New variables for classical and quantum gravity in all dimensions: V. Isolated horizon boundary degrees of freedom

N Bodendorfer\textsuperscript{1,2,3}, T Thiemann\textsuperscript{1} and A Thurn\textsuperscript{1}

\textsuperscript{1} Institute for Theoretical Physics III, FAU Erlangen-Nürnberg, Staudtstr. 7, D-91058 Erlangen, Germany
\textsuperscript{2} Institute for Gravitation and the Cosmos and Physics Department Penn State, University Park, PA 16802, USA

E-mail: norbert.bodendorfer@gravity.fau.de, norbert@gravity.psu.edu, thomas.thiemann@gravity.fau.de and andreas.thurn@gravity.fau.de

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Abstract

In this paper, we generalize the treatment of isolated horizons in loop quantum gravity, resulting in a Chern–Simons theory on the boundary in the four-dimensional case, to non-distorted isolated horizons in $2(n + 1)$-dimensional spacetimes. The key idea is to generalize the four-dimensional isolated horizon boundary condition by using the Euler topological density $E^{(2n)}$ of a spatial slice of the black hole horizon as a measure of distortion. The resulting symplectic structure on the horizon coincides with the one of higher-dimensional $SO(2(2n + 1))$-Chern–Simons theory in terms of a Peldan-type hybrid connection $F^0$ and resembles closely the usual treatment in $(3 + 1)$ dimensions. We comment briefly on a possible quantization of the horizon theory. Here, some subtleties arise since higher-dimensional non-Abelian Chern–Simons theory has local degrees of freedom. However, when replacing the natural generalization to higher dimensions of the usual boundary condition by an equally natural stronger one, it is conceivable that the problems originating from the local degrees of freedom are avoided, thus possibly resulting in a finite entropy.

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\textsuperscript{3} Author to whom any correspondence should be addressed.
1. Introduction

Black holes in higher dimensions are a subject of great interest in both general relativity and supergravity. Most prominently, the derivation of black hole entropy within string theory was first performed for a five-dimensional black hole [1]. Also, no-hair theorems familiar from \((d = 4)\) spacetime dimensions generally fail in higher dimensions, resulting in a large variety of black hole solutions with new (exotic) properties, see [2] for a review. While this fact has been appreciated in, e.g. string theory, it was not possible so far to perform these calculations in the context of loop quantum gravity (LQG), since the Ashtekar–Barbero variables [3, 4] necessary for loop quantization are restricted to \(d = 3, 4\). On the other hand, the recent extension of this type of connection formulation to higher-dimensional general relativity and supergravity [5–10] opens the window to investigate higher-dimensional black holes also with the methods of LQG.

The treatment of horizons and black hole entropy within LQG can be dated back to a remarkable paper by Smolin [11], in which it was shown that under some (natural) assumptions, boundaries of spacetime are described by a topological quantum field theory, more precisely SU(2) Chern–Simons theory. This pioneering work already contained many of the ideas which were later necessary to give a rigorous derivation of the black hole entropy within LQG. An
entropy associated to a surface which is proportional to the area was first calculated in papers by Krasnov [12] and Rovelli [13], where the important conceptual ingredient was that the punctures of horizon were distinguishable.

A rigorous technical framework for calculating black hole entropy within LQG was derived by Ashtekar and collaborators [14–17], where the notion of isolated horizon turned out to be crucial in order to have a local description of a black hole horizon. While a classical gauge fixing from SU(2) to U(1) was performed in order to derive the results of [14–17], it was later shown by Engle et al [18] that the derivation could be extended to an SU(2) invariant framework. An error in the state counting for the derivation of the black hole entropy in [17] was corrected by Domagala and Lewandowski in [19], showing that the dominance of spin 1/2 representations was incorrect. The detailed state counting has been extensively studied by Barbero and collaborators, see [20] and references therein. Non-spherical topologies in (3 + 1) dimensions were discussed in [21, 22]. Also, [23] provides a recent extensive review of the subject, including a comparison of the U(1) and SU(2) treatments.

In this paper, we are going to take first steps towards the derivation of higher-dimensional black hole entropy using LQG methods by deriving a generalization of isolated horizon boundary condition \( F \propto \Sigma \) first proposed in [11] and derived rigorously in [14]. We further show that the canonical transformation to higher-dimensional connection variables induces a higher-dimensional Chern–Simons symplectic structure on the intersection of the spatial slice with the isolated horizon. Also, we shortly comment on the quantization of the resulting theory on the black hole horizon. The derivations in this paper will be restricted to even spacetime dimensions, since the Euler topological density, which will play a key role in the construction, does not exist otherwise. In even spacetime dimensions, the isolated horizon then is odd-dimensional and a Chern–Simons theory can arise on it. A corresponding classical higher-dimensional black hole solution (with spherical symmetry) was found by Tangherlini [24] and generalizes the Schwarzschild solution to higher dimensions, see also [2] for an overview. Since, in the LQG treatment, the notion of isolated horizon is more central than that of a classical black hole solution, we will not go into details about the latter. As the notion of isolated horizon has already been generalized to higher dimensions in [25, 26], we can concentrate on deriving the isolated horizon boundary condition and the symplectic structure in this paper.

This paper is organized as follows:

We start in section 2 with an outline of the general strategy used in this paper for finding an analogue of the isolated horizon boundary condition in higher dimensions. In order to establish notation for the following calculations, we provide a comprehensive list of the notation used in this paper in section 3. Next, in section 4, we review the canonical transformation to SO(\(D + 1\)) variables and emphasize the appearance of a boundary term which will later result in the Chern–Simons symplectic structure. In section 5, the definition of a higher-dimensional isolated horizon is reviewed and its consequences are given. In the following section 6, we derive the isolated horizon boundary condition for the internal gauge group SO(1, \(D\)) starting from the Palatini action. Next, in section 7, we will develop the Hamiltonian framework and derive the Chern–Simons symplectic structure on the isolated horizon for the internal gauge group SO(1, \(D\)), also starting from the Palatini action. In order to make the connection to SO(\(D + 1\)) as the internal gauge group, we rederive the isolated horizon boundary condition and the Chern–Simons symplectic structure independently of the internal signature in section 8, this time purely within the Hamiltonian framework. We shortly comment about the generalization of the proposed framework to non-distorted horizons in section 9. Finally, we will discuss a possible quantization of the boundary degrees of freedom in section 10 and conclude in section 11. The appendices contain the construction and
generalization of the hybrid connection used in the Chern–Simons symplectic structure, further details on calculations, as well as an overview over higher-dimensional Chern–Simons theory and the higher-dimensional Newman–Penrose formalism.

2. General strategy

In this section, we will briefly comment on the general strategy of deriving the isolated horizon boundary condition. It will turn out that there is merely a single reasonable possibility for the general structure of the boundary condition for which a numerical prefactor and an expression for the connection on the horizon have to be fixed by an actual calculation. However, the connection used on the boundary is not necessarily unique as already observed in the four-dimensional case [27], where one is free to choose an independent Barbero–Immirzi-type parameter on the black hole horizon.

Let us start with some hints for the boundary condition based on the new connection variables derived in [5, 6].

- **Tensorial structure.** The $(3+1)$-dimensional SU(2) based boundary condition $F^a_{b,c} \propto E^{i} \epsilon_{abc}$ does not generalize trivially to higher dimensions due to the tensorial structure, i.e. a vector density is dual to a $(D - 1)$-form in $D$ spatial dimensions, which is a two-form only for $D = 3$. Since, in analogy to the $(3 + 1)$-dimensional case, we expect to get a theory which is purely defined in terms of a connection on the horizon, the easiest expression with the correct tensorial structure to write down is

$$\pi^{aIJ} \propto \epsilon^{aib;c_1...c_n}g^{IK_1L_1...K_n}F^{b}_{i_1L_1...I_nL_n},$$

where $a, b, c$ are spatial tensorial indices and $I, J, K, L$ are fundamental so($D + 1$) indices, $n = (D - 1)/2$, and $\pi^{aIJ}$ is the momentum conjugate to the connection on which the new variables in higher dimensions [5, 6] are based. More generally, one could also use a different invariant tensor to intertwine the adjoint so($D + 1$) representations on the momentum $\pi$ and the field strengths $F$, but the other obvious choice $\delta^{I||K_i}g^{L_i||K_i}...g^{L_{n-1}||K_i}g^{L_i||I_i}$ results in a vanishing right-hand side for even $n$ and does not allow for the construction performed in this paper in the other cases. The open question at this point is mostly on which connection the field strengths should be based.

- **Topological invariants.** Up to a constant prefactor, the derivation of the boundary condition in three spatial dimensions and spherical symmetry can be easily accomplished by appealing to special properties of curvature tensors in two dimensions. More precisely, the Riemann tensor $R_{\mu\nu\rho\sigma}$ on a two-dimensional manifold, e.g. a spatial slice of a black hole horizon in a four-dimensional spacetime, is, due to its symmetries, given by $R^{(2)}_{\mu\nu\rho\sigma} \propto \mathcal{R}_{\mu\nu\rho\sigma}$. Thus, after obtaining $F^{(4)}_{\mu\nu\rho\sigma} = R^{(4)}_{\mu\nu\rho\sigma} = R^{(4)}_{\mu\nu\rho\sigma} \Sigma^{\rho\sigma}$ from the field equations and since $R^{(4)}_{\mu\nu\rho\sigma} \Sigma^{\rho\sigma} = E^{(2)}_{\mu\nu\rho\sigma} \Sigma^{\rho\sigma}$ when using the isolated horizon (IH) boundary conditions, it directly follows that $F^{(4)}_{\mu\nu\rho\sigma} \propto R^{(2)}\Sigma^{\rho\sigma}$, where denotes the pullback from the spacetime manifold to a spatial slice of the horizon. In the further discussion of IHSs in four-dimensional LQG, it is of importance that in two dimensions, the integral over the Ricci curvature actually is a topological invariant by the Gaul–Bonnet theorem. The question thus is by which topological invariant that role will be played in higher dimensions.
From the above calculation, we expect that only the step using $R_{\mu\nu}^{(2)} \propto R_{\mu\nu}^{(2)} g_{\rho\sigma} R_{\rho\sigma}^{(2)}$ does not straightforwardly generalize to higher dimensions. However, this formula is equivalent to $R_{\mu\nu}^{(2)} \propto e^{\sigma\rho} e^{[\sigma\rho]} R^{(2)}_{\mu\nu}$, and in this form can be generalized to even dimensions and one is left to consider the Euler topological density [28]

$$E^{(D+1)} := \epsilon_{\mu_1 \nu_1 ... \mu_n \nu_n} \epsilon_{\ell_1 h_1 ... b_{1n}} R_{\mu_1 \nu_1 h_1 ... b_{1n}}^{(2)}$$

as a generalization. Although this looks already very similar to the above boundary condition (2.1), the Euler density would have to be defined on the spatial slices of black hole horizon while the internal gauge group is inherited from the bulk, thus having a representation space which is two dimensions larger than the tangent space of the spatial slice of the horizon. Later in this paper, we will chose a special connection on the boundary, the field strength of which will be inherently ‘orthogonal’ on $\pi_{all}$ and thus allowing for a precise implementation of the above idea for a boundary condition based on the Euler topological density. We remark at this point, as also stated in the notation section, that our normalization of the Euler topological density does not coincide with the standard definition leading to the Euler characteristic.

- **Higher-dimensional Chern–Simons theory.** The notion of $(2 + 1)$-dimensional Chern–Simons theory has a straightforward generalization to higher dimensions, i.e. a higher-dimensional Chern–Simons Lagrangian is defined by $d_L^{CS} = g_{h_1 h_2 ... h_n} F_{h_1}^{h_2} \wedge \ldots \wedge F_{h_n}^{h_1}$, where $d$ is the exterior derivative and $g$ intertwines $n + 1 = (D + 1)/2$ adjoint representations of $SO(D + 1)$, see [29]. The right-hand side of the previous equation can easily be seen to be the Euler topological density for $g_{h_1 h_2 ... h_n} = \epsilon_{h_1 h_2 ... h_n}$. The equations of motion derived from this Lagrangian are given by $g_{h_1 h_2 ... h_n} F_{h_1; h_2}^{h_2} \wedge \ldots \wedge F_{h_n; h_1}^{h_n} = \omega$, thus fitting nicely in the LQG quantization scheme for black holes, i.e. the straight forward generalization of $F_{h_1; h_2}^{h_2} = 0$ at points of the horizon which are not punctured by spin networks is given by $\epsilon_{h_1 h_2 ... h_n} F_{h_1; h_2}^{h_2} \wedge \ldots \wedge F_{h_n; h_1}^{h_n} = 0$.

As the canonical analysis of higher-dimensional Chern–Simons theory reveals [29], the theory has local degrees of freedom, e.g. $g_{h_1 h_2 ... h_n} F_{h_1; h_2}^{h_2} \wedge \ldots \wedge F_{h_n; h_1}^{h_n} = 0$ does not imply $F_{h_1; h_2}^{h_2} = 0$. This tension is discussed in section 10.

Based on this analysis, we will now give a precise derivation of the above proposed generalization of the isolated horizon boundary condition. The connection used will be a generalization of Peldan’s hybrid connection $\Gamma_{h_1 h_2 ... h_n}^{h_n}$ [30], which was already used in the construction of the connection variables in higher dimensions [5, 6]. We want to stress again that there might be other connections, e.g. a one-parameter family depending on a free parameter unrelated to the Barbero–Immirzi parameter, which satisfy an analogous boundary condition, as observed in [27] in the four-dimensional case.

### 3. Notation and conventions

This chapter gives an overview of the notation and conventions used in this paper. It can be skipped at first reading and should only be consulted as a reference when the notation used is unclear.

Let $\mathcal{M}$ denote a $(D + 1)$-dimensional (pseudo)-Riemannian manifold ($D \geq 2$) with metric $g_{\mu\nu}$ of signature $(-, +, \ldots, +)$. We will denote a $D$-dimensional Cauchy surface by $\Sigma$ and a $D$-dimensional null surface by $\Delta$. Equality on $\Delta$ is denoted by $\equiv$. Throughout this work, we will restrict the topology of $\Delta$ to be $S \times \mathbb{R}$, where $S$ is a $(D - 1)$-dimensional compact Riemannian manifold which allows for the isolated horizon boundary conditions defined in section 5 and has non-zero Euler characteristic. Examples are the $(D - 1)$-spheres $S^{(D - 1)}$ or hyperbolic spaces $H^{(D - 1)}$ divided by a freely acting discrete subgroup $\Gamma$, e.g. handle bodies
with genus \( g > 1 \) for \( D = 3 \) (at the level of topology) and the corresponding black hole solutions, given e.g. in [31]. For notational simplicity, we will (mostly) refer to all these manifolds as spheres in this work but keep in mind that more general topologies are allowed. We will in several sections restrict to even spacetime dimensions \( D + 1 =: 2(n + 1) \). This is necessary in the approach taken since (a) there can exist a Chern–Simons theory on the odd \((2n + 1)\)-dimensional \( \Delta \) and (b) the Euler density [28] is defined for the even \((2n)\)-dimensional intersections \( S \cong S^{D−1} \) of \( \Sigma \) and \( \Delta \). Here and in the following, we will use index conventions:

- tensorial spacetime indices will be denoted by lower Greek letters from the middle of the alphabet: \( \mu, \nu, \rho, \ldots \in \{0, \ldots, D\} \).
- tensorial spatial indices will be denoted by lower Latin letters: \( a, b, c, \ldots \in \{1, \ldots, D\} \).
- tensorial indices on \( \Delta \) will be denoted by the \( \mu, \nu, \rho \) (the pullback arrow will sometimes be omitted if there should be no confusion whether the equation is referring to \( \mathcal{M} \) or \( \Delta \)).
- tensorial indices in \((D − 1)\)-dimensional subspaces \( S \) will be denoted by lower Greek letters from the beginning of the alphabet: \( \alpha, \beta, \gamma, \ldots \in \{1, \ldots, D − 1\} \) or by \( \mu \).
- \( \text{SO}(D + 1) \) or \( \text{SO}(1, D) \) Lie algebra indices in the defining representation will be denoted by capital Latin letters: \( I, J, K, \ldots \in \{1, \ldots, D + 1\} \).
- Lower Latin letters \( i, j, k, \ldots \) will be used for \( \text{SO}(D) \) indices (and for labelling normals, only in appendix A).

The spacetime metric will be denoted by \( g_{\mu \nu} \), the spatial metric on \( \Sigma \) by \( g_{ab} \), the (degenerate) metric on \( \Delta \) by \( h_{\mu \nu} \) and on \((D − 1)\)-dimensional subspaces \( S \) by \( h_{ab} \). The corresponding Levi-Civita connections will be denoted by \( \nabla_\mu \), \( D_\mu \), \( D_\mu \) and \( D_\alpha \). For the Riemann tensor, we will use the convention \([\nabla_\mu, \nabla_\nu]u_\rho = R^{\rho}_{\nu\alpha\beta}u_\alpha\) and similar for lower dimensions. The indication of the number of dimensions on the Riemann tensor will be omitted if there is now chance of confusion. If the covariant derivate also acts on internal indices, we denote it by \( D_\mu u_\nu = D_\mu u_\nu + \Gamma_\mu_\nu_\rho u_\rho \) and its field strength as \( R_{\nu\mu\rho\sigma} \). We denote by \( E^{(D+1)} := e^{n_1, \ldots, n_{D+1}} \) the Euler topological density and remark that it coincides with other definitions in the literature only up to normalization, i.e. the integral of this density over a closed compact manifold \( S \), denoted by \( \int_S E^{(2n)} \), gives a only a multiple of the Euler characteristic \( \chi_S \) of \( S \). We choose this definition since it simplifies many formulas. Explicitly, we have

\[
\chi_S = \frac{1}{(8\pi)^n n!} \int_S E^{(2n)},
\]

which results in \( \chi_{S^{2n}} = 2 \) for spheres \( S^{2n} \). We will drop integration measures to simplify notation or work directly with differential forms.

The null normal to \( \Delta \) will be denoted by \( l \) and the vector field normal to the \((D − 1)\)-sphere cross-sections by \( k \), normalized to \( l \cdot k = -1 \) (cf. section 5). \( k \) can be extended uniquely to a spacetime 1-form at points of \( \Delta \) by requiring it to be null. Then, at points of \( \Delta \), we can decompose the metric according to \( g_{\mu \nu} = h_{\mu \nu} - 2 l_\mu k_\nu \). We will denote the \( h \)-projected vielbein by \( m, m_{\mu \nu} = h_{\mu \nu}, m_\mu \) and furthermore use the notation \( l^I = l^\mu e_\mu^I, k^I = k^\mu e_\mu^I \), and, since \( l, k \) are null and normalized, \( k^I l_\eta \eta = 0 = l^I l_\eta \eta, k^I l^I \eta = -1 \). We will call \([l, k, \{m_I\}] \) a generalized null frame. Elements of higher-dimensional Newman–Penrose formalism in this framework will be introduced in appendix D.

The future pointing timelike unit normal to a spatial slice \( \Sigma \) will be denoted by \( n_\mu \), \( n^2 = -1 \). The spacetime metric can be decomposed as follows: \( g_{\mu \nu} = q_{\mu \nu} - n_\mu n_\nu \). We refrain from using the usual notation \( n \) for this normal here, to avoid confusion with the normal to spatial slices, and also to make clear the difference between the hybrid vielbein normal \( n^I \) and \( k^I = k^\mu e_\mu^I \).
will denote the spatial vielbein as $e_{al} = (e^a)_l$ and $n^l = n^\nu e^l_{\nu l}$, $n^l n^l \eta_{ll} = -1$, $n^l e_{al} = 0$. 
Furthermore, we will introduce the notation $\tilde{\eta}_{ll} := \eta_{ll} + n_l n^l = e_{al} q^{ab} e_{bk} \tilde{\eta}_{lk} n^l n^k = 0$.

We will denote with $s$ the spacelike unit normal to the $(D-1)$-dimensional cross-sections $\Sigma \cap \Delta$, $s^2 = 1$, $s \cdot n = 0$, pointing outwards of $\sigma$. Furthermore, we define the co-normal $\tilde{s}_a := e_{al} - n_l e_{al}/s$ which is collinear with $s_a$, but normalized appropriately for usage in Stokes theorem later on. When dealing with the Hamiltonian formulation, we will choose the foliation such that $l = \frac{1}{2}(n - s)$, $k = \frac{1}{2}(n + s)$ holds, where $l$ and $k$ are the representatives of the equivalence class of the null normals to a given isolated horizon as specified in section 5. Furthermore, we will use the notation $s^l := s^\nu e^l_{\nu l}$ and introduce $\tilde{\eta}_{ll} := \eta_{ll} + n_l n^l - s s^l = \eta_{ll} + 2l k l_j = m_{al} m^{al}_j$, $\tilde{\eta}_{ll} n^l = \tilde{\eta}_{ll} s^l = \tilde{\eta}_{ll} k^l = \tilde{\eta}_{ll} k^l = 0$. An upper twiddle indicates the density weight of one w.r.t. $h_{ab}$, e.g. $\tilde{s}^l = \sqrt{\det h} s^l$.

