PERIODIC TRAVELING WAVE SOLUTIONS OF PERIODIC INTEGRODIFFERENCE SYSTEMS

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Abstract. This paper is concerned with the periodic traveling wave solutions of integrodifference systems with periodic parameters. Without the assumptions on monotonicity, the existence of periodic traveling wave solutions is deduced to the existence of generalized upper and lower solutions by fixed point theorem and an operator with multi steps. The asymptotic behavior of periodic traveling wave solutions is investigated by the stability of periodic solutions in the corresponding initial value problem or the corresponding difference systems. To illustrate our conclusions, we study the periodic traveling wave solutions of two models including a scalar equation and a competitive type system, which do not generate monotone semiflows. The existence or nonexistence of periodic traveling wave solutions with all positive wave speeds is presented, which implies the minimal wave speeds of these models.

1. Introduction. In population dynamics, there have been established a number of discrete time spatial contact models, for example, we refer to several earlier models and their analysis on changes of gene frequency by Lui [24, 25, 26, 28], Slatkin [34]. Among these models, one well studied type is the integrodifference equations that may describe populations with discrete, nonoverlapping generations and well-defined growth and dispersal stages [10, 27]. A typical integrodifference equation has the following form

$$U(x, n + 1) = \int_{\mathbb{R}} f(U(y, n))k(x - y)dy,$$

where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $U(x, n)$ denotes the population density of the $n$-th generation at location $x$, $f : \mathbb{R} \to \mathbb{R}$ is often called the birth function while $k(x)$ is a probability function on the spatial migration of individuals. In the past decades, much attention has been paid to the traveling wave solutions of (1). Here, a traveling wave solution of (1) is a special entire solution

$$U(x, n) = \phi(\xi), \xi = x + cn \in \mathbb{R},$$

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where $\phi$ is the wave profile that propagates through the one-dimensional spatial domain $\mathbb{R}$ at the constant wave speed $c$. By the above definition, we see that the corresponding wave system of (1) defines an operator, so the existence of traveling wave solutions can be studied by fixed point theorem, see Lin [17] and references cited therein for some results. Moreover, when an integrodifference equation/system satisfies proper monotone condition, the existence of traveling wave solutions may be studied by the theory of monotone dynamical systems, see Fang and Zhao [7], Liang and Zhao [16], Weinberger [39, 40], Weinberger et al. [41]. In particular, the results established for abstract monotone integrodifference equations/systems have been applied to several other parabolic type systems in these works. Note that the wave profile is a function of single variable, then the limit behavior can be studied by fluctuation method, contracting rectangles and so on when the traveling wave solution is not monotone, see Hsu and Zhao [9], Li et al. [12], Lin [17], Wang and Castillo-Chavez [36].

The above model (1) was established in the homogeneous habitat and environment, in which the inhomogeneous habitat and seasonal change were not taken into account. In some evolutionary systems, it has been proved that the inhomogeneous habitat and seasonal change may lead to some new phenomena, see a survey paper by Xin [44] from the viewpoint of spatial propagation, and Cantrell and Cosner [3] from the viewpoint of persistence and extinction of population models. We now focus on the integrodifference systems. When the spatial inhomogeneous is involved, there are some conclusions on monotone/cooperative integrodifference systems in periodic habitat, see Ding et al. [4], Fang et al. [8], Weinberger [40], Wu and Zhao [43] and references cited therein. When the time periodic is concerned, Liang et al. [15] studied the asymptotic spreading and traveling wave solutions of an abstract monotone integrodifference equation, and the results can be applied to other models generating monotone semiflows.

In this paper, we study the periodic traveling wave solutions of the following integrodifference system with time periodic parameters

$$u_i(x,n+1) = \int_{\mathbb{R}} F_i(n,u_1(y,n),\cdots,u_m(y,n))k_i(x-y)dy,$$ 

for all $i \in I, \xi \in \mathbb{R}, n \in \mathbb{Z}$. Similar to the study on (1), to formulate a desired evolutionary process, the traveling wave solution must satisfy proper asymptotic behavior when $\xi \to \pm \infty$.

Intuitively, the existence of (3) can not be investigated by an operator similar to that in (1) since it also depends on $n \in \mathbb{Z}$. Motivated by the periodic property, we try to construct an $N$–steps mapping to obtain the existence of traveling wave
solutions. More precisely, we first introduce the definition of generalized upper and lower solutions of (3) although (3) does not satisfy monotone assumptions. Then we construct a potential wave profile set by the generalized upper and lower solutions, and discuss the existence of traveling wave solutions by Schauder’s fixed point theorem and an \(N\)-steps operator. The process implies the existence of (3) can be obtained by the existence of generalized upper and lower solutions. Since we do not require the cooperative property on (3), our abstract conclusion can be applied to many models.

When the limit behavior of nonconstant solutions of (3) as \(\xi \to \pm \infty\) is concerned, there are also some difficulties since the traveling wave solutions may be nonmonotone. Because \(\Phi(\xi, n)\) depends on two variables, it is difficult to analysis the limit behavior by the methods mentioned above. In literature, there are many results on the stability of periodic difference systems, can we use the known results to study the limit behavior? Since a traveling wave solution is a special solution, we try to estimate some properties of the corresponding initial value problem to present the limit behavior of traveling wave solutions as \(\xi \to \infty\). Moreover, the initial value problem may be studied by the dynamics of the corresponding difference systems, so we achieve our purpose. In particular, we do not consider the limit behavior when \(\xi \to -\infty\) since the behavior may be obtained by the property of upper and lower solutions in many models.

To illustrate our theory, we study

\[
w(x, n + 1) = \int_{\mathbb{R}} b(n, w(y, n))k(x - y)dy,
\]

and

\[
\begin{align*}
   u_1(x, n + 1) &= \int_{\mathbb{R}} \frac{1 + r_1(n)u_1(x - y, n) + a_1(n)u_2(x - y, n)}{1 + r_2(n)u_2(x - y, n) + a_2(n)u_1(x - y, n)} k_1(y)dy, \\
u_2(x, n + 1) &= \int_{\mathbb{R}} \frac{1 + r_2(n)u_2(x - y, n) + a_2(n)u_1(x - y, n)}{1 + r_1(n)u_1(x - y, n) + a_1(n)u_2(x - y, n)} k_2(y)dy,
\end{align*}
\]

in which all the parameters are periodic and will be illustrated in later sections, and we refer to Hsu and Zhao [9], Li et al. [13], Lin [17], Zhang and Pan [45] for the models with constant coefficients. The first equation may be nonmonotone, of which the monotonicity depends on the property of \(b\). There are the so-called interspecific and intraspecific competitions in the second system, which can not be studied by the theory of monotone semiflows when the trivial steady state and coexistence state are concerned (we may refer to Fang and Zhao [6], Fang et al. [8], Weinberger et al. [40] for the dynamics between the resident and the invader, which can be changed into a cooperative system that admits ordered steady states). We shall present the existence or nonexistence of traveling wave solutions for all positive wave speed, and so obtain the minimal wave speeds. Our results could complete/improve some known conclusions in [17] when the coefficients are constant or \(N = 1\).

The rest of this paper is organized as follows. In Section 2, we shall present some preliminaries including some notations, assumptions and known results. Section 3 is devoted to an abstract result on the existence of traveling wave solutions. In Section 4, we study the asymptotic behavior of traveling wave solutions when \(\xi \to \infty\). Two examples will be presented in Sections 5-6.

2. Preliminaries. In this paper, we shall utilize the standard partial ordering in \(\mathbb{R}^m\). Define \(C\) by

\[
   C = \{U(\xi) : U(\xi) : \mathbb{R} \to \mathbb{R}^m \text{ is uniformly continuous in } \xi \text{ and bounded}\}
\]
and
\[ C_{[0, M]} = \{ U(\xi) : U(\xi) \in C \text{ and } 0 \leq U(\xi) \leq M \text{ for all } \xi \in \mathbb{R} \} , \]
in which \( M \) is a vector with positive components. Let \( \| \cdot \| \) be the supremum norm in \( \mathbb{R}^m \) and \( \mu > 0 \) be a constant. Define
\[ B_\mu = \left\{ U \in C : \sup_{x \in \mathbb{R}} \| U(x) \| e^{-\mu |x|} < \infty \right\} \]
and
\[ |U|_\mu = \sup_{x \in \mathbb{R}} \| U(x) \| e^{-\mu |x|} . \]

Then \( (B_\mu, |\cdot|_\mu) \) is a Banach space. By the decay property of the norm, we have the following conclusion.

**Proposition 1.** Assume that \( D \subset C \) is bounded in the sense of \( \| \cdot \| \). If \( D \) is equicontinuous in the sense of \( \| \cdot \| \), then it is precompact in the sense of \( |\cdot|_\mu \).

On the nonlinearity \( F \) and kernel functions, we shall make the following assumptions that will be true in Sections 3-4 of this paper.

