Operators on Lie $\infty$-algebras with respect to Lie $\infty$-actions

Raquel Caseiro and Joana Nunes da Costa
University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal

ABSTRACT
We define $\mathcal{O}$-operators on a Lie $\infty$-algebra $E$ with respect to an action of $E$ on another Lie $\infty$-algebra and we characterize them as Maurer-Cartan elements of a certain Lie $\infty$-algebra obtained by Voronov’s higher derived brackets construction. The Lie $\infty$-algebra that controls the deformation of $\mathcal{O}$-operators with respect to a fixed action is determined.

ARTICLE HISTORY
Received 13 October 2021
Communicated by Alberto Elduque

KEYWORDS
Lie $\infty$-algebra; $\mathcal{O}$-operator; Maurer Cartan element

2020 MATHEMATICS SUBJECT CLASSIFICATION
17B10; 17B40; 17B70; 55P43

0. Introduction
The first instance of Rota–Baxter operator appeared in the context of associative algebras in 1960, in a paper by Baxter [1], as a tool to study fluctuation theory in probability. Since then, these operators were widely used in many branches of mathematics and mathematical physics.

Almost forty years later, Kupershmidt [4] introduced $\mathcal{O}$-operators on Lie algebras as a kind of generalization of classical $r$-matrices, thus opening a broad application of $\mathcal{O}$-operators to integrable systems. Given a Lie algebra $(E,[\cdot,\cdot])$ and a representation $\Phi$ of $E$ on a vector space $V$, an $\mathcal{O}$-operator on $E$ with respect to $\Phi$ is a linear map $T: V \to E$ such that $[T(x),T(y)] = T(\Phi(T(x))(y) - \Phi(T(y))(x))$. When $\Phi$ is the adjoint representation of $E$, $T$ is a Rota–Baxter operator (of weight zero). $\mathcal{O}$-operators are also called relative Rota–Baxter operators or generalized Rota–Baxter operators.

In recent years Rota–Baxter and $\mathcal{O}$-operators, in different algebraic and geometric settings, have deserved a great interest by mathematical and physical communities.

In [9], a homotopy version of $\mathcal{O}$-operators on symmetric graded Lie algebras was introduced. This was the first step toward the definition of an $\mathcal{O}$-operator on a Lie $\infty$-algebra with respect to a representation on a graded vector space that was given in [6]. The current paper also deals with $\mathcal{O}$-operators on Lie $\infty$-algebras, but with a different approach which uses Lie $\infty$-actions instead of representations of Lie $\infty$-algebras. Our definition is therefore different from the one given in [6] but there is a relationship between them.

There are two equivalent definitions of Lie $\infty$-algebra structure on a graded vector space $E$, both given by collections of $n$-ary brackets which are either symmetric or skew-symmetric, depending on the definition we are considering, and must satisfy a kind of generalized Jacobi identities. One goes from one to the other by shifting the degree of $E$ and applying a décalage.
isomorphism. We use the definition in its symmetric version, where the brackets have degree +1. Equivalently, this structure can be defined by a degree +1 coderivation $M_E$ of $S(E)$, the reduced symmetric algebra of $E$, such that the commutator $[M_E, M_E]$, vanishes.

Representations of Lie $\infty$-algebras on graded vector spaces were introduced in [7]. In [6], the authors consider a representation $\Phi$ of a Lie $\infty$-algebra $E$ on a graded vector space $V$ and define an $\mathcal{O}$-operator (homotopy relative Rota–Baxter operator) on $E$ with respect to $\Phi$ as a degree zero element $T$ of Hom$(\tilde{S}(V), E)$ satisfying a family of suitable identities. Inspired by the notion of an action of a Lie $\infty$-algebra on a graded manifold [8], we define an action of a Lie $\infty$-algebra $(E, M_E)$ on a Lie $\infty$-algebra $(V, M_V)$ as a Lie $\infty$-morphism $\Phi$ between $E$ and $\text{Coder}(\tilde{S}(V))[1]$, the symmetric DGLA of coderivations of $\tilde{S}(V)$. An $\mathcal{O}$-operator on $E$ with respect to the action $\Phi$ is a comorphism between $\tilde{S}(V)$ and $\tilde{S}(E)$ that intertwines the coderivation $M_E$ and a degree +1 coderivation of $\tilde{S}(V)$ built from $M_V$ and $\Phi$, which turns out to be a Lie $\infty$-algebra structure on $V$ too.

As we said before, the two $\mathcal{O}$-operator definitions, ours and the one in [6], are different. However, since there is a close connection between Lie $\infty$-actions and representations of Lie $\infty$-algebras, the two definitions can be related. On the one hand, any representation of $(E, M_E)$ on a complex $(V, d)$ can be seen as a Lie $\infty$-action of $(E, M_E)$ on $(V, D)$, with $D$ the coderivation given by the differential $d$, and for this very “simple” Lie $\infty$-algebra structure on $V$ our $\mathcal{O}$-operator definition recovers the one given in [6]. On the other hand, any action $\Phi$ of $(E, M_E)$ on $(V, M_V)$ yields a representation $\rho$ on the graded vector space $\tilde{S}(V)$ and an $\mathcal{O}$-operator with respect to the action $\Phi$ is not the same as an $\mathcal{O}$-operator with respect to the representation $\rho$. However, there is a way to relate the two concepts.

A well-known Voronov’s construction [10] defines a Lie $\infty$-algebra structure on an abelian Lie subalgebra $\mathfrak{h}$ of $\text{Coder}(\tilde{S}(E \oplus V))$ and we show that $\mathcal{O}$-operators with respect to the action $\Phi$ are Maurer–Cartan elements of $\mathfrak{h}$.

In general, deformations of structures and morphisms are governed by DGLA’s or, more generally, by Lie $\infty$-algebras. We do not intend to deeply study deformations of $\mathcal{O}$-operators on Lie $\infty$-algebras with respect to Lie $\infty$-actions. Still, we prove that deformations of an $\mathcal{O}$-operator are controlled by the twisting of a Lie $\infty$-algebra, constructed out of a graded Lie subalgebra of $\text{Coder}(\tilde{S}(E \oplus V))$.

The paper is organized in four sections. In Section 1 we collect some basic results on graded vector spaces, graded symmetric algebras and Lie $\infty$-algebras that will be needed along the paper. In Section 2, after recalling the definition of a representation of a Lie $\infty$-algebra on a complex $(V, d)$ [7], we introduce the notion of action of a Lie $\infty$-algebra on another Lie $\infty$-algebra (Lie $\infty$-action) and we prove that a Lie $\infty$-action of $E$ on $V$ induces a Lie $\infty$-algebra structure on $E \oplus V$. We pay special attention to the adjoint action of a Lie $\infty$-algebra. In Section 3 we introduce the main notion of the paper – $\mathcal{O}$-operator on a Lie $\infty$-algebra $E$ with respect to an action of $E$ on another Lie $\infty$-algebra, and we give the explicit relation between these operators and $\mathcal{O}$-operators on $E$ with respect to a representation on a graded vector space introduced in [6]. Given an $\mathcal{O}$-operator $T$ on $E$ with respect to a Lie $\infty$-action $\Phi$ on $V$, we show that $V$ inherits a new Lie $\infty$-algebra structure given by a degree +1 coderivation which is the sum of the initial one on $V$ with a degree +1 coderivation obtained out of $\Phi$ and $T$. We prove that symmetric and invertible comorphisms $T : \tilde{S}(E') \rightarrow \tilde{S}(E)$ are $\mathcal{O}$-operators with respect to the coadjoint action if and only if a certain element of $\tilde{S}(E')$, which is defined using the inverse of $T$, is a cocycle for the Lie $\infty$-algebra cohomology of $E$. Section 3 ends with the characterization of $\mathcal{O}$-operators as Maurer–Cartan elements of a Lie $\infty$-algebra obtained by Voronov’s higher derived brackets construction. The main result in Section 4 shows that Maurer–Cartan elements of a graded Lie subalgebra of $\text{Coder}(\tilde{S}(E \oplus V))$ encode a Lie $\infty$-algebra on $E$ and an action of $E$ on $V$. Moreover, we obtain the Lie $\infty$-algebra that controls the deformation of $\mathcal{O}$-operators with respect to a fixed action.
1. Lie $\infty$-algebras

We begin by reviewing some concepts about graded vector spaces, graded symmetric algebras and Lie $\infty$-algebras.

1.1. Graded vector spaces and graded symmetric algebras

We will work with $\mathbb{Z}$-graded vector spaces with finite dimension over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $E = \bigoplus_{i \in \mathbb{Z}} E_i$ be a finite dimensional graded vector space. We call $E_i$ the homogeneous component of degree $i$. An element $x$ of $E_i$ is said to be homogeneous with degree $|x| = i$. For each $k \in \mathbb{Z}$, one may shift all the degrees by $k$ and obtain a new grading on $E$. This new graded vector space is denoted by $E[k]$ and is defined by $E[k]_i = E_{i+k}$.

A morphism $\Phi : E \to V$ between two graded vector spaces is a degree preserving linear map, that is, a collection of linear maps $\Phi_i : E_i \to V_i$, $i \in \mathbb{Z}$. We call $\Phi : E \to V$ a (homogeneous) morphism of degree $k$, for some $k \in \mathbb{Z}$, and we write $|\Phi| = k$, if it is a morphism between $E$ and $V[k]$. This way we have a natural grading in the vector space of linear maps between graded vector spaces:

$$\text{Hom}(E, V) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_i(E, V).$$

In particular, $\text{Hom}(E, E) = \text{End}(E) = \bigoplus_{i \in \mathbb{Z}} \text{End}_i(E)$.

The dual $E^*$ of $E$ is naturally a graded vector space whose component of degree $i$ is, for all $i \in \mathbb{Z}$, the dual $(E_{-i})^*$ of $E_{-i}$. In equation: $(E^*)_j = (E_{-j})^*$.

Given two graded vector spaces $E$ and $V$, their direct sum $E \oplus V$ is a vector space with grading

$$(E \oplus V)_i = E_i \oplus V_i$$

and their usual tensor product comes equipped with the grading

$$(E \otimes V)_i = \bigoplus_{j+k=i} E_j \otimes V_k.$$ 

We will adopt the Koszul sign convention, for homogeneous linear maps $f : E \to V$ and $g : F \to W$ the tensor product $f \otimes g : E \otimes F \to V \otimes W$ is the morphism of degree $|f| + |g|$ given by

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y),$$

for all homogeneous $x \in E$ and $y \in F$.

For each $k \in \mathbb{N}_0$, let $T^k(E) = \otimes^k E$, with $T^0(E) = \mathbb{K}$, and let $T(E) = \bigoplus_k T^k(E)$ be the tensor algebra over $E$. The graded symmetric algebra over $E$ is the quotient

$$S(E) = T(E) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle.$$ 

The symmetric algebra $S(E) = \bigoplus_{k \geq 0} S^k(E)$ is a graded commutative algebra, whose product we denote by $\circ$. For $x = x_1 \circ \ldots \circ x_k \in S^k(E)$, we set $|x| = \sum_{i=1}^{k} |x_i|$.

For $n \geq 1$, let $S_n$ be the permutation group of order $n$. For any homogeneous elements $x_1, \ldots, x_n \in E$ and $\sigma \in S_n$, the Koszul sign is the element in $\{-1, 1\}$ defined by

$$x_{\sigma(1)} \circ \ldots \circ x_{\sigma(n)} = \epsilon(\sigma) \cdot x_1 \circ \ldots \circ x_n.$$ 

As usual, writing $\epsilon(\sigma)$ is an abuse of notation because the Koszul sign also depends on the $x_i$.

