MAXIMUM OF DYSON BROWNIAN MOTION AND NON-COLLIDING SYSTEMS WITH A BOUNDARY

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Abstract

We prove an equality-in-law relating the maximum of GUE Dyson’s Brownian motion and the non-colliding systems with a wall. This generalizes the well known relation between the maximum of a Brownian motion and a reflected Brownian motion.

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1 Introduction and Results

Dyson’s Brownian motion model of GUE (Gaussian unitary ensemble) is a stochastic process of positions of \( m \) particles, \( X(t) = (X_1(t), \ldots, X_m(t)) \) described by the stochastic differential equation,

\[
dX_i = dB_i + \sum_{1 \leq j \leq m, j \neq i} \frac{dt}{X_i - X_j}, \quad 1 \leq i \leq m, \tag{1.1}
\]

where \( B_i, 1 \leq i \leq m \) are independent one dimensional Brownian motions\textsuperscript{[6]}. The process satisfies \( X_1(t) < X_2(t) < \cdots < X_m(t) \) for all \( t > 0 \). We remark that the process \( X \) can be started from the origin, i.e., one can take \( X_i(0) = 0, 1 \leq i \leq m \). See \textsuperscript{[11]}.

One can introduce similar non-colliding system of \( m \) particles with a wall at the origin \textsuperscript{[8, 9, 17]}. The dynamics of the positions of the \( m \) particles \( X^{(C)} = (X_1^{(C)}, \ldots, X_m^{(C)}) \) satisfying \( 0 < X_1(t) < X_2(t) < \cdots < X_m(t) \) for all \( t > 0 \) are described by the stochastic differential equation,

\[
dX_i^{(C)} = dB_i + \frac{dt}{X_i^{(C)}} + \sum_{1 \leq j \leq m, j \neq i} \left( \frac{1}{X_i^{(C)} - X_j^{(C)}} + \frac{1}{X_i^{(C)} + X_j^{(C)}} \right) dt, \quad 1 \leq i \leq m. \tag{1.2}
\]

This process is referred to as Dyson’s Brownian motion of type \( C \). It can be interpreted as a system of \( m \) Brownian particles conditioned to never collide with each other or the wall.

One can also consider the case where the wall above is replaced by a reflecting wall\textsuperscript{[9]}. The dynamics of the positions of the \( m \) particles \( X^{(D)} = (X_1^{(D)}, \ldots, X_m^{(D)}) \) satisfying \( 0 \leq X_1(t) < X_2(t) < \cdots < X_m(t) \) for all \( t > 0 \), is described by the stochastic differential equation,

\[
dX_i^{(D)} = dB_i + \frac{1}{2} 1_{(i=1)} dL(t) + \sum_{1 \leq j \leq m, j \neq i} \left( \frac{1}{X_i^{(D)} - X_j^{(D)}} + \frac{1}{X_i^{(D)} + X_j^{(D)}} \right) dt, \quad 1 \leq i \leq m, \tag{1.3}
\]

where \( L(t) \) denotes the local time of \( X_1^{(D)} \) at the origin. This process will be referred to as Dyson’s Brownian motion of type \( D \). Some authors consider a process defined by the s.d.e.s (1.3) without the local time term. In this case the first component of the process is not constrained to remain non-negative, and the process takes values in the Weyl chamber of type \( D \), \( \{ |x_1| < x_2 < x_3 \ldots < x_m \} \). The process we consider with a reflecting wall is obtained from this by replacing the first component with its absolute value, with the local time term appearing as a consequence of Tanaka’s formula.

It is known the processes \( X^{(C)} \) and \( X^{(D)} \) can be obtained using the Doob \( h \)-transform, see \textsuperscript{[8, 9]}. Let \( (P_t^{(D,C)}; t \geq 0) \), resp. \( (P_t^{D}; t \geq 0) \), be the transition semigroup for \( m \) independent Brownian motions killed on exiting \( \{ 0 < x_1 < x_2 \ldots < x_m \} \), resp. the transition semigroup for \( m \) independent Brownian motions reflected at the origin killed on exiting \( \{ 0 \leq x_1 < x_2 \ldots < x_m \} \). From the Karlin-McGregor formula, the corresponding densities can be written as

\[
\det\{\phi_t(x_i - x_j') - \phi_t(x_i + x_j')\}_{1 \leq i, j \leq m}, \quad \tag{1.4}
\]

resp.,

\[
\det\{\phi_t(x_i - x_j') + \phi_t(x_i + x_j')\}_{1 \leq i, j \leq m}, \quad \tag{1.5}
\]
where $\phi_1(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}$. Let
\[
\begin{align*}
 h^{(c)}(x) &= \prod_{i=1}^{m} x_i \prod_{1 \leq i < j \leq m} (x_j^2 - x_i^2), \\
 h^{(d)}(x) &= \prod_{1 \leq i < j \leq m} (x_j^2 - x_i^2). 
\end{align*}
\tag{1.6}
\]

