BV-generators and Lie algebroids

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Abstract

Let $A = \oplus_i A^i$ be a Gerstenhaber algebra generated by $A^0$ and $A^1$. Given a degree −1 operator $D$ on $A^0 \oplus A^1$, we find the condition on $D$ that makes $A$ a BV-algebra. Subsequently, we apply it to the Gerstenhaber or BV algebra associated to a Lie algebroid and obtain a global proof of the correspondence between BV-generators and flat connections.

1 Introduction

Batalin-Vilkovisky (BV) algebras arose originally from the quantization of gauge field theories [1]. In recent years, there has been a great deal of interest in these algebras in connection with various subjects such as string theory and operads [2, 4, 7, 11, 14, 15, 17].

The correspondence between BV or Gerstenhaber algebras and various geometric structures on a vector bundle has been studied by several authors [3, 9, 13]. The fact that Gerstenhaber algebras correspond Lie algebroids and strong differential Gestenhaber algebras correspond to Lie bialgebroids was indicated in [8]. In [16] Xu established an explicit correspondence between Lie algebroids equipped with a flat connection and BV-algebra structure on the space of their multi-sections. In the particular case of multivector fields, the correspondence was found earlier by Koszul [10]. It has also been generalized by Huebschmann to Lie-Rinehart algebras [6].

In this work we show a general result about Gerstenhaber and BV-algebras which, when applied to the Gerstenhaber or BV-algebra associated to a Lie algebroid, gives a new proof of Xu’s result.

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Let us give first the definitions of the main concepts used throughout this article.

A Gerstenhaber algebra is a triple \((\mathcal{A} = \oplus_{k \geq 0} \mathcal{A}^i, \cdot, [,])\) such that:

1. \(\mathcal{A}\) is a graded vector space,

2. The degree zero multiplication \(\cdot\) endows \(\mathcal{A}\) with a super-commutative associative algebra structure
   \[\mathcal{A}^i \cdot \mathcal{A}^j \subseteq \mathcal{A}^{i+j}.\]
   Super-commutativity means that for each \(a \in \mathcal{A}^{|a|}, b \in \mathcal{A}^{|b|},\)
   \[a \cdot b = (-1)^{|a||b|} b \cdot a.\]

3. The degree \(-1\) bracket \([,]\) endows \(\mathcal{A}\) with a super-Lie algebra structure
   \[[\mathcal{A}^i, \mathcal{A}^j] \subseteq \mathcal{A}^{i+j-1}\]
   satisfying the super-Leibniz identity: for every \(a \in \mathcal{A}^{|a|}, b \in \mathcal{A}^{|b|}, c \in \mathcal{A}^{|c|},\)
   \[ [a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)|b|} b \cdot [a, c].\]
   The bracket \([,]\) also satisfies the super-Jacobi identity: for every \(a \in \mathcal{A}^{|a|}, b \in \mathcal{A}^{|b|}, c \in \mathcal{A}^{|c|},\)
   \[ (-1)^{(|a|-1)(|c|-1)} [[a, b], c] + (-1)^{(|b|-1)(|a|-1)} [[b, c], a] + (-1)^{(|c|-1)(|b|-1)} [[c, a], b] = 0.\]

From now on all operators are assumed to be linear.

An operator \(D\) of degree \(-1\) is said to be a Gerstenhaber generator if for every \(a \in \mathcal{A}^{|a|}\) and \(b \in \mathcal{A},\)
\[ [a, b] = (-1)^{|a|} \left( D(a \cdot b) - D a \cdot b - (-1)^{|a|} a \cdot D b \right).\]

A Gerstenhaber algebra is called exact, if there is a Gerstenhaber generator \(D\) satisfying \(D^2 = 0.\) An exact Gerstenhaber algebra is often referred to as Batalin-Vilkovisky algebra (or BV-algebra for short) and a generating operator of vanishing square is called a BV-generator.

In the case that \(\mathcal{A}^i = \Gamma(\wedge^i A)\) and \(\cdot = \wedge,\) where \((A, a, [,]_A)\) is a Lie algebroid, the triple \((\mathcal{A} = \oplus_{k \geq 0} \Gamma(\wedge^i A), \wedge, [,]_A)\) is called the Gerstenhaber algebra of the Lie algebroid \((A, a, [,]_A).\)

In this work we consider a Lie algebroid \((A, a, [,]_A)\) and the Gerstenhaber algebra \((\mathcal{A} = \oplus_{k \geq 0} \Gamma(\wedge^i A), \wedge, [,]_A)\) associated to it. We suppose there exists a degree \(-1\) operator \(D\) defined on \(\mathcal{A}^i = \Gamma(A)\) and \(A^0 = C^\infty(M).\) The intent of this work is to study the necessary and sufficient conditions that make \((\mathcal{A} = \oplus_{k \geq 0} \Gamma(\wedge^i A), \wedge, [,]_A)\) exact with generating operator \(D.\)
The paper is organized as follows.

In Section 2, we define an extension $\tilde{D}$ of any degree $-1$ operator $D$ defined on $A^0 \oplus A^1$ using the bracket of the Gerstenhaber algebra ($\mathcal{A} = \oplus_{k \geq 0} \mathcal{A}^i$, $\cdot$, $[,]_A$), and check that the extension is well defined. Subsequently, we determine the conditions an operator $D$ has to satisfy for the extension to be a Gerstenhaber generator. Then we investigate the condition imposed on $D$ to make its extension a BV-generator.

In Section 3, we consider the Gerstenhaber algebra ($\mathcal{A} = \oplus_{k \geq 0} \Gamma(\wedge^i, A)$, $\wedge$, $[,]_A$) of a Lie algebroid $(\mathcal{A}, a, [\cdot, \cdot]_A)$. A generating operator was found in [16] using local coordinates. Here we define first a degree $-1$ operator on $A^0 \oplus A^1 = C^\infty(M) \oplus \Gamma(A)$ without reference to local coordinates and extend it to an operator $\tilde{D}$ defined on the whole algebra using the result of the second section. We then recover Xu’s conditions for an $A$-connection $\nabla$ on the line bundle $\wedge^n A$ that make the Gerstenhaber algebra ($\mathcal{A} = \oplus_{k \geq 0} \Gamma(\wedge^i, A)$, $\wedge$, $[,]_A$) exact. Finally, we establish isomorphism between homology and cohomology spaces

$$H_k(A, \nabla_0) \cong H^{n-k}(A, \mathbb{R})$$

for some particular flat $A$-connection $\nabla_0$ using the operator $\tilde{D}$.