(\( \text{F1} \)): \( F_i(n, \cdots, \cdot) = F_i(n + N_i, \cdots, \cdot), n \in \mathbb{Z}, i \in I; \)
(\( \text{F2} \)): \( F_i(n, 0, \cdots, 0) = 0 \), there exists \( M(n) = (m_1(n), \cdots, m_m(n)) \) with \( m_i(n) > 0, n \in \{1, 2, \cdots, N \} := J \), such that
\[ 0 \leq F_i(n, n_1, \cdots, n_m) \leq m_i(n + 1), 0 \leq n_i \leq m_i(n), i \in I, n \in J; \]
(\( \text{F3} \)): \( F_i(n, u_1, \cdots, u_m) \) is Lipschitz continuous with a Lipschitz constant \( L \) for any \( 0 \leq u_i \leq m_i(n), i \in I, n \in J; \)
(\( \text{K} \)): \( k_i : \mathbb{R} \rightarrow \mathbb{R}^+ \) is symmetric, Lebesgue measurable and integrable, and there exists \( \lambda_i^2 \in (0, \infty] \) such that \( \int_{\mathbb{R}} k_i(y)e^{\lambda y}dy < \infty, \lambda \in [0, \lambda_i^2) \) and
\[ \lim_{\lambda \rightarrow \lambda_i^2, \lambda \in (0, \lambda_i^2]} \int_{\mathbb{R}} k_i(y)e^{\lambda y}dy = +\infty, i \in I. \]

Consider the following integrodifference equation \([15, 39]\)
\[
\begin{cases}
    w(x, n + 1) = \int_{\mathbb{R}} f(n, w(y, n))k(y)dy, x \in \mathbb{R}, n = 0, 1, \cdots, \\
    w(x, 0) = \omega(x), x \in \mathbb{R},
\end{cases}
\tag{4}
\]
where \( f, k, \omega \) satisfy the following assumptions
(\( \text{f1} \)): \( f(n, \cdot) = f(n + N, \cdot), f(n, 0) = 0 \) for all \( n \in \mathbb{Z} \) and there exists \( b \in (0, \infty) \) such that \( f(n, \cdot) : [0, b] \rightarrow [0, b] \);
(\( \text{f2} \)): \( f(n, w) \) is Lipschitz continuous and nondecreasing in \( w \in [0, b] \) for \( n \in J; \)
(\( \text{f3} \)): for every \( n \in J \), \( \lim_{w \rightarrow 0^+} \frac{f(n, w)}{w} := f_n \) exists and is positive, and \( f(n, w) = w, w \in (0, b] \) has at most one root, there also exist \( L' > 0, \alpha > 1 \) such that
\[ 0 < f_nw - f(n, w) \leq L'w^\alpha, w \in (0, b]; \]
(\( \text{k} \)): \( k(x) \) satisfies (\( \text{K} \)) with some \( \lambda_i^2 \in (0, \infty] \);
(\( \text{w} \)): \( \omega(x) \geq \neq 0, x \in \mathbb{R}. \)

We shall denote \( \sqrt[n]{\prod_{i=1}^{N} f_n} := \mathcal{F} \) in what follows. Because of (\( \text{f2} \)), we have the following comparison principle.
Lemma 2.1. Assume that $0 \leq v(x, n) \leq b$, $x \in \mathbb{R}$, $n = 0, 1, \cdots$, such that

$$
\begin{align*}
  v(x, n + 1) & \geq (\leq) \int_{\mathbb{R}} f(n, v(y, n))k(y)dy, x \in \mathbb{R}, n = 0, 1, \cdots, \\
  v(x, 0) & \geq (\leq) \omega(x), x \in \mathbb{R}.
\end{align*}
$$

Then $v(x, n) \geq (\leq)w(x, n), x \in \mathbb{R}, n = 0, 1, \cdots$.

Consider the corresponding difference equation

$$w_{n+1} = f(n, w_n),
$$
which is well defined for all $n \in \mathbb{N}$ if $w_0 \in (0, b]$. Moreover, for such an equation (5), if $\bar{w}_n$ is an $N$–periodic solution, we say it is globally stable in the sense of

$$\lim_{n \to \infty} |w_n - \bar{w}_n| = 0 \text{ if } w_0 \in (0, b].$$

By the boundedness and monotonicity, we have the following conclusion.

Lemma 2.2. Assume that $\overline{T} > 1$. Then (5) has a globally asymptotic stable positive $N$–periodic solution $\bar{w}_n$.

Lemma 2.3. Assume that $\overline{T} > 1$. Define

$$\Lambda(\lambda, c) = \overline{T} \int_{\mathbb{R}} k(y)e^{\lambda(y-c)}dy, \lambda \in (0, \lambda^\dagger), c > 0$$

and

$$c' = \inf_{\lambda \in (0, \lambda^\dagger)} \frac{\ln \left( \overline{T} \int_{\mathbb{R}} e^{\lambda y} k(y)dy \right)}{\lambda}.$$

Then the following items are true.

1. $c' > 0$.
2. If $c > c'$, then $\Lambda(\lambda, c) = 1$ has two real positive roots $\lambda_1^* < \lambda_2^*$ such that $\Lambda(\lambda, c) < 1$ for all $\lambda \in (\lambda_1^*, \lambda_2^*)$.
3. If $c = c'$, then $\Lambda(\lambda, c) \geq 1$ for all $\lambda \in (0, \lambda^\dagger)$, and there exists $\lambda^*$ such that $\Lambda(\lambda^*, c) = 1$ and $\int_{\mathbb{R}} (y-c) k(y) e^{\lambda^*(y-c)} dy = 0$.
4. If $c < c'$, then $\Lambda(\lambda, c) > 1$ for all $\lambda \in (0, \lambda^\dagger)$.

In Liang et al. [15], the authors studied periodic monotone systems. By [15, Theorem 2.1], we have the following result.

Lemma 2.4. Assume that $\overline{T} > 1$. Then the following properties are true if $w(x, n)$ is defined by (4).

1. For any $c \in (0, c')$, $\lim_{n \to \infty} \sup_{|x| < cn} |w(x, n) - \bar{w}_n| = 0$.
2. When $c > c'$, $\lim_{n \to \infty} \sup_{|x| > cn} w(x, n) = 0$ if $w(x)$ admits compact support.

3. Existence of traveling wave solutions. To investigate the existence of (3) or an equivalent wave system

$$
\begin{align*}
  \phi_i(\xi, n + 1) &= \int_{\mathbb{R}} k_i(y) F_i(n, \phi_i(\xi - c - y, n), \cdots, \phi_{m}(\xi - c - y, n))dy, \\
  \phi_i(\xi, n) &= \phi_i(\xi, n + N), n \in \mathbb{Z}, \xi \in \mathbb{R}, i \in I,
\end{align*}
$$

we shall use the generalized upper and lower solutions, which is partially motivated by the invariant set.
Theorem 3.2. Assume that
Φ(ξ, n) = (Φ_1(ξ, n), ..., Φ_m(ξ, n)), Φ(ξ, n) = (φ_1(ξ, n), ..., φ_m(ξ, n)) ∈ C(0, M)
are a pair of generalized upper and lower solutions of (6) if
\[ \phi_i(ξ, n + 1) ≥ \int_R k_i(y)F_i(n, φ_1(ξ - c - y, n), ..., φ_m(ξ - c - y, n))dy, \]
and
\[ \phi_i(ξ, n + 1) ≤ \int_R k_i(y)F_i(n, φ_1(ξ - c - y, n), ..., φ_m(ξ - c - y, n))dy \]
for all i ∈ I, ξ ∈ R and any uniformly continuous functions φ_i(ξ, n) satisfying
\[ \phi_i(ξ, n) ≤ φ_i(ξ, n) ≤ \overline{φ}_i(ξ, n), i ∈ I, n ∈ J, ξ ∈ R. \]

Theorem 3.2. Assume that (6) has a pair of generalized upper and lower solutions \( \Phi(ξ, n), Φ(ξ, n) \). Then (6) has a solution \( Φ(ξ, n) \) such that
\[ \overline{Φ}(ξ, n) ≤ Φ(ξ, n) ≤ \Phi(ξ, n), ξ ∈ R, n ∈ Z. \]

Proof. Define Γ as follows
Γ = {Φ(ξ) ∈ C : Φ(ξ, 0) ≤ Φ(ξ) ≤ Φ(ξ, 0), ξ ∈ R}.

Let
\[ \mathcal{F}(Φ)(ξ) := (\mathcal{F}_1(Φ)(ξ, N), \mathcal{F}_2(Φ)(ξ, N), ..., \mathcal{F}_m(Φ)(ξ, N)) \]
be given by
\[
\begin{align*}
\mathcal{F}_i(Φ)(ξ, 1) &= \int_R k_i(y)F_i(0, φ_1(ξ - c - y), ..., φ_m(ξ - c - y))dy, \\
\mathcal{F}_i(Φ)(ξ, 2) &= \int_R k_i(y)F_i(1, \mathcal{F}_1(Φ)(ξ - c - y, 1), ..., \mathcal{F}_m(Φ)(ξ - c - y, 1))dy, \\
&\vdots \\
\mathcal{F}_i(Φ)(ξ, N) &= \int_R k_i(y)F_i(N - 1, \mathcal{F}_1(Φ)(ξ - c - y, N - 1), \mathcal{F}_m(Φ)(ξ - c - y, N - 1))dy
\end{align*}
\]
for any i ∈ I, Φ(ξ) ∈ Γ, ξ ∈ R. By the boundedness and (K), we see that \( \mathcal{F}_i(Φ)(ξ, n) \), i ∈ I, n ∈ J, are equicontinuous in ξ ∈ R. In fact, if Φ ∈ Γ, then
\[
|\mathcal{F}_i(Φ)(ξ', 1) - \mathcal{F}_i(Φ)(ξ, 1)| = \left| \int_R k_i(y)F_i(0, φ_1(ξ' - c - y), ..., φ_m(ξ' - c - y))dy \\
- \int_R k_i(y)F_i(0, φ_1(ξ - c - y), ..., φ_m(ξ - c - y))dy \right|
= \left| \int_R [k_i(ξ' - y) - k_i(ξ - y)]F_i(0, φ_1(y - c), ..., φ_m(y - c))dy \right|
≤ L \sum_{n∈J} m_i(n) \int_R |k_i(ξ' - y) - k_i(ξ - y)|dy
\]
for any ξ', ξ ∈ R, i ∈ I. By (K), we see that \( \mathcal{F}_i(Γ)(ξ, 1), i ∈ I \) are equicontinuous in the sense of super norm. Moreover, the definition of upper and lower solutions implies that
\[ \phi_i(ξ, 1) ≤ \mathcal{F}_i(Φ)(ξ, 1) ≤ \overline{φ}_i(ξ, 1), i ∈ I. \]
Repeating the above process $N$—times, we see that
\[ \mathcal{F} : \Gamma \to \Gamma. \]

and
\[ \Phi(\xi, 0) = \Phi(\xi, N) \leq \mathcal{F}(\Phi)(\xi) \leq \Phi(\xi, N) = \Phi(\xi, 0) \]
for any $\Phi(\xi) \in \Gamma, \xi \in \mathbb{R}$, and $\mathcal{F}_i(\Gamma)(\xi, N)$ is equicontinuous in the sense of super norm for all $\xi \in \mathbb{R}, i \in I$. By Proposition 1, we see that $\mathcal{F} : \Gamma \to \Gamma$ is compact in the sense of $|\cdot|_\mu$.

