gamma \left( \frac{5}{2n} \right) \Gamma \left( \frac{3}{2n} \right) ^2 \right) \Gamma \left( \frac{7}{2n} \right) ^2 \right) \tag{59}
\]

and so on. For any given $n$, these coefficients can be calculated explicitly. We present here these formulas only for the sake of reference. The substitution shows that for $n = 2$ the coefficients $b, c$ and all subsequent are set to zero, as it should be.

2.3.2 Strong coupling Green function calculation

We will denote the Green function for the obtained dual system as $\tilde{G} := \left( G + \frac{a}{g^2} \right)^{-1}$. One can provide a compact expression for it, in spirit of (30), using (20) and (13):

\[
\tilde{G}_{x,x'} = \left( G + \frac{a}{g^2} \right)^{-1} = \left( \frac{L}{1 + \frac{a}{g^2} L} \right)_{x,x'} = \frac{1}{N} \left( \prod_{i=1}^d \sum_{k_i=0}^{M-1} \frac{4\alpha \sum_{j=1}^d \sin^2 \left( \frac{p_j k_i}{2} \right) + \gamma}{1 + \frac{4\alpha \gamma}{g^2} \sum_{j=1}^d \sin^2 \left( \frac{p_j \gamma}{2} \right) + \frac{\gamma^2}{g^2}} \right) e^{-i(p, x-x')}
\]

\[
= \frac{g^2}{a} \left( \prod_{i=1}^d \frac{2\pi}{\sin^2 \left( \frac{p_i \gamma}{2} \right) + \frac{\gamma^2}{g^2} + \gamma} \right) e^{-i(q, x-x')}
\]

The last equality is the application of the spectral theorem and substitution of explicit value of lattice Laplacian, described in appendix [A]. Broadly speaking, in all the transformations we follow classical textbooks in Lattice Field Theory, e.g. [14].
We will use the same shorthand notations for \( \tilde{G} \) matrix elements as for \( G \), namely, \( \tilde{G}_x = \tilde{G}_{x,0} = \tilde{G}(x) \) and the same for the Fourier transform:

\[
\tilde{G}(q) = \frac{1}{(2\pi)^d} \frac{g^2}{a} \frac{4\alpha}{4\alpha} \sum_{j=1}^{d} \sin^2 \left( \frac{q_j}{2} \right) + \gamma \frac{q_j}{2} + \left( \frac{q_j^2}{\pi} + \gamma \right),
\]

(60)

which looks quite unusual. However, it is nothing else but lattice Laplacian Green function with some additional shift by a constant term. The physical sense of such formula is not very clear. Moreover, let us note that \( \tilde{G} \) tends to zero when \( g \to 0 \), and tends to some constant when \( g \to \infty \). This the main reason for the obtained in the following strong coupling series to be regular at zero, which makes them much more convenient for computations.

2.3.3 Dual theory partition function and correlators

Let’s now for simplicity of the next calculation denote:

\[
\frac{\tilde{V}(0)^N}{(2\pi)^{N/2}g^N \sqrt{\det (G + \frac{x}{g^2})}} = \exp(-\tilde{F}_0)
\]

which is a some kind of a "background" strong-coupling contribution to Free energy that arise after duality. For the convenience, we will denote the corresponding "background" free energy density as \( \tilde{f}_0 := \frac{\tilde{F}_0}{N} \). In the following, we will obtain a nice expression for \( f_0 \), proving that it stays finite in thermodynamic limit.

It will be also convenient to eliminate the excessive quantities from the dual actions, such as constant terms and the contributions of the source \( j(x) \). For this aim, let us define the shortened action of a dual theory, \( S_0^* [\psi] \) as:

\[
S_0^* [\psi] = \frac{1}{2} \sum_k \frac{\bar{\psi}_k^2}{\lambda_k} + \frac{a}{2} \sum_x \frac{\psi^2(x)}{g^2} + \frac{b}{24} \sum_x \frac{\psi^4(x)}{g^4} + \frac{c}{6!} \sum_x \frac{\psi^6(x)}{g^6} + \ldots,
\]

(61)

Moreover, for the future purposes it will be useful to define also the partition function (in external field \( \eta(x) \)) of the dual theory as:

\[
Z^*[\eta(x)] = \int_{\mathbb{R}^N} \frac{\Pi_k d\psi_k}{\sqrt{(2\pi)^N \det \tilde{G}}} \exp \left[ -S_0^* [\psi] + i \langle \eta(x), \psi(x) \rangle \right],
\]

(62)

where we have used the normalization factor imposed by the quadratic part of the dual action \( S^* \). This formula is a nothing else but the definition of a theory. The main thing is, that one can compute the correlators in such theory, using the common Feynman technique. So let us define the correlators in dual theory as:

\[
\langle \psi(x_1) \ldots \psi(x_k) \rangle^* = \frac{1}{Z^*[0]} \int_{\mathbb{R}^N} \frac{\Pi_k d\psi_k}{\sqrt{(2\pi)^N \det \tilde{G}}} \psi(x_1) \ldots \psi(x_k) \exp \left( -S_0^* [\psi] \right).
\]

(63)

Clearly, such a general correlation can be expressed in terms of variations of \( Z^*[\eta(x)] \) over \( \eta(x) \) in a standard way.

Proceeding, one can also express the correlators of initial theory in terms of the correlators of dual theory. It can be seen from the relation between the partition functions of the initial and dual theories, which reads as:

\[
Z[j(x)] = \exp \left[ -\tilde{F}_0 - \frac{1}{2} \sum_{x,x'} j(x) G_{x,x'} j(x') \right] Z^*[\widetilde{(Gj)(x)}],
\]

(64)

where \( (Gj)(x) = \sum_{x'} G_{x,x'} j(x') \).

This relation allows us to deduce that:

\[
\langle \phi(p) \phi(-p) \rangle = G(p) \left( 1 - G(p) \langle \psi(p) \psi(-p) \rangle^* \right),
\]

(65)
Duality between lattice theories

2.3.4 Brief discussion of the performed calculations

Let us briefly discuss the obtained result. The initial action $S$ was the simple action of $\phi^{2n}$ theory on the lattice, where $g$ is a coupling constant and $\phi$ is a field variable. And it provides a common Feynmann expansion in powers of $g$. Then we applied the constructed duality and get the interaction in a dual action $S^*$ as a power series in $\psi/g$, where $\psi$ is a new (dual) field variable. In result, the (lattice) path integral with a dual action $S^*$ can provide a Feynmann perturbation theory in the inverse powers of coupling constant $g$. Hence, the constructed duality provides a something like a roadmap for calculating strong coupling decompositions at least for power potentials.

As about the convergence of this strong coupling expansion, one can prove it for the case of finite $N$, since for $\phi^{2n}$ theory
\begin{equation}
\left|\tilde{V}(\psi)\right| \leq A_n |\psi|^{2n-2} \text{ for } \psi \text{ large enough and some fixed positive constant } A_n. \quad (15)
\end{equation}
Though, since it is a well-known fact that Euclidean $\phi^4$ has a critical point at least in 2 and 3 dimensions, these series can’t converge for all $g$ when $N \to \infty$. However, even asymptotic expansions are helpful and can provide some quantitative rather than qualitative results usually given by renormalisation group theory beyond the weak coupling limit.

We would also like to underline that we won’t claim non-tiviality of continuous $\phi^4$ theory in higher dimensions, since we consider only lattice theories without passing to a continuous limit. Besides, some recent works [17] show that this is a subtle question and we won’t touch it at all in the present work.

