Introduction to typed topological spaces *
To the memory of W.W. Comfort

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Abstract

The concept of typed topological space is introduced, for which open sets in a
topology on a finite set will be assigned types (from lattice). The neighborhood
system of a point, the closure and the connectedness can be defined according to
chain of types, which effectively avoids the situation when most singletons are closed
an open. Furthermore, statistics can be used to provide semantics of points with
statistical characteristics.

Keywords: Typed topology, Finite topology, Lattice, Chain of types, Statistics

1 Introduction

Topological spaces in general study properties of open sets under arbitrary union
and finite intersection. The concept of neighborhood, separation axioms and covering
properties help describe the local and global behaviors of a space. Spaces with separation
axioms and covering properties are useful tools to study infinite spaces. A finite space
with \( T_1 \) separation property is automatically discrete, which renders such kind of spaces
uninteresting.

Finite topological spaces have been discussed in many areas such as Psychology
(\[5\]), Social Science, Database (and big data). Because of the nature of being discrete,
there has not had any substantial applications of tools and concepts from General Topology in those areas. However, the concepts of neighborhood, approximation, and limit are still widely used.

Similarly, a database contains several tables, and each table contains a fixed number of columns and arbitrary length of rows. When treating each row as a point and each column as a predicate to define open sets, which will have a topological space. The SQL commands let us to retrieve information by selecting data using union and intersections. The space can easily be discrete.

On the other hand, one may have several different ways to define open sets on a given finite set. For example, in a community with several streets. We can use left-neighbor and right-neighbor on the same street to define open sets, which eventually becomes a discrete topology. We can also use use friendship, relative and other properties to define open sets, which will becomes a discrete topology too. The topology are the same, but there do have something different here.

To facilitate applications of general topological concepts and tools to above mentioned examples, we define a so-called typed topology on finite sets, in which each open set is associated with a type. Set inclusion of open sets will assume orders on types. Neighborhood systems of points and closure operations will be restricted to chains of types. Furthermore, statistical measures can be used to provide semantics to points and pairs of points.

In Section 2, we define the concept of typed topological spaces. In Section 3, we study neighborhood systems of points restricted to chains of types. In Section 4, we study the closure operations and density property restricted to chains of types. In Section 5, we introduce connectedness under chains of types and statistical measures.

Concepts from general topology follow from (2). For a set $P$, the binary relation $\leq$ is called a partial order (1) if for any elements $a, b, c \in P$, the following conditions hold: (1) $a \leq a$; (2) $a \leq b$ and $b \leq a$ imply $a = b$; and (3) $a \leq b$ and $b \leq c$ imply $a \leq c$. 

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The notation $a < b$ is used when $a \leq b$ and $a \neq b$. The pair $(P, \leq)$ is a lattice if for any two elements $a, b \in P$, both the greatest lower bound (the meet) $a \land b$ and least upper bound (the join) $a \lor b$ exist. A lattice is bounded if bottom 0 and top 1 exist. Furthermore, a lattice is distributive if the two operators $\land$ and $\lor$ obey the distribution law. Two elements $p, q \in P$ are called compatible if there exists an element $r \in P$ satisfying $r \leq p$ and $r \leq q$. A proper subset $F \subset P$ is called a filter if (1) $0 \not\in F$ provided 0 exists; (2) for any $p, q \in F$, there exists $r \in F$ satisfying $r \leq p$ and $r \leq q$; and (3) for any $p, q \in P$ with $p \leq q$, $q \in F$ whenever $p \in f$.

In this paper, we assume all topological spaces, partially ordered sets are finite and all lattices are bounded finite distributive lattices.

2 Typed Topology

Let $X$ be a set and let $(P, \leq)$ be a partially ordered set. For each $x \in X$, we define two elements, $x'$s and $\neg x'$s, which means belonging to $x$ and not belonging to $x$ respectively. We denote the set $\{x' : x \in X\}$ by $X'$s and $\{\neg x' : x \in X\}$ by $\neg X'$s. We consider $X'$s $\cup \neg X'$s as be unordered. Furthermore, The set $P \cup X'$s $\cup \neg X'$s is denoted $L_0$. We consider the partially ordered set $(L_0, \leq)$ as an extension of $(P, \leq)$ in the way that $\leq$ restricted to $P$ is $(P, \leq)$, and $\leq$ restricted to $X'$s $\cup \neg X'$s is only $x \leq x$ for any element $x$.

Following ([3]), the free distributive lattice $L(P, X)$ generated by $L_0$ is defined as the family of terms (or words) in $L_0$ in the form of $A_1 \lor A_2 \lor \ldots \lor A_k$, where each $A_i (1 < i \leq k)$ is a meet of elements in $L_0$. Furthermore, $(L(P, X)$ has a partial order that extends $(P, \leq)$, which is defined as follows ([4]). The order relation $A \leq B$ holds for $A, B \in L(P, X)$ when one of the following is true: (1) for some $a, b \in L_0$, $A = a, B = b$ and $a \leq b$; (2) for some $S \subseteq L_0$ and $b \in L_0$ with $A = \land\{a : a \in S\}$ and $B = b, a \leq b$ holds for some $b \in S$; (3) for some $S_1, S_2 \subseteq L_0$ with $A = \land\{a : a \in S_1\}$ and $B = \land\{b : b \in S_2\}$, $A \leq b$ holds for all $b \in S_2$; (4) for $A = A_1 \lor A_2 \lor \ldots \lor A_m$ and $B = \land\{b : b \in S\}$ for some
In particular, we have the free distributive lattice $L(P', X)$ for any subset $P' \subseteq P$. 

Since $(L(P, X), \land, \lor, \leq, 0, 1)$ is a finite distributive lattice, every element can be represented as $M_1 \lor M_2 \lor \ldots \lor M_k$, where each $M_i$ is a meet of elements from $L_0$. 