We now investigate the continuity of $\mathcal{F}$. Let $\mu > 0$ be a constant such that
\[ \int_{\mathbb{R}} k_i(y)e^{\mu|y|} dy < \infty, \ i \in I, \]
which is admissible by (K). For any $i \in I$ and
\[ \Phi(\xi) = (\phi_1(\xi), \cdots, \phi_m(\xi)) \in \Gamma, \Psi(\xi) = (\psi_1(\xi), \cdots, \psi_m(\xi)) \in \Gamma, \]
(F3) implies
\[
|\mathcal{F}_i(\Phi)(\xi, 1) - \mathcal{F}_i(\Psi)(\xi, 1)| \\
= \left| \int_{\mathbb{R}} k_i(y)\phi_j(\xi - c - y, \cdots, \phi_m(\xi - c - y)dy \\
- \int_{\mathbb{R}} k_i(y)\psi_j(\xi - c - y, \cdots, \psi_m(\xi - c - y)dy \right| \\
\leq L \sum_{j=1}^{m} \int_{\mathbb{R}} k_i(y)|\phi_j(\xi - c - y) - \psi_j(\xi - c - y)|dy, \quad (9)
\]
and so
\[
|\mathcal{F}_i(\Phi)(\xi, 1) - \mathcal{F}_i(\Psi)(\xi, 1)| e^{-\mu|\xi|} \\
\leq Le^{-\mu|\xi|} \sum_{j=1}^{m} \int_{\mathbb{R}} k_i(y)|\phi_j(\xi - c - y) - \psi_j(\xi - c - y)|dy \\
= L \sum_{j=1}^{m} \int_{\mathbb{R}} k_i(y)e^{-\mu|\xi| - \mu|\xi - c - y|}|\phi_j(\xi - c - y)e^{-\mu|\xi - c - y|} \\
- \psi_j(\xi - c - y)e^{-\mu|\xi - c - y|}|dy \\
\leq L\mu |\Phi - \Psi|_\mu \int_{\mathbb{R}} k_i(y)e^{-\mu|\xi| - \mu|\xi - c - y|}dy \\
\leq L\mu |\Phi - \Psi|_\mu \int_{\mathbb{R}} k_i(y)e^{\mu|c - y|}dy \\
\leq Lme^{\mu\gamma} |\Phi - \Psi|_\mu \int_{\mathbb{R}} k_i(y)e^{\mu|y|}dy,
\]
which further indicates that
\[
\sup_{\xi \in \mathbb{R}} \left\{ |\mathcal{F}_i(\Phi)(\xi, 1) - \mathcal{F}_i(\Psi)(\xi, 1)| e^{-\mu|\xi|} \right\} \leq Lme^{\mu\gamma} |\Phi - \Psi|_\mu \int_{\mathbb{R}} k_i(y)e^{\mu|y|}dy.
\]
So the mapping $\mathcal{F}_i(\Phi)(\xi, 1)$ is continuous in the sense of $|\cdot|_\mu$ for every $i \in I$. Repeating the above process $N$—times, then $\mathcal{F}$ is completely continuous in the sense of $|\cdot|_\mu$. 

Using Schauder’s fixed point theorem, \( F \) admits a fixed point \( \Phi^*(\xi) \in \Gamma \), and we denote it by
\[
\Phi^*(\xi) := \Phi^*(\xi, 0) = (\phi^*_1(\xi, 0), \ldots, \phi^*_m(\xi, 0)).
\]
From the definition of \( F \), we also define
\[
\phi^*_i(\xi, 1) = \int_{\mathbb{R}} k_i(y) F_i(0, \phi^*_1(\xi - c, 0), \ldots, \phi^*_m(\xi - c, 0))dy,
\]
\[
\phi^*_i(\xi, 2) = \int_{\mathbb{R}} k_i(y) F_i(1, \phi^*_1(\xi - c, 1), \ldots, \phi^*_m(\xi - c, 1))dy,
\]
\[
\cdots,
\]
\[
\phi^*_i(\xi, N) = \phi^*_i(\xi, 0) = \int_{\mathbb{R}} k_i(y) F_i(N - 1, \phi^*_1(\xi - c, N - 1), \ldots, \phi^*_m(\xi - c, N - 1))dy,
\]
and denote
\[
\Phi^*(\xi, n + kN) = (\phi^*_1(\xi, n), \ldots, \phi^*_m(\xi, n)), n \in \mathbb{J}, k \in \mathbb{Z}.
\]
Then we obtain a periodic traveling wave solution, which completes the proof. \( \square \)

**Remark 1.** When the periodic reaction-diffusion systems with proper monotone conditions are concerned, there are some results on the existence of periodic traveling wave solutions by constructing upper and lower solutions and applying monotone iteration or fixed point theorem, we may refer to Bo et al. \[1\] and Zhao and Ruan \[46, 47\] for Lotka-Volterra type competitive systems, Wang et al. \[38\] for an SIR model.

4. **Asymptotic behavior of traveling wave solutions.** In this section, we study the asymptotic behavior of traveling wave solutions. Consider the initial value problem
\[
\begin{aligned}
  u_i(x, n + 1) &= \int_{\mathbb{R}} k_i(x, y) F_i(n, u_1(y, n), \ldots, u_m(y, n))dy, \\
  u_i(x, 0) &= \psi_i(x)
\end{aligned}
\tag{10}
\]
for \( i \in I, x \in \mathbb{R}, n + 1 \in \mathbb{N} \), where the initial value
\[
0 \leq \psi_i(x) \leq M_i(0), i \in I, x \in \mathbb{R},
\]
is uniformly continuous. We assume that the corresponding difference system
\[
u_i(n + 1) = F_i(n, u_1(n), \ldots, u_m(n)), i \in I, \tag{11}\]
admits a positive \( N \)-periodic solution
\[
(U_1(n), U_2(n), \ldots, U_m(n)) = (U_1(n + N), U_2(n + N), \ldots, U_m(n + N)), n \in \mathbb{Z}
\]
such that \( U_i(n) > 0, i \in I \). For the initial value problem (10), we make the following assumption.

**(C):** Suppose that \( u_i(x, n), i \in I, x \in \mathbb{R}, n \in \mathbb{N} \) are defined by (10). If there exists \( \delta > 0 \) such that
\[
\delta \leq \psi_i(x) \leq M_i(0), i \in I, x \in \mathbb{R}, \tag{12}
\]
then
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}, i \in I} |u_i(x, n) - U_i(n)| = 0.
\]
Remark 2. In fact, the convergence condition (C) may be verified by the properties of the corresponding difference systems. For example, we shall show the property in Section 6 for a competitive system under weaker initial condition than (12).

Theorem 4.1. Assume that (C) holds. If (6) has a solution
\[ \Phi(\xi, n) = (\phi_1(\xi, n), \cdots, \phi_m(\xi, n)) \in C \]
such that
\[ \delta < \lim_{\xi \to \infty} \inf \phi_i(\xi, n) \leq \lim_{\xi \to \infty} \sup \phi_i(\xi, n) \leq M_i(n), i, i, \in I, n \in J, \] (13)
then
\[ \lim_{\xi \to \infty} \sup_{i \in I, n \in J} |\phi_i(\xi, n) - U_i(n)| = 0. \] (14)

Proof. For any given \( \epsilon > 0 \), we shall prove that there exists \( \xi_0 \) such that
\[ \sup_{\xi > \xi_0, i \in I, n \in J} |\phi_i(x, n) - U_i(n)| < \epsilon. \]
By (13), we have
\[ \delta < \phi_i(\xi, n), \xi \geq \xi_1, i, i \in I, n \in J \]
for some \( \xi_1 > 0 \). In (10), let
\[ \psi_i(x) = \begin{cases} \phi_i(x, 0), x \geq \xi_1, \\ \phi_i(\xi_1, 0), x \leq \xi_1 \end{cases} \]
for \( x \in \mathbb{R}, i \in I \). By condition (C), there exists \( N_1 > 0 \) such that
\[ \sup_{x \in \mathbb{R}, i \in I, n > N_1} |u_i(x, n) - U_i(n)| < \epsilon/2. \]
From the definition of traveling wave solutions, \( \phi_i(x + cn, n) \) satisfies
\[ \begin{aligned} \phi_i(x + c(n + 1), n + 1) &= \int_{\mathbb{R}} F_i(n, \phi_1(y + cn, n), \\ & \cdots, \phi_m(y + cn, n))k_i(x - y)dy, \end{aligned} \] (15)
for all \( i \in I, x \in \mathbb{R}, n + 1 \in \mathbb{N} \). Define
\[ w(x, n) = \sum_{i=1}^m |u_i(x, n) - \phi_i(x + cn, n)|, x \in \mathbb{R}, n = 0, 1, 2, \cdots. \]
By (F3), \( w(x, n) \) satisfies
\[ w(x, n + 1) \leq Lm \sum_{i=1}^m \int_{\mathbb{R}} w(y, n)k_i(x - y)dy, x \in \mathbb{R}, n = 0, 1, 2, \cdots, \]
of which the corresponding equality is monotone and takes the following form
\[ \overline{w}(x, n + 1) = Lm \sum_{i=1}^m \int_{\mathbb{R}} \overline{w}(y, n)k_i(x - y)dy, x \in \mathbb{R}, n = 0, 1, 2, \cdots. \]
Select \( \lambda' > 0, C > 0, D > 0, E > 0 \) such that
\[ e^{\lambda'C} > Lm \sum_{i=1}^m \int_{\mathbb{R}} e^{\lambda'y}k_i(y)dy \]
and
\[
\min\{De^{-\lambda x}, E\} \geq \sum_{i \in I} \phi_i(x,0), x < \xi_1.
\]

By direct calculation, we see that
\[
w(x, n) < \min\{De^{\lambda(-x+Cn)}, E(Lm)^n\}, n \in \mathbb{N}.
\]

Let \(x_0 > \xi_1\) such that
\[
De^{\lambda(-x+Cn)} < \epsilon/2, x > x_0, n = 1, 2, \cdots, N' + 2N.
\]

Select \(\xi_0 = x_0 + (C + |c|)(N' + 2N)\), we have
\[
\sum_{i=1}^{N} |u_i(n) - \phi_i(x + cn, n)| < \epsilon/2, x \geq x_0, n = 1, 2, \cdots, N' + 2N.
\]

The proof is complete. \(\square\)

We end this section by making the following remarks.

**Remark 3.** When the traveling wave solutions of reaction-diffusion systems with periodic parameters are concerned, Bo et al. [1], Lin [18] investigated the limit behavior of periodic traveling wave solutions without monotonicity.