An element $\sigma$ of $S_n$ is called an $(i, n-i)$-unshuffle if $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$. The set of $(i, n-i)$-unshuffles is denoted by $\text{Sh}(i, n-i)$. Similarly, $\text{Sh}(k_1, \ldots, k_j)$ is the set of $(k_1, \ldots, k_j)$-unshuffles, that is, elements of $S_n$ with $k_1 + \ldots + k_j = n$ such that the order is preserved within each block of length $k_i$, $1 \leq i \leq j$.

The reduced symmetric algebra $\bar{S}(E) = \bigoplus_{k \geq 1} S^k(E)$ has a natural coassociative and cocommutative coalgebra structure given by the coproduct $\Delta : \bar{S}(E) \to \bar{S}(E) \otimes \bar{S}(E)$,
\[ \Delta(x) = 0, \quad x \in E; \]
\[ \Delta(x_1 \odot \ldots \odot x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in S(n,n-i)} \epsilon(\sigma)(x_{\sigma(1)} \odot \ldots \odot x_{\sigma(i)}) \odot (x_{\sigma(i+1)} \odot \ldots \odot x_{\sigma(n)}), \]
for \( x_1, \ldots, x_n \in E. \)

We will mainly use Sweedler notation: given \( x \in \bar{S}(E), \)
\[ \Delta^{(1)}(x) = \Delta(x) = x_{(1)} \otimes x_{(2)}, \]
and the coassociativity yields
\[ \Delta^{(n)}(x) = (\text{id} \otimes \Delta^{(n-1)})\Delta(x) = x_{(1)} \otimes \ldots \otimes x_{(n+1)}, \quad n \geq 2. \]

Notice that
\[ \Delta^{(n)}(x) = 0, \quad x \in \tilde{S}^n(E). \]

The cocommutativity of the coproduct is expressed, for homogeneous elements of \( \bar{S}(E), \)
as
\[ x_{(1)} \otimes x_{(2)} = (-1)^{|x_{(1)}||x_{(2)}|} x_{(2)} \otimes x_{(1)}. \]

Let \( V \) be another graded vector space. A linear map \( f : \bar{S}(E) \to V \) is given by a collection of maps \( f_k : \bar{S}^k(E) \to V, k \geq 1, \) and is usually denoted by \( f = \sum_k f_k. \)

**Remark 1.1.** Every linear map \( f : \bar{S}^k(E) \to V \) corresponds to a graded symmetric \( k \)-linear map \( f \in \text{Hom}(\otimes^k E, V) \) through the quotient map \( p_k : \otimes^k E \to \bar{S}^k(E) \) that is, \( f = f \circ p_k. \) In the sequel, we shall often write
\[ f(x_1 \odot \ldots \odot x_k) = f(x_1, \ldots, x_k), \quad x_i \in E. \]

A coalgebra morphism (or comorphism) between the coalgebras \( (\bar{S}(E), \Delta_E) \) and \( (\bar{S}(V), \Delta_V) \) is a morphism \( F : \bar{S}(E) \to \bar{S}(V) \) of graded vector spaces such that
\[ (F \otimes F) \circ \Delta_E = \Delta_V \circ F. \]

There is a one-to-one correspondence between coalgebra morphisms \( F : \bar{S}(E) \to \bar{S}(V) \) and degree preserving linear maps \( f : \bar{S}(E) \to V. \) Each \( f \) determines \( F \) by
\[ F(x) = \sum_{k \geq 1} \frac{1}{k!} f(x_{(1)}) \odot \ldots \odot f(x_{(k)}), \quad x \in \bar{S}(E), \]
and \( f = p_V \circ F, \) with \( p_V : \bar{S}(V) \to V \) the projection map.

A degree \( k \) coderivation of \( \bar{S}(E), \) for some \( k \in \mathbb{Z}, \) is a linear map \( Q : \bar{S}(E) \to \bar{S}(E) \) of degree \( k \) such that
\[ \Delta \circ Q = (Q \otimes \text{id} + \text{id} \otimes Q) \circ \Delta. \]

We also have a one to one correspondence between coderivations of \( \bar{S}(E) \) and linear maps \( q = \sum_i q_i : \bar{S}(E) \to E : \)

**Proposition 1.2.** Let \( E \) be a graded vector space and \( p_E : \bar{S}(E) \to E \) the projection map. For every linear map \( q = \sum_i q_i : \bar{S}(E) \to E, \) the linear map \( Q : \bar{S}(E) \to \bar{S}(E) \) given by
\[ Q(x_1 \odot \ldots \odot x_n) = \sum_{i=1}^n \sum_{\sigma \in S(n,n-i)} \epsilon(\sigma)q_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \odot x_{\sigma(i+1)} \odot \ldots \odot x_{\sigma(n)}, \quad (1) \]
is the unique coderivation of \( \bar{S}(E) \) such that \( p_E \circ Q = q. \)
In Sweedler notation, Equation (1) is written as:
\[ Q(x) = q(x_{(1)}) \otimes x_{(2)} + q(x), \quad x \in \widetilde{S}(E). \]

When \( E \) is a finite dimensional graded vector space, we may identify \( S(E^*) \) with \((SE)^\ast\). Koszul sign conventions yield, for each homogeneous elements \( f, g \in E^* \),
\[ (f \otimes g)(x \otimes y) = (-1)^{|x||y|} f(x) g(y) + f(y) g(x), \quad x, y \in E. \]

### 1.2. Lie \( \infty \)-algebras

We briefly recall the definition of Lie \( \infty \)-algebra [5], some basic examples and related concepts.

We will consider the symmetric approach to Lie \( \infty \)-algebras.

**Definition 1.3.** A symmetric Lie \( \infty \)-algebra (or a Lie[1] \( \infty \)-algebra) is a graded vector space \( E = \bigoplus_{i \in \mathbb{Z}} E_i \) together with a family of degree +1 linear maps \( l_k : S^k(E) \to E, k \geq 1 \), satisfying
\[
\sum_{j+i=n+1} \sum_{\sigma \in S_k(j)} \epsilon(\sigma) l_j(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{2}
\]
for all \( n \in \mathbb{N} \) and all homogeneous elements \( x_1, \ldots, x_n \in E \).

The décalage isomorphism [10] establishes a one to one correspondence between skew-symmetric Lie \( \infty \)-algebra structures \( \{l_k\}_{k \in \mathbb{N}} \) on \( E \) and symmetric Lie \( \infty \)-algebra structures \( \{l_k\}_{k \in \mathbb{N}} \) on \( E[1] \):
\[
l_k(x_1, \ldots, x_k) = (-1)^{(k-1)|x_1|+(k-2)|x_2|+\ldots+|x_{k-1}|} l_k(x_1, \ldots, x_k). \]

In the sequel, we frequently write Lie \( \infty \)-algebra, omitting the term symmetric.

**Example 1.4.** (Symmetric graded Lie algebra). A symmetric graded Lie algebra is a symmetric Lie \( \infty \)-algebra \( E = \bigoplus_{i \in \mathbb{Z}} E_i \) such that \( l_n = 0 \) for \( n \neq 2 \). Then the degree 0 bilinear map on \( E[-1] \) defined by
\[
[[x, y]] := (-1)^{ij} l_2(x, y), \quad \text{for all } x \in E_i, y \in E_j, \tag{3}
\]
is a graded Lie bracket. In particular, if \( E = E_{-1} \) is concentrated on degree \(-1\), we get a Lie algebra structure.

**Example 1.5.** (Symmetric DGLA algebra). A symmetric differential graded Lie algebra (DGLA) is a symmetric Lie \( \infty \)-algebra \( E = \bigoplus_{i \in \mathbb{Z}} E_i \) such that \( l_n = 0 \) for \( n \neq 1 \) and \( n \neq 2 \).

Then, from (2), we have that \( d := l_1 \) is a degree \(+1\) linear map \( d : E \to E \) squaring zero and satisfies the following compatibility condition with the bracket \( [, , ] := l_2(\cdot, \cdot) \):
\[
\begin{align*}
\{d[x, y] + [d(x), y] + (-1)^{|x|}[x, d(y)]\} &= 0, \\
[[x, y], z] + (-1)^{|x||y|}[x, [z, y]] + (-1)^{|x||z|}[x, [y, z]] &= 0.
\end{align*}
\]

Applying the décalage isomorphism, \( (E[-1], d, [[, , ]]) \) is a (skew-symmetric) DGLA, with \([, , ]\) given by (3).

**Example 1.6.** Let \( (E = \bigoplus_{i \in \mathbb{Z}} E_i, d) \) be a cochain complex. Then \( \text{End}(E)[1] \) has a natural symmetric DGLA structure with \( l_1 = \partial \) and \( l_2 = [, , ] \) given by:
\[
\begin{align*}
\partial \phi &= -d \circ \phi + (-1)^{|\phi|+1} \phi \circ d, \\
[\phi, \psi] &= (-1)^{|\phi|+1} \left( \phi \circ \psi - (-1)^{|\phi||\psi|+1} \psi \circ \phi \right),
\end{align*}
\]
for \( \phi, \psi \) homogeneous elements of \( \text{End}(E)[1] \). In other words, \( \partial \phi = -[d, \phi] \) and \( [\phi, \psi] = (-1)^{\deg(\phi)[\phi, \psi]} \), with \([\cdot, \cdot]_c\) the graded commutator on \( \text{End}(E) \) and \( \deg(\phi) \) the degree of \( \phi \) in \( \text{End}(E) \).

The symmetric Lie bracket \([\cdot, \cdot]_c\) on \( \text{End}(S(E))[1] \) preserves \( \text{Coder}(S(E))[1] \), the space of coderivations of \( S(E) \), so that \( (\text{Coder}(S(E))[1], \partial, [\cdot, \cdot]_c) \) is a symmetric DGLA.

The isomorphism between \( \text{Hom}(S(E), E) \) and \( \text{Coder}(S(E)) \) given by Proposition 1.2, induces a Lie bracket on \( \text{Hom}(S(E), E) \) known as the Richardson-Nijenhuis bracket:

\[
[f, g]_{RN}(x) = f(G(x)) - (-1)^{|f||g|}g(F(x)), \quad x \in S(E),
\]

for each \( f, g \in \text{Hom}(S(E), E) \), where \( F \) and \( G \) denote the coderivations defined by \( f \) and \( g \), respectively. In other words, \([F, G]_c\) is the (unique) coderivation of \( S(E) \) determined by \([f, g]_{RN} \in \text{Hom}(S(E), E) \).

Degree +1 elements \( l := \sum_k l_k \) of \( \text{Hom}(S(E), E) \) satisfying \([l, l]_{RN} = 0 \) define a Lie \( \infty \)-algebra structure on \( E \). This way we have an alternative definition of Lie \( \infty \)-algebra [5]:

**Proposition 1.7.** A Lie \( \infty \)-algebra is a graded vector space \( E \) equipped with a degree +1 coderivation \( M_E \) of \( S(E) \) such that

\[
[M_E, M_E]_c = 2M_E^2 = 0.
\]

The dual of the coderivation \( M_E \) yields a differential \( d_* \) on \( S(E^*) \). The cohomology of the Lie \( \infty \)-algebra \( (E, M_E = \{l_k\}_{k \in \mathbb{N}}) \) is the cohomology defined by the differential \( d_* \).