**Definition 2.1 (Typed Topological Space)** Let $(X, T)$ be a topological space, and let $(P, \leq)$ be a partially ordered set. A function $\sigma : T \to (L(P, X), \land, \lor, \leq, 0, 1)$ is called a type mapping whenever the following conditions hold: (1) $\sigma(U) = 0$ if and only if $U = \emptyset$; (2) $\sigma(U) \neq 1$; and (3) for any two open sets $U, V \in T$, the condition $U \subseteq V$ implies $\sigma(U) \leq \sigma(V)$.

The 5-tuple $(X, T, P, \leq, \sigma)$ is called a typed topological space, and each non-empty open set $U$ is called $\sigma(U)$-typed open set, which is denoted by $\sigma(U) \models U$. The notation $\sigma(U) \models x$ is used when $x \in U$. 

Certainly, both $(T, \cap, \cup, \subseteq)$ and $(L(P, X), \land, \lor, \leq)$ are finite lattices. In general, the type mapping is not a lattice morphism which preserves meet and join.

**Proposition 2.2** Let $(X, T, P, \leq, \sigma)$ be a typed topological space. If $U, V \in T$, then $\sigma(U \cap V) \leq \sigma(U) \land \sigma(V)$, and $\sigma(U) \lor \sigma(V) \leq \sigma(U \cup V)$. \(\square\)

The concept of typed topology can be applied to situations that involves topologies on finite sets. To the memory of my academic advisor and academic father, W.W. Comfort, we first introduce the following example.

**Example 2.3 (Mathematics Genealogy)** As of December 26, 2017, the math genealogy project collects 221844 records of mathematicians in human history. Let $X$ be the set of
all those mathematicians listed there. We will define a typed topology on \( X \). Let \( P \) be the set of two elements \( \{ \text{ancestors, descendants} \} \). For each mathematician \( x \), we have \( x' \)'s and \( \neg x' \)'s. So \( L_0 = \{ \text{ancestors, descendants} \} \cup \{ x', \neg x' : x \in X \} \).

According to the semantics of \( x' \)'s, ancestors and descendants, we can define open sets on \( X \). For instance, the author’s name is W. Hu. The open set \( U_{W.Hu', \text{ancestors}} \) associated with the types of W. Hu’s and ancestors is defined as

\[
U_{W.Hu', \text{ancestors}} = \{ \text{W.W.Comfort(1), E. Hewitt(2), M.H.Stone(3), George, D. Birkhoff(4), ...} \}
\]

In fact, for any direct student \( x \) of W.W. Comfort, we have \( U_{x', \text{ancestors}} = U_{W.Hu', \text{ancestors}} \). Similarly, we define \( U_{W.W.Comfort', \text{ancestors}} \), and \( U_{W.W.Comfort', \text{ancestors}} = U_{W.Hu', \text{ancestors}} \setminus \{ \text{W.W.Comfort(1)} \} \).

Similarly, we let \( U_{W.Hu', \text{descendants}} \) be the set of all descendants of W. Hu. In this case, it is the empty set, while \( U_{W.W.Comfort', \text{descendants}} \) contains 91 names which includes W. Hu. Certainly, \( U_{W.W.Comfort', \text{ancestors}} \subseteq U_{W.Hu', \text{ancestors}} \), and \( U_{W.Hu', \text{descendants}} \subseteq U_{W.W.Comfort', \text{descendants}} \).

The type mapping \( \sigma \) follows from above definition. \( \sigma(U_{W.Hu', \text{ancestors}}) = \text{W.Hu's } \land \text{ancestors } \land \land \{ x': x \text{ is a descendant of W.W.Comfort} \} \), and \( \sigma(U_{W.Hu', \text{descendants}}) = 0 \) since \( U_{W.Hu', \text{descendants}} = \emptyset \). \( \sigma(U_{W.W.Comfort', \text{ancestors}}) = \text{W.W.Comfort's } \land \text{ancestors } \land \land \{ x': x \text{ is a descendant of E.Hewitt} \} \). \( \sigma(U_{W.W.Comfort', \text{descendants}}) = \text{W.W.Comfort's } \land \text{descendants } \land \land \{ x: x \text{ is an ancestor of W.W.Comfort} \} \).

Pierre-Simon Laplace’s advisor is Jean Le Rond, d’Alembert, who does not have ancestors in the database. Hence, \( U_{\text{Pierre-Simon Laplace', ancestors}} = \{ \text{Jean Le Rond, d’Alembert} \} \) is a singleton that is closed and open. However, when we restrict open sets to those of types involving ancestors but not descendants, \( U_{\text{Pierre-Simon Laplace', ancestors}} \) is not closed anymore. In fact, its closure is the set of all 99910 descendants of Jean Le Rond, d’Alembert. The closure of \( U_{\text{Pierre-Simon Laplace', ancestors}} \) using open sets of types involving descendants but not ancestors is itself. When a mathematician, e.g., W.Hu has no descendants (students), the singleton \( \{ W.Hu \} \) is a closed set, since for any other mathematician
\( x \), \( U_{x's, ancestors} \) is disjoint with \( \{W.Hu\} \). Whether it is open depends on other students of W.W.Comfort. When a mathematician \( x \) either has only one advisor and a student, or has an advisor and only one student, the singleton \( \{x\} \) is open and close. For instance, 
\[
U_{W.Hu's, ancestors} \cap U_{E.Hewitt's, descendants} = \{W.W.Comfort\},
\]
is an open set. It is closed, since 
\[
\{W.W.Comfort\} \cap (U_{W.W.Comfort's, ancestors} \cup U_{W.W.Comfort's, descendants}) = \emptyset.
\]
\( \square \)

**Example 2.4** (Community And Neighborhood) In a community, there are 5 streets. Residents on each street have their neighbors. Let \( X \) be the set of all residents in a given community. Let \( P \) be an unordered set that includes 5 street names. Each street name defines an open set, which consists of all residents on that street. In addition, we add two more types into \( P \), i.e., ”left-neighbor” and ”right-neighbor”. For a resident on a street \( s \), we define \( U_{x's,s, left \ neighbor} \) to be the set of all residents on the street \( s \) that are on the left-hand side of \( x \), including \( x \), and \( U_{x's,s, right \ neighbor} \) to be the set of all residents on the street \( s \) that are on the right-hand side of \( x \) including \( x \). The type mapping can be defined naturally. Hence, we have a typed topological space.

