ARITHMETIC PROGRESSIONS IN THE GRAPHS OF
SLIGHTLY CURVED SEQUENCES

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Abstract. This paper proves that arbitrarily long arithmetic progressions are contained in the graph of a slightly curved sequence with small error. More precisely, a strictly increasing sequence of positive integers is called a slightly curved sequence with small error if the sequence can be well-approximated by a function whose second derivative goes to zero faster than or equal to $1/x^\alpha$ for some $\alpha > 0$. Furthermore, we extend Szemerédi’s theorem to a theorem for slightly curved sequences. As a corollary, it follows that the graph of the sequence of the integer parts of $\{na\}_{n \in A}$ contains arbitrarily long arithmetic progressions for every $1 \leq a < 2$ and every $A \subset \mathbb{N}$ with positive upper density. We also prove that the same graph does not contain any arithmetic progressions of length 3 for every $a \geq 2$.

1. Introduction

This paper considers problems involving arithmetic progressions. Let $k \geq 3$ and $d \geq 1$ be integers. A sequence $\{a(j)\}_{j=0}^{k-1} \subset \mathbb{N}^d$ is called an arithmetic progression (AP) of length $k$ if there exists $D \in \mathbb{N}^d$ such that

$$a(j) = a(0) + jD$$

for all $j = 0, 1, \ldots, k-1$. This paper only discusses the case $d = 1$ or 2. APs are taken great interests from researchers studying arithmetic combinatorics, geometric measure theory, or fractal geometry. Most studies consider the density of sets to ensure the existence of long APs. For example, we recall Szemerédi’s celebrated result.

**Proposition 1.1 (Szemerédi [Sz]).** For every $k \geq 3$ and $0 < \delta \leq 1$ there exists a positive integer $N(k, \delta)$ such that if $N \geq N(k, \delta)$, then any set $A \subset \{1, 2, \ldots, N\}$ with $|A| \geq \delta N$ contains an AP of length $k$.

Here $|X|$ denotes the cardinality of a finite set $X$. Furthermore, Steinhaus showed that any set with positive Lebesgue measure contains arbitrarily long APs from Lebesgue’s density theorem (for instance see [J, Theorem 3]). Surprisingly we do not impose any density conditions on sets as follows, which is the first goal of this paper:

**Theorem 1.2.** Fix an integer $k \geq 3$. The graph of any slightly curved sequence with error $o((\log \log n)^{1/c_k})$ contains an AP of length $k$, where $c_k = 2^{2k+9}$.

We now define slightly curved sequences with error $o(g(n))$ and the graph of a sequence. Let $g : \mathbb{N} \to \mathbb{R}$ be an eventually positive function, and let $\mathbb{R}^+ = (0, \infty)$. A strictly increasing sequence $\{a(n)\}_{n=1}^\infty \subset \mathbb{N}$ is called a slightly curved sequence with error $o(g(n))$ if there exists a twice differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f''(x) = O(1/x^\alpha)$$

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for some $\alpha > 0$, and
\[ a(n) = f(n) + o(g(n)). \]
A slightly curved sequence with error $O(g(n))$ is also defined in the same way. For every $A \subseteq \mathbb{N}$, the \textit{graph} of a sequence $\{a(n)\}_{n \in A} \subseteq \mathbb{N}$ is defined as the set $\{(n, a(n)) \in \mathbb{N}^2 \mid n \in A\}$.

\textbf{Remark.} We write $f(x) = O(g(x))$ if there exist constants $C > 0$ and $x_0 > 0$ such that $|f(x)| \leq C g(x)$ for all $x \geq x_0$, where $g(x)$ is an eventually positive function. In this paper, the constant $C$ depends on length $k$ or $\alpha$ in (1). When we emphasize the dependence on $k$ or $\alpha$, we write $f(x) = O_{k,\alpha}(g(x))$. We also write $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$.

Section 2 will state that Theorem 1.2 with different errors can be derived from the upper bounds of $r_k(N)$. Here $r_k(N)$ is the maximal cardinality of all subsets of $\{1, 2, \ldots, N\}$ with containing no APs of length $k$. Also Section 2 will prove Theorem 1.2. Appendix A will state another result without the assumption $f''(x) = O(1/x^\alpha)$, which can be applied to the sequence $\{(c_2n^2 + c_1n + c_0)^\infty_{n=1}\}$ with a sufficiently small positive number $c_2$.

We say that a set $A \subseteq \mathbb{N}$ has \textit{positive upper density} if the condition
\[ \limsup_{N \to \infty} \frac{|A \cap [1,N]|}{N} > 0 \]
holds. The second goal of this paper is to show the following result:

\textbf{Theorem 1.3} (A generalization of Szemerédi’s theorem). If $\{a(n)\}^\infty_{n=1}$ is a slightly curved sequence with error $O(1)$ and a set $A \subseteq \mathbb{N}$ has positive upper density, then the graph of $\{a(n)\}_{n \in A}$ contains arbitrarily long APs.

As a corollary, we also obtain the following result:

\textbf{Corollary 1.4.} If a set $A \subseteq \mathbb{N}$ has positive upper density, then the graph of $\{[n^a]\}_{n \in A}$ contains arbitrarily long APs for every $1 \leq a < 2$.

Here for every $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to $x$, and $\lfloor x \rfloor$ denotes the least integer greater than or equal to $x$. Corollary 1.4 with $a = 1$ means Szemerédi’s theorem (Proposition 1.1).

\textbf{Proof of Corollary 1.4 assuming Theorem 1.3.} Fix $1 \leq a < 2$. Then $[n^a] = n^a + O(1)$. Let $f(x) = x^a$. Since $f''(x) = O(1/x^{2a-2})$, Theorem 1.3 implies Corollary 1.4. \hfill $\Box$

In particular, Corollary 1.4 with $A = \mathbb{N}$ implies the following result immediately:

\textbf{Corollary 1.5.} The graph of $\{[n^a]\}^\infty_{n=1}$ contains arbitrarily long APs for every $1 \leq a < 2$.

\textbf{Remark.} When $1 < a < 2$, the sum of the reciprocals of $\{[n^a]\}^\infty_{n=1}$ converges. Thus we cannot apply the Erdős-Turán conjecture to this sequence. Here the Erdős-Turán conjecture states that any subset of positive integers whose sum of reciprocals diverges must contain arbitrarily long APs [ET].

