Predictions of a fundamental statistical picture

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Abstract

The discovery of a Higgs boson at the electroweak scale appears to point toward supersymmetry, as the most likely mechanism for protecting a scalar boson mass from enormous radiative corrections. The earlier discovery of neutrino masses similarly appears to point toward grand unification of nongravitational forces, which permits (for neutrinos) Majorana masses, Dirac masses, and a seesaw mechanism to drive the observed masses down to low values. A third major discovery, cosmic acceleration suggesting a relatively tiny cosmological constant, appears to point toward truly revolutionary new physics. Many other problems and mysteries also indicate a need for fresh ideas at the most fundamental level. Here a picture is proposed in which standard physics and its extensions are obtained (through a nontrivial set of arguments) from statistical counting and the local geography of our universe. The unavoidable predictions include supersymmetry (at some energy scale), $SO(N)$ grand unification, a drastic diminishing of the usual cosmological constant, and a nonsupersymmetric dark matter WIMP which should be detectable within the next several years.
I. INTRODUCTION

Complexity can emerge from simplicity in amazing ways, as when most of our observed world is attributed to two quarks and two leptons (plus gauge bosons and gravity). It is worthwhile to consider the possibility that all the complexities of the Standard Model and its extensions might similarly emerge from a very simple underlying description. Here we explore the results that follow from what appears to be the simplest imaginable picture, introduced in Section III.

One motivation for a fresh perspective on fundamental physics is the remarkable mix of clarity and confusion that currently exists. The situation in the early 21st century is, in fact, similar to what it was in the late 19th century. Then most physicists were generally satisfied with the successful paradigm of classical mechanics and electrodynamics, but there were some conflicting experimental data and theoretical puzzles. Now most physicists are generally satisfied with the successful paradigm of quantum fields and gauge theories (plus Einstein gravity), but there are again mysteries that suggest the need for a deeper theory. Many excellent reviews have been given of the current situation in physics and astronomy [1], but it may be worthwhile to begin with a brief summary.

Most recently, the particle discovered by the ATLAS and CMS collaborations at the LHC is now known to be a Higgs boson [1–3]. A naive conclusion is that the Standard Model of particle physics is now complete. But the more profound interpretation is that the discovery of a scalar boson immediately points to physics beyond the Standard Model, since otherwise radiative corrections should push the mass of this particle up to an absurdly large value. The most natural candidate for such new physics is supersymmetry (susy) [4–21], for which there is already indirect experimental evidence: The coupling constants of the 3 nongravitational forces are found to converge to a common value, as they are run up to high energy in a grand unified theory, only if the calculation includes susy. So, instead of acting as an endpoint for physics, and a mere capstone of the Standard Model, the observation of a Higgs boson opens the door to a plethora of new particles and effects.

Another major advance has been the discovery and exploration of neutrino masses [22], which appear to open the door to a more fundamental understanding of forces and matter via grand unification [5, 6, 21, 23, 31]. There are two possibilities for a neutrino mass, either of which is inconsistent with the requirements of the Standard Model. For a Dirac mass, an
extra field has to be added for each generation of fermions. For a Majorana mass, lepton number conservation has to be violated. But either or both types of mass are natural with grand unification, and in addition a seesaw mechanism can explain the small observed values of neutrino masses. At the moment, it is not known whether neutrinos have Majorana masses or Dirac masses or both. This is currently an intense area of research, and any outcome will again involve rich new physics and better understanding of Nature.

There are many other mysteries and gaps in fundamental understanding. For example, the discovery and exploration of cosmic acceleration [32, 33] has suggested the need for truly revolutionary new physics. The cause of this acceleration has increasingly been found to resemble a cosmological constant \( \Lambda \), and has therefore been a strong reminder of the original cosmological constant problem [34]: Because of the various contributions to the vacuum energy, conventional general relativity predicts that \( \Lambda \) should be vastly larger than permitted by observation. A parallel astronomical mystery is the origin of dark matter [35].

Still another and even older theoretical problem is the difficulty of reconciling general relativity with quantum mechanics [36]: A new fundamental theory must somehow regularize quantum gravity near the Planck scale, in addition to reducing the value of \( \Lambda \) by many orders of magnitude.

The next level of theoretical understanding is not likely to be a “theory of everything”, since “everything” surely transcends our current observational capabilities and imagination. But the most ambitious version of a more fundamental theory might hope to include and explain the following: the absence of an enormous cosmological constant, the origin of dark matter, the origin of gravitational and gauge interactions, the origin of Lorentz invariance, the gravitational metric and its signature (which distinguishes time from space and characterizes spacetime as 4-dimensional), the action for fermionic and bosonic fields, the action for gravitational and gauge fields, the regularization of quantum gravity near the Planck scale, the origin of quantum fields, and the origin of spacetime coordinates. As will be seen below, the present theory addresses all of these issues and leads to a substantial number of predictions. These are largely qualitative, because quantitative treatments in many cases would require a detailed understanding of the very complex vacuum fields after multiple symmetry breakings at various energy scales. However, there are some specific new features, with quite quantitative predictions, which should be testable in the near future. For example, the theory predicts new fundamental particles which can be produced in pairs through
their couplings to vector bosons. The lowest-mass of these is a dark matter candidate, with precisely defined couplings and mass, which is consistent with all current experiments and which should be detectable within the next several years.

II. OVERVIEW

DeWitt has provided an elegant survey of contemporary fundamental physics [37], which is based on path-integral quantization over classical trajectories in the combined space of coordinates and fields: “A classical dynamical system is described globally by a trajectory or history. A history is a section of a fibre bundle $E$ having the manifold $M$ of spacetime as its base space. The typical fibre is known as configuration space and will be denoted by $C$. Denote by $\Phi$ the set, or space, of all possible field histories, both those that do and those that do not satisfy the dynamical equations. The nature and dynamical properties of a classical dynamical system are completely determined by specifying an action functional $S$ for it.”

This is the basic picture used in all the versions of fundamental physics that are investigated by sizable communities of physicists. Notice that coordinates and fields have essentially the same status. This is consistent with the way they are defined in the present theory, starting near the beginning of the next section. In standard field theory, the spacetime coordinates $x^\mu$ correspond to $M$ and the fields to $C$. In conventional string (or $p$-brane) theory, $M$ is a 2-dimensional worldsheer (or $(p+1)$-dimensional worldvolume) and $C$ is specified by bosonic and fermionic coordinates.

The arguments below involve many unfamiliar definitions and redefinitions of fields, with physically accessible fields emerging from more primitive “hidden” degrees of freedom. But this kind of treatment is already necessary and familiar in standard physics – for example, when the fundamental gauge fields are redefined after Higgs condensation, and when many kinds of elementary excitations are defined in condensed matter physics. More broadly, the deeper hidden origin of observed phenomena is a common theme in science.

The physical fields that have emerged by the end of this paper are interpreted as those chosen by Nature to yield a stable vacuum $|0\rangle$ satisfying $a|0\rangle = 0$, where $a$ is a typical destruction operator for one of these fields. For example, the excitations $a^\dagger|0\rangle$ must have positive rather than negative energy, and this is why the transformed bosonic fields of Section
are physically acceptable, whereas the initial fields $\psi_b$ are not.

The principal ideas and results of this paper are as follows.

(1) In standard physics, all the events of the world are simply a progression through the states of fields on a spacetime manifold. In the present picture, they are a progression through the states of a single fundamental system. This second unified picture leads back to (and beyond) the first via the arguments below.

The most primitive microstates $|m\rangle$ of the fundamental system are taken to have equal amplitudes, so we are beginning with the simplest imaginable picture, in which all possibilities are realized and have equal weights.

To avoid confusion, it should be emphasized that time, quantum mechanics, etc. are yet to be defined in this initial picture, but one may still write

$$|\text{Nature}\rangle = \sum_m a_m |m\rangle, \quad a_m = 1 \text{ for all } m$$

in the same spirit as in ordinary quantum mechanics, where this global state, expanded in “basis states” $|m\rangle$, is postulated to describe all of physical reality.

In a sense, Nature is viewed as existing in all these states “simultaneously”, just as an electron in state $|\psi\rangle$ simultaneously exists in all position states $|\vec{r}\rangle$ with amplitudes $\langle \vec{r}|\psi\rangle$:

$$|\psi\rangle = \int d^3r |\vec{r}\rangle \langle \vec{r}|\psi\rangle.$$  \hfill (2.2)

Time will be defined as a parameter describing a trajectory through the space of states. There are, of course, precedents for defining time internally, within a stationary state, using some internal parameter. For example, in minisuperspace models based on the Wheeler-DeWitt equation \cite{38,39}, this parameter is the cosmic scale factor, and time is defined by the expansion of the universe. In the present picture, all coordinates and fields are defined internally, with the progression of events in Nature regarded as progression through a space of states. In the most primitive space, each point is defined by a microstate $|m\rangle$. In the emergent space corresponding to DeWitt’s $\Phi$ (see above), each point is defined by a macrostate, as described above (3.1).

For a concrete characterization of both microstates and macrostates, we adopt the picture that the fundamental system is composed of discrete distinguishable constituents that are called “dits” because each can exist in any of $d$ states labeled by $i = 1, 2, \ldots, d$. The number of dits in the $i$th state determines the size of the field or coordinate labeled by $i$.  

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The fact that fields and coordinates have the same basic status in standard physics, as noted above, is then explained by the fact that they have the same fundamental origin.

(2) The action is defined to be essentially the negative of the entropy, in (3.29). Action conventionally has the units of $\hbar$, and entropy the units of the Boltzmann constant $k$, but here we use natural units, with $\hbar = k = c = 1$.

(3) Much later in the paper, beginning with (7.16), the resulting path integrals with Euclidean form are transformed to equivalent Lorentzian path integrals, with the action left unchanged.

One has then regained the standard formulation of quantum field theory. One can subsequently transform from path-integral quantization to canonical quantization in the usual way (since the action has a standard form).

(4) The assumption that all states of the fundamental system are realized leads inevitably to a multiverse picture. There are still many who reject the possibility of a multiverse, but one should recall that most people at the time of Galileo would have rejected the possibility of hundreds of billions of galaxies, or a single galaxy, or even a heliocentric Solar System, and that the history of physics shows a steady progression toward more expansive views of Nature.

Among the vast number of states of the fundamental system (in the full path integral), there are some trajectories through these states which can legitimately be assigned to universes, in the sense that the states or configurations can be coherently connected with high probability. This will be the case if there is a path through the space of these configurations along which a local minimum is stably maintained in the action.

In the present picture our own universe is stabilized topologically – specifically, it has an extremely stable geometry determined by two topological defects in a primordial condensate, one in 4-dimensional external spacetime and the other in a $(D-4)$-dimensional internal space. Each of these is “vortex-like” (or “instanton-like”), in the sense that they involve circulation of the primordial condensate around a central point. The stability of our universe is then analogous to the stability of a vortex in a 2-dimensional fluid.

There is an additional nuance, in that a universe can be made stable through an effect which is exhibited in the behavior of the quartic self-coupling of the recently discovered Higgs boson: Within the Standard Model, the unrenormalized value of this coupling appears to be very nearly equal to zero [40–43], but at low energies it is made appreciably finite by
radiative corrections, so that a stable Higgs condensate forms. This suggests that more generally there will be configurations, within the complete path integral of all possibilities, where a universe is “bootstrapped” into existence, because (at low enough energies) radiative corrections will similarly yield a nonzero quartic self-coupling for a primordial condensate which allows it to form. We are thus envisioning a self-consistent universe, or trajectory through the space of all possibilities, in which a condensate is stably maintained by this effect. In order to search for such a possible solution (i.e. a persistent local minimum in the action), we adopt the artifice in \((3.32)-(3.34)\) of adding an imaginary random potential to the action, proportional to a parameter \(b\) which is ultimately taken to go to zero: \(b \rightarrow 0^+\).

Known physics is regained in a very simple picture based on a primordial condensate with an \(SU(2) \times U(1)\) order parameter in external spacetime and an \(SO(D-4) \times U(1)\) (or more precisely \(Spin(D-4) \times U(1)\)) order parameter in the internal space. Each of these factors in the overall order parameter has a vortex-like topological defect at the origin. The external topological defect is interpreted as the Big Bang, and the internal topological defect gives rise to an \(SO(N)\) grand-unified gauge group, with \(N=D-4 \geq 10\).

The gravitational vierbein and the gauge fields of other forces are interpreted as “superfluid velocities”, with arbitrary curvatures permitted by a background of “rapidly fluctuating” topological defects that are analogous to vortices and vortex rings (or extended and closed flux tubes) – in roughly the same way that, as shown by Feynman and Onsager, vortices permit rotation of a superfluid. See the discussion below involving \([10.4]-[10.8]\).

(5) In Sections \(\text{V}\) and \(\text{VI}\), the fermionic fields and scalar bosonic fields are found to automatically couple in the correct way to both the gravitational field and the gauge fields of the other forces.

(6) At the same time, local Lorentz invariance automatically emerges (rather than being postulated), and external spacetime is automatically \((3+1)\)-dimensional.

(7) The present theory unavoidably predicts \(SO(N)\) grand unification. This is consistent with the fact that many regard \(SO(10)\) as the most appealing gauge group for unifying Standard Model forces. Family replication can result from \(D-4 > 10\), via a horizontal group.

Gauge symmetry results from rotational symmetry in the internal space, and this explains why forces are described by a basic gauge symmetry which is so similar to rotational symmetry.
The present theory also unavoidably predicts supersymmetry, beginning with the unphysical supersymmetry of Section IV but ending with the standard supersymmetric action of (10.2), which automatically includes the auxiliary fields $F$, after transformation to the physical fields. But, as noted below (10.2), there is a kind of “F-term susy breaking”.

The lightest supersymmetric particle (as a subdominant component) can stably coexist with the dark matter candidate of the present theory [44, 45].

The usual cosmological constant (regarded as one of the deepest problems in standard physics) automatically vanishes for two independent reasons, according to the arguments in Section XI:

(i) For fermion fields and scalar boson fields, there is no factor of $e = \sqrt{-g}$ in the integrals giving their action.

(ii) When the gauge-field action is quantized, the operators must be normal-ordered, in accordance with the interpretation of the origin of this action in Section XI. It arises from the response of vacuum fields to the curvature of the external gauge fields, and it must therefore vanish when there are no external fields. The vacuum stress-energy tensor for the gauge fields then also vanishes.

Standard physics is regained in each case:

(i) As shown in (11.7), classical matter (which follows the on-shell classical equations of motion) acts as a source for Einstein gravity in the same way as in standard physics, and all matter and fields move in the same way.

(ii) The results are consistent with experiment and observation, even though there is no vacuum zero-point energy for the gauge fields.

Many people (including some who are otherwise expert in this area) will naively object that the Casimir effect, as verified experimentally, demonstrates that the electromagnetic field does have a zero-point energy in the vacuum.

This belief is common but incorrect [46]. The experimentally-observed Casimir effect demonstrates only that the static electromagnetic field energy is changed by the modification of boundary conditions [47, 48]. In the simplest model, two metal plates are inserted and the force between them calculated. There are two ways to do the calculation: The first is indeed to assume zero-point vibrations of the electromagnetic field, whose energy is modified when the boundary conditions are modified. The second approach is instead to consider the processes involving virtual photons which mediate the interaction of the plates, with no
reference to zero-point vibrations and no need for a vacuum energy. The first method is
more popular because it is easier. But the two methods give the same answer, as they
should. (The second method regards the force as mediated by virtual photons, and the first
obtains the force from the derivative of the energy with respect to a displacement.) The
second method is more difficult, but is consistent with the way other virtual processes are
calculated in e.g. quantum electrodynamics. Of course, the second method also implies a
change in the static electromagnetic field energy (interpreted as a van der Waals interaction),
but this change does not imply an initially nonzero vacuum zero-point energy.

In summary, the observed Casimir effect is perfectly consistent with the present theory, in
which there is no vacuum zero-point energy due to the electromagnetic field or other gauge
fields.

(10) Although the usual cosmological constant vanishes, there will still be a weaker re-
sponse of the vacuum to the imposition of external fields. In Appendix E, it is shown how the
Einstein-Hilbert action (11.12) for the gravitational field can arise from a vacuum response
within the present picture.

Quantum gravity is regularized by an energy and momentum cutoff $a_0^{-1}$, where $a_0$ is the
minimum length of (3.1), which can be regarded as comparable to the Planck length $\ell_P$.

(11) The Bekenstein-Hawking entropy of black holes automatically e merges in the present
picture, for the reason given in Section XI: Gibbons and Hawking [49] have shown that the
Euclidean action $S_E$ of a black hole is equal to an expression which has the right form to be
interpreted as the Bekenstein-Hawking entropy plus a contribution from angular momentum.
Until now no convincing reason has been given for the $S$ in this expression to be identified
as a true entropy derived from microstates, but in the present picture – before the effect of
rotation is added – (11.16) implies that the Euclidean action of a black hole is its entropy.
This entropy ultimately originates from the microstates of the dits that are consistent with
the gravitational field configuration that comprises the black hole.

(12) As discussed in Sections IX and XI, the present theory predicts new particles, includ-
ing a new dark matter WIMP which is consistent with experiment and observation because it
has no couplings other than its second-order gauge couplings to $W$ and $Z$ bosons. It should
be observable in future colliders within roughly the next 12-20 years, and in direct-detection
experiments within roughly the next 2-5 years, and it may already have been detected via
the gamma rays observed by Fermi-LAT and antiprotons observed by AMS-02 [44, 45].
particle is unique among viable dark matter candidates in that both its couplings and its mass are precisely determined, making clean experimental tests possible in the near future. The favorable characteristics of this candidate are summarized at the end of Section XI.

Let us now proceed to the detailed arguments behind the above claims.

III. STATISTICAL ORIGIN OF THE INITIAL ACTION

For a theory to be viable, it must be mathematically (and philosophically) consistent, its premises must lead to testable predictions, and these predictions must be consistent with experiment and observation. The theory presented here appears to satisfy these requirements, but it starts with an extremely unfamiliar point of view: There are initially no laws, and instead all possibilities are realized with equal weight. The observed laws of nature are emergent phenomena, which result from statistical counting and the geography (i.e. specific features) of our particular universe in \( D \) dimensions. In other words, standard physics (including familiar extensions such as grand unification and supersymmetry) emerges as an effective field theory at relatively low energies.

Our starting point is a single fundamental system which consists of identical (but distinguishable) irreducible objects, which we will call “dits”. Each dit can exist in any of \( d \) states, with the number of dits in the \( i \)th state represented by \( n_i \). An unobservable microstate of the fundamental system is specified by the number of dits and the state of each dit. An observable macrostate is specified by only the occupancies \( n_i \) of the states.

As discussed below, \( D \) of the states are used to define \( D \) spacetime coordinates \( x^M \), and \( N_F \) of the states are used to define fields \( \phi_k \).

