A DOA estimation algorithm based on greedy recovery under the condition of array element failure

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Abstract. Aiming at the DOA estimation when some elements in the hydrophone array fail, the conventional beamforming algorithm is not effective in estimating the direction of arrival. This paper proposes a beamforming algorithm based on greedy recovery. The ideal conditions and the estimation effect under the failure condition of the array element, the detection performance of the method is compared and analyzed by simulation. The results show that when the failure ratio of the array element reaches 70% or more, the method always keeps the effective identification of the direction of arrival, and has better detection performance and tolerance than traditional algorithms.

1. Introduction
The signal processing of passive sonar mainly uses conventional beamformer (CBF) [1] and minimum variance distortionless response algorithm (Minimum Variance Distortionless Response, MVDR) [2]. Due to the influence of objective factors, the failure of some elements in the hydrophone will appear randomly, and it is difficult to carry out comprehensive repairs in the first time. The traditional Direction of Arrival (DOA) estimation method is difficult to deal with a large number of the demand for direction of arrival estimation when the array element fails.

The birth of Compressive Sensing (CS) theory [3] puts forward new ideas for signal processing. The CS-DOA estimation algorithm based on this theory realizes a higher precision direction of arrival estimation. However, the current development is not mature, and there are disadvantages such as slow speed and poor robustness, which are not applicable in engineering.

Based on the theoretical framework of compressed sensing, this paper uses the original signal to be restored by a suitable sensing matrix and reconstruction algorithm, and constructs the GD-BF algorithm. This method can still recognize the target signal even when the array element fails, and its performance is better than traditional methods.

2. Compression sensing theory
Compressed sensing theory includes three steps: sparse transformation, observation sampling, and signal reconstruction. The difference with the traditional theory is that the compression sensor samples and compresses the signal at the same time.

2.1. Sparse transformation
The sparseness of the signal is a priori condition of compressed sensing theory and a prerequisite for accurate signal reconstruction. For a signal $x \in \mathbb{R}^{N \times 1}$, when there are $K$ non-zero elements in the...
signal, and \( K \ll N \), the signal \( \mathbf{x} \) has sparseness. Define \( \| \mathbf{s} \|_0 = K \), say that vector \( \mathbf{x} \) is a sparse vector of order \( K \), and define the \( \ell_m \) norm of vector \( \mathbf{x} = [x_1, x_2, \ldots, x_N]^T \) as

\[
\| \mathbf{x} \|_0 = \left( \sum_{j=1}^{N} |x_j|^m \right)^{\frac{1}{m}}, 0 < m < \infty \\
\| \mathbf{x} \|_1 = \max_{j=1, \ldots, N} |x_j|
\]

(1)

The sparse transformation of the signal refers to the digital signal \( \alpha \in \mathbb{R}^{N \times 1} \) after \( N \) sampling, and there will be a normal orthogonal basis matrix \( \Psi \in \mathbb{R}^{N \times N} \), so that:

\[
\psi = \Psi \mathbf{x} = \sum_{j=1}^{N} x_j \cdot \psi_j
\]

(2)

Among them, \( \mathbf{x} = [x_1, x_2, \ldots, x_N]^T \) is the coordinates of the vector \( \alpha \) under the orthogonal basis \( \Psi = [\psi_1, \psi_2, \ldots, \psi_N] \).

2.2. Observation & sampling

For \( \alpha \in \mathbb{R}^{N \times 1} \), after introducing the canonical orthogonal basis \( \Psi \) to perform sparse transformation to obtain the vector \( \mathbf{x} \), the sparse vector \( \mathbf{x} \) is mapped to the observation matrix \( \Phi = [\phi_1, \phi_2, \ldots, \phi_M]^T \), where the observation matrix \( \Phi \) is not related to the canonical orthogonal basis \( \Psi \), and the observation vector \( \mathbf{y} \) of the signal \( \alpha \) is obtained, namely

\[
\mathbf{y} = \Phi \alpha = \langle \mathbf{a}, \phi_m \rangle, m = 1, 2, \ldots, M
\]

(3)

We know that \( M \ll N \), so

\[
\mathbf{y} = \Phi \mathbf{x} = \Phi \Psi \mathbf{x} = \Theta \mathbf{x}
\]

(4)

Where \( \mathbf{y} = [y_1, y_2, \ldots, y_M]^T \in \mathbb{R}^{M \times 1} \) is the sampling vector, \( \Phi \in \mathbb{R}^{N \times N} \) is the observation matrix, \( \Psi \in \mathbb{R}^{N \times N} \) is the sparse basis matrix, \( \mathbf{x} \in \mathbb{R}^{N \times 1} \) is the sparse vector, and \( \Theta = \Phi \Psi \in \mathbb{R}^{M \times N} \) is the sensing matrix.