Finally, a word of caution: in those parts of this work, in which the internal and external signature do not match, several of the above formulas get changed by signs ($n^l$ becomes spacelike, and the $n$ $n-$terms in the definitions of $\tilde{n}$ and $\tilde{s}$) or even become obsolete (since, to perform the signature switch, we already are in the Hamiltonian framework, $l^l$ and $k^l$ are not null anymore).

4. Introduction to the new variables

In [5, 6], a new connection formulation for general relativity in any dimensions $D + 1 \geq 3$ was introduced, which will be our starting point to extend the results on quantum black holes obtained in LQG. For completeness, we briefly review the construction of the variables from a Hamiltonian perspective, i.e. extend the ADM phase space of general relativity in such a way that we obtain a Poisson self-commuting connection as one of the canonical variables. For a comprehensive treatment including a Lagrangian derivation, see [5, 6].

The ADM Hamiltonian formulation of vacuum general relativity in $(D + 1)$ dimensions is based on a phase space with canonical coordinates $(q_{ab}, P^{ab})$, where $q_{ab}$ is the (spatial) metric of Euclidean signature on a $D$-dimensional manifold $\sigma$, $a, b, \ldots \in (1, \ldots, D)$. The images $\Sigma_\sigma$ of $\sigma$ under one parameter families of embeddings $X_t : \sigma \to \Sigma_\sigma \subset M$ into a $(D + 1)$-dimensional manifold $M$ constitute a foliation of $M$. The conjugate momentum $P^{ab}$ is related to the extrinsic curvature $K_{ab}$ via

$$P^{ab} = -s\sqrt{\det q}[K^{ab} - q^{ab} K^c_c],$$

where $s = 1$ for a Euclidean and $s = -1$ for a Lorentzian spacetime manifold. The non-vanishing Poisson brackets (we set the gravitational constant to unity for convenience) are given by

$$\{q_{ab}(x), P^{cd}(y)\} = \delta^{d}_{cb} \delta^{c}_{ab} s^{(D)}(x - y),$$

where $x, y, \ldots$ are coordinates on $\sigma$. Furthermore, we have the following constraints

$$\mathcal{H}_a = -2q_{ac} D_a P^{bc},$$

called spatial diffeomorphism constraint, and the Hamiltonian constraint

$$\mathcal{H} = -\frac{s}{\sqrt{\det(q)}} [q_{ac} q_{bd} - \frac{1}{D - 1} q_{ab} q_{cd}] P^{ac} P^{bd} - \sqrt{\det(q)} R^{(D)},$$

where $R^{(D)}$ is the Ricci scalar of $q_{ab}$ and $D_a$ denotes the unique torsion free covariant derivative compatible with $q_{ab}$.

To see in detail how the connection formulation is obtained, we split its construction in three steps:
(1) Extensions of the ADM phase space to a formulation with a densitized vielbein $\pi^{aIJ}$ ($I, J, \ldots \in \{0, \ldots, D\}$) in the \textit{adjoint} representation of $SO(D+1)$ or $SO(1, D)$.

(2) Constant Weyl rescaling on the extended phase space with a free parameter $\beta$.

(3) Perform the same extension again but now to a $SO(D+1)$ or $SO(1, D)$ connection formulation.

(1) The new phase space is coordinatized by the canonical pair $(K_{aIJ}, \pi^{bKL})$ with non-vanishing Poisson brackets

$$\{K_{aIJ}(x), \pi^{bKL}(y)\} = 2\delta_a^b\delta^I_0\delta^J_0\delta^{(D)}(x-y),$$

subject to the constraints

$$G^{IJ} := \{K_a, \pi^{a[I}J]\} := 2K_a^{IJK}\pi^{aK]\},$$

$$S_{aIJbKL} := \pi^{a[I}J]\pi^{bKL]},$$

which are called Gauß and simplicity constraint, respectively, and which form a first class constraint algebra. The symplectic reduction of the extended phase space w.r.t. Gauß and simplicity constraint leads back to the ADM phase space. More precisely, we can define a map from the extended to the ADM phase space by

$$2\xi \det q^{ab} := \pi^{aIJ}\pi^{bIJ},$$

$$P^{ab} := \frac{1}{2}G^{(a}K_{IJ}\pi^{b]IJ], S^a_I},$$

where $\xi = +1$ for $SO(D+1)$ and $\xi = -1$ for $SO(1, D)$. Then, one can easily verify that $q$ and $P$ as in (4.8), (4.9) are both, Gauß and simplicity invariant, i.e. (weak) Dirac observables, and moreover they satisfy the Poisson relations (4.2) on the surface defined by $G = S = 0$. Of course, $H_a$ and $H$ become constraints on the extended phase space by replacing $q$, $P$ in (4.3), (4.4) via (4.8), (4.9) and, by construction, the whole constraint algebra is of first class. We refrain from explicitly writing out the result since it will not be of relevance in the following.

(2) The constant scaling transformation

$$\pi^{aIJ} \rightarrow (\beta)\pi^{aIJ} := \frac{1}{\beta}\pi^{aIJ}, \quad K_{aIJ} \rightarrow (\beta)K_{aIJ} := \beta K_{aIJ} \quad (4.10)$$

for $\beta \in \mathbb{R}^+$ is, of course, canonical.

(3) The last step consist of invoking a certain connection constructed from $\pi^{aIJ}$, the \textit{hybrid–spin connection} $\Gamma_{aIJ}[\pi]$ weakly compatible with $\pi^{aIJ}$, and to redo the above extension of the ADM phase space with the role of $\pi^{aIJ}$ now played by a connection while the role of $\pi^{bKL}$ remains unchanged,

$$(\beta)K_{aIJ} \rightarrow (\beta)K_{aIJ} := \Gamma_{aIJ}[\pi] + (\beta)K_{aIJ}, (\beta)\pi^{bKL}), \quad (4.11)$$

and non-vanishing Poisson brackets given by

$$(\beta)A_{aIJ}, (\beta)\pi^{bKL} = 2\delta_a^b\delta^I_0\delta^J_0\delta^{(D)}(x-y).$$

Of course, we still have to define $\Gamma_{aIJ}[\pi]$. To this end, we have to solve the simplicity constraint [5, 32]

$$S_{aIJbKL} = 0 \Leftrightarrow \pi^{aIJ} = 2n^IE^{aIJ}, \quad (4.13)$$

where $E^{aIJ} = \sqrt{|\det q|}e^{aIJ}$ is a densitized hybrid vielbein (‘hybrid’ since the dimensions of the internal space $(D+1)$ and the spatial manifold $(D)$ do not match) and $n^I$ is the unique (up to
sign) unit normal on the hybrid vielbein, \( n' n_\ell = \zeta, n' E^\theta_\ell = 0 \). It has been shown in [30] that, like in the case of a genuine vielbein, there exists a unique, so-called hybrid–spin connection which annihilates the hybrid vielbein if the internal space is, like in our case, one dimension larger than the external one. It is given by (cf appendix A)

\[
\Gamma_{\alpha \beta \ell}[e] = \hat{e}^h_\ell D_\alpha e_{h \beta \ell} + \zeta n_\ell \partial_\ell n_\beta, \tag{4.14}
\]

where \( D_\alpha \) is the torsion-free \( q_{\alpha h} \)-compatible covariant derivative. To define \( \Gamma_{\alpha \beta \ell}[\pi] \), we demand that on the constraint surface \( S = 0 \), it should be given by (4.14), and extend it off this surface. Then, by construction, \( \Gamma_{\alpha \beta \ell}[\pi] \) is weakly compatible\( ^6 \) with \( \pi^{\alpha \beta \ell} \), i.e. it annihilates \( \pi^{\alpha \beta \ell} \) on the constraint surface \( S = 0 \), because \( \Gamma[\pi]|_{S=0} = \Gamma[e] \) and \( \pi|_{S=0} = \pi[e] \). Thus, we can rewrite the Gauß constraint to obtain its usual form up to terms which vanish if \( S = 0 \),

\[
G^{IJ} := [^{(\beta)} A_a - \Gamma[\pi] a_{\alpha} (^{(\beta)} \pi^{\alpha \beta \ell}] + \tilde{\alpha}_{a I J} + \{ G[a], \{^{(\beta)} \pi^{\alpha \beta \ell}]_I J := \{ \hat{\delta} D_a (^{(\beta)} \pi^{\alpha \beta \ell}]. \tag{4.15}
\]

For \( D = 3 \), there is a simple expression for the weakly \( \pi \)-compatible hybrid–spin connection given by

\[
\Gamma_{\alpha \beta \ell}[\pi] = \pi^{b \rho}_{[\alpha} D_\rho \pi^{k \beta \ell]}]. \tag{4.16}
\]

In higher dimensions, correction terms are necessary, for explicit expression we refer the reader to [5]. To complete step (3), we have to show that again symplectic reduction with respect to \( G, S \) leads back to the ADM phase space. The proof is similar to the case (1) above, but it becomes considerably more intricate to show that after the transformation (4.11), the ADM Poisson brackets are still weakly reproduced. The key tool in the proving this is the weak integrability of the extension \( \Gamma[\pi] \) of \( \Gamma[e] \) off the constraint surface, \( \Gamma[\pi] \approx \delta F $\delta^{\pi}[\hat{\pi}[\pi]]]$. The corresponding generating functional \( F \) has been constructed in [5] such that

\[
\delta F = \int_\sigma d^p x \left( \delta (^{(\beta)} \pi^{\alpha \beta \ell}) (\Gamma_{\alpha \beta \ell}[\pi] + S_{\alpha \beta \ell}) + \frac{2}{\beta} n^{\mu \nu} E^\mu \delta \Gamma_{\alpha \beta \ell}[e] \right)
\]

\[
\approx \int_\sigma d^p x \left( \delta (^{(\beta)} \pi^{\alpha \beta \ell}) \Gamma_{\alpha \beta \ell}[\pi] + \frac{2}{\beta} \partial_\nu (E^\mu \delta n_\nu) \right), \tag{4.17}
\]

where \( S_{\alpha \beta \ell} \) vanishes on the simplicity constraint surface. The boundary contribution to the symplectic potential can now be read off,

\[
\frac{1}{\beta} \int_\sigma d^p x \partial_\nu (2E^\mu \delta n_\nu) = \frac{1}{\beta} \int_S d^{D-1} x \tilde{s}^I \delta n_\ell, \tag{4.18}
\]

where \( \tilde{s}_e \in T^* \sigma \) denotes the unit conormal vector to \( S \) pointing outwards of \( \sigma \), \( s^I := s_e \delta^e_\ell \) and the twiddle indicates the density weight of one, \( \tilde{s}^I := \sqrt{\det \tilde{h}} \tilde{s}_e E^M \) (see section 3 for \( \tilde{s}_e \)).

In (3 + 1) dimensions, we have the possibility to introduce a Holst—like modification [6]. Repeating the above calculation then yields the modified boundary term

\[
\frac{1}{\beta} \int_\sigma d^2 x \partial_\nu \left( 2E^\mu \delta n_\nu - \frac{1}{\gamma} \epsilon^{ab} e_{a \beta \delta} \delta e^\mu_\nu \right) = \frac{1}{\beta} \int_S d^{D-1} x \left( 2\tilde{s}^I \delta n_\nu - \frac{1}{\gamma} \epsilon^{ab} e_{a \beta \delta} \delta m_\nu^\mu \right). \tag{4.19}
\]

Note that the second term in equation (4.19) corresponds to the boundary term familiar from Ashtekar–Barbero variables. The boundary terms (4.18), (4.19) will become important in sections 7 and 8.

\( ^6 \) Note that derivatives of the simplicity constraint can always be removed by partial integrations.
5. Higher-dimensional isolated horizons

The isolated horizon framework was introduced in a series of seminal papers [14–16, 33] and extended to higher dimensions in [25, 26, 34, 35]. We will therefore only briefly state the definition of undistorted, non-rotating isolated horizons in higher dimensions which we will be using, and discuss its consequences. The definition is geared towards the goal of the next section, namely to obtain the boundary condition which will lead to a higher-dimensional Chern–Simons theory on the boundary. We will start by giving the weaker definitions of near expanding and weakly isolated horizons (WIHs) and a brief discussion of their consequences in a manner very similar to [16].

**Definition 1.** A sub-manifold $\Delta$ of $(M, g)$ is said to be a non-expanding horizon (NEH) if

1. $\Delta$ is topologically $\mathbb{R} \times S^{D-1}$ and null$^7$.
2. Any null normal $l$ of $\Delta$ has vanishing expansion $\theta_l := h^{\mu \nu} \nabla_\mu l_\nu$.$^8$
3. All field equations hold at $\Delta$ and $-T^\mu_\nu l^\nu$ is a future-causal vector for any future directed null normal $l$.

We will state the consequences of definition 1. For more details on the derivations, we refer the interested reader to the standard literature cited above:

(a) **Properties of $l$.** Being a null normal to $\Delta$, $l$ is automatically twist free and geodesic. Moreover, using the vanishing of $\theta_l$, the Raychaudhuri equation and the condition on the stress–energy tensor, one can show it is additionally shear free and $R^\mu_\nu l^\mu l^\nu \hat{=} 0$.

(b) **Conditions on the Ricci tensor.** From the condition on $T^\mu_\nu$, the field equations and the relation for $R^\mu_\nu$ in (a) it follows that $R^\mu_\nu l^\nu \hat{=} 0$, or, in Newman–Penrose formalism,

$$\Phi_{00} = R^\mu_\nu l^\mu l^\nu \hat{=} 0 \quad \text{and} \quad \Phi_{0\mu} = R^\mu_\nu m^\nu_{\mu} \hat{=} 0. \quad (5.1)$$

(c) **Induced connection on $\Delta$.** Due to (a), there exists a unique intrinsic derivative operator $D$ on $\Delta$. Its action on vector fields $X \in T \Delta$ and on 1-forms $\eta \in T^* \Delta$ are given by

$$D^\mu X^\nu \hat{=} \nabla^\mu \tilde{X}^\nu \quad \text{and} \quad D^\mu \eta_\nu \hat{=} \nabla^\mu \tilde{\eta}_\nu, \quad (5.2)$$

where $\tilde{X}$ and $\tilde{\eta}$ are arbitrary extensions of $X$, $\eta$ to $M$.

(d) **Natural connection 1-form on $\Delta$.** From the properties of $l$, it follows that there exists a one-form $\omega^\mu_\nu$ such that

$$\nabla^\mu l^\nu \hat{=} \omega^\mu_\nu l^\nu, \quad (5.3)$$

which implies

$$L_l h^\mu_\nu \hat{=} 0. \quad (5.4)$$

We define the acceleration of $l$ by $l^\mu \nabla_\mu l^\nu \equiv \kappa^l l^\nu$. We infer $\kappa^l \equiv \omega^l_\mu l^\mu$.

(e) **Conditions on the Weyl tensor.** From the defining equation of the Riemann tensor, it follows that

$$2(D^\mu \omega^\nu_\mu) l^\rho \hat{=} - R^\rho_\mu \nu^\delta l^\nu \hat{=} - C^\rho_{\mu \nu} l^\nu, \quad (5.5)$$

$^7$ As explained in section 3, more general topologies are allowed without modifications of the definitions, but we restrict to spheres for notational simplicity.

$^8$ On $\Delta$, $h^{\mu \nu}$ is any tensor such that $h_{\mu \nu} = h_{\mu \nu} h^{\rho \sigma} h_{\rho \sigma}$. 
where in the last step we used (b). Contracting (5.5) with $m_{\mu l}$, we find

$$\Psi_{\mu l j} \equiv 0 \quad \text{and} \quad \Psi_{\mu l k} \equiv 0,$$

(see appendix D for notation) and therefore also

$$0 \equiv \Psi_{\mu l j} = \psi_{\mu l}.$$

Using this and (b), we find

$$0 \equiv \mathcal{L}_{\mu l} \psi_{\mu l} \equiv R\mathcal{L}_{\mu l} \psi_{\mu l} \equiv -\mathcal{L}_{\nu j} \omega_{\nu j} + D_{\mu l} k_{\mu l}.$$

**Definition 2.** A pair $(\Delta, [l])$, where $\Delta$ is a NEH and $[l]$ an equivalence class of null normals, is said to be a WIH if

(4) $\mathcal{L}_{\mu l} \psi_{\mu l} \equiv 0$

for any $l \in [l]$.

Note that, while $\omega_{\mu l}$ in general depends on the choice of null normal $l$, it is invariant under constant rescalings of $l$ and therefore depends only on the equivalence class $[l]$ we fixed. Therefore, we will drop the superscript $l$ in the following. We immediately infer from (5.8) that the 0th law holds for WIH,

$$d k_{\mu l} \equiv 0.$$

In the following, we will slightly strengthen this usual definition of WIHs in a way which is very similar to the definitions given in [15] by introducing some extra structure. Fix a foliation of $\Delta$ by $(D - 1)$--spheres. Denote by $[k]$ an equivalence class of 1-form fields normal to the foliation of $\Delta$ by $(D - 1)$--spheres. We require that any $k \in [k]$ is closed on $\Delta$. We extend them uniquely to spacetime 1-forms on $\mathcal{M}$ by requiring that they be null. Now, we introduce the equivalence class of pairs $[l, k]$ where each pair $(l^\mu, k_\nu)$ satisfies $i_2 k = -1$, i.e. we fix $l$ and $k$ up to mutually inverse and constant rescaling. We further demand $k = -d v$ for some function $v$ on $\Delta$, and each leaf $S_v \equiv S^{(D-1)}_v$ of the fixed foliation is characterized by $v = \text{constant}$. By spherically symmetric, we will in the following mean constant on the leaves $S_v$, e.g. for a spherically symmetric function $f = f(v)$.

**Definition 3.** A undistorted non-rotating isolated horizon (UDNRIH) is a WIH where to each $l \in [l]$ there is a $k$ like above, such that

(5) $k$ is shear-free with nowhere vanishing spherically symmetric expansion and vanishing Newman–Penrose coefficients $\pi_2 \equiv l^\mu \gamma_{\mu \nu} \nabla \nu k_\nu$ on $\Delta$.

(6) The Euler density $E^{(D-1)}$ of the $(D - 1)$--sphere cross sections obeys $E^{(D-1)} / \sqrt{\hbar} \equiv f(v)$ for some function $f$, i.e. the given ratio is constant on each leaf $S_v$.

Two remarks are in order: firstly, in $D = 3$, one finds for UDNRIHs [15], instead of the last condition,

(6') $T_{\mu \nu} l^\mu k^\nu$ is spherically symmetric at $\Delta$. It is only for $D = 3$ that (6) and (6') are equivalent (6') can be shown to be equivalent to demanding that the curvature scalar $R^{(2)}$ of the 2-sphere cross sections be constant. In two dimensions, we have $E^{(2)} = \text{const.} \times R^{(2)} / \sqrt{\hbar} = f(v) / \sqrt{\hbar}$ for

$9$ Two null normals $l$ and $l'$ are said to belong to the same equivalence class $[l]$ if $l = c l'$ for some positive constant $c$.

$10$ Again, two 1-forms $k, k'$ are called equivalent if $k = c k'$ for some constant $c$.

$11$ For spherical topologies, this would already follow from $S^{(D-1)} \times \mathbb{R}$ being simply connected.
some scalar function $f$. In higher dimensions, condition (6') still is equivalent to demanding that $R^{(D-1)}$ is constant on $S_c$. However, we will see that for our purposes, this condition is unnecessary, but has to be replaced by (6). This will be discussed explicitly in section 6. Apart from that, compared with [15], our definition 3 is slightly stronger (more restrictive) in that [15] does not demand (4). Furthermore, whereas we only allow for constant rescaling of $l, k$, in [15] they are fixed up to spherically symmetric and mutually inverse rescaling, but later in that paper, the gauge freedom of rescaling is fixed completely.

Secondly, the definition given above is tied to a foliation. The standard definitions of WIH are usually foliation independent, though some results rely on the existence of a so-called good cuts foliation. Moreover, when going to the Hamiltonian formulation, one usually demands that the spacetime foliation is such that at the boundary, the foliation coincides with this preferred foliation. Note that our fixed foliation is a good cuts foliation. We leave the question if all results obtained here hold in the more general context of weaker definitions of WIH or ones without reference to a fixed foliation for further research and continue by stating the consequences of definition 3:

(f) Properties of $k, \omega$ and its curvature. By the above requirements, we find for vectors $u$ tangential to $\Delta$ using $k^\mu \nabla_\mu k_\nu = 0$

\[
\nabla_u k_\nu = u^\mu \left( h^\nu_\mu h^\rho_\mu \nabla_\rho k_\nu - k_\nu \omega_\mu \right)
\]

\[
= u^\mu \left( \frac{1}{D-1} \theta^\rho_\mu - k_\nu \omega_\mu \right).
\]

(5.10)

Furthermore, we have for tangential vectors $u$ and $v$

\[
0 = u^\nu v^\rho \nabla_{[\nu} k_{\rho]} = -u^\nu v^\rho k_{[\nu} \omega_{\rho]}.
\]

(5.11)

from which we conclude that $\omega = \tilde{f} k$ for some function $\tilde{f}$. Since $i_l \omega = \kappa^l$, we have $\tilde{f} = -\kappa^l$ or

$\omega = -\kappa^l k$.