Let us begin with the coordinates:

\[
x^M = \Delta n_M a_0 \quad , \quad \Delta n_M = n_M - \overline{n} \quad , \quad M = 0, 1, ..., D - 1
\]

with \( \overline{n} \), which is defined below, specifying the initial origin of coordinates. It is convenient to include a (very small) fundamental length \( a_0 \) in this definition, so that we can later express the coordinates in conventional units. One can think of \( a_0 \) as being comparable to the Planck length \( \ell_P \).

As discussed below, we will eventually take the limit \( \overline{n} \to \infty \), with \( \Delta n_M \) finite, and there will then be no lower bound to negative coordinates. I.e., \( \Delta n_M \) can have any integer value.
(A central feature of the present theory is that both coordinates and physical fields are defined by relatively small perturbations $\Delta n_i = n_i - \bar{n}_i$, analogous to waves on a deep ocean.)

Now define a set of initial fields $\phi_k$ by

$$\phi_k^2 (x) = \rho_k (x), \quad k = 1, 2, \ldots, N_F \tag{3.2}$$

where

$$\rho_k (x) = n_k (x) / a_0^D \tag{3.3}$$

and $x$ represents all the coordinates. (To avoid awkward notation, we write $n_k$ for $n_{i=D+k}$.) These primitive bosonic fields $\phi_k$ are then real, and defined only up to a phase factor $\pm 1$.

We now set out to calculate the entropy $S$ for a given configuration of the fields $\phi_k$ at all points in spacetime. This will essentially become the action for a given path (i.e. specific classical field configuration) in the quantum path integral, beginning with the identification (3.29).

Let $S (x)$ be the entropy at a fixed point $x$, as defined by $S (x) = \log W (x)$. Here $W (x)$ is the total number of microstates for fixed occupation numbers $n_i$: $W (x) = N (x)! / \Pi_i n_i (x)!$, with

$$N (x) = \sum_i n_i (x), \quad i = 1, 2, \ldots, d. \tag{3.4}$$

The total number of available microstates for all points $x$ is $W = \Pi_x W (x)$, so the total entropy is

$$S = \sum_x S (x), \quad S (x) = \log \Gamma (N (x) + 1) - \sum_i \log \Gamma (n_i (x) + 1). \tag{3.5}$$

We will see below that $n_k (x)$ can be approximately treated as a continuous variable when it is extremely large, with

$$\frac{\partial S}{\partial n_k (x)} = \psi (N (x) + 1) - \psi (n_k (x) + 1) \tag{3.6}$$

$$\frac{\partial^2 S}{\partial n_k \partial n_{k'} (x)} = \psi^{(1)} (N (x) + 1) - \psi^{(1)} (n_k (x) + 1) \delta_{k k'}. \tag{3.7}$$

The functions $\psi (z) = d \log \Gamma (z) / dz$ and $\psi^{(1)} (z) = d^2 \log \Gamma (z) / dz^2$ have the asymptotic expansions

$$\psi (z) = \log z - \frac{1}{2z} - \sum_{l=1}^{\infty} \frac{B_{2l}}{2l z^{2l}} \quad , \quad \psi^{(1)} (z) = \frac{1}{z} + \frac{1}{2z^2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{z^{2l+1}} \tag{3.8}$$
as \( z \to \infty \). It will be assumed that each \( n_k(x) \) has some characteristic value \( \bar{n}_k(x) \) which is vastly larger than nearby values:

\[
n_k(x) = \bar{n}_k(x) + \Delta n_k(x) , \quad \bar{n}_k(x) \gg |\Delta n_k(x)| \tag{3.9}
\]

where “\( \gg \)” means “is vastly greater than”, as in \( 10^{1000} \gg 1 \). This assumption is consistent with the fact that the initial action of (3.28) and (3.29) has no lower bound as \( n_k(x) \to \infty \) before the extra stochastic term involving (3.32) is added. (To state the reasoning more cleanly, but slightly out of the order of presentation, the limit (3.33) implies the limit \( \bar{n}_k \to \infty \).) Then it is an extremely good approximation to use the asymptotic formulas above and write

\[
S = S_0 + \sum_{x,k} a_k(x) \Delta n_k(x) - \sum_{x,k} a'_k(x) [\Delta n_k(x)]^2 + \sum_{x,k,k'=k} a'_{kk'}(x) \Delta n_k(x) \Delta n_{k'}(x) \tag{3.10}
\]

\[
a_k(x) = \log N(x) - \log \bar{n}_k(x) \tag{3.11}
\]

\[
a'_k(x) = (2n_k(x))^{-1} - (2N(x))^{-1} , \quad a'_{kk'}(x) = (2N(x))^{-1} \tag{3.12}
\]

where \( N(x) \) is the value of \( N(x) \) when \( n_k(x) = \bar{n}_k(x) \) for all \( k \), and the higher-order terms have been separately neglected in \( a_k(x) \) and \( a'_k(x) \). (The above results then also follow immediately from Stirling’s approximation for factorials.) For simplicity, we will also neglect the terms involving \( (2N(x))^{-1} \). (If these small terms are retained, the conclusions below still hold with some trivial redefinitions, but the notation and algebra become much more tedious.) Since there is initially no distinction between the fields labeled by \( k \), it is consistent to assume that they all have the same \( \bar{n}_k(x) = \bar{n}(x) \), and that \( \bar{n}(x) \) is independent of \( x \): \( \bar{n}(x) = \bar{n} \) and \( N(x) = \bar{N} \), so that

\[
a_k(x) = a = \log (\bar{N}/\bar{n}) \tag{3.13}
\]

\[
a'_k(x) = a' = (2\bar{n})^{-1} . \tag{3.14}
\]

(The above assumptions are actually needed only to simplify the presentation, and they have no effect on the final results below as \( \bar{n} \to \infty \).)

It is not conventional or convenient to deal with \( \Delta n_k(x) \) and \( [\Delta n_k(x)]^2 \), so let us instead write \( S \) in terms of the fields \( \phi_k \) and their derivatives \( \partial \phi_k/\partial x^M \) via the following procedure: First, we can switch to a new set of points \( \bar{x} \), defined to be the corners of the \( D \)-dimensional
hypercubes centered on the original points \( x \). It is easy to see that

\[
S = S_0 + \sum_{\pi, k} a \{ \Delta n_k (x) \} - \sum_{\pi, k} a' \{ [\Delta n_k (x)]^2 \} \tag{3.15}
\]

where \( \langle \cdots \rangle \) in the present context indicates an average over the \( 2^D \) boxes labeled by \( x \) which have the common corner \( \pi \). Second, at each point \( x \) we can write \( \Delta n_k = \Delta \rho_k a_0^D = (\langle \Delta \rho_k \rangle + \delta \rho_k) a_0^D \), with \( \langle \delta \rho_k \rangle = 0 \):

\[
S = S_0 + \sum_{\pi, k} a \{ (\Delta \rho_k) + \delta \rho_k \} a_0^D - \sum_{\pi, k} a' \{ (\langle \Delta \rho_k \rangle + \delta \rho_k)^2 \} (a_0^D)^2 \tag{3.16}
\]

\[
= S_0 + \sum_{\pi, k} a \{ \Delta \rho_k \} a_0^D - \sum_{\pi, k} a' \{ (\Delta \rho_k)^2 + (\delta \rho_k)^2 \} (a_0^D)^2. \tag{3.17}
\]

Each of the points \( x \) surrounding \( \pi \) is displaced by \( \delta x^M = \pm a_0/2 \) along each of the \( x^M \) axes, so

\[
\{ (\delta \rho_k)^2 \} = \left\{ \left( \delta \phi_k^2 \right)^2 \right\} \tag{3.18}
\]

\[
= \left\{ \sum_M \left( \frac{\partial \phi_k^2}{\partial x^M} \delta x^M + \frac{1}{2} \frac{\partial^2 \phi_k^2}{\partial (x^M)^2} (\delta x^M)^2 \right)^2 \right\} \tag{3.19}
\]

\[
= \left\{ \sum_M \left( 2 \phi_k \frac{\partial \phi_k}{\partial x^M} \delta x^M + \left( \frac{\partial \phi_k}{\partial x^M} \right)^2 (\delta x^M)^2 + \phi_k \frac{\partial^2 \phi_k}{\partial (x^M)^2} (\delta x^M)^2 \right)^2 \right\} \tag{3.20}
\]

to lowest order, where it is now assumed that at normal energies the fields are slowly varying over the extremely small distance \( a_0 \). This assumption is justified by the prior assumption that \( \pi \) is extremely large: \( \phi_k^2 (x) = \rho_k (x) = n_k (x)/a_0^D \) implies that \( 2\delta \phi_k/\phi_k \approx \delta n_k/n_k \) and \( \phi_k = n_k^{1/2} a_0^{-D/2} \), so that \( \delta \phi_k \sim \delta n_k n_k^{-1/2} a_0^{-D/2} \). The minimum change in \( \phi_k \) is given by \( \delta n_k = 1 \):

\[
\delta \phi_k^{\text{min}} \sim n_k^{-1/2} a_0^{-D/2} \tag{3.21}
\]

which means that \( \delta \phi_k^{\text{min}} \) is extremely small if \( n_k \) is extremely large.

In other words, the fields \( \phi_k \) have effectively continuous values as \( \pi \rightarrow \infty \).

For extremely large \( \pi \) it is an extremely good approximation to neglect the middle term in (3.20), and to replace \( \phi_k^2 \) by

\[
\overline{\phi} = \overline{\rho} = \pi/a_0^D \tag{3.22}
\]

giving

\[
a' \{ (\delta \rho_k)^2 \} = \frac{1}{2a_0^D} \sum_M \left[ \left( \frac{\partial \phi_k}{\partial x^M} \right)^2 a_0^2 + \left( \frac{\partial^2 \phi_k}{\partial (x^M)^2} \right)^2 \frac{a_0^4}{16} \right]. \tag{3.23}
\]
It is similarly an extremely good approximation to neglect the term in (3.17) involving \(a'(a_0^D)^2 \langle \Delta \rho_k \rangle^2 / 2m \) in comparison to that involving \( \langle \Delta \rho_k \rangle a_0^D = \langle \Delta n_k \rangle \), so that

\[
S = S_0 + \sum_{x,k} a_0^D \mu_0 \left( \phi_k^2 - \bar{\phi}^2 \right) - \sum_{x,k} \sum_M a_0^D \frac{1}{2m_0^2} \left[ \left( \frac{\partial \phi_k}{\partial x^M} \right)^2 + \frac{a_0^2}{16} \left( \frac{\partial^2 \phi_k}{\partial (x^M)^2} \right)^2 \right]
\]

(3.24)

where

\[
m_0 = a_0^{-1}, \quad \mu_0 = m_0 \log \left( \frac{N}{n} \right).
\]

The philosophy behind the above treatment is simple: We essentially wish to replace \( \langle f^2 \rangle \) by \((\partial f / \partial x)^2\), and this can be accomplished because

\[
\langle f^2 \rangle - \langle f \rangle^2 = \left( (\partial f / \partial x)^2 \right) \approx \left( (\partial f / \partial x)^2 \right) (a_0/2)^2.
\]

(3.26)

The form of (3.24) also has a simple interpretation: The entropy \( S \) increases with the number of dits, but decreases when the dits are not uniformly distributed.

In the continuum limit,

\[
\sum_x a_0^D \rightarrow \int d^D x
\]

(3.27)

(3.24) becomes

\[
S = S_0 + \int d^D x \sum_k \left\{ \mu_0 \left( \phi_k^2 - \bar{\phi}^2 \right) - \frac{1}{2m_0^2} \sum_M \left[ \left( \frac{\partial \phi_k}{\partial x^M} \right)^2 + \frac{a_0^2}{16} \left( \frac{\partial^2 \phi_k}{\partial (x^M)^2} \right)^2 \right] \right\}.
\]

(3.28)

A physical configuration of all the fields \( \phi_k(x) \) corresponds to a specification of all the density variations \( \Delta \rho_k(x) \). In the present picture, the probability of such a configuration is proportional to \( W = e^S \). In a path integral with Euclidean form, the probability is proportional to \( e^{-\overline{S}_b} \), where \( \overline{S}_b \) is the action for these bosonic fields. We conclude that

\[
\overline{S}_b = -S + \text{constant}
\]

(3.29)

and we will choose the constant to equal \( S_0 \).

In the following it will be convenient to write the action in terms of \( \bar{\phi}_k = m_0^{-1/2} \phi_k \). For simplicity, we assume that the number of relevant \( \bar{\phi}_k \) is even, so that we can group these real fields in pairs to form \( N_f \) complex fields \( \Psi_{k,b} \). It is also convenient to subtract out the enormous contribution of \( \bar{\phi} \) by defining

\[
\Psi = \Psi_{b} - \bar{\Psi}_b
\]

(3.30)
where $\vec{\Psi}_b$ is the vector with components $\Psi_{b,k}$ and $\vec{\Psi}_b$ is similarly defined with $\phi_k \rightarrow \vec{\phi}$. Then the action can be written

$$\overline{S}_b = \int d^Dx \left\{ \frac{1}{2m_0} \left[ \frac{\partial \Psi_b^i}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^i}{\partial (x^M)^2} \frac{\partial^2 \Psi_b}{\partial (x^M)^2} \right] - \mu_0 \left( \vec{\Psi}_b^i \vec{\Psi}_b^i - \vec{\Psi}_b \vec{\Psi}_b \right) \right\}$$

(3.31)

since $\vec{\Psi}_b$ is constant, with summation now implied over repeated indices like $M$.

As described above, in Section [II] we now add an extra imaginary term $i \vec{V} \Psi_b^i \Psi_b$ in the integral giving the action. Here $\vec{V}$ is a potential which has a Gaussian distribution, with

$$\langle \vec{V} \rangle = 0 \quad , \quad \langle \vec{V} (x) \vec{V} (x') \rangle = b \delta (x - x')$$

(3.32)

where $b$ is a constant, with

$$b \rightarrow 0^+$$

(3.33)

at the end of the calculations.

Then the complete action has the form

$$\overline{S}_B [\Psi_b^i, \Psi_b] = \int d^Dx \left\{ \frac{1}{2m_0} \left[ \frac{\partial \Psi_b^i}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^i}{\partial (x^M)^2} \frac{\partial^2 \Psi_b}{\partial (x^M)^2} \right] - \mu_0 \left( \vec{\Psi}_b^i \vec{\Psi}_b^i - \vec{\Psi}_b \vec{\Psi}_b \right) + i \vec{V} \Psi_b^i \Psi_b \right\}$$

(3.34)

In the following we will assume that the only fields which make an appreciable contribution in (3.34) are those for which $\int d^Dx \Psi_b^i \Psi_b = \vec{\Psi}_b \int d^Dx \Psi_b = 0$. This assumption is justified by the fact that $\vec{\Psi}_b$ is constant with respect to all the coordinates and, in the present picture, fields $\Psi_b$ corresponding to physical gauge representations have nonzero angular momenta in the internal space of Section [VI] and Appendices [A] and [B]. Then (3.34) simplifies to

$$\overline{S}_B [\Psi_b^i, \Psi_b] = \int d^Dx \left\{ \frac{1}{2m_0} \left[ \frac{\partial \Psi_b^i}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^i}{\partial (x^M)^2} \frac{\partial^2 \Psi_b}{\partial (x^M)^2} \right] - \mu_0 \Psi_b^i \Psi_b + i \vec{V} \Psi_b^i \Psi_b \right\} \quad . (3.35)$$

**IV. PRIMITIVE SUPERSYMMETRY**

If $F$ is any functional of the fundamental fields $\Psi_b$, its average value is given by

$$\langle F \rangle = \left( \frac{\int D \Psi_b^i D \Psi_b F [\Psi_b^i, \Psi_b] e^{-\overline{S}_B [\Psi_b^i, \Psi_b]}}{\int D \Psi_b^i D \Psi_b e^{-\overline{S}_B [\Psi_b^i, \Psi_b]}} \right)$$

(4.1)
where \((\cdots)\) now represents an average over the perturbing potential \(i\widetilde{V}\) and \(\int \mathcal{D} \Psi^\dagger_b \mathcal{D} \Psi_k\) is to be interpreted as \(\prod_{x,k} \int_{-\infty}^{\infty} d \text{Re} \Psi_{b,k}(x) \int_{-\infty}^{\infty} d \text{Im} \Psi_{b,k}(x)\). The transition from the original summation over \(n_k(x)\) to this Euclidean path integral has the form (with \(\Delta n = 1\) here)

\[
\sum_{n=0}^{\infty} f(n) \Delta n \to \int_0^\infty f dn \to \int_0^\infty \int_0^\infty f d\left(a_0^D \phi^2\right) \to 2\bar{\phi} a_0^D \int_0^\infty f d\phi \to 2\bar{\phi} a_0^D m_0^{1/2} \int_{-\infty}^{\infty} f d\phi' \quad (4.2)
\]

where \(\phi' = \bar{\phi} - m_0^{-1/2} \tilde{\phi}\), since \(d(\phi^2) \approx 2\bar{\phi} d\phi\) is an extremely good approximation for physically relevant fields, and since \(\phi'\) effectively ranges from \(-\infty\) to \(+\infty\). Each \(\phi'\) then becomes a \(\text{Re} \Psi_{b,k}(x)\) or \(\text{Im} \Psi_{b,k}(x)\), and the constant factors cancel in the numerator and denominator of (4.1).