Corollary 1.5 ensures that the graph of $\{[n^a]\}^\infty_{n=1}$ with $1 \leq a < 2$ contains arbitrarily long APs, but the graph of $\{[n^a]\}^\infty_{n=1}$ with $a \geq 2$ does not contain any APs of length 3, which will be proved in Section 5. The sequence $\{n^2\}^\infty_{n=1}$ does not contain any APs of length 4, which was first shown by Euler in 1780. Moreover, for every integer $a > 2$, Darmon and Merel showed that the sequence $\{n^a\}^\infty_{n=1}$ does not contain any APs of length 3 [DM]. We do not know whether $\{[n^a]\}^\infty_{n=1}$ would contain APs of length 4 if $a > 2$ is not an integer. One might guess that such a sequence would not contain APs of length 4, but the following sequences are APs of length 4:

\begin{align*}
\{2^{2.2}, 11^{2.2}, 15^{2.2}, 18^{2.2}\}, \\
\{14^{2.655015}, 39^{2.655015}, 50^{2.655015}, 58^{2.655015}\}, \\
\{27^{2.720398}, 89^{2.720398}, 114^{2.720398}, 132^{2.720398}\}.
\end{align*}
2. Deduction to Another Proposition

For every $k \geq 3$ and $r \geq 2$, let $W(r, k)$ be the van der Waerden number which is the smallest number $N$ such that if $\{1, 2, \ldots, N\}$ are partitioned into $r$ different sets, then there exists at least one set which contains an AP of length $k$. Fix an integer $k \geq 3$. We consider a function $U_k$ satisfying the following properties:

(U1) There exist $x_k \geq 1$ and $y_k \geq 1$ such that a function $U_k : [x_k, \infty) \rightarrow [y_k, \infty)$ is increasing and bijective.

(U2) Any positive number $\alpha$ satisfies $U_k(r^\alpha) = O_{k, \alpha}(U_k(r))$.

(U3) The inverse function $U_k^{-1}$ of $U_k$ satisfies $W(r, k) = O_k(U_k^{-1}(r))$.

In order to prove Theorem 1.2, Section 3 will show the following proposition:

**Proposition 2.1.** Fix an integer $k \geq 3$. Assume that a function $U_k$ satisfies (U1), (U2), and (U3). If $\{a(n)\}_{n=1}^\infty$ is a slightly curved sequence with error $o(U_k(n))$, then there exists an arithmetic progression $P$ of length $k$ such that $a(P)$ is also an arithmetic progression of length $k$.

**Proof of Theorem 1.2 assuming Proposition 2.1.** Fix an integer $k \geq 3$. Let $U_k(r) = (\log \log r)^{1/c_k}$, where $c_k = 2^{k+9}$. It is enough to show that the function $U_k$ satisfies (U1), (U2), and (U3). The properties (U1) and (U2) follow from the definition of $U_k$. It also follows that

$$W(r, k) \leq 2^{2^{r^c_k}}$$

from Gowers’ upper bounds of the van der Waerden numbers [4, Theorem 18.6]. Therefore the function $U_k$ satisfies $W(r, k) \leq \exp(\exp(r^{c_k})) = U_k^{-1}(r)$, which means (U3). $\square$

**Remark.** Theorem 1.2 with different errors can be derived from the upper bounds of $r_k(N)$. To explain this derivation, we show that if a function $U_k$ satisfies (U1) and

$$(2.1) \quad r_k(N) < N/U_k(N) \quad (N \geq x_k),$$

then the function $U_k$ satisfies (U3). Let $r$ be a sufficiently large positive integer and $N := [U_k^{-1}(r)]$. If we partition $\{1, 2, \ldots, N\}$ into $r$ small sets, then the pigeonhole principle implies that there exists a set $A$ consisting of at least $[N/r]$ elements. Thus $A$ contains APs of length $k$ because the inequality

$$|A| \geq [N/r] \geq N/r \geq N/U_k(N) > r_k(N)$$

holds thanks to (2.1) and the definition of $N$. Hence it follows that

$$W(r, k) \leq N \leq U_k^{-1}(r) + 1 \leq 2U_k^{-1}(r)$$

from (U1). Therefore the function $U_k$ satisfies (U3).

From the above discussion, we can derive similar results to Theorem 1.2 by replacing $(\log \log n)^{1/c_k}$ in Theorem 1.2 with $U_k(n)$ satisfying (U1), (U2), and (2.1). In particular, by using known upper bounds of
Before proving Proposition 2.1 we define a semi-norm on the vector space $F = \{ f \mid f : \mathbb{R}^+ \to \mathbb{R} \}$. Let $k \geq 3$ be an integer and $P = \{ b(j) \}_{j=0}^{k-1} \subset \mathbb{R}^+$ be a strictly increasing sequence. We define

$$N_P(f) = \sum_{j=0}^{k-3} |\Delta^2 [f \circ b](j)|,$$

for every function $f \in F$, where $\Delta$ denotes the difference operator, that is,

$$\Delta f(x) = f(x + 1) - f(x),$$

and $\Delta^2 := \Delta \circ \Delta$. We can find that $N_P$ satisfies the following properties:

(N1) For every strictly increasing function $f \in F$,

$$N_P(f) = 0$$

if and only if $f(P)$ is an AP of length $k$.

(N2) Any function $f \in F$ satisfies

$$N_P(f) \geq 0.$$

(N3) Any two functions $f, g \in F$ satisfy

$$N_P(f + g) \leq N_P(f) + N_P(g).$$

All the properties above can be easily checked from the definition.