For notational simplicity we suppress the index $C, D$ for the semigroups and in $h$ in the following. Then one can show that $h(x)$ is invariant for the $P^0_t$ semigroup and we may define a Markov semigroup by
\[
P_t(x, dx') = h(x') P^0_t(x, dx')/h(x).
\tag{1.7}
\]

This is the semigroup of the Dyson non-colliding system of Brownian motions of type $C$ and $D$. Similarly to the $X$ process, the processes $X^{(c)}$ and $X^{(d)}$ can also be started from the origin (see [4] or use Lemma 4 in [9] and apply the same arguments as in [11]).

In GUE Dyson’s Brownian motion of $n$ particles, let us take the initial conditions to be $X_i(0) = 0, 1 \leq i \leq n$. The quantity we are interested in is the maximum of the position of the top particle for a finite duration of time, $\sup_{0 \leq s \leq t} X_n(s)$. In the sequel we write $\sup$ instead of max to conform with common usage in the literature. Let $m$ be the integer such that $n = 2m$ when $n$ is even and $n = 2m - 1$ when $n$ is odd. Consider the non-colliding systems of $X^{(c)}$, resp. $X^{(d)}$, of $m$ particles starting from the origin, $X^{(c)}_1(0) = 0, 1 \leq i \leq m$, resp. $X^{(d)}_1(0) = 0, 1 \leq i \leq m$.

Our main result of this note is

**Theorem 1.** Let $X$ and $X^{(c)}, X^{(d)}$ start from the origin. Then for each fixed $t \geq 0$, one has
\[
\sup_{0 \leq s \leq t} X_n(s) \overset{d}{=} \begin{cases} 
X^{(c)}_m(t), & \text{for } n = 2m, \\
X^{(d)}_m(t), & \text{for } n = 2m - 1.
\end{cases}
\tag{1.8}
\]

To prove the theorem we introduce two more processes $Z_j$ and $Y_j$. In the $Z$ process, $Z_1 \leq Z_2 \leq \ldots \leq Z_n$, $Z_1$ is a Brownian motion and $Z_{j+1}$ is reflected by $Z_j$, $1 \leq j \leq n - 1$. Here the reflection means the Skorohod construction to push $Z_{j+1}$ up from $Z_j$. More precisely,
\[
Z_1(t) = B_1(t), \\
Z_j(t) = \sup_{0 \leq s \leq t} (Z_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \leq j \leq n,
\tag{1.9}
\]

where $B_i, 1 \leq i \leq n$ are independent Brownian motions, each starting from 0. The process is the same as the process $(X_1^1(t), X_2^2(t), \ldots, X_n^n(t); t \geq 0)$ studied in section 4 of [18]. The representation (1.9) was given earlier in [2]. In the $Y$ process, $0 \leq Y_1 \leq Y_2 \leq \ldots \leq Y_n$, the interactions among $Y_j$’s are the same as in the $Z$ process, i.e., $Y_{j+1}$ is reflected by $Y_j$, $1 \leq j \leq n - 1$, but $Y_1$ is now a Brownian motion reflected at the origin (again by Skorohod construction). Similarly to (1.9),
\[
Y_1(t) = B_1(t) - \inf_{0 \leq s \leq t} B_1(s) = \sup_{0 \leq s \leq t} (B_1(t) - B_1(s)), \\
Y_j(t) = \sup_{0 \leq s \leq t} (Y_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \leq j \leq n.
\tag{1.10}
\]

From the results in [11, 5, 18], we know
\[
(X_n(t); t \geq 0) \overset{d}{=} (Z_n(t); t \geq 0)
\tag{1.11}
\]
and hence
\[ \sup_{0 \leq s \leq t} X_n(s) \overset{d}{=} \sup_{0 \leq s \leq t} Z_n(s). \] (1.12)

In this note we show

**Proposition 2.** The following equalities in law hold between processes:

\[ (Y_{2m}(t); t \geq 0) \overset{d}{=} (X^{(C)}_m(t); t \geq 0), \]
\[ (Y_{2m-1}(t); t \geq 0) \overset{d}{=} (X^{(D)}_m(t); t \geq 0), \] (1.13)

