On Unique Independence Weighted Graphs

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Abstract. An independent set in a graph \( G \) is a set of vertices no two of which are joined by an edge. A vertex-weighted graph associates a weight with every vertex in the graph. A vertex-weighted graph \( G \) is called a unique independence vertex-weighted graph if it has a unique independent set with maximum sum of weights. Although, in this paper we observe that the problem of recognizing unique independence vertex-weighted graphs is NP-hard in general and therefore no efficient characterization can be expected in general; we give, however, some combinatorial characterizations of unique independence vertex-weighted graphs. This paper introduces a motivating application of this problem in the area of combinatorial auctions, as well.

Introduction and preliminaries

In this paper, we focus on graphs whose vertices have real weights and call such graphs for simplicity, just weighted graphs. Also, we study unique independent sets in finite vertex weighted graphs. For the definition of basic concepts and notations not given here one may refer to a textbook in graph theory, for example [G] and [I]. Let \( G = (V, E) \) be a simple undirected graph with the vertex set \( V = \{1, 2, \cdots, n\} \), the edge set \( E \) and a nonnegative weight \( w(i) \) associated with each vertex \( i \in V \). The weight of \( S \subseteq V(G) \) is defined as \( w(S) = \sum_{i \in S} w(i) \). A subset \( I \) of \( V(G) \) is called an independent set (or a stable set) if the subgraph \( G[I] \) induced by \( I \) of \( G \) has no edges. A maximum weighted independent set, also called \( \alpha \)-set, is an independent set of the largest weight in \( G \). The weight of a maximum weighted independent set in \( G \) is denoted by \( \alpha(G) \). A weighted graph \( G \) is a unique independence weighted graph, if \( G \) has a unique independent set with maximum sum of weights. Characterizing unique independence graphs and various generalizations of this concept has been a subject of research in graph theory literature. As a few examples, we refer the interested reader to [B], [D], and [C]. Also, some existing papers have focused on finding or even approximating the maximum independent set problem in weighted graphs. See [A], and [E] for more details. As we will observe in this paper, this is not coincidental: we show that the problem of recognizing unique maximum independence weighted graphs is NP-hard in general and therefore no

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efficient characterization of this concept can be expected in general. To our best knowledge, this is the first paper discussing the unique maximum weighted independent set problem and gives some characterizations of it and, most importantly, defines the problem of unique combinatorial auctions. How weighted graphs and combinatorial auctions are related, is given in the last section, where we make some future research directions. For more details on combinatorial auctions, the interested reader is referred to a detailed article or a textbook both on combinatorial auctions, respectively in [F] and [J].

The rest of the paper is organized as follows: Section 1 gives a characterization of unique independence weighted graphs as generalization of the unique independence graphs. Section 2 introduces some theorems on characterization of unique independence weighted graphs most of which are based on neighborhood concept. In section 3 we show the NP-hardness of recognizing the unique independence weighted graphs and finally section 4 presents some notes on how much this article would be interesting and what lines of future researches this article may create.

1. Unique independence weighted graphs

In this section we exhibit one basic theorem in addition to a corollary obtained from the theorem, both as generalizations of unique independence graphs.

**Theorem 1.1.** Let $G$ be a weighted graph and let $I$ be an $\alpha$-set of $G$. Then the following conditions are equivalent:

(i) $G$ is a unique independence weighted graph and $I$ is the unique $\alpha$-set of $G$.

(ii) For every $x \in I$ we have $\alpha(G \setminus \{x\}) < \alpha(G)$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose there exist $x \in I$ that $\alpha(G \setminus \{x\}) \geq \alpha(G) = w(I)$. So, $G \setminus \{x\}$ contains an independent set $I'$ which differs from $I$ and also has the property $w(I') \geq w(I)$. This obviously contradicts either maximality or the uniqueness of $I$.

(ii) $\Rightarrow$ (i) Suppose $G$ has another maximum weighted independent set $I', w(I') = w(I)$, and $x \in I \setminus I'$. The set $I'$ remains a maximum weighted independent set of $G \setminus \{x\}$. So, $w(I') = \alpha(G \setminus \{x\}) < \alpha(G) = w(I)$, which is a contradiction. So, $G$ has a unique $\alpha$-set. \hfill $\square$

**Corollary 1.** Let $G$ be an edge weighted graph and let $M$ be a maximum matching of $G$. The following conditions are equivalent:

(i) $M$ is a unique maximum matching of $G$.

(ii) For every $e \in M$ we have $\alpha'(G \setminus \{x\}) < \alpha'(G)$.