3. PROOF OF PROPOSITION 2.1

Proof of Proposition 2.1 Fix an integer $k \geq 3$. Take an arbitrary slightly curved sequence $\{ a(n) \}_{n=1}^{\infty}$ with error $o(U_k(n))$. Thus there exists a twice differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying (1.1) and $a(n) = f(n) + o(U_k(n))$. Let $r$ be a sufficiently large positive integer. We define $W(r) := W(r, k)$, $R(n) := a(n) - f(n)$, and

$$A(r) = \left\{ m \in \mathbb{N} \mid \left| W(r)^{3/\alpha} \right| \leq m < \left| W(r)^{3/\alpha} \right| + W(r) \right\}.$$

Every $m \in A(r)$ satisfies

$$R(m) = o(U_k(2W(r)^{\max\{3/\alpha, 1\}})) = o(U_k(U_k^{-1}(r)))) = o(r)$$

thanks to (U1), (U2), and (U3). Thus there exists a positive function $\delta(r)$ such that

$$R(m) \in [-\delta(r), \delta(r))$$

for all $m \in A(r)$, and

$$\delta(r) = o(r) \quad \text{(3.1)}$$

as $r \to \infty$. Here we define

$$I_j = \left[ -\delta(r) + \frac{2\delta(r)}{r} j, -\delta(r) + \frac{2\delta(r)}{r} (j + 1) \right], \quad A_j(r) = \{ m \in A(r) \mid R(m) \in I_j \}$$

for all $j = 0, \ldots, r-1$. Note that the union of all the small sets $A_j(r)$ equals $A(r)$. Since $W(r) = W(r, k)$, there exists $q \in \{ 0, 1, \ldots, r-1 \}$ such that $A_q(r)$ contains an arithmetic progression $P = \{ b(j) \}_{j=0}^{k-1}$ of length
Then the triangle inequality (N3) implies $N_P(a) \leq N_P(f) + N_P(R)$.

Finally, we show $N_P(a) = o(1)$ as $r \to \infty$. Since we have \eqref{3.1} and $b(j) \in A_r(r)$ for all $j = 0, 1, \ldots, k-1$, the inequality

$$N_P(R) \leq \sum_{j=0}^{k-3} (|\Delta[R \circ b](j + 1)| + |\Delta[R \circ b](j)|) \leq \frac{4(k-2)\delta(r)}{r} = o(1)$$

holds as $r \to \infty$, that is, the relation $N_P(R) = o(1)$ holds. Next, we show $N_P(f) = o(1)$ as $r \to \infty$. The mean value theorem implies that for every $j = 0, 1, \ldots, k-3$ there exist $\theta_j, \eta_j \in (0, 1)$ such that

$$\Delta^2[f \circ b](j) = \Delta[f \circ b](j + 1) - \Delta[f \circ b](j) = (\Delta[f \circ b])'(j + \theta_j) = d\{f' \circ b(j + \theta_j + 1) - f' \circ b(j + \theta_j)\} = d^2f'' \circ b(j + \theta_j + \eta_j),$$

where $b(x) := dx + e$. Since \eqref{3.2} and the assumption $f''(x) = O(1/x^\alpha)$ hold, every $j = 0, 1, \ldots, k-3$ satisfies

$$\Delta^2[f \circ b](j) = d^2f''(d(j + \theta_j + \eta_j) + e) = O\left(W(r)^2 \cdot \frac{1}{W(r)^{\frac{\alpha}{2}}}\right) = o(1)$$

as $r \to \infty$, whence it follows that $N_P(f) = o(1)$ as $r \to \infty$. Therefore a sufficiently large positive integer $r$ satisfies $N_P(a) < 1$. Since $N_P(a)$ is a non-negative integer from the definition, the equation $N_P(a) = 0$ holds, which implies that $a(P)$ is an AP of length $k$ due to (N1).

\[\square\]

### 4. Proof of Theorem 1.3

In order to prove Theorem 1.3, this section focuses on the following condition for a set $A \subset \mathbb{N}$:

(C) For any integer $k \geq 3$ and any real number $\beta > 1$, there exists a strictly increasing sequence \(\{x_n\}_{n=1}^\infty\) of positive numbers which diverges such that for any positive integer $n$ the set $A \cap [x_n^\beta, (x_n + 1)^\beta]$ contains an AP of length $k$.

Theorem 1.3 follows from the following proposition and preliminary lemma.

**Proposition 4.1.** Let $A$ be a set of positive integers with (C), and \(\{a(n)\}_{n=1}^\infty\) be a slightly curved sequence with error $O(1)$. Then the graph of $\{a(n)\}_{n \in A}$ contains arbitrarily long APs.

**Proof.** Let $k \geq 3$ be an integer. There exists a twice differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying \(\eqref{1.1}\) and $a(n) = f(n) + O(1)$. Defining $R(n) := a(n) - f(n)$, we can take an integer $M$ satisfying $|R(n)| \leq M$. We also define $\beta := 1 + \alpha/2$ and $r := 4kM$. Then the condition (C) implies that there exists a strictly increasing sequence \(\{x_n\}_{n=1}^\infty\) which diverges such that for any positive integer $n$ the set $A \cap [x_n^\beta, (x_n + 1)^\beta]$ contains an AP of length $W(r, k)$. Let $P(n)$ be such an AP of length $W(r, k)$. Then we also define the following sets:

$$I_j := \left[-M + \frac{2M}{r}j, -M + \frac{2M}{r}(j + 1)\right] (0 \leq j \leq r - 1),$$

$$A_j(n) := \{m \in P(n) \mid R(m) \in I_j\} (0 \leq j \leq r - 1).$$

Since the union of all the small sets $A_j(n)$ equals $P(n)$, for some $0 \leq q \leq r - 1$ the set $A_q(n)$ contains an arithmetic progression $P = \{b(j)\}_{j=0}^{k-1}$. Here $b(j)$ is expressed as $b(j) = dj + e$ with two positive integers
Thus letting \( d \leq (x_n + 1)^\beta - x_n^\beta \leq \beta(x_n + 1)^{\beta - 1} \) and \( e \geq x_n^\beta \) implies

\[
N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f \circ b](j)| = \sum_{j=0}^{k-3} d^2 |f''(d(j + \theta_j + \eta_j) + e)|
\]

(4.1)

\[
= O(x_n^{2(\beta - 1) - \alpha \beta}) = O(x_n^{\alpha(1 - \beta)}) = o(1),
\]

where \( \theta_j = \theta_j(d, e) \) and \( \eta_j = \eta_j(d, e) \) are real numbers satisfying \( \theta_j, \eta_j \in (0, 1) \). Moreover, the relation \( P \subset A_q(n) \) implies

\[
N_P(R) = \sum_{j=0}^{k-3} |\Delta^2[R \circ b](j)| \leq \sum_{j=0}^{k-3} (|\Delta[R \circ b](j)| + |\Delta[R \circ b](j)| + 2(k - 2)\frac{2M}{r} = 1 - 2/k.
\]

Hence a sufficiently large \( n \) satisfies \( N_P(f) < 2/k. \) The inequality \( N_P(a) < 1 \) follows from (4.1) and (4.2). Since \( N_P(a) \) is a non-negative integer, the equation \( N_P(a) = 0 \) follows, which means that \( a(P) \) is an AP of length \( k. \) Therefore, the graph of \( \{a(n)\}_{n \in A} \) contains an AP of length \( k. \)

**Lemma 4.2.** Let \( \beta > 1. \) If a set \( A \subset \mathbb{N} \) has positive upper density, then

\[
\limsup_{x \to \infty} \frac{|A \cap [x^\beta, (x + 1)^\beta]|}{|N \cap [x^\beta, (x + 1)^\beta]|} > 0
\]

holds.