A Maurer–Cartan element of a Lie \( \infty \)-algebra \( (E, \{l_k\}_{k \in \mathbb{N}}) \) is a degree zero element \( z \) of \( E \) such that

\[
\sum_{k \geq 1} \frac{1}{k!} l_k(z, \ldots, z) = 0.
\]

The set of Maurer–Cartan elements of \( E \) is denoted by \( \text{MC}(E) \). Let \( z \) be a Maurer–Cartan element of \( (E, \{l_k\}_{k \in \mathbb{N}}) \) and set, for \( k \geq 1 \),

\[
l_k^z(x_1, \ldots, x_k) := \sum_{i \geq 0} \frac{1}{i!} l_{k+i}(z, \ldots, z, x_1, \ldots, x_k).
\]

Then, \( (E, \{l_k^z\}_{k \in \mathbb{N}}) \) is a Lie \( \infty \)-algebra, called the twisting of \( E \) by \( z \) [3]. For filtered, or even weakly filtered Lie \( \infty \)-algebras, the convergence of the infinite sums defining Maurer–Cartan elements and twisted Lie \( \infty \)-algebras (Equations (4) and (5)) is guaranteed (see [2, 3, 6]).

For a symmetric graded Lie algebra \( (E, l_2) \), the twisting by \( z \in \text{MC}(E) \) is the symmetric DGLA \( (E, l_1^z = l_2(z, \cdot), l_2^z = l_2) \).

### 1.3. Lie \( \infty \)-morphisms

A morphism of Lie \( \infty \)-algebras is a morphism between symmetric coalgebras that is compatible with the Lie \( \infty \)-structures.

**Definition 1.8.** Let \( (E, \{l_k\}_{k \in \mathbb{N}}) \) and \( (V, \{m_k\}_{k \in \mathbb{N}}) \) be Lie \( \infty \)-algebras. A **Lie \( \infty \)-morphism** \( \Phi : E \to V \) is given by a collection of degree zero linear maps:

\[
\Phi_k : S^k(E) \to V, \quad k \geq 1,
\]

such that, for each \( n \geq 1 \),
linear maps $V^*$ such that, for each $A$ representation on a complex $V$.

Remark 2.2. An adapted version of (6), for $k \neq 1$, then $\Phi$ is called a strict Lie $\infty$-morphism.

A curved Lie $\infty$-morphism $E \to V$, with $V$ a weakly filtered Lie $\infty$-algebra, is a degree zero linear map $\Phi : S(E) \to V$ satisfying, for $n \geq 0$, an adapted version of (6) where the indexes $k_i, \ldots, k_j$ on the right hand side of the equation run from 0 to $n$. The zero component $\Phi_0 : \mathbb{R} \to V$ gives rise to an element $\Phi_0(1) \in V_0$, which by abuse of notation we denote by $\Phi_0$. The curved adaptation of (6), for $n = 0$, then reads

$0 = \sum_{j=1}^n \frac{\epsilon(\sigma)}{j!} m_j(\Phi_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \Phi_{k'}(x_{\sigma(k'+1)}, \ldots, x_{\sigma(n)})),$

with $\Phi_k(\ldots, x_{\sigma(n)})$. Considering the coalgebra morphism $\Phi : S(E) \to S(V)$ defined by the collection of degree zero linear maps

$\Phi_k : S^k(E) \to V, \quad k \geq 1,$

we see that Equation (6) is equivalent to $\Phi$ preserving the Lie $\infty$-algebra structures:

$\Phi \circ M_E = M_V \circ \Phi.$

### 2. Representations of Lie $\infty$-algebras

A complex $(V, d)$ induces a natural symmetric DGLA structure in $\text{End}(V)[1]$, see Example 1.6.

**Definition 2.1.** A representation of a Lie $\infty$-algebra $(E, \{I_k\}_{k \in \mathbb{N}})$ on a complex $(V, d)$ is a Lie $\infty$-morphism

$\Phi : (E, \{I_k\}_{k \in \mathbb{N}}) \to (\text{End}(V)[1], \partial, [, [, ]]),$

that is, $\Phi \circ M_E = M_{\text{End}(V)[1]} \circ \Phi$, where $M_E$ is the coderivation determined by $\sum_k I_k$ and $M_{\text{End}(V)[1]}$ is the coderivation determined by $\partial + [, [, ]]$.

Equivalently, a representation of $E$ is defined by a collection of degree $+1$ maps

$\Phi_k : S^k(E) \to \text{End}(V), \quad k \geq 1,$

such that, for each $n \geq 1$,

$\sum_{i=1}^n \frac{\epsilon(\sigma)}{i!} \Phi_{n-i+1}(I_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)})$

$= \partial \Phi_n(x_1, \ldots, x_n) + \frac{1}{2} \sum_{j=1}^{n-1} \frac{\epsilon(\sigma)}{j!} \Phi_f(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), \Phi_{n-j}(x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})).$

**Remark 2.2.** A representation on a complex $(V, d)$ can be seen as a curved Lie $\infty$-morphism $\Phi : E \to \text{End}(V)[1]$, with $\Phi = \sum_{k \geq 0} \Phi_k$ and $\Phi_0 = d$. In fact, the first term on the right hand-side of Equations (7) is given by
and we have a curved Lie $\infty$-morphism
\[
\Phi : (E, \{l_k\}_{k \in \mathbb{N}}) \to (\text{End}(V)[1], [\cdot, \cdot])
\]
between the Lie $\infty$-algebra $E$ and the symmetric graded Lie algebra $(\text{End}(V)[1], [\cdot, \cdot])$ (see [8], Lemma 2.5). This is why sometimes a representation of a Lie $\infty$-algebra $E$ on a complex $(V, d)$ is called a representation on the graded vector space $V$ (compatible with the differential $d$ of $V$).

Any representation $\Phi : E \to \text{End}(V)[1]$ of a Lie $\infty$-algebra $E$ on a complex $(V, d)$ has a dual one. Let
\[
^{\ast} : \text{End}(V) \to \text{End}(V^*)
\]
be the Lie $\infty$-morphism given by
\[
\langle f^{\ast}(x), v \rangle = -(-1)^{|x||f|} \langle x, f(v) \rangle, \quad f \in \text{End}(V), x \in V^*, v \in V.
\]
The dual representation $^{\ast}\Phi : E \to \text{End}(V^*)[1]$ is obtained by composition of $\Phi$ with this Lie $\infty$-morphism. It is a representation on the complex $(V^*, d^\ast)$ and is given by
\[
\langle (^{\ast}\Phi)(e)(x), v \rangle = -(-1)^{|(e)|+1} \langle x, \Phi(e)(v) \rangle, \quad e \in \widetilde{S}(E), \ x \in V^*, \ v \in V.
\]

**Remark 2.3.** Given a representation $\Phi : E \to \text{End}(V)[1]$ on a complex $(V, d)$, defined by the collection of degree $+1$ linear maps $\Phi_k : S^k(E) \to \text{End}(V)$, $k \geq 1$, one may consider the collection of degree $+1$ maps $\phi_k : S^k(E) \otimes V \to V$, $k \geq 0$, where $\phi_0 = d : V \to V$ and $\phi_k(x, v) = (\Phi_k(x))(v)$, $k \geq 1$.

The embedding $\widetilde{S}(E) \oplus (S(E) \otimes V) \to \widetilde{S}(E \oplus V)$, provides a collection of maps
\[
\tilde{\Phi}_k : S^k(E \oplus V) \to E \oplus V, \quad k \geq 1,
\]
given by
\[
\tilde{\Phi}_k((x_1, v_1), \ldots, (x_k, v_k)) = (l_k(x_1, \ldots, x_k) + \sum_{i=1}^k (-1)^{|x_i|+1} \phi_{k-1}(x_1, \ldots, \widehat{x_i}, \ldots, x_k, v_i)),
\]
and we may express Equations (7) as
\[
\tilde{\Phi}_\ast\left(\tilde{\Phi}_\ast(x_{(1)}) \otimes x_{(2)}\right) + \tilde{\Phi}_1 \tilde{\Phi}_\ast(x) = 0, \quad x \in \widetilde{S}(E \oplus V).
\]

**Equation (10)** means that $\tilde{\Phi}$ equips $E \oplus V$ with a Lie $\infty$-algebra structure.

Now suppose the graded vector space $V$ has a Lie $\infty$-algebra structure $\{m_k\}_{k \in \mathbb{N}}$ given by a coderivation $M_V$ of $\widetilde{S}(V)$. By the construction in Example 1.6, the coderivation $M_V$ of $\widetilde{S}(V)$ defines a symmetric DGLA structure in Coder($\widetilde{S}(V))$:
\[
\partial_{M_V} Q = -M_V \circ Q + (-1)^{\text{deg}(Q)} Q \circ M_V,
\]
\[
[Q, P] = (-1)^{\text{deg}(Q)} (Q \circ P - (-1)^{\text{deg}(Q)\text{deg}(P)} P \circ Q),
\]
where deg($Q$) and deg($P$) are the degrees of $Q$ and $P$ in Coder($\widetilde{S}(V)$).

Generalizing the notion of an action of a graded Lie algebra on another graded Lie algebra, we have the following definition of an action of a Lie $\infty$-algebra on another Lie $\infty$-algebra:

**Definition 2.4.** An action of the Lie $\infty$-algebra $(E, M_E \equiv \{l_k\}_{k \in \mathbb{N}})$ on the Lie $\infty$-algebra $(V, M_V \equiv \{m_k\}_{k \in \mathbb{N}})$, or a Lie $\infty$-action of $E$ on $V$ is a Lie $\infty$-morphism
\[ \Phi : (E, \{ t_k \}_{k \in \mathbb{N}}) \rightarrow (\text{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot, \cdot]). \]

**Remark 2.5.** Being a Lie $\infty$-morphism, an action
\[ \Phi : E \rightarrow \text{Coder}(\bar{S}(V))[1] \]
is univocally defined by a collection of degree +1 linear maps
\[ \Phi_k : S^k(E) \rightarrow \text{Coder}(\bar{S}(V)), \quad k \geq 1. \]

By the isomorphism provided in Proposition 1.2, and since each $\Phi_k(x), x \in S^k(E)$, is a coderivation of $\bar{S}(V)$, we see that an action is completely defined by a collection of linear maps
\[ \Phi_{k, i} : S^k(E) \otimes S^i(V) \rightarrow V, \quad i, k \geq 1. \tag{11} \]

We will denote the coderivation $\Phi_k(x)$ simply by $\Phi_k$.

**Remark 2.6.** If we define $\Phi_0 := M_V$, then an action is equivalent to a curved Lie $\infty$-morphism between $E$ and the graded Lie algebra $\text{Coder}(\bar{S}(V))$ (compatible with the Lie $\infty$-structure in $V$) [8]. In this case, $\Phi = \sum_{k \geq 0} \Phi_k$ is called a *curved Lie $\infty$-action*.

There is a close relationship between representations and actions on Lie $\infty$-algebras.

First notice that each linear map $\ell : V \rightarrow V$ induces a (co)derivation of $\bar{S}(V)$. Hence we may see $\text{End}(V)[1]$ as a Lie $\infty$-subalgebra of $\text{Coder}(\bar{S}(V))[1]$. Therefore, given a representation $\Phi : E \rightarrow \text{End}(V)[1]$ of the Lie $\infty$-algebra $E$ on the complex $(V, d)$, we have a natural action of $E$ on the Lie $\infty$-algebra $(V, M_V)$, where $M_V$ is the coderivation defined by the map $d : V \rightarrow V$. In this case, we say the action is induced by a representation.