3 Perturbative calculation of Free energy

In this section, we will provide the two different perturbative expansions of partition function. The first one is a usual weak coupling expansion with standard Feynman technique. The second one is a strong coupling expansion for the initial theory, obtained as a weak coupling expansion of a dual theory, with slightly different Feynman technique, involving dual potential and inverse propagator.

\footnote{It can be proved [28] using saddle-point asymptotics of $\tilde{V}(\psi)$ for $\psi \to \infty$.}
Duality between lattice theories

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3.1 Weak coupling case

Starting from the partition function (21), we can provide its Feynman perturbation expansion:

\[ Z = \int_{\mathbb{R}^N} \frac{\prod_k d\phi_k}{(2\pi)^N \det G} \exp \left( -\frac{1}{2} \sum_{x,x'} L_{x,x'} \phi(x)\phi(x') - \frac{1}{4!} g^4 \sum_x \phi^4(x) \right) \]

\[ = \sum_{n=0}^{\infty} \left( -\frac{g^4}{n! (4!)^n} \right) \int_{\mathbb{R}^N} \frac{\prod_k d\phi_k}{(2\pi)^N \det G} \left[ \sum_x \phi^4(x) \right]^n \exp \left( -\frac{1}{2} \sum_{x,x'} L_{x,x'} \phi(x)\phi(x') \right) . \]

Its terms have usual graphical interpretation in terms of Feynman Graphs. Though, we are mostly interested in free energy density, so we need only connected diagrams. We will denote connected diagrams with \( l \) vertices as \( G_l \).

Our goal is to look on more details on both provided decompositions on relatively simple example, to plot them at one graph and, finally, compare with the results of numerical simulation. And (lattice) \( \phi^4 \) theory is the best choice for this purpose as a bright representative of well-studied theories with some non-trivial behaviour. So, all the considerations below will be performed \( \phi^4 \) theory only, though the provided techniques could be applied directly to all the lattice theories, described in the section 2.1.

3.1.1 First terms calculation in weak coupling perturbation series

In this section, we will focus on Free energy calculation. To accomplish this, we have to calculate only connected contributions. For the free energy of a system \( F \), expressing in terms of \( Z \) as:

\[ F = -\ln Z, \tag{66} \]

one can obtain Perturbation Theory on powers on \( g \) using standard Feynman Diagrams techniques, taking into the account only connected ”vacuum bubbles” (\( \circ \)). Namely, one can write, using momentum Feynman technique for free energy per cite (28):

\[ f = \frac{g^4}{8} (G_0)^2 - \frac{1}{16(2\pi)^d} g^8 (G_0)^2 \left( \prod_{i=1}^d \int_0^{2\pi} dq_i \right) G(q)^2 \]

\[ - \frac{g^8}{24(2\pi)^{3d}} \left( \prod_{n=1}^3 \left( \prod_{i=1}^d \int_0^{2\pi} dq_i^n \right) G(q^n) \right) G(q^1 + q^2 + q^3) + O(g^{12}) . \tag{67} \]

Let us note that we have obtained an expression for \( f \) with a finite limit when \( N \to \infty \). Besides, all the error terms arisen from Euler-MacLaurin summation formula become negligibly small in this limit. This fact, in particular, indirectly witnesses we’ve done everything correctly.

3.2 Strong coupling case

Now we are going to perform some decomposition with terms, which become small when \( g \to \infty \), in opposite to weak coupling expansion. For this purpose, let us initially write the dual partition function that has been described in the
where we have intentionally placed the factor $\frac{1}{N}$ before the determinant to obtain the expression finite in the limit $N \to \infty$. Though, even the notion of a mass is not clear in the dual theory, since it is essentially non-local. The propagator then becomes dependent on the coupling constant. To be honest, we have to decompose the propagator in $1/g$, since we consider only the terms with order in $1/g$ less or equal to 8. Though, we won’t perform it to avoid the cumbersome formulas, and in language of perturbation theory it means that the strong coupling limit.

Besides, we take care only about orders 4, 6 and 8 in $1/g$, since in this section we are going to provide only first terms of the perturbative expansion (up to 8th order). This expansion requires $g \gg 1$ and we will use it as the definition of a strong coupling limit.

Let us also highlight the fact that after applying the duality some “mass renormalization” occurred, namely, we have obtained some nonzero contribution to the quadratic part of action from $L_n$. That’s why we normalized integrations on $\det \left( G + \frac{a}{g^2} \right)^{-1}$, rather than only $\det G^{-1} = \det L$. Though, even the notion of a mass is not clear in the dual theory, since it is essentially non-local. The propagator then becomes dependent on the coupling constant. To be honest, we have to decompose the propagator in $1/g$, since we consider only the terms with order in $1/g$ less or equal to 8. Though, we won’t perform it to avoid the cumbersome formulas, and in language of perturbation theory it means that we “resumed” all quadratic contributions in $1/g$. Keeping this all in mind, let us proceed.

**3.2.1 Calculation of the determinant**

In the further calculations we will need the simplified expression for $\det \left( G + \frac{a}{g^2} \right)$. It is possible to write it down since we are interested in the limit $N \to \infty$. Moreover, we know the eigenvalues of $G$. So we start from the very definition of the determinant and proceed, using (13), (20) and Euler-Maclaurin summation formula, in the same way as during derivation of (30):

$$
\frac{1}{N} \ln \det \left( G + \frac{a}{g^2} \right) = \frac{1}{N} \sum_k \ln \left( \lambda_k^{-1} + \frac{a}{g^2} \right) = \frac{1}{(2\pi)^d} \left( \prod_{j=1}^d \int_0^{2\pi} dq_j \right) \ln \frac{1 + \frac{a}{g^2} \left( 4\alpha \sum_{j=1}^d \sin^2 \left( \frac{q_j}{2} \right) + \gamma \right)}{4\alpha \sum_{j=1}^d \sin^2 \left( \frac{q_j}{2} \right) + \gamma},
$$

(68)

where we have intentionally placed the factor $1/N$ before the determinant to obtain the expression finite in the limit $N \to \infty$. The expression under the logarithm is well-defined, because we require all $\lambda_k$ to be positive.
Figure 4: Plots of analytical and computational results for the free energy. The left picture corresponds to the dimension $d = 2$ and the right one to the dimension $d = 3$. Numerical results are obtained in 200 equidistant points and then have been linearly interpolated. Comparing both graphs, one can see that with the growth of the dimension the obtained strong coupling decomposition seems to converge slower.

Gathering all factors together, we obtain for "background" strong coupling free energy density $\tilde{f}_0$:

$$\tilde{f}_0 = \ln \frac{\sqrt{2\pi}}{V(0)} + \frac{1}{2(2\pi)^d} \ln \left( \prod_{j=1}^{d} \int_{0}^{2\pi} dq_j \right) \ln \left[ \frac{g^2}{4\alpha} \sum_{j=1}^{d} \sin^2 \left( \frac{q_j}{2} \right) + \gamma \right].$$

(69)

We would like to underline that this expression stays finite, when $g \to 0$, and grows logarithmically, when $g \to \infty$, namely:

$$\tilde{f}_0 = \ln g + \ln \frac{\sqrt{2\pi}}{V(0)} - \frac{1}{2(2\pi)^d} \ln \left( \prod_{j=1}^{d} \int_{0}^{2\pi} dq_j \right) \ln \left[ \frac{g^2}{4\alpha} \sum_{j=1}^{d} \sin^2 \left( \frac{q_j}{2} \right) + \gamma \right] + O \left( \frac{1}{g^2} \right), \quad g \to \infty.$$ 

(70)

It is not known for authors whether this asymptotic has been obtained before, though, it has the right sign of convexity, and provides the behaviour $Z[0] \sim 1/g$ for theories with interaction $V(\phi) = g^2n^2/2n!$, which can be obtained in some other ways (using Hoelder’s inequality in several ways) for general theories with trace-class Green functions [33], resulting in the paper coming soon.

