The Student/Project Allocation problem with group projects

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Abstract. We are given a project-based university course, where our task is to assign students to suitable projects. In the Student/Project Allocation problem, or shortly \textsc{spa}, each student submits a weighted list of their acceptable projects, and likewise, a set of assignable students is given to each project. Moreover, some projects are equipped with upper and lower quotas with regards to the number of students assigned to them. The quota requirement is strict: projects not reaching their lower quotas must be closed entirely. The challenge is to find a b-matching of maximum weight observing the lower quotas.

In our present work we show that finding such a maximum weight matching is an \textsc{np}-complete task even if all students mark at most two projects and all projects have upper and lower quota three. Moreover, we also prove that if no upper quota is larger than two, the problem immediately becomes polynomially tractable. For the general case, an approximation algorithm is presented that reaches the best possible ratio – due to inapproximability bounds – expressed in various terms.

Keywords. maximum matching, project allocation, maximum independent vertex set, inapproximability

1 Introduction

In the Student/Project Allocation (\textsc{spa}) problem, we are given a set of students and a set of projects and we seek an assignment of students to projects. Like any assignment problem, \textsc{spa} problem could also be modelled using a bipartite graph, where the set of students and the set of projects will be the two vertex sets of the bipartite graph. An edge between a student and a project exists if the student wishes to be assigned to that project and if the student is qualified to be assigned to the project. A project has an upper and lower quota restriction on the number of students who could be assigned to it.

In a valid assignment, each student has to be assigned to at most one project and the number of students assigned to each project is within its upper and

\ast Supported by the Deutsche Telekom Stiftung. Part of this work was carried out whilst visiting the University of Glasgow.
lower quota. A project with no assigned students is not required to satisfy the restrictions imposed by its upper and lower quota. The instance is also equipped with edge weights. These edge weights help in capturing the individual cardinal utilities of a student or a project and the importance of their association simultaneously. The objective is to find a maximum weight assignment.

From a theoretical point of view, the problem described above constitutes a natural generalization of the maximum weighted $b$-matching problem. We are given a bipartite graph with edge weights, where a $b$-matching satisfying two constraints is to be found: the vertices of the first vertex set are adjacent to at most one matching edge, while the degree in the matching of the remaining vertices is either 0 or it falls between the lower and upper quota of that vertex.

Student/Project Allocation is a well-studied subject. Depending on the system of specific universities and courses, various models have been explored. In many cases, the goal is to find a maximum cardinality assignment. In other cases, if preferences are expressed as ordered lists, the definition of an optimal solution becomes significantly more sophisticated. Several concepts for the optimal matching have been defined, with the prevalent one being stable matchings [2,10,11]. Several experimental studies have been conducted, covering integer programming models [3], heuristic explorations [8], and simulation-based approaches [14]. The experimental works also include implementations that permit visualization [5]. The previous frameworks, however, handled group projects catering to the specific needs of the university. The problem considered in this paper rather captures a very generic setting under which group projects are allocated.

**Organisation.** In Section 2, the complexity of the Student/Project Allocation problem with group projects is studied based on the number of applications a student submits and a project receives. Since the problem is NP-complete in general, an approximation algorithm is presented in Section 3. We also show that the guarantee provided by this algorithm is tight through an inapproximability result. All our proofs exploit the similarity between the SPA problem and the maximum independent vertex set problem, but the individual reductions are quite unique.

## 2 Complexity results for restricted cases

First, we provide a formal definition of the Student/Project Allocation (SPA) problem considered in this paper. Then, we characterize the complexity of the problem in terms of degree constraints on the two vertex sets: students and projects.