Moreover, for each action $\Phi : E \rightarrow \text{Coder}(\bar{S}(V))[1]$ of $E$ on the Lie $\infty$-algebra $(V, M_V \equiv \{ m_k \}_{k \in \mathbb{N}})$, we have a representation of $E$ on $V$ given by the collection of maps $\Phi_{k, 1} : S^k(E) \otimes V \rightarrow V, k \geq 1$, or equivalently, $\Phi_{k, 1} \equiv \rho_k : S^k(E) \rightarrow \text{End}(V), k \geq 1$. The morphism $\rho = \sum_k \rho_k$ is a representation of the Lie $\infty$-algebra $E$ on the complex $(V, d = m_1)$, called the *linear representation* defined by $\Phi$.

Finally one should notice that, given a Lie $\infty$-algebra $(V, M_V)$, the graded vector space $\text{Coder}(\bar{S}(V))[1]$ is a Lie $\infty$-subalgebra of $\text{End}(\bar{S}(V))[1]$. Therefore, any action $\Phi : E \rightarrow \text{Coder}(\bar{S}(V))[1]$ of the Lie $\infty$-algebra $E$ on $(V, M_V)$ yields a representation of $E$ on the graded vector space $\bar{S}(V)$. We call it the *representation induced by the action $\Phi$*. The coderivation $M_V$ defines a (co)derivation of $\bar{S}(\bar{S}(V))$ and the representation is compatible with this (co)derivation.

**Remark 2.7.** In [8], the authors define an action of a finite dimensional Lie $\infty$-algebra $E$ on a graded manifold $\mathcal{M}$ as a Lie $\infty$-morphism $\Phi : E \rightarrow \mathcal{X}()$.[1]. As the authors point out, when $\mathcal{M}$ is the graded manifold defined by a finite dimensional Lie $\infty$-algebra, we have an action of a Lie $\infty$-algebra on another Lie $\infty$-algebra. The definition presented here is a particular case of theirs because we are only considering coderivations of $\bar{S}(V)$, that is, coderivations of $S(V)$ vanishing on the field $S^0(E)$. This restrictive case reduces to the usual Lie algebra action on another Lie algebra (and its semi-direct product) while the definition given in [8], gives rise to general Lie algebra extensions. For our purpose, this definition is more adequate.

Next, with the identification $S^\ell(E \oplus V) \simeq \bigoplus_{k=0}^n S^{n-k}(E) \otimes S^k(V)$, we see that the action $\Phi$ determines a coderivation of $\bar{S}(E \oplus V)$. Together with $M_E$ and $M_V$ we have a Lie $\infty$-algebra structure on $E \oplus V$. Next proposition can be deduced from [8].
Proposition 2.8. Let \((E, M_E \equiv \{ l_k \}_{k \in \mathbb{N}})\) and \((V, M_V \equiv \{ m_k \}_{k \in \mathbb{N}})\) be Lie \(\infty\)-algebras. An action
\[ \Phi : E \to \text{Coder} (\tilde{S}(V))[1] \]
defines a Lie \(\infty\)-algebra structure on \(E \oplus V\).

Proof. We consider the brackets \(\{ l_n \}_{n \in \mathbb{N}}\) on \(E \oplus V\) given by:
\[
\begin{align*}
  l_n(x_1, \ldots, x_n) &= l_n(x_1, \ldots, x_n), \quad x_i \in E \\
  l_n(v_1, \ldots, v_n) &= m_n(v_1, \ldots, v_n), \quad v_i \in V \\
  l_{k+n}(x_1, \ldots, x_k, v_1, \ldots, v_n) &= \Phi_{k,n}(x_1, \ldots, x_k, v_1, \ldots, v_n),
\end{align*}
\]
with \(\Phi_{k,n} : S^k(E) \otimes S^n(V) \to V\) the collection of linear maps defining \(\Phi\) (see Remark 2.5).

The collection of linear maps \(\Phi_{k,n}\) defines a coderivation of \(\tilde{S}(E \oplus V)\),
\[
\Gamma : \tilde{S}(E \oplus V) \to (\tilde{S}(E) \otimes \tilde{S}(V)) \oplus \tilde{S}(V) \subset \tilde{S}(E \oplus V)
\]
related to the action \(\Phi\) by
\[
\Gamma(x \otimes v) = \Phi(x)(v), \quad x \in E, \ v \in \tilde{S}(V)
\]
and
\[
\Gamma(x \otimes v) = \Phi_x(v) + (-1)^{|x||v|} x(1) \otimes \Phi_{x(2)}(v), \quad x \in S^{\geq 2}(E), \ v \in \tilde{S}(V).
\]
The degree +1 coderivation of \(\tilde{S}(E \oplus V)\) determined by \(\{ l_n \}_{n \in \mathbb{N}}\) is
\[
M_{E \oplus V} = M_E + \Gamma + M_V.
\]

Let us prove that \(M^2_{E \oplus V} = 0\). For \(x \in \tilde{S}(E)\) and \(v \in \tilde{S}(V)\),
\[
M^2_{E \oplus V}(x) = M^2_E(x) = 0 \quad \text{and} \quad M^2_{E \oplus V}(v) = M^2_V(v) = 0
\]
while, for mixed terms, we have
\[
M_{E \oplus V}(x \otimes v) = M_E(x) \otimes v + (-1)^{|x|} x \otimes M_V(v) + (-1)^{|x||v|} x(1) \otimes \Phi_{x(2)}(v) + \Phi_x(v)
\]
and
\[
\Gamma(M_{E \oplus V}(x \otimes v)) = (\Phi_{M_E(x)}(v) + (-1)^{|x|} \Phi_x(v) + (-1)^{|x||v|} \Phi_{x(1)}(v) + \Phi_{x(2)}(v)) + m(x(v)).
\]

Since \(\Phi\) is a Lie \(\infty\)-morphism, we have
\[
\Phi_{M_E(x)} = -M_V \circ \Phi_x - (-1)^{|x|} \Phi_x \circ M_V + \frac{1}{2} [\Phi_{x(1)}, \Phi_{x(2)}],
\]
which implies \(M^2_{E \oplus V} = 0\).

The Lie \(\infty\)-algebra structure in \(E \oplus V\) presented in Remark 2.3, is a particular case of Proposition 2.8, with \(M_V = d\).

2.1. Adjoint representation and adjoint action

An important example of a representation is given by a Lie \(\infty\)-algebra structure.

Let \((E, M_E \equiv \{ l_k \}_{k \in \mathbb{N}})\) be a Lie \(\infty\)-algebra; thus \((E, l_1)\) is a complex. The collection of degree +1 maps
\[
\text{ad}_k : \quad S^k(E) \to \text{End}(E)
\]
\[
x_1 \otimes \ldots \otimes x_k \mapsto \text{ad}_{x_1 \otimes \ldots \otimes x_k} := l_{k+1}(x_1, \ldots, x_k, \ldots), \quad k \geq 1,
\]
satisfies Equations (7). (Note that Equations (7) are equivalent to Equations (2)). So, this collection of maps defines a representation \( \text{ad} = \sum_k \text{ad}_k \) of the Lie \( \infty \)-algebra \( E \) on \( (E, l_i) \).

**Definition 2.9.** The representation \( \text{ad} \) is called the **adjoint representation** of the Lie \( \infty \)-algebra \( (E, M_E \equiv \{ l_k \}_{k \in \mathbb{N}}) \).

Moreover, notice that for each \( x \in S(E), \ i \geq 1 \), we may consider the degree \( |x| + 1 \) coderivation \( \text{ad}_x^D \) of \( S(E) \) defined by the family of linear maps

\[
(\text{ad}_x)_k : S^k(E) \to E, \ x \mapsto l_{i+k}(x, e), \quad k \geq 1.
\]

So, we have a collection of degree +1 linear maps

\[
\text{ad}_i : S^i(E) \to \text{Coder}(S(E)), \quad x \mapsto \text{ad}_x^D, \quad i \geq 1,
\]

and we set \( \text{ad} = \sum_i \text{ad}_i \).

**Proposition 2.10.** The collection of degree +1 linear maps given by (12) defines a Lie \( \infty \)-morphism

\[
\text{ad} : (E, \{ l_k \}_{k \in \mathbb{N}}) \to \left( \text{Coder}(S(E))[1], \partial_M, [\cdot, \cdot] \right)
\]

from the Lie \( \infty \)-algebra \( E \) to the symmetric DGLA \( \text{Coder}(S(E))[1] \).

**Proof.** For each \( x \in S(E) \), let \( \text{ad}_x = \sum_k (\text{ad}_x)_k \) and set \( l = \sum_k l_k \).

If \( x \in \bigoplus_{i \geq 2} S^i(E) \) and \( e \in S(E) \), we have

\[
M_E(x \circ e) = M_E(x) \circ e + (-1)^{|x|} x \circ M_E(e) + (-1)^{|e||x|} l(x(1), e) \odot x(2) + l(x, e(1)) \odot x(2) \odot e(2) + l(x, e)
\]

and so,

\[
\text{ad}_x(M_E(e)) = l(x, M_E(e)) = (-1)^{|x|} l(M_E(x \circ e)) - (-1)^{|x|} l(M_E(x), e) - (-1)^{|x|} l(\text{ad}_x^D(e)) = 0 \quad \text{by (2)}
\]

\[
= (-1)^{|x| + |x(1)|} l(x(1), \text{ad}_x^D(e)) = (-1)^{|x|} \text{ad}_{M_E(x)} - (-1)^{|x|} l \circ \text{ad}_x^D - (-1)^{|x(2)|} \text{ad}_{x(1)} \circ \text{ad}_x^D(e),
\]

which is equivalent to

\[
\text{ad}_{M_E(x)} = -l \circ \text{ad}_x^D - (-1)^{|x|} \text{ad}_x \circ M_E - (-1)^{|x(2)|} \text{ad}_{x(1)} \circ \text{ad}_x^D,
\]

or to

\[
\text{ad}_{M_E(x)} = -[l, \text{ad}_x]_{RN} - \frac{1}{2} (-1)^{|x(1)|} [\text{ad}_{x(1)}, \text{ad}_{x(2)}]_{RN}.
\]

(13)

Note that the coderivation defined by the second member of (13) is

\[
[M_E, \text{ad}_x^D] + \frac{1}{2} [\text{ad}_x^D, \text{ad}_x^D] = \partial_M(\text{ad}_x^D) + \frac{1}{2} [\text{ad}_x^D, \text{ad}_x^D].
\]

If \( x \in E \), a similar computation gives

\[
\text{ad}_{l_i(x)} = -l \circ \text{ad}_x^D - (-1)^{|x|} \text{ad}_x \circ M_E = -[l, \text{ad}_x]_{RN}.
\]

(14)

Equations (13) and (14) mean that the map \( \text{ad} : E \to \text{Coder}(S(E))[1] \) is a Lie \( \infty \)-morphism. \( \square \)
Lemma 3.1. The linear map \( \text{ad} : E \to \text{Coder}(\bar{S}(E))[1] \) is an action of the Lie \( \infty \)-algebra \( E \) on itself, called the **adjoint action** of \( E \).

3. \( \mathcal{O} \)-operators on a Lie \( \infty \)-algebra

In this section we define \( \mathcal{O} \)-operators on a Lie \( \infty \)-algebra \( E \) with respect to an action of \( E \) on a Lie \( \infty \)-algebra \( V \). This is the main notion of the paper.