2 Gerstenhaber and BV-algebra generators

2.1 Extension of degree $-1$ operators

In this section we consider a Gerstenhaber algebra $\mathcal{A}$ generated as super-algebra by $A^0$ and $A^1$. Suppose we are given a degree $-1$ linear operator $D$ defined on $A^0 \oplus A^1$, then we define its extension $\tilde{D}$ to $\mathcal{A}$ by:

$$\tilde{D}(a \cdot b) = (-1)^{|a||b|} \tilde{D}(b \cdot a), \text{ as } \mathcal{A} \text{ is a super-commutative :}$$

$$\tilde{D}(a \cdot b) = (-1)^{|a|}|a, b| + \tilde{D}a \cdot b + (-1)^{|a|}a \cdot \tilde{D}b,$$

$$a \in A^{|a|}, b \in A^{|b|}, \text{ and } |a| + |b| > 1;$$

$$\tilde{D} = D, \text{ when restricted to } A^0 \oplus A^1.$$ (2.2)

To make sure that the preceding formula unambiguously defines a degree $-1$ operator on $\mathcal{A}$, we have to check that for any $a \in A^{|a|}, b \in A^{|b|}, c \in A^{|c|}$ we have:

1) $\tilde{D}(a \cdot b) = (-1)^{|a||b|} \tilde{D}(b \cdot a)$, as $\mathcal{A}$ is a super-commutative :

$$\tilde{D}(a \cdot b)$$

$$= (-1)^{|a|}|a, b| + \tilde{D}a \cdot b + (-1)^{|a|}a \cdot \tilde{D}b$$

$$= (-1)^{|a|}(-1)^{|a|+|b|-1}b \cdot \tilde{D}a + (1)^{|a|}(-1)^{|a|}a \cdot \tilde{D}b +$$

$$= (-1)^{|a|}b \cdot \tilde{D}a + (1)^{|a|}b \cdot \tilde{D}b +$$

$$= (-1)^{|a|}b \cdot \tilde{D}a + (1)^{|a|}b \cdot \tilde{D}b$$
2) $\tilde{D}(a \cdot b) \cdot c = \tilde{D}(a \cdot (b \cdot c))$, as $\mathcal{A}$ is an associative algebra:

$$\tilde{D}((a \cdot b) \cdot c) = (-1)^{|a|+|b|}[a \cdot b, c] + \tilde{D}(a \cdot b) \cdot c + (-1)^{|a|+|b|}(a \cdot b)\tilde{D}c.$$

Using the super-Leibniz property,

$$(-1)^{|a|+|b|}[a \cdot b, c] = (-1)^{|a|+|b|}|a, c| \cdot b + (-1)^{|a|+|b|}a \cdot [b, c],$$

and

$$\tilde{D}(a \cdot b) \cdot c = (-1)^{|a|}[a, b] \cdot c + \tilde{D}a \cdot b \cdot c + (-1)^{|a|}a \cdot \tilde{D}b \cdot c$$

so

$$\tilde{D}((a \cdot b) \cdot c) = (-1)^{|a|}[a, b] \cdot c + \tilde{D}a \cdot bc + (-1)^{|a|}a \cdot \tilde{D}(bc)$$

$$= (-1)^{|a|}[a, b] \cdot c + (-1)^{|a|+|b|}|a, c| \cdot b + (-1)^{|a|+|b|}a \cdot [b, c]$$

$$+ \tilde{D}a \cdot bc + (-1)^{|a|}a \cdot \tilde{D}b \cdot c + (-1)^{|a|+|b|}ab \cdot \tilde{D}c.$$  \hspace{1cm} (2.3)

On the other hand,

$$\tilde{D}(a \cdot (b \cdot c)) = (-1)^{|a|}[a, b, c] + \tilde{D}a \cdot bc + (-1)^{|a|}a \cdot \tilde{D}(bc)$$

$$= (-1)^{|a|}[a, b] \cdot c + (-1)^{|a|+|b|}|a, c| \cdot b + (-1)^{|a|+|b|}a \cdot [b, c]$$

$$+ \tilde{D}a \cdot bc + (-1)^{|a|}a \cdot \tilde{D}b \cdot c + (-1)^{|a|+|b|}ab \cdot \tilde{D}c.$$  \hspace{1cm} (2.4)

Expressions (2.3) and (2.4) are identical, thus $\tilde{D}((a \cdot b) \cdot c) = \tilde{D}(a \cdot (b \cdot c))$ for all $a \in \mathcal{A}^{[a]}$, $b \in \mathcal{A}^{[b]}$, $c \in \mathcal{A}^{[c]}$. Therefore the operator $\tilde{D}$ is well defined, and, since we assumed that $\mathcal{A}$ is generated by $\mathcal{A}^0$ and $\mathcal{A}^1$ as a super-subalgebra, $\tilde{D}$ is defined everywhere on $\mathcal{A}$.

### 2.2 The Gerstenhaber algebra generator condition

Here we establish the conditions that a degree $-1$ operator $D$ has to satisfy so that its extension $\tilde{D}$ is a Gerstenhaber generator.

The way we extended $D$ to $\mathcal{A}$ ensures that the Gerstenhaber generator (1.1) condition is satisfied by $\tilde{D}$ on $\mathcal{A}^2 \oplus \mathcal{A}^3 \oplus \cdots$. It remains to impose the conditions on $\mathcal{A}^0 \oplus \mathcal{A}^1$.

On $\mathcal{A}^1 = \mathcal{A}^0 \cdot \mathcal{A}^1$, condition (1.1) reads

$$\tilde{D}(a^0 \cdot a^1) = [a^0, a^1] + Da^0 \cdot b + a^0 \cdot Da^1$$

$$= [a^0, a^1] + a^0 \cdot Da^1,$$  \hspace{1cm} (2.5)

since $D$ is of degree $-1$, and on $\mathcal{A}^0 = \mathcal{A}^0 \cdot \mathcal{A}^0$ condition (1.1) is always fulfilled.

Thus we have the following result

**Proposition 2.1.** A degree $-1$ operator $D$ defined on $\mathcal{A}^0 \oplus \mathcal{A}^1$ admits an extension to whole Gerstenhaber algebra $\mathcal{A}$ (generated by $\mathcal{A}^0$ and $\mathcal{A}^1$) that is a Gerstenhaber generator if and only if it satisfies the condition

$$D(a^0 \cdot a^1) = [a^0, a^1] + a^0 \cdot Da^1.$$  \hspace{1cm} (2.6)

Furthermore the extension is given by (2.7).
2.3 BV-generator conditions

To be a BV-generator, $\tilde{D}$ must also satisfy

$$\tilde{D}^2 = 0.$$  \hfill (2.7)

The following is the necessary and sufficient condition for the extension $\tilde{D}$ defined by (2.1), (2.2) to be a BV-generator.