Without considering types, one can show that every singleton is closed and open. When using open sets that involving right-neighbor type but not left-neighbor type, the closure of each singleton \( \{x\} \) is the set of all left neighbors of \( x \) on the same street. Similarly, when using open sets that involving left-neighbor type but not right-neighbor type, the closure of the singleton \( \{x\} \) is the set of all right neighbors of \( x \) on the same street. \( \square \)

**Example 2.5** (Database) A relational database uses the so-called relational model to store data. In that model, data is represented as entity and relationships. Each table defines one entity type, of which instances (or records) of that entity type (such as customer, product) are organized as rows. Each instance is further divided into attributes, i.e., columns (fields). Tables are connected by relationships that are defined through some columns such as names shared by both tables. For instance, a table of the entity type customer and a table of the entity type office visit can be related by the same last name
and first name.

Let \( X \) be the set of all instances (rows) from all tables in a relational database. Since each table represents an entity type, we will add the table name to the partial order set \( P \). Further, each column in a table describes an attribute and each attribute is assigned a domain, which together form a type. For instance, an amount column \( fee \) may form predicates such as \( fee < 1000 \) or \( fee \geq 200 \). Similarly, a column of product name \( product \) may form predicate such as \( \text{"product likes S*"} \), which means the product name starts with \( \text{"S"} \). We will include predicates of each attribute as a subset of \( P \). The partial order on \( P \) is based on the partial order on predicates.

When using Structured Query Language (SQL) statements to operate against a database, one can use any of the eight operators, i.e., union, intersection, difference, Cartesian product, selection, projection, join, and division. The selection operator lets us define open sets according to predicates based on attributes. Typically, we use a statement such as

"SELECT *
FROM mytable
WHERE columnx > 100"

to define a subset of rows in a table. The predicate \( \text{"columnx > 100"} \) is a type and the result set of rows from the table \( \text{"mytable"} \) is an open set associated with that type. Basic predicates may use operators such as \( =, \neq, >, \geq, <, \leq, IN, BETWEEN, LIKE, IS NULL, IS NOT NULL \). Compound predicates may connect basic ones by \( \text{"AND, OR"} \) and parenthesis.

The definition of open sets automatically defines the type mapping. Hence, we have a typed topological space. Certainly, we can write an SQL statement that selects any given row in a table. Hence, each singleton is an open set and therefore a closed and open set. However, when restricted to a set of predicates, the selected set of rows may not be a singleton. \( \Box \)
For any typed topological space \((X, \mathcal{T}, P, \leq, \sigma)\), since for any non-empty open set \(U \in \mathcal{T}\), \(\sigma(U) \neq 0\), we have the following proposition.

**Proposition 2.6** Let \((X, \mathcal{T}, P, \leq, \sigma)\) be a typed topological space. For any \(x \in U \in \mathcal{T}\), if \(p, q \in L(P, X)\) are incompatible, then we cannot have both \(p \vdash x\) and \(q \vdash x\) at the same time. In particular, we cannot have both \(p \vdash x\) and \(\neg p \vdash x\) at the same time.

**Proof:** Let everything be as above. Assume for a contradiction, there are open sets \(W, V \in \mathcal{T}\) such that \(\sigma(W) = p\), \(\sigma(V) = q\) and \(x \in W \cap V\). Then by Proposition 2.2, we have \(\sigma(W \cap V) \leq \sigma(W) \land \sigma(V) = p \land q\). Since \(p\) and \(q\) are incompatible, we have \(p \land q = 0\). Hence \(\sigma(W \cap V) = 0\) and \(W \cap V = \emptyset\), which is however a contradiction with \(x \in W \cap V\). \(\Box\)

In a finite lattice \(L\), any filter \(F\) has the smallest element \(\bigwedge F\) satisfying \(\bigwedge F \leq p\) for any \(p \in F\). In general, for any other element \(q \in L\), we may not have either \(q \in F\) or \(\neg q \in F\) (or equivalently, either \(\bigwedge F \leq q\) or \(\bigwedge F \leq \neg q\)). When \(p\) is join-prime (i.e., \(p \leq q \lor r \rightarrow (p \leq q) \lor (p \leq r)\)), we have for any \(q \in L\), either \(p \leq q\) or \(p \leq \neg q\) holds. As a known fact, in a distributive lattice, join-prime is equivalent to join-irreducible, which is further equivalent to being atomic.

**Lemma 2.7** Let \((L, \land, \lor, \leq, 0, 1)\) be a finite distributive lattice. For any join-irreducible element \(p \in L\), the family \(\{q \in L : p \leq q\}\) is an ultrafilter containing \(p\). For any ultrafilter \(F\) in \(L\), \(\bigwedge F\) is a join-irreducible element.

**Proof:** When \(p\) is join-irreducible, or equivalently join-prime, the family \(F = \{q : p \leq q\}\) is a filter, since \(p \leq q \land r\) for any two elements \(q, r \in F\). It is also an ultrafilter, since for any \(q \in L\), \(p \leq q \lor \neg q = 1\), which implies either \(q \in F\) or \(\neg q \in F\).

When \(F\) is an ultrafilter in \(L\), for any \(q \in L\), we have either \(q \in F\) or \(\neg q \in F\), which implies \(\bigwedge F \leq q\) or \(\bigwedge F \leq \neg q\) respectively. Hence \(\bigwedge F\) is a join-irreducible (or join-prime) element. \(\Box\)
When a filter $F$ is not an ultrafilter, we may have element $p \in P$ satisfying neither $p \in F$ nor $\neg p \in F$. The following proposition is easy to verify.

**Lemma 2.8** Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a typed topological space. For any $U \in \mathcal{T}$, if $\sigma(U)$ is join-irreducible, then for any $p \in L(P, X)$ either $\sigma(U) \leq p$ or $\sigma(U) \leq \neg p$. Furthermore, the pair $\{p, \neg p\}$ can be replaced by any maximum antichain in $(L(P, X), \leq)$. $\blacksquare$

### 3 Typed neighborhood systems

As we mentioned above, most singletons in a finite typed topological space are closed and open. However, when limiting the open sets to certain types, they may not be that case anymore. In applications such as a database, we usually want to find a set of points (records) by a minimum set of types. Let us investigate the neighborhood systems of a point in $(X, \mathcal{T})$.