**Proof.** Any maximum matching of $G$ is corresponding to a maximum independent set of its line graph, $L(G)$. So the statement follows from Theorem 1.1 \hfill $\square$

2. Neighborhood-based characterization of unique independence weighted graphs

In this section, we try to state some theorems on characterizations of unique weighted graphs mainly based on the neighborhood concept.
DEFINITION 2.1. For any vertex \( x \in V(G) \) the open neighborhood of \( x \) in \( G \), \( N(x,G) \), is defined as:
\[
N(x,G) = N_G(x) = \{ y \in V(G) \mid xy \in E(G) \}.
\]
In addition, the extension of this concept to any subset \( I \) of vertices of a graph \( G \) is defined as: \( N_G(I) = \cup_{x \in I} N_G(x) \).

DEFINITION 2.2. For a subset \( I \) of \( V(G) \) and a vertex \( x \in I \), we define:
\[
p^G_I(x) = N_G(x) \setminus N_G(I \setminus \{x\})
\]
Furthermore, for every subset \( I \) of \( V(G) \), we define the set \( p^G_I \) as:
\[
p^G_I = \bigcup_{x \in I} p^G_I(x)
\]

The following lemma gives a sufficient neighborhood-oriented condition by which the uniqueness of a weighted graph is established.

LEMMA 2.3. Let \( G \) be a weighted graph and let \( I \) be an \( \alpha \)-set of \( G \). If for any \( I_0 \subseteq I \) we have \( w(p^G_I(I_0)) < w(I_0) \), then \( G \) is unique independence weighted graph and \( I \) is the unique \( \alpha \)-set of \( G \).

PROOF. By contradiction. Let \( I' \) be another \( \alpha \)-set of \( G \). This means, \( \emptyset \neq I \cap I' \), \( w(I) = w(I') \), and also \( w(I \setminus I') = w(I' \setminus I) \). Clearly, \( I' \setminus I \subseteq p^G_I(I \setminus I') \) and thus \( w(I' \setminus I) \leq w(p^G_I(I \setminus I')) \). Replacing the left side of this equation by its equivalent, \( w(I \setminus I') \), results \( w(I \setminus I') \leq w(p^G_I(I \setminus I')) \). Now, taking \( I_0 = I \setminus I' \) contradicts the hypothesis of the lemma. \( \square \)

The converse of Lemma 2.3 is not true for all weighted graphs. The following example gives a counter-example.

EXAMPLE 2.4. Suppose that \( G \) is the following weighted graph.

\[
\begin{align*}
&\text{A (5)} \\
&\text{E (2)} \quad \text{B (4)} \\
&\text{D (1)} \quad \text{C (2)}
\end{align*}
\]

Figure 1. A Counter-Example

The numbers enclosed in parentheses, are the vertices’ weights. Suppose \( I = \{A, C\} \). \( I \) is a unique \( \alpha \)-set of \( G \). If \( I_0 = \{A\} \), then \( p^G_I(I_0) = \{E\} \). So, \( w(p^G_I(I_0)) = 2 < 5 = w(A) \). If \( I_0 = \{C\} \) then \( p^G_I(I_0) = \{D\} \). So, \( w(p^G_I(I_0)) = 1 < 2 = w(C) \). If \( I_0 = \{A, C\} \), then \( p(I_0) = \{B, E, D\} \). So, \( w(p^G_I(I_0)) = 7 \neq 7 \). Therefore, there is a subset of \( I \) not satisfying the condition given by the Lemma 2.3.

The next theorem exhibits the fact that the converse of Lemma 2.3 is true for all trees.
Theorem 2.5. Let $T$ be a weighted tree and let $I$ be an $\alpha$-set of $T$. The following conditions are equivalent:

(i) $T$ is unique independence weighted tree and $I$ is the unique $\alpha$-set of $T$.
(ii) For every $I_0 \subseteq I$, we have $w(p_T(I_0)) < w(I_0)$.

Proof. (ii) $\Rightarrow$ (i) is implied directly from Lemma 2.3.