(5.12)

Contraction of (5.5) with $k_\rho$ yields

\[
2D \Psi_{[\nu k_{\rho]} = C_{\nu\rho\sigma} \epsilon^\rho \Psi_{\sigma k}. \]

(5.13)

where in the last step we used the trace freeness of the Weyl tensor and (5.6). We can furthermore conclude that $d\omega \equiv 0$ and $\Psi_{[0 k]} \equiv 0$, since $\omega \equiv -\kappa^l k$ and $d\kappa^l \equiv 0 \equiv d\omega$. This can be traced back to the requirement $\pi_{\nu} \equiv 0$ in the definition of ÜDNRHs, and in analogy to the $(D=3)$ case, this is why we refer to these horizons as non-rotating. (Note that $\Psi_{[0 k]}$ is the analogue of $Im \Psi_2$ in $D = 3$.)

6. Boundary condition

In this section, we will derive the boundary condition relating the bulk with the horizon degrees of freedom starting from the Palatini action. This forces us to use $SO(1, D)$ as the internal gauge group as opposed to $SO(D+1)$, which can be used in the Hamiltonian formalism even for Lorentzian signature. In a later chapter, we will rederive the boundary condition independently of the internal signature, thus allowing us to use the loop quantization based on $SO(D+1)$ connection variables for the bulk degrees of freedom.

Due to (3) of definition 1, we have at points of $\Delta$

\[
F_{\nu}^{\mu; l} \equiv R_{\nu}^{\mu; l} = R_{\nu}^{(D+1)} \epsilon^{\rho l} \epsilon^{\sigma j}.
\]

(6.1)
In the following, we will use the notation introduced in appendix D for the Weyl tensor also for the Riemann tensor, e.g. $R_{01l} = R_{i}^{(D+1)} h_{i} k^{l} m^{p} m^{o} j$. Note that therefore, the internal indices appearing on $R$ and $\Psi$ are perpendicular to $l'$ and $k'$, which will be used in several calculations in this section. Pulling back to $\Delta$, we obtain

$$F_{\mu' \nu'}^{\mu'' \nu''} = R_{\mu'' \nu''}^{\mu' \nu'} = R_{\mu'' \nu''}^{(D+1)} \epsilon_{\mu'' \nu'' \mu'} \epsilon_{\nu'' \mu'}$$

$$= \left( h_{\mu'' \nu''}^{\mu' \nu'} R_{\mu'' \nu''}^{(D+1)} - 2k_{\mu'' m}^{\mu' \nu'} R_{\mu'' m}^{(D+1)} \right)$$

$$\times \left( (m^{d} m^{p} l) - 2m^{d} l^{p} k^{l} + 2m^{d} l^{k} p^{l} + 2m^{d} p^{k} l^{l} \right)$$

$$= h_{\mu'' \nu''}^{\mu' \nu'} R_{\mu'' \nu''}^{(D+1)} m^{d} m^{p} l + m_{\mu''}^{\mu'} m_{\nu''}^{\nu'} (-2R_{KLJ}^{(l)} k^{l} - 2R_{kLJ}^{(l)} l^{l} + 2R_{KLJ}^{(l)} k^{l} l^{l})$$

$$- 2k_{\mu'' m}^{\mu' \nu'} k_{R0k}^{l} l^{l} - 2R_{0k}^{l l} l^{l} + 2R_{0k0k}^{l l} l^{l}$$

$$= h_{\mu'' \nu''}^{\mu' \nu'} R_{\mu'' \nu''}^{(D+1)} m^{d} m^{p} l + 4k_{\mu'' m}^{\mu' \nu'} k_{R0k}^{l l} l^{l}$$

$$= h_{\mu'' \nu''}^{\mu' \nu'} R_{\mu'' \nu''}^{(D+1)} m^{d} m^{p} l + \frac{4}{D - 1} k_{\mu'' m}^{\mu' \nu'} [\nabla_{l} k_{l} + k_{l}^{l} l_{l}], \quad (6.2)$$

where in the fourth line, we used that $\Phi_{0l} \equiv 0, \Phi_{00} \equiv 0$ to replace some Riemann tensor components by the corresponding Weyl tensor components, and in the fifth line we used $0 \equiv \Psi_{0l} \equiv \Psi_{00} \equiv \Psi_{00} \equiv \Psi_{00}$ and furthermore for $u_{\nu}$ that $u \cdot l = 0 = u \cdot k$,

$$R_{\mu'' \nu''}^{(D+1)} u_{\nu''} = [D_{\mu} D_{\nu}] u_{\nu''}$$

$$= h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} k_{\mu''}^{\mu'} k_{\nu''}^{\nu'} \nabla_{\nu''} u_{\nu''}$$

$$= h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} k_{\mu''}^{\mu'} k_{\nu''}^{\nu'} R_{\mu'' \nu''}^{(D+1)} u_{\nu''} + 2h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} (\nabla_{\mu''} h_{\nu''}^{\nu'} k_{\nu''}^{\nu'}) \nabla_{\nu''} u_{\nu''}$$

$$\equiv h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} k_{\mu''}^{\mu'} k_{\nu''}^{\nu'} R_{\mu'' \nu''}^{(D+1)} u_{\nu''}. \quad (6.3)$$

The second term in the second to last line vanishes due to

$$h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} (\nabla_{\mu''} h_{\nu''}^{\nu'}) \nabla_{\nu''} u_{\nu''} = h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} (\nabla_{\mu''} (l_{\nu''} k^{\nu'})) \nabla_{\nu''} u_{\nu''}$$

$$+ h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} \nabla_{\nu''} (l_{\nu''} k^{\nu'} + k_{\nu''} l^{\nu'}) \nabla_{\nu''} u_{\nu''}$$

$$\equiv h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} ((\nabla_{\mu''} l_{\nu''}) k^{\nu'} + (\nabla_{\mu''} k_{\nu''}) l^{\nu'}) \nabla_{\nu''} u_{\nu''}$$

$$+ h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} ((\nabla_{\nu''} l_{\nu''}) k^{\nu'} + (\nabla_{\nu''} k_{\nu''}) l^{\nu'}) \nabla_{\nu''} u_{\nu''}$$

$$\equiv h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} (\nabla_{\nu''} l_{\nu''}) k^{\nu'} \nabla_{\nu''} u_{\nu''}$$

$$- h_{\mu''}^{\mu'} h_{\nu''}^{\nu'} (\nabla_{\nu''} k_{\nu''}) u^{\nu'} \nabla_{\nu''} l_{\nu''}$$

$$\equiv 0, \quad (6.4)$$

where in the first line we used $\nabla_{\nu''} = 0$, in the second line that $h(k., .) = 0 = h(l., .)$, in the third that $\nabla_{\mu''} l_{\nu''} = 0$ and $l^{\nu'} \nabla_{\mu''} u_{\nu''} = -u^{\nu'} \nabla_{\mu''} l_{\nu''}$, and in the fourth line and $d l = 0$.

Finally, we have to account for the vanishing of $R_{IKL}$ in (6.2), which follows from

$$R_{IKL} = \Psi_{IKL} + \frac{2}{D - 1} \bar{n}_{IKL} \Phi_{JI}$$

$$= m^{\mu'} m^{\nu'} m^{\kappa'} R_{\mu'' \nu'' \kappa''}^{(D+1)} k^{\nu''} = m^{\mu'} m^{\nu'} m^{\kappa'} \left( \nabla_{\mu''} \nabla_{\nu''} \right) k_{\nu''}$$
From the third to the fourth line, we dropped the second two summands in the first round bracket because \( l \) and \( k \) are twist free, and the second summand in the second bracket since \( k^\alpha \nabla_k \mu = 0 \). In the fifth line, we used that \( k \) is twist and shear free and that \( \nabla \nabla \mu = \omega^\nu \). In line 6, we again invoke the twist and shear freeness of \( k \). In the last line, we used that \( \mu \theta_k = -k \nabla \theta_k \)

since it is spherical symmetric by definition 3 and that \( \omega^\nu = -k^k \).

In the last line of (6.2), we furthermore used

\[
R_{\mu\nu1J} = C_{\mu\nu1J} + \frac{1}{D - 1} (\tilde{\eta}_{1J} \Psi_{01} - \Phi_{1J}) - \frac{1}{D(D + 1)} \tilde{\eta}_{1J} R^{(D+1)} = - \frac{1}{D - 1} \tilde{\eta}_{1J} [\nabla \theta_k + k^k \theta_k],
\]

which can be shown analogously.

Since the pullback to \( H \) of the second summand in (6.2) is zero (\( k = 0 \)), we finally obtain when pulling back once more

\[
F_{\mu^1\nu^1J} = \tilde{F}_{\mu^1\nu^1J} + \frac{1}{D - 1} (\tilde{\eta}_{1J} \Psi_{01} - \Phi_{1J}) = \frac{1}{D - 1} \tilde{\eta}_{1J} R^{(D+1)} = \frac{1}{D - 1} \tilde{\eta}_{1J} \bar{S}_\alpha,
\]

and therefore, for \( D = 1 \) even,

\[
E_{K_1\ldots K_{D+1}J} = \bar{E}_{K_1\ldots K_{D+1}J} = \frac{1}{D - 1} \tilde{\eta}_{1J} R^{(D+1)} = \frac{1}{D - 1} \tilde{\eta}_{1J} \bar{S}_\alpha,
\]

where \( E^{(2n)} \) denotes the Euler density of the \( (D - 1) \)—sphere cross sections, \( \bar{S}_\alpha \) is the appropriate densitized conormal on \( S \), and \( \approx \) means equal up to the simplicity constraint.

Finally, by (6) of definition 3, \( E^{(2n)} = f(v) \sqrt{n} \). Some comment on the role of the equations (6.7), (6.8) is in order.

Firstly, notice that both of these equations are generalizations of the (3+1)-dimensional boundary conditions \( F \) known from the U(1) and SU(2) treatments. Equation (6.7) has the same left-hand side, but further manipulation of the right-hand side as in the (3+1)-dimensional case is not possible, since the Riemann tensor is in general not completely determined by the Ricci scalar in higher dimensions and the Ricci scalar also ceases to play a topological role. Equation (6.8) generalizes the right-hand side, the topological role now being played by the Euler density, while the left-hand side is more complicated than in the (3+1)-dimensional case.

\[12\] Comparing with the (3 + 1)-dimensional case, we find \( R_{JK1} = \Psi_{JK1} + \frac{1}{2} \tilde{\eta}_{EI} \Phi_{1J} = 0 \) corresponds to \( \Psi_1 = \Phi_2 = 0, \Psi_{1\alpha} = 0 \) to \( \Psi_0 = 0 \) and \( \Psi_1 = 0 \), and \( \Psi_{\alpha\beta} = 0 \) to the non-rotating condition \( \Delta \Psi_2 = 0 \).
Secondly, at the quantum level, we want to work with an independent Chern–Simons connection on the horizon from the onset and demand by constraint that the boundary connection actually is determined by the bulk fields. This constraint is in (3+1) dimensions precisely given by the boundary condition $F_{\mu
u}^{(h)} \propto \sum_{q \neq 1} \phi_q$. In higher dimensions, one can easily convince oneself that (6.8) is insufficient to determine the boundary connection and one has to impose (6.7) at the quantum level. However, (6.8) connects the momenta conjugate to the bulk connection with Chern–Simons excitations and therefore is a direct generalization of what is imposed at the quantum level in the (3+1)-dimensional case. It therefore could serve as a consistency requirement additionally to (6.7), see the discussion in section 10.

One last comment concerning (6'): assuming this condition to hold, one easily obtains that

$$G_{\mu
u} l^\mu k^\nu = \phi_{01} + \frac{D - 1}{2(D + 1)} R^{(D+1)}$$

is spherically symmetric. Moreover, taking the trace of (6.6), we infer that

$$C_{\mu1} + \frac{D - 3}{D - 1} \phi_{01} - \frac{D - 1}{D(D + 1)} R^{(D+1)} = -\nabla_i \theta_k - \kappa^i \theta_k$$

is spherically symmetric since the right-hand side is. Finally, from (6.3),

$$R^{(D-1)} = R_{ij}^{(1)} = 2C_{\mu1} + \frac{4(D - 2)}{D - 1} \phi_{01} + \frac{(D - 2)(D - 1)}{D(D + 1)} R^{(D+1)}$$

$$= 2 \left( C_{\mu1} + \frac{D - 3}{D - 1} \phi_{01} - \frac{D - 1}{D(D + 1)} R^{(D+1)} \right) + 2 \left( \phi_{01} + \frac{D - 1}{2(D + 1)} R^{(D+1)} \right),$$

where Weyl tensor component identities from appendix D were used. Since both summands in round brackets are spherically symmetric, we find that $R^{(D-1)}$ is also spherically symmetric. As we already remarked at the beginning of section 5, this property will not be needed in higher dimensions, but instead (6) will be crucial in the next section.

7. Hamiltonian framework

In this section, we will show, starting from the Palatini action in $(D+1) = 2(n+1)$ dimensions, how the symplectic structure of $(2n+1)$-dimensional Chern–Simons theory arises as boundary contribution to the symplectic structure for an internal boundary with UDNRIH conditions. We restrict to a vanishing cosmological constant. Note that the mechanics of higher-dimensional isolated horizons has already been studied in the quasi-local, the asymptotically flat [26] as well as the asymptotically anti-de Sitter [34] case. However, in all these treatments, the internal SO$(1, D)$ transformations were (partially) gauge fixed. In view of the boundary term of the generating functional for the canonical transformation to SO$(1, D)$ connection variables which we found in section 4 and which we expect to be related to the boundary symplectic structure, we are not allowed to fix the internal gauge freedom. In particular, in the usual time gauge $n' = \delta_{01}^i$, this boundary term vanishes since it is proportional to $\delta n'$. Therefore, we will rederive the Hamiltonian framework for IH in higher dimensions for our specific definition of UDNRIH without using any internal gauge fixing. Indeed, the derivation deviates from the usual treatment and we obtain the same boundary contribution to the symplectic structure we derived in 4, which (a) vanishes in time gauge and (b) can be reexpressed as SO$(1, D)$ Chern–Simons symplectic structure.

Consider a region $\mathcal{M}$ in a $(D+1)$—dimensional Lorentzian spacetime $(\mathcal{M}', g)$ bounded by two (partial) Cauchy slices $\Sigma_1$ and $\Sigma_2$, $\Delta$, and possibly an outer boundary. On $\Delta$, we
impose the UDNRIH boundary conditions and furthermore require that \( \Sigma_1, \Sigma_2 \) intersect \( \Delta \) in leaves \((D-1)-\text{spheres}\) of the preferred foliation \( \Sigma_1, \Sigma_2 \), respectively. Moreover, as usual in the IH literature, for a given history \((e, A)\) the horizon area \( A_\Sigma \) is constant in time as we will show shortly (below \((7.9)\)). We will now furthermore fix the horizon area to be a constant throughout the histories we are considering, \( \delta A_\Sigma = 0 \). The Palatini action\(^{13} \) is given by

\[
S[A, e] = \int_M \Sigma_{IJ} \wedge F^{IJ},
\]

(7.1)

where \( F = 1/2 F_{\mu \nu} dx^\mu \wedge dx^\nu, F^{IJ} = 2 \partial_{[I} A_{J]} + [A_{[I}, A_{J]}, \Sigma := -*(e \wedge e) \), or in coordinates \(-*(e \wedge e)_{\mu_1, \ldots, \mu_D} = -1/(D-1)! \varepsilon^\mu_1 \cdots \varepsilon^\mu_D, \cdots, \varepsilon^\mu_0, \varepsilon^{IJK} \cdots, \varepsilon^{K_{0-1}}, \) and as already stated, boundary terms possibly needed for \( T \) are neglected. Variation with respect to \( A \) gives rise to a surface term

\[
\int_\Delta \Sigma_{IJ} \wedge \delta A^{IJ},
\]

(7.2)

which, however, vanishes when imposing the UDNRIH boundary conditions, and therefore, the variation only yields the bulk equations of motion. This is a standard result in the IH literature, but will be derived here without any internal gauge fixing. Using \( e_{\mu I} = m_{\mu I} - k_{\mu I} \), we immediately find

\[
\Sigma_{IJ} = -\frac{1}{(D-1)!} e_{\mu_{IJKL} \cdots} k_{\mu_{D}} - (D-1)! k_{\mu L} \wedge m_{\nu} \wedge \cdots \wedge m_{D-1} = - (D-1)! k_{\mu} \wedge m^{K} \wedge \cdots \wedge m^{D-1}.
\]

(7.3)

For the pullback of the spacetime connection \( A \) we find analogous to the calculations in section 6

\[
A^{\mu}_{\mu I} = \Gamma^0_{\mu I J} = \Gamma^0_{\mu I J} + \frac{2}{D-1} \tilde{l}_{\mu I} (m_{\mu J}) \theta_k + 2 \omega_{\mu I} k_{\mu J},
\]

\[
\Gamma^0_{\mu I J} = m_{\mu I} \nabla_{m_{\nu J}} - l_{l_{\mu I}} k_{\nu J} - k_{l_{\mu I}} l_{\nu J},
\]

(7.4, 7.5)

where \( \Gamma^0 \) here denotes the connection on \( \Delta \) which annihilates \( m_{\mu K}, l_{I} \) and \( k_{J} \). Here and in the following, we will understand that \( m^{I} := h^{I \mu} m_{\mu I} \) and \( h^{I \mu} = g^{I \mu} h_{\mu \nu} g^{\nu \nu} \) such that \( h^{I \mu} k_{\nu} = 0 \).

For the variation of \( A \), we find

\[
\delta A^{\mu}_{\mu I} = \delta \Gamma^0_{\mu I J} + \frac{2}{D-1} \left[ (\delta l)_{\mu I} (m_{\mu J}) \theta_k + l_{l_{\mu I}} (\delta m_{\mu J}) \theta_k + l_{l_{\mu I}} m_{\mu J} (\delta \theta_k) \right] - 2 \left[ (\delta \omega_{\mu J}) l_{l_{\mu I}} \theta_k + \omega_{\mu I} (\delta l_{l_{\mu I}}) \right],
\]

(7.6)

which for the case at hand can be reduced to

\[
\delta A^{\mu}_{\mu I} = 2 k^{I} l_{I} k_{J} l^{J} \delta A^{K}_{K L} - 2 k_{I} \tilde{n}_{I} l^{I} \delta A^{K}_{K L} + \mathcal{R}
\]

\[
= 2 k_{I} l_{I} k_{J} l^{J} \left[ \delta A^{0}_{K L} - 2 l_{I} k_{J} \delta \omega_{I L} \right] - 2 k_{I} \tilde{n}_{I} l^{I} \delta A^{0}_{K L} + \mathcal{R}
\]

\[
= 2 k_{I} l_{I} k_{J} l^{J} \left[ \delta A^{0}_{K L} - 2 \omega_{I L} \delta A^{0}_{K L} \right] - 2 k_{I} \tilde{n}_{I} l^{I} \delta A^{0}_{K L} + \mathcal{R},
\]

(7.7)

where in the first line, we made use of the fact that only certain components of \( \delta A \) will appear when contracted with \( \Sigma \) and \( \mathcal{R} \) stands for the remaining terms which vanish in this contraction.

In the second step, several terms drop out due to \( l^{I} \delta m_{\mu I} = -m_{\mu I} l^{I} = -m_{\mu I} c_{I} l^{I} = 0 \) since \( l \) is fixed up to constant rescaling on \( \Delta \), \( l^{I} \delta l^{I} = 0 \) since \( l^{2} = 0 \) on \( \Delta \), and \( h^{I \mu} \delta \omega_{I L} \equiv 0 \). Finally, we used that \( l^{I} \delta A^{0}_{K L} = -\delta A^{0}_{K L} l_{I} + D^{0}_{K L} \delta l_{L} = D^{0}_{K L} \delta l_{L} \) since \( \Gamma^{0} \) annihilates \( l^{I} \). Putting all together, we recover for the definition of an UDNRIH as given in section 5 the result that there is no boundary term in symplectic potential for the horizon.