Recall, in a finite distributive lattice $(L, \land, \lor)$ with bottom and top, an element $p \in L$ is called join-irreducible if $p \neq 0$ and whenever $p = q \lor r$ one must have either $p = q$ or $p = r$, and meet-irreducible if $p \neq 1$ and whenever $p = q \land r$ one must have either $p = q$ or $p = r$. As a known fact, every element in such type of lattice is a meet of meet-irreducible elements or a join of join-irreducible elements.

Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a typed topological space. Then, $(\mathcal{T}, \cap, \cup)$ is a finite distributive lattice. The family of all join-irreducible open subsets of $X$ is a base of $\mathcal{T}$.

**Definition 3.1** Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a typed topological space. An open set $U \in \mathcal{T}$ is called $p$-join-irreducible for $0 \neq p \in L(P, X)$ if $U \neq \emptyset$, $p \leq \sigma(U)$ and whenever $U = W \cup V$ for some $U, V \in \mathcal{T}$ satisfying $p \leq \sigma(W)$ and $p \leq \sigma(V)$, one must have either $U = W$ or $U = V$. Similarly, we can define the concept of being $p$-meet-irreducible.

Certainly, if an open set $U$ satisfying $p \leq \sigma(U)$ is $p$-join-irreducible then it is also $q$-join-irreducible for any $q \in L(P, X)$ with $p \leq q$. Furthermore, if an open set $U$ is
join-irreducible, then it is $p$-join-irreducible for all $p \in L(P, X)$. A $p$-join irreducible open set may not be join-irreducible.

**Example 3.2** In the Example 2.3 of Mathematics Genealogy, $\sigma(U_{W.Hu's, ancestors})$ is equal to $W.Hu's \land ancestors \land \{x's: x is a descendant of W.W.Comfort\}$. That open set is the only one whose type is $W.Hu's \land ancestors \land \{x's: x is a descendant of W.W.Comfort\}$. Let $p_0 = \sigma(U_{W.Hu's, ancestors})$. If $q \in L(P, X)$ and $p_0 \preceq q$, then either $q$ is a meet of some elements from $\{W.Hu's, ancestors\} \cup \{x's: x is a descendant of W.W.Comfort\}$ or $q$ is a join of $q'$ and some $r$ not in $\{W.Hu's, ancestors\} \cup \{x's: x is a descendant of W.W.Comfort\}$, where $q'$ is a meet of some element from $\{W.Hu's, ancestors\} \cup \{x's: x is a descendant of W.W.Comfort\}$. According to Proposition 2.2 there is no open set $W$ satisfying $W \subseteq U$ and $\sigma(W) = q$. Hence, $U_{W.Hu's, ancestors}$ is $p_0$-join-irreducible. Similarly, $U_{E.Hewitt's, descendants}$ is $p_1$-join-irreducible, where $p_1 = \sigma(U_{E.Hewitt's, descendants})$.

However, if we set $p = p_0 \land p_1 \land \sigma(U_{W.W.Comfort's, ancestors})$, then $U_{W.Hu's, ancestors}$ is not $p$-join-irreducible, since $U_{W.Hu's, ancestors}$ is the disjoint union of two open sets $\{W.W.Comfort\}$ and $U_{W.W.Comfort's, ancestors}$, where both $p \preceq \sigma(\{W.W.Comfort\})$ and $p \preceq \sigma(U_{W.W.Comfort's, ancestors})$ are true.

On the other hand, the singleton $\{W.W.Comfort\}$ is an open set. Certainly, it is $p$-join-irreducible for any $p \in L(P, X)$. □

The type mapping in a typed topological space does not guarantee that $\sigma(U) \preceq \sigma(V)$ when $U$ is a proper subset of $V$.

**Definition 3.3** Let $(X, \mathcal{T}, P, \preceq, \sigma)$ be a typed topological space. When $\sigma(U) \preceq \sigma(V)$ holds for any two non-empty open sets $U$ and $V$ with $U \subset V$, the space is called strictly typed.

The spaces in Example 2.3, 2.4 and 2.5 are strictly typed topological spaces. In the rest of this article, we will assume all spaces are strictly typed.

**Lemma 3.4** Let $(X, \mathcal{T}, P, \preceq, \sigma)$ be a strictly typed topological space. For any non-empty open set $U \in \mathcal{T}$, the set $U$ is $\sigma(U)$-join-irreducible.
Proof: For any two open sets \( V, W \in \mathcal{T} \) satisfying \( U = W \cup V \), if both \( W \) and \( V \) are proper subsets of \( U \), then by the assumption of being strictly typed, we have both \( \sigma(V) \preceq \sigma(U) \) and \( \sigma(W) \preceq \sigma(U) \). Hence \( U \) cannot be the union of two non-empty proper open sets \( W, V \) satisfying \( \sigma(U) \leq \sigma(W) \) and \( \sigma(U) \leq \sigma(V) \), i.e., \( U \) is \( \sigma(U) \)-join-irreducible. \( \square \).

**Definition 3.5** Let \((X, \mathcal{T}, P, \preceq, \sigma)\) be a strictly typed topological space. For any \( p \in L(P, X) \) set \( \mathcal{T}_{\geq p}(X) = \{ U \in \mathcal{T} : p \leq \sigma(U) \} \), and \( J_{\geq p}(X) = \{ U \in \mathcal{T}_{\geq p}(X) : U \text{ is } p\text{-join-irreducible} \} \). Furthermore, for any \( x \in X \), set \( \mathcal{T}_{\geq p}(x) = \{ U \in \mathcal{T}_{\geq p}(X) : x \in U \} \) and \( J_{\geq p}(x) = \{ U \in J_{\geq p}(X) : x \in U \} \).