### 3.2.2 First terms calculation in strong coupling perturbation series

Exactly as in the case of weak coupling limit, we are interested only in free energy $F$, which is given by the connected diagrams only. However, unlike the perturbation series for the initial theory, here we will apply strong rather than weak coupling decomposition. The main difference is that now one has the interaction, containing infinitely many powers of $\phi$, so the number of different vertices in Feynman Diagrams of any fixed order grows significantly. Though, the computation is straightforward, using Feynman Diagrams in momentum space. So let us write down the first few terms for $f$ explicitly without diving into the computations:

$$f = \tilde{f}_0 + \frac{b}{8g^4} \left( \tilde{G}_0 \right)^2 + \frac{c}{2^33!g^6} \left( \tilde{G}_0 \right)^3 - \frac{b^2}{2^4(2\pi)^4g^8} \left( \tilde{G}_0 \right)^2 \left( \prod_{i=1}^{d} \int_{0}^{2\pi} dq_i \right) \tilde{G}(q)^2$$

$$+ \frac{d}{2^44!g^{10}} \left( \tilde{G}_0 \right)^4 - \frac{b^2}{4!(2\pi)^6g^{10}} \left( \prod_{i=1}^{d} \int_{0}^{2\pi} dq_i \right) \tilde{G}(q)^4$$

(71)

where $\tilde{f}_0$ is given by [69]. We will discuss this decomposition in the following section.
3.3 Discussion of the obtained results for the free energy density strong coupling decomposition

Let us mark some properties of the obtained decomposition (71). Unlike some other strong decompositions [7, 15, 19, 12], the presented one has all the terms are regular when $g \to 0$. And the main practical benefit is that these series converge much better, which will be further seen in section 4.1.

It can be shown that Feynman Perturbation Theory for dual form of partition function (53) converges for any finite number of lattice $N$ and power of initial interaction (16) $2n > 2$. However, for calculating free energy per cite in thermodynamic limit $f$ (28).

Finally, though one can express both obtained weak (67) and strong (71) coupling decompositions for the free energy density in terms of generalized hypergeometric functions [31, 27], for our purposes it is more suitable to calculate these integrals numerically. Let us only note, that one can obtain convergent series for such integrals, representing them as contour integrals over unit circle and using residue techniques for their calculation.

4 Numerical verification of Free energy perturbative expansion

Now, we are going to check the obtained duality using numerical simulation. More precisely, we are going to compare numerical and analytical (both strong and weak decompositions) plots of free energy from $g$ for several $\alpha$ and $\gamma$. We start from discussing the techniques we are going to use for numerical simulations, then we provide some results and finally compare them with our analytical perturbation results.

4.1 Methods for numerical simulations

The method we are going to apply for numerical simulations is Hamiltonian Monte Carlo method [24, 25]. This approach was firstly introduced in [10]. We prefer to use it instead of usual (Lagrangian) Monte Carlo since it usually gives more rapid convergence and not pretty comprehensive to implement.

Hamiltonian Monte-Carlo method is an improvement of the Metropolis-Hastings algorithm. The main difference is that one includes additional integrations over some momenta and consider then the obtained function in the exponent as a Hamiltonian. And then the direction of field configuration decrease is chosen not along the gradient of action, but is obtained from Hamiltonian equations of motions for this fictive Hamiltonian.

We start from rewriting the lattice action and its gradient with respect to field in a more convenient form for numerical calculation:

$$S[\phi] = -\alpha \sum_{x} \sum_{\mu} \phi(x) \phi(x + e_\mu) + \frac{1}{2} (\gamma + 2d\alpha) \sum_x \phi(x)^2 + \frac{g^n}{n!} \sum_x \phi(x)^n,$$

and:

$$(\nabla S[\phi])(x) = -\alpha \sum_{\mu} (\phi(x + e_\mu) + \phi(x - e_\mu)) + (\gamma + 2d\alpha) \phi(x) + \frac{g^n}{(n-1)!} \phi(x)^{n-1}.$$

(72)

(73)
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Figure 6: Plots of analytical and computational results for the free energy for different order of approximations in $1/g^2$ in strong coupling expansions for $d = 2$. The top left picture corresponds to zeroth order approximation (i.e $f_0$ only), the top right - to first order, the bottom left - to the second, and the bottom right - to the third. From the picture one can see that the constructed strong coupling series converge rapidly to the numerical values.

Actually, it is the form of the action which is the most commonly used in Lattice QFT. Though, for the previous analytical computations it was less convenient than the used one.

To proceed with HMC method description, let us introduce the auxiliary lattice field $\pi(x)$, which we will call momenta, and the Hamiltonian:

$$H[\phi, \pi] = \frac{1}{2} \sum_x \pi^2(x) + S[\phi].$$

(74)

Then one can rewrite the partition function (21) with the additional integrations:

$$Z[j] = \int_{\mathbb{R}^{2N}} \frac{\prod_k d\phi_k}{\sqrt{(2\pi)^N \det G}} \frac{\prod_x d\pi_x}{\sqrt{(2\pi)^N}} \exp \left[-H[\phi, \pi] + \langle j, \phi \rangle \right],$$

(75)

and let the two fields, $\phi$ and $\pi$, both acquire the dependence on $\tau$

$$\pi(x) \rightarrow \pi(x, \tau), \quad \phi(x) \rightarrow \phi(x, \tau),$$

which we will need to generate the proper distribution of fields $\phi(x)$ with probability density $\exp (-S[\phi])$. Now, to implement HMC method, we should interpret $\pi(x, \tau)$ as the conjugate momenta of $\phi(x, \tau)$ in Hamiltonian (74).

To generate proper field configurations, which we will need to compute the integral (75), we should write the equations of motion for $\phi(x)$ and $\pi(x)$, following from the Hamiltonian (74). More precisely, we obtain:
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\[
\frac{d\phi(x, \tau)}{d\tau} = \frac{\delta H[\phi, \pi]}{\delta \pi(x, \tau)} = \pi(x, \tau), \tag{76}
\]

\[
\frac{d\pi(x, \tau)}{d\tau} = -\frac{\delta H[\phi, \pi]}{\delta \phi(x, \tau)} = -\frac{\delta S[\phi]}{\delta \phi(x, \tau)} = -\langle \nabla S[\phi]\rangle(x),
\]

where \(\langle \nabla S[\phi]\rangle(x)\) is given by (73). Solving these equations to compute the evolution after the Markovian time \(\tau_0 = 1\) for randomly initialized random momenta \(\pi(x)\), we perform the step of Hamiltonian Monte Carlo method. We are going to solve them numerically with the leapfrog method with \(n_{\text{step}} = 100\) iterations, so the leapfrog time step will be \(dt = \tau_0/n_{\text{step}} = 0.01\). After all, the obtained field configuration is accepted with probability:

\[
P_A ((\phi, \pi) \rightarrow (\phi', \pi')) = \min \left\{ 1, e^{-\Delta H} \right\}, \tag{77}
\]

and the result (old or new field configuration) is appended to some list of fields configurations \(\Phi\). The obtained momenta configuration \(\pi(x)\) is then dropped off, and we repeat the described leapfrog step for \(\tau_0\) with the newly randomly initialized momenta configuration.