\(^{13} \) Note that a well defined action principle can require a boundary term, as e.g. the York–Gibbons–Hawking boundary term [36, 37] or its analogue in first order theories [38, 39]. However, such a boundary term does not enter the second variation of the action which will be relevant in this paper for deriving the Chern–Simons symplectic structure. We will thus neglect it for simplicity. For a discussion of these issues in higher dimensions, we refer the interested reader to e.g. [38, 39] and, specifically in the IH framework, [26].
\[
\int_\Delta \cdot \hat{\delta} A = \int_\Sigma \cdot \delta \hat{A} = \int_\Sigma \cdot \delta \Gamma
\]
\[
= - \frac{1}{(D - 1)!} \int_\Delta \left( \epsilon_{ljk\ldots,k_{D-1}} \left[ (-2\eta^I_{\mu\nu} k_{\mu} \hat{\delta} \eta^{IJ} + \delta \omega) + 2\eta^I_{\mu\nu} k_{\mu} \hat{\delta} \eta^{I} \right] \right) + \frac{2}{(D - 1)!} \int_\Delta \epsilon^{(D - 1)} \cdot \delta \omega
\]
\[
= - \frac{2}{(D - 2)!} \int_\Delta \left[ k \cdot \hat{\epsilon} \left( m^{k_1} \ldots m^{k_{D-1}} \right) \right] + \frac{2}{(D - 1)!} \int_\Delta \epsilon^{(D - 1)} \cdot \delta \omega
\]
\[
= 0, \quad (7.8)
\]

where in the second step, we used (7.3) and (7.7), which results in three terms in the third step, each of which vanishes separately. The first one since we can partially integrate the Lie derivative (boundary terms drop since \( \delta \tilde{l}^j = 0 \) on \( S_1, S_2 \)) and we have \( \mathcal{L}_l \epsilon^{(D - 1)} \equiv 0 \) and \( \hat{\delta} k \equiv 0 \).

Note that here, we defined
\[
\epsilon^{(D - 1)} = \epsilon_{ljk\ldots,k_{D-1}} l^j k^l \ldots m^{k_{D-1}}.
\]

To see that it is Lie dragged along \( l \), note that
\[
\mathcal{L}_l m_{ij} = l^I \nabla_i m_{jI} + m_{ij} \nabla_i l^I = l^I \nabla_i m_{jI} = -l^I \Gamma_{0i}^I m_{jI},
\]
\[
\mathcal{L}_l l^j = l^I \nabla_i l^j = -l^I \Gamma_{0i}^I l^j,
\]
\[
\mathcal{L}_l k^l = l^I \nabla_i k^l = -l^I \Gamma_{0i}^I k^l.
\]

Using this, to prove that \( \mathcal{L}_l \epsilon^{(D - 1)} = 0 \) we only need to use the invariance of \( \epsilon^{(D - 1)} \) under (infinitesimal) SO(1, \( D \)) transformations. A similar argument shows that
\[
\mathcal{L}_l d \hat{\epsilon} = 0.
\]

The second term in (7.8) is zero since \( \delta \omega \) is fixed on \( S_1, S_2 \) and also Lie dragged along \( l \), so the whole integrand is Lie dragged and vanishes at the boundary, which implies that the integral vanishes (This argument is e.g. given in [16].) The last term vanishes since the derivative \( d_{\hat{\delta} \tilde{l}} \) annihilates the whole expression (note that \( dk \equiv 0 \) and therefore leads only to a boundary contribution which vanishes again due to \( \delta l^j_{|S_1,S_2} = 0 \).)

The second variation of the action yields the symplectic current \( \delta_1 \Sigma^{IJ} \delta_2 A_{IJ} \) which is closed by standard arguments,
\[
\left( \int_{S_2} - \int_{S_1} + \int_\Delta \right) \delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} = 0.
\]

Moreover, the contribution at \( \Delta \) is a pure surface term, and we will show in the following that
\[
\int_\Delta \delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} = \Omega_{CS}^{S_1} (\delta_1, \delta_2) - \Omega_{CS}^{S_1} (\delta_1, \delta_2),
\]

where
\[
\Omega_{CS}^{S} = \frac{n A_{I}}{(E^{(2n)})^2} \int_S \epsilon^{IJKLMN \ldots} (\delta_1 A_{IJ}) \wedge (\delta_2 A_{KL}) \wedge F_{MN} \wedge \ldots \wedge F_{MN},
\]

denotes the Chern–Simons symplectic structure (cf appendix B), and therefore, the symplectic structure is given by
\[ \Omega(\delta_1, \delta_2) = \int_{\Sigma} \delta_1 \Sigma^{ij} \wedge \delta_2 A_{ij} + \frac{nA_S}{(E^{(2n)})} \int_{\Sigma} \epsilon^{ijklm} n_{ijklmn} \times (\delta_1 A_{ij}) \wedge (\delta_2 A_{KL}) \wedge F_{M,N} \wedge \cdots \wedge F_{M,N}, \]  
\tag{7.17} \]

and is independent of the choice of \( \Sigma \).

To prove (7.18), we will first show that the contribution to the symplectic structure at 0 is given by the boundary term we already found in section 4,

\[ \int_{\Delta} \delta_1 \Sigma^{ij} \wedge \delta_2 A_{ij} = \int_{S_1} 2(\delta_1 \tilde{S})(\delta_2 \eta_l) - \int_{S_1} 2(\delta_1 \tilde{S})(\delta_2 \eta_l), \]  
\tag{7.18} \]

where \( \tilde{S} = \sqrt{\sigma_1} \) and in the second step that the boundary contribution can be rewritten as

\[ \int_{S} 2(\delta_1 \tilde{S})(\delta_2 \eta_l) = \frac{A_S}{(E^{(2n)})} \int_{S} \epsilon^{ijklm} \big( \delta_1 A_{ij} \wedge (\delta_2 A_{KL}) \wedge F_{M,N} \wedge \cdots \wedge F_{M,N} \big) \]  
\tag{7.19} \]

For the variation of \( \Sigma \), we find using (7.3)

\[ -(D - 1)! \delta \Sigma^{ij} = \epsilon_{lijk,...,k_{D-1}} \left[ (D - 1)(\delta m^k) \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} \right] \]

\[ - (D - 1)(D - 2)! \epsilon^k \epsilon^{k_2} \kappa \wedge (\delta m^k) \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} \]

\[ - (D - 1)(D - 2)! \epsilon^{k_1} \epsilon^{k_2} \kappa \wedge (\delta m^k) \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} \]

\[ = \epsilon_{lijk,...,k_{D-1}} \left[ (D - 1) m_l \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} (i_m \delta m^k) \right] \]

\[ - (D - 1)(D - 2)! \epsilon^{k_1} \epsilon^{k_2} \kappa \wedge (\delta m^k) \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} (i_m \delta m^k) \]

\[ - (D - 1)(D - 2)! \epsilon^{k_1} \epsilon^{k_2} \kappa \wedge (\delta m^k) \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} \]

\[ \text{where we used} \]

\[ \delta m_l = m_l (i_m \delta m_l) - k (i_l \delta m_l) = m_l (i_m \delta m_l) + k (i_m \delta l) \]

\[ = m_l (i_m \delta m_l) + k (i_l \delta m_l) = m_l (i_m \delta m_l), \]  
\tag{7.21} \]

\[ \delta \kappa = -k (i_l \delta m_l). \]  
\tag{7.22} \]

In total, after a long calculation explained in appendix C.1, one finds for (7.18)

\[ \int_{\Delta} \delta_1 \Sigma^{ij} \wedge \delta_2 A_{ij} \]

\[ = \frac{2}{(D - 1)!} \int_{\Delta} \left\{ \delta [\delta_1 (\epsilon^{D-1} k) \delta_2 l]^{ij} + \frac{1}{(D - 1)!} \delta_1 (\epsilon^{D-1} \wedge \delta_2 [l^{ij}]) \right\} \]

\[ + (D - 1)! \delta [c^l + (i_m \delta m^l)] k \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} l^{ij} \epsilon_{lijk,...,k_{D-1}} \]

\[ + (D - 2)! \delta [k \wedge m^{k_2} \wedge \cdots \wedge m^{k_{D-1}} l^{ij} \epsilon_{lijk,...,k_{D-1}}] \]

\[ = \frac{2}{(D - 1)!} \int_{\Delta} \left\{ \delta [\delta_1 (\epsilon^{D-1} k) \delta_2 l]^{ij} + \frac{1}{(D - 1)!} \delta_1 (\epsilon^{D-1} \wedge \delta_2 [l^{ij}]) \right\}. \]  
\tag{7.23} \]

We used \( \delta \kappa = -c^l k \) and \( k = 0 \). Since we also restricted to constant area \( A_S \) throughout the phase space region we are considering \( \delta \kappa = 0 \), we furthermore find...
\[
\int_\Delta \delta_0 \epsilon_{D-1} \wedge \delta_2 \omega^I = - \int_\Delta \delta_0 \epsilon_{D-1} \wedge \delta_2 (\kappa')^k = \int_\Delta \delta_0 \epsilon_{D-1} \wedge d\delta_2 (\kappa')^v
\]
\[
= \left[ \delta_2 (\kappa')^v|_S \int_\Delta \delta_0 \epsilon_{D-1} - \delta_2 (\kappa')^v|_S \int_\Delta \delta_0 \epsilon_{D-1} \right]
\]
\[
= \left[ \delta_2 (\kappa')^v|_S \delta_2 A_S - \delta_2 (\kappa')^v|_S \delta_2 A_S \right] = 0.
\]

(7.24)

Now, since we have \( E^{(D-1)} = f(v) \epsilon_{D-1}/(D - 1)! \) for a spherically symmetric function \( f \) by the conditions for an UDNRIH, and since \( f \) are both constant in time, we have \( d^2 f = 0 \). Since also \( [K, K] = 0 \), we obtain \( F = R^0 \) which was already derived in section 6.

We now want to show that (7.19) holds, which is shown to be true in (C.5) if the connection would be given by \( \Gamma^0 \). Therefore, what needs to be checked is if

\[
\epsilon^{IJLMN} \cdots \epsilon^{MN} \left( 2 \delta_0 \Gamma^0_{IJ} \wedge \delta_2 |_{KL} + \delta_0 \Gamma^0_{IJ} \wedge \delta_2 |_{KL} \right) \wedge R^0_{MNL} \wedge \cdots \wedge R^0_{MNL} = 0.
\]

(7.28)

Using

\[
\delta K_{IJ} = \frac{2}{D - 1} \left[ -l_1 j_{ij} l^k \theta_2 \delta m^k + l_1 j_{ij} \bar{\tilde{\theta}}_2 \left( \theta_2 \delta m^k - m^k \theta_2 l^k \delta l + m^k \delta \theta_2 \right) + \bar{\tilde{\theta}}_2 \right],
\]

(7.29)

we find in a first step

\[
E^{(IJKL)}_{\perp} \wedge \delta_0 \Gamma^0_{IJ} \wedge \delta_2 |_{KL} = - \frac{8}{(D - 1)^2} E^{(IJKL)}_{\perp} \wedge l_1 j_{ij} \bar{\tilde{\theta}}_2 \left( \theta_2 \delta m^k - m^k \theta_2 l^k \delta l + m^k \delta \theta_2 \right) \delta_2 |_{MN}
\]

\[
= \frac{8}{(D - 1)^2} E^{(IJKL)}_{\perp} \wedge l_1 j_{ij} \theta_2 \delta m^k \delta_2 |_{MN} = 0.
\]

(7.30)

\( E^{(IJKL)}_{\perp} = \epsilon^{IJKLMN} \cdots \epsilon^{MN} R^0_{MNL} \wedge \cdots \wedge R^0_{MNL} \) in the above formula stands for the terms in (7.28) contracted with \( \delta K \wedge \delta K \). \( \perp \) indicates that fact that \( E^\perp \) needs to be contracted with \( l^k, l^l \) since it vanishes otherwise, therefore only one combination of terms survives when we use (7.29) in the first step. In the second line, we made use of \( l^i \delta m_l = -m_l \delta l^i \) and therefore, the expression is anti-symmetric in the index pair \( M, N \). Adding terms until all indices of the epsilon symbol in \( E^\perp \) plus the index \( M \) are totally anti-symmetric and subtracting the therefore needed terms again, we find that the whole expression vanishes: the total anti-symmetrization since there is no nonneutral rank \( (D + 2) \) anti-symmetric tensor in \( (D + 1) \) dimensions, and the subtracted terms since they are either of the form \( l^i m_l = 0 \) or \( k^l m_l = 0 \), or \( R^0_{MNL} \wedge m^k \) which vanishes due to the Bianchi identity, or \( m^k \wedge m_l = 0 \).
Furthermore, we have

\[ E^{(1) \perp}_{\perp} \wedge \delta_1 \Gamma^0_{IJ} \wedge \delta_2 K_{KL} \]

\[
= \frac{2}{D-1} E^{(1) \perp}_{\perp} \wedge \left[ -\tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \Gamma^0_{IJK} \wedge l_1 k_{LM} \eta^{LM} \delta_2 |m_M \right] \\
- 2 k_1 \tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \Gamma^0_{IJK} \wedge l_1 \tilde{\eta}^F_{LM} \left( \delta_1 \delta_1^2 m^M - m^M \theta_2 k^N \delta_2 |l_N + m^M \delta_2 \theta_2 k \right) \\
+ 2 l_1 \tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \Gamma^0_{IJK} \wedge \eta^{LM} \delta_1 \delta_2 |l_M + m^M \theta_2 k^N \delta_2 |l_N + m^M \delta_2 \theta_2 k) \\
= \frac{2}{D-1} E^{(1) \perp}_{\perp} \wedge \left[ -\tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \left( -d_{I\alpha} \delta_1 m^\alpha - m^\alpha \delta_1 \Gamma^0_{I\beta} \right) \wedge l_1 k_{LM} \eta^{LM} \delta_2 |m_M \right] \\
- 2 k_1 \tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \left( d_{I\alpha} \delta_1 m^\alpha - m^\alpha \delta_1 \Gamma^0_{I\beta} \right) \wedge l_1 \tilde{\eta}^F_{LM} \left( \delta_1 \delta_1^2 m^M - m^M \theta_2 k^N \delta_2 |l_N + m^M \delta_2 \theta_2 k \right) \\
- 2 l_1 \tilde{\eta}^F_{(I} \tilde{\eta}^F_{J)} \left( d_{I\alpha} \delta_1 m^\alpha - m^\alpha \delta_1 \Gamma^0_{I\beta} \right) \wedge \eta^{LM} \delta_1 \delta_2 |l_M + m^M \theta_2 k^N \delta_2 |l_N + m^M \delta_2 \theta_2 k) \tag{7.31}
\]

where we used \( \delta_{IJ} = m^\alpha m_{\alpha} \) in the last step as well as the fact that \( \Gamma^0 \) annihilates \( m^K, l^I, k^J \) and therefore, e.g. \( l^I \delta^0_{IJ} = \delta(d_{I\alpha} l^\alpha) = d_{I\alpha} \delta_{IJ} = d_{I\alpha} \delta_{IJ} \). In the last expression, the second summand in the second to last line and the term in the last line together just give a surface term which vanishes since the \( (D-1) \) sphere cross sections have no boundary. To see this, one needs to make use of the fact that \( d_{I\alpha} R^\alpha = d_{I\alpha} m = d_{I\alpha} l^I = d_{I\alpha} k^J = d_{I\alpha} \tilde{\eta} = d \delta_K = 0 \). Moreover, we also have \( d \delta \theta_K = 0 \) since \( \delta \theta_K \) has to be constant on the \( (D-1) \)—sphere cross sections, and therefore also the last term in the second to last line is a surface term. Using the notation \( \delta \Gamma_{*} \) to indicate that \( \delta \Gamma \) is considered as a form in the index \( \bullet \), the terms in the first line of (7.31) give

\[
\frac{2 \theta_k}{D-1} k_{LM} E_{\perp[L} E_{\perp M]} \wedge m_N m_{\alpha} \wedge \left[ \left( d_{I\alpha} \delta_1 m^\alpha_{\gamma} + m^\alpha_{\gamma} \delta_1 \Gamma^0_{I\beta} \right) \right] \delta_2 |l_N
\]

\[
= \frac{2 \theta_k}{D-1} k_{LM} E_{\perp[L} E_{\perp M]} \wedge m_N \left[ m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} + m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} M - m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} M \right] \delta_2 |l_N
\]

\[
= \frac{2 \theta_k}{D-1} k_{LM} E_{\perp[L} E_{\perp M]} \wedge m_N \times \left[ \frac{1}{3} m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} M + 2 m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} M \right] \delta_2 |l_N
\]

\[
= \frac{4 \theta_k}{D-1} k_{LM} E_{\perp[L} E_{\perp M]} \wedge m_N \left[ m^\beta \eta^{\alpha IJ} D^\gamma_{\beta} \delta_1 m_{\alpha} M \right] \delta_2 |l_N. \tag{7.32}
\]

In the third step, the term totally anti-symmetric in the indices \( M, J, I \) vanishes since

\[
k_{LM} E_{\perp[L} E_{\perp M]} \wedge m_{\alpha} \wedge m_{\alpha} = \epsilon_{(IJ)KM} \eta^{\alpha IJ} \epsilon_{MN} k_{LM} R_0 M_{\alpha} = \cdots \epsilon_{(IJ)KM} \eta^{\alpha IJ} \epsilon_{MN} k_{LM} R_0 M_{\alpha} = \cdots \wedge R_0 M_{\alpha} \wedge m_{\alpha} M\wedge m_N
\]

\[
= \frac{(D+2)}{3} \epsilon_{(IJ)KM} \eta^{\alpha IJ} \epsilon_{MN} k_{LM} R_0 M_{\alpha} = \cdots \wedge R_0 M_{\alpha} \wedge m_{\alpha} M\wedge m_N
\]

\[
= 0. \tag{7.33}
\]

since \( R_0 M_{\alpha} \wedge m_{\alpha} = 0 \) due to the Bianchi identity and \( m_{\alpha} l^I = 0 = m_{\alpha} k^J \), and the anti-symmetrization of \( (D+2) \) indices vanishes. Finally, the first term in the second to last line of (7.31) gives
is straightforward, since the construction of the connection $\Gamma_1$.

SO(D + 1) as internal gauge group

In the previous sections, we have derived the isolated horizon boundary condition relating the connection on the horizon with the bulk degrees of freedom, as well as the symplectic structure on the horizon, which coincides with the one of higher-dimensional Chern–Simons theory. Since we started from the spacetime covariant Palatini action, the internal gauge group structure on the horizon, which coincides with the one of higher-dimensional Chern–Simons, the connection on the horizon with the bulk degrees of freedom, as well as the symplectic structure so that it fits in the SO(D + 1). Thus, we are interested in reformulating the horizon boundary condition and the horizon bulk degrees of freedom can be performed with standard LQG methods as spelled out in [5] that one can change the internal gauge group to SO(D + 1) by a canonical transformation from the ADM phase space. After this reformulation, the quantization of the bulk degrees of freedom can be performed with standard LQG methods as spelled out in [7]. Thus, we are interested in reformulating the horizon boundary condition and the horizon symplectic structure so that it fits in the SO(D + 1) scheme.

As for the boundary condition, the generalization to the Euclidean internal group is straightforward, since the construction of the connection $\Gamma_1$ in appendix A works independently of the internal signature. Thus, constructing $\Gamma_0$ such that it annihilates both $n^K$ and $s^K = s_a e^{ak}$ additionally to $m^K_0 = e^k_a$, the horizon boundary conditions

\begin{equation}
R^0_{a_\beta IJ} = R^0_{a_\beta IJ} \tag{8.1}
\end{equation}

\begin{equation}
\frac{1}{\beta} e^{K_l \cdots K_{l_I}} e^{a_1 \cdots a_{\alpha_0}} e^{a_{\beta_0} \cdots a_{\beta_{l_I}}} e^{a_{\beta_{l_I}} \cdots a_{\beta_{n_I}}} = \frac{E^{(2n)}}{\sqrt{\hbar}} \hat{\delta}_a \tag{8.2}
\end{equation}

follow immediately from the fact that $R^0_{a_\beta KLI} n^K = R^0_{a_\beta KLI} s^K = 0$. We will drop the superscripts ‘bulk’ and ‘horizon’ in what follows.