In our SPA problem, a set of students $S$ and a set of projects $P$ are given. $S$ and $P$ constitute the two vertex sets of an undirected bipartite graph $G = (V,E)$ with $V = S \cup P$. Student $s$ and project $p$ are connected by an edge $e \in E$ if and only if $s$ wishes to participate in $p$ and is qualified to work in the project. For a vertex $v \in V$ we denote by $\delta(v) = \{\{v,w\} \in E\}$ the set of edges incident to
and by $\Gamma(v) = \{w \in V : \{v, w\} \in E\}$ the neighborhood of $v$, i.e., the set of vertices it is adjacent to. Each edge carries a weight $w : E \to \mathbb{R}_{\geq 0}$, representing the importance of the corresponding assignment. Each project is equipped with a lower quota $\ell : P \to \mathbb{Z}_{\geq 0}$ and an upper quota $u : P \to \mathbb{Z}_{\geq 0}$. These functions bound the number of admissible students for the project (independent of the weight of the corresponding edges). Furthermore, every student can participate in at most one project. Thus, a feasible assignment is a subset $M \subseteq E$ of the edges such that $|\delta(s) \cap M| \leq 1$ for every student $s \in S$ and $|\delta(p) \cap M| \in \{0, \ell(p), \ell(p) + 1, \ldots, u(p)\}$ for every $p \in P$. A project is said to be opened if the number of students assigned to it is greater than 0. Note that without the zero in these sets, the problem immediately becomes tractable. It is easy to see this in the unweighted case as any maximum flow algorithm can be used to determine an optimal solution in polynomial time. This could be naturally extended to the weighted case as the flow based linear program has integral extreme points due to total unimodularity property. The size of an assignment is the number of assigned students $|M|$, while the weight of the assignment is the total weight of the edges in the assignment $w(M) = \sum_{e \in M} w_e$.

In this section, we will mainly consider restricted special cases of SPA, in which the number of applications per student and applicants per project are bounded.

**Problem 1 (WEIGHTED MAX SPA$(i, j)$)**

**Input:** $I = (G, w, \ell, u)$; a weighted SPA instance where $|\Gamma(s)| \leq i$ for every student $s \in S$ and $|\Gamma(p)| \leq j$ for every project $p \in P$, lower and upper quotas.

**Task:** Find a feasible assignment of maximum weight.

The special case of WEIGHTED MAX SPA induced by unit weights on all edges is referred as MAX SPA. Note that even if $i$ and $j$ formally only restrict $|\Gamma(s)|$ and $|\Gamma(p)|$, the degree of the vertices in $G$, they affect the quota functions as well. If the upper quota $u(p)$ for a project $p$ is larger than $j$, then $u(p)$ can be replaced by $j$ without modifying the set of solutions. On the other hand, if $\ell(p) > j$, then project $p$ can immediately be deleted, since no feasible solution can satisfy the lower quota condition.

In order to establish our first $\text{NP}$-completeness result, we briefly mention the maximum independent set problem. This problem will be reduced, later in Theorem 2 to MAX SPA.

**Problem 2 (MIS)**

**Input:** $I = G$; a graph with $n$ vertices and $m$ edges.

**Task:** Find an independent vertex set of maximum size.

**Theorem 1.** [4,6,7,9,15] MIS is not approximable within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless $\text{P} = \text{NP}$. The problem remains $\text{APX}$-complete even for cubic (3-regular) graphs.

**Theorem 2.** MAX SPA$(2, 3)$ is $\text{APX}$-complete.
Proof. First of all, MAX SPA(2, 3) is in APX because feasible solutions are of polynomial size and the problem has a 3-approximation (see Theorem [4]).

To each instance I of MIS on cubic graphs we create an instance I' of MAX SPA such that there is an independent vertex set of size at least K in I if and only if I' admits an assignment of size at least 3K. The construction is as follows. To each of the n vertices of graph G in I, a project with upper and lower quota of three is created. The m edges of G are represented as m students in I'. For each student s ∈ S, |Γ(s)| = 2 and Γ(s) comprises the two projects representing the two end vertices of the corresponding edge. Since we work on cubic graphs, |Γ(p)| = 3 for every project p ∈ P.

First we show that an independent vertex set of size K can be transformed into an assignment of at least 3K students. All we need to do is to start a project with its entire neighborhood assigned to it if and only if the vertex representing that project is in the independent set. Since no two projects stand for adjacent vertices in G, their neighborhoods do not intersect. Moreover, the assignment allocates exactly three students to each of the K opened projects.

To establish the opposite direction, let us assume that an assignment of cardinality 3K is given. The projects’ upper and lower quota are both set to three, therefore, the assignment involves exactly K opened projects. No two of them can represent adjacent vertices in G, because then the student standing for the edge connecting them would be assigned to both projects at the same time.

So far we have established that if |Γ(s)| ≤ 2 for every student s ∈ S and |Γ(p)| ≤ 3 for every project p ∈ P, then MAX SPA, and therefore, WEIGHTED MAX SPA are NP-complete problems. In the following, we also show that these restrictions are the tightest possible. If |Γ(p)| ≤ 2 for every project p ∈ P, then a maximum weight matching can be found efficiently, regardless of |Γ(s)|. In our proof, we use the maximum independent set problem again, but in a different context. To each WEIGHTED MAX SPA instance, a vertex-weighted graph is created. Due to the special properties of our construction, finding a maximal weight independent set in the vertex weighted graph is equivalent to finding a maximum weight assignment in the original problem. Moreover, the graph constructed is claw-free, therefore, maximum weight independent sets can be found efficiently on it.