**Proposition 2.2.** $\tilde{D}^2 = 0$ if and only if

$$D[a, b] = \{D\alpha, b\} + \{a, D\beta\}, \forall a, b \in A^1.$$  \hfill (2.8)

**Proof.** Using a direct computation, we have

$$\tilde{D}^2(a \cdot b) = (-1)^{|a|} \left\{ \tilde{D}[a, b] - [\tilde{D}a, b] - (-1)^{|a|-1}[a, \tilde{D}b] \right\} + \tilde{D}^2a \cdot b + a \cdot \tilde{D}^2b.$$  \hfill (2.9)

Hence if $\tilde{D}^2 = 0$, then

$$\tilde{D}[a, b] = \{\tilde{D}a, b\} + (-1)^{|a|-1}[a, \tilde{D}b],$$  \hfill (2.10)

which reduces to (2.8) when $a, b \in A^1$.

Let us assume conversely that (2.8) holds, then using the following lemma and the anti-symmetry of the bracket, a double induction on the degree of $a$ and $b$ shows that (2.10) holds (namely $\tilde{D}$ is a derivation of the bracket). So (2.11) reduces to

$$\tilde{D}^2(a \cdot b) = \tilde{D}^2a \cdot b + a \cdot \tilde{D}^2b,$$  \hfill (2.11)

and by choosing $b$ in $A^1$ an induction on the degree of $a$ proves that $\tilde{D}^2 = 0$. \hfill \Box

**Lemma 2.3.** Let $a \in A^{|a|}, b \in A^{|b|}, c \in A^{|c|}$. If

$$\tilde{D}[a, b] = [\tilde{D}a, b] + (-1)^{|a|-1}[a, \tilde{D}b]$$

and

$$\tilde{D}[a, c] = [\tilde{D}a, c] + (-1)^{|a|-1}[a, \tilde{D}c],$$

then

$$\tilde{D}[a, bc] = [\tilde{D}a, bc] + (-1)^{|a|-1}[a, \tilde{D}(bc)].$$  \hfill (2.12)

**Proof.** We simply compute separately each term of (2.12):

$$\tilde{D}[a, bc] = \tilde{D}\left( [a, b] \cdot c + (-1)^{|a|-1}|b|b \cdot [a, c] \right)$$

$$= (-1)^{|a|+|b|-1}[a, b], c + \tilde{D}[a, b] \cdot c + (-1)^{|a|+|b|-1}[a, b] \cdot \tilde{D}c$$

$$+ (-1)^{|a|}b [a, c] + (-1)^{|a|-1}|b|b \cdot \tilde{D}c \cdot [a, c] + (-1)^{|a|}b \cdot \tilde{D}[a, c].$$
We can check using the super-Jacobi identity that (2.14) is zero. We thus obtain

\[ [\tilde{D}a, bc] = [\tilde{D}a, b] \cdot c + (-1)^{|a||b|}b \cdot [\tilde{D}a, c], \]

and

\[ [a, \tilde{D}(bc)] = \left[a, (-1)^{|b|}[b, c] + \tilde{D}b \cdot c + (-1)^{|b|}b \cdot \tilde{D}c\right] \]
\[ = (-1)^{|b|}[a, [b, c]] + [a, \tilde{D}b \cdot c] + (-1)^{|b|}[a, b \cdot \tilde{D}c] \]
\[ = (-1)^{|b|}[a, [b, c]] + [a, \tilde{D}b \cdot c] + (-1)^{|a| \cdot (|b| - 1)}\tilde{D}b \cdot [a, c] + (-1)^{|b|}[a, b] \cdot \tilde{D}c + (-1)^{|a||b|}b \cdot [a, \tilde{D}c]. \] (2.13)

Now we put it all together:

\[ \tilde{D}[a, bc] - [\tilde{D}a, bc] - (-1)^{|a|-1}[a, \tilde{D}(bc)] \]
\[ = \left\{ (-1)^{|a|+|b|-1}[[a, b], c] + (-1)^{|a||b|}[b, [a, c]]\right\} \]
\[ -\left\{ [\tilde{D}a, b] \cdot c + (-1)^{|a||b|}b \cdot [\tilde{D}a, c]\right\} \]
\[ -(-1)^{|a|-1}\left\{ (-1)^{|b|}[a, [b, c]] + [a, \tilde{D}b \cdot c] + (-1)^{|a||b|}b \cdot [a, \tilde{D}c]\right\} -\left\{ (-1)^{|a||b|}b \cdot \tilde{D}[a, c] - (-1)^{|a|-1}(1 - (-1)^{|a|}b \cdot [a, \tilde{D}c]\right\}. \] (2.14)

We can check using the super-Jacobi identity that (2.14) is zero. We thus obtain

\[ \tilde{D}[a, bc] - [\tilde{D}a, bc] - (-1)^{|a|-1}[a, \tilde{D}(bc)] \]
\[ = \left\{ \tilde{D}[a, b] - [\tilde{D}a, b] - (-1)^{|a|-1}[a, \tilde{D}b]\right\} \cdot c \]
\[ + (-1)^{|a||b|}b \cdot \left\{ \tilde{D}[a, c] - [\tilde{D}a, c] - (-1)^{|a|-1}[a, \tilde{D}c]\right\}. \] (2.15)

And the right-hand side is zero using the hypotheses of the lemma.

\[ \square \]

### 3 Application to Lie algebroids

In this section we consider a Lie algebroid \((A, M, a)\) of rank \(n\) over a manifold \(M\) and the associated Gerstenhaber algebra \((A = \bigoplus_{i \geq 0} A^i, \wedge, [\cdot, \cdot])\), where \(A^i = \Gamma(\wedge^i A)\) and \([\cdot, \cdot]\) is the generalized Schouten bracket.

If \(E \to M\) a vector bundle, an \textit{\(A\)-connection} on \(E\) is an \(\mathbb{R}\)-linear map \(\nabla:\)

\[
\Gamma(A) \otimes \Gamma(E) \to \Gamma(E),
\]

\[
X \otimes s \to \nabla_X s,
\]

\(6\)
satisfying axioms similar to those of the usual linear connection, that is for each \( f \in C^\infty(M), X \in \Gamma(A), s \in \Gamma(E), \)
\[
\nabla_{fX}s = f\nabla_Xs, \\
\nabla_X(fs) = (a(X)f)s + f\nabla_Xs.
\]
The curvature \( R \) of an \( A \)-connection \( \nabla \) is the element in \( \Gamma(\wedge^2 A^*) \otimes \text{End}(E) \) defined by
\[
R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad \forall X, Y \in \Gamma(A). \tag{3.1}
\]

An \( A \)-connection is flat if \( R(X, Y) = 0, \forall X, Y \in \Gamma(A) \). Next we establish the correspondence between \( A \)-connections and \( BV \)-generators.