(i) $\Rightarrow$ (ii) The proof by contradiction. Let $A = \{I' \mid I' \subseteq I\}$ and $W_A = \{w(I') \mid I' \in A\}$. Based on the contrary hypothesis, the set $W_A$ is not empty and therefore making the assumption that $s$ is an independent set so that $w(I') = w(I)$. In order to prove this claim, it suffices to show that $p_T(I_0)$ is an independent set. Let $x, y \in p_T(I_0)$ such that $xy$ be a bridge of $T$. Therefore $T \setminus \{xy\}$ has two components, say $T_1$ and $T_2$. Suppose $I_1 = I_0 \cap V(T_1)$ and $I_2 = I_0 \cap V(T_2)$. Take $x, y \in p_T(I_0)$, so there are $x', y' \in I_0$ such that $x \in p_T(x')$ and $y \in p_T(y')$. Thus $I_1, I_2 \neq \emptyset$ and $I_1 \cup I_2 = I_0$. On the other hand, $p_T(I_1) \cap p_T(I_2) = \emptyset$ and $p_T(I_1) \cup p_T(I_2) = p_T(I_0)$. So:

\[
\text{(2.1)} \quad w(p_T(I_0)) = w(p_T(I_1)) + w(p_T(I_2))
\]

But we have $I_1 \subseteq I_0$ and $I_2 \subseteq I_0$. By minimality of $w(I_0)$ we have $w(p_T(I_1)) < w(I_1)$ and $w(p_T(I_2)) < w(I_2)$. This contradicts Equation (2.1). $\square$

The following theorem gives a general condition under which the uniqueness of a weighted graph is established. Before proceeding this theorem, we make a prerequisite definition.

Definition 2.6. For any $I \subseteq V(G)$, we denote the maximum weighted independent set of $p_G(I)$ by $m(I)$.

Theorem 2.7. Let $G$ be a weighted graph and $I$ be an $\alpha$-set of $G$. The following conditions are equivalent:

(i) $G$ is unique independence weighted graph and $I$ is the unique $\alpha$-set of $G$.
(ii) For every $I_0 \subseteq I$, we have $w(m(I_0)) < w(I_0)$.

Proof. (i) $\Rightarrow$ (ii) This part is done by contradiction. Suppose there exist $I_0 \subseteq I$ such that $w(m(I_0)) \geq w(I_0)$. Let $I' = (I \setminus I_0) \cup m(I_0)$. So $I' \neq I$ and $w(I') \geq w(I)$. On the other hand, $m(I_0)$ and $I \setminus I_0$ are independent sets, $m(I_0) \subseteq p_G(I_0)$ and $p_G(I_0) \cap N(I \setminus I_0) = \emptyset$. Hence, $I'$ is an independent set which is a contradiction.

(ii) $\Rightarrow$ (i) This part is also done by contradiction. Suppose $I'$ be another $\alpha$-set of $G$. This means $w(I) = w(I')$ and also:

\[
\text{(2.2)} \quad w(I') = w(I \setminus I')
\]

In addition, $I' \setminus I \subseteq p_G(I' \setminus I')$ and $I' \setminus I$ is an independent set. Therefore, $w((I' \setminus I)) \leq w(m(I \setminus I'))$. Now, let $I_0 = I \setminus I'$ and thus, by condition (ii), we have $w(m(I \setminus I')) < w(m(I_0)) = w(I_0)$.
Let $w(I \setminus J)$. So, we obtained $w(I' \setminus J) \leq w(m(I' \setminus J)) < w(I \setminus J)$, which contradicts Equation 2.2. □

The prior theorem needs to see whether all subsets of the $\alpha$-set, have the given condition by the theorem. If so, then the uniqueness will be confirmed. What if not? Hence, it seems that Theorem 2.7 for those weighted graphs whose independent sets are decent large in size, needs a time-consuming process before making any decision. For the sake of this, we exhibit the theorem below, particularly useful and thus important for those weighted graphs including a decent large independent set.

**Theorem 2.8.** Let $G$ be a weighted graph and let $I$ be an $\alpha$-set of $G$. The following conditions are equivalent:

(i) $G$ is unique independence weighted graph and $I$ is the unique $\alpha$-set of $G$. 
(ii) For every nonempty independent subset $J$ of $V(G) \setminus I$, we have: $w(N(J) \cap I) > w(J)$.

**Proof.** (i) ⇒ (ii) If $J$ is a nonempty independent subset of $V(G) \setminus I$, then $(I \setminus N(J)) \cup J$ is an independent set in $G$. $I$ is the unique independent set of $G$, so $w((I \setminus N(J))) < w(I)$. Thus we have: $w(N(J) \cap I) > w(J)$.

(ii) ⇒ (i) Let $I'$ be an independent subset of $G$. It suffices to show that $w(I') < w(I)$. Since $I' \setminus I$ is a nonempty independent subset of $V(G) \setminus I$, we have: $w(N(I' \setminus I) \cap I) > w(I' \setminus I)$. Moreover, $N(I' \setminus I) \cap I \subseteq I$ and therefore $w(I') = w(I' \cap I) + w(I' \setminus I) < w(I' \cap I) + w(N(I' \setminus I) \cap I) \leq w(I \cap I') + w(I \setminus I') = w(I)$. So, $w(I') < w(I)$. □

The next theorem is intended to prove this our conjecture on whether or not a given weighted graph has a unique makeup? In fact, the following theorem proves that for every weighted graph, there are many other weighted graphs whose independent sets are same as the given graph, if their vertices' weights have been drawn from a predefined real interval connected to the vertices' weights of the original given graph. This theorem proves this and provides that interval, as well.