**Proof.** Assuming

\[
\limsup_{x \to \infty} \frac{|A \cap [x^\beta, (x + 1)^\beta]|}{|N \cap [x^\beta, (x + 1)^\beta]|} = 0,
\]

we will deduce a contradiction. This assumption implies that for every \( \varepsilon > 0 \) there exists \( x_0 > 0 \) such that any real number \( x \geq x_0 \) satisfies

\[
|A \cap [x^\beta, (x + 1)^\beta]| \leq \varepsilon |N \cap [x^\beta, (x + 1)^\beta]|.
\]

Thus letting \( l = \lfloor N^{1/\beta} - x_0 \rfloor \) for any integer \( N > x_0^\beta \), it follows that

\[
|A \cap [1, N]| \leq |A \cap [1, x_0^\beta]| + |A \cap [x_0^\beta, (x_0 + 1)^\beta]| + \cdots + |A \cap [(x_0 + l - 1)^\beta, (x_0 + l)^\beta]|
\]

\[
\leq x_0^\beta + \varepsilon |N \cap [x_0^\beta, (x_0 + l)^\beta]| + \varepsilon \{N(1 + N^{1/\beta}) - x_0^\beta + 1\},
\]

which implies

\[
\limsup_{N \to \infty} \frac{|A \cap [1, N]|}{N} \leq \varepsilon.
\]

Therefore, the set \( A \) does not have positive upper density, which is a contradiction.

**Proof of Theorem 1.3.** Thanks to Proposition 4.1, it is enough to show that any set \( A \subset \mathbb{N} \) with positive upper density satisfies (C). Let \( k \geq 3 \) be an integer and \( \beta > 1 \) be a real number. Lemma 4.2 implies that there exists \( 0 < \delta \leq 1 \) such that

\[
\limsup_{x \to \infty} \frac{|A \cap [x^\beta, (x + 1)^\beta]|}{|N \cap [x^\beta, (x + 1)^\beta]|} = \delta,
\]

which means that there exists a strictly increasing sequence \( \{x_n\}_{n=1}^\infty \) of positive numbers which diverges such that any positive integer \( n \) satisfies

\[
|A \cap [x_n^\beta, (x_n + 1)^\beta]| \geq \frac{\delta}{2} |N \cap [x_n^\beta, (x_n + 1)^\beta]|.
\]
Take an integer \( n_0 \) satisfying \( x_{n_0} \geq N(\delta/2, k) \). Then Proposition 1.4 implies that the set \( A \cap [x_n^2, (x_n + 1)^2] \) with \( n \geq n_0 \) contains an AP of length \( k \). Therefore, any set \( A \) with positive upper density satisfies (C). \( \square \)

5. Future Works

**Question 5.1.** Suppose that a twice differentiable function \( f : \mathbb{R}^+ \to \mathbb{R} \) satisfies (1.1). If a strictly increasing sequence \( \{a(n)\}_{n=1}^{\infty} \subset \mathbb{N} \) can be written as

\[
a(n) = f(n) + O(f'(n)),
\]

then does the sequence \( \{a(n)\}_{n=1}^{\infty} \) contain arbitrarily long arithmetic progressions?

Although we do not know the answer to this question, it is probably affirmative. Indeed, it is affirmative when the coefficients of the error \( O(f'(n)) \) are contained by a finite set \( \{c_1, c_2, \ldots, c_m\} \). See Appendix C.

Next, we remark that the sequence of all prime numbers is a slightly curved sequence. Indeed, the following asymptotic expansion holds (C):

\[
p_n = f(n) + o\left(\frac{n}{\log n}\right).
\]

Here \( p_n \) is the \( n \)-th prime and the function \( f(x) = x \{\log x + \log \log x - 1 + (\log \log x - 2)/\log x\} \), which satisfies \( f''(x) = O(1/x) \). Thus \( \{p_n\}_{n=1}^{\infty} \) is a slightly curved sequence with error \( o(n/\log n) \). If we can improve the error \( O((\log \log n)^{1/c_k}) \) in Theorem 1.2 to \( o(n/\log n) \), the set of all prime numbers contains arbitrarily long arithmetic progressions, which was shown by Green and Tao [GT1]. If assuming the Riemann hypothesis, due to [AT] Theorem 6.1, we obtain the evaluation

\[
|p_n - \text{ali}(n)| \leq \frac{1}{\pi} \sqrt{n}(\log n)^{5/2},
\]

where \( \text{ali}(x) \) is the inverse function of the logarithmic integral function \( \text{li}(x) \). Thus \( \{p_n\}_{n=1}^{\infty} \) is a slightly curved sequence with error \( O(\sqrt{n} \log n)^{5/2}) \) if we assume the Riemann hypothesis.

**Question 5.2.** Does the set of all prime numbers satisfy (C)? In particular, does the graph of \( \{a(p)\}_{p \ \text{prime}} \) contain arbitrarily long arithmetic progressions if \( \{a(n)\}_{n=1}^{\infty} \) is any slightly curved sequence with error \( O(1) \)?

The set of all prime numbers does not have positive upper density. Thus we can not apply Theorem 1.3. Nevertheless, we can answer this question if replacing \( p \) with \( n \log n + n \log \log n + O(1) \). See Theorem A.3.