### 3.1 Definition of the generating operator

Given an \( A \)-connection \( \nabla \) on the line vector bundle \( \wedge^n A \) over \( M \), we define a degree \(-1\) operator \( D \) on \( A^0 \oplus A^1 = C^\infty(M) \oplus \Gamma(A) \) by
\[
Df = 0, \quad \forall f \in C^\infty(M), \tag{3.2}
\]
\[
(DX)\Lambda = L_X\Lambda - \nabla_X\Lambda, \quad \forall X \in \Gamma(A); \tag{3.3}
\]
where \( \Lambda \) is any element of \( \Gamma(\wedge^n A) \) and \( L_X \) is the Lie derivative with respect to \( X \).

This definition of \( D \) is independent of the choice of \( \Lambda \in \Gamma(\wedge^n A) \) as for any \( \Lambda' = f\Lambda \), where \( f \) is a smooth function, we have:
\[
L_X\Lambda' - \nabla_X\Lambda' = L_X(f\Lambda) - \nabla_X(f\Lambda) \\
= \left\{ (a(X)f)\Lambda + fL_X\Lambda \right\} - \left\{ (a(X)f)\Lambda + f\nabla_X\Lambda \right\} \\
= f\left\{ L_X\Lambda - \nabla_X\Lambda \right\} \\
= f(DX)\Lambda \\
= (DX)\Lambda', \tag{3.4}
\]
as \( DX \in C^\infty(M) \).
As seen in the previous section, a Gerstenhaber generator extension \( \tilde{D} \) of \( D \) exists if and only if \( D \) satisfies condition \((2.6)\).

In the case of the Gerstenhaber algebra of a Lie algebroid \( A \) this condition reads:
\[
D(fX) = [f, X] + fDX, \quad \forall f \in C^\infty(M), X \in \Gamma(A). \tag{3.5}
\]

Now we check that it holds:
\[
D(fX)\Lambda = L_{fX}\Lambda - \nabla_{fX}\Lambda \\
= \left\{ fL_X\Lambda - X \wedge (df \downarrow \Lambda) \right\} - f\nabla_X\Lambda \\
= f\left\{ L_X\Lambda - \nabla_X\Lambda \right\} - X \wedge (df \downarrow \Lambda) \\
= fD(X)\Lambda - X \wedge (df \downarrow \Lambda). \tag{3.6}
\]
Here the contraction (or interior product) $\lrcorner$ is defined by $(\alpha \lrcorner \Lambda)(\beta) = \Lambda(\alpha, \beta)$ for each $\alpha \in \Gamma(\wedge^k A^*)$, $\beta \in \Gamma(\wedge^{n-k} A^*)$, and $d$ is the usual Lie algebroid coboundary. As

$$X \wedge (df \lrcorner \Lambda) = \langle df, X \rangle \Lambda = (a(X)f)\Lambda,$$

and $[f, X] = -[X, f] = -a(X)f$, we obtain:

$$D(fX)\Lambda = fDX \cdot \Lambda + [f, X]\Lambda.$$

As this equation is satisfied for any $\Lambda$ in $\Gamma(\wedge^n A)$, $D$ satisfies condition (3.3). We can therefore extend $D$, in a unique way, to a Gerstenhaber generator $\tilde{D}$ of the whole algebra $\mathcal{A} = \oplus_i \Gamma(\wedge^* A)$ by using (2.1).

### 3.2 Flat $A$-connections and BV-generators correspondence

The next theorem obtained by Ping Xu [16] as a generalization to any Lie algebroid of a Koszul’s result for tangent bundle Lie algebroid [10] is proved here in a coordinate-free framework.

**Theorem 3.1.** Let $(A, M, a)$ be a Lie algebroid and $\mathcal{A} = \oplus_i \Gamma(\wedge^i A)$ be its associated Gerstenhaber algebra. Then there exists a one-to-one correspondence between $A$-connections on the line bundle $\wedge^n A$ and linear operators $\tilde{D}$ generating the Gerstenhaber algebra bracket. Under this correspondence, flat connections correspond to BV-generators on $\mathcal{A}$.

In the preceding section we associated a Gerstenhaber generator to any connection on $\wedge^n A$. Here we first show in the following lemma that we can recover the connection from its associated Gerstenhaber generator, hence proving the one-to-one correspondence.

**Lemma 3.2.** Let $\nabla$ be an $A$-connection on $\wedge^n A$ and $\tilde{D}$ its associated Gerstenhaber generator. Then for $X \in \Gamma(A)$, $\Lambda \in \Gamma(\wedge^n A)$ we have:

$$\nabla_X \Lambda = -X \wedge \tilde{D}\Lambda.$$  \hfill (3.7)

**Proof.** Since $\wedge^n A$ is a line bundle, $X \wedge \Lambda = 0$ for any $\Lambda \in \Gamma(\wedge^n A)$ and $X \in \Gamma(A)$. Hence

$$L_X \Lambda = [X, \Lambda] = -\left\{ \tilde{D}(X \wedge \Lambda) - \tilde{D}X \wedge \Lambda + X \wedge \tilde{D}\Lambda \right\} = \tilde{D}X \cdot \Lambda - X \wedge \tilde{D}\Lambda,$$

Using the definition (3.3) of $DX = \tilde{D}X$, we obtain:

$$\nabla_X \Lambda = L_X \Lambda - DX \cdot \Lambda = \left\{ DX \cdot \Lambda - X \wedge \tilde{D}\Lambda \right\} - DX \cdot \Lambda = -X \wedge \tilde{D}\Lambda.$$

$\square$
To prove the correspondence between BV-generators and flat connections we simply show that the curvature $R(X,Y)$ of the connection $\nabla$ vanishes if and only if the square $\tilde{D}^2$ of its associated Gerstenhaber generator vanishes also. The following proposition shows that if $\tilde{D}^2 = 0$ then $R(X,Y) = 0$, and that if $R(X,Y) = 0$ then $\tilde{D}^2$ vanishes on $\Gamma(\wedge^n A)$.