3.1 \( \mathcal{O} \)-operators with respect to a Lie \( \infty \)-action

Let \((E, M_E \equiv \{k\}_{k \geq 1})\) and \((V, M_V \equiv \{m_k\}_{k \geq 1})\) be Lie \( \infty \)-algebras and \( \Phi : E \to \text{Coder}(\bar{S}(V))[1] \) a Lie \( \infty \)-action of \( E \) on \( V \). Remember we are using Sweedler’s notation: for each \( v \in \bar{S}(V) \),

\[
\Delta(v) = v_{(1)} \otimes v_{(2)}
\]

and

\[
\Delta^2(v) = (\text{id} \otimes \Delta)\Delta(v) = (\Delta \otimes \text{id})\Delta(v) = v_{(1)} \otimes v_{(2)} \otimes v_{(3)}.
\]

Each degree zero linear map \( T : \bar{S}(V) \to \bar{S}(E) \) defines a degree +1 linear map \( \Phi^T : \bar{S}(V) \to \bar{S}(V) \) given by

\[
\Phi^T(v) = 0, \quad v \in V,
\]

\[
\Phi^T(v) = \Phi_{T(v_{(1)})} v_{(2)}, \quad v \in \bar{S}^2(V).
\]

Lemma 3.1. The linear map \( \Phi^T : \bar{S}(V) \to \bar{S}(V) \) is a degree +1 coderivation of \( \bar{S}(V) \) and is defined by the collection of linear maps \( \sum \Phi_{\bullet, \bullet}(T \otimes \text{id})\Delta \).

Proof. For the linear map \( \Phi^T : \bar{S}(V) \to \bar{S}(V) \) to be a coderivation it must satisfy:

\[
\Delta \Phi^T(v) = (\Phi^T \otimes \text{id} + \text{id} \otimes \Phi^T)\Delta(v), \quad v \in \bar{S}(V).
\]

This equation is trivially satisfied for \( v \in V \).

For each \( v = v_1 \otimes v_2 \in \bar{S}^2(V) \) we have \( \Phi^T(v) \in V \) and consequently, \( \Delta \Phi^T(v) = 0 \). On the other hand, since \( \Phi^T|_V = 0 \), we see that

\[
(\Phi^T \otimes \text{id} + \text{id} \otimes \Phi^T)\Delta(v) = 0
\]

and the equation is satisfied in \( \bar{S}^2(V) \).

Now let \( v \in \bar{S}^{\geq 3}(V) \), then

\[
\Delta \Phi^T(v) = \Delta \Phi_{T(v_{(1)})} v_{(2)}
\]

\[
= (\Phi_{T(v_{(1)})} \otimes \text{id} + \text{id} \otimes \Phi_{T(v_{(1)})})\Delta(v_{(2)}).
\]

The coassociativity of \( \Delta \) ensures that

\[
\Delta \Phi^T(v) = \Phi_{T(v_{(1)})} v_{(2)} \otimes v_{(3)} + (-1)^{|v_{(1)}+1|} v_{(1)} \otimes \Phi_{T(v_{(1)})} v_{(3)}
\]

\[
= \Phi_{T(v_{(1)})} v_{(2)} \otimes v_{(3)} + (-1)^{|v_{(1)}|} v_{(1)} \otimes \Phi_{T(v_{(2)})} v_{(3)}
\]

\[
= (\Phi^T \otimes \text{id} + \text{id} \otimes \Phi^T)\Delta(v).
\]

\( \square \)
Definition 3.2. Let \( (E, M_E \equiv \{ l_k \}_{k \geq 1} ) \) and \( (V, M_V \equiv \{ m_k \}_{k \geq 1} ) \) be Lie ∞-algebras and \( \Phi : E \to \text{C} \text{od} (\text{S}(V))[1] \) an action. An \( \mathcal{O} \)\-operator on \( E \) with respect to the action \( \Phi \) is a (degree 0) morphism of coalgebras \( T : \text{S}(V) \to \text{S}(E) \) such that
\[
M_E \circ T = T \circ (\Phi^T + M_V).
\]

(15)

Definition 3.3. A Rota–Baxter operator (of weight 1) on a Lie ∞-algebra \( (E, M_E \equiv \{ l_k \}_{k \geq 1} ) \) is an \( \mathcal{O} \)\-operator with respect to the adjoint action.

An \( \mathcal{O} \)\-operator \( T : \text{S}(V) \to \text{S}(E) \) with respect to an action \( \Phi : E \to \text{C} \text{od} (\text{S}(V))[1] \) of \( (E, M_E \equiv \{ l_k \}_{k \geq 1} ) \) on \( (V, M_V \equiv \{ m_k \}_{k \geq 1} ) \) is defined by a linear map \( t = \sum_i t_i : \text{S}(V) \to E \) satisfying:

i. \( l_i(t_1(v)) = t_1(m_1(v)), \quad v \in V \)

ii. \( l(T(v)) = t(\Phi t_i(v_1), v_2) + m(v_1) \otimes v_2), \quad v \in \bigoplus \geq 2 \text{S}(V) \).

Remark 3.4. When \( M_v = 0 \) we are considering \( V \) simply as a graded vector space, with no Lie ∞-algebra attached and an \( \mathcal{O} \)\-operator must satisfy
\[
M_E \circ T = T \circ \Phi^T.
\]

In this case, the terms of above equations involving the brackets \( m_i \) on \( V \) vanish.

Remark 3.5. When \( (E, [\cdot, \cdot])_E \) and \( (V, [\cdot, \cdot])_V \) are Lie algebras, by degree reasons, a morphism \( T = t_1 \) must be a strict morphism. Moreover, our definition coincides with the usual definition of \( \mathcal{O} \)\-operator (of weight 1) between Lie algebras [4]:
\[ [t_1(v), t_1(w)]_E = t_1(\Phi_{t_1(v)} w - \Phi_{t_1(w)} v + [v, w]_V), \quad v, w \in V. \]

**Remark 3.6.** When \((V, d)\) is just a complex and the action \(\Phi : E \rightarrow \text{Coder}(\tilde{S}(V))[1]\) is induced by a representation \(\rho : E \rightarrow \text{End}(V)[1]\) we have that \(\Phi(x)\) is the (co)derivation defined by \(\rho(x)\). In this case, \(\mathcal{O}\)-operators with respect to \(\Phi\) coincide with \(\mathcal{O}\)-operators with respect to \(\rho\) (or relative Rota–Baxter operators) given in [6].

In [6] the authors define \(\mathcal{O}\)-operators with respect to representations of Lie \(\infty\)-algebras. Any action induces a representation and \(\mathcal{O}\)-operators with respect to an action are related with \(\mathcal{O}\)-operators of with respect to the induced representation. We shall see that this relation is given by the comorphism

\[ I = \sum_{n \geq 1} i_n : \tilde{S}(V) \rightarrow \tilde{S}(\tilde{S}(V)), \]

defined by the family of inclusion maps \(i_n : S^n(V), \rightarrow \tilde{S}(V), n \geq 1\).

Notice that any coderivation \(D\) of \(\tilde{S}(V)\) induces a (co)derivation \(D^d\) of \(\tilde{S}(\tilde{S}(V))\). The comorphism \(I\) preserves these coderivations:

**Lemma 3.7.** Let \(V\) be a graded vector space and \(D\) a coderivation of \(\tilde{S}(V)\). The map \(I : \tilde{S}(V) \rightarrow \tilde{S}(\tilde{S}(V))\) satisfies

\[ I \circ D = D^d \circ I. \]

**Proof.** We will denote by \(\cdot\) the symmetric product in \(\tilde{S}(\tilde{S}(V))\), to distinguish from the symmetric product \(\otimes\) in \(\tilde{S}(V)\).

Let \(v \in S^n(V), n \geq 1\), and denote by \(\{m_k\}_{k \geq 1}\) the family of linear maps defining the coderivation \(D\). For \(v \in V\), we immediately have \(D^d \circ I(v) = D \circ I(v) = I \circ D(v)\). For \(v \in S^{2,2}(V)\) we have

\[
D^d \circ I(v) = D^d \left( \sum_{k=1}^{n} \frac{1}{k!} v_{(1)} \cdot \ldots \cdot v_{(k)} \right)
= \sum_{k=1}^{n} \frac{1}{k!} (D(v_{(1)}) \cdot v_{(2)} \cdot \ldots \cdot v_{(k)} + \ldots + (-1)^{|D(v_{(1)}|+\ldots+|v_{(k-1)}|)} v_{(1)} \cdot \ldots \cdot v_{(k-1)} \cdot D(v_{(k)}))
= \sum_{k=1}^{n} \frac{1}{(k-1)!} D(v_{(1)}) \cdot v_{(2)} \cdot \ldots \cdot v_{(k)}
= D(v_{(1)}) \cdot I(v_{(2)}).
\]

On the other hand,

\[
I \circ D(v) = I(m_*(v_{(1)}) \otimes v_{(2)})
= m_*(v_{(1)}) \cdot I(v_{(2)}) + (m_*(v_{(1)}) \otimes v_{(2)}) \cdot I(v_{(3)})
= D(v_{(1)}) \cdot I(v_{(2)}),
\]
and the result follows. \(\square\)

**Remark 3.8.** In particular, if \(D\) defines a Lie \(\infty\)-algebra structure on \(V\), then \(D^d\) defines a Lie \(\infty\)-algebra structure on \(\tilde{S}(V)\) and \(I\) is a Lie \(\infty\)-morphism.
Proposition 3.9. Let $\Phi : E \to \text{Coder}(\tilde{S}(V))[1]$ be an action of the Lie $\infty$-algebra $(E, M_E \equiv \{l_k\}_{k \geq 1})$ on the Lie $\infty$-algebra $(V, M_V \equiv \{m_k\}_{k \geq 1})$ and $\tilde{T} : \tilde{S}(V) \to E$ be an $\mathcal{O}$-operator with respect to the induced representation $\rho : E \to \text{End}(\tilde{S}(V))[1]$. Then $T = \tilde{T} \circ I$ is an $\mathcal{O}$-operator with respect to the action $\Phi$.

Proof. For each $x \in \tilde{S}(E)$, let us denote by

$$\Phi_x^d := \Phi(x)^d = \rho(x)^d,$$

the (co)derivation of $\tilde{S}(\tilde{S}(V))$ defined by $\rho(x)$.

Let $\tilde{T}$ be an $\mathcal{O}$-operator with respect to the induced representation. This means that

$$M_E \circ \tilde{T}(w) = \tilde{T}(\Phi_x^d_{\tilde{T}(w)}) w(2) + M_V(w), \quad w \in \tilde{S}(S(V)).$$

Then, for each $w = I(v), v \in S(V)$, we have:

$$M_E \circ \tilde{T}(I(v)) = \tilde{T}(\Phi_x^d_{\tilde{T}(I(v))}) I(v)(2) + M_V(v).$$

Using the fact that $I$ is a comorphism and Lemma 3.7, we rewrite last equation as

$$M_E \circ T(v) = \tilde{T}(\Phi_x^d_{\tilde{T}(I(v))}) I(v)(2) + M_V(v) = \tilde{T}(I \circ \Phi_{T(v)}) I(v)(2) + I \circ M_V(v) = T(\Phi_{T(v)})(2) + M_V(v).$$

Taking into account this equation and that $T$ is a comorphism, because is the composition of two comorphisms, the result follows. \hfill \Box

Proposition 3.10. Let $T$ be an $\mathcal{O}$-operator on $(E, M_E \equiv \{l_k\}_{k \geq 1})$ with respect to a Lie $\infty$-action $\Phi : E \to \text{Coder}(\tilde{S}(V))[1]$ on $(V, M_V \equiv \{m_k\}_{k \geq 1})$. Then, $V$ has a new Lie $\infty$-algebra structure

$$M_V T = \Phi^T + M_V$$

and $T : (V, MVT) \to (E, M_E)$ is a Lie $\infty$-morphism.