The presence of the denominator makes it difficult to perform the average of (4.1), but there is a trick for removing the bosonic degrees of freedom \(\Psi_b\) in the denominator and replacing them with fermionic degrees of freedom \(\Psi_f\) in the numerator [50–52]: After integration by parts (with boundary terms usually assumed either to vanish or to be irrelevant in this paper), (3.35) can be written in the form \(\widetilde{S}_B = \int d^D x \Psi^\dagger_b A \Psi_b\). Then, since

\[
\int \mathcal{D} \Psi^\dagger_b \mathcal{D} \Psi_b \ e^{-\widetilde{S}_b[\Psi^\dagger_b,\Psi_b]} = C (\det \mathcal{A})^{-1} \quad (4.3)
\]

\[
\int \mathcal{D} \Psi^\dagger_f \mathcal{D} \Psi_f \ e^{-\widetilde{S}_b[\Psi^\dagger_f,\Psi_f]} = \det \mathcal{A} \quad (4.4)
\]

where the matrix \(\mathcal{A}\) corresponds to the operator \(A\) and \(C\) is a constant, it follows that

\[
\langle F \rangle = \frac{1}{C} \left( \int \mathcal{D} \Psi^\dagger_b \mathcal{D} \Psi_b \mathcal{D} \Psi^\dagger_f \mathcal{D} \Psi_f F e^{-\widetilde{S}_b[\Psi^\dagger_b,\Psi_b]} e^{-\widetilde{S}_b[\Psi^\dagger_f,\Psi_f]} \right) \quad (4.5)
\]

\[
= \frac{1}{C} \left( \int \mathcal{D} \Psi^\dagger \mathcal{D} \Psi F e^{-\widetilde{S}_b[\Psi^\dagger,\Psi]} \right) \quad (4.6)
\]

where \(\Psi_b\) and \(\Psi_f\) have been combined into

\[
\Psi = \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix} \quad (4.7)
\]

and

\[
\widetilde{S}_{bf}[\Psi^\dagger,\Psi] = \int d^D x \left\{ \frac{1}{2m_0} \left[ \frac{\partial \Psi^\dagger}{\partial x^M} \frac{\partial \Psi}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi^\dagger}{\partial (x^M)^2} \frac{\partial^2 \Psi}{\partial (x^M)^2} \right] - \mu_0 \Psi^\dagger \Psi + i\widetilde{V} \Psi^\dagger \Psi \right\}. \quad (4.8)
\]

In (4.7), \(\Psi_f\) consists of Grassmann variables \(\Psi_{f,k}\), just as \(\Psi_b\) consists of ordinary variables \(\Psi_{b,k}\), and \(\int \mathcal{D} \Psi^\dagger \mathcal{D} \Psi\) is to be interpreted as

\[
\prod_{x,k} \int_{-\infty}^{\infty} d \text{Re} \Psi_{b,k}(x) \int_{-\infty}^{\infty} d \text{Im} \Psi_{b,k} \int d \Psi^\dagger_{f,k}(x) \int d \Psi_{f,k}(x) \quad (4.9)
\]
Recall that $\Psi_b$ and $\Psi_f$ each have $N_f$ components.

In this early stage of the theory, the bosonic fields in the denominator – which are of critical importance for proper normalization – have effectively been transformed into fermionic fields in the numerator, where they perform the same function. This is the origin of supersymmetry in the present picture.

For a Gaussian random variable $v$ whose mean is zero, the result

$$\langle e^{-iv} \rangle = e^{-\frac{1}{2}v^2}$$

implies that

$$\langle e^{-\int d^Dx i\tilde{V}\Psi^\dagger\Psi} \rangle = e^{-\frac{1}{2}\int d^Dx [\Psi^\dagger(x)\Psi(x)]^2}.$$  

It follows that

$$\langle F \rangle = \frac{1}{C} \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi F e^{-S_E}$$

with

$$S_E = \int d^Dx \left\{ \frac{1}{2m_0} \left[ \frac{\partial\Psi^\dagger}{\partial x^M} \frac{\partial\Psi}{\partial x^M} + \frac{\alpha^2}{16} \frac{\partial^2\Psi^\dagger}{\partial (x^M)^2} \frac{\partial^2\Psi}{\partial (x^M)^2} \right] - \mu_0 \Psi^\dagger\Psi + \frac{1}{2}b \left( \Psi^\dagger\Psi \right)^2 \right\}.$$  

A special case (with $F = 1$) is

$$Z = \frac{1}{C} \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^{-S_E}$$

but according to (4.11) $Z = 1$, so $C = \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^{-S_E}$ and

$$\langle F \rangle = \frac{\int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi F e^{-S_E}}{\int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^{-S_E}}.$$  

Again, notice that the fermionic fields $\Psi_f$ are effectively a transformed version of the bosonic fields $\Psi_b$. The coupling between the fields $\Psi_b$ and $\Psi_f$ (or $\Psi_b$) is due to the random perturbing potential $i\tilde{V}$. In the replacement of (4.1) by (4.16), $F$ essentially serves as a test functional. The meaning of this replacement is that the action (4.14), with both bosonic and fermionic fields, must be used instead of the original action (3.35), with only bosonic fields, in treating all physical quantities and processes, if the average over random fluctuations in (4.1) is to disappear from the theory.

Notice that the two steps above serve two independent purposes: The transformation of $\Psi_b$ in the denominator to $\Psi_f$ in the numerator provides a more convenient formulation.
because all fields now have equal status in the numerator, and can be treated in the same way. The introduction of an infinitesimal perturbing potential is then preparation for the formation of a condensate at finite energy (in a cooling universe), as discussed in Section II. Of course, it is the conjunction of these two steps that makes the following developments possible.

Ordinarily we can let \( a_0 \to 0 \) in (4.14), so that

\[
S_E = \int d^Dx \left[ \frac{1}{2m_0} \partial_M \Psi^\dagger \partial_M \Psi - \mu_0 \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right].
\]

However, the higher-derivative term in (4.14) is relevant in the internal space defined below, and a finite \( a_0 \) also automatically provides an ultimate ultraviolet cutoff.

V. ORIGIN OF FERMION ACTION AND (3+1) DIMENSIONAL SPACETIME

The present theory is based on (1) statistical counting (which ultimately produced the results of the preceding two sections) and (2) the geography (or specific features) of our universe, to which we now turn.

The most central assumption is that

\[
\Psi_b = \Psi_0^\prime + \Psi_b^\prime
\]

where \( \Psi_0^\prime \) contains the order parameter \( \Psi_0 \) for a primordial bosonic condensate which forms in the very early universe, and \( \Psi_b^\prime \) contains all the other bosonic fields. I.e., in \( \Psi_0^\prime \) only one set of components is nonzero and equal to \( \Psi_0 \), and this set of components is zero in \( \Psi_b^\prime \). The treatment of Appendix A implies that

\[
\Psi_0^\prime \dagger \Psi_b^\prime = 0
\]

everywhere. (The fields in other representations do not overlap the representation containing \( \Psi_0 \). Fields in the same representation are orthogonal according to (A11) and the comments
above (A2) and (A9). The action can then be written as

\[ S_E = S_{\text{cond}} + S_b + S_f + S_{\text{int}} \] (5.3)

\[ S_{\text{cond}} = \int d^Dx \left[ \frac{1}{2m_0} \partial_M \Psi_0^\dagger \partial_M \Psi_0 - \mu_0 \Psi_0^\dagger \Psi_0 + \frac{1}{2} b (\Psi_0^\dagger \Psi_0)^2 \right] \] (5.4)

\[ S_b = \int d^Dx \left[ \frac{1}{2m_0} \partial_M \Psi_b^\dagger \partial_M \Psi_b + (V_0 - \mu_0) \Psi_b^\dagger \Psi_b + \frac{1}{2} b (\Psi_b^\dagger \Psi_b)^2 \right] \] (5.5)

\[ S_f = \int d^Dx \left[ \frac{1}{2m_0} \partial_M \Psi_f^\dagger \partial_M \Psi_f + (V_0 - \mu_0) \Psi_f^\dagger \Psi_f + \frac{1}{2} b (\Psi_f^\dagger \Psi_f)^2 \right] \] (5.6)

\[ S_{\text{int}} = \int d^Dx b (\Psi_f^\dagger \Psi_f) (\Psi_b^\dagger \Psi_b) \] (5.7)

\[ V_0 = b \Psi_0^\dagger \Psi_0. \] (5.8)

In most of the following, the last term will be neglected in (5.5) and (5.6); we are then considering the theory prior to formation of further condensates beyond the primordial \( \Psi_0 \).

For a static condensate we could write \( \Psi_0 = n_0^{1/2} \eta_0 \), where \( \eta_0 \) is constant, \( \eta_0^\dagger \eta_0 = 1 \), and \( n_0 = \Psi_0^\dagger \Psi_0 \) is the condensate density. This picture is too simplistic, however, since the order parameter can exhibit rotations that are analogous to the rotations in the complex plane of the order parameter \( \psi_s = e^{i \theta_s} n_s^{1/2} \) for an ordinary superfluid:

\[ \Psi_0(x) = U_0(x) \ n_0(x)^{1/2} \eta_0, \quad U_0^\dagger U_0 = 1. \] (5.9)

After an integration by parts in (5.4) (with boundary terms usually neglected in the present paper), extremalization of the action gives the classical equation of motion for the order parameter:

\[ - \frac{1}{2m_0} \partial_M \partial_M \Psi_0 + (V_0 - \mu_0) \Psi_0 = 0. \] (5.10)

Now an important nuance, which requires some references to the treatment below: Because the primordial condensate density is extremely large, (5.10) is assumed to always hold (at normal energies). However, consistent with this constraint, the “phase” and ”superfluid velocities” of (5.14) (5.37), (5.46), and (6.12) are allowed to vary within the path integral. This means that the gauge potentials \( A_i^\mu \) and metric tensor \( g_{\mu\nu} \) are quantized. I.e., the path integral over the original field \( \Psi_0 \) is replaced by path integrals over \( A_i^\mu \) and \( g_{\mu\nu} \). Notice that the invariance of (5.4) under a gauge transformation implies gauge-fixing within the path integrals for both \( A_i^\mu \) and \( g_{\mu\nu} \), as usual. In the present treatment, we additionally require that (5.41) always hold, so that the path integral is restricted to torsion-free spacetime geometries.
In specifying the geography of our universe, it will be assumed that $\Psi_0$ can be written as the product of a 2-component external order parameter $\Psi_{\text{ext}}$, which is a function of 4 external coordinates $x^\mu$, and an internal order parameter $\Psi_{\text{int}}$, which is primarily a function of $D-4$ internal coordinates $x^m$, but which also varies with $x^\mu$:

$$\Psi_0 = \Psi_{\text{ext}}(x^\mu) \Psi_{\text{int}}(x^m, x^\mu)$$  \hfill (5.11)

$$\Psi_{\text{ext}}(x^\mu) = U_{\text{ext}}(x^\mu) n_{\text{ext}}(x^\mu)^{1/2} \eta_{\text{ext}} \quad , \quad \mu = 0, 1, 2, 3$$  \hfill (5.12)

$$\Psi_{\text{int}}(x^m, x^\mu) = U_{\text{int}}(x^m, x^\mu) n_{\text{int}}(x^m, x^\mu)^{1/2} \eta_{\text{int}} \quad , \quad m = 4, \ldots, D - 1$$  \hfill (5.13)

where again $\eta_{\text{ext}}$ and $\eta_{\text{int}}$ are constant, and $\eta_{\text{ext}}^\dagger \eta_{\text{ext}} = \eta_{\text{int}}^\dagger \eta_{\text{int}} = 1$. Here, according to a standard notation, $x^\mu$ actually represents the set of $x^\mu$, and $x^m$ the set of $x^m$.

Let us define external and internal “superfluid velocities” by

$$m_0 v_\mu = -i U_{\text{ext}}^{-1} \partial_\mu U_{\text{ext}} \quad , \quad m_0 v_m = -i U_{\text{int}}^{-1} \partial_m U_{\text{int}} .$$  \hfill (5.14)

The fact that $U_{\text{ext}}^\dagger U_{\text{ext}} = 1$ implies that $(\partial_\mu U_{\text{ext}}^\dagger) U_{\text{ext}} = -U_{\text{ext}}^\dagger (\partial_\mu U_{\text{ext}})$ with $U_{\text{ext}}^\dagger = U_{\text{ext}}^{-1}$, or

$$m_0 v_\mu = i (\partial_\mu U_{\text{ext}}^\dagger) U_{\text{ext}} , \quad v_\mu^\dagger = v_\mu .$$  \hfill (5.15)

For simplicity, let us first consider the case

$$\partial_\mu U_{\text{int}} = 0$$  \hfill (5.16)

for which there are separate external and internal equations of motion:

$$\left( -\frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{\text{ext}} \right) \Psi_{\text{ext}} = 0 \quad , \quad \left( -\frac{1}{2m_0} \partial_m \partial_m - \mu_{\text{int}} + V_0 \right) \Psi_{\text{int}} = 0$$  \hfill (5.17)

with

$$\mu_{\text{int}} = \mu_0 - \mu_{\text{ext}} .$$  \hfill (5.18)

The quantities $\mu_{\text{int}}$ and $V_0$ have a relatively slow parametric dependence on $x^\mu$.

When (5.12) and (5.14) are used in (5.17), we obtain

$$\eta_{\text{ext}}^\dagger n_{\text{ext}}^{1/2} \left[ \left( \frac{1}{2} m_0 v_\mu v_\mu - \frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{\text{ext}} \right) - i \left( \frac{1}{2} \partial_\mu v_\mu + v_\mu \partial_\mu \right) \right] n_{\text{ext}}^{1/2} \eta_{\text{ext}} = 0$$  \hfill (5.19)

and its Hermitian conjugate

$$\eta_{\text{ext}}^\dagger n_{\text{ext}}^{1/2} \left[ \left( \frac{1}{2} m_0 v_\mu v_\mu - \frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{\text{ext}} \right) + i \left( \frac{1}{2} \partial_\mu v_\mu + v_\mu \partial_\mu \right) \right] n_{\text{ext}}^{1/2} \eta_{\text{ext}} = 0 .$$  \hfill (5.20)
Subtraction gives the equation of continuity

$$\partial_\mu j^\text{ext}_\mu = 0 \quad , \quad j^\text{ext}_\mu = n^\text{ext}_\mu \eta^\dagger_\mu v^\mu \eta^\text{ext}$$  \hspace{1cm} (5.21)

and addition gives the Bernoulli equation

$$\frac{1}{2}m_0 \bar{v}^2_\text{ext} + P^\text{ext} = \mu^\text{ext}$$  \hspace{1cm} (5.22)

where

$$\bar{v}^2_\text{ext} = \eta^\dagger_\text{ext} v^\mu \eta^\mu \eta^\text{ext} \quad , \quad P^\text{ext} = -\frac{1}{2m_0} n^{-1/2} \eta^\text{ext}_\mu \partial_\mu n^{1/2}.$$

(5.23)

Since the order parameter $\Psi^\text{ext}$ in external spacetime has 2 components, its “superfluid velocity” $v^\mu$ can be written in terms of the identity matrix $\sigma^0$ and Pauli matrices $\sigma^a$:

$$v^\mu = v^\alpha^\mu \sigma^\alpha \quad , \quad \alpha = 0, 1, 2, 3.$$  \hspace{1cm} (5.24)

Let us now transform to a coordinate system in which

$$v^0_k = v^k_0 = v^a_0 = v^0_a = 0 \quad , \quad k = 1, 2, 3 \quad \text{and} \quad a = 1, 2, 3$$  \hspace{1cm} (5.25)

(with the volume element held constant) so that (5.22) becomes

$$\frac{1}{2}m_0 v^\mu v_\alpha^\mu + P^\text{ext} = \mu^\text{ext}.$$

(5.26)

To avoid notational complexity we will still use $x^\mu$ to label the new coordinates. At this point a physically meaningful metric tensor has not yet been introduced, and the notation in (5.26) merely indicates that $v^\mu_\alpha v_\alpha^\mu$ is to be kept constant under coordinate transformations. The same is true for the other quantities in (5.33). Later in the development, where Lorentz invariance and results like those in (10.2) hold (i.e. at energies far below the Planck energy $\ell^{-1}_P$), the usual conventions are used for raising and lowering indices.

The transformation to (5.25) is trivial in, e.g., a cosmological model in which the Big Bang is at the origin of the new coordinates, with the $U(1)$ phase of $\Psi_0$ varying only with respect to the radial coordinate $x^0$, and the “$SU(2)$ phase” involving the Pauli matrices varying within successive 3-spheres with coordinates $x^k$, so that $v^a_k$ has a vortex-like (or instanton-like) configuration. More generally, the time coordinate $x^0$ is distinguished from the spatial coordinates $x^k$ in (5.25) because it is the direction of $U(1)$ rather than $SU(2)$ rotations of the order parameter.
As $v_\alpha^\mu v_{\alpha\mu}$ varies, $\mu_{\text{ext}}$ varies in response, with $\mu_{\text{int}}$ determined by (5.18).

Now expand $\Psi'_b$ in terms of a complete set of basis functions $\tilde{\psi}'_{\text{int}}$ in the internal space:

$$\Psi'_b (x^\mu, x^m) = \tilde{\psi}'_{b} (x^\mu) \tilde{\psi}'_{\text{int}} (x^m) \tag{5.27}$$

with

$$\left( -\frac{1}{2m_0} \partial_m \partial_m - \mu_{\text{int}} + V_0 \right) \tilde{\psi}'_{\text{int}} (x^m) = \varepsilon_r \tilde{\psi}'_{\text{int}} (x^m) \tag{5.28}$$

$$\int d^{D-4} x \tilde{\psi}'_{\text{int}} (x^m) \tilde{\psi}'_{\text{int}} (x^m) = \delta_{rr'} \tag{5.29}$$

and with the usual summation over repeated indices in (5.27). For reasons that will become fully apparent below, but which are already suggested by the form of the order parameter, each $\tilde{\psi}'_{b} (x^\mu)$ has two components. As usual, only the zero ($\varepsilon_r = 0$) modes will be kept. (To simplify the presentation, the higher-derivative terms are not explicitly shown in the present section; they will be restored in the next section.) When (5.27)-(5.29) are then used in (5.5) (with the last term neglected), the result is

$$S_b = \int d^4 x \tilde{\psi}'_{b} \left( -\frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{\text{ext}} \right) \tilde{\psi}'_{b} \tag{5.30}$$

where $\tilde{\psi}'_{b}$ is the vector with components $\tilde{\psi}'_{r}$.

Let $\tilde{\psi}'_{b}$ be written in the form

$$\tilde{\psi}'_{b} (x^\mu) = U_{\text{ext}} (x^\mu) \psi_{b} (x^\mu) \tag{5.31}$$

or equivalently

$$\tilde{\psi}'_{b} (x^\mu) = U_{\text{ext}} (x^\mu) \psi_{b} (x^\mu) \cdot \tag{5.32}$$

Here $\psi_{b}$ has a simple interpretation: It is the field seen by an observer in the frame of reference that is moving with the condensate. In the present theory, a (very high density) condensate $\Psi_0$ forms in the very early universe, and the other bosonic and fermionic fields are subsequently born into it. It is therefore natural to define the fields $\psi_{b}^r$ in the condensate’s frame of reference.