**Theorem 3.6** Let \((X, \mathcal{T}, P, \preceq, \sigma)\) be a strictly typed topological space. For any \( p \in L(P, X) \), every open set \( U \in \mathcal{T}_{\geq p}(X) \) is a join of some elements from \( J_{\geq p}(X) \).

**Proof:** Since \( L(P, X) \) and \( \mathcal{T} \) are finite, we can prove it by induction on open sets. We first consider three base cases. Case 1: for any open set \( U \in \mathcal{T}_{\geq p}(X) \) satisfying \( \sigma(U) = p \), by Lemma 3.4 \( U \) is \( p \)-join-irreducible. Hence \( U \in J_{\geq p}(X) \). Case 2: for any \( U \in \mathcal{T}_{\geq p}(X) \), if there exists no open set \( V \in \mathcal{T}_{\geq p}(X) \) satisfying \( V \subsetneq U \), then \( U \) is \( p \)-join-irreducible, i.e., \( U \in J_{\geq p}(X) \). Case 3: for any \( U \in \mathcal{T}_{\geq p}(X) \), if there do not exist two distinct open sets \( W, V \in \mathcal{T}_{\geq p}(X) \) satisfying \( U = W \cup V \) and \( (W \neq U) \land (V \neq U) \), then \( U \) is \( p \)-join-irreducible, i.e., \( U \in J_{\geq p}(X) \). In all three base cases, \( U \) is an element in \( J_{\geq p}(X) \).

At the induction step, let us assume that for a given open set \( U \in \mathcal{T}_{\geq p}(X) \) satisfying \( p \leq \sigma(U) \), any \( V \in \mathcal{T}_{\geq p}(X) \) with \( V \subsetneq U \) and \( p \leq \sigma(V) \leq \sigma(U) \) is a join of \( p \)-join-irreducible open sets. We will show that \( U \) is also a join of \( p \)-join-irreducible open sets. If there are no \( W, W' \in \mathcal{T}_{\geq p}(X) \) satisfying \( U = W \cup W' \) and \( p \leq \sigma(W) \land \sigma(W') \), then \( U \in J_{\geq p}(X) \). Otherwise, let \( U = W \cup W' \) with \( p \leq \sigma(W) \land \sigma(W') \). By assumption, both \( W \) and \( W' \) are joins of \( p \)-join-irreducible open sets. Therefore, \( U \) is also the join of \( p \)-join-irreducible open sets. \( \square \)

Neighborhoods of a point can be organized by chains of types, a way to sandwich
open sets. By sandwiching open sets’ types, we are able to limit those types from $P$ in the operation.

**Definition 3.7** Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. For $k > 1$, let $c = "p_0 \leq p_1 \leq \ldots \leq p_{k-1}"$ be a chain of types in $L(P, X)$. For any point $x \in X$, the $c$–neighborhood system is defined as the family $\mathcal{T}_c(x) = \{ U \in \mathcal{T} : (x \in U) \land \exists i(0 \leq i < k-1)p_i \leq \sigma(U) \leq p_{i+1}\}$. Furthermore, we define $\mathcal{J}_c(x) = \{ U \in \mathcal{T}_c(x) : U \in J_{\geq p_i}(x) \text{ for some } 0 \leq i < k-1\}$. We also set $\mathcal{T}_c(X) = \bigcup\{ \mathcal{T}_c(x) : x \in X\}$, and $\mathcal{J}_c(X) = \bigcup\{ \mathcal{J}_c(x) : x \in X\}$.

**Theorem 3.8** Let everything be as in Definition 3.7. Then $\mathcal{J}_c(x)$ is a neighborhood base of the $c$-neighborhood system $\mathcal{T}_c(x)$.

**Proof:** For any $U \in \mathcal{T}_c(x)$, we need to show that there exists $V \in \mathcal{J}_c(x)$ such that $x \in V \subseteq U$. For this $U$, there exists $i$, $0 \leq i < k-1$, such that $p_i \leq \sigma(U) \leq p_{i+1}$. Hence $U \in \mathcal{T}_{\geq p_i}(X)$. By Theorem 3.6 $U$ is the join of elements from $J_{\geq p_i}(X)$, say $U = V_1 \cup \ldots \cup V_m$ with $V_k \in J_{\geq p_i}(X)$ for each $k \leq m$. Since $x \in U$, we have $x \in V_j$ for some $j \leq m$. Hence $V_j \in J_{\geq p_i}(x)$. To complete the proof, we will show that $V_j \in \mathcal{T}_c(x)$, for which it suffices to show that $V_k \in \mathcal{T}_c(x)$ for all $k \leq m$. Since $p_i \leq \sigma(U) \leq p_{i+1}$, we have $\sigma(V_k) \leq p_{i+1}$ for all $k \leq m$. On the other hand, we have $p_i \leq \sigma(V_k)$ since $V_k \in J_{\geq p_i}(X)$. Hence $p_i \leq \sigma(V_k) \leq p_{i+1}$, i.e., $V_k \in \mathcal{T}_c(x)$. \[\Box\]

While $\mathcal{T}_{\geq p}(X)$ is the family of open sets whose types are above $p$, i.e., the family of open sets of upper types, we can also define the family of open sets whose types are below $p$. However, that family allows any kind of intersections with types including $p$, which makes it less interesting since more closed and open singletons are included. The following definition provides a modified version of lower types than $p$.

**Definition 3.9** Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. Let $p \in P$ be a type. A chain of types $c = "p_0 \leq p_1 \leq \ldots \leq p_{k-1}"$ is called a $p$-chain, if each $p_i \in L(\{p\}, X)$
and $p_{k-1} = p$. For $x \in X$, we define the $p$-chain-neighborhood system as $T_{p\text{-chain}}(x) = \bigcup \{T_c(x) : c \text{ is a p-chain } \}$. Further, we set $T_{p\text{-chain}}(X) = \bigcup \{T_{p\text{-chain}}(x) : x \in X \}$

In the Example 2.3 let $p$ be the type ancestors. Then the family $T_{p\text{-chain}}(X)$ is family of all open sets of the form $U_{x',s,ancestors}$.