**Question 5.3.** Is it true that

\[
\sup \{ a \geq 1 \mid \text{the sequence } \{\lfloor n^a \rfloor\}_{n=1}^{\infty} \text{ contains arbitrarily long APs} \} = 2?
\]

Instead of answering this question, we show that the graph of \( \{\lfloor n^a \rfloor\}_{n=1}^{\infty} \) with \( a \geq 1 \) contains an AP of length 3 if and only if \( 1 \leq a < 2 \). The if part follows from Corollary 1.5. We show the only if part, i.e., the graph of \( \{\lfloor n^a \rfloor\}_{n=1}^{\infty} \) with \( a \geq 2 \) does not contain any APs of length 3 by contradiction. Suppose that a sequence \( \{(e + dj)^a\}_{j=0}^{\infty} \) is an AP for some two positive integers \( d \) and \( e \) and some real number \( a \geq 2 \). Then the inequality \( |(e + 2d)^a + e^a - 2(e + d)^a| < 2 \) holds. The mean value theorem implies that

\[
(e + 2d)^a + e^a - 2(e + d)^a = d^2 a(a - 1)(e + d\theta + d\eta)^{a-2},
\]

where \( \theta \) and \( \eta \) are real numbers satisfying \( \theta, \eta \in (0, 1) \). Thus it follows that

\[
2 > |(e + 2d)^a + e^a - 2(e + d)^a| = d^2 a(a - 1)(e + d\theta + d\eta)^{a-2} \geq 2,
\]

which is a contradiction.

The above argument implies

\[
\sup \{ a \geq 1 \mid \text{the graph of } \{\lfloor n^a \rfloor\}_{n=1}^{\infty} \text{ contains an AP of length 3} \} = 2,
\]

but we do not achieve the answer to Question 5.3.
Appendix A. Slightly curved sequence without assumption $f''(x) = O(1/x^\alpha)$

As stated in Section 1, Theorem 1.2 requires the assumption $f''(x) = O(1/x^\alpha)$ with some positive number $\alpha$. This appendix discusses a slightly curved sequence with error $O(1)$ without this assumption. That is, the following theorem holds.

Theorem A.1. Let $f$ be a twice differentiable function satisfying $\lim_{x \to \infty} f''(x) = 0$ and $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ be a strictly increasing sequence satisfying $a(n) = f(n) + O(1)$. Then the graph of $\{a(n)\}_{n=1}^{\infty}$ contains arbitrarily long APs.

Of course, Theorem A.2 does not completely contain Theorem A.1. For example, the function $f(x) = \int_2^{x} \log(t) dt$ satisfies the assumption in Theorem A.1 but does not satisfy the assumption in Theorem 1.2 because $f''(x) = 1/\log x$. Here $\log(t)$ is the offset logarithmic integral function, i.e., $\log(t) := \int_t^{1} (1/\log s) ds$.

Theorem A.1 is derived from Theorem A.2 below immediately, which is more exact. For example, when length $k$ is given, Theorem A.2 implies that the graph of $\{a(n)\}_{n=1}^{\infty} = 27.$

\[ \frac{c_2n^2 + c_1n + c_0}{x}, \]

with a sufficiently small positive number $c_2$ contains an AP of length $k$, but Theorem A.1 does not imply.

Theorem A.2. Let $k \geq 3$ and $r \geq 1$ be integers, $f$ be a twice differentiable function, $R(n)$ be a bounded function satisfying $M_1 \leq R(n) \leq M_2$ for some two numbers $M_1$ and $M_2$, and $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ be a strictly increasing sequence satisfying $a(n) = f(n) + R(n)$. If the inequality

\[ \limsup_{x \to \infty} |f''(x)| < \left( \frac{k-1}{W(r,k)-1} \right)^2 \left( \frac{1}{k-2} \right) \left( \frac{2(M_2-M_1)}{r} \right) \]

holds, then the graph of $\{a(n)\}_{n=1}^{\infty}$ contains an AP of length $k$.

In general, the van der Waerden number $W(r,k)$ is large and so the efficient of the factor $W(r,k)$ is greater than that of the other factors of the right-hand side in (A.1). Hence, the van der Waerden number $W(r,k)$ should be small in order to make the right-hand side in (A.1) large. When putting

\[ r = [2(M_2-M_1)(k-2)] + 1 > 2(M_2-M_1)(k-2), \]

we can make $W(r,k)$ smallest. Let us use (A.2). When $M_2-M_1 = 1$ and $k = 3$, the right-hand side in (A.1) with (A.2) equals $1/(3 \cdot 13^2)$ because $r = 3$ and $W(3,3) = 27$.

Proof. Thanks to (A.1), there exists a positive integer $n_0$ such that any real number $x \geq n_0$ satisfies

\[ |f''(x)| < \left( \frac{k-1}{W-1} \right)^2 \left( \frac{1}{k-2} \right) \left( \frac{2M}{r} \right), \]

where $W = W(r,k)$ and $M = M_2-M_1$. Then we also define the following sets:

\[ I_j := [M_1 + \frac{M}{r}j, M_1 + \frac{M}{r}(j+1)] \quad (0 \leq j \leq r-1), \]

\[ A := \mathbb{N} \cap [n_0, n_0 + W - 1], \]

\[ A_j := \{ n \in A \mid R(n) \in I_j \} \quad (0 \leq j \leq r-1). \]

It can be easily checked that the interval $[M_1, M_2]$ is the union of all the small intervals $I_j$ and the set $A$ is the union of all the small sets $A_j$. Hence, there exists an integer $q \in \{0, 1, \ldots, r-1\}$ such that $A_q$ contains an arithmetic progression $P = \{b(j)\}_{j=0}^{k-1}$. Here $b(j)$ is expressed as $b(j) = dj + e$ with two
positive integers \(d\) and \(e\). The inequalities \((k-1)d \leq W - 1\) and \(e \geq n_0\) implies

\[
N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f \circ b](j)| = \sum_{j=0}^{k-3} d^2 |f''(d(j + \theta_j + \eta_j) + e)|
\]

(A.3)

\[
< \sum_{j=0}^{k-3} \left( \frac{W - 1}{k - 1} \right)^2 \left( \frac{k - 1}{W - 1} \right)^2 \left( \frac{1}{k - 2} - \frac{2M}{r} \right) = 1 - 2(k - 2) \frac{M}{r},
\]

where \(\theta_j = \theta_j(d, e)\) and \(\eta_j = \eta_j(d, e)\) are real numbers satisfying \(\theta_j, \eta_j \in (0, 1)\). Moreover, the relation \(P \subset A_q\) implies

(A.4)

\[
N_P(R) = \sum_{j=0}^{k-3} |\Delta^2[R \circ b](j)| \leq \sum_{j=0}^{k-3} (|\Delta[R \circ b](j + 1)| + |\Delta[R \circ b](j)|) \leq 2(k - 2) \frac{M}{r}.
\]

Hence the inequality \(N_P(a) < 1\) follows from \((A.3)\) and \((A.4)\). Since \(N_P(a)\) is a non-negative integer, the equation \(N_P(a) = 0\) follows, which means that \(\{a \circ b(j)\}_{j=0}^{k-1}\) is an AP of length \(k\). Therefore, the graph of \(\{a(n)\}_{n=1}^{\infty}\) contains an AP of length \(k\). \(\Box\)

We use the proof of Theorem \((A.2)\) in order to prove the next theorem. As stated in Section \(5\), we cannot answer Question \((5.2)\) but we can prove the following theorem.