Equation (5.31) is, in fact, exactly analogous to rewriting the wavefunction of a particle in an ordinary superfluid moving with velocity $v_s$: $\tilde{\psi}_{\text{par}} (x) = \exp (imv_s x) \psi_{\text{par}} (x)$ . Here $\psi_{\text{par}}$ is the wavefunction in the superfluid’s frame of reference.
When (5.31) is substituted into (5.30), the result is

\[ S_b = \int d^4x \, \psi_b^\dagger \left[ \left( \frac{1}{2} m_0 v^\mu v_\mu - \frac{1}{2m_0} \partial^\mu \partial_\mu - \mu_{\text{ext}} \right) - i \left( \frac{1}{2} \partial_\mu v^\mu + v^\mu \partial_\mu \right) \right] \psi_b. \]  

(5.33)

In the following it will be assumed that

\[ \partial_\mu v^\mu = 0 \]  

(5.34)

since, after the definition (5.37) and the introduction of a covariant derivative, the more general version of this equation (which permits general coordinate transformations and local Lorentz transformations) follows from (5.41). If the condensate density \( n_{\text{ext}} \) is slowly varying, so that \( P_{\text{ext}} \) can be neglected, (5.25) and (5.26) then lead to the simplification

\[ S_b = -\int d^4x \, \psi_b^\dagger \left( \frac{1}{2m_0} \partial^\mu \partial_\mu + iv^\mu \sigma^\alpha \partial_\mu \right) \psi_b. \]  

(5.35)

In most of the remainder of the paper it will be assumed that the first term in parentheses is negligible compared to the second for states \( \psi \) with energies \( \sim 1 \) TeV or less (as would be the case if we had, e.g., \( m_0 = a_0^{-1} \gtrsim 10^{15} \) TeV and \( v^\mu_{\alpha} \sim 1 \) for \( \mu = \alpha \), so that (5.35) reduces to just

\[ S_b = \int d^4x \, \psi_b^\dagger i e^\mu_\alpha \sigma^\alpha \partial_\mu \psi_b \]  

(5.36)

\[ e^\mu_\alpha = -v^\mu_{\alpha}. \]  

(5.37)

With this choice all fields are initially right-handed. With the choice \( e^0_\alpha = -v^0_{\alpha}, \ e^k_\alpha = v^k_{\alpha} \) all fields would be initially left-handed, as they are for fermions in conventional \( SU(5) \) and \( SO(10) \) grand-unified theories [23, 24]. It is trivial to change from one convention to the other, of course.

A central feature of the present theory is that the \textit{primitive} bosonic fields \( \psi_b \) are 2-component spinors, but, as mentioned in Section III the final \textit{physical} fields derived from these are the 1-component scalar boson fields of Section IX – the Higgs and higgson fields of an extended Higgs sector.

To permit local Lorentz transformations as well as general coordinate transformations, let us rewrite (5.36) as

\[ S_b = \int d^4x \, \overline{\psi}_b \]  

\[ \overline{\psi}_b = \psi_b^\dagger i e^\mu_\alpha \sigma^\alpha \nabla^R_\mu \psi_b \]  

(5.38)
where $\nabla^R_\mu$ gives the standard (curved spacetime) covariant derivative for a right-handed Weyl field. The full covariant derivative for a Dirac field, with right-handed and left-handed Weyl components, is given by [53, 54]

$$\nabla_\mu = \partial_\mu + i\omega^\alpha_{\mu \beta} \Sigma_{\alpha \beta}.$$  \hfill (5.39)

(The notation and conventions in this context usually follow those most common in the gravitational and string theory communities, as in Ref. [54–58], rather than the particle physics and field theory communities, as in Ref. [53]. Mainly, the metric tensor convention is (++++) throughout this paper. However, the Dirac gamma matrices are defined as in most field theory textbooks [59, 60].) The origin and meaning of curvature involving the $\omega^\alpha_{\mu \beta}$ will be considered below. For a vector field, the usual covariant derivative provides invariance under a coordinate transformation. The vierbein introduced above has both coordinate and tangent-space indices, so [53, 54]

$$\nabla_\mu e^\alpha_\nu = \partial_\mu e^\alpha_\nu - \Gamma^\rho_{\mu \nu} e^\alpha_\rho + \omega^\alpha_{\mu \beta} e^\beta_\nu.$$  \hfill (5.40)

and in the usual minimal case of a torsion-free universe

$$\nabla_\mu e^\alpha_\nu = 0.$$  \hfill (5.41)

The above arguments also hold for fermions, with (in the initial notation)

$$S_f = \int d^Dx \left( -\frac{1}{2m_0} \Psi_f^\dagger \partial^M \partial_M \Psi_f - \mu_0 \Psi_f^\dagger \Psi_f + V_0 \Psi_f^\dagger \Psi_f \right)$$  \hfill (5.42)

$$\Psi_f (x^\mu, x^m) = \overline{\psi}_f (x^\mu) \overline{\psi}_{int} (x^m)$$  \hfill (5.43)

leading to the final result

$$S_f = \int d^4x \overline{\mathcal{L}}_f , \quad \overline{\mathcal{L}}_f = \overline{\psi}_f^\dagger i e^n_\alpha \sigma^\alpha \nabla^R_\mu \psi_f.$$  \hfill (5.44)

The present theory thus yields the basic form of the standard Lagrangian for Weyl fermions, with the gravitational vierbein $e^n_\alpha$ interpreted as essentially a “superfluid velocity” associated with the condensate $\Psi_0$. The path integral still has a Euclidean form, the action for bosons is also not yet in standard form, and there is no factor of $e = |\det e^n_\alpha| = (- \det g_{\mu \nu})^{1/2}$ multiplying $\overline{\mathcal{L}}_f$ or $\overline{\mathcal{L}}_b$, but we will return to these points below.
Consistency with (5.44) is obtained if the metric tensor is related to the vierbein through
\[ g^{\mu\nu} = \eta^{\alpha\beta} e^\mu_\alpha e^\nu_\beta, \quad \eta^{\alpha\beta} = \text{diag} (-1, 1, 1, 1). \] (5.46)

In the present picture, spacetime is automatically 4-dimensional with one time coordinate, because there are 3 Pauli matrices and one \(2 \times 2\) unit matrix.

VI. GAUGE FIELDS

In this section and those following, down to and including (8.61), we will temporarily ignore the spin connection and write \(\partial_\mu\) instead of \(\nabla_\mu\) to avoid irrelevant complications in notation. Let us now relax assumption (5.16) and allow \(U_{\text{int}}\) to vary with the external coordinates \(x^\mu\). The more general version of (5.10) is satisfied if (5.17) is generalized to
\[ \left( -\frac{1}{2m_0} \partial^\mu \partial_\mu - \mu_{\text{ext}} \right) \Psi_{\text{ext}} (x^\mu) \Psi_{\text{int}} (x^m, x^\mu) = 0 \] (6.1)
with \(\Psi_{\text{int}}\) required to satisfy the internal equation of motion (at each \(x^\mu\))
\[ \left[ \sum_m \frac{1}{2m_0} \left( -\frac{\partial^2}{\partial (x^m)^2} + \frac{a_0^2}{16} \frac{\partial^4}{\partial (x^m)^4} \right) + V_0 (x^m) - \mu_{\text{int}} \right] \Psi_{\text{int}} (x^m, x^\mu) = 0. \] (6.2)
The higher-derivative term of (4.14) has been retained and two integrations by parts have been performed. (In order to simplify the notation, we do not explicitly show the weak parametric dependence of \(\mu_{\text{int}}, V_0,\) and \(n_{\text{int}}\) on \(x^\mu\).) This is a nonlinear equation because (at each \(x^\mu\)) \(V_0 (x^m)\) is mainly determined by \(n_{\text{int}} = \Psi_{\text{int}}^\dagger \Psi_{\text{int}}\).

The internal basis functions satisfy the more general version of (5.28) with \(\varepsilon_r = 0:\)
\[ \left[ \sum_m \frac{1}{2m_0} \left( -\frac{\partial^2}{\partial (x^m)^2} + \frac{a_0^2}{16} \frac{\partial^4}{\partial (x^m)^4} \right) + V_0 (x^m) - \mu_{\text{int}} \right] \tilde{\psi}_{\text{int}} (x^m, x^\mu) = 0. \] (6.3)
This is a linear equation because \(V_0 (x^m)\) is now regarded as a known function.

The full path integral involving (4.14) contains all configurations of the fields, including those with nontrivial topologies. In the present theory, the geography of our universe includes a topological defect in the \((D-4)\)-dimensional internal space which is analogous to a vortex. (See Appendix A.) The standard features of four-dimensional physics arise from the presence of this internal topological defect. For example, it compels the initial gauge symmetry to be \(SO(D-4)\).
The behavior of the condensate and basis functions in the internal space is discussed in Appendices A and B. In (A15), the parameters $\bar{\phi}_i$ specify a rotation of $\Psi_{\text{int}}(x^m, x^\mu)$ as the external coordinates $x^\mu$ are varied, and according to (A16) the $\bar{J}_i$ satisfy the $SO(D - 4)$ algebra

$$\bar{J}_i \bar{J}_j - \bar{J}_j \bar{J}_i = i c_{ij}^k \bar{J}_k.$$ (6.4)

For simplicity of notation, let

$$\langle r'| Q | r' \rangle = \int d^{D-4} x \, \bar{\psi}_{\text{int}}^r Q \psi_{\text{int}}^r$$

with

$$\langle r | r' \rangle = \delta_{rr'}$$ (6.5)

for any operator $Q$, and in particular let

$$t_{i r'}^r = \langle r | \bar{J}_i | r' \rangle$$ (6.6)

with the matrices $t_{i r'}^r$ (which are constant according to (A17)) inheriting the $SO(D - 4)$ algebra:

$$\begin{align*}
(t_i t_j - t_j t_i)^{r'}^r &= \sum_{r''} \langle r | \bar{J}_i | r'' \rangle \langle r'' | \bar{J}_j | r' \rangle - \sum_{r''} \langle r | \bar{J}_j | r'' \rangle \langle r'' | \bar{J}_i | r' \rangle \\
&= \langle r | \bar{J}_i \bar{J}_j | r' \rangle - \langle r | \bar{J}_j \bar{J}_i | r' \rangle \\
&= i c_{ij}^k t_{k r'}^r. (6.7)
\end{align*}$$

The $t_i$ are the generators in the $N_g$-dimensional reducible representation determined by the physically significant solutions to (6.3), which spans all the irreducible (physical) gauge representations.

When $x^\mu \to x^\mu + \delta x^\mu$, $\Psi_{\text{int}}$ and $\bar{\psi}_{\text{int}}^r$ rotate together, and (A15) implies that

$$\begin{align*}
\partial_\mu \bar{\psi}_{\text{int}}^r (x^m, x^\mu) &= \frac{\partial \bar{\phi}_i}{\partial x^\mu} \frac{\partial}{\partial \bar{\phi}_i} \bar{\psi}_{\text{int}}^r (x^m, x^\mu) \\
&= -i A_{\mu}^i \bar{J}_i \bar{\psi}_{\text{int}}^r (x^m, x^\mu) (6.10)
\end{align*}$$

where

$$A_{\mu}^i = \frac{\partial \bar{\phi}_i}{\partial x^\mu}.$$ (6.12)

The $A_{\mu}^i$ will be interpreted below as gauge potentials. In other words, the gauge potentials are simply the rates at which the internal order parameter $\Psi_{\text{int}}(x^m, x^\mu)$ is rotating as a function of the external coordinates $x^\mu$. 

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Let us return to the fermionic action (5.42). If (5.43) is written in the more general form

$$\Psi_f (x^\mu, x^m) = \tilde{\psi}_f (x^\mu) \tilde{\psi}^r_{\text{int}} (x^m, x^\mu) = U_{\text{ext}} (x^\mu) \psi_f (x^\mu) \tilde{\psi}^r_{\text{int}} (x^m, x^\mu)$$

we have

$$\partial_{\mu} \Psi_f = U_{\text{ext}} (x^\mu) \left( \partial'_{\mu} - im_0 e_{\alpha \mu} \sigma^\alpha - i A^\mu_{\text{int}} \right) \psi_f \tilde{\psi}^r_{\text{int}}$$

where the prime indicates that $\partial'_{\mu}$ does not operate on $\tilde{\psi}^r_{\text{int}}$, and

\[
\int d^{D-4} x \, \psi^\dagger_f \partial_{\mu} \partial_{\mu} \Psi_f \\
= \int d^{D-4} x \tilde{\psi}^r_{\text{int}} \psi_f^\dagger \left( \partial'^{\mu} - im_0 e^\alpha_{\mu} \sigma^\alpha - i A^{\mu}_{\text{int}} \right) \left( \partial'_{\mu} - im_0 e_{\alpha \mu} \sigma^\alpha - i A^\mu_{\text{int}} \right) \tilde{\psi}^r_{\text{int}} \\
= \psi_f^\dagger (r) \left( \partial'^{\mu} - im_0 e^\alpha_{\mu} \sigma^\alpha - i A^{\mu}_{\text{int}} \right) \sum_{\nu} \psi^{\nu}_{\text{int}} (r') \left( \partial'_{\mu} - im_0 e_{\alpha \mu} \sigma^\alpha - i A^\mu_{\text{int}} \right) \psi_f^{\nu}_{\text{int}}
\]

Then (5.42) becomes

$$S_f = \int d^4 x \psi_f^\dagger \left( -\frac{1}{2m_0} D^\mu D_\mu + \frac{1}{2} i e^\mu_\alpha \sigma^\alpha D_\mu + \frac{1}{2} D^\mu i e_{\alpha \mu} \sigma^\alpha + \frac{1}{2} m_0 e^\mu_\alpha \sigma^\alpha e_{\alpha \mu} \sigma^\alpha - \mu_{\text{ext}} \right) \psi_f$$

where

$$D_\mu = \partial_{\mu} - i A^\mu_{\text{int}} t_i.$$  

(6.15)

With (5.25) and the approximations above (5.35), (5.26) implies that

$$S_f = \int d^4 x \bar{\psi}_f \left( -\frac{1}{2m_0} D^\mu D_\mu + i e^\mu_\alpha \sigma^\alpha D_\mu \right) \psi_f.$$

(6.16)

This is the generalization of (5.44) when the internal order parameter is permitted to vary as a function of the external coordinates $x^\mu$. Again, for momenta and gauge potentials that are small compared to $m_0 e^\mu_\alpha$ with $\mu = \alpha$, the first term may be neglected. Furthermore, the entire treatment above can be repeated for the bosonic action, finally giving

$$S_f = \int d^4 x \bar{\psi}_f i e^\mu_\alpha \sigma^\alpha D_\mu \psi_f$$

(6.17)

$$S_b = \int d^4 x \bar{\psi}_b i e^\mu_\alpha \sigma^\alpha D_\mu \psi_b.$$

(6.18)

VII. TRANSFORMATION TO LORENTZIAN PATH INTEGRAL: FERMIONS

All of the foregoing is within a Euclidean picture, but we will now show that, in the case of fermions, there is a relatively trivial transformation to the more familiar Lorentzian
description. A key point is that the low-energy operator \( i \epsilon^\mu_\sigma \sigma^\alpha D_\mu \) in \( S_f \) is automatically in the correct Lorentzian form, even though the initial path integral is in Euclidean form. It is this fact which permits the following transformation to a Lorentzian path integral. Within the present theory, neither the fields nor the operators (nor the meaning of the time coordinate) need to be modified in performing this transformation.

The operator within \( S_f \) can be diagonalized to give

\[
S_f = \sum_s \bar{\psi}_f^*(s) \ a(s) \ \psi_f(s) \tag{7.1}
\]

where

\[
\psi_f(x) = \sum_s U(x,s) \ \bar{\psi}_f(s) \quad , \quad \bar{\psi}_f(s) = \int d^4x \ U^\dagger(x,s) \ \psi_f(x) \tag{7.2}
\]

with

\[
 i \epsilon^\mu_\sigma \sigma^\alpha D_\mu U(x,s) = a(s) \ U(x,s) \tag{7.3}
\]

\[
\int d^4x \ U^\dagger(x,s) \ U(x,s') = \delta_{ss'} \quad , \quad \sum_s U(x,s) \ U^\dagger(x',s) = \delta(x-x') \tag{7.4}
\]

Here, and in the following, \( x \) represents a point in external spacetime, and \( U(x,s) \) is a multicomponent eigenfunction. There is an implicit inner product in

\[
U^\dagger(x,s) \ \psi_f(x) = \sum_r U^\dagger_r(x,s) \ \psi_f^r(x) \tag{7.5}
\]

with the \( 2N_g \) components of \( \psi_f(x) \) labeled by \( r = 1, \ldots, N_g \) (spanning all components of all irreducible gauge representations) and \( a = 1, 2 \) (labeling the components of Weyl spinors), and with \( s \) and \( (x,r,a) \) each having \( N \) values. Also, the delta function in (7.4) implicitly multiplies the \( 2N_g \times 2N_g \) identity matrix.

Evaluation of the present Euclidean path integral (a Gaussian integral with Grassmann variables) is then trivial for fermions; as usual,

\[
Z_f = \int \mathcal{D}\psi_f^\dagger(x) \ \mathcal{D}\psi_f(x) \ e^{-S_f} \tag{7.6}
\]

\[
= \prod_{x,r,a} \int d\psi_f^{ra*}(x) \int d\psi_f^{ra}(x) \ e^{-S_f} \tag{7.7}
\]

\[
= \prod_s z_f(s) \tag{7.8}
\]

with

\[
z_f(s) = \int d\bar{\psi}_f^*(s) \int d\bar{\psi}_f(s) \ e^{-\bar{\psi}_f^{a*}(s)a(s)\bar{\psi}_f(s)} \tag{7.9}
\]

\[
= a(s) \tag{7.10}
\]
since the transformation is unitary \[59\]. Now let

\[ Z_f^L = \int D\overline{\psi}_f (s) D\overline{\psi}_f (s) \ e^{iS_f} \]
\[ = \prod_s z_f^L (s) \tag{7.12} \]

where

\[ z_f^L (s) = \int d\overline{\psi}_f (s) \int d\overline{\psi}_f (s) \ e^{i\overline{\psi}_f (s)a(s)f_j (s)} \]
\[ = -ia (s) \tag{7.13} \]

so that

\[ Z_f^L = c_f Z_f , \quad c_f = \prod_s (-i) . \tag{7.15} \]

This result holds for the path integral over an arbitrary time interval, with the fields, operator, and meaning of time left unchanged.

The transition amplitude from an initial state to a final state is equal to the path integral between these states, so transition probabilities are the same with the Lorentzian and Euclidean forms of the path integral. This result is consistent with the fact that the classical equations of motion are also the same, since they follow from extremalization of the same action. Furthermore, using the method on pp. 290-291 or 302-303 of Ref. \[59\], it is easy to show that the magnitude \(|G (x, x')|\) of the 2-point function is again the same, so particles propagate the same way in both descriptions. This result is also obtained in Appendix C with a different method.

It may seem strange that the Lorentzian and Euclidean forms of the path integral yield the same physical results, but perusal of the standard arguments in e.g. field theory textbooks shows that the physically significant features of the results derive from the Lorentzian form of the action rather than the path integral.