Proposition 3.10 Let everything be as in Definition 3.9. The $p$-chain-neighborhood system $T_{p\text{-chain}}(x)$ is equal to the family $\{U \in T : x \in U \land \sigma(U) \in L(\{p\}, X) \land \sigma(U) \leq p \}$.

Proof: For any $q \in L(\{p\}, X)$ and $q \leq p$, we can form a chain $q \leq p$, which is a $p$-chain. If an open set $U \in \{U \in T : x \in U \land \sigma(U) \in L(\{p\}, X) \land \sigma(U) \leq p \}$, then $U \in T_c(x)$, where $c = "\sigma(U) \leq p"$. Hence $U \in T_{p\text{-chain}}(x)$. The other direction of inclusion is obvious from the definition of $p$-chain. $\blacksquare$

Theorem 3.11 Let everything be as in Definition 3.9. For any $x \in X$, every open set $U \in T_{p\text{-chain}}(x)$ is an element in $J_c(x)$ for some $p$-chain $c$. Hence, $T_{p\text{-chain}}(x) = \bigcup \{J_c(x) : c \text{ is a p-chain } \}$.

Proof: Set $p_0 = \sigma(U)$, and let $c = "p_0 \leq p"$. Then, by Lemma 3.4 $U$ is $p_0$-join-irreducible. Hence $U \in J_c(x)$. $\blacksquare$

For a finite distributive lattice $L$, it is a known fact that it can be partitioned into $w(L)$-many chains, where $w(L)$ is the largest size of maximum antichain. Hence, there exists a family of $w(L)$-many chains $\{c_i : i < w(L)\}$ that covers $L$, i.e., $L = \bigcup \{c_i : i < w(L)\}$.

Theorem 3.12 Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. Let $\{c_i : i < w(L(P, X))\}$ be a family of chains that covers $L(P, X)$. Then, for any $x \in X$, $\bigcup \{T_{c_i}(x) : i < w(L(P, X))\}$ is the neighborhood system of $x$, and $\bigcup \{J_{c_i}(x) : i < w(L(P, X))\}$ is a neighborhood base.

Proof: Since $\{c_i : i < w(L(P, X))\}$ covers $L(P, X)$, for any $U \in \mathcal{T}$ satisfying $x \in U$, there exists $i < w(L(P, X))$ such that $\sigma(U) \in c_i$. Then $U \in T_{c_i}(x)$. Hence
∪{\mathcal{T}_c(x) : i < w(L(P, X))} is the neighborhood system of x. By Theorem 3.8, \mathcal{J}_c(x) is a neighborhood base of \mathcal{T}_c. Hence, there exists V \in \mathcal{J}_c(x) satisfying x \in V \subseteq U. Hence ∪{\mathcal{J}_c(x) : i < w(L(P, X))} is a neighborhood base of x. □

4 Typed Closure

As we discussed before, closure in a typed topological space will be more meaningful when it is restricted to types.

Definition 4.1 Let (X, \mathcal{T}, P, ≤, σ) be a strictly typed topological space. For any subset A ⊆ X and any chain c of types, the c-closure of A, denoted \overline{A}^c, is the set \{x ∈ X : (∀U ∈ \mathcal{J}_c(x))U ∩ A ≠ ∅\}.

Since \mathcal{J}_c(x) is finite, the following lemma is straightforward.

Lemma 4.2 Let everything be as in Definition 4.1. For any x ∈ X with \mathcal{J}_c(x) ≠ ∅, x ∈ \overline{A}^c if and only if \bigcap \mathcal{J}_c(x) ∩ A ≠ ∅. □

If x ∈ \overline{A}^c and \mathcal{J}_c(x) ≠ ∅, then \bigcap \mathcal{J}_c(x) ∩ A ≠ ∅. Let y ∈ \bigcap \mathcal{J}_c(x) ∩ A. Then \mathcal{J}_c(x) ⊆ \mathcal{J}_c(y). We have following theorem.

Theorem 4.3 Let everything be as in Definition 4.1. For any two distinct points x, y ∈ X satisfying \mathcal{J}_c(X) ≠ ∅ and \mathcal{J}_c(y) ≠ ∅, the following conditions are equivalent

1. \mathcal{J}_c(x) ⊆ \mathcal{J}_c(y);

2. x ∈ \overline{A}^c holds for all subsets A ⊆ X with y ∈ A; and

3. x ∈ \overline{\{y\}}^c.

Proof: (1)→(2). When \mathcal{J}_c(x) ⊆ \mathcal{J}_c(y), we have y ∈ \bigcap \mathcal{J}_c(x), which implies \bigcap \mathcal{J}_c(x) ∩ A ≠ ∅. Hence by Lemma 4.2 (2) holds.

(2)→(3). Trivial.
By Lemma 4.2 again, \( \bigcap J_c(x) \cap \{y\} \neq \emptyset \), which means \( y \in \bigcap J_c(x) \).
Hence (1) holds. \( \square \)

**Definition 4.4** Let everything be as in Definition 4.1. For any chain \( c \) of types, set \( E_c(X) = \{x \in X : J_c(x) = \emptyset\} \).

We have following result.

**Lemma 4.5** \( E_c(X) = X \setminus \bigcup \{ \bigcup J_c(x) : x \notin E_c(X) \} \), and \( E_c(X) \) is a \( c \)-closed subset of \((X, T)\).

**Proof:** By definition, for any \( x \notin E_c(X) \), and any \( U \in J_c(x) \), we have \( U \cap E_c(X) = \emptyset \). Hence \( \bigcup J_c(x) \cap E_c(X) = \emptyset \). \( \square \)

**Definition 4.6** Let everything be as above. For any subset \( Y \subseteq X \), a set \( D \subseteq Y \) is called \( c \)-dense in \( Y \) if for any point \( x \in Y \) either \( x \in D \) whenever \( x \in E_c(X) \) or \( U \cap D \neq \emptyset \) holds for all \( U \in J_c(x) \).

**Lemma 4.7** Let everything be as above. A set \( D \) is \( c \)-dense in \( Y \) if \( E_c(X) \cap Y \subseteq D \) and for any point \( x \in Y \setminus E_c(X) \), there exists \( y \in D \) such that \( J_c(x) \subseteq J_c(y) \).