The complete algorithm is the following:

1. Introduce the auxiliary momenta variables \(\pi(x)\) and initialize them uniformly;
2. Randomly (uniformly) initialize the fields configuration \(\phi(x)\);
3. Performing 1000 HMC steps to thermalize the states without saving the fields configurations in \(\Phi\);
4. Perform 10000 more HMC steps with remembering every 10th field configurations \(\phi(x)\) in \(\Phi\);
5. The obtained set of configurations

\[
\Phi = \{ \phi_i(x) \mid i = 1, \ldots, 1000; \ x \in \mathcal{V} \} \tag{78}
\]

is distributed with probability density:

\[
\rho(\phi) \sim \exp \left[ -S(\phi) \right],
\]

so it can be used to compute any necessary correlators in the given theory [21].

6. Estimate the error of computation with the jackknife method.

The used implementation of the described algorithm can be found at [32] (master branch) with some additional comments in README file in root directory of a project. All the presented simulations were made on square lattice of unit spacing from \(M = 32\) nodes in each direction, e. e. \(N = 32^d\) nodes in total.

So, we can’t compute the integral itself using such method, but we are able to find any correlators. However, it is not a significant obstacle, since one can note that:

\[
\frac{\partial f}{\partial (g^4)} = \frac{1}{4!} \langle \phi^4(x) \rangle \tag{79}
\]

for any fixed \(x \in \mathcal{V}\) (due to the translational invariance of the considering lattice with periodic boundary conditions). So, we can deduce that:

\[
f(g) = \frac{1}{4!} \int_0^g \langle \phi^4(x) \rangle \ d(g^4), \tag{80}
\]

using, that \(f(g = 0) = 0\).

From the other side, from the distribution of the numerically obtained field configurations \(\Phi\) it follows that:

\[
\langle \phi^4(x) \rangle \approx \frac{1}{|\Phi|} \sum_{\phi \in \Phi} \frac{1}{N} \sum_{x \in \mathcal{V}} \phi(x)^4, \tag{81}
\]

where we also used the averaging along the lattice sites to increase the precision.
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Similarly, for the two-point correlation function, one can write:

\[ \langle \phi(x_1)\phi(x_2) \rangle \approx \frac{1}{|\Phi|} \sum_{\phi \in \Phi} \phi(x_1)\phi(x_2). \]  

However, one can improve the precision of computations, using that the theory is translationally invariant, meaning that:

\[ \langle \phi(x_1)\phi(x_2) \rangle = \langle \phi(x_1 + \delta)\phi(x_2 + \delta) \rangle, \]  

for all possible lattice shifts \( \delta \). We will also use this observation in our numerical simulations, taking the average among all shifts to increase the precision of calculations.

Having all necessary background prepared, we can proceed to the comparison of the obtained results with the numerical simulations.

4.2 Discussion and comparison of numerical and analytical results

We are going to plot on the same graph both of the obtained decompositions and numerical results as a first-stage verification of the obtained duality. Namely, we present here the plots of \( f(g) \) of the decompositions \( (71), (67) \) as well as the results of numerical simulation with Hamiltonian Monte-Carlo method described in section 4.1. The results could be found in the figure 4.

Let us note that everywhere in this paper we calculate well-known weak coupling decompositions for the sake of verification of both numerical and strong coupling decompositions.

One can see that both obtained decompositions are placed reasonably and relatively close to each other. So that it is possible to draw a smoothing line between them which looks physically correct at least at qualitative level, which witnesses that both obtained decompositions are coherent with each other. Besides, as it has already been noted, we obtained the right asymptotics \( (70) \) for large \( g \) in the provided strong coupling decomposition. Proceeding, the obtained weak perturbation series \( (67) \) coincides with numerical simulation in the region of small couplings \( g \), which verifies the first one. And finally, the obtained strong coupling decomposition \( (71) \) is placed closely to numerical simulations, which testifies the correctness of this series.

The error of the provided HMC simulation was estimated using the jackknife method, which should work perfectly, since all the statistical quantities we consider are math estimations rather than medians or something more complicated \[24\]. As for the integrals in the analytical expressions \( (71), (67) \) were computed numerically using quadratures or Monte Carlo technique. The errors of Monte Carlo calculations were also estimated using the jackknife approach, the error of quadrature integration was given by python SciPy (v. 1.13.0) package \[30\]. Python Library NumPy \[29\] has been also used for arrays manipulation. The absolute errors of the performed calculations in region \( g \geq 1 \) for \( \gamma = 1 \) and \( \alpha = 1 \) is presented in the figure 4.1. For calculation of the integrals using Monte Carlo approach, we used \( 5 \cdot 10^4 \) points for \( d = 2 \) and \( 1.5 \cdot 10^5 \) points for \( d = 3 \). The graphs demonstrate that the error in the performed calculations remains
negligibly small and does not affect the convergence of the obtained strong coupling expansion. Additionally, it does not introduce any ambiguity to the convergence of the analytical expression to the numerical results.

After all, it is instructive to plot, how the described strong coupling series converge to the numerical results (figure 6). Summarizing, the obtained strong coupling fits numerical results sufficiently well in the region of medium and large $g$. This proves the "integral" correctness of the provided technique. However, we would also like to make some research of coordinate (or momentum) dependence of these perturbation series. So, we are going to study correlators, restricting ourselves to the case of two-point correlation functions.

5 Two-point function

Besides the comparison of "integral" characteristics of Lattice Field Theory, such as free energy density (27), to establish and testify the correctness of the obtained duality, one should also examine the predictions for coordinate dependence of any quantities. And the simplest, but still useful example is two-point function (41) of partition function generating functional $Z[j(x)]$ (21).

5.1 Analytical results

One can evaluate the 2-point function for the initial and the dual theory in momentum picture in terms of 39 and 41 with the help of Feynman Perturbation Technique. The calculations are the same as in the case of free energy density, except for the fact that now all the diagrams will have external vertices (in coordinate picture) or external legs (in momentum picture). It is more convenient for us to use the momenta picture, since at least in free theory ($g = 0$) the answer can be expressed in terms of elementary functions.

For the initial theory, one should sum all the diagrams from the picture 7. It is the standard computation, so let us write down the final answer for two-point function (41) without going into the details:

$$
\frac{\langle \phi(p)\phi(-p) \rangle}{G(p)^2} = \frac{1}{G(p)} - \frac{g^4}{2} G_0 + \frac{g^8}{2^3(2\pi)^d} (G_0)^2 G(p) + \frac{g^8}{2^3(2\pi)^{2d}} G_0 \left( \prod_{i=1}^{d} \int_{0}^{2\pi} dq_i \right) G(q)^2
$$

\[ + \frac{g^8}{3! (2\pi)^{3d}} \left( \prod_{i=1}^{d} \int_{0}^{2\pi} dq_i \right) \left( \prod_{i=1}^{d} \int_{0}^{2\pi} dk_i \right) G(k)G(q)G(q + k + p) + O(g^{12}), \quad g \to 0 \]  

As for the dual theory, using the one can also compute the correlation function (41), taking into the account all the contributions from the picture 8.