\( m \in \mathbb{N}. \)

The proof of this proposition is given in Section 2. The idea behind it is that the processes \((Y_i)_{i \geq 1}, (X^{(C)}_j)_{j \geq 1}\) and \((X^{(D)}_j)_{j \geq 1}\) could be realized on a common probability space consisting of Brownian motions satisfying certain interlacing conditions with a boundary \([18, 19]\). Such a system is expected to appear as a scaling limit of the discrete processes considered in \([3, 19]\). In this enlarged process, the processes \(Y_n(t)\) and \(X^{(C)}_m(t)\) or \(X^{(D)}_m(t)\) just represent two different ways of looking at the evolution of a specific particle and so the statement of Proposition 2 follows immediately. Justification of such an approach is however quite involved, and we prefer to give a simple independent proof. See also \([5]\) for another representation of \(X^{(C)}_m\) and \(X^{(D)}_m\) in terms of independent Brownian motions.

Then to prove (1.13) it is enough to show

**Proposition 3.** For each fixed \(t\) we have

\[ \sup_{0 \leq s \leq t} Z_n(s) \overset{d}{=} Y_n(t). \] (1.14)

This is shown in Section 3. For \(n = 1\) case, this is well known from the Skorokhod construction of reflected Brownian motion \([12]\). The \(n > 1\) case can also be understood graphically by reversing time direction and the order of particles. This relation could also be established as a limiting case of the last passage percolation. In fact the identities in our theorem was first anticipated from the consideration of a diffusion scaling limit of the totally asymmetric exclusion process with 2 speeds \([11]\) (in particular the last part of sections 1, 2 and section 7).

Before closing the section, we remark that similar maximization properties of Dyson’s Brownian motion have been considered for other boundary conditions in \([15, 10, 7]\).

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## 2 Proof of Proposition 2

In this section we prove the relation between \(X^{(C)}\) or \(X^{(D)}\) and \(Y\), (1.13). The following lemma is a generalization of the Rogers-Pitman criterion \([13]\) for a function of a Markov process to be Markovian. Note that it gives us a method to deduce an equality in law between two processes that need not themselves be Markov- as indeed is the case in Proposition 2.
Lemma 4. Suppose that \( \{X(t) : t \geq 0\} \) is a Markov process with state space \( E \), evolving according to a transition semigroup \( \{P_t : t \geq 0\} \) and with initial distribution \( \mu \). Suppose that \( \{Y(t) : t \geq 0\} \) is a Markov process with state space \( F \), evolving according to a transition semigroup \( \{Q_t : t \geq 0\} \) and with initial distribution \( \nu \). Suppose further that \( L \) is a Markov transition kernel from \( E \) to \( F \), such that \( \mu L = \nu \) and the intertwining \( P_t L = Q_t \) holds. Now let \( f : E \to G \) and \( g : F \to G \) be maps into a third state space \( G \), and suppose that

\[
L(x, \cdot) \text{ is carried by } \{y \in F : g(y) = f(x)\} \text{ for each } x \in E.
\]

Then we have

\[
\{f(X(t)) : t \geq 0\} \overset{d}{=} \{g(Y(t)) : t \geq 0\},
\]

in the sense of finite dimensional distributions.

Proof of Lemma 4. For any bounded function \( \alpha \) on \( G \) let \( \Gamma_1 \alpha \) be the function \( \alpha \circ f \) defined on \( E \) and let \( \Gamma_2 \alpha \) be the function \( \alpha \circ g \) defined on \( F \). Then it follows from the condition that \( L(x, \cdot) \) is carried by \( \{y \in F : g(y) = f(x)\} \) that whenever \( h \) is a bounded function defined on \( F \) then

\[
L(\Gamma_2 \alpha \times h) = \Gamma_1 \alpha \times Lh,
\]

which is shorthand for \( \int L(x, dy) \Gamma_2 \alpha (y) h(y) = \Gamma_1 \alpha \times Lh \). For any bounded test functions \( \alpha_0, \alpha_1, \ldots, \alpha_n \) defined on \( G \), and times \( 0 < t_1 < \cdots < t_n \), we have, using the previous equation and the intertwining relation repeatedly,