**Theorem A.3.** Let \(\{a(n)\}_{n=1}^{\infty}\) be a slightly curved sequences with error \(O(1)\), \(g(x)\) be the function \(x \log x + x \log \log x\), and \(A = \{\tilde{a}(n)\}_{n=1}^{\infty}\) be a strictly increasing sequence with \(\tilde{a}(n) = g(n) + O(1)\). Then the graph of \(\{a(n)\}_{n \in A}\) contains arbitrarily long APs.

**Proof.** Thanks to Proposition \((4.1)\), it is enough to show that the set \(A = \{\tilde{a}(n)\}_{n=1}^{\infty}\) satisfies (C). Let \(k \geq 3\) be an integer and \(\beta > 1\) be a real number. The first and second derivatives of \(g\) are as follows:

\[
g'(x) = \log x + 1 + \log \log x + 1/\log x, \quad g''(x) = 1/x + 1/x \log x - 1/x (\log x)^2 > 0 \quad (x \geq 3).
\]

Note that \(A = \{\tilde{a}(n)\}_{n=3}^{\infty}\) is a slightly curved sequence with error \(O(1)\). When applying Theorem \((A.2)\) to \(A = \{\tilde{a}(n)\}_{n=3}^{\infty}\), we can take two real numbers \(M_1\) and \(M_2\) satisfying \(M_1 \leq \tilde{a}(n) - g(n) \leq M_2\) and the integer \(r\) with \((A.2)\). Let \(W = W(r, k)\) and \(M = M_2 - M_1\). Any positive number \(x\) and any positive integer \(n\) satisfy

\[
\tilde{a}(n + W - 1) - \tilde{a}(n) \leq g(n + W - 1) - g(n) + M_2 - M_1 = (W - 1)g'(n + W - 1) + M,
\]

\[
(x + 1)^\beta - x^\beta \geq \beta x^{\beta - 1}.
\]

When we put \(x_n := (g(n) + M_1)^{1/\beta} \leq \tilde{a}(n)^{1/\beta}\), the inequalities \(x_n > (n \log n + M_1)^{1/\beta}\) and \(g'(n + W - 1) < 4\log(n + W - 1)\) holds. Hence there exists a positive integer \(n_0\) such that any positive integer \(n \geq n_0\) satisfies

\[
(W - 1)g'(n + W - 1) + M \leq \beta x_n^{\beta - 1}.
\]

Thus any positive integer \(n \geq n_0\) satisfies \(\tilde{a}(n + W - 1) - \tilde{a}(n) \leq (x_n + 1)^\beta - x_n^\beta\). When we take a sufficiently large integer \(n_1 \geq n_0\), the proof of Theorem \((A.2)\) implies that the set \(A \cap [x_n^\beta, (x_n + 1)^\beta]\) contains an AP of length \(k\) for any integer \(n \geq n_1\). Therefore, the sequence \(A = \{\tilde{a}(n)\}_{n=3}^{\infty}\) satisfies (C). \(\Box\)

**Appendix B. Result similar to Theorem A.2**

This appendix states a result similar to Theorem \((A.2)\) which is not completely contained by Theorem \((A.2)\). Indeed, Corollary \((B.2)\) below implies that the graph of \(\{a(n)\}_{n=1}^{\infty}\) with a positive number \(c_2 < 1/18\) contains an AP of length 4, which is better evaluation than that of Theorem \((A.2)\).
Let $\Sigma$ be a nonempty finite set of non-negative integers. In order to show Theorem B.1 we discuss words over the alphabet $\Sigma$. Here we allow words to continue infinitely on the right side such as $000 \cdots$, and to be the empty word. The length of a word $w$ (i.e., the number of all the letters of $w$) is denoted by $|w|$ and the sum of all the letters of a word $w$ is denoted by $\sum w$. Then we focus on the following condition $(C_k)$ for a word $w$: there exist $k+1$ finite length words $w_0, \ldots, w_k$ and a word $w_{k+1}$ such that

$$w = w_0 \cdots w_k w_{k+1}, \quad |w_1| = \cdots = |w_k| > 0, \quad \text{and} \quad \sum w_1 = \cdots = \sum w_k.$$ 

We also focus on the following statement: there exists a positive integer $n$ such that any word $w$ whose length is not smaller than $n$ satisfies the condition $(C_k)$. We call it the statement $(S_k)$. The statement $(S_k)$ depends on the alphabet $\Sigma$. When $k = 2, 3$, in Table 1 we exhibit integers $n$ in the statement $(S_k)$ for several alphabets $\Sigma$. In the proof of Theorem B.1 we use the statement $(S_3)$. Indeed, the statement $(S_3)$ for $\Sigma = \{0, 1\}$ can be checked by taking $n = 10$. Cassaigne et al. [CCSS] proved that the statement $(S_3)$ did not hold for $\Sigma = \{0, 1, 3, 4\}$. For details, see [CCSS].