Proof. By Lemma 3.1 we know $\Phi^T$ is a degree $+1$ coderivation of $\tilde{S}(V)$ hence so is $MVT$.

Since $\Phi$ is an action, so that $\Phi \circ M_E = M_{\text{Coder}(\tilde{S}(V))[1]} \circ \Phi$, and $T$ is a comorphism, we have, for each $v \in \tilde{S}(V)$,

$$\Phi_{M_E T(v)} v(2) = -M_V \Phi_{T(v)} v(2) - (-1)^{|v|} \Phi_{T(v)} M_V(v) + (-1)^{|v|+1} \Phi_{T(v)} \Phi T(v) v(3).$$

(16)

On the other hand, $T$ is an $\mathcal{O}$-operator:

$$M_E \circ T(v) = T \circ \Phi_{T(v)} v(2) + T \circ M_V(v)$$

and this yields

$$\Phi_{M_E T(v)} v(2) = \Phi T v(2) + \Phi T M v(2).$$

(17)
Moreover, due to the fact that both $\Phi^T$ and $M_V$ are coderivations and $M_V^2 = 0$, we have

$$M_{VT}^2(v) = (\Phi^T)^2(v) + \Phi^T \circ M_V(v) + M_V \circ \Phi^T(v)$$

$$= \Phi T(\Phi_{T(v(2))} T(v(3))) + (-1)^{|v(1)|} \Phi_{T(v(1))} T(v(3))$$

$$+ \Phi_{TMV} T(v(2)) + (-1)^{|v(1)|} M_V(v(2)) + M_V(\Phi_{T(v(1))} v(2)).$$

Taking into account Equations (16) and (17) we conclude $M_{VT}^2 = 0$. Therefore, $M_{VT}$ defines a Lie $\infty$-algebra structure on $V$ and Equation (15) means that $T : \tilde{S}(V) \to \tilde{S}(E)$ is a Lie $\infty$-morphism between the Lie $\infty$-algebras $(V, M_{VT})$ and $(E, M_E)$.

The brackets of the Lie $\infty$-algebra structure on $V$ defined by the coderivation $M_{VT}$ are given by

$$m_1^T(v) = m_1(v)$$

and, for $n > 2$,

$$m_n^T(v_1, \ldots, v_n) = m_n(v_1, \ldots, v_n)$$

$$+ \sum_{k_1, \ldots, k_n = 1} \sum_{\sigma \in S(n)} \epsilon(\sigma) \frac{1}{n!} \Phi_{k_i-k_j}(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)})$$

$$\cdots \circ t_k(v_{\sigma(k_1)}, \ldots, v_{\sigma(k_1)})$$

with $\Phi_{k_i-k_j}, i \geq 1$, the linear maps determined by the action $\Phi$ (see (11)).

$C$-operators for the coadjoint representation

Let $(E, M_E \equiv \{k_i\}_{k \geq 1})$ be a finite dimensional Lie $\infty$-algebra. Next, we consider the dual of the adjoint representation of $E$ (see (9)), called the coadjoint representation.

**Definition 3.11.** The **coadjoint representation** of $E$, $\text{ad}^* : E \to \text{End}(E^*)[1]$, is defined by

$$\langle \text{ad}_x^*(z), v \rangle = (-1)^{|z||x|+1} \langle x, \text{ad}_x v \rangle, \quad v \in E, \quad z \in E^*.$$

Notice that $E^*$ is equipped with the differential $l_1^* \equiv \text{ad}_1^*$ (see (8)).

An $C$-operator on $E$ with respect to the coadjoint representation $\text{ad}^* : E \to \text{End}(E^*)[1]$ is a coalgebra morphism $T : \tilde{S}(E^*) \to \tilde{S}(E)$ given by a collection of maps $t = \sum t_i : \tilde{S}(E^*) \to E$ satisfying

$$l(T(x)) = \sum_{i \leq i \leq n} \epsilon(\sigma) t_{n-i+1}(\text{ad}_x^* T(x_{i+1} \ldots, x_{n+1}) x_{\sigma(i+1)} x_{\sigma(i+2)} \ldots, x_{\sigma(n)})$$

$$+ \sum_{i=1}^n (-1)^{|x_{i+1}+\ldots+x_{i-1}|} t_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n),$$

for all $\alpha = x_1 \cdots x_n \in S^n(E^*)$, $n \geq 1$.

We say that $T$ is **symmetric** if

$$\langle \beta, t_n(x_1, \ldots, x_n) \rangle = (-1)^{|z||y|+|x_n||x_{n+1}|+\ldots+|x_{n-1}|} \langle x_n, t_n(x_1, \ldots, x_{n-1}, \beta) \rangle,$$

for all $x_1, \ldots, x_n, \beta \in E^*$ and $n \geq 1$.

When $T$ is invertible, its inverse $T^{-1} : \tilde{S}(E) \to \tilde{S}(E^*)$, given by $t^{-1} = \sum t^{-1}_n$, is also symmetric:

$$\langle t_n^{-1}(x_1, \ldots, x_n), y \rangle = (-1)^{|x_n|} \langle t_n^{-1}(x_1, \ldots, x_{n-1}, y), x_n \rangle,$$

for every $x_1, \ldots x_n, y \in E$, $n \geq 1$.
One should notice that $t_n^{-1}$ is not the inverse map of $t_n$. It simply denotes the $n$-component of the inverse $T^{-1}$ of $T$.

For each $n \geq 1$, let $\omega^{(n)} \in \otimes^n E^*$ be defined by $\omega^{(1)} = 0$ and

$$\langle \omega^{(n)}, x_1 \otimes \ldots \otimes x_n \rangle = \langle t_{n-1}^{-1}(x_1, \ldots, x_{n-1}), x_n \rangle, \quad x_1, \ldots, x_n \in E.$$ 

The symmetry of $T^{-1}$ guarantees that $\omega = \sum_{n \geq 1} \omega^{(n)}$ is an element of $\tilde{S}(E^*)$.

**Proposition 3.12.** Let $T : \tilde{S}(E^*) \rightarrow \tilde{S}(E)$ be an invertible symmetric comorphism. The linear map $T$ is an $\mathcal{O}$-operator with respect to the coadjoint representation if and only if $\omega \in \oplus_{n \geq 2} S^n(E^*)$, given by

$$\langle \omega, x_1 \odot \ldots \odot x_{k+1} \rangle = \langle t_k^{-1}(x_1, \ldots, x_k, x_{k+1}), x_1, \ldots, x_{k+1} \in E, \ k \geq 1,$$

is a cocycle for the Lie $\infty$-algebra cohomology.

**Proof.** When $T$ is invertible, Equation (18) is equivalent to equations

$$t_1^{-1}l_1(x) = l_1^* t_1^{-1}(x), \quad x \in E,$$

and

$$t^{-1}M_E(x) = \text{ad}_{x(1)}^* t^{-1}(x(2)) + l_1^* t_1^{-1}(x), \quad x \in S^n(E), n \geq 2.$$ 

Let $x = x_1 \odot \ldots \odot x_n \in S^n(E), n \geq 1$, and $y \in E$, such that $|y| = |x| + 1$. We have:

$$\langle \omega, M_E(x \odot y) \rangle = \langle t^{-1}(M_E(x)), y \rangle + (-1)^{|y|} \langle t^{-1}(x_1, \ldots, x_n), l_1(y) \rangle$$

$$+ (-1)^{|x(1)|} \langle t^{-1}(x(1)), \text{ad}_{x(2)} y \rangle$$

$$= \langle t^{-1}(M_E(x)), y \rangle - \langle l_1^* t_1^{-1}(x), y \rangle - \langle \text{ad}_{x(1)}^* t_1^{-1}(x(2)), y \rangle$$

and the result follows. 

---

### 3.2. $\mathcal{O}$-operators as Maurer–Cartan elements

Let $(E, M_E \equiv \{l_k\}_{k \geq 1})$ and $(V, M_V \equiv \{m_k\}_{k \geq 1})$ be Lie $\infty$-algebras.

The graded vector space of linear maps between $S(V)$ and $E$ will be denoted by $\mathfrak{h} := \text{Hom}(\tilde{S}(V), E)$. It can be identified with the space of coalgebra morphisms between $\tilde{S}(V)$ and $\tilde{S}(E)$. On the other hand, since

$$S^n(E \oplus V) \simeq \bigoplus_{k=0}^n \left( S^{n-k}(E) \otimes S^k(V) \right), \quad n \geq 1,$$

the space $\mathfrak{h}$ can be seen as a subspace of $\text{Coder}(\tilde{S}(E \oplus V))$, the space of coderivations of $\tilde{S}(E \oplus V)$. Its elements define coderivations that only act on elements of $\tilde{S}(V)$, they are $\tilde{S}(E)$-linear.

The space $S(E \oplus V)$ has a natural $S(E)$-bimodule structure. With the above identification we have:

$$e \cdot (x \odot v) = (e \odot x) \otimes v = (-1)^{|e|(|x|+|v|)} (x \odot v) \cdot e,$$

for $e \in S(E), x \odot v \in S(E \oplus V) \simeq S(E) \otimes S(V)$.

Let $t : \tilde{S}(V) \rightarrow E$ be an element of $\mathfrak{h}$ defined by the collection of maps $t_k : \tilde{S}^k(V) \rightarrow E, k \geq 1$. Let us denote by $T : \tilde{S}(V) \rightarrow \tilde{S}(E)$ the coalgebra morphism and by $t$ the coderivation of $\tilde{S}(E \oplus V)$ defined by $t$. Notice that

$$t(v) = t_1(v), \quad v \in V$$

and
\[ t(v) = t(v(1)) \otimes v(2) + t(v), \quad v \in S^{\geq 2}(V), \]

and also, for \( x \in \tilde{S}(E) \),
\[ t(x \otimes v) = (-1)^{|x||t|} x \cdot t(v), \quad v \in \tilde{S}(V). \]

**Proposition 3.13.** The space \( \mathfrak{h} \) is an abelian Lie subalgebra of \( \text{Coder}(\tilde{S}(E \oplus V)) \).

**Proof.** Let \( t = \sum_i t_i : \tilde{S}(V) \to E \) and \( w = \sum_i w_i : \tilde{S}(V) \to E \) be elements of \( \mathfrak{h} \). Denote by \( t \) and \( w \) the coderivations of \( \tilde{S}(E \oplus V) \) defined by \( t \) and \( w \), respectively.