**Proposition 3.3.** Let $\Lambda \in \Gamma(\wedge^n A)$. The extension $\tilde{D}^2$ is linked to the curvature $R$ of $\nabla$ by the following relation:

$$R(X,Y)\Lambda = -X \wedge Y \wedge \tilde{D}^2 \Lambda,$$

$$\forall X, Y \in \Gamma(A).$$

**Proof.**

$$\nabla_X \nabla_Y \Lambda = -X \wedge \tilde{D}(\nabla_Y \Lambda)$$

$$= -X \wedge \tilde{D}(-Y \wedge \tilde{D} \Lambda)$$

$$= X \wedge \tilde{D}(Y \wedge \tilde{D} \Lambda)$$

$$= X \wedge \{ -[Y, \tilde{D} \Lambda] + DY \wedge \tilde{D} \Lambda - Y \wedge \tilde{D}^2 \Lambda \}$$

$$= -X \wedge [Y, \tilde{D} \Lambda] + DY \wedge X \wedge \tilde{D} \Lambda - X \wedge Y \wedge \tilde{D}^2 \Lambda$$

$$= -X \wedge [Y, \tilde{D} \Lambda] + DY \wedge (-\nabla_X \Lambda) - X \wedge Y \wedge \tilde{D}^2 \Lambda$$

$$= -X \wedge [Y, \tilde{D} \Lambda] - (L_Y - \nabla_Y) \nabla_X \Lambda - X \wedge Y \wedge \tilde{D}^2 \Lambda$$

$$= -X \wedge [Y, \tilde{D} \Lambda] + \nabla_Y \nabla_X \Lambda - L_Y \nabla_X \Lambda - X \wedge Y \wedge \tilde{D}^2 \Lambda.$$ (3.9)

Similarly,

$$\nabla_Y \nabla_X \Lambda = -Y \wedge [X, \tilde{D} \Lambda] + \nabla_X \nabla_Y \Lambda - L_X \nabla_Y \Lambda - Y \wedge X \wedge \tilde{D}^2 \Lambda.$$ (3.10)

Now, (3.9)-(3.10) gives:

$$\left\{ \nabla_X \nabla_Y - \nabla_Y \nabla_X \right\} \Lambda = Y \wedge [X, \tilde{D} \Lambda] - X \wedge Y \wedge [Y, \tilde{D} \Lambda] - \left\{ \nabla_X \nabla_Y - \nabla_Y \nabla_X \right\} \Lambda$$

$$+ \left\{ L_X \nabla_Y - L_Y \nabla_X \right\} \Lambda - 2X \wedge Y \wedge \tilde{D}^2 \Lambda.$$ (3.11)

Thus,

$$2 \left\{ \nabla_X \nabla_Y - \nabla_Y \nabla_X \right\} \Lambda = Y \wedge [X, \tilde{D} \Lambda] - X \wedge Y \wedge [Y, \tilde{D} \Lambda] + \left\{ L_X \nabla_Y - L_Y \nabla_X \right\} \Lambda$$

$$- 2X \wedge Y \wedge \tilde{D}^2 \Lambda.$$ (3.12)

By Lemma 3.4 below,

$$(L_X \nabla_Y - L_Y \nabla_X) \Lambda = 2 \nabla_{[X,Y]} - \left\{ Y \wedge [X, \tilde{D} \Lambda] - X \wedge [Y, \tilde{D} \Lambda] \right\},$$

therefore

$$2 \left\{ \nabla_X \nabla_Y - \nabla_Y \nabla_X \right\} \Lambda = 2 \nabla_{[X,Y]} \Lambda - 2X \wedge Y \wedge \tilde{D}^2 \Lambda.$$
or
\[
\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right)\Lambda = -X \wedge Y \wedge \tilde{D}^2\Lambda. \tag{3.13}
\]
This completes the proof. \(\square\)

**Lemma 3.4.** For any \(X, Y \in \Gamma(A)\) and \(\Lambda \in \Gamma(\wedge^n A)\), we have the following identity
\[
(L_X \nabla_Y - L_Y \nabla_X)\Lambda = 2\nabla_{[X,Y]} \left\{ Y \wedge [X, \tilde{D}\Lambda] - X \wedge [Y, \tilde{D}\Lambda] \right\}. \tag{3.14}
\]

**Proof.**
\[
(L_X \nabla_Y - L_Y \nabla_X)\Lambda = [X, \nabla_Y \Lambda] - [Y, \nabla_X \Lambda]
= [X, -Y \wedge \tilde{D}\Lambda] - [Y, -X \wedge \tilde{D}\Lambda]
= [Y, X \wedge \tilde{D}\Lambda] - [X, Y \wedge \tilde{D}\Lambda]
= [Y, X] \wedge \tilde{D}\Lambda + X \wedge [Y, \tilde{D}\Lambda]
- \left\{ [X, Y] \wedge \tilde{D}\Lambda + Y \wedge [X, \tilde{D}\Lambda] \right\}
= -2[X, Y] \wedge \tilde{D}\Lambda - \left\{ Y \wedge [X, \tilde{D}\Lambda] - X \wedge [Y, \tilde{D}\Lambda] \right\}
= 2\nabla_{[X,Y]} \left\{ Y \wedge [X, \tilde{D}\Lambda] - X \wedge [Y, \tilde{D}\Lambda] \right\}. \tag{3.15}
\]

The last step in the proof of the theorem is to show that if \(\tilde{D}^2 = 0\) on \(\Gamma(\wedge^n A)\) then it vanishes on the whole Gerstenhaber algebra \(A\).

**Proposition 3.5.** Suppose \(\tilde{D}^2\Lambda = 0\), \(\forall \Lambda \in \Gamma(\wedge^n A)\). Then
\[
\tilde{D}^2 U = 0, \forall U \in A. \tag{3.16}
\]

**Proof.** We first show that for all \(X \in A^1, \Lambda \in \Gamma(\wedge^n A)\),
\[
\tilde{D}[X, \Lambda] = [\tilde{D}X, \Lambda] + [X, \tilde{D}\Lambda]. \tag{3.17}
\]
Since \(A\) is of rank \(n\), we have \(X \wedge \Lambda = 0\). Therefore by (2.11) we have
\[
\tilde{D}^2(X \wedge \Lambda) = -\left\{ \tilde{D}[X, \Lambda] - [DX, \Lambda] - [X, \tilde{D}\Lambda] \right\} + D^2 X \wedge \Lambda + X \wedge \tilde{D}^2\Lambda = 0.
\]
\(D^2 X = 0\) because \(D\) is of degree \(-1\) and \(\tilde{D}^2\Lambda = 0\) by assumption, so
\[
\tilde{D}[X, \Lambda] - [DX, \Lambda] - [X, \tilde{D}\Lambda] = 0.
\]
Now for any \( X, Y \in \Gamma(A) \) and \( \Lambda \in \Gamma(\wedge^n A) \) we also have \( Y \wedge \Lambda = 0 \), so using formula (2.15) we obtain

\[
0 = D[X, Y \wedge \Lambda] - [DX, Y \wedge \Lambda] - [X, D(Y \wedge \Lambda)] \\
= \left\{ D[X, Y] - [DX, Y] - [X, DY] \right\} \wedge \Lambda + (-1)^{1+1}Y \wedge \left\{ D[X, \Lambda] - [DX, \Lambda] - [X, D\Lambda] \right\} \\
= \left\{ D[X, Y] - [DX, Y] - [X, DY] \right\} \Lambda.
\]

As (3.18) is zero for all \( \Lambda \in \Gamma(\wedge^n A) \), we get

\[
D[X, Y] - [DX, Y] - [X, DY] = 0.
\]

Hence \( D \) is a derivation of the bracket. According to Proposition 2.2 this implies that \( D^2 \) vanishes identically.