**Theorem 2.9.** Let $G$ be a weighted graph and let $I$ be an $\alpha$-set of $G$. If $G$ is a unique independence graph and $I$ is the unique $\alpha$-set of $G$, then there is a positive real number, $\epsilon > 0$, such that if the weights of $G$'s vertices change in $(w(x) - \epsilon, w(x) + \epsilon)$, then $G$ with these new weights remains a unique independence weighted graph with the same $\alpha$-set.

**Proof.** $I$ is the unique $\alpha$-set of $G$ so by Theorem 2.7, for every $I_0 \subseteq I$, $w(m(I_0)) < w(I_0)$.

Let:

- $\sigma = \min \{ w(I_0) - w(m(I_0)) | I_0 \subseteq I \}$,
- $\eta = \min \{ w(I) - w(I_0) | I_0$ is an independent set of $G \}$,
- $\nu = \min \{ w(m(I_0)) - w(J) | I_0 \subseteq I$ and $J$ is an independent set of $p_G(I_0) \}$.

and let $\delta = \min \{ \sigma, \eta, \nu \}$ and also $\epsilon = \frac{\delta}{n+1}$, where $n$ is the number of $G$'s vertices. Suppose $G'$ is a copy of $G$, with new changed vertices weights, $w'$, such that for

\footnote{A conjecture that we had made at the early steps of this work.}
every \( x \in V(G) \): \( w(x) - \epsilon < w'(x) < w(x) + \epsilon \). Now, we make the following claim in order to complete the proof.

**Claim 1.** \( G' \) is a unique independence weighted graph and \( I \) is the unique \( \alpha \)-set of \( G' \).

**Proof of Claim 1:** By definition of \( \eta \), \( I \) is a \( \alpha \)-set of \( G' \). To prove the uniqueness of \( I \), it’s sufficient to show that for every \( I_0 \subseteq I \), \( w'(m(I_0)) < w'(I_0) \).

Proof by contradiction: Suppose there is a subset of vertices like \( J \), such that \( J \subseteq I \) and also:

\[
(2.3) \quad w'(m(J)) \geq w'(J)
\]

By definition of \( \nu \), \( m(J) \) is an \( \alpha \)-set of \( p_G(J) \) in both \( G \) and \( G' \). So we have:

\[
(2.4) \quad w(J) - |J| \epsilon \leq w'(J) \leq w(J) + |J| \epsilon
\]

\[
(2.5) \quad w(m(J)) - |m(J)| \epsilon \leq w'(m(J)) \leq w(m(J)) + |m(J)| \epsilon.
\]

By combining Equations 2.3, 2.4, 2.5 and some simple computations, we achieve:

\[
(2.6) \quad w(J) \leq w'(J) + |J| \epsilon \leq w'(m(J)) + |J| \epsilon \leq w(m(J)) + |m(J)| \epsilon + |J| \epsilon.
\]

From Equation 2.6, the following equation is obtained.

\[
(2.7) \quad w(J) \leq w(m(J)) + |m(J)| \epsilon \leq w(m(J)) + |m(J)| \epsilon + \epsilon n < w(m(J)) + \delta.
\]

Finally, we achieve: \( w(J) - w(m(J)) < \sigma \). Obviously, this contradicts the definition of \( \sigma \). This completes the proof of Claim 1 and thus the proof of Theorem 2.9 is now complete. \( \square \)

**Corollary 2.** For every given weighted graph \( G \), there are infinite number of weighted graphs whose vertices’ weights are real and their independent sets are the same as \( G \).

3. **Complexity of unique maximum weighted independent set problem**

We prove that the following problems are NP-hard. Both problems ask for detecting whether a given vertex weighted graph has a unique maximum weighted independent set; in the first problem, the input contains a candidate for the unique maximum weighted independent set in addition to the graph.

Problem \( UI_1 \):

**Input:** A weighted graph \( G \), a set \( I \) of the vertices of \( G \).

**Question:** Is \( I \) the unique maximum weighted independent set in \( G \)?

Problem \( UI_2 \):

**Input:** A weighted graph \( G \).

**Question:** Does \( G \) have a unique maximum weighted independent set?

We prove the NP-hardness of these problems by reducing the following problem to them:

Problem **WEIGHTED INDEPENDENT SET**:

**Input:** A weighted graph \( G \), an integer \( k \).
**Question:** Does $G$ contain an independent set of weight at least $k$?