\[
\begin{align*}
\mathbb{E}[\alpha_0(g(Y(0)))\alpha_1(g(Y(t_1)))\ldots\alpha_n(g(Y(t_n)))] \\
= \nu(\Gamma_2 \alpha_0 \times Q_{\tau_1}(\Gamma_2 \alpha_1 \times Q_{\tau_2-\tau_1}(\cdots (\Gamma_2 \alpha_{n-1} \times Q_{\tau_n-\tau_{n-1}} \Gamma_2 \alpha_n)))) \\
= \mu L(\Gamma_2 \alpha_0 \times Q_{\tau_1}(\Gamma_2 \alpha_1 \times Q_{\tau_2-\tau_1}(\cdots (\Gamma_2 \alpha_{n-1} \times Q_{\tau_n-\tau_{n-1}} \Gamma_2 \alpha_n)))) \\
= \mu(\Gamma_1 \alpha_0 \times P_{\tau_1}(\Gamma_1 \alpha_1 \times P_{\tau_2-\tau_1}(\cdots (\Gamma_1 \alpha_{n-1} \times P_{\tau_n-\tau_{n-1}} \Gamma_1 \alpha_n)))) \\
= \mathbb{E}[(\alpha_0(f(X(0)))\alpha_1(f(X(t_1)))\ldots\alpha_n(f(X(t_n)))]
\end{align*}
\]

which proves the equality in law.

We let \( (Y(t) : t \geq 0) \) be the process \( Y \) of \( n \) reflected Brownian motions with a wall introduced in the previous section. It is clear from the construction \([1,10]\) that the process \( Y \) is a time homogeneous Markov process. We denote its transition semigroup by \( \{Q_t : t \geq 0\} \). It turns out that there is an explicit formula for the corresponding densities. Recall \( \phi_1(z) = \frac{1}{\sqrt{2\pi} e^{-z^2/(2t)}} \). Let us define \( \phi_k^{(k)}(y) = \frac{d^k}{dz^k} \phi_1(y) \) for \( k \geq 0 \) and \( \phi_k^{(-k)}(y) = (-1)^k \int_y^\infty \frac{(z-y)^{k-1}}{(k-1)!} \phi_1(z)dz \) for \( k \geq 1 \).

Proposition 5. The transition densities \( q_t(y, y') \) from \( y = (y_1, \ldots, y_n) \) at \( t = 0 \) to \( y' = (y_1', \ldots, y_n') \) at \( t \) of the \( Y \) process can be written as

\[
q_t(y, y') = \det\{a_{i,j}(y_i, y'_j)\}_{1 \leq i, j \leq n}
\]

where \( a_{i,j} \) is given by

\[
a_{i,j}(y, y') = (-1)^{i-1} \phi_1^{(j-i)}(y+y') + (-1)^{i+j} \phi_1^{(j-i)}(y-y').
\]
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Figure 1: The set \( \mathbb{K} \). The triangle represents the intertwining relations of the variables \( z \) and the vertical line on the left indicates \( z_1^{2k+1} \geq 0 \), see (2.5), (2.6). The set of variables on the bottom line is denoted by \( b(z) \) and the one on the upper right line by \( e(z) \).

**Proof of Proposition 5** For a fixed \( y' \), define \( G(y, t) \) to be (2.3) as a function of \( y \) and \( t \). We check that \( G \) satisfies (i) the heat equation, (ii) the boundary conditions \( \frac{\partial G}{\partial y_1} |_{y_1=0} = 0; \frac{\partial G}{\partial y_i} |_{y_i=y_{i-1}} = 0, i = 2, 3, \ldots, n \) and (iii) the initial conditions \( G(y, t = 0) = \prod_{i=1}^{n} \delta(y_i - y') \).

(i) holds since \( \phi_t^{(k)}(y) \) for each \( k \) satisfies the heat equation. (ii) follows from the relations, \( \frac{\partial}{\partial y} a_{ij}(y, y') |_{y=0} = \phi_t^{(i)}(y') + (-1)^{i+1} \phi_t^{(j)}(-y') = 0 \) and \( \frac{\partial}{\partial y} a_{ij}(y, y') = -a_{i-1,j}(y, y') \). For (iii) we notice that the first term in (2.4) goes to zero as \( t \to 0 \) for \( y, y' > 0 \) and the statement for the remaining part is shown in Lemma 7 in [13].

For \( n = 2m \), resp. \( n = 2m - 1 \) we take \( (X(t), t \geq t) \) to be Dyson Brownian motion of type \( C \), resp. of type \( D \). The transition semigroup \( (P_t; t \geq 0) \) of this process is given by (1.7).