When the inverse transformation from \(\overline{\psi}_f\) to \(\psi_f\) is performed, we obtain

\[ Z_f^L = \int D\psi_f (x) D\psi_f (x) \ e^{iS_f} \]
\[ with S_f having its form \(6.17\) in the coordinate representation. \]

One may perform calculations in either the path-integral formulation or the equivalent canonical formulation, which can now be obtained in the standard way: Let us use the notation \(\int_a^b\) to indicate that the fields in a path integral are specified to begin in a state \(|a\)
at time \( t_a \) and end in state \( |b\rangle \) at time \( t_b \), and also to indicate that a path integral showing these limits has its conventional definition (so that it may differ by a normalization constant from \( Z_f^L \) as defined above). Then the Hamiltonian \( H_f \) is defined by

\[
\langle b | U_f (t_b, t_a) | a \rangle = \int_a^b \mathcal{D} \psi^\dagger_f (x) \mathcal{D} \psi_f (x) \ e^{iS_f} \tag{7.17}
\]

\[
i \frac{d}{dt} U_f (t, t_a) = H_f (t) U_f (t, t_a) \quad , \quad U_f (t_a, t_a) = 1 \ . \tag{7.18}
\]

I.e., the time evolution operator \( U_f (t_b, t_a) \) is defined to have the same effect as the path integral over intermediate states, and it is then straightforward to reverse the usual logic which leads from canonical quantization to path-integral quantization \([59, 61]\).

VIII. TRANSFORMATION TO LORENTZIAN PATH INTEGRAL: BOSONS

For bosons we can again perform the transformation \([7.2]\) to obtain

\[
S_b = \sum_s \overline{\psi}_b^* (s) a (s) \overline{\psi}_b (s) \ . \tag{8.1}
\]

We will now show how this action can be put into a form which corresponds to scalar bosonic fields plus their auxiliary fields, temporarily working in a locally inertial coordinate system, so that \( \epsilon_0^\mu \sigma^\alpha \to \sigma^\mu \). First, if the gauge potentials \( A^i_\mu \) were zero, we would have

\[
i \sigma^\mu \partial^\mu U^0 (x, s) = a_0 (s) U^0 (x, s) \ . \tag{8.2}
\]

Then

\[
U^0 (x, s) = \mathcal{V}^{-1/2} u (s) e^{ip_s \cdot x} \ , \quad p_s \cdot x = \eta_{\mu \nu} p^\mu s x^\nu \ , \quad \eta_{\mu \nu} = \text{diag} (-1, 1, 1, 1) \tag{8.3}
\]

(with \( \mathcal{V} \) a four-dimensional normalization volume) gives

\[
- \eta_{\mu \nu} \sigma^\mu p^\nu s U^0 (x, s) = a_0 (s) U^0 (x, s) \tag{8.4}
\]

where \( \sigma^\mu \) implicitly multiplies the identity matrix for the multicomponent function \( U^0 (x, s) \).

A given 2-component spinor \( u_r (s) \) has two eigenstates of \( p^k_s \sigma^k \):

\[
p^k_s \sigma^k u^+_r (s) = \left| \overrightarrow{p}_s \right| u^+_r (s) \ , \quad p^k_s \sigma^k u^-_r (s) = - \left| \overrightarrow{p}_s \right| u^-_r (s) \tag{8.5}
\]

where \( \overrightarrow{p}_s \) is the 3-momentum, with magnitude \( |\overrightarrow{p}_s| \). The multicomponent eigenstates of \( i \sigma^\mu \partial^\mu \) and their eigenvalues \( a_0 (s) = p^0_s \left| \overrightarrow{p}_s \right| \) thus come in pairs, corresponding to opposite helicities.
For nonzero $A^i_\mu$, the eigenvalues $a(s)$ will also come in pairs, with one growing out of $a_0(s)$ and the other out of its partner $a_0(s')$ as the $A^i_\mu$ are turned on. To see this, first write (7.3) as

\[ (i\partial_0 + A^i_0 t_i) U(x, s) + \sigma^k (i\partial_k + A^i_k t_i) U(x, s) = a(s) U(x, s) \]  

(8.6)

or

\[ (i\partial_0 \delta_{rr'} + A^i_0 t^r_{i'}) U_{rr'}(x, s) - P_{rr'} U_{rr'}(x, s) - a(s) \delta_{rr'} U_{rr'}(x, s) = 0 \]  

(8.7)

\[ P_{rr'} = -\sigma^k (i\partial_k \delta_{rr'} + A^i_k t^r_{i'}) \]  

(8.8)

with the usual implied summations over repeated indices. At fixed $r$, $r'$ (and $x, s$), apply a matrix $s$ which will diagonalize the $2 \times 2$ matrix $P_{rr'}$, bringing it into the form $p_{rr'} \sigma^3 + \overline{p}_{rr'} \sigma^0$, where $p_{rr'}$ and $\overline{p}_{rr'}$ are 1-component operators, while at the same time rotating the 2-component spinor $U_{rr'}$:

\[ s P_{rr'} s^{-1} = P'_{rr'} = p_{rr'} \sigma^3 + \overline{p}_{rr'} \sigma^0, \quad U'_{rr'} = sU_{rr'} \]  

(8.9)

\[ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(8.10)

But $P_{rr'}$ is traceless, and the trace is invariant under a similarity transformation, so $\overline{p}_{rr'} = 0$. Then the second term in (8.7) (for fixed $r$ and $r'$) becomes $s^{-1} p_{rr'} \sigma^3 U'_{rr'}(x, s)$. The two independent choices

\[ U'_{rr'}(x, s) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma^3 U'_{rr'}(x, s) = +U'_{rr'}(x, s) \]  

(8.11)

\[ U'_{rr'}(x, s) \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma^3 U'_{rr'}(x, s) = -U'_{rr'}(x, s) \]  

(8.12)

give $\pm s^{-1} p_{rr'} U'_{rr'}(x, s)$. Now use $s^{-1} U'_{rr'} = U_{rr'}$ to obtain for (8.7)

\[ (i\partial_0 \delta_{rr'} + A^i_0 t^r_{i'}) U_{rr'}(x, s) + p_{rr'} U_{rr'}(x, s) - a(s) \delta_{rr'} U_{rr'}(x, s) = 0 \]  

(8.13)

so (8.6) reduces to two sets of equations with different eigenvalues $a(s)$ and $a(s')$:

\[ a(s) = a_1(s) + a_2(s), \quad a(s') = a_1(s) - a_2(s) \]  

(8.14)

where these equations define $a_1(s)$ and $a_2(s)$. Notice that letting $\sigma^k \rightarrow -\sigma^k$ in (8.6) reverses the signs in (8.13), and results in $a(s) \rightarrow a(s')$:

\[ (i\partial_0 + A^i_0 t_i) U(x, s) - \sigma^k (i\partial_k + A^i_k t_i) U(x, s) = a(s') U(x, s) \]  

(8.15)
The action for a single eigenvalue \( a(s) \) and its partner \( a(s') \) is

\[
\tilde{s}_b(s) = \bar{\psi}_b'(s) a(s) \bar{\psi}_b(s) + \bar{\psi}_b'(s') a(s') \bar{\psi}_b(s') \tag{8.16}
\]

\[
= \bar{\psi}_b'(s)(a_1(s) + a_2(s)) \bar{\psi}_b(s) + \bar{\psi}_b'(s')(a_1(s) - a_2(s)) \bar{\psi}_b(s') . \tag{8.17}
\]

In the following we will circumvent singularities by implicitly following the standard prescription \( a_1 \to a_1 + i\epsilon, \epsilon \to 0^+ \), which reduces to \( \omega \to \omega + i\epsilon \) when there are no gauge fields.

For \( a_1(s) \geq 0 \), let us choose \( a_2(s) \geq 0 \) and define

\[
\bar{\psi}_b'(s') = a(s)^{1/2} \bar{\phi}_b'(s') = (a_1(s) + a_2(s))^{1/2} \bar{\phi}_b(s') \tag{8.18}
\]

\[
\bar{\psi}_b(s) = a(s)^{-1/2} \bar{F}_b(s) = (a_1(s) + a_2(s))^{-1/2} \bar{F}_b(s) \tag{8.19}
\]

so that

\[
\tilde{s}_b(s) = \bar{\phi}_b'(s') \bar{\alpha}(s) \bar{\phi}_b(s') + \bar{F}_b'(s) \bar{F}_b(s) , \quad a_1(s) \geq 0
\]

where

\[
\bar{\alpha}(s) = a(s) a(s') = a_1(s)^2 - a_2(s)^2 . \tag{8.21}
\]

For \( a_1(s) < 0 \), let us choose \( a_2(s) \leq 0 \) and write

\[
\bar{\psi}_b'(s') = (-a(s))^{1/2} \bar{\phi}_b'(s') = (-a_1(s) - a_2(s))^{1/2} \bar{\phi}_b(s') \tag{8.22}
\]

\[
\bar{\psi}_b(s) = (-a(s))^{-1/2} \bar{F}_b(s) = (-a_1(s) - a_2(s))^{-1/2} \bar{F}_b(s) \tag{8.23}
\]

so that

\[
\tilde{s}_b(s) = - \left[ \bar{\phi}_b'(s') \bar{\alpha}(s) \bar{\phi}_b(s') + \bar{F}_b'(s) \bar{F}_b(s) \right] , \quad a_1(s) < 0 . \tag{8.24}
\]

Then we have

\[
S_b = \sum_{s'} \tilde{s}_b(s) \tag{8.25}
\]

\[
= \sum_{a_1(s) \geq 0} \left[ \bar{\phi}_b'(s') \bar{\alpha}(s) \bar{\phi}_b(s') + \bar{F}_b'(s) \bar{F}_b(s) \right] - \sum_{a_1(s) < 0} \left[ \bar{\phi}_b'(s') \bar{\alpha}(s) \bar{\phi}_b(s') + \bar{F}_b'(s) \bar{F}_b(s) \right] \tag{8.26}
\]

where a prime on a summation or product over \( s \) means that only one member of an \( s, s' \) pair (as defined in (8.13) and (8.14)) is included. Let us separate the non-negative contribution
$S_+$ from the anomalous negative contribution $S_-$:

$$S_b = S_+ + S_-$$

$$S_+ = \sum_{s \geq 0} \phi^*_a(s')|\phi_a(s)| + \sum_{a_1(s) \geq 0} \bar{F}_b^*(s) \bar{F}_b(s)$$

$$S_- = -\left[ \sum_{s < 0} \phi^*_a(s')|\phi_a(s)| + \sum_{a_1(s) < 0} \bar{F}_b^*(s) \bar{F}_b(s) \right]$$

where

$$s < 0 \quad \leftrightarrow \quad \bar{a}(s) = a_1(s)^2 - a_2(s)^2 < 0 \quad \text{if} \quad a_1(s) \geq 0$$

$$s < 0 \quad \leftrightarrow \quad \bar{a}(s) = a_1(s)^2 - a_2(s)^2 > 0 \quad \text{if} \quad a_1(s) < 0$$

with $s \geq 0$ otherwise.

Recall that if the gauge potentials $A^a_\mu$ were zero, we would have $a_1 = \omega$ and $a_2 = \mp |p|$, where $\omega$ is the frequency and $p$ the 3-momentum.

The negative-action modes of (8.29) are discussed in Appendix D where it is found that their excitations can be treated in the same way as the positive-action modes. (Here and below, “positive-action modes” means those which do not have negative action in the present context, before mass and interaction terms are acquired. After the transformation to a Lorentzian path integral below, some fields may come to have negative action due to symmetry breaking and condensation, but this is permissible within a Lorentzian description.) It follows that, in the treatment below, no degrees of freedom are lost in the bosonic excitations. On-shell modes already have non-negative action, so both the excitation and condensation of scalar boson fields remain exactly the same as in standard physics.

The path integral for positive-action modes in $S_+$ is

$$Z_+ = \int \mathcal{D} \psi^\dagger_b(x) \mathcal{D} \psi_b(x) e^{-S_+}$$

$$= \prod_{x,ra} \int_{-\infty}^{\infty} d(\text{Re} \psi_{b,ra}(x)) \int_{-\infty}^{\infty} d(\text{Im} \psi_{b,ra}(x)) e^{-S_+}$$

$$= \prod_{s} \int_{-\infty}^{\infty} d(\text{Re} \bar{\psi}_b(s)) \int_{-\infty}^{\infty} d(\text{Im} \bar{\psi}_b(s)) e^{-S_+}.$$ (8.33)

Each of the transformations above from $\bar{\psi}_b$ to $\phi_b$ and $\bar{F}_b$ has the form

$$\bar{\psi}_b(s') = A(s)^{1/2} \phi_b(s'), \quad \bar{\psi}_b(s) = A(s)^{-1/2} \bar{F}_b(s)$$

(8.34)
so that \( \overline{\psi}_b(s') = A(s)^{1/2} \overline{\phi}_b(s') \), \( \overline{\psi}_b(s) = A(s)^{-1/2} \overline{F}_b(s) \), and the Jacobian is \( \Pi'_s A(s)^{1/2} A(s)^{-1/2} = 1 \). These transformations then lead to

\[
Z_{sb} = \prod_{s \geq 0} z_{\phi}(s) \cdot \prod_{s < 0} z_{\phi}(s) \cdot \prod_{a_1(s) \geq 0} \int_{a_1(s) \geq 0} z_{\phi}(s) = \prod_{s} z_{\phi}(s) \cdot \prod_{a_1(s) \geq 0} z_{F}(s) \tag{8.35}
\]

where the excitations of negative-action modes have been added and

\[
z_{\phi}(s) = \int_{-\infty}^{\infty} d(\text{Re} \overline{\phi}_b(s')) \int_{-\infty}^{\infty} d(\text{Im} \overline{\phi}_b(s')) e^{-i \overline{a}(s)}[(\text{Re} \overline{\phi}_b(s'))^2 + (\text{Im} \overline{\phi}_b(s'))^2] \tag{8.36}
\]

\[
z_{F}(s) = \int_{-\infty}^{\infty} d(\text{Re} \overline{F}_b(s)) \int_{-\infty}^{\infty} d(\text{Im} \overline{F}_b(s)) e^{-[(\text{Re} \overline{F}_b(s))^2 + (\text{Im} \overline{F}_b(s))^2]} \tag{8.37}
\]

Now let

\[
S_{sb} = \sum'_{s} \overline{\phi}_b(s') \overline{a}(s) \overline{\phi}_b(s') + \sum'_{a_1(s) > 0} \overline{F}_b(s) \overline{F}_b(s) \tag{8.40}
\]

\[
Z_{sb}^{L} = \int \mathcal{D} \overline{\phi}_b(s') \mathcal{D} \overline{\phi}_b(s') \mathcal{D} \overline{F}_b(s) \mathcal{D} \overline{F}_b(s) e^{i S_{sb}} = \prod_{s} z_{\phi}^{L}(s) \cdot \prod_{a_1(s) \geq 0} z_{F}^{L}(s) \tag{8.41}
\]

where

\[
z_{\phi}^{L}(s) = \int_{-\infty}^{\infty} d(\text{Re} \overline{\phi}_b(s')) \int_{-\infty}^{\infty} d(\text{Im} \overline{\phi}_b(s')) e^{i \overline{a}(s)}[(\text{Re} \overline{\phi}_b(s'))^2 + (\text{Im} \overline{\phi}_b(s'))^2] \tag{8.43}
\]

\[
z_{F}^{L}(s) = \int_{-\infty}^{\infty} d(\text{Re} \overline{F}_b(s)) \int_{-\infty}^{\infty} d(\text{Im} \overline{F}_b(s)) e^{i [(\text{Re} \overline{F}_b(s))^2 + (\text{Im} \overline{F}_b(s))^2]} \tag{8.45}
\]

since \( \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(i a (x^2 + y^2)) = i \pi / a \). (Nuances of Lorentzian path integrals are discussed in, e.g., Peskin and Schroeder [59], p. 286.) We have then obtained

\[
Z_{sb}^{L} = c_b Z_{sb} \tag{8.47}
\]

where \( c_b \) is a product of factors of \( i \) and \( -1 \).

To return to the coordinate representation, let us define physical fields

\[
\Phi(x) = \sum_{s} U(x, s') \overline{\phi}_b(s') \tag{8.48}
\]
and auxiliary fields

\[ F(x) = \sum_s' U(x,s) \overline{F}_b(s) . \]  

(8.49)

Recall that these fields include only positive-action modes and *excitations* of negative-action modes.

As a reminder of the notation, recall that, according to (8.6) and (8.15),

\[ i\sigma^\mu D_\mu U(x,s') = a(s') U(x,s') \quad , \quad i\overline{\sigma}^\mu D_\mu U(x,s') = a(s) U(x,s') \]  

(8.50)

with \( \overline{\sigma}^0 = \sigma^0 \), \( \overline{\sigma}^k = -\sigma^k \), \( a(s) = a_1(s) + a_2(s) \), \( a(s') = a_1(s) - a_2(s) \), \( \overline{a}(s) = a(s) a(s') = a_1(s)^2 - a_2(s)^2 \), and \( s > 0 \) or \( < 0 \) defined by (8.30)-(8.31) and the line following. Again, in the absence of gauge potentials we have \( a_1 = \omega \) and \( a_2 = \mp |\vec{p}| \).

We could return to the original coordinate system, with the action in (8.40) becoming

\[ S_{sb} = S_\Phi + S_F \]  

(8.51)

where

\[ S_\Phi = \int d^4 x \bar{\ell}_\Phi \quad , \quad S_F = \int d^4 x \bar{\ell}_F \quad , \quad \bar{\ell}_F = \bar{\ell}^\dagger(x) F(x) \]  

(8.52)

and \( \bar{\ell}_\Phi \) is obtained via \( \sigma^\mu \to e_\alpha^\mu \sigma^\alpha \), as in (7.33). However, in the following it is more convenient to remain in the locally inertial coordinate system used above, where

\[ \bar{\ell}_\Phi = \frac{1}{2} \Phi^\dagger(x) i\overline{\sigma}^\mu D_\mu i\sigma^\nu D_\nu \Phi(x) + \frac{1}{2} \Phi^\dagger(x) i\sigma^\mu D_\mu i\overline{\sigma}^\nu D_\nu \Phi(x) . \]  

(8.53)

To simplify the mathematics below, it is convenient to temporarily write

\[ \Phi_b = \begin{pmatrix} \Phi \\ \Phi \end{pmatrix} \]  

(8.54)

and to use the same Weyl representation as is used for Dirac fermions, with

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \]  

(8.55)

so that (8.53) can be written as

\[ \bar{\ell}_\Phi = -\frac{1}{2} \Phi^\dagger_b(x) \gamma^\mu D_\mu \gamma^\nu D_\nu \Phi_b(x) \]  

(8.56)

\[ = -\frac{1}{2} \Phi^\dagger_b(x) \hat{p}^2 \Phi_b(x) . \]  

(8.57)

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A result \[59, 60\] that can be extended to the nonabelian case gives

\[-\mathcal{L}^2 = D^\mu D_\mu - S^{\mu\nu} F_{\mu\nu}\] (8.58)

with the present convention for the metric tensor. (See pp. 173-174 of \[60\] for this result and those immediately below.) Here the field strength tensor $F_{\mu\nu}$ spans all the irreducible (physical) gauge representations, and the second term can be rewritten with “magnetic” and “electric” fields $B_k$ and $E_k$ defined by

\[F_{kk'} = -\varepsilon_{kk'k''} B_{k''}, \quad F_{0k} = E_k\] (8.59)

since \[60\]

\[-S^{\mu\nu} F_{\mu\nu} = \begin{pmatrix} (\vec{B} + i \vec{E}) \cdot \vec{\sigma} & 0 \\ 0 & (\vec{B} - i \vec{E}) \cdot \vec{\sigma} \end{pmatrix}\] (8.60)

where \(a \cdot b = a_k b^k\). (Recall that \(\mu = 0, 1, 2, 3\) and \(k = 1, 2, 3\).) We then obtain

\[\mathcal{L}_\Phi = \Phi^\dagger (x) D^\mu D_\mu \Phi (x) + \Phi^\dagger (x) \vec{B} \cdot \vec{\sigma} \Phi (x) .\] (8.61)

The second term above is invariant under a rotation, but not under a boost, so it breaks Lorentz invariance for the primitive bosonic fields in $\Phi$. This issue is considered in the next section, where the physical scalar boson fields are defined using arguments that are an extension of those we have given elsewhere \[44, 45\].