Figure 8: Feynman Graphs for the two-point function in momenta picture (41) in the dual theory (61) up to order $\frac{g^8}{g}$, written down in (85). The diagram technique is the same as in the picture 3.
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Two-point function comparison (weak coupling)

Figure 9: Two point function comparison for \( d = 2 \) (top) and \( d = 3 \) (bottom) for weak (84) and strong (85) coupling decompositions. From the plots, one can see a pretty well coincidence of the provided strong coupling decomposition results with the numerical results for \( g \geq 0.5 \). Oppositely, the traditional weak coupling perturbation theory crashes for \( g > 1 \).

\[
\frac{\langle \psi(p)\psi(-p) \rangle^*}{G(p)^2} = \frac{1}{G(p)} - \frac{b}{2g^4} \tilde{G}_0 - \frac{c}{23g^6} \left( \tilde{G}_0 \right)^2 - \frac{d}{3!23g^8} \left( \tilde{G}_0 \right)^3 \\
+ \frac{b^2}{g^8(2\pi)^2d} \tilde{G}_0 \left( \prod_{i=1}^{d} \int_0^{2\pi} dq_i \right) \tilde{G}(q)^2 + \frac{b^2}{23g^8(2\pi)^d} \left( \tilde{G}_0 \right)^2 \tilde{G}(p) \\
+ \frac{b^2}{3!g^8(2\pi)^3d} \left( \prod_{i=1}^{d} \int_0^{2\pi} dq_i \right) \left( \prod_{i=1}^{d} \int_0^{2\pi} dk_i \right) \tilde{G}(k)\tilde{G}(q)\tilde{G}(q+k+p) + O \left( \frac{1}{g^{10}} \right), \quad g \to \infty
\]  

(85)

where all the terms are regular when \( g \to 0 \), exactly as in the case of free energy per cite \( f \). Now, using the relation (65), we are able to obtain the strong coupling decomposition for the two-point correlation function \( \langle \phi(p)\phi(-p) \rangle \).

5.2 Comparison with the numerical results

Now, we are able to perform numerical computation of two point functions using the same HMC method, described in section 4.1. Given the set of samples \( \Phi \), as in (78), we can compute the correlator \( \langle \phi(x)\phi(y) \rangle \) in coordinate picture. After that, taking the Discrete Fourier Transform as in (24), we can derive the quantity \( \langle \phi(p)\phi(-p) \rangle \) as the function of admissible \( p \). The results of such computations together with the plots of analytical expressions are presented in the figure 5.1.

From the plots one can deduce that the results of numerical simulations are in agreement with the traditional weak coupling perturbation theory (84). However, for \( g \sim 1 \) this decomposition crashes and starts to diverge. As for the
presented strong coupling decomposition [{85}] works even for $g \sim 1$, and becomes irrelevant only for $g \ll 1$. Unlike the free energy per site, the error of the strong coupling decomposition does not seem to grow for higher dimensions $d$. Summarizing, strong coupling decomposition [{85}] suits the numerical results pretty well in the range of middle and large couplings $g$. Moreover, such decompositions can be obtained with the same efforts as the traditional weak coupling perturbation theory, putting it all together, the provided method of the obtaining the strong coupling decompositions is a good choice when one wants to obtain quantitative rather than qualitative analytical results in lattice or condensed matter theories with a finite Brillouin zone.

### 5.3 Analytical corollaries for the two-point function in strong coupling limit

It is instructive to write down explicitly the form of strong coupling Green function, imposed by [{85}] and [{65}]. Substituting, we obtain in first orders:

$$
\langle \phi(p)\phi(-p) \rangle = \frac{1}{4\alpha \sum_{j=1}^{d} \sin^2 \left( \frac{p_j}{2} \right) + \frac{g^2}{a} + \gamma} - \frac{b\bar{c}_0}{2a^2} \left( \frac{1}{\gamma + 4\alpha \sum_{j=1}^{d} \sin^2 \left( \frac{p_j}{2} \right) + \frac{g^2}{a}} \right)^2 + O \left( \frac{1}{g^6} \right), \quad g \to \infty.
$$

(86)

Surprisingly, all the excessive factors cancel out, so we obtain in result nothing else but the initial $\phi^4$ perturbation theory with Laplacian-type propagator, similar to [{13}], but with the renormalized parameters. Moreover, we obtain the formulas of renormalization. For instance, now it is reasonable to call the combination:

$$
\tilde{m}^2 = \frac{g^2}{a} + \gamma,
$$

(87)

a renormalized squared mass, compared with the weak coupling mass $m^2 = \gamma$. Physically it is an intricate result: qualitatively we haven’t obtained anything new from the Renormalization Group predictions, but got the explicit numerical formulas, suitable for practical computations, as the comparison in the previous section has shown.

Certainly, some type of embarrassment occurs, to say that there is a phase transition, when $\tilde{m}^2 = 0$. However, we don’t want to rush into the assessments, since the decomposition we use was obtained for large $g$, and the equation $\tilde{m}^2 = 0$ implies $\gamma = -\frac{g^2}{a}$, which makes the initial partition function [{21}] ill-defined. Still and all, it would be interesting to delve into this question in the separate research.

### 6 Discussion

In this study, we introduce a novel method for constructing Strong Coupling decompositions in Lattice Field Theories. This method was applied to the lattice $\phi^4$ theory, which, while being the simplest and most thoroughly examined, remains valuable for practical benchmarks. Our analysis focused on the free energy per site $f$ [{25}], essentially the partition function itself, and the two-point correlation function $\langle \phi(p)\phi(-p) \rangle$ in the momentum picture [{41}] for dimensions $d = 2$ and $d = 3$. In all instances, there was a strong concordance with the outcomes of numerical simulations. The evaluation of free energy confirms that our method accurately captures the ‘integral’ characteristics of the theory, while the examination of the two-point correlation function verifies its correctness in replicating the appropriate coordinate (or momentum) dependencies.

The authors would be eager to extend this analysis to higher dimensions; however, this was not feasible due to the limited computational resources available. The challenge lies in the exponential growth in the number of lattice vertices with the dimension $d$, which necessitates increased memory and computational time. Nevertheless, we do not anticipate significant difficulties for dimensions greater than three, as all integrals are regular and our discussion is confined solely to lattice theories.

Regarding the critique, the authors identify two significant issues.

Firstly, the triviality of the $\phi^4$ theory in the continuous limit renders the presented expansions inapplicable in this context. One counterargument is that, at the very least, the demonstrated method could be used to derive more accurate and convenient expressions for renormalized parameters in terms of bare ones, even if the theory becomes free for sufficiently large coupling constants. For instance, recalling the works [{20},{21},{19}], the method proposed could yield a Challen-Lehman representation for the two-point function from just two terms instead of an infinite series. It is conceivable that such a two-term representation might emerge if contributions from other instantonic solutions are
considered. After all, this work was initially positioned as an exploration of lattice theories without transitioning to the continuous limit, as this represents a separate and complex challenge.

Secondly, the possible unphysical nature of strong coupling modes for crystals with a finite Brillouin zone size. I.e., extremely large scales of coupling constants seem to be unrealizable in real materials. In response, it can be argued that the proposed expansions perform quite well even within the range of moderate coupling constants. Moreover, material science is advancing rapidly, and it is our task to prepare an analytical framework capable of predicting the properties of emerging materials, including the most exotic variants. Ultimately, this study does not claim to be a universal solution to all problems in quantum field theory, but rather aims to engage the reader and demonstrate the viability and utility of strong coupling expansions.

7 Conclusions

The methodologies outlined herein could readily be extended to a variety of bosonic theories, wherein interactions are additive at lattice nodes. Specifically, these include:

1. Real scalar fields with a general interaction potential $V(\phi)$;
2. Complex scalar fields with a general interaction potential $V(\phi, \phi^*)$;
3. Matrix scalar field models with matrix field variable $\Phi$, where the interaction takes the form $V(\Phi) = \text{tr}(\Phi^k)$, and at each lattice node, the matrix can be diagonalized, allowing the interaction to be expressed in terms of the pointwise eigenvalues. However, it is a gauge theory, so one should modify measure of integration measure in a proper way to fix the gauge.