Let \( v \in \tilde{S}(V) \). The Lie bracket of \( t \) and \( w \) is given by:
\[
[t, w]_t(v) = t \circ (w(v(1)) \otimes v(2)) - (-1)^{|t||w|} w \circ (t(v(1)) \otimes v(2))
= (-1)^{|t|(|v(1)|+|v(2)|)} w(v(1)) \cdot t(v(2)) - (-1)^{|t|(|w|+|v(1)|)} t(v(1)) \cdot w(v(2))
= ((-1)^{|w|(|v(3)|)} w(v(3)) \cdot t(v(2)) - (-1)^{|w|(|v(1)|)} w(v(1)) \cdot t(v(2))
\]
\[ + (-1)^{|w|(|v(3)|+|v(4)|)} w(v(3)) \cdot w(v(4)) \cdot t(v(2)) - (-1)^{|w|(|v(1)|+|v(4)|)} w(v(1)) \cdot w(v(4)) \cdot t(v(1)), \]

where we used the fact that \( t \) and \( w \) are \( \tilde{S}(E) \)-linear. Because of cocommutativity of the coproduct, the last expression vanishes. \( \square \)

Now, let \( \Phi : E \to \text{Coder}(\tilde{S}(V))[1] \) be an action of the Lie \( \infty \)-algebra \( E \) on the Lie \( \infty \)-algebra \( V \). By Proposition 3.8, \( \Phi \) induces a coderivation \( \Upsilon \) of \( \tilde{S}(E \oplus V) \) and \( M_{E \oplus V} = M_E + \Upsilon + M_V \) is a Lie \( \infty \)-algebra structure on \( E \oplus V \). Let \( P : \text{Coder}(\tilde{S}(E \oplus V)) \to \mathfrak{h} \) be the projection onto \( \mathfrak{h} \).

Then we have:

**Proposition 3.14.** The quadruple \( (\text{Coder}(\tilde{S}(E \oplus V)), \mathfrak{h}, P, M_{E \oplus V}) \) is a \( V \)-data and \( \mathfrak{h} \) has a Lie \( \infty \)-algebra structure.

**Proof.** We already know that \( \text{Coder}(\tilde{S}(E \oplus V)) \), equipped with the commutator, is a graded Lie algebra and \( \mathfrak{h} \) is an abelian Lie subalgebra.

Let \( P : \tilde{S}(E \oplus V) \to E \) be the projection and \( i : \tilde{S}(V) \to \tilde{S}(E \oplus V) \) the inclusion.

Notice that, for each \( Q \in \text{Coder}(\tilde{S}(E \oplus V)) \) we have \( P(Q) = p \circ Q \circ i \) so
\[ \ker P = \{ Q \in \text{Coder}(\tilde{S}(E \oplus V)) : Q \circ i \text{ is a coderivation of } \tilde{S}(V) \} \]
is clearly a Lie subalgebra of \( \text{Coder}(\tilde{S}(E \oplus V)) \):
\[
\mathcal{P}([Q, P])_i = P \circ [Q, P]_i \circ i = P \circ Q P \circ i - (-1)^{|Q||P|} P \circ Q \circ i
= P \circ Q \circ i \circ P \circ i - (-1)^{|Q||P|} P \circ P \circ i \circ Q \circ i = 0, \quad P, Q \in \ker \mathcal{P}. \]

Moreover
\[ M_{E \oplus V} \circ i = M_V, \quad \text{so } M_{E \oplus V} \in (\ker \mathcal{P})_1 \]
and, since \( M_{E \oplus V} \) defines a Lie \( \infty \)-structure in \( E \oplus V \), we have:
\[
[M_{E \oplus V}, M_{E \oplus V}]_c = 0.
\]

Voronov’s construction [10] guarantees that \(h\) inherits a (symmetric) \(\infty\)-structure given by:
\[
\partial_k(t_1, \ldots, t_k) = \mathcal{P} \left( \left[ \left[ M_{E \oplus V}.t_1 \right]_{RN} \ldots \right]_{RN}.t_k \right), \quad t_1, \ldots, t_k \in h, \quad k \geq 1.
\]

**Remark 3.15.** A similar proof as in [6] shows that, with the above structure, \(h\) is a filtered Lie \(\infty\)-algebra.

**Lemma 3.16.** Let \(t : \tilde{S}(V) \to E\) be a degree zero element of \(h\). For each \(v \in \tilde{S}(V)\),
\[
\partial_1 t(v) = l_1 t(v) - t \circ M_V(v)
\]
and
\[
\partial_k (t, \ldots, t)(v) = l_k (t(v(1)), \ldots, t(v(k))) - k \cdot t(\Phi_{t(v(1)) \circ \ldots \circ t(v(k-1))}v(k)), \quad k \geq 2.
\]

**Proof.** Let \(p : \tilde{S}(E \oplus V) \to E\) be the projection map and \(t\) the coderivation of \(\tilde{S}(E \oplus V)\) defined by \(t\). Notice that
\[
p \circ t = t
\]
\[
p \circ t^k = 0, \quad k \geq 2.
\]

Consequently, for \(k = 1\) we have
\[
\partial_1 t(v) = p \circ M_{E \oplus V} \circ t(v) - p \circ t \circ M_V(v) = l_1 t(v) - t \circ M_V(v), \quad v \in \tilde{S}(V)
\]
and, for \(k \geq 2\),
\[
\partial_k (t, \ldots, t) = p \circ M_{E \oplus V} \circ t^k - k \cdot p \circ t \circ M_{E \oplus V} \circ t^{k-1}
\]
\[
= l \circ t^k - k \cdot t \circ M_{E \oplus V} \circ t^{k-1}
\]
and the result follows. □

**Remark 3.17.** Notice that \(\partial_k (t, \ldots, t)(v) = 0\), for \(v \in S^c(V)\), as a consequence of \(t^k(v) = 0\) and \(\Phi \circ t^{k-1}(v) \in \tilde{S}(E)\).

Next proposition realizes \(\partial\)-operators as Maurer–Cartan elements of this Lie \(\infty\)-algebra \(h\).

**Proposition 3.18.** \(\partial\)-operators on \(E\) with respect to an action \(\Phi\) are Maurer–Cartan elements of \(h\).

**Proof.** Let \(t : \tilde{S}(V) \to E\) be a degree 0 element of \(h\). Maurer–Cartan equation yields
\[
\partial_1 t + \frac{1}{2} \partial_2 (t, t) + \ldots + \frac{1}{k!} \partial_k (t, \ldots, t) + \ldots = 0.
\]
Using Lemma 3.16 we have, for each \( v \in S^k(V) \),
\[
\partial_1 t(v) + \frac{1}{2} \partial_2 (t, t)(v) + \ldots + \frac{1}{k!} \partial_k (t, \ldots, t)(v) = \\
= l_1 t(v) - t \circ M_V(v) \\
+ \frac{1}{2} l_2 (t(v(1)), t(v(2))) - t(\Phi_{t(v(1))} v(2)) + \ldots + \\
+ \frac{1}{k!} l_k (t(v(1)), \ldots, t(v(k))) - \frac{1}{(k-1)!} t(\Phi_{t(v(1)) \otimes \ldots \otimes t(v(k-1))} v(k)).
\]

Let \( T : \tilde{S}(V) \to \tilde{S}(E) \) be the morphism of coalgebras defined by \( t : \tilde{S}(V) \to E \). Maurer–Cartan equation can be written as
\[
l_1 T(v) - l_0 M_V(v) - t \Phi_{t(v(1))} v(2) = 0,
\]
which is equivalent to \( T \) being an \( \mathcal{O} \)-operator (see Equation (15)).

\[\square\]

4. Deformation of \( \mathcal{O} \)-operators

We prove that each Maurer–Cartan element of a special graded Lie subalgebra of \( \text{Coder}(\tilde{S}(E \oplus V)) \) encondes a Lie \( \infty \)-algebra structure on \( E \) and a curved Lie \( \infty \)-action of \( E \) on \( V \). We study deformations of \( \mathcal{O} \)-operators.

4.1. Maurer–Cartan elements of \( \text{Coder}(\tilde{S}(E \oplus V)) \)

Let \( E \) and \( V \) be two graded vector spaces and consider the graded Lie algebra \( \mathfrak{L} := (\text{Coder}(\tilde{S}(E \oplus V)), [\cdot, \cdot]) \). Since \( \tilde{S}(E \oplus V) \simeq \tilde{S}(E) \oplus (\tilde{S}(E) \otimes \tilde{S}(V)) \oplus \tilde{S}(V) \), the space \( M := \text{Coder}(\tilde{S}(E)) \) of coderivations of \( \tilde{S}(E) \) can be seen as a graded Lie subalgebra of \( \mathfrak{L} \). Also, the space \( R \) of coderivations defined by linear maps of the space \( \text{Hom}(\tilde{S}(E) \otimes \tilde{S}(V)) \oplus \tilde{S}(V), V) \) can be embedded in \( \mathfrak{L} \). We will use the identifications \( M \equiv \text{Hom}(\tilde{S}(E), E) \) and \( R \equiv \text{Hom}(\tilde{S}(E) \otimes \tilde{S}(V)) \oplus \tilde{S}(V), V) \). Given \( \rho \in R \), we will denote by \( \rho_0 \) the restriction of the linear map \( \rho \) to \( \tilde{S}(V) \) and by \( \rho_x \) the linear map obtained by restriction of \( \rho \) to \( \{x\} \otimes \tilde{S}(V) \), with \( x \in \tilde{S}(E) \). We set \( \mathfrak{L}' := M \oplus R \).

**Proposition 4.1.** The space \( \mathfrak{L}' \) is a graded Lie subalgebra of \( \mathfrak{L} = \text{Coder}(\tilde{S}(E \oplus V)) \).

**Proof.** Given \( m \oplus \rho, m' \oplus \rho' \in \mathfrak{L}' \), let us see that
\[
[m \oplus \rho, m' \oplus \rho']_{RN} = [m, m']_{RN} \oplus ([m, \rho']_{RN} + [\rho, m']_{RN}) + [\rho, \rho']_{RN}
\]
is an element of \( \mathfrak{L}' \). It is obvious that \([m, m']_{RN} \in \text{Hom}(\tilde{S}(E), E) \). Consider \( m^D \) and \( \rho^D \) the coderivations of \( \tilde{S}(E \oplus V) \) defined by the morphisms \( m \) and \( \rho \), respectively. For \( x \in \tilde{S}(E) \) and \( v \in \tilde{S}(V) \) we have,
\[
[m, \rho']_{RN}(x) = [m, \rho']_{RN}(v) = 0
\]
\[
[m, \rho']_{RN}(x \otimes v) = (m \circ \rho^D - (-1)^{|m||\rho'|} \rho' \circ m^D)(x \otimes v)
\]
\[
= -(-1)^{|m||\rho'|} \rho'_m(x)(v) \in V
\]
and
which proves that $[m, \rho']_{RN} + [\rho, m']_{RN} + [\rho, \rho']_{RN} \in \text{Hom}(\tilde{S}(E) \otimes \tilde{S}(V)) \oplus \tilde{S}(V), V)$.  \hfill \Box

Next theorem shows that an element $m \oplus \rho \in \mathcal{L}'$ which is a Maurer–Cartan of $\mathcal{L} = \text{Coder}(S(E \oplus V))$ encodes a Lie $\infty$-algebra structure on $E$ and an action of $E$ on the Lie $\infty$-algebra $V$.

**Theorem 4.2.** Let $E$ and $V$ be two graded vector spaces and $m \oplus \rho \in \mathcal{L}' = M \oplus R$. Then, $m \oplus \rho$ is a Maurer–Cartan element of $\mathcal{L}'$ if and only if $m^D$ defines a Lie $\infty$-structure on $E$ and $\rho$ is a curved Lie $\infty$-action of $E$ on $V$.