**Theorem 3.** WEIGHTED MAX SPA(∞, 2) with edge weights is solvable in polynomial time.

Proof. First, we simplify the given input. If |Γ(p)| ≤ 2 for every p ∈ P, it is sufficient to consider the case when the upper and lower capacities are bounded from above by two. Naturally, all projects with |Γ(p)| < ℓ(p) are dropped. Moreover, projects with ℓ(p) = 1 behave identically to projects without a lower quota. The last case we consider is ℓ(p) = 0 (or ℓ(p) = 1) and u(p) = 2 and |Γ(p)| = 2. Then, we split p into two projects with upper quota 1. After this step, all projects with upper quota 2 also have a lower quota of 2. All remaining vertices are of upper quota 1.
From this graph we construct a helper graph $H$ with vertex weights. We show that the weight of any independent vertex set $V'$ corresponds to the weight of a feasible assignment. The construction of graph $H$ is illustrated on a sample instance in Figure 1.

The vertex set of $H$ is partitioned into three disjoint sets. Each vertex in the first group corresponds to a project with $\ell(p) = u(p) = 2$ in $G$. These vertices in $H$ have weight $w(s_1, p) + w(s_2, p)$, where $\Gamma(p) = \{s_1, s_2\}$. If such a vertex is in the chosen independent vertex set $V'$ in $H$, then $p$ is opened in $G$, and both $s_1$ and $s_2$ are assigned to it. Vertices in the second set correspond to project vertices in $G$ with $|\Gamma(p)| = 1$, and therefore, $u(p) = 1$. Vertices of $H$ in this second set get the weight $w(s, p)$, where $\Gamma(p) = \{s\}$. Trivially, such a vertex in $V'$ means to assign $s$ to $p$. The last case is when a project with $u(p) = 1$ and $|\Gamma(p)| = 2$ occurs in $G$. To ensure that at most one of the applied students $s_1, s_2$ is assigned to $p$, we create two vertices in $H$, connected by an edge. One vertex represents $s_1$ being assigned to $p$, and the other vertex represent the assignment $s_2$ to $p$. Therefore, they take the weights $w(s_1, p)$ and $w(s_2, p)$, respectively. The number of vertices in $H$ cannot exceed $2|P|$.

Having fixed all vertices and some edges of $H$, we now draw the remaining edges of the graph. Each student $s$ in $G$ contributes with a clique of edges, connecting all the projects in $\Gamma(s)$. For the projects that were split in $H$, only one copy is in the clique - the one corresponding to $s$. This ensures that a student is assigned to at most one project.

As a next step, we show that a feasible assignment $M$ of weight $W$ in $G$ corresponds to an independent vertex set $V'$ of weight $W$ in $H$ and vice versa. Let us suppose that we are given an assignment. We will now construct the independent set $V'$ for $H$ corresponding to this assignment. Each opened project in the assignment with $\ell(p) = u(p) = 2$ is chosen in $V'$, and they carry the weight of their two edges in $\Gamma(p)$. The remaining opened projects that are assigned two students appear as two unconnected vertices in $H$, and they both are included in $V'$. Similarly, the opened projects with $u(p) = 1$ and $\Gamma(p) = \{s\}$ count as a vertex of weight $w(s, p)$ in $H$ are also included in $V'$. The very last case is formed by opened projects with $u(p) = 1$ and $\Gamma(p) = \{s_1, s_2\}$, one of $\{s_1, s_2\}$ chosen in our assignment. In $H$, each application is represented by a vertex with a weight $w(s_1, p)$ or $w(s_2, p)$. We choose the vertex corresponding to the edge that is in the assignment in $V'$.

For the other direction, we show how to construct an assignment $M$ in $G$, when we are given an independent vertex set $V'$ of $H$. As described above, some vertices of $H$ represent a single edge in $G$, while other vertices of $H$ represent a set of two edges in $G$. Choosing the edges in $M$ that correspond to the vertices in $H$ results in an assignment of the same weight as the vertex set in $H$. Due to the construction of edges in $H$, no student is assigned to more than one project.