**Remark 4.** We have mentioned in the study of limit behavior, the deficiency of the monotonicity of systems often leads to the difficulty. In some nonmonotone systems, it is possible to obtain the existence of monotone traveling wave solutions, see Kwong and Ou [11], Wang et al. [37], Wu and Zou [42] for delayed reaction-diffusion systems, and Lin and Wang [22] for a diffusion equation with state-dependent delay. But there are a few results on the existence of nonmonotone traveling wave solutions. On the one hand, it has been proved that the existence of both monotone and nonmonotone traveling wave solutions in some evolutionary systems, see Lin and Ruan [20], Ni and Taniguchi [29], Tang and Fife [35] for a competitive system. On the other hand, there is not monotone traveling wave solutions in some examples, see Pan and Lin [31, Example 5.4] for a nonmonotone integrodifference equation when the kernel function admits compact support.

5. **Application to a scalar equation.** In this section, we consider the following integrodifference equation
\[
w(x, n + 1) = \int_{\mathbb{R}} b(n, w(y, n))k(x - y)dy,
\]
which satisfies
(A1): \(b(n, \cdot) = b(n + N, \cdot)\) for all \(n \in \mathbb{R}\) and some \(N \in \mathbb{N}\);
(A2): there exists \(w^* > 0\) such that \(b(\cdot, w): [0, w^*] \rightarrow [0, w^*]\) is continuous and \(b(\cdot, 0) = 0\), let \(r(n) = b'(n, w)\) \(\big|_{w=0}, n \in J\), then \(r(i) > 0, i \in J, \prod_{i=1}^{N} r(i) > 1\);
(A3): \(0 < b(n, w) \leq r(n)w, w \in (0, w^*], n \in J\) and there exist \(L > 0, \delta \in (0, 1]\) such that
\[
0 \leq r(n)w - b(n, w) \leq Lw^{1+\delta}, w \in (0, w^*], n \in J,
\]
and \(b(n, u) = u, u \in (0, w^*]\) has at most one root;
(A4): \(k\) satisfies (K) in Section 2.
In the above assumptions, we do not require the monotonicity on $b$. Due to the possible nonmonotonicity, the traveling wave solution can not be studied by Liang et al. [15]. One typical example is the periodic version of Hsu and Zhao [9, Example 4.2]

$$w(x, n + 1) = \int_{\mathbb{R}} w(y, n) e^{r_n - w(y, n)} k(x - y) dy,$$

(17)

in which $\sum_{n=1}^{N} r_n > 0$. When $r_n$ is a constant, we see its propagation dynamics by Hsu and Zhao [9] and Lin and Su [21]. For (17), let $w^* > 0$ such that

$$w^* : [0, \infty) \to [0, w^*],$$

of which the admissibility is clear since $\lim_{w \to \infty} [we^{r_n - w}] = 0$ for all $n \in J$. Then it satisfies (A1)-(A3).

If $\phi, c > 0$ are wave profile and wave speed of (16), respectively, then

$$\begin{cases}
\phi(\xi + c, n + 1) = \int_{\mathbb{R}} b(n, \phi(y, n)) k(\xi - y) dy, \xi \in \mathbb{R}, n \in \mathbb{Z}, \\
\phi(\xi, n) = \phi(\xi, n + N), \xi \in \mathbb{R}, n \in \mathbb{Z}.
\end{cases}$$

(18)

Furthermore, we also require the following asymptotic behavior

$$\lim_{\xi \to -\infty, n \in J} \phi(\xi, n) = 0, \liminf_{\xi \to \infty, n \in J} \phi(\xi, n) > 0,$$

(19)

which may formulate the successful biology invasion in population dynamics. The remainder of this section is to study the existence or nonexistence of positive solutions of (18)-(19).

Define

$$b^-(n, w) = \min_{u \in [w, w^*]} \{b(n, u)\}, n \in J,$$

then $b^-(n, w) : [0, w^*] \to [0, w^*]$ is monotone for every $n \in J$. In particular, there exists $\delta' > 0$ such that

$$b^-(n, w) = b(n, w), n \in J, w \in [0, \delta'],$$

and it satisfies (f1)-(f3) in Section 2. By the monotonicity and boundedness,

$$w_{n+1} = b^-(n, w_n)$$

has a positive $N$–periodic solution $\hat{w}_n$ that is globally asymptotic stable if $w_0 \in (0, w^*)$.

Fix

$$\bar{f} = \sqrt[N]{\prod_{i=1}^{N} r(i)}$$

in this section. Using the notations in Lemmas 2.3-2.4, if $c > c'$ is fixed, then we define an $N$–periodic sequence $\{h_n\}_{n \in \mathbb{Z}}$ by

$$h(0) = 1, h(n + 1) = r(n) h(n) \int_{\mathbb{R}} k(y) e^{r_1(y - \bar{f})} dy,$$

and a constant $\eta \in (1, 1 + \delta)$ such that

$$\eta \lambda_1^c < \lambda_2^c, \Lambda(\eta \lambda_1^c, c) < 1.$$

Further select $L' > 0$ such that

$$|b(n, w) - r(n) w| < L' u^\eta, u > 0.$$
which implies
\[ \varphi_n(x) = \max \left\{ h(n)e^{\lambda x} w^*, \phi(x) \right\}, \varphi_n(x) = \max \left\{ h(n)e^{\lambda x} - q h(n)e^{n \lambda x}, 0 \right\}, \]
where
\[ q > \max_{n \in J} \left\{ \frac{L'h(n)}{h(n + 1) - r(n) h(n) \int_\mathbb{R} e^{n \lambda y} k(y) dy} \right\} + 1 \]
such that
\[ \varphi_n(x) > \phi_n(x), x \in \mathbb{R}, n \in \mathbb{Z}. \]

**Lemma 5.1.** $\overline{\varphi}(x, n), \phi(x, n)$ are a pair of generalized upper and lower solutions of (18). Therefore, for any $c > c'$, (18)-(19) admits a nonconstant solution $\phi(x, n)$ such that
\[ \lim_{x \to -\infty} h(n)e^{\lambda x} = 1. \]

**Proof.** To prove the existence of (18), we directly verify the definition of generalized upper and lower solutions for any $\overline{\varphi}(y, n) \geq \phi(y, n) \geq \phi(y, n), y \in \mathbb{R}, n \in \mathbb{Z}$.

If $\overline{\varphi}(\xi + c, n + 1) = w^*$, then the result is clear by the definition of $w^*$. Otherwise, $\overline{\varphi}(\xi + c, n + 1) = h(n + 1)e^{\lambda y(\xi + c)}$ and
\[ \int_\mathbb{R} b(n, \phi(y, n)) k(\xi - y) dy \leq r(n) \int_\mathbb{R} \phi(y, n) k(\xi - y) dy \leq r(n) \int_\mathbb{R} h(n)e^{\lambda y} k(\xi - y) dy = h(n)r(n) \int_\mathbb{R} e^{\lambda y} k(\xi - y) dy = h(n)r(n)e^{\lambda y} \int_\mathbb{R} e^{\lambda y} k(y) dy = h(n + 1)e^{\lambda y(\xi + c)} = \overline{\varphi}(\xi + c, n + 1), \]
which implies $\overline{\varphi}(x, n)$ is an generalized upper solution.

If $\overline{\varphi}(\xi + c, n + 1) = 0$, then the result is clear. Otherwise, $\overline{\varphi}(\xi + c, n + 1) = h(n + 1)e^{\lambda y(\xi + c)} - q h(n + 1)e^{n \lambda y(\xi + c)} > 0$ such that $\xi + c < 0$ and
\[ \int_\mathbb{R} b(n, \phi(y, n)) k(\xi - y) dy \geq \int_\mathbb{R} \left[ r(n)\phi(y, n) - L'\phi(y, n) \right] k(\xi - y) dy \geq \int_\mathbb{R} \left[ r(n)\phi(y, n) - L'\overline{\varphi}(y, n) \right] k(\xi - y) dy \geq h(n)r(n) \int_\mathbb{R} [e^{\lambda y} - qe^{n \lambda y} - L'e^{n \lambda y}] k(\xi - y) dy = h(n)r(n)e^{\lambda y} \int_\mathbb{R} e^{n \lambda y} k(y) dy - (q + L')h(n)r(n)e^{n \lambda y} \int_\mathbb{R} e^{\lambda y} k(y) dy \]
large. By Lemmas 2.1 and 2.4, we see that
\[ q = \phi_{\xi}(\xi + c, n + 1), \]
which holds by the definition of \( q \). The existence is obtained by Theorem 3.2.

We now consider the limit behavior if \( \xi \to \infty \). By the definition of traveling wave solutions, \( w(x, n) = \phi(x + cn, n) \) also satisfies
\[
\begin{cases}
  w(x, n + 1) \geq \int_{\mathbb{R}} b^-(n, w(y, n))k(x - y)dy, \\
  w(x, 0) = \phi(x, 0)
\end{cases}
\]
for \( n = 0, 1, \cdots, x \in \mathbb{R} \). The lower solution \( \phi(\xi, n) \) implies that \( \phi(x, 0) > 0 \) if \( -x \) is large. By Lemmas 2.1 and 2.4, we see that
\[
\liminf_{n \to \infty} \inf_{|x| < 2c} (w(x, n) - \hat{w}_n) \geq 0.
\]
Note that \(|x| < 2c, n \in \mathbb{N}\) leads to \((0, \infty) \subset \bigcup_{n \in \mathbb{N}} (-2c + cn, 2c + cn)\), then the above implies that
\[
\liminf_{\xi \to \infty} (\phi(\xi, n) - \hat{w}_n) \geq 0, n \in J \text{ or } n \in \mathbb{Z}.
\]

The proof is complete. \(\Box\)