**Proof of Theorem B.1** We show that for any positive integer $k$, there exists a positive integer $n_0$ such that the function $|\Delta f(n)|$ takes a constant value for any integer $n$ satisfying $0 \leq n - n_0 \leq k$, by contradiction. Suppose that the assertion does not hold, namely, for some positive integer $k$, there exists no positive integer $n_0$ such that the function $|\Delta f(n)|$ takes a constant value for any integer $n$ satisfying $0 \leq n - n_0 \leq k$. Putting the function $x := [N(x)/k] - 1$, we find $N(x) = kN(x)/k > kx$. Hence the inequality $N(x) \geq k\lambda(x) + 1$ holds. Our assumption $\Delta^2 f(n) \geq 0$ yields $|\Delta f(n+1)| \geq |\Delta f(n)|$. Using this inequality and the assumption of the proof by contradiction to obtain the following (a), we have

$$x > f(k\lambda(x) + 1) - f(1) = \sum_{i=1}^{k\lambda(x)} \Delta f(i) \geq \sum_{i=1}^{k\lambda(x)} |\Delta f(i)| \geq k \sum_{i=0}^{\lambda(x)-1} i = k \left( \frac{\lambda(x) - 1}{2} \right) \left( \frac{N(x)}{k} - 1 \right) \left( \frac{N(x)}{k} - 2 \right),$$

that is, $(N(x) - k)(N(x) - 2k) < 2kx$. Since our assumption $\sup_{x>0} N(x)^2/x = \infty$ implies that $N(x) \to \infty$ when $x$ goes to infinity, we obtain $(N(x)/2)^2 < 2kx$ by taking a sufficiently large positive number $x$, which contradicts our assumption $\sup_{x>0} N(x)^2/x = \infty$.

Putting $k = 9$, we can take two integers $n_0 \geq 1$ and $c \geq 0$ satisfying $|\Delta f(n)| = c$ for $0 \leq n - n_0 \leq 9$. That is, $\Delta f(n) - c \in \{0, 1\}$ for $0 \leq n - n_0 \leq 9$. Since the statement $(S_3)$ holds for $\Sigma = \{0, 1\}$, there

| $\Sigma$ | $k$ | $n$ |
|-------|------|-----|
| $\{0, 1\}$ | 2 | 4 |
| $\{0, 1, 2\}$ | 2 | 8 |
| $\{0, 1, 2, 3\}$ | 3 | 51 |
| $\{0, 1, 3, 4\}$ | | |

(S3) does not hold [CCSS].
exist two positive integers \( e \) and \( d \) such that
\[
\sum_{i=e}^{e+d-1} (\Delta [f(i)] - c) = \sum_{i=e+d}^{e+2d-1} (\Delta [f(i)] - c) = \sum_{i=e+2d}^{e+3d-1} (\Delta [f(i)] - c).
\]
Thus the equation
\[
[f(e + d)] - [f(e)] = [f(e + 2d)] - [f(e + d)] = [f(e + 3d)] - [f(e + 2d)]
\]
holds, which means that the sequence \( \{[f(e + dn)]\}_{n=0}^{\infty} \) is an AP of length 4. That is, the graph of \( \{[f(n)]\}_{n=0}^{\infty} \) contains an AP of length 4.

**Corollary B.2.** If a function \( f : \mathbb{N} \to \mathbb{R}^+ \) satisfies \( \Delta f > 0 \), \( \Delta^2 f \geq 0 \), and \( \limsup_{x \to \infty} N(x)^2/x > 18 \), then the graph of \( \{[f(n)]\}_{n=0}^{\infty} \) contains an AP of length 4.

**Proof.** If the function \( N(x)^2/x \) is not bounded, our assertion follows from Theorem B.1. Hence we assume that the function \( N(x)^2/x \) is bounded. We show that there exists a positive integer \( n_0 \) such that the function \( [\Delta f(n)] \) takes a constant value for any integer \( n \) satisfying \( 0 \leq n - n_0 \leq k \), by contradiction. By the same way as the proof of Theorem B.1, the inequality \( (N(x) - 9)(N(x) - 18) < 18x \) can be proved. Moreover, since the function \( N(x)^2/x \) is bounded, the limit
\[
\frac{N(x)}{x} = \frac{N(x)}{x^{1/2}} \frac{1}{x^{1/2}} \to 0 \quad x \to \infty
\]
holds. Thus the inequality \( \limsup_{x \to \infty} N(x)^2/x < 18 \) follows, which contradicts our assumption. Therefore, there exists two integers \( n_0 \) and \( c \) such that \( [\Delta f(n)] = c \) for any integer \( n \) satisfying \( 0 \leq n - n_0 \leq 9 \). The remaining part can be shown in the same way as the proof of Theorem B.1. \( \square \)

**Remark.** Corollary B.2 implies that the graph of \( \{[c_n^2 + c_1n + c_0]\}_{n=1}^{\infty} \) with a positive number \( c_2 < 1/18 \) contains an AP of length 4. On the other hand, Theorem A.2 only implies that the same graph with a positive number \( c_2 < 1/(6 \cdot 13^2) \) contains an AP of length 3.

The following propositions, which provide alternative conditions of \( \limsup_{x \to \infty} N(x)^2/x > 18 \) or \( \sup_{x>0} N(x)^2/x = \infty \), can be easily proved.

**Proposition B.3.** If a function \( f : \mathbb{N} \to \mathbb{R}^+ \) satisfies \( \Delta f > 0 \) and \( \Delta^2 f \geq 0 \), then for any positive number \( c \), the following three conditions are equivalent:

(i) \( \limsup_{x \to \infty} N(x)^2/x > c \), (ii) \( \limsup_{n \to \infty} n^2/f(n) > c \), and (iii) \( \liminf_{n \to \infty} f(n)/n^2 < 1/c \).

**Proof.** First, we note that the limit \( f(n) \geq f(1) + (n - 1)\Delta f(1) \to \infty \) holds when \( n \) goes to infinity. The implication (ii)\(\Rightarrow\)(i) follows from the definition of \( N(x) \). Next, we show the implication (i)\(\Rightarrow\)(ii) by contradiction. Suppose that the condition (i) holds and the condition (ii) does not hold, i.e., \( \limsup_{x \to \infty} N(x)^2/x > c \) and \( \limsup_{n \to \infty} n^2/f(n) \leq c \). Our assumption \( \Delta f > 0 \) and the limit \( f(n) \to \infty \) stated at first imply that for any positive number \( x \), we can take only one positive integer \( n = n(x) \) satisfying \( f(n) \leq x < f(n + 1) \). Hence any positive number \( x \) satisfies
\[
N(x)^2/x < (n + 1)^2/f(n) = n^2/f(n) + 2n/f(n) + 1/f(n),
\]
where \( n = n(x) \). Now, the assumption \( \limsup_{n \to \infty} n^2/f(n) \leq c \) yields that \( \limsup_{n \to \infty} n/f(n) = 0 \) and \( \limsup_{n \to \infty} f(n)/n = 0 \). Thus, noting \( n = n(x) \to \infty \) as \( x \to \infty \) and taking the limit in \( \text{(B.1)} \) as \( x \to \infty \), we have \( c < \limsup_{x \to \infty} N(x)^2/x \leq \limsup_{n \to \infty} n^2/f(n) \leq c \), which is a contradiction. Therefore, the implication (i)\(\Rightarrow\)(ii) holds. The remaining part, i.e., the equivalence (ii)\(\iff\)(iii) is trivial. \( \square \)