**Proof:** Certainly, \( E_c(X) \cap Y \subseteq D \) must be true for \( D \) to be \( c \)-dense in \( A \). For \( x \in Y \setminus E_c(X) \), if \( Y \cap \bigcap J_c(x) = \{x\} \), then \( x \in D \) must be true; otherwise \( \bigcap J_c(x) \cap D = \emptyset \), which contradicts with the assumption that \( D \) is \( c \)-dense in \( Y \).

When \( Y \cap \bigcap J_c(x) \) contains elements other than \( x \), we have either \( x \in D \) or there exists \( y \in \bigcap J_c(x) \cap D \). In the first case, our choice of \( y \) will be \( x \) itself, which satisfying \( J_c(x) \subseteq J_c(y) \). In the second case, we also have \( J_c(x) \subseteq J_c(y) \), since \( y \in \bigcap J_c(x) \). \( \square \)

Set \( J_c = \{ J_c(x) : x \notin E_c(X) \} \). We can order the family \( J_c \) by \( \subseteq \), which becomes a partially ordered set.

It may happen that \( J_c(x) = J_c(y) \) for \( x \neq y \in X \). We can define an equivalent relation \( \equiv_c \) on \( X \setminus E_c(X) \) as \( x \equiv_c y \) whenever \( J_c(x) = J_c(y) \).
Lemma 4.8 Let everything be as above. For any $x \in X \setminus E_c(X)$, if $\mathcal{J}_c(x)$ is a maximal element in the partially ordered set $(\mathcal{J}_c, \subseteq)$, then for any $A \subseteq X$, $x \in \overline{A}$ if and only if there exists a $y \equiv_c x$ satisfying $y \in A$.

Proof: By Lemma 4.7, $x \in \overline{A}$ if and only if there exists $y \in A$ satisfying $\mathcal{J}_c(x) \subseteq \mathcal{J}_c(y)$. Since $\mathcal{J}_c(x)$ is a maximal element in the partially ordered set $(\mathcal{J}_c, \subseteq)$, we have $\mathcal{J}_c(x) = \mathcal{J}_c(y)$. Hence $x \equiv_c y$ holds.

When there exists a $y \equiv_c x$ satisfying $y \in A$, we have $\mathcal{J}_c(x) = \mathcal{J}_c(y)$ and $y \in \bigcap \mathcal{J}_c(x) \cap A \neq \emptyset$, which implies $x \in \overline{A}$. $\square$

Definition 4.9 The $c$-density of $(X, \mathcal{T})$, denoted $d_c(X, \mathcal{T})$, is the smallest size of $c$-dense subset of $(X, \mathcal{T})$.

Theorem 4.10 Let everything be as above. Then $d_c(X, \mathcal{T})$ is equal to the sum of $|E_c(X)|$ and the number of maximal elements in $(\mathcal{J}_c, \subseteq)$.

Proof: Let $D$ be a $c$-dense subset of $(X, \mathcal{T})$. Then $E_c(X) \subseteq D$. By Lemma 4.7, for any $x \in X \setminus E_c(X)$ with $\mathcal{J}_c(x)$ being a maximal element in $(\mathcal{J}_c, \subseteq)$, since $x \in X = \overline{D}$, there exists a $y \equiv_c x$ satisfying $y \in D$. Hence $|D|$ is greater than and equal to the sum of $|E_c(X)|$ and the number of maximal elements in $(\mathcal{J}_c, \subseteq)$.

If there exists $x \in D \setminus E_c(X)$ such that $\mathcal{J}_c(x)$ is not a maximal element in $(\mathcal{J}_c, \subseteq)$, then there exists $y \in D \setminus E_c(X)$ such that $\mathcal{J}_c(y)$ is maximal in $(\mathcal{J}_c, \subseteq)$ and $\mathcal{J}_c(x) \subseteq \mathcal{J}_c(y)$, which implies that $D \setminus \{x\}$ is also $c$-dense in $(X, \mathcal{T})$. However, that contradicts with the definition of $d_c(X, \mathcal{T})$ being the smallest size of $c$-dense subsets. Hence, every $x \in D \setminus E_c(X)$ satisfies that $\mathcal{J}_c(x)$ is a maximal element in $(\mathcal{J}_c, \subseteq)$, and the conclusion of the theorem holds. $\square$

Corollary 4.11 Let everything be as above. The $c$-dense subset of $(X, \mathcal{T})$ of size $d_c(X, \mathcal{T})$ is unique up to the equivalent relation $\equiv_c$. 

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Proof: For each $x$ with $J_c(x)$ being a maximal element in $(J_c, \subseteq)$, we pick one point in its equivalent classes of $\equiv_c$, and form a set $D'$. Then, by Theorem 4.10, the smallest size $c$-dense subset $D$ is the union of $E_c(X)$ and $D'$. □

5 Typed connected sets, connections and statistics

**Definition 5.1** Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. For any chain of types $c$, a subset $A \subseteq X$ is called $c$-connected if there do not exist two disjoint open sets $U, V \in \mathcal{T}_c(X)$ satisfying $A = A \cap (U \cup V)$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.

**Theorem 5.2** Let everything be as a Definition 5.1. For any $U \in J_c(X)$, if $U$ is $p_0$-join-irreducible, then $U$ is $c$-connected, where $c = "p_0 \leq p_1 \leq ... \leq p_{k-1}"$.

Proof: By definition, if $U$ is $p_0$-join-irreducible, then there do not exist any two disjoint non-empty open sets $W, V \in \mathcal{T}_{\geq p_0}(X)$ satisfying $U = W \cup V$. Since $\mathcal{T}_c(X) \subseteq \mathcal{T}_{\geq p_0}(X)$, we conclude that $U$ is $c$-connected. □

**Theorem 5.3** Let $c = "p_0 \leq p_1 \leq ... \leq p_{k-1}"$ be a chain of types. Then every open set $U \in J_{\geq p_0}(X)$ is $c$-connected.