**Lemma 5.2.** If \( c = c' \), then (18)-(19) admits a nonconstant solution \( \phi(\xi, n) \).

**Proof.** We now prove the result by a limit process. Firstly, select \( \delta > 0 \) with
\[ 2\delta < \hat{w}_n, n \in J. \]
Let \( \{c_m\}_{m \in \mathbb{N}} \) be a strictly decreasing sequence satisfying \( \lim_{m \to \infty} c_m = c' \). Then for each \( m \in \mathbb{N}, (18) \) with \( c = c_m \) admits a solution \( \phi_m(\xi, n) \) such that
\[
\lim_{\xi \to \infty} \phi_m(\xi, n) = 0, \liminf_{\xi \to \infty} \phi_m(\xi, n) > 2\delta \text{ for all } n \in J.
\]
Since a traveling wave solution is invariant in the sense of phase shift (that is, if \( \phi(\xi, n) \) is a solution of (18), then for any \( \epsilon \in \mathbb{R}, \phi(\xi + \epsilon, n) \) also satisfies (18)), we assume that
\[ \phi_m(\xi, 0) < \delta, \phi_m(0, 0) = \delta, m \in \mathbb{N}, \xi < 0. \]
By the boundedness and (k), we see that they are equicontinuous, that is, for any \( \epsilon > 0 \), there exists \( \varepsilon > 0 \) such that for any \( |\xi'| < \varepsilon \), we have
\[ |\phi_m(\xi, n) - \phi_m(\xi + \epsilon, n)| < \epsilon, n \in J, m \in \mathbb{N}. \]
Using Ascoli-Arzelà lemma, \( \{\phi_m(\xi, n)\}_{m \in \mathbb{N}} \) has a subsequence, still denoted by \( \{\phi_m(\xi, n)\}_{m \in \mathbb{N}} \), and there exists a uniform continuous function \( \phi^*(\xi, n) \) such that
\[ \lim_{m \to \infty} \phi_m(\xi, n) = \phi^*(\xi, n), \]
in which the convergence is uniform for \( n \in J \) and \( \xi \) in any compact subset of \( \mathbb{R} \), and the convergence is also pointwise in \( \xi \in \mathbb{R} \). Letting \( m \to +\infty \), we see that \( \phi^*(\xi, n) \) satisfies (18) with \( c = c' \). In particular, they also satisfy
\[ \phi^*(\xi, 0) \leq \delta, \phi^*(0, 0) = \delta, \xi < 0. \]
We now show that \( \phi^*(\xi, n) \) satisfies (19). By the continuity, then there exists \( a > 0 \) such that \( \phi^*(\xi, 0) > 0, \xi \in [-a, a] \). Then we have
\[
\liminf_{\xi \to \infty} (\phi^*(\xi, n) - \hat{w}_n) \geq 0, n \in J \text{ or } n \in \mathbb{Z}.
\]
by a discussion similar to that of (20). If \( \limsup_{\xi \to -\infty} \phi^*(\xi, n) > 0 \) for some \( n \in J \), then
\[
\limsup_{\xi \to -\infty} \phi^*(\xi, n) > 0 \text{ for all } n \in J \text{ or } n \in \mathbb{Z}
\]
by the definition and dominated convergence theorem. By the definition of \( \limsup \) and the uniform continuity, there exist \( \{\xi_m\}_{m \in \mathbb{N}}, \sigma > 0 \) and \( \rho > 0 \) such that

- (p1): \( \xi_m \to -\infty, m \to \infty \),
- (p2): \( \phi^*(\xi_m, 0) > \sigma/2 \),
- (p3): \( \phi^*(\xi_m + s, 0) > \sigma/2, |s| < \rho \).

Also consider (20) with initial value \( w(x, 0) = w(x) \) satisfying

- (w1): \( w(x) = w(-x), x \in \mathbb{R} \);
- (w2): \( w(x) = \sigma/2, x \in [-\rho/2, \rho/2] \);
- (w3): \( w(x) = 0, |x| \geq \rho \);
- (w4): \( w(x) \) is decreasing and continuous for \( x \in [\rho/2, \rho] \).

Then there exists \( N' > 0 \) such that
\[
w(0, n) > 2\delta, n \geq N'
\]
by Lemmas 2.1 and 2.4.

According to the definition of traveling wave solutions, we see that
\[
\limsup_{\xi \to -\infty} \sup_{n \in J} \phi^*(\xi, n) > 0
\]
and a contradiction occurs. The proof is complete.

**Remark 5.** Because of the periodicity, \( \{\phi_m(\xi, n)\}_{m \in \mathbb{N}} \) includes only \( N \) sequences, the process in Brown and Carr [2] and Hsu and Zhao [9, Theorem 3.2] can be applied to obtain the existence of \( \phi^*(\xi, n) \).

**Lemma 5.3.** For any \( c < c' \), (18)-(19) does not have a nonconstant solution \( \phi(\xi, n) \).

**Proof.** We prove it by contradiction. If the result is not true, then for some \( c_1 \in (0, c') \), (18) with \( c = c_1 \) admits a positive solution \( \phi(\xi, n) \) satisfying (19).

By the above notations, \( \phi(\xi, n) = w(x, n) \) satisfies (20). Let
\[
2c_1n < -2x(n) = (c_1 + c')n < 2c'n, n \in \mathbb{N}.
\]
Then Lemmas 2.1 and 2.4 imply
\[
\liminf_{n \to \infty}(w(x(n), n) - \hat{w}_n) \geq 0,
\]
and so
\[
\limsup_{\xi \to -\infty} \sup_{n \in J} \phi(\xi, n) > 0\]
by
\[
\xi_n = x(n) + c_1n \to -\infty, n \to \infty,
\]
which contradicts to (19). The proof is complete.

Summarizing what we have done, we have the following conclusion.

**Theorem 5.4.** Assume that \( \prod_{i=1}^{N} r(i) > 1 \). Then \( c' \) is the minimal wave speed of (16).
Remark 6. In population dynamics, (19) could formulate a success biology invasion. For the difference equation \( w_{n+1} = b(n, w_n) \), there are some results on the global stability of unique positive periodic solution \( w_n \), see Zhou and Zou [49] and references therein for (17). In this paper, we do not focus on the precise condition on \( \lim_{\xi \to \infty} \phi(\xi, n) = w_n, n \in J \), which at least holds for \( r_n \leq 1 \) such that (17) is monotone by selecting \( w^* = 1 \).

6. Application to a competitive system. In this section, we study the following competitive systems with periodic parameters

\[
\begin{align*}
&u_1(x, n + 1) = \int_{\mathbb{R}} \frac{(1+r_1(n))u_1(x-y,n)}{1+r_1(n)(u_1(x-y,n)+a_1(n)u_2(x-y,n))} k_1(y)dy, \\
&u_2(x, n + 1) = \int_{\mathbb{R}} \frac{(1+r_2(n))u_2(x-y,n)}{1+r_2(n)(u_2(x-y,n)+a_2(n)u_1(x-y,n))} k_2(y)dy,
\end{align*}
\]

(21)
in which \( n = 0, 1, 2, \cdots, x \in \mathbb{R} \), \( k_1(y), k_2(y) \) satisfy the assumption (K) by \( \lambda_1^2, \lambda_2^2 \), respectively, the other parameters satisfy:

(P): \( r_i(n) = r_i(n + N) \geq 0, a_i(n) = a_i(n + N) \geq 0 \) for \( n \in \mathbb{Z} \) and

\[
\prod_{n=1}^{N} (1 + r_i(n)) := \tau_i > 1, i = 1, 2.
\]

For the corresponding difference system

\[
\begin{align*}
&u_1(n + 1) = \frac{(1+r_1(n))u_1(n)}{1+r_1(n)(u_1(n)+a_1(n)u_2(n))}, \\
&u_2(n + 1) = \frac{(1+r_2(n))u_2(n)}{1+r_2(n)(u_2(n)+a_2(n)u_1(n))},
\end{align*}
\]

(22)
there are some results on the existence and stability of positive periodic solution, see, e.g., Saker [33]. In particular, it is clear that if

\[
\prod_{n=1}^{N} \frac{1 + r_i(n)}{1 + r_i(n)a_i(n)} := \tau'_i > 1, i = 1, 2, \quad \text{(23)}
\]
or

\[
\prod_{n=1}^{N} [1 + r_i(n)] \geq \prod_{n=1}^{N} [1 + r_i(n)a_i(n)], i = 1, 2,
\]
then (22) admits a positive \( N \)-periodic solution \((\hat{u}_1(n), \hat{u}_2(n))\). In what follows, we still do not focus on the precise condition on the stability of positive periodic solution \((\hat{u}_1(n), \hat{u}_2(n))\), but we shall verify the following results on the corresponding initial value problem.

Proposition 2. Assume that \((\hat{u}_1(n), \hat{u}_2(n))\) is globally stable for any given \( u_1(0) > 0, u_2(0) > 0 \). Consider the initial value problem of (21) by given nonnegative initial value \((u_1(x, 0), u_2(x, 0))\). If there exist \( \epsilon > 0, X > 0 \) such that

\[
u_i(x, 0) > \epsilon, |x| > X, i = 1, 2,
\]
then

\[
\limsup_{n \to \infty} \sup_{x \in \mathbb{R}} \{|u_1(x, n) - \hat{u}_1(n)| + |u_2(x, n) - \hat{u}_2(n)|\} = 0.
\]

Proof. By the given initial value, we may select \( \epsilon > 0, N' > 0 \) such that

\[
u_i(x, n) > \epsilon, x \in \mathbb{R}, n = N', i = 1, 2.
\]
Fix four positive constants
\[ U_i = \sup_{x \in \mathbb{R}} u_i(x, N'), U_i = \inf_{x \in \mathbb{R}} u_i(x, N'), i = 1, 2. \]

Let \((\pi_1(n), \pi_2(n))\) be defined by
\[
\begin{align*}
\pi_1(n + 1) &= \frac{(1 + r_1(n))\pi_1(n)}{1 + r_1(n)\pi_1(n) + a_1(n)\pi_2(n)}, n = N', N' + 1, \ldots, \\
\pi_2(n + 1) &= \frac{(1 + r_2(n))\pi_2(n)}{1 + r_2(n)\pi_2(n) + a_2(n)\pi_1(n)}, n = N', N' + 1, \ldots, \\
\end{align*}
\]
and \((\bar{u}_1(n), \bar{u}_2(n))\) be defined by
\[
\begin{align*}
\bar{u}_1(n + 1) &= \frac{(1 + r_1(n))\bar{u}_1(n)}{1 + r_1(n)\bar{u}_1(n) + a_1(n)\bar{u}_2(n)}, n = N', N' + 1, \ldots, \\
\bar{u}_2(n + 1) &= \frac{(1 + r_2(n))\bar{u}_2(n)}{1 + r_2(n)\bar{u}_2(n) + a_2(n)\bar{u}_1(n)}, n = N', N' + 1, \ldots, \\
\end{align*}
\]

Then we have
\[
(\bar{u}_1(n), \bar{u}_2(n)) \leq (u_1(x, n), u_2(x, n)) \leq (\pi_1(n), \pi_2(n)), x \in \mathbb{R}, n \geq N',
\]
and the result is clear by the stability of \((\bar{u}_1(n), \bar{u}_2(n))\). The proof is complete. \(\square\)

Let \(u_i(x, n) = \phi_i(x + cn, n) = \phi_i(\xi, n), i = 1, 2,\) be the traveling wave solution of (21), then \(\phi_1, \phi_2, c\) satisfy
\[
\begin{align*}
\phi_1(\xi + c, n + 1) &= \int_{\mathbb{R}} \frac{(1 + r_1(n))\phi_1(\xi - y, n)}{1 + r_1(n)\phi_1(\xi - y, n) + a_1(n)\phi_2(\xi - y, n)} k_1(y)dy, \\
\phi_2(\xi + c, n + 1) &= \int_{\mathbb{R}} \frac{(1 + r_2(n))\phi_2(\xi - y, n)}{1 + r_2(n)\phi_2(\xi - y, n) + a_2(n)\phi_1(\xi - y, n)} k_2(y)dy, \\
\phi_1(\xi, n) &= \phi_1(\xi, n + N), \phi_2(\xi, n) = \phi_2(\xi, n + N)
\end{align*}
\]
for \(\xi \in \mathbb{R}, n \in \mathbb{Z}\). Moreover, similar to that in Lin [17], we study the following asymptotic boundary behavior
\[
\lim_{\xi \to -\infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (0, 0), \lim_{\xi \to \infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (\bar{u}_1(n), \bar{u}_2(n))
\]
for every \(n \in \mathbb{Z}\) or \(n \in J\).