### 3.3 Lie Algebroid Homology

Let \((A, a, [\cdot, \cdot]_A)\) be a Lie algebroid of rank \( n \) and \( \nabla \) a flat connection on the line bundle \( \wedge^n A \). Let \( \varpi \) be the corresponding Gerstenhaber generator and \( \partial = (-1)^{n-k} \varpi \) when restricted to \( \mathcal{A}^k = \Gamma(\wedge^k A) \). As \( \nabla \) is flat, \( \partial^2 = 0 \) and we get a chain complex. We denote by \( H_*(A, \nabla) \) its homology:

\[
H_*(A, \nabla) = \ker \partial / \text{Im} \partial.
\]

We establish a relation between the Lie algebroid homology \( H_*(A, \nabla) \) and the Lie algebroid cohomology with trivial coefficients \( H^*(A, \mathbb{R}) \) in the case where the line bundle \( \wedge^n A \) is trivial.

**Definition 3.6.** Let \( \Lambda \in \Gamma(\wedge^n A) \) be a nowhere vanishing section. We define the operator \( * \) from \( \Gamma(\wedge^{n-k} A^*) \) to \( \Gamma(\wedge^k A) \) by

\[
* \omega = \omega \wedge \Lambda, \quad \forall \omega \in \Gamma(\wedge^{n-k} A^*).
\]

By assuming that \( \wedge^n A \) is a trivial line bundle, i.e., that there exists a nowhere vanishing section \( \Lambda \in \Gamma(\wedge^n A) \), we can construct a flat \( A \)-connection \( \nabla_0 \) on \( \wedge^n A \) by \( (\nabla_0)_X \Lambda = 0 \) for all \( X \in \Gamma(A) \).

The \( * \)-operator defined above becomes an intertwiner between the homological and cohomological spaces:

**Theorem 3.7.** Let \( \nabla_0 \) be a flat \( A \)-connection on \( \wedge^n A \) and \( \varpi_0 \) be the associated generating operator. Then

\[
\varpi_0 * \omega = -(-1)^{n-k} * d\omega, \quad \forall \omega \in \Gamma(\wedge^{n-k} A^*),
\]

\[3.22\]
and the following diagram is commutative:

$$
\begin{array}{c}
\Gamma(\land^{n-k}A^*) \xrightarrow{\partial} \Gamma(\land^k A) \\
\downarrow d \quad \downarrow -\partial_0 \\
\Gamma(\land^{n-k+1}A^*) \xrightarrow{\partial} \Gamma(\land^{k-1}A)
\end{array}
$$

(3.23)

where \(d\) is the usual Lie algebroid coboundary.

This leads directly to a global proof of Theorem 4.6 of [16]:

**Theorem 3.8.** Let \(\nabla_0\) be an \(A\)-connection on \(\land^n A\) that admits a global nowhere vanishing horizontal section \(\Lambda \in \Gamma(\land^n A)\). Then

$$
H_*(A, \nabla_0) \simeq H^{n-*}(A, \mathbb{R}).
$$

To prove the theorem, we need the following statements that are easily proven by induction.

**Lemma 3.9.** Let \(X_1, \ldots, X_k \in \Gamma(A)\). Then for any \(X \in \Gamma(A)\) and \(\alpha \in \Gamma(A^*)\), we have:

$$
[X, X_1 \land \ldots \land X_k] = \sum_{i=1}^{k} (-1)^{i-1} [X, X_i] \land X_1 \land \ldots \land \hat{X}_i \land \ldots \land X_k.
$$

(3.24)

$$
\alpha \downarrow X_1 \land \ldots \land X_k = \sum_{i=1}^{k} (-1)^{i-1} (\alpha \downarrow X_i) X_1 \land \ldots \hat{X}_i \ldots \land X_k.
$$

(3.25)

**Lemma 3.10.** Let \(X_1, \ldots, X_n \in \Gamma(A)\) be elements of a basis in \(\Gamma(A)\), and let \(\alpha_j = X_j^*\) be a dual basis. Then the following holds: for \(1 \leq j \leq k\),

$$
\alpha_j \downarrow X_1 \land \ldots \land X_k = (-1)^{j-1} X_1 \land \ldots \hat{X}_j \ldots \land X_k.
$$

(3.26)

For \(j > k\),

$$
\alpha_j \downarrow X_1 \land \ldots \land X_k = 0.
$$

Since the definition of the algebroid coboundary is given locally, the following general proposition, which relates the Lie algebroid coboundary to its Gerstenhaber generator, is proved using local coordinates. Its result, however, is global.

**Proposition 3.11.** Let \((A, a, [\cdot, \cdot]_A)\) be a Lie algebroid and let \(\tilde{D}\) be a generating operator of the Gerstenhaber algebra \(\mathcal{A} = \sum_{k=0}^{n} \Gamma(\land^k A)\). Then, for any section \(U \in \Gamma(\land^n A)\) and any \(\omega \in \Gamma(\land^{|w|} A)\) with \(|w| + 1 \leq n\),

$$
d\omega \downarrow U = \omega \downarrow \tilde{D} U - (-1)^{|w|} \tilde{D} (\omega \downarrow U).
$$