In order to derive the new symplectic structure, we first perform a symplectic reduction of the theory derived in the previous chapters by solving the Gauß and simplicity constraint. This leads us to the ADM phase space, from which we can perform further canonical
transformations. It is shown in section 4 that the canonical transformation to \( \text{SO}(D + 1) \) connection variables leads to the boundary symplectic structure

\[
\Omega^S(\delta_1, \delta_2) = \frac{2}{\beta} \int_S d^{D-1}x \tilde{\delta}_2 \gamma_{IJ} \eta^J.
\] (8.3)

Furthermore, under the non-distortion condition \( \delta \frac{\mathcal{E}^{(2n)}}{\sqrt{h}} = 0 \), i.e. restricting to the part of phase space where \( \delta \mathcal{E}^{(2n)} \) is constant and thus given by \( \frac{(\mathcal{E}^{(2n)})}{\sqrt{h}} \), it is shown in appendix C.2 that

\[
2 \frac{E^{(2n)}}{\sqrt{h}} (\delta_1 \phi_2) (\delta_2 \mu_I) = m e^{iKL \alpha_{1\ldots m_{-1} \beta_{-1}}} \epsilon_{\alpha_1 \beta_1 \ldots \alpha_{-1} \beta_{-1}}
\]

\[
\times (\delta_1 \Gamma^0_{aIJ}) (\delta_2 \Gamma^0_{bKL}) R^0_{a_1 \beta_1 M_1 N_1} \cdots R^0_{a_{-1} \beta_{-1} M_{-1} N_{-1}},
\] (8.4)

which results in the Chern–Simons type boundary symplectic structure

\[
\Omega_{CS}^S(\delta_1, \delta_2) = \frac{n\alpha_S}{\beta \mathcal{E}^{(2n)}} \int_S e^{iKL \alpha_{1\ldots m_{-1} \beta_{-1}}} \epsilon_{\alpha_1 \beta_1 \ldots \alpha_{-1} \beta_{-1}}
\]

\[
\times (\delta_1 \Gamma^0_{aIJ}) (\delta_2 \Gamma^0_{bKL}) R^0_{a_1 \beta_1 M_1 N_1} \cdots R^0_{a_{-1} \beta_{-1} M_{-1} N_{-1}}.
\] (8.5)

Concluding, we have shown that also for the case of \( \text{SO}(D + 1) \) as an internal gauge group, one arrives at a higher-dimensional Chern–Simons symplectic structure at the isolated horizon boundary of \( \sigma \) as well as the boundary conditions (8.1), (8.2).

A remark concerning the uniqueness of \( \Gamma^0 \) is in order. In \( D = 3 \), one easily finds that there are more connections which allow for carrying out the whole programme. Exemplarily, we can introduce a constant parameter \( \Phi \in \mathbb{R} \) and choose \( \Gamma^0_{aIJ} = \Gamma^0_{aIJ} + 2 \Phi n^I m_a^J \) as connections for the Chern–Simons theory on the boundary. We then find

\[
R^\Phi_{aIJ} = R^0_{aIJ} - 2 \Phi \hat{n}^I_m a^J m_a^J,
\] (8.6)

\[
\epsilon^{ijkl} a^I R^\Phi_{a} \epsilon_{ijk} = \frac{(E^{(2)})}{\sqrt{h}} 4 \Phi^2 \hat{n}^I \hat{s}^a,
\] (8.7)

\[
\left( \frac{(E^{(2)})}{\sqrt{h}} - 4 \Phi^2 \right)^a \epsilon^{ijkl} a^I (\delta_1 \Gamma^0_{aIJ}) (\delta_2 \Gamma^0_{bKL}) = 2 \delta \hat{n}^I \hat{s}^J m_a^J m_a^J.
\] (8.8)

A further modification of \( \Gamma^0 \), which in particular allows for generalization to distorted horizons, will be introduced in section 9.1, where a non-constant field \( \Psi \) is added to the connection. The introduction of \( \Psi \) and \( \Phi \) cannot be combined non-trivially, since otherwise there will be terms \( \propto n^I m_a^J \) contributing to \( R^\Phi_{aIJ} \).

A third possibility to change the connection in \( D = 3 \), which can be combined with both of the above methods, is as follows. As we have already seen at the end of section 4, if we introduce the Barbero–Immirzi parameter \( \gamma \) in \( D = 3 \) [6], it will appear in the boundary symplectic structure. The boundary condition in this case reads

\[
\frac{1}{\beta} \epsilon_{\alpha_1 \beta_1 \ldots \alpha_{-1} \beta_{-1}} \left( \epsilon^{ijkl} R^0_{aIJ} + \frac{1}{\gamma} \frac{(E^{(2)})}{\sqrt{h}} \right) = \frac{(E^{(2)})}{\sqrt{h}} \frac{\gamma \Phi}{\pi} a^I \hat{\xi}^a,
\] (8.9)

where

\[
(\gamma \Phi / \pi) a^I = \frac{1}{\beta} \left( \pi^{aIJ} + \frac{1}{2\gamma} \epsilon^{ijkl} a^I \right).
\] (8.10)

To show that the boundary symplectic structure can be rewritten according to

\[
2 \frac{\mathcal{E}^{(2n)}}{\sqrt{h}} (\delta_1 \phi_2) (\delta_2 \mu_I) = \frac{n\alpha_S}{\beta \mathcal{E}^{(2n)}} \int_S e^{iKL \alpha} \left( \epsilon^{ijkl} \delta_1 \Gamma^0_{aIJ} \delta_2 \Gamma^0_{bKL} + \frac{2}{\gamma} \delta_1 \Gamma^0_{aIJ} \delta_2 \Gamma^0_{bKL} \right),
\] (8.11)
it remains to verify that

$$\frac{E^{(2)}}{\sqrt{h}} \delta m^I \wedge \delta m_I = -2 \delta \Gamma^0_{IJ} \wedge \delta \Gamma^0_{IJ}. \quad (8.12)$$

Since the scalar curvature $R = \frac{E^{(2)}}{2\sqrt{h}}$ is constant on the 2-spheres, the metric $h$ is fixed up to diffeomorphism. Therefore, $m_I$, $\Gamma^0_{IJ}$ are fixed up to diffeomorphism and SO($D+1$) rotations, i.e. $\delta m_I = \Lambda_I^j \delta m_j + L_k m_I$ and $\delta \Gamma^0_{IJ} = -d\xi \Lambda_{IJ} + L_k \Gamma^0_{IJ}$. Using this for the variations, (8.12) can be proven straightforwardly using $0 = d_\xi m_I = dm_I + \Gamma^0_{IJ} \wedge m^J$, $d\Gamma^0_{IJ} + \frac{1}{2} [\Gamma^0, \Gamma^0]_{IJ} = R^0_{IJ} = \frac{1}{4} R m_I \wedge m_J$ and the properties of the exterior and Lie derivative.

In higher dimensions, it is less trivial to modify the connection $\Gamma^0$. In particular, the above constructions can at least not be applied trivially. While (8.6) continues to hold, in (8.7) mixed terms of the form $R^0 \wedge \cdots \wedge (\Phi m \wedge m)$ will appear which spoil the construction, and also the introduction of $\gamma$ is tied to $D = 3$.

9. Inclusion of distortion

In this section, we are going to comment on the generalization of the isolated horizon boundary condition derived in the non-distorted case to general isolated horizons. The seminal work on this subject has been a paper by Ashtekar et al [40], where treatment was generalized to axi-symmetric horizons. For the generalization to arbitrary spherical horizons, two methods by Perez and Pranzetti [41] and Beetle and Engle [42] exist in four dimensions. We will discuss them in the following and show that an extension of them to higher dimensions is not straightforward. Nevertheless, an extension to distorted isolated horizons seems to exist [43].

9.1. Beetle–Engle method

In order to derive the symplectic structure on a spatial slice $S$ of the horizon, it is key to the derivation that $E^{(2n)}/\sqrt{h}$ is a constant on $S$. Otherwise, unwanted terms appear due to the variation of $E^{(2n)}/\sqrt{h}$. Of course, this observation has already been made in the four-dimensional case and a solution of this problem in case of U(1) as the gauge group on $S$ has been proposed by Beetle and Engle [42]. Essentially, they construct a new U(1) connection on $S$ as

$$\tilde{V}_a := \frac{1}{2} \theta_\alpha - \epsilon_{a\beta} h^{\beta\gamma} D_\gamma \Psi, \quad (9.1)$$

where $\frac{1}{2} \theta_\alpha$ is the U(1) connection used for spherically symmetric isolated horizons and $\Psi$ is a curvature potential defined by the equation

$$\Delta \Psi = R - \langle R \rangle, \quad (9.2)$$

where $R$ is the intrinsic scalar curvature which is proportional to $E^{(2)}/\sqrt{h}$. Calculating the curvature of $\tilde{V}_a$, the terms proportional to $R$ drop out and one gets

$$d \tilde{V} = - \frac{\langle R \rangle}{4} \epsilon = - \frac{2\pi}{A_S} \Sigma s^I. \quad (9.3)$$

Thus, $\tilde{V}_a$ mimics the spin connection of a spherically symmetric horizon, although being defined for any horizon of spherical topology.

The method of Beetle and Engle can be generalized to this framework for the case of four dimensions by using the connection

$$A_{aIJ} = \Gamma^a_{aIJ} + 2 m_{aI} m_{aJ} h^{\beta\gamma} (D_\gamma \psi). \quad (9.4)$$
Insertion into the boundary condition
\[ \epsilon^{\alpha\beta} e^{IJKL} R_{\alpha\beta KL} (A) = 2\langle E^{(2)} \rangle n^{I\beta J} \]
results in a nonlinear partial differential equation.
\[ \Delta \psi = \frac{1}{4} \left( \frac{E^{(2)}}{\sqrt{h}} - \langle E^{(2)} \rangle \right). \]  
(9.6)

As shown in appendix C.3, it follows that
\[ 2\langle E^{(2)} \rangle (\delta_1^x(x)\delta_2^y(y)) = \epsilon^{IJKL} \epsilon^{\alpha\beta} (\delta_1(x) A_{\alpha KL})(\delta_2(y) A_{\beta KL}). \]  
(9.7)

The problem with generalizing this method to higher dimensions is that it leads to a nonlinear partial differential equation for \( \psi \), for which, as opposed to the Laplace operator \( \Delta \), a well developed theory ensuring the existence of a solution does not exist. Thus, although a generalization to higher dimensions seems straightforward, we cannot proceed due to the resulting nonlinear partial differential equation.

### 9.2. Perez–Pranzetti method

The basic idea of Perez and Pranzetti [41] in order to solve the problem of a varying scalar curvature on \( S \) is to use two Chern–Simons connections on \( S \), defined by
\[ A_0^i = \Gamma^i + \gamma e^i, \quad A_1^i = \Gamma^i + \sigma e^i. \]  
(9.8)

For the boundary conditions, it follows that
\[ F^i(A_0) = \Psi_2 \Sigma^i + \frac{1}{2} (\gamma^2 + c) \Sigma^i, \quad F^i(A_1) = \Psi_2 \Sigma^i + \frac{1}{2} (\sigma^2 + c) \Sigma^i, \]  
(9.9)

where the Newman–Penrose coefficient \( \Psi_2 \) is proportional to the scalar curvature and \( c \) is an extrinsic curvature scalar. Subtracting these two equations, Perez and Pranzetti find
\[ F^i(A_0) - F^i(A_1) = \frac{1}{2} (\gamma^2 - \sigma^2) \Sigma^i, \]  
(9.10)

which can be used to derive the symplectic structure of two SU(2) Chern–Simons connections on \( S \), since the scalar curvature disappeared from this new boundary condition. Furthermore, they take the additional constraint into account which follows from adding the above two field strengths, which requires to first find a suitable quantization of the scalar curvature.

The first steps of this treatment generalize to higher dimensions in a straightforward way: introduce \( N \) Chern–Simons connections of the form
\[ A_{aI}^{(i)} = \Gamma_{aI} + 2\sqrt{n} \rho_{aI} A_{aI}, \quad i \in \{1, \ldots, N\}. \]  
(9.11)

For their field strengths, it follows that
\[ F_{aI}^{(i)(j)} = R_{aI}^{\alpha\beta} - 2m_{aI} m_{\alpha\beta} a_i. \]  
(9.12)

When we insert this in the formula needed for the higher-dimensional boundary condition, we find
\[ E_{aI}^{IJ} (A^{(i)}) := \epsilon^{\beta_1\gamma_1 \cdots \beta_n\gamma_n} e^{IJKL_{\beta_1} \cdots K_{\gamma_n}} F_{aI}^{(i)(\beta_1\gamma_1)} F_{\beta_2\gamma_2}^{(i)(\beta_2\gamma_2)} \cdots F_{\beta_n\gamma_n}^{(i)(\beta_n\gamma_n)} = \sum_{k=0}^{n} d_i^k X_k, \]  
(9.13)

where, schematically, \( X_k \propto (\mathcal{R}^i)^{n-k} \wedge (m \wedge m)^k \). Only the \( k = 0 \) term, being exactly of the form \( n^{I\beta J} \), is allowed to survive when linear combining the \( E_{aI}^{IJ} \) with coefficients \( b_i \in \mathbb{R}^N, i \in \{1, \ldots, N\} \),
\[ \sum_{i=1}^{N} b_i E_{aI}^{IJ} (A^{(i)}) \propto n^{I\beta J}, \]  
(9.14)
which leads to the system of equations

\[ \sum_{i=1}^{N} b_i(a_i)^k = 0, \quad k \in \{0, \ldots, n-1\}, \]

\[ \sum_{i=1}^{N} b_i(a_i)^n = d, \] (9.15)

for some constant \( d \neq 0 \). Suppose w.l.o.g. that \( a_1 \neq 0, b_1 \neq 0 \). Introducing a new \( \tilde{d} = \frac{d}{a_1 b_1} \), we find that the above \((n+1)\) equations for fixed \( \tilde{d} \), actually only depend on the \( 2(N-1) \) unknowns \( (a_i/a_1), (b_i/b_1) \). Since \( N \) is integer and \( 2n = D - 1 \), we find that we need at least \( N = \lceil \frac{n+1}{2} \rceil + 1 = \lceil \frac{D+1}{2} \rceil + 1 \) Chern–Simons theories on the boundary, which for \( D = 3 \) reproduces the result of Perez and Pranzetti, namely \( N = 2 \). However, we now have to implement many additional constraints corresponding to (9.12) consistently, which makes a success of this route at the quantum level rather doubtful (see, however, the comments on quantization in section 10).

10. Comments on quantization

10.1. \( \text{SO}(D+1) \) as gauge group

In order to calculate the entropy associated to a spatial slice of the isolated horizon, we have to quantize the resulting theory on the horizon. In the well known \((3+1)\)-dimensional treatment [17], it is a key result that the field strength on the isolated horizon vanishes almost everywhere due to the isolated horizon boundary condition, except at points where the bulk spin network punctures the isolated horizon. Only at these points, the flux operator, which determines the field strength on \( S \) via the isolated horizon boundary condition (8.2), is non-vanishing. The resulting quantum theory on the horizon is a Chern–Simons theory with topological defects induced by these spin network punctures, which result in a finite-dimensional Hilbert space.

In higher dimensions, the situation is more complicated and we do not claim to have a satisfactory proposal for a quantization. In this section, we will comment on how such a quantization could be performed and where the problems lie. In a first attempt, one would expect to obtain a higher-dimensional Chern–Simons theory on the horizon, since the symplectic structure on the isolated horizon is exactly of this type. Due to the distributional nature of the space of generalized connections in LQG, see e.g. [44], one promotes the connection on the isolated horizon to an independent degree of freedom in the quantum theory, here called \( A_I \) with field strength \( F_{IJ} = F(A)_{IJ} \). Furthermore, a quantization of the boundary condition (8.2) (neglecting for a moment the stronger condition (8.1) and thus the fact that the connection on the isolated horizon is given by \( \Gamma_{IJ} \)) yields the quantum equations of motion of a higher-dimensional Chern–Simons theory with punctures exactly as in the \((3+1)\)-dimensional case. The immediate problem with this approach is however that higher-dimensional Chern–Simons theory admits local degrees of freedom (at least at the classical level), since the equations of motion

\[ \epsilon_{I_{1}I_{2}...I_{n}} F_{I_{1}J_{1}}^{I_{2}J_{2}} \wedge \cdots \wedge F_{I_{n}J_{n}} = 0 \] (10.1)

do not constrain the connection to be flat [29]. As a direct consequence, one would expect to obtain an infinite entropy by counting the allowed states in the Hilbert space.

Still, it seems that the objects \( \epsilon_{I_{1}I_{2}...I_{n}} F_{I_{1}J_{1}}^{I_{2}J_{2}} \wedge \cdots \wedge F_{I_{n}J_{n}} \) constitute an important subsector of the higher-dimensional Chern–Simons theory which one should consider for entropy calculations, as we will argue in the following. Using the language of Engle et al [27], the
Chern–Simons equations of motion (10.1) are modified by ‘particle degrees of freedom’ which are induced by the spin networks puncturing the horizon as

$$E^{IJ}(x) := e^{h,...,h} F_{IJ}(x) \wedge \cdots \wedge F_{IJ}(x) \propto \hat{\pi} \gamma_{IJ}(x),$$

(10.2)

where the operator on the right-hand side symbolizes to the flux operator which acts non-trivially only at points where a spin network punctures the horizon. Using the Dirac brackets obtained from solving the second class constraints of the higher-dimensional Chern–Simons theory\(^\text{14}\), we can explicitly calculate the algebra of these ‘particle excitations’ as

$$\{ E^{IJ}(x), E^{KL}(y) \} \propto \delta^{(D-1)}(x-y) f^{IJ, KL}_{MN} E^{MN}(x),$$

(10.3)

where \( f \) are the structure constants of \( SO(D+1) \). Since a representation of this algebra is just a representation of the Lie algebra \( SO(D+1) \) for each puncture, the problems which have to be discussed for the quantization are mainly connected with finding the right subspace of the tensor product of the individual \( SO(D+1) \) representation spaces which is selected by the criterion of compatibility with the bulk spin networks and the horizon topology. As opposed to the U(1) or SU(2) based constructions in four dimensions, the restrictions imposed by the simplicity constraint will also have to be taken into account properly. While the simplicity constraint is solved on the horizon at the classical level by using the variables \( n^I, s^I \) and \( e^I_0 \), to construct \( \Gamma^0_{IJ} \), there might still be non-trivial restrictions coming from imposing the quantum simplicity constraint in the bulk. One of them is to restrict the representations carried by the punctures to be the same as in the bulk, i.e. simple (spherical/class 1) \( SO(D+1) \) representations. However, the more interesting question will be if there is a restriction resulting from implementing the vertex simplicity constraints.

Despite these attractive features, we still have to deal with the local degrees of freedom. One point that we overlooked up to now is that the classical analogue of the boundary condition (10.2) does not constrain the Chern–Simons connection \( A_{aIJ} \) to be \( \Gamma^0_{aIJ} \). In section 8, it was shown that some modifications of the boundary connection parametrized by constants are allowed. Furthermore, the Beetle–Engle trick from section 9.1 suggests that further modifications are conceivable, possibly an infinite set. Thus, we should introduce a constraint which restricts the degrees of freedom of the higher-dimensional Chern–Simons theory as if the horizon connection would be given by \( \Gamma^0 \). Since the gauge invariant (local) information of a connection is contained in its field strength, we should introduce the boundary condition (8.1) in the form

$$F_{aIJ} = F_{(\Gamma^0)_{abIJ}}$$

(10.4)

on \( S \). In analogy to the \( (3+1) \)-dimensional treatment, we would quantize this boundary condition by promoting the left-hand side to an operator in the higher-dimensional Chern–Simons theory and act with a proper quantization of the right-hand side on the bulk spin network (as with a flux operator). Since we would regularize the right-hand side by fluxes and commutators involving volume operators as in [7, 45], it would automatically vanish at points where no bulk degrees of freedom are excited\(^\text{15}\). This mechanism could thus get rid of

\(^{14}\) Actually, in order to construct the Dirac brackets of the Chern–Simons theory, we would have to identify the set of second class constraints. This is non-trivial and depends on the choice of invariant tensor, as emphasized in [29]. There, it is also stated that for the epsilon tensor used in this chapter, our choice of second class constraints is correct at least in six spacetime dimensions (\( \epsilon^{IJKLMN} \) is ‘generic’ in the language of [29]). On the other hand, we can use the horizon boundary condition and the symplectic structure (8.3) to calculate the same algebra, at least under the constraint that the field strength of the Chern–Simons connection is given by \( F(\Gamma^0)_{IJ} \).

\(^{15}\) We would expect that the corresponding operator would even vanish at punctures, since the volume operator annihilates edges. On the other hand, we would demand consistency with (10.2), i.e. we would rather use (10.2) at punctures. This underlines again that the discussion here does not provide a satisfactory answer.
the local degrees of freedom and result in a finite entropy much in the same way as in \((3+1)\) dimensions. Still, there are many missing and imprecise steps in this argument, e.g. that one would first need an actual quantization of higher-dimensional Chern–Simons theory before a quantum boundary condition as \((10.4)\) could be even imposed.

This discussion leads us back to reconsider equation \((8.3)\) for the following reason: we know that the boundary symplectic structure can be written as \((8.3)\) and the boundary condition can be phrased as \(\hat{\varpi}_{\pi} = 2n^{[l}S^{l]}\). Endowed with the Poisson bracket following from \((8.3)\), the \(n^{[I}\hat{S}^{J]}\) form an SO\((D+1)\) Lie algebra in the same way as the fluxes do. On the other hand, we saw that the potentially relevant degrees of freedom \(E^{IJ}\) in the Chern–Simons theory are identical to the \(n^{[I}\hat{S}^{J]}\) and obey the same algebra. Thus, one could conclude that the Chern–Simons theory should be avoided already in the beginning. Such a system can be quantized with the same methods as the normal \(N^0\) in \([9]\) resulting from the linear simplicity constraints. However, we know from the \((3+1)\)-dimensional treatment that valuable insights were gained through the Chern–Simons treatment, e.g. logarithmic corrections resulting from a finite level, see e.g. \([27]\). If neglecting the Chern–Simons theory, these results have to be accounted for differently.