Consider
\[
\Gamma_i(\gamma, c) = \pi_i \int_{\mathbb{R}} k_i(y)e^{\gamma(y-c)}dy, i = 1, 2,
\]
then similar to that in Section 2, we have the following conclusion.

**Lemma 6.1.** There exist constants \(c_1 > 0, c_2 > 0\) satisfying the following properties.

1. **For every fixed** \(c > c_1, \Gamma_i(\gamma, c) = 1\) **has two positive roots** \(\gamma_{i,1}(c) < \gamma_{i,2}(c),\) **and** \(\gamma \in (\gamma_{i,1}(c), \gamma_{i,2}(c))\) **implies** \(\Gamma_i(\gamma, c) < 1, i = 1, 2.\)

2. **If** \(c = c_1,\) **then** \(\Gamma_i(\gamma, c) \geq 1, \gamma \geq 0,\) **and there exists** \(\gamma_i > 0\) **such that** \(\Gamma_i(\gamma, c) = 1\) **and**
\[
\int_{\mathbb{R}} k_i(y)(y-c)e^{\gamma_i(y-c)}dy = 0, i = 1, 2.
\]

3. **If** \(c < c_1,\) **then** \(\Gamma_i(\gamma, c) > 1, \gamma > 0, i = 1, 2.\)

**Remark 7.** By (2), we see that
\[
\int_{-\infty}^{c_1} k_i(y)(y-c_1)e^{\gamma_i(y-c_1)}dy = \int_{c_1}^{\infty} k_i(y)(y-c_1)e^{\gamma_i(y-c_1)}dy < 0, i = 1, 2.
\]
Therefore, \(\int_{|y|>c_i} k_i(y)dy > 0\) holds.
We define 
\[ c^* = \max\{c_1, c_2\} \]
and show the main results of this section.

**Theorem 6.2.** Assume that (P) holds.

1. If \( c > c^* \), then (24) has a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\) such that 
   \[ \lim_{\xi \to -\infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (0, 0). \]
   Furthermore, if (23) holds and \((\hat{u}_1(n), \hat{u}_2(n))\) is globally stable for any given \( u_1(0) > 0, u_2(0) > 0 \) in the corresponding initial value problem of (22), then 
   \((\phi_1(\xi, n), \phi_2(\xi, n))\) satisfies (25).

2. If \( c = c^* \) holds and both \( k_1(x), k_2(x) \) have compact supports, then (24) has a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\) such that 
   \[ \lim_{\xi \to -\infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (0, 0). \]
   Furthermore, if (23) holds and \((\hat{u}_1(n), \hat{u}_2(n))\) is globally stable for any given \( u_1(0) > 0, u_2(0) > 0 \) in the corresponding initial value problem of (22), then 
   (25) is true.

3. If \( c < c^* \), then (24) does not have a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\) such that 
   \[ \lim_{\xi \to -\infty} \phi_i(\xi, n) = 0, \liminf_{\xi \to \infty} \phi_i(\xi, n) > 0, i = 1, 2. \] (26)

In what follows, we prove the above results by several lemmas, which also implies the precise asymptotic behavior of traveling wave solutions. Since the property needs to define some constants, we do not list them in Theorem 6.2. Firstly, the following conclusion is evident.

**Lemma 6.3.** Assume that (24) has a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\) such that
\[ \phi_1(\xi_1, n_1) > 0, \phi_2(\xi_2, n_2) > 0 \]
for some \( \xi_1, \xi_2 \in \mathbb{R}, n_1, n_2 \in \mathbb{Z} \). Then
\[ \phi_1(\xi, n) > 0, \phi_2(\xi, n) > 0, \xi \in \mathbb{R}, n \in \mathbb{Z}. \]

**Lemma 6.4.** For any given \( c < c^* \), (24) with (26) does not have a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\).

**Proof.** Without loss of generality, we assume that \( c^* = c_1 \). Were the statement false, then there exists some \( c_0 < c_1 \) such that (24) has a positive solution \((\phi_1(\xi, n), \phi_2(\xi, n))\) satisfying (26), which is strictly positive by Lemma 6.3.

Let \( \epsilon > 0 \) such that 
\[ c_0 < \inf_{\lambda \in (0, \lambda_1^1)} \frac{\ln (\tau_1 \int_R e^{\lambda y} k_1(y) dy) - \ln(1 + 2\epsilon)}{\lambda} := c_3. \]
By (26), there exists \( y_1 \in \mathbb{R} \) such that 
\[ r_1(n)(\phi_1(\xi - y, n) + a_1(n)\phi_2(\xi - y, n)) < \epsilon, \xi - y < y_1, n \in J. \]
When \( \xi - y \geq y_1 \), then there exists \( M > 0 \) such that 
\[ r_1(n)(\phi_1(\xi - y, n) + a_1(n)\phi_2(\xi - y, n)) < M\phi_1(\xi - y, n), \xi - y > y_1, n \in J \]
by Lemma 6.3 and the limit behavior (26).
Thus, $\phi_1(\xi, n)$ satisfies
\[
\phi_1(\xi + c_0, n + 1) \geq \int_{\mathbb{R}} \frac{(1 + r_1(n))\phi_1(\xi - y, n)}{1 + c + M\phi_1(\xi - y, n)} k_1(y) dy, \xi \in \mathbb{R}, n \in \mathbb{Z},
\]
and so $u_1(x, n) = \phi_1(x + c_0 n, n)$ satisfies
\[
\begin{cases}
u_1(x, n + 1) \geq \int_{\mathbb{R}} \frac{(1 + r_1(n))u_1(x - y, n)}{1 + c + M u_1(x - y, n)} k_1(y) dy, x \in \mathbb{R}, n = 0, 1, \cdots, \\
u_1(x, 0) = \phi_1(x, 0), x \in \mathbb{R}.
\end{cases}
\]

Let $u_1$ be defined by
\[
u_1 = \min \{u(i) : i = 1, 2, \cdots, N\},
\]
where $u(i)$ is the unique positive periodic solution of
\[
w(n + 1) = \frac{(1 + r_1(n))w(n)}{1 + c + M w(n)}.
\]
Evidently, $u_1 > 0$ holds by the definition of $c_3$. Note that the above system is monotone, then Lemmas 2.1 and 2.4 imply
\[
\lim \inf_{n \to \infty} u_1(-c_3 n, n) \geq u_1 > 0,
\]
which indicates that
\[
\lim \sup_{n \to \infty} \phi_1(\xi(n), n) \geq u_1
\]
for
\[
\xi(n) = -c_3 n + c_0 n, n \in \mathbb{N}.
\]
Since
\[
\xi(n) = -c_3 n + c_0 n \to -\infty, n \to \infty,
\]
we obtain
\[
\lim \sup_{n \to \infty} \phi_1(\xi(n), n) = 0, n \in \mathbb{N}
\]
by (26). A contradiction occurs. The proof is complete.

**Lemma 6.5.** If $c > c^*$, then (24) has a positive solution $(\phi_1(\xi, n), \phi_2(\xi, n))$ such that
\[
\lim_{\xi \to -\infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (0, 0).
\]

**Proof.** We prove the result for any fixed $c > c^*$ by constructing proper generalized upper and lower solutions. Define two $N-$periodic sequences $\{h_i(n)\}_{n \in \mathbb{Z}}$ by
\[
h_i(0) = 1, h_i(n + 1) = (1 + r_i(n)) h_i(n) \int_{\mathbb{R}} k_i(y) e^{\gamma_{i, 1}(c)(y - c)} dy, n \in \mathbb{Z}, i = 1, 2.
\]
Select a constant $\eta \in (1, 2)$ such that
\[
\eta \gamma_{i, 1}(c) < \min \{\gamma_{i, 1}(c) + \gamma_{3-i, 1}(c), \gamma_{i, 2}(c)\}, \Lambda(\eta \gamma_{i, 1}(c), c) < 1, i = 1, 2.
\]
Define continuous functions
\[
\bar{\phi}_i(\xi, n) = \min \{h_i(n) e^{\gamma_{i, 1}(c)\xi}, 1\}, i = 1, 2,
\]
and
\[
\underline{\phi}_i(\xi, n) = \max \{h_i(n) e^{\gamma_{i, 1}(c)\xi} - q h_i(n) e^{\gamma_{i, 1}(c)\xi}, 0\}, i = 1, 2,
\]
where $q > 1$ is a constant. In particular, there exists $q > 1$ such that we obtain a pair of generalized upper and lower solutions of (24), which will be verified in the Appendix of this paper. By Theorem 3.2, we complete the proof.
Lemma 6.6. Assume that $k_1(y), k_2(y)$ admit compact supports. If $c = c^*$, then (24) has a positive solution $(\phi_1(\xi, n), \phi_2(\xi, n))$ such that

$$\lim_{\xi \to -\infty} (\phi_1(\xi, n), \phi_2(\xi, n)) = (0, 0).$$

Proof. By the property of $k_1(y), k_2(y)$, there exists $D > 0$ such that

$$k_i(y \pm c^*) = 0, |y| > D, i = 1, 2.$$ 

When $c_1 \neq c_2$, we may assume that $c_1 > c_2$ and $c^* = c_1$ without loss of generality, then we define two $N$–periodic sequences

$$e_1(0) = 1, e_1(n + 1) = (1 + r_1(n))e_1(n) \int_{\mathbb{R}} k_1(y)e^{\gamma_1(y-c)} dy, n \in \mathbb{Z},$$

and $h_2(n)$ as that in the proof of Lemma 6.5.