With additional effort, it may also be feasible to apply the described techniques to Lattice Chromodynamics. Determining the optimal application of these techniques constitutes a subject for future research due to the complexity introduced by terms such as $\partial_{\mu}A_{\nu}$ in the action of non-Abelian theories, which contain powers greater than two. Initial approaches might include expanding the exponential in third-order terms and applying duality transformations to each resulting term. For a finite number of lattice nodes $N$, such decompositions are expected to converge, as the dominant fourth power remains within the exponential. However, further speculation on this matter is reserved, with only potential research directions outlined here. It should be noted that addressing gauge theories on the lattice would likely require careful consideration of gauge invariance and appropriate modifications to the integration measure, while performing the analytical computations.

Another promising research direction involves exploring theories with fermions and the corresponding lattice integrals over Grassmann variables. Although these integrals do not possess a Percival-Plancherel identity, the approach of executing the kinetic part is expected to be effective. Nonetheless, the outcomes of such investigations remain complex and are not fully understood by the authors at this time.

Further studies might also consider the continuous limit of the lattice model. Preliminary analysis suggests that most terms in the obtained decompositions become negligible. A more meticulous and comprehensive analysis, considering potential renormalizations of different parameters, could form the basis of future investigations.

It would also be intriguing to calculate the positions of branch cuts in correlation functions as dictated by the strong coupling decompositions. These positions physically signify the minimal energies required for particle creation and comparing them with those derived from weak coupling decompositions could reveal insightful details about the analytical structure of particle interactions.

The authors are eager to engage in discussions and value any constructive criticism. We appreciate the reader’s perseverance in reaching the end of this paper and apologize for any potential inaccuracies, errors, or typos that may have occurred.

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Appendices

Appendix A. Derivation of discrete Laplacian eigenvalues

In this appendix, we are going to discuss in more details, how eigenvalues of lattice Laplacian depend on grid scale \( l \). More examples of computing Laplacian Spectrum for various lattices and boundary conditions can be found, for example, in [26].

First of all, we address the dimensionless problem. Let’s consider a graph which is a \( d \)-dimensional square lattice with “unit” spacing with \( M \) nodes in every direction and periodic boundary conditions. Let us also denote as \( N = M^d \) the total numbers of nodes or, equivalently, modes. To underline the vector nature of the performed additions, we will use the notation \( \vec{r} \) for the lattice cite position and the general function on lattice as \( f(\vec{r}) \).

Further, one can easily return the lattice spacing to formulas using dimensional analysis. So let us define the lattice Laplacian on such lattice as:

\[
(\triangle f)(\vec{r}) = \sum_{j=1}^{d} (f(\vec{r} + \vec{e}_j) - 2f(\vec{r}) + f(\vec{r} - \vec{e}_j)),
\]

(88)

where \( \vec{e}_j \) is \( j \)th vector of an orthonormal basis, corresponding to the considering cubic lattice. As one can see, for the considering lattice Laplacian it is true that:

\[
\triangle = -D + J,
\]

(89)

where:

1. \( D \) is a diagonal matrix with elements enumerated by nodes and equal to the degree of the corresponding node, i.e., the number of edges attached to it. So, in \( d \)-dimensional cubic lattice \( D_{r,r'} = 2d\delta_{r,r'} \).

2. \( J \) is an adjacency matrix of considering grid graph, i.e. cubic lattice. Evidently, for cubic lattice, its matrix elements are given by:

\[
J_{r,r'} = \sum_{j=1}^{d} (\delta_{r,r'+\vec{e}_j} + \delta_{r,r'-\vec{e}_j})
\]

(90)

So it is easy to see that we are looking for a spectrum of considering graph, or, in other words, the spectrum of its adjacency matrix. Or, equivalently, for a spectrum of graph Laplacian of cubic lattice, which is the object, defined by [28] (up to sign).
Duality between lattice theories

So, let’s diagonalize $J$. We are going to find the complete set of lattice eigenfunctions. The corresponding equation for eigenfunctions:

$$\sum_{\vec{r}'} J_{\vec{r},\vec{r}'} f(\vec{r}') = \mu f(\vec{r}), \quad \sum_{j=1}^{d} (f(\vec{r} + \vec{e}_j) + f(\vec{r} - \vec{e}_j)) = \mu f(\vec{r}),$$  \hspace{1cm} (91)

which is no more than a linear homogeneous difference equation. Its fundamental system of solutions (for general parameters) can be found in the form of exponents of linear functions of indices, i.e. $e^{i(\vec{q},\vec{r})}$, in our vector notations. This form of solution is also clear from physical intuition, since it is a plane wave solution. Hence, one can get an equation, substituting $f(\vec{r}) = e^{i(\vec{q},\vec{r})}$:

$$d \sum_{j=1}^{d} \left(e^{i(q_j \vec{e}_j)} + e^{-i(q_j \vec{e}_j)}\right) = \mu, \quad \mu = 2 \sum_{j=1}^{d} \cos q_j,$$  \hspace{1cm} (92)

where $\vec{q} = \sum_{j=1}^{d} q_j \vec{e}_j$. From boundary conditions:

$$f(\vec{r} + M \vec{e}_j) = f(\vec{r}),$$  \hspace{1cm} (93)

we find that:

$$q_j^{(k)} = \frac{2\pi k_j}{M}, \quad k_j = 0, 1, \ldots, M - 1$$  \hspace{1cm} (94)

Therefore, we have found $M^d$ distinct eigenvalues, which we have diagonalized $J$ and hence $\Delta$. As a result, the eigenvalues of discrete lattice Laplacian are

$$\lambda_{\vec{q}} = -2d + 2 \sum_{j=1}^{d} \cos q_j, \quad q_j = \frac{2\pi k_j}{M}, \quad k_j = 0, 1, \ldots, M - 1$$  \hspace{1cm} (95)

Recovering the lattice spacing $l$ from the dimensions of parameters, we get:

$$\lambda_{\vec{q}} = - \frac{2}{l^2} \sum_{j=1}^{d} \left(1 - \cos q_j\right), \quad q_j = \frac{2\pi k_j}{M}, \quad k_j = 0, 1, \ldots, M - 1.$$  \hspace{1cm} (96)

Since in this paper we don’t consider the continuous limit, so we should stop at the expression (95).

However, let us add a few words about the continuous limit of the obtained eigenvalues to validate the obtained answer. To perform the continuous limit, we would like to change variables to momenta $\vec{p} = \frac{1}{l} \vec{q}$ from the lattice quantum numbers $q_j$, since they are the real physical magnitudes which are reasonable to keep finite. So, let’s substitute $\vec{p} = \frac{1}{l} \vec{q}$, and then, simplifying, we obtain:

$$\lambda_{\vec{p}} = - \frac{4}{l^2} \sum_{j=1}^{d} \sin^2 \left(\frac{p_j l}{2}\right),$$  \hspace{1cm} (97)

where:

$$\vec{p} = \sum_{j=1}^{d} p_j \vec{e}_j, \quad p_j = \frac{2\pi k_j}{Ml}, \quad k_j = 0, 1, \ldots, M - 1,$$  \hspace{1cm} (98)

so, we get for $l \to 0$, $\vec{p} = \text{const}$ and $M \cdot l = \text{const}$:

$$\lambda_{\vec{p}} = - \frac{4}{l^2} \sum_{j=1}^{d} \sin^2 \left(\frac{p_j l}{2}\right) \to \lambda_{\vec{p}}^{(c)} = - \sum_{j=1}^{d} p_j^2 = -\vec{p}^2,$$  \hspace{1cm} (99)

and momenta values are not bounded now, after continuous limit. So, we have obtained the continuous limit of eigenvalues $\lambda_{\vec{p}}^{(c)}$, which coincide with the spectra of continuous Laplacian on a hypercube of dimension $d$, which verifies the obtained discrete answer.
Appendix B. Another way of the duality derivation

In this section we will perform another way of duality derivation, being more in spirit of standard Feynman Perturbation Theory derivation. Factually, it is the way some authors used to follow in [20, 21, 19, 7]. We will assume here also that $V(\phi)/\phi^2 \to \infty$ when $\phi \to \infty$. We will need this fact later to justify the transformations we will perform, namely - to verify the convergence of the obtained series.