To conclude, we do not have a satisfactory quantization of the resulting boundary theory and thus also no direct access to full fledged entropy calculations at the moment. The biggest uncertainty certainly is that no quantization of higher-dimensional Chern–Simons theory with a non-Abelian gauge group is known. A reduction to U\((1)\) as a gauge group would in principle facilitate the problem, but we were not able to perform this reduction as explained in the next subsection. It thus seems that to a first approximation, considering the \(n^{[I}\hat{S}^{J]}\) as the boundary degrees of freedom is sensible. A straightforward generalization of the methods developed in \([46]\) would then give an entropy proportional to the area to leading order in the same way as in \((3+1)\) dimensions. One could then fix the free parameter \(\beta\) in order to obtain the prefactor \(1/4G\) for the entropy. See \([39,47]\) for discussions about this issue.

### 10.2. Reduction to U\((1)\)

The above discussion suggests that the local degrees of freedom of the higher-dimensional Chern–Simons theory should be absent after properly implementing the boundary conditions \((10.2)\), \((10.4)\). Since U\((1)\) Chern–Simons theory on the other hand does not have local degrees of freedom to begin with, this hints that we should investigate the possibility of gauge fixing the SO\((D+1)\) theory down to SO\((2)\), as there is no obvious contradiction due to different degrees of freedom in the quantum theory. This question will be pursued in this section, but as we will see, we did not succeed in giving a satisfactory description of the boundary degrees of freedom with this structure group.

Two routes suggest themselves: (1) gauge fix the SO\((D+1)\) Chern–Simons theory we obtained in the course of this paper down to SO\((2)\), or (2) impose the gauge fixing directly at the level of the boundary symplectic structure and rewrite it in terms of an SO\((2)\) Chern–Simons symplectic structure classically. The first route fails due to the SO\((D+1)\) invariant tensor used to construct the Chern–Simons theory, namely \(\epsilon^{i_1...i_{D+1}}\), which does not admit this gauge fixing. Therefore, we will follow route (2).

We introduce the gauge fixing \(n^I = g^{i\delta i'}_I, s^I = g^{i\delta i'}_{1I}\), where \(i, j \in \{0, 1\}\) and \(g \in SO(2)\). Let us use the usual parametrization of rotations by an angle \(\phi\), \(g_{00} = g_{11} = \cos \phi, g_{01} = -g_{10} = \sin \phi\). The boundary contribution to the symplectic structure reads in this gauge

\[
\delta_{[I}S_{J]} = \delta_{[I} \sqrt{h} \delta_{J]} \phi. 
\]  
\((10.5)\)
In the SO\((D+1)\) case, to show that a Chern–Simons symplectic structure arises on the horizon cross sections, it was important that \(\sqrt{h}\) and the Euler density are proportional. Introducing an SO\(2\) connection \(A_{\alpha}\), the analogue of this requirement would read

\[
\sqrt{h} \propto e^{\alpha_1 \ldots \alpha_2} F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{2n-1} \alpha_{2n}},
\]

(10.6)

where \(F_{\alpha \beta} = 2 \partial_{[\alpha} A_{\beta]}\). It follows that \(\delta \sqrt{h} \propto 2n e^{\alpha_1 \ldots \alpha_{2n}} (\delta_{[\alpha_1} \delta A_{\alpha_{2}]} F_{\alpha_{3} \alpha_{4}} \ldots F_{\alpha_{2n-1} \alpha_{2n}}\) and therefore (upon partial integration)

\[
\delta [1 \tilde{s}_I \delta 2] n_I \propto 2n e^{\alpha_1 \ldots \alpha_{2n}} (\delta_{[1} \delta_{2]} A_{\alpha_1} ) (\delta_{2}] \delta A_{\alpha_2} \phi) F_{\alpha_3 \alpha_4} \ldots F_{\alpha_{2n-1} \alpha_{2n}}.
\]

(10.7)

With the additional requirement that \(A = d\phi\), this would become the symplectic structure of an SO\(2\) Chern–Simons theory on the boundary. However, from this requirement we also conclude that \(F = 0\), which is in contradiction with (10.6), and therefore also our second route fails. It thus seems that we have to stick to the SO\((D + 1)\) theory on the boundary and one should try to make progress with its quantization as outlined above.

10.3. Higher dimensional versus 4d entropy

As transpires from the discussion so far, the main roadblock towards a quantum treatment of the boundary conditions in higher dimensions in analogy to the procedure followed in (3+1) dimensions originates from our lack of understanding of higher dimensional quantum Chern–Simons theory. The problem here is in fact two-fold. First, the quantization of pure Chern–Simons theory with punctures by itself is an interesting open problem in mathematical physics which is unlikely to be an easy task because the theory has local degrees of freedom, gauge freedom, and is self-interacting. The second problem which is more relevant in our context is that even if quantum Chern–Simons theory was available to us, its Hilbert space (with given puncture data) is infinite dimensional due to the presence of the local degrees of freedom which suggests that a calculation in analogy to the (3+1) treatment would result in an infinite value of the entropy.

However, this may not be necessarily the case because the Chern–Simons boundary theory that we are actually interested in may in fact only be a subsector of pure Chern–Simons theory with punctures. The reason for this is that the boundary conditions at the horizon when properly translated into the degrees of freedom of Chern–Simons theory may constrain the Chern–Simons connection further to such an extent that in fact the resulting theory has no local degrees of freedom. Indeed, a phenomenon of this kind must happen if the U\((1)\) point of view sketched in the previous section is to be viable at all, because allowed gauge fixings cannot change the number of true degrees of freedom, hence it is not possible to gauge fix pure non-Abelian Chern–Simons theory in more than three dimensions to pure Abelian Chern–Simons theory (with punctures) unless there is more gauge symmetry available than that which is intrinsic to pure Chern–Simons theory (with punctures). In our case, this additional gauge symmetry might be available due to the fact that the Chern–Simons theory degrees of freedom that we are interested in are just effective degrees of freedom of a corresponding bulk theory which has more gauge symmetry. The interesting question is how much of this survives at the boundary. If this scenario would be valid, it would constitute independent support to the new point of view advertised in this paper to consider the variables \(n^{(1)}(\tilde{s})\) as independent boundary degrees of freedom and to base the boundary quantum theory on those. At least naively, this would result in finite entropy as for fixed puncture data the allowed quantum states are bounded by the relevant highest weights of the SO\((D + 1)\) representations. Notice that these are boundary degrees of freedom and thus we are not interested in which way the highest weights are constituted by bulk edges and representations ending in the punctures (for which there are infinitely many possibilities).
On the other hand, it might also be that the opposite happens and the bulk symmetry that survives in the boundary theory is too small to render the entropy calculation finite when using the Chern–Simons theory in which case the quantization based on $\hbar^{D/2}$ appears to be an attractive option which however must be substantiated by further reasoning.

We leave the investigation of these ideas for future research and conclude this section with yet another point of view which has the advantage that it makes the contact with the $(3+1)$ theory more transparent. Namely, so far we have talked about the entropy of an observer living in $(D+1)$ dimensions as we used the area of a $(D-1)$-dimensional surface. Whether or not in a $(D+1)$-dimensional world an actual observer really extends in all $(D+1)$ dimensions, using the usual Kaluza–Klein point of view the observer may argue that all but $(3+1)$ of those dimensions are not accessible to him. As such he may argue that what accounts for the entropy of a black hole is just a two-dimensional cross section of the actual $(D-1)$-dimensional horizon. This lack of knowledge due to the excess dimensions will result in an effective $(3+1)$-dimensional theory in which the actual $(D+1)$-dimensional pure states are represented by mixed states due to the usual entanglement. Ultimately, this could result in a situation identical to the $(3+1)$ theory with the following modifications: (1) the Chern–Simons theory to consider is for the gauge group $SO(D+1)$ rather than $SO(3)$ (or $SU(2)$). (2) The entropy would be computed not using the pure $(3+1)$-dimensional states but rather the $(D+1)$-dimensional mixed states. The interesting and nontrivial question is of course whether the actual higher-dimensional entropy and the effective lower-dimensional one agree with each other at least semiclassically. In certain holographic scenarios this could actually be the case, see e.g. [48] and references therein.

Fundamentally, these mixed states should be computed from the full fledged $(D+1)$ theory. While clean, this approach has the disadvantage that again quantum Chern–Simons theory in $(D>3)$ dimensions would be needed. The analysis of [29] reveals that a canonical quantization using techniques of LQG is conceivable, however, the additional constraints present (beyond gauge invariance and diffeomorphism invariance) must be accounted for in a Dirac quantization. Alternatively, a reduced phase space quantization suggests itself which has interesting connections with the WZW conformal field theory about which a lot more is known [49].

A poor man’s version of this which should be viable in the semiclassical sector of the theory (and thus in particular in our context as we are using semiclassical reasoning in many places) is to perform a classical Kaluza–Klein reduction [50] and to quantize $(3+1)$ general relativity together with matter and the Kaluza–Klein fields in the standard fashion [9, 10, 51]. In this approach one simply would have to make sure that the Kaluza–Klein modes and in particular their boundary values do not disturb the standard reasoning in $(3+1)$ dimensions, that is, without changing the Barbero–Immirzi parameter as was done explicitly, e.g. for minimally coupled Maxwell and scalar fields as well as non-minimally coupled scalar fields, see [52, 53] and references therein.

11. Concluding remarks

In this paper, we derived a generalization of the isolated horizon boundary condition to non-distorted horizons in even dimensional spacetimes and showed that the canonical transformation to $SO(D+1)$ connection variables leads to a higher-dimensional Chern–Simons symplectic structure on the boundary of the spatial slice. While the classical treatment from four spacetime dimensions generalizes rather directly, the quantization of the resulting system is less obvious, since, to the best of the authors’ knowledge, there are no known
generalizations of the quantization of $(2 + 1)$-dimensional Chern–Simons theory to higher dimensions. On the other hand, due to the stronger boundary condition (8.1), it could be the case that it is not necessary to quantize ‘all of the higher-dimensional Chern–Simons theory’, but just a subalgebra of phase space functions which result as topological defects induced by puncturing the horizon with a spin network, as discussed in the previous section. In this line of thought, it suggested itself to forget about the Chern–Simons theory entirely and to use the $n^I\tilde{\omega}^J$ along with the symplectic structure (8.3) as horizon degrees of freedom. While mathematically attractive, this goes against the idea of using a theory based on a connection for both the bulk and the horizon. Also, it is unclear whether one can recover all the insights resulting from a finite level of the Chern–Simons theory in $(3+1)$ dimensions. As opposed to the Chern–Simons treatment however, the $n^I\tilde{\omega}^J$ are not restricted to even spacetime dimensions or specific topologies, which makes them very attractive, e.g. to compare with the broad literature on five-dimensional black holes. As an alternative, we have outlined a possible route along the more standard Kaluza–Klein reduction approach. Thus, we underline again that we do not have a fully satisfactory quantization for the isolated horizon degrees of freedom.

One of the most important questions which should be answered by a suitable quantization of the theory on the black hole horizon is the treatment of the simplicity constraint. A preliminary analysis shows that the classical simplicity constraint fits nicely into the picture of a Chern–Simons theory with particles as proposed in [27]. While a quantization of the edge simplicity constraints would just restrict the group representations on the particle defects in the same way as it restricts the edge representations, it might be that a proper quantization of the horizon degrees of freedom gives us a hint on what the correct implementation of the simplicity constraint on a vertex is. The reason for this comes from the seemingly very effective treatment of a black hole as a single intertwiner, see [54] and more recently also [55]. Moreover, this question will have a direct effect on the subleading correction in the entropy formula, which makes it again very interesting to study.

Additionally, it will be interesting to check to what extent the connection on the horizon can be generalized, e.g. as in [27], where a new free parameter can be associated to the horizon connection which can rescale the entropy. The consequences of introducing a two parameter family of connections in the bulk in four dimensions as proposed in [6] should also be investigated. From a more general perspective, it is noteworthy that many ingredients of the definition of an isolated horizon were not used in the Hamiltonian treatment, were only the non-distortion condition entered and the fact that there is no boundary term in the ADM symplectic potential. It thus suggests itself to pursue the question of general boundaries of spacetime, not only isolated horizons. This is especially interesting in the context of entropy bounds for general bounded regions of spacetime [56].

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Appendix A. Hybrid connection and generalizations

In this appendix, we will introduce several connections relevant for the main text, namely Peldan’s ‘hybrid’–spin connection [30] and extensions thereof to higher-dimensional internal space.
A.1. Peldan’s hybrid connection

It is a well-known fact that, given an SO(D) vielbein $e^i_a$ in $D$ dimensions, there exists a unique spin connection $\Gamma_{\alpha j}^i[e]$ compatible with it, which is obtained by solving

$$0 = \nabla^\alpha_a e_b^j = D_a e_b^j + \Gamma^i_{\alpha j} e_b^i$$

(A.1)

for $\Gamma^i_{\alpha j}$, where $D_a$ denotes the torsion free metric compatible covariant derivative. The result is

$$\Gamma^i_{\alpha j} = e^b_{[a} D_b e_{|j]}.$$  

(A.2)

Starting from a Lagrangian formulation of general relativity on a $(D+1)$-dimensional spacetime manifold, the natural gauge group is SO$(1,D)$ or SO$(D+1)$ for the Lorentzian or Euclidean theory, respectively. When passing to the corresponding Hamiltonian system, a $(D+1)$ split is performed and we are naturally led to consider a SO$(1,D)$ or SO$(D+1)$ vielbein $e^i_a$ on the $D$-dimensional spatial slice, which we will call hybrid vielbein. However, from the Hamiltonian perspective, the signature of the internal space $\zeta$ is not necessarily tied to the spacetime signature $s$, since we can always start with an SO$(D)$ vielbein on the spatial slice and introduce gauge degrees of freedom corresponding either to SO$(1,D)$ or SO$(D+1)$. In the following, we will therefore treat internal and spacetime signature independently. Peldan [30] investigated if one could define a compatible connection also for this hybrid vielbein. We have

$$0 = \nabla^\alpha_a e_b^j = D_a e_b^j + \Gamma^{\alpha i}_{\alpha j} e_b^i,$$

(A.3)

which actually can be solved for the unique ‘hybrid’—spin connection,

$$\Gamma^{\alpha i}_{\alpha j} = e^b_{[a} D_b e_{|j]} + \zeta \eta_{ij} D_{a} n_{ij},$$

(A.4)

where $n^a$ is the unique (up to sign) unit normal to the hybrid vielbein, $n^a e_{ij} = 0$, $n^a n^b \eta_{ij} = \zeta$, and $\zeta$ again denotes the internal signature, $\zeta = -1$ for SO$(1,D)$ and $+1$ for SO$(D+1)$. Note that the sign ambiguity is absent in $\Gamma^{\alpha i}_{\alpha j}$ since $n^a$ appears quadratically.

A.2. Extensions to higher-dimensional internal space

Now we want to extend this result to a higher-dimensional internal space, which is necessary for black hole applications, since we have to deal with the vielbein on the $(D-1)$-dimensional inner boundaries of the spatial slice.

We will start quite general by introducing an $\mathbb{R}^{D+q}$—valued vielbein $e^i_a$ in $D$ dimensions (only in this section, we will have $I,J,K \ldots = 1, \ldots, D+k$), $e^i_a$ $\eta_{ij} = q_{ab}$, where $\eta_{ij} = \text{diag}(\ldots, -1, +1, \ldots)$ and $p + q = k$, and ask for a SO$(p,D+q)$ connection

$$\Gamma^{\alpha i}_{\alpha j}$$

annihilating $e^i_a$. We have

$$0 = \nabla^\alpha_a e_b^j = D_a e_b^j + \Gamma^{\alpha i}_{\alpha j} e_b^i,$$

(A.5)

corresponding to $D^i(D+k)$ equations to determine $\Gamma^{\alpha i}_{\alpha j}$. However, these equations are not all independent, since

$$0 = e^d_{[a} D^m_d e_{|b]}$$

(A.6)

is identically satisfied due to the antisymmetry of the SO$(p,D+q)$ connection and the metric compatibility of $D_a$. The result are

$$D^2(D+k) - D^2(D+1)/2 = D^2((D-1)/2 + k)$$

(A.7)
independent equations for the metric and its inverse \( \eta \):

\[
D(D + k)(D + k - 1)/2
\]

(A.8)

unknowns \( \Gamma^H_{alj} \). It is clear that \( \Gamma^H_{alj} \) cannot be determined uniquely for any \( k \), since the number of equations grows, for fixed \( D \), linearly with \( k \), while the connection components grow quadratically. More precisely, equating both, we obtain \( (A.7) \Leftrightarrow Dk(k - 1)/2 = 0 \), i.e. the connection is only uniquely determined for the gauge groups \( SO(D) \), corresponding to \( k = 0 \), and \( SO(1, D) \) or \( SO(D + 1) \) for \( k = 1 \).

Let us study the indeterminacy for \( k > 1 \) in more detail. First we ‘complete’ the vielbein by choosing an orthonormal set of unit vectors \( n_i^a \), \( i = 1, \ldots, k \), normal to the vielbein, i.e., \( n_i^a e_{al} = 0 \) \( \forall \ i = 1, \ldots, k \). and \( n_i^a n_j^a \eta_{lj} = \eta_{lj} \) \( \forall \ i, j = 1, \ldots, k \) where \( \eta_{ij} = \text{diag}(-, 1, 1, 1, \ldots, +, \ldots, +) \). The indices \( i, j, \ldots \) will be raised and lowered using this metric and its inverse \( \eta^{ij} \). Then we can decompose \( \Gamma^H_{alj} \) according to

\[
\Gamma^H_{alj} = \Gamma_{alj} + 2n^{[b}_{al} \Gamma^i_{b]lj} + n^{[b|l}[n_{b]}lj]\eta^{ij}_{alj}, \tag{A.9}
\]

where summation over repeated indices \( i, j \) is understood and \( \Gamma_{alj}n_{lj} = 0 \) \( \forall \ i = 1, \ldots, k \). \( \Gamma_{alj}n_{lj} = 0 \) \( \forall \ i, j = 1, \ldots, k \). Inserting this decomposition of \( \Gamma^H_{alj} \) into (A.5), we find that \( \Gamma^i_{aj} \) simply drops out and therefore cannot be solved for, and the number of its components, \( Dk(k - 1)/2 \) since it is anti-symmetric in \( i, j \), precisely matches the indeterminacy. For the other components, one obtains

\[
\Gamma_{ai} = e^b_{al} \tilde{\eta}_{jk} D_a e^j_b, \tag{A.10}
\]

\[
\Gamma^i_{a} = \tilde{\eta}_{jk} D_a n^j_i, \tag{A.11}
\]

where \( \tilde{\eta}_{lj} := e_{al} e^{a}_{lj} \). Inserting back into (A.9), we find

\[
\Gamma^H_{alj} = 2e^h_{alj} D_a \bar{e}^j_h - e^h_{aij} \tilde{\eta}_{jk} D_a e^j_h + n^{[b|l}[n_{b]}lj]\eta^{ij}_{alj}, \tag{A.12}
\]

and therefore a \( Dk(k - 1)/2 \) parameter family of connections annihilating \( e_{al}^i \). To obtain a unique connection, we have to add additional requirements, e.g., we could demand that \( \Gamma^i_{aj} = 0 \) \( \forall \ i, j = 1, \ldots, k \) (these requirements are independent of the choice of ‘completion’ for the vielbein \( \{n_i^a\}_{i=1}^k \)). This connection \( \Gamma^i_{alj} \) would be special in that it would only depend on \( e_{al}^j \).

\[
\Gamma^i_{alj} = 2e^h_{alj} D_a \bar{e}^j_h - e^h_{aij} \tilde{\eta}_{jk} D_a e^j_h. \tag{A.13}
\]

Having in mind the application to black holes, we will proceed differently. For a fixed extension, the extra conditions we impose are \( D_a^\alpha n_i^a = 0 \) \( \forall \ i = 1, \ldots, k - 1 \) (these requirements are sensitive to the choice of completion). Again, these conditions are not all independent. We have \( e_{al} D^a_{,\alpha} n_{lj} = 0 \) and \( n^{[b|l}[n_{b]}lj] = 0 \) already satisfied, which results in \( D(k - 1)(D + k) - (D^2(k - 1) + Dk(k - 1)/2) = Dk(k - 1)/2 \) independent equations, which equals the number of undetermined components \( \Gamma^i_{aj} \). Solving for these, we find

\[
\Gamma^i_{aj} = -n^{[b|l} D_a n_{b]}lj \tag{A.14}
\]

and

\[
\Gamma_{alj}^0 = e^b_{alj} D_a e_{bj} + n^{[b|l} D_a n_{b]}lj \tag{A.15}
\]

16 Actually, we can as well specify \((k-1)\) vectors, since the last one, \( n_i^a \), is already determined (up to sign) by the mentioned requirements.