(3.27)
Proof. In the first step we prove the theorem when $|w| + 1 = u$ and $U$ is a nowhere vanishing section.
Let us choose a basis $X_1, \ldots, X_n$ of 1-sections such that $U = X_1 \wedge \ldots \wedge X_{k+1}$ and again denote its dual basis by $\alpha_1, \ldots, \alpha_n$. For any $\omega \in \Gamma(\wedge^k A^*)$, the Lie algebroid coboundary is defined (see [13, 16]) by
\[
 d\omega \mathbin{\lrcorner} U = d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i)(\omega(X_1, \ldots, \hat{X}_i, X_{k+1})) \\
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, X_{k+1}). \tag{3.28}
\]
Using Lemma 3.10
\[
 (-1)^{i+1} a(X_i)(\omega(X_1, \ldots, \hat{X}_i, X_{k+1})) = a(X_i)(\omega(\alpha_i \mathbin{\lrcorner} U)) \\
= a(X_i)(\omega(\mathbin{\lrcorner}(\alpha_i \mathbin{\lrcorner} U))) \\
= [X_i, \omega \mathbin{\lrcorner}(\alpha_i \mathbin{\lrcorner} U)].
\]
Thus we can write
\[
 \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i)(\omega(X_1, \ldots, \hat{X}_i, X_{k+1})) = \sum_{i=1}^{k+1} [X_i, \omega \mathbin{\lrcorner}(\alpha_i \mathbin{\lrcorner} U)].
\]
Using the properties of contraction,
\[
 [X_i, \omega \mathbin{\lrcorner}(\alpha_i \mathbin{\lrcorner} U)] = (-1)^{|w|}[X_i, \alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)].
\]
Next we express the bracket using the definition of the generating operator $\tilde{D}$:
\[
 (-1)^{|w|}[X_i, \alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)] = (-1)^{|w|}(-1) \left\{ \tilde{D}(X_i \wedge (\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U))) - \tilde{D}X_i \wedge (\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)) \\
+ X_i \wedge \tilde{D}(\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)) \right\}.
\]
As $\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)$ is a function, $\tilde{D}(\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)) = 0$. Therefore we get:
\[
 (-1)^{|w|}[X_i, \alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)] = -(-1)^{|w|} \left\{ \tilde{D}(X_i \wedge (\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U))) - \tilde{D}X_i \wedge (\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)) \right\}.
\]
Since $\alpha_i$ is dual to $X_i$,
\[
 \sum_{i=1}^{k+1} X_i \wedge (\alpha_i \mathbin{\lrcorner}(\omega \mathbin{\lrcorner} U)) = \omega \mathbin{\lrcorner} U.
\]
This allows us to simplify the first term of (3.28):
\[
 \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i)(\omega(X_1, \ldots, \hat{X}_i, X_{k+1}))
\]
\[
= \sum_{i=1}^{k+1} (-1)^{|w|} \left\{ \tilde{D} \left( X_i \land (\alpha_i \lhd (\omega \rhd U)) \right) - \tilde{D} X_i \cdot (\alpha_i \lhd (\omega \rhd U)) \right\}
\]

\[
= -(-1)^{|w|} \tilde{D} (\omega \rhd U) + (-1)^{|w|} \sum_{i=1}^{k+1} \tilde{D} X_i \cdot (\alpha_i \lhd (\omega \rhd U)).
\] (3.29)

For the second term of (3.28), we can write

\[
\sum_{i=1}^{k} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i \ldots \hat{X}_j \ldots, X_{k+1})
\]

\[
= \omega \sum_{i=1}^{k} (-1)^i X_1 \land \ldots \land X_{i-1} \land [X_i, V_i],
\]

where \(V_i = X_{i+1} \land \ldots \land X_{k+1}\).

As \(X_i \land V_i = V_{i-1}\) and \(\tilde{D}\) is a generator of the bracket,

\[
(-1)^i X_1 \land \ldots \land X_{i-1} \land [X_i, V_i]
\]

\[
= (-1)^i X_1 \land \ldots \land X_{i-1} \land \left\{ \tilde{D} X_i \cdot V_i - \tilde{D} V_{i-1} - X_i \land \tilde{D} V_i \right\}
\]

\[
= (-1)^i \tilde{D} X_i \cdot X_1 \land \ldots \land X_{i-1} \land X_{i+1} \land \ldots \land X_{k+1}
\]

\[
+ (-1)^{i-1} X_1 \land \ldots \land X_{i-1} \land \tilde{D} V_{i-1}
\]

\[
- (-1)^i X_1 \land \ldots \land X_{i-1} \land X_i \land \tilde{D} V_i.
\] (3.30)

By Corollary 3.10

\[
(-1)^i \tilde{D} X_i \cdot X_1 \land \ldots \land X_{i-1} \land X_{i+1} \land \ldots \land X_{k+1} = -\tilde{D} X_i \cdot (\alpha_i \lhd U).
\] (3.31)

Thus, we obtain:

\[
\sum_{i=1}^{k} (-1)^i X_1 \land \ldots \land X_{i-1} \land [X_i, V_i]
\]

\[
= \sum_{i=1}^{k} -\tilde{D} X_i \cdot (\alpha_i \lhd U)
\]

\[
+ \sum_{i=1}^{k} \left\{ (-1)^{i-1} X_1 \land \ldots \land X_{i-1} \land \tilde{D} V_{i-1} - (-1)^i X_1 \land \ldots \land X_{i-1} \land X_i \land \tilde{D} V_i \right\}
\]

\[
= \sum_{i=1}^{k} -\tilde{D} X_i \cdot (\alpha_i \lhd U) + \tilde{D} V_0
\]

\[
+ \sum_{i=1}^{k-1} (-1)^i X_1 \land \ldots \land X_i \land \tilde{D} V_i - \sum_{i=1}^{k-1} (-1)^i X_1 \land \ldots \land X_i \land \tilde{D} V_i
\]
\[-(-1)^k \tilde{D} V_k \cdot X_1 \wedge \ldots \wedge X_k \]
\[= \tilde{D} U - \sum_{i=1}^{k+1} \tilde{D} X_i \cdot (\alpha_i \downarrow U), \quad (3.32)\]

as \( V_k = X_{k+1}, \) \((-1)^k X_1 \wedge \ldots \wedge X_k = \alpha_{k+1} \downarrow U, \) and
\[V_0 = X_1 \wedge \ldots \wedge X_{k+1} = U.\]

Therefore,
\[\omega \downarrow \sum_{i=1}^{k} \sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \hat{X}_i \ldots \hat{X}_j \wedge X_{k+1} \]
\[= \omega \downarrow \left\{ \tilde{D} U - \sum_{i=1}^{k+1} \tilde{D} X_i \cdot \alpha_i \downarrow U \right\} \]
\[= \omega \downarrow \tilde{D} U - \sum_{i=1}^{k+1} \tilde{D} X_i \cdot \omega \downarrow (\alpha_i \downarrow U). \quad (3.33)\]