**Proposition B.4.** If a function \( f : \mathbb{N} \to \mathbb{R}^+ \) satisfies \( \Delta f > 0 \) and \( \sum_{n=1}^{\infty} 1/f(n)^s = \infty \) for some real number \( s > 1/2 \), then the condition \( \sup_{x>0} N(x)^2/x = \infty \) holds.
Proof. Let $\zeta$ be the Riemann zeta function. We show our assertion by contradiction. Suppose that the function $N(x^2/x)$ is bounded. Then the inequality $N(x^2/x) \leq M$ holds for some positive integer $M$. Thus the inequality $n^2/f(n) \leq M$ holds and any real number $s > 1/2$ satisfies $\sum_{n=1}^{\infty} 1/f(n)^s \leq M\zeta(2s) < \infty$, which is a contradiction.

\section*{Appendix C. Partial Answer to Question 5.1}

This section proves that Question 5.1 is affirmative when the coefficients of the error $O(f'(n))$ are contained by a finite set $\{c_1, c_2, \ldots, c_m\}$. That is, we prove the following theorem.

\begin{thm}
Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function satisfying (1.1), and $\tilde{R} : \mathbb{N} \to \{c_1, c_2, \ldots, c_m\}$ be a bounded function. If a strictly increasing sequence $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ can be written as
\begin{equation}
\tag{C.1}
a(n) = f(n) + \tilde{R}(n)f'(n) + O(1),
\end{equation}
then the sequence $\{a(n)\}_{n=1}^{\infty}$ contains arbitrarily long APs.
\end{thm}

Proof. Fix an integer $k \geq 3$. First, the assumption $f''(x) = O(1/x^\alpha)$ implies
\begin{equation}
\tag{C.2}
a(n) = f(n) + \tilde{R}(n)f'(n) + O(1) = f(n + \tilde{R}(n)) + O(1).
\end{equation}
We set $\tilde{a}(n) = n + \tilde{R}(n)$, $W = W(m, W(r, k))$, and the following sets:
\begin{align*}
\tilde{A} &:= N \cap [[W^{3/\alpha}], [W^{3/\alpha}] + W - 1], \\
\tilde{A}_j &:= \{ n \in \tilde{A} \mid \tilde{R}(n) = c_j \} \quad (1 \leq j \leq m).
\end{align*}
Since the union of all the small sets $\tilde{A}_j$ is $\tilde{A}$, a small set $\tilde{A}_q$ contains an arithmetic progression $\tilde{P}$ of length $W(r, k)$. Thus it follows that
\begin{equation*}
N_{\tilde{P}}(\tilde{a}) \leq N_{\tilde{P}}(\text{id}) + N_{\tilde{P}}(\tilde{R}) = 0.
\end{equation*}
In other words, the sequence $\tilde{a}(\tilde{P})$ (of real numbers) is also an AP of length $W(r, k)$.

Noting that the sequence $\tilde{a}(\tilde{P})$ is a strictly increasing sequence, we define the function $R : \tilde{a}(\tilde{P}) \to \mathbb{R}$ as
\begin{equation}
\tag{C.2}
R(\tilde{a}(n)) := a(n) - f(\tilde{a}(n)) \quad (n \in \tilde{P}).
\end{equation}
Then the function $R$ is bounded. Thanks to (C.1), we can take two real numbers $M_1$ and $M_2$ satisfying $M_1 \leq R(x) \leq M_2$ such that $M_1$ and $M_2$ are independent of $r$. We set $M = M_2 - M_1$ and the following sets:
\begin{align*}
I_j &:= \left[ M_1 + \frac{M}{r}j, M_1 + \frac{M}{r}(j + 1) \right] \quad (0 \leq j \leq r - 1), \\
A_j &:= \{ x \in \tilde{a}(\tilde{P}) \mid R(n) \in I_j \} \quad (0 \leq j \leq r - 1).
\end{align*}
Since the union of all the small sets $A_j$ is $\tilde{a}(\tilde{P})$, a small set $A_q$ contains an arithmetic progression $P = \{b(j)\}_{j=0}^{k-1}$ of length $k$. Here $b(j)$ is expressed as $b(j) = dj + e$ with two positive integers $d$ and $e$. Moreover, the inequalities
\begin{align*}
d &\leq \tilde{a}([W^{3/\alpha}] + W - 1) - \tilde{a}([W^{3/\alpha}]) \leq W - 1 + M, \\
e &\geq \tilde{a}([W^{3/\alpha}]) \geq [W^{3/\alpha}] + M_1
\end{align*}
hold. These inequalities imply that
\[ N_P(R) \leq 2(k - 2) \frac{M}{r} = o(1), \]
\[ N_P(f) = \sum_{j=0}^{k-3} d^2 \left| f''(d(j + \theta_j + \eta_j) + e) \right| = O\left( \frac{(W - 1 + M)^2}{\left\lfloor \frac{W^3}{\alpha} \right\rfloor + M} \right) = O\left( \frac{1}{W} \right) = o(1) \]
as \( r \to \infty \), where \( \theta_j = \theta_j(d,e) \) and \( \eta_j = \eta_j(d,e) \) are real numbers satisfying \( \theta_j, \eta_j \in (0,1) \). Thus any sufficiently large positive integer \( r \) satisfies \( N_P(f) + N_P(R) < 1 \).

Recall (C.2). The domains of \( f \) and \( R \) are the same but the domain of \( a \) is different from them. Hence we must take a set like the inverse image of \( P \) under \( \tilde{a} \). Now, the restricted function \( \tilde{a}|\tilde{P} \) is injective. When we denote by \( P' \) the inverse image of \( P \subset \tilde{a}(\tilde{P}) \) under this restricted function, the sequence \( P' \) is also an AP of length \( k \) and the inequality \( N_{P'}(a) \leq N_P(f) + N_P(R) < 1 \) holds. Since \( N_{P'}(a) \) is a non-negative integer, the equation \( N_{P'}(a) = 0 \) follows, which means that \( a(P') \) is an AP of length \( k \). □

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