17 Note that, since \( n_i^a \) is given by \( e_{ai}^j \), \( n_i^a, i = 1, \ldots, k - 1 \), up to sign, it is automatically annihilated by \( D_a \) if the latter are.
as the unique connection annihilating the chosen completion of $e_a^I$. This connection has several nice properties, e.g. while for all connections of the family, we have

$$R_{ab}^I e^I = R_{abc}^{\phantom{abc}d} e^d,$$

$$R_{ab}^I n_i^I \eta_{KJ} = 0,$$

which follows from contraction of

$$0 = \left[J_a^{\epsilon\alpha}, J_b^{\epsilon\beta}\right] e_c^I = R_{ab}^I e_c^I + R_{abc}^\phantom{abc}d e_d^I,$$

for this connection we additionally have

$$R_{ab}^0 n_i^I = [D_a^{\epsilon\alpha}, D_b^{\epsilon\beta}] n_i^I = 0$$

and therefore

$$R_{ablj}^0 = R_{abc}^{\phantom{abc}d} \epsilon^d_{ij}.$$

From the right-hand side of (A.20), we see that, while $\Gamma_{ab}^0$ depends on the choice of $[n_i^I]_i^1$, $R_{ablj}^0$ is independent of $n$, determined completely by $e_a^I$ and its first and second derivatives. Explicitly, choosing a different completion $[\tilde{n}_i^I]_i^1$ of $e_a^I$, which is related to $[n_i^I]_i^1$ by a $\text{SO}(p, q)$ transformation $g$ via $\tilde{n}_i = g_i^j n_j$, we find

$$\Gamma_{ablj}^0[e, \tilde{n}] = \Gamma_{ablj}^0[e, n] + K_{alj},$$

$$K_{alj} := g^i_\epsilon n_l^I D_{a} g_i^I,$$

and

$$R_{ablj}^0[\Gamma^0[e, \tilde{n}]] = R_{ablj}^0[\Gamma^0[e, n]] + 2D^q [e, n]_{a} K_{blj} + [K_a, K_b]_{lj} = \cdots = R_{ablj}^0[\Gamma^0[e, n]].$$

For even dimensions $D = 2n$, it follows from (A.20)

$$e_{K_1 \cdots K_{l-1} \cdots K_{2n-1}} e^{a_1 \cdots a_{2n}} R_{a_1 b_1 \cdots b_l}^0 \cdots R_{a_{2n} b_{2n} b_{2n+1}}^0 = E^{(D)} e^{a_1 \cdots a_{2n}} n_1^I \cdots n_{2n}^I,$$

the right-hand side of which is also manifestly invariant under $\text{SO}(p, q)$ rotations and where $E^{(D)}$ denotes the $D$—dimensional Euler density

$$E^{(D)} := \frac{1}{\sqrt{\det g_{ab}}} e^{a_1 \cdots a_{2n}} e^{c_1 \cdots c_{2n}} R_{a_1 b_1 \cdots b_l}^0 \cdots R_{a_{2n} b_{2n} b_{2n+1}}^0.$$

Note that $R_{ablj}^0$ is not the only curvature tensor constructed from $e_a^I$ only. Of course, the connection $\Gamma_{ablj}^0$ we considered earlier, obtained by choosing $\Gamma_{ablj}^0 = 0$, is constructed solely from $e_a^I$ and so is the corresponding curvature tensor, but it fails to satisfy (A.20). More precisely, we find

$$R_{ablj}^I = R_{ablj}^0 + 2(\eta - \bar{\eta}) n_i^I (\eta - \bar{\eta})_{j} n_l^K (D_{a} e_c^K) (D_{b} e_d^I).$$

### Appendix B. Higher-dimensional Chern–Simons theory

In this appendix, we will review some facts about Chern–Simons theory in higher dimensions relevant for this work, with focus on the canonical formulation. In particular, we will derive the symplectic structure of the theory. We want to stress that these results are not new, but we state them here for completeness. For a more elaborate canonical treatment of higher-dimensional Chern–Simons theory, we refer the reader to [49].
The Chern–Simons action is defined for all odd dimensions \((2n + 1)\) and gauge groups \(G\) by the equation

\[
\mathcal{L}_{CS}^{2n+1} = i_{A_1 \ldots A_{n+1}} F^{A_1} \wedge \cdots \wedge F^{A_{n+1}},
\]

where \(F^A = dA^A + 1/2 [A, A]^A = dA^A + 1/2 f^{A BC} A^B \wedge A^C\) is the field strength of the connection one form \(A^A\) valued in the Lie algebra of \(G\), \(f^{A BC}\) are the structure constants of \(G, A_1, B, C \in \{1, \ldots, \text{dim}(g)\}\) are Lie algebra indices and \(i_{A_1 \ldots A_{n+1}}\) is a rank \((n + 1)\) symmetric tensor invariant under the adjoint action of the group. Explicitly,

\[
\mathcal{L}_{CS}^{2n+1} = i_{A_1 \ldots A_{n+1}} \sum_{p=0}^{n} (-1)^p \binom{2n+1}{n-p} \left( F^{A_1} \wedge \cdots \wedge F^{A_{n-p}} \wedge \left(1/2 [A, A]^{A_{n-p+1}}\right) \wedge \cdots \wedge \left(1/2 [A, A]^{A_{n+1}}\right) \right) \wedge A_{n+1}
\]

\[
=: i \sum_{p=0}^{n} (-1)^p \binom{2n+1}{n-p} (F^{n-p} \wedge (1/2 [A, A])^p) \wedge A,
\]

where the second line defines the shorthand notation we will use in the following. For our purposes, it will be sufficient to restrict attention to the groups \(SO(1, D)\) or \(SO(D + 1)\) where \(D = 2n + 1\). It is convenient to label the \(D(D+1)/2\) generators of the corresponding Lie algebras by an anti-symmetric combination of two indices in the fundamental representation \(I, J = 0, \ldots, D\) (e.g. the connection one form will be denoted by \(A^I\) with \(A^{(IJ)} = 0\)). We will furthermore restrict the invariant tensor to be the epsilon tensor \(\epsilon^{I_1 I_2 \ldots I_{n+1}}\), which is the one relevant for our application. However, we want to point out that all results of this section are independent of the choice of gauge group and invariant tensor.

In order to obtain the (pre-)symplectic structure, we invoke the covariant canonical formalism \([57–59]\), according to which the presymplectic potential is given by the boundary term of the first variation of the action, while the pre-symplectic structure is the exterior derivative of the potential.

Using the relation

\[
\delta (F^{n-p} \wedge (1/2 [A, A])^p) = \epsilon \left( (n + p + 1) \delta A \wedge F^{n-p} \wedge (1/2 [A, A])^p + (n - p) \delta A \wedge F^{n-p-1} \wedge (1/2 [A, A])^{p+1} + (n - p) d[\delta A \wedge F^{n-p-1} \wedge (1/2 [A, A])^p \wedge A] \right)
\]

the first variation of the Chern–Simons action yields

\[
\delta \mathcal{L}_{CS}^{2n+1} = \delta \int_M \mathcal{L}_{CS}^{2n+1} = \int_M \left[ \epsilon \sum_{p=0}^{n} (-1)^p \binom{2n+1}{n-p} (n + p + 1) F^{n-p} \wedge (1/2 [A, A])^p \right] \wedge \delta A
\]

\[
+ \int_M \left[ \epsilon \sum_{p=0}^{n} (-1)^p \binom{2n+1}{n-p} (n - p) F^{n-p-1} \wedge (1/2 [A, A])^{p+1} \right] \wedge \delta A
\]

\[
+ \int_M d \left[ \delta A \wedge \left( \epsilon \sum_{p=0}^{n} (-1)^p \binom{2n+1}{n-p} (n - p) F^{n-p-1} \wedge (1/2 [A, A])^p \wedge A \right) \right]
\]
Note that the two sums of the bulk contribution cancel each other term by term, and the only term surviving is the \((p = 0)\)—term of the first sum. We obtain the Chern–Simons equations of motion\(^{18}\)

\[
\epsilon \cdot F = \cdots \otimes F = 0, \tag{B.5}
\]

which in \((2+1)\) dimensions (which corresponds to \(n = 1\)) reduces to \(F = 0\). Let \(\sigma\) be a \(2n\)-dimensional Cauchy slice. The pre-symplectic potential can be read off the boundary term of the first variation and is given by

\[
\theta_\sigma(\delta) = \int_{\partial M} \epsilon \cdot F_n \wedge F - \frac{1}{2} [A, F^n] \wedge A \tag{B.6}
\]

For its variation, the equation

\[
\delta [\epsilon \cdot (\delta_1 A) \wedge F^{n-p-1} \wedge \frac{1}{2} [A, F] \wedge A] = \epsilon \cdot \left[ \frac{1}{2} (n + 1) \delta_1 A \wedge \delta_2 A \wedge F^{n-p-1} \wedge \frac{1}{2} [A, F]^p \right.
\]

\[
+ \frac{1}{2} (n - p - 1) \delta_1 A \wedge \delta_2 A \wedge F^{n-p-2} \wedge \frac{1}{2} [A, F]^{p+1} \right], \tag{B.7}
\]

is useful. Actually, in the above result, a boundary term was dropped, but in defining the symplectic current, we are allowed to drop this term since we will integrate the symplectic current we want to derive in this step over the boundary of the spacetime region we are interested in. We find for the symplectic current

\[
d\theta_\sigma(\delta_2, \delta_1) = \frac{1}{2} \sum_{p=0}^{n-1} \left[ \sum_{p=0}^{n-1} \frac{2n+1}{n-p} (-1)^p (n - p) (n + p + 1) F^{n-p-1} \wedge \frac{1}{2} [A, F]^p \right.
\]

\[
+ \sum_{p=0}^{n-2} \frac{2n+1}{n-p} (-1)^p (n - p) (n - p - 1) F^{n-p-2} \wedge \frac{1}{2} [A, F]^{p+1} \right]
\]

\[
= \frac{n(n+1)}{2} \epsilon \cdot \delta_1 A \wedge \delta_2 A \wedge F^{n-1}, \tag{B.8}
\]

where again the terms in the two sums cancel each other out, with only the \((p = 0)\)—term in the first sum remaining. Therefore, the pre-symplectic structure is given by

\[
\Omega_\sigma(\delta_2, \delta_1) = \frac{n(n+1)}{2} \int_{\partial M} \epsilon \cdot \delta_1 A \wedge \delta_2 A \wedge F^{n-1}. \tag{B.9}
\]

Usually, in order to have a meaningful phase space description, one now imposes suitable boundary conditions and checks if the pre-symplectic structure is independent of the choice of the Cauchy slice \(\sigma\) and, for noncompact \(\sigma\), if the integral is finite. However, in this paper we are only interested in a spacetime with internal isolated horizon boundary on which the Chern–Simons symplectic structure arises and we only have to answer these questions for the full spacetime.

From \(B.8\), we can also read off that the Dirac matrix of Chern–Simons theory is given, up to numerical factors, by \(\epsilon \cdot F^{n-1}\), which coincides with the result in \([49,\, \text{equation (2.7)}]\),

\(^{18}\) Note that the bulk term of the variation can be obtained within two lines by varying \(B.1\).
Appendix C. Details on calculations

C.1. Symplectic structure via the Palatini action

In this appendix, we provide calculational details for showing (7.23),

\[
\int_{\Delta} \delta_1 \sum_{\omega}^{\mu} \delta_2^\mu \eta_2 = 2 \int_{\Delta} \left\{ d \left[ \delta_1 \delta^\mu \delta_2 \eta^\mu \right] + \frac{1}{(D-1)!} \delta_1 \epsilon^{D-1} \wedge \delta_2 \omega \right\} \quad (C.1)
\]

We will contract any of the three lines of (7.20) separately with (7.6) and multiply them by \( \frac{1}{D-1} \). For the first line, we find

\[
(D-1) \epsilon_{ijk} \cdots \epsilon_{ijkl} \left[ m_L \wedge m_K \wedge \cdots \wedge m_{K^{D-1}} \left( i_m \bar{\eta}_1 m_{K^I} \right) \right] \wedge \left[ \delta_1 \left( \delta_2 \omega \right)^{D-1} \right] - 2 \epsilon_{ijk} \cdots \epsilon_{ijkl} \left[ m_L \wedge m_K \wedge \cdots \wedge m_{K^{D-1}} \left( -i_m \epsilon_{jkl} m_{K^I} \delta_1 m_{M^I} \right) \right] \wedge \left[ \delta_2 \omega \right] = 2(D-1) \epsilon_{ijk} \cdots \epsilon_{ijkl} \left[ m_L \wedge m_K \wedge \cdots \wedge m_{K^{D-1}} \left( k_l d_2 \delta_2 \omega \right) \right] \wedge \left[ \delta_1 \epsilon^{D-1} \wedge \left( \delta_2 \omega \right) \right] - 2(D-1) \epsilon_{ijk} \cdots \epsilon_{ijkl} \left[ m_L \wedge m_K \wedge \cdots \wedge m_{K^{D-1}} \left( k_l d_2 \delta_2 \omega \right) \right] \wedge \left[ \delta_1 \epsilon^{D-1} \wedge \left( \delta_2 \omega \right) \right]
\]

36
In this appendix, we provide calculational details for showing that under the assumption \(19\) contracted with (7.6) and \(1\), we have

\[
-(D-1)(D-2)\epsilon_{IJK\ldots K_{n-1}} k^{i} k \wedge m_{L} \wedge m_{K_{i}} \wedge \ldots \wedge m_{K_{n-1}} \left(i_{\mu} \delta_{11} m^{K_{i}} \right) \wedge \left[\delta_{21} \Gamma^{UJ}{\bar{\psi}}_{L} + \frac{2}{D-1} \left(\delta_{21} \delta^{UJ} m_{L} \theta_{L} \right)\right] = -2(D-2)\epsilon_{IJK\ldots K_{n-1}} d \left[\delta^{UJ} k \wedge m^{K_{i}} \wedge \ldots \wedge m_{K_{n-1}} \left(i_{\mu} \delta_{11} m^{K_{i}} \right) \delta_{21} \left| k \right| \right],
\]

(C.2)

and

\[
-(D-1)\epsilon_{IJK\ldots K_{n-1}} \left(-l^{i} k + \left(\delta_{11} l^{i} k \right)\right) k \wedge m_{K_{i}} \wedge \ldots \wedge m_{K_{n-1}} \wedge \left(\delta_{11} l^{i} m_{L} \right) \theta_{L} \wedge \left(\delta_{21} \Gamma^{UJ} \delta^{UJ} \theta_{L} \right) = -2(D-1)\epsilon_{IJK\ldots K_{n-1}} d \left[l^{i} k \wedge m_{K_{i}} \wedge \ldots \wedge m_{K_{n-1}} \left(\delta_{11} l^{i} k \right) \delta_{21} \left| k \right| \right],
\]

(C.3)

respectively. Summing up the three lines, we arrive at (7.23) rescaled by the factor \(\frac{1}{(D-1)!}\) introduced before.

**C.2. Symplectic structure independent of the internal signature**

In this appendix, we provide calculational details for showing that under the assumption\(^{19}\) \(\delta E^{(2n)}_{\phi^2} = 0\) \((2n = D - 1)\), we have

\[
2\frac{E^{(2n)}}{\sqrt{h}} \left(\delta_{11} \delta^{JL} \right) \left(\delta_{21} \gamma_{L} \right) = n \epsilon_{IJK\ldots K_{n-1} M_{n-1} \ldots M_{1}} \times e^{u_{\beta} \alpha_{\beta} \ldots \alpha_{n-1} \beta_{n-1}^{i}} \left(\delta_{11} \Gamma^{UJ}_{\alpha L} \right) \left(\delta_{21} \Gamma^{0}_{\beta} \right) R^{0}_{\alpha_{L} \beta_{L} M_{n-1} \ldots M_{1} \ldots M_{n-1}} \ldots R^{0}_{\alpha_{n-1} \beta_{n-1} \alpha_{n-1} M_{n-1} \ldots M_{1}},
\]

(C.5)

where \(\Gamma^{0}_{\alpha L} \) is the generalized hybrid connection and \(R^{0}_{\alpha_{L} \beta_{L} M_{n-1} \ldots M_{1} \ldots M_{n-1}} \) the corresponding curvature tensor which are given in appendix A.2.

\(^{19}\) Note that this requirement for an UDNRIH is equivalent to restricting to histories with a fixed value of the horizon area, \(A_{H} = 0\), which can be seen as follows: since \(E^{(2n)} = f(v) \sqrt{h}\), by integrating both sides over \(S\) we obtain \(f(v) = f = \frac{E^{(2n)}_{\phi^2}}{\sqrt{h}}\) actually is independent of \(v\) since both \(A_{H}\) and \(E^{(2n)}\) are. Therefore, we have \(\frac{\delta E^{(2n)}}{\delta A_{H}} = \frac{\delta E^{(2n)}_{\phi^2}}{\delta A_{H}} = -\frac{E^{(2n)}_{\phi^2}}{A_{H}} \delta A_{H}\), where we used that the topology of \(S\) is fixed.
Starting with (C.5), we first calculate
\[
\delta \left( \frac{E^{(2n)}}{\sqrt{h}} \right) = \delta \left( \frac{1}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}} R_{a_1 b_1} \cdots R_{a_n b_n} \right)
\]
\[
= -\left( \delta \log h \right) \frac{E^{(2n)}}{\sqrt{h}} + \frac{n}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}}
\times \left( -2 h \delta \epsilon_{\beta_1 \beta_2} D_{\alpha} \delta R_{\beta_1 \beta_2} + R_{a_1 b_1} \delta R_{a_2 b_2} \cdots R_{a_n b_n} \right)
\]
\[
= -\left( \delta \log h \right) \frac{E^{(2n)}}{\sqrt{h}} - \frac{2n}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}}
\times \left( D_{\alpha} \delta R_{a_1 b_1} \cdots R_{a_n b_n} \right)
\]
\[
= -\left( \delta \log h \right) \frac{E^{(2n)}}{2\sqrt{h}} - \frac{2n}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}}
\times \left( D_{\alpha} \delta R_{a_1 b_1} \cdots R_{a_n b_n} \right).
\]

(C.6)

In the second line, we just explicitly wrote down all variations appearing using \( \delta R_{\alpha \beta} = -2 D_{\alpha} \delta R_{\beta \gamma} \). In the third, we used \( \delta R_{\alpha \beta} = \frac{1}{2} h^{\xi} \left( 2 D_{\alpha} \delta h_{\beta \gamma} - D_{\beta} \delta h_{\alpha \gamma} \right) \) and in the last step, we used
\[
\frac{n}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}} R_{a_1 b_1} \cdots R_{a_n b_n} \delta h_{a_1 b_1} \cdots R_{a_n b_n} = \frac{E^{(2n)}}{2\sqrt{h}} \left( \delta \log h \right).
\]

(C.7)

This last identity can be verified as follows:
\[
\frac{n}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}} R_{a_1 b_1} \cdots R_{a_n b_n} \delta h_{a_1 b_1} \cdots R_{a_n b_n} = \frac{E^{(2n)}}{2\sqrt{h}} \left( \delta \log h \right) - \frac{n(n - 1)}{h} e^{a_1 b_1 \cdots a_n b_n e^{\gamma_1 \gamma_2 \cdots \gamma_n \lambda_1 \cdots \lambda_n}}
\times \left( \delta h_{a_1 b_1} \right) R_{a_2 b_2} \cdots R_{a_n b_n}.
\]

(C.8)

where in the first step, we used \( h^{\xi} \delta h_{a_1 b_1} = -h_{a_1 b_1} \delta h^{\xi} \), then we added zero by adding all terms necessary that the expression in the second line becomes antisymmetric in \( \gamma_1, \delta_1, \ldots, \gamma_n, \delta_n, \xi_1 \) and immediately subtracting them again. Since these are \( D \) indices in dimension \( (D - 1) \), the anti-symmetrization vanishes and we are left with the subtracted terms. The first of these gives, using \( h_{a_1 b_1} \delta h^{\xi} = -\delta \log h \), the first term in the fourth line, while the remaining ones, after renaming indices, reproduce up to numerical factors the expression we started with. Comparing the first and the last line of (C.8), one easily infers (C.7).