Finding a maximum weight independent vertex set is in general NP-hard. In some special cases, an optimal solution can be found efficiently. One of these special cases is when the graph is claw-free. claw is a graph comprised of four vertices, three of them with degree one and one with degree three. A claw-free
graph does not contain a claw as an induced subgraph. On claw-free graphs, maximum weight independent vertex sets can be found in $O(n^3)$ time [12,13].

All that remains is to show that $H$ does not contain any claws. Let us consider any vertex of $H$ having degree at least three. All these edges come from cliques or the edges added to avoid the assignment of two students to a project with $u(p) = 1$. If the vertex has an edge of this latter type, it has at most one student, thus, it is part of at most one clique. Otherwise, since $|\Gamma(p)| \leq 2$, the vertices representing them are in at most two cliques. Finally, at least two of the three edges must come from the same clique. In a claw, the other end vertices of the three chosen edges do not share edges.

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Fig. 1. The original instance with weights marked on the edges and quotas $(u, \ell)$ marked to the right of the vertices, the simplified instance with upper quotas and the helper graph with vertex weights.

3 Approximation in the general case

Having established the hardness of MAX SPA even for very restricted instances in Theorem 2, we turn our attention towards approximability. In this section, we give an approximation algorithm and corresponding inapproximability bounds expressed in terms of vertex set cardinalities and upper quotas in the graph.
The method, which is formally listed in Algorithm 1, is a simple greedy algorithm. We say a project $p$ is admissible if it is not yet open and $|\Gamma(p)| \geq \ell(p)$. The algorithm iteratively opens an admissible project maximizing the assignable weight, i.e., it finds a project $p' \in P$ and a set $S'$ of students in its neighborhood $\Gamma(p')$ with $\ell(p') \leq |S'| \leq u(p')$ such that $\sum_{s \in S'} w(s, p')$ is maximized among all such pairs. It then removes the assigned students from the graph (potentially rendering some project in-admissible) and re-iterates until no admissible project is left.

Algorithm 1 Greedy algorithm for weighted max spa

| Initialize $P_0 = \{p \in P : |\Gamma(p)| > \ell(p)\}$. |
| Initialize $S_0 = S$. |
| while $P_0 \neq \emptyset$ do |
| Find a pair $p' \in P_0$ and $S' \subseteq \Gamma(p')$ with $|S'| \leq u(p')$ such that $\sum_{s \in S'} w(s, p')$ is maximized among all such pairs. |
| Open $p'$ and assign all students in $S'$ to it. |
| Remove $p'$ from $P_0$ and remove the elements of $S'$ from $S_0$. |
| for $p \in P_0$ with $\ell(p) > |\Gamma(p) \cap S_0|$ do |
| Remove $p$ from $P_0$. |
| end for |
| end while |

In the following, we give a tight analysis of the algorithm, establishing approximation guarantees in terms of the number of projects $|P|$, number of students $|S|$, and the maximum upper quota $u_{\text{max}} := \max_{p \in P} u(p)$ over all projects. We remark that for the latter quantity we can assume $u_{\text{max}} \leq \max_{p \in P} |\Gamma(p)|$ without loss of generality, as for any project $p$ with $u(p) > |\Gamma(p)|$ we can decrease $u(p)$ to $|\Gamma(p)|$ without changing the set of feasible solutions.

**Theorem 4.** Algorithm 1 is an $\alpha$-approximation algorithm for weighted max spa with $\alpha = \min\{|P|, |S|, u_{\text{max}} + 1\}$. Furthermore, Algorithm 1 is a $\sqrt{|S|} + 1$-approximation algorithm for (unweighted) max spa. It can be implemented to run in time $O(|E| \log |E|)$.

**Proof.** Let $p'_1, \ldots, p'_k$ be the projects chosen by the algorithm and let $S'_1, \ldots, S'_k$ be the corresponding sets of students. Furthermore, consider an optimal solution of weight OPT, consisting of open projects $p_1, \ldots, p_k$ and the corresponding sets of students $S_1, \ldots, S_k$ assigned to those projects.