In general, since the coherence coefficient of the random matrix and any base matrix is small, the commonly used random matrix is selected as the observation matrix \([4]\).

2.3. Signal reconstruction

To reconstruct the \( N \)-dimensional signal \( \alpha \) from the sampled samples \( \mathbf{y} = \Phi \mathbf{x} \in \mathbb{R}^{M \times 1} \), it is often necessary to introduce the sparse transformation of the signal, that is, to ensure that the conditions \( \mathbf{y} = \Phi \mathbf{a} = \Phi \Psi \mathbf{x} = \Theta \mathbf{x} \) are satisfied so that the sparsity of the vector \( \mathbf{x} \) is minimized. The model is expressed as follows:

\[
\min \| \mathbf{s} \|_0, \quad \text{s.t.} \quad \mathbf{y} = \Phi \Psi \mathbf{x} = \Theta \mathbf{x}
\]

(5)

Since the l0 norm problem itself is an NP-Hard problem, it is a huge workload to solve it directly. E.Candes et al. proposed that under certain conditions, the l1 norm constraint can be used to replace the l0 norm, thereby reducing the difficulty of the solution [5]. The formula (5) is transformed into:

\[
\min \| \mathbf{x} \|_1, \quad \text{s.t.} \quad \mathbf{y} = \Phi \Psi \mathbf{x} = \Theta \mathbf{x}
\]

(6)

This condition is called the RIP (Restricted Isometry Property) [6][7]. To change equation (5) into equation (6) optimization problem, its sufficient condition is that there is a constant \( \delta \in (0, 1) \) such that
It holds for any K-order sparse vector $x$. The smallest constant $\delta_K$ that satisfies Eq. (7) is called the Restricted Isometry Constants (RIC).

3. Traditional DOA estimation

Taking a uniform linear array as an example, assuming that the number of array elements is $M$, there are $K$ narrowband signals in the far field, and the signal incident direction is $\{\theta_k\}_{k=1}^{K}\neq\emptyset$, the signal model received by the linear array at time $t$ is

$$x = A(\theta)s + e$$

Among them, $A(\theta)$ is the array flow matrix, and its column vector is the steering vector; $x, s$ and $e$ are the array received signal vector, source data vector and noise vector respectively.

The idea of conventional beamforming [1] is that when the weight takes the steering vector, that is $w = a(\theta)$, the output power of the system reaches the maximum value. The calculation formula is:

$$P_{CBF}(\theta) = w^HTw = a^H(\theta)Ra(\theta)$$

The minimum non-variance and distortion-free response algorithm, namely MVDR[2], the core idea is to minimize the output power of signals and noise in other directions under the premise that the output power of the target source direction remains unchanged. The calculation formula is:

$$P_{MVDR}(\theta) = \frac{1}{a^H(\theta)Ra(\theta)}$$

4. DOA estimation based on greedy recovery

In the signal processing of passive sonar, relative to the overall space, the number of signals is limited and sparse. The array flow matrix $A$ conforms to the strictly equidistant characteristic and can be used as the observation matrix $\Phi$ in the compression sensing theory. This ensures the feasibility of combining compression sensing with DOA estimation methods.

4.1. Raster discretization

In this paper, the unknown variables are expressed sparsely by means of equal angle division. As shown in figure 1, the entire one-dimensional angle domain is evenly divided into $K$ discrete grid points.

Among them, $\{\theta_k\}_{k=1}^{K} = \{\theta_1, \theta_2, ..., \theta_K\}$ represents the actual direction of arrival of each source, and a series of discrete grid points divided into the space of interest are represented by $\{\bar{\theta}_s\}_{s=1}^{N} = \{\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_N\}$. Due to the requirement of sparsity in the spatial domain, it needs to be satisfied $N \gg K$. 