Next, we will calculate \( \delta \Gamma^0_{\alpha \beta \gamma} \):
where in the second step we used \( \eta_{IJ} = \bar{\eta}_{IJ} + \zeta n_{IJ} + s_1 s_{IJ} \) and \( \bar{\eta}_{IJ} = m^\beta m_{\beta J} \), in the third that 
\[
(\delta \Gamma_{\alpha IJ}) m^{\beta J} = \left( \delta D^\alpha_a s_{K} \right) - \left( D^{\alpha a} \delta m_{K} \right) - \left( \delta D^{\alpha a} m_K \right) - \left( D^{\alpha a} \delta m_K \right)
\]
and in the fourth step we used \( \Gamma_{\alpha IJ}^0 \) annihilates the hybrid vielbein and \( n, s \). This way of expressing \( \delta \Gamma_{\alpha IJ} \) is convenient for several reasons. First of all, we explicitly separated the \((\bar{n} \bar{s})\), \((n \bar{s})\), \((s \bar{n})\) and \((n s)\) terms. Since the two variations of \( \Gamma_{\alpha IJ}^0 \) in (C.5) are contracted with an \( \epsilon \), which is \( \text{bar} \) projected on all other indices (remember \( R^0_{\alpha IJ} = R^0_{\alpha IJ} \), cf appendix A), the only contributions will come from \((\text{bar} \text{bar})\) - \((n s)\) and \((n s)\) - \((s s)\) terms. Secondly, many of the terms are such that covariant derivatives \( D^\alpha_a \) appear explicitly. This simplifies further manipulations like partial integrations, since almost all appearing objects are annihilated by \( D^\alpha_a \). Furthermore, since \( S \) already is a boundary, no boundary terms appear when partially integrating. Using (C.9), we thus find

\[
n \epsilon^{IJKLMN} n_{IJ} e^{a_1 b_1 c_1 d_1} (\delta_1 \Gamma_{\alpha IJ}) (\delta_2 \Gamma_{\beta K}) R^{00}_{a_1 b_1 c_1 d_1} \cdots R^{00}_{a_n b_n c_n d_n} \\
\times \left[ 8 \zeta n_{IJ} (D^\alpha_a s_I) s_K \tilde{\bar{n}}_{LJ} \left( D^{\alpha a} s_{JL} \right) + 4 \zeta n_{IJ} (D^\alpha_a s_I) s_K \tilde{\bar{n}}_{LJ} \left( D^{\alpha a} s_{JL} \right) - (\delta_1 \Gamma_{\alpha IJ}) m^\beta s_{IJ} \right]
\]

\[
= - \frac{4n}{\sqrt{\hbar}} e^{I \gamma J \beta \gamma} e^{a_1 b_1 c_1 d_1} e^{a_2 b_2 c_2 d_2} R^{00}_{a_1 b_1 c_1 d_1} \cdots R^{00}_{a_n b_n c_n d_n} \\
\times \left[ 2m_{IJ} (D^\alpha_a s_I) m_{K} \left( D^{\alpha a} s_{K} \right) - (D^\alpha_a s_I) m_{K} \left( D^{\alpha a} s_{K} \right) \right]
\]

\[
= - \frac{4n}{\sqrt{\hbar}} e^{I \gamma J \beta \gamma} e^{a_1 b_1 c_1 d_1} e^{a_2 b_2 c_2 d_2} R^{00}_{a_1 b_1 c_1 d_1} \cdots R^{00}_{a_n b_n c_n d_n} \\
\times \left[ 2m_{IJ} (D^\alpha_a s_I) m_{K} \left( D^{\alpha a} s_{K} \right) - (D^\alpha_a s_I) m_{K} \left( D^{\alpha a} s_{K} \right) \right]
\]

\[
= - \frac{8n}{\sqrt{\hbar}} e^{I \gamma J \beta \gamma} e^{a_1 b_1 c_1 d_1} e^{a_2 b_2 c_2 d_2} R^{00}_{a_1 b_1 c_1 d_1} \cdots R^{00}_{a_n b_n c_n d_n} m_{IJ} \left( D^\alpha_a s_I \right) m_{K} \left( D^{\alpha a} s_{K} \right)
\]

\[
- \left[ 2 \left( \delta_1 \left( \frac{E^{(2n)}}{\sqrt{\hbar}} \right) + \frac{E^{(2n)}}{\sqrt{\hbar}} \right) \right] \left( \delta_1 \log h \right) \left( \delta_2 \delta s \right)
\]

In the third line, note that the term containing \( D^\alpha_a \delta m^{\beta J} \) vanishes, since when partially integrating, we obtain a term of the form \( \left( D^\alpha_a D^{\alpha a} s_{\beta J} \right) \), which vanishes due to torsion-freeness. In the second step, we used \( e^{I \gamma J \beta \gamma} e^{a_1 b_1 c_1 d_1} e^{a_2 b_2 c_2 d_2} R^{00}_{a_1 b_1 c_1 d_1} \cdots R^{00}_{a_n b_n c_n d_n} = \frac{i}{\sqrt{\hbar}} e^{I \gamma J \beta \gamma} \alpha_{\bar{n}} \) and again \( \delta_1 \gamma_{\alpha \beta} = \frac{1}{2} \delta_{\alpha \beta} (2D_{\alpha IJ} h_{\beta IJ} - D_{\beta IJ} h_{\alpha IJ}) \). In the third step, we densitized \( s_{\beta J} \) (note that \( s_{\beta J} \) is always contracted such that variations on the density \( \sqrt{\hbar} \) drop out), partially integrated in the last summand and interchanged the indices \( \alpha \) and \( \beta \). In the fourth step, we replaced the second summand in square brackets using (C.6).

Now we will have a closer look at the left-hand side of (C.5).

\[
2 \frac{E^{(2n)}}{\sqrt{\hbar}} (\delta_1 s_{\beta J}) (\delta_2 m_{IJ}) = 2 E^{(2n)} (\delta_1 s_{\beta J}) (\delta_2 m_{IJ}) + \frac{E^{(2n)}}{\sqrt{\hbar}} \tilde{s}_{IJ} (\delta_1 \log h) \left( \delta_2 m_{IJ} \right)
\]

\[
= 2 E^{(2n)} (\delta_1 s_{\beta J}) (\delta_2 m_{IJ}) + \sqrt{\hbar} n^I (\delta_1 \tilde{s}_{IJ}) (\delta_2 \log h).
\]

(C.11)
Here, in the first step we varied $s'$ and the density $\sqrt{N}$ independently. In the second step, we interchanged the variations and used $\delta s_j \delta n^l = -n^l \delta s_j$ in the second summation. For the first summation, we find

$$2E^{(2n)}(\delta_1 s^j)(\delta_2 n_k)$$

$$= -\frac{2}{\sqrt{\hbar}} \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} \gamma^{I_1 ... I_n} R_{\alpha_1 \beta_1 I_1} ... R_{\alpha_n \beta_n I_n} (\delta_1 n^I + \delta_2 n^I)$$

$$= -2 \epsilon \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} e^{IJKL} e^{IJKL} R_{\alpha_1 \beta_1 I_1} ... R_{\alpha_n \beta_n I_n} (\delta_1 n^M + \delta_2 n^M)$$

$$= -4 \epsilon \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} e^{IJKL} e^{IJKL} R_{\alpha_1 \beta_1 I_1} ... R_{\alpha_n \beta_n I_n} (\delta_1 n^M + \delta_2 n^M)$$

$$= -4 \epsilon \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} (\delta_1 n^M + \delta_2 n^M)$$

$$= -2 \epsilon \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} [(\delta_1 n^M + \delta_2 n^M) + 2n(\delta_1 n^M + \delta_2 n^M) + M]$$

$$\times R_{\alpha_1 \beta_1 I_1} ... R_{\alpha_n \beta_n I_n} (\delta_1 n^M + \delta_2 n^M)$$

$$= -8 \epsilon \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} (\delta_1 n^M + \delta_2 n^M)$$

$$= -8n \epsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} \gamma^{I_1 ... I_n} R_{\alpha_1 \beta_1 I_1} ... R_{\alpha_n \beta_n I_n} m_{\gamma I} D_{\alpha_1} \delta_2 n^L$$

$$= \zeta \frac{\hbar}{N} \epsilon^{\gamma^{I_1 ... I_n}} e^{IJKL} e^{IJKL} (\delta_1 n^M + \delta_2 n^M)$$

which shows that (C.11) coincides with (C.10) iff $\delta (\frac{E^{(2n)}}{\sqrt{\hbar}}) = 0$. Here, in the first step, we used the defining equation for $E^{(2n)}$ and in the second we used

$$\frac{\zeta}{\sqrt{\hbar}} \epsilon^{\gamma^{I_1 ... I_n}} e^{IJKL} e^{IJKL} (\delta_1 n^M + \delta_2 n^M)$$

and (A.20) in the third step, we anti-symmetrize in the lower pair of indices $J$ and $M$. Note that the additional term vanishes since $s' \delta s_j = 0$ and the epsilon tensor enforces $\delta s_j$ to be projected into that direction. The fifth line is exactly the same as the fourth, we just moved $\delta n^M$ to the front and anti-symmetrized the upper indices $J$ and $M$ instead of the lower ones. Now we again anti-symmetrize the $(D + 2)$ upper indices $M, I, J, K, L, \ldots, K_L n_i$, which gives zero, and subtract the term we added for anti-symmetrization again. The first of these, the first summation in the round brackets in line 6, gives zero due to $n^l \delta n_l = 0$. The others all give the same term of the form $R_{\alpha_1 \beta_1 I_1} \delta n^M = 2D_{\alpha_1} \delta_2 n^L$, which we used in the second to last line. One more integration by parts in the last line, again using (C.13) and densitizing $s'$ gives the final result.

### C.3. Symplectic structure for the SO(4) based Beetle–Engle connection

For $D = 3$, we will show that one can bypass the restriction to spherically symmetric isolated horizons in complete analogy to the treatment of Beetle and Engle [42],

$$2E^{(2)}(\delta_1 s^j)(\delta_2 n_k) = \epsilon^{IJKL} e^{IJKL} (\delta_1 A_{\alpha \beta} + \delta_2 A_{\alpha \beta})$$

where $(E^{(2)}) := \int_S d^3 x E^{(2)}$ coincides, up to constant factors, with the Euler characteristic of the intersection of the isolated horizon with the spatial slices, and $A_{\alpha \beta}$ was defined in (9.4).

The assumption $\delta \frac{E^{(2)}}{\sqrt{\hbar}} = 0$ is then replaced by $\delta (E^{(2)}) = 0$, which however is already enforced by our choice of topology of the horizon.

To prove (C.14), we start by noting that

$$\epsilon^{IJKL} e^{IJKL} (\delta_1 A_{\alpha \beta} + \delta_2 A_{\alpha \beta})$$

$$= \epsilon^{IJKL} e^{IJKL} [\delta_1 (\delta_2) + 2 \delta_2 (\delta_2)] + (\delta_1 A_{\alpha \beta}) (\delta_2 K_{\beta \gamma}) + (\delta_1 K_{\beta \gamma}) (\delta_2 K_{\beta \gamma})$$

$$: A + B + C.$$
where we introduced the abbreviations $A, B, C$ for the three summands. The first summand in square brackets is, up to factors, the restriction to $D = 3$ of what we just calculated above,

$$A = e^{ijkl}e^{ij} \left( \delta_{1\alpha} \right) \left( \delta_{2\alpha} \right) \epsilon^{\alpha\beta\gamma\delta} \left( \delta_{2\beta} \right) \epsilon^{\beta\gamma\delta\epsilon} \left( \delta_{2\epsilon} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right).$$

(C.16)

Next, we need to calculate

$$\delta K_{\text{bar}} = \delta \left( 2m_{\text{bar}} \epsilon_{\text{bar}} \epsilon^{\alpha\beta\gamma\delta} (D_{\gamma} \psi) \right)$$

$$= 2m_{\text{bar}} \epsilon_{\text{bar}} \epsilon^{\alpha\beta\gamma\delta} (D_{\gamma} \psi) + 4\epsilon_{\text{bar}} \epsilon^{\alpha\beta\gamma\delta} \left( \delta_{2\beta} \right) \epsilon^{\beta\gamma\delta\epsilon} \left( \delta_{2\epsilon} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right)$$

$$+ 4\epsilon_{\text{bar}} \epsilon^{\alpha\beta\gamma\delta} \left( \delta_{2\epsilon} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right)$$

(C.17)

where again split the (bar bar) terms (second line) from the (bar n), (bar s) terms (third line). Since no (n s) terms appear, we find for $C$

$$C = e^{ijkl}e^{ij} \left( \delta_{1\alpha} \right) \left( \delta_{2\beta} \right) \epsilon^{\alpha\beta\gamma\delta} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\epsilon} \right) \epsilon^{\beta\gamma\delta\epsilon} \left( \delta_{2\beta} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right.$$}

where in the second step we used

$$e^{ijkl} \epsilon_{ij} \epsilon_{kl} \psi = \xi \sqrt{\epsilon} \epsilon_{ij}$$

and the last equality is easily obtained when explicitly writing out all anti-symmetrizations. For $B$, we find using (C.9) and (C.17)

$$B = 2e^{ijkl}e^{ij} \left( \delta_{1\alpha} \right) \left( \delta_{2\beta} \right) \epsilon^{\alpha\beta\gamma\delta} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\epsilon} \right) \epsilon^{\beta\gamma\delta\epsilon} \left( \delta_{2\beta} \right) \epsilon^{\epsilon\alpha\beta\gamma} \left( \delta_{2\gamma} \right) \epsilon^{\gamma\delta\epsilon\alpha} \left( \delta_{2\delta} \right.$$}
and since we assumed that $\Delta \psi = \frac{1}{2} \left( \frac{E(2)}{\sqrt{h}} \right)$ and $\delta \{E(2)\} = 0$, we find

$$\delta \Delta \psi = -2 \left\{ \left( \eta^{M} \delta_{L} \tilde{\xi}_{L} \right) \left( \frac{E(2)}{\sqrt{h}} \right) + \left( \delta_{M} \tilde{\xi}_{M} \right) \frac{E(2)}{\sqrt{h}} - \left( \frac{E(2)}{\sqrt{h}} \right) \right\}. \quad \text{(C.20)}$$

Here, in the second line, we inserted the expressions for $\delta \Gamma_{0,1}^{0}$ and $\delta \Gamma_{0,1}^{1}$ (C.9, C.17). Note that since $\delta \Gamma_{0,1}^{0}$ does not contain $(n \times)$ terms, the $(bar\ \text{bar})$ terms of $\delta \Gamma_{0,1}^{0}$ drop out. In the third step, we used (C.19) and $\tilde{\eta}_{IJ} = m_{IJ} m_{\rho J}$, and in the fourth step, epsilon identities were used and anti-symmetrizations in $(\beta, \gamma)$ were written out explicitly. When furthermore writing out the anti-symmetrizations in $(\alpha, \beta)$, we find that several terms cancel (step 5) and additionally used $(\delta m_{I}) m^{I} = -\left( \eta^{I} m_{I} \right)$, $(\delta m_{I}) m^{I} = -\left( \delta m^{I} m_{I} \right)$ and in the sixth step, the upper line is partially integrated and we used $(\delta m_{I}) m^{I} = \frac{1}{2} (h_{I} m_{I}) \delta^{I}_{\rho} = \frac{1}{\sqrt{h}} \delta \sqrt{h}$, and the two summands of the lower line are combined into one term. The seventh step consists of writing out all individual terms appearing in the square brackets explicitly and partially integrating the last term. In step 8, we used

$$\delta \Delta \psi = -(D^{\alpha} D^{\beta} \psi) \delta h_{\rho} + (\Delta \psi) - (D^{\rho} \psi) (D^{\alpha} \delta h_{\rho}) + (D^{\rho} \psi) (D_{\rho} \delta \log \sqrt{h}) \quad \text{(C.21)}$$

and the remaining steps are straightforward.

Combining (C.16), (C.18) and (C.20), we find immediately

$$\epsilon^{IJKL} \frac{r}{h_{\rho} \epsilon^{I}} \left( \delta_{1} A_{IJKL} \right) \left( \delta_{2} A_{IJKL} \right)$$

$$= -2 \left( \frac{E(2)}{\sqrt{h}} \right) \left( \delta_{I} n^{I} \right) \left( \delta_{2} \tilde{\xi}_{2} \right) + 2 \left( \delta_{1} \frac{E(2)}{\sqrt{h}} \right) n^{I} \left( \delta_{2} \tilde{\xi}_{2} \right)$$

$$- 2 \left\{ \left( \eta^{M} \delta_{1} \tilde{\xi}_{M} \right) \left( \frac{E(2)}{\sqrt{h}} \right) + \left( \delta_{M} \tilde{\xi}_{M} \right) \frac{E(2)}{\sqrt{h}} - \left( \frac{E(2)}{\sqrt{h}} \right) \right\}$$

$$= 2 \left( \frac{E(2)}{\sqrt{h}} \right) \left( \delta_{1} \tilde{\xi}_{M} \right) \left( \delta_{2} \tilde{\xi}_{M} \right). \quad \text{(C.22)}$$

### Appendix D. Higher-dimensional Newman–Penrose formalism

In this appendix, we will very briefly introduce the higher-dimensional Newman–Penrose formalism as far as it is needed for the purpose of this paper. First, the Riemann tensor can be decomposed as follows

$$R^{D+1}_{\mu \nu \rho \sigma} = C^{D+1}_{\mu \nu \rho \sigma} + \frac{2}{D-1} \left( R^{D+1}_{(\mu | \nu | \rho | \sigma) - R^{D+1}_{(\mu | \nu | \rho | \sigma)} \right) - \frac{2}{D(D-1)} g^{(\mu | \nu | \rho | \sigma)} R^{D+1}_{(\mu | \nu | \rho | \sigma)}$$

$$= C^{D+1}_{\mu \nu \rho \sigma} + \frac{2}{D-1} \left( \frac{1}{(D+1)} g^{(\mu | \nu | \rho | \sigma) - \frac{2}{D(D+1)} g^{(\mu | \nu | \rho | \sigma)} R^{D+1}_{(\mu | \nu | \rho | \sigma)} \right), \quad \text{(D.1)}$$

where $C^{D+1}_{\mu \nu \rho \sigma}$ denotes the $(D+1)$ Weyl tensor and $j^{D+1}_{\mu \nu} := R^{D+1}_{\mu \nu} - \frac{1}{D+1} g_{\mu \nu} R^{D+1}$ the trace-free Ricci tensor. In a given null frame $\{1, k, [m_{I}]\}$, $\tilde{I} = k^{I} = k \cdot m_{J} = k \cdot m_{I} = 0$, $l \cdot k = -1$, $m_{I} \cdot m_{J} = \tilde{\eta}_{IJ}$, we will use the following notation (cf [60]) for the components of the Weyl tensor

$$\Psi_{0111} := C^{D+1}_{\mu \nu \rho \sigma} k^{\mu} k^{\nu} k^{\rho} k^{\sigma}, \quad \Psi_{0111} := C^{D+1}_{\mu \nu \rho \sigma} k^{\mu} k^{\nu} k^{\rho} m_{I}^{\sigma}, \quad \Psi_{0111} := C^{D+1}_{\mu \nu \rho \sigma} k^{\mu} m_{I}^{\nu} m_{J}^{\rho} m_{K}^{\sigma}, \quad \Psi_{0111} := C^{D+1}_{\mu \nu \rho \sigma} m_{I}^{\mu} m_{J}^{\nu} m_{K}^{\rho} m_{L}^{\sigma}, \quad \Psi_{0111} := C^{D+1}_{\mu \nu \rho \sigma} m_{I}^{\mu} m_{J}^{\nu} m_{K}^{\rho} m_{L}^{\sigma}.$$
We will use analogous notation for the \((D+1)\) Riemann tensor if convenient. From curvature tensor symmetries and tracelessness, the relations
\[
\Psi_{00} \Psi^I \Psi^J = 0, \quad \Psi_{0(IJK)} = \Psi_{(IJK)} = 0, \quad \Psi_{001} = -\Psi_{0(J)}.
\]
\[
\Phi_{00} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu, \quad \Phi_{01} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu, \quad \Phi_{0J} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu.
\]
\[
\Phi_{11} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu, \quad \Phi_{IJ} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu, \quad \Phi_{IJK} = \Phi_{\mu \nu} \Phi^\mu \Phi^\nu.
\]
\begin{equation}
\label{D.3}
\end{equation}
can be derived \cite{60}. For the components of the trace-free Ricci tensor \( J_{\mu \nu} \), we introduce the notation
\[
\Phi_{00} = J_{\mu \nu} J^\mu \Phi^\nu, \quad \Phi_{01} = J_{\mu \nu} J^\mu \Phi^\nu, \quad \Phi_{0J} = J_{\mu \nu} J^\mu \Phi^\nu.
\]
\[
\Phi_{11} = J_{\mu \nu} J^\mu \Phi^\nu, \quad \Phi_{IJ} = J_{\mu \nu} J^\mu \Phi^\nu, \quad \Phi_{IJK} = J_{\mu \nu} J^\mu \Phi^\nu.
\]
\begin{equation}
\label{D.4}
\end{equation}
and, because of tracelessness, it holds that
\[
2\Phi_{01} = \Phi^J.
\]
\begin{equation}
\label{D.5}
\end{equation}

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