**Proof.** We have

$$[m \oplus \rho, m \oplus \rho]_{RN} = 0 \iff \left\{ \begin{array}{l}
[m, m]_{RN} = 0 \\
2[m, \rho]_{RN} + [\rho, \rho]_{RN} = 0.
\end{array} \right. \quad (19)$$

Similar computations to those in the proof of Proposition 4.1 give, for all $v \in \tilde{S}(V)$ and $x \in \tilde{S}(E)$,

$$\begin{cases}
(2[m, \rho]_{RN} + [\rho, \rho]_{RN})(v) = 0 \\
(2[m, \rho]_{RN} + [\rho, \rho]_{RN})(x \otimes v) = 0 \iff \left\{ \begin{array}{l}
\rho_0 \circ \rho^D_0(v) = 0 \\
\rho_{m^D}(x)(v) = \left(-\frac{(-1)^{|x|}}{2}[\rho_{x_{(1)}}, \rho_{x_{(2)}}]_{RN}\right)(v).
\end{array} \right.
\end{cases}$$

Since $m \oplus \rho$ is a degree +1 element of $\mathcal{L}'$, the right hand-side of (19) means that $m^D$ defines a Lie $\infty$-algebra structure on $E$ and $\rho = \sum_{k \geq 0} \rho_k$ is a curved Lie $\infty$-action of $E$ on $V$. Notice that $\rho^D_0 : \tilde{S}(V) \rightarrow \tilde{S}(V)$ equips $V$ with a Lie $\infty$-structure.

Reciprocally, if $(E, m^D)$ is a Lie $\infty$-algebra and $\rho$ is a curved Lie $\infty$-action of $E$ on $V$, the degree +1 element $m \oplus \rho$ of $\mathcal{L}'$ is a Maurer–Cartan element of $\mathcal{L}'$. \hfill \Box

Next proposition gives the Lie $\infty$-algebra that controls the deformations of the actions of $E$ on $V$ [3].

**Proposition 4.3.** Let $m \oplus \rho$ be a Maurer–Cartan element of $\mathcal{L}'$ and $m' \oplus \rho'$ a degree +1 element of $\mathcal{L}'$. Then, $m \oplus \rho + m' \oplus \rho'$ is a Maurer–Cartan element of $\mathcal{L}'$ if and only if $m' \oplus \rho'$ is a Maurer–Cartan element of $\mathcal{L}'$. Here, $\mathcal{L}' \oplus \rho$ denotes the DGLA which is the twisting of $\mathcal{L}'$ by $m \oplus \rho$.

### 4.2. Deformation of $\mathcal{L}$–operators

Let $\mathfrak{h}$ be the abelian Lie subalgebra of $\mathcal{L} = \text{Coder}(\tilde{S}(E \oplus V))$ considered in Proposition 3.13 and $\mathcal{P} : \mathcal{L} \rightarrow \mathfrak{h}$ the projection onto $\mathfrak{h}$. Let $\Delta \in \mathcal{L}'$ be a Maurer–Cartan element of $\mathcal{L}$. Then, $(\mathcal{L}, \mathfrak{h}, \mathcal{P}, \Delta)$ is a $V$-data and $\mathfrak{h}$ has a Lie $\infty$-structure given by the brackets:
We denote by $b_{\Delta}$ the Lie $\infty$-algebra $b$ equipped with the above structure.

The $V$-data $(\mathcal{V}, b, \mathcal{P}, \Delta)$ also provides a Lie $\infty$-algebra structure on $\mathcal{V}[1] \oplus b$, that we denote by $(\mathcal{V}[1] \oplus b)_{\Delta}$, with brackets [10]:

$$
\begin{align*}
q_{1}^A(x, a_1) &= (-[\Delta, x], \mathcal{P}(x + [\Delta, a_1]_\mathcal{V})) \\
q_{2}^A(x, x') &= (-1)^{\deg(x)[x, x']}_\mathcal{V} \\
q_{k}^A(x, a_1, \ldots, a_{k-1}) &= \mathcal{P}
\left([\ldots [x, a_1]_\mathcal{V}, a_2]_\mathcal{V}, \ldots, a_{k-1}]_\mathcal{V}\right), \quad k \geq 2,
\end{align*}
$$

where $x, x' \in \mathcal{V}[1]$ and $a_1, \ldots, a_{k-1} \in b$. Here, $\deg(x)$ is the degree of $x$ in $\mathcal{V}$.

Moreover, since $\mathcal{V}'$ is a Lie subalgebra of $\mathcal{V}$ satisfying $[\Delta, \mathcal{V}'] \subset \mathcal{V}'$, the brackets $\{q_{k}^A\}_{k \in \mathbb{N}}$ restricted to $\mathcal{V}'[1] \oplus b$ define a Lie $\infty$-algebra structure on $\mathcal{V}'[1] \oplus b$, that we denote by $(\mathcal{V}'[1] \oplus b)_{\Delta}$. Notice that the restrictions of the brackets $\{q_{k}^A\}$ to $\mathcal{V}'[1] \oplus b$ are given by the same expressions as in (20) except for $k = 1$:

$$
q_{1}^A((x, a_1)) = (-[\Delta, x], \mathcal{P}([\Delta, a_1]_\mathcal{V})) = (-[\Delta, x], \partial_1(a_1)),
$$

because $\mathcal{P}(\mathcal{V}') = 0$. Of course, $b_{\Delta}$ is a Lie $\infty$-subalgebra of $(\mathcal{V}'[1] \oplus b)_{\Delta}$.

**Remark 4.4.** The brackets (20) that define the Lie $\infty$-algebra structure of $(\mathcal{V}[1] \oplus b)_{\Delta}$ coincide with those of $b_{\Delta}$ for $x = x' = 0$. So, an easy computation yields

$$
t \in MC(b_{\Delta}) \iff (0, t) \in MC(\mathcal{V}'[1] \oplus b)_{\Delta}.
$$

Theorem 3 in [2] yields:

**Proposition 4.5.** Consider the $V$-data $(\mathcal{V}, b, \mathcal{P}, \Delta)$, with $\Delta \in MC(\mathcal{V}')$ and let $t$ be a degree zero element of $b$. Then,

$$
t \in MC(b_{\Delta}) \iff (\Delta, t) \in MC(\mathcal{V}'[1] \oplus b)_{\Delta}.
$$

Recall that, given an element $t \in b = \text{Hom}(\mathcal{S}(V), E)$, the corresponding morphism of coalgebras $T : \mathcal{S}(V) \to \mathcal{S}(E)$ is an $\mathcal{O}$-operator if and only if $t$ is a Maurer–Cartan element of $b_{\Delta}$ (Proposition 3.18). Moreover, given a Maurer–Cartan element $m \oplus \rho$ of $\mathcal{V}'$, we know from Theorem 4.2 that $(E, m^D)$ is a Lie $\infty$-algebra and $\rho$ is a curved Lie $\infty$-action of $E$ on $V$. So, an $\mathcal{O}$-operator can be seen as a Maurer–Cartan element of the Lie $\infty$-algebra $(\mathcal{V}'[1] \oplus b)_{\Delta}$:

**Proposition 4.6.** Let $E$ and $V$ be two graded vector spaces. Consider a morphism of coalgebras $T : \mathcal{S}(V) \to \mathcal{S}(E)$ defined by $t \in \text{Hom}(\mathcal{S}(V), E)$, and the $V$-data $(\mathcal{V}, b, P, \Delta)$, with $\Delta := m \oplus \rho \in MC(\mathcal{V}')$. Then, $T$ is an $\mathcal{O}$-operator on $E$ with respect to the curved Lie $\infty$-action $\rho$ if and only if $(\Delta, t)$ is a Maurer–Cartan element of the Lie $\infty$-algebra $(\mathcal{V}'[1] \oplus b)_{\Delta}$.

**Corollary 4.7.** If $T$ is an $\mathcal{O}$-operator on the Lie $\infty$-algebra $(E, m^D)$ with respect to the curved Lie $\infty$-action $\rho$ of $E$ on $V$, then $((\mathcal{V}'[1] \oplus b)_{\Delta})^m_{\Delta}$ is a Lie $\infty$-algebra.

As a consequence of Theorem 3 in [2], we obtain the Lie $\infty$-algebra that controls the deformation of $\mathcal{O}$-operators on $E$ with respect to a fixed curved Lie $\infty$-action on $V$:

**Corollary 4.8.** Let $E$ and $V$ be two graded vector spaces and consider the $V$-data $(\mathcal{V}, b, P, \Delta := m \oplus \rho)$. Let $T$ be an $\mathcal{O}$-operator on $(E, m^D)$ with respect to the curved Lie $\infty$-action $\rho$ and $T' : \mathcal{S}(V) \to \mathcal{S}(E)$ a (degree zero) morphism of coalgebras defined by $t' \in \text{Hom}(\mathcal{S}(V), E)$. Then,


\[ T + T' \] is an \( \mathcal{O} \)-operator on \( E \) with respect to the curved Lie \( \infty \)-action \( \rho \) if and only if \( (\Delta, t') \) is a Maurer–Cartan element of \( (\mathcal{U}[1] \oplus \mathfrak{h})^{(\Delta, t')}_{\Delta} \).

**Proof.** Let \( t \in \mathfrak{h} \) be the morphism defined by \( T \). Then [2],

\[
(\Delta, t + t') \in \text{MC}(\mathcal{U}[1] \oplus \mathfrak{h})_{\Delta} \iff (\Delta, t') \in \text{MC}(\mathcal{U}[1] \oplus \mathfrak{h})^{(\Delta, t')}_{\Delta}.
\]

**Funding**

The authors are partially supported by the Center for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

**References**

[1] Baxter, G. (1960). An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math.* 10(3):731–742. DOI: 10.2140/pjm.1960.10.731.

[2] Fréchet, Y., Zambon, M. (2015). Simultaneous deformations of algebras and morphisms via derived brackets. *J. Pure Appl. Algebra* 219(12):5344–5362. DOI: 10.1016/j.jpaa.2015.05.018.

[3] Getzler, E. (2009). Lie theory for nilpotent \( L_\infty \)-algebras. *Ann. Math.* 170(1):271–301. DOI: 10.4007/annals.2009.170.271.

[4] Kupershmidt, B.A. (1999). What a classical \( r \)-matrix really is. *J. Nonlinear Math. Phys.* 6(4):448–488.

[5] Lada, T., Stasheff, J. (1993). Introduction to \( SH \) Lie algebras for physicists. *Int. J. Theor. Phys.* 32(7): 1087–1103. DOI: 10.1007/BF00671791.

[6] Lazarev, A., Sheng, Y., Tang, R. (2021). Deformations and homotopy theory of relative Rota-Baxter Lie algebras. *Commun. Math. Phys.* 383(1):595–631. DOI: 10.1007/s00220-020-03881-3.

[7] Lada, T., Markl, M. (1995). Strongly homotopy Lie algebras. *Commun. Algebra* 23(6):2147–2161. DOI: 10.1080/00927879508825335.

[8] Mehta, R., Zambon, M. (2012). \( L_\infty \)-algebra actions. *Differ. Geom. Appl.* 30(6):576–587. DOI: 10.1016/j.difgeo.2012.07.006.

[9] Tang, R., Bai, C., Guo, L., Sheng, Y. Homotopy Rota-Baxter operators, homotopy \( O \)-operators and homotopy post-Lie algebras. arXiv:1907.13504.

[10] Voronov, T. (2005). Higher derived brackets and homotopy algebras. *J. Pure Appl. Algebra* 202(1-3): 133–153. DOI: 10.1016/j.jpaa.2005.01.010.