Just as the part with interaction is taken out as an exponent with variations over the currents when deriving the perturbation theory, we will take out the kinetic part in the same way:

$$Z[j(x)] = e^{-\frac{i}{2} \sum x \lambda_k \frac{\partial^2}{\partial x^2} \sum_{x} V(\phi(x))} \int d\phi \prod_{\{\lambda_p\}_{p \in \mathcal{V}}} e^{i \sum_{x} j(x)\phi(x) - \sum_{x} V(\phi(x))}$$

Let us underline here we have made a change of variables from $\{\phi(x)\}_{x \in \mathcal{V}}$ to $\{\phi_p\}_{p \in \mathcal{V}}$, in the kinetic part, exactly as in the section [2]

This transformation is convenient because it allows us to reduce the integral to a product of point-wise aren’t entangled integrals, and we will use this further. In addition, the remaining integrals are pointwise Fourier transforms of the potential exponent.

Let us prove that this equality is valid not only in the sense of asymptotic series. Since we perform all manipulations with the value $Z[j(x)]$, the number of sites $N$ is treated as a constant, and we are simply working with a finite-dimensional integral. One can use the Dominated Convergence Theorem [11] to prove the possibility of taking out the exponent. To perform this we consider an arbitrary finite sum of the exponent series of length $Q$ and obtain an estimation, that doesn’t depend on $Q$ for an absolutely integrable function. Namely:

$$\left| \sum_{\{\phi_p\}_{p \leq Q}} \prod_{p} \left\{ \int d\phi_p \frac{(-1)^p \lambda_p \phi_{2p}}{l_p!} \right\} e^{i \sum_{x} j(x)\phi(x) - \sum_{x} V(\phi(x))} \right| \leq \sum_{\{\phi_p\}_{p \leq Q}} \prod_{p} \left\{ \int d\phi_p \lambda_p |\phi_p|^{2p} \right\} e^{-\sum_{x} V(\phi(x))}$$

But the last integral converges absolutely since the integral of the potential converges, and the order of growth of the quadratic part is less than the order of growth of the potential, according to our assumptions. Note that for even potentials $|\phi|^n$ this is true when $n > 2$.

Evaluation of inner integrals gives us the next form of partition function:

$$Z[j(x)] = \frac{|\det U|}{\sqrt{(2\pi)^N \det G}} e^{-\frac{i}{2} \sum x \lambda_k \frac{\partial^2}{\partial x^2} \sum_{x} V(\phi(x))}$$

where $U$ - transition matrix from the frame of "delta functions" [10] on the grid to the frame of kinetic part operator eigenfunctions $\{\epsilon_k(x_i)\}$. It arises when the integral over the modes is replaced by the integral over the values at the nodes. Function $\tilde{V}$ is the Fourier image of the exponent of the potential. One can mention, that $\det U = \pm 1$, since $\tilde{U}$ is a transition matrix between orthonormal frames, so $|\det U| = 1$.

The idea of the next step of the derivation lies in the fact that the exponent before the integral is the product of $N$ evolution operators of one-dimensional heat equations. To emphasize our logic let us consider a one-dimensional example:

$$f(j, \lambda) = e^{\frac{i}{2} \lambda \frac{\partial^2}{\partial j^2} h(j)}$$

then differentiating both sides of the equation with respect to $\lambda$, we find that $f$ satisfies the heat equation:

$$\frac{\partial f}{\partial \lambda} = \frac{1}{2} \frac{\partial^2 f}{\partial j^2}$$

where $\lambda$ acts as a time, and $j$ - as a coordinate. Function $h(j)$ is the "initial condition", i.e. the value $f$ at $\lambda = 0$. So, we can write the answer for $f$ through the convolution product with Green’s function of the heat equation. According to it we get:

$$f(j, \lambda) = \frac{1}{\sqrt{2\pi \lambda}} \int d\psi e^{-\frac{(\psi - j)^2}{2\lambda}} h(\psi)$$
where $\psi$ corresponds to the current.

Since all the operators $\lambda_k \frac{\partial^2}{\partial j_k^2}$ commute with each other, for $N$ modes one can write:

$$Z[j(x)] = \left( \frac{1}{(2\pi)^N \sqrt{\det G \sqrt{\det L}}} \right) \int_{\mathbb{R}^N} \prod_k d\psi_k e^{-\frac{1}{2} \sum_k \frac{(\lambda_k - \psi_k)^2}{\lambda_k} + \sum_x L_n[\tilde{V}(\psi(x))]$$

Finally, cancelling the product of determinants of the operator and its inverse, we obtain:

$$Z[j(x)] = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \prod_k d\psi_k e^{-\frac{1}{2} \sum_k \frac{(\lambda_k - \psi_k)^2}{\lambda_k} + \sum_x L_n[\tilde{V}(\psi(x))]$$

The resulting integral converges, since even the integral without the first factor converges (this will be just the inverse Fourier transform). Thus, we have connected the two different lattice theories: in one theory kinetic operator was local with an integer potential and the other one, strongly nonlocal and with a worse potential. The role of the fields $\psi$ in the new theory is played by the currents of the fields $\phi$ in the theory. For convenience, we include $\tilde{V}$ into the exponent using one of the branches of the logarithm (by the definition of the logarithm, this will not affect the result of exponentiation). The value $\tilde{V}$ is always real, but can be negative, so the written complex logarithm can either be the same as the logarithm of the modulus or differ from it by $\pi i$. However, since we use this logarithm only in sense of power series being then exponentiated, the choice of the branch does not matters.

We see, that we have obtained the same expression, as in the (47), but with the completely different way. However, it we rewrite Ising model as a field theory [23]:

$$\text{det}(\beta J)$$

Thus, we have connected the two different lattice theories: in one theory kinetic operator was local with an integer potential and the other one, strongly nonlocal and with a worse potential. The role of the fields $\psi$ in the new theory is played by the currents of the fields $\phi$ in the theory. For convenience, we include $\tilde{V}$ into the exponent using one of the branches of the logarithm (by the definition of the logarithm, this will not affect the result of exponentiation). The value $\tilde{V}$ is always real, but can be negative, so the written complex logarithm can either be the same as the logarithm of the modulus or differ from it by $\pi i$. However, since we use this logarithm only in sense of power series being then exponentiated, the choice of the branch does not matters.

We see, that we have obtained the same expression, as in the (47), but with the completely different way. However, it can appear to be more natural, since it follows the philosophy of the common Feynman Expansion construction and starts from the action of taking out the kinetic part.