Let $L_1$ be a large constant such that

$$-L_1 \xi e_1(n)e^{\gamma_1 \xi} > 1, \xi_{1,2}(n) - \xi_{1,1}(n) > D, n \in J,$$

where $\xi_{1,2}(n), \xi_{1,1}(n)$ are the real roots of $-L_1 \xi e_1(n)e^{\gamma_1 \xi} = 1$. Select $P_1 > 0$ be a constant and define continuous functions as follows

$$\overline{\phi}_1(\xi, n) = \begin{cases} -L_1 \xi e_1(n)e^{\gamma_1 \xi}, & \xi < \xi_{1,1}(n), \\ 1, & \xi \geq \xi_{1,1}(n), \end{cases}$$

$$\underline{\phi}_1(\xi, n) = \begin{cases} (-L_1 \xi - P_1 \sqrt{-\xi})e_1(n)e^{\gamma_1 \xi}, & \xi < -P_1^2/L_1^2, \\ 0, & \xi \geq -P_1^2/L_1^2, \end{cases}$$

and $\overline{\phi}_2(\xi, n), \underline{\phi}_2(\xi, n)$ are similar to that in the proof of Lemma 6.5. In the Appendix, we shall confirm that we obtain a pair of generalized upper and lower solutions of (24) by selecting $P_1 > 0$ large enough.

If $c_1 = c_2$, by notations similar to the above, we may define

$$\overline{\phi}_1(\xi, n) = \begin{cases} -L_1 \xi e_1(n)e^{\gamma_1 \xi}, & \xi < \xi_{1,1}(n), \\ 1, & \xi \geq \xi_{1,1}(n), \end{cases}$$

and

$$\underline{\phi}_1(\xi, n) = \begin{cases} (-L_1 \xi - P_1 \sqrt{-\xi})e_1(n)e^{\gamma_1 \xi}, & \xi < -P_1^2/L_1^2, \\ 0, & \xi \geq -P_1^2/L_1^2, \end{cases}$$

for $i = 1, 2$. In the Appendix, we shall confirm that we obtain a pair of generalized upper and lower solutions of (24) by selecting $P_1 > 0, P_2 > 0$ large enough.

By Theorem 3.2, the proof is complete. 

According to what we have done, we may finish the proof of Theorem 6.2 by verifying the following limit behavior as $\xi \to \infty$ and applying Theorem 4.1.

Lemma 6.7. If (23) holds, then $(\phi_1(\xi, n), \phi_2(\xi, n))$ in Lemma 6.5 or 6.6 satisfies

$$\liminf_{\xi \to \infty} \phi_i(\xi, n) > 0, n \in J, i = 1, 2.$$ 

Proof. By the fact of $\phi_2(\xi, n) \leq 1$, we see that

$$\phi_1(\xi + c, n + 1) \geq \int_{\mathbb{R}} \frac{(1 + r_1(n))\phi_1(\xi - y, n)}{1 + r_1(n)(\phi_1(\xi - y, n) + a_1(n))} k_1(y) dy, \xi \in \mathbb{R}, n \in J,$$
and so \( u_1(x, n) = \phi_1(x + cn, n) \) satisfies
\[
\begin{aligned}
&u_1(x, n + 1) \geq \int_{\mathbb{R}} \frac{(1 + r_1(n)u_1(x - y, n))}{1 + r_1(n)(u_1(x - y, n) + a_1(n))} k_1(y) dy, x \in \mathbb{R}, n = 0, 1, \ldots, \\
&u_1(x, 0) = \phi_1(x, 0), x \in \mathbb{R}.
\end{aligned}
\]

Due to (23), then Lemma 2.4 leads to
\[
\liminf_{n \to \infty} [u_1(x, n) - y_1(n)] \geq 0, |x| < 2c,
\]
where \( y_1(n) \) is the unique positive periodic solution of
\[
u_1(n + 1) = \frac{(1 + r_1(n))u_1(n)}{1 + r_1(n)(u_1(n) + a_1(n))}.
\]

Note that
\[
(0, \infty) \subset \bigcup_{n \in \mathbb{N}} (-2c + nc, 2c + nc),
\]
then (27) implies
\[
\liminf_{\xi \to \infty, n \in J} [\phi_1(\xi, n) - y_1(n)] \geq 0.
\]

By a similar discussion on \( \phi_2(\xi, n) \), we complete the proof. \( \square \)

Before ending this paper, we make the following remarks.

**Remark 8.** From Lemmas 6.5-6.6, when \( \xi \to -\infty \), then the precise limit behavior of traveling wave solutions of (24) with minimal wave speed is different from that of large wave speed. In this paper, \( \lambda_1^2 \) and \( \lambda_2^2 \) may be finite. So, it is weaker than Lin [17, Section 5.2]. By similar discussion, we can improve the conditions in Lin [17, Section 5.3].

**Remark 9.** Similar to that in Lemma 6.6, if \( k(y) \) admits compact support, we may prove the conclusion of Lemma 5.2 by constructing upper and lower solutions, and such a traveling wave solution does not decay exponentially when \( \xi \to -\infty \). Moreover, it is evident that our methods can be applied to \( l \)-species competitive systems.

**Remark 10.** Besides the minimal wave speed of traveling wave solutions, another important propagation threshold is the spreading speed, which has been widely studied for monotone semiflows or local monotone semiflows. For some nonmonotone systems [5, 19, 23, 32, 30], it has been proven that the minimal wave speed may describe the spreading phenomena of the corresponding initial value problems. Very likely the minimal wave speed of (21) may play such a role, and we shall further investigate the question.

**Appendix.** In this part, we shall complete the verification of Lemmas 6.5-6.6 by two lemmas, of which the verification is partly motivated by that in nonmonotone delayed equation, and we refer to a very recent paper [14].

**Lemma A.** There exists \( q > 1 \) such that \( (\overline{\phi}_1(\xi, n), \overline{\phi}_2(\xi, n)), (\underline{\phi}_1(\xi, n), \underline{\phi}_2(\xi, n)) \) in the proof of Lemma 6.5 are a pair of generalized upper and lower solutions of (24) with the corresponding wave speed \( c > c^* \).

**Proof.** We directly verify the desired inequalities for \( i = 1 \) or 2 and
\[
\phi_i(\xi, n) \leq \psi_i(\xi, n) \leq \overline{\phi}_i(\xi, n), \xi \in \mathbb{R}, n \in \mathbb{Z}.
\]
If \( \bar{\phi}_i(\xi + c, n + 1) = 1 \), then the result is clear. Otherwise, we have
\[
(1 + r_i(n)) \phi_i(\xi - y, n) k_i(y) dy 
\]
\[
\leq \int_R (1 + r_i(n))(\phi_i(\xi - y, n) + a_i(n)\phi_{3-i}(\xi - y, n)) k_i(y) dy 
\]
\[
\leq \int_R (1 + r_i(n))\phi_i(\xi - y, n) k_i(y) dy 
\]
\[
\leq \int_R (1 + r_i(n))\bar{\phi}_i(\xi - y, n) k_i(y) dy 
\]
\[
\leq \int_R (1 + r_i(n))h(n)e^{\gamma_{i,1}(c)(\xi - y)} k_i(y) dy 
\]
\[
= h_i(n + 1)e^{\gamma_{i,1}(c)(\xi + c)} 
\]
which implies the conclusion on upper solution.

On the lower solution, it is clear if \( \phi_i(\xi + c, n + 1) = 0 \). Otherwise, let \( \epsilon_i \in (0, 1) \) such that
\[
\gamma_{i,1}(c) + \epsilon_i\gamma_{3-i,1}(c) = \eta_{i,1}(c). 
\]
In particular, we have
\[
\bar{\phi}_{3-i}(\xi, n) \leq [\bar{\phi}_{3-i}(\xi, n)]^{\epsilon_i} \leq 1, \xi \in \mathbb{R}, n \in \mathbb{Z}. 
\]

By directly direction, we have
\[
\int_R \frac{(1 + r_i(n))\phi_i(\xi - y, n) k_i(y) dy}{(1 + r_i(n))\phi_i(\xi - y, n) + a_i(n)\phi_{3-i}(\xi - y, n)} 
\]
\[
\geq \int_R (1 + r_i(n))\phi_i(\xi - y, n) k_i(y) dy 
\]
\[
\geq \int_R (1 + r_i(n))\bar{\phi}_i(\xi - y, n) k_i(y) dy 
\]
\[
\geq \int_R (1 + r_i(n))h_i(n)e^{\gamma_{i,1}(c)(\xi - y)} k_i(y) dy 
\]
\[
\geq h_i(n + 1)e^{\gamma_{i,1}(c)(\xi + c)} - q(1 + r_i(n))h_i(n)e^{\gamma_{i,1}(c)(\xi - y)} k_i(y) dy 
\]
\[
= h_i(n + 1)e^{\gamma_{i,1}(c)(\xi + c)} - qh_i(n + 1)e^{\gamma_{i,1}(c)(\xi + c)} 
\]
\[
\phi_i(\xi + c, n + 1) = \cases{\phi_i(\xi + c, n + 1)}
\]

if
\[
q \left[ h_i(n + 1)e^{\gamma_i(c)c} - (1 + r_i(n))h_i(n) \int_{\mathbb{R}} e^{q\gamma_i(c)y}k_i(y)dy \right] > r_i(n)(1 + r_i(n))[h_i^{q\gamma_i(c)}(n) + a_i(n)h_{\gamma_i(c)}(n)] \int_{\mathbb{R}} e^{-q\gamma_i(c)y}k_i(y)dy.
\]

Note that the second line of the above inequality is a bounded constant, then the above is true by selecting large but finite \( q > 1 \) if
\[
h_i(n + 1)e^{\gamma_i(c)c} - (1 + r_i(n))h_i(n) \int_{\mathbb{R}} e^{q\gamma_i(c)y}k_i(y)dy > 0,
\]
which is also a constant. In fact, (28) is true since
\[
h_i(n + 1)e^{\gamma_i(c)c} - (1 + r_i(n))h_i(n) \int_{\mathbb{R}} e^{q\gamma_i(c)y}k_i(y)dy > 0,
\]
by (1) of Lemma 6.1. The proof is complete.