Finally, we obtain:
\[d\omega \downarrow U = -(-1)^{|\omega|} \tilde{D} (\omega \downarrow U) + (-1)^{|\omega|} \sum_{i=1}^{k+1} \tilde{D} X_i \cdot (\alpha_i \downarrow (\omega \downarrow U)) \]
\[+ \omega \downarrow \tilde{D} U - \sum_{i=1}^{k+1} \tilde{D} X_i \cdot \omega \downarrow (\alpha_i \downarrow U) \]
\[= \omega \downarrow \tilde{D} U - (-1)^{|\omega|} \tilde{D} (\omega \downarrow U) \]
\[+ (-1)^{|\omega|} \sum_{i=1}^{k+1} \tilde{D} X_i \cdot (\alpha_i \downarrow (\omega \downarrow U)) - (-1)^{|\omega|} \sum_{i=1}^{k+1} \tilde{D} X_i \cdot (\alpha_i \downarrow (\omega \downarrow U)) \]
\[= \omega \downarrow \tilde{D} U - (-1)^{|\omega|} \tilde{D} (\omega \downarrow U).\]

In the second step we generalize the result to the case where \(|\omega| + 1 < u.\)

For any \( \theta \in \Gamma(\wedge^{u-1-|\omega|} A^*), \) form \( \omega \wedge \theta \) is a \( u - 1 \)-form, therefore we can apply the result proved in the first step:
\[d(\omega \wedge \theta) \downarrow U = (\omega \wedge \theta) \downarrow \tilde{D} U - (-1)^{|\omega| + |\theta|} \tilde{D} ((\omega \wedge \theta) \downarrow U). \quad (3.34)\]

For the left hand side we have
\[d(\omega \wedge \theta) \downarrow U = (d\omega \wedge \theta) \downarrow U + (-1)^{|\omega|} (\omega \wedge d\theta) \downarrow U \]
\[= \theta \downarrow (d\omega \downarrow U) + (-1)^{|\omega|} d\theta \downarrow (\omega \downarrow U).\]
The right hand side of (3.34) can be written as:

\[(\omega \wedge \theta) \mathbb{J} \tilde{D} U - (-1)^{|\omega|} \tilde{D}((\omega \wedge \theta) \mathbb{J} U) = \theta \mathbb{J} (\omega \mathbb{J} \tilde{D} U) - (-1)^{|\omega|}(-1)^{[\theta]} \tilde{D}(\theta \mathbb{J} (\omega \mathbb{J} U)).\]

Now, \(\omega \mathbb{J} U \in \Gamma(\wedge^{\theta+1} A)\), so

\[(-1)^{[\theta]} \tilde{D}(\theta \mathbb{J} (\omega \mathbb{J} U)) = \theta \mathbb{J} \tilde{D}(\omega \mathbb{J} U) - d\theta \mathbb{J} (\omega \mathbb{J} U).\]

Therefore the relation (3.34) becomes

\[\theta \mathbb{J} (d\omega \mathbb{J} U) + (-1)^{|\omega|} d\theta \mathbb{J} (\omega \mathbb{J} U) = \theta \mathbb{J} (\omega \mathbb{J} \tilde{D} U) - (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} U).\]

This gives

\[\theta \mathbb{J} (d\omega \mathbb{J} U) = \theta \mathbb{J} (\omega \mathbb{J} \tilde{D} U) - (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} U),\]

which is equivalent to

\[\theta \mathbb{J} \left\{d\omega \mathbb{J} U - \omega \mathbb{J} \tilde{D} U + (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} U)\right\} = 0.\] (3.35)

Since (3.35) holds for all \(\theta \in \Gamma(\wedge^{u-1-|\omega|} A)\), we must have

\[d\omega \mathbb{J} U - \omega \mathbb{J} \tilde{D} U + (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} U) = 0.\]

Therefore

\[d\omega \mathbb{J} U = \omega \mathbb{J} \tilde{D} U - (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} U).\] (3.36)

In the third and last step we generalize the result to the case where \(U\) may vanish at some points. For this it is sufficient to show that (3.36) holds for \(fU, f \in C^\infty(M)\), where \(U\) is nowhere vanishing, i.e. that we have

\[d\omega \mathbb{J} (fU) = \omega \mathbb{J} \tilde{D}(fU) - (-1)^{|\omega|} \tilde{D}(\omega \mathbb{J} fU).\] (3.37)

We first observe that (3.36) holds for \(f \in \Gamma(\wedge^0 A) = C^\infty(M)\), so

\[df \mathbb{J} U = f \mathbb{J} \tilde{D} U - (-1)^{|f|} \tilde{D}(f \mathbb{J} U) = f \tilde{D} U - \tilde{D}(fU),\]

and

\[\omega \mathbb{J} \tilde{D}(fU) = \omega \mathbb{J} (f \tilde{D} U) - \omega \mathbb{J} (df \mathbb{J} U) = f \omega \mathbb{J} \tilde{D} U - (df \wedge \omega) \mathbb{J} U.\]
It follows from (3.36) that
\[
d(f\omega \downarrow U) = (f\omega \downarrow \widetilde{D}U) - (-1)^{|\omega|} \widetilde{D}((f\omega) \downarrow U).
\]
As \(\widetilde{D}(\omega \downarrow fU) = \widetilde{D}((f\omega) \downarrow U)\), we can write
\[
\omega \downarrow \widetilde{D}(fU) - (-1)^{|\omega|} \widetilde{D}(\omega \downarrow fU) = (f\omega \downarrow \widetilde{D}U) - (df \wedge \omega) \downarrow U + (d(f\omega) \downarrow U - (f\omega) \downarrow \widetilde{D}U) = d(f\omega) \downarrow U - (df \wedge \omega) \downarrow U = fd\omega \downarrow U = d\omega \downarrow (fU).
\]
This finishes the proof. \(\square\)

### 3.3.1 Proof of the Theorem 3.7

**Proof.** We want to demonstrate that for all \(\omega \in \Gamma(\wedge^{n-k}A^*)\),
\[
\widetilde{D}_0(\omega \downarrow \Lambda) = -(-1)^{|\omega|} d\omega \downarrow \Lambda.
\]
By Proposition 3.11 we have
\[
d\omega \downarrow \Lambda = \omega \downarrow \widetilde{D}_0\Lambda - (-1)^{|\omega|} \widetilde{D}_0(\omega \downarrow \Lambda).
\]
As \(\widetilde{D}_0\Lambda = 0\),
\[
d\omega \downarrow \Lambda = -(-1)^{|\omega|} \widetilde{D}_0(\omega \downarrow \Lambda),
\]
which is equivalent to
\[
\widetilde{D}_0* = -(-1)^{n-k} * d.
\]
\(\square\)

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