The latest problem is NP-Complete and one may refer to [H] for a proof.

Now, the following two theorems exhibit the complexity classes to which the problems $UI_1$ and $UI_2$ are belonging.

**Theorem 3.1.** Problem $UI_1$ is coNP-complete.

**Proof.** First, we show that this problem is in coNP. To see this, it is enough to observe that a witness for the non-membership of an instance $(G, I)$ in $UI_1$ is an independent set of weight greater than or equal to $w(I)$. We now show that the problem is coNP-complete by showing a reduction from the complement of WEIGHTED INDEPENDENT SET problem to this problem. Given an instance $(G, k)$ of WEIGHTED INDEPENDENT SET, construct a graph $H$ by adding $k$ vertices to $G$ and all the edges between these $k$ vertices and the vertices of $G$ (but no edge between the $k$ new vertices) and then set the weight of each new added vertex by 1. Let $H$ denote the resulting graph, and $I$ denote the set of $k$ vertices in $V(G) \setminus V(G)$. We claim that $(G, k) \in$ WEIGHTED INDEPENDENT SET if and only if $H /\notin UI_1$. This is because by construction, every independent set of $H$ is either a subset of $I$, or an independent set in $G$. Therefore, $I$ is the unique maximum weighted independent set in $H$ if and only if $G$ does not contain an independent set whose weight exceeds $w(I)$. Therefore, the above construction is a polynomial time reduction from the complement of WEIGHTED INDEPENDENT SET to $UI_1$. This completes the proof of coNP-completeness of $UI_1$. □

For problem $UI_2$, the situation is less clear, as the problem does not seem to be in NP or coNP. It is not difficult to show that this problem is in the complexity class $\Sigma_2$, but we do not know if it is $\Sigma_2$-complete. However, we can still show that the problem is intractable, assuming $P \neq NP$.

**Theorem 3.2.** Problem $UI_2$ is NP-hard.

**Proof.** As in the proof of the previous theorem, we show a reduction from the complement of WEIGHTED INDEPENDENT SET to $UI_2$. Given an instance $(G, k)$ of WEIGHTED INDEPENDENT SET, construct a graph $H$ by adding a set $I$ of $k+1$ vertices and another set $R$ of two vertices with 1 as the weight of each vertex in both sets, to $G$. The edges of $H$ are the edges of $G$ plus edges between all vertices in $I$ and all vertices in $V(G) \cup R$, and also one edge between the two vertices of $R$. We claim that $(G, k) \in$ WEIGHTED INDEPENDENT SET if and only if $H /\notin UI_2$. This is because by construction, every weighted independent set of $H$ is either a subset of $I$, or a subset of $V(G) \cup R$. The weight of largest independent set in $V(G) \cup R$ is precisely $\alpha(G) + 1$. Therefore, the weight of the largest independent set of $H$ is $\max(k, \alpha(G)) + 1$. Therefore, if $G$ has an independent set of weight at least $k$, at least two $\alpha$-sets in $H$ can be obtained by adding either of the vertices of $R$ to a maximum weighted independent set of $G$. Thus, $H /\notin UI_2$ in this case. Conversely, if $G$ has no weighted independent set of weight $k$ or more the unique $\alpha$-set of $H$ is $I$. Therefore, the above construction is a polynomial time reduction from the complement of WEIGHTED INDEPENDENT SET to $UI_2$. This reduction completes the proof. □
4. Concluding remarks and future research directions

To our knowledge, this is the first paper beginning this line of research due to the diverse applications this problem have. Of which, let us to outline one the most important ones which has recently received the attention of researchers and has also opened many doors of research opportunities in fields like algorithmic game theory, computational economics and e-commerce. Combinatorial auctions are mechanisms for allocating a set of items between a set of (likely selfish) agents. It’s a well-known principle that every combinatorial auction can be equivalently viewed as a vertex-weighted graph where bid sets and the set of winners of the former correspond to the vertices and the maximum independent set of the latter.[E]. Hence, the most significant result of this study is to define the problem of unique combinatorial auctions, those combinatorial auctions whose the set of winners is unique, and providing some good starting points for this line of research. So, the authors believe that this article may be considered as the first step to define the problem of unique combinatorial auctions and also as the first step to characterize and classify unique combinatorial auctions not only as a specified problem but also as the first for future related studies.

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