We first observe that the first two approximation ratios of $|P|$ and $|S|$ are already achieved by the initial selection of $p'_1$ and $S'_1$ chosen in the first round of the algorithm: For every $i \in \{1, \ldots, k\}$, project $p_i$ is an admissible project in the first iteration of the algorithm. The first iteration’s choice of the pair $(p'_1, S'_1)$ implies $\sum_{s \in S'_1} w(s, p'_1) \geq \sum_{s \in S_i} w(s, p_i) \geq w(s', p_i)$ for every $s' \in S_i$. As the optimal solution opens at most $|P|$ projects and serves at most $|S|$ students, we deduce that $\sum_{s \in S'_i} w(s, p'_i) \geq \min\{|P|, |S|\} \cdot \text{OPT}$.
We now turn our attention to the remaining approximation guarantees, which are $u_{\text{max}} + 1$ for weighted max spa and $\sqrt{|S|} + 1$ for the unweighted case. For every $i \in \{1, \ldots, k\}$, let $\pi(i)$ denote the first iteration of the algorithm such that $S'_{\pi(i)} \cap S_i \neq \emptyset$ or $p'_{\pi(i)} = p_i$. This iteration is the one in which project $p_i$ is opened or a student assigned to it in the optimal solution becomes assigned. Note that such an iteration exists, because $p_i$ is not admissible after the termination of the algorithm. Furthermore, observe that $\sum_{s \in S'_{\pi(i)}} w(s, p'_{\pi(i)}) \geq \sum_{s \in S_i} w(s, p_i)$, because the pair $(p_i, S_i)$ was a valid choice for the algorithm in iteration $\pi(i)$.

Now for iteration $j$ define $P_j := \{i : \pi(i) = j\}$ and observe that $|P_j| \leq |S'_j| + 1$, because $P_j$ can only contain one index $i'$ with $p_{i'} = p'_j$, and all other $i \in P_j \setminus \{i'\}$ must have $S_i \cap S'_j \neq \emptyset$ (where the sets $S_i$ are disjoint). We conclude that

$$\begin{align*}
\text{OPT} &= \sum_{i=1}^{k} \sum_{s \in S_i} w(s, p_i) \leq \sum_{i=1}^{k} \sum_{s \in S'_{\pi(i)}} w(s, p'_{\pi(i)}) \\
&\leq \sum_{j=1}^{\ell} |P_j| \sum_{s \in S'_j} w(s, p'_j) \leq \sum_{j=1}^{\ell} (|S'_j| + 1) \sum_{s \in S'_j} w(s, p'_j).
\end{align*}$$

Note that $|S'_j| \leq u_{\text{max}}$ and therefore $\text{OPT} \leq (u_{\text{max}} + 1) \sum_{j=1}^{\ell} \sum_{s \in S'_j} w(s, p'_j)$, proving the third approximation guarantee. Now consider the case that $w(s, p) = 1$ for all $p \in P$ and $s \in S$ and define $S' = \bigcup_{j=1}^{\ell} S'_j$. If $|S'| \geq \sqrt{S}$, then $\sqrt{S}|S'| \geq |S| \geq \text{OPT}$. Therefore assume $|S'| < \sqrt{S}$. Note that in this case, the above inequalities imply $\text{OPT} \leq (|S'| + 1)|S'| \leq (\sqrt{S} + 1)|S'|$, proving the improved approximation guarantee for the unweighted case.

We now turn to proving the bound on the running time. We will describe how to implement the search for the greedy choice of the pair $(p', S')$ in each iteration efficiently using a heap data structure. Initially, for every project $p$, we sort the students in its neighborhood by non-increasing order of $w(s, p)$. This takes time at most $O(|E| \log |E|)$ as the total number of entries to sort is $\sum_{p \in P} |\Gamma(p)| = |E|$. We then introduce a heap containing all admissible projects, and associate with each project $p$ the total weight of the first $w(p)$ edges in its neighborhood list. Note that these entries can be easily kept up to date by simply replacing students assigned to other projects with the first not-yet-assigned entry in the neighborhood list (or removing the project if less than $\ell(p)$ students are available). As every edge in the graph can only trigger one such replacement, only $O(|E|)$ updates can occur and each of these requires $O(\log |P|)$ time for reinserting the project at the proper place in the heap. Now, in each iteration of the algorithm, the optimal pair $(p', S')$ can be found by retrieving the maximum element from the heap. This happens at most $|P|$ times and requires time $O(\log(|P|))$ in each step.

\[\square\]