Let \( \mathbb{K} \) denote the set with \( n \) layers \( z = (z^1, z^2, \ldots, z^n) \) where \( z^{2k} = (z^{2k}_1, z^{2k}_2, \ldots, z^{2k}_k) \in \mathbb{R}^k \), \( z^{2k-1} = (z^{2k-1}_1, z^{2k-1}_2, \ldots, z^{2k-1}_k) \in \mathbb{R}^k \) and the intertwining relations,

\[
\begin{align*}
z^{2k-1} & \leq z^k_1 \leq z^{2k-1} \leq z^k_2 \leq \cdots \leq z^{2k-1} \leq z^k_k \quad (2.5) \\
0 & \leq z^{2k+1}_1 \leq z^{2k}_1 \leq z^{2k+1}_2 \leq z^{2k}_2 \leq \cdots \leq z^{2k+1}_k \leq z^{2k+1}_k \quad (2.6)
\end{align*}
\]

hold (Fig. 1). Let \( n = 2m \) or \( n = 2m - 1 \) for some integer \( m \). We define a kernel \( L^0 \) from \( E = \{ 0 \leq x_1 \leq \ldots \leq x_m \} \) to \( F = \{ 0 \leq y_1 \leq \ldots \leq y_n \} \). For \( z \in \mathbb{K} \), define \( b(z) = z^n = (z^n_1, \ldots, z^n_m) \in E \), \( e(z) = (z^1, z^2, z^3, \ldots, z^n) \in F \) and \( \mathbb{K}(x) = \{ z \in \mathbb{K}; b(z) = x \in E \} \), \( \mathbb{K}(y) = \{ z \in \mathbb{K}; e(z) = y \in F \} \).

The kernel \( L^0 \) is defined by

\[
L^0 g(x) = \int_F L^0(x, dy) g(y) = \int_{\mathbb{K}(x)} g(e(z)) dz. \quad (2.7)
\]

where the integrals are taken with respect to Lebesgue measure but integrations with respect to \( z \) on the RHS is for \( b(z) = x \) fixed.

The function \( h \) defined at (1.6) is equal to the Euclidean volume of \( \mathbb{K}(x) \). Consequently we may define \( L \) to be the Markov kernel \( L(x, dy) = L^0(x, dy)/h(x) \). In the remaining part of this section we show

**Proposition 6.**

\[
LQ_t = P_t L. \quad (2.8)
\]
Now if we apply Lemma 4 with $f(x) = x_m$, $g(y) = y_n$ and the initial conditions starting from the origin we obtain (1.13).

**Proof of Proposition 6** The kernels $P_t(x, \cdot)$ and $L(x, \cdot)$ are continuous in $x$. Thus we may consider $x$ in the interior of $E$, and it is enough to prove

$$L^0 Q_t(x, dy) = (P^0_t L^0)(x, dy).$$

From the definition of the kernel $L^0$, this is equivalent to showing

$$\int_{x(x)} q_i(e(z), y) dz = \int_{y(y)} p^0_i(x, b(z)) dz$$

where $q_i$ and $p^0$ are densities corresponding to $Q_t$ and $P^0_t$. Integrations with respect to $z$ are on the LHS with $b(z) = x$ fixed and on the RHS with $e(z) = y$ fixed.

Let us consider the case where $n = 2m$. Using the determinantal expressions for $q_i$ and $p^0_t$ we show that both sides of (2.10) are equal to the determinant of size $2m$ whose $(i, j)$ matrix element is $a_{2i,j}(0, y_j)$ for $1 \leq i \leq m$, $1 \leq j \leq 2m$ and $a_{2m,j}(x_{i-m}, y_j)$ for $m + 1 \leq i \leq 2m, 1 \leq j \leq 2m$.

The integrand of the LHS of (2.10) is

$$q_i(e(z), y) = \det\{a_{i,j}(e(z)_l, y)_j\}_{1 \leq i, j \leq 2m}$$

with $b(z) = x$. We perform the integral with respect to $z^1, \ldots, z^{2m-1}$ in this order. After the integral up to $z^{2l-1}$, $1 \leq l \leq m$, we get the determinant of size $2m$ whose $(i, j)$ matrix element is $a_{2i,j}(0, y_j)$ for $1 \leq i \leq l$, $a_{2i,j}(x_{i-l}, y_j)$ for $l + 1 \leq i \leq 2l$ and $a_{i,j}(e(z)_l, y_j)$ for $2l + 1 \leq i \leq 2m$. Here we use a property of $a_{i,j}$,