All of the above is in a locally inertial frame of reference. Now let us return to a general coordinate system and initially assume no nongravitational gauge fields, so that $D_\mu \to \nabla_\mu$ and

\[\mathcal{L}_\Phi = \frac{1}{2} \Phi^\dagger_b (x) \gamma^\mu \nabla_\mu \Phi_b (x) = -\frac{1}{2} \Phi^\dagger_b (x) \vec{\gamma}^2 \Phi_b (x)\] (8.62)

where

\[\vec{\gamma} \equiv \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}\] (8.63)

with $\sigma^\mu = e^\mu_\alpha \sigma^\alpha$ and $\overline{\sigma}^\alpha = e^\mu_\alpha \overline{\sigma}^\alpha$. But

\[-\vec{\gamma}^2 = g^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{1}{4} R\] (8.64)
follows from e.g. (5.293) of Ref. [53] with our \((-+++)\) convention for the metric tensor, where \(R\) is the curvature scalar in 4-dimensional spacetime, so

\[
\mathcal{L}_\phi = \Phi^\dagger (x) \left( g^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{1}{4} R \right) \Phi (x) .
\]  

(8.65)

(See also (15.5.5) of Ref. [54], with the present convention for the metric tensor, but the opposite convention of their p. 274 for the Dirac gamma matrices.) It is well known that a scalar field can have a Lagrangian which includes a term \(-\xi R\) (as in (2.35) of Ref. [53]), with \(\xi\) undetermined in standard physics, but the present picture yields \(\xi = 1/4\).

IX. SCALAR BOSONS

Let \(\Phi_r\) be one of the primitive 2-component spin 1/2 bosonic fields in \(\Phi\). We can construct physical fields, which satisfy Lorentz invariance, by eliminating the anomalous second term of (8.61) in either of two ways.

**Higgs fields** can be constructed by combining 2-component fields \(\Phi_r\) and \(\Phi_{r'}\) with the same gauge quantum numbers but opposite spins (and equal amplitudes):

\[
\Phi_R = \begin{pmatrix} \Phi_r \\ \Phi_{r'} \end{pmatrix}
\]  

(9.1)

so that

\[
\Phi_R^\dagger (x) \vec{\sigma} \Phi_R (x) = \begin{pmatrix} \Phi_r^\dagger & \Phi_{r'}^\dagger \end{pmatrix} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} \Phi_r \\ \Phi_{r'} \end{pmatrix}
\]

\[
= \Phi_r^\dagger \vec{\sigma} \Phi_r + \Phi_{r'}^\dagger \vec{\sigma} \Phi_{r'}
\]

\[
= 0 .
\]  

(9.2)

(9.3)

In this case we write for each gauge component

\[
\Phi_R (x) = \phi_R (x) \xi_R \quad \text{with} \quad \xi_R^\dagger \xi_R = 1
\]  

(9.4)

where \(\xi_R\) has 4 constant components and \(\phi_R (x)\) is a 1-component complex amplitude. If this can be done for a full gauge multiplet \(\tilde{\Phi}_R\) containing \(\Phi_R\), yielding a multiplet \(\tilde{\phi}_R\) of 1-component scalar boson fields, then (8.61) implies that

\[
S_R = \int d^4 x \mathcal{L}_R , \quad \mathcal{L}_R = \tilde{\phi}_R^\dagger (x) D^\mu D_{\mu} \tilde{\phi}_R (x)
\]  

(9.5)
is the action for this multiplet (before masses and other interaction terms are added).

**Higgson fields** can be constructed by combining a 2-component field $\Phi_s$ and its charge conjugate $\Phi^c_s$ (with opposite gauge quantum numbers but the same spin):

$$\Phi_S = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_s \\ \Phi^c_s \end{pmatrix}.$$  \hfill (9.6)

The subscripts $s$ and $S$ are used in this context to avoid confusion.

$\Phi_s$ and $\Phi^c_s$ have opposite expectation values for the generators $t^j$ (which are here treated as operators rather than matrices):

$$\Phi^c_s \dagger t^j \Phi_s^c = -\Phi_s^\dagger t^j \Phi_s.$$  \hfill (9.7)

Since

$$B_{k\nu} = -\varepsilon_{k\nu k\ell} F_{k\ell}, \quad F_{k\ell} = F^j_{k\ell} t^j$$

we have

$$\Phi^\dagger_S(x) \overrightarrow{B} \Phi_S(x) = \begin{pmatrix} \Phi^\dagger_s & \Phi^c_s \end{pmatrix} \begin{pmatrix} \overrightarrow{B} & 0 \\ 0 & \overrightarrow{B} \end{pmatrix} \begin{pmatrix} \Phi_s \\ \Phi^c_s \end{pmatrix} = \Phi^\dagger_s \overrightarrow{B} \Phi_s + \Phi^c_s \overrightarrow{B} \Phi^c_s = 0.$$  \hfill (9.10)

$$= 0.$$  \hfill (9.11)

Before mass is acquired, the Lagrangian for a higgson field in the electroweak sector contains the terms [44]:

$$\mathcal{L}^0_{H_i} = \overrightarrow{H}_i^\dagger \partial\mu \partial_\mu \overrightarrow{H}_i, \quad \mathcal{L}^Z_{H_i} = -\frac{g_Z^2}{4} \overrightarrow{H}_i^\dagger Z^\mu Z_\mu \overrightarrow{H}_i, \quad \mathcal{L}^W_{H_i} = -\frac{g^2}{2} \overrightarrow{H}_i^\dagger W^{\mu+} W^{-}_\mu \overrightarrow{H}_i$$

where we have modified the notation slightly, with $H \rightarrow \overrightarrow{H}_i$ for the $i$th species. Since $\overrightarrow{H}_i$ has spin 0 and is real [44], we can write

$$\overrightarrow{H}_i(x) = H_i(x) \zeta \quad \text{with} \quad \zeta^\dagger \zeta = 1,$$

where $\zeta$ has 4 constant components which incorporate the quantum numbers of $\overrightarrow{H}_i$, and $H_i(x)$ is again an amplitude mode. Each $\Phi_S$, with four degrees of freedom, has first become four independent 4-component real fields $\overrightarrow{H}_i$, and then four 1-component real fields $H_i$.  

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Here we define \( H_i(x) \) to be a higgson field (in a slight change of nomenclature from that of previous papers). It has no interactions other than those in the reduced version of (9.12):

\[
\mathcal{L}_{H_i}^Z = -\frac{g_z^2}{4} H_i Z^\nu Z_\mu H_i, \quad \mathcal{L}_{H_i}^W = -\frac{g^2}{2} H_i W^{\mu*} W_{\mu} H_i. \tag{9.14}
\]

Each such higgson field can then be treated (and quantized) like a standard real scalar field, but with no quantum numbers and no interactions except those described in (9.14).

Some further discussion of both Higgs and higgson modes is given in Refs. [44] and [45]. Note that (i) the second term in (8.65) is obtained for all scalar boson fields and (ii) cross terms in the full \( \mathcal{L} \) of (10.3) below ultimately give zero in all physical fields, because the factor involving \( \Sigma_{\alpha\beta} \) (containing Pauli matrices) produces cancellation in Higgs fields and the factor involving \( A_\mu \) produces cancellation in higgson fields. For higgson fields we then have, in a general coordinate system,

\[
\mathcal{L}_{H_i} = H_i \left( g^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{1}{4} R \right) H_i + \mathcal{L}_{H_i}^Z + \mathcal{L}_{H_i}^W. \tag{9.15}
\]

### X. FUNDAMENTAL ACTION FOR FERMIONS, SCALAR BOSONS, AND AUXILIARY FIELDS

When the various components in the preceding sections are assembled, the total action for fermion fields \( \psi_f \), Higgs fields \( \phi_R \), auxiliary fields \( F_R \), and higgson fields \( H_i \) is

\[
S_{\text{matter}} = \int d^4x \mathcal{L}_{\text{matter}} \quad (10.1)
\]

\[
\mathcal{L}_{\text{matter}} = \bar{\psi}_f(x) \left( i e^\alpha_\sigma D_\mu \psi_f(x) + \sum_R \phi_R(x) \left( g^{\mu\nu} D_\mu D_\nu - \frac{1}{4} R \right) \phi_R(x) + \sum_R F_R^\dagger(x) F_R(x) + \sum_i \mathcal{L}_{H_i} \right) \tag{10.2}
\]

after transformation to a general coordinate system, and before masses and further interactions result from symmetry breakings and other effects. \( D_\mu \) should now be interpreted as the full covariant derivative

\[
D_\mu = \nabla_\mu - iA_\mu \quad (10.3)
\]

where \( \nabla_\mu \) and \( A_\mu \) can be regarded as operators which yield the appropriate generators and potentials for each representation.
The spin 1/2 fermion fields in $\psi_f$ span the various physical representations of the most fundamental gauge group, which must be $SO(D-4)$ in the present theory. (More precisely, the group is $Spin(D-4)$, but $SO(D-4)$ is conventional terminology.) $F_R$ is the amplitude of a component of $\mathcal{F}$, defined in the same way as $\phi_R$. The last term in (10.2) contains the action for all higgson fields, including those which lie outside the electroweak sector. We thus obtain the basic form for a Lorentz-invariant, gauge-invariant, and supersymmetric action, with the addition of a new kind of particle. Notice, however, that susy is broken by the “condensation” of the negative-action $F$ modes of (8.29).

The gravitational and gauge curvatures of (11.12) and (11.13) below must ultimately originate from a background (in the path integral) of “rapidly fluctuating” 4-dimensional topological defects (analogous to vortices and vortex rings, or extended and closed flux tubes) associated with the gauge potentials of (6.12) and the vierbein of (5.37). Here “rapidly fluctuating” means that the $A_\mu$ and $g_{\mu\nu}$ (or $e^\alpha_\mu$) are actually averages over many topological configurations of the field $\Psi_0$. As mentioned below (5.10), the path integral over all these configurations is replaced by a path integral over the $A_\mu$ and $g_{\mu\nu}$ with an effective action. The gauge and gravitational fields then vary over all possibilities, and this is how these force fields are quantized in the present theory. To discuss the topological defects in detail – with examples representing various physical phenomena associated with abelian and nonabelian gauge fields – is beyond the scope of this paper, but all that is required here is the fact that topological defects permit the curvature associated with the gauge potentials and vierbein to be nonzero. I.e., the gauge curvature (in a locally inertial frame, and with the coupling constant no longer absorbed into $A_\mu$)

$$F_{\mu\nu}^i = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f^i_{jk} A^j_\mu A^k_\nu$$

and the gravitational spin connection and curvature (see p. 274 of [54])

$$\omega^\alpha_\mu = \frac{1}{2} e^{\nu\alpha} (\partial_\mu e^\beta_\nu - \partial_\nu e^\beta_\mu) - \frac{1}{2} e^{\beta\gamma} (\partial_\mu e^\alpha_\nu - \partial_\nu e^\alpha_\mu) - \frac{1}{2} e^{\rho\alpha} e^{\sigma\beta} (\partial_\rho e_{\sigma\gamma} - \partial_\sigma e_{\rho\gamma}) e^\gamma_\mu$$

$$R^\alpha_{\mu\nu} = \partial_\mu \omega^\alpha_\nu - \partial_\nu \omega^\alpha_\mu + [\omega^\alpha_\mu, \omega^\beta_\nu]_\alpha$$

$$R^\alpha_{\mu\nu} = e^\alpha_\beta e^\gamma_\nu R^\beta_{\mu\nu}$$

originate from vortex-like configurations in the same way that the vorticity of a superfluid

$$\omega_k^s = \partial_k v^s_\ell - \partial_\ell v^s_k \quad \text{or} \quad \omega_z^s = \partial_x v^s_y - \partial_y v^s_x$$
originates from ordinary vortices, as pointed out by Feynman and Onsager. The simplest case is a magnetic field with

$$B_z = \partial_x A_y - \partial_y A_x$$  \hspace{1cm} (10.8)$$

but all the force fields above have the same basic form. In each case, the curvature is nonzero in a region penetrated by flux lines that are interpreted as vortex lines.

XI. COSMOLOGICAL CONSTANT, EINSTEIN-HILBERT ACTION, BLACK HOLE ENTROPY, AND DARK MATTER

A. Cosmological constant and Einstein-Hilbert action

In conventional physics, the contribution of fermion and scalar boson fields to the vacuum energy corresponds to a Lagrangian $\mathcal{L}_{\text{vac}}$, where $\mathcal{L}_{\text{vac}}$ is constant but $e$ is given by (5.45). The resulting gravitational energy-momentum (or stress-energy) tensor is

$$T_{\mu\nu}^{\text{vac}} = 2e^{-\frac{1}{2}} \frac{\delta \mathcal{L}_{\text{vac}}}{\delta g_{\mu\nu}} + g^{\mu\nu} \mathcal{L}_{\text{vac}}, \quad \mathcal{L}_{\text{vac}} = -\left(8\pi \ell_P^2 \right)^{-1} \Lambda$$  \hspace{1cm} (11.1)$$

since

$$\delta e = \frac{1}{2} e g^{\mu\nu} \delta g_{\mu\nu}$$  \hspace{1cm} (11.2)$$

and this produces a term $\Lambda g_{\mu\nu}$ on the left-hand side of the Einstein field equations, with a cosmological constant $\Lambda$.

In (10.2), the coupling of matter to gravity in $\mathcal{L}_{\text{matter}}$ is very nearly the same as in standard general relativity, but there is no factor of $e$ in the integrand of (10.1). This means that, for a fixed vacuum energy density due to fermions and scalar bosons,

$$T_{\mu\nu}^{\text{vac}} = 2e^{-\frac{1}{2}} \frac{\delta \mathcal{L}_{\text{vac}}}{\delta g_{\mu\nu}} = 0$$  \hspace{1cm} (11.3)$$

so there is no direct contribution to a cosmological constant from these fields.

The predictions of the present theory are identical to those of standard general relativity for the motion of all particles and waves in gravitational fields, and for gauge bosons acting as a source of gravity. We will now show that they are also the same for matter acting as
a gravitational source. The action of (10.1) begins in the initial coordinate system. Let \( \varphi \) represent any of the matter fields in (10.2), with an action having the form

\[
S_\varphi = \int d^4x \mathcal{L}_\varphi \quad , \quad \mathcal{L}_\varphi = \varphi^\dagger A_\varphi \varphi .
\] (11.4)

The contribution of \( \varphi \) to the gravitational energy-momentum tensor in the present theory is then

\[
T^\mu_\nu = 2e^{-1} \frac{\delta \mathcal{L}_\varphi}{\delta g^\mu_\nu} = 2e^{-1} \varphi^\dagger \frac{\delta A_\varphi}{\delta g^\mu_\nu} \varphi = 2 \varphi^\dagger \frac{\delta A_\varphi}{\delta g^\mu_\nu} \varphi \quad , \quad \varphi = e^{-1/2} \varphi .
\] (11.5)

The usual energy-momentum tensor, with

\[
S_\varphi' = \int d^4x \mathcal{L}_{\varphi'} \quad , \quad \mathcal{L}_{\varphi'} = e \mathcal{L}_{\varphi'} \quad , \quad \mathcal{L}_{\varphi'} = \varphi'^\dagger A_{\varphi'} \varphi'
\] (11.6)

is

\[
T'^\mu_\nu = 2e^{-1} \frac{\delta (e \mathcal{L}_{\varphi'})}{\delta g^\mu_\nu} = 2e^{-1} \frac{\delta e}{\delta g^\mu_\nu} + 2 \frac{\delta \mathcal{L}_{\varphi'}}{\delta g^\mu_\nu} = 2 \frac{\delta \mathcal{L}_{\varphi'}}{\delta g^\mu_\nu} = 2 \varphi'^\dagger \frac{\delta A_{\varphi'}}{\delta g^\mu_\nu} \varphi'
\] (11.7)

since the bilinear forms of (10.2) – or similar forms including mass and interaction terms, or for composite objects like protons or planets – imply that \( \mathcal{L}_{\varphi'} = 0 \) if the classical equations of motion are satisfied. Classical in the present context means that particles or composite bodies remain on the mass shell, and the energy-momentum tensor here corresponds to the energy and momentum of quantum fields, particles, or composite objects satisfying their quantum equations of motion, which yield classical trajectories according to Ehrenfest’s theorem.

There is agreement between (11.5) and (11.7) if \( \varphi \) can be identified with \( \varphi' \). According to (11.4), (11.5), and (11.6), this means that

\[
e \varphi'^\dagger A_{\varphi'} \varphi = \varphi^\dagger A_\varphi \varphi \quad \text{or} \quad A_\varphi e^{-1/2} \varphi = e^{-1/2} A_{\varphi'} \varphi .
\] (11.8)

The only part of an operator \( A_\varphi \) in (10.2) that does not immediately commute with \( e^{-1/2} \) is the covariant derivative \( D_\mu \). But the covariant derivative of a function of only the vierbein is zero, so

\[
D_\mu e^{-1/2} \varphi = [D_\mu e^{-1/2}] \varphi + e^{-1/2} D_\mu \varphi = e^{-1/2} D_\mu \varphi
\] (11.9)

and (11.8) is satisfied.
If the $\varphi$ are renamed and called $\phi$, then (10.2) holds with (10.1) modified to

$$S_{\text{matter}} = \int d^4x \, e^{\mathcal{L}_{\text{matter}}}$$

(11.10)

but the stress-energy tensor must be calculated from

$$T_{\phi}^{\mu\nu} = 2 \phi^i \frac{\delta A_{\phi}}{\delta g_{\mu\nu}} \phi \quad \text{or} \quad T_\phi^{\mu\nu} = -2 \phi^i \frac{\delta A_{\phi}}{\delta g_{\mu\nu}} \phi$$

(11.11)

and not from $T_{\phi}^{\mu\nu} = 2 e^{-1} \delta \left( e^{\mathcal{L}_{\phi}} \right) / \delta g_{\mu\nu}$ (unless it is recognized that $\phi$ now contains a hidden factor of $e^{-1/2}$). As shown above, the two formulas give the same result for particles satisfying their classical equations of motion, but not in general, and not for the zero-point (vacuum) action of fermions and scalar bosons (if used in the conventional way). Eq. (11.11) is, in fact, a natural definition of the energy-momentum tensor in a quantum description, and more familiar classical forms can be obtained by using the classical equation of motion and integration by parts.