Example 5 The following two examples show that our analysis of the greedy algorithm is tight for each of the described approximation factors.
Consider an instance of \textsc{max spa} with $k+1$ projects $p_0, \ldots, p_k$ and $k(k+1)$ students $s_{0,1}, \ldots, s_{0,k}, s_{1,1}, \ldots, s_{k,k}$. Let $\ell(p_i) = u(p_i) = k$ for $i \in \{0, \ldots, k\}$. Each student $s_{i,j}$ applies for projects $i$ and additionally to project $0$. For the greedy algorithm, opening project $p_0$ and assigning students $s_{1,1}, \ldots, s_{k,k}$ to it is a valid choice in its first iteration, after which now further projects are admissible. Thus, it only serves $k$ students in total. The optimal solution, however, can serve all $k(k+1)$ students by assigning students $s_{i,1}, \ldots, s_{i,k}$ to $p_i$ for each $i$. Therefore, the greedy algorithm cannot achieve an approximation factor better than $k+1$ on this family of instances, for which $|P| = k+1$, $\sqrt{|S|} < k+1$, and $u_{\text{max}} = k$.

To see that the approximation ratio of $|S|$ is tight for \textsc{weighted max spa} consider the following instance with $k$ projects $p_1, \ldots, p_k$ and $k$ students $s_1, \ldots, s_k$. Let $\ell(p_i) = 0$ and $u(p_i) = k$ for every $i$. Every student applies for every project, and $w(s_i, p_i) = 1$ for every $i$ but $w(s_i, p_i) = 0$ for every $j \neq i$. In its first iteration, the greedy algorithm might choose to open project $p_1$ and assign all students to it. This solution accumulates a weight of 1, while the weight of the optimal solution is $k = |S|$.

We further show that all approximation ratios given above for \textsc{max spa} are tight from the point of view of complexity theory.

**Theorem 6.** \textsc{max spa} is not approximable within a factor of $|P|^{1-\varepsilon}$ or $\sqrt{|S|}^{1-\varepsilon}$ or $u_{\text{max}}^{1-\varepsilon}$ for any $\varepsilon > 0$, unless $P = \text{NP}$, even when restricting to instances where $\ell(p) = u(p)$ for every $p \in P$ and $|\Gamma(s)| \leq 2$ for every $s \in S$.

**Proof.** Once again we use the maximum independent vertex set problem. Given an instance of \textsc{mis} on a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, we create a \textsc{max spa} instance with $n$ projects $p_1, \ldots, p_n$, project $p_i$ corresponding to vertex $v_i$. We also introduce $n^2 - m$ students as follows. Initially, we introduce $n$ students $s_{1,1}, s_{1,2}, \ldots, s_{1,n}$ applying for each project $p_1$. Then, for every edge $\{v_i, v_j\} \in E$, we merge the students $s_{i,j}$ and $s_{j,i}$, obtaining a single student applying for both $p_i$ and $p_j$. Furthermore, we set $\ell(p_j) = u(p_j) = n$ for every project.

Note that due to the choice of upper and lower bounds, any open project must be assigned to all the students in its neighborhood. Thus, a solution to the \textsc{max spa} instance is feasible if and only if $\Gamma(p_i) \cap \Gamma(p_j) = \emptyset$ for all open projects $p_i$ and $p_j$ with $i \neq j$, which is equivalent to $v_i$ and $v_j$ not being adjacent in $G$ by construction of the instance. Therefore, the \textsc{max spa} instance has a feasible solution opening $k$ projects (and thus serving $kn$ students) if and only if there is an independent set of size $k$ in the $G$. We conclude that $\text{OPT}_{\text{spa}} = n \cdot \text{OPT}_{\text{MIS}}$ for the two instances under consideration.

Note that in the constructed \textsc{max spa} instance, $n = |P| = u_{\text{max}} = \sqrt{|S|}$.

Therefore any approximation algorithm with a factor better than $|P|^{1-\varepsilon}$ or $\sqrt{|S|}^{1-\varepsilon}$ or $u_{\text{max}}^{1-\varepsilon}$ for $\varepsilon > 0$ yields a solution of the instance serving at least $n^{1-\varepsilon} \text{OPT}_{\text{spa}}$ students, thus opening at least $n^{-\varepsilon} \text{OPT}_{\text{spa}} = n^{1-\varepsilon} \text{OPT}_{\text{MIS}}$ projects, corresponding to an independent set of the same size. By [13], this implies $P = \text{NP}$. \hfill $\square$
4 Conclusion

In this paper, the complexity and approximation of the Student/Project Allocation problem with group projects were discussed. Regarding the complexity of the maximization problem, we drew the line between polynomialsolvable and hard cases expressed in terms of the degree of vertices. For the general case, we also provided a greedy approximation algorithm that proved to be the best reachable in various aspects.

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