![Figure 1. Spatial sparsity of signal](image-url)
4.2. Greedy algorithm

In real applications, most signals will be interfered by noise. The mathematical model is:

\[
y = \Phi \alpha + e = \Phi \Psi x + e = \Theta x + e
\]  

(11)

Where \( e \) is the noise vector, further modify the model:

\[
\min \| x \| , \quad s.t. \quad \| y - \Theta x \| \leq \varepsilon
\]

(12)

Among them \( \varepsilon \) is parameter related to noise.

Solving equation (12) requires a reasonable reconstruction algorithm, and the greedy algorithm is one of the more mature algorithms. Taking OMP[8] as a representative, the principle is to directly solve the l0 norm problem and allow a certain reconstruction error.

The current DOA estimation based on the OMP algorithm is to find the position of the grid point where the non-zero element is located in the process of reconstructing and recovering the signal, so as to obtain the corresponding angle information, which can complete the orientation estimation under the premise of a single snapshot.

Different from traditional DOA estimation methods, this type of method cannot use the coherent accumulation between multiple snapshots to improve the estimation effect [9], so the increase in the number of snapshots cannot obtain a better estimation effect. In addition, the performance is unstable under the conditions of low signal-to-noise ratio and high number of array elements, and the azimuth estimation has a large deviation.

Therefore, this paper proposes the GD algorithm and the GD-BF algorithm to complete DOA estimation in two steps: (1) Design an improved greedy algorithm GD based on OMP to make the reconstruction faster and have fewer prior conditions; (2) The received signal is reconstructed and restored by the GD algorithm, and beamforming is completed on the basis of the recovered signal.

4.3. GD algorithm

The improved algorithms based on the OMP algorithm mainly include ROMP[10] and gOMP[11] based on atomic regularization, CoSaMP[12] and SP[13] that use backtracking ideas, and StOMP[14] and SWOMP[15] that use threshold to select elements. When these algorithms are used to recover the received signal, their running speed is limited and the target number is required to be known in the input parameters.

Aiming at the problems of slow reconstruction and recovery time and difficult preset target number in practical applications, this paper proposes the GD algorithm. Input the step length \( S = \lfloor \log(N) \rfloor \), where \( N \) is the number of snapshots and \([\cdot]\) is the rounding function.

As shown in table 1, in each iteration, the GD algorithm first finds the 2S columns with the greatest correlation with the sensing matrix from the residuals, records the corresponding index positions, merges the column vectors corresponding to the indexes into the support set, and updates the residuals. After eliminating the received signal, these steps are repeated to obtain a new residual. When the number of iterations reaches \( S \), the iteration ends. At this time, the output are signal sparse representation coefficient estimation \( \hat{x} \) and residuals \( e_k = y - \Theta \hat{x} \).

| Table 1. GD algorithm flow |
|---------------------------|
| **Input**                 |
| 1. \( M \times N \) dimensional sensing matrix \( \Theta \); |  |
| 2. \( N \times 1 \) dimensional Observation vector \( y \); |  |
| 3. Step length \( S \)      |  |
| **Process**               |
| 1. Initialize residual \( e_0 = y \),Iterative index \( \Lambda_i = \emptyset \),Matrix selected by index \( \Theta_0 = \emptyset \),Number of iterations \( i = 1 \); |  |
| 2. Calculate \( u = \text{abs} \left[ e_i \right] \),choose the largest \( 2S \) values in \( u \),these values correspond to the column number \( j \) of \( \Theta \) into a set \( J_0 \); |  |
3. Let \( \Lambda = \Lambda_{0} \cup J_{0} \), \( \Theta_{0} = \Theta_{0} \cup \Theta_{j} \) (for all \( j \in J_{0} \));

4. Find the least squares solution of \( y = \Theta_{0} x \),
   \[ \hat{x}_{i} = \arg \min_{x} \| y - \Theta_{0} x \| = (\Theta_{0}' \Theta_{0})^{-1} \Theta_{0}' y ; \]

5. Update residual \( e_{i} = y - \Theta_{0} \hat{x}_{i} = y - \Theta_{0} (\Theta_{0}' \Theta_{0})^{-1} \Theta_{0}' y ; \)

6. \( i = i + 1 \), if \( i \leq S \) go back to step 2. Otherwise, stop the iteration and go to step 8;

7. The reconstructed \( \hat{x} \) has a non-zero term at \( \Lambda_{k} \), and its values are \( \hat{x}_{k} \) obtained from the last iteration.