**Lemma B.** There exist \( P_1, P_2 \) such that \((\bar{\phi}_1(\xi, n), \bar{\phi}_2(\xi, n)), (\phi_1(\xi, n), \phi_2(\xi, n))\) in Lemma 6.6 are a pair of generalized upper and lower solutions of (24).

**Proof.** We directly verify the desired inequalities for \( i = 1 \) or 2 and
\[
\phi_2(\xi, n) \leq \phi_i(\xi, n) \leq \bar{\phi}_i(\xi, n), \xi \in \mathbb{R}, n \in \mathbb{Z}.
\]

We first present the detailed verification when \( c^* = c_1 = c_2 \).

If \( \bar{\phi}_i(\xi + c^*, n + 1) = 1 \), then the result is clear. Otherwise, the selection of \( \xi_{i,1} \) implies
\[
\int_{\mathbb{R}} \frac{(1 + r_i(n))\phi_i(\xi - y, n)}{1 + r_i(n)(\phi_i(\xi - y, n) + a_i(n)\phi_{\gamma_i(c)}(\xi - y, n))}k_i(y)dy \\
\leq \int_{\mathbb{R}} (1 + r_i(n))\phi_i(\xi - y, n)k_i(y)dy \\
\leq \int_{\mathbb{R}} (1 + r_i(n))\bar{\phi}_i(\xi - y, n)k_i(y)dy \\
\leq -L_ie_i(n) \int_{\mathbb{R}} (1 + r_i(n))(\xi - y)e^{\gamma_i(c)(\xi - y)}k_i(y)dy \\
= -L_ie_i(n)(1 + r_i(n)) \int_{\mathbb{R}} (\xi - y)e^{\gamma_i(c)(\xi - y)}k_i(y)dy \\
= -L_ie_i(n)(1 + r_i(n))e^{\gamma_i(c)} \int_{\mathbb{R}} (\xi - y)e^{-\gamma_i(c)y}k_i(y)dy \\
= -L_ie_i(n)(1 + r_i(n))\xi e^{\gamma_i(c)} \int_{\mathbb{R}} e^{-\gamma_i(c)y}k_i(y)dy \\
+ L_ie_i(n)(1 + r_i(n))e^{\gamma_i(c)} \int_{\mathbb{R}} ye^{-\gamma_i(c)y}k_i(y)dy
\]
Evidently, we may fix $\lambda$ which is admissible since by (2) of Lemma 6.1, which indicates the conclusion on upper solution.

If $\phi_i(\xi+c^*, n+1) = 0$, then the conclusion on lower solution is clear. Otherwise, fix $\lambda_i \in (0, \gamma_i)$ such that

$$2\lambda_i > \gamma_i, \lambda_i + \lambda_{3-i} > \gamma_i.$$ 

Let $P_0 > 0$ be large such that

$$\phi_i(\xi, n) < e^{\lambda_i \xi}, \xi < -P_0^2/L_i^2 + D + c^*,$$

which is admissible since $\lambda_i < \gamma_i$.

By directly calculation, we have

$$\int R (1 + r_i(n)) \phi_i(\xi - y, n) k_i(y) dy
\geq (1 + r_i(n)) \int R \phi_i(\xi - y, n) k_i(y) dy
- r_i(n)(1 + r_i(n)) \int R \phi_i(\xi - y, n) [\phi_i(\xi - y, n) + a_i(n) \phi_{3-i}(\xi - y, n)] k_i(y) dy
\geq (1 + r_i(n)) \int R \phi_i(\xi - y, n) k_i(y) dy
- r_i(n)(1 + r_i(n)) \int R \phi_i(\xi - y, n) [\phi_i(\xi - y, n) + a_i(n) \phi_{3-i}(\xi - y, n)] k_i(y) dy.$$

Evidently, we may fix $K \geq 0$ such that

$$r_i(n)(1 + r_i(n)) \int R \phi_i(\xi - y, n) \phi_i(\xi - y, n) + a_i(n) \phi_{3-i}(\xi - y, n) k_i(y) dy
\leq K[e^{2\lambda_i \xi} + e^{(\lambda_i + \lambda_{3-i}) \xi}], n \in J.$$

Moreover, we have

$$\int R (1 + r_i(n)) \phi_i(\xi - y, n) k_i(y) dy
\geq e_i(n)(1 + r_i(n)) \int R (-L_i(\xi - y) - P_i \sqrt{-(\xi - y)}) e^{\gamma_i(\xi - y)} k_i(y) dy
= -L_i e_i(n)(1 + r_i(n)) \int R (\xi - y) e^{\gamma_i(\xi - y)} k_i(y) dy
- P_i e_i(n)(1 + r_i(n)) \int R \sqrt{-(\xi - y)} e^{\gamma_i(\xi - y)} k_i(y) dy
= -L_i(\xi + c^*) e_i(n + 1) e^{\gamma_i(\xi + c^*)}
- P_i e_i(n)(1 + r_i(n)) \int R \sqrt{-(\xi - y)} e^{-\gamma_i y} k_i(y) dy
= -L_i(\xi + c^*) e_i(n + 1) e^{\gamma_i(\xi + c^*)} - P_i e_i(n + 1) \sqrt{-(\xi + c^*)} e^{\gamma_i(\xi + c^*)}
+ P_i \left[ e_i(n + 1) \sqrt{-(\xi + c^*)} e^{\gamma_i(\xi + c^*)}
- e_i(n)(1 + r_i(n)) \int R \sqrt{-(\xi - y)} e^{-\gamma_i y} k_i(y) dy \right]
= \phi_i(\xi + c^*, n + 1) + P_i e^{\gamma_i \xi}$$. 

\[ e_i(n+1)\sqrt{-(\xi + c^*)}e^{\gamma y} - e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \sqrt{-(\xi - y)}e^{-\gamma y} k_i(y) dy \].

For the second line, we have

\[
e_i(n+1)\sqrt{-(\xi + c^*)}e^{\gamma y} - e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \sqrt{-(\xi - y)}e^{-\gamma y} k_i(y) dy
\]

\[= e_i(n+1)\sqrt{-(\xi + c^*)}e^{\gamma y} - e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \sqrt{-(\xi - y)}e^{-\gamma y} k_i(y) dy
\]

\[+ e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \sqrt{-(\xi + c^*)}e^{-\gamma y} k_i(y) dy
\]

\[- e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \sqrt{-(\xi - y)}e^{-\gamma y} k_i(y) dy
\]

\[= e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \left[ \sqrt{-(\xi + c^*)} - \sqrt{-(\xi - y)} \right] e^{-\gamma y} k_i(y) dy
\]

\[= e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \left[ \sqrt{-(\xi + c^*)} - \sqrt{-(\xi + y)} \right] e^{\gamma y} k_i(y) dy
\]

\[= e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \frac{y - c^*}{\sqrt{-(\xi + c^*)} + \sqrt{-(\xi + y)}} e^{\gamma y} k_i(y) dy
\]

If \( y \in (-\infty, c^*) \), then \( y - c^* < 0 \) such that

\[
\frac{y - c^*}{\sqrt{-(\xi + c^*)} + \sqrt{-(\xi + y)}} \geq \frac{y - c^*}{2\sqrt{-(\xi + c^*)}}
\]

and

\[
\int_{-\infty}^{c^*} + \int_{c^*}^{\infty} \frac{y - c^*}{\sqrt{-(\xi + c^*)} + \sqrt{-(\xi + y)}} e^{\gamma y} k_i(y) dy
\]

\[\geq \int_{-\infty}^{c^*} \frac{y - c^*}{2\sqrt{-(\xi + c^*)}} e^{\gamma y} k_i(y) dy + \int_{c^*}^{\infty} \frac{y - c^*}{2\sqrt{-(\xi + c^*)}} e^{\gamma y} k_i(y) dy.
\]

Moreover, (2) of Lemma 6.1 implies that

\[
\int_{-\infty}^{c^*} \frac{y - c^*}{2\sqrt{-(\xi + c^*)}} e^{\gamma y} k_i(y) dy + \int_{c^*}^{\infty} \frac{y - c^*}{2\sqrt{-(\xi + c^*)}} e^{\gamma y} k_i(y) dy = 0
\]

and so

\[
e_i(n)(1 + r_i(n)) \int_{\mathbb{R}} \left[ \sqrt{-(\xi + c^*)} - \sqrt{-(\xi + y)} \right] e^{\gamma y} k_i(y) dy
\]

\[\geq e_i(n)(1 + r_i(n)) \int_{c^*}^{\infty} \frac{y - c^*}{\sqrt{-(\xi + c^*)} + \sqrt{-(\xi + y)}} - \frac{y - c^*}{2\sqrt{-(\xi + c^*)}} e^{\gamma y} k_i(y) dy
\]

\[= e_i(n)(1 + r_i(n)) \int_{c^*}^{\infty} \frac{(y - c^*) \left[ \sqrt{-(\xi + c^*)} - \sqrt{-(\xi + y)} \right]}{2\sqrt{-(\xi + c^*)} \left[ \sqrt{-(\xi + c^*)} + \sqrt{-(\xi + y)} \right]} e^{\gamma y} k_i(y) dy
\]
where \( L' > 0 \) holds by \( \int_{c}^{\infty} (y - c^*)^2 e^{\gamma y} k_i(y) dy > 0 \) in Remark 7.

Summarizing what we have done, if \( \xi < -P_i^2/L_i^2 \) such that

\[
\frac{L' P e^{\gamma \xi}}{(-\xi)^2} \geq K[e^{2\lambda_i \xi} + e^{\lambda_i + \lambda_{i-1} - i} \xi], \quad n \in J,
\]

we then obtain the desired lower solution, of which the existence of \( P_i \geq P_0 \) is clear by \( 2\lambda_i > \gamma_i, \lambda_i + \lambda_{i-1} > \gamma_i \).

When \( c_1 < c_2 \) or \( c_1 > c_2 \), the verification is similar to what we have done. We omit it and complete the proof.

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