One can also try to extend this duality to the duality of continuous theories by writing something like (ignoring the infinite factor):

$$Z[j(x)] = \int D' \psi(x) \exp \left[ -\frac{1}{2} \langle j(x) - \psi(x), G(j(y) - \psi(y)) \rangle + \sum_x Ln[\tilde{V}(\psi(x))] \right],$$

Definitely, this expression is rather formal than exact and needs significant clarification. Though, let us speculate a bit about its sense. In the first term, the continuous transition is quite simple. At the same time, the continuous limit of the second term is not obvious. This all witnesses that the obtained duality is specific for the lattice theories, and its continuous limit in any sense deserves a separate and probably complicated study.

Appendix C. Application of the duality to Ising model

It is also intriguing to find out what does the presented duality mean being applied to Ising model. It is also advantageous for the additional verification of the duality.

So, let us consider the statistical lattice model with partition function and Hamiltonian of the form:

$$Z = \sum_{\{\sigma_x = \pm 1\}} e^{-\beta H(\{\sigma_x\})}, \quad H(\{\sigma_x\}) = -\frac{1}{2} \sum_{x,x'} J_{x,x'} \sigma_x \sigma_{x'},$$

where the variables $\sigma_x$ represent spins sitting in the nodes of the lattice. We will consider the ferromagnetic regime, though principally the same could be done also for the antiferromagnetic case. Hereafter, we will use the notations from the section[2.1] Using Hubbard-Stratonovich transform:

$$\frac{1}{(2\pi)^{N/2}} \int \frac{d\phi(x)}{\sqrt{\det(\beta J)}} \exp \left( -\frac{1}{2} \sum_{x,x'} (\beta J)_{x,x'}^{-1} \phi(x) \phi(x') + \sum_x \phi(x) \sigma_x \right) = \exp \left( \frac{\beta}{2} \sum_{x,x'} J_{x,x'} \sigma_x \sigma_{x'} \right),$$

we rewrite Ising model as a field theory [23]:

$$Z = \frac{1}{(2\pi)^{N/2}} \int \frac{d\phi(x)}{\sqrt{\det(\beta J)}} \exp \left( -\frac{1}{2} \sum_{x,x'} (\beta J)_{x,x'}^{-1} \phi(x) \phi(x') + \sum_x \ln (2 \cosh \phi(x)) \right).$$
So the action has a form:

\[ S[\phi] = \frac{1}{2} \sum_{x,x'} (\beta J)^{-1}_{x,x'} \phi(x) \phi(x') - \sum_x \ln (2 \cosh \phi(x)). \]

Unfortunately, it is impossible to apply the duality from section 2.2 to the action of this form, since the exponent of the interaction \( - \sum_x \ln (2 \cosh \phi(x)) \) does not have any Fourier transform even in the sense of distributions. So, let us perform some regularization by the addition of “smart zero”, namely:

\[ S[\phi] = \left( \frac{1}{2} \sum_{x,x'} (\beta J)^{-1}_{x,x'} \phi(x) \phi(x') - \frac{\epsilon}{2} \sum_x \phi(x)^2 \right) + \left( \frac{\epsilon}{2} \sum_x \phi(x)^2 - \sum_x \ln (2 \cosh \phi(x)) \right), \]

for \( \text{Re} \, \epsilon > 0 \) and small enough. And now we are able to calculate the Fourier Transform of the exponential of each bracket in the formula above. Applying Plancherel Theorem, we get to:

\[ Z = \frac{e^{\frac{N}{2} \epsilon^{-N/2}}}{(2\pi)^{N/2}} \int \frac{\prod_x d\psi(x)}{\sqrt{\det(1 - \epsilon \beta J)}} \exp \left( -\frac{1}{2} \sum_{x,x'} ((\beta J)^{-1} - \epsilon)^{-1}_{x,x'} \psi(x) \psi(x') - \frac{1}{2\epsilon} \sum_x \psi(x)^2 + \sum_x \ln \left( 2 \cosh \frac{\psi(x)}{\epsilon} \right) \right), \]

or, simplifying:

\[ Z = \frac{e^{\frac{N}{2} \epsilon^{-N/2}}}{(2\pi)^{N/2}} \int \frac{\prod_x d\psi(x)}{\sqrt{\det(1 - \epsilon \beta J)}} \exp \left( -\frac{1}{2} \sum_{x,x'} (1 - \epsilon \beta J)^{-1}_{x,x'} \psi(x) \psi(x') + \sum_x \ln \left( 2 \cosh \frac{\psi(x)}{\epsilon} \right) \right). \]

This is exactly the form of dual action for the Ising model. Nevertheless, a closer examination of the derived expression reveals that it is, once again, the Ising field theory. More precisely, if one rewrites the \( \cos \) as the sum of exponents:

\[ \exp \left[ \sum_x \ln \left( 2 \cosh \frac{\psi(x)}{\epsilon} \right) \right] = \sum_{\{\sigma_x = \pm 1\}} \exp \left( \frac{i}{\epsilon} \sum_x \sigma_x \psi(x) \right), \]

we obtain immediately:

\[ Z = \frac{e^{\frac{N}{2} \epsilon^{-N/2}}}{(2\pi)^{N/2}} \cdot (2\pi)^{N/2} e^{N/2} \sum_{\{\sigma_x = \pm 1\}} \exp \left( -\frac{1}{2} \sum_{x,x'} (1 - \epsilon \beta J)_{x,x'} \sigma_x \sigma_{x'} \right) = \exp \left( \frac{\beta}{2} \sum_{x,x'} (J_{x,x'} \sigma_x \sigma_{x'}) \right), \]

which is exactly the initial theory partition function. Firstly, we see, that the provided duality leads us to a correct, though trivial result for the case of Ising model. Secondly, its application is equivalent to the addition of a trivial term to the initial Hamiltonian. So these the reasons we wouldn’t testify the duality at Ising model. However, it is still an interesting question if it provides the trivial results for all discrete theories (such as Potts model, \( XY \)-model, etc.), or it is the miracle of Ising model only. For the case of the continuous theories, such as lattice \( \phi^4 \) it gives some kind of non-trivial transformation, as we have seen in the section 2.3

The final question regarding Ising model we would like to examine it is the structure of perturbation theory. Unfortunately, there is no any large parameter (such as coupling constant \( g \) in \( \phi^4 \) theory) we can expand over. So this question is a kind of academical interest only. We can expand potential, exactly as in the case of lattice \( \phi^4 \) and rescale \( \psi(x) \leftrightarrow \epsilon \psi(x) \):

\[ Z = \frac{e^{\frac{N}{2} \epsilon^{-N/2} 2^N}}{(2\pi)^{N/2}} \int \frac{\prod_x d\psi(x)}{\sqrt{\det(1 - \epsilon \beta J)}} \exp \left( -\frac{1}{2} \sum_{x,x'} \left( \frac{\epsilon + 1 - \epsilon \beta J}{1 - \epsilon \beta J} \right)_{x,x'} \psi(x) \psi(x') + \frac{2}{4!} \sum_x \psi(x)^4 + \ldots \right). \]

Conceivably, one can invent some applications of such decompositions, taking special values of \( \epsilon \) for given values of interest of \( \beta \). However, it all looks pretty artificially and as, we have seen, being no more than a variation of Hubbard-Stratanovich transform for the initial partition function. It is nothing basically new in this expression compared to the classical approach, so we won’t consider Ising model in this paper due to the triviality of the described duality in application to it.

\[ ^2 \text{For the antiferromagnetic case, there should be} - \sum_x \ln (2 \cos \phi(x)) \text{for some branch of the logarithm (which does not matter because of the exponentiation), however after the Fourier transform we obtain the sum of delta functions, which will lead to the same trivial result, as in the considering ferromagnetic regime.} \]