Proof: By Definition 5.1, $U \in J_{\geq p_0}(X)$ if $U$ is $p_0$-join-irreducible. Hence, there do not exist two disjoint non-empty open sets $W, V \in \mathcal{T}$ satisfying $p_0 \leq \sigma(W) \land \sigma(V)$ and $U = W \cup V$, which implies that there do not exist two disjoint non-empty open sets $W, V \in \mathcal{T}_c(X)$ satisfying $U = U \cap (W \cup V)$, $U \cap W \neq \emptyset$ and $U \cap V \neq \emptyset$. Hence $U$ is $c$-connected. □

**Theorem 5.4** Let $p \in P$ be a type. For any $U \in \mathcal{T}_{p-chain}(X)$, there exists a $p$-chain $c$ such that $U$ is $c$-connected.

Proof: By Theorem 3.11, every $U \in \mathcal{T}_{p-chain}(X)$ is an element in $J_c(X)$ for the $p$-chain $c = "\sigma(U) \leq p"$. Hence $U$ is $\sigma(U)$-join-irreducible, i.e., $U \in J_{\geq \sigma(U)}(X)$. By Theorem 5.3, $U$ is $c$-connected. □
Definition 5.5  Let everything be as above. Two elements \( x, y \in X \) are called \( c \)-connected if there exists a \( c \)-connected subset \( A \) satisfying \( x, y \in A \). The set \( A \) is called a connection between \( x \) and \( y \).

Example 5.6 (Community and Neighborhood Revisit) In Example 2.4, we defined a typed topological space of community and neighborhood. In this example, we revisit it. In addition to the types such as street names, and left-neighbor and right-neighbor, we add the type "classmate". For a resident \( x \) on a street \( s \), the open set \( U_{x',s,s,\text{left-neighbor}} \) and \( U_{x',s,s,\text{right-neighbor}} \) are the same. For the type of classmate, we define \( U_{x',s,\text{classmate}} \) to be the set of all residents in the community, including \( x \), who are classmates of \( x \) in any grades.

If two elements \( x \) and \( y \) are classmates, then let \( c \) be the chain "\( x's \wedge \text{classmate} \leq \text{classmate} \)". Then, \( O = U_{x',s,\text{classmate}} \) is a \( c \)-connected subset, and \( x \) and \( y \) are \( c \)-connected by \( O = U_{x',s,\text{classmate}} \). \( \square \)

Since our spaces are finite, statistics can play a key role in studying individual points. For instance, we can measure the neighborhood system of each individual point. We can also measure the relationship between two points.

Let \( c \) be a chain of types, Then, the set \( \{ |A| : A \subset X \text{ is a } c \text{-connected subset} \} \) is a finite set of non-negative integers. According to basic statistics, it has mean and standard deviation. In Example 5.6, for any point \( x \) and the chain \( c = "x's \wedge \text{classmate} \leq \text{classmate}" \), there are few (actually 1) sets that are \( c \)-connected. That may not be enough for statistical calculation. We can include more sets in to the calculation.

Definition 5.7  Let \((X, \mathcal{T}, P, \leq, \sigma)\) be a typed topological space. Let \( p \in P \) be a type. Then, the sample mean \( \overline{x}_p \) and sample standard deviation \( s_p \) of \( p \) is defined as the mean and standard deviation of the set \( \{ |A| : A \in \mathcal{T}_p-\text{chain}(X) \} \). Furthermore, for any member \( A \) in \( \mathcal{T}_p-\text{chain} \), its z-score (or \( p \)-standard score) is the z-score of \( |A| \) with respect to \( \overline{x}_p \) and \( s_p \).
Example 5.8 In Example 5.6 let $p$ be the type "classmate". The family $\mathcal{T}_{p\text{-chain}}(X)$ is the family that contains $c$-connected subsets with $c = "x's\text{ classmate} \leq \text{classmate}"$ for any $x \in X$. Hence, it is the family that contains subsets of the form $\{y \in X: y \text{ is one of } x's\text{ classmates in the community}\}$ for any resident $x$. In that case, $\bar{x}_p$ represents the sample mean of number of classmates residents has in the community and $s_p$ represents the corresponding standard deviation. Furthermore, for any resident $x$, the corresponding $z_p$-score of $U_{x's,\text{classmate}}$ describes how many classmates of $x$ in the community, comparing to others in the community. \qed

Individual points in a typed topological space can be evaluated by related statistics.

Definition 5.9 Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. Given a type $p \in P$, for any $x \in X$, set $L_x = \{p \in L(P, X) : p \models x\}$. Then the sample mean $\bar{x}_x$ and sample standard deviation $s_x$ of the point $x$ are defined as the mean and standard deviation of the set $\{|L_x| : x \in X\}$. Furthermore, the $z_L$-score of $x$ is the $z$-score of $|L_x|$ with respect to $\bar{x}_x$ and $s_x$.

Semantically, points with higher $z_L$-scores indicate more active in the space. In the Example 2.3 of Mathematics Genealogy, mathematicians who have most descendants have much higher $z_L$-scores. Similarly, in the Example 2.4 of Community and Neighborhood, residents living on two ends of a street have higher $z_L$-scores for an appropriate type.

The relationship between two points can be measured statistically too.

Definition 5.10 Let $(X, \mathcal{T}, P, \leq, \sigma)$ be a strictly typed topological space. For any two disjoint points $x, y$, we define $L_{x,y} = \{p \in L(P, X) : p \models x \text{ and } p \models y\}$. Then the sample mean $\bar{x}_{x,y}$ and sample standard deviation $s_{x,y}$ of the pair $\{x, y\}$ are defined as the mean and standard deviation of the set $\{|L_{x,y}| : x, y \in X\}$. Furthermore, the $z_L$-score of $\{x, y\}$ is the $z$-score of $|L_{x,y}|$ with respect to $\bar{x}_{x,y}$ and $s_{x,y}$.

Pairs with higher $z_L$-scores indicate more interactions between them. In the Example 2.4 of Community and Neighborhood, if we add more types such as "party together",...
"on a team of sports", "sleep over", etc., a pair of two points who has higher $z_L$-score is closer than other pairs. The semantics comes in naturally!

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