We might note that the shift from the initial fields of Section V to the redefined fields of (11.10) (i.e. the fields of (10.2) after reinterpretation), could have been made immediately after (5.44), with the understanding that the effect of the $e^{-1/2}$ factors absorbed into the fields cancels that of the external factor of $e$ when $\delta / \delta g_{\mu\nu}$ is applied to the action.

If the metric tensor $g_{\mu\nu}$ is taken to be fixed – e.g. if all other fields are taken to evolve on a fixed classical gravitational background – then these fields can be treated via path-integral quantization in the usual way, with only the measure changed by the extra factor of $e^{-1/2}$ in the fermion and scalar boson fields.

The predictions of general relativity are thus unchanged in the present picture, except that the cosmological constant of (11.3) is zero if the vacuum Lagrangian density $\mathcal{L}_{\text{vac}}$ is taken to be fixed.

On the other hand, $\mathcal{L}_{\text{vac}}$ is not really fixed, since the fields in the vacuum will respond to variations in the gauge potentials of (6.12) and the vierbein of (5.37) (or metric tensor of (5.36)). In Appendix E the Einstein-Hilbert action of gravity

$$\mathcal{L}_G = \left( 16 \pi \ell_p^2 \right)^{-1} e^{\mathcal{R}}$$

(11.12)

where $\ell_p^2 = G$, is obtained as the vacuum response to the curvature of the vierbein (or metric tensor).
Similarly, we conjecture that the “diamagnetic” response of the vacuum fermion and scalar-boson fields to gauge curvature gives rise to the Maxwell-Yang-Mills action, with

$$\mathcal{L}_g = -\frac{1}{4} g_0^2 \epsilon g^{\mu \rho} g^{\nu \sigma} F_{\mu \nu}^i F_{\rho \sigma}^i$$  \hspace{1cm} \text{(11.13)}

where $g_0$ is the coupling constant for the fundamental gauge group. This form is analogous to the shift in the free energy

$$\Omega(B) - \Omega(0) = \frac{e^2 A}{24 \pi mc^2} B^2,$$  \hspace{1cm} \text{(11.14)}

when electrons in a metal (with area $A$) respond to an applied magnetic field $\vec{B}$, exhibiting Landau diamagnetism.

It appears that it is nontrivial to obtain (11.13) from a proper calculation, and that a detailed knowledge of the vacuum states is required. However, (i) those states will surely be modified when they are perturbed by the curvature of gauge fields (as are the states of electrons in a metal), (ii) (11.13) has the simplest form consistent with the symmetries of the vacuum (including invariance under coordinate, Lorentz, and gauge transformations), and (iii) within the present picture (11.13) can originate only from the response of the vacuum to external gauge fields. With this interpretation, $\mathcal{L}_g$ must necessarily vanish when these fields vanish – i.e., in the vacuum itself:

$$\langle \mathcal{L}_g \rangle_{\text{vac}} = 0.$$

This means that when (11.13) is quantized, the field operators must be normal-ordered. It follows that there is no cosmological constant resulting from the gauge fields. On the other hand, virtual processes will still be affected by a change in their boundary conditions; a detailed treatment of this aspect, and of the observed Casimir effect [46–48], would be inappropriately long here, but see the discussion of this point in Section II.

B. Bekenstein-Hawking entropy of black holes

There have been many attempts to understand the Bekenstein-Hawking entropy $S_{BH}$ of black holes in terms of microscopic degrees of freedom, but none has provided a convincing explanation for even the simplest of physical black holes. On the other hand, Gibbons and Hawking [49] have shown that, for a static black hole, an expression equal to the
Euclidean action has the form required of $S_{BH}$. In the present picture, every fundamental Lorentzian action $S_L$ can alternatively be interpreted as the negative of a Boltzmann entropy $S$, determined by counting microstates (of dits) as in Sections [I] and [II]. The Lorentzian action of a general system has the form $S_L = \int dt (T - V)$, and the Euclidean action the form $S_E = \int dt (T + V)$, where $T$ includes the time derivatives. For a static system, we have $T = 0$ and the first term vanishes in both expressions. It follows that

$$S_E = -S_L = S \quad \text{for a static black hole} \ .$$

(11.16)

The entropy $S$ originates from the microstates of the gravitational field configuration called a black hole, so it has the same status as the entropy of any static thermodynamic system.

The action of a rotating black hole contains an additional contribution from the angular momentum, with the same expression for the entropy.

C. Dark matter

There are a vast number of hypothetical dark matter candidates, most of which do not have well-defined masses or couplings, and many of which have already been ruled out by experiment – or at least found to be subdominant species in a multicomponent scenario. For example, the simplest susy models which have “natural” values for the parameters, and which are also compatible with limits from the LHC, are found to be in disagreement with both the abundance of dark matter and the limits from direct-detection experiments [62–66] – if the lightest supersymmetric particle (LSP) is assumed to be the dominant constituent. But, as mentioned in Section [II] the present picture requires susy at some energy scale, and the LSP (as a subdominant component) can stably coexist with the present dark matter candidate (the lightest higgson).

Fortunately the enthusiasm of experimentalists searching for new physics has not been notably diminished by the past lack of success, and work on new facilities and capabilities has persisted even though the pandemic. For example, LZ, XENONnT, and PandaX-4T should all soon be taking data with much larger detectors than those of past experiments. They should ultimately be able to detect a dark matter WIMP with a mass of $\sim 50$ GeV/c$^2$ if the Xe collision cross-section is larger than about $1.4 \times 10^{-48}$ cm$^2$ [67, 69], within roughly the next 5 years. For the WIMP predicted by the present theory – the lowest-mass higgson of
Section IX – this cross-section is slightly below $10^{-47}$ cm$^2$, with a mass of about 72 GeV/c$^2$.

More generally, this dark matter candidate is consistent with all current experiments, and observable in the near or foreseeable future through a wide variety of direct, indirect, and collider detection experiments. To review the conclusions of Refs. [44] and [45]: This particle is unique in that it has (i) precisely defined couplings and (ii) a well-defined mass of about 72 GeV/c$^2$, providing specific cross-sections and other experimental signatures as targets for clean experimental tests. It has not yet been detected because it has no interactions other than second-order gauge couplings, to W and Z bosons. However, these weak couplings are still sufficient to enable observation by direct detection experiments which should be fully functional within the next few years, including XENONnT, LZ, and PandaX. The cross-section for collider detection at LHC energies is small – roughly 1 femtobarn – but observation may ultimately be achievable at the high-luminosity LHC, and should certainly be within reach of the even more powerful colliders now being planned. It is possible that the present dark matter candidate has already been observed via indirect detection: Several analyses of gamma rays from the Galactic center, observed by Fermi-LAT, and of antiprotons, observed by AMS-02, have shown consistency with the interpretation that these result from annihilation of dark matter particles having approximately the same mass and annihilation cross-section as the present candidate. Finally, there is consistency with the observations of Planck, which have ruled out many possible candidates with larger masses.

XII. CONCLUSION

Starting with the simplest imaginable picture, and interpreting our universe as the product of two spaces with topological singularities, we obtain the following results: 4-dimensional spacetime with one time coordinate; spin 1/2 fermion and spin zero boson fields defined on this spacetime; path-integral quantization of these fields; gauge fields and a fundamental gauge theory which is necessarily $SO(N)$; correct couplings of matter fields to the gauge fields; a gravitational vierbein; correct couplings of matter fields to gravity; Lorentz invariance; supersymmetry at some energy scale; elimination of the usual enormous cosmological constant; the Einstein-Hilbert action for gravity; the Bekenstein-Hawking entropy of black holes; and a new set of particles, including a new dark matter WIMP which should be detectable in the near future.
Appendix A: The internal space

The internal space of Section VI is \((D - 4)\)-dimensional, with an \(SO(D - 4)\) (or more precisely \(Spin(D - 4)\)) rotation group and its vector, spinor, etc. representations – for example, the 10 and 16 representations when \(D - 4 = 10\). It may be helpful to begin with an analogy, however, in which external spacetime is replaced by the \(z\)-axis. The internal space is replaced by an \(xy\)-plane, with internal states described by 2-dimensional vector fields (rather than the higher-dimension vector and spinor fields considered below). One of these states is occupied by the condensate, and is represented by a vector \(v_1\) which points radially outward from the origin at all points in the \(xy\)-plane when \(z = 0\). The other state is an additional basis function, represented by a vector \(v_2\) which is everywhere perpendicular to \(v_1\). But \(v_1\) is allowed to rotate as a function of \(z\), so it has both radial and tangential components after a displacement along the \(z\)-axis. Then \(v_2\) is forced to rotate with \(v_1\) – i.e., the condensate – in order to preserve orthogonality.

Now let us turn to the actual internal space, first considering a set of \((D - 4)\)-dimensional vector fields \(\tilde{\psi}_r^{vec}\). Let \(\tilde{\psi}_r^{vec}\) represent the state occupied by a bosonic condensate. In the simplest picture, and at some fixed \(x_\mu^0\), only the \(r\)th component of the field \(\tilde{\psi}_r^{vec}\) is nonzero along some radial direction in the internal space, making the fields trivially orthogonal in that direction. Then, with \(x^\mu\) still fixed, \(\tilde{\psi}_r^{vec}(x^m)\) in all other radial directions is obtained from the original \(\tilde{\psi}_r^{vec}(x_0^m)\) by rotating it to \(x^m\). In other words, the field at each point in the internal space is identical to the field that would be obtained at that point if the original field \(\tilde{\psi}_r^{vec}(x_0^m)\) were subjected to a rotation about the origin. This produces an isotropic configuration for the condensate and each basis function. As in (5.13) we can write

\[
\tilde{\psi}_r^{vec}(x^m) = U_{vec}(x^m, x_0^m) \tilde{\psi}_r^{vec}(x_0^m).
\] (A1)

Just as in the analogy, a field that is radial at \(x_0^m\) will also be radial at all other points \(x^m\). However, a general \(\tilde{\psi}_r^{vec}(x_0^m)\) permits a general vortex-like configuration of the condensate.

Also as in the analogy, the state \(\tilde{\psi}_r^{vec}\) of the condensate is allowed to rotate as a function of \(x^\mu\) (because such a rotation does not alter the internal action). Since the other basis functions \(\tilde{\psi}_r^{vec}\) are required to remain orthogonal to \(\tilde{\psi}_r^{vec}\) and each other, they are required to rotate with the condensate. Then (A1) becomes more generally

\[
\tilde{\psi}_r^{vec}(x^m, x^\mu) = U_{vec}(x^m, x_0^m, x^\mu, x_0^\mu) \tilde{\psi}_r^{vec}(x_0^m, x_0^\mu).
\] (A2)
with

\[ \bar{\psi}^r_{vec}(x^m, x^\mu) \bar{\psi}^{r'}_{vec}(x^m, x^\mu) = \bar{\psi}^r_{vec}(x^m_0, x^\mu_0) \bar{\psi}^{r'}_{vec}(x^m_0, x^\mu_0) = \delta_{rr'} \]  

(A3)

since

\[ U^\dagger_{vec}(x^m, x^\mu_0; x^\mu, x^\mu_0) U_{vec}(x^m, x^m_0; x^\mu, x^\mu_0) = 1. \]  

(A4)

In general (with \( x^\mu \) fixed), let \( \bar{\psi}(x) \) represent a multicomponent basis function with angular momentum \( j \) at a point \( x \) in the \((D - 4)\)-dimensional internal space. After a rotation about the origin specified by the \((D - 4) \times (D - 4)\) matrix \( R \), it is transformed to

\[ \bar{\psi}'(x) = R(R) \bar{\psi}(R^{-1}x) \]  

(A5)

where \( R(R) \) belongs to the appropriate representation of the group \( Spin(D - 4) \). However, we require that the field be isotropic, so that it is left unchanged after a rotation:

\[ \bar{\psi}'(x) = \bar{\psi}(x). \]  

(A6)

Then we can define \( \bar{\psi}(x) \) at each value of the radial coordinate \( r \) by starting with a \( \bar{\psi}(x_0) \) and requiring that

\[ \bar{\psi}(x) = R(R) \bar{\psi}(x_0), \quad x = Rx_0. \]  

(A7)

With this definition, \( \bar{\psi}(x) \) is a single-valued function of the coordinates only if \( j \) is an integer. If \( j = 1/2 \), e.g., \( \bar{\psi}(x) \) acquires a minus sign after a rotation of \( 2\pi \), but it is single-valued on the \( Spin(D - 4) \) group manifold.

Multivalued functions are well-known in other similar contexts, such as the behavior of the phase of an ordinary superfluid order parameter \( \psi_s = e^{i\theta_s n_s^{1/2}} \) around a vortex, which becomes discontinuous if it is required to be a single-valued function of the coordinates \([70]\).

In the same way, \( z^{1/2} \) exhibits a discontinuity across a branch cut if it is required to be a single-valued function and \( z \) is restricted to a single complex plane. I.e., \( z^{1/2} = |z|^{1/2} e^{i\phi/2} \) gives \( +|z|^{1/2} \) for \( \phi = 0 \) and \( -|z|^{1/2} \) for \( \phi = 2\pi \). But when defined on a pair of Riemann sheets, \( z^{1/2} \) is a continuous function, and the same is true of \( \bar{\psi}(x) \) as we have defined it above, on the group manifold. The key idea in either case is to extend the manifold over which the function is defined, so that there are no artificial discontinuities. A similar principle holds.
in condensed matter physics, where a spinor can be a multivalued function of position (but with physical expectation values single-valued).

A vectorial condensate and vectorial basis functions are appropriate for the simplest Higgs-like fields and their superpartners. Similarly, spinorial fields \( \tilde{\psi}_{sp} \) are appropriate for ordinary fermions, sfermions, and a possible primordial condensate occupying a state \( \tilde{\psi}_0 \).

(In the present context, of course, “vector” and “spinor” refer only to properties in the internal space.) Again, let \( \tilde{\psi}_{sp} (x^m) \) represent a field along some radial direction in the internal space at some fixed \( x^\mu_0 \). Then the field configuration for every point \( x^m \) is obtained by taking \( \tilde{\psi}_{sp} (x^m) \) to be identical to the field that would be obtained at that point if \( \tilde{\psi}_{sp} (x^m_0) \) were subjected to a rotation, with

\[
\tilde{\psi}_{sp} (x^m) = U_{sp} (x^m, x^m_0) \tilde{\psi}_{sp} (x^m_0)
\]

as in (A7).

Again, the state \( \tilde{\psi}_0^{sp} \) of the condensate is allowed to rotate as a function of \( x^\mu \), and since the other basis functions \( \tilde{\psi}_{sp} \) must remain orthogonal to \( \tilde{\psi}_0^{sp} \) they are required to rotate with the condensate. The general version of (A8) is then

\[
\tilde{\psi}_{sp} (x^m, x^\mu) = U_{sp} (x^m, x^m_0; x^\mu, x^{\mu}_0) \tilde{\psi}_{sp} (x^m_0, x^{\mu}_0)
\]

(A9)

The same reasoning applies to each irreducible representation, and thus to the combined set of fields \( \tilde{\psi}_{int}^r (x^m, x^\mu) \):

\[
\tilde{\psi}_{int}^r (x'^m, x'^\mu) = U_{int} (x'^m, x^m; x'^\mu, x^\mu) \tilde{\psi}_{int}^r (x^m, x^\mu)
\]

(A10)

with

\[
\tilde{\psi}_{int}^r \dagger (x'^m, x'^\mu) \tilde{\psi}_{int}^r (x^m, x^\mu) = \tilde{\psi}_{int}^r \dagger (x^m, x^\mu) \tilde{\psi}_{int}^r (x^m, x^\mu) = \delta_{rr'}.
\]

(A11)

So that the internal action will be unaffected as \( x^\mu \rightarrow x'^\mu \), we require that the order parameter experience a uniform rotation, described by a matrix \( \mathcal{R}_{int} \) which is independent of \( x^m \). Then \( U_{int} \) has the form

\[
U_{int} (x'^m, x^m; x'^\mu, x^\mu) = \mathcal{R}_{int} (x'^\mu, x^\mu) \mathcal{R}_{int} (x'^m, x^m)
\]

(A12)

(Notice that (A12) is to be distinguished from a rotation about the origin, which is given by (A5), and which according to (A6) would leave \( \tilde{\psi} (x^m) \) unchanged rather than rotated at
each point \(x^m\). It follows that

\[
\bar{\psi}_\text{int}' (x^m, x^\mu) = \mathcal{R}_{\text{int}} (x^\mu, x^m_0) \bar{\psi}_\text{int}' (x^m_0, x^\mu) .
\]  

(A13)

At each fixed \(x^m\), the order parameter has been rotated as \(x^\mu_0 \to x^\mu\). We define the parameters \(\delta \bar{\phi}_i\) by

\[
\mathcal{R}_{\text{int}} (x^\mu + \delta x^\mu, x^m_0) = \mathcal{R}_{\text{int}} (x^\mu, x^m_0) (1 - i \delta \bar{\phi}_i J_i)
\]  

(A14)
or

\[
\delta \bar{\psi}_\text{int}' (x^m) = -i \delta \bar{\phi}_i \bar{J}_i \bar{\psi}_\text{int}' (x^m) \quad \text{as} \quad x^\mu \to x^\mu + \delta x^\mu
\]  

(A15)

where the matrices \(J_i\) are the generators in the reducible representation of \(\text{Spin}(D - 4)\) corresponding to \(\bar{\psi}_\text{int}'\). The matrix elements of \(\bar{J}_i\) are independent of \(x^\mu\):

\[
\int d^{D-4} x \bar{\psi}_\text{int}'^\dagger (x^m, x^\mu) \bar{J}_i \bar{\psi}_\text{int}' (x^m, x^\mu) = \int d^{D-4} x \bar{\psi}_\text{int}'^\dagger (x^m, x^\mu_0) J_i \mathcal{R}_{\text{int}}^{-1} (x^\mu, x^m_0)
\]  

(A17)

The primordial condensate is in a specific representation, but the basis functions in other representations are chosen to rotate with it according to (A13) and (A15).