Output 1. signal sparse representation coefficient estimation \( \hat{x} \);
2. \( N \times 1 \) dimensional residuals \( e_{k} = y - \Theta_{k} \hat{x}_{k} \)

### 4.4. GD-BF algorithm

Let each grid point \( \vec{\theta}_{i} \) correspond to a steering vector \( a(\vec{\theta}_{i}) \), and its corresponding coefficient can be uniquely represented by the signal component incident from this direction. Use the steering vectors corresponding to all the azimuth grid points to construct a set of over-complete bases to sparse the received signal:

\[
\hat{x} = \Lambda(\hat{\theta}) \hat{s} + e
\]  

(13)

Comparing equation (8), \( \Lambda(\hat{\theta}) \) in equation (13) satisfies the RIP criterion in the reconstruction condition, which is the sensing matrix \( \Theta_{0} \), and the array received signal vector corresponds to the observation vector. The solution of formula (13) can be transformed into an l1 norm convex optimization problem:

\[
\hat{s} = \arg \min_{s} \| s \|_{1} \quad s.t. \quad \| x - \Lambda(\hat{\theta}) \hat{s} \|_{2} \leq \varepsilon
\]  

(14)

Through the GD algorithm, the sparse reconstruction signal estimation \( \hat{s} \) is obtained, let \( \hat{x}_{GD} = \hat{s} \), and the traditional beamforming is performed on this basis, and \( E[\hat{x}_{GD} \hat{x}_{GD}^{H}] = R_{GD} \) the autocorrelation matrix of the sparse reconstruction signal is easily obtained as. Combining formula (9), the calculation formula is obtained:

\[
P_{GD-BF}(\theta) = a^{H}(\theta)R_{GD}a(\theta)
\]  

(15)

In order to test the performance and tolerance of the GD-BF algorithm, simulation and comparative analysis of the performance under ideal conditions and element failure conditions are carried out below.

### 5. Simulation analysis

First, set the simulation conditions of the linear array DOA estimation experiment based on GD-BF under ideal conditions. The receiving array is a uniform linear array, the number of array elements \( M = 36 \), the number of snapshots \( N = 1000 \), the array element spacing is half wavelength, the number of sources is set to 1, and the direction of the incoming wave is \( 0^\circ \), the carrier frequency of the single-frequency signal is 1kHz, the sampling rate is 2kHz, the noise is Gaussian white noise, the signal and noise are not coherent, and the signal-to-noise ratio \( SNR = 0dB \). The simulation results are shown in figure 2 which also lists the results of the classic algorithms CBF and MVDR under the same conditions.
Figure 2. Comparison of CBF, MVDR and GD-BF under ideal conditions

It can be seen that, compared with CBF and MVDR, the GD-BF algorithm significantly reduces the side lobe height and improves the directivity. The width of the main lobe is the same as that of CBF but weaker than MVDR.

Then set the default simulation conditions for the individual failure experiment of the GD-BF array element, keep the settings under the ideal conditions, set the failure ratio of the array element to $Md = 70\%$, according to the set failure ratio, randomly select the number of the failed array element, and make the selected array element The received signal is 0, and the simulation result is shown in figure 3.

Figure 3. Comparison of CBF, MVDR and GD-BF when $Md=70\%$

It can be seen that when $Md = 70\%$, MVDR cannot complete the angle estimation, the CBF side lobe level is higher, and there are more false peaks, while GD-BF shows better directivity.

Next, perform 100 Monte Carlo tests under the percentage of failures of each element, and calculate the root mean square error of the angle estimation

$$RMSE = \sqrt{\frac{1}{100} \sum_{i=1}^{100} (\hat{\theta}_i - \theta)^2}$$

, and obtain the curve of RMSE with the percentage of failure of the elements, as shown in figure 4.

6
It can be seen that when the array element fails, MVDR loses the ability to estimate the angle, while CBF and GD-BF always maintain a more accurate estimation effect. CBF does not produce a large error until $Md = 90\%$.

Then we study the influence of the failure ratio of the array element on the performance of the algorithm. Because MVDR completely loses its estimation ability under the condition of element failure, its performance is not evaluated here. In this paper, side lobe level, main lobe width and directivity index are selected as evaluation indicators. Carry out 100 Monte Carlo tests on CBF and GD-BF under different percentages of element failure, and plot the average values as shown in figures 5, 6, and 7.