$$a_{i,j}(y', y') = \int_{y'}^\infty a_{i-1,j}(u, y') du,$$

and do some row operations in the determinant. The case for $l = m$ gives the desired expression. The integrand of the RHS of (2.10) is

$$p^0_t(x, z^{2m}) = \det(a_{2m,2m}(x_i, z^{2m}_j))_{1 \leq i, j \leq m}$$

with the condition $e(z) = y$. We perform the integrals with respect to $(z^{2m}_1, \ldots, z^{2m}_{m-1}), (z^{2m-1}_1, \ldots, z^{2m-1}_{m-1}), \ldots, z^{2}_1, z^{2}_1$ in this order. We use properties of $a_{i,j}$,

$$a_{i,j}(y', y') = -\int_{y'}^\infty a_{i,j+1}(y, u) du,$$

$$a_{2l,2l}(x, 0) = 0, \quad a_{2l,2l-1}(0, y) = 1, \quad a_{2l,2l}(0, 0) = 0, \quad 2l \leq j.$$ (2.15)

After each integration corresponding to a layer of $\mathbb{K}$ we simplify the determinant using column operations. We also expand the size of the determinant after an integration corresponding to $(z^{2l}_1, \ldots, z^{2l}_{l-1})$ for $1 \leq l \leq m$, by adding a new first row

$$(1, 1, \ldots, 1, 0, 0, \ldots, 0) = \left(2m-2l+1 \atop a_{2l,2l-1}(0, z^{2l-1}_1), \ldots, a_{2l,2l-1}(0, z^{2l-1}_1), a_{2l,2l}(0, e(z)_2), \ldots, a_{2l,2m}(0, e(z)_2)) \right)$$ (2.16)
together with a new column. After the integrals up to \((s_i^{2i-1}, \ldots, s_j^{2j-1})\) have been performed, we obtain the determinant of size \(2m - l + 1\),

\[
\begin{vmatrix}
    a_{2(i+i-1),2(l-1)}(0,s_i^{2(i-1)}) & a_{2(i+i-1),j+l-1}(0,e(z)_j) \\
    a_{2m,2(l-1)}(x_{i-m+l-1},s_j^{2(l-1)}) & a_{2m,j+l-1}(x_{i-m+l-1},e(z)_j)
\end{vmatrix}.
\] (2.17)

Here \(1 \leq i \leq m - l + 1\) (resp. \(m - l + 2 \leq i \leq 2m - l + 1\)) for the upper expression (resp. the lower expression) and \(1 \leq j \leq l - 1\) (resp. \(l \leq j \leq 2m - l + 1\)) for the left (resp. right) expression. For \(l = 1\) this reduces to the same determinant as for the LHS.

The case \(n = 2m - 1\) is almost identical. Similar arguments show that both sides of (2.9) are equal to a determinant of size \(2m - 1\) whose \((i, j)\) matrix element is \(a_{2i,j}(0, y_j)\) for \(1 \leq i \leq m - 1, 1 \leq j \leq 2m - 1\) and \(a_{2m-1,j}(x_{i-m+1}, y_j)\) for \(m + 1 \leq i \leq 2m - 1, 1 \leq j \leq 2m - 1\).

\[\square\]

3 Proof of Proposition 3

Using (1.10) repeatedly, one has

\[Y_n(t) = \sup_{0 \leq t_1 \leq \cdots \leq t_n \leq \bar{t}} \sum_{i=1}^{n} (B_i(t_{i+1}) - B_i(t_i))\] (3.1)

with \(t_{n+1} = \bar{t}\). By renaming \(t - t_{n+1}\) by \(t\) and changing the order of the summation, we have

\[Y_n(t) = \sup_{0 \leq t_1 \leq \cdots \leq t_n \leq \bar{t}} \sum_{i=1}^{n} (B_{n-i+1}(t - t_{i+1}) - B_{n-i+1}(t - t_i)).\] (3.2)

Since \(\bar{B}_i(s) := B_{n-i+1}(t) - B_{n-i+1}(t - s) \overset{d}{=} B_i(s)\),

\[Y_n(t) \overset{d}{=} \sup_{0 \leq t_1 \leq \cdots \leq t_n \leq \bar{t}} \sum_{i=1}^{n} (B_i(t_i) - B_i(t - t_{i-1})) = \sup_{0 \leq t_1 \leq \cdots \leq t_n \leq \bar{t}} Z_n(t).\] (3.3)

Graphically the above proof corresponds to reversing the time direction and the order of particles.

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