It may be helpful to illustrate the above ideas by returning to the 2-dimensional analogy. Equation (A7) becomes

\[
v(x) = \mathcal{R}_\text{vec} v(x_0), \quad \mathcal{R}_\text{vec} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad v(x_0) = \begin{pmatrix} R(r) \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ R(r) \end{pmatrix}
\]  

(A18)

for the vector representation and

\[
s(x) = \mathcal{R}_\text{sp} s(x_0), \quad \mathcal{R}_\text{sp} = e^{-i\sigma_3 \phi / 2}, \quad s(x_0) = \begin{pmatrix} R(r) \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ R(r) \end{pmatrix}
\]  

(A19)

for the spinor representation. The matrices corresponding to the \(J_i\) are

\[
J_\text{vec} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad J_\text{sp} = \frac{\sigma_3}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(A20)

Notice that \(\phi_i\) is an angular coordinate in the internal space, whereas \(\bar{\phi}_i\) is a parameter specifying the rotation of \(\bar{\psi}_\text{int}'\) at fixed \(x^m\) as \(x^\mu\) is varied.
Appendix B: Solutions in the internal space

Our goal in this appendix is merely to show that there are solutions with the form required in Appendix A, so we will look first for solutions with the higher-derivative terms in (6.2) and (6.3) neglected, and with $\Psi_{int}$ sufficiently small that $V_0(x^m)$ can also be neglected. Then (6.2) and (6.3) become

$$\left(-\frac{1}{2m_0} \partial_m \partial_m - \mu_{int}\right) \Psi_{int}(x^m, x^\mu) = 0$$

(B1)

$$\left(-\frac{1}{2m_0} \partial_m \partial_m - \mu_{int}\right) \tilde{\psi}_{int}(x^m, x^\mu) = 0.$$  

(B2)

For simplicity of notation, let $\tilde{\psi}_{int}(x^m, x^\mu)$ again be represented by $\tilde{\psi}(x)$, with components $\tilde{\psi}_p(x)$. Each component varies with position in the way specified by (A7) (together with the radial dependence of $\tilde{\psi}(x_0)$). It therefore has a kinetic energy given by $-\frac{1}{2m_0} \partial_m \partial_m \tilde{\psi}_p(x)$, and an orbital angular momentum given by the usual orbital angular momentum operators $\hat{J}_i$ in $\tilde{d}$ dimensions [71–76], which essentially measure how rapidly $\tilde{\psi}_p(x)$ varies as a function of the angles $\phi_i$.

The Laplacian $\partial_m \partial_m$ can be rewritten in terms of radial derivatives and the usual $\hat{J}^2$, giving [71, 73]

$$\left(-\frac{1}{r^2K} \frac{\partial}{\partial r} \left(r^{2K} \frac{\partial}{\partial r}\right) + \frac{\hat{J}^2}{r^2} - 1\right) \tilde{\psi}_p(x) = 0 , \quad K = \frac{\tilde{d} - 1}{2}$$

(B3)

after rescaling of the radial coordinate $r$, where

$$\tilde{d} = D - 4.$$  

(B4)

In addition, it is shown in Narumi and Nakau [71], Gallup [72], and Louck [73] that

$$\hat{J}^2 \tilde{\psi}_p(x) = j(j + \tilde{d} - 2) \tilde{\psi}_p(x)$$

(B5)

where $j$ is the orbital angular momentum quantum number, as defined on p. 677 of Gallup, but with this definition extended to half-integer values of $m_\alpha$ and $j$. Normally, of course, only integer values of these orbital quantum numbers are permitted. However, the functions $\tilde{\psi}_p(x)$ as defined in [A] can have $j = 1/2$ etc. (in which case they are multivalued functions of the coordinates but single-valued functions on the group manifold, as discussed below (A7)). Also, the demonstration of (B5) of Gallup [72] can be extended in the present context to half-integer $j$, because it employs raising and lowering operators. (At each $x$, $\tilde{\psi}_p$ is a linear
combination of states with different values of \( m_\alpha \), but \((B5)\) still holds.) For each \( \tilde{\psi}_p(x) \) the radial wavefunction then satisfies

\[
\begin{bmatrix}
- \frac{1}{r^{2K}} \frac{d}{dr} \left( \frac{r^{2K}}{dr} \right) + \frac{j (j + \tilde{d} - 2)}{r^2} - 1
\end{bmatrix} R(r) = 0.
\]

It may be helpful once again to consider the 2-dimensional analogy of \( A \), where the orbital angular momentum operator is

\[
\hat{\mathcal{J}} = -i \frac{\partial}{\partial \phi}.
\]

For the vector representation, \((A18)\) implies that the kinetic energy is given by

\[
\partial_m \partial_m \mathbf{v}(x) = \begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}
\end{bmatrix} \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix} \mathbf{v}(x_0)
\]

\[
= \begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \right) - \frac{1}{r^2}
\end{bmatrix} \mathbf{v}(x)
\]

in agreement with \((B6)\) for \( j = 1 \). For the spinor representation, \((A19)\) gives

\[
\partial_m \partial_m s(x) = \begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}
\end{bmatrix} e^{-i\sigma_3 \phi/2} s(x_0)
\]

\[
= \begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \right) - \frac{1/4}{r^2}
\end{bmatrix} s(x)
\]

in agreement with \((B6)\) for \( j = 1/2 \).

Equation \((B6)\) can be further reduced to \([74, 76]\)

\[
\left[ - \frac{d^2}{dr^2} + \frac{k(k-1)}{r^2} - 1 \right] \chi(r) = 0, \quad k = j + K = j + \tilde{d} - \frac{1}{2}
\]

where \( \chi(r) \equiv r^K R(r) \). It is then easy to show that

\[
\chi(r) \propto r^k \text{ as } r \to 0, \quad \chi(r) \propto \sin(r + \delta) \text{ as } r \to \infty
\]

where \( \delta \) is a phase.

The higher derivatives in the full internal wave equation \((6.3)\) permit exponentially decaying solutions which are then normalizable and have finite action. Suppose that the above equation at large \( r \) is modified to

\[
\left[ \alpha^2 \frac{d^4}{dr^4} - \frac{d^2}{dr^2} - 1 \right] \chi(r) = 0.
\]
The solutions are

$$\chi(r) \propto e^{iqr} \ , \ \ q^2 = -\frac{1}{2\alpha^2} \pm \frac{\sqrt{1+4\alpha^2}}{2\alpha^2} \ . \quad (B15)$$

There is then an exponentially decaying solution with the form $q = i/\bar{\alpha}$ and

$$\chi(r) \propto e^{-r/\bar{\alpha}} \quad (B16)$$

so both the order parameter and the basis functions fall to zero as $r \to \infty$.

**Appendix C: Euclidean and Lorentzian Propagators**

For Weyl fermions, the Euclidean 2-point function is

$$G_f(x_1, x_2) = \langle \psi_f(x_1) \overline{\psi}_f(x_2) \rangle = \frac{\int \mathcal{D}\psi_f^\dagger \mathcal{D}\psi_f \psi_f(x_1) \psi_f(x_2) e^{-S_f}}{\int \mathcal{D}\psi_f^\dagger \mathcal{D}\psi_f e^{-S_f}} \quad (C1)$$

$$= \prod_s \int d\overline{\psi}_f^*(s) \int d\psi_f(s) \ e^{-\overline{\psi}_f(s)\alpha(s)\overline{\psi}_f(s) \sum_{s_1, s_2} \overline{\psi}_f(s_1) \overline{\psi}_f(s_2) U(x_1, s_1) U^\dagger(x_2, s_2)} \prod_s \int d\overline{\psi}_f^*(s) \int d\psi_f(s) \ e^{-\overline{\psi}_f(s)\alpha(s)\overline{\psi}_f(s)}$$

where (7.1) and (7.2) have been used. In a term with $s_2 \neq s_1$, the numerator contains the factor

$$\int d\overline{\psi}_f^*(s_1) \int d\psi_f(s_1) \ e^{-\overline{\psi}_f(s_1)\alpha(s_1)\overline{\psi}_f(s_1)\overline{\psi}_f(s_1)} = 0 \quad (C2)$$

according to the rules for Berezin integration. But a term with $s_2 = s_1$ contributes

$$\frac{\int d\overline{\psi}_f^*(s_1) \int d\psi_f(s_1) \ e^{-\overline{\psi}_f(s_1)\alpha(s_1)\overline{\psi}_f(s_1)\overline{\psi}_f(s_1)}}{\int d\overline{\psi}_f(s_1) \int d\psi_f(s_1) \ e^{-\overline{\psi}_f(s_1)\alpha(s_1)\overline{\psi}_f(s_1)}} \frac{\overline{\psi}_f^*(s_1) \overline{\psi}_f(s_1) \ U(x_1, s_1) \ U^\dagger(x_2, s_1)}{\overline{\psi}_f(s_1) \overline{\psi}_f(s_1) \ U(x_1, s_1) \ U^\dagger(x_2, s_1)} = a(s_1)^{-1} U(x_1, s_1) U^\dagger(x_2, s_1) \quad (C3)$$

so

$$G_f(x_1, x_2) = \sum_s \overline{G}_f(s) U(x_1, s) U^\dagger(x_2, s) \ , \ \ \overline{G}_f(s) = a(s)^{-1} \quad (C4)$$

If the $U(x, s)$ used to represent $\psi_f(x)$ are a complete set, the propagator $G_f(x, x')$ is a true Green’s function:

$$L_f(x) U(x, s) = a(s) U(x, s) \ , \ \ \psi_f(x) = \sum_s U(x, s) \overline{\psi}_f(s) \quad (C5)$$
and \( \sum_s U(x, s) U^\dagger(x', s) = \delta(x - x') \) imply that

\[
L_f(x) G_f(x, x') = \delta(x - x')
\]

(C6)
as usual.

The treatment for scalar bosons is similar:

\[
G_b(x_1, x_2) = \left\langle \phi_b(x_1) \phi_b^\dagger(x_2) \right\rangle = \frac{\int D \phi_b^\dagger \mathcal{D} \phi_b \phi_b(x_1) \phi_b^\dagger(x_2) e^{-S_f}}{\int D \phi_b^\dagger \mathcal{D} \phi_b e^{-S_f}}
\]

(C7)

\[
= \prod_s \int_{-\infty}^{\infty} d \text{Re} \phi_b(s) \int_{-\infty}^{\infty} d \text{Im} \phi_b(s) e^{-\bar{a}(s)} \left[ (\text{Re} \phi_b(s))^2 + (\text{Im} \phi_b(s))^2 \right] \sum_{s_1, s_2} \phi_b^*(s_1) \phi_b(s_2)
\]

\[
\times U_b(x_1, s_1) U_b^\dagger(x_2, s_2)
\]

(C8)

where

\[
L_b(x) U_b(x, s) = \bar{a}(s) U_b(x, s) , \quad \phi_b(x) = \sum_s U_b(x, s) \phi_b(s).
\]

In a term with \( s_2 \neq s_1 \), the numerator contains the factor

\[
\int_{-\infty}^{\infty} d \text{Re} \phi_b(s_1) \int_{-\infty}^{\infty} d \text{Im} \phi_b(s_1) e^{-\bar{a}(s_1)} \left[ (\text{Re} \phi_b(s_1))^2 + (\text{Im} \phi_b(s_1))^2 \right] \left[ \text{Re} \phi_b(s_1) + i \text{Im} \phi_b(s_1) \right]
\]

\[
= 0
\]

(C10)
since the integrand is odd. But a term with \( s_2 = s_1 \) contains the factor

\[
\frac{\int_{-\infty}^{\infty} d \text{Re} \phi_b(s_1) e^{-\bar{a}(s_1)} (\text{Re} \phi_b(s_1))^2}{\int_{-\infty}^{\infty} d \text{Re} \phi_b(s_1) e^{-\bar{a}(s_1)} (\text{Re} \phi_b(s_1))^2} \left( \text{Re} \phi_b(s_1)^2 \right)^2
\]

\[
\int_{-\infty}^{\infty} d \text{Im} \phi_b(s_1) e^{-\bar{a}(s_1)} (\text{Im} \phi_b(s_1))^2
\]

\[
\int_{-\infty}^{\infty} d \text{Im} \phi_b(s_1) e^{-\bar{a}(s_1)} (\text{Im} \phi_b(s_1))^2
\]

\[
= \bar{a}(s_1)^{-1}
\]

(C11)

so

\[
G_b(x_1, x_2) = \sum_s \overline{G_b}(s) U_b(x_1, s) U_b^\dagger(x_2, s) , \quad \overline{G_b}(s) = \bar{a}(s)^{-1}
\]

(C12)

As usual, \( a(s) \) and \( \bar{a}(s) \) contain a \(+i\epsilon\) which is associated with a convergence factor in the path integral (and which gives a well-defined inverse).

The above are the propagators in the Euclidean formulation. The Lorentzian propagators are obtained through the same procedure with \( a(s) \rightarrow -ia(s) \) and \( \bar{a}(s) \rightarrow -i\bar{a}(s) \):

\[
\overline{G_f^L}(s) = ia(s)^{-1} , \quad \overline{G_b^L}(s) = i\bar{a}(s)^{-1}
\]

(C13)
The propagators in the Euclidean and Lorentzian formulations thus differ by only a factor of $i$. More generally, in the present picture, the action, fields, operators, classical equations of motion, quantum transition probabilities, propagation of particles, and meaning of time are the same in both formulations.

For a single noninteracting bosonic field with a mass $m_b$, the basis functions are

$$U_b(x, p) = \mathcal{V}^{-1/2} e^{ip \cdot x} = \mathcal{V}^{-1/2} e^{-i\omega t} e^{i\mathbf{p} \cdot \mathbf{x}}$$  \hfill (C14)

so with $s \rightarrow p$ we have

$$\tilde{a}(p) = \omega^2 - |p|^2 - m_b^2 + i\epsilon$$ \hfill (C15)

and

$$G_b(p) = \frac{1}{\omega^2 - |p|^2 - m_b^2 + i\epsilon}$$ \hfill (C16)

$$G_L^b(p) = \frac{i}{\omega^2 - |p|^2 - m_b^2 + i\epsilon}.$$ \hfill (C17)

Notice that (C4) and (C12) hold even when the basis functions in (C5) or (C9) are not a complete set.

Appendix D: Negative-action modes

According to (8.29), there is a negative contribution

$$S_\phi = -\sum_{s' \in \Phi_b} \overline{\phi_{b,s'}}(s') |\tilde{a}(s)| \phi_{b,s'}(s')$$ \hfill (D1)

to the action for what will become scalar boson fields. Here it will be assumed that each primitive variable $x_s = \text{Re} \phi_{b,s'}$ or $\text{Im} \phi_{b,s'}$ makes a quartic contribution $\frac{1}{2} b_s x_s^4$ to the action, stabilizing the vacuum, so that its total action is

$$-|\tilde{a}(s)| x_s^2 + \frac{1}{2} b_s x_s^4.$$ \hfill (D2)

Minimization gives the value

$$\overline{x}_s^2 = |\tilde{a}(s)|/b_s.$$ \hfill (D3)
An excitation out of the vacuum can be represented by $x_s = \bar{x}_s + \delta x_s$, and the lowest-order additional action is

$$\delta S^\phi = \left[ \frac{\partial S^\phi}{\partial \bar{x}_s} \right] \bar{x}_s + \frac{1}{2} \left[ \frac{\partial^2 S^\phi}{\partial x_s^2} \right] (\delta x_s)^2$$  \hspace{1cm} (D4)

$$= 0 + \left[ -|\bar{\alpha}(s)| + 3b_s \bar{\bar{x}}_s^2 \right] (\delta x)^2$$  \hspace{1cm} (D5)

$$= 2|\bar{\alpha}(s)| (\delta x)^2 .$$  \hspace{1cm} (D6)

Notice that the quartic coefficient $b_s$ is not present in this result. For an excitation $\delta \phi_b(s') = \text{Re} \delta \phi_b(s') + i \text{Im} \delta \phi_b(s')$, therefore, the action is

$$\delta S^\phi = 2|\bar{\alpha}(s)| \left[ (\text{Re} \delta \phi_b(s'))^2 + (\text{Im} \delta \phi_b(s'))^2 \right]$$  \hspace{1cm} (D7)

$$= 2|\bar{\alpha}(s)| \delta \phi^* b(s') \delta \phi_b(s')$$  \hspace{1cm} (D8)

$$= |\bar{\alpha}(s)| \phi^* b(s') \phi_b(s')$$  \hspace{1cm} (D9)

where

$$\phi_b(s') = \sqrt{2} \delta \phi_b(s') .$$  \hspace{1cm} (D10)

When $\phi_b(s')$ is renamed $\bar{\phi}_b(s')$, as in (8.28), the excitations of negative-action modes have the same form for the action as those of positive-action modes. Of course, there still remains a negative-action “condensate density” buried in the vacuum (and there is also a change in the measure within the path integral).

**Appendix E: Einstein-Hilbert action**

Since (8.29) has the same form as (8.28), all the subsequent steps for positive-action modes can be repeated for the negative-action modes in the second term of (8.26), with a result that differs by an overall minus sign. In the present context, the relevant part of the final action is

$$\mathcal{L}_G = \Phi^\dagger_\phi (x) \left( + \frac{1}{4} R \right) \Phi_\phi (x)$$  \hspace{1cm} (E1)

where $\Phi_\phi$ contains these negative-action modes, which all have the same interaction with $R$. (Other states in the vacuum will make contributions with the opposite sign, but we assume the modes contributing to (E1) are dominate in this context. A quantitative calculation would have to include all bosonic vacuum contributions.) There is a very large “condensate
density” for these modes, with the dimension of inverse length squared (in natural units), so we can define a very small length $\ell_P$ by

$$\ell_P = \left(4\pi n_-\right)^{-1/2}, \quad n_- = \Phi\Phi_-$$  \hspace{1cm} (E2)

and the action corresponding to (E1) is

$$S_G = \frac{1}{16\pi} \ell_P^2 \int d^4x R.$$  \hspace{1cm} (E3)

The above treatment is in a locally inertial coordinate system with coordinates $x^\mu$. (It is assumed that the parameters determining $n_-$ are themselves determined in an inertial frame.) Let $d^4x'$ be the volume element in a general coordinate system, which is held fixed as the metric tensor varies. Since $d^4x' e = d^4x$, where $e = (-\det g_{\mu\nu})^{1/2}$, the volume element in the locally inertial frame varies in proportion to $e$, and in the fixed general coordinate system $\int d^4x R$ is replaced by $\int d^4x' e R$.

With the general coordinates renamed $x^\mu$, we have

$$S_G = \frac{1}{16\pi} \ell_P^2 \int d^4x e R.$$ \hspace{1cm} (E4)

$S_G$ is interpreted as the Einstein-Hilbert action, $\ell_P$ as the Planck length, and $G = \ell_P^2$ as the gravitational constant.

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