It can be seen that as the proportion of failures increases, the CBF sidelobe level continues to increase from $Md = 15\%$, while the sidelobe level of GD-BF remains stable until the sidelobe level begins to increase after $Md = 70\%$. 
Figure 6. Main lobe width RMSE under different element failure ratios

It can be seen that before $Md = 70\%$, as the proportion of failures increased, the main lobe width of CBF and GD-BF remained stable, and after $Md = 70\%$, the main lobe width of CBF was greatly increased, while the main lobe width of GD-BF remained unchanged.

Figure 7. Directivity index RMSE under different element failure ratios

It can be seen that with the increase in the proportion of element failures, the directivity index of GD-BF is always higher than that of CBF, and the directivity index of CBF is always in a downward trend, while the directivity index of GD-BF is not until after $Md = 70\%$. Began to decline.

6. Conclusion

Based on the analysis of DOA estimation under the condition of array element failure, this paper proposes a GD-BF algorithm based on greedy recovery, and analyzes the influence of the detection performance of the method based on the ratio of element failure. The simulation compares traditional
CBF and MVDR. The DOA estimation performance of the algorithm verifies the effectiveness of the method in this paper. The following indicators can be obtained through research:

1. The performance of traditional CBF is maximized and stable when the element failure ratio is low, and MVDR cannot be used under the condition of element failure;
2. In the face of incomplete received data, GD-BF uses the GD algorithm to quickly and virtually restore the complete data, which improves the effect of linear formation and enhances the ability to identify targets;
3. The simulation analysis shows that the main lobe width, side lobe level, mean directivity index CBF and MVDR of GD-BF have good application value in actual engineering.

References
[1] Krin H, Viberg M. Two decades of array signal processing research: the parametric approach[J]. IEEE Signal Processing Magazine, 1996, 13(4): 67-94
[2] Capon J. High-resolution frequency-wavenumber spectrum analysis[J]. Processing of the IEEE, 1969, 57(8): 1408-1418.
[3] D. L. Donoho. Compressed Sensing[J]. IEEE Trans on Information Theory, 2006, 54(4): 1289-1306
[4] Baraniuk R G. Compressive sensing [lecture notes][J]. IEEE Signal Processing Magazine, 2007, 24(4): 118-121.
[5] Candes E J, Tao T. Decoding by linear programming[J]. IEEE Transactions on Information Theory, 2005, 51(12): 4203-4215.
[6] Candes E. The restricted isometry property and its implications for compressed sensing[J]. Comptes Rendus Mathematique, 2008, 346(8-9): 589-592.
[7] Baraniuk R G. Compressive sensing. IEEE Signal Processing Magazine, 2007, 24(4): 118-121.
[8] Tropp J A, Gilbert A C. Signal recovery from random measurements via orthogonal matching pursuits [J]. IEEE Transactions on Information Theory, 2007, 53(12): 4655-4666.
[9] Huang Zuzhen. Array DOA estimation based on compressed sensing [D]. Harbin, Harbin Institute of Technology, 2013
[10] NEEDELL D, VERSHYNIN R. Uniform uncertainty principle and signal recovery via regularized orthogonal matching pursuit [J]. Foundations of computational mathematics, 2009(3): 317-334.
[11] Jia Wang, Seokbeop Kwon, Byonghyo Shim. Generalized orthogonal matching pursuit, IEEE Transactions on Signal Processing, vol.60, no.12, pp.6202-6216, Dec.2012
[12] NEEDELL D, TROPP J A. CoSaMP: Iterative signal recovery from incomplete and inaccurate samples [J]. Applied and Computational Harmonic Analysis, 2009, 26(3): 301-321.
[13] Dai W, Milenkovic O. Subspace pursuit for compressive sensing signal reconstruction[J]. IEEE Transactions on Information Theory, 2009, 55(5): 2230-2249.
[14] DONOHO D L, TSAIG Y, DRORI I, et al. Sparse solution of underdetermined systems of linear equations by stagewise orthogonal matching pursuit [J]. Information Theory, IEEE Transactions on, 2012, 58(2): 1094-1121.
[15] ThomasBlumensath, Mike E.Davis. Stagewise gradient pursuits [J]. IEEE Transactions on Signal Processing, 2009, 57(11): 4333-4336.